Topology

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1 Order Theory

Definition 1 (Preorder)**.** Let X be a set. A *preorder* on X is a binary relation $\leq$ on X such that:

Reflexivity For all $x \in X$, we have $x \leq x$

Transitivity For all $x, y, z \in X$, if $x \leq y$ and $y \leq z$ then $x \leq z$.

Definition 2 (Preordered Set)**.** A *preordered set* consists of a set X and a preorder $\leq$ on X.

Definition 3 (Interval)**.** Let X be a preordered set and $Y \subseteq X$. Then Y is an *interval* if and only if, for all $a, b \in Y$ and $c \in X$, if $a \leq c \leq b$ then $c \in Y$.

2 Real Analysis

Definition 4. Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences (a_n) such that $a_n = 0$ for all but finitely many n.

3 Topological Spaces

Definition 5 (Topology)**.** A *topology* on a set X is a set $\mathcal{T} \subseteq \mathcal{P}X$ such that:

- $X \in \mathcal{T}$.
- For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$.
- For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$.

We call the elements of X *points* and the elements of $\mathcal{T}$ *open sets*.

Definition 6 (Topological Space)**.** A *topological space* X consists of a set X and a topology on X.

Definition 7 (Discrete Space)**.** For any set X, the *discrete* topology on X is $\mathcal{P}X$.

Definition 8 (Indiscrete Space)**.** For any set X, the *indiscrete* or *trivial* topology on X is $\{\emptyset, X\}$.

Definition 9 (Finite Complement Topology)**.** For any set X, the *finite complement topology* on X is $\{U \in \mathcal{P}X \mid X \setminus U \text{ is finite}\} \cup \{\emptyset\}$.

Definition 10 (Countable Complement Topology)**.** For any set X, the *countable complement topology* on X is $\{U \in \mathcal{P}X \mid X \setminus U \text{ is countable}\} \cup \{\emptyset\}$.

Definition 11 (Finer, Coarser)**.** Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a given set X. If $\mathcal{T}' \supseteq \mathcal{T}$, we say that $\mathcal{T}'$ is *finer* than $\mathcal{T}$; if $\mathcal{T}'$ *properly* contains $\mathcal{T}$, we say that $\mathcal{T}'$ is *strictly* finer than $\mathcal{T}$. We also say that $\mathcal{T}$ is *coarser* than $\mathcal{T}'$, or *stricly* coarser, in these two respective situations. We say $\mathcal{T}$ is *comparable* with $\mathcal{T}'$ if either $\mathcal{T}' \supseteq \mathcal{T}$ or $\mathcal{T} \supseteq \mathcal{T}'$.

Lemma 12. *Let X be a topological space and $U \subseteq X$. Then U is open if and only if, for all $x \in U$, there exists an open set V such that $x \in V \subseteq U$.*

PROOF:
$\langle 1 \rangle 1.$ $\Rightarrow$
  PROOF: Take $V = U$
$\langle 1 \rangle 2.$ $\Leftarrow$
  PROOF: We have $U = \bigcup \{V \text{ open in } X \mid V \subseteq U\}$.
□

Lemma 13. *Let X be a set and $\mathcal{T}$ a nonempty set of topologies on X. Then $\bigcap \mathcal{T}$ is a topology on X, and is the finest topology that is coarser than every member of $\mathcal{T}$.*

PROOF:
$\langle 1 \rangle 1.$ $X \in \bigcap \mathcal{T}$
  PROOF: Since X is in every member of $\mathcal{T}$.
$\langle 1 \rangle 2.$ $\bigcap \mathcal{T}$ is closed under union.

⟨2⟩1. Let: $\mathcal{U} \subseteq \bigcap \mathcal{T}$
⟨2⟩2. For all $T \in \mathcal{T}$ we have $\mathcal{U} \subseteq T$
⟨2⟩3. For all $T \in \mathcal{T}$ we have $\bigcup \mathcal{U} \in T$
⟨2⟩4. $\bigcup \mathcal{U} \in \bigcap \mathcal{T}$
⟨1⟩3. $\bigcap \mathcal{T}$ is closed under binary intersection.
⟨2⟩1. Let: $U, V \in \bigcap \mathcal{T}$
⟨2⟩2. For all $T \in \mathcal{T}$ we have $U, V \in T$
⟨2⟩3. For all $T \in \mathcal{T}$ we have $U \cap V \in T$
⟨2⟩4. $U \cap V \in \bigcap \mathcal{T}$
□

Lemma 14. *Let X be a set and $\mathcal{T}$ a set of topologies on X. Then there exists a unique coarsest topology that is finer than every member of $\mathcal{T}$.*

Proof: The required topology is given by
$$\bigcap \{T \in \mathcal{PP}X \mid T \textit{ is a topology on } X \textit{ that is finer than every member of } \mathcal{T}\} ,$$
The set is nonempty since it contains the discrete topology. □

Definition 15 (Neighbourhood)**.** A *neighbourhood* of a point x is an open set that contains x.

4 Closed Set

Definition 16 (Closed Set)**.** Let X be a topological space and $A \subseteq X$. Then A is *closed* if and only if $X \setminus A$ is open.

Lemma 17. *The empty set is closed.*

Proof: Since the whole space X is always open. □

Lemma 18. *The topological space X is closed.*

Proof: Since $\emptyset$ is open. □

Lemma 19. *The intersection of a nonempty set of closed sets is closed.*

Proof: Let $\mathcal{C}$ be a nonempty set of closed sets. Then $X \setminus \bigcap \mathcal{C} = \bigcup \{X \setminus C \mid C \in \mathcal{C}\}$ is open. □

Lemma 20. *The union of two closed sets is closed.*

Proof: Let C and D be closed. Then $X \setminus (C \cup D) = (X \setminus C) \cap (X \setminus D)$ is open. □

Proposition 21. *Let X be a set and $\mathcal{C} \subseteq \mathcal{P}X$ a set such that:*

1. $\emptyset \in \mathcal{C}$
2. $X \in \mathcal{C}$
3. *For all $\mathcal{A} \subseteq \mathcal{C}$ nonempty we have $\bigcap \mathcal{A} \in \mathcal{C}$*

4. For all $C, D \in \mathcal{C}$ we have $C \cup D \in \mathcal{C}$.

Then there exists a unique topology $\mathcal{T}$ such that $\mathcal{C}$ is the set of closed sets, namely

$$\mathcal{T} = \{X \setminus C \mid C \in \mathcal{C}\}$$

PROOF:
$\langle 1\rangle 1$. LET: $\mathcal{T} = \{X \setminus C \mid C \in \mathcal{C}\}$
$\langle 1\rangle 2$. $\mathcal{T}$ is a topology
  $\langle 2\rangle 1$. $X \in \mathcal{T}$
    PROOF: Since $\emptyset \in \mathcal{C}$
  $\langle 2\rangle 2$. For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$
    $\langle 3\rangle 1$. LET: $\mathcal{U} \subseteq \mathcal{T}$
    $\langle 3\rangle 2$. CASE: $\mathcal{U} = \emptyset$
      PROOF: In this case $\bigcup \mathcal{U} = \emptyset \in \mathcal{T}$ since $X \in \mathcal{C}$
    $\langle 3\rangle 3$. CASE: $\mathcal{U} \neq \emptyset$
      PROOF: In this case $X \setminus \bigcup \mathcal{U} = \bigcap \{X \setminus U \mid U \in \mathcal{U}\} \in \mathcal{C}$.
  $\langle 2\rangle 3$. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$
    PROOF: Since $X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V) \in \mathcal{C}$.
$\langle 1\rangle 3$. $\mathcal{C}$ is the set of all closed sets in $\mathcal{T}$
  PROOF:

$$\begin{aligned} & C \text{ is closed in } \mathcal{T} \\ \Leftrightarrow & X \setminus C \in \mathcal{T} \\ \Leftrightarrow & C \in \mathcal{C} \end{aligned}$$

$\langle 1\rangle 4$. If $\mathcal{T}'$ is a topology and $\mathcal{C}$ is the set of closed sets in $\mathcal{T}'$ then $\mathcal{T}' = \mathcal{T}$
  PROOF: We have

$$\begin{aligned} & U \in \mathcal{T} \\ \Leftrightarrow & X \setminus U \in \mathcal{C} \\ \Leftrightarrow & X \setminus U \text{ is closed in } \mathcal{T}' \\ \Leftrightarrow & U \in \mathcal{T}' \end{aligned}$$

□

Proposition 22. *If U is open and A is closed then $U \setminus A$ is open.*

PROOF: $U \setminus A = U \cap (X \setminus A)$ is the intersection of two open sets. □

Proposition 23. *If U is open and A is closed then $A \setminus U$ is closed.*

PROOF: $A \setminus U = A \cap (X \setminus U)$ is the intersection of two closed sets. □

5 Interior

Definition 24 (Interior)**.** Let X be a topological space and $A \subseteq X$. The *interior* of A, $\operatorname{Int} A$, is the union of all the open subsets of A.

Lemma 25. *The interior of a set is open.*

PROOF: It is a union of open sets. □

Lemma 26.

$$\operatorname{Int} A \subseteq A$$

PROOF: Immediate from definition. □

Lemma 27. *If U is open and $U \subseteq A$ then $U \subseteq \operatorname{Int} A$*

PROOF: Immediate from definition. □

Lemma 28. *A set A is open if and only if $A = \operatorname{Int} A$.*

PROOF: If $A = \operatorname{Int} A$ then A is open by Lemma 25. Conversely if A is open then $A \subseteq \operatorname{Int} A$ by the definition of interior and so $A = \operatorname{Int} A$.

6 Closure

Definition 29 (Closure)**.** Let X be a topological space and $A \subseteq X$. The *closure* of A, $\overline{A}$, is the intersection of all the closed sets that include A.

This intersection exists since X is a closed set that includes A (Lemma 18).

Lemma 30. *The closure of a set is closed.*

PROOF: Dual to Lemma 25. □

Lemma 31.

$$A \subseteq \overline{A}$$

PROOF: Immediate from definition. □

Lemma 32. *If C is closed and $A \subseteq C$ then $\overline{A} \subseteq C$.*

PROOF: Immediate from definition. □

Lemma 33. *A set A is closed if and only if $A = \overline{A}$.*

PROOF: Dual to Lemma 28. □

Theorem 34. *Let X be a topological space, $A \subseteq X$ and $x \in X$. Then $x \in \overline{A}$ if and only if every neighbourhood of x intersects A.*

PROOF: We have

$$\begin{aligned}
& x \in \overline{A} \\
\Leftrightarrow & \forall C. C \text{ closed } \wedge A \subseteq C \Rightarrow x \in C \\
\Leftrightarrow & \forall U. U \text{ open } \wedge A \cap U = \emptyset \Rightarrow x \notin U \\
\Leftrightarrow & \forall U. U \text{ open } \wedge x \in U \Rightarrow U \text{ intersects } A
\end{aligned}$$

□

Proposition 35. *If $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$.*

PROOF: This holds because $\overline{B}$ is a closed set that includes A. $\square$

Proposition 36.

$$\overline{A \cup B} = \overline{A} \cup \overline{B}$$

PROOF:
⟨1⟩1. $\overline{A} \subseteq \overline{A \cup B}$
  PROOF: By Proposition 35.
⟨1⟩2. $\overline{B} \subseteq \overline{A \cup B}$
  PROOF: By Proposition 35.
⟨1⟩3. $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$
  ⟨2⟩1. LET: $x \in \overline{A \cup B}$
  ⟨2⟩2. ASSUME: $x \notin \overline{A}$
    PROVE: $x \in \overline{B}$
  ⟨2⟩3. PICK a neighbourhood U of x that does not intersect A
  ⟨2⟩4. LET: V be any neighbourhood of x
  ⟨2⟩5. $U \cap V$ is a neighbourhood of x
  ⟨2⟩6. $U \cap V$ intersects $A \cup B$
    PROOF: From ⟨2⟩1 and Theorem 34.
  ⟨2⟩7. $U \cap V$ intersects B
    PROOF: From ⟨2⟩3
  ⟨2⟩8. V intersects B
  ⟨2⟩9. Q.E.D.
    PROOF: We have $x \in \overline{B}$ from Theorem 34.

$\square$

7 Boundary

Definition 37 (Boundary)**.** The *boundary* of a set A is the set $\partial A = \overline{A} \cap \overline{X \setminus A}$.

Proposition 38.

$$\mathrm{Int}\, A \cap \partial A = \emptyset$$

PROOF: Since $\overline{X \setminus A} = X \setminus \mathrm{Int}\, A$. $\square$

Proposition 39.

$$\overline{A} = \mathrm{Int}\, A \cup \partial A$$

PROOF:

$$\begin{aligned}
\mathrm{Int}\, A \cup \partial A &= \mathrm{Int}\, A \cup (\overline{A} \cap \overline{X \setminus A}) \\
&= (\mathrm{Int}\, A \cup \overline{A}) \cap (\mathrm{Int}\, A \cup \overline{X \setminus A}) \\
&= \overline{A} \cap X \\
&= \overline{A}
\end{aligned}$$

Proposition 40. *$\partial A = \emptyset$ if and only if A is open and closed.*

PROOF: If $\partial A = \emptyset$ then $\overline{A} = \mathrm{Int}\, A$ by Proposition 39.

Proposition 41. *A set U is open if and only if $\partial U = \overline{U} \setminus U$.*

Proof:

$$\begin{aligned} & \partial U = \overline{U} \setminus U \\ \Leftrightarrow & \overline{U} \setminus \operatorname{Int} U = \overline{U} \setminus U && \text{(Propositions 38, 39)} \\ \Leftrightarrow & \operatorname{Int} U = U && \square \end{aligned}$$

8 Limit Points

Definition 42 (Limit Point)**.** Let X be a topological space, $a \in X$ and $A \subseteq X$. Then a is a *limit point*, *cluster point* or *point of accumulation* for A if and only if every neighbourhood of a intersects A at a point other than a.

Lemma 43. *The point a is an accumulation point for A if and only if $a \in \overline{A \setminus \{a\}}$.*

Proof: From Theorem 34. $\square$

Theorem 44. *Let X be a topological space and $A \subseteq X$. Let A' be the set of all limit points of A. Then $\overline{A} = A \cup A'$.*

Proof:
$\langle 1 \rangle 1.$ For all $x \in \overline{A}$, if $x \notin A$ then $x \in A'$
  Proof: From Theorem 34.
$\langle 1 \rangle 2.$ $A \subseteq \overline{A}$
  Proof: Lemma 31.
$\langle 1 \rangle 3.$ $A' \subseteq \overline{A}$
  Proof: From Theorem 34.
$\square$

Corollary 44.1. *A set is closed if and only if it contains all its limit points.*

Proposition 45. *In an indiscrete topology, every point is a limit point of any set that has more than one point.*

Proof: Let X be an indiscrete space. Let A be a set with more than one point and x be a point. The only neighbourhood of x is X, which must intersect A at a point other than x. $\square$

9 Basis for a Topology

Definition 46 (Basis)**.** If X is a set, a *basis* for a topology on X is a set $\mathcal{B} \subseteq \mathcal{P}X$ called *basis elements* such that

1. For all $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$.
2. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

If $\mathcal{B}$ satisfies these two conditions, then we define the topology *generated* by $\mathcal{B}$ to be $\mathcal{T} = \{U \in \mathcal{P}X \mid \forall x \in U . \exists B \in \mathcal{B} . x \in B \subseteq U\}$.

We prove this is a topology.

PROOF:
$\langle 1 \rangle 1.$ $X \in \mathcal{T}$
  PROOF: For all $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B \subseteq X$ by condition 1.
$\langle 1 \rangle 2.$ For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$
  $\langle 2 \rangle 1.$ LET: $\mathcal{U} \subseteq \mathcal{T}$
  $\langle 2 \rangle 2.$ LET: $x \in \bigcup \mathcal{U}$
  $\langle 2 \rangle 3.$ PICK $U \in \mathcal{U}$ such that $x \in U$
  $\langle 2 \rangle 4.$ PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$
    PROOF: Since $U \in \mathcal{T}$ by $\langle 2 \rangle 1$ and $\langle 2 \rangle 3$.
  $\langle 2 \rangle 5.$ $x \in B \subseteq \bigcup \mathcal{U}$
$\langle 1 \rangle 3.$ For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$
  $\langle 2 \rangle 1.$ LET: $U, V \in \mathcal{T}$
  $\langle 2 \rangle 2.$ LET: $x \in U \cap V$
  $\langle 2 \rangle 3.$ PICK $B_1 \in \mathcal{B}$ such that $x \in B_1 \subseteq U$
  $\langle 2 \rangle 4.$ PICK $B_2 \in \mathcal{B}$ such that $x \in B_2 \subseteq V$
  $\langle 2 \rangle 5.$ PICK $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$
    PROOF: By condition 2.
  $\langle 2 \rangle 6.$ $x \in B_3 \subseteq U \cap V$
$\square$

Lemma 47. *Let X be a set. Let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ is the set of all unions of subsets of $\mathcal{B}$.*

PROOF:
$\langle 1 \rangle 1.$ For all $U \in \mathcal{T}$, there exists $\mathcal{A} \subseteq \mathcal{B}$ such that $U = \bigcup \mathcal{A}$
  $\langle 2 \rangle 1.$ LET: $U \in \mathcal{T}$
  $\langle 2 \rangle 2.$ LET: $\mathcal{A} = \{B \in \mathcal{B} \mid B \subseteq U\}$
  $\langle 2 \rangle 3.$ $U \subseteq \bigcup \mathcal{A}$
    $\langle 3 \rangle 1.$ LET: $x \in U$
    $\langle 3 \rangle 2.$ PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$
      PROOF: Since $\mathcal{B}$ is a basis for $\mathcal{T}$.
    $\langle 3 \rangle 3.$ $x \in B \in \mathcal{A}$
  $\langle 2 \rangle 4.$ $\bigcup \mathcal{A} \subseteq U$
    PROOF: From the definition of $\mathcal{A}$ ($\langle 2 \rangle 2$).
$\langle 1 \rangle 2.$ For all $\mathcal{A} \subseteq \mathcal{B}$ we have $\bigcup \mathcal{A} \in \mathcal{T}$
  $\langle 2 \rangle 1.$ $\mathcal{B} \subseteq \mathcal{T}$
    PROOF: If $B \in \mathcal{B}$ and $x \in B$, then there exists $B' \in \mathcal{B}$ such that $x \in B' \subseteq B$, namely $B' = B$.
  $\langle 2 \rangle 2.$ Q.E.D.
    PROOF: Since $\mathcal{T}$ is closed under union.
$\square$

Corollary 47.1. *Let X be a set. Let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ is the coarsest topology that includes $\mathcal{B}$.*

PROOF: Since every topology that includes $\mathcal{B}$ includes all unions of subsets of $\mathcal{B}$. ∎

Lemma 48. *Let X be a topological space. Suppose that $\mathcal{C}$ is a set of open sets such that, for every open set U and every point $x \in U$, there exists $C \in \mathcal{C}$ such that $x \in C \subseteq U$. Then $\mathcal{C}$ is a basis for the topology on X.*

PROOF:
⟨1⟩1. For all $x \in X$, there exists $C \in \mathcal{C}$ such that $x \in C$
  PROOF: Immediate from hypothesis.
⟨1⟩2. For all $C_1, C_2 \in \mathcal{C}$ and $x \in C_1 \cap C_2$, there exists $C_3 \in \mathcal{C}$ such that $x \in C_3 \subseteq C_1 \cap C_2$
  PROOF: Since $C_1 \cap C_2$ is open.
⟨1⟩3. Every open set is open in the topology generated by $\mathcal{C}$
  PROOF: Immediate from hypothesis.
⟨1⟩4. Every union of a subset of $\mathcal{C}$ is open.
  PROOF: Since every member of $\mathcal{C}$ is open.
∎

Lemma 49. *Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$ respectively on the set X. Then the following are equivalent.*

1. $\mathcal{T} \subseteq \mathcal{T}'$
2. *For all $B \in \mathcal{B}$ and $x \in B$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.*

PROOF:
⟨1⟩1. $1 \Rightarrow 2$
  ⟨2⟩1. ASSUME: $\mathcal{T} \subseteq \mathcal{T}'$
  ⟨2⟩2. LET: $B \in \mathcal{B}$ and $x \in B$
  ⟨2⟩3. $B \in \mathcal{T}$
    PROOF: Corollary 47.1.
  ⟨2⟩4. $B \in \mathcal{T}'$
    PROOF: By ⟨2⟩1
  ⟨2⟩5. There exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$
    PROOF: Since $\mathcal{B}'$ is a basis for $\mathcal{T}'$.
⟨1⟩2. $2 \Rightarrow 1$
  ⟨2⟩1. ASSUME: 2
  ⟨2⟩2. LET: $U \in \mathcal{T}$
    PROVE: $U \in \mathcal{T}'$
  ⟨2⟩3. LET: $x \in U$
    PROVE: There exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq U$
  ⟨2⟩4. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$
    PROOF: Since $\mathcal{B}$ is a basis for $\mathcal{T}$.
  ⟨2⟩5. PICK $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$

PROOF: By $\langle 2 \rangle 1$.
$\langle 2 \rangle 6$. $x \in B' \subseteq U$
□

Theorem 50. *Let X be a topological space, $A \subseteq X$ and $x \in X$. Let $\mathcal{B}$ be a basis for X. Then $x \in \overline{A}$ if and only if, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A.*

PROOF:
$\langle 1 \rangle 1$. If $x \in \overline{A}$ then, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A.
PROOF: This follows from Theorem 34 since every element of $\mathcal{B}$ is open (Corollary 47.1).
$\langle 1 \rangle 2$. Suppose that, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A. Then $x \in \overline{A}$.
$\langle 2 \rangle 1$. ASSUME: For all $B \in \mathcal{B}$, if $x \in B$ then B intersects A.
$\langle 2 \rangle 2$. LET: U be an open set that contains x
PROVE: U intersects A.
$\langle 2 \rangle 3$. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$.
$\langle 2 \rangle 4$. B intersects A.
PROOF: From $\langle 2 \rangle 1$.
$\langle 2 \rangle 5$. U intersects A.
$\langle 2 \rangle 6$. Q.E.D.
PROOF: By Theorem 34.
□

Definition 51 (Lower Limit Topology on the Real Line)**.** The *lower limit topology on the real line* is the topology on $\mathbb{R}$ generated by the basis consisting of all half-open intervals of the form $[a, b)$.

We write $\mathbb{R}_l$ for the topological space $\mathbb{R}$ under the lower limit topology.

We prove this is a basis for a topology.

PROOF:
$\langle 1 \rangle 1$. For all $x \in \mathbb{R}$ there exists an interval $[a, b)$ such that $x \in [a, b)$.
PROOF: Take $[a, b) = [x, x + 1)$.
$\langle 1 \rangle 2$. For any open intervals $[a, b)$, $[c, d)$ if $x \in [a, b) \cap [c, d)$, then there exists an interval $[e, f)$ such that $x \in [e, f) \subseteq [a, b) \cap [c, d)$
PROOF: Take $[e, f) = [\max(a, c), \min(b, d))$.
□

Definition 52 (K-topology on the Real Line)**.** Let $K = \{1/n \mid n \in \mathbb{Z}^+\}$.

The *K-topology on the real line* is the topology on $\mathbb{R}$ generated by the basis consisting of all open intervals (a, b) and all sets of the form $(a, b) \setminus K$.

We write $\mathbb{R}_K$ for the topological space $\mathbb{R}$ under the K-topology.

We prove this is a basis for a topology.

PROOF:
$\langle 1 \rangle 1$. For all $x \in \mathbb{R}$ there exists an open interval (a, b) such that $x \in (a, b)$.
PROOF: Take $(a, b) = (x - 1, x + 1)$.
$\langle 1 \rangle 2$. For any basic open sets B_1, B_2 if $x \in B_1 \cap B_2$, then there exists a basic open set B_3 such that $x \in B_3 \subseteq B_1 \cap B_2$

⟨2⟩1. CASE: $B_1 = (a, b)$, $B_2 = (c, d)$
PROOF: Take $B_3 = (\max(a, c), \min(b, d))$.
⟨2⟩2. CASE: $B_1 = (a, b)$ or $(a, b) \setminus K$, $B_2 = (c, d)$ or $(c, d) \setminus K$, and they are not both open intervals.
PROOF: Take $B_3 = (\max(a, c), \min(b, d)) \setminus K$.
□

Lemma 53. *The lower limit topology and the K-topology are incomparable.*

PROOF:
⟨1⟩1. The interval $[10, 11)$ is not open in the K-topology.
PROOF: There is no open interval (a, b) such that $10 \in (a, b) \subseteq [10, 11)$ or $10 \in (a, b) \setminus K \subseteq [10, 11)$.
⟨1⟩2. The set $(-1, 1) \setminus K$ is not open in the lower limit topology.
PROOF: There is no half-open interval $[a, b)$ such that $0 \in [a, b) \subseteq (-1, 1) \setminus K$, since there must be a positive integer n with $1/n \in [a, b)$.
□

Definition 54 (Subbasis)**.** A *subbasis* $\mathcal{S}$ for a topology on X is a set $\mathcal{S} \subseteq \mathcal{P}X$ such that $\bigcup \mathcal{S} = X$.

The topology *generated* by the subbasis $\mathcal{S}$ is the set of all unions of finite intersections of elements of $\mathcal{S}$.

We prove this is a topology.

PROOF:
⟨1⟩1. The set $\mathcal{B}$ of all finite intersections of elements of $\mathcal{S}$ forms a basis for a topology on X.
⟨2⟩1. $\bigcup \mathcal{B} = X$
PROOF: Since $\mathcal{S} \subseteq \mathcal{B}$.
⟨2⟩2. $\mathcal{B}$ is closed under binary intersection.
PROOF: By definition.
⟨1⟩2. Q.E.D.
PROOF: By Lemma 47.
□

We have simultaneously proved:

Proposition 55. *Let $\mathcal{S}$ be a subbasis for the topology on X. Then the set of all finite intersections of elements of $\mathcal{S}$ is a basis for the topology on X.*

Proposition 56. *Let X be a set. Let $\mathcal{S}$ be a subbasis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ is the coarsest topology that includes $\mathcal{S}$.*

PROOF: Since every topology that includes $\mathcal{S}$ includes every union of finite intersections of elements of $\mathcal{S}$. □

10 Local Basis at a Point

Definition 57 (Local Basis)**.** Let X be a topological space and $a \in X$. A *(local) basis at a* is a set $\mathcal{B}$ of neighbourhoods of a such that every neighbourhood of a

includes some member of $\mathcal{B}$.

Lemma 58. *If there exists a countable local basis at a point a, then there exists a countable local basis $\{B_n \mid n \geq 1\}$ such that $B_1 \supseteq B_2 \supseteq \cdots$.*

PROOF: Given a countable local basis $\{C_n \mid n \geq 1\}$, take $B_n = C_1 \cap \cdots \cap C_n$. ∎

11 Convergence

Definition 59 (Convergence)**.** Let X be a topological space. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of points in X and $l \in X$. Then the sequence $(a_n)_{n \in \mathbb{N}}$ *converges* to the *limit* l, $a_n \to l$ as $n \to \infty$, if and only if, for every neighbourhood U of l, there exists N such that, for all $n \geq N$, we have $a_n \in U$.

Theorem 60. *In a Hausdorff space, a sequence has at most one limit.*

PROOF:
$\langle 1 \rangle 1$. LET: X be a Hausdorff space.
$\langle 1 \rangle 2$. ASSUME: for a contradiction $a_n \to l$ as $n \to \infty$, $a_n \to m$ as $n \to \infty$, and $l \neq m$
$\langle 1 \rangle 3$. PICK disjoint neighbourhoods U of l and V of m
PROOF: By the Hausdorff axiom.
$\langle 1 \rangle 4$. PICK M and N such that $a_n \in U$ for $n \geq M$ and $a_n \in V$ for $n \geq N$
$\langle 1 \rangle 5$. $a_{\max(M,N)} \in U \cap V$
$\langle 1 \rangle 6$. Q.E.D.
PROOF: This contradicts the fact that U and V are disjoint ($\langle 1 \rangle 3$).
∎

To see this is not always true in spaces that are T_1 but not Hausdorff:

Proposition 61. *Let X be an infinite set under the finite complement topology. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence with all points distinct. Then for every $l \in X$ we have $a_n \to l$ as $n \to \infty$.*

PROOF: Let U be any neighbourhood of l. Since $X \setminus U$ is finite, there must exist N such that, for all $n \geq N$, we have $a_n \in U$. ∎

Lemma 62. *Let X be a topological space. Let $A \subseteq X$ and $l \in X$. If there is a sequence of points in A that converges to l then $l \in \overline{A}$.*

PROOF:
$\langle 1 \rangle 1$. LET: (a_n) be a sequence of points in A that converges to l.
$\langle 1 \rangle 2$. LET: U be a neighbourhood of l.
$\langle 1 \rangle 3$. PICK N such that, for all $n \geq N$, we have $a_n \in U$.
$\langle 1 \rangle 4$. $a_N \in U \cap A$
$\langle 1 \rangle 5$. Q.E.D.
PROOF: Theorem 34.
∎

Proposition 63. *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Let (a_n) be a sequence in X and $l \in X$. Then $a_n \to l$ as $n \to \infty$ if and only if, for every $B \in \mathcal{B}$ with $l \in B$, there exists N such that, for all $n \geq N$, we have $a_n \in B$.*

PROOF:
$\langle 1 \rangle 1$. If $a_n \to l$ as $n \to \infty$ then, for every $B \in \mathcal{B}$ with $l \in B$, there exists N such that, for all $n \geq N$, we have $a_n \in B$.
PROOF: Since every element of $\mathcal{B}$ is open (Corollary 47.1).
$\langle 1 \rangle 2$. If, for every $B \in \mathcal{B}$ with $l \in B$, there exists N such that, for all $n \geq N$, we have $a_n \in B$, then $a_n \to l$ as $n \to \infty$.
$\langle 2 \rangle 1$. ASSUME: for every $B \in \mathcal{B}$ with $l \in B$, there exists N such that, for all $n \geq N$, we have $a_n \in B$
$\langle 2 \rangle 2$. LET: U be a neighbourhood of l.
$\langle 2 \rangle 3$. PICK $B \in \mathcal{B}$ such that $l \in B \subseteq U$
$\langle 2 \rangle 4$. PICK N such that, for all $n \geq N$, we have $a_n \in B$
PROOF: From $\langle 2 \rangle 1$.
$\langle 2 \rangle 5$. For all $n \geq N$ we have $a_n \in U$
$\square$

Lemma 64. *If a sequence (a_n) is constant with $a_n = l$ for all n, then $a_n \to l$ as $n \to \infty$.*

PROOF: Immediate from definitions. $\square$

Theorem 65. *Let X be a linearly ordered set. Let (s_n) be a monotone increasing sequence in X with a supremum s. Then $s_n \to s$ as $n \to \infty$.*

PROOF:
$\langle 1 \rangle 1$. ASSUME: s is not least in X.
PROOF: Otherwise (s_n) is the constant sequence s and the result follows from Lemma 64.
$\langle 1 \rangle 2$. LET: U be a neighbourhood of s.
$\langle 1 \rangle 3$. PICK$a < s$ such that $(a, s] \subseteq U$
$\langle 1 \rangle 4$. PICK N such that $a < a_N$.
$\langle 1 \rangle 5$. For all $n \geq N$ we have $a_n \in (a, s]$
$\langle 1 \rangle 6$. For all $n \geq N$ we have $a_n \in U$.
$\square$

Theorem 66. *If $\sum_{i=0}^{\infty} a_i = s$ and $\sum_{i=0}^{\infty} b_i = t$ then $\sum_{i=0}^{\infty}(ca_i + b_i) = cs + t$.*

PROOF:$\sum_{i=0}^{N}(ca_i + b_i) = c\sum_{i=0}^{N} a_i + \sum_{i=0}^{N} b_i \to cs + t$ as $n \to \infty$. $\square$

Theorem 67 (Comparison Test)**.** *If $|a_i| \leq b_i$ for all i and $\sum_{i=0}^{\infty} b_i$ converges then $\sum_{i=0}^{\infty} a_i$ converges.*

PROOF:
$\langle 1 \rangle 1$. $\sum_{i=0}^{\infty} |a_i|$ converges

PROOF: The partial sums $\sum_{i=0}^{N} |a_i|$ form an increasing sequence bounded above by $\sum_{i=0}^{\infty} b_i$.

$\langle 1 \rangle 2$. LET: $c_i = |a_i| + a_i$ for all i

$\langle 1 \rangle 3$. $\sum_{i=0}^{\infty} c_i$ converges

PROOF: Each c_i is either $2|a_i|$ or 0. So the partial sums $\sum_{i=0}^{N} c_i$ form an increasing sequence bounded above by $2 \sum_{i=0}^{\infty} b_i$.

$\langle 1 \rangle 4$. Q.E.D.

PROOF: Since $a_i = c_i - |a_i|$.

□

Corollary 67.1. *If $\sum_{i=0}^{\infty} |a_i|$ converges then $\sum_{i=0}^{\infty} a_i$ converges.*

Theorem 68 (Weierstrass M-test)**.** *Let X be a set and $(f_n : X \to \mathbb{R})$ be a sequence of functions. Let*

$$s_n(x) = \sum_{i=0}^{n} f_i(x)$$

for all n, x. Suppose $|f_i(x)| \leq M_i$ for all $i \geq 0$ and $x \in X$. If the series $\sum_{i=0}^{\infty} M_i$ converges, then the sequence (s_n) converges uniformly to

$$s(x) = \sum_{i=0}^{\infty} f_i(x) \quad .$$

PROOF:

$\langle 1 \rangle 1$. LET: $r_n = \sum_{i=n+1}^{\infty} M_i$ for all n

$\langle 1 \rangle 2$. Given $0 \leq n < k$, we have $|s_k(x) - s_n(x)| \leq r_n$

PROOF:

$$\begin{aligned} |s_k(x) - s_n(x)| &= |\sum_{i=n+1}^{k} f_i(x)| \\ &\leq \sum_{i=n+1}^{k} |f_i(x)| \\ &\leq \sum_{i=n+1}^{k} M_i \\ &\leq r_n \end{aligned}$$

$\langle 1 \rangle 3$. Given $n \geq 0$ we have $|s(x) - s_n(x)| \leq r_n$

PROOF: By taking the limit $k \to \infty$ in $\langle 1 \rangle 2$.

$\langle 1 \rangle 4$. Q.E.D.

PROOF: Since $r_n \to 0$ as $n \to \infty$.

□

12 Locally Finite Sets

Definition 69 (Locally Finite)**.** Let X be a topological space and $\{A_\alpha\}$ a family of subsets of X. Then $\mathcal{A}$ is *locally finite* if and only if every point in X has a neighbourhood that intersects A_α for only finitely many α.

Theorem 70 (Pasting Lemma)**.** *Let X and Y be topological spaces and $f : X \to Y$. Let $\{A_\alpha\}$ be a locally finite family of closed subsets of X that cover X. Suppose $f \upharpoonright A_\alpha$ is continuous for all α. Then f is continuous.*

PROOF:
$\langle 1\rangle 1$. Let X and Y be topological spaces and $f : X \to Y$. Let A and B be closed subsets of X such that $X = A \cup B$. Suppose $f \upharpoonright A$ and $f \upharpoonright B$ are continuous. Then f is continuous.
  $\langle 2\rangle 1$. LET: $C \subseteq Y$ be closed.
  $\langle 2\rangle 2$. $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$
  $\langle 2\rangle 3$. $f^{-1}(C)$ and $g^{-1}(C)$ are closed in X.
    PROOF: Theorems 80 and 126.
  $\langle 2\rangle 4$. $h^{-1}(C)$ is closed in X.
    PROOF: Lemma 20.
  $\langle 2\rangle 5$. Q.E.D.
    PROOF: Theorem 80.
$\langle 1\rangle 2$. Let X and Y be topological spaces and $f : X \to Y$. Let $\{A_\alpha\}$ be a finite family of closed subsets of X that cover X. Suppose $f \upharpoonright A_\alpha$ is continuous for all α. Then f is continuous.
  PROOF: From $\langle 1\rangle 1$ by induction.
$\langle 1\rangle 3$. Let X and Y be topological spaces and $f : X \to Y$. Let $\{A_\alpha\}$ be a locally finite family of closed subsets of X that cover X. Suppose $f \upharpoonright A_\alpha$ is continuous for all α. Then f is continuous.
  $\langle 2\rangle 1$. LET: $x \in X$
    PROVE: f is continuous at x
  $\langle 2\rangle 2$. PICK a neighbourhood U of x that intersects A_α for only finitely many α.
  $\langle 2\rangle 3$. $f \upharpoonright U$ is continuous
    PROOF: By $\langle 1\rangle 2$.
  $\langle 2\rangle 4$. Q.E.D.
    PROOF: Lemma 92.
□

The following example shows that we cannot remove the assumption of local finiteness.

Example 71. Define $f : [-1, 1] \to \mathbb{R}$ by: $f(x) = 1$ if $x < -1$, $f(x) = 0$ if $x > 1$. Let $C_n = [-1, -1/n]$ for $n \geq 1$, and $D = [0, 1]$. Then $[-1, 1] = \bigcup_{n=1}^{\infty} C_n \cup D$ and f is continuous on each C_n and each D, but f is not continuous on $[-1, 1]$.

Proposition 72. *Let X be a topological space. Let Y be a linearly ordered set in the order topology. Let $f, g : X \to Y$ be continuous. Define $h : X \to Y$ by $h(x) = \min(f(x), g(x))$. Then h is continuous.*

PROOF: By the Pasting Lemma applied to $\{x \in X \mid f(x) \leq g(x)\}$ and $\{x \in X \mid g(x) \leq f(x)\}$, which are closed by Lemma 91.

13 Open Maps

Definition 73 (Open Map)**.** Let X and Y be topological spaces and $f : X \to Y$. Then f is an *open map* if and only if, for every open set U in X, the set $f(U)$ is open in Y.

Lemma 74. *Let X and Y be topological spaces and $f : X \to Y$. Let $\mathcal{B}$ be a basis for the topology on X. If $f(B)$ is open in Y for all $B \in \mathcal{B}$, then f is an open map.*

Proof: From Lemma 47. □

14 Continuous Functions

Definition 75 (Continuous)**.** Let X and Y be topological spaces. A function $f : X \to Y$ is *continuous* if and only if, for every open set V in Y, the set $f^{-1}(V)$ is open in X.

Proposition 76. *Let X and Y be topological spaces and $f : X \to Y$. Let $\mathcal{B}$ be a basis for Y. Then f is continuous if and only if, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X.*

Proof:
⟨1⟩1. If f is continuous then, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X.
  Proof: Since every element of B is open (Lemma 47).
⟨1⟩2. Suppose that, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X. Then f is continuous.
  ⟨2⟩1. Assume: For all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X.
  ⟨2⟩2. Let: V be open in Y.
  ⟨2⟩3. Pick $\mathcal{A} \subseteq \mathcal{B}$ such that $V = \bigcup \mathcal{A}$
    Proof: By Lemma 47.
  ⟨2⟩4. $f^{-1}(V)$ is open in X.
    Proof:

$$\begin{aligned} f^{-1}(V) &= f^{-1}\left(\bigcup \mathcal{A}\right) \\ &= \bigcup_{B \in \mathcal{A}} f^{-1}(B) \end{aligned}$$

□

Proposition 77. *Let X and Y be topological spaces and $f : X \to Y$. Let $\mathcal{S}$ be a subbasis for Y. Then f is continuous if and only if, for all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in X.*

Proof:
⟨1⟩1. If f is continuous then, for all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in X.
  Proof: Since every element of S is open.
⟨1⟩2. Suppose that, for all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in X. Then f is continuous.

⟨2⟩1. Assume: For all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in X.
⟨2⟩2. Let: $S_1, \ldots, S_n \in \mathcal{S}$
⟨2⟩3. $f^{-1}(S_1 \cap \cdots \cap S_n)$ is open in A
Proof: Since $f^{-1}(S_1 \cap \cdots \cap S_n) = f^{-1}(S_1) \cap \cdots \cap f^{-1}(S_n)$.
⟨2⟩4. Q.E.D.
Proof: By Propositions 76 and 55.
□

Proposition 78. *Let X and Y be topological spaces and $f : X \to Y$. Let $\mathcal{S}$ be a basis for Y. Then f is continuous if and only if, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X.*

Proof:
⟨1⟩1. If f is continuous then, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X.
Proof: Since every element of $\mathcal{S}$ is open.
⟨1⟩2. Suppose that, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X. Then f is continuous.
⟨2⟩1. Assume: For all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X.
⟨2⟩2. For every set B that is the finite intersection of elemets of $\mathcal{S}$, we have $f^{-1}(B)$ is open in X.
Proof: Because $f^{-1}(V_1 \cap \cdots \cap V_n) = f^{-1}(V_1) \cap \cdots \cap f^{-1}(V_n)$.
⟨2⟩3. Q.E.D.
Proof: From Propositions 55 and 76.
□

Definition 79 (Continuous at a Point)**.** Let X and Y be topological spaces. Let $f : X \to Y$ and $x \in X$. Then f is *continuous at x* if and only if, for every neighbourhood V of $f(x)$, there exists a neighbourhood U of x such that $f(U) \subseteq V$.

Theorem 80. *Let X and Y be topological spaces and $f : X \to Y$. Then the following are equivalent:*

1. *f is continuous.*
2. *For all $A \subseteq X$, we have $f(\overline{A}) \subseteq \overline{f(A)}$*
3. *For all $B \subseteq Y$ closed, we have $f^{-1}(B)$ is closed in X.*
4. *f is continuous at every point of X.*

Proof:
⟨1⟩1. $1 \Rightarrow 2$
⟨2⟩1. Assume: f is continuous.
⟨2⟩2. Let: $A \subseteq X$
⟨2⟩3. Let: $x \in \overline{A}$
Prove: $f(x) \in \overline{f(A)}$
⟨2⟩4. Let: V be a neighbourhood of $f(x)$
⟨2⟩5. $f^{-1}(V)$ is a neighbourhood of x

$\langle 2\rangle 6$. PICK $y \in A \cap f^{-1}(V)$

PROOF: By Theorem 34.

$\langle 2\rangle 7$. $f(y) \in V \cap f(A)$

$\langle 2\rangle 8$. Q.E.D.

PROOF: By Theorem 34.

$\langle 1\rangle 2$. $2 \Rightarrow 3$

$\langle 2\rangle 1$. ASSUME: 2

$\langle 2\rangle 2$. LET: B be closed in Y

$\langle 2\rangle 3$. LET: $x \in \overline{f^{-1}(B)}$

PROVE: $x \in f^{-1}(B)$

$\langle 2\rangle 4$. $f(x) \in B$

PROOF:

$$\begin{aligned} f(x) &\in f(\overline{f^{-1}(B)}) \\ &\subseteq \overline{f(f^{-1}(B))} && (\langle 2\rangle 1) \\ &\subseteq \overline{B} && (Proposition 35) \\ &= B \end{aligned}$$

$\langle 1\rangle 3$. $3 \Rightarrow 1$

$\langle 2\rangle 1$. ASSUME: 3

$\langle 2\rangle 2$. LET: V be open in Y

$\langle 2\rangle 3$. $Y \setminus V$ is closed in Y

$\langle 2\rangle 4$. $f^{-1}(Y \setminus V)$ is closed in X

$\langle 2\rangle 5$. $X \setminus f^{-1}(V)$ is closed in X

$\langle 2\rangle 6$. $f^{-1}(V)$ is open in X

$\langle 1\rangle 4$. $1 \Rightarrow 4$

PROOF: For any neighbourhood V of $f(x)$, the set $U = f^{-1}(V)$ is a neighbourhood of x such that $f(U) \subseteq V$.

$\langle 1\rangle 5$. $4 \Rightarrow 1$

$\langle 2\rangle 1$. ASSUME: 4

$\langle 2\rangle 2$. LET: V be open in Y

$\langle 2\rangle 3$. LET: $x \in f^{-1}(V)$

$\langle 2\rangle 4$. V is a neighbourhood of $f(x)$

$\langle 2\rangle 5$. PICK a neighbourhood U of x such that $f(U) \subseteq V$

$\langle 2\rangle 6$. $x \in U \subseteq f^{-1}(V)$

$\langle 2\rangle 7$. Q.E.D.

PROOF: By Lemma 12.

□

Theorem 81. *A constant function is continuous.*

PROOF: Let X and Y be topological spaces. Let $b \in Y$, and let $f : X \to Y$ be the constant function with value b. For any open $V \subseteq Y$, the set $f^{-1}(V)$ is either X (if $b \in V$) or $\emptyset$ (if $b \notin V$). □

Theorem 82. *If A is a subspace of X then the inclusion $j : A \to X$ is continuous.*

PROOF: For any V open in X, we have $j^{-1}(V) = V \cap A$ is open in A. □

Theorem 83. *The composite of two continuous functions is continuous.*

PROOF: Let $f : X \to Y$ and $g : Y \to Z$ be continuous. For any V open in Z, we have $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is open in X. □

Theorem 84. *Let $f : X \to Y$ be a continuous function and A be a subspace of X. Then the restriction $f \restriction A : A \to Y$ is continuous.*

PROOF: Let V be open in Y. Then $(f \restriction A)^{-1}(V) = f^{-1}(V) \cap A$ is open in A. □

Theorem 85. *Let $f : X \to Y$ be continuous. Let Z be a subspace of Y such that $f(X) \subseteq Z$. Then the corestriction $f : X \to Z$ is continuous.*

PROOF:
⟨1⟩1. LET: V be open in Z.
⟨1⟩2. PICK U open in Y such that $V = U \cap Z$.
⟨1⟩3. $f^{-1}(V) = f^{-1}(U)$
⟨1⟩4. $f^{-1}(V)$ is open in X.
□

Theorem 86. *Let $f : X \to Y$ be continuous. Let Z be a space such that Y is a subspace of Z. Then the expansion $f : X \to Z$ is continuous.*

PROOF: Let V be open in Z. Then $f^{-1}(V) = f^{-1}(V \cap Y)$ is open in X. □

Theorem 87. *Let X and Y be topological spaces. Let $f : X \to Y$. Suppose $\mathcal{U}$ is a set of open sets in X such that $X = \bigcup \mathcal{U}$ and, for all $U \in \mathcal{U}$, we have $f \restriction U : U \to Y$ is continuous. Then f is continuous.*

PROOF:
⟨1⟩1. LET: V be open in Y
⟨1⟩2. $f^{-1}(V) = \bigcup_{U \in \mathcal{U}} (f \restriction U)^{-1}(V)$
⟨1⟩3. For all $U \in \mathcal{U}$, we have $(f \restriction U)^{-1}(V)$ is open in U.
⟨1⟩4. For all $U \in \mathcal{U}$, we have $(f \restriction U)^{-1}(V)$ is open in X.
PROOF: Lemma 125.
□

Theorem 88. *Let A be a topological space and $\{X_i\}_{i \in I}$ be a family of topological spaces. Let $f : A \to \prod_{i \in I} X_i$ be a function. If $\pi_i \circ f$ is continuous for all $i \in I$ then f is continuous.*

PROOF:
⟨1⟩1. LET: $i \in I$ and U be open in X_i
⟨1⟩2. $f^{-1}({\pi_i}^{-1}(U))$ is open in A
⟨1⟩3. Q.E.D.
PROOF: Proposition 77.
□

Proposition 89. *Let X and X' be the same set X under two topologies $\mathcal{T}$ and $\mathcal{T}'$. Let $i : X \to X'$ be the identity function. Then i is continuous if and only if $\mathcal{T}' \subseteq \mathcal{T}$.*

PROOF: Immediate from definitions. □

Proposition 90. *Let $f : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$. Then f is continuous on the right at a if and only if f is continuous at a as a function $\mathbb{R}_l \to \mathbb{R}$.*

PROOF:
⟨1⟩1. If f is continuous on the right at a then f is continuous at a as a function $\mathbb{R}_l \to \mathbb{R}$.
⟨2⟩1. ASSUME: f is continuous on the right at a.
⟨2⟩2. LET: V be a neighbourhood of $f(a)$
⟨2⟩3. PICK b, c such that $f(a) \in (b, c) \subseteq V$.
⟨2⟩4. LET: $\epsilon = \min(c - f(a), f(a) - b)$
⟨2⟩5. PICK $\delta > 0$ such that, for all x, if $a < x < a + \delta$ then $|f(x) - f(a)| < \epsilon$
⟨2⟩6. LET: $U = [a, a + \delta)$
⟨2⟩7. $f(U) \subseteq V$
⟨1⟩2. If f is continuous at a as a function $\mathbb{R}_l \to \mathbb{R}$ then f is continuous on the right at a.
⟨2⟩1. ASSUME: f is continuous at a as a function $\mathbb{R}_l \to \mathbb{R}$
⟨2⟩2. LET: $\epsilon > 0$
⟨2⟩3. PICK a neighbourhood U of a such that $f(U) \subseteq (f(a) - \epsilon, f(a) + \epsilon)$
⟨2⟩4. PICK b, c such that $a \in [b, c) \subset U$
⟨2⟩5. LET: $\delta = c - a$
⟨2⟩6. For all x, if $a < x < a + \delta$ then $|f(x) - f(a)| < \epsilon$
□

Lemma 91. *Let X be a topological space. Let Y be a linearly ordered set in the order topology. Let $f, g : X \to Y$ be continuous. Then $C = \{x \in X \mid f(x) \leq g(x)\}$ is closed.*

PROOF:
⟨1⟩1. LET: $x \in X \setminus C$
⟨1⟩2. $f(x) > g(x)$
PROVE: There exists a neighbourhood U of x such that $U \subseteq X \setminus C$
⟨1⟩3. CASE: There exists y such that $g(x) < y < f(x)$
PROOF: Take $U = g^{-1}((-\infty, y)) \cup f^{-1}(y, +\infty)$.
⟨1⟩4. CASE: There is no y such that $g(x) < y < f(x)$
PROOF: Take $U = g^{-1}((-\infty, f(x))) \cup f^{-1}(g(x), +\infty)$.
□

Lemma 92. *Let $f : X \to Y$. Let Z be an open subspace of X and $a \in Z$. If $f \upharpoonright Z$ is continuous at a then f is continuous at a.*

PROOF:
⟨1⟩1. LET: V be a neighbourhood of $f(x)$
⟨1⟩2. PICK a neighbourhood W of x in Z such that $f(W) \subseteq V$
⟨1⟩3. W is a neighbourhood of x in X such that $f(W) \subseteq V$
PROOF: Lemma 125.
□

Proposition 93. *Let $f : A \to B$ and $g : C \to D$ be continuous. Define $f \times g : A \times C \to B \times D$ by*

$$(f \times g)(a, c) = (f(a), g(c)) \ .$$

Then $f \times g$ is continuous.

PROOF: $\pi_1 \circ (f \times g) = f \circ \pi_1$ and $\pi_2 \circ (f \times g) = g \circ \pi_2$ are continuous by Theorem 83. The result follows by Theorem 88.

Proposition 94. *Let X be a topological space. Let Y a Hausdorff space. Let $A \subseteq X$. Let $f, g : \overline{A} \to Y$ be continuous. If f and g agree on A then $f = g$.*

PROOF:
$\langle 1 \rangle 1$. LET: $x \in \overline{A}$
$\langle 1 \rangle 2$. ASSUME: $f(x) \neq g(x)$
$\langle 1 \rangle 3$. PICK disjoint neighbourhoods V of $f(x)$ and W of $g(x)$.
$\langle 1 \rangle 4$. PICK $y \in f^{-1}(V) \cap g^{-1}(W) \cap A$
PROOF: Since $f^{-1}(V) \cap g^{-1}(W)$ is a neighbourhood of x and hence intersects A.
$\langle 1 \rangle 5$. $f(y) = g(y) \in V \cap W$
$\langle 1 \rangle 6$. Q.E.D.
PROOF: This contradicts the fact that V and W are disjoint ($\langle 1 \rangle 3$).
$\square$

Proposition 95. *Let X and Y be topological spaces and $f : X \to Y$ be continuous. If $a_n \to l$ as $n \to \infty$ in X then $f(a_n) \to f(l)$ as $n \to \infty$.*

PROOF:
$\langle 1 \rangle 1$. LET: V be a neighbourhood of $f(l)$
$\langle 1 \rangle 2$. PICK a neighbourhood U of l such that $f(U) \subseteq V$
$\langle 1 \rangle 3$. PICK N such that, for all $n \geq N$, we have $a_n \in U$
$\langle 1 \rangle 4$. For all $n \geq N$ we have $f(n) \in V$
$\square$

15 Homeomorphisms

Definition 96 (Homeomorphism)**.** Let X and Y be topological spaces. A *Homeomorphism* f between X and Y, $f : X \cong Y$, is a bijection $f : X \to Y$ such that both f and f^{-1} are continuous.

Lemma 97. *Let X and Y be topological spaces and $f : X \to Y$ a bijection. Then the following are equivalent:*

1. *f is a homeomorphism.*
2. *f is continuous and an open map.*
3. *For any $U \subseteq X$, we have U is open if and only if $f(U)$ is open.*

PROOF: Immediate from definitions. □

Proposition 98. *Let X and X' be the same set X under two topologies $\mathcal{T}$ and $\mathcal{T}'$. Let $i : X \to X'$ be the identity function. Then i is a homeomorphism if and only if $\mathcal{T} = \mathcal{T}'$.*

PROOF: Immediate from definitions. □

Definition 99 (Topological Property)**.** Let P be a property of topological spaces. Then P is a *topological* property if and only if, for any spaces X and Y, if P holds of X and $X \cong Y$ then P holds of Y.

Definition 100 (Topological Imbedding)**.** Let X and Y be topological spaces and $f : X \to Y$. Then f is a *topological imbedding* if and only if the corestriction $f : X \to f(X)$ is a homeomorphism.

Proposition 101. *Let X and Y be topological spaces and $a \in X$. The function $i : Y \to X \times Y$ that maps y to (a, y) is an imbedding.*

PROOF:
$\langle 1 \rangle 1$. i is injective
$\langle 1 \rangle 2$. i is continuous.
  PROOF: For U open in X and V open in Y, we have $i^{-1}(U \times V)$ is V if $a \in U$, and $\emptyset$ if $a \notin U$.
$\langle 1 \rangle 3$. $i : Y \to i(Y)$ is an open map.
  PROOF: For V open in Y we have $i(V) = (X \times V) \cap i(Y)$.
□

16 The Order Topology

Definition 102 (Order Topology)**.** Let X be a linearly ordered set with at least two points. The *order topology* on X is the topology generated by the basis $\mathcal{B}$ consisting of:

- all open intervals (a, b);
- all intervals of the form $[\perp, b)$ where $\perp$ is least in X;
- all intervals of the form $(a, \top]$ where $\top$ is greatest in X.

We prove this is a basis for a topology.

PROOF:
$\langle 1 \rangle 1$. For all $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B$.
  $\langle 2 \rangle 1$. LET: $x \in X$
  $\langle 2 \rangle 2$. CASE: x is greatest in X.
    $\langle 3 \rangle 1$. PICK $y \in X$ with $y \neq x$
    $\langle 3 \rangle 2$. $x \in (y, x] \in \mathcal{B}$
  $\langle 2 \rangle 3$. CASE: x is least in X.

⟨3⟩1. Pick $y \in X$ with $y \neq x$
⟨3⟩2. $x \in [x, y) \in \mathcal{B}$
⟨2⟩4. Case: x is neither greatest nor least in X.
⟨3⟩1. Pick $a, b \in X$ with $a < x$ and $x < b$
⟨3⟩2. $x \in (a, b) \in \mathcal{B}$
⟨1⟩2. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$
⟨2⟩1. Let: $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$
⟨2⟩2. Case: $B_1 = (a, b)$, $B_2 = (c, d)$
Proof: Take $B_3 = (\max(a, c), \min(b, d))$.
⟨2⟩3. Case: $B_1 = (a, b)$, $B_2 = [\bot, d)$
Proof: Take $B_3 = (a, \min(b, d))$.
⟨2⟩4. Case: $B_1 = (a, b)$, $B_2 = (c, \top]$
Proof: Take $B_3 = (\max(a, c), b)$.
⟨2⟩5. Case: $B_1 = [\bot, b)$, $B_2 = [\bot, d)$
Proof: Take $B_3 = [\bot, \min(b, d))$.
⟨2⟩6. Case: $B_1 = [\bot, b)$, $B_2 = (c, \top]$
Proof: Take $B_3 = (c, b)$.
□

Lemma 103. *Let X be a linearly ordered set. Then the open rays form a subbasis for the order topology on X.*

Proof:
⟨1⟩1. Every open ray is open.
⟨2⟩1. For all $a \in X$, the ray $(-\infty, a)$ is open.
⟨3⟩1. Let: $x \in (-\infty, a)$
⟨3⟩2. Case: x is least in X
Proof: $xin[x, a) = (-\infty, a)$.
⟨3⟩3. Case: x is not least in X
⟨4⟩1. Pick $y < x$
⟨4⟩2. $x \in (y, a) \subseteq (-\infty, a)$
⟨2⟩2. For all $a \in X$, the ray $(a, +\infty)$ is open.
Proof: Similar.
⟨1⟩2. Every basic open set is a finite intersection of open rays.
Proof: We have $(a, b) = (a, +\infty) \cap (-\infty, b)$, $[\bot, b) = (-\infty, b)$ and $(a, \top] = (a, +\infty)$.
□

Definition 104 (Standard Topology on the Real Line)**.** The *standard topology on the real line* is the order topology on $\mathbb{R}$ generated by the standard order.

Lemma 105. *The lower limit topology is strictly finer than the standard topology on $\mathbb{R}$.*

Proof:
⟨1⟩1. Every open interval is open in the lower limit topology.
Proof: If $x \in (a, b)$ then $x \in [x, b) \subseteq (a, b)$.

⟨1⟩2. The half-open interval $[0, 1)$ is not open in the standard topology.
PROOF: There is no open interval (a, b) such that $0 \in (a, b) \subseteq [0, 1)$.
□

Lemma 106. *The K-topology is strictly finer than the standard topology on $\mathbb{R}$.*

PROOF:
⟨1⟩1. Every open interval is open in the K-topology.
PROOF: Corollary 47.1.
⟨1⟩2. The set $(-1, 1) \setminus K$ is not open in the standard topology.
PROOF: There is no open interval (a, b) such that $0 \in (a, b) \subseteq (-1, 1) \setminus K$, since there must be a positive integer n with $1/n \in (a, b)$.
□

Definition 107 (Ordered Square)**.** The *ordered square* I_o^2 is the set $[0, 1]^2$ under the order topology generated by the dictionary order.

17 The Product Topology

Definition 108 (Product Topology)**.** Let $\{A_i\}_{i \in I}$ be a family of topological spaces. The *product topology* on $\prod_{i \in I} A_i$ is the topology generated by the subbasis consisting of the sets of the form ${\pi_i}^{-1}(U)$ where $i \in I$ and U is open in A_i.

Proposition 109. *The product topology on $\prod_{i \in I} A_i$ is generated by the basis consisting of all sets of the form $\prod_{i \in I} U_i$ where $\{U_i\}_{i \in I}$ is a family such that each U_i is an open set in A_i and $U_i = A_i$ for all but finitely many i.*

PROOF: From Proposition 55. □

Proposition 110. *If A_i is closed in X_i for all $i \in I$ then $\prod_{i \in I} A_i$ is closed in $\prod_{i \in I} X_i$.*

PROOF:
$$(\prod_{i \in I} X_i) \setminus (\prod_{i \in I} A_i) = \bigcup_{j \in I} \left(\prod_{i \in I} X_i \setminus {\pi_j}^{-1}(A_j) \right) \square$$

Proposition 111. *Let $\{A_i\}_{i \in I}$ be a family of topological spaces. For $i \in I$, let $\mathcal{B}_i$ be a basis for the topology on A_i. Then $\mathcal{B} = \{\prod_{i \in I} B_i \mid \forall i \in I. B_i \in \mathcal{B}_i, B_i = A_i$ for all but finitely many $i\}$ is a basis for the box topology on $\prod_{i \in I} A_i$.*

PROOF:
⟨1⟩1. Every set in $\mathcal{B}$ is open.
⟨1⟩2. For every point $a \in \prod_{i \in I} A_i$ and every open set U with $a \in U$, there exists $B \in \mathcal{B}$ such that $a \in B \subseteq U$.
⟨2⟩1. LET: U be open and $a \in U$
⟨2⟩2. PICK a family $\{U_i\}_{i \in I}$ such that each U_i is open in A_i, such that $U_i = A_i$ except for $i = i_1, \ldots, i_n$, and such that $a \in \prod_{i \in I} U_i \subseteq U$.

⟨2⟩3. For $j = 1, \ldots, n$, Pick $B_{i_j} \in \mathcal{B}_{i_j}$ such that $a_{i_j} \in B_{i_j} \subseteq U_{i_j}$

⟨2⟩4. Let: $B = \prod_{i \in I} B_i$ where $B_i = A_i$ for $i \neq i_1, \ldots, i_n$

⟨2⟩5. $B \in \mathcal{B}$

⟨2⟩6. $a \in B \subseteq U$

⟨1⟩3. Q.E.D.

Proof: Lemma 48.

□

Proposition 112. *Let $\{A_i\}_{i \in I}$ be a family of topological spaces. Then the projections $\pi_i : \prod_{i \in I} A_i \to A_i$ are open maps.*

Proof: From Lemma 74. □

Proposition 113. *Let $\{X_i\}_{i \in I}$ be a family of sets. For $i \in I$, let $\mathcal{T}_i$ and $\mathcal{U}_i$ be topologies on X_i. Let $\mathcal{P}$ be the product topology on $\prod_{i \in I} X_i$ generated by the topologies $\mathcal{T}_i$, and $\mathcal{Q}$ the product topology on the same set generated by the topologies $\mathcal{U}_i$. Then $\mathcal{P} \subseteq \mathcal{Q}$ if and only if $\mathcal{T}_i \subseteq \mathcal{U}_i$ for all i.*

Proof:

⟨1⟩1. If $\mathcal{T}_i \subseteq \mathcal{U}_i$ for all i then $\mathcal{P} \subseteq \mathcal{Q}$

Proof: By Corollary 47.1.

⟨1⟩2. If $\mathcal{P} \subseteq \mathcal{Q}$ then $\mathcal{T}_i \subseteq \mathcal{U}_i$ for all i

⟨2⟩1. Assume: $\mathcal{P} \subseteq \mathcal{Q}$

⟨2⟩2. Let: $i \in I$

⟨2⟩3. Let: $U \in \mathcal{T}_i$

⟨2⟩4. Let: $U_i = U$ and $U_j = X_j$ for $j \neq i$

⟨2⟩5. $\prod_{i \in I} U_i \in \mathcal{P}$

⟨2⟩6. $\prod_{i \in I} U_i \in \mathcal{Q}$

⟨2⟩7. $U \in \mathcal{U}_i$

Proof: From Proposition 112.

□

Proposition 114 (AC)**.** *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let $A_i \subseteq X_i$ for all $i \in I$. Then*

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i} \enspace .$$

Proof:

⟨1⟩1. $\overline{\prod_{i \in I} A_i} \subseteq \prod_{i \in I} \overline{A_i}$

⟨2⟩1. For all $i \in I$ we have $A_i \subseteq \overline{A_i}$

Proof: Lemma 31.

⟨2⟩2. $\prod_{i \in I} A_i \subseteq \prod_{i \in I} \overline{A_i}$

⟨2⟩3. Q.E.D.

Proof: Since $\prod_{i \in I} A_i$ is closed by Proposition 110.

⟨1⟩2. $\prod_{i \in I} \overline{A_i} \subseteq \overline{\prod_{i \in I} A_i}$

⟨2⟩1. Let: $x \in \prod_{i \in I} \overline{A_i}$

⟨2⟩2. Let: U be a neighbourhood of x
⟨2⟩3. Pick V_i open in X_i such that $x \in \prod_{i \in I} V_i \subseteq U$ with $V_i = X_i$ except for $i = i_1, \ldots, i_n$
⟨2⟩4. For $i \in I$, pick $a_i \in V_i \cap A_i$
Proof: By Theorem 34 and ⟨2⟩1 using the Axiom of Choice.
⟨2⟩5. U intersects $\prod_{i \in I} A_i$
⟨2⟩6. Q.E.D.
Proof: $a \in U \cap \prod_{i \in I} A_i$
□

Example 115. The closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ is $\mathbb{R}^\omega$

Proof:
⟨1⟩1. Let: $a \in \mathbb{R}^\omega$
⟨1⟩2. Let: U be any neighbourhoods of a.
⟨1⟩3. Pick U_n open in $\mathbb{R}$ for all n such that $a \in \prod_{n \geq 0} U_n \subseteq U$ and $U_n = \mathbb{R}$ for all n except $n_1, \ldots, n_k$
⟨1⟩4. Let: $b_n = a_n$ for $n = n_1, \ldots, n_k$ and $b_n = 0$ for all other n
⟨1⟩5. $b \in \mathbb{R}^\infty \cap U$
⟨1⟩6. Q.E.D.
Proof: From Theorem 34.
□

Proposition 116. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let (a_n) be a sequence in $\prod_{i \in I} X_i$ and $l \in \prod_{i \in I} X_i$. Then $a_n \to l$ as $n \to \infty$ if and only if, for all $i \in I$, we have $\pi_i(a_n) \to \pi_i(l)$ as $n \to \infty$.*

Proof:
⟨1⟩1. If $a_n \to l$ as $n \to \infty$ then, for all $i \in I$, we have $\pi_i(a_n) \to \pi_i(l)$ as $n \to \infty$
Proof: Proposition 95.
⟨1⟩2. If, for all $i \in I$, we have $\pi_i(a_n) \to \pi_i(l)$ as $n \to \infty$, then $a_n \to l$ as $n \to \infty$
⟨2⟩1. Assume: For all $i \in I$, we have $\pi_i(a_n) \to \pi_i(l)$ as $n \to \infty$
⟨2⟩2. Let: V be a neighbourhood of l
⟨2⟩3. Pick open sets U_i in X_i such that $l \in \prod_{i \in I} U_i \subseteq V$ and $U_i = X_i$ for all i except $i = i_1, \ldots, i_k$
⟨2⟩4. For $j = 1, \ldots, k$, Pick N_j such that, for all $n \geq N_j$, we have $\pi_{i_j}(a_n) \in U_{i_j}$
⟨2⟩5. Let: $N = \max(N_1, \ldots, N_k)$
⟨2⟩6. For all $n \geq N$ we have $a_n \in V$
□

17.1 Continuous in Each Variable Separately

Definition 117 (Continuous in Each Variable Separately)**.** Let $F : X \times Y \to Z$. Then F is *continuous in each variable separately* if and only if:

- for every $a \in X$ the function $\lambda y \in Y.F(a, y)$ is continuous;

- for every $b \in Y$ the function $\lambda x \in X.F(x,b)$ is continuous.

Proposition 118. *Let $F : X \times Y \to Z$. If F is continuous then F is continuous in each variable separately.*

Proof: For $a \in X$, the function $\lambda y \in Y.F(a,y)$ is $F \circ i$ where $i : Y \to X \times Y$ maps y to (a,y). We have i is continuous by Proposition 101, hence $F \circ i$ is continuous by Theorem 83.
Similarly for $\lambda x \in X.F(x,b)$ for $b \in Y$. $\square$

Example 119. Define $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$F(x,y) = \begin{cases} xy/(x^2+y^2) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Then F is continuous in each variable separately but not continuous.

18 The Subspace Topology

Definition 120 (Subspace Topology)**.** Let X be a topological space and $Y \subseteq X$. The *subspace topology* on Y is $\mathcal{T} = \{U \cap Y \mid U \text{ is open in } X\}$.

We prove this is a topology.

Proof:
$\langle 1 \rangle 1.$ $Y \in \mathcal{T}$
  Proof: Since $Y = X \cap Y$
$\langle 1 \rangle 2.$ For all $\mathcal{U} \subseteq \mathcal{T}$, we have $\bigcup \mathcal{U} \in \mathcal{T}$
  $\langle 2 \rangle 1.$ Let: $\mathcal{U} \subseteq \mathcal{T}$
  $\langle 2 \rangle 2.$ Let: $\mathcal{V} = \{V \text{ open in } X \mid V \cap Y \in \mathcal{U}\}$
  $\langle 2 \rangle 3.$ $\bigcup \mathcal{U} = (\bigcup \mathcal{V}) \cap Y$
$\langle 1 \rangle 3.$ For all $U, V \in \mathcal{T}$, we have $U \cap V \in \mathcal{T}$
  $\langle 2 \rangle 1.$ Let: $U, V \in \mathcal{T}$
  $\langle 2 \rangle 2.$ Pick U', V' open in X such that $U = U' \cap Y$ and $V = V' \cap Y$
  $\langle 2 \rangle 3.$ $(U \cap V) = (U' \cap V') \cap Y$
$\square$

Theorem 121. *Let X be a topological space and Y a subspace of X. Let $A \subseteq Y$. Then A is closed in Y if and only if there exists a closed set C in X such that $A = C \cap Y$.*

Proof: We have

$$\begin{aligned} & A \text{ is closed in } Y \\ \Leftrightarrow & Y \setminus A \text{ is open in } Y \\ \Leftrightarrow & \exists U \text{ open in } X. Y \setminus A = Y \cap U \\ \Leftrightarrow & \exists C \text{ closed in } X. Y \setminus A = Y \cap (X \setminus U) \\ \Leftrightarrow & \exists C \text{ closed in } X. A = Y \cap U \end{aligned}$$

$\square$

Theorem 122. *Let Y be a subspace of X. Let $A \subseteq Y$. Let $\overline{A}$ be the closure of A in X. Then the closure of A in Y is $\overline{A} \cap Y$.*

PROOF: The closure of A in Y is

$$\begin{aligned}
&\bigcap\{C \text{ closed in } Y \mid A \subseteq C\} \\
=&\bigcap\{D \cap Y \mid D \text{ closed in } X, A \subseteq D \cap Y\} && \text{(Theorem 121)} \\
=&\bigcap\{D \mid D \text{ closed in } X, A \subseteq D\} \cap Y \\
=&\overline{A} \cap Y
\end{aligned}$$

∎

Lemma 123. *Let X be a topological space and $Y \subseteq X$. Let $\mathcal{B}$ be a basis for the topology on X. Then $\mathcal{B}' = \{B \cap Y \mid B \in \mathcal{B}\}$ is a basis for the subspace topology on Y.*

PROOF:
⟨1⟩1. Every element in $\mathcal{B}'$ is open in Y
⟨1⟩2. For every open set U in Y and point $y \in U$, there exists $B' \in \mathcal{B}'$ such that $y \in B' \subseteq U$
 ⟨2⟩1. LET: U be open in Y and $y \in U$
 ⟨2⟩2. PICK V open in X such that $U = V \cap Y$
 ⟨2⟩3. PICK $B \in \mathcal{B}$ such that $y \in B \subseteq V$
 ⟨2⟩4. LET: $B' = B \cap Y$
 ⟨2⟩5. $B' \in \mathcal{B}'$
 ⟨2⟩6. $y \in B' \subseteq U$
⟨1⟩3. Q.E.D.
 PROOF: By Lemma 48.
∎

Lemma 124. *Let X be a topological space and $Y \subseteq X$. Let $\mathcal{S}$ be a basis for the topology on X. Then $\mathcal{S}' = \{S \cap Y \mid S \in \mathcal{S}\}$ is a subbasis for the subspace topology on Y.*

PROOF: The set $\{B \cap Y \mid B$ is a finite intersection of elements of $\mathcal{S}\}$ is a basis for the subspace topology by Lemma 123, and this is the set of all finite intersections of elements of $\mathcal{S}'$. ∎

Lemma 125. *Let Y be a subspace of X. If U is open in Y and Y is open in X then U is open in X.*

PROOF:
⟨1⟩1. PICK V open in X such that $U = V \cap Y$
⟨1⟩2. U is open in X
 PROOF: Since it is the intersection of two open sets V and Y.
∎

Theorem 126. *Let Y be a subspace of X and $A \subseteq Y$. If A is closed in Y and Y is closed in X then A is closed in X.*

PROOF: Pick a closed set C in X such that $A = C \cap Y$ (Theorem 121). Then A is the intersection of two sets closed in X, hence A is closed in X (Lemma 19). $\square$

Theorem 127. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let A_i be a subspace of X_i for all $i \in I$. Then the product topology on $\prod_{i \in I} A_i$ is the same as the topology it inherits as a subspace of $\prod_{i \in I} X_i$.*

PROOF: The product topology is generated by the subbasis

$$\begin{aligned} &\{{\pi_i}^{-1}(U) \mid i \in I, U \text{ open in } A_i\} \\ =&\{{\pi_i}^{-1}(V) \cap A_i \mid i \in I, V \text{ open in } X_i\} \\ =&\{{\pi_i}^{-1}(V) \mid i \in I, V \text{ open in } X_i\} \cap \prod_{i \in I} A_i \end{aligned}$$

and this is a subbasis for the subspace topology by Lemma 124. $\square$

Theorem 128. *Let X be an ordered set in the order topology. Let $Y \subseteq X$ be an interval. Then the order topology on Y is the same as the subspace topology on Y.*

PROOF:
⟨1⟩1. The order topology is finer than the subspace topology.
⟨2⟩1. For every open ray R in X, the set $R \cap Y$ is open in the order topology.
⟨3⟩1. For all $a \in X$, we have $(-\infty, a) \cap Y$ is open in the order topology.
⟨4⟩1. CASE: For all $y \in Y$ we have $y < a$
PROOF: In this case $(-\infty, a) \cap Y = Y$.
⟨4⟩2. CASE: For all $y \in Y$ we have $a < y$
PROOF: In this case $(-\infty, a) \cap Y = \emptyset$.
⟨4⟩3. CASE: There exists $y \in Y$ such that $y \leq a$ and $y \in Y$ such that $a \leq y$
⟨5⟩1. $a \in Y$
PROOF: Because Y is an interval.
⟨5⟩2. $(-\infty, a) \cap Y = \{y \in Y \mid y < a\}$
⟨3⟩2. For all $a \in X$, we have $(a, +\infty) \cap Y$ is open in the order topology.
PROOF: Similar.
⟨2⟩2. Q.E.D.
PROOF: By Lemmas 103 and 124 and Proposition 56.
⟨1⟩2. The subspace topology is finer than the order topology.
⟨2⟩1. Every open ray in Y is open in the subspace topology.
PROOF: For any $a \in Y$ we have $(-\infty, a)_Y = (-\infty, a)_X \cap Y$ and $(a, +\infty)_Y = (a, +\infty)_X \cap Y$.
⟨2⟩2. Q.E.D.
PROOF: By Lemma 103 and Proposition 56

$\square$

This example shows that we cannot remove the hypothesis that Y is an interval:

Example 129. The order topology on I_o^2 is different from the subspace topology as a subspace of $\mathbb{R}^2$ under the dictionary order topology.

PROOF: The set $\{1/2\} \times (1/2, 1)$ is open in the subspace topology but not in the order topology. $\square$

Proposition 130. *Let X be a topological space, Y a subspace of X, and Z a subspace of Y. Then the subspace topology on Z inherited from X is the same as the subspace topology on Z inherited from Y.*

PROOF: The subspace topology inherited from Y is

$$\begin{aligned} & \{V \cap Z \mid V \text{ open in } Y\} \\ = & \{U \cap Y \cap Z \mid U \text{ open in } X\} \\ = & \{U \cap Z \mid U \text{ open in } X\} \end{aligned}$$

which is the subspace topology inherited from X. $\square$

Definition 131 (Unit Circle)**.** The *unit circle* S^1 is $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$

19 The Box Topology

Definition 132 (Box Topology)**.** Let $\{A_i\}_{i \in I}$ be a family of topological spaces. The *box topology* on $\prod_{i \in I} A_i$ is the topology generated by the set of all sets of the form $\prod_{i \in I} U_i$ where $\{U_i\}_{i \in I}$ is a family such that each U_i is open in A_i.

This is a basis since it covers $\prod_{i \in I} A_i$ and is closed under intersection.

Proposition 133. *The box topology is finer than the product topology.*

PROOF: From Proposition 109. $\square$

Corollary 133.1. *If A_i is closed in X_i for all $i \in I$ then $\prod_{i \in I} A_i$ is closed in $\prod_{i \in I} X_i$ under the box topology.*

PROOF: From Proposition 110.

Proposition 134 (AC)**.** *Let $\{A_i\}_{i \in I}$ be a family of topological spaces. For $i \in I$, let $\mathcal{B}_i$ be a basis for the topology on A_i. Then $\mathcal{B} = \{\prod_{i \in I} B_i \mid \forall i \in I. B_i \in \mathcal{B}_i\}$ is a basis for the box topology on $\prod_{i \in I} A_i$.*

PROOF:
$\langle 1 \rangle 1.$ Every set of the form $\prod_{i \in I} B_i$ is open.
$\langle 1 \rangle 2.$ For every point $a \in \prod_{i \in I} A_i$ and every open set U with $a \in U$, there exists $B \in \mathcal{B}$ such that $a \in B \subseteq U$.
  $\langle 2 \rangle 1.$ LET: U be open and $a \in U$
  $\langle 2 \rangle 2.$ PICK a family $\{U_i\}_{i \in I}$ such that each U_i is open in A_i and $a \in \prod_{i \in I} U_i \subseteq U$.
  $\langle 2 \rangle 3.$ For $i \in I$, PICK $B_i \in \mathcal{B}_i$ such that $a_i \in B_i \subseteq U_i$
    PROOF: Using the Axiom of Choice.
  $\langle 2 \rangle 4.$ $a \in \prod_{i \in I} B_i \subseteq U$
$\langle 1 \rangle 3.$ Q.E.D.
  PROOF: Lemma 48.

$\square$

Theorem 135. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let A_i be a subspace of X_i for all $i \in I$. Give $\prod_{i \in I} X_i$ the box topology. Then the box topology on $\prod_{i \in I} A_i$ is the same as the topology it inherits as a subspace of $\prod_{i \in I} X_i$.*

PROOF: The box topology is generated by the basis

$$\begin{aligned} &\{\prod_{i \in I} U_i \mid \forall i \in I, U_i \text{ open in } A_i\} \\ =&\{\prod_{i \in I} (V_i \cap A_i) \mid \forall i \in I, V_i \text{ open in } X_i\} \\ =&\{\prod_{i \in I} V_i \mid \forall i \in I, V_i \text{ open in } X_i\} \cap \prod_{i \in I} A_i \end{aligned}$$

and this is a basis for the subspace topology by Lemma 123. $\square$

Proposition 136. *Let $\{X_i\}_{i \in I}$ be a family of Hausdorff spaces. Then $\prod_{i \in I} X_i$ under the box topology is Hausdorff.*

PROOF:
$\langle 1 \rangle 1$. LET: $\{X_i\}_{i \in I}$ be a family of Hausdorff spaces.
$\langle 1 \rangle 2$. LET: $a, b \in \prod_{i \in I} X_i$ with $a \neq b$
$\langle 1 \rangle 3$. PICK $i \in I$ such that $a_i \neq b_i$
$\langle 1 \rangle 4$. PICK U, V disjoint open sets in X_i with $a_i \in U$ and $b_i \in V$
$\langle 1 \rangle 5$. ${\pi_i}^{-1}(U)$ and ${\pi_i}^{-1}(V)$ are disjoint open sets in $\prod_{i \in I} X_i$ with $a \in {\pi_i}^{-1}(U)$ and $b \in {\pi_i}^{-1}(V)$

$\square$

Proposition 137 (AC). *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Give $\prod_{i \in I} X_i$ the box topology. Let $A_i \subseteq X_i$ for all $i \in I$. Then*

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i} \ .$$

PROOF:
$\langle 1 \rangle 1$. $\overline{\prod_{i \in I} A_i} \subseteq \prod_{i \in I} \overline{A_i}$
  $\langle 2 \rangle 1$. For all $i \in I$ we have $A_i \subseteq \overline{A_i}$
    PROOF: Lemma 31.
  $\langle 2 \rangle 2$. $\prod_{i \in I} A_i \subseteq \prod_{i \in I} \overline{A_i}$
  $\langle 2 \rangle 3$. Q.E.D.
    PROOF: Since $\prod_{i \in I} A_i$ is closed by Corollary 133.1.
$\langle 1 \rangle 2$. $\prod_{i \in I} \overline{A_i} \subseteq \overline{\prod_{i \in I} A_i}$
  $\langle 2 \rangle 1$. LET: $x \in \prod_{i \in I} \overline{A_i}$
  $\langle 2 \rangle 2$. LET: U be a neighbourhood of x
  $\langle 2 \rangle 3$. PICK V_i open in X_i such that $x \in \prod_{i \in I} V_i \subseteq U$
  $\langle 2 \rangle 4$. For $i \in I$, pick $a_i \in V_i \cap A_i$
    PROOF: By Theorem 34 and $\langle 2 \rangle 1$ using the Axiom of Choice.

⟨2⟩5. U intersects $\prod_{i \in I} A_i$
⟨2⟩6. Q.E.D.
PROOF: $a \in U \cap \prod_{i \in I} A_i$.
□

The following example shows that Theorem 88 fails in the box topology.

Example 138. Define $f : \mathbb{R} \to \mathbb{R}^\omega$ by $f(t) = (t, t, \ldots)$. Then $\pi_n \circ f = \mathrm{id}_\mathbb{R}$ is continuous for all n. But f is not continuous when $\mathbb{R}^\omega$ is given the box topology because the inverse image of

$$(-1, 1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times \cdots$$

is $\{0\}$ which is not open.

The following example shows that Proposition 116 fails in the box topology.

Example 139. Give $\mathbb{R}^\omega$ the box topology. Let $a_n = (1/n, 1/n, \ldots)$ for $n \geq 1$ and $l = (0, 0, \ldots)$. Then $\pi_i(a_n) \to \pi_i(l)$ as $n \to \infty$ for all i, but $a_n \not\to l$ as $n \to \infty$ since the open set

$$(-1, 1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times \cdots$$

contains l but does not contain any a_n.

Example 140. The set $\mathbb{R}^\infty$ is closed in $\mathbb{R}^\omega$ under the box topology. For let (a_n) be any sequence not in $\mathbb{R}^\infty$. Let U_n be an open interval around a_n that does not contain 0 if $a_n \neq 0$, and $U_n = \mathbb{R}$ if $a_n = 0$. Then $\prod_{n \geq 0} U_n$ is a neighbourhood of (a_n) that does not intersect $\mathbb{R}^\infty$.

20 T_1 Spaces

Definition 141 (T_1 Space)**.** A topological space is T_1 if and only if every singleton is closed.

Lemma 142. *A space is T_1 if and only if every finite set is closed.*

PROOF: From Lemma 20. □

Theorem 143. *In a T_1 space, a point a is a limit point of a set A if and only if every neighbourhood of a contains infinitely many points of A.*

PROOF:
⟨1⟩1. If a is a limit point of A then every neighbourhood of a contains infinitely many points of A.
⟨2⟩1. ASSUME: a is a limit point of A.
⟨2⟩2. LET: U be a neighbourhood of a.
⟨2⟩3. ASSUME: for a contradiction U contains only finitely many points of A.
⟨2⟩4. $(U \cap A) \setminus \{a\}$ is closed.
PROOF: By the T_1 axiom.

$\langle 2\rangle 5.$ $(U \setminus A) \cup \{a\}$ is open.
PROOF: It is $U \setminus ((U \cap A) \setminus \{a\})$.
$\langle 2\rangle 6.$ $(U \setminus A) \cup \{a\}$ intersects A in a point other than a.
PROOF: From $\langle 2\rangle 1$.
$\langle 2\rangle 7.$ Q.E.D.
□
$\langle 1\rangle 2.$ If every neighbourhood of a contains infinitely many points of A then a is a limit point of A.
PROOF: Immediate from definitions.
□

(To see this does not hold in every space, see Proposition 45.)

Proposition 144. *A space is T_1 if and only if, for any two distinct points x and y, there exist neighbourhoods U of x and V of y such that $x \notin V$ and $y \notin U$.*

PROOF:
$\langle 1\rangle 1.$ LET: X be a topological space.
$\langle 1\rangle 2.$ If X is T_1 then, for any two distinct points x and y, there exist neighbourhoods U of x and V of y such that $x \notin V$ and $y \notin U$.
PROOF: This holds because $\{x\}$ and $\{y\}$ are closed.
$\langle 1\rangle 3.$ Suppose, for any two distinct points x and y, there exist neighbourhoods U of x and V of y such that $x \notin V$ and $y \notin U$. Then X is T_1.
$\langle 2\rangle 1.$ ASSUME: For any two distinct points x and y, there exist neighbourhoods U of x and V of y such that $x \notin V$ and $y \notin U$.
$\langle 2\rangle 2.$ LET: $a \in X$
$\langle 2\rangle 3.$ $\{a\}$ is closed.
PROOF: For all $b \neq a$ there exists a neighbourhood U of b such that $U \subseteq X \setminus \{a\}$.
□

21 Hausdorff Spaces

Definition 145 (Hausdorff Space)**.** A topological space is *Hausdorff* if and only if, for any points x, y with $x \neq y$, there exist disjoint open sets U and V such that $x \in U$ and $y \in V$.

Theorem 146. *Every Hausdorff space is T_1.*

PROOF:
$\langle 1\rangle 1.$ LET: X be a Hausdorff space.
$\langle 1\rangle 2.$ LET: $b \in X$
PROVE: $\overline{\{b\}} = \{b\}$
$\langle 1\rangle 3.$ ASSUME: $a \in \overline{\{b\}}$ and $a \neq b$
$\langle 1\rangle 4.$ PICK disjoint neighbourhoods U of a and V of b.
$\langle 1\rangle 5.$ U intersects $\{b\}$
PROOF: Theorem 34.

⟨1⟩6. $b \in U$
⟨1⟩7. Q.E.D.
Proof: This contradicts the fact that U and V are disjoint (⟨1⟩4).
□

Proposition 147. *An infinite set under the finite complement topology is T_1 but not Hausdorff.*

Proof:
⟨1⟩1. Let: X be an infinite set under the finite complement topology.
⟨1⟩2. Every singleton is closed.
Proof: By definition.
⟨1⟩3. Pick$a, b \in X$ with $a \neq b$
⟨1⟩4. There are no disjoint neighbourhoods U of a and V of b.
⟨2⟩1. Let: U be a neighbourhood of a and V a neighbourhood of b.
⟨2⟩2. $X \setminus U$ and $X \setminus V$ are finite.
⟨2⟩3. Pick $c \in X$ that is not in $X \setminus U$ or $X \setminus V$.
⟨2⟩4. $c \in U \cap V$
□

Proposition 148. *The product of a family of Hausdorff spaces is Hausdorff.*

Proof:
⟨1⟩1. Let: $\{X_i\}_{i \in I}$ be a family of Hausdorff spaces.
⟨1⟩2. Let: $a, b \in \prod_{i \in I} X_i$ with $a \neq b$
⟨1⟩3. Pick $i \in I$ such that $a_i \neq b_i$
⟨1⟩4. Pick U, V disjoint open sets in X_i with $a_i \in U$ and $b_i \in V$
⟨1⟩5. ${\pi_i}^{-1}(U)$ and ${\pi_i}^{-1}(V)$ are disjoint open sets in $\prod_{i \in I} X_i$ with $a \in {\pi_i}^{-1}(U)$ and $b \in {\pi_i}^{-1}(V)$
□

Theorem 149. *Every linearly ordered set under the order topology is Hausdorff.*

Proof:
⟨1⟩1. Let: X be a linearly ordered set under the order topology.
⟨1⟩2. Let: $a, b \in X$ with $a \neq b$
⟨1⟩3. Assume: w.l.o.g. $a < b$
⟨1⟩4. Case: There exists c such that $a < c < b$
Proof: The sets $(-\infty, c)$ and $(c, +\infty)$ are disjoint neighbourhoods of a and b respectively.
⟨1⟩5. Case: There is no c such that $a < c < b$
Proof: The sets $(-\infty, b)$ and $(a, +\infty)$ are disjoint neighbourhoods of a and b respectively.
□

Theorem 150. *A subspace of a Hausdorff space is Hausdorff.*

Proof:
⟨1⟩1. Let: X be a Hausdorff space and Y a subspace of X.

⟨1⟩2. LET: $x, y \in Y$ with $x \neq y$
⟨1⟩3. PICK disjoint neighbourhoods U of x and V of y in X.
⟨1⟩4. $U \cap Y$ and $V \cap Y$ are disjoint neighbourhoods of x and y respectively in Y.
□

Proposition 151. *A space X is Hausdorff if and only if the diagonal $\Delta = \{(x, x) \mid x \in X\}$ is closed in X^2.*

PROOF:

$$\begin{aligned} & X \text{ is Hausdorff} \\ \Leftrightarrow & \forall x, y \in X . x \neq y \Rightarrow \exists V, W \text{ open} . x \in V \wedge y \in W \wedge V \cap W = \emptyset \\ \Leftrightarrow & \forall (x, y) \in X^2 \setminus \Delta . \exists V, W \text{ open} . (x, y) \subseteq V \times W \subseteq X^2 \setminus \Delta \\ \Leftrightarrow & \Delta \text{ is closed} \end{aligned}$$

□

22 The First Countability Axiom

Definition 152 (First Countability Axiom)**.** A topological space X satisfies the *first countability axiom*, or is *first countable*, if and only if every point has a countable local basis.

Lemma 153 (Sequence Lemma (CC))**.** *Let X be a first countable space. Let $A \subseteq X$ and $l \in \overline{A}$. Then there exists a sequence in A that converges to l.*

PROOF:
⟨1⟩1. PICK a countable local basis $\{B_n \mid n \in \mathbb{Z}^+\}$ at l such that $B_1 \supseteq B_2 \supseteq \cdots$.
PROOF: Lemma 58.
⟨1⟩2. For all $n \geq 1$, PICK $a_n \in A \cap B_n$.
PROVE: $a_n \to l$ as $n \to \infty$
⟨1⟩3. LET: U be a neighbourhood of A
⟨1⟩4. PICK N such that $B_N \subseteq U$
⟨1⟩5. For $n \geq N$ we have $a_n \in U$
PROOF: $a_n \in B_n \subseteq B_N \subseteq U$
□

Theorem 154 (CC)**.** *Let X be a first countable space and Y a topological space. Let $f : X \to Y$. Suppose that, for every sequence (x_n) in X and $l \in X$, if $x_n \to l$ as $n \to \infty$, then $f(x_n) \to f(l)$ as $n \to \infty$. Then f is continuous.*

PROOF:
⟨1⟩1. LET: $A \subseteq X$
⟨1⟩2. LET: $a \in A$
PROVE: $f(a) \in \overline{f(A)}$
⟨1⟩3. PICK a sequence (x_n) in A that converges to a.
PROOF: By the Sequence Lemma.
⟨1⟩4. $f(x_n) \to f(a)$

$\langle 1\rangle 5.$ $f(a) \in \overline{f(A)}$
PROOF: By Lemma 62.
$\langle 1\rangle 6.$ Q.E.D.
PROOF: By Theorem 80.
□

Example 155 (CC)**.** The space $\mathbb{R}^\omega$ under the box product is not first countable.

PROOF: Let $\{B_n \mid n \geq 0\}$ be a countable set of neighbourhoods of $\vec{0}$. We will construct a neighbourhood of $\vec{0}$ that does not include any of these.
For $n \geq 0$, pick a neighbourhood U_n of 0 such that $U_n \subset \pi_n(B_p)$. Then $\prod_{n=0}^{\infty} U_n$ is a neighbourhood of $\vec{0}$ that does not include any B_n. □

Example 156. If J is an uncountable set then $\mathbb{R}^J$ is not first countable.

PROOF:
$\langle 1\rangle 1.$ LET: $\{B_n \mid n \geq 0\}$ be any countable set of neighbourhoods of $\vec{0}$.
$\langle 1\rangle 2.$ For $n \geq 0$, PICK a basis element $\prod_{\alpha \in J} U_{n\alpha}$ that contains $\vec{0}$ and is included in B_n.
PROOF: Using the Axiom of Countable Choice.
$\langle 1\rangle 3.$ For $n \geq 0$,
LET: $J_n = \{\alpha \in J \mid U_{n\alpha} \neq \mathbb{R}\}$
$\langle 1\rangle 4.$ PICK $\beta \in J$ such that $\beta \notin J_n$ for any n.
PROOF: Since each J_n is finite so $\bigcup_n J_n$ is countable.
$\langle 1\rangle 5.$ $\pi_\beta((-1, 1))$ is a neighbourhood of $\vec{0}$ that does not include any B_n.
□

Example 157. The space $\mathbb{R}_l$ is first countable.

PROOF: For any $a \in \mathbb{R}$, the set $\{[a, a + 1/n) \mid n \geq 1\}$ is a countable local basis.

Example 158. The ordered square is first countable.

PROOF: For any $(a, b) \in I_o^2$ with $b \neq 0, 1$, the set $\{(\{a\} \times (b - 1/n, b + 1/n)) \cap I_o^2 \mid n \geq 1\}$ is a countable local basis.

23 The Metric Topology

Definition 159 (Metric)**.** Let X be a set. A *metric* on X is a function $d : X^2 \to \mathbb{R}$ such that:

1. For all $x, y \in X$, $d(x, y) \geq 0$
2. For all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$
3. For all $x, y \in X$, $d(x, y) = d(y, x)$
4. (*Triangle Inequality*) For all $x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$

We call $d(x, y)$ the *distance* between x and y.

Definition 160 (Open Ball)**.** Let X be a metric space. Let $a \in X$ and $\epsilon > 0$. The *open ball* with *centre* a and *radius* ϵ is

$$B(a, \epsilon) = \{x \in X \mid d(a, x) < \epsilon\} \ .$$

Definition 161 (Metric Topology)**.** Let X be a metric space. The *metric topology* on X is the topology generated by the basis consisting of all the open balls.

We prove this is a basis for a topology.

PROOF:
$\langle 1 \rangle 1$. For every point a, there exists a ball B such that $a \in B$
PROOF: We have $a \in B(a, 1)$.
$\langle 1 \rangle 2$. For any balls B_1, B_2 and point $a \in B_1 \cap B_2$, there exists a ball B_3 such that $a \in B_3 \subseteq B_1 \cap B_2$
$\langle 2 \rangle 1$. LET: $B_1 = B(c_1, \epsilon_1)$ and $B_2 = B(c_2, \epsilon_2)$
$\langle 2 \rangle 2$. LET: $\delta = \min(\epsilon_1 - d(c_1, a), \epsilon_2 - d(c_2, a))$
PROVE: $B(a, \delta) \subseteq B_1 \cap B_2$
$\langle 2 \rangle 3$. LET: $x \in B(a, \delta)$
$\langle 2 \rangle 4$. $x \in B_1$
PROOF:

$$\begin{aligned} d(x, c_1) &= d(x, a) + d(a, c_1) \\ &< \delta + d(a, c_1) \\ &\leq \epsilon_1 \end{aligned}$$

$\langle 2 \rangle 5$. $x \in B_2$
PROOF: Similar.
$\square$

Proposition 162. *Let X be a metric space and $U \subseteq X$. Then U is open if and only if, for every $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$.*

PROOF:
$\langle 1 \rangle 1$. If U is open then, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$.
$\langle 2 \rangle 1$. ASSUME: U is open.

⟨2⟩2. LET: $x \in U$
⟨2⟩3. PICK $a \in X$ and $\delta > 0$ such that $x \in B(a, \delta) \subseteq U$
⟨2⟩4. LET: $\epsilon = \delta - d(a, x)$
PROVE: $B(x, \epsilon) \subseteq U$
⟨2⟩5. LET: $y \in B(x, \epsilon)$
⟨2⟩6. $d(y, a) < \delta$
PROOF:

$$\begin{aligned} d(y, a) &\leq d(a, x) + d(x, y) \\ &< \delta + d(x, y) \\ &= \epsilon \end{aligned}$$

⟨2⟩7. $y \in U$
⟨1⟩2. If, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$, then U is open.
PROOF: Immediate from definitions.
□

Definition 163 (Discrete Metric)**.** Let X be a set. The *discrete metric* on X is defined by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Proposition 164. *The discrete metric induces the discrete topology.*

PROOF: For any (open) set U and point $a \in U$, we have $a \in B(a, 1) \subseteq U$. □

Definition 165 (Standard Metric on $\mathbb{R}$)**.** The *standard metric* on $\mathbb{R}$ is defined by $d(x, y) = |x - y|$.

Proposition 166. *The standard metric on $\mathbb{R}$ induces the standard topology on $\mathbb{R}$.*

PROOF:
⟨1⟩1. Every open ball is open in the standard topology on $\mathbb{R}$.
PROOF: $B(a, \epsilon) = (a - \epsilon, a + \epsilon)$
⟨1⟩2. For every open set U and point $a \in U$, there exists $\epsilon > 0$ such that $B(a, \epsilon) \subseteq U$
⟨2⟩1. LET: U be an open set and $a \in U$
⟨2⟩2. PICK an open interval b, c such that $a \in (b, c) \subseteq U$
⟨2⟩3. LET: $\epsilon = \min(a - b, c - a)$
⟨2⟩4. $B(a, \epsilon) \subseteq U$
□

Definition 167 (Metrizable)**.** A topological space X is *metrizable* if and only if there exists a metric on X that induces the topology.

Definition 168 (Bounded)**.** Let X be a metric space and $A \subseteq X$. Then A is *bounded* if and only if there exists M such that, for all $x, y \in A$, we have $d(x, y) \leq M$.

Definition 169 (Diameter)**.** Let X be a metric space and $A \subseteq X$. The *diameter* of A is

$$\operatorname{diam} A = \sup_{x,y \in A} d(x,y) \enspace .$$

Definition 170 (Standard Bounded Metric)**.** Let d be a metric on X. The *standard bounded metric* corresponding to d is the metric $\overline{d}$ defined by

$$\overline{d}(x,y) = \min(d(x,y), 1) \enspace .$$

We prove this is a metric.

PROOF:
⟨1⟩1. $\overline{d}(x,y) \geq 0$
PROOF: Since $d(x,y) \geq 0$
⟨1⟩2. $\overline{d}(x,y) = 0$ if and only if $x = y$
PROOF: $\overline{d}(x,y) = 0$ if and only if $d(x,y) = 0$ if and only if $x = y$
⟨1⟩3. $\overline{d}(x,y) = \overline{d}(y,x)$
PROOF: Since $d(x,y) = d(y,x)$
⟨1⟩4. $\overline{d}(x,z) \leq \overline{d}(x,y) + \overline{d}(y,z)$
PROOF:

$$\begin{aligned}
\overline{d}(x,y) + \overline{d}(y,z) &= \min(d(x,y), 1) + \min(d(y,z), 1) \\
&= \min(d(x,y) + d(y,z), d(x,y) + 1, d(y,z) + 1, 2) \\
&\geq \min(d(x,z), 1) \\
&= \overline{d}(x,z)
\end{aligned}$$

□

Lemma 171. *In any metric space X, the set $\mathcal{B} = \{B(a,\epsilon) \mid a \in X, \epsilon < 1\}$ is a basis for the metric topology.*

PROOF:
⟨1⟩1. Every element of $\mathcal{B}$ is open.
PROOF: From Lemma 47.
⟨1⟩2. For every open set U and point $a \in U$, there exists $B \in \mathcal{B}$ such that $a \in B \subseteq U$
⟨2⟩1. LET: U be an open set and $a \in U$
⟨2⟩2. PICK$\epsilon > 0$ such that $B(a,\epsilon) \subseteq U$
⟨2⟩3. $B(a, \min(\epsilon, 1/2)) \subseteq U$
⟨1⟩3. Q.E.D.
PROOF: Lemma 48.
□

Proposition 172. *Let d be a metric on the set X. Then the standard bounded metric $\overline{d}$ induces the same metric as d.*

PROOF: This follows from Lemma 171 since the open balls with radius < 1 are the same under both metrics. □

Lemma 173. *Let d and d' be two metrics on the same set X. Let $\mathcal{T}$ and $\mathcal{T}'$ be the topologies they induce. Then $\mathcal{T} \subseteq \mathcal{T}'$ if and only if, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that*

$$B_{d'}(x, \delta) \subseteq B_d(x, \epsilon) \ .$$

PROOF:
$\langle 1 \rangle 1$. If $\mathcal{T} \subseteq \mathcal{T}'$ then, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$
PROOF: From Proposition 162 since $x \in B_d(x, \epsilon) \in \mathcal{T}'$.
$\langle 1 \rangle 2$. If, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$, then $\mathcal{T} \subseteq \mathcal{T}'$
$\langle 2 \rangle 1$. ASSUME: For all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$
$\langle 2 \rangle 2$. LET: $U \in \mathcal{T}$
$\langle 2 \rangle 3$. For all $x \in U$ there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq U$.
$\langle 3 \rangle 1$. LET: $x \in U$
$\langle 3 \rangle 2$. PICK $\epsilon > 0$ such that $B_d(x, \epsilon) \subseteq U$
PROOF: Proposition 162
$\langle 3 \rangle 3$. PICK $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$
PROOF: By $\langle 2 \rangle 1$
$\langle 3 \rangle 4$. $B_{d'}(x, \delta) \subseteq U$
$\langle 2 \rangle 4$. $U \in \mathcal{T}'$
PROOF: Proposition 162.
$\square$

Proposition 174. $\mathbb{R}^2$ *under the dictionary order topology is metrizable.*

PROOF: Define $d : \mathbb{R}^2 \to \mathbb{R}$ by

$$\begin{aligned} d((x,y),(x,z)) &= \max(|y-z|, 1) \\ d((x,y),(x',y')) &= 1 && \text{if } x \neq x' \end{aligned}$$
$\square$

$\langle 1 \rangle 1$. $x \in \bigcap_{i=1}^{N} {\pi_i}^{-1}() \subseteq B_D(a, \epsilon)$

Proposition 175. *Let $d : X^2 \to \mathbb{R}$ be a metric on X. Then the metric topology on X is the coarsest topology such that d is continuous.*

PROOF:
$\langle 1 \rangle 1$. d is continuous.
$\langle 2 \rangle 1$. LET: $a, b \in X$
$\langle 2 \rangle 2$. LET: $\epsilon > 0$
$\langle 2 \rangle 3$. LET: $\delta = \epsilon / 2$
$\langle 2 \rangle 4$. LET: $x, y \in X$
$\langle 2 \rangle 5$. ASSUME: $\rho((a,b),(x,y)) < \delta$
$\langle 2 \rangle 6$. $|d(a,b) - d(x,y)| < \epsilon$
$\langle 3 \rangle 1$. $d(a,b) - d(x,y) < \epsilon$

PROOF:

$$\begin{aligned} d(a,b) &\leq d(a,x) + d(x,y) + d(y,b) \\ &\leq d(x,y) + 2\rho((a,b),(x,y)) \\ &< d(x,y) + 2\delta \\ &= d(x,y) + \epsilon \end{aligned}$$

⟨3⟩2. $d(a.b) - d(x,y) > -\epsilon$

PROOF: Similar.

⟨2⟩7. Q.E.D.

⟨1⟩2. If $\mathcal{T}$ is any topology under which d is continuous then $\mathcal{T}$ is finer than the metric topology.

PROOF: Since $B(a,\epsilon) = {d_a}^{-1}((-\infty,\epsilon))$

□

Proposition 176. *Let X be a metric space with metric d and $A \subseteq X$. The restriction of d to A is a metric on A that induces the subspace topology.*

PROOF:

⟨1⟩1. The restriction of d to A is a metric on A.

⟨1⟩2. Every open ball under $d \restriction A$ is open under the subspace topology.

PROOF: $B_{d \restriction A}(a,\epsilon) = B_d(a,\epsilon) \cap A$.

⟨1⟩3. If U is open in the subspace topology and $x \in U$, then there exists a $d \restriction A$-ball B such that $x \in B \subseteq U$.

⟨2⟩1. PICK V open in X such that $U = V \cap A$

⟨2⟩2. PICK $\epsilon > 0$ such that $B_d(x,\epsilon) \subseteq V$

⟨2⟩3. Take $B = B_{d \restriction A}(x,\epsilon)$

□

Corollary 176.1. *A subspace of a metrizable space is metrizable.*

Proposition 177. *Every metrizable space is Hausdorff.*

PROOF:

⟨1⟩1. LET: X be a metric space

⟨1⟩2. LET: $a, b \in X$ with $a \neq b$

⟨1⟩3. LET: $\epsilon = d(a,b)/2$

⟨1⟩4. LET: $U = B(a,\epsilon)$ and $V = B(b,\epsilon)$

⟨1⟩5. U and V are disjoint neighbourhoods of a and b respectively.

□

Proposition 178 (CC)**.** *The product of a countable family of metrizable spaces is metrizable.*

PROOF:

⟨1⟩1. LET: (X_n, d_n) be a sequence of metric spaces.

⟨1⟩2. ASSUME: w.l.o.g. each d_n is bounded above by 1.

PROOF: By Proposition 172.

⟨1⟩3. LET: D be the metric on $\mathbb{R}^\omega$ defined by $D(x,y) = \sup_i(d_i(x_i,y_i)/i)$.

⟨2⟩1. $D(x, y) \geq 0$
⟨2⟩2. $D(x, y) = 0$ if and only if $x = y$
⟨2⟩3. $D(x, y) = D(y, x)$
⟨2⟩4. $D(x, z) \leq D(x, y) + D(y, z)$
PROOF:

$$\begin{aligned} D(x, z) &= \sup_i \frac{d_i(x_i, z_i)}{i} \\ &\leq \sup_i \frac{d_i(x_i, y_i) + d_i(y_i, z_i)}{i} \\ &\leq \sup_i \frac{d_i(x_i, y_i)}{i} + \sup_i \frac{d_i(y_i, z_i)}{i} \\ &= D(x, y) + D(y, z) \end{aligned}$$

⟨1⟩4. Every open ball $B_D(a, \epsilon)$ is open in the product topology.
⟨2⟩1. PICK N such that $1/\epsilon < N$
⟨2⟩2. $B_D(a, \epsilon) = \prod_{i=1}^{\infty} U_i$ where $U_i = B(a_i, i\epsilon)$ if $i \leq N$, and $U_i = X_i$ if $i > N$
⟨1⟩5. For any open set U and $a \in U$, there exists $\epsilon > 0$ such that $B_D(a, \epsilon) \subseteq U$.
⟨2⟩1. LET: $n \geq 1$, V be an open set in $\mathbb{R}$ and $a \in {\pi_n}^{-1}(V)$
⟨2⟩2. PICK $\epsilon > 0$ such that $B_{d_n}(a, \epsilon) \subseteq V$
⟨2⟩3. $B_D(a, \epsilon/n) \subseteq {\pi_n}^{-1}(V)$
□

Theorem 179. *Let X and Y be metric spaces and $f : X \to Y$. Then f is continuous if and only if, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$.*

PROOF:
⟨1⟩1. If f is continuous then, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$
⟨2⟩1. ASSUME: f is continuous.
⟨2⟩2. LET: $x \in X$ and $\epsilon > 0$
⟨2⟩3. PICK a neighbourhood U of x such that $f(U) \subseteq B(f(x), \epsilon)$
PROOF: Theorem 80.
⟨2⟩4. PICK $\delta > 0$ such that $B(x, \delta) \subseteq U$
PROOF: Proposition 162.
⟨2⟩5. For all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$
⟨1⟩2. If for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$, then f is continuous.
⟨2⟩1. ASSUME: for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$
⟨2⟩2. LET: $x \in X$ and V be a neighbourhood of $f(x)$
⟨2⟩3. PICK $\epsilon > 0$ such that $B(f(x), \epsilon) \subseteq V$
PROOF: Proposition 162.
⟨2⟩4. PICK $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$
PROOF: By ⟨2⟩1
⟨2⟩5. LET: $U = B(x, \delta)$
⟨2⟩6. U is a neighbourhood of x with $f(U) \subseteq V$

⟨2⟩7. Q.E.D.

PROOF: Theorem 80.

∎

Proposition 180. *Let X be a metric space. Let (a_n) be a sequence in X and $l \in X$. Then $a_n \to l$ as $n \to \infty$ if and only if, for all $\epsilon > 0$, there exists N such that, for all $n \geq N$, we have $d(a_n, l) < \epsilon$.*

PROOF: From Proposition 63. ∎

Proposition 181. *Every metrizable space is first countable.*

PROOF: In any metric space X, the open balls $B(a, 1/n)$ for $n \geq 1$ form a local basis at a.

Example 182. $\mathbb{R}^\omega$ under the box topology is not metrizable.

Example 183. If J is uncountable then $\mathbb{R}^J$ under the product topology is not metrizable.

24 Real Linear Algebra

Definition 184 (Square Metric)**.** The *square metric* ρ on $\mathbb{R}^n$ is defined by

$$\rho(\vec{x}, \vec{y}) = \max(|x_1 - y_1|, \ldots, |x_n - y_n|)$$

We prove this is a metric.

PROOF:

⟨1⟩1. $\rho(\vec{x}, \vec{y}) \geq 0$

PROOF: Immediate from definition.

⟨1⟩2. $\rho(\vec{x}, \vec{y}) = 0$ if and only if $\vec{x} = \vec{y}$

PROOF: Immediate from definition.

⟨1⟩3. $\rho(\vec{x}, \vec{y}) = \rho(\vec{y}, \vec{x})$

PROOF: Immediate from definition.

⟨1⟩4. $\rho(\vec{x}, \vec{z}) \leq \rho(\vec{x}, \vec{y}) + \rho(\vec{y}, \vec{z})$

PROOF: Since $|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$.

∎

Proposition 185. *The square metric induces the standard topology on $\mathbb{R}^n$.*

PROOF:

⟨1⟩1. For every $a \in X$ and $\epsilon > 0$, we have $B_\rho(a, \epsilon)$ is open in the standard product topology.

PROOF:

$$B_\rho(a, \epsilon) = (a_1 - \epsilon, a_1 + \epsilon) \times \cdots \times (a_n - \epsilon, a_n + \epsilon)$$

⟨1⟩2. For any open sets $U_1, \ldots, U_n$ in $\mathbb{R}$, we have $U_1 \times \cdots \times U_n$ is open in the square metric topology.

⟨2⟩1. LET: $\vec{a} \in U_1 \times \cdots \times U_n$

⟨2⟩2. For $i = 1, \ldots, n$, PICK $\epsilon_i > 0$ such that $(a_i - \epsilon_i, a_i + \epsilon_i) \subseteq U_i$
⟨2⟩3. LET: $\epsilon = \min(\epsilon_1, \ldots, \epsilon_n)$
⟨2⟩4. $B_\rho(\vec{a}, \epsilon) \subseteq U$
□

Definition 186. Given $\vec{x}, \vec{y} \in \mathbb{R}^n$, define the *sum* $\vec{x} + \vec{y}$ by

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n) \ .$$

Definition 187. Given $\lambda \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^n$, define the *scalar product* $\lambda\vec{x} \in \mathbb{R}^n$ by

$$\lambda(x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n)$$

Definition 188 (Inner Product)**.** Given $\vec{x}, \vec{y} \in \mathbb{R}^n$, define the *inner product* $\vec{x} \cdot \vec{y} \in \mathbb{R}$ by

$$(x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) = x_1 y_1 + \cdots + x_n y_n \ .$$

We write $\vec{x}^2$ for $\vec{x} \cdot \vec{x}$.

Definition 189 (Norm)**.** Let $n \geq 1$. The *norm* on $\mathbb{R}^n$ is the function $\| \ \| : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\|(x_1, \ldots, x_n)\| = \sqrt{x_1^2 + \cdots + x_n^2}$$

Lemma 190.

$$\|\vec{x}\|^2 = \vec{x}^2$$

PROOF: Immediate from definitions. □

Lemma 191.

$$\vec{x} \cdot (\vec{y} + \vec{z}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z}$$

PROOF: Each is equal to $(x_1 y_1 + x_1 z_1, \ldots, x_n y_n + x_n z_n)$. □

Lemma 192.

$$|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \|\vec{y}\|$$

PROOF:
⟨1⟩1. ASSUME: $\vec{x} \neq \vec{0} \neq \vec{y}$
PROOF: Otherwise both sides are 0.
⟨1⟩2. LET: $a = 1/\|\vec{x}\|$
⟨1⟩3. LET: $b = 1/\|\vec{y}\|$
⟨1⟩4. $(a\vec{x} + b\vec{y})^2 \geq 0$ and $(a\vec{x} - b\vec{y})^2 \geq 0$
⟨1⟩5. $a^2\|\vec{x}\|^2 + 2ab\vec{x} \cdot \vec{y} + b^2\|\vec{y}\|^2 \geq 0$ and $a^2\|\vec{x}\|^2 - 2ab\vec{x} \cdot \vec{y} + b^2\|\vec{y}\|^2 \geq 0$
⟨1⟩6. $2ab\vec{x} \cdot \vec{y} + 2 \geq 0$ and $-2ab\vec{x} \cdot \vec{y} + 2 \geq 0$
⟨1⟩7. $\vec{x} \cdot \vec{y} \geq -1/ab$ and $\vec{x} \cdot \vec{y} \leq 1/ab$
⟨1⟩8. $\vec{x} \cdot \vec{y} \geq -\|\vec{x}\|\|\vec{y}\|$ and $\vec{x} \cdot \vec{y} \leq \|\vec{x}\|\|\vec{y}\|$
□

Lemma 193 (Triangle Inequality)**.**

$$\|\vec{x}+\vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$$

PROOF:

$$\begin{aligned}
\|\vec{x}+\vec{y}\|^2 &= \|\vec{x}\|^2 + 2\vec{x}\cdot\vec{y} + \|\vec{y}\|^2 \\
&\leq \|\vec{x}\|^2 + 2\|\vec{x}\|\|\vec{y}\| + \|\vec{y}\|^2 && \text{(Lemma 192)} \\
&= (\|\vec{x}\| + \|\vec{y}\|)^2
\end{aligned}$$

□

Definition 194 (Euclidean Metric)**.** Let $n \geq 1$. The *Euclidean metric* on $\mathbb{R}^n$ is defined by

$$d(\vec{x},\vec{y}) = \|\vec{x}-\vec{y}\| \ .$$

We prove this is a metric.

⟨1⟩1. $d(\vec{x},\vec{y}) \geq 0$
PROOF: Immediate from definition.
⟨1⟩2. $d(\vec{x},\vec{y}) = 0$ if and only if $\vec{x}=\vec{y}$
PROOF: $d(\vec{x},\vec{y}) = 0$ if and only if $\vec{x}-\vec{y}=\vec{0}$.
⟨1⟩3. $d(\vec{x},\vec{y}) = d(\vec{y},\vec{x})$
PROOF: Immediate from definition.
⟨1⟩4. $d(\vec{x},\vec{z}) \leq d(\vec{x},\vec{y}) + d(\vec{y},\vec{z})$
PROOF:

$$\begin{aligned}
\|\vec{x}-\vec{z}\| &= \|(\vec{x}-\vec{y}) + (\vec{y}-\vec{z})\| \\
&\leq \|\vec{x}-\vec{y}\| + \|\vec{y}-\vec{z}\| && \text{(Lemma 193)}
\end{aligned}$$

□

Proposition 195. *The Euclidean metric induces the standard topology on* $\mathbb{R}^n$*.*

PROOF:
⟨1⟩1. LET: ρ be the square metric.
⟨1⟩2. For all $\vec{a} \in \mathbb{R}^n$ and $\epsilon > 0$, we have $B_d(\vec{a},\epsilon) \subseteq B_\rho(\vec{a},\epsilon)$
⟨2⟩1. LET: $\vec{x} \in B_d(\vec{a},\epsilon)$
⟨2⟩2. $\sqrt{(x_1-a_1)^2 + \cdots + (x_n-a_n)^2} < \epsilon$
⟨2⟩3. $(x_1-a_1)^2 + \cdots + (x_n-a_n)^2 < \epsilon^2$
⟨2⟩4. For all i we have $(x_i-a_i)^2 < \epsilon^2$
⟨2⟩5. For all i we have $|x_i-a_i| < \epsilon$
⟨2⟩6. $\rho(\vec{x},\vec{a}) < \epsilon$
⟨1⟩3. For all $\vec{a} \in \mathbb{R}^n$ and $\epsilon > 0$, we have $B_\rho(\vec{a},\epsilon/\sqrt{n}) \subseteq B_d(\vec{a},\epsilon)$
⟨2⟩1. LET: $\vec{x} \in B_\rho(\vec{a},\epsilon/\sqrt{n})$
⟨2⟩2. $\rho(\vec{x},\vec{a}) < \epsilon/\sqrt{n}$
⟨2⟩3. For all i we have $|x_i-x_a| < \epsilon/\sqrt{n}$
⟨2⟩4. For all i we have $(x_i-x_a)^2 < \epsilon^2/n$
⟨2⟩5. $d(\vec{x},\vec{a}) < \epsilon$
⟨1⟩4. Q.E.D.
PROOF: By Lemma 173.

□

Lemma 196. *If* $\sum_{i=0}^{\infty} x_i^2$ *and* $\sum_{i=0}^{\infty} y_i^2$ *converge then* $\sum_{i=0}^{\infty} |x_i y_i|$ *converges.*

PROOF:
$\langle 1 \rangle 1$. For all $N \geq 0$ we have $\sum_{i=0}^{N} |x_i y_i| \leq \sqrt{\sum_{i=0}^{N} |x_i|^2} \sqrt{\sum_{i=0}^{N} |y_i|^2}$
PROOF: By the Cauchy-Schwarz inequality
$\langle 1 \rangle 2$. Q.E.D.
PROOF: Since $\sum_{i=0}^{N} |x_i y_i|$ is an increasing sequence bounded above by $(\sum_{i=0}^{\infty} x_i^2)(\sum_{i=0}^{\infty} y_i^2)$.
$\square$

Corollary 196.1. *If* $\sum_{i=0}^{\infty} x_i^2$ *and* $\sum_{i=0}^{\infty} y_i^2$ *converge then* $\sum_{i=0}^{\infty} (x_i + y_i)^2$ *converges.*

PROOF: Since $\sum_{i=0}^{\infty} x_i^2$, $\sum_{i=0}^{\infty} y_i^2$ and $2 \sum_{i=0}^{\infty} x_i y_i$ all converge.

Definition 197 (l^2-metric)**.** The *l^2-metric* on

$$\left\{ (x_n) \in \mathbb{R}^{\omega} \mid \sum_{n=0}^{\infty} x_n^2 \text{ converges} \right\}$$

is defined by

$$d(x, y) = \left(\sum_{n=0}^{\infty} (x_n - y_n)^2 \right)^{1/2}$$

We prove this is a metric.

PROOF:
$\langle 1 \rangle 1$. d is well-defined.
PROOF: By Corollary 196.1.
$\langle 1 \rangle 2$. $d(x, y) \geq 0$
$\langle 1 \rangle 3$. $d(x, y) = 0$ if and only if $x = y$
$\langle 1 \rangle 4$. $d(x, y) = d(y, x)$
$\langle 1 \rangle 5$. $d(x, z) \leq d(x, y) + d(y, z)$
PROOF: By Lemma 193.
$\square$

Theorem 198. *Addition is a continuous function* $\mathbb{R}^2 \to \mathbb{R}$*.*

PROOF:
$\langle 1 \rangle 1$. LET: $a, b \in \mathbb{R}$
$\langle 1 \rangle 2$. LET: $\epsilon > 0$
$\langle 1 \rangle 3$. LET: $\delta = \epsilon / 2$
$\langle 1 \rangle 4$. LET: $x, y \in \mathbb{R}$
$\langle 1 \rangle 5$. ASSUME: $\rho((a, b), (x, y)) < \delta$
$\langle 1 \rangle 6$. $|(a + b) - (x + y)| < \epsilon$

PROOF:

$$\begin{aligned}|(a+b)-(x+y)| &= |a-x|+|b-y| \\ &\leq 2\rho((a,b),(x,y)) \\ &< 2\delta \\ &= \epsilon\end{aligned}$$

⟨1⟩7. Q.E.D.
PROOF: Theorem 179

□

Theorem 199. *Multiplication is a continuous function* $\mathbb{R}^2 \to \mathbb{R}$.

PROOF:
⟨1⟩1. LET: $a, b \in \mathbb{R}$
⟨1⟩2. LET: $\epsilon > 0$
⟨1⟩3. LET: $\delta = \min(\epsilon/(|a|+|b|+1), 1)$
⟨1⟩4. LET: $x, y \in \mathbb{R}$
⟨1⟩5. ASSUME: $\rho((a,b),(x,y)) < \delta$
⟨1⟩6. $|ab - xy| < \epsilon$
PROOF:

$$\begin{aligned}|ab-xy| &= |a(b-y)+(a-x)b-(a-x)(b-y)| \\ &\leq |a||b-y|+|b||a-x|+|a-x||b-y| \\ &< |a|\delta+|b|\delta+\delta^2 && (\langle 1\rangle 5) \\ &\leq |a|\delta+|b|\delta+\delta && (\langle 1\rangle 3) \\ &\leq \epsilon && (\langle 1\rangle 3)\end{aligned}$$

⟨1⟩7. Q.E.D.
PROOF: Theorem 179

□

Theorem 200. *The function* $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ *defined by* $f(x) = x^{-1}$ *is continuous.*

PROOF:
⟨1⟩1. For all $a \in \mathbb{R}$ we have $f^{-1}((a, +\infty))$ is open.
PROOF: The set is

$$\begin{aligned}(a^{-1}, +\infty) &\text{if } a > 0 \\ (0, +\infty) &\text{if } a = 0 \\ (-\infty, a^{-1}) \cup (0, +\infty) &\text{if } a < 0\end{aligned}$$

⟨1⟩2. For all $a \in \mathbb{R}$ we have $f^{-1}((-\infty, a))$ is open.
PROOF: Similar.
⟨1⟩3. Q.E.D.
PROOF: By Proposition 77 and Lemma 103.

□

25 The Uniform Topology

Definition 201 (Uniform Metric)**.** Let J be a set. The *uniform metric* $\overline{\rho}$ on $\mathbb{R}^J$ is defined by

$$\overline{\rho}(a,b) = \sup_{j \in J} \overline{d}(a_j, b_j)$$

where $\overline{d}$ is the standard bounded metric on $\mathbb{R}$.

The *uniform topology* on $\mathbb{R}^J$ is the topology induced by the uniform metric.

We prove this is a metric.

PROOF:
$\langle 1 \rangle 1.$ $\overline{\rho}(a,b) \geq 0$
PROOF: Immediate from definitions.
$\langle 1 \rangle 2.$ $\overline{\rho}(a,b) = 0$ if and only if $a = b$
PROOF: Immediate from definitions.
$\langle 1 \rangle 3.$ $\overline{\rho}(a,b) = \overline{\rho}(b,a)$
PROOF: Immediate from definitions.
$\langle 1 \rangle 4.$ $\overline{\rho}(a,c) \leq \overline{\rho}(a,b) + \overline{\rho}(b,c)$
PROOF:

$$\begin{aligned}
\overline{\rho}(a,c) &= \sup_{j \in J} \overline{d}(a_j, c_j) \\
&\leq \sup_{j \in J} (\overline{d}(a_j, b_j) + \overline{d}(b_j, c_j)) \\
&\leq \sup_{j \in J} \overline{d}(a_j, b_j) + \sup_{j \in J} \overline{d}(b_j, c_j) \\
&= \overline{\rho}(a,b) + \overline{\rho}(b,c)
\end{aligned}$$

□

Proposition 202. *The uniform topology on $\mathbb{R}^J$ is finer than the product topology.*

PROOF:
$\langle 1 \rangle 1.$ LET: $j \in J$ and U be open in $\mathbb{R}$
PROVE: ${\pi_j}^{-1}(U)$ is open in the uniform topology.
$\langle 1 \rangle 2.$ LET: $a \in {\pi_j}^{-1}(U)$
$\langle 1 \rangle 3.$ PICK $\epsilon > 0$ such that $(a_j - \epsilon, a_j + \epsilon) \subseteq U$
$\langle 1 \rangle 4.$ $B_{\overline{\rho}}(a, \epsilon) \subseteq {\pi_j}^{-1}(U)$
□

Proposition 203. *The uniform topology on $\mathbb{R}^J$ is coarser than the box topology.*

PROOF:
$\langle 1 \rangle 1.$ LET: $a \in \mathbb{R}^J$ and $\epsilon > 0$
PROVE: $B(a, \epsilon)$ is open in the box topology.
$\langle 1 \rangle 2.$ LET: $b \in B(a, \epsilon)$
$\langle 1 \rangle 3.$ For $j \in J$ we have $|a_j - b_j| < \epsilon$
$\langle 1 \rangle 4.$ For $j \in J$,

LET: $\delta_j = (\epsilon - |a_j - b_j|)/2$

$\langle 1 \rangle 5.$ $\prod_{j \in J}(b_j - \delta_j, b_j + \delta_j) \subseteq B(a, \epsilon)$

□

Proposition 204. *The uniform topology on $\mathbb{R}^J$ is strictly finer than the product topology if and only if J is infinite.*

PROOF:

$\langle 1 \rangle 1.$ If J is finite then the uniform and product topologies coincide.

PROOF: The uniform, box and product topologies are all the same.

$\langle 1 \rangle 2.$ If J is infinite then the uniform and product topologies are different.

PROOF: The set $B(\vec{0}, 1)$ is open in the uniform topology but not the product topology.

□

Proposition 205 (DC)**.** *The uniform topology on $\mathbb{R}^J$ is strictly coarser than the box topology if and only if J is infinite.*

PROOF:

$\langle 1 \rangle 1.$ If J is finite then the uniform and box topologies coincide.

PROOF: The uniform, box and product topologies are all the same.

$\langle 1 \rangle 2.$ If J is infinite then the uniform and box topologies are different.

PROOF: Pick an ω-sequence $(j_1, j_2, \ldots)$ in J. Let $U = \prod_{j \in J} U_j$ where $U_{j_i} = (-1/i, 1/i)$ and $U_j = (-1, 1)$ for all other j. Then $\vec{0} \in U$ but there is no $\epsilon > 0$ such that $B(\vec{0}, \epsilon) \subseteq U$.

□

Proposition 206. *The closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ under the uniform topology is $\mathbb{R}^\omega$.*

PROOF: Given any open ball $B(a, \epsilon)$, pick an integer N such that $1/\epsilon < N$. Then $B(a, \epsilon)$ includes sequences whose nth entry is 0 for all $n \geq N$. □

26 Uniform Convergence

Definition 207 (Uniform Convergence)**.** Let X be a set and Y a metric space. Let $(f_n : X \to Y)$ be a sequence of functions and $f : X \to Y$ be a function. Then f_n *converges uniformly* to f as $n \to \infty$ if and only if, for all $\epsilon > 0$, there exists N such that, for all $n \geq N$ and $x \in X$, we have $d(f_n(x), f(x)) < \epsilon$.

Example 208. Define $f_n : [0, 1] \to \mathbb{R}$ by $f_n(x) = x^n$ for $n \geq 1$, and $f : [0, 1] \to \mathbb{R}$ by $f(x) = 0$ if $x < 1$, $f(1) = 1$. Then f_n converges to f pointwise but not uniformly.

Theorem 209 (Uniform Limit Theorem)**.** *Let X be a topological space and Y a metric space. Let $(f_n : X \to Y)$ be a sequence of continuous functions and $f : X \to Y$ be a function. If f_n converges uniformly to f as $n \to \infty$, then f is continuous.*

Proof:
$\langle 1\rangle 1.$ Let: $x \in X$ and $\epsilon > 0$
$\langle 1\rangle 2.$ Pick N such that, for all $n \geq N$ and $y \in X$, we have $d(f_n(y), f(y)) < \epsilon/3$
$\langle 1\rangle 3.$ Pick a neighbourhood U of x such that $f_N(U) \subseteq B(f_N(x), \epsilon/3)$
Prove: $f(U) \subseteq B(f(x), \epsilon)$
$\langle 1\rangle 4.$ Let: $y \in U$
$\langle 1\rangle 5.$ $d(f(y), f(x)) < \epsilon$
Proof:

$$\begin{aligned} d(f(y), f(x)) &\leq d(f(y), f_N(y)) + d(f_N(y), f_N(x)) + d(f_N(x), f(x)) && \text{(Triangle Inequality)} \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 && (\langle 1\rangle 2,\ \langle 1\rangle 3) \\ &= \epsilon \end{aligned}$$

□

Proposition 210. *Let X be a topological space and Y a metric space. Let $(f_n : X \to Y)$ be a sequence of continuous functions and $f : X \to Y$ be a function. Let (a_n) be a sequence of points in X and $a \in X$. If f_n converges uniformly to f and a_n converges to a in X then $f_n(a_n)$ converges to $f(a)$ uniformly in Y.*

Proof:
$\langle 1\rangle 1.$ Let: $\epsilon > 0$
$\langle 1\rangle 2.$ Pick N_1 such that, for all $n \geq N_1$ and $x \in X$, we have $d(f_n(x), f(x)) < \epsilon/2$
$\langle 1\rangle 3.$ Pick N_2 such that, for all $n \geq N_2$, we have $a_n \in f^{-1}(B(a, \epsilon/2))$
Proof: Using the fact that f is continuous from the Uniform Limit Theorem.
$\langle 1\rangle 4.$ Let: $N = \max(N_1, N_2)$
$\langle 1\rangle 5.$ Let: $n \geq N$
$\langle 1\rangle 6.$ $d(f_n(a_n), f(a)) < \epsilon$
Proof:

$$\begin{aligned} d(f_n(a_n), f(a)) &\leq d(f_n(a_n), f(a_n)) + d(f(a_n), f(a)) && \text{(Triangle Inequality)} \\ &< \epsilon/2 + \epsilon/2 && (\langle 1\rangle 2,\ \langle 1\rangle 3) \\ &= \epsilon \end{aligned}$$

□

Proposition 211. *Let X be a set. Let $(f_n : X \to \mathbb{R})$ be a sequence of functions and $f : X \to \mathbb{R}$ be a function. Then f_n converges unifomly to f as $n \to \infty$ if and only if $f_n \to f$ as $n \to \infty$ in $\mathbb{R}^X$ under the uniform topology.*

Proof:
$\langle 1\rangle 1.$ If f_n converges uniformly to f then f_n converges to f under the uniform topology.
$\langle 2\rangle 1.$ Assume: f_n converges uniformly to f
$\langle 2\rangle 2.$ Let: $\epsilon > 0$
$\langle 2\rangle 3.$ Pick N such that, for all $n \geq N$ and $x \in X$, we have $d(f_n(x), f(x)) < \epsilon/2$
$\langle 2\rangle 4.$ For all $n \geq N$ we have $\overline{\rho}(f_n, f) \leq \epsilon/2$
$\langle 2\rangle 5.$ For all $n \geq N$ we have $\overline{\rho}(f_n, f) < \epsilon$

⟨1⟩2. If f_n converges to f under the uniform topology then f_n converges uniformly to f.

⟨2⟩1. ASSUME: f_n converges to f under the uniform topology.

⟨2⟩2. LET: $\epsilon > 0$

⟨2⟩3. PICK N such that, for all $n \geq N$, we have $\overline{\rho}(f_n, f) < \min(\epsilon, 1/2)$

⟨2⟩4. LET: $n \geq N$

⟨2⟩5. LET: $x \in X$

⟨2⟩6. $\overline{\rho}(f_n, f) < \min(\epsilon, 1/2)$

PROOF: From ⟨2⟩3.

⟨2⟩7. $d(f_n(x), f(x)) < \min(\epsilon, 1/2)$

⟨2⟩8. $d(f_n(x), f(x)) < \epsilon$

□

27 Isometric Imbeddings

Definition 212. Let X and Y be metric spaces. An *isometric imbedding* $f : X \to Y$ is a function such that, for all $x, y \in X$, we have $d(f(x), f(y)) = d(x, y)$.

Proposition 213. *Every isometric imbedding is an imbedding.*

PROOF:

⟨1⟩1. LET: $f : X \to Y$ be an isometric imbedding.

⟨1⟩2. f is injective.

PROOF: If $f(x) = f(y)$ then $d(f(x), f(y)) = 0$ hence $d(x, y) = 0$ hence $x = y$.

⟨1⟩3. f is continuous.

PROOF: For all $\epsilon > 0$, if $d(x, y) < \epsilon$ then $d(f(x), f(y)) < \epsilon$.

⟨1⟩4. $f : X \to f(X)$ is an open map.

PROOF: $f(B(a, \epsilon)) = B(f(a), \epsilon) \cap f(X)$.

□