Topology

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1 Topological Spaces

Definition 1 (Topology)**.** A *topology* on a set X is a set $\mathcal{T} \subseteq \mathcal{P}X$ such that:

- $X \in \mathcal{T}$.
- For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$.
- For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$.

We call the elements of X *points* and the elements of $\mathcal{T}$ *open sets*.

Definition 2 (Topological Space)**.** A *topological space* X consists of a set X and a topology on X.

Definition 3 (Discrete Space)**.** For any set X, the *discrete* topology on X is $\mathcal{P}X$.

Definition 4 (Indiscrete Space)**.** For any set X, the *indiscrete* or *trivial* topology on X is $\{\emptyset, X\}$.

Definition 5 (Finite Complement Topology)**.** For any set X, the *finite complement topology* on X is $\{U \in \mathcal{P}X \mid X \setminus U \text{ is finite}\} \cup \{\emptyset\}$.

Definition 6 (Countable Complement Topology)**.** For any set X, the *countable complement topology* on X is $\{U \in \mathcal{P}X \mid X \setminus U \text{ is countable}\} \cup \{\emptyset\}$.

Definition 7 (Finer, Coarser)**.** Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a given set X. If $\mathcal{T}' \supseteq \mathcal{T}$, we say that $\mathcal{T}'$ is *finer* than $\mathcal{T}$; if $\mathcal{T}'$ *properly* contains $\mathcal{T}$, we say that $\mathcal{T}'$ is *strictly* finer than $\mathcal{T}$. We also say that $\mathcal{T}$ is *coarser* than $\mathcal{T}'$, or *stricly* coarser, in these two respective situations. We say $\mathcal{T}$ is *comparable* with $\mathcal{T}'$ if either $\mathcal{T}' \supseteq \mathcal{T}$ or $\mathcal{T} \supseteq \mathcal{T}'$.

2 Basis for a Topology

Definition 8 (Basis)**.** If X is a set, a *basis* for a topology on X is a set $\mathcal{B} \subseteq \mathcal{P}X$ called *basis elements* such that

1. For all $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$.
2. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

If $\mathcal{B}$ satisfies these two conditions, then we define the topology *generated* by $\mathcal{B}$ to be $\mathcal{T} = \{U \in \mathcal{P}X \mid \forall x \in U.\exists B \in \mathcal{B}.x \in B \subseteq U\}$.

We prove this is a topology.

PROOF:
⟨1⟩1. $X \in \mathcal{T}$
  PROOF: For all $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B \subseteq X$ by condition 1.
⟨1⟩2. For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$
  ⟨2⟩1. LET: $\mathcal{U} \subseteq \mathcal{T}$
  ⟨2⟩2. LET: $x \in \bigcup \mathcal{U}$
  ⟨2⟩3. PICK $U \in \mathcal{U}$ such that $x \in U$
  ⟨2⟩4. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$
    PROOF: Since $U \in \mathcal{T}$ by ⟨2⟩1 and ⟨2⟩3.
  ⟨2⟩5. $x \in B \subseteq \bigcup \mathcal{U}$
⟨1⟩3. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$
  ⟨2⟩1. LET: $U, V \in \mathcal{T}$
  ⟨2⟩2. LET: $x \in U \cap V$
  ⟨2⟩3. PICK $B_1 \in \mathcal{B}$ such that $x \in B_1 \subseteq U$
  ⟨2⟩4. PICK $B_2 \in \mathcal{B}$ such that $x \in B_2 \subseteq V$
  ⟨2⟩5. PICK $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$
    PROOF: By condition 2.
  ⟨2⟩6. $x \in B_3 \subseteq U \cap V$

□

Lemma 1. *Let X be a set. Let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ is the set of all unions of subsets of $\mathcal{B}$.*

PROOF:
⟨1⟩1. For all $U \in \mathcal{T}$, there exists $\mathcal{A} \subseteq \mathcal{B}$ such that $U = \bigcup \mathcal{A}$
  ⟨2⟩1. LET: $U \in \mathcal{T}$
  ⟨2⟩2. LET: $\mathcal{A} = \{B \in \mathcal{B} \mid B \subseteq U\}$
  ⟨2⟩3. $U \subseteq \bigcup \mathcal{A}$
    ⟨3⟩1. LET: $x \in U$
    ⟨3⟩2. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$
      PROOF: Since $\mathcal{B}$ is a basis for $\mathcal{T}$.
    ⟨3⟩3. $x \in B \in \mathcal{A}$
  ⟨2⟩4. $\bigcup \mathcal{A} \subseteq U$
    PROOF: From the definition of $\mathcal{A}$ (⟨2⟩2).
⟨1⟩2. For all $\mathcal{A} \subseteq \mathcal{B}$ we have $\bigcup \mathcal{A} \in \mathcal{T}$
  ⟨2⟩1. $\mathcal{B} \subseteq \mathcal{T}$
    PROOF: If $B \in \mathcal{B}$ and $x \in B$, then there exists $B' \in \mathcal{B}$ such that $x \in B' \subseteq B$, namely $B' = B$.

$\langle 2 \rangle 2.$ Q.E.D.

PROOF: Since $\mathcal{T}$ is closed under union.

$\square$