Topology

Robin Adams

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1 Topological Spaces

Definition 1 (Topology)**.** A *topology* on a set X is a set $\mathcal{T} \subseteq \mathcal{P}X$ such that:

- $X \in \mathcal{T}$.
- For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$.
- For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$.

We call the elements of X *points* and the elements of $\mathcal{T}$ *open sets*.

Definition 2 (Topological Space)**.** A *topological space* X consists of a set X and a topology on X.

Definition 3 (Discrete Space)**.** For any set X, the *discrete* topology on X is $\mathcal{P}X$.

Definition 4 (Indiscrete Space)**.** For any set X, the *indiscrete* or *trivial* topology on X is $\{\emptyset, X\}$.

Definition 5 (Finite Complement Topology)**.** For any set X, the *finite complement topology* on X is $\{U \in \mathcal{P}X \mid X \setminus U \text{ is finite}\} \cup \{\emptyset\}$.

Definition 6 (Countable Complement Topology)**.** For any set X, the *countable complement topology* on X is $\{U \in \mathcal{P}X \mid X \setminus U \text{ is countable}\} \cup \{\emptyset\}$.

Definition 7 (Finer, Coarser)**.** Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a given set X. If $\mathcal{T}' \supseteq \mathcal{T}$, we say that $\mathcal{T}'$ is *finer* than $\mathcal{T}$; if $\mathcal{T}'$ *properly* contains $\mathcal{T}$, we say that $\mathcal{T}'$ is *strictly* finer than $\mathcal{T}$. We also say that $\mathcal{T}$ is *coarser* than $\mathcal{T}'$, or *stricly* coarser, in these two respective situations. We say $\mathcal{T}$ is *comparable* with $\mathcal{T}'$ if either $\mathcal{T}' \supseteq \mathcal{T}$ or $\mathcal{T} \supseteq \mathcal{T}'$.

2 Basis for a Topology

Definition 8 (Basis)**.** If X is a set, a *basis* for a topology on X is a set $\mathcal{B} \subseteq \mathcal{P}X$ called *basis elements* such that

1. For all $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$.
2. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

If $\mathcal{B}$ satisfies these two conditions, then we define the topology *generated* by $\mathcal{B}$ to be $\mathcal{T} = \{U \in \mathcal{P}X \mid \forall x \in U.\exists B \in \mathcal{B}.x \in B \subseteq U\}$.

We prove this is a topology.

PROOF:
$\langle 1 \rangle 1.$ $X \in \mathcal{T}$
PROOF: For all $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B \subseteq X$ by condition 1.
$\langle 1 \rangle 2.$ For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$
$\langle 2 \rangle 1.$ LET: $\mathcal{U} \subseteq \mathcal{T}$
$\langle 2 \rangle 2.$ LET: $x \in \bigcup \mathcal{U}$
$\langle 2 \rangle 3.$ PICK $U \in \mathcal{U}$ such that $x \in U$
$\langle 2 \rangle 4.$ PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$
PROOF: Since $U \in \mathcal{T}$ by $\langle 2 \rangle 1$ and $\langle 2 \rangle 3$.
$\langle 2 \rangle 5.$ $x \in B \subseteq \bigcup \mathcal{U}$
$\langle 1 \rangle 3.$ For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$
$\langle 2 \rangle 1.$ LET: $U, V \in \mathcal{T}$
$\langle 2 \rangle 2.$ LET: $x \in U \cap V$
$\langle 2 \rangle 3.$ PICK $B_1 \in \mathcal{B}$ such that $x \in B_1 \subseteq U$
$\langle 2 \rangle 4.$ PICK $B_2 \in \mathcal{B}$ such that $x \in B_2 \subseteq V$
$\langle 2 \rangle 5.$ PICK $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$
PROOF: By condition 2.
$\langle 2 \rangle 6.$ $x \in B_3 \subseteq U \cap V$
□

Lemma 9. *Let X be a set. Let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ is the set of all unions of subsets of $\mathcal{B}$.*

PROOF:
$\langle 1 \rangle 1.$ For all $U \in \mathcal{T}$, there exists $\mathcal{A} \subseteq \mathcal{B}$ such that $U = \bigcup \mathcal{A}$
$\langle 2 \rangle 1.$ LET: $U \in \mathcal{T}$
$\langle 2 \rangle 2.$ LET: $\mathcal{A} = \{B \in \mathcal{B} \mid B \subseteq U\}$
$\langle 2 \rangle 3.$ $U \subseteq \bigcup \mathcal{A}$
$\langle 3 \rangle 1.$ LET: $x \in U$
$\langle 3 \rangle 2.$ PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$
PROOF: Since $\mathcal{B}$ is a basis for $\mathcal{T}$.
$\langle 3 \rangle 3.$ $x \in B \in \mathcal{A}$
$\langle 2 \rangle 4.$ $\bigcup \mathcal{A} \subseteq U$
PROOF: From the definition of $\mathcal{A}$ ($\langle 2 \rangle 2$).
$\langle 1 \rangle 2.$ For all $\mathcal{A} \subseteq \mathcal{B}$ we have $\bigcup \mathcal{A} \in \mathcal{T}$
$\langle 2 \rangle 1.$ $\mathcal{B} \subseteq \mathcal{T}$
PROOF: If $B \in \mathcal{B}$ and $x \in B$, then there exists $B' \in \mathcal{B}$ such that $x \in B' \subseteq B$, namely $B' = B$.

$\langle 2 \rangle 2$. Q.E.D.
PROOF: Since $\mathcal{T}$ is closed under union.
$\square$

Corollary 9.1. *If $\mathcal{B}$ is a basis for the topology $\mathcal{T}$ then $\mathcal{B} \subseteq \mathcal{T}$.*

Lemma 10. *Let X be a topological space. Suppose that $\mathcal{C}$ is a set of open sets such that, for every open set U and every point $x \in U$, there exists $C \in \mathcal{C}$ such that $x \in C \subseteq U$. Then $\mathcal{C}$ is a basis for the topology on X.*

PROOF:
$\langle 1 \rangle 1$. For all $x \in X$, there exists $C \in \mathcal{C}$ such that $x \in C$
PROOF: Immediate from hypothesis.
$\langle 1 \rangle 2$. For all $C_1, C_2 \in \mathcal{C}$ and $x \in C_1 \cap C_2$, there exists $C_3 \in \mathcal{C}$ such that $x \in C_3 \subseteq C_1 \cap C_2$
PROOF: Since $C_1 \cap C_2$ is open.
$\langle 1 \rangle 3$. Every open set is open in the topology generated by $\mathcal{C}$
PROOF: Immediate from hypothesis.
$\langle 1 \rangle 4$. Every union of a subset of $\mathcal{C}$ is open.
PROOF: Since every member of $\mathcal{C}$ is open.
$\square$

Lemma 11. *Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$ respectively on the set X. Then the following are equivalent.*

1. $\mathcal{T} \subseteq \mathcal{T}'$
2. *For all $B \in \mathcal{B}$ and $x \in B$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.*

PROOF:
$\langle 1 \rangle 1$. $1 \Rightarrow 2$
$\langle 2 \rangle 1$. ASSUME: $\mathcal{T} \subseteq \mathcal{T}'$
$\langle 2 \rangle 2$. LET: $B \in \mathcal{B}$ and $x \in B$
$\langle 2 \rangle 3$. $B \in \mathcal{T}$
PROOF: Corollary 9.1.
$\langle 2 \rangle 4$. $B \in \mathcal{T}'$
PROOF: By $\langle 2 \rangle 1$
$\langle 2 \rangle 5$. There exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$
PROOF: Since $\mathcal{B}'$ is a basis for $\mathcal{T}'$.
$\langle 1 \rangle 2$. $2 \Rightarrow 1$
$\langle 2 \rangle 1$. ASSUME: 2
$\langle 2 \rangle 2$. LET: $U \in \mathcal{T}$
PROVE: $U \in \mathcal{T}'$
$\langle 2 \rangle 3$. LET: $x \in U$
PROVE: There exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq U$
$\langle 2 \rangle 4$. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$
PROOF: Since $\mathcal{B}$ is a basis for $\mathcal{T}$.
$\langle 2 \rangle 5$. PICK $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$

PROOF: By $\langle 2 \rangle 1$.

$\langle 2 \rangle 6$. $x \in B' \subseteq U$

$\square$

Definition 12 (Standard Topology on the Real Line)**.** The *standard topology on the real line* is the topology on $\mathbb{R}$ generated by the basis consisting of all open intervals (a, b).

We prove this is a basis for a topology.

PROOF:

$\langle 1 \rangle 1$. For all $x \in \mathbb{R}$ there exists an open interval (a, b) such that $x \in (a, b)$.

PROOF: Take $(a, b) = (x - 1, x + 1)$.

$\langle 1 \rangle 2$. For any open intervals (a, b), (c, d) if $x \in (a, b) \cap (c, d)$, then there exists an open interval (e, f) such that $x \in (e, f) \subseteq (a, b) \cap (c, d)$

PROOF: Take $(e, f) = (\max(a, c), \min(b, d))$.

$\square$