Topology

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1 Order Theory

Definition 1 (Convex)**.** Let X be a linearly ordered set and $Y \subseteq X$. Then Y is *convex* if and only if, for all $a, b \in Y$ and $c \in X$, if $a < c < b$ then $c \in Y$.

2 Topological Spaces

Definition 2 (Topology)**.** A *topology* on a set X is a set $\mathcal{T} \subseteq \mathcal{P}X$ such that:

- $X \in \mathcal{T}$.
- For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$.
- For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$.

We call the elements of X *points* and the elements of $\mathcal{T}$ *open sets*.

Definition 3 (Topological Space)**.** A *topological space* X consists of a set X and a topology on X.

Definition 4 (Discrete Space)**.** For any set X, the *discrete* topology on X is $\mathcal{P}X$.

Definition 5 (Indiscrete Space)**.** For any set X, the *indiscrete* or *trivial* topology on X is $\{\emptyset, X\}$.

Definition 6 (Finite Complement Topology)**.** For any set X, the *finite complement topology* on X is $\{U \in \mathcal{P}X \mid X \setminus U \text{ is finite}\} \cup \{\emptyset\}$.

Definition 7 (Countable Complement Topology)**.** For any set X, the *countable complement topology* on X is $\{U \in \mathcal{P}X \mid X \setminus U \text{ is countable}\} \cup \{\emptyset\}$.

Definition 8 (Finer, Coarser)**.** Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a given set X. If $\mathcal{T}' \supseteq \mathcal{T}$, we say that $\mathcal{T}'$ is *finer* than $\mathcal{T}$; if $\mathcal{T}'$ *properly* contains $\mathcal{T}$, we say that $\mathcal{T}'$ is *strictly* finer than $\mathcal{T}$. We also say that $\mathcal{T}$ is *coarser* than $\mathcal{T}'$, or *stricly* coarser, in these two respective situations. We say $\mathcal{T}$ is *comparable* with $\mathcal{T}'$ if either $\mathcal{T}' \supseteq \mathcal{T}$ or $\mathcal{T} \supseteq \mathcal{T}'$.

Lemma 9. *Let X be a topological space and $U \subseteq X$. Then U is open if and only if, for all $x \in U$, there exists an open set V such that $x \in V \subseteq U$.*

Proof:
$\langle 1 \rangle 1.$ $\Rightarrow$
  Proof: Take $V = U$
$\langle 1 \rangle 2.$ $\Leftarrow$
  Proof: We have $U = \bigcup \{V \in \mathcal{P}X \mid V \subseteq U\}$.
∎

Lemma 10. *Let X be a set and $\mathcal{T}$ a nonempty set of topologies on X. Then $\bigcap \mathcal{T}$ is a topology on X, and is the finest topology that is coarser than every member of $\mathcal{T}$.*

Proof:
$\langle 1 \rangle 1.$ $X \in \bigcap \mathcal{T}$
  Proof: Since X is in every member of $\mathcal{T}$.
$\langle 1 \rangle 2.$ $\bigcap \mathcal{T}$ is closed under union.
  $\langle 2 \rangle 1.$ Let: $\mathcal{U} \subseteq \bigcap \mathcal{T}$
  $\langle 2 \rangle 2.$ For all $T \in \mathcal{T}$ we have $\mathcal{U} \subseteq T$
  $\langle 2 \rangle 3.$ For all $T \in \mathcal{T}$ we have $\bigcup \mathcal{U} \in T$
  $\langle 2 \rangle 4.$ $\bigcup \mathcal{U} \in \bigcap \mathcal{T}$
$\langle 1 \rangle 3.$ $\bigcap \mathcal{T}$ is closed under binary intersection.
  $\langle 2 \rangle 1.$ Let: $U, V \in \bigcap \mathcal{T}$
  $\langle 2 \rangle 2.$ For all $T \in \mathcal{T}$ we have $U, V \in T$
  $\langle 2 \rangle 3.$ For all $T \in \mathcal{T}$ we have $U \cap V \in T$
  $\langle 2 \rangle 4.$ $U \cap V \in \bigcap \mathcal{T}$
∎

Lemma 11. *Let X be a set and $\mathcal{T}$ a set of topologies on X. Then there exists a unique coarsest topology that is finer than every member of $\mathcal{T}$.*

Proof: The required topology is given by

$$\bigcap \{T \in \mathcal{PP}X \mid T \textit{ is a topology on } X \textit{ that is finer than every member of } \mathcal{T}\} \enspace ,$$

The set is nonempty since it contains the discrete topology. ∎

3 Basis for a Topology

Definition 12 (Basis)**.** If X is a set, a *basis* for a topology on X is a set $\mathcal{B} \subseteq \mathcal{P}X$ called *basis elements* such that

1. For all $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$.
2. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

If $\mathcal{B}$ satisfies these two conditions, then we define the topology *generated* by $\mathcal{B}$ to be $\mathcal{T} = \{U \in \mathcal{P}X \mid \forall x \in U. \exists B \in \mathcal{B}. x \in B \subseteq U\}$.

We prove this is a topology.

PROOF:
⟨1⟩1. $X \in \mathcal{T}$
PROOF: For all $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B \subseteq X$ by condition 1.
⟨1⟩2. For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$
⟨2⟩1. LET: $\mathcal{U} \subseteq \mathcal{T}$
⟨2⟩2. LET: $x \in \bigcup \mathcal{U}$
⟨2⟩3. PICK $U \in \mathcal{U}$ such that $x \in U$
⟨2⟩4. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$
PROOF: Since $U \in \mathcal{T}$ by ⟨2⟩1 and ⟨2⟩3.
⟨2⟩5. $x \in B \subseteq \bigcup \mathcal{U}$
⟨1⟩3. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$
⟨2⟩1. LET: $U, V \in \mathcal{T}$
⟨2⟩2. LET: $x \in U \cap V$
⟨2⟩3. PICK $B_1 \in \mathcal{B}$ such that $x \in B_1 \subseteq U$
⟨2⟩4. PICK $B_2 \in \mathcal{B}$ such that $x \in B_2 \subseteq V$
⟨2⟩5. PICK $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$
PROOF: By condition 2.
⟨2⟩6. $x \in B_3 \subseteq U \cap V$
□

Lemma 13. *Let X be a set. Let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ is the set of all unions of subsets of $\mathcal{B}$.*

PROOF:
⟨1⟩1. For all $U \in \mathcal{T}$, there exists $\mathcal{A} \subseteq \mathcal{B}$ such that $U = \bigcup \mathcal{A}$
⟨2⟩1. LET: $U \in \mathcal{T}$
⟨2⟩2. LET: $\mathcal{A} = \{B \in \mathcal{B} \mid B \subseteq U\}$
⟨2⟩3. $U \subseteq \bigcup \mathcal{A}$
⟨3⟩1. LET: $x \in U$
⟨3⟩2. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$
PROOF: Since $\mathcal{B}$ is a basis for $\mathcal{T}$.
⟨3⟩3. $x \in B \in \mathcal{A}$
⟨2⟩4. $\bigcup \mathcal{A} \subseteq U$
PROOF: From the definition of $\mathcal{A}$ (⟨2⟩2).
⟨1⟩2. For all $\mathcal{A} \subseteq \mathcal{B}$ we have $\bigcup \mathcal{A} \in \mathcal{T}$
⟨2⟩1. $\mathcal{B} \subseteq \mathcal{T}$
PROOF: If $B \in \mathcal{B}$ and $x \in B$, then there exists $B' \in \mathcal{B}$ such that $x \in B' \subseteq B$, namely $B' = B$.
⟨2⟩2. Q.E.D.
PROOF: Since $\mathcal{T}$ is closed under union.
□

Corollary 13.1. *Let X be a set. Let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ is the coarsest topology that includes $\mathcal{B}$.*

PROOF: Since every topology that includes $\mathcal{B}$ includes all unions of subsets of $\mathcal{B}$. $\square$

Lemma 14. *Let X be a topological space. Suppose that $\mathcal{C}$ is a set of open sets such that, for every open set U and every point $x \in U$, there exists $C \in \mathcal{C}$ such that $x \in C \subseteq U$. Then $\mathcal{C}$ is a basis for the topology on X.*

PROOF:
$\langle 1 \rangle 1$. For all $x \in X$, there exists $C \in \mathcal{C}$ such that $x \in C$
  PROOF: Immediate from hypothesis.
$\langle 1 \rangle 2$. For all $C_1, C_2 \in \mathcal{C}$ and $x \in C_1 \cap C_2$, there exists $C_3 \in \mathcal{C}$ such that $x \in C_3 \subseteq C_1 \cap C_2$
  PROOF: Since $C_1 \cap C_2$ is open.
$\langle 1 \rangle 3$. Every open set is open in the topology generated by $\mathcal{C}$
  PROOF: Immediate from hypothesis.
$\langle 1 \rangle 4$. Every union of a subset of $\mathcal{C}$ is open.
  PROOF: Since every member of $\mathcal{C}$ is open.
$\square$

Lemma 15. *Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$ respectively on the set X. Then the following are equivalent.*

1. *$\mathcal{T} \subseteq \mathcal{T}'$*
2. *For all $B \in \mathcal{B}$ and $x \in B$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.*

PROOF:
$\langle 1 \rangle 1$. $1 \Rightarrow 2$
  $\langle 2 \rangle 1$. ASSUME: $\mathcal{T} \subseteq \mathcal{T}'$
  $\langle 2 \rangle 2$. LET: $B \in \mathcal{B}$ and $x \in B$
  $\langle 2 \rangle 3$. $B \in \mathcal{T}$
    PROOF: Corollary 13.1.
  $\langle 2 \rangle 4$. $B \in \mathcal{T}'$
    PROOF: By $\langle 2 \rangle 1$
  $\langle 2 \rangle 5$. There exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$
    PROOF: Since $\mathcal{B}'$ is a basis for $\mathcal{T}'$.
$\langle 1 \rangle 2$. $2 \Rightarrow 1$
  $\langle 2 \rangle 1$. ASSUME: 2
  $\langle 2 \rangle 2$. LET: $U \in \mathcal{T}$
    PROVE: $U \in \mathcal{T}'$
  $\langle 2 \rangle 3$. LET: $x \in U$
    PROVE: There exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq U$
  $\langle 2 \rangle 4$. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$
    PROOF: Since $\mathcal{B}$ is a basis for $\mathcal{T}$.
  $\langle 2 \rangle 5$. PICK $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$
    PROOF: By $\langle 2 \rangle 1$.
  $\langle 2 \rangle 6$. $x \in B' \subseteq U$
$\square$

Definition 16 (Lower Limit Topology on the Real Line)**.** The *lower limit topology on the real line* is the topology on $\mathbb{R}$ generated by the basis consisting of all half-open intervals of the form $[a, b)$.

We write $\mathbb{R}_l$ for the topological space $\mathbb{R}$ under the lower limit topology.

We prove this is a basis for a topology.

PROOF:
$\langle 1 \rangle 1$. For all $x \in \mathbb{R}$ there exists an interval $[a, b)$ such that $x \in [a, b)$.
PROOF: Take $[a, b) = [x, x + 1)$.
$\langle 1 \rangle 2$. For any open intervals $[a, b)$, $[c, d)$ if $x \in [a, b) \cap [c, d)$, then there exists an interval $[e, f)$ such that $x \in [e, f) \subseteq [a, b) \cap [c, d)$
PROOF: Take $[e, f) = [\max(a, c), \min(b, d))$.
$\square$

Definition 17 (K-topology on the Real Line)**.** Let $K = \{1/n \mid n \in \mathbb{Z}^+\}$.

The *K-topology on the real line* is the topology on $\mathbb{R}$ generated by the basis consisting of all open intervals (a, b) and all sets of the form $(a, b) \setminus K$.

We write $\mathbb{R}_K$ for the topological space $\mathbb{R}$ under the K-topology.

We prove this is a basis for a topology.

PROOF:
$\langle 1 \rangle 1$. For all $x \in \mathbb{R}$ there exists an open interval (a, b) such that $x \in (a, b)$.
PROOF: Take $(a, b) = (x - 1, x + 1)$.
$\langle 1 \rangle 2$. For any basic open sets B_1, B_2 if $x \in B_1 \cap B_2$, then there exists a basic open set B_3 such that $x \in B_3 \subseteq B_1 \cap B_2$
$\langle 2 \rangle 1$. CASE: $B_1 = (a, b)$, $B_2 = (c, d)$
PROOF: Take $B_3 = (\max(a, c), \min(b, d))$.
$\langle 2 \rangle 2$. CASE: $B_1 = (a, b)$ or $(a, b) \setminus K$, $B_2 = (c, d)$ or $(c, d) \setminus K$, and they are not both open intervals.
PROOF: Take $B_3 = (\max(a, c), \min(b, d)) \setminus K$.
$\square$

Lemma 18. *The lower limit topology and the K-topology are incomparable.*

PROOF:
$\langle 1 \rangle 1$. The interval $[10, 11)$ is not open in the K-topology.
PROOF: There is no open interval (a, b) such that $10 \in (a, b) \subseteq [10, 11)$ or $10 \in (a, b) \setminus K \subseteq [10, 11)$.
$\langle 1 \rangle 2$. The set $(-1, 1) \setminus K$ is not open in the lower limit topology.
PROOF: There is no half-open interval $[a, b)$ such that $0 \in [a, b) \subseteq (-1, 1) \setminus K$, since there must be a positive integer n with $1/n \in [a, b)$.
$\square$

Definition 19 (Subbasis)**.** A *subbasis* $\mathcal{S}$ for a topology on X is a set $\mathcal{S} \subseteq \mathcal{P}X$ such that $\bigcup \mathcal{S} = X$.

The topology *generated* by the subbasis $\mathcal{S}$ is the set of all unions of finite intersections of elements of $\mathcal{S}$.

We prove this is a topology.

PROOF:
⟨1⟩1. The set $\mathcal{B}$ of all finite intersections of elements of $\mathcal{S}$ forms a basis for a topology on X.
  ⟨2⟩1. $\bigcup \mathcal{B} = X$
   PROOF: Since $\mathcal{S} \subseteq \mathcal{B}$.
  ⟨2⟩2. $\mathcal{B}$ is closed under binary intersection.
   PROOF: By definition.
⟨1⟩2. Q.E.D.
  PROOF: By Lemma 13.
□

We have simultaneously proved:

Proposition 20. *Let $\mathcal{S}$ be a subbasis for the topology on X. Then the set of all finite intersections of elements of $\mathcal{S}$ is a basis for the topology on X.*

Proposition 21. *Let X be a set. Let $\mathcal{S}$ be a subbasis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ is the coarsest topology that includes $\mathcal{S}$.*

PROOF: Since every topology that includes $\mathcal{S}$ includes every union of finite intersections of elements of $\mathcal{S}$. □

4 Open Maps

Definition 22 (Open Map)**.** Let X and Y be topological spaces and $f : X \to Y$. Then f is an *open map* if and only if, for every open set U in X, the set $f(U)$ is open in Y.

5 The Order Topology

Definition 23 (Order Topology)**.** Let X be a linearly ordered set with at least two points. The *order topology* on X is the topology generated by the basis $\mathcal{B}$ consisting of:

- all open intervals (a, b);
- all intervals of the form $[\bot, b)$ where $\bot$ is least in X;
- all intervals of the form $(a, \top]$ where $\top$ is greatest in X.

We prove this is a basis for a topology.

PROOF:
⟨1⟩1. For all $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B$.
  ⟨2⟩1. LET: $x \in X$
  ⟨2⟩2. CASE: x is greatest in X.
   ⟨3⟩1. PICK $y \in X$ with $y \neq x$

⟨3⟩2. $x \in (y, x] \in \mathcal{B}$
⟨2⟩3. Case: x is least in X.
⟨3⟩1. Pick $y \in X$ with $y \neq x$
⟨3⟩2. $x \in [x, y) \in \mathcal{B}$
⟨2⟩4. Case: x is neither greatest nor least in X.
⟨3⟩1. Pick $a, b \in X$ with $a < x$ and $x < b$
⟨3⟩2. $x \in (a, b) \in \mathcal{B}$
⟨1⟩2. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$
⟨2⟩1. Let: $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$
⟨2⟩2. Case: $B_1 = (a, b)$, $B_2 = (c, d)$
Proof: Take $B_3 = (\max(a, c), \min(b, d))$.
⟨2⟩3. Case: $B_1 = (a, b)$, $B_2 = [\bot, d)$
Proof: Take $B_3 = (a, \min(b, d))$.
⟨2⟩4. Case: $B_1 = (a, b)$, $B_2 = (c, \top]$
Proof: Take $B_3 = (\max(a, c), b)$.
⟨2⟩5. Case: $B_1 = [\bot, b)$, $B_2 = [\bot, d)$
Proof: Take $B_3 = [\bot, \min(b, d))$.
⟨2⟩6. Case: $B_1 = [\bot, b)$, $B_2 = (c, \top]$
Proof: Take $B_3 = (c, b)$.
□

Lemma 24. *Let X be a linearly ordered set. Then the open rays form a subbasis for the order topology on X.*

Proof:
⟨1⟩1. Every open ray is open.
⟨2⟩1. For all $a \in X$, the ray $(-\infty, a)$ is open.
⟨3⟩1. Let: $x \in (-\infty, a)$
⟨3⟩2. Case: x is least in X
Proof: $xin[x, a) = (-\infty, a)$.
⟨3⟩3. Case: x is not least in X
⟨4⟩1. Pick $y < x$
⟨4⟩2. $x \in (y, a) \subseteq (-\infty, a)$
⟨2⟩2. For all $a \in X$, the ray $(a, +\infty)$ is open.
Proof: Similar.
⟨1⟩2. Every basic open set is a finite intersection of open rays.
Proof: We have $(a, b) = (a, +\infty) \cap (-\infty, b)$, $[\bot, b) = (-\infty, b)$ and $(a, \top] = (a, +\infty)$.
□

Definition 25 (Standard Topology on the Real Line)**.** The *standard topology on the real line* is the order topology on $\mathbb{R}$ generated by the standard order.

Lemma 26. *The lower limit topology is strictly finer than the standard topology on $\mathbb{R}$.*

Proof:

$\langle 1 \rangle 1$. Every open interval is open in the lower limit topology.
PROOF: If $x \in (a,b)$ then $x \in [x,b) \subseteq (a,b)$.
$\langle 1 \rangle 2$. The half-open interval $[0,1)$ is not open in the standard topology.
PROOF: There is no open interval (a,b) such that $0 \in (a,b) \subseteq [0,1)$.
□

Lemma 27. *The K-topology is strictly finer than the standard topology on $\mathbb{R}$.*

PROOF:
$\langle 1 \rangle 1$. Every open interval is open in the K-topology.
PROOF: Corollary 13.1.
$\langle 1 \rangle 2$. The set $(-1,1) \setminus K$ is not open in the standard topology.
PROOF: There is no open interval (a,b) such that $0 \in (a,b) \subseteq (-1,1) \setminus K$, since there must be a positive integer n with $1/n \in (a,b)$.
□

Definition 28 (Ordered Square)**.** The *ordered square* I_o^2 is the set $[0,1]^2$ under the order topology generated by the dictionary order.

6 The Product Topology

Definition 29 (Product Topology)**.** Let X and Y be topological spaces. The *product topology* on $X \times Y$ is the topology generated by the basis $\mathcal{B}$ consisting of all sets $U \times V$ such that U is open in X and V is open in Y.

We prove that this is a basis for a topology.

PROOF:
$\langle 1 \rangle 1$. For all $(x,y) \in X \times Y$ there exists $B \in \mathcal{B}$ such that $(x,y) \in B$
PROOF: Take $B = X \times Y$.
$\langle 1 \rangle 2$. For all $B_1, B_2 \in \mathcal{B}$ and $(x,y) \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $(x,y) \in B_3 \subseteq B_1 \cap B_2$
PROOF: Take $B_3 = B_1 \cap B_2$.
□

Theorem 30. *If $\mathcal{B}$ is a basis for the topology of X and $\mathcal{C}$ is a basis for the topology of Y, then*

$$\mathcal{D} = \{B \times C \mid B \in \mathcal{B}, C \in \mathcal{C}\}$$

is a basis for the product topology on $X \times Y$.

PROOF:
$\langle 1 \rangle 1$. Every member of $\mathcal{D}$ is open in the product topology.
PROOF: By definitions.
$\langle 1 \rangle 2$. For every open set U and point $(x,y) \in U$, there exists $D \in \mathcal{D}$ such that $(x,y) \in D \subseteq U$.
$\langle 2 \rangle 1$. LET: U be open and $(x,y) \in U$
$\langle 2 \rangle 2$. PICK W open in X and V open in Y with $(x,y) \in W \times V \subseteq U$

⟨2⟩3. PICK $B \in \mathcal{B}$ with $x \in B \subseteq W$
⟨2⟩4. PICK $C \in \mathcal{C}$ with $y \in C \subseteq V$
⟨2⟩5. $(x, y) \in B \times C \subseteq U$
⟨1⟩3. Q.E.D.
PROOF: By Lemma 14.
□

Definition 31 (Standard Topology on the Plane)**.** The *standard topology* on $\mathbb{R}^2$ is the product topology of the standard topology on $\mathbb{R}$ with itself.

Lemma 32. *Let X and Y be topological spaces. The set*

$$\mathcal{S} = \{U \times Y \mid U \text{ open in } X\} \cup \{X \times V \mid V \text{ open in } Y\}$$

is a subbasis for the product topology on $X \times Y$.

PROOF:
⟨1⟩1. Every element of $\mathcal{S}$ is open.
PROOF: From defintions.
⟨1⟩2. Every basic open set is a finite intersection of members of $\mathcal{S}$.
PROOF: Given U open in X and V open in Y, we have $U \times V = (U \times Y) \cap (X \times V)$.
□

Proposition 33. *For any topological spaces X and Y, the projections $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are open maps.*

PROOF:
⟨1⟩1. π_1 is an open map.
⟨2⟩1. LET: U be open in $X \times Y$.
⟨2⟩2. LET: $x \in \pi_1(U)$
⟨2⟩3. PICK $y \in Y$ such that $(x, y) \in U$.
⟨2⟩4. PICK V open in X and W open in Y such that $(x, y) \in V \times W \subseteq U$.
⟨2⟩5. $x \in V \subseteq \pi_1(U)$
⟨3⟩1. LET: $v \in V$
⟨3⟩2. $(v, y) \in V \times W$
PROOF: $y \in W$ from ⟨2⟩4.
⟨3⟩3. $(v, y) \in U$
PROOF: From ⟨2⟩4.
⟨3⟩4. $v \in \pi_1(U)$
⟨2⟩6. Q.E.D.
PROOF: Lemma 9.
⟨1⟩2. π_2 is an open map.
PROOF: Similar.
□

Proposition 34. *Let $\mathcal{T}$ and $\mathcal{T}'$ be topologies on a nonempty set X, and $\mathcal{U}$ and $\mathcal{U}'$ be topologies on a nonempty set Y. Then $\mathcal{T} \subseteq \mathcal{T}'$ and $\mathcal{U} \subseteq \mathcal{U}'$ if and only if the product topology generated by $\mathcal{T}$ and $\mathcal{U}$ is coarser than the product topology generated by $\mathcal{T}'$ and $\mathcal{U}'$.*

PROOF:
⟨1⟩1. LET: $\mathcal{P}$ be the topology generated by $\mathcal{T}$ and $\mathcal{U}$, and $\mathcal{P}'$ the product topology generated by $\mathcal{T}'$ and $\mathcal{U}'$
⟨1⟩2. If $\mathcal{T} \subseteq \mathcal{T}'$ and $\mathcal{U} \subseteq \mathcal{U}'$ then $\mathcal{P} \subseteq \mathcal{P}'$
PROOF: By Corollary 13.1.
⟨1⟩3. If $\mathcal{P} \subseteq \mathcal{P}'$ then $\mathcal{T} \subseteq \mathcal{T}'$ and $\mathcal{U} \subseteq \mathcal{U}'$
⟨2⟩1. ASSUME: $\mathcal{P} \subseteq \mathcal{P}'$
⟨2⟩2. $\mathcal{T} \subseteq \mathcal{T}'$
⟨3⟩1. LET: $U \in \mathcal{T}$
⟨3⟩2. $U \times Y \in \mathcal{P}$
⟨3⟩3. $U \times Y \in \mathcal{P}'$
PROOF: By ⟨2⟩1
⟨3⟩4. $U \in \mathcal{T}'$
PROOF: By Proposition 33.
⟨2⟩3. $\mathcal{U} \subseteq \mathcal{U}'$
PROOF: Similar.
□

7 The Subspace Topology

Definition 35 (Subspace Topology)**.** Let X be a topological space and $Y \subseteq X$. The *subspace topology* on Y is $\mathcal{T} = \{U \cap Y \mid U \text{ is open in } X\}$.

We prove this is a topology.

PROOF:
⟨1⟩1. $Y \in \mathcal{T}$
PROOF: Since $Y = X \cap Y$
⟨1⟩2. For all $\mathcal{U} \subseteq \mathcal{T}$, we have $\bigcup \mathcal{U} \in \mathcal{T}$
⟨2⟩1. LET: $\mathcal{U} \subseteq \mathcal{T}$
⟨2⟩2. LET: $\mathcal{V} = \{V \text{ open in } X \mid V \cap Y \in \mathcal{U}\}$
⟨2⟩3. $\bigcup \mathcal{U} = (\bigcup \mathcal{V}) \cap Y$
⟨1⟩3. For all $U, V \in \mathcal{T}$, we have $U \cap V \in \mathcal{T}$
⟨2⟩1. LET: $U, V \in \mathcal{T}$
⟨2⟩2. PICK U', V' open in X such that $U = U' \cap Y$ and $V = V' \cap Y$
⟨2⟩3. $(U \cap V) = (U' \cap V') \cap Y$
□

Lemma 36. *Let X be a topological space and $Y \subseteq X$. Let $\mathcal{B}$ be a basis for the topology on X. Then $\mathcal{B}' = \{B \cap Y \mid B \in \mathcal{B}\}$ is a basis for the subspace topology on Y.*

PROOF:
⟨1⟩1. Every element in $\mathcal{B}'$ is open in Y
⟨1⟩2. For every open set U in Y and point $y \in U$, there exists $B' \in \mathcal{B}'$ such that $y \in B' \subseteq U$
⟨2⟩1. LET: U be open in Y and $y \in U$

$\langle 2\rangle 2$. Pick V open in X such that $U = V \cap Y$
$\langle 2\rangle 3$. Pick $B \in \mathcal{B}$ such that $y \in B \subseteq V$
$\langle 2\rangle 4$. Let: $B' = B \cap Y$
$\langle 2\rangle 5$. $B' \in \mathcal{B}'$
$\langle 2\rangle 6$. $y \in B' \subseteq U$
$\langle 1\rangle 3$. Q.E.D.
Proof: By Lemma 14.
$\square$

Lemma 37. *Let X be a topological space and $Y \subseteq X$. Let $\mathcal{S}$ be a basis for the topology on X. Then $\mathcal{S}' = \{S \cap Y \mid S \in \mathcal{S}\}$ is a subbasis for the subspace topology on Y.*

Proof: The set $\{B \cap Y \mid B$ is a finite intersection of elements of $\mathcal{S}\}$ is a basis for the subspace topology by Lemma 36, and this is the set of all finite intersections of elements of $\mathcal{S}'$. $\square$

Lemma 38. *Let Y be a subspace of X. If U is open in Y and Y is open in X then U is open in X.*

Proof:
$\langle 1\rangle 1$. Pick V open in X such that $U = V \cap Y$
$\langle 1\rangle 2$. U is open in X
Proof: Since it is the intersection of two open sets V and Y.
$\square$

Theorem 39. *If A is a subspace of X and B is a subspace of Y then the product topology on $A \times B$ is the same as the topology it inherits as a subspace of $X \times Y$.*

Proof: The product topology is generated by

$$\begin{aligned}&\{U \times V \mid U \text{ open in } A, V \text{ open in } B\}\\ =&\{(U' \cap A) \times (V' \cap B) \mid U' \text{ open in } X, V' \text{ open in } Y\}\\ =&\{(U' \times V') \cap (A \times V) \mid U' \text{ open in } X, V' \text{ open in } Y\}\end{aligned}$$

and this is a basis for the subspace topology by Lemma 36. $\square$

Theorem 40. *Let X be an ordered set in the order topology. Let $Y \subseteq X$ be convex. Then the order topology on Y is the same as the subspace topology on Y.*

Proof:
$\langle 1\rangle 1$. The order topology is finer than the subspace topology.
$\langle 2\rangle 1$. For every open ray R in X, the set $R \cap Y$ is open in the order topology.
$\langle 3\rangle 1$. For all $a \in X$, we have $(-\infty, a) \cap Y$ is open in the order topology.
$\langle 4\rangle 1$. Case: For all $y \in Y$ we have $y < a$
Proof: In this case $(-\infty, a) \cap Y = Y$.
$\langle 4\rangle 2$. Case: For all $y \in Y$ we have $a < y$
Proof: In this case $(-\infty, a) \cap Y = \emptyset$.
$\langle 4\rangle 3$. Case: There exists $y \in Y$ such that $y \leq a$ and $y \in Y$ such that $a \leq y$

⟨5⟩1. $a \in Y$

Proof: By convexity.

⟨5⟩2. $(-\infty, a) \cap Y = \{y \in Y \mid y < a\}$

⟨3⟩2. For all $a \in X$, we have $(a, +\infty) \cap Y$ is open in the order topology.

Proof: Similar.

⟨2⟩2. Q.E.D.

Proof: By Lemmas 24 and 37 and Corollary **??**.

⟨1⟩2. The subspace topology is finer than the order topology.

⟨2⟩1. Every open ray in Y is open in the subspace topology.

Proof: For any $a \in Y$ we have $(-\infty, a)_Y = (-\infty, a)_X \cap Y$ and $(a, +\infty)_Y = (a, +\infty)_X \cap Y$.

⟨2⟩2. Q.E.D.

Proof: By Lemma 24 and Corollary **??**

□

Proposition 41. *The order topology on I_o^2 is different from the subspace topology as a subspace of $\mathbb{R}^2$ under the dictionary order topology.*

Proof: The set $\{1/2\} \times (1/2, 1)$ is open in the subspace topology but not in the order topology. □

Proposition 42. *Let X be a topological space, Y a subspace of X, and Z a subspace of Y. Then the subspace topology on Z inherited from X is the same as the subspace topology on Z inherited from Y.*

Proof: The subspace topology inherited from Y is

$$\begin{aligned} &\{V \cap Z \mid V \text{ open in } Y\} \\ =&\{U \cap Y \cap Z \mid U \text{ open in } X\} \\ =&\{U \cap Z \mid U \text{ open in } X\} \end{aligned}$$

which is the subspace topology inherited from X. □

Definition 43 (Unit Circle)**.** The *unit circle* S^1 is $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$

8 Closed Set

Definition 44 (Closed Set)**.** Let X be a topological space and $A \subseteq X$. Then A is *closed* if and only if $X \setminus A$ is open.

Lemma 45. *The empty set is closed.*

Proof: Since the whole space X is always open. □

Lemma 46. *The topological space X is closed.*

Proof: Since $\emptyset$ is open. □

Lemma 47. *The intersection of a nonempty set of closed sets is closed.*

Proof: Let $\mathcal{C}$ be a nonempty set of closed sets. Then $X \setminus \bigcap \mathcal{C} = \bigcup \{X \setminus C \mid C \in \mathcal{C}\}$ is open. ∎

Lemma 48. *The union of two closed sets is closed.*

Proof: Let C and D be closed. Then $X \setminus (C \cup D) = (X \setminus C) \cap (X \setminus D)$ is open. ∎

Theorem 49. *Let X be a topological space and Y a subspace of X. Let $A \subseteq Y$. Then A is closed in Y if and only if there exists a closed set C in X such that $A = C \cap Y$.*

Proof: We have

$$\begin{aligned} & A \text{ is closed in } Y \\ \Leftrightarrow & Y \setminus A \text{ is open in } Y \\ \Leftrightarrow & \exists U \text{ open in } X . Y \setminus A = Y \cap U \\ \Leftrightarrow & \exists C \text{ closed in } X . Y \setminus A = Y \cap (X \setminus U) \\ \Leftrightarrow & \exists C \text{ closed in } X . A = Y \cap U \end{aligned}$$

∎

Theorem 50. *Let Y be a subspace of X and $A \subseteq Y$. If A is closed in Y and Y is closed in X then A is closed in X.*

Proof: Pick a closed set C in X such that $A = C \cap Y$ (Theorem 49). Then A is the intersection of two sets closed in X, hence A is closed in X (Lemma 47). ∎

Proposition 51. *Let X be a set and $\mathcal{C} \subseteq \mathcal{P}X$ a set such that:*

1. $\emptyset \in \mathcal{C}$
2. $X \in \mathcal{C}$
3. *For all $\mathcal{A} \subseteq \mathcal{C}$ nonempty we have $\bigcap \mathcal{A} \in \mathcal{C}$*
4. *For all $C, D \in \mathcal{C}$ we have $C \cup D \in \mathcal{C}$.*

Then there exists a unique topology $\mathcal{T}$ such that $\mathcal{C}$ is the set of closed sets, namely

$$\mathcal{T} = \{X \setminus C \mid C \in \mathcal{C}\}$$

Proof:
⟨1⟩1. Let: $\mathcal{T} = \{X \setminus C \mid C \in \mathcal{C}\}$
⟨1⟩2. $\mathcal{T}$ is a topology
  ⟨2⟩1. $X \in \mathcal{T}$
    Proof: Since $\emptyset \in \mathcal{C}$
  ⟨2⟩2. For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$
    ⟨3⟩1. Let: $\mathcal{U} \subseteq \mathcal{T}$
    ⟨3⟩2. Case: $\mathcal{U} = \emptyset$
      Proof: In this case $\bigcup \mathcal{U} = \emptyset \in \mathcal{T}$ since $X \in \mathcal{C}$

⟨3⟩3. Case: $\mathcal{U} \neq \emptyset$

Proof: In this case $X \setminus \bigcup \mathcal{U} = \bigcap \{X \setminus U \mid U \in \mathcal{U}\} \in \mathcal{C}$.

⟨2⟩3. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$

Proof: Since $X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V) \in \mathcal{C}$.

⟨1⟩3. $\mathcal{C}$ is the set of all closed sets in $\mathcal{T}$

Proof:

$$\begin{aligned} & C \text{ is closed in } \mathcal{T} \\ \Leftrightarrow & X \setminus C \in \mathcal{T} \\ \Leftrightarrow & C \in \mathcal{C} \end{aligned}$$

⟨1⟩4. If $\mathcal{T}'$ is a topology and $\mathcal{C}$ is the set of closed sets in $\mathcal{T}'$ then $\mathcal{T}' = \mathcal{T}$

Proof: We have

$$\begin{aligned} & U \in \mathcal{T} \\ \Leftrightarrow & X \setminus U \in \mathcal{C} \\ \Leftrightarrow & X \setminus U \text{ is closed in } \mathcal{T}' \\ \Leftrightarrow & U \in \mathcal{T}' \end{aligned}$$

□

Proposition 52. *If A is closed in X and B is closed in Y then $A \times B$ is closed in $X \times Y$.*

Proof:

$$(X \times Y) \setminus (A \times B) = ((X \setminus A) \times Y) \cup (X \times (Y \setminus B)) \square$$

Proposition 53. *If U is open and A is closed then $U \setminus A$ is open.*

Proof: $U \setminus A = U \cap (X \setminus A)$ is the intersection of two open sets. □

Proposition 54. *If U is open and A is closed then $A \setminus U$ is closed.*

Proof: $A \setminus U = A \cap (X \setminus U)$ is the intersection of two closed sets. □

9 Interior

Definition 55 (Interior)**.** Let X be a topological space and $A \subseteq X$. The *interior* of A, $\operatorname{Int} A$, is the union of all the open subsets of A.

Lemma 56. *The interior of a set is open.*

Proof: It is a union of open sets. □

Lemma 57.

$$\operatorname{Int} A \subseteq A$$

Proof: Immediate from definition. □

Lemma 58. *A set A is open if and only if $A = \operatorname{Int} A$.*

Proof: If $A = \operatorname{Int} A$ then A is open by Lemma 56. Conversely if A is open then $A \subseteq \operatorname{Int} A$ by the definition of interior and so $A = \operatorname{Int} A$.

10 Neighbourhood

Definition 59 (Neighbourhood)**.** A *neighbourhood* of a point x is an open set that contains x.

11 Closure

Definition 60 (Closure)**.** Let X be a topological space and $A \subseteq X$. The *closure* of A, $\overline{A}$, is the intersection of all the closed sets that include A.

This intersection exists since X is a closed set that includes A (Lemma 46).

Lemma 61. *The closure of a set is closed.*

PROOF: Dual to Lemma 56. □

Lemma 62.

$$A \subseteq \overline{A}$$

PROOF: Immediate from definition. □

Lemma 63. *A set A is closed if and only if $A = \overline{A}$.*

PROOF: Dual to Lemma 58. □

Theorem 64. *Let Y be a subspace of X. Let $A \subseteq Y$. Let $\overline{A}$ be the closure of A in X. Then the closure of A in Y is $\overline{A} \cap Y$.*

PROOF: The closure of A in Y is

$$\begin{aligned} & \bigcap\{C \text{ closed in } Y \mid A \subseteq C\} \\ =& \bigcap\{D \cap Y \mid D \text{ closed in } X, A \subseteq D \cap Y\} && \text{(Theorem 49)} \\ =& \bigcap\{D \mid D \text{ closed in } X, A \subseteq D\} \cap Y \\ =& \overline{A} \cap Y \end{aligned}$$

□

Theorem 65. *Let X be a topological space, $A \subseteq X$ and $x \in X$. Then $x \in \overline{A}$ if and only if every neighbourhood of x intersects A.*

PROOF: We have

$$\begin{aligned} & x \in \overline{A} \\ \Leftrightarrow & \forall C. C \text{ closed } \wedge A \subseteq C \Rightarrow x \in C \\ \Leftrightarrow & \forall U. U \text{ open } \wedge A \cap U = \emptyset \Rightarrow x \notin U \\ \Leftrightarrow & \forall U. U \text{ open } \wedge x \in U \Rightarrow U \text{ intersects } A \end{aligned}$$

□

Theorem 66. *Let X be a topological space, $A \subseteq X$ and $x \in X$. Let $\mathcal{B}$ be a basis for X. Then $x \in \overline{A}$ if and only if, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A.*

PROOF:

⟨1⟩1. If $x \in \overline{A}$ then, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A.
PROOF: This follows from Theorem 65 since every element of $\mathcal{B}$ is open (Corollary 13.1).
⟨1⟩2. Suppose that, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A. Then $x \in \overline{A}$.
⟨2⟩1. ASSUME: For all $B \in \mathcal{B}$, if $x \in B$ then B intersects A.
⟨2⟩2. LET: U be an open set that contains x
PROVE: U intersects A.
⟨2⟩3. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$.
⟨2⟩4. B intersects A.
PROOF: From ⟨2⟩1.
⟨2⟩5. U intersects A.
⟨2⟩6. Q.E.D.
PROOF: By Theorem 65.
□

Proposition 67. *If $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$.*

PROOF: This holds because $\overline{B}$ is a closed set that includes A. □

Proposition 68.

$$\overline{A \cup B} = \overline{A} \cup \overline{B}$$

PROOF:
⟨1⟩1. $\overline{A} \subseteq \overline{A \cup B}$
PROOF: By Proposition 67.
⟨1⟩2. $\overline{B} \subseteq \overline{A \cup B}$
PROOF: By Proposition 67.
⟨1⟩3. $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$
⟨2⟩1. LET: $x \in \overline{A \cup B}$
⟨2⟩2. ASSUME: $x \notin \overline{A}$
PROVE: $x \in \overline{B}$
⟨2⟩3. PICK a neighbourhood U of x that does not intersect A
⟨2⟩4. LET: V be any neighbourhood of x
⟨2⟩5. $U \cap V$ is a neighbourhood of x
⟨2⟩6. $U \cap V$ intersects $A \cup B$
PROOF: From ⟨2⟩1 and Theorem 65.
⟨2⟩7. $U \cap V$ intersects B
PROOF: From ⟨2⟩3
⟨2⟩8. V intersects B
⟨2⟩9. Q.E.D.
PROOF: We have $x \in \overline{B}$ from Theorem 65.
□

Proposition 69. *Let X and Y be topological spaces. Let $A \subseteq X$ and $B \subseteq Y$. Then*

$$\overline{A \times B} = \overline{A} \times \overline{B}$$

PROOF:

⟨1⟩1. $\overline{A \times B} \subseteq \overline{A} \times \overline{B}$
  ⟨2⟩1. $A \subseteq \overline{A}$
    PROOF: Lemma 62.
  ⟨2⟩2. $B \subseteq \overline{B}$
    PROOF: Lemma 62.
  ⟨2⟩3. $A \times B \subseteq \overline{A} \times \overline{B}$
  ⟨2⟩4. $\overline{A \times B} \subseteq \overline{A} \times \overline{B}$
    PROOF: Since $\overline{A} \times \overline{B}$ is closed by Proposition 52.
⟨1⟩2. $\overline{A} \times \overline{B} \subseteq \overline{A \times B}$
  ⟨2⟩1. LET: $x \in \overline{A}$ and $y \in \overline{B}$
  ⟨2⟩2. LET: U be a neighbourhood of (x, y)
  ⟨2⟩3. PICK V open in X and W open in Y such that $(x, y) \in V \times W \subseteq U$
  ⟨2⟩4. V intersects A
    PROOF: By Theorem 65 and ⟨2⟩1.
  ⟨2⟩5. W intersects B
    PROOF: By Theorem 65 and ⟨2⟩1.
  ⟨2⟩6. U intersects $A \times B$
  ⟨2⟩7. Q.E.D.
    PROOF: By Theorem 65.
□

12 Limit Points

Definition 70 (Limit Point)**.** Let X be a topological space, $a \in X$ and $A \subseteq X$. Then a is a *limit point*, *cluster point* or *point of accumulation* for A if and only if every neighbourhood of a intersects A at a point other than a.

Lemma 71. *The point a is an accumulation point for A if and only if $a \in \overline{A \setminus \{a\}}$.*

PROOF: From Theorem 65. □

Theorem 72. *Let X be a topological space and $A \subseteq X$. Let A' be the set of all limit points of A. Then $\overline{A} = A \cup A'$.*

PROOF:
⟨1⟩1. For all $x \in \overline{A}$, if $x \notin A$ then $x \in A'$
  PROOF: From Theorem 65.
⟨1⟩2. $A \subseteq \overline{A}$
  PROOF: Lemma 62.
⟨1⟩3. $A' \subseteq \overline{A}$
  PROOF: From Theorem 65.
□

Corollary 72.1. *A set is closed if and only if it contains all its limit points.*

Proposition 73. *In an indiscrete topology, every point is a limit point of any set that has more than one point.*

Proof: Let X be an indiscrete space. Let A be a set with more than one point and x be a point. The only neighbourhood of x is X, which must intersect A at a point other than x. □

13 T_1 Spaces

Definition 74 (T_1 Space)**.** A topological space is T_1 if and only if every singleton is closed.

Lemma 75. *A space is T_1 if and only if every finite set is closed.*

Proof: From Lemma 48. □

Theorem 76. *In a T_1 space, a point a is a limit point of a set A if and only if every neighbourhood of a contains infinitely many points of A.*

Proof:
⟨1⟩1. If a is a limit point of A then every neighbourhood of a contains infinitely many points of A.
⟨2⟩1. Assume: a is a limit point of A.
⟨2⟩2. Let: U be a neighbourhood of a.
⟨2⟩3. Assume: for a contradiction U contains only finitely many points of A.
⟨2⟩4. $(U \cap A) \setminus \{a\}$ is closed.
Proof: By the T_1 axiom.
⟨2⟩5. $(U \setminus A) \cup \{a\}$ is open.
Proof: It is $U \setminus ((U \cap A) \setminus \{a\})$.
⟨2⟩6. $(U \setminus A) \cup \{a\}$ intersects A in a point other than a.
Proof: From ⟨2⟩1.
⟨2⟩7. Q.E.D.
□
⟨1⟩2. If every neighbourhood of a contains infinitely many points of A then a is a limit point of A.
Proof: Immediate from definitions.
□

(To see this does not hold in every space, see Proposition 73.)

Proposition 77. *A space is T_1 if and only if, for any two distinct points x and y, there exist neighbourhoods U of x and V of y such that $x \notin V$ and $y \notin U$.*

Proof:
⟨1⟩1. Let: X be a topological space.
⟨1⟩2. If X is T_1 then, for any two distinct points x and y, there exist neighbourhoods U of x and V of y such that $x \notin V$ and $y \notin U$.
Proof: This holds because $\{x\}$ and $\{y\}$ are closed.
⟨1⟩3. Suppose, for any two distinct points x and y, there exist neighbourhoods U of x and V of y such that $x \notin V$ and $y \notin U$. Then X is T_1.

⟨2⟩1. Assume: For any two distinct points x and y, there exist neighbourhoods U of x and V of y such that $x \notin V$ and $y \notin U$.
⟨2⟩2. Let: $a \in X$
⟨2⟩3. $\{a\}$ is closed.
Proof: For all $b \neq a$ there exists a neighbourhood U of b such that $U \subseteq X \setminus \{a\}$.
□

14 Hausdorff Spaces

Definition 78 (Hausdorff Space)**.** A topological space is *Hausdorff* if and only if, for any points x, y with $x \neq y$, there exist disjoint open sets U and V such that $x \in U$ and $y \in V$.

Theorem 79. *Every Hausdorff space is T_1.*

Proof:
⟨1⟩1. Let: X be a Hausdorff space.
⟨1⟩2. Let: $b \in X$
Prove: $\overline{\{b\}} = \{b\}$
⟨1⟩3. Assume: $a \in \overline{\{b\}}$ and $a \neq b$
⟨1⟩4. Pick disjoint neighbourhoods U of a and V of b.
⟨1⟩5. U intersects $\{b\}$
Proof: Theorem 65.
⟨1⟩6. $b \in U$
⟨1⟩7. Q.E.D.
Proof: This contradicts the fact that U and V are disjoint (⟨1⟩4).
□

Proposition 80. *An infinite set under the finite complement topology is T_1 but not Hausdorff.*

Proof:
⟨1⟩1. Let: X be an infinite set under the finite complement topology.
⟨1⟩2. Every singleton is closed.
Proof: By definition.
⟨1⟩3. Pick$a, b \in X$ with $a \neq b$
⟨1⟩4. There are no disjoint neighbourhoods U of a and V of b.
⟨2⟩1. Let: U be a neighbourhood of a and V a neighbourhood of b.
⟨2⟩2. $X \setminus U$ and $X \setminus V$ are finite.
⟨2⟩3. Pick $c \in X$ that is not in $X \setminus U$ or $X \setminus V$.
⟨2⟩4. $c \in U \cap V$
□

15 Convergence

Definition 81 (Convergence)**.** Let X be a topological space. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of points in X and $l \in X$. Then the sequence $(a_n)_{n\in\mathbb{N}}$ *converges* to the *limit* l, $a_n \to l$ as $n \to \infty$, if and only if, for every neighbourhood U of l, there exists N such that, for all $n \geq N$, we have $a_n \in U$.

Theorem 82. *In a Hausdorff space, a sequence has at most one limit.*

PROOF:
⟨1⟩1. LET: X be a Hausdorff space.
⟨1⟩2. ASSUME: for a contradiction $a_n \to l$ as $n \to \infty$, $a_n \to m$ as $n \to \infty$, and $l \neq m$
⟨1⟩3. PICK disjoint neighbourhoods U of l and V of m
  PROOF: By the Hausdorff axiom.
⟨1⟩4. PICK M and N such that $a_n \in U$ for $n \geq M$ and $a_n \in V$ for $n \geq N$
⟨1⟩5. $a_{\max(M,N)} \in U \cap V$
⟨1⟩6. Q.E.D.
  PROOF: This contradicts the fact that U and V are disjoint (⟨1⟩3).
□

To see this is not always true in spaces that are T_1 but not Hausdorff:

Proposition 83. *Let X be an infinite set under the finite complement topology. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence with all points distinct. Then for every $l \in X$ we have $a_n \to l$ as $n \to \infty$.*

PROOF: Let U be any neighbourhood of l. Since $X \setminus U$ is finite, there must exist N such that, for all $n \geq N$, we have $a_n \in U$. □

Theorem 84. *Every linearly ordered set under the order topology is Hausdorff.*

PROOF:
⟨1⟩1. LET: X be a linearly ordered set under the order topology.
⟨1⟩2. LET: $a, b \in X$ with $a \neq b$
⟨1⟩3. ASSUME: w.l.o.g. $a < b$
⟨1⟩4. CASE: There exists c such that $a < c < b$
  PROOF: The sets $(-\infty, c)$ and $(c, +\infty)$ are disjoint neighbourhoods of a and b respectively.
⟨1⟩5. CASE: There is no c such that $a < c < b$
  PROOF: The sets $(-\infty, b)$ and $(a, +\infty)$ are disjoint neighbourhoods of a and b respectively.
□

Theorem 85. *The product of two Hausdorff spaces is Hausdorff.*

PROOF:
⟨1⟩1. LET: X and Y be Hausdorff spaces.
⟨1⟩2. LET: (x_1, y_1) and (x_2, y_2) be distinct points in $X \times Y$

$\langle 1\rangle 3$. ASSUME: w.l.o.g. $x_1 \neq x_2$
$\langle 1\rangle 4$. PICK disjoint neighbourhoods U of x_1 and V of x_2.
$\langle 1\rangle 5$. $U \times Y$ and $V \times Y$ are disjoint neighbourhoods of (x_1, y_1) and (x_2, y_2).
$\square$

Theorem 86. *A subspace of a Hausdorff space is Hausdorff.*

PROOF:
$\langle 1\rangle 1$. LET: X be a Hausdorff space and Y a subspace of X.
$\langle 1\rangle 2$. LET: $x, y \in Y$ with $x \neq y$
$\langle 1\rangle 3$. PICK disjoint neighbourhoods U of x and V of y in X.
$\langle 1\rangle 4$. $U \cap Y$ and $V \cap Y$ are disjoint neighbourhoods of x and y respectively in Y.
$\square$

Proposition 87. *A space X is Hausdorff if and only if the diagonal $\Delta = \{(x, x) \mid x \in X\}$ is closed in X^2.*

PROOF:

$$\begin{aligned} & X \text{ is Hausdorff} \\ \Leftrightarrow & \forall x, y \in X . x \neq y \Rightarrow \exists V, W \text{ open}. x \in V \wedge y \in W \wedge V \cap W = \emptyset \\ \Leftrightarrow & \forall (x, y) \in X^2 \setminus \Delta . \exists V, W \text{ open}. (x, y) \subseteq V \times W \subseteq X^2 \setminus \Delta \\ \Leftrightarrow & \Delta \text{ is closed} \end{aligned}$$

$\square$

16 Boundary

Definition 88 (Boundary)**.** The *boundary* of a set A is the set $\partial A = \overline{A} \cap \overline{X \setminus A}$.

Proposition 89.

$$\operatorname{Int} A \cap \partial A = \emptyset$$

PROOF: Since $\overline{X \setminus A} = X \setminus \operatorname{Int} A$. $\square$

Proposition 90.

$$\overline{A} = \operatorname{Int} A \cup \partial A$$

PROOF:

$$\begin{aligned} \operatorname{Int} A \cup \partial A &= \operatorname{Int} A \cup (\overline{A} \cap \overline{X \setminus A}) \\ &= (\operatorname{Int} A \cup \overline{A}) \cap (\operatorname{Int} A \cup \overline{X \setminus A}) \\ &= \overline{A} \cap X \\ &= \overline{A} \end{aligned}$$

Proposition 91. *$\partial A = \emptyset$ if and only if A is open and closed.*

PROOF: If $\partial A = \emptyset$ then $\overline{A} = \operatorname{Int} A$ by Proposition 90.

Proposition 92. *A set U is open if and only if $\partial U = \overline{U} \setminus U$.*

PROOF:

$$\begin{aligned} & \partial U = \overline{U} \setminus U \\ \Leftrightarrow & \overline{U} \setminus \operatorname{Int} U = \overline{U} \setminus U && \text{(Propositions 89, 90)} \\ \Leftrightarrow & \operatorname{Int} U = U \end{aligned}$$

∎

17 Continuous Functions

Definition 93 (Continuous)**.** Let X and Y be topological spaces. A function $f : X \to Y$ is *continuous* if and only if, for every open set V in Y, the set $f^{-1}(V)$ is open in X.

Proposition 94. *Let X and Y be topological spaces and $f : X \to Y$. Let $\mathcal{B}$ be a basis for Y. Then f is continuous if and only if, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X.*

PROOF:
⟨1⟩1. If f is continuous then, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X.
  PROOF: Since every element of B is open (Lemma 13).
⟨1⟩2. Suppose that, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X. Then f is continuous.
  ⟨2⟩1. ASSUME: For all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X.
  ⟨2⟩2. LET: V be open in Y.
  ⟨2⟩3. PICK $\mathcal{A} \subseteq \mathcal{B}$ such that $V = \bigcup \mathcal{A}$
    PROOF: By Lemma 13.
  ⟨2⟩4. $f^{-1}(V)$ is open in X.
    PROOF:

$$\begin{aligned} f^{-1}(V) &= f^{-1}\left(\bigcup \mathcal{A}\right) \\ &= \bigcup_{B \in \mathcal{A}} f^{-1}(B) \end{aligned}$$

∎

Proposition 95. *Let X and Y be topological spaces and $f : X \to Y$. Let $\mathcal{S}$ be a basis for Y. Then f is continuous if and only if, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X.*

PROOF:
⟨1⟩1. If f is continuous then, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X.
  PROOF: Since every element of $\mathcal{S}$ is open.
⟨1⟩2. Suppose that, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X. Then f is continuous.
  ⟨2⟩1. ASSUME: For all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X.
  ⟨2⟩2. For every set B that is the finite intersection of elemets of $\mathcal{S}$, we have $f^{-1}(B)$ is open in X.
    PROOF: Because $f^{-1}(V_1 \cap \cdots \cap V_n) = f^{-1}(V_1) \cap \cdots \cap f^{-1}(V_n)$.
  ⟨2⟩3. Q.E.D.

PROOF: From Propositions 20 and 94.
∎

Definition 96 (Continuous at a Point)**.** Let X and Y be topological spaces. Let $f : X \to Y$ and $x \in X$. Then f is *continuous at* x if and only if, for every neighbourhood V of $f(x)$, there exists a neighbourhood U of x such that $f(U) \subseteq V$.

Theorem 97. *Let X and Y be topological spaces and $f : X \to Y$. Then the following are equivalent:*

1. *f is continuous.*
2. *For all $A \subseteq X$, we have $f(\overline{A}) \subseteq \overline{f(A)}$*
3. *For all $B \subseteq Y$ closed, we have $f^{-1}(B)$ is closed in X.*
4. *f is continuous at every point of X.*

PROOF:
$\langle 1 \rangle 1$. $1 \Rightarrow 2$
  $\langle 2 \rangle 1$. ASSUME: f is continuous.
  $\langle 2 \rangle 2$. LET: $A \subseteq X$
  $\langle 2 \rangle 3$. LET: $x \in \overline{A}$
    PROVE: $f(x) \in \overline{f(A)}$
  $\langle 2 \rangle 4$. LET: V be a neighbourhood of $f(x)$
  $\langle 2 \rangle 5$. $f^{-1}(V)$ is a neighbourhood of x
  $\langle 2 \rangle 6$. PICK $y \in A \cap f^{-1}(V)$
    PROOF: By Theorem 65.
  $\langle 2 \rangle 7$. $f(y) \in V \cap f(A)$
  $\langle 2 \rangle 8$. Q.E.D.
    PROOF: By Theorem 65.
$\langle 1 \rangle 2$. $2 \Rightarrow 3$
  $\langle 2 \rangle 1$. ASSUME: 2
  $\langle 2 \rangle 2$. LET: B be closed in Y
  $\langle 2 \rangle 3$. LET: $x \in \overline{f^{-1}(B)}$
    PROVE: $x \in f^{-1}(B)$
  $\langle 2 \rangle 4$. $f(x) \in B$
    PROOF:

$$\begin{aligned} f(x) &\in f(\overline{f^{-1}(B)}) \\ &\subseteq \overline{f(f^{-1}(B))} && (\langle 2 \rangle 1) \\ &\subseteq \overline{B} && (Proposition 67) \\ &= B \end{aligned}$$

$\langle 1 \rangle 3$. $3 \Rightarrow 1$
  $\langle 2 \rangle 1$. ASSUME: 3
  $\langle 2 \rangle 2$. LET: V be open in Y
  $\langle 2 \rangle 3$. $Y \setminus V$ is closed in Y

⟨2⟩4. $f^{-1}(Y \setminus V)$ is closed in X
⟨2⟩5. $X \setminus f^{-1}(V)$ is closed in X
⟨2⟩6. $f^{-1}(V)$ is open in X
⟨1⟩4. $1 \Rightarrow 4$
PROOF: For any neighbourhood V of $f(x)$, the set $U = f^{-1}(V)$ is a neighbourhood of x such that $f(U) \subseteq V$.
⟨1⟩5. $4 \Rightarrow 1$
⟨2⟩1. ASSUME: 4
⟨2⟩2. LET: V be open in Y
⟨2⟩3. LET: $x \in f^{-1}(V)$
⟨2⟩4. V is a neighbourhood of $f(x)$
⟨2⟩5. PICK a neighbourhood U of x such that $f(U) \subseteq V$
⟨2⟩6. $x \in U \subseteq f^{-1}(V)$
⟨2⟩7. Q.E.D.
PROOF: By Lemma 9.
□

Theorem 98. *A constant function is continuous.*

PROOF: Let X and Y be topological spaces. Let $b \in Y$, and let $f : X \to Y$ be the constant function with value b. For any open $V \subseteq Y$, the set $f^{-1}(V)$ is either X (if $b \in V$) or $\emptyset$ (if $b \notin V$). □

Theorem 99. *If A is a subspace of X then the inclusion $j : A \to X$ is continuous.*

PROOF: For any V open in X, we have $j^{-1}(V) = V \cap A$ is open in A. □

Theorem 100. *The composite of two continuous functions is continuous.*

PROOF: Let $f : X \to Y$ and $g : Y \to Z$ be continuous. For any V open in Z, we have $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is open in X. □

Theorem 101. *Let $f : X \to Y$ be a continuous function and A be a subspace of X. Then the restriction $f \restriction A : A \to Y$ is continuous.*

PROOF: Let V be open in Y. Then $(f \restriction A)^{-1}(V) = f^{-1}(V) \cap A$ is open in A. □

Theorem 102. *Let $f : X \to Y$ be continuous. Let Z be a subspace of Y such that $f(X) \subseteq Z$. Then the corestriction $f : X \to Z$ is continuous.*

PROOF:
⟨1⟩1. LET: V be open in Z.
⟨1⟩2. PICK U open in Y such that $V = U \cap Z$.
⟨1⟩3. $f^{-1}(V) = f^{-1}(U)$
⟨1⟩4. $f^{-1}(V)$ is open in X.
□

Theorem 103. *Let $f : X \to Y$ be continuous. Let Z be a space such that Y is a subspace of Z. Then the expansion $f : X \to Z$ is continuous.*

Proof: Let V be open in Z. Then $f^{-1}(V) = f^{-1}(V \cap Y)$ is open in X. $\square$

Theorem 104. *Let X and Y be topological spaces. Let $f : X \to Y$. Suppose $\mathcal{U}$ is a set of open sets in X such that $X = \bigcup \mathcal{U}$ and, for all $U \in \mathcal{U}$, we have $f \restriction U : U \to Y$ is continuous. Then f is continuous.*

Proof:
$\langle 1 \rangle 1$. Let: V be open in Y
$\langle 1 \rangle 2$. $f^{-1}(V) = \bigcup_{U \in \mathcal{U}} (f \restriction U)^{-1}(V)$
$\langle 1 \rangle 3$. For all $U \in \mathcal{U}$, we have $(f \restriction U)^{-1}(V)$ is open in U.
$\langle 1 \rangle 4$. For all $U \in \mathcal{U}$, we have $(f \restriction U)^{-1}(V)$ is open in X.
  Proof: Lemma 38.
$\square$

Theorem 105 (Pasting Lemma)**.** *Let X and Y be topological spaces. Let A and B be closed subspaces of X such that $X = A \cup B$. Let $f : A \to Y$ and $g : B \to Y$ be continuous. Suppose f and g agree on $A \cap B$. Define $h : X \to Y$ by*

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases}$$

Then h is continuous.

Proof:
$\langle 1 \rangle 1$. Let: $C \subseteq Y$ be closed.
$\langle 1 \rangle 2$. $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$
$\langle 1 \rangle 3$. $f^{-1}(C)$ and $g^{-1}(C)$ are closed in X.
  Proof: Theorems 97 and 50.
$\langle 1 \rangle 4$. $h^{-1}(C)$ is closed in X.
  Proof: Lemma 48.
$\langle 1 \rangle 5$. Q.E.D.
  Proof: Theorem 97.
$\square$

Theorem 106. *Let $f : X \to Y \times Z$. If $\pi_1 \circ f$ and $\pi_2 \circ f$ are continuous then f is continuous.*

Proof:
$\langle 1 \rangle 1$. Let: U be open in Y and V open in Z
$\langle 1 \rangle 2$. $f^{-1}(U \times V) = (\pi_1 \circ f)^{-1}(U) \cap (\pi_2 \circ f)^{-1}(V)$
$\langle 1 \rangle 3$. $f^{-1}(U \times V)$ is open in X.
$\langle 1 \rangle 4$. Q.E.D.
  Proof: Proposition 94.
$\square$

Proposition 107. *Let $f : \mathbb{R} \to \mathbb{R}$ and $x \in \mathbb{R}$. Then f is continuous at x if and only if, for all $\epsilon > 0$, there exists $\delta > 0$ such that. for all $y \in \mathbb{R}$, if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$.*

PROOF:
$\langle 1\rangle 1$. If f is continuous at x then, for all $\epsilon > 0$, there exists $\delta > 0$ such that. for all $y \in \mathbb{R}$, if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$
  $\langle 2\rangle 1$. ASSUME: f is continuous.
  $\langle 2\rangle 2$. LET: $\epsilon > 0$
  $\langle 2\rangle 3$. $f^{-1}((f(x) - \epsilon, f(x) + \epsilon))$ is open.
  $\langle 2\rangle 4$. PICK a, b such that $x \in (a, b) \subseteq f^{-1}((f(x) - \epsilon, f(x) + \epsilon))$
  $\langle 2\rangle 5$. LET: $\delta = \min(x - a, b - x)$
  $\langle 2\rangle 6$. LET: $y \in \mathbb{R}$ with $|x - y| < \delta$
  $\langle 2\rangle 7$. $y \in (a, b)$
  $\langle 2\rangle 8$. $f(y) \in (f(x) - \epsilon, f(x) + \epsilon)$
  $\langle 2\rangle 9$. $|f(x) - f(y)| < \epsilon$
$\langle 1\rangle 2$. Suppose, for all $\epsilon > 0$, there exists $\delta > 0$ such that. for all $y \in \mathbb{R}$, if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$. Then f is continuous at x.
  $\langle 2\rangle 1$. ASSUME: For all $\epsilon > 0$, there exists $\delta > 0$ such that. for all $y \in \mathbb{R}$, if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$
  $\langle 2\rangle 2$. LET: V be a neighbourhood of $f(x)$
  $\langle 2\rangle 3$. PICK a, b such that $f(x) \in (a, b) \subseteq V$
  $\langle 2\rangle 4$. LET: $\epsilon = \min(f(x) - a, b - f(x))$
  $\langle 2\rangle 5$. PICK $\delta > 0$ such that, for all $y \in \mathbb{R}$, if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$
  $\langle 2\rangle 6$. LET: $U = (x - \delta, x + \delta)$
  $\langle 2\rangle 7$. $x \in U \subseteq f^{-1}(V)$

$\square$

Proposition 108. *Let X and X' be the same set X under two topologies $\mathcal{T}$ and $\mathcal{T}'$. Let $i : X \to X'$ be the identity function. Then i is continuous if and only if $\mathcal{T}' \subseteq \mathcal{T}$.*

PROOF: Immediate from definitions. $\square$

Proposition 109. *Let $f : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$. Then f is continuous on the right at a if and only if f is continuous at a as a function $\mathbb{R}_l \to \mathbb{R}$.*

PROOF:
$\langle 1\rangle 1$. If f is continuous on the right at a then f is continuous at a as a function $\mathbb{R}_l \to \mathbb{R}$.
  $\langle 2\rangle 1$. ASSUME: f is continuous on the right at a.
  $\langle 2\rangle 2$. LET: V be a neighbourhood of $f(a)$
  $\langle 2\rangle 3$. PICK b, c such that $f(a) \in (b, c) \subseteq V$.
  $\langle 2\rangle 4$. LET: $\epsilon = \min(c - f(a), f(a) - b)$
  $\langle 2\rangle 5$. PICK $\delta > 0$ such that, for all x, if $a < x < a + \delta$ then $|f(x) - f(a)| < \epsilon$
  $\langle 2\rangle 6$. LET: $U = [a, a + \delta)$
  $\langle 2\rangle 7$. $f(U) \subseteq V$
$\langle 1\rangle 2$. If f is continuous at a as a function $\mathbb{R}_l \to \mathbb{R}$ then f is continuous on the right at a.
  $\langle 2\rangle 1$. ASSUME: f is continuous at a as a function $\mathbb{R}_l \to \mathbb{R}$
  $\langle 2\rangle 2$. LET: $\epsilon > 0$

⟨2⟩3. PICK a neighbourhood U of a such that $f(U) \subseteq (f(a) - \epsilon, f(a) + \epsilon)$
⟨2⟩4. PICK b, c such that $a \in [b, c) \subset U$
⟨2⟩5. LET: $\delta = c - a$
⟨2⟩6. For all x, if $a < x < a + \delta$ then $|f(x) - f(a)| < \epsilon$
□

18 Homeomorphisms

Definition 110 (Homeomorphism)**.** Let X and Y be topological spaces. A *Homeomorphism* f between X and Y, $f : X \cong Y$, is a bijection $f : X \to Y$ such that both f and f^{-1} are continuous.

Lemma 111. *Let X and Y be topological spaces and $f : X \to Y$ a bijection. Then the following are equivalent:*

1. *f is a homeomorphism.*
2. *f is continuous and an open map.*
3. *For any $U \subseteq X$, we have U is open if and only if $f(U)$ is open.*

PROOF: Immediate from definitions. □

Proposition 112. *Let X and X' be the same set X under two topologies $\mathcal{T}$ and $\mathcal{T}'$. Let $i : X \to X'$ be the identity function. Then i is a homeomorphism if and only if $\mathcal{T} = \mathcal{T}'$.*

PROOF: Immediate from definitions. □

Definition 113 (Topological Property)**.** Let P be a property of topological spaces. Then P is a *topological* property if and only if, for any spaces X and Y, if P holds of X and $X \cong Y$ then P holds of Y.

Definition 114 (Topological Imbedding)**.** Let X and Y be topological spaces and $f : X \to Y$. Then f is a *topological imbedding* if and only if the corestriction $f : X \to f(X)$ is a homeomorphism.