Topology

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May 31, 2022

Contents

1 Set Theory

Definition 1 (Cover)**.** Let X be a set and $\mathcal{A} \subseteq \mathcal{P}X$. Then $\mathcal{A}$ *covers* X, or is a *covering* of X, if and only if $\bigcup \mathcal{A} = X$.

Definition 2 (Finite Intersection Property)**.** Let X be a set and $\mathcal{A} \subseteq \mathcal{P}X$. Then $\mathcal{A}$ satisfies the *finite intersection property* if and only if every nonempty finite subset of $\mathcal{A}$ has nonempty intersection.

Lemma 3. *Let X be a set. Let $\mathcal{A} \subseteq \mathcal{P}X$ have the finite intersection property. Then there exists a maximal set $\mathcal{D}$ such that $\mathcal{A} \subseteq \mathcal{D} \subseteq \mathcal{P}X$ and $\mathcal{D}$ has the finite intersection property.*

PROOF:
⟨1⟩1. LET: $\mathbb{F} = \{\mathcal{D} \mid \mathcal{A} \subseteq \mathcal{D} \subseteq \mathcal{P}X, \mathcal{D}$ has the finite intersection property$\}$
⟨1⟩2. Every chain in $\mathbb{F}$ has an upper bound.
  ⟨2⟩1. LET: $\mathbb{C}$ be a chain in $\mathbb{F}$.
  ⟨2⟩2. ASSUME: without loss of generality $\mathbb{C} \neq \emptyset$
    PROVE: $\bigcup \mathbb{C} \in \mathbb{F}$
   PROOF: If $\mathbb{C} = \emptyset$ then $\mathcal{A}$ is an upper bound.
  ⟨2⟩3. $\mathcal{A} \subseteq \bigcup \mathbb{C} \subseteq \mathcal{P}X$
  ⟨2⟩4. LET: $C_1, \ldots, C_n \in \mathbb{C}$
    PROVE: $C_1 \cap \cdots \cap C_n \neq \emptyset$
  ⟨2⟩5. PICK $\mathcal{C}_1, \ldots, \mathcal{C}_n \in \mathbb{C}$ such that $C_i \in \mathcal{C}_i$ for all i.
  ⟨2⟩6. ASSUME: without loss of generality $\mathcal{C}_1 \subseteq \cdots \subseteq \mathcal{C}_n$
  ⟨2⟩7. $C_1, \ldots, C_n \in \mathcal{C}_n$
  ⟨2⟩8. $\mathcal{C}_n$ satisfies the finite intersection property.
  ⟨2⟩9. $C_1 \cap \cdots \cap C_n \neq \emptyset$
⟨1⟩3. Q.E.D.
  PROOF: By Zorn's Lemma.
□

Lemma 4. *Let X be a set. Let $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the finite intersection property. Then any finite intersection of elements of $\mathcal{D}$ is an element of $\mathcal{D}$.*

PROOF:
⟨1⟩1. LET: $D_1, D_2 \in \mathcal{D}$
⟨1⟩2. $\mathcal{D} \cup \{D_1 \cap D_2\}$ has the finite intersection property.
  PROOF: Any finite intersection of members of $\mathcal{D} \cup \{D_1 \cap D_2\}$ is a finite intersection of members of $\mathcal{D}$.
⟨1⟩3. $\mathcal{D} = \mathcal{D} \cup \{D_1 \cap D_2\}$
  PROOF: By maximality of $\mathcal{D}$.
⟨1⟩4. $D_1 \cap D_2 \in \mathcal{D}$.
□

Lemma 5. *Let X be a set. Let $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the finite intersection property. Let $A \subseteq X$. If A intersects every member of $\mathcal{D}$ then $A \in \mathcal{D}$.*

Proof:
⟨1⟩1. $\mathcal{D} \cup \{A\}$ has the finite intersection property.
⟨2⟩1. Let: $D_1, \ldots, D_n \in \mathcal{D}$
Prove: $D_1 \cap \cdots \cap D_n \cap A \neq \emptyset$
⟨2⟩2. $D_1 \cap \cdots \cap D_n \in \mathcal{D}$
Proof: Lemma 4.
⟨2⟩3. $D_1 \cap \cdots \cap D_n \cap A \neq \emptyset$
Proof: Since A intersects every member of $\mathcal{D}$.
⟨1⟩2. Q.E.D.
Proof: By maximality of $\mathcal{D}$.
□

Proposition 6. *Let X be a set. Let $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the finite intersection property. Let $A, D \in \mathcal{P}X$. If $D \in \mathcal{D}$ and $D \subseteq A$ then $A \in \mathcal{D}$.*

Proof:
⟨1⟩1. $\mathcal{D} \cup \{A\}$ satisfies the finite intersection property.
⟨2⟩1. Let: $D_1, \ldots, D_n \in \mathcal{D}$
⟨2⟩2. $D_1 \cap \cdots \cap D_n \cap D \neq \emptyset$
Proof: Since $\mathcal{D}$ satisfies the finite intersection property.
⟨2⟩3. $D_1 \cap \cdots \cap D_n \cap A \neq \emptyset$
⟨1⟩2. $\mathcal{D} = \mathcal{D} \cup \{A\}$
Proof: By the maximality of $\mathcal{D}$.
⟨1⟩3. $A \in \mathcal{D}$
□

Definition 7 (Graph)**.** Let $f : A \to B$. The *graph* of f is the set $\{(x, f(x)) \mid x \in A\} \subseteq A \times B$.

2 Countable Intersection Property

Definition 8 (Countable Intersection Property)**.** Let X be a set and $\mathcal{A} \subseteq \mathcal{P}X$. Then $\mathcal{A}$ satisfies the *countable intersection property* if and only if every countable subset of $\mathcal{A}$ has nonempty intersection.

Lemma 9. *Let X be a set. Let $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the countable intersection property. Then any countable intersection of elements of $\mathcal{D}$ is an element of $\mathcal{D}$.*

Proof:
⟨1⟩1. Let: $\mathcal{D}_0 \subseteq \mathcal{D}$ be countable.
⟨1⟩2. $\mathcal{D} \cup \{\bigcap \mathcal{D}_0\}$ has the countable intersection property.
Proof: Any countable intersection of members of $\mathcal{D} \cup \{\bigcap \mathcal{D}_0\}$ is a finite intersection of members of $\mathcal{D}$.
⟨1⟩3. $\mathcal{D} = \mathcal{D} \cup \{\bigcap \mathcal{D}_0\}$
Proof: By maximality of $\mathcal{D}$.
⟨1⟩4. $\bigcap \mathcal{D}_0 \in \mathcal{D}$.

□

Lemma 10. *Let X be a set. Let $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the countable intersection property. Let $A \subseteq X$. If A intersects every member of $\mathcal{D}$ then $A \in \mathcal{D}$.*

PROOF:
$\langle 1 \rangle 1$. $\mathcal{D} \cup \{A\}$ has the countable intersection property.
  $\langle 2 \rangle 1$. LET: $\mathcal{D}_0 \subseteq \mathcal{D}$ be countable.
    PROVE: $\bigcap \mathcal{D}_0 \cap A \neq \emptyset$
  $\langle 2 \rangle 2$. $\bigcap \mathcal{D}_0 \in \mathcal{D}$
    PROOF: Lemma 9.
  $\langle 2 \rangle 3$. $\bigcap \mathcal{D}_0 \cap A \neq \emptyset$
    PROOF: Since A intersects every member of $\mathcal{D}$.
$\langle 1 \rangle 2$. Q.E.D.
  PROOF: By maximality of $\mathcal{D}$.
□

3 Order Theory

Definition 11 (Preorder)**.** Let X be a set. A *preorder* on X is a binary relation $\leq$ on X such that:

Reflexivity For all $x \in X$, we have $x \leq x$

Transitivity For all $x, y, z \in X$, if $x \leq y$ and $y \leq z$ then $x \leq z$.

Definition 12 (Preordered Set)**.** A *preordered set* consists of a set X and a preorder $\leq$ on X.

Proposition 13. *Let X and Y be linearly ordered sets. Let $f : X \twoheadrightarrow Y$ be strictly monotone and surjective. Then f is a poset isomorphism.*

PROOF:
$\langle 1 \rangle 1$. f is injective.
  $\langle 2 \rangle 1$. LET: $x, y \in X$
  $\langle 2 \rangle 2$. ASSUME: $f(x) = f(y)$
  $\langle 2 \rangle 3$. $x \nless y$
    PROOF: By strong motonicity.
  $\langle 2 \rangle 4$. $y \nless x$
    PROOF: By strong motonicity.
  $\langle 2 \rangle 5$. $x = y$
    PROOF: By trichotomy.
$\langle 1 \rangle 2$. f^{-1} is monotone.
  $\langle 2 \rangle 1$. LET: $x, y \in X$
  $\langle 2 \rangle 2$. ASSUME: $x \leq y$
  $\langle 2 \rangle 3$. $f^{-1}(x) \ngtr f^{-1}(y)$
    PROOF: By strong motonicity.
  $\langle 2 \rangle 4$. $f^{-1}(x) < f^{-1}(y)$
    PROOF: By trichotomy.
□

Definition 14 (Interval)**.** Let X be a preordered set and $Y \subseteq X$. Then Y is an *interval* if and only if, for all $a, b \in Y$ and $c \in X$, if $a \leq c \leq b$ then $c \in Y$.

Definition 15 (Linear Continuum)**.** A linearly ordered set L is a *linear continuum* if and only if:

1. every nonempty subset of L that is bounded above has a supremum
2. L is dense

Proposition 16. *Every interval in a linear continuum is a linear continuum.*

PROOF:
$\langle 1 \rangle 1$. LET: L be a linear continuum and I an interval in L.
$\langle 1 \rangle 2$. Every nonempty subset of I that is bounded above has a supremum in I.
  $\langle 2 \rangle 1$. LET: $X \subseteq I$ be nonempty and bounded above by $b \in I$.

⟨2⟩2. LET: s be the supremum of X in L.
PROOF: Since L is a linear continuum.
⟨2⟩3. $s \in I$
⟨3⟩1. PICK $a \in X$
PROOF: Since X is nonempty (⟨2⟩1).
⟨3⟩2. $a \leq s \leq b$
⟨3⟩3. $s \in I$
PROOF: Since I is an interval (⟨1⟩1).
⟨2⟩4. s is the supremum of X in I
⟨1⟩3. I is dense.
⟨2⟩1. LET: $x, y \in I$ with $x < y$
⟨2⟩2. PICK $z \in L$ with $x < z < y$
PROOF: Since L is dense.
⟨2⟩3. $z \in I$
PROOF: Since I is an interval.
□

Definition 17 (Ordered Square)**.** The *ordered square* I_o^2 is the set $[0,1]^2$ under the dictionary order.

Proposition 18. *The ordered square is a linear continuum.*

PROOF:
⟨1⟩1. Every nonempty subset of I_o^2 bounded above has a supremum.
⟨2⟩1. LET: $X \subseteq I_o^2$ be nonempty and bounded above by (b, c)
⟨2⟩2. LET: $s = \sup \pi_1(X)$
PROOF: The set $\pi_1(X)$ is nonempty and bounded above by b.
⟨2⟩3. CASE: $s \in \pi_1(X)$
⟨3⟩1. LET: $t = \sup\{y \in [0,1] \mid (s, y) \in X\}$
PROOF: This set is nonempty and bounded above by c.
⟨3⟩2. (s, t) is the supremum of X.
⟨2⟩4. CASE: $s \notin \pi_1(X)$
PROOF: In this case $(s, 0)$ is the supremum of X.
⟨1⟩2. I_o^2 is dense.
⟨2⟩1. LET: $(x_1, y_1), (x_2, y_2) \in I_o^2$ with $(x_1, y_1) < (x_2, y_2)$
⟨2⟩2. CASE: $x_1 < x_2$
⟨3⟩1. PICK x_3 with $x_1 < x_3 < x_2$
⟨3⟩2. $(x_1, y_1) < (x_3, y_1) < (x_2, y_2)$
⟨2⟩3. CASE: $x_1 = x_2$ and $y_1 < y_2$
⟨3⟩1. PICK y_3 with $y_1 < y_3 < y_2$
⟨3⟩2. $(x_1, y_1) < (x_1, y_3) < (x_2, y_2)$
□

Proposition 19. *If X is a well-ordered set then $X \times [0,1)$ under the dictionary order is a linear continuum.*

PROOF:
⟨1⟩1. Every nonempty set $A \subseteq X \times [0,1)$ bounded above has a supremum.

⟨2⟩1. LET: $A \subseteq X \times [0,1)$ be nonempty and bounded above
⟨2⟩2. LET: x_0 be the supremum of $\pi_1(A)$
⟨2⟩3. CASE: $x_0 \in \pi_1(A)$
⟨3⟩1. LET: y_0 be the supremum of $\{y \in [0,1) \mid (x_0, y) \in A\}$
⟨3⟩2. (x_0, y_0) is the supremum of A.
⟨2⟩4. CASE: $x_0 \notin \pi_1(A)$
PROOF: In this case $(x_0, 0)$ is the supremum of A.
⟨1⟩2. $X \times [0,1)$ is dense.
⟨2⟩1. LET: $(x_1, y_1), (x_2, y_2) \in X \times [0,1)$ with $(x_1, y_1) < (x_2, y_2)$
⟨2⟩2. CASE: $x_1 < x_2$
⟨3⟩1. PICK y_3 such that $y_1 < y_3 < 1$
⟨3⟩2. $(x_1, y_1) < (x_1, y_3) < (x_2, y_2)$
⟨2⟩3. CASE: $x_1 = x_2$ and $y_1 < y_2$
⟨3⟩1. PICK y_3 such that $y_1 < y_3 < y_2$
⟨3⟩2. $(x_1, y_1) < (x_1, y_3) < (x_2, y_2)$
□

Lemma 20. *For all $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$, we have $[a, b) \cong [c, d)$*

PROOF: The map $\lambda t.c + (t - a)(d - c)/(b - a)$ is an order isomorphism.

Proposition 21. *Let X be a linearly ordered set. Let $a < b < c$ in X. Then $[a, c) \cong [0,1)$ if and only if $[a, b) \cong [b, c) \cong [0,1)$.*

PROOF:
⟨1⟩1. If $[a, c) \cong [0,1)$ then $[a, b) \cong [b, c) \cong [0,1)$
⟨2⟩1. ASSUME: $f : [a, c) \cong [0,1)$ is an order isomorphism
⟨2⟩2. $[a, b) \cong [0,1)$
PROOF:
$$\begin{aligned} [a, b) &\cong [0, f(b)) && \text{(by the restriction of } f\text{)} \\ &\cong [0,1) && \text{(Lemma 20)} \end{aligned}$$
⟨2⟩3. $[b, c) \cong [0,1)$
PROOF: Similar.
⟨1⟩2. If $[a, b) \cong [b, c) \cong [0,1)$ then $[a, c) \cong [0,1)$
PROOF:
$$\begin{aligned} [a, c) &= [a, b) * [b, c) \\ &\cong [0,1) * [0,1) \\ &\cong [0,1/2) * [1/2,1) && \text{(Lemma 20)} \\ &= 1 \end{aligned}$$
□

Proposition 22 (CC)**.** *Let X be a linearly ordered set. Let $x_0 < x_1 < \cdots$ be a strictly increasing sequence in X with supremum b. Then $[x_0, b) \cong [0,1)$ if and only if $[x_i, x_{i+1}) \cong [0,1)$ for all i.*

PROOF:
⟨1⟩1. If $[x_0, b) \cong [0,1)$ then $[x_i, x_{i+1}) \cong [0,1)$ for all i.

PROOF: By Lemma 20

$\langle 1 \rangle 2$. If $[x_i, x_{i+1}) \cong [0,1)$ for all i then $[x_0, b) \cong [0,1)$

$\langle 2 \rangle 1$. ASSUME: $[x_i, x_{i+1}) \cong [0,1)$ for all i

$\langle 2 \rangle 2$. PICK an order isomorphism $f_i : [x_i, x_{i+1}) \cong [1/2^i, 2/2^{i+1})$ for each i.

PROOF: By Lemma 20

$\langle 2 \rangle 3$. The union of the f_is is an order isomorphism $[x_0, b) \cong [0,1)$

$\square$

4 Real Analysis

Definition 23. Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences (a_n) such that $a_n = 0$ for all but finitely many n.

5 Group Theory

Definition 24. Given a group G and sets $A, B \subseteq G$, let $AB = \{ab \mid a \in A, b \in B\}$.

Definition 25. Given a group G and a set $A \subseteq G$, let $A^{-1} = \{a^{-1} \mid a \in A\}$.

6 Topological Spaces

Definition 26 (Topology)**.** A *topology* on a set X is a set $\mathcal{T} \subseteq \mathcal{P}X$ such that:

- $X \in \mathcal{T}$.
- For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$.
- For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$.

We call the elements of X *points* and the elements of $\mathcal{T}$ *open sets*.

Definition 27 (Topological Space)**.** A *topological space* X consists of a set X and a topology on X.

Definition 28 (Discrete Space)**.** For any set X, the *discrete* topology on X is $\mathcal{P}X$.

Definition 29 (Indiscrete Space)**.** For any set X, the *indiscrete* or *trivial* topology on X is $\{\emptyset, X\}$.

Definition 30 (Finite Complement Topology)**.** For any set X, the *finite complement topology* on X is $\{U \in \mathcal{P}X \mid X \setminus U \text{ is finite}\} \cup \{\emptyset\}$.

Definition 31 (Countable Complement Topology)**.** For any set X, the *countable complement topology* on X is $\{U \in \mathcal{P}X \mid X \setminus U \text{ is countable}\} \cup \{\emptyset\}$.

Definition 32 (Finer, Coarser)**.** Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a given set X. If $\mathcal{T}' \supseteq \mathcal{T}$, we say that $\mathcal{T}'$ is *finer* than $\mathcal{T}$; if $\mathcal{T}'$ *properly* contains $\mathcal{T}$, we say that $\mathcal{T}'$ is *strictly* finer than $\mathcal{T}$. We also say that $\mathcal{T}$ is *coarser* than $\mathcal{T}'$, or *stricly* coarser, in these two respective situations. We say $\mathcal{T}$ is *comparable* with $\mathcal{T}'$ if either $\mathcal{T}' \supseteq \mathcal{T}$ or $\mathcal{T} \supseteq \mathcal{T}'$.

Lemma 33. *Let X be a topological space and $U \subseteq X$. Then U is open if and only if, for all $x \in U$, there exists an open set V such that $x \in V \subseteq U$.*

PROOF:
$\langle 1 \rangle 1.$ $\Rightarrow$
  PROOF: Take $V = U$
$\langle 1 \rangle 2.$ $\Leftarrow$
  PROOF: We have $U = \bigcup \{V \text{ open in } X \mid V \subseteq U\}$.
□

Lemma 34. *Let X be a set and $\mathcal{T}$ a nonempty set of topologies on X. Then $\bigcap \mathcal{T}$ is a topology on X, and is the finest topology that is coarser than every member of $\mathcal{T}$.*

PROOF:
$\langle 1 \rangle 1.$ $X \in \bigcap \mathcal{T}$
  PROOF: Since X is in every member of $\mathcal{T}$.
$\langle 1 \rangle 2.$ $\bigcap \mathcal{T}$ is closed under union.

⟨2⟩1. LET: $\mathcal{U} \subseteq \bigcap \mathcal{T}$
⟨2⟩2. For all $T \in \mathcal{T}$ we have $\mathcal{U} \subseteq T$
⟨2⟩3. For all $T \in \mathcal{T}$ we have $\bigcup \mathcal{U} \in T$
⟨2⟩4. $\bigcup \mathcal{U} \in \bigcap \mathcal{T}$
⟨1⟩3. $\bigcap \mathcal{T}$ is closed under binary intersection.
⟨2⟩1. LET: $U, V \in \bigcap \mathcal{T}$
⟨2⟩2. For all $T \in \mathcal{T}$ we have $U, V \in T$
⟨2⟩3. For all $T \in \mathcal{T}$ we have $U \cap V \in T$
⟨2⟩4. $U \cap V \in \bigcap \mathcal{T}$
□

Lemma 35. *Let X be a set and $\mathcal{T}$ a set of topologies on X. Then there exists a unique coarsest topology that is finer than every member of $\mathcal{T}$.*

PROOF: The required topology is given by
$$\bigcap \{T \in \mathcal{PP}X \mid T \text{ is a topology on } X \text{ that is finer than every member of } \mathcal{T}\} ,$$
The set is nonempty since it contains the discrete topology. □

Definition 36 (Neighbourhood)**.** A *neighbourhood* of a point x is an open set that contains x.

7 Closed Set

Definition 37 (Closed Set)**.** Let X be a topological space and $A \subseteq X$. Then A is *closed* if and only if $X \setminus A$ is open.

Lemma 38. *The empty set is closed.*

PROOF: Since the whole space X is always open. □

Lemma 39. *The topological space X is closed.*

PROOF: Since $\emptyset$ is open. □

Lemma 40. *The intersection of a nonempty set of closed sets is closed.*

PROOF: Let $\mathcal{C}$ be a nonempty set of closed sets. Then $X \setminus \bigcap \mathcal{C} = \bigcup \{X \setminus C \mid C \in \mathcal{C}\}$ is open. □

Lemma 41. *The union of two closed sets is closed.*

PROOF: Let C and D be closed. Then $X \setminus (C \cup D) = (X \setminus C) \cap (X \setminus D)$ is open. □

Proposition 42. *Let X be a set and $\mathcal{C} \subseteq \mathcal{P}X$ a set such that:*

1. $\emptyset \in \mathcal{C}$
2. $X \in \mathcal{C}$
3. *For all $\mathcal{A} \subseteq \mathcal{C}$ nonempty we have $\bigcap \mathcal{A} \in \mathcal{C}$*

4. For all $C, D \in \mathcal{C}$ we have $C \cup D \in \mathcal{C}$.

Then there exists a unique topology $\mathcal{T}$ such that $\mathcal{C}$ is the set of closed sets, namely

$$\mathcal{T} = \{X \setminus C \mid C \in \mathcal{C}\}$$

PROOF:
$\langle 1 \rangle$1. LET: $\mathcal{T} = \{X \setminus C \mid C \in \mathcal{C}\}$
$\langle 1 \rangle$2. $\mathcal{T}$ is a topology
  $\langle 2 \rangle$1. $X \in \mathcal{T}$
    PROOF: Since $\emptyset \in \mathcal{C}$
  $\langle 2 \rangle$2. For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$
    $\langle 3 \rangle$1. LET: $\mathcal{U} \subseteq \mathcal{T}$
    $\langle 3 \rangle$2. CASE: $\mathcal{U} = \emptyset$
      PROOF: In this case $\bigcup \mathcal{U} = \emptyset \in \mathcal{T}$ since $X \in \mathcal{C}$
    $\langle 3 \rangle$3. CASE: $\mathcal{U} \neq \emptyset$
      PROOF: In this case $X \setminus \bigcup \mathcal{U} = \bigcap \{X \setminus U \mid U \in \mathcal{U}\} \in \mathcal{C}$.
  $\langle 2 \rangle$3. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$
    PROOF: Since $X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V) \in \mathcal{C}$.
$\langle 1 \rangle$3. $\mathcal{C}$ is the set of all closed sets in $\mathcal{T}$
  PROOF:

$$\begin{aligned} & C \text{ is closed in } \mathcal{T} \\ \Leftrightarrow & X \setminus C \in \mathcal{T} \\ \Leftrightarrow & C \in \mathcal{C} \end{aligned}$$

$\langle 1 \rangle$4. If $\mathcal{T}'$ is a topology and $\mathcal{C}$ is the set of closed sets in $\mathcal{T}'$ then $\mathcal{T}' = \mathcal{T}$
  PROOF: We have

$$\begin{aligned} & U \in \mathcal{T} \\ \Leftrightarrow & X \setminus U \in \mathcal{C} \\ \Leftrightarrow & X \setminus U \text{ is closed in } \mathcal{T}' \\ \Leftrightarrow & U \in \mathcal{T}' \end{aligned}$$

□

Proposition 43. *If U is open and A is closed then $U \setminus A$ is open.*

PROOF: $U \setminus A = U \cap (X \setminus A)$ is the intersection of two open sets. □

Proposition 44. *If U is open and A is closed then $A \setminus U$ is closed.*

PROOF: $A \setminus U = A \cap (X \setminus U)$ is the intersection of two closed sets. □

8 Interior

Definition 45 (Interior)**.** Let X be a topological space and $A \subseteq X$. The *interior* of A, $\operatorname{Int} A$, is the union of all the open subsets of A.

Lemma 46. *The interior of a set is open.*

PROOF: It is a union of open sets. ∎

Lemma 47.

$$\operatorname{Int} A \subseteq A$$

PROOF: Immediate from definition. ∎

Lemma 48. *If U is open and $U \subseteq A$ then $U \subseteq \operatorname{Int} A$*

PROOF: Immediate from definition. ∎

Lemma 49. *A set A is open if and only if $A = \operatorname{Int} A$.*

PROOF: If $A = \operatorname{Int} A$ then A is open by Lemma 46. Conversely if A is open then $A \subseteq \operatorname{Int} A$ by the definition of interior and so $A = \operatorname{Int} A$.

9 Closure

Definition 50 (Closure)**.** Let X be a topological space and $A \subseteq X$. The *closure* of A, $\overline{A}$, is the intersection of all the closed sets that include A.

This intersection exists since X is a closed set that includes A (Lemma 39).

Lemma 51. *The closure of a set is closed.*

PROOF: Dual to Lemma 46. ∎

Lemma 52.

$$A \subseteq \overline{A}$$

PROOF: Immediate from definition. ∎

Lemma 53. *If C is closed and $A \subseteq C$ then $\overline{A} \subseteq C$.*

PROOF: Immediate from definition. ∎

Lemma 54. *A set A is closed if and only if $A = \overline{A}$.*

PROOF: Dual to Lemma 49. ∎

Theorem 55. *Let X be a topological space, $A \subseteq X$ and $x \in X$. Then $x \in \overline{A}$ if and only if every neighbourhood of x intersects A.*

PROOF: We have

$$\begin{aligned}
&x \in \overline{A} \\
\Leftrightarrow &\forall C. C \text{ closed } \wedge A \subseteq C \Rightarrow x \in C \\
\Leftrightarrow &\forall U. U \text{ open } \wedge A \cap U = \emptyset \Rightarrow x \notin U \\
\Leftrightarrow &\forall U. U \text{ open } \wedge x \in U \Rightarrow U \text{ intersects } A
\end{aligned}$$

∎

Proposition 56. *If $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$.*

PROOF: This holds because $\overline{B}$ is a closed set that includes A. ∎

Proposition 57.

$$\overline{A \cup B} = \overline{A} \cup \overline{B}$$

PROOF:
⟨1⟩1. $\overline{A} \subseteq \overline{A \cup B}$
  PROOF: By Proposition 56.
⟨1⟩2. $\overline{B} \subseteq \overline{A \cup B}$
  PROOF: By Proposition 56.
⟨1⟩3. $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$
  ⟨2⟩1. LET: $x \in \overline{A \cup B}$
  ⟨2⟩2. ASSUME: $x \notin \overline{A}$
    PROVE: $x \in \overline{B}$
  ⟨2⟩3. PICK a neighbourhood U of x that does not intersect A
  ⟨2⟩4. LET: V be any neighbourhood of x
  ⟨2⟩5. $U \cap V$ is a neighbourhood of x
  ⟨2⟩6. $U \cap V$ intersects $A \cup B$
    PROOF: From ⟨2⟩1 and Theorem 55.
  ⟨2⟩7. $U \cap V$ intersects B
    PROOF: From ⟨2⟩3
  ⟨2⟩8. V intersects B
  ⟨2⟩9. Q.E.D.
    PROOF: We have $x \in \overline{B}$ from Theorem 55.

∎

Proposition 58. *Let X be a topological space. Let $\mathcal{D}$ be a set of subsets of X that is maximal with respect to the finite intersection property. Let $x \in X$. Then the following are equivalent:*

1. *For all $D \in \mathcal{D}$ we have $x \in \overline{D}$*
2. *Every neighbourhood of x is in $\mathcal{D}$.*

PROOF:
⟨1⟩1. $1 \Rightarrow 2$
  ⟨2⟩1. For all $D \in \mathcal{D}$ we have $x \in \overline{D}$
  ⟨2⟩2. LET: U be a neighbourhood of x
  ⟨2⟩3. $\mathcal{D} \cup \{U\}$ satisfies the finite intersection property.
    ⟨3⟩1. LET: $D_1, \ldots, D_n \in \mathcal{D}$
    ⟨3⟩2. $D_1 \cap \cdots \cap D_n \in \mathcal{D}$
      PROOF: Lemma 4.
    ⟨3⟩3. $x \in \overline{D_1 \cap \cdots \cap D_n}$
      PROOF: ⟨2⟩1, ⟨3⟩2
    ⟨3⟩4. $D_1 \cap \cdots \cap D_n \cap U \neq \emptyset$
      PROOF: Theorem 55, ⟨2⟩2, ⟨3⟩3.
  ⟨2⟩4. $\mathcal{D} = \mathcal{D} \cup \{U\}$
    PROOF: By the maximality of $\mathcal{D}$.

$\langle 2\rangle 5.$ $U \in \mathcal{D}$
$\langle 1\rangle 2.$ $2 \Rightarrow 1$
$\langle 2\rangle 1.$ Assume: Every neighbourhood of x is in $\mathcal{D}$.
$\langle 2\rangle 2.$ Let: $D \in \mathcal{D}$
$\langle 2\rangle 3.$ Every neighbourhood of x intersects D.
Proof: From $\langle 2\rangle 1$, $\langle 2\rangle 2$ and the fact that $\mathcal{D}$ satisfies the finite intersection property.
$\langle 2\rangle 4.$ $x \in \overline{D}$
Proof: Theorem 55, $\langle 2\rangle 3$.
$\square$

10 Boundary

Definition 59 (Boundary)**.** The *boundary* of a set A is the set $\partial A = \overline{A} \cap \overline{X \setminus A}$.

Proposition 60.

$$\operatorname{Int} A \cap \partial A = \emptyset$$

Proof: Since $\overline{X \setminus A} = X \setminus \operatorname{Int} A$. $\square$

Proposition 61.

$$\overline{A} = \operatorname{Int} A \cup \partial A$$

Proof:

$$\begin{aligned}
\operatorname{Int} A \cup \partial A &= \operatorname{Int} A \cup (\overline{A} \cap \overline{X \setminus A}) \\
&= (\operatorname{Int} A \cup \overline{A}) \cap (\operatorname{Int} A \cup \overline{X \setminus A}) \\
&= \overline{A} \cap X \\
&= \overline{A}
\end{aligned}$$

Proposition 62. *$\partial A = \emptyset$ if and only if A is open and closed.*

Proof: If $\partial A = \emptyset$ then $\overline{A} = \operatorname{Int} A$ by Proposition 61.

Proposition 63. *A set U is open if and only if $\partial U = \overline{U} \setminus U$.*

Proof:

$$\begin{aligned}
& \partial U = \overline{U} \setminus U \\
\Leftrightarrow & \overline{U} \setminus \operatorname{Int} U = \overline{U} \setminus U && \text{(Propositions 60, 61)} \\
\Leftrightarrow & \operatorname{Int} U = U && \square
\end{aligned}$$

11 Limit Points

Definition 64 (Limit Point)**.** Let X be a topological space, $a \in X$ and $A \subseteq X$. Then a is a *limit point*, *cluster point* or *point of accumulation* for A if and only if every neighbourhood of a intersects A at a point other than a.

Lemma 65. *The point a is an accumulation point for A if and only if $a \in \overline{A \setminus \{a\}}$.*

PROOF: From Theorem 55. □

Theorem 66. *Let X be a topological space and $A \subseteq X$. Let A' be the set of all limit points of A. Then $\overline{A} = A \cup A'$.*

PROOF:
$\langle 1 \rangle 1$. For all $x \in \overline{A}$, if $x \notin A$ then $x \in A'$
PROOF: From Theorem 55.
$\langle 1 \rangle 2$. $A \subseteq \overline{A}$
PROOF: Lemma 52.
$\langle 1 \rangle 3$. $A' \subseteq \overline{A}$
PROOF: From Theorem 55.
□

Corollary 66.1. *A set is closed if and only if it contains all its limit points.*

Proposition 67. *In an indiscrete topology, every point is a limit point of any set that has more than one point.*

PROOF: Let X be an indiscrete space. Let A be a set with more than one point and x be a point. The only neighbourhood of x is X, which must intersect A at a point other than x. □

Lemma 68. *Let X be a topological space and $A \subseteq B \subseteq X$. Then every limit point of A is a limit point of B.*

PROOF: Immediate from definitions. □

12 Basis for a Topology

Definition 69 (Basis)**.** If X is a set, a *basis* for a topology on X is a set $\mathcal{B} \subseteq \mathcal{P}X$ called *basis elements* such that

1. For all $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$.
2. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

If $\mathcal{B}$ satisfies these two conditions, then we define the topology *generated* by $\mathcal{B}$ to be $\mathcal{T} = \{U \in \mathcal{P}X \mid \forall x \in U. \exists B \in \mathcal{B}. x \in B \subseteq U\}$.

We prove this is a topology.

PROOF:
$\langle 1 \rangle 1$. $X \in \mathcal{T}$
PROOF: For all $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B \subseteq X$ by condition 1.

$\langle 1 \rangle 2$. For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$
  $\langle 2 \rangle 1$. Let: $\mathcal{U} \subseteq \mathcal{T}$
  $\langle 2 \rangle 2$. Let: $x \in \bigcup \mathcal{U}$
  $\langle 2 \rangle 3$. Pick $U \in \mathcal{U}$ such that $x \in U$
  $\langle 2 \rangle 4$. Pick $B \in \mathcal{B}$ such that $x \in B \subseteq U$
    Proof: Since $U \in \mathcal{T}$ by $\langle 2 \rangle 1$ and $\langle 2 \rangle 3$.
  $\langle 2 \rangle 5$. $x \in B \subseteq \bigcup \mathcal{U}$
$\langle 1 \rangle 3$. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$
  $\langle 2 \rangle 1$. Let: $U, V \in \mathcal{T}$
  $\langle 2 \rangle 2$. Let: $x \in U \cap V$
  $\langle 2 \rangle 3$. Pick $B_1 \in \mathcal{B}$ such that $x \in B_1 \subseteq U$
  $\langle 2 \rangle 4$. Pick $B_2 \in \mathcal{B}$ such that $x \in B_2 \subseteq V$
  $\langle 2 \rangle 5$. Pick $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$
    Proof: By condition 2.
  $\langle 2 \rangle 6$. $x \in B_3 \subseteq U \cap V$
□

Lemma 70. *Let X be a set. Let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ is the set of all unions of subsets of $\mathcal{B}$.*

Proof:
$\langle 1 \rangle 1$. For all $U \in \mathcal{T}$, there exists $\mathcal{A} \subseteq \mathcal{B}$ such that $U = \bigcup \mathcal{A}$
  $\langle 2 \rangle 1$. Let: $U \in \mathcal{T}$
  $\langle 2 \rangle 2$. Let: $\mathcal{A} = \{B \in \mathcal{B} \mid B \subseteq U\}$
  $\langle 2 \rangle 3$. $U \subseteq \bigcup \mathcal{A}$
    $\langle 3 \rangle 1$. Let: $x \in U$
    $\langle 3 \rangle 2$. Pick $B \in \mathcal{B}$ such that $x \in B \subseteq U$
      Proof: Since $\mathcal{B}$ is a basis for $\mathcal{T}$.
    $\langle 3 \rangle 3$. $x \in B \in \mathcal{A}$
  $\langle 2 \rangle 4$. $\bigcup \mathcal{A} \subseteq U$
    Proof: From the definition of $\mathcal{A}$ ($\langle 2 \rangle 2$).
$\langle 1 \rangle 2$. For all $\mathcal{A} \subseteq \mathcal{B}$ we have $\bigcup \mathcal{A} \in \mathcal{T}$
  $\langle 2 \rangle 1$. $\mathcal{B} \subseteq \mathcal{T}$
    Proof: If $B \in \mathcal{B}$ and $x \in B$, then there exists $B' \in \mathcal{B}$ such that $x \in B' \subseteq B$, namely $B' = B$.
  $\langle 2 \rangle 2$. Q.E.D.
    Proof: Since $\mathcal{T}$ is closed under union.
□

Corollary 70.1. *Let X be a set. Let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ is the coarsest topology that includes $\mathcal{B}$.*

Proof: Since every topology that includes $\mathcal{B}$ includes all unions of subsets of $\mathcal{B}$. □

Lemma 71. *Let X be a topological space. Suppose that $\mathcal{C}$ is a set of open sets such that, for every open set U and every point $x \in U$, there exists $C \in \mathcal{C}$ such that $x \in C \subseteq U$. Then $\mathcal{C}$ is a basis for the topology on X.*

PROOF:
$\langle 1\rangle 1$. For all $x \in X$, there exists $C \in \mathcal{C}$ such that $x \in C$
PROOF: Immediate from hypothesis.
$\langle 1\rangle 2$. For all $C_1, C_2 \in \mathcal{C}$ and $x \in C_1 \cap C_2$, there exists $C_3 \in \mathcal{C}$ such that $x \in C_3 \subseteq C_1 \cap C_2$
PROOF: Since $C_1 \cap C_2$ is open.
$\langle 1\rangle 3$. Every open set is open in the topology generated by $\mathcal{C}$
PROOF: Immediate from hypothesis.
$\langle 1\rangle 4$. Every union of a subset of $\mathcal{C}$ is open.
PROOF: Since every member of $\mathcal{C}$ is open.
□

Lemma 72. *Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$ respectively on the set X. Then the following are equivalent.*

1. $\mathcal{T} \subseteq \mathcal{T}'$
2. *For all $B \in \mathcal{B}$ and $x \in B$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.*

PROOF:
$\langle 1\rangle 1$. $1 \Rightarrow 2$
$\langle 2\rangle 1$. ASSUME: $\mathcal{T} \subseteq \mathcal{T}'$
$\langle 2\rangle 2$. LET: $B \in \mathcal{B}$ and $x \in B$
$\langle 2\rangle 3$. $B \in \mathcal{T}$
PROOF: Corollary 70.1.
$\langle 2\rangle 4$. $B \in \mathcal{T}'$
PROOF: By $\langle 2\rangle 1$
$\langle 2\rangle 5$. There exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$
PROOF: Since $\mathcal{B}'$ is a basis for $\mathcal{T}'$.
$\langle 1\rangle 2$. $2 \Rightarrow 1$
$\langle 2\rangle 1$. ASSUME: 2
$\langle 2\rangle 2$. LET: $U \in \mathcal{T}$
PROVE: $U \in \mathcal{T}'$
$\langle 2\rangle 3$. LET: $x \in U$
PROVE: There exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq U$
$\langle 2\rangle 4$. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$
PROOF: Since $\mathcal{B}$ is a basis for $\mathcal{T}$.
$\langle 2\rangle 5$. PICK $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$
PROOF: By $\langle 2\rangle 1$.
$\langle 2\rangle 6$. $x \in B' \subseteq U$
□

Theorem 73. *Let X be a topological space, $A \subseteq X$ and $x \in X$. Let $\mathcal{B}$ be a basis for X. Then $x \in \overline{A}$ if and only if, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A.*

PROOF:
$\langle 1\rangle 1$. If $x \in \overline{A}$ then, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A.
PROOF: This follows from Theorem 55 since every element of $\mathcal{B}$ is open (Corollary 70.1).

$\langle 1 \rangle 2$. Suppose that, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A. Then $x \in \overline{A}$.
  $\langle 2 \rangle 1$. ASSUME: For all $B \in \mathcal{B}$, if $x \in B$ then B intersects A.
  $\langle 2 \rangle 2$. LET: U be an open set that contains x
    PROVE: U intersects A.
  $\langle 2 \rangle 3$. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$.
  $\langle 2 \rangle 4$. B intersects A.
    PROOF: From $\langle 2 \rangle 1$.
  $\langle 2 \rangle 5$. U intersects A.
  $\langle 2 \rangle 6$. Q.E.D.
    PROOF: By Theorem 55.

□

Definition 74 (Lower Limit Topology on the Real Line)**.** The *lower limit topology on the real line* is the topology on $\mathbb{R}$ generated by the basis consisting of all half-open intervals of the form $[a, b)$.

We write $\mathbb{R}_l$ for the topological space $\mathbb{R}$ under the lower limit topology.

We prove this is a basis for a topology.

PROOF:
$\langle 1 \rangle 1$. For all $x \in \mathbb{R}$ there exists an interval $[a, b)$ such that $x \in [a, b)$.
  PROOF: Take $[a, b) = [x, x + 1)$.
$\langle 1 \rangle 2$. For any open intervals $[a, b)$, $[c, d)$ if $x \in [a, b) \cap [c, d)$, then there exists an interval $[e, f)$ such that $x \in [e, f) \subseteq [a, b) \cap [c, d)$
  PROOF: Take $[e, f) = [\max(a, c), \min(b, d))$.

□

Definition 75 (K-topology on the Real Line)**.** Let $K = \{1/n \mid n \in \mathbb{Z}^+\}$.

The *K-topology on the real line* is the topology on $\mathbb{R}$ generated by the basis consisting of all open intervals (a, b) and all sets of the form $(a, b) \setminus K$.

We write $\mathbb{R}_K$ for the topological space $\mathbb{R}$ under the K-topology.

We prove this is a basis for a topology.

PROOF:
$\langle 1 \rangle 1$. For all $x \in \mathbb{R}$ there exists an open interval (a, b) such that $x \in (a, b)$.
  PROOF: Take $(a, b) = (x - 1, x + 1)$.
$\langle 1 \rangle 2$. For any basic open sets B_1, B_2 if $x \in B_1 \cap B_2$, then there exists a basic open set B_3 such that $x \in B_3 \subseteq B_1 \cap B_2$
  $\langle 2 \rangle 1$. CASE: $B_1 = (a, b)$, $B_2 = (c, d)$
    PROOF: Take $B_3 = (\max(a, c), \min(b, d))$.
  $\langle 2 \rangle 2$. CASE: $B_1 = (a, b)$ or $(a, b) \setminus K$, $B_2 = (c, d)$ or $(c, d) \setminus K$, and they are not both open intervals.
    PROOF: Take $B_3 = (\max(a, c), \min(b, d)) \setminus K$.

□

Lemma 76. *The lower limit topology and the K-topology are incomparable.*

PROOF:

$\langle 1 \rangle 1$. The interval $[10, 11)$ is not open in the K-topology.

PROOF: There is no open interval (a, b) such that $10 \in (a, b) \subseteq [10, 11)$ or $10 \in (a, b) \setminus K \subseteq [10, 11)$.

$\langle 1 \rangle 2$. The set $(-1, 1) \setminus K$ is not open in the lower limit topology.

PROOF: There is no half-open interval $[a, b)$ such that $0 \in [a, b) \subseteq (-1, 1) \setminus K$, since there must be a positive integer n with $1/n \in [a, b)$.

□

Definition 77 (Subbasis)**.** A *subbasis* $\mathcal{S}$ for a topology on X is a set $\mathcal{S} \subseteq \mathcal{P}X$ such that $\bigcup \mathcal{S} = X$.

The topology *generated* by the subbasis $\mathcal{S}$ is the set of all unions of finite intersections of elements of $\mathcal{S}$.

We prove this is a topology.

PROOF:

$\langle 1 \rangle 1$. The set $\mathcal{B}$ of all finite intersections of elements of $\mathcal{S}$ forms a basis for a topology on X.

$\langle 2 \rangle 1$. $\bigcup \mathcal{B} = X$

PROOF: Since $\mathcal{S} \subseteq \mathcal{B}$.

$\langle 2 \rangle 2$. $\mathcal{B}$ is closed under binary intersection.

PROOF: By definition.

$\langle 1 \rangle 2$. Q.E.D.

PROOF: By Lemma 70.

□

We have simultaneously proved:

Proposition 78. *Let $\mathcal{S}$ be a subbasis for the topology on X. Then the set of all finite intersections of elements of $\mathcal{S}$ is a basis for the topology on X.*

Proposition 79. *Let X be a set. Let $\mathcal{S}$ be a subbasis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ is the coarsest topology that includes $\mathcal{S}$.*

PROOF: Since every topology that includes $\mathcal{S}$ includes every union of finite intersections of elements of $\mathcal{S}$. □

13 Local Basis at a Point

Definition 80 (Local Basis)**.** Let X be a topological space and $a \in X$. A *(local) basis at a* is a set $\mathcal{B}$ of neighbourhoods of a such that every neighbourhood of a includes some member of $\mathcal{B}$.

Lemma 81. *If there exists a countable local basis at a point a, then there exists a countable local basis $\{B_n \mid n \geq 1\}$ such that $B_1 \supseteq B_2 \supseteq \cdots$.*

PROOF: Given a countable local basis $\{C_n \mid n \geq 1\}$, take $B_n = C_1 \cap \cdots \cap C_n$. □

14 Convergence

Definition 82 (Convergence)**.** Let X be a topological space. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of points in X and $l \in X$. Then the sequence $(a_n)_{n\in\mathbb{N}}$ *converges* to the *limit* l, $a_n \to l$ as $n \to \infty$, if and only if, for every neighbourhood U of l, there exists N such that, for all $n \geq N$, we have $a_n \in U$.

Lemma 83. *Let X be a topological space. Let $A \subseteq X$ and $l \in X$. If there is a sequence of points in A that converges to l then $l \in \overline{A}$.*

Proof:
⟨1⟩1. Let: (a_n) be a sequence of points in A that converges to l.
⟨1⟩2. Let: U be a neighbourhood of l.
⟨1⟩3. Pick N such that, for all $n \geq N$, we have $a_n \in U$.
⟨1⟩4. $a_N \in U \cap A$
⟨1⟩5. Q.E.D.
  Proof: Theorem 55.
□

Proposition 84. *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Let (a_n) be a sequence in X and $l \in X$. Then $a_n \to l$ as $n \to \infty$ if and only if, for every $B \in \mathcal{B}$ with $l \in B$, there exists N such that, for all $n \geq N$, we have $a_n \in B$.*

Proof:
⟨1⟩1. If $a_n \to l$ as $n \to \infty$ then, for every $B \in \mathcal{B}$ with $l \in B$, there exists N such that, for all $n \geq N$, we have $a_n \in B$.
  Proof: Since every element of $\mathcal{B}$ is open (Corollary 70.1).
⟨1⟩2. If, for every $B \in \mathcal{B}$ with $l \in B$, there exists N such that, for all $n \geq N$, we have $a_n \in B$, then $a_n \to l$ as $n \to \infty$.
  ⟨2⟩1. Assume: for every $B \in \mathcal{B}$ with $l \in B$, there exists N such that, for all $n \geq N$, we have $a_n \in B$
  ⟨2⟩2. Let: U be a neighbourhood of l.
  ⟨2⟩3. Pick $B \in \mathcal{B}$ such that $l \in B \subseteq U$
  ⟨2⟩4. Pick N such that, for all $n \geq N$, we have $a_n \in B$
    Proof: From ⟨2⟩1.
  ⟨2⟩5. For all $n \geq N$ we have $a_n \in U$
□

Lemma 85. *If a sequence (a_n) is constant with $a_n = l$ for all n, then $a_n \to l$ as $n \to \infty$.*

Proof: Immediate from definitions. □

Theorem 86. *Let X be a linearly ordered set. Let (s_n) be a monotone increasing sequence in X with a supremum s. Then $s_n \to s$ as $n \to \infty$.*

Proof:
⟨1⟩1. Assume: s is not least in X.

PROOF: Otherwise (s_n) is the constant sequence s and the result follows from Lemma 85.
⟨1⟩2. LET: U be a neighbourhood of s.
⟨1⟩3. PICK$a < s$ such that $(a, s] \subseteq U$
⟨1⟩4. PICK N such that $a < a_N$.
⟨1⟩5. For all $n \geq N$ we have $a_n \in (a, s]$
⟨1⟩6. For all $n \geq N$ we have $a_n \in U$.
□

Theorem 87. *If* $\sum_{i=0}^{\infty} a_i = s$ *and* $\sum_{i=0}^{\infty} b_i = t$ *then* $\sum_{i=0}^{\infty}(ca_i + b_i) = cs + t$.

PROOF:$\sum_{i=0}^{N}(ca_i + b_i) = c\sum_{i=0}^{N} a_i + \sum_{i=0}^{N} b_i \to cs + t$ as $n \to \infty$. □

Theorem 88 (Comparison Test)**.** *If* $|a_i| \leq b_i$ *for all* i *and* $\sum_{i=0}^{\infty} b_i$ *converges then* $\sum_{i=0}^{\infty} a_i$ *converges.*

PROOF:
⟨1⟩1. $\sum_{i=0}^{\infty} |a_i|$ converges
PROOF: The partial sums $\sum_{i=0}^{N} |a_i|$ form an increasing sequence bounded above by $\sum_{i=0}^{\infty} b_i$.
⟨1⟩2. LET: $c_i = |a_i| + a_i$ for all i
⟨1⟩3. $\sum_{i=0}^{\infty} c_i$ converges
PROOF: Each c_i is either $2|a_i|$ or 0. So the partial sums $\sum_{i=0}^{N} c_i$ form an increasing sequence bounded above by $2\sum_{i=0}^{\infty} b_i$.
⟨1⟩4. Q.E.D.
PROOF: Since $a_i = c_i - |a_i|$.
□

Corollary 88.1. *If* $\sum_{i=0}^{\infty} |a_i|$ *converges then* $\sum_{i=0}^{\infty} a_i$ *converges.*

Theorem 89 (Weierstrass M-test)**.** *Let* X *be a set and* $(f_n : X \to \mathbb{R})$ *be a sequence of functions. Let*

$$s_n(x) = \sum_{i=0}^{n} f_i(x)$$

for all n, x. *Suppose* $|f_i(x)| \leq M_i$ *for all* $i \geq 0$ *and* $x \in X$. *If the series* $\sum_{i=0}^{\infty} M_i$ *converges, then the sequence* (s_n) *converges uniformly to*

$$s(x) = \sum_{i=0}^{\infty} f_i(x) \quad .$$

PROOF:
⟨1⟩1. LET: $r_n = \sum_{i=n+1}^{\infty} M_i$ for all n
⟨1⟩2. Given $0 \leq n < k$, we have $|s_k(x) - s_n(x)| \leq r_n$

PROOF:

$$\begin{aligned}|s_k(x) - s_n(x)| &= |\sum_{i=n+1}^{k} f_i(x)| \\ &\leq \sum_{i=n+1}^{k} |f_i(x)| \\ &\leq \sum_{i=n+1}^{k} M_i \\ &\leq r_n\end{aligned}$$

⟨1⟩3. Given $n \geq 0$ we have $|s(x) - s_n(x)| \leq r_n$
PROOF: By taking the limit $k \to \infty$ in ⟨1⟩2.
⟨1⟩4. Q.E.D.
PROOF: Since $r_n \to 0$ as $n \to \infty$.
□

15 Locally Finite Sets

Definition 90 (Locally Finite)**.** Let X be a topological space and $\{A_\alpha\}$ a family of subsets of X. Then $\mathcal{A}$ is *locally finite* if and only if every point in X has a neighbourhood that intersects A_α for only finitely many α.

Theorem 91 (Pasting Lemma)**.** *Let X and Y be topological spaces and $f : X \to Y$. Let $\{A_\alpha\}$ be a locally finite family of closed subsets of X that cover X. Suppose $f \upharpoonright A_\alpha$ is continuous for all α. Then f is continuous.*

PROOF:
⟨1⟩1. Let X and Y be topological spaces and $f : X \to Y$. Let A and B be closed subsets of X such that $X = A \cup B$. Suppose $f \upharpoonright A$ and $f \upharpoonright B$ are continuous. Then f is continuous.
⟨2⟩1. LET: $C \subseteq Y$ be closed.
⟨2⟩2. $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$
⟨2⟩3. $f^{-1}(C)$ and $g^{-1}(C)$ are closed in X.
PROOF: Theorems 101 and 152.
⟨2⟩4. $h^{-1}(C)$ is closed in X.
PROOF: Lemma 41.
⟨2⟩5. Q.E.D.
PROOF: Theorem 101.
⟨1⟩2. Let X and Y be topological spaces and $f : X \to Y$. Let $\{A_\alpha\}$ be a finite family of closed subsets of X that cover X. Suppose $f \upharpoonright A_\alpha$ is continuous for all α. Then f is continuous.
PROOF: From ⟨1⟩1 by induction.
⟨1⟩3. Let X and Y be topological spaces and $f : X \to Y$. Let $\{A_\alpha\}$ be a locally finite family of closed subsets of X that cover X. Suppose $f \upharpoonright A_\alpha$ is continuous for all α. Then f is continuous.

⟨2⟩1. LET: $x \in X$
PROVE: f is continuous at x

⟨2⟩2. PICK a neighbourhood U of x that intersects A_α for only finitely many α.

⟨2⟩3. $f \restriction U$ is continuous
PROOF: By ⟨1⟩2.

⟨2⟩4. Q.E.D.
PROOF: Lemma 111.

□

The following example shows that we cannot remove the assumption of local finiteness.

Example 92. Define $f : [-1,1] \to \mathbb{R}$ by: $f(x) = 1$ if $x < -1$, $f(x) = 0$ if $x > 1$. Let $C_n = [-1, -1/n]$ for $n \geq 1$, and $D = [0,1]$. Then $[-1,1] = \bigcup_{n=1}^{\infty} C_n \cup D$ and f is continuous on each C_n and each D, but f is not continuous on $[-1,1]$.

16 Open Maps

Definition 93 (Open Map)**.** Let X and Y be topological spaces and $f : X \to Y$. Then f is an *open map* if and only if, for every open set U in X, the set $f(U)$ is open in Y.

Lemma 94. *Let X and Y be topological spaces and $f : X \to Y$. Let $\mathcal{B}$ be a basis for the topology on X. If $f(B)$ is open in Y for all $B \in \mathcal{B}$, then f is an open map.*

PROOF: From Lemma 70. □

Proposition 95. *Let X and Y be topological spaces. Let $\mathcal{B}$ be a basis for the topology on X. Let $f : X \to Y$. Suppose that, for all $B \in \mathcal{B}$, we have $f(B)$ is open tn Y. Then f is an open map.*

PROOF: For any $\mathcal{A} \subseteq \mathcal{B}$, we have $f(\bigcup \mathcal{A}) = \bigcup_{B \in \mathcal{B}} f(B)$ is open in Y. The result follows from Lemma 70. □

17 Continuous Functions

Definition 96 (Continuous)**.** Let X and Y be topological spaces. A function $f : X \to Y$ is *continuous* if and only if, for every open set V in Y, the set $f^{-1}(V)$ is open in X.

Proposition 97. *Let X and Y be topological spaces and $f : X \to Y$. Let $\mathcal{B}$ be a basis for Y. Then f is continuous if and only if, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X.*

PROOF:
⟨1⟩1. If f is continuous then, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X.

PROOF: Since every element of B is open (Lemma 70).

$\langle 1 \rangle 2$. Suppose that, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X. Then f is continuous.

$\langle 2 \rangle 1$. ASSUME: For all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X.

$\langle 2 \rangle 2$. LET: V be open in Y.

$\langle 2 \rangle 3$. PICK $\mathcal{A} \subseteq \mathcal{B}$ such that $V = \bigcup \mathcal{A}$

PROOF: By Lemma 70.

$\langle 2 \rangle 4$. $f^{-1}(V)$ is open in X.

PROOF:

$$\begin{aligned} f^{-1}(V) &= f^{-1}\left(\bigcup \mathcal{A}\right) \\ &= \bigcup_{B \in \mathcal{A}} f^{-1}(B) \end{aligned}$$

□

Proposition 98. *Let X and Y be topological spaces and $f : X \to Y$. Let $\mathcal{S}$ be a subbasis for Y. Then f is continuous if and only if, for all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in X.*

PROOF:

$\langle 1 \rangle 1$. If f is continuous then, for all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in X.

PROOF: Since every element of S is open.

$\langle 1 \rangle 2$. Suppose that, for all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in X. Then f is continuous.

$\langle 2 \rangle 1$. ASSUME: For all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in X.

$\langle 2 \rangle 2$. LET: $S_1, \ldots, S_n \in \mathcal{S}$

$\langle 2 \rangle 3$. $f^{-1}(S_1 \cap \cdots \cap S_n)$ is open in A

PROOF: Since $f^{-1}(S_1 \cap \cdots \cap S_n) = f^{-1}(S_1) \cap \cdots \cap f^{-1}(S_n)$.

$\langle 2 \rangle 4$. Q.E.D.

PROOF: By Propositions 97 and 78.

□

Proposition 99. *Let X and Y be topological spaces and $f : X \to Y$. Let $\mathcal{S}$ be a basis for Y. Then f is continuous if and only if, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X.*

PROOF:

$\langle 1 \rangle 1$. If f is continuous then, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X.

PROOF: Since every element of $\mathcal{S}$ is open.

$\langle 1 \rangle 2$. Suppose that, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X. Then f is continuous.

$\langle 2 \rangle 1$. ASSUME: For all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X.

$\langle 2 \rangle 2$. For every set B that is the finite intersection of elemets of $\mathcal{S}$, we have $f^{-1}(B)$ is open in X.

PROOF: Because $f^{-1}(V_1 \cap \cdots \cap V_n) = f^{-1}(V_1) \cap \cdots \cap f^{-1}(V_n)$.

$\langle 2 \rangle 3$. Q.E.D.

PROOF: From Propositions 78 and 97.

□

Definition 100 (Continuous at a Point)**.** Let X and Y be topological spaces. Let $f : X \to Y$ and $x \in X$. Then f is *continuous at* x if and only if, for every neighbourhood V of $f(x)$, there exists a neighbourhood U of x such that $f(U) \subseteq V$.

Theorem 101. *Let X and Y be topological spaces and $f : X \to Y$. Then the following are equivalent:*

1. *f is continuous.*
2. *For all $A \subseteq X$, we have $f(\overline{A}) \subseteq \overline{f(A)}$*
3. *For all $B \subseteq Y$ closed, we have $f^{-1}(B)$ is closed in X.*
4. *f is continuous at every point of X.*

PROOF:
$\langle 1 \rangle$1. $1 \Rightarrow 2$
$\langle 2 \rangle$1. ASSUME: f is continuous.
$\langle 2 \rangle$2. LET: $A \subseteq X$
$\langle 2 \rangle$3. LET: $x \in \overline{A}$
PROVE: $f(x) \in \overline{f(A)}$
$\langle 2 \rangle$4. LET: V be a neighbourhood of $f(x)$
$\langle 2 \rangle$5. $f^{-1}(V)$ is a neighbourhood of x
$\langle 2 \rangle$6. PICK $y \in A \cap f^{-1}(V)$
PROOF: By Theorem 55.
$\langle 2 \rangle$7. $f(y) \in V \cap f(A)$
$\langle 2 \rangle$8. Q.E.D.
PROOF: By Theorem 55.
$\langle 1 \rangle$2. $2 \Rightarrow 3$
$\langle 2 \rangle$1. ASSUME: 2
$\langle 2 \rangle$2. LET: B be closed in Y
$\langle 2 \rangle$3. LET: $x \in \overline{f^{-1}(B)}$
PROVE: $x \in f^{-1}(B)$
$\langle 2 \rangle$4. $f(x) \in B$
PROOF:

$$\begin{aligned} f(x) &\in f(\overline{f^{-1}(B)}) \\ &\subseteq \overline{f(f^{-1}(B))} && (\langle 2 \rangle 1) \\ &\subseteq \overline{B} && (Proposition 56) \\ &= B \end{aligned}$$

$\langle 1 \rangle$3. $3 \Rightarrow 1$
$\langle 2 \rangle$1. ASSUME: 3
$\langle 2 \rangle$2. LET: V be open in Y
$\langle 2 \rangle$3. $Y \setminus V$ is closed in Y
$\langle 2 \rangle$4. $f^{-1}(Y \setminus V)$ is closed in X
$\langle 2 \rangle$5. $X \setminus f^{-1}(V)$ is closed in X
$\langle 2 \rangle$6. $f^{-1}(V)$ is open in X

⟨1⟩4. $1 \Rightarrow 4$

Proof: For any neighbourhood V of $f(x)$, the set $U = f^{-1}(V)$ is a neighbourhood of x such that $f(U) \subseteq V$.

⟨1⟩5. $4 \Rightarrow 1$

⟨2⟩1. Assume: 4

⟨2⟩2. Let: V be open in Y

⟨2⟩3. Let: $x \in f^{-1}(V)$

⟨2⟩4. V is a neighbourhood of $f(x)$

⟨2⟩5. Pick a neighbourhood U of x such that $f(U) \subseteq V$

⟨2⟩6. $x \in U \subseteq f^{-1}(V)$

⟨2⟩7. Q.E.D.

Proof: By Lemma 33.

□

Theorem 102. *A constant function is continuous.*

Proof: Let X and Y be topological spaces. Let $b \in Y$, and let $f : X \to Y$ be the constant function with value b. For any open $V \subseteq Y$, the set $f^{-1}(V)$ is either X (if $b \in V$) or $\emptyset$ (if $b \notin V$). □

Theorem 103. *If A is a subspace of X then the inclusion $j : A \to X$ is continuous.*

Proof: For any V open in X, we have $j^{-1}(V) = V \cap A$ is open in A. □

Theorem 104. *The composite of two continuous functions is continuous.*

Proof: Let $f : X \to Y$ and $g : Y \to Z$ be continuous. For any V open in Z, we have $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is open in X. □

Theorem 105. *Let $f : X \to Y$ be a continuous function and A be a subspace of X. Then the restriction $f \restriction A : A \to Y$ is continuous.*

Proof: Let V be open in Y. Then $(f \restriction A)^{-1}(V) = f^{-1}(V) \cap A$ is open in A. □

Theorem 106. *Let $f : X \to Y$ be continuous. Let Z be a subspace of Y such that $f(X) \subseteq Z$. Then the corestriction $f : X \to Z$ is continuous.*

Proof:

⟨1⟩1. Let: V be open in Z.

⟨1⟩2. Pick U open in Y such that $V = U \cap Z$.

⟨1⟩3. $f^{-1}(V) = f^{-1}(U)$

⟨1⟩4. $f^{-1}(V)$ is open in X.

□

Theorem 107. *Let $f : X \to Y$ be continuous. Let Z be a space such that Y is a subspace of Z. Then the expansion $f : X \to Z$ is continuous.*

Proof: Let V be open in Z. Then $f^{-1}(V) = f^{-1}(V \cap Y)$ is open in X. □

Theorem 108. *Let X and Y be topological spaces. Let $f : X \to Y$. Suppose $\mathcal{U}$ is a set of open sets in X such that $X = \bigcup \mathcal{U}$ and, for all $U \in \mathcal{U}$, we have $f \restriction U : U \to Y$ is continuous. Then f is continuous.*

PROOF:
⟨1⟩1. LET: V be open in Y
⟨1⟩2. $f^{-1}(V) = \bigcup_{U \in \mathcal{U}} (f \restriction U)^{-1}(V)$
⟨1⟩3. For all $U \in \mathcal{U}$, we have $(f \restriction U)^{-1}(V)$ is open in U.
⟨1⟩4. For all $U \in \mathcal{U}$, we have $(f \restriction U)^{-1}(V)$ is open in X.
  PROOF: Lemma 151.
□

Proposition 109. *Let X and X' be the same set X under two topologies $\mathcal{T}$ and $\mathcal{T}'$. Let $i : X \to X'$ be the identity function. Then i is continuous if and only if $\mathcal{T}' \subseteq \mathcal{T}$.*

PROOF: Immediate from definitions. □

Proposition 110. *Let $f : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$. Then f is continuous on the right at a if and only if f is continuous at a as a function $\mathbb{R}_l \to \mathbb{R}$.*

PROOF:
⟨1⟩1. If f is continuous on the right at a then f is continuous at a as a function $\mathbb{R}_l \to \mathbb{R}$.
  ⟨2⟩1. ASSUME: f is continuous on the right at a.
  ⟨2⟩2. LET: V be a neighbourhood of $f(a)$
  ⟨2⟩3. PICK b, c such that $f(a) \in (b, c) \subseteq V$.
  ⟨2⟩4. LET: $\epsilon = \min(c - f(a), f(a) - b)$
  ⟨2⟩5. PICK $\delta > 0$ such that, for all x, if $a < x < a + \delta$ then $|f(x) - f(a)| < \epsilon$
  ⟨2⟩6. LET: $U = [a, a + \delta)$
  ⟨2⟩7. $f(U) \subseteq V$
⟨1⟩2. If f is continuous at a as a function $\mathbb{R}_l \to \mathbb{R}$ then f is continuous on the right at a.
  ⟨2⟩1. ASSUME: f is continuous at a as a function $\mathbb{R}_l \to \mathbb{R}$
  ⟨2⟩2. LET: $\epsilon > 0$
  ⟨2⟩3. PICK a neighbourhood U of a such that $f(U) \subseteq (f(a) - \epsilon, f(a) + \epsilon)$
  ⟨2⟩4. PICK b, c such that $a \in [b, c) \subset U$
  ⟨2⟩5. LET: $\delta = c - a$
  ⟨2⟩6. For all x, if $a < x < a + \delta$ then $|f(x) - f(a)| < \epsilon$
□

Lemma 111. *Let $f : X \to Y$. Let Z be an open subspace of X and $a \in Z$. If $f \restriction Z$ is continuous at a then f is continuous at a.*

PROOF:
⟨1⟩1. LET: V be a neighbourhood of $f(x)$
⟨1⟩2. PICK a neighbourhood W of x in Z such that $f(W) \subseteq V$
⟨1⟩3. W is a neighbourhood of x in X such that $f(W) \subseteq V$

PROOF: Lemma 151.
∎

Proposition 112. *Let $f : A \to B$ and $g : C \to D$ be continuous. Define $f \times g : A \times C \to B \times D$ by*

$$(f \times g)(a, c) = (f(a), g(c)) \ .$$

Then $f \times g$ is continuous.

PROOF: $\pi_1 \circ (f \times g) = f \circ \pi_1$ and $\pi_2 \circ (f \times g) = g \circ \pi_2$ are continuous by Theorem 104. The result follows by Theorem 140.

Proposition 113. *Let X and Y be topological spaces and $f : X \to Y$ be continuous. If $a_n \to l$ as $n \to \infty$ in X then $f(a_n) \to f(l)$ as $n \to \infty$.*

PROOF:
$\langle 1 \rangle 1$. LET: V be a neighbourhood of $f(l)$
$\langle 1 \rangle 2$. PICK a neighbourhood U of l such that $f(U) \subseteq V$
$\langle 1 \rangle 3$. PICK N such that, for all $n \geq N$, we have $a_n \in U$
$\langle 1 \rangle 4$. For all $n \geq N$ we have $f(n) \in V$
∎

18 Homeomorphisms

Definition 114 (Homeomorphism)**.** Let X and Y be topological spaces. A *Homeomorphism* f between X and Y, $f : X \cong Y$, is a bijection $f : X \to Y$ such that both f and f^{-1} are continuous.

Lemma 115. *Let X and Y be topological spaces and $f : X \to Y$ a bijection. Then the following are equivalent:*

1. *f is a homeomorphism.*
2. *f is continuous and an open map.*
3. *f is continuous and a closed map.*
4. *For any $U \subseteq X$, we have U is open if and only if $f(U)$ is open.*

PROOF: Immediate from definitions. ∎

Proposition 116. *Let X and X' be the same set X under two topologies $\mathcal{T}$ and $\mathcal{T}'$. Let $i : X \to X'$ be the identity function. Then i is a homeomorphism if and only if $\mathcal{T} = \mathcal{T}'$.*

PROOF: Immediate from definitions. ∎

Definition 117 (Topological Property)**.** Let P be a property of topological spaces. Then P is a *topological* property if and only if, for any spaces X and Y, if P holds of X and $X \cong Y$ then P holds of Y.

Definition 118 (Topological Imbedding)**.** Let X and Y be topological spaces and $f : X \to Y$. Then f is a *topological imbedding* if and only if the corestriction $f : X \to f(X)$ is a homeomorphism.

Proposition 119. *Let X and Y be topological spaces and $a \in X$. The function $i : Y \to X \times Y$ that maps y to (a, y) is an imbedding.*

PROOF:
$\langle 1 \rangle 1$. i is injective
$\langle 1 \rangle 2$. i is continuous.
  PROOF: For U open in X and V open in Y, we have $i^{-1}(U \times V)$ is V if $a \in U$, and $\emptyset$ if $a \notin U$.
$\langle 1 \rangle 3$. $i : Y \to i(Y)$ is an open map.
  PROOF: For V open in Y we have $i(V) = (X \times V) \cap i(Y)$.
□

19 The Order Topology

Definition 120 (Order Topology)**.** Let X be a linearly ordered set with at least two points. The *order topology* on X is the topology generated by the basis $\mathcal{B}$ consisting of:

- all open intervals (a, b);
- all intervals of the form $[\perp, b)$ where $\perp$ is least in X;
- all intervals of the form $(a, \top]$ where $\top$ is greatest in X.

We prove this is a basis for a topology.

PROOF:
$\langle 1 \rangle 1$. For all $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B$.
  $\langle 2 \rangle 1$. LET: $x \in X$
  $\langle 2 \rangle 2$. CASE: x is greatest in X.
    $\langle 3 \rangle 1$. PICK $y \in X$ with $y \neq x$
    $\langle 3 \rangle 2$. $x \in (y, x] \in \mathcal{B}$
  $\langle 2 \rangle 3$. CASE: x is least in X.
    $\langle 3 \rangle 1$. PICK $y \in X$ with $y \neq x$
    $\langle 3 \rangle 2$. $x \in [x, y) \in \mathcal{B}$
  $\langle 2 \rangle 4$. CASE: x is neither greatest nor least in X.
    $\langle 3 \rangle 1$. PICK $a, b \in X$ with $a < x$ and $x < b$
    $\langle 3 \rangle 2$. $x \in (a, b) \in \mathcal{B}$
$\langle 1 \rangle 2$. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$
  $\langle 2 \rangle 1$. LET: $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$
  $\langle 2 \rangle 2$. CASE: $B_1 = (a, b)$, $B_2 = (c, d)$
    PROOF: Take $B_3 = (\max(a, c), \min(b, d))$.
  $\langle 2 \rangle 3$. CASE: $B_1 = (a, b)$, $B_2 = [\perp, d)$

PROOF: Take $B_3 = (a, \min(b, d))$.
⟨2⟩4. CASE: $B_1 = (a, b)$, $B_2 = (c, \top]$
PROOF: Take $B_3 = (\max(a, c), b)$.
⟨2⟩5. CASE: $B_1 = [\bot, b)$, $B_2 = [\bot, d)$
PROOF: Take $B_3 = [\bot, \min(b, d))$.
⟨2⟩6. CASE: $B_1 = [\bot, b)$, $B_2 = (c, \top]$
PROOF: Take $B_3 = (c, b)$.
$\square$

Lemma 121. *Let X be a linearly ordered set. Then the open rays form a subbasis for the order topology on X.*

PROOF:
⟨1⟩1. Every open ray is open.
⟨2⟩1. For all $a \in X$, the ray $(-\infty, a)$ is open.
⟨3⟩1. LET: $x \in (-\infty, a)$
⟨3⟩2. CASE: x is least in X
PROOF: $xin[x, a) = (-\infty, a)$.
⟨3⟩3. CASE: x is not least in X
⟨4⟩1. PICK $y < x$
⟨4⟩2. $x \in (y, a) \subseteq (-\infty, a)$
⟨2⟩2. For all $a \in X$, the ray $(a, +\infty)$ is open.
PROOF: Similar.
⟨1⟩2. Every basic open set is a finite intersection of open rays.
PROOF: We have $(a, b) = (a, +\infty) \cap (-\infty, b)$, $[\bot, b) = (-\infty, b)$ and $(a, \top] = (a, +\infty)$.
$\square$

Definition 122 (Standard Topology on the Real Line)**.** The *standard topology on the real line* is the order topology on $\mathbb{R}$ generated by the standard order.

Lemma 123. *The lower limit topology is strictly finer than the standard topology on $\mathbb{R}$.*

PROOF:
⟨1⟩1. Every open interval is open in the lower limit topology.
PROOF: If $x \in (a, b)$ then $x \in [x, b) \subseteq (a, b)$.
⟨1⟩2. The half-open interval $[0, 1)$ is not open in the standard topology.
PROOF: There is no open interval (a, b) such that $0 \in (a, b) \subseteq [0, 1)$.
$\square$

Lemma 124. *The K-topology is strictly finer than the standard topology on $\mathbb{R}$.*

PROOF:
⟨1⟩1. Every open interval is open in the K-topology.
PROOF: Corollary 70.1.
⟨1⟩2. The set $(-1, 1) \setminus K$ is not open in the standard topology.
PROOF: There is no open interval (a, b) such that $0 \in (a, b) \subseteq (-1, 1) \setminus K$, since there must be a positive integer n with $1/n \in (a, b)$.

∎

Lemma 125. *Let X be a topological space. Let Y be a linearly ordered set in the order topology. Let $f, g : X \to Y$ be continuous. Then $C = \{x \in X \mid f(x) \leq g(x)\}$ is closed.*

PROOF:
$\langle 1 \rangle 1$. LET: $x \in X \setminus C$
$\langle 1 \rangle 2$. $f(x) > g(x)$
  PROVE: There exists a neighbourhood U of x such that $U \subseteq X \setminus C$
$\langle 1 \rangle 3$. CASE: There exists y such that $g(x) < y < f(x)$
  PROOF: Take $U = g^{-1}((-\infty, y)) \cup f^{-1}(y, +\infty)$.
$\langle 1 \rangle 4$. CASE: There is no y such that $g(x) < y < f(x)$
  PROOF: Take $U = g^{-1}((-\infty, f(x))) \cup f^{-1}(g(x), +\infty)$.
∎

Proposition 126. *Let X be a topological space. Let Y be a linearly ordered set in the order topology. Let $f, g : X \to Y$ be continuous. Define $h : X \to Y$ by $h(x) = \min(f(x), g(x))$. Then h is continuous.*

PROOF: By the Pasting Lemma applied to $\{x \in X \mid f(x) \leq g(x)\}$ and $\{x \in X \mid g(x) \leq f(x)\}$, which are closed by Lemma 125.

Proposition 127. *Let X and Y be linearly ordered sets in the order topology. Let $f : X \to Y$ be strictly monotone and surjective. Then f is a homeomorphism.*

PROOF:
$\langle 1 \rangle 1$. f is bijective.
  PROOF: Proposition 13.
$\langle 1 \rangle 2$. f is continuous.
  $\langle 2 \rangle 1$. For all $y \in Y$ we have $f^{-1}((y, +\infty))$ is open.
    $\langle 3 \rangle 1$. LET: $y \in Y$
    $\langle 3 \rangle 2$. PICK$x \in X$ such that $f(x) = y$
      PROOF: Since f is surjective.
    $\langle 3 \rangle 3$. $f^{-1}((y, +\infty)) = (x, +\infty)$
      PROOF: By strict monotoncity.
  $\langle 2 \rangle 2$. For all $y \in Y$ we have $f^{-1}((-\infty, y))$ is open.
    PROOF: Similar.
$\langle 1 \rangle 3$. f^{-1} is continuous.
  $\langle 2 \rangle 1$. For all $x \in X$ we have $f((x, +\infty))$ is open.
    PROOF: $f((x, +\infty)) = (f(x), +\infty)$.
  $\langle 2 \rangle 2$. For all $x \in X$ we have $f((-\infty, x))$ is open.
    PROOF: $f((-\infty, x)) = (-\infty, f(x))$.
∎

20 The nth Root Function

Proposition 128. *For all $n \geq 1$, the function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined by $f(x) = x^n$ is a homemorphism.*

PROOF:
⟨1⟩1. f is strictly monotone.
  ⟨2⟩1. LET: $x, y \in \mathbb{R}$ with $0 \leq x < y$
  ⟨2⟩2. $x^n < y^n$

$$\begin{aligned} y^n - x^n &= (y-x)(y^{n-1} + y^{n-2}x + y^{n-3}x^2 + \cdots + x^{n-1}) \\ &> 0 \end{aligned}$$

⟨1⟩2. f is surjective.
  ⟨2⟩1. LET: $y \in \mathbb{R}_{\geq 0}$
  ⟨2⟩2. PICK $x \in \mathbb{R}$ such that $y \leq x^n$
    PROOF: If $y \leq 1$ take $x = 1$, otherwise take $x = y$.
  ⟨2⟩3. There exists $x' \in [0, x]$ such that $(x')^n = y$
    PROOF: By the Intermediate Value Theorem.
⟨1⟩3. Q.E.D.
  PROOF: Proposition 127.
□

Definition 129. For $n \geq 1$, the *nth root function* is the function $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ that is the inverse of $\lambda x.x^n$.

21 The Product Topology

Definition 130 (Product Topology)**.** Let $\{A_i\}_{i \in I}$ be a family of topological spaces. The *product topology* on $\prod_{i \in I} A_i$ is the topology generated by the subbasis consisting of the sets of the form ${\pi_i}^{-1}(U)$ where $i \in I$ and U is open in A_i.

Proposition 131. *The product topology on $\prod_{i \in I} A_i$ is generated by the basis consisting of all sets of the form $\prod_{i \in I} U_i$ where $\{U_i\}_{i \in I}$ is a family such that each U_i is an open set in A_i and $U_i = A_i$ for all but finitely many i.*

PROOF: From Proposition 78. □

Proposition 132. *If A_i is closed in X_i for all $i \in I$ then $\prod_{i \in I} A_i$ is closed in $\prod_{i \in I} X_i$.*

PROOF:

$$(\prod_{i \in I} X_i) \setminus (\prod_{i \in I} A_i) = \bigcup_{j \in I} \left(\prod_{i \in I} X_i \setminus {\pi_j}^{-1}(A_j) \right) \square$$

Proposition 133. *Let $\{A_i\}_{i \in I}$ be a family of topological spaces. For $i \in I$, let $\mathcal{B}_i$ be a basis for the topology on A_i. Then $\mathcal{B} = \{\prod_{i \in I} B_i \mid \forall i \in I. B_i \in \mathcal{B}_i, B_i = A_i$ for all but finitely many $i\}$ is a basis for the box topology on $\prod_{i \in I} A_i$.*

PROOF:
⟨1⟩1. Every set in $\mathcal{B}$ is open.
⟨1⟩2. For every point $a \in \prod_{i \in I} A_i$ and every open set U with $a \in U$, there exists $B \in \mathcal{B}$ such that $a \in B \subseteq U$.
⟨2⟩1. LET: U be open and $a \in U$
⟨2⟩2. PICK a family $\{U_i\}_{i \in I}$ such that each U_i is open in A_i, such that $U_i = A_i$ except for $i = i_1, \ldots, i_n$, and such that $a \in \prod_{i \in I} U_i \subseteq U$.
⟨2⟩3. For $j = 1, \ldots, n$, PICK $B_{i_j} \in \mathcal{B}_{i_j}$ such that $a_{i_j} \in B_{i_j} \subseteq U_{i_j}$
⟨2⟩4. LET: $B = \prod_{i \in I} B_i$ where $B_i = A_i$ for $i \neq i_1, \ldots, i_n$
⟨2⟩5. $B \in \mathcal{B}$
⟨2⟩6. $a \in B \subseteq U$
⟨1⟩3. Q.E.D.
PROOF: Lemma 71.
□

Proposition 134. *Let $\{A_i\}_{i \in I}$ be a family of topological spaces. Then the projections $\pi_i : \prod_{i \in I} A_i \to A_i$ are open maps.*

PROOF: From Lemma 94. □

Example 135. The projections are not always closed maps. For example, $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ maps the closed set $\{(x, 1/x) \mid x > 0\}$ to $(0, +\infty)$.

Proposition 136. *Let $\{X_i\}_{i \in I}$ be a family of sets. For $i \in I$, let $\mathcal{T}_i$ and $\mathcal{U}_i$ be topologies on X_i. Let $\mathcal{P}$ be the product topology on $\prod_{i \in I} X_i$ generated by the topologies $\mathcal{T}_i$, and $\mathcal{Q}$ the product topology on the same set generated by the topologies $\mathcal{U}_i$. Then $\mathcal{P} \subseteq \mathcal{Q}$ if and only if $\mathcal{T}_i \subseteq \mathcal{U}_i$ for all i.*

PROOF:
⟨1⟩1. If $\mathcal{T}_i \subseteq \mathcal{U}_i$ for all i then $\mathcal{P} \subseteq \mathcal{Q}$
PROOF: By Corollary 70.1.
⟨1⟩2. If $\mathcal{P} \subseteq \mathcal{Q}$ then $\mathcal{T}_i \subseteq \mathcal{U}_i$ for all i
⟨2⟩1. ASSUME: $\mathcal{P} \subseteq \mathcal{Q}$
⟨2⟩2. LET: $i \in I$
⟨2⟩3. LET: $U \in \mathcal{T}_i$
⟨2⟩4. LET: $U_i = U$ and $U_j = X_j$ for $j \neq i$
⟨2⟩5. $\prod_{i \in I} U_i \in \mathcal{P}$
⟨2⟩6. $\prod_{i \in I} U_i \in \mathcal{Q}$
⟨2⟩7. $U \in \mathcal{U}_i$
PROOF: From Proposition 134.
□

Proposition 137 (AC)**.** *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let $A_i \subseteq X_i$ for all $i \in I$. Then*

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i} \enspace .$$

PROOF:

⟨1⟩1. $\overline{\prod_{i \in I} A_i} \subseteq \prod_{i \in I} \overline{A_i}$

⟨2⟩1. For all $i \in I$ we have $A_i \subseteq \overline{A_i}$

PROOF: Lemma 52.

⟨2⟩2. $\prod_{i \in I} A_i \subseteq \prod_{i \in I} \overline{A_i}$

⟨2⟩3. Q.E.D.

PROOF: Since $\prod_{i \in I} A_i$ is closed by Proposition 132.

⟨1⟩2. $\prod_{i \in I} \overline{A_i} \subseteq \overline{\prod_{i \in I} A_i}$

⟨2⟩1. LET: $x \in \prod_{i \in I} \overline{A_i}$

⟨2⟩2. LET: U be a neighbourhood of x

⟨2⟩3. PICK V_i open in X_i such that $x \in \prod_{i \in I} V_i \subseteq U$ with $V_i = X_i$ except for $i = i_1, \ldots, i_n$

⟨2⟩4. For $i \in I$, pick $a_i \in V_i \cap A_i$

PROOF: By Theorem 55 and ⟨2⟩1 using the Axiom of Choice.

⟨2⟩5. U intersects $\prod_{i \in I} A_i$

⟨2⟩6. Q.E.D.

PROOF: $a \in U \cap \prod_{i \in I} A_i$

□

Example 138. The closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ is $\mathbb{R}^\omega$

PROOF:

⟨1⟩1. LET: $a \in \mathbb{R}^\omega$

⟨1⟩2. LET: U be any neighbourhoods of a.

⟨1⟩3. PICK U_n open in $\mathbb{R}$ for all n such that $a \in \prod_{n \geq 0} U_n \subseteq U$ and $U_n = \mathbb{R}$ for all n except n_1, ..., n_k

⟨1⟩4. LET: $b_n = a_n$ for $n = n_1, \ldots, n_k$ and $b_n = 0$ for all other n

⟨1⟩5. $b \in \mathbb{R}^\infty \cap U$

⟨1⟩6. Q.E.D.

PROOF: From Theorem 55.

□

Proposition 139. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let (a_n) be a sequence in $\prod_{i \in I} X_i$ and $l \in \prod_{i \in I} X_i$. Then $a_n \to l$ as $n \to \infty$ if and only if, for all $i \in I$, we have $\pi_i(a_n) \to \pi_i(l)$ as $n \to \infty$.*

PROOF:

⟨1⟩1. If $a_n \to l$ as $n \to \infty$ then, for all $i \in I$, we have $\pi_i(a_n) \to \pi_i(l)$ as $n \to \infty$

PROOF: Proposition 113.

⟨1⟩2. If, for all $i \in I$, we have $\pi_i(a_n) \to \pi_i(l)$ as $n \to \infty$, then $a_n \to l$ as $n \to \infty$

⟨2⟩1. ASSUME: For all $i \in I$, we have $\pi_i(a_n) \to \pi_i(l)$ as $n \to \infty$

⟨2⟩2. LET: V be a neighbourhood of l

⟨2⟩3. PICK open sets U_i in X_i such that $l \in \prod_{i \in I} U_i \subseteq V$ and $U_i = X_i$ for all i except $i = i_1, \ldots, i_k$

⟨2⟩4. For $j = 1, \ldots, k$, PICK N_j such that, for all $n \geq N_j$, we have $\pi_{i_j}(a_n) \in U_{i_j}$

⟨2⟩5. LET: $N = \max(N_1, \ldots, N_k)$

⟨2⟩6. For all $n \geq N$ we have $a_n \in V$

□

Theorem 140. *Let A be a topological space and $\{X_i\}_{i \in I}$ be a family of topological spaces. Let $f : A \to \prod_{i \in I} X_i$ be a function. If $\pi_i \circ f$ is continuous for all $i \in I$ then f is continuous.*

Proof:
$\langle 1 \rangle 1$. Let: $i \in I$ and U be open in X_i
$\langle 1 \rangle 2$. $f^{-1}({\pi_i}^{-1}(U))$ is open in A
$\langle 1 \rangle 3$. Q.E.D.
Proof: Proposition 98.
□

21.1 Continuous in Each Variable Separately

Definition 141 (Continuous in Each Variable Separately)**.** Let $F : X \times Y \to Z$. Then F is *continuous in each variable separately* if and only if:

- for every $a \in X$ the function $\lambda y \in Y.F(a, y)$ is continuous;
- for every $b \in Y$ the function $\lambda x \in X.F(x, b)$ is continuous.

Proposition 142. *Let $F : X \times Y \to Z$. If F is continuous then F is continuous in each variable separately.*

Proof: For $a \in X$, the function $\lambda y \in Y.F(a, y)$ is $F \circ i$ where $i : Y \to X \times Y$ maps y to (a, y). We have i is continuous by Proposition 119, hence $F \circ i$ is continuous by Theorem 104.
Similarly for $\lambda x \in X.F(x, b)$ for $b \in Y$. □

Example 143. Define $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$F(x, y) = \begin{cases} xy/(x^2 + y^2) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Then F is continuous in each variable separately but not continuous.

Proposition 144. *Let $f : A \to C$ and $g : B \to D$ be open maps. Then $f \times g : A \times B \to C \times D$ is an open map.*

Proof: Given U open in A and V open in B. Then $(f \times g)(U \times V) = f(U) \times g(V)$ is open in $C \times D$. The result follows from Proposition 95. □

Definition 145 (Sorgenfrey Plane)**.** The *Sorgenfrey plane* is $\mathbb{R}_l^2$.

22 The Subspace Topology

Definition 146 (Subspace Topology)**.** Let X be a topological space and $Y \subseteq X$. The *subspace topology* on Y is $\mathcal{T} = \{U \cap Y \mid U \text{ is open in } X\}$.

We prove this is a topology.

PROOF:
⟨1⟩1. $Y \in \mathcal{T}$
  PROOF: Since $Y = X \cap Y$
⟨1⟩2. For all $\mathcal{U} \subseteq \mathcal{T}$, we have $\bigcup \mathcal{U} \in \mathcal{T}$
  ⟨2⟩1. LET: $\mathcal{U} \subseteq \mathcal{T}$
  ⟨2⟩2. LET: $\mathcal{V} = \{V \text{ open in } X \mid V \cap Y \in \mathcal{U}\}$
  ⟨2⟩3. $\bigcup \mathcal{U} = (\bigcup \mathcal{V}) \cap Y$
⟨1⟩3. For all $U, V \in \mathcal{T}$, we have $U \cap V \in \mathcal{T}$
  ⟨2⟩1. LET: $U, V \in \mathcal{T}$
  ⟨2⟩2. PICK U', V' open in X such that $U = U' \cap Y$ and $V = V' \cap Y$
  ⟨2⟩3. $(U \cap V) = (U' \cap V') \cap Y$
□

Theorem 147. *Let X be a topological space and Y a subspace of X. Let $A \subseteq Y$. Then A is closed in Y if and only if there exists a closed set C in X such that $A = C \cap Y$.*

PROOF: We have

$$\begin{aligned}
& A \text{ is closed in } Y \\
\Leftrightarrow & Y \setminus A \text{ is open in } Y \\
\Leftrightarrow & \exists U \text{ open in } X . Y \setminus A = Y \cap U \\
\Leftrightarrow & \exists C \text{ closed in } X . Y \setminus A = Y \cap (X \setminus U) \\
\Leftrightarrow & \exists C \text{ closed in } X . A = Y \cap U
\end{aligned}$$
□

Theorem 148. *Let Y be a subspace of X. Let $A \subseteq Y$. Let $\overline{A}$ be the closure of A in X. Then the closure of A in Y is $\overline{A} \cap Y$.*

PROOF: The closure of A in Y is

$$\begin{aligned}
& \bigcap \{C \text{ closed in } Y \mid A \subseteq C\} \\
= & \bigcap \{D \cap Y \mid D \text{ closed in } X, A \subseteq D \cap Y\} && \text{(Theorem 147)} \\
= & \bigcap \{D \mid D \text{ closed in } X, A \subseteq D\} \cap Y \\
= & \overline{A} \cap Y
\end{aligned}$$
□

Lemma 149. *Let X be a topological space and $Y \subseteq X$. Let $\mathcal{B}$ be a basis for the topology on X. Then $\mathcal{B}' = \{B \cap Y \mid B \in \mathcal{B}\}$ is a basis for the subspace topology on Y.*

PROOF:
⟨1⟩1. Every element in $\mathcal{B}'$ is open in Y
⟨1⟩2. For every open set U in Y and point $y \in U$, there exists $B' \in \mathcal{B}'$ such that $y \in B' \subseteq U$
  ⟨2⟩1. LET: U be open in Y and $y \in U$
  ⟨2⟩2. PICK V open in X such that $U = V \cap Y$
  ⟨2⟩3. PICK $B \in \mathcal{B}$ such that $y \in B \subseteq V$

⟨2⟩4. LET: $B' = B \cap Y$
⟨2⟩5. $B' \in \mathcal{B}'$
⟨2⟩6. $y \in B' \subseteq U$
⟨1⟩3. Q.E.D.
PROOF: By Lemma 71.
□

Lemma 150. *Let X be a topological space and $Y \subseteq X$. Let $\mathcal{S}$ be a basis for the topology on X. Then $\mathcal{S}' = \{S \cap Y \mid S \in \mathcal{S}\}$ is a subbasis for the subspace topology on Y.*

PROOF: The set $\{B \cap Y \mid B$ is a finite intersection of elements of $\mathcal{S}\}$ is a basis for the subspace topology by Lemma 149, and this is the set of all finite intersections of elements of $\mathcal{S}'$. □

Lemma 151. *Let Y be a subspace of X. If U is open in Y and Y is open in X then U is open in X.*

PROOF:
⟨1⟩1. PICK V open in X such that $U = V \cap Y$
⟨1⟩2. U is open in X
PROOF: Since it is the intersection of two open sets V and Y.
□

Theorem 152. *Let Y be a subspace of X and $A \subseteq Y$. If A is closed in Y and Y is closed in X then A is closed in X.*

PROOF: Pick a closed set C in X such that $A = C \cap Y$ (Theorem 147). Then A is the intersection of two sets closed in X, hence A is closed in X (Lemma 40).
□

Theorem 153. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let A_i be a subspace of X_i for all $i \in I$. Then the product topology on $\prod_{i \in I} A_i$ is the same as the topology it inherits as a subspace of $\prod_{i \in I} X_i$.*

PROOF: The product topology is generated by the subbasis

$$\begin{aligned}
&\{{\pi_i}^{-1}(U) \mid i \in I, U \text{ open in } A_i\} \\
=&\{{\pi_i}^{-1}(V) \cap A_i \mid i \in I, V \text{ open in } X_i\} \\
=&\{{\pi_i}^{-1}(V) \mid i \in I, V \text{ open in } X_i\} \cap \prod_{i \in I} A_i
\end{aligned}$$

and this is a subbasis for the subspace topology by Lemma 150. □

Theorem 154. *Let X be an ordered set in the order topology. Let $Y \subseteq X$ be an interval. Then the order topology on Y is the same as the subspace topology on Y.*

PROOF:
⟨1⟩1. The order topology is finer than the subspace topology.

⟨2⟩1. For every open ray R in X, the set $R \cap Y$ is open in the order topology.
⟨3⟩1. For all $a \in X$, we have $(-\infty, a) \cap Y$ is open in the order topology.
⟨4⟩1. CASE: For all $y \in Y$ we have $y < a$
PROOF: In this case $(-\infty, a) \cap Y = Y$.
⟨4⟩2. CASE: For all $y \in Y$ we have $a < y$
PROOF: In this case $(-\infty, a) \cap Y = \emptyset$.
⟨4⟩3. CASE: There exists $y \in Y$ such that $y \leq a$ and $y \in Y$ such that $a \leq y$
⟨5⟩1. $a \in Y$
PROOF: Because Y is an interval.
⟨5⟩2. $(-\infty, a) \cap Y = \{y \in Y \mid y < a\}$
⟨3⟩2. For all $a \in X$, we have $(a, +\infty) \cap Y$ is open in the order topology.
PROOF: Similar.
⟨2⟩2. Q.E.D.
PROOF: By Lemmas 121 and 150 and Proposition 79.
⟨1⟩2. The subspace topology is finer than the order topology.
⟨2⟩1. Every open ray in Y is open in the subspace topology.
PROOF: For any $a \in Y$ we have $(-\infty, a)_Y = (-\infty, a)_X \cap Y$ and $(a, +\infty)_Y = (a, +\infty)_X \cap Y$.
⟨2⟩2. Q.E.D.
PROOF: By Lemma 121 and Proposition 79
□

This example shows that we cannot remove the hypothesis that Y is an interval:

Example 155. The order topology on I_o^2 is different from the subspace topology as a subspace of $\mathbb{R}^2$ under the dictionary order topology.

PROOF: The set $\{1/2\} \times (1/2, 1)$ is open in the subspace topology but not in the order topology. □

Proposition 156. *Let X be a topological space, Y a subspace of X, and Z a subspace of Y. Then the subspace topology on Z inherited from X is the same as the subspace topology on Z inherited from Y.*

PROOF: The subspace topology inherited from Y is

$$\begin{aligned} &\{V \cap Z \mid V \text{ open in } Y\} \\ =&\{U \cap Y \cap Z \mid U \text{ open in } X\} \\ =&\{U \cap Z \mid U \text{ open in } X\} \end{aligned}$$

which is the subspace topology inherited from X. □

Definition 157 (Unit Circle)**.** The *unit circle* S^1 is $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ as a subspace of $\mathbb{R}^2$.

Definition 158 (Unit 2-sphere)**.** The *unit 2-sphere* is $S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$ as a subspace of $\mathbb{R}^3$.

Proposition 159. *Let $f : X \to Y$ be an open map and $A \subseteq X$ be open. Then the restriction $f \restriction A : A \to f(A)$ is an open map.*

PROOF:
$\langle 1 \rangle 1$. LET: U be open in A
$\langle 1 \rangle 2$. U is open in X
PROOF: Lemma 151.
$\langle 1 \rangle 3$. $f(U)$ is open in Y
$\langle 1 \rangle 4$. $f(U)$ is open in $f(A)$
PROOF: Since $f(U) = f(U) \cap f(A)$.
□

Example 160. This example shows that we cannot remove the hypothesis that A is open.

Let $A = \{(x, y) \in \mathbb{R}^2 \mid (x > 0 \text{ and } y = 1/x) \text{ or } x = y = 0\}$. Then $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ is an open map, but $\pi_1 \restriction A : A \to [0, +\infty)$ is not, because it maps the set $\{0, 0\}$ which is open in A to $\{0\}$ which is not open in $[0, +\infty)$.

Proposition 161. *Let Y be a subspace of X. Let $A \subseteq Y$ and $l \in Y$. Then l is a limit point of A in Y if and only if l is a limit point of A in X.*

PROOF: Both are equivalent to the condition that any neighbourhood of l in X intersects A in a point other than l. □

23 The Box Topology

Definition 162 (Box Topology)**.** Let $\{A_i\}_{i \in I}$ be a family of topological spaces. The *box topology* on $\prod_{i \in I} A_i$ is the topology generated by the set of all sets of the form $\prod_{i \in I} U_i$ where $\{U_i\}_{i \in I}$ is a family such that each U_i is open in A_i.

This is a basis since it covers $\prod_{i \in I} A_i$ and is closed under intersection.

Proposition 163. *The box topology is finer than the product topology.*

PROOF: From Proposition 131. □

Corollary 163.1. *If A_i is closed in X_i for all $i \in I$ then $\prod_{i \in I} A_i$ is closed in $\prod_{i \in I} X_i$ under the box topology.*

PROOF: From Proposition 132.

Proposition 164 (AC)**.** *Let $\{A_i\}_{i \in I}$ be a family of topological spaces. For $i \in I$, let $\mathcal{B}_i$ be a basis for the topology on A_i. Then $\mathcal{B} = \{\prod_{i \in I} B_i \mid \forall i \in I . B_i \in \mathcal{B}_i\}$ is a basis for the box topology on $\prod_{i \in I} A_i$.*

PROOF:
$\langle 1 \rangle 1$. Every set of the form $\prod_{i \in I} B_i$ is open.
$\langle 1 \rangle 2$. For every point $a \in \prod_{i \in I} A_i$ and every open set U with $a \in U$, there exists $B \in \mathcal{B}$ such that $a \in B \subseteq U$.

⟨2⟩1. LET: U be open and $a \in U$

⟨2⟩2. PICK a family $\{U_i\}_{i \in I}$ such that each U_i is open in A_i and $a \in \prod_{i \in I} U_i \subseteq U$.

⟨2⟩3. For $i \in I$, PICK $B_i \in \mathcal{B}_i$ such that $a_i \in B_i \subseteq U_i$

PROOF: Using the Axiom of Choice.

⟨2⟩4. $a \in \prod_{i \in I} B_i \subseteq U$

⟨1⟩3. Q.E.D.

PROOF: Lemma 71.

□

Theorem 165. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let A_i be a subspace of X_i for all $i \in I$. Give $\prod_{i \in I} X_i$ the box topology. Then the box topology on $\prod_{i \in I} A_i$ is the same as the topology it inherits as a subspace of $\prod_{i \in I} X_i$.*

PROOF: The box topology is generated by the basis

$$\begin{aligned}
&\{\prod_{i \in I} U_i \mid \forall i \in I, U_i \text{ open in } A_i\} \\
=&\{\prod_{i \in I} (V_i \cap A_i) \mid \forall i \in I, V_i \text{ open in } X_i\} \\
=&\{\prod_{i \in I} V_i \mid \forall i \in I, V_i \text{ open in } X_i\} \cap \prod_{i \in I} A_i
\end{aligned}$$

and this is a basis for the subspace topology by Lemma 149. □

Proposition 166 (AC)**.** *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Give $\prod_{i \in I} X_i$ the box topology. Let $A_i \subseteq X_i$ for all $i \in I$. Then*

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i} \enspace .$$

PROOF:

⟨1⟩1. $\overline{\prod_{i \in I} A_i} \subseteq \prod_{i \in I} \overline{A_i}$

⟨2⟩1. For all $i \in I$ we have $A_i \subseteq \overline{A_i}$

PROOF: Lemma 52.

⟨2⟩2. $\prod_{i \in I} A_i \subseteq \prod_{i \in I} \overline{A_i}$

⟨2⟩3. Q.E.D.

PROOF: Since $\prod_{i \in I} A_i$ is closed by Corollary 163.1.

⟨1⟩2. $\prod_{i \in I} \overline{A_i} \subseteq \overline{\prod_{i \in I} A_i}$

⟨2⟩1. LET: $x \in \prod_{i \in I} \overline{A_i}$

⟨2⟩2. LET: U be a neighbourhood of x

⟨2⟩3. PICK V_i open in X_i such that $x \in \prod_{i \in I} V_i \subseteq U$

⟨2⟩4. For $i \in I$, pick $a_i \in V_i \cap A_i$

PROOF: By Theorem 55 and ⟨2⟩1 using the Axiom of Choice.

⟨2⟩5. U intersects $\prod_{i \in I} A_i$

⟨2⟩6. Q.E.D.

PROOF: $a \in U \cap \prod_{i \in I} A_i$.

□

The following example shows that Theorem 140 fails in the box topology.

Example 167. Define $f : \mathbb{R} \to \mathbb{R}^\omega$ by $f(t) = (t, t, \ldots)$. Then $\pi_n \circ f = \mathrm{id}_\mathbb{R}$ is continuous for all n. But f is not continuous when $\mathbb{R}^\omega$ is given the box topology because the inverse image of

$$(-1, 1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times \cdots$$

is $\{0\}$ which is not open.

The following example shows that Proposition 139 fails in the box topology.

Example 168. Give $\mathbb{R}^\omega$ the box topology. Let $a_n = (1/n, 1/n, \ldots)$ for $n \geq 1$ and $l = (0, 0, \ldots)$. Then $\pi_i(a_n) \to \pi_i(l)$ as $n \to \infty$ for all i, but $a_n \not\to l$ as $n \to \infty$ since the open set

$$(-1, 1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times \cdots$$

contains l but does not contain any a_n.

Example 169. The set $\mathbb{R}^\infty$ is closed in $\mathbb{R}^\omega$ under the box topology. For let (a_n) be any sequence not in $\mathbb{R}^\infty$. Let U_n be an open interval around a_n that does not contain 0 if $a_n \neq 0$, and $U_n = \mathbb{R}$ if $a_n = 0$. Then $\prod_{n \geq 0} U_n$ is a neighbourhood of (a_n) that does not intersect $\mathbb{R}^\infty$.

24 T_1 Spaces

Definition 170 (T_1 Space)**.** A topological space is T_1 if and only if every singleton is closed.

Lemma 171. *A space is T_1 if and only if every finite set is closed.*

PROOF: From Lemma 41. □

Theorem 172. *In a T_1 space, a point a is a limit point of a set A if and only if every neighbourhood of a contains infinitely many points of A.*

PROOF:
⟨1⟩1. If a is a limit point of A then every neighbourhood of a contains infinitely many points of A.
⟨2⟩1. ASSUME: a is a limit point of A.
⟨2⟩2. LET: U be a neighbourhood of a.
⟨2⟩3. ASSUME: for a contradiction U contains only finitely many points of A.
⟨2⟩4. $(U \cap A) \setminus \{a\}$ is closed.
PROOF: By the T_1 axiom.
⟨2⟩5. $(U \setminus A) \cup \{a\}$ is open.
PROOF: It is $U \setminus ((U \cap A) \setminus \{a\})$.
⟨2⟩6. $(U \setminus A) \cup \{a\}$ intersects A in a point other than a.

PROOF: From ⟨2⟩1.

⟨2⟩7. Q.E.D.

□

⟨1⟩2. If every neighbourhood of a contains infinitely many points of A then a is a limit point of A.

PROOF: Immediate from definitions.

□

(To see this does not hold in every space, see Proposition 67.)

Proposition 173. *A space is T_1 if and only if, for any two distinct points x and y, there exist neighbourhoods U of x and V of y such that $x \notin V$ and $y \notin U$.*

PROOF:

⟨1⟩1. LET: X be a topological space.

⟨1⟩2. If X is T_1 then, for any two distinct points x and y, there exist neighbourhoods U of x and V of y such that $x \notin V$ and $y \notin U$.

PROOF: This holds because $\{x\}$ and $\{y\}$ are closed.

⟨1⟩3. Suppose, for any two distinct points x and y, there exist neighbourhoods U of x and V of y such that $x \notin V$ and $y \notin U$. Then X is T_1.

⟨2⟩1. ASSUME: For any two distinct points x and y, there exist neighbourhoods U of x and V of y such that $x \notin V$ and $y \notin U$.

⟨2⟩2. LET: $a \in X$

⟨2⟩3. $\{a\}$ is closed.

PROOF: For all $b \neq a$ there exists a neighbourhood U of b such that $U \subseteq X \setminus \{a\}$.

□

Proposition 174. *A subspace of a T_1 space is T_1.*

PROOF: From Proposition 152.

25 Hausdorff Spaces

Definition 175 (Hausdorff Space)**.** A topological space is *Hausdorff* if and only if, for any points x, y with $x \neq y$, there exist disjoint open sets U and V such that $x \in U$ and $y \in V$.

Theorem 176. *Every Hausdorff space is T_1.*

PROOF:

⟨1⟩1. LET: X be a Hausdorff space.

⟨1⟩2. LET: $b \in X$

PROVE: $\overline{\{b\}} = \{b\}$

⟨1⟩3. ASSUME: $a \in \overline{\{b\}}$ and $a \neq b$

⟨1⟩4. PICK disjoint neighbourhoods U of a and V of b.

⟨1⟩5. U intersects $\{b\}$

PROOF: Theorem 55.

⟨1⟩6. $b \in U$
⟨1⟩7. Q.E.D.
Proof: This contradicts the fact that U and V are disjoint (⟨1⟩4).
□

Proposition 177. *An infinite set under the finite complement topology is T_1 but not Hausdorff.*

Proof:
⟨1⟩1. Let: X be an infinite set under the finite complement topology.
⟨1⟩2. Every singleton is closed.
Proof: By definition.
⟨1⟩3. Pick$a, b \in X$ with $a \neq b$
⟨1⟩4. There are no disjoint neighbourhoods U of a and V of b.
⟨2⟩1. Let: U be a neighbourhood of a and V a neighbourhood of b.
⟨2⟩2. $X \setminus U$ and $X \setminus V$ are finite.
⟨2⟩3. Pick $c \in X$ that is not in $X \setminus U$ or $X \setminus V$.
⟨2⟩4. $c \in U \cap V$
□

Proposition 178. *The product of a family of Hausdorff spaces is Hausdorff.*

Proof:
⟨1⟩1. Let: $\{X_i\}_{i \in I}$ be a family of Hausdorff spaces.
⟨1⟩2. Let: $a, b \in \prod_{i \in I} X_i$ with $a \neq b$
⟨1⟩3. Pick $i \in I$ such that $a_i \neq b_i$
⟨1⟩4. Pick U, V disjoint open sets in X_i with $a_i \in U$ and $b_i \in V$
⟨1⟩5. ${\pi_i}^{-1}(U)$ and ${\pi_i}^{-1}(V)$ are disjoint open sets in $\prod_{i \in I} X_i$ with $a \in {\pi_i}^{-1}(U)$ and $b \in {\pi_i}^{-1}(V)$
□

Theorem 179. *Every linearly ordered set under the order topology is Hausdorff.*

Proof:
⟨1⟩1. Let: X be a linearly ordered set under the order topology.
⟨1⟩2. Let: $a, b \in X$ with $a \neq b$
⟨1⟩3. Assume: w.l.o.g. $a < b$
⟨1⟩4. Case: There exists c such that $a < c < b$
Proof: The sets $(-\infty, c)$ and $(c, +\infty)$ are disjoint neighbourhoods of a and b respectively.
⟨1⟩5. Case: There is no c such that $a < c < b$
Proof: The sets $(-\infty, b)$ and $(a, +\infty)$ are disjoint neighbourhoods of a and b respectively.
□

Theorem 180. *A subspace of a Hausdorff space is Hausdorff.*

Proof:
⟨1⟩1. Let: X be a Hausdorff space and Y a subspace of X.

$\langle 1\rangle 2$. LET: $x, y \in Y$ with $x \neq y$
$\langle 1\rangle 3$. PICK disjoint neighbourhoods U of x and V of y in X.
$\langle 1\rangle 4$. $U \cap Y$ and $V \cap Y$ are disjoint neighbourhoods of x and y respectively in Y.
□

Proposition 181. *A space X is Hausdorff if and only if the diagonal $\Delta = \{(x, x) \mid x \in X\}$ is closed in X^2.*

PROOF:

$$\begin{aligned} & X \text{ is Hausdorff} \\ \Leftrightarrow & \forall x, y \in X. x \neq y \Rightarrow \exists V, W \text{ open}. x \in V \wedge y \in W \wedge V \cap W = \emptyset \\ \Leftrightarrow & \forall (x, y) \in X^2 \setminus \Delta. \exists V, W \text{ open}. (x, y) \subseteq V \times W \subseteq X^2 \setminus \Delta \\ \Leftrightarrow & \Delta \text{ is closed} \end{aligned}$$
□

Theorem 182. *In a Hausdorff space, a sequence has at most one limit.*

PROOF:
$\langle 1\rangle 1$. LET: X be a Hausdorff space.
$\langle 1\rangle 2$. ASSUME: for a contradiction $a_n \to l$ as $n \to \infty$, $a_n \to m$ as $n \to \infty$, and $l \neq m$
$\langle 1\rangle 3$. PICK disjoint neighbourhoods U of l and V of m
PROOF: By the Hausdorff axiom.
$\langle 1\rangle 4$. PICK M and N such that $a_n \in U$ for $n \geq M$ and $a_n \in V$ for $n \geq N$
$\langle 1\rangle 5$. $a_{\max(M,N)} \in U \cap V$
$\langle 1\rangle 6$. Q.E.D.
PROOF: This contradicts the fact that U and V are disjoint ($\langle 1\rangle 3$).
□

To see this is not always true in spaces that are T_1 but not Hausdorff:

Proposition 183. *Let X be an infinite set under the finite complement topology. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence with all points distinct. Then for every $l \in X$ we have $a_n \to l$ as $n \to \infty$.*

PROOF: Let U be any neighbourhood of l. Since $X \setminus U$ is finite, there must exist N such that, for all $n \geq N$, we have $a_n \in U$. □

Proposition 184. *Let X be a topological space. Let Y a Hausdorff space. Let $A \subseteq X$. Let $f, g : \overline{A} \to Y$ be continuous. If f and g agree on A then $f = g$.*

PROOF:
$\langle 1\rangle 1$. LET: $x \in \overline{A}$
$\langle 1\rangle 2$. ASSUME: $f(x) \neq g(x)$
$\langle 1\rangle 3$. PICK disjoint neighbourhoods V of $f(x)$ and W of $g(x)$.
$\langle 1\rangle 4$. PICK $y \in f^{-1}(V) \cap g^{-1}(W) \cap A$
PROOF: Since $f^{-1}(V) \cap g^{-1}(W)$ is a neighbourhood of x and hence intersects A.

$\langle 1\rangle 5.$ $f(y) = g(y) \in V \cap W$
$\langle 1\rangle 6.$ Q.E.D.
PROOF: This contradicts the fact that V and W are disjoint ($\langle 1\rangle 3$).
□

Proposition 185. *Let $\{X_i\}_{i \in I}$ be a family of Hausdorff spaces. Then $\prod_{i \in I} X_i$ under the box topology is Hausdorff.*

PROOF:
$\langle 1\rangle 1.$ LET: $\{X_i\}_{i \in I}$ be a family of Hausdorff spaces.
$\langle 1\rangle 2.$ LET: $a, b \in \prod_{i \in I} X_i$ with $a \neq b$
$\langle 1\rangle 3.$ PICK $i \in I$ such that $a_i \neq b_i$
$\langle 1\rangle 4.$ PICK U, V disjoint open sets in X_i with $a_i \in U$ and $b_i \in V$
$\langle 1\rangle 5.$ ${\pi_i}^{-1}(U)$ and ${\pi_i}^{-1}(V)$ are disjoint open sets in $\prod_{i \in I} X_i$ with $a \in {\pi_i}^{-1}(U)$ and $b \in {\pi_i}^{-1}(V)$
□

Proposition 186. *Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on the same set X with $\mathcal{T} \subseteq \mathcal{T}'$ If $\mathcal{T}$ is Haudorff then $\mathcal{T}'$ is Hausdorff.*

PROOF: Immediate from definitions.

Proposition 187. *Let X be a Hausdorff space. Let $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the finite intersection property. Then $\bigcap_{D \in \mathcal{D}} \overline{D}$ contains at most one point.*

PROOF:
$\langle 1\rangle 1.$ LET: $x, y \in \bigcap_{D \in \mathcal{D}} \overline{D}$
$\langle 1\rangle 2.$ ASSUME: for a contradiction $x \neq y$
$\langle 1\rangle 3.$ PICK disjoint open subsets U and V of x and y respectively.
$\langle 1\rangle 4.$ $U, V \in \mathcal{D}$
PROOF: Proposition 58.
$\langle 1\rangle 5.$ Q.E.D.
PROOF: This contradicts the fact that $\mathcal{D}$ satisfies the finite intersection property.
□

26 The First Countability Axiom

Definition 188 (First Countability Axiom)**.** A topological space X satisfies the *first countability axiom*, or is *first countable*, if and only if every point has a countable local basis.

Lemma 189 (Sequence Lemma (CC))**.** *Let X be a first countable space. Let $A \subseteq X$ and $l \in \overline{A}$. Then there exists a sequence in A that converges to l.*

PROOF:
$\langle 1\rangle 1.$ PICK a countable local basis $\{B_n \mid n \in \mathbb{Z}^+\}$ at l such that $B_1 \supseteq B_2 \supseteq \cdots$.

PROOF: Lemma 81.
⟨1⟩2. For all $n \geq 1$, PICK $a_n \in A \cap B_n$.
PROVE: $a_n \to l$ as $n \to \infty$
⟨1⟩3. LET: U be a neighbourhood of A
⟨1⟩4. PICK N such that $B_N \subseteq U$
⟨1⟩5. For $n \geq N$ we have $a_n \in U$
PROOF: $a_n \in B_n \subseteq B_N \subseteq U$
□

Theorem 190 (CC)**.** *Let X be a first countable space and Y a topological space. Let $f : X \to Y$. Suppose that, for every sequence (x_n) in X and $l \in X$, if $x_n \to l$ as $n \to \infty$, then $f(x_n) \to f(l)$ as $n \to \infty$. Then f is continuous.*

PROOF:
⟨1⟩1. LET: $A \subseteq X$
⟨1⟩2. LET: $a \in A$
PROVE: $f(a) \in \overline{f(A)}$
⟨1⟩3. PICK a sequence (x_n) in A that converges to a.
PROOF: By the Sequence Lemma.
⟨1⟩4. $f(x_n) \to f(a)$
⟨1⟩5. $f(a) \in \overline{f(A)}$
PROOF: By Lemma 83.
⟨1⟩6. Q.E.D.
PROOF: By Theorem 101.
□

Example 191 (CC)**.** The space $\mathbb{R}^\omega$ under the box product is not first countable.

PROOF: Let $\{B_n \mid n \geq 0\}$ be a countable set of neighbourhoods of $\vec{0}$. We will construct a neighbourhood of $\vec{0}$ that does not include any of these.
For $n \geq 0$, pick a neighbourhood U_n of 0 such that $U_n \subset \pi_n(B_p)$. Then $\prod_{n=0}^\infty U_n$ is a neighbourhood of $\vec{0}$ that does not include any B_n. □

Example 192. If J is an uncountable set then $\mathbb{R}^J$ is not first countable.

PROOF:
⟨1⟩1. LET: $\{B_n \mid n \geq 0\}$ be any countable set of neighbourhoods of $\vec{0}$.
⟨1⟩2. For $n \geq 0$, PICK a basis element $\prod_{\alpha \in J} U_{n\alpha}$ that contains $\vec{0}$ and is included in B_n.
PROOF: Using the Axiom of Countable Choice.
⟨1⟩3. For $n \geq 0$,
LET: $J_n = \{\alpha \in J \mid U_{n\alpha} \neq \mathbb{R}\}$
⟨1⟩4. PICK $\beta \in J$ such that $\beta \notin J_n$ for any n.
PROOF: Since each J_n is finite so $\bigcup_n J_n$ is countable.
⟨1⟩5. $\pi_\beta((-1, 1))$ is a neighbourhood of $\vec{0}$ that does not include any B_n.
□

Example 193. The space $\mathbb{R}_l$ is first countable.

PROOF: For any $a \in \mathbb{R}$, the set $\{[a, a + 1/n) \mid n \geq 1\}$ is a countable local basis.

Example 194. The ordered square is first countable.

PROOF: For any $(a, b) \in I_o^2$ with $b \neq 0, 1$, the set $\{(\{a\} \times (b-1/n, b+1/n)) \cap I_o^2 \mid n \geq 1\}$ is a countable local basis.

27 Strong Continuity

Definition 195 (Strongly Continuous)**.** Let X and Y be topological spaces and $f : X \to Y$ be a function. Then f is *strongly continuous* if and only if, for every subset $U \subseteq Y$, we have U is open in Y if and only if $f^{-1}(U)$ is open in X.

Proposition 196. *Let X and Y be topological spaces and $f : X \to Y$ be a function. Then f is* strongly continuous *if and only if, for every subset $C \subseteq Y$, we have C is closed in Y if and only if $f^{-1}(C)$ is closed in X.*

PROOF: Since $X \setminus f^{-1}(C) = f^{-1}(Y \setminus C)$. $\square$

Proposition 197. *Let X, Y and Z be topological spaces. Let $f : X \to Y$ and $g : Y \to Z$. If f and g are strongly continuous then so is $g \circ f$.*

PROOF: Since $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$. $\square$

Proposition 198. *Let X, Y and Z be topological spaces. Let $f : X \to Y$ and $g : Y \to Z$. If $g \circ f$ is continuous and f is strongly continuous then g is continuous.*

PROOF:
$\langle 1 \rangle 1$. LET: $V \subseteq Z$ be open.
$\langle 1 \rangle 2$. $f^{-1}(g^{-1}(V))$ is open in X.
  PROOF: Since $g \circ f$ is continuous.
$\langle 1 \rangle 3$. $f^{-1}(V)$ is open in Y.
  PROOF: Since g is strongly continuous.
$\square$

Proposition 199. *Let X, Y and Z be topological spaces. Let $f : X \to Y$ and $g : Y \to Z$. If $g \circ f$ is strongly continuous and f is strongly continuous then g is strongly continuous.*

PROOF: For $V \subseteq Z$, we have V is open iff $f^{-1}(g^{-1}(V))$ is open iff $g^{-1}(V)$ is open.

28 Saturated Sets

Definition 200. Let X and Y be sets and $p : X \twoheadrightarrow Y$ a surjective function. Let $C \subseteq X$. Then C is *saturated* with respect to p if and only if, for all $x, y \in X$, if $x \in C$ and $p(x) = p(y)$ then $y \in C$.

Proposition 201. *Let X and Y be sets and $p : X \twoheadrightarrow Y$ a surjective function. Let $C \subseteq X$. Then the following are equivalent:*

1. *C is saturated with respect to p.*
2. *There exists $D \subseteq Y$ such that $C = p^{-1}(D)$*
3. *$C = p^{-1}(p(C))$.*

Proof:
$\langle 1 \rangle 1$. $1 \Rightarrow 3$
  $\langle 2 \rangle 1$. Assume: C is saturated with respect to p.
  $\langle 2 \rangle 2$. $C \subseteq p^{-1}(p(C))$
    Proof: Trivial.
  $\langle 2 \rangle 3$. $p^{-1}(p(C)) \subseteq C$
    $\langle 3 \rangle 1$. Let: $x \in p^{-1}(p(C))$
    $\langle 3 \rangle 2$. $p(x) \in p(C)$
    $\langle 3 \rangle 3$. There exists $y \in C$ such that $p(x) = p(y)$
    $\langle 3 \rangle 4$. $x \in C$
      Proof: From $\langle 2 \rangle 1$.
$\langle 1 \rangle 2$. $3 \Rightarrow 2$
  Proof: Trivial.
$\langle 1 \rangle 3$. $2 \Rightarrow 1$
  Proof: This follows because if $p(x) \in D$ and $p(x) = p(y)$ then $p(y) \in D$.
□

29 Quotient Maps

Definition 202 (Quotient Map)**.** Let X and Y be topological spaces and $p : X \to Y$. Then p is a *quotient map* if and only if p is surjective and strongly continuous.

Proposition 203. *Let X and Y be topological spaces and $p : X \twoheadrightarrow Y$ be a surjective function. Then the following are equivalent.*

1. *p is a quotient map.*
2. *p is continuous and maps saturated open sets to open sets.*
3. *p is continuous and maps saturated closed sets to closed sets.*

Proof:
$\langle 1 \rangle 1$. $1 \Rightarrow 2$
  $\langle 2 \rangle 1$. Assume: p is a quotient map.
  $\langle 2 \rangle 2$. Let: U be a saturated open set in X.
  $\langle 2 \rangle 3$. $p^{-1}(p(U))$ is open in X.
    Proof: Since $U = p^{-1}(p(U))$ be Proposition 201.
  $\langle 2 \rangle 4$. $p(U)$ is open in Y.

PROOF: From $\langle 2\rangle 1$.

$\langle 1\rangle 2$. $1 \Rightarrow 3$

PROOF: Similar.

$\langle 1\rangle 3$. $2 \Rightarrow 1$

$\langle 2\rangle 1$. ASSUME: p is continuous and maps saturated open sets to open sets.

$\langle 2\rangle 2$. LET: $U \subseteq Y$

$\langle 2\rangle 3$. ASSUME: $p^{-1}(U)$ is open in X

$\langle 2\rangle 4$. $p^{-1}(U)$ is saturated.

PROOF: Proposition 201.

$\langle 2\rangle 5$. U is open in Y.

$\langle 1\rangle 4$. $3 \Rightarrow 1$

PROOF: Similar.

□

Corollary 203.1. *Every surjective continuous open map is a quotient map.*

Corollary 203.2. *Every surjective continuous closed map is a quotient map.*

Example 204. The converses of these corollaries do not hold.

Let $A = \{(x, y) \mid x \geq 0\} \cup \{(x, y) \mid y = 0\}$. Then $\pi_1 : A \to \mathbb{R}$ is a quotient map, but not an open map or a closed map.

We prove that π_1 maps saturated open sets to open sets:

$\langle 1\rangle 1$. LET: ${\pi_1}^{-1}(U)$ be a saturated open set in A

PROVE: U is open in $\mathbb{R}$

$\langle 1\rangle 2$. LET: $x \in U$

$\langle 1\rangle 3$. $(x, 0) \in \pi_1(U)^{-1}$

$\langle 1\rangle 4$. PICK W, V open in $\mathbb{R}$ such that $(x, 0) \subseteq W \times V \subseteq \pi_1(U)^{-1}$

$\langle 1\rangle 5$. $x \in W \subseteq U$

It is not an open map because it maps $((-1, 1) \times (1, 2)) \cap A$ to $[0, 1)$.

It is not a closed map because it maps $\{(x, 1/x) \mid x > 0\}$ to $(0, +\infty)$.

Proposition 205. *Let $p : X \twoheadrightarrow Y$ be a quotient map. Let $A \subseteq X$ be saturated with respect to p. Let $q : A \twoheadrightarrow p(A)$ be the restriction of p.*

1. *If A is either open or closed in X then q is a quotient map.*
2. *If p is either an open map or a closed map then q is a quotient map.*

PROOF:

$\langle 1\rangle 1$. LET: $p : X \twoheadrightarrow Y$ be a quotient map.

$\langle 1\rangle 2$. LET: $A \subseteq X$ be saturated with respect to p.

$\langle 1\rangle 3$. LET: $q : A \twoheadrightarrow p(A)$ be the restriction of p.

$\langle 1\rangle 4$. q is continuous.

PROOF: Theorem 105.

$\langle 1\rangle 5$. If A is open in X then q is a quotient map.

$\langle 2\rangle 1$. ASSUME: A is open in X.

$\langle 2\rangle 2$. q maps saturated open sets to open sets.

⟨3⟩1. LET: $U \subseteq A$ be saturated with respect to q and open in A
⟨3⟩2. U is saturated with respect to p
⟨4⟩1. LET: $x, y \in X$
⟨4⟩2. ASSUME: $x \in U$
⟨4⟩3. ASSUME: $p(x) = p(y)$
⟨4⟩4. $x \in A$
PROOF: From ⟨3⟩1 and ⟨4⟩2.
⟨4⟩5. $y \in A$
PROOF: From ⟨1⟩2 and ⟨4⟩3
⟨4⟩6. $q(x) = x(y)$
PROOF: From ⟨1⟩3, ⟨4⟩3, ⟨4⟩4, ⟨4⟩5.
⟨4⟩7. $y \in U$
PROOF: From ⟨3⟩1, ⟨4⟩2, ⟨4⟩6
⟨3⟩3. U is open in X
PROOF: Lemma 151, ⟨2⟩1, ⟨3⟩1.
⟨3⟩4. $p(U)$ is open in Y
PROOF: Proposition 203, ⟨1⟩1, ⟨3⟩2, ⟨3⟩3
⟨3⟩5. $q(U)$ is open in $p(A)$
PROOF: Since $q(U) = p(U) = p(U) \cap p(A)$.
⟨2⟩3. Q.E.D.
PROOF: By Proposition 203.
⟨1⟩6. If A is closed in X then q is a quotient map.
PROOF: Similar.
⟨1⟩7. If p is an open map then q is a quotient map.
⟨2⟩1. ASSUME: p is an open map
⟨2⟩2. q maps saturated open sets to open sets.
⟨3⟩1. LET: U be open in A and saturated with respect to q
⟨3⟩2. PICK V open in X such that $U = A \cap V$
⟨3⟩3. $p(V)$ is open in Y
⟨3⟩4. $q(U) = p(V) \cap p(A)$
⟨4⟩1. $q(U) \subseteq p(V) \cap p(A)$
PROOF: From ⟨3⟩2.
⟨4⟩2. $p(V) \cap p(A) \subseteq q(U)$
⟨5⟩1. LET: $y \in p(V) \cap p(A)$
⟨5⟩2. PICK $x \in V$ and $x' \in A$ such that $p(x) = p(x') = y$
⟨5⟩3. $x \in A$
PROOF: By ⟨1⟩2.
⟨5⟩4. $x \in U$
PROOF: From ⟨3⟩2
⟨2⟩3. Q.E.D.
PROOF: By Proposition 203.
⟨1⟩8. If p is a closed map then q is a quotient map.
PROOF: Similar.
□

Example 206. This example shows we cannot remove the hypotheses on A

and p.

Define $f : [0,1] \to [2,3] \to [0,2]$ by $f(x) = x$ if $x \leq 1$, $f(x) = x - 1$ if $x \geq 2$. Then f is a quotient map but its restriction f' to $[0,1) \cup [2,3]$ is not, because ${f'}^{-1}([1,2])$ is open but $[1,2]$ is not.

For a counterexample where A is saturated, see Example 212.

Proposition 207. *Let $p : A \twoheadrightarrow C$ and $q : B \twoheadrightarrow D$ be open quotient maps. Then $p \times q : A \times B \to C \times D$ is an open quotient map.*

Proof: From Corollary 203.1, Proposition 144 and Theorem 140. ∎

Theorem 208. *Let $p : X \twoheadrightarrow Y$ be a quotient map. Let Z be a topological space and $f : Y \to Z$ be a function. Then*

1. *$f \circ p$ is continuous if and only if f is continuous.*
2. *$f \circ p$ is a quotient map if and only if f is a quotient map.*

Proof:
⟨1⟩1. If $f \circ p$ is continuous then f is continuous.
  Proof: Proposition 198.
⟨1⟩2. If f is continuous then $f \circ p$ is continuous.
  Proof: Theorem 104.
⟨1⟩3. If $f \circ p$ is a quotient map then f is a quotient map.
  Proof: Proposition 199.
⟨1⟩4. If f is a quotient map then $f \circ p$ is a quotient map.
  Proof: From Proposition 197.
∎

Proposition 209. *Let X and Y be topological spaces. Let $p : X \to Y$ and $f : Y \to X$ be continuous maps such that $p \circ f = \mathrm{id}_Y$. Then p is a quotient map.*

Proof:
⟨1⟩1. Let: $V \subseteq Y$
⟨1⟩2. Assume: $p^{-1}(V)$ is open in X.
⟨1⟩3. $f^{-1}(p^{-1}(V))$ is open in Y.
  Proof: Because f is continuous.
⟨1⟩4. V is open in Y.
  Proof: Because $f^{-1}(p^{-1}(V)) = V$.
∎

30 Quotient Topology

Definition 210 (Quotient Topology)**.** Let X be a topological space, Y a set and $p : X \twoheadrightarrow Y$ be a surjective function. Then the *quotient topology* on Y is the unique topology on Y with respect to which p is a quotient map, namely $\mathcal{T} = \{U \in \mathcal{P}Y \mid p^{-1}(U) \text{ is open in } X\}$.

We prove this is a topology.

PROOF:
$\langle 1\rangle 1.$ $Y \in \mathcal{T}$
  PROOF: Since $p^{-1}(Y) = X$ by surjectivity.
$\langle 1\rangle 2.$ For all $\mathcal{A} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{A} \in \mathcal{T}$
  PROOF: Since $p^{-1}(\bigcup \mathcal{A}) = \bigcup_{U \in \mathcal{A}} p^{-1}(U)$
$\langle 1\rangle 3.$ For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$
  PROOF: Since $p^{-1}(U \cap V) = p^{-1}(U) \cap p^{-1}(V)$.
$\square$

Definition 211 (Quotient Space)**.** Let X be a topological space and $\sim$ an equivalence relation on X. Let $p : X \twoheadrightarrow X/\sim$ be the canonical surjection. Then $X/\sim$ under the quotient topology is called a *quotient space*, *identification space* or *decomposition space* of X.

Here is a counterexample showing we cannot remove all the hypotheses of Proposition 205 except that A is saturated.

Example 212. Let $X = (0, 1/2] \cup \{1\} \cup \{1 + 1/n : n \geq 2\}$ as a subspace of $\mathbb{R}$. Define R to be the equivalence relation on X where xRy iff ($x = y$ or $|x - y| = 1$), so we identify $1/n$ with $1 + 1/n$ for all $n \geq 2$. Let Y be the resulting quotient space X/R in the quotient topology and $p : X \twoheadrightarrow Y$ the canonical surjection.

Let $A = \{1\} \cup (0, 1/2] \setminus \{1/n : n \geq 2\} \subseteq X$. Then A is saturated under p but the restriction q of p to A is not a quotient map because it maps the saturated open set $\{1\}$ to $\{1\}$ which is not open in $p(A)$.

Proposition 213. *Let X, Y and Z be topological spaces. Let $f : X \to Y$ and $g : Y \to Z$. If f and g are quotient maps then so is $g \circ f$.*

PROOF: From Proposition 197. $\square$

Example 214. The product of two quotient maps is not necessarily a quotient map.

Let $X = \mathbb{R}$ and X^* the quotient space formed by identifying all positive integers. Let $p : X \twoheadrightarrow X^*$ be the canonical surjection.

We prove $p \times \mathrm{id}_{\mathbb{Q}} : X \times \mathbb{Q} \to X^* \times \mathbb{Q}$ is not a quotient map.

PROOF:
$\langle 1\rangle 1.$ For $n \geq 1$,
  LET: $c_n = \sqrt{2}/n$
$\langle 1\rangle 2.$ For $n \geq 1$,
  LET: $U_n = \{(x, y) \in X \times \mathbb{Q} \mid n - 1/4 < x < n + 1/4, (y + n > x + c_n \text{ and } y + n > -x + c_n) \text{ or } (y + n < x + c_n \text{ and } y + n < -x + c_n)\}$
$\langle 1\rangle 3.$ For $n \geq 1$, we have U_n is open in $X \times \mathbb{Q}$
$\langle 1\rangle 4.$ For $n \geq 1$, we have $\{n\} \times \mathbb{Q} \subseteq U_n$
$\langle 1\rangle 5.$ LET: $U = \bigcup_{n=1}^{\infty} U_n$
$\langle 1\rangle 6.$ U is open in $X \times \mathbb{Q}$
$\langle 1\rangle 7.$ U is saturated with respect to $p \times \mathrm{id}_{\mathbb{Q}}$

$\langle 1 \rangle 8$. Let: $U' = (p \times \mathrm{id}_{\mathbb{Q}})(U)$
$\langle 1 \rangle 9$. Assume: for a contradiction U' is open in $X^* \times \mathbb{Q}$
$\langle 1 \rangle 10$. $(1, 0) \in U'$
$\langle 1 \rangle 11$. Pick a neighbourhood W of 1 in X^* and $\delta > 0$ such that $W \times (-\delta, \delta) \subseteq U'$
$\langle 1 \rangle 12$. $p^{-1}(W) \times (-\delta, \delta) \subseteq U$
$\langle 1 \rangle 13$. Pick n such that $c_n < \delta$
$\langle 1 \rangle 14$. $n \in p^{-1}(W)$
$\langle 1 \rangle 15$. Pick $\epsilon > 0$ such that $\epsilon < \delta - c_n$ and $\epsilon < 1/4$ and $(n - \epsilon, n + \epsilon) \subseteq p^{-1}(W)$
$\langle 1 \rangle 16$. $(n - \epsilon, n + \epsilon) \times (-\delta, \delta) \subseteq U$
$\langle 1 \rangle 17$. Pick a rational y such that $c_n - \epsilon/2 < y < c_n + \epsilon/2$
$\langle 1 \rangle 18$. $(n + \epsilon/2, y) \notin U$
$\langle 1 \rangle 19$. Q.E.D.
Proof: This contradicts $\langle 1 \rangle 16$.
□

Proposition 215. *Let X be a topological space and $\sim$ an equivalence relation on X. Then $X/\sim$ is T_1 if and only if every equivalence class is closed in X.*

Proof: Immediate from definitions. □

31 Retractions

Definition 216 (Retraction)**.** Let X be a topological space and $A \subseteq X$. A *retraction* of X onto A is a continuous map $r : X \to A$ such that, for all $a \in A$, we have $r(a) = a$.

Proposition 217. *Every retraction is a quotient map.*

Proof: Proposition 209 with f the inclusion $A \hookrightarrow X$. □

32 Homogeneous Spaces

Definition 218 (Homogeneous)**.** A topological space X is *homogeneous* if and only if, for any points $a, b \in X$, there exists a homeomorphism $\phi : X \cong X$ such that $\phi(a) = b$.

33 Regular Spaces

Definition 219 (Regular Space)**.** A topological space X is *regular* if and only if, for any closed set A and point $a \notin A$, there exist disjoint open sets U, V such that $A \subseteq U$ and $a \in V$.

34 Connected Spaces

Definition 220 (Separation)**.** A *separation* of a topological space X is a pair of disjoint open sets U, V such that $U \cup V = \emptyset$.

Definition 221 (Connected)**.** A topological space is *connected* if and only if it has no separation; otherwise it is *disconnected*.

Proposition 222. *A topological space X is connected if and only if the only sets that are both open and closed are X and $\emptyset$.*

Immediate from defintions.

Lemma 223. *If Y is a subspace of X, a separation of Y is a pair of disjoint nonempty sets A and B whose union is Y, neither of which contains a limit point of the other.*

Proof:
$\langle 1 \rangle 1$. Let: $A, B \subseteq Y$
$\langle 1 \rangle 2$. If A and B form a separation of Y then A and B are disjoint and nonempty, $A \cup B = Y$, and neither of A and B contains a limit point of the other.
$\langle 2 \rangle 1$. Assume: A and B form a separation of Y
$\langle 2 \rangle 2$. A and B are disjoint and nonempty and $A \cup B = Y$
Proof: From $\langle 2 \rangle 1$ and the definition of separation.
$\langle 2 \rangle 3$. A does not contain a limit point of B
$\langle 3 \rangle 1$. Assume: for a contradiction $l \in A$ and l is a limit point of B in X.
$\langle 3 \rangle 2$. l is a limit point of B in Y
Proof: Proposition 161.
$\langle 3 \rangle 3$. $l \in B$
$\langle 4 \rangle 1$. B is closed in Y
Proof: Since A is open in Y and $B = Y \setminus A$ from $\langle 2 \rangle 1$.
$\langle 4 \rangle 2$. Q.E.D.
Proof: Corollary 66.1.
$\langle 3 \rangle 4$. Q.E.D.
Proof: This contradicts the fact that $A \cap B = \emptyset$ ($\langle 2 \rangle 1$).
$\langle 2 \rangle 4$. B does not contain a limit point of A
Proof: Similar.
$\langle 1 \rangle 3$. If A and B are disjoint and nonempty, $A \cup B = Y$, and neither of A and B contains a limit point of the other, then A and B form a separation of Y.
$\langle 2 \rangle 1$. Assume: A and B are disjoint and nonempty, $A \cup B = Y$, and neither of A and B contains a limit point of the other.
$\langle 2 \rangle 2$. A is open in Y
$\langle 3 \rangle 1$. B is closed in Y
$\langle 4 \rangle 1$. Let: l be a limit point of B in Y
$\langle 4 \rangle 2$. l is a limit point of B in X
Proof: Proposition 161.
$\langle 4 \rangle 3$. $l \notin A$
Proof: By $\langle 2 \rangle 1$
$\langle 4 \rangle 4$. $l \in B$
Proof: By $\langle 2 \rangle 1$ since $A \cup B = Y$
$\langle 4 \rangle 5$. Q.E.D.

PROOF: Corollary 66.1.

⟨3⟩2. Q.E.D.

PROOF: Since $A = Y \setminus B$.

⟨2⟩3. B is open in Y

PROOF: Similar.

□

Example 224. Every set under the indiscrete topology is connected.

Example 225. The discrete topology on a set X is connected if and only if $|X| \leq 1$.

Example 226. The finite complement topology on a set X is connected if and only if either $|X| \leq 1$ or X is infinite.

Example 227. The countable complement topology on a set X is connected if and only if either $|X| \leq 1$ or X is uncountable.

Example 228. The rationals $\mathbb{Q}$ are disconnected. For any irrational a, the sets $(-\infty, a) \cap \mathbb{Q}$ and $(a, +\infty) \cap \mathbb{Q}$ form a separation of $\mathbb{Q}$.

Lemma 229. *Let X be a topological space. If C and D form a separation of X, and Y is a connected subspace of X, then either $Y \subseteq C$ or $Y \subseteq D$.*

PROOF: Otherwise $Y \cap C$ and $Y \cap D$ would form a separation of Y. □

Theorem 230. *The union of a set of connected subspaces of a space X that have a point in common is connected.*

PROOF:

⟨1⟩1. LET: $\mathcal{A}$ be a set of connected subspaces of the space X that have the point a in common.

⟨1⟩2. ASSUME: for a contradiction C and D form a separation of $\bigcup \mathcal{A}$

⟨1⟩3. ASSUME: without loss of generality $a \in C$

⟨1⟩4. For all $A \in \mathcal{A}$ we have $A \subseteq C$

PROOF: Lemma 229.

⟨1⟩5. $D = \emptyset$

⟨1⟩6. Q.E.D.

PROOF: This contradicts ⟨1⟩2.

□

Theorem 231. *Let X be a topological space and A a connected subspace of X. If $A \subseteq B \subseteq \overline{A}$ then B is connected.*

PROOF:

⟨1⟩1. ASSUME: for a contradiction C and D form a separation of B.

⟨1⟩2. ASSUME: without loss of generality $A \subseteq C$

PROOF: Lemma 229.

⟨1⟩3. $B \subseteq C$

⟨2⟩1. LET: $x \in B$

⟨2⟩2. $x \in \overline{A}$

⟨2⟩3. Either $x \in A$ or x is a limit point of A.

Proof: Theorem 66.

⟨2⟩4. Either $x \in A$ or x is a limit point of C.

Proof: Lemma 68, ⟨1⟩2.

⟨2⟩5. $x \in C$

Proof: Lemma 223.

⟨1⟩4. $D = \emptyset$

⟨1⟩5. Q.E.D.

Proof: This contradicts ⟨1⟩1.

□

Theorem 232. *The image of a connected space under a continuous map is connected.*

Proof:

⟨1⟩1. Let: $f : X \twoheadrightarrow Y$ be a surjective continuous map where X is connected.

⟨1⟩2. Assume: for a contradiction C and D form a separation of Y.

⟨1⟩3. $f^{-1}(C)$ and $f^{-1}(D)$ form a separation of X.

□

Theorem 233. *The product of a family of connected spaces is connected.*

Proof:

⟨1⟩1. The product of two connected spaces is connected.

⟨2⟩1. Let: X and Y be connected spaces.

⟨2⟩2. Pick $a \in X$ and $b \in Y$

Proof: We may assume X and Y are nonempty since otherwise $X \times Y = \emptyset$ which is connected.

⟨2⟩3. $X \times \{b\}$ is connected.

Proof: It is homeomorphic to X.

⟨2⟩4. For all $x \in X$ we have $\{x\} \times Y$ is connected.

Proof: It is homeomorphic to Y.

⟨2⟩5. For any $x \in X$

Let: $T_x = (X \times \{b\}) \cup (\{x\} \times Y)$

⟨2⟩6. For all $x \in X$, T_x is connected.

Proof: Theorem 230 since $(x, b) \in (X \times \{b\}) \cap (\{x\} \times Y)$.

⟨2⟩7. $X \times Y$ is connected.

Proof: Theorem 230 since $X \times Y = \bigcup_{x \in X} T_x$ and (a, b) is a point in every T_x.

⟨1⟩2. The product of a finite family of connected spaces is connected.

Proof: From ⟨1⟩1 by induction.

⟨1⟩3. The product of any family of connected spaces is connected.

⟨2⟩1. Let: $\{X_\alpha\}_{\alpha \in J}$ be a family of connected spaces.

⟨2⟩2. Let: $X = \prod_{\alpha \in J} X_\alpha$

⟨2⟩3. Pick $a \in X$

Proof: We may assume $X \neq \emptyset$ as the empty space is connected.

⟨2⟩4. For every finite subset K of J,
LET: $X_K = \{x \in X \mid \forall \alpha \in J \setminus K . x_\alpha = a_\alpha\}$
⟨2⟩5. For every finite $K \subseteq J$, we have X_K is connected.
PROOF: From ⟨1⟩2 since $X_K \cong \prod_{\alpha \in K} X_K$.
⟨2⟩6. LET: $Y = \bigcup_K X_K$
⟨2⟩7. Y is connected
PROOF: Theorem 230 since a is a common point.
⟨2⟩8. $X = \overline{Y}$
⟨3⟩1. LET: $x \in X$
⟨3⟩2. LET: $U = \prod_{\alpha \in J} U_\alpha$ be a basic neighbourhood of x where $U_\alpha = X_\alpha$ for all α except $\alpha \in K$ for some finite $K \subseteq J$
⟨3⟩3. LET: $y \in X$ be the point with $y_\alpha = x_\alpha$ for $\alpha \in K$ and $y_\alpha = a_\alpha$ for all other α
⟨3⟩4. $y \in U \cap X_K$
⟨3⟩5. $y \in U \cap Y$
⟨2⟩9. X is connected.
PROOF: Theorem 231.
□

Example 234. The set $\mathbb{R}^\omega$ is disconnected under the box topology. The set of bounded sequences and the set of unbounded sequences form a separation.

Proposition 235. *Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on the same set X. If $\mathcal{T} \subseteq \mathcal{T}'$ and $(X, \mathcal{T}')$ is connected then $(X, \mathcal{T})$ is connected.*

PROOF: If U and V form a separation of $(X, \mathcal{T})$ then they form a separation of $(X, \mathcal{T}')$. □

Proposition 236. *Let X be a topological space and (A_n) a sequence of connected subspaces of X. If $A_n \cap A_{n+1} \neq \emptyset$ for all n then $\bigcup_n A_n$ is connected.*

PROOF:
⟨1⟩1. ASSUME: for a contradiction C and D form a separation of $\bigcup_n A_n$
⟨1⟩2. ASSUME: without loss of generality $A_0 \subseteq C$
PROOF: Lemma 229.
⟨1⟩3. For all n we gave $A_n \subseteq C$
PROOF:
⟨2⟩1. ASSUME: $A_n \subseteq C$
⟨2⟩2. PICK $x \in A_n \cap A_{n+1}$
⟨2⟩3. $x \in C$
⟨2⟩4. $A_{n+1} \subseteq C$
PROOF: Lemma 229.
⟨2⟩5. Q.E.D.
PROOF: The result follows by induction.
⟨1⟩4. $D = \emptyset$
⟨1⟩5. Q.E.D.
PROOF: This contradicts ⟨1⟩1.
□

Proposition 237. *Let X be a topological space. Let $A, C \subseteq X$. If C is connected and intersects both A and $X \setminus A$ then C intersects ∂A.*

PROOF: Otherwise $C \cap A^\circ$ and $C \setminus \overline{A}$ would form a separation of C. □

Example 238. The space $\mathbb{R}_l$ is disconnected. For any real x, the sets $(-\infty, x)$ and $[x, +\infty)$ form a separation.

Proposition 239. *Let X and Y be connected spaces. Let A be a proper subset of X and B a proper subset of Y. Then $(X \times Y) \setminus (A \times B)$ is connected.*

PROOF:
$\langle 1 \rangle 1$. PICK $a \in X \setminus A$ and $b \in Y \setminus B$
$\langle 1 \rangle 2$. For $x \in X \setminus A$ we have $(X \times \{b\}) \cup (\{x\} \times Y)$ is connected.
PROOF: Theorem 230 since (x, b) is a common point.
$\langle 1 \rangle 3$. For $y \in Y \setminus B$ we have $(X \times \{y\}) \cup (\{a\} \times Y)$ is connected.
PROOF: Theorem 230 since (a, y) is a common point.
$\langle 1 \rangle 4$. $(X \times Y) \setminus (A \times B)$ is connected.
PROOF: Theorem 230 since it is the union of the sets in $\langle 1 \rangle 2$ and $\langle 1 \rangle 3$ with (a, b) as a common point.
□

Proposition 240. *Let $p : X \twoheadrightarrow Y$ be a quotient map. If Y is connected and $p^{-1}(y)$ is connected for all $y \in Y$, then X is connected.*

PROOF:
$\langle 1 \rangle 1$. ASSUME: for a contradiction C and D form a separation of X.
$\langle 1 \rangle 2$. C is saturated.
$\langle 2 \rangle 1$. LET: $x \in C$, $y \in X$ with $p(x) = p(y) = a$, say
$\langle 2 \rangle 2$. $y \notin D$
PROOF: Otherwise $p^{-1}(a) \cap C$ and $p^{-1}(a) \cap D$ form a separation of $p^{-1}(a)$.
$\langle 2 \rangle 3$. $y \in C$
$\langle 1 \rangle 3$. D is saturated.
PROOF: Similar.
$\langle 1 \rangle 4$. $p(C)$ and $p(D)$ form a separation of Y.
□

Proposition 241. *Let X be a connected space and Y a connected subspace of X. Suppose A and B form a separation of $X \setminus Y$. Then $Y \cup A$ and $Y \cup B$ are both connected.*

PROOF:
$\langle 1 \rangle 1$. $Y \cup A$ is connected.
$\langle 2 \rangle 1$. ASSUME: for a contradiction C and D form a separation of $Y \cup A$
$\langle 2 \rangle 2$. ASSUME: without loss of generality $Y \subseteq C$
$\langle 2 \rangle 3$. PICK open sets A_1, B_1, C_1, D_1 in X with

$$A = A_1 \setminus Y$$
$$B = B_1 \setminus Y$$
$$C = C_1 \cap (Y \cup A)$$
$$D = D_1 \cap (Y \cup A)$$

⟨2⟩4. $B_1 \cup C_1$ and $A_1 \cap D_1$ form a separation of X
⟨1⟩2. $Y \cup B$ is connected.
Proof: Similar.
□

Theorem 242. *Let L be a linearly ordered set under the order topology. Then L is connected if and only if L is a linear continuum.*

Proof:
⟨1⟩1. If L is a linear continuum then L is connected.
⟨2⟩1. Let: L be a linear continuum under the order topology.
⟨2⟩2. Assume: for a contradiction C and D form a separation of L.
⟨2⟩3. Pick $a \in C$ and $b \in D$.
⟨2⟩4. Assume: without loss of generality $a < b$.
⟨2⟩5. Let: $S = \{x \in L \mid a < x \text{ and } [a, x) \subseteq C\}$
⟨2⟩6. S is nonempty.
Proof: Since $a \in C$ and C is open.
⟨2⟩7. S is bounded above by b.
Proof: Since $b \notin C$.
⟨2⟩8. Let: $s = \sup S$
⟨2⟩9. $s \in S$
⟨3⟩1. Let: $y \in [a, s)$
Prove: $y \in C$
⟨3⟩2. Pick z with $y < z \in S$
Proof: By minimality of s.
⟨3⟩3. $y \in [a, z) \subseteq C$
⟨2⟩10. Case: $s \in C$
⟨3⟩1. Pick x such that $s < x$ and $[s, x) \subseteq C$
Proof: Since C is open and s is not greatest in L because $s < b$.
⟨3⟩2. $x \in S$
Proof: Since $[a, x) = [a, s) \cup [s, x) \subseteq C$.
⟨3⟩3. Q.E.D.
Proof: This contradicts the fact that s is an upper bound for S.
⟨2⟩11. Case: $s \in D$
⟨3⟩1. Pick $x < s$ such that $(x, s] \subseteq D$
⟨3⟩2. Pick y with $x < y < s$
Proof: Since L is dense.
⟨3⟩3. $y \in C$
Proof: From ⟨2⟩9.
⟨3⟩4. $y \in D$
Proof: From ⟨3⟩1.
⟨3⟩5. Q.E.D.
⟨3⟩6. Let: L be a linear continuum under the order topology.
⟨3⟩7. Assume: for a contradiction C and D form a separation of L.
⟨3⟩8. Pick $a \in C$ and $b \in D$.
⟨3⟩9. Assume: without loss of generality $a < b$.
⟨3⟩10. Let: $S = \{x \in L \mid a < x \text{ and } [a, x) \subseteq C\}$

⟨3⟩11. S is nonempty.
PROOF: Since $a \in C$ and C is open.
⟨3⟩12. S is bounded above by b.
PROOF: Since $b \notin C$.
⟨3⟩13. LET: $s = \sup S$
⟨3⟩14. $s \in S$
⟨4⟩1. LET: $y \in [a, s)$
PROVE: $y \in C$
⟨4⟩2. PICK z with $y < z \in S$
PROOF: By minimality of s.
⟨4⟩3. $y \in [a, z) \subseteq C$
⟨3⟩15. CASE: $s \in C$
⟨4⟩1. PICK x such that $s < x$ and $[s, x) \subseteq C$
PROOF: Since C is open and s is not greatest in L because $s < b$.
⟨4⟩2. $x \in S$
PROOF: Since $[a, x) = [a, s) \cup [s, x) \subseteq C$.
⟨4⟩3. Q.E.D.
PROOF: This contradicts the fact that s is an upper bound for S.
⟨3⟩16. CASE: $s \in D$
⟨4⟩1. PICK $x < s$ such that $(x, s] \subseteq D$
⟨4⟩2. PICK y with $x < y < s$
PROOF: Since L is dense.
⟨4⟩3. $y \in C$
PROOF: From ⟨2⟩9.
⟨4⟩4. $y \in D$
PROOF: From ⟨3⟩1.
⟨4⟩5. Q.E.D.
PROOF: This contradicts ⟨2⟩2.
⟨1⟩2. If L is connected then L is a linear continuum.
⟨2⟩1. ASSUME: L is connected.
⟨2⟩2. Every nonempty subset of L that is bounded above has a supremum.
⟨3⟩1. LET: X be a nonempty subset of L bounded above by b.
⟨3⟩2. ASSUME: for a contradiction X has no supremum.
⟨3⟩3. LET: U be the set of upper bounds of X,
⟨3⟩4. U is nonempty.
PROOF: Since $b \in U$.
⟨3⟩5. U is open.
⟨4⟩1. LET: $x \in U$
⟨4⟩2. PICK an upper bound y for X such that $y < x$
⟨4⟩3. Either x is greatest in L and $(y, x] \subseteq U$, or there exists $z > x$ such that $(y, z) \subseteq U$
⟨3⟩6. LET: V be the set of lower bounds of U.
⟨3⟩7. V is nonempty.
PROOF: Since $X \subseteq V$
⟨3⟩8. V is open.
⟨4⟩1. LET: $x \in V$

⟨4⟩2. PICK $y \in X$ with $x < y$

PROOF: x cannot be an upper bound for X, because it would be the supremum of X.

⟨4⟩3. Either x least in L and $[x, y) \subseteq V$, or there exists $z < x$ such that $(z, y) \subseteq V$

⟨3⟩9. $L = U \cup V$

⟨4⟩1. LET: $x \in L \setminus U$

⟨4⟩2. PICK $y \in X$ such that $x < y$

⟨4⟩3. For all $u \in U$ we have $x < y \leq u$

⟨4⟩4. $x \in V$

⟨3⟩10. $U \cap V = \emptyset$

PROOF: Any element of $U \cap V$ would be a supremum of X.

⟨3⟩11. U and V form a separation of L.

⟨3⟩12. Q.E.D.

PROOF: This contradicts ⟨2⟩1.

⟨2⟩3. L is dense.

⟨3⟩1. LET: $x, y \in L$ with $x < y$

⟨3⟩2. There exists $z \in L$ such that $x < z < y$

PROOF: Otherwise $(-\infty, y)$ and $(x, +\infty)$ would form a separation of L.

□

Corollary 242.1. *The real line* $\mathbb{R}$ *is connected.*

Corollary 242.2. *Every interval in* $\mathbb{R}$ *is connected.*

Corollary 242.3. *The ordered square is connected.*

Theorem 243 (Intermediate Value Theorem)**.** *Let X be a connected space. Let Y be a linearly ordered set under the order topology. Let $f : X \to Y$ be continuous. Let $a, b \in X$ and $r \in Y$. Suppose $f(a) < r < f(b)$. Then there exists $c \in X$ such that $f(c) = r$.*

PROOF: Otherwise $f^{-1}((-\infty, r))$ and $f^{-1}((r, +\infty))$ would form a separation of X. □

Proposition 244. *Every function $f : [0, 1] \to [0, 1]$ has a fixed point.*

PROOF:

⟨1⟩1. LET: $g : [0, 1] \to [-1, 1]$ be the function $g(x) = f(x) - x$

PROVE: there exists $x \in [0, 1]$ such that $g(x) = 0$

⟨1⟩2. ASSUME: without loss of generality $g(0) \neq 0$ and $g(1) \neq 0$

⟨1⟩3. $g(0) > 0$

⟨1⟩4. $g(1) < 0$

⟨1⟩5. There exists $x \in (0, 1)$ such that $g(x) = 0$

PROOF: By the Intermediate Value Theorem.

Proposition 245. *Give $\mathbb{R}^\omega$ the box topology. Let $x, y \in \mathbb{R}^\omega$. Then x and y lie in the same comoponent if and only if $x - y$ is eventually zero, i.e. there exists N such that, for all $n \geq N$, we have $x_n = y_n$.*

PROOF:
$\langle 1 \rangle 1$. The component containing 0 is the set of sequences that are eventually zero.
$\langle 2 \rangle 1$. LET: B be the set of sequences that are eventually zero.
$\langle 2 \rangle 2$. B is path-connected.
$\langle 3 \rangle 1$. LET: $x, y \in B$
$\langle 3 \rangle 2$. PICK N such that, for all $n \geq N$, we have $x_n = y_n = 0$
$\langle 3 \rangle 3$. LET: $p : [0, 1] \to \mathbb{R}^\omega$, $p(t) = (1 - t)x + ty$
PROVE: p is continuous.
$\langle 3 \rangle 4$. LET: $t \in [0, 1]$ and $\prod_j U_j$ be a basic open neighbourhood of $p(t)$, where each U_j is open in $\mathbb{R}$
$\langle 3 \rangle 5$. PICK δ such that, for all $n < N$ and all $s \in [0, 1]$, if $|s - t| < \delta$ then $p(s)_n \in U_n$
$\langle 3 \rangle 6$. For all $s \in [0, 1]$, if $|s - t| < \delta$ then $p(s) \in \prod_j U_j$
$\langle 2 \rangle 3$. B is connected.
PROOF: Proposition 251.
$\langle 2 \rangle 4$. If C is connected and $B \subseteq C$ then $B = C$.
$\langle 3 \rangle 1$. ASSUME: C is connected and $B \subseteq C$
$\langle 3 \rangle 2$. ASSUME: for a contradiction $x \in C \setminus B$
$\langle 3 \rangle 3$. For $n \geq 1$,
LET: $c_n = 1$ if $x_n = 0$, $c_n = n/x_n$ otherwise
$\langle 3 \rangle 4$. LET: $h : \mathbb{R}^\omega \to \mathbb{R}^\omega$ be the function $h(x) = (c_n x_n)_{n \geq 1}$
$\langle 3 \rangle 5$. h is a homeomorphism of $\mathbb{R}^\omega$ with itself.
$\langle 3 \rangle 6$. $h(x)$ is unbounded.
PROOF: For any $b > 0$, pick $N > b$ such that $x_N \neq 0$. Then $h(x)_N > b$.
$\langle 3 \rangle 7$. $h^{-1}(\{\text{bounded sequences}\}) \cap C$ and $h^{-1}(\{\text{unbounded sequences}\}) \cap C$ form a separation of C
$\langle 3 \rangle 8$. Q.E.D.
PROOF: This contradicts $\langle 3 \rangle 1$.
$\langle 1 \rangle 2$. Q.E.D.
PROOF: Since $\lambda x.x - y$ is a homeomorphism of $\mathbb{R}^\omega$ with itself.
□

35 Totally Disconnected Spaces

Definition 246 (Totally Disconnected)**.** A topological space X is *totally disconnected* if and only if the only connected subspaces are the singletons.

Example 247. Every discrete space is totally disconnected.

Example 248. The rationals $\mathbb{Q}$ are totally disconnected.

36 Paths and Path Connectedness

Definition 249 (Path)**.** Let X be a topological space and $a, b \in X$. A *path* from a to b is a continuous function $p : [0, 1] \to X$ such that $p(0) = a$ and

$p(1) = b$.

Definition 250 (Path Connected)**.** A topological space is *path connected* if and only if there exists a path between any two points.

Proposition 251. *Every path connected space is connected.*

PROOF:
$\langle 1 \rangle 1$. LET: X be a path connected space.
$\langle 1 \rangle 2$. ASSUME: for a contradiction C and D form a separation of X.
$\langle 1 \rangle 3$. PICK $a \in C$ and $b \in D$.
$\langle 1 \rangle 4$. PICK a path $p : [0,1] \to X$ from a to b.
$\langle 1 \rangle 5$. $p^{-1}(C)$ and $p^{-1}(D)$ form a separation of $[0,1]$.
$\langle 1 \rangle 6$. Q.E.D.
PROOF: This contradicts Corollary 242.2.
□

An example that shows the converse does not hold:

Example 252. The ordered square is not path connected.

PROOF:
$\langle 1 \rangle 1$. ASSUME: for a contradiction $p : [0,1] \to I_o^2$ is a path from $(0,0)$ to $(1,1)$.
$\langle 1 \rangle 2$. p is surjective.
PROOF: By the Intermediate Value Theorem.
$\langle 1 \rangle 3$. For $x \in [0,1]$, PICK a rational $q_x \in p^{-1}((x,0),(x,1))$
PROOF: Since $p^{-1}((x,0),(x,1))$ is open and nonempty by $\langle 1 \rangle 2$.
$\langle 1 \rangle 4$. For $x, y \in [0,1]$, if $x \neq y$ then $q_x \neq q_y$
PROOF: We have $p(q_x) \neq p(q_y)$ because $((x,0),(x,1))$ and $((y,0),(y,1))$ are disjoint.
$\langle 1 \rangle 5$. $\{q_x \mid x \in [0,1]\}$ is an uncountable set of rationals.
$\langle 1 \rangle 6$. Q.E.D.
PROOF: This contradicts the fact that the rationals are countable.
□

Proposition 253. *The continuous image of a path connected space is path connected.*

PROOF:
$\langle 1 \rangle 1$. LET: X be a path connected space, Y a topological space, and $f : X \twoheadrightarrow Y$ be continuous and surjective.
$\langle 1 \rangle 2$. LET: $a, b \in Y$
$\langle 1 \rangle 3$. PICK $c, d \in X$ with $f(c) = a$ and $f(d) = b$
$\langle 1 \rangle 4$. PICK a path $p : [0,1] \to X$ from c to d.
$\langle 1 \rangle 5$. $f \circ p$ is a path from a to b in Y.
□

Proposition 254 (AC)**.** *The product of a family of path-connected spaces is path-connected.*

PROOF:
$\langle 1 \rangle 1$. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of path-connected spaces.
$\langle 1 \rangle 2$. LET: $a, b \in \prod_{\alpha \in J} X_\alpha$
$\langle 1 \rangle 3$. For $\alpha \in J$, PICK a path $p_\alpha : [0,1] \to X_\alpha$ from a_α to b_α
PROOF: Using the Axiom of Choice.
$\langle 1 \rangle 4$. Define $p : [0.1] \to \prod_{\alpha \in J} X_\alpha$ by $p(t)_\alpha = p_\alpha(t)$
$\langle 1 \rangle 5$. p is a path from a to b.
PROOF: Theorem 140.
□

Proposition 255. *The continuous image of a path-connected space is path-connected.*

PROOF:
$\langle 1 \rangle 1$. LET: $f : X \twoheadrightarrow Y$ be continuous and surjective where X is path-connected.
$\langle 1 \rangle 2$. LET: $a, b \in Y$
$\langle 1 \rangle 3$. PICK $a', b' \in X$ with $f(a') = a$ and $f(b') = b$.
$\langle 1 \rangle 4$. PICK a path $p : [0,1] \to X$ from a' to b'.
$\langle 1 \rangle 5$. $f \circ p$ is a path from a to b.
□

Proposition 256. *Let X be a topological space. The union of a set of path-connected subspaces of X that have a point in common is path-connected.*

PROOF:
$\langle 1 \rangle 1$. LET: $\mathcal{A}$ be a set of path-connected subspaces of X with the point a in common.
$\langle 1 \rangle 2$. LET: $b, c \in \bigcup \mathcal{A}$
$\langle 1 \rangle 3$. PICK $B, C \in \mathcal{A}$ with $b \in B$ and $c \in C$.
$\langle 1 \rangle 4$. PICK a path p in B from b to a.
$\langle 1 \rangle 5$. PICK a path q in C from a to c.
$\langle 1 \rangle 6$. The concatenation of p and q is a path from b to c in $\bigcup \mathcal{A}$.
□

Proposition 257. *Let $A \subseteq \mathbb{R}^2$ be countable. Then $\mathbb{R}^2 \setminus A$ is path-connected.*

PROOF:
$\langle 1 \rangle 1$. LET: $a, b \in \mathbb{R}^2 \setminus A$
$\langle 1 \rangle 2$. PICK a line l in $\mathbb{R}^2$ with a on one side and b on the other.
$\langle 1 \rangle 3$. For every point x on l,
LET: p_x be the path in $\mathbb{R}^2$ consisting of a line from a to x then a line from x to b
$\langle 1 \rangle 4$. For $x \neq y$ we have p_x and p_y have no points in common except a and b
$\langle 1 \rangle 5$. There are only countably many x such that a point of A lies on p_x.
$\langle 1 \rangle 6$. There exists x such that p_x is a path from a to b in $\mathbb{R}^2 \setminus A$.
□

Proposition 258. *Every open connected subspace of $\mathbb{R}^2$ is path-connected.*

PROOF:
$\langle 1 \rangle 1$. LET: U be an open connected subspace of $\mathbb{R}^2$.
$\langle 1 \rangle 2$. For all $x_0 \in U$,
LET: $PC(x_0) = \{y \in U \mid \text{there exists a path from } x \text{ to } y\}$
$\langle 1 \rangle 3$. For all $x_0 \in U$, the set $PC(x_0)$ is open and closed in U.
$\langle 2 \rangle 1$. LET: $x_0 \in U$
$\langle 2 \rangle 2$. $PC(x_0)$ is open in U
$\langle 3 \rangle 1$. LET: $y \in PC(x_0)$
$\langle 3 \rangle 2$. PICK $\epsilon > 0$ such that $B(y, \epsilon) \subseteq U$
PROOF: Since U is open.
$\langle 3 \rangle 3$. $B(y, \epsilon) \subseteq PC(x_0)$
PROOF: For all $z \in B(y, \epsilon)$, pick a path from x_0 to y then concatenate the straight line from y to z.
$\langle 2 \rangle 3$. $PC(x_0)$ is closed in U
$\langle 3 \rangle 1$. LET: $y \in U$ be a limit point of $PC(x_0)$
$\langle 3 \rangle 2$. PICK $\epsilon > 0$ such that $B(y, \epsilon) \subseteq U$
$\langle 3 \rangle 3$. PICK $z \in PC(x_0) \cap B(y, \epsilon)$
$\langle 3 \rangle 4$. $y \in PC(x_0)$
PROOF: Pick a path from x_0 to z then concatenate the straight line from z to y.
$\langle 1 \rangle 4$. $PC(x_0) = U$
PROOF: Proposition 222.
$\square$

Example 259. If A is a connected subspace of X, then A° is not necessarily connected.

Take two closed circles in $\mathbb{R}^2$ that touch at one point. The interior of this space is two open circles, and these two circles form a separation.

Example 260. If A is a connected subspace of X then ∂A is not necessarily connected.

We have $[0, 1]$ is connected but $\partial[0, 1] = \{0, 1\}$ is not.

Example 261. If A is a subspace of X and A° and ∂A are connected, then A is not necessarily connected.

We have $\mathbb{Q}^\circ = \emptyset$ and $\partial\mathbb{Q} = \mathbb{R}$ are connected but $\mathbb{Q}$ is not connected.

37 The Topologist's Sine Curve

Definition 262 (The Topologist's Sine Curve)**.** Let $S = \{(x, \sin 1/x) \mid 0 < x \leq 1\}$, The *topologist's sine curve* is the closure $\overline{S}$ of S in $\mathbb{R}^2$.

Proposition 263. *The topologist's sine curve is connected.*

PROOF:
$\langle 1 \rangle 1$. LET: $S = \{(x, \sin 1/x) \mid 0 < x \leq 1\}$
$\langle 1 \rangle 2$. S is connected.

PROOF: Theorem 232.
⟨1⟩3. $\overline{S}$ is connected.
PROOF: Theorem 231.
□

Proposition 264. *The topologist's sine curve is* $\{(x, \sin 1/x) \mid 0 < x \leq 1\} \cup (\{0\} \times [-1.1])$.

PROOF: Sketch proof: Given a point $(0.y)$ with $-1 \leq y \leq 1$, pick a such that $\sin a = y$. Then $((1/a, y), (1/(a + 2\pi), y), (1/(a + 4\pi), y), \ldots)$ is a sequence in S that converges to $(0, y)$.
Conversely, let (x, y) be any point not in $S \cup (\{0\} \times [-1, 1])$. If $x < 0$ or $y > 1$ or $y < -1$ then we can easily find a neighbourhood that does not intersect $S \cup (\{0\} \times [-1, 1])$. If $x > 0$ and $-1 \leq y \leq 1$, then we have $y \neq \sin 1/x$. Hence pick a neighbourhood that does not intersect S.

Proposition 265. *Every closed subset of* $\mathbb{R}$ *that is bounded above has a greatest element.*

PROOF: It has a supremum, which is a limit point of the set and hence an element. □

Proposition 266 (CC)**.** *The topologist's sine curve is not path connected.*

PROOF:
⟨1⟩1. ASSUME: For a contradction $p : [0, 1] \to \overline{S}$ is a path from $(0, 0)$ to $(1, \sin 1)$.
⟨1⟩2. $\{t \in [0, 1] \mid p(t) \in \{0\} \times [-1, 1]\}$ is closed.
PROOF: Since p is continuous and $\{0\} \times [-1, 1]$ is closed.
⟨1⟩3. LET: b be the largest number in $[0, 1]$ such that $p(b) \in \{0\} \times [-1, 1]$.
PROOF: Proposition 265.
⟨1⟩4. LET: $x : [b, 1] \to \overline{S}$ be the function $\pi_1 \circ p$
⟨1⟩5. LET: $y : [b, 1] \to \overline{S}$ be the function $\pi_2 \circ p$
⟨1⟩6. PICK a sequence t_n in $(b, 1]$ such that $t_n \to b$ and $y(t_n) = (-1)^n$ for all n
⟨2⟩1. LET: $n \geq 1$
⟨2⟩2. PICK u with $0 < u < x(1/n)$ and $\sin(1/u) = (-1)^n$
⟨2⟩3. PICK t_n with $b < t_n < 1/n$ and $x(t_n) = u$
PROOF: By the Intermediate Value Theorem
⟨1⟩7. Q.E.D.
PROOF: This contradicts Proposition 113 since y is continuous and $y(t_n)$ does not converge.
□

Corollary 266.1. *The closure of a path-connected subspace of a space is not necessarily path-connected.*

38 The Long Line

Definition 267 (The Long Line)**.** The *long line* is the space $\omega_1 \times [0, 1)$ in the dictionary order under the order topology, where ω_1 is the first uncountable ordinal.

Lemma 268. *For any ordinal α with $0 < \alpha < \omega_1$ we have* $[(0,0),(\alpha,0)) \cong [0,1)$

$\langle 1 \rangle 1$. $[(0,0),(1,0)) \cong [0,1)$
  PROOF: The map π_2 is a homeomorphism.
$\langle 1 \rangle 2$. If $[(0,0),(\alpha,0)) \cong [0,1)$ then $[(0,0),(\alpha+1,0)) \cong [0,1)$
  PROOF: Proposition 21.
$\langle 1 \rangle 3$. If λ is a limit ordinal with $\lambda < \omega_1$ and $[(0,0),(\alpha,0)) \cong [0,1)$ for all α with $0 < \alpha < \lambda$ then $[(0,0),(\lambda,0)) \cong [0,1)$
  $\langle 2 \rangle 1$. LET: λ be a limit ordinal $< \omega_1$
  $\langle 2 \rangle 2$. ASSUME: $[(0,0),(\alpha,0)) \cong [0,1)$ for all α with $0 < \alpha < \lambda$
  $\langle 2 \rangle 3$. PICK a sequence of ordinals $\alpha_0 < \alpha_1 < \cdots$ with limit λ
    PROOF: Since λ is countable.
  $\langle 2 \rangle 4$. $[(\alpha_i,0),(\alpha_{i+1},0)) \cong [0,1)$ for all i
    PROOF: Lemma 20.
  $\langle 2 \rangle 5$. Q.E.D.
    PROOF: By Proposition 22.
$\langle 1 \rangle 4$. Q.E.D.
  PROOF: By transfinite induction.

Proposition 269 (CC)**.** *The long line is path-connected.*

PROOF:
$\langle 1 \rangle 1$. LET: $(\alpha,i),(\beta,j) \in \omega_1 \times [0,1)$
$\langle 1 \rangle 2$. ASSUME: without loss of generality $(\alpha,i) < (\beta,j)$
$\langle 1 \rangle 3$. $[(0,0),(\beta+1,0)) \cong [0,1)$
  PROOF: By Lemma 268
$\langle 1 \rangle 4$. $[(\alpha,i),(\beta,j)) \cong [0,1)$
  PROOF: Lemma 20.
$\langle 1 \rangle 5$. PICK a homeomorphism $q : [0,1) \to [(\alpha,i),(\beta,j))$
$\langle 1 \rangle 6$. $q \cup \{(1,(\beta,j))\}$ is a path from (α,i) to (β,j)
□

Proposition 270. *Every point in the long line has a neighbourhood homeomorphic to an interval in $\mathbb{R}$.*

PROOF: For any (α,i) in the long line, the neighbourhood $[(0,0),(\alpha+1,0))$ satisfies the condition by Lemma 268.

Proposition 271. *The long line L is not second countable.*

PROOF:
$\langle 1 \rangle 1$. LET: $\mathcal{B}$ be a basis for L.
$\langle 1 \rangle 2$. For $\alpha < \omega_1$, PICK $B_\alpha \in \mathcal{B}$ such that $(\alpha,1/2) \in B_\alpha \subseteq ((\alpha,0),(\alpha+1,0))$
$\langle 1 \rangle 3$. $\mathcal{B}$ is uncountable.
  PROOF: The mapping $\alpha \mapsto B_\alpha$ is an injection $\omega_1 \to \mathcal{B}$.

Corollary 271.1. *The long line cannot be imbedded into $\mathbb{R}^n$ for any n.*

39 Components

Proposition 272. *Let X be a topological space. Define the relation $\sim$ on X by $a \sim b$ if and only if there exists a connected subspace $A \subseteq X$ with $a, b \in A$. Then $\sim$ is an equivalence relation on X.*

PROOF:
$\langle 1 \rangle 1$. $\sim$ is reflexive.
  PROOF: For any $a \in X$ we have $\{a\}$ is a connected subspace that contains a.
$\langle 1 \rangle 2$. $\sim$ is symmetric.
  PROOF: Trivial.
$\langle 1 \rangle 3$. $\sim$ is transitive.
  $\langle 2 \rangle 1$. LET: $a, b, c \in X$
  $\langle 2 \rangle 2$. ASSUME: $a \sim b$ and $b \sim c$
  $\langle 2 \rangle 3$. PICK connected subspaces A and B with $a, b \in A$ and $b, c \in B$
  $\langle 2 \rangle 4$. $A \cup B$ is a connected subspace that contains a and c
    PROOF: Theorem 230.
$\square$

Definition 273 ((Connected) Component)**.** Let X be a topological space. The *(connected) components* of X are the equivalence classes under the above $\sim$.

Lemma 274. *Let X be a topological space. If $A \subseteq X$ is connected and nonempty then there exists a unique component C of X such that $A \subseteq C$.*

PROOF:
$\langle 1 \rangle 1$. PICK $a \in A$
$\langle 1 \rangle 2$. LET: C be the $\sim$-equivalence class of a.
$\langle 1 \rangle 3$. $A \subseteq C$
  PROOF: For all $x \in A$ we have $x \sim a$.
$\langle 1 \rangle 4$. If C' is a component and $A \subseteq C'$ then $C = C'$
  PROOF: Since we have $a \in C'$.
$\square$

Theorem 275. *Let X be a topological space. The components of X are connected disjoint subspaces of X whose union is X such that each nonempty connected subspace of X intersects only one of them.*

PROOF:
$\langle 1 \rangle 1$. Every component of X is connected.
  PROOF: For $a \in X$, the $\sim$-equivalence class of a is $\bigcup \{A \subseteq X \mid A \text{ is connected}, a \in A\}$ which is connected by Theorem 230.
$\langle 1 \rangle 2$. The components form a partition of X.
  PROOF: Immediate from the definition.
$\langle 1 \rangle 3$. Every nonempty connected subspace of X intersects a unique component of X.
  $\langle 2 \rangle 1$. LET: $A \subseteq X$ be connected and nonempty.
  $\langle 2 \rangle 2$. LET: C be the component such that $A \subseteq C$

PROOF: Lemma 274.

⟨2⟩3. A intersects C

⟨2⟩4. If A intersects the component C' then $C' = C$

⟨3⟩1. LET: C' be a component that intersects A

⟨3⟩2. PICK $b \in A \cap C'$

⟨3⟩3. $A \subseteq C'$

PROOF: For all $x \in A$ we have $x \sim b$.

⟨3⟩4. $C = C'$

PROOF: By uniqueness in ⟨2⟩2.

□

Proposition 276. *Every component of a space is closed.*

PROOF:

⟨1⟩1. LET: X be a topological space and C a component of X.

⟨1⟩2. $\overline{C}$ is connected.

PROOF: Theorem 231.

⟨1⟩3. $C = \overline{C}$

PROOF: Lemma 229.

⟨1⟩4. C is closed.

PROOF: Lemma 54.

□

Proposition 277. *If a topological space has finitely many components then every component is open.*

PROOF: Each component is the complement of a finite union of closed sets. □

40 Path Components

Proposition 278. *Let X be a topological space. Define the relation $\sim$ on X by: $a \sim b$ if and only if there exists a path in X from a to b. Then $\sim$ is an equivalence relation on X.*

PROOF:

⟨1⟩1. $\sim$ is reflexive.

PROOF: For $a \in X$, the constant function $[0, 1] \to X$ with value a is a path from a to a.

⟨1⟩2. $\sim$ is symmetric.

PROOF: If $p : [0, 1] \to X$ is a path from a to b, then $\lambda t.p(1 - t)$ is a path from b to a.

⟨1⟩3. $\sim$ is transitive.

PROOF: Concatenate paths.

□

Definition 279 (Path Component)**.** Let X be a topological space. The *path components* of X are the equivalence relations under $\sim$.

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Theorem 280. *The path components of X are path-connected disjoint subspaces of X whose union is X such that every nonempty path-connected subspace of X intersects exactly one path component.*

PROOF:
$\langle 1\rangle 1$. Every path component is path-connected.
PROOF: If a and b are in the same path component then $a \sim b$, i.e. there exists a path from a to b.
$\langle 1\rangle 2$. The path components are disjoint and their union is X.
PROOF: Immediate from the definition.
$\langle 1\rangle 3$. Every non-empty path-cönnected subspace of X intersects exactly one path component.
$\langle 2\rangle 1$. LET: A be a nonempty path-connected subspace of X.
$\langle 2\rangle 2$. PICK $a \in A$
$\langle 2\rangle 3$. A intersects the $\sim$-equivalence class of a.
$\langle 2\rangle 4$. LET: C be any path component that intersects A.
$\langle 2\rangle 5$. PICK $b \in A \cap C$
$\langle 2\rangle 6$. $a \sim b$
PROOF: Since A is path-connected.
$\langle 2\rangle 7$. C is the $\sim$-equivalence class of a.
□

Proposition 281. *Every path component is included in a component.*

PROOF:
$\langle 1\rangle 1$. LET: X be a topological space and C a path component of X.
$\langle 1\rangle 2$. C is path-connected.
PROOF: Theorem 280.
$\langle 1\rangle 3$. C is connected.
PROOF: Proposition 251.
$\langle 1\rangle 4$. C is included in a component.
PROOF: Lemma 274.
□

41 Local Connectedness

Definition 282 (Locally Connected)**.** Let X be a topological space and $a \in X$. Then X is *locally connected* at a if and only if every neighbourhood of a includes a connected neighbourhood of a.

The space X is *locally connected* if and only if it is locally connected at every point.

Example 283. The real line is both connected and locally connected.

Example 284. The space $\mathbb{R} \setminus \{0\}$ is disconnected but locally connected.

Example 285. The topologist's sine curve is connected but not locally connected.

Example 286. The rationals $\mathbb{Q}$ are neither connected nor locally connected.

Theorem 287. *A topological space X is locally connected if and only if, for every open set U in X, every component of U is open in X.*

PROOF:
⟨1⟩1. If X is locally connected then, for every open set U in X, every component of U is open in X.
⟨2⟩1. ASSUME: X is locally connected.
⟨2⟩2. LET: U be open in X.
⟨2⟩3. LET: C be a component of U.
⟨2⟩4. LET: $a \in C$
⟨2⟩5. LET: V be a connected neighbourhood of a such that $V \subseteq U$
⟨2⟩6. $V \subseteq C$
PROOF: Lemma 274.
⟨2⟩7. Q.E.D.
PROOF: Lemma 33.
⟨1⟩2. If, for every open set U in X, every component of U is open in X, then X is locally connected.
⟨2⟩1. ASSUME: for every open set U in X, every component of U is open in X.
⟨2⟩2. LET: $a \in X$
⟨2⟩3. LET: U be a neighbourhood of a
⟨2⟩4. The component of U that contains a is a connected neighbourhood of a included in U.
□

Example 288. The ordered square is locally connected.

PROOF: Every neighbourhood of a point includes an interval around that point, which is connected by Theorem 242.

Example 289. Let X be the set of all rational points on the line segment $[0,1] \times \{0\}$, and Y the set of all rational points on the line segment $[0,1] \times \{1\}$. Let A be the space consisting of all line segments joining the point $(0,1)$ to a point of X, and all line segments joining the point $(1,0)$ to a point of Y. Then A is path-connected but is not locally connected at any point,

Proposition 290. *Let X and Y be topological spaces and $p : X \twoheadrightarrow Y$ be a quotient map. If X is locally connected then so is Y.*

PROOF:
⟨1⟩1. LET: U be an open set in Y.
⟨1⟩2. LET: C be a component of U.
⟨1⟩3. $p^{-1}(C)$ is a union of components of $p^{-1}(U)$
⟨2⟩1. LET: $x \in p^{-1}(C)$

⟨2⟩2. Let: D be the component of $p^{-1}(U)$ that contains x.
⟨2⟩3. $p(D)$ is connected.
Proof: Theorem 232.
⟨2⟩4. $p(D) \subseteq C$.
Proof: From ⟨1⟩2 since $p(x) \in p(D) \cap C$ (⟨2⟩1, ⟨2⟩2).
⟨2⟩5. $D \subseteq p^{-1}(C)$
⟨1⟩4. $p^{-1}(C)$ is open in $p^{-1}(U)$
Proof: Theorem 287.
⟨1⟩5. C is open in U
Proof: Since the restriction of p to $p : p^{-1}(U) \twoheadrightarrow U$ is a quotient map by Proposition 205.
⟨1⟩6. Q.E.D.
Proof: Theorem 287.
□

42 Local Path Connectedness

Definition 291 (Locally Path-Connected)**.** Let X be a topological space and $a \in X$. Then X is *locally path-connected* at a if and only if every neighbourhood of a includes a path-connected neighbourhood of a.

The space X is *locally path-connected* if and only if it is locally path-connected at every point.

Theorem 292. *A topological space X is locally path-connected if and only if, for every open set U in X, every path component of U is open in X.*

Proof:
⟨1⟩1. If X is locally path-connected then, for every open set U in X, every path component of U is open in X.
⟨2⟩1. Assume: X is locally path-connected.
⟨2⟩2. Let: U be open in X.
⟨2⟩3. Let: C be a path component of U.
⟨2⟩4. Let: $a \in C$
⟨2⟩5. Let: V be a path-connected neighbourhood of a such that $V \subseteq U$
⟨2⟩6. $V \subseteq C$
Proof: Lemma 274.
⟨2⟩7. Q.E.D.
Proof: Lemma 33.
⟨1⟩2. If, for every open set U in X, every component of U is open in X, then X is locally connected.
⟨2⟩1. Assume: for every open set U in X, every component of U is open in X.
⟨2⟩2. Let: $a \in X$
⟨2⟩3. Let: U be a neighbourhood of a
⟨2⟩4. The component of U that contains a is a connected neighbourhood of a included in U.

□

Theorem 293. *If a space is locally path connected then its components and its path components are the same.*

PROOF:
⟨1⟩1. LET: X be a locally path connected space.
⟨1⟩2. LET: C be a component of X.
⟨1⟩3. LET: $x \in C$
⟨1⟩4. LET: P be the path component of x
PROVE: $P = C$
⟨1⟩5. $P \subseteq C$
PROOF: Proposition 281.
⟨1⟩6. LET: Q be the union of the other path components included in C
⟨1⟩7. $C = P \cup Q$
PROOF: Proposition 281.
⟨1⟩8. P and Q are open in C
⟨2⟩1. C is open.
PROOF: Theorem 287.
⟨2⟩2. Q.E.D.
PROOF: Theorem 292.
⟨1⟩9. $Q = \emptyset$
PROOF: Otherwise P and Q would form a separation of C.
□

Example 294. The ordered square is not locally path connected, since it is connected but not path connected.

Proposition 295. *Let X be a locally path-connected space. Then every connected open subspace of X is path-connected.*

PROOF:
⟨1⟩1. LET: U be a connected open subspace of X.
⟨1⟩2. LET: P be a path component of U.
⟨1⟩3. LET: Q be the union of the other path components of U.
⟨1⟩4. P and Q are open in U.
PROOF: Theorem 292.
⟨1⟩5. $Q = \emptyset$
PROOF: Otherwise P and Q form a separation of U.
□

43 Weak Local Connectedness

Definition 296 (Weakly Locally Connected)**.** Let X be a topological space and $a \in X$. Then X is *weakly locally connected* at a if and only if every neighbourhood of a includes a connected subspace that includes a neighbourhood of a.

Proposition 297. *Let X be a topological space. If X is weakly locally connected at every point then X is locally connected.*

PROOF:
$\langle 1\rangle 1$. ASSUME: X is weakly locally connected at every point.
$\langle 1\rangle 2$. LET: U be open in X.
$\langle 1\rangle 3$. LET: C be a component of U.
$\langle 1\rangle 4$. C is open in X.
$\langle 2\rangle 1$. LET: $x \in C$
$\langle 2\rangle 2$. PICK a connected subspace D of U that includes a neighbourhood V of x.
$\langle 2\rangle 3$. $D \subseteq C$
PROOF: Lemma 274.
$\langle 2\rangle 4$. $x \in V \subseteq C$
$\langle 2\rangle 5$. Q.E.D.
PROOF: Lemma 33.
$\langle 1\rangle 5$. Q.E.D.
PROOF: Theorem 287.
□

Example 298. The 'infinite broom' (Munkres p.163) is an example of a space that is weakly locally connected at a point p but not locally connected at p.

44 Quasicomponents

Proposition 299. *Let X be a topological space. Define $\sim$ on X by $x \sim y$ if and only if there exists no separation U and V of X such that $x \in U$ and $y \in V$. Then $\sim$ is an equivalence relation on X.*

PROOF:
$\langle 1\rangle 1$. $\sim$ is reflexive.
PROOF: Since the two sets that make up a separation are disjoint.
$\langle 1\rangle 2$. $\sim$ is symmetric.
PROOF: Immediate from the defintion.
$\langle 1\rangle 3$. $\sim$ is transitive.
$\langle 2\rangle 1$. ASSUME: $x \sim y$ and $y \sim z$
$\langle 2\rangle 2$. ASSUME: for a contradiction there is a separation U and V of X with $x \in U$ and $z \in V$
$\langle 2\rangle 3$. $y \in U$ or $y \in V$
$\langle 2\rangle 4$. Q.E.D.
PROOF: Either case contradicts $\langle 2\rangle 1$.
□

Definition 300 (Quasicomponents)**.** For X a topological space, the *quasicomponents* of X are the equivalence classes under $\sim$.

Proposition 301. *Let X be a topological space. Then every component of X is included in a quasicomponent of X.*

PROOF:
$\langle 1 \rangle 1$. LET: C be a component of X.
$\langle 1 \rangle 2$. LET: $x, y \in C$
PROVE: $x \sim y$
$\langle 1 \rangle 3$. ASSUME: for a contradiction there exists a separation U and V of X with $x \in U$ and $y \in V$
$\langle 1 \rangle 4$. $C \cap U$ and $C \cap V$ form a separation of C.
$\langle 1 \rangle 5$. Q.E.D.
PROOF: This contradicts $\langle 1 \rangle 1$.

Proposition 302. *In a locally connected space, the components and the quasi-components are the same.*

PROOF:
$\langle 1 \rangle 1$. LET: X be a locally connected space and Q a quasicomponent of X.
$\langle 1 \rangle 2$. PICK a component C of X such that $C \subseteq Q$
$\langle 1 \rangle 3$. LET: D be the union of the components of X
$\langle 1 \rangle 4$. C and D are open in X.
PROOF: Theorem 287.
$\langle 1 \rangle 5$. D cannot contain any points of Q.
PROOF: If it did, then C and D would form a separation of X and there would be points $x, y \in Q$ with $x \in C$ and $y \in D$.
$\langle 1 \rangle 6$. $C = Q$
□

45 Open Coverings

Definition 303 (Open Covering)**.** Let X be a topological space. An *open covering* of X is a covering of X whose elements are all open sets.

46 Lindelöf Spaces

Definition 304 (Lindelöf Space)**.** A topological space X is *Lindelöf* if and only if every open covering has a countable subcovering.

Proposition 305. *Let X be a topological space. Then X is Lindelöf if and only if every set of closed sets that has the countable intersection property has nonempty intersection.*

PROOF: The following are equivalent.

1. X is compact.
2. Every open covering of X has a countable subcovering.
3. For any set $\mathcal{C}$ of closed sets, if $\{X \setminus C \mid C \in \mathcal{C}\}$ covers X then there is a countable subset $\mathcal{C}_0$ such that $\{X \setminus C \mid C \in \mathcal{C}_0\}$ covers X

4. For any set $\mathcal{C}$ of closed sets, if $\bigcap \mathcal{C} = \emptyset$ then there is a countable subset $\mathcal{C}_0$ with empty intersection.

5. Any set of closed sets with the countable intersection property has nonempty intersection.

□

Proposition 306 (CC)**.** *Let X be a topological space and $\mathcal{B}$ a basis for the topology on X. Then the following are equivalent.*

1. X is Lindelöf.

2. Every open covering of X by elements of $\mathcal{B}$ has a countable subcovering.

Proof:
⟨1⟩1. $1 \Rightarrow 2$
  Proof: Immediate from definitions.
⟨1⟩2. $2 \Rightarrow 1$
  ⟨2⟩1. Assume: Every open covering of X by elements of $\mathcal{B}$ has a countable subcovering.
  ⟨2⟩2. Let: $\mathcal{U}$ be an open covering of X.
  ⟨2⟩3. $\{B \in \mathcal{B} \mid \exists U \in \mathcal{U}.B \subseteq U\}$ covers X.
  ⟨2⟩4. Pick a finite subcovering $\mathcal{B}_0$.
  ⟨2⟩5. For $B \in BB$, Pick $U_B \in \mathcal{U}$ such that $B \subseteq U_B$
  ⟨2⟩6. $\{U_B \mid B \in \mathcal{B}_0\}$ covers X.
□

47 The Second Countability Axiom

Definition 307 (Second Countability Axiom)**.** A topological space satisfies the *second countability axiom*, or is *second countable*, if and only if it has a countable basis.

Example 308. The space $\mathbb{R}$ is second countable.

Proof: The set $\{(a, b) \mid a, b \in \mathbb{Q}\}$ is a basis. □

Proposition 309. *A subspace of a second countable space is second countable.*

Proof: If $\mathcal{B}$ is a countable basis for X and $Y \subseteq X$ then $\{B \cap Y \mid B \in \mathcal{B}\}$ is a countable basis for Y. □

Proposition 310 (CC)**.** *Every second countable space is Lindelöf.*

Proof: From Proposition 306.

Example 311 (CC)**.** The space $\mathbb{R}_l$ is Lindelöf.

⟨1⟩1. Let: $\mathcal{A}$ be a covering of $\mathbb{R}_l$ by basic open sets of the form $[a, b)$
⟨1⟩2. Let: $C = \bigcup\{(a, b) \mid [a, b) \in \mathcal{A}\}$

$\langle 1\rangle 3.$ $\mathbb{R} \setminus C$ is countable.
$\langle 2\rangle 1.$ For every $x \in \mathbb{R} \setminus C$, PICK a rational q_x such that $(x, q_x) \subseteq C$
$\langle 3\rangle 1.$ LET: $x \in \mathbb{R} \setminus C$
$\langle 3\rangle 2.$ PICK b such that $[x, b) \in \mathcal{A}$
$\langle 3\rangle 3.$ PICK a rational q such that $q \in (x, b)$
$\langle 2\rangle 2.$ The mapping $x \mapsto q_x$ is an injection $\mathbb{R} \setminus C \to \mathbb{Q}$
$\langle 1\rangle 4.$ PICK a countable $\mathcal{A}' \subseteq \mathcal{A}$ that covers $\mathbb{R} \setminus C$
$\langle 1\rangle 5.$ Under the standard topology on $\mathbb{R}$, C is second countable.
PROOF: Proposition 309.
$\langle 1\rangle 6.$ PICK a countable $\mathcal{A}'' \subseteq \mathcal{A}$ such that $\{(a, b) \mid [a, b) \in \mathcal{A}''\}$ covers C.
PROOF: Proposition 306.
$\langle 1\rangle 7.$ $\mathcal{A}' \cup \mathcal{A}''$ covers $\mathbb{R}_l$.
□

Example 312. The product of two Lindelöf spaces is not necessarily Lindelöf.
We prove that the Sorgenfrey plane is not Lindelöf.

PROOF:
$\langle 1\rangle 1.$ LET: $L = \{(x, -x) \mid x \in \mathbb{R}\}$
$\langle 1\rangle 2.$ L is closed in $\mathbb{R}_l^2$
$\langle 1\rangle 3.$ LET: $\mathcal{U} = \{[a, b) \times [a, -d) \mid a, b, d \in \mathbb{R}\}$
$\langle 1\rangle 4.$ $\mathcal{U} \cup \{\mathbb{R} \setminus L\}$ covers $\mathbb{R}_l^2$
$\langle 1\rangle 5.$ Every element of $\mathcal{U}$ intersects L at exactly one point.
$\langle 1\rangle 6.$ No countable subset of $\mathcal{U}$ covers $\mathbb{R}_l^2$.
□

48 Compact Spaces

Definition 313 (Compact)**.** A topological space is *compact* if and only if every open covering has a finite subcovering.

Lemma 314. *Let X be a topological space and Y a subspace of X. Then Y is compact if and only if every covering of Y by sets open in X has a finite subcovering.*

PROOF:
$\langle 1\rangle 1.$ If Y is compact then every covering of Y by sets open in X has a finite subcovering.
$\langle 2\rangle 1.$ ASSUME: Y is compact.
$\langle 2\rangle 2.$ LET: $\mathcal{U}$ be a covering of Y by sets open in X.
$\langle 2\rangle 3.$ $\{U \cap Y \mid U \in \mathcal{U}\}$ is an open covering of Y.
$\langle 2\rangle 4.$ PICK a finite subcovering $\{U_1 \cap Y, \ldots, U_n \cap Y\}$
$\langle 2\rangle 5.$ $\{U_1, \ldots, U_n\}$ is a finite subcovering of $\mathcal{U}$.
$\langle 1\rangle 2.$ If every covering of Y by sets open in X has a finite subcovering then Y is compact.
$\langle 2\rangle 1.$ LET: $\mathcal{U}$ be an open covering of Y.
$\langle 2\rangle 2.$ LET: $\mathcal{V} = \{V \text{ open in } X \mid V \cap Y \in \mathcal{U}\}$.

⟨2⟩3. $\mathcal{V}$ is a covering of Y by sets open in X.
⟨2⟩4. PICK a finite subcovering $\{V_1, \ldots, V_n\}$
⟨2⟩5. $\{V_1 \cap Y, \ldots, V_n \cap Y\}$ is a finite subcovering of $\mathcal{U}$.
□

Proposition 315. *Every closed subspace of a compact space is compact.*

PROOF:
⟨1⟩1. LET: X be a compact space and $Y \subseteq X$ be closed.
⟨1⟩2. LET: $\mathcal{U}$ be a covering of Y by sets open in X.
⟨1⟩3. $\mathcal{U} \cup \{X \setminus Y\}$ is an open covering of X.
⟨1⟩4. PICK a finite subcovering $\mathcal{U}_0$
⟨1⟩5. $\mathcal{U}_0 \cap \mathcal{U}$ is a finite subset of $\mathcal{U}$ that covers Y.
□

Theorem 316. *The continuous image of a compact space is compact.*

PROOF:
⟨1⟩1. LET: $f : X \twoheadrightarrow Y$ be continuous and surjective.
⟨1⟩2. LET: $\mathcal{V}$ be an open covering of Y
⟨1⟩3. $\{p^{-1}(V) \mid V \in \mathcal{V}\}$ is an open covering of X.
⟨1⟩4. PICK a finite subcovering $\{p^{-1}(V_1), \ldots, p^{-1}(V_n)\}$
⟨1⟩5. $\{V_1, \ldots, V_n\}$ covers Y.
□

Theorem 317. *Let A and B be compact subspaces of X and Y respectively. Let N be an open set in $X \times Y$ that includes $A \times B$. Then there exist open sets U and V in X and Y respectively such that $A \subseteq U$, $B \subseteq V$ and $U \times V \subseteq N$.*

PROOF:
⟨1⟩1. For all $x \in A$, there exist neighbourhoods U of x and V of B such that $U \times V \subseteq N$.
⟨2⟩1. LET: $x \in A$
⟨2⟩2. For all $y \in B$, there exist neighbourhoods U of x and V of y such that $U \times V \subseteq N$
⟨2⟩3. $\{V \text{ open in } Y \mid \exists \text{ neighbourhood } U \text{ of } x, U \times V \subseteq N\}$ covers B.
⟨2⟩4. PICK a finite subcover $\{V_1, \ldots, V_n\}$
⟨2⟩5. For $i = 1, \ldots, n$, PICK a neighbourhood U_i of x such that $U_i \times V_i \subseteq N$
⟨2⟩6. LET: $U = U_1 \cap \cdots \cap U_n$
⟨2⟩7. LET: $V = V_1 \cup \cdots \cup V_n$
⟨2⟩8. U is a neighbourhood of x.
⟨2⟩9. V is a neighbourhood of B.
⟨2⟩10. $U \times V \subseteq N$
⟨1⟩2. $\{U \text{ open in } X \mid \exists \text{ neighbourhood } V \text{ of } B . U \times V \subseteq N\}$ covers A.
⟨1⟩3. PICK a finite subcover $\{U_1, \ldots, U_n\}$
⟨1⟩4. For $i = 1, \ldots, n$, PICK a neighbourhood V_i of B such that $U_i \times V_i \subseteq N$
⟨1⟩5. LET: $U = U_1 \cup \cdots \cup U_n$
⟨1⟩6. LET: $V = V_1 \cap \cdots \cap V_n$

⟨1⟩7. U and V are open.
⟨1⟩8. $A \subseteq U$
⟨1⟩9. $B \subseteq V$
⟨1⟩10. $U \times V \subseteq N$
□

Corollary 317.1 (Tube Lemma)**.** *Let X and Y be topological spaces with Y compact. Let $a \in X$ and N be an open set in $X \times Y$ that includes $\{a\} \times Y$. Then there exists a neighbourhood W of a such that N includes the* tube *$W \times Y$.*

Theorem 318. *Let X be a topological space. Then X is compact if and only if every set of closed sets that has the finite intersection property has nonempty intersection.*

PROOF: The following are equivalent.

1. X is compact.
2. Every open covering of X has a finite subcovering.
3. For any set $\mathcal{C}$ of closed sets, if $\{X \setminus C \mid C \in \mathcal{C}\}$ covers X then there is a finite subset $\mathcal{C}_0$ such that $\{X \setminus C \mid C \in \mathcal{C}_0\}$ covers X
4. For any set $\mathcal{C}$ of closed sets, if $\bigcap \mathcal{C} = \emptyset$ then there is a finite subset $\mathcal{C}_0$ with empty intersection.
5. Any set of closed sets with the finite intersection property has nonempty intersection.

□

Corollary 318.1. *Let X be a topological space and $C_1 \supseteq C_2 \supseteq \cdots$ a nested sequence of nonempty closed sets. Then $\bigcap_n C_n$ is nonempty.*

Proposition 319. *Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on the same set X with $\mathcal{T} \subseteq \mathcal{T}'$. If $\mathcal{T}'$ is compact then $\mathcal{T}$ is compact.*

PROOF:
⟨1⟩1. LET: $\mathcal{U} \subseteq \mathcal{T}$ cover X
⟨1⟩2. $\mathcal{U} \subseteq \mathcal{T}'$
⟨1⟩3. A finite subset of $\mathcal{U}$ covers X.
□

Corollary 319.1. *If $\mathcal{T}$ and $\mathcal{T}'$ are two compact Hausdorff topologies on the same set X, then either $\mathcal{T} = \mathcal{T}'$ or $\mathcal{T}$ and $\mathcal{T}'$ are incomparable.*

PROOF: From the Proposition and Proposition 186. □

Example 320. Any set under the finite complement topology is compact.

Proposition 321. *Let X be a topological space. A finite union of compact subspaces of X is compact.*

PROOF:
⟨1⟩1. LET: A and B be compact subspaces of X.
⟨1⟩2. LET: $\mathcal{U}$ be a set of open sets in X that covers $A \cup B$
⟨1⟩3. PICK a finite subset $\mathcal{U}_1$ that covers A.
  PROOF: Lemma 314.
⟨1⟩4. PICK a finite subset $\mathcal{U}_2$ that covers B.
  PROOF: Lemma 314.
⟨1⟩5. $\mathcal{U}_1 \cup \mathcal{U}_2$ is a finite subset that covers $A \cup B$.
⟨1⟩6. Q.E.D.
  PROOF: Lemma 314.
□

Proposition 322. *Let A and B be disjoint compact subspaces of the Hausdorff space X. Then there exist disjoint open sets U and V that include A and B respectively.*

PROOF: From Theorem 317 with $N = X^2 \setminus \{(x,x) \mid x \in X\}$. □

Corollary 322.1. *Every compact subspace of a Hausdorff space is closed.*

Theorem 323. *Let $f : X \to Y$ be a bijective continuous function. If X is compact and Y is Hausdorff then f is a homeomorphism.*

PROOF:
⟨1⟩1. LET: $C \subseteq X$ be closed.
⟨1⟩2. C is compact.
  PROOF: Proposition 315.
⟨1⟩3. $f(C)$ is compact.
  PROOF: Theorem 316.
⟨1⟩4. $f(C)$ is closed.
  PROOF: Corollary 322.1.
⟨1⟩5. Q.E.D.
  PROOF: Lemma 115.
□

Proposition 324. *Let X be a compact space, Y a Hausdorff space, and $f : X \to Y$ a continuous map. Then f is a closed map.*

PROOF:
⟨1⟩1. LET: $C \subseteq X$ be closed.
⟨1⟩2. C is compact.
  PROOF: Proposition 315.
⟨1⟩3. $f(C)$ is compact.
  PROOF: Theorem 316.
⟨1⟩4. $f(C)$ is closed.
  PROOF: Corollary 322.1.
□

Proposition 325. *If Y is compact then the projection $\pi_1 : X \times Y \to X$ is a closed map.*

PROOF:
⟨1⟩1. LET: $A \subseteq X \times Y$ be closed.
⟨1⟩2. LET: $x \in X \setminus \pi_1(A)$
⟨1⟩3. PICK a neighbourhood U of x such that $U \times Y \subseteq (X \times Y) \setminus A$
PROOF: By the Tube Lemma.
⟨1⟩4. $x \in U \subseteq X \setminus \pi_1(A)$
⟨1⟩5. Q.E.D.
PROOF: So $X \setminus \pi_1(A)$ is open by Lemma 33.
□

Theorem 326. *Let X be a topological space and Y a compact Hausdorff space. Let $f : X \to Y$ be a function. Then f is continuous if and only if the graph of f is closed in $X \times Y$.*

PROOF:
⟨1⟩1. LET: G_f be the graph of f.
⟨1⟩2. If f is continuous then G_f is closed.
⟨2⟩1. ASSUME: f is continuous.
⟨2⟩2. LET: $(x, y) \in (X \times Y) \setminus G_f$
⟨2⟩3. PICK disjoint neighbourhoods U and V of y and $f(x)$ respectively.
⟨2⟩4. $f^{-1}(V) \times U$ is a neighbourhood of (x, y) disjoint from G_f.
⟨1⟩3. If G_f is closed then f is continuous.
⟨2⟩1. ASSUME: G_f is closed.
⟨2⟩2. LET: $x \in X$ and V be a neighbourhood of $f(x)$.
⟨2⟩3. $G_f \cap (X \times (Y \setminus V))$ is closed.
⟨2⟩4. $\pi_1(G_f \cap (X \times (Y \setminus V)))$ is closed.
PROOF: Proposition 325.
⟨2⟩5. LET: $U = X \setminus \pi_1(G_f \cap (X \times (Y \setminus V)))$
⟨2⟩6. U is a neighbourhood of x
⟨2⟩7. $f(U) \subseteq V$
□

Theorem 327. *Let X be a compact topological space. Let $(f_n : X \to \mathbb{R})$ be a monotone increasing sequence of continuous functions and $f : X \to \mathbb{R}$ a continuous function. If (f_n) converges pointwise to f, then (f_n) converges uniformly to f.*

PROOF:
⟨1⟩1. LET: $\epsilon > 0$
⟨1⟩2. For all $x \in X$, there exists N such that, for all $n \geq N$, we have $|f_n(x) - f(x)| < \epsilon$
⟨1⟩3. For $n \geq 1$,
LET: $U_n = \{x \in X \mid |f_n(x) - f(x)| < \epsilon\}$
⟨1⟩4. For $n \geq 1$, we have U_n is open in X.
⟨2⟩1. LET: $x \in X$
⟨2⟩2. LET: $\delta = \epsilon - |f_n(x) - f(x)|$
⟨2⟩3. PICK a neighbourhood U of x such that $f(U) \subseteq B(f(x), \delta/2)$

⟨2⟩4. PICK a neighbourhood V of x such that $f_n(V) \subseteq B(f_n(x), \delta/2)$

⟨2⟩5. $f(U \cap V) \subseteq U_n$

PROOF: For $y \in U \cap V$ we have

$$\begin{aligned} |f_n(y) - f(y)| &\leq |f_n(y) - f_n(x)| + |f_n(x) - f(x)| + |f(x) - y(y)| \\ &< \delta/2 + |f_n(x) - f(x)| + \delta/2 \\ &= \epsilon \end{aligned}$$

⟨1⟩5. $\{U_n \mid n \geq 1\}$ covers X

PROOF: From ⟨1⟩2

⟨1⟩6. PICK N such that $X = U_N$

⟨2⟩1. PICK $n_1, \ldots, n_k$ such that $U_{n_1}, \ldots, U_{n_k}$ cover X.

⟨2⟩2. LET: $N = \max(n_1, \ldots, n_k)$

⟨2⟩3. For all i we have $U_{n_i} \subseteq U_N$

PROOF: Since (f_n) is monotone increasing.

⟨2⟩4. $X = U_N$

⟨1⟩7. For all $x \in X$ and $n \geq N$ we have $|f_n(x) - f(x)| < \epsilon$

□

An example to show that we cannot remove the hypothesis that X is compact:

Example 328. Let $X = (0, 1)$, $f_n(x) = -x^n$ and $f(x) = 0$ for $x \in X$ and $n \geq 1$. Then $f_n \to f$ pointwise and (f_n) is monotone increasing but the convergence is not uniform since, for all $N \geq 1$, there exists $x \in (0, 1)$ such that $-x^N < -1/2$.

An example to show that we cannot remove the hypothesis that (f_n) is monotone increasing:

Example 329. Let $X = [0, 1]$, $f_n(x) = 1/(n^3(x - 1/n)^2 + 1)$ and $f(x) = 0$ for $x \in X$ and $n \geq 1$. Then X is compact and $f_n \to f$ pointwise but the convergence is not uniform since, for all $N \geq 1$, there exists $x \in [0, 1]$ such that $f_N(x) = 1$, namely $x = 1/N$.

Theorem 330. *Let X be a compact Hausdorff space. Let $\mathcal{A}$ be a chain of closed connected subsets of X. Then $\bigcap \mathcal{A}$ is connected.*

PROOF:

⟨1⟩1. ASSUME: for a contradiction C and D form a separation of $\bigcap \mathcal{A}$.

⟨1⟩2. PICK disjoint open sets U and V that include C and D respectively.

PROOF: Proposition 322.

⟨1⟩3. $\{A \setminus (U \cup V) \mid A \in \mathcal{A}\}$ is a set of closed sets with the finite intersection property.

⟨2⟩1. For all $A \in \mathcal{A}$ we have $A \setminus (U \cup V)$ is closed.

⟨2⟩2. For all $A_1, \ldots, A_n \in \mathcal{A}$ we have $(A_1 \cap \cdots \cap A_n) \setminus (U \cup V)$ is nonempty.

PROOF:

⟨3⟩1. LET: $A_1, \ldots, A_n \in \mathcal{A}$

⟨3⟩2. ASSUME: without loss of generality $A_1 \subseteq A_2, \ldots, A_n$

PROOF: Since $\mathcal{A}$ is a chain.

⟨3⟩3. $A_1 \setminus (U \cup V)$ is nonempty

PROOF: Otherwise $(A_1 \cap \cdots \cap A_n \cap U)$ and $(A_1 \cap \cdots \cap A_n \cap V)$ would form a separation of A_n.
⟨1⟩4. $\bigcap \mathcal{A} \setminus (U \cup V)$ is nonempty.
PROOF: Theorem 318.
⟨1⟩5. Q.E.D.
PROOF: This contradicts ⟨1⟩1 since $\bigcap AA \setminus (U \cup V) = \bigcap \mathcal{A} \setminus (C \cup D)$.
□

Theorem 331 (Tychonoff Theorem (AC)). *The product of a family of compact spaces is compact.*

PROOF:
⟨1⟩1. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of compact spaces.
⟨1⟩2. LET: $X = \prod_{\alpha \in J} X_\alpha$
⟨1⟩3. For any $\mathcal{A} \subseteq \mathcal{P}X$, we have $\bigcap_{A \in \mathcal{A}} \overline{A} \neq \emptyset$
⟨2⟩1. LET: $\mathcal{A} \subseteq \mathcal{P}X$
⟨2⟩2. PICK $\mathcal{D} \supseteq \mathcal{A}$ that is maximal with respect to the finite intersection property.
PROVE: $\bigcap_{D \in \mathcal{D}} \overline{D} \neq \emptyset$
PROOF: Lemma 3.
⟨2⟩3. For $\alpha \in J$, PICK $x_\alpha \in X_\alpha$ such that $x_\alpha \in \bigcap_{D \in \mathcal{D}} \overline{\pi_\alpha(D)}$
PROOF: Theorem 318 since $\{\overline{\pi_\alpha(D)} \mid D \in \mathcal{D}\}$ is a set of closed sets in X_α with the finite intersection property.
⟨2⟩4. LET: $x = (x_\alpha)_{\alpha \in J}$
PROVE: $x \in \bigcap_{D \in \mathcal{D}} \overline{D}$
⟨2⟩5. For any $\beta \in J$ and neighbourhood U of x_β in X_β, we have ${\pi_\beta}^{-1}(U)$ intersects every element of $\mathcal{D}$
⟨3⟩1. LET: $\beta \in J$
⟨3⟩2. LET: U be a neighbourhood of x_β in X_β.
⟨3⟩3. LET: $D \in \mathcal{D}$
⟨3⟩4. $x_\beta \in \overline{\pi_\beta(D)}$
PROOF: From ⟨2⟩3
⟨3⟩5. U intersects $\pi_\beta(D)$.
⟨3⟩6. ${\pi_\beta}^{-1}(U)$ intersects D.
⟨2⟩6. For any $\beta \in J$ and neighbourhood U of x_β in X_β, we have ${\pi_\beta}^{-1}(U) \in \mathcal{D}$
PROOF: Lemma 5.
⟨2⟩7. Every basic neighbourhood of x is an element of $\mathcal{D}$
PROOF: Lemma 4.
⟨2⟩8. Every basic neighbourhood of x intersects every element of $\mathcal{D}$
PROOF: Since $\mathcal{D}$ satisfies the finite intersection property.
⟨2⟩9. For all $D \in \mathcal{D}$ we have $x \in \overline{D}$
⟨1⟩4. Q.E.D.
PROOF: Theorem 318.
□

49 Perfect Maps

Definition 332 (Perfect Map)**.** Let X and Y be topological spaces and $f : X \to Y$. Then f is a *perfect map* if and only if f is a closed map, continuous, surjective and $p^{-1}(y)$ is compact for all $y \in Y$.

Proposition 333. *Let X be a topological space, Y a compact space, and $p : X \to Y$ a closed map such that, for all $y \in Y$, we have $p^{-1}(y)$ is compact. Then X is compact.*

PROOF:
$\langle 1\rangle 1.$ LET: $\mathcal{A}$ be a set of closed sets in X with the finite intersection property.
$\langle 1\rangle 2.$ $\mathcal{B} = \{p(A_1 \cap \cdots \cap A_n) \mid A_1, \ldots, A_n \in \mathcal{A}\}$ is a set of closed sets in Y with the finite intersection property.
PROOF: Since p is a closed map.
$\langle 1\rangle 3.$ PICK $y \in \bigcap \mathcal{B}$
PROOF: Theorem 318 since Y is compact.
$\langle 1\rangle 4.$ $\{A \cap p^{-1}(y) \mid A \in \mathcal{A}\}$ is a set of closed sets in $p^{-1}(y)$ with the finite intersection property.
$\langle 1\rangle 5.$ PICK $x \in \bigcap_{A \in \mathcal{A}} (A \cap p^{-1}(y))$
PROOF: Theorem 318 since $p^{-1}(y)$ is compact.
$\langle 1\rangle 6.$ $x \in \bigcap \mathcal{A}$
$\langle 1\rangle 7.$ Q.E.D.
PROOF: Theorem 318.
$\square$

50 Topological Groups

Definition 334 (Topological Group)**.** A *topological group* G consists of a T_1 space G and continuous maps $\cdot : G^2 \to G$ and $(\,)^{-1} : G \to G$ such that $(G, \cdot, (\,)^{-1})$ is a group.

Example 335.
1. The integers $\mathbb{Z}$ under addition are a topological group.
2. The real numbers $\mathbb{R}$ under addition are a topological group.
3. The positive reals under multiplication are a topological group.
4. The set $\{z \in \mathbb{C} \mid |z| = 1\}$ under multiplication and given the topology of S^1 is a topological group.
5. For any $n \geq 0$, the general linear group $GL_n(\mathbb{R})$ is a topological group under matrix multiplication, considered as a subspace of $\mathbb{R}^{n^2}$.

Lemma 336. *Let G be a T_1 space and $\cdot : G^2 \to G$, $(\,)^{-1} : G \to G$ be functions such that $(G, \cdot, (\,)^{-1})$ is a group. Then G is a topological group if and only if the function $f : G^2 \to G$ that maps (x, y) to xy^{-1} is continuous.*

PROOF:

⟨1⟩1. If G is a topological group then f is continuous.
  PROOF: From Theorem 104.
⟨1⟩2. If f is continuous then G is a topological group.
  ⟨2⟩1. ASSUME: f is continuous.
  ⟨2⟩2. $(\ )^{-1}$ is continuous.
    PROOF: Since $x^{-1} = f(e, x)$.
  ⟨2⟩3. $\cdot$ is continuous.
    PROOF: Since $xy = f(x, y^{-1})$.
□

Lemma 337. *Let G be a topological group and H a subgroup of G. Then H is a topological group under the subspace topology.*

PROOF:
⟨1⟩1. H is T_1.
  PROOF: From Proposition 174.
⟨1⟩2. multiplication and inverse on H are continuous.
  PROOF: From Theorem 105.
□

Lemma 338. *Let G be a topological group and H a subgroup of G. Then $\overline{H}$ is a subgroup of G.*

PROOF:
⟨1⟩1. LET: $x, y \in \overline{H}$
  PROVE: $xy^{-1} \in \overline{H}$
⟨1⟩2. LET: U be any neighbourhood of xy^{-1}
⟨1⟩3. LET: $f : G^2 \to G$, $f(a, b) = ab^{-1}$
⟨1⟩4. $f^{-1}(U)$ is a neighbourhood of (x, y)
⟨1⟩5. PICK neighbourhoods V, W of x and y respectively such that $f(V \times W) \subseteq U$.
⟨1⟩6. PICK $a \in V \cap H$ and $b \in W \cap H$
  PROOF: Theorem 55.
⟨1⟩7. $ab^{-1} \in U \cap H$
⟨1⟩8. Q.E.D.
  PROOF: By Theorem 55.
□

Proposition 339. *Let G be a topological group and $\alpha \in G$. Then the maps $l_\alpha, r_\alpha : G \to G$ defined by $l_\alpha(x) = \alpha x$, $r_\alpha(x) = x\alpha$ are homeomorphisms of G with itself.*

PROOF: They are continuous with continuous inverses $l_{\alpha^{-1}}$ and $r_{\alpha^{-1}}$. □

Corollary 339.1. *Every topological group is homogeneous.*

PROOF: Given a topological group G and $a, b \in G$, we have $l_{ba^{-1}}$ is a homeomorphism that maps a to b. □

Proposition 340. *Let G be a topological group and H a subgroup of G. Give G/H the quotient topology. For all $\alpha \in G$, the map $\overline{f_\alpha}$ that sends xH to αxH is a homeomorphism $G/H \cong G/H$.*

PROOF:
$\langle 1 \rangle 1.$ $\overline{f_\alpha}$ is well-defined.
PROOF: If $xy^{-1} \in H$ then $(\alpha x)(\alpha y)^{-1} = xy^{-1} \in H$.
$\langle 1 \rangle 2.$ $\overline{f_\alpha}$ is continuous.
PROOF: Theorem 208 since $\overline{f_\alpha} \circ p = p \circ f_\alpha$ is continuous, where $p : G \twoheadrightarrow G/H$ is the canonical surjection.
$\langle 1 \rangle 3.$ $\overline{f_\alpha}^{-1}$ is continuous.
PROOF: Similar since $\overline{f_\alpha}^{-1} = \overline{f_{\alpha^{-1}}}$.
∎

Corollary 340.1. *Let G be a topological group and H a subgroup of G. Give G/H the quotient topology. Then G/H is homogeneous.*

Proposition 341. *Let G be a topological group and H a subgroup of G. Give G/H the quotient topology. If H is closed in G then G/H is T_1.*

PROOF:
$\langle 1 \rangle 1.$ LET: $p : G \twoheadrightarrow G/H$ be the canonical surjection
$\langle 1 \rangle 2.$ LET: $x \in G$
$\langle 1 \rangle 3.$ $p^{-1}(xH) = f_x(H)$
$\langle 1 \rangle 4.$ $p^{-1}(xH)$ is closed in G
PROOF: Since H is closed and f_x is a homemorphism of G with itself.
$\langle 1 \rangle 5.$ $\{xH\}$ is closed in G/H
∎

Proposition 342. *Let G be a topological group and H a subgroup of G. Give G/H the quotient topology. Then the canonical surjection $p : G \twoheadrightarrow G/H$ is an open map.*

PROOF:
$\langle 1 \rangle 1.$ LET: $U \subseteq G$ be open.
$\langle 1 \rangle 2.$ $p^{-1}(p(U)) = \bigcup_{h \in H} r_h(U)$
$\langle 1 \rangle 3.$ $p^{-1}(p(U))$ is open.
$\langle 1 \rangle 4.$ $p(U)$ is open.
∎

Proposition 343. *Let G be a topological group and H a closed normal subgroup of G. Then G/H is a topological group under the quotient topology.*

PROOF:
$\langle 1 \rangle 1.$ G/H is T_1
PROOF: Proposition 341.
$\langle 1 \rangle 2.$ The map $\overline{m} : (xH, yH) \mapsto xy^{-1}H$ is continuous.
$\langle 2 \rangle 1.$ $p^2 : G^2 \to (G/H)^2$ is a quotient map.

PROOF: Propositions 207, 342.

⟨2⟩2. $\overline{m} \circ p^2$ is continuous.

PROOF: As it is $p^2 \circ m$ where $m : G^2 \to G$ with $m(x, y) = xy^{-1}$

□

Lemma 344. *Let G be a topological group and $A, B \subseteq G$. If either A or B is open then AB is open.*

PROOF: If A is open we have $AB = \bigcup_{b \in B} r_b(A)$. Similarly if B is open. □

Definition 345 (Symmetric Neighbourhood)**.** Let G be a topological group. A neighbourhood V of e is *symmetric* if and only if $V = V^{-1}$.

Lemma 346. *Let G be a topological group. Let V be a neighbourhood of e. Then V is symmetric if and only if, for all $x \in V$, we have $x^{-1} \in V$.*

PROOF:

⟨1⟩1. If V is symmetric then, for all $x \in V$, we have $x^{-1} \in V$

PROOF: Immediate from defintions.

⟨1⟩2. If, for all $x \in V$, we have $x^{-1} \in V$, then V is symmetric.

⟨2⟩1. ASSUME: for all $x \in V$ we have $x^{-1} \in V$

⟨2⟩2. $V \subseteq V^{-1}$

PROOF: If $x \in V$ then there exists $y \in V$ such that $x = y^{-1}$, namely $y = x^{-1}$

⟨2⟩3. $V^{-1} \subseteq V$

PROOF: Immediate from ⟨2⟩1.

□

Lemma 347. *Let G be a topological group. For every neighbourhood U of e, there exists a symmetric neighbourhood V of e such that $V^2 \subseteq U$.*

PROOF:

⟨1⟩1. LET: U be a neighbourhood of e.

⟨1⟩2. PICK a neighbourhood V' of e such that $V'V' \subseteq U$

PROOF: Such a neighbourhood exists because multiplication in G is continuous.

⟨1⟩3. PICK a neighbourhood W of e such that $WW^{-1} \subseteq V'$

PROOF: Such a neighbourhood exists because the function that maps (x, y) to xy^{-1} is continuous.

⟨1⟩4. LET: $V = WW^{-1}$

⟨1⟩5. V is a neighbourhood of e

⟨2⟩1. $e \in V$

PROOF: Since $e \in W$ so $e = ee^{-1} \in V$.

⟨2⟩2. V is open

PROOF: Lemma 344.

⟨1⟩6. V is symmetric

⟨2⟩1. For all $x \in V$ we have $x^{-1} \in V$

⟨3⟩1. LET: $x \in V$

⟨3⟩2. PICK$y, z \in W$ such that $x = yz^{-1}$
⟨3⟩3. $x^{-1} = zy^{-1}$
⟨3⟩4. $x^{-1} \in V$
⟨3⟩5. $x \in V^{-1}$
⟨2⟩2. Q.E.D.
PROOF: Lemma 346
⟨1⟩7. $V^2 \subseteq U$
PROOF: We have $V^2 \subseteq (V')^2 \subseteq U$
□

Proposition 348. *Every topological group is Hausdorff.*

PROOF:
⟨1⟩1. LET: G be a topological group.
⟨1⟩2. LET: $x, y \in G$ with $x \neq y$
⟨1⟩3. LET: $U = G \setminus \{x[^{-1}y]\}$
⟨1⟩4. PICK a symmetric neighbourhood V of e such that $VV \subseteq U$
⟨2⟩1. U is open
PROOF: Since G is T_1.
⟨2⟩2. $e \in U$
PROOF: Since $x \neq y$
⟨2⟩3. Q.E.D.
PROOF: Lemma 347.
⟨1⟩5. Vx and Vy are disjoint neighbourhoods of x and y respectively.
⟨2⟩1. Vx is open
PROOF: Since $Vx = r_x(V)$
⟨2⟩2. Vy is open
PROOF: Similar.
⟨2⟩3. $Vx \cap Vy = \emptyset$
⟨3⟩1. ASSUME: for a contradiction $z \in Vx \cap Vy$
⟨3⟩2. PICK $a, b \in V$ such that $z = ax = by$
⟨3⟩3. $xy^{-1} \in VV$
PROOF: Since $xy^{-1} = a^{-1}b$
⟨3⟩4. $xy^{-1} \in U$
⟨3⟩5. Q.E.D.
PROOF: From ⟨1⟩3.
□

Proposition 349. *Every topological group is regular.*

PROOF:
⟨1⟩1. LET: G be a topological group.
⟨1⟩2. LET: $A \subseteq G$ be a closed set and $a \notin A$.
⟨1⟩3. LET: $U = G \setminus Aa^{-1}$
⟨1⟩4. PICK a symmetric neighbourhood V of e such that $VV \subseteq U$
⟨2⟩1. U is open
PROOF: Since $Aa^{-1} = r_{a^{-1}}(A)$ is closed.

⟨2⟩2. $e \in U$

PROOF: Since $a \notin A$.

⟨2⟩3. Q.E.D.

PROOF: Lemma 347.

⟨1⟩5. VA and Va are disjoint open sets with $A \subseteq VA$ and $a \in Va$

⟨2⟩1. VA is open

PROOF: Lemma 344

⟨2⟩2. Va is open

PROOF: Lemma 344

⟨2⟩3. $VA \cap Va = \emptyset$

⟨3⟩1. ASSUME: for a contradiction $z \in VA \cap Va$

⟨3⟩2. PICK $b, c \in V$ and $d \in A$ with $z = bd = ca$

⟨3⟩3. $da^{-1} \in U$

PROOF: Since $da^{-1} = b^{-1}c \in VV \subseteq U$

⟨3⟩4. Q.E.D.

PROOF: This contradicts ⟨1⟩3

□

Proposition 350. *Let G be a topological group and H a subgroup of G. Give G/H the quotient topology. If H is closed in G then G/H is regular.*

PROOF:

⟨1⟩1. LET: $p : G \twoheadrightarrow G/H$ be the canonical surjection.

⟨1⟩2. LET: A be a closed set in G/H and $aH \in (G/H) \setminus A$.

⟨1⟩3. LET: $B = p^{-1}(A)$

⟨1⟩4. B is a closed saturated set in G.

⟨1⟩5. $B \cap aH = \emptyset$

⟨1⟩6. $B = BH$

⟨1⟩7. PICK a symmetric neighbourhood V of e such that VB does not intersect Va

⟨2⟩1. LET: $U = G \setminus Ba^{-1}$

⟨2⟩2. PICK a symmetric neighbourhood V of e such that $VV \subseteq U$

⟨3⟩1. U is open

PROOF: Since $Ba^{-1} = r_{a^{-1}}(B)$ is closed.

⟨3⟩2. $e \in U$

PROOF: If $e \in Ba^{-1}$ then $a \in B$

⟨3⟩3. Q.E.D.

PROOF: Lemma 347

⟨2⟩3. $VB \cap Va = \emptyset$

PROOF: If $vb = v'a$ for $v, v' \in V$ and $b \in B$ then we have $ba^{-1} = v^{-1}v' \in Ba \cap VV \subseteq Ba \cap U$.

⟨1⟩8. $p(VB)$ and $p(Va)$ are disjoint open sets

⟨2⟩1. $p(VB)$ and $p(Va)$ are open.

PROOF: Proposition 342.

⟨2⟩2. $p(VB) \cap p(Va) = \emptyset$

PROOF: If $vbH = v'aH$ for $v, v' \in V$, $b \in B$ then $v'a = vbh$ for some $h \in H$. Hence $v'a \in Va \cap VBH = Va \cap VB$.

⟨1⟩9. $A \subseteq p(VB)$
⟨1⟩10. $aH \in p(Va)$
□

Proposition 351. *Let G be a topological group. The component of G that contains e is a normal subgroup of G.*

PROOF:
⟨1⟩1. LET: C be the component of G that contains e.
⟨1⟩2. For all $x \in G$, xC is the component of G that contains x.
  ⟨2⟩1. LET: $x \in G$
  ⟨2⟩2. LET: D be the component of G that contains x.
  ⟨2⟩3. $xC \subseteq D$
    PROOF: Since xC is connected by Theorem 232.
  ⟨2⟩4. $D \subseteq xC$
    PROOF: Since $x^{-1}D \subseteq C$ similarly.
⟨1⟩3. For all $x \in G$, Cx is the component of G that contains x.
  PROOF: Similar.
⟨1⟩4. For all $x \in C$ we have $xC = Cx = C$
⟨1⟩5. For all $x \in C$ we have $x^{-1}C = C$
⟨1⟩6. For all $x \in C$ we have $x^{-1} \in C$
⟨1⟩7. For all $x, y \in C$ we have $xy \in C$
  PROOF: Since $xyC = xC = x$.
⟨1⟩8. For all $x \in G$ we have $xC = Cx$.
  PROOF: From ⟨1⟩2 and ⟨1⟩3.
□

Lemma 352. *Let G be a topological group. Let A be a closed set in G and B a compact subspace of G such that $A \cap B = \emptyset$. Then there exists a symmetric neighbourhood U of e such that $AU \cap BU = \emptyset$.*

PROOF:
⟨1⟩1. For all $b \in B$ there exists a symmetric neighbourhood V of e such that $bV^2 \cap A = \emptyset$
  ⟨2⟩1. LET: $b \in B$
  ⟨2⟩2. LET: $W = b^{-1}(G \setminus A)$
  ⟨2⟩3. W is a neighbourhood of e and $bW \cap A = \emptyset$
  ⟨2⟩4. PICK a symmetric neighbourhood V of e such that $V^2 \subseteq W$
⟨1⟩2. $\{bV^2 \mid b \in B, V \text{ is a symmetric neighbourhood of } e, bV^2 \cap A = \emptyset\}$ is an open cover of B
⟨1⟩3. PICK a finite subcover $b_1V_1^2, \ldots, b_nV_n^2$, say.
⟨1⟩4. LET: $U = V_1 \cap \cdots \cap V_n$
⟨1⟩5. $BU^2 \cap A = \emptyset$
⟨1⟩6. $AU \cap BU = \emptyset$
  PROOF: If $av \in BU$ where $a \in A$ and $v \in V$ then $a = avv^{-1} \in BU^2 \cap A$.
□

Proposition 353 (AC)**.** *Let G be a topological group. Let A be a closed set in G, and B a compact subspace of G. Then AB is closed.*

PROOF:
$\langle 1\rangle 1.$ LET: $x \in G \setminus AB$
$\langle 1\rangle 2.$ $A^{-1}x \cap B = \emptyset$
$\langle 1\rangle 3.$ $A^{-1}x$ is closed.
$\langle 1\rangle 4.$ PICK a symmetric neighbourhood U of e such that $A^{-1}xU \cap BU = \emptyset$
$\langle 1\rangle 5.$ xU^2 is open
  PROOF: Lemma 344.
$\langle 1\rangle 6.$ $x \in xU^2 \subseteq G \setminus AB$
$\square$

Corollary 353.1. *Let G be a topological group and $H \leq G$. Let $p : G \twoheadrightarrow G/H$ be the quotient map. If H is compact then p is a closed map.*

PROOF: For A closed in G, we have $p^{-1}(p(A)) = AH$ is closed, and so $p(A)$ is closed. $\square$

Corollary 353.2. *Let G be a topological group and $H \leq G$. If H and G/H are compact then G is compact.*

PROOF: From Proposition 333 since, for all $aH \in G/H$, we have $p^{-1}(aH) = aH$ is compact because it is homemorphic to H. $\square$

51 The Metric Topology

Definition 354 (Metric)**.** Let X be a set. A *metric* on X is a function $d : X^2 \to \mathbb{R}$ such that:

1. For all $x, y \in X$, $d(x, y) \geq 0$
2. For all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$
3. For all $x, y \in X$, $d(x, y) = d(y, x)$
4. (*Triangle Inequality*) For all $x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$

We call $d(x, y)$ the *distance* between x and y.

Definition 355 (Open Ball)**.** Let X be a metric space. Let $a \in X$ and $\epsilon > 0$. The *open ball* with *centre* a and *radius* ϵ is

$$B(a, \epsilon) = \{x \in X \mid d(a, x) < \epsilon\} \ .$$

Definition 356 (Metric Topology)**.** Let X be a metric space. The *metric topology* on X is the topology generated by the basis consisting of all the open balls.

We prove this is a basis for a topology.

PROOF:
$\langle 1 \rangle 1$. For every point a, there exists a ball B such that $a \in B$
PROOF: We have $a \in B(a, 1)$.
$\langle 1 \rangle 2$. For any balls B_1, B_2 and point $a \in B_1 \cap B_2$, there exists a ball B_3 such that $a \in B_3 \subseteq B_1 \cap B_2$
$\langle 2 \rangle 1$. LET: $B_1 = B(c_1, \epsilon_1)$ and $B_2 = B(c_2, \epsilon_2)$
$\langle 2 \rangle 2$. LET: $\delta = \min(\epsilon_1 - d(c_1, a), \epsilon_2 - d(c_2, a))$
PROVE: $B(a, \delta) \subseteq B_1 \cap B_2$
$\langle 2 \rangle 3$. LET: $x \in B(a, \delta)$
$\langle 2 \rangle 4$. $x \in B_1$
PROOF:

$$\begin{aligned} d(x, c_1) &= d(x, a) + d(a, c_1) \\ &< \delta + d(a, c_1) \\ &\leq \epsilon_1 \end{aligned}$$

$\langle 2 \rangle 5$. $x \in B_2$
PROOF: Similar.
$\square$

Proposition 357. *Let X be a metric space and $U \subseteq X$. Then U is open if and only if, for every $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$.*

PROOF:
$\langle 1 \rangle 1$. If U is open then, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$.
$\langle 2 \rangle 1$. ASSUME: U is open.

$\langle 2\rangle 2$. LET: $x \in U$
$\langle 2\rangle 3$. PICK $a \in X$ and $\delta > 0$ such that $x \in B(a, \delta) \subseteq U$
$\langle 2\rangle 4$. LET: $\epsilon = \delta - d(a, x)$
PROVE: $B(x, \epsilon) \subseteq U$
$\langle 2\rangle 5$. LET: $y \in B(x, \epsilon)$
$\langle 2\rangle 6$. $d(y, a) < \delta$
PROOF:

$$\begin{aligned} d(y,a) &\leq d(a,x) + d(x,y) \\ &< \delta + d(x,y) \\ &= \epsilon \end{aligned}$$

$\langle 2\rangle 7$. $y \in U$
$\langle 1\rangle 2$. If, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$, then U is open.
PROOF: Immediate from definitions.
□

Definition 358 (Discrete Metric)**.** Let X be a set. The *discrete metric* on X is defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Proposition 359. *The discrete metric induces the discrete topology.*

PROOF: For any (open) set U and point $a \in U$, we have $a \in B(a, 1) \subseteq U$. □

Definition 360 (Standard Metric on $\mathbb{R}$)**.** The *standard metric* on $\mathbb{R}$ is defined by $d(x, y) = |x - y|$.

Proposition 361. *The standard metric on $\mathbb{R}$ induces the standard topology on $\mathbb{R}$.*

PROOF:
$\langle 1\rangle 1$. Every open ball is open in the standard topology on $\mathbb{R}$.
PROOF: $B(a, \epsilon) = (a - \epsilon, a + \epsilon)$
$\langle 1\rangle 2$. For every open set U and point $a \in U$, there exists $\epsilon > 0$ such that $B(a, \epsilon) \subseteq U$
$\langle 2\rangle 1$. LET: U be an open set and $a \in U$
$\langle 2\rangle 2$. PICK an open interval b, c such that $a \in (b, c) \subseteq U$
$\langle 2\rangle 3$. LET: $\epsilon = \min(a - b, c - a)$
$\langle 2\rangle 4$. $B(a, \epsilon) \subseteq U$
□

Definition 362 (Metrizable)**.** A topological space X is *metrizable* if and only if there exists a metric on X that induces the topology.

Definition 363 (Bounded)**.** Let X be a metric space and $A \subseteq X$. Then A is *bounded* if and only if there exists M such that, for all $x, y \in A$, we have $d(x, y) \leq M$.

Definition 364 (Diameter)**.** Let X be a metric space and $A \subseteq X$. The *diameter* of A is

$$\operatorname{diam} A = \sup_{x,y \in A} d(x,y) \ .$$

Definition 365 (Standard Bounded Metric)**.** Let d be a metric on X. The *standard bounded metric* corresponding to d is the metric $\overline{d}$ defined by

$$\overline{d}(x,y) = \min(d(x,y), 1) \ .$$

We prove this is a metric.

PROOF:
$\langle 1 \rangle 1$. $\overline{d}(x,y) \geq 0$
PROOF: Since $d(x,y) \geq 0$
$\langle 1 \rangle 2$. $\overline{d}(x,y) = 0$ if and only if $x = y$
PROOF: $\overline{d}(x,y) = 0$ if and only if $d(x,y) = 0$ if and only if $x = y$
$\langle 1 \rangle 3$. $\overline{d}(x,y) = \overline{d}(y,x)$
PROOF: Since $d(x,y) = d(y,x)$
$\langle 1 \rangle 4$. $\overline{d}(x,z) \leq \overline{d}(x,y) + \overline{d}(y,z)$
PROOF:

$$\begin{aligned} \overline{d}(x,y) + \overline{d}(y,z) &= \min(d(x,y),1) + \min(d(y,z),1) \\ &= \min(d(x,y) + d(y,z), d(x,y)+1, d(y,z)+1, 2) \\ &\geq \min(d(x,z), 1) \\ &= \overline{d}(x,z) \end{aligned}$$

$\square$

Lemma 366. *In any metric space X, the set $\mathcal{B} = \{B(a,\epsilon) \mid a \in X, \epsilon < 1\}$ is a basis for the metric topology.*

PROOF:
$\langle 1 \rangle 1$. Every element of $\mathcal{B}$ is open.
PROOF: From Lemma 70.
$\langle 1 \rangle 2$. For every open set U and point $a \in U$, there exists $B \in \mathcal{B}$ such that $a \in B \subseteq U$
$\langle 2 \rangle 1$. LET: U be an open set and $a \in U$
$\langle 2 \rangle 2$. PICK$\epsilon > 0$ such that $B(a,\epsilon) \subseteq U$
$\langle 2 \rangle 3$. $B(a, \min(\epsilon, 1/2)) \subseteq U$
$\langle 1 \rangle 3$. Q.E.D.
PROOF: Lemma 71.

$\square$

Proposition 367. *Let d be a metric on the set X. Then the standard bounded metric $\overline{d}$ induces the same metric as d.*

PROOF: This follows from Lemma 366 since the open balls with radius < 1 are the same under both metrics. $\square$

Lemma 368. *Let d and d' be two metrics on the same set X. Let $\mathcal{T}$ and $\mathcal{T}'$ be the topologies they induce. Then $\mathcal{T} \subseteq \mathcal{T}'$ if and only if, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that*

$$B_{d'}(x, \delta) \subseteq B_d(x, \epsilon) \ .$$

PROOF:
⟨1⟩1. If $\mathcal{T} \subseteq \mathcal{T}'$ then, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$
  PROOF: From Proposition 357 since $x \in B_d(x, \epsilon) \in \mathcal{T}'$.
⟨1⟩2. If, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$, then $\mathcal{T} \subseteq \mathcal{T}'$
  ⟨2⟩1. ASSUME: For all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$
  ⟨2⟩2. LET: $U \in \mathcal{T}$
  ⟨2⟩3. For all $x \in U$ there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq U$.
    ⟨3⟩1. LET: $x \in U$
    ⟨3⟩2. PICK $\epsilon > 0$ such that $B_d(x, \epsilon) \subseteq U$
      PROOF: Proposition 357
    ⟨3⟩3. PICK $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$
      PROOF: By ⟨2⟩1
    ⟨3⟩4. $B_{d'}(x, \delta) \subseteq U$
  ⟨2⟩4. $U \in \mathcal{T}'$
    PROOF: Proposition 357.
□

Proposition 369. $\mathbb{R}^2$ *under the dictionary order topology is metrizable.*

PROOF: Define $d : \mathbb{R}^2 \to \mathbb{R}$ by

$$d((x, y), (x, z)) = \max(|y - z|, 1)$$

$$d((x, y), (x', y')) = 1 \qquad \text{if } x \neq x' \square$$

⟨1⟩1. $x \in \bigcap_{i=1}^{N} {\pi_i}^{-1}() \subseteq B_D(a, \epsilon)$

Proposition 370. *Let $d : X^2 \to \mathbb{R}$ be a metric on X. Then the metric topology on X is the coarsest topology such that d is continuous.*

PROOF:
⟨1⟩1. d is continuous.
  ⟨2⟩1. LET: $a, b \in X$
  ⟨2⟩2. LET: $\epsilon > 0$
  ⟨2⟩3. LET: $\delta = \epsilon / 2$
  ⟨2⟩4. LET: $x, y \in X$
  ⟨2⟩5. ASSUME: $\rho((a, b), (x, y)) < \delta$
  ⟨2⟩6. $|d(a, b) - d(x, y)| < \epsilon$
    ⟨3⟩1. $d(a, b) - d(x, y) < \epsilon$

PROOF:
$$\begin{aligned} d(a,b) &\leq d(a,x) + d(x,y) + d(y,b) \\ &\leq d(x,y) + 2\rho((a,b),(x,y)) \\ &< d(x,y) + 2\delta \\ &= d(x,y) + \epsilon \end{aligned}$$

⟨3⟩2. $d(a.b) - d(x,y) > -\epsilon$

PROOF: Similar.

⟨2⟩7. Q.E.D.

⟨1⟩2. If $\mathcal{T}$ is any topology under which d is continuous then $\mathcal{T}$ is finer than the metric topology.

PROOF: Since $B(a,\epsilon) = {d_a}^{-1}((-\infty,\epsilon))$

□

Proposition 371. *Let X be a metric space with metric d and $A \subseteq X$. The restriction of d to A is a metric on A that induces the subspace topology.*

PROOF:

⟨1⟩1. The restriction of d to A is a metric on A.

⟨1⟩2. Every open ball under $d \upharpoonright A$ is open under the subspace topology.

PROOF: $B_{d \upharpoonright A}(a,\epsilon) = B_d(a,\epsilon) \cap A$.

⟨1⟩3. If U is open in the subspace topology and $x \in U$, then there exists a $d \upharpoonright A$-ball B such that $x \in B \subseteq U$.

⟨2⟩1. PICK V open in X such that $U = V \cap A$

⟨2⟩2. PICK $\epsilon > 0$ such that $B_d(x,\epsilon) \subseteq V$

⟨2⟩3. Take $B = B_{d \upharpoonright A}(x,\epsilon)$

□

Corollary 371.1. *A subspace of a metrizable space is metrizable.*

Proposition 372. *Every metrizable space is Hausdorff.*

PROOF:

⟨1⟩1. LET: X be a metric space

⟨1⟩2. LET: $a,b \in X$ with $a \neq b$

⟨1⟩3. LET: $\epsilon = d(a,b)/2$

⟨1⟩4. LET: $U = B(a,\epsilon)$ and $V = B(b,\epsilon)$

⟨1⟩5. U and V are disjoint neighbourhoods of a and b respectively.

□

Proposition 373 (CC)**.** *The product of a countable family of metrizable spaces is metrizable.*

PROOF:

⟨1⟩1. LET: (X_n, d_n) be a sequence of metric spaces.

⟨1⟩2. ASSUME: w.l.o.g. each d_n is bounded above by 1.

PROOF: By Proposition 367.

⟨1⟩3. LET: D be the metric on $\mathbb{R}^\omega$ defined by $D(x,y) = \sup_i(d_i(x_i,y_i)/i)$.

⟨2⟩1. $D(x,y) \geq 0$
⟨2⟩2. $D(x,y) = 0$ if and only if $x = y$
⟨2⟩3. $D(x,y) = D(y,x)$
⟨2⟩4. $D(x,z) \leq D(x,y) + D(y,z)$
PROOF:

$$\begin{aligned} D(x,z) &= \sup_i \frac{d_i(x_i,z_i)}{i} \\ &\leq \sup_i \frac{d_i(x_i,y_i) + d_i(y_i,z_i)}{i} \\ &\leq \sup_i \frac{d_i(x_i,y_i)}{i} + \sup_i \frac{d_i(y_i,z_i)}{i} \\ &= D(x,y) + D(y,z) \end{aligned}$$

⟨1⟩4. Every open ball $B_D(a,\epsilon)$ is open in the product topology.
⟨2⟩1. PICK N such that $1/\epsilon < N$
⟨2⟩2. $B_D(a,\epsilon) = \prod_{i=1}^{\infty} U_i$ where $U_i = B(a_i, i\epsilon)$ if $i \leq N$, and $U_i = X_i$ if $i > N$
⟨1⟩5. For any open set U and $a \in U$, there exists $\epsilon > 0$ such that $B_D(a,\epsilon) \subseteq U$.
⟨2⟩1. LET: $n \geq 1$, V be an open set in $\mathbb{R}$ and $a \in {\pi_n}^{-1}(V)$
⟨2⟩2. PICK $\epsilon > 0$ such that $B_{d_n}(a,\epsilon) \subseteq V$
⟨2⟩3. $B_D(a,\epsilon/n) \subseteq {\pi_n}^{-1}(V)$
□

Theorem 374. *Let X and Y be metric spaces and $f : X \to Y$. Then f is continuous if and only if, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x,y) < \delta$ then $d(f(x),f(y)) < \epsilon$.*

PROOF:
⟨1⟩1. If f is continuous then, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x,y) < \delta$ then $d(f(x),f(y)) < \epsilon$
⟨2⟩1. ASSUME: f is continuous.
⟨2⟩2. LET: $x \in X$ and $\epsilon > 0$
⟨2⟩3. PICK a neighbourhood U of x such that $f(U) \subseteq B(f(x),\epsilon)$
PROOF: Theorem 101.
⟨2⟩4. PICK $\delta > 0$ such that $B(x,\delta) \subseteq U$
PROOF: Proposition 357.
⟨2⟩5. For all $y \in X$, if $d(x,y) < \delta$ then $d(f(x),f(y)) < \epsilon$
⟨1⟩2. If for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x,y) < \delta$ then $d(f(x),f(y)) < \epsilon$, then f is continuous.
⟨2⟩1. ASSUME: for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x,y) < \delta$ then $d(f(x),f(y)) < \epsilon$
⟨2⟩2. LET: $x \in X$ and V be a neighbourhood of $f(x)$
⟨2⟩3. PICK $\epsilon > 0$ such that $B(f(x),\epsilon) \subseteq V$
PROOF: Proposition 357.
⟨2⟩4. PICK $\delta > 0$ such that, for all $y \in X$, if $d(x,y) < \delta$ then $d(f(x),f(y)) < \epsilon$
PROOF: By ⟨2⟩1
⟨2⟩5. LET: $U = B(x,\delta)$
⟨2⟩6. U is a neighbourhood of x with $f(U) \subseteq V$

$\langle 2\rangle 7$. Q.E.D.
  PROOF: Theorem 101.

□

Proposition 375. *Let X be a metric space. Let (a_n) be a sequence in X and $l \in X$. Then $a_n \to l$ as $n \to \infty$ if and only if, for all $\epsilon > 0$, there exists N such that, for all $n \geq N$, we have $d(a_n, l) < \epsilon$.*

PROOF: From Proposition 84. □

Proposition 376. *Every metrizable space is first countable.*

PROOF: In any metric space X, the open balls $B(a, 1/n)$ for $n \geq 1$ form a local basis at a.

Example 377. $\mathbb{R}^\omega$ under the box topology is not metrizable.

Example 378. If J is uncountable then $\mathbb{R}^J$ under the product topology is not metrizable.

Proposition 379. *A compact subspace of a metric space is bounded.*

PROOF:
$\langle 1\rangle 1$. LET: X be a metric space and $A \subseteq X$ be compact.
$\langle 1\rangle 2$. PICK $a \in A$
$\langle 1\rangle 3$. $\{B(a, n) \mid n \in \mathbb{Z}^+\}$ covers A
$\langle 1\rangle 4$. PICK a finite subcover $\{B(a, n_1), \ldots, B(a, n_k)\}$
$\langle 1\rangle 5$. LET: $N = \max(n_1, \ldots, n_k)$
$\langle 1\rangle 6$. For all $x, y \in A$ we have $d(x, y) < 2N$
  PROOF:

$$\begin{aligned} d(x, y) &\leq d(x, a) + d(a, y) \\ &< N + N \end{aligned}$$

□

This example shows the converse does not hold:

Example 380. The space $\mathbb{R}$ under the standard bounded metric is bounded but not compact.

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Definition 381 (Square Metric)**.** The *square metric* ρ on $\mathbb{R}^n$ is defined by

$$\rho(\vec{x}, \vec{y}) = \max(|x_1 - y_1|, \ldots, |x_n - y_n|)$$

We prove this is a metric.

PROOF:
$\langle 1\rangle 1$. $\rho(\vec{x}, \vec{y}) \geq 0$
  PROOF: Immediate from definition.

$\langle 1 \rangle 2.$ $\rho(\vec{x}, \vec{y}) = 0$ if and only if $\vec{x} = \vec{y}$

PROOF: Immediate from definition.

$\langle 1 \rangle 3.$ $\rho(\vec{x}, \vec{y}) = \rho(\vec{y}, \vec{x})$

PROOF: Immediate from definition.

$\langle 1 \rangle 4.$ $\rho(\vec{x}, \vec{z}) \leq \rho(\vec{x}, \vec{y}) + \rho(\vec{y}, \vec{z})$

PROOF: Since $|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$.

□

Proposition 382. *The square metric induces the standard topology on* $\mathbb{R}^n$*.*

PROOF:

$\langle 1 \rangle 1.$ For every $a \in X$ and $\epsilon > 0$, we have $B_\rho(a, \epsilon)$ is open in the standard product topology.

PROOF:

$$B_\rho(a, \epsilon) = (a_1 - \epsilon, a_1 + \epsilon) \times \cdots \times (a_n - \epsilon, a_n + \epsilon)$$

$\langle 1 \rangle 2.$ For any open sets $U_1, \ldots, U_n$ in $\mathbb{R}$, we have $U_1 \times \cdots \times U_n$ is open in the square metric topology.

$\langle 2 \rangle 1.$ LET: $\vec{a} \in U_1 \times \cdots \times U_n$

$\langle 2 \rangle 2.$ For $i = 1, \ldots, n$, PICK $\epsilon_i > 0$ such that $(a_i - \epsilon_i, a_i + \epsilon_i) \subseteq U_i$

$\langle 2 \rangle 3.$ LET: $\epsilon = \min(\epsilon_1, \ldots, \epsilon_n)$

$\langle 2 \rangle 4.$ $B_\rho(\vec{a}, \epsilon) \subseteq U$

□

Definition 383. Given $\vec{x}, \vec{y} \in \mathbb{R}^n$, define the *sum* $\vec{x} + \vec{y}$ by

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n) \ .$$

Definition 384. Given $\lambda \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^n$, define the *scalar product* $\lambda\vec{x} \in \mathbb{R}^n$ by

$$\lambda(x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n)$$

Definition 385 (Inner Product)**.** Given $\vec{x}, \vec{y} \in \mathbb{R}^n$, define the *inner product* $\vec{x} \cdot \vec{y} \in \mathbb{R}$ by

$$(x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) = x_1 y_1 + \cdots + x_n y_n \ .$$

We write $\vec{x}^2$ for $\vec{x} \cdot \vec{x}$.

Definition 386 (Norm)**.** Let $n \geq 1$. The *norm* on $\mathbb{R}^n$ is the function $\| \ \| : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\|(x_1, \ldots, x_n)\| = \sqrt{x_1^2 + \cdots + x_n^2}$$

Lemma 387.

$$\|\vec{x}\|^2 = \vec{x}^2$$

PROOF: Immediate from definitions. □

Lemma 388.

$$\vec{x} \cdot (\vec{y} + \vec{z}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z}$$

PROOF: Each is equal to $(x_1y_1 + x_1z_1, \ldots, x_ny_n + x_nz_n)$. $\square$

Lemma 389.
$$|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\|\|\vec{y}\|$$

PROOF:
⟨1⟩1. ASSUME: $\vec{x} \neq \vec{0} \neq \vec{y}$
PROOF: Otherwise both sides are 0.
⟨1⟩2. LET: $a = 1/\|\vec{x}\|$
⟨1⟩3. LET: $b = 1/\|\vec{y}\|$
⟨1⟩4. $(a\vec{x} + b\vec{y})^2 \geq 0$ and $(a\vec{x} - b\vec{y})^2 \geq 0$
⟨1⟩5. $a^2\|\vec{x}\|^2 + 2ab\vec{x} \cdot \vec{y} + b^2\|\vec{y}\|^2 \geq 0$ and $a^2\|\vec{x}\|^2 - 2ab\vec{x} \cdot \vec{y} + b^2\|\vec{y}\|^2 \geq 0$
⟨1⟩6. $2ab\vec{x} \cdot \vec{y} + 2 \geq 0$ and $-2ab\vec{x} \cdot \vec{y} + 2 \geq 0$
⟨1⟩7. $\vec{x} \cdot \vec{y} \geq -1/ab$ and $\vec{x} \cdot \vec{y} \leq 1/ab$
⟨1⟩8. $\vec{x} \cdot \vec{y} \geq -\|\vec{x}\|\|\vec{y}\|$ and $\vec{x} \cdot \vec{y} \leq \|\vec{x}\|\|\vec{y}\|$
$\square$

Lemma 390 (Triangle Inequality)**.**

$$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$$

PROOF:
$$\begin{aligned}
\|\vec{x} + \vec{y}\|^2 &= \|\vec{x}\|^2 + 2\vec{x} \cdot \vec{y} + \|\vec{y}\|^2 \\
&\leq \|\vec{x}\|^2 + 2\|\vec{x}\|\|\vec{y}\| + \|\vec{y}\|^2 && \text{(Lemma 389)} \\
&= (\|\vec{x}\| + \|\vec{y}\|)^2 && \square
\end{aligned}$$

Definition 391 (Euclidean Metric)**.** Let $n \geq 1$. The *Euclidean metric* on $\mathbb{R}^n$ is defined by
$$d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\| \enspace .$$

We prove this is a metric.

⟨1⟩1. $d(\vec{x}, \vec{y}) \geq 0$
PROOF: Immediate from definition.
⟨1⟩2. $d(\vec{x}, \vec{y}) = 0$ if and only if $\vec{x} = \vec{y}$
PROOF: $d(\vec{x}, \vec{y}) = 0$ if and only if $\vec{x} - \vec{y} = \vec{0}$.
⟨1⟩3. $d(\vec{x}, \vec{y}) = d(\vec{y}, \vec{x})$
PROOF: Immediate from definition.
⟨1⟩4. $d(\vec{x}, \vec{z}) \leq d(\vec{x}, \vec{y}) + d(\vec{y}, \vec{z})$
PROOF:
$$\begin{aligned}
\|\vec{x} - \vec{z}\| &= \|(\vec{x} - \vec{y}) + (\vec{y} - \vec{z})\| \\
&\leq \|\vec{x} - \vec{y}\| + \|\vec{y} - \vec{z}\| && \text{(Lemma 390)}
\end{aligned}$$
$\square$

Proposition 392. *The Euclidean metric induces the standard topology on $\mathbb{R}^n$.*

PROOF:
⟨1⟩1. LET: ρ be the square metric.

⟨1⟩2. For all $\vec{a} \in \mathbb{R}^n$ and $\epsilon > 0$, we have $B_d(\vec{a}, \epsilon) \subseteq B_\rho(\vec{a}, \epsilon)$
⟨2⟩1. Let: $\vec{x} \in B_d(\vec{a}, \epsilon)$
⟨2⟩2. $\sqrt{(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2} < \epsilon$
⟨2⟩3. $(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2 < \epsilon^2$
⟨2⟩4. For all i we have $(x_i - a_i)^2 < \epsilon^2$
⟨2⟩5. For all i we have $|x_i - a_i| < \epsilon$
⟨2⟩6. $\rho(\vec{x}, \vec{a}) < \epsilon$
⟨1⟩3. For all $\vec{a} \in \mathbb{R}^n$ and $\epsilon > 0$, we have $B_\rho(\vec{a}, \epsilon/\sqrt{n}) \subseteq B_d(\vec{a}, \epsilon)$
⟨2⟩1. Let: $\vec{x} \in B_\rho(\vec{a}, \epsilon/\sqrt{n})$
⟨2⟩2. $\rho(\vec{x}, \vec{a}) < \epsilon/\sqrt{n}$
⟨2⟩3. For all i we have $|x_i - x_a| < \epsilon/\sqrt{n}$
⟨2⟩4. For all i we have $(x_i - x_a)^2 < \epsilon^2/n$
⟨2⟩5. $d(\vec{x}, \vec{a}) < \epsilon$
⟨1⟩4. Q.E.D.
Proof: By Lemma 368.
□

Proposition 393. *Let $n \geq 0$. For all $c \in \mathbb{R}^n$ and $\epsilon > 0$, the open ball $B(c, \epsilon)$ is path connected.*

Proof:
⟨1⟩1. Let: $a, b \in B(c, \epsilon)$
⟨1⟩2. Let: $p : [0, 1] \to B(c, \epsilon)$ be the function $p(t) = (1 - t)a + tb$
Proof: We have $p(t) \in B(c, \epsilon)$ for all t because

$$\begin{aligned} d(p(t), c) &= \|(1-t)a + tb - c\| \\ &= \|(1-t)(a-c) + t(b-c)\| \\ &\leq (1-t)\|a-c\| + t\|b-c\| \\ &< (1-t)\epsilon + t\epsilon \\ &= \epsilon \end{aligned}$$

⟨1⟩3. p is a path from a to b.
□

Proposition 394. *Let $n \geq 0$. For all $c \in \mathbb{R}^n$ and $\epsilon > 0$, the closed ball $\overline{B(c, \epsilon)}$ is path connected.*

Proof:
⟨1⟩1. Let: $a, b \in \overline{B(c, \epsilon)}$
⟨1⟩2. Let: $p : [0, 1] \to \overline{B(c, \epsilon)}$ be the function $p(t) = (1 - t)a + tb$
Proof: We have $p(t) \in \overline{B(c, \epsilon)}$ for all t because

$$\begin{aligned} d(p(t), c) &= \|(1-t)a + tb - c\| \\ &= \|(1-t)(a-c) + t(b-c)\| \\ &\leq (1-t)\|a-c\| + t\|b-c\| \\ &\leq (1-t)\epsilon + t\epsilon \\ &= \epsilon \end{aligned}$$

⟨1⟩3. p is a path from a to b.

□

Lemma 395. *If $\sum_{i=0}^{\infty} x_i^2$ and $\sum_{i=0}^{\infty} y_i^2$ converge then $\sum_{i=0}^{\infty} |x_i y_i|$ converges.*

PROOF:
⟨1⟩1. For all $N \geq 0$ we have $\sum_{i=0}^{N} |x_i y_i| \leq \sqrt{\sum_{i=0}^{N} |x_i|^2}\sqrt{\sum_{i=0}^{N} |y_i|^2}$
  PROOF: By the Cauchy-Schwarz inequality
⟨1⟩2. Q.E.D.
  PROOF: Since $\sum_{i=0}^{N} |x_i y_i|$ is an increasing sequence bounded above by $(\sum_{i=0}^{\infty} x_i^2)(\sum_{i=0}^{\infty} y_i^2)$.
□

Corollary 395.1. *If $\sum_{i=0}^{\infty} x_i^2$ and $\sum_{i=0}^{\infty} y_i^2$ converge then $\sum_{i=0}^{\infty}(x_i + y_i)^2$ converges.*

PROOF: Since $\sum_{i=0}^{\infty} x_i^2$, $\sum_{i=0}^{\infty} y_i^2$ and $2\sum_{i=0}^{\infty} x_i y_i$ all converge.

Definition 396 (l^2-metric)**.** The *l^2-metric* on

$$\left\{(x_n) \in \mathbb{R}^\omega \mid \sum_{n=0}^{\infty} x_n^2 \text{ converges}\right\}$$

is defined by

$$d(x, y) = \left(\sum_{n=0}^{\infty} (x_n - y_n)^2\right)^{1/2}$$

We prove this is a metric.

PROOF:
⟨1⟩1. d is well-defined.
  PROOF: By Corollary 395.1.
⟨1⟩2. $d(x, y) \geq 0$
⟨1⟩3. $d(x, y) = 0$ if and only if $x = y$
⟨1⟩4. $d(x, y) = d(y, x)$
⟨1⟩5. $d(x, z) \leq d(x, y) + d(y, z)$
  PROOF: By Lemma 390.
□

Theorem 397. *Addition is a continuous function $\mathbb{R}^2 \to \mathbb{R}$.*

PROOF:
⟨1⟩1. LET: $a, b \in \mathbb{R}$
⟨1⟩2. LET: $\epsilon > 0$
⟨1⟩3. LET: $\delta = \epsilon/2$
⟨1⟩4. LET: $x, y \in \mathbb{R}$
⟨1⟩5. ASSUME: $\rho((a, b), (x, y)) < \delta$
⟨1⟩6. $|(a + b) - (x + y)| < \epsilon$

PROOF:
$$\begin{aligned}|(a+b)-(x+y)| &= |a-x|+|b-y| \\ &\leq 2\rho((a,b),(x,y)) \\ &< 2\delta \\ &= \epsilon\end{aligned}$$
⟨1⟩7. Q.E.D.
PROOF: Theorem 374
□

Theorem 398. *Multiplication is a continuous function* $\mathbb{R}^2 \to \mathbb{R}$*.*

PROOF:
⟨1⟩1. LET: $a, b \in \mathbb{R}$
⟨1⟩2. LET: $\epsilon > 0$
⟨1⟩3. LET: $\delta = \min(\epsilon/(|a|+|b|+1), 1)$
⟨1⟩4. LET: $x, y \in \mathbb{R}$
⟨1⟩5. ASSUME: $\rho((a,b),(x,y)) < \delta$
⟨1⟩6. $|ab - xy| < \epsilon$
PROOF:
$$\begin{aligned}|ab-xy| &= |a(b-y)+(a-x)b-(a-x)(b-y)| \\ &\leq |a||b-y|+|b||a-x|+|a-x||b-y| \\ &< |a|\delta+|b|\delta+\delta^2 && (\langle 1\rangle 5) \\ &\leq |a|\delta+|b|\delta+\delta && (\langle 1\rangle 3) \\ &\leq \epsilon && (\langle 1\rangle 3)\end{aligned}$$
⟨1⟩7. Q.E.D.
PROOF: Theorem 374
□

Theorem 399. *The function* $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ *defined by* $f(x) = x^{-1}$ *is continuous.*

PROOF:
⟨1⟩1. For all $a \in \mathbb{R}$ we have $f^{-1}((a, +\infty))$ is open.
PROOF: The set is
$$\begin{aligned}(a^{-1}, +\infty) &\text{if } a > 0 \\ (0, +\infty) &\text{if } a = 0 \\ (-\infty, a^{-1}) \cup (0, +\infty) &\text{if } a < 0\end{aligned}$$
⟨1⟩2. For all $a \in \mathbb{R}$ we have $f^{-1}((-\infty, a))$ is open.
PROOF: Similar.
⟨1⟩3. Q.E.D.
PROOF: By Proposition 98 and Lemma 121.
□

Definition 400. For $n \geq 0$, the *unit ball* B^n is the space $\{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$.

Proposition 401. *For all* $n \geq 0$*, the unit ball* B^n *is path connected.*

PROOF:
⟨1⟩1. LET: $a, b \in B^n$
⟨1⟩2. LET: $p : [0,1] \to B^n$ be the function $p(t) = (1-t)a + tb$
PROOF: We have $p(t) \in B^n$ for all t because

$$\begin{aligned}\|(1-t)a + tb\| &\leq (1-t)\|a\| + t\|b\| \\ &\leq (1-t) + t \\ &= 1\end{aligned}$$

⟨1⟩3. p is a path from a to b.
□

Definition 402 (Punctured Euclidean Space)**.** For $n \geq 0$, defined *punctured Euclidean space* to be $\mathbb{R}^n \setminus \{0\}$.

Proposition 403. *For $n > 1$, punctured Euclidean space $\mathbb{R}^n \setminus \{0\}$ is path connected.*

PROOF:
⟨1⟩1. LET: $a, b \in \mathbb{R}^n \setminus \{0\}$
⟨1⟩2. CASE: 0 is on the line from a to b
⟨2⟩1. PICK a point c not on the line from a to b
⟨2⟩2. The path consisting of a straight line from a to c followed by a straight line from c to b is a path from a to b.
⟨1⟩3. CASE: 0 is not on the line from a to b
PROOF: The straight line from a to b is a path from a to b.

Corollary 403.1. *For $n > 1$, the spaces $\mathbb{R}$ and $\mathbb{R}^n$ are not homeomorphic.*

PROOF: For any point a, the space $\mathbb{R} \setminus \{a\}$ is disconnected.

Definition 404 (Unit Sphere)**.** For $n \geq 1$, the *unit sphere* S^{n-1} is the space

$$S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\} \ .$$

Proposition 405. *For $n > 1$, the unit sphere S^{n-1} is path connected.*

PROOF: The map $g : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$ defined by $g(x) = x/\|x\|$ is continuous and surjective. The result follows by Proposition 253. □

Proposition 406. *Let $f : S^1 \to \mathbb{R}$ be continuous. Then there exists $x \in S^1$ such that $f(x) = f(-x)$.*

PROOF:
⟨1⟩1. LET: $g : S^1 \to \mathbb{R}$ be the function $g(x) = f(x) - f(-x)$
PROVE: There exists $x \in S^1$ such that $g(x) = 0$
⟨1⟩2. ASSUME: without loss of generality $g((1,0)) > 0$
⟨1⟩3. $g((-1,0)) < 0$
⟨1⟩4. There exists x such that $g(x) = 0$
PROOF: By the Intermediate Value Theorem.
□

Definition 407 (Topologist's Sine Curve)**.** Let $S = \{(x, \sin 1/x) \mid 0 < x \leq 1\}$. The *topologist's sine curve* is the closure $\overline{S}$ of S.

Proposition 408.

$$\overline{S} = S \cup (\{0\} \times [-1, 1])$$

Proposition 409. *The topologist's sine curve is connected.*

Proof:
⟨1⟩1. Let: $S = \{(x, \sin 1/x) \mid 0 < x \leq 1\}$
⟨1⟩2. S is connected.
  Proof: Theorem 232.
⟨1⟩3. $\overline{S}$ is connected.
  Proof: Theorem 231.
□

Proposition 410 (CC)**.** *The topologist's sine curve is not path connected.*

Proof:
⟨1⟩1. Assume: for a contradiction $p : [0, 1] \to \overline{S}$ is a path from $(0, 0)$ to $(1, \sin 1)$.
⟨1⟩2. $p^{-1}(\{0\} \times [0, 1])$ is closed.
⟨1⟩3. Let: b be the greatest element of $p^{-1}(\{0\} \times [0, 1])$.
⟨1⟩4. $b < 1$
  Proof: Since $p(1) = (1, \sin 1)$.
⟨1⟩5. Pick a sequence $(t_n)_{n \geq 1}$ in $(b, 1]$ such that $t_n \to b$ and $\pi_2(p(t_n)) = (-1)^n$
  ⟨2⟩1. Let: $n \geq 1$
  ⟨2⟩2. Pick u with $0 < u < \pi_1(p(1/n))$ such that $\sin(1/u) = (-1)^n$
  ⟨2⟩3. Pick t_n such that $0 < t_n < 1/n$ and $\pi_1(p(t_n)) = u$
    Proof: One exists by the Intermediate Value Theorem.
⟨1⟩6. Q.E.D.
  Proof: This contradicts 113.
□

53 The Uniform Topology

Definition 411 (Uniform Metric)**.** Let J be a set. The *uniform metric* $\overline{\rho}$ on $\mathbb{R}^J$ is defined by

$$\overline{\rho}(a, b) = \sup_{j \in J} \overline{d}(a_j, b_j)$$

where $\overline{d}$ is the standard bounded metric on $\mathbb{R}$.

The *uniform topology* on $\mathbb{R}^J$ is the topology induced by the uniform metric.

We prove this is a metric.

Proof:
⟨1⟩1. $\overline{\rho}(a, b) \geq 0$
  Proof: Immediate from definitions.
⟨1⟩2. $\overline{\rho}(a, b) = 0$ if and only if $a = b$

PROOF: Immediate from definitions.
⟨1⟩3. $\overline{\rho}(a,b) = \overline{\rho}(b,a)$
PROOF: Immediate from definitions.
⟨1⟩4. $\overline{\rho}(a,c) \leq \overline{\rho}(a,b) + \overline{\rho}(b,c)$
PROOF:

$$\begin{aligned}
\overline{\rho}(a,c) &= \sup_{j \in J} \overline{d}(a_j, c_j) \\
&\leq \sup_{j \in J} (\overline{d}(a_j, b_j) + \overline{d}(b_j, c_j)) \\
&\leq \sup_{j \in J} \overline{d}(a_j, b_j) + \sup_{j \in J} \overline{d}(b_j, c_j) \\
&= \overline{\rho}(a,b) + \overline{\rho}(b,c)
\end{aligned}$$

□

Proposition 412. *The uniform topology on $\mathbb{R}^J$ is finer than the product topology.*

PROOF:
⟨1⟩1. LET: $j \in J$ and U be open in $\mathbb{R}$
PROVE: ${\pi_j}^{-1}(U)$ is open in the uniform topology.
⟨1⟩2. LET: $a \in {\pi_j}^{-1}(U)$
⟨1⟩3. PICK $\epsilon > 0$ such that $(a_j - \epsilon, a_j + \epsilon) \subseteq U$
⟨1⟩4. $B_{\overline{\rho}}(a, \epsilon) \subseteq {\pi_j}^{-1}(U)$
□

Proposition 413. *The uniform topology on $\mathbb{R}^J$ is coarser than the box topology.*

PROOF:
⟨1⟩1. LET: $a \in \mathbb{R}^J$ and $\epsilon > 0$
PROVE: $B(a, \epsilon)$ is open in the box topology.
⟨1⟩2. LET: $b \in B(a, \epsilon)$
⟨1⟩3. For $j \in J$ we have $|a_j - b_j| < \epsilon$
⟨1⟩4. For $j \in J$,
LET: $\delta_j = (\epsilon - |a_j - b_j|)/2$
⟨1⟩5. $\prod_{j \in J}(b_j - \delta_j, b_j + \delta_j) \subseteq B(a, \epsilon)$
□

Proposition 414. *The uniform topology on $\mathbb{R}^J$ is strictly finer than the product topology if and only if J is infinite.*

PROOF:
⟨1⟩1. If J is finite then the uniform and product topologies coincide.
PROOF: The uniform, box and product topologies are all the same.
⟨1⟩2. If J is infinite then the uniform and product topologies are different.
PROOF: The set $B(\vec{0}, 1)$ is open in the uniform topology but not the product topology.
□

Proposition 415 (DC)**.** *The uniform topology on $\mathbb{R}^J$ is strictly coarser than the box topology if and only if J is infinite.*

PROOF:

$\langle 1\rangle 1$. If J is finite then the uniform and box topologies coincide.

PROOF: The uniform, box and product topologies are all the same.

$\langle 1\rangle 2$. If J is infinite then the uniform and box topologies are different.

PROOF: Pick an ω-sequence $(j_1, j_2, \ldots)$ in J. Let $U = \prod_{j \in J} U_j$ where $U_{j_i} = (-1/i, 1/i)$ and $U_j = (-1, 1)$ for all other j. Then $\vec{0} \in U$ but there is no $\epsilon > 0$ such that $B(\vec{0}, \epsilon) \subseteq U$.

□

Proposition 416. *The closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ under the uniform topology is $\mathbb{R}^\omega$.*

PROOF: Given any open ball $B(a, \epsilon)$, pick an integer N such that $1/\epsilon < N$. Then $B(a, \epsilon)$ includes sequences whose nth entry is 0 for all $n \geq N$. □

Example 417. The space $\mathbb{R}^\omega$ is disconnected under the uniform topology. The set of bounded sequences and the set of unbounded sequences form a separation.

Proposition 418. *Give $\mathbb{R}^\omega$ the uniform topology. Let $x, y \in \mathbb{R}^\omega$. Then x and y are in the same component if and only if $x - y$ is bounded.*

PROOF:

$\langle 1\rangle 1$. The component containing 0 is the set of bounded sequences.

$\langle 2\rangle 1$. LET: B be the set of bounded sequences.

$\langle 2\rangle 2$. B is path-connected.

$\langle 3\rangle 1$. LET: $x.y \in B$

$\langle 3\rangle 2$. PICK $b > 0$ such that $|x_j|, |y_j| \leq b$ for all j

$\langle 3\rangle 3$. LET: $p : [0, 1] \to B$ be the function $p(t) = (1 - t)x + ty$

PROVE: p is continuous.

$\langle 3\rangle 4$. LET: $t \in [0, 1]$ and $\epsilon > 0$

$\langle 3\rangle 5$. LET: $\delta = \epsilon/2b$

$\langle 3\rangle 6$. LET: $s \in [0, 1]$ with $|s - t| < \delta$

$\langle 3\rangle 7$. $\overline{\rho}(p(s), p(t)) < \epsilon$

PROOF:

$$\begin{aligned}
\overline{\rho}(p(s), p(t)) &= \sup_j \overline{d}((1 - s)x_j + sy_j, (1 - t)x_j + ty_j) \\
&\leq |(s - t)x_j + (t - s)y_j| \\
&\leq |s - t||x_j - y_j| \\
&< 2b\delta \\
&= \epsilon
\end{aligned}$$

$\langle 2\rangle 3$. B is connected.

PROOF: Proposition 251.

$\langle 2\rangle 4$. If C is connected and $B \subseteq C$ then $B = C$.

PROOF: Otherwise $B \cap C$ and $C \setminus B$ form a separation of C.

$\langle 1\rangle 2$. Q.E.D.

PROOF: Since $\lambda x.x - y$ is a Homeomorphism of $\mathbb{R}^\omega$ with itself.

□

54 Uniform Convergence

Definition 419 (Uniform Convergence)**.** Let X be a set and Y a metric space. Let $(f_n : X \to Y)$ be a sequence of functions and $f : X \to Y$ be a function. Then f_n *converges uniformly* to f as $n \to \infty$ if and only if, for all $\epsilon > 0$, there exists N such that, for all $n \geq N$ and $x \in X$, we have $d(f_n(x), f(x)) < \epsilon$.

Example 420. Define $f_n : [0,1] \to \mathbb{R}$ by $f_n(x) = x^n$ for $n \geq 1$, and $f : [0,1] \to \mathbb{R}$ by $f(x) = 0$ if $x < 1$, $f(1) = 1$. Then f_n converges to f pointwise but not uniformly.

Theorem 421 (Uniform Limit Theorem)**.** *Let X be a topological space and Y a metric space. Let $(f_n : X \to Y)$ be a sequence of continuous functions and $f : X \to Y$ be a function. If f_n converges uniformly to f as $n \to \infty$, then f is continuous.*

PROOF:
⟨1⟩1. LET: $x \in X$ and $\epsilon > 0$
⟨1⟩2. PICK N such that, for all $n \geq N$ and $y \in X$, we have $d(f_n(y), f(y)) < \epsilon/3$
⟨1⟩3. PICK a neighbourhood U of x such that $f_N(U) \subseteq B(f_N(x), \epsilon/3)$
PROVE: $f(U) \subseteq B(f(x), \epsilon)$
⟨1⟩4. LET: $y \in U$
⟨1⟩5. $d(f(y), f(x)) < \epsilon$
PROOF:

$$\begin{aligned} d(f(y), f(x)) &\leq d(f(y), f_N(y)) + d(f_N(y), f_N(x)) + d(f_N(x), f(x)) && \text{(Triangle Inequality)} \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 && (\langle 1\rangle 2,\ \langle 1\rangle 3) \\ &= \epsilon \end{aligned}$$

∎

Proposition 422. *Let X be a topological space and Y a metric space. Let $(f_n : X \to Y)$ be a sequence of continuous functions and $f : X \to Y$ be a function. Let (a_n) be a sequence of points in X and $a \in X$. If f_n converges uniformly to f and a_n converges to a in X then $f_n(a_n)$ converges to $f(a)$ uniformly in Y.*

PROOF:
⟨1⟩1. LET: $\epsilon > 0$
⟨1⟩2. PICK N_1 such that, for all $n \geq N_1$ and $x \in X$, we have $d(f_n(x), f(x)) < \epsilon/2$
⟨1⟩3. PICK N_2 such that, for all $n \geq N_2$, we have $a_n \in f^{-1}(B(a, \epsilon/2))$
PROOF: Using the fact that f is continuous from the Uniform Limit Theorem.
⟨1⟩4. LET: $N = \max(N_1, N_2)$
⟨1⟩5. LET: $n \geq N$
⟨1⟩6. $d(f_n(a_n), f(a)) < \epsilon$
PROOF:

$$\begin{aligned} d(f_n(a_n), f(a)) &\leq d(f_n(a_n), f(a_n)) + d(f(a_n), f(a)) && \text{(Triangle Inequality)} \\ &< \epsilon/2 + \epsilon/2 && (\langle 1\rangle 2,\ \langle 1\rangle 3) \\ &= \epsilon \end{aligned}$$

□

Proposition 423. *Let X be a set. Let $(f_n : X \to \mathbb{R})$ be a sequence of functions and $f : X \to \mathbb{R}$ be a function. Then f_n converges unifomly to f as $n \to \infty$ if and only if $f_n \to f$ as $n \to \infty$ in $\mathbb{R}^X$ under the uniform topology.*

PROOF:
⟨1⟩1. If f_n converges uniformly to f then f_n converges to f under the uniform topology.
  ⟨2⟩1. ASSUME: f_n converges uniformly to f
  ⟨2⟩2. LET: $\epsilon > 0$
  ⟨2⟩3. PICK N such that, for all $n \geq N$ and $x \in X$, we have $d(f_n(x), f(x)) < \epsilon/2$
  ⟨2⟩4. For all $n \geq N$ we have $\overline{\rho}(f_n, f) \leq \epsilon/2$
  ⟨2⟩5. For all $n \geq N$ we have $\overline{\rho}(f_n, f) < \epsilon$
⟨1⟩2. If f_n converges to f under the uniform topology then f_n converges uniformly to f.
  ⟨2⟩1. ASSUME: f_n converges to f under the uniform topology.
  ⟨2⟩2. LET: $\epsilon > 0$
  ⟨2⟩3. PICK N such that, for all $n \geq N$, we have $\overline{\rho}(f_n, f) < \min(\epsilon, 1/2)$
  ⟨2⟩4. LET: $n \geq N$
  ⟨2⟩5. LET: $x \in X$
  ⟨2⟩6. $\overline{\rho}(f_n, f) < \min(\epsilon, 1/2)$
    PROOF: From ⟨2⟩3.
  ⟨2⟩7. $d(f_n(x), f(x)) < \min(\epsilon, 1/2)$
  ⟨2⟩8. $d(f_n(x), f(x)) < \epsilon$

□

55 Isometric Imbeddings

Definition 424. Let X and Y be metric spaces. An *isometric imbedding* $f : X \to Y$ is a function such that, for all $x, y \in X$, we have $d(f(x), f(y)) = d(x, y)$.

Proposition 425. *Every isometric imbedding is an imbedding.*

PROOF:
⟨1⟩1. LET: $f : X \to Y$ be an isometric imbedding.
⟨1⟩2. f is injective.
  PROOF: If $f(x) = f(y)$ then $d(f(x), f(y)) = 0$ hence $d(x, y) = 0$ hence $x = y$.
⟨1⟩3. f is continuous.
  PROOF: For all $\epsilon > 0$, if $d(x, y) < \epsilon$ then $d(f(x), f(y)) < \epsilon$.
⟨1⟩4. $f : X \to f(X)$ is an open map.
  PROOF: $f(B(a, \epsilon)) = B(f(a), \epsilon) \cap f(X)$.

□