# Topology

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# Contents

# 1 Set Theory

**Definition 1** (Cover)**.** Let $X$ be a set and $\mathcal{A} \subseteq \mathcal{P}X$. Then $\mathcal{A}$ *covers* $X$, or is a *covering* of $X$, if and only if $\bigcup \mathcal{A} = X$.

**Definition 2** (Finite Intersection Property)**.** Let $X$ be a set and $\mathcal{A} \subseteq \mathcal{P}X$. Then $\mathcal{A}$ satisfies the *finite intersection property* if and only if every nonempty finite subset of $\mathcal{A}$ has nonempty intersection.

**Definition 3** (Graph)**.** Let $f : A \to B$. The *graph* of $f$ is the set $\{(x, f(x)) \mid x \in A\} \subseteq A \times B$.

# 2    Order Theory

**Definition 4** (Preorder)**.** Let $X$ be a set. A *preorder* on $X$ is a binary relation $\leq$ on $X$ such that:

**Reflexivity** For all $x \in X$, we have $x \leq x$

**Transitivity** For all $x, y, z \in X$, if $x \leq y$ and $y \leq z$ then $x \leq z$.

**Definition 5** (Preordered Set)**.** A *preordered set* consists of a set $X$ and a preorder $\leq$ on $X$.

**Proposition 6.** *Let $X$ and $Y$ be linearly ordered sets. Let $f : X \twoheadrightarrow Y$ be strictly monotone and surjective. Then $f$ is a poset isomorphism.*

PROOF:
$\langle 1 \rangle 1$. $f$ is injective.
   $\langle 2 \rangle 1$. LET: $x, y \in X$
   $\langle 2 \rangle 2$. ASSUME: $f(x) = f(y)$
   $\langle 2 \rangle 3$. $x \not< y$
     PROOF: By strong motonicity.
   $\langle 2 \rangle 4$. $y \not< x$
     PROOF: By strong motonicity.
   $\langle 2 \rangle 5$. $x = y$
     PROOF: By trichotomy.
$\langle 1 \rangle 2$. $f^{-1}$ is monotone.
   $\langle 2 \rangle 1$. LET: $x, y \in X$
   $\langle 2 \rangle 2$. ASSUME: $x \leq y$
   $\langle 2 \rangle 3$. $f^{-1}(x) \not> f^{-1}(y)$
     PROOF: By strong motonicity.
   $\langle 2 \rangle 4$. $f^{-1}(x) < f^{-1}(y)$
     PROOF: By trichotomy.
□

**Definition 7** (Interval)**.** Let $X$ be a preordered set and $Y \subseteq X$. Then $Y$ is an *interval* if and only if, for all $a, b \in Y$ and $c \in X$, if $a \leq c \leq b$ then $c \in Y$.

**Definition 8** (Linear Continuum)**.** A linearly ordered set $L$ is a *linear continuum* if and only if:

1. every nonempty subset of $L$ that is bounded above has a supremum

2. $L$ is dense

**Proposition 9.** *Every interval in a linear continuum is a linear continuum.*

PROOF:
$\langle 1 \rangle 1$. LET: $L$ be a linear continuum and $I$ an interval in $L$.
$\langle 1 \rangle 2$. Every nonempty subset of $I$ that is bounded above has a supremum in $I$.
   $\langle 2 \rangle 1$. LET: $X \subseteq I$ be nonempty and bounded above by $b \in I$.

$\langle 2 \rangle$2. LET: $s$ be the supremum of $X$ in $L$.
    PROOF: Since $L$ is a linear continuum.
$\langle 2 \rangle$3. $s \in I$
    $\langle 3 \rangle$1. PICK $a \in X$
      PROOF: Since $X$ is nonempty ($\langle 2 \rangle$1).
    $\langle 3 \rangle$2. $a \leq s \leq b$
    $\langle 3 \rangle$3. $s \in I$
      PROOF: Since $I$ is an interval ($\langle 1 \rangle$1).
$\langle 2 \rangle$4. $s$ is the supremum of $X$ in $I$
$\langle 1 \rangle$3. $I$ is dense.
    $\langle 2 \rangle$1. LET: $x, y \in I$ with $x < y$
    $\langle 2 \rangle$2. PICK $z \in L$ with $x < z < y$
      PROOF: Since $L$ is dense.
    $\langle 2 \rangle$3. $z \in I$
      PROOF: Since $I$ is an interval.
$\square$

**Definition 10** (Ordered Square). The *ordered square* $I_o^2$ is the set $[0,1]^2$ under the dictionary order.

**Proposition 11.** *The ordered square is a linear continuum.*

PROOF:
$\langle 1 \rangle$1. Every nonempty subset of $I_o^2$ bounded above has a supremum.
    $\langle 2 \rangle$1. LET: $X \subseteq I_o^2$ be nonempty and bounded above by $(b, c)$
    $\langle 2 \rangle$2. LET: $s = \sup \pi_1(X)$
      PROOF: The set $\pi_1(X)$ is nonempty and bounded above by $b$.
    $\langle 2 \rangle$3. CASE: $s \in \pi_1(X)$
      $\langle 3 \rangle$1. LET: $t = \sup\{y \in [0,1] \mid (s, y) \in X\}$
        PROOF: This set is nonempty and bounded above by $c$.
      $\langle 3 \rangle$2. $(s, t)$ is the supremum of $X$.
    $\langle 2 \rangle$4. CASE: $s \notin \pi_1(X)$
      PROOF: In this case $(s, 0)$ is the supremum of $X$.
$\langle 1 \rangle$2. $I_o^2$ is dense.
    $\langle 2 \rangle$1. LET: $(x_1, y_1), (x_2, y_2) \in I_o^2$ with $(x_1, y_1) < (x_2, y_2)$
    $\langle 2 \rangle$2. CASE: $x_1 < x_2$
      $\langle 3 \rangle$1. PICK $x_3$ with $x_1 < x_3 < x_2$
      $\langle 3 \rangle$2. $(x_1, y_1) < (x_3, y_1) < (x_2, y_2)$
    $\langle 2 \rangle$3. CASE: $x_1 = x_2$ and $y_1 < y_2$
      $\langle 3 \rangle$1. PICK $y_3$ with $y_1 < y_3 < y_2$
      $\langle 3 \rangle$2. $(x_1, y_1) < (x_1, y_3) < (x_2, y_2)$
$\square$

**Proposition 12.** *If $X$ is a well-ordered set then $X \times [0, 1)$ under the dictionary order is a linear continuum.*

PROOF:
$\langle 1 \rangle$1. Every nonempty set $A \subseteq X \times [0, 1)$ bounded above has a supremum.

$\langle 2 \rangle 1.$ Let: $A \subseteq X \times [0, 1)$ be nonempty and bounded above
$\langle 2 \rangle 2.$ Let: $x_0$ be the supremum of $\pi_1(A)$
$\langle 2 \rangle 3.$ Case: $x_0 \in \pi_1(A)$
    $\langle 3 \rangle 1.$ Let: $y_0$ be the supremum of $\{y \in [0, 1) \mid (x_0, y) \in A\}$
    $\langle 3 \rangle 2.$ $(x_0, y_0)$ is the supremum of $A$.
$\langle 2 \rangle 4.$ Case: $x_0 \notin \pi_1(A)$
    Proof: In this case $(x_0, 0)$ is the supremum of $A$.
$\langle 1 \rangle 2.$ $X \times [0, 1)$ is dense.
$\langle 2 \rangle 1.$ Let: $(x_1, y_1), (x_2, y_2) \in X \times [0, 1)$ with $(x_1, y_1) < (x_2, y_2)$
$\langle 2 \rangle 2.$ Case: $x_1 < x_2$
    $\langle 3 \rangle 1.$ Pick $y_3$ such that $y_1 < y_3 < 1$
    $\langle 3 \rangle 2.$ $(x_1, y_1) < (x_1, y_3) < (x_2, y_2)$
$\langle 2 \rangle 3.$ Case: $x_1 = x_2$ and $y_1 < y_2$
    $\langle 3 \rangle 1.$ Pick $y_3$ such that $y_1 < y_3 < y_2$
    $\langle 3 \rangle 2.$ $(x_1, y_1) < (x_1, y_3) < (x_2, y_2)$
□

**Lemma 13.** *For all $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$, we have $[a, b) \cong [c, d)$*

Proof: The map $\lambda t.c + (t - a)(d - c)/(b - a)$ is an order isomorphism.

**Proposition 14.** *Let $X$ be a linearly ordered set. Let $a < b < c$ in $X$. Then $[a, c) \cong [0, 1)$ if and only if $[a, b) \cong [b, c) \cong [0, 1)$.*

Proof:
$\langle 1 \rangle 1.$ If $[a, c) \cong [0, 1)$ then $[a, b) \cong [b, c) \cong [0, 1)$
    $\langle 2 \rangle 1.$ Assume: $f : [a, c) \cong [0, 1)$ is an order isomorphism
    $\langle 2 \rangle 2.$ $[a, b) \cong [0, 1)$
        Proof:
$$[a, b) \cong [0, f(b)) \qquad \text{(by the restriction of } f\text{)}$$
$$\cong [0, 1) \qquad \qquad \text{(Lemma 13)}$$
    $\langle 2 \rangle 3.$ $[b, c) \cong [0, 1)$
        Proof: Similar.
$\langle 1 \rangle 2.$ If $[a, b) \cong [b, c) \cong [0, 1)$ then $[a, c) \cong [0, 1)$
    Proof:
$$[a, c) = [a, b) * [b, c)$$
$$\cong [0, 1) * [0, 1)$$
$$\cong [0, 1/2) * [1/2, 1) \qquad \text{(Lemma 13)}$$
$$= 1$$
□

**Proposition 15** (CC). *Let $X$ be a linearly ordered set. Let $x_0 < x_1 < \cdots$ be a strictly increasing sequence in $X$ with supremum $b$. Then $[x_0, b) \cong [0, 1)$ if and only if $[x_i, x_{i+1}) \cong [0, 1)$ for all $i$.*

Proof:
$\langle 1 \rangle 1.$ If $[x_0, b) \cong [0, 1)$ then $[x_i, x_{i+1}) \cong [0, 1)$ for all $i$.

7

PROOF: By Lemma 13

$\langle 1 \rangle 2$. If $[x_i, x_{i+1}) \cong [0, 1)$ for all $i$ then $[x_0, b) \cong [0, 1)$

$\quad \langle 2 \rangle 1$. ASSUME: $[x_i, x_{i+1}) \cong [0, 1)$ for all $i$

$\quad \langle 2 \rangle 2$. PICK an order isomorphism $f_i : [x_i, x_{i+1}) \cong [1/2^i, 2/2^{i+1})$ for each $i$.

$\quad\quad$ PROOF: By Lemma 13

$\quad \langle 2 \rangle 3$. The union of the $f_i$s is an order isomorphism $[x_0, b) \cong [0, 1)$

$\square$

# 3 Real Analysis

**Definition 16.** Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences $(a_n)$ such that $a_n = 0$ for all but finitely many $n$.

# 4 Group Theory

**Definition 17.** Given a group $G$ and sets $A, B \subseteq G$, let $AB = \{ab \mid a \in A, b \in B\}$.

**Definition 18.** Given a group $G$ and a set $A \subseteq G$, let $A^{-1} = \{a^{-1} \mid a \in A\}$.

# 5  Topological Spaces

**Definition 19** (Topology). A *topology* on a set $X$ is a set $\mathcal{T} \subseteq \mathcal{P}X$ such that:

- $X \in \mathcal{T}$.

- For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$.

- For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$.

We call the elements of $X$ *points* and the elements of $\mathcal{T}$ *open sets*.

**Definition 20** (Topological Space). A *topological space* $X$ consists of a set $X$ and a topology on $X$.

**Definition 21** (Discrete Space). For any set $X$, the *discrete* topology on $X$ is $\mathcal{P}X$.

**Definition 22** (Indiscrete Space). For any set $X$, the *indiscrete* or *trivial* topology on $X$ is $\{\emptyset, X\}$.

**Definition 23** (Finite Complement Topology). For any set $X$, the *finite complement topology* on $X$ is $\{U \in \mathcal{P}X \mid X \setminus U \text{ is finite}\} \cup \{\emptyset\}$.

**Definition 24** (Countable Complement Topology). For any set $X$, the *countable complement topology* on $X$ is $\{U \in \mathcal{P}X \mid X \setminus U \text{ is countable}\} \cup \{\emptyset\}$.

**Definition 25** (Finer, Coarser). Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a given set $X$. If $\mathcal{T}' \supseteq \mathcal{T}$, we say that $\mathcal{T}'$ is *finer* than $\mathcal{T}$; if $\mathcal{T}'$ *properly* contains $\mathcal{T}$, we say that $\mathcal{T}'$ is *strictly* finer than $\mathcal{T}$. We also say that $\mathcal{T}$ is *coarser* than $\mathcal{T}'$, or *stricly* coarser, in these two respective situations. We say $\mathcal{T}$ is *comparable* with $\mathcal{T}'$ if either $\mathcal{T}' \supseteq \mathcal{T}$ or $\mathcal{T} \supseteq \mathcal{T}'$.

**Lemma 26.** *Let $X$ be a topological space and $U \subseteq X$. Then $U$ is open if and only if, for all $x \in U$, there exists an open set $V$ such that $x \in V \subseteq U$.*

PROOF:
$\langle 1 \rangle 1. \Rightarrow$
  PROOF: Take $V = U$
$\langle 1 \rangle 2. \Leftarrow$
  PROOF: We have $U = \bigcup \{V \text{ open in } X \mid V \subseteq U\}$.
☐

**Lemma 27.** *Let $X$ be a set and $\mathcal{T}$ a nonempty set of topologies on $X$. Then $\bigcap \mathcal{T}$ is a topology on $X$, and is the finest topology that is coarser than every member of $\mathcal{T}$.*

PROOF:
$\langle 1 \rangle 1. \ X \in \bigcap \mathcal{T}$
  PROOF: Since $X$ is in every member of $\mathcal{T}$.
$\langle 1 \rangle 2. \ \bigcap \mathcal{T}$ is closed under union.

⟨2⟩1. LET: $\mathcal{U} \subseteq \bigcap \mathcal{T}$
⟨2⟩2. For all $T \in \mathcal{T}$ we have $\mathcal{U} \subseteq T$
⟨2⟩3. For all $T \in \mathcal{T}$ we have $\bigcup \mathcal{U} \in T$
⟨2⟩4. $\bigcup \mathcal{U} \in \bigcap \mathcal{T}$
⟨1⟩3. $\bigcap \mathcal{T}$ is closed under binary intersection.
⟨2⟩1. LET: $U, V \in \bigcap \mathcal{T}$
⟨2⟩2. For all $T \in \mathcal{T}$ we have $U, V \in T$
⟨2⟩3. For all $T \in \mathcal{T}$ we have $U \cap V \in T$
⟨2⟩4. $U \cap V \in \bigcap \mathcal{T}$
□

**Lemma 28.** *Let $X$ be a set and $\mathcal{T}$ a set of topologies on $X$. Then there exists a unique coarsest topology that is finer than every member of $\mathcal{T}$.*

PROOF: The required topology is given by
$\bigcap \{T \in \mathcal{PP}X \mid T$ *is a topology on $X$ that is finer than every member of $\mathcal{T}$*$\}$ ,
The set is nonempty since it contains the discrete topology. □

**Definition 29** (Neighbourhood)**.** A *neighbourhood* of a point $x$ is an open set that contains $x$.

# 6 Closed Set

**Definition 30** (Closed Set)**.** Let $X$ be a topological space and $A \subseteq X$. Then $A$ is *closed* if and only if $X \setminus A$ is open.

**Lemma 31.** *The empty set is closed.*

PROOF: Since the whole space $X$ is always open. □

**Lemma 32.** *The topological space $X$ is closed.*

PROOF: Since $\emptyset$ is open. □

**Lemma 33.** *The intersection of a nonempty set of closed sets is closed.*

PROOF: Let $\mathcal{C}$ be a nonempty set of closed sets. Then $X \setminus \bigcap \mathcal{C} = \bigcup \{X \setminus C \mid C \in \mathcal{C}\}$ is open. □

**Lemma 34.** *The union of two closed sets is closed.*

PROOF: Let $C$ and $D$ be closed. Then $X \setminus (C \cup D) = (X \setminus C) \cap (X \setminus D)$ is open. □

**Proposition 35.** *Let $X$ be a set and $\mathcal{C} \subseteq \mathcal{P}X$ a set such that:*

1. *$\emptyset \in \mathcal{C}$*

2. *$X \in \mathcal{C}$*

3. *For all $\mathcal{A} \subseteq \mathcal{C}$ nonempty we have $\bigcap \mathcal{A} \in \mathcal{C}$*

*4. For all $C, D \in \mathcal{C}$ we have $C \cup D \in \mathcal{C}$.*

*Then there exists a unique topology $\mathcal{T}$ such that $\mathcal{C}$ is the set of closed sets, namely*

$$\mathcal{T} = \{X \setminus C \mid C \in \mathcal{C}\}$$

PROOF:
$\langle 1 \rangle 1$. LET: $\mathcal{T} = \{X \setminus C \mid C \in \mathcal{C}\}$
$\langle 1 \rangle 2$. $\mathcal{T}$ is a topology
  $\langle 2 \rangle 1$. $X \in \mathcal{T}$
    PROOF: Since $\emptyset \in \mathcal{C}$
  $\langle 2 \rangle 2$. For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$
    $\langle 3 \rangle 1$. LET: $\mathcal{U} \subseteq \mathcal{T}$
    $\langle 3 \rangle 2$. CASE: $\mathcal{U} = \emptyset$
      PROOF: In this case $\bigcup \mathcal{U} = \emptyset \in \mathcal{T}$ since $X \in \mathcal{C}$
    $\langle 3 \rangle 3$. CASE: $\mathcal{U} \neq \emptyset$
      PROOF: In this case $X \setminus \bigcup \mathcal{U} = \bigcap \{X \setminus U \mid U \in \mathcal{U}\} \in \mathcal{C}$.
  $\langle 2 \rangle 3$. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$
    PROOF: Since $X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V) \in \mathcal{C}$.
$\langle 1 \rangle 3$. $\mathcal{C}$ is the set of all closed sets in $\mathcal{T}$
  PROOF:

$$C \text{ is closed in } \mathcal{T}$$
$$\Leftrightarrow X \setminus C \in \mathcal{T}$$
$$\Leftrightarrow C \in \mathcal{C}$$

$\langle 1 \rangle 4$. If $\mathcal{T}'$ is a topology and $\mathcal{C}$ is the set of closed sets in $\mathcal{T}'$ then $\mathcal{T}' = \mathcal{T}$
  PROOF: We have

$$U \in \mathcal{T}$$
$$\Leftrightarrow X \setminus U \in \mathcal{C}$$
$$\Leftrightarrow X \setminus U \text{ is closed in } \mathcal{T}'$$
$$\Leftrightarrow U \in \mathcal{T}'$$

□

**Proposition 36.** *If $U$ is open and $A$ is closed then $U \setminus A$ is open.*

PROOF: $U \setminus A = U \cap (X \setminus A)$ is the intersection of two open sets. □

**Proposition 37.** *If $U$ is open and $A$ is closed then $A \setminus U$ is closed.*

PROOF: $A \setminus U = A \cap (X \setminus U)$ is the intersection of two closed sets. □

# 7 Interior

**Definition 38** (Interior)**.** Let $X$ be a topological space and $A \subseteq X$. The *interior* of $A$, $\text{Int } A$, is the union of all the open subsets of $A$.

**Lemma 39.** *The interior of a set is open.*

PROOF: It is a union of open sets. □

**Lemma 40.**
$$\text{Int } A \subseteq A$$

PROOF: Immediate from definition. □

**Lemma 41.** *If $U$ is open and $U \subseteq A$ then $U \subseteq \text{Int } A$*

PROOF: Immediate from definition. □

**Lemma 42.** *A set $A$ is open if and only if $A = \text{Int } A$.*

PROOF: If $A = \text{Int } A$ then $A$ is open by Lemma 39. Conversely if $A$ is open then $A \subseteq \text{Int } A$ by the definition of interior and so $A = \text{Int } A$.

# 8 Closure

**Definition 43** (Closure)**.** Let $X$ be a topological space and $A \subseteq X$. The *closure* of $A$, $\overline{A}$, is the intersection of all the closed sets that include $A$.

This intersection exists since $X$ is a closed set that includes A (Lemma 32).

**Lemma 44.** *The closure of a set is closed.*

PROOF: Dual to Lemma 39. □

**Lemma 45.**
$$A \subseteq \overline{A}$$

PROOF: Immediate from definition. □

**Lemma 46.** *If $C$ is closed and $A \subseteq C$ then $\overline{A} \subseteq C$.*

PROOF: Immediate from definition. □

**Lemma 47.** *A set $A$ is closed if and only if $A = \overline{A}$.*

PROOF: Dual to Lemma 42. □

**Theorem 48.** *Let $X$ be a topological space, $A \subseteq X$ and $x \in X$. Then $x \in \overline{A}$ if and only if every neighbourhood of $x$ intersects $A$.*

PROOF: We have
$$x \in \overline{A}$$
$$\Leftrightarrow \forall C.C \text{ closed } \wedge A \subseteq C \Rightarrow x \in C$$
$$\Leftrightarrow \forall U.U \text{ open } \wedge A \cap U = \emptyset \Rightarrow x \notin U$$
$$\Leftrightarrow \forall U.U \text{ open } \wedge x \in U \Rightarrow U \text{ intersects } A \qquad \Box$$

**Proposition 49.** *If $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$.*

Proof: This holds because $\overline{B}$ is a closed set that includes $A$. □

**Proposition 50.**
$$\overline{A \cup B} = \overline{A} \cup \overline{B}$$

Proof:
⟨1⟩1. $\overline{A} \subseteq \overline{A \cup B}$
   Proof: By Proposition 49.
⟨1⟩2. $\overline{B} \subseteq \overline{A \cup B}$
   Proof: By Proposition 49.
⟨1⟩3. $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$
   ⟨2⟩1. Let: $x \in \overline{A \cup B}$
   ⟨2⟩2. Assume: $x \notin \overline{A}$
       Prove: $x \in \overline{B}$
   ⟨2⟩3. Pick a neighbourhood $U$ of $x$ that does not intersect $A$
   ⟨2⟩4. Let: $V$ be any neighbourhood of $x$
   ⟨2⟩5. $U \cap V$ is a neighbourhood of $x$
   ⟨2⟩6. $U \cap V$ intersects $A \cup B$
     Proof: From ⟨2⟩1 and Theorem 48.
   ⟨2⟩7. $U \cap V$ intersects $B$
     Proof: From ⟨2⟩3
   ⟨2⟩8. $V$ intersects $B$
   ⟨2⟩9. Q.E.D.
     Proof: We have $x \in \overline{B}$ from Theorem 48.
□

# 9  Boundary

**Definition 51** (Boundary)**.** The *boundary* of a set $A$ is the set $\partial A = \overline{A} \cap \overline{X \setminus A}$.

**Proposition 52.**
$$\operatorname{Int} A \cap \partial A = \emptyset$$

Proof: Since $\overline{X \setminus A} = X \setminus \operatorname{Int} A$. □

**Proposition 53.**
$$\overline{A} = \operatorname{Int} A \cup \partial A$$

Proof:
$$
\begin{aligned}
\operatorname{Int} A \cup \partial A &= \operatorname{Int} A \cup (\overline{A} \cap \overline{X \setminus A}) \\
&= (\operatorname{Int} A \cup \overline{A}) \cap (\operatorname{Int} A \cup \overline{X \setminus A}) \\
&= \overline{A} \cap X \\
&= \overline{A}
\end{aligned}
$$

**Proposition 54.** $\partial A = \emptyset$ *if and only if $A$ is open and closed.*

Proof: If $\partial A = \emptyset$ then $\overline{A} = \operatorname{Int} A$ by Proposition 53.

**Proposition 55.** *A set $U$ is open if and only if $\partial U = \overline{U} \setminus U$.*

PROOF:

$$
\begin{aligned}
&\partial U = \overline{U} \setminus U \\
\Leftrightarrow\ & \overline{U} \setminus \operatorname{Int} U = \overline{U} \setminus U && \text{(Propositions 52, 53)} \\
\Leftrightarrow\ & \operatorname{Int} U = U && \square
\end{aligned}
$$

# 10 Limit Points

**Definition 56** (Limit Point)**.** Let $X$ be a topological space, $a \in X$ and $A \subseteq X$. Then $a$ is a *limit point*, *cluster point* or *point of accumulation* for $A$ if and only if every neighbourhood of $a$ intersects $A$ at a point other than $a$.

**Lemma 57.** *The point $a$ is an accumulation point for $A$ if and only if $a \in \overline{A \setminus \{a\}}$.*

PROOF: From Theorem 48. $\square$

**Theorem 58.** *Let $X$ be a topological space and $A \subseteq X$. Let $A'$ be the set of all limit points of $A$. Then $\overline{A} = A \cup A'$.*

PROOF:
$\langle 1 \rangle 1.$ For all $x \in \overline{A}$, if $x \notin A$ then $x \in A'$
  PROOF: From Theorem 48.
$\langle 1 \rangle 2.$ $A \subseteq \overline{A}$
  PROOF: Lemma 45.
$\langle 1 \rangle 3.$ $A' \subseteq \overline{A}$
  PROOF: From Theorem 48.
$\square$

**Corollary 58.1.** *A set is closed if and only if it contains all its limit points.*

**Proposition 59.** *In an indiscrete topology, every point is a limit point of any set that has more than one point.*

PROOF: Let $X$ be an indiscrete space. Let $A$ be a set with more than one point and $x$ be a point. The only neighbourhood of $x$ is $X$, which must intersect $A$ at a point other than $x$. $\square$

**Lemma 60.** *Let $X$ be a topological space and $A \subseteq B \subseteq X$. Then every limit point of $A$ is a limit point of $B$.*

PROOF: Immediate from definitions. $\square$

# 11 Basis for a Topology

**Definition 61** (Basis)**.** If $X$ is a set, a *basis* for a topology on $X$ is a set $\mathcal{B} \subseteq \mathcal{P}X$ called *basis elements* such that

1. For all $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$.

2. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

If $\mathcal{B}$ satisfies these two conditions, then we define the topology *generated* by $\mathcal{B}$ to be $\mathcal{T} = \{U \in \mathcal{P}X \mid \forall x \in U.\exists B \in \mathcal{B}.x \in B \subseteq U\}$.

We prove this is a topology.

PROOF:
$\langle 1 \rangle 1$. $X \in \mathcal{T}$
  PROOF: For all $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B \subseteq X$ by condition 1.
$\langle 1 \rangle 2$. For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$
  $\langle 2 \rangle 1$. LET: $\mathcal{U} \subseteq \mathcal{T}$
  $\langle 2 \rangle 2$. LET: $x \in \bigcup \mathcal{U}$
  $\langle 2 \rangle 3$. PICK $U \in \mathcal{U}$ such that $x \in U$
  $\langle 2 \rangle 4$. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$
    PROOF: Since $U \in \mathcal{T}$ by $\langle 2 \rangle 1$ and $\langle 2 \rangle 3$.
  $\langle 2 \rangle 5$. $x \in B \subseteq \bigcup \mathcal{U}$
$\langle 1 \rangle 3$. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$
  $\langle 2 \rangle 1$. LET: $U, V \in \mathcal{T}$
  $\langle 2 \rangle 2$. LET: $x \in U \cap V$
  $\langle 2 \rangle 3$. PICK $B_1 \in \mathcal{B}$ such that $x \in B_1 \subseteq U$
  $\langle 2 \rangle 4$. PICK $B_2 \in \mathcal{B}$ such that $x \in B_2 \subseteq V$
  $\langle 2 \rangle 5$. PICK $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$
    PROOF: By condition 2.
  $\langle 2 \rangle 6$. $x \in B_3 \subseteq U \cap V$
□

**Lemma 62.** *Let $X$ be a set. Let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on $X$. Then $\mathcal{T}$ is the set of all unions of subsets of $\mathcal{B}$.*

PROOF:
$\langle 1 \rangle 1$. For all $U \in \mathcal{T}$, there exists $\mathcal{A} \subseteq \mathcal{B}$ such that $U = \bigcup \mathcal{A}$
  $\langle 2 \rangle 1$. LET: $U \in \mathcal{T}$
  $\langle 2 \rangle 2$. LET: $\mathcal{A} = \{B \in \mathcal{B} \mid B \subseteq U\}$
  $\langle 2 \rangle 3$. $U \subseteq \bigcup \mathcal{A}$
    $\langle 3 \rangle 1$. LET: $x \in U$
    $\langle 3 \rangle 2$. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$
      PROOF: Since $\mathcal{B}$ is a basis for $\mathcal{T}$.
    $\langle 3 \rangle 3$. $x \in B \in \mathcal{A}$
  $\langle 2 \rangle 4$. $\bigcup \mathcal{A} \subseteq U$
    PROOF: From the definition of $\mathcal{A}$ ($\langle 2 \rangle 2$).
$\langle 1 \rangle 2$. For all $\mathcal{A} \subseteq \mathcal{B}$ we have $\bigcup \mathcal{A} \in \mathcal{T}$
  $\langle 2 \rangle 1$. $\mathcal{B} \subseteq \mathcal{T}$
    PROOF: If $B \in \mathcal{B}$ and $x \in B$, then there exists $B' \in \mathcal{B}$ such that $x \in B' \subseteq B$, namely $B' = B$.

$\langle 2 \rangle 2$. Q.E.D.
    PROOF: Since $\mathcal{T}$ is closed under union.
□

**Corollary 62.1.** *Let $X$ be a set. Let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on $X$. Then $\mathcal{T}$ is the coarsest topology that includes $\mathcal{B}$.*

PROOF: Since every topology that includes $\mathcal{B}$ includes all unions of subsets of $\mathcal{B}$. □

**Lemma 63.** *Let $X$ be a topological space. Suppose that $\mathcal{C}$ is a set of open sets such that, for every open set $U$ and every point $x \in U$, there exists $C \in \mathcal{C}$ such that $x \in C \subseteq U$. Then $\mathcal{C}$ is a basis for the topology on $X$.*

PROOF:
$\langle 1 \rangle 1$. For all $x \in X$, there exists $C \in \mathcal{C}$ such that $x \in C$
    PROOF: Immediate from hypothesis.
$\langle 1 \rangle 2$. For all $C_1, C_2 \in \mathcal{C}$ and $x \in C_1 \cap C_2$, there exists $C_3 \in \mathcal{C}$ such that $x \in C_3 \subseteq C_1 \cap C_2$
    PROOF: Since $C_1 \cap C_2$ is open.
$\langle 1 \rangle 3$. Every open set is open in the topology generated by $\mathcal{C}$
    PROOF: Immediate from hypothesis.
$\langle 1 \rangle 4$. Every union of a subset of $\mathcal{C}$ is open.
    PROOF: Since every member of $\mathcal{C}$ is open.
□

**Lemma 64.** *Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$ respectively on the set $X$. Then the following are equivalent.*

  *1. $\mathcal{T} \subseteq \mathcal{T}'$*

  *2. For all $B \in \mathcal{B}$ and $x \in B$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.*

PROOF:
$\langle 1 \rangle 1$. $1 \Rightarrow 2$
    $\langle 2 \rangle 1$. ASSUME: $\mathcal{T} \subseteq \mathcal{T}'$
    $\langle 2 \rangle 2$. LET: $B \in \mathcal{B}$ and $x \in B$
    $\langle 2 \rangle 3$. $B \in \mathcal{T}$
        PROOF: Corollary 62.1.
    $\langle 2 \rangle 4$. $B \in \mathcal{T}'$
        PROOF: By $\langle 2 \rangle 1$
    $\langle 2 \rangle 5$. There exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$
        PROOF: Since $\mathcal{B}'$ is a basis for $\mathcal{T}'$.
$\langle 1 \rangle 2$. $2 \Rightarrow 1$
    $\langle 2 \rangle 1$. ASSUME: 2
    $\langle 2 \rangle 2$. LET: $U \in \mathcal{T}$
        PROVE: $U \in \mathcal{T}'$
    $\langle 2 \rangle 3$. LET: $x \in U$
        PROVE: There exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq U$

18

$\langle 2 \rangle 4$. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$
    PROOF: Since $\mathcal{B}$ is a basis for $\mathcal{T}$.
$\langle 2 \rangle 5$. PICK $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$
    PROOF: By $\langle 2 \rangle 1$.
$\langle 2 \rangle 6$. $x \in B' \subseteq U$
☐

**Theorem 65.** *Let $X$ be a topological space, $A \subseteq X$ and $x \in X$. Let $\mathcal{B}$ be a basis for $X$. Then $x \in \overline{A}$ if and only if, for all $B \in \mathcal{B}$, if $x \in B$ then $B$ intersects $A$.*

PROOF:
$\langle 1 \rangle 1$. If $x \in \overline{A}$ then, for all $B \in \mathcal{B}$, if $x \in B$ then $B$ intersects $A$.
    PROOF: This follows from Theorem 48 since every element of $\mathcal{B}$ is open (Corollary 62.1).
$\langle 1 \rangle 2$. Suppose that, for all $B \in \mathcal{B}$, if $x \in B$ then $B$ intersects $A$. Then $x \in \overline{A}$.
    $\langle 2 \rangle 1$. ASSUME: For all $B \in \mathcal{B}$, if $x \in B$ then $B$ intersects $A$.
    $\langle 2 \rangle 2$. LET: $U$ be an open set that contains $x$
        PROVE:   $U$ intersects $A$.
    $\langle 2 \rangle 3$. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$.
    $\langle 2 \rangle 4$. $B$ intersects $A$.
        PROOF: From $\langle 2 \rangle 1$.
    $\langle 2 \rangle 5$. $U$ intersects $A$.
    $\langle 2 \rangle 6$. Q.E.D.
        PROOF: By Theorem 48.
☐

**Definition 66** (Lower Limit Topology on the Real Line). The *lower limit topology on the real line* is the topology on $\mathbb{R}$ generated by the basis consisting of all half-open intervals of the form $[a, b)$.
    We write $\mathbb{R}_l$ for the topological space $\mathbb{R}$ under the lower limit topology.

    We prove this is a basis for a topology.

PROOF:
$\langle 1 \rangle 1$. For all $x \in \mathbb{R}$ there exists an interval $[a, b)$ such that $x \in [a, b)$.
    PROOF: Take $[a, b) = [x, x + 1)$.
$\langle 1 \rangle 2$. For any open intervals $[a, b)$, $[c, d)$ if $x \in [a, b) \cap [c, d)$, then there exists an interval $[e, f)$ such that $x \in [e, f) \subseteq [a, b) \cap [c, d)$
    PROOF: Take $[e, f) = [\max(a, c), \min(b, d))$.
☐

**Definition 67** ($K$-topology on the Real Line). Let $K = \{1/n \mid n \in \mathbb{Z}^+\}$.
    The $K$-*topology on the real line* is the topology on $\mathbb{R}$ generated by the basis consisting of all open intervals $(a, b)$ and all sets of the form $(a, b) \setminus K$.
    We write $\mathbb{R}_K$ for the topological space $\mathbb{R}$ under the $K$-topology.

    We prove this is a basis for a topology.

PROOF:

$\langle 1 \rangle 1$. For all $x \in \mathbb{R}$ there exists an open interval $(a,b)$ such that $x \in (a,b)$.
  PROOF: Take $(a,b) = (x-1, x+1)$.
$\langle 1 \rangle 2$. For any basic open sets $B_1$, $B_2$ if $x \in B_1 \cap B_2$, then there exists a basic
      open set $B_3$ such that $x \in B_3 \subseteq B_1 \cap B_2$
  $\langle 2 \rangle 1$. CASE: $B_1 = (a,b)$, $B_2 = (c,d)$
    PROOF: Take $B_3 = (\max(a,c), \min(b,d))$.
  $\langle 2 \rangle 2$. CASE: $B_1 = (a,b)$ or $(a,b) \setminus K$, $B_2 = (c,d)$ or $(c,d) \setminus K$, and they are
            not both open intervals.
    PROOF: Take $B_3 = (\max(a,c), \min(b,d)) \setminus K$.
☐

**Lemma 68.** *The lower limit topology and the $K$-topology are incomparable.*

PROOF:
$\langle 1 \rangle 1$. The interval $[10, 11)$ is not open in the $K$-topology.
  PROOF: There is no open interval $(a,b)$ such that $10 \in (a,b) \subseteq [10, 11)$ or
  $10 \in (a,b) \setminus K \subseteq [10, 11)$.
$\langle 1 \rangle 2$. The set $(-1, 1) \setminus K$ is not open in the lower limit topology.
  PROOF: There is no half-open interval $[a,b)$ such that $0 \in [a,b) \subseteq (-1,1) \setminus K$,
  since there must be a positive integer $n$ with $1/n \in [a,b)$.
☐

**Definition 69** (Subbasis)**.** A *subbasis* $\mathcal{S}$ for a topology on $X$ is a set $\mathcal{S} \subseteq \mathcal{P}X$
such that $\bigcup \mathcal{S} = X$.
   The topology *generated* by the subbasis $\mathcal{S}$ is the set of all unions of finite
intersections of elements of $\mathcal{S}$.

   We prove this is a topology.

PROOF:
$\langle 1 \rangle 1$. The set $\mathcal{B}$ of all finite intersections of elements of $\mathcal{S}$ forms a basis for a
      topology on $X$.
  $\langle 2 \rangle 1$. $\bigcup \mathcal{B} = X$
    PROOF: Since $\mathcal{S} \subseteq \mathcal{B}$.
  $\langle 2 \rangle 2$. $\mathcal{B}$ is closed under binary intersection.
    PROOF: By definition.
$\langle 1 \rangle 2$. Q.E.D.
  PROOF: By Lemma 62.
☐

   We have simultaneously proved:

**Proposition 70.** *Let $\mathcal{S}$ be a subbasis for the topology on $X$. Then the set of
all finite intersections of elements of $\mathcal{S}$ is a basis for the topology on $X$.*

**Proposition 71.** *Let $X$ be a set. Let $\mathcal{S}$ be a subbasis for a topology $\mathcal{T}$ on $X$.
Then $\mathcal{T}$ is the coarsest topology that includes $\mathcal{S}$.*

PROOF: Since every topology that includes $\mathcal{S}$ includes every union of finite
intersections of elements of $\mathcal{S}$. ☐

## 12   Local Basis at a Point

**Definition 72** (Local Basis)**.** Let $X$ be a topological space and $a \in X$. A *(local) basis at $a$* is a set $\mathcal{B}$ of neighbourhoods of $a$ such that every neighbourhood of $a$ includes some member of $\mathcal{B}$.

**Lemma 73.** *If there exists a countable local basis at a point $a$, then there exists a countable local basis $\{B_n \mid n \geq 1\}$ such that $B_1 \supseteq B_2 \supseteq \cdots$.*

PROOF: Given a countable local basis $\{C_n \mid n \geq 1\}$, take $B_n = C_1 \cap \cdots \cap C_n$. $\square$

## 13   Convergence

**Definition 74** (Convergence)**.** Let $X$ be a topological space. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of points in $X$ and $l \in X$. Then the sequence $(a_n)_{n \in \mathbb{N}}$ *converges* to the *limit $l$*, $a_n \to l$ as $n \to \infty$, if and only if, for every neighbourhood $U$ of $l$, there exists $N$ such that, for all $n \geq N$, we have $a_n \in U$.

**Lemma 75.** *Let $X$ be a topological space. Let $A \subseteq X$ and $l \in X$. If there is a sequence of points in $A$ that converges to $l$ then $l \in \overline{A}$.*

PROOF:
$\langle 1 \rangle 1$. LET: $(a_n)$ be a sequence of points in $A$ that converges to $l$.
$\langle 1 \rangle 2$. LET: $U$ be a neighbourhood of $l$.
$\langle 1 \rangle 3$. PICK $N$ such that, for all $n \geq N$, we have $a_n \in U$.
$\langle 1 \rangle 4$. $a_N \in U \cap A$
$\langle 1 \rangle 5$. Q.E.D.
   PROOF: Theorem 48.
$\square$

**Proposition 76.** *Let $X$ be a topological space. Let $\mathcal{B}$ be a basis for the topology on $X$. Let $(a_n)$ be a sequence in $X$ and $l \in X$. Then $a_n \to l$ as $n \to \infty$ if and only if, for every $B \in \mathcal{B}$ with $l \in B$, there exists $N$ such that, for all $n \geq N$, we have $a_n \in B$.*

PROOF:
$\langle 1 \rangle 1$. If $a_n \to l$ as $n \to \infty$ then, for every $B \in \mathcal{B}$ with $l \in B$, there exists $N$ such that, for all $n \geq N$, we have $a_n \in B$.
   PROOF: Since every element of $\mathcal{B}$ is open (Corollary 62.1).
$\langle 1 \rangle 2$. If, for every $B \in \mathcal{B}$ with $l \in B$, there exists $N$ such that, for all $n \geq N$, we have $a_n \in B$, then $a_n \to l$ as $n \to \infty$.
   $\langle 2 \rangle 1$. ASSUME: for every $B \in \mathcal{B}$ with $l \in B$, there exists $N$ such that, for all $n \geq N$, we have $a_n \in B$
   $\langle 2 \rangle 2$. LET: $U$ be a neighbourhood of $l$.
   $\langle 2 \rangle 3$. PICK $B \in \mathcal{B}$ such that $l \in B \subseteq U$
   $\langle 2 \rangle 4$. PICK $N$ such that, for all $n \geq N$, we have $a_n \in B$
      PROOF: From $\langle 2 \rangle 1$.
   $\langle 2 \rangle 5$. For all $n \geq N$ we have $a_n \in U$

☐

**Lemma 77.** *If a sequence $(a_n)$ is constant with $a_n = l$ for all $n$, then $a_n \to l$ as $n \to \infty$.*

PROOF: Immediate from definitions. ☐

**Theorem 78.** *Let $X$ be a linearly ordered set. Let $(s_n)$ be a monotone increasing sequence in $X$ with a supremum $s$. Then $s_n \to s$ as $n \to \infty$.*

PROOF:
$\langle 1 \rangle 1.$ ASSUME: $s$ is not least in $X$.
   PROOF: Otherwise $(s_n)$ is the constant sequence $s$ and the result follows from Lemma 77.
$\langle 1 \rangle 2.$ LET: $U$ be a neighbourhood of $s$.
$\langle 1 \rangle 3.$ PICK $a < s$ such that $(a, s] \subseteq U$
$\langle 1 \rangle 4.$ PICK $N$ such that $a < a_N$.
$\langle 1 \rangle 5.$ For all $n \geq N$ we have $a_n \in (a, s]$
$\langle 1 \rangle 6.$ For all $n \geq N$ we have $a_n \in U$.
☐

**Theorem 79.** *If $\sum_{i=0}^{\infty} a_i = s$ and $\sum_{i=0}^{\infty} b_i = t$ then $\sum_{i=0}^{\infty}(ca_i + b_i) = cs + t$.*

PROOF: $\sum_{i=0}^{N}(ca_i + b_i) = c\sum_{i=0}^{N} a_i + \sum_{i=0}^{N} b_i \to cs + t$ as $n \to \infty$. ☐

**Theorem 80** (Comparison Test). *If $|a_i| \leq b_i$ for all $i$ and $\sum_{i=0}^{\infty} b_i$ converges then $\sum_{i=0}^{\infty} a_i$ converges.*

PROOF:
$\langle 1 \rangle 1.$ $\sum_{i=0}^{\infty} |a_i|$ converges
   PROOF: The partial sums $\sum_{i=0}^{N} |a_i|$ form an increasing sequence bounded above by $\sum_{i=0}^{\infty} b_i$.
$\langle 1 \rangle 2.$ LET: $c_i = |a_i| + a_i$ for all $i$
$\langle 1 \rangle 3.$ $\sum_{i=0}^{\infty} c_i$ converges
   PROOF: Each $c_i$ is either $2|a_i|$ or $0$. So the partial sums $\sum_{i=0}^{N} c_i$ form an increasing sequence bounded above by $2\sum_{i=0}^{\infty} b_i$.
$\langle 1 \rangle 4.$ Q.E.D.
   PROOF: Since $a_i = c_i - |a_i|$.
☐

**Corollary 80.1.** *If $\sum_{i=0}^{\infty} |a_i|$ converges then $\sum_{i=0}^{\infty} a_i$ converges.*

**Theorem 81** (Weierstrass $M$-test). *Let $X$ be a set and $(f_n : X \to \mathbb{R})$ be a sequence of functions. Let*

$$s_n(x) = \sum_{i=0}^{n} f_i(x)$$

*for all $n$, $x$. Suppose $|f_i(x)| \leq M_i$ for all $i \geq 0$ and $x \in X$. If the series $\sum_{i=0}^{\infty} M_i$ converges, then the sequence $(s_n)$ converges uniformly to*

$$s(x) = \sum_{i=0}^{\infty} f_i(x) \ .$$

PROOF:

⟨1⟩1. LET: $r_n = \sum_{i=n+1}^{\infty} M_i$ for all $n$

⟨1⟩2. Given $0 \le n < k$, we have $|s_k(x) - s_n(x)| \le r_n$

   PROOF:

$$|s_k(x) - s_n(x)| = |\sum_{i=n+1}^{k} f_i(x)|$$

$$\le \sum_{i=n+1}^{k} |f_i(x)|$$

$$\le \sum_{i=n+1}^{k} M_i$$

$$\le r_n$$

⟨1⟩3. Given $n \ge 0$ we have $|s(x) - s_n(x)| \le r_n$

   PROOF: By taking the limit $k \to \infty$ in ⟨1⟩2.

⟨1⟩4. Q.E.D.

   PROOF: Since $r_n \to 0$ as $n \to \infty$.

▯

# 14 Locally Finite Sets

**Definition 82** (Locally Finite). Let $X$ be a topological space and $\{A_\alpha\}$ a family of subsets of $X$. Then $\mathcal{A}$ is *locally finite* if and only if every point in $X$ has a neighbourhood that intersects $A_\alpha$ for only finitely many $\alpha$.

**Theorem 83** (Pasting Lemma). *Let $X$ and $Y$ be topological spaces and $f : X \to Y$. Let $\{A_\alpha\}$ be a locally finite family of closed subsets of $X$ that cover $X$. Suppose $f \restriction A_\alpha$ is continuous for all $\alpha$. Then $f$ is continuous.*

PROOF:

⟨1⟩1. Let $X$ and $Y$ be topological spaces and $f : X \to Y$. Let $A$ and $B$ be closed subsets of $X$ such that $X = A \cup B$. Suppose $f \restriction A$ and $f \restriction B$ are continuous. Then $f$ is continuous.

  ⟨2⟩1. LET: $C \subseteq Y$ be closed.

  ⟨2⟩2. $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$

  ⟨2⟩3. $f^{-1}(C)$ and $g^{-1}(C)$ are closed in $X$.

    PROOF: Theorems 93 and 143.

  ⟨2⟩4. $h^{-1}(C)$ is closed in $X$.

    PROOF: Lemma 34.

  ⟨2⟩5. Q.E.D.

    PROOF: Theorem 93.

⟨1⟩2. Let $X$ and $Y$ be topological spaces and $f : X \to Y$. Let $\{A_\alpha\}$ be a finite family of closed subsets of $X$ that cover $X$. Suppose $f \restriction A_\alpha$ is continuous for all $\alpha$. Then $f$ is continuous.

  PROOF: From ⟨1⟩1 by induction.

⟨1⟩3. Let $X$ and $Y$ be topological spaces and $f : X \to Y$. Let $\{A_\alpha\}$ be a locally finite family of closed subsets of $X$ that cover $X$. Suppose $f \upharpoonright A_\alpha$ is continuous for all $\alpha$. Then $f$ is continuous.

    ⟨2⟩1. LET: $x \in X$
        PROVE:   $f$ is continuous at $x$
    ⟨2⟩2. PICK a neighbourhood $U$ of $x$ that intersects $A_\alpha$ for only finitely many $\alpha$.
    ⟨2⟩3. $f \upharpoonright U$ is continuous
       PROOF: By ⟨1⟩2.
    ⟨2⟩4. Q.E.D.
       PROOF: Lemma 103.

☐

The following example shows that we cannot remove the assumption of local finiteness.

**Example 84.** Define $f : [-1, 1] \to \mathbb{R}$ by: $f(x) = 1$ if $x < -1$, $f(x) = 0$ if $x > 1$. Let $C_n = [-1, -1/n]$ for $n \geq 1$, and $D = [0, 1]$. Then $[-1, 1] = \bigcup_{n=1}^{\infty} C_n \cup D$ and $f$ is continuous on each $C_n$ and each $D$, but $f$ is not continuous on $[-1, 1]$.

# 15 Open Maps

**Definition 85** (Open Map)**.** Let $X$ and $Y$ be topological spaces and $f : X \to Y$. Then $f$ is an *open map* if and only if, for every open set $U$ in $X$, the set $f(U)$ is open in $Y$.

**Lemma 86.** *Let $X$ and $Y$ be topological spaces and $f : X \to Y$. Let $\mathcal{B}$ be a basis for the topology on $X$. If $f(B)$ is open in $Y$ for all $B \in \mathcal{B}$, then $f$ is an open map.*

PROOF: From Lemma 62. ☐

**Proposition 87.** *Let $X$ and $Y$ be topological spaces. Let $\mathcal{B}$ be a basis for the topology on $X$. Let $f : X \to Y$. Suppose that, for all $B \in \mathcal{B}$, we have $f(B)$ is open tn $Y$. Then $f$ is an open map.*

PROOF: For any $\mathcal{A} \subseteq \mathcal{B}$, we have $f(\bigcup \mathcal{A}) = \bigcup_{B \in \mathcal{B}} f(B)$ is open in $Y$. The result follows from Lemma 62. ☐

# 16 Continuous Functions

**Definition 88** (Continuous)**.** Let $X$ and $Y$ be topological spaces. A function $f : X \to Y$ is *continuous* if and only if, for every open set $V$ in $Y$, the set $f^{-1}(V)$ is open in $X$.

**Proposition 89.** *Let $X$ and $Y$ be topological spaces and $f : X \to Y$. Let $\mathcal{B}$ be a basis for $Y$. Then $f$ is continuous if and only if, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in $X$.*

PROOF:

⟨1⟩1. If $f$ is continuous then, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in $X$.
  PROOF: Since every element of $B$ is open (Lemma 62).

⟨1⟩2. Suppose that, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in $X$. Then $f$ is continuous.

  ⟨2⟩1. ASSUME: For all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in $X$.
  ⟨2⟩2. LET: $V$ be open in $Y$.
  ⟨2⟩3. PICK $\mathcal{A} \subseteq \mathcal{B}$ such that $V = \bigcup \mathcal{A}$
    PROOF: By Lemma 62.
  ⟨2⟩4. $f^{-1}(V)$ is open in $X$.
    PROOF:
$$f^{-1}(V) = f^{-1}\left(\bigcup \mathcal{A}\right)$$
$$= \bigcup_{B \in \mathcal{A}} f^{-1}(B)$$

☐

**Proposition 90.** *Let $X$ and $Y$ be topological spaces and $f : X \to Y$. Let $\mathcal{S}$ be a subbasis for $Y$. Then $f$ is continuous if and only if, for all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in $X$.*

PROOF:

⟨1⟩1. If $f$ is continuous then, for all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in $X$.
  PROOF: Since every element of $S$ is open.

⟨1⟩2. Suppose that, for all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in $X$. Then $f$ is continuous.

  ⟨2⟩1. ASSUME: For all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in $X$.
  ⟨2⟩2. LET: $S_1, \ldots, S_n \in \mathcal{S}$
  ⟨2⟩3. $f^{-1}(S_1 \cap \cdots \cap S_n)$ is open in $A$
    PROOF: Since $f^{-1}(S_1 \cap \cdots \cap S_n) = f^{-1}(S_1) \cap \cdots \cap f^{-1}(S_n)$.
  ⟨2⟩4. Q.E.D.
    PROOF: By Propositions 89 and 70.

☐

**Proposition 91.** *Let $X$ and $Y$ be topological spaces and $f : X \to Y$. Let $\mathcal{S}$ be a basis for $Y$. Then $f$ is continuous if and only if, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in $X$.*

PROOF:

⟨1⟩1. If $f$ is continuous then, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in $X$.
  PROOF: Since every element of $\mathcal{S}$ is open.

⟨1⟩2. Suppose that, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in $X$. Then $f$ is continuous.

  ⟨2⟩1. ASSUME: For all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in $X$.
  ⟨2⟩2. For every set $B$ that is the finite intersection of elemets of $\mathcal{S}$, we have $f^{-1}(B)$ is open in $X$.
    PROOF: Because $f^{-1}(V_1 \cap \cdots \cap V_n) = f^{-1}(V_1) \cap \cdots \cap f^{-1}(V_n)$.
  ⟨2⟩3. Q.E.D.

PROOF: From Propositions 70 and 89.
∎

**Definition 92** (Continuous at a Point)**.** Let $X$ and $Y$ be topological spaces. Let $f : X \to Y$ and $x \in X$. Then $f$ is *continuous at $x$* if and only if, for every neighbourhood $V$ of $f(x)$, there exists a neighbourhood $U$ of $x$ such that $f(U) \subseteq V$.

**Theorem 93.** *Let $X$ and $Y$ be topological spaces and $f : X \to Y$. Then the following are equivalent:*

1. *$f$ is continuous.*

2. *For all $A \subseteq X$, we have $f(\overline{A}) \subseteq \overline{f(A)}$*

3. *For all $B \subseteq Y$ closed, we have $f^{-1}(B)$ is closed in $X$.*

4. *$f$ is continuous at every point of $X$.*

PROOF:

⟨1⟩1. $1 \Rightarrow 2$

  ⟨2⟩1. ASSUME: $f$ is continuous.

  ⟨2⟩2. LET: $A \subseteq X$

  ⟨2⟩3. LET: $x \in \overline{A}$

     PROVE: $f(x) \in \overline{f(A)}$

  ⟨2⟩4. LET: $V$ be a neighbourhood of $f(x)$

  ⟨2⟩5. $f^{-1}(V)$ is a neighbourhood of $x$

  ⟨2⟩6. PICK $y \in A \cap f^{-1}(V)$

    PROOF: By Theorem 48.

  ⟨2⟩7. $f(y) \in V \cap f(A)$

  ⟨2⟩8. Q.E.D.

    PROOF: By Theorem 48.

⟨1⟩2. $2 \Rightarrow 3$

  ⟨2⟩1. ASSUME: 2

  ⟨2⟩2. LET: $B$ be closed in $Y$

  ⟨2⟩3. LET: $x \in \overline{f^{-1}(B)}$

      PROVE: $x \in f^{-1}(B)$

  ⟨2⟩4. $f(x) \in B$

    PROOF:

$$
\begin{aligned}
f(x) &\in f(\overline{f^{-1}(B)}) \\
&\subseteq \overline{f(f^{-1}(B))} && (\langle 2 \rangle 1) \\
&\subseteq \overline{B} && (Proposition\, 49) \\
&= B
\end{aligned}
$$

⟨1⟩3. $3 \Rightarrow 1$

  ⟨2⟩1. ASSUME: 3

  ⟨2⟩2. LET: $V$ be open in $Y$

  ⟨2⟩3. $Y \setminus V$ is closed in $Y$

$\langle 2 \rangle 4.$ $f^{-1}(Y \setminus V)$ is closed in $X$
$\langle 2 \rangle 5.$ $X \setminus f^{-1}(V)$ is closed in $X$
$\langle 2 \rangle 6.$ $f^{-1}(V)$ is open in $X$
$\langle 1 \rangle 4.$ $1 \Rightarrow 4$
    PROOF: For any neighbourhood $V$ of $f(x)$, the set $U = f^{-1}(V)$ is a neighbourhood of $x$ such that $f(U) \subseteq V$.
$\langle 1 \rangle 5.$ $4 \Rightarrow 1$
    $\langle 2 \rangle 1.$ ASSUME: 4
    $\langle 2 \rangle 2.$ LET: $V$ be open in $Y$
    $\langle 2 \rangle 3.$ LET: $x \in f^{-1}(V)$
    $\langle 2 \rangle 4.$ $V$ is a neighbourhood of $f(x)$
    $\langle 2 \rangle 5.$ PICK a neighbourhood $U$ of $x$ such that $f(U) \subseteq V$
    $\langle 2 \rangle 6.$ $x \in U \subseteq f^{-1}(V)$
    $\langle 2 \rangle 7.$ Q.E.D.
      PROOF: By Lemma 26.

**Theorem 94.** *A constant function is continuous.*

PROOF: Let $X$ and $Y$ be topological spaces. Let $b \in Y$, and let $f : X \to Y$ be the constant function with value $b$. For any open $V \subseteq Y$, the set $f^{-1}(V)$ is either $X$ (if $b \in V$) or $\emptyset$ (if $b \notin V$).  □

**Theorem 95.** *If $A$ is a subspace of $X$ then the inclusion $j : A \to X$ is continuous.*

PROOF: For any $V$ open in $X$, we have $j^{-1}(V) = V \cap A$ is open in $A$.  □

**Theorem 96.** *The composite of two continuous functions is continuous.*

PROOF: Let $f : X \to Y$ and $g : Y \to Z$ be continuous. For any $V$ open in $Z$, we have $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is open in $X$.  □

**Theorem 97.** *Let $f : X \to Y$ be a continuous function and $A$ be a subspace of $X$. Then the restriction $f \restriction A : A \to Y$ is continuous.*

PROOF: Let $V$ be open in $Y$. Then $(f \restriction A)^{-1}(V) = f^{-1}(V) \cap A$ is open in $A$.  □

**Theorem 98.** *Let $f : X \to Y$ be continuous. Let $Z$ be a subspace of $Y$ such that $f(X) \subseteq Z$. Then the corestriction $f : X \to Z$ is continuous.*

PROOF:
$\langle 1 \rangle 1.$ LET: $V$ be open in $Z$.
$\langle 1 \rangle 2.$ PICK $U$ open in $Y$ such that $V = U \cap Z$.
$\langle 1 \rangle 3.$ $f^{-1}(V) = f^{-1}(U)$
$\langle 1 \rangle 4.$ $f^{-1}(V)$ is open in $X$.

**Theorem 99.** *Let $f : X \to Y$ be continuous. Let $Z$ be a space such that $Y$ is a subspace of $Z$. Then the expansion $f : X \to Z$ is continuous.*

PROOF: Let $V$ be open in $Z$. Then $f^{-1}(V) = f^{-1}(V \cap Y)$ is open in $X$. ☐

**Theorem 100.** *Let $X$ and $Y$ be topological spaces. Let $f : X \to Y$. Suppose $\mathcal{U}$ is a set of open sets in $X$ such that $X = \bigcup \mathcal{U}$ and, for all $U \in \mathcal{U}$, we have $f \upharpoonright U : U \to Y$ is continuous. Then $f$ is continuous.*

PROOF:
$\langle 1 \rangle 1.$ LET: $V$ be open in $Y$
$\langle 1 \rangle 2.$ $f^{-1}(V) = \bigcup_{U \in \mathcal{U}} (f \upharpoonright U)^{-1}(V)$
$\langle 1 \rangle 3.$ For all $U \in \mathcal{U}$, we have $(f \upharpoonright U)^{-1}(V)$ is open in $U$.
$\langle 1 \rangle 4.$ For all $U \in \mathcal{U}$, we have $(f \upharpoonright U)^{-1}(V)$ is open in $X$.
    PROOF: Lemma 142.
☐

**Proposition 101.** *Let $X$ and $X'$ be the same set $X$ under two topologies $\mathcal{T}$ and $\mathcal{T}'$. Let $i : X \to X'$ be the identity function. Then $i$ is continuous if and only if $\mathcal{T}' \subseteq \mathcal{T}$.*

PROOF: Immediate from definitions. ☐

**Proposition 102.** *Let $f : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$. Then $f$ is continuous on the right at $a$ if and only if $f$ is continuous at $a$ as a function $\mathbb{R}_l \to \mathbb{R}$.*

PROOF:
$\langle 1 \rangle 1.$ If $f$ is continuous on the right at $a$ then $f$ is continuous at $a$ as a function
        $\mathbb{R}_l \to \mathbb{R}$.
  $\langle 2 \rangle 1.$ ASSUME: $f$ is continuous on the right at $a$.
  $\langle 2 \rangle 2.$ LET: $V$ be a neighbourhood of $f(a)$
  $\langle 2 \rangle 3.$ PICK $b$, $c$ such that $f(a) \in (b, c) \subseteq V$.
  $\langle 2 \rangle 4.$ LET: $\epsilon = \min(c - f(a), f(a) - b)$
  $\langle 2 \rangle 5.$ PICK $\delta > 0$ such that, for all $x$, if $a < x < a + \delta$ then $|f(x) - f(a)| < \epsilon$
  $\langle 2 \rangle 6.$ LET: $U = [a, a + \delta)$
  $\langle 2 \rangle 7.$ $f(U) \subseteq V$
$\langle 1 \rangle 2.$ If $f$ is continuous at $a$ as a function $\mathbb{R}_l \to \mathbb{R}$ then $f$ is continuous on the
        right at $a$.
  $\langle 2 \rangle 1.$ ASSUME: $f$ is continuous at $a$ as a function $\mathbb{R}_l \to \mathbb{R}$
  $\langle 2 \rangle 2.$ LET: $\epsilon > 0$
  $\langle 2 \rangle 3.$ PICK a neighbourhood $U$ of $a$ such that $f(U) \subseteq (f(a) - \epsilon, f(a) + \epsilon)$
  $\langle 2 \rangle 4.$ PICK $b$, $c$ such that $a \in [b, c) \subset U$
  $\langle 2 \rangle 5.$ LET: $\delta = c - a$
  $\langle 2 \rangle 6.$ For all $x$, if $a < x < a + \delta$ then $|f(x) - f(a)| < \epsilon$
☐

**Lemma 103.** *Let $f : X \to Y$. Let $Z$ be an open subspace of $X$ and $a \in Z$. If $f \upharpoonright Z$ is continuous at $a$ then $f$ is continuous at $a$.*

PROOF:
$\langle 1 \rangle 1.$ LET: $V$ be a neighbourhood of $f(x)$

$\langle 1 \rangle 2$. PICK a neighbourhood $W$ of $x$ in $Z$ such that $f(W) \subseteq V$

$\langle 1 \rangle 3$. $W$ is a neighbourhood of $x$ in $X$ such that $f(W) \subseteq V$

    PROOF: Lemma 142.

□

**Proposition 104.** *Let* $f : A \to B$ *and* $g : C \to D$ *be continuous. Define* $f \times g : A \times C \to B \times D$ *by*

$$(f \times g)(a, c) = (f(a), g(c)) \ .$$

*Then* $f \times g$ *is continuous.*

PROOF: $\pi_1 \circ (f \times g) = f \circ \pi_1$ and $\pi_2 \circ (f \times g) = g \circ \pi_2$ are continuous by Theorem 96. The result follows by Theorem 132.

**Proposition 105.** *Let* $X$ *and* $Y$ *be topological spaces and* $f : X \to Y$ *be continuous. If* $a_n \to l$ *as* $n \to \infty$ *in* $X$ *then* $f(a_n) \to f(l)$ *as* $n \to \infty$.

PROOF:

$\langle 1 \rangle 1$. LET: $V$ be a neighbourhood of $f(l)$

$\langle 1 \rangle 2$. PICK a neighbourhood $U$ of $l$ such that $f(U) \subseteq V$

$\langle 1 \rangle 3$. PICK $N$ such that, for all $n \geq N$, we have $a_n \in U$

$\langle 1 \rangle 4$. For all $n \geq N$ we have $f(n) \in V$

□

# 17   Homeomorphisms

**Definition 106** (Homeomorphism)**.** Let $X$ and $Y$ be topological spaces. A *Homeomorphism* $f$ between $X$ and $Y$, $f : X \cong Y$, is a bijection $f : X \to Y$ such that both $f$ and $f^{-1}$ are continuous.

**Lemma 107.** *Let* $X$ *and* $Y$ *be topological spaces and* $f : X \to Y$ *a bijection. Then the following are equivalent:*

    *1. $f$ is a homeomorphism.*

    *2. $f$ is continuous and an open map.*

    *3. $f$ is continuous and a closed map.*

    *4. For any $U \subseteq X$, we have $U$ is open if and only if $f(U)$ is open.*

PROOF: Immediate from definitions. □

**Proposition 108.** *Let* $X$ *and* $X'$ *be the same set* $X$ *under two topologies* $\mathcal{T}$ *and* $\mathcal{T}'$. *Let* $i : X \to X'$ *be the identity function. Then* $i$ *is a homeomorphism if and only if* $\mathcal{T} = \mathcal{T}'$.

PROOF: Immediate from definitions. □

**Definition 109** (Topological Property). Let $P$ be a property of topological spaces. Then $P$ is a *topological* property if and only if, for any spaces $X$ and $Y$, if $P$ holds of $X$ and $X \cong Y$ then $P$ holds of $Y$.

**Definition 110** (Topological Imbedding). Let $X$ and $Y$ be topological spaces and $f : X \to Y$. Then $f$ is a *topological imbedding* if and only if the corestriction $f : X \to f(X)$ is a homeomorphism.

**Proposition 111.** *Let $X$ and $Y$ be topological spaces and $a \in X$. The function $i : Y \to X \times Y$ that maps $y$ to $(a, y)$ is an imbedding.*

PROOF:
$\langle 1 \rangle 1.$ $i$ is injective
$\langle 1 \rangle 2.$ $i$ is continuous.
   PROOF: For $U$ open in $X$ and $V$ open in $Y$, we have $i^{-1}(U \times V)$ is $V$ if $a \in U$, and $\emptyset$ if $a \notin U$.
$\langle 1 \rangle 3.$ $i : Y \to i(Y)$ is an open map.
   PROOF: For $V$ open in $Y$ we have $i(V) = (X \times V) \cap i(Y)$.
$\blacksquare$

# 18   The Order Topology

**Definition 112** (Order Topology). Let $X$ be a linearly ordered set with at least two points. The *order topology* on $X$ is the topology generated by the basis $\mathcal{B}$ consisting of:

- all open intervals $(a, b)$;

- all intervals of the form $[\bot, b)$ where $\bot$ is least in $X$;

- all intervals of the form $(a, \top]$ where $\top$ is greatest in $X$.

We prove this is a basis for a topology.

PROOF:
$\langle 1 \rangle 1.$ For all $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B$.
   $\langle 2 \rangle 1.$ LET: $x \in X$
   $\langle 2 \rangle 2.$ CASE: $x$ is greatest in $X$.
      $\langle 3 \rangle 1.$ PICK $y \in X$ with $y \neq x$
      $\langle 3 \rangle 2.$ $x \in (y, x] \in \mathcal{B}$
   $\langle 2 \rangle 3.$ CASE: $x$ is least in $X$.
      $\langle 3 \rangle 1.$ PICK $y \in X$ with $y \neq x$
      $\langle 3 \rangle 2.$ $x \in [x, y) \in \mathcal{B}$
   $\langle 2 \rangle 4.$ CASE: $x$ is neither greatest nor least in $X$.
      $\langle 3 \rangle 1.$ PICK $a, b \in X$ with $a < x$ and $x < b$
      $\langle 3 \rangle 2.$ $x \in (a, b) \in \mathcal{B}$
$\langle 1 \rangle 2.$ For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$

$\langle 2 \rangle$1. LET: $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$
$\langle 2 \rangle$2. CASE: $B_1 = (a, b)$, $B_2 = (c, d)$
  PROOF: Take $B_3 = (\max(a, c), \min(b, d))$.
$\langle 2 \rangle$3. CASE: $B_1 = (a, b)$, $B_2 = [\bot, d)$
  PROOF: Take $B_3 = (a, \min(b, d))$.
$\langle 2 \rangle$4. CASE: $B_1 = (a, b)$, $B_2 = (c, \top]$
  PROOF: Take $B_3 = (\max(a, c), b)$.
$\langle 2 \rangle$5. CASE: $B_1 = [\bot, b)$, $B_2 = [\bot, d)$
  PROOF: Take $B_3 = [\bot, \min(b, d))$.
$\langle 2 \rangle$6. CASE: $B_1 = [\bot, b)$, $B_2 = (c, \top]$
  PROOF: Take $B_3 = (c, b)$.
□

**Lemma 113.** *Let $X$ be a linearly ordered set. Then the open rays form a subbasis for the order topology on $X$.*

PROOF:
$\langle 1 \rangle$1. Every open ray is open.
  $\langle 2 \rangle$1. For all $a \in X$, the ray $(-\infty, a)$ is open.
    $\langle 3 \rangle$1. LET: $x \in (-\infty, a)$
    $\langle 3 \rangle$2. CASE: $x$ is least in $X$
      PROOF: $x in [x, a) = (-\infty, a)$.
    $\langle 3 \rangle$3. CASE: $x$ is not least in $X$
      $\langle 4 \rangle$1. PICK $y < x$
      $\langle 4 \rangle$2. $x \in (y, a) \subseteq (-\infty, a)$
  $\langle 2 \rangle$2. For all $a \in X$, the ray $(a, +\infty)$ is open.
    PROOF: Similar.
$\langle 1 \rangle$2. Every basic open set is a finite intersection of open rays.
  PROOF: We have $(a, b) = (a, +\infty) \cap (-\infty, b)$, $[\bot, b) = (-\infty, b)$ and $(a, \top] = (a, +\infty)$.
□

**Definition 114** (Standard Topology on the Real Line)**.** The *standard topology on the real line* is the order topology on $\mathbb{R}$ generated by the standard order.

**Lemma 115.** *The lower limit topology is strictly finer than the standard topology on $\mathbb{R}$.*

PROOF:
$\langle 1 \rangle$1. Every open interval is open in the lower limit topology.
  PROOF: If $x \in (a, b)$ then $x \in [x, b) \subseteq (a, b)$.
$\langle 1 \rangle$2. The half-open interval $[0, 1)$ is not open in the standard topology.
  PROOF: There is no open interval $(a, b)$ such that $0 \in (a, b) \subseteq [0, 1)$.
□

**Lemma 116.** *The $K$-topology is strictly finer than the standard topology on $\mathbb{R}$.*

PROOF:

31

$\langle 1 \rangle 1$. Every open interval is open in the $K$-topology.

　　PROOF: Corollary 62.1.

$\langle 1 \rangle 2$. The set $(-1, 1) \setminus K$ is not open in the standard topology.

　　PROOF: There is no open interval $(a, b)$ such that $0 \in (a, b) \subseteq (-1, 1) \setminus K$, since there must be a positive integer $n$ with $1/n \in (a, b)$.

☐

**Lemma 117.** *Let $X$ be a topological space. Let $Y$ be a linearly ordered set in the order topology. Let $f, g : X \to Y$ be continuous. Then $C = \{x \in X \mid f(x) \leq g(x)\}$ is closed.*

PROOF:

$\langle 1 \rangle 1$. LET: $x \in X \setminus C$

$\langle 1 \rangle 2$. $f(x) > g(x)$

　　　PROVE:　There exists a neighbourhood $U$ of $x$ such that $U \subseteq X \setminus C$

$\langle 1 \rangle 3$. CASE: There exists $y$ such that $g(x) < y < f(x)$

　　PROOF: Take $U = g^{-1}((-\infty, y)) \cup f^{-1}(y, +\infty)$.

$\langle 1 \rangle 4$. CASE: There is no $y$ such that $g(x) < y < f(x)$

　　PROOF: Take $U = g^{-1}((-\infty, f(x))) \cup f^{-1}(g(x), +\infty)$.

☐

**Proposition 118.** *Let $X$ be a topological space. Let $Y$ be a linearly ordered set in the order topology. Let $f, g : X \to Y$ be continuous. Define $h : X \to Y$ by $h(x) = \min(f(x), g(x))$. Then $h$ is continuous.*

PROOF: By the Pasting Lemma applied to $\{x \in X \mid f(x) \leq g(x)\}$ and $\{x \in X \mid g(x) \leq f(x)\}$, which are closed by Lemma 117.

**Proposition 119.** *Let $X$ and $Y$ be linearly ordered sets in the order topology. Let $f : X \to Y$ be strictly monotone and surjective. Then $f$ is a homeomorphism.*

PROOF:

$\langle 1 \rangle 1$. $f$ is bijective.

　　PROOF: Proposition 6.

$\langle 1 \rangle 2$. $f$ is continuous.

　　$\langle 2 \rangle 1$. For all $y \in Y$ we have $f^{-1}((y, +\infty))$ is open.

　　　$\langle 3 \rangle 1$. LET: $y \in Y$

　　　$\langle 3 \rangle 2$. PICK $x \in X$ such that $f(x) = y$

　　　　PROOF: Since $f$ is surjective.

　　　$\langle 3 \rangle 3$. $f^{-1}((y, +\infty)) = (x, +\infty)$

　　　　PROOF: By strict monotoncity.

　　$\langle 2 \rangle 2$. For all $y \in Y$ we have $f^{-1}((-\infty, y))$ is open.

　　　PROOF: Similar.

$\langle 1 \rangle 3$. $f^{-1}$ is continuous.

　　$\langle 2 \rangle 1$. For all $x \in X$ we have $f((x, +\infty))$ is open.

　　　PROOF: $f((x, +\infty)) = (f(x), +\infty)$.

　　$\langle 2 \rangle 2$. For all $x \in X$ we have $f((-\infty, x))$ is open.

　　　PROOF: $f((-\infty, x)) = (-\infty, f(x))$.

☐

# 19  The $n$th Root Function

**Proposition 120.** *For all $n \geq 1$, the function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined by $f(x) = x^n$ is a homemorphism.*

PROOF:

$\langle 1 \rangle 1$. $f$ is strictly monotone.

 $\langle 2 \rangle 1$. LET: $x, y \in \mathbb{R}$ with $0 \leq x < y$

 $\langle 2 \rangle 2$. $x^n < y^n$

$$y^n - x^n = (y - x)(y^{n-1} + y^{n-2}x + y^{n-3}x^2 + \cdots + x^{n-1})$$
$$> 0$$

$\langle 1 \rangle 2$. $f$ is surjective.

 $\langle 2 \rangle 1$. LET: $y \in \mathbb{R}_{\geq 0}$

 $\langle 2 \rangle 2$. PICK $x \in \mathbb{R}$ such that $y \leq x^n$

  PROOF: If $y \leq 1$ take $x = 1$, otherwise take $x = y$.

 $\langle 2 \rangle 3$. There exists $x' \in [0, x]$ such that $(x')^n = y$

  PROOF: By the Intermediate Value Theorem.

$\langle 1 \rangle 3$. Q.E.D.

 PROOF: Proposition 119.

□

**Definition 121.** For $n \geq 1$, the *$n$th root function* is the function $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ that is the inverse of $\lambda x.x^n$.

# 20  The Product Topology

**Definition 122** (Product Topology)**.** Let $\{A_i\}_{i \in I}$ be a family of topological spaces. The *product topology* on $\prod_{i \in I} A_i$ is the topology generated by the sub-basis consisting of the sets of the form $\pi_i{}^{-1}(U)$ where $i \in I$ and $U$ is open in $A_i$.

**Proposition 123.** *The product topology on $\prod_{i \in I} A_i$ is generated by the basis consisting of all sets of the form $\prod_{i \in I} U_i$ where $\{U_i\}_{i \in I}$ is a family such that each $U_i$ is an open set in $A_i$ and $U_i = A_i$ for all but finitely many $i$.*

PROOF: From Proposition 70. □

**Proposition 124.** *If $A_i$ is closed in $X_i$ for all $i \in I$ then $\prod_{i \in I} A_i$ is closed in $\prod_{i \in I} X_i$.*

PROOF:

$$\left( \prod_{i \in I} X_i \right) \setminus \left( \prod_{i \in I} A_i \right) = \bigcup_{j \in I} \left( \prod_{i \in I} X_i \setminus \pi_j{}^{-1}(A_j) \right) \,\square$$

**Proposition 125.** *Let $\{A_i\}_{i \in I}$ be a family of topological spaces. For $i \in I$, let $\mathcal{B}_i$ be a basis for the topology on $A_i$. Then $\mathcal{B} = \{\prod_{i \in I} B_i \mid \forall i \in I.B_i \in \mathcal{B}_i, B_i = A_i$ for all but finitely many $i\}$ is a basis for the box topology on $\prod_{i \in I} A_i$.*

PROOF:

⟨1⟩1. Every set in $\mathcal{B}$ is open.

⟨1⟩2. For every point $a \in \prod_{i \in I} A_i$ and every open set $U$ with $a \in U$, there exists
$B \in \mathcal{B}$ such that $a \in B \subseteq U$.

   ⟨2⟩1. LET: $U$ be open and $a \in U$

   ⟨2⟩2. PICK a family $\{U_i\}_{i \in I}$ such that each $U_i$ is open in $A_i$, such that $U_i = A_i$
except for $i = i_1, \ldots, i_n$, and such that $a \in \prod_{i \in I} U_i \subseteq U$.

   ⟨2⟩3. For $j = 1, \ldots, n$, PICK $B_{i_j} \in \mathcal{B}_{i_j}$ such that $a_{i_j} \in B_{i_j} \subseteq U_{i_j}$

   ⟨2⟩4. LET: $B = \prod_{i \in I} B_i$ where $B_i = A_i$ for $i \neq i_1, \ldots, i_n$

   ⟨2⟩5. $B \in \mathcal{B}$

   ⟨2⟩6. $a \in B \subseteq U$

⟨1⟩3. Q.E.D.

   PROOF: Lemma 63.

☐

**Proposition 126.** *Let $\{A_i\}_{i \in I}$ be a family of topological spaces. Then the projections $\pi_i : \prod_{i \in I} A_i \to A_i$ are open maps.*

PROOF: From Lemma 86. ☐

**Example 127.** The projections are not always closed maps. For example, $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ maps the closed set $\{(x, 1/x) \mid x > 0\}$ to $(0, +\infty)$.

**Proposition 128.** *Let $\{X_i\}_{i \in I}$ be a family of sets. For $i \in I$, let $\mathcal{T}_i$ and $\mathcal{U}_i$ be topologies on $X_i$. Let $\mathcal{P}$ be the product topology on $\prod_{i \in I} X_i$ generated by the topologies $\mathcal{T}_i$, and $\mathcal{Q}$ the product topology on the same set generated by the topologies $\mathcal{U}_i$. Then $\mathcal{P} \subseteq \mathcal{Q}$ if and only if $\mathcal{T}_i \subseteq \mathcal{U}_i$ for all $i$.*

PROOF:

⟨1⟩1. If $\mathcal{T}_i \subseteq \mathcal{U}_i$ for all $i$ then $\mathcal{P} \subseteq \mathcal{Q}$

   PROOF: By Corollary 62.1.

⟨1⟩2. If $\mathcal{P} \subseteq \mathcal{Q}$ then $\mathcal{T}_i \subseteq \mathcal{U}_i$ for all $i$

   ⟨2⟩1. ASSUME: $\mathcal{P} \subseteq \mathcal{Q}$

   ⟨2⟩2. LET: $i \in I$

   ⟨2⟩3. LET: $U \in \mathcal{T}_i$

   ⟨2⟩4. LET: $U_i = U$ and $U_j = X_j$ for $j \neq i$

   ⟨2⟩5. $\prod_{i \in I} U_i \in \mathcal{P}$

   ⟨2⟩6. $\prod_{i \in I} U_i \in \mathcal{Q}$

   ⟨2⟩7. $U \in \mathcal{U}_i$

    PROOF: From Proposition 126.

☐

**Proposition 129** (AC)**.** *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let $A_i \subseteq X_i$ for all $i \in I$. Then*

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i} \ .$$

PROOF:

$\langle 1 \rangle$1. $\overline{\prod_{i \in I} A_i} \subseteq \prod_{i \in I} \overline{A_i}$

   $\langle 2 \rangle$1. For all $i \in I$ we have $A_i \subseteq \overline{A_i}$

      PROOF: Lemma 45.

   $\langle 2 \rangle$2. $\prod_{i \in I} A_i \subseteq \prod_{i \in I} \overline{A_i}$

   $\langle 2 \rangle$3. Q.E.D.

      PROOF: Since $\prod_{i \in I} A_i$ is closed by Proposition 124.

$\langle 1 \rangle$2. $\prod_{i \in I} \overline{A_i} \subseteq \overline{\prod_{i \in I} A_i}$

   $\langle 2 \rangle$1. LET: $x \in \prod_{i \in I} \overline{A_i}$

   $\langle 2 \rangle$2. LET: $U$ be a neighbourhood of $x$

   $\langle 2 \rangle$3. PICK $V_i$ open in $X_i$ such that $x \in \prod_{i \in I} V_i \subseteq U$ with $V_i = X_i$ except for $i = i_1, \ldots, i_n$

   $\langle 2 \rangle$4. For $i \in I$, pick $a_i \in V_i \cap A_i$

      PROOF: By Theorem 48 and $\langle 2 \rangle$1 using the Axiom of Choice.

   $\langle 2 \rangle$5. $U$ intersects $\prod_{i \in I} A_i$

   $\langle 2 \rangle$6. Q.E.D.

      PROOF: $a \in U \cap \prod_{i \in I} A_i$

☐

**Example 130.** The closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ is $\mathbb{R}^\omega$

PROOF:

$\langle 1 \rangle$1. LET: $a \in \mathbb{R}^\omega$

$\langle 1 \rangle$2. LET: $U$ be any neighbourhoods of $a$.

$\langle 1 \rangle$3. PICK $U_n$ open in $\mathbb{R}$ for all $n$ such that $a \in \prod_{n \geq 0} U_n \subseteq U$ and $U_n = \mathbb{R}$ for all $n$ except $n_1, \ldots, n_k$

$\langle 1 \rangle$4. LET: $b_n = a_n$ for $n = n_1, \ldots, n_k$ and $b_n = 0$ for all other $n$

$\langle 1 \rangle$5. $b \in \mathbb{R}^\infty \cap U$

$\langle 1 \rangle$6. Q.E.D.

   PROOF: From Theorem 48.

☐

**Proposition 131.** *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let $(a_n)$ be a sequence in $\prod_{i \in I} X_i$ and $l \in \prod_{i \in I} X_i$. Then $a_n \to l$ as $n \to \infty$ if and only if, for all $i \in I$, we have $\pi_i(a_n) \to \pi_i(l)$ as $n \to \infty$.*

PROOF:

$\langle 1 \rangle$1. If $a_n \to l$ as $n \to \infty$ then, for all $i \in I$, we have $\pi_i(a_n) \to \pi_i(l)$ as $n \to \infty$

   PROOF: Proposition 105.

$\langle 1 \rangle$2. If, for all $i \in I$, we have $\pi_i(a_n) \to \pi_i(l)$ as $n \to \infty$, then $a_n \to l$ as $n \to \infty$

   $\langle 2 \rangle$1. ASSUME: For all $i \in I$, we have $\pi_i(a_n) \to \pi_i(l)$ as $n \to \infty$

   $\langle 2 \rangle$2. LET: $V$ be a neighbourhood of $l$

   $\langle 2 \rangle$3. PICK open sets $U_i$ in $X_i$ such that $l \in \prod_{i \in I} U_i \subseteq V$ and $U_i = X_i$ for all $i$ except $i = i_1, \ldots, i_k$

   $\langle 2 \rangle$4. For $j = 1, \ldots, k$, PICK $N_j$ such that, for all $n \geq N_j$, we have $\pi_{i_j}(a_n) \in U_{i_j}$

   $\langle 2 \rangle$5. LET: $N = \max(N_1, \ldots, N_k)$

   $\langle 2 \rangle$6. For all $n \geq N$ we have $a_n \in V$

□

**Theorem 132.** *Let $A$ be a topological space and $\{X_i\}_{i \in I}$ be a family of topological spaces. Let $f : A \to \prod_{i \in I} X_i$ be a function. If $\pi_i \circ f$ is continuous for all $i \in I$ then $f$ is continuous.*

PROOF:
$\langle 1 \rangle 1$. LET: $i \in I$ and $U$ be open in $X_i$
$\langle 1 \rangle 2$. $f^{-1}(\pi_i^{-1}(U))$ is open in $A$
$\langle 1 \rangle 3$. Q.E.D.
   PROOF: Proposition 90.
□

## 20.1   Continuous in Each Variable Separately

**Definition 133** (Continuous in Each Variable Separately)**.** Let $F : X \times Y \to Z$. Then $F$ is *continuous in each variable separately* if and only if:

- for every $a \in X$ the function $\lambda y \in Y.F(a, y)$ is continuous;

- for every $b \in Y$ the function $\lambda x \in X.F(x, b)$ is continuous.

**Proposition 134.** *Let $F : X \times Y \to Z$. If $F$ is continuous then $F$ is continuous in each variable separately.*

PROOF: For $a \in X$, the function $\lambda y \in Y.F(a, y)$ is $F \circ i$ where $i : Y \to X \times Y$ maps $y$ to $(a, y)$. We have $i$ is continuous by Proposition 111, hence $F \circ i$ is continuous by Theorem 96.
Similarly for $\lambda x \in X.F(x, b)$ for $b \in Y$. □

**Example 135.** Define $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$F(x, y) = \begin{cases} xy/(x^2 + y^2) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Then $F$ is continuous in each variable separately but not continuous.

**Proposition 136.** *Let $f : A \to C$ and $g : B \to D$ be open maps. Then $f \times g : A \times B \to C \times D$ is an open map.*

PROOF: Given $U$ open in $A$ and $V$ open in $B$. Then $(f \times g)(U \times V) = f(U) \times g(V)$ is open in $C \times D$. The result follows from Proposition 87. □

# 21   The Subspace Topology

**Definition 137** (Subspace Topology)**.** Let $X$ be a topological space and $Y \subseteq X$. The *subspace topology* on $Y$ is $\mathcal{T} = \{U \cap Y \mid U \text{ is open in } X\}$.

   We prove this is a topology.

PROOF:

$\langle 1 \rangle 1$. $Y \in \mathcal{T}$
   PROOF: Since $Y = X \cap Y$
$\langle 1 \rangle 2$. For all $\mathcal{U} \subseteq \mathcal{T}$, we have $\bigcup \mathcal{U} \in \mathcal{T}$
   $\langle 2 \rangle 1$. LET: $\mathcal{U} \subseteq \mathcal{T}$
   $\langle 2 \rangle 2$. LET: $\mathcal{V} = \{V \text{ open in } X \mid V \cap Y \in \mathcal{U}\}$
   $\langle 2 \rangle 3$. $\bigcup \mathcal{U} = (\bigcup \mathcal{V}) \cap Y$
$\langle 1 \rangle 3$. For all $U, V \in \mathcal{T}$, we have $U \cap V \in \mathcal{T}$
   $\langle 2 \rangle 1$. LET: $U, V \in \mathcal{T}$
   $\langle 2 \rangle 2$. PICK $U'$, $V'$ open in $X$ such that $U = U' \cap Y$ and $V = V' \cap Y$
   $\langle 2 \rangle 3$. $(U \cap V) = (U' \cap V') \cap Y$
$\square$

**Theorem 138.** *Let $X$ be a topological space and $Y$ a subspace of $X$. Let $A \subseteq Y$. Then $A$ is closed in $Y$ if and only if there exists a closed set $C$ in $X$ such that $A = C \cap Y$.*

PROOF: We have

$$A \text{ is closed in } Y$$
$$\Leftrightarrow Y \setminus A \text{ is open in } Y$$
$$\Leftrightarrow \exists U \text{ open in } X. Y \setminus A = Y \cap U$$
$$\Leftrightarrow \exists C \text{ closed in } X. Y \setminus A = Y \cap (X \setminus U)$$
$$\Leftrightarrow \exists C \text{ closed in } X. A = Y \cap U \qquad \square$$

**Theorem 139.** *Let $Y$ be a subspace of $X$. Let $A \subseteq Y$. Let $\overline{A}$ be the closure of $A$ in $X$. Then the closure of $A$ in $Y$ is $\overline{A} \cap Y$.*

PROOF: The closure of $A$ in $Y$ is
$$\bigcap \{C \text{ closed in } Y \mid A \subseteq C\}$$
$$= \bigcap \{D \cap Y \mid D \text{ closed in } X, A \subseteq D \cap Y\} \qquad \text{(Theorem 138)}$$
$$= \bigcap \{D \mid D \text{ closed in } X, A \subseteq D\} \cap Y$$
$$= \overline{A} \cap Y \qquad \square$$

**Lemma 140.** *Let $X$ be a topological space and $Y \subseteq X$. Let $\mathcal{B}$ be a basis for the topology on $X$. Then $\mathcal{B}' = \{B \cap Y \mid B \in \mathcal{B}\}$ is a basis for the subspace topology on $Y$.*

PROOF:
$\langle 1 \rangle 1$. Every element in $\mathcal{B}'$ is open in $Y$
$\langle 1 \rangle 2$. For every open set $U$ in $Y$ and point $y \in U$, there exists $B' \in \mathcal{B}'$ such that $y \in B' \subseteq U$
   $\langle 2 \rangle 1$. LET: $U$ be open in $Y$ and $y \in U$
   $\langle 2 \rangle 2$. PICK $V$ open in $X$ such that $U = V \cap Y$
   $\langle 2 \rangle 3$. PICK $B \in \mathcal{B}$ such that $y \in B \subseteq V$
   $\langle 2 \rangle 4$. LET: $B' = B \cap Y$
   $\langle 2 \rangle 5$. $B' \in \mathcal{B}'$

$\langle 2 \rangle 6.$ $y \in B' \subseteq U$
$\langle 1 \rangle 3.$ Q.E.D.
    PROOF: By Lemma 63.

□

**Lemma 141.** *Let $X$ be a topological space and $Y \subseteq X$. Let $\mathcal{S}$ be a basis for the topology on $X$. Then $\mathcal{S}' = \{S \cap Y \mid S \in \mathcal{S}\}$ is a subbasis for the subspace topology on $Y$.*

PROOF: The set $\{B \cap Y \mid B$ is a finite intersection of elements of $\mathcal{S}\}$ is a basis for the subspace topology by Lemma 140, and this is the set of all finite intersections of elements of $\mathcal{S}'$. □

**Lemma 142.** *Let $Y$ be a subspace of $X$. If $U$ is open in $Y$ and $Y$ is open in $X$ then $U$ is open in $X$.*

PROOF:
$\langle 1 \rangle 1.$ PICK $V$ open in $X$ such that $U = V \cap Y$
$\langle 1 \rangle 2.$ $U$ is open in $X$
    PROOF: Since it is the intersection of two open sets $V$ and $Y$.

□

**Theorem 143.** *Let $Y$ be a subspace of $X$ and $A \subseteq Y$. If $A$ is closed in $Y$ and $Y$ is closed in $X$ then $A$ is closed in $X$.*

PROOF: Pick a closed set $C$ in $X$ such that $A = C \cap Y$ (Theorem 138). Then $A$ is the intersection of two sets closed in $X$, hence $A$ is closed in $X$ (Lemma 33). □

**Theorem 144.** *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let $A_i$ be a subspace of $X_i$ for all $i \in I$. Then the product topology on $\prod_{i \in I} A_i$ is the same as the topology it inherits as a subspace of $\prod_{i \in I} X_i$.*

PROOF: The product topology is generated by the subbasis
$$\{\pi_i^{-1}(U) \mid i \in I, U \text{ open in } A_i\}$$
$$= \{\pi_i^{-1}(V) \cap A_i \mid i \in I, V \text{ open in } X_i\}$$
$$= \{\pi_i^{-1}(V) \mid i \in I, V \text{ open in } X_i\} \cap \prod_{i \in I} A_i$$
and this is a subbasis for the subspace topology by Lemma 141. □

**Theorem 145.** *Let $X$ be an ordered set in the order topology. Let $Y \subseteq X$ be an interval. Then the order topology on $Y$ is the same as the subspace topology on $Y$.*

PROOF:
$\langle 1 \rangle 1.$ The order topology is finer than the subspace topology.
    $\langle 2 \rangle 1.$ For every open ray $R$ in $X$, the set $R \cap Y$ is open in the order topology.
        $\langle 3 \rangle 1.$ For all $a \in X$, we have $(-\infty, a) \cap Y$ is open in the order topology.

⟨4⟩1. CASE: For all $y \in Y$ we have $y < a$
PROOF: In this case $(-\infty, a) \cap Y = Y$.
⟨4⟩2. CASE: For all $y \in Y$ we have $a < y$
PROOF: In this case $(-\infty, a) \cap Y = \emptyset$.
⟨4⟩3. CASE: There exists $y \in Y$ such that $y \le a$ and $y \in Y$ such that $a \le y$
⟨5⟩1. $a \in Y$
PROOF: Because $Y$ is an interval.
⟨5⟩2. $(-\infty, a) \cap Y = \{y \in Y \mid y < a\}$
⟨3⟩2. For all $a \in X$, we have $(a, +\infty) \cap Y$ is open in the order topology.
PROOF: Similar.
⟨2⟩2. Q.E.D.
PROOF: By Lemmas 113 and 141 and Proposition 71.
⟨1⟩2. The subspace topology is finer than the order topology.
⟨2⟩1. Every open ray in $Y$ is open in the subspace topology.
PROOF: For any $a \in Y$ we have $(-\infty, a)_Y = (-\infty, a)_X \cap Y$ and $(a, +\infty)_Y = (a, +\infty)_X \cap Y$.
⟨2⟩2. Q.E.D.
PROOF: By Lemma 113 and Proposition 71

□

This example shows that we cannot remove the hypothesis that $Y$ is an interval:

**Example 146.** The order topology on $I_o^2$ is different from the subspace topology as a subspace of $\mathbb{R}^2$ under the dictionary order topology.

PROOF: The set $\{1/2\} \times (1/2, 1)$ is open in the subspace topology but not in the order topology. □

**Proposition 147.** *Let $X$ be a topological space, $Y$ a subspace of $X$, and $Z$ a subspace of $Y$. Then the subspace topology on $Z$ inherited from $X$ is the same as the subspace topology on $Z$ inherited from $Y$.*

PROOF: The subspace topology inherited from $Y$ is
$$\{V \cap Z \mid V \text{ open in } Y\}$$
$$= \{U \cap Y \cap Z \mid U \text{ open in } X\}$$
$$= \{U \cap Z \mid U \text{ open in } X\}$$
which is the subspace topology inherited from $X$. □

**Definition 148** (Unit Circle)**.** The *unit circle* $S^1$ is $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ as a subspace of $\mathbb{R}^2$.

**Definition 149** (Unit 2-sphere)**.** The *unit 2-sphere* is $S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 \le 1\}$ as a subspace of $\mathbb{R}^3$.

**Proposition 150.** *Let $f : X \to Y$ be an open map and $A \subseteq X$ be open. Then the restriction $f \restriction A : A \to f(A)$ is an open map.*

PROOF:

⟨1⟩1. LET: $U$ be open in $A$

⟨1⟩2. $U$ is open in $X$

   PROOF: Lemma 142.

⟨1⟩3. $f(U)$ is open in $Y$

⟨1⟩4. $f(U)$ is open in $f(A)$

   PROOF: Since $f(U) = f(U) \cap f(A)$.

☐

**Example 151.** This example shows that we cannot remove the hypothesis that $A$ is open.

Let $A = \{(x, y) \in \mathbb{R}^2 \mid (x > 0$ and $y = 1/x)$ or $x = y = 0\}$. Then $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ is an open map, but $\pi_1 \restriction A : A \to [0, +\infty)$ is not, because it maps the set $\{0, 0\}$ which is open in $A$ to $\{0\}$ which is not open in $[0, +\infty)$.

**Proposition 152.** *Let $Y$ be a subspace of $X$. Let $A \subseteq Y$ and $l \in Y$. Then $l$ is a limit point of $A$ in $Y$ if and only if $l$ is a limit point of $A$ in $X$.*

PROOF: Both are equivalent to the condition that any neighbourhood of $l$ in $X$ intersects $A$ in a point other than $l$. ☐

# 22   The Box Topology

**Definition 153** (Box Topology)**.** Let $\{A_i\}_{i \in I}$ be a family of topological spaces. The *box topology* on $\prod_{i \in I} A_i$ is the topology generated by the set of all sets of the form $\prod_{i \in I} U_i$ where $\{U_i\}_{i \in I}$ is a family such that each $U_i$ is open in $A_i$.

This is a basis since it covers $\prod_{i \in I} A_i$ and is closed under intersection.

**Proposition 154.** *The box topology is finer than the product topology.*

PROOF: From Proposition 123. ☐

**Corollary 154.1.** *If $A_i$ is closed in $X_i$ for all $i \in I$ then $\prod_{i \in I} A_i$ is closed in $\prod_{i \in I} X_i$ under the box topology.*

PROOF: From Proposition 124.

**Proposition 155** (AC)**.** *Let $\{A_i\}_{i \in I}$ be a family of topological spaces. For $i \in I$, let $\mathcal{B}_i$ be a basis for the topology on $A_i$. Then $\mathcal{B} = \{\prod_{i \in I} B_i \mid \forall i \in I. B_i \in \mathcal{B}_i\}$ is a basis for the box topology on $\prod_{i \in I} A_i$.*

PROOF:

⟨1⟩1. Every set of the form $\prod_{i \in I} B_i$ is open.

⟨1⟩2. For every point $a \in \prod_{i \in I} A_i$ and every open set $U$ with $a \in U$, there exists $B \in \mathcal{B}$ such that $a \in B \subseteq U$.

  ⟨2⟩1. LET: $U$ be open and $a \in U$

  ⟨2⟩2. PICK a family $\{U_i\}_{i \in I}$ such that each $U_i$ is open in $A_i$ and $a \in \prod_{i \in I} U_i \subseteq U$.

$\langle 2 \rangle 3$. For $i \in I$, PICK $B_i \in \mathcal{B}_i$ such that $a_i \in B_i \subseteq U_i$
   PROOF: Using the Axiom of Choice.
$\langle 2 \rangle 4$. $a \in \prod_{i \in I} B_i \subseteq U$
$\langle 1 \rangle 3$. Q.E.D.
   PROOF: Lemma 63.

□

**Theorem 156.** *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let $A_i$ be a subspace of $X_i$ for all $i \in I$. Give $\prod_{i \in I} X_i$ the box topology. Then the box topology on $\prod_{i \in I} A_i$ is the same as the topology it inherits as a subspace of $\prod_{i \in I} X_i$.*

PROOF: The box topology is generated by the basis
$$\{\prod_{i \in I} U_i \mid \forall i \in I, U_i \text{ open in } A_i\}$$
$$= \{\prod_{i \in I} (V_i \cap A_i) \mid \forall i \in I, V_i \text{ open in } X_i\}$$
$$= \{\prod_{i \in I} V_i \mid \forall i \in I, V_i \text{ open in } X_i\} \cap \prod_{i \in I} A_i$$
and this is a basis for the subspace topology by Lemma 140. □

**Proposition 157** (AC). *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Give $\prod_{i \in I} X_i$ the box topology. Let $A_i \subseteq X_i$ for all $i \in I$. Then*
$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i} \ .$$

PROOF:
$\langle 1 \rangle 1$. $\overline{\prod_{i \in I} A_i} \subseteq \prod_{i \in I} \overline{A_i}$
   $\langle 2 \rangle 1$. For all $i \in I$ we have $A_i \subseteq \overline{A_i}$
      PROOF: Lemma 45.
   $\langle 2 \rangle 2$. $\prod_{i \in I} A_i \subseteq \prod_{i \in I} \overline{A_i}$
   $\langle 2 \rangle 3$. Q.E.D.
      PROOF: Since $\prod_{i \in I} A_i$ is closed by Corollary 154.1.
$\langle 1 \rangle 2$. $\prod_{i \in I} \overline{A_i} \subseteq \overline{\prod_{i \in I} A_i}$
   $\langle 2 \rangle 1$. LET: $x \in \prod_{i \in I} \overline{A_i}$
   $\langle 2 \rangle 2$. LET: $U$ be a neighbourhood of $x$
   $\langle 2 \rangle 3$. PICK $V_i$ open in $X_i$ such that $x \in \prod_{i \in I} V_i \subseteq U$
   $\langle 2 \rangle 4$. For $i \in I$, pick $a_i \in V_i \cap A_i$
      PROOF: By Theorem 48 and $\langle 2 \rangle 1$ using the Axiom of Choice.
   $\langle 2 \rangle 5$. $U$ intersects $\prod_{i \in I} A_i$
   $\langle 2 \rangle 6$. Q.E.D.
      PROOF: $a \in U \cap \prod_{i \in I} A_i$.

□

The following example shows that Theorem 132 fails in the box topology.

**Example 158.** Define $f : \mathbb{R} \to \mathbb{R}^\omega$ by $f(t) = (t, t, \ldots)$. Then $\pi_n \circ f = \mathrm{id}_{\mathbb{R}}$ is continuous for all $n$. But $f$ is not continuous when $\mathbb{R}^\omega$ is given the box topology because the inverse image of

$$(-1, 1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times \cdots$$

is $\{0\}$ which is not open.

The following example shows that Proposition 131 fails in the box topology.

**Example 159.** Give $\mathbb{R}^\omega$ the box topology. Let $a_n = (1/n, 1/n, \ldots)$ for $n \geq 1$ and $l = (0, 0, \ldots)$. Then $\pi_i(a_n) \to \pi_i(l)$ as $n \to \infty$ for all $i$, but $a_n \not\to l$ as $n \to \infty$ since the open set

$$(-1, 1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times \cdots$$

contains $l$ but does not contain any $a_n$.

**Example 160.** The set $\mathbb{R}^\infty$ is closed in $\mathbb{R}^\omega$ under the box topology. For let $(a_n)$ be any sequence not in $\mathbb{R}^\infty$. Let $U_n$ be an open interval around $a_n$ that does not contain 0 if $a_n \neq 0$, and $U_n = \mathbb{R}$ if $a_n = 0$. Then $\prod_{n \geq 0} U_n$ is a neighbourhood of $(a_n)$ that does not intersect $\mathbb{R}^\infty$.

## 23 $T_1$ Spaces

**Definition 161** ($T_1$ Space)**.** A topological space is $T_1$ if and only if every singleton is closed.

**Lemma 162.** *A space is $T_1$ if and only if every finite set is closed.*

PROOF: From Lemma 34. □

**Theorem 163.** *In a $T_1$ space, a point $a$ is a limit point of a set $A$ if and only if every neighbourhood of $a$ contains infinitely many points of $A$.*

PROOF:
$\langle 1 \rangle 1$. If $a$ is a limit point of $A$ then every neighbourhood of $a$ contains infinitely many points of $A$.
    $\langle 2 \rangle 1$. ASSUME: $a$ is a limit point of $A$.
    $\langle 2 \rangle 2$. LET: $U$ be a neighbourhood of $a$.
    $\langle 2 \rangle 3$. ASSUME: for a contradiction $U$ contains only finitely many points of $A$.
    $\langle 2 \rangle 4$. $(U \cap A) \setminus \{a\}$ is closed.
       PROOF: By the $T_1$ axiom.
    $\langle 2 \rangle 5$. $(U \setminus A) \cup \{a\}$ is open.
       PROOF: It is $U \setminus ((U \cap A) \setminus \{a\})$.
    $\langle 2 \rangle 6$. $(U \setminus A) \cup \{a\}$ intersects $A$ in a point other than $a$.
       PROOF: From $\langle 2 \rangle 1$.
    $\langle 2 \rangle 7$. Q.E.D.
□

$\langle 1 \rangle 2$. If every neighbourhood of $a$ contains infinitely many points of $A$ then $a$ is a limit point of $A$.

  PROOF: Immediate from definitions.

□

  (To see this does not hold in every space, see Proposition 59.)

**Proposition 164.** *A space is $T_1$ if and only if, for any two distinct points $x$ and $y$, there exist neighbourhoods $U$ of $x$ and $V$ of $y$ such that $x \notin V$ and $y \notin U$.*

PROOF:

$\langle 1 \rangle 1$. LET: $X$ be a topological space.

$\langle 1 \rangle 2$. If $X$ is $T_1$ then, for any two distinct points $x$ and $y$, there exist neighbourhoods $U$ of $x$ and $V$ of $y$ such that $x \notin V$ and $y \notin U$.

  PROOF: This holds because $\{x\}$ and $\{y\}$ are closed.

$\langle 1 \rangle 3$. Suppose, for any two distinct points $x$ and $y$, there exist neighbourhoods $U$ of $x$ and $V$ of $y$ such that $x \notin V$ and $y \notin U$. Then $X$ is $T_1$.

  $\langle 2 \rangle 1$. ASSUME: For any two distinct points $x$ and $y$, there exist neighbourhoods $U$ of $x$ and $V$ of $y$ such that $x \notin V$ and $y \notin U$.

  $\langle 2 \rangle 2$. LET: $a \in X$

  $\langle 2 \rangle 3$. $\{a\}$ is closed.

    PROOF: For all $b \neq a$ there exists a neighbourhood $U$ of $b$ such that $U \subseteq X \setminus \{a\}$.

□

**Proposition 165.** *A subspace of a $T_1$ space is $T_1$.*

PROOF: From Proposition 143.

# 24 Hausdorff Spaces

**Definition 166** (Hausdorff Space)**.** A topological space is *Hausdorff* if and only if, for any points $x$, $y$ with $x \neq y$, there exist disjoint open sets $U$ and $V$ such that $x \in U$ and $y \in V$.

**Theorem 167.** *Every Hausdorff space is $T_1$.*

PROOF:

$\langle 1 \rangle 1$. LET: $X$ be a Hausdorff space.

$\langle 1 \rangle 2$. LET: $b \in X$

    PROVE: $\overline{\{b\}} = \{b\}$

$\langle 1 \rangle 3$. ASSUME: $a \in \overline{\{b\}}$ and $a \neq b$

$\langle 1 \rangle 4$. PICK disjoint neighbourhoods $U$ of $a$ and $V$ of $b$.

$\langle 1 \rangle 5$. $U$ intersects $\{b\}$

  PROOF: Theorem 48.

$\langle 1 \rangle 6$. $b \in U$

$\langle 1 \rangle 7$. Q.E.D.

  PROOF: This contradicts the fact that $U$ and $V$ are disjoint ($\langle 1 \rangle 4$).

$\Box$

**Proposition 168.** *An infinite set under the finite complement topology is $T_1$ but not Hausdorff.*

PROOF:
$\langle 1 \rangle 1$. LET: $X$ be an infinite set under the finite complement topology.
$\langle 1 \rangle 2$. Every singleton is closed.
   PROOF: By definition.
$\langle 1 \rangle 3$. PICK $a, b \in X$ with $a \neq b$
$\langle 1 \rangle 4$. There are no disjoint neighbourhoods $U$ of $a$ and $V$ of $b$.
   $\langle 2 \rangle 1$. LET: $U$ be a neighbourhood of $a$ and $V$ a neighbourhood of $b$.
   $\langle 2 \rangle 2$. $X \setminus U$ and $X \setminus V$ are finite.
   $\langle 2 \rangle 3$. PICK $c \in X$ that is not in $X \setminus U$ or $X \setminus V$.
   $\langle 2 \rangle 4$. $c \in U \cap V$
$\Box$

**Proposition 169.** *The product of a family of Hausdorff spaces is Hausdorff.*

PROOF:
$\langle 1 \rangle 1$. LET: $\{X_i\}_{i \in I}$ be a family of Hausdorff spaces.
$\langle 1 \rangle 2$. LET: $a, b \in \prod_{i \in I} X_i$ with $a \neq b$
$\langle 1 \rangle 3$. PICK $i \in I$ such that $a_i \neq b_i$
$\langle 1 \rangle 4$. PICK $U, V$ disjoint open sets in $X_i$ with $a_i \in U$ and $b_i \in V$
$\langle 1 \rangle 5$. ${\pi_i}^{-1}(U)$ and ${\pi_i}^{-1}(V)$ are disjoint open sets in $\prod_{i \in I} X_i$ with $a \in {\pi_i}^{-1}(U)$ and $b \in {\pi_i}^{-1}(V)$
$\Box$

**Theorem 170.** *Every linearly ordered set under the order topology is Hausdorff.*

PROOF:
$\langle 1 \rangle 1$. LET: $X$ be a linearly ordered set under the order topology.
$\langle 1 \rangle 2$. LET: $a, b \in X$ with $a \neq b$
$\langle 1 \rangle 3$. ASSUME: w.l.o.g. $a < b$
$\langle 1 \rangle 4$. CASE: There exists $c$ such that $a < c < b$
   PROOF: The sets $(-\infty, c)$ and $(c, +\infty)$ are disjoint neighbourhoods of $a$ and $b$ respectively.
$\langle 1 \rangle 5$. CASE: There is no $c$ such that $a < c < b$
   PROOF: The sets $(-\infty, b)$ and $(a, +\infty)$ are disjoint neighbourhoods of $a$ and $b$ respectively.
$\Box$

**Theorem 171.** *A subspace of a Hausdorff space is Hausdorff.*

PROOF:
$\langle 1 \rangle 1$. LET: $X$ be a Hausdorff space and $Y$ a subspace of $X$.
$\langle 1 \rangle 2$. LET: $x, y \in Y$ with $x \neq y$
$\langle 1 \rangle 3$. PICK disjoint neighbourhoods $U$ of $x$ and $V$ of $y$ in $X$.

$\langle 1 \rangle 4$. $U \cap Y$ and $V \cap Y$ are disjoint neighbourhoods of $x$ and $y$ respectively in $Y$.

□

**Proposition 172.** *A space $X$ is Hausdorff if and only if the diagonal $\Delta = \{(x,x) \mid x \in X\}$ is closed in $X^2$.*

PROOF:

$X$ is Hausdorff

$\Leftrightarrow \forall x, y \in X.x \neq y \Rightarrow \exists V, W \text{ open}.x \in V \wedge y \in W \wedge V \cap W = \emptyset$

$\Leftrightarrow \forall (x, y) \in X^2 \setminus \Delta.\exists V, W \text{ open}.(x, y) \subseteq V \times W \subseteq X^2 \setminus \Delta$

$\Leftrightarrow \Delta \text{ is closed}$ □

**Theorem 173.** *In a Hausdorff space, a sequence has at most one limit.*

PROOF:
$\langle 1 \rangle 1$. LET: $X$ be a Hausdorff space.
$\langle 1 \rangle 2$. ASSUME: for a contradiction $a_n \to l$ as $n \to \infty$, $a_n \to m$ as $n \to \infty$, and $l \neq m$
$\langle 1 \rangle 3$. PICK disjoint neighbourhoods $U$ of $l$ and $V$ of $m$
   PROOF: By the Hausdorff axiom.
$\langle 1 \rangle 4$. PICK $M$ and $N$ such that $a_n \in U$ for $n \geq M$ and $a_n \in V$ for $n \geq N$
$\langle 1 \rangle 5$. $a_{\max(M,N)} \in U \cap V$
$\langle 1 \rangle 6$. Q.E.D.
   PROOF: This contradicts the fact that $U$ and $V$ are disjoint ($\langle 1 \rangle 3$).

□

To see this is not always true in spaces that are $T_1$ but not Hausdorff:

**Proposition 174.** *Let $X$ be an infinite set under the finite complement topology. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence with all points distinct. Then for every $l \in X$ we have $a_n \to l$ as $n \to \infty$.*

PROOF: Let $U$ be any neighbourhood of $l$. Since $X \setminus U$ is finite, there must exist $N$ such that, for all $n \geq N$, we have $a_n \in U$. □

**Proposition 175.** *Let $X$ be a topological space. Let $Y$ a Hausdorff space. Let $A \subseteq X$. Let $f, g : \overline{A} \to Y$ be continuous. If $f$ and $g$ agree on $A$ then $f = g$.*

PROOF:
$\langle 1 \rangle 1$. LET: $x \in \overline{A}$
$\langle 1 \rangle 2$. ASSUME: $f(x) \neq g(x)$
$\langle 1 \rangle 3$. PICK disjoint neighbourhoods $V$ of $f(x)$ and $W$ of $g(x)$.
$\langle 1 \rangle 4$. PICK $y \in f^{-1}(V) \cap g^{-1}(W) \cap A$
   PROOF: Since $f^{-1}(V) \cap g^{-1}(W)$ is a neighbourhood of $x$ and hence intersects $A$.
$\langle 1 \rangle 5$. $f(y) = g(y) \in V \cap W$
$\langle 1 \rangle 6$. Q.E.D.

PROOF: This contradicts the fact that $V$ and $W$ are disjoint ($\langle 1 \rangle 3$).
□

**Proposition 176.** *Let $\{X_i\}_{i \in I}$ be a family of Hausdorff spaces. Then $\prod_{i \in I} X_i$ under the box topology is Hausdorff.*

PROOF:
$\langle 1 \rangle 1$. LET: $\{X_i\}_{i \in I}$ be a family of Hausdorff spaces.
$\langle 1 \rangle 2$. LET: $a, b \in \prod_{i \in I} X_i$ with $a \neq b$
$\langle 1 \rangle 3$. PICK $i \in I$ such that $a_i \neq b_i$
$\langle 1 \rangle 4$. PICK $U, V$ disjoint open sets in $X_i$ with $a_i \in U$ and $b_i \in V$
$\langle 1 \rangle 5$. $\pi_i{}^{-1}(U)$ and $\pi_i{}^{-1}(V)$ are disjoint open sets in $\prod_{i \in I} X_i$ with $a \in \pi_i{}^{-1}(U)$ and $b \in \pi_i{}^{-1}(V)$
□

**Proposition 177.** *Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on the same set $X$ with $\mathcal{T} \subseteq \mathcal{T}'$ If $\mathcal{T}$ is Haudorff then $\mathcal{T}'$ is Hausdorff.*

PROOF: Immediate from definitions.

# 25 The First Countability Axiom

**Definition 178** (First Countability Axiom). A topological space $X$ satisfies the *first countability axiom*, or is *first countable*, if and only if every point has a countable local basis.

**Lemma 179** (Sequence Lemma (CC)). *Let $X$ be a first countable space. Let $A \subseteq X$ and $l \in \overline{A}$. Then there exists a sequence in $A$ that converges to $l$.*

PROOF:
$\langle 1 \rangle 1$. PICK a countable local basis $\{B_n \mid n \in \mathbb{Z}^+\}$ at $l$ such that $B_1 \supseteq B_2 \supseteq \cdots$.
  PROOF: Lemma 73.
$\langle 1 \rangle 2$. For all $n \geq 1$, PICK $a_n \in A \cap B_n$.
  PROVE: $a_n \to l$ as $n \to \infty$
$\langle 1 \rangle 3$. LET: $U$ be a neighbourhood of $A$
$\langle 1 \rangle 4$. PICK $N$ such that $B_N \subseteq U$
$\langle 1 \rangle 5$. For $n \geq N$ we have $a_n \in U$
  PROOF: $a_n \in B_n \subseteq B_N \subseteq U$
□

**Theorem 180** (CC). *Let $X$ be a first countable space and $Y$ a topological space. Let $f : X \to Y$. Suppose that, for every sequence $(x_n)$ in $X$ and $l \in X$, if $x_n \to l$ as $n \to \infty$, then $f(x_n) \to f(l)$ as $n \to \infty$. Then $f$ is continuous.*

PROOF:
$\langle 1 \rangle 1$. LET: $A \subseteq X$
$\langle 1 \rangle 2$. LET: $a \in A$
  PROVE: $f(a) \in \overline{f(A)}$

⟨1⟩3. PICK a sequence $(x_n)$ in $A$ that converges to $a$.
   PROOF: By the Sequence Lemma.
⟨1⟩4. $f(x_n) \to f(a)$
⟨1⟩5. $f(a) \in \overline{f(A)}$
   PROOF: By Lemma 75.
⟨1⟩6. Q.E.D.
   PROOF: By Theorem 93.
∎

**Example 181** (CC). The space $\mathbb{R}^\omega$ under the box product is not first countable.

PROOF: Let $\{B_n \mid n \geq 0\}$ be a countable set of neighbourhoods of $\vec{0}$. We will construct a neighbourhood of $\vec{0}$ that does not include any of these.
For $n \geq 0$, pick a neighbourhood $U_n$ of 0 such that $U_n \subset \pi_n(B_p)$. Then $\prod_{n=0}^\infty U_n$ is a neighbourhood of $\vec{0}$ that does not include any $B_n$. ∎

**Example 182.** If $J$ is an uncountable set then $\mathbb{R}^J$ is not first countable.

PROOF:
⟨1⟩1. LET: $\{B_n \mid n \geq 0\}$ be any countable set of neighbourhoods of $\vec{0}$.
⟨1⟩2. For $n \geq 0$, PICK a basis element $\prod_{\alpha \in J} U_{n\alpha}$ that contains $\vec{0}$ and is included in $B_n$.
   PROOF: Using the Axiom of Countable Choice.
⟨1⟩3. For $n \geq 0$,
       LET: $J_n = \{\alpha \in J \mid U_{n\alpha} \neq \mathbb{R}\}$
⟨1⟩4. PICK $\beta \in J$ such that $\beta \notin J_n$ for any $n$.
   PROOF: Since each $J_n$ is finite so $\bigcup_n J_n$ is countable.
⟨1⟩5. $\pi_\beta((-1,1))$ is a neighbourhood of $\vec{0}$ that does not include any $B_n$.
∎

**Example 183.** The space $\mathbb{R}_l$ is first countable.

PROOF: For any $a \in \mathbb{R}$, the set $\{[a, a + 1/n) \mid n \geq 1\}$ is a countable local basis.

**Example 184.** The ordered square is first countable.

PROOF: For any $(a,b) \in I_o^2$ with $b \neq 0, 1$, the set $\{(\{a\} \times (b - 1/n, b + 1/n)) \cap I_o^2 \mid n \geq 1\}$ is a countable local basis.

# 26  Strong Continuity

**Definition 185** (Strongly Continuous). Let $X$ and $Y$ be topological spaces and $f : X \to Y$ be a function. Then $f$ is *strongly continuous* if and only if, for every subset $U \subseteq Y$, we have $U$ is open in $Y$ if and only if $f^{-1}(U)$ is open in $X$.

**Proposition 186.** *Let $X$ and $Y$ be topological spaces and $f : X \to Y$ be a function. Then $f$ is strongly continuous if and only if, for every subset $C \subseteq Y$, we have $C$ is closed in $Y$ if and only if $f^{-1}(C)$ is closed in $X$.*

PROOF: Since $X \setminus f^{-1}(C) = f^{-1}(Y \setminus C)$. $\square$

**Proposition 187.** *Let $X$, $Y$ and $Z$ be topological spaces. Let $f : X \to Y$ and $g : Y \to Z$. If $f$ and $g$ are strongly continuous then so is $g \circ f$.*

PROOF: Since $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$. $\square$

**Proposition 188.** *Let $X$, $Y$ and $Z$ be topological spaces. Let $f : X \to Y$ and $g : Y \to Z$. If $g \circ f$ is continuous and $f$ is strongly continuous then $g$ is continuous.*

PROOF:
$\langle 1 \rangle 1.$ LET: $V \subseteq Z$ be open.
$\langle 1 \rangle 2.$ $f^{-1}(g^{-1}(V))$ is open in $X$.
    PROOF: Since $g \circ f$ is continuous.
$\langle 1 \rangle 3.$ $f^{-1}(V)$ is open in $Y$.
    PROOF: Since $g$ is strongly continuous.
$\square$

**Proposition 189.** *Let $X$, $Y$ and $Z$ be topological spaces. Let $f : X \to Y$ and $g : Y \to Z$. If $g \circ f$ is strongly continuous and $f$ is strongly continuous then $g$ is strongly continuous.*

PROOF: For $V \subseteq Z$, we have $V$ is open iff $f^{-1}(g^{-1}(V))$ is open iff $g^{-1}(V)$ is open.

## 27 Saturated Sets

**Definition 190.** Let $X$ and $Y$ be sets and $p : X \twoheadrightarrow Y$ a surjective function. Let $C \subseteq X$. Then $C$ is *saturated* with respect to $p$ if and only if, for all $x, y \in X$, if $x \in C$ and $p(x) = p(y)$ then $y \in C$.

**Proposition 191.** *Let $X$ and $Y$ be sets and $p : X \twoheadrightarrow Y$ a surjective function. Let $C \subseteq X$. Then the following are equivalent:*

1. *$C$ is saturated with respect to $p$.*

2. *There exists $D \subseteq Y$ such that $C = p^{-1}(D)$*

3. *$C = p^{-1}(p(C))$.*

PROOF:
$\langle 1 \rangle 1.$ $1 \Rightarrow 3$
    $\langle 2 \rangle 1.$ ASSUME: $C$ is saturated with respect to $p$.
    $\langle 2 \rangle 2.$ $C \subseteq p^{-1}(p(C))$
        PROOF: Trivial.
    $\langle 2 \rangle 3.$ $p^{-1}(p(C)) \subseteq C$
        $\langle 3 \rangle 1.$ LET: $x \in p^{-1}(p(C))$
        $\langle 3 \rangle 2.$ $p(x) \in p(C)$

48

$\langle 3\rangle 3$. There exists $y \in C$ such that $p(x) = p(y)$

$\langle 3\rangle 4$. $x \in C$

PROOF: From $\langle 2\rangle 1$.

$\langle 1\rangle 2$. $3 \Rightarrow 2$

PROOF: Trivial.

$\langle 1\rangle 3$. $2 \Rightarrow 1$

PROOF: This follows because if $p(x) \in D$ and $p(x) = p(y)$ then $p(y) \in D$.

□

## 28   Quotient Maps

**Definition 192** (Quotient Map)**.** Let $X$ and $Y$ be topological spaces and $p : X \to Y$. Then $p$ is a *quotient map* if and only if $p$ is surjective and strongly continuous.

**Proposition 193.** *Let $X$ and $Y$ be topological spaces and $p : X \twoheadrightarrow Y$ be a surjective function. Then the following are equivalent.*

1. *$p$ is a quotient map.*

2. *$p$ is continuous and maps saturated open sets to open sets.*

3. *$p$ is continuous and maps saturated closed sets to closed sets.*

PROOF:

$\langle 1\rangle 1$. $1 \Rightarrow 2$

$\langle 2\rangle 1$. ASSUME: $p$ is a quotient map.

$\langle 2\rangle 2$. LET: $U$ be a saturated open set in $X$.

$\langle 2\rangle 3$. $p^{-1}(p(U))$ is open in $X$.

PROOF: Since $U = p^{-1}(p(U))$ be Proposition 191.

$\langle 2\rangle 4$. $p(U)$ is open in $Y$.

PROOF: From $\langle 2\rangle 1$.

$\langle 1\rangle 2$. $1 \Rightarrow 3$

PROOF: Similar.

$\langle 1\rangle 3$. $2 \Rightarrow 1$

$\langle 2\rangle 1$. ASSUME: $p$ is continuous and maps saturated open sets to open sets.

$\langle 2\rangle 2$. LET: $U \subseteq Y$

$\langle 2\rangle 3$. ASSUME: $p^{-1}(U)$ is open in $X$

$\langle 2\rangle 4$. $p^{-1}(U)$ is saturated.

PROOF: Proposition 191.

$\langle 2\rangle 5$. $U$ is open in $Y$.

$\langle 1\rangle 4$. $3 \Rightarrow 1$

PROOF: Similar.

□

**Corollary 193.1.** *Every surjective continuous open map is a quotient map.*

**Corollary 193.2.** *Every surjective continuous closed map is a quotient map.*

**Example 194.** The converses of these corollaries do not hold.

Let $A = \{(x,y) \mid x \geq 0\} \cup \{(x,y) \mid y = 0\}$. Then $\pi_1 : A \to \mathbb{R}$ is a quotient map, but not an open map or a closed map.

We prove that $\pi_1$ maps saturated open sets to open sets:

$\langle 1 \rangle 1$. LET: $\pi_1{}^{-1}(U)$ be a saturated open set in $A$
 PROVE: $U$ is open in $\mathbb{R}$
$\langle 1 \rangle 2$. LET: $x \in U$
$\langle 1 \rangle 3$. $(x,0) \in \pi_1(U)^{-1}$
$\langle 1 \rangle 4$. PICK $W$, $V$ open in $\mathbb{R}$ such that $(x,0) \subseteq W \times V \subseteq \pi_1(U)^{-1}$
$\langle 1 \rangle 5$. $x \in W \subseteq U$

It is not an open map because it maps $((-1,1) \times (1,2)) \cap A$ to $[0,1)$.
It is not a closed map because it maps $\{(x, 1/x) \mid x > 0\}$ to $(0, +\infty)$.

**Proposition 195.** *Let $p : X \twoheadrightarrow Y$ be a quotient map. Let $A \subseteq X$ be saturated with respect to $p$. Let $q : A \twoheadrightarrow p(A)$ be the restriction of $p$.*

1. *If $A$ is either open or closed in $X$ then $q$ is a quotient map.*

2. *If $p$ is either an open map or a closed map then $q$ is a quotient map.*

PROOF:
$\langle 1 \rangle 1$. LET: $p : X \twoheadrightarrow Y$ be a quotient map.
$\langle 1 \rangle 2$. LET: $A \subseteq X$ be saturated with respect to $p$.
$\langle 1 \rangle 3$. LET: $q : A \twoheadrightarrow p(A)$ be the restriction of $p$.
$\langle 1 \rangle 4$. $q$ is continuous.
 PROOF: Theorem 97.
$\langle 1 \rangle 5$. If $A$ is open in $X$ then $q$ is a quotient map.
 $\langle 2 \rangle 1$. ASSUME: $A$ is open in $X$.
 $\langle 2 \rangle 2$. $q$ maps saturated open sets to open sets.
  $\langle 3 \rangle 1$. LET: $U \subseteq A$ be saturated with respect to $q$ and open in $A$
  $\langle 3 \rangle 2$. $U$ is saturated with respect to $p$
   $\langle 4 \rangle 1$. LET: $x, y \in X$
   $\langle 4 \rangle 2$. ASSUME: $x \in U$
   $\langle 4 \rangle 3$. ASSUME: $p(x) = p(y)$
   $\langle 4 \rangle 4$. $x \in A$
    PROOF: From $\langle 3 \rangle 1$ and $\langle 4 \rangle 2$.
   $\langle 4 \rangle 5$. $y \in A$
    PROOF: From $\langle 1 \rangle 2$ and $\langle 4 \rangle 3$
   $\langle 4 \rangle 6$. $q(x) = x(y)$
    PROOF: From $\langle 1 \rangle 3$, $\langle 4 \rangle 3$, $\langle 4 \rangle 4$, $\langle 4 \rangle 5$.
   $\langle 4 \rangle 7$. $y \in U$
    PROOF: From $\langle 3 \rangle 1$, $\langle 4 \rangle 2$, $\langle 4 \rangle 6$
  $\langle 3 \rangle 3$. $U$ is open in $X$
   PROOF: Lemma 142, $\langle 2 \rangle 1$, $\langle 3 \rangle 1$.
  $\langle 3 \rangle 4$. $p(U)$ is open in $Y$
   PROOF: Proposition 193, $\langle 1 \rangle 1$, $\langle 3 \rangle 2$, $\langle 3 \rangle 3$

⟨3⟩5. $q(U)$ is open in $p(A)$
   PROOF: Since $q(U) = p(U) = p(U) \cap p(A)$.
⟨2⟩3. Q.E.D.
   PROOF: By Proposition 193.
⟨1⟩6. If $A$ is closed in $X$ then $q$ is a quotient map.
   PROOF: Similar.
⟨1⟩7. If $p$ is an open map then $q$ is a quotient map.
   ⟨2⟩1. ASSUME: $p$ is an open map
   ⟨2⟩2. $q$ maps saturated open sets to open sets.
      ⟨3⟩1. LET: $U$ be open in $A$ and saturated with respect to $q$
      ⟨3⟩2. PICK $V$ open in $X$ such that $U = A \cap V$
      ⟨3⟩3. $p(V)$ is open in $Y$
      ⟨3⟩4. $q(U) = p(V) \cap p(A)$
         ⟨4⟩1. $q(U) \subseteq p(V) \cap p(A)$
            PROOF: From ⟨3⟩2.
         ⟨4⟩2. $p(V) \cap p(A) \subseteq q(U)$
            ⟨5⟩1. LET: $y \in p(V) \cap p(A)$
            ⟨5⟩2. PICK $x \in V$ and $x' \in A$ such that $p(x) = p(x') = y$
            ⟨5⟩3. $x \in A$
               PROOF: By ⟨1⟩2.
            ⟨5⟩4. $x \in U$
               PROOF: From ⟨3⟩2
   ⟨2⟩3. Q.E.D.
      PROOF: By Proposition 193.
⟨1⟩8. If $p$ is a closed map then $q$ is a quotient map.
   PROOF: Similar.
☐

**Example 196.** This example shows we cannot remove the hypotheses on $A$ and $p$.

Define $f : [0, 1] \to [2, 3] \to [0, 2]$ by $f(x) = x$ if $x \leq 1$, $f(x) = x - 1$ if $x \geq 2$. Then $f$ is a quotient map but its restriction $f'$ to $[0, 1) \cup [2, 3]$ is not, because $f'^{-1}([1, 2])$ is open but $[1, 2]$ is not.

For a counterexample where $A$ is saturated, see Example 202.

**Proposition 197.** *Let $p : A \twoheadrightarrow C$ and $q : B \twoheadrightarrow D$ be open quotient maps. Then $p \times q : A \times B \to C \times D$ is an open quotient map.*

PROOF: From Corollary 193.1, Proposition 136 and Theorem 132. ☐

**Theorem 198.** *Let $p : X \twoheadrightarrow Y$ be a quotient map. Let $Z$ be a topological space and $f : Y \to Z$ be a function. Then*

1. *$f \circ p$ is continuous if and only if $f$ is continuous.*

2. *$f \circ p$ is a quotient map if and only if $f$ is a quotient map.*

PROOF:

$\langle 1 \rangle 1$. If $f \circ p$ is continuous then $f$ is continuous.
   PROOF: Proposition 188.
$\langle 1 \rangle 2$. If $f$ is continuous then $f \circ p$ is continuous.
   PROOF: Theorem 96.
$\langle 1 \rangle 3$. If $f \circ p$ is a quotient map then $f$ is a quotient map.
   PROOF: Proposition 189.
$\langle 1 \rangle 4$. If $f$ is a quotient map then $f \circ p$ is a quotient map.
   PROOF: From Proposition 187.
$\square$

**Proposition 199.** *Let $X$ and $Y$ be topological spaces. Let $p : X \to Y$ and $f : Y \to X$ be continuous maps such that $p \circ f = \mathrm{id}_Y$. Then $p$ is a quotient map.*

PROOF:
$\langle 1 \rangle 1$. LET: $V \subseteq Y$
$\langle 1 \rangle 2$. ASSUME: $p^{-1}(V)$ is open in $X$.
$\langle 1 \rangle 3$. $f^{-1}(p^{-1}(V))$ is open in $Y$.
   PROOF: Because $f$ is continuous.
$\langle 1 \rangle 4$. $V$ is open in $Y$.
   PROOF: Because $f^{-1}(p^{-1}(V)) = V$.
$\square$


# 29   Quotient Topology

**Definition 200** (Quotient Topology)**.** Let $X$ be a topological space, $Y$ a set and $p : X \twoheadrightarrow Y$ be a surjective function. Then the *quotient topology* on $Y$ is the unique topology on $Y$ with respect to which $p$ is a quotient map, namely $\mathcal{T} = \{U \in \mathcal{P}Y \mid p^{-1}(U)$ is open in $X\}$.

   We prove this is a topology.

PROOF:
$\langle 1 \rangle 1$. $Y \in \mathcal{T}$
   PROOF: Since $p^{-1}(Y) = X$ by surjectivity.
$\langle 1 \rangle 2$. For all $\mathcal{A} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{A} \in \mathcal{T}$
   PROOF: Since $p^{-1}(\bigcup \mathcal{A}) = \bigcup_{U \in \mathcal{A}} p^{-1}(U)$
$\langle 1 \rangle 3$. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$
   PROOF: Since $p^{-1}(U \cap V) = p^{-1}(U) \cap p^{-1}(V)$.
$\square$

**Definition 201** (Quotient Space)**.** Let $X$ be a topological space and $\sim$ an equivalence relation on $X$. Let $p : X \twoheadrightarrow X/\sim$ be the canonical surjection. Then $X/\sim$ under the quotient topology is called a *quotient space*, *identification space* or *decomposition space* of $X$.

   Here is a counterexample showing we cannot remove all the hypotheses of Proposition 195 except that $A$ is saturated.

**Example 202.** Let $X = (0, 1/2] \cup \{1\} \cup \{1 + 1/n : n \geq 2\}$ as a subspace of $\mathbb{R}$. Define $R$ to be the equivalence relation on $X$ where $xRy$ iff $(x = y$ or $|x-y| = 1)$, so we identify $1/n$ with $1 + 1/n$ for all $n \geq 2$. Let $Y$ be the resulting quotient space $X/R$ in the quotient topology and $p : X \twoheadrightarrow Y$ the canonical surjection.

Let $A = \{1\} \cup (0, 1/2] \setminus \{1/n : n \geq 2\} \subseteq X$. Then $A$ is saturated under $p$ but the restriction $q$ of $p$ to $A$ is not a quotient map because it maps the saturated open set $\{1\}$ to $\{1\}$ which is not open in $p(A)$.

**Proposition 203.** *Let $X$, $Y$ and $Z$ be topological spaces. Let $f : X \to Y$ and $g : Y \to Z$. If $f$ and $g$ are quotient maps then so is $g \circ f$.*

PROOF: From Proposition 187. $\square$

**Example 204.** The product of two quotient maps is not necessarily a quotient map.

Let $X = \mathbb{R}$ and $X^*$ the quotient space formed by identifying all positive integers. Let $p : X \twoheadrightarrow X^*$ be the canonical surjection.

We prove $p \times \mathrm{id}_{\mathbb{Q}} : X \times \mathbb{Q} \to X^* \times \mathbb{Q}$ is not a quotient map.

PROOF:
⟨1⟩1. For $n \geq 1$,
        LET: $c_n = \sqrt{2}/n$
⟨1⟩2. For $n \geq 1$,
        LET: $U_n = \{(x, y) \in X \times \mathbb{Q} \mid n - 1/4 < x < n + 1/4, (y + n > x + c_n$ and $y + n > -x + c_n)$ or $(y + n < x + c_n$ and $y + n < -x + c_n)\}$
⟨1⟩3. For $n \geq 1$, we have $U_n$ is open in $X \times \mathbb{Q}$
⟨1⟩4. For $n \geq 1$, we have $\{n\} \times \mathbb{Q} \subseteq U_n$
⟨1⟩5. LET: $U = \bigcup_{n=1}^{\infty} U_n$
⟨1⟩6. $U$ is open in $X \times \mathbb{Q}$
⟨1⟩7. $U$ is saturated with respect to $p \times \mathrm{id}_{\mathbb{Q}}$
⟨1⟩8. LET: $U' = (p \times \mathrm{id}_{\mathbb{Q}})(U)$
⟨1⟩9. ASSUME: for a contradiction $U'$ is open in $X^* \times \mathbb{Q}$
⟨1⟩10. $(1, 0) \in U'$
⟨1⟩11. PICK a neighbourhood $W$ of $1$ in $X^*$ and $\delta > 0$ such that $W \times (-\delta, \delta) \subseteq U'$
⟨1⟩12. $p^{-1}(W) \times (-\delta, \delta) \subseteq U$
⟨1⟩13. PICK $n$ such that $c_n < \delta$
⟨1⟩14. $n \in p^{-1}(W)$
⟨1⟩15. PICK $\epsilon > 0$ such that $\epsilon < \delta - c_n$ and $\epsilon < 1/4$ and $(n - \epsilon, n + \epsilon) \subseteq p^{-1}(W)$
⟨1⟩16. $(n - \epsilon, n + \epsilon) \times (-\delta, \delta) \subseteq U$
⟨1⟩17. PICK a rational $y$ such that $c_n - \epsilon/2 < y < c_n + \epsilon/2$
⟨1⟩18. $(n + \epsilon/2, y) \notin U$
⟨1⟩19. Q.E.D.
    PROOF: This contradicts ⟨1⟩16.
$\square$

**Proposition 205.** *Let $X$ be a topological space and $\sim$ an equivalence relation on $X$. Then $X/\sim$ is $T_1$ if and only if every equivalence class is closed in $X$.*

PROOF: Immediate from definitions. $\square$

## 30  Retractions

**Definition 206** (Retraction)**.** Let $X$ be a topological space and $A \subseteq X$. A *retraction* of $X$ onto $A$ is a continuous map $r : X \to A$ such that, for all $a \in A$, we have $r(a) = a$.

**Proposition 207.** *Every retraction is a quotient map.*

PROOF: Proposition 199 with $f$ the inclusion $A \hookrightarrow X$. □

## 31  Homogeneous Spaces

**Definition 208** (Homogeneous)**.** A topological space $X$ is *homogeneous* if and only if, for any points $a, b \in X$, there exists a homeomorphism $\phi : X \cong X$ such that $\phi(a) = b$.

## 32  Regular Spaces

**Definition 209** (Regular Space)**.** A topological space $X$ is *regular* if and only if, for any closed set $A$ and point $a \notin A$, there exist disjoint open sets $U$, $V$ such that $A \subseteq U$ and $a \in V$.

## 33  Connected Spaces

**Definition 210** (Separation)**.** A *separation* of a topological space $X$ is a pair of disjoint open sets $U$, $V$ such that $U \cup V = \emptyset$.

**Definition 211** (Connected)**.** A topological space is *connected* if and only if it has no separation; otherwise it is *disconnected*.

**Proposition 212.** *A topological space $X$ is connected if and only if the only sets that are both open and closed are $X$ and $\emptyset$.*

Immediate from defintions.

**Lemma 213.** *If $Y$ is a subspace of $X$, a separation of $Y$ is a pair of disjoint nonempty sets $A$ and $B$ whose union is $Y$, neither of which contains a limit point of the other.*

PROOF:
⟨1⟩1. LET: $A, B \subseteq Y$
⟨1⟩2. If $A$ and $B$ form a separation of $Y$ then $A$ and $B$ are disjoint and nonempty, $A \cup B = Y$, and neither of $A$ and $B$ contains a limit point of the other.
  ⟨2⟩1. ASSUME: $A$ and $B$ form a separation of $Y$
  ⟨2⟩2. $A$ and $B$ are disjoint and nonempty and $A \cup B = Y$
    PROOF: From ⟨2⟩1 and the definition of separation.
  ⟨2⟩3. $A$ does not contain a limit point of $B$

54

⟨3⟩1. ASSUME: for a contradiction $l \in A$ and $l$ is a limit point of $B$ in $X$.

⟨3⟩2. $l$ is a limit point of $B$ in $Y$

   PROOF: Proposition 152.

⟨3⟩3. $l \in B$

  ⟨4⟩1. $B$ is closed in $Y$

    PROOF: Since $A$ is open in $Y$ and $B = Y \setminus A$ from ⟨2⟩1.

  ⟨4⟩2. Q.E.D.

    PROOF: Corollary 58.1.

⟨3⟩4. Q.E.D.

   PROOF: This contradicts the fact that $A \cap B = \emptyset$ (⟨2⟩1).

⟨2⟩4. $B$ does not contain a limit point of $A$

  PROOF: Similar.

⟨1⟩3. If $A$ and $B$ are disjoint and nonempty, $A \cup B = Y$, and neither of $A$ and $B$ contains a limit point of the other, then $A$ and $B$ form a separation of $Y$.

⟨2⟩1. ASSUME: $A$ and $B$ are disjoint and nonempty, $A \cup B = Y$, and neither of $A$ and $B$ contains a limit point of the other.

⟨2⟩2. $A$ is open in $Y$

  ⟨3⟩1. $B$ is closed in $Y$

    ⟨4⟩1. LET: $l$ be a limit point of $B$ in $Y$

    ⟨4⟩2. $l$ is a limit point of $B$ in $X$

      PROOF: Proposition 152.

    ⟨4⟩3. $l \notin A$

      PROOF: By ⟨2⟩1

    ⟨4⟩4. $l \in B$

      PROOF: By ⟨2⟩1 since $A \cup B = Y$

    ⟨4⟩5. Q.E.D.

      PROOF: Corollary 58.1.

  ⟨3⟩2. Q.E.D.

    PROOF: Since $A = Y \setminus B$.

⟨2⟩3. $B$ is open in $Y$

  PROOF: Similar.

□

**Example 214.** Every set under the indiscrete topology is connected.

**Example 215.** The discrete topology on a set $X$ is connected if and only if $|X| \leq 1$.

**Example 216.** The finite complement topology on a set $X$ is connected if and only if either $|X| \leq 1$ or $X$ is infinite.

**Example 217.** The countable complement topology on a set $X$ is connected if and only if either $|X| \leq 1$ or $X$ is uncountable.

**Example 218.** The rationals $\mathbb{Q}$ are disconnected. For any irrational $a$, the sets $(-\infty, a) \cap \mathbb{Q}$ and $(a, +\infty) \cap \mathbb{Q}$ form a separation of $\mathbb{Q}$.

**Lemma 219.** *Let $X$ be a topological space. If $C$ and $D$ form a separation of $X$, and $Y$ is a connected subspace of $X$, then either $Y \subseteq C$ or $Y \subseteq D$.*

PROOF: Otherwise $Y \cap C$ and $Y \cap D$ would form a separation of $Y$. $\square$

**Theorem 220.** *The union of a set of connected subspaces of a space $X$ that have a point in common is connected.*

PROOF:
$\langle 1 \rangle 1.$ LET: $\mathcal{A}$ be a set of connected subspaces of the space $X$ that have the point $a$ in common.
$\langle 1 \rangle 2.$ ASSUME: for a contradiction $C$ and $D$ form a separation of $\bigcup \mathcal{A}$
$\langle 1 \rangle 3.$ ASSUME: without loss of generality $a \in C$
$\langle 1 \rangle 4.$ For all $A \in \mathcal{A}$ we have $A \subseteq C$
   PROOF: Lemma 219.
$\langle 1 \rangle 5.$ $D = \emptyset$
$\langle 1 \rangle 6.$ Q.E.D.
   PROOF: This contradicts $\langle 1 \rangle 2$.
$\square$

**Theorem 221.** *Let $X$ be a topological space and $A$ a connected subspace of $X$. If $A \subseteq B \subseteq \overline{A}$ then $B$ is connected.*

PROOF:
$\langle 1 \rangle 1.$ ASSUME: for a contradiction $C$ and $D$ form a separation of $B$.
$\langle 1 \rangle 2.$ ASSUME: without loss of generality $A \subseteq C$
   PROOF: Lemma 219.
$\langle 1 \rangle 3.$ $B \subseteq C$
   $\langle 2 \rangle 1.$ LET: $x \in B$
   $\langle 2 \rangle 2.$ $x \in \overline{A}$
   $\langle 2 \rangle 3.$ Either $x \in A$ or $x$ is a limit point of $A$.
      PROOF: Theorem 58.
   $\langle 2 \rangle 4.$ Either $x \in A$ or $x$ is a limit point of $C$.
      PROOF: Lemma 60, $\langle 1 \rangle 2$.
   $\langle 2 \rangle 5.$ $x \in C$
      PROOF: Lemma 213.
$\langle 1 \rangle 4.$ $D = \emptyset$
$\langle 1 \rangle 5.$ Q.E.D.
   PROOF: This contradicts $\langle 1 \rangle 1$.
$\square$

**Theorem 222.** *The image of a connected space under a continuous map is connected.*

PROOF:
$\langle 1 \rangle 1.$ LET: $f : X \twoheadrightarrow Y$ be a surjective continuous map where $X$ is connected.
$\langle 1 \rangle 2.$ ASSUME: for a contradiction $C$ and $D$ form a separation of $Y$.
$\langle 1 \rangle 3.$ $f^{-1}(C)$ and $f^{-1}(D)$ form a separation of $X$.

⬜

**Theorem 223.** *The product of a family of connected spaces is connected.*

PROOF:

⟨1⟩1. The product of two connected spaces is connected.

   ⟨2⟩1. LET: $X$ and $Y$ be connected spaces.

   ⟨2⟩2. PICK $a \in X$ and $b \in Y$

   PROOF: We may assume $X$ and $Y$ are nonempty since otherwise $X \times Y = \emptyset$ which is connected.

   ⟨2⟩3. $X \times \{b\}$ is connected.

   PROOF: It is homeomorphic to $X$.

   ⟨2⟩4. For all $x \in X$ we have $\{x\} \times Y$ is connected.

   PROOF: It is homeomorphic to $Y$.

   ⟨2⟩5. For any $x \in X$

      LET: $T_x = (X \times \{b\}) \cup (\{x\} \times Y)$

   ⟨2⟩6. For all $x \in X$, $T_x$ is connected.

   PROOF: Theorem 220 since $(x,b) \in (X \times \{b\}) \cap (\{x\} \times Y)$.

   ⟨2⟩7. $X \times Y$ is connected.

   PROOF: Theorem 220 since $X \times Y = \bigcup_{x \in X} T_x$ and $(a,b)$ is a point in every $T_x$.

⟨1⟩2. The product of a finite family of connected spaces is connected.

   PROOF: From ⟨1⟩1 by induction.

⟨1⟩3. The product of any family of connected spaces is connected.

   ⟨2⟩1. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of connected spaces.

   ⟨2⟩2. LET: $X = \prod_{\alpha \in J} X_\alpha$

   ⟨2⟩3. PICK $a \in X$

   PROOF: We may assume $X \neq \emptyset$ as the empty space is connected.

   ⟨2⟩4. For every finite subset $K$ of $J$,

      LET: $X_K = \{x \in X \mid \forall \alpha \in J \setminus K . x_\alpha = a_\alpha\}$

   ⟨2⟩5. For every finite $K \subseteq J$, we have $X_K$ is connected.

   PROOF: From ⟨1⟩2 since $X_K \cong \prod_{\alpha \in K} X_K$.

   ⟨2⟩6. LET: $Y = \bigcup_K X_K$

   ⟨2⟩7. $Y$ is connected

   PROOF: Theorem 220 since $a$ is a common point.

   ⟨2⟩8. $X = \overline{Y}$

      ⟨3⟩1. LET: $x \in X$

      ⟨3⟩2. LET: $U = \prod_{\alpha \in J} U_\alpha$ be a basic neighbourhood of $x$ where $U_\alpha = X_\alpha$ for all $\alpha$ except $\alpha \in K$ for some finite $K \subseteq J$

      ⟨3⟩3. LET: $y \in X$ be the point with $y_\alpha = x_\alpha$ for $\alpha \in K$ and $y_\alpha = a_\alpha$ for all other $\alpha$

      ⟨3⟩4. $y \in U \cap X_K$

      ⟨3⟩5. $y \in U \cap Y$

   ⟨2⟩9. $X$ is connected.

   PROOF: Theorem 221.

⬜

**Example 224.** The set $\mathbb{R}^\omega$ is disconnected under the box topology. The set of bounded sequences and the set of unbounded sequences form a separation.

**Proposition 225.** *Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on the same set $X$. If $\mathcal{T} \subseteq \mathcal{T}'$ and $(X, \mathcal{T}')$ is connected then $(X, \mathcal{T})$ is connected.*

PROOF: If $U$ and $V$ form a separation of $(X, \mathcal{T})$ then they form a separation of $(X, \mathcal{T}')$. $\square$

**Proposition 226.** *Let $X$ be a topological space and $(A_n)$ a sequence of connected subspaces of $X$. If $A_n \cap A_{n+1} \neq \emptyset$ for all $n$ then $\bigcup_n A_n$ is connected.*

PROOF:
$\langle 1 \rangle 1$. ASSUME: for a contradiction $C$ and $D$ form a separation of $\bigcup_n A_n$
$\langle 1 \rangle 2$. ASSUME: without loss of generality $A_0 \subseteq C$
   PROOF: Lemma 219.
$\langle 1 \rangle 3$. For all $n$ we gave $A_n \subseteq C$
   PROOF:
   $\langle 2 \rangle 1$. ASSUME: $A_n \subseteq C$
   $\langle 2 \rangle 2$. PICK $x \in A_n \cap A_{n+1}$
   $\langle 2 \rangle 3$. $x \in C$
   $\langle 2 \rangle 4$. $A_{n+1} \subseteq C$
      PROOF: Lemma 219.
   $\langle 2 \rangle 5$. Q.E.D.
      PROOF: The result follows by induction.
$\langle 1 \rangle 4$. $D = \emptyset$
$\langle 1 \rangle 5$. Q.E.D.
   PROOF: This contradicts $\langle 1 \rangle 1$.
$\square$

**Proposition 227.** *Let $X$ be a topological space. Let $A, C \subseteq X$. If $C$ is connected and intersects both $A$ and $X \setminus A$ then $C$ intersects $\partial A$.*

PROOF: Otherwise $C \cap A^\circ$ and $C \setminus \overline{A}$ would form a separation of $C$. $\square$

**Example 228.** The space $\mathbb{R}_l$ is disconnected. For any real $x$, the sets $(-\infty, x)$ and $[x, +\infty)$ form a separation.

**Proposition 229.** *Let $X$ and $Y$ be connected spaces. Let $A$ be a proper subset of $X$ and $B$ a proper subset of $Y$. Then $(X \times Y) \setminus (A \times B)$ is connected.*

PROOF:
$\langle 1 \rangle 1$. PICK $a \in X \setminus A$ and $b \in Y \setminus B$
$\langle 1 \rangle 2$. For $x \in X \setminus A$ we have $(X \times \{b\}) \cup (\{x\} \times Y)$ is connected.
   PROOF: Theorem 220 since $(x, b)$ is a common point.
$\langle 1 \rangle 3$. For $y \in Y \setminus B$ we have $(X \times \{y\}) \cup (\{a\} \times Y)$ is connected.
   PROOF: Theorem 220 since $(a, y)$ is a common point.
$\langle 1 \rangle 4$. $(X \times Y) \setminus (A \times B)$ is connected.
   PROOF: Theorem 220 since it is the union of the sets in $\langle 1 \rangle 2$ and $\langle 1 \rangle 3$ with $(a, b)$ as a common point.

⬜

**Proposition 230.** *Let $p : X \twoheadrightarrow Y$ be a quotient map. If $Y$ is connected and $p^{-1}(y)$ is connected for all $y \in Y$, then $X$ is connected.*

PROOF:
$\langle 1 \rangle$1. ASSUME: for a contradiction $C$ and $D$ form a separation of $X$.
$\langle 1 \rangle$2. $C$ is saturated.
$\quad \langle 2 \rangle$1. LET: $x \in C$, $y \in X$ with $p(x) = p(y) = a$, say
$\quad \langle 2 \rangle$2. $y \notin D$
$\quad\quad$ PROOF: Otherwise $p^{-1}(a) \cap C$ and $p^{-1}(a) \cap D$ form a separation of $p^{-1}(a)$.
$\quad \langle 2 \rangle$3. $y \in C$
$\langle 1 \rangle$3. $D$ is saturated.
$\quad$ PROOF: Similar.
$\langle 1 \rangle$4. $p(C)$ and $p(D)$ form a separation of $Y$.
⬜

**Proposition 231.** *Let $X$ be a connected space and $Y$ a connected subspace of $X$. Suppose $A$ and $B$ form a separation of $X \setminus Y$. Then $Y \cup A$ and $Y \cup B$ are both connected.*

PROOF:
$\langle 1 \rangle$1. $Y \cup A$ is connected.
$\quad \langle 2 \rangle$1. ASSUME: for a contradiction $C$ and $D$ form a separation of $Y \cup A$
$\quad \langle 2 \rangle$2. ASSUME: without loss of generality $Y \subseteq C$
$\quad \langle 2 \rangle$3. PICK open sets $A_1$, $B_1$, $C_1$, $D_1$ in $X$ with
$$A = A_1 \setminus Y$$
$$B = B_1 \setminus Y$$
$$C = C_1 \cap (Y \cup A)$$
$$D = D_1 \cap (Y \cup A)$$
$\quad \langle 2 \rangle$4. $B_1 \cup C_1$ and $A_1 \cap D_1$ form a separation of $X$
$\langle 1 \rangle$2. $Y \cup B$ is connected.
$\quad$ PROOF: Similar.
⬜

**Theorem 232.** *Let $L$ be a linearly ordered set under the order topology. Then $L$ is connected if and only if $L$ is a linear continuum.*

PROOF:
$\langle 1 \rangle$1. If $L$ is a linear continuum then $L$ is connected.
$\quad \langle 2 \rangle$1. LET: $L$ be a linear continuum under the order topology.
$\quad \langle 2 \rangle$2. ASSUME: for a contradiction $C$ and $D$ form a separation of $L$.
$\quad \langle 2 \rangle$3. PICK $a \in C$ and $b \in D$.
$\quad \langle 2 \rangle$4. ASSUME: without loss of generality $a < b$.
$\quad \langle 2 \rangle$5. LET: $S = \{x \in L \mid a < x \text{ and } [a, x) \subseteq C\}$
$\quad \langle 2 \rangle$6. $S$ is nonempty.
$\quad\quad$ PROOF: Since $a \in C$ and $C$ is open.

$\langle 2 \rangle$7. $S$ is bounded above by $b$.
  PROOF: Since $b \notin C$.
$\langle 2 \rangle$8. LET: $s = \sup S$
$\langle 2 \rangle$9. $s \in S$
  $\langle 3 \rangle$1. LET: $y \in [a, s)$
      PROVE: $y \in C$
  $\langle 3 \rangle$2. PICK $z$ with $y < z \in S$
    PROOF: By minimality of $s$.
  $\langle 3 \rangle$3. $y \in [a, z) \subseteq C$
$\langle 2 \rangle$10. CASE: $s \in C$
  $\langle 3 \rangle$1. PICK $x$ such that $s < x$ and $[s, x) \subseteq C$
    PROOF: Since $C$ is open and $s$ is not greatest in $L$ because $s < b$.
  $\langle 3 \rangle$2. $x \in S$
    PROOF: Since $[a, x) = [a, s) \cup [s, x) \subseteq C$.
  $\langle 3 \rangle$3. Q.E.D.
    PROOF: This contradicts the fact that $s$ is an upper bound for $S$.
$\langle 2 \rangle$11. CASE: $s \in D$
  $\langle 3 \rangle$1. PICK $x < s$ such that $(x, s] \subseteq D$
  $\langle 3 \rangle$2. PICK $y$ with $x < y < s$
    PROOF: Since $L$ is dense.
  $\langle 3 \rangle$3. $y \in C$
    PROOF: From $\langle 2 \rangle$9.
  $\langle 3 \rangle$4. $y \in D$
    PROOF: From $\langle 3 \rangle$1.
  $\langle 3 \rangle$5. Q.E.D.
  $\langle 3 \rangle$6. LET: $L$ be a linear continuum under the order topology.
  $\langle 3 \rangle$7. ASSUME: for a contradiction $C$ and $D$ form a separation of $L$.
  $\langle 3 \rangle$8. PICK $a \in C$ and $b \in D$.
  $\langle 3 \rangle$9. ASSUME: without loss of generality $a < b$.
  $\langle 3 \rangle$10. LET: $S = \{x \in L \mid a < x \text{ and } [a, x) \subseteq C\}$
  $\langle 3 \rangle$11. $S$ is nonempty.
    PROOF: Since $a \in C$ and $C$ is open.
  $\langle 3 \rangle$12. $S$ is bounded above by $b$.
    PROOF: Since $b \notin C$.
  $\langle 3 \rangle$13. LET: $s = \sup S$
  $\langle 3 \rangle$14. $s \in S$
    $\langle 4 \rangle$1. LET: $y \in [a, s)$
        PROVE: $y \in C$
    $\langle 4 \rangle$2. PICK $z$ with $y < z \in S$
      PROOF: By minimality of $s$.
    $\langle 4 \rangle$3. $y \in [a, z) \subseteq C$
  $\langle 3 \rangle$15. CASE: $s \in C$
    $\langle 4 \rangle$1. PICK $x$ such that $s < x$ and $[s, x) \subseteq C$
      PROOF: Since $C$ is open and $s$ is not greatest in $L$ because $s < b$.
    $\langle 4 \rangle$2. $x \in S$
      PROOF: Since $[a, x) = [a, s) \cup [s, x) \subseteq C$.

60

$\langle 4 \rangle 3$. Q.E.D.
  Proof: This contradicts the fact that $s$ is an upper bound for $S$.
$\langle 3 \rangle 16$. Case: $s \in D$
  $\langle 4 \rangle 1$. Pick $x < s$ such that $(x, s] \subseteq D$
  $\langle 4 \rangle 2$. Pick $y$ with $x < y < s$
    Proof: Since $L$ is dense.
  $\langle 4 \rangle 3$. $y \in C$
    Proof: From $\langle 2 \rangle 9$.
  $\langle 4 \rangle 4$. $y \in D$
    Proof: From $\langle 3 \rangle 1$.
  $\langle 4 \rangle 5$. Q.E.D.
    Proof: This contradicts $\langle 2 \rangle 2$.
$\langle 1 \rangle 2$. If $L$ is connected then $L$ is a linear continuum.
  $\langle 2 \rangle 1$. Assume: $L$ is connected.
  $\langle 2 \rangle 2$. Every nonempty subset of $L$ that is bounded above has a supremum.
    $\langle 3 \rangle 1$. Let: $X$ be a nonempty subset of $L$ bounded above by $b$.
    $\langle 3 \rangle 2$. Assume: for a contradiction $X$ has no supremum.
    $\langle 3 \rangle 3$. Let: $U$ be the set of upper bounds of $X$,
    $\langle 3 \rangle 4$. $U$ is nonempty.
    Proof: Since $b \in U$.
    $\langle 3 \rangle 5$. $U$ is open.
      $\langle 4 \rangle 1$. Let: $x \in U$
      $\langle 4 \rangle 2$. Pick an upper bound $y$ for $X$ such that $y < x$
      $\langle 4 \rangle 3$. Either $x$ is greatest in $L$ and $(y, x] \subseteq U$, or there exists $z > x$ such
              that $(y, z) \subseteq U$
    $\langle 3 \rangle 6$. Let: $V$ be the set of lower bounds of $U$.
    $\langle 3 \rangle 7$. $V$ is nonempty.
    Proof: Since $X \subseteq V$
    $\langle 3 \rangle 8$. $V$ is open.
      $\langle 4 \rangle 1$. Let: $x \in V$
      $\langle 4 \rangle 2$. Pick $y \in X$ with $x < y$
        Proof: $x$ cannot be an upper bound for $X$, because it would be the
        supremum of $X$.
      $\langle 4 \rangle 3$. Either $x$ least in $L$ and $[x, y) \subseteq V$, or there exists $z < x$ such that
              $(z, y) \subseteq V$
    $\langle 3 \rangle 9$. $L = U \cup V$
      $\langle 4 \rangle 1$. Let: $x \in L \setminus U$
      $\langle 4 \rangle 2$. Pick $y \in X$ such that $x < y$
      $\langle 4 \rangle 3$. For all $u \in U$ we have $x < y \leq u$
      $\langle 4 \rangle 4$. $x \in V$
    $\langle 3 \rangle 10$. $U \cap V = \emptyset$
    Proof: Any element of $U \cap V$ would be a supremum of $X$.
    $\langle 3 \rangle 11$. $U$ and $V$ form a separation of $L$.
    $\langle 3 \rangle 12$. Q.E.D.
    Proof: This contradicts $\langle 2 \rangle 1$.
  $\langle 2 \rangle 3$. $L$ is dense.

61

$\langle 3 \rangle$1. LET: $x, y \in L$ with $x < y$
$\langle 3 \rangle$2. There exists $z \in L$ such that $x < z < y$
   PROOF: Otherwise $(-\infty, y)$ and $(x, +\infty)$ would form a separation of $L$.

□

**Corollary 232.1.** *The real line $\mathbb{R}$ is connected.*

**Corollary 232.2.** *Every interval in $\mathbb{R}$ is connected.*

**Corollary 232.3.** *The ordered square is connected.*

**Theorem 233** (Intermediate Value Theorem)**.** *Let $X$ be a connected space. Let $Y$ be a linearly ordered set under the order topology. Let $f : X \to Y$ be continuous. Let $a, b \in X$ and $r \in Y$. Suppose $f(a) < r < f(b)$. Then there exists $c \in X$ such that $f(c) = r$.*

PROOF: Otherwise $f^{-1}((-\infty, r))$ and $f^{-1}((r, +\infty))$ would form a separation of $X$. □

**Proposition 234.** *Every function $f : [0, 1] \to [0, 1]$ has a fixed point.*

PROOF:
$\langle 1 \rangle$1. LET: $g : [0, 1] \to [-1, 1]$ be the function $g(x) = f(x) - x$
   PROVE: there exists $x \in [0, 1]$ such that $g(x) = 0$
$\langle 1 \rangle$2. ASSUME: without loss of generality $g(0) \neq 0$ and $g(1) \neq 0$
$\langle 1 \rangle$3. $g(0) > 0$
$\langle 1 \rangle$4. $g(1) < 0$
$\langle 1 \rangle$5. There exists $x \in (0, 1)$ such that $g(x) = 0$
  PROOF: By the Intermediate Value Theorem.

**Proposition 235.** *Give $\mathbb{R}^\omega$ the box topology. Let $x, y \in \mathbb{R}^\omega$. Then $x$ and $y$ lie in the same comoponent if and only if $x - y$ is eventually zero, i.e. there exists $N$ such that, for all $n \geq N$, we have $x_n = y_n$.*

PROOF:
$\langle 1 \rangle$1. The component containing $0$ is the set of sequences that are eventually zero.
  $\langle 2 \rangle$1. LET: $B$ be the set of sequences that are eventually zero.
  $\langle 2 \rangle$2. $B$ is path-connected.
    $\langle 3 \rangle$1. LET: $x, y \in B$
    $\langle 3 \rangle$2. PICK $N$ such that, for all $n \geq N$, we have $x_n = y_n = 0$
    $\langle 3 \rangle$3. LET: $p : [0, 1] \to \mathbb{R}^\omega$, $p(t) = (1 - t)x + ty$
      PROVE: $p$ is continuous.
    $\langle 3 \rangle$4. LET: $t \in [0, 1]$ and $\prod_j U_j$ be a basic open neighbourhood of $p(t)$, where each $U_j$ is open in $\mathbb{R}$
    $\langle 3 \rangle$5. PICK $\delta$ such that, for all $n < N$ and all $s \in [0, 1]$, if $|s - t| < \delta$ then $p(s)_n \in U_n$
    $\langle 3 \rangle$6. For all $s \in [0, 1]$, if $|s - t| < \delta$ then $p(s) \in \prod_j U_j$
  $\langle 2 \rangle$3. $B$ is connected.

PROOF: Proposition 241.
$\langle 2 \rangle$4. If $C$ is connected and $B \subseteq C$ then $B = C$.
  $\langle 3 \rangle$1. ASSUME: $C$ is connected and $B \subseteq C$
  $\langle 3 \rangle$2. ASSUME: for a contradiction $x \in C \setminus B$
  $\langle 3 \rangle$3. For $n \geq 1$,
      LET: $c_n = 1$ if $x_n = 0$, $c_n = n/x_n$ otherwise
  $\langle 3 \rangle$4. LET: $h : \mathbb{R}^\omega \to \mathbb{R}^\omega$ be the function $h(x) = (c_n x_n)_{n \geq 1}$
  $\langle 3 \rangle$5. $h$ is a homeomorphism of $\mathbb{R}^\omega$ with itself.
  $\langle 3 \rangle$6. $h(x)$ is unbounded.
      PROOF: For any $b > 0$, pick $N > b$ such that $x_N \neq 0$. Then $h(x)_N > b$.
  $\langle 3 \rangle$7. $h^{-1}(\{\text{bounded sequences}\}) \cap C$ and $h^{-1}(\{\text{unbounded sequences}\}) \cap C$
      form a separation of $C$
  $\langle 3 \rangle$8. Q.E.D.
      PROOF: This contradicts $\langle 3 \rangle$1.
$\langle 1 \rangle$2. Q.E.D.
  PROOF: Since $\lambda x . x - y$ is a homeomorphism of $\mathbb{R}^\omega$ with itself.
▯

# 34 Totally Disconnected Spaces

**Definition 236** (Totally Disconnected)**.** A topological space $X$ is *totally disconnected* if and only if the only connected subspaces are the singletons.

**Example 237.** Every discrete space is totally disconnected.

**Example 238.** The rationals $\mathbb{Q}$ are totally disconnected.

# 35 Paths and Path Connectedness

**Definition 239** (Path)**.** Let $X$ be a topological space and $a, b \in X$. A *path* from $a$ to $b$ is a continuous function $p : [0, 1] \to X$ such that $p(0) = a$ and $p(1) = b$.

**Definition 240** (Path Connected)**.** A topological space is *path connected* if and only if there exists a path between any two points.

**Proposition 241.** *Every path connected space is connected.*

PROOF:
$\langle 1 \rangle$1. LET: $X$ be a path connected space.
$\langle 1 \rangle$2. ASSUME: for a contradiction $C$ and $D$ form a separation of $X$.
$\langle 1 \rangle$3. PICK $a \in C$ and $b \in D$.
$\langle 1 \rangle$4. PICK a path $p : [0, 1] \to X$ from $a$ to $b$.
$\langle 1 \rangle$5. $p^{-1}(C)$ and $p^{-1}(D)$ form a separation of $[0, 1]$.
$\langle 1 \rangle$6. Q.E.D.
  PROOF: This contradicts Corollary 232.2.

☐
An example that shows the converse does not hold:

**Example 242.** The ordered square is not path connected.

PROOF:
⟨1⟩1. ASSUME: for a contradiction $p : [0,1] \to I_o^2$ is a path from $(0,0)$ to $(1,1)$.
⟨1⟩2. $p$ is surjective.
  PROOF: By the Intermediate Value Theorem.
⟨1⟩3. For $x \in [0,1]$, PICK a rational $q_x \in p^{-1}((x,0),(x,1))$
  PROOF: Since $p^{-1}((x,0),(x,1))$ is open and nonempty by ⟨1⟩2.
⟨1⟩4. For $x, y \in [0,1]$, if $x \neq y$ then $q_x \neq q_y$
  PROOF: We have $p(q_x) \neq p(q_y)$ because $((x,0),(x,1))$ and $((y,0),(y,1))$ are disjoint.
⟨1⟩5. $\{q_x \mid x \in [0,1]\}$ is an uncountable set of rationals.
⟨1⟩6. Q.E.D.
  PROOF: This contradicts the fact that the rationals are countable.
☐

**Proposition 243.** *The continuous image of a path connected space is path connected.*

PROOF:
⟨1⟩1. LET: $X$ be a path connected space, $Y$ a topological space, and $f : X \twoheadrightarrow Y$
    be continuous and surjective.
⟨1⟩2. LET: $a, b \in Y$
⟨1⟩3. PICK $c, d \in X$ with $f(c) = a$ and $f(d) = b$
⟨1⟩4. PICK a path $p : [0,1] \to X$ from $c$ to $d$.
⟨1⟩5. $f \circ p$ is a path from $a$ to $b$ in $Y$.
☐

**Proposition 244** (AC)**.** *The product of a family of path-connected spaces is path-connected.*

PROOF:
⟨1⟩1. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of path-connected spaces.
⟨1⟩2. LET: $a, b \in \prod_{\alpha \in J} X_\alpha$
⟨1⟩3. For $\alpha \in J$, PICK a path $p_\alpha : [0,1] \to X_\alpha$ from $a_\alpha$ to $b_\alpha$
  PROOF: Using the Axiom of Choice.
⟨1⟩4. Define $p : [0.1] \to \prod_{\alpha \in J} X_\alpha$ by $p(t)_\alpha = p_\alpha(t)$
⟨1⟩5. $p$ is a path from $a$ to $b$.
  PROOF: Theorem 132.
☐

**Proposition 245.** *The continuous image of a path-connected space is path-connected.*

PROOF:
⟨1⟩1. LET: $f : X \twoheadrightarrow Y$ be continuous and surjective where $X$ is path-connected.

$\langle 1 \rangle 2$. LET: $a, b \in Y$
$\langle 1 \rangle 3$. PICK $a', b' \in X$ with $f(a') = a$ and $f(b') = b$.
$\langle 1 \rangle 4$. PICK a path $p : [0, 1] \to X$ from $a'$ to $b'$.
$\langle 1 \rangle 5$. $f \circ p$ is a path from $a$ to $b$.
□

**Proposition 246.** *Let $X$ be a topological space. The union of a set of path-connected subspaces of $X$ that have a point in common is path-connected.*

PROOF:
$\langle 1 \rangle 1$. LET: $\mathcal{A}$ be a set of path-connected subspaces of $X$ with the point $a$ in common.
$\langle 1 \rangle 2$. LET: $b, c \in \bigcup \mathcal{A}$
$\langle 1 \rangle 3$. PICK $B, C \in \mathcal{A}$ with $b \in B$ and $c \in C$.
$\langle 1 \rangle 4$. PICK a path $p$ in $B$ from $b$ to $a$.
$\langle 1 \rangle 5$. PICK a path $q$ in $C$ from $a$ to $c$.
$\langle 1 \rangle 6$. The concatenation of $p$ and $q$ is a path from $b$ to $c$ in $\bigcup \mathcal{A}$.
□

**Proposition 247.** *Let $A \subseteq \mathbb{R}^2$ be countable. Then $\mathbb{R}^2 \setminus A$ is path-connected.*

PROOF:
$\langle 1 \rangle 1$. LET: $a, b \in \mathbb{R}^2 \setminus A$
$\langle 1 \rangle 2$. PICK a line $l$ in $\mathbb{R}^2$ with $a$ on one side and $b$ on the other.
$\langle 1 \rangle 3$. For every point $x$ on $l$,
    LET: $p_x$ be the path in $\mathbb{R}^2$ consisting of a line from $a$ to $x$ then a line from $x$ to $b$
$\langle 1 \rangle 4$. For $x \neq y$ we have $p_x$ and $p_y$ have no points in common except $a$ and $b$
$\langle 1 \rangle 5$. There are only countably many $x$ such that a point of $A$ lies on $p_x$.
$\langle 1 \rangle 6$. There exists $x$ such that $p_x$ is a path from $a$ to $b$ in $\mathbb{R}^2 \setminus A$.
□

**Proposition 248.** *Every open connected subspace of $\mathbb{R}^2$ is path-connected.*

PROOF:
$\langle 1 \rangle 1$. LET: $U$ be an open connected subspace of $\mathbb{R}^2$.
$\langle 1 \rangle 2$. For all $x_0 \in U$,
    LET: $PC(x_0) = \{y \in U \mid$ there exists a path from $x$ to $y\}$
$\langle 1 \rangle 3$. For all $x_0 \in U$, the set $PC(x_0)$ is open and closed in $U$.
  $\langle 2 \rangle 1$. LET: $x_0 \in U$
  $\langle 2 \rangle 2$. $PC(x_0)$ is open in $U$
    $\langle 3 \rangle 1$. LET: $y \in PC(x_0)$
    $\langle 3 \rangle 2$. PICK $\epsilon > 0$ such that $B(y, \epsilon) \subseteq U$
      PROOF: Since $U$ is open.
    $\langle 3 \rangle 3$. $B(y, \epsilon) \subseteq PC(x_0)$
      PROOF: For all $z \in B(y, \epsilon)$, pick a path from $x_0$ to $y$ then concatenate the straight line from $y$ to $z$.
  $\langle 2 \rangle 3$. $PC(x_0)$ is closed in $U$

65

$\langle 3 \rangle 1$. LET: $y \in U$ be a limit point of $PC(x_0)$
$\langle 3 \rangle 2$. PICK $\epsilon > 0$ such that $B(y, \epsilon) \subseteq U$
$\langle 3 \rangle 3$. PICK $z \in PC(x_0) \cap B(y, \epsilon)$
$\langle 3 \rangle 4$. $y \in PC(x_0)$
    PROOF: Pick a path from $x_0$ to $z$ then concatenate the straight line from $z$ to $y$.
$\langle 1 \rangle 4$. $PC(x_0) = U$
  PROOF: Proposition 212.
□

**Example 249.** If $A$ is a connected subspace of $X$, then $A^\circ$ is not necessarily connected.

Take two closed circles in $\mathbb{R}^2$ that touch at one point. The interior of this space is two open circles, and these two circles form a separation.

**Example 250.** If $A$ is a connected subspace of $X$ then $\partial A$ is not necessarily connected.

We have $[0, 1]$ is connected but $\partial[0, 1] = \{0, 1\}$ is not.

**Example 251.** If $A$ is a subspace of $X$ and $A^\circ$ and $\partial A$ are connected, then $A$ is not necessarily connected.

We have $\mathbb{Q}^\circ = \emptyset$ and $\partial \mathbb{Q} = \mathbb{R}$ are connected but $\mathbb{Q}$ is not connected.

# 36  The Topologist's Sine Curve

**Definition 252** (The Topologist's Sine Curve)**.** Let $S = \{(x, \sin 1/x) \mid 0 < x \leq 1\}$, The *topologist's sine curve* is the closure $\overline{S}$ of $S$ in $\mathbb{R}^2$.

**Proposition 253.** *The topologist's sine curve is connected.*

PROOF:
$\langle 1 \rangle 1$. LET: $S = \{(x, \sin 1/x) \mid 0 < x \leq 1\}$
$\langle 1 \rangle 2$. $S$ is connected.
  PROOF: Theorem 222.
$\langle 1 \rangle 3$. $\overline{S}$ is connected.
  PROOF: Theorem 221.
□

**Proposition 254.** *The topologist's sine curve is* $\{(x, \sin 1/x) \mid 0 < x \leq 1\} \cup (\{0\} \times [-1.1])$.

PROOF: Sketch proof: Given a point $(0.y)$ with $-1 \leq y \leq 1$, pick $a$ such that $\sin a = y$. Then $((1/a, y), (1/(a + 2\pi), y), (1/(a + 4\pi), y), \ldots)$ is a sequence in $S$ that converges to $(0, y)$.
Conversely, let $(x, y)$ be any point not in $S \cup (\{0\} \times [-1, 1])$. If $x < 0$ or $y > 1$ or $y < -1$ then we can easily find a neighbourhood that does not intersect $S \cup (\{0\} \times [-1, 1])$. If $x > 0$ and $-1 \leq y \leq 1$, then we have $y \neq \sin 1/x$. Hence pick a neighbourhood that does not intersect $S$.

66

**Proposition 255.** *Every closed subset of $\mathbb{R}$ that is bounded above has a greatest element.*

PROOF: It has a supremum, which is a limit point of the set and hence an element. □

**Proposition 256** (CC). *The topologist's sine curve is not path connected.*

PROOF:
$\langle 1 \rangle 1$. ASSUME: For a contradction $p : [0,1] \to \overline{S}$ is a path from $(0,0)$ to $(1, \sin 1)$.
$\langle 1 \rangle 2$. $\{t \in [0,1] \mid p(t) \in \{0\} \times [-1,1]\}$ is closed.
  PROOF: Since $p$ is continuous and $\{0\} \times [-1,1]$ is closed.
$\langle 1 \rangle 3$. LET: $b$ be the largest number in $[0,1]$ such that $p(b) \in \{0\} \times [-1,1]$.
  PROOF: Proposition 255.
$\langle 1 \rangle 4$. LET: $x : [b,1] \to \overline{S}$ be the function $\pi_1 \circ p$
$\langle 1 \rangle 5$. LET: $y : [b,1] \to \overline{S}$ be the function $\pi_2 \circ p$
$\langle 1 \rangle 6$. PICK a sequence $t_n$ in $(b,1]$ such that $t_n \to b$ and $y(t_n) = (-1)^n$ for all $n$
  $\langle 2 \rangle 1$. LET: $n \geq 1$
  $\langle 2 \rangle 2$. PICK $u$ with $0 < u < x(1/n)$ and $\sin(1/u) = (-1)^n$
  $\langle 2 \rangle 3$. PICK $t_n$ with $b < t_n < 1/n$ and $x(t_n) = u$
    PROOF: By the Intermediate Value Theorem
$\langle 1 \rangle 7$. Q.E.D.
  PROOF: This contradicts Proposition 105 since $y$ is continuous and $y(t_n)$ does not converge.
□

**Corollary 256.1.** *The closure of a path-connected subspace of a space is not necessarily path-connected.*

# 37 The Long Line

**Definition 257** (The Long Line). The *long line* is the space $\omega_1 \times [0,1)$ in the dictionary order under the order topology, where $\omega_1$ is the first uncountable ordinal.

**Lemma 258.** *For any ordinal $\alpha$ with $0 < \alpha < \omega_1$ we have $[(0,0),(\alpha,0)) \cong [0,1)$*

$\langle 1 \rangle 1$. $[(0,0),(1,0)) \cong [0,1)$
  PROOF: The map $\pi_2$ is a homeomorphism.
$\langle 1 \rangle 2$. If $[(0,0),(\alpha,0)) \cong [0,1)$ then $[(0,0),(\alpha+1,0)) \cong [0,1)$
  PROOF: Proposition 14.
$\langle 1 \rangle 3$. If $\lambda$ is a limit ordinal with $\lambda < \omega_1$ and $[(0,0),(\alpha,0)) \cong [0,1)$ for all $\alpha$ with $0 < \alpha < \lambda$ then $[(0,0),(\lambda,0)) \cong [0,1)$
  $\langle 2 \rangle 1$. LET: $\lambda$ be a limit ordinal $< \omega_1$
  $\langle 2 \rangle 2$. ASSUME: $[(0,0),(\alpha,0)) \cong [0,1)$ for all $\alpha$ with $0 < \alpha < \lambda$
  $\langle 2 \rangle 3$. PICK a sequence of ordinals $\alpha_0 < \alpha_1 < \cdots$ with limit $\lambda$
    PROOF: Since $\lambda$ is countable.

$\langle 2 \rangle 4$. $[(\alpha_i, 0), (\alpha_{i+1}, 0)) \cong [0, 1)$ for all $i$
  PROOF: Lemma 13.
$\langle 2 \rangle 5$. Q.E.D.
  PROOF: By Proposition 15.
$\langle 1 \rangle 4$. Q.E.D.
  PROOF: By transfinite induction.

**Proposition 259** (CC). *The long line is path-connected.*

PROOF:
$\langle 1 \rangle 1$. LET: $(\alpha, i), (\beta, j) \in \omega_1 \times [0, 1)$
$\langle 1 \rangle 2$. ASSUME: without loss of generality $(\alpha, i) < (\beta, j)$
$\langle 1 \rangle 3$. $[(0, 0), (\beta + 1, 0)) \cong [0, 1)$
  PROOF: By Lemma 258
$\langle 1 \rangle 4$. $[(\alpha, i), (\beta, j)) \cong [0, 1)$
  PROOF: Lemma 13.
$\langle 1 \rangle 5$. PICK a homeomorphism $q : [0, 1) \to [(\alpha, i), (\beta, j))$
$\langle 1 \rangle 6$. $q \cup \{(1, (\beta, j))\}$ is a path from $(\alpha, i)$ to $(\beta, j)$
$\square$

**Proposition 260.** *Every point in the long line has a neighbourhood homeomorphic to an interval in $\mathbb{R}$.*

PROOF: For any $(\alpha, i)$ in the long line, the neighbourhood $[(0, 0), (\alpha + 1, 0))$ satisfies the condition by Lemma 258.

**Proposition 261.** *The long line $L$ is not second countable.*

PROOF:
$\langle 1 \rangle 1$. LET: $\mathcal{B}$ be a basis for $L$.
$\langle 1 \rangle 2$. For $\alpha < \omega_1$, PICK $B_\alpha \in \mathcal{B}$ such that $(\alpha, 1/2) \in B_\alpha \subseteq ((\alpha, 0), (\alpha + 1, 0))$
$\langle 1 \rangle 3$. $\mathcal{B}$ is uncountable.
  PROOF: The mapping $\alpha \mapsto B_\alpha$ is an injection $\omega_1 \to \mathcal{B}$.

**Corollary 261.1.** *The long line cannot be imbedded into $\mathbb{R}^n$ for any $n$.*

# 38   Components

**Proposition 262.** *Let $X$ be a topological space. Define the relation $\sim$ on $X$ by $a \sim b$ if and only if there exists a connected subspace $A \subseteq X$ with $a, b \in A$. Then $\sim$ is an equivalence relation on $X$.*

PROOF:
$\langle 1 \rangle 1$. $\sim$ is reflexive.
  PROOF: For any $a \in X$ we have $\{a\}$ is a connected subspace that contains $a$.
$\langle 1 \rangle 2$. $\sim$ is symmetric.
  PROOF: Trivial.
$\langle 1 \rangle 3$. $\sim$ is transitive.

$\langle 2 \rangle 1$. LET: $a, b, c \in X$
$\langle 2 \rangle 2$. ASSUME: $a \sim b$ and $b \sim c$
$\langle 2 \rangle 3$. PICK connected subspaces $A$ and $B$ with $a, b \in A$ and $b, c \in B$
$\langle 2 \rangle 4$. $A \cup B$ is a connected subspace that contains $a$ and $c$
  PROOF: Theorem 220.

$\square$

**Definition 263** ((Connected) Component). Let $X$ be a topological space. The *(connected) components* of $X$ are the equivalence classes under the above $\sim$.

**Lemma 264.** *Let $X$ be a topological space. If $A \subseteq X$ is connected and nonempty then there exists a unique component $C$ of $X$ such that $A \subseteq C$.*

PROOF:
$\langle 1 \rangle 1$. PICK $a \in A$
$\langle 1 \rangle 2$. LET: $C$ be the $\sim$-equivalence class of $a$.
$\langle 1 \rangle 3$. $A \subseteq C$
  PROOF: For all $x \in A$ we have $x \sim a$.
$\langle 1 \rangle 4$. If $C'$ is a component and $A \subseteq C'$ then $C = C'$
  PROOF: Since we have $a \in C'$.

$\square$

**Theorem 265.** *Let $X$ be a topological space. The components of $X$ are connected disjoint subspaces of $X$ whose union is $X$ such that each nonempty connected subspace of $X$ intersects only one of them.*

PROOF:
$\langle 1 \rangle 1$. Every component of $X$ is connected.
  PROOF: For $a \in X$, the $\sim$-equivalence class of $a$ is $\bigcup \{A \subseteq X \mid A$ is connected, $a \in A\}$ which is connected by Theorem 220.
$\langle 1 \rangle 2$. The components form a partition of $X$.
  PROOF: Immediate from the definition.
$\langle 1 \rangle 3$. Every nonempty connected subspace of $X$ intersects a unique component of $X$.
  $\langle 2 \rangle 1$. LET: $A \subseteq X$ be connected and nonempty.
  $\langle 2 \rangle 2$. LET: $C$ be the component such that $A \subseteq C$
    PROOF: Lemma 264.
  $\langle 2 \rangle 3$. $A$ intersects $C$
  $\langle 2 \rangle 4$. If $A$ intersects the component $C'$ then $C' = C$
    $\langle 3 \rangle 1$. LET: $C'$ be a component that intersects $A$
    $\langle 3 \rangle 2$. PICK $b \in A \cap C'$
    $\langle 3 \rangle 3$. $A \subseteq C'$
      PROOF: For all $x \in A$ we have $x \sim b$.
    $\langle 3 \rangle 4$. $C = C'$
      PROOF: By uniqueness in $\langle 2 \rangle 2$.

$\square$

**Proposition 266.** *Every component of a space is closed.*

PROOF:

⟨1⟩1. LET: $X$ be a topological space and $C$ a component of $X$.

⟨1⟩2. $\overline{C}$ is connected.

    PROOF: Theorem 221.

⟨1⟩3. $C = \overline{C}$

    PROOF: Lemma 219.

⟨1⟩4. $C$ is closed.

    PROOF: Lemma 47.

☐

**Proposition 267.** *If a topological space has finitely many components then every component is open.*

PROOF: Each component is the complement of a finite union of closed sets. ☐

## 39 Path Components

**Proposition 268.** *Let $X$ be a topological space. Define the relation $\sim$ on $X$ by: $a \sim b$ if and only if there exists a path in $X$ from $a$ to $b$. Then $\sim$ is an equivalence relation on $X$.*

PROOF:

⟨1⟩1. $\sim$ is reflexive.

    PROOF: For $a \in X$, the constant function $[0,1] \to X$ with value $a$ is a path from $a$ to $a$.

⟨1⟩2. $\sim$ is symmetric.

    PROOF: If $p : [0,1] \to X$ is a path from $a$ to $b$, then $\lambda t.p(1-t)$ is a path from $b$ to $a$.

⟨1⟩3. $\sim$ is transitive.

    PROOF: Concatenate paths.

☐

**Definition 269** (Path Component)**.** Let $X$ be a topological space. The *path components* of $X$ are the equivalence relations under $\sim$.

  .

**Theorem 270.** *The path components of $X$ are path-connected disjoint subspaces of $X$ whose union is $X$ such that every nonempty path-connected subspace of $X$ intersects exactly one path component.*

PROOF:

⟨1⟩1. Every path component is path-connected.

    PROOF: If $a$ and $b$ are in the same path component then $a \sim b$, i.e. there exists a path from $a$ to $b$.

⟨1⟩2. The path components are disjoint and their union is $X$.

    PROOF: Immediate from the definition.

⟨1⟩3. Every non-empty path-cönnected subspace of $X$ intersects exactly one path component.
 ⟨2⟩1. Let: $A$ be a nonempty path-connected subspace of $X$.
 ⟨2⟩2. Pick $a \in A$
 ⟨2⟩3. $A$ intersects the $\sim$-equivalence class of $a$.
 ⟨2⟩4. Let: $C$ be any path component that intersects $A$.
 ⟨2⟩5. Pick $b \in A \cap C$
 ⟨2⟩6. $a \sim b$
  Proof: Since $A$ is path-connected.
 ⟨2⟩7. $C$ is the $\sim$-equivalence class of $a$.
☐

**Proposition 271.** *Every path component is included in a component.*

Proof:
⟨1⟩1. Let: $X$ be a topological space and $C$ a path component of $X$.
⟨1⟩2. $C$ is path-connected.
 Proof: Theorem 270.
⟨1⟩3. $C$ is connected.
 Proof: Proposition 241.
⟨1⟩4. $C$ is included in a component.
 Proof: Lemma 264.
☐


# 40 Local Connectedness

**Definition 272** (Locally Connected)**.** Let $X$ be a topological space and $a \in X$. Then $X$ is *locally connected* at $a$ if and only if every neighbourhood of $a$ includes a connected neighbourhood of $a$.

 The space $X$ is *locally connected* if and only if it is locally connected at every point.

**Example 273.** The real line is both connected and locally connected.

**Example 274.** The space $\mathbb{R} \setminus \{0\}$ is disconnected but locally connected.

**Example 275.** The topologist's sine curve is connected but not locally connected.

**Example 276.** The rationals $\mathbb{Q}$ are neither connected nor locally connected.

**Theorem 277.** *A topological space $X$ is locally connected if and only if, for every open set $U$ in $X$, every component of $U$ is open in $X$.*

Proof:
⟨1⟩1. If $X$ is locally connected then, for every open set $U$ in $X$, every component of $U$ is open in $X$.
 ⟨2⟩1. Assume: $X$ is locally connected.

⟨2⟩2. LET: $U$ be open in $X$.
⟨2⟩3. LET: $C$ be a component of $U$.
⟨2⟩4. LET: $a \in C$
⟨2⟩5. LET: $V$ be a connected neighbourhood of $a$ such that $V \subseteq U$
⟨2⟩6. $V \subseteq C$
   PROOF: Lemma 264.
⟨2⟩7. Q.E.D.
   PROOF: Lemma 26.
⟨1⟩2. If, for every open set $U$ in $X$, every component of $U$ is open in $X$, then $X$ is locally connected.
  ⟨2⟩1. ASSUME: for every open set $U$ in $X$, every component of $U$ is open in $X$.
  ⟨2⟩2. LET: $a \in X$
  ⟨2⟩3. LET: $U$ be a neighbourhood of $a$
  ⟨2⟩4. The component of $U$ that contains $a$ is a connected neighbourhood of $a$ included in $U$.

▢

**Example 278.** The ordered square is locally connected.

PROOF: Every neighbourhood of a point includes an interval around that point, which is connected by Theorem 232.

**Example 279.** Let $X$ be the set of all rational points on the line segment $[0, 1] \times \{0\}$, and $Y$ the set of all rational points on the line segment $[0, 1] \times \{1\}$. Let $A$ be the space consisting of all line segments joining the point $(0, 1)$ to a point of $X$, and all line segments joining the point $(1, 0)$ to a point of $Y$. Then $A$ is path-connected but is not locally connected at any point,

**Proposition 280.** *Let $X$ and $Y$ be topological spaces and $p : X \twoheadrightarrow Y$ be a quotient map. If $X$ is locally connected then so is $Y$.*

PROOF:
⟨1⟩1. LET: $U$ be an open set in $Y$.
⟨1⟩2. LET: $C$ be a component of $U$.
⟨1⟩3. $p^{-1}(C)$ is a union of components of $p^{-1}(U)$
  ⟨2⟩1. LET: $x \in p^{-1}(C)$
  ⟨2⟩2. LET: $D$ be the component of $p^{-1}(U)$ that contains $x$.
  ⟨2⟩3. $p(D)$ is connected.
    PROOF: Theorem 222.
  ⟨2⟩4. $p(D) \subseteq C$.
    PROOF: From ⟨1⟩2 since $p(x) \in p(D) \cap C$ (⟨2⟩1, ⟨2⟩2).
  ⟨2⟩5. $D \subseteq p^{-1}(C)$
⟨1⟩4. $p^{-1}(C)$ is open in $p^{-1}(U)$
  PROOF: Theorem 277.
⟨1⟩5. $C$ is open in $U$
  PROOF: Since the restriction of $p$ to $p : p^{-1}(U) \twoheadrightarrow U$ is a quotient map by Proposition 195.

⟨1⟩6. Q.E.D.
    Proof: Theorem 277.
▯

# 41   Local Path Connectedness

**Definition 281** (Locally Path-Connected). Let $X$ be a topological space and $a \in X$. Then $X$ is *locally path-connected* at $a$ if and only if every neighbourhood of $a$ includes a path-connected neighbourhood of $a$.

    The space $X$ is *locally path-connected* if and only if it is locally path-connected at every point.

**Theorem 282.** *A topological space $X$ is locally path-connected if and only if, for every open set $U$ in $X$, every path component of $U$ is open in $X$.*

Proof:
⟨1⟩1. If $X$ is locally path-connected then, for every open set $U$ in $X$, every path component of $U$ is open in $X$.
  ⟨2⟩1. Assume: $X$ is locally path-connected.
  ⟨2⟩2. Let: $U$ be open in $X$.
  ⟨2⟩3. Let: $C$ be a path component of $U$.
  ⟨2⟩4. Let: $a \in C$
  ⟨2⟩5. Let: $V$ be a path-connected neighbourhood of $a$ such that $V \subseteq U$
  ⟨2⟩6. $V \subseteq C$
    Proof: Lemma 264.
  ⟨2⟩7. Q.E.D.
    Proof: Lemma 26.
⟨1⟩2. If, for every open set $U$ in $X$, every component of $U$ is open in $X$, then $X$ is locally connected.
  ⟨2⟩1. Assume: for every open set $U$ in $X$, every component of $U$ is open in $X$.
  ⟨2⟩2. Let: $a \in X$
  ⟨2⟩3. Let: $U$ be a neighbourhood of $a$
  ⟨2⟩4. The component of $U$ that contains $a$ is a connected neighbourhood of $a$ included in $U$.
▯

**Theorem 283.** *If a space is locally path connected then its components and its path components are the same.*

Proof:
⟨1⟩1. Let: $X$ be a locally path connected space.
⟨1⟩2. Let: $C$ be a component of $X$.
⟨1⟩3. Let: $x \in C$
⟨1⟩4. Let: $P$ be the path component of $x$
    Prove:   $P = C$
⟨1⟩5. $P \subseteq C$

Proof: Proposition 271.

⟨1⟩6. Let: $Q$ be the union of the other path components included in $C$

⟨1⟩7. $C = P \cup Q$

Proof: Proposition 271.

⟨1⟩8. $P$ and $Q$ are open in $C$

   ⟨2⟩1. $C$ is open.

     Proof: Theorem 277.

   ⟨2⟩2. Q.E.D.

     Proof: Theorem 282.

⟨1⟩9. $Q = \emptyset$

Proof: Otherwise $P$ and $Q$ would form a separation of $C$.

☐

**Example 284.** The ordered square is not locally path connected, since it is connected but not path connected.

**Proposition 285.** *Let $X$ be a locally path-connected space. Then every connected open subspace of $X$ is path-connected.*

Proof:

⟨1⟩1. Let: $U$ be a connected open subspace of $X$.

⟨1⟩2. Let: $P$ be a path component of $U$.

⟨1⟩3. Let: $Q$ be the union of the other path components of $U$.

⟨1⟩4. $P$ and $Q$ are open in $U$.

Proof: Theorem 282.

⟨1⟩5. $Q = \emptyset$

Proof: Otherwise $P$ and $Q$ form a separation of $U$.

☐

# 42 Weak Local Connectedness

**Definition 286** (Weakly Locally Connected). Let $X$ be a topological space and $a \in X$. Then $X$ is *weakly locally connected* at $a$ if and only if every neighbourhood of $a$ includes a connected subspace that includes a neighbourhood of $a$.

**Proposition 287.** *Let $X$ be a topological space. If $X$ is weakly locally connected at every point then $X$ is locally connected.*

Proof:

⟨1⟩1. Assume: $X$ is weakly locally connected at every point.

⟨1⟩2. Let: $U$ be open in $X$.

⟨1⟩3. Let: $C$ be a component of $U$.

⟨1⟩4. $C$ is open in $X$.

   ⟨2⟩1. Let: $x \in C$

   ⟨2⟩2. Pick a connected subspace $D$ of $U$ that includes a neighbourhood $V$ of $x$.

$\langle 2 \rangle 3.$ $D \subseteq C$
    Proof: Lemma 264.
$\langle 2 \rangle 4.$ $x \in V \subseteq C$
$\langle 2 \rangle 5.$ Q.E.D.
    Proof: Lemma 26.
$\langle 1 \rangle 5.$ Q.E.D.
  Proof: Theorem 277.

∎

**Example 288.** The 'infinite broom' (Munkres p.163) is an example of a space that is weakly locally connected at a point $p$ but not locally connected at $p$.

# 43 Quasicomponents

**Proposition 289.** *Let $X$ be a topological space. Define $\sim$ on $X$ by $x \sim y$ if and only if there exists no separation $U$ and $V$ of $X$ such that $x \in U$ and $y \in V$. Then $\sim$ is an equivalence relation on $X$.*

Proof:
$\langle 1 \rangle 1.$ $\sim$ is reflexive.
  Proof: Since the two sets that make up a separation are disjoint.
$\langle 1 \rangle 2.$ $\sim$ is symmetric.
  Proof: Immediate from the defintion.
$\langle 1 \rangle 3.$ $\sim$ is transitive.
  $\langle 2 \rangle 1.$ Assume: $x \sim y$ and $y \sim z$
  $\langle 2 \rangle 2.$ Assume: for a contradiction there is a separation $U$ and $V$ of $X$ with
            $x \in U$ and $z \in V$
  $\langle 2 \rangle 3.$ $y \in U$ or $y \in V$
  $\langle 2 \rangle 4.$ Q.E.D.
    Proof: Either case contradicts $\langle 2 \rangle 1$.

∎

**Definition 290** (Quasicomponents)**.** For $X$ a topological space, the *quasicomponents* of $X$ are the equivalence classes under $\sim$.

**Proposition 291.** *Let $X$ be a topological space. Then every component of $X$ is included in a quasicomponent of $X$.*

Proof:
$\langle 1 \rangle 1.$ Let: $C$ be a component of $X$.
$\langle 1 \rangle 2.$ Let: $x, y \in C$
    Prove: $x \sim y$
$\langle 1 \rangle 3.$ Assume: for a contradiction there exists a separation $U$ and $V$ of $X$ with
            $x \in U$ and $y \in V$
$\langle 1 \rangle 4.$ $C \cap U$ and $C \cap V$ form a separation of $C$.
$\langle 1 \rangle 5.$ Q.E.D.
  Proof: This contradicts $\langle 1 \rangle 1$.

**Proposition 292.** *In a locally connected space, the components and the quasi-components are the same.*

PROOF:

⟨1⟩1. LET: $X$ be a locally connected space and $Q$ a quasicomponent of $X$.

⟨1⟩2. PICK a component $C$ of $X$ such that $C \subseteq Q$

⟨1⟩3. LET: $D$ be the union of the components of $X$

⟨1⟩4. $C$ and $D$ are open in $X$.

   PROOF: Theorem 277.

⟨1⟩5. $D$ cannot contain any points of $Q$.

   PROOF: If it did, then $C$ and $D$ would form a separation of $X$ and there would be points $x, y \in Q$ with $x \in C$ and $y \in D$.

⟨1⟩6. $C = Q$

□

# 44   Open Coverings

**Definition 293** (Open Covering). Let $X$ be a topological space. An *open covering* of $X$ is a covering of $X$ whose elements are all open sets.

# 45   Compact Spaces

**Definition 294** (Compact). A topological space is *compact* if and only if every open covering has a finite subcovering.

**Lemma 295.** *Let $X$ be a topological space and $Y$ a subspace of $X$. Then $Y$ is compact if and only if every covering of $Y$ by sets open in $X$ has a finite subcovering.*

PROOF:

⟨1⟩1. If $Y$ is compact then every covering of $Y$ by sets open in $X$ has a finite subcovering.

  ⟨2⟩1. ASSUME: $Y$ is compact.

  ⟨2⟩2. LET: $\mathcal{U}$ be a covering of $Y$ by sets open in $X$.

  ⟨2⟩3. $\{U \cap Y \mid U \in \mathcal{U}\}$ is an open covering of $Y$.

  ⟨2⟩4. PICK a finite subcovering $\{U_1 \cap Y, \ldots, U_n \cap Y\}$

  ⟨2⟩5. $\{U_1, \ldots, U_n\}$ is a finite subcovering of $\mathcal{U}$.

⟨1⟩2. If every covering of $Y$ by sets open in $X$ has a finite subcovering then $Y$ is compact.

  ⟨2⟩1. LET: $\mathcal{U}$ be an open covering of $Y$.

  ⟨2⟩2. LET: $\mathcal{V} = \{V \text{ open in } X \mid V \cap Y \in \mathcal{U}\}$.

  ⟨2⟩3. $\mathcal{V}$ is a covering of $Y$ by sets open in $X$.

  ⟨2⟩4. PICK a finite subcovering $\{V_1, \ldots, V_n\}$

  ⟨2⟩5. $\{V_1 \cap Y, \ldots, V_n \cap Y\}$ is a finite subcovering of $\mathcal{U}$.

□

**Proposition 296.** *Every closed subspace of a compact space is compact.*

PROOF:

$\langle 1 \rangle 1$. LET: $X$ be a compact space and $Y \subseteq X$ be closed.

$\langle 1 \rangle 2$. LET: $\mathcal{U}$ be a covering of $Y$ by sets open in $X$.

$\langle 1 \rangle 3$. $\mathcal{U} \cup \{X \setminus Y\}$ is an open covering of $X$.

$\langle 1 \rangle 4$. PICK a finite subcovering $\mathcal{U}_0$

$\langle 1 \rangle 5$. $\mathcal{U}_0 \cap \mathcal{U}$ is a finite subset of $\mathcal{U}$ that covers $Y$.

□

**Theorem 297.** *The continuous image of a compact space is compact.*

PROOF:

$\langle 1 \rangle 1$. LET: $f : X \twoheadrightarrow Y$ be continuous and surjective.

$\langle 1 \rangle 2$. LET: $\mathcal{V}$ be an open covering of $Y$

$\langle 1 \rangle 3$. $\{p^{-1}(V) \mid V \in \mathcal{V}\}$ is an open covering of $X$.

$\langle 1 \rangle 4$. PICK a finite subcovering $\{p^{-1}(V_1), \ldots, p^{-1}(V_n)\}$

$\langle 1 \rangle 5$. $\{V_1, \ldots, V_n\}$ covers $Y$.

□

**Theorem 298.** *Let $A$ and $B$ be compact subspaces of $X$ and $Y$ respectively. Let $N$ be an open set in $X \times Y$ that includes $A \times B$. Then there exist open sets $U$ and $V$ in $X$ and $Y$ respectively such that $A \subseteq U$, $B \subseteq V$ and $U \times V \subseteq N$.*

PROOF:

$\langle 1 \rangle 1$. For all $x \in A$, there exist neighbourhoods $U$ of $x$ and $V$ of $B$ such that $U \times V \subseteq N$.

$\quad \langle 2 \rangle 1$. LET: $x \in A$

$\quad \langle 2 \rangle 2$. For all $y \in B$, there exist neighbourhoods $U$ of $x$ and $V$ of $y$ such that $U \times V \subseteq N$

$\quad \langle 2 \rangle 3$. $\{V$ open in $Y \mid \exists$ neighbourhood $U$ of $x, U \times V \subseteq N\}$ covers $B$.

$\quad \langle 2 \rangle 4$. PICK a finite subcover $\{V_1, \ldots, V_n\}$

$\quad \langle 2 \rangle 5$. For $i = 1, \ldots, n$, PICK a neighbourhood $U_i$ of $x$ such that $U_i \times V_i \subseteq N$

$\quad \langle 2 \rangle 6$. LET: $U = U_1 \cap \cdots \cap U_n$

$\quad \langle 2 \rangle 7$. LET: $V = V_1 \cup \cdots \cup V_n$

$\quad \langle 2 \rangle 8$. $U$ is a neighbourhood of $x$.

$\quad \langle 2 \rangle 9$. $V$ is a neighbourhood of $B$.

$\quad \langle 2 \rangle 10$. $U \times V \subseteq N$

$\langle 1 \rangle 2$. $\{U$ open in $X \mid \exists$ neighbourhood $V$ of $B.U \times V \subseteq N\}$ covers $A$.

$\langle 1 \rangle 3$. PICK a finite subcover $\{U_1, \ldots, U_n\}$

$\langle 1 \rangle 4$. For $i = 1, \ldots, n$, PICK a neighbourhood $V_i$ of $B$ such that $U_i \times V_i \subseteq N$

$\langle 1 \rangle 5$. LET: $U = U_1 \cup \cdots \cup U_n$

$\langle 1 \rangle 6$. LET: $V = V_1 \cap \cdots \cap V_n$

$\langle 1 \rangle 7$. $U$ and $V$ are open.

$\langle 1 \rangle 8$. $A \subseteq U$

$\langle 1 \rangle 9$. $B \subseteq V$

$\langle 1 \rangle 10$. $U \times V \subseteq N$

□

**Corollary 298.1** (Tube Lemma). *Let $X$ and $Y$ be topological spaces with $Y$ compact. Let $a \in X$ and $N$ be an open set in $X \times Y$ that includes $\{a\} \times Y$. Then there exists a neighbourhood $W$ of $a$ such that $N$ includes the tube $W \times Y$.*

**Theorem 299.** *Let $X$ be a topological space. Then $X$ is compact if and only if every set of closed sets that has the finite intersection property has nonempty intersection.*

PROOF: The following are equivalent.

1. $X$ is compact.

2. Every open covering of $X$ has a finite subcovering.

3. For any set $\mathcal{C}$ of closed sets, if $\{X \setminus C \mid C \in \mathcal{C}\}$ covers $X$ then there is a finite subset $\mathcal{C}_0$ such that $\{X \setminus C \mid C \in \mathcal{C}_0\}$ covers $X$

4. For any set $\mathcal{C}$ of closed sets, if $\bigcap \mathcal{C} = \emptyset$ then there is a finite subset $\mathcal{C}_0$ with empty intersection.

5. Any set of closed sets with the finite intersection property has nonempty intersection.

$\square$

**Corollary 299.1.** *Let $X$ be a topological space and $C_1 \supseteq C_2 \supseteq \cdots$ a nested sequence of nonempty closed sets. Then $\bigcap_n C_n$ is nonempty.*

**Proposition 300.** *Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on the same set $X$ with $\mathcal{T} \subseteq \mathcal{T}'$. If $\mathcal{T}'$ is compact then $\mathcal{T}$ is compact.*

PROOF:
$\langle 1 \rangle 1$. LET: $\mathcal{U} \subseteq \mathcal{T}$ cover $X$
$\langle 1 \rangle 2$. $\mathcal{U} \subseteq \mathcal{T}'$
$\langle 1 \rangle 3$. A finite subset of $\mathcal{U}$ covers $X$.
$\square$

**Corollary 300.1.** *If $\mathcal{T}$ and $\mathcal{T}'$ are two compact Hausdorff topologies on the same set $X$, then either $\mathcal{T} = \mathcal{T}'$ or $\mathcal{T}$ and $\mathcal{T}'$ are incomparable.*

PROOF: From the Proposition and Proposition 177. $\square$

**Example 301.** Any set under the finite complement topology is compact.

**Proposition 302.** *Let $X$ be a topological space. A finite union of compact subspaces of $X$ is compact.*

PROOF:
$\langle 1 \rangle 1$. LET: $A$ and $B$ be compact subspaces of $X$.
$\langle 1 \rangle 2$. LET: $\mathcal{U}$ be a set of open sets in $X$ that covers $A \cup B$
$\langle 1 \rangle 3$. PICK a finite subset $\mathcal{U}_1$ that covers $A$.
  PROOF: Lemma 295.

⟨1⟩4. PICK a finite subset $\mathcal{U}_2$ that covers $B$.
  PROOF: Lemma 295.
⟨1⟩5. $\mathcal{U}_1 \cup \mathcal{U}_2$ is a finite subset that covers $A \cup B$.
⟨1⟩6. Q.E.D.
  PROOF: Lemma 295.
▯

**Proposition 303.** *Let $A$ and $B$ be disjoint compact subspaces of the Hausdorff space $X$. Then there exist disjoint open sets $U$ and $V$ that include $A$ and $B$ respectively.*

PROOF: From Theorem 298 with $N = X^2 \setminus \{(x,x) \mid x \in X\}$. ▯

**Corollary 303.1.** *Every compact subspace of a Hausdorff space is closed.*

**Theorem 304.** *Let $f : X \to Y$ be a bijective continuous function. If $X$ is compact and $Y$ is Hausdorff then $f$ is a homeomorphism.*

PROOF:
⟨1⟩1. LET: $C \subseteq X$ be closed.
⟨1⟩2. $C$ is compact.
  PROOF: Proposition 296.
⟨1⟩3. $f(C)$ is compact.
  PROOF: Theorem 297.
⟨1⟩4. $f(C)$ is closed.
  PROOF: Corollary 303.1.
⟨1⟩5. Q.E.D.
  PROOF: Lemma 107.
▯

**Proposition 305.** *Let $X$ be a compact space, $Y$ a Hausdorff space, and $f : X \to Y$ a continuous map. Then $f$ is a closed map.*

PROOF:
⟨1⟩1. LET: $C \subseteq X$ be closed.
⟨1⟩2. $C$ is compact.
  PROOF: Proposition 296.
⟨1⟩3. $f(C)$ is compact.
  PROOF: Theorem 297.
⟨1⟩4. $f(C)$ is closed.
  PROOF: Corollary 303.1.
▯

**Proposition 306.** *If $Y$ is compact then the projection $\pi_1 : X \times Y \to X$ is a closed map.*

PROOF:
⟨1⟩1. LET: $A \subseteq X \times Y$ be closed.
⟨1⟩2. LET: $x \in X \setminus \pi_1(A)$

⟨1⟩3. PICK a neighbourhood $U$ of $x$ such that $U \times Y \subseteq (X \times Y) \setminus A$
    PROOF: By the Tube Lemma.
⟨1⟩4. $x \in U \subseteq X \setminus \pi_1(A)$
⟨1⟩5. Q.E.D.
    PROOF: So $X \setminus \pi_1(A)$ is open by Lemma 26.
□

**Theorem 307.** *Let $X$ be a topological space and $Y$ a compact Hausdorff space. Let $f : X \to Y$ be a function. Then $f$ is continuous if and only if the graph of $f$ is closed in $X \times Y$.*

PROOF:
⟨1⟩1. LET: $G_f$ be the graph of $f$.
⟨1⟩2. If $f$ is continuous then $G_f$ is closed.
    ⟨2⟩1. ASSUME: $f$ is continuous.
    ⟨2⟩2. LET: $(x, y) \in (X \times Y) \setminus G_f$
    ⟨2⟩3. PICK disjoint neighbourhoods $U$ and $V$ of $y$ and $f(x)$ respectively.
    ⟨2⟩4. $f^{-1}(V) \times U$ is a neighbourhood of $(x, y)$ disjoint from $G_f$.
⟨1⟩3. If $G_f$ is closed then $f$ is continuous.
    ⟨2⟩1. ASSUME: $G_f$ is closed.
    ⟨2⟩2. LET: $x \in X$ and $V$ be a neighbourhood of $f(x)$.
    ⟨2⟩3. $G_f \cap (X \times (Y \setminus V))$ is closed.
    ⟨2⟩4. $\pi_1(G_f \cap (X \times (Y \setminus V)))$ is closed.
      PROOF: Proposition 306.
    ⟨2⟩5. LET: $U = X \setminus \pi_1(G_f \cap (X \times (Y \setminus V)))$
    ⟨2⟩6. $U$ is a neighbourhood of $x$
    ⟨2⟩7. $f(U) \subseteq V$
□

**Theorem 308.** *Let $X$ be a compact topological space. Let $(f_n : X \to \mathbb{R})$ be a monotone increasing sequence of continuous functions and $f : X \to \mathbb{R}$ a continuous function. If $(f_n)$ converges pointwise to $f$, then $(f_n)$ converges uniformly to $f$.*

PROOF:
⟨1⟩1. LET: $\epsilon > 0$
⟨1⟩2. For all $x \in X$, there exists $N$ such that, for all $n \geq N$, we have $|f_n(x) - f(x)| < \epsilon$
⟨1⟩3. For $n \geq 1$,
    LET: $U_n = \{x \in X \mid |f_n(x) - f(x)| < \epsilon\}$
⟨1⟩4. For $n \geq 1$, we have $U_n$ is open in $X$.
    ⟨2⟩1. LET: $x \in X$
    ⟨2⟩2. LET: $\delta = \epsilon - |f_n(x) - f(x)|$
    ⟨2⟩3. PICK a neighbourhood $U$ of $x$ such that $f(U) \subseteq B(f(x), \delta/2)$
    ⟨2⟩4. PICK a neighbourhood $V$ of $x$ such that $f_n(V) \subseteq B(f_n(x), \delta/2)$
    ⟨2⟩5. $f(U \cap V) \subseteq U_n$

PROOF: For $y \in U \cap V$ we have
$$|f_n(y) - f(y)| \leq |f_n(y) - f_n(x)| + |f_n(x) - f(x)| + |f(x) - y(y)|$$
$$< \delta/2 + |f_n(x) - f(x)| + \delta/2$$
$$= \epsilon$$

$\langle 1 \rangle 5.$ $\{U_n \mid n \geq 1\}$ covers $X$
  PROOF: From $\langle 1 \rangle 2$
$\langle 1 \rangle 6.$ PICK $N$ such that $X = U_N$
  $\langle 2 \rangle 1.$ PICK $n_1, \ldots, n_k$ such that $U_{n_1}, \ldots, U_{n_k}$ cover $X$.
  $\langle 2 \rangle 2.$ LET: $N = \max(n_1, \ldots, n_k)$
  $\langle 2 \rangle 3.$ For all $i$ we have $U_{n_i} \subseteq U_N$
    PROOF: Since $(f_n)$ is monotone increasing.
  $\langle 2 \rangle 4.$ $X = U_N$
$\langle 1 \rangle 7.$ For all $x \in X$ and $n \geq N$ we have $|f_n(x) - f(x)| < \epsilon$
$\square$

An example to show that we cannot remove the hypothesis that $X$ is compact:

**Example 309.** Let $X = (0, 1)$, $f_n(x) = -x^n$ and $f(x) = 0$ for $x \in X$ and $n \geq 1$. Then $f_n \to f$ pointwise and $(f_n)$ is monotone increasing but the convergence is not uniform since, for all $N \geq 1$, there exists $x \in (0, 1)$ such that $-x^N < -1/2$.

An example to show that we cannot remove the hypothesis that $(f_n)$ is monotone increasing:

**Example 310.** Let $X = [0, 1]$, $f_n(x) = 1/(n^3(x - 1/n)^2 + 1)$ and $f(x) = 0$ for $x \in X$ and $n \geq 1$. Then $X$ is compact and $f_n \to f$ pointwise but the convergence is not uniform since, for all $N \geq 1$, there exists $x \in [0, 1]$ such that $f_N(x) = 1$, namely $x = 1/N$.

**Theorem 311.** *Let $X$ be a compact Hausdorff space. Let $\mathcal{A}$ be a chain of closed connected subsets of $X$. Then $\bigcap \mathcal{A}$ is connected.*

PROOF:
$\langle 1 \rangle 1.$ ASSUME: for a contradiction $C$ and $D$ form a separation of $\bigcap \mathcal{A}$.
$\langle 1 \rangle 2.$ PICK disjoint open sets $U$ and $V$ that include $C$ and $D$ respectively.
  PROOF: Proposition 303.
$\langle 1 \rangle 3.$ $\{A \setminus (U \cup V) \mid A \in \mathcal{A}\}$ is a set of closed sets with the finite intersection property.
  $\langle 2 \rangle 1.$ For all $A \in \mathcal{A}$ we have $A \setminus (U \cup V)$ is closed.
  $\langle 2 \rangle 2.$ For all $A_1, \ldots, A_n \in \mathcal{A}$ we have $(A_1 \cap \cdots \cap A_n) \setminus (U \cup V)$ is nonempty.
    PROOF:
    $\langle 3 \rangle 1.$ LET: $A_1, \ldots, A_n \in \mathcal{A}$
    $\langle 3 \rangle 2.$ ASSUME: without loss of generality $A_1 \subseteq A_2, \ldots, A_n$
      PROOF: Since $\mathcal{A}$ is a chain.
    $\langle 3 \rangle 3.$ $A_1 \setminus (U \cup V)$ is nonempty
      PROOF: Otherwise $(A_1 \cap \cdots \cap A_n \cap U)$ and $(A_1 \cap \cdots \cap A_n \cap V)$ would form a separation of $A_n$.

$\langle 1 \rangle 4.$ $\bigcap \mathcal{A} \setminus (U \cup V)$ is nonempty.
   PROOF: Theorem 299.
$\langle 1 \rangle 5.$ Q.E.D.
   PROOF: This contradicts $\langle 1 \rangle 1$ since $\bigcap AA \setminus (U \cup V) = \bigcap \mathcal{A} \setminus (C \cup D)$.
□


# 46   Perfect Maps

**Definition 312** (Perfect Map). Let $X$ and $Y$ be topological spaces and $f :$ $X \to Y$. Then $f$ is a *perfect map* if and only if $f$ is a closed map, continuous, surjective and $p^{-1}(y)$ is compact for all $y \in Y$.

**Proposition 313.** *Let $X$ be a topological space, $Y$ a compact space, and $p :$ $X \to Y$ a closed map such that, for all $y \in Y$, we have $p^{-1}(y)$ is compact. Then $X$ is compact.*

PROOF:
$\langle 1 \rangle 1.$ LET: $\mathcal{A}$ be a set of closed sets in $X$ with the finite intersection property.
$\langle 1 \rangle 2.$ $\mathcal{B} = \{p(A_1 \cap \cdots \cap A_n) \mid A_1, \ldots, A_n \in \mathcal{A}\}$ is a set of closed sets in $Y$ with the finite intersection property.
   PROOF: Since $p$ is a closed map.
$\langle 1 \rangle 3.$ PICK $y \in \bigcap \mathcal{B}$
   PROOF: Theorem 299 since $Y$ is compact.
$\langle 1 \rangle 4.$ $\{A \cap p^{-1}(y) \mid A \in \mathcal{A}\}$ is a set of closed sets in $p^{-1}(y)$ with the finite intersection property.
$\langle 1 \rangle 5.$ PICK $x \in \bigcap_{A \in \mathcal{A}} (A \cap p^{-1}(y))$
   PROOF: Theorem 299 since $p^{-1}(y)$ is compact.
$\langle 1 \rangle 6.$ $x \in \bigcap \mathcal{A}$
$\langle 1 \rangle 7.$ Q.E.D.
   PROOF: Theorem 299.
□


# 47   Topological Groups

**Definition 314** (Topological Group). A *topological group $G$* consists of a $T_1$ space $G$ and continuous maps $\cdot : G^2 \to G$ and $(\ )^{-1} : G \to G$ such that $(G, \cdot, (\ )^{-1})$ is a group.

**Example 315.**   1. The integers $\mathbb{Z}$ under addition are a topological group.

   2. The real numbers $\mathbb{R}$ under addition are a topological group.

   3. The positive reals under multiplication are a topological group.

   4. The set $\{z \in \mathbb{C} \mid |z| = 1\}$ under multiplication and given the topology of $S^1$ is a topological group.

5. For any $n \geq 0$, the general linear group $GL_n(\mathbb{R})$ is a topological group under matrix multiplication, considered as a subspace of $\mathbb{R}^{n^2}$.

**Lemma 316.** *Let $G$ be a $T_1$ space and $\cdot : G^2 \to G$, $(\ )^{-1} : G \to G$ be functions such that $(G, \cdot, (\ )^{-1})$ is a group. Then $G$ is a topological group if and only if the function $f : G^2 \to G$ that maps $(x, y)$ to $xy^{-1}$ is continuous.*

PROOF:
$\langle 1 \rangle 1$. If $G$ is a topological group then $f$ is continuous.
    PROOF: From Theorem 96.
$\langle 1 \rangle 2$. If $f$ is continuous then $G$ is a topological group.
    $\langle 2 \rangle 1$. ASSUME: $f$ is continuous.
    $\langle 2 \rangle 2$. $(\ )^{-1}$ is continuous.
        PROOF: Since $x^{-1} = f(e, x)$.
    $\langle 2 \rangle 3$. $\cdot$ is continuous.
        PROOF: Since $xy = f(x, y^{-1})$.
☐

**Lemma 317.** *Let $G$ be a topological group and $H$ a subgroup of $G$. Then $H$ is a topological group under the subspace topology.*

PROOF:
$\langle 1 \rangle 1$. $H$ is $T_1$.
    PROOF: From Proposition 165.
$\langle 1 \rangle 2$. multiplication and inverse on $H$ are continuous.
    PROOF: From Theorem 97.
☐

**Lemma 318.** *Let $G$ be a topological group and $H$ a subgroup of $G$. Then $\overline{H}$ is a subgroup of $G$.*

PROOF:
$\langle 1 \rangle 1$. LET: $x, y \in \overline{H}$
    PROVE: $xy^{-1} \in \overline{H}$
$\langle 1 \rangle 2$. LET: $U$ be any neighbourhood of $xy^{-1}$
$\langle 1 \rangle 3$. LET: $f : G^2 \to G$, $f(a, b) = ab^{-1}$
$\langle 1 \rangle 4$. $f^{-1}(U)$ is a neighbourhood of $(x, y)$
$\langle 1 \rangle 5$. PICK neighbourhoods $V$, $W$ of $x$ and $y$ respectively such that $f(V \times W) \subseteq U$.
$\langle 1 \rangle 6$. PICK $a \in V \cap H$ and $b \in W \cap H$
    PROOF: Theorem 48.
$\langle 1 \rangle 7$. $ab^{-1} \in U \cap H$
$\langle 1 \rangle 8$. Q.E.D.
    PROOF: By Theorem 48.
☐

**Proposition 319.** *Let $G$ be a topological group and $\alpha \in G$. Then the maps $l_\alpha, r_\alpha : G \to G$ defined by $l_\alpha(x) = \alpha x$, $r_\alpha(x) = x\alpha$ are homeomorphisms of $G$ with itself.*

PROOF: They are continuous with continuous inverses $l_{\alpha^{-1}}$ and $r_{\alpha^{-1}}$. $\square$

**Corollary 319.1.** *Every topological group is homogeneous.*

PROOF: Given a topological group $G$ and $a, b \in G$, we have $l_{ba^{-1}}$ is a homeomorphism that maps $a$ to $b$. $\square$

**Proposition 320.** *Let $G$ be a topological group and $H$ a subgroup of $G$. Give $G/H$ the quotient topology. For all $\alpha \in G$, the map $\overline{f_\alpha}$ that sends $xH$ to $\alpha xH$ is a homeomorphism $G/H \cong G/H$.*

PROOF:
$\langle 1 \rangle 1.$ $\overline{f_\alpha}$ is well-defined.
    PROOF: If $xy^{-1} \in H$ then $(\alpha x)(\alpha y)^{-1} = xy^{-1} \in H$.
$\langle 1 \rangle 2.$ $\overline{f_\alpha}$ is continuous.
    PROOF: Theorem 198 since $\overline{f_\alpha} \circ p = p \circ f_\alpha$ is continuous, where $p : G \twoheadrightarrow G/H$ is the canonical surjection.
$\langle 1 \rangle 3.$ $\overline{f_\alpha}^{-1}$ is continuous.
    PROOF: Similar since $\overline{f_\alpha}^{-1} = \overline{f_{\alpha^{-1}}}$.
$\square$

**Corollary 320.1.** *Let $G$ be a topological group and $H$ a subgroup of $G$. Give $G/H$ the quotient topology. Then $G/H$ is homogeneous.*

**Proposition 321.** *Let $G$ be a topological group and $H$ a subgroup of $G$. Give $G/H$ the quotient topology. If $H$ is closed in $G$ then $G/H$ is $T_1$.*

PROOF:
$\langle 1 \rangle 1.$ LET: $p : G \twoheadrightarrow G/H$ be the canonical surjection
$\langle 1 \rangle 2.$ LET: $x \in G$
$\langle 1 \rangle 3.$ $p^{-1}(xH) = f_x(H)$
$\langle 1 \rangle 4.$ $p^{-1}(xH)$ is closed in $G$
    PROOF: Since $H$ is closed and $f_x$ is a homemorphism of $G$ with itself.
$\langle 1 \rangle 5.$ $\{xH\}$ is closed in $G/H$
$\square$

**Proposition 322.** *Let $G$ be a topological group and $H$ a subgroup of $G$. Give $G/H$ the quotient topology. Then the canonical surjection $p : G \twoheadrightarrow G/H$ is an open map.*

PROOF:
$\langle 1 \rangle 1.$ LET: $U \subseteq G$ be open.
$\langle 1 \rangle 2.$ $p^{-1}(p(U)) = \bigcup_{h \in H} r_h(U)$
$\langle 1 \rangle 3.$ $p^{-1}(p(U))$ is open.
$\langle 1 \rangle 4.$ $p(U)$ is open.
$\square$

**Proposition 323.** *Let $G$ be a topological group and $H$ a closed normal subgroup of $G$. Then $G/H$ is a topological group under the quotient topology.*

PROOF:

$\langle 1 \rangle 1.$ $G/H$ is $T_1$

    PROOF: Proposition 321.

$\langle 1 \rangle 2.$ The map $\overline{m} : (xH, yH) \mapsto xy^{-1}H$ is continuous.

    $\langle 2 \rangle 1.$ $p^2 : G^2 \to (G/H)^2$ is a quotient map.

      PROOF: Propositions 197, 322.

    $\langle 2 \rangle 2.$ $\overline{m} \circ p^2$ is continuous.

      PROOF: As it is $p^2 \circ m$ where $m : G^2 \to G$ with $m(x, y) = xy^{-1}$

▯

**Lemma 324.** *Let $G$ be a topological group and $A, B \subseteq G$. If either $A$ or $B$ is open then $AB$ is open.*

PROOF: If $A$ is open we have $AB = \bigcup_{b \in B} r_b(A)$. Similarly if $B$ is open. ▯

**Definition 325** (Symmetric Neighbourhood)**.** Let $G$ be a topological group. A neighbourhood $V$ of $e$ is *symmetric* if and only if $V = V^{-1}$.

**Lemma 326.** *Let $G$ be a topological group. Let $V$ be a neighbourhood of $e$. Then $V$ is symmetric if and only if, for all $x \in V$, we have $x^{-1} \in V$.*

PROOF:

$\langle 1 \rangle 1.$ If $V$ is symmetric then, for all $x \in V$, we have $x^{-1} \in V$

    PROOF: Immediate from defintions.

$\langle 1 \rangle 2.$ If, for all $x \in V$, we have $x^{-1} \in V$, then $V$ is symmetric.

    $\langle 2 \rangle 1.$ ASSUME: for all $x \in V$ we have $x^{-1} \in V$

    $\langle 2 \rangle 2.$ $V \subseteq V^{-1}$

      PROOF: If $x \in V$ then there exists $y \in V$ such that $x = y^{-1}$, namely $y = x^{-1}$

    $\langle 2 \rangle 3.$ $V^{-1} \subseteq V$

      PROOF: Immediate from $\langle 2 \rangle 1$.

▯

**Lemma 327.** *Let $G$ be a topological group. For every neighbourhood $U$ of $e$, there exists a symmetric neighbourhood $V$ of $e$ such that $V^2 \subseteq U$.*

PROOF:

$\langle 1 \rangle 1.$ LET: $U$ be a neighbourhood of $e$.

$\langle 1 \rangle 2.$ PICK a neighbourhood $V'$ of $e$ such that $V'V' \subseteq U$

    PROOF: Such a neighbourhood exists because multiplication in $G$ is continuous.

$\langle 1 \rangle 3.$ PICK a neighbourhood $W$ of $e$ such that $WW^{-1} \subseteq V'$

    PROOF: Such a neighbourhood exists because the function that maps $(x, y)$ to $xy^{-1}$ is continuous.

$\langle 1 \rangle 4.$ LET: $V = WW^{-1}$

$\langle 1 \rangle 5.$ $V$ is a neighbourhood of $e$

    $\langle 2 \rangle 1.$ $e \in V$

      PROOF: Since $e \in W$ so $e = ee^{-1} \in V$.

$\langle 2 \rangle 2$. $V$ is open

 PROOF: Lemma 324.

$\langle 1 \rangle 6$. $V$ is symmetric

 $\langle 2 \rangle 1$. For all $x \in V$ we have $x^{-1} \in V$

  $\langle 3 \rangle 1$. LET: $x \in V$

  $\langle 3 \rangle 2$. PICK $y, z \in W$ such that $x = yz^{-1}$

  $\langle 3 \rangle 3$. $x^{-1} = zy^{-1}$

  $\langle 3 \rangle 4$. $x^{-1} \in V$

  $\langle 3 \rangle 5$. $x \in V^{-1}$

 $\langle 2 \rangle 2$. Q.E.D.

 PROOF: Lemma 326

$\langle 1 \rangle 7$. $V^2 \subseteq U$

 PROOF: We have $V^2 \subseteq (V')^2 \subseteq U$

☐

**Proposition 328.** *Every topological group is Hausdorff.*

PROOF:

$\langle 1 \rangle 1$. LET: $G$ be a topological group.

$\langle 1 \rangle 2$. LET: $x, y \in G$ with $x \neq y$

$\langle 1 \rangle 3$. LET: $U = G \setminus \{x[^{-1}y]\}$

$\langle 1 \rangle 4$. PICK a symmetric neighbourhood $V$ of $e$ such that $VV \subseteq U$

 $\langle 2 \rangle 1$. $U$ is open

 PROOF: Since $G$ is $T_1$.

 $\langle 2 \rangle 2$. $e \in U$

 PROOF: Since $x \neq y$

 $\langle 2 \rangle 3$. Q.E.D.

 PROOF: Lemma 327.

$\langle 1 \rangle 5$. $Vx$ and $Vy$ are disjoint neighbourhoods of $x$ and $y$ respectively.

 $\langle 2 \rangle 1$. $Vx$ is open

 PROOF: Since $Vx = r_x(V)$

 $\langle 2 \rangle 2$. $Vy$ is open

 PROOF: Similar.

 $\langle 2 \rangle 3$. $Vx \cap Vy = \emptyset$

  $\langle 3 \rangle 1$. ASSUME: for a contradiction $z \in Vx \cap Vy$

  $\langle 3 \rangle 2$. PICK $a, b \in V$ such that $z = ax = by$

  $\langle 3 \rangle 3$. $xy^{-1} \in VV$

   PROOF: Since $xy^{-1} = a^{-1}b$

  $\langle 3 \rangle 4$. $xy^{-1} \in U$

  $\langle 3 \rangle 5$. Q.E.D.

   PROOF: From $\langle 1 \rangle 3$.

☐

**Proposition 329.** *Every topological group is regular.*

PROOF:

$\langle 1 \rangle 1$. LET: $G$ be a topological group.

$\langle 1 \rangle 2$. LET: $A \subseteq G$ be a closed set and $a \notin A$.

$\langle 1 \rangle 3$. LET: $U = G \setminus Aa^{-1}$

$\langle 1 \rangle 4$. PICK a symmetric neighbourhood $V$ of $e$ such that $VV \subseteq U$

  $\langle 2 \rangle 1$. $U$ is open

    PROOF: Since $Aa^{-1} = r_{a^{-1}}(A)$ is closed.

  $\langle 2 \rangle 2$. $e \in U$

    PROOF: Since $a \notin A$.

  $\langle 2 \rangle 3$. Q.E.D.

    PROOF: Lemma 327.

$\langle 1 \rangle 5$. $VA$ and $Va$ are disjoint open sets with $A \subseteq VA$ and $a \in Va$

  $\langle 2 \rangle 1$. $VA$ is open

    PROOF: Lemma 324

  $\langle 2 \rangle 2$. $Va$ is open

    PROOF: Lemma 324

  $\langle 2 \rangle 3$. $VA \cap Va = \emptyset$

    $\langle 3 \rangle 1$. ASSUME: for a contradiction $z \in VA \cap Va$

    $\langle 3 \rangle 2$. PICK $b, c \in V$ and $d \in A$ with $z = bd = ca$

    $\langle 3 \rangle 3$. $da^{-1} \in U$

      PROOF: Since $da^{-1} = b^{-1}c \in VV \subseteq U$

    $\langle 3 \rangle 4$. Q.E.D.

      PROOF: This contradicts $\langle 1 \rangle 3$

$\square$

**Proposition 330.** *Let $G$ be a topological group and $H$ a subgroup of $G$. Give $G/H$ the quotient topology. If $H$ is closed in $G$ then $G/H$ is regular.*

PROOF:

$\langle 1 \rangle 1$. LET: $p : G \twoheadrightarrow G/H$ be the canonical surjection.

$\langle 1 \rangle 2$. LET: $A$ be a closed set in $G/H$ and $aH \in (G/H) \setminus A$.

$\langle 1 \rangle 3$. LET: $B = p^{-1}(A)$

$\langle 1 \rangle 4$. $B$ is a closed saturated set in $G$.

$\langle 1 \rangle 5$. $B \cap aH = \emptyset$

$\langle 1 \rangle 6$. $B = BH$

$\langle 1 \rangle 7$. PICK a symmetric neighbourhood $V$ of $e$ such that $VB$ does not intersect $Va$

  $\langle 2 \rangle 1$. LET: $U = G \setminus Ba^{-1}$

  $\langle 2 \rangle 2$. PICK a symmetric neighbourhood $V$ of $e$ such that $VV \subseteq U$

    $\langle 3 \rangle 1$. $U$ is open

      PROOF: Since $Ba^{-1} = r_{a^{-1}}(B)$ is closed.

    $\langle 3 \rangle 2$. $e \in U$

      PROOF: If $e \in Ba^{-1}$ then $a \in B$

    $\langle 3 \rangle 3$. Q.E.D.

      PROOF: Lemma 327

  $\langle 2 \rangle 3$. $VB \cap Va = \emptyset$

    PROOF: If $vb = v'a$ for $v, v' \in V$ and $b \in B$ then we have $ba^{-1} = v^{-1}v' \in Ba \cap VV \subseteq Ba \cap U$.

$\langle 1 \rangle 8$. $p(VB)$ and $p(Va)$ are disjoint open sets

$\langle 2 \rangle$1. $p(VB)$ and $p(Va)$ are open.
  PROOF: Proposition 322.
$\langle 2 \rangle$2. $p(VB) \cap p(Va) = \emptyset$
  PROOF: If $vbH = v'aH$ for $v, v' \in V$, $b \in B$ then $v'a = vbh$ for some $h \in H$.
  Hence $v'a \in Va \cap VBH = Va \cap VB$.
$\langle 1 \rangle$9. $A \subseteq p(VB)$
$\langle 1 \rangle$10. $aH \in p(Va)$
$\square$

**Proposition 331.** *Let $G$ be a topological group. The component of $G$ that contains $e$ is a normal subgroup of $G$.*

PROOF:
$\langle 1 \rangle$1. LET: $C$ be the component of $G$ that contains $e$.
$\langle 1 \rangle$2. For all $x \in G$, $xC$ is the component of $G$ that contains $x$.
  $\langle 2 \rangle$1. LET: $x \in G$
  $\langle 2 \rangle$2. LET: $D$ be the component of $G$ that contains $x$.
  $\langle 2 \rangle$3. $xC \subseteq D$
    PROOF: Since $xC$ is connected by Theorem 222.
  $\langle 2 \rangle$4. $D \subseteq xC$
    PROOF: Since $x^{-1}D \subseteq C$ similarly.
$\langle 1 \rangle$3. For all $x \in G$, $Cx$ is the component of $G$ that contains $x$.
  PROOF: Similar.
$\langle 1 \rangle$4. For all $x \in C$ we have $xC = Cx = C$
$\langle 1 \rangle$5. For all $x \in C$ we have $x^{-1}C = C$
$\langle 1 \rangle$6. For all $x \in C$ we have $x^{-1} \in C$
$\langle 1 \rangle$7. For all $x, y \in C$ we have $xy \in C$
  PROOF: Since $xyC = xC = x$.
$\langle 1 \rangle$8. For all $x \in G$ we have $xC = Cx$.
  PROOF: From $\langle 1 \rangle$2 and $\langle 1 \rangle$3.
$\square$

**Lemma 332.** *Let $G$ be a topological group. Let $A$ be a closed set in $G$ and $B$ a compact subspace of $G$ such that $A \cap B = \emptyset$. Then there exists a symmetric neighbourhood $U$ of $e$ such that $AU \cap BU = \emptyset$.*

PROOF:
$\langle 1 \rangle$1. For all $b \in B$ there exists a symmetric neighbourhood $V$ of $e$ such that $bV^2 \cap A = \emptyset$
  $\langle 2 \rangle$1. LET: $b \in B$
  $\langle 2 \rangle$2. LET: $W = b^{-1}(G \setminus A)$
  $\langle 2 \rangle$3. $W$ is a neighbourhood of $e$ and $bW \cap A = \emptyset$
  $\langle 2 \rangle$4. PICK a symmetric neighbourhood $V$ of $e$ such that $V^2 \subseteq W$
$\langle 1 \rangle$2. $\{bV^2 \mid b \in B, V$ is a symmetric neighbourhood of $e$, $bV^2 \cap A = \emptyset\}$ is an open cover of $B$
$\langle 1 \rangle$3. PICK a finite subcover $b_1 V_1^2, \ldots, b_n V_n^2$, say.
$\langle 1 \rangle$4. LET: $U = V_1 \cap \cdots \cap V_n$

$\langle 1 \rangle 5.$ $BU^2 \cap A = \emptyset$
$\langle 1 \rangle 6.$ $AU \cap BU = \emptyset$
    PROOF: If $av \in BU$ where $a \in A$ and $v \in V$ then $a = avv^{-1} \in BU^2 \cap A.$
☐

**Proposition 333** (AC). *Let $G$ be a topological group. Let $A$ be a closed set in $G$, and $B$ a compact subspace of $G$. Then $AB$ is closed.*

PROOF:
$\langle 1 \rangle 1.$ LET: $x \in G \setminus AB$
$\langle 1 \rangle 2.$ $A^{-1}x \cap B = \emptyset$
$\langle 1 \rangle 3.$ $A^{-1}x$ is closed.
$\langle 1 \rangle 4.$ PICK a symmetric neighbourhood $U$ of $e$ such that $A^{-1}xU \cap BU = \emptyset$
$\langle 1 \rangle 5.$ $xU^2$ is open
    PROOF: Lemma 324.
$\langle 1 \rangle 6.$ $x \in xU^2 \subseteq G \setminus AB$
☐

**Corollary 333.1.** *Let $G$ be a topological group and $H \leq G$. Let $p : G \twoheadrightarrow G/H$ be the quotient map. If $H$ is compact then $p$ is a closed map.*

PROOF: For $A$ closed in $G$, we have $p^{-1}(p(A)) = AH$ is closed, and so $p(A)$ is closed. ☐

**Corollary 333.2.** *Let $G$ be a topological group and $H \leq G$. If $H$ and $G/H$ are compact then $G$ is compact.*

PROOF: From Proposition 313 since, for all $aH \in G/H$, we have $p^{-1}(aH) = aH$ is compact because it is homemorphic to $H$. ☐

# 48  The Metric Topology

**Definition 334** (Metric). Let $X$ be a set. A *metric* on $X$ is a function $d : X^2 \to \mathbb{R}$ such that:

1. For all $x, y \in X$, $d(x, y) \geq 0$

2. For all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$

3. For all $x, y \in X$, $d(x, y) = d(y, x)$

4. (*Triangle Inequality*) For all $x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$

We call $d(x, y)$ the *distance* between $x$ and $y$.

**Definition 335** (Open Ball). Let $X$ be a metric space. Let $a \in X$ and $\epsilon > 0$. The *open ball* with *centre* $a$ and *radius* $\epsilon$ is

$$B(a, \epsilon) = \{x \in X \mid d(a, x) < \epsilon\} \ .$$

**Definition 336** (Metric Topology). Let $X$ be a metric space. The *metric topology* on $X$ is the topology generated by the basis consisting of all the open balls.

We prove this is a basis for a topology.

PROOF:
$\langle 1 \rangle 1$. For every point $a$, there exists a ball $B$ such that $a \in B$
   PROOF: We have $a \in B(a, 1)$.
$\langle 1 \rangle 2$. For any balls $B_1$, $B_2$ and point $a \in B_1 \cap B_2$, there exists a ball $B_3$ such that $a \in B_3 \subseteq B_1 \cap B_2$
   $\langle 2 \rangle 1$. LET: $B_1 = B(c_1, \epsilon_1)$ and $B_2 = B(c_2, \epsilon_2)$
   $\langle 2 \rangle 2$. LET: $\delta = \min(\epsilon_1 - d(c_1, a), \epsilon_2 - d(c_2, a))$
      PROVE: $B(a, \delta) \subseteq B_1 \cap B_2$
   $\langle 2 \rangle 3$. LET: $x \in B(a, \delta)$
   $\langle 2 \rangle 4$. $x \in B_1$
     PROOF:
$$d(x, c_1) = d(x, a) + d(a, c_1)$$
$$< \delta + d(a, c_1)$$
$$\leq \epsilon_1$$
   $\langle 2 \rangle 5$. $x \in B_2$
     PROOF: Similar.
$\square$

**Proposition 337.** *Let $X$ be a metric space and $U \subseteq X$. Then $U$ is open if and only if, for every $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$.*

PROOF:
$\langle 1 \rangle 1$. If $U$ is open then, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$.
   $\langle 2 \rangle 1$. ASSUME: $U$ is open.

⟨2⟩2. Let: $x \in U$
⟨2⟩3. Pick $a \in X$ and $\delta > 0$ such that $x \in B(a, \delta) \subseteq U$
⟨2⟩4. Let: $\epsilon = \delta - d(a, x)$
      Prove:  $B(x, \epsilon) \subseteq U$
⟨2⟩5. Let: $y \in B(x, \epsilon)$
⟨2⟩6. $d(y, a) < \delta$
  Proof:

$$d(y, a) \leq d(a, x) + d(x, y)$$
$$< \delta + d(x, y)$$
$$= \epsilon$$

⟨2⟩7. $y \in U$
⟨1⟩2. If, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$, then $U$ is open.
   Proof: Immediate from definitions.
□

**Definition 338** (Discrete Metric)**.** Let $X$ be a set. The *discrete metric* on $X$ is defined by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

**Proposition 339.** *The discrete metric induces the discrete topology.*

Proof: For any (open) set $U$ and point $a \in U$, we have $a \in B(a, 1) \subseteq U$. □

**Definition 340** (Standard Metric on $\mathbb{R}$)**.** The *standard metric* on $\mathbb{R}$ is defined by $d(x, y) = |x - y|$.

**Proposition 341.** *The standard metric on $\mathbb{R}$ induces the standard topology on $\mathbb{R}$.*

Proof:
⟨1⟩1. Every open ball is open in the standard topology on $\mathbb{R}$.
   Proof: $B(a, \epsilon) = (a - \epsilon, a + \epsilon)$
⟨1⟩2. For every open set $U$ and point $a \in U$, there exists $\epsilon > 0$ such that $B(a, \epsilon) \subseteq U$
  ⟨2⟩1. Let: $U$ be an open set and $a \in U$
  ⟨2⟩2. Pick an open interval $b$, $c$ such that $a \in (b, c) \subseteq U$
  ⟨2⟩3. Let: $\epsilon = \min(a - b, c - a)$
  ⟨2⟩4. $B(a, \epsilon) \subseteq U$
□

**Definition 342** (Metrizable)**.** A topological space $X$ is *metrizable* if and only if there exists a metric on $X$ that induces the topology.

**Definition 343** (Bounded)**.** Let $X$ be a metric space and $A \subseteq X$. Then $A$ is *bounded* if and only if there exists $M$ such that, for all $x, y \in A$, we have $d(x, y) \leq M$.

**Definition 344** (Diameter)**.** Let $X$ be a metric space and $A \subseteq X$. The *diameter* of $A$ is

$$\operatorname{diam} A = \sup_{x,y \in A} d(x,y) \ .$$

**Definition 345** (Standard Bounded Metric)**.** Let $d$ be a metric on $X$. The *standard bounded metric* corresponding to $d$ is the metric $\overline{d}$ defined by

$$\overline{d}(x,y) = \min(d(x,y), 1) \ .$$

We prove this is a metric.

PROOF:
$\langle 1 \rangle 1.$ $\overline{d}(x,y) \geq 0$
  PROOF: Since $d(x,y) \geq 0$
$\langle 1 \rangle 2.$ $\overline{d}(x,y) = 0$ if and only if $x = y$
  PROOF: $\overline{d}(x,y) = 0$ if and only if $d(x,y) = 0$ if and only if $x = y$
$\langle 1 \rangle 3.$ $\overline{d}(x,y) = \overline{d}(y,x)$
  PROOF: Since $d(x,y) = d(y,x)$
$\langle 1 \rangle 4.$ $\overline{d}(x,z) \leq \overline{d}(x,y) + \overline{d}(y,z)$
  PROOF:
$$\begin{aligned}
\overline{d}(x,y) + \overline{d}(y,z) &= \min(d(x,y), 1) + \min(d(y,z), 1) \\
&= \min(d(x,y) + d(y,z), d(x,y) + 1, d(y,z) + 1, 2) \\
&\geq \min(d(x,z), 1) \\
&= \overline{d}(x,z)
\end{aligned}$$

□

**Lemma 346.** *In any metric space $X$, the set $\mathcal{B} = \{B(a, \epsilon) \mid a \in X, \epsilon < 1\}$ is a basis for the metric topology.*

PROOF:
$\langle 1 \rangle 1.$ Every element of $\mathcal{B}$ is open.
  PROOF: From Lemma 62.
$\langle 1 \rangle 2.$ For every open set $U$ and point $a \in U$, there exists $B \in \mathcal{B}$ such that $a \in B \subseteq U$
  $\langle 2 \rangle 1.$ LET: $U$ be an open set and $a \in U$
  $\langle 2 \rangle 2.$ PICK $\epsilon > 0$ such that $B(a, \epsilon) \subseteq U$
  $\langle 2 \rangle 3.$ $B(a, \min(\epsilon, 1/2)) \subseteq U$
$\langle 1 \rangle 3.$ Q.E.D.
  PROOF: Lemma 63.

□

**Proposition 347.** *Let $d$ be a metric on the set $X$. Then the standard bounded metric $\overline{d}$ induces the same metric as $d$.*

PROOF: This follows from Lemma 346 since the open balls with radius $< 1$ are the same under both metrics. □

**Lemma 348.** *Let $d$ and $d'$ be two metrics on the same set $X$. Let $\mathcal{T}$ and $\mathcal{T}'$ be the topologies they induce. Then $\mathcal{T} \subseteq \mathcal{T}'$ if and only if, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that*

$$B_{d'}(x, \delta) \subseteq B_d(x, \epsilon) \ .$$

PROOF:

$\langle 1 \rangle 1.$ If $\mathcal{T} \subseteq \mathcal{T}'$ then, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$

PROOF: From Proposition 337 since $x \in B_d(x, \epsilon) \in \mathcal{T}'$.

$\langle 1 \rangle 2.$ If, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$, then $\mathcal{T} \subseteq \mathcal{T}'$

$\quad \langle 2 \rangle 1.$ ASSUME: For all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$

$\quad \langle 2 \rangle 2.$ LET: $U \in \mathcal{T}$

$\quad \langle 2 \rangle 3.$ For all $x \in U$ there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq U$.

$\quad \quad \langle 3 \rangle 1.$ LET: $x \in U$

$\quad \quad \langle 3 \rangle 2.$ PICK $\epsilon > 0$ such that $B_d(x, \epsilon) \subseteq U$

$\quad \quad \quad$ PROOF: Proposition 337

$\quad \quad \langle 3 \rangle 3.$ PICK $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$

$\quad \quad \quad$ PROOF: By $\langle 2 \rangle 1$

$\quad \quad \langle 3 \rangle 4.$ $B_{d'}(x, \delta) \subseteq U$

$\quad \langle 2 \rangle 4.$ $U \in \mathcal{T}'$

$\quad \quad$ PROOF: Proposition 337.

$\square$

**Proposition 349.** $\mathbb{R}^2$ *under the dictionary order topology is metrizable.*

PROOF: Define $d : \mathbb{R}^2 \to \mathbb{R}$ by
$$d((x, y), (x, z)) = \max(|y - z|, 1)$$

$$d((x, y), (x', y')) = 1 \qquad \qquad \text{if } x \neq x' \square$$

$\langle 1 \rangle 1.$ $x \in \bigcap_{i=1}^{N} \pi_i{}^{-1}() \subseteq B_D(a, \epsilon)$

**Proposition 350.** *Let $d : X^2 \to \mathbb{R}$ be a metric on $X$. Then the metric topology on $X$ is the coarsest topology such that $d$ is continuous.*

PROOF:

$\langle 1 \rangle 1.$ $d$ is continuous.

$\quad \langle 2 \rangle 1.$ LET: $a, b \in X$

$\quad \langle 2 \rangle 2.$ LET: $\epsilon > 0$

$\quad \langle 2 \rangle 3.$ LET: $\delta = \epsilon/2$

$\quad \langle 2 \rangle 4.$ LET: $x, y \in X$

$\quad \langle 2 \rangle 5.$ ASSUME: $\rho((a, b), (x, y)) < \delta$

$\quad \langle 2 \rangle 6.$ $|d(a, b) - d(x, y)| < \epsilon$

$\quad \quad \langle 3 \rangle 1.$ $d(a, b) - d(x, y) < \epsilon$

PROOF:

$$d(a, b) \leq d(a, x) + d(x, y) + d(y, b)$$
$$\leq d(x, y) + 2\rho((a, b), (x, y))$$
$$< d(x, y) + 2\delta$$
$$= d(x, y) + \epsilon$$

$\langle 3 \rangle 2.$ $d(a.b) - d(x, y) > -\epsilon$
    PROOF: Similar.
  $\langle 2 \rangle 7.$ Q.E.D.
$\langle 1 \rangle 2.$ If $\mathcal{T}$ is any topology under which $d$ is continuous then $\mathcal{T}$ is finer than the metric topology.
    PROOF: Since $B(a, \epsilon) = d_a{}^{-1}((-\infty, \epsilon))$

▯

**Proposition 351.** *Let $X$ be a metric space with metric $d$ and $A \subseteq X$. The restriction of $d$ to $A$ is a metric on $A$ that induces the subspace topology.*

PROOF:
$\langle 1 \rangle 1.$ The restriction of $d$ to $A$ is a metric on $A$.
$\langle 1 \rangle 2.$ Every open ball under $d \upharpoonright A$ is open under the subspace topology.
    PROOF: $B_{d \upharpoonright A}(a, \epsilon) = B_d(a, \epsilon) \cap A$.
$\langle 1 \rangle 3.$ If $U$ is open in the subspace topology and $x \in U$, then there exists a $d \upharpoonright A$-ball $B$ such that $x \in B \subseteq U$.
  $\langle 2 \rangle 1.$ PICK $V$ open in $X$ such that $U = V \cap A$
  $\langle 2 \rangle 2.$ PICK $\epsilon > 0$ such that $B_d(x, \epsilon) \subseteq V$
  $\langle 2 \rangle 3.$ Take $B = B_{d \upharpoonright A}(x, \epsilon)$

▯

**Corollary 351.1.** *A subspace of a metrizable space is metrizable.*

**Proposition 352.** *Every metrizable space is Hausdorff.*

PROOF:
$\langle 1 \rangle 1.$ LET: $X$ be a metric space
$\langle 1 \rangle 2.$ LET: $a, b \in X$ with $a \neq b$
$\langle 1 \rangle 3.$ LET: $\epsilon = d(a, b)/2$
$\langle 1 \rangle 4.$ LET: $U = B(a, \epsilon)$ and $V = B(b, \epsilon)$
$\langle 1 \rangle 5.$ $U$ and $V$ are disjoint neighbourhoods of $a$ and $b$ respectively.

▯

**Proposition 353** (CC)**.** *The product of a countable family of metrizable spaces is metrizable.*

PROOF:
$\langle 1 \rangle 1.$ LET: $(X_n, d_n)$ be a sequence of metric spaces.
$\langle 1 \rangle 2.$ ASSUME: w.l.o.g. each $d_n$ is bounded above by 1.
    PROOF: By Proposition 347.
$\langle 1 \rangle 3.$ LET: $D$ be the metric on $\mathbb{R}^\omega$ defined by $D(x, y) = \sup_i (d_i(x_i, y_i)/i)$.

$\langle 2 \rangle 1.$ $D(x,y) \geq 0$

$\langle 2 \rangle 2.$ $D(x,y) = 0$ if and only if $x = y$

$\langle 2 \rangle 3.$ $D(x,y) = D(y,x)$

$\langle 2 \rangle 4.$ $D(x,z) \leq D(x,y) + D(y,z)$

PROOF:

$$D(x,z) = \sup_i \frac{d_i(x_i, z_i)}{i}$$
$$\leq \sup_i \frac{d_i(x_i, y_i) + d_i(y_i, z_i)}{i}$$
$$\leq \sup_i \frac{d_i(x_i, y_i)}{i} + \sup_i \frac{d_i(y_i, z_i)}{i}$$
$$= D(x,y) + D(y,z)$$

$\langle 1 \rangle 4.$ Every open ball $B_D(a, \epsilon)$ is open in the product topology.

$\quad \langle 2 \rangle 1.$ PICK $N$ such that $1/\epsilon < N$

$\quad \langle 2 \rangle 2.$ $B_D(a, \epsilon) = \prod_{i=1}^{\infty} U_i$ where $U_i = B(a_i, i\epsilon)$ if $i \leq N$, and $U_i = X_i$ if $i > N$

$\langle 1 \rangle 5.$ For any open set $U$ and $a \in U$, there exists $\epsilon > 0$ such that $B_D(a, \epsilon) \subseteq U$.

$\quad \langle 2 \rangle 1.$ LET: $n \geq 1$, $V$ be an open set in $\mathbb{R}$ and $a \in \pi_n{}^{-1}(V)$

$\quad \langle 2 \rangle 2.$ PICK $\epsilon > 0$ such that $B_{d_n}(a, \epsilon) \subseteq V$

$\quad \langle 2 \rangle 3.$ $B_D(a, \epsilon/n) \subseteq \pi_n{}^{-1}(V)$

$\square$

**Theorem 354.** *Let $X$ and $Y$ be metric spaces and $f : X \to Y$. Then $f$ is continuous if and only if, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x,y) < \delta$ then $d(f(x), f(y)) < \epsilon$.*

PROOF:

$\langle 1 \rangle 1.$ If $f$ is continuous then, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x,y) < \delta$ then $d(f(x), f(y)) < \epsilon$

$\quad \langle 2 \rangle 1.$ ASSUME: $f$ is continuous.

$\quad \langle 2 \rangle 2.$ LET: $x \in X$ and $\epsilon > 0$

$\quad \langle 2 \rangle 3.$ PICK a neighbourhood $U$ of $x$ such that $f(U) \subseteq B(f(x), \epsilon)$

$\qquad$ PROOF: Theorem 93.

$\quad \langle 2 \rangle 4.$ PICK $\delta > 0$ such that $B(x, \delta) \subseteq U$

$\qquad$ PROOF: Proposition 337.

$\quad \langle 2 \rangle 5.$ For all $y \in X$, if $d(x,y) < \delta$ then $d(f(x), f(y)) < \epsilon$

$\langle 1 \rangle 2.$ If for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x,y) < \delta$ then $d(f(x), f(y)) < \epsilon$, then $f$ is continuous.

$\quad \langle 2 \rangle 1.$ ASSUME: for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x,y) < \delta$ then $d(f(x), f(y)) < \epsilon$

$\quad \langle 2 \rangle 2.$ LET: $x \in X$ and $V$ be a neighbourhood of $f(x)$

$\quad \langle 2 \rangle 3.$ PICK $\epsilon > 0$ such that $B(f(x), \epsilon) \subseteq V$

$\qquad$ PROOF: Proposition 337.

$\quad \langle 2 \rangle 4.$ PICK $\delta > 0$ such that, for all $y \in X$, if $d(x,y) < \delta$ then $d(f(x), f(y)) < \epsilon$

$\qquad$ PROOF: By $\langle 2 \rangle 1$

$\quad \langle 2 \rangle 5.$ LET: $U = B(x, \delta)$

$\quad \langle 2 \rangle 6.$ $U$ is a neighbourhood of $x$ with $f(U) \subseteq V$

⟨2⟩7. Q.E.D.
    PROOF: Theorem 93.
☐

**Proposition 355.** *Let $X$ be a metric space. Let $(a_n)$ be a sequence in $X$ and $l \in X$. Then $a_n \to l$ as $n \to \infty$ if and only if, for all $\epsilon > 0$, there exists $N$ such that, for all $n \geq N$, we have $d(a_n, l) < \epsilon$.*

PROOF: From Proposition 76. ☐

**Proposition 356.** *Every metrizable space is first countable.*

PROOF: In any metric space $X$, the open balls $B(a, 1/n)$ for $n \geq 1$ form a local basis at $a$.

**Example 357.** $\mathbb{R}^\omega$ under the box topology is not metrizable.

**Example 358.** If $J$ is uncountable then $\mathbb{R}^J$ under the product topology is not metrizable.

**Proposition 359.** *A compact subspace of a metric space is bounded.*

PROOF:
⟨1⟩1. LET: $X$ be a metric space and $A \subseteq X$ be compact.
⟨1⟩2. PICK $a \in A$
⟨1⟩3. $\{B(a, n) \mid n \in \mathbb{Z}^+\}$ covers $A$
⟨1⟩4. PICK a finite subcover $\{B(a, n_1), \ldots, B(a, n_k)\}$
⟨1⟩5. LET: $N = \max(n_1, \ldots, n_k)$
⟨1⟩6. For all $x, y \in A$ we have $d(x, y) < 2N$
    PROOF:
$$d(x, y) \leq d(x, a) + d(a, y)$$
$$< N + N$$
☐

    This example shows the converse does not hold:

**Example 360.** The space $\mathbb{R}$ under the standard bounded metric is bounded but not compact.

# 49   Real Linear Algebra

**Definition 361** (Square Metric). The *square metric* $\rho$ on $\mathbb{R}^n$ is defined by

$$\rho(\vec{x}, \vec{y}) = \max(|x_1 - y_1|, \ldots, |x_n - y_n|)$$

    We prove this is a metric.

PROOF:
⟨1⟩1. $\rho(\vec{x}, \vec{y}) \geq 0$
    PROOF: Immediate from definition.

$\langle 1 \rangle 2$. $\rho(\vec{x}, \vec{y}) = 0$ if and only if $\vec{x} = \vec{y}$
   PROOF: Immediate from definition.
$\langle 1 \rangle 3$. $\rho(\vec{x}, \vec{y}) = \rho(\vec{y}, \vec{x})$
   PROOF: Immediate from definition.
$\langle 1 \rangle 4$. $\rho(\vec{x}, \vec{z}) \leq \rho(\vec{x}, \vec{y}) + \rho(\vec{y}, \vec{z})$
   PROOF: Since $|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$.
$\square$

**Proposition 362.** *The square metric induces the standard topology on $\mathbb{R}^n$.*

PROOF:
$\langle 1 \rangle 1$. For every $a \in X$ and $\epsilon > 0$, we have $B_\rho(a, \epsilon)$ is open in the standard product topology.
   PROOF:
$$B_\rho(a, \epsilon) = (a_1 - \epsilon, a_1 + \epsilon) \times \cdots \times (a_n - \epsilon, a_n + \epsilon)$$
$\langle 1 \rangle 2$. For any open sets $U_1, \ldots, U_n$ in $\mathbb{R}$, we have $U_1 \times \cdots \times U_n$ is open in the square metric topology.
   $\langle 2 \rangle 1$. LET: $\vec{a} \in U_1 \times \cdots \times U_n$
   $\langle 2 \rangle 2$. For $i = 1, \ldots, n$, PICK $\epsilon_i > 0$ such that $(a_i - \epsilon_i, a_i + \epsilon_i) \subseteq U_i$
   $\langle 2 \rangle 3$. LET: $\epsilon = \min(\epsilon_1, \ldots, \epsilon_n)$
   $\langle 2 \rangle 4$. $B_\rho(\vec{a}, \epsilon) \subseteq U$
$\square$

**Definition 363.** Given $\vec{x}, \vec{y} \in \mathbb{R}^n$, define the *sum* $\vec{x} + \vec{y}$ by

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n) \ .$$

**Definition 364.** Given $\lambda \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^n$, define the *scalar product* $\lambda \vec{x} \in \mathbb{R}^n$ by

$$\lambda(x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n)$$

**Definition 365** (Inner Product)**.** Given $\vec{x}, \vec{y} \in \mathbb{R}^n$, define the *inner product* $\vec{x} \cdot \vec{y} \in \mathbb{R}$ by

$$(x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) = x_1 y_1 + \cdots + x_n y_n \ .$$

We write $\vec{x}^2$ for $\vec{x} \cdot \vec{x}$.

**Definition 366** (Norm)**.** Let $n \geq 1$. The *norm* on $\mathbb{R}^n$ is the function $\| \ \| : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\|(x_1, \ldots, x_n)\| = \sqrt{x_1^2 + \cdots + x_n^2}$$

**Lemma 367.**
$$\|\vec{x}\|^2 = \vec{x}^2$$

PROOF: Immediate from definitions. $\square$

**Lemma 368.**
$$\vec{x} \cdot (\vec{y} + \vec{z}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z}$$

PROOF: Each is equal to $(x_1y_1 + x_1z_1, \ldots, x_ny_n + x_nz_n)$. $\Box$

**Lemma 369.**
$$|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\|\|\vec{y}\|$$

PROOF:
$\langle 1 \rangle 1.$ ASSUME: $\vec{x} \neq \vec{0} \neq \vec{y}$
  PROOF: Otherwise both sides are 0.
$\langle 1 \rangle 2.$ LET: $a = 1/\|\vec{x}\|$
$\langle 1 \rangle 3.$ LET: $b = 1/\|\vec{y}\|$
$\langle 1 \rangle 4.$ $(a\vec{x} + b\vec{y})^2 \geq 0$ and $(a\vec{x} - b\vec{y})^2 \geq 0$
$\langle 1 \rangle 5.$ $a^2\|\vec{x}\|^2 + 2ab\vec{x} \cdot \vec{y} + b^2\|\vec{y}\|^2 \geq 0$ and $a^2\|\vec{x}\|^2 - 2ab\vec{x} \cdot \vec{y} + b^2\|\vec{y}\|^2 \geq 0$
$\langle 1 \rangle 6.$ $2ab\vec{x} \cdot \vec{y} + 2 \geq 0$ and $-2ab\vec{x} \cdot \vec{y} + 2 \geq 0$
$\langle 1 \rangle 7.$ $\vec{x} \cdot \vec{y} \geq -1/ab$ and $\vec{x} \cdot \vec{y} \leq 1/ab$
$\langle 1 \rangle 8.$ $\vec{x} \cdot \vec{y} \geq -\|\vec{x}\|\|\vec{y}\|$ and $\vec{x} \cdot \vec{y} \leq \|\vec{x}\|\|\vec{y}\|$
$\Box$

**Lemma 370** (Triangle Inequality)**.**

$$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$$

PROOF:
$$\begin{aligned}
\|\vec{x} + \vec{y}\|^2 &= \|\vec{x}\|^2 + 2\vec{x} \cdot \vec{y} + \|\vec{y}\|^2 \\
&\leq \|\vec{x}\|^2 + 2\|\vec{x}\|\|\vec{y}\| + \|\vec{y}\|^2 \qquad \text{(Lemma 369)} \\
&= (\|\vec{x}\| + \|\vec{y}\|)^2 \qquad\qquad\qquad\quad \Box
\end{aligned}$$

**Definition 371** (Euclidean Metric)**.** Let $n \geq 1$. The *Euclidean metric* on $\mathbb{R}^n$ is defined by
$$d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\| \ .$$

We prove this is a metric.

$\langle 1 \rangle 1.$ $d(\vec{x}, \vec{y}) \geq 0$
  PROOF: Immediate from definition.
$\langle 1 \rangle 2.$ $d(\vec{x}, \vec{y}) = 0$ if and only if $\vec{x} = \vec{y}$
  PROOF: $d(\vec{x}, \vec{y}) = 0$ if and only if $\vec{x} - \vec{y} = \vec{0}$.
$\langle 1 \rangle 3.$ $d(\vec{x}, \vec{y}) = d(\vec{y}, \vec{x})$
  PROOF: Immediate from definition.
$\langle 1 \rangle 4.$ $d(\vec{x}, \vec{z}) \leq d(\vec{x}, \vec{y}) + d(\vec{y}, \vec{z})$
    PROOF:
$$\begin{aligned}
\|\vec{x} - \vec{z}\| &= \|(\vec{x} - \vec{y}) + (\vec{y} - \vec{z})\| \\
&\leq \|\vec{x} - \vec{y}\| + \|\vec{y} - \vec{z}\| \qquad \text{(Lemma 370)}
\end{aligned}$$
$\Box$

**Proposition 372.** *The Euclidean metric induces the standard topology on $\mathbb{R}^n$.*

PROOF:
$\langle 1 \rangle 1.$ LET: $\rho$ be the square metric.

$\langle 1 \rangle 2$. For all $\vec{a} \in \mathbb{R}^n$ and $\epsilon > 0$, we have $B_d(\vec{a}, \epsilon) \subseteq B_\rho(\vec{a}, \epsilon)$
  $\langle 2 \rangle 1$. LET: $\vec{x} \in B_d(\vec{a}, \epsilon)$
  $\langle 2 \rangle 2$. $\sqrt{(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2} < \epsilon$
  $\langle 2 \rangle 3$. $(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2 < \epsilon^2$
  $\langle 2 \rangle 4$. For all $i$ we have $(x_i - a_i)^2 < \epsilon^2$
  $\langle 2 \rangle 5$. For all $i$ we have $|x_i - a_i| < \epsilon$
  $\langle 2 \rangle 6$. $\rho(\vec{x}, \vec{a}) < \epsilon$
$\langle 1 \rangle 3$. For all $\vec{a} \in \mathbb{R}^n$ and $\epsilon > 0$, we have $B_\rho(\vec{a}, \epsilon/\sqrt{n}) \subseteq B_d(\vec{a}, \epsilon)$
  $\langle 2 \rangle 1$. LET: $\vec{x} \in B_\rho(\vec{a}, \epsilon/\sqrt{n})$
  $\langle 2 \rangle 2$. $\rho(\vec{x}, \vec{a}) < \epsilon/\sqrt{n}$
  $\langle 2 \rangle 3$. For all $i$ we have $|x_i - x_a| < \epsilon/\sqrt{n}$
  $\langle 2 \rangle 4$. For all $i$ we have $(x_i - x_a)^2 < \epsilon^2/n$
  $\langle 2 \rangle 5$. $d(\vec{x}, \vec{a}) < \epsilon$
$\langle 1 \rangle 4$. Q.E.D.
  PROOF: By Lemma 348.
□

**Proposition 373.** *Let $n \geq 0$. For all $c \in \mathbb{R}^n$ and $\epsilon > 0$, the open ball $B(c, \epsilon)$ is path connected.*

PROOF:
$\langle 1 \rangle 1$. LET: $a, b \in B(c, \epsilon)$
$\langle 1 \rangle 2$. LET: $p : [0, 1] \to B(c, \epsilon)$ be the function $p(t) = (1 - t)a + tb$
  PROOF: We have $p(t) \in B(c, \epsilon)$ for all $t$ because
$$d(p(t), c) = \|(1 - t)a + tb - c\|$$
$$= \|(1 - t)(a - c) + t(b - c)\|$$
$$\leq (1 - t)\|a - c\| + t\|b - c\|$$
$$< (1 - t)\epsilon + t\epsilon$$
$$= \epsilon$$
$\langle 1 \rangle 3$. $p$ is a path from $a$ to $b$.
□

**Proposition 374.** *Let $n \geq 0$. For all $c \in \mathbb{R}^n$ and $\epsilon > 0$, the closed ball $\overline{B(c, \epsilon)}$ is path connected.*

PROOF:
$\langle 1 \rangle 1$. LET: $a, b \in \overline{B(c, \epsilon)}$
$\langle 1 \rangle 2$. LET: $p : [0, 1] \to \overline{B(c, \epsilon)}$ be the function $p(t) = (1 - t)a + tb$
  PROOF: We have $p(t) \in \overline{B(c, \epsilon)}$ for all $t$ because
$$d(p(t), c) = \|(1 - t)a + tb - c\|$$
$$= \|(1 - t)(a - c) + t(b - c)\|$$
$$\leq (1 - t)\|a - c\| + t\|b - c\|$$
$$\leq (1 - t)\epsilon + t\epsilon$$
$$= \epsilon$$
$\langle 1 \rangle 3$. $p$ is a path from $a$ to $b$.

□

**Lemma 375.** *If $\sum_{i=0}^{\infty} x_i^2$ and $\sum_{i=0}^{\infty} y_i^2$ converge then $\sum_{i=0}^{\infty} |x_i y_i|$ converges.*

PROOF:

$\langle 1 \rangle 1$. For all $N \geq 0$ we have $\sum_{i=0}^{N} |x_i y_i| \leq \sqrt{\sum_{i=0}^{N} |x_i|^2} \sqrt{\sum_{i=0}^{N} |y_i|^2}$
   PROOF: By the Cauchy-Schwarz inequality

$\langle 1 \rangle 2$. Q.E.D.
   PROOF: Since $\sum_{i=0}^{N} |x_i y_i|$ is an increasing sequence bounded above by
   $(\sum_{i=0}^{\infty} x_i^2)(\sum_{i=0}^{\infty} y_i^2)$.

□

**Corollary 375.1.** *If $\sum_{i=0}^{\infty} x_i^2$ and $\sum_{i=0}^{\infty} y_i^2$ converge then $\sum_{i=0}^{\infty} (x_i + y_i)^2$ converges.*

PROOF: Since $\sum_{i=0}^{\infty} x_i^2$, $\sum_{i=0}^{\infty} y_i^2$ and $2 \sum_{i=0}^{\infty} x_i y_i$ all converge.

**Definition 376** ($l^2$-metric)**.** The $l^2$-*metric* on

$$\left\{ (x_n) \in \mathbb{R}^\omega \mid \sum_{n=0}^{\infty} x_n^2 \text{ converges} \right\}$$

is defined by

$$d(x, y) = \left( \sum_{n=0}^{\infty} (x_n - y_n)^2 \right)^{1/2}$$

   We prove this is a metric.

PROOF:

$\langle 1 \rangle 1$. $d$ is well-defined.
   PROOF: By Corollary 375.1.
$\langle 1 \rangle 2$. $d(x, y) \geq 0$
$\langle 1 \rangle 3$. $d(x, y) = 0$ if and only if $x = y$
$\langle 1 \rangle 4$. $d(x, y) = d(y, x)$
$\langle 1 \rangle 5$. $d(x, z) \leq d(x, y) + d(y, z)$
   PROOF: By Lemma 370.

□

**Theorem 377.** *Addition is a continuous function $\mathbb{R}^2 \to \mathbb{R}$.*

PROOF:

$\langle 1 \rangle 1$. LET: $a, b \in \mathbb{R}$
$\langle 1 \rangle 2$. LET: $\epsilon > 0$
$\langle 1 \rangle 3$. LET: $\delta = \epsilon/2$
$\langle 1 \rangle 4$. LET: $x, y \in \mathbb{R}$
$\langle 1 \rangle 5$. ASSUME: $\rho((a, b), (x, y)) < \delta$
$\langle 1 \rangle 6$. $|(a + b) - (x + y)| < \epsilon$

PROOF:

$$|(a + b) - (x + y)| = |a - x| + |b - y|$$
$$\leq 2\rho((a, b), (x, y))$$
$$< 2\delta$$
$$= \epsilon$$

$\langle 1 \rangle 7$. Q.E.D.

PROOF: Theorem 354

□

**Theorem 378.** *Multiplication is a continuous function* $\mathbb{R}^2 \to \mathbb{R}$.

PROOF:

$\langle 1 \rangle 1$. LET: $a, b \in \mathbb{R}$

$\langle 1 \rangle 2$. LET: $\epsilon > 0$

$\langle 1 \rangle 3$. LET: $\delta = \min(\epsilon/(|a| + |b| + 1), 1)$

$\langle 1 \rangle 4$. LET: $x, y \in \mathbb{R}$

$\langle 1 \rangle 5$. ASSUME: $\rho((a, b), (x, y)) < \delta$

$\langle 1 \rangle 6$. $|ab - xy| < \epsilon$

PROOF:

$$|ab - xy| = |a(b - y) + (a - x)b - (a - x)(b - y)|$$
$$\leq |a||b - y| + |b||a - x| + |a - x||b - y|$$
$$< |a|\delta + |b|\delta + \delta^2 \qquad (\langle 1 \rangle 5)$$
$$\leq |a|\delta + |b|\delta + \delta \qquad (\langle 1 \rangle 3)$$
$$\leq \epsilon \qquad (\langle 1 \rangle 3)$$

$\langle 1 \rangle 7$. Q.E.D.

PROOF: Theorem 354

□

**Theorem 379.** *The function* $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ *defined by* $f(x) = x^{-1}$ *is continuous.*

PROOF:

$\langle 1 \rangle 1$. For all $a \in \mathbb{R}$ we have $f^{-1}((a, +\infty))$ is open.

PROOF: The set is

$$(a^{-1}, +\infty) \text{if } a > 0$$
$$(0, +\infty) \text{if } a = 0$$
$$(-\infty, a^{-1}) \cup (0, +\infty) \text{if } a < 0$$

$\langle 1 \rangle 2$. For all $a \in \mathbb{R}$ we have $f^{-1}((-\infty, a))$ is open.

PROOF: Similar.

$\langle 1 \rangle 3$. Q.E.D.

PROOF: By Proposition 90 and Lemma 113.

□

**Definition 380.** For $n \geq 0$, the *unit ball* $B^n$ is the space $\{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$.

**Proposition 381.** *For all* $n \geq 0$, *the unit ball* $B^n$ *is path connected.*

PROOF:

⟨1⟩1. LET: $a, b \in B^n$

⟨1⟩2. LET: $p : [0, 1] \to B^n$ be the function $p(t) = (1 - t)a + tb$

PROOF: We have $p(t) \in B^n$ for all $t$ because
$$\|(1 - t)a + tb\| \leq (1 - t)\|a\| + t\|b\|$$
$$\leq (1 - t) + t$$
$$= 1$$

⟨1⟩3. $p$ is a path from $a$ to $b$.

□

**Definition 382** (Punctured Euclidean Space). For $n \geq 0$, defined *punctured Euclidean space* to be $\mathbb{R}^n \setminus \{0\}$.

**Proposition 383.** *For $n > 1$, punctured Euclidean space $\mathbb{R}^n \setminus \{0\}$ is path connected.*

PROOF:

⟨1⟩1. LET: $a, b \in \mathbb{R}^n \setminus \{0\}$

⟨1⟩2. CASE: 0 is on the line from $a$ to $b$

⟨2⟩1. PICK a point $c$ not on the line from $a$ to $b$

⟨2⟩2. The path consisting of a straight line from $a$ to $c$ followed by a straight line from $c$ to $b$ is a path from $a$ to $b$.

⟨1⟩3. CASE: 0 is not on the line from $a$ to $b$

PROOF: The straight line from $a$ to $b$ is a path from $a$ to $b$.

**Corollary 383.1.** *For $n > 1$, the spaces $\mathbb{R}$ and $\mathbb{R}^n$ are not homeomorphic.*

PROOF: For any point $a$, the space $\mathbb{R} \setminus \{a\}$ is disconnected.

**Definition 384** (Unit Sphere). For $n \geq 1$, the *unit sphere $S^{n-1}$* is the space

$$S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\} \ .$$

**Proposition 385.** *For $n > 1$, the unit sphere $S^{n-1}$ is path connected.*

PROOF: The map $g : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$ defined by $g(x) = x/\|x\|$ is continuous and surjective. The result follows by Proposition 243. □

**Proposition 386.** *Let $f : S^1 \to \mathbb{R}$ be continuous. Then there exists $x \in S^1$ such that $f(x) = f(-x)$.*

PROOF:

⟨1⟩1. LET: $g : S^1 \to \mathbb{R}$ be the function $g(x) = f(x) - f(-x)$

PROVE: There exists $x \in S^1$ such that $g(x) = 0$

⟨1⟩2. ASSUME: without loss of generality $g((1, 0)) > 0$

⟨1⟩3. $g((-1, 0)) < 0$

⟨1⟩4. There exists $x$ such that $g(x) = 0$

PROOF: By the Intermediate Value Theorem.

□

**Definition 387** (Topologist's Sine Curve)**.** Let $S = \{(x, \sin 1/x) \mid 0 < x \leq 1\}$. The *topologist's sine curve* is the closure $\overline{S}$ of $S$.

**Proposition 388.**
$$\overline{S} = S \cup (\{0\} \times [-1, 1])$$

**Proposition 389.** *The topologist's sine curve is connected.*

PROOF:
$\langle 1 \rangle 1$. LET: $S = \{(x, \sin 1/x) \mid 0 < x \leq 1\}$
$\langle 1 \rangle 2$. $S$ is connected.
    PROOF: Theorem 222.
$\langle 1 \rangle 3$. $\overline{S}$ is connected.
    PROOF: Theorem 221.
□

**Proposition 390** (CC)**.** *The topologist's sine curve is not path connected.*

PROOF:
$\langle 1 \rangle 1$. ASSUME: for a contradiction $p : [0, 1] \to \overline{S}$ is a path from $(0, 0)$ to $(1, \sin 1)$.
$\langle 1 \rangle 2$. $p^{-1}(\{0\} \times [0, 1])$ is closed.
$\langle 1 \rangle 3$. LET: $b$ be the greatest element of $p^{-1}(\{0\} \times [0, 1])$.
$\langle 1 \rangle 4$. $b < 1$
    PROOF: Since $p(1) = (1, \sin 1)$.
$\langle 1 \rangle 5$. PICK a sequence $(t_n)_{n \geq 1}$ in $(b, 1]$ such that $t_n \to b$ and $\pi_2(p(t_n)) = (-1)^n$
    $\langle 2 \rangle 1$. LET: $n \geq 1$
    $\langle 2 \rangle 2$. PICK $u$ with $0 < u < \pi_1(p(1/n))$ such that $\sin(1/u) = (-1)^n$
    $\langle 2 \rangle 3$. PICK $t_n$ such that $0 < t_n < 1/n$ and $\pi_1(p(t_n)) = u$
        PROOF: One exists by the Intermediate Value Theorem.
$\langle 1 \rangle 6$. Q.E.D.
    PROOF: This contradicts 105.
□

# 50   The Uniform Topology

**Definition 391** (Uniform Metric)**.** Let $J$ be a set. The *uniform metric* $\overline{\rho}$ on $\mathbb{R}^J$ is defined by
$$\overline{\rho}(a, b) = \sup_{j \in J} \overline{d}(a_j, b_j)$$

where $\overline{d}$ is the standard bounded metric on $\mathbb{R}$.

The *uniform topology* on $\mathbb{R}^J$ is the topology induced by the uniform metric.

We prove this is a metric.

PROOF:
$\langle 1 \rangle 1$. $\overline{\rho}(a, b) \geq 0$
    PROOF: Immediate from definitions.
$\langle 1 \rangle 2$. $\overline{\rho}(a, b) = 0$ if and only if $a = b$

Proof: Immediate from definitions.

$\langle 1 \rangle 3.$ $\overline{\rho}(a, b) = \overline{\rho}(b, a)$

Proof: Immediate from definitions.

$\langle 1 \rangle 4.$ $\overline{\rho}(a, c) \leq \overline{\rho}(a, b) + \overline{\rho}(b, c)$

Proof:

$$\begin{aligned} \overline{\rho}(a, c) &= \sup_{j \in J} \overline{d}(a_j, c_j) \\ &\leq \sup_{j \in J} (\overline{d}(a_j, b_j) + \overline{d}(b_j, c_j)) \\ &\leq \sup_{j \in J} \overline{d}(a_j, b_j) + \sup_{j \in J} \overline{d}(b_j, c_j) \\ &= \overline{\rho}(a, b) + \overline{\rho}(b, c) \end{aligned}$$

$\square$

**Proposition 392.** *The uniform topology on $\mathbb{R}^J$ is finer than the product topology.*

Proof:

$\langle 1 \rangle 1.$ Let: $j \in J$ and $U$ be open in $\mathbb{R}$

Prove: $\pi_j{}^{-1}(U)$ is open in the uniform topology.

$\langle 1 \rangle 2.$ Let: $a \in \pi_j{}^{-1}(U)$

$\langle 1 \rangle 3.$ Pick $\epsilon > 0$ such that $(a_j - \epsilon, a_j + \epsilon) \subseteq U$

$\langle 1 \rangle 4.$ $B_{\overline{\rho}}(a, \epsilon) \subseteq \pi_j{}^{-1}(U)$

$\square$

**Proposition 393.** *The uniform topology on $\mathbb{R}^J$ is coarser than the box topology.*

Proof:

$\langle 1 \rangle 1.$ Let: $a \in \mathbb{R}^J$ and $\epsilon > 0$

Prove: $B(a, \epsilon)$ is open in the box topology.

$\langle 1 \rangle 2.$ Let: $b \in B(a, \epsilon)$

$\langle 1 \rangle 3.$ For $j \in J$ we have $|a_j - b_j| < \epsilon$

$\langle 1 \rangle 4.$ For $j \in J$,

Let: $\delta_j = (\epsilon - |a_j - b_j|)/2$

$\langle 1 \rangle 5.$ $\prod_{j \in J}(b_j - \delta_j, b_j + \delta_j) \subseteq B(a, \epsilon)$

$\square$

**Proposition 394.** *The uniform topology on $\mathbb{R}^J$ is strictly finer than the product topology if and only if $J$ is infinite.*

Proof:

$\langle 1 \rangle 1.$ If $J$ is finite then the uniform and product topologies coincide.

Proof: The uniform, box and product topologies are all the same.

$\langle 1 \rangle 2.$ If $J$ is infinite then the uniform and product topologies are different.

Proof: The set $B(\vec{0}, 1)$ is open in the uniform topology but not the product topology.

$\square$

**Proposition 395** (DC)**.** *The uniform topology on $\mathbb{R}^J$ is strictly coarser than the box topology if and only if $J$ is infinite.*

PROOF:

$\langle 1 \rangle 1$. If $J$ is finite then the uniform and box topologies coincide.

   PROOF: The uniform, box and product topologies are all the same.

$\langle 1 \rangle 2$. If $J$ is infinite then the uniform and box topologies are different.

   PROOF: Pick an $\omega$-sequence $(j_1, j_2, \ldots)$ in $J$. Let $U = \prod_{j \in J} U_j$ where $U_{j_i} = (-1/i, 1/i)$ and $U_j = (-1, 1)$ for all other $j$. Then $\vec{0} \in U$ but there is no $\epsilon > 0$ such that $B(\vec{0}, \epsilon) \subseteq U$.

$\Box$

**Proposition 396.** *The closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ under the uniform topology is $\mathbb{R}^\omega$.*

PROOF: Given any open ball $B(a, \epsilon)$, pick an integer $N$ such that $1/\epsilon < N$. Then $B(a, \epsilon)$ includes sequences whose $n$th entry is 0 for all $n \geq N$. $\Box$

**Example 397.** The space $\mathbb{R}^\omega$ is disconnected under the uniform topology. The set of bounded sequences and the set of unbounded sequences form a separation.

**Proposition 398.** *Give $\mathbb{R}^\omega$ the uniform topology. Let $x, y \in \mathbb{R}^\omega$. Then $x$ and $y$ are in the same component if and only if $x - y$ is bounded.*

PROOF:

$\langle 1 \rangle 1$. The component containing 0 is the set of bounded sequences.

   $\langle 2 \rangle 1$. LET: $B$ be the set of bounded sequences.

   $\langle 2 \rangle 2$. $B$ is path-connected.

      $\langle 3 \rangle 1$. LET: $x.y \in B$

      $\langle 3 \rangle 2$. PICK $b > 0$ such that $|x_j|, |y_j| \leq b$ for all $j$

      $\langle 3 \rangle 3$. LET: $p : [0, 1] \to B$ be the function $p(t) = (1 - t)x + ty$

         PROVE: $p$ is continuous.

      $\langle 3 \rangle 4$. LET: $t \in [0, 1]$ and $\epsilon > 0$

      $\langle 3 \rangle 5$. LET: $\delta = \epsilon/2b$

      $\langle 3 \rangle 6$. LET: $s \in [0, 1]$ with $|s - t| < \delta$

      $\langle 3 \rangle 7$. $\overline{\rho}(p(s), p(t)) < \epsilon$

         PROOF:
$$\overline{\rho}(p(s), p(t)) = \sup_j \overline{d}((1 - s)x_j + sy_j, (1 - t)x_j + ty_j)$$
$$\leq |(s - t)x_j + (t - s)y_j|$$
$$\leq |s - t||x_j - y_j|$$
$$< 2b\delta$$
$$= \epsilon$$

   $\langle 2 \rangle 3$. $B$ is connected.

      PROOF: Proposition 241.

   $\langle 2 \rangle 4$. If $C$ is connected and $B \subseteq C$ then $B = C$.

      PROOF: Otherwise $B \cap C$ and $C \setminus B$ form a separation of $C$.

$\langle 1 \rangle 2$. Q.E.D.

   PROOF: Since $\lambda x.x - y$ is a Homeomorphism of $\mathbb{R}^\omega$ with itself.

$\Box$

# 51   Uniform Convergence

**Definition 399** (Uniform Convergence). Let $X$ be a set and $Y$ a metric space. Let $(f_n : X \to Y)$ be a sequence of functions and $f : X \to Y$ be a function. Then $f_n$ *converges uniformly* to $f$ as $n \to \infty$ if and only if, for all $\epsilon > 0$, there exists $N$ such that, for all $n \geq N$ and $x \in X$, we have $d(f_n(x), f(x)) < \epsilon$.

**Example 400.** Define $f_n : [0,1] \to \mathbb{R}$ by $f_n(x) = x^n$ for $n \geq 1$, and $f : [0,1] \to \mathbb{R}$ by $f(x) = 0$ if $x < 1$, $f(1) = 1$. Then $f_n$ converges to $f$ pointwise but not uniformly.

**Theorem 401** (Uniform Limit Theorem). *Let $X$ be a topological space and $Y$ a metric space. Let $(f_n : X \to Y)$ be a sequence of continuous functions and $f : X \to Y$ be a function. If $f_n$ converges uniformly to $f$ as $n \to \infty$, then $f$ is continuous.*

PROOF:
$\langle 1 \rangle 1$. LET: $x \in X$ and $\epsilon > 0$
$\langle 1 \rangle 2$. PICK $N$ such that, for all $n \geq N$ and $y \in X$, we have $d(f_n(y), f(y)) < \epsilon/3$
$\langle 1 \rangle 3$. PICK a neighbourhood $U$ of $x$ such that $f_N(U) \subseteq B(f_N(x), \epsilon/3)$
    PROVE:  $f(U) \subseteq B(f(x), \epsilon)$
$\langle 1 \rangle 4$. LET: $y \in U$
$\langle 1 \rangle 5$. $d(f(y), f(x)) < \epsilon$
    PROOF:

$$d(f(y), f(x)) \leq d(f(y), f_N(y)) + d(f_N(y), f_N(x)) + d(f_N(x), f(x)) \quad \text{(Triangle Inequality)}$$
$$< \epsilon/3 + \epsilon/3 + \epsilon/3 \qquad (\langle 1 \rangle 2, \langle 1 \rangle 3)$$
$$= \epsilon$$

$\square$

**Proposition 402.** *Let $X$ be a topological space and $Y$ a metric space. Let $(f_n : X \to Y)$ be a sequence of continuous functions and $f : X \to Y$ be a function. Let $(a_n)$ be a sequence of points in $X$ and $a \in X$. If $f_n$ converges uniformly to $f$ and $a_n$ converges to $a$ in $X$ then $f_n(a_n)$ converges to $f(a)$ uniformly in $Y$.*

PROOF:
$\langle 1 \rangle 1$. LET: $\epsilon > 0$
$\langle 1 \rangle 2$. PICK $N_1$ such that, for all $n \geq N_1$ and $x \in X$, we have $d(f_n(x), f(x)) < \epsilon/2$
$\langle 1 \rangle 3$. PICK $N_2$ such that, for all $n \geq N_2$, we have $a_n \in f^{-1}(B(a, \epsilon/2))$
    PROOF: Using the fact that $f$ is continuous from the Uniform Limit Theorem.
$\langle 1 \rangle 4$. LET: $N = \max(N_1, N_2)$
$\langle 1 \rangle 5$. LET: $n \geq N$
$\langle 1 \rangle 6$. $d(f_n(a_n), f(a)) < \epsilon$
    PROOF:

$$d(f_n(a_n), f(a)) \leq d(f_n(a_n), f(a_n)) + d(f(a_n), f(a)) \quad \text{(Triangle Inequality)}$$
$$< \epsilon/2 + \epsilon/2 \qquad (\langle 1 \rangle 2, \langle 1 \rangle 3)$$
$$= \epsilon$$

□

**Proposition 403.** *Let $X$ be a set. Let $(f_n : X \to \mathbb{R})$ be a sequence of functions and $f : X \to \mathbb{R}$ be a function. Then $f_n$ converges uniformly to $f$ as $n \to \infty$ if and only if $f_n \to f$ as $n \to \infty$ in $\mathbb{R}^X$ under the uniform topology.*

PROOF:

$\langle 1 \rangle 1$. If $f_n$ converges uniformly to $f$ then $f_n$ converges to $f$ under the uniform topology.

$\quad \langle 2 \rangle 1$. ASSUME: $f_n$ converges uniformly to $f$

$\quad \langle 2 \rangle 2$. LET: $\epsilon > 0$

$\quad \langle 2 \rangle 3$. PICK $N$ such that, for all $n \geq N$ and $x \in X$, we have $d(f_n(x), f(x)) < \epsilon/2$

$\quad \langle 2 \rangle 4$. For all $n \geq N$ we have $\overline{\rho}(f_n, f) \leq \epsilon/2$

$\quad \langle 2 \rangle 5$. For all $n \geq N$ we have $\overline{\rho}(f_n, f) < \epsilon$

$\langle 1 \rangle 2$. If $f_n$ converges to $f$ under the uniform topology then $f_n$ converges uniformly to $f$.

$\quad \langle 2 \rangle 1$. ASSUME: $f_n$ converges to $f$ under the uniform topology.

$\quad \langle 2 \rangle 2$. LET: $\epsilon > 0$

$\quad \langle 2 \rangle 3$. PICK $N$ such that, for all $n \geq N$, we have $\overline{\rho}(f_n, f) < \min(\epsilon, 1/2)$

$\quad \langle 2 \rangle 4$. LET: $n \geq N$

$\quad \langle 2 \rangle 5$. LET: $x \in X$

$\quad \langle 2 \rangle 6$. $\overline{\rho}(f_n, f) < \min(\epsilon, 1/2)$

$\quad \quad$ PROOF: From $\langle 2 \rangle 3$.

$\quad \langle 2 \rangle 7$. $d(f_n(x), f(x)) < \min(\epsilon, 1/2)$

$\quad \langle 2 \rangle 8$. $d(f_n(x), f(x)) < \epsilon$

□

# 52　Isometric Imbeddings

**Definition 404.** Let $X$ and $Y$ be metric spaces. An *isometric imbedding* $f : X \to Y$ is a function such that, for all $x, y \in X$, we have $d(f(x), f(y)) = d(x, y)$.

**Proposition 405.** *Every isometric imbedding is an imbedding.*

PROOF:

$\langle 1 \rangle 1$. LET: $f : X \to Y$ be an isometric imbedding.

$\langle 1 \rangle 2$. $f$ is injective.

$\quad$ PROOF: If $f(x) = f(y)$ then $d(f(x), f(y)) = 0$ hence $d(x, y) = 0$ hence $x = y$.

$\langle 1 \rangle 3$. $f$ is continuous.

$\quad$ PROOF: For all $\epsilon > 0$, if $d(x, y) < \epsilon$ then $d(f(x), f(y)) < \epsilon$.

$\langle 1 \rangle 4$. $f : X \to f(X)$ is an open map.

$\quad$ PROOF: $f(B(a, \epsilon)) = B(f(a), \epsilon) \cap f(X)$.

□