

# Topology

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## 1 Order Theory

**Definition 1** (Preorder). Let  $X$  be a set. A *preorder* on  $X$  is a binary relation  $\leq$  on  $X$  such that:

**Reflexivity** For all  $x \in X$ , we have  $x \leq x$

**Transitivity** For all  $x, y, z \in X$ , if  $x \leq y$  and  $y \leq z$  then  $x \leq z$ .

**Definition 2** (Preordered Set). A *preordered set* consists of a set  $X$  and a preorder  $\leq$  on  $X$ .

**Definition 3** (Interval). Let  $X$  be a preordered set and  $Y \subseteq X$ . Then  $Y$  is an *interval* if and only if, for all  $a, b \in Y$  and  $c \in X$ , if  $a \leq c \leq b$  then  $c \in Y$ .

## 2 Real Analysis

**Definition 4.** Let  $\mathbb{R}^\infty$  be the subset of  $\mathbb{R}^\omega$  consisting of all sequences  $(a_n)$  such that  $a_n = 0$  for all but finitely many  $n$ .

### 3 Topological Spaces

**Definition 5** (Topology). A *topology* on a set  $X$  is a set  $\mathcal{T} \subseteq \mathcal{P}X$  such that:

- $X \in \mathcal{T}$ .
- For all  $\mathcal{U} \subseteq \mathcal{T}$  we have  $\bigcup \mathcal{U} \in \mathcal{T}$ .
- For all  $U, V \in \mathcal{T}$  we have  $U \cap V \in \mathcal{T}$ .

We call the elements of  $X$  *points* and the elements of  $\mathcal{T}$  *open sets*.

**Definition 6** (Topological Space). A *topological space*  $X$  consists of a set  $X$  and a topology on  $X$ .

**Definition 7** (Discrete Space). For any set  $X$ , the *discrete* topology on  $X$  is  $\mathcal{P}X$ .

**Definition 8** (Indiscrete Space). For any set  $X$ , the *indiscrete* or *trivial* topology on  $X$  is  $\{\emptyset, X\}$ .

**Definition 9** (Finite Complement Topology). For any set  $X$ , the *finite complement topology* on  $X$  is  $\{U \in \mathcal{P}X \mid X \setminus U \text{ is finite}\} \cup \{\emptyset\}$ .

**Definition 10** (Countable Complement Topology). For any set  $X$ , the *countable complement topology* on  $X$  is  $\{U \in \mathcal{P}X \mid X \setminus U \text{ is countable}\} \cup \{\emptyset\}$ .

**Definition 11** (Finer, Coarser). Suppose that  $\mathcal{T}$  and  $\mathcal{T}'$  are two topologies on a given set  $X$ . If  $\mathcal{T}' \supseteq \mathcal{T}$ , we say that  $\mathcal{T}'$  is *finer* than  $\mathcal{T}$ ; if  $\mathcal{T}'$  *properly* contains  $\mathcal{T}$ , we say that  $\mathcal{T}'$  is *strictly finer* than  $\mathcal{T}$ . We also say that  $\mathcal{T}$  is *coarser* than  $\mathcal{T}'$ , or *strictly coarser*, in these two respective situations. We say  $\mathcal{T}$  is *comparable* with  $\mathcal{T}'$  if either  $\mathcal{T}' \supseteq \mathcal{T}$  or  $\mathcal{T} \supseteq \mathcal{T}'$ .

**Lemma 12.** Let  $X$  be a topological space and  $U \subseteq X$ . Then  $U$  is open if and only if, for all  $x \in U$ , there exists an open set  $V$  such that  $x \in V \subseteq U$ .

PROOF:

$\langle 1 \rangle 1. \Rightarrow$

PROOF: Take  $V = U$

$\langle 1 \rangle 2. \Leftarrow$

PROOF: We have  $U = \bigcup \{V \text{ open in } X \mid V \subseteq U\}$ .

□

**Lemma 13.** Let  $X$  be a set and  $\mathcal{T}$  a nonempty set of topologies on  $X$ . Then  $\bigcap \mathcal{T}$  is a topology on  $X$ , and is the finest topology that is coarser than every member of  $\mathcal{T}$ .

PROOF:

$\langle 1 \rangle 1. X \in \bigcap \mathcal{T}$

PROOF: Since  $X$  is in every member of  $\mathcal{T}$ .

$\langle 1 \rangle 2. \bigcap \mathcal{T}$  is closed under union.

- ⟨2⟩1. LET:  $\mathcal{U} \subseteq \bigcap \mathcal{T}$
- ⟨2⟩2. For all  $T \in \mathcal{T}$  we have  $\mathcal{U} \subseteq T$
- ⟨2⟩3. For all  $T \in \mathcal{T}$  we have  $\bigcup \mathcal{U} \in T$
- ⟨2⟩4.  $\bigcup \mathcal{U} \in \bigcap \mathcal{T}$
- ⟨1⟩3.  $\bigcap \mathcal{T}$  is closed under binary intersection.
- ⟨2⟩1. LET:  $U, V \in \bigcap \mathcal{T}$
- ⟨2⟩2. For all  $T \in \mathcal{T}$  we have  $U, V \in T$
- ⟨2⟩3. For all  $T \in \mathcal{T}$  we have  $U \cap V \in T$
- ⟨2⟩4.  $U \cap V \in \bigcap \mathcal{T}$

□

**Lemma 14.** *Let  $X$  be a set and  $\mathcal{T}$  a set of topologies on  $X$ . Then there exists a unique coarsest topology that is finer than every member of  $\mathcal{T}$ .*

PROOF: The required topology is given by

$$\bigcap \{T \in \mathcal{P}X \mid T \text{ is a topology on } X \text{ that is finer than every member of } \mathcal{T}\},$$

The set is nonempty since it contains the discrete topology. □

**Definition 15** (Neighbourhood). A *neighbourhood* of a point  $x$  is an open set that contains  $x$ .

## 4 Closed Set

**Definition 16** (Closed Set). Let  $X$  be a topological space and  $A \subseteq X$ . Then  $A$  is *closed* if and only if  $X \setminus A$  is open.

**Lemma 17.** *The empty set is closed.*

PROOF: Since the whole space  $X$  is always open. □

**Lemma 18.** *The topological space  $X$  is closed.*

PROOF: Since  $\emptyset$  is open. □

**Lemma 19.** *The intersection of a nonempty set of closed sets is closed.*

PROOF: Let  $\mathcal{C}$  be a nonempty set of closed sets. Then  $X \setminus \bigcap \mathcal{C} = \bigcup \{X \setminus C \mid C \in \mathcal{C}\}$  is open. □

**Lemma 20.** *The union of two closed sets is closed.*

PROOF: Let  $C$  and  $D$  be closed. Then  $X \setminus (C \cup D) = (X \setminus C) \cap (X \setminus D)$  is open. □

**Proposition 21.** *Let  $X$  be a set and  $\mathcal{C} \subseteq \mathcal{P}X$  a set such that:*

1.  $\emptyset \in \mathcal{C}$
2.  $X \in \mathcal{C}$
3. For all  $\mathcal{A} \subseteq \mathcal{C}$  nonempty we have  $\bigcap \mathcal{A} \in \mathcal{C}$

4. For all  $C, D \in \mathcal{C}$  we have  $C \cup D \in \mathcal{C}$ .

Then there exists a unique topology  $\mathcal{T}$  such that  $\mathcal{C}$  is the set of closed sets, namely

$$\mathcal{T} = \{X \setminus C \mid C \in \mathcal{C}\}$$

PROOF:

$\langle 1 \rangle 1$ . LET:  $\mathcal{T} = \{X \setminus C \mid C \in \mathcal{C}\}$

$\langle 1 \rangle 2$ .  $\mathcal{T}$  is a topology

$\langle 2 \rangle 1$ .  $X \in \mathcal{T}$

PROOF: Since  $\emptyset \in \mathcal{C}$

$\langle 2 \rangle 2$ . For all  $\mathcal{U} \subseteq \mathcal{T}$  we have  $\bigcup \mathcal{U} \in \mathcal{T}$

$\langle 3 \rangle 1$ . LET:  $\mathcal{U} \subseteq \mathcal{T}$

$\langle 3 \rangle 2$ . CASE:  $\mathcal{U} = \emptyset$

PROOF: In this case  $\bigcup \mathcal{U} = \emptyset \in \mathcal{T}$  since  $X \in \mathcal{C}$

$\langle 3 \rangle 3$ . CASE:  $\mathcal{U} \neq \emptyset$

PROOF: In this case  $X \setminus \bigcup \mathcal{U} = \bigcap \{X \setminus U \mid U \in \mathcal{U}\} \in \mathcal{C}$ .

$\langle 2 \rangle 3$ . For all  $U, V \in \mathcal{T}$  we have  $U \cap V \in \mathcal{T}$

PROOF: Since  $X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V) \in \mathcal{C}$ .

$\langle 1 \rangle 3$ .  $\mathcal{C}$  is the set of all closed sets in  $\mathcal{T}$

PROOF:

$C$  is closed in  $\mathcal{T}$

$$\Leftrightarrow X \setminus C \in \mathcal{T}$$

$$\Leftrightarrow C \in \mathcal{C}$$

$\langle 1 \rangle 4$ . If  $\mathcal{T}'$  is a topology and  $\mathcal{C}$  is the set of closed sets in  $\mathcal{T}'$  then  $\mathcal{T}' = \mathcal{T}$

PROOF: We have

$$U \in \mathcal{T}$$

$$\Leftrightarrow X \setminus U \in \mathcal{C}$$

$$\Leftrightarrow X \setminus U \text{ is closed in } \mathcal{T}'$$

$$\Leftrightarrow U \in \mathcal{T}'$$

□

**Proposition 22.** If  $U$  is open and  $A$  is closed then  $U \setminus A$  is open.

PROOF:  $U \setminus A = U \cap (X \setminus A)$  is the intersection of two open sets. □

**Proposition 23.** If  $U$  is open and  $A$  is closed then  $A \setminus U$  is closed.

PROOF:  $A \setminus U = A \cap (X \setminus U)$  is the intersection of two closed sets. □

## 5 Interior

**Definition 24** (Interior). Let  $X$  be a topological space and  $A \subseteq X$ . The *interior* of  $A$ ,  $\text{Int } A$ , is the union of all the open subsets of  $A$ .

**Lemma 25.** The interior of a set is open.

PROOF: It is a union of open sets.  $\square$

**Lemma 26.**

$$\text{Int } A \subseteq A$$

PROOF: Immediate from definition.  $\square$

**Lemma 27.** *If  $U$  is open and  $U \subseteq A$  then  $U \subseteq \text{Int } A$*

PROOF: Immediate from definition.  $\square$

**Lemma 28.** *A set  $A$  is open if and only if  $A = \text{Int } A$ .*

PROOF: If  $A = \text{Int } A$  then  $A$  is open by Lemma 25. Conversely if  $A$  is open then  $A \subseteq \text{Int } A$  by the definition of interior and so  $A = \text{Int } A$ .

## 6 Closure

**Definition 29** (Closure). Let  $X$  be a topological space and  $A \subseteq X$ . The *closure* of  $A$ ,  $\overline{A}$ , is the intersection of all the closed sets that include  $A$ .

This intersection exists since  $X$  is a closed set that includes  $A$  (Lemma 18).

**Lemma 30.** *The closure of a set is closed.*

PROOF: Dual to Lemma 25.  $\square$

**Lemma 31.**

$$A \subseteq \overline{A}$$

PROOF: Immediate from definition.  $\square$

**Lemma 32.** *If  $C$  is closed and  $A \subseteq C$  then  $\overline{A} \subseteq C$ .*

PROOF: Immediate from definition.  $\square$

**Lemma 33.** *A set  $A$  is closed if and only if  $A = \overline{A}$ .*

PROOF: Dual to Lemma 28.  $\square$

**Theorem 34.** *Let  $X$  be a topological space,  $A \subseteq X$  and  $x \in X$ . Then  $x \in \overline{A}$  if and only if every neighbourhood of  $x$  intersects  $A$ .*

PROOF: We have

$$\begin{aligned} x \in \overline{A} \\ \Leftrightarrow \forall C. C \text{ closed} \wedge A \subseteq C \Rightarrow x \in C \\ \Leftrightarrow \forall U. U \text{ open} \wedge A \cap U = \emptyset \Rightarrow x \notin U \\ \Leftrightarrow \forall U. U \text{ open} \wedge x \in U \Rightarrow U \text{ intersects } A \end{aligned} \quad \square$$

**Proposition 35.** *If  $A \subseteq B$  then  $\overline{A} \subseteq \overline{B}$ .*

PROOF: This holds because  $\overline{B}$  is a closed set that includes  $A$ .  $\square$

**Proposition 36.**

$$\overline{A \cup B} = \overline{A} \cup \overline{B}$$

PROOF:

$\langle 1 \rangle 1. \overline{A} \subseteq \overline{A \cup B}$

PROOF: By Proposition 35.

$\langle 1 \rangle 2. \overline{B} \subseteq \overline{A \cup B}$

PROOF: By Proposition 35.

$\langle 1 \rangle 3. \overline{A \cup B} \subseteq \overline{A \cup B}$

$\langle 2 \rangle 1. \text{ LET: } x \in \overline{A \cup B}$

$\langle 2 \rangle 2. \text{ ASSUME: } x \notin \overline{A}$

PROVE:  $x \in \overline{B}$

$\langle 2 \rangle 3. \text{ PICK a neighbourhood } U \text{ of } x \text{ that does not intersect } A$

$\langle 2 \rangle 4. \text{ LET: } V \text{ be any neighbourhood of } x$

$\langle 2 \rangle 5. U \cap V \text{ is a neighbourhood of } x$

$\langle 2 \rangle 6. U \cap V \text{ intersects } A \cup B$

PROOF: From  $\langle 2 \rangle 1$  and Theorem 34.

$\langle 2 \rangle 7. U \cap V \text{ intersects } B$

PROOF: From  $\langle 2 \rangle 3$

$\langle 2 \rangle 8. V \text{ intersects } B$

$\langle 2 \rangle 9. \text{ Q.E.D.}$

PROOF: We have  $x \in \overline{B}$  from Theorem 34.

$\square$

## 7 Boundary

**Definition 37** (Boundary). The *boundary* of a set  $A$  is the set  $\partial A = \overline{A} \cap \overline{X \setminus A}$ .

**Proposition 38.**

$$\text{Int } A \cap \partial A = \emptyset$$

PROOF: Since  $\overline{X \setminus A} = X \setminus \text{Int } A$ .  $\square$

**Proposition 39.**

$$\overline{A} = \text{Int } A \cup \partial A$$

PROOF:

$$\begin{aligned} \text{Int } A \cup \partial A &= \text{Int } A \cup (\overline{A} \cap \overline{X \setminus A}) \\ &= (\text{Int } A \cup \overline{A}) \cap (\text{Int } A \cup \overline{X \setminus A}) \\ &= \overline{A} \cap X \\ &= \overline{A} \end{aligned}$$

**Proposition 40.**  $\partial A = \emptyset$  if and only if  $A$  is open and closed.

PROOF: If  $\partial A = \emptyset$  then  $\overline{A} = \text{Int } A$  by Proposition 39.



**Proposition 41.** *A set  $U$  is open if and only if  $\partial U = \overline{U} \setminus U$ .*

PROOF:

$$\begin{aligned} \partial U &= \overline{U} \setminus U \\ \Leftrightarrow \overline{U} \setminus \text{Int } U &= \overline{U} \setminus U && (\text{Propositions 38, 39}) \\ \Leftrightarrow \text{Int } U &= U && \square \end{aligned}$$

## 8 Limit Points

**Definition 42** (Limit Point). Let  $X$  be a topological space,  $a \in X$  and  $A \subseteq X$ . Then  $a$  is a *limit point*, *cluster point* or *point of accumulation* for  $A$  if and only if every neighbourhood of  $a$  intersects  $A$  at a point other than  $a$ .

**Lemma 43.** *The point  $a$  is an accumulation point for  $A$  if and only if  $a \in \overline{A} \setminus \{a\}$ .*

PROOF: From Theorem 34.  $\square$

**Theorem 44.** *Let  $X$  be a topological space and  $A \subseteq X$ . Let  $A'$  be the set of all limit points of  $A$ . Then  $\overline{A} = A \cup A'$ .*

PROOF:

$\langle 1 \rangle 1$ . For all  $x \in \overline{A}$ , if  $x \notin A$  then  $x \in A'$

PROOF: From Theorem 34.

$\langle 1 \rangle 2$ .  $A \subseteq \overline{A}$

PROOF: Lemma 31.

$\langle 1 \rangle 3$ .  $A' \subseteq \overline{A}$

PROOF: From Theorem 34.

$\square$

**Corollary 44.1.** *A set is closed if and only if it contains all its limit points.*

**Proposition 45.** *In an indiscrete topology, every point is a limit point of any set that has more than one point.*

PROOF: Let  $X$  be an indiscrete space. Let  $A$  be a set with more than one point and  $x$  be a point. The only neighbourhood of  $x$  is  $X$ , which must intersect  $A$  at a point other than  $x$ .  $\square$

## 9 Basis for a Topology

**Definition 46** (Basis). If  $X$  is a set, a *basis* for a topology on  $X$  is a set  $\mathcal{B} \subseteq \mathcal{P}X$  called *basis elements* such that

1. For all  $x \in X$ , there exists  $B \in \mathcal{B}$  such that  $x \in B$ .
2. For all  $B_1, B_2 \in \mathcal{B}$  and  $x \in B_1 \cap B_2$ , there exists  $B_3 \in \mathcal{B}$  such that  $x \in B_3 \subseteq B_1 \cap B_2$ .

If  $\mathcal{B}$  satisfies these two conditions, then we define the topology *generated* by  $\mathcal{B}$  to be  $\mathcal{T} = \{U \in \mathcal{P}X \mid \forall x \in U. \exists B \in \mathcal{B}. x \in B \subseteq U\}$ .

We prove this is a topology.

PROOF:

$\langle 1 \rangle 1.$   $X \in \mathcal{T}$

PROOF: For all  $x \in X$  there exists  $B \in \mathcal{B}$  such that  $x \in B \subseteq X$  by condition 1.

$\langle 1 \rangle 2.$  For all  $\mathcal{U} \subseteq \mathcal{T}$  we have  $\bigcup \mathcal{U} \in \mathcal{T}$

$\langle 2 \rangle 1.$  LET:  $\mathcal{U} \subseteq \mathcal{T}$

$\langle 2 \rangle 2.$  LET:  $x \in \bigcup \mathcal{U}$

$\langle 2 \rangle 3.$  PICK  $U \in \mathcal{U}$  such that  $x \in U$

$\langle 2 \rangle 4.$  PICK  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$

PROOF: Since  $U \in \mathcal{T}$  by  $\langle 2 \rangle 1$  and  $\langle 2 \rangle 3$ .

$\langle 2 \rangle 5.$   $x \in B \subseteq \bigcup \mathcal{U}$

$\langle 1 \rangle 3.$  For all  $U, V \in \mathcal{T}$  we have  $U \cap V \in \mathcal{T}$

$\langle 2 \rangle 1.$  LET:  $U, V \in \mathcal{T}$

$\langle 2 \rangle 2.$  LET:  $x \in U \cap V$

$\langle 2 \rangle 3.$  PICK  $B_1 \in \mathcal{B}$  such that  $x \in B_1 \subseteq U$

$\langle 2 \rangle 4.$  PICK  $B_2 \in \mathcal{B}$  such that  $x \in B_2 \subseteq V$

$\langle 2 \rangle 5.$  PICK  $B_3 \in \mathcal{B}$  such that  $x \in B_3 \subseteq B_1 \cap B_2$

PROOF: By condition 2.

$\langle 2 \rangle 6.$   $x \in B_3 \subseteq U \cap V$

□

**Lemma 47.** *Let  $X$  be a set. Let  $\mathcal{B}$  be a basis for a topology  $\mathcal{T}$  on  $X$ . Then  $\mathcal{T}$  is the set of all unions of subsets of  $\mathcal{B}$ .*

PROOF:

$\langle 1 \rangle 1.$  For all  $U \in \mathcal{T}$ , there exists  $\mathcal{A} \subseteq \mathcal{B}$  such that  $U = \bigcup \mathcal{A}$

$\langle 2 \rangle 1.$  LET:  $U \in \mathcal{T}$

$\langle 2 \rangle 2.$  LET:  $\mathcal{A} = \{B \in \mathcal{B} \mid B \subseteq U\}$

$\langle 2 \rangle 3.$   $U \subseteq \bigcup \mathcal{A}$

$\langle 3 \rangle 1.$  LET:  $x \in U$

$\langle 3 \rangle 2.$  PICK  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$

PROOF: Since  $\mathcal{B}$  is a basis for  $\mathcal{T}$ .

$\langle 3 \rangle 3.$   $x \in B \in \mathcal{A}$

$\langle 2 \rangle 4.$   $\bigcup \mathcal{A} \subseteq U$

PROOF: From the definition of  $\mathcal{A}$  ( $\langle 2 \rangle 2$ ).

$\langle 1 \rangle 2.$  For all  $\mathcal{A} \subseteq \mathcal{B}$  we have  $\bigcup \mathcal{A} \in \mathcal{T}$

$\langle 2 \rangle 1.$   $\mathcal{B} \subseteq \mathcal{T}$

PROOF: If  $B \in \mathcal{B}$  and  $x \in B$ , then there exists  $B' \in \mathcal{B}$  such that  $x \in B' \subseteq B$ , namely  $B' = B$ .

$\langle 2 \rangle 2.$  Q.E.D.

PROOF: Since  $\mathcal{T}$  is closed under union.

□

**Corollary 47.1.** *Let  $X$  be a set. Let  $\mathcal{B}$  be a basis for a topology  $\mathcal{T}$  on  $X$ . Then  $\mathcal{T}$  is the coarsest topology that includes  $\mathcal{B}$ .*

PROOF: Since every topology that includes  $\mathcal{B}$  includes all unions of subsets of  $\mathcal{B}$ .  $\square$

**Lemma 48.** *Let  $X$  be a topological space. Suppose that  $\mathcal{C}$  is a set of open sets such that, for every open set  $U$  and every point  $x \in U$ , there exists  $C \in \mathcal{C}$  such that  $x \in C \subseteq U$ . Then  $\mathcal{C}$  is a basis for the topology on  $X$ .*

PROOF:

$\langle 1 \rangle 1$ . For all  $x \in X$ , there exists  $C \in \mathcal{C}$  such that  $x \in C$

PROOF: Immediate from hypothesis.

$\langle 1 \rangle 2$ . For all  $C_1, C_2 \in \mathcal{C}$  and  $x \in C_1 \cap C_2$ , there exists  $C_3 \in \mathcal{C}$  such that  $x \in C_3 \subseteq C_1 \cap C_2$

PROOF: Since  $C_1 \cap C_2$  is open.

$\langle 1 \rangle 3$ . Every open set is open in the topology generated by  $\mathcal{C}$

PROOF: Immediate from hypothesis.

$\langle 1 \rangle 4$ . Every union of a subset of  $\mathcal{C}$  is open.

PROOF: Since every member of  $\mathcal{C}$  is open.

$\square$

**Lemma 49.** *Let  $\mathcal{B}$  and  $\mathcal{B}'$  be bases for the topologies  $\mathcal{T}$  and  $\mathcal{T}'$  respectively on the set  $X$ . Then the following are equivalent.*

1.  $\mathcal{T} \subseteq \mathcal{T}'$

2. For all  $B \in \mathcal{B}$  and  $x \in B$ , there exists  $B' \in \mathcal{B}'$  such that  $x \in B' \subseteq B$ .

PROOF:

$\langle 1 \rangle 1$ .  $1 \Rightarrow 2$

$\langle 2 \rangle 1$ . ASSUME:  $\mathcal{T} \subseteq \mathcal{T}'$

$\langle 2 \rangle 2$ . LET:  $B \in \mathcal{B}$  and  $x \in B$

$\langle 2 \rangle 3$ .  $B \in \mathcal{T}$

PROOF: Corollary 47.1.

$\langle 2 \rangle 4$ .  $B \in \mathcal{T}'$

PROOF: By  $\langle 2 \rangle 1$

$\langle 2 \rangle 5$ . There exists  $B' \in \mathcal{B}'$  such that  $x \in B' \subseteq B$

PROOF: Since  $\mathcal{B}'$  is a basis for  $\mathcal{T}'$ .

$\langle 1 \rangle 2$ .  $2 \Rightarrow 1$

$\langle 2 \rangle 1$ . ASSUME: 2

$\langle 2 \rangle 2$ . LET:  $U \in \mathcal{T}$

PROVE:  $U \in \mathcal{T}'$

$\langle 2 \rangle 3$ . LET:  $x \in U$

PROVE: There exists  $B' \in \mathcal{B}'$  such that  $x \in B' \subseteq U$

$\langle 2 \rangle 4$ . PICK  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$

PROOF: Since  $\mathcal{B}$  is a basis for  $\mathcal{T}$ .

$\langle 2 \rangle 5$ . PICK  $B' \in \mathcal{B}'$  such that  $x \in B' \subseteq B$

PROOF: By  $\langle 2 \rangle 1$ .  
 $\langle 2 \rangle 6$ .  $x \in B' \subseteq U$

□

**Theorem 50.** *Let  $X$  be a topological space,  $A \subseteq X$  and  $x \in X$ . Let  $\mathcal{B}$  be a basis for  $X$ . Then  $x \in \overline{A}$  if and only if, for all  $B \in \mathcal{B}$ , if  $x \in B$  then  $B$  intersects  $A$ .*

PROOF:

$\langle 1 \rangle 1$ . If  $x \in \overline{A}$  then, for all  $B \in \mathcal{B}$ , if  $x \in B$  then  $B$  intersects  $A$ .

PROOF: This follows from Theorem 34 since every element of  $\mathcal{B}$  is open (Corollary 47.1).

$\langle 1 \rangle 2$ . Suppose that, for all  $B \in \mathcal{B}$ , if  $x \in B$  then  $B$  intersects  $A$ . Then  $x \in \overline{A}$ .

$\langle 2 \rangle 1$ . ASSUME: For all  $B \in \mathcal{B}$ , if  $x \in B$  then  $B$  intersects  $A$ .

$\langle 2 \rangle 2$ . LET:  $U$  be an open set that contains  $x$

PROVE:  $U$  intersects  $A$ .

$\langle 2 \rangle 3$ . PICK  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$ .

$\langle 2 \rangle 4$ .  $B$  intersects  $A$ .

PROOF: From  $\langle 2 \rangle 1$ .

$\langle 2 \rangle 5$ .  $U$  intersects  $A$ .

$\langle 2 \rangle 6$ . Q.E.D.

PROOF: By Theorem 34.

□

**Definition 51** (Lower Limit Topology on the Real Line). The *lower limit topology on the real line* is the topology on  $\mathbb{R}$  generated by the basis consisting of all half-open intervals of the form  $[a, b)$ .

We write  $\mathbb{R}_l$  for the topological space  $\mathbb{R}$  under the lower limit topology.

We prove this is a basis for a topology.

PROOF:

$\langle 1 \rangle 1$ . For all  $x \in \mathbb{R}$  there exists an interval  $[a, b)$  such that  $x \in [a, b)$ .

PROOF: Take  $[a, b) = [x, x + 1)$ .

$\langle 1 \rangle 2$ . For any open intervals  $[a, b)$ ,  $[c, d)$  if  $x \in [a, b) \cap [c, d)$ , then there exists an interval  $[e, f)$  such that  $x \in [e, f) \subseteq [a, b) \cap [c, d)$

PROOF: Take  $[e, f) = [\max(a, c), \min(b, d))$ .

□

**Definition 52** ( $K$ -topology on the Real Line). Let  $K = \{1/n \mid n \in \mathbb{Z}^+\}$ .

The  *$K$ -topology on the real line* is the topology on  $\mathbb{R}$  generated by the basis consisting of all open intervals  $(a, b)$  and all sets of the form  $(a, b) \setminus K$ .

We write  $\mathbb{R}_K$  for the topological space  $\mathbb{R}$  under the  $K$ -topology.

We prove this is a basis for a topology.

PROOF:

$\langle 1 \rangle 1$ . For all  $x \in \mathbb{R}$  there exists an open interval  $(a, b)$  such that  $x \in (a, b)$ .

PROOF: Take  $(a, b) = (x - 1, x + 1)$ .

$\langle 1 \rangle 2$ . For any basic open sets  $B_1, B_2$  if  $x \in B_1 \cap B_2$ , then there exists a basic open set  $B_3$  such that  $x \in B_3 \subseteq B_1 \cap B_2$

⟨2⟩1. CASE:  $B_1 = (a, b)$ ,  $B_2 = (c, d)$

PROOF: Take  $B_3 = (\max(a, c), \min(b, d))$ .

⟨2⟩2. CASE:  $B_1 = (a, b)$  or  $(a, b) \setminus K$ ,  $B_2 = (c, d)$  or  $(c, d) \setminus K$ , and they are not both open intervals.

PROOF: Take  $B_3 = (\max(a, c), \min(b, d)) \setminus K$ .

□

**Lemma 53.** *The lower limit topology and the  $K$ -topology are incomparable.*

PROOF:

⟨1⟩1. The interval  $[10, 11)$  is not open in the  $K$ -topology.

PROOF: There is no open interval  $(a, b)$  such that  $10 \in (a, b) \subseteq [10, 11)$  or  $10 \in (a, b) \setminus K \subseteq [10, 11)$ .

⟨1⟩2. The set  $(-1, 1) \setminus K$  is not open in the lower limit topology.

PROOF: There is no half-open interval  $[a, b)$  such that  $0 \in [a, b) \subseteq (-1, 1) \setminus K$ , since there must be a positive integer  $n$  with  $1/n \in [a, b)$ .

□

**Definition 54** (Subbasis). A *subbasis*  $\mathcal{S}$  for a topology on  $X$  is a set  $\mathcal{S} \subseteq \mathcal{P}X$  such that  $\bigcup \mathcal{S} = X$ .

The topology *generated* by the subbasis  $\mathcal{S}$  is the set of all unions of finite intersections of elements of  $\mathcal{S}$ .

We prove this is a topology.

PROOF:

⟨1⟩1. The set  $\mathcal{B}$  of all finite intersections of elements of  $\mathcal{S}$  forms a basis for a topology on  $X$ .

⟨2⟩1.  $\bigcup \mathcal{B} = X$

PROOF: Since  $\mathcal{S} \subseteq \mathcal{B}$ .

⟨2⟩2.  $\mathcal{B}$  is closed under binary intersection.

PROOF: By definition.

⟨1⟩2. Q.E.D.

PROOF: By Lemma 47.

□

We have simultaneously proved:

**Proposition 55.** *Let  $\mathcal{S}$  be a subbasis for the topology on  $X$ . Then the set of all finite intersections of elements of  $\mathcal{S}$  is a basis for the topology on  $X$ .*

**Proposition 56.** *Let  $X$  be a set. Let  $\mathcal{S}$  be a subbasis for a topology  $\mathcal{T}$  on  $X$ . Then  $\mathcal{T}$  is the coarsest topology that includes  $\mathcal{S}$ .*

PROOF: Since every topology that includes  $\mathcal{S}$  includes every union of finite intersections of elements of  $\mathcal{S}$ . □

## 10 Local Basis at a Point

**Definition 57** (Local Basis). Let  $X$  be a topological space and  $a \in X$ . A *(local) basis at  $a$*  is a set  $\mathcal{B}$  of neighbourhoods of  $a$  such that every neighbourhood of  $a$

includes some member of  $\mathcal{B}$ .

**Lemma 58.** *If there exists a countable local basis at a point  $a$ , then there exists a countable local basis  $\{B_n \mid n \geq 1\}$  such that  $B_1 \supseteq B_2 \supseteq \dots$ .*

PROOF: Given a countable local basis  $\{C_n \mid n \geq 1\}$ , take  $B_n = C_1 \cap \dots \cap C_n$ .  $\square$

## 11 Convergence

**Definition 59** (Convergence). Let  $X$  be a topological space. Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence of points in  $X$  and  $l \in X$ . Then the sequence  $(a_n)_{n \in \mathbb{N}}$  *converges* to the *limit*  $l$ ,  $a_n \rightarrow l$  as  $n \rightarrow \infty$ , if and only if, for every neighbourhood  $U$  of  $l$ , there exists  $N$  such that, for all  $n \geq N$ , we have  $a_n \in U$ .

**Theorem 60.** *In a Hausdorff space, a sequence has at most one limit.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $X$  be a Hausdorff space.

$\langle 1 \rangle 2$ . ASSUME: for a contradiction  $a_n \rightarrow l$  as  $n \rightarrow \infty$ ,  $a_n \rightarrow m$  as  $n \rightarrow \infty$ , and  $l \neq m$

$\langle 1 \rangle 3$ . PICK disjoint neighbourhoods  $U$  of  $l$  and  $V$  of  $m$

PROOF: By the Hausdorff axiom.

$\langle 1 \rangle 4$ . PICK  $M$  and  $N$  such that  $a_n \in U$  for  $n \geq M$  and  $a_n \in V$  for  $n \geq N$

$\langle 1 \rangle 5$ .  $a_{\max(M,N)} \in U \cap V$

$\langle 1 \rangle 6$ . Q.E.D.

PROOF: This contradicts the fact that  $U$  and  $V$  are disjoint ( $\langle 1 \rangle 3$ ).

$\square$

To see this is not always true in spaces that are  $T_1$  but not Hausdorff:

**Proposition 61.** *Let  $X$  be an infinite set under the finite complement topology. Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence with all points distinct. Then for every  $l \in X$  we have  $a_n \rightarrow l$  as  $n \rightarrow \infty$ .*

PROOF: Let  $U$  be any neighbourhood of  $l$ . Since  $X \setminus U$  is finite, there must exist  $N$  such that, for all  $n \geq N$ , we have  $a_n \in U$ .  $\square$

**Lemma 62.** *Let  $X$  be a topological space. Let  $A \subseteq X$  and  $l \in X$ . If there is a sequence of points in  $A$  that converges to  $l$  then  $l \in \bar{A}$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $(a_n)$  be a sequence of points in  $A$  that converges to  $l$ .

$\langle 1 \rangle 2$ . LET:  $U$  be a neighbourhood of  $l$ .

$\langle 1 \rangle 3$ . PICK  $N$  such that, for all  $n \geq N$ , we have  $a_n \in U$ .

$\langle 1 \rangle 4$ .  $a_N \in U \cap A$

$\langle 1 \rangle 5$ . Q.E.D.

PROOF: Theorem 34.

$\square$

**Proposition 63.** *Let  $X$  be a topological space. Let  $\mathcal{B}$  be a basis for the topology on  $X$ . Let  $(a_n)$  be a sequence in  $X$  and  $l \in X$ . Then  $a_n \rightarrow l$  as  $n \rightarrow \infty$  if and only if, for every  $B \in \mathcal{B}$  with  $l \in B$ , there exists  $N$  such that, for all  $n \geq N$ , we have  $a_n \in B$ .*

PROOF:

$\langle 1 \rangle 1$ . If  $a_n \rightarrow l$  as  $n \rightarrow \infty$  then, for every  $B \in \mathcal{B}$  with  $l \in B$ , there exists  $N$  such that, for all  $n \geq N$ , we have  $a_n \in B$ .

PROOF: Since every element of  $\mathcal{B}$  is open (Corollary 47.1).

$\langle 1 \rangle 2$ . If, for every  $B \in \mathcal{B}$  with  $l \in B$ , there exists  $N$  such that, for all  $n \geq N$ , we have  $a_n \in B$ , then  $a_n \rightarrow l$  as  $n \rightarrow \infty$ .

$\langle 2 \rangle 1$ . ASSUME: for every  $B \in \mathcal{B}$  with  $l \in B$ , there exists  $N$  such that, for all  $n \geq N$ , we have  $a_n \in B$

$\langle 2 \rangle 2$ . LET:  $U$  be a neighbourhood of  $l$ .

$\langle 2 \rangle 3$ . PICK  $B \in \mathcal{B}$  such that  $l \in B \subseteq U$

$\langle 2 \rangle 4$ . PICK  $N$  such that, for all  $n \geq N$ , we have  $a_n \in B$

PROOF: From  $\langle 2 \rangle 1$ .

$\langle 2 \rangle 5$ . For all  $n \geq N$  we have  $a_n \in U$

□

**Lemma 64.** *If a sequence  $(a_n)$  is constant with  $a_n = l$  for all  $n$ , then  $a_n \rightarrow l$  as  $n \rightarrow \infty$ .*

PROOF: Immediate from definitions. □

**Theorem 65.** *Let  $X$  be a linearly ordered set. Let  $(s_n)$  be a monotone increasing sequence in  $X$  with a supremum  $s$ . Then  $s_n \rightarrow s$  as  $n \rightarrow \infty$ .*

PROOF:

$\langle 1 \rangle 1$ . ASSUME:  $s$  is not least in  $X$ .

PROOF: Otherwise  $(s_n)$  is the constant sequence  $s$  and the result follows from Lemma 64.

$\langle 1 \rangle 2$ . LET:  $U$  be a neighbourhood of  $s$ .

$\langle 1 \rangle 3$ . PICK  $a < s$  such that  $(a, s] \subseteq U$

$\langle 1 \rangle 4$ . PICK  $N$  such that  $a < a_N$ .

$\langle 1 \rangle 5$ . For all  $n \geq N$  we have  $a_n \in (a, s]$

$\langle 1 \rangle 6$ . For all  $n \geq N$  we have  $a_n \in U$ .

□

**Theorem 66.** *If  $\sum_{i=0}^{\infty} a_i = s$  and  $\sum_{i=0}^{\infty} b_i = t$  then  $\sum_{i=0}^{\infty} (ca_i + b_i) = cs + t$ .*

PROOF:  $\sum_{i=0}^N (ca_i + b_i) = c \sum_{i=0}^N a_i + \sum_{i=0}^N b_i \rightarrow cs + t$  as  $n \rightarrow \infty$ . □

**Theorem 67** (Comparison Test). *If  $|a_i| \leq b_i$  for all  $i$  and  $\sum_{i=0}^{\infty} b_i$  converges then  $\sum_{i=0}^{\infty} a_i$  converges.*

PROOF:

$\langle 1 \rangle 1$ .  $\sum_{i=0}^{\infty} |a_i|$  converges

PROOF: The partial sums  $\sum_{i=0}^N |a_i|$  form an increasing sequence bounded above by  $\sum_{i=0}^{\infty} b_i$ .

$\langle 1 \rangle 2$ . LET:  $c_i = |a_i| + a_i$  for all  $i$

$\langle 1 \rangle 3$ .  $\sum_{i=0}^{\infty} c_i$  converges

PROOF: Each  $c_i$  is either  $2|a_i|$  or 0. So the partial sums  $\sum_{i=0}^N c_i$  form an increasing sequence bounded above by  $2 \sum_{i=0}^{\infty} b_i$ .

$\langle 1 \rangle 4$ . Q.E.D.

PROOF: Since  $a_i = c_i - |a_i|$ .

□

**Corollary 67.1.** *If  $\sum_{i=0}^{\infty} |a_i|$  converges then  $\sum_{i=0}^{\infty} a_i$  converges.*

**Theorem 68** (Weierstrass  $M$ -test). *Let  $X$  be a set and  $(f_n : X \rightarrow \mathbb{R})$  be a sequence of functions. Let*

$$s_n(x) = \sum_{i=0}^n f_i(x)$$

*for all  $n, x$ . Suppose  $|f_i(x)| \leq M_i$  for all  $i \geq 0$  and  $x \in X$ . If the series  $\sum_{i=0}^{\infty} M_i$  converges, then the sequence  $(s_n)$  converges uniformly to*

$$s(x) = \sum_{i=0}^{\infty} f_i(x) .$$

PROOF:

$\langle 1 \rangle 1$ . LET:  $r_n = \sum_{i=n+1}^{\infty} M_i$  for all  $n$

$\langle 1 \rangle 2$ . Given  $0 \leq n < k$ , we have  $|s_k(x) - s_n(x)| \leq r_n$

PROOF:

$$\begin{aligned} |s_k(x) - s_n(x)| &= \left| \sum_{i=n+1}^k f_i(x) \right| \\ &\leq \sum_{i=n+1}^k |f_i(x)| \\ &\leq \sum_{i=n+1}^k M_i \\ &\leq r_n \end{aligned}$$

$\langle 1 \rangle 3$ . Given  $n \geq 0$  we have  $|s(x) - s_n(x)| \leq r_n$

PROOF: By taking the limit  $k \rightarrow \infty$  in  $\langle 1 \rangle 2$ .

$\langle 1 \rangle 4$ . Q.E.D.

PROOF: Since  $r_n \rightarrow 0$  as  $n \rightarrow \infty$ .

□

## 12 Locally Finite Sets

**Definition 69** (Locally Finite). Let  $X$  be a topological space and  $\{A_\alpha\}$  a family of subsets of  $X$ . Then  $\mathcal{A}$  is *locally finite* if and only if every point in  $X$  has a



neighbourhood that intersects  $A_\alpha$  for only finitely many  $\alpha$ .

**Theorem 70** (Pasting Lemma). *Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Let  $\{A_\alpha\}$  be a locally finite family of closed subsets of  $X$  that cover  $X$ . Suppose  $f \upharpoonright A_\alpha$  is continuous for all  $\alpha$ . Then  $f$  is continuous.*

PROOF:

$\langle 1 \rangle 1$ . Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Let  $A$  and  $B$  be closed subsets of  $X$  such that  $X = A \cup B$ . Suppose  $f \upharpoonright A$  and  $f \upharpoonright B$  are continuous. Then  $f$  is continuous.

$\langle 2 \rangle 1$ . LET:  $C \subseteq Y$  be closed.

$\langle 2 \rangle 2$ .  $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$

$\langle 2 \rangle 3$ .  $f^{-1}(C)$  and  $g^{-1}(C)$  are closed in  $X$ .

PROOF: Theorems 80 and 127.

$\langle 2 \rangle 4$ .  $h^{-1}(C)$  is closed in  $X$ .

PROOF: Lemma 20.

$\langle 2 \rangle 5$ . Q.E.D.

PROOF: Theorem 80.

$\langle 1 \rangle 2$ . Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Let  $\{A_\alpha\}$  be a finite family of closed subsets of  $X$  that cover  $X$ . Suppose  $f \upharpoonright A_\alpha$  is continuous for all  $\alpha$ . Then  $f$  is continuous.

PROOF: From  $\langle 1 \rangle 1$  by induction.

$\langle 1 \rangle 3$ . Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Let  $\{A_\alpha\}$  be a locally finite family of closed subsets of  $X$  that cover  $X$ . Suppose  $f \upharpoonright A_\alpha$  is continuous for all  $\alpha$ . Then  $f$  is continuous.

$\langle 2 \rangle 1$ . LET:  $x \in X$

PROVE:  $f$  is continuous at  $x$

$\langle 2 \rangle 2$ . PICK a neighbourhood  $U$  of  $x$  that intersects  $A_\alpha$  for only finitely many  $\alpha$ .

$\langle 2 \rangle 3$ .  $f \upharpoonright U$  is continuous

PROOF: By  $\langle 1 \rangle 2$ .

$\langle 2 \rangle 4$ . Q.E.D.

PROOF: Lemma 92.

□

The following example shows that we cannot remove the assumption of local finiteness.

**Example 71.** Define  $f : [-1, 1] \rightarrow \mathbb{R}$  by:  $f(x) = 1$  if  $x < -1$ ,  $f(x) = 0$  if  $x > 1$ . Let  $C_n = [-1, -1/n]$  for  $n \geq 1$ , and  $D = [0, 1]$ . Then  $[-1, 1] = \bigcup_{n=1}^{\infty} C_n \cup D$  and  $f$  is continuous on each  $C_n$  and each  $D$ , but  $f$  is not continuous on  $[-1, 1]$ .

**Proposition 72.** *Let  $X$  be a topological space. Let  $Y$  be a linearly ordered set in the order topology. Let  $f, g : X \rightarrow Y$  be continuous. Define  $h : X \rightarrow Y$  by  $h(x) = \min(f(x), g(x))$ . Then  $h$  is continuous.*

PROOF: By the Pasting Lemma applied to  $\{x \in X \mid f(x) \leq g(x)\}$  and  $\{x \in X \mid g(x) \leq f(x)\}$ , which are closed by Lemma 91.

## 13 Open Maps

**Definition 73** (Open Map). Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Then  $f$  is an *open map* if and only if, for every open set  $U$  in  $X$ , the set  $f(U)$  is open in  $Y$ .

**Lemma 74.** *Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Let  $\mathcal{B}$  be a basis for the topology on  $X$ . If  $f(B)$  is open in  $Y$  for all  $B \in \mathcal{B}$ , then  $f$  is an open map.*

PROOF: From Lemma 47.  $\square$

## 14 Continuous Functions

**Definition 75** (Continuous). Let  $X$  and  $Y$  be topological spaces. A function  $f : X \rightarrow Y$  is *continuous* if and only if, for every open set  $V$  in  $Y$ , the set  $f^{-1}(V)$  is open in  $X$ .

**Proposition 76.** *Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Let  $\mathcal{B}$  be a basis for  $Y$ . Then  $f$  is continuous if and only if, for all  $B \in \mathcal{B}$ , we have  $f^{-1}(B)$  is open in  $X$ .*

PROOF:

$\langle 1 \rangle 1$ . If  $f$  is continuous then, for all  $B \in \mathcal{B}$ , we have  $f^{-1}(B)$  is open in  $X$ .

PROOF: Since every element of  $B$  is open (Lemma 47).

$\langle 1 \rangle 2$ . Suppose that, for all  $B \in \mathcal{B}$ , we have  $f^{-1}(B)$  is open in  $X$ . Then  $f$  is continuous.

$\langle 2 \rangle 1$ . ASSUME: For all  $B \in \mathcal{B}$ , we have  $f^{-1}(B)$  is open in  $X$ .

$\langle 2 \rangle 2$ . LET:  $V$  be open in  $Y$ .

$\langle 2 \rangle 3$ . PICK  $\mathcal{A} \subseteq \mathcal{B}$  such that  $V = \bigcup \mathcal{A}$

PROOF: By Lemma 47.

$\langle 2 \rangle 4$ .  $f^{-1}(V)$  is open in  $X$ .

PROOF:

$$\begin{aligned} f^{-1}(V) &= f^{-1}\left(\bigcup \mathcal{A}\right) \\ &= \bigcup_{B \in \mathcal{A}} f^{-1}(B) \end{aligned}$$

$\square$

**Proposition 77.** *Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Let  $\mathcal{S}$  be a subbasis for  $Y$ . Then  $f$  is continuous if and only if, for all  $S \in \mathcal{S}$ , we have  $f^{-1}(S)$  is open in  $X$ .*

PROOF:

$\langle 1 \rangle 1$ . If  $f$  is continuous then, for all  $S \in \mathcal{S}$ , we have  $f^{-1}(S)$  is open in  $X$ .

PROOF: Since every element of  $S$  is open.

$\langle 1 \rangle 2$ . Suppose that, for all  $S \in \mathcal{S}$ , we have  $f^{-1}(S)$  is open in  $X$ . Then  $f$  is continuous.

- ⟨2⟩1. ASSUME: For all  $S \in \mathcal{S}$ , we have  $f^{-1}(S)$  is open in  $X$ .  
 ⟨2⟩2. LET:  $S_1, \dots, S_n \in \mathcal{S}$   
 ⟨2⟩3.  $f^{-1}(S_1 \cap \dots \cap S_n)$  is open in  $A$   
 PROOF: Since  $f^{-1}(S_1 \cap \dots \cap S_n) = f^{-1}(S_1) \cap \dots \cap f^{-1}(S_n)$ .  
 ⟨2⟩4. Q.E.D.  
 PROOF: By Propositions 76 and 55.

□

**Proposition 78.** *Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Let  $\mathcal{S}$  be a basis for  $Y$ . Then  $f$  is continuous if and only if, for all  $V \in \mathcal{S}$ , we have  $f^{-1}(V)$  is open in  $X$ .*

PROOF:

- ⟨1⟩1. If  $f$  is continuous then, for all  $V \in \mathcal{S}$ , we have  $f^{-1}(V)$  is open in  $X$ .  
 PROOF: Since every element of  $\mathcal{S}$  is open.  
 ⟨1⟩2. Suppose that, for all  $V \in \mathcal{S}$ , we have  $f^{-1}(V)$  is open in  $X$ . Then  $f$  is continuous.  
 ⟨2⟩1. ASSUME: For all  $V \in \mathcal{S}$ , we have  $f^{-1}(V)$  is open in  $X$ .  
 ⟨2⟩2. For every set  $B$  that is the finite intersection of elements of  $\mathcal{S}$ , we have  $f^{-1}(B)$  is open in  $X$ .  
 PROOF: Because  $f^{-1}(V_1 \cap \dots \cap V_n) = f^{-1}(V_1) \cap \dots \cap f^{-1}(V_n)$ .  
 ⟨2⟩3. Q.E.D.  
 PROOF: From Propositions 55 and 76.

□

**Definition 79** (Continuous at a Point). *Let  $X$  and  $Y$  be topological spaces. Let  $f : X \rightarrow Y$  and  $x \in X$ . Then  $f$  is *continuous at  $x$*  if and only if, for every neighbourhood  $V$  of  $f(x)$ , there exists a neighbourhood  $U$  of  $x$  such that  $f(U) \subseteq V$ .*

**Theorem 80.** *Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Then the following are equivalent:*

1.  $f$  is continuous.
2. For all  $A \subseteq X$ , we have  $f(\overline{A}) \subseteq \overline{f(A)}$
3. For all  $B \subseteq Y$  closed, we have  $f^{-1}(B)$  is closed in  $X$ .
4.  $f$  is continuous at every point of  $X$ .

PROOF:

- ⟨1⟩1.  $1 \Rightarrow 2$   
 ⟨2⟩1. ASSUME:  $f$  is continuous.  
 ⟨2⟩2. LET:  $A \subseteq X$   
 ⟨2⟩3. LET:  $x \in \overline{A}$   
 PROVE:  $f(x) \in \overline{f(A)}$   
 ⟨2⟩4. LET:  $V$  be a neighbourhood of  $f(x)$   
 ⟨2⟩5.  $f^{-1}(V)$  is a neighbourhood of  $x$

⟨2⟩6. PICK  $y \in A \cap f^{-1}(V)$

PROOF: By Theorem 34.

⟨2⟩7.  $f(y) \in V \cap f(A)$

⟨2⟩8. Q.E.D.

PROOF: By Theorem 34.

⟨1⟩2.  $2 \Rightarrow 3$

⟨2⟩1. ASSUME: 2

⟨2⟩2. LET:  $B$  be closed in  $Y$

⟨2⟩3. LET:  $x \in \overline{f^{-1}(B)}$

PROVE:  $x \in f^{-1}(B)$

⟨2⟩4.  $f(x) \in B$

PROOF:

$$\begin{aligned} f(x) &\in f(\overline{f^{-1}(B)}) \\ &\subseteq \overline{f(f^{-1}(B))} && (\langle 2 \rangle 1) \\ &\subseteq \overline{B} && (Proposition 35) \\ &= B \end{aligned}$$

⟨1⟩3.  $3 \Rightarrow 1$

⟨2⟩1. ASSUME: 3

⟨2⟩2. LET:  $V$  be open in  $Y$

⟨2⟩3.  $Y \setminus V$  is closed in  $Y$

⟨2⟩4.  $f^{-1}(Y \setminus V)$  is closed in  $X$

⟨2⟩5.  $X \setminus f^{-1}(V)$  is closed in  $X$

⟨2⟩6.  $f^{-1}(V)$  is open in  $X$

⟨1⟩4.  $1 \Rightarrow 4$

PROOF: For any neighbourhood  $V$  of  $f(x)$ , the set  $U = f^{-1}(V)$  is a neighbourhood of  $x$  such that  $f(U) \subseteq V$ .

⟨1⟩5.  $4 \Rightarrow 1$

⟨2⟩1. ASSUME: 4

⟨2⟩2. LET:  $V$  be open in  $Y$

⟨2⟩3. LET:  $x \in f^{-1}(V)$

⟨2⟩4.  $V$  is a neighbourhood of  $f(x)$

⟨2⟩5. PICK a neighbourhood  $U$  of  $x$  such that  $f(U) \subseteq V$

⟨2⟩6.  $x \in U \subseteq f^{-1}(V)$

⟨2⟩7. Q.E.D.

PROOF: By Lemma 12.

□

**Theorem 81.** *A constant function is continuous.*

PROOF: Let  $X$  and  $Y$  be topological spaces. Let  $b \in Y$ , and let  $f : X \rightarrow Y$  be the constant function with value  $b$ . For any open  $V \subseteq Y$ , the set  $f^{-1}(V)$  is either  $X$  (if  $b \in V$ ) or  $\emptyset$  (if  $b \notin V$ ). □

**Theorem 82.** *If  $A$  is a subspace of  $X$  then the inclusion  $j : A \rightarrow X$  is continuous.*

PROOF: For any  $V$  open in  $X$ , we have  $j^{-1}(V) = V \cap A$  is open in  $A$ . □

**Theorem 83.** *The composite of two continuous functions is continuous.*

PROOF: Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  be continuous. For any  $V$  open in  $Z$ , we have  $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$  is open in  $X$ .  $\square$

**Theorem 84.** *Let  $f : X \rightarrow Y$  be a continuous function and  $A$  be a subspace of  $X$ . Then the restriction  $f \upharpoonright A : A \rightarrow Y$  is continuous.*

PROOF: Let  $V$  be open in  $Y$ . Then  $(f \upharpoonright A)^{-1}(V) = f^{-1}(V) \cap A$  is open in  $A$ .  $\square$

**Theorem 85.** *Let  $f : X \rightarrow Y$  be continuous. Let  $Z$  be a subspace of  $Y$  such that  $f(X) \subseteq Z$ . Then the corestriction  $f : X \rightarrow Z$  is continuous.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $V$  be open in  $Z$ .
- $\langle 1 \rangle 2$ . PICK  $U$  open in  $Y$  such that  $V = U \cap Z$ .
- $\langle 1 \rangle 3$ .  $f^{-1}(V) = f^{-1}(U)$
- $\langle 1 \rangle 4$ .  $f^{-1}(V)$  is open in  $X$ .

$\square$

**Theorem 86.** *Let  $f : X \rightarrow Y$  be continuous. Let  $Z$  be a space such that  $Y$  is a subspace of  $Z$ . Then the expansion  $f : X \rightarrow Z$  is continuous.*

PROOF: Let  $V$  be open in  $Z$ . Then  $f^{-1}(V) = f^{-1}(V \cap Y)$  is open in  $X$ .  $\square$

**Theorem 87.** *Let  $X$  and  $Y$  be topological spaces. Let  $f : X \rightarrow Y$ . Suppose  $\mathcal{U}$  is a set of open sets in  $X$  such that  $X = \bigcup \mathcal{U}$  and, for all  $U \in \mathcal{U}$ , we have  $f \upharpoonright U : U \rightarrow Y$  is continuous. Then  $f$  is continuous.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $V$  be open in  $Y$
- $\langle 1 \rangle 2$ .  $f^{-1}(V) = \bigcup_{U \in \mathcal{U}} (f \upharpoonright U)^{-1}(V)$
- $\langle 1 \rangle 3$ . For all  $U \in \mathcal{U}$ , we have  $(f \upharpoonright U)^{-1}(V)$  is open in  $U$ .
- $\langle 1 \rangle 4$ . For all  $U \in \mathcal{U}$ , we have  $(f \upharpoonright U)^{-1}(V)$  is open in  $X$ .

PROOF: Lemma 126.

$\square$

**Theorem 88.** *Let  $A$  be a topological space and  $\{X_i\}_{i \in I}$  be a family of topological spaces. Let  $f : A \rightarrow \prod_{i \in I} X_i$  be a function. If  $\pi_i \circ f$  is continuous for all  $i \in I$  then  $f$  is continuous.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $i \in I$  and  $U$  be open in  $X_i$
- $\langle 1 \rangle 2$ .  $f^{-1}(\pi_i^{-1}(U))$  is open in  $A$
- $\langle 1 \rangle 3$ . Q.E.D.

PROOF: Proposition 77.

$\square$

**Proposition 89.** *Let  $X$  and  $X'$  be the same set  $X$  under two topologies  $\mathcal{T}$  and  $\mathcal{T}'$ . Let  $i : X \rightarrow X'$  be the identity function. Then  $i$  is continuous if and only if  $\mathcal{T}' \subseteq \mathcal{T}$ .*

PROOF: Immediate from definitions.  $\square$

**Proposition 90.** *Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  and  $a \in \mathbb{R}$ . Then  $f$  is continuous on the right at  $a$  if and only if  $f$  is continuous at  $a$  as a function  $\mathbb{R}_l \rightarrow \mathbb{R}$ .*

PROOF:

- $\langle 1 \rangle 1$ . If  $f$  is continuous on the right at  $a$  then  $f$  is continuous at  $a$  as a function  $\mathbb{R}_l \rightarrow \mathbb{R}$ .
  - $\langle 2 \rangle 1$ . ASSUME:  $f$  is continuous on the right at  $a$ .
  - $\langle 2 \rangle 2$ . LET:  $V$  be a neighbourhood of  $f(a)$
  - $\langle 2 \rangle 3$ . PICK  $b, c$  such that  $f(a) \in (b, c) \subseteq V$ .
  - $\langle 2 \rangle 4$ . LET:  $\epsilon = \min(c - f(a), f(a) - b)$
  - $\langle 2 \rangle 5$ . PICK  $\delta > 0$  such that, for all  $x$ , if  $a < x < a + \delta$  then  $|f(x) - f(a)| < \epsilon$
  - $\langle 2 \rangle 6$ . LET:  $U = [a, a + \delta)$
  - $\langle 2 \rangle 7$ .  $f(U) \subseteq V$
- $\langle 1 \rangle 2$ . If  $f$  is continuous at  $a$  as a function  $\mathbb{R}_l \rightarrow \mathbb{R}$  then  $f$  is continuous on the right at  $a$ .
  - $\langle 2 \rangle 1$ . ASSUME:  $f$  is continuous at  $a$  as a function  $\mathbb{R}_l \rightarrow \mathbb{R}$
  - $\langle 2 \rangle 2$ . LET:  $\epsilon > 0$
  - $\langle 2 \rangle 3$ . PICK a neighbourhood  $U$  of  $a$  such that  $f(U) \subseteq (f(a) - \epsilon, f(a) + \epsilon)$
  - $\langle 2 \rangle 4$ . PICK  $b, c$  such that  $a \in [b, c) \subset U$
  - $\langle 2 \rangle 5$ . LET:  $\delta = c - a$
  - $\langle 2 \rangle 6$ . For all  $x$ , if  $a < x < a + \delta$  then  $|f(x) - f(a)| < \epsilon$

$\square$

**Lemma 91.** *Let  $X$  be a topological space. Let  $Y$  be a linearly ordered set in the order topology. Let  $f, g : X \rightarrow Y$  be continuous. Then  $C = \{x \in X \mid f(x) \leq g(x)\}$  is closed.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $x \in X \setminus C$
- $\langle 1 \rangle 2$ .  $f(x) > g(x)$ 
  - PROVE: There exists a neighbourhood  $U$  of  $x$  such that  $U \subseteq X \setminus C$
- $\langle 1 \rangle 3$ . CASE: There exists  $y$  such that  $g(x) < y < f(x)$ 
  - PROOF: Take  $U = g^{-1}((-\infty, y)) \cup f^{-1}(y, +\infty)$ .
- $\langle 1 \rangle 4$ . CASE: There is no  $y$  such that  $g(x) < y < f(x)$ 
  - PROOF: Take  $U = g^{-1}((-\infty, f(x))) \cup f^{-1}(g(x), +\infty)$ .

$\square$

**Lemma 92.** *Let  $f : X \rightarrow Y$ . Let  $Z$  be an open subspace of  $X$  and  $a \in Z$ . If  $f \upharpoonright Z$  is continuous at  $a$  then  $f$  is continuous at  $a$ .*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $V$  be a neighbourhood of  $f(a)$
- $\langle 1 \rangle 2$ . PICK a neighbourhood  $W$  of  $a$  in  $Z$  such that  $f(W) \subseteq V$
- $\langle 1 \rangle 3$ .  $W$  is a neighbourhood of  $a$  in  $X$  such that  $f(W) \subseteq V$
- PROOF: Lemma 126.

$\square$

**Proposition 93.** Let  $f : A \rightarrow B$  and  $g : C \rightarrow D$  be continuous. Define  $f \times g : A \times C \rightarrow B \times D$  by

$$(f \times g)(a, c) = (f(a), g(c)) .$$

Then  $f \times g$  is continuous.

PROOF:  $\pi_1 \circ (f \times g) = f \circ \pi_1$  and  $\pi_2 \circ (f \times g) = g \circ \pi_2$  are continuous by Theorem 83. The result follows by Theorem 88.

**Proposition 94.** Let  $X$  be a topological space. Let  $Y$  a Hausdorff space. Let  $A \subseteq X$ . Let  $f, g : \overline{A} \rightarrow Y$  be continuous. If  $f$  and  $g$  agree on  $A$  then  $f = g$ .

PROOF:

$\langle 1 \rangle 1$ . LET:  $x \in \overline{A}$

$\langle 1 \rangle 2$ . ASSUME:  $f(x) \neq g(x)$

$\langle 1 \rangle 3$ . PICK disjoint neighbourhoods  $V$  of  $f(x)$  and  $W$  of  $g(x)$ .

$\langle 1 \rangle 4$ . PICK  $y \in f^{-1}(V) \cap g^{-1}(W) \cap A$

PROOF: Since  $f^{-1}(V) \cap g^{-1}(W)$  is a neighbourhood of  $x$  and hence intersects  $A$ .

$\langle 1 \rangle 5$ .  $f(y) = g(y) \in V \cap W$

$\langle 1 \rangle 6$ . Q.E.D.

PROOF: This contradicts the fact that  $V$  and  $W$  are disjoint ( $\langle 1 \rangle 3$ ).

□

**Proposition 95.** Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$  be continuous. If  $a_n \rightarrow l$  as  $n \rightarrow \infty$  in  $X$  then  $f(a_n) \rightarrow f(l)$  as  $n \rightarrow \infty$ .

PROOF:

$\langle 1 \rangle 1$ . LET:  $V$  be a neighbourhood of  $f(l)$

$\langle 1 \rangle 2$ . PICK a neighbourhood  $U$  of  $l$  such that  $f(U) \subseteq V$

$\langle 1 \rangle 3$ . PICK  $N$  such that, for all  $n \geq N$ , we have  $a_n \in U$

$\langle 1 \rangle 4$ . For all  $n \geq N$  we have  $f(a_n) \in V$

□

## 15 Homeomorphisms

**Definition 96** (Homeomorphism). Let  $X$  and  $Y$  be topological spaces. A *Homeomorphism*  $f$  between  $X$  and  $Y$ ,  $f : X \cong Y$ , is a bijection  $f : X \rightarrow Y$  such that both  $f$  and  $f^{-1}$  are continuous.

**Lemma 97.** Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$  a bijection. Then the following are equivalent:

1.  $f$  is a homeomorphism.
2.  $f$  is continuous and an open map.
3. For any  $U \subseteq X$ , we have  $U$  is open if and only if  $f(U)$  is open.

PROOF: Immediate from definitions.  $\square$

**Proposition 98.** *Let  $X$  and  $X'$  be the same set  $X$  under two topologies  $\mathcal{T}$  and  $\mathcal{T}'$ . Let  $i : X \rightarrow X'$  be the identity function. Then  $i$  is a homeomorphism if and only if  $\mathcal{T} = \mathcal{T}'$ .*

PROOF: Immediate from definitions.  $\square$

**Definition 99** (Topological Property). Let  $P$  be a property of topological spaces. Then  $P$  is a *topological* property if and only if, for any spaces  $X$  and  $Y$ , if  $P$  holds of  $X$  and  $X \cong Y$  then  $P$  holds of  $Y$ .

**Definition 100** (Topological Imbedding). Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Then  $f$  is a *topological imbedding* if and only if the corestriction  $f : X \rightarrow f(X)$  is a homeomorphism.

**Proposition 101.** *Let  $X$  and  $Y$  be topological spaces and  $a \in X$ . The function  $i : Y \rightarrow X \times Y$  that maps  $y$  to  $(a, y)$  is an imbedding.*

PROOF:

$\langle 1 \rangle 1$ .  $i$  is injective

$\langle 1 \rangle 2$ .  $i$  is continuous.

PROOF: For  $U$  open in  $X$  and  $V$  open in  $Y$ , we have  $i^{-1}(U \times V)$  is  $V$  if  $a \in U$ , and  $\emptyset$  if  $a \notin U$ .

$\langle 1 \rangle 3$ .  $i : Y \rightarrow i(Y)$  is an open map.

PROOF: For  $V$  open in  $Y$  we have  $i(V) = (X \times V) \cap i(Y)$ .

$\square$

## 16 The Order Topology

**Definition 102** (Order Topology). Let  $X$  be a linearly ordered set with at least two points. The *order topology* on  $X$  is the topology generated by the basis  $\mathcal{B}$  consisting of:

- all open intervals  $(a, b)$ ;
- all intervals of the form  $[\perp, b)$  where  $\perp$  is least in  $X$ ;
- all intervals of the form  $(a, \top]$  where  $\top$  is greatest in  $X$ .

We prove this is a basis for a topology.

PROOF:

$\langle 1 \rangle 1$ . For all  $x \in X$  there exists  $B \in \mathcal{B}$  such that  $x \in B$ .

$\langle 2 \rangle 1$ . LET:  $x \in X$

$\langle 2 \rangle 2$ . CASE:  $x$  is greatest in  $X$ .

$\langle 3 \rangle 1$ . PICK  $y \in X$  with  $y \neq x$

$\langle 3 \rangle 2$ .  $x \in (y, x] \in \mathcal{B}$

$\langle 2 \rangle 3$ . CASE:  $x$  is least in  $X$ .





⟨1⟩2. The half-open interval  $[0, 1)$  is not open in the standard topology.

PROOF: There is no open interval  $(a, b)$  such that  $0 \in (a, b) \subseteq [0, 1)$ .

□

**Lemma 106.** *The  $K$ -topology is strictly finer than the standard topology on  $\mathbb{R}$ .*

PROOF:

⟨1⟩1. Every open interval is open in the  $K$ -topology.

PROOF: Corollary 47.1.

⟨1⟩2. The set  $(-1, 1) \setminus K$  is not open in the standard topology.

PROOF: There is no open interval  $(a, b)$  such that  $0 \in (a, b) \subseteq (-1, 1) \setminus K$ , since there must be a positive integer  $n$  with  $1/n \in (a, b)$ .

□

**Definition 107** (Ordered Square). The *ordered square*  $I_o^2$  is the set  $[0, 1]^2$  under the order topology generated by the dictionary order.

## 17 The Product Topology

**Definition 108** (Product Topology). Let  $\{A_i\}_{i \in I}$  be a family of topological spaces. The *product topology* on  $\prod_{i \in I} A_i$  is the topology generated by the sub-basis consisting of the sets of the form  $\pi_i^{-1}(U)$  where  $i \in I$  and  $U$  is open in  $A_i$ .

**Proposition 109.** *The product topology on  $\prod_{i \in I} A_i$  is generated by the basis consisting of all sets of the form  $\prod_{i \in I} U_i$  where  $\{U_i\}_{i \in I}$  is a family such that each  $U_i$  is an open set in  $A_i$  and  $U_i = A_i$  for all but finitely many  $i$ .*

PROOF: From Proposition 55. □

**Proposition 110.** *If  $A_i$  is closed in  $X_i$  for all  $i \in I$  then  $\prod_{i \in I} A_i$  is closed in  $\prod_{i \in I} X_i$ .*

PROOF:

$$\left(\prod_{i \in I} X_i\right) \setminus \left(\prod_{i \in I} A_i\right) = \bigcup_{j \in I} \left(\prod_{i \in I} X_i \setminus \pi_j^{-1}(A_j)\right) \quad \square$$

**Proposition 111.** *Let  $\{A_i\}_{i \in I}$  be a family of topological spaces. For  $i \in I$ , let  $\mathcal{B}_i$  be a basis for the topology on  $A_i$ . Then  $\mathcal{B} = \{\prod_{i \in I} B_i \mid \forall i \in I, B_i \in \mathcal{B}_i, B_i = A_i \text{ for all but finitely many } i\}$  is a basis for the box topology on  $\prod_{i \in I} A_i$ .*

PROOF:

⟨1⟩1. Every set in  $\mathcal{B}$  is open.

⟨1⟩2. For every point  $a \in \prod_{i \in I} A_i$  and every open set  $U$  with  $a \in U$ , there exists  $B \in \mathcal{B}$  such that  $a \in B \subseteq U$ .

⟨2⟩1. LET:  $U$  be open and  $a \in U$

⟨2⟩2. PICK a family  $\{U_i\}_{i \in I}$  such that each  $U_i$  is open in  $A_i$ , such that  $U_i = A_i$  except for  $i = i_1, \dots, i_n$ , and such that  $a \in \prod_{i \in I} U_i \subseteq U$ .

- ⟨2⟩3. For  $j = 1, \dots, n$ , PICK  $B_{i_j} \in \mathcal{B}_{i_j}$  such that  $a_{i_j} \in B_{i_j} \subseteq U_{i_j}$
- ⟨2⟩4. LET:  $B = \prod_{i \in I} B_i$  where  $B_i = A_i$  for  $i \neq i_1, \dots, i_n$
- ⟨2⟩5.  $B \in \mathcal{B}$
- ⟨2⟩6.  $a \in B \subseteq U$
- ⟨1⟩3. Q.E.D.

PROOF: Lemma 48.

□

**Proposition 112.** *Let  $\{A_i\}_{i \in I}$  be a family of topological spaces. Then the projections  $\pi_i : \prod_{i \in I} A_i \rightarrow A_i$  are open maps.*

PROOF: From Lemma 74. □

**Example 113.** The projections are not always closed maps. For example,  $\pi_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$  maps the closed set  $\{(x, 1/x) \mid x > 0\}$  to  $(0, +\infty)$ .

**Proposition 114.** *Let  $\{X_i\}_{i \in I}$  be a family of sets. For  $i \in I$ , let  $\mathcal{T}_i$  and  $\mathcal{U}_i$  be topologies on  $X_i$ . Let  $\mathcal{P}$  be the product topology on  $\prod_{i \in I} X_i$  generated by the topologies  $\mathcal{T}_i$ , and  $\mathcal{Q}$  the product topology on the same set generated by the topologies  $\mathcal{U}_i$ . Then  $\mathcal{P} \subseteq \mathcal{Q}$  if and only if  $\mathcal{T}_i \subseteq \mathcal{U}_i$  for all  $i$ .*

PROOF:

- ⟨1⟩1. If  $\mathcal{T}_i \subseteq \mathcal{U}_i$  for all  $i$  then  $\mathcal{P} \subseteq \mathcal{Q}$   
PROOF: By Corollary 47.1.
- ⟨1⟩2. If  $\mathcal{P} \subseteq \mathcal{Q}$  then  $\mathcal{T}_i \subseteq \mathcal{U}_i$  for all  $i$ 
  - ⟨2⟩1. ASSUME:  $\mathcal{P} \subseteq \mathcal{Q}$
  - ⟨2⟩2. LET:  $i \in I$
  - ⟨2⟩3. LET:  $U \in \mathcal{T}_i$
  - ⟨2⟩4. LET:  $U_i = U$  and  $U_j = X_j$  for  $j \neq i$
  - ⟨2⟩5.  $\prod_{i \in I} U_i \in \mathcal{P}$
  - ⟨2⟩6.  $\prod_{i \in I} U_i \in \mathcal{Q}$
  - ⟨2⟩7.  $U \in \mathcal{U}_i$

PROOF: From Proposition 112.

□

**Proposition 115 (AC).** *Let  $\{X_i\}_{i \in I}$  be a family of topological spaces. Let  $A_i \subseteq X_i$  for all  $i \in I$ . Then*

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i} .$$

PROOF:

- ⟨1⟩1.  $\overline{\prod_{i \in I} A_i} \subseteq \prod_{i \in I} \overline{A_i}$ 
  - ⟨2⟩1. For all  $i \in I$  we have  $A_i \subseteq \overline{A_i}$   
PROOF: Lemma 31.
  - ⟨2⟩2.  $\prod_{i \in I} A_i \subseteq \prod_{i \in I} \overline{A_i}$
  - ⟨2⟩3. Q.E.D.
- PROOF: Since  $\prod_{i \in I} A_i$  is closed by Proposition 110.

- $\langle 1 \rangle 2.$   $\prod_{i \in I} \overline{A_i} \subseteq \overline{\prod_{i \in I} A_i}$   
 $\langle 2 \rangle 1.$  LET:  $x \in \prod_{i \in I} \overline{A_i}$   
 $\langle 2 \rangle 2.$  LET:  $U$  be a neighbourhood of  $x$   
 $\langle 2 \rangle 3.$  PICK  $V_i$  open in  $X_i$  such that  $x \in \prod_{i \in I} V_i \subseteq U$  with  $V_i = X_i$  except for  $i = i_1, \dots, i_n$   
 $\langle 2 \rangle 4.$  For  $i \in I$ , pick  $a_i \in V_i \cap A_i$   
 PROOF: By Theorem 34 and  $\langle 2 \rangle 1$  using the Axiom of Choice.  
 $\langle 2 \rangle 5.$   $U$  intersects  $\prod_{i \in I} A_i$   
 $\langle 2 \rangle 6.$  Q.E.D.  
 PROOF:  $a \in U \cap \prod_{i \in I} A_i$

□

**Example 116.** The closure of  $\mathbb{R}^\infty$  in  $\mathbb{R}^\omega$  is  $\mathbb{R}^\omega$

PROOF:

- $\langle 1 \rangle 1.$  LET:  $a \in \mathbb{R}^\omega$   
 $\langle 1 \rangle 2.$  LET:  $U$  be any neighbourhoods of  $a$ .  
 $\langle 1 \rangle 3.$  PICK  $U_n$  open in  $\mathbb{R}$  for all  $n$  such that  $a \in \prod_{n \geq 0} U_n \subseteq U$  and  $U_n = \mathbb{R}$  for all  $n$  except  $n_1, \dots, n_k$   
 $\langle 1 \rangle 4.$  LET:  $b_n = a_n$  for  $n = n_1, \dots, n_k$  and  $b_n = 0$  for all other  $n$   
 $\langle 1 \rangle 5.$   $b \in \mathbb{R}^\infty \cap U$   
 $\langle 1 \rangle 6.$  Q.E.D.

PROOF: From Theorem 34.

□

**Proposition 117.** Let  $\{X_i\}_{i \in I}$  be a family of topological spaces. Let  $(a_n)$  be a sequence in  $\prod_{i \in I} X_i$  and  $l \in \prod_{i \in I} X_i$ . Then  $a_n \rightarrow l$  as  $n \rightarrow \infty$  if and only if, for all  $i \in I$ , we have  $\pi_i(a_n) \rightarrow \pi_i(l)$  as  $n \rightarrow \infty$ .

PROOF:

- $\langle 1 \rangle 1.$  If  $a_n \rightarrow l$  as  $n \rightarrow \infty$  then, for all  $i \in I$ , we have  $\pi_i(a_n) \rightarrow \pi_i(l)$  as  $n \rightarrow \infty$   
 PROOF: Proposition 95.  
 $\langle 1 \rangle 2.$  If, for all  $i \in I$ , we have  $\pi_i(a_n) \rightarrow \pi_i(l)$  as  $n \rightarrow \infty$ , then  $a_n \rightarrow l$  as  $n \rightarrow \infty$   
 $\langle 2 \rangle 1.$  ASSUME: For all  $i \in I$ , we have  $\pi_i(a_n) \rightarrow \pi_i(l)$  as  $n \rightarrow \infty$   
 $\langle 2 \rangle 2.$  LET:  $V$  be a neighbourhood of  $l$   
 $\langle 2 \rangle 3.$  PICK open sets  $U_i$  in  $X_i$  such that  $l \in \prod_{i \in I} U_i \subseteq V$  and  $U_i = X_i$  for all  $i$  except  $i = i_1, \dots, i_k$   
 $\langle 2 \rangle 4.$  For  $j = 1, \dots, k$ , PICK  $N_j$  such that, for all  $n \geq N_j$ , we have  $\pi_{i_j}(a_n) \in U_{i_j}$   
 $\langle 2 \rangle 5.$  LET:  $N = \max(N_1, \dots, N_k)$   
 $\langle 2 \rangle 6.$  For all  $n \geq N$  we have  $a_n \in V$

□

## 17.1 Continuous in Each Variable Separately

**Definition 118** (Continuous in Each Variable Separately). Let  $F : X \times Y \rightarrow Z$ . Then  $F$  is *continuous in each variable separately* if and only if:

- for every  $a \in X$  the function  $\lambda y \in Y. F(a, y)$  is continuous;
- for every  $b \in Y$  the function  $\lambda x \in X. F(x, b)$  is continuous.

**Proposition 119.** *Let  $F : X \times Y \rightarrow Z$ . If  $F$  is continuous then  $F$  is continuous in each variable separately.*

PROOF: For  $a \in X$ , the function  $\lambda y \in Y. F(a, y)$  is  $F \circ i$  where  $i : Y \rightarrow X \times Y$  maps  $y$  to  $(a, y)$ . We have  $i$  is continuous by Proposition 101, hence  $F \circ i$  is continuous by Theorem 83.

Similarly for  $\lambda x \in X. F(x, b)$  for  $b \in Y$ .  $\square$

**Example 120.** Define  $F : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  by

$$F(x, y) = \begin{cases} xy/(x^2 + y^2) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Then  $F$  is continuous in each variable separately but not continuous.

## 18 The Subspace Topology

**Definition 121** (Subspace Topology). Let  $X$  be a topological space and  $Y \subseteq X$ . The *subspace topology* on  $Y$  is  $\mathcal{T} = \{U \cap Y \mid U \text{ is open in } X\}$ .

We prove this is a topology.

PROOF:

$\langle 1 \rangle 1. Y \in \mathcal{T}$

PROOF: Since  $Y = X \cap Y$

$\langle 1 \rangle 2. \text{ For all } \mathcal{U} \subseteq \mathcal{T}, \text{ we have } \bigcup \mathcal{U} \in \mathcal{T}$

$\langle 2 \rangle 1. \text{ LET: } \mathcal{U} \subseteq \mathcal{T}$

$\langle 2 \rangle 2. \text{ LET: } \mathcal{V} = \{V \text{ open in } X \mid V \cap Y \in \mathcal{U}\}$

$\langle 2 \rangle 3. \bigcup \mathcal{U} = (\bigcup \mathcal{V}) \cap Y$

$\langle 1 \rangle 3. \text{ For all } U, V \in \mathcal{T}, \text{ we have } U \cap V \in \mathcal{T}$

$\langle 2 \rangle 1. \text{ LET: } U, V \in \mathcal{T}$

$\langle 2 \rangle 2. \text{ PICK } U', V' \text{ open in } X \text{ such that } U = U' \cap Y \text{ and } V = V' \cap Y$

$\langle 2 \rangle 3. (U \cap V) = (U' \cap V') \cap Y$

$\square$

**Theorem 122.** *Let  $X$  be a topological space and  $Y$  a subspace of  $X$ . Let  $A \subseteq Y$ . Then  $A$  is closed in  $Y$  if and only if there exists a closed set  $C$  in  $X$  such that  $A = C \cap Y$ .*

PROOF: We have

$$\begin{aligned} & A \text{ is closed in } Y \\ \Leftrightarrow & Y \setminus A \text{ is open in } Y \\ \Leftrightarrow & \exists U \text{ open in } X. Y \setminus A = Y \cap U \\ \Leftrightarrow & \exists C \text{ closed in } X. Y \setminus A = Y \cap (X \setminus U) \\ \Leftrightarrow & \exists C \text{ closed in } X. A = Y \cap U \end{aligned}$$

$\square$

**Theorem 123.** Let  $Y$  be a subspace of  $X$ . Let  $A \subseteq Y$ . Let  $\bar{A}$  be the closure of  $A$  in  $X$ . Then the closure of  $A$  in  $Y$  is  $\bar{A} \cap Y$ .

PROOF: The closure of  $A$  in  $Y$  is

$$\begin{aligned} & \bigcap \{C \text{ closed in } Y \mid A \subseteq C\} \\ &= \bigcap \{D \cap Y \mid D \text{ closed in } X, A \subseteq D \cap Y\} \quad (\text{Theorem 122}) \\ &= \bigcap \{D \mid D \text{ closed in } X, A \subseteq D\} \cap Y \\ &= \bar{A} \cap Y \quad \square \end{aligned}$$

**Lemma 124.** Let  $X$  be a topological space and  $Y \subseteq X$ . Let  $\mathcal{B}$  be a basis for the topology on  $X$ . Then  $\mathcal{B}' = \{B \cap Y \mid B \in \mathcal{B}\}$  is a basis for the subspace topology on  $Y$ .

PROOF:

- $\langle 1 \rangle 1$ . Every element in  $\mathcal{B}'$  is open in  $Y$
- $\langle 1 \rangle 2$ . For every open set  $U$  in  $Y$  and point  $y \in U$ , there exists  $B' \in \mathcal{B}'$  such that  $y \in B' \subseteq U$ 
  - $\langle 2 \rangle 1$ . LET:  $U$  be open in  $Y$  and  $y \in U$
  - $\langle 2 \rangle 2$ . PICK  $V$  open in  $X$  such that  $U = V \cap Y$
  - $\langle 2 \rangle 3$ . PICK  $B \in \mathcal{B}$  such that  $y \in B \subseteq V$
  - $\langle 2 \rangle 4$ . LET:  $B' = B \cap Y$
  - $\langle 2 \rangle 5$ .  $B' \in \mathcal{B}'$
  - $\langle 2 \rangle 6$ .  $y \in B' \subseteq U$
- $\langle 1 \rangle 3$ . Q.E.D.

PROOF: By Lemma 48.

□

**Lemma 125.** Let  $X$  be a topological space and  $Y \subseteq X$ . Let  $\mathcal{S}$  be a basis for the topology on  $X$ . Then  $\mathcal{S}' = \{S \cap Y \mid S \in \mathcal{S}\}$  is a subbasis for the subspace topology on  $Y$ .

PROOF: The set  $\{B \cap Y \mid B \text{ is a finite intersection of elements of } \mathcal{S}\}$  is a basis for the subspace topology by Lemma 124, and this is the set of all finite intersections of elements of  $\mathcal{S}'$ . □

**Lemma 126.** Let  $Y$  be a subspace of  $X$ . If  $U$  is open in  $Y$  and  $Y$  is open in  $X$  then  $U$  is open in  $X$ .

PROOF:

- $\langle 1 \rangle 1$ . PICK  $V$  open in  $X$  such that  $U = V \cap Y$
- $\langle 1 \rangle 2$ .  $U$  is open in  $X$

PROOF: Since it is the intersection of two open sets  $V$  and  $Y$ .

□

**Theorem 127.** Let  $Y$  be a subspace of  $X$  and  $A \subseteq Y$ . If  $A$  is closed in  $Y$  and  $Y$  is closed in  $X$  then  $A$  is closed in  $X$ .

PROOF: Pick a closed set  $C$  in  $X$  such that  $A = C \cap Y$  (Theorem 122). Then  $A$  is the intersection of two sets closed in  $X$ , hence  $A$  is closed in  $X$  (Lemma 19).  $\square$

**Theorem 128.** *Let  $\{X_i\}_{i \in I}$  be a family of topological spaces. Let  $A_i$  be a subspace of  $X_i$  for all  $i \in I$ . Then the product topology on  $\prod_{i \in I} A_i$  is the same as the topology it inherits as a subspace of  $\prod_{i \in I} X_i$ .*

PROOF: The product topology is generated by the subbasis

$$\begin{aligned} & \{\pi_i^{-1}(U) \mid i \in I, U \text{ open in } A_i\} \\ &= \{\pi_i^{-1}(V) \cap A_i \mid i \in I, V \text{ open in } X_i\} \\ &= \{\pi_i^{-1}(V) \mid i \in I, V \text{ open in } X_i\} \cap \prod_{i \in I} A_i \end{aligned}$$

and this is a subbasis for the subspace topology by Lemma 125.  $\square$

**Theorem 129.** *Let  $X$  be an ordered set in the order topology. Let  $Y \subseteq X$  be an interval. Then the order topology on  $Y$  is the same as the subspace topology on  $Y$ .*

PROOF:

- $\langle 1 \rangle 1$ . The order topology is finer than the subspace topology.
- $\langle 2 \rangle 1$ . For every open ray  $R$  in  $X$ , the set  $R \cap Y$  is open in the order topology.
- $\langle 3 \rangle 1$ . For all  $a \in X$ , we have  $(-\infty, a) \cap Y$  is open in the order topology.
- $\langle 4 \rangle 1$ . CASE: For all  $y \in Y$  we have  $y < a$   
PROOF: In this case  $(-\infty, a) \cap Y = Y$ .
- $\langle 4 \rangle 2$ . CASE: For all  $y \in Y$  we have  $a < y$   
PROOF: In this case  $(-\infty, a) \cap Y = \emptyset$ .
- $\langle 4 \rangle 3$ . CASE: There exists  $y \in Y$  such that  $y \leq a$  and  $y \in Y$  such that  $a \leq y$
- $\langle 5 \rangle 1$ .  $a \in Y$   
PROOF: Because  $Y$  is an interval.
- $\langle 5 \rangle 2$ .  $(-\infty, a) \cap Y = \{y \in Y \mid y < a\}$
- $\langle 3 \rangle 2$ . For all  $a \in X$ , we have  $(a, +\infty) \cap Y$  is open in the order topology.  
PROOF: Similar.
- $\langle 2 \rangle 2$ . Q.E.D.  
PROOF: By Lemmas 103 and 125 and Proposition 56.
- $\langle 1 \rangle 2$ . The subspace topology is finer than the order topology.
- $\langle 2 \rangle 1$ . Every open ray in  $Y$  is open in the subspace topology.  
PROOF: For any  $a \in Y$  we have  $(-\infty, a)_Y = (-\infty, a)_X \cap Y$  and  $(a, +\infty)_Y = (a, +\infty)_X \cap Y$ .
- $\langle 2 \rangle 2$ . Q.E.D.  
PROOF: By Lemma 103 and Proposition 56

$\square$

This example shows that we cannot remove the hypothesis that  $Y$  is an interval:

**Example 130.** The order topology on  $I_o^2$  is different from the subspace topology as a subspace of  $\mathbb{R}^2$  under the dictionary order topology.

PROOF: The set  $\{1/2\} \times (1/2, 1)$  is open in the subspace topology but not in the order topology.  $\square$

**Proposition 131.** *Let  $X$  be a topological space,  $Y$  a subspace of  $X$ , and  $Z$  a subspace of  $Y$ . Then the subspace topology on  $Z$  inherited from  $X$  is the same as the subspace topology on  $Z$  inherited from  $Y$ .*

PROOF: The subspace topology inherited from  $Y$  is

$$\begin{aligned} & \{V \cap Z \mid V \text{ open in } Y\} \\ &= \{U \cap Y \cap Z \mid U \text{ open in } X\} \\ &= \{U \cap Z \mid U \text{ open in } X\} \end{aligned}$$

which is the subspace topology inherited from  $X$ .  $\square$

**Definition 132** (Unit Circle). The *unit circle*  $S^1$  is  $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$  as a subspace of  $\mathbb{R}^2$ .

**Definition 133** (Unit 2-sphere). The *unit 2-sphere* is  $S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$  as a subspace of  $\mathbb{R}^3$ .

## 19 The Box Topology

**Definition 134** (Box Topology). Let  $\{A_i\}_{i \in I}$  be a family of topological spaces. The *box topology* on  $\prod_{i \in I} A_i$  is the topology generated by the set of all sets of the form  $\prod_{i \in I} U_i$  where  $\{U_i\}_{i \in I}$  is a family such that each  $U_i$  is open in  $A_i$ .

This is a basis since it covers  $\prod_{i \in I} A_i$  and is closed under intersection.

**Proposition 135.** *The box topology is finer than the product topology.*

PROOF: From Proposition 109.  $\square$

**Corollary 135.1.** *If  $A_i$  is closed in  $X_i$  for all  $i \in I$  then  $\prod_{i \in I} A_i$  is closed in  $\prod_{i \in I} X_i$  under the box topology.*

PROOF: From Proposition 110.

**Proposition 136** (AC). *Let  $\{A_i\}_{i \in I}$  be a family of topological spaces. For  $i \in I$ , let  $\mathcal{B}_i$  be a basis for the topology on  $A_i$ . Then  $\mathcal{B} = \{\prod_{i \in I} B_i \mid \forall i \in I, B_i \in \mathcal{B}_i\}$  is a basis for the box topology on  $\prod_{i \in I} A_i$ .*

PROOF:

- $\langle 1 \rangle 1$ . Every set of the form  $\prod_{i \in I} B_i$  is open.
- $\langle 1 \rangle 2$ . For every point  $a \in \prod_{i \in I} A_i$  and every open set  $U$  with  $a \in U$ , there exists  $B \in \mathcal{B}$  such that  $a \in B \subseteq U$ .
  - $\langle 2 \rangle 1$ . LET:  $U$  be open and  $a \in U$
  - $\langle 2 \rangle 2$ . PICK a family  $\{U_i\}_{i \in I}$  such that each  $U_i$  is open in  $A_i$  and  $a \in \prod_{i \in I} U_i \subseteq U$ .
  - $\langle 2 \rangle 3$ . For  $i \in I$ , PICK  $B_i \in \mathcal{B}_i$  such that  $a_i \in B_i \subseteq U_i$

PROOF: Using the Axiom of Choice.



- ⟨2⟩4.  $a \in \prod_{i \in I} B_i \subseteq U$   
 ⟨1⟩3. Q.E.D.

PROOF: Lemma 48.

□

**Theorem 137.** *Let  $\{X_i\}_{i \in I}$  be a family of topological spaces. Let  $A_i$  be a subspace of  $X_i$  for all  $i \in I$ . Give  $\prod_{i \in I} X_i$  the box topology. Then the box topology on  $\prod_{i \in I} A_i$  is the same as the topology it inherits as a subspace of  $\prod_{i \in I} X_i$ .*

PROOF: The box topology is generated by the basis

$$\begin{aligned} & \left\{ \prod_{i \in I} U_i \mid \forall i \in I, U_i \text{ open in } A_i \right\} \\ &= \left\{ \prod_{i \in I} (V_i \cap A_i) \mid \forall i \in I, V_i \text{ open in } X_i \right\} \\ &= \left\{ \prod_{i \in I} V_i \mid \forall i \in I, V_i \text{ open in } X_i \right\} \cap \prod_{i \in I} A_i \end{aligned}$$

and this is a basis for the subspace topology by Lemma 124. □

**Proposition 138.** *Let  $\{X_i\}_{i \in I}$  be a family of Hausdorff spaces. Then  $\prod_{i \in I} X_i$  under the box topology is Hausdorff.*

PROOF:

- ⟨1⟩1. LET:  $\{X_i\}_{i \in I}$  be a family of Hausdorff spaces.  
 ⟨1⟩2. LET:  $a, b \in \prod_{i \in I} X_i$  with  $a \neq b$   
 ⟨1⟩3. PICK  $i \in I$  such that  $a_i \neq b_i$   
 ⟨1⟩4. PICK  $U, V$  disjoint open sets in  $X_i$  with  $a_i \in U$  and  $b_i \in V$   
 ⟨1⟩5.  $\pi_i^{-1}(U)$  and  $\pi_i^{-1}(V)$  are disjoint open sets in  $\prod_{i \in I} X_i$  with  $a \in \pi_i^{-1}(U)$  and  $b \in \pi_i^{-1}(V)$

□

**Proposition 139 (AC).** *Let  $\{X_i\}_{i \in I}$  be a family of topological spaces. Give  $\prod_{i \in I} X_i$  the box topology. Let  $A_i \subseteq X_i$  for all  $i \in I$ . Then*

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i}.$$

PROOF:

- ⟨1⟩1.  $\overline{\prod_{i \in I} A_i} \subseteq \prod_{i \in I} \overline{A_i}$   
 ⟨2⟩1. For all  $i \in I$  we have  $A_i \subseteq \overline{A_i}$   
 PROOF: Lemma 31.  
 ⟨2⟩2.  $\prod_{i \in I} A_i \subseteq \prod_{i \in I} \overline{A_i}$   
 ⟨2⟩3. Q.E.D.  
 PROOF: Since  $\prod_{i \in I} A_i$  is closed by Corollary 135.1.  
 ⟨1⟩2.  $\prod_{i \in I} \overline{A_i} \subseteq \overline{\prod_{i \in I} A_i}$   
 ⟨2⟩1. LET:  $x \in \prod_{i \in I} \overline{A_i}$   
 ⟨2⟩2. LET:  $U$  be a neighbourhood of  $x$

⟨2⟩3. PICK  $V_i$  open in  $X_i$  such that  $x \in \prod_{i \in I} V_i \subseteq U$

⟨2⟩4. For  $i \in I$ , pick  $a_i \in V_i \cap A_i$

PROOF: By Theorem 34 and ⟨2⟩1 using the Axiom of Choice.

⟨2⟩5.  $U$  intersects  $\prod_{i \in I} A_i$

⟨2⟩6. Q.E.D.

PROOF:  $a \in U \cap \prod_{i \in I} A_i$ .

□

The following example shows that Theorem 88 fails in the box topology.

**Example 140.** Define  $f : \mathbb{R} \rightarrow \mathbb{R}^\omega$  by  $f(t) = (t, t, \dots)$ . Then  $\pi_n \circ f = \text{id}_{\mathbb{R}}$  is continuous for all  $n$ . But  $f$  is not continuous when  $\mathbb{R}^\omega$  is given the box topology because the inverse image of

$$(-1, 1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times \dots$$

is  $\{0\}$  which is not open.

The following example shows that Proposition 117 fails in the box topology.

**Example 141.** Give  $\mathbb{R}^\omega$  the box topology. Let  $a_n = (1/n, 1/n, \dots)$  for  $n \geq 1$  and  $l = (0, 0, \dots)$ . Then  $\pi_i(a_n) \rightarrow \pi_i(l)$  as  $n \rightarrow \infty$  for all  $i$ , but  $a_n \not\rightarrow l$  as  $n \rightarrow \infty$  since the open set

$$(-1, 1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times \dots$$

contains  $l$  but does not contain any  $a_n$ .

**Example 142.** The set  $\mathbb{R}^\infty$  is closed in  $\mathbb{R}^\omega$  under the box topology. For let  $(a_n)$  be any sequence not in  $\mathbb{R}^\infty$ . Let  $U_n$  be an open interval around  $a_n$  that does not contain 0 if  $a_n \neq 0$ , and  $U_n = \mathbb{R}$  if  $a_n = 0$ . Then  $\prod_{n \geq 0} U_n$  is a neighbourhood of  $(a_n)$  that does not intersect  $\mathbb{R}^\infty$ .

## 20 $T_1$ Spaces

**Definition 143** ( $T_1$  Space). A topological space is  $T_1$  if and only if every singleton is closed.

**Lemma 144.** A space is  $T_1$  if and only if every finite set is closed.

PROOF: From Lemma 20. □

**Theorem 145.** In a  $T_1$  space, a point  $a$  is a limit point of a set  $A$  if and only if every neighbourhood of  $a$  contains infinitely many points of  $A$ .

PROOF:

⟨1⟩1. If  $a$  is a limit point of  $A$  then every neighbourhood of  $a$  contains infinitely many points of  $A$ .

⟨2⟩1. ASSUME:  $a$  is a limit point of  $A$ .

⟨2⟩2. LET:  $U$  be a neighbourhood of  $a$ .

⟨2⟩3. ASSUME: for a contradiction  $U$  contains only finitely many points of  $A$ .

⟨2⟩4.  $(U \cap A) \setminus \{a\}$  is closed.

PROOF: By the  $T_1$  axiom.

⟨2⟩5.  $(U \setminus A) \cup \{a\}$  is open.

PROOF: It is  $U \setminus ((U \cap A) \setminus \{a\})$ .

⟨2⟩6.  $(U \setminus A) \cup \{a\}$  intersects  $A$  in a point other than  $a$ .

PROOF: From ⟨2⟩1.

⟨2⟩7. Q.E.D.

□

⟨1⟩2. If every neighbourhood of  $a$  contains infinitely many points of  $A$  then  $a$  is a limit point of  $A$ .

PROOF: Immediate from definitions.

□

(To see this does not hold in every space, see Proposition 45.)

**Proposition 146.** *A space is  $T_1$  if and only if, for any two distinct points  $x$  and  $y$ , there exist neighbourhoods  $U$  of  $x$  and  $V$  of  $y$  such that  $x \notin V$  and  $y \notin U$ .*

PROOF:

⟨1⟩1. LET:  $X$  be a topological space.

⟨1⟩2. If  $X$  is  $T_1$  then, for any two distinct points  $x$  and  $y$ , there exist neighbourhoods  $U$  of  $x$  and  $V$  of  $y$  such that  $x \notin V$  and  $y \notin U$ .

PROOF: This holds because  $\{x\}$  and  $\{y\}$  are closed.

⟨1⟩3. Suppose, for any two distinct points  $x$  and  $y$ , there exist neighbourhoods  $U$  of  $x$  and  $V$  of  $y$  such that  $x \notin V$  and  $y \notin U$ . Then  $X$  is  $T_1$ .

⟨2⟩1. ASSUME: For any two distinct points  $x$  and  $y$ , there exist neighbourhoods  $U$  of  $x$  and  $V$  of  $y$  such that  $x \notin V$  and  $y \notin U$ .

⟨2⟩2. LET:  $a \in X$

⟨2⟩3.  $\{a\}$  is closed.

PROOF: For all  $b \neq a$  there exists a neighbourhood  $U$  of  $b$  such that  $U \subseteq X \setminus \{a\}$ .

□

## 21 Hausdorff Spaces

**Definition 147** (Hausdorff Space). A topological space is *Hausdorff* if and only if, for any points  $x, y$  with  $x \neq y$ , there exist disjoint open sets  $U$  and  $V$  such that  $x \in U$  and  $y \in V$ .

**Theorem 148.** *Every Hausdorff space is  $T_1$ .*

PROOF:

⟨1⟩1. LET:  $X$  be a Hausdorff space.

⟨1⟩2. LET:  $b \in X$

PROVE:  $\overline{\{b\}} = \{b\}$

⟨1⟩3. ASSUME:  $a \in \overline{\{b\}}$  and  $a \neq b$

⟨1⟩4. PICK disjoint neighbourhoods  $U$  of  $a$  and  $V$  of  $b$ .

⟨1⟩5.  $U$  intersects  $\{b\}$

PROOF: Theorem 34.

⟨1⟩6.  $b \in U$

⟨1⟩7. Q.E.D.

PROOF: This contradicts the fact that  $U$  and  $V$  are disjoint (⟨1⟩4).

□

**Proposition 149.** *An infinite set under the finite complement topology is  $T_1$  but not Hausdorff.*

PROOF:

⟨1⟩1. LET:  $X$  be an infinite set under the finite complement topology.

⟨1⟩2. Every singleton is closed.

PROOF: By definition.

⟨1⟩3. PICK  $a, b \in X$  with  $a \neq b$

⟨1⟩4. There are no disjoint neighbourhoods  $U$  of  $a$  and  $V$  of  $b$ .

⟨2⟩1. LET:  $U$  be a neighbourhood of  $a$  and  $V$  a neighbourhood of  $b$ .

⟨2⟩2.  $X \setminus U$  and  $X \setminus V$  are finite.

⟨2⟩3. PICK  $c \in X$  that is not in  $X \setminus U$  or  $X \setminus V$ .

⟨2⟩4.  $c \in U \cap V$

□

**Proposition 150.** *The product of a family of Hausdorff spaces is Hausdorff.*

PROOF:

⟨1⟩1. LET:  $\{X_i\}_{i \in I}$  be a family of Hausdorff spaces.

⟨1⟩2. LET:  $a, b \in \prod_{i \in I} X_i$  with  $a \neq b$

⟨1⟩3. PICK  $i \in I$  such that  $a_i \neq b_i$

⟨1⟩4. PICK  $U, V$  disjoint open sets in  $X_i$  with  $a_i \in U$  and  $b_i \in V$

⟨1⟩5.  $\pi_i^{-1}(U)$  and  $\pi_i^{-1}(V)$  are disjoint open sets in  $\prod_{i \in I} X_i$  with  $a \in \pi_i^{-1}(U)$  and  $b \in \pi_i^{-1}(V)$

□

**Theorem 151.** *Every linearly ordered set under the order topology is Hausdorff.*

PROOF:

⟨1⟩1. LET:  $X$  be a linearly ordered set under the order topology.

⟨1⟩2. LET:  $a, b \in X$  with  $a \neq b$

⟨1⟩3. ASSUME: w.l.o.g.  $a < b$

⟨1⟩4. CASE: There exists  $c$  such that  $a < c < b$

PROOF: The sets  $(-\infty, c)$  and  $(c, +\infty)$  are disjoint neighbourhoods of  $a$  and  $b$  respectively.

⟨1⟩5. CASE: There is no  $c$  such that  $a < c < b$

PROOF: The sets  $(-\infty, b)$  and  $(a, +\infty)$  are disjoint neighbourhoods of  $a$  and  $b$  respectively.

□

**Theorem 152.** *A subspace of a Hausdorff space is Hausdorff.*

PROOF:

- ⟨1⟩1. LET:  $X$  be a Hausdorff space and  $Y$  a subspace of  $X$ .
- ⟨1⟩2. LET:  $x, y \in Y$  with  $x \neq y$
- ⟨1⟩3. PICK disjoint neighbourhoods  $U$  of  $x$  and  $V$  of  $y$  in  $X$ .
- ⟨1⟩4.  $U \cap Y$  and  $V \cap Y$  are disjoint neighbourhoods of  $x$  and  $y$  respectively in  $Y$ .

□

**Proposition 153.** *A space  $X$  is Hausdorff if and only if the diagonal  $\Delta = \{(x, x) \mid x \in X\}$  is closed in  $X^2$ .*

PROOF:

$X$  is Hausdorff

$$\Leftrightarrow \forall x, y \in X. x \neq y \Rightarrow \exists V, W \text{ open. } x \in V \wedge y \in W \wedge V \cap W = \emptyset$$

$$\Leftrightarrow \forall (x, y) \in X^2 \setminus \Delta. \exists V, W \text{ open. } (x, y) \subseteq V \times W \subseteq X^2 \setminus \Delta$$

$$\Leftrightarrow \Delta \text{ is closed}$$

□

## 22 The First Countability Axiom

**Definition 154** (First Countability Axiom). A topological space  $X$  satisfies the *first countability axiom*, or is *first countable*, if and only if every point has a countable local basis.

**Lemma 155** (Sequence Lemma (CC)). *Let  $X$  be a first countable space. Let  $A \subseteq X$  and  $l \in A$ . Then there exists a sequence in  $A$  that converges to  $l$ .*

PROOF:

- ⟨1⟩1. PICK a countable local basis  $\{B_n \mid n \in \mathbb{Z}^+\}$  at  $l$  such that  $B_1 \supseteq B_2 \supseteq \dots$ .

PROOF: Lemma 58.

- ⟨1⟩2. For all  $n \geq 1$ , PICK  $a_n \in A \cap B_n$ .

PROVE:  $a_n \rightarrow l$  as  $n \rightarrow \infty$

- ⟨1⟩3. LET:  $U$  be a neighbourhood of  $l$

- ⟨1⟩4. PICK  $N$  such that  $B_N \subseteq U$

- ⟨1⟩5. For  $n \geq N$  we have  $a_n \in U$

PROOF:  $a_n \in B_n \subseteq B_N \subseteq U$

□

**Theorem 156** (CC). *Let  $X$  be a first countable space and  $Y$  a topological space. Let  $f : X \rightarrow Y$ . Suppose that, for every sequence  $(x_n)$  in  $X$  and  $l \in X$ , if  $x_n \rightarrow l$  as  $n \rightarrow \infty$ , then  $f(x_n) \rightarrow f(l)$  as  $n \rightarrow \infty$ . Then  $f$  is continuous.*

PROOF:

- ⟨1⟩1. LET:  $A \subseteq X$

- ⟨1⟩2. LET:  $a \in A$

PROVE:  $f(a) \in \overline{f(A)}$

- ⟨1⟩3. PICK a sequence  $(x_n)$  in  $A$  that converges to  $a$ .

PROOF: By the Sequence Lemma.

⟨1⟩4.  $f(x_n) \rightarrow f(a)$

⟨1⟩5.  $f(a) \in f(A)$

PROOF: By Lemma 62.

⟨1⟩6. Q.E.D.

PROOF: By Theorem 80.

□

**Example 157 (CC).** The space  $\mathbb{R}^\omega$  under the box product is not first countable.

PROOF: Let  $\{B_n \mid n \geq 0\}$  be a countable set of neighbourhoods of  $\vec{0}$ . We will construct a neighbourhood of  $\vec{0}$  that does not include any of these.

For  $n \geq 0$ , pick a neighbourhood  $U_n$  of 0 such that  $U_n \subset \pi_n(B_p)$ . Then  $\prod_{n=0}^\infty U_n$  is a neighbourhood of  $\vec{0}$  that does not include any  $B_n$ . □

**Example 158.** If  $J$  is an uncountable set then  $\mathbb{R}^J$  is not first countable.

PROOF:

⟨1⟩1. LET:  $\{B_n \mid n \geq 0\}$  be any countable set of neighbourhoods of  $\vec{0}$ .

⟨1⟩2. For  $n \geq 0$ , PICK a basis element  $\prod_{\alpha \in J} U_{n\alpha}$  that contains  $\vec{0}$  and is included in  $B_n$ .

PROOF: Using the Axiom of Countable Choice.

⟨1⟩3. For  $n \geq 0$ ,

LET:  $J_n = \{\alpha \in J \mid U_{n\alpha} \neq \mathbb{R}\}$

⟨1⟩4. PICK  $\beta \in J$  such that  $\beta \notin J_n$  for any  $n$ .

PROOF: Since each  $J_n$  is finite so  $\bigcup_n J_n$  is countable.

⟨1⟩5.  $\pi_\beta((-1, 1))$  is a neighbourhood of  $\vec{0}$  that does not include any  $B_n$ .

□

**Example 159.** The space  $\mathbb{R}_l$  is first countable.

PROOF: For any  $a \in \mathbb{R}$ , the set  $\{[a, a + 1/n) \mid n \geq 1\}$  is a countable local basis.

**Example 160.** The ordered square is first countable.

PROOF: For any  $(a, b) \in I_o^2$  with  $b \neq 0, 1$ , the set  $\{(\{a\} \times (b - 1/n, b + 1/n)) \cap I_o^2 \mid n \geq 1\}$  is a countable local basis.

## 23 Strong Continuity

**Definition 161 (Strongly Continuous).** Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$  be a function. Then  $f$  is *strongly continuous* if and only if, for every subset  $U \subseteq Y$ , we have  $U$  is open in  $Y$  if and only if  $f^{-1}(U)$  is open in  $X$ .

**Proposition 162.** Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$  be a function. Then  $f$  is strongly continuous if and only if, for every subset  $C \subseteq Y$ , we have  $C$  is closed in  $Y$  if and only if  $f^{-1}(C)$  is closed in  $X$ .

PROOF: Since  $X \setminus f^{-1}(C) = f^{-1}(Y \setminus C)$ . □

## 24 Saturated Sets

**Definition 163.** Let  $X$  and  $Y$  be sets and  $p : X \rightarrow Y$  a surjective function. Let  $C \subseteq X$ . Then  $C$  is *saturated* with respect to  $p$  if and only if, for all  $x, y \in X$ , if  $x \in C$  and  $p(x) = p(y)$  then  $y \in C$ .

**Proposition 164.** Let  $X$  and  $Y$  be sets and  $p : X \rightarrow Y$  a surjective function. Let  $C \subseteq X$ . Then the following are equivalent:

1.  $C$  is saturated with respect to  $p$ .
2. There exists  $D \subseteq Y$  such that  $C = p^{-1}(D)$
3.  $C = p^{-1}(p(C))$ .

PROOF:

$\langle 1 \rangle 1. 1 \Rightarrow 3$

$\langle 2 \rangle 1.$  ASSUME:  $C$  is saturated with respect to  $p$ .

$\langle 2 \rangle 2. C \subseteq p^{-1}(p(C))$

PROOF: Trivial.

$\langle 2 \rangle 3. p^{-1}(p(C)) \subseteq C$

$\langle 3 \rangle 1.$  LET:  $x \in p^{-1}(p(C))$

$\langle 3 \rangle 2. p(x) \in p(C)$

$\langle 3 \rangle 3.$  There exists  $y \in C$  such that  $p(x) = p(y)$

$\langle 3 \rangle 4. x \in C$

PROOF: From  $\langle 2 \rangle 1$ .

$\langle 1 \rangle 2. 3 \Rightarrow 2$

PROOF: Trivial.

$\langle 1 \rangle 3. 2 \Rightarrow 1$

PROOF: This follows because if  $p(x) \in D$  and  $p(x) = p(y)$  then  $p(y) \in D$ .

□

## 25 Quotient Maps

**Definition 165** (Quotient Map). Let  $X$  and  $Y$  be topological spaces and  $p : X \rightarrow Y$ . Then  $p$  is a *quotient map* if and only if  $p$  is surjective and strongly continuous.

**Proposition 166.** Let  $X$  and  $Y$  be topological spaces and  $p : X \rightarrow Y$  be a surjective function. Then the following are equivalent.

1.  $p$  is a quotient map.
2.  $p$  is continuous and maps saturated open sets to open sets.
3.  $p$  is continuous and maps saturated closed sets to closed sets.

PROOF:

$\langle 1 \rangle 1. 1 \Rightarrow 2$

⟨2⟩1. ASSUME:  $p$  is a quotient map.  
 ⟨2⟩2. LET:  $U$  be a saturated open set in  $X$ .  
 ⟨2⟩3.  $p^{-1}(p(U))$  is open in  $X$ .  
 PROOF: Since  $U = p^{-1}(p(U))$  be Proposition 164.  
 ⟨2⟩4.  $p(U)$  is open in  $Y$ .  
 PROOF: From ⟨2⟩1.  
 ⟨1⟩2.  $1 \Rightarrow 3$   
 PROOF: Similar.  
 ⟨1⟩3.  $2 \Rightarrow 1$   
 ⟨2⟩1. ASSUME:  $p$  is continuous and maps saturated open sets to open sets.  
 ⟨2⟩2. LET:  $U \subseteq Y$   
 ⟨2⟩3. ASSUME:  $p^{-1}(U)$  is open in  $X$   
 ⟨2⟩4.  $p^{-1}(U)$  is saturated.  
 PROOF: Proposition 164.  
 ⟨2⟩5.  $U$  is open in  $Y$ .  
 ⟨1⟩4.  $3 \Rightarrow 1$   
 PROOF: Similar.  
 □

**Corollary 166.1.** *Every surjective continuous open map is a quotient map.*

**Corollary 166.2.** *Every surjective continuous closed map is a quotient map.*

**Example 167.** The converses of these corollaries do not hold.

Let  $A = \{(x, y) \mid x \geq 0\} \cup \{(x, y) \mid y = 0\}$ . Then  $\pi_1 : A \rightarrow \mathbb{R}$  is a quotient map, but not an open map or a closed map.

We prove that  $\pi_1$  maps saturated open sets to open sets:

⟨1⟩1. LET:  $\pi_1^{-1}(U)$  be a saturated open set in  $A$   
 PROVE:  $U$  is open in  $\mathbb{R}$   
 ⟨1⟩2. LET:  $x \in U$   
 ⟨1⟩3.  $(x, 0) \in \pi_1^{-1}(U)$   
 ⟨1⟩4. PICK  $W, V$  open in  $\mathbb{R}$  such that  $(x, 0) \in W \times V \subseteq \pi_1^{-1}(U)$   
 ⟨1⟩5.  $x \in W \subseteq U$

It is not an open map because it maps  $((-1, 1) \times (1, 2)) \cap A$  to  $[0, 1)$ .

It is not a closed map because it maps  $\{(x, 1/x) \mid x > 0\}$  to  $(0, +\infty)$ .

**Proposition 168.** *Let  $p : X \rightarrow Y$  be a quotient map. Let  $A \subseteq X$  be saturated with respect to  $p$ . Let  $q : A \rightarrow p(A)$  be the restriction of  $p$ .*

1. *If  $A$  is either open or closed in  $X$  then  $q$  is a quotient map.*

2. *If  $p$  is either an open map or a closed map then  $q$  is a quotient map.*

PROOF:

⟨1⟩1. LET:  $p : X \rightarrow Y$  be a quotient map.  
 ⟨1⟩2. LET:  $A \subseteq X$  be saturated with respect to  $p$ .  
 ⟨1⟩3. LET:  $q : A \rightarrow p(A)$  be the restriction of  $p$ .



$\langle 1 \rangle 4$ .  $q$  is continuous.  
 PROOF: Theorem 84.

$\langle 1 \rangle 5$ . If  $A$  is open in  $X$  then  $q$  is a quotient map.

$\langle 2 \rangle 1$ . ASSUME:  $A$  is open in  $X$ .

$\langle 2 \rangle 2$ .  $q$  maps saturated open sets to open sets.

$\langle 3 \rangle 1$ . LET:  $U \subseteq A$  be saturated with respect to  $q$  and open in  $A$

$\langle 3 \rangle 2$ .  $U$  is saturated with respect to  $p$

$\langle 4 \rangle 1$ . LET:  $x, y \in X$

$\langle 4 \rangle 2$ . ASSUME:  $x \in U$

$\langle 4 \rangle 3$ . ASSUME:  $p(x) = p(y)$

$\langle 4 \rangle 4$ .  $x \in A$

PROOF: From  $\langle 3 \rangle 1$  and  $\langle 4 \rangle 2$ .

$\langle 4 \rangle 5$ .  $y \in A$

PROOF: From  $\langle 1 \rangle 2$  and  $\langle 4 \rangle 3$

$\langle 4 \rangle 6$ .  $q(x) = x(y)$

PROOF: From  $\langle 1 \rangle 3$ ,  $\langle 4 \rangle 3$ ,  $\langle 4 \rangle 4$ ,  $\langle 4 \rangle 5$ .

$\langle 4 \rangle 7$ .  $y \in U$

PROOF: From  $\langle 3 \rangle 1$ ,  $\langle 4 \rangle 2$ ,  $\langle 4 \rangle 6$

$\langle 3 \rangle 3$ .  $U$  is open in  $X$

PROOF: Lemma 126,  $\langle 2 \rangle 1$ ,  $\langle 3 \rangle 1$ .

$\langle 3 \rangle 4$ .  $p(U)$  is open in  $Y$

PROOF: Proposition 166,  $\langle 1 \rangle 1$ ,  $\langle 3 \rangle 2$ ,  $\langle 3 \rangle 3$

$\langle 3 \rangle 5$ .  $q(U)$  is open in  $p(A)$

PROOF: Since  $q(U) = p(U) = p(U) \cap p(A)$ .

$\langle 2 \rangle 3$ . Q.E.D.

PROOF: By Proposition 166.

$\langle 1 \rangle 6$ . If  $A$  is closed in  $X$  then  $q$  is a quotient map.

PROOF: Similar.

$\langle 1 \rangle 7$ . If  $p$  is an open map then  $q$  is a quotient map.

$\langle 2 \rangle 1$ . ASSUME:  $p$  is an open map

$\langle 2 \rangle 2$ .  $q$  maps saturated open sets to open sets.

$\langle 3 \rangle 1$ . LET:  $U$  be open in  $A$  and saturated with respect to  $q$

$\langle 3 \rangle 2$ . PICK  $V$  open in  $X$  such that  $U = A \cap V$

$\langle 3 \rangle 3$ .  $p(V)$  is open in  $Y$

$\langle 3 \rangle 4$ .  $q(U) = p(V) \cap p(A)$

$\langle 4 \rangle 1$ .  $q(U) \subseteq p(V) \cap p(A)$

PROOF: From  $\langle 3 \rangle 2$ .

$\langle 4 \rangle 2$ .  $p(V) \cap p(A) \subseteq q(U)$

$\langle 5 \rangle 1$ . LET:  $y \in p(V) \cap p(A)$

$\langle 5 \rangle 2$ . PICK  $x \in V$  and  $x' \in A$  such that  $p(x) = p(x') = y$

$\langle 5 \rangle 3$ .  $x \in A$

PROOF: By  $\langle 1 \rangle 2$ .

$\langle 5 \rangle 4$ .  $x \in U$

PROOF: From  $\langle 3 \rangle 2$

$\langle 2 \rangle 3$ . Q.E.D.

PROOF: By Proposition 166.

⟨1⟩8. If  $p$  is a closed map then  $q$  is a quotient map.

PROOF: Similar.

□

**Example 169.** This example shows we cannot remove the hypotheses on  $A$  and  $p$ .

Define  $f : [0, 1] \rightarrow [2, 3] \rightarrow [0, 2]$  by  $f(x) = x$  if  $x \leq 1$ ,  $f(x) = x - 1$  if  $x \geq 2$ . Then  $f$  is a quotient map but its restriction  $f'$  to  $[0, 1] \cup [2, 3]$  is not, because  $f'^{-1}([1, 2])$  is open but  $[1, 2]$  is not.

For a counterexample where  $A$  is saturated, see Example 172.

## 26 Quotient Topology

**Definition 170** (Quotient Topology). Let  $X$  be a topological space,  $Y$  a set and  $p : X \twoheadrightarrow Y$  be a surjective function. Then the *quotient topology* on  $Y$  is the unique topology on  $Y$  with respect to which  $p$  is a quotient map, namely  $\mathcal{T} = \{U \in \mathcal{P}Y \mid p^{-1}(U) \text{ is open in } X\}$ .

We prove this is a topology.

PROOF:

⟨1⟩1.  $Y \in \mathcal{T}$

PROOF: Since  $p^{-1}(Y) = X$  by surjectivity.

⟨1⟩2. For all  $\mathcal{A} \subseteq \mathcal{T}$  we have  $\bigcup \mathcal{A} \in \mathcal{T}$

PROOF: Since  $p^{-1}(\bigcup \mathcal{A}) = \bigcup_{U \in \mathcal{A}} p^{-1}(U)$

⟨1⟩3. For all  $U, V \in \mathcal{T}$  we have  $U \cap V \in \mathcal{T}$

PROOF: Since  $p^{-1}(U \cap V) = p^{-1}(U) \cap p^{-1}(V)$ .

□

**Definition 171** (Quotient Space). Let  $X$  be a topological space and  $\sim$  an equivalence relation on  $X$ . Let  $p : X \twoheadrightarrow X/\sim$  be the canonical surjection. Then  $X/\sim$  under the quotient topology is called a *quotient space*, *identification space* or *decomposition space* of  $X$ .

Here is a counterexample showing we cannot remove all the hypotheses of Proposition 168 except that  $A$  is saturated.

**Example 172.** Let  $X = (0, 1/2] \cup \{1\} \cup \{1 + 1/n : n \geq 2\}$  as a subspace of  $\mathbb{R}$ . Define  $R$  to be the equivalence relation on  $X$  where  $xRy$  iff  $(x = y \text{ or } |x - y| = 1)$ , so we identify  $1/n$  with  $1 + 1/n$  for all  $n \geq 2$ . Let  $Y$  be the resulting quotient space  $X/R$  in the quotient topology and  $p : X \twoheadrightarrow Y$  the canonical surjection.

Let  $A = \{1\} \cup (0, 1/2] \setminus \{1/n : n \geq 2\} \subseteq X$ . Then  $A$  is saturated under  $p$  but the restriction  $q$  of  $p$  to  $A$  is not a quotient map because it maps the saturated open set  $\{1\}$  to  $\{1\}$  which is not open in  $p(A)$ .

## 27 The Metric Topology

**Definition 173** (Metric). Let  $X$  be a set. A *metric* on  $X$  is a function  $d : X^2 \rightarrow \mathbb{R}$  such that:

1. For all  $x, y \in X$ ,  $d(x, y) \geq 0$
2. For all  $x, y \in X$ ,  $d(x, y) = 0$  if and only if  $x = y$
3. For all  $x, y \in X$ ,  $d(x, y) = d(y, x)$
4. (*Triangle Inequality*) For all  $x, y, z \in X$ ,  $d(x, z) \leq d(x, y) + d(y, z)$

We call  $d(x, y)$  the *distance* between  $x$  and  $y$ .

**Definition 174** (Open Ball). Let  $X$  be a metric space. Let  $a \in X$  and  $\epsilon > 0$ . The *open ball* with *centre*  $a$  and *radius*  $\epsilon$  is

$$B(a, \epsilon) = \{x \in X \mid d(a, x) < \epsilon\} .$$

**Definition 175** (Metric Topology). Let  $X$  be a metric space. The *metric topology* on  $X$  is the topology generated by the basis consisting of all the open balls.

We prove this is a basis for a topology.

PROOF:

$\langle 1 \rangle 1$ . For every point  $a$ , there exists a ball  $B$  such that  $a \in B$

PROOF: We have  $a \in B(a, 1)$ .

$\langle 1 \rangle 2$ . For any balls  $B_1, B_2$  and point  $a \in B_1 \cap B_2$ , there exists a ball  $B_3$  such that  $a \in B_3 \subseteq B_1 \cap B_2$

$\langle 2 \rangle 1$ . LET:  $B_1 = B(c_1, \epsilon_1)$  and  $B_2 = B(c_2, \epsilon_2)$

$\langle 2 \rangle 2$ . LET:  $\delta = \min(\epsilon_1 - d(c_1, a), \epsilon_2 - d(c_2, a))$

PROVE:  $B(a, \delta) \subseteq B_1 \cap B_2$

$\langle 2 \rangle 3$ . LET:  $x \in B(a, \delta)$

$\langle 2 \rangle 4$ .  $x \in B_1$

PROOF:

$$\begin{aligned} d(x, c_1) &= d(x, a) + d(a, c_1) \\ &< \delta + d(a, c_1) \\ &\leq \epsilon_1 \end{aligned}$$

$\langle 2 \rangle 5$ .  $x \in B_2$

PROOF: Similar.

□

**Proposition 176.** Let  $X$  be a metric space and  $U \subseteq X$ . Then  $U$  is open if and only if, for every  $x \in U$ , there exists  $\epsilon > 0$  such that  $B(x, \epsilon) \subseteq U$ .

PROOF:

$\langle 1 \rangle 1$ . If  $U$  is open then, for all  $x \in U$ , there exists  $\epsilon > 0$  such that  $B(x, \epsilon) \subseteq U$ .

$\langle 2 \rangle 1$ . ASSUME:  $U$  is open.

- ⟨2⟩2. LET:  $x \in U$   
 ⟨2⟩3. PICK  $a \in X$  and  $\delta > 0$  such that  $x \in B(a, \delta) \subseteq U$   
 ⟨2⟩4. LET:  $\epsilon = \delta - d(a, x)$   
         PROVE:  $B(x, \epsilon) \subseteq U$   
 ⟨2⟩5. LET:  $y \in B(x, \epsilon)$   
 ⟨2⟩6.  $d(y, a) < \delta$   
         PROOF:

$$\begin{aligned}
 d(y, a) &\leq d(a, x) + d(x, y) \\
 &< \delta + d(x, y) \\
 &= \epsilon
 \end{aligned}$$

- ⟨2⟩7.  $y \in U$   
 ⟨1⟩2. If, for all  $x \in U$ , there exists  $\epsilon > 0$  such that  $B(x, \epsilon) \subseteq U$ , then  $U$  is open.  
 PROOF: Immediate from definitions.

□

**Definition 177** (Discrete Metric). Let  $X$  be a set. The *discrete metric* on  $X$  is defined by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

**Proposition 178.** *The discrete metric induces the discrete topology.*

PROOF: For any (open) set  $U$  and point  $a \in U$ , we have  $a \in B(a, 1) \subseteq U$ . □

**Definition 179** (Standard Metric on  $\mathbb{R}$ ). The *standard metric* on  $\mathbb{R}$  is defined by  $d(x, y) = |x - y|$ .

**Proposition 180.** *The standard metric on  $\mathbb{R}$  induces the standard topology on  $\mathbb{R}$ .*

PROOF:

- ⟨1⟩1. Every open ball is open in the standard topology on  $\mathbb{R}$ .  
         PROOF:  $B(a, \epsilon) = (a - \epsilon, a + \epsilon)$   
 ⟨1⟩2. For every open set  $U$  and point  $a \in U$ , there exists  $\epsilon > 0$  such that  
          $B(a, \epsilon) \subseteq U$   
 ⟨2⟩1. LET:  $U$  be an open set and  $a \in U$   
 ⟨2⟩2. PICK an open interval  $b, c$  such that  $a \in (b, c) \subseteq U$   
 ⟨2⟩3. LET:  $\epsilon = \min(a - b, c - a)$   
 ⟨2⟩4.  $B(a, \epsilon) \subseteq U$

□

**Definition 181** (Metriizable). A topological space  $X$  is *metrizable* if and only if there exists a metric on  $X$  that induces the topology.

**Definition 182** (Bounded). Let  $X$  be a metric space and  $A \subseteq X$ . Then  $A$  is *bounded* if and only if there exists  $M$  such that, for all  $x, y \in A$ , we have  $d(x, y) \leq M$ .

**Definition 183** (Diameter). Let  $X$  be a metric space and  $A \subseteq X$ . The *diameter* of  $A$  is

$$\text{diam } A = \sup_{x,y \in A} d(x,y) .$$

**Definition 184** (Standard Bounded Metric). Let  $d$  be a metric on  $X$ . The *standard bounded metric* corresponding to  $d$  is the metric  $\bar{d}$  defined by

$$\bar{d}(x,y) = \min(d(x,y), 1) .$$

We prove this is a metric.

PROOF:

$\langle 1 \rangle 1. \bar{d}(x,y) \geq 0$

PROOF: Since  $d(x,y) \geq 0$

$\langle 1 \rangle 2. \bar{d}(x,y) = 0$  if and only if  $x = y$

PROOF:  $\bar{d}(x,y) = 0$  if and only if  $d(x,y) = 0$  if and only if  $x = y$

$\langle 1 \rangle 3. \bar{d}(x,y) = \bar{d}(y,x)$

PROOF: Since  $d(x,y) = d(y,x)$

$\langle 1 \rangle 4. \bar{d}(x,z) \leq \bar{d}(x,y) + \bar{d}(y,z)$

PROOF:

$$\begin{aligned} \bar{d}(x,y) + \bar{d}(y,z) &= \min(d(x,y), 1) + \min(d(y,z), 1) \\ &= \min(d(x,y) + d(y,z), d(x,y) + 1, d(y,z) + 1, 2) \\ &\geq \min(d(x,z), 1) \\ &= \bar{d}(x,z) \end{aligned}$$

□

**Lemma 185.** In any metric space  $X$ , the set  $\mathcal{B} = \{B(a, \epsilon) \mid a \in X, \epsilon < 1\}$  is a basis for the metric topology.

PROOF:

$\langle 1 \rangle 1.$  Every element of  $\mathcal{B}$  is open.

PROOF: From Lemma 47.

$\langle 1 \rangle 2.$  For every open set  $U$  and point  $a \in U$ , there exists  $B \in \mathcal{B}$  such that  $a \in B \subseteq U$

$\langle 2 \rangle 1.$  LET:  $U$  be an open set and  $a \in U$

$\langle 2 \rangle 2.$  PICK  $\epsilon > 0$  such that  $B(a, \epsilon) \subseteq U$

$\langle 2 \rangle 3.$   $B(a, \min(\epsilon, 1/2)) \subseteq U$

$\langle 1 \rangle 3.$  Q.E.D.

PROOF: Lemma 48.

□

**Proposition 186.** Let  $d$  be a metric on the set  $X$ . Then the standard bounded metric  $\bar{d}$  induces the same metric as  $d$ .

PROOF: This follows from Lemma 185 since the open balls with radius  $< 1$  are the same under both metrics. □

**Lemma 187.** *Let  $d$  and  $d'$  be two metrics on the same set  $X$ . Let  $\mathcal{T}$  and  $\mathcal{T}'$  be the topologies they induce. Then  $\mathcal{T} \subseteq \mathcal{T}'$  if and only if, for all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that*

$$B_{d'}(x, \delta) \subseteq B_d(x, \epsilon) .$$

PROOF:

$\langle 1 \rangle 1$ . If  $\mathcal{T} \subseteq \mathcal{T}'$  then, for all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$

PROOF: From Proposition 176 since  $x \in B_d(x, \epsilon) \in \mathcal{T}'$ .

$\langle 1 \rangle 2$ . If, for all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$ , then  $\mathcal{T} \subseteq \mathcal{T}'$

$\langle 2 \rangle 1$ . ASSUME: For all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$

$\langle 2 \rangle 2$ . LET:  $U \in \mathcal{T}$

$\langle 2 \rangle 3$ . For all  $x \in U$  there exists  $\delta > 0$  such that  $B_{d'}(x, \delta) \subseteq U$ .

$\langle 3 \rangle 1$ . LET:  $x \in U$

$\langle 3 \rangle 2$ . PICK  $\epsilon > 0$  such that  $B_d(x, \epsilon) \subseteq U$

PROOF: Proposition 176

$\langle 3 \rangle 3$ . PICK  $\delta > 0$  such that  $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$

PROOF: By  $\langle 2 \rangle 1$

$\langle 3 \rangle 4$ .  $B_{d'}(x, \delta) \subseteq U$

$\langle 2 \rangle 4$ .  $U \in \mathcal{T}'$

PROOF: Proposition 176.

□

**Proposition 188.**  $\mathbb{R}^2$  under the dictionary order topology is metrizable.

PROOF: Define  $d : \mathbb{R}^2 \rightarrow \mathbb{R}$  by

$$d((x, y), (x, z)) = \max(|y - z|, 1)$$

$$d((x, y), (x', y')) = 1$$

$$\text{if } x \neq x' \square$$

$\langle 1 \rangle 1$ .  $x \in \bigcap_{i=1}^N \pi_i^{-1}() \subseteq B_D(a, \epsilon)$

**Proposition 189.** *Let  $d : X^2 \rightarrow \mathbb{R}$  be a metric on  $X$ . Then the metric topology on  $X$  is the coarsest topology such that  $d$  is continuous.*

PROOF:

$\langle 1 \rangle 1$ .  $d$  is continuous.

$\langle 2 \rangle 1$ . LET:  $a, b \in X$

$\langle 2 \rangle 2$ . LET:  $\epsilon > 0$

$\langle 2 \rangle 3$ . LET:  $\delta = \epsilon/2$

$\langle 2 \rangle 4$ . LET:  $x, y \in X$

$\langle 2 \rangle 5$ . ASSUME:  $\rho((a, b), (x, y)) < \delta$

$\langle 2 \rangle 6$ .  $|d(a, b) - d(x, y)| < \epsilon$

$\langle 3 \rangle 1$ .  $d(a, b) - d(x, y) < \epsilon$

PROOF:

$$\begin{aligned}
d(a, b) &\leq d(a, x) + d(x, y) + d(y, b) \\
&\leq d(x, y) + 2\rho((a, b), (x, y)) \\
&< d(x, y) + 2\delta \\
&= d(x, y) + \epsilon
\end{aligned}$$

$$\langle 3 \rangle 2. \ d(a, b) - d(x, y) > -\epsilon$$

PROOF: Similar.

$\langle 2 \rangle 7.$  Q.E.D.

$\langle 1 \rangle 2.$  If  $\mathcal{T}$  is any topology under which  $d$  is continuous then  $\mathcal{T}$  is finer than the metric topology.

PROOF: Since  $B(a, \epsilon) = d_a^{-1}((-\infty, \epsilon))$

□

**Proposition 190.** *Let  $X$  be a metric space with metric  $d$  and  $A \subseteq X$ . The restriction of  $d$  to  $A$  is a metric on  $A$  that induces the subspace topology.*

PROOF:

$\langle 1 \rangle 1.$  The restriction of  $d$  to  $A$  is a metric on  $A$ .

$\langle 1 \rangle 2.$  Every open ball under  $d \upharpoonright A$  is open under the subspace topology.

PROOF:  $B_{d \upharpoonright A}(a, \epsilon) = B_d(a, \epsilon) \cap A$ .

$\langle 1 \rangle 3.$  If  $U$  is open in the subspace topology and  $x \in U$ , then there exists a  $d \upharpoonright A$ -ball  $B$  such that  $x \in B \subseteq U$ .

$\langle 2 \rangle 1.$  PICK  $V$  open in  $X$  such that  $U = V \cap A$

$\langle 2 \rangle 2.$  PICK  $\epsilon > 0$  such that  $B_d(x, \epsilon) \subseteq V$

$\langle 2 \rangle 3.$  Take  $B = B_{d \upharpoonright A}(x, \epsilon)$

□

**Corollary 190.1.** *A subspace of a metrizable space is metrizable.*

**Proposition 191.** *Every metrizable space is Hausdorff.*

PROOF:

$\langle 1 \rangle 1.$  LET:  $X$  be a metric space

$\langle 1 \rangle 2.$  LET:  $a, b \in X$  with  $a \neq b$

$\langle 1 \rangle 3.$  LET:  $\epsilon = d(a, b)/2$

$\langle 1 \rangle 4.$  LET:  $U = B(a, \epsilon)$  and  $V = B(b, \epsilon)$

$\langle 1 \rangle 5.$   $U$  and  $V$  are disjoint neighbourhoods of  $a$  and  $b$  respectively.

□

**Proposition 192 (CC).** *The product of a countable family of metrizable spaces is metrizable.*

PROOF:

$\langle 1 \rangle 1.$  LET:  $(X_n, d_n)$  be a sequence of metric spaces.

$\langle 1 \rangle 2.$  ASSUME: w.l.o.g. each  $d_n$  is bounded above by 1.

PROOF: By Proposition 186.

$\langle 1 \rangle 3.$  LET:  $D$  be the metric on  $\mathbb{R}^\omega$  defined by  $D(x, y) = \sup_i (d_i(x_i, y_i)/i)$ .

- ⟨2⟩1.  $D(x, y) \geq 0$
- ⟨2⟩2.  $D(x, y) = 0$  if and only if  $x = y$
- ⟨2⟩3.  $D(x, y) = D(y, x)$
- ⟨2⟩4.  $D(x, z) \leq D(x, y) + D(y, z)$

PROOF:

$$\begin{aligned}
D(x, z) &= \sup_i \frac{d_i(x_i, z_i)}{i} \\
&\leq \sup_i \frac{d_i(x_i, y_i) + d_i(y_i, z_i)}{i} \\
&\leq \sup_i \frac{d_i(x_i, y_i)}{i} + \sup_i \frac{d_i(y_i, z_i)}{i} \\
&= D(x, y) + D(y, z)
\end{aligned}$$

- ⟨1⟩4. Every open ball  $B_D(a, \epsilon)$  is open in the product topology.
- ⟨2⟩1. PICK  $N$  such that  $1/\epsilon < N$
- ⟨2⟩2.  $B_D(a, \epsilon) = \prod_{i=1}^{\infty} U_i$  where  $U_i = B(a_i, i\epsilon)$  if  $i \leq N$ , and  $U_i = X_i$  if  $i > N$
- ⟨1⟩5. For any open set  $U$  and  $a \in U$ , there exists  $\epsilon > 0$  such that  $B_D(a, \epsilon) \subseteq U$ .
- ⟨2⟩1. LET:  $n \geq 1$ ,  $V$  be an open set in  $\mathbb{R}$  and  $a \in \pi_n^{-1}(V)$
- ⟨2⟩2. PICK  $\epsilon > 0$  such that  $B_{d_n}(a, \epsilon) \subseteq V$
- ⟨2⟩3.  $B_D(a, \epsilon/n) \subseteq \pi_n^{-1}(V)$

□

**Theorem 193.** *Let  $X$  and  $Y$  be metric spaces and  $f : X \rightarrow Y$ . Then  $f$  is continuous if and only if, for all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that, for all  $y \in X$ , if  $d(x, y) < \delta$  then  $d(f(x), f(y)) < \epsilon$ .*

PROOF:

- ⟨1⟩1. If  $f$  is continuous then, for all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that, for all  $y \in X$ , if  $d(x, y) < \delta$  then  $d(f(x), f(y)) < \epsilon$
- ⟨2⟩1. ASSUME:  $f$  is continuous.
- ⟨2⟩2. LET:  $x \in X$  and  $\epsilon > 0$
- ⟨2⟩3. PICK a neighbourhood  $U$  of  $x$  such that  $f(U) \subseteq B(f(x), \epsilon)$
- PROOF: Theorem 80.
- ⟨2⟩4. PICK  $\delta > 0$  such that  $B(x, \delta) \subseteq U$
- PROOF: Proposition 176.
- ⟨2⟩5. For all  $y \in X$ , if  $d(x, y) < \delta$  then  $d(f(x), f(y)) < \epsilon$
- ⟨1⟩2. If for all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that, for all  $y \in X$ , if  $d(x, y) < \delta$  then  $d(f(x), f(y)) < \epsilon$ , then  $f$  is continuous.
- ⟨2⟩1. ASSUME: for all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that, for all  $y \in X$ , if  $d(x, y) < \delta$  then  $d(f(x), f(y)) < \epsilon$
- ⟨2⟩2. LET:  $x \in X$  and  $V$  be a neighbourhood of  $f(x)$
- ⟨2⟩3. PICK  $\epsilon > 0$  such that  $B(f(x), \epsilon) \subseteq V$
- PROOF: Proposition 176.
- ⟨2⟩4. PICK  $\delta > 0$  such that, for all  $y \in X$ , if  $d(x, y) < \delta$  then  $d(f(x), f(y)) < \epsilon$
- PROOF: By ⟨2⟩1
- ⟨2⟩5. LET:  $U = B(x, \delta)$
- ⟨2⟩6.  $U$  is a neighbourhood of  $x$  with  $f(U) \subseteq V$



⟨2⟩7. Q.E.D.

PROOF: Theorem 80.

□

**Proposition 194.** *Let  $X$  be a metric space. Let  $(a_n)$  be a sequence in  $X$  and  $l \in X$ . Then  $a_n \rightarrow l$  as  $n \rightarrow \infty$  if and only if, for all  $\epsilon > 0$ , there exists  $N$  such that, for all  $n \geq N$ , we have  $d(a_n, l) < \epsilon$ .*

PROOF: From Proposition 63. □

**Proposition 195.** *Every metrizable space is first countable.*

PROOF: In any metric space  $X$ , the open balls  $B(a, 1/n)$  for  $n \geq 1$  form a local basis at  $a$ .

**Example 196.**  $\mathbb{R}^\omega$  under the box topology is not metrizable.

**Example 197.** If  $J$  is uncountable then  $\mathbb{R}^J$  under the product topology is not metrizable.

## 28 Real Linear Algebra

**Definition 198** (Square Metric). The *square metric*  $\rho$  on  $\mathbb{R}^n$  is defined by

$$\rho(\vec{x}, \vec{y}) = \max(|x_1 - y_1|, \dots, |x_n - y_n|)$$

We prove this is a metric.

PROOF:

⟨1⟩1.  $\rho(\vec{x}, \vec{y}) \geq 0$

PROOF: Immediate from definition.

⟨1⟩2.  $\rho(\vec{x}, \vec{y}) = 0$  if and only if  $\vec{x} = \vec{y}$

PROOF: Immediate from definition.

⟨1⟩3.  $\rho(\vec{x}, \vec{y}) = \rho(\vec{y}, \vec{x})$

PROOF: Immediate from definition.

⟨1⟩4.  $\rho(\vec{x}, \vec{z}) \leq \rho(\vec{x}, \vec{y}) + \rho(\vec{y}, \vec{z})$

PROOF: Since  $|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$ .

□

**Proposition 199.** *The square metric induces the standard topology on  $\mathbb{R}^n$ .*

PROOF:

⟨1⟩1. For every  $a \in X$  and  $\epsilon > 0$ , we have  $B_\rho(a, \epsilon)$  is open in the standard product topology.

PROOF:

$$B_\rho(a, \epsilon) = (a_1 - \epsilon, a_1 + \epsilon) \times \cdots \times (a_n - \epsilon, a_n + \epsilon)$$

⟨1⟩2. For any open sets  $U_1, \dots, U_n$  in  $\mathbb{R}$ , we have  $U_1 \times \cdots \times U_n$  is open in the square metric topology.

⟨2⟩1. LET:  $\vec{a} \in U_1 \times \cdots \times U_n$

- $\langle 2 \rangle 2$ . For  $i = 1, \dots, n$ , PICK  $\epsilon_i > 0$  such that  $(a_i - \epsilon_i, a_i + \epsilon_i) \subseteq U_i$   
 $\langle 2 \rangle 3$ . LET:  $\epsilon = \min(\epsilon_1, \dots, \epsilon_n)$   
 $\langle 2 \rangle 4$ .  $B_\rho(\vec{a}, \epsilon) \subseteq U$

□

**Definition 200.** Given  $\vec{x}, \vec{y} \in \mathbb{R}^n$ , define the *sum*  $\vec{x} + \vec{y}$  by

$$(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n) .$$

**Definition 201.** Given  $\lambda \in \mathbb{R}$  and  $\vec{x} \in \mathbb{R}^n$ , define the *scalar product*  $\lambda \vec{x} \in \mathbb{R}^n$  by

$$\lambda(x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n)$$

**Definition 202** (Inner Product). Given  $\vec{x}, \vec{y} \in \mathbb{R}^n$ , define the *inner product*  $\vec{x} \cdot \vec{y} \in \mathbb{R}$  by

$$(x_1, \dots, x_n) \cdot (y_1, \dots, y_n) = x_1 y_1 + \dots + x_n y_n .$$

We write  $\vec{x}^2$  for  $\vec{x} \cdot \vec{x}$ .

**Definition 203** (Norm). Let  $n \geq 1$ . The *norm* on  $\mathbb{R}^n$  is the function  $\| \cdot \| : \mathbb{R}^n \rightarrow \mathbb{R}$  defined by

$$\|(x_1, \dots, x_n)\| = \sqrt{x_1^2 + \dots + x_n^2}$$

**Lemma 204.**

$$\|\vec{x}\|^2 = \vec{x}^2$$

PROOF: Immediate from definitions. □

**Lemma 205.**

$$\vec{x} \cdot (\vec{y} + \vec{z}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z}$$

PROOF: Each is equal to  $(x_1 y_1 + x_1 z_1, \dots, x_n y_n + x_n z_n)$ . □

**Lemma 206.**

$$|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \|\vec{y}\|$$

PROOF:

$\langle 1 \rangle 1$ . ASSUME:  $\vec{x} \neq \vec{0} \neq \vec{y}$

PROOF: Otherwise both sides are 0.

$\langle 1 \rangle 2$ . LET:  $a = 1/\|\vec{x}\|$

$\langle 1 \rangle 3$ . LET:  $b = 1/\|\vec{y}\|$

$\langle 1 \rangle 4$ .  $(a\vec{x} + b\vec{y})^2 \geq 0$  and  $(a\vec{x} - b\vec{y})^2 \geq 0$

$\langle 1 \rangle 5$ .  $a^2 \|\vec{x}\|^2 + 2ab\vec{x} \cdot \vec{y} + b^2 \|\vec{y}\|^2 \geq 0$  and  $a^2 \|\vec{x}\|^2 - 2ab\vec{x} \cdot \vec{y} + b^2 \|\vec{y}\|^2 \geq 0$

$\langle 1 \rangle 6$ .  $2ab\vec{x} \cdot \vec{y} + 2 \geq 0$  and  $-2ab\vec{x} \cdot \vec{y} + 2 \geq 0$

$\langle 1 \rangle 7$ .  $\vec{x} \cdot \vec{y} \geq -1/ab$  and  $\vec{x} \cdot \vec{y} \leq 1/ab$

$\langle 1 \rangle 8$ .  $\vec{x} \cdot \vec{y} \geq -\|\vec{x}\| \|\vec{y}\|$  and  $\vec{x} \cdot \vec{y} \leq \|\vec{x}\| \|\vec{y}\|$

□

**Lemma 207** (Triangle Inequality).

$$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$$

PROOF:

$$\begin{aligned} \|\vec{x} + \vec{y}\|^2 &= \|\vec{x}\|^2 + 2\vec{x} \cdot \vec{y} + \|\vec{y}\|^2 \\ &\leq \|\vec{x}\|^2 + 2\|\vec{x}\|\|\vec{y}\| + \|\vec{y}\|^2 && \text{(Lemma 206)} \\ &= (\|\vec{x}\| + \|\vec{y}\|)^2 \end{aligned} \quad \square$$

**Definition 208** (Euclidean Metric). Let  $n \geq 1$ . The *Euclidean metric* on  $\mathbb{R}^n$  is defined by

$$d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|.$$

We prove this is a metric.

$\langle 1 \rangle 1$ .  $d(\vec{x}, \vec{y}) \geq 0$

PROOF: Immediate from definition.

$\langle 1 \rangle 2$ .  $d(\vec{x}, \vec{y}) = 0$  if and only if  $\vec{x} = \vec{y}$

PROOF:  $d(\vec{x}, \vec{y}) = 0$  if and only if  $\vec{x} - \vec{y} = \vec{0}$ .

$\langle 1 \rangle 3$ .  $d(\vec{x}, \vec{y}) = d(\vec{y}, \vec{x})$

PROOF: Immediate from definition.

$\langle 1 \rangle 4$ .  $d(\vec{x}, \vec{z}) \leq d(\vec{x}, \vec{y}) + d(\vec{y}, \vec{z})$

PROOF:

$$\begin{aligned} \|\vec{x} - \vec{z}\| &= \|(\vec{x} - \vec{y}) + (\vec{y} - \vec{z})\| \\ &\leq \|\vec{x} - \vec{y}\| + \|\vec{y} - \vec{z}\| \end{aligned} \quad \text{(Lemma 207)}$$

$\square$

**Proposition 209.** The Euclidean metric induces the standard topology on  $\mathbb{R}^n$ .

PROOF:

$\langle 1 \rangle 1$ . LET:  $\rho$  be the square metric.

$\langle 1 \rangle 2$ . For all  $\vec{a} \in \mathbb{R}^n$  and  $\epsilon > 0$ , we have  $B_d(\vec{a}, \epsilon) \subseteq B_\rho(\vec{a}, \epsilon)$

$\langle 2 \rangle 1$ . LET:  $\vec{x} \in B_d(\vec{a}, \epsilon)$

$\langle 2 \rangle 2$ .  $\sqrt{(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2} < \epsilon$

$\langle 2 \rangle 3$ .  $(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2 < \epsilon^2$

$\langle 2 \rangle 4$ . For all  $i$  we have  $(x_i - a_i)^2 < \epsilon^2$

$\langle 2 \rangle 5$ . For all  $i$  we have  $|x_i - a_i| < \epsilon$

$\langle 2 \rangle 6$ .  $\rho(\vec{x}, \vec{a}) < \epsilon$

$\langle 1 \rangle 3$ . For all  $\vec{a} \in \mathbb{R}^n$  and  $\epsilon > 0$ , we have  $B_\rho(\vec{a}, \epsilon/\sqrt{n}) \subseteq B_d(\vec{a}, \epsilon)$

$\langle 2 \rangle 1$ . LET:  $\vec{x} \in B_\rho(\vec{a}, \epsilon/\sqrt{n})$

$\langle 2 \rangle 2$ .  $\rho(\vec{x}, \vec{a}) < \epsilon/\sqrt{n}$

$\langle 2 \rangle 3$ . For all  $i$  we have  $|x_i - a_i| < \epsilon/\sqrt{n}$

$\langle 2 \rangle 4$ . For all  $i$  we have  $(x_i - a_i)^2 < \epsilon^2/n$

$\langle 2 \rangle 5$ .  $d(\vec{x}, \vec{a}) < \epsilon$

$\langle 1 \rangle 4$ . Q.E.D.

PROOF: By Lemma 187.

$\square$

**Lemma 210.** If  $\sum_{i=0}^{\infty} x_i^2$  and  $\sum_{i=0}^{\infty} y_i^2$  converge then  $\sum_{i=0}^{\infty} |x_i y_i|$  converges.

PROOF:

$\langle 1 \rangle 1$ . For all  $N \geq 0$  we have  $\sum_{i=0}^N |x_i y_i| \leq \sqrt{\sum_{i=0}^N |x_i|^2} \sqrt{\sum_{i=0}^N |y_i|^2}$

PROOF: By the Cauchy-Schwarz inequality

$\langle 1 \rangle 2$ . Q.E.D.

PROOF: Since  $\sum_{i=0}^N |x_i y_i|$  is an increasing sequence bounded above by  $(\sum_{i=0}^{\infty} x_i^2)(\sum_{i=0}^{\infty} y_i^2)$ .

□

**Corollary 210.1.** If  $\sum_{i=0}^{\infty} x_i^2$  and  $\sum_{i=0}^{\infty} y_i^2$  converge then  $\sum_{i=0}^{\infty} (x_i + y_i)^2$  converges.

PROOF: Since  $\sum_{i=0}^{\infty} x_i^2$ ,  $\sum_{i=0}^{\infty} y_i^2$  and  $2 \sum_{i=0}^{\infty} x_i y_i$  all converge.

**Definition 211** ( $l^2$ -metric). The  $l^2$ -metric on

$$\left\{ (x_n) \in \mathbb{R}^{\omega} \mid \sum_{n=0}^{\infty} x_n^2 \text{ converges} \right\}$$

is defined by

$$d(x, y) = \left( \sum_{n=0}^{\infty} (x_n - y_n)^2 \right)^{1/2}$$

We prove this is a metric.

PROOF:

$\langle 1 \rangle 1$ .  $d$  is well-defined.

PROOF: By Corollary 210.1.

$\langle 1 \rangle 2$ .  $d(x, y) \geq 0$

$\langle 1 \rangle 3$ .  $d(x, y) = 0$  if and only if  $x = y$

$\langle 1 \rangle 4$ .  $d(x, y) = d(y, x)$

$\langle 1 \rangle 5$ .  $d(x, z) \leq d(x, y) + d(y, z)$

PROOF: By Lemma 207.

□

**Theorem 212.** Addition is a continuous function  $\mathbb{R}^2 \rightarrow \mathbb{R}$ .

PROOF:

$\langle 1 \rangle 1$ . LET:  $a, b \in \mathbb{R}$

$\langle 1 \rangle 2$ . LET:  $\epsilon > 0$

$\langle 1 \rangle 3$ . LET:  $\delta = \epsilon/2$

$\langle 1 \rangle 4$ . LET:  $x, y \in \mathbb{R}$

$\langle 1 \rangle 5$ . ASSUME:  $\rho((a, b), (x, y)) < \delta$

$\langle 1 \rangle 6$ .  $|(a + b) - (x + y)| < \epsilon$

PROOF:

$$\begin{aligned}
|(a+b) - (x+y)| &= |a-x| + |b-y| \\
&\leq 2\rho((a,b), (x,y)) \\
&< 2\delta \\
&= \epsilon
\end{aligned}$$

$\langle 1 \rangle 7$ . Q.E.D.

PROOF: Theorem 193

□

**Theorem 213.** *Multiplication is a continuous function  $\mathbb{R}^2 \rightarrow \mathbb{R}$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $a, b \in \mathbb{R}$

$\langle 1 \rangle 2$ . LET:  $\epsilon > 0$

$\langle 1 \rangle 3$ . LET:  $\delta = \min(\epsilon/(|a| + |b| + 1), 1)$

$\langle 1 \rangle 4$ . LET:  $x, y \in \mathbb{R}$

$\langle 1 \rangle 5$ . ASSUME:  $\rho((a,b), (x,y)) < \delta$

$\langle 1 \rangle 6$ .  $|ab - xy| < \epsilon$

PROOF:

$$\begin{aligned}
|ab - xy| &= |a(b-y) + (a-x)b - (a-x)(b-y)| \\
&\leq |a||b-y| + |b||a-x| + |a-x||b-y| \\
&< |a|\delta + |b|\delta + \delta^2 & (\langle 1 \rangle 5) \\
&\leq |a|\delta + |b|\delta + \delta & (\langle 1 \rangle 3) \\
&\leq \epsilon & (\langle 1 \rangle 3)
\end{aligned}$$

$\langle 1 \rangle 7$ . Q.E.D.

PROOF: Theorem 193

□

**Theorem 214.** *The function  $f : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$  defined by  $f(x) = x^{-1}$  is continuous.*

PROOF:

$\langle 1 \rangle 1$ . For all  $a \in \mathbb{R}$  we have  $f^{-1}((a, +\infty))$  is open.

PROOF: The set is

$$(a^{-1}, +\infty) \text{ if } a > 0$$

$$(0, +\infty) \text{ if } a = 0$$

$$(-\infty, a^{-1}) \cup (0, +\infty) \text{ if } a < 0$$

$\langle 1 \rangle 2$ . For all  $a \in \mathbb{R}$  we have  $f^{-1}((-\infty, a))$  is open.

PROOF: Similar.

$\langle 1 \rangle 3$ . Q.E.D.

PROOF: By Proposition 77 and Lemma 103.

□

## 29 The Uniform Topology

**Definition 215** (Uniform Metric). Let  $J$  be a set. The *uniform metric*  $\bar{\rho}$  on  $\mathbb{R}^J$  is defined by

$$\bar{\rho}(a, b) = \sup_{j \in J} \bar{d}(a_j, b_j)$$

where  $\bar{d}$  is the standard bounded metric on  $\mathbb{R}$ .

The *uniform topology* on  $\mathbb{R}^J$  is the topology induced by the uniform metric.

We prove this is a metric.

PROOF:

$\langle 1 \rangle 1. \bar{\rho}(a, b) \geq 0$

PROOF: Immediate from definitions.

$\langle 1 \rangle 2. \bar{\rho}(a, b) = 0$  if and only if  $a = b$

PROOF: Immediate from definitions.

$\langle 1 \rangle 3. \bar{\rho}(a, b) = \bar{\rho}(b, a)$

PROOF: Immediate from definitions.

$\langle 1 \rangle 4. \bar{\rho}(a, c) \leq \bar{\rho}(a, b) + \bar{\rho}(b, c)$

PROOF:

$$\begin{aligned} \bar{\rho}(a, c) &= \sup_{j \in J} \bar{d}(a_j, c_j) \\ &\leq \sup_{j \in J} (\bar{d}(a_j, b_j) + \bar{d}(b_j, c_j)) \\ &\leq \sup_{j \in J} \bar{d}(a_j, b_j) + \sup_{j \in J} \bar{d}(b_j, c_j) \\ &= \bar{\rho}(a, b) + \bar{\rho}(b, c) \end{aligned}$$

□

**Proposition 216.** The uniform topology on  $\mathbb{R}^J$  is finer than the product topology.

PROOF:

$\langle 1 \rangle 1.$  LET:  $j \in J$  and  $U$  be open in  $\mathbb{R}$

PROVE:  $\pi_j^{-1}(U)$  is open in the uniform topology.

$\langle 1 \rangle 2.$  LET:  $a \in \pi_j^{-1}(U)$

$\langle 1 \rangle 3.$  PICK  $\epsilon > 0$  such that  $(a_j - \epsilon, a_j + \epsilon) \subseteq U$

$\langle 1 \rangle 4. B_{\bar{\rho}}(a, \epsilon) \subseteq \pi_j^{-1}(U)$

□

**Proposition 217.** The uniform topology on  $\mathbb{R}^J$  is coarser than the box topology.

PROOF:

$\langle 1 \rangle 1.$  LET:  $a \in \mathbb{R}^J$  and  $\epsilon > 0$

PROVE:  $B(a, \epsilon)$  is open in the box topology.

$\langle 1 \rangle 2.$  LET:  $b \in B(a, \epsilon)$

$\langle 1 \rangle 3.$  For  $j \in J$  we have  $|a_j - b_j| < \epsilon$

$\langle 1 \rangle 4.$  For  $j \in J$ ,

LET:  $\delta_j = (\epsilon - |a_j - b_j|)/2$   
 $\langle 1 \rangle 5. \prod_{j \in J} (b_j - \delta_j, b_j + \delta_j) \subseteq B(a, \epsilon)$   
 $\square$

**Proposition 218.** *The uniform topology on  $\mathbb{R}^J$  is strictly finer than the product topology if and only if  $J$  is infinite.*

PROOF:

$\langle 1 \rangle 1.$  If  $J$  is finite then the uniform and product topologies coincide.

PROOF: The uniform, box and product topologies are all the same.

$\langle 1 \rangle 2.$  If  $J$  is infinite then the uniform and product topologies are different.

PROOF: The set  $B(\vec{0}, 1)$  is open in the uniform topology but not the product topology.

$\square$

**Proposition 219 (DC).** *The uniform topology on  $\mathbb{R}^J$  is strictly coarser than the box topology if and only if  $J$  is infinite.*

PROOF:

$\langle 1 \rangle 1.$  If  $J$  is finite then the uniform and box topologies coincide.

PROOF: The uniform, box and product topologies are all the same.

$\langle 1 \rangle 2.$  If  $J$  is infinite then the uniform and box topologies are different.

PROOF: Pick an  $\omega$ -sequence  $(j_1, j_2, \dots)$  in  $J$ . Let  $U = \prod_{j \in J} U_j$  where  $U_{j_i} = (-1/i, 1/i)$  and  $U_j = (-1, 1)$  for all other  $j$ . Then  $\vec{0} \in U$  but there is no  $\epsilon > 0$  such that  $B(\vec{0}, \epsilon) \subseteq U$ .

$\square$

**Proposition 220.** *The closure of  $\mathbb{R}^\infty$  in  $\mathbb{R}^\omega$  under the uniform topology is  $\mathbb{R}^\omega$ .*

PROOF: Given any open ball  $B(a, \epsilon)$ , pick an integer  $N$  such that  $1/\epsilon < N$ . Then  $B(a, \epsilon)$  includes sequences whose  $n$ th entry is 0 for all  $n \geq N$ .  $\square$

## 30 Uniform Convergence

**Definition 221** (Uniform Convergence). Let  $X$  be a set and  $Y$  a metric space. Let  $(f_n : X \rightarrow Y)$  be a sequence of functions and  $f : X \rightarrow Y$  be a function. Then  $f_n$  converges uniformly to  $f$  as  $n \rightarrow \infty$  if and only if, for all  $\epsilon > 0$ , there exists  $N$  such that, for all  $n \geq N$  and  $x \in X$ , we have  $d(f_n(x), f(x)) < \epsilon$ .

**Example 222.** Define  $f_n : [0, 1] \rightarrow \mathbb{R}$  by  $f_n(x) = x^n$  for  $n \geq 1$ , and  $f : [0, 1] \rightarrow \mathbb{R}$  by  $f(x) = 0$  if  $x < 1$ ,  $f(1) = 1$ . Then  $f_n$  converges to  $f$  pointwise but not uniformly.

**Theorem 223** (Uniform Limit Theorem). *Let  $X$  be a topological space and  $Y$  a metric space. Let  $(f_n : X \rightarrow Y)$  be a sequence of continuous functions and  $f : X \rightarrow Y$  be a function. If  $f_n$  converges uniformly to  $f$  as  $n \rightarrow \infty$ , then  $f$  is continuous.*

PROOF:

⟨1⟩1. LET:  $x \in X$  and  $\epsilon > 0$

⟨1⟩2. PICK  $N$  such that, for all  $n \geq N$  and  $y \in X$ , we have  $d(f_n(y), f(y)) < \epsilon/3$

⟨1⟩3. PICK a neighbourhood  $U$  of  $x$  such that  $f_N(U) \subseteq B(f_N(x), \epsilon/3)$

PROVE:  $f(U) \subseteq B(f(x), \epsilon)$

⟨1⟩4. LET:  $y \in U$

⟨1⟩5.  $d(f(y), f(x)) < \epsilon$

PROOF:

$d(f(y), f(x)) \leq d(f(y), f_N(y)) + d(f_N(y), f_N(x)) + d(f_N(x), f(x))$  (Triangle Inequality)

$< \epsilon/3 + \epsilon/3 + \epsilon/3$

((⟨1⟩2, ⟨1⟩3)

$= \epsilon$

□

**Proposition 224.** *Let  $X$  be a topological space and  $Y$  a metric space. Let  $(f_n : X \rightarrow Y)$  be a sequence of continuous functions and  $f : X \rightarrow Y$  be a function. Let  $(a_n)$  be a sequence of points in  $X$  and  $a \in X$ . If  $f_n$  converges uniformly to  $f$  and  $a_n$  converges to  $a$  in  $X$  then  $f_n(a_n)$  converges to  $f(a)$  uniformly in  $Y$ .*

PROOF:

⟨1⟩1. LET:  $\epsilon > 0$

⟨1⟩2. PICK  $N_1$  such that, for all  $n \geq N_1$  and  $x \in X$ , we have  $d(f_n(x), f(x)) < \epsilon/2$

⟨1⟩3. PICK  $N_2$  such that, for all  $n \geq N_2$ , we have  $a_n \in f^{-1}(B(a, \epsilon/2))$

PROOF: Using the fact that  $f$  is continuous from the Uniform Limit Theorem.

⟨1⟩4. LET:  $N = \max(N_1, N_2)$

⟨1⟩5. LET:  $n \geq N$

⟨1⟩6.  $d(f_n(a_n), f(a)) < \epsilon$

PROOF:

$d(f_n(a_n), f(a)) \leq d(f_n(a_n), f(a_n)) + d(f(a_n), f(a))$  (Triangle Inequality)

$< \epsilon/2 + \epsilon/2$

((⟨1⟩2, ⟨1⟩3)

$= \epsilon$

□

**Proposition 225.** *Let  $X$  be a set. Let  $(f_n : X \rightarrow \mathbb{R})$  be a sequence of functions and  $f : X \rightarrow \mathbb{R}$  be a function. Then  $f_n$  converges uniformly to  $f$  as  $n \rightarrow \infty$  if and only if  $f_n \rightarrow f$  as  $n \rightarrow \infty$  in  $\mathbb{R}^X$  under the uniform topology.*

PROOF:

⟨1⟩1. If  $f_n$  converges uniformly to  $f$  then  $f_n$  converges to  $f$  under the uniform topology.

⟨2⟩1. ASSUME:  $f_n$  converges uniformly to  $f$

⟨2⟩2. LET:  $\epsilon > 0$

⟨2⟩3. PICK  $N$  such that, for all  $n \geq N$  and  $x \in X$ , we have  $d(f_n(x), f(x)) < \epsilon/2$

⟨2⟩4. For all  $n \geq N$  we have  $\bar{\rho}(f_n, f) \leq \epsilon/2$

⟨2⟩5. For all  $n \geq N$  we have  $\bar{\rho}(f_n, f) < \epsilon$



- ⟨1⟩2. If  $f_n$  converges to  $f$  under the uniform topology then  $f_n$  converges uniformly to  $f$ .  
 ⟨2⟩1. ASSUME:  $f_n$  converges to  $f$  under the uniform topology.  
 ⟨2⟩2. LET:  $\epsilon > 0$   
 ⟨2⟩3. PICK  $N$  such that, for all  $n \geq N$ , we have  $\bar{\rho}(f_n, f) < \min(\epsilon, 1/2)$   
 ⟨2⟩4. LET:  $n \geq N$   
 ⟨2⟩5. LET:  $x \in X$   
 ⟨2⟩6.  $\bar{\rho}(f_n, f) < \min(\epsilon, 1/2)$   
 PROOF: From ⟨2⟩3.  
 ⟨2⟩7.  $d(f_n(x), f(x)) < \min(\epsilon, 1/2)$   
 ⟨2⟩8.  $d(f_n(x), f(x)) < \epsilon$

□

## 31 Isometric Imbeddings

**Definition 226.** Let  $X$  and  $Y$  be metric spaces. An *isometric imbedding*  $f : X \rightarrow Y$  is a function such that, for all  $x, y \in X$ , we have  $d(f(x), f(y)) = d(x, y)$ .

**Proposition 227.** *Every isometric imbedding is an imbedding.*

PROOF:

- ⟨1⟩1. LET:  $f : X \rightarrow Y$  be an isometric imbedding.  
 ⟨1⟩2.  $f$  is injective.  
 PROOF: If  $f(x) = f(y)$  then  $d(f(x), f(y)) = 0$  hence  $d(x, y) = 0$  hence  $x = y$ .  
 ⟨1⟩3.  $f$  is continuous.  
 PROOF: For all  $\epsilon > 0$ , if  $d(x, y) < \epsilon$  then  $d(f(x), f(y)) < \epsilon$ .  
 ⟨1⟩4.  $f : X \rightarrow f(X)$  is an open map.  
 PROOF:  $f(B(a, \epsilon)) = B(f(a), \epsilon) \cap f(X)$ .

□