Topology

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Contents

I Set Theory **4**

1 Set Theory **5**

1.1 Membership . . . 5
1.2 Subsets . . . 5
1.3 Abstraction Notation . . . 5
1.4 The Empty Set . . . 5
1.5 Pair Sets . . . 6
1.6 Unions . . . 6
1.7 Power Set . . . 6
1.8 Singletons . . . 6
1.9 Finite Sets . . . 6
1.10 Subset Axioms . . . 6
1.11 Intersection . . . 7
1.12 Relative Complement . . . 7
1.13 Covers . . . 7

2 Relations **8**

2.1 Ordered Pairs . . . 8
2.2 Cartesian Product . . . 9
2.3 Relations . . . 9
2.4 Domain . . . 9
2.5 Range . . . 10
2.6 Field . . . 10
2.7 Functions . . . 10
2.8 Single-Rooted . . . 10
2.9 Surjective . . . 10
2.10 Inverse . . . 10
2.11 Composition . . . 11
2.12 Identity Function . . . 12
2.13 Restriction . . . 13
2.14 Image . . . 14
2.15 Infinite Cartesian Product . . . 15
2.16 Reflexive Relations . . . 16
2.17 Symmetric . . . 16

2.18 Transitivity 16
2.19 Equivalence Relations 17
2.20 Equivalence Class 17
2.21 Disjoint 18
2.22 Partitions 18
2.23 Quotient Sets 18
2.24 The Finite Intersection Property 19
2.25 Countable Intersection Property 21
2.26 The Axiom of Choice 22
2.27 Choice Functions 22
2.28 Order Theory 23
2.29 Partially Ordered Sets 26
2.30 Strict Linear Orders 26
2.31 Real Analysis 30
2.32 Group Theory 31
2.33 Topological Spaces 32
2.34 Closed Set 33
2.35 Interior 34
2.36 Closure 35
2.37 Boundary 37
2.38 Limit Points 37
2.39 Basis for a Topology 38
2.40 Local Basis at a Point 42
2.41 Convergence 43
2.42 Locally Finite Sets 45
2.43 Open Maps 46
2.44 Continuous Functions 46
2.45 Homeomorphisms 51
2.46 The Order Topology 52
2.47 The nth Root Function 55
2.48 The Product Topology 55
2.48.1 Continuous in Each Variable Separately 58
2.49 The Subspace Topology 58
2.50 The Box Topology 62
2.51 T_1 Spaces 64
2.52 Hausdorff Spaces 65
2.53 The First Countability Axiom 68
2.54 Strong Continuity 70
2.55 Saturated Sets 70
2.56 Quotient Maps 71
2.57 Quotient Topology 74
2.58 Retractions 76
2.59 Homogeneous Spaces 76
2.60 Regular Spaces 76
2.61 Connected Spaces 76
2.62 Totally Disconnected Spaces 85

2.63 Paths and Path Connectedness 85
2.64 The Topologist's Sine Curve 88
2.65 The Long Line 89
2.66 Components 91
2.67 Path Components 92
2.68 Local Connectedness 93
2.69 Local Path Connectedness 95
2.70 Weak Local Connectedness 96
2.71 Quasicomponents 97
2.72 Open Coverings 98
2.73 Lindelöf Spaces 98
2.74 The Second Countability Axiom 99
2.75 Compact Spaces 100
2.76 Perfect Maps 110
2.77 Topological Groups 111
2.78 The Metric Topology 118
2.79 Real Linear Algebra 124
2.80 The Uniform Topology 132
2.81 Uniform Convergence 134
2.82 Isometric Imbeddings 136
2.83 Distance to a Set 136

Part I

Set Theory

Chapter 1

Set Theory

1.1 Membership

We take as undefined the binary relation of *membership*, $\in$. If $a \in A$ we say a is a *member* or *element* of A. If this does not hold, we write $a \notin A$.

Axiom 1.1 (Axiom of Extensionality)**.** *Two sets with exactly the same elements are equal.*

1.2 Subsets

Definition 1.2 (Subset)**.** Let A and B be sets. We say A is a *subset* of B, $A \subseteq B$, if and only if every member of A is a member of B.

1.3 Abstraction Notation

Definition Schema 1.3 (Extensionality)**.** Let $P(x)$ be a property. If there is a set whose members are exactly the sets x such that $P(x)$, then we denote this set by $\{x \mid P(x)\}$.

It is unique by the Axiom of Extensionality.

1.4 The Empty Set

Axiom 1.4 (Empty Set Axiom)**.** *There exists a set with no members.*

Definition 1.5 (Empty Set (Extensionality, Empty Set Axiom))**.** The *empty set* $\emptyset$ is the set with no members $\{x \mid \bot\}$.

1.5 Pair Sets

Axiom 1.6 (Pairing Axiom)**.** *For any sets u and v, there exists a set having as members just u and v.*

Definition 1.7 (Pair Set (Extensionality, Pairing Axiom))**.** For any sets u and v, the *pair set* $\{u, v\}$ is the set $\{x \mid x = u \vee x = v\}$.

1.6 Unions

Axiom 1.8 (Union Axiom)**.** *For any set A, there exists a set whose elements are exactly the members of the members of A.*

Definition 1.9 (Union (Extensionality, Union))**.** For any set A, the *union* $\bigcup A$ is the set $\{x \mid \exists b \in A.x \in b\}$.

Definition 1.10 (Union (Extensionality, Pair Set, Union))**.** For any sets a and b, the *union* $a \cup b$ is the set $\bigcup\{a, b\}$.

1.7 Power Set

Axiom 1.11 (Power Set Axiom)**.** *For any set a, there is a set whose members are exactly the subsets of a.*

Definition 1.12 (Power Set (Extensionality, Power Set))**.** For any set a, the *power set* $\mathcal{P}a$ is the set $\{x \mid x \subseteq a\}$.

1.8 Singletons

Definition 1.13 (Singleton (Extensionality, Pair Set))**.** Given any x, define the *singleton* $\{x\}$ to be $\{x, x\}$.

1.9 Finite Sets

Definition Schema 1.14 (Extensionality, Pair Set, Union)**.** Given any objects a_1, ..., a_n, define the set $\{a_1, \ldots, a_n\}$ as follows:

$$\{a_1, \ldots, a_n, a_{n+1}\} = \{a_1, \ldots, a_n\} \cup \{a_{n+1}\} \ .$$

1.10 Subset Axioms

Axiom Schema 1.15 (Subset Axioms, Aussonderung Axioms)**.** *For any property $P(x)$ and any set B, there exists a set whose members are exactly the sets $x \in B$ such that $P(x)$.*

Definition Schema 1.16 (Extensionality, Subset)**.** For any property $P(x)$ and any set B, we write $\{x \in B \mid P(x)\}$ for $\{x \mid x \in B \wedge P(x)\}$.

Theorem 1.17 (Subset)**.** *There is no set to which every set belongs.*

Proof:
$\langle 1 \rangle 1.$ Let: A be a set.
Prove: There exists a set that does not belong to A.
$\langle 1 \rangle 2.$ Pick a set B whose members are exactly the sets $x \in A$ such that $x \notin x$.
Proof: By a Subset Axiom.
$\langle 1 \rangle 3.$ If $B \in A$ then we have $B \in B \Leftrightarrow B \notin B$
$\langle 1 \rangle 4.$ $B \notin A$
□

1.11 Intersection

Definition 1.18 (Intersection (Extensionality, Subset))**.** For any sets a and b, the *intersection* $a \cap b$ is $\{x \in a \mid x \in b\}$.

Theorem 1.19 (Extensionality, Subset)**.** *For any nonempty set A, there exists a unique set B such that, for any x, we have $x \in B$ if and only if x belongs to every member of A.*

Proof:
$\langle 1 \rangle 1.$ Let: A be a nonempty set.
$\langle 1 \rangle 2.$ Pick $a \in A$
$\langle 1 \rangle 3.$ Let: $B = \{x \in a \mid \forall y \in A.x \in y\}$
$\langle 1 \rangle 4.$ B is the unique set such that, for any x, we have $x \in B$ if and only if x belongs to every member of A.
□

Definition 1.20 (Intersection (Extensionality, Subset))**.** For any nonempty set A, the *intersection* $\bigcap A$ is the set whose elements are those sets that belong to every member of A.

1.12 Relative Complement

Definition 1.21 (Relative Complement (Extensionality, Subset))**.** For any sets A and B, the *relative complement* $A - B$ is $\{x \in A \mid x \notin B\}$.

1.13 Covers

Definition 1.22 (Cover)**.** Let X be a set and $\mathcal{A} \subseteq \mathcal{P}X$. Then $\mathcal{A}$ *covers* X, or is a *covering* of X, if and only if $\bigcup \mathcal{A} = X$.

Chapter 2

Relations

2.1 Ordered Pairs

Definition 2.1 (Ordered Pair (Extensionality, Pairing))**.** For any sets x and y, the *ordered pair* (x, y) is defined to be $\{\{x\}, \{x, y\}\}$.

Theorem 2.2 (Extensionality, Pairing)**.** *For any sets u, v, x, y, we have $(u, v) = (x, y)$ if and only if $u = x$ and $v = y$*

PROOF:
$\langle 1 \rangle 1$. ASSUME: $\{\{u\}, \{u, v\}\} = \{\{x\}, \{x, y\}\}$
$\langle 1 \rangle 2$. $\{u\} \in \{\{x\}, \{x, y\}\}$
$\langle 1 \rangle 3$. $\{u, v\} \in \{\{x\}, \{x, y\}\}$
$\langle 1 \rangle 4$. $\{u\} = \{x\}$ or $\{u\} = \{x, y\}$
$\langle 1 \rangle 5$. $\{u, v\} = \{x\}$ or $\{u, v\} = \{x, y\}$
$\langle 1 \rangle 6$. CASE: $\{u\} = \{x, y\}$
  $\langle 2 \rangle 1$. $u = x = y$
  $\langle 2 \rangle 2$. $u = v = x = y$
    PROOF: From $\langle 1 \rangle 5$
$\langle 1 \rangle 7$. CASE: $\{u, v\} = \{x\}$
  PROOF: Similar.
$\langle 1 \rangle 8$. CASE: $\{u\} = \{x\}$ and $\{u, v\} = \{x, y\}$
  $\langle 2 \rangle 1$. $u = x$
  $\langle 2 \rangle 2$. $u = y$ or $v = y$
  $\langle 2 \rangle 3$. CASE: $u = y$
    PROOF: This case is the case considered in $\langle 1 \rangle 6$.
  $\langle 2 \rangle 4$. CASE: $v = y$
    PROOF: We have $u = x$ and $v = y$ as required.
$\square$

Lemma 2.3 (Extensionality, Pairing, Power Set)**.** *Let x, y and C be sets. If $x \in C$ and $y \in C$ then $(x, y) \in \mathcal{P}\mathcal{P}C$.*

PROOF:

$\langle 1 \rangle 1$. LET: x, y and C be sets.
$\langle 1 \rangle 2$. ASSUME: $x \in C$
$\langle 1 \rangle 3$. ASSUME: $y \in C$
$\langle 1 \rangle 4$. $\{x\} \subseteq C$
$\langle 1 \rangle 5$. $\{x, y\} \subseteq C$
$\langle 1 \rangle 6$. $\{x\} \in \mathcal{P}C$
$\langle 1 \rangle 7$. $\{x, y\} \in \mathcal{P}C$
$\langle 1 \rangle 8$. $\{\{x\}, \{x, y\}\} \subseteq \mathcal{P}C$
$\langle 1 \rangle 9$. $\{\{x\}, \{x, y\}\} \in \mathcal{P}\mathcal{P}C$
$\square$

Lemma 2.4 (Extensionality, Pairing, Union)**.** *Let x, y and A be sets. If $(x, y) \in A$ then x and y belong to $\bigcup\bigcup A$.*

PROOF:
$\langle 1 \rangle 1$. LET: x, y and A be sets.
$\langle 1 \rangle 2$. ASSUME: $(x, y) \in A$
$\langle 1 \rangle 3$. $\{x, y\} \in \bigcup A$
$\langle 1 \rangle 4$. $x \in \bigcup\bigcup A$
$\langle 1 \rangle 5$. $y \in \bigcup\bigcup A$
$\square$

2.2 Cartesian Product

Definition 2.5 (Cartesian Product (Extensionality, Pairing, Union, Power Set, Subset))**.** Let A and B be sets. The *Cartesian product* $A \times B$ is the set $\{(x, y) \mid x \in A, y \in B\}$.

This is a set since, if $x \in A$ and $y \in B$, then $(x, y) \in \mathcal{P}\mathcal{P}(A \cup B)$ by Lemma 2.3.

2.3 Relations

Definition 2.6 (Relation (Extensionality, Pairing))**.** A *relation* is a set of ordered pairs.

Given a relation R, we write xRy for $(x, y) \in R$.

2.4 Domain

Definition 2.7 (Domain (Extensionality, Pairing, Union, Subset))**.** Let R be a set. The *domain* of R is $\operatorname{dom} R = \{x \mid \exists y.(x, y) \in R\}$.

This is a set by Lemma 2.4.

2.5 Range

Definition 2.8 (Range (Extensionality, Pairing, Union, Subset))**.** Let R be a set. The *range* of R is $\operatorname{ran} R = \{y \mid \exists x.(x, y) \in R\}$.

This is a set by Lemma 2.4.

2.6 Field

Definition 2.9 (Field (Extensionality, Pairing, Union, Subset))**.** Let R be a set. The *field* of R is $\operatorname{fld} R = \operatorname{dom} R \cup \operatorname{ran} R$.

2.7 Functions

Definition 2.10 (Extensionality, Pairing)**.** A *function* is a relation F such that, for all x, y, y', if xFy and xFy' then $y = y'$.

If there exists x such that xFy, then we write $F(x)$ for the unique such y, and call $F(x)$ the *value* of F at x.

Definition 2.11 (Extensionality, Pairing, Union, Subset)**.** We write $F : A \to B$ iff F is a function, $\operatorname{dom} F = A$ and $\operatorname{ran} F \subseteq B$.

Axiom 2.12 (Axiom of Choice, First Form)**.** *For any relation R, there exists a function $H \subseteq R$ such that* $\operatorname{dom} H = \operatorname{dom} R$.

2.8 Single-Rooted

Definition 2.13 (Extensionality, Pairing)**.** A set R is *single-rooted* if and only if, for all x, x', y, if xRy and $x'Ry$ then $x = x'$.

We call a function *one-to-one*, *injective* or an *injection* if and only if it is single-rooted.

2.9 Surjective

Definition 2.14 (Surjective)**.** Let $F : A \to B$. Then F is *surjective* if and only if $\operatorname{ran} F = B$.

2.10 Inverse

Definition 2.15 (Inverse (Extensionality, Pairing, Union, Power Set, Subset))**.** Let R be a set. The *inverse* of R is $R^{-1} = \{(y, x) \mid (x, y) \in R\}$.

This is a set because if $(x, y) \in R$ then $(y, x) \in \operatorname{ran} R \times \operatorname{dom} R$.

Theorem 2.16 (Extensionality, Pairing, Union, Power Set, Subset)**.** *For any set F, we have* $\operatorname{dom} F^{-1} = \operatorname{ran} F$.

PROOF: For any x, we have

$$\begin{aligned} x \in \operatorname{dom} F^{-1} &\Leftrightarrow \exists y.(x,y) \in F^{-1} \\ &\Leftrightarrow \exists y.(y,x) \in F \\ &\Leftrightarrow x \in \operatorname{ran} F \end{aligned}$$

The result follows by the Axiom of Extensionality. ∎

Theorem 2.17 (Extensionality, Pairing, Union, Power Set, Subset)**.** *For any set F, we have* $\operatorname{ran} F^{-1} = \operatorname{dom} F$.

PROOF: For any x, we have

$$\begin{aligned} x \in \operatorname{ran} F^{-1} &\Leftrightarrow \exists y.(y,x) \in F^{-1} \\ &\Leftrightarrow \exists y.(x,y) \in F \\ &\Leftrightarrow x \in \operatorname{dom} F \end{aligned}$$

The result follows by the Axiom of Extensionality. ∎

Theorem 2.18 (Extensionality, Pairing, Union, Power Set, Subset)**.** *For any relation F, we have* $(F^{-1})^{-1} = F$.

PROOF: For any z we have

$$\begin{aligned} z \in (F^{-1})^{-1} &\Leftrightarrow \exists x,y.z = (x,y) \land (y,x) \in F^{-1} \\ &\Leftrightarrow \exists x,y.z = (x,y) \land (x,y) \in F \\ &\Leftrightarrow z \in F && (F \text{ is a relation}) \end{aligned}$$

The result follows by the Axiom of Extensionality.

Theorem 2.19 (Extensionality, Pairing, Union, Power Set, Subset)**.** *For any set F, we have F^{-1} is a function if and only if F is single-rooted.*

PROOF: Immediate from definitions. ∎

Theorem 2.20 (Extensionality, Pairing, Union, Power Set, Subset)**.** *Let F be a relation. Then F is a function if and only if F^{-1} is single-rooted.*

PROOF: Immediate from definitions. ∎

Theorem 2.21 (Extensionality, Pairing, Union, Power Set, Subset)**.** *Let F be a one-to-one function and $x \in \operatorname{dom} F$. Then $F^{-1}(F(x)) = x$.*

PROOF: We have $(x, F(x)) \in F$ and so $(F(x), x) \in F^{-1}$. ∎

Theorem 2.22 (Extensionality, Pairing, Union, Power Set, Subset)**.** *Let F be a one-to-one function and $y \in \operatorname{ran} F$. Then $F(F^{-1}(y)) = y$.*

PROOF: From Theorems 2.16, 2.18 and 2.21. ∎

2.11 Composition

Definition 2.23 (Composition (Extensionality, Pairing, Union, Power Set, Subset))**.** Let R and S be relations. The *composition* of R and S is $S \circ R = \{(x,z) \mid \exists y.xRy \land ySz\}$.

This is a set because if xRy and ySz then $(x,z) \in \operatorname{dom} R \times \operatorname{ran} S$.

Theorem 2.24 (Extensionality, Pairing, Union, Power Set, Subset)**.** *Let F and G be functions. Then $G \circ F$ is a function, its domain is $\{x \in \operatorname{dom} F \mid F(x) \in \operatorname{dom} G\}$, and for x in this domain, we have $(F \circ G)(x) = F(G(x))$.*

PROOF:
$\langle 1 \rangle 1.$ $G \circ F$ is a function.
  $\langle 2 \rangle 1.$ LET: $x(G \circ F)z$ and $x(G \circ F)z'$
  $\langle 2 \rangle 2.$ PICK y, y' such that xFy, xFy', yGz and $y'Gz'$
  $\langle 2 \rangle 3.$ $y = y'$
    PROOF: Since F is a function.
  $\langle 2 \rangle 4.$ $z = z'$
    PROOF: Since G is a function.
$\langle 1 \rangle 2.$ $\operatorname{dom}(G \circ F) = \{x \in \operatorname{dom} F \mid F(x) \in \operatorname{dom} G\}$
  PROOF:

$$\begin{aligned} x \in \operatorname{dom}(G \circ F) &\Leftrightarrow \exists z. x(G \circ F)z \\ &\Leftrightarrow \exists y, z. xFy \wedge yGz \\ &\Leftrightarrow x \in \operatorname{dom} F \wedge F(x) \in \operatorname{dom} G \end{aligned}$$

$\langle 1 \rangle 3.$ For x in this domain, we have $(F \circ G)(x) = F(G(x))$.
  PROOF: Since $(x, F(x)) \in F$ and $(F(x), G(F(x))) \in G$.
□

Theorem 2.25 (Extensionality, Pairing, Union, Power Set, Subset)**.** *For any sets F and G, we have $(G \circ F)^{-1} = F^{-1} \circ G^{-1}$.*

PROOF:

$$\begin{aligned} (x, z) \in (G \circ F)^{-1} &\Leftrightarrow (z, x) \in G \circ F \\ &\Leftrightarrow \exists y. zFy \wedge yGx \\ &\Leftrightarrow \exists y. (y, z) \in F^{-1} \wedge (x, y) \in G^{-1} \\ &\Leftrightarrow (x, z) \in F^{-1} \circ G^{-1} \end{aligned}$$
□

2.12 Identity Function

Definition 2.26 (Identity Function (Extensionality, Pairing, Union, Power Set, Subset))**.** Let A be a set. The *identity function* id_A on A is $\{(x, x) \mid x \in A\}$.

This is a set because it is a subset of $A \times A$.

Theorem 2.27 (Extensionality, Pairing, Union, Power Set, Subset)**.** *Let $F : A \to B$ and A be nonempty. Then there exists a function $G : B \to A$ such that $G \circ F = \mathrm{id}_A$ if and only if F is one-to-one.*

PROOF:
$\langle 1 \rangle 1.$ LET: $F : A \to B$
$\langle 1 \rangle 2.$ ASSUME: A is nonempty
$\langle 1 \rangle 3.$ If there exists $G : B \to A$ such that $G \circ F = \mathrm{id}_A$ then F is one-to-one.
  $\langle 2 \rangle 1.$ ASSUME: $G : B \to A$ and $G \circ F = \mathrm{id}_A$

$\langle 2 \rangle 2$. LET: $x, y \in A$
$\langle 2 \rangle 3$. ASSUME: $F(x) = F(y)$
$\langle 2 \rangle 4$. $x = y$
PROOF: $x = G(F(x)) = G(F(y)) = y$.
$\langle 1 \rangle 4$. If F is one-to-one then there exists $G : B \to A$ such that $G \circ F = \mathrm{id}_A$.
$\langle 2 \rangle 1$. ASSUME: F is one-to-one.
$\langle 2 \rangle 2$. PICK $a \in A$
$\langle 2 \rangle 3$. Define $G : B \to A$ by: $G(y)$ is the x such that $F(x) = y$ if $y \in \operatorname{ran} F$, otherwise $G(y) = a$
$\langle 2 \rangle 4$. $G \circ F = \mathrm{id}_A$
PROOF: For $x \in A$ we have $(G \circ F)(x) = G(F(x)) = x$ by Theorem 2.24.
□

Theorem 2.28 (Extensionality, Pairing, Union, Power Set, Subset)**.** *Let $F : A \to B$ and A be nonempty. If there exists a function $H : B \to A$ such that $F \circ H = \mathrm{id}_B$ then F is surjective.*

PROOF:
$\langle 1 \rangle 1$. LET: $F : A \to B$
$\langle 1 \rangle 2$. ASSUME: A is nonempty.
$\langle 1 \rangle 3$. LET: $H : B \to A$ satisfy $F \circ H = \mathrm{id}_B$
$\langle 1 \rangle 4$. LET: $y \in B$
$\langle 1 \rangle 5$. $F(H(y)) = y$.
□

Theorem 2.29 (Extensionality, Pairing, Union, Power Set, Subset, Choice)**.** *Let $F : A \to B$ and A be nonempty. If F is surjective then there exists a function $H : B \to A$ such that $F \circ H = \mathrm{id}_B$.*

PROOF:
$\langle 1 \rangle 1$. ASSUME: F is surjective.
$\langle 1 \rangle 2$. PICK a function $H \subseteq F^{-1}$ with $\operatorname{dom} H = B$
PROOF: By the Axiom of Choice.
$\langle 1 \rangle 3$. $H : B \to A$
$\langle 1 \rangle 4$. $F \circ H = \mathrm{id}_B$
$\langle 2 \rangle 1$. LET: $y \in B$
$\langle 2 \rangle 2$. $(y, H(y)) \in F^{-1}$
$\langle 2 \rangle 3$. $(H(y), y) \in F$
$\langle 2 \rangle 4$. $F(H(y)) = y$
□

2.13 Restriction

Definition 2.30 (Restriction (Extensionality, Pairing, Subset))**.** Let R be a relation and A a set. The *restriction* of R to A is $R \restriction A = \{(x, y) \mid x \in A \wedge xRy\}$.

This is a set because it is a subset of R.

2.14 Image

Definition 2.31 (Image (Extensionality, Pairing, Union, Subset))**.** Let F be a function and $A \subseteq \operatorname{dom} F$. The *image* of A under F is $\{F(x) \mid x \in A\}$.

This is a set because it is a subset of $\operatorname{ran} F$.

Theorem 2.32 (Extensionality, Pairing, Union, Power Set, Subset)**.** *For any sets F and $\mathcal{A}$ we have*

$$F\left(\bigcup \mathcal{A}\right) = \bigcup_{A \in \mathcal{A}} F(A)$$

Proof: Each is the set of all y such that $\exists x. \exists A. x \in A \in \mathcal{A} \wedge y = F(x)$. ∎

Corollary 2.32.1. *For any sets F, A_1, …, A_n, we have*

$$F(A_1 \cup \cdots \cup A_n) = F(A_1) \cup \cdots \cup F(A_n) \ .$$

Theorem 2.33 (Extensionality, Pairing, Union, Power Set, Subset)**.** *For any sets F and $\mathcal{A}$ with $\mathcal{A}$ nonempty, we have*

$$F\left(\bigcap \mathcal{A}\right) \subseteq \bigcap_{A \in \mathcal{A}} F(A) \ .$$

Equality holds if F is single-rooted.

Proof:
⟨1⟩1. $F\left(\bigcap \mathcal{A}\right) \subseteq \bigcap_{A \in \mathcal{A}} F(A)$
  ⟨2⟩1. Let: $y \in F\left(\bigcap \mathcal{A}\right)$
  ⟨2⟩2. Pick $x \in \bigcap \mathcal{A}$ such that $y = F(x)$
  ⟨2⟩3. Let: $A \in \mathcal{A}$
  ⟨2⟩4. $x \in A$
  ⟨2⟩5. $y \in F(A)$
⟨1⟩2. If F is single-rooted then $F\left(\bigcap \mathcal{A}\right) = \bigcap_{A \in \mathcal{A}} F(A)$
  ⟨2⟩1. Assume: F is single-rooted.
  ⟨2⟩2. Let: $y \in \bigcap_{A \in \mathcal{A}} F(A)$
  ⟨2⟩3. Pick $A \in \mathcal{A}$
  ⟨2⟩4. Pick $x \in A$ such that $y = F(x)$
  ⟨2⟩5. $x \in \bigcap \mathcal{A}$
    ⟨3⟩1. Let: $A' \in \mathcal{A}$
    ⟨3⟩2. Pick $x' \in A'$ such that $y = F(x')$
    ⟨3⟩3. $x = x'$
      Proof: By ⟨2⟩1.
    ⟨3⟩4. $x \in A'$
∎

Corollary 2.33.1 (Extensionality, Pairing, Union, Power Set, Subset)**.** *For any function F and set $\mathcal{A}$ with $\mathcal{A}$ nonempty, we have*

$$F^{-1}\left(\bigcap \mathcal{A}\right) = \bigcap_{A \in \mathcal{A}} F^{-1}(A) \ .$$

Corollary 2.33.2 (Extensionality, Pairing, Union, Power Set, Subset)**.** *For any sets F, A_1, ..., A_n, we have*

$$F(A_1 \cap \cdots \cap A_n) \subseteq F(A_1) \cap \cdots \cap F(A_n) \ .$$

Equality holds if F is single-rooted.

Corollary 2.33.3 (Extensionality, Pairing, Union, Power Set, Subset)**.** *For any function F and sets A_1, ..., A_n, we have*

$$F^{-1}(A_1 \cap \cdots \cap A_n) = F^{-1}(A_1) \cap \cdots \cap F^{-1}(A_n) \ .$$

Theorem 2.34 (Extensionality, Pairing, Union, Power Set, Subset)**.** *For any sets F, A and B, we have*

$$F(A) - F(B) \subseteq F(A - B) \ .$$

Equality holds if F is single-rooted.

Proof:
⟨1⟩1. Let: F, A and B be sets.
⟨1⟩2. $F(A) - F(B) \subseteq F(A - B)$
  ⟨2⟩1. Let: $y \in F(A) - F(B)$
  ⟨2⟩2. Pick $x \in A$ such that xFy
  ⟨2⟩3. $x \in A - B$
⟨1⟩3. If F is single-rooted then $F(A - B) = F(A) - F(B)$.
  ⟨2⟩1. Assume: F is single-rooted.
  ⟨2⟩2. Let: $y \in F(A - B)$
  ⟨2⟩3. Pick $x \in A - B$ such that $y = F(x)$
  ⟨2⟩4. $y \in F(A)$
  ⟨2⟩5. $y \notin F(B)$
    ⟨3⟩1. Assume: for a contradiction $x' \in B$ and $x'Fy$
    ⟨3⟩2. $x' = x$
      Proof: From ⟨2⟩1
    ⟨3⟩3. $x \in B$
    ⟨3⟩4. Q.E.D.
      Proof: This contradicts ⟨2⟩3.
□

Corollary 2.34.1 (Extensionality, Pairing, Union, Power Set, Subset)**.** *For any function F and sets A and B, we have*

$$F^{-1}(A) - F^{-1}(B) = F^{-1}(A - B) \ .$$

2.15 Infinite Cartesian Product

Definition 2.35 (Infinite Cartesian Product (Extensionality, Pairing, Union, Power Set, Subset))**.** Let H be a function with domain I. The *Cartesian product*

$\prod_{i \in I} H(i)$ is the set of all functions f with domain I such that, for all $i \in I$, we have $f(i) \in H(i)$.

This is a set because it is a subset of $\mathcal{P}(I \times \bigcup \operatorname{ran} H)$.

Theorem 2.36 (Axiom of Choice, Second Version (Extensionality, Pairing, Union, Power Set, Subset))**.** *The Axiom of Choice is equivalent to the statement: for any function H with domain I, if $H(i)$ is nonempty for all $i \in I$, then $\prod_{i \in I} H(i)$ is nonempty.*

PROOF:

$\langle 1 \rangle 1.$ If the Axiom of Choice is true then, for any function H with domain I, if $H(i)$ is nonempty for all $i \in I$, then $\prod_{i \in I} H(i)$ is nonempty.

$\langle 2 \rangle 1.$ ASSUME: The Axiom of Choice

$\langle 2 \rangle 2.$ LET: H be a function with domain I such that $H(i)$ is nonempty for all $i \in I$.

$\langle 2 \rangle 3.$ PICK a function $f \subseteq \{(i, x) \mid x \in H(i)\}$

$\langle 2 \rangle 4.$ $f \in \prod_{i \in I} H(i)$

$\langle 1 \rangle 2.$ If, for any function H with domain I, if $H(i)$ is nonempty for all $i \in I$, then $\prod_{i \in I} H(i)$ is nonempty, then the Axiom of Choice is true.

$\langle 2 \rangle 1.$ ASSUME: for any function H with domain I, if $H(i)$ is nonempty for all $i \in I$, then $\prod_{i \in I} H(i)$ is nonempty

$\langle 2 \rangle 2.$ LET: R be a relation.

$\langle 2 \rangle 3.$ LET: $I = \operatorname{dom} R$

$\langle 2 \rangle 4.$ LET: H be the function with domain I such that $H(i) = \{y \mid iRy\}$ for all i.

$\langle 2 \rangle 5.$ PICK $f \in \prod_{i \in I} H(i)$

$\langle 2 \rangle 6.$ $f \subseteq R$

$\square$

2.16 Reflexive Relations

Definition 2.37 (Reflexive (Extensionality, Pairing))**.** Let R be a relation on A. Then R is *reflexive* on A if and only if, for all $x \in A$, we have xRx.

2.17 Symmetric

Definition 2.38 (Symmetric (Extensionality, Pairing))**.** Let R be a relation. Then R is *symmetric* if and only if, whenever xRy, then yRx.

2.18 Transitivity

Definition 2.39 (Transitivity (Extensionality, Pairing))**.** Let R be a relation. Then R is *transitive* if and only if, whenever xRy and yRz, then xRz.

2.19 Equivalence Relations

Definition 2.40 (Equivalence Relation (Extensionality, Pairing))**.** Let R be a relation on A. Then R is an *equivalence relation* on A if and only if R is reflexive on A, symmetric and transitive.

Theorem 2.41 (Extensionality, Pairing, Union, Subset)**.** *If R is a symmetric and transitive relation then R is an equivalence relation on* $\operatorname{fld} R$*.*

PROOF:
$\langle 1\rangle 1$. LET: R be a symmetric and transitive relation.
$\langle 1\rangle 2$. LET: $x \in \operatorname{fld} R$
$\langle 1\rangle 3$. PICK y such that xRy or yRx
$\langle 1\rangle 4$. xRy and yRx
  PROOF: By symmetry.
$\langle 1\rangle 5$. xRx
  PROOF: By transitivity.
□

2.20 Equivalence Class

Definition 2.42 (Equivalence Class (Extensionality, Pairing, Subset))**.** Let R be an equivalence relation on A and $a \in A$. Then the *equivalence class* of *a modulo R* is

$$[a]_R = \{x \in A \mid aRx\} \enspace .$$

Lemma 2.43 (Extensionality, Pairing, Subset)**.** *Let R be an equivalence relation on A and $x, y \in A$. Then $[x]_R = [y]_R$ if and only if xRy.*

PROOF:
$\langle 1\rangle 1$. If $[x]_R = [y]_R$ then xRy.
  $\langle 2\rangle 1$. ASSUME: $[x]_R = [y]_R$
  $\langle 2\rangle 2$. $y \in [y]_R$
    PROOF: Since yRy by reflexivity.
  $\langle 2\rangle 3$. $y \in [x]_R$
  $\langle 2\rangle 4$. xRy
$\langle 1\rangle 2$. If xRy then $[x]_R = [y]_R$.
  $\langle 2\rangle 1$. ASSUME: xRy
  $\langle 2\rangle 2$. $[y]_R \subseteq [x]_R$
    PROOF: If yRz then xRz by transitivity.
  $\langle 2\rangle 3$. $[x]_R \subseteq [y]_R$
    PROOF: Similar since yRx by symmetry.
□

2.21 Disjoint

Definition 2.44 (Disjoint)**.** Two sets A and B are *disjoint* if and only if there is no x such that $x \in A$ and $x \in B$.

2.22 Partitions

Definition 2.45 (Partition)**.** A *partition* P of a set A is a set of nonempty subsets of A such that:

1. For all $x \in A$ there exists $S \in P$ such that $x \in S$.
2. Any two distinct elements of P are disjoint.

2.23 Quotient Sets

Definition 2.46 (Quotient Set (Extensionality, Pairing, Power Set, Subset))**.** Let R be an equivalence relation on A. The *quotient set* A/R is the set of all equivalence classes modulo R.

This is a set because it is a subset of $\mathcal{P}A$.

Theorem 2.47 (Extensionality, Pairing, Power Set, Subset)**.** *Let R be an equivalence relation on A. Then the quotient set A/R is a partition of A.*

PROOF:
$\langle 1 \rangle 1$. For all $x \in A$ there exists $y \in A$ such that $x \in [y]_R$
PROOF: Take $y = x$.
$\langle 1 \rangle 2$. Any two distinct equivalence classes are disjoint.
$\langle 2 \rangle 1$. ASSUME: $z \in [x]_R$ and $z \in [y]_R$
$\langle 2 \rangle 2$. xRz and yRz
$\langle 2 \rangle 3$. $[x]_R = [z]_R = [y]_R$
PROOF: Lemma 2.43.
□

Definition 2.48 (Canonical Map (Extensionality, Pairing, Power Set, Subset))**.** Let R be an equivalence relation on A. The *canonical map* $\phi : A \to A/R$ is the function defined by $\phi(a) = [a]_R$.

Theorem 2.49. *Let R be an equivalence relation on A and $F : A \to B$. Then the following are equivalent:*

1. *For all $x, y \in A$, if xRy then $F(x) = F(y)$.*
2. *There exists $G : A/R \to B$ such that $F = G \circ \phi$, where $\phi : A \to A/R$ is the canonical map.*

In this case, G is unique.

PROOF:

$\langle 1 \rangle 1.$ $1 \Rightarrow 2$
$\langle 2 \rangle 1.$ ASSUME: 1
$\langle 2 \rangle 2.$ Let $G = \{([a]_R, b) \mid F(a) = b\}$
$\langle 2 \rangle 3.$ G is a function.
$\langle 3 \rangle 1.$ LET: $(c, b), (c, b') \in G$
$\langle 3 \rangle 2.$ PICK $a, a' \in A$ such that $c = [a]_R = [a']_R$ with $F(a) = b$ and $F(a') = b'$
$\langle 3 \rangle 3.$ aRa'
PROOF: Lemma 2.43.
$\langle 3 \rangle 4.$ $F(a) = F(a')$
PROOF: From $\langle 2 \rangle 1$.
$\langle 3 \rangle 5.$ $b = b'$
PROOF: From $\langle 3 \rangle 2$.
$\langle 2 \rangle 4.$ $F = G \circ \phi$
PROOF: For $a \in A$ we have $G(\phi(a)) = G([a]) = F(a)$.
$\langle 1 \rangle 2.$ $2 \Rightarrow 1$
$\langle 2 \rangle 1.$ LET: $G : A/R \to B$ be such that $F = G \circ \phi$
$\langle 2 \rangle 2.$ LET: $x, y \in A$
$\langle 2 \rangle 3.$ ASSUME: xRy
$\langle 2 \rangle 4.$ $G([x]) = G([y])$
PROOF: Lemma 2.43
$\langle 2 \rangle 5.$ $F(x) = F(y)$
PROOF: From $\langle 2 \rangle 1$.
$\langle 1 \rangle 3.$ If $G, G' : A/R \to B$ and $G \circ \phi = G' \circ \phi$ then $G = G'$
PROOF: For any $a \in A$ we have $G([a]) = G'([a])$.
$\square$

2.24 The Finite Intersection Property

Definition 2.50 (Finite Intersection Property)**.** Let X be a set and $\mathcal{A} \subseteq \mathcal{P}X$. Then $\mathcal{A}$ satisfies the *finite intersection property* if and only if every nonempty finite subset of $\mathcal{A}$ has nonempty intersection.

Lemma 2.51. *Let X be a set. Let $\mathcal{A} \subseteq \mathcal{P}X$ have the finite intersection property. Then there exists a maximal set $\mathcal{D}$ such that $\mathcal{A} \subseteq \mathcal{D} \subseteq \mathcal{P}X$ and $\mathcal{D}$ has the finite intersection property.*

PROOF:
$\langle 1 \rangle 1.$ LET: $\mathbb{F} = \{\mathcal{D} \mid \mathcal{A} \subseteq \mathcal{D} \subseteq \mathcal{P}X, \mathcal{D} \text{ has the finite intersection property}\}$
$\langle 1 \rangle 2.$ Every chain in $\mathbb{F}$ has an upper bound.
$\langle 2 \rangle 1.$ LET: $\mathbb{C}$ be a chain in $\mathbb{F}$.
$\langle 2 \rangle 2.$ ASSUME: without loss of generality $\mathbb{C} \neq \emptyset$
PROVE: $\bigcup \mathbb{C} \in \mathbb{F}$
PROOF: If $\mathbb{C} = \emptyset$ then $\mathcal{A}$ is an upper bound.
$\langle 2 \rangle 3.$ $\mathcal{A} \subseteq \bigcup \mathbb{C} \subseteq \mathcal{P}X$
$\langle 2 \rangle 4.$ LET: $C_1, \ldots, C_n \in \mathbb{C}$

PROVE: $C_1 \cap \cdots \cap C_n \neq \emptyset$
$\langle 2 \rangle 5.$ PICK $\mathcal{C}_1, \ldots, \mathcal{C}_n \in \mathbb{C}$ such that $C_i \in \mathcal{C}_i$ for all i.
$\langle 2 \rangle 6.$ ASSUME: without loss of generality $\mathcal{C}_1 \subseteq \cdots \subseteq \mathcal{C}_n$
$\langle 2 \rangle 7.$ $C_1, \ldots, C_n \in \mathcal{C}_n$
$\langle 2 \rangle 8.$ $\mathcal{C}_n$ satisfies the finite intersection property.
$\langle 2 \rangle 9.$ $C_1 \cap \cdots \cap C_n \neq \emptyset$
$\langle 1 \rangle 3.$ Q.E.D.
PROOF: By Zorn's Lemma.
□

Lemma 2.52. *Let X be a set. Let $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the finite intersection property. Then any finite intersection of elements of $\mathcal{D}$ is an element of $\mathcal{D}$.*

PROOF:
$\langle 1 \rangle 1.$ LET: $D_1, D_2 \in \mathcal{D}$
$\langle 1 \rangle 2.$ $\mathcal{D} \cup \{D_1 \cap D_2\}$ has the finite intersection property.
PROOF: Any finite intersection of members of $\mathcal{D} \cup \{D_1 \cap D_2\}$ is a finite intersection of members of $\mathcal{D}$.
$\langle 1 \rangle 3.$ $\mathcal{D} = \mathcal{D} \cup \{D_1 \cap D_2\}$
PROOF: By maximality of $\mathcal{D}$.
$\langle 1 \rangle 4.$ $D_1 \cap D_2 \in \mathcal{D}$.
□

Lemma 2.53. *Let X be a set. Let $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the finite intersection property. Let $A \subseteq X$. If A intersects every member of $\mathcal{D}$ then $A \in \mathcal{D}$.*

PROOF:
$\langle 1 \rangle 1.$ $\mathcal{D} \cup \{A\}$ has the finite intersection property.
$\langle 2 \rangle 1.$ LET: $D_1, \ldots, D_n \in \mathcal{D}$
PROVE: $D_1 \cap \cdots \cap D_n \cap A \neq \emptyset$
$\langle 2 \rangle 2.$ $D_1 \cap \cdots \cap D_n \in \mathcal{D}$
PROOF: Lemma 2.52.
$\langle 2 \rangle 3.$ $D_1 \cap \cdots \cap D_n \cap A \neq \emptyset$
PROOF: Since A intersects every member of $\mathcal{D}$.
$\langle 1 \rangle 2.$ Q.E.D.
PROOF: By maximality of $\mathcal{D}$.
□

Proposition 2.54. *Let X be a set. Let $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the finite intersection property. Let $A, D \in \mathcal{P}X$. If $D \in \mathcal{D}$ and $D \subseteq A$ then $A \in \mathcal{D}$.*

PROOF:
$\langle 1 \rangle 1.$ $\mathcal{D} \cup \{A\}$ satisfies the finite intersection property.
$\langle 2 \rangle 1.$ LET: $D_1, \ldots, D_n \in \mathcal{D}$
$\langle 2 \rangle 2.$ $D_1 \cap \cdots \cap D_n \cap D \neq \emptyset$

PROOF: Since $\mathcal{D}$ satisfies the finite intersection property.
⟨2⟩3. $D_1 \cap \cdots \cap D_n \cap A \neq \emptyset$
⟨1⟩2. $\mathcal{D} = \mathcal{D} \cup \{A\}$
PROOF: By the maximality of $\mathcal{D}$.
⟨1⟩3. $A \in \mathcal{D}$
□

Definition 2.55 (Graph)**.** Let $f : A \to B$. The *graph* of f is the set $\{(x, f(x)) \mid x \in A\} \subseteq A \times B$.

2.25 Countable Intersection Property

Definition 2.56 (Countable Intersection Property)**.** Let X be a set and $\mathcal{A} \subseteq \mathcal{P}X$. Then $\mathcal{A}$ satisfies the *countable intersection property* if and only if every countable subset of $\mathcal{A}$ has nonempty intersection.

Lemma 2.57. *Let X be a set. Let $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the countable intersection property. Then any countable intersection of elements of $\mathcal{D}$ is an element of $\mathcal{D}$.*

PROOF:
⟨1⟩1. LET: $\mathcal{D}_0 \subseteq \mathcal{D}$ be countable.
⟨1⟩2. $\mathcal{D} \cup \{\bigcap \mathcal{D}_0\}$ has the countable intersection property.
PROOF: Any countable intersection of members of $\mathcal{D} \cup \{\bigcap \mathcal{D}_0\}$ is a finite intersection of members of $\mathcal{D}$.
⟨1⟩3. $\mathcal{D} = \mathcal{D} \cup \{\bigcap \mathcal{D}_0\}$
PROOF: By maximality of $\mathcal{D}$.
⟨1⟩4. $\bigcap \mathcal{D}_0 \in \mathcal{D}$.
□

Lemma 2.58. *Let X be a set. Let $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the countable intersection property. Let $A \subseteq X$. If A intersects every member of $\mathcal{D}$ then $A \in \mathcal{D}$.*

PROOF:
⟨1⟩1. $\mathcal{D} \cup \{A\}$ has the countable intersection property.
⟨2⟩1. LET: $\mathcal{D}_0 \subseteq \mathcal{D}$ be countable.
PROVE: $\bigcap \mathcal{D}_0 \cap A \neq \emptyset$
⟨2⟩2. $\bigcap \mathcal{D}_0 \in \mathcal{D}$
PROOF: Lemma 2.57.
⟨2⟩3. $\bigcap \mathcal{D}_0 \cap A \neq \emptyset$
PROOF: Since A intersects every member of $\mathcal{D}$.
⟨1⟩2. Q.E.D.
PROOF: By maximality of $\mathcal{D}$.
□

2.26 The Axiom of Choice

Axiom 2.59 (Axiom of Choice)**.** *Let $\mathcal{A}$ be a set of disjoint nonempty sets. Then there exists a set C consisting of exactly one element from each member of $\mathcal{A}$.*

2.27 Choice Functions

Definition 2.60 (Choice Function)**.** Let $\mathcal{B}$ be a set of nonempty sets. A *choice function* for $\mathcal{B}$ is a function $c : \mathcal{B} \to \bigcup \mathcal{B}$ such that, for all $B \in \mathcal{B}$, we have $c(B) \in B$.

Lemma 2.61 (Existence of a Choice Function (AC))**.** *Every set of nonempty sets has a choice function.*

PROOF:
$\langle 1 \rangle 1$. LET: $\mathcal{B}$ be a set of nonempty sets.
$\langle 1 \rangle 2$. For $B \in \mathcal{B}$,
LET: $B' = \{B\} \times B$
$\langle 1 \rangle 3$. $\{B' \mid B \in \mathcal{B}\}$ is a set of disjoint nonempty sets.
$\langle 1 \rangle 4$. PICK a set c consisting of exactly one element from each B' for $B \in \mathcal{B}$.
$\langle 1 \rangle 5$. c is a choice function for $\mathcal{B}$.
□

2.28 Order Theory

Definition 2.62 (Preorder)**.** Let X be a set. A *preorder* on X is a binary relation $\leq$ on X such that:

Reflexivity For all $x \in X$, we have $x \leq x$

Transitivity For all $x, y, z \in X$, if $x \leq y$ and $y \leq z$ then $x \leq z$.

Definition 2.63 (Preordered Set)**.** A *preordered set* consists of a set X and a preorder $\leq$ on X.

Proposition 2.64. *Let X and Y be linearly ordered sets. Let $f : X \twoheadrightarrow Y$ be strictly monotone and surjective. Then f is a poset isomorphism.*

PROOF:
⟨1⟩1. f is injective.
  ⟨2⟩1. LET: $x, y \in X$
  ⟨2⟩2. ASSUME: $f(x) = f(y)$
  ⟨2⟩3. $x \not< y$
    PROOF: By strong motonicity.
  ⟨2⟩4. $y \not< x$
    PROOF: By strong motonicity.
  ⟨2⟩5. $x = y$
    PROOF: By trichotomy.
⟨1⟩2. f^{-1} is monotone.
  ⟨2⟩1. LET: $x, y \in X$
  ⟨2⟩2. ASSUME: $x \leq y$
  ⟨2⟩3. $f^{-1}(x) \not> f^{-1}(y)$
    PROOF: By strong motonicity.
  ⟨2⟩4. $f^{-1}(x) < f^{-1}(y)$
    PROOF: By trichotomy.
□

Definition 2.65 (Interval)**.** Let X be a preordered set and $Y \subseteq X$. Then Y is an *interval* if and only if, for all $a, b \in Y$ and $c \in X$, if $a \leq c \leq b$ then $c \in Y$.

Definition 2.66 (Linear Continuum)**.** A linearly ordered set L is a *linear continuum* if and only if:

1. every nonempty subset of L that is bounded above has a supremum

2. L is dense

Proposition 2.67. *Every interval in a linear continuum is a linear continuum.*

PROOF:
⟨1⟩1. LET: L be a linear continuum and I an interval in L.
⟨1⟩2. Every nonempty subset of I that is bounded above has a supremum in I.
  ⟨2⟩1. LET: $X \subseteq I$ be nonempty and bounded above by $b \in I$.

$\langle 2\rangle 2$. LET: s be the supremum of X in L.
PROOF: Since L is a linear continuum.
$\langle 2\rangle 3$. $s \in I$
$\langle 3\rangle 1$. PICK $a \in X$
PROOF: Since X is nonempty ($\langle 2\rangle 1$).
$\langle 3\rangle 2$. $a \leq s \leq b$
$\langle 3\rangle 3$. $s \in I$
PROOF: Since I is an interval ($\langle 1\rangle 1$).
$\langle 2\rangle 4$. s is the supremum of X in I
$\langle 1\rangle 3$. I is dense.
$\langle 2\rangle 1$. LET: $x, y \in I$ with $x < y$
$\langle 2\rangle 2$. PICK $z \in L$ with $x < z < y$
PROOF: Since L is dense.
$\langle 2\rangle 3$. $z \in I$
PROOF: Since I is an interval.
□

Definition 2.68 (Ordered Square)**.** The *ordered square* I_o^2 is the set $[0,1]^2$ under the dictionary order.

Proposition 2.69. *The ordered square is a linear continuum.*

PROOF:
$\langle 1\rangle 1$. Every nonempty subset of I_o^2 bounded above has a supremum.
$\langle 2\rangle 1$. LET: $X \subseteq I_o^2$ be nonempty and bounded above by (b, c)
$\langle 2\rangle 2$. LET: $s = \sup \pi_1(X)$
PROOF: The set $\pi_1(X)$ is nonempty and bounded above by b.
$\langle 2\rangle 3$. CASE: $s \in \pi_1(X)$
$\langle 3\rangle 1$. LET: $t = \sup\{y \in [0,1] \mid (s, y) \in X\}$
PROOF: This set is nonempty and bounded above by c.
$\langle 3\rangle 2$. (s, t) is the supremum of X.
$\langle 2\rangle 4$. CASE: $s \notin \pi_1(X)$
PROOF: In this case $(s, 0)$ is the supremum of X.
$\langle 1\rangle 2$. I_o^2 is dense.
$\langle 2\rangle 1$. LET: $(x_1, y_1), (x_2, y_2) \in I_o^2$ with $(x_1, y_1) < (x_2, y_2)$
$\langle 2\rangle 2$. CASE: $x_1 < x_2$
$\langle 3\rangle 1$. PICK x_3 with $x_1 < x_3 < x_2$
$\langle 3\rangle 2$. $(x_1, y_1) < (x_3, y_1) < (x_2, y_2)$
$\langle 2\rangle 3$. CASE: $x_1 = x_2$ and $y_1 < y_2$
$\langle 3\rangle 1$. PICK y_3 with $y_1 < y_3 < y_2$
$\langle 3\rangle 2$. $(x_1, y_1) < (x_1, y_3) < (x_2, y_2)$
□

Proposition 2.70. *If X is a well-ordered set then $X \times [0,1)$ under the dictionary order is a linear continuum.*

PROOF:
$\langle 1\rangle 1$. Every nonempty set $A \subseteq X \times [0,1)$ bounded above has a supremum.

⟨2⟩1. LET: $A \subseteq X \times [0,1)$ be nonempty and bounded above
⟨2⟩2. LET: x_0 be the supremum of $\pi_1(A)$
⟨2⟩3. CASE: $x_0 \in \pi_1(A)$
⟨3⟩1. LET: y_0 be the supremum of $\{y \in [0,1) \mid (x_0, y) \in A\}$
⟨3⟩2. (x_0, y_0) is the supremum of A.
⟨2⟩4. CASE: $x_0 \notin \pi_1(A)$
PROOF: In this case $(x_0, 0)$ is the supremum of A.
⟨1⟩2. $X \times [0,1)$ is dense.
⟨2⟩1. LET: $(x_1, y_1), (x_2, y_2) \in X \times [0,1)$ with $(x_1, y_1) < (x_2, y_2)$
⟨2⟩2. CASE: $x_1 < x_2$
⟨3⟩1. PICK y_3 such that $y_1 < y_3 < 1$
⟨3⟩2. $(x_1, y_1) < (x_1, y_3) < (x_2, y_2)$
⟨2⟩3. CASE: $x_1 = x_2$ and $y_1 < y_2$
⟨3⟩1. PICK y_3 such that $y_1 < y_3 < y_2$
⟨3⟩2. $(x_1, y_1) < (x_1, y_3) < (x_2, y_2)$
□

Lemma 2.71. *For all $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$, we have $[a,b) \cong [c,d)$*

PROOF: The map $\lambda t.c + (t-a)(d-c)/(b-a)$ is an order isomorphism.

Proposition 2.72. *Let X be a linearly ordered set. Let $a < b < c$ in X. Then $[a,c) \cong [0,1)$ if and only if $[a,b) \cong [b,c) \cong [0,1)$.*

PROOF:
⟨1⟩1. If $[a,c) \cong [0,1)$ then $[a,b) \cong [b,c) \cong [0,1)$
⟨2⟩1. ASSUME: $f : [a,c) \cong [0,1)$ is an order isomorphism
⟨2⟩2. $[a,b) \cong [0,1)$
PROOF:
$$\begin{aligned} [a,b) &\cong [0, f(b)) && \text{(by the restriction of } f\text{)} \\ &\cong [0,1) && \text{(Lemma 2.71)} \end{aligned}$$
⟨2⟩3. $[b,c) \cong [0,1)$
PROOF: Similar.
⟨1⟩2. If $[a,b) \cong [b,c) \cong [0,1)$ then $[a,c) \cong [0,1)$
PROOF:
$$\begin{aligned} [a,c) &= [a,b) * [b,c) \\ &\cong [0,1) * [0,1) \\ &\cong [0,1/2) * [1/2,1) && \text{(Lemma 2.71)} \\ &= 1 \end{aligned}$$
□

Proposition 2.73 (CC)**.** *Let X be a linearly ordered set. Let $x_0 < x_1 < \cdots$ be a strictly increasing sequence in X with supremum b. Then $[x_0, b) \cong [0,1)$ if and only if $[x_i, x_{i+1}) \cong [0,1)$ for all i.*

PROOF:
⟨1⟩1. If $[x_0, b) \cong [0,1)$ then $[x_i, x_{i+1}) \cong [0,1)$ for all i.

PROOF: By Lemma 2.71
$\langle 1\rangle 2$. If $[x_i, x_{i+1}) \cong [0,1)$ for all i then $[x_0, b) \cong [0,1)$
$\langle 2\rangle 1$. ASSUME: $[x_i, x_{i+1}) \cong [0,1)$ for all i
$\langle 2\rangle 2$. PICK an order isomorphism $f_i : [x_i, x_{i+1}) \cong [1/2^i, 2/2^{i+1})$ for each i.
PROOF: By Lemma 2.71
$\langle 2\rangle 3$. The union of the f_is is an order isomorphism $[x_0, b) \cong [0,1)$
□

2.29 Partially Ordered Sets

Definition 2.74 (Partial Order). A *partial order* on a set X is a preorder $\leq$ that is *anti-symmetric*, i.e. whenever $x \leq y$ and $y \leq x$ then $x = y$.

Definition 2.75 (Linear Order). A *linear order* on a set X is a partial order such that, for any $x, y \in X$, either $x \leq y$ or $y \leq x$.

2.30 Strict Linear Orders

Definition 2.76 (Strict Linear Order (Extensionality, Pairing)). Let A be a set. A *strict linear order* on A is a binary relation R on A that is transitive and satisfies *trichotomy*: for any $x, y \in A$, exactly one of xRy, $x = y$, yRx holds.

Theorem 2.77. *Let R be a strict linear order on A. Then there is no $x \in A$ such that xRx.*

PROOF: Immediate from trichotomy.

Definition 2.78 (Well-ordering). A *well-order* on a set X is a linear order such that every nonempty set has a least element.

Definition 2.79 (Section). Given a well-ordered set X and $\alpha \in X$, the *section* of X by α is $S_\alpha = \{x \in X \mid x < \alpha\}$.

Theorem 2.80 (Transfinite Induction). *Let J be a well-ordered set and $J_0 \subseteq J$. Suppose that, for all $\alpha \in J$, if $S_\alpha \subseteq J_0$ then $\alpha \in J_0$. Then $J_0 = J$.*

PROOF:
$\langle 1\rangle 1$. ASSUME: for a contradiction $J_0 \neq J$
$\langle 1\rangle 2$. LET: α be the least element of $J \setminus J_0$
$\langle 1\rangle 3$. $S_\alpha \subseteq J_0$
$\langle 1\rangle 4$. $\alpha \in J_0$
$\langle 1\rangle 5$. Q.E.D.
PROOF: This contradicts $\langle 1\rangle 2$.
□

Theorem 2.81 (Transfinite Recursion). *Let J be a well-ordered set and C a set. Let $\mathcal{F}$ be the set of all functions from a section of J to C. Let $\rho : \mathcal{F} \to C$. Then there exists a unique function $h : J \to C$ such that, for all $\alpha \in J$, we have $h(\alpha) = \rho(h \upharpoonright S_\alpha)$.*

PROOF:

⟨1⟩1. For every $\beta \in J$, there exists a unique $h_\beta : S_\beta \to J$ such that, for all $\alpha < \beta$, we have $h_\beta(\alpha) = \rho(h_\beta \restriction \alpha)$

⟨2⟩1. LET: $\beta \in J$

⟨2⟩2. ASSUME: for all $\gamma < \beta$ there exists a unique $h : S_\gamma \to J$ such that, for all $\alpha < \gamma$, we have $h(\alpha) = \rho(h \restriction S_\alpha)$

⟨2⟩3. For $\gamma < \beta$,
LET: $h_\gamma : S_\gamma \to J$ be the function such that, for all $\alpha < \gamma$, we have $h_\gamma(\alpha) = \rho(h_\gamma \restriction S_\alpha)$

⟨2⟩4. LET: $h : S_\beta \to J$ be the function $h(\gamma) = \rho(h_\gamma)$ for $\gamma < \beta$

⟨2⟩5. For $\gamma < \beta$ we have $h \restriction S_\gamma = h_\gamma$

⟨3⟩1. LET: $\gamma < \beta$

⟨3⟩2. ASSUME: For all $\alpha < \gamma$ we have $h \restriction S_\alpha = h_\alpha$

⟨3⟩3. For all $\alpha < \gamma$ we have $(h \restriction S_\gamma)(\alpha) = \rho((h \restriction S_\gamma) \restriction S_\alpha)$

PROOF:

$$\begin{aligned}(h \restriction S_\gamma)(\alpha) &= h(\alpha) \\ &= \rho(h_\alpha) && (\langle 2\rangle 4) \\ &= \rho(h \restriction S_\alpha) && (\langle 3\rangle 2) \\ &= \rho((h \restriction S_\gamma) \restriction S_\alpha)\end{aligned}$$

⟨3⟩4. $h \restriction S_\gamma = h_\gamma$

PROOF: From ⟨2⟩4

⟨3⟩5. Q.E.D.

PROOF: By transfinite induction.

⟨2⟩6. For $\alpha < \beta$ we have $h(\alpha) = \rho(h \restriction S_\alpha)$

⟨2⟩7. If $h' : S_\beta \to J$ and $h'(\alpha) = \rho(h' \restriction S_\alpha)$ for all $\alpha < \beta$, then $h' = h$)

⟨3⟩1. LET: $h' : S_\beta \to J$ and $h'(\alpha) = \rho(h' \restriction S_\alpha)$ for all $\alpha < \beta$

⟨3⟩2. For all $\gamma < \beta$ we have $h' \restriction S_\gamma = h_\gamma$

⟨4⟩1. For all $\alpha < \gamma$ we have $(h' \restriction S_\gamma)(\alpha) = \rho((h' \restriction S_\gamma) \restriction S_\alpha)$

PROOF:

$$\begin{aligned}(h' \restriction S_\gamma)(\alpha) &= h'(\alpha) \\ &= \rho(h' \restriction S_\alpha) && (\langle 3\rangle 1) \\ &= \rho((h' \restriction S_\gamma) \restriction S_\alpha)\end{aligned}$$

⟨4⟩2. Q.E.D.

PROOF: From ⟨2⟩4

⟨3⟩3. For all $\alpha < \beta$ we have $h'(\alpha) = \rho(h_\alpha)$

⟨1⟩2. There exists $h : J \to C$ such that, for all $\alpha \in J$, we have $h(\alpha) = \rho(h \restriction S_\alpha)$

⟨2⟩1. For $\alpha \in J$,
LET: $h(\alpha) = \rho(h_\alpha)$

⟨2⟩2. For $\alpha \in J$ we have $h \restriction S_\alpha = h_\alpha$

⟨3⟩1. LET: $\alpha \in J$

⟨3⟩2. ASSUME: For all $\beta < \alpha$ we have $h \restriction S_\beta = h_\beta$

⟨3⟩3. For all $\beta < \alpha$ we have $(h \restriction S_\alpha)(\beta) = \rho((h \restriction S_\alpha) \restriction S_\beta)$

PROOF:

$$\begin{aligned}(h \upharpoonright S_\alpha)(\beta) &= h(\beta) \\ &= \rho(h_\beta) && (\langle 2\rangle 1) \\ &= \rho(h \upharpoonright S_\beta) && (\langle 3\rangle 2) \\ &= \rho((h \upharpoonright S_\alpha) \upharpoonright S_\beta)\end{aligned}$$

⟨3⟩4. $h \upharpoonright S_\alpha = h_\alpha$

PROOF: From ⟨1⟩1

⟨3⟩5. Q.E.D.

PROOF: By transfinite induction.

⟨2⟩3. For $\alpha \in J$ we have $h(\alpha) = \rho(h \upharpoonright S_\alpha)$

⟨1⟩3. If $h, h' : J \to C$ and, for all $\alpha \in J$, we have $h(\alpha) = \rho(h \upharpoonright S_\alpha)$ and $h'(\alpha) = \rho(h' \upharpoonright S_\alpha)$, then $h = h'$

⟨2⟩1. ASSUME: $h, h' : J \to C$ and, for all $\alpha \in J$, we have $h(\alpha) = \rho(h \upharpoonright S_\alpha)$ and $h'(\alpha) = \rho(h' \upharpoonright S_\alpha)$

⟨2⟩2. LET: $\alpha \in J$

⟨2⟩3. ASSUME: for all $\beta < \alpha$ we have $h(\beta) = h'(\beta)$

⟨2⟩4. $h(\alpha) = h'(\alpha)$

PROOF:

$$\begin{aligned}h(\alpha) &= \rho(h \upharpoonright S_\alpha) \\ &= \rho(h' \upharpoonright S_\alpha) && (\langle 2\rangle 3) \\ &= h'(\alpha)\end{aligned}$$

⟨2⟩5. Q.E.D.

PROOF: By transfinite induction.

□

Theorem 2.82 (Well-Ordering Theorem (AC)). *Every set has a well-ordering.*

PROOF:

⟨1⟩1. LET: X be a set.

⟨1⟩2. PICK a choice function for $\mathcal{P}X \setminus \{\emptyset\}$

PROOF: Lemma 2.61.

⟨1⟩3. LET: a *tower* in X be a pair $(T, <)$ where $T \subseteq X$, $<$ is a well-ordering of T, and $x = c(X \setminus \{y \in T \mid y < x\})$.

⟨1⟩4. For any two towers $(T_1, <_1)$ and $(T_2, <_2)$, either these two posets are equal or one is a section of the other.

⟨2⟩1.

⟨1⟩5. For any tower $(T, <)$ in X with $T \neq X$, there exists a tower in X of which $(T, <)$ is a section.

⟨1⟩6. LET: $T = \bigcup\{T' \subseteq X \mid \exists R.(T, R) \text{ is a tower in } X\}$

⟨1⟩7. Define $<$ on T by: $x < y$ iff there exists a tower (T, R) in X such that $x, y \in T$ and xRy.

⟨1⟩8. $(T, <)$ is a tower in X.

⟨1⟩9. $T = X$

⟨1⟩10. $<$ is a well-ordering of X.

□

Theorem 2.83 (Maximum Principle (AC))**.** *Every poset has a maximal chain.*

Lemma 2.84 (Zorn's Lemma (AC))**.** *Let A be a poset. If every chain in A has an upper bound in A, then A has a maximal element.*

2.31 Real Analysis

Definition 2.85. Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences (a_n) such that $a_n = 0$ for all but finitely many n.

2.32 Group Theory

Definition 2.86. Given a group G and sets $A, B \subseteq G$, let $AB = \{ab \mid a \in A, b \in B\}$.

Definition 2.87. Given a group G and a set $A \subseteq G$, let $A^{-1} = \{a^{-1} \mid a \in A\}$.

2.33 Topological Spaces

Definition 2.88 (Topology)**.** A *topology* on a set X is a set $\mathcal{T} \subseteq \mathcal{P}X$ such that:

- $X \in \mathcal{T}$.
- For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$.
- For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$.

We call the elements of X *points* and the elements of $\mathcal{T}$ *open sets*.

Definition 2.89 (Topological Space)**.** A *topological space* X consists of a set X and a topology on X.

Definition 2.90 (Discrete Space)**.** For any set X, the *discrete* topology on X is $\mathcal{P}X$.

Definition 2.91 (Indiscrete Space)**.** For any set X, the *indiscrete* or *trivial* topology on X is $\{\emptyset, X\}$.

Definition 2.92 (Finite Complement Topology)**.** For any set X, the *finite complement topology* on X is $\{U \in \mathcal{P}X \mid X \setminus U \text{ is finite}\} \cup \{\emptyset\}$.

Definition 2.93 (Countable Complement Topology)**.** For any set X, the *countable complement topology* on X is $\{U \in \mathcal{P}X \mid X \setminus U \text{ is countable}\} \cup \{\emptyset\}$.

Definition 2.94 (Finer, Coarser)**.** Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a given set X. If $\mathcal{T}' \supseteq \mathcal{T}$, we say that $\mathcal{T}'$ is *finer* than $\mathcal{T}$; if $\mathcal{T}'$ *properly* contains $\mathcal{T}$, we say that $\mathcal{T}'$ is *strictly* finer than $\mathcal{T}$. We also say that $\mathcal{T}$ is *coarser* than $\mathcal{T}'$, or *stricly* coarser, in these two respective situations. We say $\mathcal{T}$ is *comparable* with $\mathcal{T}'$ if either $\mathcal{T}' \supseteq \mathcal{T}$ or $\mathcal{T} \supseteq \mathcal{T}'$.

Lemma 2.95. *Let X be a topological space and $U \subseteq X$. Then U is open if and only if, for all $x \in U$, there exists an open set V such that $x \in V \subseteq U$.*

Proof:
$\langle 1 \rangle 1.$ $\Rightarrow$
  Proof: Take $V = U$
$\langle 1 \rangle 2.$ $\Leftarrow$
  Proof: We have $U = \bigcup \{V \text{ open in } X \mid V \subseteq U\}$.
$\square$

Lemma 2.96. *Let X be a set and $\mathcal{T}$ a nonempty set of topologies on X. Then $\bigcap \mathcal{T}$ is a topology on X, and is the finest topology that is coarser than every member of $\mathcal{T}$.*

Proof:
$\langle 1 \rangle 1.$ $X \in \bigcap \mathcal{T}$
  Proof: Since X is in every member of $\mathcal{T}$.
$\langle 1 \rangle 2.$ $\bigcap \mathcal{T}$ is closed under union.

$\langle 2\rangle 1.$ LET: $\mathcal{U} \subseteq \bigcap \mathcal{T}$
$\langle 2\rangle 2.$ For all $T \in \mathcal{T}$ we have $\mathcal{U} \subseteq T$
$\langle 2\rangle 3.$ For all $T \in \mathcal{T}$ we have $\bigcup \mathcal{U} \in T$
$\langle 2\rangle 4.$ $\bigcup \mathcal{U} \in \bigcap \mathcal{T}$
$\langle 1\rangle 3.$ $\bigcap \mathcal{T}$ is closed under binary intersection.
$\langle 2\rangle 1.$ LET: $U, V \in \bigcap \mathcal{T}$
$\langle 2\rangle 2.$ For all $T \in \mathcal{T}$ we have $U, V \in T$
$\langle 2\rangle 3.$ For all $T \in \mathcal{T}$ we have $U \cap V \in T$
$\langle 2\rangle 4.$ $U \cap V \in \bigcap \mathcal{T}$
□

Lemma 2.97. *Let X be a set and $\mathcal{T}$ a set of topologies on X. Then there exists a unique coarsest topology that is finer than every member of $\mathcal{T}$.*

PROOF: The required topology is given by
$\bigcap\{T \in \mathcal{PP}X \mid T \textit{ is a topology on } X \textit{ that is finer than every member of } \mathcal{T}\}$,
The set is nonempty since it contains the discrete topology. □

Definition 2.98 (Neighbourhood)**.** A *neighbourhood* of a point x is an open set that contains x.

2.34 Closed Set

Definition 2.99 (Closed Set)**.** Let X be a topological space and $A \subseteq X$. Then A is *closed* if and only if $X \setminus A$ is open.

Lemma 2.100. *The empty set is closed.*

PROOF: Since the whole space X is always open. □

Lemma 2.101. *The topological space X is closed.*

PROOF: Since $\emptyset$ is open. □

Lemma 2.102. *The intersection of a nonempty set of closed sets is closed.*

PROOF: Let $\mathcal{C}$ be a nonempty set of closed sets. Then $X \setminus \bigcap \mathcal{C} = \bigcup\{X \setminus C \mid C \in \mathcal{C}\}$ is open. □

Lemma 2.103. *The union of two closed sets is closed.*

PROOF: Let C and D be closed. Then $X \setminus (C \cup D) = (X \setminus C) \cap (X \setminus D)$ is open.
□

Proposition 2.104. *Let X be a set and $\mathcal{C} \subseteq \mathcal{P}X$ a set such that:*

1. $\emptyset \in \mathcal{C}$
2. $X \in \mathcal{C}$
3. *For all $\mathcal{A} \subseteq \mathcal{C}$ nonempty we have $\bigcap \mathcal{A} \in \mathcal{C}$*

4. For all $C, D \in \mathcal{C}$ we have $C \cup D \in \mathcal{C}$.

Then there exists a unique topology $\mathcal{T}$ such that $\mathcal{C}$ is the set of closed sets, namely

$$\mathcal{T} = \{X \setminus C \mid C \in \mathcal{C}\}$$

PROOF:
$\langle 1 \rangle 1.$ LET: $\mathcal{T} = \{X \setminus C \mid C \in \mathcal{C}\}$
$\langle 1 \rangle 2.$ $\mathcal{T}$ is a topology
$\langle 2 \rangle 1.$ $X \in \mathcal{T}$
PROOF: Since $\emptyset \in \mathcal{C}$
$\langle 2 \rangle 2.$ For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$
$\langle 3 \rangle 1.$ LET: $\mathcal{U} \subseteq \mathcal{T}$
$\langle 3 \rangle 2.$ CASE: $\mathcal{U} = \emptyset$
PROOF: In this case $\bigcup \mathcal{U} = \emptyset \in \mathcal{T}$ since $X \in \mathcal{C}$
$\langle 3 \rangle 3.$ CASE: $\mathcal{U} \neq \emptyset$
PROOF: In this case $X \setminus \bigcup \mathcal{U} = \bigcap \{X \setminus U \mid U \in \mathcal{U}\} \in \mathcal{C}$.
$\langle 2 \rangle 3.$ For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$
PROOF: Since $X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V) \in \mathcal{C}$.
$\langle 1 \rangle 3.$ $\mathcal{C}$ is the set of all closed sets in $\mathcal{T}$
PROOF:

$$\begin{aligned} & C \text{ is closed in } \mathcal{T} \\ \Leftrightarrow & X \setminus C \in \mathcal{T} \\ \Leftrightarrow & C \in \mathcal{C} \end{aligned}$$

$\langle 1 \rangle 4.$ If $\mathcal{T}'$ is a topology and $\mathcal{C}$ is the set of closed sets in $\mathcal{T}'$ then $\mathcal{T}' = \mathcal{T}$
PROOF: We have

$$\begin{aligned} & U \in \mathcal{T} \\ \Leftrightarrow & X \setminus U \in \mathcal{C} \\ \Leftrightarrow & X \setminus U \text{ is closed in } \mathcal{T}' \\ \Leftrightarrow & U \in \mathcal{T}' \end{aligned}$$

□

Proposition 2.105. *If U is open and A is closed then $U \setminus A$ is open.*

PROOF: $U \setminus A = U \cap (X \setminus A)$ is the intersection of two open sets. □

Proposition 2.106. *If U is open and A is closed then $A \setminus U$ is closed.*

PROOF: $A \setminus U = A \cap (X \setminus U)$ is the intersection of two closed sets. □

2.35 Interior

Definition 2.107 (Interior)**.** Let X be a topological space and $A \subseteq X$. The *interior* of A, $\operatorname{Int} A$, is the union of all the open subsets of A.

Lemma 2.108. *The interior of a set is open.*

PROOF: It is a union of open sets. □

Lemma 2.109.

$$\operatorname{Int} A \subseteq A$$

PROOF: Immediate from definition. □

Lemma 2.110. *If U is open and $U \subseteq A$ then $U \subseteq \operatorname{Int} A$*

PROOF: Immediate from definition. □

Lemma 2.111. *A set A is open if and only if $A = \operatorname{Int} A$.*

PROOF: If $A = \operatorname{Int} A$ then A is open by Lemma 2.108. Conversely if A is open then $A \subseteq \operatorname{Int} A$ by the definition of interior and so $A = \operatorname{Int} A$.

2.36 Closure

Definition 2.112 (Closure)**.** Let X be a topological space and $A \subseteq X$. The *closure* of A, $\overline{A}$, is the intersection of all the closed sets that include A.

This intersection exists since X is a closed set that includes A (Lemma 2.101).

Lemma 2.113. *The closure of a set is closed.*

PROOF: Dual to Lemma 2.108. □

Lemma 2.114.

$$A \subseteq \overline{A}$$

PROOF: Immediate from definition. □

Lemma 2.115. *If C is closed and $A \subseteq C$ then $\overline{A} \subseteq C$.*

PROOF: Immediate from definition. □

Lemma 2.116. *A set A is closed if and only if $A = \overline{A}$.*

PROOF: Dual to Lemma 2.111. □

Theorem 2.117. *Let X be a topological space, $A \subseteq X$ and $x \in X$. Then $x \in \overline{A}$ if and only if every neighbourhood of x intersects A.*

PROOF: We have

$$\begin{aligned} & x \in \overline{A} \\ \Leftrightarrow & \forall C. C \text{ closed } \wedge A \subseteq C \Rightarrow x \in C \\ \Leftrightarrow & \forall U. U \text{ open } \wedge A \cap U = \emptyset \Rightarrow x \notin U \\ \Leftrightarrow & \forall U. U \text{ open } \wedge x \in U \Rightarrow U \text{ intersects } A \end{aligned}$$

□

Proposition 2.118. *If $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$.*

PROOF: This holds because $\overline{B}$ is a closed set that includes A. □

Proposition 2.119.

$$\overline{A \cup B} = \overline{A} \cup \overline{B}$$

PROOF:
⟨1⟩1. $\overline{A} \subseteq \overline{A \cup B}$
  PROOF: By Proposition 2.118.
⟨1⟩2. $\overline{B} \subseteq \overline{A \cup B}$
  PROOF: By Proposition 2.118.
⟨1⟩3. $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$
  ⟨2⟩1. LET: $x \in \overline{A \cup B}$
  ⟨2⟩2. ASSUME: $x \notin \overline{A}$
     PROVE: $x \in \overline{B}$
  ⟨2⟩3. PICK a neighbourhood U of x that does not intersect A
  ⟨2⟩4. LET: V be any neighbourhood of x
  ⟨2⟩5. $U \cap V$ is a neighbourhood of x
  ⟨2⟩6. $U \cap V$ intersects $A \cup B$
   PROOF: From ⟨2⟩1 and Theorem 2.117.
  ⟨2⟩7. $U \cap V$ intersects B
   PROOF: From ⟨2⟩3
  ⟨2⟩8. V intersects B
  ⟨2⟩9. Q.E.D.
   PROOF: We have $x \in \overline{B}$ from Theorem 2.117.
□

Proposition 2.120. *Let X be a topological space. Let $\mathcal{D}$ be a set of subsets of X that is maximal with respect to the finite intersection property. Let $x \in X$. Then the following are equivalent:*

1. *For all $D \in \mathcal{D}$ we have $x \in \overline{D}$*
2. *Every neighbourhood of x is in $\mathcal{D}$.*

PROOF:
⟨1⟩1. $1 \Rightarrow 2$
  ⟨2⟩1. For all $D \in \mathcal{D}$ we have $x \in \overline{D}$
  ⟨2⟩2. LET: U be a neighbourhood of x
  ⟨2⟩3. $\mathcal{D} \cup \{U\}$ satisfies the finite intersection property.
   ⟨3⟩1. LET: $D_1, \ldots, D_n \in \mathcal{D}$
   ⟨3⟩2. $D_1 \cap \cdots \cap D_n \in \mathcal{D}$
    PROOF: Lemma 2.52.
   ⟨3⟩3. $x \in \overline{D_1 \cap \cdots \cap D_n}$
    PROOF: ⟨2⟩1, ⟨3⟩2
   ⟨3⟩4. $D_1 \cap \cdots \cap D_n \cap U \neq \emptyset$
    PROOF: Theorem 2.117, ⟨2⟩2, ⟨3⟩3.
  ⟨2⟩4. $\mathcal{D} = \mathcal{D} \cup \{U\}$
   PROOF: By the maximality of $\mathcal{D}$.

⟨2⟩5. $U \in \mathcal{D}$
⟨1⟩2. $2 \Rightarrow 1$
⟨2⟩1. Assume: Every neighbourhood of x is in $\mathcal{D}$.
⟨2⟩2. Let: $D \in \mathcal{D}$
⟨2⟩3. Every neighbourhood of x intersects D.
Proof: From ⟨2⟩1, ⟨2⟩2 and the fact that $\mathcal{D}$ satisfies the finite intersection property.
⟨2⟩4. $x \in \overline{D}$
Proof: Theorem 2.117, ⟨2⟩3.
□

2.37 Boundary

Definition 2.121 (Boundary)**.** The *boundary* of a set A is the set $\partial A = \overline{A} \cap \overline{X \setminus A}$.

Proposition 2.122.

$$\mathrm{Int}\, A \cap \partial A = \emptyset$$

Proof: Since $\overline{X \setminus A} = X \setminus \mathrm{Int}\, A$. □

Proposition 2.123.

$$\overline{A} = \mathrm{Int}\, A \cup \partial A$$

Proof:

$$\begin{aligned} \mathrm{Int}\, A \cup \partial A &= \mathrm{Int}\, A \cup (\overline{A} \cap \overline{X \setminus A}) \\ &= (\mathrm{Int}\, A \cup \overline{A}) \cap (\mathrm{Int}\, A \cup \overline{X \setminus A}) \\ &= \overline{A} \cap X \\ &= \overline{A} \end{aligned}$$

Proposition 2.124. *$\partial A = \emptyset$ if and only if A is open and closed.*

Proof: If $\partial A = \emptyset$ then $\overline{A} = \mathrm{Int}\, A$ by Proposition 2.123.

Proposition 2.125. *A set U is open if and only if $\partial U = \overline{U} \setminus U$.*

Proof:

$$\begin{aligned} & \partial U = \overline{U} \setminus U \\ \Leftrightarrow & \overline{U} \setminus \mathrm{Int}\, U = \overline{U} \setminus U && \text{(Propositions 2.122, 2.123)} \\ \Leftrightarrow & \mathrm{Int}\, U = U \end{aligned}$$

□

2.38 Limit Points

Definition 2.126 (Limit Point)**.** Let X be a topological space, $a \in X$ and $A \subseteq X$. Then a is a *limit point*, *cluster point* or *point of accumulation* for A if and only if every neighbourhood of a intersects A at a point other than a.

Lemma 2.127. *The point a is an accumulation point for A if and only if $a \in \overline{A \setminus \{a\}}$.*

PROOF: From Theorem 2.117. ∎

Theorem 2.128. *Let X be a topological space and $A \subseteq X$. Let A' be the set of all limit points of A. Then $\overline{A} = A \cup A'$.*

PROOF:
⟨1⟩1. For all $x \in \overline{A}$, if $x \notin A$ then $x \in A'$
  PROOF: From Theorem 2.117.
⟨1⟩2. $A \subseteq \overline{A}$
  PROOF: Lemma 2.114.
⟨1⟩3. $A' \subseteq \overline{A}$
  PROOF: From Theorem 2.117.
∎

Corollary 2.128.1. *A set is closed if and only if it contains all its limit points.*

Proposition 2.129. *In an indiscrete topology, every point is a limit point of any set that has more than one point.*

PROOF: Let X be an indiscrete space. Let A be a set with more than one point and x be a point. The only neighbourhood of x is X, which must intersect A at a point other than x. ∎

Lemma 2.130. *Let X be a topological space and $A \subseteq B \subseteq X$. Then every limit point of A is a limit point of B.*

PROOF: Immediate from definitions. ∎

2.39 Basis for a Topology

Definition 2.131 (Basis)**.** If X is a set, a *basis* for a topology on X is a set $\mathcal{B} \subseteq \mathcal{P}X$ called *basis elements* such that

1. For all $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$.
2. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

If $\mathcal{B}$ satisfies these two conditions, then we define the topology *generated* by $\mathcal{B}$ to be $\mathcal{T} = \{U \in \mathcal{P}X \mid \forall x \in U. \exists B \in \mathcal{B}. x \in B \subseteq U\}$.

We prove this is a topology.

PROOF:
⟨1⟩1. $X \in \mathcal{T}$
  PROOF: For all $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B \subseteq X$ by condition 1.

$\langle 1 \rangle 2$. For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$
$\langle 2 \rangle 1$. Let: $\mathcal{U} \subseteq \mathcal{T}$
$\langle 2 \rangle 2$. Let: $x \in \bigcup \mathcal{U}$
$\langle 2 \rangle 3$. Pick $U \in \mathcal{U}$ such that $x \in U$
$\langle 2 \rangle 4$. Pick $B \in \mathcal{B}$ such that $x \in B \subseteq U$
Proof: Since $U \in \mathcal{T}$ by $\langle 2 \rangle 1$ and $\langle 2 \rangle 3$.
$\langle 2 \rangle 5$. $x \in B \subseteq \bigcup \mathcal{U}$
$\langle 1 \rangle 3$. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$
$\langle 2 \rangle 1$. Let: $U, V \in \mathcal{T}$
$\langle 2 \rangle 2$. Let: $x \in U \cap V$
$\langle 2 \rangle 3$. Pick $B_1 \in \mathcal{B}$ such that $x \in B_1 \subseteq U$
$\langle 2 \rangle 4$. Pick $B_2 \in \mathcal{B}$ such that $x \in B_2 \subseteq V$
$\langle 2 \rangle 5$. Pick $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$
Proof: By condition 2.
$\langle 2 \rangle 6$. $x \in B_3 \subseteq U \cap V$
□

Lemma 2.132. *Let X be a set. Let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ is the set of all unions of subsets of $\mathcal{B}$.*

Proof:
$\langle 1 \rangle 1$. For all $U \in \mathcal{T}$, there exists $\mathcal{A} \subseteq \mathcal{B}$ such that $U = \bigcup \mathcal{A}$
$\langle 2 \rangle 1$. Let: $U \in \mathcal{T}$
$\langle 2 \rangle 2$. Let: $\mathcal{A} = \{B \in \mathcal{B} \mid B \subseteq U\}$
$\langle 2 \rangle 3$. $U \subseteq \bigcup \mathcal{A}$
$\langle 3 \rangle 1$. Let: $x \in U$
$\langle 3 \rangle 2$. Pick $B \in \mathcal{B}$ such that $x \in B \subseteq U$
Proof: Since $\mathcal{B}$ is a basis for $\mathcal{T}$.
$\langle 3 \rangle 3$. $x \in B \in \mathcal{A}$
$\langle 2 \rangle 4$. $\bigcup \mathcal{A} \subseteq U$
Proof: From the definition of $\mathcal{A}$ ($\langle 2 \rangle 2$).
$\langle 1 \rangle 2$. For all $\mathcal{A} \subseteq \mathcal{B}$ we have $\bigcup \mathcal{A} \in \mathcal{T}$
$\langle 2 \rangle 1$. $\mathcal{B} \subseteq \mathcal{T}$
Proof: If $B \in \mathcal{B}$ and $x \in B$, then there exists $B' \in \mathcal{B}$ such that $x \in B' \subseteq B$, namely $B' = B$.
$\langle 2 \rangle 2$. Q.E.D.
Proof: Since $\mathcal{T}$ is closed under union.
□

Corollary 2.132.1. *Let X be a set. Let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ is the coarsest topology that includes $\mathcal{B}$.*

Proof: Since every topology that includes $\mathcal{B}$ includes all unions of subsets of $\mathcal{B}$. □

Lemma 2.133. *Let X be a topological space. Suppose that $\mathcal{C}$ is a set of open sets such that, for every open set U and every point $x \in U$, there exists $C \in \mathcal{C}$ such that $x \in C \subseteq U$. Then $\mathcal{C}$ is a basis for the topology on X.*

PROOF:
$\langle 1\rangle 1$. For all $x \in X$, there exists $C \in \mathcal{C}$ such that $x \in C$
PROOF: Immediate from hypothesis.
$\langle 1\rangle 2$. For all $C_1, C_2 \in \mathcal{C}$ and $x \in C_1 \cap C_2$, there exists $C_3 \in \mathcal{C}$ such that $x \in C_3 \subseteq C_1 \cap C_2$
PROOF: Since $C_1 \cap C_2$ is open.
$\langle 1\rangle 3$. Every open set is open in the topology generated by $\mathcal{C}$
PROOF: Immediate from hypothesis.
$\langle 1\rangle 4$. Every union of a subset of $\mathcal{C}$ is open.
PROOF: Since every member of $\mathcal{C}$ is open.
□

Lemma 2.134. *Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$ respectively on the set X. Then the following are equivalent.*

1. $\mathcal{T} \subseteq \mathcal{T}'$
2. *For all $B \in \mathcal{B}$ and $x \in B$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.*

PROOF:
$\langle 1\rangle 1$. $1 \Rightarrow 2$
$\langle 2\rangle 1$. ASSUME: $\mathcal{T} \subseteq \mathcal{T}'$
$\langle 2\rangle 2$. LET: $B \in \mathcal{B}$ and $x \in B$
$\langle 2\rangle 3$. $B \in \mathcal{T}$
PROOF: Corollary 2.132.1.
$\langle 2\rangle 4$. $B \in \mathcal{T}'$
PROOF: By $\langle 2\rangle 1$
$\langle 2\rangle 5$. There exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$
PROOF: Since $\mathcal{B}'$ is a basis for $\mathcal{T}'$.
$\langle 1\rangle 2$. $2 \Rightarrow 1$
$\langle 2\rangle 1$. ASSUME: 2
$\langle 2\rangle 2$. LET: $U \in \mathcal{T}$
PROVE: $U \in \mathcal{T}'$
$\langle 2\rangle 3$. LET: $x \in U$
PROVE: There exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq U$
$\langle 2\rangle 4$. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$
PROOF: Since $\mathcal{B}$ is a basis for $\mathcal{T}$.
$\langle 2\rangle 5$. PICK $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$
PROOF: By $\langle 2\rangle 1$.
$\langle 2\rangle 6$. $x \in B' \subseteq U$
□

Theorem 2.135. *Let X be a topological space, $A \subseteq X$ and $x \in X$. Let $\mathcal{B}$ be a basis for X. Then $x \in \overline{A}$ if and only if, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A.*

PROOF:
$\langle 1\rangle 1$. If $x \in \overline{A}$ then, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A.

PROOF: This follows from Theorem 2.117 since every element of $\mathcal{B}$ is open (Corollary 2.132.1).

⟨1⟩2. Suppose that, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A. Then $x \in \overline{A}$.

⟨2⟩1. ASSUME: For all $B \in \mathcal{B}$, if $x \in B$ then B intersects A.

⟨2⟩2. LET: U be an open set that contains x

PROVE: U intersects A.

⟨2⟩3. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

⟨2⟩4. B intersects A.

PROOF: From ⟨2⟩1.

⟨2⟩5. U intersects A.

⟨2⟩6. Q.E.D.

PROOF: By Theorem 2.117.

□

Definition 2.136 (Lower Limit Topology on the Real Line)**.** The *lower limit topology on the real line* is the topology on $\mathbb{R}$ generated by the basis consisting of all half-open intervals of the form $[a, b)$.

We write $\mathbb{R}_l$ for the topological space $\mathbb{R}$ under the lower limit topology.

We prove this is a basis for a topology.

PROOF:

⟨1⟩1. For all $x \in \mathbb{R}$ there exists an interval $[a, b)$ such that $x \in [a, b)$.

PROOF: Take $[a, b) = [x, x + 1)$.

⟨1⟩2. For any open intervals $[a, b)$, $[c, d)$ if $x \in [a, b) \cap [c, d)$, then there exists an interval $[e, f)$ such that $x \in [e, f) \subseteq [a, b) \cap [c, d)$

PROOF: Take $[e, f) = [\max(a, c), \min(b, d))$.

□

Definition 2.137 (K-topology on the Real Line)**.** Let $K = \{1/n \mid n \in \mathbb{Z}^+\}$.

The *K-topology on the real line* is the topology on $\mathbb{R}$ generated by the basis consisting of all open intervals (a, b) and all sets of the form $(a, b) \setminus K$.

We write $\mathbb{R}_K$ for the topological space $\mathbb{R}$ under the K-topology.

We prove this is a basis for a topology.

PROOF:

⟨1⟩1. For all $x \in \mathbb{R}$ there exists an open interval (a, b) such that $x \in (a, b)$.

PROOF: Take $(a, b) = (x - 1, x + 1)$.

⟨1⟩2. For any basic open sets B_1, B_2 if $x \in B_1 \cap B_2$, then there exists a basic open set B_3 such that $x \in B_3 \subseteq B_1 \cap B_2$

⟨2⟩1. CASE: $B_1 = (a, b)$, $B_2 = (c, d)$

PROOF: Take $B_3 = (\max(a, c), \min(b, d))$.

⟨2⟩2. CASE: $B_1 = (a, b)$ or $(a, b) \setminus K$, $B_2 = (c, d)$ or $(c, d) \setminus K$, and they are not both open intervals.

PROOF: Take $B_3 = (\max(a, c), \min(b, d)) \setminus K$.

□

Lemma 2.138. *The lower limit topology and the K-topology are incomparable.*

PROOF:
⟨1⟩1. The interval $[10, 11)$ is not open in the K-topology.
PROOF: There is no open interval (a, b) such that $10 \in (a, b) \subseteq [10, 11)$ or $10 \in (a, b) \setminus K \subseteq [10, 11)$.
⟨1⟩2. The set $(-1, 1) \setminus K$ is not open in the lower limit topology.
PROOF: There is no half-open interval $[a, b)$ such that $0 \in [a, b) \subseteq (-1, 1) \setminus K$, since there must be a positive integer n with $1/n \in [a, b)$.
□

Definition 2.139 (Subbasis)**.** A *subbasis* $\mathcal{S}$ for a topology on X is a set $\mathcal{S} \subseteq \mathcal{P}X$ such that $\bigcup \mathcal{S} = X$.

The topology *generated* by the subbasis $\mathcal{S}$ is the set of all unions of finite intersections of elements of $\mathcal{S}$.

We prove this is a topology.

PROOF:
⟨1⟩1. The set $\mathcal{B}$ of all finite intersections of elements of $\mathcal{S}$ forms a basis for a topology on X.
⟨2⟩1. $\bigcup \mathcal{B} = X$
PROOF: Since $\mathcal{S} \subseteq \mathcal{B}$.
⟨2⟩2. $\mathcal{B}$ is closed under binary intersection.
PROOF: By definition.
⟨1⟩2. Q.E.D.
PROOF: By Lemma 2.132.
□

We have simultaneously proved:

Proposition 2.140. *Let $\mathcal{S}$ be a subbasis for the topology on X. Then the set of all finite intersections of elements of $\mathcal{S}$ is a basis for the topology on X.*

Proposition 2.141. *Let X be a set. Let $\mathcal{S}$ be a subbasis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ is the coarsest topology that includes $\mathcal{S}$.*

PROOF: Since every topology that includes $\mathcal{S}$ includes every union of finite intersections of elements of $\mathcal{S}$. □

2.40 Local Basis at a Point

Definition 2.142 (Local Basis)**.** Let X be a topological space and $a \in X$. A *(local) basis at a* is a set $\mathcal{B}$ of neighbourhoods of a such that every neighbourhood of a includes some member of $\mathcal{B}$.

Lemma 2.143. *If there exists a countable local basis at a point a, then there exists a countable local basis $\{B_n \mid n \geq 1\}$ such that $B_1 \supseteq B_2 \supseteq \cdots$.*

PROOF: Given a countable local basis $\{C_n \mid n \geq 1\}$, take $B_n = C_1 \cap \cdots \cap C_n$. □

2.41 Convergence

Definition 2.144 (Convergence)**.** Let X be a topological space. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of points in X and $l \in X$. Then the sequence $(a_n)_{n\in\mathbb{N}}$ *converges* to the *limit* l, $a_n \to l$ as $n \to \infty$, if and only if, for every neighbourhood U of l, there exists N such that, for all $n \geq N$, we have $a_n \in U$.

Lemma 2.145. *Let X be a topological space. Let $A \subseteq X$ and $l \in X$. If there is a sequence of points in A that converges to l then $l \in \overline{A}$.*

Proof:
⟨1⟩1. Let: (a_n) be a sequence of points in A that converges to l.
⟨1⟩2. Let: U be a neighbourhood of l.
⟨1⟩3. Pick N such that, for all $n \geq N$, we have $a_n \in U$.
⟨1⟩4. $a_N \in U \cap A$
⟨1⟩5. Q.E.D.
  Proof: Theorem 2.117.
□

Proposition 2.146. *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Let (a_n) be a sequence in X and $l \in X$. Then $a_n \to l$ as $n \to \infty$ if and only if, for every $B \in \mathcal{B}$ with $l \in B$, there exists N such that, for all $n \geq N$, we have $a_n \in B$.*

Proof:
⟨1⟩1. If $a_n \to l$ as $n \to \infty$ then, for every $B \in \mathcal{B}$ with $l \in B$, there exists N such that, for all $n \geq N$, we have $a_n \in B$.
  Proof: Since every element of $\mathcal{B}$ is open (Corollary 2.132.1).
⟨1⟩2. If, for every $B \in \mathcal{B}$ with $l \in B$, there exists N such that, for all $n \geq N$, we have $a_n \in B$, then $a_n \to l$ as $n \to \infty$.
  ⟨2⟩1. Assume: for every $B \in \mathcal{B}$ with $l \in B$, there exists N such that, for all $n \geq N$, we have $a_n \in B$
  ⟨2⟩2. Let: U be a neighbourhood of l.
  ⟨2⟩3. Pick $B \in \mathcal{B}$ such that $l \in B \subseteq U$
  ⟨2⟩4. Pick N such that, for all $n \geq N$, we have $a_n \in B$
    Proof: From ⟨2⟩1.
  ⟨2⟩5. For all $n \geq N$ we have $a_n \in U$
□

Lemma 2.147. *If a sequence (a_n) is constant with $a_n = l$ for all n, then $a_n \to l$ as $n \to \infty$.*

Proof: Immediate from definitions. □

Theorem 2.148. *Let X be a linearly ordered set. Let (s_n) be a monotone increasing sequence in X with a supremum s. Then $s_n \to s$ as $n \to \infty$.*

Proof:
⟨1⟩1. Assume: s is not least in X.

PROOF: Otherwise (s_n) is the constant sequence s and the result follows from Lemma 2.147.

⟨1⟩2. LET: U be a neighbourhood of s.

⟨1⟩3. PICK$a < s$ such that $(a, s] \subseteq U$

⟨1⟩4. PICK N such that $a < a_N$.

⟨1⟩5. For all $n \geq N$ we have $a_n \in (a, s]$

⟨1⟩6. For all $n \geq N$ we have $a_n \in U$.

□

Theorem 2.149. *If* $\sum_{i=0}^{\infty} a_i = s$ *and* $\sum_{i=0}^{\infty} b_i = t$ *then* $\sum_{i=0}^{\infty}(ca_i + b_i) = cs + t$.

PROOF:$\sum_{i=0}^{N}(ca_i + b_i) = c\sum_{i=0}^{N} a_i + \sum_{i=0}^{N} b_i \to cs + t$ as $n \to \infty$. □

Theorem 2.150 (Comparison Test)**.** *If* $|a_i| \leq b_i$ *for all* i *and* $\sum_{i=0}^{\infty} b_i$ *converges then* $\sum_{i=0}^{\infty} a_i$ *converges.*

PROOF:

⟨1⟩1. $\sum_{i=0}^{\infty} |a_i|$ converges

PROOF: The partial sums $\sum_{i=0}^{N} |a_i|$ form an increasing sequence bounded above by $\sum_{i=0}^{\infty} b_i$.

⟨1⟩2. LET: $c_i = |a_i| + a_i$ for all i

⟨1⟩3. $\sum_{i=0}^{\infty} c_i$ converges

PROOF: Each c_i is either $2|a_i|$ or 0. So the partial sums $\sum_{i=0}^{N} c_i$ form an increasing sequence bounded above by $2\sum_{i=0}^{\infty} b_i$.

⟨1⟩4. Q.E.D.

PROOF: Since $a_i = c_i - |a_i|$.

□

Corollary 2.150.1. *If* $\sum_{i=0}^{\infty} |a_i|$ *converges then* $\sum_{i=0}^{\infty} a_i$ *converges.*

Theorem 2.151 (Weierstrass M-test)**.** *Let* X *be a set and* $(f_n : X \to \mathbb{R})$ *be a sequence of functions. Let*

$$s_n(x) = \sum_{i=0}^{n} f_i(x)$$

for all n, x. *Suppose* $|f_i(x)| \leq M_i$ *for all* $i \geq 0$ *and* $x \in X$. *If the series* $\sum_{i=0}^{\infty} M_i$ *converges, then the sequence* (s_n) *converges uniformly to*

$$s(x) = \sum_{i=0}^{\infty} f_i(x) \quad .$$

PROOF:

⟨1⟩1. LET: $r_n = \sum_{i=n+1}^{\infty} M_i$ for all n

⟨1⟩2. Given $0 \leq n < k$, we have $|s_k(x) - s_n(x)| \leq r_n$

PROOF:

$$\begin{aligned}|s_k(x) - s_n(x)| &= |\sum_{i=n+1}^{k} f_i(x)| \\ &\leq \sum_{i=n+1}^{k} |f_i(x)| \\ &\leq \sum_{i=n+1}^{k} M_i \\ &\leq r_n\end{aligned}$$

⟨1⟩3. Given $n \geq 0$ we have $|s(x) - s_n(x)| \leq r_n$

PROOF: By taking the limit $k \to \infty$ in ⟨1⟩2.

⟨1⟩4. Q.E.D.

PROOF: Since $r_n \to 0$ as $n \to \infty$.

□

2.42 Locally Finite Sets

Definition 2.152 (Locally Finite)**.** Let X be a topological space and $\{A_\alpha\}$ a family of subsets of X. Then $\mathcal{A}$ is *locally finite* if and only if every point in X has a neighbourhood that intersects A_α for only finitely many α.

Theorem 2.153 (Pasting Lemma)**.** *Let X and Y be topological spaces and $f : X \to Y$. Let $\{A_\alpha\}$ be a locally finite family of closed subsets of X that cover X. Suppose $f \upharpoonright A_\alpha$ is continuous for all α. Then f is continuous.*

PROOF:

⟨1⟩1. Let X and Y be topological spaces and $f : X \to Y$. Let A and B be closed subsets of X such that $X = A \cup B$. Suppose $f \upharpoonright A$ and $f \upharpoonright B$ are continuous. Then f is continuous.

⟨2⟩1. LET: $C \subseteq Y$ be closed.

⟨2⟩2. $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$

⟨2⟩3. $f^{-1}(C)$ and $g^{-1}(C)$ are closed in X.

PROOF: Theorems 2.163 and 2.214.

⟨2⟩4. $h^{-1}(C)$ is closed in X.

PROOF: Lemma 2.103.

⟨2⟩5. Q.E.D.

PROOF: Theorem 2.163.

⟨1⟩2. Let X and Y be topological spaces and $f : X \to Y$. Let $\{A_\alpha\}$ be a finite family of closed subsets of X that cover X. Suppose $f \upharpoonright A_\alpha$ is continuous for all α. Then f is continuous.

PROOF: From ⟨1⟩1 by induction.

⟨1⟩3. Let X and Y be topological spaces and $f : X \to Y$. Let $\{A_\alpha\}$ be a locally finite family of closed subsets of X that cover X. Suppose $f \upharpoonright A_\alpha$ is continuous for all α. Then f is continuous.

⟨2⟩1. LET: $x \in X$
PROVE: f is continuous at x
⟨2⟩2. PICK a neighbourhood U of x that intersects A_α for only finitely many α.
⟨2⟩3. $f \upharpoonright U$ is continuous
PROOF: By ⟨1⟩2.
⟨2⟩4. Q.E.D.
PROOF: Lemma 2.173.
□

The following example shows that we cannot remove the assumption of local finiteness.

Example 2.154. Define $f : [-1, 1] \to \mathbb{R}$ by: $f(x) = 1$ if $x < -1$, $f(x) = 0$ if $x > 1$. Let $C_n = [-1, -1/n]$ for $n \geq 1$, and $D = [0, 1]$. Then $[-1, 1] = \bigcup_{n=1}^{\infty} C_n \cup D$ and f is continuous on each C_n and each D, but f is not continuous on $[-1, 1]$.

2.43 Open Maps

Definition 2.155 (Open Map)**.** Let X and Y be topological spaces and $f : X \to Y$. Then f is an *open map* if and only if, for every open set U in X, the set $f(U)$ is open in Y.

Lemma 2.156. *Let X and Y be topological spaces and $f : X \to Y$. Let $\mathcal{B}$ be a basis for the topology on X. If $f(B)$ is open in Y for all $B \in \mathcal{B}$, then f is an open map.*

PROOF: From Lemma 2.132. □

Proposition 2.157. *Let X and Y be topological spaces. Let $\mathcal{B}$ be a basis for the topology on X. Let $f : X \to Y$. Suppose that, for all $B \in \mathcal{B}$, we have $f(B)$ is open tn Y. Then f is an open map.*

PROOF: For any $\mathcal{A} \subseteq \mathcal{B}$, we have $f(\bigcup \mathcal{A}) = \bigcup_{B \in \mathcal{B}} f(B)$ is open in Y. The result follows from Lemma 2.132. □

2.44 Continuous Functions

Definition 2.158 (Continuous)**.** Let X and Y be topological spaces. A function $f : X \to Y$ is *continuous* if and only if, for every open set V in Y, the set $f^{-1}(V)$ is open in X.

Proposition 2.159. *Let X and Y be topological spaces and $f : X \to Y$. Let $\mathcal{B}$ be a basis for Y. Then f is continuous if and only if, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X.*

PROOF:
⟨1⟩1. If f is continuous then, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X.

PROOF: Since every element of B is open (Lemma 2.132).

$\langle 1 \rangle 2$. Suppose that, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X. Then f is continuous.

$\langle 2 \rangle 1$. ASSUME: For all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X.

$\langle 2 \rangle 2$. LET: V be open in Y.

$\langle 2 \rangle 3$. PICK $\mathcal{A} \subseteq \mathcal{B}$ such that $V = \bigcup \mathcal{A}$

PROOF: By Lemma 2.132.

$\langle 2 \rangle 4$. $f^{-1}(V)$ is open in X.

PROOF:

$$\begin{aligned} f^{-1}(V) &= f^{-1}\left(\bigcup \mathcal{A}\right) \\ &= \bigcup_{B \in \mathcal{A}} f^{-1}(B) \end{aligned}$$

□

Proposition 2.160. *Let X and Y be topological spaces and $f : X \to Y$. Let $\mathcal{S}$ be a subbasis for Y. Then f is continuous if and only if, for all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in X.*

PROOF:

$\langle 1 \rangle 1$. If f is continuous then, for all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in X.

PROOF: Since every element of S is open.

$\langle 1 \rangle 2$. Suppose that, for all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in X. Then f is continuous.

$\langle 2 \rangle 1$. ASSUME: For all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in X.

$\langle 2 \rangle 2$. LET: $S_1, \ldots, S_n \in \mathcal{S}$

$\langle 2 \rangle 3$. $f^{-1}(S_1 \cap \cdots \cap S_n)$ is open in A

PROOF: Since $f^{-1}(S_1 \cap \cdots \cap S_n) = f^{-1}(S_1) \cap \cdots \cap f^{-1}(S_n)$.

$\langle 2 \rangle 4$. Q.E.D.

PROOF: By Propositions 2.159 and 2.140.

□

Proposition 2.161. *Let X and Y be topological spaces and $f : X \to Y$. Let $\mathcal{S}$ be a basis for Y. Then f is continuous if and only if, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X.*

PROOF:

$\langle 1 \rangle 1$. If f is continuous then, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X.

PROOF: Since every element of $\mathcal{S}$ is open.

$\langle 1 \rangle 2$. Suppose that, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X. Then f is continuous.

$\langle 2 \rangle 1$. ASSUME: For all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X.

$\langle 2 \rangle 2$. For every set B that is the finite intersection of elemets of $\mathcal{S}$, we have $f^{-1}(B)$ is open in X.

PROOF: Because $f^{-1}(V_1 \cap \cdots \cap V_n) = f^{-1}(V_1) \cap \cdots \cap f^{-1}(V_n)$.

$\langle 2 \rangle 3$. Q.E.D.

PROOF: From Propositions 2.140 and 2.159.

□

Definition 2.162 (Continuous at a Point)**.** Let X and Y be topological spaces. Let $f : X \to Y$ and $x \in X$. Then f is *continuous at* x if and only if, for every neighbourhood V of $f(x)$, there exists a neighbourhood U of x such that $f(U) \subseteq V$.

Theorem 2.163. *Let X and Y be topological spaces and $f : X \to Y$. Then the following are equivalent:*

1. *f is continuous.*
2. *For all $A \subseteq X$, we have $f(\overline{A}) \subseteq \overline{f(A)}$*
3. *For all $B \subseteq Y$ closed, we have $f^{-1}(B)$ is closed in X.*
4. *f is continuous at every point of X.*

PROOF:
⟨1⟩1. $1 \Rightarrow 2$
  ⟨2⟩1. ASSUME: f is continuous.
  ⟨2⟩2. LET: $A \subseteq X$
  ⟨2⟩3. LET: $x \in \overline{A}$
    PROVE: $f(x) \in \overline{f(A)}$
  ⟨2⟩4. LET: V be a neighbourhood of $f(x)$
  ⟨2⟩5. $f^{-1}(V)$ is a neighbourhood of x
  ⟨2⟩6. PICK $y \in A \cap f^{-1}(V)$
    PROOF: By Theorem 2.117.
  ⟨2⟩7. $f(y) \in V \cap f(A)$
  ⟨2⟩8. Q.E.D.
    PROOF: By Theorem 2.117.
⟨1⟩2. $2 \Rightarrow 3$
  ⟨2⟩1. ASSUME: 2
  ⟨2⟩2. LET: B be closed in Y
  ⟨2⟩3. LET: $x \in \overline{f^{-1}(B)}$
    PROVE: $x \in f^{-1}(B)$
  ⟨2⟩4. $f(x) \in B$
    PROOF:

$$\begin{aligned} f(x) &\in f(\overline{f^{-1}(B)}) \\ &\subseteq \overline{f(f^{-1}(B))} && (\langle 2\rangle 1) \\ &\subseteq \overline{B} && (Proposition 2.118) \\ &= B \end{aligned}$$

⟨1⟩3. $3 \Rightarrow 1$
  ⟨2⟩1. ASSUME: 3
  ⟨2⟩2. LET: V be open in Y
  ⟨2⟩3. $Y \setminus V$ is closed in Y
  ⟨2⟩4. $f^{-1}(Y \setminus V)$ is closed in X
  ⟨2⟩5. $X \setminus f^{-1}(V)$ is closed in X
  ⟨2⟩6. $f^{-1}(V)$ is open in X

$\langle 1 \rangle 4.$ $1 \Rightarrow 4$

PROOF: For any neighbourhood V of $f(x)$, the set $U = f^{-1}(V)$ is a neighbourhood of x such that $f(U) \subseteq V$.

$\langle 1 \rangle 5.$ $4 \Rightarrow 1$

$\langle 2 \rangle 1.$ ASSUME: 4

$\langle 2 \rangle 2.$ LET: V be open in Y

$\langle 2 \rangle 3.$ LET: $x \in f^{-1}(V)$

$\langle 2 \rangle 4.$ V is a neighbourhood of $f(x)$

$\langle 2 \rangle 5.$ PICK a neighbourhood U of x such that $f(U) \subseteq V$

$\langle 2 \rangle 6.$ $x \in U \subseteq f^{-1}(V)$

$\langle 2 \rangle 7.$ Q.E.D.

PROOF: By Lemma 2.95.

$\square$

Theorem 2.164. *A constant function is continuous.*

PROOF: Let X and Y be topological spaces. Let $b \in Y$, and let $f : X \to Y$ be the constant function with value b. For any open $V \subseteq Y$, the set $f^{-1}(V)$ is either X (if $b \in V$) or $\emptyset$ (if $b \notin V$). $\square$

Theorem 2.165. *If A is a subspace of X then the inclusion $j : A \to X$ is continuous.*

PROOF: For any V open in X, we have $j^{-1}(V) = V \cap A$ is open in A. $\square$

Theorem 2.166. *The composite of two continuous functions is continuous.*

PROOF: Let $f : X \to Y$ and $g : Y \to Z$ be continuous. For any V open in Z, we have $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is open in X. $\square$

Theorem 2.167. *Let $f : X \to Y$ be a continuous function and A be a subspace of X. Then the restriction $f \restriction A : A \to Y$ is continuous.*

PROOF: Let V be open in Y. Then $(f \restriction A)^{-1}(V) = f^{-1}(V) \cap A$ is open in A. $\square$

Theorem 2.168. *Let $f : X \to Y$ be continuous. Let Z be a subspace of Y such that $f(X) \subseteq Z$. Then the corestriction $f : X \to Z$ is continuous.*

PROOF:

$\langle 1 \rangle 1.$ LET: V be open in Z.

$\langle 1 \rangle 2.$ PICK U open in Y such that $V = U \cap Z$.

$\langle 1 \rangle 3.$ $f^{-1}(V) = f^{-1}(U)$

$\langle 1 \rangle 4.$ $f^{-1}(V)$ is open in X.

$\square$

Theorem 2.169. *Let $f : X \to Y$ be continuous. Let Z be a space such that Y is a subspace of Z. Then the expansion $f : X \to Z$ is continuous.*

PROOF: Let V be open in Z. Then $f^{-1}(V) = f^{-1}(V \cap Y)$ is open in X. $\square$

Theorem 2.170. *Let X and Y be topological spaces. Let $f : X \to Y$. Suppose $\mathcal{U}$ is a set of open sets in X such that $X = \bigcup \mathcal{U}$ and, for all $U \in \mathcal{U}$, we have $f \upharpoonright U : U \to Y$ is continuous. Then f is continuous.*

PROOF:
⟨1⟩1. LET: V be open in Y
⟨1⟩2. $f^{-1}(V) = \bigcup_{U \in \mathcal{U}} (f \upharpoonright U)^{-1}(V)$
⟨1⟩3. For all $U \in \mathcal{U}$, we have $(f \upharpoonright U)^{-1}(V)$ is open in U.
⟨1⟩4. For all $U \in \mathcal{U}$, we have $(f \upharpoonright U)^{-1}(V)$ is open in X.
PROOF: Lemma 2.213.
□

Proposition 2.171. *Let X and X' be the same set X under two topologies $\mathcal{T}$ and $\mathcal{T}'$. Let $i : X \to X'$ be the identity function. Then i is continuous if and only if $\mathcal{T}' \subseteq \mathcal{T}$.*

PROOF: Immediate from definitions. □

Proposition 2.172. *Let $f : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$. Then f is continuous on the right at a if and only if f is continuous at a as a function $\mathbb{R}_l \to \mathbb{R}$.*

PROOF:
⟨1⟩1. If f is continuous on the right at a then f is continuous at a as a function $\mathbb{R}_l \to \mathbb{R}$.
⟨2⟩1. ASSUME: f is continuous on the right at a.
⟨2⟩2. LET: V be a neighbourhood of $f(a)$
⟨2⟩3. PICK b, c such that $f(a) \in (b, c) \subseteq V$.
⟨2⟩4. LET: $\epsilon = \min(c - f(a), f(a) - b)$
⟨2⟩5. PICK $\delta > 0$ such that, for all x, if $a < x < a + \delta$ then $|f(x) - f(a)| < \epsilon$
⟨2⟩6. LET: $U = [a, a + \delta)$
⟨2⟩7. $f(U) \subseteq V$
⟨1⟩2. If f is continuous at a as a function $\mathbb{R}_l \to \mathbb{R}$ then f is continuous on the right at a.
⟨2⟩1. ASSUME: f is continuous at a as a function $\mathbb{R}_l \to \mathbb{R}$
⟨2⟩2. LET: $\epsilon > 0$
⟨2⟩3. PICK a neighbourhood U of a such that $f(U) \subseteq (f(a) - \epsilon, f(a) + \epsilon)$
⟨2⟩4. PICK b, c such that $a \in [b, c) \subset U$
⟨2⟩5. LET: $\delta = c - a$
⟨2⟩6. For all x, if $a < x < a + \delta$ then $|f(x) - f(a)| < \epsilon$
□

Lemma 2.173. *Let $f : X \to Y$. Let Z be an open subspace of X and $a \in Z$. If $f \upharpoonright Z$ is continuous at a then f is continuous at a.*

PROOF:
⟨1⟩1. LET: V be a neighbourhood of $f(x)$
⟨1⟩2. PICK a neighbourhood W of x in Z such that $f(W) \subseteq V$
⟨1⟩3. W is a neighbourhood of x in X such that $f(W) \subseteq V$

PROOF: Lemma 2.213.
∎

Proposition 2.174. *Let $f : A \to B$ and $g : C \to D$ be continuous. Define $f \times g : A \times C \to B \times D$ by*

$$(f \times g)(a, c) = (f(a), g(c)) \ .$$

Then $f \times g$ is continuous.

PROOF: $\pi_1 \circ (f \times g) = f \circ \pi_1$ and $\pi_2 \circ (f \times g) = g \circ \pi_2$ are continuous by Theorem 2.166. The result follows by Theorem 2.202.

Proposition 2.175. *Let X and Y be topological spaces and $f : X \to Y$ be continuous. If $a_n \to l$ as $n \to \infty$ in X then $f(a_n) \to f(l)$ as $n \to \infty$.*

PROOF:
$\langle 1 \rangle 1$. LET: V be a neighbourhood of $f(l)$
$\langle 1 \rangle 2$. PICK a neighbourhood U of l such that $f(U) \subseteq V$
$\langle 1 \rangle 3$. PICK N such that, for all $n \geq N$, we have $a_n \in U$
$\langle 1 \rangle 4$. For all $n \geq N$ we have $f(n) \in V$
∎

2.45 Homeomorphisms

Definition 2.176 (Homeomorphism)**.** Let X and Y be topological spaces. A *Homeomorphism* f between X and Y, $f : X \cong Y$, is a bijection $f : X \to Y$ such that both f and f^{-1} are continuous.

Lemma 2.177. *Let X and Y be topological spaces and $f : X \to Y$ a bijection. Then the following are equivalent:*

1. *f is a homeomorphism.*
2. *f is continuous and an open map.*
3. *f is continuous and a closed map.*
4. *For any $U \subseteq X$, we have U is open if and only if $f(U)$ is open.*

PROOF: Immediate from definitions. ∎

Proposition 2.178. *Let X and X' be the same set X under two topologies $\mathcal{T}$ and $\mathcal{T}'$. Let $i : X \to X'$ be the identity function. Then i is a homeomorphism if and only if $\mathcal{T} = \mathcal{T}'$.*

PROOF: Immediate from definitions. ∎

Definition 2.179 (Topological Property)**.** Let P be a property of topological spaces. Then P is a *topological* property if and only if, for any spaces X and Y, if P holds of X and $X \cong Y$ then P holds of Y.

Definition 2.180 (Topological Imbedding)**.** Let X and Y be topological spaces and $f : X \to Y$. Then f is a *topological imbedding* if and only if the corestriction $f : X \to f(X)$ is a homeomorphism.

Proposition 2.181. *Let X and Y be topological spaces and $a \in X$. The function $i : Y \to X \times Y$ that maps y to (a, y) is an imbedding.*

PROOF:
$\langle 1 \rangle 1$. i is injective
$\langle 1 \rangle 2$. i is continuous.
PROOF: For U open in X and V open in Y, we have $i^{-1}(U \times V)$ is V if $a \in U$, and $\emptyset$ if $a \notin U$.
$\langle 1 \rangle 3$. $i : Y \to i(Y)$ is an open map.
PROOF: For V open in Y we have $i(V) = (X \times V) \cap i(Y)$.
$\square$

2.46 The Order Topology

Definition 2.182 (Order Topology)**.** Let X be a linearly ordered set with at least two points. The *order topology* on X is the topology generated by the basis $\mathcal{B}$ consisting of:

- all open intervals (a, b);
- all intervals of the form $[\perp, b)$ where $\perp$ is least in X;
- all intervals of the form $(a, \top]$ where $\top$ is greatest in X.

We prove this is a basis for a topology.

PROOF:
$\langle 1 \rangle 1$. For all $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B$.
$\langle 2 \rangle 1$. LET: $x \in X$
$\langle 2 \rangle 2$. CASE: x is greatest in X.
$\langle 3 \rangle 1$. PICK $y \in X$ with $y \neq x$
$\langle 3 \rangle 2$. $x \in (y, x] \in \mathcal{B}$
$\langle 2 \rangle 3$. CASE: x is least in X.
$\langle 3 \rangle 1$. PICK $y \in X$ with $y \neq x$
$\langle 3 \rangle 2$. $x \in [x, y) \in \mathcal{B}$
$\langle 2 \rangle 4$. CASE: x is neither greatest nor least in X.
$\langle 3 \rangle 1$. PICK $a, b \in X$ with $a < x$ and $x < b$
$\langle 3 \rangle 2$. $x \in (a, b) \in \mathcal{B}$
$\langle 1 \rangle 2$. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$
$\langle 2 \rangle 1$. LET: $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$
$\langle 2 \rangle 2$. CASE: $B_1 = (a, b)$, $B_2 = (c, d)$
PROOF: Take $B_3 = (\max(a, c), \min(b, d))$.
$\langle 2 \rangle 3$. CASE: $B_1 = (a, b)$, $B_2 = [\perp, d)$

PROOF: Take $B_3 = (a, \min(b, d))$.
⟨2⟩4. CASE: $B_1 = (a, b)$, $B_2 = (c, \top]$
PROOF: Take $B_3 = (\max(a, c), b)$.
⟨2⟩5. CASE: $B_1 = [\perp, b)$, $B_2 = [\perp, d)$
PROOF: Take $B_3 = [\perp, \min(b, d))$.
⟨2⟩6. CASE: $B_1 = [\perp, b)$, $B_2 = (c, \top]$
PROOF: Take $B_3 = (c, b)$.
□

Lemma 2.183. *Let X be a linearly ordered set. Then the open rays form a subbasis for the order topology on X.*

PROOF:
⟨1⟩1. Every open ray is open.
⟨2⟩1. For all $a \in X$, the ray $(-\infty, a)$ is open.
⟨3⟩1. LET: $x \in (-\infty, a)$
⟨3⟩2. CASE: x is least in X
PROOF: $xin[x, a) = (-\infty, a)$.
⟨3⟩3. CASE: x is not least in X
⟨4⟩1. PICK $y < x$
⟨4⟩2. $x \in (y, a) \subseteq (-\infty, a)$
⟨2⟩2. For all $a \in X$, the ray $(a, +\infty)$ is open.
PROOF: Similar.
⟨1⟩2. Every basic open set is a finite intersection of open rays.
PROOF: We have $(a, b) = (a, +\infty) \cap (-\infty, b)$, $[\perp, b) = (-\infty, b)$ and $(a, \top] = (a, +\infty)$.
□

Definition 2.184 (Standard Topology on the Real Line)**.** The *standard topology on the real line* is the order topology on $\mathbb{R}$ generated by the standard order.

Lemma 2.185. *The lower limit topology is strictly finer than the standard topology on $\mathbb{R}$.*

PROOF:
⟨1⟩1. Every open interval is open in the lower limit topology.
PROOF: If $x \in (a, b)$ then $x \in [x, b) \subseteq (a, b)$.
⟨1⟩2. The half-open interval $[0, 1)$ is not open in the standard topology.
PROOF: There is no open interval (a, b) such that $0 \in (a, b) \subseteq [0, 1)$.
□

Lemma 2.186. *The K-topology is strictly finer than the standard topology on $\mathbb{R}$.*

PROOF:
⟨1⟩1. Every open interval is open in the K-topology.
PROOF: Corollary 2.132.1.
⟨1⟩2. The set $(-1, 1) \setminus K$ is not open in the standard topology.

PROOF: There is no open interval (a, b) such that $0 \in (a, b) \subseteq (-1, 1) \setminus K$, since there must be a positive integer n with $1/n \in (a, b)$.
∎

Lemma 2.187. *Let X be a topological space. Let Y be a linearly ordered set in the order topology. Let $f, g : X \to Y$ be continuous. Then $C = \{x \in X \mid f(x) \leq g(x)\}$ is closed.*

PROOF:
⟨1⟩1. LET: $x \in X \setminus C$
⟨1⟩2. $f(x) > g(x)$
PROVE: There exists a neighbourhood U of x such that $U \subseteq X \setminus C$
⟨1⟩3. CASE: There exists y such that $g(x) < y < f(x)$
PROOF: Take $U = g^{-1}((-\infty, y)) \cup f^{-1}(y, +\infty)$.
⟨1⟩4. CASE: There is no y such that $g(x) < y < f(x)$
PROOF: Take $U = g^{-1}((-\infty, f(x))) \cup f^{-1}(g(x), +\infty)$.
∎

Proposition 2.188. *Let X be a topological space. Let Y be a linearly ordered set in the order topology. Let $f, g : X \to Y$ be continuous. Define $h : X \to Y$ by $h(x) = \min(f(x), g(x))$. Then h is continuous.*

PROOF: By the Pasting Lemma applied to $\{x \in X \mid f(x) \leq g(x)\}$ and $\{x \in X \mid g(x) \leq f(x)\}$, which are closed by Lemma 2.187.

Proposition 2.189. *Let X and Y be linearly ordered sets in the order topology. Let $f : X \to Y$ be strictly monotone and surjective. Then f is a homeomorphism.*

PROOF:
⟨1⟩1. f is bijective.
PROOF: Proposition 2.64.
⟨1⟩2. f is continuous.
⟨2⟩1. For all $y \in Y$ we have $f^{-1}((y, +\infty))$ is open.
⟨3⟩1. LET: $y \in Y$
⟨3⟩2. PICK$x \in X$ such that $f(x) = y$
PROOF: Since f is surjective.
⟨3⟩3. $f^{-1}((y, +\infty)) = (x, +\infty)$
PROOF: By strict monotoncity.
⟨2⟩2. For all $y \in Y$ we have $f^{-1}((-\infty, y))$ is open.
PROOF: Similar.
⟨1⟩3. f^{-1} is continuous.
⟨2⟩1. For all $x \in X$ we have $f((x, +\infty))$ is open.
PROOF: $f((x, +\infty)) = (f(x), +\infty)$.
⟨2⟩2. For all $x \in X$ we have $f((-\infty, x))$ is open.
PROOF: $f((-\infty, x)) = (-\infty, f(x))$.
∎

2.47 The nth Root Function

Proposition 2.190. *For all $n \geq 1$, the function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined by $f(x) = x^n$ is a homemorphism.*

PROOF:
$\langle 1 \rangle 1.$ f is strictly monotone.
  $\langle 2 \rangle 1.$ LET: $x, y \in \mathbb{R}$ with $0 \leq x < y$
  $\langle 2 \rangle 2.$ $x^n < y^n$

$$\begin{aligned} y^n - x^n &= (y - x)(y^{n-1} + y^{n-2}x + y^{n-3}x^2 + \cdots + x^{n-1}) \\ &> 0 \end{aligned}$$

$\langle 1 \rangle 2.$ f is surjective.
  $\langle 2 \rangle 1.$ LET: $y \in \mathbb{R}_{\geq 0}$
  $\langle 2 \rangle 2.$ PICK $x \in \mathbb{R}$ such that $y \leq x^n$
    PROOF: If $y \leq 1$ take $x = 1$, otherwise take $x = y$.
  $\langle 2 \rangle 3.$ There exists $x' \in [0, x]$ such that $(x')^n = y$
    PROOF: By the Intermediate Value Theorem.
$\langle 1 \rangle 3.$ Q.E.D.
  PROOF: Proposition 2.189.
$\square$

Definition 2.191. For $n \geq 1$, the *nth root function* is the function $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ that is the inverse of $\lambda x.x^n$.

2.48 The Product Topology

Definition 2.192 (Product Topology)**.** Let $\{A_i\}_{i \in I}$ be a family of topological spaces. The *product topology* on $\prod_{i \in I} A_i$ is the topology generated by the subbasis consisting of the sets of the form $\pi_i{}^{-1}(U)$ where $i \in I$ and U is open in A_i.

Proposition 2.193. *The product topology on $\prod_{i \in I} A_i$ is generated by the basis consisting of all sets of the form $\prod_{i \in I} U_i$ where $\{U_i\}_{i \in I}$ is a family such that each U_i is an open set in A_i and $U_i = A_i$ for all but finitely many i.*

PROOF: From Proposition 2.140. $\square$

Proposition 2.194. *If A_i is closed in X_i for all $i \in I$ then $\prod_{i \in I} A_i$ is closed in $\prod_{i \in I} X_i$.*

PROOF:

$$(\prod_{i \in I} X_i) \setminus (\prod_{i \in I} A_i) = \bigcup_{j \in I} \left(\prod_{i \in I} X_i \setminus \pi_j{}^{-1}(A_j) \right) \square$$

Proposition 2.195. *Let $\{A_i\}_{i \in I}$ be a family of topological spaces. For $i \in I$, let $\mathcal{B}_i$ be a basis for the topology on A_i. Then $\mathcal{B} = \{\prod_{i \in I} B_i \mid \forall i \in I. B_i \in \mathcal{B}_i, B_i = A_i$ for all but finitely many $i\}$ is a basis for the box topology on $\prod_{i \in I} A_i$.*

PROOF:
⟨1⟩1. Every set in $\mathcal{B}$ is open.
⟨1⟩2. For every point $a \in \prod_{i \in I} A_i$ and every open set U with $a \in U$, there exists $B \in \mathcal{B}$ such that $a \in B \subseteq U$.
⟨2⟩1. LET: U be open and $a \in U$
⟨2⟩2. PICK a family $\{U_i\}_{i \in I}$ such that each U_i is open in A_i, such that $U_i = A_i$ except for $i = i_1, \ldots, i_n$, and such that $a \in \prod_{i \in I} U_i \subseteq U$.
⟨2⟩3. For $j = 1, \ldots, n$, PICK $B_{i_j} \in \mathcal{B}_{i_j}$ such that $a_{i_j} \in B_{i_j} \subseteq U_{i_j}$
⟨2⟩4. LET: $B = \prod_{i \in I} B_i$ where $B_i = A_i$ for $i \neq i_1, \ldots, i_n$
⟨2⟩5. $B \in \mathcal{B}$
⟨2⟩6. $a \in B \subseteq U$
⟨1⟩3. Q.E.D.
PROOF: Lemma 2.133.
□

Proposition 2.196. *Let $\{A_i\}_{i \in I}$ be a family of topological spaces. Then the projections $\pi_i : \prod_{i \in I} A_i \to A_i$ are open maps.*

PROOF: From Lemma 2.156. □

Example 2.197. The projections are not always closed maps. For example, $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ maps the closed set $\{(x, 1/x) \mid x > 0\}$ to $(0, +\infty)$.

Proposition 2.198. *Let $\{X_i\}_{i \in I}$ be a family of sets. For $i \in I$, let $\mathcal{T}_i$ and $\mathcal{U}_i$ be topologies on X_i. Let $\mathcal{P}$ be the product topology on $\prod_{i \in I} X_i$ generated by the topologies $\mathcal{T}_i$, and $\mathcal{Q}$ the product topology on the same set generated by the topologies $\mathcal{U}_i$. Then $\mathcal{P} \subseteq \mathcal{Q}$ if and only if $\mathcal{T}_i \subseteq \mathcal{U}_i$ for all i.*

PROOF:
⟨1⟩1. If $\mathcal{T}_i \subseteq \mathcal{U}_i$ for all i then $\mathcal{P} \subseteq \mathcal{Q}$
PROOF: By Corollary 2.132.1.
⟨1⟩2. If $\mathcal{P} \subseteq \mathcal{Q}$ then $\mathcal{T}_i \subseteq \mathcal{U}_i$ for all i
⟨2⟩1. ASSUME: $\mathcal{P} \subseteq \mathcal{Q}$
⟨2⟩2. LET: $i \in I$
⟨2⟩3. LET: $U \in \mathcal{T}_i$
⟨2⟩4. LET: $U_i = U$ and $U_j = X_j$ for $j \neq i$
⟨2⟩5. $\prod_{i \in I} U_i \in \mathcal{P}$
⟨2⟩6. $\prod_{i \in I} U_i \in \mathcal{Q}$
⟨2⟩7. $U \in \mathcal{U}_i$
PROOF: From Proposition 2.196.
□

Proposition 2.199 (AC)**.** *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let $A_i \subseteq X_i$ for all $i \in I$. Then*

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i} \ .$$

PROOF:

⟨1⟩1. $\overline{\prod_{i \in I} A_i} \subseteq \prod_{i \in I} \overline{A_i}$

⟨2⟩1. For all $i \in I$ we have $A_i \subseteq \overline{A_i}$

PROOF: Lemma 2.114.

⟨2⟩2. $\prod_{i \in I} A_i \subseteq \prod_{i \in I} \overline{A_i}$

⟨2⟩3. Q.E.D.

PROOF: Since $\prod_{i \in I} \overline{A_i}$ is closed by Proposition 2.194.

⟨1⟩2. $\prod_{i \in I} \overline{A_i} \subseteq \overline{\prod_{i \in I} A_i}$

⟨2⟩1. LET: $x \in \prod_{i \in I} \overline{A_i}$

⟨2⟩2. LET: U be a neighbourhood of x

⟨2⟩3. PICK V_i open in X_i such that $x \in \prod_{i \in I} V_i \subseteq U$ with $V_i = X_i$ except for $i = i_1, \ldots, i_n$

⟨2⟩4. For $i \in I$, pick $a_i \in V_i \cap A_i$

PROOF: By Theorem 2.117 and ⟨2⟩1 using the Axiom of Choice.

⟨2⟩5. U intersects $\prod_{i \in I} A_i$

⟨2⟩6. Q.E.D.

PROOF: $a \in U \cap \prod_{i \in I} A_i$

□

Example 2.200. The closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ is $\mathbb{R}^\omega$

PROOF:

⟨1⟩1. LET: $a \in \mathbb{R}^\omega$

⟨1⟩2. LET: U be any neighbourhoods of a.

⟨1⟩3. PICK U_n open in $\mathbb{R}$ for all n such that $a \in \prod_{n \geq 0} U_n \subseteq U$ and $U_n = \mathbb{R}$ for all n except $n_1, \ldots, n_k$

⟨1⟩4. LET: $b_n = a_n$ for $n = n_1, \ldots, n_k$ and $b_n = 0$ for all other n

⟨1⟩5. $b \in \mathbb{R}^\infty \cap U$

⟨1⟩6. Q.E.D.

PROOF: From Theorem 2.117.

□

Proposition 2.201. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let (a_n) be a sequence in $\prod_{i \in I} X_i$ and $l \in \prod_{i \in I} X_i$. Then $a_n \to l$ as $n \to \infty$ if and only if, for all $i \in I$, we have $\pi_i(a_n) \to \pi_i(l)$ as $n \to \infty$.*

PROOF:

⟨1⟩1. If $a_n \to l$ as $n \to \infty$ then, for all $i \in I$, we have $\pi_i(a_n) \to \pi_i(l)$ as $n \to \infty$

PROOF: Proposition 2.175.

⟨1⟩2. If, for all $i \in I$, we have $\pi_i(a_n) \to \pi_i(l)$ as $n \to \infty$, then $a_n \to l$ as $n \to \infty$

⟨2⟩1. ASSUME: For all $i \in I$, we have $\pi_i(a_n) \to \pi_i(l)$ as $n \to \infty$

⟨2⟩2. LET: V be a neighbourhood of l

⟨2⟩3. PICK open sets U_i in X_i such that $l \in \prod_{i \in I} U_i \subseteq V$ and $U_i = X_i$ for all i except $i = i_1, \ldots, i_k$

⟨2⟩4. For $j = 1, \ldots, k$, PICK N_j such that, for all $n \geq N_j$, we have $\pi_{i_j}(a_n) \in U_{i_j}$

⟨2⟩5. LET: $N = \max(N_1, \ldots, N_k)$

⟨2⟩6. For all $n \geq N$ we have $a_n \in V$

□

Theorem 2.202. *Let A be a topological space and $\{X_i\}_{i \in I}$ be a family of topological spaces. Let $f : A \to \prod_{i \in I} X_i$ be a function. If $\pi_i \circ f$ is continuous for all $i \in I$ then f is continuous.*

Proof:
$\langle 1 \rangle 1.$ Let: $i \in I$ and U be open in X_i
$\langle 1 \rangle 2.$ $f^{-1}({\pi_i}^{-1}(U))$ is open in A
$\langle 1 \rangle 3.$ Q.E.D.
Proof: Proposition 2.160.
□

2.48.1 Continuous in Each Variable Separately

Definition 2.203 (Continuous in Each Variable Separately)**.** Let $F : X \times Y \to Z$. Then F is *continuous in each variable separately* if and only if:

- for every $a \in X$ the function $\lambda y \in Y.F(a, y)$ is continuous;
- for every $b \in Y$ the function $\lambda x \in X.F(x, b)$ is continuous.

Proposition 2.204. *Let $F : X \times Y \to Z$. If F is continuous then F is continuous in each variable separately.*

Proof: For $a \in X$, the function $\lambda y \in Y.F(a, y)$ is $F \circ i$ where $i : Y \to X \times Y$ maps y to (a, y). We have i is continuous by Proposition 2.181, hence $F \circ i$ is continuous by Theorem 2.166.
Similarly for $\lambda x \in X.F(x, b)$ for $b \in Y$. □

Example 2.205. Define $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$F(x, y) = \begin{cases} xy/(x^2 + y^2) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Then F is continuous in each variable separately but not continuous.

Proposition 2.206. *Let $f : A \to C$ and $g : B \to D$ be open maps. Then $f \times g : A \times B \to C \times D$ is an open map.*

Proof: Given U open in A and V open in B. Then $(f \times g)(U \times V) = f(U) \times g(V)$ is open in $C \times D$. The result follows from Proposition 2.157. □

Definition 2.207 (Sorgenfrey Plane)**.** The *Sorgenfrey plane* is $\mathbb{R}_l^2$.

2.49 The Subspace Topology

Definition 2.208 (Subspace Topology)**.** Let X be a topological space and $Y \subseteq X$. The *subspace topology* on Y is $\mathcal{T} = \{U \cap Y \mid U \text{ is open in } X\}$.

We prove this is a topology.

PROOF:
⟨1⟩1. $Y \in \mathcal{T}$
  PROOF: Since $Y = X \cap Y$
⟨1⟩2. For all $\mathcal{U} \subseteq \mathcal{T}$, we have $\bigcup \mathcal{U} \in \mathcal{T}$
  ⟨2⟩1. LET: $\mathcal{U} \subseteq \mathcal{T}$
  ⟨2⟩2. LET: $\mathcal{V} = \{V \text{ open in } X \mid V \cap Y \in \mathcal{U}\}$
  ⟨2⟩3. $\bigcup \mathcal{U} = (\bigcup \mathcal{V}) \cap Y$
⟨1⟩3. For all $U, V \in \mathcal{T}$, we have $U \cap V \in \mathcal{T}$
  ⟨2⟩1. LET: $U, V \in \mathcal{T}$
  ⟨2⟩2. PICK U', V' open in X such that $U = U' \cap Y$ and $V = V' \cap Y$
  ⟨2⟩3. $(U \cap V) = (U' \cap V') \cap Y$
□

Theorem 2.209. *Let X be a topological space and Y a subspace of X. Let $A \subseteq Y$. Then A is closed in Y if and only if there exists a closed set C in X such that $A = C \cap Y$.*

PROOF: We have

$$\begin{aligned} & A \text{ is closed in } Y \\ \Leftrightarrow & Y \setminus A \text{ is open in } Y \\ \Leftrightarrow & \exists U \text{ open in } X. Y \setminus A = Y \cap U \\ \Leftrightarrow & \exists C \text{ closed in } X. Y \setminus A = Y \cap (X \setminus U) \\ \Leftrightarrow & \exists C \text{ closed in } X. A = Y \cap U \end{aligned}$$

□

Theorem 2.210. *Let Y be a subspace of X. Let $A \subseteq Y$. Let $\overline{A}$ be the closure of A in X. Then the closure of A in Y is $\overline{A} \cap Y$.*

PROOF: The closure of A in Y is

$$\begin{aligned} & \bigcap \{C \text{ closed in } Y \mid A \subseteq C\} \\ = & \bigcap \{D \cap Y \mid D \text{ closed in } X, A \subseteq D \cap Y\} && \text{(Theorem 2.209)} \\ = & \bigcap \{D \mid D \text{ closed in } X, A \subseteq D\} \cap Y \\ = & \overline{A} \cap Y \end{aligned}$$

□

Lemma 2.211. *Let X be a topological space and $Y \subseteq X$. Let $\mathcal{B}$ be a basis for the topology on X. Then $\mathcal{B}' = \{B \cap Y \mid B \in \mathcal{B}\}$ is a basis for the subspace topology on Y.*

PROOF:
⟨1⟩1. Every element in $\mathcal{B}'$ is open in Y
⟨1⟩2. For every open set U in Y and point $y \in U$, there exists $B' \in \mathcal{B}'$ such that $y \in B' \subseteq U$
  ⟨2⟩1. LET: U be open in Y and $y \in U$
  ⟨2⟩2. PICK V open in X such that $U = V \cap Y$
  ⟨2⟩3. PICK $B \in \mathcal{B}$ such that $y \in B \subseteq V$

$\langle 2 \rangle 4$. LET: $B' = B \cap Y$
$\langle 2 \rangle 5$. $B' \in \mathcal{B}'$
$\langle 2 \rangle 6$. $y \in B' \subseteq U$
$\langle 1 \rangle 3$. Q.E.D.
PROOF: By Lemma 2.133.
□

Lemma 2.212. *Let X be a topological space and $Y \subseteq X$. Let $\mathcal{S}$ be a basis for the topology on X. Then $\mathcal{S}' = \{S \cap Y \mid S \in \mathcal{S}\}$ is a subbasis for the subspace topology on Y.*

PROOF: The set $\{B \cap Y \mid B$ is a finite intersection of elements of $\mathcal{S}\}$ is a basis for the subspace topology by Lemma 2.211, and this is the set of all finite intersections of elements of $\mathcal{S}'$. □

Lemma 2.213. *Let Y be a subspace of X. If U is open in Y and Y is open in X then U is open in X.*

PROOF:
$\langle 1 \rangle 1$. PICK V open in X such that $U = V \cap Y$
$\langle 1 \rangle 2$. U is open in X
PROOF: Since it is the intersection of two open sets V and Y.
□

Theorem 2.214. *Let Y be a subspace of X and $A \subseteq Y$. If A is closed in Y and Y is closed in X then A is closed in X.*

PROOF: Pick a closed set C in X such that $A = C \cap Y$ (Theorem 2.209). Then A is the intersection of two sets closed in X, hence A is closed in X (Lemma 2.102). □

Theorem 2.215. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let A_i be a subspace of X_i for all $i \in I$. Then the product topology on $\prod_{i \in I} A_i$ is the same as the topology it inherits as a subspace of $\prod_{i \in I} X_i$.*

PROOF: The product topology is generated by the subbasis

$$\begin{aligned} & \{{\pi_i}^{-1}(U) \mid i \in I, U \text{ open in } A_i\} \\ =& \{{\pi_i}^{-1}(V) \cap A_i \mid i \in I, V \text{ open in } X_i\} \\ =& \{{\pi_i}^{-1}(V) \mid i \in I, V \text{ open in } X_i\} \cap \prod_{i \in I} A_i \end{aligned}$$

and this is a subbasis for the subspace topology by Lemma 2.212. □

Theorem 2.216. *Let X be an ordered set in the order topology. Let $Y \subseteq X$ be an interval. Then the order topology on Y is the same as the subspace topology on Y.*

PROOF:
$\langle 1 \rangle 1$. The order topology is finer than the subspace topology.

⟨2⟩1. For every open ray R in X, the set $R \cap Y$ is open in the order topology.
⟨3⟩1. For all $a \in X$, we have $(-\infty, a) \cap Y$ is open in the order topology.
⟨4⟩1. CASE: For all $y \in Y$ we have $y < a$
PROOF: In this case $(-\infty, a) \cap Y = Y$.
⟨4⟩2. CASE: For all $y \in Y$ we have $a < y$
PROOF: In this case $(-\infty, a) \cap Y = \emptyset$.
⟨4⟩3. CASE: There exists $y \in Y$ such that $y \leq a$ and $y \in Y$ such that $a \leq y$
⟨5⟩1. $a \in Y$
PROOF: Because Y is an interval.
⟨5⟩2. $(-\infty, a) \cap Y = \{y \in Y \mid y < a\}$
⟨3⟩2. For all $a \in X$, we have $(a, +\infty) \cap Y$ is open in the order topology.
PROOF: Similar.
⟨2⟩2. Q.E.D.
PROOF: By Lemmas 2.183 and 2.212 and Proposition 2.141.
⟨1⟩2. The subspace topology is finer than the order topology.
⟨2⟩1. Every open ray in Y is open in the subspace topology.
PROOF: For any $a \in Y$ we have $(-\infty, a)_Y = (-\infty, a)_X \cap Y$ and $(a, +\infty)_Y = (a, +\infty)_X \cap Y$.
⟨2⟩2. Q.E.D.
PROOF: By Lemma 2.183 and Proposition 2.141
□

This example shows that we cannot remove the hypothesis that Y is an interval:

Example 2.217. The order topology on I_o^2 is different from the subspace topology as a subspace of $\mathbb{R}^2$ under the dictionary order topology.

PROOF: The set $\{1/2\} \times (1/2, 1)$ is open in the subspace topology but not in the order topology. □

Proposition 2.218. *Let X be a topological space, Y a subspace of X, and Z a subspace of Y. Then the subspace topology on Z inherited from X is the same as the subspace topology on Z inherited from Y.*

PROOF: The subspace topology inherited from Y is

$$\begin{aligned} &\{V \cap Z \mid V \text{ open in } Y\} \\ =&\{U \cap Y \cap Z \mid U \text{ open in } X\} \\ =&\{U \cap Z \mid U \text{ open in } X\} \end{aligned}$$

which is the subspace topology inherited from X. □

Definition 2.219 (Unit Circle)**.** The *unit circle* S^1 is $\{(x, y) \in \mathbb{R}^2 \mid x^2+y^2 = 1\}$ as a subspace of $\mathbb{R}^2$.

Definition 2.220 (Unit 2-sphere)**.** The *unit 2-sphere* is $S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$ as a subspace of $\mathbb{R}^3$.

Proposition 2.221. *Let $f : X \to Y$ be an open map and $A \subseteq X$ be open. Then the restriction $f \upharpoonright A : A \to f(A)$ is an open map.*

Proof:
⟨1⟩1. Let: U be open in A
⟨1⟩2. U is open in X
  Proof: Lemma 2.213.
⟨1⟩3. $f(U)$ is open in Y
⟨1⟩4. $f(U)$ is open in $f(A)$
  Proof: Since $f(U) = f(U) \cap f(A)$.
□

Example 2.222. This example shows that we cannot remove the hypothesis that A is open.

Let $A = \{(x, y) \in \mathbb{R}^2 \mid (x > 0 \text{ and } y = 1/x) \text{ or } x = y = 0\}$. Then $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ is an open map, but $\pi_1 \upharpoonright A : A \to [0, +\infty)$ is not, because it maps the set $\{0, 0\}$ which is open in A to $\{0\}$ which is not open in $[0, +\infty)$.

Proposition 2.223. *Let Y be a subspace of X. Let $A \subseteq Y$ and $l \in Y$. Then l is a limit point of A in Y if and only if l is a limit point of A in X.*

Proof: Both are equivalent to the condition that any neighbourhood of l in X intersects A in a point other than l. □

2.50 The Box Topology

Definition 2.224 (Box Topology)**.** Let $\{A_i\}_{i \in I}$ be a family of topological spaces. The *box topology* on $\prod_{i \in I} A_i$ is the topology generated by the set of all sets of the form $\prod_{i \in I} U_i$ where $\{U_i\}_{i \in I}$ is a family such that each U_i is open in A_i.

This is a basis since it covers $\prod_{i \in I} A_i$ and is closed under intersection.

Proposition 2.225. *The box topology is finer than the product topology.*

Proof: From Proposition 2.193. □

Corollary 2.225.1. *If A_i is closed in X_i for all $i \in I$ then $\prod_{i \in I} A_i$ is closed in $\prod_{i \in I} X_i$ under the box topology.*

Proof: From Proposition 2.194.

Proposition 2.226 (AC)**.** *Let $\{A_i\}_{i \in I}$ be a family of topological spaces. For $i \in I$, let $\mathcal{B}_i$ be a basis for the topology on A_i. Then $\mathcal{B} = \{\prod_{i \in I} B_i \mid \forall i \in I. B_i \in \mathcal{B}_i\}$ is a basis for the box topology on $\prod_{i \in I} A_i$.*

Proof:
⟨1⟩1. Every set of the form $\prod_{i \in I} B_i$ is open.
⟨1⟩2. For every point $a \in \prod_{i \in I} A_i$ and every open set U with $a \in U$, there exists $B \in \mathcal{B}$ such that $a \in B \subseteq U$.

⟨2⟩1. LET: U be open and $a \in U$

⟨2⟩2. PICK a family $\{U_i\}_{i \in I}$ such that each U_i is open in A_i and $a \in \prod_{i \in I} U_i \subseteq U$.

⟨2⟩3. For $i \in I$, PICK $B_i \in \mathcal{B}_i$ such that $a_i \in B_i \subseteq U_i$

PROOF: Using the Axiom of Choice.

⟨2⟩4. $a \in \prod_{i \in I} B_i \subseteq U$

⟨1⟩3. Q.E.D.

PROOF: Lemma 2.133.

□

Theorem 2.227. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let A_i be a subspace of X_i for all $i \in I$. Give $\prod_{i \in I} X_i$ the box topology. Then the box topology on $\prod_{i \in I} A_i$ is the same as the topology it inherits as a subspace of $\prod_{i \in I} X_i$.*

PROOF: The box topology is generated by the basis

$$\begin{aligned} &\{\prod_{i \in I} U_i \mid \forall i \in I, U_i \text{ open in } A_i\} \\ =&\{\prod_{i \in I} (V_i \cap A_i) \mid \forall i \in I, V_i \text{ open in } X_i\} \\ =&\{\prod_{i \in I} V_i \mid \forall i \in I, V_i \text{ open in } X_i\} \cap \prod_{i \in I} A_i \end{aligned}$$

and this is a basis for the subspace topology by Lemma 2.211. □

Proposition 2.228 (AC)**.** *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Give $\prod_{i \in I} X_i$ the box topology. Let $A_i \subseteq X_i$ for all $i \in I$. Then*

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i} \enspace .$$

PROOF:

⟨1⟩1. $\overline{\prod_{i \in I} A_i} \subseteq \prod_{i \in I} \overline{A_i}$

⟨2⟩1. For all $i \in I$ we have $A_i \subseteq \overline{A_i}$

PROOF: Lemma 2.114.

⟨2⟩2. $\prod_{i \in I} A_i \subseteq \prod_{i \in I} \overline{A_i}$

⟨2⟩3. Q.E.D.

PROOF: Since $\prod_{i \in I} A_i$ is closed by Corollary 2.225.1.

⟨1⟩2. $\prod_{i \in I} \overline{A_i} \subseteq \overline{\prod_{i \in I} A_i}$

⟨2⟩1. LET: $x \in \prod_{i \in I} \overline{A_i}$

⟨2⟩2. LET: U be a neighbourhood of x

⟨2⟩3. PICK V_i open in X_i such that $x \in \prod_{i \in I} V_i \subseteq U$

⟨2⟩4. For $i \in I$, pick $a_i \in V_i \cap A_i$

PROOF: By Theorem 2.117 and ⟨2⟩1 using the Axiom of Choice.

⟨2⟩5. U intersects $\prod_{i \in I} A_i$

⟨2⟩6. Q.E.D.

PROOF: $a \in U \cap \prod_{i \in I} A_i$.

□

The following example shows that Theorem 2.202 fails in the box topology.

Example 2.229. Define $f : \mathbb{R} \to \mathbb{R}^\omega$ by $f(t) = (t, t, \ldots)$. Then $\pi_n \circ f = \mathrm{id}_\mathbb{R}$ is continuous for all n. But f is not continuous when $\mathbb{R}^\omega$ is given the box topology because the inverse image of

$$(-1, 1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times \cdots$$

is $\{0\}$ which is not open.

The following example shows that Proposition 2.201 fails in the box topology.

Example 2.230. Give $\mathbb{R}^\omega$ the box topology. Let $a_n = (1/n, 1/n, \ldots)$ for $n \geq 1$ and $l = (0, 0, \ldots)$. Then $\pi_i(a_n) \to \pi_i(l)$ as $n \to \infty$ for all i, but $a_n \not\to l$ as $n \to \infty$ since the open set

$$(-1, 1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times \cdots$$

contains l but does not contain any a_n.

Example 2.231. The set $\mathbb{R}^\infty$ is closed in $\mathbb{R}^\omega$ under the box topology. For let (a_n) be any sequence not in $\mathbb{R}^\infty$. Let U_n be an open interval around a_n that does not contain 0 if $a_n \neq 0$, and $U_n = \mathbb{R}$ if $a_n = 0$. Then $\prod_{n \geq 0} U_n$ is a neighbourhood of (a_n) that does not intersect $\mathbb{R}^\infty$.

2.51 T_1 Spaces

Definition 2.232 (T_1 Space)**.** A topological space is T_1 if and only if every singleton is closed.

Lemma 2.233. *A space is T_1 if and only if every finite set is closed.*

PROOF: From Lemma 2.103. □

Theorem 2.234. *In a T_1 space, a point a is a limit point of a set A if and only if every neighbourhood of a contains infinitely many points of A.*

PROOF:
$\langle 1 \rangle$1. If a is a limit point of A then every neighbourhood of a contains infinitely many points of A.
  $\langle 2 \rangle$1. ASSUME: a is a limit point of A.
  $\langle 2 \rangle$2. LET: U be a neighbourhood of a.
  $\langle 2 \rangle$3. ASSUME: for a contradiction U contains only finitely many points of A.
  $\langle 2 \rangle$4. $(U \cap A) \setminus \{a\}$ is closed.
    PROOF: By the T_1 axiom.
  $\langle 2 \rangle$5. $(U \setminus A) \cup \{a\}$ is open.
    PROOF: It is $U \setminus ((U \cap A) \setminus \{a\})$.
  $\langle 2 \rangle$6. $(U \setminus A) \cup \{a\}$ intersects A in a point other than a.

PROOF: From ⟨2⟩1.

⟨2⟩7. Q.E.D.

□

⟨1⟩2. If every neighbourhood of a contains infinitely many points of A then a is a limit point of A.

PROOF: Immediate from definitions.

□

(To see this does not hold in every space, see Proposition 2.129.)

Proposition 2.235. *A space is T_1 if and only if, for any two distinct points x and y, there exist neighbourhoods U of x and V of y such that $x \notin V$ and $y \notin U$.*

PROOF:

⟨1⟩1. LET: X be a topological space.

⟨1⟩2. If X is T_1 then, for any two distinct points x and y, there exist neighbourhoods U of x and V of y such that $x \notin V$ and $y \notin U$.

PROOF: This holds because $\{x\}$ and $\{y\}$ are closed.

⟨1⟩3. Suppose, for any two distinct points x and y, there exist neighbourhoods U of x and V of y such that $x \notin V$ and $y \notin U$. Then X is T_1.

⟨2⟩1. ASSUME: For any two distinct points x and y, there exist neighbourhoods U of x and V of y such that $x \notin V$ and $y \notin U$.

⟨2⟩2. LET: $a \in X$

⟨2⟩3. $\{a\}$ is closed.

PROOF: For all $b \neq a$ there exists a neighbourhood U of b such that $U \subseteq X \setminus \{a\}$.

□

Proposition 2.236. *A subspace of a T_1 space is T_1.*

PROOF: From Proposition 2.214.

2.52 Hausdorff Spaces

Definition 2.237 (Hausdorff Space)**.** A topological space is *Hausdorff* if and only if, for any points x, y with $x \neq y$, there exist disjoint open sets U and V such that $x \in U$ and $y \in V$.

Theorem 2.238. *Every Hausdorff space is T_1.*

PROOF:

⟨1⟩1. LET: X be a Hausdorff space.

⟨1⟩2. LET: $b \in X$

PROVE: $\overline{\{b\}} = \{b\}$

⟨1⟩3. ASSUME: $a \in \overline{\{b\}}$ and $a \neq b$

⟨1⟩4. PICK disjoint neighbourhoods U of a and V of b.

⟨1⟩5. U intersects $\{b\}$

PROOF: Theorem 2.117.

$\langle 1\rangle 6.$ $b \in U$
$\langle 1\rangle 7.$ Q.E.D.
PROOF: This contradicts the fact that U and V are disjoint ($\langle 1\rangle 4$).
□

Proposition 2.239. *An infinite set under the finite complement topology is T_1 but not Hausdorff.*

PROOF:
$\langle 1\rangle 1.$ LET: X be an infinite set under the finite complement topology.
$\langle 1\rangle 2.$ Every singleton is closed.
PROOF: By definition.
$\langle 1\rangle 3.$ PICK$a, b \in X$ with $a \neq b$
$\langle 1\rangle 4.$ There are no disjoint neighbourhoods U of a and V of b.
$\langle 2\rangle 1.$ LET: U be a neighbourhood of a and V a neighbourhood of b.
$\langle 2\rangle 2.$ $X \setminus U$ and $X \setminus V$ are finite.
$\langle 2\rangle 3.$ PICK $c \in X$ that is not in $X \setminus U$ or $X \setminus V$.
$\langle 2\rangle 4.$ $c \in U \cap V$
□

Proposition 2.240. *The product of a family of Hausdorff spaces is Hausdorff.*

PROOF:
$\langle 1\rangle 1.$ LET: $\{X_i\}_{i \in I}$ be a family of Hausdorff spaces.
$\langle 1\rangle 2.$ LET: $a, b \in \prod_{i \in I} X_i$ with $a \neq b$
$\langle 1\rangle 3.$ PICK $i \in I$ such that $a_i \neq b_i$
$\langle 1\rangle 4.$ PICK U, V disjoint open sets in X_i with $a_i \in U$ and $b_i \in V$
$\langle 1\rangle 5.$ ${\pi_i}^{-1}(U)$ and ${\pi_i}^{-1}(V)$ are disjoint open sets in $\prod_{i \in I} X_i$ with $a \in {\pi_i}^{-1}(U)$ and $b \in {\pi_i}^{-1}(V)$
□

Theorem 2.241. *Every linearly ordered set under the order topology is Hausdorff.*

PROOF:
$\langle 1\rangle 1.$ LET: X be a linearly ordered set under the order topology.
$\langle 1\rangle 2.$ LET: $a, b \in X$ with $a \neq b$
$\langle 1\rangle 3.$ ASSUME: w.l.o.g. $a < b$
$\langle 1\rangle 4.$ CASE: There exists c such that $a < c < b$
PROOF: The sets $(-\infty, c)$ and $(c, +\infty)$ are disjoint neighbourhoods of a and b respectively.
$\langle 1\rangle 5.$ CASE: There is no c such that $a < c < b$
PROOF: The sets $(-\infty, b)$ and $(a, +\infty)$ are disjoint neighbourhoods of a and b respectively.
□

Theorem 2.242. *A subspace of a Hausdorff space is Hausdorff.*

PROOF:

$\langle 1 \rangle 1$. LET: X be a Hausdorff space and Y a subspace of X.
$\langle 1 \rangle 2$. LET: $x, y \in Y$ with $x \neq y$
$\langle 1 \rangle 3$. PICK disjoint neighbourhoods U of x and V of y in X.
$\langle 1 \rangle 4$. $U \cap Y$ and $V \cap Y$ are disjoint neighbourhoods of x and y respectively in Y.
□

Proposition 2.243. *A space X is Hausdorff if and only if the diagonal $\Delta = \{(x,x) \mid x \in X\}$ is closed in X^2.*

PROOF:

$$\begin{aligned} & X \text{ is Hausdorff} \\ \Leftrightarrow & \forall x, y \in X. x \neq y \Rightarrow \exists V, W \text{ open}. x \in V \wedge y \in W \wedge V \cap W = \emptyset \\ \Leftrightarrow & \forall (x,y) \in X^2 \setminus \Delta. \exists V, W \text{ open}. (x,y) \subseteq V \times W \subseteq X^2 \setminus \Delta \\ \Leftrightarrow & \Delta \text{ is closed} \end{aligned}$$
□

Theorem 2.244. *In a Hausdorff space, a sequence has at most one limit.*

PROOF:
$\langle 1 \rangle 1$. LET: X be a Hausdorff space.
$\langle 1 \rangle 2$. ASSUME: for a contradiction $a_n \to l$ as $n \to \infty$, $a_n \to m$ as $n \to \infty$, and $l \neq m$
$\langle 1 \rangle 3$. PICK disjoint neighbourhoods U of l and V of m
PROOF: By the Hausdorff axiom.
$\langle 1 \rangle 4$. PICK M and N such that $a_n \in U$ for $n \geq M$ and $a_n \in V$ for $n \geq N$
$\langle 1 \rangle 5$. $a_{\max(M,N)} \in U \cap V$
$\langle 1 \rangle 6$. Q.E.D.
PROOF: This contradicts the fact that U and V are disjoint ($\langle 1 \rangle 3$).
□

To see this is not always true in spaces that are T_1 but not Hausdorff:

Proposition 2.245. *Let X be an infinite set under the finite complement topology. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence with all points distinct. Then for every $l \in X$ we have $a_n \to l$ as $n \to \infty$.*

PROOF: Let U be any neighbourhood of l. Since $X \setminus U$ is finite, there must exist N such that, for all $n \geq N$, we have $a_n \in U$. □

Proposition 2.246. *Let X be a topological space. Let Y a Hausdorff space. Let $A \subseteq X$. Let $f, g : \overline{A} \to Y$ be continuous. If f and g agree on A then $f = g$.*

PROOF:
$\langle 1 \rangle 1$. LET: $x \in \overline{A}$
$\langle 1 \rangle 2$. ASSUME: $f(x) \neq g(x)$
$\langle 1 \rangle 3$. PICK disjoint neighbourhoods V of $f(x)$ and W of $g(x)$.
$\langle 1 \rangle 4$. PICK $y \in f^{-1}(V) \cap g^{-1}(W) \cap A$
PROOF: Since $f^{-1}(V) \cap g^{-1}(W)$ is a neighbourhood of x and hence intersects A.

$\langle 1\rangle 5.$ $f(y) = g(y) \in V \cap W$
$\langle 1\rangle 6.$ Q.E.D.
PROOF: This contradicts the fact that V and W are disjoint ($\langle 1\rangle 3$).
□

Proposition 2.247. *Let $\{X_i\}_{i \in I}$ be a family of Hausdorff spaces. Then $\prod_{i \in I} X_i$ under the box topology is Hausdorff.*

PROOF:
$\langle 1\rangle 1.$ LET: $\{X_i\}_{i \in I}$ be a family of Hausdorff spaces.
$\langle 1\rangle 2.$ LET: $a, b \in \prod_{i \in I} X_i$ with $a \neq b$
$\langle 1\rangle 3.$ PICK $i \in I$ such that $a_i \neq b_i$
$\langle 1\rangle 4.$ PICK U, V disjoint open sets in X_i with $a_i \in U$ and $b_i \in V$
$\langle 1\rangle 5.$ ${\pi_i}^{-1}(U)$ and ${\pi_i}^{-1}(V)$ are disjoint open sets in $\prod_{i \in I} X_i$ with $a \in {\pi_i}^{-1}(U)$ and $b \in {\pi_i}^{-1}(V)$
□

Proposition 2.248. *Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on the same set X with $\mathcal{T} \subseteq \mathcal{T}'$ If $\mathcal{T}$ is Haudorff then $\mathcal{T}'$ is Hausdorff.*

PROOF: Immediate from definitions.

Proposition 2.249. *Let X be a Hausdorff space. Let $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the finite intersection property. Then $\bigcap_{D \in \mathcal{D}} \overline{D}$ contains at most one point.*

PROOF:
$\langle 1\rangle 1.$ LET: $x, y \in \bigcap_{D \in \mathcal{D}} \overline{D}$
$\langle 1\rangle 2.$ ASSUME: for a contradiction $x \neq y$
$\langle 1\rangle 3.$ PICK disjoint open subsets U and V of x and y respectively.
$\langle 1\rangle 4.$ $U, V \in \mathcal{D}$
PROOF: Proposition 2.120.
$\langle 1\rangle 5.$ Q.E.D.
PROOF: This contradicts the fact that $\mathcal{D}$ satisfies the finite intersection property.
□

2.53 The First Countability Axiom

Definition 2.250 (First Countability Axiom)**.** A topological space X satisfies the *first countability axiom*, or is *first countable*, if and only if every point has a countable local basis.

Lemma 2.251 (Sequence Lemma (CC))**.** *Let X be a first countable space. Let $A \subseteq X$ and $l \in \overline{A}$. Then there exists a sequence in A that converges to l.*

PROOF:
$\langle 1\rangle 1.$ PICK a countable local basis $\{B_n \mid n \in \mathbb{Z}^+\}$ at l such that $B_1 \supseteq B_2 \supseteq \cdots$.

PROOF: Lemma 2.143.
⟨1⟩2. For all $n \geq 1$, PICK $a_n \in A \cap B_n$.
PROVE: $a_n \to l$ as $n \to \infty$
⟨1⟩3. LET: U be a neighbourhood of A
⟨1⟩4. PICK N such that $B_N \subseteq U$
⟨1⟩5. For $n \geq N$ we have $a_n \in U$
PROOF: $a_n \in B_n \subseteq B_N \subseteq U$
□

Theorem 2.252 (CC)**.** *Let X be a first countable space and Y a topological space. Let $f : X \to Y$. Suppose that, for every sequence (x_n) in X and $l \in X$, if $x_n \to l$ as $n \to \infty$, then $f(x_n) \to f(l)$ as $n \to \infty$. Then f is continuous.*

PROOF:
⟨1⟩1. LET: $A \subseteq X$
⟨1⟩2. LET: $a \in A$
PROVE: $f(a) \in \overline{f(A)}$
⟨1⟩3. PICK a sequence (x_n) in A that converges to a.
PROOF: By the Sequence Lemma.
⟨1⟩4. $f(x_n) \to f(a)$
⟨1⟩5. $f(a) \in \overline{f(A)}$
PROOF: By Lemma 2.145.
⟨1⟩6. Q.E.D.
PROOF: By Theorem 2.163.
□

Example 2.253 (CC)**.** The space $\mathbb{R}^\omega$ under the box product is not first countable.

PROOF: Let $\{B_n \mid n \geq 0\}$ be a countable set of neighbourhoods of $\vec{0}$. We will construct a neighbourhood of $\vec{0}$ that does not include any of these.
For $n \geq 0$, pick a neighbourhood U_n of 0 such that $U_n \subset \pi_n(B_p)$. Then $\prod_{n=0}^\infty U_n$ is a neighbourhood of $\vec{0}$ that does not include any B_n. □

Example 2.254. If J is an uncountable set then $\mathbb{R}^J$ is not first countable.

PROOF:
⟨1⟩1. LET: $\{B_n \mid n \geq 0\}$ be any countable set of neighbourhoods of $\vec{0}$.
⟨1⟩2. For $n \geq 0$, PICK a basis element $\prod_{\alpha \in J} U_{n\alpha}$ that contains $\vec{0}$ and is included in B_n.
PROOF: Using the Axiom of Countable Choice.
⟨1⟩3. For $n \geq 0$,
LET: $J_n = \{\alpha \in J \mid U_{n\alpha} \neq \mathbb{R}\}$
⟨1⟩4. PICK $\beta \in J$ such that $\beta \notin J_n$ for any n.
PROOF: Since each J_n is finite so $\bigcup_n J_n$ is countable.
⟨1⟩5. $\pi_\beta((-1, 1))$ is a neighbourhood of $\vec{0}$ that does not include any B_n.
□

Example 2.255. The space $\mathbb{R}_l$ is first countable.

PROOF: For any $a \in \mathbb{R}$, the set $\{[a, a+1/n) \mid n \geq 1\}$ is a countable local basis.

Example 2.256. The ordered square is first countable.

PROOF: For any $(a,b) \in I_o^2$ with $b \neq 0, 1$, the set $\{(\{a\} \times (b-1/n, b+1/n)) \cap I_o^2 \mid n \geq 1\}$ is a countable local basis.

2.54 Strong Continuity

Definition 2.257 (Strongly Continuous)**.** Let X and Y be topological spaces and $f : X \to Y$ be a function. Then f is *strongly continuous* if and only if, for every subset $U \subseteq Y$, we have U is open in Y if and only if $f^{-1}(U)$ is open in X.

Proposition 2.258. *Let X and Y be topological spaces and $f : X \to Y$ be a function. Then f is* strongly continuous *if and only if, for every subset $C \subseteq Y$, we have C is closed in Y if and only if $f^{-1}(C)$ is closed in X.*

PROOF: Since $X \setminus f^{-1}(C) = f^{-1}(Y \setminus C)$. ∎

Proposition 2.259. *Let X, Y and Z be topological spaces. Let $f : X \to Y$ and $g : Y \to Z$. If f and g are strongly continuous then so is $g \circ f$.*

PROOF: Since $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$. ∎

Proposition 2.260. *Let X, Y and Z be topological spaces. Let $f : X \to Y$ and $g : Y \to Z$. If $g \circ f$ is continuous and f is strongly continuous then g is continuous.*

PROOF:
⟨1⟩1. LET: $V \subseteq Z$ be open.
⟨1⟩2. $f^{-1}(g^{-1}(V))$ is open in X.
  PROOF: Since $g \circ f$ is continuous.
⟨1⟩3. $f^{-1}(V)$ is open in Y.
  PROOF: Since g is strongly continuous.
∎

Proposition 2.261. *Let X, Y and Z be topological spaces. Let $f : X \to Y$ and $g : Y \to Z$. If $g \circ f$ is strongly continuous and f is strongly continuous then g is strongly continuous.*

PROOF: For $V \subseteq Z$, we have V is open iff $f^{-1}(g^{-1}(V))$ is open iff $g^{-1}(V)$ is open.

2.55 Saturated Sets

Definition 2.262. Let X and Y be sets and $p : X \twoheadrightarrow Y$ a surjective function. Let $C \subseteq X$. Then C is *saturated* with respect to p if and only if, for all $x, y \in X$, if $x \in C$ and $p(x) = p(y)$ then $y \in C$.

Proposition 2.263. *Let X and Y be sets and $p : X \twoheadrightarrow Y$ a surjective function. Let $C \subseteq X$. Then the following are equivalent:*

1. *C is saturated with respect to p.*
2. *There exists $D \subseteq Y$ such that $C = p^{-1}(D)$*
3. *$C = p^{-1}(p(C))$.*

PROOF:
$\langle 1 \rangle 1$. $1 \Rightarrow 3$
  $\langle 2 \rangle 1$. ASSUME: C is saturated with respect to p.
  $\langle 2 \rangle 2$. $C \subseteq p^{-1}(p(C))$
    PROOF: Trivial.
  $\langle 2 \rangle 3$. $p^{-1}(p(C)) \subseteq C$
    $\langle 3 \rangle 1$. LET: $x \in p^{-1}(p(C))$
    $\langle 3 \rangle 2$. $p(x) \in p(C)$
    $\langle 3 \rangle 3$. There exists $y \in C$ such that $p(x) = p(y)$
    $\langle 3 \rangle 4$. $x \in C$
      PROOF: From $\langle 2 \rangle 1$.
$\langle 1 \rangle 2$. $3 \Rightarrow 2$
  PROOF: Trivial.
$\langle 1 \rangle 3$. $2 \Rightarrow 1$
  PROOF: This follows because if $p(x) \in D$ and $p(x) = p(y)$ then $p(y) \in D$.
$\square$

2.56 Quotient Maps

Definition 2.264 (Quotient Map)**.** Let X and Y be topological spaces and $p : X \to Y$. Then p is a *quotient map* if and only if p is surjective and strongly continuous.

Proposition 2.265. *Let X and Y be topological spaces and $p : X \twoheadrightarrow Y$ be a surjective function. Then the following are equivalent.*

1. *p is a quotient map.*
2. *p is continuous and maps saturated open sets to open sets.*
3. *p is continuous and maps saturated closed sets to closed sets.*

PROOF:
$\langle 1 \rangle 1$. $1 \Rightarrow 2$
  $\langle 2 \rangle 1$. ASSUME: p is a quotient map.
  $\langle 2 \rangle 2$. LET: U be a saturated open set in X.
  $\langle 2 \rangle 3$. $p^{-1}(p(U))$ is open in X.
    PROOF: Since $U = p^{-1}(p(U))$ be Proposition 2.263.
  $\langle 2 \rangle 4$. $p(U)$ is open in Y.

PROOF: From ⟨2⟩1.

⟨1⟩2. $1 \Rightarrow 3$

PROOF: Similar.

⟨1⟩3. $2 \Rightarrow 1$

⟨2⟩1. ASSUME: p is continuous and maps saturated open sets to open sets.

⟨2⟩2. LET: $U \subseteq Y$

⟨2⟩3. ASSUME: $p^{-1}(U)$ is open in X

⟨2⟩4. $p^{-1}(U)$ is saturated.

PROOF: Proposition 2.263.

⟨2⟩5. U is open in Y.

⟨1⟩4. $3 \Rightarrow 1$

PROOF: Similar.

□

Corollary 2.265.1. *Every surjective continuous open map is a quotient map.*

Corollary 2.265.2. *Every surjective continuous closed map is a quotient map.*

Example 2.266. The converses of these corollaries do not hold.

Let $A = \{(x, y) \mid x \geq 0\} \cup \{(x, y) \mid y = 0\}$. Then $\pi_1 : A \to \mathbb{R}$ is a quotient map, but not an open map or a closed map.

We prove that π_1 maps saturated open sets to open sets:

⟨1⟩1. LET: ${\pi_1}^{-1}(U)$ be a saturated open set in A

PROVE: U is open in $\mathbb{R}$

⟨1⟩2. LET: $x \in U$

⟨1⟩3. $(x, 0) \in \pi_1(U)^{-1}$

⟨1⟩4. PICK W, V open in $\mathbb{R}$ such that $(x, 0) \subseteq W \times V \subseteq \pi_1(U)^{-1}$

⟨1⟩5. $x \in W \subseteq U$

It is not an open map because it maps $((-1, 1) \times (1, 2)) \cap A$ to $[0, 1)$.

It is not a closed map because it maps $\{(x, 1/x) \mid x > 0\}$ to $(0, +\infty)$.

Proposition 2.267. *Let $p : X \twoheadrightarrow Y$ be a quotient map. Let $A \subseteq X$ be saturated with respect to p. Let $q : A \twoheadrightarrow p(A)$ be the restriction of p.*

1. *If A is either open or closed in X then q is a quotient map.*
2. *If p is either an open map or a closed map then q is a quotient map.*

PROOF:

⟨1⟩1. LET: $p : X \twoheadrightarrow Y$ be a quotient map.

⟨1⟩2. LET: $A \subseteq X$ be saturated with respect to p.

⟨1⟩3. LET: $q : A \twoheadrightarrow p(A)$ be the restriction of p.

⟨1⟩4. q is continuous.

PROOF: Theorem 2.167.

⟨1⟩5. If A is open in X then q is a quotient map.

⟨2⟩1. ASSUME: A is open in X.

⟨2⟩2. q maps saturated open sets to open sets.

$\langle 3\rangle 1$. Let: $U \subseteq A$ be saturated with respect to q and open in A
$\langle 3\rangle 2$. U is saturated with respect to p
$\langle 4\rangle 1$. Let: $x, y \in X$
$\langle 4\rangle 2$. Assume: $x \in U$
$\langle 4\rangle 3$. Assume: $p(x) = p(y)$
$\langle 4\rangle 4$. $x \in A$
Proof: From $\langle 3\rangle 1$ and $\langle 4\rangle 2$.
$\langle 4\rangle 5$. $y \in A$
Proof: From $\langle 1\rangle 2$ and $\langle 4\rangle 3$
$\langle 4\rangle 6$. $q(x) = x(y)$
Proof: From $\langle 1\rangle 3$, $\langle 4\rangle 3$, $\langle 4\rangle 4$, $\langle 4\rangle 5$.
$\langle 4\rangle 7$. $y \in U$
Proof: From $\langle 3\rangle 1$, $\langle 4\rangle 2$, $\langle 4\rangle 6$
$\langle 3\rangle 3$. U is open in X
Proof: Lemma 2.213, $\langle 2\rangle 1$, $\langle 3\rangle 1$.
$\langle 3\rangle 4$. $p(U)$ is open in Y
Proof: Proposition 2.265, $\langle 1\rangle 1$, $\langle 3\rangle 2$, $\langle 3\rangle 3$
$\langle 3\rangle 5$. $q(U)$ is open in $p(A)$
Proof: Since $q(U) = p(U) = p(U) \cap p(A)$.
$\langle 2\rangle 3$. Q.E.D.
Proof: By Proposition 2.265.
$\langle 1\rangle 6$. If A is closed in X then q is a quotient map.
Proof: Similar.
$\langle 1\rangle 7$. If p is an open map then q is a quotient map.
$\langle 2\rangle 1$. Assume: p is an open map
$\langle 2\rangle 2$. q maps saturated open sets to open sets.
$\langle 3\rangle 1$. Let: U be open in A and saturated with respect to q
$\langle 3\rangle 2$. Pick V open in X such that $U = A \cap V$
$\langle 3\rangle 3$. $p(V)$ is open in Y
$\langle 3\rangle 4$. $q(U) = p(V) \cap p(A)$
$\langle 4\rangle 1$. $q(U) \subseteq p(V) \cap p(A)$
Proof: From $\langle 3\rangle 2$.
$\langle 4\rangle 2$. $p(V) \cap p(A) \subseteq q(U)$
$\langle 5\rangle 1$. Let: $y \in p(V) \cap p(A)$
$\langle 5\rangle 2$. Pick $x \in V$ and $x' \in A$ such that $p(x) = p(x') = y$
$\langle 5\rangle 3$. $x \in A$
Proof: By $\langle 1\rangle 2$.
$\langle 5\rangle 4$. $x \in U$
Proof: From $\langle 3\rangle 2$
$\langle 2\rangle 3$. Q.E.D.
Proof: By Proposition 2.265.
$\langle 1\rangle 8$. If p is a closed map then q is a quotient map.
Proof: Similar.
□

Example 2.268. This example shows we cannot remove the hypotheses on A

and p.

Define $f : [0,1] \to [2,3] \to [0,2]$ by $f(x) = x$ if $x \leq 1$, $f(x) = x - 1$ if $x \geq 2$. Then f is a quotient map but its restriction f' to $[0,1) \cup [2,3]$ is not, because ${f'}^{-1}([1,2])$ is open but $[1,2]$ is not.

For a counterexample where A is saturated, see Example 2.274.

Proposition 2.269. *Let $p : A \twoheadrightarrow C$ and $q : B \twoheadrightarrow D$ be open quotient maps. Then $p \times q : A \times B \to C \times D$ is an open quotient map.*

PROOF: From Corollary 2.265.1, Proposition 2.206 and Theorem 2.202. ∎

Theorem 2.270. *Let $p : X \twoheadrightarrow Y$ be a quotient map. Let Z be a topological space and $f : Y \to Z$ be a function. Then*

1. *$f \circ p$ is continuous if and only if f is continuous.*
2. *$f \circ p$ is a quotient map if and only if f is a quotient map.*

PROOF:
$\langle 1 \rangle 1$. If $f \circ p$ is continuous then f is continuous.
PROOF: Proposition 2.260.
$\langle 1 \rangle 2$. If f is continuous then $f \circ p$ is continuous.
PROOF: Theorem 2.166.
$\langle 1 \rangle 3$. If $f \circ p$ is a quotient map then f is a quotient map.
PROOF: Proposition 2.261.
$\langle 1 \rangle 4$. If f is a quotient map then $f \circ p$ is a quotient map.
PROOF: From Proposition 2.259.
∎

Proposition 2.271. *Let X and Y be topological spaces. Let $p : X \to Y$ and $f : Y \to X$ be continuous maps such that $p \circ f = \mathrm{id}_Y$. Then p is a quotient map.*

PROOF:
$\langle 1 \rangle 1$. LET: $V \subseteq Y$
$\langle 1 \rangle 2$. ASSUME: $p^{-1}(V)$ is open in X.
$\langle 1 \rangle 3$. $f^{-1}(p^{-1}(V))$ is open in Y.
PROOF: Because f is continuous.
$\langle 1 \rangle 4$. V is open in Y.
PROOF: Because $f^{-1}(p^{-1}(V)) = V$.
∎

2.57 Quotient Topology

Definition 2.272 (Quotient Topology)**.** Let X be a topological space, Y a set and $p : X \twoheadrightarrow Y$ be a surjective function. Then the *quotient topology* on Y is the unique topology on Y with respect to which p is a quotient map, namely $\mathcal{T} = \{U \in \mathcal{P}Y \mid p^{-1}(U) \text{ is open in } X\}$.

We prove this is a topology.

PROOF:
⟨1⟩1. $Y \in \mathcal{T}$
  PROOF: Since $p^{-1}(Y) = X$ by surjectivity.
⟨1⟩2. For all $\mathcal{A} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{A} \in \mathcal{T}$
  PROOF: Since $p^{-1}(\bigcup \mathcal{A}) = \bigcup_{U \in \mathcal{A}} p^{-1}(U)$
⟨1⟩3. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$
  PROOF: Since $p^{-1}(U \cap V) = p^{-1}(U) \cap p^{-1}(V)$.
□

Definition 2.273 (Quotient Space)**.** Let X be a topological space and $\sim$ an equivalence relation on X. Let $p : X \twoheadrightarrow X/\sim$ be the canonical surjection. Then $X/\sim$ under the quotient topology is called a *quotient space*, *identification space* or *decomposition space* of X.

Here is a counterexample showing we cannot remove all the hypotheses of Proposition 2.267 except that A is saturated.

Example 2.274. Let $X = (0, 1/2] \cup \{1\} \cup \{1 + 1/n : n \geq 2\}$ as a subspace of $\mathbb{R}$. Define R to be the equivalence relation on X where xRy iff ($x = y$ or $|x - y| = 1$), so we identify $1/n$ with $1 + 1/n$ for all $n \geq 2$. Let Y be the resulting quotient space X/R in the quotient topology and $p : X \twoheadrightarrow Y$ the canonical surjection.

Let $A = \{1\} \cup (0, 1/2] \setminus \{1/n : n \geq 2\} \subseteq X$. Then A is saturated under p but the restriction q of p to A is not a quotient map because it maps the saturated open set $\{1\}$ to $\{1\}$ which is not open in $p(A)$.

Proposition 2.275. *Let X, Y and Z be topological spaces. Let $f : X \to Y$ and $g : Y \to Z$. If f and g are quotient maps then so is $g \circ f$.*

PROOF: From Proposition 2.259. □

Example 2.276. The product of two quotient maps is not necessarily a quotient map.

Let $X = \mathbb{R}$ and X^* the quotient space formed by identifying all positive integers. Let $p : X \twoheadrightarrow X^*$ be the canonical surjection.

We prove $p \times \mathrm{id}_{\mathbb{Q}} : X \times \mathbb{Q} \to X^* \times \mathbb{Q}$ is not a quotient map.

PROOF:
⟨1⟩1. For $n \geq 1$,
    LET: $c_n = \sqrt{2}/n$
⟨1⟩2. For $n \geq 1$,
    LET: $U_n = \{(x, y) \in X \times \mathbb{Q} \mid n - 1/4 < x < n + 1/4, (y + n > x + c_n \text{ and } y + n > -x + c_n) \text{ or } (y + n < x + c_n \text{ and } y + n < -x + c_n)\}$
⟨1⟩3. For $n \geq 1$, we have U_n is open in $X \times \mathbb{Q}$
⟨1⟩4. For $n \geq 1$, we have $\{n\} \times \mathbb{Q} \subseteq U_n$
⟨1⟩5. LET: $U = \bigcup_{n=1}^{\infty} U_n$
⟨1⟩6. U is open in $X \times \mathbb{Q}$
⟨1⟩7. U is saturated with respect to $p \times \mathrm{id}_{\mathbb{Q}}$

$\langle 1 \rangle 8$. LET: $U' = (p \times \mathrm{id}_{\mathbb{Q}})(U)$
$\langle 1 \rangle 9$. ASSUME: for a contradiction U' is open in $X^* \times \mathbb{Q}$
$\langle 1 \rangle 10$. $(1, 0) \in U'$
$\langle 1 \rangle 11$. PICK a neighbourhood W of 1 in X^* and $\delta > 0$ such that $W \times (-\delta, \delta) \subseteq U'$
$\langle 1 \rangle 12$. $p^{-1}(W) \times (-\delta, \delta) \subseteq U$
$\langle 1 \rangle 13$. PICK n such that $c_n < \delta$
$\langle 1 \rangle 14$. $n \in p^{-1}(W)$
$\langle 1 \rangle 15$. PICK $\epsilon > 0$ such that $\epsilon < \delta - c_n$ and $\epsilon < 1/4$ and $(n - \epsilon, n + \epsilon) \subseteq p^{-1}(W)$
$\langle 1 \rangle 16$. $(n - \epsilon, n + \epsilon) \times (-\delta, \delta) \subseteq U$
$\langle 1 \rangle 17$. PICK a rational y such that $c_n - \epsilon/2 < y < c_n + \epsilon/2$
$\langle 1 \rangle 18$. $(n + \epsilon/2, y) \notin U$
$\langle 1 \rangle 19$. Q.E.D.
PROOF: This contradicts $\langle 1 \rangle 16$.
□

Proposition 2.277. *Let X be a topological space and $\sim$ an equivalence relation on X. Then $X/\sim$ is T_1 if and only if every equivalence class is closed in X.*

PROOF: Immediate from definitions. □

2.58 Retractions

Definition 2.278 (Retraction)**.** Let X be a topological space and $A \subseteq X$. A *retraction* of X onto A is a continuous map $r : X \to A$ such that, for all $a \in A$, we have $r(a) = a$.

Proposition 2.279. *Every retraction is a quotient map.*

PROOF: Proposition 2.271 with f the inclusion $A \hookrightarrow X$. □

2.59 Homogeneous Spaces

Definition 2.280 (Homogeneous)**.** A topological space X is *homogeneous* if and only if, for any points $a, b \in X$, there exists a homeomorphism $\phi : X \cong X$ such that $\phi(a) = b$.

2.60 Regular Spaces

Definition 2.281 (Regular Space)**.** A topological space X is *regular* if and only if, for any closed set A and point $a \notin A$, there exist disjoint open sets U, V such that $A \subseteq U$ and $a \in V$.

2.61 Connected Spaces

Definition 2.282 (Separation)**.** A *separation* of a topological space X is a pair of disjoint open sets U, V such that $U \cup V = \emptyset$.

Definition 2.283 (Connected)**.** A topological space is *connected* if and only if it has no separation; otherwise it is *disconnected*.

Proposition 2.284. *A topological space X is connected if and only if the only sets that are both open and closed are X and $\emptyset$.*

Immediate from defintions.

Lemma 2.285. *If Y is a subspace of X, a separation of Y is a pair of disjoint nonempty sets A and B whose union is Y, neither of which contains a limit point of the other.*

PROOF:
⟨1⟩1. LET: $A, B \subseteq Y$
⟨1⟩2. If A and B form a separation of Y then A and B are disjoint and nonempty, $A \cup B = Y$, and neither of A and B contains a limit point of the other.
⟨2⟩1. ASSUME: A and B form a separation of Y
⟨2⟩2. A and B are disjoint and nonempty and $A \cup B = Y$
PROOF: From ⟨2⟩1 and the definition of separation.
⟨2⟩3. A does not contain a limit point of B
⟨3⟩1. ASSUME: for a contradiction $l \in A$ and l is a limit point of B in X.
⟨3⟩2. l is a limit point of B in Y
PROOF: Proposition 2.223.
⟨3⟩3. $l \in B$
⟨4⟩1. B is closed in Y
PROOF: Since A is open in Y and $B = Y \setminus A$ from ⟨2⟩1.
⟨4⟩2. Q.E.D.
PROOF: Corollary 2.128.1.
⟨3⟩4. Q.E.D.
PROOF: This contradicts the fact that $A \cap B = \emptyset$ (⟨2⟩1).
⟨2⟩4. B does not contain a limit point of A
PROOF: Similar.
⟨1⟩3. If A and B are disjoint and nonempty, $A \cup B = Y$, and neither of A and B contains a limit point of the other, then A and B form a separation of Y.
⟨2⟩1. ASSUME: A and B are disjoint and nonempty, $A \cup B = Y$, and neither of A and B contains a limit point of the other.
⟨2⟩2. A is open in Y
⟨3⟩1. B is closed in Y
⟨4⟩1. LET: l be a limit point of B in Y
⟨4⟩2. l is a limit point of B in X
PROOF: Proposition 2.223.
⟨4⟩3. $l \notin A$
PROOF: By ⟨2⟩1
⟨4⟩4. $l \in B$
PROOF: By ⟨2⟩1 since $A \cup B = Y$
⟨4⟩5. Q.E.D.

PROOF: Corollary 2.128.1.
⟨3⟩2. Q.E.D.
PROOF: Since $A = Y \setminus B$.
⟨2⟩3. B is open in Y
PROOF: Similar.
□

Example 2.286. Every set under the indiscrete topology is connected.

Example 2.287. The discrete topology on a set X is connected if and only if $|X| \leq 1$.

Example 2.288. The finite complement topology on a set X is connected if and only if either $|X| \leq 1$ or X is infinite.

Example 2.289. The countable complement topology on a set X is connected if and only if either $|X| \leq 1$ or X is uncountable.

Example 2.290. The rationals $\mathbb{Q}$ are disconnected. For any irrational a, the sets $(-\infty, a) \cap \mathbb{Q}$ and $(a, +\infty) \cap \mathbb{Q}$ form a separation of $\mathbb{Q}$.

Lemma 2.291. *Let X be a topological space. If C and D form a separation of X, and Y is a connected subspace of X, then either $Y \subseteq C$ or $Y \subseteq D$.*

PROOF: Otherwise $Y \cap C$ and $Y \cap D$ would form a separation of Y. □

Theorem 2.292. *The union of a set of connected subspaces of a space X that have a point in common is connected.*

PROOF:
⟨1⟩1. LET: $\mathcal{A}$ be a set of connected subspaces of the space X that have the point a in common.
⟨1⟩2. ASSUME: for a contradiction C and D form a separation of $\bigcup \mathcal{A}$
⟨1⟩3. ASSUME: without loss of generality $a \in C$
⟨1⟩4. For all $A \in \mathcal{A}$ we have $A \subseteq C$
PROOF: Lemma 2.291.
⟨1⟩5. $D = \emptyset$
⟨1⟩6. Q.E.D.
PROOF: This contradicts ⟨1⟩2.
□

Theorem 2.293. *Let X be a topological space and A a connected subspace of X. If $A \subseteq B \subseteq \overline{A}$ then B is connected.*

PROOF:
⟨1⟩1. ASSUME: for a contradiction C and D form a separation of B.
⟨1⟩2. ASSUME: without loss of generality $A \subseteq C$
PROOF: Lemma 2.291.
⟨1⟩3. $B \subseteq C$
⟨2⟩1. LET: $x \in B$

⟨2⟩2. $x \in \overline{A}$
⟨2⟩3. Either $x \in A$ or x is a limit point of A.
PROOF: Theorem 2.128.
⟨2⟩4. Either $x \in A$ or x is a limit point of C.
PROOF: Lemma 2.130, ⟨1⟩2.
⟨2⟩5. $x \in C$
PROOF: Lemma 2.285.
⟨1⟩4. $D = \emptyset$
⟨1⟩5. Q.E.D.
PROOF: This contradicts ⟨1⟩1.
□

Theorem 2.294. *The image of a connected space under a continuous map is connected.*

PROOF:
⟨1⟩1. LET: $f : X \twoheadrightarrow Y$ be a surjective continuous map where X is connected.
⟨1⟩2. ASSUME: for a contradiction C and D form a separation of Y.
⟨1⟩3. $f^{-1}(C)$ and $f^{-1}(D)$ form a separation of X.
□

Theorem 2.295. *The product of a family of connected spaces is connected.*

PROOF:
⟨1⟩1. The product of two connected spaces is connected.
⟨2⟩1. LET: X and Y be connected spaces.
⟨2⟩2. PICK $a \in X$ and $b \in Y$
PROOF: We may assume X and Y are nonempty since otherwise $X \times Y = \emptyset$ which is connected.
⟨2⟩3. $X \times \{b\}$ is connected.
PROOF: It is homeomorphic to X.
⟨2⟩4. For all $x \in X$ we have $\{x\} \times Y$ is connected.
PROOF: It is homeomorphic to Y.
⟨2⟩5. For any $x \in X$
LET: $T_x = (X \times \{b\}) \cup (\{x\} \times Y)$
⟨2⟩6. For all $x \in X$, T_x is connected.
PROOF: Theorem 2.292 since $(x, b) \in (X \times \{b\}) \cap (\{x\} \times Y)$.
⟨2⟩7. $X \times Y$ is connected.
PROOF: Theorem 2.292 since $X \times Y = \bigcup_{x \in X} T_x$ and (a, b) is a point in every T_x.
⟨1⟩2. The product of a finite family of connected spaces is connected.
PROOF: From ⟨1⟩1 by induction.
⟨1⟩3. The product of any family of connected spaces is connected.
⟨2⟩1. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of connected spaces.
⟨2⟩2. LET: $X = \prod_{\alpha \in J} X_\alpha$
⟨2⟩3. PICK $a \in X$
PROOF: We may assume $X \neq \emptyset$ as the empty space is connected.

⟨2⟩4. For every finite subset K of J,
LET: $X_K = \{x \in X \mid \forall \alpha \in J \setminus K . x_\alpha = a_\alpha\}$
⟨2⟩5. For every finite $K \subseteq J$, we have X_K is connected.
PROOF: From ⟨1⟩2 since $X_K \cong \prod_{\alpha \in K} X_K$.
⟨2⟩6. LET: $Y = \bigcup_K X_K$
⟨2⟩7. Y is connected
PROOF: Theorem 2.292 since a is a common point.
⟨2⟩8. $X = \overline{Y}$
⟨3⟩1. LET: $x \in X$
⟨3⟩2. LET: $U = \prod_{\alpha \in J} U_\alpha$ be a basic neighbourhood of x where $U_\alpha = X_\alpha$ for all α except $\alpha \in K$ for some finite $K \subseteq J$
⟨3⟩3. LET: $y \in X$ be the point with $y_\alpha = x_\alpha$ for $\alpha \in K$ and $y_\alpha = a_\alpha$ for all other α
⟨3⟩4. $y \in U \cap X_K$
⟨3⟩5. $y \in U \cap Y$
⟨2⟩9. X is connected.
PROOF: Theorem 2.293.
□

Example 2.296. The set $\mathbb{R}^\omega$ is disconnected under the box topology. The set of bounded sequences and the set of unbounded sequences form a separation.

Proposition 2.297. *Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on the same set X. If $\mathcal{T} \subseteq \mathcal{T}'$ and $(X, \mathcal{T}')$ is connected then $(X, \mathcal{T})$ is connected.*

PROOF: If U and V form a separation of $(X, \mathcal{T})$ then they form a separation of $(X, \mathcal{T}')$. □

Proposition 2.298. *Let X be a topological space and (A_n) a sequence of connected subspaces of X. If $A_n \cap A_{n+1} \neq \emptyset$ for all n then $\bigcup_n A_n$ is connected.*

PROOF:
⟨1⟩1. ASSUME: for a contradiction C and D form a separation of $\bigcup_n A_n$
⟨1⟩2. ASSUME: without loss of generality $A_0 \subseteq C$
PROOF: Lemma 2.291.
⟨1⟩3. For all n we gave $A_n \subseteq C$
PROOF:
⟨2⟩1. ASSUME: $A_n \subseteq C$
⟨2⟩2. PICK $x \in A_n \cap A_{n+1}$
⟨2⟩3. $x \in C$
⟨2⟩4. $A_{n+1} \subseteq C$
PROOF: Lemma 2.291.
⟨2⟩5. Q.E.D.
PROOF: The result follows by induction.
⟨1⟩4. $D = \emptyset$
⟨1⟩5. Q.E.D.
PROOF: This contradicts ⟨1⟩1.
□

Proposition 2.299. *Let X be a topological space. Let $A, C \subseteq X$. If C is connected and intersects both A and $X \setminus A$ then C intersects ∂A.*

PROOF: Otherwise $C \cap A^\circ$ and $C \setminus \overline{A}$ would form a separation of C. ∎

Example 2.300. The space $\mathbb{R}_l$ is disconnected. For any real x, the sets $(-\infty, x)$ and $[x, +\infty)$ form a separation.

Proposition 2.301. *Let X and Y be connected spaces. Let A be a proper subset of X and B a proper subset of Y. Then $(X \times Y) \setminus (A \times B)$ is connected.*

PROOF:
$\langle 1 \rangle 1$. PICK $a \in X \setminus A$ and $b \in Y \setminus B$
$\langle 1 \rangle 2$. For $x \in X \setminus A$ we have $(X \times \{b\}) \cup (\{x\} \times Y)$ is connected.
  PROOF: Theorem 2.292 since (x, b) is a common point.
$\langle 1 \rangle 3$. For $y \in Y \setminus B$ we have $(X \times \{y\}) \cup (\{a\} \times Y)$ is connected.
  PROOF: Theorem 2.292 since (a, y) is a common point.
$\langle 1 \rangle 4$. $(X \times Y) \setminus (A \times B)$ is connected.
  PROOF: Theorem 2.292 since it is the union of the sets in $\langle 1 \rangle 2$ and $\langle 1 \rangle 3$ with (a, b) as a common point.
∎

Proposition 2.302. *Let $p : X \twoheadrightarrow Y$ be a quotient map. If Y is connected and $p^{-1}(y)$ is connected for all $y \in Y$, then X is connected.*

PROOF:
$\langle 1 \rangle 1$. ASSUME: for a contradiction C and D form a separation of X.
$\langle 1 \rangle 2$. C is saturated.
  $\langle 2 \rangle 1$. LET: $x \in C$, $y \in X$ with $p(x) = p(y) = a$, say
  $\langle 2 \rangle 2$. $y \notin D$
    PROOF: Otherwise $p^{-1}(a) \cap C$ and $p^{-1}(a) \cap D$ form a separation of $p^{-1}(a)$.
  $\langle 2 \rangle 3$. $y \in C$
$\langle 1 \rangle 3$. D is saturated.
  PROOF: Similar.
$\langle 1 \rangle 4$. $p(C)$ and $p(D)$ form a separation of Y.
∎

Proposition 2.303. *Let X be a connected space and Y a connected subspace of X. Suppose A and B form a separation of $X \setminus Y$. Then $Y \cup A$ and $Y \cup B$ are both connected.*

PROOF:
$\langle 1 \rangle 1$. $Y \cup A$ is connected.
  $\langle 2 \rangle 1$. ASSUME: for a contradiction C and D form a separation of $Y \cup A$
  $\langle 2 \rangle 2$. ASSUME: without loss of generality $Y \subseteq C$
  $\langle 2 \rangle 3$. PICK open sets A_1, B_1, C_1, D_1 in X with

$$A = A_1 \setminus Y$$
$$B = B_1 \setminus Y$$
$$C = C_1 \cap (Y \cup A)$$
$$D = D_1 \cap (Y \cup A)$$

⟨2⟩4. $B_1 \cup C_1$ and $A_1 \cap D_1$ form a separation of X

⟨1⟩2. $Y \cup B$ is connected.

PROOF: Similar.

□

Theorem 2.304. *Let L be a linearly ordered set under the order topology. Then L is connected if and only if L is a linear continuum.*

PROOF:

⟨1⟩1. If L is a linear continuum then L is connected.

⟨2⟩1. LET: L be a linear continuum under the order topology.

⟨2⟩2. ASSUME: for a contradiction C and D form a separation of L.

⟨2⟩3. PICK $a \in C$ and $b \in D$.

⟨2⟩4. ASSUME: without loss of generality $a < b$.

⟨2⟩5. LET: $S = \{x \in L \mid a < x \text{ and } [a, x) \subseteq C\}$

⟨2⟩6. S is nonempty.

PROOF: Since $a \in C$ and C is open.

⟨2⟩7. S is bounded above by b.

PROOF: Since $b \notin C$.

⟨2⟩8. LET: $s = \sup S$

⟨2⟩9. $s \in S$

⟨3⟩1. LET: $y \in [a, s)$

PROVE: $y \in C$

⟨3⟩2. PICK z with $y < z \in S$

PROOF: By minimality of s.

⟨3⟩3. $y \in [a, z) \subseteq C$

⟨2⟩10. CASE: $s \in C$

⟨3⟩1. PICK x such that $s < x$ and $[s, x) \subseteq C$

PROOF: Since C is open and s is not greatest in L because $s < b$.

⟨3⟩2. $x \in S$

PROOF: Since $[a, x) = [a, s) \cup [s, x) \subseteq C$.

⟨3⟩3. Q.E.D.

PROOF: This contradicts the fact that s is an upper bound for S.

⟨2⟩11. CASE: $s \in D$

⟨3⟩1. PICK $x < s$ such that $(x, s] \subseteq D$

⟨3⟩2. PICK y with $x < y < s$

PROOF: Since L is dense.

⟨3⟩3. $y \in C$

PROOF: From ⟨2⟩9.

⟨3⟩4. $y \in D$

PROOF: From ⟨3⟩1.

⟨3⟩5. Q.E.D.

⟨3⟩6. LET: L be a linear continuum under the order topology.

⟨3⟩7. ASSUME: for a contradiction C and D form a separation of L.

⟨3⟩8. PICK $a \in C$ and $b \in D$.

⟨3⟩9. ASSUME: without loss of generality $a < b$.

⟨3⟩10. LET: $S = \{x \in L \mid a < x \text{ and } [a, x) \subseteq C\}$

⟨3⟩11. S is nonempty.
Proof: Since $a \in C$ and C is open.
⟨3⟩12. S is bounded above by b.
Proof: Since $b \notin C$.
⟨3⟩13. Let: $s = \sup S$
⟨3⟩14. $s \in S$
⟨4⟩1. Let: $y \in [a, s)$
Prove: $y \in C$
⟨4⟩2. Pick z with $y < z \in S$
Proof: By minimality of s.
⟨4⟩3. $y \in [a, z) \subseteq C$
⟨3⟩15. Case: $s \in C$
⟨4⟩1. Pick x such that $s < x$ and $[s, x) \subseteq C$
Proof: Since C is open and s is not greatest in L because $s < b$.
⟨4⟩2. $x \in S$
Proof: Since $[a, x) = [a, s) \cup [s, x) \subseteq C$.
⟨4⟩3. Q.E.D.
Proof: This contradicts the fact that s is an upper bound for S.
⟨3⟩16. Case: $s \in D$
⟨4⟩1. Pick $x < s$ such that $(x, s] \subseteq D$
⟨4⟩2. Pick y with $x < y < s$
Proof: Since L is dense.
⟨4⟩3. $y \in C$
Proof: From ⟨2⟩9.
⟨4⟩4. $y \in D$
Proof: From ⟨3⟩1.
⟨4⟩5. Q.E.D.
Proof: This contradicts ⟨2⟩2.
⟨1⟩2. If L is connected then L is a linear continuum.
⟨2⟩1. Assume: L is connected.
⟨2⟩2. Every nonempty subset of L that is bounded above has a supremum.
⟨3⟩1. Let: X be a nonempty subset of L bounded above by b.
⟨3⟩2. Assume: for a contradiction X has no supremum.
⟨3⟩3. Let: U be the set of upper bounds of X,
⟨3⟩4. U is nonempty.
Proof: Since $b \in U$.
⟨3⟩5. U is open.
⟨4⟩1. Let: $x \in U$
⟨4⟩2. Pick an upper bound y for X such that $y < x$
⟨4⟩3. Either x is greatest in L and $(y, x] \subseteq U$, or there exists $z > x$ such that $(y, z) \subseteq U$
⟨3⟩6. Let: V be the set of lower bounds of U.
⟨3⟩7. V is nonempty.
Proof: Since $X \subseteq V$
⟨3⟩8. V is open.
⟨4⟩1. Let: $x \in V$

⟨4⟩2. Pick $y \in X$ with $x < y$

Proof: x cannot be an upper bound for X, because it would be the supremum of X.

⟨4⟩3. Either x least in L and $[x, y) \subseteq V$, or there exists $z < x$ such that $(z, y) \subseteq V$

⟨3⟩9. $L = U \cup V$

⟨4⟩1. Let: $x \in L \setminus U$

⟨4⟩2. Pick $y \in X$ such that $x < y$

⟨4⟩3. For all $u \in U$ we have $x < y \leq u$

⟨4⟩4. $x \in V$

⟨3⟩10. $U \cap V = \emptyset$

Proof: Any element of $U \cap V$ would be a supremum of X.

⟨3⟩11. U and V form a separation of L.

⟨3⟩12. Q.E.D.

Proof: This contradicts ⟨2⟩1.

⟨2⟩3. L is dense.

⟨3⟩1. Let: $x, y \in L$ with $x < y$

⟨3⟩2. There exists $z \in L$ such that $x < z < y$

Proof: Otherwise $(-\infty, y)$ and $(x, +\infty)$ would form a separation of L.

□

Corollary 2.304.1. *The real line $\mathbb{R}$ is connected.*

Corollary 2.304.2. *Every interval in $\mathbb{R}$ is connected.*

Corollary 2.304.3. *The ordered square is connected.*

Theorem 2.305 (Intermediate Value Theorem)**.** *Let X be a connected space. Let Y be a linearly ordered set under the order topology. Let $f : X \to Y$ be continuous. Let $a, b \in X$ and $r \in Y$. Suppose $f(a) < r < f(b)$. Then there exists $c \in X$ such that $f(c) = r$.*

Proof: Otherwise $f^{-1}((-\infty, r))$ and $f^{-1}((r, +\infty))$ would form a separation of X. □

Proposition 2.306. *Every function $f : [0, 1] \to [0, 1]$ has a fixed point.*

Proof:

⟨1⟩1. Let: $g : [0, 1] \to [-1, 1]$ be the function $g(x) = f(x) - x$

Prove: there exists $x \in [0, 1]$ such that $g(x) = 0$

⟨1⟩2. Assume: without loss of generality $g(0) \neq 0$ and $g(1) \neq 0$

⟨1⟩3. $g(0) > 0$

⟨1⟩4. $g(1) < 0$

⟨1⟩5. There exists $x \in (0, 1)$ such that $g(x) = 0$

Proof: By the Intermediate Value Theorem.

Proposition 2.307. *Give $\mathbb{R}^\omega$ the box topology. Let $x, y \in \mathbb{R}^\omega$. Then x and y lie in the same comoponent if and only if $x - y$ is eventually zero, i.e. there exists N such that, for all $n \geq N$, we have $x_n = y_n$.*

PROOF:
⟨1⟩1. The component containing 0 is the set of sequences that are eventually zero.
⟨2⟩1. LET: B be the set of sequences that are eventually zero.
⟨2⟩2. B is path-connected.
⟨3⟩1. LET: $x, y \in B$
⟨3⟩2. PICK N such that, for all $n \geq N$, we have $x_n = y_n = 0$
⟨3⟩3. LET: $p : [0, 1] \to \mathbb{R}^\omega$, $p(t) = (1 - t)x + ty$
PROVE: p is continuous.
⟨3⟩4. LET: $t \in [0, 1]$ and $\prod_j U_j$ be a basic open neighbourhood of $p(t)$, where each U_j is open in $\mathbb{R}$
⟨3⟩5. PICK δ such that, for all $n < N$ and all $s \in [0, 1]$, if $|s - t| < \delta$ then $p(s)_n \in U_n$
⟨3⟩6. For all $s \in [0, 1]$, if $|s - t| < \delta$ then $p(s) \in \prod_j U_j$
⟨2⟩3. B is connected.
PROOF: Proposition 2.313.
⟨2⟩4. If C is connected and $B \subseteq C$ then $B = C$.
⟨3⟩1. ASSUME: C is connected and $B \subseteq C$
⟨3⟩2. ASSUME: for a contradiction $x \in C \setminus B$
⟨3⟩3. For $n \geq 1$,
LET: $c_n = 1$ if $x_n = 0$, $c_n = n/x_n$ otherwise
⟨3⟩4. LET: $h : \mathbb{R}^\omega \to \mathbb{R}^\omega$ be the function $h(x) = (c_n x_n)_{n \geq 1}$
⟨3⟩5. h is a homeomorphism of $\mathbb{R}^\omega$ with itself.
⟨3⟩6. $h(x)$ is unbounded.
PROOF: For any $b > 0$, pick $N > b$ such that $x_N \neq 0$. Then $h(x)_N > b$.
⟨3⟩7. $h^{-1}(\{\text{bounded sequences}\}) \cap C$ and $h^{-1}(\{\text{unbounded sequences}\}) \cap C$ form a separation of C
⟨3⟩8. Q.E.D.
PROOF: This contradicts ⟨3⟩1.
⟨1⟩2. Q.E.D.
PROOF: Since $\lambda x.x - y$ is a homeomorphism of $\mathbb{R}^\omega$ with itself.
□

2.62 Totally Disconnected Spaces

Definition 2.308 (Totally Disconnected)**.** A topological space X is *totally disconnected* if and only if the only connected subspaces are the singletons.

Example 2.309. Every discrete space is totally disconnected.

Example 2.310. The rationals $\mathbb{Q}$ are totally disconnected.

2.63 Paths and Path Connectedness

Definition 2.311 (Path)**.** Let X be a topological space and $a, b \in X$. A *path* from a to b is a continuous function $p : [0, 1] \to X$ such that $p(0) = a$ and

$p(1) = b$.

Definition 2.312 (Path Connected)**.** A topological space is *path connected* if and only if there exists a path between any two points.

Proposition 2.313. *Every path connected space is connected.*

PROOF:
$\langle 1 \rangle 1$. LET: X be a path connected space.
$\langle 1 \rangle 2$. ASSUME: for a contradiction C and D form a separation of X.
$\langle 1 \rangle 3$. PICK $a \in C$ and $b \in D$.
$\langle 1 \rangle 4$. PICK a path $p : [0, 1] \to X$ from a to b.
$\langle 1 \rangle 5$. $p^{-1}(C)$ and $p^{-1}(D)$ form a separation of $[0, 1]$.
$\langle 1 \rangle 6$. Q.E.D.
PROOF: This contradicts Corollary 2.304.2.
□

An example that shows the converse does not hold:

Example 2.314. The ordered square is not path connected.

PROOF:
$\langle 1 \rangle 1$. ASSUME: for a contradiction $p : [0, 1] \to I_o^2$ is a path from $(0, 0)$ to $(1, 1)$.
$\langle 1 \rangle 2$. p is surjective.
PROOF: By the Intermediate Value Theorem.
$\langle 1 \rangle 3$. For $x \in [0, 1]$, PICK a rational $q_x \in p^{-1}((x, 0), (x, 1))$
PROOF: Since $p^{-1}((x, 0), (x, 1))$ is open and nonempty by $\langle 1 \rangle 2$.
$\langle 1 \rangle 4$. For $x, y \in [0, 1]$, if $x \neq y$ then $q_x \neq q_y$
PROOF: We have $p(q_x) \neq p(q_y)$ because $((x, 0), (x, 1))$ and $((y, 0), (y, 1))$ are disjoint.
$\langle 1 \rangle 5$. $\{q_x \mid x \in [0, 1]\}$ is an uncountable set of rationals.
$\langle 1 \rangle 6$. Q.E.D.
PROOF: This contradicts the fact that the rationals are countable.
□

Proposition 2.315. *The continuous image of a path connected space is path connected.*

PROOF:
$\langle 1 \rangle 1$. LET: X be a path connected space, Y a topological space, and $f : X \twoheadrightarrow Y$ be continuous and surjective.
$\langle 1 \rangle 2$. LET: $a, b \in Y$
$\langle 1 \rangle 3$. PICK $c, d \in X$ with $f(c) = a$ and $f(d) = b$
$\langle 1 \rangle 4$. PICK a path $p : [0, 1] \to X$ from c to d.
$\langle 1 \rangle 5$. $f \circ p$ is a path from a to b in Y.
□

Proposition 2.316 (AC)**.** *The product of a family of path-connected spaces is path-connected.*

PROOF:
$\langle 1\rangle 1$. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of path-connected spaces.
$\langle 1\rangle 2$. LET: $a, b \in \prod_{\alpha \in J} X_\alpha$
$\langle 1\rangle 3$. For $\alpha \in J$, PICK a path $p_\alpha : [0,1] \to X_\alpha$ from a_α to b_α
PROOF: Using the Axiom of Choice.
$\langle 1\rangle 4$. Define $p : [0.1] \to \prod_{\alpha \in J} X_\alpha$ by $p(t)_\alpha = p_\alpha(t)$
$\langle 1\rangle 5$. p is a path from a to b.
PROOF: Theorem 2.202.
$\square$

Proposition 2.317. *The continuous image of a path-connected space is path-connected.*

PROOF:
$\langle 1\rangle 1$. LET: $f : X \twoheadrightarrow Y$ be continuous and surjective where X is path-connected.
$\langle 1\rangle 2$. LET: $a, b \in Y$
$\langle 1\rangle 3$. PICK $a', b' \in X$ with $f(a') = a$ and $f(b') = b$.
$\langle 1\rangle 4$. PICK a path $p : [0,1] \to X$ from a' to b'.
$\langle 1\rangle 5$. $f \circ p$ is a path from a to b.
$\square$

Proposition 2.318. *Let X be a topological space. The union of a set of path-connected subspaces of X that have a point in common is path-connected.*

PROOF:
$\langle 1\rangle 1$. LET: $\mathcal{A}$ be a set of path-connected subspaces of X with the point a in common.
$\langle 1\rangle 2$. LET: $b, c \in \bigcup \mathcal{A}$
$\langle 1\rangle 3$. PICK $B, C \in \mathcal{A}$ with $b \in B$ and $c \in C$.
$\langle 1\rangle 4$. PICK a path p in B from b to a.
$\langle 1\rangle 5$. PICK a path q in C from a to c.
$\langle 1\rangle 6$. The concatenation of p and q is a path from b to c in $\bigcup \mathcal{A}$.
$\square$

Proposition 2.319. *Let $A \subseteq \mathbb{R}^2$ be countable. Then $\mathbb{R}^2 \setminus A$ is path-connected.*

PROOF:
$\langle 1\rangle 1$. LET: $a, b \in \mathbb{R}^2 \setminus A$
$\langle 1\rangle 2$. PICK a line l in $\mathbb{R}^2$ with a on one side and b on the other.
$\langle 1\rangle 3$. For every point x on l,
LET: p_x be the path in $\mathbb{R}^2$ consisting of a line from a to x then a line from x to b
$\langle 1\rangle 4$. For $x \neq y$ we have p_x and p_y have no points in common except a and b
$\langle 1\rangle 5$. There are only countably many x such that a point of A lies on p_x.
$\langle 1\rangle 6$. There exists x such that p_x is a path from a to b in $\mathbb{R}^2 \setminus A$.
$\square$

Proposition 2.320. *Every open connected subspace of $\mathbb{R}^2$ is path-connected.*

PROOF:
$\langle 1 \rangle 1$. LET: U be an open connected subspace of $\mathbb{R}^2$.
$\langle 1 \rangle 2$. For all $x_0 \in U$,
LET: $PC(x_0) = \{y \in U \mid \text{there exists a path from } x \text{ to } y\}$
$\langle 1 \rangle 3$. For all $x_0 \in U$, the set $PC(x_0)$ is open and closed in U.
$\langle 2 \rangle 1$. LET: $x_0 \in U$
$\langle 2 \rangle 2$. $PC(x_0)$ is open in U
$\langle 3 \rangle 1$. LET: $y \in PC(x_0)$
$\langle 3 \rangle 2$. PICK $\epsilon > 0$ such that $B(y, \epsilon) \subseteq U$
PROOF: Since U is open.
$\langle 3 \rangle 3$. $B(y, \epsilon) \subseteq PC(x_0)$
PROOF: For all $z \in B(y, \epsilon)$, pick a path from x_0 to y then concatenate the straight line from y to z.
$\langle 2 \rangle 3$. $PC(x_0)$ is closed in U
$\langle 3 \rangle 1$. LET: $y \in U$ be a limit point of $PC(x_0)$
$\langle 3 \rangle 2$. PICK $\epsilon > 0$ such that $B(y, \epsilon) \subseteq U$
$\langle 3 \rangle 3$. PICK $z \in PC(x_0) \cap B(y, \epsilon)$
$\langle 3 \rangle 4$. $y \in PC(x_0)$
PROOF: Pick a path from x_0 to z then concatenate the straight line from z to y.
$\langle 1 \rangle 4$. $PC(x_0) = U$
PROOF: Proposition 2.284.
□

Example 2.321. If A is a connected subspace of X, then A° is not necessarily connected.

Take two closed circles in $\mathbb{R}^2$ that touch at one point. The interior of this space is two open circles, and these two circles form a separation.

Example 2.322. If A is a connected subspace of X then ∂A is not necessarily connected.

We have $[0, 1]$ is connected but $\partial[0, 1] = \{0, 1\}$ is not.

Example 2.323. If A is a subspace of X and A° and ∂A are connected, then A is not necessarily connected.

We have $\mathbb{Q}^\circ = \emptyset$ and $\partial\mathbb{Q} = \mathbb{R}$ are connected but $\mathbb{Q}$ is not connected.

2.64 The Topologist's Sine Curve

Definition 2.324 (The Topologist's Sine Curve)**.** Let $S = \{(x, \sin 1/x) \mid 0 < x \leq 1\}$, The *topologist's sine curve* is the closure $\overline{S}$ of S in $\mathbb{R}^2$.

Proposition 2.325. *The topologist's sine curve is connected.*

PROOF:
$\langle 1 \rangle 1$. LET: $S = \{(x, \sin 1/x) \mid 0 < x \leq 1\}$
$\langle 1 \rangle 2$. S is connected.

PROOF: Theorem 2.294.

⟨1⟩3. $\overline{S}$ is connected.

PROOF: Theorem 2.293.

□

Proposition 2.326. *The topologist's sine curve is* $\{(x, \sin 1/x) \mid 0 < x \leq 1\} \cup (\{0\} \times [-1.1])$.

PROOF: Sketch proof: Given a point $(0.y)$ with $-1 \leq y \leq 1$, pick a such that $\sin a = y$. Then $((1/a, y), (1/(a + 2\pi), y), (1/(a + 4\pi), y), \ldots)$ is a sequence in S that converges to $(0, y)$.
Conversely, let (x, y) be any point not in $S \cup (\{0\} \times [-1, 1])$. If $x < 0$ or $y > 1$ or $y < -1$ then we can easily find a neighbourhood that does not intersect $S \cup (\{0\} \times [-1, 1])$. If $x > 0$ and $-1 \leq y \leq 1$, then we have $y \neq \sin 1/x$. Hence pick a neighbourhood that does not intersect S.

Proposition 2.327. *Every closed subset of* $\mathbb{R}$ *that is bounded above has a greatest element.*

PROOF: It has a supremum, which is a limit point of the set and hence an element. □

Proposition 2.328 (CC)**.** *The topologist's sine curve is not path connected.*

PROOF:

⟨1⟩1. ASSUME: For a contradction $p : [0, 1] \to \overline{S}$ is a path from $(0, 0)$ to $(1, \sin 1)$.

⟨1⟩2. $\{t \in [0, 1] \mid p(t) \in \{0\} \times [-1, 1]\}$ is closed.

PROOF: Since p is continuous and $\{0\} \times [-1, 1]$ is closed.

⟨1⟩3. LET: b be the largest number in $[0, 1]$ such that $p(b) \in \{0\} \times [-1, 1]$.

PROOF: Proposition 2.327.

⟨1⟩4. LET: $x : [b, 1] \to \overline{S}$ be the function $\pi_1 \circ p$

⟨1⟩5. LET: $y : [b, 1] \to \overline{S}$ be the function $\pi_2 \circ p$

⟨1⟩6. PICK a sequence t_n in $(b, 1]$ such that $t_n \to b$ and $y(t_n) = (-1)^n$ for all n

⟨2⟩1. LET: $n \geq 1$

⟨2⟩2. PICK u with $0 < u < x(1/n)$ and $\sin(1/u) = (-1)^n$

⟨2⟩3. PICK t_n with $b < t_n < 1/n$ and $x(t_n) = u$

PROOF: By the Intermediate Value Theorem

⟨1⟩7. Q.E.D.

PROOF: This contradicts Proposition 2.175 since y is continuous and $y(t_n)$ does not converge.

□

Corollary 2.328.1. *The closure of a path-connected subspace of a space is not necessarily path-connected.*

2.65 The Long Line

Definition 2.329 (The Long Line)**.** The *long line* is the space $\omega_1 \times [0, 1)$ in the dictionary order under the order topology, where ω_1 is the first uncountable ordinal.

Lemma 2.330. *For any ordinal α with $0 < \alpha < \omega_1$ we have $[(0,0),(\alpha,0)) \cong [0,1)$*

$\langle 1 \rangle 1.$ $[(0,0),(1,0)) \cong [0,1)$
  Proof: The map π_2 is a homeomorphism.
$\langle 1 \rangle 2.$ If $[(0,0),(\alpha,0)) \cong [0,1)$ then $[(0,0),(\alpha+1,0)) \cong [0,1)$
  Proof: Proposition 2.72.
$\langle 1 \rangle 3.$ If λ is a limit ordinal with $\lambda < \omega_1$ and $[(0,0),(\alpha,0)) \cong [0,1)$ for all α with $0 < \alpha < \lambda$ then $[(0,0),(\lambda,0)) \cong [0,1)$
  $\langle 2 \rangle 1.$ Let: λ be a limit ordinal $< \omega_1$
  $\langle 2 \rangle 2.$ Assume: $[(0,0),(\alpha,0)) \cong [0,1)$ for all α with $0 < \alpha < \lambda$
  $\langle 2 \rangle 3.$ Pick a sequence of ordinals $\alpha_0 < \alpha_1 < \cdots$ with limit λ
    Proof: Since λ is countable.
  $\langle 2 \rangle 4.$ $[(\alpha_i,0),(\alpha_{i+1},0)) \cong [0,1)$ for all i
    Proof: Lemma 2.71.
  $\langle 2 \rangle 5.$ Q.E.D.
    Proof: By Proposition 2.73.
$\langle 1 \rangle 4.$ Q.E.D.
  Proof: By transfinite induction.

Proposition 2.331 (CC)**.** *The long line is path-connected.*

Proof:
$\langle 1 \rangle 1.$ Let: $(\alpha,i),(\beta,j) \in \omega_1 \times [0,1)$
$\langle 1 \rangle 2.$ Assume: without loss of generality $(\alpha,i) < (\beta,j)$
$\langle 1 \rangle 3.$ $[(0,0),(\beta+1,0)) \cong [0,1)$
  Proof: By Lemma 2.330
$\langle 1 \rangle 4.$ $[(\alpha,i),(\beta,j)) \cong [0,1)$
  Proof: Lemma 2.71.
$\langle 1 \rangle 5.$ Pick a homeomorphism $q : [0,1) \to [(\alpha,i),(\beta,j))$
$\langle 1 \rangle 6.$ $q \cup \{(1,(\beta,j))\}$ is a path from (α,i) to (β,j)
□

Proposition 2.332. *Every point in the long line has a neighbourhood homeomorphic to an interval in $\mathbb{R}$.*

Proof: For any (α,i) in the long line, the neighbourhood $[(0,0),(\alpha+1,0))$ satisfies the condition by Lemma 2.330.

Proposition 2.333. *The long line L is not second countable.*

Proof:
$\langle 1 \rangle 1.$ Let: $\mathcal{B}$ be a basis for L.
$\langle 1 \rangle 2.$ For $\alpha < \omega_1$, Pick $B_\alpha \in \mathcal{B}$ such that $(\alpha,1/2) \in B_\alpha \subseteq ((\alpha,0),(\alpha+1,0))$
$\langle 1 \rangle 3.$ $\mathcal{B}$ is uncountable.
  Proof: The mapping $\alpha \mapsto B_\alpha$ is an injection $\omega_1 \to \mathcal{B}$.

Corollary 2.333.1. *The long line cannot be imbedded into $\mathbb{R}^n$ for any n.*

2.66 Components

Proposition 2.334. *Let X be a topological space. Define the relation $\sim$ on X by $a \sim b$ if and only if there exists a connected subspace $A \subseteq X$ with $a, b \in A$. Then $\sim$ is an equivalence relation on X.*

Proof:
$\langle 1 \rangle 1$. $\sim$ is reflexive.
  Proof: For any $a \in X$ we have $\{a\}$ is a connected subspace that contains a.
$\langle 1 \rangle 2$. $\sim$ is symmetric.
  Proof: Trivial.
$\langle 1 \rangle 3$. $\sim$ is transitive.
  $\langle 2 \rangle 1$. Let: $a, b, c \in X$
  $\langle 2 \rangle 2$. Assume: $a \sim b$ and $b \sim c$
  $\langle 2 \rangle 3$. Pick connected subspaces A and B with $a, b \in A$ and $b, c \in B$
  $\langle 2 \rangle 4$. $A \cup B$ is a connected subspace that contains a and c
    Proof: Theorem 2.292.
□

Definition 2.335 ((Connected) Component)**.** Let X be a topological space. The *(connected) components* of X are the equivalence classes under the above $\sim$.

Lemma 2.336. *Let X be a topological space. If $A \subseteq X$ is connected and nonempty then there exists a unique component C of X such that $A \subseteq C$.*

Proof:
$\langle 1 \rangle 1$. Pick $a \in A$
$\langle 1 \rangle 2$. Let: C be the $\sim$-equivalence class of a.
$\langle 1 \rangle 3$. $A \subseteq C$
  Proof: For all $x \in A$ we have $x \sim a$.
$\langle 1 \rangle 4$. If C' is a component and $A \subseteq C'$ then $C = C'$
  Proof: Since we have $a \in C'$.
□

Theorem 2.337. *Let X be a topological space. The components of X are connected disjoint subspaces of X whose union is X such that each nonempty connected subspace of X intersects only one of them.*

Proof:
$\langle 1 \rangle 1$. Every component of X is connected.
  Proof: For $a \in X$, the $\sim$-equivalence class of a is $\bigcup\{A \subseteq X \mid A \text{ is connected}, a \in A\}$ which is connected by Theorem 2.292.
$\langle 1 \rangle 2$. The components form a partition of X.
  Proof: Immediate from the definition.
$\langle 1 \rangle 3$. Every nonempty connected subspace of X intersects a unique component of X.
  $\langle 2 \rangle 1$. Let: $A \subseteq X$ be connected and nonempty.

$\langle 2 \rangle 2$. LET: C be the component such that $A \subseteq C$
PROOF: Lemma 2.336.
$\langle 2 \rangle 3$. A intersects C
$\langle 2 \rangle 4$. If A intersects the component C' then $C' = C$
$\langle 3 \rangle 1$. LET: C' be a component that intersects A
$\langle 3 \rangle 2$. PICK $b \in A \cap C'$
$\langle 3 \rangle 3$. $A \subseteq C'$
PROOF: For all $x \in A$ we have $x \sim b$.
$\langle 3 \rangle 4$. $C = C'$
PROOF: By uniqueness in $\langle 2 \rangle 2$.
□

Proposition 2.338. *Every component of a space is closed.*

PROOF:
$\langle 1 \rangle 1$. LET: X be a topological space and C a component of X.
$\langle 1 \rangle 2$. $\overline{C}$ is connected.
PROOF: Theorem 2.293.
$\langle 1 \rangle 3$. $C = \overline{C}$
PROOF: Lemma 2.291.
$\langle 1 \rangle 4$. C is closed.
PROOF: Lemma 2.116.
□

Proposition 2.339. *If a topological space has finitely many components then every component is open.*

PROOF: Each component is the complement of a finite union of closed sets. □

2.67 Path Components

Proposition 2.340. *Let X be a topological space. Define the relation $\sim$ on X by: $a \sim b$ if and only if there exists a path in X from a to b. Then $\sim$ is an equivalence relation on X.*

PROOF:
$\langle 1 \rangle 1$. $\sim$ is reflexive.
PROOF: For $a \in X$, the constant function $[0, 1] \to X$ with value a is a path from a to a.
$\langle 1 \rangle 2$. $\sim$ is symmetric.
PROOF: If $p : [0, 1] \to X$ is a path from a to b, then $\lambda t.p(1 - t)$ is a path from b to a.
$\langle 1 \rangle 3$. $\sim$ is transitive.
PROOF: Concatenate paths.
□

Definition 2.341 (Path Component)**.** Let X be a topological space. The *path components* of X are the equivalence relations under $\sim$.

.

Theorem 2.342. *The path components of X are path-connected disjoint subspaces of X whose union is X such that every nonempty path-connected subspace of X intersects exactly one path component.*

PROOF:
⟨1⟩1. Every path component is path-connected.
PROOF: If a and b are in the same path component then $a \sim b$, i.e. there exists a path from a to b.
⟨1⟩2. The path components are disjoint and their union is X.
PROOF: Immediate from the definition.
⟨1⟩3. Every non-empty path-cönnected subspace of X intersects exactly one path component.
⟨2⟩1. LET: A be a nonempty path-connected subspace of X.
⟨2⟩2. PICK $a \in A$
⟨2⟩3. A intersects the $\sim$-equivalence class of a.
⟨2⟩4. LET: C be any path component that intersects A.
⟨2⟩5. PICK $b \in A \cap C$
⟨2⟩6. $a \sim b$
PROOF: Since A is path-connected.
⟨2⟩7. C is the $\sim$-equivalence class of a.
□

Proposition 2.343. *Every path component is included in a component.*

PROOF:
⟨1⟩1. LET: X be a topological space and C a path component of X.
⟨1⟩2. C is path-connected.
PROOF: Theorem 2.342.
⟨1⟩3. C is connected.
PROOF: Proposition 2.313.
⟨1⟩4. C is included in a component.
PROOF: Lemma 2.336.
□

2.68 Local Connectedness

Definition 2.344 (Locally Connected)**.** Let X be a topological space and $a \in X$. Then X is *locally connected* at a if and only if every neighbourhood of a includes a connected neighbourhood of a.

The space X is *locally connected* if and only if it is locally connected at every point.

Example 2.345. The real line is both connected and locally connected.

Example 2.346. The space $\mathbb{R} \setminus \{0\}$ is disconnected but locally connected.

Example 2.347. The topologist's sine curve is connected but not locally connected.

Example 2.348. The rationals $\mathbb{Q}$ are neither connected nor locally connected.

Theorem 2.349. *A topological space X is locally connected if and only if, for every open set U in X, every component of U is open in X.*

PROOF:
$\langle 1 \rangle 1$. If X is locally connected then, for every open set U in X, every component of U is open in X.
  $\langle 2 \rangle 1$. ASSUME: X is locally connected.
  $\langle 2 \rangle 2$. LET: U be open in X.
  $\langle 2 \rangle 3$. LET: C be a component of U.
  $\langle 2 \rangle 4$. LET: $a \in C$
  $\langle 2 \rangle 5$. LET: V be a connected neighbourhood of a such that $V \subseteq U$
  $\langle 2 \rangle 6$. $V \subseteq C$
    PROOF: Lemma 2.336.
  $\langle 2 \rangle 7$. Q.E.D.
    PROOF: Lemma 2.95.
$\langle 1 \rangle 2$. If, for every open set U in X, every component of U is open in X, then X is locally connected.
  $\langle 2 \rangle 1$. ASSUME: for every open set U in X, every component of U is open in X.
  $\langle 2 \rangle 2$. LET: $a \in X$
  $\langle 2 \rangle 3$. LET: U be a neighbourhood of a
  $\langle 2 \rangle 4$. The component of U that contains a is a connected neighbourhood of a included in U.

□

Example 2.350. The ordered square is locally connected.

PROOF: Every neighbourhood of a point includes an interval around that point, which is connected by Theorem 2.304.

Example 2.351. Let X be the set of all rational points on the line segment $[0,1] \times \{0\}$, and Y the set of all rational points on the line segment $[0,1] \times \{1\}$. Let A be the space consisting of all line segments joining the point $(0,1)$ to a point of X, and all line segments joining the point $(1,0)$ to a point of Y. Then A is path-connected but is not locally connected at any point,

Proposition 2.352. *Let X and Y be topological spaces and $p : X \twoheadrightarrow Y$ be a quotient map. If X is locally connected then so is Y.*

PROOF:
$\langle 1 \rangle 1$. LET: U be an open set in Y.
$\langle 1 \rangle 2$. LET: C be a component of U.
$\langle 1 \rangle 3$. $p^{-1}(C)$ is a union of components of $p^{-1}(U)$
  $\langle 2 \rangle 1$. LET: $x \in p^{-1}(C)$

⟨2⟩2. LET: D be the component of $p^{-1}(U)$ that contains x.
⟨2⟩3. $p(D)$ is connected.
PROOF: Theorem 2.294.
⟨2⟩4. $p(D) \subseteq C$.
PROOF: From ⟨1⟩2 since $p(x) \in p(D) \cap C$ (⟨2⟩1, ⟨2⟩2).
⟨2⟩5. $D \subseteq p^{-1}(C)$
⟨1⟩4. $p^{-1}(C)$ is open in $p^{-1}(U)$
PROOF: Theorem 2.349.
⟨1⟩5. C is open in U
PROOF: Since the restriction of p to $p : p^{-1}(U) \twoheadrightarrow U$ is a quotient map by Proposition 2.267.
⟨1⟩6. Q.E.D.
PROOF: Theorem 2.349.
□

2.69 Local Path Connectedness

Definition 2.353 (Locally Path-Connected)**.** Let X be a topological space and $a \in X$. Then X is *locally path-connected* at a if and only if every neighbourhood of a includes a path-connected neighbourhood of a.

The space X is *locally path-connected* if and only if it is locally path-connected at every point.

Theorem 2.354. *A topological space X is locally path-connected if and only if, for every open set U in X, every path component of U is open in X.*

PROOF:
⟨1⟩1. If X is locally path-connected then, for every open set U in X, every path component of U is open in X.
⟨2⟩1. ASSUME: X is locally path-connected.
⟨2⟩2. LET: U be open in X.
⟨2⟩3. LET: C be a path component of U.
⟨2⟩4. LET: $a \in C$
⟨2⟩5. LET: V be a path-connected neighbourhood of a such that $V \subseteq U$
⟨2⟩6. $V \subseteq C$
PROOF: Lemma 2.336.
⟨2⟩7. Q.E.D.
PROOF: Lemma 2.95.
⟨1⟩2. If, for every open set U in X, every component of U is open in X, then X is locally connected.
⟨2⟩1. ASSUME: for every open set U in X, every component of U is open in X.
⟨2⟩2. LET: $a \in X$
⟨2⟩3. LET: U be a neighbourhood of a
⟨2⟩4. The component of U that contains a is a connected neighbourhood of a included in U.

∎

Theorem 2.355. *If a space is locally path connected then its components and its path components are the same.*

PROOF:
⟨1⟩1. LET: X be a locally path connected space.
⟨1⟩2. LET: C be a component of X.
⟨1⟩3. LET: $x \in C$
⟨1⟩4. LET: P be the path component of x
PROVE: $P = C$
⟨1⟩5. $P \subseteq C$
PROOF: Proposition 2.343.
⟨1⟩6. LET: Q be the union of the other path components included in C
⟨1⟩7. $C = P \cup Q$
PROOF: Proposition 2.343.
⟨1⟩8. P and Q are open in C
⟨2⟩1. C is open.
PROOF: Theorem 2.349.
⟨2⟩2. Q.E.D.
PROOF: Theorem 2.354.
⟨1⟩9. $Q = \emptyset$
PROOF: Otherwise P and Q would form a separation of C.
∎

Example 2.356. The ordered square is not locally path connected, since it is connected but not path connected.

Proposition 2.357. *Let X be a locally path-connected space. Then every connected open subspace of X is path-connected.*

PROOF:
⟨1⟩1. LET: U be a connected open subspace of X.
⟨1⟩2. LET: P be a path component of U.
⟨1⟩3. LET: Q be the union of the other path components of U.
⟨1⟩4. P and Q are open in U.
PROOF: Theorem 2.354.
⟨1⟩5. $Q = \emptyset$
PROOF: Otherwise P and Q form a separation of U.
∎

2.70 Weak Local Connectedness

Definition 2.358 (Weakly Locally Connected)**.** Let X be a topological space and $a \in X$. Then X is *weakly locally connected* at a if and only if every neighbourhood of a includes a connected subspace that includes a neighbourhood of a.

Proposition 2.359. *Let X be a topological space. If X is weakly locally connected at every point then X is locally connected.*

PROOF:
⟨1⟩1. ASSUME: X is weakly locally connected at every point.
⟨1⟩2. LET: U be open in X.
⟨1⟩3. LET: C be a component of U.
⟨1⟩4. C is open in X.
⟨2⟩1. LET: $x \in C$
⟨2⟩2. PICK a connected subspace D of U that includes a neighbourhood V of x.
⟨2⟩3. $D \subseteq C$
PROOF: Lemma 2.336.
⟨2⟩4. $x \in V \subseteq C$
⟨2⟩5. Q.E.D.
PROOF: Lemma 2.95.
⟨1⟩5. Q.E.D.
PROOF: Theorem 2.349.
□

Example 2.360. The 'infinite broom' (Munkres p.163) is an example of a space that is weakly locally connected at a point p but not locally connected at p.

2.71 Quasicomponents

Proposition 2.361. *Let X be a topological space. Define $\sim$ on X by $x \sim y$ if and only if there exists no separation U and V of X such that $x \in U$ and $y \in V$. Then $\sim$ is an equivalence relation on X.*

PROOF:
⟨1⟩1. $\sim$ is reflexive.
PROOF: Since the two sets that make up a separation are disjoint.
⟨1⟩2. $\sim$ is symmetric.
PROOF: Immediate from the defintion.
⟨1⟩3. $\sim$ is transitive.
⟨2⟩1. ASSUME: $x \sim y$ and $y \sim z$
⟨2⟩2. ASSUME: for a contradiction there is a separation U and V of X with $x \in U$ and $z \in V$
⟨2⟩3. $y \in U$ or $y \in V$
⟨2⟩4. Q.E.D.
PROOF: Either case contradicts ⟨2⟩1.
□

Definition 2.362 (Quasicomponents)**.** For X a topological space, the *quasicomponents* of X are the equivalence classes under $\sim$.

Proposition 2.363. *Let X be a topological space. Then every component of X is included in a quasicomponent of X.*

PROOF:
$\langle 1\rangle 1$. LET: C be a component of X.
$\langle 1\rangle 2$. LET: $x, y \in C$
PROVE: $x \sim y$
$\langle 1\rangle 3$. ASSUME: for a contradiction there exists a separation U and V of X with $x \in U$ and $y \in V$
$\langle 1\rangle 4$. $C \cap U$ and $C \cap V$ form a separation of C.
$\langle 1\rangle 5$. Q.E.D.
PROOF: This contradicts $\langle 1\rangle 1$.

Proposition 2.364. *In a locally connected space, the components and the quasicomponents are the same.*

PROOF:
$\langle 1\rangle 1$. LET: X be a locally connected space and Q a quasicomponent of X.
$\langle 1\rangle 2$. PICK a component C of X such that $C \subseteq Q$
$\langle 1\rangle 3$. LET: D be the union of the components of X
$\langle 1\rangle 4$. C and D are open in X.
PROOF: Theorem 2.349.
$\langle 1\rangle 5$. D cannot contain any points of Q.
PROOF: If it did, then C and D would form a separation of X and there would be points $x, y \in Q$ with $x \in C$ and $y \in D$.
$\langle 1\rangle 6$. $C = Q$
□

2.72 Open Coverings

Definition 2.365 (Open Covering)**.** Let X be a topological space. An *open covering* of X is a covering of X whose elements are all open sets.

2.73 Lindelöf Spaces

Definition 2.366 (Lindelöf Space)**.** A topological space X is *Lindelöf* if and only if every open covering has a countable subcovering.

Proposition 2.367. *Let X be a topological space. Then X is Lindelöf if and only if every set of closed sets that has the countable intersection property has nonempty intersection.*

PROOF: The following are equivalent.

1. X is compact.
2. Every open covering of X has a countable subcovering.
3. For any set $\mathcal{C}$ of closed sets, if $\{X \setminus C \mid C \in \mathcal{C}\}$ covers X then there is a countable subset $\mathcal{C}_0$ such that $\{X \setminus C \mid C \in \mathcal{C}_0\}$ covers X

4. For any set $\mathcal{C}$ of closed sets, if $\bigcap \mathcal{C} = \emptyset$ then there is a countable subset $\mathcal{C}_0$ with empty intersection.

5. Any set of closed sets with the countable intersection property has nonempty intersection.

□

Proposition 2.368 (CC)**.** *Let X be a topological space and $\mathcal{B}$ a basis for the topology on X. Then the following are equivalent.*

1. X is Lindelöf.

2. Every open covering of X by elements of $\mathcal{B}$ has a countable subcovering.

PROOF:
⟨1⟩1. $1 \Rightarrow 2$
PROOF: Immediate from definitions.
⟨1⟩2. $2 \Rightarrow 1$
⟨2⟩1. ASSUME: Every open covering of X by elements of $\mathcal{B}$ has a countable subcovering.
⟨2⟩2. LET: $\mathcal{U}$ be an open covering of X.
⟨2⟩3. $\{B \in \mathcal{B} \mid \exists U \in \mathcal{U}. B \subseteq U\}$ covers X.
⟨2⟩4. PICK a finite subcovering $\mathcal{B}_0$.
⟨2⟩5. For $B \in BB$, PICK $U_B \in \mathcal{U}$ such that $B \subseteq U_B$
⟨2⟩6. $\{U_B \mid B \in \mathcal{B}_0\}$ covers X.
□

2.74 The Second Countability Axiom

Definition 2.369 (Second Countability Axiom)**.** A topological space satisfies the *second countability axiom*, or is *second countable*, if and only if it has a countable basis.

Example 2.370. The space $\mathbb{R}$ is second countable.

PROOF: The set $\{(a, b) \mid a, b \in \mathbb{Q}\}$ is a basis. □

Proposition 2.371. *A subspace of a second countable space is second countable.*

PROOF: If $\mathcal{B}$ is a countable basis for X and $Y \subseteq X$ then $\{B \cap Y \mid B \in \mathcal{B}\}$ is a countable basis for Y. □

Proposition 2.372 (CC)**.** *Every second countable space is Lindelöf.*

PROOF: From Proposition 2.368.

Example 2.373 (CC)**.** The space $\mathbb{R}_l$ is Lindelöf.

⟨1⟩1. LET: $\mathcal{A}$ be a covering of $\mathbb{R}_l$ by basic open sets of the form $[a, b)$
⟨1⟩2. LET: $C = \bigcup\{(a, b) \mid [a, b) \in \mathcal{A}\}$

$\langle 1\rangle 3.$ $\mathbb{R} \setminus C$ is countable.

$\langle 2\rangle 1.$ For every $x \in \mathbb{R} \setminus C$, PICK a rational q_x such that $(x, q_x) \subseteq C$

$\langle 3\rangle 1.$ LET: $x \in \mathbb{R} \setminus C$

$\langle 3\rangle 2.$ PICK b such that $[x, b) \in \mathcal{A}$

$\langle 3\rangle 3.$ PICK a rational q such that $q \in (x, b)$

$\langle 2\rangle 2.$ The mapping $x \mapsto q_x$ is an injection $\mathbb{R} \setminus C \to \mathbb{Q}$

$\langle 1\rangle 4.$ PICK a countable $\mathcal{A}' \subseteq \mathcal{A}$ that covers $\mathbb{R} \setminus C$

$\langle 1\rangle 5.$ Under the standard topology on $\mathbb{R}$, C is second countable.

PROOF: Proposition 2.371.

$\langle 1\rangle 6.$ PICK a countable $\mathcal{A}'' \subseteq \mathcal{A}$ such that $\{(a, b) \mid [a, b) \in \mathcal{A}''\}$ covers C.

PROOF: Proposition 2.368.

$\langle 1\rangle 7.$ $\mathcal{A}' \cup \mathcal{A}''$ covers $\mathbb{R}_l$.

□

Example 2.374. The product of two Lindelöf spaces is not necessarily Lindelöf.

We prove that the Sorgenfrey plane is not Lindelöf.

PROOF:

$\langle 1\rangle 1.$ LET: $L = \{(x, -x) \mid x \in \mathbb{R}\}$

$\langle 1\rangle 2.$ L is closed in $\mathbb{R}_l^2$

$\langle 1\rangle 3.$ LET: $\mathcal{U} = \{[a, b) \times [a, -d) \mid a, b, d \in \mathbb{R}\}$

$\langle 1\rangle 4.$ $\mathcal{U} \cup \{\mathbb{R} \setminus L\}$ covers $\mathbb{R}_l^2$

$\langle 1\rangle 5.$ Every element of $\mathcal{U}$ intersects L at exactly one point.

$\langle 1\rangle 6.$ No countable subset of $\mathcal{U}$ covers $\mathbb{R}_l^2$.

□

2.75 Compact Spaces

Definition 2.375 (Compact)**.** A topological space is *compact* if and only if every open covering has a finite subcovering.

Lemma 2.376. *Let X be a topological space and Y a subspace of X. Then Y is compact if and only if every covering of Y by sets open in X has a finite subcovering.*

PROOF:

$\langle 1\rangle 1.$ If Y is compact then every covering of Y by sets open in X has a finite subcovering.

$\langle 2\rangle 1.$ ASSUME: Y is compact.

$\langle 2\rangle 2.$ LET: $\mathcal{U}$ be a covering of Y by sets open in X.

$\langle 2\rangle 3.$ $\{U \cap Y \mid U \in \mathcal{U}\}$ is an open covering of Y.

$\langle 2\rangle 4.$ PICK a finite subcovering $\{U_1 \cap Y, \ldots, U_n \cap Y\}$

$\langle 2\rangle 5.$ $\{U_1, \ldots, U_n\}$ is a finite subcovering of $\mathcal{U}$.

$\langle 1\rangle 2.$ If every covering of Y by sets open in X has a finite subcovering then Y is compact.

$\langle 2\rangle 1.$ LET: $\mathcal{U}$ be an open covering of Y.

$\langle 2\rangle 2.$ LET: $\mathcal{V} = \{V \text{ open in } X \mid V \cap Y \in \mathcal{U}\}$.

$\langle 2\rangle 3$. $\mathcal{V}$ is a covering of Y by sets open in X.
$\langle 2\rangle 4$. Pick a finite subcovering $\{V_1, \ldots, V_n\}$
$\langle 2\rangle 5$. $\{V_1 \cap Y, \ldots, V_n \cap Y\}$ is a finite subcovering of $\mathcal{U}$.
□

Proposition 2.377. *Every closed subspace of a compact space is compact.*

Proof:
$\langle 1\rangle 1$. Let: X be a compact space and $Y \subseteq X$ be closed.
$\langle 1\rangle 2$. Let: $\mathcal{U}$ be a covering of Y by sets open in X.
$\langle 1\rangle 3$. $\mathcal{U} \cup \{X \setminus Y\}$ is an open covering of X.
$\langle 1\rangle 4$. Pick a finite subcovering $\mathcal{U}_0$
$\langle 1\rangle 5$. $\mathcal{U}_0 \cap \mathcal{U}$ is a finite subset of $\mathcal{U}$ that covers Y.
□

Theorem 2.378. *The continuous image of a compact space is compact.*

Proof:
$\langle 1\rangle 1$. Let: $f : X \twoheadrightarrow Y$ be continuous and surjective.
$\langle 1\rangle 2$. Let: $\mathcal{V}$ be an open covering of Y
$\langle 1\rangle 3$. $\{p^{-1}(V) \mid V \in \mathcal{V}\}$ is an open covering of X.
$\langle 1\rangle 4$. Pick a finite subcovering $\{p^{-1}(V_1), \ldots, p^{-1}(V_n)\}$
$\langle 1\rangle 5$. $\{V_1, \ldots, V_n\}$ covers Y.
□

Theorem 2.379. *Let A and B be compact subspaces of X and Y respectively. Let N be an open set in $X \times Y$ that includes $A \times B$. Then there exist open sets U and V in X and Y respectively such that $A \subseteq U$, $B \subseteq V$ and $U \times V \subseteq N$.*

Proof:
$\langle 1\rangle 1$. For all $x \in A$, there exist neighbourhoods U of x and V of B such that $U \times V \subseteq N$.
$\langle 2\rangle 1$. Let: $x \in A$
$\langle 2\rangle 2$. For all $y \in B$, there exist neighbourhoods U of x and V of y such that $U \times V \subseteq N$
$\langle 2\rangle 3$. $\{V \text{ open in } Y \mid \exists \text{ neighbourhood } U \text{ of } x, U \times V \subseteq N\}$ covers B.
$\langle 2\rangle 4$. Pick a finite subcover $\{V_1, \ldots, V_n\}$
$\langle 2\rangle 5$. For $i = 1, \ldots, n$, Pick a neighbourhood U_i of x such that $U_i \times V_i \subseteq N$
$\langle 2\rangle 6$. Let: $U = U_1 \cap \cdots \cap U_n$
$\langle 2\rangle 7$. Let: $V = V_1 \cup \cdots \cup V_n$
$\langle 2\rangle 8$. U is a neighbourhood of x.
$\langle 2\rangle 9$. V is a neighbourhood of B.
$\langle 2\rangle 10$. $U \times V \subseteq N$
$\langle 1\rangle 2$. $\{U \text{ open in } X \mid \exists \text{ neighbourhood } V \text{ of } B. U \times V \subseteq N\}$ covers A.
$\langle 1\rangle 3$. Pick a finite subcover $\{U_1, \ldots, U_n\}$
$\langle 1\rangle 4$. For $i = 1, \ldots, n$, Pick a neighbourhood V_i of B such that $U_i \times V_i \subseteq N$
$\langle 1\rangle 5$. Let: $U = U_1 \cup \cdots \cup U_n$
$\langle 1\rangle 6$. Let: $V = V_1 \cap \cdots \cap V_n$

$\langle 1 \rangle 7.$ U and V are open.
$\langle 1 \rangle 8.$ $A \subseteq U$
$\langle 1 \rangle 9.$ $B \subseteq V$
$\langle 1 \rangle 10.$ $U \times V \subseteq N$
□

Corollary 2.379.1 (Tube Lemma)**.** *Let X and Y be topological spaces with Y compact. Let $a \in X$ and N be an open set in $X \times Y$ that includes $\{a\} \times Y$. Then there exists a neighbourhood W of a such that N includes the* tube *$W \times Y$.*

Theorem 2.380. *Let X be a topological space. Then X is compact if and only if every set of closed sets that has the finite intersection property has nonempty intersection.*

Proof: The following are equivalent.

1. X is compact.
2. Every open covering of X has a finite subcovering.
3. For any set $\mathcal{C}$ of closed sets, if $\{X \setminus C \mid C \in \mathcal{C}\}$ covers X then there is a finite subset $\mathcal{C}_0$ such that $\{X \setminus C \mid C \in \mathcal{C}_0\}$ covers X
4. For any set $\mathcal{C}$ of closed sets, if $\bigcap \mathcal{C} = \emptyset$ then there is a finite subset $\mathcal{C}_0$ with empty intersection.
5. Any set of closed sets with the finite intersection property has nonempty intersection.

□

Corollary 2.380.1. *Let X be a topological space and $C_1 \supseteq C_2 \supseteq \cdots$ a nested sequence of nonempty closed sets. Then $\bigcap_n C_n$ is nonempty.*

Proposition 2.381. *Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on the same set X with $\mathcal{T} \subseteq \mathcal{T}'$. If $\mathcal{T}'$ is compact then $\mathcal{T}$ is compact.*

Proof:
$\langle 1 \rangle 1.$ Let: $\mathcal{U} \subseteq \mathcal{T}$ cover X
$\langle 1 \rangle 2.$ $\mathcal{U} \subseteq \mathcal{T}'$
$\langle 1 \rangle 3.$ A finite subset of $\mathcal{U}$ covers X.
□

Corollary 2.381.1. *If $\mathcal{T}$ and $\mathcal{T}'$ are two compact Hausdorff topologies on the same set X, then either $\mathcal{T} = \mathcal{T}'$ or $\mathcal{T}$ and $\mathcal{T}'$ are incomparable.*

Proof: From the Proposition and Proposition 2.248. □

Example 2.382. Any set under the finite complement topology is compact.

Proposition 2.383. *Let X be a topological space. A finite union of compact subspaces of X is compact.*

Proof:
⟨1⟩1. Let: A and B be compact subspaces of X.
⟨1⟩2. Let: $\mathcal{U}$ be a set of open sets in X that covers $A \cup B$
⟨1⟩3. Pick a finite subset $\mathcal{U}_1$ that covers A.
Proof: Lemma 2.376.
⟨1⟩4. Pick a finite subset $\mathcal{U}_2$ that covers B.
Proof: Lemma 2.376.
⟨1⟩5. $\mathcal{U}_1 \cup \mathcal{U}_2$ is a finite subset that covers $A \cup B$.
⟨1⟩6. Q.E.D.
Proof: Lemma 2.376.
□

Proposition 2.384. *Let A and B be disjoint compact subspaces of the Hausdorff space X. Then there exist disjoint open sets U and V that include A and B respectively.*

Proof: From Theorem 2.379 with $N = X^2 \setminus \{(x, x) \mid x \in X\}$. □

Corollary 2.384.1. *Every compact subspace of a Hausdorff space is closed.*

Theorem 2.385. *Let $f : X \to Y$ be a bijective continuous function. If X is compact and Y is Hausdorff then f is a homeomorphism.*

Proof:
⟨1⟩1. Let: $C \subseteq X$ be closed.
⟨1⟩2. C is compact.
Proof: Proposition 2.377.
⟨1⟩3. $f(C)$ is compact.
Proof: Theorem 2.378.
⟨1⟩4. $f(C)$ is closed.
Proof: Corollary 2.384.1.
⟨1⟩5. Q.E.D.
Proof: Lemma 2.177.
□

Proposition 2.386. *Let X be a compact space, Y a Hausdorff space, and $f : X \to Y$ a continuous map. Then f is a closed map.*

Proof:
⟨1⟩1. Let: $C \subseteq X$ be closed.
⟨1⟩2. C is compact.
Proof: Proposition 2.377.
⟨1⟩3. $f(C)$ is compact.
Proof: Theorem 2.378.
⟨1⟩4. $f(C)$ is closed.
Proof: Corollary 2.384.1.
□

Proposition 2.387. *If Y is compact then the projection $\pi_1 : X \times Y \to X$ is a closed map.*

PROOF:
⟨1⟩1. LET: $A \subseteq X \times Y$ be closed.
⟨1⟩2. LET: $x \in X \setminus \pi_1(A)$
⟨1⟩3. PICK a neighbourhood U of x such that $U \times Y \subseteq (X \times Y) \setminus A$
PROOF: By the Tube Lemma.
⟨1⟩4. $x \in U \subseteq X \setminus \pi_1(A)$
⟨1⟩5. Q.E.D.
PROOF: So $X \setminus \pi_1(A)$ is open by Lemma 2.95.
□

Theorem 2.388. *Let X be a topological space and Y a compact Hausdorff space. Let $f : X \to Y$ be a function. Then f is continuous if and only if the graph of f is closed in $X \times Y$.*

PROOF:
⟨1⟩1. LET: G_f be the graph of f.
⟨1⟩2. If f is continuous then G_f is closed.
⟨2⟩1. ASSUME: f is continuous.
⟨2⟩2. LET: $(x, y) \in (X \times Y) \setminus G_f$
⟨2⟩3. PICK disjoint neighbourhoods U and V of y and $f(x)$ respectively.
⟨2⟩4. $f^{-1}(V) \times U$ is a neighbourhood of (x, y) disjoint from G_f.
⟨1⟩3. If G_f is closed then f is continuous.
⟨2⟩1. ASSUME: G_f is closed.
⟨2⟩2. LET: $x \in X$ and V be a neighbourhood of $f(x)$.
⟨2⟩3. $G_f \cap (X \times (Y \setminus V))$ is closed.
⟨2⟩4. $\pi_1(G_f \cap (X \times (Y \setminus V)))$ is closed.
PROOF: Proposition 2.387.
⟨2⟩5. LET: $U = X \setminus \pi_1(G_f \cap (X \times (Y \setminus V)))$
⟨2⟩6. U is a neighbourhood of x
⟨2⟩7. $f(U) \subseteq V$
□

Theorem 2.389. *Let X be a compact topological space. Let $(f_n : X \to \mathbb{R})$ be a monotone increasing sequence of continuous functions and $f : X \to \mathbb{R}$ a continuous function. If (f_n) converges pointwise to f, then (f_n) converges uniformly to f.*

PROOF:
⟨1⟩1. LET: $\epsilon > 0$
⟨1⟩2. For all $x \in X$, there exists N such that, for all $n \geq N$, we have $|f_n(x) - f(x)| < \epsilon$
⟨1⟩3. For $n \geq 1$,
LET: $U_n = \{x \in X \mid |f_n(x) - f(x)| < \epsilon\}$
⟨1⟩4. For $n \geq 1$, we have U_n is open in X.
⟨2⟩1. LET: $x \in X$
⟨2⟩2. LET: $\delta = \epsilon - |f_n(x) - f(x)|$
⟨2⟩3. PICK a neighbourhood U of x such that $f(U) \subseteq B(f(x), \delta/2)$

⟨2⟩4. PICK a neighbourhood V of x such that $f_n(V) \subseteq B(f_n(x), \delta/2)$
⟨2⟩5. $f(U \cap V) \subseteq U_n$
PROOF: For $y \in U \cap V$ we have

$$\begin{aligned}|f_n(y) - f(y)| &\leq |f_n(y) - f_n(x)| + |f_n(x) - f(x)| + |f(x) - y(y)| \\ &< \delta/2 + |f_n(x) - f(x)| + \delta/2 \\ &= \epsilon\end{aligned}$$

⟨1⟩5. $\{U_n \mid n \geq 1\}$ covers X
PROOF: From ⟨1⟩2
⟨1⟩6. PICK N such that $X = U_N$
⟨2⟩1. PICK $n_1, \ldots, n_k$ such that $U_{n_1}, \ldots, U_{n_k}$ cover X.
⟨2⟩2. LET: $N = \max(n_1, \ldots, n_k)$
⟨2⟩3. For all i we have $U_{n_i} \subseteq U_N$
PROOF: Since (f_n) is monotone increasing.
⟨2⟩4. $X = U_N$
⟨1⟩7. For all $x \in X$ and $n \geq N$ we have $|f_n(x) - f(x)| < \epsilon$
□

An example to show that we cannot remove the hypothesis that X is compact:

Example 2.390. Let $X = (0, 1)$, $f_n(x) = -x^n$ and $f(x) = 0$ for $x \in X$ and $n \geq 1$. Then $f_n \to f$ pointwise and (f_n) is monotone increasing but the convergence is not uniform since, for all $N \geq 1$, there exists $x \in (0, 1)$ such that $-x^N < -1/2$.

An example to show that we cannot remove the hypothesis that (f_n) is monotone increasing:

Example 2.391. Let $X = [0, 1]$, $f_n(x) = 1/(n^3(x - 1/n)^2 + 1)$ and $f(x) = 0$ for $x \in X$ and $n \geq 1$. Then X is compact and $f_n \to f$ pointwise but the convergence is not uniform since, for all $N \geq 1$, there exists $x \in [0, 1]$ such that $f_N(x) = 1$, namely $x = 1/N$.

Theorem 2.392. *Let X be a compact Hausdorff space. Let $\mathcal{A}$ be a chain of closed connected subsets of X. Then $\bigcap \mathcal{A}$ is connected.*

PROOF:
⟨1⟩1. ASSUME: for a contradiction C and D form a separation of $\bigcap \mathcal{A}$.
⟨1⟩2. PICK disjoint open sets U and V that include C and D respectively.
PROOF: Proposition 2.384.
⟨1⟩3. $\{A \setminus (U \cup V) \mid A \in \mathcal{A}\}$ is a set of closed sets with the finite intersection property.
⟨2⟩1. For all $A \in \mathcal{A}$ we have $A \setminus (U \cup V)$ is closed.
⟨2⟩2. For all $A_1, \ldots, A_n \in \mathcal{A}$ we have $(A_1 \cap \cdots \cap A_n) \setminus (U \cup V)$ is nonempty.
PROOF:
⟨3⟩1. LET: $A_1, \ldots, A_n \in \mathcal{A}$
⟨3⟩2. ASSUME: without loss of generality $A_1 \subseteq A_2, \ldots, A_n$
PROOF: Since $\mathcal{A}$ is a chain.

$\langle 3\rangle 3.$ $A_1 \setminus (U \cup V)$ is nonempty

PROOF: Otherwise $(A_1 \cap \cdots \cap A_n \cap U)$ and $(A_1 \cap \cdots \cap A_n \cap V)$ would form a separation of A_n.

$\langle 1\rangle 4.$ $\bigcap \mathcal{A} \setminus (U \cup V)$ is nonempty.

PROOF: Theorem 2.380.

$\langle 1\rangle 5.$ Q.E.D.

PROOF: This contradicts $\langle 1\rangle 1$ since $\bigcap AA \setminus (U \cup V) = \bigcap \mathcal{A} \setminus (C \cup D)$.

□

Theorem 2.393 (Tychonoff Theorem (AC))**.** *The product of a family of compact spaces is compact.*

PROOF:

$\langle 1\rangle 1.$ LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of compact spaces.

$\langle 1\rangle 2.$ LET: $X = \prod_{\alpha \in J} X_\alpha$

$\langle 1\rangle 3.$ For any $\mathcal{A} \subseteq \mathcal{P}X$, we have $\bigcap_{A \in \mathcal{A}} \overline{A} \neq \emptyset$

$\langle 2\rangle 1.$ LET: $\mathcal{A} \subseteq \mathcal{P}X$

$\langle 2\rangle 2.$ PICK $\mathcal{D} \supseteq \mathcal{A}$ that is maximal with respect to the finite intersection property.

PROVE: $\bigcap_{D \in \mathcal{D}} \overline{D} \neq \emptyset$

PROOF: Lemma 2.51.

$\langle 2\rangle 3.$ For $\alpha \in J$, PICK $x_\alpha \in X_\alpha$ such that $x_\alpha \in \bigcap_{D \in \mathcal{D}} \overline{\pi_\alpha(D)}$

PROOF: Theorem 2.380 since $\{\overline{\pi_\alpha(D)} \mid D \in \mathcal{D}\}$ is a set of closed sets in X_α with the finite intersection property.

$\langle 2\rangle 4.$ LET: $x = (x_\alpha)_{\alpha \in J}$

PROVE: $x \in \bigcap_{D \in \mathcal{D}} \overline{D}$

$\langle 2\rangle 5.$ For any $\beta \in J$ and neighbourhood U of x_β in X_β, we have ${\pi_\beta}^{-1}(U)$ intersects every element of $\mathcal{D}$

$\langle 3\rangle 1.$ LET: $\beta \in J$

$\langle 3\rangle 2.$ LET: U be a neighbourhood of x_β in X_β.

$\langle 3\rangle 3.$ LET: $D \in \mathcal{D}$

$\langle 3\rangle 4.$ $x_\beta \in \overline{\pi_\beta(D)}$

PROOF: From $\langle 2\rangle 3$

$\langle 3\rangle 5.$ U intersects $\pi_\beta(D)$.

$\langle 3\rangle 6.$ ${\pi_\beta}^{-1}(U)$ intersects D.

$\langle 2\rangle 6.$ For any $\beta \in J$ and neighbourhood U of x_β in X_β, we have ${\pi_\beta}^{-1}(U) \in \mathcal{D}$

PROOF: Lemma 2.53.

$\langle 2\rangle 7.$ Every basic neighbourhood of x is an element of $\mathcal{D}$

PROOF: Lemma 2.52.

$\langle 2\rangle 8.$ Every basic neighbourhood of x intersects every element of $\mathcal{D}$

PROOF: Since $\mathcal{D}$ satisfies the finite intersection property.

$\langle 2\rangle 9.$ For all $D \in \mathcal{D}$ we have $x \in \overline{D}$

$\langle 1\rangle 4.$ Q.E.D.

PROOF: Theorem 2.380.

□

Lemma 2.394. *Let X and Y be topological spaces. Let $\mathcal{A}$ be a set of basis* *elements for the product topology on $X \times Y$ such that no finite subset of $\mathcal{A}$ covers $X \times Y$. If X is compact, then there exists $x \in X$ such that no finite subset of $\mathcal{A}$ covers the slice $\{x\} \times Y$.*

PROOF:
$\langle 1 \rangle 1$. ASSUME: for every $x \in X$, there exists a finite subset of $\mathcal{A}$ that covers $\{x\} \times Y$
PROVE: A finite subset of $\mathcal{A}$ covers $X \times Y$
$\langle 1 \rangle 2$. $\{U \mid \exists U_1 \times V_1, \ldots, U_n \times V_n \in \mathcal{A}. U = U_1 \cap \cdots \cap U_n, V_1 \cup \cdots \cup V_n = Y\}$ covers X
$\langle 1 \rangle 3$. PICK a finite subcover $U_1, \ldots, U_m$
$\langle 1 \rangle 4$. PICK $U_{ij} \times V_{ij} \in \mathcal{A}$ such that, for every i, we have $U_i = \bigcap_j U_{ij}$ and $Y = \bigcup_j V_{ij}$
$\langle 1 \rangle 5$. The collection of all $U_{ij} \times V_{ij}$ covers $X \times Y$
□

Theorem 2.395 (AC). *Let X be a compact Hausdorff space. Then the quasi-components and the components of X are the same.*

PROOF:
$\langle 1 \rangle 1$. LET: $x, y \in X$
$\langle 1 \rangle 2$. ASSUME: x and y are in the same quasicomponent.
PROVE: x and y are in the same component.
$\langle 1 \rangle 3$. LET: $\mathcal{A}$ be the set of all closed subsets A of X such that x and y are in the same quasicomponent of A.
$\langle 1 \rangle 4$. For every chain $\mathcal{B} \subseteq \mathcal{A}$ we have $\bigcap \mathcal{B} \in \mathcal{A}$
$\langle 2 \rangle 1$. LET: $BB \subseteq \mathcal{A}$ be a chain.
$\langle 2 \rangle 2$. ASSUME: for a contradiction U and V form a separation of $\bigcap \mathcal{B}$ with $x \in U$ and $y \in V$
$\langle 2 \rangle 3$. PICK disjoint open sets U', V' in X such that $U \subseteq U'$ and $V \subseteq V'$
$\langle 2 \rangle 4$. $\{B \setminus (U' \cup V') \mid B \in \mathcal{B}\}$ satisfies the finite intersection property.
$\langle 3 \rangle 1$. LET: $B_1, \ldots, B_n \in \mathcal{B}$
$\langle 3 \rangle 2$. ASSUME: without loss of generality $B_1 \subseteq \cdots \subseteq B_n$
PROOF: Since $\mathcal{B}$ is a chain.
$\langle 3 \rangle 3$. $\bigcap\{B_1 \setminus (U' \cup V'), \ldots, B_n \setminus (U' \cup V')\} = B_1 \setminus (U' \cup V')$
$\langle 3 \rangle 4$. $B_1 \setminus (U' \cup V')$ is nonempty
PROOF: Otherwise $B_1 \cap U'$ and $B_1 \cap V'$ would form a separation of B_1, contradicting the fact that x and y are in the same quasicomponent of B_1.
$\langle 2 \rangle 5$. $\bigcap \mathcal{B} \setminus (U \cup V)$ is nonempty
PROOF: Theorem 2.380.
$\langle 2 \rangle 6$. Q.E.D.
PROOF: This contradicts $\langle 2 \rangle 2$.
$\langle 1 \rangle 5$. PICK a minimal element D in $\mathcal{A}$.
PROVE: D is connected.
PROOF: By Zorn's Lemma.
$\langle 1 \rangle 6$. ASSUME: for a contradiction U and V form a separation of D.

⟨1⟩7. ASSUME: without loss of generality $x, y \in U$

PROOF: We cannot have that one of x, y is in U and the other in V sicnce $D \in \mathcal{A}$.

⟨1⟩8. $U \in \mathcal{A}$

PROOF: If X and Y form a separation of U with $x \in X$ and $y \in Y$, then X and $Y \cup V$ form a separation of D with $x \in X$ and $y \in Y \cup V$.

⟨1⟩9. Q.E.D.

PROOF: There is a connected set D that contains both x and y.

□

PROOF:

⟨1⟩1. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of compact spaces.

⟨1⟩2. LET: $X = \prod_{\alpha \in J} X_\alpha$

⟨1⟩3. PICK a well-ordering $<$ on J such that J has a greatest element.

⟨1⟩4. For $\alpha \in J$ and $p = \{p_i \in X_i\}_{i \leq \alpha}$ a family of points,
LET: $Y(p) = \{x \in X \mid \forall i \leq \alpha . x_i = p_i\}$

⟨1⟩5. If $\alpha < \alpha'$ and p is an α'-indexed family of points then $Y(p) \subseteq Y(p \restriction \alpha)$

PROOF: From definition.

⟨1⟩6. Given $\beta \in J$ and $p = \{p_i \in X_i\}_{i<\beta}$ a family of points,
LET: $Z(p) = \bigcap_{\alpha<\beta} Y(p \restriction \alpha)$

⟨1⟩7. Given $\beta \in J$ and $p = \{p_i \in X_i\}_{i<\beta}$ a family of points, if $\mathcal{A}$ is a finite set of basic open spaces for X that covers $Z(p)$, then there exists $\alpha < \beta$ such that $\mathcal{A}$ covers $Y(p \restriction \alpha)$

⟨2⟩1. ASSUME: without loss of generality β has no immediate predecessor.

⟨2⟩2. For $A \in \mathcal{A}$,
LET: $J_A = \{i < \beta \mid \pi_i(A) \neq X_i\}$

⟨2⟩3. LET: $\alpha = \max \bigcup_{A \in \mathcal{A}} J_A$

⟨2⟩4. LET: $x \in Y(p \restriction \alpha)$

⟨2⟩5. LET: $y \in Z(p)$ be the point with $y_i = p_i$ for $i < \beta$ and $y_i = x_i$ for $i \geq \beta$

⟨2⟩6. PICK $A \in \mathcal{A}$ such that $y \in A$

PROOF: Since $\mathcal{A}$ covers $Z(p)$.

⟨2⟩7. For $i \in J_A$ we have $x_i \in \pi_i(A)$

PROOF: Since $i \leq \alpha$ so $x_i = p_i$

⟨2⟩8. For $i \in J \setminus J_A$ we have $x_i \in \pi_i(A)$

PROOF: Since $\pi_i(A) = X_i$

⟨2⟩9. $x \in A$

⟨1⟩8. ASSUME: for a contraction $\mathcal{A}$ is a set of basic open sets for X that covers X but such that no finite subset of $\mathcal{A}$ covers X

⟨1⟩9. PICK a set of points $\{p_i\}_{i \in J}$ such that, for all $\alpha \in J$, we have $Y(p \restriction \alpha)$ is not finitely covered by $\mathcal{A}$

⟨2⟩1. ASSUME: as transfinite induction hypothesis $\alpha \in J$ and $\{p_i\}_{i<\alpha}$ is a family of points such that, for all $\alpha' < \alpha$, we have $Y(p \restriction \alpha')$ is not finitely covered by $\mathcal{A}$

⟨2⟩2. $Z(p)$ is not finitely covered by $\mathcal{A}$

PROOF: By ⟨1⟩7.

⟨2⟩3. PICK $p_\alpha \in X_\alpha$ such that $Y(p)$ is not finitely covered by $\mathcal{A}$

PROOF: By Lemma 2.394 since there is a homeomorphism $\phi : Z(p) \cong X_\alpha \times \prod_{\alpha' > \alpha} X_{\alpha'}$ and, given p_α, this homemorphism ϕ restricts to a homeomorphism $Y(p) \cong \{p_\alpha\} \times \prod_{\alpha' > \alpha} X_{\alpha'}$.

⟨1⟩10. Q.E.D.

PROOF: If ω is the greatest element of J then $Y(p \restriction \omega)$ is a singleton.

∎

Theorem 2.396. *Every complete linearly ordered set in the order topology is compact.*

PROOF:

⟨1⟩1. LET: X be a complete linearly ordered set with least element a and greatest element b.

⟨1⟩2. LET: $\mathcal{A}$ be an open covering of X.

⟨1⟩3. For all $x < b$, there exists $y > x$ such that $[x, y]$ can be covered by at most two elements of $\mathcal{A}$.

⟨2⟩1. LET: $x \in X$

⟨2⟩2. PICK $A \in \mathcal{A}$ with $x \in A$

⟨2⟩3. PICK $y > x$ such that $[x, y) \subseteq A$

⟨2⟩4. PICK $B \in \mathcal{A}$ with $y \in B$

⟨2⟩5. $[x, y]$ is covered by A and B

⟨1⟩4. LET: $C = \{y \in X \mid [a, y] \text{ can be covered by finitely many elements of } \mathcal{A}\}$

⟨1⟩5. LET: $c = \sup C$

⟨1⟩6. $c > a$

⟨2⟩1. PICK $x > a$ such that $[a, x]$ can be covered by at most two elements of $\mathcal{A}$.

PROOF: From ⟨1⟩3.

⟨2⟩2. $x \in C$

⟨1⟩7. $c \in C$

⟨2⟩1. PICK $A \in \mathcal{A}$

⟨2⟩2. PICK $x < c$ such that $(x, c] \subseteq A$

⟨2⟩3. PICK $y > x$ such that $y \in C$

⟨2⟩4. PICK $\mathcal{A}_0 \subseteq^{\text{fin}} \mathcal{A}$ that covers $[a, y]$

⟨2⟩5. $\mathcal{A}_0 \cup \{A\}$ covers $[a, c]$

⟨1⟩8. $c = b$

⟨2⟩1. ASSUME: for a contradiction $c < b$

⟨2⟩2. PICK $x > c$ such that $[c, x]$ can be covered by at most two elements of $\mathcal{A}$

PROOF: From ⟨1⟩3.

⟨2⟩3. $[a, x]$ can be finitely covered by $\mathcal{A}$

PROOF: From ⟨1⟩7.

⟨2⟩4. Q.E.D.

PROOF: This contradicts the maximality of c.

∎

Corollary 2.396.1. *Let X be a linearly ordered set with the least upper bound property. Then every closed interval in X is compact.*

Corollary 2.396.2. *Every closed interval in $\mathbb{R}$ is compact.*

Theorem 2.397 (Extreme Value Theorem)**.** *Any linearly ordered set under the order topology that is compact has a greatest and a least element.*

PROOF:
$\langle 1 \rangle 1$. LET: X be a linearly ordered set under the order topology that is compact.
$\langle 1 \rangle 2$. X has a greatest element.
  $\langle 2 \rangle 1$. ASSUME: for a contradiction X has no greatest element.
  $\langle 2 \rangle 2$. $\{(-\infty, a) \mid a \in X\}$ covers X.
  $\langle 2 \rangle 3$. PICK a finite subcover $\{(-\infty, a_1), \ldots, (-\infty, a_n)\}$, say.
  $\langle 2 \rangle 4$. ASSUME: without loss of generality $a_1 \leq \cdots \leq a_n$
  $\langle 2 \rangle 5$. $X \subseteq (-\infty, a_n)$
  $\langle 2 \rangle 6$. $a_n < a_n$
$\langle 1 \rangle 3$. X has a least element.
  PROOF: Similar.
□

2.76 Perfect Maps

Definition 2.398 (Perfect Map)**.** Let X and Y be topological spaces and $f : X \to Y$. Then f is a *perfect map* if and only if f is a closed map, continuous, surjective and $p^{-1}(y)$ is compact for all $y \in Y$.

Proposition 2.399. *Let X be a topological space, Y a compact space, and $p : X \to Y$ a closed map such that, for all $y \in Y$, we have $p^{-1}(y)$ is compact. Then X is compact.*

PROOF:
$\langle 1 \rangle 1$. LET: $\mathcal{A}$ be a set of closed sets in X with the finite intersection property.
$\langle 1 \rangle 2$. $\mathcal{B} = \{p(A_1 \cap \cdots \cap A_n) \mid A_1, \ldots, A_n \in \mathcal{A}\}$ is a set of closed sets in Y with the finite intersection property.
  PROOF: Since p is a closed map.
$\langle 1 \rangle 3$. PICK $y \in \bigcap \mathcal{B}$
  PROOF: Theorem 2.380 since Y is compact.
$\langle 1 \rangle 4$. $\{A \cap p^{-1}(y) \mid A \in \mathcal{A}\}$ is a set of closed sets in $p^{-1}(y)$ with the finite intersection property.
$\langle 1 \rangle 5$. PICK $x \in \bigcap_{A \in \mathcal{A}} (A \cap p^{-1}(y))$
  PROOF: Theorem 2.380 since $p^{-1}(y)$ is compact.
$\langle 1 \rangle 6$. $x \in \bigcap \mathcal{A}$
$\langle 1 \rangle 7$. Q.E.D.
  PROOF: Theorem 2.380.
□

2.77 Topological Groups

Definition 2.400 (Topological Group)**.** A *topological group* G consists of a T_1 space G and continuous maps $\cdot : G^2 \to G$ and $(\ )^{-1} : G \to G$ such that $(G, \cdot, (\ )^{-1})$ is a group.

Example 2.401. 1. The integers $\mathbb{Z}$ under addition are a topological group.

2. The real numbers $\mathbb{R}$ under addition are a topological group.

3. The positive reals under multiplication are a topological group.

4. The set $\{z \in \mathbb{C} \mid |z| = 1\}$ under multiplication and given the topology of S^1 is a topological group.

5. For any $n \geq 0$, the general linear group $GL_n(\mathbb{R})$ is a topological group under matrix multiplication, considered as a subspace of $\mathbb{R}^{n^2}$.

Lemma 2.402. *Let G be a T_1 space and $\cdot : G^2 \to G$, $(\ )^{-1} : G \to G$ be functions such that $(G, \cdot, (\ )^{-1})$ is a group. Then G is a topological group if and only if the function $f : G^2 \to G$ that maps (x, y) to xy^{-1} is continuous.*

PROOF:
⟨1⟩1. If G is a topological group then f is continuous.
  PROOF: From Theorem 2.166.
⟨1⟩2. If f is continuous then G is a topological group.
  ⟨2⟩1. ASSUME: f is continuous.
  ⟨2⟩2. $(\ )^{-1}$ is continuous.
    PROOF: Since $x^{-1} = f(e, x)$.
  ⟨2⟩3. $\cdot$ is continuous.
    PROOF: Since $xy = f(x, y^{-1})$.
□

Lemma 2.403. *Let G be a topological group and H a subgroup of G. Then H is a topological group under the subspace topology.*

PROOF:
⟨1⟩1. H is T_1.
  PROOF: From Proposition 2.236.
⟨1⟩2. multiplication and inverse on H are continuous.
  PROOF: From Theorem 2.167.
□

Lemma 2.404. *Let G be a topological group and H a subgroup of G. Then $\overline{H}$ is a subgroup of G.*

PROOF:
⟨1⟩1. LET: $x, y \in \overline{H}$
    PROVE: $xy^{-1} \in \overline{H}$

$\langle 1\rangle 2.$ LET: U be any neighbourhood of xy^{-1}
$\langle 1\rangle 3.$ LET: $f : G^2 \to G$, $f(a,b) = ab^{-1}$
$\langle 1\rangle 4.$ $f^{-1}(U)$ is a neighbourhood of (x,y)
$\langle 1\rangle 5.$ PICK neighbourhoods V, W of x and y respectively such that $f(V \times W) \subseteq U$.
$\langle 1\rangle 6.$ PICK $a \in V \cap H$ and $b \in W \cap H$
PROOF: Theorem 2.117.
$\langle 1\rangle 7.$ $ab^{-1} \in U \cap H$
$\langle 1\rangle 8.$ Q.E.D.
PROOF: By Theorem 2.117.
□

Proposition 2.405. *Let G be a topological group and $\alpha \in G$. Then the maps $l_\alpha, r_\alpha : G \to G$ defined by $l_\alpha(x) = \alpha x$, $r_\alpha(x) = x\alpha$ are homeomorphisms of G with itself.*

PROOF: They are continuous with continuous inverses $l_{\alpha^{-1}}$ and $r_{\alpha^{-1}}$. □

Corollary 2.405.1. *Every topological group is homogeneous.*

PROOF: Given a topological group G and $a, b \in G$, we have $l_{ba^{-1}}$ is a homeomorphism that maps a to b. □

Proposition 2.406. *Let G be a topological group and H a subgroup of G. Give G/H the quotient topology. For all $\alpha \in G$, the map $\overline{f_\alpha}$ that sends xH to αxH is a homeomorphism $G/H \cong G/H$.*

PROOF:
$\langle 1\rangle 1.$ $\overline{f_\alpha}$ is well-defined.
PROOF: If $xy^{-1} \in H$ then $(\alpha x)(\alpha y)^{-1} = xy^{-1} \in H$.
$\langle 1\rangle 2.$ $\overline{f_\alpha}$ is continuous.
PROOF: Theorem 2.270 since $\overline{f_\alpha} \circ p = p \circ f_\alpha$ is continuous, where $p : G \twoheadrightarrow G/H$ is the canonical surjection.
$\langle 1\rangle 3.$ $\overline{f_\alpha}^{-1}$ is continuous.
PROOF: Similar since $\overline{f_\alpha}^{-1} = \overline{f_{\alpha^{-1}}}$.
□

Corollary 2.406.1. *Let G be a topological group and H a subgroup of G. Give G/H the quotient topology. Then G/H is homogeneous.*

Proposition 2.407. *Let G be a topological group and H a subgroup of G. Give G/H the quotient topology. If H is closed in G then G/H is T_1.*

PROOF:
$\langle 1\rangle 1.$ LET: $p : G \twoheadrightarrow G/H$ be the canonical surjection
$\langle 1\rangle 2.$ LET: $x \in G$
$\langle 1\rangle 3.$ $p^{-1}(xH) = f_x(H)$
$\langle 1\rangle 4.$ $p^{-1}(xH)$ is closed in G
PROOF: Since H is closed and f_x is a homemorphism of G with itself.

⟨1⟩5. $\{xH\}$ is closed in G/H

∎

Proposition 2.408. *Let G be a topological group and H a subgroup of G. Give G/H the quotient topology. Then the canonical surjection $p : G \twoheadrightarrow G/H$ is an open map.*

PROOF:
⟨1⟩1. LET: $U \subseteq G$ be open.
⟨1⟩2. $p^{-1}(p(U)) = \bigcup_{h \in H} r_h(U)$
⟨1⟩3. $p^{-1}(p(U))$ is open.
⟨1⟩4. $p(U)$ is open.

∎

Proposition 2.409. *Let G be a topological group and H a closed normal subgroup of G. Then G/H is a topological group under the quotient topology.*

PROOF:
⟨1⟩1. G/H is T_1
PROOF: Proposition 2.407.
⟨1⟩2. The map $\overline{m} : (xH, yH) \mapsto xy^{-1}H$ is continuous.
⟨2⟩1. $p^2 : G^2 \to (G/H)^2$ is a quotient map.
PROOF: Propositions 2.269, 2.408.
⟨2⟩2. $\overline{m} \circ p^2$ is continuous.
PROOF: As it is $p^2 \circ m$ where $m : G^2 \to G$ with $m(x, y) = xy^{-1}$

∎

Lemma 2.410. *Let G be a topological group and $A, B \subseteq G$. If either A or B is open then AB is open.*

PROOF: If A is open we have $AB = \bigcup_{b \in B} r_b(A)$. Similarly if B is open. ∎

Definition 2.411 (Symmetric Neighbourhood)**.** Let G be a topological group. A neighbourhood V of e is *symmetric* if and only if $V = V^{-1}$.

Lemma 2.412. *Let G be a topological group. Let V be a neighbourhood of e. Then V is symmetric if and only if, for all $x \in V$, we have $x^{-1} \in V$.*

PROOF:
⟨1⟩1. If V is symmetric then, for all $x \in V$, we have $x^{-1} \in V$
PROOF: Immediate from defintions.
⟨1⟩2. If, for all $x \in V$, we have $x^{-1} \in V$, then V is symmetric.
⟨2⟩1. ASSUME: for all $x \in V$ we have $x^{-1} \in V$
⟨2⟩2. $V \subseteq V^{-1}$
PROOF: If $x \in V$ then there exists $y \in V$ such that $x = y^{-1}$, namely $y = x^{-1}$
⟨2⟩3. $V^{-1} \subseteq V$
PROOF: Immediate from ⟨2⟩1.

∎

Lemma 2.413. *Let G be a topological group. For every neighbourhood U of e, there exists a symmetric neighbourhood V of e such that $V^2 \subseteq U$.*

PROOF:
⟨1⟩1. LET: U be a neighbourhood of e.
⟨1⟩2. PICK a neighbourhood V' of e such that $V'V' \subseteq U$
PROOF: Such a neighbourhood exists because multiplication in G is continuous.
⟨1⟩3. PICK a neighbourhood W of e such that $WW^{-1} \subseteq V'$
PROOF: Such a neighbourhood exists because the function that maps (x, y) to xy^{-1} is continuous.
⟨1⟩4. LET: $V = WW^{-1}$
⟨1⟩5. V is a neighbourhood of e
⟨2⟩1. $e \in V$
PROOF: Since $e \in W$ so $e = ee^{-1} \in V$.
⟨2⟩2. V is open
PROOF: Lemma 2.410.
⟨1⟩6. V is symmetric
⟨2⟩1. For all $x \in V$ we have $x^{-1} \in V$
⟨3⟩1. LET: $x \in V$
⟨3⟩2. PICK$y, z \in W$ such that $x = yz^{-1}$
⟨3⟩3. $x^{-1} = zy^{-1}$
⟨3⟩4. $x^{-1} \in V$
⟨3⟩5. $x \in V^{-1}$
⟨2⟩2. Q.E.D.
PROOF: Lemma 2.412
⟨1⟩7. $V^2 \subseteq U$
PROOF: We have $V^2 \subseteq (V')^2 \subseteq U$
□

Proposition 2.414. *Every topological group is Hausdorff.*

PROOF:
⟨1⟩1. LET: G be a topological group.
⟨1⟩2. LET: $x, y \in G$ with $x \neq y$
⟨1⟩3. LET: $U = G \setminus \{x[^{-1}y]\}$
⟨1⟩4. PICK a symmetric neighbourhood V of e such that $VV \subseteq U$
⟨2⟩1. U is open
PROOF: Since G is T_1.
⟨2⟩2. $e \in U$
PROOF: Since $x \neq y$
⟨2⟩3. Q.E.D.
PROOF: Lemma 2.413.
⟨1⟩5. Vx and Vy are disjoint neighbourhoods of x and y respectively.
⟨2⟩1. Vx is open
PROOF: Since $Vx = r_x(V)$
⟨2⟩2. Vy is open

PROOF: Similar.

$\langle 2\rangle 3.$ $Vx \cap Vy = \emptyset$

$\langle 3\rangle 1.$ ASSUME: for a contradiction $z \in Vx \cap Vy$

$\langle 3\rangle 2.$ PICK $a, b \in V$ such that $z = ax = by$

$\langle 3\rangle 3.$ $xy^{-1} \in VV$

PROOF: Since $xy^{-1} = a^{-1}b$

$\langle 3\rangle 4.$ $xy^{-1} \in U$

$\langle 3\rangle 5.$ Q.E.D.

PROOF: From $\langle 1\rangle 3$.

□

Proposition 2.415. *Every topological group is regular.*

PROOF:

$\langle 1\rangle 1.$ LET: G be a topological group.

$\langle 1\rangle 2.$ LET: $A \subseteq G$ be a closed set and $a \notin A$.

$\langle 1\rangle 3.$ LET: $U = G \setminus Aa^{-1}$

$\langle 1\rangle 4.$ PICK a symmetric neighbourhood V of e such that $VV \subseteq U$

$\langle 2\rangle 1.$ U is open

PROOF: Since $Aa^{-1} = r_{a^{-1}}(A)$ is closed.

$\langle 2\rangle 2.$ $e \in U$

PROOF: Since $a \notin A$.

$\langle 2\rangle 3.$ Q.E.D.

PROOF: Lemma 2.413.

$\langle 1\rangle 5.$ VA and Va are disjoint open sets with $A \subseteq VA$ and $a \in Va$

$\langle 2\rangle 1.$ VA is open

PROOF: Lemma 2.410

$\langle 2\rangle 2.$ Va is open

PROOF: Lemma 2.410

$\langle 2\rangle 3.$ $VA \cap Va = \emptyset$

$\langle 3\rangle 1.$ ASSUME: for a contradiction $z \in VA \cap Va$

$\langle 3\rangle 2.$ PICK $b, c \in V$ and $d \in A$ with $z = bd = ca$

$\langle 3\rangle 3.$ $da^{-1} \in U$

PROOF: Since $da^{-1} = b^{-1}c \in VV \subseteq U$

$\langle 3\rangle 4.$ Q.E.D.

PROOF: This contradicts $\langle 1\rangle 3$

□

Proposition 2.416. *Let G be a topological group and H a subgroup of G. Give G/H the quotient topology. If H is closed in G then G/H is regular.*

PROOF:

$\langle 1\rangle 1.$ LET: $p : G \twoheadrightarrow G/H$ be the canonical surjection.

$\langle 1\rangle 2.$ LET: A be a closed set in G/H and $aH \in (G/H) \setminus A$.

$\langle 1\rangle 3.$ LET: $B = p^{-1}(A)$

$\langle 1\rangle 4.$ B is a closed saturated set in G.

$\langle 1\rangle 5.$ $B \cap aH = \emptyset$

$\langle 1\rangle 6.$ $B = BH$
$\langle 1\rangle 7.$ Pick a symmetric neighbourhood V of e such that VB does not intersect Va
$\langle 2\rangle 1.$ Let: $U = G \setminus Ba^{-1}$
$\langle 2\rangle 2.$ Pick a symmetric neighbourhood V of e such that $VV \subseteq U$
$\langle 3\rangle 1.$ U is open
Proof: Since $Ba^{-1} = r_{a^{-1}}(B)$ is closed.
$\langle 3\rangle 2.$ $e \in U$
Proof: If $e \in Ba^{-1}$ then $a \in B$
$\langle 3\rangle 3.$ Q.E.D.
Proof: Lemma 2.413
$\langle 2\rangle 3.$ $VB \cap Va = \emptyset$
Proof: If $vb = v'a$ for $v, v' \in V$ and $b \in B$ then we have $ba^{-1} = v^{-1}v' \in Ba \cap VV \subseteq Ba \cap U$.
$\langle 1\rangle 8.$ $p(VB)$ and $p(Va)$ are disjoint open sets
$\langle 2\rangle 1.$ $p(VB)$ and $p(Va)$ are open.
Proof: Proposition 2.408.
$\langle 2\rangle 2.$ $p(VB) \cap p(Va) = \emptyset$
Proof: If $vbH = v'aH$ for $v, v' \in V$, $b \in B$ then $v'a = vbh$ for some $h \in H$. Hence $v'a \in Va \cap VBH = Va \cap VB$.
$\langle 1\rangle 9.$ $A \subseteq p(VB)$
$\langle 1\rangle 10.$ $aH \in p(Va)$
□

Proposition 2.417. *Let G be a topological group. The component of G that contains e is a normal subgroup of G.*

Proof:
$\langle 1\rangle 1.$ Let: C be the component of G that contains e.
$\langle 1\rangle 2.$ For all $x \in G$, xC is the component of G that contains x.
$\langle 2\rangle 1.$ Let: $x \in G$
$\langle 2\rangle 2.$ Let: D be the component of G that contains x.
$\langle 2\rangle 3.$ $xC \subseteq D$
Proof: Since xC is connected by Theorem 2.294.
$\langle 2\rangle 4.$ $D \subseteq xC$
Proof: Since $x^{-1}D \subseteq C$ similarly.
$\langle 1\rangle 3.$ For all $x \in G$, Cx is the component of G that contains x.
Proof: Similar.
$\langle 1\rangle 4.$ For all $x \in C$ we have $xC = Cx = C$
$\langle 1\rangle 5.$ For all $x \in C$ we have $x^{-1}C = C$
$\langle 1\rangle 6.$ For all $x \in C$ we have $x^{-1} \in C$
$\langle 1\rangle 7.$ For all $x, y \in C$ we have $xy \in C$
Proof: Since $xyC = xC = x$.
$\langle 1\rangle 8.$ For all $x \in G$ we have $xC = Cx$.
Proof: From $\langle 1\rangle 2$ and $\langle 1\rangle 3$.
□

Lemma 2.418. *Let G be a topological group. Let A be a closed set in G and B*

a compact subspace of G such that $A \cap B = \emptyset$. Then there exists a symmetric neighbourhood U of e such that $AU \cap BU = \emptyset$.

PROOF:
$\langle 1 \rangle 1$. For all $b \in B$ there exists a symmetric neighbourhood V of e such that $bV^2 \cap A = \emptyset$
$\langle 2 \rangle 1$. LET: $b \in B$
$\langle 2 \rangle 2$. LET: $W = b^{-1}(G \setminus A)$
$\langle 2 \rangle 3$. W is a neighbourhood of e and $bW \cap A = \emptyset$
$\langle 2 \rangle 4$. PICK a symmetric neighbourhood V of e such that $V^2 \subseteq W$
$\langle 1 \rangle 2$. $\{bV^2 \mid b \in B, V \text{ is a symmetric neighbourhood of } e, bV^2 \cap A = \emptyset\}$ is an open cover of B
$\langle 1 \rangle 3$. PICK a finite subcover $b_1 V_1^2, \ldots, b_n V_n^2$, say.
$\langle 1 \rangle 4$. LET: $U = V_1 \cap \cdots \cap V_n$
$\langle 1 \rangle 5$. $BU^2 \cap A = \emptyset$
$\langle 1 \rangle 6$. $AU \cap BU = \emptyset$
PROOF: If $av \in BU$ where $a \in A$ and $v \in V$ then $a = avv^{-1} \in BU^2 \cap A$.
□

Proposition 2.419 (AC)**.** *Let G be a topological group. Let A be a closed set in G, and B a compact subspace of G. Then AB is closed.*

PROOF:
$\langle 1 \rangle 1$. LET: $x \in G \setminus AB$
$\langle 1 \rangle 2$. $A^{-1}x \cap B = \emptyset$
$\langle 1 \rangle 3$. $A^{-1}x$ is closed.
$\langle 1 \rangle 4$. PICK a symmetric neighbourhood U of e such that $A^{-1}xU \cap BU = \emptyset$
$\langle 1 \rangle 5$. xU^2 is open
PROOF: Lemma 2.410.
$\langle 1 \rangle 6$. $x \in xU^2 \subseteq G \setminus AB$
□

Corollary 2.419.1. *Let G be a topological group and $H \leq G$. Let $p : G \twoheadrightarrow G/H$ be the quotient map. If H is compact then p is a closed map.*

PROOF: For A closed in G, we have $p^{-1}(p(A)) = AH$ is closed, and so $p(A)$ is closed. □

Corollary 2.419.2. *Let G be a topological group and $H \leq G$. If H and G/H are compact then G is compact.*

PROOF: From Proposition 2.399 since, for all $aH \in G/H$, we have $p^{-1}(aH) = aH$ is compact because it is homemorphic to H. □

2.78 The Metric Topology

Definition 2.420 (Metric)**.** Let X be a set. A *metric* on X is a function $d : X^2 \to \mathbb{R}$ such that:

1. For all $x, y \in X$, $d(x, y) \geq 0$
2. For all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$
3. For all $x, y \in X$, $d(x, y) = d(y, x)$
4. (*Triangle Inequality*) For all $x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$

We call $d(x, y)$ the *distance* between x and y.

Definition 2.421 (Open Ball)**.** Let X be a metric space. Let $a \in X$ and $\epsilon > 0$. The *open ball* with *centre* a and *radius* ϵ is

$$B(a, \epsilon) = \{x \in X \mid d(a, x) < \epsilon\} \ .$$

Definition 2.422 (Metric Topology)**.** Let X be a metric space. The *metric topology* on X is the topology generated by the basis consisting of all the open balls.

We prove this is a basis for a topology.

PROOF:
$\langle 1 \rangle$1. For every point a, there exists a ball B such that $a \in B$
PROOF: We have $a \in B(a, 1)$.
$\langle 1 \rangle$2. For any balls B_1, B_2 and point $a \in B_1 \cap B_2$, there exists a ball B_3 such that $a \in B_3 \subseteq B_1 \cap B_2$
$\langle 2 \rangle$1. LET: $B_1 = B(c_1, \epsilon_1)$ and $B_2 = B(c_2, \epsilon_2)$
$\langle 2 \rangle$2. LET: $\delta = \min(\epsilon_1 - d(c_1, a), \epsilon_2 - d(c_2, a))$
PROVE: $B(a, \delta) \subseteq B_1 \cap B_2$
$\langle 2 \rangle$3. LET: $x \in B(a, \delta)$
$\langle 2 \rangle$4. $x \in B_1$
PROOF:

$$\begin{aligned} d(x, c_1) &= d(x, a) + d(a, c_1) \\ &< \delta + d(a, c_1) \\ &\leq \epsilon_1 \end{aligned}$$

$\langle 2 \rangle$5. $x \in B_2$
PROOF: Similar.
□

Proposition 2.423. *Let X be a metric space and $U \subseteq X$. Then U is open if and only if, for every $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$.*

PROOF:
$\langle 1 \rangle$1. If U is open then, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$.
$\langle 2 \rangle$1. ASSUME: U is open.

⟨2⟩2. LET: $x \in U$
⟨2⟩3. PICK $a \in X$ and $\delta > 0$ such that $x \in B(a, \delta) \subseteq U$
⟨2⟩4. LET: $\epsilon = \delta - d(a, x)$
PROVE: $B(x, \epsilon) \subseteq U$
⟨2⟩5. LET: $y \in B(x, \epsilon)$
⟨2⟩6. $d(y, a) < \delta$
PROOF:

$$\begin{aligned} d(y,a) &\leq d(a,x) + d(x,y) \\ &< \delta + d(x,y) \\ &= \epsilon \end{aligned}$$

⟨2⟩7. $y \in U$
⟨1⟩2. If, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$, then U is open.
PROOF: Immediate from definitions.
□

Definition 2.424 (Discrete Metric)**.** Let X be a set. The *discrete metric* on X is defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Proposition 2.425. *The discrete metric induces the discrete topology.*

PROOF: For any (open) set U and point $a \in U$, we have $a \in B(a, 1) \subseteq U$. □

Definition 2.426 (Standard Metric on $\mathbb{R}$)**.** The *standard metric* on $\mathbb{R}$ is defined by $d(x, y) = |x - y|$.

Proposition 2.427. *The standard metric on $\mathbb{R}$ induces the standard topology on $\mathbb{R}$.*

PROOF:
⟨1⟩1. Every open ball is open in the standard topology on $\mathbb{R}$.
PROOF: $B(a, \epsilon) = (a - \epsilon, a + \epsilon)$
⟨1⟩2. For every open set U and point $a \in U$, there exists $\epsilon > 0$ such that $B(a, \epsilon) \subseteq U$
⟨2⟩1. LET: U be an open set and $a \in U$
⟨2⟩2. PICK an open interval b, c such that $a \in (b, c) \subseteq U$
⟨2⟩3. LET: $\epsilon = \min(a - b, c - a)$
⟨2⟩4. $B(a, \epsilon) \subseteq U$
□

Definition 2.428 (Metrizable)**.** A topological space X is *metrizable* if and only if there exists a metric on X that induces the topology.

Definition 2.429 (Bounded)**.** Let X be a metric space and $A \subseteq X$. Then A is *bounded* if and only if there exists M such that, for all $x, y \in A$, we have $d(x, y) \leq M$.

Definition 2.430 (Diameter)**.** Let X be a metric space and $A \subseteq X$. The *diameter* of A is

$$\operatorname{diam} A = \sup_{x,y \in A} d(x,y) \ .$$

Definition 2.431 (Standard Bounded Metric)**.** Let d be a metric on X. The *standard bounded metric* corresponding to d is the metric $\overline{d}$ defined by

$$\overline{d}(x,y) = \min(d(x,y), 1) \ .$$

We prove this is a metric.

PROOF:
⟨1⟩1. $\overline{d}(x,y) \geq 0$
  PROOF: Since $d(x,y) \geq 0$
⟨1⟩2. $\overline{d}(x,y) = 0$ if and only if $x = y$
  PROOF: $\overline{d}(x,y) = 0$ if and only if $d(x,y) = 0$ if and only if $x = y$
⟨1⟩3. $\overline{d}(x,y) = \overline{d}(y,x)$
  PROOF: Since $d(x,y) = d(y,x)$
⟨1⟩4. $\overline{d}(x,z) \leq \overline{d}(x,y) + \overline{d}(y,z)$
  PROOF:

$$\begin{aligned}\overline{d}(x,y) + \overline{d}(y,z) &= \min(d(x,y),1) + \min(d(y,z),1) \\ &= \min(d(x,y) + d(y,z), d(x,y)+1, d(y,z)+1, 2) \\ &\geq \min(d(x,z),1) \\ &= \overline{d}(x,z)\end{aligned}$$

□

Lemma 2.432. *In any metric space X, the set $\mathcal{B} = \{B(a,\epsilon) \mid a \in X, \epsilon < 1\}$ is a basis for the metric topology.*

PROOF:
⟨1⟩1. Every element of $\mathcal{B}$ is open.
  PROOF: From Lemma 2.132.
⟨1⟩2. For every open set U and point $a \in U$, there exists $B \in \mathcal{B}$ such that $a \in B \subseteq U$
  ⟨2⟩1. LET: U be an open set and $a \in U$
  ⟨2⟩2. PICK$\epsilon > 0$ such that $B(a,\epsilon) \subseteq U$
  ⟨2⟩3. $B(a, \min(\epsilon, 1/2)) \subseteq U$
⟨1⟩3. Q.E.D.
  PROOF: Lemma 2.133.
□

Proposition 2.433. *Let d be a metric on the set X. Then the standard bounded metric $\overline{d}$ induces the same metric as d.*

PROOF: This follows from Lemma 2.432 since the open balls with radius < 1 are the same under both metrics. □

Lemma 2.434. *Let d and d' be two metrics on the same set X. Let $\mathcal{T}$ and $\mathcal{T}'$ be the topologies they induce. Then $\mathcal{T} \subseteq \mathcal{T}'$ if and only if, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that*

$$B_{d'}(x, \delta) \subseteq B_d(x, \epsilon) \ .$$

PROOF:
⟨1⟩1. If $\mathcal{T} \subseteq \mathcal{T}'$ then, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$
  PROOF: From Proposition 2.423 since $x \in B_d(x, \epsilon) \in \mathcal{T}'$.
⟨1⟩2. If, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$, then $\mathcal{T} \subseteq \mathcal{T}'$
  ⟨2⟩1. ASSUME: For all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$
  ⟨2⟩2. LET: $U \in \mathcal{T}$
  ⟨2⟩3. For all $x \in U$ there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq U$.
    ⟨3⟩1. LET: $x \in U$
    ⟨3⟩2. PICK $\epsilon > 0$ such that $B_d(x, \epsilon) \subseteq U$
      PROOF: Proposition 2.423
    ⟨3⟩3. PICK $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$
      PROOF: By ⟨2⟩1
    ⟨3⟩4. $B_{d'}(x, \delta) \subseteq U$
  ⟨2⟩4. $U \in \mathcal{T}'$
    PROOF: Proposition 2.423.
□

Proposition 2.435. $\mathbb{R}^2$ *under the dictionary order topology is metrizable.*

PROOF: Define $d : \mathbb{R}^2 \to \mathbb{R}$ by

$$\begin{aligned} d((x,y),(x,z)) &= \max(|y-z|, 1) \\ d((x,y),(x',y')) &= 1 && \text{if } x \neq x' \end{aligned}$$ □

⟨1⟩1. $x \in \bigcap_{i=1}^{N} {\pi_i}^{-1}() \subseteq B_D(a, \epsilon)$

Proposition 2.436. *Let $d : X^2 \to \mathbb{R}$ be a metric on X. Then the metric topology on X is the coarsest topology such that d is continuous.*

PROOF:
⟨1⟩1. d is continuous.
  ⟨2⟩1. LET: $a, b \in X$
  ⟨2⟩2. LET: $\epsilon > 0$
  ⟨2⟩3. LET: $\delta = \epsilon/2$
  ⟨2⟩4. LET: $x, y \in X$
  ⟨2⟩5. ASSUME: $\rho((a,b),(x,y)) < \delta$
  ⟨2⟩6. $|d(a,b) - d(x,y)| < \epsilon$
    ⟨3⟩1. $d(a,b) - d(x,y) < \epsilon$

PROOF:

$$\begin{aligned} d(a,b) &\leq d(a,x) + d(x,y) + d(y,b) \\ &\leq d(x,y) + 2\rho((a,b),(x,y)) \\ &< d(x,y) + 2\delta \\ &= d(x,y) + \epsilon \end{aligned}$$

⟨3⟩2. $d(a.b) - d(x,y) > -\epsilon$

PROOF: Similar.

⟨2⟩7. Q.E.D.

⟨1⟩2. If $\mathcal{T}$ is any topology under which d is continuous then $\mathcal{T}$ is finer than the metric topology.

PROOF: Since $B(a,\epsilon) = {d_a}^{-1}((-\infty,\epsilon))$

□

Proposition 2.437. *Let X be a metric space with metric d and $A \subseteq X$. The restriction of d to A is a metric on A that induces the subspace topology.*

PROOF:

⟨1⟩1. The restriction of d to A is a metric on A.

⟨1⟩2. Every open ball under $d \upharpoonright A$ is open under the subspace topology.

PROOF: $B_{d\upharpoonright A}(a,\epsilon) = B_d(a,\epsilon) \cap A$.

⟨1⟩3. If U is open in the subspace topology and $x \in U$, then there exists a $d \upharpoonright A$-ball B such that $x \in B \subseteq U$.

⟨2⟩1. PICK V open in X such that $U = V \cap A$

⟨2⟩2. PICK $\epsilon > 0$ such that $B_d(x,\epsilon) \subseteq V$

⟨2⟩3. Take $B = B_{d\upharpoonright A}(x,\epsilon)$

□

Corollary 2.437.1. *A subspace of a metrizable space is metrizable.*

Proposition 2.438. *Every metrizable space is Hausdorff.*

PROOF:

⟨1⟩1. LET: X be a metric space

⟨1⟩2. LET: $a, b \in X$ with $a \neq b$

⟨1⟩3. LET: $\epsilon = d(a,b)/2$

⟨1⟩4. LET: $U = B(a,\epsilon)$ and $V = B(b,\epsilon)$

⟨1⟩5. U and V are disjoint neighbourhoods of a and b respectively.

□

Proposition 2.439 (CC)**.** *The product of a countable family of metrizable spaces is metrizable.*

PROOF:

⟨1⟩1. LET: (X_n, d_n) be a sequence of metric spaces.

⟨1⟩2. ASSUME: w.l.o.g. each d_n is bounded above by 1.

PROOF: By Proposition 2.433.

⟨1⟩3. LET: D be the metric on $\mathbb{R}^\omega$ defined by $D(x,y) = \sup_i(d_i(x_i,y_i)/i)$.

⟨2⟩1. $D(x, y) \geq 0$
⟨2⟩2. $D(x, y) = 0$ if and only if $x = y$
⟨2⟩3. $D(x, y) = D(y, x)$
⟨2⟩4. $D(x, z) \leq D(x, y) + D(y, z)$
PROOF:

$$\begin{aligned} D(x, z) &= \sup_i \frac{d_i(x_i, z_i)}{i} \\ &\leq \sup_i \frac{d_i(x_i, y_i) + d_i(y_i, z_i)}{i} \\ &\leq \sup_i \frac{d_i(x_i, y_i)}{i} + \sup_i \frac{d_i(y_i, z_i)}{i} \\ &= D(x, y) + D(y, z) \end{aligned}$$

⟨1⟩4. Every open ball $B_D(a, \epsilon)$ is open in the product topology.
⟨2⟩1. PICK N such that $1/\epsilon < N$
⟨2⟩2. $B_D(a, \epsilon) = \prod_{i=1}^{\infty} U_i$ where $U_i = B(a_i, i\epsilon)$ if $i \leq N$, and $U_i = X_i$ if $i > N$
⟨1⟩5. For any open set U and $a \in U$, there exists $\epsilon > 0$ such that $B_D(a, \epsilon) \subseteq U$.
⟨2⟩1. LET: $n \geq 1$, V be an open set in $\mathbb{R}$ and $a \in {\pi_n}^{-1}(V)$
⟨2⟩2. PICK $\epsilon > 0$ such that $B_{d_n}(a, \epsilon) \subseteq V$
⟨2⟩3. $B_D(a, \epsilon/n) \subseteq {\pi_n}^{-1}(V)$
□

Theorem 2.440. *Let X and Y be metric spaces and $f : X \to Y$. Then f is continuous if and only if, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$.*

PROOF:
⟨1⟩1. If f is continuous then, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$
⟨2⟩1. ASSUME: f is continuous.
⟨2⟩2. LET: $x \in X$ and $\epsilon > 0$
⟨2⟩3. PICK a neighbourhood U of x such that $f(U) \subseteq B(f(x), \epsilon)$
PROOF: Theorem 2.163.
⟨2⟩4. PICK $\delta > 0$ such that $B(x, \delta) \subseteq U$
PROOF: Proposition 2.423.
⟨2⟩5. For all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$
⟨1⟩2. If for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$, then f is continuous.
⟨2⟩1. ASSUME: for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$
⟨2⟩2. LET: $x \in X$ and V be a neighbourhood of $f(x)$
⟨2⟩3. PICK $\epsilon > 0$ such that $B(f(x), \epsilon) \subseteq V$
PROOF: Proposition 2.423.
⟨2⟩4. PICK $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$
PROOF: By ⟨2⟩1
⟨2⟩5. LET: $U = B(x, \delta)$
⟨2⟩6. U is a neighbourhood of x with $f(U) \subseteq V$

$\langle 2\rangle 7$. Q.E.D.
PROOF: Theorem 2.163.
∎

Proposition 2.441. *Let X be a metric space. Let (a_n) be a sequence in X and $l \in X$. Then $a_n \to l$ as $n \to \infty$ if and only if, for all $\epsilon > 0$, there exists N such that, for all $n \geq N$, we have $d(a_n, l) < \epsilon$.*

PROOF: From Proposition 2.146. ∎

Proposition 2.442. *Every metrizable space is first countable.*

PROOF: In any metric space X, the open balls $B(a, 1/n)$ for $n \geq 1$ form a local basis at a.

Example 2.443. $\mathbb{R}^\omega$ under the box topology is not metrizable.

Example 2.444. If J is uncountable then $\mathbb{R}^J$ under the product topology is not metrizable.

Proposition 2.445. *A compact subspace of a metric space is bounded.*

PROOF:
$\langle 1\rangle 1$. LET: X be a metric space and $A \subseteq X$ be compact.
$\langle 1\rangle 2$. PICK $a \in A$
$\langle 1\rangle 3$. $\{B(a, n) \mid n \in \mathbb{Z}^+\}$ covers A
$\langle 1\rangle 4$. PICK a finite subcover $\{B(a, n_1), \ldots, B(a, n_k)\}$
$\langle 1\rangle 5$. LET: $N = \max(n_1, \ldots, n_k)$
$\langle 1\rangle 6$. For all $x, y \in A$ we have $d(x, y) < 2N$
PROOF:

$$\begin{aligned} d(x, y) &\leq d(x, a) + d(a, y) \\ &< N + N \end{aligned}$$

∎

This example shows the converse does not hold:

Example 2.446. The space $\mathbb{R}$ under the standard bounded metric is bounded but not compact.

2.79 Real Linear Algebra

Definition 2.447 (Square Metric)**.** The *square metric* ρ on $\mathbb{R}^n$ is defined by

$$\rho(\vec{x}, \vec{y}) = \max(|x_1 - y_1|, \ldots, |x_n - y_n|)$$

We prove this is a metric.

PROOF:
$\langle 1\rangle 1$. $\rho(\vec{x}, \vec{y}) \geq 0$
PROOF: Immediate from definition.

⟨1⟩2. $\rho(\vec{x}, \vec{y}) = 0$ if and only if $\vec{x} = \vec{y}$

PROOF: Immediate from definition.

⟨1⟩3. $\rho(\vec{x}, \vec{y}) = \rho(\vec{y}, \vec{x})$

PROOF: Immediate from definition.

⟨1⟩4. $\rho(\vec{x}, \vec{z}) \leq \rho(\vec{x}, \vec{y}) + \rho(\vec{y}, \vec{z})$

PROOF: Since $|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$.

□

Proposition 2.448. *The square metric induces the standard topology on* $\mathbb{R}^n$*.*

PROOF:

⟨1⟩1. For every $a \in X$ and $\epsilon > 0$, we have $B_\rho(a, \epsilon)$ is open in the standard product topology.

PROOF:

$$B_\rho(a, \epsilon) = (a_1 - \epsilon, a_1 + \epsilon) \times \cdots \times (a_n - \epsilon, a_n + \epsilon)$$

⟨1⟩2. For any open sets $U_1, \ldots, U_n$ in $\mathbb{R}$, we have $U_1 \times \cdots \times U_n$ is open in the square metric topology.

⟨2⟩1. LET: $\vec{a} \in U_1 \times \cdots \times U_n$

⟨2⟩2. For $i = 1, \ldots, n$, PICK $\epsilon_i > 0$ such that $(a_i - \epsilon_i, a_i + \epsilon_i) \subseteq U_i$

⟨2⟩3. LET: $\epsilon = \min(\epsilon_1, \ldots, \epsilon_n)$

⟨2⟩4. $B_\rho(\vec{a}, \epsilon) \subseteq U$

□

Definition 2.449. Given $\vec{x}, \vec{y} \in \mathbb{R}^n$, define the *sum* $\vec{x} + \vec{y}$ by

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n) \ .$$

Definition 2.450. Given $\lambda \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^n$, define the *scalar product* $\lambda\vec{x} \in \mathbb{R}^n$ by

$$\lambda(x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n)$$

Definition 2.451 (Inner Product)**.** Given $\vec{x}, \vec{y} \in \mathbb{R}^n$, define the *inner product* $\vec{x} \cdot \vec{y} \in \mathbb{R}$ by

$$(x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) = x_1 y_1 + \cdots + x_n y_n \ .$$

We write $\vec{x}^2$ for $\vec{x} \cdot \vec{x}$.

Definition 2.452 (Norm)**.** Let $n \geq 1$. The *norm* on $\mathbb{R}^n$ is the function $\| \ \| : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\|(x_1, \ldots, x_n)\| = \sqrt{x_1^2 + \cdots + x_n^2}$$

Lemma 2.453.

$$\|\vec{x}\|^2 = \vec{x}^2$$

PROOF: Immediate from definitions. □

Lemma 2.454.

$$\vec{x} \cdot (\vec{y} + \vec{z}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z}$$

PROOF: Each is equal to $(x_1 y_1 + x_1 z_1, \ldots, x_n y_n + x_n z_n)$. ∎

Lemma 2.455.

$$|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \|\vec{y}\|$$

PROOF:
⟨1⟩1. ASSUME: $\vec{x} \neq \vec{0} \neq \vec{y}$
PROOF: Otherwise both sides are 0.
⟨1⟩2. LET: $a = 1/\|\vec{x}\|$
⟨1⟩3. LET: $b = 1/\|\vec{y}\|$
⟨1⟩4. $(a\vec{x} + b\vec{y})^2 \geq 0$ and $(a\vec{x} - b\vec{y})^2 \geq 0$
⟨1⟩5. $a^2\|\vec{x}\|^2 + 2ab\vec{x} \cdot \vec{y} + b^2\|\vec{y}\|^2 \geq 0$ and $a^2\|\vec{x}\|^2 - 2ab\vec{x} \cdot \vec{y} + b^2\|\vec{y}\|^2 \geq 0$
⟨1⟩6. $2ab\vec{x} \cdot \vec{y} + 2 \geq 0$ and $-2ab\vec{x} \cdot \vec{y} + 2 \geq 0$
⟨1⟩7. $\vec{x} \cdot \vec{y} \geq -1/ab$ and $\vec{x} \cdot \vec{y} \leq 1/ab$
⟨1⟩8. $\vec{x} \cdot \vec{y} \geq -\|\vec{x}\|\|\vec{y}\|$ and $\vec{x} \cdot \vec{y} \leq \|\vec{x}\|\|\vec{y}\|$
∎

Lemma 2.456 (Triangle Inequality)**.**

$$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$$

PROOF:

$$\begin{aligned} \|\vec{x} + \vec{y}\|^2 &= \|\vec{x}\|^2 + 2\vec{x} \cdot \vec{y} + \|\vec{y}\|^2 \\ &\leq \|\vec{x}\|^2 + 2\|\vec{x}\|\|\vec{y}\| + \|\vec{y}\|^2 && \text{(Lemma 2.455)} \\ &= (\|\vec{x}\| + \|\vec{y}\|)^2 \end{aligned}$$ ∎

Definition 2.457 (Euclidean Metric)**.** Let $n \geq 1$. The *Euclidean metric* on $\mathbb{R}^n$ is defined by

$$d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\| \ .$$

We prove this is a metric.

⟨1⟩1. $d(\vec{x}, \vec{y}) \geq 0$
PROOF: Immediate from definition.
⟨1⟩2. $d(\vec{x}, \vec{y}) = 0$ if and only if $\vec{x} = \vec{y}$
PROOF: $d(\vec{x}, \vec{y}) = 0$ if and only if $\vec{x} - \vec{y} = \vec{0}$.
⟨1⟩3. $d(\vec{x}, \vec{y}) = d(\vec{y}, \vec{x})$
PROOF: Immediate from definition.
⟨1⟩4. $d(\vec{x}, \vec{z}) \leq d(\vec{x}, \vec{y}) + d(\vec{y}, \vec{z})$
PROOF:

$$\begin{aligned} \|\vec{x} - \vec{z}\| &= \|(\vec{x} - \vec{y}) + (\vec{y} - \vec{z})\| \\ &\leq \|\vec{x} - \vec{y}\| + \|\vec{y} - \vec{z}\| && \text{(Lemma 2.456)} \end{aligned}$$

∎

Proposition 2.458. *The Euclidean metric induces the standard topology on* $\mathbb{R}^n$.

PROOF:

⟨1⟩1. LET: ρ be the square metric.
⟨1⟩2. For all $\vec{a} \in \mathbb{R}^n$ and $\epsilon > 0$, we have $B_d(\vec{a}, \epsilon) \subseteq B_\rho(\vec{a}, \epsilon)$
⟨2⟩1. LET: $\vec{x} \in B_d(\vec{a}, \epsilon)$
⟨2⟩2. $\sqrt{(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2} < \epsilon$
⟨2⟩3. $(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2 < \epsilon^2$
⟨2⟩4. For all i we have $(x_i - a_i)^2 < \epsilon^2$
⟨2⟩5. For all i we have $|x_i - a_i| < \epsilon$
⟨2⟩6. $\rho(\vec{x}, \vec{a}) < \epsilon$
⟨1⟩3. For all $\vec{a} \in \mathbb{R}^n$ and $\epsilon > 0$, we have $B_\rho(\vec{a}, \epsilon/\sqrt{n}) \subseteq B_d(\vec{a}, \epsilon)$
⟨2⟩1. LET: $\vec{x} \in B_\rho(\vec{a}, \epsilon/\sqrt{n})$
⟨2⟩2. $\rho(\vec{x}, \vec{a}) < \epsilon/\sqrt{n}$
⟨2⟩3. For all i we have $|x_i - x_a| < \epsilon/\sqrt{n}$
⟨2⟩4. For all i we have $(x_i - x_a)^2 < \epsilon^2/n$
⟨2⟩5. $d(\vec{x}, \vec{a}) < \epsilon$
⟨1⟩4. Q.E.D.
PROOF: By Lemma 2.434.
□

Proposition 2.459. *Let $n \geq 0$. For all $c \in \mathbb{R}^n$ and $\epsilon > 0$, the open ball $B(c, \epsilon)$ is path connected.*

PROOF:
⟨1⟩1. LET: $a, b \in B(c, \epsilon)$
⟨1⟩2. LET: $p : [0, 1] \to B(c, \epsilon)$ be the function $p(t) = (1 - t)a + tb$
PROOF: We have $p(t) \in B(c, \epsilon)$ for all t because

$$\begin{aligned} d(p(t), c) &= \|(1-t)a + tb - c\| \\ &= \|(1-t)(a-c) + t(b-c)\| \\ &\leq (1-t)\|a-c\| + t\|b-c\| \\ &< (1-t)\epsilon + t\epsilon \\ &= \epsilon \end{aligned}$$

⟨1⟩3. p is a path from a to b.
□

Proposition 2.460. *Let $n \geq 0$. For all $c \in \mathbb{R}^n$ and $\epsilon > 0$, the closed ball $\overline{B(c, \epsilon)}$ is path connected.*

PROOF:
⟨1⟩1. LET: $a, b \in \overline{B(c, \epsilon)}$
⟨1⟩2. LET: $p : [0, 1] \to \overline{B(c, \epsilon)}$ be the function $p(t) = (1 - t)a + tb$
PROOF: We have $p(t) \in \overline{B(c, \epsilon)}$ for all t because

$$\begin{aligned} d(p(t), c) &= \|(1-t)a + tb - c\| \\ &= \|(1-t)(a-c) + t(b-c)\| \\ &\leq (1-t)\|a-c\| + t\|b-c\| \\ &\leq (1-t)\epsilon + t\epsilon \\ &= \epsilon \end{aligned}$$

$\langle 1\rangle 3.$ p is a path from a to b.
□

Lemma 2.461. *If $\sum_{i=0}^{\infty} x_i^2$ and $\sum_{i=0}^{\infty} y_i^2$ converge then $\sum_{i=0}^{\infty} |x_i y_i|$ converges.*

PROOF:
$\langle 1\rangle 1.$ For all $N \geq 0$ we have $\sum_{i=0}^{N} |x_i y_i| \leq \sqrt{\sum_{i=0}^{N} |x_i|^2}\sqrt{\sum_{i=0}^{N} |y_i|^2}$
PROOF: By the Cauchy-Schwarz inequality
$\langle 1\rangle 2.$ Q.E.D.
PROOF: Since $\sum_{i=0}^{N} |x_i y_i|$ is an increasing sequence bounded above by $(\sum_{i=0}^{\infty} x_i^2)(\sum_{i=0}^{\infty} y_i^2)$.
□

Corollary 2.461.1. *If $\sum_{i=0}^{\infty} x_i^2$ and $\sum_{i=0}^{\infty} y_i^2$ converge then $\sum_{i=0}^{\infty}(x_i + y_i)^2$ converges.*

PROOF: Since $\sum_{i=0}^{\infty} x_i^2$, $\sum_{i=0}^{\infty} y_i^2$ and $2\sum_{i=0}^{\infty} x_i y_i$ all converge.

Definition 2.462 (l^2-metric)**.** The l^2-*metric* on

$$\left\{(x_n) \in \mathbb{R}^{\omega} \mid \sum_{n=0}^{\infty} x_n^2 \text{ converges}\right\}$$

is defined by

$$d(x,y) = \left(\sum_{n=0}^{\infty}(x_n - y_n)^2\right)^{1/2}$$

We prove this is a metric.

PROOF:
$\langle 1\rangle 1.$ d is well-defined.
PROOF: By Corollary 2.461.1.
$\langle 1\rangle 2.$ $d(x,y) \geq 0$
$\langle 1\rangle 3.$ $d(x,y) = 0$ if and only if $x = y$
$\langle 1\rangle 4.$ $d(x,y) = d(y,x)$
$\langle 1\rangle 5.$ $d(x,z) \leq d(x,y) + d(y,z)$
PROOF: By Lemma 2.456.
□

Theorem 2.463. *Addition is a continuous function $\mathbb{R}^2 \to \mathbb{R}$.*

PROOF:
$\langle 1\rangle 1.$ LET: $a, b \in \mathbb{R}$
$\langle 1\rangle 2.$ LET: $\epsilon > 0$
$\langle 1\rangle 3.$ LET: $\delta = \epsilon/2$
$\langle 1\rangle 4.$ LET: $x, y \in \mathbb{R}$
$\langle 1\rangle 5.$ ASSUME: $\rho((a,b),(x,y)) < \delta$
$\langle 1\rangle 6.$ $|(a+b) - (x+y)| < \epsilon$

PROOF:

$$\begin{aligned}|(a+b)-(x+y)| &= |a-x|+|b-y| \\ &\leq 2\rho((a,b),(x,y)) \\ &< 2\delta \\ &= \epsilon\end{aligned}$$

⟨1⟩7. Q.E.D.
PROOF: Theorem 2.440
□

Theorem 2.464. *Multiplication is a continuous function* $\mathbb{R}^2 \to \mathbb{R}$.

PROOF:
⟨1⟩1. LET: $a, b \in \mathbb{R}$
⟨1⟩2. LET: $\epsilon > 0$
⟨1⟩3. LET: $\delta = \min(\epsilon/(|a|+|b|+1), 1)$
⟨1⟩4. LET: $x, y \in \mathbb{R}$
⟨1⟩5. ASSUME: $\rho((a,b),(x,y)) < \delta$
⟨1⟩6. $|ab - xy| < \epsilon$
PROOF:

$$\begin{aligned}|ab-xy| &= |a(b-y)+(a-x)b-(a-x)(b-y)| \\ &\leq |a||b-y|+|b||a-x|+|a-x||b-y| \\ &< |a|\delta+|b|\delta+\delta^2 && (\langle 1\rangle 5) \\ &\leq |a|\delta+|b|\delta+\delta && (\langle 1\rangle 3) \\ &\leq \epsilon && (\langle 1\rangle 3)\end{aligned}$$

⟨1⟩7. Q.E.D.
PROOF: Theorem 2.440
□

Theorem 2.465. *The function* $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ *defined by* $f(x) = x^{-1}$ *is continuous.*

PROOF:
⟨1⟩1. For all $a \in \mathbb{R}$ we have $f^{-1}((a, +\infty))$ is open.
PROOF: The set is

$$\begin{aligned}(a^{-1}, +\infty) &\text{if } a > 0 \\ (0, +\infty) &\text{if } a = 0 \\ (-\infty, a^{-1}) \cup (0, +\infty) &\text{if } a < 0\end{aligned}$$

⟨1⟩2. For all $a \in \mathbb{R}$ we have $f^{-1}((-\infty, a))$ is open.
PROOF: Similar.
⟨1⟩3. Q.E.D.
PROOF: By Proposition 2.160 and Lemma 2.183.
□

Definition 2.466. For $n \geq 0$, the *unit ball* B^n is the space $\{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$.

Proposition 2.467. *For all* $n \geq 0$, *the unit ball* B^n *is path connected.*

PROOF:
$\langle 1 \rangle 1$. LET: $a, b \in B^n$
$\langle 1 \rangle 2$. LET: $p : [0,1] \to B^n$ be the function $p(t) = (1-t)a + tb$
PROOF: We have $p(t) \in B^n$ for all t because

$$\begin{aligned} \|(1-t)a + tb\| &\leq (1-t)\|a\| + t\|b\| \\ &\leq (1-t) + t \\ &= 1 \end{aligned}$$

$\langle 1 \rangle 3$. p is a path from a to b.
$\square$

Definition 2.468 (Punctured Euclidean Space)**.** For $n \geq 0$, defined *punctured Euclidean space* to be $\mathbb{R}^n \setminus \{0\}$.

Proposition 2.469. *For $n > 1$, punctured Euclidean space $\mathbb{R}^n \setminus \{0\}$ is path connected.*

PROOF:
$\langle 1 \rangle 1$. LET: $a, b \in \mathbb{R}^n \setminus \{0\}$
$\langle 1 \rangle 2$. CASE: 0 is on the line from a to b
$\langle 2 \rangle 1$. PICK a point c not on the line from a to b
$\langle 2 \rangle 2$. The path consisting of a straight line from a to c followed by a straight line from c to b is a path from a to b.
$\langle 1 \rangle 3$. CASE: 0 is not on the line from a to b
PROOF: The straight line from a to b is a path from a to b.

Corollary 2.469.1. *For $n > 1$, the spaces $\mathbb{R}$ and $\mathbb{R}^n$ are not homeomorphic.*

PROOF: For any point a, the space $\mathbb{R} \setminus \{a\}$ is disconnected.

Definition 2.470 (Unit Sphere)**.** For $n \geq 1$, the *unit sphere* S^{n-1} is the space

$$S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\} .$$

Proposition 2.471. *For $n > 1$, the unit sphere S^{n-1} is path connected.*

PROOF: The map $g : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$ defined by $g(x) = x/\|x\|$ is continuous and surjective. The result follows by Proposition 2.315. $\square$

Proposition 2.472. *Let $f : S^1 \to \mathbb{R}$ be continuous. Then there exists $x \in S^1$ such that $f(x) = f(-x)$.*

PROOF:
$\langle 1 \rangle 1$. LET: $g : S^1 \to \mathbb{R}$ be the function $g(x) = f(x) - f(-x)$
PROVE: There exists $x \in S^1$ such that $g(x) = 0$
$\langle 1 \rangle 2$. ASSUME: without loss of generality $g((1,0)) > 0$
$\langle 1 \rangle 3$. $g((-1,0)) < 0$
$\langle 1 \rangle 4$. There exists x such that $g(x) = 0$
PROOF: By the Intermediate Value Theorem.
$\square$

Definition 2.473 (Topologist's Sine Curve)**.** Let $S = \{(x, \sin 1/x) \mid 0 < x \leq 1\}$. The *topologist's sine curve* is the closure $\overline{S}$ of S.

Proposition 2.474.

$$\overline{S} = S \cup (\{0\} \times [-1, 1])$$

Proposition 2.475. *The topologist's sine curve is connected.*

PROOF:
$\langle 1 \rangle 1$. LET: $S = \{(x, \sin 1/x) \mid 0 < x \leq 1\}$
$\langle 1 \rangle 2$. S is connected.
  PROOF: Theorem 2.294.
$\langle 1 \rangle 3$. $\overline{S}$ is connected.
  PROOF: Theorem 2.293.
□

Proposition 2.476 (CC)**.** *The topologist's sine curve is not path connected.*

PROOF:
$\langle 1 \rangle 1$. ASSUME: for a contradiction $p : [0, 1] \to \overline{S}$ is a path from $(0, 0)$ to $(1, \sin 1)$.
$\langle 1 \rangle 2$. $p^{-1}(\{0\} \times [0, 1])$ is closed.
$\langle 1 \rangle 3$. LET: b be the greatest element of $p^{-1}(\{0\} \times [0, 1])$.
$\langle 1 \rangle 4$. $b < 1$
  PROOF: Since $p(1) = (1, \sin 1)$.
$\langle 1 \rangle 5$. PICK a sequence $(t_n)_{n \geq 1}$ in $(b, 1]$ such that $t_n \to b$ and $\pi_2(p(t_n)) = (-1)^n$
  $\langle 2 \rangle 1$. LET: $n \geq 1$
  $\langle 2 \rangle 2$. PICK u with $0 < u < \pi_1(p(1/n))$ such that $\sin(1/u) = (-1)^n$
  $\langle 2 \rangle 3$. PICK t_n such that $0 < t_n < 1/n$ and $\pi_1(p(t_n)) = u$
    PROOF: One exists by the Intermediate Value Theorem.
$\langle 1 \rangle 6$. Q.E.D.
  PROOF: This contradicts 2.175.
□

Theorem 2.477. *Let A be a subspace of $\mathbb{R}^n$. Then the following are equivalent:*

1. *A is compact.*
2. *A is closed and bounded under the Euclidean metric.*
3. *A is closed and bounded under the square metric.*

PROOF:
$\langle 1 \rangle 1$. $1 \Rightarrow 2$
  PROOF: By Corollary 2.384.1 and Proposition 2.445.
$\langle 1 \rangle 2$. $2 \Rightarrow 3$
  PROOF: If $d(x, y) \leq M$ for all $x, y \in A$ then $\rho(x, y) \leq M/\sqrt{2}$.
$\langle 1 \rangle 3$. $3 \Rightarrow 1$
  $\langle 2 \rangle 1$. ASSUME: A is closed and $\rho(x, y) \leq M$ for all $x, y \in A$
  $\langle 2 \rangle 2$. PICK $a \in A$

PROOF: We may assume w.l.o.g. A is nonempty since the empty space is compact.

$\langle 2\rangle 3.$ A is a closed subspace of $[a_1 - M, a_1 + M] \times \cdots \times [a_n - M, a_n + M]$

$\langle 2\rangle 4.$ A is compact

PROOF: Proposition 2.377.

□

Corollary 2.477.1. *The unit sphere S^{n-1} and the closed unit ball B^n are compact for any n.*

2.80 The Uniform Topology

Definition 2.478 (Uniform Metric)**.** Let J be a set. The *uniform metric* $\overline{\rho}$ on $\mathbb{R}^J$ is defined by

$$\overline{\rho}(a, b) = \sup_{j \in J} \overline{d}(a_j, b_j)$$

where $\overline{d}$ is the standard bounded metric on $\mathbb{R}$.

The *uniform topology* on $\mathbb{R}^J$ is the topology induced by the uniform metric.

We prove this is a metric.

PROOF:

$\langle 1\rangle 1.$ $\overline{\rho}(a, b) \geq 0$

PROOF: Immediate from definitions.

$\langle 1\rangle 2.$ $\overline{\rho}(a, b) = 0$ if and only if $a = b$

PROOF: Immediate from definitions.

$\langle 1\rangle 3.$ $\overline{\rho}(a, b) = \overline{\rho}(b, a)$

PROOF: Immediate from definitions.

$\langle 1\rangle 4.$ $\overline{\rho}(a, c) \leq \overline{\rho}(a, b) + \overline{\rho}(b, c)$

PROOF:

$$\begin{aligned}
\overline{\rho}(a, c) &= \sup_{j \in J} \overline{d}(a_j, c_j) \\
&\leq \sup_{j \in J} (\overline{d}(a_j, b_j) + \overline{d}(b_j, c_j)) \\
&\leq \sup_{j \in J} \overline{d}(a_j, b_j) + \sup_{j \in J} \overline{d}(b_j, c_j) \\
&= \overline{\rho}(a, b) + \overline{\rho}(b, c)
\end{aligned}$$

□

Proposition 2.479. *The uniform topology on $\mathbb{R}^J$ is finer than the product topology.*

PROOF:

$\langle 1\rangle 1.$ LET: $j \in J$ and U be open in $\mathbb{R}$

PROVE: ${\pi_j}^{-1}(U)$ is open in the uniform topology.

$\langle 1\rangle 2.$ LET: $a \in {\pi_j}^{-1}(U)$

$\langle 1\rangle 3.$ PICK $\epsilon > 0$ such that $(a_j - \epsilon, a_j + \epsilon) \subseteq U$

⟨1⟩4. $B_{\overline{\rho}}(a, \epsilon) \subseteq {\pi_j}^{-1}(U)$
□

Proposition 2.480. *The uniform topology on $\mathbb{R}^J$ is coarser than the box topology.*

PROOF:
⟨1⟩1. LET: $a \in \mathbb{R}^J$ and $\epsilon > 0$
PROVE: $B(a, \epsilon)$ is open in the box topology.
⟨1⟩2. LET: $b \in B(a, \epsilon)$
⟨1⟩3. For $j \in J$ we have $|a_j - b_j| < \epsilon$
⟨1⟩4. For $j \in J$,
LET: $\delta_j = (\epsilon - |a_j - b_j|)/2$
⟨1⟩5. $\prod_{j \in J}(b_j - \delta_j, b_j + \delta_j) \subseteq B(a, \epsilon)$
□

Proposition 2.481. *The uniform topology on $\mathbb{R}^J$ is strictly finer than the product topology if and only if J is infinite.*

PROOF:
⟨1⟩1. If J is finite then the uniform and product topologies coincide.
PROOF: The uniform, box and product topologies are all the same.
⟨1⟩2. If J is infinite then the uniform and product topologies are different.
PROOF: The set $B(\vec{0}, 1)$ is open in the uniform topology but not the product topology.
□

Proposition 2.482 (DC)**.** *The uniform topology on $\mathbb{R}^J$ is strictly coarser than the box topology if and only if J is infinite.*

PROOF:
⟨1⟩1. If J is finite then the uniform and box topologies coincide.
PROOF: The uniform, box and product topologies are all the same.
⟨1⟩2. If J is infinite then the uniform and box topologies are different.
PROOF: Pick an ω-sequence $(j_1, j_2, \ldots)$ in J. Let $U = \prod_{j \in J} U_j$ where $U_{j_i} = (-1/i, 1/i)$ and $U_j = (-1, 1)$ for all other j. Then $\vec{0} \in U$ but there is no $\epsilon > 0$ such that $B(\vec{0}, \epsilon) \subseteq U$.
□

Proposition 2.483. *The closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ under the uniform topology is $\mathbb{R}^\omega$.*

PROOF: Given any open ball $B(a, \epsilon)$, pick an integer N such that $1/\epsilon < N$. Then $B(a, \epsilon)$ includes sequences whose nth entry is 0 for all $n \geq N$. □

Example 2.484. The space $\mathbb{R}^\omega$ is disconnected under the uniform topology. The set of bounded sequences and the set of unbounded sequences form a separation.

Proposition 2.485. *Give $\mathbb{R}^\omega$ the uniform topology. Let $x, y \in \mathbb{R}^\omega$. Then x and y are in the same component if and only if $x - y$ is bounded.*

PROOF:
⟨1⟩1. The component containing 0 is the set of bounded sequences.
⟨2⟩1. LET: B be the set of bounded sequences.
⟨2⟩2. B is path-connected.
⟨3⟩1. LET: $x.y \in B$
⟨3⟩2. PICK $b > 0$ such that $|x_j|, |y_j| \leq b$ for all j
⟨3⟩3. LET: $p : [0,1] \to B$ be the function $p(t) = (1-t)x + ty$
PROVE: p is continuous.
⟨3⟩4. LET: $t \in [0,1]$ and $\epsilon > 0$
⟨3⟩5. LET: $\delta = \epsilon/2b$
⟨3⟩6. LET: $s \in [0,1]$ with $|s - t| < \delta$
⟨3⟩7. $\overline{\rho}(p(s), p(t)) < \epsilon$
PROOF:

$$\begin{aligned}
\overline{\rho}(p(s), p(t)) &= \sup_j \overline{d}((1-s)x_j + sy_j, (1-t)x_j + ty_j) \\
&\leq |(s-t)x_j + (t-s)y_j| \\
&\leq |s-t||x_j - y_j| \\
&< 2b\delta \\
&= \epsilon
\end{aligned}$$

⟨2⟩3. B is connected.
PROOF: Proposition 2.313.
⟨2⟩4. If C is connected and $B \subseteq C$ then $B = C$.
PROOF: Otherwise $B \cap C$ and $C \setminus B$ form a separation of C.
⟨1⟩2. Q.E.D.
PROOF: Since $\lambda x.x - y$ is a Homeomorphism of $\mathbb{R}^\omega$ with itself.
□

2.81 Uniform Convergence

Definition 2.486 (Uniform Convergence)**.** Let X be a set and Y a metric space. Let $(f_n : X \to Y)$ be a sequence of functions and $f : X \to Y$ be a function. Then f_n *converges uniformly* to f as $n \to \infty$ if and only if, for all $\epsilon > 0$, there exists N such that, for all $n \geq N$ and $x \in X$, we have $d(f_n(x), f(x)) < \epsilon$.

Example 2.487. Define $f_n : [0,1] \to \mathbb{R}$ by $f_n(x) = x^n$ for $n \geq 1$, and $f : [0,1] \to \mathbb{R}$ by $f(x) = 0$ if $x < 1$, $f(1) = 1$. Then f_n converges to f pointwise but not uniformly.

Theorem 2.488 (Uniform Limit Theorem)**.** *Let X be a topological space and Y a metric space. Let $(f_n : X \to Y)$ be a sequence of continuous functions and $f : X \to Y$ be a function. If f_n converges uniformly to f as $n \to \infty$, then f is continuous.*

PROOF:
⟨1⟩1. LET: $x \in X$ and $\epsilon > 0$
⟨1⟩2. PICK N such that, for all $n \geq N$ and $y \in X$, we have $d(f_n(y), f(y)) < \epsilon/3$
⟨1⟩3. PICK a neighbourhood U of x such that $f_N(U) \subseteq B(f_N(x), \epsilon/3)$
PROVE: $f(U) \subseteq B(f(x), \epsilon)$
⟨1⟩4. LET: $y \in U$
⟨1⟩5. $d(f(y), f(x)) < \epsilon$
PROOF:

$$\begin{aligned} d(f(y), f(x)) &\leq d(f(y), f_N(y)) + d(f_N(y), f_N(x)) + d(f_N(x), f(x)) && \text{(Triangle Inequality)} \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 && (\langle 1\rangle 2,\ \langle 1\rangle 3) \\ &= \epsilon \end{aligned}$$

□

Proposition 2.489. *Let X be a topological space and Y a metric space. Let $(f_n : X \to Y)$ be a sequence of continuous functions and $f : X \to Y$ be a function. Let (a_n) be a sequence of points in X and $a \in X$. If f_n converges uniformly to f and a_n converges to a in X then $f_n(a_n)$ converges to $f(a)$ uniformly in Y.*

PROOF:
⟨1⟩1. LET: $\epsilon > 0$
⟨1⟩2. PICK N_1 such that, for all $n \geq N_1$ and $x \in X$, we have $d(f_n(x), f(x)) < \epsilon/2$
⟨1⟩3. PICK N_2 such that, for all $n \geq N_2$, we have $a_n \in f^{-1}(B(a, \epsilon/2))$
PROOF: Using the fact that f is continuous from the Uniform Limit Theorem.
⟨1⟩4. LET: $N = \max(N_1, N_2)$
⟨1⟩5. LET: $n \geq N$
⟨1⟩6. $d(f_n(a_n), f(a)) < \epsilon$
PROOF:

$$\begin{aligned} d(f_n(a_n), f(a)) &\leq d(f_n(a_n), f(a_n)) + d(f(a_n), f(a)) && \text{(Triangle Inequality)} \\ &< \epsilon/2 + \epsilon/2 && (\langle 1\rangle 2,\ \langle 1\rangle 3) \\ &= \epsilon \end{aligned}$$

□

Proposition 2.490. *Let X be a set. Let $(f_n : X \to \mathbb{R})$ be a sequence of functions and $f : X \to \mathbb{R}$ be a function. Then f_n converges unifomly to f as $n \to \infty$ if and only if $f_n \to f$ as $n \to \infty$ in $\mathbb{R}^X$ under the uniform topology.*

PROOF:
⟨1⟩1. If f_n converges uniformly to f then f_n converges to f under the uniform topology.
⟨2⟩1. ASSUME: f_n converges uniformly to f
⟨2⟩2. LET: $\epsilon > 0$
⟨2⟩3. PICK N such that, for all $n \geq N$ and $x \in X$, we have $d(f_n(x), f(x)) < \epsilon/2$
⟨2⟩4. For all $n \geq N$ we have $\overline{\rho}(f_n, f) \leq \epsilon/2$

⟨2⟩5. For all $n \geq N$ we have $\overline{\rho}(f_n, f) < \epsilon$

⟨1⟩2. If f_n converges to f under the uniform topology then f_n converges uniformly to f.

⟨2⟩1. ASSUME: f_n converges to f under the uniform topology.

⟨2⟩2. LET: $\epsilon > 0$

⟨2⟩3. PICK N such that, for all $n \geq N$, we have $\overline{\rho}(f_n, f) < \min(\epsilon, 1/2)$

⟨2⟩4. LET: $n \geq N$

⟨2⟩5. LET: $x \in X$

⟨2⟩6. $\overline{\rho}(f_n, f) < \min(\epsilon, 1/2)$

PROOF: From ⟨2⟩3.

⟨2⟩7. $d(f_n(x), f(x)) < \min(\epsilon, 1/2)$

⟨2⟩8. $d(f_n(x), f(x)) < \epsilon$

□

2.82 Isometric Imbeddings

Definition 2.491. Let X and Y be metric spaces. An *isometric imbedding* $f : X \to Y$ is a function such that, for all $x, y \in X$, we have $d(f(x), f(y)) = d(x, y)$.

Proposition 2.492. *Every isometric imbedding is an imbedding.*

PROOF:

⟨1⟩1. LET: $f : X \to Y$ be an isometric imbedding.

⟨1⟩2. f is injective.

PROOF: If $f(x) = f(y)$ then $d(f(x), f(y)) = 0$ hence $d(x, y) = 0$ hence $x = y$.

⟨1⟩3. f is continuous.

PROOF: For all $\epsilon > 0$, if $d(x, y) < \epsilon$ then $d(f(x), f(y)) < \epsilon$.

⟨1⟩4. $f : X \to f(X)$ is an open map.

PROOF: $f(B(a, \epsilon)) = B(f(a), \epsilon) \cap f(X)$.

□

2.83 Distance to a Set

Definition 2.493. Let X be a metric space, $x \in X$ and $A \subseteq X$ be nonempty. The *distance* from x to A is defined as

$$d(x, A) = \inf_{a \in A} d(x, a) \ .$$