Topology

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Contents

Part I

Set Theory

Chapter 1

Classes

1.1 Classes

We speak informally of *classes*. A class is specified by a unary predicate. We write $\{x \mid P(x)\}$ for the class determined by the predicate $P(x)$.

Definition 1.1.1 (Membership)**.** Let a be an object and $\mathbf{A}$ a class. We define the proposition $a \in \mathbf{A}$ (a is a *member* or *element* of A) as follows:

The proposition $a \in \{x \mid P(x)\}$ is the proposition $P(a)$.

Definition 1.1.2 (Equality of Classes)**.** Let $\mathbf{A}$ and $\mathbf{B}$ be classes. We say $\mathbf{A}$ and $\mathbf{B}$ are *equal*, $\mathbf{A} = \mathbf{B}$, if and only if they have exactly the same elements.

1.2 Subclasses

Definition 1.2.1 (Subclass)**.** Let $\mathbf{A}$ and $\mathbf{B}$ be classes. We say $\mathbf{A}$ is a *subclass* of $\mathbf{B}$, $\mathbf{A} \subseteq \mathbf{B}$, if and only if every member of $\mathbf{A}$ is a member of $\mathbf{B}$.

We say $\mathbf{A}$ is a *proper* subclass of $\mathbf{B}$, $\mathbf{A} \subset \mathbf{B}$, if and only if $\mathbf{A} \subseteq \mathbf{B}$ and $\mathbf{A} \neq \mathbf{B}$.

1.3 The Empty Class

Definition 1.3.1 (Empty Class)**.** The *empty* class $\emptyset$ is $\{x \mid \bot\}$.

1.4 Finite Classes

Definition 1.4.1. For any objects $a_1, \ldots, a_n$, we write $\{a_1, \ldots, a_n\}$ for the class $\{x \mid x = a_1 \vee \cdots \vee x = a_n\}$.

1.5 Universal Class

Definition 1.5.1 (Universal Class)**.** The *universal class* $\mathbf{V}$ is the class $\{x \mid \top\}$.

1.6 Union

Definition 1.6.1 (Union)**.** For any classes $\mathbf{A}$ and $\mathbf{B}$, the *union* $\mathbf{A} \cup \mathbf{B}$ is the class $\{x \mid x \in \mathbf{A} \vee x \in \mathbf{B}\}$.

1.7 Intersection

Definition 1.7.1 (Intersection)**.** For any classes $\mathbf{A}$ and $\mathbf{B}$, the *intersection* $\mathbf{A} \cap \mathbf{B}$ is the class $\{x \mid x \in \mathbf{A} \wedge x \in \mathbf{B}\}$.

1.8 Disjoint Classes

Definition 1.8.1 (Disjoint)**.** Classes $\mathbf{A}$ and $\mathbf{B}$ are *disjoint* if and only if $\mathbf{A} \cap \mathbf{B} = \emptyset$.

1.9 Relative Complement

Definition 1.9.1 (Relative Complement)**.** For any classes $\mathbf{A}$ and $\mathbf{B}$, the *relative complement* $\mathbf{A} - \mathbf{B}$ is $\{x \in \mathbf{A} \mid x \notin \mathbf{B}\}$.

Chapter 2

Sets

2.1 Membership

We take as undefined the notion of *set*.

We take as undefined the binary relation of *membership*, $\in$. If $a \in A$ we say a is a *member* or *element* of A. If this does not hold, we write $a \notin A$.

Axiom 2.1.1 (Axiom of Extensionality)**.** *Two sets with exactly the same elements are equal.*

We may therefore identify the set A with the class $\{x \mid x \in A\}$.

We say a class $\mathbf{A}$ *is a set* iff there exists a set A such that $A = \mathbf{A}$. That is, $\{x \mid P(x)\}$ is a set if and only if there exists a set A such that, for all x, we have $x \in A$ if and only if $P(x)$.

2.2 The Empty Set

Axiom 2.2.1 (Empty Set Axiom)**.** *The empty class $\emptyset$ is a set.*

2.3 Pair Sets

Axiom 2.3.1 (Pairing Axiom)**.** *For any objects u and v, the class $\{u, v\}$ is a set.*

Theorem 2.3.2. *For any object a, the class $\{a\}$ is a set.*

PROOF: It is $\{a, a\}$. $\square$

2.4 Unions

Definition 2.4.1 (Union)**.** For any class of sets $\mathbf{A}$, the *union* $\bigcup \mathbf{A}$ is the class $\{x \mid \exists A \in \mathbf{A}. x \in A\}$.

Axiom 2.4.2 (Union Axiom)**.** *For any set A, the union $\bigcup A$ is a set.*

Theorem 2.4.3. *For any sets A and B, the class $A \cup B$ is a set.*

PROOF: It is $\bigcup\{A, B\}$. □

Theorem Schema 2.4.4. *For any objects a_1, …, a_n, the class $\{a_1, \ldots, a_n\}$ is a set.*

PROOF: It is $\{a_1\} \cup \cdots \cup \{a_n\}$. □

2.5 Power Set

Definition 2.5.1 (Power Class)**.** For any class $\mathbf{A}$, the *power* class $\mathcal{P}\mathbf{A}$ is the class of all subsets of $\mathbf{A}$.

Axiom 2.5.2 (Power Set Axiom)**.** *For any set A, the power class $\mathcal{P}A$ is a set.*

2.6 Covers

Definition 2.6.1 (Cover)**.** Let $\mathbf{X}$ be a class and $\mathcal{A} \subseteq \mathcal{P}\mathbf{X}$. Then $\mathcal{A}$ *covers* $\mathbf{X}$, or is a *covering* of $\mathbf{X}$, if and only if $\bigcup \mathcal{A} = \mathbf{X}$.

2.7 Subset Axioms

Axiom Schema 2.7.1 (Subset Axioms, Aussonderung Axioms)**.** *For any classes $\mathbf{A}$ and set B, if $\mathbf{A} \subseteq B$ then $\mathbf{A}$ is a set.*

Theorem 2.7.2. *The universal class $\mathbf{V}$ is not a set.*

PROOF:
$\langle 1 \rangle 1.$ ASSUME: $\mathbf{V}$ is a set.
$\langle 1 \rangle 2.$ LET: $R = \{x \in \mathbf{V} \mid x \notin x\}$
$\langle 1 \rangle 3.$ $R \in R$ if and only if $R \notin R$
$\langle 1 \rangle 4.$ Q.E.D.
PROOF: This is a contradiction.
□

Theorem 2.7.3. *If A is a set and $\mathbf{B}$ is a class then $A - \mathbf{B}$ is a set.*

PROOF: It is a subset of A. □

Theorem 2.7.4. *For any set A and class $\mathbf{B}$, the class $A \cap \mathbf{B}$ is a set.*

PROOF: It is a subset of A.

2.8 Intersection

Definition 2.8.1 (Intersection)**.** For any class $\mathbf{A}$ of sets, the *intersection* $\bigcap \mathbf{A}$ is the class $\{x \mid \forall A \in \mathbf{A}. x \in A\}$.

Theorem 2.8.2. *For any nonempty class* $\mathbf{A}$ *of sets, we have* $\bigcap \mathbf{A}$ *is a set.*

Proof:
$\langle 1 \rangle 1.$ PickA $\in \mathbf{A}$
$\langle 1 \rangle 2.$ $\bigcap \mathbf{A} \subseteq A$
$\langle 1 \rangle 3.$ Q.E.D.
Proof: By a Subset Axiom.
□

2.9 Pairwise Disjoint Classes

Definition 2.9.1 (Pairwise Disjoint)**.** Let $\mathbf{A}$ be a class of sets. Then $\mathbf{A}$ is *pairwise disjoint* iff any two distinct elements of $\mathbf{A}$ are disjoint.

2.10 Axiom of Choice

Axiom 2.10.1 (Axiom of Choice)**.** *Let* $\mathcal{A}$ *be a set of pairwise disjoint nonempty sets. Then there exists a set* C *containing exactly one element from each member of* $\mathcal{A}$.

2.11 Axiom of Regularity

Axiom 2.11.1 (Regularity)**.** *For any nonempty set* A*, there exists* $m \in A$ *such that* m *and* A *are disjoint.*

Theorem 2.11.2. *No set is a member of itself.*

Proof: From the Axiom of Regularity, for any set A, we have A and $\{A\}$ are disjoint, i.e. $A \notin A$. □

Theorem 2.11.3. *There are no sets* A *and* B *such that* $A \in B$ *and* $B \in A$.

Proof:
$\langle 1 \rangle 1.$ Let: A and B be sets.
$\langle 1 \rangle 2.$ Pick $m \in \{A, B\}$ such that $m \cap \{A, B\} = \emptyset$
$\langle 1 \rangle 3.$ Case: $m = A$
Proof: In this case $B \notin A$.
$\langle 1 \rangle 4.$ Case: $m = B$
Proof: In this case $A \notin B$.
□

2.12 Transitive Sets

Definition 2.12.1 (Transitive Set)**.** A set A is *transitive* if and only if, whenever $x \in y \in A$ then $x \in A$.

Theorem 2.12.2. *Let A be a set. Then the following are equivalent.*

1. *A is transitive.*
2. $\bigcup A \subseteq A$
3. *For all $a \in A$ we have $a \subseteq A$*
4. $A \subseteq \mathcal{P}A$

Proof: From definitions. □

2.13 Partitions

Definition 2.13.1 (Partition)**.** A *partition* P of a set A is a set of nonempty subsets of A such that:

1. For all $x \in A$ there exists $S \in P$ such that $x \in S$.
2. Any two distinct elements of P are disjoint.

Chapter 3

Relations

3.1 Ordered Pairs

Definition 3.1.1 (Ordered Pair)**.** For any sets x and y, the *ordered pair* (x, y) is defined to be $\{\{x\}, \{x, y\}\}$.

Theorem 3.1.2. *For any sets u, v, x, y, we have $(u, v) = (x, y)$ if and only if $u = x$ and $v = y$*

PROOF:
$\langle 1 \rangle 1$. ASSUME: $\{\{u\}, \{u, v\}\} = \{\{x\}, \{x, y\}\}$
$\langle 1 \rangle 2$. $\{u\} \in \{\{x\}, \{x, y\}\}$
$\langle 1 \rangle 3$. $\{u, v\} \in \{\{x\}, \{x, y\}\}$
$\langle 1 \rangle 4$. $\{u\} = \{x\}$ or $\{u\} = \{x, y\}$
$\langle 1 \rangle 5$. $\{u, v\} = \{x\}$ or $\{u, v\} = \{x, y\}$
$\langle 1 \rangle 6$. CASE: $\{u\} = \{x, y\}$
  $\langle 2 \rangle 1$. $u = x = y$
  $\langle 2 \rangle 2$. $u = v = x = y$
    PROOF: From $\langle 1 \rangle 5$
$\langle 1 \rangle 7$. CASE: $\{u, v\} = \{x\}$
  PROOF: Similar.
$\langle 1 \rangle 8$. CASE: $\{u\} = \{x\}$ and $\{u, v\} = \{x, y\}$
  $\langle 2 \rangle 1$. $u = x$
  $\langle 2 \rangle 2$. $u = y$ or $v = y$
  $\langle 2 \rangle 3$. CASE: $u = y$
    PROOF: This case is the case considered in $\langle 1 \rangle 6$.
  $\langle 2 \rangle 4$. CASE: $v = y$
    PROOF: We have $u = x$ and $v = y$ as required.
□

Lemma 3.1.3. *Let x, y and C be sets. If $x \in C$ and $y \in C$ then $(x, y) \in \mathcal{PP}C$.*

PROOF:

$\langle 1\rangle 1.$ LET: x, y and C be sets.
$\langle 1\rangle 2.$ ASSUME: $x \in C$
$\langle 1\rangle 3.$ ASSUME: $y \in C$
$\langle 1\rangle 4.$ $\{x\} \in \mathcal{P}C$
$\langle 1\rangle 5.$ $\{x, y\} \in \mathcal{P}C$
$\langle 1\rangle 6.$ $\{\{x\}, \{x, y\}\} \in \mathcal{P}\mathcal{P}C$
$\square$

Lemma 3.1.4. *Let x, y and A be sets. If $(x, y) \in A$ then x and y belong to $\bigcup\bigcup A$.*

PROOF:
$\langle 1\rangle 1.$ LET: x, y and A be sets.
$\langle 1\rangle 2.$ ASSUME: $(x, y) \in A$
$\langle 1\rangle 3.$ $\{x, y\} \in \bigcup A$
$\langle 1\rangle 4.$ $x \in \bigcup\bigcup A$
$\langle 1\rangle 5.$ $y \in \bigcup\bigcup A$
$\square$

3.2 Cartesian Product

Definition 3.2.1 (Cartesian Product)**.** Let $\mathbf{A}$ and $\mathbf{B}$ be classes. The *Cartesian product* $\mathbf{A} \times \mathbf{B}$ is the class $\{(x, y) \mid x \in \mathbf{A}, y \in \mathbf{B}\}$.

Theorem 3.2.2. *For any sets A and B, the Cartesian product $A \times B$ is a set.*

PROOF: It is a subset of $\mathcal{P}\mathcal{P}(A \cup B)$ by Lemma 3.1.3. $\square$

3.3 Relations

Definition 3.3.1 (Relation)**.** A *relation* is a class of ordered pairs.

Given a relation $\mathbf{R}$, we write $x\mathbf{R}y$ for $(x, y) \in \mathbf{R}$.

A relation is *small* iff it is a set.

3.4 Domain

Definition 3.4.1 (Domain)**.** Let $\mathbf{R}$ be a relation. The *domain* of $\mathbf{R}$ is $\operatorname{dom} \mathbf{R} = \{x \mid \exists y. x\mathbf{R}y\}$.

Theorem 3.4.2. *For any set R, the domain $\operatorname{dom} R$ is a set.*

PROOF: It is a subset of $\bigcup\bigcup R$ by Lemma 3.1.4. $\square$

3.5 Range

Definition 3.5.1 (Range). Let $\mathbf{R}$ be a relation. The *range* of $\mathbf{R}$ is $\operatorname{ran}\mathbf{R} = \{y \mid \exists x. x\mathbf{R}y\}$.

Theorem 3.5.2. *For any set R, the range $\operatorname{ran} R$ is a set.*

PROOF: It is a subset of $\bigcup\bigcup R$ by Lemma 3.1.4. ∎

3.6 Single-Rooted

Definition 3.6.1 (Single-Rooted). A relation $\mathbf{R}$ is *single-rooted* if and only if, for all x, x', y, if $x\mathbf{R}y$ and $x'\mathbf{R}y$ then $x = x'$.

3.7 Inverse

Definition 3.7.1 (Inverse). Let $\mathbf{R}$ be a class. The *inverse* of $\mathbf{R}$ is $\mathbf{R}^{-1} = \{(y, x) \mid x\mathbf{R}y\}$.

Theorem 3.7.2. *For any small relation R, the inverse R^{-1} is small.*

PROOF: It is a subset of $\operatorname{ran} R \times \operatorname{dom} R$. ∎

Theorem 3.7.3. *For any relation $\mathbf{F}$, we have $\operatorname{dom}\mathbf{F}^{-1} = \operatorname{ran}\mathbf{F}$.*

PROOF: For any x, we have

$$\begin{aligned} x \in \operatorname{dom}\mathbf{F}^{-1} &\Leftrightarrow \exists y.(x, y) \in \mathbf{F}^{-1} \\ &\Leftrightarrow \exists y.(y, x) \in \mathbf{F} \\ &\Leftrightarrow x \in \operatorname{ran}\mathbf{F} \end{aligned}$$

∎

Theorem 3.7.4. *For any relation $\mathbf{F}$, we have $\operatorname{ran}\mathbf{F}^{-1} = \operatorname{dom}\mathbf{F}$.*

PROOF: For any x, we have

$$\begin{aligned} x \in \operatorname{ran}\mathbf{F}^{-1} &\Leftrightarrow \exists y.(y, x) \in \mathbf{F}^{-1} \\ &\Leftrightarrow \exists y.(x, y) \in \mathbf{F} \\ &\Leftrightarrow x \in \operatorname{dom}\mathbf{F} \end{aligned}$$

∎

Theorem 3.7.5. *For any relation $\mathbf{F}$, we have $(\mathbf{F}^{-1})^{-1} = \mathbf{F}$.*

PROOF: For any z we have

$$\begin{aligned} z \in (\mathbf{F}^{-1})^{-1} &\Leftrightarrow \exists x, y.z = (x, y) \land (y, x) \in \mathbf{F}^{-1} \\ &\Leftrightarrow \exists x, y.z = (x, y) \land (x, y) \in \mathbf{F} \\ &\Leftrightarrow z \in \mathbf{F} && (\mathbf{F} \text{ is a relation}) \end{aligned}$$

∎

3.8 Composition

Definition 3.8.1 (Composition)**.** Let $\mathbf{R}$ and $\mathbf{S}$ be relations. The *composition* of $\mathbf{R}$ and $\mathbf{S}$ is $\mathbf{S} \circ \mathbf{R} = \{(x, z) \mid \exists y.x\mathbf{R}y \wedge y\mathbf{S}z\}$.

Theorem 3.8.2. *If R and S are small relations then $S \circ R$ is small.*

Proof: It is a subset of $\operatorname{dom} R \times \operatorname{ran} S$. □

Theorem 3.8.3. *For any relations $\mathbf{F}$ and $\mathbf{G}$, we have $(\mathbf{G} \circ \mathbf{F})^{-1} = \mathbf{F}^{-1} \circ \mathbf{G}^{-1}$.*

Proof:

$$\begin{aligned}
(x, z) \in (\mathbf{G} \circ \mathbf{F})^{-1} &\Leftrightarrow (z, x) \in \mathbf{G} \circ \mathbf{F} \\
&\Leftrightarrow \exists y.z\mathbf{F}y \wedge y\mathbf{G}x \\
&\Leftrightarrow \exists y.(y, z) \in \mathbf{F}^{-1} \wedge (x, y) \in \mathbf{G}^{-1} \\
&\Leftrightarrow (x, z) \in \mathbf{F}^{-1} \circ \mathbf{G}^{-1} \qquad \square
\end{aligned}$$

3.9 Restriction

Definition 3.9.1 (Restriction)**.** Let $\mathbf{R}$ be a relation and $\mathbf{A}$ a class. The *restriction* of $\mathbf{R}$ to $\mathbf{A}$ is $\mathbf{R} \upharpoonright \mathbf{A} = \{(x, y) \mid x \in \mathbf{A} \wedge x\mathbf{R}y\}$.

Theorem 3.9.2. *If R is a small relation then $R \upharpoonright \mathbf{A}$ is small.*

Proof: Since it is a subset of R. □

3.10 Image

Definition 3.10.1 (Image)**.** Let $\mathbf{F}$ be a relation and $\mathbf{A}$ a class. The *image* of $\mathbf{A}$ under $\mathbf{F}$ is $\mathbf{F}(\mathbf{A}) = \{\mathbf{F}(x) \mid x \in \mathbf{A}\}$.

Theorem 3.10.2. *If F is small then $F(\mathbf{A})$ is a set.*

Proof: Since it is a subset of $\operatorname{ran} F$. □

Theorem 3.10.3. *For any relation $\mathbf{F}$ and class of sets $\mathcal{A}$ we have*

$$\mathbf{F}\left(\bigcup \mathcal{A}\right) = \bigcup_{A \in \mathcal{A}} \mathbf{F}(A)$$

Proof: Each is the class of all y such that $\exists x.\exists A.x \in A \in \mathcal{A} \wedge y = \mathbf{F}(x)$. □

Theorem 3.10.4. *For any relation $\mathbf{F}$ and classes $\mathbf{A_1}$, ..., $\mathbf{A_n}$, we have*

$$\mathbf{F}(\mathbf{A_1} \cup \cdots \cup \mathbf{A_n}) = \mathbf{F}(\mathbf{A_1}) \cup \cdots \cup \mathbf{F}(\mathbf{A_n}) \ .$$

Proof: Similar. □

Theorem 3.10.5. *For any relation* $\mathbf{F}$ *and class of sets* $\mathcal{A}$*, we have*

$$\mathbf{F}\left(\bigcap \mathcal{A}\right) \subseteq \bigcap_{A \in \mathcal{A}} \mathbf{F}(A) \enspace .$$

Equality holds if $\mathbf{F}$ *is single-rooted and* $\mathcal{A}$ *is nonempty.*

PROOF:
$\langle 1\rangle 1.$ $\mathbf{F}\left(\bigcap \mathcal{A}\right) \subseteq \bigcap_{A \in \mathcal{A}} \mathbf{F}(A)$
 $\langle 2\rangle 1.$ LET: $y \in \mathbf{F}\left(\bigcap \mathcal{A}\right)$
 $\langle 2\rangle 2.$ PICK $x \in \bigcap \mathcal{A}$ such that $y = \mathbf{F}(x)$
 $\langle 2\rangle 3.$ LET: $A \in \mathcal{A}$
 $\langle 2\rangle 4.$ $x \in A$
 $\langle 2\rangle 5.$ $y \in \mathbf{F}(A)$
$\langle 1\rangle 2.$ If $\mathbf{F}$ is single-rooted then $\mathbf{F}\left(\bigcap \mathcal{A}\right) = \bigcap_{A \in \mathcal{A}} \mathbf{F}(A)$
 $\langle 2\rangle 1.$ ASSUME: $\mathbf{F}$ is single-rooted and $\mathcal{A}$ is nonempty.
 $\langle 2\rangle 2.$ LET: $y \in \bigcap_{A \in \mathcal{A}} \mathbf{F}(A)$
 $\langle 2\rangle 3.$ PICK $A \in \mathcal{A}$
 $\langle 2\rangle 4.$ PICK $x \in A$ such that $y = \mathbf{F}(x)$
 $\langle 2\rangle 5.$ $x \in \bigcap \mathcal{A}$
  $\langle 3\rangle 1.$ LET: $A' \in \mathcal{A}$
  $\langle 3\rangle 2.$ PICK $x' \in A'$ such that $y = \mathbf{F}(x')$
  $\langle 3\rangle 3.$ $x = x'$
   PROOF: By $\langle 2\rangle 1$.
  $\langle 3\rangle 4.$ $x \in A'$
$\square$

Theorem 3.10.6. *For any relation* $\mathbf{F}$ *and classes* $\mathbf{A_1}, \ldots, \mathbf{A_n}$*, we have*

$$\mathbf{F}(\mathbf{A_1} \cap \cdots \cap \mathbf{A_n}) \subseteq \mathbf{F}(\mathbf{A_1}) \cap \cdots \cap \mathbf{F}(\mathbf{A_n}) \enspace .$$

Equality holds if $\mathbf{F}$ *is single-rooted.*

PROOF: Similar.

Theorem 3.10.7. *For any relation* $\mathbf{F}$ *and classes* $\mathbf{A}$ *and* $\mathbf{B}$*, we have*

$$\mathbf{F}(\mathbf{A}) - \mathbf{F}(\mathbf{B}) \subseteq \mathbf{F}(\mathbf{A} - \mathbf{B}) \enspace .$$

Equality holds if $\mathbf{F}$ *is single-rooted.*

PROOF:
$\langle 1\rangle 1.$ LET: $\mathbf{F}$, $\mathbf{A}$ and $\mathbf{B}$ be sets.
$\langle 1\rangle 2.$ $\mathbf{F}(\mathbf{A}) - \mathbf{F}(\mathbf{B}) \subseteq \mathbf{F}(\mathbf{A} - \mathbf{B})$
 $\langle 2\rangle 1.$ LET: $y \in \mathbf{F}(\mathbf{A}) - \mathbf{F}(\mathbf{B})$
 $\langle 2\rangle 2.$ PICK $x \in \mathbf{A}$ such that $x\mathbf{F}y$
 $\langle 2\rangle 3.$ $x \in \mathbf{A} - \mathbf{B}$
$\langle 1\rangle 3.$ If $\mathbf{F}$ is single-rooted then $\mathbf{F}(\mathbf{A} - \mathbf{B}) = \mathbf{F}(\mathbf{A}) - \mathbf{F}(\mathbf{B})$.
 $\langle 2\rangle 1.$ ASSUME: $\mathbf{F}$ is single-rooted.

$\langle 2 \rangle 2$. LET: $y \in \mathbf{F}(\mathbf{A} - \mathbf{B})$
$\langle 2 \rangle 3$. PICK $x \in \mathbf{A} - \mathbf{B}$ such that $y = \mathbf{F}(x)$
$\langle 2 \rangle 4$. $y \in \mathbf{F}(\mathbf{A})$
$\langle 2 \rangle 5$. $y \notin \mathbf{F}(\mathbf{B})$
$\langle 3 \rangle 1$. ASSUME: for a contradiction $x' \in \mathbf{B}$ and $x' \mathbf{F} y$
$\langle 3 \rangle 2$. $x' = x$
PROOF: From $\langle 2 \rangle 1$
$\langle 3 \rangle 3$. $x \in \mathbf{B}$
$\langle 3 \rangle 4$. Q.E.D.
PROOF: This contradicts $\langle 2 \rangle 3$.
□

3.11 Reflexive Relations

Definition 3.11.1 (Reflexive)**.** Let $\mathbf{R}$ be a relation on $\mathbf{A}$. Then $\mathbf{R}$ is *reflexive* on A if and only if, for all $x \in \mathbf{A}$, we have $x\mathbf{R}x$.

3.12 Symmetric

Definition 3.12.1 (Symmetric (Pairing))**.** Let $\mathbf{R}$ be a relation. Then $\mathbf{R}$ is *symmetric* if and only if, whenever $x\mathbf{R}y$, then $y\mathbf{R}x$.

3.13 Transitivity

Definition 3.13.1 (Transitivity (Pairing))**.** Let $\mathbf{R}$ be a relation. Then $\mathbf{R}$ is *transitive* if and only if, whenever $x\mathbf{R}y$ and $y\mathbf{R}z$, then $x\mathbf{R}z$.

3.14 Equivalence Relations

Definition 3.14.1 (Equivalence Relation (Pairing))**.** Let $\mathbf{R}$ be a relation on $\mathbf{A}$. Then $\mathbf{R}$ is an *equivalence relation* on $\mathbf{A}$ if and only if $\mathbf{R}$ is reflexive on $\mathbf{A}$, symmetric and transitive.

Theorem 3.14.2. *If $\mathbf{R}$ is a symmetric and transitive relation then $\mathbf{R}$ is an equivalence relation on $\operatorname{fld} \mathbf{R}$.*

PROOF:
$\langle 1 \rangle 1$. LET: $\mathbf{R}$ be a symmetric and transitive relation.
$\langle 1 \rangle 2$. LET: $x \in \operatorname{fld} \mathbf{R}$
$\langle 1 \rangle 3$. PICK y such that $x\mathbf{R}y$ or $y\mathbf{R}x$
$\langle 1 \rangle 4$. $x\mathbf{R}y$ and $y\mathbf{R}x$
PROOF: By symmetry.
$\langle 1 \rangle 5$. $x\mathbf{R}x$
PROOF: By transitivity.
□

3.15 Equivalence Class

Definition 3.15.1 (Equivalence Class)**.** Let $\mathbf{R}$ be an equivalence relation on $\mathbf{A}$ and $a \in \mathbf{A}$. Then the *equivalence class* of a *modulo* $\mathbf{R}$ is

$$[a]_{\mathbf{R}} = \{x \in \mathbf{A} \mid a\mathbf{R}x\} \enspace .$$

Lemma 3.15.2. *Let* $\mathbf{R}$ *be an equivalence relation on* $\mathbf{A}$ *and* $x, y \in \mathbf{A}$*. Then* $[x]_{\mathbf{R}} = [y]_{\mathbf{R}}$ *if and only if* $x\mathbf{R}y$*.*

PROOF:
⟨1⟩1. If $[x]_{\mathbf{R}} = [y]_{\mathbf{R}}$ then $x\mathbf{R}y$.
  ⟨2⟩1. ASSUME: $[x]_{\mathbf{R}} = [y]_{\mathbf{R}}$
  ⟨2⟩2. $y \in [y]_{\mathbf{R}}$
    PROOF: Since $y\mathbf{R}y$ by reflexivity.
  ⟨2⟩3. $y \in [x]_{\mathbf{R}}$
  ⟨2⟩4. $x\mathbf{R}y$
⟨1⟩2. If $x\mathbf{R}y$ then $[x]_{\mathbf{R}} = [y]_{\mathbf{R}}$.
  ⟨2⟩1. ASSUME: $x\mathbf{R}y$
  ⟨2⟩2. $[y]_{\mathbf{R}} \subseteq [x]_{\mathbf{R}}$
    PROOF: If $y\mathbf{R}z$ then $x\mathbf{R}z$ by transitivity.
  ⟨2⟩3. $[x]_{\mathbf{R}} \subseteq [y]_{\mathbf{R}}$
    PROOF: Similar since $y\mathbf{R}x$ by symmetry.
□

3.16 Quotient Sets

Definition 3.16.1 (Quotient Set)**.** Let R be an equivalence relation on A. The *quotient set* A/R is the set of all equivalence classes modulo R.

This is a set because it is a subset of $\mathcal{P}A$.

Theorem 3.16.2. *Let* R *be an equivalence relation on* A*. Then the quotient set* A/R *is a partition of* A*.*

PROOF:
⟨1⟩1. For all $x \in A$ there exists $y \in A$ such that $x \in [y]_R$
  PROOF: Take $y = x$.
⟨1⟩2. Any two distinct equivalence classes are disjoint.
  ⟨2⟩1. ASSUME: $z \in [x]_R$ and $z \in [y]_R$
  ⟨2⟩2. xRz and yRz
  ⟨2⟩3. $[x]_R = [z]_R = [y]_R$
    PROOF: Lemma 3.15.2.
□

3.17 Minimal Elements

Definition 3.17.1 (Minimal)**.** Let R be a binary relation and A a set. An element $a \in A$ is *minimal* w.r.t. R iff there is no $x \in A$ such that xRa.

3.18 Well-Founded Relations

Definition 3.18.1 (Well-Founded)**.** Let R be a relation on A. Then R is *well-founded* iff every nonempty subset of A has an R-minimal element.

Theorem 3.18.2 (Transfinite Induction)**.** *Let R be a well-founded relation on A and $B \subseteq A$. Assume that, for every $t \in A$, if $\{x \in A \mid xRt\} \subseteq B$ then $t \in B$. Then we have $B = A$.*

Proof:
$\langle 1 \rangle 1$. Assume: for a contradiction $B \neq A$
$\langle 1 \rangle 2$. Pick an R-minimal element t of $A - B$
$\langle 1 \rangle 3$. For all $x \in A$, if xRt then $x \in B$
$\langle 1 \rangle 4$. $t \in B$
$\langle 1 \rangle 5$. Q.E.D.
  Proof: This contradicts $\langle 1 \rangle 2$.
□

3.19 Transitive Closure

Theorem 3.19.1. *Let R be a relation. Then there exists a unique relation R^t such that R^t is transitive, $R \subseteq R^t$, and for every transitive relation S with $R \subseteq S$ we have $R^t \subseteq S$.*

Proof:
$\langle 1 \rangle 1$. Let: $R^t = \bigcap\{S \mid R \subseteq S, S \text{ is a transitive relation}\}$
$\langle 1 \rangle 2$. R^t is transitive
  $\langle 2 \rangle 1$. Let: $(x, y), (y, z) \in R^t$
    Prove: $(x, z) \in R^t$
  $\langle 2 \rangle 2$. Let: S be a transitive relation with $R \subseteq S$
  $\langle 2 \rangle 3$. xSy and ySz
  $\langle 2 \rangle 4$. xSz
$\langle 1 \rangle 3$. $R \subseteq R^t$
$\langle 1 \rangle 4$. For any transitive relation S with $R \subseteq S$ we have $R^t \subseteq S$
$\langle 1 \rangle 5$. R^t is unique.
  Proof: If S satisfies the same properties then $R^t \subseteq S$ and $S \subseteq R^t$.
□

Definition 3.19.2 (Transitive Closure)**.** The *transitive closure* of a relation R is this relation R^t.

Theorem 3.19.3. *If R is well-founded then R^t is well-founded.*

Proof:
$\langle 1 \rangle 1$. Let: R be a well-founded relation on A
$\langle 1 \rangle 2$. For all $x, y \in A$, if xR^ty then there exists z such that zR^ty
  $\langle 2 \rangle 1$. Let: $S = \{(x, y) \mid \exists z.zRy\}$
    Prove: $R^t \subseteq S$

$\langle 2\rangle 2.$ S is transitive

$\langle 3\rangle 1.$ ASSUME: xSy and ySz

$\langle 3\rangle 2.$ There exists t such that tRz

$\langle 3\rangle 3.$ xSz

$\langle 2\rangle 3.$ $R \subseteq S$

$\langle 1\rangle 3.$ LET: $B \subseteq A$ be nonempty.

$\langle 1\rangle 4.$ PICK an R-minimal element b of B

$\langle 1\rangle 5.$ b is R^t-minimal.

PROOF: From $\langle 1\rangle 2$.

□

Chapter 4

Functions

4.1 Functions

Definition 4.1.1 (Class Function)**.** A *class function* is a relation $\mathbf{F}$ such that, for all x, y, y', if $x\mathbf{F}y$ and $x\mathbf{F}y'$ then $y = y'$.

If $\mathbf{F}$ is a class function and $x \in \operatorname{dom} \mathbf{F}$, then we write $\mathbf{F}(x)$ for the unique y such that $x\mathbf{F}y$.

We write $\mathbf{F} : \mathbf{A} \to \mathbf{B}$ iff $\mathbf{F}$ is a class function, $\operatorname{dom} \mathbf{F} = \mathbf{A}$ and $\operatorname{ran} \mathbf{F} \subseteq \mathbf{B}$.

A *function* is a class function that is a set.

Theorem 4.1.2. *The Axiom of Choice is equivalent to the following statement:*

For any small relation R, there exists a function $H \subseteq R$ such that $\operatorname{dom} H = \operatorname{dom} R$.

PROOF:

⟨1⟩1. If, for any small relation R, there exists a function $H \subseteq R$ such that $\operatorname{dom} H = \operatorname{dom} R$, then the Axiom of Choice is true.

⟨2⟩1. ASSUME: for any small relation R, there exists a function $H \subseteq R$ such that $\operatorname{dom} H = \operatorname{dom} R$

⟨2⟩2. LET: $\mathcal{A}$ be a set of pairwise disjoint nonempty sets.

⟨2⟩3. LET: $R = \{(A, a) \mid A \in \mathcal{A}, a \in A\}$

⟨2⟩4. PICK a function $H \subseteq R$ such that $\operatorname{dom} H = \operatorname{dom} R$

PROOF: By ⟨2⟩1.

⟨2⟩5. LET: $C = \operatorname{ran} H$

⟨2⟩6. C contains exactly one element from each $A \in \mathcal{A}$, namely $H(A)$.

⟨1⟩2. If the Axiom of Choice is true then, for any small relation R, there exists a function $H \subseteq R$ such that $\operatorname{dom} H = \operatorname{dom} R$.

⟨2⟩1. ASSUME: the Axiom of Choice

⟨2⟩2. LET: R be a small relation.

⟨2⟩3. For $a \in \operatorname{dom} R$,

LET: $R_a = \{(a, b) \mid aRb\}$

⟨2⟩4. LET: $\mathcal{A} = \{R_a \mid a \in \operatorname{dom} R\}$

$\langle 2 \rangle 5$. PICK a set H that contains exactly one element from each R_a.
PROOF: By the Axiom of Choice ($\langle 2 \rangle 1$).
$\langle 2 \rangle 6$. H is a function, $H \subseteq R$ and $\operatorname{dom} H = \operatorname{dom} R$.
□

Theorem 4.1.3. *For any relation $\mathbf{F}$, we have $\mathbf{F}^{-1}$ is a class function if and only if $\mathbf{F}$ is single-rooted.*

PROOF: Immediate from definitions. □

Theorem 4.1.4. *Let $\mathbf{F}$ be a relation. Then $\mathbf{F}$ is a class function if and only if $\mathbf{F}^{-1}$ is single-rooted.*

PROOF: Immediate from definitions. □

Theorem 4.1.5. *Let $\mathbf{F}$ and $\mathbf{G}$ be class functions. Then $\mathbf{G} \circ \mathbf{F}$ is a function, its domain is $\{x \in \operatorname{dom} \mathbf{F} \mid \mathbf{F}(x) \in \operatorname{dom} \mathbf{G}\}$, and for x in this domain, we have $(\mathbf{F} \circ \mathbf{G})(x) = \mathbf{F}(\mathbf{G}(x))$.*

PROOF:
$\langle 1 \rangle 1$. $\mathbf{G} \circ \mathbf{F}$ is a class function.
$\langle 2 \rangle 1$. LET: $x(\mathbf{G} \circ \mathbf{F})z$ and $x(\mathbf{G} \circ \mathbf{F})z'$
$\langle 2 \rangle 2$. PICK y, y' such that $x\mathbf{F}y$, $x\mathbf{F}y'$, $y\mathbf{G}z$ and $y'\mathbf{G}z'$
$\langle 2 \rangle 3$. $y = y'$
PROOF: Since $\mathbf{F}$ is a class function.
$\langle 2 \rangle 4$. $z = z'$
PROOF: Since $\mathbf{G}$ is a class function.
$\langle 1 \rangle 2$. $\operatorname{dom}(\mathbf{G} \circ \mathbf{F}) = \{x \in \operatorname{dom} \mathbf{F} \mid \mathbf{F}(x) \in \operatorname{dom} \mathbf{G}\}$
PROOF:

$$\begin{aligned} x \in \operatorname{dom}(\mathbf{G} \circ \mathbf{F}) &\Leftrightarrow \exists z.x(\mathbf{G} \circ \mathbf{F})z \\ &\Leftrightarrow \exists y, z.x\mathbf{F}y \wedge y\mathbf{G}z \\ &\Leftrightarrow x \in \operatorname{dom} \mathbf{F} \wedge \mathbf{F}(x) \in \operatorname{dom} \mathbf{G} \end{aligned}$$

$\langle 1 \rangle 3$. For x in this domain, we have $(\mathbf{F} \circ \mathbf{G})(x) = \mathbf{F}(\mathbf{G}(x))$.
PROOF: Since $(x, \mathbf{F}(x)) \in \mathbf{F}$ and $(\mathbf{F}(x), \mathbf{G}(\mathbf{F}(x))) \in \mathbf{G}$.
□

Axiom Schema 4.1.6 (Replacement)**.** *Let $\mathbf{H}$ be a class function. If $\operatorname{dom} \mathbf{H}$ is a set then $\mathbf{H}$ is a set.*

4.2 Choice Functions

Definition 4.2.1 (Choice Function)**.** Let $\mathcal{B}$ be a set of nonempty sets. A *choice function* for $\mathcal{B}$ is a function $c : \mathcal{B} \to \bigcup \mathcal{B}$ such that, for all $B \in \mathcal{B}$, we have $c(B) \in B$.

Theorem 4.2.2. *The Axiom of Choice is equivalent to the statement: every set of nonempty sets has a choice function.*

PROOF:
$\langle 1\rangle 1$. If the Axiom of Choice is true then every set of nonempty sets has a choice function.
  $\langle 2\rangle 1$. ASSUME: The Axiom of Choice
  $\langle 2\rangle 2$. LET: $\mathcal{B}$ be a set of nonempty sets.
  $\langle 2\rangle 3$. LET: $R = \{(A, a) \mid A \in \mathcal{B}, a \in A\}$
  $\langle 2\rangle 4$. R is a set.
    PROOF: It is a subset of $\mathcal{B} \times \bigcup \mathcal{B}$.
  $\langle 2\rangle 5$. PICK a function $c \subseteq R$ with $\operatorname{dom} c = \operatorname{dom} R$
    PROOF: Theorem 4.1.2.
  $\langle 2\rangle 6$. $\operatorname{dom} c = \mathcal{B}$
    $\langle 3\rangle 1$. LET: $A \in \mathcal{B}$
    $\langle 3\rangle 2$. PICK $a \in A$
      PROOF: A is nonempty ($\langle 2\rangle 2$)
    $\langle 3\rangle 3$. ARa
      PROOF: By $\langle 2\rangle 3$.
    $\langle 3\rangle 4$. $A \in \operatorname{dom} R$
    $\langle 3\rangle 5$. $A \in \operatorname{dom} c$
      PROOF: By $\langle 2\rangle 5$.
  $\langle 2\rangle 7$. For all $A \in \mathcal{B}$ we have $c(A) \in A$
    PROOF: From $\langle 2\rangle 5$.
$\langle 1\rangle 2$. If every set of nonempty sets has a choice function then the Axiom of Choice is true.
  $\langle 2\rangle 1$. ASSUME: Every set of nonempty sets has a choice function.
  $\langle 2\rangle 2$. LET: $\mathcal{A}$ be a set of pairwise disjoint nonempty sets.
  $\langle 2\rangle 3$. PICK a choice function c for $\mathcal{A}$
  $\langle 2\rangle 4$. LET: $C = \operatorname{ran} c$
  $\langle 2\rangle 5$. C contains exactly one element from each $A \in \mathcal{A}$, namely $c(A)$
□

4.3 Injective Functions

Definition 4.3.1 (Injective)**.** We call a class function *one-to-one*, *injective* or an *injection* if and only if it is single-rooted.

Theorem 4.3.2. *Let* $\mathbf{F}$ *be a one-to-one class function and* $x \in \operatorname{dom} \mathbf{F}$. *Then* $\mathbf{F}^{-1}(\mathbf{F}(x)) = x$.

PROOF: We have $(x, \mathbf{F}(x)) \in \mathbf{F}$ and so $(\mathbf{F}(x), x) \in \mathbf{F}^{-1}$. □

Theorem 4.3.3. *Let* $\mathbf{F}$ *be a one-to-one function and* $y \in \operatorname{ran} \mathbf{F}$. *Then* $\mathbf{F}(\mathbf{F}^{-1}(y)) = y$.

PROOF: From Theorems 3.7.3, 3.7.5 and 4.3.2. □

4.4 Surjective Functions

Definition 4.4.1 (Surjective)**.** Let $\mathbf{F} : \mathbf{A} \to \mathbf{B}$. Then $\mathbf{F}$ is *surjective* if and only if $\operatorname{ran} \mathbf{F} = \mathbf{B}$.

4.5 Bijective Functions

Definition 4.5.1 (Bijective)**.** A class function $\mathbf{F} : \mathbf{A} \to \mathbf{B}$ is *bijective* or a *bijection* if and only if it is injective and surjective.

4.6 Identity Function

Definition 4.6.1 (Identity class function)**.** Let $\mathbf{A}$ be a class. The *identity class function* $\mathrm{id}_{\mathbf{A}}$ on $\mathbf{A}$ is $\{(x, x) \mid x \in \mathbf{A}\}$.

Theorem 4.6.2. *For any set A, we have id_A is a function.*

PROOF: It is a subset of $A \times A$. ∎

Theorem 4.6.3. *Let $F : A \to B$ and A be nonempty. Then there exists a function $G : B \to A$ such that $G \circ F = \mathrm{id}_A$ if and only if F is one-to-one.*

PROOF:
⟨1⟩1. LET: $F : A \to B$
⟨1⟩2. ASSUME: A is nonempty
⟨1⟩3. If there exists $G : B \to A$ such that $G \circ F = \mathrm{id}_A$ then F is one-to-one.
  ⟨2⟩1. ASSUME: $G : B \to A$ and $G \circ F = \mathrm{id}_A$
  ⟨2⟩2. LET: $x, y \in A$
  ⟨2⟩3. ASSUME: $F(x) = F(y)$
  ⟨2⟩4. $x = y$
    PROOF: $x = G(F(x)) = G(F(y)) = y$.
⟨1⟩4. If F is one-to-one then there exists $G : B \to A$ such that $G \circ F = \mathrm{id}_A$.
  ⟨2⟩1. ASSUME: F is one-to-one.
  ⟨2⟩2. PICK $a \in A$
  ⟨2⟩3. Define $G : B \to A$ by: $G(y)$ is the x such that $F(x) = y$ if $y \in \operatorname{ran} F$, otherwise $G(y) = a$
  ⟨2⟩4. $G \circ F = \mathrm{id}_A$
    PROOF: For $x \in A$ we have $(G \circ F)(x) = G(F(x)) = x$ by Theorem 4.1.5.
∎

Theorem 4.6.4. *Let $F : A \to B$ and A be nonempty. If there exists a function $H : B \to A$ such that $F \circ H = \mathrm{id}_B$ then F is surjective.*

PROOF:
⟨1⟩1. LET: $F : A \to B$
⟨1⟩2. ASSUME: A is nonempty.
⟨1⟩3. LET: $H : B \to A$ satisfy $F \circ H = \mathrm{id}_B$

⟨1⟩4. LET: $y \in B$
⟨1⟩5. $F(H(y)) = y$.
□

Theorem 4.6.5 (Choice)**.** *Let $F : A \to B$ and A be nonempty. If F is surjective then there exists a function $H : B \to A$ such that $F \circ H = \mathrm{id}_B$.*

PROOF:
⟨1⟩1. ASSUME: F is surjective.
⟨1⟩2. PICK a function $H \subseteq F^{-1}$ with $\mathrm{dom}\, H = B$
PROOF: By the Axiom of Choice.
⟨1⟩3. $H : B \to A$
⟨1⟩4. $F \circ H = \mathrm{id}_B$
⟨2⟩1. LET: $y \in B$
⟨2⟩2. $(y, H(y)) \in F^{-1}$
⟨2⟩3. $(H(y), y) \in F$
⟨2⟩4. $F(H(y)) = y$
□

4.7 Infinite Cartesian Product

Definition 4.7.1 (Infinite Cartesian Product)**.** Let H be a function with domain I such that, for all $i \in I$, $H(i)$ is a set. The *Cartesian product* $\prod_{i \in I} H(i)$ is the class of all functions f with domain I such that, for all $i \in I$, we have $f(i) \in H(i)$.

Theorem 4.7.2. *If H is a function with domain I and $H(i)$ is a set for all $i \in I$, then $\prod_{i \in I} H(i)$ is a set.*

PROOF: It is a subset of $\mathcal{P}(I \times \bigcup \mathrm{ran}\, H)$. □

Theorem 4.7.3 (Multiplicative Axiom)**.** *The Axiom of Choice is equivalent to the* Multiplicative Axiom*: for any function H with domain I, if $H(i)$ is nonempty for all $i \in I$, then $\prod_{i \in I} H(i)$ is nonempty.*

PROOF:
⟨1⟩1. If the Axiom of Choice is true then the Multiplicative Axiom is true.
⟨2⟩1. ASSUME: The Axiom of Choice
⟨2⟩2. LET: H be a function with domain I such that $H(i)$ is nonempty for all $i \in I$.
⟨2⟩3. PICK a function $f \subseteq \{(i, x) \mid x \in H(i)\}$
⟨2⟩4. $f \in \prod_{i \in I} H(i)$
⟨1⟩2. If the Multiplicative Axiom is true then the Axiom of Choice is true.
⟨2⟩1. ASSUME: for any function H with domain I, if $H(i)$ is nonempty for all $i \in I$, then $\prod_{i \in I} H(i)$ is nonempty
⟨2⟩2. LET: R be a relation.
⟨2⟩3. LET: $I = \mathrm{dom}\, R$

$\langle 2\rangle 4.$ LET: H be the function with domain I such that $H(i) = \{y \mid iRy\}$ for all i.
$\langle 2\rangle 5.$ PICK $f \in \prod_{i \in I} H(i)$
$\langle 2\rangle 6.$ $f \subseteq R$
□

4.8 Quotient Sets

Definition 4.8.1 (Canonical Map)**.** Let R be an equivalence relation on A. The *canonical map* $\phi : A \to A/R$ is the function defined by $\phi(a) = [a]_R$.

Theorem 4.8.2. *Let R be an equivalence relation on A and $F : A \to B$. Then the following are equivalent:*

1. *For all $x, y \in A$, if xRy then $F(x) = F(y)$.*
2. *There exists $G : A/R \to B$ such that $F = G \circ \phi$, where $\phi : A \to A/R$ is the canonical map.*

In this case, G is unique.

PROOF:
$\langle 1\rangle 1.$ $1 \Rightarrow 2$
$\langle 2\rangle 1.$ ASSUME: 1
$\langle 2\rangle 2.$ Let $G = \{([a]_R, b) \mid F(a) = b\}$
$\langle 2\rangle 3.$ G is a function.
$\langle 3\rangle 1.$ LET: $(c, b), (c, b') \in G$
$\langle 3\rangle 2.$ PICK $a, a' \in A$ such that $c = [a]_R = [a']_R$ with $F(a) = b$ and $F(a') = b'$
$\langle 3\rangle 3.$ aRa'
PROOF: Lemma 3.15.2.
$\langle 3\rangle 4.$ $F(a) = F(a')$
PROOF: From $\langle 2\rangle 1$.
$\langle 3\rangle 5.$ $b = b'$
PROOF: From $\langle 3\rangle 2$.
$\langle 2\rangle 4.$ $F = G \circ \phi$
PROOF: For $a \in A$ we have $G(\phi(a)) = G([a]) = F(a)$.
$\langle 1\rangle 2.$ $2 \Rightarrow 1$
$\langle 2\rangle 1.$ LET: $G : A/R \to B$ be such that $F = G \circ \phi$
$\langle 2\rangle 2.$ LET: $x, y \in A$
$\langle 2\rangle 3.$ ASSUME: xRy
$\langle 2\rangle 4.$ $G([x]) = G([y])$
PROOF: Lemma 3.15.2
$\langle 2\rangle 5.$ $F(x) = F(y)$
PROOF: From $\langle 2\rangle 1$.
$\langle 1\rangle 3.$ If $G, G' : A/R \to B$ and $G \circ \phi = G' \circ \phi$ then $G = G'$
PROOF: For any $a \in A$ we have $G([a]) = G'([a])$.
□

4.9 Transfinite Recursion

Theorem 4.9.1 (Transfinite Recursion)**.** *Let R be a well-founded relation on a set C.*

Let $\mathbf{A}$ be a class. Let $\mathbf{B}$ be the class of all functions from a subset of C to $\mathbf{A}$. Let $\mathbf{F} : \mathbf{B} \times C \to \mathbf{A}$ be a class function.

Then there exists a unique function $f : C \to \mathbf{A}$ such that, for all $t \in C$, we have $f(t) = \mathbf{F}(f \restriction \{x \in C \mid xRt\}, t)$.

PROOF:

$\langle 1 \rangle 1$. Let us say a function v is *acceptable* if and only if $\operatorname{dom} v \subseteq C$, $\operatorname{ran} v \subseteq \mathbf{A}$ and, for all $t \in \operatorname{dom} v$, we have $\{x \in C \mid xRt\} \subseteq \operatorname{dom} v$ and $v(t) = \mathbf{F}(v \restriction \{x \in C \mid xRt\})$

$\langle 1 \rangle 2$. LET: $\mathcal{K}$ be the set of all acceptable functions.

PROOF: This is a set by an Axiom of Replacement.

$\langle 1 \rangle 3$. LET: $h = \bigcup \mathcal{K}$

$\langle 1 \rangle 4$. h is a function.

$\langle 2 \rangle 1$. LET: $x \in C$

$\langle 2 \rangle 2$. ASSUME: as induction hypothesis, for all yRx we have that, whenever (y, a) and (y, b) in h then $a = b$

$\langle 2 \rangle 3$. ASSUME: (x, a) and (x, b) are in h

$\langle 2 \rangle 4$. PICK acceptable u and v such that $u(x) = a$ and $v(x) = b$

$\langle 2 \rangle 5$. For all yRx we have $u(y) = v(y)$

PROOF: From $\langle 2 \rangle 2$ since $(y, u(y))$ and $(y, v(y))$ are in h.

$\langle 2 \rangle 6$. $a = b$

PROOF:

$$\begin{aligned}
a &= u(x) && (\langle 2 \rangle 4) \\
&= \mathbf{F}(u \restriction \{y \in C \mid yRx\}) \\
&= \mathbf{F}(v \restriction \{y \in C \mid yRx\}) && (\langle 2 \rangle 5) \\
&= v(x) \\
&= b && (\langle 2 \rangle 4)
\end{aligned}$$

$\langle 2 \rangle 7$. Q.E.D.

PROOF: By transfinite induction, for all $x \in C$, if (x, a) and (x, b) are in h then $a = b$.

$\langle 1 \rangle 5$. h is acceptable.

$\langle 2 \rangle 1$. LET: $t \in \operatorname{dom} h$

$\langle 2 \rangle 2$. PICK v acceptable such that $(t, h(t)) \in v$

$\langle 2 \rangle 3$. $\{x \in C \mid xRt\} \subseteq \operatorname{dom} v$ and $v(t) = \mathbf{F}(v \restriction \{x \in C \mid xRt\})$

$\langle 2 \rangle 4$. $v \restriction \{x \in C \mid xRt\} = h \restriction \{x \in C \mid xRt\}$

PROOF: By $\langle 1 \rangle 4$.

$\langle 2 \rangle 5$. $h(t) = \mathbf{F}(h \restriction \{x \in C \mid xRt\})$

$\langle 1 \rangle 6$. $\operatorname{dom} h = C$

$\langle 2 \rangle 1$. LET: $t \in C$

$\langle 2 \rangle 2$. ASSUME: as induction hypothesis, for all xRt, we have $x \in \operatorname{dom} h$

$\langle 2 \rangle 3$. ASSUME: for a contradiction $t \notin \operatorname{dom} h$

⟨2⟩4. $h \cup (t, \mathbf{F}(h \restriction \{x \in C \mid xRt\})$ is acceptable)
⟨2⟩5. $h \cup (t, \mathbf{F}(h \restriction \{x \in C \mid xRt\}) \subseteq h$ is acceptable)
⟨2⟩6. Q.E.D.
PROOF: This contradicts ⟨2⟩3. Thus, by transfinite induction, for all $t \in C$ we have $t \in \operatorname{dom} h$.

⟨1⟩7. If $h' : C \to \mathbf{A}$ is acceptable then $h' = h$.
⟨2⟩1. LET: $t \in C$
⟨2⟩2. ASSUME: as induction hypothesis, for all xRt, we have $h'(x) = h(x)$
⟨2⟩3. $h'(t) = h(t)$
PROOF:

$$\begin{aligned} h'(t) &= \mathbf{F}(h' \restriction \{x \in C \mid xRt\}) \\ &= \mathbf{F}(h \restriction \{x \in C \mid xRt\}) && (\langle 2\rangle 2) \\ &= h(t) \end{aligned}$$

⟨2⟩4. Q.E.D.
PROOF: By transfinite induction, for all $t \in C$, we have $h'(t) = h(t)$.

∎

4.10 Fixed Points

Definition 4.10.1 (Fixed Point)**.** Let X be a set. Let $f : X \to X$. Then a *fixed point* of f is an element $a \in X$ such that $f(a) = a$.

Chapter 5

Cardinal Numbers

5.1 Equinumerosity

Definition 5.1.1 (Equinumerous)**.** Two sets A and B are *equinumerous* if and only if there exists a bijection between them.

Theorem 5.1.2. *Equinumerosity is an equivalence relation on the class of all sets.*

Theorem 5.1.3 (Cantor)**.** *No set is equinumerous with its power set.*

Definition 5.1.4. We say a set A is *dominated* by B, $A \preccurlyeq B$, iff A is equinumerous with a subset of B.

Theorem 5.1.5. $A \preccurlyeq A$

Theorem 5.1.6. *If* $A \preccurlyeq B \preccurlyeq C$ *then* $A \preccurlyeq C$.

Theorem 5.1.7 (Schröder-Bernstein Theorem)**.** *If* $A \preccurlyeq B$ *and* $B \preccurlyeq A$ *then* $A \equiv B$.

PROOF:
$\langle 1 \rangle 1.$ LET: $f : A \to B$ and $g : B \to A$ be injections.
$\langle 1 \rangle 2.$ Define a sequence of sets $C_n \subseteq A$ by

$$C_0 = A - \operatorname{ran} g$$
$$C_{n+1} = g(f(C_n))$$

$\langle 1 \rangle 3.$ Define $h : A \to B$ by

$$h(x) = \begin{cases} f(x) & \text{if } x \in \bigcup_n C_n \\ g^{-1}(x) & \text{if not} \end{cases}$$

$\langle 1 \rangle 4.$ h is a bijection.
□

Theorem 5.1.8 (AC)**.** *For any infinite set A we have* $\mathbb{N} \preccurlyeq A$.

PROOF: Given a choice funtion f for A, choose a sequence (a_n) in A by $a_n = f(A - \{a_0, \dots, a_{n-1}\})$. $\square$

Corollary 5.1.8.1 (AC)**.** *A set is infinite if and only if it is equinumerous with a proper subset.*

5.2 Countability

Definition 5.2.1 (Countable)**.** A set A is *countable* iff $A \preccurlyeq \mathbb{N}$.

Theorem 5.2.2 (AC)**.** *A countable union of countable sets is countable.*

Proposition 5.2.3 (AC)**.** *Every infinite set has a countable subset.*

5.3 Order Theory

Definition 5.3.1 (Preorder)**.** Let X be a set. A *preorder* on X is a binary relation $\leq$ on X such that:

Reflexivity For all $x \in X$, we have $x \leq x$

Transitivity For all $x, y, z \in X$, if $x \leq y$ and $y \leq z$ then $x \leq z$.

Definition 5.3.2 (Preordered Set)**.** A *preordered set* consists of a set X and a preorder $\leq$ on X.

Proposition 5.3.3. *Let X and Y be linearly ordered sets. Let $f : X \twoheadrightarrow Y$ be strictly monotone and surjective. Then f is a poset isomorphism.*

PROOF:
⟨1⟩1. f is injective.
  ⟨2⟩1. LET: $x, y \in X$
  ⟨2⟩2. ASSUME: $f(x) = f(y)$
  ⟨2⟩3. $x \not< y$
    PROOF: By strong motonicity.
  ⟨2⟩4. $y \not< x$
    PROOF: By strong motonicity.
  ⟨2⟩5. $x = y$
    PROOF: By trichotomy.
⟨1⟩2. f^{-1} is monotone.
  ⟨2⟩1. LET: $x, y \in X$
  ⟨2⟩2. ASSUME: $x \leq y$
  ⟨2⟩3. $f^{-1}(x) \not> f^{-1}(y)$
    PROOF: By strong motonicity.
  ⟨2⟩4. $f^{-1}(x) < f^{-1}(y)$
    PROOF: By trichotomy.
□

Definition 5.3.4 (Interval)**.** Let X be a preordered set and $Y \subseteq X$. Then Y is an *interval* if and only if, for all $a, b \in Y$ and $c \in X$, if $a \leq c \leq b$ then $c \in Y$.

Definition 5.3.5 (Linear Continuum)**.** A linearly ordered set L is a *linear continuum* if and only if:

1. every nonempty subset of L that is bounded above has a supremum
2. L is dense

Proposition 5.3.6. *Every interval in a linear continuum is a linear continuum.*

PROOF:
⟨1⟩1. LET: L be a linear continuum and I an interval in L.
⟨1⟩2. Every nonempty subset of I that is bounded above has a supremum in I.
  ⟨2⟩1. LET: $X \subseteq I$ be nonempty and bounded above by $b \in I$.

$\langle 2 \rangle 2$. LET: s be the supremum of X in L.
PROOF: Since L is a linear continuum.
$\langle 2 \rangle 3$. $s \in I$
$\langle 3 \rangle 1$. PICK $a \in X$
PROOF: Since X is nonempty ($\langle 2 \rangle 1$).
$\langle 3 \rangle 2$. $a \leq s \leq b$
$\langle 3 \rangle 3$. $s \in I$
PROOF: Since I is an interval ($\langle 1 \rangle 1$).
$\langle 2 \rangle 4$. s is the supremum of X in I
$\langle 1 \rangle 3$. I is dense.
$\langle 2 \rangle 1$. LET: $x, y \in I$ with $x < y$
$\langle 2 \rangle 2$. PICK $z \in L$ with $x < z < y$
PROOF: Since L is dense.
$\langle 2 \rangle 3$. $z \in I$
PROOF: Since I is an interval.
□

Definition 5.3.7 (Ordered Square)**.** The *ordered square* I_o^2 is the set $[0,1]^2$ under the dictionary order.

Proposition 5.3.8. *The ordered square is a linear continuum.*

PROOF:
$\langle 1 \rangle 1$. Every nonempty subset of I_o^2 bounded above has a supremum.
$\langle 2 \rangle 1$. LET: $X \subseteq I_o^2$ be nonempty and bounded above by (b, c)
$\langle 2 \rangle 2$. LET: $s = \sup \pi_1(X)$
PROOF: The set $\pi_1(X)$ is nonempty and bounded above by b.
$\langle 2 \rangle 3$. CASE: $s \in \pi_1(X)$
$\langle 3 \rangle 1$. LET: $t = \sup\{y \in [0,1] \mid (s, y) \in X\}$
PROOF: This set is nonempty and bounded above by c.
$\langle 3 \rangle 2$. (s, t) is the supremum of X.
$\langle 2 \rangle 4$. CASE: $s \notin \pi_1(X)$
PROOF: In this case $(s, 0)$ is the supremum of X.
$\langle 1 \rangle 2$. I_o^2 is dense.
$\langle 2 \rangle 1$. LET: $(x_1, y_1), (x_2, y_2) \in I_o^2$ with $(x_1, y_1) < (x_2, y_2)$
$\langle 2 \rangle 2$. CASE: $x_1 < x_2$
$\langle 3 \rangle 1$. PICK x_3 with $x_1 < x_3 < x_2$
$\langle 3 \rangle 2$. $(x_1, y_1) < (x_3, y_1) < (x_2, y_2)$
$\langle 2 \rangle 3$. CASE: $x_1 = x_2$ and $y_1 < y_2$
$\langle 3 \rangle 1$. PICK y_3 with $y_1 < y_3 < y_2$
$\langle 3 \rangle 2$. $(x_1, y_1) < (x_1, y_3) < (x_2, y_2)$
□

Proposition 5.3.9. *If X is a well-ordered set then $X \times [0,1)$ under the dictionary order is a linear continuum.*

PROOF:
$\langle 1 \rangle 1$. Every nonempty set $A \subseteq X \times [0,1)$ bounded above has a supremum.

⟨2⟩1. LET: $A \subseteq X \times [0, 1)$ be nonempty and bounded above
⟨2⟩2. LET: x_0 be the supremum of $\pi_1(A)$
⟨2⟩3. CASE: $x_0 \in \pi_1(A)$
⟨3⟩1. LET: y_0 be the supremum of $\{y \in [0, 1) \mid (x_0, y) \in A\}$
⟨3⟩2. (x_0, y_0) is the supremum of A.
⟨2⟩4. CASE: $x_0 \notin \pi_1(A)$
PROOF: In this case $(x_0, 0)$ is the supremum of A.
⟨1⟩2. $X \times [0, 1)$ is dense.
⟨2⟩1. LET: $(x_1, y_1), (x_2, y_2) \in X \times [0, 1)$ with $(x_1, y_1) < (x_2, y_2)$
⟨2⟩2. CASE: $x_1 < x_2$
⟨3⟩1. PICK y_3 such that $y_1 < y_3 < 1$
⟨3⟩2. $(x_1, y_1) < (x_1, y_3) < (x_2, y_2)$
⟨2⟩3. CASE: $x_1 = x_2$ and $y_1 < y_2$
⟨3⟩1. PICK y_3 such that $y_1 < y_3 < y_2$
⟨3⟩2. $(x_1, y_1) < (x_1, y_3) < (x_2, y_2)$
□

Lemma 5.3.10. *For all $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$, we have $[a, b) \cong [c, d)$*

PROOF: The map $\lambda t.c + (t - a)(d - c)/(b - a)$ is an order isomorphism.

Proposition 5.3.11. *Let X be a linearly ordered set. Let $a < b < c$ in X. Then $[a, c) \cong [0, 1)$ if and only if $[a, b) \cong [b, c) \cong [0, 1)$.*

PROOF:
⟨1⟩1. If $[a, c) \cong [0, 1)$ then $[a, b) \cong [b, c) \cong [0, 1)$
⟨2⟩1. ASSUME: $f : [a, c) \cong [0, 1)$ is an order isomorphism
⟨2⟩2. $[a, b) \cong [0, 1)$
PROOF:

$$\begin{aligned} [a, b) &\cong [0, f(b)) && \text{(by the restriction of } f\text{)} \\ &\cong [0, 1) && \text{(Lemma 5.3.10)} \end{aligned}$$

⟨2⟩3. $[b, c) \cong [0, 1)$
PROOF: Similar.
⟨1⟩2. If $[a, b) \cong [b, c) \cong [0, 1)$ then $[a, c) \cong [0, 1)$
PROOF:

$$\begin{aligned} [a, c) &= [a, b) * [b, c) \\ &\cong [0, 1) * [0, 1) \\ &\cong [0, 1/2) * [1/2, 1) && \text{(Lemma 5.3.10)} \\ &= 1 \end{aligned}$$

□

Proposition 5.3.12 (CC)**.** *Let X be a linearly ordered set. Let $x_0 < x_1 < \cdots$ be a strictly increasing sequence in X with supremum b. Then $[x_0, b) \cong [0, 1)$ if and only if $[x_i, x_{i+1}) \cong [0, 1)$ for all i.*

PROOF:

⟨1⟩1. If $[x_0, b) \cong [0, 1)$ then $[x_i, x_{i+1}) \cong [0, 1)$ for all i.
  PROOF: By Lemma 5.3.10
⟨1⟩2. If $[x_i, x_{i+1}) \cong [0, 1)$ for all i then $[x_0, b) \cong [0, 1)$
  ⟨2⟩1. ASSUME: $[x_i, x_{i+1}) \cong [0, 1)$ for all i
  ⟨2⟩2. PICK an order isomorphism $f_i : [x_i, x_{i+1}) \cong [1/2^i, 2/2^{i+1})$ for each i.
    PROOF: By Lemma 5.3.10
  ⟨2⟩3. The union of the f_is is an order isomorphism $[x_0, b) \cong [0, 1)$
□

5.4 Partially Ordered Sets

Definition 5.4.1 (Partial Order)**.** A *partial order* on a set X is a preorder $\leq$ that is *anti-symmetric*, i.e. whenever $x \leq y$ and $y \leq x$ then $x = y$.

5.5 Strict Partial Order

Definition 5.5.1 (Strict Partial Order)**.** A *strict partial order* on a set X is a relation on X that is transitive and irreflexive.

Proposition 5.5.2. *If $<$ is a strict partial order on X and $x, y \in X$, then at most one of $x < y$, $y < x$, $x = y$ holds.*

Proposition 5.5.3. *If $<$ is a strict partial order then the relation $\leq$ defined by: $x \leq y$ iff $x < y$ or $x = y$, is a partial order.*

Theorem 5.5.4. *If R is a well-founded relation then its transitive closure is a partial order.*

Definition 5.5.5 (Linear Order)**.** A *linear order* on a set X is a partial order such that, for any $x, y \in X$, either $x \leq y$ or $y \leq x$.

5.6 Strict Linear Orders

Definition 5.6.1 (Strict Linear Order (Extensionality, Pairing))**.** Let A be a set. A *strict linear order* on A is a binary relation R on A that is transitive and satisfies *trichotomy*: for any $x, y \in A$, exactly one of xRy, $x = y$, yRx holds.

Theorem 5.6.2. *Let R be a strict linear order on A. Then there is no $x \in A$ such that xRx.*

PROOF: Immediate from trichotomy.

5.7 Well Orderings

Definition 5.7.1 (Well-ordering)**.** A *well-order* on a set X is a linear order such that every nonempty set has a least element.

Proposition 5.7.2. *Let $\leq$ be a linear order on X. Then $\leq$ is a well-order iff there is no function $f : \mathbb{N} \to X$ such that $f(n+1) < f(n)$ for all n.*

Definition 5.7.3 (Initial Segment)**.** Given a well-ordered set X and $\alpha \in X$, the *initial segment* of X up to α is $\operatorname{seg} \alpha = \{x \in X \mid x < \alpha\}$.

Theorem 5.7.4 (Transfinite Induction)**.** *Let $\leq$ be a linear order on J. Then the following are equivalent:*

1. *$\leq$ is a well-order on J.*
2. *For every subset $J_0 \subseteq J$, if the following condition holds:*
   - *For every $\alpha \in J$, if $\operatorname{seg} \alpha \subseteq J_0$ then $\alpha \in J$.*

   then $J_0 = J$.

Theorem 5.7.5 (Transfinite Recursion)**.** *Let J be a well-ordered set and C a set. Let $\mathcal{F}$ be the set of all functions from a section of J to C. Let G be a function with domain $\mathcal{F}$. Then there exists a unique function h with domain J such that, for all $\alpha \in J$, we have $h(\alpha) = \rho(h \restriction \operatorname{seg} \alpha)$.*

PROOF:
$\langle 1 \rangle 1$. If v is a function and $t \in J$, we say v is *ρ-constructed up to t* iff $\operatorname{dom} v = \{x \in J \mid x \leq t\}$ and, for all $x \in \operatorname{dom} v$, we have $v(x) = \rho(v \restriction \operatorname{seg} x)$
$\langle 1 \rangle 2$. If $t_1 \leq t_2$, v_1 is ρ-constructed up to t_1, and v_2 is ρ-constructed up to t_2, then $v_1 = v_2 \restriction \{x \in J \mid x \leq t_1\}$
$\langle 1 \rangle 3$. LET: $\mathcal{K}$ be the set of all functions that are ρ-constructed up to some $t \in J$
  PROOF: $\mathcal{K}$ is a set by a Replacement Axiom.
$\langle 1 \rangle 4$. LET: $F = \bigcup \mathcal{K}$
$\langle 1 \rangle 5$. F is a function
$\langle 1 \rangle 6$. For all $x \in \operatorname{dom} F$ we have $F(x) = \rho(F \restriction \operatorname{seg} x)$
$\langle 1 \rangle 7$. $\operatorname{dom} F = J$
$\langle 1 \rangle 8$. F is unique
$\square$

Theorem 5.7.6. *The following are equivalent.*

1. *The Axiom of Choice*
2. *(*Well-Ordering Theorem*) Every set has a well-ordering.*
3. *(*Zorn's Lemma*) Let X be a poset. If every chain in X has an upper bound in X, then X has a maximal element.*

PROOF:
$\langle 1 \rangle 1$. $1 \Rightarrow 2$
  PROOF:
  $\langle 2 \rangle 1$. ASSUME: The Axiom of Choice
  $\langle 2 \rangle 2$. LET: X be a set.

⟨2⟩3. Pick a choice function for $\mathcal{P}X \setminus \{\emptyset\}$
Proof: Lemma **??**.
⟨2⟩4. Let: a *tower* in X be a pair $(T, <)$ where $T \subseteq X$, $<$ is a well-ordering of T, and $x = c(X \setminus \{y \in T \mid y < x\})$.
⟨2⟩5. For any two towers $(T_1, <_1)$ and $(T_2, <_2)$, either these two posets are equal or one is a section of the other.
⟨3⟩1.
⟨2⟩6. For any tower $(T, <)$ in X with $T \neq X$, there exists a tower in X of which $(T, <)$ is a section.
⟨2⟩7. Let: $T = \bigcup\{T' \subseteq X \mid \exists R.(T, R) \text{ is a tower in } X\}$
⟨2⟩8. Define $<$ on T by: $x < y$ iff there exists a tower (T, R) in X such that $x, y \in T$ and xRy.
⟨2⟩9. $(T, <)$ is a tower in X.
⟨2⟩10. $T = X$
⟨2⟩11. $<$ is a well-ordering of X.
⟨1⟩2. $2 \Rightarrow 3$
⟨2⟩1. Assume: The Well-Ordering Theorem
⟨2⟩2. Let: X be a poset in which every chain has an upper bound.
⟨2⟩3. Pick a well-ordering R of X
⟨2⟩4. Define $F : X \to \{0, 1\}$ by transfinite R-recursion by:

$$F(a) = \begin{cases} 1 & \text{if } b < a \text{ for all } b \text{ such that } bRa \text{ and } f(b) = 1 \\ 0 & \text{otherwise} \end{cases}$$

⟨2⟩5. Let: $C = \{a \in X \mid f(a) = 1\}$
⟨2⟩6. C is a chain in X
⟨3⟩1. Let: $x, y \in C$
⟨3⟩2. Assume: without loss of generality xRy
⟨3⟩3. $f(y) = 1$
⟨3⟩4. for all z such that zRy and $f(z) = 1$ we have $z < y$
⟨3⟩5. $x < y$
⟨2⟩7. Pick an upper bound u for C
⟨2⟩8. u is maximal in X
⟨3⟩1. Let: $x \in X$ with $u \leq x$
⟨3⟩2. for all b such that bRx and $f(b) = 1$ we have $b < x$
Proof: Since $b \in C$ so $b \leq u \leq x$
⟨3⟩3. $f(u) = 1$
⟨3⟩4. $u \leq x$
⟨3⟩5. $u = x$
⟨2⟩9. $3 \Rightarrow 1$
⟨3⟩1. Assume: Zorn's Lemma
⟨3⟩2. Let: R be a relation
⟨3⟩3. Let: $\mathcal{A}$ be the poset of functions that are subsets of R under $\subseteq$
⟨3⟩4. Every chain in $\mathcal{A}$ has an upper bound
⟨4⟩1. Let: $\mathcal{C} \subseteq \mathcal{A}$ be a chain.
Prove: $\bigcup \mathcal{C} \in \mathcal{A}$
⟨4⟩2. Assume: $(x, y), (x, z) \in \bigcup \mathcal{C}$

⟨4⟩3. PICK $f, g \in \mathcal{C}$ such that $f(x) = y$ and $g(x) = z$
⟨4⟩4. ASSUME: without loss of generality $f \subseteq g$
⟨4⟩5. $g(x) = y$
⟨4⟩6. $y = z$
⟨3⟩5. PICK F maximal in $\mathcal{A}$
⟨3⟩6. $\operatorname{dom} F = \operatorname{dom} R$
⟨4⟩1. ASSUME: for a contradiction $x \in \operatorname{dom} R - \operatorname{dom} F$
⟨4⟩2. PICK y such that xRy
⟨4⟩3. LET: $G = F \cup \{(x, y)\}$
⟨4⟩4. $G \in \mathcal{A}$
⟨4⟩5. $F \subset G$
⟨4⟩6. Q.E.D.
PROOF: This contradicts the maximality of F.

□

Lemma 5.7.7 (Choice)**.** *Let X be a set. Let $\mathcal{A} \subseteq \mathcal{P}X$ have the finite intersection property. Then there exists a maximal set $\mathcal{D}$ such that $\mathcal{A} \subseteq \mathcal{D} \subseteq \mathcal{P}X$ and $\mathcal{D}$ has the finite intersection property.*

PROOF:
⟨1⟩1. LET: $\mathbb{F} = \{\mathcal{D} \mid \mathcal{A} \subseteq \mathcal{D} \subseteq \mathcal{P}X, \mathcal{D}$ has the finite intersection property$\}$
⟨1⟩2. Every chain in $\mathbb{F}$ has an upper bound.
⟨2⟩1. LET: $\mathbb{C}$ be a chain in $\mathbb{F}$.
⟨2⟩2. ASSUME: without loss of generality $\mathbb{C} \neq \emptyset$
PROVE: $\bigcup \mathbb{C} \in \mathbb{F}$
PROOF: If $\mathbb{C} = \emptyset$ then $\mathcal{A}$ is an upper bound.
⟨2⟩3. $\mathcal{A} \subseteq \bigcup \mathbb{C} \subseteq \mathcal{P}X$
⟨2⟩4. LET: $C_1, \ldots, C_n \in \mathbb{C}$
PROVE: $C_1 \cap \cdots \cap C_n \neq \emptyset$
⟨2⟩5. PICK $\mathcal{C}_1, \ldots, \mathcal{C}_n \in \mathbb{C}$ such that $C_i \in \mathcal{C}_i$ for all i.
⟨2⟩6. ASSUME: without loss of generality $\mathcal{C}_1 \subseteq \cdots \subseteq \mathcal{C}_n$
⟨2⟩7. $C_1, \ldots, C_n \in \mathcal{C}_n$
⟨2⟩8. $\mathcal{C}_n$ satisfies the finite intersection property.
⟨2⟩9. $C_1 \cap \cdots \cap C_n \neq \emptyset$
⟨1⟩3. Q.E.D.
PROOF: By Zorn's Lemma.

□

Theorem 5.7.8 (Cardinal Comparability)**.** *The Axiom of Choice is equivalent to the* Cardinal Comparability Theorem*: for any two sets A and B, either $A \preccurlyeq B$ or $B \preccurlyeq A$.*

PROOF:
⟨1⟩1. Zorn's Lemma implies Cardinal Comparability
⟨2⟩1. ASSUME: Zorn's Lemma
⟨2⟩2. LET: A and B be sets.
⟨2⟩3. LET: $\mathcal{A}$ be the poset of all injective functions f such that $\operatorname{dom} f \subseteq C$ and $\operatorname{ran} f \subseteq D$ under $\subseteq$

⟨2⟩4. Every chain in $\mathcal{A}$ has an upper bound.
⟨3⟩1. LET: $\mathcal{C} \subseteq \mathcal{A}$ be a chain.
PROVE: $\bigcup \mathcal{C} \in \mathcal{A}$
⟨3⟩2. $\bigcup \mathcal{C}$ is a function.
⟨4⟩1. LET: $(x, y), (x, z) \in \bigcup \mathcal{C}$
⟨4⟩2. PICK $f, g \in \mathcal{C}$ such that $f(x) = y$ and $g(x) = z$
⟨4⟩3. ASSUME: without loss of generality $f \subseteq g$
⟨4⟩4. $g(x) = y$
⟨4⟩5. $y = z$
⟨3⟩3. $\bigcup \mathcal{C}$ is injective.
PROOF: Similar.
⟨2⟩5. PICK $\hat{f}$ maximal in $\mathcal{A}$
PROOF: By Zorn's Lemma.
⟨2⟩6. Either $\mathrm{dom}\, \hat{f} = C$ or $\mathrm{ran}\, \hat{f} = D$
⟨3⟩1. ASSUME: for a contradiction $\mathrm{dom}\, \hat{f} \subset C$ and $\mathrm{ran}\, \hat{f} \subset D$
⟨3⟩2. PICK $x \in C - \mathrm{dom}\, \hat{f}$ and $y \in D - \mathrm{ran}\, \hat{f}$
⟨3⟩3. LET: $g = \hat{f} \cup \{(x, y)\}$
⟨3⟩4. $g \in \mathcal{A}$
⟨3⟩5. $\hat{f} \subset g$
⟨3⟩6. Q.E.D.
PROOF: This contradicts the maximality of $\hat{f}$.
⟨2⟩7. If $\mathrm{dom}\, \hat{f} = C$ then $C \preccurlyeq D$
⟨2⟩8. If $\mathrm{ran}\, \hat{f} = D$ then $D \preccurlyeq C$
⟨1⟩2. Cardinal Comparability implies the Well-Ordering Theorem
⟨2⟩1. ASSUME: Cardinal Comparability
⟨2⟩2. LET: A be a set
⟨2⟩3. PICK an ordinal α such that $\alpha \not\preccurlyeq A$
⟨2⟩4. $A \preccurlyeq \alpha$
PROOF: By Cardinal Comparability.
⟨2⟩5. PICK an injection $f : A \to \alpha$
⟨2⟩6. Define $<$ on A by $x < y$ iff $f(x) \in f(y)$
⟨2⟩7. $<$ is a well-ordering on A.
□

Theorem 5.7.9. *Given two well-ordered sets A and B, either $A \cong B$ or one of A, B is isomorphic to an initial segment of the other.*

5.8 Ordinal Numbers

Definition 5.8.1. Let $(A, \leq)$ be a well-ordered set. The *ordinal number* of $(A, \leq)$ is the range of E, where E is the unique function with domain A such that $E(t) = \mathrm{ran}(E \upharpoonright \mathrm{seg}\, t)$ for all $t \in A$.

Theorem 5.8.2. *Let $(A, \leq)$ be a well-ordered set and $E : A \to \alpha$ be the canonical function onto the ordinal of A. Then:*

1. *For all $t \in A$ we have $E(t) \notin E(t)$.*

2. *E is a bijection.*

3. *For any $s, t \in A$, we have $s < t$ if and only if $E(s) \in E(t)$.*

4. *α is a transitive set.*

5. *α is well-ordered by $\in$*

6. *E is an order isomorphism between $(A, \leq)$ and $(\alpha, \underline{\in})$.*

Theorem 5.8.3. *Two well-ordered sets are isomorphic if and only if they have the same ordinal number.*

Theorem 5.8.4. *A set is an ordinal number if and only if it is a transitive set well-ordered by $\in$.*

Theorem 5.8.5. *Every member of an ordinal number is an ordinal number.*

Theorem 5.8.6. *Any transitive set of ordinal numbers is an ordinal number.*

Theorem 5.8.7. *The empty set is an ordinal number.*

Theorem 5.8.8. *The successor of an ordinal number is an ordinal number.*

Theorem 5.8.9. *If A is a set of ordinal numbers then $\bigcup A$ is an ordinal number.*

Theorem 5.8.10. *Any nonempty set of ordinal numbers has a least element.*

Theorem 5.8.11 (Burali-Forti Paradox)**.** *The class of ordinal numbers is a proper class.*

Theorem 5.8.12 (Hartogs' Theorem)**.** *For any set A, there exists an ordinal that is not dominated by A.*

PROOF:
$\langle 1 \rangle$1. LET: α be the class of all ordinals β such that $\beta \preccurlyeq A$
$\langle 1 \rangle$2. α is a set.
  $\langle 2 \rangle$1. LET: W be the set of all pairs $(B, \leq)$ such that $B \subseteq A$ and $\leq$ is a well-ordering on B.
  $\langle 2 \rangle$2. Every member of α is the ordinal number of a member of W
  $\langle 2 \rangle$3. Q.E.D.
    PROOF: By a Replacement Axiom.
$\langle 1 \rangle$3. α is an ordinal.
$\langle 1 \rangle$4. α is not dominated by A.
$\square$

Definition 5.8.13. A class term $\mathbf{F} : \mathbf{On} \to \mathbf{On}$ is *continuous* iff, for every limit ordinal λ, we have $\mathbf{F}(\lambda) = \sup_{\alpha < \lambda} F(\alpha)$.

Theorem 5.8.14. *Let $\mathbf{F} : \mathbf{On} \to \mathbf{On}$. If $\mathbf{F}$ is continuous and $\mathbf{F}(\alpha) < \mathbf{F}(\alpha + 1)$ for every ordinal α, then $\mathbf{F}$ is strictly monotone.*

Definition 5.8.15. A class term $\mathbf{F} : \mathbf{On} \to \mathbf{On}$ is *normal* iff it is strictly monotone and continuous.

Theorem 5.8.16. *Let $\mathbf{F} : \mathbf{On} \to \mathbf{On}$ be normal. For every ordinal $\beta \geq \mathbf{F}(0)$, there exists a greatest ordinal α such that $\mathbf{F}(\alpha) \leq \beta$.*

Theorem 5.8.17. *Let $\mathbf{F} : \mathbf{On} \to \mathbf{On}$ be normal. Let S be a set of ordinals. Then $\mathbf{F}(\sup S) = \sup_{\alpha \in S} \mathbf{F}(\alpha)$.*

Theorem 5.8.18 (Veblen Fixed-Point Theorem)**.** *Let $\mathbf{F} : \mathbf{On} \to \mathbf{On}$ be normal. For every ordinal α, there exists $\beta \geq \alpha$ such that $\mathbf{F}(\beta) = \beta$.*

PROOF: Let β be the supremum of α, $\mathbf{F}(\alpha)$, $\mathbf{F}^2(\alpha)$, ... ∎

Lemma 5.8.19. *Let α be an ordinal. Let $(f(\gamma))_{\gamma<\alpha}$ be an α-sequence of ordinals. Then there exists $\beta \leq \alpha$ and an increasing sequence of ordinals $(g(\gamma))_{\gamma<\beta}$ such that $\sup_{\gamma<\alpha} f(\gamma) = \sup_{\gamma<\beta} g(\gamma)$.*

5.9 Cardinal Numbers

Definition 5.9.1 (Cardinal Number (AC))**.** For any set A, the *cardinal number* of A, $\operatorname{card} A$, is the least ordinal equinumerous with A.

There exists some ordinal equinumerous with A by the Well-Ordering Theorem.

Theorem 5.9.2. *For any sets A and B, we have $A \equiv B$ if and only if $\operatorname{card} A = \operatorname{card} B$.*

Theorem 5.9.3. *A set A is finite if and only if $\operatorname{card} A$ is a natural number.*

Theorem 5.9.4. *The supremum of a set of cardinal numbers is a cardinal number.*

5.10 Cardinal Arithmetic

Definition 5.10.1. For cardinal numbers κ and λ, the *sum* $\kappa + \lambda$ is the cardinal number of $A \cup B$, where A and B are disjoint sets of cardinality κ and λ respectively.

Theorem 5.10.2. $\kappa + \lambda = \lambda + \kappa$

Theorem 5.10.3. $\kappa + (\lambda + \mu) = (\kappa + \lambda) + \mu$

Theorem 5.10.4. *The definition of addition agrees with the definition on natural numbers.*

Definition 5.10.5. For cardinal numbers κ and λ, the *product* $\kappa\lambda$ is the cardinality of $\kappa \times \lambda$.

Theorem 5.10.6. $\kappa\lambda = \lambda\kappa$

Theorem 5.10.7. $\kappa(\lambda\mu) = (\kappa\lambda)\mu$

Theorem 5.10.8. $\kappa(\lambda + \mu) = \kappa\lambda + \kappa\mu$

Theorem 5.10.9. *The definition of multiplication agrees with the definition on natural numbers.*

Theorem 5.10.10 (AC)**.** *For any infinite cardinal κ we have $\kappa\kappa = \kappa$.*

PROOF:
$\langle 1 \rangle 1$. LET: B be a set with cardinality κ
$\langle 1 \rangle 2$. LET: $\mathcal{H} = \{\emptyset\} \cup \{f \mid \exists A \subseteq B . A$ is infinite and f is a bijection between $A \times A$ and $A\}$
$\langle 1 \rangle 3$. For every chain $\mathcal{C} \subseteq \mathcal{H}$ we have $\bigcup \mathcal{C} \in \mathcal{H}$
$\langle 1 \rangle 4$. PICK a maximal f_0 in $\mathcal{H}$
$\langle 1 \rangle 5$. $f_0 \neq \emptyset$
PROOF: B has a subset of cardinality $\aleph_0$ and $\aleph_0 \aleph_0 = \aleph_0$.
$\langle 1 \rangle 6$. LET: A_0 be the set such that f_0 is a bijection between $A_0 \times A_0$ and A_0
$\langle 1 \rangle 7$. LET: $\lambda = \operatorname{card} A_0$
$\langle 1 \rangle 8$. $\operatorname{card}(B - A_0) < \lambda$
$\langle 1 \rangle 9$. $\kappa = \lambda$
PROOF:

$$\begin{aligned} \kappa &= \operatorname{card} A_0 + \operatorname{card}(B - A_0) \\ &\leq \lambda + \lambda \\ &= 2\lambda \\ &\leq \lambda\lambda \\ &= \lambda && (\langle 1 \rangle 6) \\ &\leq \kappa && \square \end{aligned}$$

Theorem 5.10.11 (Absorption Law)**.** *Let κ and λ be cardinal numbers such that $0 < \kappa \leq \lambda$ and λ is infinite. Then*

$$\kappa + \lambda = \lambda \ .$$

Theorem 5.10.12 (Absorption Law)**.** *Let κ and λ be cardinal numbers such that $0 < \kappa \leq \lambda$ and λ is infinite. Then*

$$\kappa\lambda = \lambda \ .$$

Definition 5.10.13. For cardinal numbers κ and λ, we write κ^λ for the cardinality of the set of functions from λ to κ.

Theorem 5.10.14. $\kappa^{\lambda+\mu} = \kappa^\lambda + \kappa^\mu$

Theorem 5.10.15. $(\kappa\lambda)^\mu = \kappa^\mu \lambda^\mu$

Theorem 5.10.16. $(\kappa^\lambda)^\mu = \kappa^{\lambda\mu}$

Theorem 5.10.17. *The definition of exponentiation agrees with the definition on natural numbers.*

Theorem 5.10.18. *Given sets A and B, we have* $\operatorname{card} A \leq \operatorname{card} B$ *if and only if* $A \preccurlyeq B$.

Definition 5.10.19. Let $\aleph_0 = \operatorname{card} \mathbb{N}$.

Theorem 5.10.20 (AC)**.** *For any infinite cardinal κ we have $\aleph_0 \leq \kappa$.*

Theorem 5.10.21 (Maximum Principle (AC))**.** *Every poset has a maximal chain.*

5.11 Rank of a Set

Definition 5.11.1 (Cumulative Hierarchy of Sets)**.** For every ordinal α, define the *rank* V_α by transfinite recursion thus:

$$\begin{aligned} V_0 &= \emptyset \\ V_{\alpha+1} &= \mathcal{P} V_\alpha \\ V_\lambda &= \bigcup_{\alpha < \lambda} V_\alpha \end{aligned}$$

for λ a limit ordinal.

The *von Neumann universe* is the class $\mathbf{V} = \bigcup_{\alpha \in \mathbf{On}} V_\alpha$.

Theorem 5.11.2. *If λ is a limit ordinal and $\lambda > \omega$ then V_λ is a model of Zermelo set theory.*

Lemma 5.11.3 (AC)**.** *There exists a well-ordered set in $V_{\omega 2}$ whose ordinal is not in $V_{\omega 2}$.*

Proof: Pick a well-ordering $<$ of $\mathcal{P}\mathbb{N}$. Then $(\mathcal{P}\mathbb{N}, <) \in V_{\omega 2}$ but its ordinal is not because its ordinal is uncountable. □

Theorem 5.11.4. *The set $V_{\omega 2}$ is not a model of Zermelo-Fraenkel set theory.*

Thus, the Replacement Axioms cannot be proven from the other axioms.

Definition 5.11.5 (Well-Founded Set)**.** A set A is *well-founded* iff $A \in V_\alpha$ for some $\alpha \in \mathbf{On}$.

Definition 5.11.6 (Rank)**.** The *rank* of a well-founded set A, $\operatorname{rank} A$, is the least ordinal α such that $A \in V_\alpha$.

Theorem 5.11.7. *If $A \in B$ and B is well-founded then A is well-founded and* $\operatorname{rank} A < \operatorname{rank} B$.

Theorem 5.11.8. *If A is a set and every member of A is well-founded then A is well-founded and* $\operatorname{rank} A = \sup_{B \in A} (\operatorname{rank} B + 1)$.

Theorem 5.11.9. *The Axiom of Regularity is equivalent to the statement that every set is well-founded.*

5.12 Transfinite Recursion Again

Theorem 5.12.1. *Let* $\mathbf{A}$ *be a class. Let* $\mathbf{B}$ *be the class of all functions* $f : \alpha \to \mathbf{A}$ *for some ordinal* α. *Let* $\mathbf{F} : \mathbf{B} \to \mathbf{A}$ *be a class term. Then there exists a unique class term* $\mathbf{G} : \mathbf{On} \to \mathbf{A}$ *such that, for all* $\alpha \in \mathbf{On}$, *we have* $\mathbf{G}(\alpha) = \mathbf{F}(\mathbf{G} \restriction \alpha)$.

5.13 Alephs

Definition 5.13.1. Define the cardinal number $\aleph_\alpha$ for every ordinal α by transfinite recursion on α thus: $\aleph_\alpha$ is the least infinite cardinal different from $\aleph_\beta$ for all $\beta < \alpha$.

Theorem 5.13.2. *If* $\alpha < \beta$ *then* $\aleph_\alpha < \aleph_\beta$.

Theorem 5.13.3. *Every infinite cardinal has the form* $\aleph_\alpha$ *for some ordinal* α.

5.14 Ordinal Arithmetic

Definition 5.14.1 (Sum)**.** Let α and β be ordinals. The *sum* $\alpha + \beta$ is the ordinal of the concatenation of A followed by B, where A is a well-ordered set of ordinal α and B a well-ordered set of ordinal β.

Theorem 5.14.2. *Addition is associative.*

Theorem 5.14.3. $\alpha + 0 = \alpha$

Theorem 5.14.4. $0 + \alpha = \alpha$

Theorem 5.14.5. *For* λ *a limit ordinal we have* $\alpha + \lambda = \sup_{\beta<\lambda}(\alpha + \beta)$

Theorem 5.14.6. *For any* α, *the class term that maps* β *to* $\alpha + \beta$ *is normal.*

Theorem 5.14.7. $\beta < \gamma$ *iff* $\alpha + \beta < \alpha + \gamma$.

Theorem 5.14.8. *If* $\beta \leq \gamma$ *then* $\beta + \alpha \leq \gamma + \alpha$.

Theorem 5.14.9 (Subtraction Theorem)**.** *If* $\alpha < \beta$ *then there exists a unique* δ *such that* $\alpha + \delta < \beta$.

Definition 5.14.10 (Product)**.** Let α and β be ordinals. The *sum* $\alpha + \beta$ is the ordinal of $A \times B$ ordered under the Hebrew lexicographic order, where A is a well-ordered set of ordinal α and B a well-ordered set of ordinal β.

Theorem 5.14.11. *Multiplication is associative.*

Theorem 5.14.12. *Multiplication distributes over addition on the left.*

Theorem 5.14.13. $\alpha 1 = \alpha$

Theorem 5.14.14. $1\alpha = \alpha$

Theorem 5.14.15. $\alpha 0 = 0$

Theorem 5.14.16. $0\alpha = 0$

Theorem 5.14.17. *For λ a limit ordinal, we have $\alpha\lambda = \sup_{\beta<\lambda}(\alpha\beta)$.*

Theorem 5.14.18. *For $\alpha > 0$, the class term that maps β to $\alpha\beta$ is normal.*

Theorem 5.14.19. *If $\alpha > 0$, then $\beta < \gamma$ iff $\alpha\beta < \alpha\gamma$.*

Theorem 5.14.20. *If $\beta \leq \gamma$ then $\beta\alpha \leq \gamma\alpha$.*

Theorem 5.14.21 (Division Theorem)**.** *For any ordinals α and δ with $\delta \neq 0$, there exist unique ordinals β and γ with $\gamma < \delta$ and $\alpha = \delta\beta + \gamma$.*

Definition 5.14.22 (Exponentiation)**.** For ordinals α and β, define the ordinal α^β by transfinite recursion on β by:

$$\begin{aligned} \alpha^0 &= 1 \\ \alpha^{\beta+1} &= \alpha^\beta + \alpha \\ \alpha^\lambda &= \sup_{\beta<\lambda} \alpha^\beta \end{aligned}$$

for λ a limit ordinal.

Theorem 5.14.23. *For $\alpha > 1$, the class term that maps β to α^β is normal.*

Theorem 5.14.24. *If $\alpha > 1$, then $\beta < \gamma$ iff $\alpha^\beta < \alpha^\gamma$.*

Theorem 5.14.25. *If $\beta \leq \gamma$ then $\beta^\alpha \leq \gamma^\alpha$.*

Theorem 5.14.26 (Logarithm Theorem)**.** *Let α and β be ordinals with $\alpha \neq 0$ and $\beta > 1$. Then there exist unique ordinals γ, δ and ρ such that $\delta \neq 0$, $\delta < \beta$, $\rho < \beta^\gamma$, and $\alpha = \beta^\gamma\delta + \rho$.*

Theorem 5.14.27.

$$\alpha^{\beta+\gamma} = \alpha^\beta\alpha^\gamma$$

Theorem 5.14.28.

$$(\alpha^\beta)^\gamma = \alpha^{\beta\gamma}$$

5.15 Beth Cardinals

Definition 5.15.1. Define the cardinal $\beth_\alpha$ for every ordinal α by:

$$\begin{aligned} \beth_0 &= \aleph_0 \\ \beth_{\alpha+1} &= 2^{\beth_\alpha} \\ \beth_\lambda &= \sup_{\alpha<\lambda} \beth_\alpha \end{aligned}$$

for λ a limit ordinal.

Lemma 5.15.2. *For any ordinal α we have* $\operatorname{card} V_{\omega+\alpha} = \beth_\alpha$.

5.16 Cofinality

Definition 5.16.1 (Cofinality)**.** For λ a limit ordinal, the *cofinality* of λ, $\operatorname{cf}\lambda$, is the least cardinal κ such that λ is the supremum of a set of κ smaller ordinals.

We extend cf to all the ordinals by setting $\operatorname{cf} 0 = 0$ and $\operatorname{cf}(\alpha+1) = 1$.

Theorem 5.16.2. *For any limit ordinal λ we have $\operatorname{cf}\aleph_\lambda = \operatorname{cf}\lambda$.*

Lemma 5.16.3. *Let λ be a limit ordinal. Then $\operatorname{cf}\lambda$ is the least ordinal α such that there exists an increasing α-sequence of ordinals with limit λ.*

Theorem 5.16.4. *Let λ be an infinite cardinal. Then $\operatorname{cf}\lambda$ is the least cardinal number κ such that λ can be partitioned into κ sets each of cardinality $< \lambda$.*

Theorem 5.16.5 (König's Theorem)**.** *Let κ be an infinite cardinal. Then $\kappa < 2^{\operatorname{cf}\kappa}$.*

Corollary 5.16.5.1. $2^{\aleph_0} \neq \aleph_\omega$.

Definition 5.16.6 (Regular)**.** A cardinal κ is *regular* iff $\operatorname{cf}\kappa = \kappa$.

Theorem 5.16.7. *For any ordinal λ, we have $\operatorname{cf}\lambda$ is a regular cardinal.*

Definition 5.16.8 (Singular)**.** A cardinal κ is *singular* iff $\operatorname{cf}\kappa < \kappa$.

Theorem 5.16.9. *For any ordinal α we have $\aleph_{\alpha+1}$ is a regular cardinal.*

5.17 Inaccessible Cardinals

Definition 5.17.1 (Inaccessible)**.** A cardinal number κ is *inaccessible* iff:

- $\kappa > \aleph_0$
- For every cardinal $\lambda < \kappa$ we have $2^\lambda < \kappa$
- κ is regular.

Lemma 5.17.2. *If κ is inaccessible and $\alpha < \kappa$ then $\beth_\alpha < \kappa$.*

Lemma 5.17.3. *If κ is inaccessible and $A \in V_\kappa$ then $\operatorname{card} A < \kappa$.*

Theorem 5.17.4. *If κ is inaccessible then V_κ is a model of ZF.*

5.18 Directed Set

Definition 5.18.1 (Directed Set)**.** A preodered set P is *directed* iff, for all $a, b \in P$, there exists $c \in P$ such that $a \leq c$ and $b \leq c$.

Proposition 5.18.2. *Every linearly ordered set is directed.*

Proposition 5.18.3. *For any set A, the $\mathcal{P}A$ under $\subseteq$ is directed.*

5.19 Cofinal Set

Definition 5.19.1 (Cofinal)**.** Let A be a preordered set and $B \subseteq A$. Then B is *cofinal* if and only if, for every $x \in A$, there exists $y \in B$ such that $x \leq y$.

Proposition 5.19.2. *If A is a directed preordered set and $B \subseteq A$ is cofinal then B is directed.*

PROOF:
$\langle 1 \rangle 1.$ LET: $x, y \in B$
$\langle 1 \rangle 2.$ PICK $z \in A$ such that $x \leq z$ and $y \leq z$
$\langle 1 \rangle 3.$ PICK $z' \in B$ such that $z \leq z'$
$\langle 1 \rangle 4.$ $x \leq z'$ and $y \leq z'$
$\square$

Chapter 6

Natural Numbers

6.1 Successors

Definition 6.1.1 (Successor (Pairing, Union))**.** For any set a, its *Successor* a^+ is the set $a \cup \{a\}$

Theorem 6.1.2 (Pairing, Union)**.** *If a is a transitive set then $\bigcup(a^+) = a$.*

PROOF:

$$\begin{aligned}\bigcup(a^+) &= \bigcup(a \cup \{a\}) \\ &= \bigcup a \cup \bigcup\{a\} \\ &= \bigcup a \cup a \\ &= a & (\bigcup a \subseteq a)\square\end{aligned}$$

Theorem 6.1.3. *If A is a transitive set then A^+ is transitive.*

PROOF: If A is transitive then $\bigcup(A^+) = A \subseteq A^+$. $\square$

6.2 Inductive Sets

Definition 6.2.1 (Inductive (Extensionality, Empty Set, Pairing, Union))**.** A set A is *inductive* iff $\emptyset \in A$ and, for every $a \in A$, we have $a^+ \in A$.

Axiom 6.2.2 (Axiom of Infinity (Extensionality, Empty Set, Pairing, Union))**.** *There exists an inductive set.*

6.3 Natural Numbers

Definition 6.3.1 (Natural Number (Extensionality, Empty Set, Pairing, Union))**.** A *natural number* is a set that belongs to every inductive set.

We write $\mathbb{N}$ for the class of all natural numbers.

Theorem 6.3.2 (Extensionality, Empty Set, Pairing, Union, Infinity, Subset)**.** *The class of natural numbers is a set.*

PROOF:
$\langle 1 \rangle 1.$ PICK an inductive set I.
  PROOF: By the Axiom of Infinity.
$\langle 1 \rangle 2.$ $\mathbb{N} \subseteq I$
$\langle 1 \rangle 3.$ Q.E.D.
  PROOF: By a Subset Axiom.
□

Theorem 6.3.3 (Extensionality, Empty Set, Pairing, Union, Infinity, Subset)**.** *The set* $\mathbb{N}$ *is inductive.*

PROOF:
$\langle 1 \rangle 1.$ $\emptyset \in \mathbb{N}$
  PROOF: Since $\emptyset$ is a member of every inductive set.
$\langle 1 \rangle 2.$ For all $n \in \mathbb{N}$ we have $n^+ \in \mathbb{N}$
  PROOF: If n is a member of every inductive set then so is n^+.
□

Theorem 6.3.4 (Extensionality, Empty Set, Pairing, Union, Infinity, Subset)**.** *The set* $\mathbb{N}$ *is a subset of every inductive set.*

PROOF: Immediate from definition. □

Corollary 6.3.4.1 (Proof by Induction (Extensionality, Empty Set, Pairing, Union, Infinity, Subset))**.** *If* $A \subseteq \mathbb{N}$ *and* A *is inductive then* $A = \mathbb{N}$*.*

Definition 6.3.5 (Zero (Empty Set))**.** The natural number *zero*, 0, is defined to be $\emptyset$.

Theorem 6.3.6 (Extensionality, Empty Set, Pairing, Union, Infinity, Subset)**.** *Every natural number except 0 is a successor of a natural number.*

PROOF: The set $\{x \in \mathbb{N} \mid x = 0 \vee \exists y \in \mathbb{N}. x = y^+\}$ is inductive. □

Theorem 6.3.7 (Extensionality, Empty Set, Pairing, Union, Infinity, Subset)**.** *Every natural number is transitive.*

PROOF: By induction using Theorem 6.1.3. □

Theorem 6.3.8 (Extensionality, Empty Set, Pairing, Union, Infinity, Subset)**.** *The set* $\mathbb{N}$ *is transitive.*

PROOF:
$\langle 1 \rangle 1.$ For every natural number n and every $m \in n$ then m is a natural number.
  $\langle 2 \rangle 1.$ Every member of $\emptyset$ is a natural number.
    PROOF: Vacuous.
  $\langle 2 \rangle 2.$ If n is a natural number and a set of natural numbers then n^+ is a set of natural numbers.

PROOF: From the definition of n^+.

$\langle 2\rangle 3.$ Q.E.D.

PROOF: By induction.

□

Theorem 6.3.9 (Extensionality, Empty Set, Pairing, Union, Infinity, Subset). *Let A be a set, $a \in A$, and $F : A \to A$. Then there exists a unique function $h : \mathbb{N} \to A$ such that $h(0) = a$ and, for all $n \in \mathbb{N}$, we have $h(n^+) = F(h(n))$.*

PROOF:

$\langle 1\rangle 1.$ Call a function v *acceptable* iff $\operatorname{dom} v \subseteq \mathbb{N}$, $\operatorname{ran} v \subseteq A$, and:

1. If $0 \in \operatorname{dom} v$ then $v(0) = a$.
2. For all $n \in \mathbb{N}$, if $n^+ \in \operatorname{dom} v$ then $n \in \operatorname{dom} v$ and $v(n^+) = F(v(n))$.

$\langle 1\rangle 2.$ LET: $\mathcal{K}$ be the set of all acceptable functions.

$\langle 1\rangle 3.$ LET: $h = \bigcup \mathcal{K}$

$\langle 1\rangle 4.$ h is a function.

$\langle 2\rangle 1.$ If $(0, y) \in h$ and $(0, y') \in h$ then $y = y'$

PROOF: We have $y = y' = a$.

$\langle 2\rangle 2.$ For any natural number n, if there is at most one y such that $(n, y) \in h$, then there is at most one y such that $(n^+, y) \in h$

$\langle 3\rangle 1.$ LET: n be a natural number.

$\langle 3\rangle 2.$ ASSUME: there is at most one y such that $(n, y) \in h$

$\langle 3\rangle 3.$ ASSUME: (n^+, y) and (n^+, y') are in h)

$\langle 3\rangle 4.$ PICK acceptable functions u and v such that $u(n^+) = y$ and $v(n^+) = y'$

$\langle 3\rangle 5.$ $n \in \operatorname{dom} u$, $n \in \operatorname{dom} v$ and $y = F(u(n))$, $y' = F(v(n))$

$\langle 3\rangle 6.$ $u(n) = v(n)$

PROOF: By the induction hypothesis $\langle 3\rangle 2$

$\langle 3\rangle 7.$ $y = y'$

$\langle 2\rangle 3.$ Q.E.D.

PROOF: By induction.

$\langle 1\rangle 5.$ h is acceptable.

$\langle 2\rangle 1.$ If $0 \in \operatorname{dom} h$ then $h(0) = a$

$\langle 2\rangle 2.$ If $n^+ \in \operatorname{dom} h$ then $n \in \operatorname{dom} h$ and $h(n^+) = F(h(n))$

$\langle 3\rangle 1.$ ASSUME: $n^+ \in \operatorname{dom} h$

$\langle 3\rangle 2.$ PICK an acceptable v such that $n^+ \in \operatorname{dom} v$

$\langle 3\rangle 3.$ $v(n^+) = F(v(n))$

$\langle 3\rangle 4.$ $h(n^+) = F(h(n))$

$\langle 1\rangle 6.$ $\operatorname{dom} h = \mathbb{N}$

$\langle 2\rangle 1.$ $0 \in \operatorname{dom} h$

PROOF: Since $\{(0, a)\}$ is an acceptable function.

$\langle 2\rangle 2.$ For all $n \in \operatorname{dom} h$ we have $n^+ \in \operatorname{dom} h$

$\langle 3\rangle 1.$ ASSUME: $n \in \operatorname{dom} h$

$\langle 3\rangle 2.$ LET: v be an acceptable function with $n \in \operatorname{dom} v$

$\langle 3\rangle 3.$ ASSUME: without loss of generality $n^+ \notin \operatorname{dom} v$

$\langle 3\rangle 4.$ $v \cup \{(n^+, F(v(n)))\}$ is acceptable

$\langle 3 \rangle 5.$ $n^+ \in \operatorname{dom} v$

$\langle 1 \rangle 7.$ If $h' : \mathbb{N} \to A$, $h'(0) = a$ and, for all $n \in \mathbb{N}$, we have $h'(n^+) = F(h'(n))$, then $h' = h$

PROOF: Prove $h(n) = h'(n)$ by induction on n.

□

6.4 Peano Systems

Definition 6.4.1 (Peano System)**.** A *Peano system* consists of a set N, an element $z \in N$, and a function $S : N \to N$ such that:

- S is one-to-one
- $z \notin \operatorname{ran} S$
- For any set $A \subseteq N$, if $z \in A$ and $S(A) \subseteq A$ then $A = N$.

Theorem 6.4.2. *$\mathbb{N}$ is a Peano system with zero 0 and successor $n \mapsto n^+$.*

Theorem 6.4.3. *For any Peano system (N, z, S), there exists a unique bijection $h : \mathbb{N} \cong N$ such that $h(0) = z$ and $S(h(n)) = h(n^+)$ for all n.*

6.5 Arithmetic

Definition 6.5.1 (Addition)**.** Define *addition* $+ : \mathbb{N}^2 \to \mathbb{N}$ recursively by

$$m + 0 = m$$
$$m + n^+ = (m + n)^+$$

for any $m, n \in \mathbb{N}$.

Theorem 6.5.2. *Addition is associative.*

Theorem 6.5.3. *Addition is commutative*

Definition 6.5.4 (Multiplication)**.** Define *multiplication* $\cdot : \mathbb{N}^2 \to \mathbb{N}$ recursively by

$$m0 = 0$$
$$mn^+ = mn + m$$

for any $m, n \in \mathbb{N}$

Theorem 6.5.5. *Multiplication is associative.*

Theorem 6.5.6. *Multiplication is commutative.*

Theorem 6.5.7. *Multiplication distributes over addition.*

Definition 6.5.8. For natural numbers m and n, we write $m < n$ iff $m \in n$. We write $m \leq n$ iff $m < n$ or $m = n$.

Theorem 6.5.9. *We have $m < n$ iff $m^+ < n^+$.*

Theorem 6.5.10. *We never have $n < n$.*

Theorem 6.5.11. *The ordering on $\mathbb{N}$ satisfies trichotomy; that is, for any m, n, exactly one of $m < n$, $m = n$, $n < m$ holds.*

Theorem 6.5.12. *For any natural numbers m and n, we have $m \leq n$ iff $m \subseteq n$.*

Theorem 6.5.13. *We have $m < n$ iff $m + p < n + p$.*

Corollary 6.5.13.1. *If $m + p = n + p$ then $m = n$.*

Theorem 6.5.14. *If $p \neq 0$ then $m < n$ iff $mp < np$.*

Corollary 6.5.14.1. *If $mp = np$ and $p \neq 0$ then $m = n$.*

Theorem 6.5.15 (Well-Ordering of $\mathbb{N}$)**.** *Any nonempty set $A \subseteq \mathbb{N}$ has a least element.*

Corollary 6.5.15.1. *There is no function $f : \mathbb{N} \to \mathbb{N}$ such that $f(n^+) < f(n)$ for all n.*

Theorem 6.5.16 (Strong Induction)**.** *Let $A \subseteq \mathbb{N}$. Suppose that, for every natural number n, if $\forall m < n . m \in A$ then $n \in A$. Then $A = \mathbb{N}$.*

Theorem 6.5.17 (Pigeonhole Principle)**.** *No natural number is equinumerous with a proper subset of itself.*

Proof: Prove by induction on n that if $f : n \to n$ is injective then it is surjective.
□

Chapter 7

Integers

Lemma 7.0.1. *Define* $\sim$ *on* $\mathbb{N}^2$ *by:* $(m,n) \sim (p,q)$ *iff* $m+q = n+p$. *Then* $\sim$ *is an equivalence relation on* $\mathbb{N}^2$.

Definition 7.0.2 (Integers)**.** The set $\mathbb{Z}$ of *integers* is $\mathbb{N}^2/\sim$.

Definition 7.0.3. Define *addition* $+ : \mathbb{Z}^2 \to \mathbb{Z}$ by: $(m,n)+(p,q) = (m+p, n+q)$.

Prove this is well-defined.

Theorem 7.0.4. *Addition is associative and commutative.*

Definition 7.0.5 (Zero)**.** The integer *zero* is $0 = (0,0)$.

Theorem 7.0.6. *For any integer* a*, we have* $a + 0 = a$.

Theorem 7.0.7. *For any integer* a*, there exists a unique integer* b *such that* $a + b = 0$.

Definition 7.0.8 (Multiplication)**.** Define multiplication on $\mathbb{Z}$ by $(m,n)(p,q) = (mp+nq, mq+np)$.

Theorem 7.0.9. *Multiplication is associative, commutative and distributive over addition.*

Definition 7.0.10. The integer *one* is $1 = (1,0)$.

Theorem 7.0.11. *For any integer* a *we have* $a1 = a$.

Theorem 7.0.12. $1 \neq 0$

Theorem 7.0.13. *Whenever* $ab = 0$ *then either* $a = 0$ *or* $b = 0$.

Definition 7.0.14. Define $<$ on $\mathbb{Z}$ by: $(m,n) < (p,q)$ iff $m+q < n+p$.

Theorem 7.0.15. *The relation* $<$ *is a strict linear ordering on* $\mathbb{Z}$.

Theorem 7.0.16. *We have* $a < b$ *iff* $< +c < b + c$.

Corollary 7.0.16.1. *If* $a + c = b + c$ *then* $a = b$.

Theorem 7.0.17. *If* $0 < c$ *then* $a < b$ *iff* $ac < bc$.

Corollary 7.0.17.1. *If* $ac = bc$ *and* $c \neq 0$ *then* $a = b$.

Definition 7.0.18. We identify any natural number n with the integer $(n, 0)$.

Theorem 7.0.19. *This embedding preserves 0, 1, addition, multiplication and the ordering.*

Chapter 8

Rational Numbers

Definition 8.0.1 (Rational Numbers)**.** The set of *rationals* $\mathbb{Q}$ is $\mathbb{Z}\times(\mathbb{Z}\setminus\{0\})/\sim$, where $(a,b)\sim(c,d)$ iff $ad=bc$.

Definition 8.0.2 (Addition)**.** Define addition on $\mathbb{Q}$ by: $(a,b)+(c,d)=(ad+bc,bd)$.

Theorem 8.0.3. *Addition is commutative and associative*

Definition 8.0.4. The rational number 0 is $(0,1)$.

Theorem 8.0.5. *For any rational q we have $q+0=q$.*

Theorem 8.0.6. *For any rational q, there exists a unique rational r such that $q+r=0$.*

Definition 8.0.7. Define multiplication on $\mathbb{Q}$ by: $(a,b)(c,d)=(ac,bd)$.

Theorem 8.0.8. *Multiplication is commutative, associative and distributive over addition.*

Definition 8.0.9. The rational number 1 is $(1,1)$.

Theorem 8.0.10. *For every nonzero rational r, there exists a nonzero rational q such that $rq=1$.*

Corollary 8.0.10.1. *If $qr=0$ then either $q=0$ or $r=0$.*

Definition 8.0.11. Define $<$ on $\mathbb{Q}$ by: for b and d positive, $(a,b)<(c,d)$ iff $ad<bc$.

Theorem 8.0.12. *The relation $<$ is a strict linear ordering on $\mathbb{Q}$.*

Theorem 8.0.13. *We have $q<r$ iff $q+s<r+s$*

Corollary 8.0.13.1. *If $q+s=r+s$ then $q=r$.*

Theorem 8.0.14. *If $s>0$ then we have $q<r$ iff $qs<rs$.*

Corollary 8.0.14.1. *If* $qs = rs$ *and* $s \neq 0$ *then* $q = r$.

Definition 8.0.15. We identify an integer n with the rational $(n, 1)$.

Theorem 8.0.16. *This embedding preserves zero, one, addition, multiplication and the ordering.*

Chapter 9

Real Numbers

Definition 9.0.1 (Dedekind Cut)**.** A *Dedekind cut* is a subset $X \subseteq \mathbb{Q}$ such that:

- X is nonempty
- $X \neq \mathbb{Q}$
- X is closed downward
- X has no largest element.

Definition 9.0.2 (Real Numbers)**.** The set of *real numbers* $\mathbb{R}$ is the set of all Dedekind cuts.

Definition 9.0.3. Define $<$ on $\mathbb{R}$ by: $x < y$ iff x is a proper subset of y.

Theorem 9.0.4. *The relation $<$ is a strict linear ordering on $\mathbb{R}$.*

Theorem 9.0.5. *Any bounded nonempty subset of $\mathbb{R}$ has a least upper bound.*

Definition 9.0.6. Define addition on $\mathbb{R}$ by: $x + y = \{q + r \mid q \in x, r \in y\}$.

Theorem 9.0.7. *Addition is associative and commutative.*

Definition 9.0.8. The zero real 0 is $\{q \in \mathbb{Q} \mid q < 0\}$.

Theorem 9.0.9. *For any $x \in \mathbb{R}$ we have $x + 0 = x$.*

Definition 9.0.10. Given a real x, define $-x = \{q \in \mathbb{Q} \mid \exists r > q. -r \notin x\}$.

Theorem 9.0.11. *For any real x we have $x + (-x) = 0$.*

Corollary 9.0.11.1. *If $x + z = y + z$ then $x = y$.*

Theorem 9.0.12. *We have $x < y$ iff $x + z < y + z$.*

Definition 9.0.13. Define the *absolute value* of a real x by $|x| = x \cup -x$.

Theorem 9.0.14. *For any real x we have $0 \leq |x|$.*

Definition 9.0.15. Define multiplication on $\mathbb{R}$ by:

- If x and y are nonnegative then

$$xy = 0 \cup \{qr \mid 0 \leq q, 0 \leq r, q \in x, r \in y\}$$

- If x and y are both negative then $xy = |x||y|$
- If one of x and y is negaive and the other not then $xy = -|x||y|$.

Theorem 9.0.16. *Multiplication is associative, commutative and distributive over addition.*

Definition 9.0.17. The real number 1 is $\{q \in \mathbb{Q} \mid q < 1\}$.

Theorem 9.0.18. $0 \neq 1$

Theorem 9.0.19. *For any real x we have $x1 = x$*

Theorem 9.0.20. *For any nonzero x, there exists a real y with $xy = 1$.*

Theorem 9.0.21. *If $0 < x$ then $y < z$ iff $xy < xz$.*

Definition 9.0.22. Identify a rational q with $\{r \in \mathbb{Q} \mid r < q\}$.

Theorem 9.0.23. *This embedding preserves zero, one, addition, multiplication and the ordering.*

9.1 The Cantor Set

Definition 9.1.1 (Cantor Set)**.** Define the sequence of sets $A_n \subseteq \mathbb{R}$ by

$$\begin{aligned} A_0 &= [0,1] \\ A_n &= A_{n-1} - \bigcup_{k=0}^{3^{n-1}-1} ((3k+1)/3^n, (3k+2)/3^n) \end{aligned}$$

The *Cantor set* is $\bigcap_{n=0}^{\infty} A_n$.

Proposition 9.1.2. *The set A_n is a union of finitely many disjoint closed intervals of length $1/3^n$, and the endpoints of these intervals lie in C.*

Proof: An easy induction on n. $\square$

Chapter 10

Finite Sets

Definition 10.0.1 (Finite)**.** A set is *finite* iff it is equinumerous with a natural number; otherwise it is *infinite*.

Theorem 10.0.2. *No finite set is equinumerous with a proper subset of itself.*

PROOF: From the Pigeonhole Principle.

Corollary 10.0.2.1. *The set* $\mathbb{N}$ *is infinite.*

Corollary 10.0.2.2. *A finite set is equinumerous with a unique natural number.*

Lemma 10.0.3. *If* A *is a proper subset of a natural number* n *then there exists* $m < n$ *such that* $C \equiv m$*.*

Corollary 10.0.3.1. *A subset of a finite set is finite.*

Theorem 10.0.4 (Regularity)**.** *There is no function* f *with domain* $\mathbb{N}$ *such that* $f(n+1) \in f(n)$ *for all* n*.*

PROOF:
$\langle 1 \rangle 1$. ASSUME: for a contradiction f is a function with domain $\mathbb{N}$ such that $f(n+1) \in f(n)$ for all n.
$\langle 1 \rangle 2$. PICK $m \in \operatorname{ran} f$ such that $m \cap \operatorname{ran} f = \emptyset$
PROOF: By the Axiom of Regularity.
$\langle 1 \rangle 3$. PICK $n \in \mathbb{N}$ such that $f(n) = m$
$\langle 1 \rangle 4$. $f(n+1) \in m \cap \operatorname{ran} f$
$\langle 1 \rangle 5$. Q.E.D.
PROOF: This is a contradiction.
□

Theorem 10.0.5. *A relation* R *is well-founded if and only if there is no function* f *with domain* $\mathbb{N}$ *such that, for all* $n \in \mathbb{N}$*, we have* $f(n+1) R f(n)$*.*

10.1 The Finite Intersection Property

Definition 10.1.1 (Finite Intersection Property)**.** Let X be a set and $\mathcal{A} \subseteq \mathcal{P}X$. Then $\mathcal{A}$ satisfies the *finite intersection property* if and only if every nonempty finite subset of $\mathcal{A}$ has nonempty intersection.

Lemma 10.1.2. *Let X be a set. Let $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the finite intersection property. Then any finite intersection of elements of $\mathcal{D}$ is an element of $\mathcal{D}$.*

PROOF:
⟨1⟩1. LET: $D_1, D_2 \in \mathcal{D}$
⟨1⟩2. $\mathcal{D} \cup \{D_1 \cap D_2\}$ has the finite intersection property.
  PROOF: Any finite intersection of members of $\mathcal{D} \cup \{D_1 \cap D_2\}$ is a finite intersection of members of $\mathcal{D}$.
⟨1⟩3. $\mathcal{D} = \mathcal{D} \cup \{D_1 \cap D_2\}$
  PROOF: By maximality of $\mathcal{D}$.
⟨1⟩4. $D_1 \cap D_2 \in \mathcal{D}$.
□

Lemma 10.1.3. *Let X be a set. Let $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the finite intersection property. Let $A \subseteq X$. If A intersects every member of $\mathcal{D}$ then $A \in \mathcal{D}$.*

PROOF:
⟨1⟩1. $\mathcal{D} \cup \{A\}$ has the finite intersection property.
  ⟨2⟩1. LET: $D_1, \ldots, D_n \in \mathcal{D}$
    PROVE: $D_1 \cap \cdots \cap D_n \cap A \neq \emptyset$
  ⟨2⟩2. $D_1 \cap \cdots \cap D_n \in \mathcal{D}$
    PROOF: Lemma 10.1.2.
  ⟨2⟩3. $D_1 \cap \cdots \cap D_n \cap A \neq \emptyset$
    PROOF: Since A intersects every member of $\mathcal{D}$.
⟨1⟩2. Q.E.D.
  PROOF: By maximality of $\mathcal{D}$.
□

Proposition 10.1.4. *Let X be a set. Let $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the finite intersection property. Let $A, D \in \mathcal{P}X$. If $D \in \mathcal{D}$ and $D \subseteq A$ then $A \in \mathcal{D}$.*

PROOF:
⟨1⟩1. $\mathcal{D} \cup \{A\}$ satisfies the finite intersection property.
  ⟨2⟩1. LET: $D_1, \ldots, D_n \in \mathcal{D}$
  ⟨2⟩2. $D_1 \cap \cdots \cap D_n \cap D \neq \emptyset$
    PROOF: Since $\mathcal{D}$ satisfies the finite intersection property.
  ⟨2⟩3. $D_1 \cap \cdots \cap D_n \cap A \neq \emptyset$
⟨1⟩2. $\mathcal{D} = \mathcal{D} \cup \{A\}$
  PROOF: By the maximality of $\mathcal{D}$.

$\langle 1 \rangle 3.$ $A \in \mathcal{D}$

$\square$

10.2 Real Analysis

Definition 10.2.1. Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences (a_n) such that $a_n = 0$ for all but finitely many n.

10.3 Group Theory

Definition 10.3.1. Given a group G and sets $A, B \subseteq G$, let $AB = \{ab \mid a \in A, b \in B\}$.

Definition 10.3.2. Given a group G and a set $A \subseteq G$, let $A^{-1} = \{a^{-1} \mid a \in A\}$.

Chapter 11

Topological Spaces

11.1 Topologies

Definition 11.1.1 (Topology)**.** A *topology* on a set X is a set $\mathcal{T} \subseteq \mathcal{P}X$ such that:

- $X \in \mathcal{T}$.
- For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$.
- For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$.

We call the elements of X *points* and the elements of $\mathcal{T}$ *open sets*.

Definition 11.1.2 (Topological Space)**.** A *topological space* X consists of a set X and a topology on X.

Definition 11.1.3 (Discrete Space)**.** For any set X, the *discrete* topology on X is $\mathcal{P}X$.

Definition 11.1.4 (Indiscrete Space)**.** For any set X, the *indiscrete* or *trivial* topology on X is $\{\emptyset, X\}$.

Definition 11.1.5 (Finite Complement Topology)**.** For any set X, the *finite complement topology* on X is $\{U \in \mathcal{P}X \mid X \setminus U \text{ is finite}\} \cup \{\emptyset\}$.

Definition 11.1.6 (Countable Complement Topology)**.** For any set X, the *countable complement topology* on X is $\{U \in \mathcal{P}X \mid X \setminus U \text{ is countable}\} \cup \{\emptyset\}$.

Definition 11.1.7 (Finer, Coarser)**.** Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a given set X. If $\mathcal{T}' \supseteq \mathcal{T}$, we say that $\mathcal{T}'$ is *finer* than $\mathcal{T}$; if $\mathcal{T}'$ *properly* contains $\mathcal{T}$, we say that $\mathcal{T}'$ is *strictly* finer than $\mathcal{T}$. We also say that $\mathcal{T}$ is *coarser* than $\mathcal{T}'$, or *stricly* coarser, in these two respective situations. We say $\mathcal{T}$ is *comparable* with $\mathcal{T}'$ if either $\mathcal{T}' \supseteq \mathcal{T}$ or $\mathcal{T} \supseteq \mathcal{T}'$.

Lemma 11.1.8. *Let X be a topological space and $U \subseteq X$. Then U is open if and only if, for all $x \in U$, there exists an open set V such that $x \in V \subseteq U$.*

PROOF:
⟨1⟩1. $\Rightarrow$
PROOF: Take $V = U$
⟨1⟩2. $\Leftarrow$
PROOF: We have $U = \bigcup\{V \text{ open in } X \mid V \subseteq U\}$.
□

Lemma 11.1.9. *Let X be a set and $\mathcal{T}$ a nonempty set of topologies on X. Then $\bigcap\mathcal{T}$ is a topology on X, and is the finest topology that is coarser than every member of $\mathcal{T}$.*

PROOF:
⟨1⟩1. $X \in \bigcap\mathcal{T}$
PROOF: Since X is in every member of $\mathcal{T}$.
⟨1⟩2. $\bigcap\mathcal{T}$ is closed under union.
⟨2⟩1. LET: $\mathcal{U} \subseteq \bigcap\mathcal{T}$
⟨2⟩2. For all $T \in \mathcal{T}$ we have $\mathcal{U} \subseteq T$
⟨2⟩3. For all $T \in \mathcal{T}$ we have $\bigcup\mathcal{U} \in T$
⟨2⟩4. $\bigcup\mathcal{U} \in \bigcap\mathcal{T}$
⟨1⟩3. $\bigcap\mathcal{T}$ is closed under binary intersection.
⟨2⟩1. LET: $U, V \in \bigcap\mathcal{T}$
⟨2⟩2. For all $T \in \mathcal{T}$ we have $U, V \in T$
⟨2⟩3. For all $T \in \mathcal{T}$ we have $U \cap V \in T$
⟨2⟩4. $U \cap V \in \bigcap\mathcal{T}$
□

Lemma 11.1.10. *Let X be a set and $\mathcal{T}$ a set of topologies on X. Then there exists a unique coarsest topology that is finer than every member of $\mathcal{T}$.*

PROOF: The required topology is given by
$\bigcap\{T \in \mathcal{PP}X \mid T \textit{ is a topology on } X \textit{ that is finer than every member of } \mathcal{T}\}$,
The set is nonempty since it contains the discrete topology. □

Definition 11.1.11 (Neighbourhood)**.** A *neighbourhood* of a point x is an open set that contains x.

11.2 Closed Set

Definition 11.2.1 (Closed Set)**.** Let X be a topological space and $A \subseteq X$. Then A is *closed* if and only if $X \setminus A$ is open.

Lemma 11.2.2. *The empty set is closed.*

PROOF: Since the whole space X is always open. □

Lemma 11.2.3. *The topological space X is closed.*

PROOF: Since $\emptyset$ is open. □

Lemma 11.2.4. *The intersection of a nonempty set of closed sets is closed.*

PROOF: Let $\mathcal{C}$ be a nonempty set of closed sets. Then $X \setminus \bigcap \mathcal{C} = \bigcup \{X \setminus C \mid C \in \mathcal{C}\}$ is open. □

Lemma 11.2.5. *The union of two closed sets is closed.*

PROOF: Let C and D be closed. Then $X \setminus (C \cup D) = (X \setminus C) \cap (X \setminus D)$ is open. □

Proposition 11.2.6. *Let X be a set and $\mathcal{C} \subseteq \mathcal{P}X$ a set such that:*

1. $\emptyset \in \mathcal{C}$
2. $X \in \mathcal{C}$
3. *For all* $\mathcal{A} \subseteq \mathcal{C}$ *nonempty we have* $\bigcap \mathcal{A} \in \mathcal{C}$
4. *For all* $C, D \in \mathcal{C}$ *we have* $C \cup D \in \mathcal{C}$.

Then there exists a unique topology $\mathcal{T}$ such that $\mathcal{C}$ is the set of closed sets, namely

$$\mathcal{T} = \{X \setminus C \mid C \in \mathcal{C}\}$$

PROOF:
$\langle 1 \rangle 1$. LET: $\mathcal{T} = \{X \setminus C \mid C \in \mathcal{C}\}$
$\langle 1 \rangle 2$. $\mathcal{T}$ is a topology
  $\langle 2 \rangle 1$. $X \in \mathcal{T}$
    PROOF: Since $\emptyset \in \mathcal{C}$
  $\langle 2 \rangle 2$. For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$
    $\langle 3 \rangle 1$. LET: $\mathcal{U} \subseteq \mathcal{T}$
    $\langle 3 \rangle 2$. CASE: $\mathcal{U} = \emptyset$
      PROOF: In this case $\bigcup \mathcal{U} = \emptyset \in \mathcal{T}$ since $X \in \mathcal{C}$
    $\langle 3 \rangle 3$. CASE: $\mathcal{U} \neq \emptyset$
      PROOF: In this case $X \setminus \bigcup \mathcal{U} = \bigcap \{X \setminus U \mid U \in \mathcal{U}\} \in \mathcal{C}$.
  $\langle 2 \rangle 3$. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$
    PROOF: Since $X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V) \in \mathcal{C}$.
$\langle 1 \rangle 3$. $\mathcal{C}$ is the set of all closed sets in $\mathcal{T}$
  PROOF:

$$\begin{aligned} & C \text{ is closed in } \mathcal{T} \\ \Leftrightarrow & X \setminus C \in \mathcal{T} \\ \Leftrightarrow & C \in \mathcal{C} \end{aligned}$$

$\langle 1 \rangle 4$. If $\mathcal{T}'$ is a topology and $\mathcal{C}$ is the set of closed sets in $\mathcal{T}'$ then $\mathcal{T}' = \mathcal{T}$
  PROOF: We have

$$\begin{aligned} & U \in \mathcal{T} \\ \Leftrightarrow & X \setminus U \in \mathcal{C} \\ \Leftrightarrow & X \setminus U \text{ is closed in } \mathcal{T}' \\ \Leftrightarrow & U \in \mathcal{T}' \end{aligned}$$

□

Proposition 11.2.7. *If U is open and A is closed then $U \setminus A$ is open.*

PROOF: $U \setminus A = U \cap (X \setminus A)$ is the intersection of two open sets. □

Proposition 11.2.8. *If U is open and A is closed then $A \setminus U$ is closed.*

PROOF: $A \setminus U = A \cap (X \setminus U)$ is the intersection of two closed sets. □

11.3 Interior

Definition 11.3.1 (Interior)**.** Let X be a topological space and $A \subseteq X$. The *interior* of A, $\operatorname{Int} A$, is the union of all the open subsets of A.

Lemma 11.3.2. *The interior of a set is open.*

PROOF: It is a union of open sets. □

Lemma 11.3.3.

$$\operatorname{Int} A \subseteq A$$

PROOF: Immediate from definition. □

Lemma 11.3.4. *If U is open and $U \subseteq A$ then $U \subseteq \operatorname{Int} A$*

PROOF: Immediate from definition. □

Lemma 11.3.5. *A set A is open if and only if $A = \operatorname{Int} A$.*

PROOF: If $A = \operatorname{Int} A$ then A is open by Lemma 11.3.2. Conversely if A is open then $A \subseteq \operatorname{Int} A$ by the definition of interior and so $A = \operatorname{Int} A$.

11.4 Closure

Definition 11.4.1 (Closure)**.** Let X be a topological space and $A \subseteq X$. The *closure* of A, $\overline{A}$, is the intersection of all the closed sets that include A.

This intersection exists since X is a closed set that includes A (Lemma 11.2.3).

Lemma 11.4.2. *The closure of a set is closed.*

PROOF: Dual to Lemma 11.3.2. □

Lemma 11.4.3.

$$A \subseteq \overline{A}$$

PROOF: Immediate from definition. □

Lemma 11.4.4. *If C is closed and $A \subseteq C$ then $\overline{A} \subseteq C$.*

PROOF: Immediate from definition. ∎

Lemma 11.4.5. *A set A is closed if and only if $A = \overline{A}$.*

PROOF: Dual to Lemma 11.3.5. ∎

Theorem 11.4.6. *Let X be a topological space, $A \subseteq X$ and $x \in X$. Then $x \in \overline{A}$ if and only if every neighbourhood of x intersects A.*

PROOF: We have

$$\begin{aligned} & x \in \overline{A} \\ \Leftrightarrow & \forall C.C \text{ closed } \wedge A \subseteq C \Rightarrow x \in C \\ \Leftrightarrow & \forall U.U \text{ open } \wedge A \cap U = \emptyset \Rightarrow x \notin U \\ \Leftrightarrow & \forall U.U \text{ open } \wedge x \in U \Rightarrow U \text{ intersects } A \end{aligned}$$ ∎

Proposition 11.4.7. *If $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$.*

PROOF: This holds because $\overline{B}$ is a closed set that includes A. ∎

Proposition 11.4.8.

$$\overline{A \cup B} = \overline{A} \cup \overline{B}$$

PROOF:
⟨1⟩1. $\overline{A} \subseteq \overline{A \cup B}$
  PROOF: By Proposition 11.4.7.
⟨1⟩2. $\overline{B} \subseteq \overline{A \cup B}$
  PROOF: By Proposition 11.4.7.
⟨1⟩3. $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$
  ⟨2⟩1. LET: $x \in \overline{A \cup B}$
  ⟨2⟩2. ASSUME: $x \notin \overline{A}$
    PROVE: $x \in \overline{B}$
  ⟨2⟩3. PICK a neighbourhood U of x that does not intersect A
  ⟨2⟩4. LET: V be any neighbourhood of x
  ⟨2⟩5. $U \cap V$ is a neighbourhood of x
  ⟨2⟩6. $U \cap V$ intersects $A \cup B$
    PROOF: From ⟨2⟩1 and Theorem 11.4.6.
  ⟨2⟩7. $U \cap V$ intersects B
    PROOF: From ⟨2⟩3
  ⟨2⟩8. V intersects B
  ⟨2⟩9. Q.E.D.
    PROOF: We have $x \in \overline{B}$ from Theorem 11.4.6.
∎

Proposition 11.4.9. *Let X be a topological space. Let $\mathcal{D}$ be a set of subsets of X that is maximal with respect to the finite intersection property. Let $x \in X$. Then the following are equivalent:*

1. *For all $D \in \mathcal{D}$ we have $x \in \overline{D}$*

2. *Every neighbourhood of x is in $\mathcal{D}$.*

PROOF:
⟨1⟩1. $1 \Rightarrow 2$
⟨2⟩1. For all $D \in \mathcal{D}$ we have $x \in \overline{D}$
⟨2⟩2. LET: U be a neighbourhood of x
⟨2⟩3. $\mathcal{D} \cup \{U\}$ satisfies the finite intersection property.
⟨3⟩1. LET: $D_1, \ldots, D_n \in \mathcal{D}$
⟨3⟩2. $D_1 \cap \cdots \cap D_n \in \mathcal{D}$
PROOF: Lemma 10.1.2.
⟨3⟩3. $x \in \overline{D_1 \cap \cdots \cap D_n}$
PROOF: ⟨2⟩1, ⟨3⟩2
⟨3⟩4. $D_1 \cap \cdots \cap D_n \cap U \neq \emptyset$
PROOF: Theorem 11.4.6, ⟨2⟩2, ⟨3⟩3.
⟨2⟩4. $\mathcal{D} = \mathcal{D} \cup \{U\}$
PROOF: By the maximality of $\mathcal{D}$.
⟨2⟩5. $U \in \mathcal{D}$
⟨1⟩2. $2 \Rightarrow 1$
⟨2⟩1. ASSUME: Every neighbourhood of x is in $\mathcal{D}$.
⟨2⟩2. LET: $D \in \mathcal{D}$
⟨2⟩3. Every neighbourhood of x intersects D.
PROOF: From ⟨2⟩1, ⟨2⟩2 and the fact that $\mathcal{D}$ satisfies the finite intersection property.
⟨2⟩4. $x \in \overline{D}$
PROOF: Theorem 11.4.6, ⟨2⟩3.
□

11.5 Boundary

Definition 11.5.1 (Boundary)**.** The *boundary* of a set A is the set $\partial A = \overline{A} \cap \overline{X \setminus A}$.

Proposition 11.5.2.
$$\text{Int}\, A \cap \partial A = \emptyset$$

PROOF: Since $\overline{X \setminus A} = X \setminus \text{Int}\, A$. □

Proposition 11.5.3.
$$\overline{A} = \text{Int}\, A \cup \partial A$$

PROOF:
$$\begin{aligned} \text{Int}\, A \cup \partial A &= \text{Int}\, A \cup (\overline{A} \cap \overline{X \setminus A}) \\ &= (\text{Int}\, A \cup \overline{A}) \cap (\text{Int}\, A \cup \overline{X \setminus A}) \\ &= \overline{A} \cap X \\ &= \overline{A} \end{aligned}$$

Proposition 11.5.4. *$\partial A = \emptyset$ if and only if A is open and closed.*

PROOF: If $\partial A = \emptyset$ then $\overline{A} = \text{Int}\, A$ by Proposition 11.5.3.

Proposition 11.5.5. *A set U is open if and only if $\partial U = \overline{U} \setminus U$.*

PROOF:

$$\begin{aligned} & \partial U = \overline{U} \setminus U \\ \Leftrightarrow & \overline{U} \setminus \text{Int}\, U = \overline{U} \setminus U & \text{(Propositions 11.5.2, 11.5.3)} \\ \Leftrightarrow & \text{Int}\, U = U & \square \end{aligned}$$

11.6 Limit Points

Definition 11.6.1 (Limit Point)**.** Let X be a topological space, $a \in X$ and $A \subseteq X$. Then a is a *limit point*, *cluster point* or *point of accumulation* for A if and only if every neighbourhood of a intersects A at a point other than a.

Lemma 11.6.2. *The point a is an accumulation point for A if and only if $a \in \overline{A \setminus \{a\}}$.*

PROOF: From Theorem 11.4.6. $\square$

Theorem 11.6.3. *Let X be a topological space and $A \subseteq X$. Let A' be the set of all limit points of A. Then $\overline{A} = A \cup A'$.*

PROOF:
$\langle 1 \rangle 1.$ For all $x \in \overline{A}$, if $x \notin A$ then $x \in A'$
  PROOF: From Theorem 11.4.6.
$\langle 1 \rangle 2.$ $A \subseteq \overline{A}$
  PROOF: Lemma 11.4.3.
$\langle 1 \rangle 3.$ $A' \subseteq \overline{A}$
  PROOF: From Theorem 11.4.6.
$\square$

Corollary 11.6.3.1. *A set is closed if and only if it contains all its limit points.*

Proposition 11.6.4. *In an indiscrete topology, every point is a limit point of any set that has more than one point.*

PROOF: Let X be an indiscrete space. Let A be a set with more than one point and x be a point. The only neighbourhood of x is X, which must intersect A at a point other than x. $\square$

Lemma 11.6.5. *Let X be a topological space and $A \subseteq B \subseteq X$. Then every limit point of A is a limit point of B.*

PROOF: Immediate from definitions. $\square$

11.7 Basis for a Topology

Definition 11.7.1 (Basis)**.** If X is a set, a *basis* for a topology on X is a set $\mathcal{B} \subseteq \mathcal{P}X$ called *basis elements* such that

1. For all $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$.
2. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

If $\mathcal{B}$ satisfies these two conditions, then we define the topology *generated* by $\mathcal{B}$ to be $\mathcal{T} = \{U \in \mathcal{P}X \mid \forall x \in U.\exists B \in \mathcal{B}.x \in B \subseteq U\}$.

We prove this is a topology.

PROOF:
⟨1⟩1. $X \in \mathcal{T}$
  PROOF: For all $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B \subseteq X$ by condition 1.
⟨1⟩2. For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$
  ⟨2⟩1. LET: $\mathcal{U} \subseteq \mathcal{T}$
  ⟨2⟩2. LET: $x \in \bigcup \mathcal{U}$
  ⟨2⟩3. PICK $U \in \mathcal{U}$ such that $x \in U$
  ⟨2⟩4. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$
    PROOF: Since $U \in \mathcal{T}$ by ⟨2⟩1 and ⟨2⟩3.
  ⟨2⟩5. $x \in B \subseteq \bigcup \mathcal{U}$
⟨1⟩3. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$
  ⟨2⟩1. LET: $U, V \in \mathcal{T}$
  ⟨2⟩2. LET: $x \in U \cap V$
  ⟨2⟩3. PICK $B_1 \in \mathcal{B}$ such that $x \in B_1 \subseteq U$
  ⟨2⟩4. PICK $B_2 \in \mathcal{B}$ such that $x \in B_2 \subseteq V$
  ⟨2⟩5. PICK $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$
    PROOF: By condition 2.
  ⟨2⟩6. $x \in B_3 \subseteq U \cap V$
□

Lemma 11.7.2. *Let X be a set. Let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ is the set of all unions of subsets of $\mathcal{B}$.*

PROOF:
⟨1⟩1. For all $U \in \mathcal{T}$, there exists $\mathcal{A} \subseteq \mathcal{B}$ such that $U = \bigcup \mathcal{A}$
  ⟨2⟩1. LET: $U \in \mathcal{T}$
  ⟨2⟩2. LET: $\mathcal{A} = \{B \in \mathcal{B} \mid B \subseteq U\}$
  ⟨2⟩3. $U \subseteq \bigcup \mathcal{A}$
    ⟨3⟩1. LET: $x \in U$
    ⟨3⟩2. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$
      PROOF: Since $\mathcal{B}$ is a basis for $\mathcal{T}$.
    ⟨3⟩3. $x \in B \in \mathcal{A}$
  ⟨2⟩4. $\bigcup \mathcal{A} \subseteq U$

PROOF: From the definition of $\mathcal{A}$ ($\langle 2\rangle 2$).
$\langle 1\rangle 2$. For all $\mathcal{A} \subseteq \mathcal{B}$ we have $\bigcup \mathcal{A} \in \mathcal{T}$
$\langle 2\rangle 1$. $\mathcal{B} \subseteq \mathcal{T}$
PROOF: If $B \in \mathcal{B}$ and $x \in B$, then there exists $B' \in \mathcal{B}$ such that $x \in B' \subseteq B$, namely $B' = B$.
$\langle 2\rangle 2$. Q.E.D.
PROOF: Since $\mathcal{T}$ is closed under union.
□

Corollary 11.7.2.1. *Let X be a set. Let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ is the coarsest topology that includes $\mathcal{B}$.*

PROOF: Since every topology that includes $\mathcal{B}$ includes all unions of subsets of $\mathcal{B}$. □

Lemma 11.7.3. *Let X be a topological space. Suppose that $\mathcal{C}$ is a set of open sets such that, for every open set U and every point $x \in U$, there exists $C \in \mathcal{C}$ such that $x \in C \subseteq U$. Then $\mathcal{C}$ is a basis for the topology on X.*

PROOF:
$\langle 1\rangle 1$. For all $x \in X$, there exists $C \in \mathcal{C}$ such that $x \in C$
PROOF: Immediate from hypothesis.
$\langle 1\rangle 2$. For all $C_1, C_2 \in \mathcal{C}$ and $x \in C_1 \cap C_2$, there exists $C_3 \in \mathcal{C}$ such that $x \in C_3 \subseteq C_1 \cap C_2$
PROOF: Since $C_1 \cap C_2$ is open.
$\langle 1\rangle 3$. Every open set is open in the topology generated by $\mathcal{C}$
PROOF: Immediate from hypothesis.
$\langle 1\rangle 4$. Every union of a subset of $\mathcal{C}$ is open.
PROOF: Since every member of $\mathcal{C}$ is open.
□

Lemma 11.7.4. *Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$ respectively on the set X. Then the following are equivalent.*

1. *$\mathcal{T} \subseteq \mathcal{T}'$*
2. *For all $B \in \mathcal{B}$ and $x \in B$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.*

PROOF:
$\langle 1\rangle 1$. $1 \Rightarrow 2$
$\langle 2\rangle 1$. ASSUME: $\mathcal{T} \subseteq \mathcal{T}'$
$\langle 2\rangle 2$. LET: $B \in \mathcal{B}$ and $x \in B$
$\langle 2\rangle 3$. $B \in \mathcal{T}$
PROOF: Corollary 11.7.2.1.
$\langle 2\rangle 4$. $B \in \mathcal{T}'$
PROOF: By $\langle 2\rangle 1$
$\langle 2\rangle 5$. There exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$
PROOF: Since $\mathcal{B}'$ is a basis for $\mathcal{T}'$.
$\langle 1\rangle 2$. $2 \Rightarrow 1$

⟨2⟩1. ASSUME: 2
⟨2⟩2. LET: $U \in \mathcal{T}$
PROVE: $U \in \mathcal{T}'$
⟨2⟩3. LET: $x \in U$
PROVE: There exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq U$
⟨2⟩4. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$
PROOF: Since $\mathcal{B}$ is a basis for $\mathcal{T}$.
⟨2⟩5. PICK $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$
PROOF: By ⟨2⟩1.
⟨2⟩6. $x \in B' \subseteq U$
□

Theorem 11.7.5. *Let X be a topological space, $A \subseteq X$ and $x \in X$. Let $\mathcal{B}$ be a basis for X. Then $x \in \overline{A}$ if and only if, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A.*

PROOF:
⟨1⟩1. If $x \in \overline{A}$ then, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A.
PROOF: This follows from Theorem 11.4.6 since every element of $\mathcal{B}$ is open (Corollary 11.7.2.1).
⟨1⟩2. Suppose that, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A. Then $x \in \overline{A}$.
⟨2⟩1. ASSUME: For all $B \in \mathcal{B}$, if $x \in B$ then B intersects A.
⟨2⟩2. LET: U be an open set that contains x
PROVE: U intersects A.
⟨2⟩3. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$.
⟨2⟩4. B intersects A.
PROOF: From ⟨2⟩1.
⟨2⟩5. U intersects A.
⟨2⟩6. Q.E.D.
PROOF: By Theorem 11.4.6.
□

Definition 11.7.6 (Lower Limit Topology on the Real Line)**.** The *lower limit topology on the real line* is the topology on $\mathbb{R}$ generated by the basis consisting of all half-open intervals of the form $[a, b)$.

We write $\mathbb{R}_l$ for the topological space $\mathbb{R}$ under the lower limit topology.

We prove this is a basis for a topology.

PROOF:
⟨1⟩1. For all $x \in \mathbb{R}$ there exists an interval $[a, b)$ such that $x \in [a, b)$.
PROOF: Take $[a, b) = [x, x + 1)$.
⟨1⟩2. For any open intervals $[a, b)$, $[c, d)$ if $x \in [a, b) \cap [c, d)$, then there exists an interval $[e, f)$ such that $x \in [e, f) \subseteq [a, b) \cap [c, d)$
PROOF: Take $[e, f) = [\max(a, c), \min(b, d))$.
□

Definition 11.7.7 (K-topology on the Real Line)**.** Let $K = \{1/n \mid n \in \mathbb{Z}^+\}$.

The K*-topology on the real line* is the topology on $\mathbb{R}$ generated by the basis consisting of all open intervals (a, b) and all sets of the form $(a, b) \setminus K$.

We write $\mathbb{R}_K$ for the topological space $\mathbb{R}$ under the K-topology.

We prove this is a basis for a topology.

PROOF:
⟨1⟩1. For all $x \in \mathbb{R}$ there exists an open interval (a, b) such that $x \in (a, b)$.
PROOF: Take $(a, b) = (x - 1, x + 1)$.
⟨1⟩2. For any basic open sets B_1, B_2 if $x \in B_1 \cap B_2$, then there exists a basic open set B_3 such that $x \in B_3 \subseteq B_1 \cap B_2$
⟨2⟩1. CASE: $B_1 = (a, b)$, $B_2 = (c, d)$
PROOF: Take $B_3 = (\max(a, c), \min(b, d))$.
⟨2⟩2. CASE: $B_1 = (a, b)$ or $(a, b) \setminus K$, $B_2 = (c, d)$ or $(c, d) \setminus K$, and they are not both open intervals.
PROOF: Take $B_3 = (\max(a, c), \min(b, d)) \setminus K$.
□

Lemma 11.7.8. *The lower limit topology and the* K*-topology are incomparable.*

PROOF:
⟨1⟩1. The interval $[10, 11)$ is not open in the K-topology.
PROOF: There is no open interval (a, b) such that $10 \in (a, b) \subseteq [10, 11)$ or $10 \in (a, b) \setminus K \subseteq [10, 11)$.
⟨1⟩2. The set $(-1, 1) \setminus K$ is not open in the lower limit topology.
PROOF: There is no half-open interval $[a, b)$ such that $0 \in [a, b) \subseteq (-1, 1) \setminus K$, since there must be a positive integer n with $1/n \in [a, b)$.
□

Definition 11.7.9 (Subbasis)**.** A *subbasis* $\mathcal{S}$ for a topology on X is a set $\mathcal{S} \subseteq \mathcal{P}X$ such that $\bigcup \mathcal{S} = X$.

The topology *generated* by the subbasis $\mathcal{S}$ is the set of all unions of finite intersections of elements of $\mathcal{S}$.

We prove this is a topology.

PROOF:
⟨1⟩1. The set $\mathcal{B}$ of all finite intersections of elements of $\mathcal{S}$ forms a basis for a topology on X.
⟨2⟩1. $\bigcup \mathcal{B} = X$
PROOF: Since $\mathcal{S} \subseteq \mathcal{B}$.
⟨2⟩2. $\mathcal{B}$ is closed under binary intersection.
PROOF: By definition.
⟨1⟩2. Q.E.D.
PROOF: By Lemma 11.7.2.
□

We have simultaneously proved:

Proposition 11.7.10. *Let* $\mathcal{S}$ *be a subbasis for the topology on* X*. Then the set of all finite intersections of elements of* $\mathcal{S}$ *is a basis for the topology on* X*.*

Proposition 11.7.11. *Let X be a set. Let $\mathcal{S}$ be a subbasis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ is the coarsest topology that includes $\mathcal{S}$.*

PROOF: Since every topology that includes $\mathcal{S}$ includes every union of finite intersections of elements of $\mathcal{S}$. ∎

11.8 Local Basis at a Point

Definition 11.8.1 (Local Basis)**.** Let X be a topological space and $a \in X$. A *(local) basis at a* is a set $\mathcal{B}$ of neighbourhoods of a such that every neighbourhood of a includes some member of $\mathcal{B}$.

Lemma 11.8.2. *If there exists a countable local basis at a point a, then there exists a countable local basis $\{B_n \mid n \geq 1\}$ such that $B_1 \supseteq B_2 \supseteq \cdots$.*

PROOF: Given a countable local basis $\{C_n \mid n \geq 1\}$, take $B_n = C_1 \cap \cdots \cap C_n$. ∎

11.9 Nets

Definition 11.9.1 (Net)**.** Let X be a topological space. A *net* in X consists of a directed poset J and a family $(x_\alpha)_{\alpha \in J}$ of points of X indexed by J.

Definition 11.9.2 (Convergence)**.** Let X be a topological space. Let $(x_\alpha)_{\alpha \in J}$ be a net in X and $l \in X$. Then (x_α) *converges* to the *limit* l iff, for every limit U of l, there exists $\alpha \in J$ such that, for all $\beta \geq \alpha$, we have $x_\beta \in U$.

Lemma 11.9.3. *Let X be a topological space. Let $A \subseteq X$ and $l \in X$. Then $l \in \overline{A}$ if and only if there exists a net of points in A that converges to l.*

PROOF:
⟨1⟩1. If $l \in \overline{A}$ then there exists a net of points in A that converges to l.
  ⟨2⟩1. ASSUME: $l \in \overline{A}$
  ⟨2⟩2. LET: J be the set of neighbourhoods of l under $\supseteq$
  ⟨2⟩3. For $U \in J$, PICK $a_U \in U \cap A$
    PROVE: $a_U \to l$ as $U \to \infty$
   PROOF: Theorem 11.4.6.
  ⟨2⟩4. LET: U be a neighbourhood of l.
  ⟨2⟩5. For any $V \subseteq U$ we have $a_V \in V$.
⟨1⟩2. If there exists a net of points in A that coverges to l, then $l \in \overline{A}$.
  ⟨2⟩1. LET: $(a_\alpha)_{\alpha \in J}$ be a sequence of points in A that converges to l.
  ⟨2⟩2. LET: U be a neighbourhood of l.
  ⟨2⟩3. PICK $\alpha \in J$ such that, for all $\beta \geq \alpha$, we have $a_\beta \in U$.
  ⟨2⟩4. $a_\alpha \in U \cap A$
  ⟨2⟩5. Q.E.D.
   PROOF: Theorem 11.4.6.
∎

Proposition 11.9.4. *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Let (a_n) be a sequence in X and $l \in X$. Then $a_n \to l$ as $n \to \infty$ if and only if, for every $B \in \mathcal{B}$ with $l \in B$, there exists N such that, for all $n \geq N$, we have $a_n \in B$.*

PROOF:
⟨1⟩1. If $a_n \to l$ as $n \to \infty$ then, for every $B \in \mathcal{B}$ with $l \in B$, there exists N such that, for all $n \geq N$, we have $a_n \in B$.
PROOF: Since every element of $\mathcal{B}$ is open (Corollary 11.7.2.1).
⟨1⟩2. If, for every $B \in \mathcal{B}$ with $l \in B$, there exists N such that, for all $n \geq N$, we have $a_n \in B$, then $a_n \to l$ as $n \to \infty$.
⟨2⟩1. ASSUME: for every $B \in \mathcal{B}$ with $l \in B$, there exists N such that, for all $n \geq N$, we have $a_n \in B$
⟨2⟩2. LET: U be a neighbourhood of l.
⟨2⟩3. PICK $B \in \mathcal{B}$ such that $l \in B \subseteq U$
⟨2⟩4. PICK N such that, for all $n \geq N$, we have $a_n \in B$
PROOF: From ⟨2⟩1.
⟨2⟩5. For all $n \geq N$ we have $a_n \in U$
□

Lemma 11.9.5. *If a sequence (a_n) is constant with $a_n = l$ for all n, then $a_n \to l$ as $n \to \infty$.*

PROOF: Immediate from definitions. □

Theorem 11.9.6. *Let X be a linearly ordered set. Let (s_n) be a monotone increasing sequence in X with a supremum s. Then $s_n \to s$ as $n \to \infty$.*

PROOF:
⟨1⟩1. ASSUME: s is not least in X.
PROOF: Otherwise (s_n) is the constant sequence s and the result follows from Lemma 11.9.5.
⟨1⟩2. LET: U be a neighbourhood of s.
⟨1⟩3. PICK$a < s$ such that $(a, s] \subseteq U$
⟨1⟩4. PICK N such that $a < a_N$.
⟨1⟩5. For all $n \geq N$ we have $a_n \in (a, s]$
⟨1⟩6. For all $n \geq N$ we have $a_n \in U$.
□

Theorem 11.9.7. *If $\sum_{i=0}^{\infty} a_i = s$ and $\sum_{i=0}^{\infty} b_i = t$ then $\sum_{i=0}^{\infty}(ca_i + b_i) = cs + t$.*

PROOF:$\sum_{i=0}^{N}(ca_i + b_i) = c\sum_{i=0}^{N} a_i + \sum_{i=0}^{N} b_i \to cs + t$ as $n \to \infty$. □

Theorem 11.9.8 (Comparison Test)**.** *If $|a_i| \leq b_i$ for all i and $\sum_{i=0}^{\infty} b_i$ converges then $\sum_{i=0}^{\infty} a_i$ converges.*

PROOF:
⟨1⟩1. $\sum_{i=0}^{\infty} |a_i|$ converges

PROOF: The partial sums $\sum_{i=0}^{N} |a_i|$ form an increasing sequence bounded above by $\sum_{i=0}^{\infty} b_i$.

$\langle 1 \rangle 2$. LET: $c_i = |a_i| + a_i$ for all i

$\langle 1 \rangle 3$. $\sum_{i=0}^{\infty} c_i$ converges

PROOF: Each c_i is either $2|a_i|$ or 0. So the partial sums $\sum_{i=0}^{N} c_i$ form an increasing sequence bounded above by $2 \sum_{i=0}^{\infty} b_i$.

$\langle 1 \rangle 4$. Q.E.D.

PROOF: Since $a_i = c_i - |a_i|$.

□

Corollary 11.9.8.1. *If $\sum_{i=0}^{\infty} |a_i|$ converges then $\sum_{i=0}^{\infty} a_i$ converges.*

Theorem 11.9.9 (Weierstrass M-test)**.** *Let X be a set and $(f_n : X \to \mathbb{R})$ be a sequence of functions. Let*

$$s_n(x) = \sum_{i=0}^{n} f_i(x)$$

for all n, x. Suppose $|f_i(x)| \leq M_i$ for all $i \geq 0$ and $x \in X$. If the series $\sum_{i=0}^{\infty} M_i$ converges, then the sequence (s_n) converges uniformly to

$$s(x) = \sum_{i=0}^{\infty} f_i(x) \quad .$$

PROOF:

$\langle 1 \rangle 1$. LET: $r_n = \sum_{i=n+1}^{\infty} M_i$ for all n

$\langle 1 \rangle 2$. Given $0 \leq n < k$, we have $|s_k(x) - s_n(x)| \leq r_n$

PROOF:

$$\begin{aligned} |s_k(x) - s_n(x)| &= |\sum_{i=n+1}^{k} f_i(x)| \\ &\leq \sum_{i=n+1}^{k} |f_i(x)| \\ &\leq \sum_{i=n+1}^{k} M_i \\ &\leq r_n \end{aligned}$$

$\langle 1 \rangle 3$. Given $n \geq 0$ we have $|s(x) - s_n(x)| \leq r_n$

PROOF: By taking the limit $k \to \infty$ in $\langle 1 \rangle 2$.

$\langle 1 \rangle 4$. Q.E.D.

PROOF: Since $r_n \to 0$ as $n \to \infty$.

□

11.10 Locally Finite Sets

Definition 11.10.1 (Locally Finite)**.** Let X be a topological space and $\{A_\alpha\}$ a family of subsets of X. Then $\mathcal{A}$ is *locally finite* if and only if every point in X

has a neighbourhood that intersects A_α for only finitely many α.

The following example shows that we cannot remove the assumption of local finiteness.

Example 11.10.2. Define $f : [-1,1] \to \mathbb{R}$ by: $f(x) = 1$ if $x < -1$, $f(x) = 0$ if $x > 1$. Let $C_n = [-1, -1/n]$ for $n \geq 1$, and $D = [0,1]$. Then $[-1,1] = \bigcup_{n=1}^{\infty} C_n \cup D$ and f is continuous on each C_n and each D, but f is not continuous on $[-1,1]$.

11.11 Open Maps

Definition 11.11.1 (Open Map)**.** Let X and Y be topological spaces and $f : X \to Y$. Then f is an *open map* if and only if, for every open set U in X, the set $f(U)$ is open in Y.

Lemma 11.11.2. *Let X and Y be topological spaces and $f : X \to Y$. Let $\mathcal{B}$ be a basis for the topology on X. If $f(B)$ is open in Y for all $B \in \mathcal{B}$, then f is an open map.*

Proof: From Lemma 11.7.2. □

Proposition 11.11.3. *Let X and Y be topological spaces. Let $\mathcal{B}$ be a basis for the topology on X. Let $f : X \to Y$. Suppose that, for all $B \in \mathcal{B}$, we have $f(B)$ is open tn Y. Then f is an open map.*

Proof: For any $\mathcal{A} \subseteq \mathcal{B}$, we have $f(\bigcup \mathcal{A}) = \bigcup_{B \in \mathcal{B}} f(B)$ is open in Y. The result follows from Lemma 11.7.2. □

11.12 Continuous Functions

Definition 11.12.1 (Continuous)**.** Let X and Y be topological spaces. A function $f : X \to Y$ is *continuous* if and only if, for every open set V in Y, the set $f^{-1}(V)$ is open in X.

Proposition 11.12.2. *Let X and Y be topological spaces and $f : X \to Y$. Let $\mathcal{B}$ be a basis for Y. Then f is continuous if and only if, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X.*

Proof:

$\langle 1 \rangle 1$. If f is continuous then, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X.
  Proof: Since every element of B is open (Lemma 11.7.2).

$\langle 1 \rangle 2$. Suppose that, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X. Then f is continuous.
  $\langle 2 \rangle 1$. Assume: For all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X.
  $\langle 2 \rangle 2$. Let: V be open in Y.
  $\langle 2 \rangle 3$. Pick $\mathcal{A} \subseteq \mathcal{B}$ such that $V = \bigcup \mathcal{A}$

Proof: By Lemma 11.7.2.

⟨2⟩4. $f^{-1}(V)$ is open in X.

Proof:

$$f^{-1}(V) = f^{-1}\left(\bigcup \mathcal{A}\right) = \bigcup_{B \in \mathcal{A}} f^{-1}(B)$$

□

Proposition 11.12.3. *Let X and Y be topological spaces and $f : X \to Y$. Let $\mathcal{S}$ be a subbasis for Y. Then f is continuous if and only if, for all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in X.*

Proof:

⟨1⟩1. If f is continuous then, for all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in X.

Proof: Since every element of S is open.

⟨1⟩2. Suppose that, for all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in X. Then f is continuous.

⟨2⟩1. Assume: For all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in X.

⟨2⟩2. Let: $S_1, \ldots, S_n \in \mathcal{S}$

⟨2⟩3. $f^{-1}(S_1 \cap \cdots \cap S_n)$ is open in A

Proof: Since $f^{-1}(S_1 \cap \cdots \cap S_n) = f^{-1}(S_1) \cap \cdots \cap f^{-1}(S_n)$.

⟨2⟩4. Q.E.D.

Proof: By Propositions 11.12.2 and 11.7.10.

□

Proposition 11.12.4. *Let X and Y be topological spaces and $f : X \to Y$. Let $\mathcal{S}$ be a basis for Y. Then f is continuous if and only if, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X.*

Proof:

⟨1⟩1. If f is continuous then, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X.

Proof: Since every element of $\mathcal{S}$ is open.

⟨1⟩2. Suppose that, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X. Then f is continuous.

⟨2⟩1. Assume: For all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X.

⟨2⟩2. For every set B that is the finite intersection of elemets of $\mathcal{S}$, we have $f^{-1}(B)$ is open in X.

Proof: Because $f^{-1}(V_1 \cap \cdots \cap V_n) = f^{-1}(V_1) \cap \cdots \cap f^{-1}(V_n)$.

⟨2⟩3. Q.E.D.

Proof: From Propositions 11.7.10 and 11.12.2.

□

Definition 11.12.5 (Continuous at a Point)**.** Let X and Y be topological spaces. Let $f : X \to Y$ and $x \in X$. Then f is *continuous at x* if and only if, for every neighbourhood V of $f(x)$, there exists a neighbourhood U of x such that $f(U) \subseteq V$.

Theorem 11.12.6. *Let X and Y be topological spaces and $f : X \to Y$. Then the following are equivalent:*

1. *f is continuous.*
2. *For all $A \subseteq X$, we have $f(\overline{A}) \subseteq \overline{f(A)}$*
3. *For all $B \subseteq Y$ closed, we have $f^{-1}(B)$ is closed in X.*
4. *f is continuous at every point of X.*

Proof:
$\langle 1 \rangle 1$. $1 \Rightarrow 2$
$\langle 2 \rangle 1$. Assume: f is continuous.
$\langle 2 \rangle 2$. Let: $A \subseteq X$
$\langle 2 \rangle 3$. Let: $x \in \overline{A}$
Prove: $f(x) \in \overline{f(A)}$
$\langle 2 \rangle 4$. Let: V be a neighbourhood of $f(x)$
$\langle 2 \rangle 5$. $f^{-1}(V)$ is a neighbourhood of x
$\langle 2 \rangle 6$. Pick $y \in A \cap f^{-1}(V)$
Proof: By Theorem 11.4.6.
$\langle 2 \rangle 7$. $f(y) \in V \cap f(A)$
$\langle 2 \rangle 8$. Q.E.D.
Proof: By Theorem 11.4.6.
$\langle 1 \rangle 2$. $2 \Rightarrow 3$
$\langle 2 \rangle 1$. Assume: 2
$\langle 2 \rangle 2$. Let: B be closed in Y
$\langle 2 \rangle 3$. Let: $x \in \overline{f^{-1}(B)}$
Prove: $x \in f^{-1}(B)$
$\langle 2 \rangle 4$. $f(x) \in B$
Proof:

$$\begin{aligned} f(x) &\in f(\overline{f^{-1}(B)}) \\ &\subseteq \overline{f(f^{-1}(B))} && (\langle 2 \rangle 1) \\ &\subseteq \overline{B} && (Proposition 11.4.7) \\ &= B \end{aligned}$$

$\langle 1 \rangle 3$. $3 \Rightarrow 1$
$\langle 2 \rangle 1$. Assume: 3
$\langle 2 \rangle 2$. Let: V be open in Y
$\langle 2 \rangle 3$. $Y \setminus V$ is closed in Y
$\langle 2 \rangle 4$. $f^{-1}(Y \setminus V)$ is closed in X
$\langle 2 \rangle 5$. $X \setminus f^{-1}(V)$ is closed in X
$\langle 2 \rangle 6$. $f^{-1}(V)$ is open in X
$\langle 1 \rangle 4$. $1 \Rightarrow 4$
Proof: For any neighbourhood V of $f(x)$, the set $U = f^{-1}(V)$ is a neighbourhood of x such that $f(U) \subseteq V$.
$\langle 1 \rangle 5$. $4 \Rightarrow 1$

$\langle 2 \rangle 1$. ASSUME: 4
$\langle 2 \rangle 2$. LET: V be open in Y
$\langle 2 \rangle 3$. LET: $x \in f^{-1}(V)$
$\langle 2 \rangle 4$. V is a neighbourhood of $f(x)$
$\langle 2 \rangle 5$. PICK a neighbourhood U of x such that $f(U) \subseteq V$
$\langle 2 \rangle 6$. $x \in U \subseteq f^{-1}(V)$
$\langle 2 \rangle 7$. Q.E.D.
PROOF: By Lemma 11.1.8.
$\square$

Theorem 11.12.7. *A constant function is continuous.*

PROOF: Let X and Y be topological spaces. Let $b \in Y$, and let $f : X \to Y$ be the constant function with value b. For any open $V \subseteq Y$, the set $f^{-1}(V)$ is either X (if $b \in V$) or $\emptyset$ (if $b \notin V$). $\square$

Theorem 11.12.8. *If A is a subspace of X then the inclusion $j : A \to X$ is continuous.*

PROOF: For any V open in X, we have $j^{-1}(V) = V \cap A$ is open in A. $\square$

Theorem 11.12.9. *The composite of two continuous functions is continuous.*

PROOF: Let $f : X \to Y$ and $g : Y \to Z$ be continuous. For any V open in Z, we have $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is open in X. $\square$

Theorem 11.12.10. *Let $f : X \to Y$ be a continuous function and A be a subspace of X. Then the restriction $f \restriction A : A \to Y$ is continuous.*

PROOF: Let V be open in Y. Then $(f \restriction A)^{-1}(V) = f^{-1}(V) \cap A$ is open in A. $\square$

Theorem 11.12.11. *Let $f : X \to Y$ be continuous. Let Z be a subspace of Y such that $f(X) \subseteq Z$. Then the corestriction $f : X \to Z$ is continuous.*

PROOF:
$\langle 1 \rangle 1$. LET: V be open in Z.
$\langle 1 \rangle 2$. PICK U open in Y such that $V = U \cap Z$.
$\langle 1 \rangle 3$. $f^{-1}(V) = f^{-1}(U)$
$\langle 1 \rangle 4$. $f^{-1}(V)$ is open in X.
$\square$

Theorem 11.12.12. *Let $f : X \to Y$ be continuous. Let Z be a space such that Y is a subspace of Z. Then the expansion $f : X \to Z$ is continuous.*

PROOF: Let V be open in Z. Then $f^{-1}(V) = f^{-1}(V \cap Y)$ is open in X. $\square$

Theorem 11.12.13. *Let X and Y be topological spaces. Let $f : X \to Y$. Suppose $\mathcal{U}$ is a set of open sets in X such that $X = \bigcup \mathcal{U}$ and, for all $U \in \mathcal{U}$, we have $f \restriction U : U \to Y$ is continuous. Then f is continuous.*

PROOF:

⟨1⟩1. LET: V be open in Y
⟨1⟩2. $f^{-1}(V) = \bigcup_{U \in \mathcal{U}} (f \restriction U)^{-1}(V)$
⟨1⟩3. For all $U \in \mathcal{U}$, we have $(f \restriction U)^{-1}(V)$ is open in U.
⟨1⟩4. For all $U \in \mathcal{U}$, we have $(f \restriction U)^{-1}(V)$ is open in X.
  PROOF: Lemma 11.17.6.
□

Proposition 11.12.14. *Let X and X' be the same set X under two topologies $\mathcal{T}$ and $\mathcal{T}'$. Let $i : X \to X'$ be the identity function. Then i is continuous if and only if $\mathcal{T}' \subseteq \mathcal{T}$.*

PROOF: Immediate from definitions. □

Proposition 11.12.15. *Let $f : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$. Then f is continuous on the right at a if and only if f is continuous at a as a function $\mathbb{R}_l \to \mathbb{R}$.*

PROOF:
⟨1⟩1. If f is continuous on the right at a then f is continuous at a as a function $\mathbb{R}_l \to \mathbb{R}$.
  ⟨2⟩1. ASSUME: f is continuous on the right at a.
  ⟨2⟩2. LET: V be a neighbourhood of $f(a)$
  ⟨2⟩3. PICK b, c such that $f(a) \in (b, c) \subseteq V$.
  ⟨2⟩4. LET: $\epsilon = \min(c - f(a), f(a) - b)$
  ⟨2⟩5. PICK $\delta > 0$ such that, for all x, if $a < x < a + \delta$ then $|f(x) - f(a)| < \epsilon$
  ⟨2⟩6. LET: $U = [a, a + \delta)$
  ⟨2⟩7. $f(U) \subseteq V$
⟨1⟩2. If f is continuous at a as a function $\mathbb{R}_l \to \mathbb{R}$ then f is continuous on the right at a.
  ⟨2⟩1. ASSUME: f is continuous at a as a function $\mathbb{R}_l \to \mathbb{R}$
  ⟨2⟩2. LET: $\epsilon > 0$
  ⟨2⟩3. PICK a neighbourhood U of a such that $f(U) \subseteq (f(a) - \epsilon, f(a) + \epsilon)$
  ⟨2⟩4. PICK b, c such that $a \in [b, c) \subset U$
  ⟨2⟩5. LET: $\delta = c - a$
  ⟨2⟩6. For all x, if $a < x < a + \delta$ then $|f(x) - f(a)| < \epsilon$
□

Lemma 11.12.16. *Let $f : X \to Y$. Let Z be an open subspace of X and $a \in Z$. If $f \restriction Z$ is continuous at a then f is continuous at a.*

PROOF:
⟨1⟩1. LET: V be a neighbourhood of $f(x)$
⟨1⟩2. PICK a neighbourhood W of x in Z such that $f(W) \subseteq V$
⟨1⟩3. W is a neighbourhood of x in X such that $f(W) \subseteq V$
  PROOF: Lemma 11.17.6.
□

Proposition 11.12.17. *Let $f : A \to B$ and $g : C \to D$ be continuous. Define $f \times g : A \times C \to B \times D$ by*

$$(f \times g)(a, c) = (f(a), g(c)) \ .$$

Then $f \times g$ is continuous.

PROOF: $\pi_1 \circ (f \times g) = f \circ \pi_1$ and $\pi_2 \circ (f \times g) = g \circ \pi_2$ are continuous by Theorem 11.12.9. The result follows by Theorem 11.16.11.

Proposition 11.12.18. *Let X and Y be topological spaces and $f : X \to Y$. Then f is continuous if and only if, for any net $(a_\alpha)_{\alpha \in J}$ in X and $l \in X$, if $a_\alpha \to l$ as $\alpha \to \infty$ in X then $f(a_\alpha) \to f(l)$ as $\alpha \to \infty$.*

PROOF:
⟨1⟩1. If f is continuous then, for every net $(a_\alpha)_{\alpha \in J}$ in X and $l \in X$, if $a_\alpha \to l$ as $\alpha \to \infty$ then $f(a_\alpha) \to f(l)$ as $\alpha \to \infty$
  ⟨2⟩1. ASSUME: f is continuous.
  ⟨2⟩2. LET: $(a_\alpha)_{\alpha \in J}$ be a net in X
  ⟨2⟩3. LET: $l \in X$
  ⟨2⟩4. ASSUME: $a_\alpha \to l$ as $\alpha \to \infty$
  ⟨2⟩5. LET: V be a neighbourhood of $f(l)$
  ⟨2⟩6. PICK a neighbourhood U of l such that $f(U) \subseteq V$
  ⟨2⟩7. PICK $\alpha \in J$ such that, for all $\beta \geq \alpha$, we have $a_\beta \in U$
  ⟨2⟩8. For all $\beta \geq \alpha$ we have $f(a_\beta) \in V$
⟨1⟩2. If, for every net $(a_\alpha)_{\alpha \in J}$ in X and $l \in X$, if $a_\alpha \to l$ as $\alpha \to \infty$ then $f(a_\alpha) \to f(l)$ as $\alpha \to \infty$, then f is continuous.
  ⟨2⟩1. ASSUME: for every net $(a_\alpha)_{\alpha \in J}$ in X and $l \in X$, if $a_\alpha \to l$ as $\alpha \to \infty$ then $f(a_\alpha) \to f(l)$ as $\alpha \to \infty$
    PROVE: For every $A \subseteq X$, we have $f(\overline{A}) \subseteq \overline{f(A)}$
  ⟨2⟩2. LET: $A \subseteq X$
  ⟨2⟩3. LET: $x \in \overline{A}$
    PROVE: $f(x) \in \overline{f(A)}$
  ⟨2⟩4. PICK a net $(a_\alpha)_{\alpha \in J}$ of points in A that converges to x
    PROOF: Lemma 11.9.3.
  ⟨2⟩5. $(f(a_\alpha))_{\alpha \in J}$ is a net of points in $f(A)$ that converges to $f(x)$
    PROOF: From ⟨2⟩1.
  ⟨2⟩6. $f(x) \in \overline{f(A)}$
    PROOF: Lemma 11.9.3.
  ⟨2⟩7. Q.E.D.
    PROOF: Theorem 11.12.6.
□

Theorem 11.12.19 (Pasting Lemma)**.** *Let X and Y be topological spaces and $f : X \to Y$. Let $\{A_\alpha\}$ be a locally finite family of closed subsets of X that cover X. Suppose $f \restriction A_\alpha$ is continuous for all α. Then f is continuous.*

PROOF:
⟨1⟩1. Let X and Y be topological spaces and $f : X \to Y$. Let A and B be closed subsets of X such that $X = A \cup B$. Suppose $f \restriction A$ and $f \restriction B$ are continuous. Then f is continuous.
  ⟨2⟩1. LET: $C \subseteq Y$ be closed.
  ⟨2⟩2. $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$

⟨2⟩3. $f^{-1}(C)$ and $g^{-1}(C)$ are closed in X.
PROOF: Theorems 11.12.6 and 11.17.7.
⟨2⟩4. $h^{-1}(C)$ is closed in X.
PROOF: Lemma 11.2.5.
⟨2⟩5. Q.E.D.
PROOF: Theorem 11.12.6.
⟨1⟩2. Let X and Y be topological spaces and $f : X \to Y$. Let $\{A_\alpha\}$ be a finite family of closed subsets of X that cover X. Suppose $f \upharpoonright A_\alpha$ is continuous for all α. Then f is continuous.
PROOF: From ⟨1⟩1 by induction.
⟨1⟩3. Let X and Y be topological spaces and $f : X \to Y$. Let $\{A_\alpha\}$ be a locally finite family of closed subsets of X that cover X. Suppose $f \upharpoonright A_\alpha$ is continuous for all α. Then f is continuous.
⟨2⟩1. LET: $x \in X$
PROVE: f is continuous at x
⟨2⟩2. PICK a neighbourhood U of x that intersects A_α for only finitely many α.
⟨2⟩3. $f \upharpoonright U$ is continuous
PROOF: By ⟨1⟩2.
⟨2⟩4. Q.E.D.
PROOF: Lemma 11.12.16.
□

11.13 Homeomorphisms

Definition 11.13.1 (Homeomorphism)**.** Let X and Y be topological spaces. A *Homeomorphism* f between X and Y, $f : X \cong Y$, is a bijection $f : X \to Y$ such that both f and f^{-1} are continuous.

Lemma 11.13.2. *Let X and Y be topological spaces and $f : X \to Y$ a bijection. Then the following are equivalent:*

1. *f is a homeomorphism.*
2. *f is continuous and an open map.*
3. *f is continuous and a closed map.*
4. *For any $U \subseteq X$, we have U is open if and only if $f(U)$ is open.*

PROOF: Immediate from definitions. □

Proposition 11.13.3. *Let X and X' be the same set X under two topologies $\mathcal{T}$ and $\mathcal{T}'$. Let $i : X \to X'$ be the identity function. Then i is a homeomorphism if and only if $\mathcal{T} = \mathcal{T}'$.*

PROOF: Immediate from definitions. □

Definition 11.13.4 (Topological Property)**.** Let P be a property of topological spaces. Then P is a *topological* property if and only if, for any spaces X and Y, if P holds of X and $X \cong Y$ then P holds of Y.

Definition 11.13.5 (Topological Imbedding)**.** Let X and Y be topological spaces and $f : X \to Y$. Then f is a *topological imbedding* if and only if the corestriction $f : X \to f(X)$ is a homeomorphism.

Proposition 11.13.6. *Let X and Y be topological spaces and $a \in X$. The function $i : Y \to X \times Y$ that maps y to (a, y) is an imbedding.*

PROOF:
$\langle 1 \rangle 1.$ i is injective
$\langle 1 \rangle 2.$ i is continuous.
PROOF: For U open in X and V open in Y, we have $i^{-1}(U \times V)$ is V if $a \in U$, and $\emptyset$ if $a \notin U$.
$\langle 1 \rangle 3.$ $i : Y \to i(Y)$ is an open map.
PROOF: For V open in Y we have $i(V) = (X \times V) \cap i(Y)$.
□

11.14 The Order Topology

Definition 11.14.1 (Order Topology)**.** Let X be a linearly ordered set with at least two points. The *order topology* on X is the topology generated by the basis $\mathcal{B}$ consisting of:

- all open intervals (a, b);
- all intervals of the form $[\bot, b)$ where $\bot$ is least in X;
- all intervals of the form $(a, \top]$ where $\top$ is greatest in X.

We prove this is a basis for a topology.

PROOF:
$\langle 1 \rangle 1.$ For all $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B$.
$\langle 2 \rangle 1.$ LET: $x \in X$
$\langle 2 \rangle 2.$ CASE: x is greatest in X.
$\langle 3 \rangle 1.$ PICK $y \in X$ with $y \neq x$
$\langle 3 \rangle 2.$ $x \in (y, x] \in \mathcal{B}$
$\langle 2 \rangle 3.$ CASE: x is least in X.
$\langle 3 \rangle 1.$ PICK $y \in X$ with $y \neq x$
$\langle 3 \rangle 2.$ $x \in [x, y) \in \mathcal{B}$
$\langle 2 \rangle 4.$ CASE: x is neither greatest nor least in X.
$\langle 3 \rangle 1.$ PICK $a, b \in X$ with $a < x$ and $x < b$
$\langle 3 \rangle 2.$ $x \in (a, b) \in \mathcal{B}$
$\langle 1 \rangle 2.$ For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$

⟨2⟩1. Let: $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$
⟨2⟩2. Case: $B_1 = (a, b)$, $B_2 = (c, d)$
Proof: Take $B_3 = (\max(a, c), \min(b, d))$.
⟨2⟩3. Case: $B_1 = (a, b)$, $B_2 = [\perp, d)$
Proof: Take $B_3 = (a, \min(b, d))$.
⟨2⟩4. Case: $B_1 = (a, b)$, $B_2 = (c, \top]$
Proof: Take $B_3 = (\max(a, c), b)$.
⟨2⟩5. Case: $B_1 = [\perp, b)$, $B_2 = [\perp, d)$
Proof: Take $B_3 = [\perp, \min(b, d))$.
⟨2⟩6. Case: $B_1 = [\perp, b)$, $B_2 = (c, \top]$
Proof: Take $B_3 = (c, b)$.
□

Lemma 11.14.2. *Let X be a linearly ordered set. Then the open rays form a subbasis for the order topology on X.*

Proof:
⟨1⟩1. Every open ray is open.
⟨2⟩1. For all $a \in X$, the ray $(-\infty, a)$ is open.
⟨3⟩1. Let: $x \in (-\infty, a)$
⟨3⟩2. Case: x is least in X
Proof: $xin[x, a) = (-\infty, a)$.
⟨3⟩3. Case: x is not least in X
⟨4⟩1. Pick $y < x$
⟨4⟩2. $x \in (y, a) \subseteq (-\infty, a)$
⟨2⟩2. For all $a \in X$, the ray $(a, +\infty)$ is open.
Proof: Similar.
⟨1⟩2. Every basic open set is a finite intersection of open rays.
Proof: We have $(a, b) = (a, +\infty) \cap (-\infty, b)$, $[\perp, b) = (-\infty, b)$ and $(a, \top] = (a, +\infty)$.
□

Definition 11.14.3 (Standard Topology on the Real Line)**.** The *standard topology on the real line* is the order topology on $\mathbb{R}$ generated by the standard order.

Lemma 11.14.4. *The lower limit topology is strictly finer than the standard topology on $\mathbb{R}$.*

Proof:
⟨1⟩1. Every open interval is open in the lower limit topology.
Proof: If $x \in (a, b)$ then $x \in [x, b) \subseteq (a, b)$.
⟨1⟩2. The half-open interval $[0, 1)$ is not open in the standard topology.
Proof: There is no open interval (a, b) such that $0 \in (a, b) \subseteq [0, 1)$.
□

Lemma 11.14.5. *The K-topology is strictly finer than the standard topology on $\mathbb{R}$.*

Proof:

$\langle 1\rangle 1$. Every open interval is open in the K-topology.

PROOF: Corollary 11.7.2.1.

$\langle 1\rangle 2$. The set $(-1,1) \setminus K$ is not open in the standard topology.

PROOF: There is no open interval (a,b) such that $0 \in (a,b) \subseteq (-1,1) \setminus K$, since there must be a positive integer n with $1/n \in (a,b)$.

□

Lemma 11.14.6. *Let X be a topological space. Let Y be a linearly ordered set in the order topology. Let $f, g : X \to Y$ be continuous. Then $C = \{x \in X \mid f(x) \leq g(x)\}$ is closed.*

PROOF:

$\langle 1\rangle 1$. LET: $x \in X \setminus C$

$\langle 1\rangle 2$. $f(x) > g(x)$

PROVE: There exists a neighbourhood U of x such that $U \subseteq X \setminus C$

$\langle 1\rangle 3$. CASE: There exists y such that $g(x) < y < f(x)$

PROOF: Take $U = g^{-1}((-\infty, y)) \cup f^{-1}(y, +\infty)$.

$\langle 1\rangle 4$. CASE: There is no y such that $g(x) < y < f(x)$

PROOF: Take $U = g^{-1}((-\infty, f(x))) \cup f^{-1}(g(x), +\infty)$.

□

Proposition 11.14.7. *Let X be a topological space. Let Y be a linearly ordered set in the order topology. Let $f, g : X \to Y$ be continuous. Define $h : X \to Y$ by $h(x) = \min(f(x), g(x))$. Then h is continuous.*

PROOF: By the Pasting Lemma applied to $\{x \in X \mid f(x) \leq g(x)\}$ and $\{x \in X \mid g(x) \leq f(x)\}$, which are closed by Lemma 11.14.6.

Proposition 11.14.8. *Let X and Y be linearly ordered sets in the order topology. Let $f : X \to Y$ be strictly monotone and surjective. Then f is a homeomorphism.*

PROOF:

$\langle 1\rangle 1$. f is bijective.

PROOF: Proposition 5.3.3.

$\langle 1\rangle 2$. f is continuous.

$\langle 2\rangle 1$. For all $y \in Y$ we have $f^{-1}((y, +\infty))$ is open.

$\langle 3\rangle 1$. LET: $y \in Y$

$\langle 3\rangle 2$. PICK$x \in X$ such that $f(x) = y$

PROOF: Since f is surjective.

$\langle 3\rangle 3$. $f^{-1}((y, +\infty)) = (x, +\infty)$

PROOF: By strict monotoncity.

$\langle 2\rangle 2$. For all $y \in Y$ we have $f^{-1}((-\infty, y))$ is open.

PROOF: Similar.

$\langle 1\rangle 3$. f^{-1} is continuous.

$\langle 2\rangle 1$. For all $x \in X$ we have $f((x, +\infty))$ is open.

PROOF: $f((x, +\infty)) = (f(x), +\infty)$.

$\langle 2\rangle 2$. For all $x \in X$ we have $f((-\infty, x))$ is open.

PROOF: $f((-\infty, x)) = (-\infty, f(x))$.

□

11.15 The nth Root Function

Proposition 11.15.1. *For all $n \geq 1$, the function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined by $f(x) = x^n$ is a homemorphism.*

PROOF:
$\langle 1 \rangle 1$. f is strictly monotone.
  $\langle 2 \rangle 1$. LET: $x, y \in \mathbb{R}$ with $0 \leq x < y$
  $\langle 2 \rangle 2$. $x^n < y^n$

$$\begin{aligned} y^n - x^n &= (y-x)(y^{n-1} + y^{n-2}x + y^{n-3}x^2 + \cdots + x^{n-1}) \\ &> 0 \end{aligned}$$

$\langle 1 \rangle 2$. f is surjective.
  $\langle 2 \rangle 1$. LET: $y \in \mathbb{R}_{\geq 0}$
  $\langle 2 \rangle 2$. PICK $x \in \mathbb{R}$ such that $y \leq x^n$
    PROOF: If $y \leq 1$ take $x = 1$, otherwise take $x = y$.
  $\langle 2 \rangle 3$. There exists $x' \in [0, x]$ such that $(x')^n = y$
    PROOF: By the Intermediate Value Theorem.
$\langle 1 \rangle 3$. Q.E.D.
  PROOF: Proposition 11.14.8.
$\square$

Definition 11.15.2. For $n \geq 1$, the *nth root function* is the function $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ that is the inverse of $\lambda x.x^n$.

11.16 The Product Topology

Definition 11.16.1 (Product Topology)**.** Let $\{A_i\}_{i \in I}$ be a family of topological spaces. The *product topology* on $\prod_{i \in I} A_i$ is the topology generated by the subbasis consisting of the sets of the form ${\pi_i}^{-1}(U)$ where $i \in I$ and U is open in A_i.

Proposition 11.16.2. *The product topology on $\prod_{i \in I} A_i$ is generated by the basis consisting of all sets of the form $\prod_{i \in I} U_i$ where $\{U_i\}_{i \in I}$ is a family such that each U_i is an open set in A_i and $U_i = A_i$ for all but finitely many i.*

PROOF: From Proposition 11.7.10. $\square$

Proposition 11.16.3. *If A_i is closed in X_i for all $i \in I$ then $\prod_{i \in I} A_i$ is closed in $\prod_{i \in I} X_i$.*

PROOF:

$$(\prod_{i \in I} X_i) \setminus (\prod_{i \in I} A_i) = \bigcup_{j \in I} \left(\prod_{i \in I} X_i \setminus {\pi_j}^{-1}(A_j) \right) \square$$

Proposition 11.16.4. *Let $\{A_i\}_{i \in I}$ be a family of topological spaces. For $i \in I$, let $\mathcal{B}_i$ be a basis for the topology on A_i. Then $\mathcal{B} = \{\prod_{i \in I} B_i \mid \forall i \in I.B_i \in \mathcal{B}_i, B_i = A_i$ for all but finitely many $i\}$ is a basis for the box topology on $\prod_{i \in I} A_i$.*

PROOF:
⟨1⟩1. Every set in $\mathcal{B}$ is open.
⟨1⟩2. For every point $a \in \prod_{i \in I} A_i$ and every open set U with $a \in U$, there exists $B \in \mathcal{B}$ such that $a \in B \subseteq U$.
⟨2⟩1. LET: U be open and $a \in U$
⟨2⟩2. PICK a family $\{U_i\}_{i \in I}$ such that each U_i is open in A_i, such that $U_i = A_i$ except for $i = i_1, \ldots, i_n$, and such that $a \in \prod_{i \in I} U_i \subseteq U$.
⟨2⟩3. For $j = 1, \ldots, n$, PICK $B_{i_j} \in \mathcal{B}_{i_j}$ such that $a_{i_j} \in B_{i_j} \subseteq U_{i_j}$
⟨2⟩4. LET: $B = \prod_{i \in I} B_i$ where $B_i = A_i$ for $i \neq i_1, \ldots, i_n$
⟨2⟩5. $B \in \mathcal{B}$
⟨2⟩6. $a \in B \subseteq U$
⟨1⟩3. Q.E.D.
PROOF: Lemma 11.7.3.
□

Proposition 11.16.5. *Let $\{A_i\}_{i \in I}$ be a family of topological spaces. Then the projections $\pi_i : \prod_{i \in I} A_i \to A_i$ are open maps.*

PROOF: From Lemma 11.11.2. □

Example 11.16.6. The projections are not always closed maps. For example, $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ maps the closed set $\{(x, 1/x) \mid x > 0\}$ to $(0, +\infty)$.

Proposition 11.16.7. *Let $\{X_i\}_{i \in I}$ be a family of sets. For $i \in I$, let $\mathcal{T}_i$ and $\mathcal{U}_i$ be topologies on X_i. Let $\mathcal{P}$ be the product topology on $\prod_{i \in I} X_i$ generated by the topologies $\mathcal{T}_i$, and $\mathcal{Q}$ the product topology on the same set generated by the topologies $\mathcal{U}_i$. Then $\mathcal{P} \subseteq \mathcal{Q}$ if and only if $\mathcal{T}_i \subseteq \mathcal{U}_i$ for all i.*

PROOF:
⟨1⟩1. If $\mathcal{T}_i \subseteq \mathcal{U}_i$ for all i then $\mathcal{P} \subseteq \mathcal{Q}$
PROOF: By Corollary 11.7.2.1.
⟨1⟩2. If $\mathcal{P} \subseteq \mathcal{Q}$ then $\mathcal{T}_i \subseteq \mathcal{U}_i$ for all i
⟨2⟩1. ASSUME: $\mathcal{P} \subseteq \mathcal{Q}$
⟨2⟩2. LET: $i \in I$
⟨2⟩3. LET: $U \in \mathcal{T}_i$
⟨2⟩4. LET: $U_i = U$ and $U_j = X_j$ for $j \neq i$
⟨2⟩5. $\prod_{i \in I} U_i \in \mathcal{P}$
⟨2⟩6. $\prod_{i \in I} U_i \in \mathcal{Q}$
⟨2⟩7. $U \in \mathcal{U}_i$
PROOF: From Proposition 11.16.5.
□

Proposition 11.16.8 (AC)**.** *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let $A_i \subseteq X_i$ for all $i \in I$. Then*

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i} \ .$$

PROOF:

⟨1⟩1. $\overline{\prod_{i \in I} A_i} \subseteq \prod_{i \in I} \overline{A_i}$

⟨2⟩1. For all $i \in I$ we have $A_i \subseteq \overline{A_i}$

PROOF: Lemma 11.4.3.

⟨2⟩2. $\prod_{i \in I} A_i \subseteq \prod_{i \in I} \overline{A_i}$

⟨2⟩3. Q.E.D.

PROOF: Since $\prod_{i \in I} A_i$ is closed by Proposition 11.16.3.

⟨1⟩2. $\prod_{i \in I} \overline{A_i} \subseteq \overline{\prod_{i \in I} A_i}$

⟨2⟩1. LET: $x \in \prod_{i \in I} \overline{A_i}$

⟨2⟩2. LET: U be a neighbourhood of x

⟨2⟩3. PICK V_i open in X_i such that $x \in \prod_{i \in I} V_i \subseteq U$ with $V_i = X_i$ except for $i = i_1, \ldots, i_n$

⟨2⟩4. For $i \in I$, pick $a_i \in V_i \cap A_i$

PROOF: By Theorem 11.4.6 and ⟨2⟩1 using the Axiom of Choice.

⟨2⟩5. U intersects $\prod_{i \in I} A_i$

⟨2⟩6. Q.E.D.

PROOF: $a \in U \cap \prod_{i \in I} A_i$

□

Example 11.16.9. The closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ is $\mathbb{R}^\omega$

PROOF:

⟨1⟩1. LET: $a \in \mathbb{R}^\omega$

⟨1⟩2. LET: U be any neighbourhoods of a.

⟨1⟩3. PICK U_n open in $\mathbb{R}$ for all n such that $a \in \prod_{n \geq 0} U_n \subseteq U$ and $U_n = \mathbb{R}$ for all n except n_1, …, n_k

⟨1⟩4. LET: $b_n = a_n$ for $n = n_1, \ldots, n_k$ and $b_n = 0$ for all other n

⟨1⟩5. $b \in \mathbb{R}^\infty \cap U$

⟨1⟩6. Q.E.D.

PROOF: From Theorem 11.4.6.

□

Proposition 11.16.10. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let $(a_\alpha)_{\alpha \in J}$ be a net in $\prod_{i \in I} X_i$ and $l \in \prod_{i \in I} X_i$. Then $a_\alpha \to l$ as $\alpha \to \infty$ if and only if, for all $i \in I$, we have $\pi_i(a_\alpha) \to \pi_i(l)$ as $\alpha \to \infty$.*

PROOF:

⟨1⟩1. If $a_\alpha \to l$ as $\alpha \to \infty$ then, for all $i \in I$, we have $\pi_i(a_\alpha) \to \pi_i(l)$ as $n \to \infty$

PROOF: Proposition 11.12.18.

⟨1⟩2. If, for all $i \in I$, we have $\pi_i(a_\alpha) \to \pi_i(l)$ as $\alpha \to \infty$, then $a_\alpha \to l$ as $\alpha \to \infty$

⟨2⟩1. ASSUME: For all $i \in I$, we have $\pi_i(a_\alpha) \to \pi_i(l)$ as $\alpha \to \infty$

⟨2⟩2. LET: V be a neighbourhood of l

⟨2⟩3. PICK open sets U_i in X_i such that $l \in \prod_{i \in I} U_i \subseteq V$ and $U_i = X_i$ for all i except $i = i_1, \ldots, i_k$

⟨2⟩4. For $j = 1, \ldots, k$, PICK α_j such that, for all $\beta \geq \alpha_j$, we have $\pi_{i_j}(a_\beta) \in U_{i_j}$

⟨2⟩5. PICK $\alpha \in J$ such that $\alpha_1, \ldots, \alpha_k \leq \alpha$

⟨2⟩6. For all $\beta \geq \alpha$ we have $a_\beta \in V$

□

Theorem 11.16.11. *Let A be a topological space and $\{X_i\}_{i \in I}$ be a family of topological spaces. Let $f : A \to \prod_{i \in I} X_i$ be a function. If $\pi_i \circ f$ is continuous for all $i \in I$ then f is continuous.*

PROOF:
$\langle 1 \rangle 1$. LET: $i \in I$ and U be open in X_i
$\langle 1 \rangle 2$. $f^{-1}({\pi_i}^{-1}(U))$ is open in A
$\langle 1 \rangle 3$. Q.E.D.
PROOF: Proposition 11.12.3.
□

11.16.1 Continuous in Each Variable Separately

Definition 11.16.12 (Continuous in Each Variable Separately)**.** Let $F : X \times Y \to Z$. Then F is *continuous in each variable separately* if and only if:

- for every $a \in X$ the function $\lambda y \in Y.F(a, y)$ is continuous;
- for every $b \in Y$ the function $\lambda x \in X.F(x, b)$ is continuous.

Proposition 11.16.13. *Let $F : X \times Y \to Z$. If F is continuous then F is continuous in each variable separately.*

PROOF: For $a \in X$, the function $\lambda y \in Y.F(a, y)$ is $F \circ i$ where $i : Y \to X \times Y$ maps y to (a, y). We have i is continuous by Proposition 11.13.6, hence $F \circ i$ is continuous by Theorem 11.12.9.
Similarly for $\lambda x \in X.F(x, b)$ for $b \in Y$. □

Example 11.16.14. Define $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$F(x, y) = \begin{cases} xy/(x^2 + y^2) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Then F is continuous in each variable separately but not continuous.

Proposition 11.16.15. *Let $f : A \to C$ and $g : B \to D$ be open maps. Then $f \times g : A \times B \to C \times D$ is an open map.*

PROOF: Given U open in A and V open in B. Then $(f \times g)(U \times V) = f(U) \times g(V)$ is open in $C \times D$. The result follows from Proposition 11.11.3. □

Definition 11.16.16 (Sorgenfrey Plane)**.** The *Sorgenfrey plane* is $\mathbb{R}_l^2$.

11.17 The Subspace Topology

Definition 11.17.1 (Subspace Topology)**.** Let X be a topological space and $Y \subseteq X$. The *subspace topology* on Y is $\mathcal{T} = \{U \cap Y \mid U \text{ is open in } X\}$.

We prove this is a topology.

PROOF:
⟨1⟩1. $Y \in \mathcal{T}$
PROOF: Since $Y = X \cap Y$
⟨1⟩2. For all $\mathcal{U} \subseteq \mathcal{T}$, we have $\bigcup \mathcal{U} \in \mathcal{T}$
⟨2⟩1. LET: $\mathcal{U} \subseteq \mathcal{T}$
⟨2⟩2. LET: $\mathcal{V} = \{V \text{ open in } X \mid V \cap Y \in \mathcal{U}\}$
⟨2⟩3. $\bigcup \mathcal{U} = (\bigcup \mathcal{V}) \cap Y$
⟨1⟩3. For all $U, V \in \mathcal{T}$, we have $U \cap V \in \mathcal{T}$
⟨2⟩1. LET: $U, V \in \mathcal{T}$
⟨2⟩2. PICK U', V' open in X such that $U = U' \cap Y$ and $V = V' \cap Y$
⟨2⟩3. $(U \cap V) = (U' \cap V') \cap Y$
□

Theorem 11.17.2. *Let X be a topological space and Y a subspace of X. Let $A \subseteq Y$. Then A is closed in Y if and only if there exists a closed set C in X such that $A = C \cap Y$.*

PROOF: We have

$$\begin{aligned} & A \text{ is closed in } Y \\ \Leftrightarrow & Y \setminus A \text{ is open in } Y \\ \Leftrightarrow & \exists U \text{ open in } X. Y \setminus A = Y \cap U \\ \Leftrightarrow & \exists C \text{ closed in } X. Y \setminus A = Y \cap (X \setminus U) \\ \Leftrightarrow & \exists C \text{ closed in } X. A = Y \cap U \end{aligned}$$ □

Theorem 11.17.3. *Let Y be a subspace of X. Let $A \subseteq Y$. Let $\overline{A}$ be the closure of A in X. Then the closure of A in Y is $\overline{A} \cap Y$.*

PROOF: The closure of A in Y is

$$\begin{aligned} & \bigcap \{C \text{ closed in } Y \mid A \subseteq C\} \\ = & \bigcap \{D \cap Y \mid D \text{ closed in } X, A \subseteq D \cap Y\} && \text{(Theorem 11.17.2)} \\ = & \bigcap \{D \mid D \text{ closed in } X, A \subseteq D\} \cap Y \\ = & \overline{A} \cap Y \end{aligned}$$ □

Lemma 11.17.4. *Let X be a topological space and $Y \subseteq X$. Let $\mathcal{B}$ be a basis for the topology on X. Then $\mathcal{B}' = \{B \cap Y \mid B \in \mathcal{B}\}$ is a basis for the subspace topology on Y.*

PROOF:
⟨1⟩1. Every element in $\mathcal{B}'$ is open in Y
⟨1⟩2. For every open set U in Y and point $y \in U$, there exists $B' \in \mathcal{B}'$ such that $y \in B' \subseteq U$
⟨2⟩1. LET: U be open in Y and $y \in U$
⟨2⟩2. PICK V open in X such that $U = V \cap Y$
⟨2⟩3. PICK $B \in \mathcal{B}$ such that $y \in B \subseteq V$

⟨2⟩4. LET: $B' = B \cap Y$
⟨2⟩5. $B' \in \mathcal{B}'$
⟨2⟩6. $y \in B' \subseteq U$
⟨1⟩3. Q.E.D.
PROOF: By Lemma 11.7.3.
□

Lemma 11.17.5. *Let X be a topological space and $Y \subseteq X$. Let $\mathcal{S}$ be a basis for the topology on X. Then $\mathcal{S}' = \{S \cap Y \mid S \in \mathcal{S}\}$ is a subbasis for the subspace topology on Y.*

PROOF: The set $\{B \cap Y \mid B$ is a finite intersection of elements of $\mathcal{S}\}$ is a basis for the subspace topology by Lemma 11.17.4, and this is the set of all finite intersections of elements of $\mathcal{S}'$. □

Lemma 11.17.6. *Let Y be a subspace of X. If U is open in Y and Y is open in X then U is open in X.*

PROOF:
⟨1⟩1. PICK V open in X such that $U = V \cap Y$
⟨1⟩2. U is open in X
PROOF: Since it is the intersection of two open sets V and Y.
□

Theorem 11.17.7. *Let Y be a subspace of X and $A \subseteq Y$. If A is closed in Y and Y is closed in X then A is closed in X.*

PROOF: Pick a closed set C in X such that $A = C \cap Y$ (Theorem 11.17.2). Then A is the intersection of two sets closed in X, hence A is closed in X (Lemma 11.2.4). □

Theorem 11.17.8. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let A_i be a subspace of X_i for all $i \in I$. Then the product topology on $\prod_{i \in I} A_i$ is the same as the topology it inherits as a subspace of $\prod_{i \in I} X_i$.*

PROOF: The product topology is generated by the subbasis

$$\begin{aligned} &\{{\pi_i}^{-1}(U) \mid i \in I, U \text{ open in } A_i\} \\ =&\{{\pi_i}^{-1}(V) \cap A_i \mid i \in I, V \text{ open in } X_i\} \\ =&\{{\pi_i}^{-1}(V) \mid i \in I, V \text{ open in } X_i\} \cap \prod_{i \in I} A_i \end{aligned}$$

and this is a subbasis for the subspace topology by Lemma 11.17.5. □

Theorem 11.17.9. *Let X be an ordered set in the order topology. Let $Y \subseteq X$ be an interval. Then the order topology on Y is the same as the subspace topology on Y.*

PROOF:
⟨1⟩1. The order topology is finer than the subspace topology.

⟨2⟩1. For every open ray R in X, the set $R \cap Y$ is open in the order topology.
⟨3⟩1. For all $a \in X$, we have $(-\infty, a) \cap Y$ is open in the order topology.
⟨4⟩1. Case: For all $y \in Y$ we have $y < a$
Proof: In this case $(-\infty, a) \cap Y = Y$.
⟨4⟩2. Case: For all $y \in Y$ we have $a < y$
Proof: In this case $(-\infty, a) \cap Y = \emptyset$.
⟨4⟩3. Case: There exists $y \in Y$ such that $y \leq a$ and $y \in Y$ such that $a \leq y$
⟨5⟩1. $a \in Y$
Proof: Because Y is an interval.
⟨5⟩2. $(-\infty, a) \cap Y = \{y \in Y \mid y < a\}$
⟨3⟩2. For all $a \in X$, we have $(a, +\infty) \cap Y$ is open in the order topology.
Proof: Similar.
⟨2⟩2. Q.E.D.
Proof: By Lemmas 11.14.2 and 11.17.5 and Proposition 11.7.11.
⟨1⟩2. The subspace topology is finer than the order topology.
⟨2⟩1. Every open ray in Y is open in the subspace topology.
Proof: For any $a \in Y$ we have $(-\infty, a)_Y = (-\infty, a)_X \cap Y$ and $(a, +\infty)_Y = (a, +\infty)_X \cap Y$.
⟨2⟩2. Q.E.D.
Proof: By Lemma 11.14.2 and Proposition 11.7.11
□

This example shows that we cannot remove the hypothesis that Y is an interval:

Example 11.17.10. The order topology on I_o^2 is different from the subspace topology as a subspace of $\mathbb{R}^2$ under the dictionary order topology.

Proof: The set $\{1/2\} \times (1/2, 1)$ is open in the subspace topology but not in the order topology. □

Proposition 11.17.11. *Let X be a topological space, Y a subspace of X, and Z a subspace of Y. Then the subspace topology on Z inherited from X is the same as the subspace topology on Z inherited from Y.*

Proof: The subspace topology inherited from Y is

$$\begin{aligned} &\{V \cap Z \mid V \text{ open in } Y\} \\ =&\{U \cap Y \cap Z \mid U \text{ open in } X\} \\ =&\{U \cap Z \mid U \text{ open in } X\} \end{aligned}$$

which is the subspace topology inherited from X. □

Definition 11.17.12 (Unit Circle)**.** The *unit circle* S^1 is $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ as a subspace of $\mathbb{R}^2$.

Definition 11.17.13 (Unit 2-sphere)**.** The *unit 2-sphere* is $S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$ as a subspace of $\mathbb{R}^3$.

Proposition 11.17.14. *Let $f : X \to Y$ be an open map and $A \subseteq X$ be open. Then the restriction $f \restriction A : A \to f(A)$ is an open map.*

PROOF:
$\langle 1 \rangle 1$. LET: U be open in A
$\langle 1 \rangle 2$. U is open in X
 PROOF: Lemma 11.17.6.
$\langle 1 \rangle 3$. $f(U)$ is open in Y
$\langle 1 \rangle 4$. $f(U)$ is open in $f(A)$
 PROOF: Since $f(U) = f(U) \cap f(A)$.
□

Example 11.17.15. This example shows that we cannot remove the hypothesis that A is open.

Let $A = \{(x, y) \in \mathbb{R}^2 \mid (x > 0 \text{ and } y = 1/x) \text{ or } x = y = 0\}$. Then $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ is an open map, but $\pi_1 \restriction A : A \to [0, +\infty)$ is not, because it maps the set $\{0, 0\}$ which is open in A to $\{0\}$ which is not open in $[0, +\infty)$.

Proposition 11.17.16. *Let Y be a subspace of X. Let $A \subseteq Y$ and $l \in Y$. Then l is a limit point of A in Y if and only if l is a limit point of A in X.*

PROOF: Both are equivalent to the condition that any neighbourhood of l in X intersects A in a point other than l. □

11.18 The Box Topology

Definition 11.18.1 (Box Topology)**.** Let $\{A_i\}_{i \in I}$ be a family of topological spaces. The *box topology* on $\prod_{i \in I} A_i$ is the topology generated by the set of all sets of the form $\prod_{i \in I} U_i$ where $\{U_i\}_{i \in I}$ is a family such that each U_i is open in A_i.

This is a basis since it covers $\prod_{i \in I} A_i$ and is closed under intersection.

Proposition 11.18.2. *The box topology is finer than the product topology.*

PROOF: From Proposition 11.16.2. □

Corollary 11.18.2.1. *If A_i is closed in X_i for all $i \in I$ then $\prod_{i \in I} A_i$ is closed in $\prod_{i \in I} X_i$ under the box topology.*

PROOF: From Proposition 11.16.3.

Proposition 11.18.3 (AC)**.** *Let $\{A_i\}_{i \in I}$ be a family of topological spaces. For $i \in I$, let $\mathcal{B}_i$ be a basis for the topology on A_i. Then $\mathcal{B} = \{\prod_{i \in I} B_i \mid \forall i \in I. B_i \in \mathcal{B}_i\}$ is a basis for the box topology on $\prod_{i \in I} A_i$.*

PROOF:
$\langle 1 \rangle 1$. Every set of the form $\prod_{i \in I} B_i$ is open.
$\langle 1 \rangle 2$. For every point $a \in \prod_{i \in I} A_i$ and every open set U with $a \in U$, there exists $B \in \mathcal{B}$ such that $a \in B \subseteq U$.

⟨2⟩1. Let: U be open and $a \in U$

⟨2⟩2. Pick a family $\{U_i\}_{i \in I}$ such that each U_i is open in A_i and $a \in \prod_{i \in I} U_i \subseteq U$.

⟨2⟩3. For $i \in I$, Pick $B_i \in \mathcal{B}_i$ such that $a_i \in B_i \subseteq U_i$

Proof: Using the Axiom of Choice.

⟨2⟩4. $a \in \prod_{i \in I} B_i \subseteq U$

⟨1⟩3. Q.E.D.

Proof: Lemma 11.7.3.

□

Theorem 11.18.4. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let A_i be a subspace of X_i for all $i \in I$. Give $\prod_{i \in I} X_i$ the box topology. Then the box topology on $\prod_{i \in I} A_i$ is the same as the topology it inherits as a subspace of $\prod_{i \in I} X_i$.*

Proof: The box topology is generated by the basis

$$\begin{aligned} & \{\prod_{i \in I} U_i \mid \forall i \in I, U_i \text{ open in } A_i\} \\ = & \{\prod_{i \in I} (V_i \cap A_i) \mid \forall i \in I, V_i \text{ open in } X_i\} \\ = & \{\prod_{i \in I} V_i \mid \forall i \in I, V_i \text{ open in } X_i\} \cap \prod_{i \in I} A_i \end{aligned}$$

and this is a basis for the subspace topology by Lemma 11.17.4. □

Proposition 11.18.5 (AC)**.** *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Give $\prod_{i \in I} X_i$ the box topology. Let $A_i \subseteq X_i$ for all $i \in I$. Then*

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i} \ .$$

Proof:

⟨1⟩1. $\overline{\prod_{i \in I} A_i} \subseteq \prod_{i \in I} \overline{A_i}$

⟨2⟩1. For all $i \in I$ we have $A_i \subseteq \overline{A_i}$

Proof: Lemma 11.4.3.

⟨2⟩2. $\prod_{i \in I} A_i \subseteq \prod_{i \in I} \overline{A_i}$

⟨2⟩3. Q.E.D.

Proof: Since $\prod_{i \in I} A_i$ is closed by Corollary 11.18.2.1.

⟨1⟩2. $\prod_{i \in I} \overline{A_i} \subseteq \overline{\prod_{i \in I} A_i}$

⟨2⟩1. Let: $x \in \prod_{i \in I} \overline{A_i}$

⟨2⟩2. Let: U be a neighbourhood of x

⟨2⟩3. Pick V_i open in X_i such that $x \in \prod_{i \in I} V_i \subseteq U$

⟨2⟩4. For $i \in I$, pick $a_i \in V_i \cap A_i$

Proof: By Theorem 11.4.6 and ⟨2⟩1 using the Axiom of Choice.

⟨2⟩5. U intersects $\prod_{i \in I} A_i$

⟨2⟩6. Q.E.D.

Proof: $a \in U \cap \prod_{i \in I} A_i$.

∎

The following example shows that Theorem 11.16.11 fails in the box topology.

Example 11.18.6. Define $f : \mathbb{R} \to \mathbb{R}^\omega$ by $f(t) = (t, t, \ldots)$. Then $\pi_n \circ f = \mathrm{id}_\mathbb{R}$ is continuous for all n. But f is not continuous when $\mathbb{R}^\omega$ is given the box topology because the inverse image of

$$(-1, 1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times \cdots$$

is $\{0\}$ which is not open.

The following example shows that Proposition 11.16.10 fails in the box topology.

Example 11.18.7. Give $\mathbb{R}^\omega$ the box topology. Let $a_n = (1/n, 1/n, \ldots)$ for $n \geq 1$ and $l = (0, 0, \ldots)$. Then $\pi_i(a_n) \to \pi_i(l)$ as $n \to \infty$ for all i, but $a_n \not\to l$ as $n \to \infty$ since the open set

$$(-1, 1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times \cdots$$

contains l but does not contain any a_n.

Example 11.18.8. The set $\mathbb{R}^\infty$ is closed in $\mathbb{R}^\omega$ under the box topology. For let (a_n) be any sequence not in $\mathbb{R}^\infty$. Let U_n be an open interval around a_n that does not contain 0 if $a_n \neq 0$, and $U_n = \mathbb{R}$ if $a_n = 0$. Then $\prod_{n \geq 0} U_n$ is a neighbourhood of (a_n) that does not intersect $\mathbb{R}^\infty$.

11.19 T_1 Spaces

Definition 11.19.1 (T_1 Space)**.** A topological space is T_1 if and only if every singleton is closed.

Lemma 11.19.2. *A space is T_1 if and only if every finite set is closed.*

PROOF: From Lemma 11.2.5. ∎

Theorem 11.19.3. *In a T_1 space, a point a is a limit point of a set A if and only if every neighbourhood of a contains infinitely many points of A.*

PROOF:
⟨1⟩1. If a is a limit point of A then every neighbourhood of a contains infinitely many points of A.
  ⟨2⟩1. ASSUME: a is a limit point of A.
  ⟨2⟩2. LET: U be a neighbourhood of a.
  ⟨2⟩3. ASSUME: for a contradiction U contains only finitely many points of A.
  ⟨2⟩4. $(U \cap A) \setminus \{a\}$ is closed.
    PROOF: By the T_1 axiom.
  ⟨2⟩5. $(U \setminus A) \cup \{a\}$ is open.
    PROOF: It is $U \setminus ((U \cap A) \setminus \{a\})$.

⟨2⟩6. $(U \setminus A) \cup \{a\}$ intersects A in a point other than a.
PROOF: From ⟨2⟩1.
⟨2⟩7. Q.E.D.
□
⟨1⟩2. If every neighbourhood of a contains infinitely many points of A then a is a limit point of A.
PROOF: Immediate from definitions.
□

(To see this does not hold in every space, see Proposition 11.6.4.)

Proposition 11.19.4. *A space is T_1 if and only if, for any two distinct points x and y, there exist neighbourhoods U of x and V of y such that $x \notin V$ and $y \notin U$.*

PROOF:
⟨1⟩1. LET: X be a topological space.
⟨1⟩2. If X is T_1 then, for any two distinct points x and y, there exist neighbourhoods U of x and V of y such that $x \notin V$ and $y \notin U$.
PROOF: This holds because $\{x\}$ and $\{y\}$ are closed.
⟨1⟩3. Suppose, for any two distinct points x and y, there exist neighbourhoods U of x and V of y such that $x \notin V$ and $y \notin U$. Then X is T_1.
⟨2⟩1. ASSUME: For any two distinct points x and y, there exist neighbourhoods U of x and V of y such that $x \notin V$ and $y \notin U$.
⟨2⟩2. LET: $a \in X$
⟨2⟩3. $\{a\}$ is closed.
PROOF: For all $b \neq a$ there exists a neighbourhood U of b such that $U \subseteq X \setminus \{a\}$.
□

Proposition 11.19.5. *A subspace of a T_1 space is T_1.*

PROOF: From Proposition 11.17.7.

11.20 Hausdorff Spaces

Definition 11.20.1 (Hausdorff Space)**.** A topological space is *Hausdorff* if and only if, for any points x, y with $x \neq y$, there exist disjoint open sets U and V such that $x \in U$ and $y \in V$.

Theorem 11.20.2. *Every Hausdorff space is T_1.*

PROOF:
⟨1⟩1. LET: X be a Hausdorff space.
⟨1⟩2. LET: $b \in X$
PROVE: $\overline{\{b\}} = \{b\}$
⟨1⟩3. ASSUME: $a \in \overline{\{b\}}$ and $a \neq b$
⟨1⟩4. PICK disjoint neighbourhoods U of a and V of b.

$\langle 1\rangle 5.$ U intersects $\{b\}$
PROOF: Theorem 11.4.6.
$\langle 1\rangle 6.$ $b \in U$
$\langle 1\rangle 7.$ Q.E.D.
PROOF: This contradicts the fact that U and V are disjoint ($\langle 1\rangle 4$).
□

Proposition 11.20.3. *An infinite set under the finite complement topology is T_1 but not Hausdorff.*

PROOF:
$\langle 1\rangle 1.$ LET: X be an infinite set under the finite complement topology.
$\langle 1\rangle 2.$ Every singleton is closed.
PROOF: By definition.
$\langle 1\rangle 3.$ PICK$a, b \in X$ with $a \neq b$
$\langle 1\rangle 4.$ There are no disjoint neighbourhoods U of a and V of b.
$\langle 2\rangle 1.$ LET: U be a neighbourhood of a and V a neighbourhood of b.
$\langle 2\rangle 2.$ $X \setminus U$ and $X \setminus V$ are finite.
$\langle 2\rangle 3.$ PICK $c \in X$ that is not in $X \setminus U$ or $X \setminus V$.
$\langle 2\rangle 4.$ $c \in U \cap V$
□

Proposition 11.20.4. *The product of a family of Hausdorff spaces is Hausdorff.*

PROOF:
$\langle 1\rangle 1.$ LET: $\{X_i\}_{i \in I}$ be a family of Hausdorff spaces.
$\langle 1\rangle 2.$ LET: $a, b \in \prod_{i \in I} X_i$ with $a \neq b$
$\langle 1\rangle 3.$ PICK $i \in I$ such that $a_i \neq b_i$
$\langle 1\rangle 4.$ PICK U, V disjoint open sets in X_i with $a_i \in U$ and $b_i \in V$
$\langle 1\rangle 5.$ ${\pi_i}^{-1}(U)$ and ${\pi_i}^{-1}(V)$ are disjoint open sets in $\prod_{i \in I} X_i$ with $a \in {\pi_i}^{-1}(U)$ and $b \in {\pi_i}^{-1}(V)$
□

Theorem 11.20.5. *Every linearly ordered set under the order topology is Hausdorff.*

PROOF:
$\langle 1\rangle 1.$ LET: X be a linearly ordered set under the order topology.
$\langle 1\rangle 2.$ LET: $a, b \in X$ with $a \neq b$
$\langle 1\rangle 3.$ ASSUME: w.l.o.g. $a < b$
$\langle 1\rangle 4.$ CASE: There exists c such that $a < c < b$
PROOF: The sets $(-\infty, c)$ and $(c, +\infty)$ are disjoint neighbourhoods of a and b respectively.
$\langle 1\rangle 5.$ CASE: There is no c such that $a < c < b$
PROOF: The sets $(-\infty, b)$ and $(a, +\infty)$ are disjoint neighbourhoods of a and b respectively.
□

Theorem 11.20.6. *A subspace of a Hausdorff space is Hausdorff.*

PROOF:
⟨1⟩1. LET: X be a Hausdorff space and Y a subspace of X.
⟨1⟩2. LET: $x, y \in Y$ with $x \neq y$
⟨1⟩3. PICK disjoint neighbourhoods U of x and V of y in X.
⟨1⟩4. $U \cap Y$ and $V \cap Y$ are disjoint neighbourhoods of x and y respectively in Y.
□

Proposition 11.20.7. *A space X is Hausdorff if and only if the diagonal $\Delta = \{(x, x) \mid x \in X\}$ is closed in X^2.*

PROOF:

$$\begin{aligned}&X \text{ is Hausdorff}\\ \Leftrightarrow&\forall x, y \in X. x \neq y \Rightarrow \exists V, W \text{ open}. x \in V \wedge y \in W \wedge V \cap W = \emptyset\\ \Leftrightarrow&\forall (x, y) \in X^2 \setminus \Delta. \exists V, W \text{ open}. (x, y) \subseteq V \times W \subseteq X^2 \setminus \Delta\\ \Leftrightarrow&\Delta \text{ is closed}\end{aligned}$$

□

Theorem 11.20.8. *In a Hausdorff space, a net has at most one limit.*

PROOF:
⟨1⟩1. LET: X be a Hausdorff space.
⟨1⟩2. ASSUME: for a contradiction $(a_\alpha)_{\alpha \in J}$ is a net with limits l and m.
⟨1⟩3. PICK disjoint neighbourhoods U of l and V of m
PROOF: By the Hausdorff axiom.
⟨1⟩4. PICK α and β such that $a_\gamma \in U$ for $\gamma \geq \alpha$ and $a_\gamma \in V$ for $\gamma \geq \beta$
⟨1⟩5. PICK $\gamma \in J$ with $\gamma \geq \alpha$ and $\gamma \geq \beta$
⟨1⟩6. $a_\gamma \in U \cap V$
⟨1⟩7. Q.E.D.
PROOF: This contradicts the fact that U and V are disjoint (⟨1⟩3).
□

To see this is not always true in spaces that are T_1 but not Hausdorff:

Proposition 11.20.9. *Let X be an infinite set under the finite complement topology. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence with all points distinct. Then for every $l \in X$ we have $a_n \to l$ as $n \to \infty$.*

PROOF: Let U be any neighbourhood of l. Since $X \setminus U$ is finite, there must exist N such that, for all $n \geq N$, we have $a_n \in U$. □

Proposition 11.20.10. *Let X be a topological space. Let Y a Hausdorff space. Let $A \subseteq X$. Let $f, g : \overline{A} \to Y$ be continuous. If f and g agree on A then $f = g$.*

PROOF:
⟨1⟩1. LET: $x \in \overline{A}$
⟨1⟩2. ASSUME: $f(x) \neq g(x)$
⟨1⟩3. PICK disjoint neighbourhoods V of $f(x)$ and W of $g(x)$.

⟨1⟩4. Pick $y \in f^{-1}(V) \cap g^{-1}(W) \cap A$

Proof: Since $f^{-1}(V) \cap g^{-1}(W)$ is a neighbourhood of x and hence intersects A.

⟨1⟩5. $f(y) = g(y) \in V \cap W$

⟨1⟩6. Q.E.D.

Proof: This contradicts the fact that V and W are disjoint (⟨1⟩3).

□

Proposition 11.20.11. *Let $\{X_i\}_{i \in I}$ be a family of Hausdorff spaces. Then $\prod_{i \in I} X_i$ under the box topology is Hausdorff.*

Proof:

⟨1⟩1. Let: $\{X_i\}_{i \in I}$ be a family of Hausdorff spaces.

⟨1⟩2. Let: $a, b \in \prod_{i \in I} X_i$ with $a \neq b$

⟨1⟩3. Pick $i \in I$ such that $a_i \neq b_i$

⟨1⟩4. Pick U, V disjoint open sets in X_i with $a_i \in U$ and $b_i \in V$

⟨1⟩5. ${\pi_i}^{-1}(U)$ and ${\pi_i}^{-1}(V)$ are disjoint open sets in $\prod_{i \in I} X_i$ with $a \in {\pi_i}^{-1}(U)$ and $b \in {\pi_i}^{-1}(V)$

□

Proposition 11.20.12. *Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on the same set X with $\mathcal{T} \subseteq \mathcal{T}'$ If $\mathcal{T}$ is Haudorff then $\mathcal{T}'$ is Hausdorff.*

Proof: Immediate from definitions.

Proposition 11.20.13. *Let X be a Hausdorff space. Let $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the finite intersection property. Then $\bigcap_{D \in \mathcal{D}} \overline{D}$ contains at most one point.*

Proof:

⟨1⟩1. Let: $x, y \in \bigcap_{D \in \mathcal{D}} \overline{D}$

⟨1⟩2. Assume: for a contradiction $x \neq y$

⟨1⟩3. Pick disjoint open subsets U and V of x and y respectively.

⟨1⟩4. $U, V \in \mathcal{D}$

Proof: Proposition 11.4.9.

⟨1⟩5. Q.E.D.

Proof: This contradicts the fact that $\mathcal{D}$ satisfies the finite intersection property.

□

11.21 The First Countability Axiom

Definition 11.21.1 (First Countability Axiom)**.** A topological space X satisfies the *first countability axiom*, or is *first countable*, if and only if every point has a countable local basis.

Example 11.21.2. The space S_Ω is first countable. For any $\alpha \in S_\Omega$, the set $\{(\beta, \alpha + 1) \mid \beta < \alpha\} \cup \{[0, \alpha + 1)\}$ is a local basis at α.

Lemma 11.21.3 (Sequence Lemma (CC)). *Let X be a first countable space. Let $A \subseteq X$ and $l \in \overline{A}$. Then there exists a sequence in A that converges to l.*

PROOF:
$\langle 1 \rangle 1$. PICK a countable local basis $\{B_n \mid n \in \mathbb{Z}^+\}$ at l such that $B_1 \supseteq B_2 \supseteq \cdots$.
PROOF: Lemma 11.8.2.
$\langle 1 \rangle 2$. For all $n \geq 1$, PICK $a_n \in A \cap B_n$.
PROVE: $a_n \rightarrow l$ as $n \rightarrow \infty$
$\langle 1 \rangle 3$. LET: U be a neighbourhood of A
$\langle 1 \rangle 4$. PICK N such that $B_N \subseteq U$
$\langle 1 \rangle 5$. For $n \geq N$ we have $a_n \in U$
PROOF: $a_n \in B_n \subseteq B_N \subseteq U$
□

Example 11.21.4. The space $\overline{S_\Omega}$ is not first countable, since Ω is a limit point for S_Ω but there is no sequence of points in S_Ω that converges to Ω.

Theorem 11.21.5 (CC). *Let X be a first countable space and Y a topological space. Let $f : X \rightarrow Y$. Suppose that, for every sequence (x_n) in X and $l \in X$, if $x_n \rightarrow l$ as $n \rightarrow \infty$, then $f(x_n) \rightarrow f(l)$ as $n \rightarrow \infty$. Then f is continuous.*

PROOF:
$\langle 1 \rangle 1$. LET: $A \subseteq X$
$\langle 1 \rangle 2$. LET: $a \in A$
PROVE: $f(a) \in \overline{f(A)}$
$\langle 1 \rangle 3$. PICK a sequence (x_n) in A that converges to a.
PROOF: By the Sequence Lemma.
$\langle 1 \rangle 4$. $f(x_n) \rightarrow f(a)$
$\langle 1 \rangle 5$. $f(a) \in \overline{f(A)}$
PROOF: By Lemma 11.9.3.
$\langle 1 \rangle 6$. Q.E.D.
PROOF: By Theorem 11.12.6.
□

Example 11.21.6 (CC). The space $\mathbb{R}^\omega$ under the box product is not first countable.

PROOF: Let $\{B_n \mid n \geq 0\}$ be a countable set of neighbourhoods of $\vec{0}$. We will construct a neighbourhood of $\vec{0}$ that does not include any of these.
For $n \geq 0$, pick a neighbourhood U_n of 0 such that $U_n \subset \pi_n(B_p)$. Then $\prod_{n=0}^{\infty} U_n$ is a neighbourhood of $\vec{0}$ that does not include any B_n. □

Example 11.21.7. If J is an uncountable set then $\mathbb{R}^J$ is not first countable.

PROOF:
$\langle 1 \rangle 1$. LET: $\{B_n \mid n \geq 0\}$ be any countable set of neighbourhoods of $\vec{0}$.
$\langle 1 \rangle 2$. For $n \geq 0$, PICK a basis element $\prod_{\alpha \in J} U_{n\alpha}$ that contains $\vec{0}$ and is included in B_n.
PROOF: Using the Axiom of Countable Choice.

⟨1⟩3. For $n \geq 0$,
LET: $J_n = \{\alpha \in J \mid U_{n\alpha} \neq \mathbb{R}\}$
⟨1⟩4. PICK $\beta \in J$ such that $\beta \notin J_n$ for any n.
PROOF: Since each J_n is finite so $\bigcup_n J_n$ is countable.
⟨1⟩5. $\pi_\beta((-1,1))$ is a neighbourhood of $\vec{0}$ that does not include any B_n.
□

Example 11.21.8. The space $\mathbb{R}_l$ is first countable.

PROOF: For any real number x, the set $\{[x,q) \mid q \in \mathbb{Q}, q > x\}$ is a countable local basis at x. □

PROOF: For any $a \in \mathbb{R}$, the set $\{[a, a+1/n) \mid n \geq 1\}$ is a countable local basis.

Example 11.21.9. The ordered square is first countable.

PROOF: For any $(a,b) \in I_o^2$ with $b \neq 0,1$, the set $\{(\{a\} \times (b-1/n, b+1/n)) \cap I_o^2 \mid n \geq 1\}$ is a countable local basis.

Proposition 11.21.10. *A subspace of a first countable space is first countable.*

PROOF:
⟨1⟩1. LET: X be a first countable space.
⟨1⟩2. LET: $Y \subseteq X$
⟨1⟩3. LET: $y \in Y$
⟨1⟩4. PICK a countable local basis $\mathcal{B}$ for y in X
⟨1⟩5. $\{B \cap Y \mid B \in \mathcal{B}\}$ is a countable local basis for y in Y
□

Proposition 11.21.11 (AC)**.** *A countable product of first countable spaces is first countable.*

PROOF:
⟨1⟩1. LET: (X_n) be a sequence of first countable spaces.
⟨1⟩2. LET: $(x_n) \in \prod_n X_n$
⟨1⟩3. For all n, PICK a countable local basis $\mathcal{B}_n$ for x_n in X_n
⟨1⟩4. LET: $\mathcal{B}$ be the set of all sets of the form $\prod_n U_n$ where (U_n) is a family such that $U_n \in \mathcal{B}_n$ for finitely many n and $U_n = X_n$ for all other n
⟨1⟩5. $\mathcal{B}$ is a countable local basis for (x_n)
□

Example 11.21.12. The space S_Ω is first countable. For any $x \in S_\Omega$, the set $\{[0, x+1)\} \cup \{(y, x+1) \mid y < x\}$ is a countable local basis at x.

Example 11.21.13. The space $\overline{S_\Omega}$ is not first countable.

PROOF:
⟨1⟩1. ASSUME: for a contradiction $\mathcal{B}$ is a countable local basis at Ω.
⟨1⟩2. For $B \in \mathcal{B}$,
LET: a_B be least such that $(a_B, \Omega] \subseteq B$.
⟨1⟩3. LET: $b = \sup_{B \in \mathcal{B}} a_B$

⟨1⟩4. $b < \Omega$
⟨1⟩5. There is no $B \in \mathcal{B}$ such that $\Omega \in B \subseteq (b+1, \Omega]$
∎

Proposition 11.21.14. *The image of a first countable space under a continuous open map is first countable.*

PROOF:
⟨1⟩1. LET: X be a first countable space.
⟨1⟩2. LET: Y be a topological space.
⟨1⟩3. LET: $f : X \twoheadrightarrow Y$ be a surjective, continuous open map.
⟨1⟩4. LET: $y \in Y$
⟨1⟩5. PICK $x \in X$ such that $f(x) = y$
⟨1⟩6. PICK a countable local basis $\mathcal{B}$ at x.
PROVE: $\{f(B) \mid B \in \mathcal{B}\}$ is a local basis at y.
⟨1⟩7. LET: V be a neighbourhood of y
⟨1⟩8. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq f^{-1}(V)$
⟨1⟩9. $y \in f(B) \subseteq V$
∎

11.22 Strong Continuity

Definition 11.22.1 (Strongly Continuous)**.** Let X and Y be topological spaces and $f : X \to Y$ be a function. Then f is *strongly continuous* if and only if, for every subset $U \subseteq Y$, we have U is open in Y if and only if $f^{-1}(U)$ is open in X.

Proposition 11.22.2. *Let X and Y be topological spaces and $f : X \to Y$ be a function. Then f is* strongly continuous *if and only if, for every subset $C \subseteq Y$, we have C is closed in Y if and only if $f^{-1}(C)$ is closed in X.*

PROOF: Since $X \setminus f^{-1}(C) = f^{-1}(Y \setminus C)$. ∎

Proposition 11.22.3. *Let X, Y and Z be topological spaces. Let $f : X \to Y$ and $g : Y \to Z$. If f and g are strongly continuous then so is $g \circ f$.*

PROOF: Since $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$. ∎

Proposition 11.22.4. *Let X, Y and Z be topological spaces. Let $f : X \to Y$ and $g : Y \to Z$. If $g \circ f$ is continuous and f is strongly continuous then g is continuous.*

PROOF:
⟨1⟩1. LET: $V \subseteq Z$ be open.
⟨1⟩2. $f^{-1}(g^{-1}(V))$ is open in X.
PROOF: Since $g \circ f$ is continuous.
⟨1⟩3. $f^{-1}(V)$ is open in Y.
PROOF: Since g is strongly continuous.
∎

Proposition 11.22.5. *Let X, Y and Z be topological spaces. Let $f : X \to Y$ and $g : Y \to Z$. If $g \circ f$ is strongly continuous and f is strongly continuous then g is strongly continuous.*

Proof: For $V \subseteq Z$, we have V is open iff $f^{-1}(g^{-1}(V))$ is open iff $g^{-1}(V)$ is open.

11.23 Saturated Sets

Definition 11.23.1. Let X and Y be sets and $p : X \twoheadrightarrow Y$ a surjective function. Let $C \subseteq X$. Then C is *saturated* with respect to p if and only if, for all $x, y \in X$, if $x \in C$ and $p(x) = p(y)$ then $y \in C$.

Proposition 11.23.2. *Let X and Y be sets and $p : X \twoheadrightarrow Y$ a surjective function. Let $C \subseteq X$. Then the following are equivalent:*

1. *C is saturated with respect to p.*
2. *There exists $D \subseteq Y$ such that $C = p^{-1}(D)$*
3. *$C = p^{-1}(p(C))$.*

Proof:
⟨1⟩1. $1 \Rightarrow 3$
  ⟨2⟩1. Assume: C is saturated with respect to p.
  ⟨2⟩2. $C \subseteq p^{-1}(p(C))$
    Proof: Trivial.
  ⟨2⟩3. $p^{-1}(p(C)) \subseteq C$
    ⟨3⟩1. Let: $x \in p^{-1}(p(C))$
    ⟨3⟩2. $p(x) \in p(C)$
    ⟨3⟩3. There exists $y \in C$ such that $p(x) = p(y)$
    ⟨3⟩4. $x \in C$
      Proof: From ⟨2⟩1.
⟨1⟩2. $3 \Rightarrow 2$
  Proof: Trivial.
⟨1⟩3. $2 \Rightarrow 1$
  Proof: This follows because if $p(x) \in D$ and $p(x) = p(y)$ then $p(y) \in D$.
□

11.24 Quotient Maps

Definition 11.24.1 (Quotient Map)**.** Let X and Y be topological spaces and $p : X \to Y$. Then p is a *quotient map* if and only if p is surjective and strongly continuous.

Proposition 11.24.2. *Let X and Y be topological spaces and $p : X \twoheadrightarrow Y$ be a surjective function. Then the following are equivalent.*

1. *p is a quotient map.*

2. *p is continuous and maps saturated open sets to open sets.*

3. *p is continuous and maps saturated closed sets to closed sets.*

PROOF:
$\langle 1\rangle 1.$ $1 \Rightarrow 2$
  $\langle 2\rangle 1.$ ASSUME: p is a quotient map.
  $\langle 2\rangle 2.$ LET: U be a saturated open set in X.
  $\langle 2\rangle 3.$ $p^{-1}(p(U))$ is open in X.
    PROOF: Since $U = p^{-1}(p(U))$ be Proposition 11.23.2.
  $\langle 2\rangle 4.$ $p(U)$ is open in Y.
    PROOF: From $\langle 2\rangle 1$.
$\langle 1\rangle 2.$ $1 \Rightarrow 3$
  PROOF: Similar.
$\langle 1\rangle 3.$ $2 \Rightarrow 1$
  $\langle 2\rangle 1.$ ASSUME: p is continuous and maps saturated open sets to open sets.
  $\langle 2\rangle 2.$ LET: $U \subseteq Y$
  $\langle 2\rangle 3.$ ASSUME: $p^{-1}(U)$ is open in X
  $\langle 2\rangle 4.$ $p^{-1}(U)$ is saturated.
    PROOF: Proposition 11.23.2.
  $\langle 2\rangle 5.$ U is open in Y.
$\langle 1\rangle 4.$ $3 \Rightarrow 1$
  PROOF: Similar.
$\square$

Corollary 11.24.2.1. *Every surjective continuous open map is a quotient map.*

Corollary 11.24.2.2. *Every surjective continuous closed map is a quotient map.*

Example 11.24.3. The converses of these corollaries do not hold.

Let $A = \{(x, y) \mid x \geq 0\} \cup \{(x, y) \mid y = 0\}$. Then $\pi_1 : A \to \mathbb{R}$ is a quotient map, but not an open map or a closed map.

We prove that π_1 maps saturated open sets to open sets:

$\langle 1\rangle 1.$ LET: ${\pi_1}^{-1}(U)$ be a saturated open set in A
    PROVE: U is open in $\mathbb{R}$
$\langle 1\rangle 2.$ LET: $x \in U$
$\langle 1\rangle 3.$ $(x, 0) \in \pi_1(U)^{-1}$
$\langle 1\rangle 4.$ PICK W, V open in $\mathbb{R}$ such that $(x, 0) \subseteq W \times V \subseteq \pi_1(U)^{-1}$
$\langle 1\rangle 5.$ $x \in W \subseteq U$

It is not an open map because it maps $((-1, 1) \times (1, 2)) \cap A$ to $[0, 1)$.

It is not a closed map because it maps $\{(x, 1/x) \mid x > 0\}$ to $(0, +\infty)$.

Proposition 11.24.4. *Let $p : X \twoheadrightarrow Y$ be a quotient map. Let $A \subseteq X$ be saturated with respect to p. Let $q : A \twoheadrightarrow p(A)$ be the restriction of p.*

1. *If A is either open or closed in X then q is a quotient map.*

2. *If p is either an open map or a closed map then q is a quotient map.*

PROOF:
$\langle 1\rangle 1$. LET: $p : X \twoheadrightarrow Y$ be a quotient map.
$\langle 1\rangle 2$. LET: $A \subseteq X$ be saturated with respect to p.
$\langle 1\rangle 3$. LET: $q : A \twoheadrightarrow p(A)$ be the restriction of p.
$\langle 1\rangle 4$. q is continuous.
PROOF: Theorem 11.12.10.
$\langle 1\rangle 5$. If A is open in X then q is a quotient map.
$\langle 2\rangle 1$. ASSUME: A is open in X.
$\langle 2\rangle 2$. q maps saturated open sets to open sets.
$\langle 3\rangle 1$. LET: $U \subseteq A$ be saturated with respect to q and open in A
$\langle 3\rangle 2$. U is saturated with respect to p
$\langle 4\rangle 1$. LET: $x, y \in X$
$\langle 4\rangle 2$. ASSUME: $x \in U$
$\langle 4\rangle 3$. ASSUME: $p(x) = p(y)$
$\langle 4\rangle 4$. $x \in A$
PROOF: From $\langle 3\rangle 1$ and $\langle 4\rangle 2$.
$\langle 4\rangle 5$. $y \in A$
PROOF: From $\langle 1\rangle 2$ and $\langle 4\rangle 3$
$\langle 4\rangle 6$. $q(x) = x(y)$
PROOF: From $\langle 1\rangle 3$, $\langle 4\rangle 3$, $\langle 4\rangle 4$, $\langle 4\rangle 5$.
$\langle 4\rangle 7$. $y \in U$
PROOF: From $\langle 3\rangle 1$, $\langle 4\rangle 2$, $\langle 4\rangle 6$
$\langle 3\rangle 3$. U is open in X
PROOF: Lemma 11.17.6, $\langle 2\rangle 1$, $\langle 3\rangle 1$.
$\langle 3\rangle 4$. $p(U)$ is open in Y
PROOF: Proposition 11.24.2, $\langle 1\rangle 1$, $\langle 3\rangle 2$, $\langle 3\rangle 3$
$\langle 3\rangle 5$. $q(U)$ is open in $p(A)$
PROOF: Since $q(U) = p(U) = p(U) \cap p(A)$.
$\langle 2\rangle 3$. Q.E.D.
PROOF: By Proposition 11.24.2.
$\langle 1\rangle 6$. If A is closed in X then q is a quotient map.
PROOF: Similar.
$\langle 1\rangle 7$. If p is an open map then q is a quotient map.
$\langle 2\rangle 1$. ASSUME: p is an open map
$\langle 2\rangle 2$. q maps saturated open sets to open sets.
$\langle 3\rangle 1$. LET: U be open in A and saturated with respect to q
$\langle 3\rangle 2$. PICK V open in X such that $U = A \cap V$
$\langle 3\rangle 3$. $p(V)$ is open in Y
$\langle 3\rangle 4$. $q(U) = p(V) \cap p(A)$
$\langle 4\rangle 1$. $q(U) \subseteq p(V) \cap p(A)$
PROOF: From $\langle 3\rangle 2$.
$\langle 4\rangle 2$. $p(V) \cap p(A) \subseteq q(U)$

$\langle 5\rangle 1$. LET: $y \in p(V) \cap p(A)$
$\langle 5\rangle 2$. PICK $x \in V$ and $x' \in A$ such that $p(x) = p(x') = y$
$\langle 5\rangle 3$. $x \in A$
PROOF: By $\langle 1\rangle 2$.
$\langle 5\rangle 4$. $x \in U$
PROOF: From $\langle 3\rangle 2$
$\langle 2\rangle 3$. Q.E.D.
PROOF: By Proposition 11.24.2.
$\langle 1\rangle 8$. If p is a closed map then q is a quotient map.
PROOF: Similar.
□

Example 11.24.5. This example shows we cannot remove the hypotheses on A and p.

Define $f : [0,1] \to [2,3] \to [0,2]$ by $f(x) = x$ if $x \leq 1$, $f(x) = x - 1$ if $x \geq 2$. Then f is a quotient map but its restriction f' to $[0,1) \cup [2,3]$ is not, because ${f'}^{-1}([1,2])$ is open but $[1,2]$ is not.

For a counterexample where A is saturated, see Example 11.25.3.

Proposition 11.24.6. *Let $p : A \twoheadrightarrow C$ and $q : B \twoheadrightarrow D$ be open quotient maps. Then $p \times q : A \times B \to C \times D$ is an open quotient map.*

PROOF: From Corollary 11.24.2.1, Proposition 11.16.15 and Theorem 11.16.11.
□

Theorem 11.24.7. *Let $p : X \twoheadrightarrow Y$ be a quotient map. Let Z be a topological space and $f : Y \to Z$ be a function. Then*

1. *$f \circ p$ is continuous if and only if f is continuous.*
2. *$f \circ p$ is a quotient map if and only if f is a quotient map.*

PROOF:
$\langle 1\rangle 1$. If $f \circ p$ is continuous then f is continuous.
PROOF: Proposition 11.22.4.
$\langle 1\rangle 2$. If f is continuous then $f \circ p$ is continuous.
PROOF: Theorem 11.12.9.
$\langle 1\rangle 3$. If $f \circ p$ is a quotient map then f is a quotient map.
PROOF: Proposition 11.22.5.
$\langle 1\rangle 4$. If f is a quotient map then $f \circ p$ is a quotient map.
PROOF: From Proposition 11.22.3.
□

Proposition 11.24.8. *Let X and Y be topological spaces. Let $p : X \to Y$ and $f : Y \to X$ be continuous maps such that $p \circ f = \mathrm{id}_Y$. Then p is a quotient map.*

PROOF:

$\langle 1\rangle 1$. LET: $V \subseteq Y$
$\langle 1\rangle 2$. ASSUME: $p^{-1}(V)$ is open in X.
$\langle 1\rangle 3$. $f^{-1}(p^{-1}(V))$ is open in Y.
PROOF: Because f is continuous.
$\langle 1\rangle 4$. V is open in Y.
PROOF: Because $f^{-1}(p^{-1}(V)) = V$.
□

11.25 Quotient Topology

Definition 11.25.1 (Quotient Topology)**.** Let X be a topological space, Y a set and $p : X \twoheadrightarrow Y$ be a surjective function. Then the *quotient topology* on Y is the unique topology on Y with respect to which p is a quotient map, namely $\mathcal{T} = \{U \in \mathcal{P}Y \mid p^{-1}(U) \text{ is open in } X\}$.

We prove this is a topology.

PROOF:
$\langle 1\rangle 1$. $Y \in \mathcal{T}$
PROOF: Since $p^{-1}(Y) = X$ by surjectivity.
$\langle 1\rangle 2$. For all $\mathcal{A} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{A} \in \mathcal{T}$
PROOF: Since $p^{-1}(\bigcup \mathcal{A}) = \bigcup_{U \in \mathcal{A}} p^{-1}(U)$
$\langle 1\rangle 3$. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$
PROOF: Since $p^{-1}(U \cap V) = p^{-1}(U) \cap p^{-1}(V)$.
□

Definition 11.25.2 (Quotient Space)**.** Let X be a topological space and $\sim$ an equivalence relation on X. Let $p : X \twoheadrightarrow X/\sim$ be the canonical surjection. Then $X/\sim$ under the quotient topology is called a *quotient space*, *identification space* or *decomposition space* of X.

Here is a counterexample showing we cannot remove all the hypotheses of Proposition 11.24.4 except that A is saturated.

Example 11.25.3. Let $X = (0, 1/2] \cup \{1\} \cup \{1 + 1/n : n \geq 2\}$ as a subspace of $\mathbb{R}$. Define R to be the equivalence relation on X where xRy iff ($x = y$ or $|x - y| = 1$), so we identify $1/n$ with $1 + 1/n$ for all $n \geq 2$. Let Y be the resulting quotient space X/R in the quotient topology and $p : X \twoheadrightarrow Y$ the canonical surjection.

Let $A = \{1\} \cup (0, 1/2] \setminus \{1/n : n \geq 2\} \subseteq X$. Then A is saturated under p but the restriction q of p to A is not a quotient map because it maps the saturated open set $\{1\}$ to $\{1\}$ which is not open in $p(A)$.

Proposition 11.25.4. *Let X, Y and Z be topological spaces. Let $f : X \to Y$ and $g : Y \to Z$. If f and g are quotient maps then so is $g \circ f$.*

PROOF: From Proposition 11.22.3. □

Example 11.25.5. The product of two quotient maps is not necessarily a quotient map.

Let $X = \mathbb{R}$ and X^* the quotient space formed by identifying all positive integers. Let $p : X \twoheadrightarrow X^*$ be the canonical surjection.

We prove $p \times \mathrm{id}_{\mathbb{Q}} : X \times \mathbb{Q} \to X^* \times \mathbb{Q}$ is not a quotient map.

PROOF:
⟨1⟩1. For $n \geq 1$,
LET: $c_n = \sqrt{2}/n$
⟨1⟩2. For $n \geq 1$,
LET: $U_n = \{(x,y) \in X \times \mathbb{Q} \mid n - 1/4 < x < n + 1/4, (y + n > x + c_n \text{ and } y + n > -x + c_n) \text{ or } (y + n < x + c_n \text{ and } y + n < -x + c_n)\}$
⟨1⟩3. For $n \geq 1$, we have U_n is open in $X \times \mathbb{Q}$
⟨1⟩4. For $n \geq 1$, we have $\{n\} \times \mathbb{Q} \subseteq U_n$
⟨1⟩5. LET: $U = \bigcup_{n=1}^{\infty} U_n$
⟨1⟩6. U is open in $X \times \mathbb{Q}$
⟨1⟩7. U is saturated with respect to $p \times \mathrm{id}_{\mathbb{Q}}$
⟨1⟩8. LET: $U' = (p \times \mathrm{id}_{\mathbb{Q}})(U)$
⟨1⟩9. ASSUME: for a contradiction U' is open in $X^* \times \mathbb{Q}$
⟨1⟩10. $(1, 0) \in U'$
⟨1⟩11. PICK a neighbourhood W of 1 in X^* and $\delta > 0$ such that $W \times (-\delta, \delta) \subseteq U'$
⟨1⟩12. $p^{-1}(W) \times (-\delta, \delta) \subseteq U$
⟨1⟩13. PICK n such that $c_n < \delta$
⟨1⟩14. $n \in p^{-1}(W)$
⟨1⟩15. PICK $\epsilon > 0$ such that $\epsilon < \delta - c_n$ and $\epsilon < 1/4$ and $(n - \epsilon, n + \epsilon) \subseteq p^{-1}(W)$
⟨1⟩16. $(n - \epsilon, n + \epsilon) \times (-\delta, \delta) \subseteq U$
⟨1⟩17. PICK a rational y such that $c_n - \epsilon/2 < y < c_n + \epsilon/2$
⟨1⟩18. $(n + \epsilon/2, y) \notin U$
⟨1⟩19. Q.E.D.
PROOF: This contradicts ⟨1⟩16.
□

Proposition 11.25.6. *Let X be a topological space and $\sim$ an equivalence relation on X. Then $X/\sim$ is T_1 if and only if every equivalence class is closed in X.*

PROOF: Immediate from definitions. □

11.26 Retractions

Definition 11.26.1 (Retraction)**.** Let X be a topological space and $A \subseteq X$. A *retraction* of X onto A is a continuous map $r : X \to A$ such that, for all $a \in A$, we have $r(a) = a$.

Proposition 11.26.2. *Every retraction is a quotient map.*

PROOF: Proposition 11.24.8 with f the inclusion $A \hookrightarrow X$. □

11.27 Homogeneous Spaces

Definition 11.27.1 (Homogeneous)**.** A topological space X is *homogeneous* if and only if, for any points $a, b \in X$, there exists a homeomorphism $\phi : X \cong X$ such that $\phi(a) = b$.

11.28 Regular Spaces

Definition 11.28.1 (Regular Space)**.** A topological space X is *regular* if and only if, for any closed set A and point $a \notin A$, there exist disjoint open sets U, V such that $A \subseteq U$ and $a \in V$.

11.29 Dense Sets

Definition 11.29.1 (Dense)**.** Let X be a topological space and $A \subseteq X$. Then A is *dense* if and only if $\overline{A} = X$.

11.30 Connected Spaces

Definition 11.30.1 (Separation)**.** A *separation* of a topological space X is a pair of disjoint open sets U, V such that $U \cup V = \emptyset$.

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Definition 11.30.2 (Connected)**.** A topological space is *connected* if and only if it has no separation; otherwise it is *disconnected*.

Proposition 11.30.3. *A topological space X is connected if and only if the only sets that are both open and closed are X and $\emptyset$.*

Immediate from defintions.

Lemma 11.30.4. *If Y is a subspace of X, a separation of Y is a pair of disjoint nonempty sets A and B whose union is Y, neither of which contains a limit point of the other.*

PROOF:
⟨1⟩1. LET: $A, B \subseteq Y$
⟨1⟩2. If A and B form a separation of Y then A and B are disjoint and nonempty, $A \cup B = Y$, and neither of A and B contains a limit point of the other.
⟨2⟩1. ASSUME: A and B form a separation of Y
⟨2⟩2. A and B are disjoint and nonempty and $A \cup B = Y$
PROOF: From ⟨2⟩1 and the definition of separation.
⟨2⟩3. A does not contain a limit point of B
⟨3⟩1. ASSUME: for a contradiction $l \in A$ and l is a limit point of B in X.
⟨3⟩2. l is a limit point of B in Y

PROOF: Proposition 11.17.16.

⟨3⟩3. $l \in B$

⟨4⟩1. B is closed in Y

PROOF: Since A is open in Y and $B = Y \setminus A$ from ⟨2⟩1.

⟨4⟩2. Q.E.D.

PROOF: Corollary 11.6.3.1.

⟨3⟩4. Q.E.D.

PROOF: This contradicts the fact that $A \cap B = \emptyset$ (⟨2⟩1).

⟨2⟩4. B does not contain a limit point of A

PROOF: Similar.

⟨1⟩3. If A and B are disjoint and nonempty, $A \cup B = Y$, and neither of A and B contains a limit point of the other, then A and B form a separation of Y.

⟨2⟩1. ASSUME: A and B are disjoint and nonempty, $A \cup B = Y$, and neither of A and B contains a limit point of the other.

⟨2⟩2. A is open in Y

⟨3⟩1. B is closed in Y

⟨4⟩1. LET: l be a limit point of B in Y

⟨4⟩2. l is a limit point of B in X

PROOF: Proposition 11.17.16.

⟨4⟩3. $l \notin A$

PROOF: By ⟨2⟩1

⟨4⟩4. $l \in B$

PROOF: By ⟨2⟩1 since $A \cup B = Y$

⟨4⟩5. Q.E.D.

PROOF: Corollary 11.6.3.1.

⟨3⟩2. Q.E.D.

PROOF: Since $A = Y \setminus B$.

⟨2⟩3. B is open in Y

PROOF: Similar.

□

Example 11.30.5. Every set under the indiscrete topology is connected.

Example 11.30.6. The discrete topology on a set X is connected if and only if $|X| \leq 1$.

Example 11.30.7. The finite complement topology on a set X is connected if and only if either $|X| \leq 1$ or X is infinite.

Example 11.30.8. The countable complement topology on a set X is connected if and only if either $|X| \leq 1$ or X is uncountable.

Example 11.30.9. The rationals $\mathbb{Q}$ are disconnected. For any irrational a, the sets $(-\infty, a) \cap \mathbb{Q}$ and $(a, +\infty) \cap \mathbb{Q}$ form a separation of $\mathbb{Q}$.

Lemma 11.30.10. *Let X be a topological space. If C and D form a separation of X, and Y is a connected subspace of X, then either $Y \subseteq C$ or $Y \subseteq D$.*

PROOF: Otherwise $Y \cap C$ and $Y \cap D$ would form a separation of Y. $\square$

Theorem 11.30.11. *The union of a set of connected subspaces of a space X that have a point in common is connected.*

PROOF:
$\langle 1 \rangle 1$. LET: $\mathcal{A}$ be a set of connected subspaces of the space X that have the point a in common.
$\langle 1 \rangle 2$. ASSUME: for a contradiction C and D form a separation of $\bigcup \mathcal{A}$
$\langle 1 \rangle 3$. ASSUME: without loss of generality $a \in C$
$\langle 1 \rangle 4$. For all $A \in \mathcal{A}$ we have $A \subseteq C$
PROOF: Lemma 11.30.10.
$\langle 1 \rangle 5$. $D = \emptyset$
$\langle 1 \rangle 6$. Q.E.D.
PROOF: This contradicts $\langle 1 \rangle 2$.
$\square$

Theorem 11.30.12. *Let X be a topological space and A a connected subspace of X. If $A \subseteq B \subseteq \overline{A}$ then B is connected.*

PROOF:
$\langle 1 \rangle 1$. ASSUME: for a contradiction C and D form a separation of B.
$\langle 1 \rangle 2$. ASSUME: without loss of generality $A \subseteq C$
PROOF: Lemma 11.30.10.
$\langle 1 \rangle 3$. $B \subseteq C$
$\langle 2 \rangle 1$. LET: $x \in B$
$\langle 2 \rangle 2$. $x \in \overline{A}$
$\langle 2 \rangle 3$. Either $x \in A$ or x is a limit point of A.
PROOF: Theorem 11.6.3.
$\langle 2 \rangle 4$. Either $x \in A$ or x is a limit point of C.
PROOF: Lemma 11.6.5, $\langle 1 \rangle 2$.
$\langle 2 \rangle 5$. $x \in C$
PROOF: Lemma 11.30.4.
$\langle 1 \rangle 4$. $D = \emptyset$
$\langle 1 \rangle 5$. Q.E.D.
PROOF: This contradicts $\langle 1 \rangle 1$.
$\square$

Theorem 11.30.13. *The image of a connected space under a continuous map is connected.*

PROOF:
$\langle 1 \rangle 1$. LET: $f : X \twoheadrightarrow Y$ be a surjective continuous map where X is connected.
$\langle 1 \rangle 2$. ASSUME: for a contradiction C and D form a separation of Y.
$\langle 1 \rangle 3$. $f^{-1}(C)$ and $f^{-1}(D)$ form a separation of X.
$\square$

Theorem 11.30.14. *The product of a family of connected spaces is connected.*

PROOF:
⟨1⟩1. The product of two connected spaces is connected.
⟨2⟩1. LET: X and Y be connected spaces.
⟨2⟩2. PICK $a \in X$ and $b \in Y$
PROOF: We may assume X and Y are nonempty since otherwise $X \times Y = \emptyset$ which is connected.
⟨2⟩3. $X \times \{b\}$ is connected.
PROOF: It is homeomorphic to X.
⟨2⟩4. For all $x \in X$ we have $\{x\} \times Y$ is connected.
PROOF: It is homeomorphic to Y.
⟨2⟩5. For any $x \in X$
LET: $T_x = (X \times \{b\}) \cup (\{x\} \times Y)$
⟨2⟩6. For all $x \in X$, T_x is connected.
PROOF: Theorem 11.30.11 since $(x, b) \in (X \times \{b\}) \cap (\{x\} \times Y)$.
⟨2⟩7. $X \times Y$ is connected.
PROOF: Theorem 11.30.11 since $X \times Y = \bigcup_{x \in X} T_x$ and (a, b) is a point in every T_x.
⟨1⟩2. The product of a finite family of connected spaces is connected.
PROOF: From ⟨1⟩1 by induction.
⟨1⟩3. The product of any family of connected spaces is connected.
⟨2⟩1. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of connected spaces.
⟨2⟩2. LET: $X = \prod_{\alpha \in J} X_\alpha$
⟨2⟩3. PICK $a \in X$
PROOF: We may assume $X \neq \emptyset$ as the empty space is connected.
⟨2⟩4. For every finite subset K of J,
LET: $X_K = \{x \in X \mid \forall \alpha \in J \setminus K . x_\alpha = a_\alpha\}$
⟨2⟩5. For every finite $K \subseteq J$, we have X_K is connected.
PROOF: From ⟨1⟩2 since $X_K \cong \prod_{\alpha \in K} X_K$.
⟨2⟩6. LET: $Y = \bigcup_K X_K$
⟨2⟩7. Y is connected
PROOF: Theorem 11.30.11 since a is a common point.
⟨2⟩8. $X = \overline{Y}$
⟨3⟩1. LET: $x \in X$
⟨3⟩2. LET: $U = \prod_{\alpha \in J} U_\alpha$ be a basic neighbourhood of x where $U_\alpha = X_\alpha$ for all α except $\alpha \in K$ for some finite $K \subseteq J$
⟨3⟩3. LET: $y \in X$ be the point with $y_\alpha = x_\alpha$ for $\alpha \in K$ and $y_\alpha = a_\alpha$ for all other α
⟨3⟩4. $y \in U \cap X_K$
⟨3⟩5. $y \in U \cap Y$
⟨2⟩9. X is connected.
PROOF: Theorem 11.30.12.
□

Example 11.30.15. The set $\mathbb{R}^\omega$ is disconnected under the box topology. The set of bounded sequences and the set of unbounded sequences form a separation.

Proposition 11.30.16. *Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on the same set X. If*

$\mathcal{T} \subseteq \mathcal{T}'$ *and* $(X, \mathcal{T}')$ *is connected then* $(X, \mathcal{T})$ *is connected.*

PROOF: If U and V form a separation of $(X, \mathcal{T})$ then they form a separation of $(X, \mathcal{T}')$. ∎

Proposition 11.30.17. *Let* X *be a topological space and* (A_n) *a sequence of connected subspaces of* X*. If* $A_n \cap A_{n+1} \neq \emptyset$ *for all* n *then* $\bigcup_n A_n$ *is connected.*

PROOF:
⟨1⟩1. ASSUME: for a contradiction C and D form a separation of $\bigcup_n A_n$
⟨1⟩2. ASSUME: without loss of generality $A_0 \subseteq C$
PROOF: Lemma 11.30.10.
⟨1⟩3. For all n we gave $A_n \subseteq C$
PROOF:
⟨2⟩1. ASSUME: $A_n \subseteq C$
⟨2⟩2. PICK $x \in A_n \cap A_{n+1}$
⟨2⟩3. $x \in C$
⟨2⟩4. $A_{n+1} \subseteq C$
PROOF: Lemma 11.30.10.
⟨2⟩5. Q.E.D.
PROOF: The result follows by induction.
⟨1⟩4. $D = \emptyset$
⟨1⟩5. Q.E.D.
PROOF: This contradicts ⟨1⟩1.
∎

Proposition 11.30.18. *Let* X *be a topological space. Let* $A, C \subseteq X$*. If* C *is connected and intersects both* A *and* $X \setminus A$ *then* C *intersects* ∂A*.*

PROOF: Otherwise $C \cap A^\circ$ and $C \setminus \overline{A}$ would form a separation of C. ∎

Example 11.30.19. The space $\mathbb{R}_l$ is disconnected. For any real x, the sets $(-\infty, x)$ and $[x, +\infty)$ form a separation.

Proposition 11.30.20. *Let* X *and* Y *be connected spaces. Let* A *be a proper subset of* X *and* B *a proper subset of* Y*. Then* $(X \times Y) \setminus (A \times B)$ *is connected.*

PROOF:
⟨1⟩1. PICK $a \in X \setminus A$ and $b \in Y \setminus B$
⟨1⟩2. For $x \in X \setminus A$ we have $(X \times \{b\}) \cup (\{x\} \times Y)$ is connected.
PROOF: Theorem 11.30.11 since (x, b) is a common point.
⟨1⟩3. For $y \in Y \setminus B$ we have $(X \times \{y\}) \cup (\{a\} \times Y)$ is connected.
PROOF: Theorem 11.30.11 since (a, y) is a common point.
⟨1⟩4. $(X \times Y) \setminus (A \times B)$ is connected.
PROOF: Theorem 11.30.11 since it is the union of the sets in ⟨1⟩2 and ⟨1⟩3 with (a, b) as a common point.
∎

Proposition 11.30.21. *Let* $p : X \twoheadrightarrow Y$ *be a quotient map. If* Y *is connected and* $p^{-1}(y)$ *is connected for all* $y \in Y$*, then* X *is connected.*

PROOF:
$\langle 1 \rangle 1$. ASSUME: for a contradiction C and D form a separation of X.
$\langle 1 \rangle 2$. C is saturated.
$\langle 2 \rangle 1$. LET: $x \in C$, $y \in X$ with $p(x) = p(y) = a$, say
$\langle 2 \rangle 2$. $y \notin D$
PROOF: Otherwise $p^{-1}(a) \cap C$ and $p^{-1}(a) \cap D$ form a separation of $p^{-1}(a)$.
$\langle 2 \rangle 3$. $y \in C$
$\langle 1 \rangle 3$. D is saturated.
PROOF: Similar.
$\langle 1 \rangle 4$. $p(C)$ and $p(D)$ form a separation of Y.
□

Proposition 11.30.22. *Let X be a connected space and Y a connected subspace of X. Suppose A and B form a separation of $X \setminus Y$. Then $Y \cup A$ and $Y \cup B$ are both connected.*

PROOF:
$\langle 1 \rangle 1$. $Y \cup A$ is connected.
$\langle 2 \rangle 1$. ASSUME: for a contradiction C and D form a separation of $Y \cup A$
$\langle 2 \rangle 2$. ASSUME: without loss of generality $Y \subseteq C$
$\langle 2 \rangle 3$. PICK open sets A_1, B_1, C_1, D_1 in X with

$$A = A_1 \setminus Y$$
$$B = B_1 \setminus Y$$
$$C = C_1 \cap (Y \cup A)$$
$$D = D_1 \cap (Y \cup A)$$

$\langle 2 \rangle 4$. $B_1 \cup C_1$ and $A_1 \cap D_1$ form a separation of X
$\langle 1 \rangle 2$. $Y \cup B$ is connected.
PROOF: Similar.
□

Theorem 11.30.23. *Let L be a linearly ordered set under the order topology. Then L is connected if and only if L is a linear continuum.*

PROOF:
$\langle 1 \rangle 1$. If L is a linear continuum then L is connected.
$\langle 2 \rangle 1$. LET: L be a linear continuum under the order topology.
$\langle 2 \rangle 2$. ASSUME: for a contradiction C and D form a separation of L.
$\langle 2 \rangle 3$. PICK $a \in C$ and $b \in D$.
$\langle 2 \rangle 4$. ASSUME: without loss of generality $a < b$.
$\langle 2 \rangle 5$. LET: $S = \{x \in L \mid a < x \text{ and } [a, x) \subseteq C\}$
$\langle 2 \rangle 6$. S is nonempty.
PROOF: Since $a \in C$ and C is open.
$\langle 2 \rangle 7$. S is bounded above by b.
PROOF: Since $b \notin C$.
$\langle 2 \rangle 8$. LET: $s = \sup S$
$\langle 2 \rangle 9$. $s \in S$
$\langle 3 \rangle 1$. LET: $y \in [a, s)$

PROVE: $y \in C$

$\langle 3 \rangle 2$. PICK z with $y < z \in S$

PROOF: By minimality of s.

$\langle 3 \rangle 3$. $y \in [a, z) \subseteq C$

$\langle 2 \rangle 10$. CASE: $s \in C$

$\langle 3 \rangle 1$. PICK x such that $s < x$ and $[s, x) \subseteq C$

PROOF: Since C is open and s is not greatest in L because $s < b$.

$\langle 3 \rangle 2$. $x \in S$

PROOF: Since $[a, x) = [a, s) \cup [s, x) \subseteq C$.

$\langle 3 \rangle 3$. Q.E.D.

PROOF: This contradicts the fact that s is an upper bound for S.

$\langle 2 \rangle 11$. CASE: $s \in D$

$\langle 3 \rangle 1$. PICK $x < s$ such that $(x, s] \subseteq D$

$\langle 3 \rangle 2$. PICK y with $x < y < s$

PROOF: Since L is dense.

$\langle 3 \rangle 3$. $y \in C$

PROOF: From $\langle 2 \rangle 9$.

$\langle 3 \rangle 4$. $y \in D$

PROOF: From $\langle 3 \rangle 1$.

$\langle 3 \rangle 5$. Q.E.D.

$\langle 3 \rangle 6$. LET: L be a linear continuum under the order topology.

$\langle 3 \rangle 7$. ASSUME: for a contradiction C and D form a separation of L.

$\langle 3 \rangle 8$. PICK $a \in C$ and $b \in D$.

$\langle 3 \rangle 9$. ASSUME: without loss of generality $a < b$.

$\langle 3 \rangle 10$. LET: $S = \{x \in L \mid a < x \text{ and } [a, x) \subseteq C\}$

$\langle 3 \rangle 11$. S is nonempty.

PROOF: Since $a \in C$ and C is open.

$\langle 3 \rangle 12$. S is bounded above by b.

PROOF: Since $b \notin C$.

$\langle 3 \rangle 13$. LET: $s = \sup S$

$\langle 3 \rangle 14$. $s \in S$

$\langle 4 \rangle 1$. LET: $y \in [a, s)$

PROVE: $y \in C$

$\langle 4 \rangle 2$. PICK z with $y < z \in S$

PROOF: By minimality of s.

$\langle 4 \rangle 3$. $y \in [a, z) \subseteq C$

$\langle 3 \rangle 15$. CASE: $s \in C$

$\langle 4 \rangle 1$. PICK x such that $s < x$ and $[s, x) \subseteq C$

PROOF: Since C is open and s is not greatest in L because $s < b$.

$\langle 4 \rangle 2$. $x \in S$

PROOF: Since $[a, x) = [a, s) \cup [s, x) \subseteq C$.

$\langle 4 \rangle 3$. Q.E.D.

PROOF: This contradicts the fact that s is an upper bound for S.

$\langle 3 \rangle 16$. CASE: $s \in D$

$\langle 4 \rangle 1$. PICK $x < s$ such that $(x, s] \subseteq D$

$\langle 4 \rangle 2$. PICK y with $x < y < s$

PROOF: Since L is dense.

$\langle 4\rangle 3$. $y \in C$

PROOF: From $\langle 2\rangle 9$.

$\langle 4\rangle 4$. $y \in D$

PROOF: From $\langle 3\rangle 1$.

$\langle 4\rangle 5$. Q.E.D.

PROOF: This contradicts $\langle 2\rangle 2$.

$\langle 1\rangle 2$. If L is connected then L is a linear continuum.

$\langle 2\rangle 1$. ASSUME: L is connected.

$\langle 2\rangle 2$. Every nonempty subset of L that is bounded above has a supremum.

$\langle 3\rangle 1$. LET: X be a nonempty subset of L bounded above by b.

$\langle 3\rangle 2$. ASSUME: for a contradiction X has no supremum.

$\langle 3\rangle 3$. LET: U be the set of upper bounds of X,

$\langle 3\rangle 4$. U is nonempty.

PROOF: Since $b \in U$.

$\langle 3\rangle 5$. U is open.

$\langle 4\rangle 1$. LET: $x \in U$

$\langle 4\rangle 2$. PICK an upper bound y for X such that $y < x$

$\langle 4\rangle 3$. Either x is greatest in L and $(y, x] \subseteq U$, or there exists $z > x$ such that $(y, z) \subseteq U$

$\langle 3\rangle 6$. LET: V be the set of lower bounds of U.

$\langle 3\rangle 7$. V is nonempty.

PROOF: Since $X \subseteq V$

$\langle 3\rangle 8$. V is open.

$\langle 4\rangle 1$. LET: $x \in V$

$\langle 4\rangle 2$. PICK $y \in X$ with $x < y$

PROOF: x cannot be an upper bound for X, because it would be the supremum of X.

$\langle 4\rangle 3$. Either x least in L and $[x, y) \subseteq V$, or there exists $z < x$ such that $(z, y) \subseteq V$

$\langle 3\rangle 9$. $L = U \cup V$

$\langle 4\rangle 1$. LET: $x \in L \setminus U$

$\langle 4\rangle 2$. PICK $y \in X$ such that $x < y$

$\langle 4\rangle 3$. For all $u \in U$ we have $x < y \leq u$

$\langle 4\rangle 4$. $x \in V$

$\langle 3\rangle 10$. $U \cap V = \emptyset$

PROOF: Any element of $U \cap V$ would be a supremum of X.

$\langle 3\rangle 11$. U and V form a separation of L.

$\langle 3\rangle 12$. Q.E.D.

PROOF: This contradicts $\langle 2\rangle 1$.

$\langle 2\rangle 3$. L is dense.

$\langle 3\rangle 1$. LET: $x, y \in L$ with $x < y$

$\langle 3\rangle 2$. There exists $z \in L$ such that $x < z < y$

PROOF: Otherwise $(-\infty, y)$ and $(x, +\infty)$ would form a separation of L.

□

Corollary 11.30.23.1. *The real line $\mathbb{R}$ is connected.*

Corollary 11.30.23.2. *Every interval in $\mathbb{R}$ is connected.*

Corollary 11.30.23.3. *The ordered square is connected.*

Theorem 11.30.24 (Intermediate Value Theorem)**.** *Let X be a connected space. Let Y be a linearly ordered set under the order topology. Let $f : X \to Y$ be continuous. Let $a, b \in X$ and $r \in Y$. Suppose $f(a) < r < f(b)$. Then there exists $c \in X$ such that $f(c) = r$.*

Proof: Otherwise $f^{-1}((-\infty, r))$ and $f^{-1}((r, +\infty))$ would form a separation of X. □

Proposition 11.30.25. *Every function $f : [0, 1] \to [0, 1]$ has a fixed point.*

Proof:
⟨1⟩1. Let: $g : [0, 1] \to [-1, 1]$ be the function $g(x) = f(x) - x$
Prove: there exists $x \in [0, 1]$ such that $g(x) = 0$
⟨1⟩2. Assume: without loss of generality $g(0) \neq 0$ and $g(1) \neq 0$
⟨1⟩3. $g(0) > 0$
⟨1⟩4. $g(1) < 0$
⟨1⟩5. There exists $x \in (0, 1)$ such that $g(x) = 0$
Proof: By the Intermediate Value Theorem.

Proposition 11.30.26. *Give $\mathbb{R}^\omega$ the box topology. Let $x, y \in \mathbb{R}^\omega$. Then x and y lie in the same comoponent if and only if $x - y$ is eventually zero, i.e. there exists N such that, for all $n \geq N$, we have $x_n = y_n$.*

Proof:
⟨1⟩1. The component containing 0 is the set of sequences that are eventually zero.
⟨2⟩1. Let: B be the set of sequences that are eventually zero.
⟨2⟩2. B is path-connected.
⟨3⟩1. Let: $x, y \in B$
⟨3⟩2. Pick N such that, for all $n \geq N$, we have $x_n = y_n = 0$
⟨3⟩3. Let: $p : [0, 1] \to \mathbb{R}^\omega$, $p(t) = (1 - t)x + ty$
Prove: p is continuous.
⟨3⟩4. Let: $t \in [0, 1]$ and $\prod_j U_j$ be a basic open neighbourhood of $p(t)$, where each U_j is open in $\mathbb{R}$
⟨3⟩5. Pick δ such that, for all $n < N$ and all $s \in [0, 1]$, if $|s - t| < \delta$ then $p(s)_n \in U_n$
⟨3⟩6. For all $s \in [0, 1]$, if $|s - t| < \delta$ then $p(s) \in \prod_j U_j$
⟨2⟩3. B is connected.
Proof: Proposition 11.32.3.
⟨2⟩4. If C is connected and $B \subseteq C$ then $B = C$.
⟨3⟩1. Assume: C is connected and $B \subseteq C$
⟨3⟩2. Assume: for a contradiction $x \in C \setminus B$
⟨3⟩3. For $n \geq 1$,

LET: $c_n = 1$ if $x_n = 0$, $c_n = n/x_n$ otherwise

$\langle 3\rangle 4.$ LET: $h : \mathbb{R}^\omega \to \mathbb{R}^\omega$ be the function $h(x) = (c_n x_n)_{n \geq 1}$

$\langle 3\rangle 5.$ h is a homeomorphism of $\mathbb{R}^\omega$ with itself.

$\langle 3\rangle 6.$ $h(x)$ is unbounded.

PROOF: For any $b > 0$, pick $N > b$ such that $x_N \neq 0$. Then $h(x)_N > b$.

$\langle 3\rangle 7.$ $h^{-1}(\{\text{bounded sequences}\}) \cap C$ and $h^{-1}(\{\text{unbounded sequences}\}) \cap C$ form a separation of C

$\langle 3\rangle 8.$ Q.E.D.

PROOF: This contradicts $\langle 3\rangle 1$.

$\langle 1\rangle 2.$ Q.E.D.

PROOF: Since $\lambda x.x - y$ is a homeomorphism of $\mathbb{R}^\omega$ with itself.

□

Example 11.30.27. The space $\mathbb{R}_K$ is connected.

PROOF:

$\langle 1\rangle 1.$ ASSUME: for a contradiction U and V form a separation of $\mathbb{R}_K$

$\langle 1\rangle 2.$ ASSUME: without loss of generality $0 \in U$

$\langle 1\rangle 3.$ There exists an open interval (a, b) such that $(a, b) - K \subseteq U$ and $(a, b) \nsubseteq U$

PROOF: Otherwise U and V would form a separation of $\mathbb{R}$.

$\langle 1\rangle 4.$ PICK $1/n \in (a, b) - U$

$\langle 1\rangle 5.$ $1/n \in V$

$\langle 1\rangle 6.$ There exists an open interval (c, d) around $1/n$ that is included in V

$\langle 1\rangle 7.$ Q.E.D.

PROOF: This is a contradiction since $(a, b) - K$ and (c, d) must intersect.

□

11.31 Totally Disconnected Spaces

Definition 11.31.1 (Totally Disconnected)**.** A topological space X is *totally disconnected* if and only if the only connected subspaces are the singletons.

Example 11.31.2. Every discrete space is totally disconnected.

Example 11.31.3. The rationals $\mathbb{Q}$ are totally disconnected.

Example 11.31.4. The Cantor set is totally disconnected.

PROOF:

$\langle 1\rangle 1.$ LET: (A_n) be the sequence of sets in Definition 9.1.1.

$\langle 1\rangle 2.$ LET: C be the Cantor set $\bigcap_n A_n$

$\langle 1\rangle 3.$ ASSUME:

for a contradiction $D \subseteq C$ is connected and has more than one point.

$\langle 1\rangle 4.$ LET: $x, y \in D$ with $x < y$

$\langle 1\rangle 5.$ PICK n such that $|x - y| > 1/3^n$

$\langle 1\rangle 6.$ A_n is a sequence of disjoint intervals of length $1/3^n$

$\langle 1\rangle 7.$ x and y are in two different intervals out of the intervals that make up A_n

⟨1⟩8. There exists z with $x < z < y$ such that $z \notin A_n$
⟨1⟩9. $(-\infty, z) \cap D$ and $(z, +\infty) \cap D$ form a separation of D.
□

11.32 Paths and Path Connectedness

Definition 11.32.1 (Path)**.** Let X be a topological space and $a, b \in X$. A *path* from a to b is a continuous function $p : [0, 1] \to X$ such that $p(0) = a$ and $p(1) = b$.

Definition 11.32.2 (Path Connected)**.** A topological space is *path connected* if and only if there exists a path between any two points.

Proposition 11.32.3. *Every path connected space is connected.*

PROOF:
⟨1⟩1. LET: X be a path connected space.
⟨1⟩2. ASSUME: for a contradiction C and D form a separation of X.
⟨1⟩3. PICK $a \in C$ and $b \in D$.
⟨1⟩4. PICK a path $p : [0, 1] \to X$ from a to b.
⟨1⟩5. $p^{-1}(C)$ and $p^{-1}(D)$ form a separation of $[0, 1]$.
⟨1⟩6. Q.E.D.
PROOF: This contradicts Corollary 11.30.23.2.
□

An example that shows the converse does not hold:

Example 11.32.4. The ordered square is not path connected.

PROOF:
⟨1⟩1. ASSUME: for a contradiction $p : [0, 1] \to I_o^2$ is a path from $(0, 0)$ to $(1, 1)$.
⟨1⟩2. p is surjective.
PROOF: By the Intermediate Value Theorem.
⟨1⟩3. For $x \in [0, 1]$, PICK a rational $q_x \in p^{-1}((x, 0), (x, 1))$
PROOF: Since $p^{-1}((x, 0), (x, 1))$ is open and nonempty by ⟨1⟩2.
⟨1⟩4. For $x, y \in [0, 1]$, if $x \neq y$ then $q_x \neq q_y$
PROOF: We have $p(q_x) \neq p(q_y)$ because $((x, 0), (x, 1))$ and $((y, 0), (y, 1))$ are disjoint.
⟨1⟩5. $\{q_x \mid x \in [0, 1]\}$ is an uncountable set of rationals.
⟨1⟩6. Q.E.D.
PROOF: This contradicts the fact that the rationals are countable.
□

Proposition 11.32.5. *The continuous image of a path connected space is path connected.*

PROOF:
⟨1⟩1. LET: X be a path connected space, Y a topological space, and $f : X \twoheadrightarrow Y$ be continuous and surjective.

$\langle 1\rangle 2$. LET: $a, b \in Y$
$\langle 1\rangle 3$. PICK $c, d \in X$ with $f(c) = a$ and $f(d) = b$
$\langle 1\rangle 4$. PICK a path $p : [0, 1] \to X$ from c to d.
$\langle 1\rangle 5$. $f \circ p$ is a path from a to b in Y.
$\square$

Proposition 11.32.6 (AC)**.** *The product of a family of path-connected spaces is path-connected.*

PROOF:
$\langle 1\rangle 1$. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of path-connected spaces.
$\langle 1\rangle 2$. LET: $a, b \in \prod_{\alpha \in J} X_\alpha$
$\langle 1\rangle 3$. For $\alpha \in J$, PICK a path $p_\alpha : [0, 1] \to X_\alpha$ from a_α to b_α
PROOF: Using the Axiom of Choice.
$\langle 1\rangle 4$. Define $p : [0.1] \to \prod_{\alpha \in J} X_\alpha$ by $p(t)_\alpha = p_\alpha(t)$
$\langle 1\rangle 5$. p is a path from a to b.
PROOF: Theorem 11.16.11.
$\square$

Proposition 11.32.7. *The continuous image of a path-connected space is path-connected.*

PROOF:
$\langle 1\rangle 1$. LET: $f : X \twoheadrightarrow Y$ be continuous and surjective where X is path-connected.
$\langle 1\rangle 2$. LET: $a, b \in Y$
$\langle 1\rangle 3$. PICK $a', b' \in X$ with $f(a') = a$ and $f(b') = b$.
$\langle 1\rangle 4$. PICK a path $p : [0, 1] \to X$ from a' to b'.
$\langle 1\rangle 5$. $f \circ p$ is a path from a to b.
$\square$

Proposition 11.32.8. *Let X be a topological space. The union of a set of path-connected subspaces of X that have a point in common is path-connected.*

PROOF:
$\langle 1\rangle 1$. LET: $\mathcal{A}$ be a set of path-connected subspaces of X with the point a in common.
$\langle 1\rangle 2$. LET: $b, c \in \bigcup \mathcal{A}$
$\langle 1\rangle 3$. PICK $B, C \in \mathcal{A}$ with $b \in B$ and $c \in C$.
$\langle 1\rangle 4$. PICK a path p in B from b to a.
$\langle 1\rangle 5$. PICK a path q in C from a to c.
$\langle 1\rangle 6$. The concatenation of p and q is a path from b to c in $\bigcup \mathcal{A}$.
$\square$

Proposition 11.32.9. *Let $A \subseteq \mathbb{R}^2$ be countable. Then $\mathbb{R}^2 \setminus A$ is path-connected.*

PROOF:
$\langle 1\rangle 1$. LET: $a, b \in \mathbb{R}^2 \setminus A$
$\langle 1\rangle 2$. PICK a line l in $\mathbb{R}^2$ with a on one side and b on the other.
$\langle 1\rangle 3$. For every point x on l,

LET: p_x be the path in $\mathbb{R}^2$ consisting of a line from a to x then a line from x to b

$\langle 1 \rangle 4$. For $x \neq y$ we have p_x and p_y have no points in common except a and b

$\langle 1 \rangle 5$. There are only countably many x such that a point of A lies on p_x.

$\langle 1 \rangle 6$. There exists x such that p_x is a path from a to b in $\mathbb{R}^2 \setminus A$.

□

Proposition 11.32.10. *Every open connected subspace of $\mathbb{R}^2$ is path-connected.*

PROOF:

$\langle 1 \rangle 1$. LET: U be an open connected subspace of $\mathbb{R}^2$.

$\langle 1 \rangle 2$. For all $x_0 \in U$,

LET: $PC(x_0) = \{y \in U \mid \text{there exists a path from } x \text{ to } y\}$

$\langle 1 \rangle 3$. For all $x_0 \in U$, the set $PC(x_0)$ is open and closed in U.

$\langle 2 \rangle 1$. LET: $x_0 \in U$

$\langle 2 \rangle 2$. $PC(x_0)$ is open in U

$\langle 3 \rangle 1$. LET: $y \in PC(x_0)$

$\langle 3 \rangle 2$. PICK $\epsilon > 0$ such that $B(y, \epsilon) \subseteq U$

PROOF: Since U is open.

$\langle 3 \rangle 3$. $B(y, \epsilon) \subseteq PC(x_0)$

PROOF: For all $z \in B(y, \epsilon)$, pick a path from x_0 to y then concatenate the straight line from y to z.

$\langle 2 \rangle 3$. $PC(x_0)$ is closed in U

$\langle 3 \rangle 1$. LET: $y \in U$ be a limit point of $PC(x_0)$

$\langle 3 \rangle 2$. PICK $\epsilon > 0$ such that $B(y, \epsilon) \subseteq U$

$\langle 3 \rangle 3$. PICK $z \in PC(x_0) \cap B(y, \epsilon)$

$\langle 3 \rangle 4$. $y \in PC(x_0)$

PROOF: Pick a path from x_0 to z then concatenate the straight line from z to y.

$\langle 1 \rangle 4$. $PC(x_0) = U$

PROOF: Proposition 11.30.3.

□

Example 11.32.11. If A is a connected subspace of X, then A° is not necessarily connected.

Take two closed circles in $\mathbb{R}^2$ that touch at one point. The interior of this space is two open circles, and these two circles form a separation.

Example 11.32.12. If A is a connected subspace of X then ∂A is not necessarily connected.

We have $[0, 1]$ is connected but $\partial[0, 1] = \{0, 1\}$ is not.

Example 11.32.13. If A is a subspace of X and A° and ∂A are connected, then A is not necessarily connected.

We have $\mathbb{Q}^\circ = \emptyset$ and $\partial\mathbb{Q} = \mathbb{R}$ are connected but $\mathbb{Q}$ is not connected.

11.33 The Topologist's Sine Curve

Definition 11.33.1 (The Topologist's Sine Curve)**.** Let $S = \{(x, \sin 1/x) \mid 0 < x \leq 1\}$, The *topologist's sine curve* is the closure $\overline{S}$ of S in $\mathbb{R}^2$.

Proposition 11.33.2. *The topologist's sine curve is connected.*

PROOF:
$\langle 1 \rangle 1$. LET: $S = \{(x, \sin 1/x) \mid 0 < x \leq 1\}$
$\langle 1 \rangle 2$. S is connected.
  PROOF: Theorem 11.30.13.
$\langle 1 \rangle 3$. $\overline{S}$ is connected.
  PROOF: Theorem 11.30.12.
□

Proposition 11.33.3. *The topologist's sine curve is* $\{(x, \sin 1/x) \mid 0 < x \leq 1\} \cup (\{0\} \times [-1.1])$.

PROOF: Sketch proof: Given a point $(0.y)$ with $-1 \leq y \leq 1$, pick a such that $\sin a = y$. Then $((1/a, y), (1/(a + 2\pi), y), (1/(a + 4\pi), y), \ldots)$ is a sequence in S that converges to $(0, y)$.
Conversely, let (x, y) be any point not in $S \cup (\{0\} \times [-1, 1])$. If $x < 0$ or $y > 1$ or $y < -1$ then we can easily find a neighbourhood that does not intersect $S \cup (\{0\} \times [-1, 1])$. If $x > 0$ and $-1 \leq y \leq 1$, then we have $y \neq \sin 1/x$. Hence pick a neighbourhood that does not intersect S.

Proposition 11.33.4. *Every closed subset of* $\mathbb{R}$ *that is bounded above has a greatest element.*

PROOF: It has a supremum, which is a limit point of the set and hence an element. □

Proposition 11.33.5 (CC)**.** *The topologist's sine curve is not path connected.*

PROOF:
$\langle 1 \rangle 1$. ASSUME: For a contradction $p : [0, 1] \to \overline{S}$ is a path from $(0, 0)$ to $(1, \sin 1)$.
$\langle 1 \rangle 2$. $\{t \in [0, 1] \mid p(t) \in \{0\} \times [-1, 1]\}$ is closed.
  PROOF: Since p is continuous and $\{0\} \times [-1, 1]$ is closed.
$\langle 1 \rangle 3$. LET: b be the largest number in $[0, 1]$ such that $p(b) \in \{0\} \times [-1, 1]$.
  PROOF: Proposition 11.33.4.
$\langle 1 \rangle 4$. LET: $x : [b, 1] \to \overline{S}$ be the function $\pi_1 \circ p$
$\langle 1 \rangle 5$. LET: $y : [b, 1] \to \overline{S}$ be the function $\pi_2 \circ p$
$\langle 1 \rangle 6$. PICK a sequence t_n in $(b, 1]$ such that $t_n \to b$ and $y(t_n) = (-1)^n$ for all n
  $\langle 2 \rangle 1$. LET: $n \geq 1$
  $\langle 2 \rangle 2$. PICK u with $0 < u < x(1/n)$ and $\sin(1/u) = (-1)^n$
  $\langle 2 \rangle 3$. PICK t_n with $b < t_n < 1/n$ and $x(t_n) = u$
    PROOF: By the Intermediate Value Theorem
$\langle 1 \rangle 7$. Q.E.D.

PROOF: This contradicts Proposition 11.12.18 since y is continuous and $y(t_n)$ does not converge.
□

Corollary 11.33.5.1. *The closure of a path-connected subspace of a space is not necessarily path-connected.*

11.34 The Long Line

Definition 11.34.1 (The Long Line)**.** The *long line* is the space $\omega_1 \times [0,1)$ in the dictionary order under the order topology, where ω_1 is the first uncountable ordinal.

Lemma 11.34.2. *For any ordinal α with $0 < \alpha < \omega_1$ we have $[(0,0),(\alpha,0)) \cong [0,1)$*

$\langle 1 \rangle$1. $[(0,0),(1,0)) \cong [0,1)$
PROOF: The map π_2 is a homeomorphism.
$\langle 1 \rangle$2. If $[(0,0),(\alpha,0)) \cong [0,1)$ then $[(0,0),(\alpha+1,0)) \cong [0,1)$
PROOF: Proposition 5.3.11.
$\langle 1 \rangle$3. If λ is a limit ordinal with $\lambda < \omega_1$ and $[(0,0),(\alpha,0)) \cong [0,1)$ for all α with $0 < \alpha < \lambda$ then $[(0,0),(\lambda,0)) \cong [0,1)$
$\langle 2 \rangle$1. LET: λ be a limit ordinal $< \omega_1$
$\langle 2 \rangle$2. ASSUME: $[(0,0),(\alpha,0)) \cong [0,1)$ for all α with $0 < \alpha < \lambda$
$\langle 2 \rangle$3. PICK a sequence of ordinals $\alpha_0 < \alpha_1 < \cdots$ with limit λ
PROOF: Since λ is countable.
$\langle 2 \rangle$4. $[(\alpha_i,0),(\alpha_{i+1},0)) \cong [0,1)$ for all i
PROOF: Lemma 5.3.10.
$\langle 2 \rangle$5. Q.E.D.
PROOF: By Proposition 5.3.12.
$\langle 1 \rangle$4. Q.E.D.
PROOF: By transfinite induction.

Proposition 11.34.3 (CC)**.** *The long line is path-connected.*

PROOF:
$\langle 1 \rangle$1. LET: $(\alpha,i),(\beta,j) \in \omega_1 \times [0,1)$
$\langle 1 \rangle$2. ASSUME: without loss of generality $(\alpha,i) < (\beta,j)$
$\langle 1 \rangle$3. $[(0,0),(\beta+1,0)) \cong [0,1)$
PROOF: By Lemma 11.34.2
$\langle 1 \rangle$4. $[(\alpha,i),(\beta,j)) \cong [0,1)$
PROOF: Lemma 5.3.10.
$\langle 1 \rangle$5. PICK a homeomorphism $q : [0,1) \to [(\alpha,i),(\beta,j))$
$\langle 1 \rangle$6. $q \cup \{(1,(\beta,j))\}$ is a path from (α,i) to (β,j)
□

Proposition 11.34.4. *Every point in the long line has a neighbourhood homeomorphic to an interval in $\mathbb{R}$.*

Proof: For any (α, i) in the long line, the neighbourhood $[(0,0),(\alpha+1,0))$ satisfies the condition by Lemma 11.34.2.

11.35 Components

Proposition 11.35.1. *Let X be a topological space. Define the relation $\sim$ on X by $a \sim b$ if and only if there exists a connected subspace $A \subseteq X$ with $a, b \in A$. Then $\sim$ is an equivalence relation on X.*

Proof:
$\langle 1 \rangle 1.$ $\sim$ is reflexive.
Proof: For any $a \in X$ we have $\{a\}$ is a connected subspace that contains a.
$\langle 1 \rangle 2.$ $\sim$ is symmetric.
Proof: Trivial.
$\langle 1 \rangle 3.$ $\sim$ is transitive.
$\langle 2 \rangle 1.$ Let: $a, b, c \in X$
$\langle 2 \rangle 2.$ Assume: $a \sim b$ and $b \sim c$
$\langle 2 \rangle 3.$ Pick connected subspaces A and B with $a, b \in A$ and $b, c \in B$
$\langle 2 \rangle 4.$ $A \cup B$ is a connected subspace that contains a and c
Proof: Theorem 11.30.11.
∎

Definition 11.35.2 ((Connected) Component)**.** Let X be a topological space. The *(connected) components* of X are the equivalence classes under the above $\sim$.

Lemma 11.35.3. *Let X be a topological space. If $A \subseteq X$ is connected and nonempty then there exists a unique component C of X such that $A \subseteq C$.*

Proof:
$\langle 1 \rangle 1.$ Pick $a \in A$
$\langle 1 \rangle 2.$ Let: C be the $\sim$-equivalence class of a.
$\langle 1 \rangle 3.$ $A \subseteq C$
Proof: For all $x \in A$ we have $x \sim a$.
$\langle 1 \rangle 4.$ If C' is a component and $A \subseteq C'$ then $C = C'$
Proof: Since we have $a \in C'$.
∎

Theorem 11.35.4. *Let X be a topological space. The components of X are connected disjoint subspaces of X whose union is X such that each nonempty connected subspace of X intersects only one of them.*

Proof:
$\langle 1 \rangle 1.$ Every component of X is connected.
Proof: For $a \in X$, the $\sim$-equivalence class of a is $\bigcup\{A \subseteq X \mid A \text{ is connected}, a \in A\}$ which is connected by Theorem 11.30.11.
$\langle 1 \rangle 2.$ The components form a partition of X.

PROOF: Immediate from the definition.
⟨1⟩3. Every nonempty connected subspace of X intersects a unique component of X.
⟨2⟩1. LET: $A \subseteq X$ be connected and nonempty.
⟨2⟩2. LET: C be the component such that $A \subseteq C$
PROOF: Lemma 11.35.3.
⟨2⟩3. A intersects C
⟨2⟩4. If A intersects the component C' then $C' = C$
⟨3⟩1. LET: C' be a component that intersects A
⟨3⟩2. PICK $b \in A \cap C'$
⟨3⟩3. $A \subseteq C'$
PROOF: For all $x \in A$ we have $x \sim b$.
⟨3⟩4. $C = C'$
PROOF: By uniqueness in ⟨2⟩2.
□

Proposition 11.35.5. *Every component of a space is closed.*

PROOF:
⟨1⟩1. LET: X be a topological space and C a component of X.
⟨1⟩2. $\overline{C}$ is connected.
PROOF: Theorem 11.30.12.
⟨1⟩3. $C = \overline{C}$
PROOF: Lemma 11.30.10.
⟨1⟩4. C is closed.
PROOF: Lemma 11.4.5.
□

Proposition 11.35.6. *If a topological space has finitely many components then every component is open.*

PROOF: Each component is the complement of a finite union of closed sets. □

11.36 Path Components

Proposition 11.36.1. *Let X be a topological space. Define the relation $\sim$ on X by: $a \sim b$ if and only if there exists a path in X from a to b. Then $\sim$ is an equivalence relation on X.*

PROOF:
⟨1⟩1. $\sim$ is reflexive.
PROOF: For $a \in X$, the constant function $[0,1] \to X$ with value a is a path from a to a.
⟨1⟩2. $\sim$ is symmetric.
PROOF: If $p : [0,1] \to X$ is a path from a to b, then $\lambda t.p(1-t)$ is a path from b to a.
⟨1⟩3. $\sim$ is transitive.

PROOF: Concatenate paths.
∎

Definition 11.36.2 (Path Component)**.** Let X be a topological space. The *path components* of X are the equivalence relations under $\sim$.

.

Theorem 11.36.3. *The path components of X are path-connected disjoint subspaces of X whose union is X such that every nonempty path-connected subspace of X intersects exactly one path component.*

PROOF:
⟨1⟩1. Every path component is path-connected.
PROOF: If a and b are in the same path component then $a \sim b$, i.e. there exists a path from a to b.
⟨1⟩2. The path components are disjoint and their union is X.
PROOF: Immediate from the definition.
⟨1⟩3. Every non-empty path-cönnected subspace of X intersects exactly one path component.
⟨2⟩1. LET: A be a nonempty path-connected subspace of X.
⟨2⟩2. PICK $a \in A$
⟨2⟩3. A intersects the $\sim$-equivalence class of a.
⟨2⟩4. LET: C be any path component that intersects A.
⟨2⟩5. PICK $b \in A \cap C$
⟨2⟩6. $a \sim b$
PROOF: Since A is path-connected.
⟨2⟩7. C is the $\sim$-equivalence class of a.
∎

Proposition 11.36.4. *Every path component is included in a component.*

PROOF:
⟨1⟩1. LET: X be a topological space and C a path component of X.
⟨1⟩2. C is path-connected.
PROOF: Theorem 11.36.3.
⟨1⟩3. C is connected.
PROOF: Proposition 11.32.3.
⟨1⟩4. C is included in a component.
PROOF: Lemma 11.35.3.
∎

11.37 Local Connectedness

Definition 11.37.1 (Locally Connected)**.** Let X be a topological space and $a \in X$. Then X is *locally connected* at a if and only if every neighbourhood of a includes a connected neighbourhood of a.

The space X is *locally connected* if and only if it is locally connected at every point.

Example 11.37.2. The real line is both connected and locally connected.

Example 11.37.3. The space $\mathbb{R} \setminus \{0\}$ is disconnected but locally connected.

Example 11.37.4. The topologist's sine curve is connected but not locally connected.

Example 11.37.5. The rationals $\mathbb{Q}$ are neither connected nor locally connected.

Theorem 11.37.6. *A topological space X is locally connected if and only if, for every open set U in X, every component of U is open in X.*

PROOF:
$\langle 1 \rangle 1$. If X is locally connected then, for every open set U in X, every component of U is open in X.
 $\langle 2 \rangle 1$. ASSUME: X is locally connected.
 $\langle 2 \rangle 2$. LET: U be open in X.
 $\langle 2 \rangle 3$. LET: C be a component of U.
 $\langle 2 \rangle 4$. LET: $a \in C$
 $\langle 2 \rangle 5$. LET: V be a connected neighbourhood of a such that $V \subseteq U$
 $\langle 2 \rangle 6$. $V \subseteq C$
  PROOF: Lemma 11.35.3.
 $\langle 2 \rangle 7$. Q.E.D.
  PROOF: Lemma 11.1.8.
$\langle 1 \rangle 2$. If, for every open set U in X, every component of U is open in X, then X is locally connected.
 $\langle 2 \rangle 1$. ASSUME: for every open set U in X, every component of U is open in X.
 $\langle 2 \rangle 2$. LET: $a \in X$
 $\langle 2 \rangle 3$. LET: U be a neighbourhood of a
 $\langle 2 \rangle 4$. The component of U that contains a is a connected neighbourhood of a included in U.
□

Example 11.37.7. The ordered square is locally connected.

PROOF: Every neighbourhood of a point includes an interval around that point, which is connected by Theorem 11.30.23.

Example 11.37.8. Let X be the set of all rational points on the line segment $[0,1] \times \{0\}$, and Y the set of all rational points on the line segment $[0,1] \times \{1\}$. Let A be the space consisting of all line segments joining the point $(0,1)$ to a point of X, and all line segments joining the point $(1,0)$ to a point of Y. Then A is path-connected but is not locally connected at any point,

Proposition 11.37.9. *Let X and Y be topological spaces and $p : X \twoheadrightarrow Y$ be a quotient map. If X is locally connected then so is Y.*

PROOF:

⟨1⟩1. LET: U be an open set in Y.
⟨1⟩2. LET: C be a component of U.
⟨1⟩3. $p^{-1}(C)$ is a union of components of $p^{-1}(U)$
⟨2⟩1. LET: $x \in p^{-1}(C)$
⟨2⟩2. LET: D be the component of $p^{-1}(U)$ that contains x.
⟨2⟩3. $p(D)$ is connected.
PROOF: Theorem 11.30.13.
⟨2⟩4. $p(D) \subseteq C$.
PROOF: From ⟨1⟩2 since $p(x) \in p(D) \cap C$ (⟨2⟩1, ⟨2⟩2).
⟨2⟩5. $D \subseteq p^{-1}(C)$
⟨1⟩4. $p^{-1}(C)$ is open in $p^{-1}(U)$
PROOF: Theorem 11.37.6.
⟨1⟩5. C is open in U
PROOF: Since the restriction of p to $p : p^{-1}(U) \twoheadrightarrow U$ is a quotient map by Proposition 11.24.4.
⟨1⟩6. Q.E.D.
PROOF: Theorem 11.37.6.
□

11.38 Local Path Connectedness

Definition 11.38.1 (Locally Path-Connected)**.** Let X be a topological space and $a \in X$. Then X is *locally path-connected* at a if and only if every neighbourhood of a includes a path-connected neighbourhood of a.

The space X is *locally path-connected* if and only if it is locally path-connected at every point.

Theorem 11.38.2. *A topological space X is locally path-connected if and only if, for every open set U in X, every path component of U is open in X.*

PROOF:
⟨1⟩1. If X is locally path-connected then, for every open set U in X, every path component of U is open in X.
⟨2⟩1. ASSUME: X is locally path-connected.
⟨2⟩2. LET: U be open in X.
⟨2⟩3. LET: C be a path component of U.
⟨2⟩4. LET: $a \in C$
⟨2⟩5. LET: V be a path-connected neighbourhood of a such that $V \subseteq U$
⟨2⟩6. $V \subseteq C$
PROOF: Lemma 11.35.3.
⟨2⟩7. Q.E.D.
PROOF: Lemma 11.1.8.
⟨1⟩2. If, for every open set U in X, every component of U is open in X, then X is locally connected.
⟨2⟩1. ASSUME: for every open set U in X, every component of U is open in X.

⟨2⟩2. LET: $a \in X$

⟨2⟩3. LET: U be a neighbourhood of a

⟨2⟩4. The component of U that contains a is a connected neighbourhood of a included in U.

□

Theorem 11.38.3. *If a space is locally path connected then its components and its path components are the same.*

PROOF:

⟨1⟩1. LET: X be a locally path connected space.

⟨1⟩2. LET: C be a component of X.

⟨1⟩3. LET: $x \in C$

⟨1⟩4. LET: P be the path component of x

PROVE: $P = C$

⟨1⟩5. $P \subseteq C$

PROOF: Proposition 11.36.4.

⟨1⟩6. LET: Q be the union of the other path components included in C

⟨1⟩7. $C = P \cup Q$

PROOF: Proposition 11.36.4.

⟨1⟩8. P and Q are open in C

⟨2⟩1. C is open.

PROOF: Theorem 11.37.6.

⟨2⟩2. Q.E.D.

PROOF: Theorem 11.38.2.

⟨1⟩9. $Q = \emptyset$

PROOF: Otherwise P and Q would form a separation of C.

□

Example 11.38.4. The ordered square is not locally path connected, since it is connected but not path connected.

Proposition 11.38.5. *Let X be a locally path-connected space. Then every connected open subspace of X is path-connected.*

PROOF:

⟨1⟩1. LET: U be a connected open subspace of X.

⟨1⟩2. LET: P be a path component of U.

⟨1⟩3. LET: Q be the union of the other path components of U.

⟨1⟩4. P and Q are open in U.

PROOF: Theorem 11.38.2.

⟨1⟩5. $Q = \emptyset$

PROOF: Otherwise P and Q form a separation of U.

□

11.39 Weak Local Connectedness

Definition 11.39.1 (Weakly Locally Connected)**.** Let X be a topological space and $a \in X$. Then X is *weakly locally connected* at a if and only if every neighbourhood of a includes a connected subspace that includes a neighbourhood of a.

Proposition 11.39.2. *Let X be a topological space. If X is weakly locally connected at every point then X is locally connected.*

Proof:
$\langle 1\rangle 1$. Assume: X is weakly locally connected at every point.
$\langle 1\rangle 2$. Let: U be open in X.
$\langle 1\rangle 3$. Let: C be a component of U.
$\langle 1\rangle 4$. C is open in X.
$\langle 2\rangle 1$. Let: $x \in C$
$\langle 2\rangle 2$. Pick a connected subspace D of U that includes a neighbourhood V of x.
$\langle 2\rangle 3$. $D \subseteq C$
Proof: Lemma 11.35.3.
$\langle 2\rangle 4$. $x \in V \subseteq C$
$\langle 2\rangle 5$. Q.E.D.
Proof: Lemma 11.1.8.
$\langle 1\rangle 5$. Q.E.D.
Proof: Theorem 11.37.6.
□

Example 11.39.3. The 'infinite broom' (Munkres p.163) is an example of a space that is weakly locally connected at a point p but not locally connected at p.

11.40 Quasicomponents

Proposition 11.40.1. *Let X be a topological space. Define $\sim$ on X by $x \sim y$ if and only if there exists no separation U and V of X such that $x \in U$ and $y \in V$. Then $\sim$ is an equivalence relation on X.*

Proof:
$\langle 1\rangle 1$. $\sim$ is reflexive.
Proof: Since the two sets that make up a separation are disjoint.
$\langle 1\rangle 2$. $\sim$ is symmetric.
Proof: Immediate from the defintion.
$\langle 1\rangle 3$. $\sim$ is transitive.
$\langle 2\rangle 1$. Assume: $x \sim y$ and $y \sim z$
$\langle 2\rangle 2$. Assume: for a contradiction there is a separation U and V of X with $x \in U$ and $z \in V$
$\langle 2\rangle 3$. $y \in U$ or $y \in V$

$\langle 2\rangle 4$. Q.E.D.

PROOF: Either case contradicts $\langle 2\rangle 1$.

□

Definition 11.40.2 (Quasicomponents)**.** For X a topological space, the *quasicomponents* of X are the equivalence classes under $\sim$.

Proposition 11.40.3. *Let X be a topological space. Then every component of X is included in a quasicomponent of X.*

PROOF:

$\langle 1\rangle 1$. LET: C be a component of X.

$\langle 1\rangle 2$. LET: $x, y \in C$

PROVE: $x \sim y$

$\langle 1\rangle 3$. ASSUME: for a contradiction there exists a separation U and V of X with $x \in U$ and $y \in V$

$\langle 1\rangle 4$. $C \cap U$ and $C \cap V$ form a separation of C.

$\langle 1\rangle 5$. Q.E.D.

PROOF: This contradicts $\langle 1\rangle 1$.

Proposition 11.40.4. *In a locally connected space, the components and the quasicomponents are the same.*

PROOF:

$\langle 1\rangle 1$. LET: X be a locally connected space and Q a quasicomponent of X.

$\langle 1\rangle 2$. PICK a component C of X such that $C \subseteq Q$

$\langle 1\rangle 3$. LET: D be the union of the components of X

$\langle 1\rangle 4$. C and D are open in X.

PROOF: Theorem 11.37.6.

$\langle 1\rangle 5$. D cannot contain any points of Q.

PROOF: If it did, then C and D would form a separation of X and there would be points $x, y \in Q$ with $x \in C$ and $y \in D$.

$\langle 1\rangle 6$. $C = Q$

□

11.41 Open Coverings

Definition 11.41.1 (Open Covering)**.** Let X be a topological space. An *open covering* of X is a covering of X whose elements are all open sets.

11.42 Lindelöf Spaces

Definition 11.42.1 (Lindelöf Space)**.** A topological space X is *Lindelöf* if and only if every open covering has a countable subcovering.

Proposition 11.42.2. *Let X be a topological space. Then X is Lindelöf if and only if every set of closed sets that has the countable intersection property has nonempty intersection.*

PROOF: The following are equivalent.

1. X is compact.
2. Every open covering of X has a countable subcovering.
3. For any set $\mathcal{C}$ of closed sets, if $\{X \setminus C \mid C \in \mathcal{C}\}$ covers X then there is a countable subset $\mathcal{C}_0$ such that $\{X \setminus C \mid C \in \mathcal{C}_0\}$ covers X
4. For any set $\mathcal{C}$ of closed sets, if $\bigcap \mathcal{C} = \emptyset$ then there is a countable subset $\mathcal{C}_0$ with empty intersection.
5. Any set of closed sets with the countable intersection property has nonempty intersection.

□

Proposition 11.42.3 (CC)**.** *Let X be a topological space and $\mathcal{B}$ a basis for the topology on X. Then the following are equivalent.*

1. X is Lindelöf.

2. Every open covering of X by elements of $\mathcal{B}$ has a countable subcovering.

PROOF:
$\langle 1 \rangle 1.$ $1 \Rightarrow 2$
  PROOF: Immediate from definitions.
$\langle 1 \rangle 2.$ $2 \Rightarrow 1$
  $\langle 2 \rangle 1.$ ASSUME: Every open covering of X by elements of $\mathcal{B}$ has a countable subcovering.
  $\langle 2 \rangle 2.$ LET: $\mathcal{U}$ be an open covering of X.
  $\langle 2 \rangle 3.$ $\{B \in \mathcal{B} \mid \exists U \in \mathcal{U}. B \subseteq U\}$ covers X.
  $\langle 2 \rangle 4.$ PICK a finite subcovering $\mathcal{B}_0$.
  $\langle 2 \rangle 5.$ For $B \in BB$, PICK $U_B \in \mathcal{U}$ such that $B \subseteq U_B$
  $\langle 2 \rangle 6.$ $\{U_B \mid B \in \mathcal{B}_0\}$ covers X.
□

Example 11.42.4 (AC)**.** The space $\overline{S_\Omega}$ is Lindelöf.

PROOF:
$\langle 1 \rangle 1.$ LET: $\mathcal{A}$ be any open cover of $\overline{S_\Omega}$
$\langle 1 \rangle 2.$ PICK $U \in \mathcal{A}$ such that $\Omega \in U$
$\langle 1 \rangle 3.$ PICK $\alpha < \Omega$ such that $(\alpha, \Omega] \subseteq U$
$\langle 1 \rangle 4.$ For $\beta < \alpha$, PICK $U_\beta \in \mathcal{A}$ such that $\beta \in U_\beta$
$\langle 1 \rangle 5.$ $\{U_\beta \mid \beta < \alpha\} \cup \{U\}$ covers $\overline{S_\Omega}$
□

Proposition 11.42.5. *Every closed subspace of a Lindelöf space is Lindelöf.*

PROOF:
$\langle 1 \rangle 1.$ LET: X be a Lindelöf space.

⟨1⟩2. LET: $Y \subseteq X$ be closed.
⟨1⟩3. LET: $\mathcal{A}$ be an open covering of Y.
⟨1⟩4. LET: $\mathcal{B} = \{U \text{ open in } X \mid U \cap Y \in \mathcal{A}\} \cup \{X - Y\}$
⟨1⟩5. $\mathcal{B}$ is an open covering of X.
⟨1⟩6. PICK a countable subcovering $\mathcal{B}_0$
⟨1⟩7. $\{U \cap Y \mid U \in \mathcal{B}_0\} - \{\emptyset\}$ is a countable subset of $\mathcal{A}$ that covers Y.
□

The following examples show that an open subspace of a Lindelöf space is not necessarily Lindelöf.

Example 11.42.6. The space S_Ω is not Lindelöf, because the open cover $\{[0, x) \mid x \in S_\Omega\}$ has no countable subcover.

Example 11.42.7. The set $[0, 1] \times (0, 1)$ as a subspace of the ordered square is not Lindelöf.

The open cover $\{\{x\} \times (0, 1) \mid x \in [0, 1]\}$ has no countable subcover.

Proposition 11.42.8 (Choice)**.** *The continuous image of a Lindelöf space is Lindelöf.*

PROOF:
⟨1⟩1. LET: X be a Lindelöf space.
⟨1⟩2. LET: Y be a topological space.
⟨1⟩3. LET: $f : X \twoheadrightarrow Y$ be continuous and surjective.
⟨1⟩4. LET: $\mathcal{A}$ be an open cover of Y.
⟨1⟩5. $\{f^{-1}(V) \mid V \in \mathcal{A}\}$ is an open cover of X.
⟨1⟩6. PICK a countable subcover $\mathcal{B}$
⟨1⟩7. For $U \in \mathcal{B}$, PICK $V_U \in \mathcal{A}$ such that $U = f^{-1}(V_U)$
⟨1⟩8. $\{V_U \mid U \in \mathcal{B}\}$ covers Y.
□

11.43 Separable Spaces

Definition 11.43.1 (Separable)**.** A topological space is *separable* if and only if it has a countable dense subset.

Proposition 11.43.2 (AC)**.** *A countable product of separable spaces is separable.*

PROOF:
⟨1⟩1. LET: (X_n) be a sequence of separable spaces.
⟨1⟩2. For each n, PICK a countable dense set D_n in X_n
PROVE: $\prod_n D_n$ is dense in $\prod_n X_n$
⟨1⟩3. LET: $\prod_n U_n$ be a nonempty basic open set where each U_n is open in X_n.
⟨1⟩4. For each n, PICK $a_n \in D_n \cap U_n$
⟨1⟩5. $(a_n) \in \prod_n D_n \cap \prod_n U_n$
□

Example 11.43.3. The space $\mathbb{R}_l$ is separable. The set $\mathbb{Q}$ is dense.

The following example shows that a closed subspace of a separable space is not necessarily separable.

Example 11.43.4 (AC)**.** The space $\mathbb{R}_l^2$ is separable, but $\{(x, -x) \mid x \in \mathbb{R}\}$ as a subspace is uncountable and discrete, and hence not separable.

Example 11.43.5. The space S_Ω is not separable. For any countable $D \subseteq S_\Omega$, we have $\sup D + 1 \notin \overline{D}$.

Example 11.43.6. The space $\overline{S_\Omega}$ is not separable. For any countable $D \subseteq \overline{S_\Omega}$, we have $\sup(D - \{\Omega\}) + 1 \notin \overline{D}$.

Proposition 11.43.7. *The continuous image of a separable space is separable.*

PROOF:
$\langle 1\rangle 1$. LET: X be a separable space.
$\langle 1\rangle 2$. LET: Y be a topological space.
$\langle 1\rangle 3$. LET: $f : X \twoheadrightarrow Y$ be continuous and separable.
$\langle 1\rangle 4$. PICK a countable dense D in X
PROVE: $f(D)$ is dense in Y.
$\langle 1\rangle 5$. LET: V be open in Y and nonempty.
$\langle 1\rangle 6$. PICK $a \in f^{-1}(V) \cap D$
$\langle 1\rangle 7$. $f(a) \in V \cap f(D)$
□

Proposition 11.43.8 (Choice)**.** *In a separable space, every set of disjoint open sets is countable.*

PROOF:
$\langle 1\rangle 1$. LET: X be a separable space.
$\langle 1\rangle 2$. PICK a countable dense set D.
$\langle 1\rangle 3$. LET: $\mathcal{A}$ be a set of disjoint open sets.
$\langle 1\rangle 4$. For $U \in \mathcal{A}$ nonempty, PICK $a_U \in U \cap D$
$\langle 1\rangle 5$. The mapping $\mathcal{A} \to D$ that maps U to a_U is injective.
$\langle 1\rangle 6$. $\mathcal{A}$ is countable.
□

11.44 The Second Countability Axiom

Definition 11.44.1 (Second Countability Axiom)**.** A topological space satisfies the *second countability axiom*, or is *second countable*, if and only if it has a countable basis.

Example 11.44.2. The space $\mathbb{R}$ is second countable.
The set $\{(a, b) \mid a, b \in \mathbb{Q}\}$ is a basis.

Proposition 11.44.3. *A subspace of a second countable space is second countable.*

PROOF: If $\mathcal{B}$ is a countable basis for X and $Y \subseteq X$ then $\{B \cap Y \mid B \in \mathcal{B}\}$ is a countable basis for Y. $\square$

Proposition 11.44.4 (CC)**.** *Every second countable space is Lindelöf.*

PROOF: From Proposition 11.42.3.

Example 11.44.5. The space S_Ω is not second countable, because it is not Lindelöf.

The following example shows that the product of two Lindelöf spaces is not necessarily Lindelöf.

Example 11.44.6. The Sorgenfrey plane is not Lindelöf.

PROOF:
$\langle 1 \rangle 1$. LET: $L = \{(x, -x) \mid x \in \mathbb{R}\}$
$\langle 1 \rangle 2$. L is closed in $\mathbb{R}_l^2$
$\langle 1 \rangle 3$. LET: $\mathcal{U} = \{[a, b) \times [a, -d) \mid a, b, d \in \mathbb{R}\}$
$\langle 1 \rangle 4$. $\mathcal{U} \cup \{\mathbb{R} \setminus L\}$ covers $\mathbb{R}_l^2$
$\langle 1 \rangle 5$. Every element of $\mathcal{U}$ intersects L at exactly one point.
$\langle 1 \rangle 6$. No countable subset of $\mathcal{U}$ covers $\mathbb{R}_l^2$.
$\square$

Proposition 11.44.7. *The long line L is not second countable.*

PROOF:
$\langle 1 \rangle 1$. LET: $\mathcal{B}$ be a basis for L.
$\langle 1 \rangle 2$. For $\alpha < \omega_1$, PICK $B_\alpha \in \mathcal{B}$ such that $(\alpha, 1/2) \in B_\alpha \subseteq ((\alpha, 0), (\alpha + 1, 0))$
$\langle 1 \rangle 3$. $\mathcal{B}$ is uncountable.
PROOF: The mapping $\alpha \mapsto B_\alpha$ is an injection $\omega_1 \to \mathcal{B}$.

Corollary 11.44.7.1. *The long line cannot be imbedded into $\mathbb{R}^n$ for any n.*

Proposition 11.44.8. *Every second countable space is first countable.*

PROOF:
$\langle 1 \rangle 1$. LET: X be a second countable space.
$\langle 1 \rangle 2$. PICK a countable bases $\mathcal{B}$ for X.
$\langle 1 \rangle 3$. LET: $x \in X$
$\langle 1 \rangle 4$. $\{B \in \mathcal{B} \mid x \in B\}$ is a countable local basis at x.
$\square$

Proposition 11.44.9 (AC)**.** *A countable product of second countable spaces is second countable.*

PROOF:
$\langle 1 \rangle 1$. LET: (X_n) be a sequence of second countable spaces.
$\langle 1 \rangle 2$. For each n, PICK a countable basis $\mathcal{B}_n$ of X_n
$\langle 1 \rangle 3$. LET: $\mathcal{B} = \{\prod_i U_i \mid U_i \in \mathcal{B}_i$ for finitely many $i, U_i = X_i$ for all other $i\}$
$\langle 1 \rangle 4$. $\mathcal{B}$ is a countable basis for $\prod_n X_n$

∎

Proposition 11.44.10 (AC)**.** *Any discrete subspace of a second countable space is countable.*

Proof:
⟨1⟩1. Let: X be a second countable space.
⟨1⟩2. Let: $A \subseteq X$ be discrete.
⟨1⟩3. Pick a countable basis $\mathcal{B}$ for X.
⟨1⟩4. For all $a \in A$, Pick $B_a \in \mathcal{B}$ such that $B_a \cap A = \{a\}$
  ⟨2⟩1. Let: $a \in A$
  ⟨2⟩2. Pick U open in X such that $U \cap A = \{a\}$
  ⟨2⟩3. Pick $B \in \mathcal{B}$ such that $a \in B \subseteq U$
⟨1⟩5. The mapping $A \to \mathcal{B}$ that maps a to B_a is injective.
⟨1⟩6. A is countable.
∎

Proposition 11.44.11 (AC)**.** *Every second countable space is separable.*

Proof:
⟨1⟩1. Let: X be a second countable space.
⟨1⟩2. Pick a countable basis $\mathcal{B}$ for X.
⟨1⟩3. For all $B \in \mathcal{B}$ nonempty Pick $a_B \in B$.
⟨1⟩4. Let: $A = \{a_B \mid B \in \mathcal{B}, B \neq \emptyset\}$
  Prove: A is dense
⟨1⟩5. Let: $x \in X$
  Prove: $x \in \overline{A}$
⟨1⟩6. Let: U be a neighbourhood of x
  Prove: U intersects A
⟨1⟩7. Pick $B \in \mathcal{B}$ such that $x \in B \subseteq U$
⟨1⟩8. $a_B \in U \cap A$
∎

Example 11.44.12 (AC)**.** The space $\mathbb{R}_l$ is not second countable.

Proof:
⟨1⟩1. Let: $\mathcal{B}$ be any basis for $\mathbb{R}_l$
⟨1⟩2. For $x \in \mathbb{R}$, Pick $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq [x, x+1)$
⟨1⟩3. The mapping $\mathbb{R} \to \mathcal{B}$ that maps x to B_x is injective.
  Proof: If $x < y$ then $x \in B_x$ and $x \notin B_y$.
⟨1⟩4. $\mathcal{B}$ is uncountable.
∎

Example 11.44.13 (CC)**.** The space $\mathbb{R}_l$ is Lindelöf.

⟨1⟩1. Let: $\mathcal{A}$ be a covering of $\mathbb{R}_l$ by basic open sets of the form $[a, b)$
⟨1⟩2. Let: $C = \bigcup\{(a, b) \mid [a, b) \in \mathcal{A}\}$
⟨1⟩3. $\mathbb{R} \setminus C$ is countable.
  ⟨2⟩1. For every $x \in \mathbb{R} \setminus C$, Pick a rational q_x such that $(x, q_x) \subseteq C$

⟨3⟩1. Let: $x \in \mathbb{R} \setminus C$
⟨3⟩2. Pick b such that $[x, b) \in \mathcal{A}$
⟨3⟩3. Pick a rational q such that $q \in (x, b)$
⟨2⟩2. The mapping $x \mapsto q_x$ is an injection $\mathbb{R} \setminus C \to \mathbb{Q}$
⟨1⟩4. Pick a countable $\mathcal{A}' \subseteq \mathcal{A}$ that covers $\mathbb{R} \setminus C$
⟨1⟩5. Under the standard topology on $\mathbb{R}$, C is second countable.
Proof: Proposition 11.44.3.
⟨1⟩6. Pick a countable $\mathcal{A}'' \subseteq \mathcal{A}$ such that $\{(a, b) \mid [a, b) \in \mathcal{A}''\}$ covers C.
Proof: Proposition 11.42.3.
⟨1⟩7. $\mathcal{A}' \cup \mathcal{A}''$ covers $\mathbb{R}_l$.
□

Proposition 11.44.14 (AC)**.** *A topological space is second countable if and only if every basis includes a countable basis.*

Proof:
⟨1⟩1. Let: X be a topological space.
⟨1⟩2. If X is second countable then every basis includes a countable basis.
⟨2⟩1. Assume: X is second countable.
⟨2⟩2. Let: $\mathcal{B}$ be a basis.
⟨2⟩3. Pick a countable basis $\mathcal{C}$.
⟨2⟩4. For every pair of basis elements $C, C' \in \mathcal{C}$ such that there exists $B \in \mathcal{B}$ with $C \subseteq B \subseteq C'$, Pick $B_{CC'} \in \mathcal{B}$ such that $C \subseteq B_{CC'} \subseteq C'$
Prove: The set of all $B_{CC'}$ form a basis for X.
⟨2⟩5. Let: $x \in X$
⟨2⟩6. Let: U be a neighbourhood of x.
⟨2⟩7. Pick $C' \in \mathcal{C}$ such that $x \in C' \subseteq U$
⟨2⟩8. Pick $B \in \mathcal{B}$ such that $x \in B \subseteq C'$
⟨2⟩9. Pick $C \in \mathcal{C}$ such that $x \in C \subseteq B$
⟨2⟩10. $x \in B_{CC'} \subseteq U$
⟨1⟩3. If every basis includes a countable basis then X is second countable.
Proof: The set of all open sets is a basis and so includes a countable basis.
□

Proposition 11.44.15 (AC)**.** *Let X be a second countable space. Let $A \subseteq X$ be uncountable. Then A contains uncountably many of its own limit points.*

Proof:
⟨1⟩1. Let: X be a second countable space.
⟨1⟩2. Pick a countable basis $\mathcal{B}$ for X.
⟨1⟩3. Let: $A \subseteq X$
⟨1⟩4. Assume: only countably many points of A are limit points of A.
⟨1⟩5. For every point x of A that is not a limit point of A, Pick $B_x \in \mathcal{B}$ such that $B_x \cap A = \{x\}$.
⟨1⟩6. The mapping $A - A' \to \mathcal{B}$ that maps x to B_x is injective.
⟨1⟩7. A is countable.
□

Example 11.44.16. Thes space $\overline{S_\Omega}$ is not second countable because it is neither first countable nor separable.

Proposition 11.44.17. *The image of a first countable space under a continuous open map is first countable.*

PROOF:
$\langle 1\rangle 1$. LET: X be a first countable space.
$\langle 1\rangle 2$. LET: Y be a topological space.
$\langle 1\rangle 3$. LET: $f : X \twoheadrightarrow Y$ be a surjective, continuous open map.
$\langle 1\rangle 4$. LET: $y \in Y$
$\langle 1\rangle 5$. PICK $x \in X$ such that $f(x) = y$
$\langle 1\rangle 6$. PICK a countable local basis $\mathcal{B}$ at x.
PROVE: $\{f(B) \mid B \in \mathcal{B}\}$ is a local basis at y.
$\langle 1\rangle 7$. LET: V be a neighbourhood of y
$\langle 1\rangle 8$. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq f^{-1}(V)$
$\langle 1\rangle 9$. $y \in f(B) \subseteq V$
□

Proposition 11.44.18. *The image of a second countable space under a continuous open map is second countable.*

PROOF:
$\langle 1\rangle 1$. LET: X be a second countable space.
$\langle 1\rangle 2$. LET: Y be a topological space.
$\langle 1\rangle 3$. LET: $f : X \twoheadrightarrow Y$ be a surjective, continuous open map.
$\langle 1\rangle 4$. PICK a countable basis $\mathcal{B}$.
PROVE: $\{f(B) \mid B \in \mathcal{B}\}$ is a basis.
$\langle 1\rangle 5$. LET: V be a neighbourhood of y
$\langle 1\rangle 6$. PICK $x \in X$ such that $f(x) = y$
$\langle 1\rangle 7$. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq f^{-1}(V)$
$\langle 1\rangle 8$. $y \in f(B) \subseteq V$
□

11.45 Sequential Compactness

Definition 11.45.1 (Sequentially Compact)**.** A topological space is *sequentially compact* if and only if every sequence has a convergent subsequence.

11.46 Limit Point Compactness

Definition 11.46.1 (Limit Point Compact Space)**.** A topological space is *limit point compact* if and only if every infinite set has a limit point.

Proposition 11.46.2. *Every limit point compact T_1 space is sequentially compact.*

PROOF:
⟨1⟩1. LET: X be a limit point compact T_1 space.
⟨1⟩2. LET: (x_n) be a sequence in X.
⟨1⟩3. CASE: $\{x_n \mid n \geq 1\}$ is finite.
  ⟨2⟩1. PICK n such that x_n occurs infintely often in the sequence (x_n)
  ⟨2⟩2. The subsequence consisting of all the terms equal to x_n is convergent.
⟨1⟩4. CASE: $\{x_n \mid n \geq 1\}$ is infinite.
  ⟨2⟩1. PICK a limit point l for $\{x_n \mid n \geq 1\}$
  ⟨2⟩2. PICK an increasing sequence n_r with $x_{n_r} \in B(x, 1/r)$ for all r
    PROOF: This is always possible by Theorem 11.19.3.
  ⟨2⟩3. (x_{n_r}) converges to l.
□

Corollary 11.46.2.1. *Every compact T_1 spact is sequentially compact.*

Example 11.46.3. The space $[0, 1]^\omega$ under the uniform topology is not limit point compact.

The infinite set $\{0, 1\}^\omega$ has no limit point.

Example 11.46.4. The space $[0, 1]$ under the lower limit topology is not limit point compact.

The infinite set $A = \{1 - 1/n \mid n \geq 1\}$ has no limit point. 1 is not a limit point because the neighbourhood $\{1\}$ does not intersect A.

Proposition 11.46.5. *A closed subspace of a limit point compact space is limit point compact.*

PROOF:
⟨1⟩1. LET: X be a limit point compact space.
⟨1⟩2. LET: $A \subseteq X$ be closed.
⟨1⟩3. LET: $B \subseteq A$ be infinite.
⟨1⟩4. PICK a limit point l of B in X.
⟨1⟩5. $l \in A$
⟨1⟩6. l is a limit point of B in A.
□

Example 11.46.6. An open subspace of a limit point compact space is not necessarily limit point compact.

The space $[0, 1]$ is limit point compact but $(0, 1)$ is not.

Example 11.46.7. The continuous image of a limit point compact space is not necessarily limit point compact.

Let Y be the indiscrete space with two points. Then $\mathbb{Z}^+ \times Y$ is limit point compact but $\mathbb{Z}^+$ is not.

Example 11.46.8. A limit point compact subspace of a Hausdorff space is not necessarily closed.

The space S_Ω is limit point compact but is not closed in $\overline{S_\Omega}$.

For an example that shows that the product of two limit point compact spaces is not necessarily limit point compact, see L. A. Steen and J. A. Seerbach Jr. *Counterexamples in Topology* Example 112.

11.47 Countable Compactness

Definition 11.47.1 (Countably Compact)**.** A topological space is *countably compact* if and only if every countable open covering has a finite subcovering.

Proposition 11.47.2 (AC)**.** *Every closed subspace of a countably compact space is countably compact.*

PROOF:
⟨1⟩1. LET: X be a countably compact space.
⟨1⟩2. LET: $A \subseteq X$ be closed.
⟨1⟩3. LET: $\mathcal{U}$ be a countable open cover of A.
⟨1⟩4. For $U \in \mathcal{U}$, PICK an open set V_U is X such that $U = V_U \cap A$
⟨1⟩5. $\{V_U \mid U \in \mathcal{U}\} \cup \{X - A\}$ is a countable open cover of X
⟨1⟩6. PICK a finite subcover $\{V_{U_1}, \ldots, V_{U_n}, X - A\}$
⟨1⟩7. $\{U_1, \ldots, U_n\}$ covers A.
□

Proposition 11.47.3 (AC)**.** *Every countably compact space is limit point compact.*

PROOF:
⟨1⟩1. ASSUME: X is countably compact.
⟨1⟩2. LET: $A \subseteq X$ be infinite.
⟨1⟩3. ASSUME: for a contradiction A has no limit point.
⟨1⟩4. PICK a countably infinite $B \subseteq A$
⟨1⟩5. B is discrete.
PROOF: For all $b \in B$, there exists U_b open in X such that $U_b \cap B = \{b\}$.
⟨1⟩6. $\{\{b\} \mid b \in B\}$ is a countable cover of B that has no finite subcover.
⟨1⟩7. B is not countably compact.
⟨1⟩8. B is not closed in X
⟨1⟩9. B has a limit point.
⟨1⟩10. A has a limit point.
⟨1⟩11. Q.E.D.
PROOF: This contradicts ⟨1⟩3.
□

Proposition 11.47.4 (AC)**.** *Every limit point compact T_1 space is countably compact.*

PROOF:
⟨1⟩1. LET: X be a limit point compact T_1 space.
⟨1⟩2. LET: $\{U_n \mid n \in \mathbb{Z}^+\}$ be a countable open cover of X.
⟨1⟩3. For $n \in \mathbb{Z}^+$,
LET: $V_n = U_1 \cup \cdots \cup V_n$
⟨1⟩4. ASSUME: for a contradiction none of the V_n covers X
⟨1⟩5. For $n \in \mathbb{Z}^+$, PICK $a_n \in X - V_n$
⟨1⟩6. PICK a limit point l for $\{a_n \mid n \in \mathbb{Z}^+\}$

$\langle 1\rangle 7$. PICK n such that $l \in U_n$
$\langle 1\rangle 8$. CASE: $l = a_m$ for some $m \leq n$
PROOF: $U_n - \{a_1, \ldots, a_{m-1}, a_{m+1}, \ldots, a_n\}$ is a neighbourhood of l that intersects $\{a_n \mid n \in \mathbb{Z}^+\}$ only at l, contradicting $\langle 1\rangle 6$.
$\langle 1\rangle 9$. CASE: $l \neq a_m$ for any $m \leq n$
PROOF: $U_n - \{a_1, \ldots, a_n\}$ is a neighbourhood of l that does not intersect $\{a_n \mid n \in \mathbb{Z}^+\}$, which contradicts $\langle 1\rangle 6$.
□

The following example shows we cannot remove the hypothesis that the space is T_1.

Example 11.47.5. Let Y be the indiscrete space with two points. Then $\mathbb{Z}^+ \times Y$ is a limit point compact space that is not countably compact, since $\{\{n\} \times Y \mid n \in \mathbb{Z}^+\}$ is a countable open cover that has no finite subcover.

Proposition 11.47.6. *A topological space is countably compact if and only if every nested sequence $C_1 \supseteq C_2 \supseteq \cdots$ of nonempty closed sets has nonempty intersection.*

PROOF:
$\langle 1\rangle 1$. LET: X be a topological space.
$\langle 1\rangle 2$. If X is countably compact then every nested sequence of nonempty closed sets has nonempty intersection.
$\langle 2\rangle 1$. ASSUME: X is countably compact.
$\langle 2\rangle 2$. LET: $C_1 \supseteq C_2 \supseteq \cdots$ be a nested sequence of nonempty closed sets.
$\langle 2\rangle 3$. ASSUME: for a contradiction $\bigcap_n C_n = \emptyset$
$\langle 2\rangle 4$. $\{X - C_n \mid n \in \mathbb{Z}^+\}$ covers X
$\langle 2\rangle 5$. PICK a finite subcover $\{X - C_{n_1}, \ldots, X - C_{n_k}\}$ where $n_1 < \cdots < n_k$
$\langle 2\rangle 6$. $C_{n_k} = \emptyset$
$\langle 2\rangle 7$. Q.E.D.
PROOF: This contradicts $\langle 2\rangle 2$.
□
$\langle 1\rangle 3$. If every nested sequence of nonempty closed sets has nonempty intersection then X is countably compact.
$\langle 2\rangle 1$. ASSUME: Every nested sequence of nonempty closed sets has nonempty intersection.
$\langle 2\rangle 2$. LET: $\{U_n \mid n \geq 1\}$ is a countable open cover of X.
$\langle 2\rangle 3$. $X - U_1 \supseteq X - (U_1 \cup U_2) \supseteq \cdots$ is a nested sequence of closed sets with empty intersection.
$\langle 2\rangle 4$. PICK k such that $X - (U_1 \cup \cdots \cup U_k) = \emptyset$
$\langle 2\rangle 5$. $\{U_1, \ldots, U_k\}$ covers X.
□

11.48 Subnets

Definition 11.48.1 (Subnet)**.** Let X be a topological space. Let $(a_\alpha)_{\alpha \in J}$ be a net in X. A *subnet* of $(a_\alpha)_{\alpha \in J}$ is a net of the form $(a_{g(\beta)})_{\beta \in K}$ where K is a directed set, $g : K \to J$ is monotone, and $g(K)$ is cofinal in J.

Proposition 11.48.2. *Let X be a topological space. Let $(a_\alpha)_{\alpha \in J}$ be a net in X. Let $l \in X$. If (a_α) converges to l then any subnet of (a_α) converges to l.*

PROOF:
$\langle 1 \rangle 1$. LET: X be a topological space.
$\langle 1 \rangle 2$. LET: $(a_\alpha)_{\alpha \in J}$ be a net in X.
$\langle 1 \rangle 3$. LET: $l \in X$
$\langle 1 \rangle 4$. ASSUME: $a_\alpha \to l$ as $\alpha \to \infty$
$\langle 1 \rangle 5$. LET: $(a_{g(\beta)})_{\beta \in K}$ be a subnet of $(a_\alpha)_{\alpha \in J}$
$\langle 1 \rangle 6$. LET: U be a neighbourhood of l.
$\langle 1 \rangle 7$. PICK $\alpha \in J$ be such that, for all $\alpha' \geq \alpha$, we have $a_{\alpha'} \in U$
$\langle 1 \rangle 8$. PICK $\beta \in K$ such that $g(\beta) \geq \alpha$.
$\langle 1 \rangle 9$. For all $\beta' \geq \beta$ we have $a_{g(\beta')} \in U$.
□

11.49 Accumulation Points

Definition 11.49.1 (Accumulation Point)**.** Let X be a topological space. Let $(a_\alpha)_{\alpha \in J}$ be a net in X. Let $l \in X$. Then l is an *accumulation point* of $(a_\alpha)_{\alpha \in J}$ if and only if, for every neighbourhood U of l, the set $\{\alpha \in J \mid a_\alpha \in U\}$ is cofinal in J.

Lemma 11.49.2. *Let X be a topological space. Let $(a_\alpha)_{\alpha \in J}$ be a net in X. Let $l \in X$. Then l is an accumulation point of $(a_\alpha)_{\alpha \in J}$ if and only if there exists a subnet of $(a_\alpha)_{\alpha \in J}$ that converges to l.*

PROOF:
$\langle 1 \rangle 1$. LET: X be a topological space.
$\langle 1 \rangle 2$. LET: $(a_\alpha)_{\alpha \in J}$ be a net in X.
$\langle 1 \rangle 3$. LET: $l \in X$
$\langle 1 \rangle 4$. If l is an accumulation point of $(a_\alpha)_{\alpha \in J}$ then there exists a subnet of $(a_\alpha)_{\alpha \in J}$ that converges to l.
$\langle 2 \rangle 1$. ASSUME: l is an accumulation point of $(a_\alpha)_{\alpha \in J}$.
$\langle 2 \rangle 2$. LET: $K = \{(\alpha, U) \mid \alpha \in J, U \text{ is a neighbourhood of } l, a_\alpha \in U\}$ with $(\alpha, U) \leq (\beta, V)$ if and only if $\alpha \leq \beta$ and $V \subseteq U$
$\langle 2 \rangle 3$. K is a directed set
$\langle 3 \rangle 1$. $\leq$ is reflexive on K.
$\langle 3 \rangle 2$. $\leq$ is transitive on K.
$\langle 3 \rangle 3$. $\leq$ is antisymmetric on K.
$\langle 3 \rangle 4$. For all $(\alpha, U), (\beta, V) \in K$, there exists (γ, W) such that $(\alpha, U) \leq (\gamma, W)$ and $(\beta, V) \leq (\gamma, W)$
$\langle 4 \rangle 1$. LET: $(\alpha, U), (\beta, V) \in K$
$\langle 4 \rangle 2$. PICK $\gamma \in J$ with $\alpha \leq \gamma$ and $\beta \leq \gamma$
$\langle 4 \rangle 3$. PICK $\delta \in J$ with $\gamma \leq \delta$ and $a_\delta \in U \cap V$
$\langle 4 \rangle 4$. $(\alpha, U) \leq (\delta, U \cap V)$ and $(\beta, V) \leq (\delta, U \cap V)$

⟨2⟩4. LET: $g : K \to J$, $g(\alpha, U) = \alpha$
⟨2⟩5. g is monotone
⟨2⟩6. $g(K)$ is cofinal in J
PROOF: For all $\alpha \in J$ we have $\alpha = g(\alpha, X)$.
⟨2⟩7. $(a_{g(\alpha,U)})_{(\alpha,U)\in K}$ converges to l.
⟨3⟩1. LET: U be a neighbourhood of l
⟨3⟩2. PICK $\alpha \in J$ such that $a_\alpha \in U$
⟨3⟩3. For all $(\beta, V) \geq (\alpha, U)$ we have $a_\beta \in U$
PROOF: Since $a_\beta \in V \subseteq U$
⟨1⟩5. If there exists a subnet of $(a_\alpha)_{\alpha\in J}$ that converges to l then l is an accumulation point of $(a_\alpha)_{\alpha\in J}$.
⟨2⟩1. ASSUME: $(a_{g(\beta)})_{\beta\in K}$ converges to l
⟨2⟩2. LET: U be a neighbourhood of l
⟨2⟩3. LET: $\alpha \in J$
⟨2⟩4. PICK $\beta \in K$ such that, for all $\beta' \geq \beta$, we have $a_{g(\beta')} \in U$
⟨2⟩5. PICK $\gamma \in K$ such that $g(\gamma) \geq \alpha$
⟨2⟩6. PICK $\delta \in K$ with $\beta \leq \delta$ and $\gamma \leq \delta$
⟨2⟩7. $\alpha \leq g(\delta)$
⟨2⟩8. $a_{g(\delta)} \in U$
□

11.50 Compact Spaces

Definition 11.50.1 (Compact)**.** A topological space is *compact* if and only if every open covering has a finite subcovering.

Lemma 11.50.2. *Let X be a topological space and Y a subspace of X. Then Y is compact if and only if every covering of Y by sets open in X has a finite subcovering.*

PROOF:
⟨1⟩1. If Y is compact then every covering of Y by sets open in X has a finite subcovering.
⟨2⟩1. ASSUME: Y is compact.
⟨2⟩2. LET: $\mathcal{U}$ be a covering of Y by sets open in X.
⟨2⟩3. $\{U \cap Y \mid U \in \mathcal{U}\}$ is an open covering of Y.
⟨2⟩4. PICK a finite subcovering $\{U_1 \cap Y, \ldots, U_n \cap Y\}$
⟨2⟩5. $\{U_1, \ldots, U_n\}$ is a finite subcovering of $\mathcal{U}$.
⟨1⟩2. If every covering of Y by sets open in X has a finite subcovering then Y is compact.
⟨2⟩1. LET: $\mathcal{U}$ be an open covering of Y.
⟨2⟩2. LET: $\mathcal{V} = \{V \text{ open in } X \mid V \cap Y \in \mathcal{U}\}$.
⟨2⟩3. $\mathcal{V}$ is a covering of Y by sets open in X.
⟨2⟩4. PICK a finite subcovering $\{V_1, \ldots, V_n\}$
⟨2⟩5. $\{V_1 \cap Y, \ldots, V_n \cap Y\}$ is a finite subcovering of $\mathcal{U}$.
□

Proposition 11.50.3. *Every closed subspace of a compact space is compact.*

PROOF:
⟨1⟩1. LET: X be a compact space and $Y \subseteq X$ be closed.
⟨1⟩2. LET: $\mathcal{U}$ be a covering of Y by sets open in X.
⟨1⟩3. $\mathcal{U} \cup \{X \setminus Y\}$ is an open covering of X.
⟨1⟩4. PICK a finite subcovering $\mathcal{U}_0$
⟨1⟩5. $\mathcal{U}_0 \cap \mathcal{U}$ is a finite subset of $\mathcal{U}$ that covers Y.
□

Theorem 11.50.4. *The continuous image of a compact space is compact.*

PROOF:
⟨1⟩1. LET: $f : X \twoheadrightarrow Y$ be continuous and surjective.
⟨1⟩2. LET: $\mathcal{V}$ be an open covering of Y
⟨1⟩3. $\{p^{-1}(V) \mid V \in \mathcal{V}\}$ is an open covering of X.
⟨1⟩4. PICK a finite subcovering $\{p^{-1}(V_1), \ldots, p^{-1}(V_n)\}$
⟨1⟩5. $\{V_1, \ldots, V_n\}$ covers Y.
□

Theorem 11.50.5. *Let A and B be compact subspaces of X and Y respectively. Let N be an open set in $X \times Y$ that includes $A \times B$. Then there exist open sets U and V in X and Y respectively such that $A \subseteq U$, $B \subseteq V$ and $U \times V \subseteq N$.*

PROOF:
⟨1⟩1. For all $x \in A$, there exist neighbourhoods U of x and V of B such that $U \times V \subseteq N$.
  ⟨2⟩1. LET: $x \in A$
  ⟨2⟩2. For all $y \in B$, there exist neighbourhoods U of x and V of y such that $U \times V \subseteq N$
  ⟨2⟩3. $\{V \text{ open in } Y \mid \exists \text{ neighbourhood } U \text{ of } x, U \times V \subseteq N\}$ covers B.
  ⟨2⟩4. PICK a finite subcover $\{V_1, \ldots, V_n\}$
  ⟨2⟩5. For $i = 1, \ldots, n$, PICK a neighbourhood U_i of x such that $U_i \times V_i \subseteq N$
  ⟨2⟩6. LET: $U = U_1 \cap \cdots \cap U_n$
  ⟨2⟩7. LET: $V = V_1 \cup \cdots \cup V_n$
  ⟨2⟩8. U is a neighbourhood of x.
  ⟨2⟩9. V is a neighbourhood of B.
  ⟨2⟩10. $U \times V \subseteq N$
⟨1⟩2. $\{U \text{ open in } X \mid \exists \text{ neighbourhood } V \text{ of } B. U \times V \subseteq N\}$ covers A.
⟨1⟩3. PICK a finite subcover $\{U_1, \ldots, U_n\}$
⟨1⟩4. For $i = 1, \ldots, n$, PICK a neighbourhood V_i of B such that $U_i \times V_i \subseteq N$
⟨1⟩5. LET: $U = U_1 \cup \cdots \cup U_n$
⟨1⟩6. LET: $V = V_1 \cap \cdots \cap V_n$
⟨1⟩7. U and V are open.
⟨1⟩8. $A \subseteq U$
⟨1⟩9. $B \subseteq V$
⟨1⟩10. $U \times V \subseteq N$
□

Corollary 11.50.5.1 (Tube Lemma). *Let X and Y be topological spaces with Y compact. Let $a \in X$ and N be an open set in $X \times Y$ that includes $\{a\} \times Y$. Then there exists a neighbourhood W of a such that N includes the* tube *$W \times Y$.*

Theorem 11.50.6. *Let X be a topological space. Then X is compact if and only if every set of closed sets that has the finite intersection property has nonempty intersection.*

Proof: The following are equivalent.

1. X is compact.
2. Every open covering of X has a finite subcovering.
3. For any set $\mathcal{C}$ of closed sets, if $\{X \setminus C \mid C \in \mathcal{C}\}$ covers X then there is a finite subset $\mathcal{C}_0$ such that $\{X \setminus C \mid C \in \mathcal{C}_0\}$ covers X
4. For any set $\mathcal{C}$ of closed sets, if $\bigcap \mathcal{C} = \emptyset$ then there is a finite subset $\mathcal{C}_0$ with empty intersection.
5. Any set of closed sets with the finite intersection property has nonempty intersection.

□

Corollary 11.50.6.1. *Let X be a topological space and $C_1 \supseteq C_2 \supseteq \cdots$ a nested sequence of nonempty closed sets. Then $\bigcap_n C_n$ is nonempty.*

Proposition 11.50.7. *Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on the same set X with $\mathcal{T} \subseteq \mathcal{T}'$. If $\mathcal{T}'$ is compact then $\mathcal{T}$ is compact.*

Proof:
$\langle 1 \rangle 1$. Let: $\mathcal{U} \subseteq \mathcal{T}$ cover X
$\langle 1 \rangle 2$. $\mathcal{U} \subseteq \mathcal{T}'$
$\langle 1 \rangle 3$. A finite subset of $\mathcal{U}$ covers X.
□

Corollary 11.50.7.1. *If $\mathcal{T}$ and $\mathcal{T}'$ are two compact Hausdorff topologies on the same set X, then either $\mathcal{T} = \mathcal{T}'$ or $\mathcal{T}$ and $\mathcal{T}'$ are incomparable.*

Proof: From the Proposition and Proposition 11.20.12. □

Example 11.50.8. Any set under the finite complement topology is compact.

Proposition 11.50.9. *Let X be a topological space. A finite union of compact subspaces of X is compact.*

Proof:
$\langle 1 \rangle 1$. Let: A and B be compact subspaces of X.
$\langle 1 \rangle 2$. Let: $\mathcal{U}$ be a set of open sets in X that covers $A \cup B$
$\langle 1 \rangle 3$. Pick a finite subset $\mathcal{U}_1$ that covers A.
Proof: Lemma 11.50.2.

$\langle 1 \rangle 4.$ Pick a finite subset $\mathcal{U}_2$ that covers B.
Proof: Lemma 11.50.2.
$\langle 1 \rangle 5.$ $\mathcal{U}_1 \cup \mathcal{U}_2$ is a finite subset that covers $A \cup B$.
$\langle 1 \rangle 6.$ Q.E.D.
Proof: Lemma 11.50.2.
□

Proposition 11.50.10. *Let A and B be disjoint compact subspaces of the Hausdorff space X. Then there exist disjoint open sets U and V that include A and B respectively.*

Proof: From Theorem 11.50.5 with $N = X^2 \setminus \{(x, x) \mid x \in X\}$. □

Corollary 11.50.10.1. *Every compact subspace of a Hausdorff space is closed.*

Theorem 11.50.11. *Let $f : X \to Y$ be a bijective continuous function. If X is compact and Y is Hausdorff then f is a homeomorphism.*

Proof:
$\langle 1 \rangle 1.$ Let: $C \subseteq X$ be closed.
$\langle 1 \rangle 2.$ C is compact.
Proof: Proposition 11.50.3.
$\langle 1 \rangle 3.$ $f(C)$ is compact.
Proof: Theorem 11.50.4.
$\langle 1 \rangle 4.$ $f(C)$ is closed.
Proof: Corollary 11.50.10.1.
$\langle 1 \rangle 5.$ Q.E.D.
Proof: Lemma 11.13.2.
□

Proposition 11.50.12. *Let X be a compact space, Y a Hausdorff space, and $f : X \to Y$ a continuous map. Then f is a closed map.*

Proof:
$\langle 1 \rangle 1.$ Let: $C \subseteq X$ be closed.
$\langle 1 \rangle 2.$ C is compact.
Proof: Proposition 11.50.3.
$\langle 1 \rangle 3.$ $f(C)$ is compact.
Proof: Theorem 11.50.4.
$\langle 1 \rangle 4.$ $f(C)$ is closed.
Proof: Corollary 11.50.10.1.
□

Proposition 11.50.13. *If Y is compact then the projection $\pi_1 : X \times Y \to X$ is a closed map.*

Proof:
$\langle 1 \rangle 1.$ Let: $A \subseteq X \times Y$ be closed.
$\langle 1 \rangle 2.$ Let: $x \in X \setminus \pi_1(A)$

⟨1⟩3. PICK a neighbourhood U of x such that $U \times Y \subseteq (X \times Y) \setminus A$
PROOF: By the Tube Lemma.
⟨1⟩4. $x \in U \subseteq X \setminus \pi_1(A)$
⟨1⟩5. Q.E.D.
PROOF: So $X \setminus \pi_1(A)$ is open by Lemma 11.1.8.
□

Proposition 11.50.14. *Let X be a topological space and Y a Hausdorff space. Let $f : X \to Y$ be continuous. Then the graph of f is closed in $X \times Y$.*

⟨1⟩1. ASSUME: f is continuous.
⟨1⟩2. LET: $(x, y) \in (X \times Y) \setminus G_f$
⟨1⟩3. PICK disjoint neighbourhoods U and V of y and $f(x)$ respectively.
⟨1⟩4. $f^{-1}(V) \times U$ is a neighbourhood of (x, y) disjoint from G_f.
□

Theorem 11.50.15. *Let X be a topological space and Y a compact space. Let $f : X \to Y$ be a function. If the granh of f is closed in $X \times Y$ then f is continuous.*

PROOF:
⟨1⟩1. ASSUME: G_f is closed.
⟨1⟩2. LET: $x \in X$ and V be a neighbourhood of $f(x)$.
⟨1⟩3. $G_f \cap (X \times (Y \setminus V))$ is closed.
⟨1⟩4. $\pi_1(G_f \cap (X \times (Y \setminus V)))$ is closed.
PROOF: Proposition 11.50.13.
⟨1⟩5. LET: $U = X \setminus \pi_1(G_f \cap (X \times (Y \setminus V)))$
⟨1⟩6. U is a neighbourhood of x
⟨1⟩7. $f(U) \subseteq V$
□

Theorem 11.50.16. *Let X be a compact topological space. Let $(f_n : X \to \mathbb{R})$ be a monotone increasing sequence of continuous functions and $f : X \to \mathbb{R}$ a continuous function. If (f_n) converges pointwise to f, then (f_n) converges uniformly to f.*

PROOF:
⟨1⟩1. LET: $\epsilon > 0$
⟨1⟩2. For all $x \in X$, there exists N such that, for all $n \geq N$, we have $|f_n(x) - f(x)| < \epsilon$
⟨1⟩3. For $n \geq 1$,
LET: $U_n = \{x \in X \mid |f_n(x) - f(x)| < \epsilon\}$
⟨1⟩4. For $n \geq 1$, we have U_n is open in X.
⟨2⟩1. LET: $x \in X$
⟨2⟩2. LET: $\delta = \epsilon - |f_n(x) - f(x)|$
⟨2⟩3. PICK a neighbourhood U of x such that $f(U) \subseteq B(f(x), \delta/2)$
⟨2⟩4. PICK a neighbourhood V of x such that $f_n(V) \subseteq B(f_n(x), \delta/2)$
⟨2⟩5. $f(U \cap V) \subseteq U_n$

PROOF: For $y \in U \cap V$ we have

$$\begin{aligned}|f_n(y) - f(y)| &\leq |f_n(y) - f_n(x)| + |f_n(x) - f(x)| + |f(x) - y(y)| \\ &< \delta/2 + |f_n(x) - f(x)| + \delta/2 \\ &= \epsilon\end{aligned}$$

⟨1⟩5. $\{U_n \mid n \geq 1\}$ covers X

PROOF: From ⟨1⟩2

⟨1⟩6. PICK N such that $X = U_N$

⟨2⟩1. PICK $n_1, \ldots, n_k$ such that $U_{n_1}, \ldots, U_{n_k}$ cover X.

⟨2⟩2. LET: $N = \max(n_1, \ldots, n_k)$

⟨2⟩3. For all i we have $U_{n_i} \subseteq U_N$

PROOF: Since (f_n) is monotone increasing.

⟨2⟩4. $X = U_N$

⟨1⟩7. For all $x \in X$ and $n \geq N$ we have $|f_n(x) - f(x)| < \epsilon$

□

An example to show that we cannot remove the hypothesis that X is compact:

Example 11.50.17. Let $X = (0, 1)$, $f_n(x) = -x^n$ and $f(x) = 0$ for $x \in X$ and $n \geq 1$. Then $f_n \to f$ pointwise and (f_n) is monotone increasing but the convergence is not uniform since, for all $N \geq 1$, there exists $x \in (0, 1)$ such that $-x^N < -1/2$.

An example to show that we cannot remove the hypothesis that (f_n) is monotone increasing:

Example 11.50.18. Let $X = [0, 1]$, $f_n(x) = 1/(n^3(x-1/n)^2+1)$ and $f(x) = 0$ for $x \in X$ and $n \geq 1$. Then X is compact and $f_n \to f$ pointwise but the convergence is not uniform since, for all $N \geq 1$, there exists $x \in [0, 1]$ such that $f_N(x) = 1$, namely $x = 1/N$.

Theorem 11.50.19. *Let X be a compact Hausdorff space. Let $\mathcal{A}$ be a chain of closed connected subsets of X. Then $\bigcap \mathcal{A}$ is connected.*

PROOF:

⟨1⟩1. ASSUME: for a contradiction C and D form a separation of $\bigcap \mathcal{A}$.

⟨1⟩2. PICK disjoint open sets U and V that include C and D respectively.

PROOF: Proposition 11.50.10.

⟨1⟩3. $\{A \setminus (U \cup V) \mid A \in \mathcal{A}\}$ is a set of closed sets with the finite intersection property.

⟨2⟩1. For all $A \in \mathcal{A}$ we have $A \setminus (U \cup V)$ is closed.

⟨2⟩2. For all $A_1, \ldots, A_n \in \mathcal{A}$ we have $(A_1 \cap \cdots \cap A_n) \setminus (U \cup V)$ is nonempty.

PROOF:

⟨3⟩1. LET: $A_1, \ldots, A_n \in \mathcal{A}$

⟨3⟩2. ASSUME: without loss of generality $A_1 \subseteq A_2, \ldots, A_n$

PROOF: Since $\mathcal{A}$ is a chain.

⟨3⟩3. $A_1 \setminus (U \cup V)$ is nonempty

PROOF: Otherwise $(A_1 \cap \cdots \cap A_n \cap U)$ and $(A_1 \cap \cdots \cap A_n \cap V)$ would form a separation of A_n.

⟨1⟩4. $\bigcap \mathcal{A} \setminus (U \cup V)$ is nonempty.

PROOF: Theorem 11.50.6.

⟨1⟩5. Q.E.D.

PROOF: This contradicts ⟨1⟩1 since $\bigcap AA \setminus (U \cup V) = \bigcap \mathcal{A} \setminus (C \cup D)$.

□

Theorem 11.50.20 (Tychonoff Theorem (AC)). *The product of a family of compact spaces is compact.*

PROOF:

⟨1⟩1. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of compact spaces.

⟨1⟩2. LET: $X = \prod_{\alpha \in J} X_\alpha$

⟨1⟩3. For any $\mathcal{A} \subseteq \mathcal{P}X$, we have $\bigcap_{A \in \mathcal{A}} \overline{A} \neq \emptyset$

⟨2⟩1. LET: $\mathcal{A} \subseteq \mathcal{P}X$

⟨2⟩2. PICK $\mathcal{D} \supseteq \mathcal{A}$ that is maximal with respect to the finite intersection property.

PROVE: $\bigcap_{D \in \mathcal{D}} \overline{D} \neq \emptyset$

PROOF: Lemma 5.7.7.

⟨2⟩3. For $\alpha \in J$, PICK $x_\alpha \in X_\alpha$ such that $x_\alpha \in \bigcap_{D \in \mathcal{D}} \overline{\pi_\alpha(D)}$

PROOF: Theorem 11.50.6 since $\{\overline{\pi_\alpha(D)} \mid D \in \mathcal{D}\}$ is a set of closed sets in X_α with the finite intersection property.

⟨2⟩4. LET: $x = (x_\alpha)_{\alpha \in J}$

PROVE: $x \in \bigcap_{D \in \mathcal{D}} \overline{D}$

⟨2⟩5. For any $\beta \in J$ and neighbourhood U of x_β in X_β, we have ${\pi_\beta}^{-1}(U)$ intersects every element of $\mathcal{D}$

⟨3⟩1. LET: $\beta \in J$

⟨3⟩2. LET: U be a neighbourhood of x_β in X_β.

⟨3⟩3. LET: $D \in \mathcal{D}$

⟨3⟩4. $x_\beta \in \overline{\pi_\beta(D)}$

PROOF: From ⟨2⟩3

⟨3⟩5. U intersects $\pi_\beta(D)$.

⟨3⟩6. ${\pi_\beta}^{-1}(U)$ intersects D.

⟨2⟩6. For any $\beta \in J$ and neighbourhood U of x_β in X_β, we have ${\pi_\beta}^{-1}(U) \in \mathcal{D}$

PROOF: Lemma 10.1.3.

⟨2⟩7. Every basic neighbourhood of x is an element of $\mathcal{D}$

PROOF: Lemma 10.1.2.

⟨2⟩8. Every basic neighbourhood of x intersects every element of $\mathcal{D}$

PROOF: Since $\mathcal{D}$ satisfies the finite intersection property.

⟨2⟩9. For all $D \in \mathcal{D}$ we have $x \in \overline{D}$

⟨1⟩4. Q.E.D.

PROOF: Theorem 11.50.6.

□

Lemma 11.50.21. *Let X and Y be topological spaces. Let $\mathcal{A}$ be a set of basis elements for the product topology on $X \times Y$ such that no finite subset of $\mathcal{A}$ covers*

$X \times Y$. If X is compact, then there exists $x \in X$ such that no finite subset of $\mathcal{A}$ covers the slice $\{x\} \times Y$.

PROOF:
⟨1⟩1. ASSUME: for every $x \in X$, there exists a finite subset of $\mathcal{A}$ that covers $\{x\} \times Y$
PROVE: A finite subset of $\mathcal{A}$ covers $X \times Y$
⟨1⟩2. $\{U \mid \exists U_1 \times V_1, \ldots, U_n \times V_n \in \mathcal{A}. U = U_1 \cap \cdots \cap U_n, V_1 \cup \cdots \cup V_n = Y\}$ covers X
⟨1⟩3. PICK a finite subcover $U_1, \ldots, U_m$
⟨1⟩4. PICK $U_{ij} \times V_{ij} \in \mathcal{A}$ such that, for every i, we have $U_i = \bigcap_j U_{ij}$ and $Y = \bigcup_j V_{ij}$
⟨1⟩5. The collection of all $U_{ij} \times V_{ij}$ covers $X \times Y$
□

Theorem 11.50.22 (AC). *Let X be a compact Hausdorff space. Then the quasicomponents and the components of X are the same.*

PROOF:
⟨1⟩1. LET: $x, y \in X$
⟨1⟩2. ASSUME: x and y are in the same quasicomponent.
PROVE: x and y are in the same component.
⟨1⟩3. LET: $\mathcal{A}$ be the set of all closed subsets A of X such that x and y are in the same quasicomponent of A.
⟨1⟩4. For every chain $\mathcal{B} \subseteq \mathcal{A}$ we have $\bigcap \mathcal{B} \in \mathcal{A}$
⟨2⟩1. LET: $BB \subseteq \mathcal{A}$ be a chain.
⟨2⟩2. ASSUME: for a contradiction U and V form a separation of $\bigcap \mathcal{B}$ with $x \in U$ and $y \in V$
⟨2⟩3. PICK disjoint open sets U', V' in X such that $U \subseteq U'$ and $V \subseteq V'$
⟨2⟩4. $\{B \setminus (U' \cup V') \mid B \in \mathcal{B}\}$ satisfies the finite intersection property.
⟨3⟩1. LET: $B_1, \ldots, B_n \in \mathcal{B}$
⟨3⟩2. ASSUME: without loss of generality $B_1 \subseteq \cdots \subseteq B_n$
PROOF: Since $\mathcal{B}$ is a chain.
⟨3⟩3. $\bigcap \{B_1 \setminus (U' \cup V'), \ldots, B_n \setminus (U' \cup V')\} = B_1 \setminus (U' \cup V')$
⟨3⟩4. $B_1 \setminus (U' \cup V')$ is nonempty
PROOF: Otherwise $B_1 \cap U'$ and $B_1 \cap V'$ would form a separation of B_1, contradicting the fact that x and y are in the same quasicomponent of B_1.
⟨2⟩5. $\bigcap \mathcal{B} \setminus (U \cup V)$ is nonempty
PROOF: Theorem 11.50.6.
⟨2⟩6. Q.E.D.
PROOF: This contradicts ⟨2⟩2.
⟨1⟩5. PICK a minimal element D in $\mathcal{A}$.
PROVE: D is connected.
PROOF: By Zorn's Lemma.
⟨1⟩6. ASSUME: for a contradiction U and V form a separation of D.
⟨1⟩7. ASSUME: without loss of generality $x, y \in U$

PROOF: We cannot have that one of x, y is in U and the other in V sicnce $D \in \mathcal{A}$.

$\langle 1 \rangle 8.$ $U \in \mathcal{A}$

PROOF: If X and Y form a separation of U with $x \in X$ and $y \in Y$, then X and $Y \cup V$ form a separation of D with $x \in X$ and $y \in Y \cup V$.

$\langle 1 \rangle 9.$ Q.E.D.

PROOF: There is a connected set D that contains both x and y.

$\square$

PROOF:

$\langle 1 \rangle 1.$ LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of compact spaces.

$\langle 1 \rangle 2.$ LET: $X = \prod_{\alpha \in J} X_\alpha$

$\langle 1 \rangle 3.$ PICK a well-ordering $<$ on J such that J has a greatest element.

$\langle 1 \rangle 4.$ For $\alpha \in J$ and $p = \{p_i \in X_i\}_{i \leq \alpha}$ a family of points,
LET: $Y(p) = \{x \in X \mid \forall i \leq \alpha . x_i = p_i\}$

$\langle 1 \rangle 5.$ If $\alpha < \alpha'$ and p is an α'-indexed family of points then $Y(p) \subseteq Y(p \restriction \alpha)$

PROOF: From definition.

$\langle 1 \rangle 6.$ Given $\beta \in J$ and $p = \{p_i \in X_i\}_{i < \beta}$ a family of points,
LET: $Z(p) = \bigcap_{\alpha < \beta} Y(p \restriction \alpha)$

$\langle 1 \rangle 7.$ Given $\beta \in J$ and $p = \{p_i \in X_i\}_{i < \beta}$ a family of points, if $\mathcal{A}$ is a finite set of basic open spaces for X that covers $Z(p)$, then there exists $\alpha < \beta$ such that $\mathcal{A}$ covers $Y(p \restriction \alpha)$

$\langle 2 \rangle 1.$ ASSUME: without loss of generality β has no immediate predecessor.

$\langle 2 \rangle 2.$ For $A \in \mathcal{A}$,
LET: $J_A = \{i < \beta \mid \pi_i(A) \neq X_i\}$

$\langle 2 \rangle 3.$ LET: $\alpha = \max \bigcup_{A \in \mathcal{A}} J_A$

$\langle 2 \rangle 4.$ LET: $x \in Y(p \restriction \alpha)$

$\langle 2 \rangle 5.$ LET: $y \in Z(p)$ be the point with $y_i = p_i$ for $i < \beta$ and $y_i = x_i$ for $i \geq \beta$

$\langle 2 \rangle 6.$ PICK $A \in \mathcal{A}$ such that $y \in A$

PROOF: Since $\mathcal{A}$ covers $Z(p)$.

$\langle 2 \rangle 7.$ For $i \in J_A$ we have $x_i \in \pi_i(A)$

PROOF: Since $i \leq \alpha$ so $x_i = p_i$

$\langle 2 \rangle 8.$ For $i \in J \setminus J_A$ we have $x_i \in \pi_i(A)$

PROOF: Since $\pi_i(A) = X_i$

$\langle 2 \rangle 9.$ $x \in A$

$\langle 1 \rangle 8.$ ASSUME: for a contraction $\mathcal{A}$ is a set of basic open sets for X that covers X but such that no finite subset of $\mathcal{A}$ covers X

$\langle 1 \rangle 9.$ PICK a set of points $\{p_i\}_{i \in J}$ such that, for all $\alpha \in J$, we have $Y(p \restriction \alpha)$ is not finitely covered by $\mathcal{A}$

$\langle 2 \rangle 1.$ ASSUME: as transfinite induction hypothesis $\alpha \in J$ and $\{p_i\}_{i < \alpha}$ is a family of points such that, for all $\alpha' < \alpha$, we have $Y(p \restriction \alpha')$ is not finitely covered by $\mathcal{A}$

$\langle 2 \rangle 2.$ $Z(p)$ is not finitely covered by $\mathcal{A}$

PROOF: By $\langle 1 \rangle 7$.

$\langle 2 \rangle 3.$ PICK $p_\alpha \in X_\alpha$ such that $Y(p)$ is not finitely covered by $\mathcal{A}$

PROOF: By Lemma 11.50.21 since there is a homeomorphism $\phi : Z(p) \cong$

$X_\alpha \times \prod_{\alpha' > \alpha} X_{\alpha'}$ and, given p_α, this homemorphism ϕ restricts to a homeomorphism $Y(p) \cong \{p_\alpha\} \times \prod_{\alpha' > \alpha} X_{\alpha'}$.

⟨1⟩10. Q.E.D.

PROOF: If ω is the greatest element of J then $Y(p \restriction \omega)$ is a singleton.

□

Theorem 11.50.23. *Every complete linearly ordered set in the order topology is compact.*

PROOF:

⟨1⟩1. LET: X be a complete linearly ordered set with least element a and greatest element b.

⟨1⟩2. LET: $\mathcal{A}$ be an open covering of X.

⟨1⟩3. For all $x < b$, there exists $y > x$ such that $[x, y]$ can be covered by at most two elements of $\mathcal{A}$.

⟨2⟩1. LET: $x \in X$

⟨2⟩2. PICK $A \in \mathcal{A}$ with $x \in A$

⟨2⟩3. PICK $y > x$ such that $[x, y) \subseteq A$

⟨2⟩4. PICK $B \in \mathcal{A}$ with $y \in B$

⟨2⟩5. $[x, y]$ is covered by A and B

⟨1⟩4. LET: $C = \{y \in X \mid [a, y]$ can be covered by finitely many elements of $\mathcal{A}\}$

⟨1⟩5. LET: $c = \sup C$

⟨1⟩6. $c > a$

⟨2⟩1. PICK $x > a$ such that $[a, x]$ can be covered by at most two elements of $\mathcal{A}$.

PROOF: From ⟨1⟩3.

⟨2⟩2. $x \in C$

⟨1⟩7. $c \in C$

⟨2⟩1. PICK $A \in \mathcal{A}$

⟨2⟩2. PICK $x < c$ such that $(x, c] \subseteq A$

⟨2⟩3. PICK $y > x$ such that $y \in C$

⟨2⟩4. PICK $\mathcal{A}_0 \subseteq^{\text{fin}} \mathcal{A}$ that covers $[a, y]$

⟨2⟩5. $\mathcal{A}_0 \cup \{A\}$ covers $[a, c]$

⟨1⟩8. $c = b$

⟨2⟩1. ASSUME: for a contradiction $c < b$

⟨2⟩2. PICK $x > c$ such that $[c, x]$ can be covered by at most two elements of $\mathcal{A}$

PROOF: From ⟨1⟩3.

⟨2⟩3. $[a, x]$ can be finitely covered by $\mathcal{A}$

PROOF: From ⟨1⟩7.

⟨2⟩4. Q.E.D.

PROOF: This contradicts the maximality of c.

□

Corollary 11.50.23.1. *Let X be a linearly ordered set with the least upper bound property. Then every closed interval in X is compact.*

Corollary 11.50.23.2. *Every closed interval in $\mathbb{R}$ is compact.*

Theorem 11.50.24 (Extreme Value Theorem)**.** *Any linearly ordered set under the order topology that is compact has a greatest and a least element.*

Proof:
⟨1⟩1. Let: X be a linearly ordered set under the order topology that is compact.
⟨1⟩2. X has a greatest element.
⟨2⟩1. Assume: for a contradiction X has no greatest element.
⟨2⟩2. $\{(-\infty, a) \mid a \in X\}$ covers X.
⟨2⟩3. Pick a finite subcover $\{(-\infty, a_1), \ldots, (-\infty, a_n)\}$, say.
⟨2⟩4. Assume: without loss of generality $a_1 \leq \cdots \leq a_n$
⟨2⟩5. $X \subseteq (-\infty, a_n)$
⟨2⟩6. $a_n < a_n$
⟨1⟩3. X has a least element.
Proof: Similar.
□

Proposition 11.50.25. *Every linearly ordered set in which every closed interval is compact satisfies the least upper bound property.*

Proof:
⟨1⟩1. Let: X be a linearly ordered set in which every closed interval is compact.
⟨1⟩2. Let: $A \subseteq X$ be nonempty with upper bound u
⟨1⟩3. Pick $a \in A$
⟨1⟩4. The closed interval $[a, u]$ is compact.
⟨1⟩5. Assume: for a contradiction A has no supremum.
⟨1⟩6. $\{(-\infty, x) \mid x \in A\} \cup \{(x, +\infty) \mid x \text{ is an upper bound of } A\}$ covers $[a, u]$.
⟨2⟩1. Let: $x \in [a, u]$
⟨2⟩2. Assume: for all $y \in A$ we have $x \notin (-\infty, y)$
⟨2⟩3. x is an upper bound for A
⟨2⟩4. Pick an upper bound y for A with $y < x$
⟨2⟩5. $x \in (y, +\infty)$
⟨1⟩7. Pick a finite subcover $\{(-\infty, x_1), \ldots, (-\infty, x_m), (y_1, +\infty), \ldots, (y_n, +\infty)\}$
⟨1⟩8. Assume: $x_m = \max(x_1, \ldots, x_m)$ and $y_1 = \min(y_1, \ldots, y_n)$
⟨1⟩9. $x_m \notin (-\infty, x_i)$ for any i
Proof: Since $x_i \leq x_m$
⟨1⟩10. $x_m \notin (y_i, +\infty)$ for any i
Proof: Since $x_m \in A$ so $x_m \leq y_i$
⟨1⟩11. $x_m \in [a, u]$
⟨2⟩1. $a \notin (y_i, +\infty)$ for any i
Proof: Since y_i is an upper bound for A and $a \in A$.
⟨2⟩2. $a \in (-\infty, x_i)$ for some i
Proof: From ⟨1⟩7.
⟨2⟩3. $a < x_m$
Proof: Since $x_i \leq x_m$
⟨2⟩4. $x_m \leq u$
Proof: Since u is an upper bound for A and $x_m \in A$.
⟨1⟩12. Q.E.D.

PROOF: This contradicts ⟨1⟩7.
∎

Example 11.50.26. The set $[0,1]$ is not compact under the K-topology.

PROOF: For every $n \geq 1$, pick an open interval U_n such that $U_n \cap K = \{1/n\}$. Then the open cover $\{[0,1] - K\} \cup \{U_n \mid n \in \mathbb{Z}^+\}$ has no finite subcover. ∎

Proposition 11.50.27 (AC)**.** *Let X be a compact Hausdorff space. Let $\mathcal{A}$ be a countable set of closed sets in X. If every element of $\mathcal{A}$ has empty interior, then $\bigcup \mathcal{A}$ has empty interior.*

PROOF:
⟨1⟩1. LET: X be a compact Hausdorff space.
⟨1⟩2. For every closed set A in X and open U in X with $U \nsubseteq A$, there exists a nonempty open set V such that $\overline{V} \subseteq U - A$.
⟨2⟩1. LET: A be a closed set in X
⟨2⟩2. LET: U be an open set in X with $U \nsubseteq A$
⟨2⟩3. PICK $x \in U - A$
⟨2⟩4. PICK disjoint neighbourhoods W and V of $A \cup (X - U)$ and x respectively.
PROOF: Proposition 11.50.10.
⟨2⟩5. $\overline{V} \subseteq U - A$
PROOF:

$$\begin{aligned}
\overline{V} &\subseteq X - W && \text{(since } V \subseteq X - W\text{)} \\
&\subseteq X - (A \cup (X - U)) \\
&= (x - A) \cap U \\
&= U - A
\end{aligned}$$

⟨1⟩3. PICK an enumeration $\{A_1, A_2, \ldots\}$ of $\mathcal{A}$
⟨1⟩4. LET: U_0 be any nonempty open set
PROVE: $U_0 \nsubseteq \bigcup \mathcal{A}$
⟨1⟩5. PICK a sequence of nonempty open sets U_1, U_2, ... such that, for $n \geq 1$, we have $\overline{U_n} \subseteq U_{n-1} - A_n$
⟨2⟩1. ASSUME: we have picked U_0, U_1, ..., U_n
⟨2⟩2. $U_n \nsubseteq A_{n+1}$
PROOF: Since A_{n+1} has empty interior.
⟨2⟩3. PICK a nonempty open set U_{n+1} such that $\overline{U_{n+1}} \subseteq U_n - A_{n+1}$
PROOF: By ⟨1⟩2
⟨1⟩6. PICK $a \in \bigcap_{n=0}^{\infty} \overline{U_n}$
PROOF: Corollary 11.50.6.1.
⟨1⟩7. $a \in U_0$
PROOF: Since $a \in \overline{U_1} \subseteq U_0$.
⟨1⟩8. $a \notin \bigcup \mathcal{A}$
PROOF: For all n, we have $a \in \overline{U_n} \subseteq U_{n-1} - A_n$.
∎

Example 11.50.28. The Cantor set is compact.

Proof: It is a closed subset of the compact set $[0,1]$. ∎

Proposition 11.50.29. *Every compact space is limit point compact.*

Proof:
$\langle 1 \rangle 1$. Let: X be a compact space.
$\langle 1 \rangle 2$. Let: $A \subseteq X$ have no limit points.
Prove: A is finite.
$\langle 1 \rangle 3$. A is closed.
Proof: Corollary 11.6.3.1.
$\langle 1 \rangle 4$. A is compact.
Proof: Proposition 11.50.3.
$\langle 1 \rangle 5$. $\{U \mid U \text{ open }, |U \cap A| = 1\}$ covers A.
Proof: From $\langle 1 \rangle 2$, for all $a \in A$, there is a neighbourhood U of a that intersects A in a only.
$\langle 1 \rangle 6$. Pick a finite subcover $\{U_1, \ldots, U_n\}$
$\langle 1 \rangle 7$. For $i = 1, \ldots, n$,
Let: $U_i \cap A = \{x_i\}$.
$\langle 1 \rangle 8$. $A = \{x_1, \ldots, x_n\}$
∎

The following examples show that not every limit point compact space is compact.

Example 11.50.30. Let Y be a set with two elements under the indiscrete topology. Then $\mathbb{Z}^+ \times Y$ is limit point compact, since every nonempty set has a limit point. It is not compact, since $\{\{n\} \times Y \mid n \in \mathbb{Z}^+\}$ has no finite subcover.

Example 11.50.31. The space S_Ω is limit point compact but not compact.

Proof:
$\langle 1 \rangle 1$. S_Ω is not compact.
Proof: From the Extreme Value Theorem, since S_Ω has no greatest element.
$\langle 1 \rangle 2$. Let: A be an infinite subset of S_Ω.
$\langle 1 \rangle 3$. Pick $B \subseteq A$ that is countably infinite.
Proof: Proposition **??**.
$\langle 1 \rangle 4$. Let: $b = \sup B$
$\langle 1 \rangle 5$. $B \subseteq [0, b]$
$\langle 1 \rangle 6$. $[0, b]$ is compact.
Proof: Corollary 11.50.23.1.
$\langle 1 \rangle 7$. Pick a limit point x of B in $[0, b]$.
Proof: Proposition 11.50.29.
$\langle 1 \rangle 8$. x is a limit point of A.
Proof: Lemma 11.6.5.
∎

Proposition 11.50.32 (AC)**.** *A topological space is compact if and only if every net has a convergent subnet.*

Proof:

⟨1⟩1. LET: X be a topological space.
⟨1⟩2. If X is compact then every net has a convergent subnet.
⟨2⟩1. ASSUME: X is compact.
⟨2⟩2. LET: $(a_\alpha)_{\alpha \in J}$ be a net in X.
⟨2⟩3. For $\alpha \in J$,
LET: $B_\alpha = \{a_\beta \mid \alpha \leq \beta\}$
⟨2⟩4. $\{B_\alpha \mid \alpha \in J\}$ has the finite intersection property.
⟨2⟩5. PICK $x \in \bigcap_{\alpha \in J} \overline{B_\alpha}$
⟨2⟩6. x is an accumulation point of $(a_\alpha)_{\alpha \in J}$
⟨3⟩1. LET: U be a neighbourhood of x.
⟨3⟩2. LET: $\alpha \in J$
⟨3⟩3. $x \in \overline{B_\alpha}$
⟨3⟩4. There exists $\beta \geq \alpha$ such that $a_\beta \in U$
⟨2⟩7. Q.E.D.
PROOF: Lemma 11.49.2.
⟨1⟩3. If every net in X has a convergent subnet then X is compact.
⟨2⟩1. ASSUME: Every net in X has a convergent subnet.
⟨2⟩2. LET: $\mathcal{A}$ be a set of closed sets with the finite intersection property.
⟨2⟩3. LET: $\mathcal{B}$ be the set of all finite intersections of elements of $\mathcal{A}$ under $\supseteq$.
⟨2⟩4. For $B \in \mathcal{B}$, PICK $a_B \in B$
⟨2⟩5. PICK a convergent subnet $(a_{g(\alpha)})_{\alpha \in K}$ with limit l.
PROVE: $l \in \bigcap \mathcal{A}$
PROOF: From ⟨2⟩1.
⟨2⟩6. LET: $A \in \mathcal{A}$
⟨2⟩7. ASSUME: for a contradiction $l \notin A$
⟨2⟩8. PICK $\alpha \in K$ such that, for all $\beta \geq \alpha$, we have $a_{g(\beta)} \in X - A$
⟨2⟩9. PICK $\beta \in K$ such that $g(\beta) \geq A$
⟨2⟩10. PICK $\gamma \in K$ with $\alpha \leq \gamma$ and $\beta \leq \gamma$
⟨2⟩11. $a_{g(\gamma)} \in A$ and $a_{g(\gamma)} \in X - A$
⟨2⟩12. Q.E.D.
PROOF: This is a contradiction.
□

Example 11.50.33. The space $\mathbb{R}_K$ is not path connected.

PROOF:
⟨1⟩1. ASSUME: for a contradiction $p : [0, 1] \to \mathbb{R}_K$ was a path from 0 to 1.
⟨1⟩2. $p([0, 1])$ as a subspace of $\mathbb{R}_K$ is compact.
PROOF: Theorem 11.50.4.
⟨1⟩3. $p([0, 1])$ as a subspace of $\mathbb{R}_K$ is connected.
PROOF: Theorem 11.30.13.
⟨1⟩4. $p([0, 1])$ is connected as a subspace of $\mathbb{R}$.
PROOF: Theorem 11.30.13 as the identity map is continuous as a map $\mathbb{R}_K \to \mathbb{R}$.
⟨1⟩5. $p([0, 1])$ is convex.
⟨2⟩1. LET: $a, b \in p([0, 1])$ and $a < c < b$
⟨2⟩2. ASSUME: for a contradiction $c \notin p([0, 1])$

⟨2⟩3. $(-\infty, c) \cap p([0,1])$ and $(c, +\infty) \cap p([0,1])$ form a separation of $p([0,1])$ as a subspace of $\mathbb{R}$.

⟨2⟩4. Q.E.D.

PROOF: This contradicts ⟨1⟩4.

⟨1⟩6. $[0,1] \subseteq p([0,1])$

⟨1⟩7. $[0,1]$ as a subspace of $\mathbb{R}_K$ is compact.

PROOF: By Proposition 11.50.3 and ⟨1⟩2.

⟨1⟩8. Q.E.D.

PROOF: This contradicts Example 11.50.26.

□

Proposition 11.50.34 (Choice)**.** *The product of a Lindelöf and a compact space is Lindelöf.*

PROOF:

⟨1⟩1. LET: X be a Lindelöf space.

⟨1⟩2. LET: Y be a compact space.

⟨1⟩3. LET: $\mathcal{A}$ be an open cover of $X \times Y$.

⟨1⟩4. $\{W \text{ open in } X \mid W \times Y \text{ can be covered by finitely many elements of } \mathcal{A}\}$ is an open cover of X.

⟨2⟩1. LET: $x \in X$

⟨2⟩2. $\{V \text{ open in } Y \mid \exists U \text{ open in } X . \exists A \in \mathcal{A} . x \in U \text{ and } U \times V \subseteq A\}$ covers Y

⟨2⟩3. PICK a finite subcover $V_1, \ldots, V_n$

⟨2⟩4. $\{x\} \times Y \subseteq V_1 \cup \cdots \cup V_n$

⟨2⟩5. PICK a neighbourhood W of x such that $W \times Y \subseteq V_1 \cup \cdots \cup V_n$

PROOF: By the Tube Lemma.

⟨1⟩5. PICK a countable subcover $\mathcal{B}$

⟨1⟩6. For $W \in \mathcal{B}$, PICK $U_{W1}, \ldots, U_{Wn_W} \in \mathcal{A}$ such that $W \times Y \subseteq U_{W1} \cup \cdots \cup U_{Wn_W}$

⟨1⟩7. $\{U_{Wi} \mid W \in \mathcal{B}, 1 \leq i \leq n_W\}$ covers $X \times Y$

□

11.51 Perfect Maps

Definition 11.51.1 (Perfect Map)**.** Let X and Y be topological spaces and $f : X \to Y$. Then f is a *perfect map* if and only if f is a closed map, continuous, surjective and $p^{-1}(y)$ is compact for all $y \in Y$.

Proposition 11.51.2. *Let X be a topological space, Y a compact space, and $p : X \to Y$ a closed map such that, for all $y \in Y$, we have $p^{-1}(y)$ is compact. Then X is compact.*

PROOF:

⟨1⟩1. LET: $\mathcal{A}$ be a set of closed sets in X with the finite intersection property.

⟨1⟩2. $\mathcal{B} = \{p(A_1 \cap \cdots \cap A_n) \mid A_1, \ldots, A_n \in \mathcal{A}\}$ is a set of closed sets in Y with the finite intersection property.

PROOF: Since p is a closed map.
⟨1⟩3. PICK $y \in \bigcap \mathcal{B}$
PROOF: Theorem 11.50.6 since Y is compact.
⟨1⟩4. $\{A \cap p^{-1}(y) \mid A \in \mathcal{A}\}$ is a set of closed sets in $p^{-1}(y)$ with the finite intersection property.
⟨1⟩5. PICK $x \in \bigcap_{A \in \mathcal{A}}(A \cap p^{-1}(y))$
PROOF: Theorem 11.50.6 since $p^{-1}(y)$ is compact.
⟨1⟩6. $x \in \bigcap \mathcal{A}$
⟨1⟩7. Q.E.D.
PROOF: Theorem 11.50.6.
□

11.52 Isolated Points

Definition 11.52.1 (Isolated Point)**.** Let X be a topolgical space and $x \in X$. Then x is an *isolated point* if and only if $\{x\}$ is open.

Theorem 11.52.2 (AC)**.** *A nonempty compact Hausdorff space with no isolated points is uncountable.*

PROOF:
⟨1⟩1. LET: X be a nonempty compact Hausdorff space with no isolated points.
⟨1⟩2. For every nonempty open set U and every point $x \in X$, there exists a nonempty open set $V \subseteq U$ such that $x \notin \overline{V}$.
⟨2⟩1. LET: U be a nonempty open set.
⟨2⟩2. LET: $x \in X$
⟨2⟩3. PICK $y \in U - \{x\}$
PROOF: This is possible because U cannot be $\{x\}$.
⟨2⟩4. PICK disjoint open neighbourhoods W_1 of x and W_2 of y
⟨2⟩5. LET: $V = W_2 \cap U$
⟨2⟩6. V is nonempty
PROOF: Since $y \in V$
⟨2⟩7. V is open
PROOF: From ⟨2⟩1, ⟨2⟩4, ⟨2⟩5.
⟨2⟩8. $V \subseteq U$
PROOF: From ⟨2⟩5
⟨2⟩9. $x \notin V$
PROOF: From ⟨2⟩4 and ⟨2⟩5
⟨1⟩3. LET: (a_n) be any sequence of points in X.
PROVE: The set $X - \{a_1, a_2, \ldots\}$ is nonempty.
⟨1⟩4. PICK a sequence of nonempty open sets $V_1, V_2, \ldots,$ such that $V_1 \supseteq V_2 \supseteq \cdots$ and $a_n \notin \overline{V_n}$ for all n.
PROOF: From ⟨1⟩2.
⟨1⟩5. PICK $a \in \bigcap_{n=1}^{\infty} \overline{V_n}$
PROOF: Corollary 11.50.6.1.
⟨1⟩6. $a \in X - \{a_1, a_2, \ldots\}$

PROOF: We cannot have $a = a_n$ because $a \in \overline{V_n}$.
∎

Corollary 11.52.2.1. *For all $a, b \in \mathbb{R}$ with $a < b$, the closed interval $[a, b]$ is uncountable.*

Example 11.52.3. The Cantor set has no isolated points, and is therefore uncountable.

PROOF:
⟨1⟩1. LET: (A_n) be the sets in Definition 9.1.1.
⟨1⟩2. LET: $x \in C$
⟨1⟩3. LET: A_n be the first set such that x is an endpoint of one of the intervals that make up A_n
⟨1⟩4. LET: $(a_m)_{m \geq n}$ be the sequence of points defined by: a_m is the point such that either $[a_m, x]$ or $[x, a_m]$ is one of the intervals that make up A_m.
⟨1⟩5. (a_m) is a sequence of points of C distinct from x that converges to x.
PROOF: Since $|a_m - x| = 1/3^m$ for all m.
⟨1⟩6. x is a limit point of C.
∎

11.53 Local Compactness

Definition 11.53.1 (Locally Compact)**.** Let X be a topological space and $x \in X$. Then X is *locally compact* at x if and only if there exists a compact subspace of X that includes a neighbourhood of x.

A space is *locally compact* if and only if it is locally compact at every point.

Example 11.53.2. The real line is locally compact, because for every real number x we have $x \in (x - 1, x + 1) \subseteq [x - 1, x + 1]$.

Example 11.53.3. For all $n \geq 1$, we have $\mathbb{R}^n$ is locally compact. For any point $x = (x_1, \ldots, x_n)$, we have $x \in (x_1 - 1, x_1 + 1) \times \cdots \times (x_n - 1, x_n + 1) \subseteq [x_1 - 1, x_1 + 1] \times \cdots \times [x_n - 1, x_n + 1]$.

The following example shows that a countable product of locally compact spaces is not necessarily locally compact.

Example 11.53.4. The space $\mathbb{R}^\omega$ is not locally compact.

PROOF:
⟨1⟩1. ASSUME: for a contradiction $0 \in U \subseteq C$ where U is open and C is compact.
⟨1⟩2. PICK a basic open set $B = (a_1, b_1) \times \cdots \times (a_n, b_n) \times \mathbb{R} \times \mathbb{R} \times \cdots$ such that $0 \in B \subseteq U$
⟨1⟩3. $\overline{B} = [a_1, b_1] \times \cdots \times [a_n, b_n] \times \mathbb{R} \times \mathbb{R} \times \cdots$ is compact.
PROOF:Proposition 11.50.3.

⟨1⟩4. Q.E.D.

Proof: This is a contradiction.

□

Example 11.53.5. Every linearly ordered set X with the least upper bound property is locally compact under the order topology.

For any point x, pick a basic open set B such that $x \in B$. Then $x \in B \subseteq \overline{B}$ and $\overline{B}$ is a closed interval, hence compact (Corollary 11.50.23.1).

Proposition 11.53.6. *Any closed subspace of a locally compact space is locally compact.*

Proof:

⟨1⟩1. Let: X be a locally compact space and $Y \subseteq X$ be closed.

⟨1⟩2. Let: $y \in Y$.

⟨1⟩3. Pick a compact subspace C of X and neighbourhood U of y in X such that $U \subseteq C$

⟨1⟩4. $y \in U \cap Y \subseteq C \cap Y$

⟨1⟩5. $C \cap Y$ is compact.

Proof:Proposition 11.50.3.

□

Proposition 11.53.7. *Let X be a Hausdorff space. Let $x \in X$. Then X is locally compact at x if and only if, for every neighbourhood U of x, there exists a neighbourhood V of x such that $\overline{V}$ is compact and $\overline{V} \subseteq U$.*

Corollary 11.53.7.1. *Every open subspace of a locally compact Hausdorff space is locally compact Hausdorff.*

This example shows that a subspace of a locally compact Hausdorff space is not necessarily locally compact Hausdorff.

Example 11.53.8. The rationals $\mathbb{Q}$ are not locally compact.

Assume for a contradiction $C \subseteq \mathbb{Q}$ is compact and includes $(-\epsilon, \epsilon) \cap \mathbb{Q}$. Pick an irrational $\xi \in (-\epsilon, \epsilon)$. Then $\{(-\infty, q) \cap \mathbb{Q} \mid q \in \mathbb{Q}, q < \xi\} \cup \{(q, +\infty) \cap \mathbb{Q} \mid q \in \mathbb{Q}, q > \xi\}$ covers C but no finite subcover does.

Proposition 11.53.9. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of nonempty spaces. If $\prod_{\alpha \in J} X_\alpha$ is locally compact under the box topology then each X_α is locally compact.*

Proof:

⟨1⟩1. Let: $\alpha \in J$

⟨1⟩2. Let: $x_\alpha \in X_\alpha$

⟨1⟩3. Extend x_α to a family $(x_\beta)_{\beta \in J} \in \prod_{\beta \in J} X_\beta$

⟨1⟩4. Pick a compact $C \subseteq \prod_{\beta \in J} X_\beta$ that includes a basic open neighbourhood $\prod_{\beta \in J} U_\beta$ of (x_β) such that each U_β is open in X_β.

⟨1⟩5. $x_\alpha \in U_\alpha \subseteq \pi_\alpha(C)$

⟨1⟩6. $\pi_\alpha(C)$ is compact.

Proof: Theorem 11.50.4.

□

Proposition 11.53.10 (AC)**.** *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of nonempty spaces. Then $\prod_{\alpha \in J} X_\alpha$ is locally compact if and only if each X_α is locally compact, and X_α is compact for all but finitely many $\alpha \in J$.*

PROOF:
⟨1⟩1. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of nonempty spaces.
⟨1⟩2. If $\prod_{\alpha \in J} X_\alpha$ is locally compact then each X_α is locally compact.
  ⟨2⟩1. ASSUME: $\prod_{\alpha \in J} X_\alpha$ is locally compact.
  ⟨2⟩2. For all $\alpha \in J$ we have X_α is locally compact.
    ⟨3⟩1. LET: $\alpha \in J$
    ⟨3⟩2. LET: $x_\alpha \in X_\alpha$
    ⟨3⟩3. Extend x_α to a family $(x_\beta)_{\beta \in J} \in \prod_{\beta \in J} X_\beta$
    ⟨3⟩4. PICK a compact $C \subseteq \prod_{\beta \in J} X_\beta$ that includes a basic open neighbourhood $\prod_{\beta \in J} U_\beta$ of (x_β) such that each U_β is open in X_β.
    ⟨3⟩5. $x_\alpha \in U_\alpha \subseteq \pi_\alpha(C)$
    ⟨3⟩6. $\pi_\alpha(C)$ is compact.
      PROOF: Theorem 11.50.4.
⟨1⟩3. If $\prod_{\alpha \in J} X_\alpha$ is locally compact then X_α is compact for all but finitely many $\alpha \in J$.
  ⟨2⟩1. ASSUME: $\prod_{\alpha \in J} X_\alpha$ is locally compact.
  ⟨2⟩2. PICK $x_\alpha \in X_\alpha$ for all α.
  ⟨2⟩3. PICK a compact $C \subseteq \prod_{\alpha \in J} X_\alpha$ that includes a basic open neighbourhood $\prod_{\alpha \in J} U_\alpha$ of (x_α) such that each U_α is open in X_α, and $U_\alpha = X_\alpha$ for all but finitely many α.
  ⟨2⟩4. For all but finitely many $\alpha \in J$, we have $X_\alpha = \pi_\alpha(C)$
  ⟨2⟩5. For all but finitely many $\alpha \in J$, we have X_α is compact.
    PROOF: Theorem 11.50.4.
⟨1⟩4. If each X_α is locally compact and X_α is compact for all but finitely many $\alpha \in J$ then $\prod_{\alpha \in J} X_\alpha$ is locally compact.
  ⟨2⟩1. ASSUME: X_α is compact for all α except $\alpha_1, \ldots, \alpha_n$
  ⟨2⟩2. ASSUME: $X_{\alpha_1}, \ldots, X_{\alpha_n}$ are locally compact.
  ⟨2⟩3. LET: $(x_\alpha) \in \prod X_\alpha$
  ⟨2⟩4. For $i = 1, \ldots, n$, PICK a compact $C_{\alpha_i} \subseteq X_{\alpha_i}$ that includes the neighbourhood U_{α_i} of x_{α_i}.
  ⟨2⟩5. For $\alpha \neq \alpha_1, \ldots, \alpha_n$,
    LET: $C_\alpha = U_\alpha = X_\alpha$
  ⟨2⟩6. $\prod_{\alpha \in J} C_\alpha$ is compact.
    PROOF: Tychonoff's Theorem.
  ⟨2⟩7. $(x_\alpha) \in \prod U_\alpha \subseteq \prod C_\alpha$

□

The following example shows that the continuous image of a locally compact space is not necessarily locally compact.

Example 11.53.11. Pick an enumeration $\{q_1, q_2, \ldots\}$ of $\mathbb{Q}$. Let $X = \bigcup_{n \in \mathbb{Z}^+} (n, n+1)$. Define $f : X \to \mathbb{Q}$ by $f(x) = q_n$ if $x \in (n, n+1)$. Then f is continuous, X

is locally compact, but $f(X) = \mathbb{Q}$ is not locally compact.

Proposition 11.53.12. *The image of a locally compact space under a continuous open map is locally compact.*

PROOF:
$\langle 1 \rangle 1$. LET: X be locally compact and $f : X \twoheadrightarrow Y$ be a surjective continuous open map.
$\langle 1 \rangle 2$. LET: $y \in Y$
$\langle 1 \rangle 3$. PICK $x \in X$ such that $f(x) = y$
$\langle 1 \rangle 4$. PICK a compact $C \subseteq X$ that includes a neighbourhood U of x
$\langle 1 \rangle 5$. $y \in f(U) \subseteq f(C)$ and $f(U)$ is open, $f(C)$ is compact.
□

Lemma 11.53.13. *Let X, Y and Z be topological spaces and $p : X \to Y$. If p is a quotient map and Z is locally compact Hausdorff, then $p \times \mathrm{id}_Z : X \times Z \to Y \times Z$ is a quotient map.*

PROOF:
$\langle 1 \rangle 1$. LET: X, Y and Z be topological spaces and $p : X \to Y$.
$\langle 1 \rangle 2$. ASSUME: p is a quotient map and Z is locally compact Hausdorff.
$\langle 1 \rangle 3$. LET: $\pi = p \times \mathrm{id}_Z$
$\langle 1 \rangle 4$. π is sujective.
$\langle 1 \rangle 5$. π is continuous.
$\langle 1 \rangle 6$. π is strongly continuous.
$\langle 2 \rangle 1$. LET: $A \subseteq Y \times Z$
$\langle 2 \rangle 2$. ASSUME: $\pi^{-1}(A)$ is open.
$\langle 2 \rangle 3$. LET: $(y, z) \in A$
$\langle 2 \rangle 4$. PICK $x \in X$ such that $p(x) = y$
$\langle 2 \rangle 5$. PICK open sets U_1 in X and V in Z such that $x \in U_1$, $z \in V$, $\overline{V}$ is compact, and $U_1 \times \overline{V} \subseteq \pi^{-1}(A)$
$\langle 3 \rangle 1$. PICK open sets U_1 in X and V' in Z such that $x \in U_1$, $z \in V'$ and $U' \times V' \subseteq \pi^{-1}(A)$
$\langle 3 \rangle 2$. PICK V open in Z such that $z \in V$, $\overline{V}$ is compact and $\overline{V} \subseteq V'$
PROOF: Proposition 11.53.7.
$\langle 2 \rangle 6$. LET: $U = \bigcup \{U' \text{ open in } X \mid U' \times \overline{V} \subseteq \pi^{-1}(A)\}$
$\langle 2 \rangle 7$. U is saturated
$\langle 3 \rangle 1$. LET: $a \in U$, $b \in X$ with $p(a) = p(b)$
$\langle 3 \rangle 2$. $\{b\} \times \overline{V} \subseteq \pi^{-1}(A)$
$\langle 3 \rangle 3$. PICK U' open in X such that $b \in U'$ and $U' \times \overline{V} \subseteq \pi^{-1}(A)$
PROOF: By the Tube Lemma.
$\langle 3 \rangle 4$. $b \in U' \subseteq U$
$\langle 2 \rangle 8$. $\pi(U \times V)$ is open
PROOF: Since $\pi(U \times V) = p(U) \times V$.
$\langle 2 \rangle 9$. $(y, z) \in \pi(U \times V)$
$\langle 2 \rangle 10$. $\pi(U \times V) \subseteq A$
□

Theorem 11.53.14. *Let A, B, C and D be topological spaces with B and C locally compact Hausdorff. Let $p : A \twoheadrightarrow B$ and $q : C \twoheadrightarrow D$ be quotient maps. Then $p \times q : A \times C \twoheadrightarrow B \times D$.*

PROOF: By Lemma 11.53.13 since $p \times q = (\mathrm{id}_B \times q) \circ (p \times \mathrm{id}_C)$. □

11.54 Compactifications

Definition 11.54.1 (Compactification)**.** Let X be a topological space. A *compactification* of X consists of a compact Hausdorff space Y and an imbedding $X \to Y$.

Definition 11.54.2 (One-Point Compactification)**.** Let X be a topological space. A *one-point compactification* of X is a compactification $i : X \to Y$ such that $Y - i(x)$ consists of a single point.

Theorem 11.54.3. *Let X be a topological space. Then X is locally compact Hausdorff if and only if there exists a one-point compactification $i : X \to Y$. In this case, Y is unique up to unique homeomorphism that commutes with i.*

PROOF:
⟨1⟩1. For any compact Hausdorff space Y and point $a \in Y$, the space $Y - \{a\}$ is locally compact Hausdorff.
⟨2⟩1. LET: Y be a compact Hausdorff space.
⟨2⟩2. LET: $a \in Y$
⟨2⟩3. $Y - \{a\}$ is closed.
⟨2⟩4. $Y - \{a\}$ is locally compact.
PROOF: Proposition 11.53.6.
⟨2⟩5. $Y - \{a\}$ is Hausdorff.
PROOF: Theorem 11.20.6.
⟨1⟩2. For any locally compact Hausdorff space X, there exists a compact Hausdorff space Y and imbedding $i : X \to Y$ such that $Y - i(X)$ is a single point.
⟨2⟩1. LET: X be a locally compact Hausdorff space.
⟨2⟩2. LET: $Y = X \cup \{\infty\}$
⟨2⟩3. Define a topology on Y by: $U \subseteq Y$ is open if and only if U is an open set in X or $U = Y - C$ where C is a compact subspace of X.
⟨3⟩1. Y is open.
PROOF: Since $Y = Y - \emptyset$ and $\emptyset$ is a compact subspace of X.
⟨3⟩2. For any set of open sets $\mathcal{U}$ we have $\bigcup \mathcal{U}$ is open.
PROOF: We have $\bigcup \mathcal{U} = Y - (\bigcap \{C \subseteq X \mid C \text{ is compact}, Y - C \in \mathcal{U}\} - \bigcup \{U \subseteq X \mid U \text{ is open in } X, U \in \mathcal{U}\})$, where we take the empty intersection to be Y.
⟨3⟩3. For any open sets U and V we have $U \cap V$ is open.
⟨4⟩1. LET: U and V be open sets.
⟨4⟩2. CASE: U and V are open sets in X.
PROOF: In this case $U \cap V$ is open in X.

⟨4⟩3. CASE: C_1 and C_2 are compact subspaces of X and $U = X - C_1$, $V = X - C_2$

PROOF: In this case $C_1 \cup C_2$ is compact and $U \cap V = X - (C_1 \cup C_2)$.

⟨4⟩4. CASE: U is open in X, C is a compact subspace of X and $V = X - C$

PROOF: In this case $U \cap V = U - C$ which is open since C is closed.

⟨2⟩4. Y is compact.

⟨3⟩1. LET: $\mathcal{A}$ be an open cover of Y.

⟨3⟩2. PICK C compact in X such that $Y - C \in \mathcal{A}$

PROOF: There must be at least one such member of $\mathcal{A}$ since $\infty \in \bigcup \mathcal{A}$.

⟨3⟩3. $\{U \cap X \mid U \in \mathcal{A} - \{Y - C\}\}$ is a set of open sets in X that covers C.

⟨3⟩4. PICK a finite subcover $\{U_1 \cap X, \ldots, U_n \cap X\}$

⟨3⟩5. $\{U_1 \cap X, \ldots, U_n \cap X, Y - C\}$ covers Y.

⟨2⟩5. Y is Hausdorff.

⟨3⟩1. LET: $x, y \in Y$ with $x \neq y$

⟨3⟩2. CASE: $x, y \in X$

PROOF: There are disjoint open sets U, V in X such that $x \in U$, $y \in V$.

⟨3⟩3. CASE: $x \in X$, $y = \infty$

⟨4⟩1. PICK a compact C that includes a neighbourhood U of x

PROOF: Since X is locally compact.

⟨4⟩2. U and $Y - C$ are disjoint open sets in Y with $x \in U$ and $\infty \in Y - C$

⟨2⟩6. Let $i : X \to Y$ be the inclusion.

⟨2⟩7. i is an imbedding.

⟨3⟩1. i is continuous

⟨3⟩2. i is an open map.

⟨2⟩8. $Y - i(X) = \{\infty\}$

⟨1⟩3. If X is locally compact Hausdorff, Y and Y' are compact Hausdorff, and $i : X \to Y$, $i' :\to Y'$ are imbeddings such that $Y - i(X)$ and $Y' - i'(X)$ each have just one point, then there exists a unique homeomorphism $\theta : Y \cong Y'$ such that $\theta \circ i = i'$.

⟨2⟩1. LET: $Y - i(X) = \{a\}$ and $Y' - i'(X) = \{b\}$

⟨2⟩2. LET: $\theta : Y \to Y'$ be the function with $\theta(a) = b$ and $\theta(i(x)) = i'(x)$

⟨2⟩3. θ is a bijection

⟨2⟩4. θ is continuous.

⟨3⟩1. LET: $U \subseteq Y'$ be open.

PROVE: $\theta^{-1}(U)$ is open.

⟨3⟩2. CASE: $b \in U$

⟨4⟩1. $Y' - U$ is compact

⟨4⟩2. $i({i'}^{-1}(Y' - U))$ is compact.

⟨4⟩3. $i({i'}^{-1}(Y' - U))$ is closed.

⟨4⟩4. $\theta^{-1}(U) = X - i({i'}^{-1}(Y' - U))$

⟨3⟩3. CASE: $b \notin U$

PROOF: $U = i'(V)$ for some V open in X and $\theta^{-1}(U) = i(V)$.

⟨2⟩5. θ is an open map.

PROOF: Similar.

⟨2⟩6. θ is unique.
□

Example 11.54.4. S^1 is the one-point compactification of $\mathbb{R}$.

Example 11.54.5. S^2 is the one-point compactification of $\mathbb{R}^2$.

Definition 11.54.6 (Riemann Sphere)**.** The *Riemann sphere* or *extended complex plane* is $\mathcal{C} \cup \{\infty\}$ topologized as the one-point compactification of $\mathcal{C}$. It is homeomorphic to S^2.

Example 11.54.7. The one-point compactification of $\mathbb{Z}^+$ is $\{0\} \cup \{1/n \mid n \in \mathbb{Z}^+\}$.

11.55 G_δ Sets

Definition 11.55.1 (G_δ Set)**.** Let X be a topological space and $A \subseteq X$. Then A is G_δ if and only if it is the intersection of a countable set of open sets.

Proposition 11.55.2. *In a first countable T_1 space, every singleton is G_δ.*

Proof:
⟨1⟩1. Let: X be a first countable T_1 space.
⟨1⟩2. Let: $a \in X$
⟨1⟩3. Pick a countable local basis $\mathcal{B}$ at a.
⟨1⟩4. $\bigcap \mathcal{B} = \{a\}$
 ⟨2⟩1. Let: $b \in X - \{a\}$
  Prove: $b \notin \bigcap \mathcal{B}$
 ⟨2⟩2. Pick $B \in \mathcal{B}$ with $a \in B \subseteq X - \{b\}$
 ⟨2⟩3. $b \notin B$
□

Example 11.55.3. In the space $\mathbb{R}^\omega$ under the box topology, every singleton is G_δ. However, $\mathbb{R}^\omega$ is not first countable.

Chapter 12

Topological Groups

Definition 12.0.1 (Topological Group)**.** A *topological group* G consists of a T_1 space G and continuous maps $\cdot : G^2 \to G$ and $(\ )^{-1} : G \to G$ such that $(G, \cdot, (\ )^{-1})$ is a group.

Example 12.0.2.

1. The integers $\mathbb{Z}$ under addition are a topological group.
2. The real numbers $\mathbb{R}$ under addition are a topological group.
3. The positive reals under multiplication are a topological group.
4. The set $\{z \in \mathbb{C} \mid |z| = 1\}$ under multiplication and given the topology of S^1 is a topological group.
5. For any $n \geq 0$, the general linear group $GL_n(\mathbb{R})$ is a topological group under matrix multiplication, considered as a subspace of $\mathbb{R}^{n^2}$.

Lemma 12.0.3. *Let G be a T_1 space and $\cdot : G^2 \to G$, $(\ )^{-1} : G \to G$ be functions such that $(G, \cdot, (\ )^{-1})$ is a group. Then G is a topological group if and only if the function $f : G^2 \to G$ that maps (x, y) to xy^{-1} is continuous.*

PROOF:
⟨1⟩1. If G is a topological group then f is continuous.
  PROOF: From Theorem 11.12.9.
⟨1⟩2. If f is continuous then G is a topological group.
  ⟨2⟩1. ASSUME: f is continuous.
  ⟨2⟩2. $(\ )^{-1}$ is continuous.
    PROOF: Since $x^{-1} = f(e, x)$.
  ⟨2⟩3. $\cdot$ is continuous.
    PROOF: Since $xy = f(x, y^{-1})$.
□

Lemma 12.0.4. *Let G be a topological group and H a subgroup of G. Then H is a topological group under the subspace topology.*

PROOF:
⟨1⟩1. H is T_1.
PROOF: From Proposition 11.19.5.
⟨1⟩2. multiplication and inverse on H are continuous.
PROOF: From Theorem 11.12.10.
□

Lemma 12.0.5. *Let G be a topological group and H a subgroup of G. Then $\overline{H}$ is a subgroup of G.*

PROOF:
⟨1⟩1. LET: $x, y \in \overline{H}$
PROVE: $xy^{-1} \in \overline{H}$
⟨1⟩2. LET: U be any neighbourhood of xy^{-1}
⟨1⟩3. LET: $f : G^2 \to G$, $f(a, b) = ab^{-1}$
⟨1⟩4. $f^{-1}(U)$ is a neighbourhood of (x, y)
⟨1⟩5. PICK neighbourhoods V, W of x and y respectively such that $f(V \times W) \subseteq U$.
⟨1⟩6. PICK $a \in V \cap H$ and $b \in W \cap H$
PROOF: Theorem 11.4.6.
⟨1⟩7. $ab^{-1} \in U \cap H$
⟨1⟩8. Q.E.D.
PROOF: By Theorem 11.4.6.
□

Proposition 12.0.6. *Let G be a topological group and $\alpha \in G$. Then the maps $l_\alpha, r_\alpha : G \to G$ defined by $l_\alpha(x) = \alpha x$, $r_\alpha(x) = x\alpha$ are homeomorphisms of G with itself.*

PROOF: They are continuous with continuous inverses $l_{\alpha^{-1}}$ and $r_{\alpha^{-1}}$. □

Corollary 12.0.6.1. *Every topological group is homogeneous.*

PROOF: Given a topological group G and $a, b \in G$, we have $l_{ba^{-1}}$ is a homeomorphism that maps a to b. □

Proposition 12.0.7. *Let G be a topological group and H a subgroup of G. Give G/H the quotient topology. For all $\alpha \in G$, the map $\overline{f_\alpha}$ that sends xH to αxH is a homeomorphism $G/H \cong G/H$.*

PROOF:
⟨1⟩1. $\overline{f_\alpha}$ is well-defined.
PROOF: If $xy^{-1} \in H$ then $(\alpha x)(\alpha y)^{-1} = xy^{-1} \in H$.
⟨1⟩2. $\overline{f_\alpha}$ is continuous.
PROOF: Theorem 11.24.7 since $\overline{f_\alpha} \circ p = p \circ f_\alpha$ is continuous, where $p : G \twoheadrightarrow G/H$ is the canonical surjection.
⟨1⟩3. $\overline{f_\alpha}^{-1}$ is continuous.
PROOF: Similar since $\overline{f_\alpha}^{-1} = \overline{f_{\alpha^{-1}}}$.

□

Corollary 12.0.7.1. *Let G be a topological group and H a subgroup of G. Give G/H the quotient topology. Then G/H is homogeneous.*

Proposition 12.0.8. *Let G be a topological group and H a subgroup of G. Give G/H the quotient topology. If H is closed in G then G/H is T_1.*

PROOF:
$\langle 1\rangle 1$. LET: $p : G \twoheadrightarrow G/H$ be the canonical surjection
$\langle 1\rangle 2$. LET: $x \in G$
$\langle 1\rangle 3$. $p^{-1}(xH) = f_x(H)$
$\langle 1\rangle 4$. $p^{-1}(xH)$ is closed in G
PROOF: Since H is closed and f_x is a homemorphism of G with itself.
$\langle 1\rangle 5$. $\{xH\}$ is closed in G/H
□

Proposition 12.0.9. *Let G be a topological group and H a subgroup of G. Give G/H the quotient topology. Then the canonical surjection $p : G \twoheadrightarrow G/H$ is an open map.*

PROOF:
$\langle 1\rangle 1$. LET: $U \subseteq G$ be open.
$\langle 1\rangle 2$. $p^{-1}(p(U)) = \bigcup_{h \in H} r_h(U)$
$\langle 1\rangle 3$. $p^{-1}(p(U))$ is open.
$\langle 1\rangle 4$. $p(U)$ is open.
□

Proposition 12.0.10. *Let G be a topological group and H a closed normal subgroup of G. Then G/H is a topological group under the quotient topology.*

PROOF:
$\langle 1\rangle 1$. G/H is T_1
PROOF: Proposition 12.0.8.
$\langle 1\rangle 2$. The map $\overline{m} : (xH, yH) \mapsto xy^{-1}H$ is continuous.
$\langle 2\rangle 1$. $p^2 : G^2 \to (G/H)^2$ is a quotient map.
PROOF: Propositions 11.24.6, 12.0.9.
$\langle 2\rangle 2$. $\overline{m} \circ p^2$ is continuous.
PROOF: As it is $p^2 \circ m$ where $m : G^2 \to G$ with $m(x, y) = xy^{-1}$
□

Lemma 12.0.11. *Let G be a topological group and $A, B \subseteq G$. If either A or B is open then AB is open.*

PROOF: If A is open we have $AB = \bigcup_{b \in B} r_b(A)$. Similarly if B is open. □

Definition 12.0.12 (Symmetric Neighbourhood)**.** Let G be a topological group. A neighbourhood V of e is *symmetric* if and only if $V = V^{-1}$.

Lemma 12.0.13. *Let G be a topological group. Let V be a neighbourhood of e. Then V is symmetric if and only if, for all $x \in V$, we have $x^{-1} \in V$.*

PROOF:
⟨1⟩1. If V is symmetric then, for all $x \in V$, we have $x^{-1} \in V$
PROOF: Immediate from defintions.
⟨1⟩2. If, for all $x \in V$, we have $x^{-1} \in V$, then V is symmetric.
⟨2⟩1. ASSUME: for all $x \in V$ we have $x^{-1} \in V$
⟨2⟩2. $V \subseteq V^{-1}$
PROOF: If $x \in V$ then there exists $y \in V$ such that $x = y^{-1}$, namely $y = x^{-1}$
⟨2⟩3. $V^{-1} \subseteq V$
PROOF: Immediate from ⟨2⟩1.
□

Lemma 12.0.14. *Let G be a topological group. For every neighbourhood U of e, there exists a symmetric neighbourhood V of e such that $V^2 \subseteq U$.*

PROOF:
⟨1⟩1. LET: U be a neighbourhood of e.
⟨1⟩2. PICK a neighbourhood V' of e such that $V'V' \subseteq U$
PROOF: Such a neighbourhood exists because multiplication in G is continuous.
⟨1⟩3. PICK a neighbourhood W of e such that $WW^{-1} \subseteq V'$
PROOF: Such a neighbourhood exists because the function that maps (x, y) to xy^{-1} is continuous.
⟨1⟩4. LET: $V = WW^{-1}$
⟨1⟩5. V is a neighbourhood of e
⟨2⟩1. $e \in V$
PROOF: Since $e \in W$ so $e = ee^{-1} \in V$.
⟨2⟩2. V is open
PROOF: Lemma 12.0.11.
⟨1⟩6. V is symmetric
⟨2⟩1. For all $x \in V$ we have $x^{-1} \in V$
⟨3⟩1. LET: $x \in V$
⟨3⟩2. PICK$y, z \in W$ such that $x = yz^{-1}$
⟨3⟩3. $x^{-1} = zy^{-1}$
⟨3⟩4. $x^{-1} \in V$
⟨3⟩5. $x \in V^{-1}$
⟨2⟩2. Q.E.D.
PROOF: Lemma 12.0.13
⟨1⟩7. $V^2 \subseteq U$
PROOF: We have $V^2 \subseteq (V')^2 \subseteq U$
□

Proposition 12.0.15. *Every topological group is Hausdorff.*

PROOF:
⟨1⟩1. LET: G be a topological group.
⟨1⟩2. LET: $x, y \in G$ with $x \neq y$

⟨1⟩3. Let: $U = G \setminus \{x[^{-1}y]\}$
⟨1⟩4. Pick a symmetric neighbourhood V of e such that $VV \subseteq U$
⟨2⟩1. U is open
Proof: Since G is T_1.
⟨2⟩2. $e \in U$
Proof: Since $x \neq y$
⟨2⟩3. Q.E.D.
Proof: Lemma 12.0.14.
⟨1⟩5. Vx and Vy are disjoint neighbourhoods of x and y respectively.
⟨2⟩1. Vx is open
Proof: Since $Vx = r_x(V)$
⟨2⟩2. Vy is open
Proof: Similar.
⟨2⟩3. $Vx \cap Vy = \emptyset$
⟨3⟩1. Assume: for a contradiction $z \in Vx \cap Vy$
⟨3⟩2. Pick $a, b \in V$ such that $z = ax = by$
⟨3⟩3. $xy^{-1} \in VV$
Proof: Since $xy^{-1} = a^{-1}b$
⟨3⟩4. $xy^{-1} \in U$
⟨3⟩5. Q.E.D.
Proof: From ⟨1⟩3.
□

Proposition 12.0.16. *Every topological group is regular.*

Proof:
⟨1⟩1. Let: G be a topological group.
⟨1⟩2. Let: $A \subseteq G$ be a closed set and $a \notin A$.
⟨1⟩3. Let: $U = G \setminus Aa^{-1}$
⟨1⟩4. Pick a symmetric neighbourhood V of e such that $VV \subseteq U$
⟨2⟩1. U is open
Proof: Since $Aa^{-1} = r_{a^{-1}}(A)$ is closed.
⟨2⟩2. $e \in U$
Proof: Since $a \notin A$.
⟨2⟩3. Q.E.D.
Proof: Lemma 12.0.14.
⟨1⟩5. VA and Va are disjoint open sets with $A \subseteq VA$ and $a \in Va$
⟨2⟩1. VA is open
Proof: Lemma 12.0.11
⟨2⟩2. Va is open
Proof: Lemma 12.0.11
⟨2⟩3. $VA \cap Va = \emptyset$
⟨3⟩1. Assume: for a contradiction $z \in VA \cap Va$
⟨3⟩2. Pick $b, c \in V$ and $d \in A$ with $z = bd = ca$
⟨3⟩3. $da^{-1} \in U$
Proof: Since $da^{-1} = b^{-1}c \in VV \subseteq U$
⟨3⟩4. Q.E.D.

PROOF: This contradicts ⟨1⟩3

∎

Proposition 12.0.17. *Let G be a topological group and H a subgroup of G. Give G/H the quotient topology. If H is closed in G then G/H is regular.*

PROOF:
⟨1⟩1. LET: $p : G \twoheadrightarrow G/H$ be the canonical surjection.
⟨1⟩2. LET: A be a closed set in G/H and $aH \in (G/H) \setminus A$.
⟨1⟩3. LET: $B = p^{-1}(A)$
⟨1⟩4. B is a closed saturated set in G.
⟨1⟩5. $B \cap aH = \emptyset$
⟨1⟩6. $B = BH$
⟨1⟩7. PICK a symmetric neighbourhood V of e such that VB does not intersect Va
⟨2⟩1. LET: $U = G \setminus Ba^{-1}$
⟨2⟩2. PICK a symmetric neighbourhood V of e such that $VV \subseteq U$
⟨3⟩1. U is open
PROOF: Since $Ba^{-1} = r_{a^{-1}}(B)$ is closed.
⟨3⟩2. $e \in U$
PROOF: If $e \in Ba^{-1}$ then $a \in B$
⟨3⟩3. Q.E.D.
PROOF: Lemma 12.0.14
⟨2⟩3. $VB \cap Va = \emptyset$
PROOF: If $vb = v'a$ for $v, v' \in V$ and $b \in B$ then we have $ba^{-1} = v^{-1}v' \in Ba \cap VV \subseteq Ba \cap U$.
⟨1⟩8. $p(VB)$ and $p(Va)$ are disjoint open sets
⟨2⟩1. $p(VB)$ and $p(Va)$ are open.
PROOF: Proposition 12.0.9.
⟨2⟩2. $p(VB) \cap p(Va) = \emptyset$
PROOF: If $vbH = v'aH$ for $v, v' \in V$, $b \in B$ then $v'a = vbh$ for some $h \in H$. Hence $v'a \in Va \cap VBH = Va \cap VB$.
⟨1⟩9. $A \subseteq p(VB)$
⟨1⟩10. $aH \in p(Va)$

∎

Proposition 12.0.18. *Let G be a topological group. The component of G that contains e is a normal subgroup of G.*

PROOF:
⟨1⟩1. LET: C be the component of G that contains e.
⟨1⟩2. For all $x \in G$, xC is the component of G that contains x.
⟨2⟩1. LET: $x \in G$
⟨2⟩2. LET: D be the component of G that contains x.
⟨2⟩3. $xC \subseteq D$
PROOF: Since xC is connected by Theorem 11.30.13.
⟨2⟩4. $D \subseteq xC$

PROOF: Since $x^{-1}D \subseteq C$ similarly.
$\langle 1 \rangle 3$. For all $x \in G$, Cx is the component of G that contains x.
PROOF: Similar.
$\langle 1 \rangle 4$. For all $x \in C$ we have $xC = Cx = C$
$\langle 1 \rangle 5$. For all $x \in C$ we have $x^{-1}C = C$
$\langle 1 \rangle 6$. For all $x \in C$ we have $x^{-1} \in C$
$\langle 1 \rangle 7$. For all $x, y \in C$ we have $xy \in C$
PROOF: Since $xyC = xC = x$.
$\langle 1 \rangle 8$. For all $x \in G$ we have $xC = Cx$.
PROOF: From $\langle 1 \rangle 2$ and $\langle 1 \rangle 3$.
∎

Lemma 12.0.19. *Let G be a topological group. Let A be a closed set in G and B a compact subspace of G such that $A \cap B = \emptyset$. Then there exists a symmetric neighbourhood U of e such that $AU \cap BU = \emptyset$.*

PROOF:
$\langle 1 \rangle 1$. For all $b \in B$ there exists a symmetric neighbourhood V of e such that $bV^2 \cap A = \emptyset$
$\langle 2 \rangle 1$. LET: $b \in B$
$\langle 2 \rangle 2$. LET: $W = b^{-1}(G \setminus A)$
$\langle 2 \rangle 3$. W is a neighbourhood of e and $bW \cap A = \emptyset$
$\langle 2 \rangle 4$. PICK a symmetric neighbourhood V of e such that $V^2 \subseteq W$
$\langle 1 \rangle 2$. $\{bV^2 \mid b \in B, V \text{ is a symmetric neighbourhood of } e, bV^2 \cap A = \emptyset\}$ is an open cover of B
$\langle 1 \rangle 3$. PICK a finite subcover $b_1 V_1^2, \ldots, b_n V_n^2$, say.
$\langle 1 \rangle 4$. LET: $U = V_1 \cap \cdots \cap V_n$
$\langle 1 \rangle 5$. $BU^2 \cap A = \emptyset$
$\langle 1 \rangle 6$. $AU \cap BU = \emptyset$
PROOF: If $av \in BU$ where $a \in A$ and $v \in V$ then $a = avv^{-1} \in BU^2 \cap A$.
∎

Proposition 12.0.20 (AC)**.** *Let G be a topological group. Let A be a closed set in G, and B a compact subspace of G. Then AB is closed.*

PROOF:
$\langle 1 \rangle 1$. LET: $x \in G \setminus AB$
$\langle 1 \rangle 2$. $A^{-1}x \cap B = \emptyset$
$\langle 1 \rangle 3$. $A^{-1}x$ is closed.
$\langle 1 \rangle 4$. PICK a symmetric neighbourhood U of e such that $A^{-1}xU \cap BU = \emptyset$
$\langle 1 \rangle 5$. xU^2 is open
PROOF: Lemma 12.0.11.
$\langle 1 \rangle 6$. $x \in xU^2 \subseteq G \setminus AB$
∎

PROOF:
$\langle 1 \rangle 1$. LET: $x \in \overline{AB}$
PROVE: $x \in AB$

$\langle 1 \rangle 2$. PICK a net $(a_\alpha b_\alpha)_{\alpha \in J}$ in AB that converges to x.
$\langle 1 \rangle 3$. PICK a convergent subnet $(b_{g(\beta)})_{\beta \in K}$ of $(b_\alpha)_{\alpha \in J}$ with limit l.
$\langle 1 \rangle 4$. $a_{g(\beta)} \to xl^{-1}$ as $\beta \to \infty$
PROOF:

$$\begin{aligned} a_{g(\beta)} &= a_{g(\beta)} b_{g(\beta)} {b_{g(\beta)}}^{-1} \\ &\to xl^{-1} \end{aligned}$$

$\langle 1 \rangle 5$. $xl^{-1} \in A$
$\langle 1 \rangle 6$. $l \in B$
PROOF: B is closed because it is compact.
$\langle 1 \rangle 7$. $x \in AB$
□

Corollary 12.0.20.1. *Let G be a topological group and $H \leq G$. Let $p : G \twoheadrightarrow G/H$ be the quotient map. If H is compact then p is a closed map.*

PROOF: For A closed in G, we have $p^{-1}(p(A)) = AH$ is closed, and so $p(A)$ is closed. □

Corollary 12.0.20.2. *Let G be a topological group and $H \leq G$. If H and G/H are compact then G is compact.*

PROOF: From Proposition 11.51.2 since, for all $aH \in G/H$, we have $p^{-1}(aH) = aH$ is compact because it is homemorphic to H. □

Proposition 12.0.21. *Let G be a locally compact topological group. Let $H \leq G$. Then G/H is locally compact.*

PROOF: From Propositions 11.53.12 and 12.0.9. □

12.1 The Metric Topology

Definition 12.1.1 (Metric)**.** Let X be a set. A *metric* on X is a function $d : X^2 \to \mathbb{R}$ such that:

1. For all $x, y \in X$, $d(x, y) \geq 0$
2. For all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$
3. For all $x, y \in X$, $d(x, y) = d(y, x)$
4. (*Triangle Inequality*) For all $x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$

We call $d(x, y)$ the *distance* between x and y.

Definition 12.1.2 (Open Ball)**.** Let X be a metric space. Let $a \in X$ and $\epsilon > 0$. The *open ball* with *centre* a and *radius* ϵ is

$$B(a, \epsilon) = \{x \in X \mid d(a, x) < \epsilon\} \ .$$

Definition 12.1.3 (Metric Topology)**.** Let X be a metric space. The *metric topology* on X is the topology generated by the basis consisting of all the open balls.

We prove this is a basis for a topology.

PROOF:
⟨1⟩1. For every point a, there exists a ball B such that $a \in B$
PROOF: We have $a \in B(a, 1)$.
⟨1⟩2. For any balls B_1, B_2 and point $a \in B_1 \cap B_2$, there exists a ball B_3 such that $a \in B_3 \subseteq B_1 \cap B_2$
⟨2⟩1. LET: $B_1 = B(c_1, \epsilon_1)$ and $B_2 = B(c_2, \epsilon_2)$
⟨2⟩2. LET: $\delta = \min(\epsilon_1 - d(c_1, a), \epsilon_2 - d(c_2, a))$
PROVE: $B(a, \delta) \subseteq B_1 \cap B_2$
⟨2⟩3. LET: $x \in B(a, \delta)$
⟨2⟩4. $x \in B_1$
PROOF:

$$\begin{aligned} d(x, c_1) &= d(x, a) + d(a, c_1) \\ &< \delta + d(a, c_1) \\ &\leq \epsilon_1 \end{aligned}$$

⟨2⟩5. $x \in B_2$
PROOF: Similar.
□

Proposition 12.1.4. *Let X be a metric space and $U \subseteq X$. Then U is open if and only if, for every $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$.*

PROOF:
⟨1⟩1. If U is open then, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$.
⟨2⟩1. ASSUME: U is open.

⟨2⟩2. LET: $x \in U$
⟨2⟩3. PICK $a \in X$ and $\delta > 0$ such that $x \in B(a, \delta) \subseteq U$
⟨2⟩4. LET: $\epsilon = \delta - d(a, x)$
PROVE: $B(x, \epsilon) \subseteq U$
⟨2⟩5. LET: $y \in B(x, \epsilon)$
⟨2⟩6. $d(y, a) < \delta$
PROOF:

$$\begin{aligned} d(y,a) &\leq d(a,x) + d(x,y) \\ &< \delta + d(x,y) \\ &= \epsilon \end{aligned}$$

⟨2⟩7. $y \in U$
⟨1⟩2. If, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$, then U is open.
PROOF: Immediate from definitions.
□

Definition 12.1.5 (Discrete Metric)**.** Let X be a set. The *discrete metric* on X is defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Proposition 12.1.6. *The discrete metric induces the discrete topology.*

PROOF: For any (open) set U and point $a \in U$, we have $a \in B(a, 1) \subseteq U$. □

Definition 12.1.7 (Standard Metric on $\mathbb{R}$)**.** The *standard metric* on $\mathbb{R}$ is defined by $d(x, y) = |x - y|$.

Proposition 12.1.8. *The standard metric on $\mathbb{R}$ induces the standard topology on $\mathbb{R}$.*

PROOF:
⟨1⟩1. Every open ball is open in the standard topology on $\mathbb{R}$.
PROOF: $B(a, \epsilon) = (a - \epsilon, a + \epsilon)$
⟨1⟩2. For every open set U and point $a \in U$, there exists $\epsilon > 0$ such that $B(a, \epsilon) \subseteq U$
⟨2⟩1. LET: U be an open set and $a \in U$
⟨2⟩2. PICK an open interval b, c such that $a \in (b, c) \subseteq U$
⟨2⟩3. LET: $\epsilon = \min(a - b, c - a)$
⟨2⟩4. $B(a, \epsilon) \subseteq U$
□

Definition 12.1.9 (Metrizable)**.** A topological space X is *metrizable* if and only if there exists a metric on X that induces the topology.

Definition 12.1.10 (Bounded)**.** Let X be a metric space and $A \subseteq X$. Then A is *bounded* if and only if there exists M such that, for all $x, y \in A$, we have $d(x, y) \leq M$.

Definition 12.1.11 (Diameter)**.** Let X be a metric space and $A \subseteq X$. The *diameter* of A is

$$\operatorname{diam} A = \sup_{x,y \in A} d(x,y) \enspace .$$

Definition 12.1.12 (Standard Bounded Metric)**.** Let d be a metric on X. The *standard bounded metric* corresponding to d is the metric $\overline{d}$ defined by

$$\overline{d}(x,y) = \min(d(x,y), 1) \enspace .$$

We prove this is a metric.

PROOF:
⟨1⟩1. $\overline{d}(x,y) \geq 0$
  PROOF: Since $d(x,y) \geq 0$
⟨1⟩2. $\overline{d}(x,y) = 0$ if and only if $x = y$
  PROOF: $\overline{d}(x,y) = 0$ if and only if $d(x,y) = 0$ if and only if $x = y$
⟨1⟩3. $\overline{d}(x,y) = \overline{d}(y,x)$
  PROOF: Since $d(x,y) = d(y,x)$
⟨1⟩4. $\overline{d}(x,z) \leq \overline{d}(x,y) + \overline{d}(y,z)$
  PROOF:

$$\begin{aligned}
\overline{d}(x,y) + \overline{d}(y,z) &= \min(d(x,y),1) + \min(d(y,z),1) \\
&= \min(d(x,y) + d(y,z), d(x,y) + 1, d(y,z) + 1, 2) \\
&\geq \min(d(x,z), 1) \\
&= \overline{d}(x,z)
\end{aligned}$$

□

Lemma 12.1.13. *In any metric space X, the set $\mathcal{B} = \{B(a,\epsilon) \mid a \in X, \epsilon < 1\}$ is a basis for the metric topology.*

PROOF:
⟨1⟩1. Every element of $\mathcal{B}$ is open.
  PROOF: From Lemma 11.7.2.
⟨1⟩2. For every open set U and point $a \in U$, there exists $B \in \mathcal{B}$ such that $a \in B \subseteq U$
  ⟨2⟩1. LET: U be an open set and $a \in U$
  ⟨2⟩2. PICK$\epsilon > 0$ such that $B(a,\epsilon) \subseteq U$
  ⟨2⟩3. $B(a, \min(\epsilon, 1/2)) \subseteq U$
⟨1⟩3. Q.E.D.
  PROOF: Lemma 11.7.3.
□

Proposition 12.1.14. *Let d be a metric on the set X. Then the standard bounded metric $\overline{d}$ induces the same metric as d.*

PROOF: This follows from Lemma 12.1.13 since the open balls with radius < 1 are the same under both metrics. □

Lemma 12.1.15. *Let d and d' be two metrics on the same set X. Let $\mathcal{T}$ and $\mathcal{T}'$ be the topologies they induce. Then $\mathcal{T} \subseteq \mathcal{T}'$ if and only if, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that*

$$B_{d'}(x, \delta) \subseteq B_d(x, \epsilon) \ .$$

PROOF:
$\langle 1 \rangle 1$. If $\mathcal{T} \subseteq \mathcal{T}'$ then, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$
  PROOF: From Proposition 12.1.4 since $x \in B_d(x, \epsilon) \in \mathcal{T}'$.
$\langle 1 \rangle 2$. If, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$, then $\mathcal{T} \subseteq \mathcal{T}'$
  $\langle 2 \rangle 1$. ASSUME: For all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$
  $\langle 2 \rangle 2$. LET: $U \in \mathcal{T}$
  $\langle 2 \rangle 3$. For all $x \in U$ there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq U$.
    $\langle 3 \rangle 1$. LET: $x \in U$
    $\langle 3 \rangle 2$. PICK $\epsilon > 0$ such that $B_d(x, \epsilon) \subseteq U$
      PROOF: Proposition 12.1.4
    $\langle 3 \rangle 3$. PICK $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$
      PROOF: By $\langle 2 \rangle 1$
    $\langle 3 \rangle 4$. $B_{d'}(x, \delta) \subseteq U$
  $\langle 2 \rangle 4$. $U \in \mathcal{T}'$
    PROOF: Proposition 12.1.4.
□

Proposition 12.1.16. $\mathbb{R}^2$ *under the dictionary order topology is metrizable.*

PROOF: Define $d : \mathbb{R}^2 \to \mathbb{R}$ by

$$d((x, y), (x, z)) = \max(|y - z|, 1)$$
$$d((x, y), (x', y')) = 1 \qquad \text{if } x \neq x'$$ □

$\langle 1 \rangle 1$. $x \in \bigcap_{i=1}^{N} {\pi_i}^{-1}() \subseteq B_D(a, \epsilon)$

Proposition 12.1.17. *Let $d : X^2 \to \mathbb{R}$ be a metric on X. Then the metric topology on X is the coarsest topology such that d is continuous.*

PROOF:
$\langle 1 \rangle 1$. d is continuous.
  $\langle 2 \rangle 1$. LET: $a, b \in X$
  $\langle 2 \rangle 2$. LET: $\epsilon > 0$
  $\langle 2 \rangle 3$. LET: $\delta = \epsilon / 2$
  $\langle 2 \rangle 4$. LET: $x, y \in X$
  $\langle 2 \rangle 5$. ASSUME: $\rho((a, b), (x, y)) < \delta$
  $\langle 2 \rangle 6$. $|d(a, b) - d(x, y)| < \epsilon$
    $\langle 3 \rangle 1$. $d(a, b) - d(x, y) < \epsilon$

PROOF:

$$\begin{aligned} d(a,b) &\leq d(a,x) + d(x,y) + d(y,b) \\ &\leq d(x,y) + 2\rho((a,b),(x,y)) \\ &< d(x,y) + 2\delta \\ &= d(x,y) + \epsilon \end{aligned}$$

⟨3⟩2. $d(a.b) - d(x,y) > -\epsilon$

PROOF: Similar.

⟨2⟩7. Q.E.D.

⟨1⟩2. If $\mathcal{T}$ is any topology under which d is continuous then $\mathcal{T}$ is finer than the metric topology.

PROOF: Since $B(a,\epsilon) = {d_a}^{-1}((-\infty,\epsilon))$

□

Proposition 12.1.18. *Let X be a metric space with metric d and $A \subseteq X$. The restriction of d to A is a metric on A that induces the subspace topology.*

PROOF:

⟨1⟩1. The restriction of d to A is a metric on A.

⟨1⟩2. Every open ball under $d \upharpoonright A$ is open under the subspace topology.

PROOF: $B_{d \upharpoonright A}(a,\epsilon) = B_d(a,\epsilon) \cap A$.

⟨1⟩3. If U is open in the subspace topology and $x \in U$, then there exists a $d \upharpoonright A$-ball B such that $x \in B \subseteq U$.

⟨2⟩1. PICK V open in X such that $U = V \cap A$

⟨2⟩2. PICK $\epsilon > 0$ such that $B_d(x,\epsilon) \subseteq V$

⟨2⟩3. Take $B = B_{d \upharpoonright A}(x,\epsilon)$

□

Corollary 12.1.18.1. *A subspace of a metrizable space is metrizable.*

Proposition 12.1.19. *Every metrizable space is Hausdorff.*

PROOF:

⟨1⟩1. LET: X be a metric space

⟨1⟩2. LET: $a, b \in X$ with $a \neq b$

⟨1⟩3. LET: $\epsilon = d(a,b)/2$

⟨1⟩4. LET: $U = B(a,\epsilon)$ and $V = B(b,\epsilon)$

⟨1⟩5. U and V are disjoint neighbourhoods of a and b respectively.

□

Corollary 12.1.19.1. *Every metrizable space is T_1.*

Proposition 12.1.20 (CC)**.** *The product of a countable family of metrizable spaces is metrizable.*

PROOF:

⟨1⟩1. LET: (X_n, d_n) be a sequence of metric spaces.

⟨1⟩2. ASSUME: w.l.o.g. each d_n is bounded above by 1.

PROOF: By Proposition 12.1.14.

⟨1⟩3. LET: D be the metric on $\mathbb{R}^\omega$ defined by $D(x, y) = \sup_i(d_i(x_i, y_i)/i)$.

⟨2⟩1. $D(x, y) \geq 0$

⟨2⟩2. $D(x, y) = 0$ if and only if $x = y$

⟨2⟩3. $D(x, y) = D(y, x)$

⟨2⟩4. $D(x, z) \leq D(x, y) + D(y, z)$

PROOF:

$$\begin{aligned} D(x,z) &= \sup_i \frac{d_i(x_i, z_i)}{i} \\ &\leq \sup_i \frac{d_i(x_i, y_i) + d_i(y_i, z_i)}{i} \\ &\leq \sup_i \frac{d_i(x_i, y_i)}{i} + \sup_i \frac{d_i(y_i, z_i)}{i} \\ &= D(x, y) + D(y, z) \end{aligned}$$

⟨1⟩4. Every open ball $B_D(a, \epsilon)$ is open in the product topology.

⟨2⟩1. PICK N such that $1/\epsilon < N$

⟨2⟩2. $B_D(a, \epsilon) = \prod_{i=1}^{\infty} U_i$ where $U_i = B(a_i, i\epsilon)$ if $i \leq N$, and $U_i = X_i$ if $i > N$

⟨1⟩5. For any open set U and $a \in U$, there exists $\epsilon > 0$ such that $B_D(a, \epsilon) \subseteq U$.

⟨2⟩1. LET: $n \geq 1$, V be an open set in $\mathbb{R}$ and $a \in {\pi_n}^{-1}(V)$

⟨2⟩2. PICK $\epsilon > 0$ such that $B_{d_n}(a, \epsilon) \subseteq V$

⟨2⟩3. $B_D(a, \epsilon/n) \subseteq {\pi_n}^{-1}(V)$

□

Theorem 12.1.21. *Let X and Y be metric spaces and $f : X \to Y$. Then f is continuous if and only if, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$.*

PROOF:

⟨1⟩1. If f is continuous then, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$

⟨2⟩1. ASSUME: f is continuous.

⟨2⟩2. LET: $x \in X$ and $\epsilon > 0$

⟨2⟩3. PICK a neighbourhood U of x such that $f(U) \subseteq B(f(x), \epsilon)$

PROOF: Theorem 11.12.6.

⟨2⟩4. PICK $\delta > 0$ such that $B(x, \delta) \subseteq U$

PROOF: Proposition 12.1.4.

⟨2⟩5. For all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$

⟨1⟩2. If for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$, then f is continuous.

⟨2⟩1. ASSUME: for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$

⟨2⟩2. LET: $x \in X$ and V be a neighbourhood of $f(x)$

⟨2⟩3. PICK $\epsilon > 0$ such that $B(f(x), \epsilon) \subseteq V$

PROOF: Proposition 12.1.4.

⟨2⟩4. PICK $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$

PROOF: By ⟨2⟩1

⟨2⟩5. LET: $U = B(x, \delta)$

⟨2⟩6. U is a neighbourhood of x with $f(U) \subseteq V$
⟨2⟩7. Q.E.D.
PROOF: Theorem 11.12.6.
□

Proposition 12.1.22. *Let X be a metric space. Let (a_n) be a sequence in X and $l \in X$. Then $a_n \to l$ as $n \to \infty$ if and only if, for all $\epsilon > 0$, there exists N such that, for all $n \geq N$, we have $d(a_n, l) < \epsilon$.*

PROOF: From Proposition 11.9.4. □

Proposition 12.1.23. *Every metrizable space is first countable.*

PROOF: In any metric space X, the open balls $B(a, 1/n)$ for $n \geq 1$ form a local basis at a.

Example 12.1.24. $\mathbb{R}^\omega$ under the box topology is not metrizable.

Example 12.1.25. If J is uncountable then $\mathbb{R}^J$ under the product topology is not metrizable.

Example 12.1.26. The space $\overline{S_\Omega}$ is not metrizable by Example 11.21.4.

Proposition 12.1.27. *A compact subspace of a metric space is bounded.*

PROOF:
⟨1⟩1. LET: X be a metric space and $A \subseteq X$ be compact.
⟨1⟩2. PICK $a \in A$
⟨1⟩3. $\{B(a, n) \mid n \in \mathbb{Z}^+\}$ covers A
⟨1⟩4. PICK a finite subcover $\{B(a, n_1), \ldots, B(a, n_k)\}$
⟨1⟩5. LET: $N = \max(n_1, \ldots, n_k)$
⟨1⟩6. For all $x, y \in A$ we have $d(x, y) < 2N$
PROOF:

$$\begin{aligned} d(x,y) &\leq d(x,a) + d(a,y) \\ &< N + N \end{aligned}$$

□

This example shows the converse does not hold:

Example 12.1.28. The space $\mathbb{R}$ under the standard bounded metric is bounded but not compact.

Proposition 12.1.29. *A connected metric space with more than one point is uncountable.*

PROOF:
⟨1⟩1. LET: X be a connected metric space with more than one point.
⟨1⟩2. PICK $a \in X$
⟨1⟩3. $d(a, -) : X \to \mathbb{R}$ is continuous.
PROOF: Proposition 12.1.17.

⟨1⟩4. $\{d(a,x) \mid x \in X\}$ is a connected subspace of $\mathbb{R}$ that includes 0.
PROOF: Theorem 11.30.13.
⟨1⟩5. $\{d(a,x) \mid x \in X\} \neq \{0\}$
PROOF: Since X has more than one point.
⟨1⟩6. $\{d(a,x) \mid x \in X\}$ is uncountable.
PROOF: Since it includes a closed interval (Corollary 11.52.2.1).
□

12.2 Real Linear Algebra

Definition 12.2.1 (Square Metric)**.** The *square metric* ρ on $\mathbb{R}^n$ is defined by

$$\rho(\vec{x}, \vec{y}) = \max(|x_1 - y_1|, \ldots, |x_n - y_n|)$$

We prove this is a metric.

PROOF:
⟨1⟩1. $\rho(\vec{x}, \vec{y}) \geq 0$
PROOF: Immediate from definition.
⟨1⟩2. $\rho(\vec{x}, \vec{y}) = 0$ if and only if $\vec{x} = \vec{y}$
PROOF: Immediate from definition.
⟨1⟩3. $\rho(\vec{x}, \vec{y}) = \rho(\vec{y}, \vec{x})$
PROOF: Immediate from definition.
⟨1⟩4. $\rho(\vec{x}, \vec{z}) \leq \rho(\vec{x}, \vec{y}) + \rho(\vec{y}, \vec{z})$
PROOF: Since $|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$.
□

Proposition 12.2.2. *The square metric induces the standard topology on $\mathbb{R}^n$.*

PROOF:
⟨1⟩1. For every $a \in X$ and $\epsilon > 0$, we have $B_\rho(a, \epsilon)$ is open in the standard product topology.
PROOF:

$$B_\rho(a, \epsilon) = (a_1 - \epsilon, a_1 + \epsilon) \times \cdots \times (a_n - \epsilon, a_n + \epsilon)$$

⟨1⟩2. For any open sets $U_1, \ldots, U_n$ in $\mathbb{R}$, we have $U_1 \times \cdots \times U_n$ is open in the square metric topology.
⟨2⟩1. LET: $\vec{a} \in U_1 \times \cdots \times U_n$
⟨2⟩2. For $i = 1, \ldots, n$, PICK $\epsilon_i > 0$ such that $(a_i - \epsilon_i, a_i + \epsilon_i) \subseteq U_i$
⟨2⟩3. LET: $\epsilon = \min(\epsilon_1, \ldots, \epsilon_n)$
⟨2⟩4. $B_\rho(\vec{a}, \epsilon) \subseteq U$
□

Definition 12.2.3. Given $\vec{x}, \vec{y} \in \mathbb{R}^n$, define the *sum* $\vec{x} + \vec{y}$ by

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n) \ .$$

Definition 12.2.4. Given $\lambda \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^n$, define the *scalar product* $\lambda\vec{x} \in \mathbb{R}^n$ by

$$\lambda(x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n)$$

Definition 12.2.5 (Inner Product)**.** Given $\vec{x}, \vec{y} \in \mathbb{R}^n$, define the *inner product* $\vec{x} \cdot \vec{y} \in \mathbb{R}$ by

$$(x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) = x_1 y_1 + \cdots + x_n y_n \ .$$

We write $\vec{x}^2$ for $\vec{x} \cdot \vec{x}$.

Definition 12.2.6 (Norm)**.** Let $n \geq 1$. The *norm* on $\mathbb{R}^n$ is the function $\| \ \| : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\|(x_1, \ldots, x_n)\| = \sqrt{x_1^2 + \cdots + x_n^2}$$

Lemma 12.2.7.

$$\|\vec{x}\|^2 = \vec{x}^2$$

Proof: Immediate from definitions. □

Lemma 12.2.8.

$$\vec{x} \cdot (\vec{y} + \vec{z}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z}$$

Proof: Each is equal to $(x_1 y_1 + x_1 z_1, \ldots, x_n y_n + x_n z_n)$. □

Lemma 12.2.9.

$$|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \|\vec{y}\|$$

Proof:
⟨1⟩1. Assume: $\vec{x} \neq \vec{0} \neq \vec{y}$
  Proof: Otherwise both sides are 0.
⟨1⟩2. Let: $a = 1/\|\vec{x}\|$
⟨1⟩3. Let: $b = 1/\|\vec{y}\|$
⟨1⟩4. $(a\vec{x} + b\vec{y})^2 \geq 0$ and $(a\vec{x} - b\vec{y})^2 \geq 0$
⟨1⟩5. $a^2\|\vec{x}\|^2 + 2ab\vec{x} \cdot \vec{y} + b^2\|\vec{y}\|^2 \geq 0$ and $a^2\|\vec{x}\|^2 - 2ab\vec{x} \cdot \vec{y} + b^2\|\vec{y}\|^2 \geq 0$
⟨1⟩6. $2ab\vec{x} \cdot \vec{y} + 2 \geq 0$ and $-2ab\vec{x} \cdot \vec{y} + 2 \geq 0$
⟨1⟩7. $\vec{x} \cdot \vec{y} \geq -1/ab$ and $\vec{x} \cdot \vec{y} \leq 1/ab$
⟨1⟩8. $\vec{x} \cdot \vec{y} \geq -\|\vec{x}\|\|\vec{y}\|$ and $\vec{x} \cdot \vec{y} \leq \|\vec{x}\|\|\vec{y}\|$
□

Lemma 12.2.10 (Triangle Inequality)**.**

$$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$$

Proof:

$$\begin{aligned} \|\vec{x} + \vec{y}\|^2 &= \|\vec{x}\|^2 + 2\vec{x} \cdot \vec{y} + \|\vec{y}\|^2 \\ &\leq \|\vec{x}\|^2 + 2\|\vec{x}\|\|\vec{y}\| + \|\vec{y}\|^2 && \text{(Lemma 12.2.9)} \\ &= (\|\vec{x}\| + \|\vec{y}\|)^2 && \square \end{aligned}$$

Definition 12.2.11 (Euclidean Metric)**.** Let $n \geq 1$. The *Euclidean metric* on $\mathbb{R}^n$ is defined by

$$d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\| \ .$$

We prove this is a metric.

⟨1⟩1. $d(\vec{x}, \vec{y}) \geq 0$

Proof: Immediate from definition.

⟨1⟩2. $d(\vec{x}, \vec{y}) = 0$ if and only if $\vec{x} = \vec{y}$

Proof: $d(\vec{x}, \vec{y}) = 0$ if and only if $\vec{x} - \vec{y} = \vec{0}$.

⟨1⟩3. $d(\vec{x}, \vec{y}) = d(\vec{y}, \vec{x})$

Proof: Immediate from definition.

⟨1⟩4. $d(\vec{x}, \vec{z}) \leq d(\vec{x}, \vec{y}) + d(\vec{y}, \vec{z})$

Proof:

$$\begin{aligned} \|\vec{x} - \vec{z}\| &= \|(\vec{x} - \vec{y}) + (\vec{y} - \vec{z})\| \\ &\leq \|\vec{x} - \vec{y}\| + \|\vec{y} - \vec{z}\| \qquad \text{(Lemma 12.2.10)} \end{aligned}$$

□

Proposition 12.2.12. *The Euclidean metric induces the standard topology on* $\mathbb{R}^n$.

Proof:

⟨1⟩1. Let: ρ be the square metric.

⟨1⟩2. For all $\vec{a} \in \mathbb{R}^n$ and $\epsilon > 0$, we have $B_d(\vec{a}, \epsilon) \subseteq B_\rho(\vec{a}, \epsilon)$

⟨2⟩1. Let: $\vec{x} \in B_d(\vec{a}, \epsilon)$

⟨2⟩2. $\sqrt{(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2} < \epsilon$

⟨2⟩3. $(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2 < \epsilon^2$

⟨2⟩4. For all i we have $(x_i - a_i)^2 < \epsilon^2$

⟨2⟩5. For all i we have $|x_i - a_i| < \epsilon$

⟨2⟩6. $\rho(\vec{x}, \vec{a}) < \epsilon$

⟨1⟩3. For all $\vec{a} \in \mathbb{R}^n$ and $\epsilon > 0$, we have $B_\rho(\vec{a}, \epsilon/\sqrt{n}) \subseteq B_d(\vec{a}, \epsilon)$

⟨2⟩1. Let: $\vec{x} \in B_\rho(\vec{a}, \epsilon/\sqrt{n})$

⟨2⟩2. $\rho(\vec{x}, \vec{a}) < \epsilon/\sqrt{n}$

⟨2⟩3. For all i we have $|x_i - x_a| < \epsilon/\sqrt{n}$

⟨2⟩4. For all i we have $(x_i - x_a)^2 < \epsilon^2/n$

⟨2⟩5. $d(\vec{x}, \vec{a}) < \epsilon$

⟨1⟩4. Q.E.D.

Proof: By Lemma 12.1.15.

□

Proposition 12.2.13. *Let $n \geq 0$. For all $c \in \mathbb{R}^n$ and $\epsilon > 0$, the open ball $B(c, \epsilon)$ is path connected.*

Proof:

⟨1⟩1. Let: $a, b \in B(c, \epsilon)$

⟨1⟩2. Let: $p : [0, 1] \to B(c, \epsilon)$ be the function $p(t) = (1 - t)a + tb$

PROOF: We have $p(t) \in B(c, \epsilon)$ for all t because

$$\begin{aligned} d(p(t), c) &= \|(1-t)a + tb - c\| \\ &= \|(1-t)(a-c) + t(b-c)\| \\ &\leq (1-t)\|a-c\| + t\|b-c\| \\ &< (1-t)\epsilon + t\epsilon \\ &= \epsilon \end{aligned}$$

$\langle 1 \rangle 3$. p is a path from a to b.
□

Proposition 12.2.14. *Let $n \geq 0$. For all $c \in \mathbb{R}^n$ and $\epsilon > 0$, the closed ball $\overline{B(c, \epsilon)}$ is path connected.*

PROOF:
$\langle 1 \rangle 1$. LET: $a, b \in \overline{B(c, \epsilon)}$
$\langle 1 \rangle 2$. LET: $p : [0,1] \to \overline{B(c, \epsilon)}$ be the function $p(t) = (1-t)a + tb$
PROOF: We have $p(t) \in \overline{B(c, \epsilon)}$ for all t because

$$\begin{aligned} d(p(t), c) &= \|(1-t)a + tb - c\| \\ &= \|(1-t)(a-c) + t(b-c)\| \\ &\leq (1-t)\|a-c\| + t\|b-c\| \\ &\leq (1-t)\epsilon + t\epsilon \\ &= \epsilon \end{aligned}$$

$\langle 1 \rangle 3$. p is a path from a to b.
□

Lemma 12.2.15. *If $\sum_{i=0}^{\infty} x_i^2$ and $\sum_{i=0}^{\infty} y_i^2$ converge then $\sum_{i=0}^{\infty} |x_i y_i|$ converges.*

PROOF:
$\langle 1 \rangle 1$. For all $N \geq 0$ we have $\sum_{i=0}^{N} |x_i y_i| \leq \sqrt{\sum_{i=0}^{N} |x_i|^2} \sqrt{\sum_{i=0}^{N} |y_i|^2}$
PROOF: By the Cauchy-Schwarz inequality
$\langle 1 \rangle 2$. Q.E.D.
PROOF: Since $\sum_{i=0}^{N} |x_i y_i|$ is an increasing sequence bounded above by $(\sum_{i=0}^{\infty} x_i^2)(\sum_{i=0}^{\infty} y_i^2)$.
□

Corollary 12.2.15.1. *If $\sum_{i=0}^{\infty} x_i^2$ and $\sum_{i=0}^{\infty} y_i^2$ converge then $\sum_{i=0}^{\infty} (x_i + y_i)^2$ converges.*

PROOF: Since $\sum_{i=0}^{\infty} x_i^2$, $\sum_{i=0}^{\infty} y_i^2$ and $2 \sum_{i=0}^{\infty} x_i y_i$ all converge.

Definition 12.2.16 (l^2-metric)**.** The *l^2-metric* on

$$\left\{ (x_n) \in \mathbb{R}^{\omega} \mid \sum_{n=0}^{\infty} x_n^2 \text{ converges} \right\}$$

is defined by

$$d(x, y) = \left(\sum_{n=0}^{\infty} (x_n - y_n)^2 \right)^{1/2}$$

We prove this is a metric.

Proof:
$\langle 1\rangle 1.$ d is well-defined.
Proof: By Corollary 12.2.15.1.
$\langle 1\rangle 2.$ $d(x, y) \geq 0$
$\langle 1\rangle 3.$ $d(x, y) = 0$ if and only if $x = y$
$\langle 1\rangle 4.$ $d(x, y) = d(y, x)$
$\langle 1\rangle 5.$ $d(x, z) \leq d(x, y) + d(y, z)$
Proof: By Lemma 12.2.10.
□

Theorem 12.2.17. *Addition is a continuous function $\mathbb{R}^2 \to \mathbb{R}$.*

Proof:
$\langle 1\rangle 1.$ Let: $a, b \in \mathbb{R}$
$\langle 1\rangle 2.$ Let: $\epsilon > 0$
$\langle 1\rangle 3.$ Let: $\delta = \epsilon/2$
$\langle 1\rangle 4.$ Let: $x, y \in \mathbb{R}$
$\langle 1\rangle 5.$ Assume: $\rho((a, b), (x, y)) < \delta$
$\langle 1\rangle 6.$ $|(a + b) - (x + y)| < \epsilon$
Proof:
$$\begin{aligned} |(a+b) - (x+y)| &= |a - x| + |b - y| \\ &\leq 2\rho((a, b), (x, y)) \\ &< 2\delta \\ &= \epsilon \end{aligned}$$
$\langle 1\rangle 7.$ Q.E.D.
Proof: Theorem 12.1.21
□

Theorem 12.2.18. *Multiplication is a continuous function $\mathbb{R}^2 \to \mathbb{R}$.*

Proof:
$\langle 1\rangle 1.$ Let: $a, b \in \mathbb{R}$
$\langle 1\rangle 2.$ Let: $\epsilon > 0$
$\langle 1\rangle 3.$ Let: $\delta = \min(\epsilon/(|a| + |b| + 1), 1)$
$\langle 1\rangle 4.$ Let: $x, y \in \mathbb{R}$
$\langle 1\rangle 5.$ Assume: $\rho((a, b), (x, y)) < \delta$
$\langle 1\rangle 6.$ $|ab - xy| < \epsilon$
Proof:
$$\begin{aligned} |ab - xy| &= |a(b - y) + (a - x)b - (a - x)(b - y)| \\ &\leq |a||b - y| + |b||a - x| + |a - x||b - y| \\ &< |a|\delta + |b|\delta + \delta^2 && (\langle 1\rangle 5) \\ &\leq |a|\delta + |b|\delta + \delta && (\langle 1\rangle 3) \\ &\leq \epsilon && (\langle 1\rangle 3) \end{aligned}$$
$\langle 1\rangle 7.$ Q.E.D.

PROOF: Theorem 12.1.21
∎

Theorem 12.2.19. *The function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by $f(x) = x^{-1}$ is continuous.*

PROOF:
⟨1⟩1. For all $a \in \mathbb{R}$ we have $f^{-1}((a, +\infty))$ is open.
PROOF: The set is

$$\begin{aligned} (a^{-1}, +\infty) & \text{if } a > 0 \\ (0, +\infty) & \text{if } a = 0 \\ (-\infty, a^{-1}) \cup (0, +\infty) & \text{if } a < 0 \end{aligned}$$

⟨1⟩2. For all $a \in \mathbb{R}$ we have $f^{-1}((-\infty, a))$ is open.
PROOF: Similar.
⟨1⟩3. Q.E.D.
PROOF: By Proposition 11.12.3 and Lemma 11.14.2.
∎

Definition 12.2.20. For $n \geq 0$, the *unit ball* B^n is the space $\{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$.

Proposition 12.2.21. *For all $n \geq 0$, the unit ball B^n is path connected.*

PROOF:
⟨1⟩1. LET: $a, b \in B^n$
⟨1⟩2. LET: $p : [0, 1] \to B^n$ be the function $p(t) = (1 - t)a + tb$
PROOF: We have $p(t) \in B^n$ for all t because

$$\begin{aligned} \|(1-t)a + tb\| &\leq (1-t)\|a\| + t\|b\| \\ &\leq (1-t) + t \\ &= 1 \end{aligned}$$

⟨1⟩3. p is a path from a to b.
∎

Definition 12.2.22 (Punctured Euclidean Space)**.** For $n \geq 0$, defined *punctured Euclidean space* to be $\mathbb{R}^n \setminus \{0\}$.

Proposition 12.2.23. *For $n > 1$, punctured Euclidean space $\mathbb{R}^n \setminus \{0\}$ is path connected.*

PROOF:
⟨1⟩1. LET: $a, b \in \mathbb{R}^n \setminus \{0\}$
⟨1⟩2. CASE: 0 is on the line from a to b
⟨2⟩1. PICK a point c not on the line from a to b
⟨2⟩2. The path consisting of a straight line from a to c followed by a straight line from c to b is a path from a to b.
⟨1⟩3. CASE: 0 is not on the line from a to b
PROOF: The straight line from a to b is a path from a to b.

Corollary 12.2.23.1. *For $n > 1$, the spaces $\mathbb{R}$ and $\mathbb{R}^n$ are not homeomorphic.*

Proof: For any point a, the space $\mathbb{R} \setminus \{a\}$ is disconnected.

Definition 12.2.24 (Unit Sphere)**.** For $n \geq 1$, the *unit sphere* S^{n-1} is the space

$$S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\} \ .$$

Proposition 12.2.25. *For $n > 1$, the unit sphere S^{n-1} is path connected.*

Proof: The map $g : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$ defined by $g(x) = x/\|x\|$ is continuous and surjective. The result follows by Proposition 11.32.5. ∎

Proposition 12.2.26. *Let $f : S^1 \to \mathbb{R}$ be continuous. Then there exists $x \in S^1$ such that $f(x) = f(-x)$.*

Proof:
⟨1⟩1. Let: $g : S^1 \to \mathbb{R}$ be the function $g(x) = f(x) - f(-x)$
Prove: There exists $x \in S^1$ such that $g(x) = 0$
⟨1⟩2. Assume: without loss of generality $g((1,0)) > 0$
⟨1⟩3. $g((-1,0)) < 0$
⟨1⟩4. There exists x such that $g(x) = 0$
Proof: By the Intermediate Value Theorem.
∎

Definition 12.2.27 (Topologist's Sine Curve)**.** Let $S = \{(x, \sin 1/x) \mid 0 < x \leq 1\}$. The *topologist's sine curve* is the closure $\overline{S}$ of S.

Proposition 12.2.28.

$$\overline{S} = S \cup (\{0\} \times [-1,1])$$

Proposition 12.2.29. *The topologist's sine curve is connected.*

Proof:
⟨1⟩1. Let: $S = \{(x, \sin 1/x) \mid 0 < x \leq 1\}$
⟨1⟩2. S is connected.
Proof: Theorem 11.30.13.
⟨1⟩3. $\overline{S}$ is connected.
Proof: Theorem 11.30.12.
∎

Proposition 12.2.30 (CC)**.** *The topologist's sine curve is not path connected.*

Proof:
⟨1⟩1. Assume: for a contradiction $p : [0,1] \to \overline{S}$ is a path from $(0,0)$ to $(1, \sin 1)$.
⟨1⟩2. $p^{-1}(\{0\} \times [0,1])$ is closed.
⟨1⟩3. Let: b be the greatest element of $p^{-1}(\{0\} \times [0,1])$.
⟨1⟩4. $b < 1$
Proof: Since $p(1) = (1, \sin 1)$.
⟨1⟩5. Pick a sequence $(t_n)_{n \geq 1}$ in $(b, 1]$ such that $t_n \to b$ and $\pi_2(p(t_n)) = (-1)^n$

⟨2⟩1. Let: $n \geq 1$
⟨2⟩2. Pick u with $0 < u < \pi_1(p(1/n))$ such that $\sin(1/u) = (-1)^n$
⟨2⟩3. Pick t_n such that $0 < t_n < 1/n$ and $\pi_1(p(t_n)) = u$
Proof: One exists by the Intermediate Value Theorem.
⟨1⟩6. Q.E.D.
Proof: This contradicts 11.12.18.
□

Theorem 12.2.31. *Let A be a subspace of $\mathbb{R}^n$. Then the following are equivalent:*

1. *A is compact.*
2. *A is closed and bounded under the Euclidean metric.*
3. *A is closed and bounded under the square metric.*

Proof:
⟨1⟩1. $1 \Rightarrow 2$
Proof: By Corollary 11.50.10.1 and Proposition 12.1.27.
⟨1⟩2. $2 \Rightarrow 3$
Proof: If $d(x, y) \leq M$ for all $x, y \in A$ then $\rho(x, y) \leq M/\sqrt{2}$.
⟨1⟩3. $3 \Rightarrow 1$
⟨2⟩1. Assume: A is closed and $\rho(x, y) \leq M$ for all $x, y \in A$
⟨2⟩2. Pick $a \in A$
Proof: We may assume w.l.o.g. A is nonempty since the empty space is compact.
⟨2⟩3. A is a closed subspace of $[a_1 - M, a_1 + M] \times \cdots \times [a_n - M, a_n + M]$
⟨2⟩4. A is compact
Proof: Proposition 11.50.3.
□

Corollary 12.2.31.1. *The unit sphere S^{n-1} and the closed unit ball B^n are compact for any n.*

12.3 The Uniform Topology

Definition 12.3.1 (Uniform Metric)**.** Let J be a set. The *uniform metric* $\overline{\rho}$ on $\mathbb{R}^J$ is defined by

$$\overline{\rho}(a, b) = \sup_{j \in J} \overline{d}(a_j, b_j)$$

where $\overline{d}$ is the standard bounded metric on $\mathbb{R}$.

The *uniform topology* on $\mathbb{R}^J$ is the topology induced by the uniform metric.

We prove this is a metric.

Proof:
⟨1⟩1. $\overline{\rho}(a, b) \geq 0$

PROOF: Immediate from definitions.
⟨1⟩2. $\overline{\rho}(a,b) = 0$ if and only if $a = b$
PROOF: Immediate from definitions.
⟨1⟩3. $\overline{\rho}(a,b) = \overline{\rho}(b,a)$
PROOF: Immediate from definitions.
⟨1⟩4. $\overline{\rho}(a,c) \leq \overline{\rho}(a,b) + \overline{\rho}(b,c)$
PROOF:

$$\begin{aligned}
\overline{\rho}(a,c) &= \sup_{j \in J} \overline{d}(a_j, c_j) \\
&\leq \sup_{j \in J} (\overline{d}(a_j, b_j) + \overline{d}(b_j, c_j)) \\
&\leq \sup_{j \in J} \overline{d}(a_j, b_j) + \sup_{j \in J} \overline{d}(b_j, c_j) \\
&= \overline{\rho}(a,b) + \overline{\rho}(b,c)
\end{aligned}$$

□

Proposition 12.3.2. *The uniform topology on $\mathbb{R}^J$ is finer than the product topology.*

PROOF:
⟨1⟩1. LET: $j \in J$ and U be open in $\mathbb{R}$
PROVE: ${\pi_j}^{-1}(U)$ is open in the uniform topology.
⟨1⟩2. LET: $a \in {\pi_j}^{-1}(U)$
⟨1⟩3. PICK $\epsilon > 0$ such that $(a_j - \epsilon, a_j + \epsilon) \subseteq U$
⟨1⟩4. $B_{\overline{\rho}}(a, \epsilon) \subseteq {\pi_j}^{-1}(U)$
□

Proposition 12.3.3. *The uniform topology on $\mathbb{R}^J$ is coarser than the box topology.*

PROOF:
⟨1⟩1. LET: $a \in \mathbb{R}^J$ and $\epsilon > 0$
PROVE: $B(a, \epsilon)$ is open in the box topology.
⟨1⟩2. LET: $b \in B(a, \epsilon)$
⟨1⟩3. For $j \in J$ we have $|a_j - b_j| < \epsilon$
⟨1⟩4. For $j \in J$,
LET: $\delta_j = (\epsilon - |a_j - b_j|)/2$
⟨1⟩5. $\prod_{j \in J} (b_j - \delta_j, b_j + \delta_j) \subseteq B(a, \epsilon)$
□

Proposition 12.3.4. *The uniform topology on $\mathbb{R}^J$ is strictly finer than the product topology if and only if J is infinite.*

PROOF:
⟨1⟩1. If J is finite then the uniform and product topologies coincide.
PROOF: The uniform, box and product topologies are all the same.
⟨1⟩2. If J is infinite then the uniform and product topologies are different.

PROOF: The set $B(\vec{0}, 1)$ is open in the uniform topology but not the product topology.
□

Proposition 12.3.5 (DC)**.** *The uniform topology on $\mathbb{R}^J$ is strictly coarser than the box topology if and only if J is infinite.*

PROOF:
$\langle 1 \rangle 1$. If J is finite then the uniform and box topologies coincide.
PROOF: The uniform, box and product topologies are all the same.
$\langle 1 \rangle 2$. If J is infinite then the uniform and box topologies are different.
PROOF: Pick an ω-sequence $(j_1, j_2, \ldots)$ in J. Let $U = \prod_{j \in J} U_j$ where $U_{j_i} = (-1/i, 1/i)$ and $U_j = (-1, 1)$ for all other j. Then $\vec{0} \in U$ but there is no $\epsilon > 0$ such that $B(\vec{0}, \epsilon) \subseteq U$.
□

Proposition 12.3.6. *The closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ under the uniform topology is $\mathbb{R}^\omega$.*

PROOF: Given any open ball $B(a, \epsilon)$, pick an integer N such that $1/\epsilon < N$. Then $B(a, \epsilon)$ includes sequences whose nth entry is 0 for all $n \geq N$. □

Example 12.3.7. The space $\mathbb{R}^\omega$ is disconnected under the uniform topology. The set of bounded sequences and the set of unbounded sequences form a separation.

Proposition 12.3.8. *Give $\mathbb{R}^\omega$ the uniform topology. Let $x, y \in \mathbb{R}^\omega$. Then x and y are in the same component if and only if $x - y$ is bounded.*

PROOF:
$\langle 1 \rangle 1$. The component containing 0 is the set of bounded sequences.
$\langle 2 \rangle 1$. LET: B be the set of bounded sequences.
$\langle 2 \rangle 2$. B is path-connected.
$\langle 3 \rangle 1$. LET: $x.y \in B$
$\langle 3 \rangle 2$. PICK $b > 0$ such that $|x_j|, |y_j| \leq b$ for all j
$\langle 3 \rangle 3$. LET: $p : [0, 1] \to B$ be the function $p(t) = (1 - t)x + ty$
PROVE: p is continuous.
$\langle 3 \rangle 4$. LET: $t \in [0, 1]$ and $\epsilon > 0$
$\langle 3 \rangle 5$. LET: $\delta = \epsilon/2b$
$\langle 3 \rangle 6$. LET: $s \in [0, 1]$ with $|s - t| < \delta$
$\langle 3 \rangle 7$. $\overline{\rho}(p(s), p(t)) < \epsilon$
PROOF:

$$\begin{aligned}
\overline{\rho}(p(s), p(t)) &= \sup_j \overline{d}((1 - s)x_j + sy_j, (1 - t)x_j + ty_j) \\
&\leq |(s - t)x_j + (t - s)y_j| \\
&\leq |s - t||x_j - y_j| \\
&< 2b\delta \\
&= \epsilon
\end{aligned}$$

$\langle 2\rangle 3$. B is connected.

PROOF: Proposition 11.32.3.

$\langle 2\rangle 4$. If C is connected and $B \subseteq C$ then $B = C$.

PROOF: Otherwise $B \cap C$ and $C \setminus B$ form a separation of C.

$\langle 1\rangle 2$. Q.E.D.

PROOF: Since $\lambda x.x - y$ is a Homeomorphism of $\mathbb{R}^\omega$ with itself.

□

Example 12.3.9. The space $[0,1]^\omega$ under the uniform topology is not locally compact.

It is not compact because the set $\{0,1\}^\omega$ has no limit point.

Now, assume for a contradiction $[0,1]^\omega$ is locally compact. Pick $\epsilon > 0$ such that $B(0,\epsilon)$ is included in a compact subspace. Then $\overline{B(0,\epsilon)}$ is compact. But $\overline{B(0,\epsilon)} = [0,1]^\omega$ if $\epsilon \geq 1$, or $[0,\epsilon]^\omega$ if $\epsilon < 1$. In either case $\overline{B(0,\epsilon)} \cong [0,1]^\epsilon$ which is not compact.

Example 12.3.10. The space $\mathbb{R}^\omega$ under the uniform topology is not second countable.

PROOF: The set $\{0,1\}^\omega$ is an uncountable discrete subspace. □

Proposition 12.3.11. *Every separable metrizable space is second countable.*

PROOF:

$\langle 1\rangle 1$. LET: X be a separable metrizable space.

$\langle 1\rangle 2$. PICK a countable dense subset D.

PROVE: $\{B(x, 1/n) \mid x \in D, n \in \mathbb{Z}^+\}$ is a basis for X

$\langle 1\rangle 3$. LET: $x \in X$

$\langle 1\rangle 4$. LET: U be a neighbourhood of x.

$\langle 1\rangle 5$. PICK n such that $B(x, 1/n) \subseteq U$.

$\langle 1\rangle 6$. PICK $d \in D \cap B(x, 1/2n)$

$\langle 1\rangle 7$. $x \in B(d, 1/2n) \subseteq U$.

□

Proposition 12.3.12 (AC)**.** *Every Lindelöf metrizable space is second countable.*

PROOF:

$\langle 1\rangle 1$. LET: X be a Lindelöf metrizable space.

$\langle 1\rangle 2$. For $n \in \mathbb{Z}^+$, PICK a countable set $\mathcal{A}_n$ of open balls of radius $1/n$ that covers X.

$\langle 1\rangle 3$. LET: $\mathcal{B} = \bigcup_{n=1}^\infty \mathcal{A}_n$

PROVE: $\mathcal{B}$ is a basis for X.

$\langle 1\rangle 4$. LET: $x \in X$

$\langle 1\rangle 5$. LET: U be a neighbourhood of x.

$\langle 1\rangle 6$. PICK n such that $B(x, 1/n) \subseteq U$

$\langle 1\rangle 7$. PICK $B \in \mathcal{A}_{2n}$ such that $x \in B$

$\langle 1\rangle 8$. $B \subseteq U$

PROOF: Since $\operatorname{diam} B \leq n$.

□

Example 12.3.13. The space $\mathbb{R}_l$ is not metrizable, because it is Lindelöf but not second countable.

Example 12.3.14. The ordered square is not metrizable, because it is compact but not separable.

12.4 Uniform Convergence

Definition 12.4.1 (Uniform Convergence)**.** Let X be a set and Y a metric space. Let $(f_n : X \to Y)$ be a sequence of functions and $f : X \to Y$ be a function. Then f_n *converges uniformly* to f as $n \to \infty$ if and only if, for all $\epsilon > 0$, there exists N such that, for all $n \geq N$ and $x \in X$, we have $d(f_n(x), f(x)) < \epsilon$.

Example 12.4.2. Define $f_n : [0,1] \to \mathbb{R}$ by $f_n(x) = x^n$ for $n \geq 1$, and $f : [0,1] \to \mathbb{R}$ by $f(x) = 0$ if $x < 1$, $f(1) = 1$. Then f_n converges to f pointwise but not uniformly.

Theorem 12.4.3 (Uniform Limit Theorem)**.** *Let X be a topological space and Y a metric space. Let $(f_n : X \to Y)$ be a sequence of continuous functions and $f : X \to Y$ be a function. If f_n converges uniformly to f as $n \to \infty$, then f is continuous.*

PROOF:
$\langle 1 \rangle 1.$ LET: $x \in X$ and $\epsilon > 0$
$\langle 1 \rangle 2.$ PICK N such that, for all $n \geq N$ and $y \in X$, we have $d(f_n(y), f(y)) < \epsilon/3$
$\langle 1 \rangle 3.$ PICK a neighbourhood U of x such that $f_N(U) \subseteq B(f_N(x), \epsilon/3)$
PROVE: $f(U) \subseteq B(f(x), \epsilon)$
$\langle 1 \rangle 4.$ LET: $y \in U$
$\langle 1 \rangle 5.$ $d(f(y), f(x)) < \epsilon$
PROOF:

$$\begin{aligned} d(f(y), f(x)) &\leq d(f(y), f_N(y)) + d(f_N(y), f_N(x)) + d(f_N(x), f(x)) && \text{(Triangle Inequality)} \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 && (\langle 1 \rangle 2,\ \langle 1 \rangle 3) \\ &= \epsilon \end{aligned}$$

□

Proposition 12.4.4. *Let X be a topological space and Y a metric space. Let $(f_n : X \to Y)$ be a sequence of continuous functions and $f : X \to Y$ be a function. Let (a_n) be a sequence of points in X and $a \in X$. If f_n converges uniformly to f and a_n converges to a in X then $f_n(a_n)$ converges to $f(a)$ uniformly in Y.*

PROOF:
$\langle 1 \rangle 1.$ LET: $\epsilon > 0$
$\langle 1 \rangle 2.$ PICK N_1 such that, for all $n \geq N_1$ and $x \in X$, we have $d(f_n(x), f(x)) < \epsilon/2$

$\langle 1 \rangle 3$. Pick N_2 such that, for all $n \geq N_2$, we have $a_n \in f^{-1}(B(a, \epsilon/2))$
Proof: Using the fact that f is continuous from the Uniform Limit Theorem.
$\langle 1 \rangle 4$. Let: $N = \max(N_1, N_2)$
$\langle 1 \rangle 5$. Let: $n \geq N$
$\langle 1 \rangle 6$. $d(f_n(a_n), f(a)) < \epsilon$
Proof:

$$\begin{aligned} d(f_n(a_n), f(a)) &\leq d(f_n(a_n), f(a_n)) + d(f(a_n), f(a)) && \text{(Triangle Inequality)} \\ &< \epsilon/2 + \epsilon/2 && (\langle 1 \rangle 2,\ \langle 1 \rangle 3) \\ &= \epsilon \end{aligned}$$

□

Proposition 12.4.5. *Let X be a set. Let $(f_n : X \to \mathbb{R})$ be a sequence of functions and $f : X \to \mathbb{R}$ be a function. Then f_n converges unifomly to f as $n \to \infty$ if and only if $f_n \to f$ as $n \to \infty$ in $\mathbb{R}^X$ under the uniform topology.*

Proof:
$\langle 1 \rangle 1$. If f_n converges uniformly to f then f_n converges to f under the uniform topology.
$\langle 2 \rangle 1$. Assume: f_n converges uniformly to f
$\langle 2 \rangle 2$. Let: $\epsilon > 0$
$\langle 2 \rangle 3$. Pick N such that, for all $n \geq N$ and $x \in X$, we have $d(f_n(x), f(x)) < \epsilon/2$
$\langle 2 \rangle 4$. For all $n \geq N$ we have $\overline{\rho}(f_n, f) \leq \epsilon/2$
$\langle 2 \rangle 5$. For all $n \geq N$ we have $\overline{\rho}(f_n, f) < \epsilon$
$\langle 1 \rangle 2$. If f_n converges to f under the uniform topology then f_n converges uniformly to f.
$\langle 2 \rangle 1$. Assume: f_n converges to f under the uniform topology.
$\langle 2 \rangle 2$. Let: $\epsilon > 0$
$\langle 2 \rangle 3$. Pick N such that, for all $n \geq N$, we have $\overline{\rho}(f_n, f) < \min(\epsilon, 1/2)$
$\langle 2 \rangle 4$. Let: $n \geq N$
$\langle 2 \rangle 5$. Let: $x \in X$
$\langle 2 \rangle 6$. $\overline{\rho}(f_n, f) < \min(\epsilon, 1/2)$
Proof: From $\langle 2 \rangle 3$.
$\langle 2 \rangle 7$. $d(f_n(x), f(x)) < \min(\epsilon, 1/2)$
$\langle 2 \rangle 8$. $d(f_n(x), f(x)) < \epsilon$
□

12.5 Isometric Imbeddings

Definition 12.5.1. Let X and Y be metric spaces. An *isometric imbedding* $f : X \to Y$ is a function such that, for all $x, y \in X$, we have $d(f(x), f(y)) = d(x, y)$.

Proposition 12.5.2. *Every isometric imbedding is an imbedding.*

Proof:

$\langle 1\rangle 1$. LET: $f : X \to Y$ be an isometric imbedding.
$\langle 1\rangle 2$. f is injective.
PROOF: If $f(x) = f(y)$ then $d(f(x), f(y)) = 0$ hence $d(x, y) = 0$ hence $x = y$.
$\langle 1\rangle 3$. f is continuous.
PROOF: For all $\epsilon > 0$, if $d(x, y) < \epsilon$ then $d(f(x), f(y)) < \epsilon$.
$\langle 1\rangle 4$. $f : X \to f(X)$ is an open map.
PROOF: $f(B(a, \epsilon)) = B(f(a), \epsilon) \cap f(X)$.
□

12.6 Distance to a Set

Definition 12.6.1. Let X be a metric space, $x \in X$ and $A \subseteq X$ be nonempty. The *distance* from x to A is defined as

$$d(x, A) = \inf_{a \in A} d(x, a) \ .$$

Proposition 12.6.2. *Let X be a metric space and $A \subseteq X$ be nonempty. Then the function $d(-, A) : X \to \mathbb{R}$ is continuous.*

PROOF:
$\langle 1\rangle 1$. LET: X be a metric space.
$\langle 1\rangle 2$. LET: $A \subseteq X$ be nonempty.
$\langle 1\rangle 3$. LET: $x \in X$ and $\epsilon > 0$
$\langle 1\rangle 4$. LET: $\delta = \epsilon$
$\langle 1\rangle 5$. LET: $y \in B(x, \delta)$
$\langle 1\rangle 6$. $|d(x, A) - d(y, A)| < \epsilon$
$\langle 2\rangle 1$. $d(x, A) - d(y, A) < \epsilon$
PROOF:
$\langle 3\rangle 1$. For all $a \in A$ we have $d(x, A) \leq d(x, y) + d(y, a)$
PROOF:

$$\begin{aligned} d(x, A) &\leq d(x, a) && \text{(definition of } d(x, A)\text{)} \\ &\leq d(x, y) + d(y, a) && \text{(Triangle Inequality)} \end{aligned}$$

$\langle 3\rangle 2$. $d(x, A) - d(x, y) \leq d(y, A)$
$\langle 2\rangle 2$. $d(y, A) - d(x, A) < \epsilon$
PROOF: Similar.
$\langle 1\rangle 7$. Q.E.D.
PROOF: Theorem 12.1.21.
□

Theorem 12.6.3. *Let X be a metric space, $A \subseteq X$ be nonempty, and $x \in X$. Then $d(x, A) = 0$ if and only if $x \in \overline{A}$.*

PROOF:
$\langle 1\rangle 1$. LET: X be a metric space.
$\langle 1\rangle 2$. LET: $A \subseteq X$ be nonempty.
$\langle 1\rangle 3$. LET: $x \in X$

⟨1⟩4. If $d(x, A) = 0$ then $x \in \overline{A}$
⟨2⟩1. Assume: $d(x, A) = 0$
⟨2⟩2. Let: U be any neighbourhood of x.
⟨2⟩3. Pick $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$
Proof: Proposition 12.1.4, ⟨1⟩1, ⟨2⟩2.
⟨2⟩4. Pick $a \in A$ such that $d(x, a) < \epsilon$
Proof: From ⟨2⟩1, ⟨2⟩3.
⟨2⟩5. $a \in A \cap U$
Proof: From ⟨2⟩3, ⟨2⟩4.
⟨2⟩6. Q.E.D.
Proof: Theorem 11.4.6.
⟨1⟩5. If $x \in \overline{A}$ then $d(x, A) = 0$
□

Theorem 12.6.4. *Let X be a metric space. Let $A \subseteq X$ be nonempty and compact. Let $x \in X$. Then there exists $a \in A$ such that $d(x, A) = d(x, a)$.*

Proof: By the Extreme Value Theorem, the function $d(x, -) : A \to \mathbb{R}$ attains its minimum. □

12.7 Lebesgue Numbers

Definition 12.7.1 (Lebesgue Number)**.** Let X be a metric space. Let $\mathcal{U}$ be an open covering of X. A *Lebesgue number* for $\mathcal{U}$ is a real number $\delta > 0$ such that, for every subset $A \subseteq X$ with diameter diameter $< \delta$, there exists $U \in \mathcal{U}$ such that $A \subseteq U$.

Theorem 12.7.2 (Lebesgue Number Lemma)**.** *Every open covering of a compact metric space has a Lebesgue number.*

Proof:
⟨1⟩1. Let: X be a compact metric space.
⟨1⟩2. Let: $\mathcal{U}$ be an open covering of X.
⟨1⟩3. Pick a finite subset $\{U_1, \ldots, U_n\}$ of $\mathcal{U}$ that covers X.
⟨1⟩4. For $i = 1, \ldots, n$,
Let: $C_i = X - U_i$
⟨1⟩5. Let: $f : X \to \mathbb{R}$,

$$f(x) = 1/n \sum_{i=1}^{n} d(x, C_i)$$

⟨1⟩6. For all $x \in X$ we have $f(x) > 0$
⟨2⟩1. Let: $x \in X$
⟨2⟩2. Pick i such that $x \in U_i$
Proof: From ⟨1⟩3.
⟨2⟩3. Pick $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U_i$
Proof: Proposition 12.1.4.
⟨2⟩4. $d(x, C_i) \geq \epsilon$

⟨2⟩5. $f(x) \geq \epsilon/n$

⟨1⟩7. f is continuous.

PROOF: Proposition 12.6.2.

⟨1⟩8. LET: δ be the minimum value of $f(X)$

PROOF: By the Extreme Value Theorem

⟨1⟩9. $\delta > 0$

PROOF: From ⟨1⟩6

⟨1⟩10. For every subset $A \subseteq X$ with diameter $< \delta$, there exists $U \in \mathcal{U}$ such that $A \subseteq U$

⟨2⟩1. LET: $A \subseteq X$ with $\operatorname{diam} A < \delta$

⟨2⟩2. PICK $x_0 \in A$

⟨2⟩3. $A \subseteq B(x_0, \delta)$

⟨2⟩4. $f(x_0) \geq \delta$

⟨2⟩5. PICK m such that $d(x_0, C_m)$ is the largest out of $d(x_0, C_1), \ldots, d(x_0, C_n)$

⟨2⟩6. $d(x_0, C_m) \geq f(x_0)$

⟨2⟩7. $B(x_0, \delta) \subseteq U_m$

⟨2⟩8. $A \subseteq U_m$

⟨1⟩11. δ is a Lebesgue number for $\mathcal{U}$

□

Theorem 12.7.3 (AC)**.** *Every sequentially compact metric space is compact.*

PROOF:

⟨1⟩1. LET: X be a sequentially comapct metric space.

⟨1⟩2. Every open covering of X has a Lebesgue number.

⟨2⟩1. LET: $\mathcal{A}$ be an open covering of X.

⟨2⟩2. ASSUME: for a contradiction $\mathcal{A}$ has no Lebesgue number.

⟨2⟩3. For $n \geq 1$, PICK a set C_n with diameter $< 1/n$ that is not included in any member of $\mathcal{A}$.

⟨2⟩4. For $n \geq 1$, PICK $x_n \in C_n$.

⟨2⟩5. PICK a convergent subsequence (C_{n_r}) of (C_n) with limit a.

⟨2⟩6. PICK $A \in \mathcal{A}$ such that $a \in A$

⟨2⟩7. PICK $\epsilon > 0$ such that $B(a, \epsilon) \subseteq A$.

⟨2⟩8. PICK r such that $1/n_r < \epsilon/2$ and $d(x_{n_r}, a) < \epsilon/2$

⟨2⟩9. $C_{n_r} \subseteq B(a, \epsilon)$

⟨2⟩10. $C_{n_r} \subseteq A$

⟨2⟩11. Q.E.D.

PROOF: This contradicts ⟨2⟩3.

⟨1⟩3. For every $\epsilon > 0$, there exists a finite covering of X by ϵ-balls.

⟨2⟩1. ASSUME: for a contradiction that there exists $\epsilon > 0$ such that X cannot be finitely covered by ϵ-balls.

⟨2⟩2. PICK a sequence of points (x_n) such that $x_n \in X - (B(x_1, \epsilon) \cup \cdots \cup B(x_{n-1}, \epsilon))$

⟨2⟩3. $d(x_m, x_n) \geq \epsilon$ for all m, n distinct

⟨2⟩4. (x_n) has no convergent subsequence

⟨2⟩5. Q.E.D.

PROOF: This contradicts ⟨1⟩1.

$\langle 1\rangle 4$. LET: $\mathcal{A}$ be an open covering of X.
$\langle 1\rangle 5$. PICK a Lebesgue number δ for $\mathcal{A}$.
  PROOF: By $\langle 1\rangle 2$.
$\langle 1\rangle 6$. LET: $\epsilon = \delta/3$
$\langle 1\rangle 7$. PICK a finite covering $\{B_1, \ldots, B_n\}$ of X be ϵ-balls.
  PROOF: By $\langle 1\rangle 3$.
$\langle 1\rangle 8$. For $i = 1, \ldots, n$, PICK $U_i \in \mathcal{A}$ such that $B_i \subseteq A_i$
  PROOF: By $\langle 1\rangle 5$ since $\operatorname{diam} B_i = 2\epsilon < \delta$.
$\langle 1\rangle 9$. $\{U_1, \ldots, U_n\}$ covers X.
□

Example 12.7.4. The space S_Ω is not metrizable, because it is sequentially compact but not compact.

12.8 Uniform Continuity

Definition 12.8.1 (Uniformly Continuous)**.** Let X and Y be metric spaces. Let $f : X \to Y$. Then f is *uniformly continuous* if and only if, for every $\epsilon > 0$, there exists $\delta > 0$ such that, for all $x, y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$.

Theorem 12.8.2 (Uniform Continuity Theorem)**.** *Every continuous function from a compact metric space to a metric space is uniformly continuous.*

PROOF:
$\langle 1\rangle 1$. LET: X be a compact metric space.
$\langle 1\rangle 2$. LET: Y be a metric space.
$\langle 1\rangle 3$. LET: $f : X \to Y$ be a continuous function.
$\langle 1\rangle 4$. LET: $\epsilon > 0$
$\langle 1\rangle 5$. LET: $\mathcal{U} = \{f^{-1}(B(y, \epsilon/2)) \mid y \in Y\}$
$\langle 1\rangle 6$. PICK a Lebesgue number $\delta > 0$ for $\mathcal{U}$.
  PROOF: By the Lebesgue Number Lemma.
$\langle 1\rangle 7$. LET: $x, x' \in X$
$\langle 1\rangle 8$. ASSUME: $d(x, x') < \delta$
$\langle 1\rangle 9$. PICK $y \in Y$ such that $\{x, x'\} \subseteq f^{-1}(B(y, \epsilon/2))$
  PROOF: Since $\operatorname{diam}\{x, x'\} < \delta$.
$\langle 1\rangle 10$. $d(f(x), f(x')) < \epsilon$
  PROOF:

$$\begin{aligned} d(f(x), f(x')) &\leq d(f(x), y) + d(y, f(x')) && \text{(Triangle Inequality)} \\ &< \epsilon/2 + \epsilon/2 && (\langle 1\rangle 9) \\ &= \epsilon \end{aligned}$$

□

12.9 Epsilon-neighbourhoods

Definition 12.9.1 (ϵ-neighbourhood)**.** Let X be a metric space. Let $A \subseteq X$ be nonempty. Let $\epsilon > 0$. Then the *ϵ-neighbourhood* of A, $U(A, \epsilon)$, is the set

$$U(A, \epsilon) = \{x \in X \mid d(x, A) < \epsilon\} \ .$$

Proposition 12.9.2. *Let X be a metric space. Let $A \subseteq X$ be nonempty. Let $\epsilon > 0$. Then $U(A, \epsilon) = \bigcup_{a \in A} B(a, \epsilon)$.*

Proof:
⟨1⟩1. Let: X be a metric space.
⟨1⟩2. Let: $A \subseteq X$ be nonempty.
⟨1⟩3. Let: $\epsilon > 0$
⟨1⟩4. $U(A, \epsilon) \subseteq \bigcup_{a \in A} B(a, \epsilon)$
  ⟨2⟩1. Let: $x \in U(A, \epsilon)$
  ⟨2⟩2. $d(x, A) < \epsilon$
  ⟨2⟩3. ϵ is not a lower bound for $\{d(x, a) \mid a \in A\}$
  ⟨2⟩4. Pick $a \in A$ such that $d(x, a) < \epsilon$
  ⟨2⟩5. $x \in B(a, \epsilon)$
⟨1⟩5. $\bigcup_{a \in A} B(a, \epsilon) \subseteq U(A, \epsilon)$
  ⟨2⟩1. Let: $a \in A$ and $x \in B(a, \epsilon)$
  ⟨2⟩2. $d(x, A) \leq d(x, a)$
  ⟨2⟩3. $d(x, A) < \epsilon$
  ⟨2⟩4. $x \in U(A, \epsilon)$
□

Proposition 12.9.3. *Let X be a metric space. Let $A \subseteq X$ be nonempty and compact. Let U be an open set such that $A \subseteq U$. Then there exists $\epsilon > 0$ such that $U(A, \epsilon) \subseteq U$.*

Proof:
⟨1⟩1. Let: X be a metric space.
⟨1⟩2. Let: $A \subseteq X$ be nonempty and compact.
⟨1⟩3. Let: U be an open set such that $A \subseteq U$
⟨1⟩4. $\{B(a, \epsilon) \mid a \in A, \epsilon > 0, B(a, 2\epsilon) \subseteq U\}$ covers A.
  Proof: By Proposition 12.1.4.
⟨1⟩5. Pick a finite subcover $\{B(a_1, \epsilon_1), \ldots, B(a_n, \epsilon_n)\}$
  Proof: Since A is compact (⟨1⟩2).
⟨1⟩6. Let: $\epsilon = \min(\epsilon_1, \ldots, \epsilon_n)$
    Prove: $U(A, \epsilon) \subseteq U$
⟨1⟩7. Let: $x \in U(A, \epsilon)$
⟨1⟩8. Pick $a \in A$ such that $d(x, a) < \epsilon$
  Proof: Proposition 12.9.2.
⟨1⟩9. Pick i such that $a \in B(a_i, \epsilon_i)$
  Proof: By ⟨1⟩5.
⟨1⟩10. $d(x, a_i) < 2\epsilon$
  Proof: By the Triangle Inequality.

$\langle 1\rangle 11.$ $x \in U$
PROOF: From $\langle 1\rangle 4$.
□

This example shows that we cannot weaken the hypothesis that A is compact to A being closed:

Example 12.9.4. Let $X = \mathbb{R}^2$. Let $A = \{(x, 1/x) \mid x > 0\}$. Let $U = \{(x, y) \mid x > 0, y > 0\}$. Then A is nonempty and closed (Proposition 11.50.14). The set U is open and $A \subseteq U$. But there is no $\epsilon > 0$ such that $U(A, \epsilon) \subseteq U$.

PROOF:
$\langle 1\rangle 1.$ LET: $\epsilon > 0$
$\langle 1\rangle 2.$ $(2/\epsilon, \epsilon/2) \in A$
$\langle 1\rangle 3.$ $(2/\epsilon, 0) \in U(A, \epsilon)$
$\langle 1\rangle 4.$ $(2/\epsilon, 0) \notin U$
□

12.10 Isometry

Definition 12.10.1 (Isometry)**.** Let X be a metric space. An *isometry* of X is a function $f : X \to X$ such that, for all $x, y \in X$, we have $d(x, y) = d(f(x), f(y))$.

Proposition 12.10.2. *An isometry on a compact metric space is a homeomorphism.*

PROOF:
$\langle 1\rangle 1.$ LET: X be a compact metric space.
$\langle 1\rangle 2.$ LET: $f : X \to X$ be an isometry.
$\langle 1\rangle 3.$ f is an imbedding
PROOF: Proposition 12.5.2.
$\langle 1\rangle 4.$ f is surjective.
$\langle 2\rangle 1.$ ASSUME: for a contradiction $a \notin f(X)$
$\langle 2\rangle 2.$ $f(X)$ is closed
PROOF: Proposition 11.50.12.
$\langle 2\rangle 3.$ PICK $\epsilon > 0$ such that $B(a, \epsilon) \cap f(X) = \emptyset$
$\langle 2\rangle 4.$ For $m, n \in \mathbb{N}$ with $m \neq n$, we have $d(f^m(a), f^n(a)) \geq \epsilon$
$\langle 3\rangle 1.$ ASSUME: without loss of generality $m < n$
$\langle 3\rangle 2.$ $d(a, f^{n-m}(a)) \geq \epsilon$
PROOF: $\langle 2\rangle 3$
$\langle 3\rangle 3.$ $d(f^m(a), f^n(a)) \geq \epsilon$
PROOF: $\langle 1\rangle 2$
$\langle 2\rangle 5.$ The sequence $(f^n(a))$ has a convergent subsequence.
PROOF: Corollary 11.46.2.1, $\langle 1\rangle 1$, Corollary 12.1.19.1.
$\langle 2\rangle 6.$ Q.E.D.
PROOF: $\langle 2\rangle 4$ and $\langle 2\rangle 5$ form a contradiction.
□
□

12.11 Shrinking Maps

Definition 12.11.1 (Shrinking Map)**.** Let X be a metric space. Let $f : X \to X$. Then f is a *shrinking map* if and only if, for all $x, y \in X$ with $x \neq y$, we have $d(f(x), f(y)) < d(x, y)$.

Proposition 12.11.2. *Let X be a compact metric space. Let $f : X \to X$ be a contraction. Then f has a unique fixed point.*

PROOF:
$\langle 1 \rangle 1.$ LET: $A_n = f^n(X)$ for $n \geq 1$
$\langle 1 \rangle 2.$ For all $n \geq 1$ we have A_n is closed.
PROOF: Proposition 11.50.12.
$\langle 1 \rangle 3.$ LET: $A = \bigcap_{n=1}^{\infty} A_n$
$\langle 1 \rangle 4.$ PICK $a \in A$
PROOF: Proposition 11.47.6.
$\langle 1 \rangle 5.$ $f(A) = A$
$\langle 2 \rangle 1.$ $f(A) \subseteq A$
$\langle 2 \rangle 2.$ $A \subseteq f(A)$
$\langle 3 \rangle 1.$ LET: $x \in A$
$\langle 3 \rangle 2.$ For $n \geq 1$, PICK x_n such that $x = f^n(x_n)$
$\langle 3 \rangle 3.$ PICK a convergent subsequence $(f^{n_r - 1}(x_{n_r}))$ of $(f^{n-1}(x_n))$ with limit l
PROOF: Corollary 11.46.2.1.
$\langle 3 \rangle 4.$ $f(l) = x$
PROOF: Both are the limit of $f(f^{n_r - 1}(a_{n_r})) = f^{n_r}(a_{n_r})$.
$\langle 3 \rangle 5.$ $l \in A$
$\langle 4 \rangle 1.$ ASSUME: for a contradiction $l \notin A$
$\langle 4 \rangle 2.$ PICK N such that $l \notin A_N$
$\langle 4 \rangle 3.$ PICK R such that $n_R > N$
$\langle 4 \rangle 4.$ For $r \geq R$ we have $f^{n_r - 1}(a_{n_r}) \in A_N$
$\langle 4 \rangle 5.$ Q.E.D.
PROOF: This is a contradiction.
$\langle 1 \rangle 6.$ $\operatorname{diam} A = A$
$\langle 2 \rangle 1.$ PICK $x, y \in A$ such that $d(x, y) = \operatorname{diam} A$
PROOF: By the Extreme Value Theorem.
$\langle 2 \rangle 2.$ PICK $x', y' \in A$ such that $x = f(x')$ and $y = f(y')$
PROOF: By $\langle 1 \rangle 5$.
$\langle 2 \rangle 3.$ $x' = y'$
PROOF: If $x' \neq y'$ then $\operatorname{diam} A = d(x, y) < d(x', y')$ which is a contradiction.
$\langle 2 \rangle 4.$ $x = y$
$\langle 1 \rangle 7.$ $f(a) = a$
PROOF: Since $a, f(a) \in A$
$\langle 1 \rangle 8.$ If $f(b) = b$ then $b = a$
PROOF: If $f(b) = b$ then $b \in A$.
□

The following example shows that we cannot weaken the hypothesis from 'X is a compact metric space' to 'X is a complete metric space'.

Example 12.11.3. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = [x+(x^2+1)^{1/2}]/2$ is a shrinking map with no fixed point.

12.12 Contractions

Definition 12.12.1 (Contraction)**.** Let X be a metric space. Let $f : X \to X$. Then f is a *contraction* if and only if there exists $\alpha < 1$ such that, for all $x, y \in X$, we have $d(f(x), f(y)) \leq \alpha d(x, y)$.