Topology

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Contents

Part I

Set Theory

Chapter 1

Set Theory

1.1 Membership

We take as undefined the binary relation of *membership*, $\in$. If $a \in A$ we say a is a *member* or *element* of A. If this does not hold, we write $a \notin A$.

Axiom 1.1 (Axiom of Extensionality)**.** *Two sets with exactly the same elements are equal.*

1.2 Subsets

Definition 1.2 (Subset)**.** Let A and B be sets. We say A is a *subset* of B, $A \subseteq B$, if and only if every member of A is a member of B.

1.3 Abstraction Notation

Definition Schema 1.3 (Extensionality)**.** Let $P(x)$ be a property. If there is a set whose members are exactly the sets x such that $P(x)$, then we denote this set by $\{x \mid P(x)\}$.

It is unique by the Axiom of Extensionality.

1.4 The Empty Set

Axiom 1.4 (Empty Set Axiom)**.** *There exists a set with no members.*

Definition 1.5 (Empty Set (Extensionality, Empty Set Axiom))**.** The *empty set* $\emptyset$ is the set with no members $\{x \mid \bot\}$.

1.5 Pair Sets

Axiom 1.6 (Pairing Axiom)**.** *For any sets u and v, there exists a set having as members just u and v.*

Definition 1.7 (Pair Set (Extensionality, Pairing Axiom))**.** For any sets u and v, the *pair set* $\{u, v\}$ is the set $\{x \mid x = u \vee x = v\}$.

1.6 Unions

Axiom 1.8 (Union Axiom)**.** *For any set A, there exists a set whose elements are exactly the members of the members of A.*

Definition 1.9 (Union (Extensionality, Union))**.** For any set A, the *union* $\bigcup A$ is the set $\{x \mid \exists b \in A . x \in b\}$.

Definition 1.10 (Union (Extensionality, Pair Set, Union))**.** For any sets a and b, the *union* $a \cup b$ is the set $\bigcup\{a, b\}$.

1.7 Power Set

Axiom 1.11 (Power Set Axiom)**.** *For any set a, there is a set whose members are exactly the subsets of a.*

Definition 1.12 (Power Set (Extensionality, Power Set))**.** For any set a, the *power set* $\mathcal{P}a$ is the set $\{x \mid x \subseteq a\}$.

1.8 Singletons

Definition 1.13 (Singleton (Extensionality, Pair Set))**.** Given any x, define the *singleton* $\{x\}$ to be $\{x, x\}$.

1.9 Finite Sets

Definition Schema 1.14 (Extensionality, Pair Set, Union)**.** Given any objects $a_1, \ldots, a_n$, define the set $\{a_1, \ldots, a_n\}$ as follows:

$$\{a_1, \ldots, a_n, a_{n+1}\} = \{a_1, \ldots, a_n\} \cup \{a_{n+1}\} \ .$$

1.10 Subset Axioms

Axiom Schema 1.15 (Subset Axioms, Aussonderung Axioms)**.** *For any property $P(x)$ and any set B, there exists a set whose members are exactly the sets $x \in B$ such that $P(x)$.*

Definition Schema 1.16 (Extensionality, Subset)**.** For any property $P(x)$ and any set B, we write $\{x \in B \mid P(x)\}$ for $\{x \mid x \in B \wedge P(x)\}$.

Theorem 1.17 (Subset)**.** *There is no set to which every set belongs.*

PROOF:
$\langle 1 \rangle 1$. LET: A be a set.
PROVE: There exists a set that does not belong to A.
$\langle 1 \rangle 2$. PICK a set B whose members are exactly the sets $x \in A$ such that $x \notin x$.
PROOF: By a Subset Axiom.
$\langle 1 \rangle 3$. If $B \in A$ then we have $B \in B \Leftrightarrow B \notin B$
$\langle 1 \rangle 4$. $B \notin A$
□

1.11 Intersection

Definition 1.18 (Intersection (Extensionality, Subset))**.** For any sets a and b, the *intersection* $a \cap b$ is $\{x \in a \mid x \in b\}$.

Theorem 1.19 (Extensionality, Subset)**.** *For any nonempty set A, there exists a unique set B such that, for any x, we have $x \in B$ if and only if x belongs to every member of A.*

PROOF:
$\langle 1 \rangle 1$. LET: A be a nonempty set.
$\langle 1 \rangle 2$. PICK $a \in A$
$\langle 1 \rangle 3$. LET: $B = \{x \in a \mid \forall y \in A. x \in y\}$
$\langle 1 \rangle 4$. B is the unique set such that, for any x, we have $x \in B$ if and only if x belongs to every member of A.
□

Definition 1.20 (Intersection (Extensionality, Subset))**.** For any nonempty set A, the *intersection* $\bigcap A$ is the set whose elements are those sets that belong to every member of A.

1.12 Relative Complement

Definition 1.21 (Relative Complement (Extensionality, Subset))**.** For any sets A and B, the *relative complement* $A - B$ is $\{x \in A \mid x \notin B\}$.

1.13 Covers

Definition 1.22 (Cover)**.** Let X be a set and $\mathcal{A} \subseteq \mathcal{P}X$. Then $\mathcal{A}$ *covers* X, or is a *covering* of X, if and only if $\bigcup \mathcal{A} = X$.

Chapter 2

Relations

2.1 Ordered Pairs

Definition 2.1 (Ordered Pair (Extensionality, Pairing))**.** For any sets x and y, the *ordered pair* (x, y) is defined to be $\{\{x\}, \{x, y\}\}$.

Theorem 2.2 (Extensionality, Pairing)**.** *For any sets u, v, x, y, we have $(u, v) = (x, y)$ if and only if $u = x$ and $v = y$*

PROOF:
$\langle 1 \rangle 1$. ASSUME: $\{\{u\}, \{u, v\}\} = \{\{x\}, \{x, y\}\}$
$\langle 1 \rangle 2$. $\{u\} \in \{\{x\}, \{x, y\}\}$
$\langle 1 \rangle 3$. $\{u, v\} \in \{\{x\}, \{x, y\}\}$
$\langle 1 \rangle 4$. $\{u\} = \{x\}$ or $\{u\} = \{x, y\}$
$\langle 1 \rangle 5$. $\{u, v\} = \{x\}$ or $\{u, v\} = \{x, y\}$
$\langle 1 \rangle 6$. CASE: $\{u\} = \{x, y\}$
  $\langle 2 \rangle 1$. $u = x = y$
  $\langle 2 \rangle 2$. $u = v = x = y$
    PROOF: From $\langle 1 \rangle 5$
$\langle 1 \rangle 7$. CASE: $\{u, v\} = \{x\}$
  PROOF: Similar.
$\langle 1 \rangle 8$. CASE: $\{u\} = \{x\}$ and $\{u, v\} = \{x, y\}$
  $\langle 2 \rangle 1$. $u = x$
  $\langle 2 \rangle 2$. $u = y$ or $v = y$
  $\langle 2 \rangle 3$. CASE: $u = y$
    PROOF: This case is the case considered in $\langle 1 \rangle 6$.
  $\langle 2 \rangle 4$. CASE: $v = y$
    PROOF: We have $u = x$ and $v = y$ as required.
$\square$

Lemma 2.3 (Extensionality, Pairing, Power Set)**.** *Let x, y and C be sets. If $x \in C$ and $y \in C$ then $(x, y) \in \mathcal{P}\mathcal{P}C$.*

PROOF:

$\langle 1 \rangle$1. LET: x, y and C be sets.
$\langle 1 \rangle$2. ASSUME: $x \in C$
$\langle 1 \rangle$3. ASSUME: $y \in C$
$\langle 1 \rangle$4. $\{x\} \subseteq C$
$\langle 1 \rangle$5. $\{x, y\} \subseteq C$
$\langle 1 \rangle$6. $\{x\} \in \mathcal{P}C$
$\langle 1 \rangle$7. $\{x, y\} \in \mathcal{P}C$
$\langle 1 \rangle$8. $\{\{x\}, \{x, y\}\} \subseteq \mathcal{P}C$
$\langle 1 \rangle$9. $\{\{x\}, \{x, y\}\} \in \mathcal{P}\mathcal{P}C$
□

Lemma 2.4 (Extensionality, Pairing, Union)**.** *Let x, y and A be sets. If $(x, y) \in A$ then x and y belong to $\bigcup\bigcup A$.*

PROOF:
$\langle 1 \rangle$1. LET: x, y and A be sets.
$\langle 1 \rangle$2. ASSUME: $(x, y) \in A$
$\langle 1 \rangle$3. $\{x, y\} \in \bigcup A$
$\langle 1 \rangle$4. $x \in \bigcup\bigcup A$
$\langle 1 \rangle$5. $y \in \bigcup\bigcup A$
□

2.2 Cartesian Product

Definition 2.5 (Cartesian Product (Extensionality, Pairing, Union, Power Set, Subset))**.** Let A and B be sets. The *Cartesian product* $A \times B$ is the set $\{(x, y) \mid x \in A, y \in B\}$.

This is a set since, if $x \in A$ and $y \in B$, then $(x, y) \in \mathcal{P}\mathcal{P}(A \cup B)$ by Lemma 2.3.

2.3 Relations

Definition 2.6 (Relation (Extensionality, Pairing))**.** A *relation* is a set of ordered pairs.

Given a relation R, we write xRy for $(x, y) \in R$.

2.4 Domain

Definition 2.7 (Domain (Extensionality, Pairing, Union, Subset))**.** Let R be a set. The *domain* of R is $\operatorname{dom} R = \{x \mid \exists y.(x, y) \in R\}$.

This is a set by Lemma 2.4.

2.5 Range

Definition 2.8 (Range (Extensionality, Pairing, Union, Subset)). Let R be a set. The *range* of R is $\operatorname{ran} R = \{y \mid \exists x.(x, y) \in R\}$.

This is a set by Lemma 2.4.

2.6 Functions

Definition 2.9 (Extensionality, Pairing). A *function* is a relation F such that, for all x, y, y', if xFy and xFy' then $y = y'$.

If there exists x such that xFy, then we write $F(x)$ for the unique such y, and call $F(x)$ the *value* of F at x.

Definition 2.10 (Extensionality, Pairing, Union, Subset). We write $F : A \to B$ iff F is a function, $\operatorname{dom} F = A$ and $\operatorname{ran} F \subseteq B$.

Axiom 2.11 (Axiom of Choice, First Form). *For any relation R, there exists a function $H \subseteq R$ such that* $\operatorname{dom} H = \operatorname{dom} R$.

2.7 Single-Rooted

Definition 2.12 (Extensionality, Pairing). A set R is *single-rooted* if and only if, for all x, x', y, if xRy and $x'Ry$ then $x = x'$.

We call a function *one-to-one*, *injective* or an *injection* if and only if it is single-rooted.

2.8 Surjective

Definition 2.13 (Surjective). Let $F : A \to B$. Then F is *surjective* if and only if $\operatorname{ran} F = B$.

2.9 Inverse

Definition 2.14 (Inverse (Extensionality, Pairing, Union, Power Set, Subset)). Let R be a set. The *inverse* of R is $R^{-1} = \{(y, x) \mid (x, y) \in R\}$.

This is a set because if $(x, y) \in R$ then $(y, x) \in \operatorname{ran} R \times \operatorname{dom} R$.

Theorem 2.15 (Extensionality, Pairing, Union, Power Set, Subset). *For any set F, we have* $\operatorname{dom} F^{-1} = \operatorname{ran} F$.

PROOF: For any x, we have

$$\begin{aligned} x \in \operatorname{dom} F^{-1} &\Leftrightarrow \exists y.(x, y) \in F^{-1} \\ &\Leftrightarrow \exists y.(y, x) \in F \\ &\Leftrightarrow x \in \operatorname{ran} F \end{aligned}$$

The result follows by the Axiom of Extensionality. □

Theorem 2.16 (Extensionality, Pairing, Union, Power Set, Subset)**.** *For any set F, we have* $\operatorname{ran} F^{-1} = \operatorname{dom} F$.

PROOF: For any x, we have

$$\begin{aligned} x \in \operatorname{ran} F^{-1} &\Leftrightarrow \exists y.(y,x) \in F^{-1} \\ &\Leftrightarrow \exists y.(x,y) \in F \\ &\Leftrightarrow x \in \operatorname{dom} F \end{aligned}$$

The result follows by the Axiom of Extensionality. ∎

Theorem 2.17 (Extensionality, Pairing, Union, Power Set, Subset)**.** *For any relation F, we have* $(F^{-1})^{-1} = F$.

PROOF: For any z we have

$$\begin{aligned} z \in (F^{-1})^{-1} &\Leftrightarrow \exists x,y.z = (x,y) \wedge (y,x) \in F^{-1} \\ &\Leftrightarrow \exists x,y.z = (x,y) \wedge (x,y) \in F \\ &\Leftrightarrow z \in F & (F \text{ is a relation}) \end{aligned}$$

The result follows by the Axiom of Extensionality.

Theorem 2.18 (Extensionality, Pairing, Union, Power Set, Subset)**.** *For any set F, we have F^{-1} is a function if and only if F is single-rooted.*

PROOF: Immediate from definitions. ∎

Theorem 2.19 (Extensionality, Pairing, Union, Power Set, Subset)**.** *Let F be a relation. Then F is a function if and only if F^{-1} is single-rooted.*

PROOF: Immediate from definitions. ∎

Theorem 2.20 (Extensionality, Pairing, Union, Power Set, Subset)**.** *Let F be a one-to-one function and $x \in \operatorname{dom} F$. Then $F^{-1}(F(x)) = x$.*

PROOF: We have $(x, F(x)) \in F$ and so $(F(x), x) \in F^{-1}$. ∎

Theorem 2.21 (Extensionality, Pairing, Union, Power Set, Subset)**.** *Let F be a one-to-one function and $y \in \operatorname{ran} F$. Then $F(F^{-1}(y)) = y$.*

PROOF: From Theorems 2.15, 2.17 and 2.20. ∎

2.10 Composition

Definition 2.22 (Composition (Extensionality, Pairing, Union, Power Set, Subset))**.** Let R and S be relations. The *composition* of R and S is $S \circ R = \{(x,z) \mid \exists y.xRy \wedge ySz\}$.

This is a set because if xRy and ySz then $(x,z) \in \operatorname{dom} R \times \operatorname{ran} S$.

Theorem 2.23 (Extensionality, Pairing, Union, Power Set, Subset)**.** *Let F and G be functions. Then $G \circ F$ is a function, its domain is $\{x \in \operatorname{dom} F \mid F(x) \in \operatorname{dom} G\}$, and for x in this domain, we have $(F \circ G)(x) = F(G(x))$.*

PROOF:
$\langle 1\rangle 1.$ $G \circ F$ is a function.
  $\langle 2\rangle 1.$ LET: $x(G \circ F)z$ and $x(G \circ F)z'$
  $\langle 2\rangle 2.$ PICK y, y' such that xFy, xFy', yGz and $y'Gz'$
  $\langle 2\rangle 3.$ $y = y'$
    PROOF: Since F is a function.
  $\langle 2\rangle 4.$ $z = z'$
    PROOF: Since G is a function.
$\langle 1\rangle 2.$ $\mathrm{dom}(G \circ F) = \{x \in \mathrm{dom}\, F \mid F(x) \in \mathrm{dom}\, G\}$
  PROOF:

$$\begin{aligned} x \in \mathrm{dom}(G \circ F) &\Leftrightarrow \exists z. x(G \circ F)z \\ &\Leftrightarrow \exists y, z. xFy \wedge yGz \\ &\Leftrightarrow x \in \mathrm{dom}\, F \wedge F(x) \in \mathrm{dom}\, G \end{aligned}$$

$\langle 1\rangle 3.$ For x in this domain, we have $(F \circ G)(x) = F(G(x))$.
  PROOF: Since $(x, F(x)) \in F$ and $(F(x), G(F(x))) \in G$.
$\square$

Theorem 2.24 (Extensionality, Pairing, Union, Power Set, Subset)**.** *For any sets F and G, we have $(G \circ F)^{-1} = F^{-1} \circ G^{-1}$.*

PROOF:

$$\begin{aligned} (x, z) \in (G \circ F)^{-1} &\Leftrightarrow (z, x) \in G \circ F \\ &\Leftrightarrow \exists y. zFy \wedge yGx \\ &\Leftrightarrow \exists y. (y, z) \in F^{-1} \wedge (x, y) \in G^{-1} \\ &\Leftrightarrow (x, z) \in F^{-1} \circ G^{-1} \end{aligned} \qquad \square$$

2.11 Identity Function

Definition 2.25 (Identity Function (Extensionality, Pairing, Union, Power Set, Subset))**.** Let A be a set. The *identity function* id_A on A is $\{(x, x) \mid x \in A\}$.

This is a set because it is a subset of $A \times A$.

Theorem 2.26 (Extensionality, Pairing, Union, Power Set, Subset)**.** *Let $F : A \to B$ and A be nonempty. Then there exists a function $G : B \to A$ such that $G \circ F = \mathrm{id}_A$ if and only if F is one-to-one.*

PROOF:
$\langle 1\rangle 1.$ LET: $F : A \to B$
$\langle 1\rangle 2.$ ASSUME: A is nonempty
$\langle 1\rangle 3.$ If there exists $G : B \to A$ such that $G \circ F = \mathrm{id}_A$ then F is one-to-one.
  $\langle 2\rangle 1.$ ASSUME: $G : B \to A$ and $G \circ F = \mathrm{id}_A$
  $\langle 2\rangle 2.$ LET: $x, y \in A$
  $\langle 2\rangle 3.$ ASSUME: $F(x) = F(y)$
  $\langle 2\rangle 4.$ $x = y$
    PROOF: $x = G(F(x)) = G(F(y)) = y$.

$\langle 1 \rangle 4.$ If F is one-to-one then there exists $G : B \to A$ such that $G \circ F = \mathrm{id}_A$.
  $\langle 2 \rangle 1.$ ASSUME: F is one-to-one.
  $\langle 2 \rangle 2.$ PICK $a \in A$
  $\langle 2 \rangle 3.$ Define $G : B \to A$ by: $G(y)$ is the x such that $F(x) = y$ if $y \in \operatorname{ran} F$, otherwise $G(y) = a$
  $\langle 2 \rangle 4.$ $G \circ F = \mathrm{id}_A$
    PROOF: For $x \in A$ we have $(G \circ F)(x) = G(F(x)) = x$ by Theorem 2.23.
□

Theorem 2.27 (Extensionality, Pairing, Union, Power Set, Subset)**.** *Let $F : A \to B$ and A be nonempty. If there exists a function $H : B \to A$ such that $F \circ H = \mathrm{id}_B$ then F is surjective.*

PROOF:
$\langle 1 \rangle 1.$ LET: $F : A \to B$
$\langle 1 \rangle 2.$ ASSUME: A is nonempty.
$\langle 1 \rangle 3.$ LET: $H : B \to A$ satisfy $F \circ H = \mathrm{id}_B$
$\langle 1 \rangle 4.$ LET: $y \in B$
$\langle 1 \rangle 5.$ $F(H(y)) = y$.
□

Theorem 2.28 (Extensionality, Pairing, Union, Power Set, Subset, Choice)**.** *Let $F : A \to B$ and A be nonempty. If F is surjective then there exists a function $H : B \to A$ such that $F \circ H = \mathrm{id}_B$.*

PROOF:
$\langle 1 \rangle 1.$ ASSUME: F is surjective.
$\langle 1 \rangle 2.$ PICK a function $H \subseteq F^{-1}$ with $\operatorname{dom} H = B$
  PROOF: By the Axiom of Choice.
$\langle 1 \rangle 3.$ $H : B \to A$
$\langle 1 \rangle 4.$ $F \circ H = \mathrm{id}_B$
  $\langle 2 \rangle 1.$ LET: $y \in B$
  $\langle 2 \rangle 2.$ $(y, H(y)) \in F^{-1}$
  $\langle 2 \rangle 3.$ $(H(y), y) \in F$
  $\langle 2 \rangle 4.$ $F(H(y)) = y$
□

2.12 Restriction

Definition 2.29 (Restriction (Extensionality, Pairing, Subset))**.** Let R be a relation and A a set. The *restriction* of R to A is $R \restriction A = \{(x, y) \mid x \in A \wedge xRy\}$.

This is a set because it is a subset of R.

2.13 Image

Definition 2.30 (Image (Extensionality, Pairing, Union, Subset))**.** Let F be a function and $A \subseteq \operatorname{dom} F$. The *image* of A under F is $\{F(x) \mid x \in A\}$.

This is a set because it is a subset of ran F.

2.14 The Finite Intersection Property

Definition 2.31 (Finite Intersection Property)**.** Let X be a set and $\mathcal{A} \subseteq \mathcal{P}X$. Then $\mathcal{A}$ satisfies the *finite intersection property* if and only if every nonempty finite subset of $\mathcal{A}$ has nonempty intersection.

Lemma 2.32. *Let X be a set. Let $\mathcal{A} \subseteq \mathcal{P}X$ have the finite intersection property. Then there exists a maximal set $\mathcal{D}$ such that $\mathcal{A} \subseteq \mathcal{D} \subseteq \mathcal{P}X$ and $\mathcal{D}$ has the finite intersection property.*

Proof:
$\langle 1 \rangle 1.$ Let: $\mathbb{F} = \{\mathcal{D} \mid \mathcal{A} \subseteq \mathcal{D} \subseteq \mathcal{P}X, \mathcal{D}$ has the finite intersection property$\}$
$\langle 1 \rangle 2.$ Every chain in $\mathbb{F}$ has an upper bound.
  $\langle 2 \rangle 1.$ Let: $\mathbb{C}$ be a chain in $\mathbb{F}$.
  $\langle 2 \rangle 2.$ Assume: without loss of generality $\mathbb{C} \neq \emptyset$
    Prove: $\bigcup \mathbb{C} \in \mathbb{F}$
   Proof: If $\mathbb{C} = \emptyset$ then $\mathcal{A}$ is an upper bound.
  $\langle 2 \rangle 3.$ $\mathcal{A} \subseteq \bigcup \mathbb{C} \subseteq \mathcal{P}X$
  $\langle 2 \rangle 4.$ Let: $C_1, \ldots, C_n \in \mathbb{C}$
    Prove: $C_1 \cap \cdots \cap C_n \neq \emptyset$
  $\langle 2 \rangle 5.$ Pick $\mathcal{C}_1, \ldots, \mathcal{C}_n \in \mathbb{C}$ such that $C_i \in \mathcal{C}_i$ for all i.
  $\langle 2 \rangle 6.$ Assume: without loss of generality $\mathcal{C}_1 \subseteq \cdots \subseteq \mathcal{C}_n$
  $\langle 2 \rangle 7.$ $C_1, \ldots, C_n \in \mathcal{C}_n$
  $\langle 2 \rangle 8.$ $\mathcal{C}_n$ satisfies the finite intersection property.
  $\langle 2 \rangle 9.$ $C_1 \cap \cdots \cap C_n \neq \emptyset$
$\langle 1 \rangle 3.$ Q.E.D.
  Proof: By Zorn's Lemma.
□

Lemma 2.33. *Let X be a set. Let $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the finite intersection property. Then any finite intersection of elements of $\mathcal{D}$ is an element of $\mathcal{D}$.*

Proof:
$\langle 1 \rangle 1.$ Let: $D_1, D_2 \in \mathcal{D}$
$\langle 1 \rangle 2.$ $\mathcal{D} \cup \{D_1 \cap D_2\}$ has the finite intersection property.
  Proof: Any finite intersection of members of $\mathcal{D} \cup \{D_1 \cap D_2\}$ is a finite intersection of members of $\mathcal{D}$.
$\langle 1 \rangle 3.$ $\mathcal{D} = \mathcal{D} \cup \{D_1 \cap D_2\}$
  Proof: By maximality of $\mathcal{D}$.
$\langle 1 \rangle 4.$ $D_1 \cap D_2 \in \mathcal{D}$.
□

Lemma 2.34. *Let X be a set. Let $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the finite intersection property. Let $A \subseteq X$. If A intersects every member of $\mathcal{D}$ then $A \in \mathcal{D}$.*

PROOF:
⟨1⟩1. $\mathcal{D} \cup \{A\}$ has the finite intersection property.
  ⟨2⟩1. LET: $D_1, \ldots, D_n \in \mathcal{D}$
      PROVE: $D_1 \cap \cdots \cap D_n \cap A \neq \emptyset$
  ⟨2⟩2. $D_1 \cap \cdots \cap D_n \in \mathcal{D}$
    PROOF: Lemma 2.33.
  ⟨2⟩3. $D_1 \cap \cdots \cap D_n \cap A \neq \emptyset$
    PROOF: Since A intersects every member of $\mathcal{D}$.
⟨1⟩2. Q.E.D.
  PROOF: By maximality of $\mathcal{D}$.
□

Proposition 2.35. *Let X be a set. Let $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the finite intersection property. Let $A, D \in \mathcal{P}X$. If $D \in \mathcal{D}$ and $D \subseteq A$ then $A \in \mathcal{D}$.*

PROOF:
⟨1⟩1. $\mathcal{D} \cup \{A\}$ satisfies the finite intersection property.
  ⟨2⟩1. LET: $D_1, \ldots, D_n \in \mathcal{D}$
  ⟨2⟩2. $D_1 \cap \cdots \cap D_n \cap D \neq \emptyset$
    PROOF: Since $\mathcal{D}$ satisfies the finite intersection property.
  ⟨2⟩3. $D_1 \cap \cdots \cap D_n \cap A \neq \emptyset$
⟨1⟩2. $\mathcal{D} = \mathcal{D} \cup \{A\}$
  PROOF: By the maximality of $\mathcal{D}$.
⟨1⟩3. $A \in \mathcal{D}$
□

Definition 2.36 (Graph)**.** Let $f : A \to B$. The *graph* of f is the set $\{(x, f(x)) \mid x \in A\} \subseteq A \times B$.

2.15 Countable Intersection Property

Definition 2.37 (Countable Intersection Property)**.** Let X be a set and $\mathcal{A} \subseteq \mathcal{P}X$. Then $\mathcal{A}$ satisfies the *countable intersection property* if and only if every countable subset of $\mathcal{A}$ has nonempty intersection.

Lemma 2.38. *Let X be a set. Let $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the countable intersection property. Then any countable intersection of elements of $\mathcal{D}$ is an element of $\mathcal{D}$.*

PROOF:
⟨1⟩1. LET: $\mathcal{D}_0 \subseteq \mathcal{D}$ be countable.
⟨1⟩2. $\mathcal{D} \cup \{\bigcap \mathcal{D}_0\}$ has the countable intersection property.
  PROOF: Any countable intersection of members of $\mathcal{D} \cup \{\bigcap \mathcal{D}_0\}$ is a finite intersection of members of $\mathcal{D}$.
⟨1⟩3. $\mathcal{D} = \mathcal{D} \cup \{\bigcap \mathcal{D}_0\}$
  PROOF: By maximality of $\mathcal{D}$.

$\langle 1\rangle 4.$ $\bigcap \mathcal{D}_0 \in \mathcal{D}$.
□

Lemma 2.39. *Let X be a set. Let $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the countable intersection property. Let $A \subseteq X$. If A intersects every member of $\mathcal{D}$ then $A \in \mathcal{D}$.*

PROOF:
$\langle 1\rangle 1.$ $\mathcal{D} \cup \{A\}$ has the countable intersection property.
  $\langle 2\rangle 1.$ LET: $\mathcal{D}_0 \subseteq \mathcal{D}$ be countable.
    PROVE: $\bigcap \mathcal{D}_0 \cap A \neq \emptyset$
  $\langle 2\rangle 2.$ $\bigcap \mathcal{D}_0 \in \mathcal{D}$
    PROOF: Lemma 2.38.
  $\langle 2\rangle 3.$ $\bigcap \mathcal{D}_0 \cap A \neq \emptyset$
    PROOF: Since A intersects every member of $\mathcal{D}$.
$\langle 1\rangle 2.$ Q.E.D.
  PROOF: By maximality of $\mathcal{D}$.
□

2.16 The Axiom of Choice

Axiom 2.40 (Axiom of Choice)**.** *Let $\mathcal{A}$ be a set of disjoint nonempty sets. Then there exists a set C consisting of exactly one element from each member of $\mathcal{A}$.*

2.17 Choice Functions

Definition 2.41 (Choice Function)**.** Let $\mathcal{B}$ be a set of nonempty sets. A *choice function* for $\mathcal{B}$ is a function $c : \mathcal{B} \to \bigcup \mathcal{B}$ such that, for all $B \in \mathcal{B}$, we have $c(B) \in B$.

Lemma 2.42 (Existence of a Choice Function (AC))**.** *Every set of nonempty sets has a choice function.*

PROOF:
$\langle 1\rangle 1.$ LET: $\mathcal{B}$ be a set of nonempty sets.
$\langle 1\rangle 2.$ For $B \in \mathcal{B}$,
    LET: $B' = \{B\} \times B$
$\langle 1\rangle 3.$ $\{B' \mid B \in \mathcal{B}\}$ is a set of disjoint nonempty sets.
$\langle 1\rangle 4.$ PICK a set c consisting of exactly one element from each B' for $B \in \mathcal{B}$.
$\langle 1\rangle 5.$ c is a choice function for $\mathcal{B}$.
□

2.18 Order Theory

Definition 2.43 (Preorder)**.** Let X be a set. A *preorder* on X is a binary relation $\leq$ on X such that:

Reflexivity For all $x \in X$, we have $x \leq x$

Transitivity For all $x, y, z \in X$, if $x \leq y$ and $y \leq z$ then $x \leq z$.

Definition 2.44 (Preordered Set)**.** A *preordered set* consists of a set X and a preorder $\leq$ on X.

Proposition 2.45. *Let X and Y be linearly ordered sets. Let $f : X \twoheadrightarrow Y$ be strictly monotone and surjective. Then f is a poset isomorphism.*

PROOF:
⟨1⟩1. f is injective.
  ⟨2⟩1. LET: $x, y \in X$
  ⟨2⟩2. ASSUME: $f(x) = f(y)$
  ⟨2⟩3. $x \not< y$
    PROOF: By strong motonicity.
  ⟨2⟩4. $y \not< x$
    PROOF: By strong motonicity.
  ⟨2⟩5. $x = y$
    PROOF: By trichotomy.
⟨1⟩2. f^{-1} is monotone.
  ⟨2⟩1. LET: $x, y \in X$
  ⟨2⟩2. ASSUME: $x \leq y$
  ⟨2⟩3. $f^{-1}(x) \not> f^{-1}(y)$
    PROOF: By strong motonicity.
  ⟨2⟩4. $f^{-1}(x) < f^{-1}(y)$
    PROOF: By trichotomy.
□

Definition 2.46 (Interval)**.** Let X be a preordered set and $Y \subseteq X$. Then Y is an *interval* if and only if, for all $a, b \in Y$ and $c \in X$, if $a \leq c \leq b$ then $c \in Y$.

Definition 2.47 (Linear Continuum)**.** A linearly ordered set L is a *linear continuum* if and only if:

1. every nonempty subset of L that is bounded above has a supremum
2. L is dense

Proposition 2.48. *Every interval in a linear continuum is a linear continuum.*

PROOF:
⟨1⟩1. LET: L be a linear continuum and I an interval in L.
⟨1⟩2. Every nonempty subset of I that is bounded above has a supremum in I.
  ⟨2⟩1. LET: $X \subseteq I$ be nonempty and bounded above by $b \in I$.

⟨2⟩2. LET: s be the supremum of X in L.
PROOF: Since L is a linear continuum.
⟨2⟩3. $s \in I$
⟨3⟩1. PICK $a \in X$
PROOF: Since X is nonempty (⟨2⟩1).
⟨3⟩2. $a \leq s \leq b$
⟨3⟩3. $s \in I$
PROOF: Since I is an interval (⟨1⟩1).
⟨2⟩4. s is the supremum of X in I
⟨1⟩3. I is dense.
⟨2⟩1. LET: $x, y \in I$ with $x < y$
⟨2⟩2. PICK $z \in L$ with $x < z < y$
PROOF: Since L is dense.
⟨2⟩3. $z \in I$
PROOF: Since I is an interval.
□

Definition 2.49 (Ordered Square)**.** The *ordered square* I_o^2 is the set $[0,1]^2$ under the dictionary order.

Proposition 2.50. *The ordered square is a linear continuum.*

PROOF:
⟨1⟩1. Every nonempty subset of I_o^2 bounded above has a supremum.
⟨2⟩1. LET: $X \subseteq I_o^2$ be nonempty and bounded above by (b, c)
⟨2⟩2. LET: $s = \sup \pi_1(X)$
PROOF: The set $\pi_1(X)$ is nonempty and bounded above by b.
⟨2⟩3. CASE: $s \in \pi_1(X)$
⟨3⟩1. LET: $t = \sup\{y \in [0,1] \mid (s, y) \in X\}$
PROOF: This set is nonempty and bounded above by c.
⟨3⟩2. (s, t) is the supremum of X.
⟨2⟩4. CASE: $s \notin \pi_1(X)$
PROOF: In this case $(s, 0)$ is the supremum of X.
⟨1⟩2. I_o^2 is dense.
⟨2⟩1. LET: $(x_1, y_1), (x_2, y_2) \in I_o^2$ with $(x_1, y_1) < (x_2, y_2)$
⟨2⟩2. CASE: $x_1 < x_2$
⟨3⟩1. PICK x_3 with $x_1 < x_3 < x_2$
⟨3⟩2. $(x_1, y_1) < (x_3, y_1) < (x_2, y_2)$
⟨2⟩3. CASE: $x_1 = x_2$ and $y_1 < y_2$
⟨3⟩1. PICK y_3 with $y_1 < y_3 < y_2$
⟨3⟩2. $(x_1, y_1) < (x_1, y_3) < (x_2, y_2)$
□

Proposition 2.51. *If X is a well-ordered set then $X \times [0,1)$ under the dictionary order is a linear continuum.*

PROOF:
⟨1⟩1. Every nonempty set $A \subseteq X \times [0,1)$ bounded above has a supremum.

⟨2⟩1. LET: $A \subseteq X \times [0,1)$ be nonempty and bounded above
⟨2⟩2. LET: x_0 be the supremum of $\pi_1(A)$
⟨2⟩3. CASE: $x_0 \in \pi_1(A)$
⟨3⟩1. LET: y_0 be the supremum of $\{y \in [0,1) \mid (x_0, y) \in A\}$
⟨3⟩2. (x_0, y_0) is the supremum of A.
⟨2⟩4. CASE: $x_0 \notin \pi_1(A)$
PROOF: In this case $(x_0, 0)$ is the supremum of A.
⟨1⟩2. $X \times [0,1)$ is dense.
⟨2⟩1. LET: $(x_1, y_1), (x_2, y_2) \in X \times [0,1)$ with $(x_1, y_1) < (x_2, y_2)$
⟨2⟩2. CASE: $x_1 < x_2$
⟨3⟩1. PICK y_3 such that $y_1 < y_3 < 1$
⟨3⟩2. $(x_1, y_1) < (x_1, y_3) < (x_2, y_2)$
⟨2⟩3. CASE: $x_1 = x_2$ and $y_1 < y_2$
⟨3⟩1. PICK y_3 such that $y_1 < y_3 < y_2$
⟨3⟩2. $(x_1, y_1) < (x_1, y_3) < (x_2, y_2)$
□

Lemma 2.52. *For all $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$, we have $[a, b) \cong [c, d)$*

PROOF: The map $\lambda t.c + (t - a)(d - c)/(b - a)$ is an order isomorphism.

Proposition 2.53. *Let X be a linearly ordered set. Let $a < b < c$ in X. Then $[a, c) \cong [0, 1)$ if and only if $[a, b) \cong [b, c) \cong [0, 1)$.*

PROOF:
⟨1⟩1. If $[a, c) \cong [0, 1)$ then $[a, b) \cong [b, c) \cong [0, 1)$
⟨2⟩1. ASSUME: $f : [a, c) \cong [0, 1)$ is an order isomorphism
⟨2⟩2. $[a, b) \cong [0, 1)$
PROOF:

$$\begin{aligned} [a, b) &\cong [0, f(b)) && \text{(by the restriction of } f\text{)} \\ &\cong [0, 1) && \text{(Lemma 2.52)} \end{aligned}$$

⟨2⟩3. $[b, c) \cong [0, 1)$
PROOF: Similar.
⟨1⟩2. If $[a, b) \cong [b, c) \cong [0, 1)$ then $[a, c) \cong [0, 1)$
PROOF:

$$\begin{aligned} [a, c) &= [a, b) * [b, c) \\ &\cong [0, 1) * [0, 1) \\ &\cong [0, 1/2) * [1/2, 1) && \text{(Lemma 2.52)} \\ &= 1 \end{aligned}$$

□

Proposition 2.54 (CC)**.** *Let X be a linearly ordered set. Let $x_0 < x_1 < \cdots$ be a strictly increasing sequence in X with supremum b. Then $[x_0, b) \cong [0, 1)$ if and only if $[x_i, x_{i+1}) \cong [0, 1)$ for all i.*

PROOF:
⟨1⟩1. If $[x_0, b) \cong [0, 1)$ then $[x_i, x_{i+1}) \cong [0, 1)$ for all i.

PROOF: By Lemma 2.52

⟨1⟩2. If $[x_i, x_{i+1}) \cong [0, 1)$ for all i then $[x_0, b) \cong [0, 1)$

⟨2⟩1. ASSUME: $[x_i, x_{i+1}) \cong [0, 1)$ for all i

⟨2⟩2. PICK an order isomorphism $f_i : [x_i, x_{i+1}) \cong [1/2^i, 2/2^{i+1})$ for each i.

PROOF: By Lemma 2.52

⟨2⟩3. The union of the f_is is an order isomorphism $[x_0, b) \cong [0, 1)$

□

2.19 Partially Ordered Sets

Definition 2.55 (Partial Order)**.** A *partial order* on a set X is a preorder $\leq$ that is *anti-symmetric*, i.e. whenever $x \leq y$ and $y \leq x$ then $x = y$.

Definition 2.56 (Linear Order)**.** A *linear order* on a set X is a partial order such that, for any $x, y \in X$, either $x \leq y$ or $y \leq x$.

Definition 2.57 (Well-ordering)**.** A *well-order* on a set X is a linear order such that every nonempty set has a least element.

Definition 2.58 (Section)**.** Given a well-ordered set X and $\alpha \in X$, the *section* of X by α is $S_\alpha = \{x \in X \mid x < \alpha\}$.

Theorem 2.59 (Transfinite Induction)**.** *Let J be a well-ordered set and $J_0 \subseteq J$. Suppose that, for all $\alpha \in J$, if $S_\alpha \subseteq J_0$ then $\alpha \in J_0$. Then $J_0 = J$.*

PROOF:

⟨1⟩1. ASSUME: for a contradiction $J_0 \neq J$

⟨1⟩2. LET: α be the least element of $J \setminus J_0$

⟨1⟩3. $S_\alpha \subseteq J_0$

⟨1⟩4. $\alpha \in J_0$

⟨1⟩5. Q.E.D.

PROOF: This contradicts ⟨1⟩2.

□

Theorem 2.60 (Transfinite Recursion)**.** *Let J be a well-ordered set and C a set. Let $\mathcal{F}$ be the set of all functions from a section of J to C. Let $\rho : \mathcal{F} \to C$. Then there exists a unique function $h : J \to C$ such that, for all $\alpha \in J$, we have $h(\alpha) = \rho(h \restriction S_\alpha)$.*

PROOF:

⟨1⟩1. For every $\beta \in J$, there exists a unique $h_\beta : S_\beta \to J$ such that, for all $\alpha < \beta$, we have $h_\beta(\alpha) = \rho(h_\beta \restriction \alpha)$

⟨2⟩1. LET: $\beta \in J$

⟨2⟩2. ASSUME: for all $\gamma < \beta$ there exists a unique $h : S_\gamma \to J$ such that, for all $\alpha < \gamma$, we have $h(\alpha) = \rho(h \restriction S_\alpha)$

⟨2⟩3. For $\gamma < \beta$,

LET: $h_\gamma : S_\gamma \to J$ be the function such that, for all $\alpha < \gamma$, we have $h_\gamma(\alpha) = \rho(h_\gamma \restriction S_\alpha)$

$\langle 2\rangle 4$. LET: $h : S_\beta \to J$ be the function $h(\gamma) = \rho(h_\gamma)$ for $\gamma < \beta$

$\langle 2\rangle 5$. For $\gamma < \beta$ we have $h \restriction S_\gamma = h_\gamma$

$\langle 3\rangle 1$. LET: $\gamma < \beta$

$\langle 3\rangle 2$. ASSUME: For all $\alpha < \gamma$ we have $h \restriction S_\alpha = h_\alpha$

$\langle 3\rangle 3$. For all $\alpha < \gamma$ we have $(h \restriction S_\gamma)(\alpha) = \rho((h \restriction S_\gamma) \restriction S_\alpha)$

PROOF:

$$\begin{aligned}(h \restriction S_\gamma)(\alpha) &= h(\alpha) \\ &= \rho(h_\alpha) && (\langle 2\rangle 4) \\ &= \rho(h \restriction S_\alpha) && (\langle 3\rangle 2) \\ &= \rho((h \restriction S_\gamma) \restriction S_\alpha)\end{aligned}$$

$\langle 3\rangle 4$. $h \restriction S_\gamma = h_\gamma$

PROOF: From $\langle 2\rangle 4$

$\langle 3\rangle 5$. Q.E.D.

PROOF: By transfinite induction.

$\langle 2\rangle 6$. For $\alpha < \beta$ we have $h(\alpha) = \rho(h \restriction S_\alpha)$

$\langle 2\rangle 7$. If $h' : S_\beta \to J$ and $h'(\alpha) = \rho(h' \restriction S_\alpha)$ for all $\alpha < \beta$, then $h' = h$)

$\langle 3\rangle 1$. LET: $h' : S_\beta \to J$ and $h'(\alpha) = \rho(h' \restriction S_\alpha)$ for all $\alpha < \beta$

$\langle 3\rangle 2$. For all $\gamma < \beta$ we have $h' \restriction S_\gamma = h_\gamma$

$\langle 4\rangle 1$. For all $\alpha < \gamma$ we have $(h' \restriction S_\gamma)(\alpha) = \rho((h' \restriction S_\gamma) \restriction S_\alpha)$

PROOF:

$$\begin{aligned}(h' \restriction S_\gamma)(\alpha) &= h'(\alpha) \\ &= \rho(h' \restriction S_\alpha) && (\langle 3\rangle 1) \\ &= \rho((h' \restriction S_\gamma) \restriction S_\alpha)\end{aligned}$$

$\langle 4\rangle 2$. Q.E.D.

PROOF: From $\langle 2\rangle 4$

$\langle 3\rangle 3$. For all $\alpha < \beta$ we have $h'(\alpha) = \rho(h_\alpha)$

$\langle 1\rangle 2$. There exists $h : J \to C$ such that, for all $\alpha \in J$, we have $h(\alpha) = \rho(h \restriction S_\alpha)$

$\langle 2\rangle 1$. For $\alpha \in J$,

LET: $h(\alpha) = \rho(h_\alpha)$

$\langle 2\rangle 2$. For $\alpha \in J$ we have $h \restriction S_\alpha = h_\alpha$

$\langle 3\rangle 1$. LET: $\alpha \in J$

$\langle 3\rangle 2$. ASSUME: For all $\beta < \alpha$ we have $h \restriction S_\beta = h_\beta$

$\langle 3\rangle 3$. For all $\beta < \alpha$ we have $(h \restriction S_\alpha)(\beta) = \rho((h \restriction S_\alpha) \restriction S_\beta)$

PROOF:

$$\begin{aligned}(h \restriction S_\alpha)(\beta) &= h(\beta) \\ &= \rho(h_\beta) && (\langle 2\rangle 1) \\ &= \rho(h \restriction S_\beta) && (\langle 3\rangle 2) \\ &= \rho((h \restriction S_\alpha) \restriction S_\beta)\end{aligned}$$

$\langle 3\rangle 4$. $h \restriction S_\alpha = h_\alpha$

PROOF: From $\langle 1\rangle 1$

$\langle 3\rangle 5$. Q.E.D.

PROOF: By transfinite induction.

$\langle 2\rangle 3$. For $\alpha \in J$ we have $h(\alpha) = \rho(h \restriction S_\alpha)$

$\langle 1\rangle 3$. If $h, h' : J \to C$ and, for all $\alpha \in J$, we have $h(\alpha) = \rho(h \restriction S_\alpha)$ and

$h'(\alpha) = \rho(h' \upharpoonright S_\alpha)$, then $h = h'$

$\langle 2 \rangle 1$. ASSUME: $h, h' : J \to C$ and, for all $\alpha \in J$, we have $h(\alpha) = \rho(h \upharpoonright S_\alpha)$ and $h'(\alpha) = \rho(h' \upharpoonright S_\alpha)$

$\langle 2 \rangle 2$. LET: $\alpha \in J$

$\langle 2 \rangle 3$. ASSUME: for all $\beta < \alpha$ we have $h(\beta) = h'(\beta)$

$\langle 2 \rangle 4$. $h(\alpha) = h'(\alpha)$

PROOF:

$$\begin{aligned} h(\alpha) &= \rho(h \upharpoonright S_\alpha) \\ &= \rho(h' \upharpoonright S_\alpha) && (\langle 2 \rangle 3) \\ &= h'(\alpha) \end{aligned}$$

$\langle 2 \rangle 5$. Q.E.D.

PROOF: By transfinite induction.

□

Theorem 2.61 (Well-Ordering Theorem (AC)). *Every set has a well-ordering.*

PROOF:

$\langle 1 \rangle 1$. LET: X be a set.

$\langle 1 \rangle 2$. PICK a choice function for $\mathcal{P}X \setminus \{\emptyset\}$

PROOF: Lemma 2.42.

$\langle 1 \rangle 3$. LET: a *tower* in X be a pair $(T, <)$ where $T \subseteq X$, $<$ is a well-ordering of T, and $x = c(X \setminus \{y \in T \mid y < x\})$.

$\langle 1 \rangle 4$. For any two towers $(T_1, <_1)$ and $(T_2, <_2)$, either these two posets are equal or one is a section of the other.

$\langle 2 \rangle 1$.

$\langle 1 \rangle 5$. For any tower $(T, <)$ in X with $T \neq X$, there exists a tower in X of which $(T, <)$ is a section.

$\langle 1 \rangle 6$. LET: $T = \bigcup \{T' \subseteq X \mid \exists R.(T, R) \text{ is a tower in } X\}$

$\langle 1 \rangle 7$. Define $<$ on T by: $x < y$ iff there exists a tower (T, R) in X such that $x, y \in T$ and xRy.

$\langle 1 \rangle 8$. $(T, <)$ is a tower in X.

$\langle 1 \rangle 9$. $T = X$

$\langle 1 \rangle 10$. $<$ is a well-ordering of X.

□

Theorem 2.62 (Maximum Principle (AC)). *Every poset has a maximal chain.*

Lemma 2.63 (Zorn's Lemma (AC)). *Let A be a poset. If every chain in A has an upper bound in A, then A has a maximal element.*

2.20 Real Analysis

Definition 2.64. Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences (a_n) such that $a_n = 0$ for all but finitely many n.

2.21 Group Theory

Definition 2.65. Given a group G and sets $A, B \subseteq G$, let $AB = \{ab \mid a \in A, b \in B\}$.

Definition 2.66. Given a group G and a set $A \subseteq G$, let $A^{-1} = \{a^{-1} \mid a \in A\}$.

2.22 Topological Spaces

Definition 2.67 (Topology)**.** A *topology* on a set X is a set $\mathcal{T} \subseteq \mathcal{P}X$ such that:

- $X \in \mathcal{T}$.
- For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$.
- For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$.

We call the elements of X *points* and the elements of $\mathcal{T}$ *open sets*.

Definition 2.68 (Topological Space)**.** A *topological space* X consists of a set X and a topology on X.

Definition 2.69 (Discrete Space)**.** For any set X, the *discrete* topology on X is $\mathcal{P}X$.

Definition 2.70 (Indiscrete Space)**.** For any set X, the *indiscrete* or *trivial* topology on X is $\{\emptyset, X\}$.

Definition 2.71 (Finite Complement Topology)**.** For any set X, the *finite complement topology* on X is $\{U \in \mathcal{P}X \mid X \setminus U \text{ is finite}\} \cup \{\emptyset\}$.

Definition 2.72 (Countable Complement Topology)**.** For any set X, the *countable complement topology* on X is $\{U \in \mathcal{P}X \mid X \setminus U \text{ is countable}\} \cup \{\emptyset\}$.

Definition 2.73 (Finer, Coarser)**.** Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a given set X. If $\mathcal{T}' \supseteq \mathcal{T}$, we say that $\mathcal{T}'$ is *finer* than $\mathcal{T}$; if $\mathcal{T}'$ *properly* contains $\mathcal{T}$, we say that $\mathcal{T}'$ is *strictly* finer than $\mathcal{T}$. We also say that $\mathcal{T}$ is *coarser* than $\mathcal{T}'$, or *stricly* coarser, in these two respective situations. We say $\mathcal{T}$ is *comparable* with $\mathcal{T}'$ if either $\mathcal{T}' \supseteq \mathcal{T}$ or $\mathcal{T} \supseteq \mathcal{T}'$.

Lemma 2.74. *Let X be a topological space and $U \subseteq X$. Then U is open if and only if, for all $x \in U$, there exists an open set V such that $x \in V \subseteq U$.*

Proof:
$\langle 1 \rangle 1.$ $\Rightarrow$
  Proof: Take $V = U$
$\langle 1 \rangle 2.$ $\Leftarrow$
  Proof: We have $U = \bigcup \{V \text{ open in } X \mid V \subseteq U\}$.
$\square$

Lemma 2.75. *Let X be a set and $\mathcal{T}$ a nonempty set of topologies on X. Then $\bigcap \mathcal{T}$ is a topology on X, and is the finest topology that is coarser than every member of $\mathcal{T}$.*

Proof:
$\langle 1 \rangle 1.$ $X \in \bigcap \mathcal{T}$
  Proof: Since X is in every member of $\mathcal{T}$.
$\langle 1 \rangle 2.$ $\bigcap \mathcal{T}$ is closed under union.

⟨2⟩1. LET: $\mathcal{U} \subseteq \bigcap \mathcal{T}$
⟨2⟩2. For all $T \in \mathcal{T}$ we have $\mathcal{U} \subseteq T$
⟨2⟩3. For all $T \in \mathcal{T}$ we have $\bigcup \mathcal{U} \in T$
⟨2⟩4. $\bigcup \mathcal{U} \in \bigcap \mathcal{T}$
⟨1⟩3. $\bigcap \mathcal{T}$ is closed under binary intersection.
⟨2⟩1. LET: $U, V \in \bigcap \mathcal{T}$
⟨2⟩2. For all $T \in \mathcal{T}$ we have $U, V \in T$
⟨2⟩3. For all $T \in \mathcal{T}$ we have $U \cap V \in T$
⟨2⟩4. $U \cap V \in \bigcap \mathcal{T}$
□

Lemma 2.76. *Let X be a set and $\mathcal{T}$ a set of topologies on X. Then there exists a unique coarsest topology that is finer than every member of $\mathcal{T}$.*

PROOF: The required topology is given by
$\bigcap \{T \in \mathcal{PP}X \mid T \textit{ is a topology on } X \textit{ that is finer than every member of } \mathcal{T}\}$,
The set is nonempty since it contains the discrete topology. □

Definition 2.77 (Neighbourhood)**.** A *neighbourhood* of a point x is an open set that contains x.

2.23 Closed Set

Definition 2.78 (Closed Set)**.** Let X be a topological space and $A \subseteq X$. Then A is *closed* if and only if $X \setminus A$ is open.

Lemma 2.79. *The empty set is closed.*

PROOF: Since the whole space X is always open. □

Lemma 2.80. *The topological space X is closed.*

PROOF: Since $\emptyset$ is open. □

Lemma 2.81. *The intersection of a nonempty set of closed sets is closed.*

PROOF: Let $\mathcal{C}$ be a nonempty set of closed sets. Then $X \setminus \bigcap \mathcal{C} = \bigcup \{X \setminus C \mid C \in \mathcal{C}\}$ is open. □

Lemma 2.82. *The union of two closed sets is closed.*

PROOF: Let C and D be closed. Then $X \setminus (C \cup D) = (X \setminus C) \cap (X \setminus D)$ is open. □

Proposition 2.83. *Let X be a set and $\mathcal{C} \subseteq \mathcal{P}X$ a set such that:*

1. $\emptyset \in \mathcal{C}$
2. $X \in \mathcal{C}$
3. *For all $\mathcal{A} \subseteq \mathcal{C}$ nonempty we have $\bigcap \mathcal{A} \in \mathcal{C}$*

4. For all $C, D \in \mathcal{C}$ we have $C \cup D \in \mathcal{C}$.

Then there exists a unique topology $\mathcal{T}$ such that $\mathcal{C}$ is the set of closed sets, namely

$$\mathcal{T} = \{X \setminus C \mid C \in \mathcal{C}\}$$

PROOF:
$\langle 1\rangle 1.$ LET: $\mathcal{T} = \{X \setminus C \mid C \in \mathcal{C}\}$
$\langle 1\rangle 2.$ $\mathcal{T}$ is a topology
  $\langle 2\rangle 1.$ $X \in \mathcal{T}$
    PROOF: Since $\emptyset \in \mathcal{C}$
  $\langle 2\rangle 2.$ For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$
    $\langle 3\rangle 1.$ LET: $\mathcal{U} \subseteq \mathcal{T}$
    $\langle 3\rangle 2.$ CASE: $\mathcal{U} = \emptyset$
      PROOF: In this case $\bigcup \mathcal{U} = \emptyset \in \mathcal{T}$ since $X \in \mathcal{C}$
    $\langle 3\rangle 3.$ CASE: $\mathcal{U} \neq \emptyset$
      PROOF: In this case $X \setminus \bigcup \mathcal{U} = \bigcap \{X \setminus U \mid U \in \mathcal{U}\} \in \mathcal{C}$.
  $\langle 2\rangle 3.$ For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$
    PROOF: Since $X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V) \in \mathcal{C}$.
$\langle 1\rangle 3.$ $\mathcal{C}$ is the set of all closed sets in $\mathcal{T}$
  PROOF:

$$\begin{aligned} & C \text{ is closed in } \mathcal{T} \\ \Leftrightarrow & X \setminus C \in \mathcal{T} \\ \Leftrightarrow & C \in \mathcal{C} \end{aligned}$$

$\langle 1\rangle 4.$ If $\mathcal{T}'$ is a topology and $\mathcal{C}$ is the set of closed sets in $\mathcal{T}'$ then $\mathcal{T}' = \mathcal{T}$
  PROOF: We have

$$\begin{aligned} & U \in \mathcal{T} \\ \Leftrightarrow & X \setminus U \in \mathcal{C} \\ \Leftrightarrow & X \setminus U \text{ is closed in } \mathcal{T}' \\ \Leftrightarrow & U \in \mathcal{T}' \end{aligned}$$

□

Proposition 2.84. *If U is open and A is closed then $U \setminus A$ is open.*

PROOF: $U \setminus A = U \cap (X \setminus A)$ is the intersection of two open sets. □

Proposition 2.85. *If U is open and A is closed then $A \setminus U$ is closed.*

PROOF: $A \setminus U = A \cap (X \setminus U)$ is the intersection of two closed sets. □

2.24 Interior

Definition 2.86 (Interior)**.** Let X be a topological space and $A \subseteq X$. The *interior* of A, $\operatorname{Int} A$, is the union of all the open subsets of A.

Lemma 2.87. *The interior of a set is open.*

PROOF: It is a union of open sets. ∎

Lemma 2.88.

$$\operatorname{Int} A \subseteq A$$

PROOF: Immediate from definition. ∎

Lemma 2.89. *If U is open and $U \subseteq A$ then $U \subseteq \operatorname{Int} A$*

PROOF: Immediate from definition. ∎

Lemma 2.90. *A set A is open if and only if $A = \operatorname{Int} A$.*

PROOF: If $A = \operatorname{Int} A$ then A is open by Lemma 2.87. Conversely if A is open then $A \subseteq \operatorname{Int} A$ by the definition of interior and so $A = \operatorname{Int} A$.

2.25 Closure

Definition 2.91 (Closure)**.** Let X be a topological space and $A \subseteq X$. The *closure* of A, $\overline{A}$, is the intersection of all the closed sets that include A.

This intersection exists since X is a closed set that includes A (Lemma 2.80).

Lemma 2.92. *The closure of a set is closed.*

PROOF: Dual to Lemma 2.87. ∎

Lemma 2.93.

$$A \subseteq \overline{A}$$

PROOF: Immediate from definition. ∎

Lemma 2.94. *If C is closed and $A \subseteq C$ then $\overline{A} \subseteq C$.*

PROOF: Immediate from definition. ∎

Lemma 2.95. *A set A is closed if and only if $A = \overline{A}$.*

PROOF: Dual to Lemma 2.90. ∎

Theorem 2.96. *Let X be a topological space, $A \subseteq X$ and $x \in X$. Then $x \in \overline{A}$ if and only if every neighbourhood of x intersects A.*

PROOF: We have

$$\begin{aligned}
& x \in \overline{A} \\
\Leftrightarrow & \forall C. C \text{ closed } \wedge A \subseteq C \Rightarrow x \in C \\
\Leftrightarrow & \forall U. U \text{ open } \wedge A \cap U = \emptyset \Rightarrow x \notin U \\
\Leftrightarrow & \forall U. U \text{ open } \wedge x \in U \Rightarrow U \text{ intersects } A
\end{aligned}$$

∎

Proposition 2.97. *If $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$.*

PROOF: This holds because $\overline{B}$ is a closed set that includes A. ∎

Proposition 2.98.

$$\overline{A \cup B} = \overline{A} \cup \overline{B}$$

PROOF:
⟨1⟩1. $\overline{A} \subseteq \overline{A \cup B}$
  PROOF: By Proposition 2.97.
⟨1⟩2. $\overline{B} \subseteq \overline{A \cup B}$
  PROOF: By Proposition 2.97.
⟨1⟩3. $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$
  ⟨2⟩1. LET: $x \in \overline{A \cup B}$
  ⟨2⟩2. ASSUME: $x \notin \overline{A}$
     PROVE: $x \in \overline{B}$
  ⟨2⟩3. PICK a neighbourhood U of x that does not intersect A
  ⟨2⟩4. LET: V be any neighbourhood of x
  ⟨2⟩5. $U \cap V$ is a neighbourhood of x
  ⟨2⟩6. $U \cap V$ intersects $A \cup B$
    PROOF: From ⟨2⟩1 and Theorem 2.96.
  ⟨2⟩7. $U \cap V$ intersects B
    PROOF: From ⟨2⟩3
  ⟨2⟩8. V intersects B
  ⟨2⟩9. Q.E.D.
    PROOF: We have $x \in \overline{B}$ from Theorem 2.96.

∎

Proposition 2.99. *Let X be a topological space. Let $\mathcal{D}$ be a set of subsets of X that is maximal with respect to the finite intersection property. Let $x \in X$. Then the following are equivalent:*

1. *For all $D \in \mathcal{D}$ we have $x \in \overline{D}$*
2. *Every neighbourhood of x is in $\mathcal{D}$.*

PROOF:
⟨1⟩1. $1 \Rightarrow 2$
  ⟨2⟩1. For all $D \in \mathcal{D}$ we have $x \in \overline{D}$
  ⟨2⟩2. LET: U be a neighbourhood of x
  ⟨2⟩3. $\mathcal{D} \cup \{U\}$ satisfies the finite intersection property.
    ⟨3⟩1. LET: $D_1, \dots, D_n \in \mathcal{D}$
    ⟨3⟩2. $D_1 \cap \cdots \cap D_n \in \mathcal{D}$
      PROOF: Lemma 2.33.
    ⟨3⟩3. $x \in \overline{D_1 \cap \cdots \cap D_n}$
      PROOF: ⟨2⟩1, ⟨3⟩2
    ⟨3⟩4. $D_1 \cap \cdots \cap D_n \cap U \neq \emptyset$
      PROOF: Theorem 2.96, ⟨2⟩2, ⟨3⟩3.
  ⟨2⟩4. $\mathcal{D} = \mathcal{D} \cup \{U\}$
    PROOF: By the maximality of $\mathcal{D}$.

$\langle 2\rangle 5.$ $U \in \mathcal{D}$

$\langle 1\rangle 2.$ $2 \Rightarrow 1$

$\langle 2\rangle 1.$ Assume: Every neighbourhood of x is in $\mathcal{D}$.

$\langle 2\rangle 2.$ Let: $D \in \mathcal{D}$

$\langle 2\rangle 3.$ Every neighbourhood of x intersects D.

Proof: From $\langle 2\rangle 1$, $\langle 2\rangle 2$ and the fact that $\mathcal{D}$ satisfies the finite intersection property.

$\langle 2\rangle 4.$ $x \in \overline{D}$

Proof: Theorem 2.96, $\langle 2\rangle 3$.

□

2.26 Boundary

Definition 2.100 (Boundary)**.** The *boundary* of a set A is the set $\partial A = \overline{A} \cap \overline{X \setminus A}$.

Proposition 2.101.

$$\operatorname{Int} A \cap \partial A = \emptyset$$

Proof: Since $\overline{X \setminus A} = X \setminus \operatorname{Int} A$. □

Proposition 2.102.

$$\overline{A} = \operatorname{Int} A \cup \partial A$$

Proof:

$$\begin{aligned} \operatorname{Int} A \cup \partial A &= \operatorname{Int} A \cup (\overline{A} \cap \overline{X \setminus A}) \\ &= (\operatorname{Int} A \cup \overline{A}) \cap (\operatorname{Int} A \cup \overline{X \setminus A}) \\ &= \overline{A} \cap X \\ &= \overline{A} \end{aligned}$$

Proposition 2.103. *$\partial A = \emptyset$ if and only if A is open and closed.*

Proof: If $\partial A = \emptyset$ then $\overline{A} = \operatorname{Int} A$ by Proposition 2.102.

Proposition 2.104. *A set U is open if and only if $\partial U = \overline{U} \setminus U$.*

Proof:

$$\begin{aligned} &\partial U = \overline{U} \setminus U \\ \Leftrightarrow &\overline{U} \setminus \operatorname{Int} U = \overline{U} \setminus U && \text{(Propositions 2.101, 2.102)} \\ \Leftrightarrow &\operatorname{Int} U = U \end{aligned}$$

□

2.27 Limit Points

Definition 2.105 (Limit Point)**.** Let X be a topological space, $a \in X$ and $A \subseteq X$. Then a is a *limit point*, *cluster point* or *point of accumulation* for A if and only if every neighbourhood of a intersects A at a point other than a.

Lemma 2.106. *The point a is an accumulation point for A if and only if $a \in \overline{A \setminus \{a\}}$.*

Proof: From Theorem 2.96. $\square$

Theorem 2.107. *Let X be a topological space and $A \subseteq X$. Let A' be the set of all limit points of A. Then $\overline{A} = A \cup A'$.*

Proof:
⟨1⟩1. For all $x \in \overline{A}$, if $x \notin A$ then $x \in A'$
  Proof: From Theorem 2.96.
⟨1⟩2. $A \subseteq \overline{A}$
  Proof: Lemma 2.93.
⟨1⟩3. $A' \subseteq \overline{A}$
  Proof: From Theorem 2.96.
$\square$

Corollary 2.107.1. *A set is closed if and only if it contains all its limit points.*

Proposition 2.108. *In an indiscrete topology, every point is a limit point of any set that has more than one point.*

Proof: Let X be an indiscrete space. Let A be a set with more than one point and x be a point. The only neighbourhood of x is X, which must intersect A at a point other than x. $\square$

Lemma 2.109. *Let X be a topological space and $A \subseteq B \subseteq X$. Then every limit point of A is a limit point of B.*

Proof: Immediate from definitions. $\square$

2.28 Basis for a Topology

Definition 2.110 (Basis)**.** If X is a set, a *basis* for a topology on X is a set $\mathcal{B} \subseteq \mathcal{P}X$ called *basis elements* such that

1. For all $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$.
2. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

If $\mathcal{B}$ satisfies these two conditions, then we define the topology *generated* by $\mathcal{B}$ to be $\mathcal{T} = \{U \in \mathcal{P}X \mid \forall x \in U. \exists B \in \mathcal{B}. x \in B \subseteq U\}$.

We prove this is a topology.

Proof:
⟨1⟩1. $X \in \mathcal{T}$
  Proof: For all $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B \subseteq X$ by condition 1.

$\langle 1 \rangle 2$. For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$
$\langle 2 \rangle 1$. LET: $\mathcal{U} \subseteq \mathcal{T}$
$\langle 2 \rangle 2$. LET: $x \in \bigcup \mathcal{U}$
$\langle 2 \rangle 3$. PICK $U \in \mathcal{U}$ such that $x \in U$
$\langle 2 \rangle 4$. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$
PROOF: Since $U \in \mathcal{T}$ by $\langle 2 \rangle 1$ and $\langle 2 \rangle 3$.
$\langle 2 \rangle 5$. $x \in B \subseteq \bigcup \mathcal{U}$
$\langle 1 \rangle 3$. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$
$\langle 2 \rangle 1$. LET: $U, V \in \mathcal{T}$
$\langle 2 \rangle 2$. LET: $x \in U \cap V$
$\langle 2 \rangle 3$. PICK $B_1 \in \mathcal{B}$ such that $x \in B_1 \subseteq U$
$\langle 2 \rangle 4$. PICK $B_2 \in \mathcal{B}$ such that $x \in B_2 \subseteq V$
$\langle 2 \rangle 5$. PICK $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$
PROOF: By condition 2.
$\langle 2 \rangle 6$. $x \in B_3 \subseteq U \cap V$
$\square$

Lemma 2.111. *Let X be a set. Let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ is the set of all unions of subsets of $\mathcal{B}$.*

PROOF:
$\langle 1 \rangle 1$. For all $U \in \mathcal{T}$, there exists $\mathcal{A} \subseteq \mathcal{B}$ such that $U = \bigcup \mathcal{A}$
$\langle 2 \rangle 1$. LET: $U \in \mathcal{T}$
$\langle 2 \rangle 2$. LET: $\mathcal{A} = \{B \in \mathcal{B} \mid B \subseteq U\}$
$\langle 2 \rangle 3$. $U \subseteq \bigcup \mathcal{A}$
$\langle 3 \rangle 1$. LET: $x \in U$
$\langle 3 \rangle 2$. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$
PROOF: Since $\mathcal{B}$ is a basis for $\mathcal{T}$.
$\langle 3 \rangle 3$. $x \in B \in \mathcal{A}$
$\langle 2 \rangle 4$. $\bigcup \mathcal{A} \subseteq U$
PROOF: From the definition of $\mathcal{A}$ ($\langle 2 \rangle 2$).
$\langle 1 \rangle 2$. For all $\mathcal{A} \subseteq \mathcal{B}$ we have $\bigcup \mathcal{A} \in \mathcal{T}$
$\langle 2 \rangle 1$. $\mathcal{B} \subseteq \mathcal{T}$
PROOF: If $B \in \mathcal{B}$ and $x \in B$, then there exists $B' \in \mathcal{B}$ such that $x \in B' \subseteq B$, namely $B' = B$.
$\langle 2 \rangle 2$. Q.E.D.
PROOF: Since $\mathcal{T}$ is closed under union.
$\square$

Corollary 2.111.1. *Let X be a set. Let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ is the coarsest topology that includes $\mathcal{B}$.*

PROOF: Since every topology that includes $\mathcal{B}$ includes all unions of subsets of $\mathcal{B}$. $\square$

Lemma 2.112. *Let X be a topological space. Suppose that $\mathcal{C}$ is a set of open sets such that, for every open set U and every point $x \in U$, there exists $C \in \mathcal{C}$ such that $x \in C \subseteq U$. Then $\mathcal{C}$ is a basis for the topology on X.*

Proof:
$\langle 1\rangle 1$. For all $x \in X$, there exists $C \in \mathcal{C}$ such that $x \in C$
Proof: Immediate from hypothesis.
$\langle 1\rangle 2$. For all $C_1, C_2 \in \mathcal{C}$ and $x \in C_1 \cap C_2$, there exists $C_3 \in \mathcal{C}$ such that $x \in C_3 \subseteq C_1 \cap C_2$
Proof: Since $C_1 \cap C_2$ is open.
$\langle 1\rangle 3$. Every open set is open in the topology generated by $\mathcal{C}$
Proof: Immediate from hypothesis.
$\langle 1\rangle 4$. Every union of a subset of $\mathcal{C}$ is open.
Proof: Since every member of $\mathcal{C}$ is open.
$\square$

Lemma 2.113. *Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$ respectively on the set X. Then the following are equivalent.*

1. $\mathcal{T} \subseteq \mathcal{T}'$
2. *For all $B \in \mathcal{B}$ and $x \in B$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.*

Proof:
$\langle 1\rangle 1$. $1 \Rightarrow 2$
$\langle 2\rangle 1$. Assume: $\mathcal{T} \subseteq \mathcal{T}'$
$\langle 2\rangle 2$. Let: $B \in \mathcal{B}$ and $x \in B$
$\langle 2\rangle 3$. $B \in \mathcal{T}$
Proof: Corollary 2.111.1.
$\langle 2\rangle 4$. $B \in \mathcal{T}'$
Proof: By $\langle 2\rangle 1$
$\langle 2\rangle 5$. There exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$
Proof: Since $\mathcal{B}'$ is a basis for $\mathcal{T}'$.
$\langle 1\rangle 2$. $2 \Rightarrow 1$
$\langle 2\rangle 1$. Assume: 2
$\langle 2\rangle 2$. Let: $U \in \mathcal{T}$
Prove: $U \in \mathcal{T}'$
$\langle 2\rangle 3$. Let: $x \in U$
Prove: There exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq U$
$\langle 2\rangle 4$. Pick $B \in \mathcal{B}$ such that $x \in B \subseteq U$
Proof: Since $\mathcal{B}$ is a basis for $\mathcal{T}$.
$\langle 2\rangle 5$. Pick $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$
Proof: By $\langle 2\rangle 1$.
$\langle 2\rangle 6$. $x \in B' \subseteq U$
$\square$

Theorem 2.114. *Let X be a topological space, $A \subseteq X$ and $x \in X$. Let $\mathcal{B}$ be a basis for X. Then $x \in \overline{A}$ if and only if, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A.*

Proof:
$\langle 1\rangle 1$. If $x \in \overline{A}$ then, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A.

PROOF: This follows from Theorem 2.96 since every element of $\mathcal{B}$ is open (Corollary 2.111.1).

$\langle 1 \rangle 2$. Suppose that, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A. Then $x \in \overline{A}$.

$\langle 2 \rangle 1$. ASSUME: For all $B \in \mathcal{B}$, if $x \in B$ then B intersects A.

$\langle 2 \rangle 2$. LET: U be an open set that contains x
PROVE: U intersects A.

$\langle 2 \rangle 3$. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

$\langle 2 \rangle 4$. B intersects A.
PROOF: From $\langle 2 \rangle 1$.

$\langle 2 \rangle 5$. U intersects A.

$\langle 2 \rangle 6$. Q.E.D.
PROOF: By Theorem 2.96.

□

Definition 2.115 (Lower Limit Topology on the Real Line)**.** The *lower limit topology on the real line* is the topology on $\mathbb{R}$ generated by the basis consisting of all half-open intervals of the form $[a, b)$.

We write $\mathbb{R}_l$ for the topological space $\mathbb{R}$ under the lower limit topology.

We prove this is a basis for a topology.

PROOF:

$\langle 1 \rangle 1$. For all $x \in \mathbb{R}$ there exists an interval $[a, b)$ such that $x \in [a, b)$.
PROOF: Take $[a, b) = [x, x + 1)$.

$\langle 1 \rangle 2$. For any open intervals $[a, b)$, $[c, d)$ if $x \in [a, b) \cap [c, d)$, then there exists an interval $[e, f)$ such that $x \in [e, f) \subseteq [a, b) \cap [c, d)$
PROOF: Take $[e, f) = [\max(a, c), \min(b, d))$.

□

Definition 2.116 (K-topology on the Real Line)**.** Let $K = \{1/n \mid n \in \mathbb{Z}^+\}$.

The *K-topology on the real line* is the topology on $\mathbb{R}$ generated by the basis consisting of all open intervals (a, b) and all sets of the form $(a, b) \setminus K$.

We write $\mathbb{R}_K$ for the topological space $\mathbb{R}$ under the K-topology.

We prove this is a basis for a topology.

PROOF:

$\langle 1 \rangle 1$. For all $x \in \mathbb{R}$ there exists an open interval (a, b) such that $x \in (a, b)$.
PROOF: Take $(a, b) = (x - 1, x + 1)$.

$\langle 1 \rangle 2$. For any basic open sets B_1, B_2 if $x \in B_1 \cap B_2$, then there exists a basic open set B_3 such that $x \in B_3 \subseteq B_1 \cap B_2$

$\langle 2 \rangle 1$. CASE: $B_1 = (a, b)$, $B_2 = (c, d)$
PROOF: Take $B_3 = (\max(a, c), \min(b, d))$.

$\langle 2 \rangle 2$. CASE: $B_1 = (a, b)$ or $(a, b) \setminus K$, $B_2 = (c, d)$ or $(c, d) \setminus K$, and they are not both open intervals.
PROOF: Take $B_3 = (\max(a, c), \min(b, d)) \setminus K$.

□

Lemma 2.117. *The lower limit topology and the K-topology are incomparable.*

PROOF:
$\langle 1 \rangle 1$. The interval $[10, 11)$ is not open in the K-topology.
PROOF: There is no open interval (a, b) such that $10 \in (a, b) \subseteq [10, 11)$ or $10 \in (a, b) \setminus K \subseteq [10, 11)$.
$\langle 1 \rangle 2$. The set $(-1, 1) \setminus K$ is not open in the lower limit topology.
PROOF: There is no half-open interval $[a, b)$ such that $0 \in [a, b) \subseteq (-1, 1) \setminus K$, since there must be a positive integer n with $1/n \in [a, b)$.
□

Definition 2.118 (Subbasis)**.** A *subbasis* $\mathcal{S}$ for a topology on X is a set $\mathcal{S} \subseteq \mathcal{P}X$ such that $\bigcup \mathcal{S} = X$.

The topology *generated* by the subbasis $\mathcal{S}$ is the set of all unions of finite intersections of elements of $\mathcal{S}$.

We prove this is a topology.

PROOF:
$\langle 1 \rangle 1$. The set $\mathcal{B}$ of all finite intersections of elements of $\mathcal{S}$ forms a basis for a topology on X.
$\langle 2 \rangle 1$. $\bigcup \mathcal{B} = X$
PROOF: Since $\mathcal{S} \subseteq \mathcal{B}$.
$\langle 2 \rangle 2$. $\mathcal{B}$ is closed under binary intersection.
PROOF: By definition.
$\langle 1 \rangle 2$. Q.E.D.
PROOF: By Lemma 2.111.
□

We have simultaneously proved:

Proposition 2.119. *Let $\mathcal{S}$ be a subbasis for the topology on X. Then the set of all finite intersections of elements of $\mathcal{S}$ is a basis for the topology on X.*

Proposition 2.120. *Let X be a set. Let $\mathcal{S}$ be a subbasis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ is the coarsest topology that includes $\mathcal{S}$.*

PROOF: Since every topology that includes $\mathcal{S}$ includes every union of finite intersections of elements of $\mathcal{S}$. □

2.29 Local Basis at a Point

Definition 2.121 (Local Basis)**.** Let X be a topological space and $a \in X$. A *(local) basis at a* is a set $\mathcal{B}$ of neighbourhoods of a such that every neighbourhood of a includes some member of $\mathcal{B}$.

Lemma 2.122. *If there exists a countable local basis at a point a, then there exists a countable local basis $\{B_n \mid n \geq 1\}$ such that $B_1 \supseteq B_2 \supseteq \cdots$.*

PROOF: Given a countable local basis $\{C_n \mid n \geq 1\}$, take $B_n = C_1 \cap \cdots \cap C_n$. □

2.30 Convergence

Definition 2.123 (Convergence)**.** Let X be a topological space. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of points in X and $l \in X$. Then the sequence $(a_n)_{n\in\mathbb{N}}$ *converges* to the *limit* l, $a_n \to l$ as $n \to \infty$, if and only if, for every neighbourhood U of l, there exists N such that, for all $n \geq N$, we have $a_n \in U$.

Lemma 2.124. *Let X be a topological space. Let $A \subseteq X$ and $l \in X$. If there is a sequence of points in A that converges to l then $l \in \overline{A}$.*

PROOF:
⟨1⟩1. LET: (a_n) be a sequence of points in A that converges to l.
⟨1⟩2. LET: U be a neighbourhood of l.
⟨1⟩3. PICK N such that, for all $n \geq N$, we have $a_n \in U$.
⟨1⟩4. $a_N \in U \cap A$
⟨1⟩5. Q.E.D.
PROOF: Theorem 2.96.
□

Proposition 2.125. *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Let (a_n) be a sequence in X and $l \in X$. Then $a_n \to l$ as $n \to \infty$ if and only if, for every $B \in \mathcal{B}$ with $l \in B$, there exists N such that, for all $n \geq N$, we have $a_n \in B$.*

PROOF:
⟨1⟩1. If $a_n \to l$ as $n \to \infty$ then, for every $B \in \mathcal{B}$ with $l \in B$, there exists N such that, for all $n \geq N$, we have $a_n \in B$.
PROOF: Since every element of $\mathcal{B}$ is open (Corollary 2.111.1).
⟨1⟩2. If, for every $B \in \mathcal{B}$ with $l \in B$, there exists N such that, for all $n \geq N$, we have $a_n \in B$, then $a_n \to l$ as $n \to \infty$.
⟨2⟩1. ASSUME: for every $B \in \mathcal{B}$ with $l \in B$, there exists N such that, for all $n \geq N$, we have $a_n \in B$
⟨2⟩2. LET: U be a neighbourhood of l.
⟨2⟩3. PICK $B \in \mathcal{B}$ such that $l \in B \subseteq U$
⟨2⟩4. PICK N such that, for all $n \geq N$, we have $a_n \in B$
PROOF: From ⟨2⟩1.
⟨2⟩5. For all $n \geq N$ we have $a_n \in U$
□

Lemma 2.126. *If a sequence (a_n) is constant with $a_n = l$ for all n, then $a_n \to l$ as $n \to \infty$.*

PROOF: Immediate from definitions. □

Theorem 2.127. *Let X be a linearly ordered set. Let (s_n) be a monotone increasing sequence in X with a supremum s. Then $s_n \to s$ as $n \to \infty$.*

PROOF:
⟨1⟩1. ASSUME: s is not least in X.

PROOF: Otherwise (s_n) is the constant sequence s and the result follows from Lemma 2.126.
⟨1⟩2. LET: U be a neighbourhood of s.
⟨1⟩3. PICK$a < s$ such that $(a, s] \subseteq U$
⟨1⟩4. PICK N such that $a < a_N$.
⟨1⟩5. For all $n \geq N$ we have $a_n \in (a, s]$
⟨1⟩6. For all $n \geq N$ we have $a_n \in U$.
□

Theorem 2.128. *If* $\sum_{i=0}^{\infty} a_i = s$ *and* $\sum_{i=0}^{\infty} b_i = t$ *then* $\sum_{i=0}^{\infty}(ca_i + b_i) = cs + t$.

PROOF:$\sum_{i=0}^{N}(ca_i + b_i) = c\sum_{i=0}^{N} a_i + \sum_{i=0}^{N} b_i \to cs + t$ as $n \to \infty$. □

Theorem 2.129 (Comparison Test)**.** *If* $|a_i| \leq b_i$ *for all* i *and* $\sum_{i=0}^{\infty} b_i$ *converges then* $\sum_{i=0}^{\infty} a_i$ *converges.*

PROOF:
⟨1⟩1. $\sum_{i=0}^{\infty} |a_i|$ converges
PROOF: The partial sums $\sum_{i=0}^{N} |a_i|$ form an increasing sequence bounded above by $\sum_{i=0}^{\infty} b_i$.
⟨1⟩2. LET: $c_i = |a_i| + a_i$ for all i
⟨1⟩3. $\sum_{i=0}^{\infty} c_i$ converges
PROOF: Each c_i is either $2|a_i|$ or 0. So the partial sums $\sum_{i=0}^{N} c_i$ form an increasing sequence bounded above by $2\sum_{i=0}^{\infty} b_i$.
⟨1⟩4. Q.E.D.
PROOF: Since $a_i = c_i - |a_i|$.
□

Corollary 2.129.1. *If* $\sum_{i=0}^{\infty} |a_i|$ *converges then* $\sum_{i=0}^{\infty} a_i$ *converges.*

Theorem 2.130 (Weierstrass M-test)**.** *Let* X *be a set and* $(f_n : X \to \mathbb{R})$ *be a sequence of functions. Let*

$$s_n(x) = \sum_{i=0}^{n} f_i(x)$$

for all n, x. *Suppose* $|f_i(x)| \leq M_i$ *for all* $i \geq 0$ *and* $x \in X$. *If the series* $\sum_{i=0}^{\infty} M_i$ *converges, then the sequence* (s_n) *converges uniformly to*

$$s(x) = \sum_{i=0}^{\infty} f_i(x) \ .$$

PROOF:
⟨1⟩1. LET: $r_n = \sum_{i=n+1}^{\infty} M_i$ for all n
⟨1⟩2. Given $0 \leq n < k$, we have $|s_k(x) - s_n(x)| \leq r_n$

PROOF:

$$
\begin{aligned}
|s_k(x) - s_n(x)| &= |\sum_{i=n+1}^{k} f_i(x)| \\
&\leq \sum_{i=n+1}^{k} |f_i(x)| \\
&\leq \sum_{i=n+1}^{k} M_i \\
&\leq r_n
\end{aligned}
$$

⟨1⟩3. Given $n \geq 0$ we have $|s(x) - s_n(x)| \leq r_n$

PROOF: By taking the limit $k \to \infty$ in ⟨1⟩2.

⟨1⟩4. Q.E.D.

PROOF: Since $r_n \to 0$ as $n \to \infty$.

□

2.31 Locally Finite Sets

Definition 2.131 (Locally Finite)**.** Let X be a topological space and $\{A_\alpha\}$ a family of subsets of X. Then $\mathcal{A}$ is *locally finite* if and only if every point in X has a neighbourhood that intersects A_α for only finitely many α.

Theorem 2.132 (Pasting Lemma)**.** *Let X and Y be topological spaces and $f : X \to Y$. Let $\{A_\alpha\}$ be a locally finite family of closed subsets of X that cover X. Suppose $f \restriction A_\alpha$ is continuous for all α. Then f is continuous.*

PROOF:

⟨1⟩1. Let X and Y be topological spaces and $f : X \to Y$. Let A and B be closed subsets of X such that $X = A \cup B$. Suppose $f \restriction A$ and $f \restriction B$ are continuous. Then f is continuous.

⟨2⟩1. LET: $C \subseteq Y$ be closed.

⟨2⟩2. $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$

⟨2⟩3. $f^{-1}(C)$ and $g^{-1}(C)$ are closed in X.

PROOF: Theorems 2.142 and 2.193.

⟨2⟩4. $h^{-1}(C)$ is closed in X.

PROOF: Lemma 2.82.

⟨2⟩5. Q.E.D.

PROOF: Theorem 2.142.

⟨1⟩2. Let X and Y be topological spaces and $f : X \to Y$. Let $\{A_\alpha\}$ be a finite family of closed subsets of X that cover X. Suppose $f \restriction A_\alpha$ is continuous for all α. Then f is continuous.

PROOF: From ⟨1⟩1 by induction.

⟨1⟩3. Let X and Y be topological spaces and $f : X \to Y$. Let $\{A_\alpha\}$ be a locally finite family of closed subsets of X that cover X. Suppose $f \restriction A_\alpha$ is continuous for all α. Then f is continuous.

$\langle 2\rangle 1$. LET: $x \in X$
PROVE: f is continuous at x
$\langle 2\rangle 2$. PICK a neighbourhood U of x that intersects A_α for only finitely many α.
$\langle 2\rangle 3$. $f \upharpoonright U$ is continuous
PROOF: By $\langle 1\rangle 2$.
$\langle 2\rangle 4$. Q.E.D.
PROOF: Lemma 2.152.
□

The following example shows that we cannot remove the assumption of local finiteness.

Example 2.133. Define $f : [-1, 1] \to \mathbb{R}$ by: $f(x) = 1$ if $x < -1$, $f(x) = 0$ if $x > 1$. Let $C_n = [-1, -1/n]$ for $n \geq 1$, and $D = [0, 1]$. Then $[-1, 1] = \bigcup_{n=1}^{\infty} C_n \cup D$ and f is continuous on each C_n and each D, but f is not continuous on $[-1, 1]$.

2.32 Open Maps

Definition 2.134 (Open Map)**.** Let X and Y be topological spaces and $f : X \to Y$. Then f is an *open map* if and only if, for every open set U in X, the set $f(U)$ is open in Y.

Lemma 2.135. *Let X and Y be topological spaces and $f : X \to Y$. Let $\mathcal{B}$ be a basis for the topology on X. If $f(B)$ is open in Y for all $B \in \mathcal{B}$, then f is an open map.*

PROOF: From Lemma 2.111. □

Proposition 2.136. *Let X and Y be topological spaces. Let $\mathcal{B}$ be a basis for the topology on X. Let $f : X \to Y$. Suppose that, for all $B \in \mathcal{B}$, we have $f(B)$ is open tn Y. Then f is an open map.*

PROOF: For any $\mathcal{A} \subseteq \mathcal{B}$, we have $f(\bigcup \mathcal{A}) = \bigcup_{B \in \mathcal{B}} f(B)$ is open in Y. The result follows from Lemma 2.111. □

2.33 Continuous Functions

Definition 2.137 (Continuous)**.** Let X and Y be topological spaces. A function $f : X \to Y$ is *continuous* if and only if, for every open set V in Y, the set $f^{-1}(V)$ is open in X.

Proposition 2.138. *Let X and Y be topological spaces and $f : X \to Y$. Let $\mathcal{B}$ be a basis for Y. Then f is continuous if and only if, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X.*

PROOF:
$\langle 1\rangle 1$. If f is continuous then, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X.

PROOF: Since every element of B is open (Lemma 2.111).
⟨1⟩2. Suppose that, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X. Then f is continuous.
⟨2⟩1. ASSUME: For all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X.
⟨2⟩2. LET: V be open in Y.
⟨2⟩3. PICK $\mathcal{A} \subseteq \mathcal{B}$ such that $V = \bigcup \mathcal{A}$
PROOF: By Lemma 2.111.
⟨2⟩4. $f^{-1}(V)$ is open in X.
PROOF:

$$\begin{aligned} f^{-1}(V) &= f^{-1}\left(\bigcup \mathcal{A}\right) \\ &= \bigcup_{B \in \mathcal{A}} f^{-1}(B) \end{aligned}$$

□

Proposition 2.139. *Let X and Y be topological spaces and $f : X \to Y$. Let $\mathcal{S}$ be a subbasis for Y. Then f is continuous if and only if, for all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in X.*

PROOF:
⟨1⟩1. If f is continuous then, for all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in X.
PROOF: Since every element of S is open.
⟨1⟩2. Suppose that, for all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in X. Then f is continuous.
⟨2⟩1. ASSUME: For all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in X.
⟨2⟩2. LET: $S_1, \ldots, S_n \in \mathcal{S}$
⟨2⟩3. $f^{-1}(S_1 \cap \cdots \cap S_n)$ is open in A
PROOF: Since $f^{-1}(S_1 \cap \cdots \cap S_n) = f^{-1}(S_1) \cap \cdots \cap f^{-1}(S_n)$.
⟨2⟩4. Q.E.D.
PROOF: By Propositions 2.138 and 2.119.

□

Proposition 2.140. *Let X and Y be topological spaces and $f : X \to Y$. Let $\mathcal{S}$ be a basis for Y. Then f is continuous if and only if, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X.*

PROOF:
⟨1⟩1. If f is continuous then, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X.
PROOF: Since every element of $\mathcal{S}$ is open.
⟨1⟩2. Suppose that, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X. Then f is continuous.
⟨2⟩1. ASSUME: For all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X.
⟨2⟩2. For every set B that is the finite intersection of elemets of $\mathcal{S}$, we have $f^{-1}(B)$ is open in X.
PROOF: Because $f^{-1}(V_1 \cap \cdots \cap V_n) = f^{-1}(V_1) \cap \cdots \cap f^{-1}(V_n)$.
⟨2⟩3. Q.E.D.
PROOF: From Propositions 2.119 and 2.138.

□

Definition 2.141 (Continuous at a Point)**.** Let X and Y be topological spaces. Let $f : X \to Y$ and $x \in X$. Then f is *continuous at* x if and only if, for every neighbourhood V of $f(x)$, there exists a neighbourhood U of x such that $f(U) \subseteq V$.

Theorem 2.142. *Let X and Y be topological spaces and $f : X \to Y$. Then the following are equivalent:*

1. *f is continuous.*
2. *For all $A \subseteq X$, we have $f(\overline{A}) \subseteq \overline{f(A)}$*
3. *For all $B \subseteq Y$ closed, we have $f^{-1}(B)$ is closed in X.*
4. *f is continuous at every point of X.*

PROOF:
⟨1⟩1. $1 \Rightarrow 2$
 ⟨2⟩1. ASSUME: f is continuous.
 ⟨2⟩2. LET: $A \subseteq X$
 ⟨2⟩3. LET: $x \in \overline{A}$
  PROVE: $f(x) \in \overline{f(A)}$
 ⟨2⟩4. LET: V be a neighbourhood of $f(x)$
 ⟨2⟩5. $f^{-1}(V)$ is a neighbourhood of x
 ⟨2⟩6. PICK $y \in A \cap f^{-1}(V)$
  PROOF: By Theorem 2.96.
 ⟨2⟩7. $f(y) \in V \cap f(A)$
 ⟨2⟩8. Q.E.D.
  PROOF: By Theorem 2.96.
⟨1⟩2. $2 \Rightarrow 3$
 ⟨2⟩1. ASSUME: 2
 ⟨2⟩2. LET: B be closed in Y
 ⟨2⟩3. LET: $x \in \overline{f^{-1}(B)}$
  PROVE: $x \in f^{-1}(B)$
 ⟨2⟩4. $f(x) \in B$
  PROOF:

$$\begin{aligned} f(x) &\in f(\overline{f^{-1}(B)}) \\ &\subseteq \overline{f(f^{-1}(B))} && (\langle 2\rangle 1) \\ &\subseteq \overline{B} && (Proposition 2.97) \\ &= B \end{aligned}$$

⟨1⟩3. $3 \Rightarrow 1$
 ⟨2⟩1. ASSUME: 3
 ⟨2⟩2. LET: V be open in Y
 ⟨2⟩3. $Y \setminus V$ is closed in Y
 ⟨2⟩4. $f^{-1}(Y \setminus V)$ is closed in X
 ⟨2⟩5. $X \setminus f^{-1}(V)$ is closed in X
 ⟨2⟩6. $f^{-1}(V)$ is open in X

$\langle 1 \rangle 4.$ $1 \Rightarrow 4$

PROOF: For any neighbourhood V of $f(x)$, the set $U = f^{-1}(V)$ is a neighbourhood of x such that $f(U) \subseteq V$.

$\langle 1 \rangle 5.$ $4 \Rightarrow 1$

$\langle 2 \rangle 1.$ ASSUME: 4

$\langle 2 \rangle 2.$ LET: V be open in Y

$\langle 2 \rangle 3.$ LET: $x \in f^{-1}(V)$

$\langle 2 \rangle 4.$ V is a neighbourhood of $f(x)$

$\langle 2 \rangle 5.$ PICK a neighbourhood U of x such that $f(U) \subseteq V$

$\langle 2 \rangle 6.$ $x \in U \subseteq f^{-1}(V)$

$\langle 2 \rangle 7.$ Q.E.D.

PROOF: By Lemma 2.74.

$\square$

Theorem 2.143. *A constant function is continuous.*

PROOF: Let X and Y be topological spaces. Let $b \in Y$, and let $f : X \to Y$ be the constant function with value b. For any open $V \subseteq Y$, the set $f^{-1}(V)$ is either X (if $b \in V$) or $\emptyset$ (if $b \notin V$). $\square$

Theorem 2.144. *If A is a subspace of X then the inclusion $j : A \to X$ is continuous.*

PROOF: For any V open in X, we have $j^{-1}(V) = V \cap A$ is open in A. $\square$

Theorem 2.145. *The composite of two continuous functions is continuous.*

PROOF: Let $f : X \to Y$ and $g : Y \to Z$ be continuous. For any V open in Z, we have $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is open in X. $\square$

Theorem 2.146. *Let $f : X \to Y$ be a continuous function and A be a subspace of X. Then the restriction $f \upharpoonright A : A \to Y$ is continuous.*

PROOF: Let V be open in Y. Then $(f \upharpoonright A)^{-1}(V) = f^{-1}(V) \cap A$ is open in A. $\square$

Theorem 2.147. *Let $f : X \to Y$ be continuous. Let Z be a subspace of Y such that $f(X) \subseteq Z$. Then the corestriction $f : X \to Z$ is continuous.*

PROOF:

$\langle 1 \rangle 1.$ LET: V be open in Z.

$\langle 1 \rangle 2.$ PICK U open in Y such that $V = U \cap Z$.

$\langle 1 \rangle 3.$ $f^{-1}(V) = f^{-1}(U)$

$\langle 1 \rangle 4.$ $f^{-1}(V)$ is open in X.

$\square$

Theorem 2.148. *Let $f : X \to Y$ be continuous. Let Z be a space such that Y is a subspace of Z. Then the expansion $f : X \to Z$ is continuous.*

PROOF: Let V be open in Z. Then $f^{-1}(V) = f^{-1}(V \cap Y)$ is open in X. $\square$

Theorem 2.149. *Let X and Y be topological spaces. Let $f : X \to Y$. Suppose $\mathcal{U}$ is a set of open sets in X such that $X = \bigcup \mathcal{U}$ and, for all $U \in \mathcal{U}$, we have $f \restriction U : U \to Y$ is continuous. Then f is continuous.*

PROOF:
⟨1⟩1. LET: V be open in Y
⟨1⟩2. $f^{-1}(V) = \bigcup_{U \in \mathcal{U}} (f \restriction U)^{-1}(V)$
⟨1⟩3. For all $U \in \mathcal{U}$, we have $(f \restriction U)^{-1}(V)$ is open in U.
⟨1⟩4. For all $U \in \mathcal{U}$, we have $(f \restriction U)^{-1}(V)$ is open in X.
  PROOF: Lemma 2.192.
□

Proposition 2.150. *Let X and X' be the same set X under two topologies $\mathcal{T}$ and $\mathcal{T}'$. Let $i : X \to X'$ be the identity function. Then i is continuous if and only if $\mathcal{T}' \subseteq \mathcal{T}$.*

PROOF: Immediate from definitions. □

Proposition 2.151. *Let $f : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$. Then f is continuous on the right at a if and only if f is continuous at a as a function $\mathbb{R}_l \to \mathbb{R}$.*

PROOF:
⟨1⟩1. If f is continuous on the right at a then f is continuous at a as a function $\mathbb{R}_l \to \mathbb{R}$.
  ⟨2⟩1. ASSUME: f is continuous on the right at a.
  ⟨2⟩2. LET: V be a neighbourhood of $f(a)$
  ⟨2⟩3. PICK b, c such that $f(a) \in (b, c) \subseteq V$.
  ⟨2⟩4. LET: $\epsilon = \min(c - f(a), f(a) - b)$
  ⟨2⟩5. PICK $\delta > 0$ such that, for all x, if $a < x < a + \delta$ then $|f(x) - f(a)| < \epsilon$
  ⟨2⟩6. LET: $U = [a, a + \delta)$
  ⟨2⟩7. $f(U) \subseteq V$
⟨1⟩2. If f is continuous at a as a function $\mathbb{R}_l \to \mathbb{R}$ then f is continuous on the right at a.
  ⟨2⟩1. ASSUME: f is continuous at a as a function $\mathbb{R}_l \to \mathbb{R}$
  ⟨2⟩2. LET: $\epsilon > 0$
  ⟨2⟩3. PICK a neighbourhood U of a such that $f(U) \subseteq (f(a) - \epsilon, f(a) + \epsilon)$
  ⟨2⟩4. PICK b, c such that $a \in [b, c) \subset U$
  ⟨2⟩5. LET: $\delta = c - a$
  ⟨2⟩6. For all x, if $a < x < a + \delta$ then $|f(x) - f(a)| < \epsilon$
□

Lemma 2.152. *Let $f : X \to Y$. Let Z be an open subspace of X and $a \in Z$. If $f \restriction Z$ is continuous at a then f is continuous at a.*

PROOF:
⟨1⟩1. LET: V be a neighbourhood of $f(x)$
⟨1⟩2. PICK a neighbourhood W of x in Z such that $f(W) \subseteq V$
⟨1⟩3. W is a neighbourhood of x in X such that $f(W) \subseteq V$

PROOF: Lemma 2.192.
∎

Proposition 2.153. *Let $f : A \to B$ and $g : C \to D$ be continuous. Define $f \times g : A \times C \to B \times D$ by*

$$(f \times g)(a, c) = (f(a), g(c)) \ .$$

Then $f \times g$ is continuous.

PROOF: $\pi_1 \circ (f \times g) = f \circ \pi_1$ and $\pi_2 \circ (f \times g) = g \circ \pi_2$ are continuous by Theorem 2.145. The result follows by Theorem 2.181.

Proposition 2.154. *Let X and Y be topological spaces and $f : X \to Y$ be continuous. If $a_n \to l$ as $n \to \infty$ in X then $f(a_n) \to f(l)$ as $n \to \infty$.*

PROOF:
$\langle 1 \rangle 1$. LET: V be a neighbourhood of $f(l)$
$\langle 1 \rangle 2$. PICK a neighbourhood U of l such that $f(U) \subseteq V$
$\langle 1 \rangle 3$. PICK N such that, for all $n \geq N$, we have $a_n \in U$
$\langle 1 \rangle 4$. For all $n \geq N$ we have $f(n) \in V$
∎

2.34 Homeomorphisms

Definition 2.155 (Homeomorphism)**.** Let X and Y be topological spaces. A *Homeomorphism* f between X and Y, $f : X \cong Y$, is a bijection $f : X \to Y$ such that both f and f^{-1} are continuous.

Lemma 2.156. *Let X and Y be topological spaces and $f : X \to Y$ a bijection. Then the following are equivalent:*

1. *f is a homeomorphism.*
2. *f is continuous and an open map.*
3. *f is continuous and a closed map.*
4. *For any $U \subseteq X$, we have U is open if and only if $f(U)$ is open.*

PROOF: Immediate from definitions. ∎

Proposition 2.157. *Let X and X' be the same set X under two topologies $\mathcal{T}$ and $\mathcal{T}'$. Let $i : X \to X'$ be the identity function. Then i is a homeomorphism if and only if $\mathcal{T} = \mathcal{T}'$.*

PROOF: Immediate from definitions. ∎

Definition 2.158 (Topological Property)**.** Let P be a property of topological spaces. Then P is a *topological* property if and only if, for any spaces X and Y, if P holds of X and $X \cong Y$ then P holds of Y.

Definition 2.159 (Topological Imbedding)**.** Let X and Y be topological spaces and $f : X \to Y$. Then f is a *topological imbedding* if and only if the corestriction $f : X \to f(X)$ is a homeomorphism.

Proposition 2.160. *Let X and Y be topological spaces and $a \in X$. The function $i : Y \to X \times Y$ that maps y to (a, y) is an imbedding.*

PROOF:
$\langle 1 \rangle 1$. i is injective
$\langle 1 \rangle 2$. i is continuous.
PROOF: For U open in X and V open in Y, we have $i^{-1}(U \times V)$ is V if $a \in U$, and $\emptyset$ if $a \notin U$.
$\langle 1 \rangle 3$. $i : Y \to i(Y)$ is an open map.
PROOF: For V open in Y we have $i(V) = (X \times V) \cap i(Y)$.
□

2.35 The Order Topology

Definition 2.161 (Order Topology)**.** Let X be a linearly ordered set with at least two points. The *order topology* on X is the topology generated by the basis $\mathcal{B}$ consisting of:

- all open intervals (a, b);
- all intervals of the form $[\bot, b)$ where $\bot$ is least in X;
- all intervals of the form $(a, \top]$ where $\top$ is greatest in X.

We prove this is a basis for a topology.

PROOF:
$\langle 1 \rangle 1$. For all $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B$.
$\langle 2 \rangle 1$. LET: $x \in X$
$\langle 2 \rangle 2$. CASE: x is greatest in X.
$\langle 3 \rangle 1$. PICK $y \in X$ with $y \neq x$
$\langle 3 \rangle 2$. $x \in (y, x] \in \mathcal{B}$
$\langle 2 \rangle 3$. CASE: x is least in X.
$\langle 3 \rangle 1$. PICK $y \in X$ with $y \neq x$
$\langle 3 \rangle 2$. $x \in [x, y) \in \mathcal{B}$
$\langle 2 \rangle 4$. CASE: x is neither greatest nor least in X.
$\langle 3 \rangle 1$. PICK $a, b \in X$ with $a < x$ and $x < b$
$\langle 3 \rangle 2$. $x \in (a, b) \in \mathcal{B}$
$\langle 1 \rangle 2$. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$
$\langle 2 \rangle 1$. LET: $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$
$\langle 2 \rangle 2$. CASE: $B_1 = (a, b)$, $B_2 = (c, d)$
PROOF: Take $B_3 = (\max(a, c), \min(b, d))$.
$\langle 2 \rangle 3$. CASE: $B_1 = (a, b)$, $B_2 = [\bot, d)$

PROOF: Take $B_3 = (a, \min(b, d))$.
⟨2⟩4. CASE: $B_1 = (a, b)$, $B_2 = (c, \top]$
PROOF: Take $B_3 = (\max(a, c), b)$.
⟨2⟩5. CASE: $B_1 = [\perp, b)$, $B_2 = [\perp, d)$
PROOF: Take $B_3 = [\perp, \min(b, d))$.
⟨2⟩6. CASE: $B_1 = [\perp, b)$, $B_2 = (c, \top]$
PROOF: Take $B_3 = (c, b)$.
□

Lemma 2.162. *Let X be a linearly ordered set. Then the open rays form a subbasis for the order topology on X.*

PROOF:
⟨1⟩1. Every open ray is open.
⟨2⟩1. For all $a \in X$, the ray $(-\infty, a)$ is open.
⟨3⟩1. LET: $x \in (-\infty, a)$
⟨3⟩2. CASE: x is least in X
PROOF: $xin[x, a) = (-\infty, a)$.
⟨3⟩3. CASE: x is not least in X
⟨4⟩1. PICK $y < x$
⟨4⟩2. $x \in (y, a) \subseteq (-\infty, a)$
⟨2⟩2. For all $a \in X$, the ray $(a, +\infty)$ is open.
PROOF: Similar.
⟨1⟩2. Every basic open set is a finite intersection of open rays.
PROOF: We have $(a, b) = (a, +\infty) \cap (-\infty, b)$, $[\perp, b) = (-\infty, b)$ and $(a, \top] = (a, +\infty)$.
□

Definition 2.163 (Standard Topology on the Real Line)**.** The *standard topology on the real line* is the order topology on $\mathbb{R}$ generated by the standard order.

Lemma 2.164. *The lower limit topology is strictly finer than the standard topology on $\mathbb{R}$.*

PROOF:
⟨1⟩1. Every open interval is open in the lower limit topology.
PROOF: If $x \in (a, b)$ then $x \in [x, b) \subseteq (a, b)$.
⟨1⟩2. The half-open interval $[0, 1)$ is not open in the standard topology.
PROOF: There is no open interval (a, b) such that $0 \in (a, b) \subseteq [0, 1)$.
□

Lemma 2.165. *The K-topology is strictly finer than the standard topology on $\mathbb{R}$.*

PROOF:
⟨1⟩1. Every open interval is open in the K-topology.
PROOF: Corollary 2.111.1.
⟨1⟩2. The set $(-1, 1) \setminus K$ is not open in the standard topology.

PROOF: There is no open interval (a, b) such that $0 \in (a, b) \subseteq (-1, 1) \setminus K$, since there must be a positive integer n with $1/n \in (a, b)$.
∎

Lemma 2.166. *Let X be a topological space. Let Y be a linearly ordered set in the order topology. Let $f, g : X \to Y$ be continuous. Then $C = \{x \in X \mid f(x) \leq g(x)\}$ is closed.*

PROOF:
$\langle 1 \rangle 1$. LET: $x \in X \setminus C$
$\langle 1 \rangle 2$. $f(x) > g(x)$
PROVE: There exists a neighbourhood U of x such that $U \subseteq X \setminus C$
$\langle 1 \rangle 3$. CASE: There exists y such that $g(x) < y < f(x)$
PROOF: Take $U = g^{-1}((-\infty, y)) \cup f^{-1}(y, +\infty)$.
$\langle 1 \rangle 4$. CASE: There is no y such that $g(x) < y < f(x)$
PROOF: Take $U = g^{-1}((-\infty, f(x))) \cup f^{-1}(g(x), +\infty)$.
∎

Proposition 2.167. *Let X be a topological space. Let Y be a linearly ordered set in the order topology. Let $f, g : X \to Y$ be continuous. Define $h : X \to Y$ by $h(x) = \min(f(x), g(x))$. Then h is continuous.*

PROOF: By the Pasting Lemma applied to $\{x \in X \mid f(x) \leq g(x)\}$ and $\{x \in X \mid g(x) \leq f(x)\}$, which are closed by Lemma 2.166.

Proposition 2.168. *Let X and Y be linearly ordered sets in the order topology. Let $f : X \to Y$ be strictly monotone and surjective. Then f is a homeomorphism.*

PROOF:
$\langle 1 \rangle 1$. f is bijective.
PROOF: Proposition 2.45.
$\langle 1 \rangle 2$. f is continuous.
$\langle 2 \rangle 1$. For all $y \in Y$ we have $f^{-1}((y, +\infty))$ is open.
$\langle 3 \rangle 1$. LET: $y \in Y$
$\langle 3 \rangle 2$. PICK$x \in X$ such that $f(x) = y$
PROOF: Since f is surjective.
$\langle 3 \rangle 3$. $f^{-1}((y, +\infty)) = (x, +\infty)$
PROOF: By strict monotoncity.
$\langle 2 \rangle 2$. For all $y \in Y$ we have $f^{-1}((-\infty, y))$ is open.
PROOF: Similar.
$\langle 1 \rangle 3$. f^{-1} is continuous.
$\langle 2 \rangle 1$. For all $x \in X$ we have $f((x, +\infty))$ is open.
PROOF: $f((x, +\infty)) = (f(x), +\infty)$.
$\langle 2 \rangle 2$. For all $x \in X$ we have $f((-\infty, x))$ is open.
PROOF: $f((-\infty, x)) = (-\infty, f(x))$.
∎

2.36 The nth Root Function

Proposition 2.169. *For all $n \geq 1$, the function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined by $f(x) = x^n$ is a homemorphism.*

PROOF:
$\langle 1 \rangle 1$. f is strictly monotone.
  $\langle 2 \rangle 1$. LET: $x, y \in \mathbb{R}$ with $0 \leq x < y$
  $\langle 2 \rangle 2$. $x^n < y^n$

$$\begin{aligned} y^n - x^n &= (y-x)(y^{n-1} + y^{n-2}x + y^{n-3}x^2 + \cdots + x^{n-1}) \\ &> 0 \end{aligned}$$

$\langle 1 \rangle 2$. f is surjective.
  $\langle 2 \rangle 1$. LET: $y \in \mathbb{R}_{\geq 0}$
  $\langle 2 \rangle 2$. PICK $x \in \mathbb{R}$ such that $y \leq x^n$
    PROOF: If $y \leq 1$ take $x = 1$, otherwise take $x = y$.
  $\langle 2 \rangle 3$. There exists $x' \in [0, x]$ such that $(x')^n = y$
    PROOF: By the Intermediate Value Theorem.
$\langle 1 \rangle 3$. Q.E.D.
  PROOF: Proposition 2.168.
□

Definition 2.170. For $n \geq 1$, the *nth root function* is the function $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ that is the inverse of $\lambda x.x^n$.

2.37 The Product Topology

Definition 2.171 (Product Topology)**.** Let $\{A_i\}_{i \in I}$ be a family of topological spaces. The *product topology* on $\prod_{i \in I} A_i$ is the topology generated by the subbasis consisting of the sets of the form ${\pi_i}^{-1}(U)$ where $i \in I$ and U is open in A_i.

Proposition 2.172. *The product topology on $\prod_{i \in I} A_i$ is generated by the basis consisting of all sets of the form $\prod_{i \in I} U_i$ where $\{U_i\}_{i \in I}$ is a family such that each U_i is an open set in A_i and $U_i = A_i$ for all but finitely many i.*

PROOF: From Proposition 2.119. □

Proposition 2.173. *If A_i is closed in X_i for all $i \in I$ then $\prod_{i \in I} A_i$ is closed in $\prod_{i \in I} X_i$.*

PROOF:

$$(\prod_{i \in I} X_i) \setminus (\prod_{i \in I} A_i) = \bigcup_{j \in I} \left(\prod_{i \in I} X_i \setminus {\pi_j}^{-1}(A_j) \right) \square$$

Proposition 2.174. *Let $\{A_i\}_{i \in I}$ be a family of topological spaces. For $i \in I$, let $\mathcal{B}_i$ be a basis for the topology on A_i. Then $\mathcal{B} = \{\prod_{i \in I} B_i \mid \forall i \in I. B_i \in \mathcal{B}_i, B_i = A_i$ for all but finitely many $i\}$ is a basis for the box topology on $\prod_{i \in I} A_i$.*

PROOF:
⟨1⟩1. Every set in $\mathcal{B}$ is open.
⟨1⟩2. For every point $a \in \prod_{i \in I} A_i$ and every open set U with $a \in U$, there exists $B \in \mathcal{B}$ such that $a \in B \subseteq U$.
⟨2⟩1. LET: U be open and $a \in U$
⟨2⟩2. PICK a family $\{U_i\}_{i \in I}$ such that each U_i is open in A_i, such that $U_i = A_i$ except for $i = i_1, \ldots, i_n$, and such that $a \in \prod_{i \in I} U_i \subseteq U$.
⟨2⟩3. For $j = 1, \ldots, n$, PICK $B_{i_j} \in \mathcal{B}_{i_j}$ such that $a_{i_j} \in B_{i_j} \subseteq U_{i_j}$
⟨2⟩4. LET: $B = \prod_{i \in I} B_i$ where $B_i = A_i$ for $i \neq i_1, \ldots, i_n$
⟨2⟩5. $B \in \mathcal{B}$
⟨2⟩6. $a \in B \subseteq U$
⟨1⟩3. Q.E.D.
PROOF: Lemma 2.112.
□

Proposition 2.175. *Let $\{A_i\}_{i \in I}$ be a family of topological spaces. Then the projections $\pi_i : \prod_{i \in I} A_i \to A_i$ are open maps.*

PROOF: From Lemma 2.135. □

Example 2.176. The projections are not always closed maps. For example, $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ maps the closed set $\{(x, 1/x) \mid x > 0\}$ to $(0, +\infty)$.

Proposition 2.177. *Let $\{X_i\}_{i \in I}$ be a family of sets. For $i \in I$, let $\mathcal{T}_i$ and $\mathcal{U}_i$ be topologies on X_i. Let $\mathcal{P}$ be the product topology on $\prod_{i \in I} X_i$ generated by the topologies $\mathcal{T}_i$, and $\mathcal{Q}$ the product topology on the same set generated by the topologies $\mathcal{U}_i$. Then $\mathcal{P} \subseteq \mathcal{Q}$ if and only if $\mathcal{T}_i \subseteq \mathcal{U}_i$ for all i.*

PROOF:
⟨1⟩1. If $\mathcal{T}_i \subseteq \mathcal{U}_i$ for all i then $\mathcal{P} \subseteq \mathcal{Q}$
PROOF: By Corollary 2.111.1.
⟨1⟩2. If $\mathcal{P} \subseteq \mathcal{Q}$ then $\mathcal{T}_i \subseteq \mathcal{U}_i$ for all i
⟨2⟩1. ASSUME: $\mathcal{P} \subseteq \mathcal{Q}$
⟨2⟩2. LET: $i \in I$
⟨2⟩3. LET: $U \in \mathcal{T}_i$
⟨2⟩4. LET: $U_i = U$ and $U_j = X_j$ for $j \neq i$
⟨2⟩5. $\prod_{i \in I} U_i \in \mathcal{P}$
⟨2⟩6. $\prod_{i \in I} U_i \in \mathcal{Q}$
⟨2⟩7. $U \in \mathcal{U}_i$
PROOF: From Proposition 2.175.
□

Proposition 2.178 (AC)**.** *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let $A_i \subseteq X_i$ for all $i \in I$. Then*

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i} \ .$$

PROOF:

⟨1⟩1. $\overline{\prod_{i \in I} A_i} \subseteq \prod_{i \in I} \overline{A_i}$

⟨2⟩1. For all $i \in I$ we have $A_i \subseteq \overline{A_i}$

PROOF: Lemma 2.93.

⟨2⟩2. $\prod_{i \in I} A_i \subseteq \prod_{i \in I} \overline{A_i}$

⟨2⟩3. Q.E.D.

PROOF: Since $\prod_{i \in I} A_i$ is closed by Proposition 2.173.

⟨1⟩2. $\prod_{i \in I} \overline{A_i} \subseteq \overline{\prod_{i \in I} A_i}$

⟨2⟩1. LET: $x \in \prod_{i \in I} \overline{A_i}$

⟨2⟩2. LET: U be a neighbourhood of x

⟨2⟩3. PICK V_i open in X_i such that $x \in \prod_{i \in I} V_i \subseteq U$ with $V_i = X_i$ except for $i = i_1, \ldots, i_n$

⟨2⟩4. For $i \in I$, pick $a_i \in V_i \cap A_i$

PROOF: By Theorem 2.96 and ⟨2⟩1 using the Axiom of Choice.

⟨2⟩5. U intersects $\prod_{i \in I} A_i$

⟨2⟩6. Q.E.D.

PROOF: $a \in U \cap \prod_{i \in I} A_i$

□

Example 2.179. The closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ is $\mathbb{R}^\omega$

PROOF:

⟨1⟩1. LET: $a \in \mathbb{R}^\omega$

⟨1⟩2. LET: U be any neighbourhoods of a.

⟨1⟩3. PICK U_n open in $\mathbb{R}$ for all n such that $a \in \prod_{n \geq 0} U_n \subseteq U$ and $U_n = \mathbb{R}$ for all n except $n_1, \ldots, n_k$

⟨1⟩4. LET: $b_n = a_n$ for $n = n_1, \ldots, n_k$ and $b_n = 0$ for all other n

⟨1⟩5. $b \in \mathbb{R}^\infty \cap U$

⟨1⟩6. Q.E.D.

PROOF: From Theorem 2.96.

□

Proposition 2.180. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let (a_n) be a sequence in $\prod_{i \in I} X_i$ and $l \in \prod_{i \in I} X_i$. Then $a_n \to l$ as $n \to \infty$ if and only if, for all $i \in I$, we have $\pi_i(a_n) \to \pi_i(l)$ as $n \to \infty$.*

PROOF:

⟨1⟩1. If $a_n \to l$ as $n \to \infty$ then, for all $i \in I$, we have $\pi_i(a_n) \to \pi_i(l)$ as $n \to \infty$

PROOF: Proposition 2.154.

⟨1⟩2. If, for all $i \in I$, we have $\pi_i(a_n) \to \pi_i(l)$ as $n \to \infty$, then $a_n \to l$ as $n \to \infty$

⟨2⟩1. ASSUME: For all $i \in I$, we have $\pi_i(a_n) \to \pi_i(l)$ as $n \to \infty$

⟨2⟩2. LET: V be a neighbourhood of l

⟨2⟩3. PICK open sets U_i in X_i such that $l \in \prod_{i \in I} U_i \subseteq V$ and $U_i = X_i$ for all i except $i = i_1, \ldots, i_k$

⟨2⟩4. For $j = 1, \ldots, k$, PICK N_j such that, for all $n \geq N_j$, we have $\pi_{i_j}(a_n) \in U_{i_j}$

⟨2⟩5. LET: $N = \max(N_1, \ldots, N_k)$

⟨2⟩6. For all $n \geq N$ we have $a_n \in V$

∎

Theorem 2.181. *Let A be a topological space and $\{X_i\}_{i \in I}$ be a family of topological spaces. Let $f : A \to \prod_{i \in I} X_i$ be a function. If $\pi_i \circ f$ is continuous for all $i \in I$ then f is continuous.*

Proof:
$\langle 1 \rangle 1.$ Let: $i \in I$ and U be open in X_i
$\langle 1 \rangle 2.$ $f^{-1}({\pi_i}^{-1}(U))$ is open in A
$\langle 1 \rangle 3.$ Q.E.D.
Proof: Proposition 2.139.
∎

2.37.1 Continuous in Each Variable Separately

Definition 2.182 (Continuous in Each Variable Separately)**.** Let $F : X \times Y \to Z$. Then F is *continuous in each variable separately* if and only if:

- for every $a \in X$ the function $\lambda y \in Y.F(a, y)$ is continuous;
- for every $b \in Y$ the function $\lambda x \in X.F(x, b)$ is continuous.

Proposition 2.183. *Let $F : X \times Y \to Z$. If F is continuous then F is continuous in each variable separately.*

Proof: For $a \in X$, the function $\lambda y \in Y.F(a, y)$ is $F \circ i$ where $i : Y \to X \times Y$ maps y to (a, y). We have i is continuous by Proposition 2.160, hence $F \circ i$ is continuous by Theorem 2.145.
Similarly for $\lambda x \in X.F(x, b)$ for $b \in Y$. ∎

Example 2.184. Define $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$F(x, y) = \begin{cases} xy/(x^2 + y^2) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Then F is continuous in each variable separately but not continuous.

Proposition 2.185. *Let $f : A \to C$ and $g : B \to D$ be open maps. Then $f \times g : A \times B \to C \times D$ is an open map.*

Proof: Given U open in A and V open in B. Then $(f \times g)(U \times V) = f(U) \times g(V)$ is open in $C \times D$. The result follows from Proposition 2.136. ∎

Definition 2.186 (Sorgenfrey Plane)**.** The *Sorgenfrey plane* is $\mathbb{R}_l^2$.

2.38 The Subspace Topology

Definition 2.187 (Subspace Topology)**.** Let X be a topological space and $Y \subseteq X$. The *subspace topology* on Y is $\mathcal{T} = \{U \cap Y \mid U \text{ is open in } X\}$.

We prove this is a topology.

PROOF:
⟨1⟩1. $Y \in \mathcal{T}$
  PROOF: Since $Y = X \cap Y$
⟨1⟩2. For all $\mathcal{U} \subseteq \mathcal{T}$, we have $\bigcup \mathcal{U} \in \mathcal{T}$
  ⟨2⟩1. LET: $\mathcal{U} \subseteq \mathcal{T}$
  ⟨2⟩2. LET: $\mathcal{V} = \{V \text{ open in } X \mid V \cap Y \in \mathcal{U}\}$
  ⟨2⟩3. $\bigcup \mathcal{U} = (\bigcup \mathcal{V}) \cap Y$
⟨1⟩3. For all $U, V \in \mathcal{T}$, we have $U \cap V \in \mathcal{T}$
  ⟨2⟩1. LET: $U, V \in \mathcal{T}$
  ⟨2⟩2. PICK U', V' open in X such that $U = U' \cap Y$ and $V = V' \cap Y$
  ⟨2⟩3. $(U \cap V) = (U' \cap V') \cap Y$
□

Theorem 2.188. *Let X be a topological space and Y a subspace of X. Let $A \subseteq Y$. Then A is closed in Y if and only if there exists a closed set C in X such that $A = C \cap Y$.*

PROOF: We have

$$\begin{aligned} & A \text{ is closed in } Y \\ \Leftrightarrow & Y \setminus A \text{ is open in } Y \\ \Leftrightarrow & \exists U \text{ open in } X . Y \setminus A = Y \cap U \\ \Leftrightarrow & \exists C \text{ closed in } X . Y \setminus A = Y \cap (X \setminus U) \\ \Leftrightarrow & \exists C \text{ closed in } X . A = Y \cap U \end{aligned} \qquad \square$$

Theorem 2.189. *Let Y be a subspace of X. Let $A \subseteq Y$. Let $\overline{A}$ be the closure of A in X. Then the closure of A in Y is $\overline{A} \cap Y$.*

PROOF: The closure of A in Y is

$$\begin{aligned} & \bigcap \{C \text{ closed in } Y \mid A \subseteq C\} \\ = & \bigcap \{D \cap Y \mid D \text{ closed in } X, A \subseteq D \cap Y\} && \text{(Theorem 2.188)} \\ = & \bigcap \{D \mid D \text{ closed in } X, A \subseteq D\} \cap Y \\ = & \overline{A} \cap Y \end{aligned} \qquad \square$$

Lemma 2.190. *Let X be a topological space and $Y \subseteq X$. Let $\mathcal{B}$ be a basis for the topology on X. Then $\mathcal{B}' = \{B \cap Y \mid B \in \mathcal{B}\}$ is a basis for the subspace topology on Y.*

PROOF:
⟨1⟩1. Every element in $\mathcal{B}'$ is open in Y
⟨1⟩2. For every open set U in Y and point $y \in U$, there exists $B' \in \mathcal{B}'$ such that $y \in B' \subseteq U$
  ⟨2⟩1. LET: U be open in Y and $y \in U$
  ⟨2⟩2. PICK V open in X such that $U = V \cap Y$
  ⟨2⟩3. PICK $B \in \mathcal{B}$ such that $y \in B \subseteq V$

⟨2⟩4. LET: $B' = B \cap Y$
⟨2⟩5. $B' \in \mathcal{B}'$
⟨2⟩6. $y \in B' \subseteq U$
⟨1⟩3. Q.E.D.
PROOF: By Lemma 2.112.
□

Lemma 2.191. *Let X be a topological space and $Y \subseteq X$. Let $\mathcal{S}$ be a basis for the topology on X. Then $\mathcal{S}' = \{S \cap Y \mid S \in \mathcal{S}\}$ is a subbasis for the subspace topology on Y.*

PROOF: The set $\{B \cap Y \mid B$ is a finite intersection of elements of $\mathcal{S}\}$ is a basis for the subspace topology by Lemma 2.190, and this is the set of all finite intersections of elements of $\mathcal{S}'$. □

Lemma 2.192. *Let Y be a subspace of X. If U is open in Y and Y is open in X then U is open in X.*

PROOF:
⟨1⟩1. PICK V open in X such that $U = V \cap Y$
⟨1⟩2. U is open in X
PROOF: Since it is the intersection of two open sets V and Y.
□

Theorem 2.193. *Let Y be a subspace of X and $A \subseteq Y$. If A is closed in Y and Y is closed in X then A is closed in X.*

PROOF: Pick a closed set C in X such that $A = C \cap Y$ (Theorem 2.188). Then A is the intersection of two sets closed in X, hence A is closed in X (Lemma 2.81). □

Theorem 2.194. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let A_i be a subspace of X_i for all $i \in I$. Then the product topology on $\prod_{i \in I} A_i$ is the same as the topology it inherits as a subspace of $\prod_{i \in I} X_i$.*

PROOF: The product topology is generated by the subbasis

$$\begin{aligned} & \{{\pi_i}^{-1}(U) \mid i \in I, U \text{ open in } A_i\} \\ =& \{{\pi_i}^{-1}(V) \cap A_i \mid i \in I, V \text{ open in } X_i\} \\ =& \{{\pi_i}^{-1}(V) \mid i \in I, V \text{ open in } X_i\} \cap \prod_{i \in I} A_i \end{aligned}$$

and this is a subbasis for the subspace topology by Lemma 2.191. □

Theorem 2.195. *Let X be an ordered set in the order topology. Let $Y \subseteq X$ be an interval. Then the order topology on Y is the same as the subspace topology on Y.*

PROOF:
⟨1⟩1. The order topology is finer than the subspace topology.

⟨2⟩1. For every open ray R in X, the set $R \cap Y$ is open in the order topology.
⟨3⟩1. For all $a \in X$, we have $(-\infty, a) \cap Y$ is open in the order topology.
⟨4⟩1. CASE: For all $y \in Y$ we have $y < a$
PROOF: In this case $(-\infty, a) \cap Y = Y$.
⟨4⟩2. CASE: For all $y \in Y$ we have $a < y$
PROOF: In this case $(-\infty, a) \cap Y = \emptyset$.
⟨4⟩3. CASE: There exists $y \in Y$ such that $y \leq a$ and $y \in Y$ such that $a \leq y$
⟨5⟩1. $a \in Y$
PROOF: Because Y is an interval.
⟨5⟩2. $(-\infty, a) \cap Y = \{y \in Y \mid y < a\}$
⟨3⟩2. For all $a \in X$, we have $(a, +\infty) \cap Y$ is open in the order topology.
PROOF: Similar.
⟨2⟩2. Q.E.D.
PROOF: By Lemmas 2.162 and 2.191 and Proposition 2.120.
⟨1⟩2. The subspace topology is finer than the order topology.
⟨2⟩1. Every open ray in Y is open in the subspace topology.
PROOF: For any $a \in Y$ we have $(-\infty, a)_Y = (-\infty, a)_X \cap Y$ and $(a, +\infty)_Y = (a, +\infty)_X \cap Y$.
⟨2⟩2. Q.E.D.
PROOF: By Lemma 2.162 and Proposition 2.120
□

This example shows that we cannot remove the hypothesis that Y is an interval:

Example 2.196. The order topology on I_o^2 is different from the subspace topology as a subspace of $\mathbb{R}^2$ under the dictionary order topology.

PROOF: The set $\{1/2\} \times (1/2, 1)$ is open in the subspace topology but not in the order topology. □

Proposition 2.197. *Let X be a topological space, Y a subspace of X, and Z a subspace of Y. Then the subspace topology on Z inherited from X is the same as the subspace topology on Z inherited from Y.*

PROOF: The subspace topology inherited from Y is

$$\begin{aligned}
&\{V \cap Z \mid V \text{ open in } Y\} \\
=&\{U \cap Y \cap Z \mid U \text{ open in } X\} \\
=&\{U \cap Z \mid U \text{ open in } X\}
\end{aligned}$$

which is the subspace topology inherited from X. □

Definition 2.198 (Unit Circle)**.** The *unit circle* S^1 is $\{(x, y) \in \mathbb{R}^2 \mid x^2+y^2 = 1\}$ as a subspace of $\mathbb{R}^2$.

Definition 2.199 (Unit 2-sphere)**.** The *unit 2-sphere* is $S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$ as a subspace of $\mathbb{R}^3$.

Proposition 2.200. *Let $f : X \to Y$ be an open map and $A \subseteq X$ be open. Then the restriction $f \restriction A : A \to f(A)$ is an open map.*

PROOF:
$\langle 1 \rangle 1$. LET: U be open in A
$\langle 1 \rangle 2$. U is open in X
  PROOF: Lemma 2.192.
$\langle 1 \rangle 3$. $f(U)$ is open in Y
$\langle 1 \rangle 4$. $f(U)$ is open in $f(A)$
  PROOF: Since $f(U) = f(U) \cap f(A)$.
□

Example 2.201. This example shows that we cannot remove the hypothesis that A is open.

Let $A = \{(x, y) \in \mathbb{R}^2 \mid (x > 0 \text{ and } y = 1/x) \text{ or } x = y = 0\}$. Then $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ is an open map, but $\pi_1 \restriction A : A \to [0, +\infty)$ is not, because it maps the set $\{0, 0\}$ which is open in A to $\{0\}$ which is not open in $[0, +\infty)$.

Proposition 2.202. *Let Y be a subspace of X. Let $A \subseteq Y$ and $l \in Y$. Then l is a limit point of A in Y if and only if l is a limit point of A in X.*

PROOF: Both are equivalent to the condition that any neighbourhood of l in X intersects A in a point other than l. □

2.39 The Box Topology

Definition 2.203 (Box Topology)**.** Let $\{A_i\}_{i \in I}$ be a family of topological spaces. The *box topology* on $\prod_{i \in I} A_i$ is the topology generated by the set of all sets of the form $\prod_{i \in I} U_i$ where $\{U_i\}_{i \in I}$ is a family such that each U_i is open in A_i.

This is a basis since it covers $\prod_{i \in I} A_i$ and is closed under intersection.

Proposition 2.204. *The box topology is finer than the product topology.*

PROOF: From Proposition 2.172. □

Corollary 2.204.1. *If A_i is closed in X_i for all $i \in I$ then $\prod_{i \in I} A_i$ is closed in $\prod_{i \in I} X_i$ under the box topology.*

PROOF: From Proposition 2.173.

Proposition 2.205 (AC)**.** *Let $\{A_i\}_{i \in I}$ be a family of topological spaces. For $i \in I$, let $\mathcal{B}_i$ be a basis for the topology on A_i. Then $\mathcal{B} = \{\prod_{i \in I} B_i \mid \forall i \in I. B_i \in \mathcal{B}_i\}$ is a basis for the box topology on $\prod_{i \in I} A_i$.*

PROOF:
$\langle 1 \rangle 1$. Every set of the form $\prod_{i \in I} B_i$ is open.
$\langle 1 \rangle 2$. For every point $a \in \prod_{i \in I} A_i$ and every open set U with $a \in U$, there exists $B \in \mathcal{B}$ such that $a \in B \subseteq U$.

⟨2⟩1. Let: U be open and $a \in U$

⟨2⟩2. Pick a family $\{U_i\}_{i \in I}$ such that each U_i is open in A_i and $a \in \prod_{i \in I} U_i \subseteq U$.

⟨2⟩3. For $i \in I$, Pick $B_i \in \mathcal{B}_i$ such that $a_i \in B_i \subseteq U_i$

Proof: Using the Axiom of Choice.

⟨2⟩4. $a \in \prod_{i \in I} B_i \subseteq U$

⟨1⟩3. Q.E.D.

Proof: Lemma 2.112.

□

Theorem 2.206. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let A_i be a subspace of X_i for all $i \in I$. Give $\prod_{i \in I} X_i$ the box topology. Then the box topology on $\prod_{i \in I} A_i$ is the same as the topology it inherits as a subspace of $\prod_{i \in I} X_i$.*

Proof: The box topology is generated by the basis

$$\begin{aligned} &\{\prod_{i \in I} U_i \mid \forall i \in I, U_i \text{ open in } A_i\} \\ =&\{\prod_{i \in I}(V_i \cap A_i) \mid \forall i \in I, V_i \text{ open in } X_i\} \\ =&\{\prod_{i \in I} V_i \mid \forall i \in I, V_i \text{ open in } X_i\} \cap \prod_{i \in I} A_i \end{aligned}$$

and this is a basis for the subspace topology by Lemma 2.190. □

Proposition 2.207 (AC)**.** *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Give $\prod_{i \in I} X_i$ the box topology. Let $A_i \subseteq X_i$ for all $i \in I$. Then*

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i} \enspace .$$

Proof:

⟨1⟩1. $\overline{\prod_{i \in I} A_i} \subseteq \prod_{i \in I} \overline{A_i}$

⟨2⟩1. For all $i \in I$ we have $A_i \subseteq \overline{A_i}$

Proof: Lemma 2.93.

⟨2⟩2. $\prod_{i \in I} A_i \subseteq \prod_{i \in I} \overline{A_i}$

⟨2⟩3. Q.E.D.

Proof: Since $\prod_{i \in I} A_i$ is closed by Corollary 2.204.1.

⟨1⟩2. $\prod_{i \in I} \overline{A_i} \subseteq \overline{\prod_{i \in I} A_i}$

⟨2⟩1. Let: $x \in \prod_{i \in I} \overline{A_i}$

⟨2⟩2. Let: U be a neighbourhood of x

⟨2⟩3. Pick V_i open in X_i such that $x \in \prod_{i \in I} V_i \subseteq U$

⟨2⟩4. For $i \in I$, pick $a_i \in V_i \cap A_i$

Proof: By Theorem 2.96 and ⟨2⟩1 using the Axiom of Choice.

⟨2⟩5. U intersects $\prod_{i \in I} A_i$

⟨2⟩6. Q.E.D.

Proof: $a \in U \cap \prod_{i \in I} A_i$.

$\square$

The following example shows that Theorem 2.181 fails in the box topology.

Example 2.208. Define $f : \mathbb{R} \to \mathbb{R}^\omega$ by $f(t) = (t, t, \ldots)$. Then $\pi_n \circ f = \mathrm{id}_\mathbb{R}$ is continuous for all n. But f is not continuous when $\mathbb{R}^\omega$ is given the box topology because the inverse image of

$$(-1, 1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times \cdots$$

is $\{0\}$ which is not open.

The following example shows that Proposition 2.180 fails in the box topology.

Example 2.209. Give $\mathbb{R}^\omega$ the box topology. Let $a_n = (1/n, 1/n, \ldots)$ for $n \geq 1$ and $l = (0, 0, \ldots)$. Then $\pi_i(a_n) \to \pi_i(l)$ as $n \to \infty$ for all i, but $a_n \not\to l$ as $n \to \infty$ since the open set

$$(-1, 1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times \cdots$$

contains l but does not contain any a_n.

Example 2.210. The set $\mathbb{R}^\infty$ is closed in $\mathbb{R}^\omega$ under the box topology. For let (a_n) be any sequence not in $\mathbb{R}^\infty$. Let U_n be an open interval around a_n that does not contain 0 if $a_n \neq 0$, and $U_n = \mathbb{R}$ if $a_n = 0$. Then $\prod_{n \geq 0} U_n$ is a neighbourhood of (a_n) that does not intersect $\mathbb{R}^\infty$.

2.40 T_1 Spaces

Definition 2.211 (T_1 Space)**.** A topological space is T_1 if and only if every singleton is closed.

Lemma 2.212. *A space is T_1 if and only if every finite set is closed.*

PROOF: From Lemma 2.82. $\square$

Theorem 2.213. *In a T_1 space, a point a is a limit point of a set A if and only if every neighbourhood of a contains infinitely many points of A.*

PROOF:
$\langle 1 \rangle$1. If a is a limit point of A then every neighbourhood of a contains infinitely many points of A.
  $\langle 2 \rangle$1. ASSUME: a is a limit point of A.
  $\langle 2 \rangle$2. LET: U be a neighbourhood of a.
  $\langle 2 \rangle$3. ASSUME: for a contradiction U contains only finitely many points of A.
  $\langle 2 \rangle$4. $(U \cap A) \setminus \{a\}$ is closed.
    PROOF: By the T_1 axiom.
  $\langle 2 \rangle$5. $(U \setminus A) \cup \{a\}$ is open.
    PROOF: It is $U \setminus ((U \cap A) \setminus \{a\})$.
  $\langle 2 \rangle$6. $(U \setminus A) \cup \{a\}$ intersects A in a point other than a.

PROOF: From $\langle 2 \rangle 1$.

$\langle 2 \rangle 7$. Q.E.D.

□

$\langle 1 \rangle 2$. If every neighbourhood of a contains infinitely many points of A then a is a limit point of A.

PROOF: Immediate from definitions.

□

(To see this does not hold in every space, see Proposition 2.108.)

Proposition 2.214. *A space is T_1 if and only if, for any two distinct points x and y, there exist neighbourhoods U of x and V of y such that $x \notin V$ and $y \notin U$.*

PROOF:

$\langle 1 \rangle 1$. LET: X be a topological space.

$\langle 1 \rangle 2$. If X is T_1 then, for any two distinct points x and y, there exist neighbourhoods U of x and V of y such that $x \notin V$ and $y \notin U$.

PROOF: This holds because $\{x\}$ and $\{y\}$ are closed.

$\langle 1 \rangle 3$. Suppose, for any two distinct points x and y, there exist neighbourhoods U of x and V of y such that $x \notin V$ and $y \notin U$. Then X is T_1.

$\langle 2 \rangle 1$. ASSUME: For any two distinct points x and y, there exist neighbourhoods U of x and V of y such that $x \notin V$ and $y \notin U$.

$\langle 2 \rangle 2$. LET: $a \in X$

$\langle 2 \rangle 3$. $\{a\}$ is closed.

PROOF: For all $b \neq a$ there exists a neighbourhood U of b such that $U \subseteq X \setminus \{a\}$.

□

Proposition 2.215. *A subspace of a T_1 space is T_1.*

PROOF: From Proposition 2.193.

2.41 Hausdorff Spaces

Definition 2.216 (Hausdorff Space)**.** A topological space is *Hausdorff* if and only if, for any points x, y with $x \neq y$, there exist disjoint open sets U and V such that $x \in U$ and $y \in V$.

Theorem 2.217. *Every Hausdorff space is T_1.*

PROOF:

$\langle 1 \rangle 1$. LET: X be a Hausdorff space.

$\langle 1 \rangle 2$. LET: $b \in X$

PROVE: $\overline{\{b\}} = \{b\}$

$\langle 1 \rangle 3$. ASSUME: $a \in \overline{\{b\}}$ and $a \neq b$

$\langle 1 \rangle 4$. PICK disjoint neighbourhoods U of a and V of b.

$\langle 1 \rangle 5$. U intersects $\{b\}$

PROOF: Theorem 2.96.

$\langle 1\rangle 6.$ $b \in U$
$\langle 1\rangle 7.$ Q.E.D.
PROOF: This contradicts the fact that U and V are disjoint ($\langle 1\rangle 4$).
□

Proposition 2.218. *An infinite set under the finite complement topology is T_1 but not Hausdorff.*

PROOF:
$\langle 1\rangle 1.$ LET: X be an infinite set under the finite complement topology.
$\langle 1\rangle 2.$ Every singleton is closed.
PROOF: By definition.
$\langle 1\rangle 3.$ PICK$a, b \in X$ with $a \neq b$
$\langle 1\rangle 4.$ There are no disjoint neighbourhoods U of a and V of b.
$\langle 2\rangle 1.$ LET: U be a neighbourhood of a and V a neighbourhood of b.
$\langle 2\rangle 2.$ $X \setminus U$ and $X \setminus V$ are finite.
$\langle 2\rangle 3.$ PICK $c \in X$ that is not in $X \setminus U$ or $X \setminus V$.
$\langle 2\rangle 4.$ $c \in U \cap V$
□

Proposition 2.219. *The product of a family of Hausdorff spaces is Hausdorff.*

PROOF:
$\langle 1\rangle 1.$ LET: $\{X_i\}_{i \in I}$ be a family of Hausdorff spaces.
$\langle 1\rangle 2.$ LET: $a, b \in \prod_{i \in I} X_i$ with $a \neq b$
$\langle 1\rangle 3.$ PICK $i \in I$ such that $a_i \neq b_i$
$\langle 1\rangle 4.$ PICK U, V disjoint open sets in X_i with $a_i \in U$ and $b_i \in V$
$\langle 1\rangle 5.$ ${\pi_i}^{-1}(U)$ and ${\pi_i}^{-1}(V)$ are disjoint open sets in $\prod_{i \in I} X_i$ with $a \in {\pi_i}^{-1}(U)$ and $b \in {\pi_i}^{-1}(V)$
□

Theorem 2.220. *Every linearly ordered set under the order topology is Hausdorff.*

PROOF:
$\langle 1\rangle 1.$ LET: X be a linearly ordered set under the order topology.
$\langle 1\rangle 2.$ LET: $a, b \in X$ with $a \neq b$
$\langle 1\rangle 3.$ ASSUME: w.l.o.g. $a < b$
$\langle 1\rangle 4.$ CASE: There exists c such that $a < c < b$
PROOF: The sets $(-\infty, c)$ and $(c, +\infty)$ are disjoint neighbourhoods of a and b respectively.
$\langle 1\rangle 5.$ CASE: There is no c such that $a < c < b$
PROOF: The sets $(-\infty, b)$ and $(a, +\infty)$ are disjoint neighbourhoods of a and b respectively.
□

Theorem 2.221. *A subspace of a Hausdorff space is Hausdorff.*

PROOF:

⟨1⟩1. LET: X be a Hausdorff space and Y a subspace of X.
⟨1⟩2. LET: $x, y \in Y$ with $x \neq y$
⟨1⟩3. PICK disjoint neighbourhoods U of x and V of y in X.
⟨1⟩4. $U \cap Y$ and $V \cap Y$ are disjoint neighbourhoods of x and y respectively in Y.
□

Proposition 2.222. *A space X is Hausdorff if and only if the diagonal $\Delta = \{(x, x) \mid x \in X\}$ is closed in X^2.*

PROOF:

$$\begin{aligned} & X \text{ is Hausdorff} \\ \Leftrightarrow & \forall x, y \in X. x \neq y \Rightarrow \exists V, W \text{ open}. x \in V \wedge y \in W \wedge V \cap W = \emptyset \\ \Leftrightarrow & \forall (x, y) \in X^2 \setminus \Delta . \exists V, W \text{ open}. (x, y) \subseteq V \times W \subseteq X^2 \setminus \Delta \\ \Leftrightarrow & \Delta \text{ is closed} \end{aligned}$$

□

Theorem 2.223. *In a Hausdorff space, a sequence has at most one limit.*

PROOF:
⟨1⟩1. LET: X be a Hausdorff space.
⟨1⟩2. ASSUME: for a contradiction $a_n \to l$ as $n \to \infty$, $a_n \to m$ as $n \to \infty$, and $l \neq m$
⟨1⟩3. PICK disjoint neighbourhoods U of l and V of m
PROOF: By the Hausdorff axiom.
⟨1⟩4. PICK M and N such that $a_n \in U$ for $n \geq M$ and $a_n \in V$ for $n \geq N$
⟨1⟩5. $a_{\max(M,N)} \in U \cap V$
⟨1⟩6. Q.E.D.
PROOF: This contradicts the fact that U and V are disjoint (⟨1⟩3).
□

To see this is not always true in spaces that are T_1 but not Hausdorff:

Proposition 2.224. *Let X be an infinite set under the finite complement topology. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence with all points distinct. Then for every $l \in X$ we have $a_n \to l$ as $n \to \infty$.*

PROOF: Let U be any neighbourhood of l. Since $X \setminus U$ is finite, there must exist N such that, for all $n \geq N$, we have $a_n \in U$. □

Proposition 2.225. *Let X be a topological space. Let Y a Hausdorff space. Let $A \subseteq X$. Let $f, g : \overline{A} \to Y$ be continuous. If f and g agree on A then $f = g$.*

PROOF:
⟨1⟩1. LET: $x \in \overline{A}$
⟨1⟩2. ASSUME: $f(x) \neq g(x)$
⟨1⟩3. PICK disjoint neighbourhoods V of $f(x)$ and W of $g(x)$.
⟨1⟩4. PICK $y \in f^{-1}(V) \cap g^{-1}(W) \cap A$
PROOF: Since $f^{-1}(V) \cap g^{-1}(W)$ is a neighbourhood of x and hence intersects A.

⟨1⟩5. $f(y) = g(y) \in V \cap W$
⟨1⟩6. Q.E.D.
PROOF: This contradicts the fact that V and W are disjoint (⟨1⟩3).
□

Proposition 2.226. *Let $\{X_i\}_{i \in I}$ be a family of Hausdorff spaces. Then $\prod_{i \in I} X_i$ under the box topology is Hausdorff.*

PROOF:
⟨1⟩1. LET: $\{X_i\}_{i \in I}$ be a family of Hausdorff spaces.
⟨1⟩2. LET: $a, b \in \prod_{i \in I} X_i$ with $a \neq b$
⟨1⟩3. PICK $i \in I$ such that $a_i \neq b_i$
⟨1⟩4. PICK U, V disjoint open sets in X_i with $a_i \in U$ and $b_i \in V$
⟨1⟩5. ${\pi_i}^{-1}(U)$ and ${\pi_i}^{-1}(V)$ are disjoint open sets in $\prod_{i \in I} X_i$ with $a \in {\pi_i}^{-1}(U)$ and $b \in {\pi_i}^{-1}(V)$
□

Proposition 2.227. *Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on the same set X with $\mathcal{T} \subseteq \mathcal{T}'$ If $\mathcal{T}$ is Haudorff then $\mathcal{T}'$ is Hausdorff.*

PROOF: Immediate from definitions.

Proposition 2.228. *Let X be a Hausdorff space. Let $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the finite intersection property. Then $\bigcap_{D \in \mathcal{D}} \overline{D}$ contains at most one point.*

PROOF:
⟨1⟩1. LET: $x, y \in \bigcap_{D \in \mathcal{D}} \overline{D}$
⟨1⟩2. ASSUME: for a contradiction $x \neq y$
⟨1⟩3. PICK disjoint open subsets U and V of x and y respectively.
⟨1⟩4. $U, V \in \mathcal{D}$
PROOF: Proposition 2.99.
⟨1⟩5. Q.E.D.
PROOF: This contradicts the fact that $\mathcal{D}$ satisfies the finite intersection property.
□

2.42 The First Countability Axiom

Definition 2.229 (First Countability Axiom)**.** A topological space X satisfies the *first countability axiom*, or is *first countable*, if and only if every point has a countable local basis.

Lemma 2.230 (Sequence Lemma (CC))**.** *Let X be a first countable space. Let $A \subseteq X$ and $l \in \overline{A}$. Then there exists a sequence in A that converges to l.*

PROOF:
⟨1⟩1. PICK a countable local basis $\{B_n \mid n \in \mathbb{Z}^+\}$ at l such that $B_1 \supseteq B_2 \supseteq \cdots$.

PROOF: Lemma 2.122.
⟨1⟩2. For all $n \geq 1$, PICK $a_n \in A \cap B_n$.
PROVE: $a_n \to l$ as $n \to \infty$
⟨1⟩3. LET: U be a neighbourhood of A
⟨1⟩4. PICK N such that $B_N \subseteq U$
⟨1⟩5. For $n \geq N$ we have $a_n \in U$
PROOF: $a_n \in B_n \subseteq B_N \subseteq U$
□

Theorem 2.231 (CC)**.** *Let X be a first countable space and Y a topological space. Let $f : X \to Y$. Suppose that, for every sequence (x_n) in X and $l \in X$, if $x_n \to l$ as $n \to \infty$, then $f(x_n) \to f(l)$ as $n \to \infty$. Then f is continuous.*

PROOF:
⟨1⟩1. LET: $A \subseteq X$
⟨1⟩2. LET: $a \in A$
PROVE: $f(a) \in \overline{f(A)}$
⟨1⟩3. PICK a sequence (x_n) in A that converges to a.
PROOF: By the Sequence Lemma.
⟨1⟩4. $f(x_n) \to f(a)$
⟨1⟩5. $f(a) \in \overline{f(A)}$
PROOF: By Lemma 2.124.
⟨1⟩6. Q.E.D.
PROOF: By Theorem 2.142.
□

Example 2.232 (CC)**.** The space $\mathbb{R}^\omega$ under the box product is not first countable.

PROOF: Let $\{B_n \mid n \geq 0\}$ be a countable set of neighbourhoods of $\vec{0}$. We will construct a neighbourhood of $\vec{0}$ that does not include any of these.
For $n \geq 0$, pick a neighbourhood U_n of 0 such that $U_n \subset \pi_n(B_p)$. Then $\prod_{n=0}^{\infty} U_n$ is a neighbourhood of $\vec{0}$ that does not include any B_n. □

Example 2.233. If J is an uncountable set then $\mathbb{R}^J$ is not first countable.

PROOF:
⟨1⟩1. LET: $\{B_n \mid n \geq 0\}$ be any countable set of neighbourhoods of $\vec{0}$.
⟨1⟩2. For $n \geq 0$, PICK a basis element $\prod_{\alpha \in J} U_{n\alpha}$ that contains $\vec{0}$ and is included in B_n.
PROOF: Using the Axiom of Countable Choice.
⟨1⟩3. For $n \geq 0$,
LET: $J_n = \{\alpha \in J \mid U_{n\alpha} \neq \mathbb{R}\}$
⟨1⟩4. PICK $\beta \in J$ such that $\beta \notin J_n$ for any n.
PROOF: Since each J_n is finite so $\bigcup_n J_n$ is countable.
⟨1⟩5. $\pi_\beta((-1, 1))$ is a neighbourhood of $\vec{0}$ that does not include any B_n.
□

Example 2.234. The space $\mathbb{R}_l$ is first countable.

PROOF: For any $a \in \mathbb{R}$, the set $\{[a, a + 1/n) \mid n \geq 1\}$ is a countable local basis.

Example 2.235. The ordered square is first countable.

PROOF: For any $(a, b) \in I_o^2$ with $b \neq 0, 1$, the set $\{(\{a\} \times (b-1/n, b+1/n)) \cap I_o^2 \mid n \geq 1\}$ is a countable local basis.

2.43 Strong Continuity

Definition 2.236 (Strongly Continuous)**.** Let X and Y be topological spaces and $f : X \to Y$ be a function. Then f is *strongly continuous* if and only if, for every subset $U \subseteq Y$, we have U is open in Y if and only if $f^{-1}(U)$ is open in X.

Proposition 2.237. *Let X and Y be topological spaces and $f : X \to Y$ be a function. Then f is* strongly continuous *if and only if, for every subset $C \subseteq Y$, we have C is closed in Y if and only if $f^{-1}(C)$ is closed in X.*

PROOF: Since $X \setminus f^{-1}(C) = f^{-1}(Y \setminus C)$. ◻

Proposition 2.238. *Let X, Y and Z be topological spaces. Let $f : X \to Y$ and $g : Y \to Z$. If f and g are strongly continuous then so is $g \circ f$.*

PROOF: Since $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$. ◻

Proposition 2.239. *Let X, Y and Z be topological spaces. Let $f : X \to Y$ and $g : Y \to Z$. If $g \circ f$ is continuous and f is strongly continuous then g is continuous.*

PROOF:
$\langle 1 \rangle$1. LET: $V \subseteq Z$ be open.
$\langle 1 \rangle$2. $f^{-1}(g^{-1}(V))$ is open in X.
  PROOF: Since $g \circ f$ is continuous.
$\langle 1 \rangle$3. $f^{-1}(V)$ is open in Y.
  PROOF: Since g is strongly continuous.
◻

Proposition 2.240. *Let X, Y and Z be topological spaces. Let $f : X \to Y$ and $g : Y \to Z$. If $g \circ f$ is strongly continuous and f is strongly continuous then g is strongly continuous.*

PROOF: For $V \subseteq Z$, we have V is open iff $f^{-1}(g^{-1}(V))$ is open iff $g^{-1}(V)$ is open.

2.44 Saturated Sets

Definition 2.241. Let X and Y be sets and $p : X \twoheadrightarrow Y$ a surjective function. Let $C \subseteq X$. Then C is *saturated* with respect to p if and only if, for all $x, y \in X$, if $x \in C$ and $p(x) = p(y)$ then $y \in C$.

Proposition 2.242. *Let X and Y be sets and $p : X \twoheadrightarrow Y$ a surjective function. Let $C \subseteq X$. Then the following are equivalent:*

1. *C is saturated with respect to p.*
2. *There exists $D \subseteq Y$ such that $C = p^{-1}(D)$*
3. *$C = p^{-1}(p(C))$.*

Proof:
$\langle 1 \rangle 1.$ $1 \Rightarrow 3$
  $\langle 2 \rangle 1.$ Assume: C is saturated with respect to p.
  $\langle 2 \rangle 2.$ $C \subseteq p^{-1}(p(C))$
    Proof: Trivial.
  $\langle 2 \rangle 3.$ $p^{-1}(p(C)) \subseteq C$
    $\langle 3 \rangle 1.$ Let: $x \in p^{-1}(p(C))$
    $\langle 3 \rangle 2.$ $p(x) \in p(C)$
    $\langle 3 \rangle 3.$ There exists $y \in C$ such that $p(x) = p(y)$
    $\langle 3 \rangle 4.$ $x \in C$
      Proof: From $\langle 2 \rangle 1$.
$\langle 1 \rangle 2.$ $3 \Rightarrow 2$
  Proof: Trivial.
$\langle 1 \rangle 3.$ $2 \Rightarrow 1$
  Proof: This follows because if $p(x) \in D$ and $p(x) = p(y)$ then $p(y) \in D$.
□

2.45 Quotient Maps

Definition 2.243 (Quotient Map)**.** Let X and Y be topological spaces and $p : X \to Y$. Then p is a *quotient map* if and only if p is surjective and strongly continuous.

Proposition 2.244. *Let X and Y be topological spaces and $p : X \twoheadrightarrow Y$ be a surjective function. Then the following are equivalent.*

1. *p is a quotient map.*
2. *p is continuous and maps saturated open sets to open sets.*
3. *p is continuous and maps saturated closed sets to closed sets.*

Proof:
$\langle 1 \rangle 1.$ $1 \Rightarrow 2$
  $\langle 2 \rangle 1.$ Assume: p is a quotient map.
  $\langle 2 \rangle 2.$ Let: U be a saturated open set in X.
  $\langle 2 \rangle 3.$ $p^{-1}(p(U))$ is open in X.
    Proof: Since $U = p^{-1}(p(U))$ be Proposition 2.242.
  $\langle 2 \rangle 4.$ $p(U)$ is open in Y.

PROOF: From $\langle 2\rangle 1$.

$\langle 1\rangle 2$. $1 \Rightarrow 3$

PROOF: Similar.

$\langle 1\rangle 3$. $2 \Rightarrow 1$

$\langle 2\rangle 1$. ASSUME: p is continuous and maps saturated open sets to open sets.

$\langle 2\rangle 2$. LET: $U \subseteq Y$

$\langle 2\rangle 3$. ASSUME: $p^{-1}(U)$ is open in X

$\langle 2\rangle 4$. $p^{-1}(U)$ is saturated.

PROOF: Proposition 2.242.

$\langle 2\rangle 5$. U is open in Y.

$\langle 1\rangle 4$. $3 \Rightarrow 1$

PROOF: Similar.

$\square$

Corollary 2.244.1. *Every surjective continuous open map is a quotient map.*

Corollary 2.244.2. *Every surjective continuous closed map is a quotient map.*

Example 2.245. The converses of these corollaries do not hold.

Let $A = \{(x, y) \mid x \geq 0\} \cup \{(x, y) \mid y = 0\}$. Then $\pi_1 : A \to \mathbb{R}$ is a quotient map, but not an open map or a closed map.

We prove that π_1 maps saturated open sets to open sets:

$\langle 1\rangle 1$. LET: ${\pi_1}^{-1}(U)$ be a saturated open set in A

PROVE: U is open in $\mathbb{R}$

$\langle 1\rangle 2$. LET: $x \in U$

$\langle 1\rangle 3$. $(x, 0) \in \pi_1(U)^{-1}$

$\langle 1\rangle 4$. PICK W, V open in $\mathbb{R}$ such that $(x, 0) \subseteq W \times V \subseteq \pi_1(U)^{-1}$

$\langle 1\rangle 5$. $x \in W \subseteq U$

It is not an open map because it maps $((-1, 1) \times (1, 2)) \cap A$ to $[0, 1)$.

It is not a closed map because it maps $\{(x, 1/x) \mid x > 0\}$ to $(0, +\infty)$.

Proposition 2.246. *Let $p : X \twoheadrightarrow Y$ be a quotient map. Let $A \subseteq X$ be saturated with respect to p. Let $q : A \twoheadrightarrow p(A)$ be the restriction of p.*

1. *If A is either open or closed in X then q is a quotient map.*
2. *If p is either an open map or a closed map then q is a quotient map.*

PROOF:

$\langle 1\rangle 1$. LET: $p : X \twoheadrightarrow Y$ be a quotient map.

$\langle 1\rangle 2$. LET: $A \subseteq X$ be saturated with respect to p.

$\langle 1\rangle 3$. LET: $q : A \twoheadrightarrow p(A)$ be the restriction of p.

$\langle 1\rangle 4$. q is continuous.

PROOF: Theorem 2.146.

$\langle 1\rangle 5$. If A is open in X then q is a quotient map.

$\langle 2\rangle 1$. ASSUME: A is open in X.

$\langle 2\rangle 2$. q maps saturated open sets to open sets.

⟨3⟩1. LET: $U \subseteq A$ be saturated with respect to q and open in A
⟨3⟩2. U is saturated with respect to p
⟨4⟩1. LET: $x, y \in X$
⟨4⟩2. ASSUME: $x \in U$
⟨4⟩3. ASSUME: $p(x) = p(y)$
⟨4⟩4. $x \in A$
PROOF: From ⟨3⟩1 and ⟨4⟩2.
⟨4⟩5. $y \in A$
PROOF: From ⟨1⟩2 and ⟨4⟩3
⟨4⟩6. $q(x) = x(y)$
PROOF: From ⟨1⟩3, ⟨4⟩3, ⟨4⟩4, ⟨4⟩5.
⟨4⟩7. $y \in U$
PROOF: From ⟨3⟩1, ⟨4⟩2, ⟨4⟩6
⟨3⟩3. U is open in X
PROOF: Lemma 2.192, ⟨2⟩1, ⟨3⟩1.
⟨3⟩4. $p(U)$ is open in Y
PROOF: Proposition 2.244, ⟨1⟩1, ⟨3⟩2, ⟨3⟩3
⟨3⟩5. $q(U)$ is open in $p(A)$
PROOF: Since $q(U) = p(U) = p(U) \cap p(A)$.
⟨2⟩3. Q.E.D.
PROOF: By Proposition 2.244.
⟨1⟩6. If A is closed in X then q is a quotient map.
PROOF: Similar.
⟨1⟩7. If p is an open map then q is a quotient map.
⟨2⟩1. ASSUME: p is an open map
⟨2⟩2. q maps saturated open sets to open sets.
⟨3⟩1. LET: U be open in A and saturated with respect to q
⟨3⟩2. PICK V open in X such that $U = A \cap V$
⟨3⟩3. $p(V)$ is open in Y
⟨3⟩4. $q(U) = p(V) \cap p(A)$
⟨4⟩1. $q(U) \subseteq p(V) \cap p(A)$
PROOF: From ⟨3⟩2.
⟨4⟩2. $p(V) \cap p(A) \subseteq q(U)$
⟨5⟩1. LET: $y \in p(V) \cap p(A)$
⟨5⟩2. PICK $x \in V$ and $x' \in A$ such that $p(x) = p(x') = y$
⟨5⟩3. $x \in A$
PROOF: By ⟨1⟩2.
⟨5⟩4. $x \in U$
PROOF: From ⟨3⟩2
⟨2⟩3. Q.E.D.
PROOF: By Proposition 2.244.
⟨1⟩8. If p is a closed map then q is a quotient map.
PROOF: Similar.
□

Example 2.247. This example shows we cannot remove the hypotheses on A

and p.

Define $f : [0,1] \to [2,3] \to [0,2]$ by $f(x) = x$ if $x \leq 1$, $f(x) = x - 1$ if $x \geq 2$. Then f is a quotient map but its restriction f' to $[0,1) \cup [2,3]$ is not, because ${f'}^{-1}([1,2])$ is open but $[1,2]$ is not.

For a counterexample where A is saturated, see Example 2.253.

Proposition 2.248. *Let $p : A \twoheadrightarrow C$ and $q : B \twoheadrightarrow D$ be open quotient maps. Then $p \times q : A \times B \to C \times D$ is an open quotient map.*

PROOF: From Corollary 2.244.1, Proposition 2.185 and Theorem 2.181. ∎

Theorem 2.249. *Let $p : X \twoheadrightarrow Y$ be a quotient map. Let Z be a topological space and $f : Y \to Z$ be a function. Then*

1. *$f \circ p$ is continuous if and only if f is continuous.*
2. *$f \circ p$ is a quotient map if and only if f is a quotient map.*

PROOF:
$\langle 1 \rangle 1$. If $f \circ p$ is continuous then f is continuous.
PROOF: Proposition 2.239.
$\langle 1 \rangle 2$. If f is continuous then $f \circ p$ is continuous.
PROOF: Theorem 2.145.
$\langle 1 \rangle 3$. If $f \circ p$ is a quotient map then f is a quotient map.
PROOF: Proposition 2.240.
$\langle 1 \rangle 4$. If f is a quotient map then $f \circ p$ is a quotient map.
PROOF: From Proposition 2.238.
∎

Proposition 2.250. *Let X and Y be topological spaces. Let $p : X \to Y$ and $f : Y \to X$ be continuous maps such that $p \circ f = \mathrm{id}_Y$. Then p is a quotient map.*

PROOF:
$\langle 1 \rangle 1$. LET: $V \subseteq Y$
$\langle 1 \rangle 2$. ASSUME: $p^{-1}(V)$ is open in X.
$\langle 1 \rangle 3$. $f^{-1}(p^{-1}(V))$ is open in Y.
PROOF: Because f is continuous.
$\langle 1 \rangle 4$. V is open in Y.
PROOF: Because $f^{-1}(p^{-1}(V)) = V$.
∎

2.46 Quotient Topology

Definition 2.251 (Quotient Topology)**.** Let X be a topological space, Y a set and $p : X \twoheadrightarrow Y$ be a surjective function. Then the *quotient topology* on Y is the unique topology on Y with respect to which p is a quotient map, namely $\mathcal{T} = \{U \in \mathcal{P}Y \mid p^{-1}(U) \text{ is open in } X\}$.

We prove this is a topology.

PROOF:
$\langle 1\rangle 1.$ $Y \in \mathcal{T}$
PROOF: Since $p^{-1}(Y) = X$ by surjectivity.
$\langle 1\rangle 2.$ For all $\mathcal{A} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{A} \in \mathcal{T}$
PROOF: Since $p^{-1}(\bigcup \mathcal{A}) = \bigcup_{U \in \mathcal{A}} p^{-1}(U)$
$\langle 1\rangle 3.$ For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$
PROOF: Since $p^{-1}(U \cap V) = p^{-1}(U) \cap p^{-1}(V)$.
$\square$

Definition 2.252 (Quotient Space)**.** Let X be a topological space and $\sim$ an equivalence relation on X. Let $p : X \twoheadrightarrow X/\sim$ be the canonical surjection. Then $X/\sim$ under the quotient topology is called a *quotient space*, *identification space* or *decomposition space* of X.

Here is a counterexample showing we cannot remove all the hypotheses of Proposition 2.246 except that A is saturated.

Example 2.253. Let $X = (0, 1/2] \cup \{1\} \cup \{1 + 1/n : n \geq 2\}$ as a subspace of $\mathbb{R}$. Define R to be the equivalence relation on X where xRy iff ($x = y$ or $|x - y| = 1$), so we identify $1/n$ with $1 + 1/n$ for all $n \geq 2$. Let Y be the resulting quotient space X/R in the quotient topology and $p : X \twoheadrightarrow Y$ the canonical surjection.

Let $A = \{1\} \cup (0, 1/2] \setminus \{1/n : n \geq 2\} \subseteq X$. Then A is saturated under p but the restriction q of p to A is not a quotient map because it maps the saturated open set $\{1\}$ to $\{1\}$ which is not open in $p(A)$.

Proposition 2.254. *Let X, Y and Z be topological spaces. Let $f : X \to Y$ and $g : Y \to Z$. If f and g are quotient maps then so is $g \circ f$.*

PROOF: From Proposition 2.238. $\square$

Example 2.255. The product of two quotient maps is not necessarily a quotient map.

Let $X = \mathbb{R}$ and X^* the quotient space formed by identifying all positive integers. Let $p : X \twoheadrightarrow X^*$ be the canonical surjection.

We prove $p \times \mathrm{id}_{\mathbb{Q}} : X \times \mathbb{Q} \to X^* \times \mathbb{Q}$ is not a quotient map.

PROOF:
$\langle 1\rangle 1.$ For $n \geq 1$,
LET: $c_n = \sqrt{2}/n$
$\langle 1\rangle 2.$ For $n \geq 1$,
LET: $U_n = \{(x, y) \in X \times \mathbb{Q} \mid n - 1/4 < x < n + 1/4, (y + n > x + c_n \text{ and } y + n > -x + c_n) \text{ or } (y + n < x + c_n \text{ and } y + n < -x + c_n)\}$
$\langle 1\rangle 3.$ For $n \geq 1$, we have U_n is open in $X \times \mathbb{Q}$
$\langle 1\rangle 4.$ For $n \geq 1$, we have $\{n\} \times \mathbb{Q} \subseteq U_n$
$\langle 1\rangle 5.$ LET: $U = \bigcup_{n=1}^{\infty} U_n$
$\langle 1\rangle 6.$ U is open in $X \times \mathbb{Q}$
$\langle 1\rangle 7.$ U is saturated with respect to $p \times \mathrm{id}_{\mathbb{Q}}$

$\langle 1 \rangle 8.$ LET: $U' = (p \times \mathrm{id}_{\mathbb{Q}})(U)$
$\langle 1 \rangle 9.$ ASSUME: for a contradiction U' is open in $X^* \times \mathbb{Q}$
$\langle 1 \rangle 10.$ $(1, 0) \in U'$
$\langle 1 \rangle 11.$ PICK a neighbourhood W of 1 in X^* and $\delta > 0$ such that $W \times (-\delta, \delta) \subseteq U'$
$\langle 1 \rangle 12.$ $p^{-1}(W) \times (-\delta, \delta) \subseteq U$
$\langle 1 \rangle 13.$ PICK n such that $c_n < \delta$
$\langle 1 \rangle 14.$ $n \in p^{-1}(W)$
$\langle 1 \rangle 15.$ PICK $\epsilon > 0$ such that $\epsilon < \delta - c_n$ and $\epsilon < 1/4$ and $(n - \epsilon, n + \epsilon) \subseteq p^{-1}(W)$
$\langle 1 \rangle 16.$ $(n - \epsilon, n + \epsilon) \times (-\delta, \delta) \subseteq U$
$\langle 1 \rangle 17.$ PICK a rational y such that $c_n - \epsilon/2 < y < c_n + \epsilon/2$
$\langle 1 \rangle 18.$ $(n + \epsilon/2, y) \notin U$
$\langle 1 \rangle 19.$ Q.E.D.
PROOF: This contradicts $\langle 1 \rangle 16$.
□

Proposition 2.256. *Let X be a topological space and $\sim$ an equivalence relation on X. Then $X/\sim$ is T_1 if and only if every equivalence class is closed in X.*

PROOF: Immediate from definitions. □

2.47 Retractions

Definition 2.257 (Retraction)**.** Let X be a topological space and $A \subseteq X$. A *retraction* of X onto A is a continuous map $r : X \to A$ such that, for all $a \in A$, we have $r(a) = a$.

Proposition 2.258. *Every retraction is a quotient map.*

PROOF: Proposition 2.250 with f the inclusion $A \hookrightarrow X$. □

2.48 Homogeneous Spaces

Definition 2.259 (Homogeneous)**.** A topological space X is *homogeneous* if and only if, for any points $a, b \in X$, there exists a homeomorphism $\phi : X \cong X$ such that $\phi(a) = b$.

2.49 Regular Spaces

Definition 2.260 (Regular Space)**.** A topological space X is *regular* if and only if, for any closed set A and point $a \notin A$, there exist disjoint open sets U, V such that $A \subseteq U$ and $a \in V$.

2.50 Connected Spaces

Definition 2.261 (Separation)**.** A *separation* of a topological space X is a pair of disjoint open sets U, V such that $U \cup V = \emptyset$.

Definition 2.262 (Connected)**.** A topological space is *connected* if and only if it has no separation; otherwise it is *disconnected*.

Proposition 2.263. *A topological space X is connected if and only if the only sets that are both open and closed are X and $\emptyset$.*

Immediate from defintions.

Lemma 2.264. *If Y is a subspace of X, a separation of Y is a pair of disjoint nonempty sets A and B whose union is Y, neither of which contains a limit point of the other.*

PROOF:
$\langle 1 \rangle 1$. LET: $A, B \subseteq Y$
$\langle 1 \rangle 2$. If A and B form a separation of Y then A and B are disjoint and nonempty, $A \cup B = Y$, and neither of A and B contains a limit point of the other.
$\langle 2 \rangle 1$. ASSUME: A and B form a separation of Y
$\langle 2 \rangle 2$. A and B are disjoint and nonempty and $A \cup B = Y$
PROOF: From $\langle 2 \rangle 1$ and the definition of separation.
$\langle 2 \rangle 3$. A does not contain a limit point of B
$\langle 3 \rangle 1$. ASSUME: for a contradiction $l \in A$ and l is a limit point of B in X.
$\langle 3 \rangle 2$. l is a limit point of B in Y
PROOF: Proposition 2.202.
$\langle 3 \rangle 3$. $l \in B$
$\langle 4 \rangle 1$. B is closed in Y
PROOF: Since A is open in Y and $B = Y \setminus A$ from $\langle 2 \rangle 1$.
$\langle 4 \rangle 2$. Q.E.D.
PROOF: Corollary 2.107.1.
$\langle 3 \rangle 4$. Q.E.D.
PROOF: This contradicts the fact that $A \cap B = \emptyset$ ($\langle 2 \rangle 1$).
$\langle 2 \rangle 4$. B does not contain a limit point of A
PROOF: Similar.
$\langle 1 \rangle 3$. If A and B are disjoint and nonempty, $A \cup B = Y$, and neither of A and B contains a limit point of the other, then A and B form a separation of Y.
$\langle 2 \rangle 1$. ASSUME: A and B are disjoint and nonempty, $A \cup B = Y$, and neither of A and B contains a limit point of the other.
$\langle 2 \rangle 2$. A is open in Y
$\langle 3 \rangle 1$. B is closed in Y
$\langle 4 \rangle 1$. LET: l be a limit point of B in Y
$\langle 4 \rangle 2$. l is a limit point of B in X
PROOF: Proposition 2.202.
$\langle 4 \rangle 3$. $l \notin A$
PROOF: By $\langle 2 \rangle 1$
$\langle 4 \rangle 4$. $l \in B$
PROOF: By $\langle 2 \rangle 1$ since $A \cup B = Y$
$\langle 4 \rangle 5$. Q.E.D.

PROOF: Corollary 2.107.1.

$\langle 3\rangle 2$. Q.E.D.

PROOF: Since $A = Y \setminus B$.

$\langle 2\rangle 3$. B is open in Y

PROOF: Similar.

∎

Example 2.265. Every set under the indiscrete topology is connected.

Example 2.266. The discrete topology on a set X is connected if and only if $|X| \leq 1$.

Example 2.267. The finite complement topology on a set X is connected if and only if either $|X| \leq 1$ or X is infinite.

Example 2.268. The countable complement topology on a set X is connected if and only if either $|X| \leq 1$ or X is uncountable.

Example 2.269. The rationals $\mathbb{Q}$ are disconnected. For any irrational a, the sets $(-\infty, a) \cap \mathbb{Q}$ and $(a, +\infty) \cap \mathbb{Q}$ form a separation of $\mathbb{Q}$.

Lemma 2.270. *Let X be a topological space. If C and D form a separation of X, and Y is a connected subspace of X, then either $Y \subseteq C$ or $Y \subseteq D$.*

PROOF: Otherwise $Y \cap C$ and $Y \cap D$ would form a separation of Y. ∎

Theorem 2.271. *The union of a set of connected subspaces of a space X that have a point in common is connected.*

PROOF:

$\langle 1\rangle 1$. LET: $\mathcal{A}$ be a set of connected subspaces of the space X that have the point a in common.

$\langle 1\rangle 2$. ASSUME: for a contradiction C and D form a separation of $\bigcup \mathcal{A}$

$\langle 1\rangle 3$. ASSUME: without loss of generality $a \in C$

$\langle 1\rangle 4$. For all $A \in \mathcal{A}$ we have $A \subseteq C$

PROOF: Lemma 2.270.

$\langle 1\rangle 5$. $D = \emptyset$

$\langle 1\rangle 6$. Q.E.D.

PROOF: This contradicts $\langle 1\rangle 2$.

∎

Theorem 2.272. *Let X be a topological space and A a connected subspace of X. If $A \subseteq B \subseteq \overline{A}$ then B is connected.*

PROOF:

$\langle 1\rangle 1$. ASSUME: for a contradiction C and D form a separation of B.

$\langle 1\rangle 2$. ASSUME: without loss of generality $A \subseteq C$

PROOF: Lemma 2.270.

$\langle 1\rangle 3$. $B \subseteq C$

$\langle 2\rangle 1$. LET: $x \in B$

⟨2⟩2. $x \in \overline{A}$
⟨2⟩3. Either $x \in A$ or x is a limit point of A.
PROOF: Theorem 2.107.
⟨2⟩4. Either $x \in A$ or x is a limit point of C.
PROOF: Lemma 2.109, ⟨1⟩2.
⟨2⟩5. $x \in C$
PROOF: Lemma 2.264.
⟨1⟩4. $D = \emptyset$
⟨1⟩5. Q.E.D.
PROOF: This contradicts ⟨1⟩1.
□

Theorem 2.273. *The image of a connected space under a continuous map is connected.*

PROOF:
⟨1⟩1. LET: $f : X \twoheadrightarrow Y$ be a surjective continuous map where X is connected.
⟨1⟩2. ASSUME: for a contradiction C and D form a separation of Y.
⟨1⟩3. $f^{-1}(C)$ and $f^{-1}(D)$ form a separation of X.
□

Theorem 2.274. *The product of a family of connected spaces is connected.*

PROOF:
⟨1⟩1. The product of two connected spaces is connected.
⟨2⟩1. LET: X and Y be connected spaces.
⟨2⟩2. PICK $a \in X$ and $b \in Y$
PROOF: We may assume X and Y are nonempty since otherwise $X \times Y = \emptyset$ which is connected.
⟨2⟩3. $X \times \{b\}$ is connected.
PROOF: It is homeomorphic to X.
⟨2⟩4. For all $x \in X$ we have $\{x\} \times Y$ is connected.
PROOF: It is homeomorphic to Y.
⟨2⟩5. For any $x \in X$
LET: $T_x = (X \times \{b\}) \cup (\{x\} \times Y)$
⟨2⟩6. For all $x \in X$, T_x is connected.
PROOF: Theorem 2.271 since $(x, b) \in (X \times \{b\}) \cap (\{x\} \times Y)$.
⟨2⟩7. $X \times Y$ is connected.
PROOF: Theorem 2.271 since $X \times Y = \bigcup_{x \in X} T_x$ and (a, b) is a point in every T_x.
⟨1⟩2. The product of a finite family of connected spaces is connected.
PROOF: From ⟨1⟩1 by induction.
⟨1⟩3. The product of any family of connected spaces is connected.
⟨2⟩1. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of connected spaces.
⟨2⟩2. LET: $X = \prod_{\alpha \in J} X_\alpha$
⟨2⟩3. PICK $a \in X$
PROOF: We may assume $X \neq \emptyset$ as the empty space is connected.

⟨2⟩4. For every finite subset K of J,
LET: $X_K = \{x \in X \mid \forall \alpha \in J \setminus K . x_\alpha = a_\alpha\}$
⟨2⟩5. For every finite $K \subseteq J$, we have X_K is connected.
PROOF: From ⟨1⟩2 since $X_K \cong \prod_{\alpha \in K} X_K$.
⟨2⟩6. LET: $Y = \bigcup_K X_K$
⟨2⟩7. Y is connected
PROOF: Theorem 2.271 since a is a common point.
⟨2⟩8. $X = \overline{Y}$
⟨3⟩1. LET: $x \in X$
⟨3⟩2. LET: $U = \prod_{\alpha \in J} U_\alpha$ be a basic neighbourhood of x where $U_\alpha = X_\alpha$ for all α except $\alpha \in K$ for some finite $K \subseteq J$
⟨3⟩3. LET: $y \in X$ be the point with $y_\alpha = x_\alpha$ for $\alpha \in K$ and $y_\alpha = a_\alpha$ for all other α
⟨3⟩4. $y \in U \cap X_K$
⟨3⟩5. $y \in U \cap Y$
⟨2⟩9. X is connected.
PROOF: Theorem 2.272.
□

Example 2.275. The set $\mathbb{R}^\omega$ is disconnected under the box topology. The set of bounded sequences and the set of unbounded sequences form a separation.

Proposition 2.276. *Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on the same set X. If $\mathcal{T} \subseteq \mathcal{T}'$ and $(X, \mathcal{T}')$ is connected then $(X, \mathcal{T})$ is connected.*

PROOF: If U and V form a separation of $(X, \mathcal{T})$ then they form a separation of $(X, \mathcal{T}')$. □

Proposition 2.277. *Let X be a topological space and (A_n) a sequence of connected subspaces of X. If $A_n \cap A_{n+1} \neq \emptyset$ for all n then $\bigcup_n A_n$ is connected.*

PROOF:
⟨1⟩1. ASSUME: for a contradiction C and D form a separation of $\bigcup_n A_n$
⟨1⟩2. ASSUME: without loss of generality $A_0 \subseteq C$
PROOF: Lemma 2.270.
⟨1⟩3. For all n we gave $A_n \subseteq C$
PROOF:
⟨2⟩1. ASSUME: $A_n \subseteq C$
⟨2⟩2. PICK $x \in A_n \cap A_{n+1}$
⟨2⟩3. $x \in C$
⟨2⟩4. $A_{n+1} \subseteq C$
PROOF: Lemma 2.270.
⟨2⟩5. Q.E.D.
PROOF: The result follows by induction.
⟨1⟩4. $D = \emptyset$
⟨1⟩5. Q.E.D.
PROOF: This contradicts ⟨1⟩1.
□

Proposition 2.278. *Let X be a topological space. Let $A, C \subseteq X$. If C is connected and intersects both A and $X \setminus A$ then C intersects ∂A.*

PROOF: Otherwise $C \cap A^\circ$ and $C \setminus \overline{A}$ would form a separation of C. □

Example 2.279. The space $\mathbb{R}_l$ is disconnected. For any real x, the sets $(-\infty, x)$ and $[x, +\infty)$ form a separation.

Proposition 2.280. *Let X and Y be connected spaces. Let A be a proper subset of X and B a proper subset of Y. Then $(X \times Y) \setminus (A \times B)$ is connected.*

PROOF:
$\langle 1 \rangle 1$. PICK $a \in X \setminus A$ and $b \in Y \setminus B$
$\langle 1 \rangle 2$. For $x \in X \setminus A$ we have $(X \times \{b\}) \cup (\{x\} \times Y)$ is connected.
  PROOF: Theorem 2.271 since (x, b) is a common point.
$\langle 1 \rangle 3$. For $y \in Y \setminus B$ we have $(X \times \{y\}) \cup (\{a\} \times Y)$ is connected.
  PROOF: Theorem 2.271 since (a, y) is a common point.
$\langle 1 \rangle 4$. $(X \times Y) \setminus (A \times B)$ is connected.
  PROOF: Theorem 2.271 since it is the union of the sets in $\langle 1 \rangle 2$ and $\langle 1 \rangle 3$ with (a, b) as a common point.
□

Proposition 2.281. *Let $p : X \twoheadrightarrow Y$ be a quotient map. If Y is connected and $p^{-1}(y)$ is connected for all $y \in Y$, then X is connected.*

PROOF:
$\langle 1 \rangle 1$. ASSUME: for a contradiction C and D form a separation of X.
$\langle 1 \rangle 2$. C is saturated.
  $\langle 2 \rangle 1$. LET: $x \in C$, $y \in X$ with $p(x) = p(y) = a$, say
  $\langle 2 \rangle 2$. $y \notin D$
    PROOF: Otherwise $p^{-1}(a) \cap C$ and $p^{-1}(a) \cap D$ form a separation of $p^{-1}(a)$.
  $\langle 2 \rangle 3$. $y \in C$
$\langle 1 \rangle 3$. D is saturated.
  PROOF: Similar.
$\langle 1 \rangle 4$. $p(C)$ and $p(D)$ form a separation of Y.
□

Proposition 2.282. *Let X be a connected space and Y a connected subspace of X. Suppose A and B form a separation of $X \setminus Y$. Then $Y \cup A$ and $Y \cup B$ are both connected.*

PROOF:
$\langle 1 \rangle 1$. $Y \cup A$ is connected.
  $\langle 2 \rangle 1$. ASSUME: for a contradiction C and D form a separation of $Y \cup A$
  $\langle 2 \rangle 2$. ASSUME: without loss of generality $Y \subseteq C$
  $\langle 2 \rangle 3$. PICK open sets A_1, B_1, C_1, D_1 in X with

$$A = A_1 \setminus Y$$
$$B = B_1 \setminus Y$$
$$C = C_1 \cap (Y \cup A)$$
$$D = D_1 \cap (Y \cup A)$$

⟨2⟩4. $B_1 \cup C_1$ and $A_1 \cap D_1$ form a separation of X
⟨1⟩2. $Y \cup B$ is connected.
PROOF: Similar.
□

Theorem 2.283. *Let L be a linearly ordered set under the order topology. Then L is connected if and only if L is a linear continuum.*

PROOF:
⟨1⟩1. If L is a linear continuum then L is connected.
⟨2⟩1. LET: L be a linear continuum under the order topology.
⟨2⟩2. ASSUME: for a contradiction C and D form a separation of L.
⟨2⟩3. PICK $a \in C$ and $b \in D$.
⟨2⟩4. ASSUME: without loss of generality $a < b$.
⟨2⟩5. LET: $S = \{x \in L \mid a < x \text{ and } [a, x) \subseteq C\}$
⟨2⟩6. S is nonempty.
PROOF: Since $a \in C$ and C is open.
⟨2⟩7. S is bounded above by b.
PROOF: Since $b \notin C$.
⟨2⟩8. LET: $s = \sup S$
⟨2⟩9. $s \in S$
⟨3⟩1. LET: $y \in [a, s)$
PROVE: $y \in C$
⟨3⟩2. PICK z with $y < z \in S$
PROOF: By minimality of s.
⟨3⟩3. $y \in [a, z) \subseteq C$
⟨2⟩10. CASE: $s \in C$
⟨3⟩1. PICK x such that $s < x$ and $[s, x) \subseteq C$
PROOF: Since C is open and s is not greatest in L because $s < b$.
⟨3⟩2. $x \in S$
PROOF: Since $[a, x) = [a, s) \cup [s, x) \subseteq C$.
⟨3⟩3. Q.E.D.
PROOF: This contradicts the fact that s is an upper bound for S.
⟨2⟩11. CASE: $s \in D$
⟨3⟩1. PICK $x < s$ such that $(x, s] \subseteq D$
⟨3⟩2. PICK y with $x < y < s$
PROOF: Since L is dense.
⟨3⟩3. $y \in C$
PROOF: From ⟨2⟩9.
⟨3⟩4. $y \in D$
PROOF: From ⟨3⟩1.
⟨3⟩5. Q.E.D.
⟨3⟩6. LET: L be a linear continuum under the order topology.
⟨3⟩7. ASSUME: for a contradiction C and D form a separation of L.
⟨3⟩8. PICK $a \in C$ and $b \in D$.
⟨3⟩9. ASSUME: without loss of generality $a < b$.
⟨3⟩10. LET: $S = \{x \in L \mid a < x \text{ and } [a, x) \subseteq C\}$

$\langle 3\rangle 11$. S is nonempty.

PROOF: Since $a \in C$ and C is open.

$\langle 3\rangle 12$. S is bounded above by b.

PROOF: Since $b \notin C$.

$\langle 3\rangle 13$. LET: $s = \sup S$

$\langle 3\rangle 14$. $s \in S$

$\langle 4\rangle 1$. LET: $y \in [a, s)$

PROVE: $y \in C$

$\langle 4\rangle 2$. PICK z with $y < z \in S$

PROOF: By minimality of s.

$\langle 4\rangle 3$. $y \in [a, z) \subseteq C$

$\langle 3\rangle 15$. CASE: $s \in C$

$\langle 4\rangle 1$. PICK x such that $s < x$ and $[s, x) \subseteq C$

PROOF: Since C is open and s is not greatest in L because $s < b$.

$\langle 4\rangle 2$. $x \in S$

PROOF: Since $[a, x) = [a, s) \cup [s, x) \subseteq C$.

$\langle 4\rangle 3$. Q.E.D.

PROOF: This contradicts the fact that s is an upper bound for S.

$\langle 3\rangle 16$. CASE: $s \in D$

$\langle 4\rangle 1$. PICK $x < s$ such that $(x, s] \subseteq D$

$\langle 4\rangle 2$. PICK y with $x < y < s$

PROOF: Since L is dense.

$\langle 4\rangle 3$. $y \in C$

PROOF: From $\langle 2\rangle 9$.

$\langle 4\rangle 4$. $y \in D$

PROOF: From $\langle 3\rangle 1$.

$\langle 4\rangle 5$. Q.E.D.

PROOF: This contradicts $\langle 2\rangle 2$.

$\langle 1\rangle 2$. If L is connected then L is a linear continuum.

$\langle 2\rangle 1$. ASSUME: L is connected.

$\langle 2\rangle 2$. Every nonempty subset of L that is bounded above has a supremum.

$\langle 3\rangle 1$. LET: X be a nonempty subset of L bounded above by b.

$\langle 3\rangle 2$. ASSUME: for a contradiction X has no supremum.

$\langle 3\rangle 3$. LET: U be the set of upper bounds of X,

$\langle 3\rangle 4$. U is nonempty.

PROOF: Since $b \in U$.

$\langle 3\rangle 5$. U is open.

$\langle 4\rangle 1$. LET: $x \in U$

$\langle 4\rangle 2$. PICK an upper bound y for X such that $y < x$

$\langle 4\rangle 3$. Either x is greatest in L and $(y, x] \subseteq U$, or there exists $z > x$ such that $(y, z) \subseteq U$

$\langle 3\rangle 6$. LET: V be the set of lower bounds of U.

$\langle 3\rangle 7$. V is nonempty.

PROOF: Since $X \subseteq V$

$\langle 3\rangle 8$. V is open.

$\langle 4\rangle 1$. LET: $x \in V$

⟨4⟩2. Pick $y \in X$ with $x < y$

Proof: x cannot be an upper bound for X, because it would be the supremum of X.

⟨4⟩3. Either x least in L and $[x, y) \subseteq V$, or there exists $z < x$ such that $(z, y) \subseteq V$

⟨3⟩9. $L = U \cup V$

⟨4⟩1. Let: $x \in L \setminus U$

⟨4⟩2. Pick $y \in X$ such that $x < y$

⟨4⟩3. For all $u \in U$ we have $x < y \leq u$

⟨4⟩4. $x \in V$

⟨3⟩10. $U \cap V = \emptyset$

Proof: Any element of $U \cap V$ would be a supremum of X.

⟨3⟩11. U and V form a separation of L.

⟨3⟩12. Q.E.D.

Proof: This contradicts ⟨2⟩1.

⟨2⟩3. L is dense.

⟨3⟩1. Let: $x, y \in L$ with $x < y$

⟨3⟩2. There exists $z \in L$ such that $x < z < y$

Proof: Otherwise $(-\infty, y)$ and $(x, +\infty)$ would form a separation of L.

□

Corollary 2.283.1. *The real line $\mathbb{R}$ is connected.*

Corollary 2.283.2. *Every interval in $\mathbb{R}$ is connected.*

Corollary 2.283.3. *The ordered square is connected.*

Theorem 2.284 (Intermediate Value Theorem)**.** *Let X be a connected space. Let Y be a linearly ordered set under the order topology. Let $f : X \to Y$ be continuous. Let $a, b \in X$ and $r \in Y$. Suppose $f(a) < r < f(b)$. Then there exists $c \in X$ such that $f(c) = r$.*

Proof: Otherwise $f^{-1}((-\infty, r))$ and $f^{-1}((r, +\infty))$ would form a separation of X. □

Proposition 2.285. *Every function $f : [0, 1] \to [0, 1]$ has a fixed point.*

Proof:

⟨1⟩1. Let: $g : [0, 1] \to [-1, 1]$ be the function $g(x) = f(x) - x$

Prove: there exists $x \in [0, 1]$ such that $g(x) = 0$

⟨1⟩2. Assume: without loss of generality $g(0) \neq 0$ and $g(1) \neq 0$

⟨1⟩3. $g(0) > 0$

⟨1⟩4. $g(1) < 0$

⟨1⟩5. There exists $x \in (0, 1)$ such that $g(x) = 0$

Proof: By the Intermediate Value Theorem.

Proposition 2.286. *Give $\mathbb{R}^\omega$ the box topology. Let $x, y \in \mathbb{R}^\omega$. Then x and y lie in the same comoponent if and only if $x - y$ is eventually zero, i.e. there exists N such that, for all $n \geq N$, we have $x_n = y_n$.*

PROOF:
⟨1⟩1. The component containing 0 is the set of sequences that are eventually zero.
⟨2⟩1. LET: B be the set of sequences that are eventually zero.
⟨2⟩2. B is path-connected.
⟨3⟩1. LET: $x, y \in B$
⟨3⟩2. PICK N such that, for all $n \geq N$, we have $x_n = y_n = 0$
⟨3⟩3. LET: $p : [0, 1] \to \mathbb{R}^\omega$, $p(t) = (1 - t)x + ty$
PROVE: p is continuous.
⟨3⟩4. LET: $t \in [0, 1]$ and $\prod_j U_j$ be a basic open neighbourhood of $p(t)$, where each U_j is open in $\mathbb{R}$
⟨3⟩5. PICK δ such that, for all $n < N$ and all $s \in [0, 1]$, if $|s - t| < \delta$ then $p(s)_n \in U_n$
⟨3⟩6. For all $s \in [0, 1]$, if $|s - t| < \delta$ then $p(s) \in \prod_j U_j$
⟨2⟩3. B is connected.
PROOF: Proposition 2.292.
⟨2⟩4. If C is connected and $B \subseteq C$ then $B = C$.
⟨3⟩1. ASSUME: C is connected and $B \subseteq C$
⟨3⟩2. ASSUME: for a contradiction $x \in C \setminus B$
⟨3⟩3. For $n \geq 1$,
LET: $c_n = 1$ if $x_n = 0$, $c_n = n/x_n$ otherwise
⟨3⟩4. LET: $h : \mathbb{R}^\omega \to \mathbb{R}^\omega$ be the function $h(x) = (c_n x_n)_{n \geq 1}$
⟨3⟩5. h is a homeomorphism of $\mathbb{R}^\omega$ with itself.
⟨3⟩6. $h(x)$ is unbounded.
PROOF: For any $b > 0$, pick $N > b$ such that $x_N \neq 0$. Then $h(x)_N > b$.
⟨3⟩7. $h^{-1}(\{\text{bounded sequences}\}) \cap C$ and $h^{-1}(\{\text{unbounded sequences}\}) \cap C$ form a separation of C
⟨3⟩8. Q.E.D.
PROOF: This contradicts ⟨3⟩1.
⟨1⟩2. Q.E.D.
PROOF: Since $\lambda x.x - y$ is a homeomorphism of $\mathbb{R}^\omega$ with itself.
□

2.51 Totally Disconnected Spaces

Definition 2.287 (Totally Disconnected)**.** A topological space X is *totally disconnected* if and only if the only connected subspaces are the singletons.

Example 2.288. Every discrete space is totally disconnected.

Example 2.289. The rationals $\mathbb{Q}$ are totally disconnected.

2.52 Paths and Path Connectedness

Definition 2.290 (Path)**.** Let X be a topological space and $a, b \in X$. A *path* from a to b is a continuous function $p : [0, 1] \to X$ such that $p(0) = a$ and

$p(1) = b$.

Definition 2.291 (Path Connected)**.** A topological space is *path connected* if and only if there exists a path between any two points.

Proposition 2.292. *Every path connected space is connected.*

PROOF:
⟨1⟩1. LET: X be a path connected space.
⟨1⟩2. ASSUME: for a contradiction C and D form a separation of X.
⟨1⟩3. PICK $a \in C$ and $b \in D$.
⟨1⟩4. PICK a path $p : [0,1] \to X$ from a to b.
⟨1⟩5. $p^{-1}(C)$ and $p^{-1}(D)$ form a separation of $[0,1]$.
⟨1⟩6. Q.E.D.
PROOF: This contradicts Corollary 2.283.2.
□

An example that shows the converse does not hold:

Example 2.293. The ordered square is not path connected.

PROOF:
⟨1⟩1. ASSUME: for a contradiction $p : [0,1] \to I_o^2$ is a path from $(0,0)$ to $(1,1)$.
⟨1⟩2. p is surjective.
PROOF: By the Intermediate Value Theorem.
⟨1⟩3. For $x \in [0,1]$, PICK a rational $q_x \in p^{-1}((x,0),(x,1))$
PROOF: Since $p^{-1}((x,0),(x,1))$ is open and nonempty by ⟨1⟩2.
⟨1⟩4. For $x, y \in [0,1]$, if $x \neq y$ then $q_x \neq q_y$
PROOF: We have $p(q_x) \neq p(q_y)$ because $((x,0),(x,1))$ and $((y,0),(y,1))$ are disjoint.
⟨1⟩5. $\{q_x \mid x \in [0,1]\}$ is an uncountable set of rationals.
⟨1⟩6. Q.E.D.
PROOF: This contradicts the fact that the rationals are countable.
□

Proposition 2.294. *The continuous image of a path connected space is path connected.*

PROOF:
⟨1⟩1. LET: X be a path connected space, Y a topological space, and $f : X \twoheadrightarrow Y$ be continuous and surjective.
⟨1⟩2. LET: $a, b \in Y$
⟨1⟩3. PICK $c, d \in X$ with $f(c) = a$ and $f(d) = b$
⟨1⟩4. PICK a path $p : [0,1] \to X$ from c to d.
⟨1⟩5. $f \circ p$ is a path from a to b in Y.
□

Proposition 2.295 (AC)**.** *The product of a family of path-connected spaces is path-connected.*

PROOF:
⟨1⟩1. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of path-connected spaces.
⟨1⟩2. LET: $a, b \in \prod_{\alpha \in J} X_\alpha$
⟨1⟩3. For $\alpha \in J$, PICK a path $p_\alpha : [0,1] \to X_\alpha$ from a_α to b_α
PROOF: Using the Axiom of Choice.
⟨1⟩4. Define $p : [0.1] \to \prod_{\alpha \in J} X_\alpha$ by $p(t)_\alpha = p_\alpha(t)$
⟨1⟩5. p is a path from a to b.
PROOF: Theorem 2.181.
□

Proposition 2.296. *The continuous image of a path-connected space is path-connected.*

PROOF:
⟨1⟩1. LET: $f : X \twoheadrightarrow Y$ be continuous and surjective where X is path-connected.
⟨1⟩2. LET: $a, b \in Y$
⟨1⟩3. PICK $a', b' \in X$ with $f(a') = a$ and $f(b') = b$.
⟨1⟩4. PICK a path $p : [0,1] \to X$ from a' to b'.
⟨1⟩5. $f \circ p$ is a path from a to b.
□

Proposition 2.297. *Let X be a topological space. The union of a set of path-connected subspaces of X that have a point in common is path-connected.*

PROOF:
⟨1⟩1. LET: $\mathcal{A}$ be a set of path-connected subspaces of X with the point a in common.
⟨1⟩2. LET: $b, c \in \bigcup \mathcal{A}$
⟨1⟩3. PICK $B, C \in \mathcal{A}$ with $b \in B$ and $c \in C$.
⟨1⟩4. PICK a path p in B from b to a.
⟨1⟩5. PICK a path q in C from a to c.
⟨1⟩6. The concatenation of p and q is a path from b to c in $\bigcup \mathcal{A}$.
□

Proposition 2.298. *Let $A \subseteq \mathbb{R}^2$ be countable. Then $\mathbb{R}^2 \setminus A$ is path-connected.*

PROOF:
⟨1⟩1. LET: $a, b \in \mathbb{R}^2 \setminus A$
⟨1⟩2. PICK a line l in $\mathbb{R}^2$ with a on one side and b on the other.
⟨1⟩3. For every point x on l,
LET: p_x be the path in $\mathbb{R}^2$ consisting of a line from a to x then a line from x to b
⟨1⟩4. For $x \neq y$ we have p_x and p_y have no points in common except a and b
⟨1⟩5. There are only countably many x such that a point of A lies on p_x.
⟨1⟩6. There exists x such that p_x is a path from a to b in $\mathbb{R}^2 \setminus A$.
□

Proposition 2.299. *Every open connected subspace of $\mathbb{R}^2$ is path-connected.*

Proof:
$\langle 1\rangle 1$. Let: U be an open connected subspace of $\mathbb{R}^2$.
$\langle 1\rangle 2$. For all $x_0 \in U$,
Let: $PC(x_0) = \{y \in U \mid \text{there exists a path from } x \text{ to } y\}$
$\langle 1\rangle 3$. For all $x_0 \in U$, the set $PC(x_0)$ is open and closed in U.
$\langle 2\rangle 1$. Let: $x_0 \in U$
$\langle 2\rangle 2$. $PC(x_0)$ is open in U
$\langle 3\rangle 1$. Let: $y \in PC(x_0)$
$\langle 3\rangle 2$. Pick $\epsilon > 0$ such that $B(y, \epsilon) \subseteq U$
Proof: Since U is open.
$\langle 3\rangle 3$. $B(y, \epsilon) \subseteq PC(x_0)$
Proof: For all $z \in B(y, \epsilon)$, pick a path from x_0 to y then concatenate the straight line from y to z.
$\langle 2\rangle 3$. $PC(x_0)$ is closed in U
$\langle 3\rangle 1$. Let: $y \in U$ be a limit point of $PC(x_0)$
$\langle 3\rangle 2$. Pick $\epsilon > 0$ such that $B(y, \epsilon) \subseteq U$
$\langle 3\rangle 3$. Pick $z \in PC(x_0) \cap B(y, \epsilon)$
$\langle 3\rangle 4$. $y \in PC(x_0)$
Proof: Pick a path from x_0 to z then concatenate the straight line from z to y.
$\langle 1\rangle 4$. $PC(x_0) = U$
Proof: Proposition 2.263.
□

Example 2.300. If A is a connected subspace of X, then A° is not necessarily connected.

Take two closed circles in $\mathbb{R}^2$ that touch at one point. The interior of this space is two open circles, and these two circles form a separation.

Example 2.301. If A is a connected subspace of X then ∂A is not necessarily connected.

We have $[0, 1]$ is connected but $\partial[0, 1] = \{0, 1\}$ is not.

Example 2.302. If A is a subspace of X and A° and ∂A are connected, then A is not necessarily connected.

We have $\mathbb{Q}^\circ = \emptyset$ and $\partial\mathbb{Q} = \mathbb{R}$ are connected but $\mathbb{Q}$ is not connected.

2.53 The Topologist's Sine Curve

Definition 2.303 (The Topologist's Sine Curve)**.** Let $S = \{(x, \sin 1/x) \mid 0 < x \leq 1\}$, The *topologist's sine curve* is the closure $\overline{S}$ of S in $\mathbb{R}^2$.

Proposition 2.304. *The topologist's sine curve is connected.*

Proof:
$\langle 1\rangle 1$. Let: $S = \{(x, \sin 1/x) \mid 0 < x \leq 1\}$
$\langle 1\rangle 2$. S is connected.

PROOF: Theorem 2.273.
⟨1⟩3. $\overline{S}$ is connected.
PROOF: Theorem 2.272.
□

Proposition 2.305. *The topologist's sine curve is* $\{(x, \sin 1/x) \mid 0 < x \leq 1\} \cup (\{0\} \times [-1.1])$.

PROOF: Sketch proof: Given a point $(0.y)$ with $-1 \leq y \leq 1$, pick a such that $\sin a = y$. Then $((1/a, y), (1/(a + 2\pi), y), (1/(a + 4\pi), y), \ldots)$ is a sequence in S that converges to $(0, y)$.
Conversely, let (x, y) be any point not in $S \cup (\{0\} \times [-1, 1])$. If $x < 0$ or $y > 1$ or $y < -1$ then we can easily find a neighbourhood that does not intersect $S \cup (\{0\} \times [-1, 1])$. If $x > 0$ and $-1 \leq y \leq 1$, then we have $y \neq \sin 1/x$. Hence pick a neighbourhood that does not intersect S.

Proposition 2.306. *Every closed subset of* $\mathbb{R}$ *that is bounded above has a greatest element.*

PROOF: It has a supremum, which is a limit point of the set and hence an element. □

Proposition 2.307 (CC)**.** *The topologist's sine curve is not path connected.*

PROOF:
⟨1⟩1. ASSUME: For a contradction $p : [0, 1] \to \overline{S}$ is a path from $(0, 0)$ to $(1, \sin 1)$.
⟨1⟩2. $\{t \in [0, 1] \mid p(t) \in \{0\} \times [-1, 1]\}$ is closed.
PROOF: Since p is continuous and $\{0\} \times [-1, 1]$ is closed.
⟨1⟩3. LET: b be the largest number in $[0, 1]$ such that $p(b) \in \{0\} \times [-1, 1]$.
PROOF: Proposition 2.306.
⟨1⟩4. LET: $x : [b, 1] \to \overline{S}$ be the function $\pi_1 \circ p$
⟨1⟩5. LET: $y : [b, 1] \to \overline{S}$ be the function $\pi_2 \circ p$
⟨1⟩6. PICK a sequence t_n in $(b, 1]$ such that $t_n \to b$ and $y(t_n) = (-1)^n$ for all n
⟨2⟩1. LET: $n \geq 1$
⟨2⟩2. PICK u with $0 < u < x(1/n)$ and $\sin(1/u) = (-1)^n$
⟨2⟩3. PICK t_n with $b < t_n < 1/n$ and $x(t_n) = u$
PROOF: By the Intermediate Value Theorem
⟨1⟩7. Q.E.D.
PROOF: This contradicts Proposition 2.154 since y is continuous and $y(t_n)$ does not converge.
□

Corollary 2.307.1. *The closure of a path-connected subspace of a space is not necessarily path-connected.*

2.54 The Long Line

Definition 2.308 (The Long Line)**.** The *long line* is the space $\omega_1 \times [0, 1)$ in the dictionary order under the order topology, where ω_1 is the first uncountable ordinal.

Lemma 2.309. *For any ordinal α with $0 < \alpha < \omega_1$ we have $[(0,0),(\alpha,0)) \cong [0,1)$*

$\langle 1\rangle 1.$ $[(0,0),(1,0)) \cong [0,1)$
  PROOF: The map π_2 is a homeomorphism.
$\langle 1\rangle 2.$ If $[(0,0),(\alpha,0)) \cong [0,1)$ then $[(0,0),(\alpha+1,0)) \cong [0,1)$
  PROOF: Proposition 2.53.
$\langle 1\rangle 3.$ If λ is a limit ordinal with $\lambda < \omega_1$ and $[(0,0),(\alpha,0)) \cong [0,1)$ for all α with $0 < \alpha < \lambda$ then $[(0,0),(\lambda,0)) \cong [0,1)$
  $\langle 2\rangle 1.$ LET: λ be a limit ordinal $< \omega_1$
  $\langle 2\rangle 2.$ ASSUME: $[(0,0),(\alpha,0)) \cong [0,1)$ for all α with $0 < \alpha < \lambda$
  $\langle 2\rangle 3.$ PICK a sequence of ordinals $\alpha_0 < \alpha_1 < \cdots$ with limit λ
    PROOF: Since λ is countable.
  $\langle 2\rangle 4.$ $[(\alpha_i,0),(\alpha_{i+1},0)) \cong [0,1)$ for all i
    PROOF: Lemma 2.52.
  $\langle 2\rangle 5.$ Q.E.D.
    PROOF: By Proposition 2.54.
$\langle 1\rangle 4.$ Q.E.D.
  PROOF: By transfinite induction.

Proposition 2.310 (CC)**.** *The long line is path-connected.*

PROOF:
$\langle 1\rangle 1.$ LET: $(\alpha,i),(\beta,j) \in \omega_1 \times [0,1)$
$\langle 1\rangle 2.$ ASSUME: without loss of generality $(\alpha,i) < (\beta,j)$
$\langle 1\rangle 3.$ $[(0,0),(\beta+1,0)) \cong [0,1)$
  PROOF: By Lemma 2.309
$\langle 1\rangle 4.$ $[(\alpha,i),(\beta,j)) \cong [0,1)$
  PROOF: Lemma 2.52.
$\langle 1\rangle 5.$ PICK a homeomorphism $q : [0,1) \to [(\alpha,i),(\beta,j))$
$\langle 1\rangle 6.$ $q \cup \{(1,(\beta,j))\}$ is a path from (α,i) to (β,j)
$\square$

Proposition 2.311. *Every point in the long line has a neighbourhood homeomorphic to an interval in $\mathbb{R}$.*

PROOF: For any (α,i) in the long line, the neighbourhood $[(0,0),(\alpha+1,0))$ satisfies the condition by Lemma 2.309.

Proposition 2.312. *The long line L is not second countable.*

PROOF:
$\langle 1\rangle 1.$ LET: $\mathcal{B}$ be a basis for L.
$\langle 1\rangle 2.$ For $\alpha < \omega_1$, PICK $B_\alpha \in \mathcal{B}$ such that $(\alpha,1/2) \in B_\alpha \subseteq ((\alpha,0),(\alpha+1,0))$
$\langle 1\rangle 3.$ $\mathcal{B}$ is uncountable.
  PROOF: The mapping $\alpha \mapsto B_\alpha$ is an injection $\omega_1 \to \mathcal{B}$.

Corollary 2.312.1. *The long line cannot be imbedded into $\mathbb{R}^n$ for any n.*

2.55 Components

Proposition 2.313. *Let X be a topological space. Define the relation $\sim$ on X by $a \sim b$ if and only if there exists a connected subspace $A \subseteq X$ with $a, b \in A$. Then $\sim$ is an equivalence relation on X.*

PROOF:
⟨1⟩1. $\sim$ is reflexive.
PROOF: For any $a \in X$ we have $\{a\}$ is a connected subspace that contains a.
⟨1⟩2. $\sim$ is symmetric.
PROOF: Trivial.
⟨1⟩3. $\sim$ is transitive.
⟨2⟩1. LET: $a, b, c \in X$
⟨2⟩2. ASSUME: $a \sim b$ and $b \sim c$
⟨2⟩3. PICK connected subspaces A and B with $a, b \in A$ and $b, c \in B$
⟨2⟩4. $A \cup B$ is a connected subspace that contains a and c
PROOF: Theorem 2.271.
□

Definition 2.314 ((Connected) Component)**.** Let X be a topological space. The *(connected) components* of X are the equivalence classes under the above $\sim$.

Lemma 2.315. *Let X be a topological space. If $A \subseteq X$ is connected and nonempty then there exists a unique component C of X such that $A \subseteq C$.*

PROOF:
⟨1⟩1. PICK $a \in A$
⟨1⟩2. LET: C be the $\sim$-equivalence class of a.
⟨1⟩3. $A \subseteq C$
PROOF: For all $x \in A$ we have $x \sim a$.
⟨1⟩4. If C' is a component and $A \subseteq C'$ then $C = C'$
PROOF: Since we have $a \in C'$.
□

Theorem 2.316. *Let X be a topological space. The components of X are connected disjoint subspaces of X whose union is X such that each nonempty connected subspace of X intersects only one of them.*

PROOF:
⟨1⟩1. Every component of X is connected.
PROOF: For $a \in X$, the $\sim$-equivalence class of a is $\bigcup\{A \subseteq X \mid A \text{ is connected}, a \in A\}$ which is connected by Theorem 2.271.
⟨1⟩2. The components form a partition of X.
PROOF: Immediate from the definition.
⟨1⟩3. Every nonempty connected subspace of X intersects a unique component of X.
⟨2⟩1. LET: $A \subseteq X$ be connected and nonempty.

⟨2⟩2. LET: C be the component such that $A \subseteq C$
PROOF: Lemma 2.315.
⟨2⟩3. A intersects C
⟨2⟩4. If A intersects the component C' then $C' = C$
⟨3⟩1. LET: C' be a component that intersects A
⟨3⟩2. PICK $b \in A \cap C'$
⟨3⟩3. $A \subseteq C'$
PROOF: For all $x \in A$ we have $x \sim b$.
⟨3⟩4. $C = C'$
PROOF: By uniqueness in ⟨2⟩2.
□

Proposition 2.317. *Every component of a space is closed.*

PROOF:
⟨1⟩1. LET: X be a topological space and C a component of X.
⟨1⟩2. $\overline{C}$ is connected.
PROOF: Theorem 2.272.
⟨1⟩3. $C = \overline{C}$
PROOF: Lemma 2.270.
⟨1⟩4. C is closed.
PROOF: Lemma 2.95.
□

Proposition 2.318. *If a topological space has finitely many components then every component is open.*

PROOF: Each component is the complement of a finite union of closed sets. □

2.56 Path Components

Proposition 2.319. *Let X be a topological space. Define the relation $\sim$ on X by: $a \sim b$ if and only if there exists a path in X from a to b. Then $\sim$ is an equivalence relation on X.*

PROOF:
⟨1⟩1. $\sim$ is reflexive.
PROOF: For $a \in X$, the constant function $[0, 1] \to X$ with value a is a path from a to a.
⟨1⟩2. $\sim$ is symmetric.
PROOF: If $p : [0, 1] \to X$ is a path from a to b, then $\lambda t.p(1 - t)$ is a path from b to a.
⟨1⟩3. $\sim$ is transitive.
PROOF: Concatenate paths.
□

Definition 2.320 (Path Component)**.** Let X be a topological space. The *path components* of X are the equivalence relations under $\sim$.

.

Theorem 2.321. *The path components of X are path-connected disjoint subspaces of X whose union is X such that every nonempty path-connected subspace of X intersects exactly one path component.*

Proof:
⟨1⟩1. Every path component is path-connected.
  Proof: If a and b are in the same path component then $a \sim b$, i.e. there exists a path from a to b.
⟨1⟩2. The path components are disjoint and their union is X.
  Proof: Immediate from the definition.
⟨1⟩3. Every non-empty path-cönnected subspace of X intersects exactly one path component.
  ⟨2⟩1. Let: A be a nonempty path-connected subspace of X.
  ⟨2⟩2. Pick $a \in A$
  ⟨2⟩3. A intersects the $\sim$-equivalence class of a.
  ⟨2⟩4. Let: C be any path component that intersects A.
  ⟨2⟩5. Pick $b \in A \cap C$
  ⟨2⟩6. $a \sim b$
    Proof: Since A is path-connected.
  ⟨2⟩7. C is the $\sim$-equivalence class of a.
□

Proposition 2.322. *Every path component is included in a component.*

Proof:
⟨1⟩1. Let: X be a topological space and C a path component of X.
⟨1⟩2. C is path-connected.
  Proof: Theorem 2.321.
⟨1⟩3. C is connected.
  Proof: Proposition 2.292.
⟨1⟩4. C is included in a component.
  Proof: Lemma 2.315.
□

2.57 Local Connectedness

Definition 2.323 (Locally Connected)**.** Let X be a topological space and $a \in X$. Then X is *locally connected* at a if and only if every neighbourhood of a includes a connected neighbourhood of a.

The space X is *locally connected* if and only if it is locally connected at every point.

Example 2.324. The real line is both connected and locally connected.

Example 2.325. The space $\mathbb{R} \setminus \{0\}$ is disconnected but locally connected.

Example 2.326. The topologist's sine curve is connected but not locally connected.

Example 2.327. The rationals $\mathbb{Q}$ are neither connected nor locally connected.

Theorem 2.328. *A topological space X is locally connected if and only if, for every open set U in X, every component of U is open in X.*

PROOF:
$\langle 1 \rangle 1$. If X is locally connected then, for every open set U in X, every component of U is open in X.
  $\langle 2 \rangle 1$. ASSUME: X is locally connected.
  $\langle 2 \rangle 2$. LET: U be open in X.
  $\langle 2 \rangle 3$. LET: C be a component of U.
  $\langle 2 \rangle 4$. LET: $a \in C$
  $\langle 2 \rangle 5$. LET: V be a connected neighbourhood of a such that $V \subseteq U$
  $\langle 2 \rangle 6$. $V \subseteq C$
    PROOF: Lemma 2.315.
  $\langle 2 \rangle 7$. Q.E.D.
    PROOF: Lemma 2.74.
$\langle 1 \rangle 2$. If, for every open set U in X, every component of U is open in X, then X is locally connected.
  $\langle 2 \rangle 1$. ASSUME: for every open set U in X, every component of U is open in X.
  $\langle 2 \rangle 2$. LET: $a \in X$
  $\langle 2 \rangle 3$. LET: U be a neighbourhood of a
  $\langle 2 \rangle 4$. The component of U that contains a is a connected neighbourhood of a included in U.

$\square$

Example 2.329. The ordered square is locally connected.

PROOF: Every neighbourhood of a point includes an interval around that point, which is connected by Theorem 2.283.

Example 2.330. Let X be the set of all rational points on the line segment $[0,1] \times \{0\}$, and Y the set of all rational points on the line segment $[0,1] \times \{1\}$. Let A be the space consisting of all line segments joining the point $(0,1)$ to a point of X, and all line segments joining the point $(1,0)$ to a point of Y. Then A is path-connected but is not locally connected at any point,

Proposition 2.331. *Let X and Y be topological spaces and $p : X \twoheadrightarrow Y$ be a quotient map. If X is locally connected then so is Y.*

PROOF:
$\langle 1 \rangle 1$. LET: U be an open set in Y.
$\langle 1 \rangle 2$. LET: C be a component of U.
$\langle 1 \rangle 3$. $p^{-1}(C)$ is a union of components of $p^{-1}(U)$
  $\langle 2 \rangle 1$. LET: $x \in p^{-1}(C)$

⟨2⟩2. Let: D be the component of $p^{-1}(U)$ that contains x.
⟨2⟩3. $p(D)$ is connected.
Proof: Theorem 2.273.
⟨2⟩4. $p(D) \subseteq C$.
Proof: From ⟨1⟩2 since $p(x) \in p(D) \cap C$ (⟨2⟩1, ⟨2⟩2).
⟨2⟩5. $D \subseteq p^{-1}(C)$
⟨1⟩4. $p^{-1}(C)$ is open in $p^{-1}(U)$
Proof: Theorem 2.328.
⟨1⟩5. C is open in U
Proof: Since the restriction of p to $p : p^{-1}(U) \twoheadrightarrow U$ is a quotient map by Proposition 2.246.
⟨1⟩6. Q.E.D.
Proof: Theorem 2.328.
□

2.58 Local Path Connectedness

Definition 2.332 (Locally Path-Connected)**.** Let X be a topological space and $a \in X$. Then X is *locally path-connected* at a if and only if every neighbourhood of a includes a path-connected neighbourhood of a.

The space X is *locally path-connected* if and only if it is locally path-connected at every point.

Theorem 2.333. *A topological space X is locally path-connected if and only if, for every open set U in X, every path component of U is open in X.*

Proof:
⟨1⟩1. If X is locally path-connected then, for every open set U in X, every path component of U is open in X.
⟨2⟩1. Assume: X is locally path-connected.
⟨2⟩2. Let: U be open in X.
⟨2⟩3. Let: C be a path component of U.
⟨2⟩4. Let: $a \in C$
⟨2⟩5. Let: V be a path-connected neighbourhood of a such that $V \subseteq U$
⟨2⟩6. $V \subseteq C$
Proof: Lemma 2.315.
⟨2⟩7. Q.E.D.
Proof: Lemma 2.74.
⟨1⟩2. If, for every open set U in X, every component of U is open in X, then X is locally connected.
⟨2⟩1. Assume: for every open set U in X, every component of U is open in X.
⟨2⟩2. Let: $a \in X$
⟨2⟩3. Let: U be a neighbourhood of a
⟨2⟩4. The component of U that contains a is a connected neighbourhood of a included in U.

□

Theorem 2.334. *If a space is locally path connected then its components and its path components are the same.*

PROOF:
⟨1⟩1. LET: X be a locally path connected space.
⟨1⟩2. LET: C be a component of X.
⟨1⟩3. LET: $x \in C$
⟨1⟩4. LET: P be the path component of x
PROVE: $P = C$
⟨1⟩5. $P \subseteq C$
PROOF: Proposition 2.322.
⟨1⟩6. LET: Q be the union of the other path components included in C
⟨1⟩7. $C = P \cup Q$
PROOF: Proposition 2.322.
⟨1⟩8. P and Q are open in C
⟨2⟩1. C is open.
PROOF: Theorem 2.328.
⟨2⟩2. Q.E.D.
PROOF: Theorem 2.333.
⟨1⟩9. $Q = \emptyset$
PROOF: Otherwise P and Q would form a separation of C.
□

Example 2.335. The ordered square is not locally path connected, since it is connected but not path connected.

Proposition 2.336. *Let X be a locally path-connected space. Then every connected open subspace of X is path-connected.*

PROOF:
⟨1⟩1. LET: U be a connected open subspace of X.
⟨1⟩2. LET: P be a path component of U.
⟨1⟩3. LET: Q be the union of the other path components of U.
⟨1⟩4. P and Q are open in U.
PROOF: Theorem 2.333.
⟨1⟩5. $Q = \emptyset$
PROOF: Otherwise P and Q form a separation of U.
□

2.59 Weak Local Connectedness

Definition 2.337 (Weakly Locally Connected)**.** Let X be a topological space and $a \in X$. Then X is *weakly locally connected* at a if and only if every neighbourhood of a includes a connected subspace that includes a neighbourhood of a.

Proposition 2.338. *Let X be a topological space. If X is weakly locally connected at every point then X is locally connected.*

PROOF:
$\langle 1 \rangle 1$. ASSUME: X is weakly locally connected at every point.
$\langle 1 \rangle 2$. LET: U be open in X.
$\langle 1 \rangle 3$. LET: C be a component of U.
$\langle 1 \rangle 4$. C is open in X.
  $\langle 2 \rangle 1$. LET: $x \in C$
  $\langle 2 \rangle 2$. PICK a connected subspace D of U that includes a neighbourhood V of x.
  $\langle 2 \rangle 3$. $D \subseteq C$
    PROOF: Lemma 2.315.
  $\langle 2 \rangle 4$. $x \in V \subseteq C$
  $\langle 2 \rangle 5$. Q.E.D.
    PROOF: Lemma 2.74.
$\langle 1 \rangle 5$. Q.E.D.
  PROOF: Theorem 2.328.
□

Example 2.339. The 'infinite broom' (Munkres p.163) is an example of a space that is weakly locally connected at a point p but not locally connected at p.

2.60 Quasicomponents

Proposition 2.340. *Let X be a topological space. Define $\sim$ on X by $x \sim y$ if and only if there exists no separation U and V of X such that $x \in U$ and $y \in V$. Then $\sim$ is an equivalence relation on X.*

PROOF:
$\langle 1 \rangle 1$. $\sim$ is reflexive.
  PROOF: Since the two sets that make up a separation are disjoint.
$\langle 1 \rangle 2$. $\sim$ is symmetric.
  PROOF: Immediate from the defintion.
$\langle 1 \rangle 3$. $\sim$ is transitive.
  $\langle 2 \rangle 1$. ASSUME: $x \sim y$ and $y \sim z$
  $\langle 2 \rangle 2$. ASSUME: for a contradiction there is a separation U and V of X with $x \in U$ and $z \in V$
  $\langle 2 \rangle 3$. $y \in U$ or $y \in V$
  $\langle 2 \rangle 4$. Q.E.D.
    PROOF: Either case contradicts $\langle 2 \rangle 1$.
□

Definition 2.341 (Quasicomponents)**.** For X a topological space, the *quasicomponents* of X are the equivalence classes under $\sim$.

Proposition 2.342. *Let X be a topological space. Then every component of X is included in a quasicomponent of X.*

PROOF:
⟨1⟩1. LET: C be a component of X.
⟨1⟩2. LET: $x, y \in C$
PROVE: $x \sim y$
⟨1⟩3. ASSUME: for a contradiction there exists a separation U and V of X with $x \in U$ and $y \in V$
⟨1⟩4. $C \cap U$ and $C \cap V$ form a separation of C.
⟨1⟩5. Q.E.D.
PROOF: This contradicts ⟨1⟩1.

Proposition 2.343. *In a locally connected space, the components and the quasicomponents are the same.*

PROOF:
⟨1⟩1. LET: X be a locally connected space and Q a quasicomponent of X.
⟨1⟩2. PICK a component C of X such that $C \subseteq Q$
⟨1⟩3. LET: D be the union of the components of X
⟨1⟩4. C and D are open in X.
PROOF: Theorem 2.328.
⟨1⟩5. D cannot contain any points of Q.
PROOF: If it did, then C and D would form a separation of X and there would be points $x, y \in Q$ with $x \in C$ and $y \in D$.
⟨1⟩6. $C = Q$
□

2.61 Open Coverings

Definition 2.344 (Open Covering)**.** Let X be a topological space. An *open covering* of X is a covering of X whose elements are all open sets.

2.62 Lindelöf Spaces

Definition 2.345 (Lindelöf Space)**.** A topological space X is *Lindelöf* if and only if every open covering has a countable subcovering.

Proposition 2.346. *Let X be a topological space. Then X is Lindelöf if and only if every set of closed sets that has the countable intersection property has nonempty intersection.*

PROOF: The following are equivalent.

1. X is compact.
2. Every open covering of X has a countable subcovering.
3. For any set $\mathcal{C}$ of closed sets, if $\{X \setminus C \mid C \in \mathcal{C}\}$ covers X then there is a countable subset $\mathcal{C}_0$ such that $\{X \setminus C \mid C \in \mathcal{C}_0\}$ covers X

4. For any set $\mathcal{C}$ of closed sets, if $\bigcap \mathcal{C} = \emptyset$ then there is a countable subset $\mathcal{C}_0$ with empty intersection.

5. Any set of closed sets with the countable intersection property has nonempty intersection.

∎

Proposition 2.347 (CC)**.** *Let X be a topological space and $\mathcal{B}$ a basis for the topology on X. Then the following are equivalent.*

1. *X is Lindelöf.*

2. *Every open covering of X by elements of $\mathcal{B}$ has a countable subcovering.*

PROOF:
⟨1⟩1. $1 \Rightarrow 2$
PROOF: Immediate from definitions.
⟨1⟩2. $2 \Rightarrow 1$
⟨2⟩1. ASSUME: Every open covering of X by elements of $\mathcal{B}$ has a countable subcovering.
⟨2⟩2. LET: $\mathcal{U}$ be an open covering of X.
⟨2⟩3. $\{B \in \mathcal{B} \mid \exists U \in \mathcal{U}. B \subseteq U\}$ covers X.
⟨2⟩4. PICK a finite subcovering $\mathcal{B}_0$.
⟨2⟩5. For $B \in BB$, PICK $U_B \in \mathcal{U}$ such that $B \subseteq U_B$
⟨2⟩6. $\{U_B \mid B \in \mathcal{B}_0\}$ covers X.
∎

2.63 The Second Countability Axiom

Definition 2.348 (Second Countability Axiom)**.** A topological space satisfies the *second countability axiom*, or is *second countable*, if and only if it has a countable basis.

Example 2.349. The space $\mathbb{R}$ is second countable.

PROOF: The set $\{(a, b) \mid a, b \in \mathbb{Q}\}$ is a basis. ∎

Proposition 2.350. *A subspace of a second countable space is second countable.*

PROOF: If $\mathcal{B}$ is a countable basis for X and $Y \subseteq X$ then $\{B \cap Y \mid B \in \mathcal{B}\}$ is a countable basis for Y. ∎

Proposition 2.351 (CC)**.** *Every second countable space is Lindelöf.*

PROOF: From Proposition 2.347.

Example 2.352 (CC)**.** The space $\mathbb{R}_l$ is Lindelöf.

⟨1⟩1. LET: $\mathcal{A}$ be a covering of $\mathbb{R}_l$ by basic open sets of the form $[a, b)$
⟨1⟩2. LET: $C = \bigcup\{(a, b) \mid [a, b) \in \mathcal{A}\}$

$\langle 1\rangle 3.$ $\mathbb{R} \setminus C$ is countable.

$\langle 2\rangle 1.$ For every $x \in \mathbb{R} \setminus C$, Pick a rational q_x such that $(x, q_x) \subseteq C$

$\langle 3\rangle 1.$ Let: $x \in \mathbb{R} \setminus C$

$\langle 3\rangle 2.$ Pick b such that $[x, b) \in \mathcal{A}$

$\langle 3\rangle 3.$ Pick a rational q such that $q \in (x, b)$

$\langle 2\rangle 2.$ The mapping $x \mapsto q_x$ is an injection $\mathbb{R} \setminus C \to \mathbb{Q}$

$\langle 1\rangle 4.$ Pick a countable $\mathcal{A}' \subseteq \mathcal{A}$ that covers $\mathbb{R} \setminus C$

$\langle 1\rangle 5.$ Under the standard topology on $\mathbb{R}$, C is second countable.

Proof: Proposition 2.350.

$\langle 1\rangle 6.$ Pick a countable $\mathcal{A}'' \subseteq \mathcal{A}$ such that $\{(a, b) \mid [a, b) \in \mathcal{A}''\}$ covers C.

Proof: Proposition 2.347.

$\langle 1\rangle 7.$ $\mathcal{A}' \cup \mathcal{A}''$ covers $\mathbb{R}_l$.

□

Example 2.353. The product of two Lindelöf spaces is not necessarily Lindelöf.
We prove that the Sorgenfrey plane is not Lindelöf.

Proof:

$\langle 1\rangle 1.$ Let: $L = \{(x, -x) \mid x \in \mathbb{R}\}$

$\langle 1\rangle 2.$ L is closed in $\mathbb{R}_l^2$

$\langle 1\rangle 3.$ Let: $\mathcal{U} = \{[a, b) \times [a, -d) \mid a, b, d \in \mathbb{R}\}$

$\langle 1\rangle 4.$ $\mathcal{U} \cup \{\mathbb{R} \setminus L\}$ covers $\mathbb{R}_l^2$

$\langle 1\rangle 5.$ Every element of $\mathcal{U}$ intersects L at exactly one point.

$\langle 1\rangle 6.$ No countable subset of $\mathcal{U}$ covers $\mathbb{R}_l^2$.

□

2.64 Compact Spaces

Definition 2.354 (Compact)**.** A topological space is *compact* if and only if every open covering has a finite subcovering.

Lemma 2.355. *Let X be a topological space and Y a subspace of X. Then Y is compact if and only if every covering of Y by sets open in X has a finite subcovering.*

Proof:

$\langle 1\rangle 1.$ If Y is compact then every covering of Y by sets open in X has a finite subcovering.

$\langle 2\rangle 1.$ Assume: Y is compact.

$\langle 2\rangle 2.$ Let: $\mathcal{U}$ be a covering of Y by sets open in X.

$\langle 2\rangle 3.$ $\{U \cap Y \mid U \in \mathcal{U}\}$ is an open covering of Y.

$\langle 2\rangle 4.$ Pick a finite subcovering $\{U_1 \cap Y, \ldots, U_n \cap Y\}$

$\langle 2\rangle 5.$ $\{U_1, \ldots, U_n\}$ is a finite subcovering of $\mathcal{U}$.

$\langle 1\rangle 2.$ If every covering of Y by sets open in X has a finite subcovering then Y is compact.

$\langle 2\rangle 1.$ Let: $\mathcal{U}$ be an open covering of Y.

$\langle 2\rangle 2.$ Let: $\mathcal{V} = \{V \text{ open in } X \mid V \cap Y \in \mathcal{U}\}$.

⟨2⟩3. $\mathcal{V}$ is a covering of Y by sets open in X.
⟨2⟩4. Pick a finite subcovering $\{V_1, \ldots, V_n\}$
⟨2⟩5. $\{V_1 \cap Y, \ldots, V_n \cap Y\}$ is a finite subcovering of $\mathcal{U}$.
□

Proposition 2.356. *Every closed subspace of a compact space is compact.*

Proof:
⟨1⟩1. Let: X be a compact space and $Y \subseteq X$ be closed.
⟨1⟩2. Let: $\mathcal{U}$ be a covering of Y by sets open in X.
⟨1⟩3. $\mathcal{U} \cup \{X \setminus Y\}$ is an open covering of X.
⟨1⟩4. Pick a finite subcovering $\mathcal{U}_0$
⟨1⟩5. $\mathcal{U}_0 \cap \mathcal{U}$ is a finite subset of $\mathcal{U}$ that covers Y.
□

Theorem 2.357. *The continuous image of a compact space is compact.*

Proof:
⟨1⟩1. Let: $f : X \twoheadrightarrow Y$ be continuous and surjective.
⟨1⟩2. Let: $\mathcal{V}$ be an open covering of Y
⟨1⟩3. $\{p^{-1}(V) \mid V \in \mathcal{V}\}$ is an open covering of X.
⟨1⟩4. Pick a finite subcovering $\{p^{-1}(V_1), \ldots, p^{-1}(V_n)\}$
⟨1⟩5. $\{V_1, \ldots, V_n\}$ covers Y.
□

Theorem 2.358. *Let A and B be compact subspaces of X and Y respectively. Let N be an open set in $X \times Y$ that includes $A \times B$. Then there exist open sets U and V in X and Y respectively such that $A \subseteq U$, $B \subseteq V$ and $U \times V \subseteq N$.*

Proof:
⟨1⟩1. For all $x \in A$, there exist neighbourhoods U of x and V of B such that $U \times V \subseteq N$.
⟨2⟩1. Let: $x \in A$
⟨2⟩2. For all $y \in B$, there exist neighbourhoods U of x and V of y such that $U \times V \subseteq N$
⟨2⟩3. $\{V \text{ open in } Y \mid \exists \text{ neighbourhood } U \text{ of } x, U \times V \subseteq N\}$ covers B.
⟨2⟩4. Pick a finite subcover $\{V_1, \ldots, V_n\}$
⟨2⟩5. For $i = 1, \ldots, n$, Pick a neighbourhood U_i of x such that $U_i \times V_i \subseteq N$
⟨2⟩6. Let: $U = U_1 \cap \cdots \cap U_n$
⟨2⟩7. Let: $V = V_1 \cup \cdots \cup V_n$
⟨2⟩8. U is a neighbourhood of x.
⟨2⟩9. V is a neighbourhood of B.
⟨2⟩10. $U \times V \subseteq N$
⟨1⟩2. $\{U \text{ open in } X \mid \exists \text{ neighbourhood } V \text{ of } B.U \times V \subseteq N\}$ covers A.
⟨1⟩3. Pick a finite subcover $\{U_1, \ldots, U_n\}$
⟨1⟩4. For $i = 1, \ldots, n$, Pick a neighbourhood V_i of B such that $U_i \times V_i \subseteq N$
⟨1⟩5. Let: $U = U_1 \cup \cdots \cup U_n$
⟨1⟩6. Let: $V = V_1 \cap \cdots \cap V_n$

⟨1⟩7. U and V are open.
⟨1⟩8. $A \subseteq U$
⟨1⟩9. $B \subseteq V$
⟨1⟩10. $U \times V \subseteq N$
□

Corollary 2.358.1 (Tube Lemma)**.** *Let X and Y be topological spaces with Y compact. Let $a \in X$ and N be an open set in $X \times Y$ that includes $\{a\} \times Y$. Then there exists a neighbourhood W of a such that N includes the* tube *$W \times Y$.*

Theorem 2.359. *Let X be a topological space. Then X is compact if and only if every set of closed sets that has the finite intersection property has nonempty intersection.*

Proof: The following are equivalent.

1. X is compact.
2. Every open covering of X has a finite subcovering.
3. For any set $\mathcal{C}$ of closed sets, if $\{X \setminus C \mid C \in \mathcal{C}\}$ covers X then there is a finite subset $\mathcal{C}_0$ such that $\{X \setminus C \mid C \in \mathcal{C}_0\}$ covers X
4. For any set $\mathcal{C}$ of closed sets, if $\bigcap \mathcal{C} = \emptyset$ then there is a finite subset $\mathcal{C}_0$ with empty intersection.
5. Any set of closed sets with the finite intersection property has nonempty intersection.

□

Corollary 2.359.1. *Let X be a topological space and $C_1 \supseteq C_2 \supseteq \cdots$ a nested sequence of nonempty closed sets. Then $\bigcap_n C_n$ is nonempty.*

Proposition 2.360. *Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on the same set X with $\mathcal{T} \subseteq \mathcal{T}'$. If $\mathcal{T}'$ is compact then $\mathcal{T}$ is compact.*

Proof:
⟨1⟩1. Let: $\mathcal{U} \subseteq \mathcal{T}$ cover X
⟨1⟩2. $\mathcal{U} \subseteq \mathcal{T}'$
⟨1⟩3. A finite subset of $\mathcal{U}$ covers X.
□

Corollary 2.360.1. *If $\mathcal{T}$ and $\mathcal{T}'$ are two compact Hausdorff topologies on the same set X, then either $\mathcal{T} = \mathcal{T}'$ or $\mathcal{T}$ and $\mathcal{T}'$ are incomparable.*

Proof: From the Proposition and Proposition 2.227. □

Example 2.361. Any set under the finite complement topology is compact.

Proposition 2.362. *Let X be a topological space. A finite union of compact subspaces of X is compact.*

PROOF:
⟨1⟩1. LET: A and B be compact subspaces of X.
⟨1⟩2. LET: $\mathcal{U}$ be a set of open sets in X that covers $A \cup B$
⟨1⟩3. PICK a finite subset $\mathcal{U}_1$ that covers A.
PROOF: Lemma 2.355.
⟨1⟩4. PICK a finite subset $\mathcal{U}_2$ that covers B.
PROOF: Lemma 2.355.
⟨1⟩5. $\mathcal{U}_1 \cup \mathcal{U}_2$ is a finite subset that covers $A \cup B$.
⟨1⟩6. Q.E.D.
PROOF: Lemma 2.355.
□

Proposition 2.363. *Let A and B be disjoint compact subspaces of the Hausdorff space X. Then there exist disjoint open sets U and V that include A and B respectively.*

PROOF: From Theorem 2.358 with $N = X^2 \setminus \{(x, x) \mid x \in X\}$. □

Corollary 2.363.1. *Every compact subspace of a Hausdorff space is closed.*

Theorem 2.364. *Let $f : X \to Y$ be a bijective continuous function. If X is compact and Y is Hausdorff then f is a homeomorphism.*

PROOF:
⟨1⟩1. LET: $C \subseteq X$ be closed.
⟨1⟩2. C is compact.
PROOF: Proposition 2.356.
⟨1⟩3. $f(C)$ is compact.
PROOF: Theorem 2.357.
⟨1⟩4. $f(C)$ is closed.
PROOF: Corollary 2.363.1.
⟨1⟩5. Q.E.D.
PROOF: Lemma 2.156.
□

Proposition 2.365. *Let X be a compact space, Y a Hausdorff space, and $f : X \to Y$ a continuous map. Then f is a closed map.*

PROOF:
⟨1⟩1. LET: $C \subseteq X$ be closed.
⟨1⟩2. C is compact.
PROOF: Proposition 2.356.
⟨1⟩3. $f(C)$ is compact.
PROOF: Theorem 2.357.
⟨1⟩4. $f(C)$ is closed.
PROOF: Corollary 2.363.1.
□

Proposition 2.366. *If Y is compact then the projection $\pi_1 : X \times Y \to X$ is a closed map.*

PROOF:
$\langle 1\rangle 1$. LET: $A \subseteq X \times Y$ be closed.
$\langle 1\rangle 2$. LET: $x \in X \setminus \pi_1(A)$
$\langle 1\rangle 3$. PICK a neighbourhood U of x such that $U \times Y \subseteq (X \times Y) \setminus A$
PROOF: By the Tube Lemma.
$\langle 1\rangle 4$. $x \in U \subseteq X \setminus \pi_1(A)$
$\langle 1\rangle 5$. Q.E.D.
PROOF: So $X \setminus \pi_1(A)$ is open by Lemma 2.74.
□

Theorem 2.367. *Let X be a topological space and Y a compact Hausdorff space. Let $f : X \to Y$ be a function. Then f is continuous if and only if the graph of f is closed in $X \times Y$.*

PROOF:
$\langle 1\rangle 1$. LET: G_f be the graph of f.
$\langle 1\rangle 2$. If f is continuous then G_f is closed.
$\langle 2\rangle 1$. ASSUME: f is continuous.
$\langle 2\rangle 2$. LET: $(x, y) \in (X \times Y) \setminus G_f$
$\langle 2\rangle 3$. PICK disjoint neighbourhoods U and V of y and $f(x)$ respectively.
$\langle 2\rangle 4$. $f^{-1}(V) \times U$ is a neighbourhood of (x, y) disjoint from G_f.
$\langle 1\rangle 3$. If G_f is closed then f is continuous.
$\langle 2\rangle 1$. ASSUME: G_f is closed.
$\langle 2\rangle 2$. LET: $x \in X$ and V be a neighbourhood of $f(x)$.
$\langle 2\rangle 3$. $G_f \cap (X \times (Y \setminus V))$ is closed.
$\langle 2\rangle 4$. $\pi_1(G_f \cap (X \times (Y \setminus V)))$ is closed.
PROOF: Proposition 2.366.
$\langle 2\rangle 5$. LET: $U = X \setminus \pi_1(G_f \cap (X \times (Y \setminus V)))$
$\langle 2\rangle 6$. U is a neighbourhood of x
$\langle 2\rangle 7$. $f(U) \subseteq V$
□

Theorem 2.368. *Let X be a compact topological space. Let $(f_n : X \to \mathbb{R})$ be a monotone increasing sequence of continuous functions and $f : X \to \mathbb{R}$ a continuous function. If (f_n) converges pointwise to f, then (f_n) converges uniformly to f.*

PROOF:
$\langle 1\rangle 1$. LET: $\epsilon > 0$
$\langle 1\rangle 2$. For all $x \in X$, there exists N such that, for all $n \geq N$, we have $|f_n(x) - f(x)| < \epsilon$
$\langle 1\rangle 3$. For $n \geq 1$,
LET: $U_n = \{x \in X \mid |f_n(x) - f(x)| < \epsilon\}$
$\langle 1\rangle 4$. For $n \geq 1$, we have U_n is open in X.
$\langle 2\rangle 1$. LET: $x \in X$
$\langle 2\rangle 2$. LET: $\delta = \epsilon - |f_n(x) - f(x)|$
$\langle 2\rangle 3$. PICK a neighbourhood U of x such that $f(U) \subseteq B(f(x), \delta/2)$

⟨2⟩4. Pick a neighbourhood V of x such that $f_n(V) \subseteq B(f_n(x), \delta/2)$
⟨2⟩5. $f(U \cap V) \subseteq U_n$
Proof: For $y \in U \cap V$ we have
$$\begin{aligned} |f_n(y) - f(y)| &\leq |f_n(y) - f_n(x)| + |f_n(x) - f(x)| + |f(x) - y(y)| \\ &< \delta/2 + |f_n(x) - f(x)| + \delta/2 \\ &= \epsilon \end{aligned}$$
⟨1⟩5. $\{U_n \mid n \geq 1\}$ covers X
Proof: From ⟨1⟩2
⟨1⟩6. Pick N such that $X = U_N$
⟨2⟩1. Pick $n_1, \ldots, n_k$ such that $U_{n_1}, \ldots, U_{n_k}$ cover X.
⟨2⟩2. Let: $N = \max(n_1, \ldots, n_k)$
⟨2⟩3. For all i we have $U_{n_i} \subseteq U_N$
Proof: Since (f_n) is monotone increasing.
⟨2⟩4. $X = U_N$
⟨1⟩7. For all $x \in X$ and $n \geq N$ we have $|f_n(x) - f(x)| < \epsilon$
□

An example to show that we cannot remove the hypothesis that X is compact:

Example 2.369. Let $X = (0, 1)$, $f_n(x) = -x^n$ and $f(x) = 0$ for $x \in X$ and $n \geq 1$. Then $f_n \to f$ pointwise and (f_n) is monotone increasing but the convergence is not uniform since, for all $N \geq 1$, there exists $x \in (0, 1)$ such that $-x^N < -1/2$.

An example to show that we cannot remove the hypothesis that (f_n) is monotone increasing:

Example 2.370. Let $X = [0, 1]$, $f_n(x) = 1/(n^3(x - 1/n)^2 + 1)$ and $f(x) = 0$ for $x \in X$ and $n \geq 1$. Then X is compact and $f_n \to f$ pointwise but the convergence is not uniform since, for all $N \geq 1$, there exists $x \in [0, 1]$ such that $f_N(x) = 1$, namely $x = 1/N$.

Theorem 2.371. *Let X be a compact Hausdorff space. Let $\mathcal{A}$ be a chain of closed connected subsets of X. Then $\bigcap \mathcal{A}$ is connected.*

Proof:
⟨1⟩1. Assume: for a contradiction C and D form a separation of $\bigcap \mathcal{A}$.
⟨1⟩2. Pick disjoint open sets U and V that include C and D respectively.
Proof: Proposition 2.363.
⟨1⟩3. $\{A \setminus (U \cup V) \mid A \in \mathcal{A}\}$ is a set of closed sets with the finite intersection property.
⟨2⟩1. For all $A \in \mathcal{A}$ we have $A \setminus (U \cup V)$ is closed.
⟨2⟩2. For all $A_1, \ldots, A_n \in \mathcal{A}$ we have $(A_1 \cap \cdots \cap A_n) \setminus (U \cup V)$ is nonempty.
Proof:
⟨3⟩1. Let: $A_1, \ldots, A_n \in \mathcal{A}$
⟨3⟩2. Assume: without loss of generality $A_1 \subseteq A_2, \ldots, A_n$
Proof: Since $\mathcal{A}$ is a chain.

⟨3⟩3. $A_1 \setminus (U \cup V)$ is nonempty

Proof: Otherwise $(A_1 \cap \cdots \cap A_n \cap U)$ and $(A_1 \cap \cdots \cap A_n \cap V)$ would form a separation of A_n.

⟨1⟩4. $\bigcap \mathcal{A} \setminus (U \cup V)$ is nonempty.

Proof: Theorem 2.359.

⟨1⟩5. Q.E.D.

Proof: This contradicts ⟨1⟩1 since $\bigcap AA \setminus (U \cup V) = \bigcap \mathcal{A} \setminus (C \cup D)$.

□

Theorem 2.372 (Tychonoff Theorem (AC)). *The product of a family of compact spaces is compact.*

Proof:

⟨1⟩1. Let: $\{X_\alpha\}_{\alpha \in J}$ be a family of compact spaces.

⟨1⟩2. Let: $X = \prod_{\alpha \in J} X_\alpha$

⟨1⟩3. For any $\mathcal{A} \subseteq \mathcal{P}X$, we have $\bigcap_{A \in \mathcal{A}} \overline{A} \neq \emptyset$

⟨2⟩1. Let: $\mathcal{A} \subseteq \mathcal{P}X$

⟨2⟩2. Pick $\mathcal{D} \supseteq \mathcal{A}$ that is maximal with respect to the finite intersection property.

Prove: $\bigcap_{D \in \mathcal{D}} \overline{D} \neq \emptyset$

Proof: Lemma 2.32.

⟨2⟩3. For $\alpha \in J$, Pick $x_\alpha \in X_\alpha$ such that $x_\alpha \in \bigcap_{D \in \mathcal{D}} \overline{\pi_\alpha(D)}$

Proof: Theorem 2.359 since $\{\overline{\pi_\alpha(D)} \mid D \in \mathcal{D}\}$ is a set of closed sets in X_α with the finite intersection property.

⟨2⟩4. Let: $x = (x_\alpha)_{\alpha \in J}$

Prove: $x \in \bigcap_{D \in \mathcal{D}} \overline{D}$

⟨2⟩5. For any $\beta \in J$ and neighbourhood U of x_β in X_β, we have ${\pi_\beta}^{-1}(U)$ intersects every element of $\mathcal{D}$

⟨3⟩1. Let: $\beta \in J$

⟨3⟩2. Let: U be a neighbourhood of x_β in X_β.

⟨3⟩3. Let: $D \in \mathcal{D}$

⟨3⟩4. $x_\beta \in \overline{\pi_\beta(D)}$

Proof: From ⟨2⟩3

⟨3⟩5. U intersects $\pi_\beta(D)$.

⟨3⟩6. ${\pi_\beta}^{-1}(U)$ intersects D.

⟨2⟩6. For any $\beta \in J$ and neighbourhood U of x_β in X_β, we have ${\pi_\beta}^{-1}(U) \in \mathcal{D}$

Proof: Lemma 2.34.

⟨2⟩7. Every basic neighbourhood of x is an element of $\mathcal{D}$

Proof: Lemma 2.33.

⟨2⟩8. Every basic neighbourhood of x intersects every element of $\mathcal{D}$

Proof: Since $\mathcal{D}$ satisfies the finite intersection property.

⟨2⟩9. For all $D \in \mathcal{D}$ we have $x \in \overline{D}$

⟨1⟩4. Q.E.D.

Proof: Theorem 2.359.

□

Lemma 2.373. *Let X and Y be topological spaces. Let $\mathcal{A}$ be a set of basis* *elements for the product topology on $X \times Y$ such that no finite subset of $\mathcal{A}$ covers $X \times Y$. If X is compact, then there exists $x \in X$ such that no finite subset of $\mathcal{A}$ covers the slice $\{x\} \times Y$.*

PROOF:
⟨1⟩1. ASSUME: for every $x \in X$, there exists a finite subset of $\mathcal{A}$ that covers $\{x\} \times Y$
PROVE: A finite subset of $\mathcal{A}$ covers $X \times Y$
⟨1⟩2. $\{U \mid \exists U_1 \times V_1, \ldots, U_n \times V_n \in \mathcal{A}. U = U_1 \cap \cdots \cap U_n, V_1 \cup \cdots \cup V_n = Y\}$ covers X
⟨1⟩3. PICK a finite subcover $U_1, \ldots, U_m$
⟨1⟩4. PICK $U_{ij} \times V_{ij} \in \mathcal{A}$ such that, for every i, we have $U_i = \bigcap_j U_{ij}$ and $Y = \bigcup_j V_{ij}$
⟨1⟩5. The collection of all $U_{ij} \times V_{ij}$ covers $X \times Y$
□

Theorem 2.374 (AC)**.** *Let X be a compact Hausdorff space. Then the quasicomponents and the components of X are the same.*

PROOF:
⟨1⟩1. LET: $x, y \in X$
⟨1⟩2. ASSUME: x and y are in the same quasicomponent.
PROVE: x and y are in the same component.
⟨1⟩3. LET: $\mathcal{A}$ be the set of all closed subsets A of X such that x and y are in the same quasicomponent of A.
⟨1⟩4. For every chain $\mathcal{B} \subseteq \mathcal{A}$ we have $\bigcap \mathcal{B} \in \mathcal{A}$
⟨2⟩1. LET: $BB \subseteq \mathcal{A}$ be a chain.
⟨2⟩2. ASSUME: for a contradiction U and V form a separation of $\bigcap \mathcal{B}$ with $x \in U$ and $y \in V$
⟨2⟩3. PICK disjoint open sets U', V' in X such that $U \subseteq U'$ and $V \subseteq V'$
⟨2⟩4. $\{B \setminus (U' \cup V') \mid B \in \mathcal{B}\}$ satisfies the finite intersection property.
⟨3⟩1. LET: $B_1, \ldots, B_n \in \mathcal{B}$
⟨3⟩2. ASSUME: without loss of generality $B_1 \subseteq \cdots \subseteq B_n$
PROOF: Since $\mathcal{B}$ is a chain.
⟨3⟩3. $\bigcap\{B_1 \setminus (U' \cup V'), \ldots, B_n \setminus (U' \cup V')\} = B_1 \setminus (U' \cup V')$
⟨3⟩4. $B_1 \setminus (U' \cup V')$ is nonempty
PROOF: Otherwise $B_1 \cap U'$ and $B_1 \cap V'$ would form a separation of B_1, contradicting the fact that x and y are in the same quasicomponent of B_1.
⟨2⟩5. $\bigcap \mathcal{B} \setminus (U \cup V)$ is nonempty
PROOF: Theorem 2.359.
⟨2⟩6. Q.E.D.
PROOF: This contradicts ⟨2⟩2.
⟨1⟩5. PICK a minimal element D in $\mathcal{A}$.
PROVE: D is connected.
PROOF: By Zorn's Lemma.
⟨1⟩6. ASSUME: for a contradiction U and V form a separation of D.

⟨1⟩7. ASSUME: without loss of generality $x, y \in U$

PROOF: We cannot have that one of x, y is in U and the other in V sicnce $D \in \mathcal{A}$.

⟨1⟩8. $U \in \mathcal{A}$

PROOF: If X and Y form a separation of U with $x \in X$ and $y \in Y$, then X and $Y \cup V$ form a separation of D with $x \in X$ and $y \in Y \cup V$.

⟨1⟩9. Q.E.D.

PROOF: There is a connected set D that contains both x and y.

□

PROOF:

⟨1⟩1. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of compact spaces.

⟨1⟩2. LET: $X = \prod_{\alpha \in J} X_\alpha$

⟨1⟩3. PICK a well-ordering $<$ on J such that J has a greatest element.

⟨1⟩4. For $\alpha \in J$ and $p = \{p_i \in X_i\}_{i \leq \alpha}$ a family of points,
LET: $Y(p) = \{x \in X \mid \forall i \leq \alpha . x_i = p_i\}$

⟨1⟩5. If $\alpha < \alpha'$ and p is an α'-indexed family of points then $Y(p) \subseteq Y(p \upharpoonright \alpha)$

PROOF: From definition.

⟨1⟩6. Given $\beta \in J$ and $p = \{p_i \in X_i\}_{i < \beta}$ a family of points,
LET: $Z(p) = \bigcap_{\alpha < \beta} Y(p \upharpoonright \alpha)$

⟨1⟩7. Given $\beta \in J$ and $p = \{p_i \in X_i\}_{i < \beta}$ a family of points, if $\mathcal{A}$ is a finite set of basic open spaces for X that covers $Z(p)$, then there exists $\alpha < \beta$ such that $\mathcal{A}$ covers $Y(p \upharpoonright \alpha)$

⟨2⟩1. ASSUME: without loss of generality β has no immediate predecessor.

⟨2⟩2. For $A \in \mathcal{A}$,
LET: $J_A = \{i < \beta \mid \pi_i(A) \neq X_i\}$

⟨2⟩3. LET: $\alpha = \max \bigcup_{A \in \mathcal{A}} J_A$

⟨2⟩4. LET: $x \in Y(p \upharpoonright \alpha)$

⟨2⟩5. LET: $y \in Z(p)$ be the point with $y_i = p_i$ for $i < \beta$ and $y_i = x_i$ for $i \geq \beta$

⟨2⟩6. PICK $A \in \mathcal{A}$ such that $y \in A$

PROOF: Since $\mathcal{A}$ covers $Z(p)$.

⟨2⟩7. For $i \in J_A$ we have $x_i \in \pi_i(A)$

PROOF: Since $i \leq \alpha$ so $x_i = p_i$

⟨2⟩8. For $i \in J \setminus J_A$ we have $x_i \in \pi_i(A)$

PROOF: Since $\pi_i(A) = X_i$

⟨2⟩9. $x \in A$

⟨1⟩8. ASSUME: for a contraction $\mathcal{A}$ is a set of basic open sets for X that covers X but such that no finite subset of $\mathcal{A}$ covers X

⟨1⟩9. PICK a set of points $\{p_i\}_{i \in J}$ such that, for all $\alpha \in J$, we have $Y(p \upharpoonright \alpha)$ is not finitely covered by $\mathcal{A}$

⟨2⟩1. ASSUME: as transfinite induction hypothesis $\alpha \in J$ and $\{p_i\}_{i < \alpha}$ is a family of points such that, for all $\alpha' < \alpha$, we have $Y(p \upharpoonright \alpha')$ is not finitely covered by $\mathcal{A}$

⟨2⟩2. $Z(p)$ is not finitely covered by $\mathcal{A}$

PROOF: By ⟨1⟩7.

⟨2⟩3. PICK $p_\alpha \in X_\alpha$ such that $Y(p)$ is not finitely covered by $\mathcal{A}$

PROOF: By Lemma 2.373 since there is a homeomorphism $\phi : Z(p) \cong X_\alpha \times \prod_{\alpha' > \alpha} X_{\alpha'}$ and, given p_α, this homemorphism ϕ restricts to a homeomorphism $Y(p) \cong \{p_\alpha\} \times \prod_{\alpha' > \alpha} X_{\alpha'}$.

⟨1⟩10. Q.E.D.

PROOF: If ω is the greatest element of J then $Y(p \upharpoonright \omega)$ is a singleton.

□

Theorem 2.375. *Every complete linearly ordered set in the order topology is compact.*

PROOF:

⟨1⟩1. LET: X be a complete linearly ordered set with least element a and greatest element b.

⟨1⟩2. LET: $\mathcal{A}$ be an open covering of X.

⟨1⟩3. For all $x < b$, there exists $y > x$ such that $[x, y]$ can be covered by at most two elements of $\mathcal{A}$.

⟨2⟩1. LET: $x \in X$

⟨2⟩2. PICK $A \in \mathcal{A}$ with $x \in A$

⟨2⟩3. PICK $y > x$ such that $[x, y) \subseteq A$

⟨2⟩4. PICK $B \in \mathcal{A}$ with $y \in B$

⟨2⟩5. $[x, y]$ is covered by A and B

⟨1⟩4. LET: $C = \{y \in X \mid [a, y]$ can be covered by finitely many elements of $\mathcal{A}\}$

⟨1⟩5. LET: $c = \sup C$

⟨1⟩6. $c > a$

⟨2⟩1. PICK $x > a$ such that $[a, x]$ can be covered by at most two elements of $\mathcal{A}$.

PROOF: From ⟨1⟩3.

⟨2⟩2. $x \in C$

⟨1⟩7. $c \in C$

⟨2⟩1. PICK $A \in \mathcal{A}$

⟨2⟩2. PICK $x < c$ such that $(x, c] \subseteq A$

⟨2⟩3. PICK $y > x$ such that $y \in C$

⟨2⟩4. PICK $\mathcal{A}_0 \subseteq^{\text{fin}} \mathcal{A}$ that covers $[a, y]$

⟨2⟩5. $\mathcal{A}_0 \cup \{A\}$ covers $[a, c]$

⟨1⟩8. $c = b$

⟨2⟩1. ASSUME: for a contradiction $c < b$

⟨2⟩2. PICK $x > c$ such that $[c, x]$ can be covered by at most two elements of $\mathcal{A}$

PROOF: From ⟨1⟩3.

⟨2⟩3. $[a, x]$ can be finitely covered by $\mathcal{A}$

PROOF: From ⟨1⟩7.

⟨2⟩4. Q.E.D.

PROOF: This contradicts the maximality of c.

□

Corollary 2.375.1. *Let X be a linearly ordered set with the least upper bound property. Then every closed interval in X is compact.*

Corollary 2.375.2. *Every closed interval in $\mathbb{R}$ is compact.*

Theorem 2.376 (Extreme Value Theorem)**.** *Any linearly ordered set under the order topology that is compact has a greatest and a least element.*

Proof:
$\langle 1 \rangle 1$. Let: X be a linearly ordered set under the order topology that is compact.
$\langle 1 \rangle 2$. X has a greatest element.
  $\langle 2 \rangle 1$. Assume: for a contradiction X has no greatest element.
  $\langle 2 \rangle 2$. $\{(-\infty, a) \mid a \in X\}$ covers X.
  $\langle 2 \rangle 3$. Pick a finite subcover $\{(-\infty, a_1), \ldots, (-\infty, a_n)\}$, say.
  $\langle 2 \rangle 4$. Assume: without loss of generality $a_1 \leq \cdots \leq a_n$
  $\langle 2 \rangle 5$. $X \subseteq (-\infty, a_n)$
  $\langle 2 \rangle 6$. $a_n < a_n$
$\langle 1 \rangle 3$. X has a least element.
  Proof: Similar.
□

2.65 Perfect Maps

Definition 2.377 (Perfect Map)**.** Let X and Y be topological spaces and $f : X \to Y$. Then f is a *perfect map* if and only if f is a closed map, continuous, surjective and $p^{-1}(y)$ is compact for all $y \in Y$.

Proposition 2.378. *Let X be a topological space, Y a compact space, and $p : X \to Y$ a closed map such that, for all $y \in Y$, we have $p^{-1}(y)$ is compact. Then X is compact.*

Proof:
$\langle 1 \rangle 1$. Let: $\mathcal{A}$ be a set of closed sets in X with the finite intersection property.
$\langle 1 \rangle 2$. $\mathcal{B} = \{p(A_1 \cap \cdots \cap A_n) \mid A_1, \ldots, A_n \in \mathcal{A}\}$ is a set of closed sets in Y with the finite intersection property.
  Proof: Since p is a closed map.
$\langle 1 \rangle 3$. Pick $y \in \bigcap \mathcal{B}$
  Proof: Theorem 2.359 since Y is compact.
$\langle 1 \rangle 4$. $\{A \cap p^{-1}(y) \mid A \in \mathcal{A}\}$ is a set of closed sets in $p^{-1}(y)$ with the finite intersection property.
$\langle 1 \rangle 5$. Pick $x \in \bigcap_{A \in \mathcal{A}} (A \cap p^{-1}(y))$
  Proof: Theorem 2.359 since $p^{-1}(y)$ is compact.
$\langle 1 \rangle 6$. $x \in \bigcap \mathcal{A}$
$\langle 1 \rangle 7$. Q.E.D.
  Proof: Theorem 2.359.
□

2.66 Topological Groups

Definition 2.379 (Topological Group)**.** A *topological group* G consists of a T_1 space G and continuous maps $\cdot : G^2 \to G$ and $(\ )^{-1} : G \to G$ such that $(G, \cdot, (\ )^{-1})$ is a group.

Example 2.380.

1. The integers $\mathbb{Z}$ under addition are a topological group.
2. The real numbers $\mathbb{R}$ under addition are a topological group.
3. The positive reals under multiplication are a topological group.
4. The set $\{z \in \mathbb{C} \mid |z| = 1\}$ under multiplication and given the topology of S^1 is a topological group.
5. For any $n \geq 0$, the general linear group $GL_n(\mathbb{R})$ is a topological group under matrix multiplication, considered as a subspace of $\mathbb{R}^{n^2}$.

Lemma 2.381. *Let G be a T_1 space and $\cdot : G^2 \to G$, $(\ )^{-1} : G \to G$ be functions such that $(G, \cdot, (\ )^{-1})$ is a group. Then G is a topological group if and only if the function $f : G^2 \to G$ that maps (x, y) to xy^{-1} is continuous.*

PROOF:
⟨1⟩1. If G is a topological group then f is continuous.
  PROOF: From Theorem 2.145.
⟨1⟩2. If f is continuous then G is a topological group.
  ⟨2⟩1. ASSUME: f is continuous.
  ⟨2⟩2. $(\ )^{-1}$ is continuous.
    PROOF: Since $x^{-1} = f(e, x)$.
  ⟨2⟩3. $\cdot$ is continuous.
    PROOF: Since $xy = f(x, y^{-1})$.
□

Lemma 2.382. *Let G be a topological group and H a subgroup of G. Then H is a topological group under the subspace topology.*

PROOF:
⟨1⟩1. H is T_1.
  PROOF: From Proposition 2.215.
⟨1⟩2. multiplication and inverse on H are continuous.
  PROOF: From Theorem 2.146.
□

Lemma 2.383. *Let G be a topological group and H a subgroup of G. Then $\overline{H}$ is a subgroup of G.*

PROOF:
⟨1⟩1. LET: $x, y \in \overline{H}$
    PROVE: $xy^{-1} \in \overline{H}$

$\langle 1\rangle 2$. LET: U be any neighbourhood of xy^{-1}
$\langle 1\rangle 3$. LET: $f : G^2 \to G$, $f(a,b) = ab^{-1}$
$\langle 1\rangle 4$. $f^{-1}(U)$ is a neighbourhood of (x,y)
$\langle 1\rangle 5$. PICK neighbourhoods V, W of x and y respectively such that $f(V \times W) \subseteq U$.
$\langle 1\rangle 6$. PICK $a \in V \cap H$ and $b \in W \cap H$
PROOF: Theorem 2.96.
$\langle 1\rangle 7$. $ab^{-1} \in U \cap H$
$\langle 1\rangle 8$. Q.E.D.
PROOF: By Theorem 2.96.
□

Proposition 2.384. *Let G be a topological group and $\alpha \in G$. Then the maps $l_\alpha, r_\alpha : G \to G$ defined by $l_\alpha(x) = \alpha x$, $r_\alpha(x) = x\alpha$ are homeomorphisms of G with itself.*

PROOF: They are continuous with continuous inverses $l_{\alpha^{-1}}$ and $r_{\alpha^{-1}}$. □

Corollary 2.384.1. *Every topological group is homogeneous.*

PROOF: Given a topological group G and $a, b \in G$, we have $l_{ba^{-1}}$ is a homeomorphism that maps a to b. □

Proposition 2.385. *Let G be a topological group and H a subgroup of G. Give G/H the quotient topology. For all $\alpha \in G$, the map $\overline{f_\alpha}$ that sends xH to αxH is a homeomorphism $G/H \cong G/H$.*

PROOF:
$\langle 1\rangle 1$. $\overline{f_\alpha}$ is well-defined.
PROOF: If $xy^{-1} \in H$ then $(\alpha x)(\alpha y)^{-1} = xy^{-1} \in H$.
$\langle 1\rangle 2$. $\overline{f_\alpha}$ is continuous.
PROOF: Theorem 2.249 since $\overline{f_\alpha} \circ p = p \circ f_\alpha$ is continuous, where $p : G \twoheadrightarrow G/H$ is the canonical surjection.
$\langle 1\rangle 3$. $\overline{f_\alpha}^{-1}$ is continuous.
PROOF: Similar since $\overline{f_\alpha}^{-1} = \overline{f_{\alpha^{-1}}}$.
□

Corollary 2.385.1. *Let G be a topological group and H a subgroup of G. Give G/H the quotient topology. Then G/H is homogeneous.*

Proposition 2.386. *Let G be a topological group and H a subgroup of G. Give G/H the quotient topology. If H is closed in G then G/H is T_1.*

PROOF:
$\langle 1\rangle 1$. LET: $p : G \twoheadrightarrow G/H$ be the canonical surjection
$\langle 1\rangle 2$. LET: $x \in G$
$\langle 1\rangle 3$. $p^{-1}(xH) = f_x(H)$
$\langle 1\rangle 4$. $p^{-1}(xH)$ is closed in G
PROOF: Since H is closed and f_x is a homemorphism of G with itself.

$\langle 1 \rangle 5$. $\{xH\}$ is closed in G/H
$\square$

Proposition 2.387. *Let G be a topological group and H a subgroup of G. Give G/H the quotient topology. Then the canonical surjection $p : G \twoheadrightarrow G/H$ is an open map.*

PROOF:
$\langle 1 \rangle 1$. LET: $U \subseteq G$ be open.
$\langle 1 \rangle 2$. $p^{-1}(p(U)) = \bigcup_{h \in H} r_h(U)$
$\langle 1 \rangle 3$. $p^{-1}(p(U))$ is open.
$\langle 1 \rangle 4$. $p(U)$ is open.
$\square$

Proposition 2.388. *Let G be a topological group and H a closed normal subgroup of G. Then G/H is a topological group under the quotient topology.*

PROOF:
$\langle 1 \rangle 1$. G/H is T_1
  PROOF: Proposition 2.386.
$\langle 1 \rangle 2$. The map $\overline{m} : (xH, yH) \mapsto xy^{-1}H$ is continuous.
  $\langle 2 \rangle 1$. $p^2 : G^2 \to (G/H)^2$ is a quotient map.
    PROOF: Propositions 2.248, 2.387.
  $\langle 2 \rangle 2$. $\overline{m} \circ p^2$ is continuous.
    PROOF: As it is $p^2 \circ m$ where $m : G^2 \to G$ with $m(x, y) = xy^{-1}$
$\square$

Lemma 2.389. *Let G be a topological group and $A, B \subseteq G$. If either A or B is open then AB is open.*

PROOF: If A is open we have $AB = \bigcup_{b \in B} r_b(A)$. Similarly if B is open. $\square$

Definition 2.390 (Symmetric Neighbourhood)**.** Let G be a topological group. A neighbourhood V of e is *symmetric* if and only if $V = V^{-1}$.

Lemma 2.391. *Let G be a topological group. Let V be a neighbourhood of e. Then V is symmetric if and only if, for all $x \in V$, we have $x^{-1} \in V$.*

PROOF:
$\langle 1 \rangle 1$. If V is symmetric then, for all $x \in V$, we have $x^{-1} \in V$
  PROOF: Immediate from defintions.
$\langle 1 \rangle 2$. If, for all $x \in V$, we have $x^{-1} \in V$, then V is symmetric.
  $\langle 2 \rangle 1$. ASSUME: for all $x \in V$ we have $x^{-1} \in V$
  $\langle 2 \rangle 2$. $V \subseteq V^{-1}$
    PROOF: If $x \in V$ then there exists $y \in V$ such that $x = y^{-1}$, namely $y = x^{-1}$
  $\langle 2 \rangle 3$. $V^{-1} \subseteq V$
    PROOF: Immediate from $\langle 2 \rangle 1$.
$\square$

Lemma 2.392. *Let G be a topological group. For every neighbourhood U of e, there exists a symmetric neighbourhood V of e such that $V^2 \subseteq U$.*

PROOF:
⟨1⟩1. LET: U be a neighbourhood of e.
⟨1⟩2. PICK a neighbourhood V' of e such that $V'V' \subseteq U$
PROOF: Such a neighbourhood exists because multiplication in G is continuous.
⟨1⟩3. PICK a neighbourhood W of e such that $WW^{-1} \subseteq V'$
PROOF: Such a neighbourhood exists because the function that maps (x, y) to xy^{-1} is continuous.
⟨1⟩4. LET: $V = WW^{-1}$
⟨1⟩5. V is a neighbourhood of e
⟨2⟩1. $e \in V$
PROOF: Since $e \in W$ so $e = ee^{-1} \in V$.
⟨2⟩2. V is open
PROOF: Lemma 2.389.
⟨1⟩6. V is symmetric
⟨2⟩1. For all $x \in V$ we have $x^{-1} \in V$
⟨3⟩1. LET: $x \in V$
⟨3⟩2. PICK$y, z \in W$ such that $x = yz^{-1}$
⟨3⟩3. $x^{-1} = zy^{-1}$
⟨3⟩4. $x^{-1} \in V$
⟨3⟩5. $x \in V^{-1}$
⟨2⟩2. Q.E.D.
PROOF: Lemma 2.391
⟨1⟩7. $V^2 \subseteq U$
PROOF: We have $V^2 \subseteq (V')^2 \subseteq U$
□

Proposition 2.393. *Every topological group is Hausdorff.*

PROOF:
⟨1⟩1. LET: G be a topological group.
⟨1⟩2. LET: $x, y \in G$ with $x \neq y$
⟨1⟩3. LET: $U = G \setminus \{x[^{-1}y]\}$
⟨1⟩4. PICK a symmetric neighbourhood V of e such that $VV \subseteq U$
⟨2⟩1. U is open
PROOF: Since G is T_1.
⟨2⟩2. $e \in U$
PROOF: Since $x \neq y$
⟨2⟩3. Q.E.D.
PROOF: Lemma 2.392.
⟨1⟩5. Vx and Vy are disjoint neighbourhoods of x and y respectively.
⟨2⟩1. Vx is open
PROOF: Since $Vx = r_x(V)$
⟨2⟩2. Vy is open

PROOF: Similar.

⟨2⟩3. $Vx \cap Vy = \emptyset$

⟨3⟩1. ASSUME: for a contradiction $z \in Vx \cap Vy$

⟨3⟩2. PICK $a, b \in V$ such that $z = ax = by$

⟨3⟩3. $xy^{-1} \in VV$

PROOF: Since $xy^{-1} = a^{-1}b$

⟨3⟩4. $xy^{-1} \in U$

⟨3⟩5. Q.E.D.

PROOF: From ⟨1⟩3.

□

Proposition 2.394. *Every topological group is regular.*

PROOF:

⟨1⟩1. LET: G be a topological group.

⟨1⟩2. LET: $A \subseteq G$ be a closed set and $a \notin A$.

⟨1⟩3. LET: $U = G \setminus Aa^{-1}$

⟨1⟩4. PICK a symmetric neighbourhood V of e such that $VV \subseteq U$

⟨2⟩1. U is open

PROOF: Since $Aa^{-1} = r_{a^{-1}}(A)$ is closed.

⟨2⟩2. $e \in U$

PROOF: Since $a \notin A$.

⟨2⟩3. Q.E.D.

PROOF: Lemma 2.392.

⟨1⟩5. VA and Va are disjoint open sets with $A \subseteq VA$ and $a \in Va$

⟨2⟩1. VA is open

PROOF: Lemma 2.389

⟨2⟩2. Va is open

PROOF: Lemma 2.389

⟨2⟩3. $VA \cap Va = \emptyset$

⟨3⟩1. ASSUME: for a contradiction $z \in VA \cap Va$

⟨3⟩2. PICK $b, c \in V$ and $d \in A$ with $z = bd = ca$

⟨3⟩3. $da^{-1} \in U$

PROOF: Since $da^{-1} = b^{-1}c \in VV \subseteq U$

⟨3⟩4. Q.E.D.

PROOF: This contradicts ⟨1⟩3

□

Proposition 2.395. *Let G be a topological group and H a subgroup of G. Give G/H the quotient topology. If H is closed in G then G/H is regular.*

PROOF:

⟨1⟩1. LET: $p : G \twoheadrightarrow G/H$ be the canonical surjection.

⟨1⟩2. LET: A be a closed set in G/H and $aH \in (G/H) \setminus A$.

⟨1⟩3. LET: $B = p^{-1}(A)$

⟨1⟩4. B is a closed saturated set in G.

⟨1⟩5. $B \cap aH = \emptyset$

⟨1⟩6. $B = BH$
⟨1⟩7. PICK a symmetric neighbourhood V of e such that VB does not intersect Va
⟨2⟩1. LET: $U = G \setminus Ba^{-1}$
⟨2⟩2. PICK a symmetric neighbourhood V of e such that $VV \subseteq U$
⟨3⟩1. U is open
PROOF: Since $Ba^{-1} = r_{a^{-1}}(B)$ is closed.
⟨3⟩2. $e \in U$
PROOF: If $e \in Ba^{-1}$ then $a \in B$
⟨3⟩3. Q.E.D.
PROOF: Lemma 2.392
⟨2⟩3. $VB \cap Va = \emptyset$
PROOF: If $vb = v'a$ for $v, v' \in V$ and $b \in B$ then we have $ba^{-1} = v^{-1}v' \in Ba \cap VV \subseteq Ba \cap U$.
⟨1⟩8. $p(VB)$ and $p(Va)$ are disjoint open sets
⟨2⟩1. $p(VB)$ and $p(Va)$ are open.
PROOF: Proposition 2.387.
⟨2⟩2. $p(VB) \cap p(Va) = \emptyset$
PROOF: If $vbH = v'aH$ for $v, v' \in V$, $b \in B$ then $v'a = vbh$ for some $h \in H$. Hence $v'a \in Va \cap VBH = Va \cap VB$.
⟨1⟩9. $A \subseteq p(VB)$
⟨1⟩10. $aH \in p(Va)$
□

Proposition 2.396. *Let G be a topological group. The component of G that contains e is a normal subgroup of G.*

PROOF:
⟨1⟩1. LET: C be the component of G that contains e.
⟨1⟩2. For all $x \in G$, xC is the component of G that contains x.
⟨2⟩1. LET: $x \in G$
⟨2⟩2. LET: D be the component of G that contains x.
⟨2⟩3. $xC \subseteq D$
PROOF: Since xC is connected by Theorem 2.273.
⟨2⟩4. $D \subseteq xC$
PROOF: Since $x^{-1}D \subseteq C$ similarly.
⟨1⟩3. For all $x \in G$, Cx is the component of G that contains x.
PROOF: Similar.
⟨1⟩4. For all $x \in C$ we have $xC = Cx = C$
⟨1⟩5. For all $x \in C$ we have $x^{-1}C = C$
⟨1⟩6. For all $x \in C$ we have $x^{-1} \in C$
⟨1⟩7. For all $x, y \in C$ we have $xy \in C$
PROOF: Since $xyC = xC = x$.
⟨1⟩8. For all $x \in G$ we have $xC = Cx$.
PROOF: From ⟨1⟩2 and ⟨1⟩3.
□

Lemma 2.397. *Let G be a topological group. Let A be a closed set in G and B*

a compact subspace of G such that $A \cap B = \emptyset$. Then there exists a symmetric neighbourhood U of e such that $AU \cap BU = \emptyset$.

PROOF:
$\langle 1 \rangle 1$. For all $b \in B$ there exists a symmetric neighbourhood V of e such that $bV^2 \cap A = \emptyset$
  $\langle 2 \rangle 1$. LET: $b \in B$
  $\langle 2 \rangle 2$. LET: $W = b^{-1}(G \setminus A)$
  $\langle 2 \rangle 3$. W is a neighbourhood of e and $bW \cap A = \emptyset$
  $\langle 2 \rangle 4$. PICK a symmetric neighbourhood V of e such that $V^2 \subseteq W$
$\langle 1 \rangle 2$. $\{bV^2 \mid b \in B, V \text{ is a symmetric neighbourhood of } e, bV^2 \cap A = \emptyset\}$ is an open cover of B
$\langle 1 \rangle 3$. PICK a finite subcover $b_1 V_1^2, \ldots, b_n V_n^2$, say.
$\langle 1 \rangle 4$. LET: $U = V_1 \cap \cdots \cap V_n$
$\langle 1 \rangle 5$. $BU^2 \cap A = \emptyset$
$\langle 1 \rangle 6$. $AU \cap BU = \emptyset$
  PROOF: If $av \in BU$ where $a \in A$ and $v \in V$ then $a = avv^{-1} \in BU^2 \cap A$.
□

Proposition 2.398 (AC). *Let G be a topological group. Let A be a closed set in G, and B a compact subspace of G. Then AB is closed.*

PROOF:
$\langle 1 \rangle 1$. LET: $x \in G \setminus AB$
$\langle 1 \rangle 2$. $A^{-1}x \cap B = \emptyset$
$\langle 1 \rangle 3$. $A^{-1}x$ is closed.
$\langle 1 \rangle 4$. PICK a symmetric neighbourhood U of e such that $A^{-1}xU \cap BU = \emptyset$
$\langle 1 \rangle 5$. xU^2 is open
  PROOF: Lemma 2.389.
$\langle 1 \rangle 6$. $x \in xU^2 \subseteq G \setminus AB$
□

Corollary 2.398.1. *Let G be a topological group and $H \leq G$. Let $p : G \twoheadrightarrow G/H$ be the quotient map. If H is compact then p is a closed map.*

PROOF: For A closed in G, we have $p^{-1}(p(A)) = AH$ is closed, and so $p(A)$ is closed. □

Corollary 2.398.2. *Let G be a topological group and $H \leq G$. If H and G/H are compact then G is compact.*

PROOF: From Proposition 2.378 since, for all $aH \in G/H$, we have $p^{-1}(aH) = aH$ is compact because it is homemorphic to H. □

2.67 The Metric Topology

Definition 2.399 (Metric)**.** Let X be a set. A *metric* on X is a function $d : X^2 \to \mathbb{R}$ such that:

1. For all $x, y \in X$, $d(x, y) \geq 0$
2. For all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$
3. For all $x, y \in X$, $d(x, y) = d(y, x)$
4. (*Triangle Inequality*) For all $x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$

We call $d(x, y)$ the *distance* between x and y.

Definition 2.400 (Open Ball)**.** Let X be a metric space. Let $a \in X$ and $\epsilon > 0$. The *open ball* with *centre* a and *radius* ϵ is

$$B(a, \epsilon) = \{x \in X \mid d(a, x) < \epsilon\} \ .$$

Definition 2.401 (Metric Topology)**.** Let X be a metric space. The *metric topology* on X is the topology generated by the basis consisting of all the open balls.

We prove this is a basis for a topology.

PROOF:
⟨1⟩1. For every point a, there exists a ball B such that $a \in B$
PROOF: We have $a \in B(a, 1)$.
⟨1⟩2. For any balls B_1, B_2 and point $a \in B_1 \cap B_2$, there exists a ball B_3 such that $a \in B_3 \subseteq B_1 \cap B_2$
⟨2⟩1. LET: $B_1 = B(c_1, \epsilon_1)$ and $B_2 = B(c_2, \epsilon_2)$
⟨2⟩2. LET: $\delta = \min(\epsilon_1 - d(c_1, a), \epsilon_2 - d(c_2, a))$
PROVE: $B(a, \delta) \subseteq B_1 \cap B_2$
⟨2⟩3. LET: $x \in B(a, \delta)$
⟨2⟩4. $x \in B_1$
PROOF:

$$\begin{aligned} d(x, c_1) &= d(x, a) + d(a, c_1) \\ &< \delta + d(a, c_1) \\ &\leq \epsilon_1 \end{aligned}$$

⟨2⟩5. $x \in B_2$
PROOF: Similar.
$\square$

Proposition 2.402. *Let X be a metric space and $U \subseteq X$. Then U is open if and only if, for every $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$.*

PROOF:
⟨1⟩1. If U is open then, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$.
⟨2⟩1. ASSUME: U is open.

⟨2⟩2. LET: $x \in U$
⟨2⟩3. PICK $a \in X$ and $\delta > 0$ such that $x \in B(a, \delta) \subseteq U$
⟨2⟩4. LET: $\epsilon = \delta - d(a, x)$
PROVE: $B(x, \epsilon) \subseteq U$
⟨2⟩5. LET: $y \in B(x, \epsilon)$
⟨2⟩6. $d(y, a) < \delta$
PROOF:

$$\begin{aligned} d(y,a) &\leq d(a,x) + d(x,y) \\ &< \delta + d(x,y) \\ &= \epsilon \end{aligned}$$

⟨2⟩7. $y \in U$
⟨1⟩2. If, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$, then U is open.
PROOF: Immediate from definitions.
□

Definition 2.403 (Discrete Metric)**.** Let X be a set. The *discrete metric* on X is defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Proposition 2.404. *The discrete metric induces the discrete topology.*

PROOF: For any (open) set U and point $a \in U$, we have $a \in B(a, 1) \subseteq U$. □

Definition 2.405 (Standard Metric on $\mathbb{R}$)**.** The *standard metric* on $\mathbb{R}$ is defined by $d(x, y) = |x - y|$.

Proposition 2.406. *The standard metric on* $\mathbb{R}$ *induces the standard topology on* $\mathbb{R}$.

PROOF:
⟨1⟩1. Every open ball is open in the standard topology on $\mathbb{R}$.
PROOF: $B(a, \epsilon) = (a - \epsilon, a + \epsilon)$
⟨1⟩2. For every open set U and point $a \in U$, there exists $\epsilon > 0$ such that $B(a, \epsilon) \subseteq U$
⟨2⟩1. LET: U be an open set and $a \in U$
⟨2⟩2. PICK an open interval b, c such that $a \in (b, c) \subseteq U$
⟨2⟩3. LET: $\epsilon = \min(a - b, c - a)$
⟨2⟩4. $B(a, \epsilon) \subseteq U$
□

Definition 2.407 (Metrizable)**.** A topological space X is *metrizable* if and only if there exists a metric on X that induces the topology.

Definition 2.408 (Bounded)**.** Let X be a metric space and $A \subseteq X$. Then A is *bounded* if and only if there exists M such that, for all $x, y \in A$, we have $d(x, y) \leq M$.

Definition 2.409 (Diameter)**.** Let X be a metric space and $A \subseteq X$. The *diameter* of A is

$$\operatorname{diam} A = \sup_{x,y \in A} d(x,y) \enspace .$$

Definition 2.410 (Standard Bounded Metric)**.** Let d be a metric on X. The *standard bounded metric* corresponding to d is the metric $\overline{d}$ defined by

$$\overline{d}(x,y) = \min(d(x,y), 1) \enspace .$$

We prove this is a metric.

PROOF:
$\langle 1 \rangle 1$. $\overline{d}(x,y) \geq 0$
  PROOF: Since $d(x,y) \geq 0$
$\langle 1 \rangle 2$. $\overline{d}(x,y) = 0$ if and only if $x = y$
  PROOF: $\overline{d}(x,y) = 0$ if and only if $d(x,y) = 0$ if and only if $x = y$
$\langle 1 \rangle 3$. $\overline{d}(x,y) = \overline{d}(y,x)$
  PROOF: Since $d(x,y) = d(y,x)$
$\langle 1 \rangle 4$. $\overline{d}(x,z) \leq \overline{d}(x,y) + \overline{d}(y,z)$
  PROOF:

$$\begin{aligned} \overline{d}(x,y) + \overline{d}(y,z) &= \min(d(x,y),1) + \min(d(y,z),1) \\ &= \min(d(x,y) + d(y,z), d(x,y)+1, d(y,z)+1, 2) \\ &\geq \min(d(x,z),1) \\ &= \overline{d}(x,z) \end{aligned}$$

□

Lemma 2.411. *In any metric space X, the set $\mathcal{B} = \{B(a,\epsilon) \mid a \in X, \epsilon < 1\}$ is a basis for the metric topology.*

PROOF:
$\langle 1 \rangle 1$. Every element of $\mathcal{B}$ is open.
  PROOF: From Lemma 2.111.
$\langle 1 \rangle 2$. For every open set U and point $a \in U$, there exists $B \in \mathcal{B}$ such that $a \in B \subseteq U$
  $\langle 2 \rangle 1$. LET: U be an open set and $a \in U$
  $\langle 2 \rangle 2$. PICK$\epsilon > 0$ such that $B(a,\epsilon) \subseteq U$
  $\langle 2 \rangle 3$. $B(a, \min(\epsilon, 1/2)) \subseteq U$
$\langle 1 \rangle 3$. Q.E.D.
  PROOF: Lemma 2.112.
□

Proposition 2.412. *Let d be a metric on the set X. Then the standard bounded metric $\overline{d}$ induces the same metric as d.*

PROOF: This follows from Lemma 2.411 since the open balls with radius < 1 are the same under both metrics. □

Lemma 2.413. *Let d and d' be two metrics on the same set X. Let $\mathcal{T}$ and $\mathcal{T}'$ be the topologies they induce. Then $\mathcal{T} \subseteq \mathcal{T}'$ if and only if, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that*

$$B_{d'}(x, \delta) \subseteq B_d(x, \epsilon) \ .$$

PROOF:
$\langle 1 \rangle 1$. If $\mathcal{T} \subseteq \mathcal{T}'$ then, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$

PROOF: From Proposition 2.402 since $x \in B_d(x, \epsilon) \in \mathcal{T}'$.

$\langle 1 \rangle 2$. If, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$, then $\mathcal{T} \subseteq \mathcal{T}'$

$\langle 2 \rangle 1$. ASSUME: For all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$

$\langle 2 \rangle 2$. LET: $U \in \mathcal{T}$

$\langle 2 \rangle 3$. For all $x \in U$ there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq U$.

$\langle 3 \rangle 1$. LET: $x \in U$

$\langle 3 \rangle 2$. PICK $\epsilon > 0$ such that $B_d(x, \epsilon) \subseteq U$

PROOF: Proposition 2.402

$\langle 3 \rangle 3$. PICK $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$

PROOF: By $\langle 2 \rangle 1$

$\langle 3 \rangle 4$. $B_{d'}(x, \delta) \subseteq U$

$\langle 2 \rangle 4$. $U \in \mathcal{T}'$

PROOF: Proposition 2.402.

□

Proposition 2.414. *$\mathbb{R}^2$ under the dictionary order topology is metrizable.*

PROOF: Define $d : \mathbb{R}^2 \to \mathbb{R}$ by

$$d((x, y), (x, z)) = \max(|y - z|, 1)$$

$$d((x, y), (x', y')) = 1 \qquad \text{if } x \neq x' \square$$

$\langle 1 \rangle 1$. $x \in \bigcap_{i=1}^{N} {\pi_i}^{-1}() \subseteq B_D(a, \epsilon)$

Proposition 2.415. *Let $d : X^2 \to \mathbb{R}$ be a metric on X. Then the metric topology on X is the coarsest topology such that d is continuous.*

PROOF:
$\langle 1 \rangle 1$. d is continuous.

$\langle 2 \rangle 1$. LET: $a, b \in X$

$\langle 2 \rangle 2$. LET: $\epsilon > 0$

$\langle 2 \rangle 3$. LET: $\delta = \epsilon / 2$

$\langle 2 \rangle 4$. LET: $x, y \in X$

$\langle 2 \rangle 5$. ASSUME: $\rho((a, b), (x, y)) < \delta$

$\langle 2 \rangle 6$. $|d(a, b) - d(x, y)| < \epsilon$

$\langle 3 \rangle 1$. $d(a, b) - d(x, y) < \epsilon$

PROOF:

$$\begin{aligned} d(a,b) &\leq d(a,x) + d(x,y) + d(y,b) \\ &\leq d(x,y) + 2\rho((a,b),(x,y)) \\ &< d(x,y) + 2\delta \\ &= d(x,y) + \epsilon \end{aligned}$$

⟨3⟩2. $d(a.b) - d(x,y) > -\epsilon$

PROOF: Similar.

⟨2⟩7. Q.E.D.

⟨1⟩2. If $\mathcal{T}$ is any topology under which d is continuous then $\mathcal{T}$ is finer than the metric topology.

PROOF: Since $B(a,\epsilon) = {d_a}^{-1}((-\infty,\epsilon))$

□

Proposition 2.416. *Let X be a metric space with metric d and $A \subseteq X$. The restriction of d to A is a metric on A that induces the subspace topology.*

PROOF:

⟨1⟩1. The restriction of d to A is a metric on A.

⟨1⟩2. Every open ball under $d \upharpoonright A$ is open under the subspace topology.

PROOF: $B_{d\upharpoonright A}(a,\epsilon) = B_d(a,\epsilon) \cap A$.

⟨1⟩3. If U is open in the subspace topology and $x \in U$, then there exists a $d \upharpoonright A$-ball B such that $x \in B \subseteq U$.

⟨2⟩1. PICK V open in X such that $U = V \cap A$

⟨2⟩2. PICK $\epsilon > 0$ such that $B_d(x,\epsilon) \subseteq V$

⟨2⟩3. Take $B = B_{d\upharpoonright A}(x,\epsilon)$

□

Corollary 2.416.1. *A subspace of a metrizable space is metrizable.*

Proposition 2.417. *Every metrizable space is Hausdorff.*

PROOF:

⟨1⟩1. LET: X be a metric space

⟨1⟩2. LET: $a, b \in X$ with $a \neq b$

⟨1⟩3. LET: $\epsilon = d(a,b)/2$

⟨1⟩4. LET: $U = B(a,\epsilon)$ and $V = B(b,\epsilon)$

⟨1⟩5. U and V are disjoint neighbourhoods of a and b respectively.

□

Proposition 2.418 (CC)**.** *The product of a countable family of metrizable spaces is metrizable.*

PROOF:

⟨1⟩1. LET: (X_n, d_n) be a sequence of metric spaces.

⟨1⟩2. ASSUME: w.l.o.g. each d_n is bounded above by 1.

PROOF: By Proposition 2.412.

⟨1⟩3. LET: D be the metric on $\mathbb{R}^\omega$ defined by $D(x,y) = \sup_i(d_i(x_i,y_i)/i)$.

$\langle 2 \rangle 1$. $D(x, y) \geq 0$
$\langle 2 \rangle 2$. $D(x, y) = 0$ if and only if $x = y$
$\langle 2 \rangle 3$. $D(x, y) = D(y, x)$
$\langle 2 \rangle 4$. $D(x, z) \leq D(x, y) + D(y, z)$
PROOF:

$$\begin{aligned} D(x, z) &= \sup_i \frac{d_i(x_i, z_i)}{i} \\ &\leq \sup_i \frac{d_i(x_i, y_i) + d_i(y_i, z_i)}{i} \\ &\leq \sup_i \frac{d_i(x_i, y_i)}{i} + \sup_i \frac{d_i(y_i, z_i)}{i} \\ &= D(x, y) + D(y, z) \end{aligned}$$

$\langle 1 \rangle 4$. Every open ball $B_D(a, \epsilon)$ is open in the product topology.
$\langle 2 \rangle 1$. PICK N such that $1/\epsilon < N$
$\langle 2 \rangle 2$. $B_D(a, \epsilon) = \prod_{i=1}^{\infty} U_i$ where $U_i = B(a_i, i\epsilon)$ if $i \leq N$, and $U_i = X_i$ if $i > N$
$\langle 1 \rangle 5$. For any open set U and $a \in U$, there exists $\epsilon > 0$ such that $B_D(a, \epsilon) \subseteq U$.
$\langle 2 \rangle 1$. LET: $n \geq 1$, V be an open set in $\mathbb{R}$ and $a \in {\pi_n}^{-1}(V)$
$\langle 2 \rangle 2$. PICK $\epsilon > 0$ such that $B_{d_n}(a, \epsilon) \subseteq V$
$\langle 2 \rangle 3$. $B_D(a, \epsilon/n) \subseteq {\pi_n}^{-1}(V)$
□

Theorem 2.419. *Let X and Y be metric spaces and $f : X \to Y$. Then f is continuous if and only if, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$.*

PROOF:
$\langle 1 \rangle 1$. If f is continuous then, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$
$\langle 2 \rangle 1$. ASSUME: f is continuous.
$\langle 2 \rangle 2$. LET: $x \in X$ and $\epsilon > 0$
$\langle 2 \rangle 3$. PICK a neighbourhood U of x such that $f(U) \subseteq B(f(x), \epsilon)$
PROOF: Theorem 2.142.
$\langle 2 \rangle 4$. PICK $\delta > 0$ such that $B(x, \delta) \subseteq U$
PROOF: Proposition 2.402.
$\langle 2 \rangle 5$. For all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$
$\langle 1 \rangle 2$. If for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$, then f is continuous.
$\langle 2 \rangle 1$. ASSUME: for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$
$\langle 2 \rangle 2$. LET: $x \in X$ and V be a neighbourhood of $f(x)$
$\langle 2 \rangle 3$. PICK $\epsilon > 0$ such that $B(f(x), \epsilon) \subseteq V$
PROOF: Proposition 2.402.
$\langle 2 \rangle 4$. PICK $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$
PROOF: By $\langle 2 \rangle 1$
$\langle 2 \rangle 5$. LET: $U = B(x, \delta)$
$\langle 2 \rangle 6$. U is a neighbourhood of x with $f(U) \subseteq V$

$\langle 2\rangle 7$. Q.E.D.
PROOF: Theorem 2.142.
□

Proposition 2.420. *Let X be a metric space. Let (a_n) be a sequence in X and $l \in X$. Then $a_n \to l$ as $n \to \infty$ if and only if, for all $\epsilon > 0$, there exists N such that, for all $n \geq N$, we have $d(a_n, l) < \epsilon$.*

PROOF: From Proposition 2.125. □

Proposition 2.421. *Every metrizable space is first countable.*

PROOF: In any metric space X, the open balls $B(a, 1/n)$ for $n \geq 1$ form a local basis at a.

Example 2.422. $\mathbb{R}^\omega$ under the box topology is not metrizable.

Example 2.423. If J is uncountable then $\mathbb{R}^J$ under the product topology is not metrizable.

Proposition 2.424. *A compact subspace of a metric space is bounded.*

PROOF:
$\langle 1\rangle 1$. LET: X be a metric space and $A \subseteq X$ be compact.
$\langle 1\rangle 2$. PICK $a \in A$
$\langle 1\rangle 3$. $\{B(a, n) \mid n \in \mathbb{Z}^+\}$ covers A
$\langle 1\rangle 4$. PICK a finite subcover $\{B(a, n_1), \ldots, B(a, n_k)\}$
$\langle 1\rangle 5$. LET: $N = \max(n_1, \ldots, n_k)$
$\langle 1\rangle 6$. For all $x, y \in A$ we have $d(x, y) < 2N$
PROOF:

$$\begin{aligned} d(x, y) &\leq d(x, a) + d(a, y) \\ &< N + N \end{aligned}$$

□

This example shows the converse does not hold:

Example 2.425. The space $\mathbb{R}$ under the standard bounded metric is bounded but not compact.

2.68 Real Linear Algebra

Definition 2.426 (Square Metric)**.** The *square metric* ρ on $\mathbb{R}^n$ is defined by

$$\rho(\vec{x}, \vec{y}) = \max(|x_1 - y_1|, \ldots, |x_n - y_n|)$$

We prove this is a metric.

PROOF:
$\langle 1\rangle 1$. $\rho(\vec{x}, \vec{y}) \geq 0$
PROOF: Immediate from definition.

$\langle 1 \rangle 2.$ $\rho(\vec{x}, \vec{y}) = 0$ if and only if $\vec{x} = \vec{y}$
PROOF: Immediate from definition.
$\langle 1 \rangle 3.$ $\rho(\vec{x}, \vec{y}) = \rho(\vec{y}, \vec{x})$
PROOF: Immediate from definition.
$\langle 1 \rangle 4.$ $\rho(\vec{x}, \vec{z}) \leq \rho(\vec{x}, \vec{y}) + \rho(\vec{y}, \vec{z})$
PROOF: Since $|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$.
□

Proposition 2.427. *The square metric induces the standard topology on* $\mathbb{R}^n$.

PROOF:
$\langle 1 \rangle 1.$ For every $a \in X$ and $\epsilon > 0$, we have $B_\rho(a, \epsilon)$ is open in the standard product topology.
PROOF:

$$B_\rho(a, \epsilon) = (a_1 - \epsilon, a_1 + \epsilon) \times \cdots \times (a_n - \epsilon, a_n + \epsilon)$$

$\langle 1 \rangle 2.$ For any open sets $U_1, \ldots, U_n$ in $\mathbb{R}$, we have $U_1 \times \cdots \times U_n$ is open in the square metric topology.
$\langle 2 \rangle 1.$ LET: $\vec{a} \in U_1 \times \cdots \times U_n$
$\langle 2 \rangle 2.$ For $i = 1, \ldots, n$, PICK $\epsilon_i > 0$ such that $(a_i - \epsilon_i, a_i + \epsilon_i) \subseteq U_i$
$\langle 2 \rangle 3.$ LET: $\epsilon = \min(\epsilon_1, \ldots, \epsilon_n)$
$\langle 2 \rangle 4.$ $B_\rho(\vec{a}, \epsilon) \subseteq U$
□

Definition 2.428. Given $\vec{x}, \vec{y} \in \mathbb{R}^n$, define the *sum* $\vec{x} + \vec{y}$ by

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n) \ .$$

Definition 2.429. Given $\lambda \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^n$, define the *scalar product* $\lambda\vec{x} \in \mathbb{R}^n$ by

$$\lambda(x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n)$$

Definition 2.430 (Inner Product)**.** Given $\vec{x}, \vec{y} \in \mathbb{R}^n$, define the *inner product* $\vec{x} \cdot \vec{y} \in \mathbb{R}$ by

$$(x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) = x_1 y_1 + \cdots + x_n y_n \ .$$

We write $\vec{x}^2$ for $\vec{x} \cdot \vec{x}$.

Definition 2.431 (Norm)**.** Let $n \geq 1$. The *norm* on $\mathbb{R}^n$ is the function $\| \ \| : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\|(x_1, \ldots, x_n)\| = \sqrt{x_1^2 + \cdots + x_n^2}$$

Lemma 2.432.

$$\|\vec{x}\|^2 = \vec{x}^2$$

PROOF: Immediate from definitions. □

Lemma 2.433.

$$\vec{x} \cdot (\vec{y} + \vec{z}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z}$$

PROOF: Each is equal to $(x_1y_1 + x_1z_1, \ldots, x_ny_n + x_nz_n)$. $\square$

Lemma 2.434.

$$|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\|\|\vec{y}\|$$

PROOF:
⟨1⟩1. ASSUME: $\vec{x} \neq \vec{0} \neq \vec{y}$
PROOF: Otherwise both sides are 0.
⟨1⟩2. LET: $a = 1/\|\vec{x}\|$
⟨1⟩3. LET: $b = 1/\|\vec{y}\|$
⟨1⟩4. $(a\vec{x} + b\vec{y})^2 \geq 0$ and $(a\vec{x} - b\vec{y})^2 \geq 0$
⟨1⟩5. $a^2\|\vec{x}\|^2 + 2ab\vec{x} \cdot \vec{y} + b^2\|\vec{y}\|^2 \geq 0$ and $a^2\|\vec{x}\|^2 - 2ab\vec{x} \cdot \vec{y} + b^2\|\vec{y}\|^2 \geq 0$
⟨1⟩6. $2ab\vec{x} \cdot \vec{y} + 2 \geq 0$ and $-2ab\vec{x} \cdot \vec{y} + 2 \geq 0$
⟨1⟩7. $\vec{x} \cdot \vec{y} \geq -1/ab$ and $\vec{x} \cdot \vec{y} \leq 1/ab$
⟨1⟩8. $\vec{x} \cdot \vec{y} \geq -\|\vec{x}\|\|\vec{y}\|$ and $\vec{x} \cdot \vec{y} \leq \|\vec{x}\|\|\vec{y}\|$
$\square$

Lemma 2.435 (Triangle Inequality)**.**

$$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$$

PROOF:

$$\begin{aligned} \|\vec{x} + \vec{y}\|^2 &= \|\vec{x}\|^2 + 2\vec{x} \cdot \vec{y} + \|\vec{y}\|^2 \\ &\leq \|\vec{x}\|^2 + 2\|\vec{x}\|\|\vec{y}\| + \|\vec{y}\|^2 && \text{(Lemma 2.434)} \\ &= (\|\vec{x}\| + \|\vec{y}\|)^2 && \square \end{aligned}$$

Definition 2.436 (Euclidean Metric)**.** Let $n \geq 1$. The *Euclidean metric* on $\mathbb{R}^n$ is defined by

$$d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\| \ .$$

We prove this is a metric.

⟨1⟩1. $d(\vec{x}, \vec{y}) \geq 0$
PROOF: Immediate from definition.
⟨1⟩2. $d(\vec{x}, \vec{y}) = 0$ if and only if $\vec{x} = \vec{y}$
PROOF: $d(\vec{x}, \vec{y}) = 0$ if and only if $\vec{x} - \vec{y} = \vec{0}$.
⟨1⟩3. $d(\vec{x}, \vec{y}) = d(\vec{y}, \vec{x})$
PROOF: Immediate from definition.
⟨1⟩4. $d(\vec{x}, \vec{z}) \leq d(\vec{x}, \vec{y}) + d(\vec{y}, \vec{z})$
PROOF:

$$\begin{aligned} \|\vec{x} - \vec{z}\| &= \|(\vec{x} - \vec{y}) + (\vec{y} - \vec{z})\| \\ &\leq \|\vec{x} - \vec{y}\| + \|\vec{y} - \vec{z}\| && \text{(Lemma 2.435)} \end{aligned}$$

$\square$

Proposition 2.437. *The Euclidean metric induces the standard topology on* $\mathbb{R}^n$.

PROOF:

⟨1⟩1. LET: ρ be the square metric.
⟨1⟩2. For all $\vec{a} \in \mathbb{R}^n$ and $\epsilon > 0$, we have $B_d(\vec{a}, \epsilon) \subseteq B_\rho(\vec{a}, \epsilon)$
⟨2⟩1. LET: $\vec{x} \in B_d(\vec{a}, \epsilon)$
⟨2⟩2. $\sqrt{(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2} < \epsilon$
⟨2⟩3. $(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2 < \epsilon^2$
⟨2⟩4. For all i we have $(x_i - a_i)^2 < \epsilon^2$
⟨2⟩5. For all i we have $|x_i - a_i| < \epsilon$
⟨2⟩6. $\rho(\vec{x}, \vec{a}) < \epsilon$
⟨1⟩3. For all $\vec{a} \in \mathbb{R}^n$ and $\epsilon > 0$, we have $B_\rho(\vec{a}, \epsilon/\sqrt{n}) \subseteq B_d(\vec{a}, \epsilon)$
⟨2⟩1. LET: $\vec{x} \in B_\rho(\vec{a}, \epsilon/\sqrt{n})$
⟨2⟩2. $\rho(\vec{x}, \vec{a}) < \epsilon/\sqrt{n}$
⟨2⟩3. For all i we have $|x_i - x_a| < \epsilon/\sqrt{n}$
⟨2⟩4. For all i we have $(x_i - x_a)^2 < \epsilon^2/n$
⟨2⟩5. $d(\vec{x}, \vec{a}) < \epsilon$
⟨1⟩4. Q.E.D.
PROOF: By Lemma 2.413.
□

Proposition 2.438. *Let $n \geq 0$. For all $c \in \mathbb{R}^n$ and $\epsilon > 0$, the open ball $B(c, \epsilon)$ is path connected.*

PROOF:
⟨1⟩1. LET: $a, b \in B(c, \epsilon)$
⟨1⟩2. LET: $p : [0, 1] \to B(c, \epsilon)$ be the function $p(t) = (1 - t)a + tb$
PROOF: We have $p(t) \in B(c, \epsilon)$ for all t because

$$\begin{aligned} d(p(t), c) &= \|(1-t)a + tb - c\| \\ &= \|(1-t)(a-c) + t(b-c)\| \\ &\leq (1-t)\|a-c\| + t\|b-c\| \\ &< (1-t)\epsilon + t\epsilon \\ &= \epsilon \end{aligned}$$

⟨1⟩3. p is a path from a to b.
□

Proposition 2.439. *Let $n \geq 0$. For all $c \in \mathbb{R}^n$ and $\epsilon > 0$, the closed ball $\overline{B(c, \epsilon)}$ is path connected.*

PROOF:
⟨1⟩1. LET: $a, b \in \overline{B(c, \epsilon)}$
⟨1⟩2. LET: $p : [0, 1] \to \overline{B(c, \epsilon)}$ be the function $p(t) = (1 - t)a + tb$
PROOF: We have $p(t) \in \overline{B(c, \epsilon)}$ for all t because

$$\begin{aligned} d(p(t), c) &= \|(1-t)a + tb - c\| \\ &= \|(1-t)(a-c) + t(b-c)\| \\ &\leq (1-t)\|a-c\| + t\|b-c\| \\ &\leq (1-t)\epsilon + t\epsilon \\ &= \epsilon \end{aligned}$$

$\langle 1 \rangle 3.$ p is a path from a to b.
$\square$

Lemma 2.440. *If* $\sum_{i=0}^{\infty} x_i^2$ *and* $\sum_{i=0}^{\infty} y_i^2$ *converge then* $\sum_{i=0}^{\infty} |x_i y_i|$ *converges.*

PROOF:
$\langle 1 \rangle 1.$ For all $N \geq 0$ we have $\sum_{i=0}^{N} |x_i y_i| \leq \sqrt{\sum_{i=0}^{N} |x_i|^2} \sqrt{\sum_{i=0}^{N} |y_i|^2}$
PROOF: By the Cauchy-Schwarz inequality
$\langle 1 \rangle 2.$ Q.E.D.
PROOF: Since $\sum_{i=0}^{N} |x_i y_i|$ is an increasing sequence bounded above by $(\sum_{i=0}^{\infty} x_i^2)(\sum_{i=0}^{\infty} y_i^2)$.
$\square$

Corollary 2.440.1. *If* $\sum_{i=0}^{\infty} x_i^2$ *and* $\sum_{i=0}^{\infty} y_i^2$ *converge then* $\sum_{i=0}^{\infty} (x_i + y_i)^2$ *converges.*

PROOF: Since $\sum_{i=0}^{\infty} x_i^2$, $\sum_{i=0}^{\infty} y_i^2$ and $2 \sum_{i=0}^{\infty} x_i y_i$ all converge.

Definition 2.441 (l^2-metric)**.** The l^2-*metric* on

$$\left\{ (x_n) \in \mathbb{R}^\omega \mid \sum_{n=0}^{\infty} x_n^2 \text{ converges} \right\}$$

is defined by

$$d(x, y) = \left(\sum_{n=0}^{\infty} (x_n - y_n)^2 \right)^{1/2}$$

We prove this is a metric.

PROOF:
$\langle 1 \rangle 1.$ d is well-defined.
PROOF: By Corollary 2.440.1.
$\langle 1 \rangle 2.$ $d(x, y) \geq 0$
$\langle 1 \rangle 3.$ $d(x, y) = 0$ if and only if $x = y$
$\langle 1 \rangle 4.$ $d(x, y) = d(y, x)$
$\langle 1 \rangle 5.$ $d(x, z) \leq d(x, y) + d(y, z)$
PROOF: By Lemma 2.435.
$\square$

Theorem 2.442. *Addition is a continuous function* $\mathbb{R}^2 \to \mathbb{R}$*.*

PROOF:
$\langle 1 \rangle 1.$ LET: $a, b \in \mathbb{R}$
$\langle 1 \rangle 2.$ LET: $\epsilon > 0$
$\langle 1 \rangle 3.$ LET: $\delta = \epsilon / 2$
$\langle 1 \rangle 4.$ LET: $x, y \in \mathbb{R}$
$\langle 1 \rangle 5.$ ASSUME: $\rho((a, b), (x, y)) < \delta$
$\langle 1 \rangle 6.$ $|(a + b) - (x + y)| < \epsilon$

PROOF:

$$\begin{aligned}|(a+b)-(x+y)| &= |a-x|+|b-y| \\ &\leq 2\rho((a,b),(x,y)) \\ &< 2\delta \\ &= \epsilon\end{aligned}$$

⟨1⟩7. Q.E.D.
PROOF: Theorem 2.419
□

Theorem 2.443. *Multiplication is a continuous function* $\mathbb{R}^2 \to \mathbb{R}$.

PROOF:
⟨1⟩1. LET: $a, b \in \mathbb{R}$
⟨1⟩2. LET: $\epsilon > 0$
⟨1⟩3. LET: $\delta = \min(\epsilon/(|a|+|b|+1), 1)$
⟨1⟩4. LET: $x, y \in \mathbb{R}$
⟨1⟩5. ASSUME: $\rho((a,b),(x,y)) < \delta$
⟨1⟩6. $|ab - xy| < \epsilon$
PROOF:

$$\begin{aligned}|ab-xy| &= |a(b-y)+(a-x)b-(a-x)(b-y)| \\ &\leq |a||b-y|+|b||a-x|+|a-x||b-y| \\ &< |a|\delta+|b|\delta+\delta^2 && (\langle 1\rangle 5) \\ &\leq |a|\delta+|b|\delta+\delta && (\langle 1\rangle 3) \\ &\leq \epsilon && (\langle 1\rangle 3)\end{aligned}$$

⟨1⟩7. Q.E.D.
PROOF: Theorem 2.419
□

Theorem 2.444. *The function* $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ *defined by* $f(x) = x^{-1}$ *is continuous.*

PROOF:
⟨1⟩1. For all $a \in \mathbb{R}$ we have $f^{-1}((a, +\infty))$ is open.
PROOF: The set is

$$\begin{aligned}(a^{-1}, +\infty) &\text{if } a > 0 \\ (0, +\infty) &\text{if } a = 0 \\ (-\infty, a^{-1}) \cup (0, +\infty) &\text{if } a < 0\end{aligned}$$

⟨1⟩2. For all $a \in \mathbb{R}$ we have $f^{-1}((-\infty, a))$ is open.
PROOF: Similar.
⟨1⟩3. Q.E.D.
PROOF: By Proposition 2.139 and Lemma 2.162.
□

Definition 2.445. For $n \geq 0$, the *unit ball* B^n is the space $\{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$.

Proposition 2.446. *For all* $n \geq 0$, *the unit ball* B^n *is path connected.*

PROOF:
⟨1⟩1. LET: $a, b \in B^n$
⟨1⟩2. LET: $p : [0,1] \to B^n$ be the function $p(t) = (1-t)a + tb$
PROOF: We have $p(t) \in B^n$ for all t because

$$\begin{aligned} \|(1-t)a + tb\| &\leq (1-t)\|a\| + t\|b\| \\ &\leq (1-t) + t \\ &= 1 \end{aligned}$$

⟨1⟩3. p is a path from a to b.
□

Definition 2.447 (Punctured Euclidean Space)**.** For $n \geq 0$, defined *punctured Euclidean space* to be $\mathbb{R}^n \setminus \{0\}$.

Proposition 2.448. *For $n > 1$, punctured Euclidean space $\mathbb{R}^n \setminus \{0\}$ is path connected.*

PROOF:
⟨1⟩1. LET: $a, b \in \mathbb{R}^n \setminus \{0\}$
⟨1⟩2. CASE: 0 is on the line from a to b
⟨2⟩1. PICK a point c not on the line from a to b
⟨2⟩2. The path consisting of a straight line from a to c followed by a straight line from c to b is a path from a to b.
⟨1⟩3. CASE: 0 is not on the line from a to b
PROOF: The straight line from a to b is a path from a to b.

Corollary 2.448.1. *For $n > 1$, the spaces $\mathbb{R}$ and $\mathbb{R}^n$ are not homeomorphic.*

PROOF: For any point a, the space $\mathbb{R} \setminus \{a\}$ is disconnected.

Definition 2.449 (Unit Sphere)**.** For $n \geq 1$, the *unit sphere* S^{n-1} is the space

$$S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\} \ .$$

Proposition 2.450. *For $n > 1$, the unit sphere S^{n-1} is path connected.*

PROOF: The map $g : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$ defined by $g(x) = x/\|x\|$ is continuous and surjective. The result follows by Proposition 2.294. □

Proposition 2.451. *Let $f : S^1 \to \mathbb{R}$ be continuous. Then there exists $x \in S^1$ such that $f(x) = f(-x)$.*

PROOF:
⟨1⟩1. LET: $g : S^1 \to \mathbb{R}$ be the function $g(x) = f(x) - f(-x)$
PROVE: There exists $x \in S^1$ such that $g(x) = 0$
⟨1⟩2. ASSUME: without loss of generality $g((1,0)) > 0$
⟨1⟩3. $g((-1,0)) < 0$
⟨1⟩4. There exists x such that $g(x) = 0$
PROOF: By the Intermediate Value Theorem.
□

Definition 2.452 (Topologist's Sine Curve)**.** Let $S = \{(x, \sin 1/x) \mid 0 < x \leq 1\}$. The *topologist's sine curve* is the closure $\overline{S}$ of S.

Proposition 2.453.
$$\overline{S} = S \cup (\{0\} \times [-1, 1])$$

Proposition 2.454. *The topologist's sine curve is connected.*

PROOF:
$\langle 1 \rangle 1$. LET: $S = \{(x, \sin 1/x) \mid 0 < x \leq 1\}$
$\langle 1 \rangle 2$. S is connected.
  PROOF: Theorem 2.273.
$\langle 1 \rangle 3$. $\overline{S}$ is connected.
  PROOF: Theorem 2.272.
$\square$

Proposition 2.455 (CC)**.** *The topologist's sine curve is not path connected.*

PROOF:
$\langle 1 \rangle 1$. ASSUME: for a contradiction $p : [0, 1] \to \overline{S}$ is a path from $(0, 0)$ to $(1, \sin 1)$.
$\langle 1 \rangle 2$. $p^{-1}(\{0\} \times [0, 1])$ is closed.
$\langle 1 \rangle 3$. LET: b be the greatest element of $p^{-1}(\{0\} \times [0, 1])$.
$\langle 1 \rangle 4$. $b < 1$
  PROOF: Since $p(1) = (1, \sin 1)$.
$\langle 1 \rangle 5$. PICK a sequence $(t_n)_{n \geq 1}$ in $(b, 1]$ such that $t_n \to b$ and $\pi_2(p(t_n)) = (-1)^n$
  $\langle 2 \rangle 1$. LET: $n \geq 1$
  $\langle 2 \rangle 2$. PICK u with $0 < u < \pi_1(p(1/n))$ such that $\sin(1/u) = (-1)^n$
  $\langle 2 \rangle 3$. PICK t_n such that $0 < t_n < 1/n$ and $\pi_1(p(t_n)) = u$
    PROOF: One exists by the Intermediate Value Theorem.
$\langle 1 \rangle 6$. Q.E.D.
  PROOF: This contradicts 2.154.
$\square$

Theorem 2.456. *Let A be a subspace of $\mathbb{R}^n$. Then the following are equivalent:*

1. *A is compact.*
2. *A is closed and bounded under the Euclidean metric.*
3. *A is closed and bounded under the square metric.*

PROOF:
$\langle 1 \rangle 1$. $1 \Rightarrow 2$
  PROOF: By Corollary 2.363.1 and Proposition 2.424.
$\langle 1 \rangle 2$. $2 \Rightarrow 3$
  PROOF: If $d(x, y) \leq M$ for all $x, y \in A$ then $\rho(x, y) \leq M/\sqrt{2}$.
$\langle 1 \rangle 3$. $3 \Rightarrow 1$
  $\langle 2 \rangle 1$. ASSUME: A is closed and $\rho(x, y) \leq M$ for all $x, y \in A$
  $\langle 2 \rangle 2$. PICK $a \in A$

PROOF: We may assume w.l.o.g. A is nonempty since the empty space is compact.

⟨2⟩3. A is a closed subspace of $[a_1 - M, a_1 + M] \times \cdots \times [a_n - M, a_n + M]$

⟨2⟩4. A is compact

PROOF: Proposition 2.356.

□

Corollary 2.456.1. *The unit sphere S^{n-1} and the closed unit ball B^n are compact for any n.*

2.69 The Uniform Topology

Definition 2.457 (Uniform Metric)**.** Let J be a set. The *uniform metric* $\overline{\rho}$ on $\mathbb{R}^J$ is defined by

$$\overline{\rho}(a,b) = \sup_{j \in J} \overline{d}(a_j, b_j)$$

where $\overline{d}$ is the standard bounded metric on $\mathbb{R}$.

The *uniform topology* on $\mathbb{R}^J$ is the topology induced by the uniform metric.

We prove this is a metric.

PROOF:

⟨1⟩1. $\overline{\rho}(a,b) \geq 0$

PROOF: Immediate from definitions.

⟨1⟩2. $\overline{\rho}(a,b) = 0$ if and only if $a = b$

PROOF: Immediate from definitions.

⟨1⟩3. $\overline{\rho}(a,b) = \overline{\rho}(b,a)$

PROOF: Immediate from definitions.

⟨1⟩4. $\overline{\rho}(a,c) \leq \overline{\rho}(a,b) + \overline{\rho}(b,c)$

PROOF:

$$\begin{aligned}
\overline{\rho}(a,c) &= \sup_{j \in J} \overline{d}(a_j, c_j) \\
&\leq \sup_{j \in J} (\overline{d}(a_j, b_j) + \overline{d}(b_j, c_j)) \\
&\leq \sup_{j \in J} \overline{d}(a_j, b_j) + \sup_{j \in J} \overline{d}(b_j, c_j) \\
&= \overline{\rho}(a,b) + \overline{\rho}(b,c)
\end{aligned}$$

□

Proposition 2.458. *The uniform topology on $\mathbb{R}^J$ is finer than the product topology.*

PROOF:

⟨1⟩1. LET: $j \in J$ and U be open in $\mathbb{R}$

PROVE: ${\pi_j}^{-1}(U)$ is open in the uniform topology.

⟨1⟩2. LET: $a \in {\pi_j}^{-1}(U)$

⟨1⟩3. PICK $\epsilon > 0$ such that $(a_j - \epsilon, a_j + \epsilon) \subseteq U$

⟨1⟩4. $B_{\overline{\rho}}(a, \epsilon) \subseteq {\pi_j}^{-1}(U)$
□

Proposition 2.459. *The uniform topology on $\mathbb{R}^J$ is coarser than the box topology.*

PROOF:
⟨1⟩1. LET: $a \in \mathbb{R}^J$ and $\epsilon > 0$
PROVE: $B(a, \epsilon)$ is open in the box topology.
⟨1⟩2. LET: $b \in B(a, \epsilon)$
⟨1⟩3. For $j \in J$ we have $|a_j - b_j| < \epsilon$
⟨1⟩4. For $j \in J$,
LET: $\delta_j = (\epsilon - |a_j - b_j|)/2$
⟨1⟩5. $\prod_{j \in J}(b_j - \delta_j, b_j + \delta_j) \subseteq B(a, \epsilon)$
□

Proposition 2.460. *The uniform topology on $\mathbb{R}^J$ is strictly finer than the product topology if and only if J is infinite.*

PROOF:
⟨1⟩1. If J is finite then the uniform and product topologies coincide.
PROOF: The uniform, box and product topologies are all the same.
⟨1⟩2. If J is infinite then the uniform and product topologies are different.
PROOF: The set $B(\vec{0}, 1)$ is open in the uniform topology but not the product topology.
□

Proposition 2.461 (DC)**.** *The uniform topology on $\mathbb{R}^J$ is strictly coarser than the box topology if and only if J is infinite.*

PROOF:
⟨1⟩1. If J is finite then the uniform and box topologies coincide.
PROOF: The uniform, box and product topologies are all the same.
⟨1⟩2. If J is infinite then the uniform and box topologies are different.
PROOF: Pick an ω-sequence $(j_1, j_2, \ldots)$ in J. Let $U = \prod_{j \in J} U_j$ where $U_{j_i} = (-1/i, 1/i)$ and $U_j = (-1, 1)$ for all other j. Then $\vec{0} \in U$ but there is no $\epsilon > 0$ such that $B(\vec{0}, \epsilon) \subseteq U$.
□

Proposition 2.462. *The closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ under the uniform topology is $\mathbb{R}^\omega$.*

PROOF: Given any open ball $B(a, \epsilon)$, pick an integer N such that $1/\epsilon < N$. Then $B(a, \epsilon)$ includes sequences whose nth entry is 0 for all $n \geq N$. □

Example 2.463. The space $\mathbb{R}^\omega$ is disconnected under the uniform topology. The set of bounded sequences and the set of unbounded sequences form a separation.

Proposition 2.464. *Give $\mathbb{R}^\omega$ the uniform topology. Let $x, y \in \mathbb{R}^\omega$. Then x and y are in the same component if and only if $x - y$ is bounded.*

PROOF:
⟨1⟩1. The component containing 0 is the set of bounded sequences.
⟨2⟩1. LET: B be the set of bounded sequences.
⟨2⟩2. B is path-connected.
⟨3⟩1. LET: $x.y \in B$
⟨3⟩2. PICK $b > 0$ such that $|x_j|, |y_j| \leq b$ for all j
⟨3⟩3. LET: $p : [0,1] \to B$ be the function $p(t) = (1-t)x + ty$
PROVE: p is continuous.
⟨3⟩4. LET: $t \in [0,1]$ and $\epsilon > 0$
⟨3⟩5. LET: $\delta = \epsilon/2b$
⟨3⟩6. LET: $s \in [0,1]$ with $|s - t| < \delta$
⟨3⟩7. $\overline{\rho}(p(s), p(t)) < \epsilon$
PROOF:

$$\begin{aligned}\overline{\rho}(p(s), p(t)) &= \sup_j \overline{d}((1-s)x_j + sy_j, (1-t)x_j + ty_j) \\ &\leq |(s-t)x_j + (t-s)y_j| \\ &\leq |s-t||x_j - y_j| \\ &< 2b\delta \\ &= \epsilon\end{aligned}$$

⟨2⟩3. B is connected.
PROOF: Proposition 2.292.
⟨2⟩4. If C is connected and $B \subseteq C$ then $B = C$.
PROOF: Otherwise $B \cap C$ and $C \setminus B$ form a separation of C.
⟨1⟩2. Q.E.D.
PROOF: Since $\lambda x.x - y$ is a Homeomorphism of $\mathbb{R}^\omega$ with itself.
□

2.70 Uniform Convergence

Definition 2.465 (Uniform Convergence)**.** Let X be a set and Y a metric space. Let $(f_n : X \to Y)$ be a sequence of functions and $f : X \to Y$ be a function. Then f_n *converges uniformly* to f as $n \to \infty$ if and only if, for all $\epsilon > 0$, there exists N such that, for all $n \geq N$ and $x \in X$, we have $d(f_n(x), f(x)) < \epsilon$.

Example 2.466. Define $f_n : [0,1] \to \mathbb{R}$ by $f_n(x) = x^n$ for $n \geq 1$, and $f : [0,1] \to \mathbb{R}$ by $f(x) = 0$ if $x < 1$, $f(1) = 1$. Then f_n converges to f pointwise but not uniformly.

Theorem 2.467 (Uniform Limit Theorem)**.** *Let X be a topological space and Y a metric space. Let $(f_n : X \to Y)$ be a sequence of continuous functions and $f : X \to Y$ be a function. If f_n converges uniformly to f as $n \to \infty$, then f is continuous.*

PROOF:
⟨1⟩1. LET: $x \in X$ and $\epsilon > 0$
⟨1⟩2. PICK N such that, for all $n \geq N$ and $y \in X$, we have $d(f_n(y), f(y)) < \epsilon/3$
⟨1⟩3. PICK a neighbourhood U of x such that $f_N(U) \subseteq B(f_N(x), \epsilon/3)$
PROVE: $f(U) \subseteq B(f(x), \epsilon)$
⟨1⟩4. LET: $y \in U$
⟨1⟩5. $d(f(y), f(x)) < \epsilon$
PROOF:

$$\begin{aligned} d(f(y), f(x)) &\leq d(f(y), f_N(y)) + d(f_N(y), f_N(x)) + d(f_N(x), f(x)) && \text{(Triangle Inequality)} \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 && (\langle 1\rangle 2,\ \langle 1\rangle 3) \\ &= \epsilon \end{aligned}$$

□

Proposition 2.468. *Let X be a topological space and Y a metric space. Let $(f_n : X \to Y)$ be a sequence of continuous functions and $f : X \to Y$ be a function. Let (a_n) be a sequence of points in X and $a \in X$. If f_n converges uniformly to f and a_n converges to a in X then $f_n(a_n)$ converges to $f(a)$ uniformly in Y.*

PROOF:
⟨1⟩1. LET: $\epsilon > 0$
⟨1⟩2. PICK N_1 such that, for all $n \geq N_1$ and $x \in X$, we have $d(f_n(x), f(x)) < \epsilon/2$
⟨1⟩3. PICK N_2 such that, for all $n \geq N_2$, we have $a_n \in f^{-1}(B(a, \epsilon/2))$
PROOF: Using the fact that f is continuous from the Uniform Limit Theorem.
⟨1⟩4. LET: $N = \max(N_1, N_2)$
⟨1⟩5. LET: $n \geq N$
⟨1⟩6. $d(f_n(a_n), f(a)) < \epsilon$
PROOF:

$$\begin{aligned} d(f_n(a_n), f(a)) &\leq d(f_n(a_n), f(a_n)) + d(f(a_n), f(a)) && \text{(Triangle Inequality)} \\ &< \epsilon/2 + \epsilon/2 && (\langle 1\rangle 2,\ \langle 1\rangle 3) \\ &= \epsilon \end{aligned}$$

□

Proposition 2.469. *Let X be a set. Let $(f_n : X \to \mathbb{R})$ be a sequence of functions and $f : X \to \mathbb{R}$ be a function. Then f_n converges unifomly to f as $n \to \infty$ if and only if $f_n \to f$ as $n \to \infty$ in $\mathbb{R}^X$ under the uniform topology.*

PROOF:
⟨1⟩1. If f_n converges uniformly to f then f_n converges to f under the uniform topology.
⟨2⟩1. ASSUME: f_n converges uniformly to f
⟨2⟩2. LET: $\epsilon > 0$
⟨2⟩3. PICK N such that, for all $n \geq N$ and $x \in X$, we have $d(f_n(x), f(x)) < \epsilon/2$
⟨2⟩4. For all $n \geq N$ we have $\overline{\rho}(f_n, f) \leq \epsilon/2$

⟨2⟩5. For all $n \geq N$ we have $\overline{\rho}(f_n, f) < \epsilon$

⟨1⟩2. If f_n converges to f under the uniform topology then f_n converges uniformly to f.

⟨2⟩1. Assume: f_n converges to f under the uniform topology.

⟨2⟩2. Let: $\epsilon > 0$

⟨2⟩3. Pick N such that, for all $n \geq N$, we have $\overline{\rho}(f_n, f) < \min(\epsilon, 1/2)$

⟨2⟩4. Let: $n \geq N$

⟨2⟩5. Let: $x \in X$

⟨2⟩6. $\overline{\rho}(f_n, f) < \min(\epsilon, 1/2)$

Proof: From ⟨2⟩3.

⟨2⟩7. $d(f_n(x), f(x)) < \min(\epsilon, 1/2)$

⟨2⟩8. $d(f_n(x), f(x)) < \epsilon$

□

2.71 Isometric Imbeddings

Definition 2.470. Let X and Y be metric spaces. An *isometric imbedding* $f : X \to Y$ is a function such that, for all $x, y \in X$, we have $d(f(x), f(y)) = d(x, y)$.

Proposition 2.471. *Every isometric imbedding is an imbedding.*

Proof:

⟨1⟩1. Let: $f : X \to Y$ be an isometric imbedding.

⟨1⟩2. f is injective.

Proof: If $f(x) = f(y)$ then $d(f(x), f(y)) = 0$ hence $d(x, y) = 0$ hence $x = y$.

⟨1⟩3. f is continuous.

Proof: For all $\epsilon > 0$, if $d(x, y) < \epsilon$ then $d(f(x), f(y)) < \epsilon$.

⟨1⟩4. $f : X \to f(X)$ is an open map.

Proof: $f(B(a, \epsilon)) = B(f(a), \epsilon) \cap f(X)$.

□

2.72 Distance to a Set

Definition 2.472. Let X be a metric space, $x \in X$ and $A \subseteq X$ be nonempty. The *distance* from x to A is defined as

$$d(x, A) = \inf_{a \in A} d(x, a) \ .$$