The Universe

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Contents

Chapter 1

Category Theory

1.1 Categories

Definition 1.1 (Category)**.** A *category* $\mathcal{C}$ consists of:

- a class $|\mathcal{C}|$ of *objects*;
- for any objects $X, Y \in \mathcal{C}$, a set $\mathcal{C}[X, Y]$ of *morphisms*. We write $f : X \to Y$ for $f \in \mathcal{C}[X, Y]$
- for any object $X \in \mathcal{C}$, an *identity* morphism $\mathrm{id}_X : X \to X$
- for any morphisms $f : X \to Y$ and $g : Y \to Z$, a morphism $g \circ f : X \to Z$, the *composite* of f and g

such that:

Unit Laws For any $f : X \to Y$ we have $f = \mathrm{id}_Y \circ f = f \circ \mathrm{id}_X$

Associativity For any $f : X \to Y$, $g : Y \to Z$ and $h : Z \to W$, we have

$$h \circ (g \circ f) = (h \circ g) \circ f$$

Definition 1.2 (Category of Sets)**.** The *category of sets* **Set** is the category with objects all sets and morphisms all functions.

Definition 1.3 (Category of Pointed Sets)**.** The *category of pointed sets* $\mathbf{Set}_*$ is the category with objects all pairs (A, a) such that A is a set and $a \in A$; and morphisms $f : (A, a) \to (B, b)$ all functions $f : A \to B$ such that $f(a) = b$.

Definition 1.4 (Opposite Category)**.** Let $\mathcal{C}$ be a category. The *opposite* category $[\mathcal{C}]^{\mathrm{op}}$ is the category with $|[\mathcal{C}]^{\mathrm{op}}| = |\mathcal{C}|$ and $[\mathcal{C}]^{\mathrm{op}}[X, Y] = \mathcal{C}[Y, X]$.

Definition 1.5 (Section, Retraction)**.** Let $\mathcal{C}$ be a category. Let $r : A \to B$ and $s : B \to A$ in $\mathcal{C}$. Then r is a *retraction* of s, and s is a *section* of r, if and only if $r \circ s = \mathrm{id}_B$.

Definition 1.6 (Isomorphism)**.** Let $\mathcal{C}$ be a category. A morphism $f : A \to B$ is an *isomorphism* in $\mathcal{C}$, $f : A \cong B$, if and only if there exists a morphism $f^{-1} : B \to A$, the *inverse* of f, that is both a section and a retraction of f.

Two objects A and B are *isomorphic*, $A \cong B$, if and only if there exists an isomorphism between them.

Proposition 1.7. *The inverse of an isomorphism is unique.*

Proposition 1.8. *A morphism is an isomorphism if and only if it is both a section and a retraction.*

Proposition 1.9. *For any object X, we have $\mathrm{id}_X : X \cong X$ and ${\mathrm{id}_X}^{-1} = \mathrm{id}_X$.*

Proposition 1.10. *For any isomorphism $f : X \cong Y$ we have $f^{-1} : Y \cong X$ and $(f^{-1})^{-1} = f$.*

Proposition 1.11. *For any isomorphisms $f : X \cong Y$ and $g : Y \cong Z$ we have $g \circ f : X \cong Z$ and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.*

Proposition 1.12. *A function is an isomorphism in* **Set** *if and only if it is bijective.*

Theorem 1.13. *Let $\mathcal{C}$ be a category. Let $f : A \to B$ in $\mathcal{C}$. Then the following are equivalent.*

1. *f is an isomorphism.*
2. *For all X, the function $\mathcal{C}[f, -] : \mathcal{C}[B, X] \to \mathcal{C}[A, X]$ is a bijection.*
3. *For all X, the function $\mathcal{C}[-, f] : \mathcal{C}[X, A] \to \mathcal{C}[X, B]$ is a bijection.*

1.2 Functors

Definition 1.14 (Functor)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor* $F : \mathcal{C} \to \mathcal{D}$ consists of:

- for every object $A \in \mathcal{C}$, an object $FA \in \mathcal{D}$
- for every morphism $f : A \to B : \mathcal{C}$, a morphism $Ff : FA \to FB : \mathcal{D}$

such that

- for every object $A \in \mathcal{C}$, we have $F\mathrm{id}_A = \mathrm{id}_{FA}$
- for any morphisms $f : A \to B$ and $g : B \to C$ in $\mathcal{C}$, we have $F(g \circ f) = Fg \circ Ff$.

Definition 1.15 (Hom-Set Functor)**.** Let $\mathcal{C}$ be a category. The *hom-set functor* $\mathcal{C}[-, -] : [\mathcal{C}]^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ is the functor defined by:

- Given objects $A, B \in \mathcal{C}$, we have $\mathcal{C}[A, B]$ is the set of all morphisms from A to B.
- Given morphisms $f : A \to A'$ and $g : B \to B'$, then $\mathcal{C}[f, g] : \mathcal{C}[A', B] \to \mathcal{C}[A, B']$ is defined by
$$\mathcal{C}[f, g](h) = g \circ h \circ f$$

Chapter 2

Topology

2.1 Topologies and Topological Spaces

Definition 2.1 (Topology)**.** Let X be a set. A *topology* on X is a set $\mathcal{T} \subseteq \mathcal{P}X$ such that:

1. $X \in \mathcal{T}$
2. $\forall \mathcal{U} \subseteq \mathcal{T}. \bigcup \mathcal{U} \in \mathcal{T}$
3. $\forall U, V \in \mathcal{T}. U \cap V \in \mathcal{T}$

Definition 2.2 (Topological Space)**.** A *topological space* X consists of a set X and a topology $\mathcal{T}$ on X. We call the elements of X *points* and the elements of $\mathcal{T}$ *open sets*.

Definition 2.3 (Discrete Topology)**.** Let X be a set. The *discrete* topology on X is $\mathcal{P}X$.

Definition 2.4 (Indiscrete Topology)**.** Let X be a set. The *indiscrete* or *trivial* topology on X is $\{\emptyset, X\}$.

Definition 2.5 (Open Neighbourhood)**.** Let X be a topological space. Let $x \in X$ and $U \subseteq X$. Then U is an *open Neighbourhood* of x if and only if $x \in U$ and U is open.

Definition 2.6 (Coarser, Finer)**.** Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on the same set X. Then $\mathcal{T}$ is *coarser*, *smaller* or *weaker* than $\mathcal{T}'$, and $\mathcal{T}'$ is *finer*, *larger* or *stronger* than $\mathcal{T}$, if and only if $\mathcal{T} \subseteq \mathcal{T}'$.

Proposition 2.7. *Let X be a set. The intersection of a set of topologies on X is a topology on X.*

Corollary 2.7.1. *Let X be a set. The poset of topologies on X is a complete lattice.*

2.2 Closed Sets

Definition 2.8 (Closed Set)**.** Let X be a topological space and $C \subseteq X$. Then C is *closed* if and only if $X - C$ is open.

2.3 Basis for a Topology

Definition 2.9 (Basis for a Topology)**.** Let X be a set. A *basis* for a topology on X is a set $\mathcal{B} \subseteq \mathcal{P}X$ such that:

1. $\bigcup \mathcal{B} = X$
2. $\forall B_1, B_2 \in \mathcal{B}. \forall x \in B_1 \cap B_2. \exists B_3 \in \mathcal{B}. x \in B_3 \subseteq B_1 \cap B_2$

The topology *generated* by $\mathcal{B}$ is then the coarsest topology that includes $\mathcal{B}$.

Given $x \in X$, a *basic open neighbourhood* of x is a set $B \in \mathcal{B}$ such that $x \in B$.

Proposition 2.10. *Let X be a set and $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Let $U \subseteq X$. Then $U \in \mathcal{T}$ if and only if $\forall x \in U. \exists B \in \mathcal{B}. x \in B \subseteq U$.*

2.4 Continuous Functions

Definition 2.11 (Continuous)**.** Let X and Y be topological spaces and $f : X \to Y$. Then f is *continuous* if and only if, for any open set V in Y, we have $f^{-1}(V)$ is open in X.

Proposition 2.12. *For any topological space X, the identity function on X is continuous.*

Proposition 2.13. *Let X, Y and Z be topological spaces. Let $f : X \to Y$ and $g : Y \to Z$ be continuous functions. Then $g \circ f$ is continuous.*

Definition 2.14 (Category of Topological Spaces)**.** The *category of topological spaces* **Top** is the category with objects all topological spaces and morphisms all continuous functions.

Definition 2.15 (Category of Pointed Topological Spaces)**.** The *category of pointed topological spaces* $\mathbf{Top}_*$ is the category with objects all pairs (X, x) where X is a topological space and $x \in X$; and morphisms $f : (X, x) \to (Y, y)$ all continuous functions $f : X \to Y$ such that $f(x) = y$.

Definition 2.16 (Homeomorphism)**.** A *homeomorphism* is an isomorphism in **Top**.

Topological spaces are *homeomorphic* if and only if they are isomorphic in **Top**.

2.5 Homotopy Theory

Definition 2.17. Let **hTop** be the category with objects all topological spaces and morphisms $X \to Y$ all continuous functions $X \to Y$ quotiented by homotopy.

Definition 2.18 (Homotopy Equivalence)**.** A *homotopy equivalence* is an isomorphism in **hTop**.

Topological spaces are *homotopic* if and only if they are isomorphic in **hTop**.

Chapter 3

Metric Spaces

3.1 Metrics

Definition 3.1 (Metric, Metric Space)**.** Let X be a set. A *metric* on X is a function $d : X^2 \to \mathbb{R}$ such that:

1. $\forall x, y \in X . d(x, y) \geq 0$

2. $\forall x, y \in X . d(x, y) = 0 \Leftrightarrow x = y$

3. $\forall x, y \in X . d(x, y) = d(y, x)$

4. $\forall x, y, z \in X . d(x, z) \leq d(x, y) + d(y, z)$

A *metric space* X consists of a set X and a metric on X.

Definition 3.2 (Open Ball)**.** Let X be a metric space. Let $x \in X$ and $\epsilon > 0$. The *open ball* with *center* x and *radius* ϵ is $B(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon\}$.

Definition 3.3 (Metric Topology)**.** On any metric space, the *metric topology* is the topology generated by the basis consisting of the open balls.

Definition 3.4 (Metrizable)**.** A topological space X is *metrizable* if and only if there exists a metric d on X such that the topology on X is the metric topology induced by d.

Definition 3.5 (Euclidean Metric)**.** The *Euclidean metric* on $\mathbb{R}^n$ is defined by

$$d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2} \ .$$

We write just $\mathbb{R}^n$ for the metric space $\mathbb{R}^n$ under the Euclidean metric.

3.2 Subspaces

Proposition 3.6. *Let X be a set and $Y \subseteq X$. Let d be a metric on X. Then $d \upharpoonright Y^2$ is a metric on Y.*

Given a metric space (X, d) and a set $Y \subseteq X$, we will write just Y for the metric space $(Y, d \upharpoonright Y^2)$.

Definition 3.7 (Interval)**.** The *interval* I is the metric space $I = [0, 1]$ as a subspace of $\mathbb{R}$.

Definition 3.8 (Disk)**.** Let $n \in \mathbb{Z}^+$. The *n-disk* D^n is the metric space

$$D^n = \{x \in \mathbb{R}^n \mid d(x, 0) \leq 1\}$$

as a subspace of $\mathbb{R}^n$.

Definition 3.9 (Sphere)**.** Let $n \in \mathbb{Z}^+$. The *n-sphere* S^n is the metric space

$$D^n = \{x \in \mathbb{R}^{n+1} \mid d(x, 0) = 1\}$$

as a subspace of $\mathbb{R}^{n+1}$.

Proposition 3.10. *Boundedness is not a topological property. That is, there exist homeomorphic metric spaces such that one is bounded and the other is not.*

PROOF: We have $\mathbb{R}$ is complete but $(-1, 1)$ is not. $\square$

3.3 Complete Metric Spaces

Definition 3.11 (Complete)**.** A metric space is *complete* if and only if every Cauchy sequence converges.

Proposition 3.12. *Completeness is not a topological property. That is, there exist homeomorphic metric spaces such that one is complete and the other is not.*

PROOF: We have $\mathbb{R}$ is complete but $(-1, 1)$ is not. $\square$

Chapter 4

Group Theory

Definition 4.1 (Category of Groups)**.** The *category of groups* **Grp** is the category with objects all groups and morphisms all group homomorphisms.

Definition 4.2. We identify any group G with the category with one object $\bullet$ such that $G[\bullet, \bullet]$ is the set of elements of G and composition is the group multiplication.

Chapter 5

Ring Theory

5.1 Modules

Definition 5.1 (Category of Modules)**.** Let R be a ring. The *category of R-modules* $R-\mathbf{Mod}$ is the category with objects the modules over R and morphisms the R-linear maps.

Chapter 6

Linear Algebra

6.1 Vector Spaces

Definition 6.1 (Category of Vector Spaces)**.** Let K be a field. The *category of vector spaces* over K, $\mathbf{Vect}_K$, is the category with objects all vector spaces over K and morphisms all linear transformations.