

Topology

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1 Real Analysis

Definition 1. Let \mathbb{R}^ω be the subset of \mathbb{R}^ω consisting of all sequences (a_n) such that $a_n = 0$ for all but finitely many n .

Theorem 2. If (s_n) is a bounded sequence of real numbers and $s_n \leq s_{n+1}$ for all n then (s_n) converges to its supremum.

PROOF: Let $l = \sup_n s_n$ and let $\epsilon > 0$. Then $l - \epsilon$ is not an upper bound for (s_n) . Pick N such that $s_N > l - \epsilon$. Then $|s_n - l| < \epsilon$ for all $n \geq N$. \square

Theorem 3. If $\sum_{i=0}^\infty a_i = s$ and $\sum_{i=0}^\infty b_i = t$ then $\sum_{i=0}^\infty (ca_i + b_i) = cs + t$.

PROOF: $\sum_{i=0}^N (ca_i + b_i) = c \sum_{i=0}^N a_i + \sum_{i=0}^N b_i \rightarrow cs + t$ as $n \rightarrow \infty$. \square

Theorem 4 (Comparison Test). If $|a_i| \leq b_i$ for all i and $\sum_{i=0}^\infty b_i$ converges then $\sum_{i=0}^\infty a_i$ converges.

PROOF:

$\langle 1 \rangle 1$. $\sum_{i=0}^\infty |a_i|$ converges

PROOF: The partial sums $\sum_{i=0}^N |a_i|$ form an increasing sequence bounded above by $\sum_{i=0}^\infty b_i$.

$\langle 1 \rangle 2$. LET: $c_i = |a_i| + a_i$ for all i

$\langle 1 \rangle 3$. $\sum_{i=0}^\infty c_i$ converges

PROOF: Each c_i is either $2|a_i|$ or 0. So the partial sums $\sum_{i=0}^N c_i$ form an increasing sequence bounded above by $2 \sum_{i=0}^\infty b_i$.

$\langle 1 \rangle 4$. Q.E.D.

PROOF: Since $a_i = c_i - |a_i|$.

\square

Corollary 4.1. If $\sum_{i=0}^\infty |a_i|$ converges then $\sum_{i=0}^\infty a_i$ converges.

Theorem 5 (Weierstrass M-test). Let X be a set and $(f_n : X \rightarrow \mathbb{R})$ be a sequence of functions. Let

$$s_n(x) = \sum_{i=0}^n f_i(x)$$

for all n, x . Suppose $|f_i(x)| \leq M_i$ for all $i \geq 0$ and $x \in X$. If the series $\sum_{i=0}^{\infty} M_i$ converges, then the sequence (s_n) converges uniformly to

$$s(x) = \sum_{i=0}^{\infty} f_i(x) .$$

PROOF:

$\langle 1 \rangle 1$. LET: $r_n = \sum_{i=n+1}^{\infty} M_i$ for all n

$\langle 1 \rangle 2$. Given $0 \leq n < k$, we have $|s_k(x) - s_n(x)| \leq r_n$

PROOF:

$$\begin{aligned} |s_k(x) - s_n(x)| &= \left| \sum_{i=n+1}^k f_i(x) \right| \\ &\leq \sum_{i=n+1}^k |f_i(x)| \\ &\leq \sum_{i=n+1}^k M_i \\ &\leq r_n \end{aligned}$$

$\langle 1 \rangle 3$. Given $n \geq 0$ we have $|s(x) - s_n(x)| \leq r_n$

PROOF: By taking the limit $k \rightarrow \infty$ in $\langle 1 \rangle 2$.

$\langle 1 \rangle 4$. Q.E.D.

PROOF: Since $r_n \rightarrow 0$ as $n \rightarrow \infty$.

□

2 Order Theory

Definition 6 (Convex). Let X be a linearly ordered set and $Y \subseteq X$. Then Y is *convex* if and only if, for all $a, b \in Y$ and $c \in X$, if $a < c < b$ then $c \in Y$.

3 Topological Spaces

Definition 7 (Topology). A *topology* on a set X is a set $\mathcal{T} \subseteq \mathcal{P}X$ such that:

- $X \in \mathcal{T}$.
- For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$.
- For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$.

We call the elements of X *points* and the elements of \mathcal{T} *open sets*.

Definition 8 (Topological Space). A *topological space* X consists of a set X and a topology on X .

Definition 9 (Discrete Space). For any set X , the *discrete* topology on X is $\mathcal{P}X$.

Definition 10 (Indiscrete Space). For any set X , the *indiscrete* or *trivial* topology on X is $\{\emptyset, X\}$.

Definition 11 (Finite Complement Topology). For any set X , the *finite complement topology* on X is $\{U \in \mathcal{P}X \mid X \setminus U \text{ is finite}\} \cup \{\emptyset\}$.

Definition 12 (Countable Complement Topology). For any set X , the *countable complement topology* on X is $\{U \in \mathcal{P}X \mid X \setminus U \text{ is countable}\} \cup \{\emptyset\}$.

Definition 13 (Finer, Coarser). Suppose that \mathcal{T} and \mathcal{T}' are two topologies on a given set X . If $\mathcal{T}' \supseteq \mathcal{T}$, we say that \mathcal{T}' is *finer* than \mathcal{T} ; if \mathcal{T}' properly contains \mathcal{T} , we say that \mathcal{T}' is *strictly finer* than \mathcal{T} . We also say that \mathcal{T} is *coarser* than \mathcal{T}' , or *strictly coarser*, in these two respective situations. We say \mathcal{T} is *comparable* with \mathcal{T}' if either $\mathcal{T}' \supseteq \mathcal{T}$ or $\mathcal{T} \supseteq \mathcal{T}'$.

Lemma 14. Let X be a topological space and $U \subseteq X$. Then U is open if and only if, for all $x \in U$, there exists an open set V such that $x \in V \subseteq U$.

PROOF:

$\langle 1 \rangle 1. \Rightarrow$

PROOF: Take $V = U$

$\langle 1 \rangle 2. \Leftarrow$

PROOF: We have $U = \bigcup \{V \in \mathcal{P}X \mid V \subseteq U\}$.

□

Lemma 15. Let X be a set and \mathcal{T} a nonempty set of topologies on X . Then $\bigcap \mathcal{T}$ is a topology on X , and is the finest topology that is coarser than every member of \mathcal{T} .

PROOF:

$\langle 1 \rangle 1. X \in \bigcap \mathcal{T}$

PROOF: Since X is in every member of \mathcal{T} .

$\langle 1 \rangle 2. \bigcap \mathcal{T}$ is closed under union.

$\langle 2 \rangle 1. \text{ LET: } \mathcal{U} \subseteq \bigcap \mathcal{T}$

$\langle 2 \rangle 2. \text{ For all } T \in \mathcal{T} \text{ we have } \mathcal{U} \subseteq T$

$\langle 2 \rangle 3. \text{ For all } T \in \mathcal{T} \text{ we have } \bigcup \mathcal{U} \in T$

$\langle 2 \rangle 4. \bigcup \mathcal{U} \in \bigcap \mathcal{T}$

$\langle 1 \rangle 3. \bigcap \mathcal{T}$ is closed under binary intersection.

$\langle 2 \rangle 1. \text{ LET: } U, V \in \bigcap \mathcal{T}$

$\langle 2 \rangle 2. \text{ For all } T \in \mathcal{T} \text{ we have } U, V \in T$

$\langle 2 \rangle 3. \text{ For all } T \in \mathcal{T} \text{ we have } U \cap V \in T$

$\langle 2 \rangle 4. U \cap V \in \bigcap \mathcal{T}$

□

Lemma 16. Let X be a set and \mathcal{T} a set of topologies on X . Then there exists a unique coarsest topology that is finer than every member of \mathcal{T} .

PROOF: The required topology is given by

$\bigcap \{T \in \mathcal{P}X \mid T \text{ is a topology on } X \text{ that is finer than every member of } \mathcal{T}\}$,

The set is nonempty since it contains the discrete topology. □

4 Basis for a Topology

Definition 17 (Basis). If X is a set, a *basis* for a topology on X is a set $\mathcal{B} \subseteq \mathcal{P}X$ called *basis elements* such that

1. For all $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$.
2. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

If \mathcal{B} satisfies these two conditions, then we define the topology *generated* by \mathcal{B} to be $\mathcal{T} = \{U \in \mathcal{P}X \mid \forall x \in U. \exists B \in \mathcal{B}. x \in B \subseteq U\}$.

We prove this is a topology.

PROOF:

$\langle 1 \rangle 1.$ $X \in \mathcal{T}$

PROOF: For all $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B \subseteq X$ by condition 1.

$\langle 1 \rangle 2.$ For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$

$\langle 2 \rangle 1.$ LET: $\mathcal{U} \subseteq \mathcal{T}$

$\langle 2 \rangle 2.$ LET: $x \in \bigcup \mathcal{U}$

$\langle 2 \rangle 3.$ PICK $U \in \mathcal{U}$ such that $x \in U$

$\langle 2 \rangle 4.$ PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$

PROOF: Since $U \in \mathcal{T}$ by $\langle 2 \rangle 1$ and $\langle 2 \rangle 3$.

$\langle 2 \rangle 5.$ $x \in B \subseteq \bigcup \mathcal{U}$

$\langle 1 \rangle 3.$ For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$

$\langle 2 \rangle 1.$ LET: $U, V \in \mathcal{T}$

$\langle 2 \rangle 2.$ LET: $x \in U \cap V$

$\langle 2 \rangle 3.$ PICK $B_1 \in \mathcal{B}$ such that $x \in B_1 \subseteq U$

$\langle 2 \rangle 4.$ PICK $B_2 \in \mathcal{B}$ such that $x \in B_2 \subseteq V$

$\langle 2 \rangle 5.$ PICK $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$

PROOF: By condition 2.

$\langle 2 \rangle 6.$ $x \in B_3 \subseteq U \cap V$

□

Lemma 18. Let X be a set. Let \mathcal{B} be a basis for a topology \mathcal{T} on X . Then \mathcal{T} is the set of all unions of subsets of \mathcal{B} .

PROOF:

$\langle 1 \rangle 1.$ For all $U \in \mathcal{T}$, there exists $\mathcal{A} \subseteq \mathcal{B}$ such that $U = \bigcup \mathcal{A}$

$\langle 2 \rangle 1.$ LET: $U \in \mathcal{T}$

$\langle 2 \rangle 2.$ LET: $\mathcal{A} = \{B \in \mathcal{B} \mid B \subseteq U\}$

$\langle 2 \rangle 3.$ $U \subseteq \bigcup \mathcal{A}$

$\langle 3 \rangle 1.$ LET: $x \in U$

$\langle 3 \rangle 2.$ PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$

PROOF: Since \mathcal{B} is a basis for \mathcal{T} .

$\langle 3 \rangle 3.$ $x \in B \in \mathcal{A}$

$\langle 2 \rangle 4.$ $\bigcup \mathcal{A} \subseteq U$

PROOF: From the definition of \mathcal{A} ($\langle 2 \rangle 2$).

$\langle 1 \rangle 2$. For all $\mathcal{A} \subseteq \mathcal{B}$ we have $\bigcup \mathcal{A} \in \mathcal{T}$

$\langle 2 \rangle 1$. $\mathcal{B} \subseteq \mathcal{T}$

PROOF: If $B \in \mathcal{B}$ and $x \in B$, then there exists $B' \in \mathcal{B}$ such that $x \in B' \subseteq B$, namely $B' = B$.

$\langle 2 \rangle 2$. Q.E.D.

PROOF: Since \mathcal{T} is closed under union.

□

Corollary 18.1. *Let X be a set. Let \mathcal{B} be a basis for a topology \mathcal{T} on X . Then \mathcal{T} is the coarsest topology that includes \mathcal{B} .*

PROOF: Since every topology that includes \mathcal{B} includes all unions of subsets of \mathcal{B} . □

Lemma 19. *Let X be a topological space. Suppose that \mathcal{C} is a set of open sets such that, for every open set U and every point $x \in U$, there exists $C \in \mathcal{C}$ such that $x \in C \subseteq U$. Then \mathcal{C} is a basis for the topology on X .*

PROOF:

$\langle 1 \rangle 1$. For all $x \in X$, there exists $C \in \mathcal{C}$ such that $x \in C$

PROOF: Immediate from hypothesis.

$\langle 1 \rangle 2$. For all $C_1, C_2 \in \mathcal{C}$ and $x \in C_1 \cap C_2$, there exists $C_3 \in \mathcal{C}$ such that $x \in C_3 \subseteq C_1 \cap C_2$

PROOF: Since $C_1 \cap C_2$ is open.

$\langle 1 \rangle 3$. Every open set is open in the topology generated by \mathcal{C}

PROOF: Immediate from hypothesis.

$\langle 1 \rangle 4$. Every union of a subset of \mathcal{C} is open.

PROOF: Since every member of \mathcal{C} is open.

□

Lemma 20. *Let \mathcal{B} and \mathcal{B}' be bases for the topologies \mathcal{T} and \mathcal{T}' respectively on the set X . Then the following are equivalent.*

1. $\mathcal{T} \subseteq \mathcal{T}'$

2. For all $B \in \mathcal{B}$ and $x \in B$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.

PROOF:

$\langle 1 \rangle 1$. $1 \Rightarrow 2$

$\langle 2 \rangle 1$. ASSUME: $\mathcal{T} \subseteq \mathcal{T}'$

$\langle 2 \rangle 2$. LET: $B \in \mathcal{B}$ and $x \in B$

$\langle 2 \rangle 3$. $B \in \mathcal{T}$

PROOF: Corollary 18.1.

$\langle 2 \rangle 4$. $B \in \mathcal{T}'$

PROOF: By $\langle 2 \rangle 1$

$\langle 2 \rangle 5$. There exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$

PROOF: Since \mathcal{B}' is a basis for \mathcal{T}' .

$\langle 1 \rangle 2$. $2 \Rightarrow 1$

- ⟨2⟩1. ASSUME: 2
- ⟨2⟩2. LET: $U \in \mathcal{T}$
PROVE: $U \in \mathcal{T}'$
- ⟨2⟩3. LET: $x \in U$
PROVE: There exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq U$
- ⟨2⟩4. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$
PROOF: Since \mathcal{B} is a basis for \mathcal{T} .
- ⟨2⟩5. PICK $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$
PROOF: By ⟨2⟩1.
- ⟨2⟩6. $x \in B' \subseteq U$

□

Definition 21 (Lower Limit Topology on the Real Line). The *lower limit topology on the real line* is the topology on \mathbb{R} generated by the basis consisting of all half-open intervals of the form $[a, b)$.

We write \mathbb{R}_l for the topological space \mathbb{R} under the lower limit topology.

We prove this is a basis for a topology.

PROOF:

- ⟨1⟩1. For all $x \in \mathbb{R}$ there exists an interval $[a, b)$ such that $x \in [a, b)$.
PROOF: Take $[a, b) = [x, x + 1)$.
- ⟨1⟩2. For any open intervals $[a, b)$, $[c, d)$ if $x \in [a, b) \cap [c, d)$, then there exists an interval $[e, f)$ such that $x \in [e, f) \subseteq [a, b) \cap [c, d)$
PROOF: Take $[e, f) = [\max(a, c), \min(b, d))$.

□

Definition 22 (K -topology on the Real Line). Let $K = \{1/n \mid n \in \mathbb{Z}^+\}$.

The *K -topology on the real line* is the topology on \mathbb{R} generated by the basis consisting of all open intervals (a, b) and all sets of the form $(a, b) \setminus K$.

We write \mathbb{R}_K for the topological space \mathbb{R} under the K -topology.

We prove this is a basis for a topology.

PROOF:

- ⟨1⟩1. For all $x \in \mathbb{R}$ there exists an open interval (a, b) such that $x \in (a, b)$.
PROOF: Take $(a, b) = (x - 1, x + 1)$.
- ⟨1⟩2. For any basic open sets B_1, B_2 if $x \in B_1 \cap B_2$, then there exists a basic open set B_3 such that $x \in B_3 \subseteq B_1 \cap B_2$
 - ⟨2⟩1. CASE: $B_1 = (a, b)$, $B_2 = (c, d)$
PROOF: Take $B_3 = (\max(a, c), \min(b, d))$.
 - ⟨2⟩2. CASE: $B_1 = (a, b)$ or $(a, b) \setminus K$, $B_2 = (c, d)$ or $(c, d) \setminus K$, and they are not both open intervals.
PROOF: Take $B_3 = (\max(a, c), \min(b, d)) \setminus K$.

□

Lemma 23. The lower limit topology and the K -topology are incomparable.

PROOF:

⟨1⟩1. The interval $[10, 11)$ is not open in the K -topology.

PROOF: There is no open interval (a, b) such that $10 \in (a, b) \subseteq [10, 11)$ or $10 \in (a, b) \setminus K \subseteq [10, 11)$.

⟨1⟩2. The set $(-1, 1) \setminus K$ is not open in the lower limit topology.

PROOF: There is no half-open interval $[a, b)$ such that $0 \in [a, b) \subseteq (-1, 1) \setminus K$, since there must be a positive integer n with $1/n \in [a, b)$.

□

Definition 24 (Subbasis). A *subbasis* \mathcal{S} for a topology on X is a set $\mathcal{S} \subseteq \mathcal{P}X$ such that $\bigcup \mathcal{S} = X$.

The topology *generated* by the subbasis \mathcal{S} is the set of all unions of finite intersections of elements of \mathcal{S} .

We prove this is a topology.

PROOF:

⟨2⟩1. The set \mathcal{B} of all finite intersections of elements of \mathcal{S} forms a basis for a topology on X .

⟨2⟩1. $\bigcup \mathcal{B} = X$

PROOF: Since $\mathcal{S} \subseteq \mathcal{B}$.

⟨2⟩2. \mathcal{B} is closed under binary intersection.

PROOF: By definition.

⟨1⟩2. Q.E.D.

PROOF: By Lemma 18.

□

We have simultaneously proved:

Proposition 25. Let \mathcal{S} be a subbasis for the topology on X . Then the set of all finite intersections of elements of \mathcal{S} is a basis for the topology on X .

Proposition 26. Let X be a set. Let \mathcal{S} be a subbasis for a topology \mathcal{T} on X . Then \mathcal{T} is the coarsest topology that includes \mathcal{S} .

PROOF: Since every topology that includes \mathcal{S} includes every union of finite intersections of elements of \mathcal{S} . □

5 Open Maps

Definition 27 (Open Map). Let X and Y be topological spaces and $f : X \rightarrow Y$. Then f is an *open map* if and only if, for every open set U in X , the set $f(U)$ is open in Y .

Lemma 28. Let X and Y be topological spaces and $f : X \rightarrow Y$. Let \mathcal{B} be a basis for the topology on X . If $f(B)$ is open in Y for all $B \in \mathcal{B}$, then f is an open map.

PROOF: From Lemma 18. □

6 The Order Topology

Definition 29 (Order Topology). Let X be a linearly ordered set with at least two points. The *order topology* on X is the topology generated by the basis \mathcal{B} consisting of:

- all open intervals (a, b) ;
- all intervals of the form $[\perp, b)$ where \perp is least in X ;
- all intervals of the form $(a, \top]$ where \top is greatest in X .

We prove this is a basis for a topology.

PROOF:

- $\langle 1 \rangle 1$. For all $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B$.
 - $\langle 2 \rangle 1$. LET: $x \in X$
 - $\langle 2 \rangle 2$. CASE: x is greatest in X .
 - $\langle 3 \rangle 1$. PICK $y \in X$ with $y \neq x$
 - $\langle 3 \rangle 2$. $x \in (y, x] \in \mathcal{B}$
 - $\langle 2 \rangle 3$. CASE: x is least in X .
 - $\langle 3 \rangle 1$. PICK $y \in X$ with $y \neq x$
 - $\langle 3 \rangle 2$. $x \in [x, y) \in \mathcal{B}$
 - $\langle 2 \rangle 4$. CASE: x is neither greatest nor least in X .
 - $\langle 3 \rangle 1$. PICK $a, b \in X$ with $a < x$ and $x < b$
 - $\langle 3 \rangle 2$. $x \in (a, b) \in \mathcal{B}$
- $\langle 1 \rangle 2$. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$
 - $\langle 2 \rangle 1$. LET: $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$
 - $\langle 2 \rangle 2$. CASE: $B_1 = (a, b)$, $B_2 = (c, d)$
 - PROOF: Take $B_3 = (\max(a, c), \min(b, d))$.
 - $\langle 2 \rangle 3$. CASE: $B_1 = (a, b)$, $B_2 = [\perp, d)$
 - PROOF: Take $B_3 = (a, \min(b, d))$.
 - $\langle 2 \rangle 4$. CASE: $B_1 = (a, b)$, $B_2 = (c, \top]$
 - PROOF: Take $B_3 = (\max(a, c), b)$.
 - $\langle 2 \rangle 5$. CASE: $B_1 = [\perp, b)$, $B_2 = [\perp, d)$
 - PROOF: Take $B_3 = [\perp, \min(b, d))$.
 - $\langle 2 \rangle 6$. CASE: $B_1 = [\perp, b)$, $B_2 = (c, \top]$
 - PROOF: Take $B_3 = (c, b)$.

□

Lemma 30. Let X be a linearly ordered set. Then the open rays form a subbasis for the order topology on X .

PROOF:

- $\langle 1 \rangle 1$. Every open ray is open.
 - $\langle 2 \rangle 1$. For all $a \in X$, the ray $(-\infty, a)$ is open.
 - $\langle 3 \rangle 1$. LET: $x \in (-\infty, a)$
 - $\langle 3 \rangle 2$. CASE: x is least in X

PROOF: $x \in [x, a) = (-\infty, a)$.
 (3)3. CASE: x is not least in X
 (4)1. PICK $y < x$
 (4)2. $x \in (y, a) \subseteq (-\infty, a)$
 (2)2. For all $a \in X$, the ray $(a, +\infty)$ is open.
 PROOF: Similar.
 (1)2. Every basic open set is a finite intersection of open rays.
 PROOF: We have $(a, b) = (a, +\infty) \cap (-\infty, b)$, $[\perp, b) = (-\infty, b)$ and $(a, \top] = (a, +\infty)$.
 \square

Definition 31 (Standard Topology on the Real Line). The *standard topology on the real line* is the order topology on \mathbb{R} generated by the standard order.

Lemma 32. *The lower limit topology is strictly finer than the standard topology on \mathbb{R} .*

PROOF:
 (1)1. Every open interval is open in the lower limit topology.
 PROOF: If $x \in (a, b)$ then $x \in [x, b) \subseteq (a, b)$.
 (1)2. The half-open interval $[0, 1)$ is not open in the standard topology.
 PROOF: There is no open interval (a, b) such that $0 \in (a, b) \subseteq [0, 1)$.
 \square

Lemma 33. *The K -topology is strictly finer than the standard topology on \mathbb{R} .*

PROOF:
 (1)1. Every open interval is open in the K -topology.
 PROOF: Corollary 18.1.
 (1)2. The set $(-1, 1) \setminus K$ is not open in the standard topology.
 PROOF: There is no open interval (a, b) such that $0 \in (a, b) \subseteq (-1, 1) \setminus K$, since there must be a positive integer n with $1/n \in (a, b)$.
 \square

Definition 34 (Ordered Square). The *ordered square* I_o^2 is the set $[0, 1]^2$ under the order topology generated by the dictionary order.

7 The Product Topology

Definition 35 (Product Topology). Let $\{A_i\}_{i \in I}$ be a family of topological spaces. The *product topology* on $\prod_{i \in I} A_i$ is the topology generated by the sub-basis consisting of the sets of the form $\pi_i^{-1}(U)$ where $i \in I$ and U is open in A_i .

Proposition 36. *The product topology on $\prod_{i \in I} A_i$ is generated by the basis consisting of all sets of the form $\prod_{i \in I} U_i$ where $\{U_i\}_{i \in I}$ is a family such that each U_i is an open set in A_i and $U_i = A_i$ for all but finitely many i .*

PROOF: From Proposition 25. \square

Proposition 37. Let $\{A_i\}_{i \in I}$ be a family of topological spaces. For $i \in I$, let \mathcal{B}_i be a basis for the topology on A_i . Then $\mathcal{B} = \{\prod_{i \in I} B_i \mid \forall i \in I, B_i \in \mathcal{B}_i, B_i = A_i \text{ for all but finitely many } i\}$ is a basis for the box topology on $\prod_{i \in I} A_i$.

PROOF:

- $\langle 1 \rangle 1$. Every set in \mathcal{B} is open.
- $\langle 1 \rangle 2$. For every point $a \in \prod_{i \in I} A_i$ and every open set U with $a \in U$, there exists $B \in \mathcal{B}$ such that $a \in B \subseteq U$.
 - $\langle 2 \rangle 1$. LET: U be open and $a \in U$
 - $\langle 2 \rangle 2$. PICK a family $\{U_i\}_{i \in I}$ such that each U_i is open in A_i , such that $U_i = A_i$ except for $i = i_1, \dots, i_n$, and such that $a \in \prod_{i \in I} U_i \subseteq U$.
 - $\langle 2 \rangle 3$. For $j = 1, \dots, n$, PICK $B_{i_j} \in \mathcal{B}_{i_j}$ such that $a_{i_j} \in B_{i_j} \subseteq U_{i_j}$
 - $\langle 2 \rangle 4$. LET: $B = \prod_{i \in I} B_i$ where $B_i = A_i$ for $i \neq i_1, \dots, i_n$
 - $\langle 2 \rangle 5$. $B \in \mathcal{B}$
 - $\langle 2 \rangle 6$. $a \in B \subseteq U$
- $\langle 1 \rangle 3$. Q.E.D.

PROOF: Lemma 19.

□

Proposition 38. Let $\{A_i\}_{i \in I}$ be a family of topological spaces. Then the projections $\pi_i : \prod_{i \in I} A_i \rightarrow A_i$ are open maps.

PROOF: From Lemma 28. □

Proposition 39. Let $\{X_i\}_{i \in I}$ be a family of sets. For $i \in I$, let \mathcal{T}_i and \mathcal{U}_i be topologies on X_i . Let \mathcal{P} be the product topology on $\prod_{i \in I} X_i$ generated by the topologies \mathcal{T}_i , and \mathcal{Q} the product topology on the same set generated by the topologies \mathcal{U}_i . Then $\mathcal{P} \subseteq \mathcal{Q}$ if and only if $\mathcal{T}_i \subseteq \mathcal{U}_i$ for all i .

PROOF:

- $\langle 1 \rangle 1$. If $\mathcal{T}_i \subseteq \mathcal{U}_i$ for all i then $\mathcal{P} \subseteq \mathcal{Q}$
PROOF: By Corollary 18.1.
- $\langle 1 \rangle 2$. If $\mathcal{P} \subseteq \mathcal{Q}$ then $\mathcal{T}_i \subseteq \mathcal{U}_i$ for all i
 - $\langle 2 \rangle 1$. ASSUME: $\mathcal{P} \subseteq \mathcal{Q}$
 - $\langle 2 \rangle 2$. LET: $i \in I$
 - $\langle 2 \rangle 3$. LET: $U \in \mathcal{T}_i$
 - $\langle 2 \rangle 4$. LET: $U_i = U$ and $U_j = X_j$ for $j \neq i$
 - $\langle 2 \rangle 5$. $\prod_{i \in I} U_i \in \mathcal{P}$
 - $\langle 2 \rangle 6$. $\prod_{i \in I} U_i \in \mathcal{Q}$
 - $\langle 2 \rangle 7$. $U \in \mathcal{U}_i$

PROOF: From Proposition 38.

□

8 The Subspace Topology

Definition 40 (Subspace Topology). Let X be a topological space and $Y \subseteq X$. The *subspace topology* on Y is $\mathcal{T} = \{U \cap Y \mid U \text{ is open in } X\}$.

We prove this is a topology.

PROOF:

$\langle 1 \rangle 1. Y \in \mathcal{T}$

PROOF: Since $Y = X \cap Y$

$\langle 1 \rangle 2. \text{ For all } \mathcal{U} \subseteq \mathcal{T}, \text{ we have } \bigcup \mathcal{U} \in \mathcal{T}$

$\langle 2 \rangle 1. \text{ LET: } \mathcal{U} \subseteq \mathcal{T}$

$\langle 2 \rangle 2. \text{ LET: } \mathcal{V} = \{V \text{ open in } X \mid V \cap Y \in \mathcal{U}\}$

$\langle 2 \rangle 3. \bigcup \mathcal{U} = (\bigcup \mathcal{V}) \cap Y$

$\langle 1 \rangle 3. \text{ For all } U, V \in \mathcal{T}, \text{ we have } U \cap V \in \mathcal{T}$

$\langle 2 \rangle 1. \text{ LET: } U, V \in \mathcal{T}$

$\langle 2 \rangle 2. \text{ PICK } U', V' \text{ open in } X \text{ such that } U = U' \cap Y \text{ and } V = V' \cap Y$

$\langle 2 \rangle 3. (U \cap V) = (U' \cap V') \cap Y$

□

Lemma 41. *Let X be a topological space and $Y \subseteq X$. Let \mathcal{B} be a basis for the topology on X . Then $\mathcal{B}' = \{B \cap Y \mid B \in \mathcal{B}\}$ is a basis for the subspace topology on Y .*

PROOF:

$\langle 1 \rangle 1. \text{ Every element in } \mathcal{B}' \text{ is open in } Y$

$\langle 1 \rangle 2. \text{ For every open set } U \text{ in } Y \text{ and point } y \in U, \text{ there exists } B' \in \mathcal{B}' \text{ such that } y \in B' \subseteq U$

$\langle 2 \rangle 1. \text{ LET: } U \text{ be open in } Y \text{ and } y \in U$

$\langle 2 \rangle 2. \text{ PICK } V \text{ open in } X \text{ such that } U = V \cap Y$

$\langle 2 \rangle 3. \text{ PICK } B \in \mathcal{B} \text{ such that } y \in B \subseteq V$

$\langle 2 \rangle 4. \text{ LET: } B' = B \cap Y$

$\langle 2 \rangle 5. B' \in \mathcal{B}'$

$\langle 2 \rangle 6. y \in B' \subseteq U$

$\langle 1 \rangle 3. \text{ Q.E.D.}$

PROOF: By Lemma 19.

□

Lemma 42. *Let X be a topological space and $Y \subseteq X$. Let \mathcal{S} be a basis for the topology on X . Then $\mathcal{S}' = \{S \cap Y \mid S \in \mathcal{S}\}$ is a subbasis for the subspace topology on Y .*

PROOF: The set $\{B \cap Y \mid B \text{ is a finite intersection of elements of } \mathcal{S}\}$ is a basis for the subspace topology by Lemma 41, and this is the set of all finite intersections of elements of \mathcal{S}' . □

Lemma 43. *Let Y be a subspace of X . If U is open in Y and Y is open in X then U is open in X .*

PROOF:

$\langle 1 \rangle 1. \text{ PICK } V \text{ open in } X \text{ such that } U = V \cap Y$

$\langle 1 \rangle 2. U \text{ is open in } X$

PROOF: Since it is the intersection of two open sets V and Y .

□

Theorem 44. Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let A_i be a subspace of X_i for all $i \in I$. Then the product topology on $\prod_{i \in I} A_i$ is the same as the topology it inherits as a subspace of $\prod_{i \in I} X_i$.

PROOF: The product topology is generated by the subbasis

$$\begin{aligned} & \{\pi_i^{-1}(U) \mid i \in I, U \text{ open in } A_i\} \\ &= \{\pi_i^{-1}(V) \cap A_i \mid i \in I, V \text{ open in } X_i\} \\ &= \{\pi_i^{-1}(V) \mid i \in I, V \text{ open in } X_i\} \cap \prod_{i \in I} A_i \end{aligned}$$

and this is a subbasis for the subspace topology by Lemma 42. \square

Theorem 45. Let X be an ordered set in the order topology. Let $Y \subseteq X$ be convex. Then the order topology on Y is the same as the subspace topology on Y .

PROOF:

- $\langle 1 \rangle 1$. The order topology is finer than the subspace topology.
 - $\langle 2 \rangle 1$. For every open ray R in X , the set $R \cap Y$ is open in the order topology.
 - $\langle 3 \rangle 1$. For all $a \in X$, we have $(-\infty, a) \cap Y$ is open in the order topology.
 - $\langle 4 \rangle 1$. CASE: For all $y \in Y$ we have $y < a$
 PROOF: In this case $(-\infty, a) \cap Y = Y$.
 - $\langle 4 \rangle 2$. CASE: For all $y \in Y$ we have $a < y$
 PROOF: In this case $(-\infty, a) \cap Y = \emptyset$.
 - $\langle 4 \rangle 3$. CASE: There exists $y \in Y$ such that $y \leq a$ and $y \in Y$ such that $a \leq y$
 $\langle 5 \rangle 1$. $a \in Y$
 PROOF: By convexity.
 - $\langle 5 \rangle 2$. $(-\infty, a) \cap Y = \{y \in Y \mid y < a\}$
 - $\langle 3 \rangle 2$. For all $a \in X$, we have $(a, +\infty) \cap Y$ is open in the order topology.
 PROOF: Similar.
 - $\langle 2 \rangle 2$. Q.E.D.
 PROOF: By Lemmas 30 and 42 and Proposition 26.
- $\langle 1 \rangle 2$. The subspace topology is finer than the order topology.
 - $\langle 2 \rangle 1$. Every open ray in Y is open in the subspace topology.
 PROOF: For any $a \in Y$ we have $(-\infty, a)_Y = (-\infty, a)_X \cap Y$ and $(a, +\infty)_Y = (a, +\infty)_X \cap Y$.
 - $\langle 2 \rangle 2$. Q.E.D.
 PROOF: By Lemma 30 and Proposition 26

\square

Proposition 46. The order topology on I_o^2 is different from the subspace topology as a subspace of \mathbb{R}^2 under the dictionary order topology.

PROOF: The set $\{1/2\} \times (1/2, 1)$ is open in the subspace topology but not in the order topology. \square

Proposition 47. Let X be a topological space, Y a subspace of X , and Z a subspace of Y . Then the subspace topology on Z inherited from X is the same as the subspace topology on Z inherited from Y .

PROOF: The subspace topology inherited from Y is

$$\begin{aligned} & \{V \cap Z \mid V \text{ open in } Y\} \\ &= \{U \cap Y \cap Z \mid U \text{ open in } X\} \\ &= \{U \cap Z \mid U \text{ open in } X\} \end{aligned}$$

which is the subspace topology inherited from X . \square

Definition 48 (Unit Circle). The *unit circle* S^1 is $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$

9 Closed Set

Definition 49 (Closed Set). Let X be a topological space and $A \subseteq X$. Then A is *closed* if and only if $X \setminus A$ is open.

Lemma 50. *The empty set is closed.*

PROOF: Since the whole space X is always open. \square

Lemma 51. *The topological space X is closed.*

PROOF: Since \emptyset is open. \square

Lemma 52. *The intersection of a nonempty set of closed sets is closed.*

PROOF: Let \mathcal{C} be a nonempty set of closed sets. Then $X \setminus \bigcap \mathcal{C} = \bigcup \{X \setminus C \mid C \in \mathcal{C}\}$ is open. \square

Lemma 53. *The union of two closed sets is closed.*

PROOF: Let C and D be closed. Then $X \setminus (C \cup D) = (X \setminus C) \cap (X \setminus D)$ is open. \square

Theorem 54. *Let X be a topological space and Y a subspace of X . Let $A \subseteq Y$. Then A is closed in Y if and only if there exists a closed set C in X such that $A = C \cap Y$.*

PROOF: We have

$$\begin{aligned} & A \text{ is closed in } Y \\ \Leftrightarrow & Y \setminus A \text{ is open in } Y \\ \Leftrightarrow & \exists U \text{ open in } X. Y \setminus A = Y \cap U \\ \Leftrightarrow & \exists C \text{ closed in } X. Y \setminus A = Y \cap (X \setminus U) \\ \Leftrightarrow & \exists C \text{ closed in } X. A = Y \cap C \end{aligned} \quad \square$$

Theorem 55. *Let Y be a subspace of X and $A \subseteq Y$. If A is closed in Y and Y is closed in X then A is closed in X .*

PROOF: Pick a closed set C in X such that $A = C \cap Y$ (Theorem 54). Then A is the intersection of two sets closed in X , hence A is closed in X (Lemma 52). \square

Proposition 56. Let X be a set and $\mathcal{C} \subseteq \mathcal{P}X$ a set such that:

1. $\emptyset \in \mathcal{C}$
2. $X \in \mathcal{C}$
3. For all $\mathcal{A} \subseteq \mathcal{C}$ nonempty we have $\bigcap \mathcal{A} \in \mathcal{C}$
4. For all $C, D \in \mathcal{C}$ we have $C \cup D \in \mathcal{C}$.

Then there exists a unique topology \mathcal{T} such that \mathcal{C} is the set of closed sets, namely

$$\mathcal{T} = \{X \setminus C \mid C \in \mathcal{C}\}$$

PROOF:

$\langle 1 \rangle 1.$ LET: $\mathcal{T} = \{X \setminus C \mid C \in \mathcal{C}\}$

$\langle 1 \rangle 2.$ \mathcal{T} is a topology

$\langle 2 \rangle 1.$ $X \in \mathcal{T}$

PROOF: Since $\emptyset \in \mathcal{C}$

$\langle 2 \rangle 2.$ For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$

$\langle 3 \rangle 1.$ LET: $\mathcal{U} \subseteq \mathcal{T}$

$\langle 3 \rangle 2.$ CASE: $\mathcal{U} = \emptyset$

PROOF: In this case $\bigcup \mathcal{U} = \emptyset \in \mathcal{T}$ since $X \in \mathcal{C}$

$\langle 3 \rangle 3.$ CASE: $\mathcal{U} \neq \emptyset$

PROOF: In this case $X \setminus \bigcup \mathcal{U} = \bigcap \{X \setminus U \mid U \in \mathcal{U}\} \in \mathcal{C}$.

$\langle 2 \rangle 3.$ For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$

PROOF: Since $X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V) \in \mathcal{C}$.

$\langle 1 \rangle 3.$ \mathcal{C} is the set of all closed sets in \mathcal{T}

PROOF:

C is closed in \mathcal{T}

$$\Leftrightarrow X \setminus C \in \mathcal{T}$$

$$\Leftrightarrow C \in \mathcal{C}$$

$\langle 1 \rangle 4.$ If \mathcal{T}' is a topology and \mathcal{C} is the set of closed sets in \mathcal{T}' then $\mathcal{T}' = \mathcal{T}$

PROOF: We have

$$U \in \mathcal{T}$$

$$\Leftrightarrow X \setminus U \in \mathcal{C}$$

$$\Leftrightarrow X \setminus U \text{ is closed in } \mathcal{T}'$$

$$\Leftrightarrow U \in \mathcal{T}'$$

□

Proposition 57. If A_i is closed in X_i for all $i \in I$ then $\prod_{i \in I} A_i$ is closed in $\prod_{i \in I} X_i$.

PROOF:

$$\left(\prod_{i \in I} X_i \right) \setminus \left(\prod_{i \in I} A_i \right) = \bigcup_{j \in I} \left(\prod_{i \in I} X_i \setminus \pi_j^{-1}(A_j) \right) \square$$

Proposition 58. If U is open and A is closed then $U \setminus A$ is open.

PROOF: $U \setminus A = U \cap (X \setminus A)$ is the intersection of two open sets. \square

Proposition 59. *If U is open and A is closed then $A \setminus U$ is closed.*

PROOF: $A \setminus U = A \cap (X \setminus U)$ is the intersection of two closed sets. \square

10 Interior

Definition 60 (Interior). Let X be a topological space and $A \subseteq X$. The *interior* of A , $\text{Int } A$, is the union of all the open subsets of A .

Lemma 61. *The interior of a set is open.*

PROOF: It is a union of open sets. \square

Lemma 62.

$$\text{Int } A \subseteq A$$

PROOF: Immediate from definition. \square

Lemma 63. *A set A is open if and only if $A = \text{Int } A$.*

PROOF: If $A = \text{Int } A$ then A is open by Lemma 61. Conversely if A is open then $A \subseteq \text{Int } A$ by the definition of interior and so $A = \text{Int } A$.

11 Neighbourhood

Definition 64 (Neighbourhood). A *neighbourhood* of a point x is an open set that contains x .

12 Closure

Definition 65 (Closure). Let X be a topological space and $A \subseteq X$. The *closure* of A , \overline{A} , is the intersection of all the closed sets that include A .

This intersection exists since X is a closed set that includes A (Lemma 51).

Lemma 66. *The closure of a set is closed.*

PROOF: Dual to Lemma 61. \square

Lemma 67.

$$A \subseteq \overline{A}$$

PROOF: Immediate from definition. \square

Lemma 68. *A set A is closed if and only if $A = \overline{A}$.*

PROOF: Dual to Lemma 63. \square

Theorem 69. Let Y be a subspace of X . Let $A \subseteq Y$. Let \bar{A} be the closure of A in X . Then the closure of A in Y is $\bar{A} \cap Y$.

PROOF: The closure of A in Y is

$$\begin{aligned} & \bigcap \{C \text{ closed in } Y \mid A \subseteq C\} \\ &= \bigcap \{D \cap Y \mid D \text{ closed in } X, A \subseteq D \cap Y\} \quad (\text{Theorem 54}) \\ &= \bigcap \{D \mid D \text{ closed in } X, A \subseteq D\} \cap Y \\ &= \bar{A} \cap Y \quad \square \end{aligned}$$

Theorem 70. Let X be a topological space, $A \subseteq X$ and $x \in X$. Then $x \in \bar{A}$ if and only if every neighbourhood of x intersects A .

PROOF: We have

$$\begin{aligned} x \in \bar{A} & \Leftrightarrow \forall C. C \text{ closed} \wedge A \subseteq C \Rightarrow x \in C \\ & \Leftrightarrow \forall U. U \text{ open} \wedge A \cap U = \emptyset \Rightarrow x \notin U \\ & \Leftrightarrow \forall U. U \text{ open} \wedge x \in U \Rightarrow U \text{ intersects } A \quad \square \end{aligned}$$

Theorem 71. Let X be a topological space, $A \subseteq X$ and $x \in X$. Let \mathcal{B} be a basis for X . Then $x \in \bar{A}$ if and only if, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A .

PROOF:

$\langle 1 \rangle 1$. If $x \in \bar{A}$ then, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A .

PROOF: This follows from Theorem 70 since every element of \mathcal{B} is open (Corollary 18.1).

$\langle 1 \rangle 2$. Suppose that, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A . Then $x \in \bar{A}$.

$\langle 2 \rangle 1$. ASSUME: For all $B \in \mathcal{B}$, if $x \in B$ then B intersects A .

$\langle 2 \rangle 2$. LET: U be an open set that contains x

PROVE: U intersects A .

$\langle 2 \rangle 3$. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

$\langle 2 \rangle 4$. B intersects A .

PROOF: From $\langle 2 \rangle 1$.

$\langle 2 \rangle 5$. U intersects A .

$\langle 2 \rangle 6$. Q.E.D.

PROOF: By Theorem 70.

\square

Proposition 72. If $A \subseteq B$ then $\bar{A} \subseteq \bar{B}$.

PROOF: This holds because \bar{B} is a closed set that includes A . \square

Proposition 73.

$$\overline{A \cup B} = \bar{A} \cup \bar{B}$$

PROOF:

$\langle 1 \rangle 1$. $\bar{A} \subseteq \overline{A \cup B}$

PROOF: By Proposition 72.

$\langle 1 \rangle 2. \overline{B} \subseteq \overline{A \cup B}$

PROOF: By Proposition 72.

$\langle 1 \rangle 3. \overline{A \cup B} \subseteq \overline{A \cup B}$

$\langle 2 \rangle 1. \text{ LET: } x \in \overline{A \cup B}$

$\langle 2 \rangle 2. \text{ ASSUME: } x \notin \overline{A}$

PROVE: $x \in \overline{B}$

$\langle 2 \rangle 3. \text{ PICK a neighbourhood } U \text{ of } x \text{ that does not intersect } A$

$\langle 2 \rangle 4. \text{ LET: } V \text{ be any neighbourhood of } x$

$\langle 2 \rangle 5. U \cap V \text{ is a neighbourhood of } x$

$\langle 2 \rangle 6. U \cap V \text{ intersects } A \cup B$

PROOF: From $\langle 2 \rangle 1$ and Theorem 70.

$\langle 2 \rangle 7. U \cap V \text{ intersects } B$

PROOF: From $\langle 2 \rangle 3$

$\langle 2 \rangle 8. V \text{ intersects } B$

$\langle 2 \rangle 9. \text{ Q.E.D.}$

PROOF: We have $x \in \overline{B}$ from Theorem 70.

□

Proposition 74 (AC). *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let $A_i \subseteq X_i$ for all $i \in I$. Then*

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i} .$$

PROOF:

$\langle 1 \rangle 1. \overline{\prod_{i \in I} A_i} \subseteq \prod_{i \in I} \overline{A_i}$

$\langle 2 \rangle 1. \text{ For all } i \in I \text{ we have } A_i \subseteq \overline{A_i}$

PROOF: Lemma 67.

$\langle 2 \rangle 2. \prod_{i \in I} A_i \subseteq \prod_{i \in I} \overline{A_i}$

$\langle 2 \rangle 3. \text{ Q.E.D.}$

PROOF: Since $\prod_{i \in I} \overline{A_i}$ is closed by Proposition 57.

$\langle 1 \rangle 2. \prod_{i \in I} \overline{A_i} \subseteq \overline{\prod_{i \in I} A_i}$

$\langle 2 \rangle 1. \text{ LET: } x \in \prod_{i \in I} \overline{A_i}$

$\langle 2 \rangle 2. \text{ LET: } U \text{ be a neighbourhood of } x$

$\langle 2 \rangle 3. \text{ PICK } V_i \text{ open in } X_i \text{ such that } x \in \prod_{i \in I} V_i \subseteq U \text{ with } V_i = X_i \text{ except for } i = i_1, \dots, i_n$

$\langle 2 \rangle 4. \text{ For } i \in I, \text{ pick } a_i \in V_i \cap A_i$

PROOF: By Theorem 70 and $\langle 2 \rangle 1$ using the Axiom of Choice.

$\langle 2 \rangle 5. U \text{ intersects } \prod_{i \in I} A_i$

$\langle 2 \rangle 6. \text{ Q.E.D.}$

PROOF: $a \in U \cap \prod_{i \in I} A_i$

□

Example 75. The closure of \mathbb{R}^∞ in \mathbb{R}^ω is \mathbb{R}^ω

PROOF:

$\langle 1 \rangle 1. \text{ LET: } a \in \mathbb{R}^\omega$

- <1>2. LET: U be any neighbourhoods of a .
 <1>3. PICK U_n open in \mathbb{R} for all n such that $a \in \prod_{n \geq 0} U_n \subseteq U$ and $U_n = \mathbb{R}$ for all n except n_1, \dots, n_k
 <1>4. LET: $b_n = a_n$ for $n = n_1, \dots, n_k$ and $b_n = 0$ for all other n
 <1>5. $b \in \mathbb{R}^\infty \cap U$
 <1>6. Q.E.D.
 PROOF: From Theorem 70.
 \square

13 Limit Points

Definition 76 (Limit Point). Let X be a topological space, $a \in X$ and $A \subseteq X$. Then a is a *limit point*, *cluster point* or *point of accumulation* for A if and only if every neighbourhood of a intersects A at a point other than a .

Lemma 77. *The point a is an accumulation point for A if and only if $a \in \overline{A \setminus \{a\}}$.*

PROOF: From Theorem 70. \square

Theorem 78. *Let X be a topological space and $A \subseteq X$. Let A' be the set of all limit points of A . Then $\overline{A} = A \cup A'$.*

PROOF:

- <1>1. For all $x \in \overline{A}$, if $x \notin A$ then $x \in A'$

PROOF: From Theorem 70.

- <1>2. $A \subseteq \overline{A}$

PROOF: Lemma 67.

- <1>3. $A' \subseteq \overline{A}$

PROOF: From Theorem 70.

\square

Corollary 78.1. *A set is closed if and only if it contains all its limit points.*

Proposition 79. *In an indiscrete topology, every point is a limit point of any set that has more than one point.*

PROOF: Let X be an indiscrete space. Let A be a set with more than one point and x be a point. The only neighbourhood of x is X , which must intersect A at a point other than x . \square

14 T_1 Spaces

Definition 80 (T_1 Space). A topological space is T_1 if and only if every singleton is closed.

Lemma 81. *A space is T_1 if and only if every finite set is closed.*

PROOF: From Lemma 53. \square

Theorem 82. *In a T_1 space, a point a is a limit point of a set A if and only if every neighbourhood of a contains infinitely many points of A .*

PROOF:

$\langle 1 \rangle 1$. If a is a limit point of A then every neighbourhood of a contains infinitely many points of A .

$\langle 2 \rangle 1$. ASSUME: a is a limit point of A .

$\langle 2 \rangle 2$. LET: U be a neighbourhood of a .

$\langle 2 \rangle 3$. ASSUME: for a contradiction U contains only finitely many points of A .

$\langle 2 \rangle 4$. $(U \cap A) \setminus \{a\}$ is closed.

PROOF: By the T_1 axiom.

$\langle 2 \rangle 5$. $(U \setminus A) \cup \{a\}$ is open.

PROOF: It is $U \setminus ((U \cap A) \setminus \{a\})$.

$\langle 2 \rangle 6$. $(U \setminus A) \cup \{a\}$ intersects A in a point other than a .

PROOF: From $\langle 2 \rangle 1$.

$\langle 2 \rangle 7$. Q.E.D.

\square

$\langle 1 \rangle 2$. If every neighbourhood of a contains infinitely many points of A then a is a limit point of A .

PROOF: Immediate from definitions.

\square

(To see this does not hold in every space, see Proposition 79.)

Proposition 83. *A space is T_1 if and only if, for any two distinct points x and y , there exist neighbourhoods U of x and V of y such that $x \notin V$ and $y \notin U$.*

PROOF:

$\langle 1 \rangle 1$. LET: X be a topological space.

$\langle 1 \rangle 2$. If X is T_1 then, for any two distinct points x and y , there exist neighbourhoods U of x and V of y such that $x \notin V$ and $y \notin U$.

PROOF: This holds because $\{x\}$ and $\{y\}$ are closed.

$\langle 1 \rangle 3$. Suppose, for any two distinct points x and y , there exist neighbourhoods U of x and V of y such that $x \notin V$ and $y \notin U$. Then X is T_1 .

$\langle 2 \rangle 1$. ASSUME: For any two distinct points x and y , there exist neighbourhoods U of x and V of y such that $x \notin V$ and $y \notin U$.

$\langle 2 \rangle 2$. LET: $a \in X$

$\langle 2 \rangle 3$. $\{a\}$ is closed.

PROOF: For all $b \neq a$ there exists a neighbourhood U of b such that $U \subseteq X \setminus \{a\}$.

\square

15 Hausdorff Spaces

Definition 84 (Hausdorff Space). A topological space is *Hausdorff* if and only if, for any points x, y with $x \neq y$, there exist disjoint open sets U and V such

that $x \in U$ and $y \in V$.

Theorem 85. *Every Hausdorff space is T_1 .*

PROOF:

$\langle 1 \rangle 1$. LET: X be a Hausdorff space.

$\langle 1 \rangle 2$. LET: $b \in X$

PROVE: $\overline{\{b\}} = \{b\}$

$\langle 1 \rangle 3$. ASSUME: $a \in \overline{\{b\}}$ and $a \neq b$

$\langle 1 \rangle 4$. PICK disjoint neighbourhoods U of a and V of b .

$\langle 1 \rangle 5$. U intersects $\{b\}$

PROOF: Theorem 70.

$\langle 1 \rangle 6$. $b \in U$

$\langle 1 \rangle 7$. Q.E.D.

PROOF: This contradicts the fact that U and V are disjoint ($\langle 1 \rangle 4$).

□

Proposition 86. *An infinite set under the finite complement topology is T_1 but not Hausdorff.*

PROOF:

$\langle 1 \rangle 1$. LET: X be an infinite set under the finite complement topology.

$\langle 1 \rangle 2$. Every singleton is closed.

PROOF: By definition.

$\langle 1 \rangle 3$. PICK $a, b \in X$ with $a \neq b$

$\langle 1 \rangle 4$. There are no disjoint neighbourhoods U of a and V of b .

$\langle 2 \rangle 1$. LET: U be a neighbourhood of a and V a neighbourhood of b .

$\langle 2 \rangle 2$. $X \setminus U$ and $X \setminus V$ are finite.

$\langle 2 \rangle 3$. PICK $c \in X$ that is not in $X \setminus U$ or $X \setminus V$.

$\langle 2 \rangle 4$. $c \in U \cap V$

□

Proposition 87. *The product of a family of Hausdorff spaces is Hausdorff.*

PROOF:

$\langle 1 \rangle 1$. LET: $\{X_i\}_{i \in I}$ be a family of Hausdorff spaces.

$\langle 1 \rangle 2$. LET: $a, b \in \prod_{i \in I} X_i$ with $a \neq b$

$\langle 1 \rangle 3$. PICK $i \in I$ such that $a_i \neq b_i$

$\langle 1 \rangle 4$. PICK U, V disjoint open sets in X_i with $a_i \in U$ and $b_i \in V$

$\langle 1 \rangle 5$. $\pi_i^{-1}(U)$ and $\pi_i^{-1}(V)$ are disjoint open sets in $\prod_{i \in I} X_i$ with $a \in \pi_i^{-1}(U)$ and $b \in \pi_i^{-1}(V)$

□

Theorem 88. *Every linearly ordered set under the order topology is Hausdorff.*

PROOF:

$\langle 1 \rangle 1$. LET: X be a linearly ordered set under the order topology.

$\langle 1 \rangle 2$. LET: $a, b \in X$ with $a \neq b$

$\langle 1 \rangle 3$. ASSUME: w.l.o.g. $a < b$

⟨1⟩4. CASE: There exists c such that $a < c < b$

PROOF: The sets $(-\infty, c)$ and $(c, +\infty)$ are disjoint neighbourhoods of a and b respectively.

⟨1⟩5. CASE: There is no c such that $a < c < b$

PROOF: The sets $(-\infty, b)$ and $(a, +\infty)$ are disjoint neighbourhoods of a and b respectively.

□

Theorem 89. *A subspace of a Hausdorff space is Hausdorff.*

PROOF:

⟨1⟩1. LET: X be a Hausdorff space and Y a subspace of X .

⟨1⟩2. LET: $x, y \in Y$ with $x \neq y$

⟨1⟩3. PICK disjoint neighbourhoods U of x and V of y in X .

⟨1⟩4. $U \cap Y$ and $V \cap Y$ are disjoint neighbourhoods of x and y respectively in Y .

□

Proposition 90. *A space X is Hausdorff if and only if the diagonal $\Delta = \{(x, x) \mid x \in X\}$ is closed in X^2 .*

PROOF:

X is Hausdorff

$$\Leftrightarrow \forall x, y \in X. x \neq y \Rightarrow \exists V, W \text{ open. } x \in V \wedge y \in W \wedge V \cap W = \emptyset$$

$$\Leftrightarrow \forall (x, y) \in X^2 \setminus \Delta. \exists V, W \text{ open. } (x, y) \in V \times W \subseteq X^2 \setminus \Delta$$

$$\Leftrightarrow \Delta \text{ is closed}$$

□

16 Convergence

Definition 91 (Convergence). Let X be a topological space. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of points in X and $l \in X$. Then the sequence $(a_n)_{n \in \mathbb{N}}$ *converges* to the *limit* l , $a_n \rightarrow l$ as $n \rightarrow \infty$, if and only if, for every neighbourhood U of l , there exists N such that, for all $n \geq N$, we have $a_n \in U$.

Theorem 92. *In a Hausdorff space, a sequence has at most one limit.*

PROOF:

⟨1⟩1. LET: X be a Hausdorff space.

⟨1⟩2. ASSUME: for a contradiction $a_n \rightarrow l$ as $n \rightarrow \infty$, $a_n \rightarrow m$ as $n \rightarrow \infty$, and $l \neq m$

⟨1⟩3. PICK disjoint neighbourhoods U of l and V of m

PROOF: By the Hausdorff axiom.

⟨1⟩4. PICK M and N such that $a_n \in U$ for $n \geq M$ and $a_n \in V$ for $n \geq N$

⟨1⟩5. $a_{\max(M, N)} \in U \cap V$

⟨1⟩6. Q.E.D.

PROOF: This contradicts the fact that U and V are disjoint (⟨1⟩3).

□

To see this is not always true in spaces that are T_1 but not Hausdorff:

Proposition 93. *Let X be an infinite set under the finite complement topology. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence with all points distinct. Then for every $l \in X$ we have $a_n \rightarrow l$ as $n \rightarrow \infty$.*

PROOF: Let U be any neighbourhood of l . Since $X \setminus U$ is finite, there must exist N such that, for all $n \geq N$, we have $a_n \in U$. □

17 Boundary

Definition 94 (Boundary). The *boundary* of a set A is the set $\partial A = \overline{A} \cap \overline{X \setminus A}$.

Proposition 95.

$$\text{Int } A \cap \partial A = \emptyset$$

PROOF: Since $\overline{X \setminus A} = X \setminus \text{Int } A$. □

Proposition 96.

$$\overline{A} = \text{Int } A \cup \partial A$$

PROOF:

$$\begin{aligned} \text{Int } A \cup \partial A &= \text{Int } A \cup (\overline{A} \cap \overline{X \setminus A}) \\ &= (\text{Int } A \cup \overline{A}) \cap (\text{Int } A \cup \overline{X \setminus A}) \\ &= \overline{A} \cap X \\ &= \overline{A} \end{aligned}$$

Proposition 97. $\partial A = \emptyset$ if and only if A is open and closed.

PROOF: If $\partial A = \emptyset$ then $\overline{A} = \text{Int } A$ by Proposition 96.

Proposition 98. A set U is open if and only if $\partial U = \overline{U} \setminus U$.

PROOF:

$$\begin{aligned} \partial U &= \overline{U} \setminus U \\ \Leftrightarrow \overline{U} \setminus \text{Int } U &= \overline{U} \setminus U && \text{(Propositions 95, 96)} \\ \Leftrightarrow \text{Int } U &= U && \square \end{aligned}$$

18 Continuous Functions

Definition 99 (Continuous). Let X and Y be topological spaces. A function $f : X \rightarrow Y$ is *continuous* if and only if, for every open set V in Y , the set $f^{-1}(V)$ is open in X .

Proposition 100. *Let X and Y be topological spaces and $f : X \rightarrow Y$. Let \mathcal{B} be a basis for Y . Then f is continuous if and only if, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X .*

PROOF:

$\langle 1 \rangle 1$. If f is continuous then, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X .

PROOF: Since every element of \mathcal{B} is open (Lemma 18).

$\langle 1 \rangle 2$. Suppose that, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X . Then f is continuous.

$\langle 2 \rangle 1$. ASSUME: For all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X .

$\langle 2 \rangle 2$. LET: V be open in Y .

$\langle 2 \rangle 3$. PICK $\mathcal{A} \subseteq \mathcal{B}$ such that $V = \bigcup \mathcal{A}$

PROOF: By Lemma 18.

$\langle 2 \rangle 4$. $f^{-1}(V)$ is open in X .

PROOF:

$$\begin{aligned} f^{-1}(V) &= f^{-1}\left(\bigcup \mathcal{A}\right) \\ &= \bigcup_{B \in \mathcal{A}} f^{-1}(B) \end{aligned}$$

□

Proposition 101. *Let X and Y be topological spaces and $f : X \rightarrow Y$. Let \mathcal{S} be a subbasis for Y . Then f is continuous if and only if, for all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in X .*

PROOF:

$\langle 1 \rangle 1$. If f is continuous then, for all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in X .

PROOF: Since every element of \mathcal{S} is open.

$\langle 1 \rangle 2$. Suppose that, for all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in X . Then f is continuous.

$\langle 2 \rangle 1$. ASSUME: For all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in X .

$\langle 2 \rangle 2$. LET: $S_1, \dots, S_n \in \mathcal{S}$

$\langle 2 \rangle 3$. $f^{-1}(S_1 \cap \dots \cap S_n)$ is open in X

PROOF: Since $f^{-1}(S_1 \cap \dots \cap S_n) = f^{-1}(S_1) \cap \dots \cap f^{-1}(S_n)$.

$\langle 2 \rangle 4$. Q.E.D.

PROOF: By Propositions 100 and 25.

□

Proposition 102. *Let X and Y be topological spaces and $f : X \rightarrow Y$. Let \mathcal{S} be a subbasis for Y . Then f is continuous if and only if, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X .*

PROOF:

$\langle 1 \rangle 1$. If f is continuous then, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X .

PROOF: Since every element of \mathcal{S} is open.

$\langle 1 \rangle 2$. Suppose that, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X . Then f is continuous.

$\langle 2 \rangle 1$. ASSUME: For all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X .

⟨2⟩2. For every set B that is the finite intersection of elements of \mathcal{S} , we have $f^{-1}(B)$ is open in X .

PROOF: Because $f^{-1}(V_1 \cap \dots \cap V_n) = f^{-1}(V_1) \cap \dots \cap f^{-1}(V_n)$.

⟨2⟩3. Q.E.D.

PROOF: From Propositions 25 and 100.

□

Definition 103 (Continuous at a Point). Let X and Y be topological spaces. Let $f : X \rightarrow Y$ and $x \in X$. Then f is *continuous at x* if and only if, for every neighbourhood V of $f(x)$, there exists a neighbourhood U of x such that $f(U) \subseteq V$.

Theorem 104. Let X and Y be topological spaces and $f : X \rightarrow Y$. Then the following are equivalent:

1. f is continuous.
2. For all $A \subseteq X$, we have $f(\overline{A}) \subseteq \overline{f(A)}$
3. For all $B \subseteq Y$ closed, we have $f^{-1}(B)$ is closed in X .
4. f is continuous at every point of X .

PROOF:

⟨1⟩1. $1 \Rightarrow 2$

⟨2⟩1. ASSUME: f is continuous.

⟨2⟩2. LET: $A \subseteq X$

⟨2⟩3. LET: $x \in \overline{A}$

PROVE: $f(x) \in \overline{f(A)}$

⟨2⟩4. LET: V be a neighbourhood of $f(x)$

⟨2⟩5. $f^{-1}(V)$ is a neighbourhood of x

⟨2⟩6. PICK $y \in A \cap f^{-1}(V)$

PROOF: By Theorem 70.

⟨2⟩7. $f(y) \in V \cap f(A)$

⟨2⟩8. Q.E.D.

PROOF: By Theorem 70.

⟨1⟩2. $2 \Rightarrow 3$

⟨2⟩1. ASSUME: 2

⟨2⟩2. LET: B be closed in Y

⟨2⟩3. LET: $x \in \overline{f^{-1}(B)}$

PROVE: $x \in f^{-1}(B)$

⟨2⟩4. $f(x) \in B$

PROOF:

$$f(x) \in \overline{f(f^{-1}(B))}$$

$$\subseteq \overline{f(f^{-1}(B))}$$

(⟨2⟩1)

$$\subseteq \overline{B}$$

(Proposition 72)

$$= B$$

⟨1⟩3. $3 \Rightarrow 1$

⟨2⟩1. ASSUME: 3

⟨2⟩2. LET: V be open in Y

⟨2⟩3. $Y \setminus V$ is closed in Y

⟨2⟩4. $f^{-1}(Y \setminus V)$ is closed in X

⟨2⟩5. $X \setminus f^{-1}(V)$ is closed in X

⟨2⟩6. $f^{-1}(V)$ is open in X

⟨1⟩4. $1 \Rightarrow 4$

PROOF: For any neighbourhood V of $f(x)$, the set $U = f^{-1}(V)$ is a neighbourhood of x such that $f(U) \subseteq V$.

⟨1⟩5. $4 \Rightarrow 1$

⟨2⟩1. ASSUME: 4

⟨2⟩2. LET: V be open in Y

⟨2⟩3. LET: $x \in f^{-1}(V)$

⟨2⟩4. V is a neighbourhood of $f(x)$

⟨2⟩5. PICK a neighbourhood U of x such that $f(U) \subseteq V$

⟨2⟩6. $x \in U \subseteq f^{-1}(V)$

⟨2⟩7. Q.E.D.

PROOF: By Lemma 14.

□

Theorem 105. *A constant function is continuous.*

PROOF: Let X and Y be topological spaces. Let $b \in Y$, and let $f : X \rightarrow Y$ be the constant function with value b . For any open $V \subseteq Y$, the set $f^{-1}(V)$ is either X (if $b \in V$) or \emptyset (if $b \notin V$). □

Theorem 106. *If A is a subspace of X then the inclusion $j : A \rightarrow X$ is continuous.*

PROOF: For any V open in X , we have $j^{-1}(V) = V \cap A$ is open in A . □

Theorem 107. *The composite of two continuous functions is continuous.*

PROOF: Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be continuous. For any V open in Z , we have $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is open in X . □

Theorem 108. *Let $f : X \rightarrow Y$ be a continuous function and A be a subspace of X . Then the restriction $f \upharpoonright A : A \rightarrow Y$ is continuous.*

PROOF: Let V be open in Y . Then $(f \upharpoonright A)^{-1}(V) = f^{-1}(V) \cap A$ is open in A . □

Theorem 109. *Let $f : X \rightarrow Y$ be continuous. Let Z be a subspace of Y such that $f(X) \subseteq Z$. Then the corestriction $f : X \rightarrow Z$ is continuous.*

PROOF:

⟨1⟩1. LET: V be open in Z .

⟨1⟩2. PICK U open in Y such that $V = U \cap Z$.

⟨1⟩3. $f^{-1}(V) = f^{-1}(U)$

$\langle 1 \rangle 4.$ $f^{-1}(V)$ is open in X .

□

Theorem 110. *Let $f : X \rightarrow Y$ be continuous. Let Z be a space such that Y is a subspace of Z . Then the expansion $f : X \rightarrow Z$ is continuous.*

PROOF: Let V be open in Z . Then $f^{-1}(V) = f^{-1}(V \cap Y)$ is open in X . □

Theorem 111. *Let X and Y be topological spaces. Let $f : X \rightarrow Y$. Suppose \mathcal{U} is a set of open sets in X such that $X = \bigcup \mathcal{U}$ and, for all $U \in \mathcal{U}$, we have $f \upharpoonright U : U \rightarrow Y$ is continuous. Then f is continuous.*

PROOF:

$\langle 1 \rangle 1.$ LET: V be open in Y

$\langle 1 \rangle 2.$ $f^{-1}(V) = \bigcup_{U \in \mathcal{U}} (f \upharpoonright U)^{-1}(V)$

$\langle 1 \rangle 3.$ For all $U \in \mathcal{U}$, we have $(f \upharpoonright U)^{-1}(V)$ is open in U .

$\langle 1 \rangle 4.$ For all $U \in \mathcal{U}$, we have $(f \upharpoonright U)^{-1}(V)$ is open in X .

PROOF: Lemma 43.

□

Theorem 112. *Let A be a topological space and $\{X_i\}_{i \in I}$ be a family of topological spaces. Let $f : A \rightarrow \prod_{i \in I} X_i$ be a function. If $\pi_i \circ f$ is continuous for all $i \in I$ then f is continuous.*

PROOF:

$\langle 1 \rangle 1.$ LET: $i \in I$ and U be open in X_i

$\langle 1 \rangle 2.$ $f^{-1}(\pi_i^{-1}(U))$ is open in A

$\langle 1 \rangle 3.$ Q.E.D.

PROOF: Proposition 101.

□

Proposition 113. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ and $x \in \mathbb{R}$. Then f is continuous at x if and only if, for all $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in \mathbb{R}$, if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$.*

PROOF:

$\langle 1 \rangle 1.$ If f is continuous at x then, for all $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in \mathbb{R}$, if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$

$\langle 2 \rangle 1.$ ASSUME: f is continuous.

$\langle 2 \rangle 2.$ LET: $\epsilon > 0$

$\langle 2 \rangle 3.$ $f^{-1}((f(x) - \epsilon, f(x) + \epsilon))$ is open.

$\langle 2 \rangle 4.$ PICK a, b such that $x \in (a, b) \subseteq f^{-1}((f(x) - \epsilon, f(x) + \epsilon))$

$\langle 2 \rangle 5.$ LET: $\delta = \min(x - a, b - x)$

$\langle 2 \rangle 6.$ LET: $y \in \mathbb{R}$ with $|x - y| < \delta$

$\langle 2 \rangle 7.$ $y \in (a, b)$

$\langle 2 \rangle 8.$ $f(y) \in (f(x) - \epsilon, f(x) + \epsilon)$

$\langle 2 \rangle 9.$ $|f(x) - f(y)| < \epsilon$

$\langle 1 \rangle 2.$ Suppose, for all $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in \mathbb{R}$, if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$. Then f is continuous at x .

- ⟨2⟩1. ASSUME: For all $\epsilon > 0$, there exists $\delta > 0$ such that. for all $y \in \mathbb{R}$, if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$
- ⟨2⟩2. LET: V be a neighbourhood of $f(x)$
- ⟨2⟩3. PICK a, b such that $f(x) \in (a, b) \subseteq V$
- ⟨2⟩4. LET: $\epsilon = \min(f(x) - a, b - f(x))$
- ⟨2⟩5. PICK $\delta > 0$ such that, for all $y \in \mathbb{R}$, if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$
- ⟨2⟩6. LET: $U = (x - \delta, x + \delta)$
- ⟨2⟩7. $x \in U \subseteq f^{-1}(V)$

□

Proposition 114. *Let X and X' be the same set X under two topologies \mathcal{T} and \mathcal{T}' . Let $i : X \rightarrow X'$ be the identity function. Then i is continuous if and only if $\mathcal{T}' \subseteq \mathcal{T}$.*

PROOF: Immediate from definitions. □

Proposition 115. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ and $a \in \mathbb{R}$. Then f is continuous on the right at a if and only if f is continuous at a as a function $\mathbb{R}_l \rightarrow \mathbb{R}$.*

PROOF:

- ⟨1⟩1. If f is continuous on the right at a then f is continuous at a as a function $\mathbb{R}_l \rightarrow \mathbb{R}$.
 - ⟨2⟩1. ASSUME: f is continuous on the right at a .
 - ⟨2⟩2. LET: V be a neighbourhood of $f(a)$
 - ⟨2⟩3. PICK b, c such that $f(a) \in (b, c) \subseteq V$.
 - ⟨2⟩4. LET: $\epsilon = \min(c - f(a), f(a) - b)$
 - ⟨2⟩5. PICK $\delta > 0$ such that, for all x , if $a < x < a + \delta$ then $|f(x) - f(a)| < \epsilon$
 - ⟨2⟩6. LET: $U = [a, a + \delta)$
 - ⟨2⟩7. $f(U) \subseteq V$
- ⟨1⟩2. If f is continuous at a as a function $\mathbb{R}_l \rightarrow \mathbb{R}$ then f is continuous on the right at a .
 - ⟨2⟩1. ASSUME: f is continuous at a as a function $\mathbb{R}_l \rightarrow \mathbb{R}$
 - ⟨2⟩2. LET: $\epsilon > 0$
 - ⟨2⟩3. PICK a neighbourhood U of a such that $f(U) \subseteq (f(a) - \epsilon, f(a) + \epsilon)$
 - ⟨2⟩4. PICK b, c such that $a \in [b, c) \subset U$
 - ⟨2⟩5. LET: $\delta = c - a$
 - ⟨2⟩6. For all x , if $a < x < a + \delta$ then $|f(x) - f(a)| < \epsilon$

□

Lemma 116. *Let X be a topological space. Let Y be a linearly ordered set in the order topology. Let $f, g : X \rightarrow Y$ be continuous. Then $C = \{x \in X \mid f(x) \leq g(x)\}$ is closed.*

PROOF:

- ⟨1⟩1. LET: $x \in X \setminus C$
- ⟨1⟩2. $f(x) > g(x)$
 - PROVE: There exists a neighbourhood U of x such that $U \subseteq X \setminus C$
- ⟨1⟩3. CASE: There exists y such that $g(x) < y < f(x)$

PROOF: Take $U = g^{-1}((-\infty, y)) \cup f^{-1}(y, +\infty)$.
 (1)4. CASE: There is no y such that $g(x) < y < f(x)$
 PROOF: Take $U = g^{-1}((-\infty, f(x))) \cup f^{-1}(g(x), +\infty)$.
 \square

Lemma 117. *Let $f : X \rightarrow Y$. Let Z be an open subspace of X and $a \in Z$. If $f \upharpoonright Z$ is continuous at a then f is continuous at a .*

PROOF:
 (1)1. LET: V be a neighbourhood of $f(x)$
 (1)2. PICK a neighbourhood W of x in Z such that $f(W) \subseteq V$
 (1)3. W is a neighbourhood of x in X such that $f(W) \subseteq V$
 PROOF: Lemma 43.
 \square

Proposition 118. *Let $f : A \rightarrow B$ and $g : C \rightarrow D$ be continuous. Define $f \times g : A \times C \rightarrow B \times D$ by*

$$(f \times g)(a, c) = (f(a), g(c)) .$$

Then $f \times g$ is continuous.

PROOF: $\pi_1 \circ (f \times g) = f \circ \pi_1$ and $\pi_2 \circ (f \times g) = g \circ \pi_2$ are continuous by Theorem 107. The result follows by Theorem 112.

Proposition 119. *Let X be a topological space. Let Y a Hausdorff space. Let $A \subseteq X$. Let $f, g : \overline{A} \rightarrow Y$ be continuous. If f and g agree on A then $f = g$.*

PROOF:
 (1)1. LET: $x \in \overline{A}$
 (1)2. ASSUME: $f(x) \neq g(x)$
 (1)3. PICK disjoint neighbourhoods V of $f(x)$ and W of $g(x)$.
 (1)4. PICK $y \in f^{-1}(V) \cap g^{-1}(W) \cap A$
 PROOF: Since $f^{-1}(V) \cap g^{-1}(W)$ is a neighbourhood of x and hence intersects A .
 (1)5. $f(y) = g(y) \in V \cap W$
 (1)6. Q.E.D.
 PROOF: This contradicts the fact that V and W are disjoint ((1)3).
 \square

Proposition 120. *Let X and Y be topological spaces and $f : X \rightarrow Y$ be continuous. If $a_n \rightarrow l$ as $n \rightarrow \infty$ in X then $f(a_n) \rightarrow f(l)$ as $n \rightarrow \infty$.*

PROOF:
 (1)1. LET: V be a neighbourhood of $f(l)$
 (1)2. PICK a neighbourhood U of l such that $f(U) \subseteq V$
 (1)3. PICK N such that, for all $n \geq N$, we have $a_n \in U$
 (1)4. For all $n \geq N$ we have $f(a_n) \in V$
 \square

Proposition 121. Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let (a_n) be a sequence in $\prod_{i \in I} X_i$ and $l \in \prod_{i \in I} X_i$. Then $a_n \rightarrow l$ as $n \rightarrow \infty$ if and only if, for all $i \in I$, we have $\pi_i(a_n) \rightarrow \pi_i(l)$ as $n \rightarrow \infty$.

PROOF:

$\langle 1 \rangle 1$. If $a_n \rightarrow l$ as $n \rightarrow \infty$ then, for all $i \in I$, we have $\pi_i(a_n) \rightarrow \pi_i(l)$ as $n \rightarrow \infty$

PROOF: Proposition 120.

$\langle 1 \rangle 2$. If, for all $i \in I$, we have $\pi_i(a_n) \rightarrow \pi_i(l)$ as $n \rightarrow \infty$, then $a_n \rightarrow l$ as $n \rightarrow \infty$

$\langle 2 \rangle 1$. ASSUME: For all $i \in I$, we have $\pi_i(a_n) \rightarrow \pi_i(l)$ as $n \rightarrow \infty$

$\langle 2 \rangle 2$. LET: V be a neighbourhood of l

$\langle 2 \rangle 3$. PICK open sets U_i in X_i such that $l \in \prod_{i \in I} U_i \subseteq V$ and $U_i = X_i$ for all i except $i = i_1, \dots, i_k$

$\langle 2 \rangle 4$. For $j = 1, \dots, k$, PICK N_j such that, for all $n \geq N_j$, we have $\pi_{i_j}(a_n) \in U_{i_j}$

$\langle 2 \rangle 5$. LET: $N = \max(N_1, \dots, N_k)$

$\langle 2 \rangle 6$. For all $n \geq N$ we have $a_n \in V$

□

19 Homeomorphisms

Definition 122 (Homeomorphism). Let X and Y be topological spaces. A *Homeomorphism* f between X and Y , $f : X \cong Y$, is a bijection $f : X \rightarrow Y$ such that both f and f^{-1} are continuous.

Lemma 123. Let X and Y be topological spaces and $f : X \rightarrow Y$ a bijection. Then the following are equivalent:

1. f is a homeomorphism.
2. f is continuous and an open map.
3. For any $U \subseteq X$, we have U is open if and only if $f(U)$ is open.

PROOF: Immediate from definitions. □

Proposition 124. Let X and X' be the same set X under two topologies \mathcal{T} and \mathcal{T}' . Let $i : X \rightarrow X'$ be the identity function. Then i is a homeomorphism if and only if $\mathcal{T} = \mathcal{T}'$.

PROOF: Immediate from definitions. □

Definition 125 (Topological Property). Let P be a property of topological spaces. Then P is a *topological* property if and only if, for any spaces X and Y , if P holds of X and $X \cong Y$ then P holds of Y .

Definition 126 (Topological Imbedding). Let X and Y be topological spaces and $f : X \rightarrow Y$. Then f is a *topological imbedding* if and only if the corestriction $f : X \rightarrow f(X)$ is a homeomorphism.

Proposition 127. *Let X and Y be topological spaces and $a \in X$. The function $i : Y \rightarrow X \times Y$ that maps y to (a, y) is an imbedding.*

PROOF:

$\langle 1 \rangle 1$. i is injective

$\langle 1 \rangle 2$. i is continuous.

PROOF: For U open in X and V open in Y , we have $i^{-1}(U \times V)$ is V if $a \in U$, and \emptyset if $a \notin U$.

$\langle 1 \rangle 3$. $i : Y \rightarrow i(Y)$ is an open map.

PROOF: For V open in Y we have $i(V) = (X \times V) \cap i(Y)$.

□

20 Locally Finite Sets

Definition 128 (Locally Finite). Let X be a topological space and $\{A_\alpha\}$ a family of subsets of X . Then \mathcal{A} is *locally finite* if and only if every point in X has a neighbourhood that intersects A_α for only finitely many α .

Theorem 129 (Pasting Lemma). *Let X and Y be topological spaces and $f : X \rightarrow Y$. Let $\{A_\alpha\}$ be a locally finite family of closed subsets of X that cover X . Suppose $f \upharpoonright A_\alpha$ is continuous for all α . Then f is continuous.*

PROOF:

$\langle 1 \rangle 1$. Let X and Y be topological spaces and $f : X \rightarrow Y$. Let A and B be closed subsets of X such that $X = A \cup B$. Suppose $f \upharpoonright A$ and $f \upharpoonright B$ are continuous. Then f is continuous.

$\langle 2 \rangle 1$. LET: $C \subseteq Y$ be closed.

$\langle 2 \rangle 2$. $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$

$\langle 2 \rangle 3$. $f^{-1}(C)$ and $g^{-1}(C)$ are closed in X .

PROOF: Theorems 104 and 55.

$\langle 2 \rangle 4$. $h^{-1}(C)$ is closed in X .

PROOF: Lemma 53.

$\langle 2 \rangle 5$. Q.E.D.

PROOF: Theorem 104.

$\langle 1 \rangle 2$. Let X and Y be topological spaces and $f : X \rightarrow Y$. Let $\{A_\alpha\}$ be a finite family of closed subsets of X that cover X . Suppose $f \upharpoonright A_\alpha$ is continuous for all α . Then f is continuous.

PROOF: From $\langle 1 \rangle 1$ by induction.

$\langle 1 \rangle 3$. Let X and Y be topological spaces and $f : X \rightarrow Y$. Let $\{A_\alpha\}$ be a locally finite family of closed subsets of X that cover X . Suppose $f \upharpoonright A_\alpha$ is continuous for all α . Then f is continuous.

$\langle 2 \rangle 1$. LET: $x \in X$

PROVE: f is continuous at x

$\langle 2 \rangle 2$. PICK a neighbourhood U of x that intersects A_α for only finitely many α .

$\langle 2 \rangle 3$. $f \upharpoonright U$ is continuous

PROOF: By $\langle 1 \rangle 2$.
 $\langle 2 \rangle 4$. Q.E.D.
 PROOF: Lemma 117.

□

The following example shows that we cannot remove the assumption of local finiteness.

Example 130. Define $f : [-1, 1] \rightarrow \mathbb{R}$ by: $f(x) = 1$ if $x < -1$, $f(x) = 0$ if $x > 1$. Let $C_n = [-1, -1/n]$ for $n \geq 1$, and $D = [0, 1]$. Then $[-1, 1] = \bigcup_{n=1}^{\infty} C_n \cup D$ and f is continuous on each C_n and each D , but f is not continuous on $[-1, 1]$.

Proposition 131. Let X be a topological space. Let Y be a linearly ordered set in the order topology. Let $f, g : X \rightarrow Y$ be continuous. Define $h : X \rightarrow Y$ by $h(x) = \min(f(x), g(x))$. Then h is continuous.

PROOF: By the Pasting Lemma applied to $\{x \in X \mid f(x) \leq g(x)\}$ and $\{x \in X \mid g(x) \leq f(x)\}$, which are closed by Lemma 116.

21 Continuous in Each Variable Separately

Definition 132 (Continuous in Each Variable Separately). Let $F : X \times Y \rightarrow Z$. Then F is *continuous in each variable separately* if and only if:

- for every $a \in X$ the function $\lambda y \in Y. F(a, y)$ is continuous;
- for every $b \in Y$ the function $\lambda x \in X. F(x, b)$ is continuous.

Proposition 133. Let $F : X \times Y \rightarrow Z$. If F is continuous then F is continuous in each variable separately.

PROOF: For $a \in X$, the function $\lambda y \in Y. F(a, y)$ is $F \circ i$ where $i : Y \rightarrow X \times Y$ maps y to (a, y) . We have i is continuous by Proposition 127, hence $F \circ i$ is continuous by Theorem 107.

Similarly for $\lambda x \in X. F(x, b)$ for $b \in Y$. □

Example 134. Define $F : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$F(x, y) = \begin{cases} xy/(x^2 + y^2) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Then F is continuous in each variable separately but not continuous.

22 The Box Topology

Definition 135 (Box Topology). Let $\{A_i\}_{i \in I}$ be a family of topological spaces. The *box topology* on $\prod_{i \in I} A_i$ is the topology generated by the set of all sets of the form $\prod_{i \in I} U_i$ where $\{U_i\}_{i \in I}$ is a family such that each U_i is open in A_i .

This is a basis since it covers $\prod_{i \in I} A_i$ and is closed under intersection.

Proposition 136. *The box topology is finer than the product topology.*

PROOF: From Proposition 36. \square

Corollary 136.1. *If A_i is closed in X_i for all $i \in I$ then $\prod_{i \in I} A_i$ is closed in $\prod_{i \in I} X_i$ under the box topology.*

PROOF: From Proposition 57.

Proposition 137 (AC). *Let $\{A_i\}_{i \in I}$ be a family of topological spaces. For $i \in I$, let \mathcal{B}_i be a basis for the topology on A_i . Then $\mathcal{B} = \{\prod_{i \in I} B_i \mid \forall i \in I, B_i \in \mathcal{B}_i\}$ is a basis for the box topology on $\prod_{i \in I} A_i$.*

PROOF:

- $\langle 1 \rangle 1$. Every set of the form $\prod_{i \in I} B_i$ is open.
- $\langle 1 \rangle 2$. For every point $a \in \prod_{i \in I} A_i$ and every open set U with $a \in U$, there exists $B \in \mathcal{B}$ such that $a \in B \subseteq U$.
 - $\langle 2 \rangle 1$. LET: U be open and $a \in U$
 - $\langle 2 \rangle 2$. PICK a family $\{U_i\}_{i \in I}$ such that each U_i is open in A_i and $a \in \prod_{i \in I} U_i \subseteq U$.
 - $\langle 2 \rangle 3$. For $i \in I$, PICK $B_i \in \mathcal{B}_i$ such that $a_i \in B_i \subseteq U_i$
- PROOF: Using the Axiom of Choice.
- $\langle 2 \rangle 4$. $a \in \prod_{i \in I} B_i \subseteq U$
- $\langle 1 \rangle 3$. Q.E.D.

PROOF: Lemma 19.

\square

Theorem 138. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let A_i be a subspace of X_i for all $i \in I$. Give $\prod_{i \in I} X_i$ the box topology. Then the box topology on $\prod_{i \in I} A_i$ is the same as the topology it inherits as a subspace of $\prod_{i \in I} X_i$.*

PROOF: The box topology is generated by the basis

$$\begin{aligned}
& \left\{ \prod_{i \in I} U_i \mid \forall i \in I, U_i \text{ open in } A_i \right\} \\
&= \left\{ \prod_{i \in I} (V_i \cap A_i) \mid \forall i \in I, V_i \text{ open in } X_i \right\} \\
&= \left\{ \prod_{i \in I} V_i \mid \forall i \in I, V_i \text{ open in } X_i \right\} \cap \prod_{i \in I} A_i
\end{aligned}$$

and this is a basis for the subspace topology by Lemma 41. \square

Proposition 139. *Let $\{X_i\}_{i \in I}$ be a family of Hausdorff spaces. Then $\prod_{i \in I} X_i$ under the box topology is Hausdorff.*

PROOF:

- $\langle 1 \rangle 1$. LET: $\{X_i\}_{i \in I}$ be a family of Hausdorff spaces.
- $\langle 1 \rangle 2$. LET: $a, b \in \prod_{i \in I} X_i$ with $a \neq b$

- ⟨1⟩3. PICK $i \in I$ such that $a_i \neq b_i$
 ⟨1⟩4. PICK U, V disjoint open sets in X_i with $a_i \in U$ and $b_i \in V$
 ⟨1⟩5. $\pi_i^{-1}(U)$ and $\pi_i^{-1}(V)$ are disjoint open sets in $\prod_{i \in I} X_i$ with $a \in \pi_i^{-1}(U)$
 and $b \in \pi_i^{-1}(V)$

□

Proposition 140 (AC). *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Give $\prod_{i \in I} X_i$ the box topology. Let $A_i \subseteq X_i$ for all $i \in I$. Then*

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i} .$$

PROOF:

⟨1⟩1. $\overline{\prod_{i \in I} A_i} \subseteq \prod_{i \in I} \overline{A_i}$

⟨2⟩1. For all $i \in I$ we have $A_i \subseteq \overline{A_i}$

PROOF: Lemma 67.

⟨2⟩2. $\prod_{i \in I} A_i \subseteq \prod_{i \in I} \overline{A_i}$

⟨2⟩3. Q.E.D.

PROOF: Since $\prod_{i \in I} \overline{A_i}$ is closed by Corollary 136.1.

⟨1⟩2. $\prod_{i \in I} \overline{A_i} \subseteq \overline{\prod_{i \in I} A_i}$

⟨2⟩1. LET: $x \in \prod_{i \in I} \overline{A_i}$

⟨2⟩2. LET: U be a neighbourhood of x

⟨2⟩3. PICK V_i open in X_i such that $x \in \prod_{i \in I} V_i \subseteq U$

⟨2⟩4. For $i \in I$, pick $a_i \in V_i \cap A_i$

PROOF: By Theorem 70 and ⟨2⟩1 using the Axiom of Choice.

⟨2⟩5. U intersects $\prod_{i \in I} A_i$

⟨2⟩6. Q.E.D.

PROOF: $a \in U \cap \prod_{i \in I} A_i$.

□

The following example shows that Theorem 112 fails in the box topology.

Example 141. Define $f : \mathbb{R} \rightarrow \mathbb{R}^\omega$ by $f(t) = (t, t, \dots)$. Then $\pi_n \circ f = \text{id}_{\mathbb{R}}$ is continuous for all n . But f is not continuous when \mathbb{R}^ω is given the box topology because the inverse image of

$$(-1, 1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times \dots$$

is $\{0\}$ which is not open.

The following example shows that Proposition 121 fails in the box topology.

Example 142. Give \mathbb{R}^ω the box topology. Let $a_n = (1/n, 1/n, \dots)$ for $n \geq 1$ and $l = (0, 0, \dots)$. Then $\pi_i(a_n) \rightarrow \pi_i(l)$ as $n \rightarrow \infty$ for all i , but $a_n \not\rightarrow l$ as $n \rightarrow \infty$ since the open set

$$(-1, 1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times \dots$$

contains l but does not contain any a_n .

Example 143. The set \mathbb{R}^∞ is closed in \mathbb{R}^ω under the box topology. For let (a_n) be any sequence not in \mathbb{R}^∞ . Let U_n be an open interval around a_n that does not contain 0 if $a_n \neq 0$, and $U_n = \mathbb{R}$ if $a_n = 0$. Then $\prod_{n \geq 0} U_n$ is a neighbourhood of (a_n) that does not intersect \mathbb{R}^∞ .

23 The Metric Topology

Definition 144 (Metric). Let X be a set. A *metric* on X is a function $d : X^2 \rightarrow \mathbb{R}$ such that:

1. For all $x, y \in X$, $d(x, y) \geq 0$
2. For all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$
3. For all $x, y \in X$, $d(x, y) = d(y, x)$
4. (*Triangle Inequality*) For all $x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$

We call $d(x, y)$ the *distance* between x and y .

Definition 145 (Open Ball). Let X be a metric space. Let $a \in X$ and $\epsilon > 0$. The *open ball* with *centre* a and *radius* ϵ is

$$B(a, \epsilon) = \{x \in X \mid d(a, x) < \epsilon\} .$$

Definition 146 (Metric Topology). Let X be a metric space. The *metric topology* on X is the topology generated by the basis consisting of all the open balls.

We prove this is a basis for a topology.

PROOF:

$\langle 1 \rangle 1$. For every point a , there exists a ball B such that $a \in B$

PROOF: We have $a \in B(a, 1)$.

$\langle 1 \rangle 2$. For any balls B_1, B_2 and point $a \in B_1 \cap B_2$, there exists a ball B_3 such that $a \in B_3 \subseteq B_1 \cap B_2$

$\langle 2 \rangle 1$. LET: $B_1 = B(c_1, \epsilon_1)$ and $B_2 = B(c_2, \epsilon_2)$

$\langle 2 \rangle 2$. LET: $\delta = \min(\epsilon_1 - d(c_1, a), \epsilon_2 - d(c_2, a))$

PROVE: $B(a, \delta) \subseteq B_1 \cap B_2$

$\langle 2 \rangle 3$. LET: $x \in B(a, \delta)$

$\langle 2 \rangle 4$. $x \in B_1$

PROOF:

$$\begin{aligned} d(x, c_1) &= d(x, a) + d(a, c_1) \\ &< \delta + d(a, c_1) \\ &\leq \epsilon_1 \end{aligned}$$

$\langle 2 \rangle 5$. $x \in B_2$

PROOF: Similar.

□

Proposition 147. *Let X be a metric space and $U \subseteq X$. Then U is open if and only if, for every $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$.*

PROOF:

$\langle 1 \rangle 1$. If U is open then, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$.

$\langle 2 \rangle 1$. ASSUME: U is open.

$\langle 2 \rangle 2$. LET: $x \in U$

$\langle 2 \rangle 3$. PICK $a \in X$ and $\delta > 0$ such that $x \in B(a, \delta) \subseteq U$

$\langle 2 \rangle 4$. LET: $\epsilon = \delta - d(a, x)$

PROVE: $B(x, \epsilon) \subseteq U$

$\langle 2 \rangle 5$. LET: $y \in B(x, \epsilon)$

$\langle 2 \rangle 6$. $d(y, a) < \delta$

PROOF:

$$\begin{aligned} d(y, a) &\leq d(a, x) + d(x, y) \\ &< \delta + d(x, y) \\ &= \epsilon \end{aligned}$$

$\langle 2 \rangle 7$. $y \in U$

$\langle 1 \rangle 2$. If, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$, then U is open.

PROOF: Immediate from definitions.

□

Definition 148 (Discrete Metric). Let X be a set. The *discrete metric* on X is defined by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Proposition 149. *The discrete metric induces the discrete topology.*

PROOF: For any (open) set U and point $a \in U$, we have $a \in B(a, 1) \subseteq U$. □

Definition 150 (Standard Metric on \mathbb{R}). The *standard metric* on \mathbb{R} is defined by $d(x, y) = |x - y|$.

Proposition 151. *The standard metric on \mathbb{R} induces the standard topology on \mathbb{R} .*

PROOF:

$\langle 1 \rangle 1$. Every open ball is open in the standard topology on \mathbb{R} .

PROOF: $B(a, \epsilon) = (a - \epsilon, a + \epsilon)$

$\langle 1 \rangle 2$. For every open set U and point $a \in U$, there exists $\epsilon > 0$ such that

$$B(a, \epsilon) \subseteq U$$

$\langle 2 \rangle 1$. LET: U be an open set and $a \in U$

$\langle 2 \rangle 2$. PICK an open interval b, c such that $a \in (b, c) \subseteq U$

$\langle 2 \rangle 3$. LET: $\epsilon = \min(a - b, c - a)$

$\langle 2 \rangle 4$. $B(a, \epsilon) \subseteq U$

□

Definition 152 (Metriizable). A topological space X is *metriizable* if and only if there exists a metric on X that induces the topology.

Definition 153 (Bounded). Let X be a metric space and $A \subseteq X$. Then A is *bounded* if and only if there exists M such that, for all $x, y \in A$, we have $d(x, y) \leq M$.

Definition 154 (Diameter). Let X be a metric space and $A \subseteq X$. The *diameter* of A is

$$\text{diam } A = \sup_{x, y \in A} d(x, y) .$$

Definition 155 (Standard Bounded Metric). Let d be a metric on X . The *standard bounded metric* corresponding to d is the metric \bar{d} defined by

$$\bar{d}(x, y) = \min(d(x, y), 1) .$$

We prove this is a metric.

PROOF:

$\langle 1 \rangle 1. \bar{d}(x, y) \geq 0$

PROOF: Since $d(x, y) \geq 0$

$\langle 1 \rangle 2. \bar{d}(x, y) = 0$ if and only if $x = y$

PROOF: $\bar{d}(x, y) = 0$ if and only if $d(x, y) = 0$ if and only if $x = y$

$\langle 1 \rangle 3. \bar{d}(x, y) = \bar{d}(y, x)$

PROOF: Since $d(x, y) = d(y, x)$

$\langle 1 \rangle 4. \bar{d}(x, z) \leq \bar{d}(x, y) + \bar{d}(y, z)$

PROOF:

$$\begin{aligned} \bar{d}(x, y) + \bar{d}(y, z) &= \min(d(x, y), 1) + \min(d(y, z), 1) \\ &= \min(d(x, y) + d(y, z), d(x, y) + 1, d(y, z) + 1, 2) \\ &\geq \min(d(x, z), 1) \\ &= \bar{d}(x, z) \end{aligned}$$

□

Lemma 156. In any metric space X , the set $\mathcal{B} = \{B(a, \epsilon) \mid a \in X, \epsilon < 1\}$ is a basis for the metric topology.

PROOF:

$\langle 1 \rangle 1.$ Every element of \mathcal{B} is open.

PROOF: From Lemma 18.

$\langle 1 \rangle 2.$ For every open set U and point $a \in U$, there exists $B \in \mathcal{B}$ such that $a \in B \subseteq U$

$\langle 2 \rangle 1.$ LET: U be an open set and $a \in U$

$\langle 2 \rangle 2.$ PICK $\epsilon > 0$ such that $B(a, \epsilon) \subseteq U$

$\langle 2 \rangle 3.$ $B(a, \min(\epsilon, 1/2)) \subseteq U$

$\langle 1 \rangle 3.$ Q.E.D.

PROOF: Lemma 19.

□

Proposition 157. Let d be a metric on the set X . Then the standard bounded metric \bar{d} induces the same metric as d .

PROOF: This follows from Lemma 156 since the open balls with radius < 1 are the same under both metrics. \square

Lemma 158. *Let d and d' be two metrics on the same set X . Let \mathcal{T} and \mathcal{T}' be the topologies they induce. Then $\mathcal{T} \subseteq \mathcal{T}'$ if and only if, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that*

$$B_{d'}(x, \delta) \subseteq B_d(x, \epsilon) .$$

PROOF:

$\langle 1 \rangle 1$. If $\mathcal{T} \subseteq \mathcal{T}'$ then, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$

PROOF: From Proposition 147 since $x \in B_d(x, \epsilon) \in \mathcal{T}'$.

$\langle 1 \rangle 2$. If, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$, then $\mathcal{T} \subseteq \mathcal{T}'$

$\langle 2 \rangle 1$. ASSUME: For all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$

$\langle 2 \rangle 2$. LET: $U \in \mathcal{T}$

$\langle 2 \rangle 3$. For all $x \in U$ there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq U$.

$\langle 3 \rangle 1$. LET: $x \in U$

$\langle 3 \rangle 2$. PICK $\epsilon > 0$ such that $B_d(x, \epsilon) \subseteq U$

PROOF: Proposition 147

$\langle 3 \rangle 3$. PICK $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$

PROOF: By $\langle 2 \rangle 1$

$\langle 3 \rangle 4$. $B_{d'}(x, \delta) \subseteq U$

$\langle 2 \rangle 4$. $U \in \mathcal{T}'$

PROOF: Proposition 147.

\square

Proposition 159. \mathbb{R}^2 under the dictionary order topology is metrizable.

PROOF: Define $d : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$d((x, y), (x, z)) = \max(|y - z|, 1)$$

$$d((x, y), (x', y')) = 1 \quad \text{if } x \neq x' \square$$

$\langle 1 \rangle 1$. $x \in \bigcap_{i=1}^N \pi_i^{-1}() \subseteq B_D(a, \epsilon)$

Proposition 160. *Let $d : X^2 \rightarrow \mathbb{R}$ be a metric on X . Then the metric topology on X is the coarsest topology such that d is continuous.*

PROOF:

$\langle 1 \rangle 1$. d is continuous in each variable separately.

$\langle 2 \rangle 1$. LET: $a \in X$ and $d_a : X \rightarrow \mathbb{R}$ be the function $d(a, -)$

$\langle 2 \rangle 2$. LET: $b \in X$ and $\epsilon > 0$

$\langle 2 \rangle 3$. For all $x \in X$, if $d(b, x) < \epsilon$ then $|d(a, b) - d(a, x)| < \epsilon$

$\langle 1 \rangle 2$. If \mathcal{T} is any topology under which d is continuous then \mathcal{T} is finer than the metric topology.

PROOF: Since $B(a, \epsilon) = d_a^{-1}((-\infty, \epsilon))$

\square

24 Real Linear Algebra

Definition 161 (Square Metric). The *square metric* ρ on \mathbb{R}^n is defined by

$$\rho(\vec{x}, \vec{y}) = \max(|x_1 - y_1|, \dots, |x_n - y_n|)$$

We prove this is a metric.

PROOF:

$\langle 1 \rangle 1. \rho(\vec{x}, \vec{y}) \geq 0$

PROOF: Immediate from definition.

$\langle 1 \rangle 2. \rho(\vec{x}, \vec{y}) = 0$ if and only if $\vec{x} = \vec{y}$

PROOF: Immediate from definition.

$\langle 1 \rangle 3. \rho(\vec{x}, \vec{y}) = \rho(\vec{y}, \vec{x})$

PROOF: Immediate from definition.

$\langle 1 \rangle 4. \rho(\vec{x}, \vec{z}) \leq \rho(\vec{x}, \vec{y}) + \rho(\vec{y}, \vec{z})$

PROOF: Since $|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$.

□

Proposition 162. The square metric induces the standard topology on \mathbb{R}^n .

PROOF:

$\langle 1 \rangle 1.$ For every $a \in X$ and $\epsilon > 0$, we have $B_\rho(a, \epsilon)$ is open in the standard product topology.

PROOF:

$$B_\rho(a, \epsilon) = (a_1 - \epsilon, a_1 + \epsilon) \times \cdots \times (a_n - \epsilon, a_n + \epsilon)$$

$\langle 1 \rangle 2.$ For any open sets U_1, \dots, U_n in \mathbb{R} , we have $U_1 \times \cdots \times U_n$ is open in the square metric topology.

$\langle 2 \rangle 1.$ LET: $\vec{a} \in U_1 \times \cdots \times U_n$

$\langle 2 \rangle 2.$ For $i = 1, \dots, n$, PICK $\epsilon_i > 0$ such that $(a_i - \epsilon_i, a_i + \epsilon_i) \subseteq U_i$

$\langle 2 \rangle 3.$ LET: $\epsilon = \min(\epsilon_1, \dots, \epsilon_n)$

$\langle 2 \rangle 4.$ $B_\rho(\vec{a}, \epsilon) \subseteq U$

□

Definition 163. Given $\vec{x}, \vec{y} \in \mathbb{R}^n$, define the *sum* $\vec{x} + \vec{y}$ by

$$(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n) .$$

Definition 164. Given $\lambda \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^n$, define the *scalar product* $\lambda \vec{x} \in \mathbb{R}^n$ by

$$\lambda(x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n)$$

Definition 165 (Inner Product). Given $\vec{x}, \vec{y} \in \mathbb{R}^n$, define the *inner product* $\vec{x} \cdot \vec{y} \in \mathbb{R}$ by

$$(x_1, \dots, x_n) \cdot (y_1, \dots, y_n) = x_1 y_1 + \cdots + x_n y_n .$$

We write \vec{x}^2 for $\vec{x} \cdot \vec{x}$.

Definition 166 (Norm). Let $n \geq 1$. The *norm* on \mathbb{R}^n is the function $\| \cdot \| : \mathbb{R}^n \rightarrow \mathbb{R}$ defined by

$$\|(x_1, \dots, x_n)\| = \sqrt{x_1^2 + \dots + x_n^2}$$

Lemma 167.

$$\|\vec{x}\|^2 = \vec{x} \cdot \vec{x}$$

PROOF: Immediate from definitions. \square

Lemma 168.

$$\vec{x} \cdot (\vec{y} + \vec{z}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z}$$

PROOF: Each is equal to $(x_1y_1 + x_1z_1, \dots, x_ny_n + x_nz_n)$. \square

Lemma 169.

$$|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \|\vec{y}\|$$

PROOF:

$\langle 1 \rangle 1$. ASSUME: $\vec{x} \neq \vec{0} \neq \vec{y}$

PROOF: Otherwise both sides are 0.

$\langle 1 \rangle 2$. LET: $a = 1/\|\vec{x}\|$

$\langle 1 \rangle 3$. LET: $b = 1/\|\vec{y}\|$

$\langle 1 \rangle 4$. $(a\vec{x} + b\vec{y})^2 \geq 0$ and $(a\vec{x} - b\vec{y})^2 \geq 0$

$\langle 1 \rangle 5$. $a^2\|\vec{x}\|^2 + 2ab\vec{x} \cdot \vec{y} + b^2\|\vec{y}\|^2 \geq 0$ and $a^2\|\vec{x}\|^2 - 2ab\vec{x} \cdot \vec{y} + b^2\|\vec{y}\|^2 \geq 0$

$\langle 1 \rangle 6$. $2ab\vec{x} \cdot \vec{y} + 2 \geq 0$ and $-2ab\vec{x} \cdot \vec{y} + 2 \geq 0$

$\langle 1 \rangle 7$. $\vec{x} \cdot \vec{y} \geq -1/ab$ and $\vec{x} \cdot \vec{y} \leq 1/ab$

$\langle 1 \rangle 8$. $\vec{x} \cdot \vec{y} \geq -\|\vec{x}\|\|\vec{y}\|$ and $\vec{x} \cdot \vec{y} \leq \|\vec{x}\|\|\vec{y}\|$

\square

Lemma 170 (Triangle Inequality).

$$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$$

PROOF:

$$\begin{aligned} \|\vec{x} + \vec{y}\|^2 &= \|\vec{x}\|^2 + 2\vec{x} \cdot \vec{y} + \|\vec{y}\|^2 \\ &\leq \|\vec{x}\|^2 + 2\|\vec{x}\|\|\vec{y}\| + \|\vec{y}\|^2 && \text{(Lemma 169)} \\ &= (\|\vec{x}\| + \|\vec{y}\|)^2 && \square \end{aligned}$$

Definition 171 (Euclidean Metric). Let $n \geq 1$. The *Euclidean metric* on \mathbb{R}^n is defined by

$$d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\| .$$

We prove this is a metric.

$\langle 1 \rangle 1$. $d(\vec{x}, \vec{y}) \geq 0$

PROOF: Immediate from definition.

$\langle 1 \rangle 2$. $d(\vec{x}, \vec{y}) = 0$ if and only if $\vec{x} = \vec{y}$

PROOF: $d(\vec{x}, \vec{y}) = 0$ if and only if $\vec{x} - \vec{y} = \vec{0}$.

⟨1⟩3. $d(\vec{x}, \vec{y}) = d(\vec{y}, \vec{x})$

PROOF: Immediate from definition.

⟨1⟩4. $d(\vec{x}, \vec{z}) \leq d(\vec{x}, \vec{y}) + d(\vec{y}, \vec{z})$

PROOF:

$$\begin{aligned} \|\vec{x} - \vec{z}\| &= \|(\vec{x} - \vec{y}) + (\vec{y} - \vec{z})\| \\ &\leq \|\vec{x} - \vec{y}\| + \|\vec{y} - \vec{z}\| \end{aligned} \quad (\text{Lemma 170})$$

□

Proposition 172. *The Euclidean metric induces the standard topology on \mathbb{R}^n .*

PROOF:

⟨1⟩1. LET: ρ be the square metric.

⟨1⟩2. For all $\vec{a} \in \mathbb{R}^n$ and $\epsilon > 0$, we have $B_d(\vec{a}, \epsilon) \subseteq B_\rho(\vec{a}, \epsilon)$

⟨2⟩1. LET: $\vec{x} \in B_d(\vec{a}, \epsilon)$

⟨2⟩2. $\sqrt{(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2} < \epsilon$

⟨2⟩3. $(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2 < \epsilon^2$

⟨2⟩4. For all i we have $(x_i - a_i)^2 < \epsilon^2$

⟨2⟩5. For all i we have $|x_i - a_i| < \epsilon$

⟨2⟩6. $\rho(\vec{x}, \vec{a}) < \epsilon$

⟨1⟩3. For all $\vec{a} \in \mathbb{R}^n$ and $\epsilon > 0$, we have $B_\rho(\vec{a}, \epsilon/\sqrt{n}) \subseteq B_d(\vec{a}, \epsilon)$

⟨2⟩1. LET: $\vec{x} \in B_\rho(\vec{a}, \epsilon/\sqrt{n})$

⟨2⟩2. $\rho(\vec{x}, \vec{a}) < \epsilon/\sqrt{n}$

⟨2⟩3. For all i we have $|x_i - a_i| < \epsilon/\sqrt{n}$

⟨2⟩4. For all i we have $(x_i - a_i)^2 < \epsilon^2/n$

⟨2⟩5. $d(\vec{x}, \vec{a}) < \epsilon$

⟨1⟩4. Q.E.D.

PROOF: By Lemma 158.

□

Proposition 173. *The space \mathbb{R}^ω is metrizable.*

PROOF:

⟨1⟩1. LET: D be the metric on \mathbb{R}^ω defined by $D(x, y) = \sup_i (\bar{d}(x_i, y_i)/i)$ where \bar{d} is the standard bounded metric.

⟨2⟩1. $D(x, y) \geq 0$

⟨2⟩2. $D(x, y) = 0$ if and only if $x = y$

⟨2⟩3. $D(x, y) = D(y, x)$

⟨2⟩4. $D(x, z) \leq D(x, y) + D(y, z)$

PROOF:

$$\begin{aligned} D(x, z) &= \sup_i \frac{\bar{d}(x_i, z_i)}{i} \\ &\leq \sup_i \frac{\bar{d}(x_i, y_i) + \bar{d}(y_i, z_i)}{i} \\ &\leq \sup_i \frac{\bar{d}(x_i, y_i)}{i} + \sup_i \frac{\bar{d}(y_i, z_i)}{i} \\ &= D(x, y) + D(y, z) \end{aligned}$$

- (1)2. Every open ball $B_D(a, \epsilon)$ is open in the product topology.
 (2)1. PICK N such that $1/\epsilon < N$
 (2)2. $B_D(a, \epsilon) = \prod_{i=1}^{\infty} U_i$ where $U_i = (a_i - i\epsilon, a_i + i\epsilon)$ if $i \leq N$, and $U_i = \mathbb{R}$ if $i > N$
 (1)3. For any open set U and $a \in U$, there exists $\epsilon > 0$ such that $B(a, \epsilon) \subseteq U$.
 (2)1. LET: $n \geq 1$, V be an open set in \mathbb{R} and $a \in \pi_n^{-1}(V)$
 (2)2. PICK $\epsilon > 0$ such that $(a - \epsilon, a + \epsilon) \subseteq V$
 (2)3. $B(a, \epsilon/n) \subseteq \pi_n^{-1}(V)$

□

Lemma 174. *If $\sum_{i=0}^{\infty} x_i^2$ and $\sum_{i=0}^{\infty} y_i^2$ converge then $\sum_{i=0}^{\infty} |x_i y_i|$ converges.*

PROOF:

- (1)1. For all $N \geq 0$ we have $\sum_{i=0}^N |x_i y_i| \leq \sqrt{\sum_{i=0}^N |x_i|^2} \sqrt{\sum_{i=0}^N |y_i|^2}$

PROOF: By the Cauchy-Schwarz inequality

- (1)2. Q.E.D.

PROOF: Since $\sum_{i=0}^N |x_i y_i|$ is an increasing sequence bounded above by $(\sum_{i=0}^{\infty} x_i^2)(\sum_{i=0}^{\infty} y_i^2)$.

□

Corollary 174.1. *If $\sum_{i=0}^{\infty} x_i^2$ and $\sum_{i=0}^{\infty} y_i^2$ converge then $\sum_{i=0}^{\infty} (x_i + y_i)^2$ converges.*

PROOF: Since $\sum_{i=0}^{\infty} x_i^2$, $\sum_{i=0}^{\infty} y_i^2$ and $2 \sum_{i=0}^{\infty} x_i y_i$ all converge.

Definition 175 (l^2 -metric). The l^2 -metric on

$$\left\{ (x_n) \in \mathbb{R}^{\omega} \mid \sum_{n=0}^{\infty} x_n^2 \text{ converges} \right\}$$

is defined by

$$d(x, y) = \left(\sum_{n=0}^{\infty} (x_n - y_n)^2 \right)^{1/2}$$

We prove this is a metric.

PROOF:

- (1)1. d is well-defined.

PROOF: By Corollary 174.1.

- (1)2. $d(x, y) \geq 0$

- (1)3. $d(x, y) = 0$ if and only if $x = y$

- (1)4. $d(x, y) = d(y, x)$

- (1)5. $d(x, z) \leq d(x, y) + d(y, z)$

PROOF: By Lemma 170.

□

25 The Uniform Topology

Definition 176 (Uniform Metric). Let J be a set. The *uniform metric* $\bar{\rho}$ on \mathbb{R}^J is defined by

$$\bar{\rho}(a, b) = \sup_{j \in J} \bar{d}(a_j, b_j)$$

where \bar{d} is the standard bounded metric on \mathbb{R} .

The *uniform topology* on \mathbb{R}^J is the topology induced by the uniform metric.

We prove this is a metric.

PROOF:

$\langle 1 \rangle 1. \bar{\rho}(a, b) \geq 0$

PROOF: Immediate from definitions.

$\langle 1 \rangle 2. \bar{\rho}(a, b) = 0$ if and only if $a = b$

PROOF: Immediate from definitions.

$\langle 1 \rangle 3. \bar{\rho}(a, b) = \bar{\rho}(b, a)$

PROOF: Immediate from definitions.

$\langle 1 \rangle 4. \bar{\rho}(a, c) \leq \bar{\rho}(a, b) + \bar{\rho}(b, c)$

PROOF:

$$\begin{aligned} \bar{\rho}(a, c) &= \sup_{j \in J} \bar{d}(a_j, c_j) \\ &\leq \sup_{j \in J} (\bar{d}(a_j, b_j) + \bar{d}(b_j, c_j)) \\ &\leq \sup_{j \in J} \bar{d}(a_j, b_j) + \sup_{j \in J} \bar{d}(b_j, c_j) \\ &= \bar{\rho}(a, b) + \bar{\rho}(b, c) \end{aligned}$$

□

Proposition 177. The uniform topology on \mathbb{R}^J is finer than the product topology.

PROOF:

$\langle 1 \rangle 1.$ LET: $j \in J$ and U be open in \mathbb{R}

PROVE: $\pi_j^{-1}(U)$ is open in the uniform topology.

$\langle 1 \rangle 2.$ LET: $a \in \pi_j^{-1}(U)$

$\langle 1 \rangle 3.$ PICK $\epsilon > 0$ such that $(a_j - \epsilon, a_j + \epsilon) \subseteq U$

$\langle 1 \rangle 4. B_{\bar{\rho}}(a, \epsilon) \subseteq \pi_j^{-1}(U)$

□

Proposition 178. The uniform topology on \mathbb{R}^J is coarser than the box topology.

PROOF:

$\langle 1 \rangle 1.$ LET: $a \in \mathbb{R}^J$ and $\epsilon > 0$

PROVE: $B(a, \epsilon)$ is open in the box topology.

$\langle 1 \rangle 2.$ LET: $b \in B(a, \epsilon)$

$\langle 1 \rangle 3.$ For $j \in J$ we have $|a_j - b_j| < \epsilon$

$\langle 1 \rangle 4.$ For $j \in J$,

LET: $\delta_j = (\epsilon - |a_j - b_j|)/2$
 $\langle 1 \rangle 5. \prod_{j \in J} (b_j - \delta_j, b_j + \delta_j) \subseteq B(a, \epsilon)$
 \square

Proposition 179. *The uniform topology on \mathbb{R}^J is strictly finer than the product topology if and only if J is infinite.*

PROOF:

$\langle 1 \rangle 1.$ If J is finite then the uniform and product topologies coincide.

PROOF: The uniform, box and product topologies are all the same.

$\langle 1 \rangle 2.$ If J is infinite then the uniform and product topologies are different.

PROOF: The set $B(\vec{0}, 1)$ is open in the uniform topology but not the product topology.

\square

Proposition 180 (DC). *The uniform topology on \mathbb{R}^J is strictly coarser than the box topology if and only if J is infinite.*

PROOF:

$\langle 1 \rangle 1.$ If J is finite then the uniform and box topologies coincide.

PROOF: The uniform, box and product topologies are all the same.

$\langle 1 \rangle 2.$ If J is infinite then the uniform and box topologies are different.

PROOF: Pick an ω -sequence (j_1, j_2, \dots) in J . Let $U = \prod_{j \in J} U_j$ where $U_{j_i} = (-1/i, 1/i)$ and $U_j = (-1, 1)$ for all other j . Then $\vec{0} \in U$ but there is no $\epsilon > 0$ such that $B(\vec{0}, \epsilon) \subseteq U$.

\square

Proposition 181. *The closure of \mathbb{R}^∞ in \mathbb{R}^ω under the uniform topology is \mathbb{R}^ω .*

PROOF: Given any open ball $B(a, \epsilon)$, pick an integer N such that $1/\epsilon < N$. Then $B(a, \epsilon)$ includes sequences whose n th entry is 0 for all $n \geq N$. \square