

Topology

Robin Adams

May 22, 2022

1 Order Theory

Definition 1 (Preorder). Let X be a set. A *preorder* on X is a binary relation \leq on X such that:

Reflexivity For all $x \in X$, we have $x \leq x$

Transitivity For all $x, y, z \in X$, if $x \leq y$ and $y \leq z$ then $x \leq z$.

Definition 2 (Preordered Set). A *preordered set* consists of a set X and a preorder \leq on X .

Proposition 3. Let X and Y be linearly ordered sets. Let $f : X \rightarrow Y$ be strictly monotone and surjective. Then f is a poset isomorphism.

PROOF:

$\langle 1 \rangle 1.$ f is injective.

$\langle 2 \rangle 1.$ LET: $x, y \in X$

$\langle 2 \rangle 2.$ ASSUME: $f(x) = f(y)$

$\langle 2 \rangle 3.$ $x \not\leq y$

PROOF: By strong monotonicity.

$\langle 2 \rangle 4.$ $y \not\leq x$

PROOF: By strong monotonicity.

$\langle 2 \rangle 5.$ $x = y$

PROOF: By trichotomy.

$\langle 1 \rangle 2.$ f^{-1} is monotone.

$\langle 2 \rangle 1.$ LET: $x, y \in X$

$\langle 2 \rangle 2.$ ASSUME: $x \leq y$

$\langle 2 \rangle 3.$ $f^{-1}(x) \not\leq f^{-1}(y)$

PROOF: By strong monotonicity.

$\langle 2 \rangle 4.$ $f^{-1}(x) < f^{-1}(y)$

PROOF: By trichotomy.

□

Definition 4 (Interval). Let X be a preordered set and $Y \subseteq X$. Then Y is an *interval* if and only if, for all $a, b \in Y$ and $c \in X$, if $a \leq c \leq b$ then $c \in Y$.

Definition 5 (Linear Continuum). A linearly ordered set L is a *linear continuum* if and only if:

1. every nonempty subset of L that is bounded above has a supremum
2. L is dense

Proposition 6. Every interval in a linear continuum is a linear continuum.

PROOF:

$\langle 1 \rangle 1.$ LET: L be a linear continuum and I an interval in L .

$\langle 1 \rangle 2.$ Every nonempty subset of I that is bounded above has a supremum in I .

$\langle 2 \rangle 1.$ LET: $X \subseteq I$ be nonempty and bounded above by $b \in I$.

$\langle 2 \rangle 2$. LET: s be the supremum of X in L .
 PROOF: Since L is a linear continuum.
 $\langle 2 \rangle 3$. $s \in I$
 $\langle 3 \rangle 1$. PICK $a \in X$
 PROOF: Since X is nonempty ($\langle 2 \rangle 1$).
 $\langle 3 \rangle 2$. $a \leq s \leq b$
 $\langle 3 \rangle 3$. $s \in I$
 PROOF: Since I is an interval ($\langle 1 \rangle 1$).
 $\langle 2 \rangle 4$. s is the supremum of X in I
 $\langle 1 \rangle 3$. I is dense.
 $\langle 2 \rangle 1$. LET: $x, y \in I$ with $x < y$
 $\langle 2 \rangle 2$. PICK $z \in L$ with $x < z < y$
 PROOF: Since L is dense.
 $\langle 2 \rangle 3$. $z \in I$
 PROOF: Since I is an interval.

□

Definition 7 (Ordered Square). The *ordered square* I_o^2 is the set $[0, 1]^2$ under the dictionary order.

Proposition 8. *The ordered square is a linear continuum.*

PROOF:

$\langle 1 \rangle 1$. Every nonempty subset of I_o^2 bounded above has a supremum.
 $\langle 2 \rangle 1$. LET: $X \subseteq I_o^2$ be nonempty and bounded above by (b, c)
 $\langle 2 \rangle 2$. LET: $s = \sup \pi_1(X)$
 PROOF: The set $\pi_1(X)$ is nonempty and bounded above by b .
 $\langle 2 \rangle 3$. CASE: $s \in \pi_1(X)$
 $\langle 3 \rangle 1$. LET: $t = \sup \{y \in [0, 1] \mid (s, y) \in X\}$
 PROOF: This set is nonempty and bounded above by c .
 $\langle 3 \rangle 2$. (s, t) is the supremum of X .
 $\langle 2 \rangle 4$. CASE: $s \notin \pi_1(X)$
 PROOF: In this case $(s, 0)$ is the supremum of X .
 $\langle 1 \rangle 2$. I_o^2 is dense.
 $\langle 2 \rangle 1$. LET: $(x_1, y_1), (x_2, y_2) \in I_o^2$ with $(x_1, y_1) < (x_2, y_2)$
 $\langle 2 \rangle 2$. CASE: $x_1 < x_2$
 $\langle 3 \rangle 1$. PICK x_3 with $x_1 < x_3 < x_2$
 $\langle 3 \rangle 2$. $(x_1, y_1) < (x_3, y_1) < (x_2, y_2)$
 $\langle 2 \rangle 3$. CASE: $x_1 = x_2$ and $y_1 < y_2$
 $\langle 3 \rangle 1$. PICK y_3 with $y_1 < y_3 < y_2$
 $\langle 3 \rangle 2$. $(x_1, y_1) < (x_1, y_3) < (x_2, y_2)$

□

Proposition 9. *If X is a well-ordered set then $X \times [0, 1)$ under the dictionary order is a linear continuum.*

PROOF:

$\langle 1 \rangle 1$. Every nonempty set $A \subseteq X \times [0, 1)$ bounded above has a supremum.

- ⟨2⟩1. LET: $A \subseteq X \times [0, 1)$ be nonempty and bounded above
- ⟨2⟩2. LET: x_0 be the supremum of $\pi_1(A)$
- ⟨2⟩3. CASE: $x_0 \in \pi_1(A)$
 - ⟨3⟩1. LET: y_0 be the supremum of $\{y \in [0, 1) \mid (x_0, y) \in A\}$
 - ⟨3⟩2. (x_0, y_0) is the supremum of A .
- ⟨2⟩4. CASE: $x_0 \notin \pi_1(A)$
 - PROOF: In this case $(x_0, 0)$ is the supremum of A .
- ⟨1⟩2. $X \times [0, 1)$ is dense.
 - ⟨2⟩1. LET: $(x_1, y_1), (x_2, y_2) \in X \times [0, 1)$ with $(x_1, y_1) < (x_2, y_2)$
 - ⟨2⟩2. CASE: $x_1 < x_2$
 - ⟨3⟩1. PICK y_3 such that $y_1 < y_3 < 1$
 - ⟨3⟩2. $(x_1, y_1) < (x_1, y_3) < (x_2, y_2)$
 - ⟨2⟩3. CASE: $x_1 = x_2$ and $y_1 < y_2$
 - ⟨3⟩1. PICK y_3 such that $y_1 < y_3 < y_2$
 - ⟨3⟩2. $(x_1, y_1) < (x_1, y_3) < (x_2, y_2)$

□

Lemma 10. For all $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$, we have $[a, b) \cong [c, d)$

PROOF: The map $\lambda t.c + (t - a)(d - c)/(b - a)$ is an order isomorphism.

Proposition 11. Let X be a linearly ordered set. Let $a < b < c$ in X . Then $[a, c) \cong [0, 1)$ if and only if $[a, b) \cong [b, c) \cong [0, 1)$.

PROOF:

- ⟨1⟩1. If $[a, c) \cong [0, 1)$ then $[a, b) \cong [b, c) \cong [0, 1)$
- ⟨2⟩1. ASSUME: $f : [a, c) \cong [0, 1)$ is an order isomorphism
- ⟨2⟩2. $[a, b) \cong [0, 1)$
 - PROOF:
$$\begin{aligned} [a, b) &\cong [0, f(b)) && \text{(by the restriction of } f) \\ &\cong [0, 1) && \text{(Lemma 10)} \end{aligned}$$
- ⟨2⟩3. $[b, c) \cong [0, 1)$
 - PROOF: Similar.
- ⟨1⟩2. If $[a, b) \cong [b, c) \cong [0, 1)$ then $[a, c) \cong [0, 1)$
 - PROOF:
$$\begin{aligned} [a, c) &= [a, b) * [b, c) \\ &\cong [0, 1) * [0, 1) \\ &\cong [0, 1/2) * [1/2, 1) && \text{(Lemma 10)} \\ &= 1 \end{aligned}$$

□

Proposition 12 (CC). Let X be a linearly ordered set. Let $x_0 < x_1 < \dots$ be a strictly increasing sequence in X with supremum b . Then $[x_0, b) \cong [0, 1)$ if and only if $[x_i, x_{i+1}) \cong [0, 1)$ for all i .

PROOF:

- ⟨1⟩1. If $[x_0, b) \cong [0, 1)$ then $[x_i, x_{i+1}) \cong [0, 1)$ for all i .

PROOF: By Lemma 10

$\langle 1 \rangle 2$. If $[x_i, x_{i+1}) \cong [0, 1)$ for all i then $[x_0, b) \cong [0, 1)$

$\langle 2 \rangle 1$. ASSUME: $[x_i, x_{i+1}) \cong [0, 1)$ for all i

$\langle 2 \rangle 2$. PICK an order isomorphism $f_i : [x_i, x_{i+1}) \cong [1/2^i, 2/2^{i+1})$ for each i .

PROOF: By Lemma 10

$\langle 2 \rangle 3$. The union of the f_i s is an order isomorphism $[x_0, b) \cong [0, 1)$

□

2 Real Analysis

Definition 13. Let \mathbb{R}^∞ be the subset of \mathbb{R}^ω consisting of all sequences (a_n) such that $a_n = 0$ for all but finitely many n .

3 Group Theory

Definition 14. Given a group G and sets $A, B \subseteq G$, let $AB = \{ab \mid a \in A, b \in B\}$.

Definition 15. Given a group G and a set $A \subseteq G$, let $A^{-1} = \{a^{-1} \mid a \in A\}$.

4 Topological Spaces

Definition 16 (Topology). A *topology* on a set X is a set $\mathcal{T} \subseteq \mathcal{P}X$ such that:

- $X \in \mathcal{T}$.
- For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$.
- For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$.

We call the elements of X *points* and the elements of \mathcal{T} *open sets*.

Definition 17 (Topological Space). A *topological space* X consists of a set X and a topology on X .

Definition 18 (Discrete Space). For any set X , the *discrete* topology on X is $\mathcal{P}X$.

Definition 19 (Indiscrete Space). For any set X , the *indiscrete* or *trivial* topology on X is $\{\emptyset, X\}$.

Definition 20 (Finite Complement Topology). For any set X , the *finite complement topology* on X is $\{U \in \mathcal{P}X \mid X \setminus U \text{ is finite}\} \cup \{\emptyset\}$.

Definition 21 (Countable Complement Topology). For any set X , the *countable complement topology* on X is $\{U \in \mathcal{P}X \mid X \setminus U \text{ is countable}\} \cup \{\emptyset\}$.

Definition 22 (Finer, Coarser). Suppose that \mathcal{T} and \mathcal{T}' are two topologies on a given set X . If $\mathcal{T}' \supseteq \mathcal{T}$, we say that \mathcal{T}' is *finer* than \mathcal{T} ; if \mathcal{T}' *properly* contains \mathcal{T} , we say that \mathcal{T}' is *strictly finer* than \mathcal{T} . We also say that \mathcal{T} is *coarser* than \mathcal{T}' , or *strictly coarser*, in these two respective situations. We say \mathcal{T} is *comparable* with \mathcal{T}' if either $\mathcal{T}' \supseteq \mathcal{T}$ or $\mathcal{T} \supseteq \mathcal{T}'$.

Lemma 23. Let X be a topological space and $U \subseteq X$. Then U is open if and only if, for all $x \in U$, there exists an open set V such that $x \in V \subseteq U$.

PROOF:

$\langle 1 \rangle 1. \Rightarrow$

PROOF: Take $V = U$

$\langle 1 \rangle 2. \Leftarrow$

PROOF: We have $U = \bigcup \{V \text{ open in } X \mid V \subseteq U\}$.

□

Lemma 24. Let X be a set and \mathcal{T} a nonempty set of topologies on X . Then $\bigcap \mathcal{T}$ is a topology on X , and is the finest topology that is coarser than every member of \mathcal{T} .

PROOF:

$\langle 1 \rangle 1. X \in \bigcap \mathcal{T}$

PROOF: Since X is in every member of \mathcal{T} .

$\langle 1 \rangle 2. \bigcap \mathcal{T}$ is closed under union.

- ⟨2⟩1. LET: $\mathcal{U} \subseteq \bigcap \mathcal{T}$
- ⟨2⟩2. For all $T \in \mathcal{T}$ we have $\mathcal{U} \subseteq T$
- ⟨2⟩3. For all $T \in \mathcal{T}$ we have $\bigcup \mathcal{U} \in T$
- ⟨2⟩4. $\bigcup \mathcal{U} \in \bigcap \mathcal{T}$
- ⟨1⟩3. $\bigcap \mathcal{T}$ is closed under binary intersection.
- ⟨2⟩1. LET: $U, V \in \bigcap \mathcal{T}$
- ⟨2⟩2. For all $T \in \mathcal{T}$ we have $U, V \in T$
- ⟨2⟩3. For all $T \in \mathcal{T}$ we have $U \cap V \in T$
- ⟨2⟩4. $U \cap V \in \bigcap \mathcal{T}$

□

Lemma 25. *Let X be a set and \mathcal{T} a set of topologies on X . Then there exists a unique coarsest topology that is finer than every member of \mathcal{T} .*

PROOF: The required topology is given by

$$\bigcap \{T \in \mathcal{P}X \mid T \text{ is a topology on } X \text{ that is finer than every member of } \mathcal{T}\},$$

The set is nonempty since it contains the discrete topology. □

Definition 26 (Neighbourhood). A *neighbourhood* of a point x is an open set that contains x .

5 Closed Set

Definition 27 (Closed Set). Let X be a topological space and $A \subseteq X$. Then A is *closed* if and only if $X \setminus A$ is open.

Lemma 28. *The empty set is closed.*

PROOF: Since the whole space X is always open. □

Lemma 29. *The topological space X is closed.*

PROOF: Since \emptyset is open. □

Lemma 30. *The intersection of a nonempty set of closed sets is closed.*

PROOF: Let \mathcal{C} be a nonempty set of closed sets. Then $X \setminus \bigcap \mathcal{C} = \bigcup \{X \setminus C \mid C \in \mathcal{C}\}$ is open. □

Lemma 31. *The union of two closed sets is closed.*

PROOF: Let C and D be closed. Then $X \setminus (C \cup D) = (X \setminus C) \cap (X \setminus D)$ is open. □

Proposition 32. *Let X be a set and $\mathcal{C} \subseteq \mathcal{P}X$ a set such that:*

1. $\emptyset \in \mathcal{C}$
2. $X \in \mathcal{C}$
3. For all $\mathcal{A} \subseteq \mathcal{C}$ nonempty we have $\bigcap \mathcal{A} \in \mathcal{C}$

4. For all $C, D \in \mathcal{C}$ we have $C \cup D \in \mathcal{C}$.

Then there exists a unique topology \mathcal{T} such that \mathcal{C} is the set of closed sets, namely

$$\mathcal{T} = \{X \setminus C \mid C \in \mathcal{C}\}$$

PROOF:

$\langle 1 \rangle 1$. LET: $\mathcal{T} = \{X \setminus C \mid C \in \mathcal{C}\}$

$\langle 1 \rangle 2$. \mathcal{T} is a topology

$\langle 2 \rangle 1$. $X \in \mathcal{T}$

PROOF: Since $\emptyset \in \mathcal{C}$

$\langle 2 \rangle 2$. For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$

$\langle 3 \rangle 1$. LET: $\mathcal{U} \subseteq \mathcal{T}$

$\langle 3 \rangle 2$. CASE: $\mathcal{U} = \emptyset$

PROOF: In this case $\bigcup \mathcal{U} = \emptyset \in \mathcal{T}$ since $X \in \mathcal{C}$

$\langle 3 \rangle 3$. CASE: $\mathcal{U} \neq \emptyset$

PROOF: In this case $X \setminus \bigcup \mathcal{U} = \bigcap \{X \setminus U \mid U \in \mathcal{U}\} \in \mathcal{C}$.

$\langle 2 \rangle 3$. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$

PROOF: Since $X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V) \in \mathcal{C}$.

$\langle 1 \rangle 3$. \mathcal{C} is the set of all closed sets in \mathcal{T}

PROOF:

C is closed in \mathcal{T}

$$\Leftrightarrow X \setminus C \in \mathcal{T}$$

$$\Leftrightarrow C \in \mathcal{C}$$

$\langle 1 \rangle 4$. If \mathcal{T}' is a topology and \mathcal{C} is the set of closed sets in \mathcal{T}' then $\mathcal{T}' = \mathcal{T}$

PROOF: We have

$$U \in \mathcal{T}$$

$$\Leftrightarrow X \setminus U \in \mathcal{C}$$

$$\Leftrightarrow X \setminus U \text{ is closed in } \mathcal{T}'$$

$$\Leftrightarrow U \in \mathcal{T}'$$

□

Proposition 33. If U is open and A is closed then $U \setminus A$ is open.

PROOF: $U \setminus A = U \cap (X \setminus A)$ is the intersection of two open sets. □

Proposition 34. If U is open and A is closed then $A \setminus U$ is closed.

PROOF: $A \setminus U = A \cap (X \setminus U)$ is the intersection of two closed sets. □

6 Interior

Definition 35 (Interior). Let X be a topological space and $A \subseteq X$. The *interior* of A , $\text{Int } A$, is the union of all the open subsets of A .

Lemma 36. The interior of a set is open.

PROOF: It is a union of open sets. \square

Lemma 37.

$$\text{Int } A \subseteq A$$

PROOF: Immediate from definition. \square

Lemma 38. *If U is open and $U \subseteq A$ then $U \subseteq \text{Int } A$*

PROOF: Immediate from definition. \square

Lemma 39. *A set A is open if and only if $A = \text{Int } A$.*

PROOF: If $A = \text{Int } A$ then A is open by Lemma 36. Conversely if A is open then $A \subseteq \text{Int } A$ by the definition of interior and so $A = \text{Int } A$.

7 Closure

Definition 40 (Closure). Let X be a topological space and $A \subseteq X$. The *closure* of A , \overline{A} , is the intersection of all the closed sets that include A .

This intersection exists since X is a closed set that includes A (Lemma 29).

Lemma 41. *The closure of a set is closed.*

PROOF: Dual to Lemma 36. \square

Lemma 42.

$$A \subseteq \overline{A}$$

PROOF: Immediate from definition. \square

Lemma 43. *If C is closed and $A \subseteq C$ then $\overline{A} \subseteq C$.*

PROOF: Immediate from definition. \square

Lemma 44. *A set A is closed if and only if $A = \overline{A}$.*

PROOF: Dual to Lemma 39. \square

Theorem 45. *Let X be a topological space, $A \subseteq X$ and $x \in X$. Then $x \in \overline{A}$ if and only if every neighbourhood of x intersects A .*

PROOF: We have

$$\begin{aligned} x \in \overline{A} \\ \Leftrightarrow \forall C. C \text{ closed} \wedge A \subseteq C \Rightarrow x \in C \\ \Leftrightarrow \forall U. U \text{ open} \wedge A \cap U = \emptyset \Rightarrow x \notin U \\ \Leftrightarrow \forall U. U \text{ open} \wedge x \in U \Rightarrow U \text{ intersects } A \end{aligned} \quad \square$$

Proposition 46. *If $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$.*

PROOF: This holds because \overline{B} is a closed set that includes A . \square

Proposition 47.

$$\overline{A \cup B} = \overline{A} \cup \overline{B}$$

PROOF:

$\langle 1 \rangle 1. \overline{A} \subseteq \overline{A \cup B}$

PROOF: By Proposition 46.

$\langle 1 \rangle 2. \overline{B} \subseteq \overline{A \cup B}$

PROOF: By Proposition 46.

$\langle 1 \rangle 3. \overline{A \cup B} \subseteq \overline{A \cup B}$

$\langle 2 \rangle 1. \text{ LET: } x \in \overline{A \cup B}$

$\langle 2 \rangle 2. \text{ ASSUME: } x \notin \overline{A}$

PROVE: $x \in \overline{B}$

$\langle 2 \rangle 3. \text{ PICK a neighbourhood } U \text{ of } x \text{ that does not intersect } A$

$\langle 2 \rangle 4. \text{ LET: } V \text{ be any neighbourhood of } x$

$\langle 2 \rangle 5. U \cap V \text{ is a neighbourhood of } x$

$\langle 2 \rangle 6. U \cap V \text{ intersects } A \cup B$

PROOF: From $\langle 2 \rangle 1$ and Theorem 45.

$\langle 2 \rangle 7. U \cap V \text{ intersects } B$

PROOF: From $\langle 2 \rangle 3$

$\langle 2 \rangle 8. V \text{ intersects } B$

$\langle 2 \rangle 9. \text{ Q.E.D.}$

PROOF: We have $x \in \overline{B}$ from Theorem 45.

\square

8 Boundary

Definition 48 (Boundary). The *boundary* of a set A is the set $\partial A = \overline{A} \cap \overline{X \setminus A}$.

Proposition 49.

$$\text{Int } A \cap \partial A = \emptyset$$

PROOF: Since $\overline{X \setminus A} = X \setminus \text{Int } A$. \square

Proposition 50.

$$\overline{A} = \text{Int } A \cup \partial A$$

PROOF:

$$\begin{aligned} \text{Int } A \cup \partial A &= \text{Int } A \cup (\overline{A} \cap \overline{X \setminus A}) \\ &= (\text{Int } A \cup \overline{A}) \cap (\text{Int } A \cup \overline{X \setminus A}) \\ &= \overline{A} \cap X \\ &= \overline{A} \end{aligned}$$

Proposition 51. $\partial A = \emptyset$ if and only if A is open and closed.

PROOF: If $\partial A = \emptyset$ then $\overline{A} = \text{Int } A$ by Proposition 50.

Proposition 52. *A set U is open if and only if $\partial U = \overline{U} \setminus U$.*

PROOF:

$$\begin{aligned} \partial U &= \overline{U} \setminus U \\ \Leftrightarrow \overline{U} \setminus \text{Int } U &= \overline{U} \setminus U && (\text{Propositions 49, 50}) \\ \Leftrightarrow \text{Int } U &= U && \square \end{aligned}$$

9 Limit Points

Definition 53 (Limit Point). Let X be a topological space, $a \in X$ and $A \subseteq X$. Then a is a *limit point*, *cluster point* or *point of accumulation* for A if and only if every neighbourhood of a intersects A at a point other than a .

Lemma 54. *The point a is an accumulation point for A if and only if $a \in \overline{A \setminus \{a\}}$.*

PROOF: From Theorem 45. \square

Theorem 55. *Let X be a topological space and $A \subseteq X$. Let A' be the set of all limit points of A . Then $\overline{A} = A \cup A'$.*

PROOF:

$\langle 1 \rangle 1$. For all $x \in \overline{A}$, if $x \notin A$ then $x \in A'$

PROOF: From Theorem 45.

$\langle 1 \rangle 2$. $A \subseteq \overline{A}$

PROOF: Lemma 42.

$\langle 1 \rangle 3$. $A' \subseteq \overline{A}$

PROOF: From Theorem 45.

\square

Corollary 55.1. *A set is closed if and only if it contains all its limit points.*

Proposition 56. *In an indiscrete topology, every point is a limit point of any set that has more than one point.*

PROOF: Let X be an indiscrete space. Let A be a set with more than one point and x be a point. The only neighbourhood of x is X , which must intersect A at a point other than x . \square

Lemma 57. *Let X be a topological space and $A \subseteq B \subseteq X$. Then every limit point of A is a limit point of B .*

PROOF: Immediate from definitions. \square

10 Basis for a Topology

Definition 58 (Basis). If X is a set, a *basis* for a topology on X is a set $\mathcal{B} \subseteq \mathcal{P}X$ called *basis elements* such that

1. For all $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$.
2. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

If \mathcal{B} satisfies these two conditions, then we define the topology *generated* by \mathcal{B} to be $\mathcal{T} = \{U \in \mathcal{P}X \mid \forall x \in U. \exists B \in \mathcal{B}. x \in B \subseteq U\}$.

We prove this is a topology.

PROOF:

$\langle 1 \rangle 1.$ $X \in \mathcal{T}$

PROOF: For all $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B \subseteq X$ by condition 1.

$\langle 1 \rangle 2.$ For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$

$\langle 2 \rangle 1.$ LET: $\mathcal{U} \subseteq \mathcal{T}$

$\langle 2 \rangle 2.$ LET: $x \in \bigcup \mathcal{U}$

$\langle 2 \rangle 3.$ PICK $U \in \mathcal{U}$ such that $x \in U$

$\langle 2 \rangle 4.$ PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$

PROOF: Since $U \in \mathcal{T}$ by $\langle 2 \rangle 1$ and $\langle 2 \rangle 3$.

$\langle 2 \rangle 5.$ $x \in B \subseteq \bigcup \mathcal{U}$

$\langle 1 \rangle 3.$ For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$

$\langle 2 \rangle 1.$ LET: $U, V \in \mathcal{T}$

$\langle 2 \rangle 2.$ LET: $x \in U \cap V$

$\langle 2 \rangle 3.$ PICK $B_1 \in \mathcal{B}$ such that $x \in B_1 \subseteq U$

$\langle 2 \rangle 4.$ PICK $B_2 \in \mathcal{B}$ such that $x \in B_2 \subseteq V$

$\langle 2 \rangle 5.$ PICK $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$

PROOF: By condition 2.

$\langle 2 \rangle 6.$ $x \in B_3 \subseteq U \cap V$

□

Lemma 59. *Let X be a set. Let \mathcal{B} be a basis for a topology \mathcal{T} on X . Then \mathcal{T} is the set of all unions of subsets of \mathcal{B} .*

PROOF:

$\langle 1 \rangle 1.$ For all $U \in \mathcal{T}$, there exists $\mathcal{A} \subseteq \mathcal{B}$ such that $U = \bigcup \mathcal{A}$

$\langle 2 \rangle 1.$ LET: $U \in \mathcal{T}$

$\langle 2 \rangle 2.$ LET: $\mathcal{A} = \{B \in \mathcal{B} \mid B \subseteq U\}$

$\langle 2 \rangle 3.$ $U \subseteq \bigcup \mathcal{A}$

$\langle 3 \rangle 1.$ LET: $x \in U$

$\langle 3 \rangle 2.$ PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$

PROOF: Since \mathcal{B} is a basis for \mathcal{T} .

$\langle 3 \rangle 3.$ $x \in B \in \mathcal{A}$

$\langle 2 \rangle 4.$ $\bigcup \mathcal{A} \subseteq U$

PROOF: From the definition of \mathcal{A} ($\langle 2 \rangle 2$).

$\langle 1 \rangle 2.$ For all $\mathcal{A} \subseteq \mathcal{B}$ we have $\bigcup \mathcal{A} \in \mathcal{T}$

$\langle 2 \rangle 1.$ $\bigcup \mathcal{A} \in \mathcal{T}$

PROOF: If $B \in \mathcal{B}$ and $x \in B$, then there exists $B' \in \mathcal{B}$ such that $x \in B' \subseteq B$, namely $B' = B$.

⟨2⟩2. Q.E.D.

PROOF: Since \mathcal{T} is closed under union.

□

Corollary 59.1. *Let X be a set. Let \mathcal{B} be a basis for a topology \mathcal{T} on X . Then \mathcal{T} is the coarsest topology that includes \mathcal{B} .*

PROOF: Since every topology that includes \mathcal{B} includes all unions of subsets of \mathcal{B} . □

Lemma 60. *Let X be a topological space. Suppose that \mathcal{C} is a set of open sets such that, for every open set U and every point $x \in U$, there exists $C \in \mathcal{C}$ such that $x \in C \subseteq U$. Then \mathcal{C} is a basis for the topology on X .*

PROOF:

⟨1⟩1. For all $x \in X$, there exists $C \in \mathcal{C}$ such that $x \in C$

PROOF: Immediate from hypothesis.

⟨1⟩2. For all $C_1, C_2 \in \mathcal{C}$ and $x \in C_1 \cap C_2$, there exists $C_3 \in \mathcal{C}$ such that $x \in C_3 \subseteq C_1 \cap C_2$

PROOF: Since $C_1 \cap C_2$ is open.

⟨1⟩3. Every open set is open in the topology generated by \mathcal{C}

PROOF: Immediate from hypothesis.

⟨1⟩4. Every union of a subset of \mathcal{C} is open.

PROOF: Since every member of \mathcal{C} is open.

□

Lemma 61. *Let \mathcal{B} and \mathcal{B}' be bases for the topologies \mathcal{T} and \mathcal{T}' respectively on the set X . Then the following are equivalent.*

1. $\mathcal{T} \subseteq \mathcal{T}'$

2. For all $B \in \mathcal{B}$ and $x \in B$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.

PROOF:

⟨1⟩1. $1 \Rightarrow 2$

⟨2⟩1. ASSUME: $\mathcal{T} \subseteq \mathcal{T}'$

⟨2⟩2. LET: $B \in \mathcal{B}$ and $x \in B$

⟨2⟩3. $B \in \mathcal{T}$

PROOF: Corollary 59.1.

⟨2⟩4. $B \in \mathcal{T}'$

PROOF: By ⟨2⟩1

⟨2⟩5. There exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$

PROOF: Since \mathcal{B}' is a basis for \mathcal{T}' .

⟨1⟩2. $2 \Rightarrow 1$

⟨2⟩1. ASSUME: 2

⟨2⟩2. LET: $U \in \mathcal{T}$

PROVE: $U \in \mathcal{T}'$

⟨2⟩3. LET: $x \in U$

PROVE: There exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq U$

- $\langle 2 \rangle 4$. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$
 PROOF: Since \mathcal{B} is a basis for \mathcal{T} .
 $\langle 2 \rangle 5$. PICK $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$
 PROOF: By $\langle 2 \rangle 1$.
 $\langle 2 \rangle 6$. $x \in B' \subseteq U$

□

Theorem 62. *Let X be a topological space, $A \subseteq X$ and $x \in X$. Let \mathcal{B} be a basis for X . Then $x \in \overline{A}$ if and only if, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A .*

PROOF:

- $\langle 1 \rangle 1$. If $x \in \overline{A}$ then, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A .
 PROOF: This follows from Theorem 45 since every element of \mathcal{B} is open (Corollary 59.1).
 $\langle 1 \rangle 2$. Suppose that, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A . Then $x \in \overline{A}$.
 $\langle 2 \rangle 1$. ASSUME: For all $B \in \mathcal{B}$, if $x \in B$ then B intersects A .
 $\langle 2 \rangle 2$. LET: U be an open set that contains x
 PROVE: U intersects A .
 $\langle 2 \rangle 3$. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$.
 $\langle 2 \rangle 4$. B intersects A .
 PROOF: From $\langle 2 \rangle 1$.
 $\langle 2 \rangle 5$. U intersects A .
 $\langle 2 \rangle 6$. Q.E.D.
 PROOF: By Theorem 45.

□

Definition 63 (Lower Limit Topology on the Real Line). The *lower limit topology on the real line* is the topology on \mathbb{R} generated by the basis consisting of all half-open intervals of the form $[a, b)$.

We write \mathbb{R}_l for the topological space \mathbb{R} under the lower limit topology.

We prove this is a basis for a topology.

PROOF:

- $\langle 1 \rangle 1$. For all $x \in \mathbb{R}$ there exists an interval $[a, b)$ such that $x \in [a, b)$.
 PROOF: Take $[a, b) = [x, x + 1)$.
 $\langle 1 \rangle 2$. For any open intervals $[a, b)$, $[c, d)$ if $x \in [a, b) \cap [c, d)$, then there exists an interval $[e, f)$ such that $x \in [e, f) \subseteq [a, b) \cap [c, d)$
 PROOF: Take $[e, f) = [\max(a, c), \min(b, d))$.

□

Definition 64 (K -topology on the Real Line). Let $K = \{1/n \mid n \in \mathbb{Z}^+\}$.

The *K -topology on the real line* is the topology on \mathbb{R} generated by the basis consisting of all open intervals (a, b) and all sets of the form $(a, b) \setminus K$.

We write \mathbb{R}_K for the topological space \mathbb{R} under the K -topology.

We prove this is a basis for a topology.

PROOF:

- ⟨1⟩1. For all $x \in \mathbb{R}$ there exists an open interval (a, b) such that $x \in (a, b)$.
PROOF: Take $(a, b) = (x - 1, x + 1)$.
- ⟨1⟩2. For any basic open sets B_1, B_2 if $x \in B_1 \cap B_2$, then there exists a basic open set B_3 such that $x \in B_3 \subseteq B_1 \cap B_2$
- ⟨2⟩1. CASE: $B_1 = (a, b), B_2 = (c, d)$
PROOF: Take $B_3 = (\max(a, c), \min(b, d))$.
- ⟨2⟩2. CASE: $B_1 = (a, b)$ or $(a, b) \setminus K, B_2 = (c, d)$ or $(c, d) \setminus K$, and they are not both open intervals.
PROOF: Take $B_3 = (\max(a, c), \min(b, d)) \setminus K$.

□

Lemma 65. *The lower limit topology and the K -topology are incomparable.*

PROOF:

- ⟨1⟩1. The interval $[10, 11)$ is not open in the K -topology.
PROOF: There is no open interval (a, b) such that $10 \in (a, b) \subseteq [10, 11)$ or $10 \in (a, b) \setminus K \subseteq [10, 11)$.
- ⟨1⟩2. The set $(-1, 1) \setminus K$ is not open in the lower limit topology.
PROOF: There is no half-open interval $[a, b)$ such that $0 \in [a, b) \subseteq (-1, 1) \setminus K$, since there must be a positive integer n with $1/n \in [a, b)$.

□

Definition 66 (Subbasis). A *subbasis* \mathcal{S} for a topology on X is a set $\mathcal{S} \subseteq \mathcal{P}X$ such that $\bigcup \mathcal{S} = X$.

The topology *generated* by the subbasis \mathcal{S} is the set of all unions of finite intersections of elements of \mathcal{S} .

We prove this is a topology.

PROOF:

- ⟨1⟩1. The set \mathcal{B} of all finite intersections of elements of \mathcal{S} forms a basis for a topology on X .
⟨2⟩1. $\bigcup \mathcal{B} = X$
PROOF: Since $\mathcal{S} \subseteq \mathcal{B}$.
- ⟨2⟩2. \mathcal{B} is closed under binary intersection.
PROOF: By definition.
- ⟨1⟩2. Q.E.D.
PROOF: By Lemma 59.

□

We have simultaneously proved:

Proposition 67. *Let \mathcal{S} be a subbasis for the topology on X . Then the set of all finite intersections of elements of \mathcal{S} is a basis for the topology on X .*

Proposition 68. *Let X be a set. Let \mathcal{S} be a subbasis for a topology \mathcal{T} on X . Then \mathcal{T} is the coarsest topology that includes \mathcal{S} .*

PROOF: Since every topology that includes \mathcal{S} includes every union of finite intersections of elements of \mathcal{S} . □

11 Local Basis at a Point

Definition 69 (Local Basis). Let X be a topological space and $a \in X$. A *(local) basis at a* is a set \mathcal{B} of neighbourhoods of a such that every neighbourhood of a includes some member of \mathcal{B} .

Lemma 70. *If there exists a countable local basis at a point a , then there exists a countable local basis $\{B_n \mid n \geq 1\}$ such that $B_1 \supseteq B_2 \supseteq \dots$.*

PROOF: Given a countable local basis $\{C_n \mid n \geq 1\}$, take $B_n = C_1 \cap \dots \cap C_n$. \square

12 Convergence

Definition 71 (Convergence). Let X be a topological space. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of points in X and $l \in X$. Then the sequence $(a_n)_{n \in \mathbb{N}}$ *converges* to the *limit* l , $a_n \rightarrow l$ as $n \rightarrow \infty$, if and only if, for every neighbourhood U of l , there exists N such that, for all $n \geq N$, we have $a_n \in U$.

Theorem 72. *In a Hausdorff space, a sequence has at most one limit.*

PROOF:

$\langle 1 \rangle 1$. LET: X be a Hausdorff space.

$\langle 1 \rangle 2$. ASSUME: for a contradiction $a_n \rightarrow l$ as $n \rightarrow \infty$, $a_n \rightarrow m$ as $n \rightarrow \infty$, and $l \neq m$

$\langle 1 \rangle 3$. PICK disjoint neighbourhoods U of l and V of m

PROOF: By the Hausdorff axiom.

$\langle 1 \rangle 4$. PICK M and N such that $a_n \in U$ for $n \geq M$ and $a_n \in V$ for $n \geq N$

$\langle 1 \rangle 5$. $a_{\max(M,N)} \in U \cap V$

$\langle 1 \rangle 6$. Q.E.D.

PROOF: This contradicts the fact that U and V are disjoint ($\langle 1 \rangle 3$).

\square

To see this is not always true in spaces that are T_1 but not Hausdorff:

Proposition 73. *Let X be an infinite set under the finite complement topology. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence with all points distinct. Then for every $l \in X$ we have $a_n \rightarrow l$ as $n \rightarrow \infty$.*

PROOF: Let U be any neighbourhood of l . Since $X \setminus U$ is finite, there must exist N such that, for all $n \geq N$, we have $a_n \in U$. \square

Lemma 74. *Let X be a topological space. Let $A \subseteq X$ and $l \in X$. If there is a sequence of points in A that converges to l then $l \in \bar{A}$.*

PROOF:

$\langle 1 \rangle 1$. LET: (a_n) be a sequence of points in A that converges to l .

$\langle 1 \rangle 2$. LET: U be a neighbourhood of l .

$\langle 1 \rangle 3$. PICK N such that, for all $n \geq N$, we have $a_n \in U$.

$\langle 1 \rangle 4$. $a_N \in U \cap A$

⟨1⟩5. Q.E.D.

PROOF: Theorem 45.

□

Proposition 75. *Let X be a topological space. Let \mathcal{B} be a basis for the topology on X . Let (a_n) be a sequence in X and $l \in X$. Then $a_n \rightarrow l$ as $n \rightarrow \infty$ if and only if, for every $B \in \mathcal{B}$ with $l \in B$, there exists N such that, for all $n \geq N$, we have $a_n \in B$.*

PROOF:

⟨1⟩1. If $a_n \rightarrow l$ as $n \rightarrow \infty$ then, for every $B \in \mathcal{B}$ with $l \in B$, there exists N such that, for all $n \geq N$, we have $a_n \in B$.

PROOF: Since every element of \mathcal{B} is open (Corollary 59.1).

⟨1⟩2. If, for every $B \in \mathcal{B}$ with $l \in B$, there exists N such that, for all $n \geq N$, we have $a_n \in B$, then $a_n \rightarrow l$ as $n \rightarrow \infty$.

⟨2⟩1. ASSUME: for every $B \in \mathcal{B}$ with $l \in B$, there exists N such that, for all $n \geq N$, we have $a_n \in B$

⟨2⟩2. LET: U be a neighbourhood of l .

⟨2⟩3. PICK $B \in \mathcal{B}$ such that $l \in B \subseteq U$

⟨2⟩4. PICK N such that, for all $n \geq N$, we have $a_n \in B$

PROOF: From ⟨2⟩1.

⟨2⟩5. For all $n \geq N$ we have $a_n \in U$

□

Lemma 76. *If a sequence (a_n) is constant with $a_n = l$ for all n , then $a_n \rightarrow l$ as $n \rightarrow \infty$.*

PROOF: Immediate from definitions. □

Theorem 77. *Let X be a linearly ordered set. Let (s_n) be a monotone increasing sequence in X with a supremum s . Then $s_n \rightarrow s$ as $n \rightarrow \infty$.*

PROOF:

⟨1⟩1. ASSUME: s is not least in X .

PROOF: Otherwise (s_n) is the constant sequence s and the result follows from Lemma 76.

⟨1⟩2. LET: U be a neighbourhood of s .

⟨1⟩3. PICK $a < s$ such that $(a, s] \subseteq U$

⟨1⟩4. PICK N such that $a < a_N$.

⟨1⟩5. For all $n \geq N$ we have $a_n \in (a, s]$

⟨1⟩6. For all $n \geq N$ we have $a_n \in U$.

□

Theorem 78. *If $\sum_{i=0}^{\infty} a_i = s$ and $\sum_{i=0}^{\infty} b_i = t$ then $\sum_{i=0}^{\infty} (ca_i + b_i) = cs + t$.*

PROOF: $\sum_{i=0}^N (ca_i + b_i) = c \sum_{i=0}^N a_i + \sum_{i=0}^N b_i \rightarrow cs + t$ as $n \rightarrow \infty$. □

Theorem 79 (Comparison Test). *If $|a_i| \leq b_i$ for all i and $\sum_{i=0}^{\infty} b_i$ converges then $\sum_{i=0}^{\infty} a_i$ converges.*

PROOF:

⟨1⟩1. $\sum_{i=0}^{\infty} |a_i|$ converges

PROOF: The partial sums $\sum_{i=0}^N |a_i|$ form an increasing sequence bounded above by $\sum_{i=0}^{\infty} b_i$.

⟨1⟩2. LET: $c_i = |a_i| + a_i$ for all i

⟨1⟩3. $\sum_{i=0}^{\infty} c_i$ converges

PROOF: Each c_i is either $2|a_i|$ or 0. So the partial sums $\sum_{i=0}^N c_i$ form an increasing sequence bounded above by $2 \sum_{i=0}^{\infty} b_i$.

⟨1⟩4. Q.E.D.

PROOF: Since $a_i = c_i - |a_i|$.

□

Corollary 79.1. *If $\sum_{i=0}^{\infty} |a_i|$ converges then $\sum_{i=0}^{\infty} a_i$ converges.*

Theorem 80 (Weierstrass M -test). *Let X be a set and $(f_n : X \rightarrow \mathbb{R})$ be a sequence of functions. Let*

$$s_n(x) = \sum_{i=0}^n f_i(x)$$

for all n, x . Suppose $|f_i(x)| \leq M_i$ for all $i \geq 0$ and $x \in X$. If the series $\sum_{i=0}^{\infty} M_i$ converges, then the sequence (s_n) converges uniformly to

$$s(x) = \sum_{i=0}^{\infty} f_i(x) .$$

PROOF:

⟨1⟩1. LET: $r_n = \sum_{i=n+1}^{\infty} M_i$ for all n

⟨1⟩2. Given $0 \leq n < k$, we have $|s_k(x) - s_n(x)| \leq r_n$

PROOF:

$$\begin{aligned} |s_k(x) - s_n(x)| &= \left| \sum_{i=n+1}^k f_i(x) \right| \\ &\leq \sum_{i=n+1}^k |f_i(x)| \\ &\leq \sum_{i=n+1}^k M_i \\ &\leq r_n \end{aligned}$$

⟨1⟩3. Given $n \geq 0$ we have $|s(x) - s_n(x)| \leq r_n$

PROOF: By taking the limit $k \rightarrow \infty$ in ⟨1⟩2.

⟨1⟩4. Q.E.D.

PROOF: Since $r_n \rightarrow 0$ as $n \rightarrow \infty$.

□

13 Locally Finite Sets

Definition 81 (Locally Finite). Let X be a topological space and $\{A_\alpha\}$ a family of subsets of X . Then \mathcal{A} is *locally finite* if and only if every point in X has a neighbourhood that intersects A_α for only finitely many α .

Theorem 82 (Pasting Lemma). Let X and Y be topological spaces and $f : X \rightarrow Y$. Let $\{A_\alpha\}$ be a locally finite family of closed subsets of X that cover X . Suppose $f \upharpoonright A_\alpha$ is continuous for all α . Then f is continuous.

PROOF:

$\langle 1 \rangle 1$. Let X and Y be topological spaces and $f : X \rightarrow Y$. Let A and B be closed subsets of X such that $X = A \cup B$. Suppose $f \upharpoonright A$ and $f \upharpoonright B$ are continuous. Then f is continuous.

$\langle 2 \rangle 1$. LET: $C \subseteq Y$ be closed.

$\langle 2 \rangle 2$. $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$

$\langle 2 \rangle 3$. $f^{-1}(C)$ and $g^{-1}(C)$ are closed in X .

PROOF: Theorems 92 and 143.

$\langle 2 \rangle 4$. $h^{-1}(C)$ is closed in X .

PROOF: Lemma 31.

$\langle 2 \rangle 5$. Q.E.D.

PROOF: Theorem 92.

$\langle 1 \rangle 2$. Let X and Y be topological spaces and $f : X \rightarrow Y$. Let $\{A_\alpha\}$ be a finite family of closed subsets of X that cover X . Suppose $f \upharpoonright A_\alpha$ is continuous for all α . Then f is continuous.

PROOF: From $\langle 1 \rangle 1$ by induction.

$\langle 1 \rangle 3$. Let X and Y be topological spaces and $f : X \rightarrow Y$. Let $\{A_\alpha\}$ be a locally finite family of closed subsets of X that cover X . Suppose $f \upharpoonright A_\alpha$ is continuous for all α . Then f is continuous.

$\langle 2 \rangle 1$. LET: $x \in X$

PROVE: f is continuous at x

$\langle 2 \rangle 2$. PICK a neighbourhood U of x that intersects A_α for only finitely many α .

$\langle 2 \rangle 3$. $f \upharpoonright U$ is continuous

PROOF: By $\langle 1 \rangle 2$.

$\langle 2 \rangle 4$. Q.E.D.

PROOF: Lemma 102.

□

The following example shows that we cannot remove the assumption of local finiteness.

Example 83. Define $f : [-1, 1] \rightarrow \mathbb{R}$ by: $f(x) = 1$ if $x < -1$, $f(x) = 0$ if $x > 1$. Let $C_n = [-1, -1/n]$ for $n \geq 1$, and $D = [0, 1]$. Then $[-1, 1] = \bigcup_{n=1}^{\infty} C_n \cup D$ and f is continuous on each C_n and each D , but f is not continuous on $[-1, 1]$.

14 Open Maps

Definition 84 (Open Map). Let X and Y be topological spaces and $f : X \rightarrow Y$. Then f is an *open map* if and only if, for every open set U in X , the set $f(U)$ is open in Y .

Lemma 85. Let X and Y be topological spaces and $f : X \rightarrow Y$. Let \mathcal{B} be a basis for the topology on X . If $f(B)$ is open in Y for all $B \in \mathcal{B}$, then f is an open map.

PROOF: From Lemma 59. \square

Proposition 86. Let X and Y be topological spaces. Let \mathcal{B} be a basis for the topology on X . Let $f : X \rightarrow Y$. Suppose that, for all $B \in \mathcal{B}$, we have $f(B)$ is open in Y . Then f is an open map.

PROOF: For any $\mathcal{A} \subseteq \mathcal{B}$, we have $f(\bigcup \mathcal{A}) = \bigcup_{B \in \mathcal{A}} f(B)$ is open in Y . The result follows from Lemma 59. \square

15 Continuous Functions

Definition 87 (Continuous). Let X and Y be topological spaces. A function $f : X \rightarrow Y$ is *continuous* if and only if, for every open set V in Y , the set $f^{-1}(V)$ is open in X .

Proposition 88. Let X and Y be topological spaces and $f : X \rightarrow Y$. Let \mathcal{B} be a basis for Y . Then f is continuous if and only if, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X .

PROOF:

$\langle 1 \rangle 1$. If f is continuous then, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X .

PROOF: Since every element of \mathcal{B} is open (Lemma 59).

$\langle 1 \rangle 2$. Suppose that, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X . Then f is continuous.

$\langle 2 \rangle 1$. ASSUME: For all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X .

$\langle 2 \rangle 2$. LET: V be open in Y .

$\langle 2 \rangle 3$. PICK $\mathcal{A} \subseteq \mathcal{B}$ such that $V = \bigcup \mathcal{A}$

PROOF: By Lemma 59.

$\langle 2 \rangle 4$. $f^{-1}(V)$ is open in X .

PROOF:

$$\begin{aligned} f^{-1}(V) &= f^{-1}\left(\bigcup \mathcal{A}\right) \\ &= \bigcup_{B \in \mathcal{A}} f^{-1}(B) \end{aligned}$$

\square

Proposition 89. Let X and Y be topological spaces and $f : X \rightarrow Y$. Let \mathcal{S} be a subbasis for Y . Then f is continuous if and only if, for all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in X .

PROOF:

⟨1⟩1. If f is continuous then, for all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in X .

PROOF: Since every element of \mathcal{S} is open.

⟨1⟩2. Suppose that, for all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in X . Then f is continuous.

⟨2⟩1. ASSUME: For all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in X .

⟨2⟩2. LET: $S_1, \dots, S_n \in \mathcal{S}$

⟨2⟩3. $f^{-1}(S_1 \cap \dots \cap S_n)$ is open in A

PROOF: Since $f^{-1}(S_1 \cap \dots \cap S_n) = f^{-1}(S_1) \cap \dots \cap f^{-1}(S_n)$.

⟨2⟩4. Q.E.D.

PROOF: By Propositions 88 and 67.

□

Proposition 90. *Let X and Y be topological spaces and $f : X \rightarrow Y$. Let \mathcal{S} be a basis for Y . Then f is continuous if and only if, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X .*

PROOF:

⟨1⟩1. If f is continuous then, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X .

PROOF: Since every element of \mathcal{S} is open.

⟨1⟩2. Suppose that, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X . Then f is continuous.

⟨2⟩1. ASSUME: For all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X .

⟨2⟩2. For every set B that is the finite intersection of elements of \mathcal{S} , we have $f^{-1}(B)$ is open in X .

PROOF: Because $f^{-1}(V_1 \cap \dots \cap V_n) = f^{-1}(V_1) \cap \dots \cap f^{-1}(V_n)$.

⟨2⟩3. Q.E.D.

PROOF: From Propositions 67 and 88.

□

Definition 91 (Continuous at a Point). Let X and Y be topological spaces. Let $f : X \rightarrow Y$ and $x \in X$. Then f is *continuous at x* if and only if, for every neighbourhood V of $f(x)$, there exists a neighbourhood U of x such that $f(U) \subseteq V$.

Theorem 92. *Let X and Y be topological spaces and $f : X \rightarrow Y$. Then the following are equivalent:*

1. f is continuous.
2. For all $A \subseteq X$, we have $f(\overline{A}) \subseteq \overline{f(A)}$
3. For all $B \subseteq Y$ closed, we have $f^{-1}(B)$ is closed in X .
4. f is continuous at every point of X .

PROOF:

⟨1⟩1. $1 \Rightarrow 2$

⟨2⟩1. ASSUME: f is continuous.

(2)2. LET: $A \subseteq X$
 (2)3. LET: $x \in \overline{A}$
 PROVE: $f(x) \in \overline{f(A)}$
 (2)4. LET: V be a neighbourhood of $f(x)$
 (2)5. $f^{-1}(V)$ is a neighbourhood of x
 (2)6. PICK $y \in A \cap f^{-1}(V)$
 PROOF: By Theorem 45.
 (2)7. $f(y) \in V \cap f(A)$
 (2)8. Q.E.D.
 PROOF: By Theorem 45.
 (1)2. $2 \Rightarrow 3$
 (2)1. ASSUME: 2
 (2)2. LET: B be closed in Y
 (2)3. LET: $x \in \overline{f^{-1}(B)}$
 PROVE: $x \in f^{-1}(B)$
 (2)4. $f(x) \in B$
 PROOF:

$$\begin{aligned}
 f(x) &\in f(\overline{f^{-1}(B)}) \\
 &\subseteq \overline{f(f^{-1}(B))} && ((2)1) \\
 &\subseteq \overline{B} && (Proposition 46) \\
 &= B
 \end{aligned}$$

 (1)3. $3 \Rightarrow 1$
 (2)1. ASSUME: 3
 (2)2. LET: V be open in Y
 (2)3. $Y \setminus V$ is closed in Y
 (2)4. $f^{-1}(Y \setminus V)$ is closed in X
 (2)5. $X \setminus f^{-1}(V)$ is closed in X
 (2)6. $f^{-1}(V)$ is open in X
 (1)4. $1 \Rightarrow 4$
 PROOF: For any neighbourhood V of $f(x)$, the set $U = f^{-1}(V)$ is a neighbourhood of x such that $f(U) \subseteq V$.
 (1)5. $4 \Rightarrow 1$
 (2)1. ASSUME: 4
 (2)2. LET: V be open in Y
 (2)3. LET: $x \in f^{-1}(V)$
 (2)4. V is a neighbourhood of $f(x)$
 (2)5. PICK a neighbourhood U of x such that $f(U) \subseteq V$
 (2)6. $x \in U \subseteq f^{-1}(V)$
 (2)7. Q.E.D.
 PROOF: By Lemma 23.

□

Theorem 93. *A constant function is continuous.*

PROOF: Let X and Y be topological spaces. Let $b \in Y$, and let $f : X \rightarrow Y$ be the constant function with value b . For any open $V \subseteq Y$, the set $f^{-1}(V)$ is

either X (if $b \in V$) or \emptyset (if $b \notin V$). \square

Theorem 94. *If A is a subspace of X then the inclusion $j : A \rightarrow X$ is continuous.*

PROOF: For any V open in X , we have $j^{-1}(V) = V \cap A$ is open in A . \square

Theorem 95. *The composite of two continuous functions is continuous.*

PROOF: Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be continuous. For any V open in Z , we have $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is open in X . \square

Theorem 96. *Let $f : X \rightarrow Y$ be a continuous function and A be a subspace of X . Then the restriction $f \upharpoonright A : A \rightarrow Y$ is continuous.*

PROOF: Let V be open in Y . Then $(f \upharpoonright A)^{-1}(V) = f^{-1}(V) \cap A$ is open in A . \square

Theorem 97. *Let $f : X \rightarrow Y$ be continuous. Let Z be a subspace of Y such that $f(X) \subseteq Z$. Then the corestriction $f : X \rightarrow Z$ is continuous.*

PROOF:

- $\langle 1 \rangle 1$. LET: V be open in Z .
- $\langle 1 \rangle 2$. PICK U open in Y such that $V = U \cap Z$.
- $\langle 1 \rangle 3$. $f^{-1}(V) = f^{-1}(U)$
- $\langle 1 \rangle 4$. $f^{-1}(V)$ is open in X .

\square

Theorem 98. *Let $f : X \rightarrow Y$ be continuous. Let Z be a space such that Y is a subspace of Z . Then the expansion $f : X \rightarrow Z$ is continuous.*

PROOF: Let V be open in Z . Then $f^{-1}(V) = f^{-1}(V \cap Y)$ is open in X . \square

Theorem 99. *Let X and Y be topological spaces. Let $f : X \rightarrow Y$. Suppose \mathcal{U} is a set of open sets in X such that $X = \bigcup \mathcal{U}$ and, for all $U \in \mathcal{U}$, we have $f \upharpoonright U : U \rightarrow Y$ is continuous. Then f is continuous.*

PROOF:

- $\langle 1 \rangle 1$. LET: V be open in Y
- $\langle 1 \rangle 2$. $f^{-1}(V) = \bigcup_{U \in \mathcal{U}} (f \upharpoonright U)^{-1}(V)$
- $\langle 1 \rangle 3$. For all $U \in \mathcal{U}$, we have $(f \upharpoonright U)^{-1}(V)$ is open in U .
- $\langle 1 \rangle 4$. For all $U \in \mathcal{U}$, we have $(f \upharpoonright U)^{-1}(V)$ is open in X .

PROOF: Lemma 142.

\square

Proposition 100. *Let X and X' be the same set X under two topologies \mathcal{T} and \mathcal{T}' . Let $i : X \rightarrow X'$ be the identity function. Then i is continuous if and only if $\mathcal{T}' \subseteq \mathcal{T}$.*

PROOF: Immediate from definitions. \square

Proposition 101. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ and $a \in \mathbb{R}$. Then f is continuous on the right at a if and only if f is continuous at a as a function $\mathbb{R}_l \rightarrow \mathbb{R}$.*

PROOF:

- ⟨1⟩1. If f is continuous on the right at a then f is continuous at a as a function $\mathbb{R}_l \rightarrow \mathbb{R}$.
 - ⟨2⟩1. ASSUME: f is continuous on the right at a .
 - ⟨2⟩2. LET: V be a neighbourhood of $f(a)$
 - ⟨2⟩3. PICK b, c such that $f(a) \in (b, c) \subseteq V$.
 - ⟨2⟩4. LET: $\epsilon = \min(c - f(a), f(a) - b)$
 - ⟨2⟩5. PICK $\delta > 0$ such that, for all x , if $a < x < a + \delta$ then $|f(x) - f(a)| < \epsilon$
 - ⟨2⟩6. LET: $U = [a, a + \delta)$
 - ⟨2⟩7. $f(U) \subseteq V$
- ⟨1⟩2. If f is continuous at a as a function $\mathbb{R}_l \rightarrow \mathbb{R}$ then f is continuous on the right at a .
 - ⟨2⟩1. ASSUME: f is continuous at a as a function $\mathbb{R}_l \rightarrow \mathbb{R}$
 - ⟨2⟩2. LET: $\epsilon > 0$
 - ⟨2⟩3. PICK a neighbourhood U of a such that $f(U) \subseteq (f(a) - \epsilon, f(a) + \epsilon)$
 - ⟨2⟩4. PICK b, c such that $a \in [b, c) \subset U$
 - ⟨2⟩5. LET: $\delta = c - a$
 - ⟨2⟩6. For all x , if $a < x < a + \delta$ then $|f(x) - f(a)| < \epsilon$

□

Lemma 102. *Let $f : X \rightarrow Y$. Let Z be an open subspace of X and $a \in Z$. If $f \upharpoonright Z$ is continuous at a then f is continuous at a .*

PROOF:

- ⟨1⟩1. LET: V be a neighbourhood of $f(x)$
- ⟨1⟩2. PICK a neighbourhood W of x in Z such that $f(W) \subseteq V$
- ⟨1⟩3. W is a neighbourhood of x in X such that $f(W) \subseteq V$

PROOF: Lemma 142.

□

Proposition 103. *Let $f : A \rightarrow B$ and $g : C \rightarrow D$ be continuous. Define $f \times g : A \times C \rightarrow B \times D$ by*

$$(f \times g)(a, c) = (f(a), g(c)) .$$

Then $f \times g$ is continuous.

PROOF: $\pi_1 \circ (f \times g) = f \circ \pi_1$ and $\pi_2 \circ (f \times g) = g \circ \pi_2$ are continuous by Theorem 95. The result follows by Theorem 132.

Proposition 104. *Let X be a topological space. Let Y a Hausdorff space. Let $A \subseteq X$. Let $f, g : \overline{A} \rightarrow Y$ be continuous. If f and g agree on A then $f = g$.*

PROOF:

- ⟨1⟩1. LET: $x \in \overline{A}$
- ⟨1⟩2. ASSUME: $f(x) \neq g(x)$
- ⟨1⟩3. PICK disjoint neighbourhoods V of $f(x)$ and W of $g(x)$.
- ⟨1⟩4. PICK $y \in f^{-1}(V) \cap g^{-1}(W) \cap A$

PROOF: Since $f^{-1}(V) \cap g^{-1}(W)$ is a neighbourhood of x and hence intersects A .

$\langle 1 \rangle 5$. $f(y) = g(y) \in V \cap W$

$\langle 1 \rangle 6$. Q.E.D.

PROOF: This contradicts the fact that V and W are disjoint ($\langle 1 \rangle 3$).

□

Proposition 105. *Let X and Y be topological spaces and $f : X \rightarrow Y$ be continuous. If $a_n \rightarrow l$ as $n \rightarrow \infty$ in X then $f(a_n) \rightarrow f(l)$ as $n \rightarrow \infty$.*

PROOF:

$\langle 1 \rangle 1$. LET: V be a neighbourhood of $f(l)$

$\langle 1 \rangle 2$. PICK a neighbourhood U of l such that $f(U) \subseteq V$

$\langle 1 \rangle 3$. PICK N such that, for all $n \geq N$, we have $a_n \in U$

$\langle 1 \rangle 4$. For all $n \geq N$ we have $f(a_n) \in V$

□

16 Homeomorphisms

Definition 106 (Homeomorphism). Let X and Y be topological spaces. A *Homeomorphism* f between X and Y , $f : X \cong Y$, is a bijection $f : X \rightarrow Y$ such that both f and f^{-1} are continuous.

Lemma 107. *Let X and Y be topological spaces and $f : X \rightarrow Y$ a bijection. Then the following are equivalent:*

1. f is a homeomorphism.
2. f is continuous and an open map.
3. For any $U \subseteq X$, we have U is open if and only if $f(U)$ is open.

PROOF: Immediate from definitions. □

Proposition 108. *Let X and X' be the same set X under two topologies \mathcal{T} and \mathcal{T}' . Let $i : X \rightarrow X'$ be the identity function. Then i is a homeomorphism if and only if $\mathcal{T} = \mathcal{T}'$.*

PROOF: Immediate from definitions. □

Definition 109 (Topological Property). Let P be a property of topological spaces. Then P is a *topological* property if and only if, for any spaces X and Y , if P holds of X and $X \cong Y$ then P holds of Y .

Definition 110 (Topological Imbedding). Let X and Y be topological spaces and $f : X \rightarrow Y$. Then f is a *topological imbedding* if and only if the corestriction $f : X \rightarrow f(X)$ is a homeomorphism.

Proposition 111. *Let X and Y be topological spaces and $a \in X$. The function $i : Y \rightarrow X \times Y$ that maps y to (a, y) is an imbedding.*

PROOF:

⟨1⟩1. i is injective

⟨1⟩2. i is continuous.

PROOF: For U open in X and V open in Y , we have $i^{-1}(U \times V)$ is V if $a \in U$, and \emptyset if $a \notin U$.

⟨1⟩3. $i : Y \rightarrow i(Y)$ is an open map.

PROOF: For V open in Y we have $i(V) = (X \times V) \cap i(Y)$.

□

17 The Order Topology

Definition 112 (Order Topology). Let X be a linearly ordered set with at least two points. The *order topology* on X is the topology generated by the basis \mathcal{B} consisting of:

- all open intervals (a, b) ;
- all intervals of the form $[\perp, b)$ where \perp is least in X ;
- all intervals of the form $(a, \top]$ where \top is greatest in X .

We prove this is a basis for a topology.

PROOF:

⟨1⟩1. For all $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B$.

⟨2⟩1. LET: $x \in X$

⟨2⟩2. CASE: x is greatest in X .

⟨3⟩1. PICK $y \in X$ with $y \neq x$

⟨3⟩2. $x \in (y, x] \in \mathcal{B}$

⟨2⟩3. CASE: x is least in X .

⟨3⟩1. PICK $y \in X$ with $y \neq x$

⟨3⟩2. $x \in [x, y) \in \mathcal{B}$

⟨2⟩4. CASE: x is neither greatest nor least in X .

⟨3⟩1. PICK $a, b \in X$ with $a < x$ and $x < b$

⟨3⟩2. $x \in (a, b) \in \mathcal{B}$

⟨1⟩2. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$

⟨2⟩1. LET: $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$

⟨2⟩2. CASE: $B_1 = (a, b)$, $B_2 = (c, d)$

PROOF: Take $B_3 = (\max(a, c), \min(b, d))$.

⟨2⟩3. CASE: $B_1 = (a, b)$, $B_2 = [\perp, d)$

PROOF: Take $B_3 = (a, \min(b, d))$.

⟨2⟩4. CASE: $B_1 = (a, b)$, $B_2 = (c, \top]$

PROOF: Take $B_3 = (\max(a, c), b)$.

⟨2⟩5. CASE: $B_1 = [\perp, b)$, $B_2 = [\perp, d)$

PROOF: Take $B_3 = [\perp, \min(b, d))$.

⟨2⟩6. CASE: $B_1 = [\perp, b)$, $B_2 = (c, \top]$

PROOF: Take $B_3 = (c, b)$.

□

Lemma 113. *Let X be a linearly ordered set. Then the open rays form a subbasis for the order topology on X .*

PROOF:

⟨1⟩1. Every open ray is open.

⟨2⟩1. For all $a \in X$, the ray $(-\infty, a)$ is open.

⟨3⟩1. LET: $x \in (-\infty, a)$

⟨3⟩2. CASE: x is least in X

PROOF: $x \in [x, a) = (-\infty, a)$.

⟨3⟩3. CASE: x is not least in X

⟨4⟩1. PICK $y < x$

⟨4⟩2. $x \in (y, a) \subseteq (-\infty, a)$

⟨2⟩2. For all $a \in X$, the ray $(a, +\infty)$ is open.

PROOF: Similar.

⟨1⟩2. Every basic open set is a finite intersection of open rays.

PROOF: We have $(a, b) = (a, +\infty) \cap (-\infty, b)$, $[\perp, b) = (-\infty, b)$ and $(a, \top] = (a, +\infty)$.

□

Definition 114 (Standard Topology on the Real Line). The *standard topology on the real line* is the order topology on \mathbb{R} generated by the standard order.

Lemma 115. *The lower limit topology is strictly finer than the standard topology on \mathbb{R} .*

PROOF:

⟨1⟩1. Every open interval is open in the lower limit topology.

PROOF: If $x \in (a, b)$ then $x \in [x, b) \subseteq (a, b)$.

⟨1⟩2. The half-open interval $[0, 1)$ is not open in the standard topology.

PROOF: There is no open interval (a, b) such that $0 \in (a, b) \subseteq [0, 1)$.

□

Lemma 116. *The K -topology is strictly finer than the standard topology on \mathbb{R} .*

PROOF:

⟨1⟩1. Every open interval is open in the K -topology.

PROOF: Corollary 59.1.

⟨1⟩2. The set $(-1, 1) \setminus K$ is not open in the standard topology.

PROOF: There is no open interval (a, b) such that $0 \in (a, b) \subseteq (-1, 1) \setminus K$, since there must be a positive integer n with $1/n \in (a, b)$.

□

Lemma 117. *Let X be a topological space. Let Y be a linearly ordered set in the order topology. Let $f, g : X \rightarrow Y$ be continuous. Then $C = \{x \in X \mid f(x) \leq g(x)\}$ is closed.*

PROOF:

⟨1⟩1. LET: $x \in X \setminus C$

⟨1⟩2. $f(x) > g(x)$

PROVE: There exists a neighbourhood U of x such that $U \subseteq X \setminus C$

⟨1⟩3. CASE: There exists y such that $g(x) < y < f(x)$

PROOF: Take $U = g^{-1}((-\infty, y)) \cup f^{-1}(y, +\infty)$.

⟨1⟩4. CASE: There is no y such that $g(x) < y < f(x)$

PROOF: Take $U = g^{-1}((-\infty, f(x))) \cup f^{-1}(g(x), +\infty)$.

□

Proposition 118. *Let X be a topological space. Let Y be a linearly ordered set in the order topology. Let $f, g : X \rightarrow Y$ be continuous. Define $h : X \rightarrow Y$ by $h(x) = \min(f(x), g(x))$. Then h is continuous.*

PROOF: By the Pasting Lemma applied to $\{x \in X \mid f(x) \leq g(x)\}$ and $\{x \in X \mid g(x) \leq f(x)\}$, which are closed by Lemma 117.

Proposition 119. *Let X and Y be linearly ordered sets in the order topology. Let $f : X \rightarrow Y$ be strictly monotone and surjective. Then f is a homeomorphism.*

PROOF:

⟨1⟩1. f is bijective.

PROOF: Proposition 3.

⟨1⟩2. f is continuous.

⟨2⟩1. For all $y \in Y$ we have $f^{-1}((y, +\infty))$ is open.

⟨3⟩1. LET: $y \in Y$

⟨3⟩2. PICK $x \in X$ such that $f(x) = y$

PROOF: Since f is surjective.

⟨3⟩3. $f^{-1}((y, +\infty)) = (x, +\infty)$

PROOF: By strict monotonicity.

⟨2⟩2. For all $y \in Y$ we have $f^{-1}((-\infty, y))$ is open.

PROOF: Similar.

⟨1⟩3. f^{-1} is continuous.

⟨2⟩1. For all $x \in X$ we have $f((x, +\infty))$ is open.

PROOF: $f((x, +\infty)) = (f(x), +\infty)$.

⟨2⟩2. For all $x \in X$ we have $f((-\infty, x))$ is open.

PROOF: $f((-\infty, x)) = (-\infty, f(x))$.

□

18 The n th Root Function

Proposition 120. *For all $n \geq 1$, the function $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ defined by $f(x) = x^n$ is a homeomorphism.*

PROOF:

⟨1⟩1. f is strictly monotone.

⟨2⟩1. LET: $x, y \in \mathbb{R}$ with $0 \leq x < y$

⟨2⟩2. $x^n < y^n$

$$y^n - x^n = (y - x)(y^{n-1} + y^{n-2}x + y^{n-3}x^2 + \cdots + x^{n-1}) > 0$$

⟨1⟩2. f is surjective.

⟨2⟩1. LET: $y \in \mathbb{R}_{\geq 0}$

⟨2⟩2. PICK $x \in \mathbb{R}$ such that $y \leq x^n$

PROOF: If $y \leq 1$ take $x = 1$, otherwise take $x = y$.

⟨2⟩3. There exists $x' \in [0, x]$ such that $(x')^n = y$

PROOF: By the Intermediate Value Theorem.

⟨1⟩3. Q.E.D.

PROOF: Proposition 119.

□

Definition 121. For $n \geq 1$, the n th root function is the function $\mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ that is the inverse of $\lambda x.x^n$.

19 The Product Topology

Definition 122 (Product Topology). Let $\{A_i\}_{i \in I}$ be a family of topological spaces. The *product topology* on $\prod_{i \in I} A_i$ is the topology generated by the sub-basis consisting of the sets of the form $\pi_i^{-1}(U)$ where $i \in I$ and U is open in A_i .

Proposition 123. The product topology on $\prod_{i \in I} A_i$ is generated by the basis consisting of all sets of the form $\prod_{i \in I} U_i$ where $\{U_i\}_{i \in I}$ is a family such that each U_i is an open set in A_i and $U_i = A_i$ for all but finitely many i .

PROOF: From Proposition 67. □

Proposition 124. If A_i is closed in X_i for all $i \in I$ then $\prod_{i \in I} A_i$ is closed in $\prod_{i \in I} X_i$.

PROOF:

$$\left(\prod_{i \in I} X_i\right) \setminus \left(\prod_{i \in I} A_i\right) = \bigcup_{j \in I} \left(\prod_{i \in I} X_i \setminus \pi_j^{-1}(A_j)\right) \square$$

Proposition 125. Let $\{A_i\}_{i \in I}$ be a family of topological spaces. For $i \in I$, let \mathcal{B}_i be a basis for the topology on A_i . Then $\mathcal{B} = \{\prod_{i \in I} B_i \mid \forall i \in I. B_i \in \mathcal{B}_i, B_i = A_i \text{ for all but finitely many } i\}$ is a basis for the box topology on $\prod_{i \in I} A_i$.

PROOF:

⟨1⟩1. Every set in \mathcal{B} is open.

⟨1⟩2. For every point $a \in \prod_{i \in I} A_i$ and every open set U with $a \in U$, there exists $B \in \mathcal{B}$ such that $a \in B \subseteq U$.

⟨2⟩1. LET: U be open and $a \in U$

- ⟨2⟩2. PICK a family $\{U_i\}_{i \in I}$ such that each U_i is open in A_i , such that $U_i = A_i$ except for $i = i_1, \dots, i_n$, and such that $a \in \prod_{i \in I} U_i \subseteq U$.
- ⟨2⟩3. For $j = 1, \dots, n$, PICK $B_{i_j} \in \mathcal{B}_{i_j}$ such that $a_{i_j} \in B_{i_j} \subseteq U_{i_j}$.
- ⟨2⟩4. LET: $B = \prod_{i \in I} B_i$ where $B_i = A_i$ for $i \neq i_1, \dots, i_n$.
- ⟨2⟩5. $B \in \mathcal{B}$
- ⟨2⟩6. $a \in B \subseteq U$
- ⟨1⟩3. Q.E.D.

PROOF: Lemma 60.

□

Proposition 126. *Let $\{A_i\}_{i \in I}$ be a family of topological spaces. Then the projections $\pi_i : \prod_{i \in I} A_i \rightarrow A_i$ are open maps.*

PROOF: From Lemma 85. □

Example 127. The projections are not always closed maps. For example, $\pi_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$ maps the closed set $\{(x, 1/x) \mid x > 0\}$ to $(0, +\infty)$.

Proposition 128. *Let $\{X_i\}_{i \in I}$ be a family of sets. For $i \in I$, let \mathcal{T}_i and \mathcal{U}_i be topologies on X_i . Let \mathcal{P} be the product topology on $\prod_{i \in I} X_i$ generated by the topologies \mathcal{T}_i , and \mathcal{Q} the product topology on the same set generated by the topologies \mathcal{U}_i . Then $\mathcal{P} \subseteq \mathcal{Q}$ if and only if $\mathcal{T}_i \subseteq \mathcal{U}_i$ for all i .*

PROOF:

- ⟨1⟩1. If $\mathcal{T}_i \subseteq \mathcal{U}_i$ for all i then $\mathcal{P} \subseteq \mathcal{Q}$
PROOF: By Corollary 59.1.
- ⟨1⟩2. If $\mathcal{P} \subseteq \mathcal{Q}$ then $\mathcal{T}_i \subseteq \mathcal{U}_i$ for all i
 - ⟨2⟩1. ASSUME: $\mathcal{P} \subseteq \mathcal{Q}$
 - ⟨2⟩2. LET: $i \in I$
 - ⟨2⟩3. LET: $U \in \mathcal{T}_i$
 - ⟨2⟩4. LET: $U_i = U$ and $U_j = X_j$ for $j \neq i$
 - ⟨2⟩5. $\prod_{i \in I} U_i \in \mathcal{P}$
 - ⟨2⟩6. $\prod_{i \in I} U_i \in \mathcal{Q}$
 - ⟨2⟩7. $U \in \mathcal{U}_i$

PROOF: From Proposition 126.

□

Proposition 129 (AC). *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let $A_i \subseteq X_i$ for all $i \in I$. Then*

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i} .$$

PROOF:

- ⟨1⟩1. $\overline{\prod_{i \in I} A_i} \subseteq \prod_{i \in I} \overline{A_i}$
 - ⟨2⟩1. For all $i \in I$ we have $A_i \subseteq \overline{A_i}$
PROOF: Lemma 42.
 - ⟨2⟩2. $\prod_{i \in I} A_i \subseteq \prod_{i \in I} \overline{A_i}$

⟨2⟩3. Q.E.D.

PROOF: Since $\prod_{i \in I} A_i$ is closed by Proposition 124.

⟨1⟩2. $\prod_{i \in I} \overline{A_i} \subseteq \overline{\prod_{i \in I} A_i}$

⟨2⟩1. LET: $x \in \prod_{i \in I} \overline{A_i}$

⟨2⟩2. LET: U be a neighbourhood of x

⟨2⟩3. PICK V_i open in X_i such that $x \in \prod_{i \in I} V_i \subseteq U$ with $V_i = X_i$ except for $i = i_1, \dots, i_n$

⟨2⟩4. For $i \in I$, pick $a_i \in V_i \cap A_i$

PROOF: By Theorem 45 and ⟨2⟩1 using the Axiom of Choice.

⟨2⟩5. U intersects $\prod_{i \in I} A_i$

⟨2⟩6. Q.E.D.

PROOF: $a \in U \cap \prod_{i \in I} A_i$

□

Example 130. The closure of \mathbb{R}^∞ in \mathbb{R}^ω is \mathbb{R}^ω

PROOF:

⟨1⟩1. LET: $a \in \mathbb{R}^\omega$

⟨1⟩2. LET: U be any neighbourhoods of a .

⟨1⟩3. PICK U_n open in \mathbb{R} for all n such that $a \in \prod_{n \geq 0} U_n \subseteq U$ and $U_n = \mathbb{R}$ for all n except n_1, \dots, n_k

⟨1⟩4. LET: $b_n = a_n$ for $n = n_1, \dots, n_k$ and $b_n = 0$ for all other n

⟨1⟩5. $b \in \mathbb{R}^\infty \cap U$

⟨1⟩6. Q.E.D.

PROOF: From Theorem 45.

□

Proposition 131. Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let (a_n) be a sequence in $\prod_{i \in I} X_i$ and $l \in \prod_{i \in I} X_i$. Then $a_n \rightarrow l$ as $n \rightarrow \infty$ if and only if, for all $i \in I$, we have $\pi_i(a_n) \rightarrow \pi_i(l)$ as $n \rightarrow \infty$.

PROOF:

⟨1⟩1. If $a_n \rightarrow l$ as $n \rightarrow \infty$ then, for all $i \in I$, we have $\pi_i(a_n) \rightarrow \pi_i(l)$ as $n \rightarrow \infty$

PROOF: Proposition 105.

⟨1⟩2. If, for all $i \in I$, we have $\pi_i(a_n) \rightarrow \pi_i(l)$ as $n \rightarrow \infty$, then $a_n \rightarrow l$ as $n \rightarrow \infty$

⟨2⟩1. ASSUME: For all $i \in I$, we have $\pi_i(a_n) \rightarrow \pi_i(l)$ as $n \rightarrow \infty$

⟨2⟩2. LET: V be a neighbourhood of l

⟨2⟩3. PICK open sets U_i in X_i such that $l \in \prod_{i \in I} U_i \subseteq V$ and $U_i = X_i$ for all i except $i = i_1, \dots, i_k$

⟨2⟩4. For $j = 1, \dots, k$, PICK N_j such that, for all $n \geq N_j$, we have $\pi_{i_j}(a_n) \in U_{i_j}$

⟨2⟩5. LET: $N = \max(N_1, \dots, N_k)$

⟨2⟩6. For all $n \geq N$ we have $a_n \in V$

□

Theorem 132. Let A be a topological space and $\{X_i\}_{i \in I}$ be a family of topological spaces. Let $f : A \rightarrow \prod_{i \in I} X_i$ be a function. If $\pi_i \circ f$ is continuous for all $i \in I$ then f is continuous.

PROOF:

$\langle 1 \rangle 1$. LET: $i \in I$ and U be open in X_i

$\langle 1 \rangle 2$. $f^{-1}(\pi_i^{-1}(U))$ is open in A

$\langle 1 \rangle 3$. Q.E.D.

PROOF: Proposition 89.

□

19.1 Continuous in Each Variable Separately

Definition 133 (Continuous in Each Variable Separately). Let $F : X \times Y \rightarrow Z$. Then F is *continuous in each variable separately* if and only if:

- for every $a \in X$ the function $\lambda y \in Y. F(a, y)$ is continuous;
- for every $b \in Y$ the function $\lambda x \in X. F(x, b)$ is continuous.

Proposition 134. Let $F : X \times Y \rightarrow Z$. If F is continuous then F is continuous in each variable separately.

PROOF: For $a \in X$, the function $\lambda y \in Y. F(a, y)$ is $F \circ i$ where $i : Y \rightarrow X \times Y$ maps y to (a, y) . We have i is continuous by Proposition 111, hence $F \circ i$ is continuous by Theorem 95.

Similarly for $\lambda x \in X. F(x, b)$ for $b \in Y$. □

Example 135. Define $F : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$F(x, y) = \begin{cases} xy/(x^2 + y^2) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Then F is continuous in each variable separately but not continuous.

Proposition 136. Let $f : A \rightarrow C$ and $g : B \rightarrow D$ be open maps. Then $f \times g : A \times B \rightarrow C \times D$ is an open map.

PROOF: Given U open in A and V open in B . Then $(f \times g)(U \times V) = f(U) \times g(V)$ is open in $C \times D$. The result follows from Proposition 86. □

20 The Subspace Topology

Definition 137 (Subspace Topology). Let X be a topological space and $Y \subseteq X$. The *subspace topology* on Y is $\mathcal{T} = \{U \cap Y \mid U \text{ is open in } X\}$.

We prove this is a topology.

PROOF:

$\langle 1 \rangle 1$. $Y \in \mathcal{T}$

PROOF: Since $Y = X \cap Y$

$\langle 1 \rangle 2$. For all $\mathcal{U} \subseteq \mathcal{T}$, we have $\bigcup \mathcal{U} \in \mathcal{T}$

$\langle 2 \rangle 1$. LET: $\mathcal{U} \subseteq \mathcal{T}$

- ⟨2⟩2. LET: $\mathcal{V} = \{V \text{ open in } X \mid V \cap Y \in \mathcal{U}\}$
- ⟨2⟩3. $\bigcup \mathcal{U} = (\bigcup \mathcal{V}) \cap Y$
- ⟨1⟩3. For all $U, V \in \mathcal{T}$, we have $U \cap V \in \mathcal{T}$
- ⟨2⟩1. LET: $U, V \in \mathcal{T}$
- ⟨2⟩2. PICK U', V' open in X such that $U = U' \cap Y$ and $V = V' \cap Y$
- ⟨2⟩3. $(U \cap V) = (U' \cap V') \cap Y$

□

Theorem 138. *Let X be a topological space and Y a subspace of X . Let $A \subseteq Y$. Then A is closed in Y if and only if there exists a closed set C in X such that $A = C \cap Y$.*

PROOF: We have

$$\begin{aligned}
& A \text{ is closed in } Y \\
& \Leftrightarrow Y \setminus A \text{ is open in } Y \\
& \Leftrightarrow \exists U \text{ open in } X. Y \setminus A = Y \cap U \\
& \Leftrightarrow \exists C \text{ closed in } X. Y \setminus A = Y \cap (X \setminus U) \\
& \Leftrightarrow \exists C \text{ closed in } X. A = Y \cap U
\end{aligned}$$

□

Theorem 139. *Let Y be a subspace of X . Let $A \subseteq Y$. Let \bar{A} be the closure of A in X . Then the closure of A in Y is $\bar{A} \cap Y$.*

PROOF: The closure of A in Y is

$$\begin{aligned}
& \bigcap \{C \text{ closed in } Y \mid A \subseteq C\} \\
& = \bigcap \{D \cap Y \mid D \text{ closed in } X, A \subseteq D \cap Y\} \quad (\text{Theorem 138}) \\
& = \bigcap \{D \mid D \text{ closed in } X, A \subseteq D\} \cap Y \\
& = \bar{A} \cap Y
\end{aligned}$$

□

Lemma 140. *Let X be a topological space and $Y \subseteq X$. Let \mathcal{B} be a basis for the topology on X . Then $\mathcal{B}' = \{B \cap Y \mid B \in \mathcal{B}\}$ is a basis for the subspace topology on Y .*

PROOF:

- ⟨1⟩1. Every element in \mathcal{B}' is open in Y
- ⟨1⟩2. For every open set U in Y and point $y \in U$, there exists $B' \in \mathcal{B}'$ such that $y \in B' \subseteq U$
 - ⟨2⟩1. LET: U be open in Y and $y \in U$
 - ⟨2⟩2. PICK V open in X such that $U = V \cap Y$
 - ⟨2⟩3. PICK $B \in \mathcal{B}$ such that $y \in B \subseteq V$
 - ⟨2⟩4. LET: $B' = B \cap Y$
 - ⟨2⟩5. $B' \in \mathcal{B}'$
 - ⟨2⟩6. $y \in B' \subseteq U$
- ⟨1⟩3. Q.E.D.

PROOF: By Lemma 60.

□

Lemma 141. *Let X be a topological space and $Y \subseteq X$. Let \mathcal{S} be a basis for the topology on X . Then $\mathcal{S}' = \{S \cap Y \mid S \in \mathcal{S}\}$ is a subbasis for the subspace topology on Y .*

PROOF: The set $\{B \cap Y \mid B \text{ is a finite intersection of elements of } \mathcal{S}\}$ is a basis for the subspace topology by Lemma 140, and this is the set of all finite intersections of elements of \mathcal{S}' . \square

Lemma 142. *Let Y be a subspace of X . If U is open in Y and Y is open in X then U is open in X .*

PROOF:

$\langle 1 \rangle 1$. PICK V open in X such that $U = V \cap Y$

$\langle 1 \rangle 2$. U is open in X

PROOF: Since it is the intersection of two open sets V and Y .

\square

Theorem 143. *Let Y be a subspace of X and $A \subseteq Y$. If A is closed in Y and Y is closed in X then A is closed in X .*

PROOF: Pick a closed set C in X such that $A = C \cap Y$ (Theorem 138). Then A is the intersection of two sets closed in X , hence A is closed in X (Lemma 30).

\square

Theorem 144. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let A_i be a subspace of X_i for all $i \in I$. Then the product topology on $\prod_{i \in I} A_i$ is the same as the topology it inherits as a subspace of $\prod_{i \in I} X_i$.*

PROOF: The product topology is generated by the subbasis

$$\begin{aligned} & \{\pi_i^{-1}(U) \mid i \in I, U \text{ open in } A_i\} \\ &= \{\pi_i^{-1}(V) \cap A_i \mid i \in I, V \text{ open in } X_i\} \\ &= \{\pi_i^{-1}(V) \mid i \in I, V \text{ open in } X_i\} \cap \prod_{i \in I} A_i \end{aligned}$$

and this is a subbasis for the subspace topology by Lemma 141. \square

Theorem 145. *Let X be an ordered set in the order topology. Let $Y \subseteq X$ be an interval. Then the order topology on Y is the same as the subspace topology on Y .*

PROOF:

$\langle 1 \rangle 1$. The order topology is finer than the subspace topology.

$\langle 2 \rangle 1$. For every open ray R in X , the set $R \cap Y$ is open in the order topology.

$\langle 3 \rangle 1$. For all $a \in X$, we have $(-\infty, a) \cap Y$ is open in the order topology.

$\langle 4 \rangle 1$. CASE: For all $y \in Y$ we have $y < a$

PROOF: In this case $(-\infty, a) \cap Y = Y$.

$\langle 4 \rangle 2$. CASE: For all $y \in Y$ we have $a < y$

PROOF: In this case $(-\infty, a) \cap Y = \emptyset$.

$\langle 4 \rangle 3$. CASE: There exists $y \in Y$ such that $y \leq a$ and $y \in Y$ such that $a \leq y$

⟨5⟩1. $a \in Y$

PROOF: Because Y is an interval.

⟨5⟩2. $(-\infty, a) \cap Y = \{y \in Y \mid y < a\}$

⟨3⟩2. For all $a \in X$, we have $(a, +\infty) \cap Y$ is open in the order topology.

PROOF: Similar.

⟨2⟩2. Q.E.D.

PROOF: By Lemmas 113 and 141 and Proposition 68.

⟨1⟩2. The subspace topology is finer than the order topology.

⟨2⟩1. Every open ray in Y is open in the subspace topology.

PROOF: For any $a \in Y$ we have $(-\infty, a)_Y = (-\infty, a)_X \cap Y$ and $(a, +\infty)_Y = (a, +\infty)_X \cap Y$.

⟨2⟩2. Q.E.D.

PROOF: By Lemma 113 and Proposition 68

□

This example shows that we cannot remove the hypothesis that Y is an interval:

Example 146. The order topology on I_o^2 is different from the subspace topology as a subspace of \mathbb{R}^2 under the dictionary order topology.

PROOF: The set $\{1/2\} \times (1/2, 1)$ is open in the subspace topology but not in the order topology. □

Proposition 147. Let X be a topological space, Y a subspace of X , and Z a subspace of Y . Then the subspace topology on Z inherited from X is the same as the subspace topology on Z inherited from Y .

PROOF: The subspace topology inherited from Y is

$$\begin{aligned} & \{V \cap Z \mid V \text{ open in } Y\} \\ &= \{U \cap Y \cap Z \mid U \text{ open in } X\} \\ &= \{U \cap Z \mid U \text{ open in } X\} \end{aligned}$$

which is the subspace topology inherited from X . □

Definition 148 (Unit Circle). The *unit circle* S^1 is $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ as a subspace of \mathbb{R}^2 .

Definition 149 (Unit 2-sphere). The *unit 2-sphere* is $S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$ as a subspace of \mathbb{R}^3 .

Proposition 150. Let $f : X \rightarrow Y$ be an open map and $A \subseteq X$ be open. Then the restriction $f \upharpoonright A : A \rightarrow f(A)$ is an open map.

PROOF:

⟨1⟩1. LET: U be open in A

⟨1⟩2. U is open in X

PROOF: Lemma 142.

⟨1⟩3. $f(U)$ is open in Y

⟨1⟩4. $f(U)$ is open in $f(A)$

PROOF: Since $f(U) = f(U) \cap f(A)$.
 \square

Example 151. This example shows that we cannot remove the hypothesis that A is open.

Let $A = \{(x, y) \in \mathbb{R}^2 \mid (x > 0 \text{ and } y = 1/x) \text{ or } x = y = 0\}$. Then $\pi_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$ is an open map, but $\pi_1 \upharpoonright A : A \rightarrow [0, +\infty)$ is not, because it maps the set $\{0, 0\}$ which is open in A to $\{0\}$ which is not open in $[0, +\infty)$.

Proposition 152. *Let Y be a subspace of X . Let $A \subseteq Y$ and $l \in Y$. Then l is a limit point of A in Y if and only if l is a limit point of A in X .*

PROOF: Both are equivalent to the condition that any neighbourhood of l in X intersects A in a point other than l . \square

21 The Box Topology

Definition 153 (Box Topology). Let $\{A_i\}_{i \in I}$ be a family of topological spaces. The *box topology* on $\prod_{i \in I} A_i$ is the topology generated by the set of all sets of the form $\prod_{i \in I} U_i$ where $\{U_i\}_{i \in I}$ is a family such that each U_i is open in A_i .

This is a basis since it covers $\prod_{i \in I} A_i$ and is closed under intersection.

Proposition 154. *The box topology is finer than the product topology.*

PROOF: From Proposition 123. \square

Corollary 154.1. *If A_i is closed in X_i for all $i \in I$ then $\prod_{i \in I} A_i$ is closed in $\prod_{i \in I} X_i$ under the box topology.*

PROOF: From Proposition 124.

Proposition 155 (AC). *Let $\{A_i\}_{i \in I}$ be a family of topological spaces. For $i \in I$, let \mathcal{B}_i be a basis for the topology on A_i . Then $\mathcal{B} = \{\prod_{i \in I} B_i \mid \forall i \in I, B_i \in \mathcal{B}_i\}$ is a basis for the box topology on $\prod_{i \in I} A_i$.*

PROOF:

- $\langle 1 \rangle 1$. Every set of the form $\prod_{i \in I} B_i$ is open.
- $\langle 1 \rangle 2$. For every point $a \in \prod_{i \in I} A_i$ and every open set U with $a \in U$, there exists $B \in \mathcal{B}$ such that $a \in B \subseteq U$.
 - $\langle 2 \rangle 1$. LET: U be open and $a \in U$
 - $\langle 2 \rangle 2$. PICK a family $\{U_i\}_{i \in I}$ such that each U_i is open in A_i and $a \in \prod_{i \in I} U_i \subseteq U$.
 - $\langle 2 \rangle 3$. For $i \in I$, PICK $B_i \in \mathcal{B}_i$ such that $a_i \in B_i \subseteq U_i$
- PROOF: Using the Axiom of Choice.
- $\langle 2 \rangle 4$. $a \in \prod_{i \in I} B_i \subseteq U$
- $\langle 1 \rangle 3$. Q.E.D.

PROOF: Lemma 60.
 \square

Theorem 156. Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let A_i be a subspace of X_i for all $i \in I$. Give $\prod_{i \in I} X_i$ the box topology. Then the box topology on $\prod_{i \in I} A_i$ is the same as the topology it inherits as a subspace of $\prod_{i \in I} X_i$.

PROOF: The box topology is generated by the basis

$$\begin{aligned} & \left\{ \prod_{i \in I} U_i \mid \forall i \in I, U_i \text{ open in } A_i \right\} \\ &= \left\{ \prod_{i \in I} (V_i \cap A_i) \mid \forall i \in I, V_i \text{ open in } X_i \right\} \\ &= \left\{ \prod_{i \in I} V_i \mid \forall i \in I, V_i \text{ open in } X_i \right\} \cap \prod_{i \in I} A_i \end{aligned}$$

and this is a basis for the subspace topology by Lemma 140. \square

Proposition 157. Let $\{X_i\}_{i \in I}$ be a family of Hausdorff spaces. Then $\prod_{i \in I} X_i$ under the box topology is Hausdorff.

PROOF:

- $\langle 1 \rangle 1$. LET: $\{X_i\}_{i \in I}$ be a family of Hausdorff spaces.
- $\langle 1 \rangle 2$. LET: $a, b \in \prod_{i \in I} X_i$ with $a \neq b$
- $\langle 1 \rangle 3$. PICK $i \in I$ such that $a_i \neq b_i$
- $\langle 1 \rangle 4$. PICK U, V disjoint open sets in X_i with $a_i \in U$ and $b_i \in V$
- $\langle 1 \rangle 5$. $\pi_i^{-1}(U)$ and $\pi_i^{-1}(V)$ are disjoint open sets in $\prod_{i \in I} X_i$ with $a \in \pi_i^{-1}(U)$ and $b \in \pi_i^{-1}(V)$

\square

Proposition 158 (AC). Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Give $\prod_{i \in I} X_i$ the box topology. Let $A_i \subseteq X_i$ for all $i \in I$. Then

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i}.$$

PROOF:

- $\langle 1 \rangle 1$. $\overline{\prod_{i \in I} A_i} \subseteq \prod_{i \in I} \overline{A_i}$
- $\langle 2 \rangle 1$. For all $i \in I$ we have $A_i \subseteq \overline{A_i}$
PROOF: Lemma 42.
- $\langle 2 \rangle 2$. $\prod_{i \in I} A_i \subseteq \prod_{i \in I} \overline{A_i}$
- $\langle 2 \rangle 3$. Q.E.D.
- PROOF: Since $\prod_{i \in I} A_i$ is closed by Corollary 154.1.
- $\langle 1 \rangle 2$. $\prod_{i \in I} \overline{A_i} \subseteq \overline{\prod_{i \in I} A_i}$
- $\langle 2 \rangle 1$. LET: $x \in \prod_{i \in I} \overline{A_i}$
- $\langle 2 \rangle 2$. LET: U be a neighbourhood of x
- $\langle 2 \rangle 3$. PICK V_i open in X_i such that $x \in \prod_{i \in I} V_i \subseteq U$
- $\langle 2 \rangle 4$. For $i \in I$, pick $a_i \in V_i \cap A_i$
PROOF: By Theorem 45 and $\langle 2 \rangle 1$ using the Axiom of Choice.
- $\langle 2 \rangle 5$. U intersects $\prod_{i \in I} A_i$

⟨2⟩6. Q.E.D.

PROOF: $a \in U \cap \prod_{i \in I} A_i$.

□

The following example shows that Theorem 132 fails in the box topology.

Example 159. Define $f : \mathbb{R} \rightarrow \mathbb{R}^\omega$ by $f(t) = (t, t, \dots)$. Then $\pi_n \circ f = \text{id}_{\mathbb{R}}$ is continuous for all n . But f is not continuous when \mathbb{R}^ω is given the box topology because the inverse image of

$$(-1, 1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times \dots$$

is $\{0\}$ which is not open.

The following example shows that Proposition 131 fails in the box topology.

Example 160. Give \mathbb{R}^ω the box topology. Let $a_n = (1/n, 1/n, \dots)$ for $n \geq 1$ and $l = (0, 0, \dots)$. Then $\pi_i(a_n) \rightarrow \pi_i(l)$ as $n \rightarrow \infty$ for all i , but $a_n \not\rightarrow l$ as $n \rightarrow \infty$ since the open set

$$(-1, 1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times \dots$$

contains l but does not contain any a_n .

Example 161. The set \mathbb{R}^∞ is closed in \mathbb{R}^ω under the box topology. For let (a_n) be any sequence not in \mathbb{R}^∞ . Let U_n be an open interval around a_n that does not contain 0 if $a_n \neq 0$, and $U_n = \mathbb{R}$ if $a_n = 0$. Then $\prod_{n \geq 0} U_n$ is a neighbourhood of (a_n) that does not intersect \mathbb{R}^∞ .

22 T_1 Spaces

Definition 162 (T_1 Space). A topological space is T_1 if and only if every singleton is closed.

Lemma 163. A space is T_1 if and only if every finite set is closed.

PROOF: From Lemma 31. □

Theorem 164. In a T_1 space, a point a is a limit point of a set A if and only if every neighbourhood of a contains infinitely many points of A .

PROOF:

⟨1⟩1. If a is a limit point of A then every neighbourhood of a contains infinitely many points of A .

⟨2⟩1. ASSUME: a is a limit point of A .

⟨2⟩2. LET: U be a neighbourhood of a .

⟨2⟩3. ASSUME: for a contradiction U contains only finitely many points of A .

⟨2⟩4. $(U \cap A) \setminus \{a\}$ is closed.

PROOF: By the T_1 axiom.

⟨2⟩5. $(U \setminus A) \cup \{a\}$ is open.

PROOF: It is $U \setminus ((U \cap A) \setminus \{a\})$.

⟨2⟩6. $(U \setminus A) \cup \{a\}$ intersects A in a point other than a .

PROOF: From ⟨2⟩1.

⟨2⟩7. Q.E.D.

□

⟨1⟩2. If every neighbourhood of a contains infinitely many points of A then a is a limit point of A .

PROOF: Immediate from definitions.

□

(To see this does not hold in every space, see Proposition 56.)

Proposition 165. *A space is T_1 if and only if, for any two distinct points x and y , there exist neighbourhoods U of x and V of y such that $x \notin V$ and $y \notin U$.*

PROOF:

⟨1⟩1. LET: X be a topological space.

⟨1⟩2. If X is T_1 then, for any two distinct points x and y , there exist neighbourhoods U of x and V of y such that $x \notin V$ and $y \notin U$.

PROOF: This holds because $\{x\}$ and $\{y\}$ are closed.

⟨1⟩3. Suppose, for any two distinct points x and y , there exist neighbourhoods U of x and V of y such that $x \notin V$ and $y \notin U$. Then X is T_1 .

⟨2⟩1. ASSUME: For any two distinct points x and y , there exist neighbourhoods U of x and V of y such that $x \notin V$ and $y \notin U$.

⟨2⟩2. LET: $a \in X$

⟨2⟩3. $\{a\}$ is closed.

PROOF: For all $b \neq a$ there exists a neighbourhood U of b such that $U \subseteq X \setminus \{a\}$.

□

Proposition 166. *A subspace of a T_1 space is T_1 .*

PROOF: From Proposition 143.

23 Hausdorff Spaces

Definition 167 (Hausdorff Space). A topological space is *Hausdorff* if and only if, for any points x, y with $x \neq y$, there exist disjoint open sets U and V such that $x \in U$ and $y \in V$.

Theorem 168. *Every Hausdorff space is T_1 .*

PROOF:

⟨1⟩1. LET: X be a Hausdorff space.

⟨1⟩2. LET: $b \in X$

PROVE: $\overline{\{b\}} = \{b\}$

⟨1⟩3. ASSUME: $a \in \overline{\{b\}}$ and $a \neq b$

⟨1⟩4. PICK disjoint neighbourhoods U of a and V of b .

⟨1⟩5. U intersects $\{b\}$

PROOF: Theorem 45.

⟨1⟩6. $b \in U$

⟨1⟩7. Q.E.D.

PROOF: This contradicts the fact that U and V are disjoint (⟨1⟩4).

□

Proposition 169. *An infinite set under the finite complement topology is T_1 but not Hausdorff.*

PROOF:

⟨1⟩1. LET: X be an infinite set under the finite complement topology.

⟨1⟩2. Every singleton is closed.

PROOF: By definition.

⟨1⟩3. PICK $a, b \in X$ with $a \neq b$

⟨1⟩4. There are no disjoint neighbourhoods U of a and V of b .

⟨2⟩1. LET: U be a neighbourhood of a and V a neighbourhood of b .

⟨2⟩2. $X \setminus U$ and $X \setminus V$ are finite.

⟨2⟩3. PICK $c \in X$ that is not in $X \setminus U$ or $X \setminus V$.

⟨2⟩4. $c \in U \cap V$

□

Proposition 170. *The product of a family of Hausdorff spaces is Hausdorff.*

PROOF:

⟨1⟩1. LET: $\{X_i\}_{i \in I}$ be a family of Hausdorff spaces.

⟨1⟩2. LET: $a, b \in \prod_{i \in I} X_i$ with $a \neq b$

⟨1⟩3. PICK $i \in I$ such that $a_i \neq b_i$

⟨1⟩4. PICK U, V disjoint open sets in X_i with $a_i \in U$ and $b_i \in V$

⟨1⟩5. $\pi_i^{-1}(U)$ and $\pi_i^{-1}(V)$ are disjoint open sets in $\prod_{i \in I} X_i$ with $a \in \pi_i^{-1}(U)$ and $b \in \pi_i^{-1}(V)$

□

Theorem 171. *Every linearly ordered set under the order topology is Hausdorff.*

PROOF:

⟨1⟩1. LET: X be a linearly ordered set under the order topology.

⟨1⟩2. LET: $a, b \in X$ with $a \neq b$

⟨1⟩3. ASSUME: w.l.o.g. $a < b$

⟨1⟩4. CASE: There exists c such that $a < c < b$

PROOF: The sets $(-\infty, c)$ and $(c, +\infty)$ are disjoint neighbourhoods of a and b respectively.

⟨1⟩5. CASE: There is no c such that $a < c < b$

PROOF: The sets $(-\infty, b)$ and $(a, +\infty)$ are disjoint neighbourhoods of a and b respectively.

□

Theorem 172. *A subspace of a Hausdorff space is Hausdorff.*

PROOF:

- ⟨1⟩1. LET: X be a Hausdorff space and Y a subspace of X .
- ⟨1⟩2. LET: $x, y \in Y$ with $x \neq y$
- ⟨1⟩3. PICK disjoint neighbourhoods U of x and V of y in X .
- ⟨1⟩4. $U \cap Y$ and $V \cap Y$ are disjoint neighbourhoods of x and y respectively in Y .

□

Proposition 173. *A space X is Hausdorff if and only if the diagonal $\Delta = \{(x, x) \mid x \in X\}$ is closed in X^2 .*

PROOF:

X is Hausdorff

$$\Leftrightarrow \forall x, y \in X. x \neq y \Rightarrow \exists V, W \text{ open. } x \in V \wedge y \in W \wedge V \cap W = \emptyset$$

$$\Leftrightarrow \forall (x, y) \in X^2 \setminus \Delta. \exists V, W \text{ open. } (x, y) \subseteq V \times W \subseteq X^2 \setminus \Delta$$

$$\Leftrightarrow \Delta \text{ is closed}$$

□

24 The First Countability Axiom

Definition 174 (First Countability Axiom). A topological space X satisfies the *first countability axiom*, or is *first countable*, if and only if every point has a countable local basis.

Lemma 175 (Sequence Lemma (CC)). *Let X be a first countable space. Let $A \subseteq X$ and $l \in A$. Then there exists a sequence in A that converges to l .*

PROOF:

- ⟨1⟩1. PICK a countable local basis $\{B_n \mid n \in \mathbb{Z}^+\}$ at l such that $B_1 \supseteq B_2 \supseteq \dots$.

PROOF: Lemma 70.

- ⟨1⟩2. For all $n \geq 1$, PICK $a_n \in A \cap B_n$.

PROVE: $a_n \rightarrow l$ as $n \rightarrow \infty$

- ⟨1⟩3. LET: U be a neighbourhood of l

- ⟨1⟩4. PICK N such that $B_N \subseteq U$

- ⟨1⟩5. For $n \geq N$ we have $a_n \in U$

PROOF: $a_n \in B_n \subseteq B_N \subseteq U$

□

Theorem 176 (CC). *Let X be a first countable space and Y a topological space. Let $f : X \rightarrow Y$. Suppose that, for every sequence (x_n) in X and $l \in X$, if $x_n \rightarrow l$ as $n \rightarrow \infty$, then $f(x_n) \rightarrow f(l)$ as $n \rightarrow \infty$. Then f is continuous.*

PROOF:

- ⟨1⟩1. LET: $A \subseteq X$

- ⟨1⟩2. LET: $a \in A$

PROVE: $f(a) \in \overline{f(A)}$

- ⟨1⟩3. PICK a sequence (x_n) in A that converges to a .

PROOF: By the Sequence Lemma.

⟨1⟩4. $f(x_n) \rightarrow f(a)$

⟨1⟩5. $f(a) \in f(A)$

PROOF: By Lemma 74.

⟨1⟩6. Q.E.D.

PROOF: By Theorem 92.

□

Example 177 (CC). The space \mathbb{R}^ω under the box product is not first countable.

PROOF: Let $\{B_n \mid n \geq 0\}$ be a countable set of neighbourhoods of $\vec{0}$. We will construct a neighbourhood of $\vec{0}$ that does not include any of these.

For $n \geq 0$, pick a neighbourhood U_n of 0 such that $U_n \subset \pi_n(B_p)$. Then $\prod_{n=0}^\infty U_n$ is a neighbourhood of $\vec{0}$ that does not include any B_n . □

Example 178. If J is an uncountable set then \mathbb{R}^J is not first countable.

PROOF:

⟨1⟩1. LET: $\{B_n \mid n \geq 0\}$ be any countable set of neighbourhoods of $\vec{0}$.

⟨1⟩2. For $n \geq 0$, PICK a basis element $\prod_{\alpha \in J} U_{n\alpha}$ that contains $\vec{0}$ and is included in B_n .

PROOF: Using the Axiom of Countable Choice.

⟨1⟩3. For $n \geq 0$,

LET: $J_n = \{\alpha \in J \mid U_{n\alpha} \neq \mathbb{R}\}$

⟨1⟩4. PICK $\beta \in J$ such that $\beta \notin J_n$ for any n .

PROOF: Since each J_n is finite so $\bigcup_n J_n$ is countable.

⟨1⟩5. $\pi_\beta((-1, 1))$ is a neighbourhood of $\vec{0}$ that does not include any B_n .

□

Example 179. The space \mathbb{R}_l is first countable.

PROOF: For any $a \in \mathbb{R}$, the set $\{[a, a + 1/n) \mid n \geq 1\}$ is a countable local basis.

Example 180. The ordered square is first countable.

PROOF: For any $(a, b) \in I_o^2$ with $b \neq 0, 1$, the set $\{(\{a\} \times (b - 1/n, b + 1/n)) \cap I_o^2 \mid n \geq 1\}$ is a countable local basis.

25 Strong Continuity

Definition 181 (Strongly Continuous). Let X and Y be topological spaces and $f : X \rightarrow Y$ be a function. Then f is *strongly continuous* if and only if, for every subset $U \subseteq Y$, we have U is open in Y if and only if $f^{-1}(U)$ is open in X .

Proposition 182. Let X and Y be topological spaces and $f : X \rightarrow Y$ be a function. Then f is strongly continuous if and only if, for every subset $C \subseteq Y$, we have C is closed in Y if and only if $f^{-1}(C)$ is closed in X .

PROOF: Since $X \setminus f^{-1}(C) = f^{-1}(Y \setminus C)$. □

Proposition 183. *Let X, Y and Z be topological spaces. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$. If f and g are strongly continuous then so is $g \circ f$.*

PROOF: Since $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$. \square

Proposition 184. *Let X, Y and Z be topological spaces. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$. If $g \circ f$ is continuous and f is strongly continuous then g is continuous.*

PROOF:

$\langle 1 \rangle 1$. LET: $V \subseteq Z$ be open.

$\langle 1 \rangle 2$. $f^{-1}(g^{-1}(V))$ is open in X .

PROOF: Since $g \circ f$ is continuous.

$\langle 1 \rangle 3$. $f^{-1}(V)$ is open in Y .

PROOF: Since g is strongly continuous.

\square

Proposition 185. *Let X, Y and Z be topological spaces. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$. If $g \circ f$ is strongly continuous and f is strongly continuous then g is strongly continuous.*

PROOF: For $V \subseteq Z$, we have V is open iff $f^{-1}(g^{-1}(V))$ is open iff $g^{-1}(V)$ is open.

26 Saturated Sets

Definition 186. Let X and Y be sets and $p : X \rightarrow Y$ a surjective function. Let $C \subseteq X$. Then C is *saturated* with respect to p if and only if, for all $x, y \in X$, if $x \in C$ and $p(x) = p(y)$ then $y \in C$.

Proposition 187. *Let X and Y be sets and $p : X \rightarrow Y$ a surjective function. Let $C \subseteq X$. Then the following are equivalent:*

1. C is saturated with respect to p .
2. There exists $D \subseteq Y$ such that $C = p^{-1}(D)$
3. $C = p^{-1}(p(C))$.

PROOF:

$\langle 1 \rangle 1$. $1 \Rightarrow 3$

$\langle 2 \rangle 1$. ASSUME: C is saturated with respect to p .

$\langle 2 \rangle 2$. $C \subseteq p^{-1}(p(C))$

PROOF: Trivial.

$\langle 2 \rangle 3$. $p^{-1}(p(C)) \subseteq C$

$\langle 3 \rangle 1$. LET: $x \in p^{-1}(p(C))$

$\langle 3 \rangle 2$. $p(x) \in p(C)$

$\langle 3 \rangle 3$. There exists $y \in C$ such that $p(x) = p(y)$

$\langle 3 \rangle 4$. $x \in C$

PROOF: From $\langle 2 \rangle 1$.

$\langle 1 \rangle 2$. $3 \Rightarrow 2$

PROOF: Trivial.

$\langle 1 \rangle 3$. $2 \Rightarrow 1$

PROOF: This follows because if $p(x) \in D$ and $p(x) = p(y)$ then $p(y) \in D$.

□

27 Quotient Maps

Definition 188 (Quotient Map). Let X and Y be topological spaces and $p : X \rightarrow Y$. Then p is a *quotient map* if and only if p is surjective and strongly continuous.

Proposition 189. Let X and Y be topological spaces and $p : X \rightarrow Y$ be a surjective function. Then the following are equivalent.

1. p is a quotient map.
2. p is continuous and maps saturated open sets to open sets.
3. p is continuous and maps saturated closed sets to closed sets.

PROOF:

$\langle 1 \rangle 1$. $1 \Rightarrow 2$

$\langle 2 \rangle 1$. ASSUME: p is a quotient map.

$\langle 2 \rangle 2$. LET: U be a saturated open set in X .

$\langle 2 \rangle 3$. $p^{-1}(p(U))$ is open in X .

PROOF: Since $U = p^{-1}(p(U))$ by Proposition 187.

$\langle 2 \rangle 4$. $p(U)$ is open in Y .

PROOF: From $\langle 2 \rangle 1$.

$\langle 1 \rangle 2$. $1 \Rightarrow 3$

PROOF: Similar.

$\langle 1 \rangle 3$. $2 \Rightarrow 1$

$\langle 2 \rangle 1$. ASSUME: p is continuous and maps saturated open sets to open sets.

$\langle 2 \rangle 2$. LET: $U \subseteq Y$

$\langle 2 \rangle 3$. ASSUME: $p^{-1}(U)$ is open in X

$\langle 2 \rangle 4$. $p^{-1}(U)$ is saturated.

PROOF: Proposition 187.

$\langle 2 \rangle 5$. U is open in Y .

$\langle 1 \rangle 4$. $3 \Rightarrow 1$

PROOF: Similar.

□

Corollary 189.1. Every surjective continuous open map is a quotient map.

Corollary 189.2. Every surjective continuous closed map is a quotient map.

Example 190. The converses of these corollaries do not hold.

Let $A = \{(x, y) \mid x \geq 0\} \cup \{(x, y) \mid y = 0\}$. Then $\pi_1 : A \rightarrow \mathbb{R}$ is a quotient map, but not an open map or a closed map.

We prove that π_1 maps saturated open sets to open sets:

- $\langle 1 \rangle 1$. LET: $\pi_1^{-1}(U)$ be a saturated open set in A
PROVE: U is open in \mathbb{R}
- $\langle 1 \rangle 2$. LET: $x \in U$
- $\langle 1 \rangle 3$. $(x, 0) \in \pi_1(U)^{-1}$
- $\langle 1 \rangle 4$. PICK W, V open in \mathbb{R} such that $(x, 0) \subseteq W \times V \subseteq \pi_1(U)^{-1}$
- $\langle 1 \rangle 5$. $x \in W \subseteq U$

It is not an open map because it maps $((-1, 1) \times (1, 2)) \cap A$ to $[0, 1)$.

It is not a closed map because it maps $\{(x, 1/x) \mid x > 0\}$ to $(0, +\infty)$.

Proposition 191. Let $p : X \twoheadrightarrow Y$ be a quotient map. Let $A \subseteq X$ be saturated with respect to p . Let $q : A \twoheadrightarrow p(A)$ be the restriction of p .

- 1. If A is either open or closed in X then q is a quotient map.
- 2. If p is either an open map or a closed map then q is a quotient map.

PROOF:

- $\langle 1 \rangle 1$. LET: $p : X \twoheadrightarrow Y$ be a quotient map.
- $\langle 1 \rangle 2$. LET: $A \subseteq X$ be saturated with respect to p .
- $\langle 1 \rangle 3$. LET: $q : A \twoheadrightarrow p(A)$ be the restriction of p .
- $\langle 1 \rangle 4$. q is continuous.
PROOF: Theorem 96.
- $\langle 1 \rangle 5$. If A is open in X then q is a quotient map.
 - $\langle 2 \rangle 1$. ASSUME: A is open in X .
 - $\langle 2 \rangle 2$. q maps saturated open sets to open sets.
 - $\langle 3 \rangle 1$. LET: $U \subseteq A$ be saturated with respect to q and open in A
 - $\langle 3 \rangle 2$. U is saturated with respect to p
 - $\langle 4 \rangle 1$. LET: $x, y \in X$
 - $\langle 4 \rangle 2$. ASSUME: $x \in U$
 - $\langle 4 \rangle 3$. ASSUME: $p(x) = p(y)$
 - $\langle 4 \rangle 4$. $x \in A$
PROOF: From $\langle 3 \rangle 1$ and $\langle 4 \rangle 2$.
 - $\langle 4 \rangle 5$. $y \in A$
PROOF: From $\langle 1 \rangle 2$ and $\langle 4 \rangle 3$
 - $\langle 4 \rangle 6$. $q(x) = x(y)$
PROOF: From $\langle 1 \rangle 3$, $\langle 4 \rangle 3$, $\langle 4 \rangle 4$, $\langle 4 \rangle 5$.
 - $\langle 4 \rangle 7$. $y \in U$
PROOF: From $\langle 3 \rangle 1$, $\langle 4 \rangle 2$, $\langle 4 \rangle 6$
 - $\langle 3 \rangle 3$. U is open in X
PROOF: Lemma 142, $\langle 2 \rangle 1$, $\langle 3 \rangle 1$.
 - $\langle 3 \rangle 4$. $p(U)$ is open in Y
PROOF: Proposition 189, $\langle 1 \rangle 1$, $\langle 3 \rangle 2$, $\langle 3 \rangle 3$

⟨3⟩5. $q(U)$ is open in $p(A)$

PROOF: Since $q(U) = p(U) = p(U) \cap p(A)$.

⟨2⟩3. Q.E.D.

PROOF: By Proposition 189.

⟨1⟩6. If A is closed in X then q is a quotient map.

PROOF: Similar.

⟨1⟩7. If p is an open map then q is a quotient map.

⟨2⟩1. ASSUME: p is an open map

⟨2⟩2. q maps saturated open sets to open sets.

⟨3⟩1. LET: U be open in A and saturated with respect to q

⟨3⟩2. PICK V open in X such that $U = A \cap V$

⟨3⟩3. $p(V)$ is open in Y

⟨3⟩4. $q(U) = p(V) \cap p(A)$

⟨4⟩1. $q(U) \subseteq p(V) \cap p(A)$

PROOF: From ⟨3⟩2.

⟨4⟩2. $p(V) \cap p(A) \subseteq q(U)$

⟨5⟩1. LET: $y \in p(V) \cap p(A)$

⟨5⟩2. PICK $x \in V$ and $x' \in A$ such that $p(x) = p(x') = y$

⟨5⟩3. $x \in A$

PROOF: By ⟨1⟩2.

⟨5⟩4. $x \in U$

PROOF: From ⟨3⟩2

⟨2⟩3. Q.E.D.

PROOF: By Proposition 189.

⟨1⟩8. If p is a closed map then q is a quotient map.

PROOF: Similar.

□

Example 192. This example shows we cannot remove the hypotheses on A and p .

Define $f : [0, 1] \rightarrow [2, 3] \rightarrow [0, 2]$ by $f(x) = x$ if $x \leq 1$, $f(x) = x - 1$ if $x \geq 2$. Then f is a quotient map but its restriction f' to $[0, 1] \cup [2, 3]$ is not, because $f'^{-1}([1, 2])$ is open but $[1, 2]$ is not.

For a counterexample where A is saturated, see Example 198.

Proposition 193. Let $p : A \rightarrow C$ and $q : B \rightarrow D$ be open quotient maps. Then $p \times q : A \times B \rightarrow C \times D$ is an open quotient map.

PROOF: From Corollary 189.1, Proposition 136 and Theorem 132. □

Theorem 194. Let $p : X \rightarrow Y$ be a quotient map. Let Z be a topological space and $f : Y \rightarrow Z$ be a function. Then

1. $f \circ p$ is continuous if and only if f is continuous.

2. $f \circ p$ is a quotient map if and only if f is a quotient map.

PROOF:

- $\langle 1 \rangle 1$. If $f \circ p$ is continuous then f is continuous.
 PROOF: Proposition 184.
 $\langle 1 \rangle 2$. If f is continuous then $f \circ p$ is continuous.
 PROOF: Theorem 95.
 $\langle 1 \rangle 3$. If $f \circ p$ is a quotient map then f is a quotient map.
 PROOF: Proposition 185.
 $\langle 1 \rangle 4$. If f is a quotient map then $f \circ p$ is a quotient map.
 PROOF: From Proposition 183.

□

Proposition 195. *Let X and Y be topological spaces. Let $p : X \rightarrow Y$ and $f : Y \rightarrow X$ be continuous maps such that $p \circ f = \text{id}_Y$. Then p is a quotient map.*

PROOF:

- $\langle 1 \rangle 1$. LET: $V \subseteq Y$
 $\langle 1 \rangle 2$. ASSUME: $p^{-1}(V)$ is open in X .
 $\langle 1 \rangle 3$. $f^{-1}(p^{-1}(V))$ is open in Y .
 PROOF: Because f is continuous.
 $\langle 1 \rangle 4$. V is open in Y .
 PROOF: Because $f^{-1}(p^{-1}(V)) = V$.

□

28 Quotient Topology

Definition 196 (Quotient Topology). Let X be a topological space, Y a set and $p : X \rightarrow Y$ be a surjective function. Then the *quotient topology* on Y is the unique topology on Y with respect to which p is a quotient map, namely $\mathcal{T} = \{U \in \mathcal{P}Y \mid p^{-1}(U) \text{ is open in } X\}$.

We prove this is a topology.

PROOF:

- $\langle 1 \rangle 1$. $Y \in \mathcal{T}$
 PROOF: Since $p^{-1}(Y) = X$ by surjectivity.
 $\langle 1 \rangle 2$. For all $\mathcal{A} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{A} \in \mathcal{T}$
 PROOF: Since $p^{-1}(\bigcup \mathcal{A}) = \bigcup_{U \in \mathcal{A}} p^{-1}(U)$
 $\langle 1 \rangle 3$. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$
 PROOF: Since $p^{-1}(U \cap V) = p^{-1}(U) \cap p^{-1}(V)$.

□

Definition 197 (Quotient Space). Let X be a topological space and \sim an equivalence relation on X . Let $p : X \rightarrow X/\sim$ be the canonical surjection. Then X/\sim under the quotient topology is called a *quotient space*, *identification space* or *decomposition space* of X .

Here is a counterexample showing we cannot remove all the hypotheses of Proposition 191 except that A is saturated.

Example 198. Let $X = (0, 1/2] \cup \{1\} \cup \{1 + 1/n : n \geq 2\}$ as a subspace of \mathbb{R} . Define R to be the equivalence relation on X where xRy iff $(x = y \text{ or } |x - y| = 1)$, so we identify $1/n$ with $1 + 1/n$ for all $n \geq 2$. Let Y be the resulting quotient space X/R in the quotient topology and $p : X \twoheadrightarrow Y$ the canonical surjection.

Let $A = \{1\} \cup (0, 1/2] \setminus \{1/n : n \geq 2\} \subseteq X$. Then A is saturated under p but the restriction q of p to A is not a quotient map because it maps the saturated open set $\{1\}$ to $\{1\}$ which is not open in $p(A)$.

Proposition 199. Let X, Y and Z be topological spaces. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$. If f and g are quotient maps then so is $g \circ f$.

PROOF: From Proposition 183. \square

Example 200. The product of two quotient maps is not necessarily a quotient map.

Let $X = \mathbb{R}$ and X^* the quotient space formed by identifying all positive integers. Let $p : X \twoheadrightarrow X^*$ be the canonical surjection.

We prove $p \times \text{id}_{\mathbb{Q}} : X \times \mathbb{Q} \rightarrow X^* \times \mathbb{Q}$ is not a quotient map.

PROOF:

$\langle 1 \rangle 1$. For $n \geq 1$,

LET: $c_n = \sqrt{2}/n$

$\langle 1 \rangle 2$. For $n \geq 1$,

LET: $U_n = \{(x, y) \in X \times \mathbb{Q} \mid n - 1/4 < x < n + 1/4, (y + n > x + c_n \text{ and } y + n > -x + c_n) \text{ or } (y + n < x + c_n \text{ and } y + n < -x + c_n)\}$

$\langle 1 \rangle 3$. For $n \geq 1$, we have U_n is open in $X \times \mathbb{Q}$

$\langle 1 \rangle 4$. For $n \geq 1$, we have $\{n\} \times \mathbb{Q} \subseteq U_n$

$\langle 1 \rangle 5$. LET: $U = \bigcup_{n=1}^{\infty} U_n$

$\langle 1 \rangle 6$. U is open in $X \times \mathbb{Q}$

$\langle 1 \rangle 7$. U is saturated with respect to $p \times \text{id}_{\mathbb{Q}}$

$\langle 1 \rangle 8$. LET: $U' = (p \times \text{id}_{\mathbb{Q}})(U)$

$\langle 1 \rangle 9$. ASSUME: for a contradiction U' is open in $X^* \times \mathbb{Q}$

$\langle 1 \rangle 10$. $(1, 0) \in U'$

$\langle 1 \rangle 11$. PICK a neighbourhood W of 1 in X^* and $\delta > 0$ such that $W \times (-\delta, \delta) \subseteq U'$

$\langle 1 \rangle 12$. $p^{-1}(W) \times (-\delta, \delta) \subseteq U$

$\langle 1 \rangle 13$. PICK n such that $c_n < \delta$

$\langle 1 \rangle 14$. $n \in p^{-1}(W)$

$\langle 1 \rangle 15$. PICK $\epsilon > 0$ such that $\epsilon < \delta - c_n$ and $\epsilon < 1/4$ and $(n - \epsilon, n + \epsilon) \subseteq p^{-1}(W)$

$\langle 1 \rangle 16$. $(n - \epsilon, n + \epsilon) \times (-\delta, \delta) \subseteq U$

$\langle 1 \rangle 17$. PICK a rational y such that $c_n - \epsilon/2 < y < c_n + \epsilon/2$

$\langle 1 \rangle 18$. $(n + \epsilon/2, y) \notin U$

$\langle 1 \rangle 19$. Q.E.D.

PROOF: This contradicts $\langle 1 \rangle 16$.

\square

Proposition 201. Let X be a topological space and \sim an equivalence relation on X . Then X/\sim is T_1 if and only if every equivalence class is closed in X .

PROOF: Immediate from definitions. \square

29 Retractions

Definition 202 (Retraction). Let X be a topological space and $A \subseteq X$. A *retraction* of X onto A is a continuous map $r : X \rightarrow A$ such that, for all $a \in A$, we have $r(a) = a$.

Proposition 203. *Every retraction is a quotient map.*

PROOF: Proposition 195 with f the inclusion $A \hookrightarrow X$. \square

30 Homogeneous Spaces

Definition 204 (Homogeneous). A topological space X is *homogeneous* if and only if, for any points $a, b \in X$, there exists a homeomorphism $\phi : X \cong X$ such that $\phi(a) = b$.

31 Regular Spaces

Definition 205 (Regular Space). A topological space X is *regular* if and only if, for any closed set A and point $a \notin A$, there exist disjoint open sets U, V such that $A \subseteq U$ and $a \in V$.

32 Connected Spaces

Definition 206 (Separation). A *separation* of a topological space X is a pair of disjoint open sets U, V such that $U \cup V = \emptyset$.

Definition 207 (Connected). A topological space is *connected* if and only if it has no separation; otherwise it is *disconnected*.

Proposition 208. *A topological space X is connected if and only if the only sets that are both open and closed are X and \emptyset .*

Immediate from definitions.

Lemma 209. *If Y is a subspace of X , a separation of Y is a pair of disjoint nonempty sets A and B whose union is Y , neither of which contains a limit point of the other.*

PROOF:

$\langle 1 \rangle 1$. LET: $A, B \subseteq Y$

$\langle 1 \rangle 2$. If A and B form a separation of Y then A and B are disjoint and nonempty, $A \cup B = Y$, and neither of A and B contains a limit point of the other.

$\langle 2 \rangle 1$. ASSUME: A and B form a separation of Y

$\langle 2 \rangle 2$. A and B are disjoint and nonempty and $A \cup B = Y$

PROOF: From $\langle 2 \rangle 1$ and the definition of separation.

$\langle 2 \rangle 3$. A does not contain a limit point of B

$\langle 3 \rangle 1$. ASSUME: for a contradiction $l \in A$ and l is a limit point of B in X .
 $\langle 3 \rangle 2$. l is a limit point of B in Y
 PROOF: Proposition 152.
 $\langle 3 \rangle 3$. $l \in B$
 $\langle 4 \rangle 1$. B is closed in Y
 PROOF: Since A is open in Y and $B = Y \setminus A$ from $\langle 2 \rangle 1$.
 $\langle 4 \rangle 2$. Q.E.D.
 PROOF: Corollary 55.1.
 $\langle 3 \rangle 4$. Q.E.D.
 PROOF: This contradicts the fact that $A \cap B = \emptyset$ ($\langle 2 \rangle 1$).
 $\langle 2 \rangle 4$. B does not contain a limit point of A
 PROOF: Similar.
 $\langle 1 \rangle 3$. If A and B are disjoint and nonempty, $A \cup B = Y$, and neither of A and B contains a limit point of the other, then A and B form a separation of Y .
 $\langle 2 \rangle 1$. ASSUME: A and B are disjoint and nonempty, $A \cup B = Y$, and neither of A and B contains a limit point of the other.
 $\langle 2 \rangle 2$. A is open in Y
 $\langle 3 \rangle 1$. B is closed in Y
 $\langle 4 \rangle 1$. LET: l be a limit point of B in Y
 $\langle 4 \rangle 2$. l is a limit point of B in X
 PROOF: Proposition 152.
 $\langle 4 \rangle 3$. $l \notin A$
 PROOF: By $\langle 2 \rangle 1$
 $\langle 4 \rangle 4$. $l \in B$
 PROOF: By $\langle 2 \rangle 1$ since $A \cup B = Y$
 $\langle 4 \rangle 5$. Q.E.D.
 PROOF: Corollary 55.1.
 $\langle 3 \rangle 2$. Q.E.D.
 PROOF: Since $A = Y \setminus B$.
 $\langle 2 \rangle 3$. B is open in Y
 PROOF: Similar.

□

Example 210. Every set under the indiscrete topology is connected.

Example 211. The discrete topology on a set X is connected if and only if $|X| \leq 1$.

Example 212. The finite complement topology on a set X is connected if and only if either $|X| \leq 1$ or X is infinite.

Example 213. The countable complement topology on a set X is connected if and only if either $|X| \leq 1$ or X is uncountable.

Example 214. The rationals \mathbb{Q} are disconnected. For any irrational a , the sets $(-\infty, a) \cap \mathbb{Q}$ and $(a, +\infty) \cap \mathbb{Q}$ form a separation of \mathbb{Q} .

Lemma 215. *Let X be a topological space. If C and D form a separation of X , and Y is a connected subspace of X , then either $Y \subseteq C$ or $Y \subseteq D$.*

PROOF: Otherwise $Y \cap C$ and $Y \cap D$ would form a separation of Y . \square

Theorem 216. *The union of a set of connected subspaces of a space X that have a point in common is connected.*

PROOF:

$\langle 1 \rangle 1$. LET: \mathcal{A} be a set of connected subspaces of the space X that have the point a in common.

$\langle 1 \rangle 2$. ASSUME: for a contradiction C and D form a separation of $\bigcup \mathcal{A}$

$\langle 1 \rangle 3$. ASSUME: without loss of generality $a \in C$

$\langle 1 \rangle 4$. For all $A \in \mathcal{A}$ we have $A \subseteq C$

PROOF: Lemma 215.

$\langle 1 \rangle 5$. $D = \emptyset$

$\langle 1 \rangle 6$. Q.E.D.

PROOF: This contradicts $\langle 1 \rangle 2$.

\square

Theorem 217. *Let X be a topological space and A a connected subspace of X . If $A \subseteq B \subseteq \overline{A}$ then B is connected.*

PROOF:

$\langle 1 \rangle 1$. ASSUME: for a contradiction C and D form a separation of B .

$\langle 1 \rangle 2$. ASSUME: without loss of generality $A \subseteq C$

PROOF: Lemma 215.

$\langle 1 \rangle 3$. $B \subseteq C$

$\langle 2 \rangle 1$. LET: $x \in B$

$\langle 2 \rangle 2$. $x \in \overline{A}$

$\langle 2 \rangle 3$. Either $x \in A$ or x is a limit point of A .

PROOF: Theorem 55.

$\langle 2 \rangle 4$. Either $x \in A$ or x is a limit point of C .

PROOF: Lemma 57, $\langle 1 \rangle 2$.

$\langle 2 \rangle 5$. $x \in C$

PROOF: Lemma 209.

$\langle 1 \rangle 4$. $D = \emptyset$

$\langle 1 \rangle 5$. Q.E.D.

PROOF: This contradicts $\langle 1 \rangle 1$.

\square

Theorem 218. *The image of a connected space under a continuous map is connected.*

PROOF:

$\langle 1 \rangle 1$. LET: $f : X \rightarrow Y$ be a surjective continuous map where X is connected.

$\langle 1 \rangle 2$. ASSUME: for a contradiction C and D form a separation of Y .

$\langle 1 \rangle 3$. $f^{-1}(C)$ and $f^{-1}(D)$ form a separation of X .

□

Theorem 219. *The product of a family of connected spaces is connected.*

PROOF:

⟨1⟩1. The product of two connected spaces is connected.

⟨2⟩1. LET: X and Y be connected spaces.

⟨2⟩2. PICK $a \in X$ and $b \in Y$

PROOF: We may assume X and Y are nonempty since otherwise $X \times Y = \emptyset$ which is connected.

⟨2⟩3. $X \times \{b\}$ is connected.

PROOF: It is homeomorphic to X .

⟨2⟩4. For all $x \in X$ we have $\{x\} \times Y$ is connected.

PROOF: It is homeomorphic to Y .

⟨2⟩5. For any $x \in X$

LET: $T_x = (X \times \{b\}) \cup (\{x\} \times Y)$

⟨2⟩6. For all $x \in X$, T_x is connected.

PROOF: Theorem 216 since $(x, b) \in (X \times \{b\}) \cap (\{x\} \times Y)$.

⟨2⟩7. $X \times Y$ is connected.

PROOF: Theorem 216 since $X \times Y = \bigcup_{x \in X} T_x$ and (a, b) is a point in every T_x .

⟨1⟩2. The product of a finite family of connected spaces is connected.

PROOF: From ⟨1⟩1 by induction.

⟨1⟩3. The product of any family of connected spaces is connected.

⟨2⟩1. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of connected spaces.

⟨2⟩2. LET: $X = \prod_{\alpha \in J} X_\alpha$

⟨2⟩3. PICK $a \in X$

PROOF: We may assume $X \neq \emptyset$ as the empty space is connected.

⟨2⟩4. For every finite subset K of J ,

LET: $X_K = \{x \in X \mid \forall \alpha \in J \setminus K. x_\alpha = a_\alpha\}$

⟨2⟩5. For every finite $K \subseteq J$, we have X_K is connected.

PROOF: From ⟨1⟩2 since $X_K \cong \prod_{\alpha \in K} X_\alpha$.

⟨2⟩6. LET: $Y = \bigcup_K X_K$

⟨2⟩7. Y is connected

PROOF: Theorem 216 since a is a common point.

⟨2⟩8. $X = \bar{Y}$

⟨3⟩1. LET: $x \in X$

⟨3⟩2. LET: $U = \prod_{\alpha \in J} U_\alpha$ be a basic neighbourhood of x where $U_\alpha = X_\alpha$ for all α except $\alpha \in K$ for some finite $K \subseteq J$

⟨3⟩3. LET: $y \in X$ be the point with $y_\alpha = x_\alpha$ for $\alpha \in K$ and $y_\alpha = a_\alpha$ for all other α

⟨3⟩4. $y \in U \cap X_K$

⟨3⟩5. $y \in U \cap Y$

⟨2⟩9. X is connected.

PROOF: Theorem 217.

□

Example 220. The set \mathbb{R}^ω is disconnected under the box topology. The set of bounded sequences and the set of unbounded sequences form a separation.

Proposition 221. Let \mathcal{T} and \mathcal{T}' be two topologies on the same set X . If $\mathcal{T} \subseteq \mathcal{T}'$ and (X, \mathcal{T}') is connected then (X, \mathcal{T}) is connected.

PROOF: If U and V form a separation of (X, \mathcal{T}) then they form a separation of (X, \mathcal{T}') . \square

Proposition 222. Let X be a topological space and (A_n) a sequence of connected subspaces of X . If $A_n \cap A_{n+1} \neq \emptyset$ for all n then $\bigcup_n A_n$ is connected.

PROOF:

$\langle 1 \rangle 1$. ASSUME: for a contradiction C and D form a separation of $\bigcup_n A_n$

$\langle 1 \rangle 2$. ASSUME: without loss of generality $A_0 \subseteq C$

PROOF: Lemma 215.

$\langle 1 \rangle 3$. For all n we have $A_n \subseteq C$

PROOF:

$\langle 2 \rangle 1$. ASSUME: $A_n \subseteq C$

$\langle 2 \rangle 2$. PICK $x \in A_n \cap A_{n+1}$

$\langle 2 \rangle 3$. $x \in C$

$\langle 2 \rangle 4$. $A_{n+1} \subseteq C$

PROOF: Lemma 215.

$\langle 2 \rangle 5$. Q.E.D.

PROOF: The result follows by induction.

$\langle 1 \rangle 4$. $D = \emptyset$

$\langle 1 \rangle 5$. Q.E.D.

PROOF: This contradicts $\langle 1 \rangle 1$.

\square

Proposition 223. Let X be a topological space. Let $A, C \subseteq X$. If C is connected and intersects both A and $X \setminus A$ then C intersects ∂A .

PROOF: Otherwise $C \cap A^\circ$ and $C \setminus \overline{A}$ would form a separation of C . \square

Example 224. The space \mathbb{R}_l is disconnected. For any real x , the sets $(-\infty, x)$ and $[x, +\infty)$ form a separation.

Proposition 225. Let X and Y be connected spaces. Let A be a proper subset of X and B a proper subset of Y . Then $(X \times Y) \setminus (A \times B)$ is connected.

PROOF:

$\langle 1 \rangle 1$. PICK $a \in X \setminus A$ and $b \in Y \setminus B$

$\langle 1 \rangle 2$. For $x \in X \setminus A$ we have $(X \times \{b\}) \cup (\{x\} \times Y)$ is connected.

PROOF: Theorem 216 since (x, b) is a common point.

$\langle 1 \rangle 3$. For $y \in Y \setminus B$ we have $(X \times \{y\}) \cup (\{a\} \times Y)$ is connected.

PROOF: Theorem 216 since (a, y) is a common point.

$\langle 1 \rangle 4$. $(X \times Y) \setminus (A \times B)$ is connected.

PROOF: Theorem 216 since it is the union of the sets in $\langle 1 \rangle 2$ and $\langle 1 \rangle 3$ with (a, b) as a common point.

□

Proposition 226. *Let $p : X \rightarrow Y$ be a quotient map. If Y is connected and $p^{-1}(y)$ is connected for all $y \in Y$, then X is connected.*

PROOF:

⟨1⟩1. ASSUME: for a contradiction C and D form a separation of X .

⟨1⟩2. C is saturated.

⟨2⟩1. LET: $x \in C, y \in X$ with $p(x) = p(y) = a$, say

⟨2⟩2. $y \notin D$

PROOF: Otherwise $p^{-1}(a) \cap C$ and $p^{-1}(a) \cap D$ form a separation of $p^{-1}(a)$.

⟨2⟩3. $y \in C$

⟨1⟩3. D is saturated.

PROOF: Similar.

⟨1⟩4. $p(C)$ and $p(D)$ form a separation of Y .

□

Proposition 227. *Let X be a connected space and Y a connected subspace of X . Suppose A and B form a separation of $X \setminus Y$. Then $Y \cup A$ and $Y \cup B$ are both connected.*

PROOF:

⟨1⟩1. $Y \cup A$ is connected.

⟨2⟩1. ASSUME: for a contradiction C and D form a separation of $Y \cup A$

⟨2⟩2. ASSUME: without loss of generality $Y \subseteq C$

⟨2⟩3. PICK open sets A_1, B_1, C_1, D_1 in X with

$$A = A_1 \setminus Y$$

$$B = B_1 \setminus Y$$

$$C = C_1 \cap (Y \cup A)$$

$$D = D_1 \cap (Y \cup A)$$

⟨2⟩4. $B_1 \cup C_1$ and $A_1 \cap D_1$ form a separation of X

⟨1⟩2. $Y \cup B$ is connected.

PROOF: Similar.

□

Theorem 228. *Let L be a linearly ordered set under the order topology. Then L is connected if and only if L is a linear continuum.*

PROOF:

⟨1⟩1. If L is a linear continuum then L is connected.

⟨2⟩1. LET: L be a linear continuum under the order topology.

⟨2⟩2. ASSUME: for a contradiction C and D form a separation of L .

⟨2⟩3. PICK $a \in C$ and $b \in D$.

⟨2⟩4. ASSUME: without loss of generality $a < b$.

⟨2⟩5. LET: $S = \{x \in L \mid a < x \text{ and } [a, x) \subseteq C\}$

⟨2⟩6. S is nonempty.

PROOF: Since $a \in C$ and C is open.

(2)7. S is bounded above by b .
 PROOF: Since $b \notin C$.
 (2)8. LET: $s = \sup S$
 (2)9. $s \in S$
 (3)1. LET: $y \in [a, s)$
 PROVE: $y \in C$
 (3)2. PICK z with $y < z \in S$
 PROOF: By minimality of s .
 (3)3. $y \in [a, z) \subseteq C$
 (2)10. CASE: $s \in C$
 (3)1. PICK x such that $s < x$ and $[s, x) \subseteq C$
 PROOF: Since C is open and s is not greatest in L because $s < b$.
 (3)2. $x \in S$
 PROOF: Since $[a, x) = [a, s) \cup [s, x) \subseteq C$.
 (3)3. Q.E.D.
 PROOF: This contradicts the fact that s is an upper bound for S .
 (2)11. CASE: $s \in D$
 (3)1. PICK $x < s$ such that $(x, s] \subseteq D$
 (3)2. PICK y with $x < y < s$
 PROOF: Since L is dense.
 (3)3. $y \in C$
 PROOF: From (2)9.
 (3)4. $y \in D$
 PROOF: From (3)1.
 (3)5. Q.E.D.
 (3)6. LET: L be a linear continuum under the order topology.
 (3)7. ASSUME: for a contradiction C and D form a separation of L .
 (3)8. PICK $a \in C$ and $b \in D$.
 (3)9. ASSUME: without loss of generality $a < b$.
 (3)10. LET: $S = \{x \in L \mid a < x \text{ and } [a, x) \subseteq C\}$
 (3)11. S is nonempty.
 PROOF: Since $a \in C$ and C is open.
 (3)12. S is bounded above by b .
 PROOF: Since $b \notin C$.
 (3)13. LET: $s = \sup S$
 (3)14. $s \in S$
 (4)1. LET: $y \in [a, s)$
 PROVE: $y \in C$
 (4)2. PICK z with $y < z \in S$
 PROOF: By minimality of s .
 (4)3. $y \in [a, z) \subseteq C$
 (3)15. CASE: $s \in C$
 (4)1. PICK x such that $s < x$ and $[s, x) \subseteq C$
 PROOF: Since C is open and s is not greatest in L because $s < b$.
 (4)2. $x \in S$
 PROOF: Since $[a, x) = [a, s) \cup [s, x) \subseteq C$.

⟨4⟩3. Q.E.D.

PROOF: This contradicts the fact that s is an upper bound for S .

$\langle 3 \rangle 16$. CASE: $s \in D$

⟨4⟩1. PICK $x < s$ such that $(x, s] \subseteq D$

⟨4⟩2. PICK y with $x < y < s$

PROOF: Since L is dense.

⟨4⟩3. $y \in C$

PROOF: From $\langle 2 \rangle 9$.

$\langle 4 \rangle 4.$ $y \in D$

PROOF: From $\langle 3 \rangle 1$.

⟨4⟩5. Q.E.D.

PROOF: This contradicts $\langle 2 \rangle 2$.

$\langle 1 \rangle 2$. If L is connected then L is a linear continuum.

$\langle 2 \rangle 1$. ASSUME: L is connected.

$\langle 2 \rangle 2$. Every nonempty subset of L that is bounded above has a supremum.

(3)1. LET: X be a nonempty subset of L bounded above by b .

⟨3⟩2. ASSUME: for a contradiction X has no supremum.

3. LET: U be the set of upper bounds of X ,

$\langle 3 \rangle 4$. U is nonempty.

PROOF: Since $b \in U$.

$\langle 3 \rangle 5$. U is open.

$\langle 4 \rangle 1$. LET: $x \in U$

4)2. PICK an upper bound y for X such that $y < x$

⟨4⟩3. Either x is greatest in L and $(y, x] \subseteq U$, or there exists $z > x$ such that $(y, z) \subseteq U$

⟨3⟩6. LET: V be the set of lower bounds of U .

$\langle 3 \rangle 7$. V is nonempty.

PROOF: Since $X \subset V$

$\langle 3 \rangle 8$. V is open.

$\langle 4 \rangle 1$. LET: $x \in V$

⟨4⟩2. PICK $y \in X$ with $x < y$

PROOF: x cannot be an upper bound for X , because it would be the supremum of X .

(4)3. Either x least in L and $[x, y) \subseteq V$, or there exists $z < x$ such that $(z, y) \subseteq V$

⟨3⟩9. $L = U \cup V$

$\langle 4 \rangle 1$. LET: $x \in L \setminus U$

⟨4⟩2. PICK $y \in X$ such that $x < y$

⟨4⟩3. For all $u \in U$ we have $x < y \leq u$

$\langle 4 \rangle 4. \ x \in V$

⟨3⟩10. $U \cap V = \emptyset$

PROOF: Any element of $U \cap V$ would be a supremum of X .

⟨3⟩11. U and V form a separation of L .

$\langle 3 \rangle 12$. Q.E.D.

PROOF: This contradicts $\langle 2 \rangle 1$.

$\langle 2 \rangle 3$. L is dense.

⟨3⟩1. LET: $x, y \in L$ with $x < y$

⟨3⟩2. There exists $z \in L$ such that $x < z < y$

PROOF: Otherwise $(-\infty, y)$ and $(x, +\infty)$ would form a separation of L .

□

Corollary 228.1. *The real line \mathbb{R} is connected.*

Corollary 228.2. *Every interval in \mathbb{R} is connected.*

Corollary 228.3. *The ordered square is connected.*

Theorem 229 (Intermediate Value Theorem). *Let X be a connected space. Let Y be a linearly ordered set under the order topology. Let $f : X \rightarrow Y$ be continuous. Let $a, b \in X$ and $r \in Y$. Suppose $f(a) < r < f(b)$. Then there exists $c \in X$ such that $f(c) = r$.*

PROOF: Otherwise $f^{-1}((-\infty, r))$ and $f^{-1}((r, +\infty))$ would form a separation of X . □

Proposition 230. *Every function $f : [0, 1] \rightarrow [0, 1]$ has a fixed point.*

PROOF:

⟨1⟩1. LET: $g : [0, 1] \rightarrow [-1, 1]$ be the function $g(x) = f(x) - x$

PROVE: there exists $x \in [0, 1]$ such that $g(x) = 0$

⟨1⟩2. ASSUME: without loss of generality $g(0) \neq 0$ and $g(1) \neq 0$

⟨1⟩3. $g(0) > 0$

⟨1⟩4. $g(1) < 0$

⟨1⟩5. There exists $x \in (0, 1)$ such that $g(x) = 0$

PROOF: By the Intermediate Value Theorem.

33 Totally Disconnected Spaces

Definition 231 (Totally Disconnected). A topological space X is *totally disconnected* if and only if the only connected subspaces are the singletons.

Example 232. Every discrete space is totally disconnected.

Example 233. The rationals \mathbb{Q} are totally disconnected.

34 Paths and Path Connectedness

Definition 234 (Path). Let X be a topological space and $a, b \in X$. A *path* from a to b is a continuous function $p : [0, 1] \rightarrow X$ such that $p(0) = a$ and $p(1) = b$.

Definition 235 (Path Connected). A topological space is *path connected* if and only if there exists a path between any two points.

Proposition 236. *Every path connected space is connected.*

PROOF:

- ⟨1⟩1. LET: X be a path connected space.
- ⟨1⟩2. ASSUME: for a contradiction C and D form a separation of X .
- ⟨1⟩3. PICK $a \in C$ and $b \in D$.
- ⟨1⟩4. PICK a path $p : [0, 1] \rightarrow X$ from a to b .
- ⟨1⟩5. $p^{-1}(C)$ and $p^{-1}(D)$ form a separation of $[0, 1]$.
- ⟨1⟩6. Q.E.D.

PROOF: This contradicts Corollary 228.2.

□

An example that shows the converse does not hold:

Example 237. The ordered square is not path connected.

PROOF:

- ⟨1⟩1. ASSUME: for a contradiction $p : [0, 1] \rightarrow I_o^2$ is a path from $(0, 0)$ to $(1, 1)$.
- ⟨1⟩2. p is surjective.
PROOF: By the Intermediate Value Theorem.
- ⟨1⟩3. For $x \in [0, 1]$, PICK a rational $q_x \in p^{-1}((x, 0), (x, 1))$
PROOF: Since $p^{-1}((x, 0), (x, 1))$ is open and nonempty by ⟨1⟩2.
- ⟨1⟩4. For $x, y \in [0, 1]$, if $x \neq y$ then $q_x \neq q_y$
PROOF: We have $p(q_x) \neq p(q_y)$ because $((x, 0), (x, 1))$ and $((y, 0), (y, 1))$ are disjoint.
- ⟨1⟩5. $\{q_x \mid x \in [0, 1]\}$ is an uncountable set of rationals.
- ⟨1⟩6. Q.E.D.

PROOF: This contradicts the fact that the rationals are countable.

□

Proposition 238. *The continuous image of a path connected space is path connected.*

PROOF:

- ⟨1⟩1. LET: X be a path connected space, Y a topological space, and $f : X \rightarrow Y$ be continuous and surjective.
- ⟨1⟩2. LET: $a, b \in Y$
- ⟨1⟩3. PICK $c, d \in X$ with $f(c) = a$ and $f(d) = b$
- ⟨1⟩4. PICK a path $p : [0, 1] \rightarrow X$ from c to d .
- ⟨1⟩5. $f \circ p$ is a path from a to b in Y .

□

Proposition 239 (AC). *The product of a family of path-connected spaces is path-connected.*

PROOF:

- ⟨1⟩1. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of path-connected spaces.
- ⟨1⟩2. LET: $a, b \in \prod_{\alpha \in J} X_\alpha$
- ⟨1⟩3. For $\alpha \in J$, PICK a path $p_\alpha : [0, 1] \rightarrow X_\alpha$ from a_α to b_α
PROOF: Using the Axiom of Choice.
- ⟨1⟩4. Define $p : [0, 1] \rightarrow \prod_{\alpha \in J} X_\alpha$ by $p(t)_\alpha = p_\alpha(t)$

⟨1⟩5. p is a path from a to b .

PROOF: Theorem 132.

□

Proposition 240. *The continuous image of a path-connected space is path-connected.*

PROOF:

⟨1⟩1. LET: $f : X \rightarrow Y$ be continuous and surjective where X is path-connected.

⟨1⟩2. LET: $a, b \in Y$

⟨1⟩3. PICK $a', b' \in X$ with $f(a') = a$ and $f(b') = b$.

⟨1⟩4. PICK a path $p : [0, 1] \rightarrow X$ from a' to b' .

⟨1⟩5. $f \circ p$ is a path from a to b .

□

Proposition 241. *Let X be a topological space. The union of a set of path-connected subspaces of X that have a point in common is path-connected.*

PROOF:

⟨1⟩1. LET: \mathcal{A} be a set of path-connected subspaces of X with the point a in common.

⟨1⟩2. LET: $b, c \in \bigcup \mathcal{A}$

⟨1⟩3. PICK $B, C \in \mathcal{A}$ with $b \in B$ and $c \in C$.

⟨1⟩4. PICK a path p in B from b to a .

⟨1⟩5. PICK a path q in C from a to c .

⟨1⟩6. The concatenation of p and q is a path from b to c in $\bigcup \mathcal{A}$.

□

Proposition 242. *Let $A \subseteq \mathbb{R}^2$ be countable. Then $\mathbb{R}^2 \setminus A$ is path-connected.*

PROOF:

⟨1⟩1. LET: $a, b \in \mathbb{R}^2 \setminus A$

⟨1⟩2. PICK a line l in \mathbb{R}^2 with a on one side and b on the other.

⟨1⟩3. For every point x on l ,

LET: p_x be the path in \mathbb{R}^2 consisting of a line from a to x then a line from x to b

⟨1⟩4. For $x \neq y$ we have p_x and p_y have no points in common except a and b

⟨1⟩5. There are only countably many x such that a point of A lies on p_x .

⟨1⟩6. There exists x such that p_x is a path from a to b in $\mathbb{R}^2 \setminus A$.

□

Proposition 243. *Every open connected subspace of \mathbb{R}^2 is path-connected.*

PROOF:

⟨1⟩1. LET: U be an open connected subspace of \mathbb{R}^2 .

⟨1⟩2. For all $x_0 \in U$,

LET: $PC(x_0) = \{y \in U \mid \text{there exists a path from } x_0 \text{ to } y\}$

⟨1⟩3. For all $x_0 \in U$, the set $PC(x_0)$ is open and closed in U .

⟨2⟩1. LET: $x_0 \in U$

$\langle 2 \rangle 2$. $PC(x_0)$ is open in U
 $\langle 3 \rangle 1$. LET: $y \in PC(x_0)$
 $\langle 3 \rangle 2$. PICK $\epsilon > 0$ such that $B(y, \epsilon) \subseteq U$
PROOF: Since U is open.
 $\langle 3 \rangle 3$. $B(y, \epsilon) \subseteq PC(x_0)$
PROOF: For all $z \in B(y, \epsilon)$, pick a path from x_0 to y then concatenate the straight line from y to z .
 $\langle 2 \rangle 3$. $PC(x_0)$ is closed in U
 $\langle 3 \rangle 1$. LET: $y \in U$ be a limit point of $PC(x_0)$
 $\langle 3 \rangle 2$. PICK $\epsilon > 0$ such that $B(y, \epsilon) \subseteq U$
 $\langle 3 \rangle 3$. PICK $z \in PC(x_0) \cap B(y, \epsilon)$
 $\langle 3 \rangle 4$. $y \in PC(x_0)$
PROOF: Pick a path from x_0 to z then concatenate the straight line from z to y .
 $\langle 1 \rangle 4$. $PC(x_0) = U$
PROOF: Proposition 208.

□

Example 244. If A is a connected subspace of X , then A° is not necessarily connected.

Take two closed circles in \mathbb{R}^2 that touch at one point. The interior of this space is two open circles, and these two circles form a separation.

Example 245. If A is a connected subspace of X then ∂A is not necessarily connected.

We have $[0, 1]$ is connected but $\partial[0, 1] = \{0, 1\}$ is not.

Example 246. If A is a subspace of X and A° and ∂A are connected, then A is not necessarily connected.

We have $\mathbb{Q}^\circ = \emptyset$ and $\partial\mathbb{Q} = \mathbb{R}$ are connected but \mathbb{Q} is not connected.

35 The Topologist's Sine Curve

Definition 247 (The Topologist's Sine Curve). Let $S = \{(x, \sin 1/x) \mid 0 < x \leq 1\}$, The *topologist's sine curve* is the closure \bar{S} of S in \mathbb{R}^2 .

Proposition 248. *The topologist's sine curve is connected.*

PROOF:

$\langle 1 \rangle 1$. LET: $S = \{(x, \sin 1/x) \mid 0 < x \leq 1\}$

$\langle 1 \rangle 2$. S is connected.

PROOF: Theorem 218.

$\langle 1 \rangle 3$. \bar{S} is connected.

PROOF: Theorem 217.

□

Proposition 249. *The topologist's sine curve is $\{(x, \sin 1/x) \mid 0 < x \leq 1\} \cup (\{0\} \times [-1, 1])$.*

PROOF: Sketch proof: Given a point $(0, y)$ with $-1 \leq y \leq 1$, pick a such that $\sin a = y$. Then $((1/a, y), (1/(a + 2\pi), y), (1/(a + 4\pi), y), \dots)$ is a sequence in S that converges to $(0, y)$.

Conversely, let (x, y) be any point not in $S \cup (\{0\} \times [-1, 1])$. If $x < 0$ or $y > 1$ or $y < -1$ then we can easily find a neighbourhood that does not intersect $S \cup (\{0\} \times [-1, 1])$. If $x > 0$ and $-1 \leq y \leq 1$, then we have $y \neq \sin 1/x$. Hence pick a neighbourhood that does not intersect S .

Proposition 250. *Every closed subset of \mathbb{R} that is bounded above has a greatest element.*

PROOF: It has a supremum, which is a limit point of the set and hence an element. \square

Proposition 251 (CC). *The topologist's sine curve is not path connected.*

PROOF:

$\langle 1 \rangle 1$. ASSUME: For a contradiction $p : [0, 1] \rightarrow \overline{S}$ is a path from $(0, 0)$ to $(1, \sin 1)$.

$\langle 1 \rangle 2$. $\{t \in [0, 1] \mid p(t) \in \{0\} \times [-1, 1]\}$ is closed.

PROOF: Since p is continuous and $\{0\} \times [-1, 1]$ is closed.

$\langle 1 \rangle 3$. LET: b be the largest number in $[0, 1]$ such that $p(b) \in \{0\} \times [-1, 1]$.

PROOF: Proposition 250.

$\langle 1 \rangle 4$. LET: $x : [b, 1] \rightarrow \overline{S}$ be the function $\pi_1 \circ p$

$\langle 1 \rangle 5$. LET: $y : [b, 1] \rightarrow \overline{S}$ be the function $\pi_2 \circ p$

$\langle 1 \rangle 6$. PICK a sequence t_n in $(b, 1]$ such that $t_n \rightarrow b$ and $y(t_n) = (-1)^n$ for all n

$\langle 2 \rangle 1$. LET: $n \geq 1$

$\langle 2 \rangle 2$. PICK u with $0 < u < x(1/n)$ and $\sin(1/u) = (-1)^n$

$\langle 2 \rangle 3$. PICK t_n with $b < t_n < 1/n$ and $x(t_n) = u$

PROOF: By the Intermediate Value Theorem

$\langle 1 \rangle 7$. Q.E.D.

PROOF: This contradicts Proposition 105 since y is continuous and $y(t_n)$ does not converge.

\square

Corollary 251.1. *The closure of a path-connected subspace of a space is not necessarily path-connected.*

36 The Long Line

Definition 252 (The Long Line). The *long line* is the space $\omega_1 \times [0, 1]$ in the dictionary order under the order topology, where ω_1 is the first uncountable ordinal.

Lemma 253. *For any ordinal α with $0 < \alpha < \omega_1$ we have $[(0, 0), (\alpha, 0)) \cong [0, 1]$*

$\langle 1 \rangle 1$. $[(0, 0), (1, 0)) \cong [0, 1]$

PROOF: The map π_2 is a homeomorphism.

$\langle 1 \rangle 2$. If $[(0, 0), (\alpha, 0)) \cong [0, 1]$ then $[(0, 0), (\alpha + 1, 0)) \cong [0, 1]$

PROOF: Proposition 11.

$\langle 1 \rangle 3$. If λ is a limit ordinal with $\lambda < \omega_1$ and $[(0, 0), (\alpha, 0)) \cong [0, 1)$ for all α with $0 < \alpha < \lambda$ then $[(0, 0), (\lambda, 0)) \cong [0, 1)$

$\langle 2 \rangle 1$. LET: λ be a limit ordinal $< \omega_1$

$\langle 2 \rangle 2$. ASSUME: $[(0, 0), (\alpha, 0)) \cong [0, 1)$ for all α with $0 < \alpha < \lambda$

$\langle 2 \rangle 3$. PICK a sequence of ordinals $\alpha_0 < \alpha_1 < \dots$ with limit λ

PROOF: Since λ is countable.

$\langle 2 \rangle 4$. $[(\alpha_i, 0), (\alpha_{i+1}, 0)) \cong [0, 1)$ for all i

PROOF: Lemma 10.

$\langle 2 \rangle 5$. Q.E.D.

PROOF: By Proposition 12.

$\langle 1 \rangle 4$. Q.E.D.

PROOF: By transfinite induction.

Proposition 254 (CC). *The long line is path-connected.*

PROOF:

$\langle 1 \rangle 1$. LET: $(\alpha, i), (\beta, j) \in \omega_1 \times [0, 1)$

$\langle 1 \rangle 2$. ASSUME: without loss of generality $(\alpha, i) < (\beta, j)$

$\langle 1 \rangle 3$. $[(0, 0), (\beta + 1, 0)) \cong [0, 1)$

PROOF: By Lemma 253

$\langle 1 \rangle 4$. $[(\alpha, i), (\beta, j)) \cong [0, 1)$

PROOF: Lemma 10.

$\langle 1 \rangle 5$. PICK a homeomorphism $q : [0, 1) \rightarrow [(\alpha, i), (\beta, j))$

$\langle 1 \rangle 6$. $q \cup \{(1, (\beta, j))\}$ is a path from (α, i) to (β, j)

□

Proposition 255. *Every point in the long line has a neighbourhood homeomorphic to an interval in \mathbb{R} .*

PROOF: For any (α, i) in the long line, the neighbourhood $[(0, 0), (\alpha + 1, 0))$ satisfies the condition by Lemma 253.

Proposition 256. *The long line L is not second countable.*

PROOF:

$\langle 1 \rangle 1$. LET: \mathcal{B} be a basis for L .

$\langle 1 \rangle 2$. For $\alpha < \omega_1$, PICK $B_\alpha \in \mathcal{B}$ such that $(\alpha, 1/2) \in B_\alpha \subseteq ((\alpha, 0), (\alpha + 1, 0))$

$\langle 1 \rangle 3$. \mathcal{B} is uncountable.

PROOF: The mapping $\alpha \mapsto B_\alpha$ is an injection $\omega_1 \rightarrow \mathcal{B}$.

Corollary 256.1. *The long line cannot be imbedded into \mathbb{R}^n for any n .*

37 Components

Proposition 257. *Let X be a topological space. Define the relation \sim on X by $a \sim b$ if and only if there exists a connected subspace $A \subseteq X$ with $a, b \in A$. Then \sim is an equivalence relation on X .*

PROOF:

⟨1⟩1. \sim is reflexive.

PROOF: For any $a \in X$ we have $\{a\}$ is a connected subspace that contains a .

⟨1⟩2. \sim is symmetric.

PROOF: Trivial.

⟨1⟩3. \sim is transitive.

⟨2⟩1. LET: $a, b, c \in X$

⟨2⟩2. ASSUME: $a \sim b$ and $b \sim c$

⟨2⟩3. PICK connected subspaces A and B with $a, b \in A$ and $b, c \in B$

⟨2⟩4. $A \cup B$ is a connected subspace that contains a and c

PROOF: Theorem 216.

□

Definition 258 ((Connected) Component). Let X be a topological space. The *(connected) components* of X are the equivalence classes under the above \sim .

Lemma 259. *Let X be a topological space. If $A \subseteq X$ is connected and nonempty then there exists a unique component C of X such that $A \subseteq C$.*

PROOF:

⟨1⟩1. PICK $a \in A$

⟨1⟩2. LET: C be the \sim -equivalence class of a .

⟨1⟩3. $A \subseteq C$

PROOF: For all $x \in A$ we have $x \sim a$.

⟨1⟩4. If C' is a component and $A \subseteq C'$ then $C = C'$

PROOF: Since we have $a \in C'$.

□

Theorem 260. *Let X be a topological space. The components of X are connected disjoint subspaces of X whose union is X such that each nonempty connected subspace of X intersects only one of them.*

PROOF:

⟨1⟩1. Every component of X is connected.

PROOF: For $a \in X$, the \sim -equivalence class of a is $\bigcup \{A \subseteq X \mid A \text{ is connected, } a \in A\}$ which is connected by Theorem 216.

⟨1⟩2. The components form a partition of X .

PROOF: Immediate from the definition.

⟨1⟩3. Every nonempty connected subspace of X intersects a unique component of X .

⟨2⟩1. LET: $A \subseteq X$ be connected and nonempty.

⟨2⟩2. LET: C be the component such that $A \subseteq C$

PROOF: Lemma 259.

⟨2⟩3. A intersects C

⟨2⟩4. If A intersects the component C' then $C' = C$

⟨3⟩1. LET: C' be a component that intersects A

⟨3⟩2. PICK $b \in A \cap C'$

⟨3⟩3. $A \subseteq C'$

PROOF: For all $x \in A$ we have $x \sim b$.

$\langle 3 \rangle 4$. $C = C'$

PROOF: By uniqueness in $\langle 2 \rangle 2$.

□

Proposition 261. *Every component of a space is closed.*

PROOF:

$\langle 1 \rangle 1$. LET: X be a topological space and C a component of X .

$\langle 1 \rangle 2$. \overline{C} is connected.

PROOF: Theorem 217.

$\langle 1 \rangle 3$. $C = \overline{C}$

PROOF: Lemma 215.

$\langle 1 \rangle 4$. C is closed.

PROOF: Lemma 44.

□

Proposition 262. *If a topological space has finitely many components then every component is open.*

PROOF: Each component is the complement of a finite union of closed sets. □

38 Path Components

Proposition 263. *Let X be a topological space. Define the relation \sim on X by: $a \sim b$ if and only if there exists a path in X from a to b . Then \sim is an equivalence relation on X .*

PROOF:

$\langle 1 \rangle 1$. \sim is reflexive.

PROOF: For $a \in X$, the constant function $[0, 1] \rightarrow X$ with value a is a path from a to a .

$\langle 1 \rangle 2$. \sim is symmetric.

PROOF: If $p : [0, 1] \rightarrow X$ is a path from a to b , then $\lambda t.p(1-t)$ is a path from b to a .

$\langle 1 \rangle 3$. \sim is transitive.

PROOF: Concatenate paths.

□

Definition 264 (Path Component). Let X be a topological space. The *path components* of X are the equivalence relations under \sim .

Theorem 265. *The path components of X are path-connected disjoint subspaces of X whose union is X such that every nonempty path-connected subspace of X intersects exactly one path component.*

PROOF:

- ⟨1⟩1. Every path component is path-connected.
 PROOF: If a and b are in the same path component then $a \sim b$, i.e. there exists a path from a to b .
- ⟨1⟩2. The path components are disjoint and their union is X .
 PROOF: Immediate from the definition.
- ⟨1⟩3. Every non-empty path-connected subspace of X intersects exactly one path component.
 ⟨2⟩1. LET: A be a nonempty path-connected subspace of X .
 ⟨2⟩2. PICK $a \in A$
 ⟨2⟩3. A intersects the \sim -equivalence class of a .
 ⟨2⟩4. LET: C be any path component that intersects A .
 ⟨2⟩5. PICK $b \in A \cap C$
 ⟨2⟩6. $a \sim b$
 PROOF: Since A is path-connected.
 ⟨2⟩7. C is the \sim -equivalence class of a .

□

Proposition 266. *Every path component is included in a component.*

PROOF:

- ⟨1⟩1. LET: X be a topological space and C a path component of X .
 ⟨1⟩2. C is path-connected.
 PROOF: Theorem 265.
 ⟨1⟩3. C is connected.
 PROOF: Proposition 236.
 ⟨1⟩4. C is included in a component.
 PROOF: Lemma 259.

□

39 Local Connectedness

Definition 267 (Locally Connected). Let X be a topological space and $a \in X$. Then X is *locally connected* at a if and only if every neighbourhood of a includes a connected neighbourhood of a .

The space X is *locally connected* if and only if it is locally connected at every point.

Example 268. The real line is both connected and locally connected.

Example 269. The space $\mathbb{R} \setminus \{0\}$ is disconnected but locally connected.

Example 270. The topologist's sine curve is connected but not locally connected.

Example 271. The rationals \mathbb{Q} are neither connected nor locally connected.

Theorem 272. *A topological space X is locally connected if and only if, for every open set U in X , every component of U is open in X .*

PROOF:

- ⟨1⟩1. If X is locally connected then, for every open set U in X , every component of U is open in X .
- ⟨2⟩1. ASSUME: X is locally connected.
- ⟨2⟩2. LET: U be open in X .
- ⟨2⟩3. LET: C be a component of U .
- ⟨2⟩4. LET: $a \in C$
- ⟨2⟩5. LET: V be a connected neighbourhood of a such that $V \subseteq U$
- ⟨2⟩6. $V \subseteq C$
PROOF: Lemma 259.
- ⟨2⟩7. Q.E.D.
PROOF: Lemma 23.
- ⟨1⟩2. If, for every open set U in X , every component of U is open in X , then X is locally connected.
- ⟨2⟩1. ASSUME: for every open set U in X , every component of U is open in X .
- ⟨2⟩2. LET: $a \in X$
- ⟨2⟩3. LET: U be a neighbourhood of a
- ⟨2⟩4. The component of U that contains a is a connected neighbourhood of a included in U .

□

40 Local Path Connectedness

Definition 273 (Locally Path-Connected). Let X be a topological space and $a \in X$. Then X is *locally path-connected* at a if and only if every neighbourhood of a includes a path-connected neighbourhood of a .

The space X is *locally path-connected* if and only if it is locally path-connected at every point.

Theorem 274. *A topological space X is locally path-connected if and only if, for every open set U in X , every path component of U is open in X .*

PROOF:

- ⟨1⟩1. If X is locally path-connected then, for every open set U in X , every path component of U is open in X .
- ⟨2⟩1. ASSUME: X is locally path-connected.
- ⟨2⟩2. LET: U be open in X .
- ⟨2⟩3. LET: C be a path component of U .
- ⟨2⟩4. LET: $a \in C$
- ⟨2⟩5. LET: V be a path-connected neighbourhood of a such that $V \subseteq U$
- ⟨2⟩6. $V \subseteq C$
PROOF: Lemma 259.
- ⟨2⟩7. Q.E.D.
PROOF: Lemma 23.

- ⟨1⟩2. If, for every open set U in X , every component of U is open in X , then X is locally connected.
 ⟨2⟩1. ASSUME: for every open set U in X , every component of U is open in X .
 ⟨2⟩2. LET: $a \in X$
 ⟨2⟩3. LET: U be a neighbourhood of a
 ⟨2⟩4. The component of U that contains a is a connected neighbourhood of a included in U .

□

Theorem 275. *If a space is locally path connected then its components and its path components are the same.*

PROOF:

- ⟨1⟩1. LET: X be a locally path connected space.
 ⟨1⟩2. LET: C be a component of X .
 ⟨1⟩3. LET: $x \in C$
 ⟨1⟩4. LET: P be the path component of x
 PROVE: $P = C$
 ⟨1⟩5. $P \subseteq C$
 PROOF: Proposition 266.
 ⟨1⟩6. LET: Q be the union of the other path components included in C
 ⟨1⟩7. $C = P \cup Q$
 PROOF: Proposition 266.
 ⟨1⟩8. P and Q are open in C
 ⟨2⟩1. C is open.
 PROOF: Theorem 272.
 ⟨2⟩2. Q.E.D.
 PROOF: Theorem 274.
 ⟨1⟩9. $Q = \emptyset$
 PROOF: Otherwise P and Q would form a separation of C .

□

41 Topological Groups

Definition 276 (Topological Group). A *topological group* G consists of a T_1 space G and continuous maps $\cdot : G^2 \rightarrow G$ and $(\)^{-1} : G \rightarrow G$ such that $(G, \cdot, (\)^{-1})$ is a group.

- Example 277.**
1. The integers \mathbb{Z} under addition are a topological group.
 2. The real numbers \mathbb{R} under addition are a topological group.
 3. The positive reals under multiplication are a topological group.
 4. The set $\{z \in \mathbb{C} \mid |z| = 1\}$ under multiplication and given the topology of S^1 is a topological group.

5. For any $n \geq 0$, the general linear group $GL_n(\mathbb{R})$ is a topological group under matrix multiplication, considered as a subspace of \mathbb{R}^{n^2} .

Lemma 278. *Let G be a T_1 space and $\cdot : G^2 \rightarrow G$, $(\)^{-1} : G \rightarrow G$ be functions such that $(G, \cdot, (\)^{-1})$ is a group. Then G is a topological group if and only if the function $f : G^2 \rightarrow G$ that maps (x, y) to xy^{-1} is continuous.*

PROOF:

- $\langle 1 \rangle 1$. If G is a topological group then f is continuous.

PROOF: From Theorem 95.

- $\langle 1 \rangle 2$. If f is continuous then G is a topological group.

- $\langle 2 \rangle 1$. ASSUME: f is continuous.

- $\langle 2 \rangle 2$. $(\)^{-1}$ is continuous.

PROOF: Since $x^{-1} = f(e, x)$.

- $\langle 2 \rangle 3$. \cdot is continuous.

PROOF: Since $xy = f(x, y^{-1})$.

□

Lemma 279. *Let G be a topological group and H a subgroup of G . Then H is a topological group under the subspace topology.*

PROOF:

- $\langle 1 \rangle 1$. H is T_1 .

PROOF: From Proposition 166.

- $\langle 1 \rangle 2$. multiplication and inverse on H are continuous.

PROOF: From Theorem 96.

□

Lemma 280. *Let G be a topological group and H a subgroup of G . Then \overline{H} is a subgroup of G .*

PROOF:

- $\langle 1 \rangle 1$. LET: $x, y \in \overline{H}$

PROVE: $xy^{-1} \in \overline{H}$

- $\langle 1 \rangle 2$. LET: U be any neighbourhood of xy^{-1}

- $\langle 1 \rangle 3$. LET: $f : G^2 \rightarrow G$, $f(a, b) = ab^{-1}$

- $\langle 1 \rangle 4$. $f^{-1}(U)$ is a neighbourhood of (x, y)

- $\langle 1 \rangle 5$. PICK neighbourhoods V, W of x and y respectively such that $f(V \times W) \subseteq U$.

- $\langle 1 \rangle 6$. PICK $a \in V \cap H$ and $b \in W \cap H$

PROOF: Theorem 45.

- $\langle 1 \rangle 7$. $ab^{-1} \in U \cap H$

- $\langle 1 \rangle 8$. Q.E.D.

PROOF: By Theorem 45.

□

Proposition 281. *Let G be a topological group and $\alpha \in G$. Then the maps $l_\alpha, r_\alpha : G \rightarrow G$ defined by $l_\alpha(x) = \alpha x$, $r_\alpha(x) = x\alpha$ are homeomorphisms of G with itself.*

PROOF: They are continuous with continuous inverses $l_{\alpha^{-1}}$ and $r_{\alpha^{-1}}$. \square

Corollary 281.1. *Every topological group is homogeneous.*

PROOF: Given a topological group G and $a, b \in G$, we have $l_{ba^{-1}}$ is a homeomorphism that maps a to b . \square

Proposition 282. *Let G be a topological group and H a subgroup of G . Give G/H the quotient topology. For all $\alpha \in G$, the map $\overline{f_\alpha}$ that sends xH to αxH is a homeomorphism $G/H \cong G/H$.*

PROOF:

$\langle 1 \rangle 1$. $\overline{f_\alpha}$ is well-defined.

PROOF: If $xy^{-1} \in H$ then $(\alpha x)(\alpha y)^{-1} = xy^{-1} \in H$.

$\langle 1 \rangle 2$. $\overline{f_\alpha}$ is continuous.

PROOF: Theorem 194 since $\overline{f_\alpha} \circ p = p \circ f_\alpha$ is continuous, where $p : G \twoheadrightarrow G/H$ is the canonical surjection.

$\langle 1 \rangle 3$. $\overline{f_\alpha}^{-1}$ is continuous.

PROOF: Similar since $\overline{f_\alpha}^{-1} = \overline{f_{\alpha^{-1}}}$.

\square

Corollary 282.1. *Let G be a topological group and H a subgroup of G . Give G/H the quotient topology. Then G/H is homogeneous.*

Proposition 283. *Let G be a topological group and H a subgroup of G . Give G/H the quotient topology. If H is closed in G then G/H is T_1 .*

PROOF:

$\langle 1 \rangle 1$. LET: $p : G \twoheadrightarrow G/H$ be the canonical surjection

$\langle 1 \rangle 2$. LET: $x \in G$

$\langle 1 \rangle 3$. $p^{-1}(xH) = f_x(H)$

$\langle 1 \rangle 4$. $p^{-1}(xH)$ is closed in G

PROOF: Since H is closed and f_x is a homeomorphism of G with itself.

$\langle 1 \rangle 5$. $\{xH\}$ is closed in G/H

\square

Proposition 284. *Let G be a topological group and H a subgroup of G . Give G/H the quotient topology. Then the canonical surjection $p : G \twoheadrightarrow G/H$ is an open map.*

PROOF:

$\langle 1 \rangle 1$. LET: $U \subseteq G$ be open.

$\langle 1 \rangle 2$. $p^{-1}(p(U)) = \bigcup_{h \in H} r_h(U)$

$\langle 1 \rangle 3$. $p^{-1}(p(U))$ is open.

$\langle 1 \rangle 4$. $p(U)$ is open.

\square

Proposition 285. *Let G be a topological group and H a closed normal subgroup of G . Then G/H is a topological group under the quotient topology.*

PROOF:

$\langle 1 \rangle 1.$ G/H is T_1

PROOF: Proposition 283.

$\langle 1 \rangle 2.$ The map $\bar{m} : (xH, yH) \mapsto xy^{-1}H$ is continuous.

$\langle 2 \rangle 1.$ $p^2 : G^2 \rightarrow (G/H)^2$ is a quotient map.

PROOF: Propositions 193, 284.

$\langle 2 \rangle 2.$ $\bar{m} \circ p^2$ is continuous.

PROOF: As it is $p^2 \circ m$ where $m : G^2 \rightarrow G$ with $m(x, y) = xy^{-1}$

□

Lemma 286. *Let G be a topological group and $A, B \subseteq G$. If either A or B is open then AB is open.*

PROOF: If A is open we have $AB = \bigcup_{b \in B} r_b(A)$. Similarly if B is open. □

Definition 287 (Symmetric Neighbourhood). Let G be a topological group. A neighbourhood V of e is *symmetric* if and only if $V = V^{-1}$.

Lemma 288. *Let G be a topological group. Let V be a neighbourhood of e . Then V is symmetric if and only if, for all $x \in V$, we have $x^{-1} \in V$.*

PROOF:

$\langle 1 \rangle 1.$ If V is symmetric then, for all $x \in V$, we have $x^{-1} \in V$

PROOF: Immediate from definitions.

$\langle 1 \rangle 2.$ If, for all $x \in V$, we have $x^{-1} \in V$, then V is symmetric.

$\langle 2 \rangle 1.$ ASSUME: for all $x \in V$ we have $x^{-1} \in V$

$\langle 2 \rangle 2.$ $V \subseteq V^{-1}$

PROOF: If $x \in V$ then there exists $y \in V$ such that $x = y^{-1}$, namely $y = x^{-1}$

$\langle 2 \rangle 3.$ $V^{-1} \subseteq V$

PROOF: Immediate from $\langle 2 \rangle 1.$

□

Lemma 289. *Let G be a topological group. For every neighbourhood U of e , there exists a symmetric neighbourhood V of e such that $V^2 \subseteq U$.*

PROOF:

$\langle 1 \rangle 1.$ LET: U be a neighbourhood of e .

$\langle 1 \rangle 2.$ PICK a neighbourhood V' of e such that $V'V' \subseteq U$

PROOF: Such a neighbourhood exists because multiplication in G is continuous.

$\langle 1 \rangle 3.$ PICK a neighbourhood W of e such that $WW^{-1} \subseteq V'$

PROOF: Such a neighbourhood exists because the function that maps (x, y) to xy^{-1} is continuous.

$\langle 1 \rangle 4.$ LET: $V = WW^{-1}$

$\langle 1 \rangle 5.$ V is a neighbourhood of e

$\langle 2 \rangle 1.$ $e \in V$

PROOF: Since $e \in W$ so $e = ee^{-1} \in V$.

PROOF: Lemma 286.

$\langle 1 \rangle 6$. V is symmetric

31. LET: $x \in V$

$\langle 3 \rangle 3. \ x^{-1} = zy^{-1}$

$\langle 3 \rangle 5$. $x \in V^{-1}$

PROOF: Lemma 288

PROOF: We have $V^2 \subseteq (V')^2 \subseteq U$

Proposition 290. *Every topological group is Hausdorff.*

$\langle 1 \rangle 1$. LET: G be a topological group.

$\langle 1 \rangle 3$. LET: $U = G \setminus \{x[{}^{-1}y]\}$

⟨2⟩1. U is open

$\langle 2 \rangle 2. \quad e \in U$

PROOF: Since $x \neq y$

PROOF: Lemma 289.

$\langle 2 \rangle 1.$ Vx is open

PROOF: Since $Vx = r_x(V)$

$\langle 2 \rangle 2$. V_y is open

PROOF: Similar.

$\langle 2 \rangle 3. \quad Vx \cap Vy = \emptyset$

⟨3⟩1. ASSUME: for a contradiction $z \in Vx \cap Vy$

⟨3⟩2. PICK $a, b \in V$ such that $z = ax = by$

$\langle 3 \rangle 3. \ xy^{-1} \in VV$

PROOF: Since $xy^{-1} = a^{-1}b$

$\langle 3 \rangle 4. \ xy^{-1} \in U$

⟨3⟩5. Q.E.D.

PROOF: From $\langle 1 \rangle 3$.

Proposition 291. *Every topological group is regular.*

$\langle 1 \rangle 1$. LET: G be a topological group.

- ⟨1⟩2. LET: $A \subseteq G$ be a closed set and $a \notin A$.
- ⟨1⟩3. LET: $U = G \setminus Aa^{-1}$
- ⟨1⟩4. PICK a symmetric neighbourhood V of e such that $VV \subseteq U$
 - ⟨2⟩1. U is open
PROOF: Since $Aa^{-1} = r_{a^{-1}}(A)$ is closed.
 - ⟨2⟩2. $e \in U$
PROOF: Since $a \notin A$.
 - ⟨2⟩3. Q.E.D.
PROOF: Lemma 289.
- ⟨1⟩5. VA and Va are disjoint open sets with $A \subseteq VA$ and $a \in Va$
 - ⟨2⟩1. VA is open
PROOF: Lemma 286
 - ⟨2⟩2. Va is open
PROOF: Lemma 286
 - ⟨2⟩3. $VA \cap Va = \emptyset$
 - ⟨3⟩1. ASSUME: for a contradiction $z \in VA \cap Va$
 - ⟨3⟩2. PICK $b, c \in V$ and $d \in A$ with $z = bd = ca$
 - ⟨3⟩3. $da^{-1} \in U$
PROOF: Since $da^{-1} = b^{-1}c \in VV \subseteq U$
 - ⟨3⟩4. Q.E.D.
PROOF: This contradicts ⟨1⟩3

□

Proposition 292. *Let G be a topological group and H a subgroup of G . Give G/H the quotient topology. If H is closed in G then G/H is regular.*

PROOF:

- ⟨1⟩1. LET: $p : G \rightarrow G/H$ be the canonical surjection.
- ⟨1⟩2. LET: A be a closed set in G/H and $aH \in (G/H) \setminus A$.
- ⟨1⟩3. LET: $B = p^{-1}(A)$
- ⟨1⟩4. B is a closed saturated set in G .
- ⟨1⟩5. $B \cap aH = \emptyset$
- ⟨1⟩6. $B = BH$
- ⟨1⟩7. PICK a symmetric neighbourhood V of e such that VB does not intersect Va
 - ⟨2⟩1. LET: $U = G \setminus Ba^{-1}$
 - ⟨2⟩2. PICK a symmetric neighbourhood V of e such that $VV \subseteq U$
 - ⟨3⟩1. U is open
PROOF: Since $Ba^{-1} = r_{a^{-1}}(B)$ is closed.
 - ⟨3⟩2. $e \in U$
PROOF: If $e \in Ba^{-1}$ then $a \in B$
 - ⟨3⟩3. Q.E.D.
PROOF: Lemma 289
 - ⟨2⟩3. $VB \cap Va = \emptyset$
PROOF: If $vb = v'a$ for $v, v' \in V$ and $b \in B$ then we have $ba^{-1} = v^{-1}v' \in Ba \cap VV \subseteq Ba \cap U$.
- ⟨1⟩8. $p(VB)$ and $p(Va)$ are disjoint open sets

$\langle 2 \rangle 1.$ $p(VB)$ and $p(Va)$ are open.

PROOF: Proposition 284.

$\langle 2 \rangle 2.$ $p(VB) \cap p(Va) = \emptyset$

PROOF: If $vbH = v'aH$ for $v, v' \in V, b \in B$ then $v'a = vbh$ for some $h \in H$.

Hence $v'a \in Va \cap VBH = Va \cap VB$.

$\langle 1 \rangle 9.$ $A \subseteq p(VB)$

$\langle 1 \rangle 10.$ $aH \in p(Va)$

□

42 The Metric Topology

Definition 293 (Metric). Let X be a set. A *metric* on X is a function $d : X^2 \rightarrow \mathbb{R}$ such that:

1. For all $x, y \in X$, $d(x, y) \geq 0$
2. For all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$
3. For all $x, y \in X$, $d(x, y) = d(y, x)$
4. (*Triangle Inequality*) For all $x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$

We call $d(x, y)$ the *distance* between x and y .

Definition 294 (Open Ball). Let X be a metric space. Let $a \in X$ and $\epsilon > 0$. The *open ball* with *centre* a and *radius* ϵ is

$$B(a, \epsilon) = \{x \in X \mid d(a, x) < \epsilon\} .$$

Definition 295 (Metric Topology). Let X be a metric space. The *metric topology* on X is the topology generated by the basis consisting of all the open balls.

We prove this is a basis for a topology.

PROOF:

$\langle 1 \rangle 1$. For every point a , there exists a ball B such that $a \in B$

PROOF: We have $a \in B(a, 1)$.

$\langle 1 \rangle 2$. For any balls B_1, B_2 and point $a \in B_1 \cap B_2$, there exists a ball B_3 such that $a \in B_3 \subseteq B_1 \cap B_2$

$\langle 2 \rangle 1$. LET: $B_1 = B(c_1, \epsilon_1)$ and $B_2 = B(c_2, \epsilon_2)$

$\langle 2 \rangle 2$. LET: $\delta = \min(\epsilon_1 - d(c_1, a), \epsilon_2 - d(c_2, a))$

PROVE: $B(a, \delta) \subseteq B_1 \cap B_2$

$\langle 2 \rangle 3$. LET: $x \in B(a, \delta)$

$\langle 2 \rangle 4$. $x \in B_1$

PROOF:

$$\begin{aligned} d(x, c_1) &= d(x, a) + d(a, c_1) \\ &< \delta + d(a, c_1) \\ &\leq \epsilon_1 \end{aligned}$$

$\langle 2 \rangle 5$. $x \in B_2$

PROOF: Similar.

□

Proposition 296. Let X be a metric space and $U \subseteq X$. Then U is open if and only if, for every $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$.

PROOF:

$\langle 1 \rangle 1$. If U is open then, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$.

$\langle 2 \rangle 1$. ASSUME: U is open.

⟨2⟩2. LET: $x \in U$
 ⟨2⟩3. PICK $a \in X$ and $\delta > 0$ such that $x \in B(a, \delta) \subseteq U$
 ⟨2⟩4. LET: $\epsilon = \delta - d(a, x)$
 PROVE: $B(x, \epsilon) \subseteq U$
 ⟨2⟩5. LET: $y \in B(x, \epsilon)$
 ⟨2⟩6. $d(y, a) < \delta$
 PROOF:

$$\begin{aligned}
 d(y, a) &\leq d(a, x) + d(x, y) \\
 &< \delta + d(x, y) \\
 &= \epsilon
 \end{aligned}$$

⟨2⟩7. $y \in U$
 ⟨1⟩2. If, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$, then U is open.
 PROOF: Immediate from definitions.

□

Definition 297 (Discrete Metric). Let X be a set. The *discrete metric* on X is defined by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Proposition 298. *The discrete metric induces the discrete topology.*

PROOF: For any (open) set U and point $a \in U$, we have $a \in B(a, 1) \subseteq U$. □

Definition 299 (Standard Metric on \mathbb{R}). The *standard metric* on \mathbb{R} is defined by $d(x, y) = |x - y|$.

Proposition 300. *The standard metric on \mathbb{R} induces the standard topology on \mathbb{R} .*

PROOF:

⟨1⟩1. Every open ball is open in the standard topology on \mathbb{R} .

PROOF: $B(a, \epsilon) = (a - \epsilon, a + \epsilon)$

⟨1⟩2. For every open set U and point $a \in U$, there exists $\epsilon > 0$ such that $B(a, \epsilon) \subseteq U$

⟨2⟩1. LET: U be an open set and $a \in U$

⟨2⟩2. PICK an open interval b, c such that $a \in (b, c) \subseteq U$

⟨2⟩3. LET: $\epsilon = \min(a - b, c - a)$

⟨2⟩4. $B(a, \epsilon) \subseteq U$

□

Definition 301 (Metriizable). A topological space X is *metrizable* if and only if there exists a metric on X that induces the topology.

Definition 302 (Bounded). Let X be a metric space and $A \subseteq X$. Then A is *bounded* if and only if there exists M such that, for all $x, y \in A$, we have $d(x, y) \leq M$.

Definition 303 (Diameter). Let X be a metric space and $A \subseteq X$. The *diameter* of A is

$$\text{diam } A = \sup_{x,y \in A} d(x,y) .$$

Definition 304 (Standard Bounded Metric). Let d be a metric on X . The *standard bounded metric* corresponding to d is the metric \bar{d} defined by

$$\bar{d}(x,y) = \min(d(x,y), 1) .$$

We prove this is a metric.

PROOF:

$\langle 1 \rangle 1. \bar{d}(x,y) \geq 0$

PROOF: Since $d(x,y) \geq 0$

$\langle 1 \rangle 2. \bar{d}(x,y) = 0$ if and only if $x = y$

PROOF: $\bar{d}(x,y) = 0$ if and only if $d(x,y) = 0$ if and only if $x = y$

$\langle 1 \rangle 3. \bar{d}(x,y) = \bar{d}(y,x)$

PROOF: Since $d(x,y) = d(y,x)$

$\langle 1 \rangle 4. \bar{d}(x,z) \leq \bar{d}(x,y) + \bar{d}(y,z)$

PROOF:

$$\begin{aligned} \bar{d}(x,y) + \bar{d}(y,z) &= \min(d(x,y), 1) + \min(d(y,z), 1) \\ &= \min(d(x,y) + d(y,z), d(x,y) + 1, d(y,z) + 1, 2) \\ &\geq \min(d(x,z), 1) \\ &= \bar{d}(x,z) \end{aligned}$$

□

Lemma 305. In any metric space X , the set $\mathcal{B} = \{B(a, \epsilon) \mid a \in X, \epsilon < 1\}$ is a basis for the metric topology.

PROOF:

$\langle 1 \rangle 1.$ Every element of \mathcal{B} is open.

PROOF: From Lemma 59.

$\langle 1 \rangle 2.$ For every open set U and point $a \in U$, there exists $B \in \mathcal{B}$ such that $a \in B \subseteq U$

$\langle 2 \rangle 1.$ LET: U be an open set and $a \in U$

$\langle 2 \rangle 2.$ PICK $\epsilon > 0$ such that $B(a, \epsilon) \subseteq U$

$\langle 2 \rangle 3.$ $B(a, \min(\epsilon, 1/2)) \subseteq U$

$\langle 1 \rangle 3.$ Q.E.D.

PROOF: Lemma 60.

□

Proposition 306. Let d be a metric on the set X . Then the standard bounded metric \bar{d} induces the same metric as d .

PROOF: This follows from Lemma 305 since the open balls with radius < 1 are the same under both metrics. □

Lemma 307. *Let d and d' be two metrics on the same set X . Let \mathcal{T} and \mathcal{T}' be the topologies they induce. Then $\mathcal{T} \subseteq \mathcal{T}'$ if and only if, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that*

$$B_{d'}(x, \delta) \subseteq B_d(x, \epsilon) .$$

PROOF:

$\langle 1 \rangle 1$. If $\mathcal{T} \subseteq \mathcal{T}'$ then, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$

PROOF: From Proposition 296 since $x \in B_d(x, \epsilon) \in \mathcal{T}'$.

$\langle 1 \rangle 2$. If, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$, then $\mathcal{T} \subseteq \mathcal{T}'$

$\langle 2 \rangle 1$. ASSUME: For all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$

$\langle 2 \rangle 2$. LET: $U \in \mathcal{T}$

$\langle 2 \rangle 3$. For all $x \in U$ there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq U$.

$\langle 3 \rangle 1$. LET: $x \in U$

$\langle 3 \rangle 2$. PICK $\epsilon > 0$ such that $B_d(x, \epsilon) \subseteq U$

PROOF: Proposition 296

$\langle 3 \rangle 3$. PICK $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$

PROOF: By $\langle 2 \rangle 1$

$\langle 3 \rangle 4$. $B_{d'}(x, \delta) \subseteq U$

$\langle 2 \rangle 4$. $U \in \mathcal{T}'$

PROOF: Proposition 296.

□

Proposition 308. \mathbb{R}^2 under the dictionary order topology is metrizable.

PROOF: Define $d : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$d((x, y), (x, z)) = \max(|y - z|, 1)$$

$$d((x, y), (x', y')) = 1$$

$$\text{if } x \neq x' \square$$

$\langle 1 \rangle 1$. $x \in \bigcap_{i=1}^N \pi_i^{-1}() \subseteq B_D(a, \epsilon)$

Proposition 309. *Let $d : X^2 \rightarrow \mathbb{R}$ be a metric on X . Then the metric topology on X is the coarsest topology such that d is continuous.*

PROOF:

$\langle 1 \rangle 1$. d is continuous.

$\langle 2 \rangle 1$. LET: $a, b \in X$

$\langle 2 \rangle 2$. LET: $\epsilon > 0$

$\langle 2 \rangle 3$. LET: $\delta = \epsilon/2$

$\langle 2 \rangle 4$. LET: $x, y \in X$

$\langle 2 \rangle 5$. ASSUME: $\rho((a, b), (x, y)) < \delta$

$\langle 2 \rangle 6$. $|d(a, b) - d(x, y)| < \epsilon$

$\langle 3 \rangle 1$. $d(a, b) - d(x, y) < \epsilon$

PROOF:

$$\begin{aligned}
d(a, b) &\leq d(a, x) + d(x, y) + d(y, b) \\
&\leq d(x, y) + 2\rho((a, b), (x, y)) \\
&< d(x, y) + 2\delta \\
&= d(x, y) + \epsilon
\end{aligned}$$

$$\langle 3 \rangle 2. \ d(a, b) - d(x, y) > -\epsilon$$

PROOF: Similar.

$\langle 2 \rangle 7.$ Q.E.D.

$\langle 1 \rangle 2.$ If \mathcal{T} is any topology under which d is continuous then \mathcal{T} is finer than the metric topology.

PROOF: Since $B(a, \epsilon) = d_a^{-1}((-\infty, \epsilon))$

□

Proposition 310. *Let X be a metric space with metric d and $A \subseteq X$. The restriction of d to A is a metric on A that induces the subspace topology.*

PROOF:

$\langle 1 \rangle 1.$ The restriction of d to A is a metric on A .

$\langle 1 \rangle 2.$ Every open ball under $d \upharpoonright A$ is open under the subspace topology.

PROOF: $B_{d \upharpoonright A}(a, \epsilon) = B_d(a, \epsilon) \cap A$.

$\langle 1 \rangle 3.$ If U is open in the subspace topology and $x \in U$, then there exists a $d \upharpoonright A$ -ball B such that $x \in B \subseteq U$.

$\langle 2 \rangle 1.$ PICK V open in X such that $U = V \cap A$

$\langle 2 \rangle 2.$ PICK $\epsilon > 0$ such that $B_d(x, \epsilon) \subseteq V$

$\langle 2 \rangle 3.$ Take $B = B_{d \upharpoonright A}(x, \epsilon)$

□

Corollary 310.1. *A subspace of a metrizable space is metrizable.*

Proposition 311. *Every metrizable space is Hausdorff.*

PROOF:

$\langle 1 \rangle 1.$ LET: X be a metric space

$\langle 1 \rangle 2.$ LET: $a, b \in X$ with $a \neq b$

$\langle 1 \rangle 3.$ LET: $\epsilon = d(a, b)/2$

$\langle 1 \rangle 4.$ LET: $U = B(a, \epsilon)$ and $V = B(b, \epsilon)$

$\langle 1 \rangle 5.$ U and V are disjoint neighbourhoods of a and b respectively.

□

Proposition 312 (CC). *The product of a countable family of metrizable spaces is metrizable.*

PROOF:

$\langle 1 \rangle 1.$ LET: (X_n, d_n) be a sequence of metric spaces.

$\langle 1 \rangle 2.$ ASSUME: w.l.o.g. each d_n is bounded above by 1.

PROOF: By Proposition 306.

$\langle 1 \rangle 3.$ LET: D be the metric on \mathbb{R}^ω defined by $D(x, y) = \sup_i (d_i(x_i, y_i)/i)$.

- ⟨2⟩1. $D(x, y) \geq 0$
- ⟨2⟩2. $D(x, y) = 0$ if and only if $x = y$
- ⟨2⟩3. $D(x, y) = D(y, x)$
- ⟨2⟩4. $D(x, z) \leq D(x, y) + D(y, z)$

PROOF:

$$\begin{aligned}
 D(x, z) &= \sup_i \frac{d_i(x_i, z_i)}{i} \\
 &\leq \sup_i \frac{d_i(x_i, y_i) + d_i(y_i, z_i)}{i} \\
 &\leq \sup_i \frac{d_i(x_i, y_i)}{i} + \sup_i \frac{d_i(y_i, z_i)}{i} \\
 &= D(x, y) + D(y, z)
 \end{aligned}$$

- ⟨1⟩4. Every open ball $B_D(a, \epsilon)$ is open in the product topology.
- ⟨2⟩1. PICK N such that $1/\epsilon < N$
- ⟨2⟩2. $B_D(a, \epsilon) = \prod_{i=1}^{\infty} U_i$ where $U_i = B(a_i, i\epsilon)$ if $i \leq N$, and $U_i = X_i$ if $i > N$
- ⟨1⟩5. For any open set U and $a \in U$, there exists $\epsilon > 0$ such that $B_D(a, \epsilon) \subseteq U$.
- ⟨2⟩1. LET: $n \geq 1$, V be an open set in \mathbb{R} and $a \in \pi_n^{-1}(V)$
- ⟨2⟩2. PICK $\epsilon > 0$ such that $B_{d_n}(a, \epsilon) \subseteq V$
- ⟨2⟩3. $B_D(a, \epsilon/n) \subseteq \pi_n^{-1}(V)$

□

Theorem 313. *Let X and Y be metric spaces and $f : X \rightarrow Y$. Then f is continuous if and only if, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$.*

PROOF:

- ⟨1⟩1. If f is continuous then, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$
- ⟨2⟩1. ASSUME: f is continuous.
- ⟨2⟩2. LET: $x \in X$ and $\epsilon > 0$
- ⟨2⟩3. PICK a neighbourhood U of x such that $f(U) \subseteq B(f(x), \epsilon)$
- PROOF: Theorem 92.
- ⟨2⟩4. PICK $\delta > 0$ such that $B(x, \delta) \subseteq U$
- PROOF: Proposition 296.
- ⟨2⟩5. For all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$
- ⟨1⟩2. If for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$, then f is continuous.
- ⟨2⟩1. ASSUME: for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$
- ⟨2⟩2. LET: $x \in X$ and V be a neighbourhood of $f(x)$
- ⟨2⟩3. PICK $\epsilon > 0$ such that $B(f(x), \epsilon) \subseteq V$
- PROOF: Proposition 296.
- ⟨2⟩4. PICK $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$
- PROOF: By ⟨2⟩1
- ⟨2⟩5. LET: $U = B(x, \delta)$
- ⟨2⟩6. U is a neighbourhood of x with $f(U) \subseteq V$

⟨2⟩7. Q.E.D.

PROOF: Theorem 92.

□

Proposition 314. *Let X be a metric space. Let (a_n) be a sequence in X and $l \in X$. Then $a_n \rightarrow l$ as $n \rightarrow \infty$ if and only if, for all $\epsilon > 0$, there exists N such that, for all $n \geq N$, we have $d(a_n, l) < \epsilon$.*

PROOF: From Proposition 75. □

Proposition 315. *Every metrizable space is first countable.*

PROOF: In any metric space X , the open balls $B(a, 1/n)$ for $n \geq 1$ form a local basis at a .

Example 316. \mathbb{R}^ω under the box topology is not metrizable.

Example 317. If J is uncountable then \mathbb{R}^J under the product topology is not metrizable.

43 Real Linear Algebra

Definition 318 (Square Metric). The *square metric* ρ on \mathbb{R}^n is defined by

$$\rho(\vec{x}, \vec{y}) = \max(|x_1 - y_1|, \dots, |x_n - y_n|)$$

We prove this is a metric.

PROOF:

⟨1⟩1. $\rho(\vec{x}, \vec{y}) \geq 0$

PROOF: Immediate from definition.

⟨1⟩2. $\rho(\vec{x}, \vec{y}) = 0$ if and only if $\vec{x} = \vec{y}$

PROOF: Immediate from definition.

⟨1⟩3. $\rho(\vec{x}, \vec{y}) = \rho(\vec{y}, \vec{x})$

PROOF: Immediate from definition.

⟨1⟩4. $\rho(\vec{x}, \vec{z}) \leq \rho(\vec{x}, \vec{y}) + \rho(\vec{y}, \vec{z})$

PROOF: Since $|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$.

□

Proposition 319. *The square metric induces the standard topology on \mathbb{R}^n .*

PROOF:

⟨1⟩1. For every $a \in X$ and $\epsilon > 0$, we have $B_\rho(a, \epsilon)$ is open in the standard product topology.

PROOF:

$$B_\rho(a, \epsilon) = (a_1 - \epsilon, a_1 + \epsilon) \times \cdots \times (a_n - \epsilon, a_n + \epsilon)$$

⟨1⟩2. For any open sets U_1, \dots, U_n in \mathbb{R} , we have $U_1 \times \cdots \times U_n$ is open in the square metric topology.

⟨2⟩1. LET: $\vec{a} \in U_1 \times \cdots \times U_n$

- $\langle 2 \rangle 2$. For $i = 1, \dots, n$, PICK $\epsilon_i > 0$ such that $(a_i - \epsilon_i, a_i + \epsilon_i) \subseteq U_i$
 $\langle 2 \rangle 3$. LET: $\epsilon = \min(\epsilon_1, \dots, \epsilon_n)$
 $\langle 2 \rangle 4$. $B_\rho(\vec{a}, \epsilon) \subseteq U$

□

Definition 320. Given $\vec{x}, \vec{y} \in \mathbb{R}^n$, define the *sum* $\vec{x} + \vec{y}$ by

$$(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n) .$$

Definition 321. Given $\lambda \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^n$, define the *scalar product* $\lambda \vec{x} \in \mathbb{R}^n$ by

$$\lambda(x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n)$$

Definition 322 (Inner Product). Given $\vec{x}, \vec{y} \in \mathbb{R}^n$, define the *inner product* $\vec{x} \cdot \vec{y} \in \mathbb{R}$ by

$$(x_1, \dots, x_n) \cdot (y_1, \dots, y_n) = x_1 y_1 + \dots + x_n y_n .$$

We write \vec{x}^2 for $\vec{x} \cdot \vec{x}$.

Definition 323 (Norm). Let $n \geq 1$. The *norm* on \mathbb{R}^n is the function $\| \cdot \| : \mathbb{R}^n \rightarrow \mathbb{R}$ defined by

$$\|(x_1, \dots, x_n)\| = \sqrt{x_1^2 + \dots + x_n^2}$$

Lemma 324.

$$\|\vec{x}\|^2 = \vec{x}^2$$

PROOF: Immediate from definitions. □

Lemma 325.

$$\vec{x} \cdot (\vec{y} + \vec{z}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z}$$

PROOF: Each is equal to $(x_1 y_1 + x_1 z_1, \dots, x_n y_n + x_n z_n)$. □

Lemma 326.

$$|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \|\vec{y}\|$$

PROOF:

$\langle 1 \rangle 1$. ASSUME: $\vec{x} \neq \vec{0} \neq \vec{y}$

PROOF: Otherwise both sides are 0.

$\langle 1 \rangle 2$. LET: $a = 1/\|\vec{x}\|$

$\langle 1 \rangle 3$. LET: $b = 1/\|\vec{y}\|$

$\langle 1 \rangle 4$. $(a\vec{x} + b\vec{y})^2 \geq 0$ and $(a\vec{x} - b\vec{y})^2 \geq 0$

$\langle 1 \rangle 5$. $a^2 \|\vec{x}\|^2 + 2ab\vec{x} \cdot \vec{y} + b^2 \|\vec{y}\|^2 \geq 0$ and $a^2 \|\vec{x}\|^2 - 2ab\vec{x} \cdot \vec{y} + b^2 \|\vec{y}\|^2 \geq 0$

$\langle 1 \rangle 6$. $2ab\vec{x} \cdot \vec{y} + 2 \geq 0$ and $-2ab\vec{x} \cdot \vec{y} + 2 \geq 0$

$\langle 1 \rangle 7$. $\vec{x} \cdot \vec{y} \geq -1/ab$ and $\vec{x} \cdot \vec{y} \leq 1/ab$

$\langle 1 \rangle 8$. $\vec{x} \cdot \vec{y} \geq -\|\vec{x}\| \|\vec{y}\|$ and $\vec{x} \cdot \vec{y} \leq \|\vec{x}\| \|\vec{y}\|$

□

Lemma 327 (Triangle Inequality).

$$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$$

PROOF:

$$\begin{aligned} \|\vec{x} + \vec{y}\|^2 &= \|\vec{x}\|^2 + 2\vec{x} \cdot \vec{y} + \|\vec{y}\|^2 \\ &\leq \|\vec{x}\|^2 + 2\|\vec{x}\|\|\vec{y}\| + \|\vec{y}\|^2 && \text{(Lemma 326)} \\ &= (\|\vec{x}\| + \|\vec{y}\|)^2 \end{aligned} \quad \square$$

Definition 328 (Euclidean Metric). Let $n \geq 1$. The *Euclidean metric* on \mathbb{R}^n is defined by

$$d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|.$$

We prove this is a metric.

$\langle 1 \rangle 1$. $d(\vec{x}, \vec{y}) \geq 0$

PROOF: Immediate from definition.

$\langle 1 \rangle 2$. $d(\vec{x}, \vec{y}) = 0$ if and only if $\vec{x} = \vec{y}$

PROOF: $d(\vec{x}, \vec{y}) = 0$ if and only if $\vec{x} - \vec{y} = \vec{0}$.

$\langle 1 \rangle 3$. $d(\vec{x}, \vec{y}) = d(\vec{y}, \vec{x})$

PROOF: Immediate from definition.

$\langle 1 \rangle 4$. $d(\vec{x}, \vec{z}) \leq d(\vec{x}, \vec{y}) + d(\vec{y}, \vec{z})$

PROOF:

$$\begin{aligned} \|\vec{x} - \vec{z}\| &= \|(\vec{x} - \vec{y}) + (\vec{y} - \vec{z})\| \\ &\leq \|\vec{x} - \vec{y}\| + \|\vec{y} - \vec{z}\| \end{aligned} \quad \text{(Lemma 327)}$$

\square

Proposition 329. The Euclidean metric induces the standard topology on \mathbb{R}^n .

PROOF:

$\langle 1 \rangle 1$. LET: ρ be the square metric.

$\langle 1 \rangle 2$. For all $\vec{a} \in \mathbb{R}^n$ and $\epsilon > 0$, we have $B_d(\vec{a}, \epsilon) \subseteq B_\rho(\vec{a}, \epsilon)$

$\langle 2 \rangle 1$. LET: $\vec{x} \in B_d(\vec{a}, \epsilon)$

$\langle 2 \rangle 2$. $\sqrt{(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2} < \epsilon$

$\langle 2 \rangle 3$. $(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2 < \epsilon^2$

$\langle 2 \rangle 4$. For all i we have $(x_i - a_i)^2 < \epsilon^2$

$\langle 2 \rangle 5$. For all i we have $|x_i - a_i| < \epsilon$

$\langle 2 \rangle 6$. $\rho(\vec{x}, \vec{a}) < \epsilon$

$\langle 1 \rangle 3$. For all $\vec{a} \in \mathbb{R}^n$ and $\epsilon > 0$, we have $B_\rho(\vec{a}, \epsilon/\sqrt{n}) \subseteq B_d(\vec{a}, \epsilon)$

$\langle 2 \rangle 1$. LET: $\vec{x} \in B_\rho(\vec{a}, \epsilon/\sqrt{n})$

$\langle 2 \rangle 2$. $\rho(\vec{x}, \vec{a}) < \epsilon/\sqrt{n}$

$\langle 2 \rangle 3$. For all i we have $|x_i - a_i| < \epsilon/\sqrt{n}$

$\langle 2 \rangle 4$. For all i we have $(x_i - a_i)^2 < \epsilon^2/n$

$\langle 2 \rangle 5$. $d(\vec{x}, \vec{a}) < \epsilon$

$\langle 1 \rangle 4$. Q.E.D.

PROOF: By Lemma 307.

\square

Proposition 330. Let $n \geq 0$. For all $c \in \mathbb{R}^n$ and $\epsilon > 0$, the open ball $B(c, \epsilon)$ is path connected.

PROOF:

$\langle 1 \rangle 1$. LET: $a, b \in B(c, \epsilon)$

$\langle 1 \rangle 2$. LET: $p : [0, 1] \rightarrow B(c, \epsilon)$ be the function $p(t) = (1 - t)a + tb$

PROOF: We have $p(t) \in B(c, \epsilon)$ for all t because

$$\begin{aligned} d(p(t), c) &= \|(1 - t)a + tb - c\| \\ &= \|(1 - t)(a - c) + t(b - c)\| \\ &\leq (1 - t)\|a - c\| + t\|b - c\| \\ &< (1 - t)\epsilon + t\epsilon \\ &= \epsilon \end{aligned}$$

$\langle 1 \rangle 3$. p is a path from a to b .

□

Proposition 331. Let $n \geq 0$. For all $c \in \mathbb{R}^n$ and $\epsilon > 0$, the closed ball $\overline{B(c, \epsilon)}$ is path connected.

PROOF:

$\langle 1 \rangle 1$. LET: $a, b \in \overline{B(c, \epsilon)}$

$\langle 1 \rangle 2$. LET: $p : [0, 1] \rightarrow \overline{B(c, \epsilon)}$ be the function $p(t) = (1 - t)a + tb$

PROOF: We have $p(t) \in \overline{B(c, \epsilon)}$ for all t because

$$\begin{aligned} d(p(t), c) &= \|(1 - t)a + tb - c\| \\ &= \|(1 - t)(a - c) + t(b - c)\| \\ &\leq (1 - t)\|a - c\| + t\|b - c\| \\ &\leq (1 - t)\epsilon + t\epsilon \\ &= \epsilon \end{aligned}$$

$\langle 1 \rangle 3$. p is a path from a to b .

□

Lemma 332. If $\sum_{i=0}^{\infty} x_i^2$ and $\sum_{i=0}^{\infty} y_i^2$ converge then $\sum_{i=0}^{\infty} |x_i y_i|$ converges.

PROOF:

$\langle 1 \rangle 1$. For all $N \geq 0$ we have $\sum_{i=0}^N |x_i y_i| \leq \sqrt{\sum_{i=0}^N |x_i|^2} \sqrt{\sum_{i=0}^N |y_i|^2}$

PROOF: By the Cauchy-Schwarz inequality

$\langle 1 \rangle 2$. Q.E.D.

PROOF: Since $\sum_{i=0}^N |x_i y_i|$ is an increasing sequence bounded above by $(\sum_{i=0}^{\infty} x_i^2)(\sum_{i=0}^{\infty} y_i^2)$.

□

Corollary 332.1. If $\sum_{i=0}^{\infty} x_i^2$ and $\sum_{i=0}^{\infty} y_i^2$ converge then $\sum_{i=0}^{\infty} (x_i + y_i)^2$ converges.

PROOF: Since $\sum_{i=0}^{\infty} x_i^2$, $\sum_{i=0}^{\infty} y_i^2$ and $2 \sum_{i=0}^{\infty} x_i y_i$ all converge.

Definition 333 (l^2 -metric). The l^2 -metric on

$$\left\{ (x_n) \in \mathbb{R}^\omega \mid \sum_{n=0}^{\infty} x_n^2 \text{ converges} \right\}$$

is defined by

$$d(x, y) = \left(\sum_{n=0}^{\infty} (x_n - y_n)^2 \right)^{1/2}$$

We prove this is a metric.

PROOF:

$\langle 1 \rangle 1.$ d is well-defined.

PROOF: By Corollary 332.1.

$\langle 1 \rangle 2.$ $d(x, y) \geq 0$

$\langle 1 \rangle 3.$ $d(x, y) = 0$ if and only if $x = y$

$\langle 1 \rangle 4.$ $d(x, y) = d(y, x)$

$\langle 1 \rangle 5.$ $d(x, z) \leq d(x, y) + d(y, z)$

PROOF: By Lemma 327.

□

Theorem 334. Addition is a continuous function $\mathbb{R}^2 \rightarrow \mathbb{R}$.

PROOF:

$\langle 1 \rangle 1.$ LET: $a, b \in \mathbb{R}$

$\langle 1 \rangle 2.$ LET: $\epsilon > 0$

$\langle 1 \rangle 3.$ LET: $\delta = \epsilon/2$

$\langle 1 \rangle 4.$ LET: $x, y \in \mathbb{R}$

$\langle 1 \rangle 5.$ ASSUME: $\rho((a, b), (x, y)) < \delta$

$\langle 1 \rangle 6.$ $|(a + b) - (x + y)| < \epsilon$

PROOF:

$$\begin{aligned} |(a + b) - (x + y)| &= |a - x| + |b - y| \\ &\leq 2\rho((a, b), (x, y)) \\ &< 2\delta \\ &= \epsilon \end{aligned}$$

$\langle 1 \rangle 7.$ Q.E.D.

PROOF: Theorem 313

□

Theorem 335. Multiplication is a continuous function $\mathbb{R}^2 \rightarrow \mathbb{R}$.

PROOF:

$\langle 1 \rangle 1.$ LET: $a, b \in \mathbb{R}$

$\langle 1 \rangle 2.$ LET: $\epsilon > 0$

$\langle 1 \rangle 3.$ LET: $\delta = \min(\epsilon/(|a| + |b| + 1), 1)$

$\langle 1 \rangle 4.$ LET: $x, y \in \mathbb{R}$

$\langle 1 \rangle 5.$ ASSUME: $\rho((a, b), (x, y)) < \delta$

⟨1⟩6. $|ab - xy| < \epsilon$

PROOF:

$$\begin{aligned}
 |ab - xy| &= |a(b - y) + (a - x)b - (a - x)(b - y)| \\
 &\leq |a||b - y| + |b||a - x| + |a - x||b - y| \\
 &< |a|\delta + |b|\delta + \delta^2 & (\langle 1 \rangle 5) \\
 &\leq |a|\delta + |b|\delta + \delta & (\langle 1 \rangle 3) \\
 &\leq \epsilon & (\langle 1 \rangle 3)
 \end{aligned}$$

⟨1⟩7. Q.E.D.

PROOF: Theorem 313

□

Theorem 336. *The function $f : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ defined by $f(x) = x^{-1}$ is continuous.*

PROOF:

⟨1⟩1. For all $a \in \mathbb{R}$ we have $f^{-1}((a, +\infty))$ is open.

PROOF: The set is

$$(a^{-1}, +\infty) \text{ if } a > 0$$

$$(0, +\infty) \text{ if } a = 0$$

$$(-\infty, a^{-1}) \cup (0, +\infty) \text{ if } a < 0$$

⟨1⟩2. For all $a \in \mathbb{R}$ we have $f^{-1}((-\infty, a))$ is open.

PROOF: Similar.

⟨1⟩3. Q.E.D.

PROOF: By Proposition 89 and Lemma 113.

□

Definition 337. For $n \geq 0$, the *unit ball* B^n is the space $\{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$.

Proposition 338. *For all $n \geq 0$, the unit ball B^n is path connected.*

PROOF:

⟨1⟩1. LET: $a, b \in B^n$

⟨1⟩2. LET: $p : [0, 1] \rightarrow B^n$ be the function $p(t) = (1 - t)a + tb$

PROOF: We have $p(t) \in B^n$ for all t because

$$\begin{aligned}
 \|(1 - t)a + tb\| &\leq (1 - t)\|a\| + t\|b\| \\
 &\leq (1 - t) + t \\
 &= 1
 \end{aligned}$$

⟨1⟩3. p is a path from a to b .

□

Definition 339 (Punctured Euclidean Space). For $n \geq 0$, defined *punctured Euclidean space* to be $\mathbb{R}^n \setminus \{0\}$.

Proposition 340. *For $n > 1$, punctured Euclidean space $\mathbb{R}^n \setminus \{0\}$ is path connected.*

PROOF:

- ⟨1⟩1. LET: $a, b \in \mathbb{R}^n \setminus \{0\}$
- ⟨1⟩2. CASE: 0 is on the line from a to b
 - ⟨2⟩1. PICK a point c not on the line from a to b
 - ⟨2⟩2. The path consisting of a straight line from a to c followed by a straight line from c to b is a path from a to b .
- ⟨1⟩3. CASE: 0 is not on the line from a to b
 - PROOF: The straight line from a to b is a path from a to b .

Corollary 340.1. *For $n > 1$, the spaces \mathbb{R} and \mathbb{R}^n are not homeomorphic.*

PROOF: For any point a , the space $\mathbb{R} \setminus \{a\}$ is disconnected.

Definition 341 (Unit Sphere). For $n \geq 1$, the *unit sphere* S^{n-1} is the space

$$S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\} .$$

Proposition 342. *For $n > 1$, the unit sphere S^{n-1} is path connected.*

PROOF: The map $g : \mathbb{R}^n \setminus \{0\} \rightarrow S^{n-1}$ defined by $g(x) = x/\|x\|$ is continuous and surjective. The result follows by Proposition 238. \square

Proposition 343. *Let $f : S^1 \rightarrow \mathbb{R}$ be continuous. Then there exists $x \in S^1$ such that $f(x) = f(-x)$.*

PROOF:

- ⟨1⟩1. LET: $g : S^1 \rightarrow \mathbb{R}$ be the function $g(x) = f(x) - f(-x)$
 - PROVE: There exists $x \in S^1$ such that $g(x) = 0$
- ⟨1⟩2. ASSUME: without loss of generality $g((1, 0)) > 0$
- ⟨1⟩3. $g((-1, 0)) < 0$
- ⟨1⟩4. There exists x such that $g(x) = 0$

PROOF: By the Intermediate Value Theorem.

\square

Definition 344 (Topologist's Sine Curve). Let $S = \{(x, \sin 1/x) \mid 0 < x \leq 1\}$. The *topologist's sine curve* is the closure \overline{S} of S .

Proposition 345.

$$\overline{S} = S \cup (\{0\} \times [-1, 1])$$

Proposition 346. *The topologist's sine curve is connected.*

PROOF:

- ⟨1⟩1. LET: $S = \{(x, \sin 1/x) \mid 0 < x \leq 1\}$
- ⟨1⟩2. S is connected.
 - PROOF: Theorem 218.
- ⟨1⟩3. \overline{S} is connected.
 - PROOF: Theorem 217.

\square

Proposition 347 (CC). *The topologist's sine curve is not path connected.*

PROOF:

⟨1⟩1. ASSUME: for a contradiction $p : [0, 1] \rightarrow \bar{S}$ is a path from $(0, 0)$ to $(1, \sin 1)$.

⟨1⟩2. $p^{-1}(\{0\} \times [0, 1])$ is closed.

⟨1⟩3. LET: b be the greatest element of $p^{-1}(\{0\} \times [0, 1])$.

⟨1⟩4. $b < 1$

PROOF: Since $p(1) = (1, \sin 1)$.

⟨1⟩5. PICK a sequence $(t_n)_{n \geq 1}$ in $(b, 1]$ such that $t_n \rightarrow b$ and $\pi_2(p(t_n)) = (-1)^n$

⟨2⟩1. LET: $n \geq 1$

⟨2⟩2. PICK u with $0 < u < \pi_1(p(1/n))$ such that $\sin(1/u) = (-1)^n$

⟨2⟩3. PICK t_n such that $0 < t_n < 1/n$ and $\pi_1(p(t_n)) = u$

PROOF: One exists by the Intermediate Value Theorem.

⟨1⟩6. Q.E.D.

PROOF: This contradicts 105.

□

44 The Uniform Topology

Definition 348 (Uniform Metric). Let J be a set. The *uniform metric* $\bar{\rho}$ on \mathbb{R}^J is defined by

$$\bar{\rho}(a, b) = \sup_{j \in J} \bar{d}(a_j, b_j)$$

where \bar{d} is the standard bounded metric on \mathbb{R} .

The *uniform topology* on \mathbb{R}^J is the topology induced by the uniform metric.

We prove this is a metric.

PROOF:

⟨1⟩1. $\bar{\rho}(a, b) \geq 0$

PROOF: Immediate from definitions.

⟨1⟩2. $\bar{\rho}(a, b) = 0$ if and only if $a = b$

PROOF: Immediate from definitions.

⟨1⟩3. $\bar{\rho}(a, b) = \bar{\rho}(b, a)$

PROOF: Immediate from definitions.

⟨1⟩4. $\bar{\rho}(a, c) \leq \bar{\rho}(a, b) + \bar{\rho}(b, c)$

PROOF:

$$\begin{aligned} \bar{\rho}(a, c) &= \sup_{j \in J} \bar{d}(a_j, c_j) \\ &\leq \sup_{j \in J} (\bar{d}(a_j, b_j) + \bar{d}(b_j, c_j)) \\ &\leq \sup_{j \in J} \bar{d}(a_j, b_j) + \sup_{j \in J} \bar{d}(b_j, c_j) \\ &= \bar{\rho}(a, b) + \bar{\rho}(b, c) \end{aligned}$$

□

Proposition 349. The uniform topology on \mathbb{R}^J is finer than the product topology.

PROOF:

- ⟨1⟩1. LET: $j \in J$ and U be open in \mathbb{R}
 PROVE: $\pi_j^{-1}(U)$ is open in the uniform topology.
- ⟨1⟩2. LET: $a \in \pi_j^{-1}(U)$
- ⟨1⟩3. PICK $\epsilon > 0$ such that $(a_j - \epsilon, a_j + \epsilon) \subseteq U$
- ⟨1⟩4. $B_{\bar{\rho}}(a, \epsilon) \subseteq \pi_j^{-1}(U)$

□

Proposition 350. *The uniform topology on \mathbb{R}^J is coarser than the box topology.*

PROOF:

- ⟨1⟩1. LET: $a \in \mathbb{R}^J$ and $\epsilon > 0$
 PROVE: $B(a, \epsilon)$ is open in the box topology.
- ⟨1⟩2. LET: $b \in B(a, \epsilon)$
- ⟨1⟩3. For $j \in J$ we have $|a_j - b_j| < \epsilon$
- ⟨1⟩4. For $j \in J$,
 LET: $\delta_j = (\epsilon - |a_j - b_j|)/2$
- ⟨1⟩5. $\prod_{j \in J} (b_j - \delta_j, b_j + \delta_j) \subseteq B(a, \epsilon)$

□

Proposition 351. *The uniform topology on \mathbb{R}^J is strictly finer than the product topology if and only if J is infinite.*

PROOF:

- ⟨1⟩1. If J is finite then the uniform and product topologies coincide.
 PROOF: The uniform, box and product topologies are all the same.
- ⟨1⟩2. If J is infinite then the uniform and product topologies are different.
 PROOF: The set $B(\vec{0}, 1)$ is open in the uniform topology but not the product topology.

□

Proposition 352 (DC). *The uniform topology on \mathbb{R}^J is strictly coarser than the box topology if and only if J is infinite.*

PROOF:

- ⟨1⟩1. If J is finite then the uniform and box topologies coincide.
 PROOF: The uniform, box and product topologies are all the same.
- ⟨1⟩2. If J is infinite then the uniform and box topologies are different.
 PROOF: Pick an ω -sequence (j_1, j_2, \dots) in J . Let $U = \prod_{j \in J} U_j$ where $U_{j_i} = (-1/i, 1/i)$ and $U_j = (-1, 1)$ for all other j . Then $\vec{0} \in U$ but there is no $\epsilon > 0$ such that $B(\vec{0}, \epsilon) \subseteq U$.

□

Proposition 353. *The closure of \mathbb{R}^ω in \mathbb{R}^ω under the uniform topology is \mathbb{R}^ω .*

PROOF: Given any open ball $B(a, \epsilon)$, pick an integer N such that $1/\epsilon < N$. Then $B(a, \epsilon)$ includes sequences whose n th entry is 0 for all $n \geq N$. □

Example 354. The space \mathbb{R}^ω is disconnected under the uniform topology. The set of bounded sequences and the set of unbounded sequences form a separation.

Proposition 355. Give \mathbb{R}^ω the uniform topology. Let $x, y \in \mathbb{R}^\omega$. Then x and y are in the same component if and only if $x - y$ is bounded.

PROOF:

$\langle 1 \rangle 1$. The component containing 0 is the set of bounded sequences.

$\langle 2 \rangle 1$. LET: B be the set of bounded sequences.

$\langle 2 \rangle 2$. B is path-connected.

$\langle 3 \rangle 1$. LET: $x, y \in B$

$\langle 3 \rangle 2$. PICK $b > 0$ such that $|x_j|, |y_j| \leq b$ for all j

$\langle 3 \rangle 3$. LET: $p : [0, 1] \rightarrow B$ be the function $p(t) = (1 - t)x + ty$

PROVE: p is continuous.

$\langle 3 \rangle 4$. LET: $t \in [0, 1]$ and $\epsilon > 0$

$\langle 3 \rangle 5$. LET: $\delta = \epsilon/2b$

$\langle 3 \rangle 6$. LET: $s \in [0, 1]$ with $|s - t| < \delta$

$\langle 3 \rangle 7$. $\bar{\rho}(p(s), p(t)) < \epsilon$

PROOF:

$$\begin{aligned} \bar{\rho}(p(s), p(t)) &= \sup_j \bar{d}((1 - s)x_j + sy_j, (1 - t)x_j + ty_j) \\ &\leq |(s - t)x_j + (t - s)y_j| \\ &\leq |s - t||x_j - y_j| \\ &< 2b\delta \\ &= \epsilon \end{aligned}$$

$\langle 2 \rangle 3$. B is connected.

PROOF: Proposition 236.

$\langle 2 \rangle 4$. If C is connected and $B \subseteq C$ then $B = C$.

PROOF: Otherwise $B \cap C$ and $C \setminus B$ form a separation of C .

$\langle 1 \rangle 2$. Q.E.D.

PROOF: Since $\lambda x. x - y$ is a Homeomorphism of \mathbb{R}^ω with itself.

□

45 Uniform Convergence

Definition 356 (Uniform Convergence). Let X be a set and Y a metric space. Let $(f_n : X \rightarrow Y)$ be a sequence of functions and $f : X \rightarrow Y$ be a function. Then f_n converges uniformly to f as $n \rightarrow \infty$ if and only if, for all $\epsilon > 0$, there exists N such that, for all $n \geq N$ and $x \in X$, we have $d(f_n(x), f(x)) < \epsilon$.

Example 357. Define $f_n : [0, 1] \rightarrow \mathbb{R}$ by $f_n(x) = x^n$ for $n \geq 1$, and $f : [0, 1] \rightarrow \mathbb{R}$ by $f(x) = 0$ if $x < 1$, $f(1) = 1$. Then f_n converges to f pointwise but not uniformly.

Theorem 358 (Uniform Limit Theorem). Let X be a topological space and Y a metric space. Let $(f_n : X \rightarrow Y)$ be a sequence of continuous functions and $f : X \rightarrow Y$ be a function. If f_n converges uniformly to f as $n \rightarrow \infty$, then f is continuous.

PROOF:

⟨1⟩1. LET: $x \in X$ and $\epsilon > 0$

⟨1⟩2. PICK N such that, for all $n \geq N$ and $y \in X$, we have $d(f_n(y), f(y)) < \epsilon/3$

⟨1⟩3. PICK a neighbourhood U of x such that $f_N(U) \subseteq B(f_N(x), \epsilon/3)$

PROVE: $f(U) \subseteq B(f(x), \epsilon)$

⟨1⟩4. LET: $y \in U$

⟨1⟩5. $d(f(y), f(x)) < \epsilon$

PROOF:

$$d(f(y), f(x)) \leq d(f(y), f_N(y)) + d(f_N(y), f_N(x)) + d(f_N(x), f(x)) \quad (\text{Triangle Inequality})$$

$$< \epsilon/3 + \epsilon/3 + \epsilon/3$$

((⟨1⟩2, ⟨1⟩3)

$$= \epsilon$$

□

Proposition 359. *Let X be a topological space and Y a metric space. Let $(f_n : X \rightarrow Y)$ be a sequence of continuous functions and $f : X \rightarrow Y$ be a function. Let (a_n) be a sequence of points in X and $a \in X$. If f_n converges uniformly to f and a_n converges to a in X then $f_n(a_n)$ converges to $f(a)$ uniformly in Y .*

PROOF:

⟨1⟩1. LET: $\epsilon > 0$

⟨1⟩2. PICK N_1 such that, for all $n \geq N_1$ and $x \in X$, we have $d(f_n(x), f(x)) < \epsilon/2$

⟨1⟩3. PICK N_2 such that, for all $n \geq N_2$, we have $a_n \in f^{-1}(B(a, \epsilon/2))$

PROOF: Using the fact that f is continuous from the Uniform Limit Theorem.

⟨1⟩4. LET: $N = \max(N_1, N_2)$

⟨1⟩5. LET: $n \geq N$

⟨1⟩6. $d(f_n(a_n), f(a)) < \epsilon$

PROOF:

$$d(f_n(a_n), f(a)) \leq d(f_n(a_n), f(a_n)) + d(f(a_n), f(a)) \quad (\text{Triangle Inequality})$$

$$< \epsilon/2 + \epsilon/2$$

((⟨1⟩2, ⟨1⟩3)

$$= \epsilon$$

□

Proposition 360. *Let X be a set. Let $(f_n : X \rightarrow \mathbb{R})$ be a sequence of functions and $f : X \rightarrow \mathbb{R}$ be a function. Then f_n converges uniformly to f as $n \rightarrow \infty$ if and only if $f_n \rightarrow f$ as $n \rightarrow \infty$ in \mathbb{R}^X under the uniform topology.*

PROOF:

⟨1⟩1. If f_n converges uniformly to f then f_n converges to f under the uniform topology.

⟨2⟩1. ASSUME: f_n converges uniformly to f

⟨2⟩2. LET: $\epsilon > 0$

⟨2⟩3. PICK N such that, for all $n \geq N$ and $x \in X$, we have $d(f_n(x), f(x)) < \epsilon/2$

⟨2⟩4. For all $n \geq N$ we have $\bar{\rho}(f_n, f) \leq \epsilon/2$

⟨2⟩5. For all $n \geq N$ we have $\bar{\rho}(f_n, f) < \epsilon$

- ⟨1⟩2. If f_n converges to f under the uniform topology then f_n converges uniformly to f .
 ⟨2⟩1. ASSUME: f_n converges to f under the uniform topology.
 ⟨2⟩2. LET: $\epsilon > 0$
 ⟨2⟩3. PICK N such that, for all $n \geq N$, we have $\bar{\rho}(f_n, f) < \min(\epsilon, 1/2)$
 ⟨2⟩4. LET: $n \geq N$
 ⟨2⟩5. LET: $x \in X$
 ⟨2⟩6. $\bar{\rho}(f_n, f) < \min(\epsilon, 1/2)$
 PROOF: From ⟨2⟩3.
 ⟨2⟩7. $d(f_n(x), f(x)) < \min(\epsilon, 1/2)$
 ⟨2⟩8. $d(f_n(x), f(x)) < \epsilon$

□

46 Isometric Imbeddings

Definition 361. Let X and Y be metric spaces. An *isometric imbedding* $f : X \rightarrow Y$ is a function such that, for all $x, y \in X$, we have $d(f(x), f(y)) = d(x, y)$.

Proposition 362. *Every isometric imbedding is an imbedding.*

PROOF:

- ⟨1⟩1. LET: $f : X \rightarrow Y$ be an isometric imbedding.
 ⟨1⟩2. f is injective.
 PROOF: If $f(x) = f(y)$ then $d(f(x), f(y)) = 0$ hence $d(x, y) = 0$ hence $x = y$.
 ⟨1⟩3. f is continuous.
 PROOF: For all $\epsilon > 0$, if $d(x, y) < \epsilon$ then $d(f(x), f(y)) < \epsilon$.
 ⟨1⟩4. $f : X \rightarrow f(X)$ is an open map.
 PROOF: $f(B(a, \epsilon)) = B(f(a), \epsilon) \cap f(X)$.

□