The Universe

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Contents

Chapter 1

Category Theory

1.1 Categories

Definition 1.1 (Category)**.** A *category* $\mathcal{C}$ consists of:

- a class $\mathcal{C}$ of *objects*;
- for any objects $X, Y \in \mathcal{C}$, a set $\mathcal{C}[X, Y]$ of *morphisms*. We write $f : X \to Y$ for $f \in \mathcal{C}[X, Y]$
- for any object $X \in \mathcal{C}$, an *identity* morphism $\mathrm{id}_X : X \to X$
- for any morphisms $f : X \to Y$ and $g : Y \to Z$, a morphism $g \circ f : X \to Z$, the *composite* of f and g

such that:

Unit Laws For any $f : X \to Y$ we have $f = \mathrm{id}_Y \circ f = f \circ \mathrm{id}_X$

Associativity For any $f : X \to Y$, $g : Y \to Z$ and $h : Z \to W$, we have

$$h \circ (g \circ f) = (h \circ g) \circ f$$

Chapter 2

Topology

2.1 Topologies and Topological Spaces

Definition 2.1 (Topology)**.** Let X be a set. A *topology* on X is a set $\mathcal{T} \subseteq \mathcal{P}X$ such that:

1. $X \in \mathcal{T}$
2. $\forall \mathcal{U} \subseteq \mathcal{T}. \bigcup \mathcal{U} \in \mathcal{T}$
3. $\forall U, V \in \mathcal{T}. U \cap V \in \mathcal{T}$

Definition 2.2 (Topological Space)**.** A *topological space* X consists of a set X and a topology $\mathcal{T}$ on X. We call the elements of X *points* and the elements of $\mathcal{T}$ *open sets*.

Definition 2.3 (Discrete Topology)**.** Let X be a set. The *discrete* topology on X is $\mathcal{P}X$.

Definition 2.4 (Indiscrete Topology)**.** Let X be a set. The *indiscrete* or *trivial* topology on X is $\{\emptyset, X\}$.

Definition 2.5 (Open Neighbourhood)**.** Let X be a topological space. Let $x \in X$ and $U \subseteq X$. Then U is an *open Neighbourhood* of x if and only if $x \in U$ and U is open.

Definition 2.6 (Coarser, Finer)**.** Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on the same set X. Then $\mathcal{T}$ is *coarser*, *smaller* or *weaker* than $\mathcal{T}'$, and $\mathcal{T}'$ is *finer*, *larger* or *stronger* than $\mathcal{T}$, if and only if $\mathcal{T} \subseteq \mathcal{T}'$.

Proposition 2.7. *Let X be a set. The intersection of a set of topologies on X is a topology on X.*

Corollary 2.7.1. *Let X be a set. The poset of topologies on X is a complete lattice.*

2.2 Closed Sets

Definition 2.8 (Closed Set)**.** Let X be a topological space and $C \subseteq X$. Then C is *closed* if and only if $X - C$ is open.

2.3 Basis for a Topology

Definition 2.9 (Basis for a Topology)**.** Let X be a set. A *basis* for a topology on X is a set $\mathcal{B} \subseteq \mathcal{P}X$ such that:

1. $\bigcup \mathcal{B} = X$

2. $\forall B_1, B_2 \in \mathcal{B}. \forall x \in B_1 \cap B_2. \exists B_3 \in \mathcal{B}. x \in B_3 \subseteq B_1 \cap B_2$

The topology *generated* by $\mathcal{B}$ is then the coarsest topology that includes $\mathcal{B}$.

Given $x \in X$, a *basic open neighbourhood* of x is a set $B \in \mathcal{B}$ such that $x \in B$.

Proposition 2.10. *Let X be a set and $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Let $U \subseteq X$. Then $U \in \mathcal{T}$ if and only if $\forall x \in U. \exists B \in \mathcal{B}. x \in B \subseteq U$.*

2.4 Continuous Functions

Definition 2.11 (Continuous)**.** Let X and Y be topological spaces and $f : X \to Y$. Then f is *continuous* if and only if, for any open set V in Y, we have $f^{-1}(V)$ is open in X.

Proposition 2.12. *For any topological space X, the identity function on X is continuous.*

Proposition 2.13. *Let X, Y and Z be topological spaces. Let $f : X \to Y$ and $g : Y \to Z$ be continuous functions. Then $g \circ f$ is continuous.*

Chapter 3

Metric Spaces

3.1 Metrics

Definition 3.1 (Metric, Metric Space)**.** Let X be a set. A *metric* on X is a function $d : X^2 \to \mathbb{R}$ such that:

1. $\forall x, y \in X . d(x, y) \geq 0$

2. $\forall x, y \in X . d(x, y) = 0 \Leftrightarrow x = y$

3. $\forall x, y \in X . d(x, y) = d(y, x)$

4. $\forall x, y, z \in X . d(x, z) \leq d(x, y) + d(y, z)$

A *metric space* X consists of a set X and a metric on X.

Definition 3.2 (Open Ball)**.** Let X be a metric space. Let $x \in X$ and $\epsilon > 0$. The *open ball* with *center* x and *radius* ϵ is $B(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon\}$.

Definition 3.3 (Metric Topology)**.** On any metric space, the *metric topology* is the topology generated by the basis consisting of the open balls.

Definition 3.4 (Metrizable)**.** A topological space X is *metrizable* if and only if there exists a metric d on X such that the topology on X is the metric topology induced by d.

Definition 3.5 (Euclidean Metric)**.** The *Euclidean metric* on $\mathbb{R}^n$ is defined by

$$d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2} \; .$$

We write just $\mathbb{R}^n$ for the metric space $\mathbb{R}^n$ under the Euclidean metric.

3.2 Subspaces

Proposition 3.6. *Let X be a set and $Y \subseteq X$. Let d be a metric on X. Then $d \restriction Y^2$ is a metric on Y.*

Given a metric space (X, d) and a set $Y \subseteq X$, we will write just Y for the metric space $(Y, d \restriction Y^2)$.

Definition 3.7 (Interval)**.** The *interval* I is the metric space $I = [0, 1]$ as a subspace of $\mathbb{R}$.

Definition 3.8 (Disk)**.** Let $n \in \mathbb{Z}^+$. The *n-disk* D^n is the metric space

$$D^n = \{x \in \mathbb{R}^n \mid d(x, 0) \leq 1\}$$

as a subspace of $\mathbb{R}^n$.

Definition 3.9 (Sphere)**.** Let $n \in \mathbb{Z}^+$. The *n-sphere* S^n is the metric space

$$D^n = \{x \in \mathbb{R}^{n+1} \mid d(x, 0) = 1\}$$

as a subspace of $\mathbb{R}^{n+1}$.