

# Topology

Robin Adams

June 21, 2022

# Contents

<b>I</b>	<b>Set Theory</b>	<b>5</b>
<b>1</b>	<b>Classes</b>	<b>6</b>
1.1	Classes . . . . .	6
1.2	Subclasses . . . . .	6
1.3	The Empty Class . . . . .	6
1.4	Finite Classes . . . . .	6
1.5	Universal Class . . . . .	7
1.6	Union . . . . .	7
1.7	Intersection . . . . .	7
1.8	Relative Complement . . . . .	7
<b>2</b>	<b>Sets</b>	<b>8</b>
2.1	Membership . . . . .	8
2.2	The Empty Set . . . . .	8
2.3	Pair Sets . . . . .	8
2.4	Unions . . . . .	8
2.5	Power Set . . . . .	9
2.6	Covers . . . . .	9
2.7	Subset Axioms . . . . .	9
2.8	Intersection . . . . .	10
<b>3</b>	<b>Relations</b>	<b>11</b>
3.1	Ordered Pairs . . . . .	11
3.2	Cartesian Product . . . . .	12
3.3	Relations . . . . .	12
3.4	Domain . . . . .	12
3.5	Range . . . . .	13
3.6	Field . . . . .	13
3.7	Functions . . . . .	13
3.8	Single-Rooted . . . . .	13
3.9	Surjective . . . . .	14
3.10	Inverse . . . . .	14
3.11	Composition . . . . .	15
3.12	Identity Function . . . . .	16

3.13	Restriction . . . . .	17
3.14	Image . . . . .	17
3.15	Infinite Cartesian Product . . . . .	19
3.16	Reflexive Relations . . . . .	20
3.17	Symmetric . . . . .	20
3.18	Transitivity . . . . .	20
3.19	Equivalence Relations . . . . .	20
3.20	Equivalence Class . . . . .	21
3.21	Disjoint . . . . .	21
3.22	Partitions . . . . .	21
3.23	Quotient Sets . . . . .	22
3.24	The Finite Intersection Property . . . . .	23
3.25	Countable Intersection Property . . . . .	24
3.26	The Axiom of Choice . . . . .	25
3.27	Choice Functions . . . . .	25
3.28	Transitive . . . . .	26
3.29	Minimal Elements . . . . .	26
3.30	Well-Founded Relations . . . . .	26
3.31	Transitive Closure . . . . .	27
<b>4</b>	<b>Cardinal Numbers</b>	<b>28</b>
4.1	Countability . . . . .	29
4.2	Order Theory . . . . .	30
4.3	Partially Ordered Sets . . . . .	33
4.4	Strict Partial Order . . . . .	33
4.5	Strict Linear Orders . . . . .	33
4.6	Well Orderings . . . . .	33
4.7	Ordinal Numbers . . . . .	37
4.8	Cardinal Numbers . . . . .	39
4.9	Cardinal Arithmetic . . . . .	39
4.10	Rank of a Set . . . . .	40
4.11	Transfinite Recursion Again . . . . .	41
4.12	Alephs . . . . .	41
4.13	Ordinal Arithmetic . . . . .	42
4.14	Beth Cardinals . . . . .	43
4.15	Cofinality . . . . .	43
4.16	Inaccessible Cardinals . . . . .	44
<b>5</b>	<b>Natural Numbers</b>	<b>45</b>
5.1	Successors . . . . .	45
5.2	Inductive Sets . . . . .	45
5.3	Natural Numbers . . . . .	45
5.4	Peano Systems . . . . .	48
5.5	Arithmetic . . . . .	48
<b>6</b>	<b>Integers</b>	<b>50</b>

<b>7</b>	<b>Rational Numbers</b>	<b>52</b>
<b>8</b>	<b>Real Numbers</b>	<b>54</b>
8.1	The Cantor Set . . . . .	55
<b>9</b>	<b>Finite Sets</b>	<b>56</b>
9.1	Real Analysis . . . . .	57
9.2	Group Theory . . . . .	58
9.3	Topological Spaces . . . . .	59
9.4	Closed Set . . . . .	60
9.5	Interior . . . . .	61
9.6	Closure . . . . .	62
9.7	Boundary . . . . .	64
9.8	Limit Points . . . . .	64
9.9	Basis for a Topology . . . . .	65
9.10	Local Basis at a Point . . . . .	69
9.11	Convergence . . . . .	70
9.12	Locally Finite Sets . . . . .	72
9.13	Open Maps . . . . .	73
9.14	Continuous Functions . . . . .	73
9.15	Homeomorphisms . . . . .	78
9.16	The Order Topology . . . . .	79
9.17	The nth Root Function . . . . .	82
9.18	The Product Topology . . . . .	82
9.18.1	Continuous in Each Variable Separately . . . . .	85
9.19	The Subspace Topology . . . . .	85
9.20	The Box Topology . . . . .	89
9.21	$T_1$ Spaces . . . . .	91
9.22	Hausdorff Spaces . . . . .	92
9.23	The First Countability Axiom . . . . .	95
9.24	Strong Continuity . . . . .	97
9.25	Saturated Sets . . . . .	97
9.26	Quotient Maps . . . . .	98
9.27	Quotient Topology . . . . .	101
9.28	Retractions . . . . .	103
9.29	Homogeneous Spaces . . . . .	103
9.30	Regular Spaces . . . . .	103
9.31	Connected Spaces . . . . .	103
9.32	Totally Disconnected Spaces . . . . .	113
9.33	Paths and Path Connectedness . . . . .	113
9.34	The Topologist's Sine Curve . . . . .	116
9.35	The Long Line . . . . .	118
9.36	Components . . . . .	119
9.37	Path Components . . . . .	120
9.38	Local Connectedness . . . . .	121
9.39	Local Path Connectedness . . . . .	123

9.40	Weak Local Connectedness . . . . .	125
9.41	Quasicomponents . . . . .	125
9.42	Open Coverings . . . . .	126
9.43	Lindelöf Spaces . . . . .	126
9.44	The Second Countability Axiom . . . . .	127
9.45	Sequential Compactness . . . . .	128
9.46	Limit Point Compactness . . . . .	128
9.47	Compact Spaces . . . . .	129
9.48	Perfect Maps . . . . .	141
9.49	Isolated Points . . . . .	142
<b>10</b>	<b>Topological Groups</b>	<b>144</b>
10.1	The Metric Topology . . . . .	152
10.2	Real Linear Algebra . . . . .	159
10.3	The Uniform Topology . . . . .	166
10.4	Uniform Convergence . . . . .	169
10.5	Isometric Imbeddings . . . . .	170
10.6	Distance to a Set . . . . .	171
10.7	Lebesgue Numbers . . . . .	172
10.8	Uniform Continuity . . . . .	174
10.9	Epsilon-neighbourhoods . . . . .	174

Part I

Set Theory

# Chapter 1

## Classes

### 1.1 Classes

We speak informally of *classes*. A class is specified by a unary predicate. We write  $\{x \mid P(x)\}$  for the class determined by the predicate  $P(x)$ .

**Definition 1.1** (Membership). Let  $a$  be an object and  $\mathbf{A}$  a class. We define the proposition  $a \in \mathbf{A}$  ( $a$  is a *member* or *element* of  $\mathbf{A}$ ) as follows:

The proposition  $a \in \{x \mid P(x)\}$  is the proposition  $P(a)$ .

**Definition 1.2** (Equality of Classes). Let  $\mathbf{A}$  and  $\mathbf{B}$  be classes. We say  $\mathbf{A}$  and  $\mathbf{B}$  are *equal*,  $\mathbf{A} = \mathbf{B}$ , if and only if they have exactly the same elements.

### 1.2 Subclasses

**Definition 1.3** (Subclass). Let  $\mathbf{A}$  and  $\mathbf{B}$  be classes. We say  $\mathbf{A}$  is a *subclass* of  $\mathbf{B}$ ,  $\mathbf{A} \subseteq \mathbf{B}$ , if and only if every member of  $\mathbf{A}$  is a member of  $\mathbf{B}$ .

We say  $\mathbf{A}$  is a *proper* subclass of  $\mathbf{B}$ ,  $\mathbf{A} \subset \mathbf{B}$ , if and only if  $\mathbf{A} \subseteq \mathbf{B}$  and  $\mathbf{A} \neq \mathbf{B}$ .

### 1.3 The Empty Class

**Definition 1.4** (Empty Class). The *empty* class  $\emptyset$  is  $\{x \mid \perp\}$ .

### 1.4 Finite Classes

**Definition 1.5.** For any objects  $a_1, \dots, a_n$ , we write  $\{a_1, \dots, a_n\}$  for the class  $\{x \mid x = a_1 \vee \dots \vee x = a_n\}$ .

## 1.5 Universal Class

**Definition 1.6** (Universal Class). The *universal class*  $\mathbf{V}$  is the class  $\{x \mid \top\}$ .

## 1.6 Union

**Definition 1.7** (Union). For any classes  $\mathbf{A}$  and  $\mathbf{B}$ , the *union*  $\mathbf{A} \cup \mathbf{B}$  is the class  $\{x \mid x \in \mathbf{A} \vee x \in \mathbf{B}\}$ .

## 1.7 Intersection

**Definition 1.8** (Intersection). For any classes  $\mathbf{A}$  and  $\mathbf{B}$ , the *intersection*  $\mathbf{A} \cap \mathbf{B}$  is the class  $\{x \mid x \in \mathbf{A} \wedge x \in \mathbf{B}\}$ .

## 1.8 Relative Complement

**Definition 1.9** (Relative Complement). For any classes  $\mathbf{A}$  and  $\mathbf{B}$ , the *relative complement*  $\mathbf{A} - \mathbf{B}$  is  $\{x \in \mathbf{A} \mid x \notin \mathbf{B}\}$ .



# Chapter 2

## Sets

### 2.1 Membership

We take as undefined the notion of *set*.

We take as undefined the binary relation of *membership*,  $\in$ . If  $a \in A$  we say  $a$  is a *member* or *element* of  $A$ . If this does not hold, we write  $a \notin A$ .

**Axiom 2.1** (Axiom of Extensionality). *Two sets with exactly the same elements are equal.*

We may therefore identify the set  $A$  with the class  $\{x \mid x \in A\}$ .

We say a class  $\mathbf{A}$  is a *set* iff there exists a set  $A$  such that  $A = \mathbf{A}$ . That is,  $\{x \mid P(x)\}$  is a set if and only if there exists a set  $A$  such that, for all  $x$ , we have  $x \in A$  if and only if  $P(x)$ .

### 2.2 The Empty Set

**Axiom 2.2** (Empty Set Axiom). *The empty class  $\emptyset$  is a set.*

### 2.3 Pair Sets

**Axiom 2.3** (Pairing Axiom). *For any objects  $u$  and  $v$ , the class  $\{u, v\}$  is a set.*

**Theorem 2.4** (Pairing). *For any object  $a$ , the class  $\{a\}$  is a set.*

PROOF: It is  $\{a, a\}$ .  $\square$

### 2.4 Unions

**Definition 2.5** (Union). For any class of sets  $\mathbf{A}$ , the *union*  $\bigcup \mathbf{A}$  is the class  $\{x \mid \exists A \in \mathbf{A}. x \in A\}$ .

**Axiom 2.6** (Union Axiom). *For any set  $A$ , the union  $\bigcup A$  is a set.*

**Theorem 2.7** (Union, Pairing). *For any sets  $A$  and  $B$ , the class  $A \cup B$  is a set.*

PROOF: It is  $\bigcup\{A, B\}$ .  $\square$

**Theorem Schema 2.8** (Union, Pairing). *For any objects  $a_1, \dots, a_n$ , the class  $\{a_1, \dots, a_n\}$  is a set.*

PROOF: We prove each theorem using the last since  $\{a_1, \dots, a_n, a_{n+1}\} = \{a_1, \dots, a_n\} \cup \{a_{n+1}\}$ .  $\square$

## 2.5 Power Set

**Definition 2.9** (Power Class). For any class  $\mathbf{A}$ , the *power class*  $\mathcal{P}\mathbf{A}$  is the class of all subsets of  $\mathbf{A}$ .

**Axiom 2.10** (Power Set Axiom). *For any set  $A$ , the power class  $\mathcal{P}A$  is a set.*

## 2.6 Covers

**Definition 2.11** (Cover). Let  $\mathbf{X}$  be a class and  $\mathcal{A} \subseteq \mathcal{P}\mathbf{X}$ . Then  $\mathcal{A}$  *covers*  $\mathbf{X}$ , or is a *covering* of  $\mathbf{X}$ , if and only if  $\bigcup \mathcal{A} = \mathbf{X}$ .

## 2.7 Subset Axioms

**Axiom Schema 2.12** (Subset Axioms, Aussonderung Axioms). *For any classes  $\mathbf{A}$  and  $\mathbf{B}$ , if  $\mathbf{A} \subseteq \mathbf{B}$  and  $\mathbf{B}$  is a set then  $\mathbf{A}$  is a set.*

**Theorem 2.13** (Subset). *The universal class  $\mathbf{V}$  is not a set.*

PROOF:

$\langle 1 \rangle 1$ . ASSUME:  $\mathbf{V}$  is a set.

$\langle 1 \rangle 2$ . LET:  $R = \{x \in \mathbf{V} \mid x \notin x\}$

$\langle 1 \rangle 3$ .  $R \in R$  if and only if  $R \notin R$

$\langle 1 \rangle 4$ . Q.E.D.

PROOF: This is a contradiction.

$\square$

**Theorem 2.14** (Subset). *If  $A$  is a set and  $\mathbf{B}$  is a class then  $A - \mathbf{B}$  is a set.*

PROOF: It is a subset of  $A$ .  $\square$

## 2.8 Intersection

**Definition 2.15** (Intersection). For any class  $\mathbf{A}$  of sets, the *intersection*  $\bigcap \mathbf{A}$  is the class  $\{x \mid \forall A \in \mathbf{A}. x \in A\}$ .

**Theorem 2.16** (Subset). *For any nonempty class  $\mathbf{A}$  of sets, we have  $\bigcap \mathbf{A}$  is a set.*

PROOF:

$\langle 1 \rangle 1.$  PICK  $A \in \mathbf{A}$

$\langle 1 \rangle 2.$   $\bigcap \mathbf{A} \subseteq A$

$\langle 1 \rangle 3.$  Q.E.D.

PROOF: By a Subset Axiom.

□

**Theorem 2.17** (Subset). *For any sets  $A$  and  $B$ , the class  $A \cap B$  is a set.*

PROOF: From a Subset Axiom since  $A \cap B \subseteq A$ . □

# Chapter 3

## Relations

### 3.1 Ordered Pairs

**Definition 3.1** (Ordered Pair (Pairing)). For any sets  $x$  and  $y$ , the *ordered pair*  $(x, y)$  is defined to be  $\{\{x\}, \{x, y\}\}$ .

**Theorem 3.2** (Pairing). For any sets  $u, v, x, y$ , we have  $(u, v) = (x, y)$  if and only if  $u = x$  and  $v = y$

PROOF:

$\langle 1 \rangle 1.$  ASSUME:  $\{\{u\}, \{u, v\}\} = \{\{x\}, \{x, y\}\}$

$\langle 1 \rangle 2.$   $\{u\} \in \{\{x\}, \{x, y\}\}$

$\langle 1 \rangle 3.$   $\{u, v\} \in \{\{x\}, \{x, y\}\}$

$\langle 1 \rangle 4.$   $\{u\} = \{x\}$  or  $\{u\} = \{x, y\}$

$\langle 1 \rangle 5.$   $\{u, v\} = \{x\}$  or  $\{u, v\} = \{x, y\}$

$\langle 1 \rangle 6.$  CASE:  $\{u\} = \{x, y\}$

$\langle 2 \rangle 1.$   $u = x = y$

$\langle 2 \rangle 2.$   $u = v = x = y$

PROOF: From  $\langle 1 \rangle 5$

$\langle 1 \rangle 7.$  CASE:  $\{u, v\} = \{x\}$

PROOF: Similar.

$\langle 1 \rangle 8.$  CASE:  $\{u\} = \{x\}$  and  $\{u, v\} = \{x, y\}$

$\langle 2 \rangle 1.$   $u = x$

$\langle 2 \rangle 2.$   $u = y$  or  $v = y$

$\langle 2 \rangle 3.$  CASE:  $u = y$

PROOF: This case is the case considered in  $\langle 1 \rangle 6$ .

$\langle 2 \rangle 4.$  CASE:  $v = y$

PROOF: We have  $u = x$  and  $v = y$  as required.

□

**Lemma 3.3** (Pairing, Power Set). Let  $x, y$  and  $C$  be sets. If  $x \in C$  and  $y \in C$  then  $(x, y) \in \mathcal{P}C$ .

PROOF:

- $\langle 1 \rangle 1.$  LET:  $x, y$  and  $C$  be sets.
- $\langle 1 \rangle 2.$  ASSUME:  $x \in C$
- $\langle 1 \rangle 3.$  ASSUME:  $y \in C$
- $\langle 1 \rangle 4.$   $\{x\} \subseteq C$
- $\langle 1 \rangle 5.$   $\{x, y\} \subseteq C$
- $\langle 1 \rangle 6.$   $\{x\} \in \mathcal{P}C$
- $\langle 1 \rangle 7.$   $\{x, y\} \in \mathcal{P}C$
- $\langle 1 \rangle 8.$   $\{\{x\}, \{x, y\}\} \subseteq \mathcal{P}C$
- $\langle 1 \rangle 9.$   $\{\{x\}, \{x, y\}\} \in \mathcal{P}\mathcal{P}C$

□

**Lemma 3.4** (Pairing, Union). *Let  $x, y$  and  $A$  be sets. If  $(x, y) \in A$  then  $x$  and  $y$  belong to  $\bigcup \bigcup A$ .*

PROOF:

- $\langle 1 \rangle 1.$  LET:  $x, y$  and  $A$  be sets.
- $\langle 1 \rangle 2.$  ASSUME:  $(x, y) \in A$
- $\langle 1 \rangle 3.$   $\{x, y\} \in \bigcup A$
- $\langle 1 \rangle 4.$   $x \in \bigcup \bigcup A$
- $\langle 1 \rangle 5.$   $y \in \bigcup \bigcup A$

□

## 3.2 Cartesian Product

**Definition 3.5** (Cartesian Product (Pairing)). Let  $\mathbf{A}$  and  $\mathbf{B}$  be classes. The *Cartesian product*  $\mathbf{A} \times \mathbf{B}$  is the class  $\{(x, y) \mid x \in \mathbf{A}, y \in \mathbf{B}\}$ .

**Theorem 3.6** (Pairing, Union, Power Set, Subset). *For any sets  $A$  and  $B$ , the Cartesian product  $A \times B$  is a set.*

PROOF: It is a subset of  $\mathcal{P}\mathcal{P}(A \cup B)$  by Lemma 3.3. □

## 3.3 Relations

**Definition 3.7** (Relation (Pairing)). A *relation* is a class of ordered pairs.

Given a relation  $\mathbf{R}$ , we write  $x\mathbf{R}y$  for  $(x, y) \in \mathbf{R}$ .

A relation is *small* iff it is a set.

## 3.4 Domain

**Definition 3.8** (Domain (Pairing)). Let  $\mathbf{R}$  be a class. The *domain* of  $\mathbf{R}$  is  $\text{dom } \mathbf{R} = \{x \mid \exists y. x\mathbf{R}y\}$ .

**Theorem 3.9** (Pairing, Union, Subset). *For any set  $R$ , the domain  $\text{dom } R$  is a set.*

PROOF: It is a subset of  $\bigcup \bigcup R$  by Lemma 3.4. □

### 3.5 Range

**Definition 3.10** (Domain (Pairing)). Let  $\mathbf{R}$  be a class. The *range* of  $\mathbf{R}$  is  $\text{ran } \mathbf{R} = \{y \mid \exists x. x\mathbf{R}y\}$ .

**Theorem 3.11** (Pairing, Union, Subset). *For any set  $R$ , the range  $\text{ran } R$  is a set.*

PROOF: It is a subset of  $\bigcup \bigcup R$  by Lemma 3.4.  $\square$

### 3.6 Field

**Definition 3.12** (Field). Let  $\mathbf{R}$  be a class. The *field* of  $\mathbf{R}$  is  $\text{fld } \mathbf{R} = \text{dom } \mathbf{R} \cup \text{ran } \mathbf{R}$ .

**Theorem 3.13** (Pairing, Union, Subset). *For any set  $R$ , the field  $\text{fld } R$  is a set.*

PROOF: Theorems 2.7, 3.9 and 3.11.  $\square$

### 3.7 Functions

**Definition 3.14** (Class Term (Pairing)). A *class term* is a relation  $\mathbf{F}$  such that, for all  $x, y, y'$ , if  $x\mathbf{F}y$  and  $x\mathbf{F}y'$  then  $y = y'$ .

If  $\mathbf{F}$  is a class term and  $x \in \text{dom } \mathbf{F}$ , then we write  $\mathbf{F}(x)$  for the unique  $y$  such that  $x\mathbf{F}y$ .

We write  $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$  iff  $\mathbf{F}$  is a class term,  $\text{dom } \mathbf{F} = \mathbf{A}$  and  $\text{ran } \mathbf{F} \subseteq \mathbf{B}$ .

A *function* is a class term that is a set.

**Axiom 3.15** (Axiom of Choice, First Form (Pairing)). *For any relation  $R$ , there exists a function  $H \subseteq R$  such that  $\text{dom } H = \text{dom } R$ .*

**Theorem 3.16.** *The following are equivalent.*

1. *The Axiom of Choice*
2. **(Multiplicative Axiom)** *For any function  $H$  with domain  $I$  such that  $H(i)$  is nonempty for all  $i \in I$ , there exists a function  $f$  with domain  $I$  such that, for all  $i \in I$ , we have  $f(i) \in H(i)$ .*
3. *Every set has a choice function.*
4. *Let  $\mathcal{A}$  be a set of pairwise disjoint nonempty sets. Then there exists a set  $C$  containing exactly one element from each member of  $\mathcal{A}$ .*

### 3.8 Single-Rooted

**Definition 3.17** (Single-Rooted (Pairing)). A class  $\mathbf{R}$  is *single-rooted* if and only if, for all  $x, x', y$ , if  $x\mathbf{R}y$  and  $x'\mathbf{R}y$  then  $x = x'$ .

We call a class term *one-to-one*, *injective* or an *injection* if and only if it is single-rooted.

### 3.9 Surjective

**Definition 3.18** (Surjective (Pairing)). Let  $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$ . Then  $\mathbf{F}$  is *surjective* if and only if  $\text{ran } \mathbf{F} = \mathbf{B}$ .

### 3.10 Inverse

**Definition 3.19** (Inverse (Pairing)). Let  $\mathbf{R}$  be a class. The *inverse* of  $\mathbf{R}$  is  $\mathbf{R}^{-1} = \{(y, x) \mid x\mathbf{R}y\}$ .

**Theorem 3.20** (Pairing, Union, Power Set, Subset). *For any set  $R$ , the inverse  $R^{-1}$  is a set.*

PROOF: It is a subset of  $\text{ran } R \times \text{dom } R$ .  $\square$

**Theorem 3.21** (Pairing). *For any class  $\mathbf{F}$ , we have  $\text{dom } \mathbf{F}^{-1} = \text{ran } \mathbf{F}$ .*

PROOF: For any  $x$ , we have

$$\begin{aligned} x \in \text{dom } \mathbf{F}^{-1} &\Leftrightarrow \exists y.(x, y) \in \mathbf{F}^{-1} \\ &\Leftrightarrow \exists y.(y, x) \in \mathbf{F} \\ &\Leftrightarrow x \in \text{ran } \mathbf{F} \end{aligned} \quad \square$$

**Theorem 3.22** (Pairing). *For any set  $\mathbf{F}$ , we have  $\text{ran } \mathbf{F}^{-1} = \text{dom } \mathbf{F}$ .*

PROOF: For any  $x$ , we have

$$\begin{aligned} x \in \text{ran } \mathbf{F}^{-1} &\Leftrightarrow \exists y.(y, x) \in \mathbf{F}^{-1} \\ &\Leftrightarrow \exists y.(x, y) \in \mathbf{F} \\ &\Leftrightarrow x \in \text{dom } \mathbf{F} \end{aligned} \quad \square$$

**Theorem 3.23** (Pairing). *For any relation  $\mathbf{F}$ , we have  $(\mathbf{F}^{-1})^{-1} = \mathbf{F}$ .*

PROOF: For any  $z$  we have

$$\begin{aligned} z \in (\mathbf{F}^{-1})^{-1} &\Leftrightarrow \exists x, y. z = (x, y) \wedge (y, x) \in \mathbf{F}^{-1} \\ &\Leftrightarrow \exists x, y. z = (x, y) \wedge (x, y) \in \mathbf{F} \\ &\Leftrightarrow z \in \mathbf{F} \end{aligned} \quad (\mathbf{F} \text{ is a relation}) \square$$

**Theorem 3.24** (Pairing). *For any class  $\mathbf{F}$ , we have  $\mathbf{F}^{-1}$  is a class term if and only if  $\mathbf{F}$  is single-rooted.*

PROOF: Immediate from definitions.  $\square$

**Theorem 3.25** (Pairing). *Let  $\mathbf{F}$  be a relation. Then  $\mathbf{F}$  is a class term if and only if  $\mathbf{F}^{-1}$  is single-rooted.*

PROOF: Immediate from definitions.  $\square$

**Theorem 3.26** (Pairing). *Let  $\mathbf{F}$  be a one-to-one class term and  $x \in \text{dom } \mathbf{F}$ . Then  $\mathbf{F}^{-1}(\mathbf{F}(x)) = x$ .*

PROOF: We have  $(x, \mathbf{F}(x)) \in \mathbf{F}$  and so  $(\mathbf{F}(x), x) \in \mathbf{F}^{-1}$ .  $\square$

**Theorem 3.27** (Pairing). *Let  $\mathbf{F}$  be a one-to-one function and  $y \in \text{ran } \mathbf{F}$ . Then  $\mathbf{F}(\mathbf{F}^{-1}(y)) = y$ .*

PROOF: From Theorems 3.21, 3.23 and 3.26.  $\square$

### 3.11 Composition

**Definition 3.28** (Composition (Pairing)). Let  $\mathbf{R}$  and  $\mathbf{S}$  be relations. The *composition* of  $\mathbf{R}$  and  $\mathbf{S}$  is  $\mathbf{S} \circ \mathbf{R} = \{(x, z) \mid \exists y. x\mathbf{R}y \wedge y\mathbf{S}z\}$ .

**Theorem 3.29** (Pairing, Union, Power Set, Subset). *If  $R$  and  $S$  are small relations then  $S \circ R$  is small.*

PROOF: It is a subset of  $\text{dom } R \times \text{ran } S$ .  $\square$

**Theorem 3.30** (Pairing). *Let  $\mathbf{F}$  and  $\mathbf{G}$  be class terms. Then  $\mathbf{G} \circ \mathbf{F}$  is a function, its domain is  $\{x \in \text{dom } \mathbf{F} \mid \mathbf{F}(x) \in \text{dom } \mathbf{G}\}$ , and for  $x$  in this domain, we have  $(\mathbf{F} \circ \mathbf{G})(x) = \mathbf{F}(\mathbf{G}(x))$ .*

PROOF:

$\langle 1 \rangle 1$ .  $\mathbf{G} \circ \mathbf{F}$  is a class term.

$\langle 2 \rangle 1$ . LET:  $x(\mathbf{G} \circ \mathbf{F})z$  and  $x(\mathbf{G} \circ \mathbf{F})z'$

$\langle 2 \rangle 2$ . PICK  $y, y'$  such that  $x\mathbf{F}y, x\mathbf{F}y', y\mathbf{G}z$  and  $y'\mathbf{G}z'$

$\langle 2 \rangle 3$ .  $y = y'$

PROOF: Since  $\mathbf{F}$  is a class term.

$\langle 2 \rangle 4$ .  $z = z'$

PROOF: Since  $\mathbf{G}$  is a class term.

$\langle 1 \rangle 2$ .  $\text{dom}(\mathbf{G} \circ \mathbf{F}) = \{x \in \text{dom } \mathbf{F} \mid \mathbf{F}(x) \in \text{dom } \mathbf{G}\}$

PROOF:

$$\begin{aligned} x \in \text{dom}(\mathbf{G} \circ \mathbf{F}) &\Leftrightarrow \exists z. x(\mathbf{G} \circ \mathbf{F})z \\ &\Leftrightarrow \exists y, z. x\mathbf{F}y \wedge y\mathbf{G}z \\ &\Leftrightarrow x \in \text{dom } \mathbf{F} \wedge \mathbf{F}(x) \in \text{dom } \mathbf{G} \end{aligned}$$

$\langle 1 \rangle 3$ . For  $x$  in this domain, we have  $(\mathbf{F} \circ \mathbf{G})(x) = \mathbf{F}(\mathbf{G}(x))$ .

PROOF: Since  $(x, \mathbf{F}(x)) \in \mathbf{F}$  and  $(\mathbf{F}(x), \mathbf{G}(\mathbf{F}(x))) \in \mathbf{G}$ .

$\square$

**Theorem 3.31** (Pairing). *For any classes  $\mathbf{F}$  and  $\mathbf{G}$ , we have  $(\mathbf{G} \circ \mathbf{F})^{-1} = \mathbf{F}^{-1} \circ \mathbf{G}^{-1}$ .*

PROOF:

$$\begin{aligned} (x, z) \in (\mathbf{G} \circ \mathbf{F})^{-1} &\Leftrightarrow (z, x) \in \mathbf{G} \circ \mathbf{F} \\ &\Leftrightarrow \exists y. z\mathbf{F}y \wedge y\mathbf{G}x \\ &\Leftrightarrow \exists y. (y, z) \in \mathbf{F}^{-1} \wedge (x, y) \in \mathbf{G}^{-1} \\ &\Leftrightarrow (x, z) \in \mathbf{F}^{-1} \circ \mathbf{G}^{-1} \end{aligned}$$

$\square$



## 3.12 Identity Function

**Definition 3.32** (Identity Class Term (Pairing)). Let  $\mathbf{A}$  be a set. The *identity class term*  $\text{id}_{\mathbf{A}}$  on  $\mathbf{A}$  is  $\{(x, x) \mid x \in \mathbf{A}\}$ .

**Theorem 3.33** (Pairing, Power Set, Subset). *For any set  $A$ , we have  $\text{id}_A$  is a function.*

PROOF: It is a subset of  $\mathcal{P}A$ .  $\square$

**Theorem 3.34** (Extensionality, Pairing, Union, Power Set, Subset). *Let  $F : A \rightarrow B$  and  $A$  be nonempty. Then there exists a function  $G : B \rightarrow A$  such that  $G \circ F = \text{id}_A$  if and only if  $F$  is one-to-one.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $F : A \rightarrow B$
- $\langle 1 \rangle 2$ . ASSUME:  $A$  is nonempty
- $\langle 1 \rangle 3$ . If there exists  $G : B \rightarrow A$  such that  $G \circ F = \text{id}_A$  then  $F$  is one-to-one.
  - $\langle 2 \rangle 1$ . ASSUME:  $G : B \rightarrow A$  and  $G \circ F = \text{id}_A$
  - $\langle 2 \rangle 2$ . LET:  $x, y \in A$
  - $\langle 2 \rangle 3$ . ASSUME:  $F(x) = F(y)$
  - $\langle 2 \rangle 4$ .  $x = y$
- PROOF:  $x = G(F(x)) = G(F(y)) = y$ .
- $\langle 1 \rangle 4$ . If  $F$  is one-to-one then there exists  $G : B \rightarrow A$  such that  $G \circ F = \text{id}_A$ .
  - $\langle 2 \rangle 1$ . ASSUME:  $F$  is one-to-one.
  - $\langle 2 \rangle 2$ . PICK  $a \in A$
  - $\langle 2 \rangle 3$ . Define  $G : B \rightarrow A$  by:  $G(y)$  is the  $x$  such that  $F(x) = y$  if  $y \in \text{ran } F$ , otherwise  $G(y) = a$
  - $\langle 2 \rangle 4$ .  $G \circ F = \text{id}_A$

PROOF: For  $x \in A$  we have  $(G \circ F)(x) = G(F(x)) = x$  by Theorem 3.30.

$\square$

**Theorem 3.35** (Extensionality, Pairing, Union, Power Set, Subset). *Let  $F : A \rightarrow B$  and  $A$  be nonempty. If there exists a function  $H : B \rightarrow A$  such that  $F \circ H = \text{id}_B$  then  $F$  is surjective.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $F : A \rightarrow B$
- $\langle 1 \rangle 2$ . ASSUME:  $A$  is nonempty.
- $\langle 1 \rangle 3$ . LET:  $H : B \rightarrow A$  satisfy  $F \circ H = \text{id}_B$
- $\langle 1 \rangle 4$ . LET:  $y \in B$
- $\langle 1 \rangle 5$ .  $F(H(y)) = y$ .

$\square$

**Theorem 3.36** (Extensionality, Pairing, Union, Power Set, Subset, Choice). *Let  $F : A \rightarrow B$  and  $A$  be nonempty. If  $F$  is surjective then there exists a function  $H : B \rightarrow A$  such that  $F \circ H = \text{id}_B$ .*

PROOF:

- ⟨1⟩1. ASSUME:  $F$  is surjective.  
 ⟨1⟩2. PICK a function  $H \subseteq F^{-1}$  with  $\text{dom } H = B$   
 PROOF: By the Axiom of Choice.  
 ⟨1⟩3.  $H : B \rightarrow A$   
 ⟨1⟩4.  $F \circ H = \text{id}_B$   
 ⟨2⟩1. LET:  $y \in B$   
 ⟨2⟩2.  $(y, H(y)) \in F^{-1}$   
 ⟨2⟩3.  $(H(y), y) \in F$   
 ⟨2⟩4.  $F(H(y)) = y$

□

### 3.13 Restriction

**Definition 3.37** (Restriction (Pairing)). Let  $\mathbf{R}$  be a relation and  $\mathbf{A}$  a class. The *restriction* of  $\mathbf{R}$  to  $\mathbf{A}$  is  $\mathbf{R} \upharpoonright \mathbf{A} = \{(x, y) \mid x \in \mathbf{A} \wedge x\mathbf{R}y\}$ .

**Theorem 3.38** (Pairing, Subset). *If  $R$  is a small relation then  $R \upharpoonright \mathbf{A}$  is small.*

PROOF: Since it is a subset of  $R$ . □

### 3.14 Image

**Definition 3.39** (Image (Pairing)). Let  $\mathbf{F}$  and  $\mathbf{A}$  be classes. The *image* of  $\mathbf{A}$  under  $\mathbf{F}$  is  $\mathbf{F}(\mathbf{A}) = \{\mathbf{F}(x) \mid x \in \mathbf{A}\}$ .

**Theorem 3.40** (Pairing, Union, Subset). *If  $F$  is a set then  $F(\mathbf{A})$  is a set.*

PROOF: Since it is a subset of  $\text{ran } F$ . □

**Theorem 3.41** (Pairing). *For any classes  $\mathbf{F}$  and  $\mathcal{A}$  we have*

$$\mathbf{F}\left(\bigcup \mathcal{A}\right) = \bigcup_{A \in \mathcal{A}} \mathbf{F}(A)$$

PROOF: Each is the class of all  $y$  such that  $\exists x. \exists A. x \in A \in \mathcal{A} \wedge y = \mathbf{F}(x)$ . □

**Theorem 3.42** (Pairing). *For any classes  $\mathbf{F}$ ,  $\mathbf{A}_1, \dots, \mathbf{A}_n$ , we have*

$$\mathbf{F}(\mathbf{A}_1 \cup \dots \cup \mathbf{A}_n) = \mathbf{F}(\mathbf{A}_1) \cup \dots \cup \mathbf{F}(\mathbf{A}_n) .$$

PROOF: Similar. □

**Theorem 3.43** (Pairing). *For any classes  $\mathbf{F}$  and  $\mathcal{A}$ , we have*

$$\mathbf{F}\left(\bigcap \mathcal{A}\right) \subseteq \bigcap_{A \in \mathcal{A}} \mathbf{F}(A) .$$

*Equality holds if  $\mathbf{F}$  is single-rooted and  $\mathcal{A}$  is nonempty.*

PROOF:

- ⟨1⟩1.  $\mathbf{F}(\bigcap \mathcal{A}) \subseteq \bigcap_{A \in \mathcal{A}} \mathbf{F}(A)$
- ⟨2⟩1. LET:  $y \in \mathbf{F}(\bigcap \mathcal{A})$
- ⟨2⟩2. PICK  $x \in \bigcap \mathcal{A}$  such that  $y = \mathbf{F}(x)$
- ⟨2⟩3. LET:  $A \in \mathcal{A}$
- ⟨2⟩4.  $x \in A$
- ⟨2⟩5.  $y \in \mathbf{F}(A)$
- ⟨1⟩2. If  $\mathbf{F}$  is single-rooted then  $\mathbf{F}(\bigcap \mathcal{A}) = \bigcap_{A \in \mathcal{A}} \mathbf{F}(A)$
- ⟨2⟩1. ASSUME:  $\mathbf{F}$  is single-rooted and  $\mathcal{A}$  is nonempty.
- ⟨2⟩2. LET:  $y \in \bigcap_{A \in \mathcal{A}} \mathbf{F}(A)$
- ⟨2⟩3. PICK  $A \in \mathcal{A}$
- ⟨2⟩4. PICK  $x \in A$  such that  $y = \mathbf{F}(x)$
- ⟨2⟩5.  $x \in \bigcap \mathcal{A}$
- ⟨3⟩1. LET:  $A' \in \mathcal{A}$
- ⟨3⟩2. PICK  $x' \in A'$  such that  $y = \mathbf{F}(x')$
- ⟨3⟩3.  $x = x'$
- PROOF: By ⟨2⟩1.
- ⟨3⟩4.  $x \in A'$

□

**Corollary 3.43.1** (Pairing). *For any class  $\mathbf{F}$  and nonempty class  $\mathcal{A}$ , we have*

$$\mathbf{F}^{-1}\left(\bigcap \mathcal{A}\right) = \bigcap_{A \in \mathcal{A}} \mathbf{F}^{-1}(A) .$$

**Theorem 3.44** (Pairing). *For any classes  $\mathbf{F}$ ,  $\mathbf{A}_1, \dots, \mathbf{A}_n$ , we have*

$$\mathbf{F}(\mathbf{A}_1 \cap \dots \cap \mathbf{A}_n) \subseteq \mathbf{F}(\mathbf{A}_1) \cap \dots \cap \mathbf{F}(\mathbf{A}_n) .$$

*Equality holds if  $\mathbf{F}$  is single-rooted.*

PROOF: Similar.

**Corollary 3.44.1** (Pairing). *For any classes  $\mathbf{F}$ ,  $\mathbf{A}_1, \dots, \mathbf{A}_n$ , we have*

$$\mathbf{F}^{-1}(\mathbf{A}_1 \cap \dots \cap \mathbf{A}_n) = \mathbf{F}^{-1}(\mathbf{A}_1) \cap \dots \cap \mathbf{F}^{-1}(\mathbf{A}_n) .$$

**Theorem 3.45** (Pairing). *For any classes  $\mathbf{F}$ ,  $\mathbf{A}$  and  $\mathbf{B}$ , we have*

$$\mathbf{F}(\mathbf{A}) - \mathbf{F}(\mathbf{B}) \subseteq \mathbf{F}(\mathbf{A} - \mathbf{B}) .$$

*Equality holds if  $\mathbf{F}$  is single-rooted.*

PROOF:

- ⟨1⟩1. LET:  $\mathbf{F}$ ,  $\mathbf{A}$  and  $\mathbf{B}$  be sets.
- ⟨1⟩2.  $\mathbf{F}(\mathbf{A}) - \mathbf{F}(\mathbf{B}) \subseteq \mathbf{F}(\mathbf{A} - \mathbf{B})$
- ⟨2⟩1. LET:  $y \in \mathbf{F}(\mathbf{A}) - \mathbf{F}(\mathbf{B})$
- ⟨2⟩2. PICK  $x \in \mathbf{A}$  such that  $x\mathbf{F}y$
- ⟨2⟩3.  $x \in \mathbf{A} - \mathbf{B}$

- ⟨1⟩3. If  $\mathbf{F}$  is single-rooted then  $\mathbf{F}(\mathbf{A} - \mathbf{B}) = \mathbf{F}(\mathbf{A}) - \mathbf{F}(\mathbf{B})$ .
- ⟨2⟩1. ASSUME:  $\mathbf{F}$  is single-rooted.
- ⟨2⟩2. LET:  $y \in \mathbf{F}(\mathbf{A} - \mathbf{B})$
- ⟨2⟩3. PICK  $x \in \mathbf{A} - \mathbf{B}$  such that  $y = \mathbf{F}(x)$
- ⟨2⟩4.  $y \in \mathbf{F}(\mathbf{A})$
- ⟨2⟩5.  $y \notin \mathbf{F}(\mathbf{B})$ 
  - ⟨3⟩1. ASSUME: for a contradiction  $x' \in \mathbf{B}$  and  $x' \mathbf{F} y$
  - ⟨3⟩2.  $x' = x$
  - PROOF: From ⟨2⟩1
  - ⟨3⟩3.  $x \in \mathbf{B}$
  - ⟨3⟩4. Q.E.D.
  - PROOF: This contradicts ⟨2⟩3.

□

**Corollary 3.45.1** (Pairing). *For any classes  $\mathbf{F}$  and sets  $\mathbf{A}$  and  $\mathbf{B}$ , we have*

$$\mathbf{F}^{-1}(\mathbf{A}) - \mathbf{F}^{-1}(\mathbf{B}) = \mathbf{F}^{-1}(\mathbf{A} - \mathbf{B}) .$$

### 3.15 Infinite Cartesian Product

**Definition 3.46** (Infinite Cartesian Product (Pairing)). Let  $H$  be a function with domain  $I$ . The *Cartesian product*  $\prod_{i \in I} H(i)$  is the class of all functions  $f$  with domain  $I$  such that, for all  $i \in I$ , we have  $f(i) \in H(i)$ .

**Theorem 3.47** (Pairing, Union, Power Set, Subset). *If  $H$  is a function with domain  $I$  then  $\prod_{i \in I} H(i)$  is a set.*

PROOF: It is a subset of  $\mathcal{P}(I \times \bigcup \text{ran } H)$ . □

**Theorem 3.48** (Axiom of Choice, Second Version (Pairing, Union, Power Set, Subset)). *The Axiom of Choice is equivalent to the statement: for any function  $H$  with domain  $I$ , if  $H(i)$  is nonempty for all  $i \in I$ , then  $\prod_{i \in I} H(i)$  is nonempty.*

PROOF:

- ⟨1⟩1. If the Axiom of Choice is true then, for any function  $H$  with domain  $I$ , if  $H(i)$  is nonempty for all  $i \in I$ , then  $\prod_{i \in I} H(i)$  is nonempty.
- ⟨2⟩1. ASSUME: The Axiom of Choice
- ⟨2⟩2. LET:  $H$  be a function with domain  $I$  such that  $H(i)$  is nonempty for all  $i \in I$ .
- ⟨2⟩3. PICK a function  $f \subseteq \{(i, x) \mid x \in H(i)\}$
- ⟨2⟩4.  $f \in \prod_{i \in I} H(i)$
- ⟨1⟩2. If, for any function  $H$  with domain  $I$ , if  $H(i)$  is nonempty for all  $i \in I$ , then  $\prod_{i \in I} H(i)$  is nonempty, then the Axiom of Choice is true.
- ⟨2⟩1. ASSUME: for any function  $H$  with domain  $I$ , if  $H(i)$  is nonempty for all  $i \in I$ , then  $\prod_{i \in I} H(i)$  is nonempty
- ⟨2⟩2. LET:  $R$  be a relation.
- ⟨2⟩3. LET:  $I = \text{dom } R$

- (2)4. LET:  $H$  be the function with domain  $I$  such that  $H(i) = \{y \mid iRy\}$  for all  $i$ .  
 (2)5. PICK  $f \in \prod_{i \in I} H(i)$   
 (2)6.  $f \subseteq R$   
 $\square$

### 3.16 Reflexive Relations

**Definition 3.49** (Reflexive (Pairing)). Let  $\mathbf{R}$  be a relation on  $\mathbf{A}$ . Then  $\mathbf{R}$  is *reflexive* on  $A$  if and only if, for all  $x \in \mathbf{A}$ , we have  $x\mathbf{R}x$ .

### 3.17 Symmetric

**Definition 3.50** (Symmetric (Pairing)). Let  $\mathbf{R}$  be a relation. Then  $\mathbf{R}$  is *symmetric* if and only if, whenever  $x\mathbf{R}y$ , then  $y\mathbf{R}x$ .

### 3.18 Transitivity

**Definition 3.51** (Transitivity (Pairing)). Let  $\mathbf{R}$  be a relation. Then  $\mathbf{R}$  is *transitive* if and only if, whenever  $x\mathbf{R}y$  and  $y\mathbf{R}z$ , then  $x\mathbf{R}z$ .

### 3.19 Equivalence Relations

**Definition 3.52** (Equivalence Relation (Pairing)). Let  $\mathbf{R}$  be a relation on  $\mathbf{A}$ . Then  $\mathbf{R}$  is an *equivalence relation* on  $\mathbf{A}$  if and only if  $\mathbf{R}$  is reflexive on  $\mathbf{A}$ , symmetric and transitive.

**Theorem 3.53** (Pairing). *If  $\mathbf{R}$  is a symmetric and transitive relation then  $\mathbf{R}$  is an equivalence relation on  $\text{fld } \mathbf{R}$ .*

PROOF:

- (1)1. LET:  $\mathbf{R}$  be a symmetric and transitive relation.  
 (1)2. LET:  $x \in \text{fld } \mathbf{R}$   
 (1)3. PICK  $y$  such that  $x\mathbf{R}y$  or  $y\mathbf{R}x$   
 (1)4.  $x\mathbf{R}y$  and  $y\mathbf{R}x$

PROOF: By symmetry.

- (1)5.  $x\mathbf{R}x$

PROOF: By transitivity.

$\square$

### 3.20 Equivalence Class

**Definition 3.54** (Equivalence Class (Pairing)). Let  $\mathbf{R}$  be an equivalence relation on  $\mathbf{A}$  and  $a \in \mathbf{A}$ . Then the *equivalence class of  $a$  modulo  $\mathbf{R}$*  is

$$[a]_{\mathbf{R}} = \{x \in \mathbf{A} \mid a\mathbf{R}x\} .$$

**Lemma 3.55** (Extensionality, Pairing, Subset). *Let  $\mathbf{R}$  be an equivalence relation on  $\mathbf{A}$  and  $x, y \in \mathbf{A}$ . Then  $[x]_{\mathbf{R}} = [y]_{\mathbf{R}}$  if and only if  $x\mathbf{R}y$ .*

PROOF:

$\langle 1 \rangle 1$ . If  $[x]_{\mathbf{R}} = [y]_{\mathbf{R}}$  then  $x\mathbf{R}y$ .

$\langle 2 \rangle 1$ . ASSUME:  $[x]_{\mathbf{R}} = [y]_{\mathbf{R}}$

$\langle 2 \rangle 2$ .  $y \in [y]_{\mathbf{R}}$

PROOF: Since  $y\mathbf{R}y$  by reflexivity.

$\langle 2 \rangle 3$ .  $y \in [x]_{\mathbf{R}}$

$\langle 2 \rangle 4$ .  $x\mathbf{R}y$

$\langle 1 \rangle 2$ . If  $x\mathbf{R}y$  then  $[x]_{\mathbf{R}} = [y]_{\mathbf{R}}$ .

$\langle 2 \rangle 1$ . ASSUME:  $x\mathbf{R}y$

$\langle 2 \rangle 2$ .  $[y]_{\mathbf{R}} \subseteq [x]_{\mathbf{R}}$

PROOF: If  $y\mathbf{R}z$  then  $x\mathbf{R}z$  by transitivity.

$\langle 2 \rangle 3$ .  $[x]_{\mathbf{R}} \subseteq [y]_{\mathbf{R}}$

PROOF: Similar since  $y\mathbf{R}x$  by symmetry.

□

### 3.21 Disjoint

**Definition 3.56** (Disjoint). Two classes  $\mathbf{A}$  and  $\mathbf{B}$  are *disjoint* if and only if there is no  $x$  such that  $x \in \mathbf{A}$  and  $x \in \mathbf{B}$ .

**Axiom 3.57** (Regularity). *For any nonempty set  $A$ , there exists  $m \in A$  such that  $m$  and  $A$  are disjoint.*

**Theorem 3.58** (Regularity). *No set is a member of itself.*

**Theorem 3.59** (Regularity). *There are no sets  $A$  and  $B$  such that  $A \in B$  and  $B \in A$ .*

### 3.22 Partitions

**Definition 3.60** (Partition). A *partition*  $P$  of a set  $A$  is a set of nonempty subsets of  $A$  such that:

1. For all  $x \in A$  there exists  $S \in P$  such that  $x \in S$ .
2. Any two distinct elements of  $P$  are disjoint.

### 3.23 Quotient Sets

**Definition 3.61** (Quotient Set (Pairing, Power Set, Subset)). Let  $R$  be an equivalence relation on  $A$ . The *quotient set*  $A/R$  is the set of all equivalence classes modulo  $R$ .

This is a set because it is a subset of  $\mathcal{P}A$ .

**Theorem 3.62** (Extensionality, Pairing, Power Set, Subset). *Let  $R$  be an equivalence relation on  $A$ . Then the quotient set  $A/R$  is a partition of  $A$ .*

PROOF:

$\langle 1 \rangle 1$ . For all  $x \in A$  there exists  $y \in A$  such that  $x \in [y]_R$

PROOF: Take  $y = x$ .

$\langle 1 \rangle 2$ . Any two distinct equivalence classes are disjoint.

$\langle 2 \rangle 1$ . ASSUME:  $z \in [x]_R$  and  $z \in [y]_R$

$\langle 2 \rangle 2$ .  $xRz$  and  $yRz$

$\langle 2 \rangle 3$ .  $[x]_R = [z]_R = [y]_R$

PROOF: Lemma 3.55.

□

**Definition 3.63** (Canonical Map (Pairing, Power Set, Subset)). Let  $R$  be an equivalence relation on  $A$ . The *canonical map*  $\phi : A \rightarrow A/R$  is the function defined by  $\phi(a) = [a]_R$ .

**Theorem 3.64.** *Let  $R$  be an equivalence relation on  $A$  and  $F : A \rightarrow B$ . Then the following are equivalent:*

1. For all  $x, y \in A$ , if  $xRy$  then  $F(x) = F(y)$ .

2. There exists  $G : A/R \rightarrow B$  such that  $F = G \circ \phi$ , where  $\phi : A \rightarrow A/R$  is the canonical map.

In this case,  $G$  is unique.

PROOF:

$\langle 1 \rangle 1$ .  $1 \Rightarrow 2$

$\langle 2 \rangle 1$ . ASSUME: 1

$\langle 2 \rangle 2$ . Let  $G = \{([a]_R, b) \mid F(a) = b\}$

$\langle 2 \rangle 3$ .  $G$  is a function.

$\langle 3 \rangle 1$ . LET:  $(c, b), (c, b') \in G$

$\langle 3 \rangle 2$ . PICK  $a, a' \in A$  such that  $c = [a]_R = [a']_R$  with  $F(a) = b$  and  $F(a') = b'$

$\langle 3 \rangle 3$ .  $aRa'$

PROOF: Lemma 3.55.

$\langle 3 \rangle 4$ .  $F(a) = F(a')$

PROOF: From  $\langle 2 \rangle 1$ .

$\langle 3 \rangle 5$ .  $b = b'$

PROOF: From  $\langle 3 \rangle 2$ .

$\langle 2 \rangle 4$ .  $F = G \circ \phi$

PROOF: For  $a \in A$  we have  $G(\phi(a)) = G([a]) = F(a)$ .

$\langle 1 \rangle 2$ .  $2 \Rightarrow 1$

$\langle 2 \rangle 1$ . LET:  $G : A/R \rightarrow B$  be such that  $F = G \circ \phi$

$\langle 2 \rangle 2$ . LET:  $x, y \in A$

$\langle 2 \rangle 3$ . ASSUME:  $xRy$

$\langle 2 \rangle 4$ .  $G([x]) = G([y])$

PROOF: Lemma 3.55

$\langle 2 \rangle 5$ .  $F(x) = F(y)$

PROOF: From  $\langle 2 \rangle 1$ .

$\langle 1 \rangle 3$ . If  $G, G' : A/R \rightarrow B$  and  $G \circ \phi = G' \circ \phi$  then  $G = G'$

PROOF: For any  $a \in A$  we have  $G([a]) = G'([a])$ .

□

### 3.24 The Finite Intersection Property

**Definition 3.65** (Finite Intersection Property). Let  $X$  be a set and  $\mathcal{A} \subseteq \mathcal{P}X$ . Then  $\mathcal{A}$  satisfies the *finite intersection property* if and only if every nonempty finite subset of  $\mathcal{A}$  has nonempty intersection.

**Lemma 3.66.** Let  $X$  be a set. Let  $\mathcal{A} \subseteq \mathcal{P}X$  have the finite intersection property. Then there exists a maximal set  $\mathcal{D}$  such that  $\mathcal{A} \subseteq \mathcal{D} \subseteq \mathcal{P}X$  and  $\mathcal{D}$  has the finite intersection property.

PROOF:

$\langle 1 \rangle 1$ . LET:  $\mathbb{F} = \{\mathcal{D} \mid \mathcal{A} \subseteq \mathcal{D} \subseteq \mathcal{P}X, \mathcal{D} \text{ has the finite intersection property}\}$

$\langle 1 \rangle 2$ . Every chain in  $\mathbb{F}$  has an upper bound.

$\langle 2 \rangle 1$ . LET:  $\mathbb{C}$  be a chain in  $\mathbb{F}$ .

$\langle 2 \rangle 2$ . ASSUME: without loss of generality  $\mathbb{C} \neq \emptyset$

PROVE:  $\bigcup \mathbb{C} \in \mathbb{F}$

PROOF: If  $\mathbb{C} = \emptyset$  then  $\mathcal{A}$  is an upper bound.

$\langle 2 \rangle 3$ .  $\mathcal{A} \subseteq \bigcup \mathbb{C} \subseteq \mathcal{P}X$

$\langle 2 \rangle 4$ . LET:  $C_1, \dots, C_n \in \mathbb{C}$

PROVE:  $C_1 \cap \dots \cap C_n \neq \emptyset$

$\langle 2 \rangle 5$ . PICK  $\mathcal{C}_1, \dots, \mathcal{C}_n \in \mathbb{C}$  such that  $C_i \in \mathcal{C}_i$  for all  $i$ .

$\langle 2 \rangle 6$ . ASSUME: without loss of generality  $\mathcal{C}_1 \subseteq \dots \subseteq \mathcal{C}_n$

$\langle 2 \rangle 7$ .  $C_1, \dots, C_n \in \mathcal{C}_n$

$\langle 2 \rangle 8$ .  $\mathcal{C}_n$  satisfies the finite intersection property.

$\langle 2 \rangle 9$ .  $C_1 \cap \dots \cap C_n \neq \emptyset$

$\langle 1 \rangle 3$ . Q.E.D.

PROOF: By Zorn's Lemma.

□

**Lemma 3.67.** Let  $X$  be a set. Let  $\mathcal{D} \subseteq \mathcal{P}X$  be maximal with respect to the finite intersection property. Then any finite intersection of elements of  $\mathcal{D}$  is an element of  $\mathcal{D}$ .

PROOF:



- (1)1. LET:  $D_1, D_2 \in \mathcal{D}$   
 (1)2.  $\mathcal{D} \cup \{D_1 \cap D_2\}$  has the finite intersection property.  
 PROOF: Any finite intersection of members of  $\mathcal{D} \cup \{D_1 \cap D_2\}$  is a finite intersection of members of  $\mathcal{D}$ .  
 (1)3.  $\mathcal{D} = \mathcal{D} \cup \{D_1 \cap D_2\}$   
 PROOF: By maximality of  $\mathcal{D}$ .  
 (1)4.  $D_1 \cap D_2 \in \mathcal{D}$ .  
 $\square$

**Lemma 3.68.** *Let  $X$  be a set. Let  $\mathcal{D} \subseteq \mathcal{P}X$  be maximal with respect to the finite intersection property. Let  $A \subseteq X$ . If  $A$  intersects every member of  $\mathcal{D}$  then  $A \in \mathcal{D}$ .*

PROOF:

- (1)1.  $\mathcal{D} \cup \{A\}$  has the finite intersection property.  
 (2)1. LET:  $D_1, \dots, D_n \in \mathcal{D}$   
 PROVE:  $D_1 \cap \dots \cap D_n \cap A \neq \emptyset$   
 (2)2.  $D_1 \cap \dots \cap D_n \in \mathcal{D}$   
 PROOF: Lemma 3.67.  
 (2)3.  $D_1 \cap \dots \cap D_n \cap A \neq \emptyset$   
 PROOF: Since  $A$  intersects every member of  $\mathcal{D}$ .  
 (1)2. Q.E.D.  
 PROOF: By maximality of  $\mathcal{D}$ .  
 $\square$

**Proposition 3.69.** *Let  $X$  be a set. Let  $\mathcal{D} \subseteq \mathcal{P}X$  be maximal with respect to the finite intersection property. Let  $A, D \in \mathcal{P}X$ . If  $D \in \mathcal{D}$  and  $D \subseteq A$  then  $A \in \mathcal{D}$ .*

PROOF:

- (1)1.  $\mathcal{D} \cup \{A\}$  satisfies the finite intersection property.  
 (2)1. LET:  $D_1, \dots, D_n \in \mathcal{D}$   
 (2)2.  $D_1 \cap \dots \cap D_n \cap D \neq \emptyset$   
 PROOF: Since  $\mathcal{D}$  satisfies the finite intersection property.  
 (2)3.  $D_1 \cap \dots \cap D_n \cap A \neq \emptyset$   
 (1)2.  $\mathcal{D} = \mathcal{D} \cup \{A\}$   
 PROOF: By the maximality of  $\mathcal{D}$ .  
 (1)3.  $A \in \mathcal{D}$   
 $\square$

**Definition 3.70** (Graph). Let  $f : A \rightarrow B$ . The *graph* of  $f$  is the set  $\{(x, f(x)) \mid x \in A\} \subseteq A \times B$ .

## 3.25 Countable Intersection Property

**Definition 3.71** (Countable Intersection Property). Let  $X$  be a set and  $\mathcal{A} \subseteq \mathcal{P}X$ . Then  $\mathcal{A}$  satisfies the *countable intersection property* if and only if every countable subset of  $\mathcal{A}$  has nonempty intersection.

**Lemma 3.72.** *Let  $X$  be a set. Let  $\mathcal{D} \subseteq \mathcal{P}X$  be maximal with respect to the countable intersection property. Then any countable intersection of elements of  $\mathcal{D}$  is an element of  $\mathcal{D}$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $\mathcal{D}_0 \subseteq \mathcal{D}$  be countable.

$\langle 1 \rangle 2$ .  $\mathcal{D} \cup \{\bigcap \mathcal{D}_0\}$  has the countable intersection property.

PROOF: Any countable intersection of members of  $\mathcal{D} \cup \{\bigcap \mathcal{D}_0\}$  is a finite intersection of members of  $\mathcal{D}$ .

$\langle 1 \rangle 3$ .  $\mathcal{D} = \mathcal{D} \cup \{\bigcap \mathcal{D}_0\}$

PROOF: By maximality of  $\mathcal{D}$ .

$\langle 1 \rangle 4$ .  $\bigcap \mathcal{D}_0 \in \mathcal{D}$ .

□

**Lemma 3.73.** *Let  $X$  be a set. Let  $\mathcal{D} \subseteq \mathcal{P}X$  be maximal with respect to the countable intersection property. Let  $A \subseteq X$ . If  $A$  intersects every member of  $\mathcal{D}$  then  $A \in \mathcal{D}$ .*

PROOF:

$\langle 1 \rangle 1$ .  $\mathcal{D} \cup \{A\}$  has the countable intersection property.

$\langle 2 \rangle 1$ . LET:  $\mathcal{D}_0 \subseteq \mathcal{D}$  be countable.

PROVE:  $\bigcap \mathcal{D}_0 \cap A \neq \emptyset$

$\langle 2 \rangle 2$ .  $\bigcap \mathcal{D}_0 \in \mathcal{D}$

PROOF: Lemma 3.72.

$\langle 2 \rangle 3$ .  $\bigcap \mathcal{D}_0 \cap A \neq \emptyset$

PROOF: Since  $A$  intersects every member of  $\mathcal{D}$ .

$\langle 1 \rangle 2$ . Q.E.D.

PROOF: By maximality of  $\mathcal{D}$ .

□

## 3.26 The Axiom of Choice

**Axiom 3.74** (Axiom of Choice). *Let  $\mathcal{A}$  be a set of disjoint nonempty sets. Then there exists a set  $C$  consisting of exactly one element from each member of  $\mathcal{A}$ .*

## 3.27 Choice Functions

**Definition 3.75** (Choice Function). Let  $\mathcal{B}$  be a set of nonempty sets. A *choice function* for  $\mathcal{B}$  is a function  $c : \mathcal{B} \rightarrow \bigcup \mathcal{B}$  such that, for all  $B \in \mathcal{B}$ , we have  $c(B) \in B$ .

**Lemma 3.76** (Existence of a Choice Function (AC)). *Every set of nonempty sets has a choice function.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $\mathcal{B}$  be a set of nonempty sets.

- $\langle 1 \rangle 2.$  For  $B \in \mathcal{B}$ ,  
     LET:  $B' = \{B\} \times B$   
 $\langle 1 \rangle 3.$   $\{B' \mid B \in \mathcal{B}\}$  is a set of disjoint nonempty sets.  
 $\langle 1 \rangle 4.$  PICK a set  $c$  consisting of exactly one element from each  $B'$  for  $B \in \mathcal{B}$ .  
 $\langle 1 \rangle 5.$   $c$  is a choice function for  $\mathcal{B}$ .  
 $\square$

### 3.28 Transitive

**Definition 3.77** (Transitive Set). A set  $A$  is *transitive* if and only if, whenever  $x \in y \in A$  then  $x \in A$ .

**Theorem 3.78** (Union, Power Set). *Let  $A$  be a set. Then the following are equivalent.*

1.  $A$  is transitive.
2.  $\bigcup A \subseteq A$
3. For all  $a \in A$  we have  $a \subseteq A$
4.  $A \subseteq \mathcal{P}A$

PROOF: From definitions.  $\square$

### 3.29 Minimal Elements

**Definition 3.79** (Minimal). Let  $R$  be a binary relation and  $A$  a set. An element  $a \in A$  is *minimal* w.r.t.  $R$  iff there is no  $x \in A$  such that  $xRa$ .

### 3.30 Well-Founded Relations

**Definition 3.80** (Well-Founded). Let  $R$  be a relation on  $A$ . Then  $R$  is *well-founded* iff every nonempty subset of  $A$  has an  $R$ -minimal element.

**Theorem 3.81** (Transfinite Induction). *Let  $R$  be a well-founded relation on  $A$  and  $B \subseteq A$ . Assume that, for every  $t \in A$ , if  $\{x \in A \mid xRt\} \subseteq B$  then  $t \in B$ . Then we have  $B = A$ .*

**Theorem 3.82** (Transfinite Recursion). *Let  $R$  be a well-founded relation on a set  $C$ .*

*Let  $\mathbf{A}$  be a class. Let  $\mathbf{B}$  be the class of all functions from a subset of  $C$  to  $\mathbf{A}$ . Let  $\mathbf{F} : \mathbf{B} \times C \rightarrow \mathbf{A}$  be a class term.*

*Then there exists a unique function  $f : C \rightarrow \mathbf{A}$  such that, for all  $t \in C$ , we have  $f(t) = \mathbf{F}(f \upharpoonright \{x \in C \mid xRt\}, t)$ .*

### 3.31 Transitive Closure

**Theorem 3.83.** *Let  $R$  be a relation. Then there exists a unique relation  $R^t$  such that  $R^t$  is transitive,  $R \subseteq R^t$ , and for every transitive relation  $S$  with  $R \subseteq S$  we have  $R^t \subseteq S$ .*

**Definition 3.84** (Transitive Closure). The *transitive closure* of a relation  $R$  is this relation  $R^t$ .

**Theorem 3.85.** *If  $R$  is well-founded then  $R^t$  is well-founded.*

## Chapter 4

# Cardinal Numbers

**Definition 4.1** (Equinumerous). Two sets  $A$  and  $B$  are *equinumerous* if and only if there exists a bijection between them.

**Theorem 4.2.** *Equinumerosity is an equivalence relation on the class of all sets.*

**Theorem 4.3** (Cantor). *No set is equinumerous with its power set.*

**Definition 4.4.** We say a set  $A$  is *dominated* by  $B$ ,  $A \preccurlyeq B$ , iff  $A$  is equinumerous with a subset of  $B$ .

**Theorem 4.5.**  $A \preccurlyeq A$

**Theorem 4.6.** *If  $A \preccurlyeq B \preccurlyeq C$  then  $A \preccurlyeq C$ .*

**Theorem 4.7** (Schröder-Bernstein Theorem). *If  $A \preccurlyeq B$  and  $B \preccurlyeq A$  then  $A \equiv B$ .*

PROOF:

⟨1⟩1. LET:  $f : A \rightarrow B$  and  $g : B \rightarrow A$  be injections.

⟨1⟩2. Define a sequence of sets  $C_n \subseteq A$  by

$$\begin{aligned} C_0 &= A - \text{ran } g \\ C_{n+1} &= g(f(C_n)) \end{aligned}$$

⟨1⟩3. Define  $h : A \rightarrow B$  by

$$h(x) = \begin{cases} f(x) & \text{if } x \in \bigcup_n C_n \\ g^{-1}(x) & \text{if not} \end{cases}$$

⟨1⟩4.  $h$  is a bijection.

□

**Theorem 4.8** (AC). *For any infinite set  $A$  we have  $\mathbb{N} \preccurlyeq A$ .*

PROOF: Given a choice function  $f$  for  $A$ , choose a sequence  $(a_n)$  in  $A$  by  $a_n = f(A - \{a_0, \dots, a_{n-1}\})$ . □

**Corollary 4.8.1** (AC). *A set is infinite if and only if it is equinumerous with a proper subset.*

## 4.1 Countability

**Definition 4.9** (Countable). A set  $A$  is *countable* iff  $A \preceq \mathbb{N}$ .

**Theorem 4.10** (AC). *A countable union of countable sets is countable.*

**Proposition 4.11** (AC). *Every infinite set has a countable subset.*

## 4.2 Order Theory

**Definition 4.12** (Preorder). Let  $X$  be a set. A *preorder* on  $X$  is a binary relation  $\leq$  on  $X$  such that:

**Reflexivity** For all  $x \in X$ , we have  $x \leq x$

**Transitivity** For all  $x, y, z \in X$ , if  $x \leq y$  and  $y \leq z$  then  $x \leq z$ .

**Definition 4.13** (Preordered Set). A *preordered set* consists of a set  $X$  and a preorder  $\leq$  on  $X$ .

**Proposition 4.14.** Let  $X$  and  $Y$  be linearly ordered sets. Let  $f : X \rightarrow Y$  be strictly monotone and surjective. Then  $f$  is a poset isomorphism.

PROOF:

$\langle 1 \rangle 1.$   $f$  is injective.

$\langle 2 \rangle 1.$  LET:  $x, y \in X$

$\langle 2 \rangle 2.$  ASSUME:  $f(x) = f(y)$

$\langle 2 \rangle 3.$   $x \not\leq y$

PROOF: By strong monotonicity.

$\langle 2 \rangle 4.$   $y \not\leq x$

PROOF: By strong monotonicity.

$\langle 2 \rangle 5.$   $x = y$

PROOF: By trichotomy.

$\langle 1 \rangle 2.$   $f^{-1}$  is monotone.

$\langle 2 \rangle 1.$  LET:  $x, y \in X$

$\langle 2 \rangle 2.$  ASSUME:  $x \leq y$

$\langle 2 \rangle 3.$   $f^{-1}(x) \not\leq f^{-1}(y)$

PROOF: By strong monotonicity.

$\langle 2 \rangle 4.$   $f^{-1}(x) < f^{-1}(y)$

PROOF: By trichotomy.

□

**Definition 4.15** (Interval). Let  $X$  be a preordered set and  $Y \subseteq X$ . Then  $Y$  is an *interval* if and only if, for all  $a, b \in Y$  and  $c \in X$ , if  $a \leq c \leq b$  then  $c \in Y$ .

**Definition 4.16** (Linear Continuum). A linearly ordered set  $L$  is a *linear continuum* if and only if:

1. every nonempty subset of  $L$  that is bounded above has a supremum
2.  $L$  is dense

**Proposition 4.17.** Every interval in a linear continuum is a linear continuum.

PROOF:

$\langle 1 \rangle 1.$  LET:  $L$  be a linear continuum and  $I$  an interval in  $L$ .

$\langle 1 \rangle 2.$  Every nonempty subset of  $I$  that is bounded above has a supremum in  $I$ .

$\langle 2 \rangle 1.$  LET:  $X \subseteq I$  be nonempty and bounded above by  $b \in I$ .

(2)2. LET:  $s$  be the supremum of  $X$  in  $L$ .  
 PROOF: Since  $L$  is a linear continuum.  
 (2)3.  $s \in I$   
 (3)1. PICK  $a \in X$   
 PROOF: Since  $X$  is nonempty ((2)1).  
 (3)2.  $a \leq s \leq b$   
 (3)3.  $s \in I$   
 PROOF: Since  $I$  is an interval ((1)1).  
 (2)4.  $s$  is the supremum of  $X$  in  $I$   
 (1)3.  $I$  is dense.  
 (2)1. LET:  $x, y \in I$  with  $x < y$   
 (2)2. PICK  $z \in L$  with  $x < z < y$   
 PROOF: Since  $L$  is dense.  
 (2)3.  $z \in I$   
 PROOF: Since  $I$  is an interval.

□

**Definition 4.18** (Ordered Square). The *ordered square*  $I_o^2$  is the set  $[0, 1]^2$  under the dictionary order.

**Proposition 4.19.** *The ordered square is a linear continuum.*

PROOF:

(1)1. Every nonempty subset of  $I_o^2$  bounded above has a supremum.  
 (2)1. LET:  $X \subseteq I_o^2$  be nonempty and bounded above by  $(b, c)$   
 (2)2. LET:  $s = \sup \pi_1(X)$   
 PROOF: The set  $\pi_1(X)$  is nonempty and bounded above by  $b$ .  
 (2)3. CASE:  $s \in \pi_1(X)$   
 (3)1. LET:  $t = \sup\{y \in [0, 1] \mid (s, y) \in X\}$   
 PROOF: This set is nonempty and bounded above by  $c$ .  
 (3)2.  $(s, t)$  is the supremum of  $X$ .  
 (2)4. CASE:  $s \notin \pi_1(X)$   
 PROOF: In this case  $(s, 0)$  is the supremum of  $X$ .  
 (1)2.  $I_o^2$  is dense.  
 (2)1. LET:  $(x_1, y_1), (x_2, y_2) \in I_o^2$  with  $(x_1, y_1) < (x_2, y_2)$   
 (2)2. CASE:  $x_1 < x_2$   
 (3)1. PICK  $x_3$  with  $x_1 < x_3 < x_2$   
 (3)2.  $(x_1, y_1) < (x_3, y_1) < (x_2, y_2)$   
 (2)3. CASE:  $x_1 = x_2$  and  $y_1 < y_2$   
 (3)1. PICK  $y_3$  with  $y_1 < y_3 < y_2$   
 (3)2.  $(x_1, y_1) < (x_1, y_3) < (x_2, y_2)$

□

**Proposition 4.20.** *If  $X$  is a well-ordered set then  $X \times [0, 1)$  under the dictionary order is a linear continuum.*

PROOF:

(1)1. Every nonempty set  $A \subseteq X \times [0, 1)$  bounded above has a supremum.



- ⟨2⟩1. LET:  $A \subseteq X \times [0, 1)$  be nonempty and bounded above
- ⟨2⟩2. LET:  $x_0$  be the supremum of  $\pi_1(A)$
- ⟨2⟩3. CASE:  $x_0 \in \pi_1(A)$ 
  - ⟨3⟩1. LET:  $y_0$  be the supremum of  $\{y \in [0, 1) \mid (x_0, y) \in A\}$
  - ⟨3⟩2.  $(x_0, y_0)$  is the supremum of  $A$ .
- ⟨2⟩4. CASE:  $x_0 \notin \pi_1(A)$ 
  - PROOF: In this case  $(x_0, 0)$  is the supremum of  $A$ .
- ⟨1⟩2.  $X \times [0, 1)$  is dense.
  - ⟨2⟩1. LET:  $(x_1, y_1), (x_2, y_2) \in X \times [0, 1)$  with  $(x_1, y_1) < (x_2, y_2)$
  - ⟨2⟩2. CASE:  $x_1 < x_2$ 
    - ⟨3⟩1. PICK  $y_3$  such that  $y_1 < y_3 < 1$
    - ⟨3⟩2.  $(x_1, y_1) < (x_1, y_3) < (x_2, y_2)$
  - ⟨2⟩3. CASE:  $x_1 = x_2$  and  $y_1 < y_2$ 
    - ⟨3⟩1. PICK  $y_3$  such that  $y_1 < y_3 < y_2$
    - ⟨3⟩2.  $(x_1, y_1) < (x_1, y_3) < (x_2, y_2)$

□

**Lemma 4.21.** For all  $a, b, c, d \in \mathbb{R}$  with  $a < b$  and  $c < d$ , we have  $[a, b) \cong [c, d)$

PROOF: The map  $\lambda t.c + (t - a)(d - c)/(b - a)$  is an order isomorphism.

**Proposition 4.22.** Let  $X$  be a linearly ordered set. Let  $a < b < c$  in  $X$ . Then  $[a, c) \cong [0, 1)$  if and only if  $[a, b) \cong [b, c) \cong [0, 1)$ .

PROOF:

- ⟨1⟩1. If  $[a, c) \cong [0, 1)$  then  $[a, b) \cong [b, c) \cong [0, 1)$
- ⟨2⟩1. ASSUME:  $f : [a, c) \cong [0, 1)$  is an order isomorphism
- ⟨2⟩2.  $[a, b) \cong [0, 1)$

PROOF:

$$\begin{aligned} [a, b) &\cong [0, f(b)) && \text{(by the restriction of } f) \\ &\cong [0, 1) && \text{(Lemma 4.21)} \end{aligned}$$

- ⟨2⟩3.  $[b, c) \cong [0, 1)$

PROOF: Similar.

- ⟨1⟩2. If  $[a, b) \cong [b, c) \cong [0, 1)$  then  $[a, c) \cong [0, 1)$

PROOF:

$$\begin{aligned} [a, c) &= [a, b) * [b, c) \\ &\cong [0, 1) * [0, 1) \\ &\cong [0, 1/2) * [1/2, 1) && \text{(Lemma 4.21)} \\ &= 1 \end{aligned}$$

□

**Proposition 4.23 (CC).** Let  $X$  be a linearly ordered set. Let  $x_0 < x_1 < \dots$  be a strictly increasing sequence in  $X$  with supremum  $b$ . Then  $[x_0, b) \cong [0, 1)$  if and only if  $[x_i, x_{i+1}) \cong [0, 1)$  for all  $i$ .

PROOF:

- ⟨1⟩1. If  $[x_0, b) \cong [0, 1)$  then  $[x_i, x_{i+1}) \cong [0, 1)$  for all  $i$ .

PROOF: By Lemma 4.21

- (1)2. If  $[x_i, x_{i+1}) \cong [0, 1)$  for all  $i$  then  $[x_0, b) \cong [0, 1)$
- (2)1. ASSUME:  $[x_i, x_{i+1}) \cong [0, 1)$  for all  $i$
- (2)2. PICK an order isomorphism  $f_i : [x_i, x_{i+1}) \cong [1/2^i, 2/2^{i+1})$  for each  $i$ .
- PROOF: By Lemma 4.21
- (2)3. The union of the  $f_i$ s is an order isomorphism  $[x_0, b) \cong [0, 1)$

□

### 4.3 Partially Ordered Sets

**Definition 4.24** (Partial Order). A *partial order* on a set  $X$  is a preorder  $\leq$  that is *anti-symmetric*, i.e. whenever  $x \leq y$  and  $y \leq x$  then  $x = y$ .

### 4.4 Strict Partial Order

**Definition 4.25** (Strict Partial Order). A *strict partial order* on a set  $X$  is a relation on  $X$  that is transitive and irreflexive.

**Proposition 4.26.** If  $<$  is a strict partial order on  $X$  and  $x, y \in X$ , then at most one of  $x < y$ ,  $y < x$ ,  $x = y$  holds.

**Proposition 4.27.** If  $<$  is a strict partial order then the relation  $\leq$  defined by:  $x \leq y$  iff  $x < y$  or  $x = y$ , is a partial order.

**Theorem 4.28.** If  $R$  is a well-founded relation then its transitive closure is a partial order.

**Definition 4.29** (Linear Order). A *linear order* on a set  $X$  is a partial order such that, for any  $x, y \in X$ , either  $x \leq y$  or  $y \leq x$ .

### 4.5 Strict Linear Orders

**Definition 4.30** (Strict Linear Order (Extensionality, Pairing)). Let  $A$  be a set. A *strict linear order* on  $A$  is a binary relation  $R$  on  $A$  that is transitive and satisfies *trichotomy*: for any  $x, y \in A$ , exactly one of  $xRy$ ,  $x = y$ ,  $yRx$  holds.

**Theorem 4.31.** Let  $R$  be a strict linear order on  $A$ . Then there is no  $x \in A$  such that  $xRx$ .

PROOF: Immediate from trichotomy.

### 4.6 Well Orderings

**Definition 4.32** (Well-ordering). A *well-order* on a set  $X$  is a linear order such that every nonempty set has a least element.

**Proposition 4.33.** *Let  $\leq$  be a linear order on  $X$ . Then  $\leq$  is a well-order iff there is no function  $f : \mathbb{N} \rightarrow X$  such that  $f(n+1) < f(n)$  for all  $n$ .*

**Definition 4.34** (Initial Segment). Given a well-ordered set  $X$  and  $\alpha \in X$ , the initial segment of  $X$  up to  $\alpha$  is  $\text{seg } \alpha = \{x \in X \mid x < \alpha\}$ .

**Theorem 4.35** (Transfinite Induction). *Let  $\leq$  be a linear order on  $J$ . Then the following are equivalent:*

1.  $\leq$  is a well-order on  $J$ .
2. For every subset  $J_0 \subseteq J$ , if the following condition holds:
  - For every  $\alpha \in J$ , if  $\text{seg } \alpha \subseteq J_0$  then  $\alpha \in J_0$ .
then  $J_0 = J$ .

**Axiom Schema 4.36** (Replacement). *Let  $\mathbf{H}$  be a class term. If  $\text{dom } \mathbf{H}$  is a set then  $\mathbf{H}$  is a set.*

**Theorem 4.37** (Transfinite Recursion). *Let  $J$  be a well-ordered set and  $C$  a set. Let  $\mathcal{F}$  be the set of all functions from a section of  $J$  to  $C$ . Let  $G$  be a function with domain  $\mathcal{F}$ . Then there exists a unique function  $h$  with domain  $J$  such that, for all  $\alpha \in J$ , we have  $h(\alpha) = \rho(h \upharpoonright \text{seg } \alpha)$ .*

PROOF:

- $\langle 1 \rangle 1$ . If  $v$  is a function and  $t \in J$ , we say  $v$  is  $\rho$ -constructed up to  $t$  iff  $\text{dom } v = \{x \in J \mid x \leq t\}$  and, for all  $x \in \text{dom } v$ , we have  $v(x) = \rho(v \upharpoonright \text{seg } x)$
- $\langle 1 \rangle 2$ . If  $t_1 \leq t_2$ ,  $v_1$  is  $\rho$ -constructed up to  $t_1$ , and  $v_2$  is  $\rho$ -constructed up to  $t_2$ , then  $v_1 = v_2 \upharpoonright \{x \in J \mid x \leq t_1\}$
- $\langle 1 \rangle 3$ . LET:  $\mathcal{K}$  be the set of all functions that are  $\rho$ -constructed up to some  $t \in J$   
PROOF:  $\mathcal{K}$  is a set by a Replacement Axiom.
- $\langle 1 \rangle 4$ . LET:  $F = \bigcup \mathcal{K}$
- $\langle 1 \rangle 5$ .  $F$  is a function
- $\langle 1 \rangle 6$ . For all  $x \in \text{dom } F$  we have  $F(x) = \rho(F \upharpoonright \text{seg } x)$
- $\langle 1 \rangle 7$ .  $\text{dom } F = J$
- $\langle 1 \rangle 8$ .  $F$  is unique

□

**Theorem 4.38.** *The following are equivalent.*

1. The Axiom of Choice
2. (Well-Ordering Theorem) Every set has a well-ordering.
3. (Zorn's Lemma) Let  $X$  be a poset. If every chain in  $X$  has an upper bound in  $X$ , then  $X$  has a maximal element.

PROOF:

- $\langle 1 \rangle 1$ .  $1 \Rightarrow 2$

PROOF:

- (2)1. ASSUME: The Axiom of Choice
- (2)2. LET:  $X$  be a set.
- (2)3. PICK a choice function for  $\mathcal{P}X \setminus \{\emptyset\}$   
PROOF: Lemma 3.76.
- (2)4. LET: a *tower* in  $X$  be a pair  $(T, <)$  where  $T \subseteq X$ ,  $<$  is a well-ordering of  $T$ , and  $x = c(X \setminus \{y \in T \mid y < x\})$ .
- (2)5. For any two towers  $(T_1, <_1)$  and  $(T_2, <_2)$ , either these two posets are equal or one is a section of the other.
- (3)1.
- (2)6. For any tower  $(T, <)$  in  $X$  with  $T \neq X$ , there exists a tower in  $X$  of which  $(T, <)$  is a section.
- (2)7. LET:  $T = \bigcup \{T' \subseteq X \mid \exists R. (T, R) \text{ is a tower in } X\}$
- (2)8. Define  $<$  on  $T$  by:  $x < y$  iff there exists a tower  $(T, R)$  in  $X$  such that  $x, y \in T$  and  $xRy$ .
- (2)9.  $(T, <)$  is a tower in  $X$ .
- (2)10.  $T = X$
- (2)11.  $<$  is a well-ordering of  $X$ .
- (1)2.  $2 \Rightarrow 3$
- (2)1. ASSUME: The Well-Ordering Theorem
- (2)2. LET:  $X$  be a poset in which every chain has an upper bound.
- (2)3. PICK a well-ordering  $R$  of  $X$
- (2)4. Define  $F : X \rightarrow \{0, 1\}$  by transfinite  $R$ -recursion by:
 
$$F(a) = \begin{cases} 1 & \text{if } b < a \text{ for all } b \text{ such that } bRa \text{ and } f(b) = 1 \\ 0 & \text{otherwise} \end{cases}$$
- (2)5. LET:  $C = \{a \in X \mid f(a) = 1\}$
- (2)6.  $C$  is a chain in  $X$ 
  - (3)1. LET:  $x, y \in C$
  - (3)2. ASSUME: without loss of generality  $xRy$
  - (3)3.  $f(y) = 1$
  - (3)4. for all  $z$  such that  $zRy$  and  $f(z) = 1$  we have  $z < y$
  - (3)5.  $x < y$
- (2)7. PICK an upper bound  $u$  for  $C$
- (2)8.  $u$  is maximal in  $X$ 
  - (3)1. LET:  $x \in X$  with  $u \leq x$
  - (3)2. for all  $b$  such that  $bRx$  and  $f(b) = 1$  we have  $b < x$   
PROOF: Since  $b \in C$  so  $b \leq u \leq x$
  - (3)3.  $f(u) = 1$
  - (3)4.  $u \leq x$
  - (3)5.  $u = x$
- (2)9.  $3 \Rightarrow 1$ 
  - (3)1. ASSUME: Zorn's Lemma
  - (3)2. LET:  $R$  be a relation
  - (3)3. LET:  $\mathcal{A}$  be the poset of functions that are subsets of  $R$  under  $\subseteq$
  - (3)4. Every chain in  $\mathcal{A}$  has an upper bound
    - (4)1. LET:  $\mathcal{C} \subseteq \mathcal{A}$  be a chain.

- PROVE:  $\bigcup \mathcal{C} \in \mathcal{A}$
- $\langle 4 \rangle 2$ . ASSUME:  $(x, y), (x, z) \in \bigcup \mathcal{C}$
  - $\langle 4 \rangle 3$ . PICK  $f, g \in \mathcal{C}$  such that  $f(x) = y$  and  $g(x) = z$
  - $\langle 4 \rangle 4$ . ASSUME: without loss of generality  $f \subseteq g$
  - $\langle 4 \rangle 5$ .  $g(x) = y$
  - $\langle 4 \rangle 6$ .  $y = z$
  - $\langle 3 \rangle 5$ . PICK  $F$  maximal in  $\mathcal{A}$
  - $\langle 3 \rangle 6$ .  $\text{dom } F = \text{dom } R$
  - $\langle 4 \rangle 1$ . ASSUME: for a contradiction  $x \in \text{dom } R - \text{dom } F$
  - $\langle 4 \rangle 2$ . PICK  $y$  such that  $xRy$
  - $\langle 4 \rangle 3$ . LET:  $G = F \cup \{(x, y)\}$
  - $\langle 4 \rangle 4$ .  $G \in \mathcal{A}$
  - $\langle 4 \rangle 5$ .  $F \subset G$
  - $\langle 4 \rangle 6$ . Q.E.D.
- PROOF: This contradicts the maximality of  $F$ .

□

**Theorem 4.39** (Cardinal Comparability). *The Axiom of Choice is equivalent to the Cardinal Comparability Theorem: for any two sets  $A$  and  $B$ , either  $A \preceq B$  or  $B \preceq A$ .*

PROOF:

- $\langle 1 \rangle 1$ . Zorn's Lemma implies Cardinal Comparability
- $\langle 2 \rangle 1$ . ASSUME: Zorn's Lemma
- $\langle 2 \rangle 2$ . LET:  $A$  and  $B$  be sets.
- $\langle 2 \rangle 3$ . LET:  $\mathcal{A}$  be the poset of all injective functions  $f$  such that  $\text{dom } f \subseteq C$  and  $\text{ran } f \subseteq D$  under  $\subseteq$
- $\langle 2 \rangle 4$ . Every chain in  $\mathcal{A}$  has an upper bound.
- $\langle 3 \rangle 1$ . LET:  $\mathcal{C} \subseteq \mathcal{A}$  be a chain.
- PROVE:  $\bigcup \mathcal{C} \in \mathcal{A}$
- $\langle 3 \rangle 2$ .  $\bigcup \mathcal{C}$  is a function.
- $\langle 4 \rangle 1$ . LET:  $(x, y), (x, z) \in \bigcup \mathcal{C}$
- $\langle 4 \rangle 2$ . PICK  $f, g \in \mathcal{C}$  such that  $f(x) = y$  and  $g(x) = z$
- $\langle 4 \rangle 3$ . ASSUME: without loss of generality  $f \subseteq g$
- $\langle 4 \rangle 4$ .  $g(x) = y$
- $\langle 4 \rangle 5$ .  $y = z$
- $\langle 3 \rangle 3$ .  $\bigcup \mathcal{C}$  is injective.
- PROOF: Similar.
- $\langle 2 \rangle 5$ . PICK  $\hat{f}$  maximal in  $\mathcal{A}$
- PROOF: By Zorn's Lemma.
- $\langle 2 \rangle 6$ . Either  $\text{dom } \hat{f} = C$  or  $\text{ran } \hat{f} = D$
- $\langle 3 \rangle 1$ . ASSUME: for a contradiction  $\text{dom } \hat{f} \subset C$  and  $\text{ran } \hat{f} \subset D$
- $\langle 3 \rangle 2$ . PICK  $x \in C - \text{dom } \hat{f}$  and  $y \in D - \text{ran } \hat{f}$
- $\langle 3 \rangle 3$ . LET:  $g = \hat{f} \cup \{(x, y)\}$
- $\langle 3 \rangle 4$ .  $g \in \mathcal{A}$
- $\langle 3 \rangle 5$ .  $\hat{f} \subset g$

⟨3⟩6. Q.E.D.

PROOF: This contradicts the maximality of  $\hat{f}$ .

⟨2⟩7. If  $\text{dom } \hat{f} = C$  then  $C \preceq D$

⟨2⟩8. If  $\text{ran } \hat{f} = D$  then  $D \preceq C$

⟨1⟩2. Cardinal Comparability implies the Well-Ordering Theorem

⟨2⟩1. ASSUME: Cardinal Comparability

⟨2⟩2. LET:  $A$  be a set

⟨2⟩3. PICK an ordinal  $\alpha$  such that  $\alpha \not\preceq A$

⟨2⟩4.  $A \preceq \alpha$

PROOF: By Cardinal Comparability.

⟨2⟩5. PICK an injection  $f : A \rightarrow \alpha$

⟨2⟩6. Define  $<$  on  $A$  by  $x < y$  iff  $f(x) \in f(y)$

⟨2⟩7.  $<$  is a well-ordering on  $A$ .

□

**Theorem 4.40.** *Given two well-ordered sets  $A$  and  $B$ , either  $A \cong B$  or one of  $A, B$  is isomorphic to an initial segment of the other.*

## 4.7 Ordinal Numbers

**Definition 4.41.** Let  $(A, \leq)$  be a well-ordered set. The *ordinal number* of  $(A, \leq)$  is the range of  $E$ , where  $E$  is the unique function with domain  $A$  such that  $E(t) = \text{ran}(E \upharpoonright \text{seg } t)$  for all  $t \in A$ .

**Theorem 4.42.** *Let  $(A, \leq)$  be a well-ordered set and  $E : A \rightarrow \alpha$  be the canonical function onto the ordinal of  $A$ . Then:*

1. *For all  $t \in A$  we have  $E(t) \notin E(t)$ .*
2.  *$E$  is a bijection.*
3. *For any  $s, t \in A$ , we have  $s < t$  if and only if  $E(s) \in E(t)$ .*
4.  *$\alpha$  is a transitive set.*
5.  *$\alpha$  is well-ordered by  $\in$*
6.  *$E$  is an order isomorphism between  $(A, \leq)$  and  $(\alpha, \in)$ .*

**Theorem 4.43.** *Two well-ordered sets are isomorphic if and only if they have the same ordinal number.*

**Theorem 4.44.** *A set is an ordinal number if and only if it is a transitive set well-ordered by  $\in$ .*

**Theorem 4.45.** *Every member of an ordinal number is an ordinal number.*

**Theorem 4.46.** *Any transitive set of ordinal numbers is an ordinal number.*

**Theorem 4.47.** *The empty set is an ordinal number.*

**Theorem 4.48.** *The successor of an ordinal number is an ordinal number.*

**Theorem 4.49.** *If  $A$  is a set of ordinal numbers then  $\bigcup A$  is an ordinal number.*

**Theorem 4.50.** *Any nonempty set of ordinal numbers has a least element.*

**Theorem 4.51** (Burali-Forti Paradox). *The class of ordinal numbers is a proper class.*

**Theorem 4.52** (Hartogs' Theorem). *For any set  $A$ , there exists an ordinal that is not dominated by  $A$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $\alpha$  be the class of all ordinals  $\beta$  such that  $\beta \preccurlyeq A$

$\langle 1 \rangle 2$ .  $\alpha$  is a set.

$\langle 2 \rangle 1$ . LET:  $W$  be the set of all pairs  $(B, \leq)$  such that  $B \subseteq A$  and  $\leq$  is a well-ordering on  $B$ .

$\langle 2 \rangle 2$ . Every member of  $\alpha$  is the ordinal number of a member of  $W$

$\langle 2 \rangle 3$ . Q.E.D.

PROOF: By a Replacement Axiom.

$\langle 1 \rangle 3$ .  $\alpha$  is an ordinal.

$\langle 1 \rangle 4$ .  $\alpha$  is not dominated by  $A$ .

□

**Definition 4.53.** A class term  $\mathbf{F} : \mathbf{On} \rightarrow \mathbf{On}$  is *continuous* iff, for every limit ordinal  $\lambda$ , we have  $\mathbf{F}(\lambda) = \sup_{\alpha < \lambda} \mathbf{F}(\alpha)$ .

**Theorem 4.54.** *Let  $\mathbf{F} : \mathbf{On} \rightarrow \mathbf{On}$ . If  $\mathbf{F}$  is continuous and  $\mathbf{F}(\alpha) < \mathbf{F}(\alpha + 1)$  for every ordinal  $\alpha$ , then  $\mathbf{F}$  is strictly monotone.*

**Definition 4.55.** A class term  $\mathbf{F} : \mathbf{On} \rightarrow \mathbf{On}$  is *normal* iff it is strictly monotone and continuous.

**Theorem 4.56.** *Let  $\mathbf{F} : \mathbf{On} \rightarrow \mathbf{On}$  be normal. For every ordinal  $\beta \geq \mathbf{F}(0)$ , there exists a greatest ordinal  $\alpha$  such that  $\mathbf{F}(\alpha) \leq \beta$ .*

**Theorem 4.57.** *Let  $\mathbf{F} : \mathbf{On} \rightarrow \mathbf{On}$  be normal. Let  $S$  be a set of ordinals. Then  $\mathbf{F}(\sup S) = \sup_{\alpha \in S} \mathbf{F}(\alpha)$ .*

**Theorem 4.58** (Veblen Fixed-Point Theorem). *Let  $\mathbf{F} : \mathbf{On} \rightarrow \mathbf{On}$  be normal. For every ordinal  $\alpha$ , there exists  $\beta \geq \alpha$  such that  $\mathbf{F}(\beta) = \beta$ .*

PROOF: Let  $\beta$  be the supremum of  $\alpha, \mathbf{F}(\alpha), \mathbf{F}^2(\alpha), \dots$  □

**Lemma 4.59.** *Let  $\alpha$  be an ordinal. Let  $(f(\gamma))_{\gamma < \alpha}$  be an  $\alpha$ -sequence of ordinals. Then there exists  $\beta \leq \alpha$  and an increasing sequence of ordinals  $(g(\gamma))_{\gamma < \beta}$  such that  $\sup_{\gamma < \alpha} f(\gamma) = \sup_{\gamma < \beta} g(\gamma)$ .*

## 4.8 Cardinal Numbers

**Definition 4.60** (Cardinal Number (AC)). For any set  $A$ , the *cardinal number* of  $A$ ,  $\text{card } A$ , is the least ordinal equinumerous with  $A$ .

There exists some ordinal equinumerous with  $A$  by the Well-Ordering Theorem.

**Theorem 4.61.** For any sets  $A$  and  $B$ , we have  $A \equiv B$  if and only if  $\text{card } A = \text{card } B$ .

**Theorem 4.62.** A set  $A$  is finite if and only if  $\text{card } A$  is a natural number.

**Theorem 4.63.** The supremum of a set of cardinal numbers is a cardinal number.

## 4.9 Cardinal Arithmetic

**Definition 4.64.** For cardinal numbers  $\kappa$  and  $\lambda$ , the *sum*  $\kappa + \lambda$  is the cardinal number of  $A \cup B$ , where  $A$  and  $B$  are disjoint sets of cardinality  $\kappa$  and  $\lambda$  respectively.

**Theorem 4.65.**  $\kappa + \lambda = \lambda + \kappa$

**Theorem 4.66.**  $\kappa + (\lambda + \mu) = (\kappa + \lambda) + \mu$

**Theorem 4.67.** The definition of addition agrees with the definition on natural numbers.

**Definition 4.68.** For cardinal numbers  $\kappa$  and  $\lambda$ , the *product*  $\kappa\lambda$  is the cardinality of  $\kappa \times \lambda$ .

**Theorem 4.69.**  $\kappa\lambda = \lambda\kappa$

**Theorem 4.70.**  $\kappa(\lambda\mu) = (\kappa\lambda)\mu$

**Theorem 4.71.**  $\kappa(\lambda + \mu) = \kappa\lambda + \kappa\mu$

**Theorem 4.72.** The definition of multiplication agrees with the definition on natural numbers.

**Theorem 4.73** (AC). For any infinite cardinal  $\kappa$  we have  $\kappa\kappa = \kappa$ .

PROOF:

- $\langle 1 \rangle 1$ . LET:  $B$  be a set with cardinality  $\kappa$
- $\langle 1 \rangle 2$ . LET:  $\mathcal{H} = \{\emptyset\} \cup \{f \mid \exists A \subseteq B. A \text{ is infinite and } f \text{ is a bijection between } A \times A \text{ and } A\}$
- $\langle 1 \rangle 3$ . For every chain  $\mathcal{C} \subseteq \mathcal{H}$  we have  $\bigcup \mathcal{C} \in \mathcal{H}$
- $\langle 1 \rangle 4$ . PICK a maximal  $f_0$  in  $\mathcal{H}$
- $\langle 1 \rangle 5$ .  $f_0 \neq \emptyset$

PROOF:  $B$  has a subset of cardinality  $\aleph_0$  and  $\aleph_0\aleph_0 = \aleph_0$ .

- $\langle 1 \rangle 6$ . LET:  $A_0$  be the set such that  $f_0$  is a bijection between  $A_0 \times A_0$  and  $A_0$



$\langle 1 \rangle 7.$  LET:  $\lambda = \text{card } A_0$

$\langle 1 \rangle 8.$   $\text{card}(B - A_0) < \lambda$

$\langle 1 \rangle 9.$   $\kappa = \lambda$

PROOF:

$$\begin{aligned}
 \kappa &= \text{card } A_0 + \text{card}(B - A_0) \\
 &\leq \lambda + \lambda \\
 &= 2\lambda \\
 &\leq \lambda\lambda \\
 &= \lambda & (\langle 1 \rangle 6) \\
 &\leq \kappa & \square
 \end{aligned}$$

**Theorem 4.74** (Absorption Law). *Let  $\kappa$  and  $\lambda$  be cardinal numbers such that  $0 < \kappa \leq \lambda$  and  $\lambda$  is infinite. Then*

$$\kappa + \lambda = \lambda .$$

**Theorem 4.75** (Absorption Law). *Let  $\kappa$  and  $\lambda$  be cardinal numbers such that  $0 < \kappa \leq \lambda$  and  $\lambda$  is infinite. Then*

$$\kappa\lambda = \lambda .$$

**Definition 4.76.** For cardinal numbers  $\kappa$  and  $\lambda$ , we write  $\kappa^\lambda$  for the cardinality of the set of functions from  $\lambda$  to  $\kappa$ .

**Theorem 4.77.**  $\kappa^{\lambda+\mu} = \kappa^\lambda + \kappa^\mu$

**Theorem 4.78.**  $(\kappa\lambda)^\mu = \kappa^\mu\lambda^\mu$

**Theorem 4.79.**  $(\kappa^\lambda)^\mu = \kappa^{\lambda\mu}$

**Theorem 4.80.** *The definition of exponentiation agrees with the definition on natural numbers.*

**Theorem 4.81.** *Given sets  $A$  and  $B$ , we have  $\text{card } A \leq \text{card } B$  if and only if  $A \preceq B$ .*

**Definition 4.82.** Let  $\aleph_0 = \text{card } \mathbb{N}$ .

**Theorem 4.83** (AC). *For any infinite cardinal  $\kappa$  we have  $\aleph_0 \leq \kappa$ .*

**Theorem 4.84** (Maximum Principle (AC)). *Every poset has a maximal chain.*

## 4.10 Rank of a Set

**Definition 4.85** (Cumulative Hierarchy of Sets). For every ordinal  $\alpha$ , define the *rank*  $V_\alpha$  by transfinite recursion thus:

$$\begin{aligned}
 V_0 &= \emptyset \\
 V_{\alpha+1} &= \mathcal{P}V_\alpha \\
 V_\lambda &= \bigcup_{\alpha < \lambda} V_\alpha
 \end{aligned}$$

for  $\lambda$  a limit ordinal.

The *von Neumann universe* is the class  $\mathbf{V} = \bigcup_{\alpha \in \mathbf{On}} V_\alpha$ .

**Theorem 4.86.** *If  $\lambda$  is a limit ordinal and  $\lambda > \omega$  then  $V_\lambda$  is a model of Zermelo set theory.*

**Lemma 4.87 (AC).** *There exists a well-ordered set in  $V_{\omega_2}$  whose ordinal is not in  $V_{\omega_2}$ .*

PROOF: Pick a well-ordering  $<$  of  $\mathcal{PN}$ . Then  $(\mathcal{PN}, <) \in V_{\omega_2}$  but its ordinal is not because its ordinal is uncountable.  $\square$

**Theorem 4.88.** *The set  $V_{\omega_2}$  is not a model of Zermelo-Fraenkel set theory.*

Thus, the Replacement Axioms cannot be proven from the other axioms.

**Definition 4.89** (Well-Founded Set). A set  $A$  is *well-founded* iff  $A \in V_\alpha$  for some  $\alpha \in \mathbf{On}$ .

**Definition 4.90** (Rank). The *rank* of a well-founded set  $A$ ,  $\text{rank } A$ , is the least ordinal  $\alpha$  such that  $A \in V_\alpha$ .

**Theorem 4.91.** *If  $A \in B$  and  $B$  is well-founded then  $A$  is well-founded and  $\text{rank } A < \text{rank } B$ .*

**Theorem 4.92.** *If  $A$  is a set and every member of  $A$  is well-founded then  $A$  is well-founded and  $\text{rank } A = \sup_{B \in A} (\text{rank } B + 1)$ .*

**Theorem 4.93.** *The Axiom of Regularity is equivalent to the statement that every set is well-founded.*

## 4.11 Transfinite Recursion Again

**Theorem 4.94.** *Let  $\mathbf{A}$  be a class. Let  $\mathbf{B}$  be the class of all functions  $f : \alpha \rightarrow \mathbf{A}$  for some ordinal  $\alpha$ . Let  $\mathbf{F} : \mathbf{B} \rightarrow \mathbf{A}$  be a class term. Then there exists a unique class term  $\mathbf{G} : \mathbf{On} \rightarrow \mathbf{A}$  such that, for all  $\alpha \in \mathbf{On}$ , we have  $\mathbf{G}(\alpha) = \mathbf{F}(\mathbf{G} \upharpoonright \alpha)$ .*

## 4.12 Alephs

**Definition 4.95.** Define the cardinal number  $\aleph_\alpha$  for every ordinal  $\alpha$  by transfinite recursion on  $\alpha$  thus:  $\aleph_\alpha$  is the least infinite cardinal different from  $\aleph_\beta$  for all  $\beta < \alpha$ .

**Theorem 4.96.** *If  $\alpha < \beta$  then  $\aleph_\alpha < \aleph_\beta$ .*

**Theorem 4.97.** *Every infinite cardinal has the form  $\aleph_\alpha$  for some ordinal  $\alpha$ .*

## 4.13 Ordinal Arithmetic

**Definition 4.98** (Sum). Let  $\alpha$  and  $\beta$  be ordinals. The *sum*  $\alpha + \beta$  is the ordinal of the concatenation of  $A$  followed by  $B$ , where  $A$  is a well-ordered set of ordinal  $\alpha$  and  $B$  a well-ordered set of ordinal  $\beta$ .

**Theorem 4.99.** *Addition is associative.*

**Theorem 4.100.**  $\alpha + 0 = \alpha$

**Theorem 4.101.**  $0 + \alpha = \alpha$

**Theorem 4.102.** *For  $\lambda$  a limit ordinal we have  $\alpha + \lambda = \sup_{\beta < \lambda} (\alpha + \beta)$*

**Theorem 4.103.** *For any  $\alpha$ , the class term that maps  $\beta$  to  $\alpha + \beta$  is normal.*

**Theorem 4.104.**  $\beta < \gamma$  iff  $\alpha + \beta < \alpha + \gamma$ .

**Theorem 4.105.** *If  $\beta \leq \gamma$  then  $\beta + \alpha \leq \gamma + \alpha$ .*

**Theorem 4.106** (Subtraction Theorem). *If  $\alpha < \beta$  then there exists a unique  $\delta$  such that  $\alpha + \delta < \beta$ .*

**Definition 4.107** (Product). Let  $\alpha$  and  $\beta$  be ordinals. The *sum*  $\alpha \times \beta$  is the ordinal of  $A \times B$  ordered under the Hebrew lexicographic order, where  $A$  is a well-ordered set of ordinal  $\alpha$  and  $B$  a well-ordered set of ordinal  $\beta$ .

**Theorem 4.108.** *Multiplication is associative.*

**Theorem 4.109.** *Multiplication distributes over addition on the left.*

**Theorem 4.110.**  $\alpha 1 = \alpha$

**Theorem 4.111.**  $1\alpha = \alpha$

**Theorem 4.112.**  $\alpha 0 = 0$

**Theorem 4.113.**  $0\alpha = 0$

**Theorem 4.114.** *For  $\lambda$  a limit ordinal, we have  $\alpha\lambda = \sup_{\beta < \lambda} (\alpha\beta)$ .*

**Theorem 4.115.** *For  $\alpha > 0$ , the class term that maps  $\beta$  to  $\alpha\beta$  is normal.*

**Theorem 4.116.** *If  $\alpha > 0$ , then  $\beta < \gamma$  iff  $\alpha\beta < \alpha\gamma$ .*

**Theorem 4.117.** *If  $\beta \leq \gamma$  then  $\beta\alpha \leq \gamma\alpha$ .*

**Theorem 4.118** (Division Theorem). *For any ordinals  $\alpha$  and  $\delta$  with  $\delta \neq 0$ , there exist unique ordinals  $\beta$  and  $\gamma$  with  $\gamma < \delta$  and  $\alpha = \delta\beta + \gamma$ .*

**Definition 4.119** (Exponentiation). For ordinals  $\alpha$  and  $\beta$ , define the ordinal  $\alpha^\beta$  by transfinite recursion on  $\beta$  by:

$$\begin{aligned}\alpha^0 &= 1 \\ \alpha^{\beta+1} &= \alpha^\beta + \alpha \\ \alpha^\lambda &= \sup_{\beta < \lambda} \alpha^\beta\end{aligned}$$

for  $\lambda$  a limit ordinal.

**Theorem 4.120.** For  $\alpha > 1$ , the class term that maps  $\beta$  to  $\alpha^\beta$  is normal.

**Theorem 4.121.** If  $\alpha > 1$ , then  $\beta < \gamma$  iff  $\alpha^\beta < \alpha^\gamma$ .

**Theorem 4.122.** If  $\beta \leq \gamma$  then  $\beta^\alpha \leq \gamma^\alpha$ .

**Theorem 4.123** (Logarithm Theorem). Let  $\alpha$  and  $\beta$  be ordinals with  $\alpha \neq 0$  and  $\beta > 1$ . Then there exist unique ordinals  $\gamma$ ,  $\delta$  and  $\rho$  such that  $\delta \neq 0$ ,  $\delta < \beta$ ,  $\rho < \beta^\gamma$ , and  $\alpha = \beta^\gamma \delta + \rho$ .

**Theorem 4.124.**

$$\alpha^{\beta+\gamma} = \alpha^\beta \alpha^\gamma$$

**Theorem 4.125.**

$$(\alpha^\beta)^\gamma = \alpha^{\beta\gamma}$$

## 4.14 Beth Cardinals

**Definition 4.126.** Define the cardinal  $\beth_\alpha$  for every ordinal  $\alpha$  by:

$$\begin{aligned}\beth_0 &= \aleph_0 \\ \beth_{\alpha+1} &= 2^{\beth_\alpha} \\ \beth_\lambda &= \sup_{\alpha < \lambda} \beth_\alpha\end{aligned}$$

for  $\lambda$  a limit ordinal.

**Lemma 4.127.** For any ordinal  $\alpha$  we have  $\text{card } V_{\omega+\alpha} = \beth_\alpha$ .

## 4.15 Cofinality

**Definition 4.128** (Cofinality). For  $\lambda$  a limit ordinal, the *cofinality* of  $\lambda$ ,  $\text{cf } \lambda$ , is the least cardinal  $\kappa$  such that  $\lambda$  is the supremum of a set of  $\kappa$  smaller ordinals.

We extend  $\text{cf}$  to all the ordinals by setting  $\text{cf } 0 = 0$  and  $\text{cf }(\alpha + 1) = 1$ .

**Theorem 4.129.** For any limit ordinal  $\lambda$  we have  $\text{cf } \aleph_\lambda = \text{cf } \lambda$ .

**Lemma 4.130.** Let  $\lambda$  be a limit ordinal. Then  $\text{cf } \lambda$  is the least ordinal  $\alpha$  such that there exists an increasing  $\alpha$ -sequence of ordinals with limit  $\lambda$ .

**Theorem 4.131.** Let  $\lambda$  be an infinite cardinal. Then  $\text{cf } \lambda$  is the least cardinal number  $\kappa$  such that  $\lambda$  can be partitioned into  $\kappa$  sets each of cardinality  $< \lambda$ .

**Theorem 4.132** (König's Theorem). Let  $\kappa$  be an infinite cardinal. Then  $\kappa < 2^{\text{cf } \kappa}$ .

**Corollary 4.132.1.**  $2^{\aleph_0} \neq \aleph_\omega$ .

**Definition 4.133** (Regular). A cardinal  $\kappa$  is *regular* iff  $\text{cf } \kappa = \kappa$ .

**Theorem 4.134.** For any ordinal  $\lambda$ , we have  $\text{cf } \lambda$  is a regular cardinal.

**Definition 4.135** (Singular). A cardinal  $\kappa$  is *singular* iff  $\text{cf } \kappa < \kappa$ .

**Theorem 4.136.** For any ordinal  $\alpha$  we have  $\aleph_{\alpha+1}$  is a regular cardinal.

## 4.16 Inaccessible Cardinals

**Definition 4.137** (Inaccessible). A cardinal number  $\kappa$  is *inaccessible* iff:

- $\kappa > \aleph_0$
- For every cardinal  $\lambda < \kappa$  we have  $2^\lambda < \kappa$
- $\kappa$  is regular.

**Lemma 4.138.** *If  $\kappa$  is inaccessible and  $\alpha < \kappa$  then  $\beth_\alpha < \kappa$ .*

**Lemma 4.139.** *If  $\kappa$  is inaccessible and  $A \in V_\kappa$  then  $\text{card } A < \kappa$ .*

**Theorem 4.140.** *If  $\kappa$  is inaccessible then  $V_\kappa$  is a model of ZF.*

## Chapter 5

# Natural Numbers

### 5.1 Successors

**Definition 5.1** (Successor (Pairing, Union)). For any set  $a$ , its *Successor*  $a^+$  is the set  $a \cup \{a\}$

**Theorem 5.2** (Pairing, Union). *If  $a$  is a transitive set then  $\bigcup(a^+) = a$ .*

PROOF:

$$\begin{aligned}\bigcup(a^+) &= \bigcup(a \cup \{a\}) \\ &= \bigcup a \cup \bigcup \{a\} \\ &= \bigcup a \cup a \\ &= a \qquad \qquad \qquad (\bigcup a \subseteq a) \square\end{aligned}$$

**Theorem 5.3.** *If  $A$  is a transitive set then  $A^+$  is transitive.*

PROOF: If  $A$  is transitive then  $\bigcup(A^+) = A \subseteq A^+$ .  $\square$

### 5.2 Inductive Sets

**Definition 5.4** (Inductive (Extensionality, Empty Set, Pairing, Union)). A set  $A$  is *inductive* iff  $\emptyset \in A$  and, for every  $a \in A$ , we have  $a^+ \in A$ .

**Axiom 5.5** (Axiom of Infinity (Extensionality, Empty Set, Pairing, Union)). *There exists an inductive set.*

### 5.3 Natural Numbers

**Definition 5.6** (Natural Number (Extensionality, Empty Set, Pairing, Union)). A *natural number* is a set that belongs to every inductive set.

We write  $\mathbb{N}$  for the class of all natural numbers.

**Theorem 5.7** (Extensionality, Empty Set, Pairing, Union, Infinity, Subset).  
*The class of natural numbers is a set.*

PROOF:

⟨1⟩1. PICK an inductive set  $I$ .

PROOF: By the Axiom of Infinity.

⟨1⟩2.  $\mathbb{N} \subseteq I$

⟨1⟩3. Q.E.D.

PROOF: By a Subset Axiom.

□

**Theorem 5.8** (Extensionality, Empty Set, Pairing, Union, Infinity, Subset).  
*The set  $\mathbb{N}$  is inductive.*

PROOF:

⟨1⟩1.  $\emptyset \in \mathbb{N}$

PROOF: Since  $\emptyset$  is a member of every inductive set.

⟨1⟩2. For all  $n \in \mathbb{N}$  we have  $n^+ \in \mathbb{N}$

PROOF: If  $n$  is a member of every inductive set then so is  $n^+$ .

□

**Theorem 5.9** (Extensionality, Empty Set, Pairing, Union, Infinity, Subset).  
*The set  $\mathbb{N}$  is a subset of every inductive set.*

PROOF: Immediate from definition. □

**Corollary 5.9.1** (Proof by Induction (Extensionality, Empty Set, Pairing, Union, Infinity, Subset)). *If  $A \subseteq \mathbb{N}$  and  $A$  is inductive then  $A = \mathbb{N}$ .*

**Definition 5.10** (Zero (Empty Set)). The natural number *zero*,  $0$ , is defined to be  $\emptyset$ .

**Theorem 5.11** (Extensionality, Empty Set, Pairing, Union, Infinity, Subset).  
*Every natural number except  $0$  is a successor of a natural number.*

PROOF: The set  $\{x \in \mathbb{N} \mid x = 0 \vee \exists y \in \mathbb{N}. x = y^+\}$  is inductive. □

**Theorem 5.12** (Extensionality, Empty Set, Pairing, Union, Infinity, Subset).  
*Every natural number is transitive.*

PROOF: By induction using Theorem 5.3. □

**Theorem 5.13** (Extensionality, Empty Set, Pairing, Union, Infinity, Subset).  
*The set  $\mathbb{N}$  is transitive.*

PROOF:

⟨1⟩1. For every natural number  $n$  and every  $m \in n$  then  $m$  is a natural number.

⟨2⟩1. Every member of  $\emptyset$  is a natural number.

PROOF: Vacuous.

⟨2⟩2. If  $n$  is a natural number and a set of natural numbers then  $n^+$  is a set of natural numbers.

PROOF: From the definition of  $n^+$ .

(2)3. Q.E.D.

PROOF: By induction.

□

**Theorem 5.14** (Extensionality, Empty Set, Pairing, Union, Infinity, Subset).  
*Let  $A$  be a set,  $a \in A$ , and  $F : A \rightarrow A$ . Then there exists a unique function  $h : \mathbb{N} \rightarrow A$  such that  $h(0) = a$  and, for all  $n \in \mathbb{N}$ , we have  $h(n^+) = F(h(n))$ .*

PROOF:

(1)1. Call a function  $v$  *acceptable* iff  $\text{dom } v \subseteq \mathbb{N}$ ,  $\text{ran } v \subseteq A$ , and:

1. If  $0 \in \text{dom } v$  then  $v(0) = a$ .

2. For all  $n \in \mathbb{N}$ , if  $n^+ \in \text{dom } v$  then  $n \in \text{dom } v$  and  $v(n^+) = F(v(n))$ .

(1)2. LET:  $\mathcal{K}$  be the set of all acceptable functions.

(1)3. LET:  $h = \bigcup \mathcal{K}$

(1)4.  $h$  is a function.

(2)1. If  $(0, y) \in h$  and  $(0, y') \in h$  then  $y = y'$

PROOF: We have  $y = y' = a$ .

(2)2. For any natural number  $n$ , if there is at most one  $y$  such that  $(n, y) \in h$ , then there is at most one  $y$  such that  $(n^+, y) \in h$

(3)1. LET:  $n$  be a natural number.

(3)2. ASSUME: there is at most one  $y$  such that  $(n, y) \in h$

(3)3. ASSUME:  $(n^+, y)$  and  $(n^+, y')$  are in  $h$

(3)4. PICK acceptable functions  $u$  and  $v$  such that  $u(n^+) = y$  and  $v(n^+) = y'$

(3)5.  $n \in \text{dom } u$ ,  $n \in \text{dom } v$  and  $y = F(u(n))$ ,  $y' = F(v(n))$

(3)6.  $u(n) = v(n)$

PROOF: By the induction hypothesis (3)2

(3)7.  $y = y'$

(2)3. Q.E.D.

PROOF: By induction.

(1)5.  $h$  is acceptable.

(2)1. If  $0 \in \text{dom } h$  then  $h(0) = a$

(2)2. If  $n^+ \in \text{dom } h$  then  $n \in \text{dom } h$  and  $h(n^+) = F(h(n))$

(3)1. ASSUME:  $n^+ \in \text{dom } h$

(3)2. PICK an acceptable  $v$  such that  $n^+ \in \text{dom } v$

(3)3.  $v(n^+) = F(v(n))$

(3)4.  $h(n^+) = F(h(n))$

(1)6.  $\text{dom } h = \mathbb{N}$

(2)1.  $0 \in \text{dom } h$

PROOF: Since  $\{(0, a)\}$  is an acceptable function.

(2)2. For all  $n \in \text{dom } h$  we have  $n^+ \in \text{dom } h$

(3)1. ASSUME:  $n \in \text{dom } h$

(3)2. LET:  $v$  be an acceptable function with  $n \in \text{dom } v$

(3)3. ASSUME: without loss of generality  $n^+ \notin \text{dom } v$

(3)4.  $v \cup \{(n^+, F(v(n)))\}$  is acceptable



$\langle 3 \rangle 5. \ n^+ \in \text{dom } v$   
 $\langle 1 \rangle 7. \text{ If } h' : \mathbb{N} \rightarrow A, h'(0) = a \text{ and, for all } n \in \mathbb{N}, \text{ we have } h'(n^+) = F(h'(n)),$   
 $\text{then } h' = h$   
 PROOF: Prove  $h(n) = h'(n)$  by induction on  $n$ .  
 $\square$

## 5.4 Peano Systems

**Definition 5.15** (Peano System). A *Peano system* consists of a set  $N$ , an element  $z \in N$ , and a function  $S : N \rightarrow N$  such that:

- $S$  is one-to-one
- $z \notin \text{ran } S$
- For any set  $A \subseteq N$ , if  $z \in A$  and  $S(A) \subseteq A$  then  $A = N$ .

**Theorem 5.16.**  $\mathbb{N}$  is a Peano system with zero  $0$  and successor  $n \mapsto n^+$ .

**Theorem 5.17.** For any Peano system  $(N, z, S)$ , there exists a unique bijection  $h : \mathbb{N} \cong N$  such that  $h(0) = z$  and  $S(h(n)) = h(n^+)$  for all  $n$ .

## 5.5 Arithmetic

**Definition 5.18** (Addition). Define *addition*  $+ : \mathbb{N}^2 \rightarrow \mathbb{N}$  recursively by

$$\begin{aligned}
 m + 0 &= m \\
 m + n^+ &= (m + n)^+
 \end{aligned}$$

for any  $m, n \in \mathbb{N}$ .

**Theorem 5.19.** *Addition is associative.*

**Theorem 5.20.** *Addition is commutative*

**Definition 5.21** (Multiplication). Define *multiplication*  $\cdot : \mathbb{N}^2 \rightarrow \mathbb{N}$  recursively by

$$\begin{aligned}
 m0 &= 0 \\
 mn^+ &= mn + m
 \end{aligned}$$

for any  $m, n \in \mathbb{N}$

**Theorem 5.22.** *Multiplication is associative.*

**Theorem 5.23.** *Multiplication is commutative.*

**Theorem 5.24.** *Multiplication distributes over addition.*

**Definition 5.25.** For natural numbers  $m$  and  $n$ , we write  $m < n$  iff  $m \in n$ . We write  $m \leq n$  iff  $m < n$  or  $m = n$ .

**Theorem 5.26.** We have  $m < n$  iff  $m^+ < n^+$ .

**Theorem 5.27.** We never have  $n < n$ .

**Theorem 5.28.** The ordering on  $\mathbb{N}$  satisfies trichotomy; that is, for any  $m, n$ , exactly one of  $m < n$ ,  $m = n$ ,  $n < m$  holds.

**Theorem 5.29.** For any natural numbers  $m$  and  $n$ , we have  $m \leq n$  iff  $m \subseteq n$ .

**Theorem 5.30.** We have  $m < n$  iff  $m + p < n + p$ .

**Corollary 5.30.1.** If  $m + p = n + p$  then  $m = n$ .

**Theorem 5.31.** If  $p \neq 0$  then  $m < n$  iff  $mp < np$ .

**Corollary 5.31.1.** If  $mp = np$  and  $p \neq 0$  then  $m = n$ .

**Theorem 5.32** (Well-Ordering of  $\mathbb{N}$ ). Any nonempty set  $A \subseteq \mathbb{N}$  has a least element.

**Corollary 5.32.1.** There is no function  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that  $f(n^+) < f(n)$  for all  $n$ .

**Theorem 5.33** (Strong Induction). Let  $A \subseteq \mathbb{N}$ . Suppose that, for every natural number  $n$ , if  $\forall m < n. m \in A$  then  $n \in A$ . Then  $A = \mathbb{N}$ .

**Theorem 5.34** (Pigeonhole Principle). No natural number is equinumerous with a proper subset of itself.

PROOF: Prove by induction on  $n$  that if  $f : n \rightarrow n$  is injective then it is surjective.  
 $\square$

## Chapter 6

# Integers

**Lemma 6.1.** Define  $\sim$  on  $\mathbb{N}^2$  by:  $(m, n) \sim (p, q)$  iff  $m + q = n + p$ . Then  $\sim$  is an equivalence relation on  $\mathbb{N}^2$ .

**Definition 6.2** (Integers). The set  $\mathbb{Z}$  of *integers* is  $\mathbb{N}^2 / \sim$ .

**Definition 6.3.** Define *addition*  $+: \mathbb{Z}^2 \rightarrow \mathbb{Z}$  by:  $(m, n) + (p, q) = (m + p, n + q)$ .  
Prove this is well-defined.

**Theorem 6.4.** *Addition is associative and commutative.*

**Definition 6.5** (Zero). The integer *zero* is  $0 = (0, 0)$ .

**Theorem 6.6.** *For any integer  $a$ , we have  $a + 0 = a$ .*

**Theorem 6.7.** *For any integer  $a$ , there exists a unique integer  $b$  such that  $a + b = 0$ .*

**Definition 6.8** (Multiplication). Define *multiplication* on  $\mathbb{Z}$  by  $(m, n)(p, q) = (mp + nq, mq + np)$ .

**Theorem 6.9.** *Multiplication is associative, commutative and distributive over addition.*

**Definition 6.10.** The integer *one* is  $1 = (1, 0)$ .

**Theorem 6.11.** *For any integer  $a$  we have  $a1 = a$ .*

**Theorem 6.12.**  $1 \neq 0$

**Theorem 6.13.** *Whenever  $ab = 0$  then either  $a = 0$  or  $b = 0$ .*

**Definition 6.14.** Define  $<$  on  $\mathbb{Z}$  by:  $(m, n) < (p, q)$  iff  $m + q < n + p$ .

**Theorem 6.15.** *The relation  $<$  is a strict linear ordering on  $\mathbb{Z}$ .*

**Theorem 6.16.** *We have  $a < b$  iff  $a + c < b + c$ .*

**Corollary 6.16.1.** *If  $a + c = b + c$  then  $a = b$ .*

**Theorem 6.17.** *If  $0 < c$  then  $a < b$  iff  $ac < bc$ .*

**Corollary 6.17.1.** *If  $ac = bc$  and  $c \neq 0$  then  $a = b$ .*

**Definition 6.18.** We identify any natural number  $n$  with the integer  $(n, 0)$ .

**Theorem 6.19.** *This embedding preserves 0, 1, addition, multiplication and the ordering.*

## Chapter 7

# Rational Numbers

**Definition 7.1** (Rational Numbers). The set of *rational numbers*  $\mathbb{Q}$  is  $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) / \sim$ , where  $(a, b) \sim (c, d)$  iff  $ad = bc$ .

**Definition 7.2** (Addition). Define addition on  $\mathbb{Q}$  by:  $(a, b) + (c, d) = (ad + bc, bd)$ .

**Theorem 7.3.** *Addition is commutative and associative*

**Definition 7.4.** The rational number 0 is  $(0, 1)$ .

**Theorem 7.5.** *For any rational  $q$  we have  $q + 0 = q$ .*

**Theorem 7.6.** *For any rational  $q$ , there exists a unique rational  $r$  such that  $q + r = 0$ .*

**Definition 7.7.** Define multiplication on  $\mathbb{Q}$  by:  $(a, b)(c, d) = (ac, bd)$ .

**Theorem 7.8.** *Multiplication is commutative, associative and distributive over addition.*

**Definition 7.9.** The rational number 1 is  $(1, 1)$ .

**Theorem 7.10.** *For every nonzero rational  $r$ , there exists a nonzero rational  $q$  such that  $rq = 1$ .*

**Corollary 7.10.1.** *If  $qr = 0$  then either  $q = 0$  or  $r = 0$ .*

**Definition 7.11.** Define  $<$  on  $\mathbb{Q}$  by: for  $b$  and  $d$  positive,  $(a, b) < (c, d)$  iff  $ad < bc$ .

**Theorem 7.12.** *The relation  $<$  is a strict linear ordering on  $\mathbb{Q}$ .*

**Theorem 7.13.** *We have  $q < r$  iff  $q + s < r + s$*

**Corollary 7.13.1.** *If  $q + s = r + s$  then  $q = r$ .*

**Theorem 7.14.** *If  $s > 0$  then we have  $q < r$  iff  $qs < rs$ .*

**Corollary 7.14.1.** *If  $qs = rs$  and  $s \neq 0$  then  $q = r$ .*

**Definition 7.15.** We identify an integer  $n$  with the rational  $(n, 1)$ .

**Theorem 7.16.** *This embedding preserves zero, one, addition, multiplication and the ordering.*

## Chapter 8

# Real Numbers

**Definition 8.1** (Dedekind Cut). A *Dedekind cut* is a subset  $X \subseteq \mathbb{Q}$  such that:

- $X$  is nonempty
- $X \neq \mathbb{Q}$
- $X$  is closed downward
- $X$  has no largest element.

**Definition 8.2** (Real Numbers). The set of *real numbers*  $\mathbb{R}$  is the set of all Dedekind cuts.

**Definition 8.3.** Define  $<$  on  $\mathbb{R}$  by:  $x < y$  iff  $x$  is a proper subset of  $y$ .

**Theorem 8.4.** *The relation  $<$  is a strict linear ordering on  $\mathbb{R}$ .*

**Theorem 8.5.** *Any bounded nonempty subset of  $\mathbb{R}$  has a least upper bound.*

**Definition 8.6.** Define addition on  $\mathbb{R}$  by:  $x + y = \{q + r \mid q \in x, r \in y\}$ .

**Theorem 8.7.** *Addition is associative and commutative.*

**Definition 8.8.** The zero real  $0$  is  $\{q \in \mathbb{Q} \mid q < 0\}$ .

**Theorem 8.9.** *For any  $x \in \mathbb{R}$  we have  $x + 0 = x$ .*

**Definition 8.10.** Given a real  $x$ , define  $-x = \{q \in \mathbb{Q} \mid \exists r > q. -r \notin x\}$ .

**Theorem 8.11.** *For any real  $x$  we have  $x + (-x) = 0$ .*

**Corollary 8.11.1.** *If  $x + z = y + z$  then  $x = y$ .*

**Theorem 8.12.** *We have  $x < y$  iff  $x + z < y + z$ .*

**Definition 8.13.** Define the *absolute value* of a real  $x$  by  $|x| = x \cup -x$ .

**Theorem 8.14.** *For any real  $x$  we have  $0 \leq |x|$ .*

**Definition 8.15.** Define multiplication on  $\mathbb{R}$  by:

- If  $x$  and  $y$  are nonnegative then

$$xy = 0 \cup \{qr \mid 0 \leq q, 0 \leq r, q \in x, r \in y\}$$

- If  $x$  and  $y$  are both negative then  $xy = |x||y|$
- If one of  $x$  and  $y$  is negative and the other not then  $xy = -|x||y|$ .

**Theorem 8.16.** *Multiplication is associative, commutative and distributive over addition.*

**Definition 8.17.** The real number 1 is  $\{q \in \mathbb{Q} \mid q < 1\}$ .

**Theorem 8.18.**  $0 \neq 1$

**Theorem 8.19.** *For any real  $x$  we have  $x1 = x$*

**Theorem 8.20.** *For any nonzero  $x$ , there exists a real  $y$  with  $xy = 1$ .*

**Theorem 8.21.** *If  $0 < x$  then  $y < z$  iff  $xy < xz$ .*

**Definition 8.22.** Identify a rational  $q$  with  $\{r \in \mathbb{Q} \mid r < q\}$ .

**Theorem 8.23.** *This embedding preserves zero, one, addition, multiplication and the ordering.*

## 8.1 The Cantor Set

**Definition 8.24** (Cantor Set). Define the sequence of sets  $A_n \subseteq \mathbb{R}$  by

$$A_0 = [0, 1]$$

$$A_n = A_{n-1} - \bigcup_{k=0}^{3^{n-1}-1} ((3k+1)/3^n, (3k+2)/3^n)$$

The *Cantor set* is  $\bigcap_{n=0}^{\infty} A_n$ .

**Proposition 8.25.** *The set  $A_n$  is a union of finitely many disjoint closed intervals of length  $1/3^n$ , and the endpoints of these intervals lie in  $C$ .*

PROOF: An easy induction on  $n$ .  $\square$



## Chapter 9

# Finite Sets

**Definition 9.1** (Finite). A set is *finite* iff it is equinumerous with a natural number; otherwise it is *infinite*.

**Theorem 9.2.** *No finite set is equinumerous with a proper subset of itself.*

PROOF: From the Pigeonhole Principle.

**Corollary 9.2.1.** *The set  $\mathbb{N}$  is infinite.*

**Corollary 9.2.2.** *A finite set is equinumerous with a unique natural number.*

**Lemma 9.3.** *If  $A$  is a proper subset of a natural number  $n$  then there exists  $m < n$  such that  $C \equiv m$ .*

**Corollary 9.3.1.** *A subset of a finite set is finite.*

**Theorem 9.4** (Regularity). *There is no function  $f$  with domain  $\mathbb{N}$  such that  $f(n+1) \in f(n)$  for all  $n$ .*

PROOF:

$\langle 1 \rangle 1$ . ASSUME: for a contradiction  $f$  is a function with domain  $\mathbb{N}$  such that  $f(n+1) \in f(n)$  for all  $n$ .

$\langle 1 \rangle 2$ . PICK  $m \in \text{ran } f$  such that  $m \cap \text{ran } f = \emptyset$

PROOF: By the Axiom of Regularity.

$\langle 1 \rangle 3$ . PICK  $n \in \mathbb{N}$  such that  $f(n) = m$

$\langle 1 \rangle 4$ .  $f(n+1) \in m \cap \text{ran } f$

$\langle 1 \rangle 5$ . Q.E.D.

PROOF: This is a contradiction.

□

**Theorem 9.5.** *A relation  $R$  is well-founded if and only if there is no function  $f$  with domain  $\mathbb{N}$  such that, for all  $n \in \mathbb{N}$ , we have  $f(n+1)Rf(n)$ .*

## 9.1 Real Analysis

**Definition 9.6.** Let  $\mathbb{R}^\infty$  be the subset of  $\mathbb{R}^\omega$  consisting of all sequences  $(a_n)$  such that  $a_n = 0$  for all but finitely many  $n$ .

## 9.2 Group Theory

**Definition 9.7.** Given a group  $G$  and sets  $A, B \subseteq G$ , let  $AB = \{ab \mid a \in A, b \in B\}$ .

**Definition 9.8.** Given a group  $G$  and a set  $A \subseteq G$ , let  $A^{-1} = \{a^{-1} \mid a \in A\}$ .

### 9.3 Topological Spaces

**Definition 9.9** (Topology). A *topology* on a set  $X$  is a set  $\mathcal{T} \subseteq \mathcal{P}X$  such that:

- $X \in \mathcal{T}$ .
- For all  $\mathcal{U} \subseteq \mathcal{T}$  we have  $\bigcup \mathcal{U} \in \mathcal{T}$ .
- For all  $U, V \in \mathcal{T}$  we have  $U \cap V \in \mathcal{T}$ .

We call the elements of  $X$  *points* and the elements of  $\mathcal{T}$  *open sets*.

**Definition 9.10** (Topological Space). A *topological space*  $X$  consists of a set  $X$  and a topology on  $X$ .

**Definition 9.11** (Discrete Space). For any set  $X$ , the *discrete* topology on  $X$  is  $\mathcal{P}X$ .

**Definition 9.12** (Indiscrete Space). For any set  $X$ , the *indiscrete* or *trivial* topology on  $X$  is  $\{\emptyset, X\}$ .

**Definition 9.13** (Finite Complement Topology). For any set  $X$ , the *finite complement topology* on  $X$  is  $\{U \in \mathcal{P}X \mid X \setminus U \text{ is finite}\} \cup \{\emptyset\}$ .

**Definition 9.14** (Countable Complement Topology). For any set  $X$ , the *countable complement topology* on  $X$  is  $\{U \in \mathcal{P}X \mid X \setminus U \text{ is countable}\} \cup \{\emptyset\}$ .

**Definition 9.15** (Finer, Coarser). Suppose that  $\mathcal{T}$  and  $\mathcal{T}'$  are two topologies on a given set  $X$ . If  $\mathcal{T}' \supseteq \mathcal{T}$ , we say that  $\mathcal{T}'$  is *finer* than  $\mathcal{T}$ ; if  $\mathcal{T}'$  *properly* contains  $\mathcal{T}$ , we say that  $\mathcal{T}'$  is *strictly finer* than  $\mathcal{T}$ . We also say that  $\mathcal{T}$  is *coarser* than  $\mathcal{T}'$ , or *strictly coarser*, in these two respective situations. We say  $\mathcal{T}$  is *comparable* with  $\mathcal{T}'$  if either  $\mathcal{T}' \supseteq \mathcal{T}$  or  $\mathcal{T} \supseteq \mathcal{T}'$ .

**Lemma 9.16.** Let  $X$  be a topological space and  $U \subseteq X$ . Then  $U$  is open if and only if, for all  $x \in U$ , there exists an open set  $V$  such that  $x \in V \subseteq U$ .

PROOF:

$\langle 1 \rangle 1. \Rightarrow$

PROOF: Take  $V = U$

$\langle 1 \rangle 2. \Leftarrow$

PROOF: We have  $U = \bigcup \{V \text{ open in } X \mid V \subseteq U\}$ .

□

**Lemma 9.17.** Let  $X$  be a set and  $\mathcal{T}$  a nonempty set of topologies on  $X$ . Then  $\bigcap \mathcal{T}$  is a topology on  $X$ , and is the finest topology that is coarser than every member of  $\mathcal{T}$ .

PROOF:

$\langle 1 \rangle 1. X \in \bigcap \mathcal{T}$

PROOF: Since  $X$  is in every member of  $\mathcal{T}$ .

$\langle 1 \rangle 2. \bigcap \mathcal{T}$  is closed under union.

- (2)1. LET:  $\mathcal{U} \subseteq \bigcap \mathcal{T}$
- (2)2. For all  $T \in \mathcal{T}$  we have  $\mathcal{U} \subseteq T$
- (2)3. For all  $T \in \mathcal{T}$  we have  $\bigcup \mathcal{U} \in T$
- (2)4.  $\bigcup \mathcal{U} \in \bigcap \mathcal{T}$
- (1)3.  $\bigcap \mathcal{T}$  is closed under binary intersection.
- (2)1. LET:  $U, V \in \bigcap \mathcal{T}$
- (2)2. For all  $T \in \mathcal{T}$  we have  $U, V \in T$
- (2)3. For all  $T \in \mathcal{T}$  we have  $U \cap V \in T$
- (2)4.  $U \cap V \in \bigcap \mathcal{T}$

□

**Lemma 9.18.** *Let  $X$  be a set and  $\mathcal{T}$  a set of topologies on  $X$ . Then there exists a unique coarsest topology that is finer than every member of  $\mathcal{T}$ .*

PROOF: The required topology is given by

$$\bigcap \{T \in \mathcal{P}X \mid T \text{ is a topology on } X \text{ that is finer than every member of } \mathcal{T}\},$$

The set is nonempty since it contains the discrete topology. □

**Definition 9.19** (Neighbourhood). A *neighbourhood* of a point  $x$  is an open set that contains  $x$ .

## 9.4 Closed Set

**Definition 9.20** (Closed Set). Let  $X$  be a topological space and  $A \subseteq X$ . Then  $A$  is *closed* if and only if  $X \setminus A$  is open.

**Lemma 9.21.** *The empty set is closed.*

PROOF: Since the whole space  $X$  is always open. □

**Lemma 9.22.** *The topological space  $X$  is closed.*

PROOF: Since  $\emptyset$  is open. □

**Lemma 9.23.** *The intersection of a nonempty set of closed sets is closed.*

PROOF: Let  $\mathcal{C}$  be a nonempty set of closed sets. Then  $X \setminus \bigcap \mathcal{C} = \bigcup \{X \setminus C \mid C \in \mathcal{C}\}$  is open. □

**Lemma 9.24.** *The union of two closed sets is closed.*

PROOF: Let  $C$  and  $D$  be closed. Then  $X \setminus (C \cup D) = (X \setminus C) \cap (X \setminus D)$  is open. □

**Proposition 9.25.** *Let  $X$  be a set and  $\mathcal{C} \subseteq \mathcal{P}X$  a set such that:*

1.  $\emptyset \in \mathcal{C}$
2.  $X \in \mathcal{C}$
3. For all  $\mathcal{A} \subseteq \mathcal{C}$  nonempty we have  $\bigcap \mathcal{A} \in \mathcal{C}$

4. For all  $C, D \in \mathcal{C}$  we have  $C \cup D \in \mathcal{C}$ .

Then there exists a unique topology  $\mathcal{T}$  such that  $\mathcal{C}$  is the set of closed sets, namely

$$\mathcal{T} = \{X \setminus C \mid C \in \mathcal{C}\}$$

PROOF:

$\langle 1 \rangle 1$ . LET:  $\mathcal{T} = \{X \setminus C \mid C \in \mathcal{C}\}$

$\langle 1 \rangle 2$ .  $\mathcal{T}$  is a topology

$\langle 2 \rangle 1$ .  $X \in \mathcal{T}$

PROOF: Since  $\emptyset \in \mathcal{C}$

$\langle 2 \rangle 2$ . For all  $\mathcal{U} \subseteq \mathcal{T}$  we have  $\bigcup \mathcal{U} \in \mathcal{T}$

$\langle 3 \rangle 1$ . LET:  $\mathcal{U} \subseteq \mathcal{T}$

$\langle 3 \rangle 2$ . CASE:  $\mathcal{U} = \emptyset$

PROOF: In this case  $\bigcup \mathcal{U} = \emptyset \in \mathcal{T}$  since  $X \in \mathcal{C}$

$\langle 3 \rangle 3$ . CASE:  $\mathcal{U} \neq \emptyset$

PROOF: In this case  $X \setminus \bigcup \mathcal{U} = \bigcap \{X \setminus U \mid U \in \mathcal{U}\} \in \mathcal{C}$ .

$\langle 2 \rangle 3$ . For all  $U, V \in \mathcal{T}$  we have  $U \cap V \in \mathcal{T}$

PROOF: Since  $X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V) \in \mathcal{C}$ .

$\langle 1 \rangle 3$ .  $\mathcal{C}$  is the set of all closed sets in  $\mathcal{T}$

PROOF:

$C$  is closed in  $\mathcal{T}$

$$\Leftrightarrow X \setminus C \in \mathcal{T}$$

$$\Leftrightarrow C \in \mathcal{C}$$

$\langle 1 \rangle 4$ . If  $\mathcal{T}'$  is a topology and  $\mathcal{C}$  is the set of closed sets in  $\mathcal{T}'$  then  $\mathcal{T}' = \mathcal{T}$

PROOF: We have

$$U \in \mathcal{T}$$

$$\Leftrightarrow X \setminus U \in \mathcal{C}$$

$$\Leftrightarrow X \setminus U \text{ is closed in } \mathcal{T}'$$

$$\Leftrightarrow U \in \mathcal{T}'$$

□

**Proposition 9.26.** If  $U$  is open and  $A$  is closed then  $U \setminus A$  is open.

PROOF:  $U \setminus A = U \cap (X \setminus A)$  is the intersection of two open sets. □

**Proposition 9.27.** If  $U$  is open and  $A$  is closed then  $A \setminus U$  is closed.

PROOF:  $A \setminus U = A \cap (X \setminus U)$  is the intersection of two closed sets. □

## 9.5 Interior

**Definition 9.28** (Interior). Let  $X$  be a topological space and  $A \subseteq X$ . The *interior* of  $A$ ,  $\text{Int } A$ , is the union of all the open subsets of  $A$ .

**Lemma 9.29.** The interior of a set is open.

PROOF: It is a union of open sets.  $\square$

**Lemma 9.30.**

$$\text{Int } A \subseteq A$$

PROOF: Immediate from definition.  $\square$

**Lemma 9.31.** *If  $U$  is open and  $U \subseteq A$  then  $U \subseteq \text{Int } A$*

PROOF: Immediate from definition.  $\square$

**Lemma 9.32.** *A set  $A$  is open if and only if  $A = \text{Int } A$ .*

PROOF: If  $A = \text{Int } A$  then  $A$  is open by Lemma 9.29. Conversely if  $A$  is open then  $A \subseteq \text{Int } A$  by the definition of interior and so  $A = \text{Int } A$ .

## 9.6 Closure

**Definition 9.33** (Closure). Let  $X$  be a topological space and  $A \subseteq X$ . The *closure* of  $A$ ,  $\bar{A}$ , is the intersection of all the closed sets that include  $A$ .

This intersection exists since  $X$  is a closed set that includes  $A$  (Lemma 9.22).

**Lemma 9.34.** *The closure of a set is closed.*

PROOF: Dual to Lemma 9.29.  $\square$

**Lemma 9.35.**

$$A \subseteq \bar{A}$$

PROOF: Immediate from definition.  $\square$

**Lemma 9.36.** *If  $C$  is closed and  $A \subseteq C$  then  $\bar{A} \subseteq C$ .*

PROOF: Immediate from definition.  $\square$

**Lemma 9.37.** *A set  $A$  is closed if and only if  $A = \bar{A}$ .*

PROOF: Dual to Lemma 9.32.  $\square$

**Theorem 9.38.** *Let  $X$  be a topological space,  $A \subseteq X$  and  $x \in X$ . Then  $x \in \bar{A}$  if and only if every neighbourhood of  $x$  intersects  $A$ .*

PROOF: We have

$$x \in \bar{A}$$

$$\Leftrightarrow \forall C. C \text{ closed} \wedge A \subseteq C \Rightarrow x \in C$$

$$\Leftrightarrow \forall U. U \text{ open} \wedge A \cap U = \emptyset \Rightarrow x \notin U$$

$$\Leftrightarrow \forall U. U \text{ open} \wedge x \in U \Rightarrow U \text{ intersects } A \quad \square$$

**Proposition 9.39.** *If  $A \subseteq B$  then  $\bar{A} \subseteq \bar{B}$ .*

PROOF: This holds because  $\overline{B}$  is a closed set that includes  $A$ .  $\square$

**Proposition 9.40.**

$$\overline{A \cup B} = \overline{A} \cup \overline{B}$$

PROOF:

$\langle 1 \rangle 1. \overline{A} \subseteq \overline{A \cup B}$

PROOF: By Proposition 9.39.

$\langle 1 \rangle 2. \overline{B} \subseteq \overline{A \cup B}$

PROOF: By Proposition 9.39.

$\langle 1 \rangle 3. \overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$

$\langle 2 \rangle 1. \text{ LET: } x \in \overline{A \cup B}$

$\langle 2 \rangle 2. \text{ ASSUME: } x \notin \overline{A}$

PROVE:  $x \in \overline{B}$

$\langle 2 \rangle 3. \text{ PICK a neighbourhood } U \text{ of } x \text{ that does not intersect } A$

$\langle 2 \rangle 4. \text{ LET: } V \text{ be any neighbourhood of } x$

$\langle 2 \rangle 5. U \cap V \text{ is a neighbourhood of } x$

$\langle 2 \rangle 6. U \cap V \text{ intersects } A \cup B$

PROOF: From  $\langle 2 \rangle 1$  and Theorem 9.38.

$\langle 2 \rangle 7. U \cap V \text{ intersects } B$

PROOF: From  $\langle 2 \rangle 3$

$\langle 2 \rangle 8. V \text{ intersects } B$

$\langle 2 \rangle 9. \text{ Q.E.D.}$

PROOF: We have  $x \in \overline{B}$  from Theorem 9.38.

$\square$

**Proposition 9.41.** *Let  $X$  be a topological space. Let  $\mathcal{D}$  be a set of subsets of  $X$  that is maximal with respect to the finite intersection property. Let  $x \in X$ . Then the following are equivalent:*

1. *For all  $D \in \mathcal{D}$  we have  $x \in \overline{D}$*
2. *Every neighbourhood of  $x$  is in  $\mathcal{D}$ .*

PROOF:

$\langle 1 \rangle 1. 1 \Rightarrow 2$

$\langle 2 \rangle 1. \text{ For all } D \in \mathcal{D} \text{ we have } x \in \overline{D}$

$\langle 2 \rangle 2. \text{ LET: } U \text{ be a neighbourhood of } x$

$\langle 2 \rangle 3. \mathcal{D} \cup \{U\} \text{ satisfies the finite intersection property.}$

$\langle 3 \rangle 1. \text{ LET: } D_1, \dots, D_n \in \mathcal{D}$

$\langle 3 \rangle 2. D_1 \cap \dots \cap D_n \in \mathcal{D}$

PROOF: Lemma 3.67.

$\langle 3 \rangle 3. x \in \overline{D_1 \cap \dots \cap D_n}$

PROOF:  $\langle 2 \rangle 1, \langle 3 \rangle 2$

$\langle 3 \rangle 4. D_1 \cap \dots \cap D_n \cap U \neq \emptyset$

PROOF: Theorem 9.38,  $\langle 2 \rangle 2, \langle 3 \rangle 3$ .

$\langle 2 \rangle 4. \mathcal{D} = \mathcal{D} \cup \{U\}$

PROOF: By the maximality of  $\mathcal{D}$ .



$\langle 2 \rangle 5.$   $U \in \mathcal{D}$   
 $\langle 1 \rangle 2.$   $2 \Rightarrow 1$   
 $\langle 2 \rangle 1.$  ASSUME: Every neighbourhood of  $x$  is in  $\mathcal{D}$ .  
 $\langle 2 \rangle 2.$  LET:  $D \in \mathcal{D}$   
 $\langle 2 \rangle 3.$  Every neighbourhood of  $x$  intersects  $D$ .  
 PROOF: From  $\langle 2 \rangle 1$ ,  $\langle 2 \rangle 2$  and the fact that  $\mathcal{D}$  satisfies the finite intersection property.  
 $\langle 2 \rangle 4.$   $x \in \overline{D}$   
 PROOF: Theorem 9.38,  $\langle 2 \rangle 3$ .  
 $\square$

## 9.7 Boundary

**Definition 9.42** (Boundary). The *boundary* of a set  $A$  is the set  $\partial A = \overline{A} \cap \overline{X \setminus A}$ .

**Proposition 9.43.**

$$\text{Int } A \cap \partial A = \emptyset$$

PROOF: Since  $\overline{X \setminus A} = X \setminus \text{Int } A$ .  $\square$

**Proposition 9.44.**

$$\overline{A} = \text{Int } A \cup \partial A$$

PROOF:

$$\begin{aligned}
 \text{Int } A \cup \partial A &= \text{Int } A \cup (\overline{A} \cap \overline{X \setminus A}) \\
 &= (\text{Int } A \cup \overline{A}) \cap (\text{Int } A \cup \overline{X \setminus A}) \\
 &= \overline{A} \cap X \\
 &= \overline{A}
 \end{aligned}$$

**Proposition 9.45.**  $\partial A = \emptyset$  if and only if  $A$  is open and closed.

PROOF: If  $\partial A = \emptyset$  then  $\overline{A} = \text{Int } A$  by Proposition 9.44.

**Proposition 9.46.** A set  $U$  is open if and only if  $\partial U = \overline{U} \setminus U$ .

PROOF:

$$\begin{aligned}
 \partial U &= \overline{U} \setminus U \\
 \Leftrightarrow \overline{U} \setminus \text{Int } U &= \overline{U} \setminus U && (\text{Propositions 9.43, 9.44}) \\
 \Leftrightarrow \text{Int } U &= U && \square
 \end{aligned}$$

## 9.8 Limit Points

**Definition 9.47** (Limit Point). Let  $X$  be a topological space,  $a \in X$  and  $A \subseteq X$ . Then  $a$  is a *limit point*, *cluster point* or *point of accumulation* for  $A$  if and only if every neighbourhood of  $a$  intersects  $A$  at a point other than  $a$ .

**Lemma 9.48.** *The point  $a$  is an accumulation point for  $A$  if and only if  $a \in \overline{A \setminus \{a\}}$ .*

PROOF: From Theorem 9.38.  $\square$

**Theorem 9.49.** *Let  $X$  be a topological space and  $A \subseteq X$ . Let  $A'$  be the set of all limit points of  $A$ . Then  $\overline{A} = A \cup A'$ .*

PROOF:

$\langle 1 \rangle 1.$  For all  $x \in \overline{A}$ , if  $x \notin A$  then  $x \in A'$

PROOF: From Theorem 9.38.

$\langle 1 \rangle 2.$   $A \subseteq \overline{A}$

PROOF: Lemma 9.35.

$\langle 1 \rangle 3.$   $A' \subseteq \overline{A}$

PROOF: From Theorem 9.38.

$\square$

**Corollary 9.49.1.** *A set is closed if and only if it contains all its limit points.*

**Proposition 9.50.** *In an indiscrete topology, every point is a limit point of any set that has more than one point.*

PROOF: Let  $X$  be an indiscrete space. Let  $A$  be a set with more than one point and  $x$  be a point. The only neighbourhood of  $x$  is  $X$ , which must intersect  $A$  at a point other than  $x$ .  $\square$

**Lemma 9.51.** *Let  $X$  be a topological space and  $A \subseteq B \subseteq X$ . Then every limit point of  $A$  is a limit point of  $B$ .*

PROOF: Immediate from definitions.  $\square$

## 9.9 Basis for a Topology

**Definition 9.52** (Basis). If  $X$  is a set, a *basis* for a topology on  $X$  is a set  $\mathcal{B} \subseteq \mathcal{P}X$  called *basis elements* such that

1. For all  $x \in X$ , there exists  $B \in \mathcal{B}$  such that  $x \in B$ .
2. For all  $B_1, B_2 \in \mathcal{B}$  and  $x \in B_1 \cap B_2$ , there exists  $B_3 \in \mathcal{B}$  such that  $x \in B_3 \subseteq B_1 \cap B_2$ .

If  $\mathcal{B}$  satisfies these two conditions, then we define the topology *generated* by  $\mathcal{B}$  to be  $\mathcal{T} = \{U \in \mathcal{P}X \mid \forall x \in U. \exists B \in \mathcal{B}. x \in B \subseteq U\}$ .

We prove this is a topology.

PROOF:

$\langle 1 \rangle 1.$   $X \in \mathcal{T}$

PROOF: For all  $x \in X$  there exists  $B \in \mathcal{B}$  such that  $x \in B \subseteq X$  by condition 1.

$\langle 1 \rangle 2$ . For all  $\mathcal{U} \subseteq \mathcal{T}$  we have  $\bigcup \mathcal{U} \in \mathcal{T}$   
 $\langle 2 \rangle 1$ . LET:  $\mathcal{U} \subseteq \mathcal{T}$   
 $\langle 2 \rangle 2$ . LET:  $x \in \bigcup \mathcal{U}$   
 $\langle 2 \rangle 3$ . PICK  $U \in \mathcal{U}$  such that  $x \in U$   
 $\langle 2 \rangle 4$ . PICK  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$   
 PROOF: Since  $U \in \mathcal{T}$  by  $\langle 2 \rangle 1$  and  $\langle 2 \rangle 3$ .  
 $\langle 2 \rangle 5$ .  $x \in B \subseteq \bigcup \mathcal{U}$   
 $\langle 1 \rangle 3$ . For all  $U, V \in \mathcal{T}$  we have  $U \cap V \in \mathcal{T}$   
 $\langle 2 \rangle 1$ . LET:  $U, V \in \mathcal{T}$   
 $\langle 2 \rangle 2$ . LET:  $x \in U \cap V$   
 $\langle 2 \rangle 3$ . PICK  $B_1 \in \mathcal{B}$  such that  $x \in B_1 \subseteq U$   
 $\langle 2 \rangle 4$ . PICK  $B_2 \in \mathcal{B}$  such that  $x \in B_2 \subseteq V$   
 $\langle 2 \rangle 5$ . PICK  $B_3 \in \mathcal{B}$  such that  $x \in B_3 \subseteq B_1 \cap B_2$   
 PROOF: By condition 2.  
 $\langle 2 \rangle 6$ .  $x \in B_3 \subseteq U \cap V$

□

**Lemma 9.53.** *Let  $X$  be a set. Let  $\mathcal{B}$  be a basis for a topology  $\mathcal{T}$  on  $X$ . Then  $\mathcal{T}$  is the set of all unions of subsets of  $\mathcal{B}$ .*

PROOF:

$\langle 1 \rangle 1$ . For all  $U \in \mathcal{T}$ , there exists  $\mathcal{A} \subseteq \mathcal{B}$  such that  $U = \bigcup \mathcal{A}$   
 $\langle 2 \rangle 1$ . LET:  $U \in \mathcal{T}$   
 $\langle 2 \rangle 2$ . LET:  $\mathcal{A} = \{B \in \mathcal{B} \mid B \subseteq U\}$   
 $\langle 2 \rangle 3$ .  $U \subseteq \bigcup \mathcal{A}$   
 $\langle 3 \rangle 1$ . LET:  $x \in U$   
 $\langle 3 \rangle 2$ . PICK  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$   
 PROOF: Since  $\mathcal{B}$  is a basis for  $\mathcal{T}$ .  
 $\langle 3 \rangle 3$ .  $x \in B \in \mathcal{A}$   
 $\langle 2 \rangle 4$ .  $\bigcup \mathcal{A} \subseteq U$   
 PROOF: From the definition of  $\mathcal{A}$  ( $\langle 2 \rangle 2$ ).  
 $\langle 1 \rangle 2$ . For all  $\mathcal{A} \subseteq \mathcal{B}$  we have  $\bigcup \mathcal{A} \in \mathcal{T}$   
 $\langle 2 \rangle 1$ .  $\mathcal{B} \subseteq \mathcal{T}$   
 PROOF: If  $B \in \mathcal{B}$  and  $x \in B$ , then there exists  $B' \in \mathcal{B}$  such that  $x \in B' \subseteq B$ , namely  $B' = B$ .  
 $\langle 2 \rangle 2$ . Q.E.D.  
 PROOF: Since  $\mathcal{T}$  is closed under union.

□

**Corollary 9.53.1.** *Let  $X$  be a set. Let  $\mathcal{B}$  be a basis for a topology  $\mathcal{T}$  on  $X$ . Then  $\mathcal{T}$  is the coarsest topology that includes  $\mathcal{B}$ .*

PROOF: Since every topology that includes  $\mathcal{B}$  includes all unions of subsets of  $\mathcal{B}$ . □

**Lemma 9.54.** *Let  $X$  be a topological space. Suppose that  $\mathcal{C}$  is a set of open sets such that, for every open set  $U$  and every point  $x \in U$ , there exists  $C \in \mathcal{C}$  such that  $x \in C \subseteq U$ . Then  $\mathcal{C}$  is a basis for the topology on  $X$ .*

PROOF:

⟨1⟩1. For all  $x \in X$ , there exists  $C \in \mathcal{C}$  such that  $x \in C$

PROOF: Immediate from hypothesis.

⟨1⟩2. For all  $C_1, C_2 \in \mathcal{C}$  and  $x \in C_1 \cap C_2$ , there exists  $C_3 \in \mathcal{C}$  such that  $x \in C_3 \subseteq C_1 \cap C_2$

PROOF: Since  $C_1 \cap C_2$  is open.

⟨1⟩3. Every open set is open in the topology generated by  $\mathcal{C}$

PROOF: Immediate from hypothesis.

⟨1⟩4. Every union of a subset of  $\mathcal{C}$  is open.

PROOF: Since every member of  $\mathcal{C}$  is open.

□

**Lemma 9.55.** *Let  $\mathcal{B}$  and  $\mathcal{B}'$  be bases for the topologies  $\mathcal{T}$  and  $\mathcal{T}'$  respectively on the set  $X$ . Then the following are equivalent.*

1.  $\mathcal{T} \subseteq \mathcal{T}'$

2. For all  $B \in \mathcal{B}$  and  $x \in B$ , there exists  $B' \in \mathcal{B}'$  such that  $x \in B' \subseteq B$ .

PROOF:

⟨1⟩1.  $1 \Rightarrow 2$

⟨2⟩1. ASSUME:  $\mathcal{T} \subseteq \mathcal{T}'$

⟨2⟩2. LET:  $B \in \mathcal{B}$  and  $x \in B$

⟨2⟩3.  $B \in \mathcal{T}$

PROOF: Corollary 9.53.1.

⟨2⟩4.  $B \in \mathcal{T}'$

PROOF: By ⟨2⟩1

⟨2⟩5. There exists  $B' \in \mathcal{B}'$  such that  $x \in B' \subseteq B$

PROOF: Since  $\mathcal{B}'$  is a basis for  $\mathcal{T}'$ .

⟨1⟩2.  $2 \Rightarrow 1$

⟨2⟩1. ASSUME: 2

⟨2⟩2. LET:  $U \in \mathcal{T}$

PROVE:  $U \in \mathcal{T}'$

⟨2⟩3. LET:  $x \in U$

PROVE: There exists  $B' \in \mathcal{B}'$  such that  $x \in B' \subseteq U$

⟨2⟩4. PICK  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$

PROOF: Since  $\mathcal{B}$  is a basis for  $\mathcal{T}$ .

⟨2⟩5. PICK  $B' \in \mathcal{B}'$  such that  $x \in B' \subseteq B$

PROOF: By ⟨2⟩1.

⟨2⟩6.  $x \in B' \subseteq U$

□

**Theorem 9.56.** *Let  $X$  be a topological space,  $A \subseteq X$  and  $x \in X$ . Let  $\mathcal{B}$  be a basis for  $X$ . Then  $x \in \overline{A}$  if and only if, for all  $B \in \mathcal{B}$ , if  $x \in B$  then  $B$  intersects  $A$ .*

PROOF:

⟨1⟩1. If  $x \in \overline{A}$  then, for all  $B \in \mathcal{B}$ , if  $x \in B$  then  $B$  intersects  $A$ .

PROOF: This follows from Theorem 9.38 since every element of  $\mathcal{B}$  is open (Corollary 9.53.1).

$\langle 1 \rangle 2$ . Suppose that, for all  $B \in \mathcal{B}$ , if  $x \in B$  then  $B$  intersects  $A$ . Then  $x \in \overline{A}$ .

$\langle 2 \rangle 1$ . ASSUME: For all  $B \in \mathcal{B}$ , if  $x \in B$  then  $B$  intersects  $A$ .

$\langle 2 \rangle 2$ . LET:  $U$  be an open set that contains  $x$

PROVE:  $U$  intersects  $A$ .

$\langle 2 \rangle 3$ . PICK  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$ .

$\langle 2 \rangle 4$ .  $B$  intersects  $A$ .

PROOF: From  $\langle 2 \rangle 1$ .

$\langle 2 \rangle 5$ .  $U$  intersects  $A$ .

$\langle 2 \rangle 6$ . Q.E.D.

PROOF: By Theorem 9.38.

□

**Definition 9.57** (Lower Limit Topology on the Real Line). The *lower limit topology on the real line* is the topology on  $\mathbb{R}$  generated by the basis consisting of all half-open intervals of the form  $[a, b)$ .

We write  $\mathbb{R}_l$  for the topological space  $\mathbb{R}$  under the lower limit topology.

We prove this is a basis for a topology.

PROOF:

$\langle 1 \rangle 1$ . For all  $x \in \mathbb{R}$  there exists an interval  $[a, b)$  such that  $x \in [a, b)$ .

PROOF: Take  $[a, b) = [x, x + 1)$ .

$\langle 1 \rangle 2$ . For any open intervals  $[a, b)$ ,  $[c, d)$  if  $x \in [a, b) \cap [c, d)$ , then there exists an interval  $[e, f)$  such that  $x \in [e, f) \subseteq [a, b) \cap [c, d)$

PROOF: Take  $[e, f) = [\max(a, c), \min(b, d))$ .

□

**Definition 9.58** ( $K$ -topology on the Real Line). Let  $K = \{1/n \mid n \in \mathbb{Z}^+\}$ .

The  *$K$ -topology on the real line* is the topology on  $\mathbb{R}$  generated by the basis consisting of all open intervals  $(a, b)$  and all sets of the form  $(a, b) \setminus K$ .

We write  $\mathbb{R}_K$  for the topological space  $\mathbb{R}$  under the  $K$ -topology.

We prove this is a basis for a topology.

PROOF:

$\langle 1 \rangle 1$ . For all  $x \in \mathbb{R}$  there exists an open interval  $(a, b)$  such that  $x \in (a, b)$ .

PROOF: Take  $(a, b) = (x - 1, x + 1)$ .

$\langle 1 \rangle 2$ . For any basic open sets  $B_1, B_2$  if  $x \in B_1 \cap B_2$ , then there exists a basic open set  $B_3$  such that  $x \in B_3 \subseteq B_1 \cap B_2$

$\langle 2 \rangle 1$ . CASE:  $B_1 = (a, b)$ ,  $B_2 = (c, d)$

PROOF: Take  $B_3 = (\max(a, c), \min(b, d))$ .

$\langle 2 \rangle 2$ . CASE:  $B_1 = (a, b)$  or  $(a, b) \setminus K$ ,  $B_2 = (c, d)$  or  $(c, d) \setminus K$ , and they are not both open intervals.

PROOF: Take  $B_3 = (\max(a, c), \min(b, d)) \setminus K$ .

□

**Lemma 9.59.** *The lower limit topology and the  $K$ -topology are incomparable.*

PROOF:

$\langle 1 \rangle 1$ . The interval  $[10, 11)$  is not open in the  $K$ -topology.

PROOF: There is no open interval  $(a, b)$  such that  $10 \in (a, b) \subseteq [10, 11)$  or  $10 \in (a, b) \setminus K \subseteq [10, 11)$ .

$\langle 1 \rangle 2$ . The set  $(-1, 1) \setminus K$  is not open in the lower limit topology.

PROOF: There is no half-open interval  $[a, b)$  such that  $0 \in [a, b) \subseteq (-1, 1) \setminus K$ , since there must be a positive integer  $n$  with  $1/n \in [a, b)$ .

□

**Definition 9.60** (Subbasis). A *subbasis*  $\mathcal{S}$  for a topology on  $X$  is a set  $\mathcal{S} \subseteq \mathcal{P}X$  such that  $\bigcup \mathcal{S} = X$ .

The topology *generated* by the subbasis  $\mathcal{S}$  is the set of all unions of finite intersections of elements of  $\mathcal{S}$ .

We prove this is a topology.

PROOF:

$\langle 1 \rangle 1$ . The set  $\mathcal{B}$  of all finite intersections of elements of  $\mathcal{S}$  forms a basis for a topology on  $X$ .

$\langle 2 \rangle 1$ .  $\bigcup \mathcal{B} = X$

PROOF: Since  $\mathcal{S} \subseteq \mathcal{B}$ .

$\langle 2 \rangle 2$ .  $\mathcal{B}$  is closed under binary intersection.

PROOF: By definition.

$\langle 1 \rangle 2$ . Q.E.D.

PROOF: By Lemma 9.53.

□

We have simultaneously proved:

**Proposition 9.61.** *Let  $\mathcal{S}$  be a subbasis for the topology on  $X$ . Then the set of all finite intersections of elements of  $\mathcal{S}$  is a basis for the topology on  $X$ .*

**Proposition 9.62.** *Let  $X$  be a set. Let  $\mathcal{S}$  be a subbasis for a topology  $\mathcal{T}$  on  $X$ . Then  $\mathcal{T}$  is the coarsest topology that includes  $\mathcal{S}$ .*

PROOF: Since every topology that includes  $\mathcal{S}$  includes every union of finite intersections of elements of  $\mathcal{S}$ . □

## 9.10 Local Basis at a Point

**Definition 9.63** (Local Basis). Let  $X$  be a topological space and  $a \in X$ . A *(local) basis at  $a$*  is a set  $\mathcal{B}$  of neighbourhoods of  $a$  such that every neighbourhood of  $a$  includes some member of  $\mathcal{B}$ .

**Lemma 9.64.** *If there exists a countable local basis at a point  $a$ , then there exists a countable local basis  $\{B_n \mid n \geq 1\}$  such that  $B_1 \supseteq B_2 \supseteq \dots$ .*

PROOF: Given a countable local basis  $\{C_n \mid n \geq 1\}$ , take  $B_n = C_1 \cap \dots \cap C_n$ . □

## 9.11 Convergence

**Definition 9.65** (Convergence). Let  $X$  be a topological space. Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence of points in  $X$  and  $l \in X$ . Then the sequence  $(a_n)_{n \in \mathbb{N}}$  converges to the limit  $l$ ,  $a_n \rightarrow l$  as  $n \rightarrow \infty$ , if and only if, for every neighbourhood  $U$  of  $l$ , there exists  $N$  such that, for all  $n \geq N$ , we have  $a_n \in U$ .

**Lemma 9.66.** Let  $X$  be a topological space. Let  $A \subseteq X$  and  $l \in X$ . If there is a sequence of points in  $A$  that converges to  $l$  then  $l \in \bar{A}$ .

PROOF:

- $\langle 1 \rangle 1$ . LET:  $(a_n)$  be a sequence of points in  $A$  that converges to  $l$ .
- $\langle 1 \rangle 2$ . LET:  $U$  be a neighbourhood of  $l$ .
- $\langle 1 \rangle 3$ . PICK  $N$  such that, for all  $n \geq N$ , we have  $a_n \in U$ .
- $\langle 1 \rangle 4$ .  $a_N \in U \cap A$
- $\langle 1 \rangle 5$ . Q.E.D.

PROOF: Theorem 9.38.

□

**Proposition 9.67.** Let  $X$  be a topological space. Let  $\mathcal{B}$  be a basis for the topology on  $X$ . Let  $(a_n)$  be a sequence in  $X$  and  $l \in X$ . Then  $a_n \rightarrow l$  as  $n \rightarrow \infty$  if and only if, for every  $B \in \mathcal{B}$  with  $l \in B$ , there exists  $N$  such that, for all  $n \geq N$ , we have  $a_n \in B$ .

PROOF:

- $\langle 1 \rangle 1$ . If  $a_n \rightarrow l$  as  $n \rightarrow \infty$  then, for every  $B \in \mathcal{B}$  with  $l \in B$ , there exists  $N$  such that, for all  $n \geq N$ , we have  $a_n \in B$ .

PROOF: Since every element of  $\mathcal{B}$  is open (Corollary 9.53.1).

- $\langle 1 \rangle 2$ . If, for every  $B \in \mathcal{B}$  with  $l \in B$ , there exists  $N$  such that, for all  $n \geq N$ , we have  $a_n \in B$ , then  $a_n \rightarrow l$  as  $n \rightarrow \infty$ .
  - $\langle 2 \rangle 1$ . ASSUME: for every  $B \in \mathcal{B}$  with  $l \in B$ , there exists  $N$  such that, for all  $n \geq N$ , we have  $a_n \in B$
  - $\langle 2 \rangle 2$ . LET:  $U$  be a neighbourhood of  $l$ .
  - $\langle 2 \rangle 3$ . PICK  $B \in \mathcal{B}$  such that  $l \in B \subseteq U$
  - $\langle 2 \rangle 4$ . PICK  $N$  such that, for all  $n \geq N$ , we have  $a_n \in B$

PROOF: From  $\langle 2 \rangle 1$ .

- $\langle 2 \rangle 5$ . For all  $n \geq N$  we have  $a_n \in U$

□

**Lemma 9.68.** If a sequence  $(a_n)$  is constant with  $a_n = l$  for all  $n$ , then  $a_n \rightarrow l$  as  $n \rightarrow \infty$ .

PROOF: Immediate from definitions. □

**Theorem 9.69.** Let  $X$  be a linearly ordered set. Let  $(s_n)$  be a monotone increasing sequence in  $X$  with a supremum  $s$ . Then  $s_n \rightarrow s$  as  $n \rightarrow \infty$ .

PROOF:

- $\langle 1 \rangle 1$ . ASSUME:  $s$  is not least in  $X$ .

PROOF: Otherwise  $(s_n)$  is the constant sequence  $s$  and the result follows from Lemma 9.68.

- ⟨1⟩2. LET:  $U$  be a neighbourhood of  $s$ .
- ⟨1⟩3. PICK  $a < s$  such that  $(a, s] \subseteq U$
- ⟨1⟩4. PICK  $N$  such that  $a < a_N$ .
- ⟨1⟩5. For all  $n \geq N$  we have  $a_n \in (a, s]$
- ⟨1⟩6. For all  $n \geq N$  we have  $a_n \in U$ .

□

**Theorem 9.70.** *If  $\sum_{i=0}^{\infty} a_i = s$  and  $\sum_{i=0}^{\infty} b_i = t$  then  $\sum_{i=0}^{\infty} (ca_i + b_i) = cs + t$ .*

PROOF:  $\sum_{i=0}^N (ca_i + b_i) = c \sum_{i=0}^N a_i + \sum_{i=0}^N b_i \rightarrow cs + t$  as  $n \rightarrow \infty$ . □

**Theorem 9.71** (Comparison Test). *If  $|a_i| \leq b_i$  for all  $i$  and  $\sum_{i=0}^{\infty} b_i$  converges then  $\sum_{i=0}^{\infty} a_i$  converges.*

PROOF:

- ⟨1⟩1.  $\sum_{i=0}^{\infty} |a_i|$  converges

PROOF: The partial sums  $\sum_{i=0}^N |a_i|$  form an increasing sequence bounded above by  $\sum_{i=0}^{\infty} b_i$ .

- ⟨1⟩2. LET:  $c_i = |a_i| + a_i$  for all  $i$

- ⟨1⟩3.  $\sum_{i=0}^{\infty} c_i$  converges

PROOF: Each  $c_i$  is either  $2|a_i|$  or 0. So the partial sums  $\sum_{i=0}^N c_i$  form an increasing sequence bounded above by  $2 \sum_{i=0}^{\infty} b_i$ .

- ⟨1⟩4. Q.E.D.

PROOF: Since  $a_i = c_i - |a_i|$ .

□

**Corollary 9.71.1.** *If  $\sum_{i=0}^{\infty} |a_i|$  converges then  $\sum_{i=0}^{\infty} a_i$  converges.*

**Theorem 9.72** (Weierstrass  $M$ -test). *Let  $X$  be a set and  $(f_n : X \rightarrow \mathbb{R})$  be a sequence of functions. Let*

$$s_n(x) = \sum_{i=0}^n f_i(x)$$

*for all  $n, x$ . Suppose  $|f_i(x)| \leq M_i$  for all  $i \geq 0$  and  $x \in X$ . If the series  $\sum_{i=0}^{\infty} M_i$  converges, then the sequence  $(s_n)$  converges uniformly to*

$$s(x) = \sum_{i=0}^{\infty} f_i(x) .$$

PROOF:

- ⟨1⟩1. LET:  $r_n = \sum_{i=n+1}^{\infty} M_i$  for all  $n$

- ⟨1⟩2. Given  $0 \leq n < k$ , we have  $|s_k(x) - s_n(x)| \leq r_n$



PROOF:

$$\begin{aligned}
|s_k(x) - s_n(x)| &= \left| \sum_{i=n+1}^k f_i(x) \right| \\
&\leq \sum_{i=n+1}^k |f_i(x)| \\
&\leq \sum_{i=n+1}^k M_i \\
&\leq r_n
\end{aligned}$$

$\langle 1 \rangle 3$ . Given  $n \geq 0$  we have  $|s(x) - s_n(x)| \leq r_n$

PROOF: By taking the limit  $k \rightarrow \infty$  in  $\langle 1 \rangle 2$ .

$\langle 1 \rangle 4$ . Q.E.D.

PROOF: Since  $r_n \rightarrow 0$  as  $n \rightarrow \infty$ .

□

## 9.12 Locally Finite Sets

**Definition 9.73** (Locally Finite). Let  $X$  be a topological space and  $\{A_\alpha\}$  a family of subsets of  $X$ . Then  $\mathcal{A}$  is *locally finite* if and only if every point in  $X$  has a neighbourhood that intersects  $A_\alpha$  for only finitely many  $\alpha$ .

**Theorem 9.74** (Pasting Lemma). Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Let  $\{A_\alpha\}$  be a locally finite family of closed subsets of  $X$  that cover  $X$ . Suppose  $f \upharpoonright A_\alpha$  is continuous for all  $\alpha$ . Then  $f$  is continuous.

PROOF:

$\langle 1 \rangle 1$ . Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Let  $A$  and  $B$  be closed subsets of  $X$  such that  $X = A \cup B$ . Suppose  $f \upharpoonright A$  and  $f \upharpoonright B$  are continuous. Then  $f$  is continuous.

$\langle 2 \rangle 1$ . LET:  $C \subseteq Y$  be closed.

$\langle 2 \rangle 2$ .  $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$

$\langle 2 \rangle 3$ .  $f^{-1}(C)$  and  $g^{-1}(C)$  are closed in  $X$ .

PROOF: Theorems 9.84 and 9.135.

$\langle 2 \rangle 4$ .  $h^{-1}(C)$  is closed in  $X$ .

PROOF: Lemma 9.24.

$\langle 2 \rangle 5$ . Q.E.D.

PROOF: Theorem 9.84.

$\langle 1 \rangle 2$ . Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Let  $\{A_\alpha\}$  be a finite family of closed subsets of  $X$  that cover  $X$ . Suppose  $f \upharpoonright A_\alpha$  is continuous for all  $\alpha$ . Then  $f$  is continuous.

PROOF: From  $\langle 1 \rangle 1$  by induction.

$\langle 1 \rangle 3$ . Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Let  $\{A_\alpha\}$  be a locally finite family of closed subsets of  $X$  that cover  $X$ . Suppose  $f \upharpoonright A_\alpha$  is continuous for all  $\alpha$ . Then  $f$  is continuous.

(2)1. LET:  $x \in X$   
 PROVE:  $f$  is continuous at  $x$   
 (2)2. PICK a neighbourhood  $U$  of  $x$  that intersects  $A_\alpha$  for only finitely many  $\alpha$ .  
 (2)3.  $f \upharpoonright U$  is continuous  
 PROOF: By (1)2.  
 (2)4. Q.E.D.  
 PROOF: Lemma 9.94.

□

The following example shows that we cannot remove the assumption of local finiteness.

**Example 9.75.** Define  $f : [-1, 1] \rightarrow \mathbb{R}$  by:  $f(x) = 1$  if  $x < -1$ ,  $f(x) = 0$  if  $x > 1$ . Let  $C_n = [-1, -1/n]$  for  $n \geq 1$ , and  $D = [0, 1]$ . Then  $[-1, 1] = \bigcup_{n=1}^{\infty} C_n \cup D$  and  $f$  is continuous on each  $C_n$  and each  $D$ , but  $f$  is not continuous on  $[-1, 1]$ .

## 9.13 Open Maps

**Definition 9.76** (Open Map). Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Then  $f$  is an *open map* if and only if, for every open set  $U$  in  $X$ , the set  $f(U)$  is open in  $Y$ .

**Lemma 9.77.** Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Let  $\mathcal{B}$  be a basis for the topology on  $X$ . If  $f(B)$  is open in  $Y$  for all  $B \in \mathcal{B}$ , then  $f$  is an open map.

PROOF: From Lemma 9.53. □

**Proposition 9.78.** Let  $X$  and  $Y$  be topological spaces. Let  $\mathcal{B}$  be a basis for the topology on  $X$ . Let  $f : X \rightarrow Y$ . Suppose that, for all  $B \in \mathcal{B}$ , we have  $f(B)$  is open in  $Y$ . Then  $f$  is an open map.

PROOF: For any  $\mathcal{A} \subseteq \mathcal{B}$ , we have  $f(\bigcup \mathcal{A}) = \bigcup_{B \in \mathcal{A}} f(B)$  is open in  $Y$ . The result follows from Lemma 9.53. □

## 9.14 Continuous Functions

**Definition 9.79** (Continuous). Let  $X$  and  $Y$  be topological spaces. A function  $f : X \rightarrow Y$  is *continuous* if and only if, for every open set  $V$  in  $Y$ , the set  $f^{-1}(V)$  is open in  $X$ .

**Proposition 9.80.** Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Let  $\mathcal{B}$  be a basis for  $Y$ . Then  $f$  is continuous if and only if, for all  $B \in \mathcal{B}$ , we have  $f^{-1}(B)$  is open in  $X$ .

PROOF:

(1)1. If  $f$  is continuous then, for all  $B \in \mathcal{B}$ , we have  $f^{-1}(B)$  is open in  $X$ .

PROOF: Since every element of  $\mathcal{B}$  is open (Lemma 9.53).

(1)2. Suppose that, for all  $B \in \mathcal{B}$ , we have  $f^{-1}(B)$  is open in  $X$ . Then  $f$  is continuous.

(2)1. ASSUME: For all  $B \in \mathcal{B}$ , we have  $f^{-1}(B)$  is open in  $X$ .

(2)2. LET:  $V$  be open in  $Y$ .

(2)3. PICK  $\mathcal{A} \subseteq \mathcal{B}$  such that  $V = \bigcup \mathcal{A}$

PROOF: By Lemma 9.53.

(2)4.  $f^{-1}(V)$  is open in  $X$ .

PROOF:

$$\begin{aligned} f^{-1}(V) &= f^{-1}\left(\bigcup \mathcal{A}\right) \\ &= \bigcup_{B \in \mathcal{A}} f^{-1}(B) \end{aligned}$$

□

**Proposition 9.81.** *Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Let  $\mathcal{S}$  be a subbasis for  $Y$ . Then  $f$  is continuous if and only if, for all  $S \in \mathcal{S}$ , we have  $f^{-1}(S)$  is open in  $X$ .*

PROOF:

(1)1. If  $f$  is continuous then, for all  $S \in \mathcal{S}$ , we have  $f^{-1}(S)$  is open in  $X$ .

PROOF: Since every element of  $\mathcal{S}$  is open.

(1)2. Suppose that, for all  $S \in \mathcal{S}$ , we have  $f^{-1}(S)$  is open in  $X$ . Then  $f$  is continuous.

(2)1. ASSUME: For all  $S \in \mathcal{S}$ , we have  $f^{-1}(S)$  is open in  $X$ .

(2)2. LET:  $S_1, \dots, S_n \in \mathcal{S}$

(2)3.  $f^{-1}(S_1 \cap \dots \cap S_n)$  is open in  $X$

PROOF: Since  $f^{-1}(S_1 \cap \dots \cap S_n) = f^{-1}(S_1) \cap \dots \cap f^{-1}(S_n)$ .

(2)4. Q.E.D.

PROOF: By Propositions 9.80 and 9.61.

□

**Proposition 9.82.** *Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Let  $\mathcal{S}$  be a basis for  $Y$ . Then  $f$  is continuous if and only if, for all  $V \in \mathcal{S}$ , we have  $f^{-1}(V)$  is open in  $X$ .*

PROOF:

(1)1. If  $f$  is continuous then, for all  $V \in \mathcal{S}$ , we have  $f^{-1}(V)$  is open in  $X$ .

PROOF: Since every element of  $\mathcal{S}$  is open.

(1)2. Suppose that, for all  $V \in \mathcal{S}$ , we have  $f^{-1}(V)$  is open in  $X$ . Then  $f$  is continuous.

(2)1. ASSUME: For all  $V \in \mathcal{S}$ , we have  $f^{-1}(V)$  is open in  $X$ .

(2)2. For every set  $B$  that is the finite intersection of elements of  $\mathcal{S}$ , we have  $f^{-1}(B)$  is open in  $X$ .

PROOF: Because  $f^{-1}(V_1 \cap \dots \cap V_n) = f^{-1}(V_1) \cap \dots \cap f^{-1}(V_n)$ .

(2)3. Q.E.D.

PROOF: From Propositions 9.61 and 9.80.

□

**Definition 9.83** (Continuous at a Point). Let  $X$  and  $Y$  be topological spaces. Let  $f : X \rightarrow Y$  and  $x \in X$ . Then  $f$  is *continuous at  $x$*  if and only if, for every neighbourhood  $V$  of  $f(x)$ , there exists a neighbourhood  $U$  of  $x$  such that  $f(U) \subseteq V$ .

**Theorem 9.84.** Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Then the following are equivalent:

1.  $f$  is continuous.
2. For all  $A \subseteq X$ , we have  $f(\overline{A}) \subseteq \overline{f(A)}$
3. For all  $B \subseteq Y$  closed, we have  $f^{-1}(B)$  is closed in  $X$ .
4.  $f$  is continuous at every point of  $X$ .

PROOF:

$\langle 1 \rangle 1. 1 \Rightarrow 2$

$\langle 2 \rangle 1.$  ASSUME:  $f$  is continuous.

$\langle 2 \rangle 2.$  LET:  $A \subseteq X$

$\langle 2 \rangle 3.$  LET:  $x \in \overline{A}$

PROVE:  $f(x) \in \overline{f(A)}$

$\langle 2 \rangle 4.$  LET:  $V$  be a neighbourhood of  $f(x)$

$\langle 2 \rangle 5.$   $f^{-1}(V)$  is a neighbourhood of  $x$

$\langle 2 \rangle 6.$  PICK  $y \in A \cap f^{-1}(V)$

PROOF: By Theorem 9.38.

$\langle 2 \rangle 7.$   $f(y) \in V \cap f(A)$

$\langle 2 \rangle 8.$  Q.E.D.

PROOF: By Theorem 9.38.

$\langle 1 \rangle 2. 2 \Rightarrow 3$

$\langle 2 \rangle 1.$  ASSUME: 2

$\langle 2 \rangle 2.$  LET:  $B$  be closed in  $Y$

$\langle 2 \rangle 3.$  LET:  $x \in \overline{f^{-1}(B)}$

PROVE:  $x \in f^{-1}(B)$

$\langle 2 \rangle 4.$   $f(x) \in B$

PROOF:

$$f(x) \in f(\overline{f^{-1}(B)})$$

$$\subseteq \overline{f(f^{-1}(B))}$$

$(\langle 2 \rangle 1)$

$$\subseteq \overline{B}$$

$(\text{Proposition 9.39})$

$$= B$$

$\langle 1 \rangle 3. 3 \Rightarrow 1$

$\langle 2 \rangle 1.$  ASSUME: 3

$\langle 2 \rangle 2.$  LET:  $V$  be open in  $Y$

$\langle 2 \rangle 3.$   $Y \setminus V$  is closed in  $Y$

$\langle 2 \rangle 4.$   $f^{-1}(Y \setminus V)$  is closed in  $X$

$\langle 2 \rangle 5.$   $X \setminus f^{-1}(V)$  is closed in  $X$

$\langle 2 \rangle 6.$   $f^{-1}(V)$  is open in  $X$

⟨1⟩4.  $1 \Rightarrow 4$

PROOF: For any neighbourhood  $V$  of  $f(x)$ , the set  $U = f^{-1}(V)$  is a neighbourhood of  $x$  such that  $f(U) \subseteq V$ .

⟨1⟩5.  $4 \Rightarrow 1$

⟨2⟩1. ASSUME: 4

⟨2⟩2. LET:  $V$  be open in  $Y$

⟨2⟩3. LET:  $x \in f^{-1}(V)$

⟨2⟩4.  $V$  is a neighbourhood of  $f(x)$

⟨2⟩5. PICK a neighbourhood  $U$  of  $x$  such that  $f(U) \subseteq V$

⟨2⟩6.  $x \in U \subseteq f^{-1}(V)$

⟨2⟩7. Q.E.D.

PROOF: By Lemma 9.16.

□

**Theorem 9.85.** *A constant function is continuous.*

PROOF: Let  $X$  and  $Y$  be topological spaces. Let  $b \in Y$ , and let  $f : X \rightarrow Y$  be the constant function with value  $b$ . For any open  $V \subseteq Y$ , the set  $f^{-1}(V)$  is either  $X$  (if  $b \in V$ ) or  $\emptyset$  (if  $b \notin V$ ). □

**Theorem 9.86.** *If  $A$  is a subspace of  $X$  then the inclusion  $j : A \rightarrow X$  is continuous.*

PROOF: For any  $V$  open in  $X$ , we have  $j^{-1}(V) = V \cap A$  is open in  $A$ . □

**Theorem 9.87.** *The composite of two continuous functions is continuous.*

PROOF: Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  be continuous. For any  $V$  open in  $Z$ , we have  $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$  is open in  $X$ . □

**Theorem 9.88.** *Let  $f : X \rightarrow Y$  be a continuous function and  $A$  be a subspace of  $X$ . Then the restriction  $f \upharpoonright A : A \rightarrow Y$  is continuous.*

PROOF: Let  $V$  be open in  $Y$ . Then  $(f \upharpoonright A)^{-1}(V) = f^{-1}(V) \cap A$  is open in  $A$ . □

**Theorem 9.89.** *Let  $f : X \rightarrow Y$  be continuous. Let  $Z$  be a subspace of  $Y$  such that  $f(X) \subseteq Z$ . Then the corestriction  $f : X \rightarrow Z$  is continuous.*

PROOF:

⟨1⟩1. LET:  $V$  be open in  $Z$ .

⟨1⟩2. PICK  $U$  open in  $Y$  such that  $V = U \cap Z$ .

⟨1⟩3.  $f^{-1}(V) = f^{-1}(U)$

⟨1⟩4.  $f^{-1}(V)$  is open in  $X$ .

□

**Theorem 9.90.** *Let  $f : X \rightarrow Y$  be continuous. Let  $Z$  be a space such that  $Y$  is a subspace of  $Z$ . Then the expansion  $f : X \rightarrow Z$  is continuous.*

PROOF: Let  $V$  be open in  $Z$ . Then  $f^{-1}(V) = f^{-1}(V \cap Y)$  is open in  $X$ . □

**Theorem 9.91.** *Let  $X$  and  $Y$  be topological spaces. Let  $f : X \rightarrow Y$ . Suppose  $\mathcal{U}$  is a set of open sets in  $X$  such that  $X = \bigcup \mathcal{U}$  and, for all  $U \in \mathcal{U}$ , we have  $f \upharpoonright U : U \rightarrow Y$  is continuous. Then  $f$  is continuous.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $V$  be open in  $Y$
- $\langle 1 \rangle 2$ .  $f^{-1}(V) = \bigcup_{U \in \mathcal{U}} (f \upharpoonright U)^{-1}(V)$
- $\langle 1 \rangle 3$ . For all  $U \in \mathcal{U}$ , we have  $(f \upharpoonright U)^{-1}(V)$  is open in  $U$ .
- $\langle 1 \rangle 4$ . For all  $U \in \mathcal{U}$ , we have  $(f \upharpoonright U)^{-1}(V)$  is open in  $X$ .

PROOF: Lemma 9.134.

□

**Proposition 9.92.** *Let  $X$  and  $X'$  be the same set  $X$  under two topologies  $\mathcal{T}$  and  $\mathcal{T}'$ . Let  $i : X \rightarrow X'$  be the identity function. Then  $i$  is continuous if and only if  $\mathcal{T}' \subseteq \mathcal{T}$ .*

PROOF: Immediate from definitions. □

**Proposition 9.93.** *Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  and  $a \in \mathbb{R}$ . Then  $f$  is continuous on the right at  $a$  if and only if  $f$  is continuous at  $a$  as a function  $\mathbb{R}_l \rightarrow \mathbb{R}$ .*

PROOF:

- $\langle 1 \rangle 1$ . If  $f$  is continuous on the right at  $a$  then  $f$  is continuous at  $a$  as a function  $\mathbb{R}_l \rightarrow \mathbb{R}$ .
  - $\langle 2 \rangle 1$ . ASSUME:  $f$  is continuous on the right at  $a$ .
  - $\langle 2 \rangle 2$ . LET:  $V$  be a neighbourhood of  $f(a)$
  - $\langle 2 \rangle 3$ . PICK  $b, c$  such that  $f(a) \in (b, c) \subseteq V$ .
  - $\langle 2 \rangle 4$ . LET:  $\epsilon = \min(c - f(a), f(a) - b)$
  - $\langle 2 \rangle 5$ . PICK  $\delta > 0$  such that, for all  $x$ , if  $a < x < a + \delta$  then  $|f(x) - f(a)| < \epsilon$
  - $\langle 2 \rangle 6$ . LET:  $U = [a, a + \delta)$
  - $\langle 2 \rangle 7$ .  $f(U) \subseteq V$
- $\langle 1 \rangle 2$ . If  $f$  is continuous at  $a$  as a function  $\mathbb{R}_l \rightarrow \mathbb{R}$  then  $f$  is continuous on the right at  $a$ .
  - $\langle 2 \rangle 1$ . ASSUME:  $f$  is continuous at  $a$  as a function  $\mathbb{R}_l \rightarrow \mathbb{R}$
  - $\langle 2 \rangle 2$ . LET:  $\epsilon > 0$
  - $\langle 2 \rangle 3$ . PICK a neighbourhood  $U$  of  $a$  such that  $f(U) \subseteq (f(a) - \epsilon, f(a) + \epsilon)$
  - $\langle 2 \rangle 4$ . PICK  $b, c$  such that  $a \in [b, c) \subset U$
  - $\langle 2 \rangle 5$ . LET:  $\delta = c - a$
  - $\langle 2 \rangle 6$ . For all  $x$ , if  $a < x < a + \delta$  then  $|f(x) - f(a)| < \epsilon$

□

**Lemma 9.94.** *Let  $f : X \rightarrow Y$ . Let  $Z$  be an open subspace of  $X$  and  $a \in Z$ . If  $f \upharpoonright Z$  is continuous at  $a$  then  $f$  is continuous at  $a$ .*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $V$  be a neighbourhood of  $f(a)$
- $\langle 1 \rangle 2$ . PICK a neighbourhood  $W$  of  $a$  in  $Z$  such that  $f(W) \subseteq V$
- $\langle 1 \rangle 3$ .  $W$  is a neighbourhood of  $a$  in  $X$  such that  $f(W) \subseteq V$

PROOF: Lemma 9.134.

□

**Proposition 9.95.** *Let  $f : A \rightarrow B$  and  $g : C \rightarrow D$  be continuous. Define  $f \times g : A \times C \rightarrow B \times D$  by*

$$(f \times g)(a, c) = (f(a), g(c)) .$$

*Then  $f \times g$  is continuous.*

PROOF:  $\pi_1 \circ (f \times g) = f \circ \pi_1$  and  $\pi_2 \circ (f \times g) = g \circ \pi_2$  are continuous by Theorem 9.87. The result follows by Theorem 9.123.

**Proposition 9.96.** *Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$  be continuous. If  $a_n \rightarrow l$  as  $n \rightarrow \infty$  in  $X$  then  $f(a_n) \rightarrow f(l)$  as  $n \rightarrow \infty$ .*

PROOF:

- ⟨1⟩1. LET:  $V$  be a neighbourhood of  $f(l)$
- ⟨1⟩2. PICK a neighbourhood  $U$  of  $l$  such that  $f(U) \subseteq V$
- ⟨1⟩3. PICK  $N$  such that, for all  $n \geq N$ , we have  $a_n \in U$
- ⟨1⟩4. For all  $n \geq N$  we have  $f(a_n) \in V$

□

## 9.15 Homeomorphisms

**Definition 9.97** (Homeomorphism). Let  $X$  and  $Y$  be topological spaces. A *Homeomorphism*  $f$  between  $X$  and  $Y$ ,  $f : X \cong Y$ , is a bijection  $f : X \rightarrow Y$  such that both  $f$  and  $f^{-1}$  are continuous.

**Lemma 9.98.** *Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$  a bijection. Then the following are equivalent:*

1.  $f$  is a homeomorphism.
2.  $f$  is continuous and an open map.
3.  $f$  is continuous and a closed map.
4. For any  $U \subseteq X$ , we have  $U$  is open if and only if  $f(U)$  is open.

PROOF: Immediate from definitions. □

**Proposition 9.99.** *Let  $X$  and  $X'$  be the same set  $X$  under two topologies  $\mathcal{T}$  and  $\mathcal{T}'$ . Let  $i : X \rightarrow X'$  be the identity function. Then  $i$  is a homeomorphism if and only if  $\mathcal{T} = \mathcal{T}'$ .*

PROOF: Immediate from definitions. □

**Definition 9.100** (Topological Property). Let  $P$  be a property of topological spaces. Then  $P$  is a *topological property* if and only if, for any spaces  $X$  and  $Y$ , if  $P$  holds of  $X$  and  $X \cong Y$  then  $P$  holds of  $Y$ .

**Definition 9.101** (Topological Imbedding). Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Then  $f$  is a *topological imbedding* if and only if the corestriction  $f : X \rightarrow f(X)$  is a homeomorphism.

**Proposition 9.102.** Let  $X$  and  $Y$  be topological spaces and  $a \in X$ . The function  $i : Y \rightarrow X \times Y$  that maps  $y$  to  $(a, y)$  is an imbedding.

PROOF:

$\langle 1 \rangle 1$ .  $i$  is injective

$\langle 1 \rangle 2$ .  $i$  is continuous.

PROOF: For  $U$  open in  $X$  and  $V$  open in  $Y$ , we have  $i^{-1}(U \times V)$  is  $V$  if  $a \in U$ , and  $\emptyset$  if  $a \notin U$ .

$\langle 1 \rangle 3$ .  $i : Y \rightarrow i(Y)$  is an open map.

PROOF: For  $V$  open in  $Y$  we have  $i(V) = (X \times V) \cap i(Y)$ .

□

## 9.16 The Order Topology

**Definition 9.103** (Order Topology). Let  $X$  be a linearly ordered set with at least two points. The *order topology* on  $X$  is the topology generated by the basis  $\mathcal{B}$  consisting of:

- all open intervals  $(a, b)$ ;
- all intervals of the form  $[\perp, b)$  where  $\perp$  is least in  $X$ ;
- all intervals of the form  $(a, \top]$  where  $\top$  is greatest in  $X$ .

We prove this is a basis for a topology.

PROOF:

$\langle 1 \rangle 1$ . For all  $x \in X$  there exists  $B \in \mathcal{B}$  such that  $x \in B$ .

$\langle 2 \rangle 1$ . LET:  $x \in X$

$\langle 2 \rangle 2$ . CASE:  $x$  is greatest in  $X$ .

$\langle 3 \rangle 1$ . PICK  $y \in X$  with  $y \neq x$

$\langle 3 \rangle 2$ .  $x \in (y, x] \in \mathcal{B}$

$\langle 2 \rangle 3$ . CASE:  $x$  is least in  $X$ .

$\langle 3 \rangle 1$ . PICK  $y \in X$  with  $y \neq x$

$\langle 3 \rangle 2$ .  $x \in [x, y) \in \mathcal{B}$

$\langle 2 \rangle 4$ . CASE:  $x$  is neither greatest nor least in  $X$ .

$\langle 3 \rangle 1$ . PICK  $a, b \in X$  with  $a < x$  and  $x < b$

$\langle 3 \rangle 2$ .  $x \in (a, b) \in \mathcal{B}$

$\langle 1 \rangle 2$ . For all  $B_1, B_2 \in \mathcal{B}$  and  $x \in B_1 \cap B_2$ , there exists  $B_3 \in \mathcal{B}$  such that  $x \in B_3 \subseteq B_1 \cap B_2$

$\langle 2 \rangle 1$ . LET:  $B_1, B_2 \in \mathcal{B}$  and  $x \in B_1 \cap B_2$

$\langle 2 \rangle 2$ . CASE:  $B_1 = (a, b)$ ,  $B_2 = (c, d)$

PROOF: Take  $B_3 = (\max(a, c), \min(b, d))$ .

$\langle 2 \rangle 3$ . CASE:  $B_1 = (a, b)$ ,  $B_2 = [\perp, d)$



PROOF: Take  $B_3 = (a, \min(b, d))$ .

(2)4. CASE:  $B_1 = (a, b)$ ,  $B_2 = (c, \top]$

PROOF: Take  $B_3 = (\max(a, c), b)$ .

(2)5. CASE:  $B_1 = [\perp, b)$ ,  $B_2 = [\perp, d)$

PROOF: Take  $B_3 = [\perp, \min(b, d))$ .

(2)6. CASE:  $B_1 = [\perp, b)$ ,  $B_2 = (c, \top]$

PROOF: Take  $B_3 = (c, b)$ .

□

**Lemma 9.104.** *Let  $X$  be a linearly ordered set. Then the open rays form a subbasis for the order topology on  $X$ .*

PROOF:

(1)1. Every open ray is open.

(2)1. For all  $a \in X$ , the ray  $(-\infty, a)$  is open.

(3)1. LET:  $x \in (-\infty, a)$

(3)2. CASE:  $x$  is least in  $X$

PROOF:  $x \in \text{min}[x, a) = (-\infty, a)$ .

(3)3. CASE:  $x$  is not least in  $X$

(4)1. PICK  $y < x$

(4)2.  $x \in (y, a) \subseteq (-\infty, a)$

(2)2. For all  $a \in X$ , the ray  $(a, +\infty)$  is open.

PROOF: Similar.

(1)2. Every basic open set is a finite intersection of open rays.

PROOF: We have  $(a, b) = (a, +\infty) \cap (-\infty, b)$ ,  $[\perp, b) = (-\infty, b)$  and  $(a, \top] = (a, +\infty)$ .

□

**Definition 9.105** (Standard Topology on the Real Line). The *standard topology on the real line* is the order topology on  $\mathbb{R}$  generated by the standard order.

**Lemma 9.106.** *The lower limit topology is strictly finer than the standard topology on  $\mathbb{R}$ .*

PROOF:

(1)1. Every open interval is open in the lower limit topology.

PROOF: If  $x \in (a, b)$  then  $x \in [x, b) \subseteq (a, b)$ .

(1)2. The half-open interval  $[0, 1)$  is not open in the standard topology.

PROOF: There is no open interval  $(a, b)$  such that  $0 \in (a, b) \subseteq [0, 1)$ .

□

**Lemma 9.107.** *The  $K$ -topology is strictly finer than the standard topology on  $\mathbb{R}$ .*

PROOF:

(1)1. Every open interval is open in the  $K$ -topology.

PROOF: Corollary 9.53.1.

(1)2. The set  $(-1, 1) \setminus K$  is not open in the standard topology.

PROOF: There is no open interval  $(a, b)$  such that  $0 \in (a, b) \subseteq (-1, 1) \setminus K$ , since there must be a positive integer  $n$  with  $1/n \in (a, b)$ .

□

**Lemma 9.108.** *Let  $X$  be a topological space. Let  $Y$  be a linearly ordered set in the order topology. Let  $f, g : X \rightarrow Y$  be continuous. Then  $C = \{x \in X \mid f(x) \leq g(x)\}$  is closed.*

PROOF:

⟨1⟩1. LET:  $x \in X \setminus C$

⟨1⟩2.  $f(x) > g(x)$

PROVE: There exists a neighbourhood  $U$  of  $x$  such that  $U \subseteq X \setminus C$

⟨1⟩3. CASE: There exists  $y$  such that  $g(x) < y < f(x)$

PROOF: Take  $U = g^{-1}((-\infty, y)) \cup f^{-1}(y, +\infty)$ .

⟨1⟩4. CASE: There is no  $y$  such that  $g(x) < y < f(x)$

PROOF: Take  $U = g^{-1}((-\infty, f(x))) \cup f^{-1}(g(x), +\infty)$ .

□

**Proposition 9.109.** *Let  $X$  be a topological space. Let  $Y$  be a linearly ordered set in the order topology. Let  $f, g : X \rightarrow Y$  be continuous. Define  $h : X \rightarrow Y$  by  $h(x) = \min(f(x), g(x))$ . Then  $h$  is continuous.*

PROOF: By the Pasting Lemma applied to  $\{x \in X \mid f(x) \leq g(x)\}$  and  $\{x \in X \mid g(x) \leq f(x)\}$ , which are closed by Lemma 9.108.

**Proposition 9.110.** *Let  $X$  and  $Y$  be linearly ordered sets in the order topology. Let  $f : X \rightarrow Y$  be strictly monotone and surjective. Then  $f$  is a homeomorphism.*

PROOF:

⟨1⟩1.  $f$  is bijective.

PROOF: Proposition 4.14.

⟨1⟩2.  $f$  is continuous.

⟨2⟩1. For all  $y \in Y$  we have  $f^{-1}((y, +\infty))$  is open.

⟨3⟩1. LET:  $y \in Y$

⟨3⟩2. PICK  $x \in X$  such that  $f(x) = y$

PROOF: Since  $f$  is surjective.

⟨3⟩3.  $f^{-1}((y, +\infty)) = (x, +\infty)$

PROOF: By strict monotonicity.

⟨2⟩2. For all  $y \in Y$  we have  $f^{-1}((-\infty, y))$  is open.

PROOF: Similar.

⟨1⟩3.  $f^{-1}$  is continuous.

⟨2⟩1. For all  $x \in X$  we have  $f((x, +\infty))$  is open.

PROOF:  $f((x, +\infty)) = (f(x), +\infty)$ .

⟨2⟩2. For all  $x \in X$  we have  $f((-\infty, x))$  is open.

PROOF:  $f((-\infty, x)) = (-\infty, f(x))$ .

□

## 9.17 The nth Root Function

**Proposition 9.111.** For all  $n \geq 1$ , the function  $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  defined by  $f(x) = x^n$  is a homomorphism.

PROOF:

$\langle 1 \rangle 1$ .  $f$  is strictly monotone.

$\langle 2 \rangle 1$ . LET:  $x, y \in \mathbb{R}$  with  $0 \leq x < y$

$\langle 2 \rangle 2$ .  $x^n < y^n$

$$\begin{aligned} y^n - x^n &= (y - x)(y^{n-1} + y^{n-2}x + y^{n-3}x^2 + \cdots + x^{n-1}) \\ &> 0 \end{aligned}$$

$\langle 1 \rangle 2$ .  $f$  is surjective.

$\langle 2 \rangle 1$ . LET:  $y \in \mathbb{R}_{\geq 0}$

$\langle 2 \rangle 2$ . PICK  $x \in \mathbb{R}$  such that  $y \leq x^n$

PROOF: If  $y \leq 1$  take  $x = 1$ , otherwise take  $x = y$ .

$\langle 2 \rangle 3$ . There exists  $x' \in [0, x]$  such that  $(x')^n = y$

PROOF: By the Intermediate Value Theorem.

$\langle 1 \rangle 3$ . Q.E.D.

PROOF: Proposition 9.110.

□

**Definition 9.112.** For  $n \geq 1$ , the  $n$ th root function is the function  $\mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  that is the inverse of  $\lambda x.x^n$ .

## 9.18 The Product Topology

**Definition 9.113** (Product Topology). Let  $\{A_i\}_{i \in I}$  be a family of topological spaces. The *product topology* on  $\prod_{i \in I} A_i$  is the topology generated by the sub-basis consisting of the sets of the form  $\pi_i^{-1}(U)$  where  $i \in I$  and  $U$  is open in  $A_i$ .

**Proposition 9.114.** The product topology on  $\prod_{i \in I} A_i$  is generated by the basis consisting of all sets of the form  $\prod_{i \in I} U_i$  where  $\{U_i\}_{i \in I}$  is a family such that each  $U_i$  is an open set in  $A_i$  and  $U_i = A_i$  for all but finitely many  $i$ .

PROOF: From Proposition 9.61. □

**Proposition 9.115.** If  $A_i$  is closed in  $X_i$  for all  $i \in I$  then  $\prod_{i \in I} A_i$  is closed in  $\prod_{i \in I} X_i$ .

PROOF:

$$\left( \prod_{i \in I} X_i \right) \setminus \left( \prod_{i \in I} A_i \right) = \bigcup_{j \in I} \left( \prod_{i \in I} X_i \setminus \pi_j^{-1}(A_j) \right) \square$$

**Proposition 9.116.** Let  $\{A_i\}_{i \in I}$  be a family of topological spaces. For  $i \in I$ , let  $\mathcal{B}_i$  be a basis for the topology on  $A_i$ . Then  $\mathcal{B} = \{\prod_{i \in I} B_i \mid \forall i \in I. B_i \in \mathcal{B}_i, B_i = A_i \text{ for all but finitely many } i\}$  is a basis for the box topology on  $\prod_{i \in I} A_i$ .

PROOF:

- ⟨1⟩1. Every set in  $\mathcal{B}$  is open.
- ⟨1⟩2. For every point  $a \in \prod_{i \in I} A_i$  and every open set  $U$  with  $a \in U$ , there exists  $B \in \mathcal{B}$  such that  $a \in B \subseteq U$ .
  - ⟨2⟩1. LET:  $U$  be open and  $a \in U$
  - ⟨2⟩2. PICK a family  $\{U_i\}_{i \in I}$  such that each  $U_i$  is open in  $A_i$ , such that  $U_i = A_i$  except for  $i = i_1, \dots, i_n$ , and such that  $a \in \prod_{i \in I} U_i \subseteq U$ .
  - ⟨2⟩3. For  $j = 1, \dots, n$ , PICK  $B_{i_j} \in \mathcal{B}_{i_j}$  such that  $a_{i_j} \in B_{i_j} \subseteq U_{i_j}$
  - ⟨2⟩4. LET:  $B = \prod_{i \in I} B_i$  where  $B_i = A_i$  for  $i \neq i_1, \dots, i_n$
  - ⟨2⟩5.  $B \in \mathcal{B}$
  - ⟨2⟩6.  $a \in B \subseteq U$
- ⟨1⟩3. Q.E.D.

PROOF: Lemma 9.54.

□

**Proposition 9.117.** *Let  $\{A_i\}_{i \in I}$  be a family of topological spaces. Then the projections  $\pi_i : \prod_{i \in I} A_i \rightarrow A_i$  are open maps.*

PROOF: From Lemma 9.77. □

**Example 9.118.** The projections are not always closed maps. For example,  $\pi_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$  maps the closed set  $\{(x, 1/x) \mid x > 0\}$  to  $(0, +\infty)$ .

**Proposition 9.119.** *Let  $\{X_i\}_{i \in I}$  be a family of sets. For  $i \in I$ , let  $\mathcal{T}_i$  and  $\mathcal{U}_i$  be topologies on  $X_i$ . Let  $\mathcal{P}$  be the product topology on  $\prod_{i \in I} X_i$  generated by the topologies  $\mathcal{T}_i$ , and  $\mathcal{Q}$  the product topology on the same set generated by the topologies  $\mathcal{U}_i$ . Then  $\mathcal{P} \subseteq \mathcal{Q}$  if and only if  $\mathcal{T}_i \subseteq \mathcal{U}_i$  for all  $i$ .*

PROOF:

- ⟨1⟩1. If  $\mathcal{T}_i \subseteq \mathcal{U}_i$  for all  $i$  then  $\mathcal{P} \subseteq \mathcal{Q}$   
 PROOF: By Corollary 9.53.1.
- ⟨1⟩2. If  $\mathcal{P} \subseteq \mathcal{Q}$  then  $\mathcal{T}_i \subseteq \mathcal{U}_i$  for all  $i$ 
  - ⟨2⟩1. ASSUME:  $\mathcal{P} \subseteq \mathcal{Q}$
  - ⟨2⟩2. LET:  $i \in I$
  - ⟨2⟩3. LET:  $U \in \mathcal{T}_i$
  - ⟨2⟩4. LET:  $U_i = U$  and  $U_j = X_j$  for  $j \neq i$
  - ⟨2⟩5.  $\prod_{i \in I} U_i \in \mathcal{P}$
  - ⟨2⟩6.  $\prod_{i \in I} U_i \in \mathcal{Q}$
  - ⟨2⟩7.  $U \in \mathcal{U}_i$

PROOF: From Proposition 9.117.

□

**Proposition 9.120 (AC).** *Let  $\{X_i\}_{i \in I}$  be a family of topological spaces. Let  $A_i \subseteq X_i$  for all  $i \in I$ . Then*

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i} .$$

PROOF:

- (1)1.  $\overline{\prod_{i \in I} A_i} \subseteq \prod_{i \in I} \overline{A_i}$   
 (2)1. For all  $i \in I$  we have  $A_i \subseteq \overline{A_i}$   
 PROOF: Lemma 9.35.  
 (2)2.  $\prod_{i \in I} A_i \subseteq \prod_{i \in I} \overline{A_i}$   
 (2)3. Q.E.D.  
 PROOF: Since  $\prod_{i \in I} A_i$  is closed by Proposition 9.115.  
 (1)2.  $\prod_{i \in I} \overline{A_i} \subseteq \overline{\prod_{i \in I} A_i}$   
 (2)1. LET:  $x \in \prod_{i \in I} \overline{A_i}$   
 (2)2. LET:  $U$  be a neighbourhood of  $x$   
 (2)3. PICK  $V_i$  open in  $X_i$  such that  $x \in \prod_{i \in I} V_i \subseteq U$  with  $V_i = X_i$  except for  
 $i = i_1, \dots, i_n$   
 (2)4. For  $i \in I$ , pick  $a_i \in V_i \cap A_i$   
 PROOF: By Theorem 9.38 and (2)1 using the Axiom of Choice.  
 (2)5.  $U$  intersects  $\prod_{i \in I} A_i$   
 (2)6. Q.E.D.  
 PROOF:  $a \in U \cap \prod_{i \in I} A_i$   
 □

**Example 9.121.** The closure of  $\mathbb{R}^\infty$  in  $\mathbb{R}^\omega$  is  $\mathbb{R}^\omega$

PROOF:

- (1)1. LET:  $a \in \mathbb{R}^\omega$   
 (1)2. LET:  $U$  be any neighbourhoods of  $a$ .  
 (1)3. PICK  $U_n$  open in  $\mathbb{R}$  for all  $n$  such that  $a \in \prod_{n \geq 0} U_n \subseteq U$  and  $U_n = \mathbb{R}$  for  
 all  $n$  except  $n_1, \dots, n_k$   
 (1)4. LET:  $b_n = a_n$  for  $n = n_1, \dots, n_k$  and  $b_n = 0$  for all other  $n$   
 (1)5.  $b \in \mathbb{R}^\infty \cap U$   
 (1)6. Q.E.D.  
 PROOF: From Theorem 9.38.  
 □

**Proposition 9.122.** Let  $\{X_i\}_{i \in I}$  be a family of topological spaces. Let  $(a_n)$  be a sequence in  $\prod_{i \in I} X_i$  and  $l \in \prod_{i \in I} X_i$ . Then  $a_n \rightarrow l$  as  $n \rightarrow \infty$  if and only if, for all  $i \in I$ , we have  $\pi_i(a_n) \rightarrow \pi_i(l)$  as  $n \rightarrow \infty$ .

PROOF:

- (1)1. If  $a_n \rightarrow l$  as  $n \rightarrow \infty$  then, for all  $i \in I$ , we have  $\pi_i(a_n) \rightarrow \pi_i(l)$  as  $n \rightarrow \infty$   
 PROOF: Proposition 9.96.  
 (1)2. If, for all  $i \in I$ , we have  $\pi_i(a_n) \rightarrow \pi_i(l)$  as  $n \rightarrow \infty$ , then  $a_n \rightarrow l$  as  $n \rightarrow \infty$   
 (2)1. ASSUME: For all  $i \in I$ , we have  $\pi_i(a_n) \rightarrow \pi_i(l)$  as  $n \rightarrow \infty$   
 (2)2. LET:  $V$  be a neighbourhood of  $l$   
 (2)3. PICK open sets  $U_i$  in  $X_i$  such that  $l \in \prod_{i \in I} U_i \subseteq V$  and  $U_i = X_i$  for all  
 $i$  except  $i = i_1, \dots, i_k$   
 (2)4. For  $j = 1, \dots, k$ , PICK  $N_j$  such that, for all  $n \geq N_j$ , we have  $\pi_{i_j}(a_n) \in$   
 $U_{i_j}$   
 (2)5. LET:  $N = \max(N_1, \dots, N_k)$   
 (2)6. For all  $n \geq N$  we have  $a_n \in V$

□

**Theorem 9.123.** *Let  $A$  be a topological space and  $\{X_i\}_{i \in I}$  be a family of topological spaces. Let  $f : A \rightarrow \prod_{i \in I} X_i$  be a function. If  $\pi_i \circ f$  is continuous for all  $i \in I$  then  $f$  is continuous.*

PROOF:

<1>1. LET:  $i \in I$  and  $U$  be open in  $X_i$

<1>2.  $f^{-1}(\pi_i^{-1}(U))$  is open in  $A$

<1>3. Q.E.D.

PROOF: Proposition 9.81.

□

### 9.18.1 Continuous in Each Variable Separately

**Definition 9.124** (Continuous in Each Variable Separately). Let  $F : X \times Y \rightarrow Z$ . Then  $F$  is *continuous in each variable separately* if and only if:

- for every  $a \in X$  the function  $\lambda y \in Y. F(a, y)$  is continuous;
- for every  $b \in Y$  the function  $\lambda x \in X. F(x, b)$  is continuous.

**Proposition 9.125.** *Let  $F : X \times Y \rightarrow Z$ . If  $F$  is continuous then  $F$  is continuous in each variable separately.*

PROOF: For  $a \in X$ , the function  $\lambda y \in Y. F(a, y)$  is  $F \circ i$  where  $i : Y \rightarrow X \times Y$  maps  $y$  to  $(a, y)$ . We have  $i$  is continuous by Proposition 9.102, hence  $F \circ i$  is continuous by Theorem 9.87.

Similarly for  $\lambda x \in X. F(x, b)$  for  $b \in Y$ . □

**Example 9.126.** Define  $F : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  by

$$F(x, y) = \begin{cases} xy/(x^2 + y^2) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Then  $F$  is continuous in each variable separately but not continuous.

**Proposition 9.127.** *Let  $f : A \rightarrow C$  and  $g : B \rightarrow D$  be open maps. Then  $f \times g : A \times B \rightarrow C \times D$  is an open map.*

PROOF: Given  $U$  open in  $A$  and  $V$  open in  $B$ . Then  $(f \times g)(U \times V) = f(U) \times g(V)$  is open in  $C \times D$ . The result follows from Proposition 9.78. □

**Definition 9.128** (Sorgenfrey Plane). The *Sorgenfrey plane* is  $\mathbb{R}_l^2$ .

## 9.19 The Subspace Topology

**Definition 9.129** (Subspace Topology). Let  $X$  be a topological space and  $Y \subseteq X$ . The *subspace topology* on  $Y$  is  $\mathcal{T} = \{U \cap Y \mid U \text{ is open in } X\}$ .

We prove this is a topology.

PROOF:

(1)1.  $Y \in \mathcal{T}$

PROOF: Since  $Y = X \cap Y$

(1)2. For all  $\mathcal{U} \subseteq \mathcal{T}$ , we have  $\bigcup \mathcal{U} \in \mathcal{T}$

(2)1. LET:  $\mathcal{U} \subseteq \mathcal{T}$

(2)2. LET:  $\mathcal{V} = \{V \text{ open in } X \mid V \cap Y \in \mathcal{U}\}$

(2)3.  $\bigcup \mathcal{U} = (\bigcup \mathcal{V}) \cap Y$

(1)3. For all  $U, V \in \mathcal{T}$ , we have  $U \cap V \in \mathcal{T}$

(2)1. LET:  $U, V \in \mathcal{T}$

(2)2. PICK  $U', V'$  open in  $X$  such that  $U = U' \cap Y$  and  $V = V' \cap Y$

(2)3.  $(U \cap V) = (U' \cap V') \cap Y$

□

**Theorem 9.130.** *Let  $X$  be a topological space and  $Y$  a subspace of  $X$ . Let  $A \subseteq Y$ . Then  $A$  is closed in  $Y$  if and only if there exists a closed set  $C$  in  $X$  such that  $A = C \cap Y$ .*

PROOF: We have

$A$  is closed in  $Y$

$\Leftrightarrow Y \setminus A$  is open in  $Y$

$\Leftrightarrow \exists U$  open in  $X. Y \setminus A = Y \cap U$

$\Leftrightarrow \exists C$  closed in  $X. Y \setminus A = Y \cap (X \setminus U)$

$\Leftrightarrow \exists C$  closed in  $X. A = Y \cap U$

□

**Theorem 9.131.** *Let  $Y$  be a subspace of  $X$ . Let  $A \subseteq Y$ . Let  $\bar{A}$  be the closure of  $A$  in  $X$ . Then the closure of  $A$  in  $Y$  is  $\bar{A} \cap Y$ .*

PROOF: The closure of  $A$  in  $Y$  is

$\bigcap \{C \text{ closed in } Y \mid A \subseteq C\}$

$= \bigcap \{D \cap Y \mid D \text{ closed in } X, A \subseteq D \cap Y\}$  (Theorem 9.130)

$= \bigcap \{D \mid D \text{ closed in } X, A \subseteq D\} \cap Y$

$= \bar{A} \cap Y$

□

**Lemma 9.132.** *Let  $X$  be a topological space and  $Y \subseteq X$ . Let  $\mathcal{B}$  be a basis for the topology on  $X$ . Then  $\mathcal{B}' = \{B \cap Y \mid B \in \mathcal{B}\}$  is a basis for the subspace topology on  $Y$ .*

PROOF:

(1)1. Every element in  $\mathcal{B}'$  is open in  $Y$

(1)2. For every open set  $U$  in  $Y$  and point  $y \in U$ , there exists  $B' \in \mathcal{B}'$  such that  $y \in B' \subseteq U$

(2)1. LET:  $U$  be open in  $Y$  and  $y \in U$

(2)2. PICK  $V$  open in  $X$  such that  $U = V \cap Y$

(2)3. PICK  $B \in \mathcal{B}$  such that  $y \in B \subseteq V$

- (2)4. LET:  $B' = B \cap Y$   
 (2)5.  $B' \in \mathcal{B}'$   
 (2)6.  $y \in B' \subseteq U$   
 (1)3. Q.E.D.

PROOF: By Lemma 9.54.

□

**Lemma 9.133.** *Let  $X$  be a topological space and  $Y \subseteq X$ . Let  $\mathcal{S}$  be a basis for the topology on  $X$ . Then  $\mathcal{S}' = \{S \cap Y \mid S \in \mathcal{S}\}$  is a subbasis for the subspace topology on  $Y$ .*

PROOF: The set  $\{B \cap Y \mid B \text{ is a finite intersection of elements of } \mathcal{S}\}$  is a basis for the subspace topology by Lemma 9.132, and this is the set of all finite intersections of elements of  $\mathcal{S}'$ . □

**Lemma 9.134.** *Let  $Y$  be a subspace of  $X$ . If  $U$  is open in  $Y$  and  $Y$  is open in  $X$  then  $U$  is open in  $X$ .*

PROOF:

- (1)1. PICK  $V$  open in  $X$  such that  $U = V \cap Y$   
 (1)2.  $U$  is open in  $X$

PROOF: Since it is the intersection of two open sets  $V$  and  $Y$ .

□

**Theorem 9.135.** *Let  $Y$  be a subspace of  $X$  and  $A \subseteq Y$ . If  $A$  is closed in  $Y$  and  $Y$  is closed in  $X$  then  $A$  is closed in  $X$ .*

PROOF: Pick a closed set  $C$  in  $X$  such that  $A = C \cap Y$  (Theorem 9.130). Then  $A$  is the intersection of two sets closed in  $X$ , hence  $A$  is closed in  $X$  (Lemma 9.23). □

**Theorem 9.136.** *Let  $\{X_i\}_{i \in I}$  be a family of topological spaces. Let  $A_i$  be a subspace of  $X_i$  for all  $i \in I$ . Then the product topology on  $\prod_{i \in I} A_i$  is the same as the topology it inherits as a subspace of  $\prod_{i \in I} X_i$ .*

PROOF: The product topology is generated by the subbasis

$$\begin{aligned}
 & \{\pi_i^{-1}(U) \mid i \in I, U \text{ open in } A_i\} \\
 &= \{\pi_i^{-1}(V) \cap A_i \mid i \in I, V \text{ open in } X_i\} \\
 &= \{\pi_i^{-1}(V) \mid i \in I, V \text{ open in } X_i\} \cap \prod_{i \in I} A_i
 \end{aligned}$$

and this is a subbasis for the subspace topology by Lemma 9.133. □

**Theorem 9.137.** *Let  $X$  be an ordered set in the order topology. Let  $Y \subseteq X$  be an interval. Then the order topology on  $Y$  is the same as the subspace topology on  $Y$ .*

PROOF:

- (1)1. The order topology is finer than the subspace topology.



- (2)1. For every open ray  $R$  in  $X$ , the set  $R \cap Y$  is open in the order topology.  
 (3)1. For all  $a \in X$ , we have  $(-\infty, a) \cap Y$  is open in the order topology.  
 (4)1. CASE: For all  $y \in Y$  we have  $y < a$   
 PROOF: In this case  $(-\infty, a) \cap Y = Y$ .  
 (4)2. CASE: For all  $y \in Y$  we have  $a < y$   
 PROOF: In this case  $(-\infty, a) \cap Y = \emptyset$ .  
 (4)3. CASE: There exists  $y \in Y$  such that  $y \leq a$  and  $y \in Y$  such that  

$$a \leq y$$
  
 (5)1.  $a \in Y$   
 PROOF: Because  $Y$  is an interval.  
 (5)2.  $(-\infty, a) \cap Y = \{y \in Y \mid y < a\}$   
 (3)2. For all  $a \in X$ , we have  $(a, +\infty) \cap Y$  is open in the order topology.  
 PROOF: Similar.  
 (2)2. Q.E.D.  
 PROOF: By Lemmas 9.104 and 9.133 and Proposition 9.62.  
 (1)2. The subspace topology is finer than the order topology.  
 (2)1. Every open ray in  $Y$  is open in the subspace topology.  
 PROOF: For any  $a \in Y$  we have  $(-\infty, a)_Y = (-\infty, a)_X \cap Y$  and  $(a, +\infty)_Y = (a, +\infty)_X \cap Y$ .  
 (2)2. Q.E.D.  
 PROOF: By Lemma 9.104 and Proposition 9.62

□

This example shows that we cannot remove the hypothesis that  $Y$  is an interval:

**Example 9.138.** The order topology on  $I_o^2$  is different from the subspace topology as a subspace of  $\mathbb{R}^2$  under the dictionary order topology.

PROOF: The set  $\{1/2\} \times (1/2, 1)$  is open in the subspace topology but not in the order topology. □

**Proposition 9.139.** Let  $X$  be a topological space,  $Y$  a subspace of  $X$ , and  $Z$  a subspace of  $Y$ . Then the subspace topology on  $Z$  inherited from  $X$  is the same as the subspace topology on  $Z$  inherited from  $Y$ .

PROOF: The subspace topology inherited from  $Y$  is

$$\begin{aligned}
 & \{V \cap Z \mid V \text{ open in } Y\} \\
 &= \{U \cap Y \cap Z \mid U \text{ open in } X\} \\
 &= \{U \cap Z \mid U \text{ open in } X\}
 \end{aligned}$$

which is the subspace topology inherited from  $X$ . □

**Definition 9.140** (Unit Circle). The *unit circle*  $S^1$  is  $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$  as a subspace of  $\mathbb{R}^2$ .

**Definition 9.141** (Unit 2-sphere). The *unit 2-sphere* is  $S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$  as a subspace of  $\mathbb{R}^3$ .

**Proposition 9.142.** *Let  $f : X \rightarrow Y$  be an open map and  $A \subseteq X$  be open. Then the restriction  $f \upharpoonright A : A \rightarrow f(A)$  is an open map.*

PROOF:

$\langle 1 \rangle 1.$  LET:  $U$  be open in  $A$

$\langle 1 \rangle 2.$   $U$  is open in  $X$

PROOF: Lemma 9.134.

$\langle 1 \rangle 3.$   $f(U)$  is open in  $Y$

$\langle 1 \rangle 4.$   $f(U)$  is open in  $f(A)$

PROOF: Since  $f(U) = f(U) \cap f(A)$ .

□

**Example 9.143.** This example shows that we cannot remove the hypothesis that  $A$  is open.

Let  $A = \{(x, y) \in \mathbb{R}^2 \mid (x > 0 \text{ and } y = 1/x) \text{ or } x = y = 0\}$ . Then  $\pi_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$  is an open map, but  $\pi_1 \upharpoonright A : A \rightarrow [0, +\infty)$  is not, because it maps the set  $\{0, 0\}$  which is open in  $A$  to  $\{0\}$  which is not open in  $[0, +\infty)$ .

**Proposition 9.144.** *Let  $Y$  be a subspace of  $X$ . Let  $A \subseteq Y$  and  $l \in Y$ . Then  $l$  is a limit point of  $A$  in  $Y$  if and only if  $l$  is a limit point of  $A$  in  $X$ .*

PROOF: Both are equivalent to the condition that any neighbourhood of  $l$  in  $X$  intersects  $A$  in a point other than  $l$ . □

## 9.20 The Box Topology

**Definition 9.145** (Box Topology). Let  $\{A_i\}_{i \in I}$  be a family of topological spaces. The *box topology* on  $\prod_{i \in I} A_i$  is the topology generated by the set of all sets of the form  $\prod_{i \in I} U_i$  where  $\{U_i\}_{i \in I}$  is a family such that each  $U_i$  is open in  $A_i$ .

This is a basis since it covers  $\prod_{i \in I} A_i$  and is closed under intersection.

**Proposition 9.146.** *The box topology is finer than the product topology.*

PROOF: From Proposition 9.114. □

**Corollary 9.146.1.** *If  $A_i$  is closed in  $X_i$  for all  $i \in I$  then  $\prod_{i \in I} A_i$  is closed in  $\prod_{i \in I} X_i$  under the box topology.*

PROOF: From Proposition 9.115.

**Proposition 9.147** (AC). *Let  $\{A_i\}_{i \in I}$  be a family of topological spaces. For  $i \in I$ , let  $\mathcal{B}_i$  be a basis for the topology on  $A_i$ . Then  $\mathcal{B} = \{\prod_{i \in I} B_i \mid \forall i \in I. B_i \in \mathcal{B}_i\}$  is a basis for the box topology on  $\prod_{i \in I} A_i$ .*

PROOF:

$\langle 1 \rangle 1.$  Every set of the form  $\prod_{i \in I} B_i$  is open.

$\langle 1 \rangle 2.$  For every point  $a \in \prod_{i \in I} A_i$  and every open set  $U$  with  $a \in U$ , there exists  $B \in \mathcal{B}$  such that  $a \in B \subseteq U$ .

- (2)1. LET:  $U$  be open and  $a \in U$   
 (2)2. PICK a family  $\{U_i\}_{i \in I}$  such that each  $U_i$  is open in  $A_i$  and  $a \in \prod_{i \in I} U_i \subseteq U$ .  
 (2)3. For  $i \in I$ , PICK  $B_i \in \mathcal{B}_i$  such that  $a_i \in B_i \subseteq U_i$   
 PROOF: Using the Axiom of Choice.  
 (2)4.  $a \in \prod_{i \in I} B_i \subseteq U$   
 (1)3. Q.E.D.  
 PROOF: Lemma 9.54.

□

**Theorem 9.148.** *Let  $\{X_i\}_{i \in I}$  be a family of topological spaces. Let  $A_i$  be a subspace of  $X_i$  for all  $i \in I$ . Give  $\prod_{i \in I} X_i$  the box topology. Then the box topology on  $\prod_{i \in I} A_i$  is the same as the topology it inherits as a subspace of  $\prod_{i \in I} X_i$ .*

PROOF: The box topology is generated by the basis

$$\begin{aligned}
 & \left\{ \prod_{i \in I} U_i \mid \forall i \in I, U_i \text{ open in } A_i \right\} \\
 &= \left\{ \prod_{i \in I} (V_i \cap A_i) \mid \forall i \in I, V_i \text{ open in } X_i \right\} \\
 &= \left\{ \prod_{i \in I} V_i \mid \forall i \in I, V_i \text{ open in } X_i \right\} \cap \prod_{i \in I} A_i
 \end{aligned}$$

and this is a basis for the subspace topology by Lemma 9.132. □

**Proposition 9.149 (AC).** *Let  $\{X_i\}_{i \in I}$  be a family of topological spaces. Give  $\prod_{i \in I} X_i$  the box topology. Let  $A_i \subseteq X_i$  for all  $i \in I$ . Then*

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i}.$$

PROOF:

- (1)1.  $\overline{\prod_{i \in I} A_i} \subseteq \prod_{i \in I} \overline{A_i}$   
 (2)1. For all  $i \in I$  we have  $A_i \subseteq \overline{A_i}$   
 PROOF: Lemma 9.35.  
 (2)2.  $\prod_{i \in I} A_i \subseteq \prod_{i \in I} \overline{A_i}$   
 (2)3. Q.E.D.  
 PROOF: Since  $\prod_{i \in I} A_i$  is closed by Corollary 9.146.1.  
 (1)2.  $\prod_{i \in I} \overline{A_i} \subseteq \overline{\prod_{i \in I} A_i}$   
 (2)1. LET:  $x \in \prod_{i \in I} \overline{A_i}$   
 (2)2. LET:  $U$  be a neighbourhood of  $x$   
 (2)3. PICK  $V_i$  open in  $X_i$  such that  $x \in \prod_{i \in I} V_i \subseteq U$   
 (2)4. For  $i \in I$ , pick  $a_i \in V_i \cap A_i$   
 PROOF: By Theorem 9.38 and (2)1 using the Axiom of Choice.  
 (2)5.  $U$  intersects  $\prod_{i \in I} A_i$   
 (2)6. Q.E.D.  
 PROOF:  $a \in U \cap \prod_{i \in I} A_i$ .

□

The following example shows that Theorem 9.123 fails in the box topology.

**Example 9.150.** Define  $f : \mathbb{R} \rightarrow \mathbb{R}^\omega$  by  $f(t) = (t, t, \dots)$ . Then  $\pi_n \circ f = \text{id}_{\mathbb{R}}$  is continuous for all  $n$ . But  $f$  is not continuous when  $\mathbb{R}^\omega$  is given the box topology because the inverse image of

$$(-1, 1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times \dots$$

is  $\{0\}$  which is not open.

The following example shows that Proposition 9.122 fails in the box topology.

**Example 9.151.** Give  $\mathbb{R}^\omega$  the box topology. Let  $a_n = (1/n, 1/n, \dots)$  for  $n \geq 1$  and  $l = (0, 0, \dots)$ . Then  $\pi_i(a_n) \rightarrow \pi_i(l)$  as  $n \rightarrow \infty$  for all  $i$ , but  $a_n \not\rightarrow l$  as  $n \rightarrow \infty$  since the open set

$$(-1, 1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times \dots$$

contains  $l$  but does not contain any  $a_n$ .

**Example 9.152.** The set  $\mathbb{R}^\infty$  is closed in  $\mathbb{R}^\omega$  under the box topology. For let  $(a_n)$  be any sequence not in  $\mathbb{R}^\infty$ . Let  $U_n$  be an open interval around  $a_n$  that does not contain 0 if  $a_n \neq 0$ , and  $U_n = \mathbb{R}$  if  $a_n = 0$ . Then  $\prod_{n \geq 0} U_n$  is a neighbourhood of  $(a_n)$  that does not intersect  $\mathbb{R}^\infty$ .

## 9.21 $T_1$ Spaces

**Definition 9.153** ( $T_1$  Space). A topological space is  $T_1$  if and only if every singleton is closed.

**Lemma 9.154.** A space is  $T_1$  if and only if every finite set is closed.

PROOF: From Lemma 9.24. □

**Theorem 9.155.** In a  $T_1$  space, a point  $a$  is a limit point of a set  $A$  if and only if every neighbourhood of  $a$  contains infinitely many points of  $A$ .

PROOF:

⟨1⟩1. If  $a$  is a limit point of  $A$  then every neighbourhood of  $a$  contains infinitely many points of  $A$ .

⟨2⟩1. ASSUME:  $a$  is a limit point of  $A$ .

⟨2⟩2. LET:  $U$  be a neighbourhood of  $a$ .

⟨2⟩3. ASSUME: for a contradiction  $U$  contains only finitely many points of  $A$ .

⟨2⟩4.  $(U \cap A) \setminus \{a\}$  is closed.

PROOF: By the  $T_1$  axiom.

⟨2⟩5.  $(U \setminus A) \cup \{a\}$  is open.

PROOF: It is  $U \setminus ((U \cap A) \setminus \{a\})$ .

⟨2⟩6.  $(U \setminus A) \cup \{a\}$  intersects  $A$  in a point other than  $a$ .

PROOF: From  $\langle 2 \rangle 1$ .

$\langle 2 \rangle 7$ . Q.E.D.

□

$\langle 1 \rangle 2$ . If every neighbourhood of  $a$  contains infinitely many points of  $A$  then  $a$  is a limit point of  $A$ .

PROOF: Immediate from definitions.

□

(To see this does not hold in every space, see Proposition 9.50.)

**Proposition 9.156.** *A space is  $T_1$  if and only if, for any two distinct points  $x$  and  $y$ , there exist neighbourhoods  $U$  of  $x$  and  $V$  of  $y$  such that  $x \notin V$  and  $y \notin U$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $X$  be a topological space.

$\langle 1 \rangle 2$ . If  $X$  is  $T_1$  then, for any two distinct points  $x$  and  $y$ , there exist neighbourhoods  $U$  of  $x$  and  $V$  of  $y$  such that  $x \notin V$  and  $y \notin U$ .

PROOF: This holds because  $\{x\}$  and  $\{y\}$  are closed.

$\langle 1 \rangle 3$ . Suppose, for any two distinct points  $x$  and  $y$ , there exist neighbourhoods  $U$  of  $x$  and  $V$  of  $y$  such that  $x \notin V$  and  $y \notin U$ . Then  $X$  is  $T_1$ .

$\langle 2 \rangle 1$ . ASSUME: For any two distinct points  $x$  and  $y$ , there exist neighbourhoods  $U$  of  $x$  and  $V$  of  $y$  such that  $x \notin V$  and  $y \notin U$ .

$\langle 2 \rangle 2$ . LET:  $a \in X$

$\langle 2 \rangle 3$ .  $\{a\}$  is closed.

PROOF: For all  $b \neq a$  there exists a neighbourhood  $U$  of  $b$  such that  $U \subseteq X \setminus \{a\}$ .

□

**Proposition 9.157.** *A subspace of a  $T_1$  space is  $T_1$ .*

PROOF: From Proposition 9.135.

## 9.22 Hausdorff Spaces

**Definition 9.158** (Hausdorff Space). A topological space is *Hausdorff* if and only if, for any points  $x, y$  with  $x \neq y$ , there exist disjoint open sets  $U$  and  $V$  such that  $x \in U$  and  $y \in V$ .

**Theorem 9.159.** *Every Hausdorff space is  $T_1$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $X$  be a Hausdorff space.

$\langle 1 \rangle 2$ . LET:  $b \in X$

PROVE:  $\overline{\{b\}} = \{b\}$

$\langle 1 \rangle 3$ . ASSUME:  $a \in \overline{\{b\}}$  and  $a \neq b$

$\langle 1 \rangle 4$ . PICK disjoint neighbourhoods  $U$  of  $a$  and  $V$  of  $b$ .

$\langle 1 \rangle 5$ .  $U$  intersects  $\{b\}$

PROOF: Theorem 9.38.

⟨1⟩6.  $b \in U$

⟨1⟩7. Q.E.D.

PROOF: This contradicts the fact that  $U$  and  $V$  are disjoint (⟨1⟩4).

□

**Proposition 9.160.** *An infinite set under the finite complement topology is  $T_1$  but not Hausdorff.*

PROOF:

⟨1⟩1. LET:  $X$  be an infinite set under the finite complement topology.

⟨1⟩2. Every singleton is closed.

PROOF: By definition.

⟨1⟩3. PICK  $a, b \in X$  with  $a \neq b$

⟨1⟩4. There are no disjoint neighbourhoods  $U$  of  $a$  and  $V$  of  $b$ .

⟨2⟩1. LET:  $U$  be a neighbourhood of  $a$  and  $V$  a neighbourhood of  $b$ .

⟨2⟩2.  $X \setminus U$  and  $X \setminus V$  are finite.

⟨2⟩3. PICK  $c \in X$  that is not in  $X \setminus U$  or  $X \setminus V$ .

⟨2⟩4.  $c \in U \cap V$

□

**Proposition 9.161.** *The product of a family of Hausdorff spaces is Hausdorff.*

PROOF:

⟨1⟩1. LET:  $\{X_i\}_{i \in I}$  be a family of Hausdorff spaces.

⟨1⟩2. LET:  $a, b \in \prod_{i \in I} X_i$  with  $a \neq b$

⟨1⟩3. PICK  $i \in I$  such that  $a_i \neq b_i$

⟨1⟩4. PICK  $U, V$  disjoint open sets in  $X_i$  with  $a_i \in U$  and  $b_i \in V$

⟨1⟩5.  $\pi_i^{-1}(U)$  and  $\pi_i^{-1}(V)$  are disjoint open sets in  $\prod_{i \in I} X_i$  with  $a \in \pi_i^{-1}(U)$  and  $b \in \pi_i^{-1}(V)$

□

**Theorem 9.162.** *Every linearly ordered set under the order topology is Hausdorff.*

PROOF:

⟨1⟩1. LET:  $X$  be a linearly ordered set under the order topology.

⟨1⟩2. LET:  $a, b \in X$  with  $a \neq b$

⟨1⟩3. ASSUME: w.l.o.g.  $a < b$

⟨1⟩4. CASE: There exists  $c$  such that  $a < c < b$

PROOF: The sets  $(-\infty, c)$  and  $(c, +\infty)$  are disjoint neighbourhoods of  $a$  and  $b$  respectively.

⟨1⟩5. CASE: There is no  $c$  such that  $a < c < b$

PROOF: The sets  $(-\infty, b)$  and  $(a, +\infty)$  are disjoint neighbourhoods of  $a$  and  $b$  respectively.

□

**Theorem 9.163.** *A subspace of a Hausdorff space is Hausdorff.*

PROOF:

- <1>1. LET:  $X$  be a Hausdorff space and  $Y$  a subspace of  $X$ .
- <1>2. LET:  $x, y \in Y$  with  $x \neq y$
- <1>3. PICK disjoint neighbourhoods  $U$  of  $x$  and  $V$  of  $y$  in  $X$ .
- <1>4.  $U \cap Y$  and  $V \cap Y$  are disjoint neighbourhoods of  $x$  and  $y$  respectively in  $Y$ .

□

**Proposition 9.164.** *A space  $X$  is Hausdorff if and only if the diagonal  $\Delta = \{(x, x) \mid x \in X\}$  is closed in  $X^2$ .*

PROOF:

$X$  is Hausdorff

$$\Leftrightarrow \forall x, y \in X. x \neq y \Rightarrow \exists V, W \text{ open. } x \in V \wedge y \in W \wedge V \cap W = \emptyset$$

$$\Leftrightarrow \forall (x, y) \in X^2 \setminus \Delta. \exists V, W \text{ open. } (x, y) \subseteq V \times W \subseteq X^2 \setminus \Delta$$

$$\Leftrightarrow \Delta \text{ is closed}$$

□

**Theorem 9.165.** *In a Hausdorff space, a sequence has at most one limit.*

PROOF:

- <1>1. LET:  $X$  be a Hausdorff space.
- <1>2. ASSUME: for a contradiction  $a_n \rightarrow l$  as  $n \rightarrow \infty$ ,  $a_n \rightarrow m$  as  $n \rightarrow \infty$ , and  $l \neq m$
- <1>3. PICK disjoint neighbourhoods  $U$  of  $l$  and  $V$  of  $m$
- PROOF: By the Hausdorff axiom.
- <1>4. PICK  $M$  and  $N$  such that  $a_n \in U$  for  $n \geq M$  and  $a_n \in V$  for  $n \geq N$
- <1>5.  $a_{\max(M, N)} \in U \cap V$
- <1>6. Q.E.D.

PROOF: This contradicts the fact that  $U$  and  $V$  are disjoint (<1>3).

□

To see this is not always true in spaces that are  $T_1$  but not Hausdorff:

**Proposition 9.166.** *Let  $X$  be an infinite set under the finite complement topology. Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence with all points distinct. Then for every  $l \in X$  we have  $a_n \rightarrow l$  as  $n \rightarrow \infty$ .*

PROOF: Let  $U$  be any neighbourhood of  $l$ . Since  $X \setminus U$  is finite, there must exist  $N$  such that, for all  $n \geq N$ , we have  $a_n \in U$ . □

**Proposition 9.167.** *Let  $X$  be a topological space. Let  $Y$  a Hausdorff space. Let  $A \subseteq X$ . Let  $f, g : \overline{A} \rightarrow Y$  be continuous. If  $f$  and  $g$  agree on  $A$  then  $f = g$ .*

PROOF:

- <1>1. LET:  $x \in \overline{A}$
- <1>2. ASSUME:  $f(x) \neq g(x)$
- <1>3. PICK disjoint neighbourhoods  $V$  of  $f(x)$  and  $W$  of  $g(x)$ .
- <1>4. PICK  $y \in f^{-1}(V) \cap g^{-1}(W) \cap A$

PROOF: Since  $f^{-1}(V) \cap g^{-1}(W)$  is a neighbourhood of  $x$  and hence intersects  $A$ .

⟨1⟩5.  $f(y) = g(y) \in V \cap W$

⟨1⟩6. Q.E.D.

PROOF: This contradicts the fact that  $V$  and  $W$  are disjoint (⟨1⟩3).

□

**Proposition 9.168.** *Let  $\{X_i\}_{i \in I}$  be a family of Hausdorff spaces. Then  $\prod_{i \in I} X_i$  under the box topology is Hausdorff.*

PROOF:

⟨1⟩1. LET:  $\{X_i\}_{i \in I}$  be a family of Hausdorff spaces.

⟨1⟩2. LET:  $a, b \in \prod_{i \in I} X_i$  with  $a \neq b$

⟨1⟩3. PICK  $i \in I$  such that  $a_i \neq b_i$

⟨1⟩4. PICK  $U, V$  disjoint open sets in  $X_i$  with  $a_i \in U$  and  $b_i \in V$

⟨1⟩5.  $\pi_i^{-1}(U)$  and  $\pi_i^{-1}(V)$  are disjoint open sets in  $\prod_{i \in I} X_i$  with  $a \in \pi_i^{-1}(U)$  and  $b \in \pi_i^{-1}(V)$

□

**Proposition 9.169.** *Let  $\mathcal{T}$  and  $\mathcal{T}'$  be two topologies on the same set  $X$  with  $\mathcal{T} \subseteq \mathcal{T}'$ . If  $\mathcal{T}$  is Hausdorff then  $\mathcal{T}'$  is Hausdorff.*

PROOF: Immediate from definitions.

**Proposition 9.170.** *Let  $X$  be a Hausdorff space. Let  $\mathcal{D} \subseteq \mathcal{P}X$  be maximal with respect to the finite intersection property. Then  $\bigcap_{D \in \mathcal{D}} \overline{D}$  contains at most one point.*

PROOF:

⟨1⟩1. LET:  $x, y \in \bigcap_{D \in \mathcal{D}} \overline{D}$

⟨1⟩2. ASSUME: for a contradiction  $x \neq y$

⟨1⟩3. PICK disjoint open subsets  $U$  and  $V$  of  $x$  and  $y$  respectively.

⟨1⟩4.  $U, V \in \mathcal{D}$

PROOF: Proposition 9.41.

⟨1⟩5. Q.E.D.

PROOF: This contradicts the fact that  $\mathcal{D}$  satisfies the finite intersection property.

□

## 9.23 The First Countability Axiom

**Definition 9.171** (First Countability Axiom). A topological space  $X$  satisfies the *first countability axiom*, or is *first countable*, if and only if every point has a countable local basis.

**Lemma 9.172** (Sequence Lemma (CC)). *Let  $X$  be a first countable space. Let  $A \subseteq X$  and  $l \in \overline{A}$ . Then there exists a sequence in  $A$  that converges to  $l$ .*

PROOF:

⟨1⟩1. PICK a countable local basis  $\{B_n \mid n \in \mathbb{Z}^+\}$  at  $l$  such that  $B_1 \supseteq B_2 \supseteq \dots$ .



PROOF: Lemma 9.64.

⟨1⟩2. For all  $n \geq 1$ , PICK  $a_n \in A \cap B_n$ .

PROVE:  $a_n \rightarrow l$  as  $n \rightarrow \infty$

⟨1⟩3. LET:  $U$  be a neighbourhood of  $A$

⟨1⟩4. PICK  $N$  such that  $B_N \subseteq U$

⟨1⟩5. For  $n \geq N$  we have  $a_n \in U$

PROOF:  $a_n \in B_n \subseteq B_N \subseteq U$

□

**Theorem 9.173 (CC).** *Let  $X$  be a first countable space and  $Y$  a topological space. Let  $f : X \rightarrow Y$ . Suppose that, for every sequence  $(x_n)$  in  $X$  and  $l \in X$ , if  $x_n \rightarrow l$  as  $n \rightarrow \infty$ , then  $f(x_n) \rightarrow f(l)$  as  $n \rightarrow \infty$ . Then  $f$  is continuous.*

PROOF:

⟨1⟩1. LET:  $A \subseteq X$

⟨1⟩2. LET:  $a \in A$

PROVE:  $f(a) \in \overline{f(A)}$

⟨1⟩3. PICK a sequence  $(x_n)$  in  $A$  that converges to  $a$ .

PROOF: By the Sequence Lemma.

⟨1⟩4.  $f(x_n) \rightarrow f(a)$

⟨1⟩5.  $f(a) \in \overline{f(A)}$

PROOF: By Lemma 9.66.

⟨1⟩6. Q.E.D.

PROOF: By Theorem 9.84.

□

**Example 9.174 (CC).** The space  $\mathbb{R}^\omega$  under the box product is not first countable.

PROOF: Let  $\{B_n \mid n \geq 0\}$  be a countable set of neighbourhoods of  $\vec{0}$ . We will construct a neighbourhood of  $\vec{0}$  that does not include any of these.

For  $n \geq 0$ , pick a neighbourhood  $U_n$  of 0 such that  $U_n \subset \pi_n(B_p)$ . Then  $\prod_{n=0}^\infty U_n$  is a neighbourhood of  $\vec{0}$  that does not include any  $B_n$ . □

**Example 9.175.** If  $J$  is an uncountable set then  $\mathbb{R}^J$  is not first countable.

PROOF:

⟨1⟩1. LET:  $\{B_n \mid n \geq 0\}$  be any countable set of neighbourhoods of  $\vec{0}$ .

⟨1⟩2. For  $n \geq 0$ , PICK a basis element  $\prod_{\alpha \in J} U_{n\alpha}$  that contains  $\vec{0}$  and is included in  $B_n$ .

PROOF: Using the Axiom of Countable Choice.

⟨1⟩3. For  $n \geq 0$ ,

LET:  $J_n = \{\alpha \in J \mid U_{n\alpha} \neq \mathbb{R}\}$

⟨1⟩4. PICK  $\beta \in J$  such that  $\beta \notin J_n$  for any  $n$ .

PROOF: Since each  $J_n$  is finite so  $\bigcup_n J_n$  is countable.

⟨1⟩5.  $\pi_\beta((-1, 1))$  is a neighbourhood of  $\vec{0}$  that does not include any  $B_n$ .

□

**Example 9.176.** The space  $\mathbb{R}_l$  is first countable.

PROOF: For any  $a \in \mathbb{R}$ , the set  $\{[a, a + 1/n) \mid n \geq 1\}$  is a countable local basis.

**Example 9.177.** The ordered square is first countable.

PROOF: For any  $(a, b) \in I_o^2$  with  $b \neq 0, 1$ , the set  $\{(\{a\} \times (b - 1/n, b + 1/n)) \cap I_o^2 \mid n \geq 1\}$  is a countable local basis.

## 9.24 Strong Continuity

**Definition 9.178** (Strongly Continuous). Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$  be a function. Then  $f$  is *strongly continuous* if and only if, for every subset  $U \subseteq Y$ , we have  $U$  is open in  $Y$  if and only if  $f^{-1}(U)$  is open in  $X$ .

**Proposition 9.179.** Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$  be a function. Then  $f$  is strongly continuous if and only if, for every subset  $C \subseteq Y$ , we have  $C$  is closed in  $Y$  if and only if  $f^{-1}(C)$  is closed in  $X$ .

PROOF: Since  $X \setminus f^{-1}(C) = f^{-1}(Y \setminus C)$ .  $\square$

**Proposition 9.180.** Let  $X, Y$  and  $Z$  be topological spaces. Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$ . If  $f$  and  $g$  are strongly continuous then so is  $g \circ f$ .

PROOF: Since  $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ .  $\square$

**Proposition 9.181.** Let  $X, Y$  and  $Z$  be topological spaces. Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$ . If  $g \circ f$  is continuous and  $f$  is strongly continuous then  $g$  is continuous.

PROOF:

$\langle 1 \rangle 1$ . LET:  $V \subseteq Z$  be open.

$\langle 1 \rangle 2$ .  $f^{-1}(g^{-1}(V))$  is open in  $X$ .

PROOF: Since  $g \circ f$  is continuous.

$\langle 1 \rangle 3$ .  $f^{-1}(V)$  is open in  $Y$ .

PROOF: Since  $g$  is strongly continuous.

$\square$

**Proposition 9.182.** Let  $X, Y$  and  $Z$  be topological spaces. Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$ . If  $g \circ f$  is strongly continuous and  $f$  is strongly continuous then  $g$  is strongly continuous.

PROOF: For  $V \subseteq Z$ , we have  $V$  is open iff  $f^{-1}(g^{-1}(V))$  is open iff  $g^{-1}(V)$  is open.

## 9.25 Saturated Sets

**Definition 9.183.** Let  $X$  and  $Y$  be sets and  $p : X \rightarrow Y$  a surjective function. Let  $C \subseteq X$ . Then  $C$  is *saturated* with respect to  $p$  if and only if, for all  $x, y \in X$ , if  $x \in C$  and  $p(x) = p(y)$  then  $y \in C$ .

**Proposition 9.184.** *Let  $X$  and  $Y$  be sets and  $p : X \twoheadrightarrow Y$  a surjective function. Let  $C \subseteq X$ . Then the following are equivalent:*

1.  $C$  is saturated with respect to  $p$ .
2. There exists  $D \subseteq Y$  such that  $C = p^{-1}(D)$
3.  $C = p^{-1}(p(C))$ .

PROOF:

$\langle 1 \rangle 1. 1 \Rightarrow 3$

$\langle 2 \rangle 1.$  ASSUME:  $C$  is saturated with respect to  $p$ .

$\langle 2 \rangle 2. C \subseteq p^{-1}(p(C))$

PROOF: Trivial.

$\langle 2 \rangle 3. p^{-1}(p(C)) \subseteq C$

$\langle 3 \rangle 1.$  LET:  $x \in p^{-1}(p(C))$

$\langle 3 \rangle 2. p(x) \in p(C)$

$\langle 3 \rangle 3.$  There exists  $y \in C$  such that  $p(x) = p(y)$

$\langle 3 \rangle 4. x \in C$

PROOF: From  $\langle 2 \rangle 1$ .

$\langle 1 \rangle 2. 3 \Rightarrow 2$

PROOF: Trivial.

$\langle 1 \rangle 3. 2 \Rightarrow 1$

PROOF: This follows because if  $p(x) \in D$  and  $p(x) = p(y)$  then  $p(y) \in D$ .

□

## 9.26 Quotient Maps

**Definition 9.185** (Quotient Map). Let  $X$  and  $Y$  be topological spaces and  $p : X \rightarrow Y$ . Then  $p$  is a *quotient map* if and only if  $p$  is surjective and strongly continuous.

**Proposition 9.186.** *Let  $X$  and  $Y$  be topological spaces and  $p : X \twoheadrightarrow Y$  be a surjective function. Then the following are equivalent.*

1.  $p$  is a quotient map.
2.  $p$  is continuous and maps saturated open sets to open sets.
3.  $p$  is continuous and maps saturated closed sets to closed sets.

PROOF:

$\langle 1 \rangle 1. 1 \Rightarrow 2$

$\langle 2 \rangle 1.$  ASSUME:  $p$  is a quotient map.

$\langle 2 \rangle 2.$  LET:  $U$  be a saturated open set in  $X$ .

$\langle 2 \rangle 3. p^{-1}(p(U))$  is open in  $X$ .

PROOF: Since  $U = p^{-1}(p(U))$  be Proposition 9.184.

$\langle 2 \rangle 4. p(U)$  is open in  $Y$ .

PROOF: From  $\langle 2 \rangle 1$ .  
 $\langle 1 \rangle 2$ .  $1 \Rightarrow 3$   
PROOF: Similar.  
 $\langle 1 \rangle 3$ .  $2 \Rightarrow 1$   
 $\langle 2 \rangle 1$ . ASSUME:  $p$  is continuous and maps saturated open sets to open sets.  
 $\langle 2 \rangle 2$ . LET:  $U \subseteq Y$   
 $\langle 2 \rangle 3$ . ASSUME:  $p^{-1}(U)$  is open in  $X$   
 $\langle 2 \rangle 4$ .  $p^{-1}(U)$  is saturated.  
PROOF: Proposition 9.184.  
 $\langle 2 \rangle 5$ .  $U$  is open in  $Y$ .  
 $\langle 1 \rangle 4$ .  $3 \Rightarrow 1$   
PROOF: Similar.  
 $\square$

**Corollary 9.186.1.** *Every surjective continuous open map is a quotient map.*

**Corollary 9.186.2.** *Every surjective continuous closed map is a quotient map.*

**Example 9.187.** The converses of these corollaries do not hold.

Let  $A = \{(x, y) \mid x \geq 0\} \cup \{(x, y) \mid y = 0\}$ . Then  $\pi_1 : A \rightarrow \mathbb{R}$  is a quotient map, but not an open map or a closed map.

We prove that  $\pi_1$  maps saturated open sets to open sets:

- $\langle 1 \rangle 1$ . LET:  $\pi_1^{-1}(U)$  be a saturated open set in  $A$   
PROVE:  $U$  is open in  $\mathbb{R}$
- $\langle 1 \rangle 2$ . LET:  $x \in U$
- $\langle 1 \rangle 3$ .  $(x, 0) \in \pi_1(U)^{-1}$
- $\langle 1 \rangle 4$ . PICK  $W, V$  open in  $\mathbb{R}$  such that  $(x, 0) \subseteq W \times V \subseteq \pi_1(U)^{-1}$
- $\langle 1 \rangle 5$ .  $x \in W \subseteq U$

It is not an open map because it maps  $((-1, 1) \times (1, 2)) \cap A$  to  $[0, 1)$ .

It is not a closed map because it maps  $\{(x, 1/x) \mid x > 0\}$  to  $(0, +\infty)$ .

**Proposition 9.188.** *Let  $p : X \twoheadrightarrow Y$  be a quotient map. Let  $A \subseteq X$  be saturated with respect to  $p$ . Let  $q : A \twoheadrightarrow p(A)$  be the restriction of  $p$ .*

- 1. *If  $A$  is either open or closed in  $X$  then  $q$  is a quotient map.*
- 2. *If  $p$  is either an open map or a closed map then  $q$  is a quotient map.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $p : X \twoheadrightarrow Y$  be a quotient map.
- $\langle 1 \rangle 2$ . LET:  $A \subseteq X$  be saturated with respect to  $p$ .
- $\langle 1 \rangle 3$ . LET:  $q : A \twoheadrightarrow p(A)$  be the restriction of  $p$ .
- $\langle 1 \rangle 4$ .  $q$  is continuous.  
PROOF: Theorem 9.88.
- $\langle 1 \rangle 5$ . If  $A$  is open in  $X$  then  $q$  is a quotient map.  
 $\langle 2 \rangle 1$ . ASSUME:  $A$  is open in  $X$ .  
 $\langle 2 \rangle 2$ .  $q$  maps saturated open sets to open sets.

100

and  $p$ .

Define  $f : [0, 1] \rightarrow [2, 3] \rightarrow [0, 2]$  by  $f(x) = x$  if  $x \leq 1$ ,  $f(x) = x - 1$  if  $x \geq 2$ . Then  $f$  is a quotient map but its restriction  $f'$  to  $[0, 1] \cup [2, 3]$  is not, because  $f'^{-1}([1, 2])$  is open but  $[1, 2]$  is not.

For a counterexample where  $A$  is saturated, see Example 9.195.

**Proposition 9.190.** *Let  $p : A \twoheadrightarrow C$  and  $q : B \twoheadrightarrow D$  be open quotient maps. Then  $p \times q : A \times B \rightarrow C \times D$  is an open quotient map.*

PROOF: From Corollary 9.186.1, Proposition 9.127 and Theorem 9.123.  $\square$

**Theorem 9.191.** *Let  $p : X \twoheadrightarrow Y$  be a quotient map. Let  $Z$  be a topological space and  $f : Y \rightarrow Z$  be a function. Then*

1.  $f \circ p$  is continuous if and only if  $f$  is continuous.
2.  $f \circ p$  is a quotient map if and only if  $f$  is a quotient map.

PROOF:

$\langle 1 \rangle 1$ . If  $f \circ p$  is continuous then  $f$  is continuous.

PROOF: Proposition 9.181.

$\langle 1 \rangle 2$ . If  $f$  is continuous then  $f \circ p$  is continuous.

PROOF: Theorem 9.87.

$\langle 1 \rangle 3$ . If  $f \circ p$  is a quotient map then  $f$  is a quotient map.

PROOF: Proposition 9.182.

$\langle 1 \rangle 4$ . If  $f$  is a quotient map then  $f \circ p$  is a quotient map.

PROOF: From Proposition 9.180.

$\square$

**Proposition 9.192.** *Let  $X$  and  $Y$  be topological spaces. Let  $p : X \rightarrow Y$  and  $f : Y \rightarrow X$  be continuous maps such that  $p \circ f = \text{id}_Y$ . Then  $p$  is a quotient map.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $V \subseteq Y$

$\langle 1 \rangle 2$ . ASSUME:  $p^{-1}(V)$  is open in  $X$ .

$\langle 1 \rangle 3$ .  $f^{-1}(p^{-1}(V))$  is open in  $Y$ .

PROOF: Because  $f$  is continuous.

$\langle 1 \rangle 4$ .  $V$  is open in  $Y$ .

PROOF: Because  $f^{-1}(p^{-1}(V)) = V$ .

$\square$

## 9.27 Quotient Topology

**Definition 9.193** (Quotient Topology). Let  $X$  be a topological space,  $Y$  a set and  $p : X \twoheadrightarrow Y$  be a surjective function. Then the *quotient topology* on  $Y$  is the unique topology on  $Y$  with respect to which  $p$  is a quotient map, namely  $\mathcal{T} = \{U \in \mathcal{P}Y \mid p^{-1}(U) \text{ is open in } X\}$ .

We prove this is a topology.

PROOF:

$\langle 1 \rangle 1. Y \in \mathcal{T}$

PROOF: Since  $p^{-1}(Y) = X$  by surjectivity.

$\langle 1 \rangle 2. \text{ For all } \mathcal{A} \subseteq \mathcal{T} \text{ we have } \bigcup \mathcal{A} \in \mathcal{T}$

PROOF: Since  $p^{-1}(\bigcup \mathcal{A}) = \bigcup_{U \in \mathcal{A}} p^{-1}(U)$

$\langle 1 \rangle 3. \text{ For all } U, V \in \mathcal{T} \text{ we have } U \cap V \in \mathcal{T}$

PROOF: Since  $p^{-1}(U \cap V) = p^{-1}(U) \cap p^{-1}(V)$ .

□

**Definition 9.194** (Quotient Space). Let  $X$  be a topological space and  $\sim$  an equivalence relation on  $X$ . Let  $p : X \twoheadrightarrow X/\sim$  be the canonical surjection. Then  $X/\sim$  under the quotient topology is called a *quotient space*, *identification space* or *decomposition space* of  $X$ .

Here is a counterexample showing we cannot remove all the hypotheses of Proposition 9.188 except that  $A$  is saturated.

**Example 9.195.** Let  $X = (0, 1/2] \cup \{1\} \cup \{1 + 1/n : n \geq 2\}$  as a subspace of  $\mathbb{R}$ . Define  $R$  to be the equivalence relation on  $X$  where  $xRy$  iff  $(x = y \text{ or } |x - y| = 1)$ , so we identify  $1/n$  with  $1 + 1/n$  for all  $n \geq 2$ . Let  $Y$  be the resulting quotient space  $X/R$  in the quotient topology and  $p : X \twoheadrightarrow Y$  the canonical surjection.

Let  $A = \{1\} \cup (0, 1/2] \setminus \{1/n : n \geq 2\} \subseteq X$ . Then  $A$  is saturated under  $p$  but the restriction  $q$  of  $p$  to  $A$  is not a quotient map because it maps the saturated open set  $\{1\}$  to  $\{1\}$  which is not open in  $p(A)$ .

**Proposition 9.196.** Let  $X, Y$  and  $Z$  be topological spaces. Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$ . If  $f$  and  $g$  are quotient maps then so is  $g \circ f$ .

PROOF: From Proposition 9.180. □

**Example 9.197.** The product of two quotient maps is not necessarily a quotient map.

Let  $X = \mathbb{R}$  and  $X^*$  the quotient space formed by identifying all positive integers. Let  $p : X \twoheadrightarrow X^*$  be the canonical surjection.

We prove  $p \times \text{id}_{\mathbb{Q}} : X \times \mathbb{Q} \rightarrow X^* \times \mathbb{Q}$  is not a quotient map.

PROOF:

$\langle 1 \rangle 1. \text{ For } n \geq 1,$

LET:  $c_n = \sqrt{2}/n$

$\langle 1 \rangle 2. \text{ For } n \geq 1,$

LET:  $U_n = \{(x, y) \in X \times \mathbb{Q} \mid n - 1/4 < x < n + 1/4, (y + n > x + c_n \text{ and } y + n > -x + c_n) \text{ or } (y + n < x + c_n \text{ and } y + n < -x + c_n)\}$

$\langle 1 \rangle 3. \text{ For } n \geq 1, \text{ we have } U_n \text{ is open in } X \times \mathbb{Q}$

$\langle 1 \rangle 4. \text{ For } n \geq 1, \text{ we have } \{n\} \times \mathbb{Q} \subseteq U_n$

$\langle 1 \rangle 5. \text{ LET: } U = \bigcup_{n=1}^{\infty} U_n$

$\langle 1 \rangle 6. U \text{ is open in } X \times \mathbb{Q}$

$\langle 1 \rangle 7. U \text{ is saturated with respect to } p \times \text{id}_{\mathbb{Q}}$

- ⟨1⟩8. LET:  $U' = (p \times \text{id}_{\mathbb{Q}})(U)$
- ⟨1⟩9. ASSUME: for a contradiction  $U'$  is open in  $X^* \times \mathbb{Q}$
- ⟨1⟩10.  $(1, 0) \in U'$
- ⟨1⟩11. PICK a neighbourhood  $W$  of 1 in  $X^*$  and  $\delta > 0$  such that  $W \times (-\delta, \delta) \subseteq U'$
- ⟨1⟩12.  $p^{-1}(W) \times (-\delta, \delta) \subseteq U$
- ⟨1⟩13. PICK  $n$  such that  $c_n < \delta$
- ⟨1⟩14.  $n \in p^{-1}(W)$
- ⟨1⟩15. PICK  $\epsilon > 0$  such that  $\epsilon < \delta - c_n$  and  $\epsilon < 1/4$  and  $(n - \epsilon, n + \epsilon) \subseteq p^{-1}(W)$
- ⟨1⟩16.  $(n - \epsilon, n + \epsilon) \times (-\delta, \delta) \subseteq U$
- ⟨1⟩17. PICK a rational  $y$  such that  $c_n - \epsilon/2 < y < c_n + \epsilon/2$
- ⟨1⟩18.  $(n + \epsilon/2, y) \notin U$
- ⟨1⟩19. Q.E.D.

PROOF: This contradicts ⟨1⟩16.

□

**Proposition 9.198.** *Let  $X$  be a topological space and  $\sim$  an equivalence relation on  $X$ . Then  $X/\sim$  is  $T_1$  if and only if every equivalence class is closed in  $X$ .*

PROOF: Immediate from definitions. □

## 9.28 Retractions

**Definition 9.199** (Retraction). Let  $X$  be a topological space and  $A \subseteq X$ . A *retraction* of  $X$  onto  $A$  is a continuous map  $r : X \rightarrow A$  such that, for all  $a \in A$ , we have  $r(a) = a$ .

**Proposition 9.200.** *Every retraction is a quotient map.*

PROOF: Proposition 9.192 with  $f$  the inclusion  $A \hookrightarrow X$ . □

## 9.29 Homogeneous Spaces

**Definition 9.201** (Homogeneous). A topological space  $X$  is *homogeneous* if and only if, for any points  $a, b \in X$ , there exists a homeomorphism  $\phi : X \cong X$  such that  $\phi(a) = b$ .

## 9.30 Regular Spaces

**Definition 9.202** (Regular Space). A topological space  $X$  is *regular* if and only if, for any closed set  $A$  and point  $a \notin A$ , there exist disjoint open sets  $U, V$  such that  $A \subseteq U$  and  $a \in V$ .

## 9.31 Connected Spaces

**Definition 9.203** (Separation). A *separation* of a topological space  $X$  is a pair of disjoint open sets  $U, V$  such that  $U \cup V = \emptyset$ .



**Definition 9.204** (Connected). A topological space is *connected* if and only if it has no separation; otherwise it is *disconnected*.

**Proposition 9.205.** *A topological space  $X$  is connected if and only if the only sets that are both open and closed are  $X$  and  $\emptyset$ .*

Immediate from definitions.

**Lemma 9.206.** *If  $Y$  is a subspace of  $X$ , a separation of  $Y$  is a pair of disjoint nonempty sets  $A$  and  $B$  whose union is  $Y$ , neither of which contains a limit point of the other.*

PROOF:

- ⟨1⟩1. LET:  $A, B \subseteq Y$
- ⟨1⟩2. If  $A$  and  $B$  form a separation of  $Y$  then  $A$  and  $B$  are disjoint and nonempty,  $A \cup B = Y$ , and neither of  $A$  and  $B$  contains a limit point of the other.
  - ⟨2⟩1. ASSUME:  $A$  and  $B$  form a separation of  $Y$
  - ⟨2⟩2.  $A$  and  $B$  are disjoint and nonempty and  $A \cup B = Y$ 
    - PROOF: From ⟨2⟩1 and the definition of separation.
  - ⟨2⟩3.  $A$  does not contain a limit point of  $B$ 
    - ⟨3⟩1. ASSUME: for a contradiction  $l \in A$  and  $l$  is a limit point of  $B$  in  $X$ .
    - ⟨3⟩2.  $l$  is a limit point of  $B$  in  $Y$ 
      - PROOF: Proposition 9.144.
    - ⟨3⟩3.  $l \in B$ 
      - ⟨4⟩1.  $B$  is closed in  $Y$ 
        - PROOF: Since  $A$  is open in  $Y$  and  $B = Y \setminus A$  from ⟨2⟩1.
      - ⟨4⟩2. Q.E.D.
        - PROOF: Corollary 9.49.1.
    - ⟨3⟩4. Q.E.D.
      - PROOF: This contradicts the fact that  $A \cap B = \emptyset$  (⟨2⟩1).
  - ⟨2⟩4.  $B$  does not contain a limit point of  $A$ 
    - PROOF: Similar.
- ⟨1⟩3. If  $A$  and  $B$  are disjoint and nonempty,  $A \cup B = Y$ , and neither of  $A$  and  $B$  contains a limit point of the other, then  $A$  and  $B$  form a separation of  $Y$ .
  - ⟨2⟩1. ASSUME:  $A$  and  $B$  are disjoint and nonempty,  $A \cup B = Y$ , and neither of  $A$  and  $B$  contains a limit point of the other.
  - ⟨2⟩2.  $A$  is open in  $Y$ 
    - ⟨3⟩1.  $B$  is closed in  $Y$ 
      - ⟨4⟩1. LET:  $l$  be a limit point of  $B$  in  $Y$
      - ⟨4⟩2.  $l$  is a limit point of  $B$  in  $X$ 
        - PROOF: Proposition 9.144.
      - ⟨4⟩3.  $l \notin A$ 
        - PROOF: By ⟨2⟩1
      - ⟨4⟩4.  $l \in B$ 
        - PROOF: By ⟨2⟩1 since  $A \cup B = Y$
      - ⟨4⟩5. Q.E.D.

PROOF: Corollary 9.49.1.

⟨3⟩2. Q.E.D.

PROOF: Since  $A = Y \setminus B$ .

⟨2⟩3.  $B$  is open in  $Y$

PROOF: Similar.

□

**Example 9.207.** Every set under the indiscrete topology is connected.

**Example 9.208.** The discrete topology on a set  $X$  is connected if and only if  $|X| \leq 1$ .

**Example 9.209.** The finite complement topology on a set  $X$  is connected if and only if either  $|X| \leq 1$  or  $X$  is infinite.

**Example 9.210.** The countable complement topology on a set  $X$  is connected if and only if either  $|X| \leq 1$  or  $X$  is uncountable.

**Example 9.211.** The rationals  $\mathbb{Q}$  are disconnected. For any irrational  $a$ , the sets  $(-\infty, a) \cap \mathbb{Q}$  and  $(a, +\infty) \cap \mathbb{Q}$  form a separation of  $\mathbb{Q}$ .

**Lemma 9.212.** *Let  $X$  be a topological space. If  $C$  and  $D$  form a separation of  $X$ , and  $Y$  is a connected subspace of  $X$ , then either  $Y \subseteq C$  or  $Y \subseteq D$ .*

PROOF: Otherwise  $Y \cap C$  and  $Y \cap D$  would form a separation of  $Y$ . □

**Theorem 9.213.** *The union of a set of connected subspaces of a space  $X$  that have a point in common is connected.*

PROOF:

⟨1⟩1. LET:  $\mathcal{A}$  be a set of connected subspaces of the space  $X$  that have the point  $a$  in common.

⟨1⟩2. ASSUME: for a contradiction  $C$  and  $D$  form a separation of  $\bigcup \mathcal{A}$

⟨1⟩3. ASSUME: without loss of generality  $a \in C$

⟨1⟩4. For all  $A \in \mathcal{A}$  we have  $A \subseteq C$

PROOF: Lemma 9.212.

⟨1⟩5.  $D = \emptyset$

⟨1⟩6. Q.E.D.

PROOF: This contradicts ⟨1⟩2.

□

**Theorem 9.214.** *Let  $X$  be a topological space and  $A$  a connected subspace of  $X$ . If  $A \subseteq B \subseteq \overline{A}$  then  $B$  is connected.*

PROOF:

⟨1⟩1. ASSUME: for a contradiction  $C$  and  $D$  form a separation of  $B$ .

⟨1⟩2. ASSUME: without loss of generality  $A \subseteq C$

PROOF: Lemma 9.212.

⟨1⟩3.  $B \subseteq C$

⟨2⟩1. LET:  $x \in B$

- (2)2.  $x \in \overline{A}$
- (2)3. Either  $x \in A$  or  $x$  is a limit point of  $A$ .  
PROOF: Theorem 9.49.
- (2)4. Either  $x \in A$  or  $x$  is a limit point of  $C$ .  
PROOF: Lemma 9.51, (1)2.
- (2)5.  $x \in C$   
PROOF: Lemma 9.206.
- (1)4.  $D = \emptyset$
- (1)5. Q.E.D.

PROOF: This contradicts (1)1.  
□

**Theorem 9.215.** *The image of a connected space under a continuous map is connected.*

PROOF:

- (1)1. LET:  $f : X \rightarrow Y$  be a surjective continuous map where  $X$  is connected.
- (1)2. ASSUME: for a contradiction  $C$  and  $D$  form a separation of  $Y$ .
- (1)3.  $f^{-1}(C)$  and  $f^{-1}(D)$  form a separation of  $X$ .

□

**Theorem 9.216.** *The product of a family of connected spaces is connected.*

PROOF:

- (1)1. The product of two connected spaces is connected.
  - (2)1. LET:  $X$  and  $Y$  be connected spaces.
  - (2)2. PICK  $a \in X$  and  $b \in Y$   
PROOF: We may assume  $X$  and  $Y$  are nonempty since otherwise  $X \times Y = \emptyset$  which is connected.
  - (2)3.  $X \times \{b\}$  is connected.  
PROOF: It is homeomorphic to  $X$ .
  - (2)4. For all  $x \in X$  we have  $\{x\} \times Y$  is connected.  
PROOF: It is homeomorphic to  $Y$ .
  - (2)5. For any  $x \in X$   
LET:  $T_x = (X \times \{b\}) \cup (\{x\} \times Y)$
  - (2)6. For all  $x \in X$ ,  $T_x$  is connected.  
PROOF: Theorem 9.213 since  $(x, b) \in (X \times \{b\}) \cap (\{x\} \times Y)$ .
  - (2)7.  $X \times Y$  is connected.  
PROOF: Theorem 9.213 since  $X \times Y = \bigcup_{x \in X} T_x$  and  $(a, b)$  is a point in every  $T_x$ .
- (1)2. The product of a finite family of connected spaces is connected.  
PROOF: From (1)1 by induction.
- (1)3. The product of any family of connected spaces is connected.
  - (2)1. LET:  $\{X_\alpha\}_{\alpha \in J}$  be a family of connected spaces.
  - (2)2. LET:  $X = \prod_{\alpha \in J} X_\alpha$
  - (2)3. PICK  $a \in X$   
PROOF: We may assume  $X \neq \emptyset$  as the empty space is connected.

- (2)4. For every finite subset  $K$  of  $J$ ,  
 LET:  $X_K = \{x \in X \mid \forall \alpha \in J \setminus K. x_\alpha = a_\alpha\}$   
 (2)5. For every finite  $K \subseteq J$ , we have  $X_K$  is connected.  
 PROOF: From (1)2 since  $X_K \cong \prod_{\alpha \in K} X_K$ .  
 (2)6. LET:  $Y = \bigcup_K X_K$   
 (2)7.  $Y$  is connected  
 PROOF: Theorem 9.213 since  $a$  is a common point.  
 (2)8.  $X = \bar{Y}$   
 (3)1. LET:  $x \in X$   
 (3)2. LET:  $U = \prod_{\alpha \in J} U_\alpha$  be a basic neighbourhood of  $x$  where  $U_\alpha = X_\alpha$   
 for all  $\alpha$  except  $\alpha \in K$  for some finite  $K \subseteq J$   
 (3)3. LET:  $y \in X$  be the point with  $y_\alpha = x_\alpha$  for  $\alpha \in K$  and  $y_\alpha = a_\alpha$  for  
 all other  $\alpha$   
 (3)4.  $y \in U \cap X_K$   
 (3)5.  $y \in U \cap Y$   
 (2)9.  $X$  is connected.  
 PROOF: Theorem 9.214.

□

**Example 9.217.** The set  $\mathbb{R}^\omega$  is disconnected under the box topology. The set of bounded sequences and the set of unbounded sequences form a separation.

**Proposition 9.218.** Let  $\mathcal{T}$  and  $\mathcal{T}'$  be two topologies on the same set  $X$ . If  $\mathcal{T} \subseteq \mathcal{T}'$  and  $(X, \mathcal{T}')$  is connected then  $(X, \mathcal{T})$  is connected.

PROOF: If  $U$  and  $V$  form a separation of  $(X, \mathcal{T})$  then they form a separation of  $(X, \mathcal{T}')$ . □

**Proposition 9.219.** Let  $X$  be a topological space and  $(A_n)$  a sequence of connected subspaces of  $X$ . If  $A_n \cap A_{n+1} \neq \emptyset$  for all  $n$  then  $\bigcup_n A_n$  is connected.

PROOF:

- (1)1. ASSUME: for a contradiction  $C$  and  $D$  form a separation of  $\bigcup_n A_n$   
 (1)2. ASSUME: without loss of generality  $A_0 \subseteq C$

PROOF: Lemma 9.212.

- (1)3. For all  $n$  we have  $A_n \subseteq C$

PROOF:

- (2)1. ASSUME:  $A_n \subseteq C$   
 (2)2. PICK  $x \in A_n \cap A_{n+1}$   
 (2)3.  $x \in C$   
 (2)4.  $A_{n+1} \subseteq C$

PROOF: Lemma 9.212.

- (2)5. Q.E.D.

PROOF: The result follows by induction.

- (1)4.  $D = \emptyset$

- (1)5. Q.E.D.

PROOF: This contradicts (1)1.

□

**Proposition 9.220.** *Let  $X$  be a topological space. Let  $A, C \subseteq X$ . If  $C$  is connected and intersects both  $A$  and  $X \setminus A$  then  $C$  intersects  $\partial A$ .*

PROOF: Otherwise  $C \cap A^\circ$  and  $C \setminus \bar{A}$  would form a separation of  $C$ .  $\square$

**Example 9.221.** The space  $\mathbb{R}_l$  is disconnected. For any real  $x$ , the sets  $(-\infty, x)$  and  $[x, +\infty)$  form a separation.

**Proposition 9.222.** *Let  $X$  and  $Y$  be connected spaces. Let  $A$  be a proper subset of  $X$  and  $B$  a proper subset of  $Y$ . Then  $(X \times Y) \setminus (A \times B)$  is connected.*

PROOF:

$\langle 1 \rangle 1$ . PICK  $a \in X \setminus A$  and  $b \in Y \setminus B$

$\langle 1 \rangle 2$ . For  $x \in X \setminus A$  we have  $(X \times \{b\}) \cup (\{x\} \times Y)$  is connected.

PROOF: Theorem 9.213 since  $(x, b)$  is a common point.

$\langle 1 \rangle 3$ . For  $y \in Y \setminus B$  we have  $(X \times \{y\}) \cup (\{a\} \times Y)$  is connected.

PROOF: Theorem 9.213 since  $(a, y)$  is a common point.

$\langle 1 \rangle 4$ .  $(X \times Y) \setminus (A \times B)$  is connected.

PROOF: Theorem 9.213 since it is the union of the sets in  $\langle 1 \rangle 2$  and  $\langle 1 \rangle 3$  with  $(a, b)$  as a common point.

$\square$

**Proposition 9.223.** *Let  $p : X \twoheadrightarrow Y$  be a quotient map. If  $Y$  is connected and  $p^{-1}(y)$  is connected for all  $y \in Y$ , then  $X$  is connected.*

PROOF:

$\langle 1 \rangle 1$ . ASSUME: for a contradiction  $C$  and  $D$  form a separation of  $X$ .

$\langle 1 \rangle 2$ .  $C$  is saturated.

$\langle 2 \rangle 1$ . LET:  $x \in C$ ,  $y \in X$  with  $p(x) = p(y) = a$ , say

$\langle 2 \rangle 2$ .  $y \notin D$

PROOF: Otherwise  $p^{-1}(a) \cap C$  and  $p^{-1}(a) \cap D$  form a separation of  $p^{-1}(a)$ .

$\langle 2 \rangle 3$ .  $y \in C$

$\langle 1 \rangle 3$ .  $D$  is saturated.

PROOF: Similar.

$\langle 1 \rangle 4$ .  $p(C)$  and  $p(D)$  form a separation of  $Y$ .

$\square$

**Proposition 9.224.** *Let  $X$  be a connected space and  $Y$  a connected subspace of  $X$ . Suppose  $A$  and  $B$  form a separation of  $X \setminus Y$ . Then  $Y \cup A$  and  $Y \cup B$  are both connected.*

PROOF:

$\langle 1 \rangle 1$ .  $Y \cup A$  is connected.

$\langle 2 \rangle 1$ . ASSUME: for a contradiction  $C$  and  $D$  form a separation of  $Y \cup A$

$\langle 2 \rangle 2$ . ASSUME: without loss of generality  $Y \subseteq C$

$\langle 2 \rangle 3$ . PICK open sets  $A_1, B_1, C_1, D_1$  in  $X$  with

$$A = A_1 \setminus Y$$

$$B = B_1 \setminus Y$$

$$C = C_1 \cap (Y \cup A)$$

$$D = D_1 \cap (Y \cup A)$$

- ⟨2⟩4.  $B_1 \cup C_1$  and  $A_1 \cap D_1$  form a separation of  $X$   
 ⟨1⟩2.  $Y \cup B$  is connected.

PROOF: Similar.

□

**Theorem 9.225.** *Let  $L$  be a linearly ordered set under the order topology. Then  $L$  is connected if and only if  $L$  is a linear continuum.*

PROOF:

- ⟨1⟩1. If  $L$  is a linear continuum then  $L$  is connected.  
 ⟨2⟩1. LET:  $L$  be a linear continuum under the order topology.  
 ⟨2⟩2. ASSUME: for a contradiction  $C$  and  $D$  form a separation of  $L$ .  
 ⟨2⟩3. PICK  $a \in C$  and  $b \in D$ .  
 ⟨2⟩4. ASSUME: without loss of generality  $a < b$ .  
 ⟨2⟩5. LET:  $S = \{x \in L \mid a < x \text{ and } [a, x) \subseteq C\}$   
 ⟨2⟩6.  $S$  is nonempty.  
   PROOF: Since  $a \in C$  and  $C$  is open.  
 ⟨2⟩7.  $S$  is bounded above by  $b$ .  
   PROOF: Since  $b \notin C$ .  
 ⟨2⟩8. LET:  $s = \sup S$   
 ⟨2⟩9.  $s \in S$   
   ⟨3⟩1. LET:  $y \in [a, s)$   
     PROVE:  $y \in C$   
   ⟨3⟩2. PICK  $z$  with  $y < z \in S$   
     PROOF: By minimality of  $s$ .  
   ⟨3⟩3.  $y \in [a, z) \subseteq C$   
 ⟨2⟩10. CASE:  $s \in C$   
   ⟨3⟩1. PICK  $x$  such that  $s < x$  and  $[s, x) \subseteq C$   
     PROOF: Since  $C$  is open and  $s$  is not greatest in  $L$  because  $s < b$ .  
   ⟨3⟩2.  $x \in S$   
     PROOF: Since  $[a, x) = [a, s) \cup [s, x) \subseteq C$ .  
   ⟨3⟩3. Q.E.D.  
     PROOF: This contradicts the fact that  $s$  is an upper bound for  $S$ .  
 ⟨2⟩11. CASE:  $s \in D$   
   ⟨3⟩1. PICK  $x < s$  such that  $(x, s] \subseteq D$   
   ⟨3⟩2. PICK  $y$  with  $x < y < s$   
     PROOF: Since  $L$  is dense.  
   ⟨3⟩3.  $y \in C$   
     PROOF: From ⟨2⟩9.  
   ⟨3⟩4.  $y \in D$   
     PROOF: From ⟨3⟩1.  
   ⟨3⟩5. Q.E.D.  
   ⟨3⟩6. LET:  $L$  be a linear continuum under the order topology.  
   ⟨3⟩7. ASSUME: for a contradiction  $C$  and  $D$  form a separation of  $L$ .  
   ⟨3⟩8. PICK  $a \in C$  and  $b \in D$ .  
   ⟨3⟩9. ASSUME: without loss of generality  $a < b$ .  
   ⟨3⟩10. LET:  $S = \{x \in L \mid a < x \text{ and } [a, x) \subseteq C\}$



⟨4⟩2. PICK  $y \in X$  with  $x < y$

PROOF:  $x$  cannot be an upper bound for  $X$ , because it would be the supremum of  $X$ .

⟨4⟩3. Either  $x$  least in  $L$  and  $[x, y) \subseteq V$ , or there exists  $z < x$  such that  $(z, y) \subseteq V$

⟨3⟩9.  $L = U \cup V$

⟨4⟩1. LET:  $x \in L \setminus U$

⟨4⟩2. PICK  $y \in X$  such that  $x < y$

⟨4⟩3. For all  $u \in U$  we have  $x < y \leq u$

⟨4⟩4.  $x \in V$

⟨3⟩10.  $U \cap V = \emptyset$

PROOF: Any element of  $U \cap V$  would be a supremum of  $X$ .

⟨3⟩11.  $U$  and  $V$  form a separation of  $L$ .

⟨3⟩12. Q.E.D.

PROOF: This contradicts ⟨2⟩1.

⟨2⟩3.  $L$  is dense.

⟨3⟩1. LET:  $x, y \in L$  with  $x < y$

⟨3⟩2. There exists  $z \in L$  such that  $x < z < y$

PROOF: Otherwise  $(-\infty, y)$  and  $(x, +\infty)$  would form a separation of  $L$ .

□

**Corollary 9.225.1.** *The real line  $\mathbb{R}$  is connected.*

**Corollary 9.225.2.** *Every interval in  $\mathbb{R}$  is connected.*

**Corollary 9.225.3.** *The ordered square is connected.*

**Theorem 9.226** (Intermediate Value Theorem). *Let  $X$  be a connected space. Let  $Y$  be a linearly ordered set under the order topology. Let  $f : X \rightarrow Y$  be continuous. Let  $a, b \in X$  and  $r \in Y$ . Suppose  $f(a) < r < f(b)$ . Then there exists  $c \in X$  such that  $f(c) = r$ .*

PROOF: Otherwise  $f^{-1}((-\infty, r))$  and  $f^{-1}((r, +\infty))$  would form a separation of  $X$ . □

**Proposition 9.227.** *Every function  $f : [0, 1] \rightarrow [0, 1]$  has a fixed point.*

PROOF:

⟨1⟩1. LET:  $g : [0, 1] \rightarrow [-1, 1]$  be the function  $g(x) = f(x) - x$

PROVE: there exists  $x \in [0, 1]$  such that  $g(x) = 0$

⟨1⟩2. ASSUME: without loss of generality  $g(0) \neq 0$  and  $g(1) \neq 0$

⟨1⟩3.  $g(0) > 0$

⟨1⟩4.  $g(1) < 0$

⟨1⟩5. There exists  $x \in (0, 1)$  such that  $g(x) = 0$

PROOF: By the Intermediate Value Theorem.

**Proposition 9.228.** *Give  $\mathbb{R}^\omega$  the box topology. Let  $x, y \in \mathbb{R}^\omega$ . Then  $x$  and  $y$  lie in the same component if and only if  $x - y$  is eventually zero, i.e. there exists  $N$  such that, for all  $n \geq N$ , we have  $x_n = y_n$ .*



PROOF:

- ⟨1⟩1. The component containing 0 is the set of sequences that are eventually zero.
- ⟨2⟩1. LET:  $B$  be the set of sequences that are eventually zero.
- ⟨2⟩2.  $B$  is path-connected.
  - ⟨3⟩1. LET:  $x, y \in B$
  - ⟨3⟩2. PICK  $N$  such that, for all  $n \geq N$ , we have  $x_n = y_n = 0$
  - ⟨3⟩3. LET:  $p : [0, 1] \rightarrow \mathbb{R}^\omega$ ,  $p(t) = (1 - t)x + ty$   
PROVE:  $p$  is continuous.
  - ⟨3⟩4. LET:  $t \in [0, 1]$  and  $\prod_j U_j$  be a basic open neighbourhood of  $p(t)$ , where each  $U_j$  is open in  $\mathbb{R}$
  - ⟨3⟩5. PICK  $\delta$  such that, for all  $n < N$  and all  $s \in [0, 1]$ , if  $|s - t| < \delta$  then  $p(s)_n \in U_n$
  - ⟨3⟩6. For all  $s \in [0, 1]$ , if  $|s - t| < \delta$  then  $p(s) \in \prod_j U_j$
- ⟨2⟩3.  $B$  is connected.  
PROOF: Proposition 9.236.
- ⟨2⟩4. If  $C$  is connected and  $B \subseteq C$  then  $B = C$ .
  - ⟨3⟩1. ASSUME:  $C$  is connected and  $B \subseteq C$
  - ⟨3⟩2. ASSUME: for a contradiction  $x \in C \setminus B$
  - ⟨3⟩3. For  $n \geq 1$ ,  
LET:  $c_n = 1$  if  $x_n = 0$ ,  $c_n = n/x_n$  otherwise
  - ⟨3⟩4. LET:  $h : \mathbb{R}^\omega \rightarrow \mathbb{R}^\omega$  be the function  $h(x) = (c_n x_n)_{n \geq 1}$
  - ⟨3⟩5.  $h$  is a homeomorphism of  $\mathbb{R}^\omega$  with itself.
  - ⟨3⟩6.  $h(x)$  is unbounded.  
PROOF: For any  $b > 0$ , pick  $N > b$  such that  $x_N \neq 0$ . Then  $h(x)_N > b$ .
  - ⟨3⟩7.  $h^{-1}(\{\text{bounded sequences}\}) \cap C$  and  $h^{-1}(\{\text{unbounded sequences}\}) \cap C$  form a separation of  $C$
  - ⟨3⟩8. Q.E.D.  
PROOF: This contradicts ⟨3⟩1.
- ⟨1⟩2. Q.E.D.

□

**Example 9.229.** The space  $\mathbb{R}_K$  is connected.

PROOF:

- ⟨1⟩1. ASSUME: for a contradiction  $U$  and  $V$  form a separation of  $\mathbb{R}_K$
- ⟨1⟩2. ASSUME: without loss of generality  $0 \in U$
- ⟨1⟩3. There exists an open interval  $(a, b)$  such that  $(a, b) - K \subseteq U$  and  $(a, b) \not\subseteq U$   
PROOF: Otherwise  $U$  and  $V$  would form a separation of  $\mathbb{R}$ .
- ⟨1⟩4. PICK  $1/n \in (a, b) - U$
- ⟨1⟩5.  $1/n \in V$
- ⟨1⟩6. There exists an open interval  $(c, d)$  around  $1/n$  that is included in  $V$
- ⟨1⟩7. Q.E.D.  
PROOF: This is a contradiction since  $(a, b) - K$  and  $(c, d)$  must intersect.

□

## 9.32 Totally Disconnected Spaces

**Definition 9.230** (Totally Disconnected). A topological space  $X$  is *totally disconnected* if and only if the only connected subspaces are the singletons.

**Example 9.231.** Every discrete space is totally disconnected.

**Example 9.232.** The rationals  $\mathbb{Q}$  are totally disconnected.

**Example 9.233.** The Cantor set is totally disconnected.

PROOF:

- ⟨1⟩1. LET:  $(A_n)$  be the sequence of sets in Definition 8.24.
- ⟨1⟩2. LET:  $C$  be the Cantor set  $\bigcap_n A_n$
- ⟨1⟩3. ASSUME:  
for a contradiction  $D \subseteq C$  is connected and has more than one point.
- ⟨1⟩4. LET:  $x, y \in D$  with  $x < y$
- ⟨1⟩5. PICK  $n$  such that  $|x - y| > 1/3^n$
- ⟨1⟩6.  $A_n$  is a sequence of disjoint intervals of length  $1/3^n$
- ⟨1⟩7.  $x$  and  $y$  are in two different intervals out of the intervals that make up  $A_n$
- ⟨1⟩8. There exists  $z$  with  $x < z < y$  such that  $z \notin A_n$
- ⟨1⟩9.  $(-\infty, z) \cap D$  and  $(z, +\infty) \cap D$  form a separation of  $D$ .

□

## 9.33 Paths and Path Connectedness

**Definition 9.234** (Path). Let  $X$  be a topological space and  $a, b \in X$ . A *path* from  $a$  to  $b$  is a continuous function  $p : [0, 1] \rightarrow X$  such that  $p(0) = a$  and  $p(1) = b$ .

**Definition 9.235** (Path Connected). A topological space is *path connected* if and only if there exists a path between any two points.

**Proposition 9.236.** *Every path connected space is connected.*

PROOF:

- ⟨1⟩1. LET:  $X$  be a path connected space.
- ⟨1⟩2. ASSUME: for a contradiction  $C$  and  $D$  form a separation of  $X$ .
- ⟨1⟩3. PICK  $a \in C$  and  $b \in D$ .
- ⟨1⟩4. PICK a path  $p : [0, 1] \rightarrow X$  from  $a$  to  $b$ .
- ⟨1⟩5.  $p^{-1}(C)$  and  $p^{-1}(D)$  form a separation of  $[0, 1]$ .
- ⟨1⟩6. Q.E.D.

PROOF: This contradicts Corollary 9.225.2.

□

An example that shows the converse does not hold:

**Example 9.237.** The ordered square is not path connected.

PROOF:

- (1)1. ASSUME: for a contradiction  $p : [0, 1] \rightarrow I_o^2$  is a path from  $(0, 0)$  to  $(1, 1)$ .  
 (1)2.  $p$  is surjective.  
 PROOF: By the Intermediate Value Theorem.  
 (1)3. For  $x \in [0, 1]$ , PICK a rational  $q_x \in p^{-1}((x, 0), (x, 1))$   
 PROOF: Since  $p^{-1}((x, 0), (x, 1))$  is open and nonempty by (1)2.  
 (1)4. For  $x, y \in [0, 1]$ , if  $x \neq y$  then  $q_x \neq q_y$   
 PROOF: We have  $p(q_x) \neq p(q_y)$  because  $((x, 0), (x, 1))$  and  $((y, 0), (y, 1))$  are disjoint.  
 (1)5.  $\{q_x \mid x \in [0, 1]\}$  is an uncountable set of rationals.  
 (1)6. Q.E.D.  
 PROOF: This contradicts the fact that the rationals are countable.

□

**Proposition 9.238.** *The continuous image of a path connected space is path connected.*

PROOF:

- (1)1. LET:  $X$  be a path connected space,  $Y$  a topological space, and  $f : X \rightarrow Y$  be continuous and surjective.  
 (1)2. LET:  $a, b \in Y$   
 (1)3. PICK  $c, d \in X$  with  $f(c) = a$  and  $f(d) = b$   
 (1)4. PICK a path  $p : [0, 1] \rightarrow X$  from  $c$  to  $d$ .  
 (1)5.  $f \circ p$  is a path from  $a$  to  $b$  in  $Y$ .

□

**Proposition 9.239 (AC).** *The product of a family of path-connected spaces is path-connected.*

PROOF:

- (1)1. LET:  $\{X_\alpha\}_{\alpha \in J}$  be a family of path-connected spaces.  
 (1)2. LET:  $a, b \in \prod_{\alpha \in J} X_\alpha$   
 (1)3. For  $\alpha \in J$ , PICK a path  $p_\alpha : [0, 1] \rightarrow X_\alpha$  from  $a_\alpha$  to  $b_\alpha$   
 PROOF: Using the Axiom of Choice.  
 (1)4. Define  $p : [0, 1] \rightarrow \prod_{\alpha \in J} X_\alpha$  by  $p(t)_\alpha = p_\alpha(t)$   
 (1)5.  $p$  is a path from  $a$  to  $b$ .

PROOF: Theorem 9.123.

□

**Proposition 9.240.** *The continuous image of a path-connected space is path-connected.*

PROOF:

- (1)1. LET:  $f : X \rightarrow Y$  be continuous and surjective where  $X$  is path-connected.  
 (1)2. LET:  $a, b \in Y$   
 (1)3. PICK  $a', b' \in X$  with  $f(a') = a$  and  $f(b') = b$ .  
 (1)4. PICK a path  $p : [0, 1] \rightarrow X$  from  $a'$  to  $b'$ .  
 (1)5.  $f \circ p$  is a path from  $a$  to  $b$ .

□

**Proposition 9.241.** *Let  $X$  be a topological space. The union of a set of path-connected subspaces of  $X$  that have a point in common is path-connected.*

PROOF:

- ⟨1⟩1. LET:  $\mathcal{A}$  be a set of path-connected subspaces of  $X$  with the point  $a$  in common.
  - ⟨1⟩2. LET:  $b, c \in \bigcup \mathcal{A}$
  - ⟨1⟩3. PICK  $B, C \in \mathcal{A}$  with  $b \in B$  and  $c \in C$ .
  - ⟨1⟩4. PICK a path  $p$  in  $B$  from  $b$  to  $a$ .
  - ⟨1⟩5. PICK a path  $q$  in  $C$  from  $a$  to  $c$ .
  - ⟨1⟩6. The concatenation of  $p$  and  $q$  is a path from  $b$  to  $c$  in  $\bigcup \mathcal{A}$ .
- 

**Proposition 9.242.** *Let  $A \subseteq \mathbb{R}^2$  be countable. Then  $\mathbb{R}^2 \setminus A$  is path-connected.*

PROOF:

- ⟨1⟩1. LET:  $a, b \in \mathbb{R}^2 \setminus A$
  - ⟨1⟩2. PICK a line  $l$  in  $\mathbb{R}^2$  with  $a$  on one side and  $b$  on the other.
  - ⟨1⟩3. For every point  $x$  on  $l$ ,  
LET:  $p_x$  be the path in  $\mathbb{R}^2$  consisting of a line from  $a$  to  $x$  then a line from  $x$  to  $b$
  - ⟨1⟩4. For  $x \neq y$  we have  $p_x$  and  $p_y$  have no points in common except  $a$  and  $b$
  - ⟨1⟩5. There are only countably many  $x$  such that a point of  $A$  lies on  $p_x$ .
  - ⟨1⟩6. There exists  $x$  such that  $p_x$  is a path from  $a$  to  $b$  in  $\mathbb{R}^2 \setminus A$ .
- 

**Proposition 9.243.** *Every open connected subspace of  $\mathbb{R}^2$  is path-connected.*

PROOF:

- ⟨1⟩1. LET:  $U$  be an open connected subspace of  $\mathbb{R}^2$ .
- ⟨1⟩2. For all  $x_0 \in U$ ,  
LET:  $PC(x_0) = \{y \in U \mid \text{there exists a path from } x_0 \text{ to } y\}$
- ⟨1⟩3. For all  $x_0 \in U$ , the set  $PC(x_0)$  is open and closed in  $U$ .
  - ⟨2⟩1. LET:  $x_0 \in U$
  - ⟨2⟩2.  $PC(x_0)$  is open in  $U$ 
    - ⟨3⟩1. LET:  $y \in PC(x_0)$
    - ⟨3⟩2. PICK  $\epsilon > 0$  such that  $B(y, \epsilon) \subseteq U$   
PROOF: Since  $U$  is open.
    - ⟨3⟩3.  $B(y, \epsilon) \subseteq PC(x_0)$   
PROOF: For all  $z \in B(y, \epsilon)$ , pick a path from  $x_0$  to  $y$  then concatenate the straight line from  $y$  to  $z$ .
- ⟨2⟩3.  $PC(x_0)$  is closed in  $U$ 
  - ⟨3⟩1. LET:  $y \in U$  be a limit point of  $PC(x_0)$
  - ⟨3⟩2. PICK  $\epsilon > 0$  such that  $B(y, \epsilon) \subseteq U$
  - ⟨3⟩3. PICK  $z \in PC(x_0) \cap B(y, \epsilon)$
  - ⟨3⟩4.  $y \in PC(x_0)$   
PROOF: Pick a path from  $x_0$  to  $z$  then concatenate the straight line from  $z$  to  $y$ .

⟨1⟩4.  $PC(x_0) = U$

PROOF: Proposition 9.205.

□

**Example 9.244.** If  $A$  is a connected subspace of  $X$ , then  $A^\circ$  is not necessarily connected.

Take two closed circles in  $\mathbb{R}^2$  that touch at one point. The interior of this space is two open circles, and these two circles form a separation.

**Example 9.245.** If  $A$  is a connected subspace of  $X$  then  $\partial A$  is not necessarily connected.

We have  $[0, 1]$  is connected but  $\partial[0, 1] = \{0, 1\}$  is not.

**Example 9.246.** If  $A$  is a subspace of  $X$  and  $A^\circ$  and  $\partial A$  are connected, then  $A$  is not necessarily connected.

We have  $\mathbb{Q}^\circ = \emptyset$  and  $\partial\mathbb{Q} = \mathbb{R}$  are connected but  $\mathbb{Q}$  is not connected.

**Example 9.247.** The space  $\mathbb{R}_K$  is not path connected.

PROOF:

⟨1⟩1. ASSUME: for a contradiction  $p : [0, 1] \rightarrow \mathbb{R}_K$  was a path from 0 to 1.

⟨1⟩2.  $p([0, 1])$  as a subspace of  $\mathbb{R}_K$  is compact.

PROOF: Theorem 9.305.

⟨1⟩3.  $p([0, 1])$  as a subspace of  $\mathbb{R}_K$  is connected.

PROOF: Theorem 9.215.

⟨1⟩4.  $p([0, 1])$  is connected as a subspace of  $\mathbb{R}$ .

PROOF: Theorem 9.215 as the identity map is continuous as a map  $\mathbb{R}_K \rightarrow \mathbb{R}$ .

⟨1⟩5.  $p([0, 1])$  is convex.

⟨2⟩1. LET:  $a, b \in p([0, 1])$  and  $a < c < b$

⟨2⟩2. ASSUME: for a contradiction  $c \notin p([0, 1])$

⟨2⟩3.  $(-\infty, c) \cap p([0, 1])$  and  $(c, +\infty) \cap p([0, 1])$  form a separation of  $p([0, 1])$  as a subspace of  $\mathbb{R}$ .

⟨2⟩4. Q.E.D.

PROOF: This contradicts ⟨1⟩4.

⟨1⟩6.  $[0, 1] \subseteq p([0, 1])$

⟨1⟩7.  $[0, 1]$  as a subspace of  $\mathbb{R}_K$  is compact.

PROOF: By Proposition 9.304 and ⟨1⟩2.

⟨1⟩8. Q.E.D.

PROOF: This contradicts Example 9.327.

□

## 9.34 The Topologist's Sine Curve

**Definition 9.248** (The Topologist's Sine Curve). Let  $S = \{(x, \sin 1/x) \mid 0 < x \leq 1\}$ , The *topologist's sine curve* is the closure  $\overline{S}$  of  $S$  in  $\mathbb{R}^2$ .

**Proposition 9.249.** The topologist's sine curve is connected.

PROOF:

<1>1. LET:  $S = \{(x, \sin 1/x) \mid 0 < x \leq 1\}$

<1>2.  $S$  is connected.

PROOF: Theorem 9.215.

<1>3.  $\bar{S}$  is connected.

PROOF: Theorem 9.214.

□

**Proposition 9.250.** *The topologist's sine curve is  $\{(x, \sin 1/x) \mid 0 < x \leq 1\} \cup (\{0\} \times [-1, 1])$ .*

PROOF: Sketch proof: Given a point  $(0, y)$  with  $-1 \leq y \leq 1$ , pick  $a$  such that  $\sin a = y$ . Then  $((1/a, y), (1/(a + 2\pi), y), (1/(a + 4\pi), y), \dots)$  is a sequence in  $S$  that converges to  $(0, y)$ .

Conversely, let  $(x, y)$  be any point not in  $S \cup (\{0\} \times [-1, 1])$ . If  $x < 0$  or  $y > 1$  or  $y < -1$  then we can easily find a neighbourhood that does not intersect  $S \cup (\{0\} \times [-1, 1])$ . If  $x > 0$  and  $-1 \leq y \leq 1$ , then we have  $y \neq \sin 1/x$ . Hence pick a neighbourhood that does not intersect  $S$ .

**Proposition 9.251.** *Every closed subset of  $\mathbb{R}$  that is bounded above has a greatest element.*

PROOF: It has a supremum, which is a limit point of the set and hence an element. □

**Proposition 9.252 (CC).** *The topologist's sine curve is not path connected.*

PROOF:

<1>1. ASSUME: For a contradiction  $p : [0, 1] \rightarrow \bar{S}$  is a path from  $(0, 0)$  to  $(1, \sin 1)$ .

<1>2.  $\{t \in [0, 1] \mid p(t) \in \{0\} \times [-1, 1]\}$  is closed.

PROOF: Since  $p$  is continuous and  $\{0\} \times [-1, 1]$  is closed.

<1>3. LET:  $b$  be the largest number in  $[0, 1]$  such that  $p(b) \in \{0\} \times [-1, 1]$ .

PROOF: Proposition 9.251.

<1>4. LET:  $x : [b, 1] \rightarrow \bar{S}$  be the function  $\pi_1 \circ p$

<1>5. LET:  $y : [b, 1] \rightarrow \bar{S}$  be the function  $\pi_2 \circ p$

<1>6. PICK a sequence  $t_n$  in  $(b, 1]$  such that  $t_n \rightarrow b$  and  $y(t_n) = (-1)^n$  for all  $n$

<2>1. LET:  $n \geq 1$

<2>2. PICK  $u$  with  $0 < u < x(1/n)$  and  $\sin(1/u) = (-1)^n$

<2>3. PICK  $t_n$  with  $b < t_n < 1/n$  and  $x(t_n) = u$

PROOF: By the Intermediate Value Theorem

<1>7. Q.E.D.

PROOF: This contradicts Proposition 9.96 since  $y$  is continuous and  $y(t_n)$  does not converge.

□

**Corollary 9.252.1.** *The closure of a path-connected subspace of a space is not necessarily path-connected.*

## 9.35 The Long Line

**Definition 9.253** (The Long Line). The *long line* is the space  $\omega_1 \times [0, 1)$  in the dictionary order under the order topology, where  $\omega_1$  is the first uncountable ordinal.

**Lemma 9.254.** For any ordinal  $\alpha$  with  $0 < \alpha < \omega_1$  we have  $[(0, 0), (\alpha, 0)) \cong [0, 1)$

$\langle 1 \rangle 1$ .  $[(0, 0), (1, 0)) \cong [0, 1)$

PROOF: The map  $\pi_2$  is a homeomorphism.

$\langle 1 \rangle 2$ . If  $[(0, 0), (\alpha, 0)) \cong [0, 1)$  then  $[(0, 0), (\alpha + 1, 0)) \cong [0, 1)$

PROOF: Proposition 4.22.

$\langle 1 \rangle 3$ . If  $\lambda$  is a limit ordinal with  $\lambda < \omega_1$  and  $[(0, 0), (\alpha, 0)) \cong [0, 1)$  for all  $\alpha$  with  $0 < \alpha < \lambda$  then  $[(0, 0), (\lambda, 0)) \cong [0, 1)$

$\langle 2 \rangle 1$ . LET:  $\lambda$  be a limit ordinal  $< \omega_1$

$\langle 2 \rangle 2$ . ASSUME:  $[(0, 0), (\alpha, 0)) \cong [0, 1)$  for all  $\alpha$  with  $0 < \alpha < \lambda$

$\langle 2 \rangle 3$ . PICK a sequence of ordinals  $\alpha_0 < \alpha_1 < \dots$  with limit  $\lambda$

PROOF: Since  $\lambda$  is countable.

$\langle 2 \rangle 4$ .  $[(\alpha_i, 0), (\alpha_{i+1}, 0)) \cong [0, 1)$  for all  $i$

PROOF: Lemma 4.21.

$\langle 2 \rangle 5$ . Q.E.D.

PROOF: By Proposition 4.23.

$\langle 1 \rangle 4$ . Q.E.D.

PROOF: By transfinite induction.

**Proposition 9.255** (CC). The long line is path-connected.

PROOF:

$\langle 1 \rangle 1$ . LET:  $(\alpha, i), (\beta, j) \in \omega_1 \times [0, 1)$

$\langle 1 \rangle 2$ . ASSUME: without loss of generality  $(\alpha, i) < (\beta, j)$

$\langle 1 \rangle 3$ .  $[(0, 0), (\beta + 1, 0)) \cong [0, 1)$

PROOF: By Lemma 9.254

$\langle 1 \rangle 4$ .  $[(\alpha, i), (\beta, j)) \cong [0, 1)$

PROOF: Lemma 4.21.

$\langle 1 \rangle 5$ . PICK a homeomorphism  $q : [0, 1) \rightarrow [(\alpha, i), (\beta, j))$

$\langle 1 \rangle 6$ .  $q \cup \{(1, (\beta, j))\}$  is a path from  $(\alpha, i)$  to  $(\beta, j)$

□

**Proposition 9.256.** Every point in the long line has a neighbourhood homeomorphic to an interval in  $\mathbb{R}$ .

PROOF: For any  $(\alpha, i)$  in the long line, the neighbourhood  $[(0, 0), (\alpha + 1, 0))$  satisfies the condition by Lemma 9.254.

**Proposition 9.257.** The long line  $L$  is not second countable.

PROOF:

$\langle 1 \rangle 1$ . LET:  $\mathcal{B}$  be a basis for  $L$ .

⟨1⟩2. For  $\alpha < \omega_1$ , PICK  $B_\alpha \in \mathcal{B}$  such that  $(\alpha, 1/2) \in B_\alpha \subseteq ((\alpha, 0), (\alpha + 1, 0))$

⟨1⟩3.  $\mathcal{B}$  is uncountable.

PROOF: The mapping  $\alpha \mapsto B_\alpha$  is an injection  $\omega_1 \rightarrow \mathcal{B}$ .

**Corollary 9.257.1.** *The long line cannot be imbedded into  $\mathbb{R}^n$  for any  $n$ .*

## 9.36 Components

**Proposition 9.258.** *Let  $X$  be a topological space. Define the relation  $\sim$  on  $X$  by  $a \sim b$  if and only if there exists a connected subspace  $A \subseteq X$  with  $a, b \in A$ . Then  $\sim$  is an equivalence relation on  $X$ .*

PROOF:

⟨1⟩1.  $\sim$  is reflexive.

PROOF: For any  $a \in X$  we have  $\{a\}$  is a connected subspace that contains  $a$ .

⟨1⟩2.  $\sim$  is symmetric.

PROOF: Trivial.

⟨1⟩3.  $\sim$  is transitive.

⟨2⟩1. LET:  $a, b, c \in X$

⟨2⟩2. ASSUME:  $a \sim b$  and  $b \sim c$

⟨2⟩3. PICK connected subspaces  $A$  and  $B$  with  $a, b \in A$  and  $b, c \in B$

⟨2⟩4.  $A \cup B$  is a connected subspace that contains  $a$  and  $c$

PROOF: Theorem 9.213.

□

**Definition 9.259** ((Connected) Component). Let  $X$  be a topological space. The *(connected) components* of  $X$  are the equivalence classes under the above  $\sim$ .

**Lemma 9.260.** *Let  $X$  be a topological space. If  $A \subseteq X$  is connected and nonempty then there exists a unique component  $C$  of  $X$  such that  $A \subseteq C$ .*

PROOF:

⟨1⟩1. PICK  $a \in A$

⟨1⟩2. LET:  $C$  be the  $\sim$ -equivalence class of  $a$ .

⟨1⟩3.  $A \subseteq C$

PROOF: For all  $x \in A$  we have  $x \sim a$ .

⟨1⟩4. If  $C'$  is a component and  $A \subseteq C'$  then  $C = C'$

PROOF: Since we have  $a \in C'$ .

□

**Theorem 9.261.** *Let  $X$  be a topological space. The components of  $X$  are connected disjoint subspaces of  $X$  whose union is  $X$  such that each nonempty connected subspace of  $X$  intersects only one of them.*

PROOF:

⟨1⟩1. Every component of  $X$  is connected.



PROOF: For  $a \in X$ , the  $\sim$ -equivalence class of  $a$  is  $\bigcup\{A \subseteq X \mid A \text{ is connected, } a \in A\}$  which is connected by Theorem 9.213.

$\langle 1 \rangle 2$ . The components form a partition of  $X$ .  
PROOF: Immediate from the definition.

$\langle 1 \rangle 3$ . Every nonempty connected subspace of  $X$  intersects a unique component of  $X$ .  
 $\langle 2 \rangle 1$ . LET:  $A \subseteq X$  be connected and nonempty.  
 $\langle 2 \rangle 2$ . LET:  $C$  be the component such that  $A \subseteq C$   
PROOF: Lemma 9.260.  
 $\langle 2 \rangle 3$ .  $A$  intersects  $C$   
 $\langle 2 \rangle 4$ . If  $A$  intersects the component  $C'$  then  $C' = C$   
 $\langle 3 \rangle 1$ . LET:  $C'$  be a component that intersects  $A$   
 $\langle 3 \rangle 2$ . PICK  $b \in A \cap C'$   
 $\langle 3 \rangle 3$ .  $A \subseteq C'$   
PROOF: For all  $x \in A$  we have  $x \sim b$ .  
 $\langle 3 \rangle 4$ .  $C = C'$   
PROOF: By uniqueness in  $\langle 2 \rangle 2$ .

□

**Proposition 9.262.** *Every component of a space is closed.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $X$  be a topological space and  $C$  a component of  $X$ .  
 $\langle 1 \rangle 2$ .  $\overline{C}$  is connected.  
PROOF: Theorem 9.214.  
 $\langle 1 \rangle 3$ .  $C = \overline{C}$   
PROOF: Lemma 9.212.  
 $\langle 1 \rangle 4$ .  $C$  is closed.  
PROOF: Lemma 9.37.

□

**Proposition 9.263.** *If a topological space has finitely many components then every component is open.*

PROOF: Each component is the complement of a finite union of closed sets. □

## 9.37 Path Components

**Proposition 9.264.** *Let  $X$  be a topological space. Define the relation  $\sim$  on  $X$  by:  $a \sim b$  if and only if there exists a path in  $X$  from  $a$  to  $b$ . Then  $\sim$  is an equivalence relation on  $X$ .*

PROOF:

$\langle 1 \rangle 1$ .  $\sim$  is reflexive.  
PROOF: For  $a \in X$ , the constant function  $[0, 1] \rightarrow X$  with value  $a$  is a path from  $a$  to  $a$ .  
 $\langle 1 \rangle 2$ .  $\sim$  is symmetric.

PROOF: If  $p : [0, 1] \rightarrow X$  is a path from  $a$  to  $b$ , then  $\lambda t.p(1-t)$  is a path from  $b$  to  $a$ .

$\langle 1 \rangle 3$ .  $\sim$  is transitive.

PROOF: Concatenate paths.

□

**Definition 9.265** (Path Component). Let  $X$  be a topological space. The *path components* of  $X$  are the equivalence relations under  $\sim$ .

**Theorem 9.266.** *The path components of  $X$  are path-connected disjoint subspaces of  $X$  whose union is  $X$  such that every nonempty path-connected subspace of  $X$  intersects exactly one path component.*

PROOF:

$\langle 1 \rangle 1$ . Every path component is path-connected.

PROOF: If  $a$  and  $b$  are in the same path component then  $a \sim b$ , i.e. there exists a path from  $a$  to  $b$ .

$\langle 1 \rangle 2$ . The path components are disjoint and their union is  $X$ .

PROOF: Immediate from the definition.

$\langle 1 \rangle 3$ . Every non-empty path-connected subspace of  $X$  intersects exactly one path component.

$\langle 2 \rangle 1$ . LET:  $A$  be a nonempty path-connected subspace of  $X$ .

$\langle 2 \rangle 2$ . PICK  $a \in A$

$\langle 2 \rangle 3$ .  $A$  intersects the  $\sim$ -equivalence class of  $a$ .

$\langle 2 \rangle 4$ . LET:  $C$  be any path component that intersects  $A$ .

$\langle 2 \rangle 5$ . PICK  $b \in A \cap C$

$\langle 2 \rangle 6$ .  $a \sim b$

PROOF: Since  $A$  is path-connected.

$\langle 2 \rangle 7$ .  $C$  is the  $\sim$ -equivalence class of  $a$ .

□

**Proposition 9.267.** *Every path component is included in a component.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $X$  be a topological space and  $C$  a path component of  $X$ .

$\langle 1 \rangle 2$ .  $C$  is path-connected.

PROOF: Theorem 9.266.

$\langle 1 \rangle 3$ .  $C$  is connected.

PROOF: Proposition 9.236.

$\langle 1 \rangle 4$ .  $C$  is included in a component.

PROOF: Lemma 9.260.

□

## 9.38 Local Connectedness

**Definition 9.268** (Locally Connected). Let  $X$  be a topological space and  $a \in X$ . Then  $X$  is *locally connected* at  $a$  if and only if every neighbourhood of  $a$

includes a connected neighbourhood of  $a$ .

The space  $X$  is *locally connected* if and only if it is locally connected at every point.

**Example 9.269.** The real line is both connected and locally connected.

**Example 9.270.** The space  $\mathbb{R} \setminus \{0\}$  is disconnected but locally connected.

**Example 9.271.** The topologist's sine curve is connected but not locally connected.

**Example 9.272.** The rationals  $\mathbb{Q}$  are neither connected nor locally connected.

**Theorem 9.273.** *A topological space  $X$  is locally connected if and only if, for every open set  $U$  in  $X$ , every component of  $U$  is open in  $X$ .*

PROOF:

(1)1. If  $X$  is locally connected then, for every open set  $U$  in  $X$ , every component of  $U$  is open in  $X$ .

(2)1. ASSUME:  $X$  is locally connected.

(2)2. LET:  $U$  be open in  $X$ .

(2)3. LET:  $C$  be a component of  $U$ .

(2)4. LET:  $a \in C$

(2)5. LET:  $V$  be a connected neighbourhood of  $a$  such that  $V \subseteq U$

(2)6.  $V \subseteq C$

PROOF: Lemma 9.260.

(2)7. Q.E.D.

PROOF: Lemma 9.16.

(1)2. If, for every open set  $U$  in  $X$ , every component of  $U$  is open in  $X$ , then  $X$  is locally connected.

(2)1. ASSUME: for every open set  $U$  in  $X$ , every component of  $U$  is open in  $X$ .

(2)2. LET:  $a \in X$

(2)3. LET:  $U$  be a neighbourhood of  $a$

(2)4. The component of  $U$  that contains  $a$  is a connected neighbourhood of  $a$  included in  $U$ .

□

**Example 9.274.** The ordered square is locally connected.

PROOF: Every neighbourhood of a point includes an interval around that point, which is connected by Theorem 9.225.

**Example 9.275.** Let  $X$  be the set of all rational points on the line segment  $[0, 1] \times \{0\}$ , and  $Y$  the set of all rational points on the line segment  $[0, 1] \times \{1\}$ . Let  $A$  be the space consisting of all line segments joining the point  $(0, 1)$  to a point of  $X$ , and all line segments joining the point  $(1, 0)$  to a point of  $Y$ . Then  $A$  is path-connected but is not locally connected at any point,

**Proposition 9.276.** *Let  $X$  and  $Y$  be topological spaces and  $p : X \twoheadrightarrow Y$  be a quotient map. If  $X$  is locally connected then so is  $Y$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $U$  be an open set in  $Y$ .

$\langle 1 \rangle 2$ . LET:  $C$  be a component of  $U$ .

$\langle 1 \rangle 3$ .  $p^{-1}(C)$  is a union of components of  $p^{-1}(U)$

$\langle 2 \rangle 1$ . LET:  $x \in p^{-1}(C)$

$\langle 2 \rangle 2$ . LET:  $D$  be the component of  $p^{-1}(U)$  that contains  $x$ .

$\langle 2 \rangle 3$ .  $p(D)$  is connected.

PROOF: Theorem 9.215.

$\langle 2 \rangle 4$ .  $p(D) \subseteq C$ .

PROOF: From  $\langle 1 \rangle 2$  since  $p(x) \in p(D) \cap C$  ( $\langle 2 \rangle 1$ ,  $\langle 2 \rangle 2$ ).

$\langle 2 \rangle 5$ .  $D \subseteq p^{-1}(C)$

$\langle 1 \rangle 4$ .  $p^{-1}(C)$  is open in  $p^{-1}(U)$

PROOF: Theorem 9.273.

$\langle 1 \rangle 5$ .  $C$  is open in  $U$

PROOF: Since the restriction of  $p$  to  $p : p^{-1}(U) \twoheadrightarrow U$  is a quotient map by Proposition 9.188.

$\langle 1 \rangle 6$ . Q.E.D.

PROOF: Theorem 9.273.

□

### 9.39 Local Path Connectedness

**Definition 9.277** (Locally Path-Connected). Let  $X$  be a topological space and  $a \in X$ . Then  $X$  is *locally path-connected* at  $a$  if and only if every neighbourhood of  $a$  includes a path-connected neighbourhood of  $a$ .

The space  $X$  is *locally path-connected* if and only if it is locally path-connected at every point.

**Theorem 9.278.** *A topological space  $X$  is locally path-connected if and only if, for every open set  $U$  in  $X$ , every path component of  $U$  is open in  $X$ .*

PROOF:

$\langle 1 \rangle 1$ . If  $X$  is locally path-connected then, for every open set  $U$  in  $X$ , every path component of  $U$  is open in  $X$ .

$\langle 2 \rangle 1$ . ASSUME:  $X$  is locally path-connected.

$\langle 2 \rangle 2$ . LET:  $U$  be open in  $X$ .

$\langle 2 \rangle 3$ . LET:  $C$  be a path component of  $U$ .

$\langle 2 \rangle 4$ . LET:  $a \in C$

$\langle 2 \rangle 5$ . LET:  $V$  be a path-connected neighbourhood of  $a$  such that  $V \subseteq U$

$\langle 2 \rangle 6$ .  $V \subseteq C$

PROOF: Lemma 9.260.

$\langle 2 \rangle 7$ . Q.E.D.

PROOF: Lemma 9.16.

- ⟨1⟩2. If, for every open set  $U$  in  $X$ , every component of  $U$  is open in  $X$ , then  $X$  is locally connected.
- ⟨2⟩1. ASSUME: for every open set  $U$  in  $X$ , every component of  $U$  is open in  $X$ .
- ⟨2⟩2. LET:  $a \in X$
- ⟨2⟩3. LET:  $U$  be a neighbourhood of  $a$
- ⟨2⟩4. The component of  $U$  that contains  $a$  is a connected neighbourhood of  $a$  included in  $U$ .

□

**Theorem 9.279.** *If a space is locally path connected then its components and its path components are the same.*

PROOF:

- ⟨1⟩1. LET:  $X$  be a locally path connected space.
- ⟨1⟩2. LET:  $C$  be a component of  $X$ .
- ⟨1⟩3. LET:  $x \in C$
- ⟨1⟩4. LET:  $P$  be the path component of  $x$   
PROVE:  $P = C$
- ⟨1⟩5.  $P \subseteq C$   
PROOF: Proposition 9.267.
- ⟨1⟩6. LET:  $Q$  be the union of the other path components included in  $C$
- ⟨1⟩7.  $C = P \cup Q$   
PROOF: Proposition 9.267.
- ⟨1⟩8.  $P$  and  $Q$  are open in  $C$   
⟨2⟩1.  $C$  is open.  
PROOF: Theorem 9.273.
- ⟨2⟩2. Q.E.D.  
PROOF: Theorem 9.278.
- ⟨1⟩9.  $Q = \emptyset$   
PROOF: Otherwise  $P$  and  $Q$  would form a separation of  $C$ .

□

**Example 9.280.** The ordered square is not locally path connected, since it is connected but not path connected.

**Proposition 9.281.** *Let  $X$  be a locally path-connected space. Then every connected open subspace of  $X$  is path-connected.*

PROOF:

- ⟨1⟩1. LET:  $U$  be a connected open subspace of  $X$ .
- ⟨1⟩2. LET:  $P$  be a path component of  $U$ .
- ⟨1⟩3. LET:  $Q$  be the union of the other path components of  $U$ .
- ⟨1⟩4.  $P$  and  $Q$  are open in  $U$ .  
PROOF: Theorem 9.278.
- ⟨1⟩5.  $Q = \emptyset$   
PROOF: Otherwise  $P$  and  $Q$  form a separation of  $U$ .

□

## 9.40 Weak Local Connectedness

**Definition 9.282** (Weakly Locally Connected). Let  $X$  be a topological space and  $a \in X$ . Then  $X$  is *weakly locally connected* at  $a$  if and only if every neighbourhood of  $a$  includes a connected subspace that includes a neighbourhood of  $a$ .

**Proposition 9.283.** *Let  $X$  be a topological space. If  $X$  is weakly locally connected at every point then  $X$  is locally connected.*

PROOF:

$\langle 1 \rangle 1$ . ASSUME:  $X$  is weakly locally connected at every point.

$\langle 1 \rangle 2$ . LET:  $U$  be open in  $X$ .

$\langle 1 \rangle 3$ . LET:  $C$  be a component of  $U$ .

$\langle 1 \rangle 4$ .  $C$  is open in  $X$ .

$\langle 2 \rangle 1$ . LET:  $x \in C$

$\langle 2 \rangle 2$ . PICK a connected subspace  $D$  of  $U$  that includes a neighbourhood  $V$  of  $x$ .

$\langle 2 \rangle 3$ .  $D \subseteq C$

PROOF: Lemma 9.260.

$\langle 2 \rangle 4$ .  $x \in V \subseteq C$

$\langle 2 \rangle 5$ . Q.E.D.

PROOF: Lemma 9.16.

$\langle 1 \rangle 5$ . Q.E.D.

PROOF: Theorem 9.273.

□

**Example 9.284.** The 'infinite broom' (Munkres p.163) is an example of a space that is weakly locally connected at a point  $p$  but not locally connected at  $p$ .

## 9.41 Quasicomponents

**Proposition 9.285.** *Let  $X$  be a topological space. Define  $\sim$  on  $X$  by  $x \sim y$  if and only if there exists no separation  $U$  and  $V$  of  $X$  such that  $x \in U$  and  $y \in V$ . Then  $\sim$  is an equivalence relation on  $X$ .*

PROOF:

$\langle 1 \rangle 1$ .  $\sim$  is reflexive.

PROOF: Since the two sets that make up a separation are disjoint.

$\langle 1 \rangle 2$ .  $\sim$  is symmetric.

PROOF: Immediate from the definition.

$\langle 1 \rangle 3$ .  $\sim$  is transitive.

$\langle 2 \rangle 1$ . ASSUME:  $x \sim y$  and  $y \sim z$

$\langle 2 \rangle 2$ . ASSUME: for a contradiction there is a separation  $U$  and  $V$  of  $X$  with  $x \in U$  and  $z \in V$

$\langle 2 \rangle 3$ .  $y \in U$  or  $y \in V$

$\langle 2 \rangle 4$ . Q.E.D.

PROOF: Either case contradicts  $\langle 2 \rangle 1$ .

□

**Definition 9.286** (Quasicomponents). For  $X$  a topological space, the *quasi-components* of  $X$  are the equivalence classes under  $\sim$ .

**Proposition 9.287.** *Let  $X$  be a topological space. Then every component of  $X$  is included in a quasicomponent of  $X$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $C$  be a component of  $X$ .

$\langle 1 \rangle 2$ . LET:  $x, y \in C$

PROVE:  $x \sim y$

$\langle 1 \rangle 3$ . ASSUME: for a contradiction there exists a separation  $U$  and  $V$  of  $X$  with  $x \in U$  and  $y \in V$

$\langle 1 \rangle 4$ .  $C \cap U$  and  $C \cap V$  form a separation of  $C$ .

$\langle 1 \rangle 5$ . Q.E.D.

PROOF: This contradicts  $\langle 1 \rangle 1$ .

**Proposition 9.288.** *In a locally connected space, the components and the quasicomponents are the same.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $X$  be a locally connected space and  $Q$  a quasicomponent of  $X$ .

$\langle 1 \rangle 2$ . PICK a component  $C$  of  $X$  such that  $C \subseteq Q$

$\langle 1 \rangle 3$ . LET:  $D$  be the union of the components of  $X$

$\langle 1 \rangle 4$ .  $C$  and  $D$  are open in  $X$ .

PROOF: Theorem 9.273.

$\langle 1 \rangle 5$ .  $D$  cannot contain any points of  $Q$ .

PROOF: If it did, then  $C$  and  $D$  would form a separation of  $X$  and there would be points  $x, y \in Q$  with  $x \in C$  and  $y \in D$ .

$\langle 1 \rangle 6$ .  $C = Q$

□

## 9.42 Open Coverings

**Definition 9.289** (Open Covering). Let  $X$  be a topological space. An *open covering* of  $X$  is a covering of  $X$  whose elements are all open sets.

## 9.43 Lindelöf Spaces

**Definition 9.290** (Lindelöf Space). A topological space  $X$  is *Lindelöf* if and only if every open covering has a countable subcovering.

**Proposition 9.291.** *Let  $X$  be a topological space. Then  $X$  is Lindelöf if and only if every set of closed sets that has the countable intersection property has nonempty intersection.*

PROOF: The following are equivalent.

1.  $X$  is compact.
2. Every open covering of  $X$  has a countable subcovering.
3. For any set  $\mathcal{C}$  of closed sets, if  $\{X \setminus C \mid C \in \mathcal{C}\}$  covers  $X$  then there is a countable subset  $\mathcal{C}_0$  such that  $\{X \setminus C \mid C \in \mathcal{C}_0\}$  covers  $X$ .
4. For any set  $\mathcal{C}$  of closed sets, if  $\bigcap \mathcal{C} = \emptyset$  then there is a countable subset  $\mathcal{C}_0$  with empty intersection.
5. Any set of closed sets with the countable intersection property has nonempty intersection.

□

**Proposition 9.292 (CC).** *Let  $X$  be a topological space and  $\mathcal{B}$  a basis for the topology on  $X$ . Then the following are equivalent.*

1.  $X$  is Lindelöf.
2. Every open covering of  $X$  by elements of  $\mathcal{B}$  has a countable subcovering.

PROOF:

⟨1⟩1.  $1 \Rightarrow 2$

PROOF: Immediate from definitions.

⟨1⟩2.  $2 \Rightarrow 1$

⟨2⟩1. ASSUME: Every open covering of  $X$  by elements of  $\mathcal{B}$  has a countable subcovering.

⟨2⟩2. LET:  $\mathcal{U}$  be an open covering of  $X$ .

⟨2⟩3.  $\{B \in \mathcal{B} \mid \exists U \in \mathcal{U}. B \subseteq U\}$  covers  $X$ .

⟨2⟩4. PICK a finite subcovering  $\mathcal{B}_0$ .

⟨2⟩5. For  $B \in \mathcal{B}_0$ , PICK  $U_B \in \mathcal{U}$  such that  $B \subseteq U_B$ .

⟨2⟩6.  $\{U_B \mid B \in \mathcal{B}_0\}$  covers  $X$ .

□

## 9.44 The Second Countability Axiom

**Definition 9.293** (Second Countability Axiom). A topological space satisfies the *second countability axiom*, or is *second countable*, if and only if it has a countable basis.

**Example 9.294.** The space  $\mathbb{R}$  is second countable.

PROOF: The set  $\{(a, b) \mid a, b \in \mathbb{Q}\}$  is a basis. □

**Proposition 9.295.** *A subspace of a second countable space is second countable.*

PROOF: If  $\mathcal{B}$  is a countable basis for  $X$  and  $Y \subseteq X$  then  $\{B \cap Y \mid B \in \mathcal{B}\}$  is a countable basis for  $Y$ . □



**Proposition 9.296** (CC). *Every second countable space is Lindelöf.*

PROOF: From Proposition 9.292.

**Example 9.297** (CC). The space  $\mathbb{R}_l$  is Lindelöf.

$\langle 1 \rangle 1$ . LET:  $\mathcal{A}$  be a covering of  $\mathbb{R}_l$  by basic open sets of the form  $[a, b)$

$\langle 1 \rangle 2$ . LET:  $C = \bigcup \{(a, b) \mid [a, b) \in \mathcal{A}\}$

$\langle 1 \rangle 3$ .  $\mathbb{R} \setminus C$  is countable.

$\langle 2 \rangle 1$ . For every  $x \in \mathbb{R} \setminus C$ , PICK a rational  $q_x$  such that  $(x, q_x) \subseteq C$

$\langle 3 \rangle 1$ . LET:  $x \in \mathbb{R} \setminus C$

$\langle 3 \rangle 2$ . PICK  $b$  such that  $[x, b) \in \mathcal{A}$

$\langle 3 \rangle 3$ . PICK a rational  $q$  such that  $q \in (x, b)$

$\langle 2 \rangle 2$ . The mapping  $x \mapsto q_x$  is an injection  $\mathbb{R} \setminus C \rightarrow \mathbb{Q}$

$\langle 1 \rangle 4$ . PICK a countable  $\mathcal{A}' \subseteq \mathcal{A}$  that covers  $\mathbb{R} \setminus C$

$\langle 1 \rangle 5$ . Under the standard topology on  $\mathbb{R}$ ,  $C$  is second countable.

PROOF: Proposition 9.295.

$\langle 1 \rangle 6$ . PICK a countable  $\mathcal{A}'' \subseteq \mathcal{A}$  such that  $\{(a, b) \mid [a, b) \in \mathcal{A}''\}$  covers  $C$ .

PROOF: Proposition 9.292.

$\langle 1 \rangle 7$ .  $\mathcal{A}' \cup \mathcal{A}''$  covers  $\mathbb{R}_l$ .

□

**Example 9.298**. The product of two Lindelöf spaces is not necessarily Lindelöf.

We prove that the Sorgenfrey plane is not Lindelöf.

PROOF:

$\langle 1 \rangle 1$ . LET:  $L = \{(x, -x) \mid x \in \mathbb{R}\}$

$\langle 1 \rangle 2$ .  $L$  is closed in  $\mathbb{R}_l^2$

$\langle 1 \rangle 3$ . LET:  $\mathcal{U} = \{[a, b) \times [a, -d) \mid a, b, d \in \mathbb{R}\}$

$\langle 1 \rangle 4$ .  $\mathcal{U} \cup \{\mathbb{R} \setminus L\}$  covers  $\mathbb{R}_l^2$

$\langle 1 \rangle 5$ . Every element of  $\mathcal{U}$  intersects  $L$  at exactly one point.

$\langle 1 \rangle 6$ . No countable subset of  $\mathcal{U}$  covers  $\mathbb{R}_l^2$ .

□

## 9.45 Sequential Compactness

**Definition 9.299** (Sequentially Compact). A topological space is *sequentially compact* if and only if every sequence has a convergent subsequence.

## 9.46 Limit Point Compactness

**Definition 9.300** (Limit Point Compact Space). A topological space is *limit point compact* if and only if every infinite set has a limit point.

**Proposition 9.301**. *Every limit point compact  $T_1$  space is sequentially compact.*

PROOF:

- ⟨1⟩1. LET:  $X$  be a limit point compact  $T_1$  space.
  - ⟨1⟩2. LET:  $(x_n)$  be a sequence in  $X$ .
  - ⟨1⟩3. CASE:  $\{x_n \mid n \geq 1\}$  is finite.
    - ⟨2⟩1. PICK  $n$  such that  $x_n$  occurs infinitely often in the sequence  $(x_n)$
    - ⟨2⟩2. The subsequence consisting of all the terms equal to  $x_n$  is convergent.
  - ⟨1⟩4. CASE:  $\{x_n \mid n \geq 1\}$  is infinite.
    - ⟨2⟩1. PICK a limit point  $l$  for  $\{x_n \mid n \geq 1\}$
    - ⟨2⟩2. PICK an increasing sequence  $n_r$  with  $x_{n_r} \in B(x, 1/r)$  for all  $r$
    - PROOF: This is always possible by Theorem 9.155.
    - ⟨2⟩3.  $(x_{n_r})$  converges to  $l$ .
- 

## 9.47 Compact Spaces

**Definition 9.302** (Compact). A topological space is *compact* if and only if every open covering has a finite subcovering.

**Lemma 9.303.** *Let  $X$  be a topological space and  $Y$  a subspace of  $X$ . Then  $Y$  is compact if and only if every covering of  $Y$  by sets open in  $X$  has a finite subcovering.*

PROOF:

- ⟨1⟩1. If  $Y$  is compact then every covering of  $Y$  by sets open in  $X$  has a finite subcovering.
  - ⟨2⟩1. ASSUME:  $Y$  is compact.
  - ⟨2⟩2. LET:  $\mathcal{U}$  be a covering of  $Y$  by sets open in  $X$ .
  - ⟨2⟩3.  $\{U \cap Y \mid U \in \mathcal{U}\}$  is an open covering of  $Y$ .
  - ⟨2⟩4. PICK a finite subcovering  $\{U_1 \cap Y, \dots, U_n \cap Y\}$
  - ⟨2⟩5.  $\{U_1, \dots, U_n\}$  is a finite subcovering of  $\mathcal{U}$ .
- ⟨1⟩2. If every covering of  $Y$  by sets open in  $X$  has a finite subcovering then  $Y$  is compact.
  - ⟨2⟩1. LET:  $\mathcal{U}$  be an open covering of  $Y$ .
  - ⟨2⟩2. LET:  $\mathcal{V} = \{V \text{ open in } X \mid V \cap Y \in \mathcal{U}\}$ .
  - ⟨2⟩3.  $\mathcal{V}$  is a covering of  $Y$  by sets open in  $X$ .
  - ⟨2⟩4. PICK a finite subcovering  $\{V_1, \dots, V_n\}$
  - ⟨2⟩5.  $\{V_1 \cap Y, \dots, V_n \cap Y\}$  is a finite subcovering of  $\mathcal{U}$ .

□

**Proposition 9.304.** *Every closed subspace of a compact space is compact.*

PROOF:

- ⟨1⟩1. LET:  $X$  be a compact space and  $Y \subseteq X$  be closed.
- ⟨1⟩2. LET:  $\mathcal{U}$  be a covering of  $Y$  by sets open in  $X$ .
- ⟨1⟩3.  $\mathcal{U} \cup \{X \setminus Y\}$  is an open covering of  $X$ .
- ⟨1⟩4. PICK a finite subcovering  $\mathcal{U}_0$
- ⟨1⟩5.  $\mathcal{U}_0 \cap \mathcal{U}$  is a finite subset of  $\mathcal{U}$  that covers  $Y$ .

□

**Theorem 9.305.** *The continuous image of a compact space is compact.*

PROOF:

- ⟨1⟩1. LET:  $f : X \rightarrow Y$  be continuous and surjective.
- ⟨1⟩2. LET:  $\mathcal{V}$  be an open covering of  $Y$
- ⟨1⟩3.  $\{p^{-1}(V) \mid V \in \mathcal{V}\}$  is an open covering of  $X$ .
- ⟨1⟩4. PICK a finite subcovering  $\{p^{-1}(V_1), \dots, p^{-1}(V_n)\}$
- ⟨1⟩5.  $\{V_1, \dots, V_n\}$  covers  $Y$ .

□

**Theorem 9.306.** *Let  $A$  and  $B$  be compact subspaces of  $X$  and  $Y$  respectively. Let  $N$  be an open set in  $X \times Y$  that includes  $A \times B$ . Then there exist open sets  $U$  and  $V$  in  $X$  and  $Y$  respectively such that  $A \subseteq U$ ,  $B \subseteq V$  and  $U \times V \subseteq N$ .*

PROOF:

- ⟨1⟩1. For all  $x \in A$ , there exist neighbourhoods  $U$  of  $x$  and  $V$  of  $B$  such that  $U \times V \subseteq N$ .
- ⟨2⟩1. LET:  $x \in A$
- ⟨2⟩2. For all  $y \in B$ , there exist neighbourhoods  $U$  of  $x$  and  $V$  of  $y$  such that  $U \times V \subseteq N$
- ⟨2⟩3.  $\{V \text{ open in } Y \mid \exists \text{ neighbourhood } U \text{ of } x, U \times V \subseteq N\}$  covers  $B$ .
- ⟨2⟩4. PICK a finite subcover  $\{V_1, \dots, V_n\}$
- ⟨2⟩5. For  $i = 1, \dots, n$ , PICK a neighbourhood  $U_i$  of  $x$  such that  $U_i \times V_i \subseteq N$
- ⟨2⟩6. LET:  $U = U_1 \cap \dots \cap U_n$
- ⟨2⟩7. LET:  $V = V_1 \cup \dots \cup V_n$
- ⟨2⟩8.  $U$  is a neighbourhood of  $x$ .
- ⟨2⟩9.  $V$  is a neighbourhood of  $B$ .
- ⟨2⟩10.  $U \times V \subseteq N$
- ⟨1⟩2.  $\{U \text{ open in } X \mid \exists \text{ neighbourhood } V \text{ of } B, U \times V \subseteq N\}$  covers  $A$ .
- ⟨1⟩3. PICK a finite subcover  $\{U_1, \dots, U_n\}$
- ⟨1⟩4. For  $i = 1, \dots, n$ , PICK a neighbourhood  $V_i$  of  $B$  such that  $U_i \times V_i \subseteq N$
- ⟨1⟩5. LET:  $U = U_1 \cup \dots \cup U_n$
- ⟨1⟩6. LET:  $V = V_1 \cap \dots \cap V_n$
- ⟨1⟩7.  $U$  and  $V$  are open.
- ⟨1⟩8.  $A \subseteq U$
- ⟨1⟩9.  $B \subseteq V$
- ⟨1⟩10.  $U \times V \subseteq N$

□

**Corollary 9.306.1** (Tube Lemma). *Let  $X$  and  $Y$  be topological spaces with  $Y$  compact. Let  $a \in X$  and  $N$  be an open set in  $X \times Y$  that includes  $\{a\} \times Y$ . Then there exists a neighbourhood  $W$  of  $a$  such that  $N$  includes the tube  $W \times Y$ .*

**Theorem 9.307.** *Let  $X$  be a topological space. Then  $X$  is compact if and only if every set of closed sets that has the finite intersection property has nonempty intersection.*

PROOF: The following are equivalent.

1.  $X$  is compact.
2. Every open covering of  $X$  has a finite subcovering.
3. For any set  $\mathcal{C}$  of closed sets, if  $\{X \setminus C \mid C \in \mathcal{C}\}$  covers  $X$  then there is a finite subset  $\mathcal{C}_0$  such that  $\{X \setminus C \mid C \in \mathcal{C}_0\}$  covers  $X$ .
4. For any set  $\mathcal{C}$  of closed sets, if  $\bigcap \mathcal{C} = \emptyset$  then there is a finite subset  $\mathcal{C}_0$  with empty intersection.
5. Any set of closed sets with the finite intersection property has nonempty intersection.

□

**Corollary 9.307.1.** *Let  $X$  be a topological space and  $C_1 \supseteq C_2 \supseteq \cdots$  a nested sequence of nonempty closed sets. Then  $\bigcap_n C_n$  is nonempty.*

**Proposition 9.308.** *Let  $\mathcal{T}$  and  $\mathcal{T}'$  be two topologies on the same set  $X$  with  $\mathcal{T} \subseteq \mathcal{T}'$ . If  $\mathcal{T}'$  is compact then  $\mathcal{T}$  is compact.*

PROOF:

- ⟨1⟩1. LET:  $\mathcal{U} \subseteq \mathcal{T}$  cover  $X$
- ⟨1⟩2.  $\mathcal{U} \subseteq \mathcal{T}'$
- ⟨1⟩3. A finite subset of  $\mathcal{U}$  covers  $X$ .

□

**Corollary 9.308.1.** *If  $\mathcal{T}$  and  $\mathcal{T}'$  are two compact Hausdorff topologies on the same set  $X$ , then either  $\mathcal{T} = \mathcal{T}'$  or  $\mathcal{T}$  and  $\mathcal{T}'$  are incomparable.*

PROOF: From the Proposition and Proposition 9.169. □

**Example 9.309.** Any set under the finite complement topology is compact.

**Proposition 9.310.** *Let  $X$  be a topological space. A finite union of compact subspaces of  $X$  is compact.*

PROOF:

- ⟨1⟩1. LET:  $A$  and  $B$  be compact subspaces of  $X$ .
- ⟨1⟩2. LET:  $\mathcal{U}$  be a set of open sets in  $X$  that covers  $A \cup B$
- ⟨1⟩3. PICK a finite subset  $\mathcal{U}_1$  that covers  $A$ .

PROOF: Lemma 9.303.

- ⟨1⟩4. PICK a finite subset  $\mathcal{U}_2$  that covers  $B$ .

PROOF: Lemma 9.303.

- ⟨1⟩5.  $\mathcal{U}_1 \cup \mathcal{U}_2$  is a finite subset that covers  $A \cup B$ .

- ⟨1⟩6. Q.E.D.

PROOF: Lemma 9.303.

□

**Proposition 9.311.** *Let  $A$  and  $B$  be disjoint compact subspaces of the Hausdorff space  $X$ . Then there exist disjoint open sets  $U$  and  $V$  that include  $A$  and  $B$  respectively.*

PROOF: From Theorem 9.306 with  $N = X^2 \setminus \{(x, x) \mid x \in X\}$ .  $\square$

**Corollary 9.311.1.** *Every compact subspace of a Hausdorff space is closed.*

**Theorem 9.312.** *Let  $f : X \rightarrow Y$  be a bijective continuous function. If  $X$  is compact and  $Y$  is Hausdorff then  $f$  is a homeomorphism.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $C \subseteq X$  be closed.

$\langle 1 \rangle 2$ .  $C$  is compact.

PROOF: Proposition 9.304.

$\langle 1 \rangle 3$ .  $f(C)$  is compact.

PROOF: Theorem 9.305.

$\langle 1 \rangle 4$ .  $f(C)$  is closed.

PROOF: Corollary 9.311.1.

$\langle 1 \rangle 5$ . Q.E.D.

PROOF: Lemma 9.98.

$\square$

**Proposition 9.313.** *Let  $X$  be a compact space,  $Y$  a Hausdorff space, and  $f : X \rightarrow Y$  a continuous map. Then  $f$  is a closed map.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $C \subseteq X$  be closed.

$\langle 1 \rangle 2$ .  $C$  is compact.

PROOF: Proposition 9.304.

$\langle 1 \rangle 3$ .  $f(C)$  is compact.

PROOF: Theorem 9.305.

$\langle 1 \rangle 4$ .  $f(C)$  is closed.

PROOF: Corollary 9.311.1.

$\square$

**Proposition 9.314.** *If  $Y$  is compact then the projection  $\pi_1 : X \times Y \rightarrow X$  is a closed map.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $A \subseteq X \times Y$  be closed.

$\langle 1 \rangle 2$ . LET:  $x \in X \setminus \pi_1(A)$

$\langle 1 \rangle 3$ . PICK a neighbourhood  $U$  of  $x$  such that  $U \times Y \subseteq (X \times Y) \setminus A$

PROOF: By the Tube Lemma.

$\langle 1 \rangle 4$ .  $x \in U \subseteq X \setminus \pi_1(A)$

$\langle 1 \rangle 5$ . Q.E.D.

PROOF: So  $X \setminus \pi_1(A)$  is open by Lemma 9.16.

$\square$

**Proposition 9.315.** *Let  $X$  be a topological space and  $Y$  a Hausdorff space. Let  $f : X \rightarrow Y$  be continuous. Then the graph of  $f$  is closed in  $X \times Y$ .*

$\langle 1 \rangle 1$ . ASSUME:  $f$  is continuous.

- ⟨1⟩2. LET:  $(x, y) \in (X \times Y) \setminus G_f$
  - ⟨1⟩3. PICK disjoint neighbourhoods  $U$  and  $V$  of  $y$  and  $f(x)$  respectively.
  - ⟨1⟩4.  $f^{-1}(V) \times U$  is a neighbourhood of  $(x, y)$  disjoint from  $G_f$ .
- 

**Theorem 9.316.** *Let  $X$  be a topological space and  $Y$  a compact space. Let  $f : X \rightarrow Y$  be a function. If the graph of  $f$  is closed in  $X \times Y$  then  $f$  is continuous.*

PROOF:

- ⟨1⟩1. ASSUME:  $G_f$  is closed.
  - ⟨1⟩2. LET:  $x \in X$  and  $V$  be a neighbourhood of  $f(x)$ .
  - ⟨1⟩3.  $G_f \cap (X \times (Y \setminus V))$  is closed.
  - ⟨1⟩4.  $\pi_1(G_f \cap (X \times (Y \setminus V)))$  is closed.
  - PROOF: Proposition 9.314.
  - ⟨1⟩5. LET:  $U = X \setminus \pi_1(G_f \cap (X \times (Y \setminus V)))$
  - ⟨1⟩6.  $U$  is a neighbourhood of  $x$
  - ⟨1⟩7.  $f(U) \subseteq V$
- 

**Theorem 9.317.** *Let  $X$  be a compact topological space. Let  $(f_n : X \rightarrow \mathbb{R})$  be a monotone increasing sequence of continuous functions and  $f : X \rightarrow \mathbb{R}$  a continuous function. If  $(f_n)$  converges pointwise to  $f$ , then  $(f_n)$  converges uniformly to  $f$ .*

PROOF:

- ⟨1⟩1. LET:  $\epsilon > 0$
- ⟨1⟩2. For all  $x \in X$ , there exists  $N$  such that, for all  $n \geq N$ , we have  $|f_n(x) - f(x)| < \epsilon$
- ⟨1⟩3. For  $n \geq 1$ ,  
LET:  $U_n = \{x \in X \mid |f_n(x) - f(x)| < \epsilon\}$
- ⟨1⟩4. For  $n \geq 1$ , we have  $U_n$  is open in  $X$ .  
  - ⟨2⟩1. LET:  $x \in X$
  - ⟨2⟩2. LET:  $\delta = \epsilon - |f_n(x) - f(x)|$
  - ⟨2⟩3. PICK a neighbourhood  $U$  of  $x$  such that  $f(U) \subseteq B(f(x), \delta/2)$
  - ⟨2⟩4. PICK a neighbourhood  $V$  of  $x$  such that  $f_n(V) \subseteq B(f_n(x), \delta/2)$
  - ⟨2⟩5.  $f(U \cap V) \subseteq U_n$
- PROOF: For  $y \in U \cap V$  we have
 
$$\begin{aligned}
 |f_n(y) - f(y)| &\leq |f_n(y) - f_n(x)| + |f_n(x) - f(x)| + |f(x) - f(y)| \\
 &< \delta/2 + |f_n(x) - f(x)| + \delta/2 \\
 &= \epsilon
 \end{aligned}$$
- ⟨1⟩5.  $\{U_n \mid n \geq 1\}$  covers  $X$   
PROOF: From ⟨1⟩2
- ⟨1⟩6. PICK  $N$  such that  $X = U_N$   
  - ⟨2⟩1. PICK  $n_1, \dots, n_k$  such that  $U_{n_1}, \dots, U_{n_k}$  cover  $X$ .
  - ⟨2⟩2. LET:  $N = \max(n_1, \dots, n_k)$
  - ⟨2⟩3. For all  $i$  we have  $U_{n_i} \subseteq U_N$

PROOF: Since  $(f_n)$  is monotone increasing.

⟨2⟩4.  $X = U_N$

⟨1⟩7. For all  $x \in X$  and  $n \geq N$  we have  $|f_n(x) - f(x)| < \epsilon$

□

An example to show that we cannot remove the hypothesis that  $X$  is compact:

**Example 9.318.** Let  $X = (0, 1)$ ,  $f_n(x) = -x^n$  and  $f(x) = 0$  for  $x \in X$  and  $n \geq 1$ . Then  $f_n \rightarrow f$  pointwise and  $(f_n)$  is monotone increasing but the convergence is not uniform since, for all  $N \geq 1$ , there exists  $x \in (0, 1)$  such that  $-x^N < -1/2$ .

An example to show that we cannot remove the hypothesis that  $(f_n)$  is monotone increasing:

**Example 9.319.** Let  $X = [0, 1]$ ,  $f_n(x) = 1/(n^3(x - 1/n)^2 + 1)$  and  $f(x) = 0$  for  $x \in X$  and  $n \geq 1$ . Then  $X$  is compact and  $f_n \rightarrow f$  pointwise but the convergence is not uniform since, for all  $N \geq 1$ , there exists  $x \in [0, 1]$  such that  $f_N(x) = 1$ , namely  $x = 1/N$ .

**Theorem 9.320.** Let  $X$  be a compact Hausdorff space. Let  $\mathcal{A}$  be a chain of closed connected subsets of  $X$ . Then  $\bigcap \mathcal{A}$  is connected.

PROOF:

⟨1⟩1. ASSUME: for a contradiction  $C$  and  $D$  form a separation of  $\bigcap \mathcal{A}$ .

⟨1⟩2. PICK disjoint open sets  $U$  and  $V$  that include  $C$  and  $D$  respectively.

PROOF: Proposition 9.311.

⟨1⟩3.  $\{A \setminus (U \cup V) \mid A \in \mathcal{A}\}$  is a set of closed sets with the finite intersection property.

⟨2⟩1. For all  $A \in \mathcal{A}$  we have  $A \setminus (U \cup V)$  is closed.

⟨2⟩2. For all  $A_1, \dots, A_n \in \mathcal{A}$  we have  $(A_1 \cap \dots \cap A_n) \setminus (U \cup V)$  is nonempty.

PROOF:

⟨3⟩1. LET:  $A_1, \dots, A_n \in \mathcal{A}$

⟨3⟩2. ASSUME: without loss of generality  $A_1 \subseteq A_2, \dots, A_n$

PROOF: Since  $\mathcal{A}$  is a chain.

⟨3⟩3.  $A_1 \setminus (U \cup V)$  is nonempty

PROOF: Otherwise  $(A_1 \cap \dots \cap A_n \cap U)$  and  $(A_1 \cap \dots \cap A_n \cap V)$  would form a separation of  $A_n$ .

⟨1⟩4.  $\bigcap \mathcal{A} \setminus (U \cup V)$  is nonempty.

PROOF: Theorem 9.307.

⟨1⟩5. Q.E.D.

PROOF: This contradicts ⟨1⟩1 since  $\bigcap \mathcal{A} \setminus (U \cup V) = \bigcap \mathcal{A} \setminus (C \cup D)$ .

□

**Theorem 9.321** (Tychonoff Theorem (AC)). The product of a family of compact spaces is compact.

PROOF:

- (1)1. LET:  $\{X_\alpha\}_{\alpha \in J}$  be a family of compact spaces.  
 (1)2. LET:  $X = \prod_{\alpha \in J} X_\alpha$   
 (1)3. For any  $\mathcal{A} \subseteq \mathcal{P}X$ , we have  $\bigcap_{A \in \mathcal{A}} \overline{A} \neq \emptyset$   
 (2)1. LET:  $\mathcal{A} \subseteq \mathcal{P}X$   
 (2)2. PICK  $\mathcal{D} \supseteq \mathcal{A}$  that is maximal with respect to the finite intersection property.  
     PROVE:  $\bigcap_{D \in \mathcal{D}} \overline{D} \neq \emptyset$   
     PROOF: Lemma 3.66.  
 (2)3. For  $\alpha \in J$ , PICK  $x_\alpha \in X_\alpha$  such that  $x_\alpha \in \bigcap_{D \in \mathcal{D}} \overline{\pi_\alpha(D)}$   
     PROOF: Theorem 9.307 since  $\{\overline{\pi_\alpha(D)} \mid D \in \mathcal{D}\}$  is a set of closed sets in  $X_\alpha$  with the finite intersection property.  
 (2)4. LET:  $x = (x_\alpha)_{\alpha \in J}$   
     PROVE:  $x \in \bigcap_{D \in \mathcal{D}} \overline{D}$   
 (2)5. For any  $\beta \in J$  and neighbourhood  $U$  of  $x_\beta$  in  $X_\beta$ , we have  $\pi_\beta^{-1}(U)$  intersects every element of  $\mathcal{D}$   
     (3)1. LET:  $\beta \in J$   
     (3)2. LET:  $U$  be a neighbourhood of  $x_\beta$  in  $X_\beta$ .  
     (3)3. LET:  $D \in \mathcal{D}$   
     (3)4.  $x_\beta \in \overline{\pi_\beta(D)}$   
         PROOF: From (2)3  
     (3)5.  $U$  intersects  $\pi_\beta(D)$ .  
     (3)6.  $\pi_\beta^{-1}(U)$  intersects  $D$ .  
 (2)6. For any  $\beta \in J$  and neighbourhood  $U$  of  $x_\beta$  in  $X_\beta$ , we have  $\pi_\beta^{-1}(U) \in \mathcal{D}$   
     PROOF: Lemma 3.68.  
 (2)7. Every basic neighbourhood of  $x$  is an element of  $\mathcal{D}$   
     PROOF: Lemma 3.67.  
 (2)8. Every basic neighbourhood of  $x$  intersects every element of  $\mathcal{D}$   
     PROOF: Since  $\mathcal{D}$  satisfies the finite intersection property.  
 (2)9. For all  $D \in \mathcal{D}$  we have  $x \in \overline{D}$   
 (1)4. Q.E.D.  
     PROOF: Theorem 9.307.

□

**Lemma 9.322.** *Let  $X$  and  $Y$  be topological spaces. Let  $\mathcal{A}$  be a set of basis elements for the product topology on  $X \times Y$  such that no finite subset of  $\mathcal{A}$  covers  $X \times Y$ . If  $X$  is compact, then there exists  $x \in X$  such that no finite subset of  $\mathcal{A}$  covers the slice  $\{x\} \times Y$ .*

PROOF:

- (1)1. ASSUME: for every  $x \in X$ , there exists a finite subset of  $\mathcal{A}$  that covers  $\{x\} \times Y$   
     PROVE: A finite subset of  $\mathcal{A}$  covers  $X \times Y$   
 (1)2.  $\{U \mid \exists U_1 \times V_1, \dots, U_n \times V_n \in \mathcal{A}. U = U_1 \cap \dots \cap U_n, V_1 \cup \dots \cup V_n = Y\}$  covers  $X$   
 (1)3. PICK a finite subcover  $U_1, \dots, U_m$   
 (1)4. PICK  $U_{ij} \times V_{ij} \in \mathcal{A}$  such that, for every  $i$ , we have  $U_i = \bigcap_j U_{ij}$  and  $Y = \bigcup_j V_{ij}$



⟨1⟩5. The collection of all  $U_{ij} \times V_{ij}$  covers  $X \times Y$

□

**Theorem 9.323 (AC).** *Let  $X$  be a compact Hausdorff space. Then the quasi-components and the components of  $X$  are the same.*

PROOF:

⟨1⟩1. LET:  $x, y \in X$

⟨1⟩2. ASSUME:  $x$  and  $y$  are in the same quasicomponent.

PROVE:  $x$  and  $y$  are in the same component.

⟨1⟩3. LET:  $\mathcal{A}$  be the set of all closed subsets  $A$  of  $X$  such that  $x$  and  $y$  are in the same quasicomponent of  $A$ .

⟨1⟩4. For every chain  $\mathcal{B} \subseteq \mathcal{A}$  we have  $\bigcap \mathcal{B} \in \mathcal{A}$

⟨2⟩1. LET:  $\mathcal{B} \subseteq \mathcal{A}$  be a chain.

⟨2⟩2. ASSUME: for a contradiction  $U$  and  $V$  form a separation of  $\bigcap \mathcal{B}$  with  $x \in U$  and  $y \in V$

⟨2⟩3. PICK disjoint open sets  $U', V'$  in  $X$  such that  $U \subseteq U'$  and  $V \subseteq V'$

⟨2⟩4.  $\{B \setminus (U' \cup V') \mid B \in \mathcal{B}\}$  satisfies the finite intersection property.

⟨3⟩1. LET:  $B_1, \dots, B_n \in \mathcal{B}$

⟨3⟩2. ASSUME: without loss of generality  $B_1 \subseteq \dots \subseteq B_n$

PROOF: Since  $\mathcal{B}$  is a chain.

⟨3⟩3.  $\bigcap \{B_1 \setminus (U' \cup V'), \dots, B_n \setminus (U' \cup V')\} = B_1 \setminus (U' \cup V')$

⟨3⟩4.  $B_1 \setminus (U' \cup V')$  is nonempty

PROOF: Otherwise  $B_1 \cap U'$  and  $B_1 \cap V'$  would form a separation of  $B_1$ , contradicting the fact that  $x$  and  $y$  are in the same quasicomponent of  $B_1$ .

⟨2⟩5.  $\bigcap \mathcal{B} \setminus (U \cup V)$  is nonempty

PROOF: Theorem 9.307.

⟨2⟩6. Q.E.D.

PROOF: This contradicts ⟨2⟩2.

⟨1⟩5. PICK a minimal element  $D$  in  $\mathcal{A}$ .

PROVE:  $D$  is connected.

PROOF: By Zorn's Lemma.

⟨1⟩6. ASSUME: for a contradiction  $U$  and  $V$  form a separation of  $D$ .

⟨1⟩7. ASSUME: without loss of generality  $x, y \in U$

PROOF: We cannot have that one of  $x, y$  is in  $U$  and the other in  $V$  since  $D \in \mathcal{A}$ .

⟨1⟩8.  $U \in \mathcal{A}$

PROOF: If  $X$  and  $Y$  form a separation of  $U$  with  $x \in X$  and  $y \in Y$ , then  $X$  and  $Y \cup V$  form a separation of  $D$  with  $x \in X$  and  $y \in Y \cup V$ .

⟨1⟩9. Q.E.D.

PROOF: There is a connected set  $D$  that contains both  $x$  and  $y$ .

□

PROOF:

⟨1⟩1. LET:  $\{X_\alpha\}_{\alpha \in J}$  be a family of compact spaces.

⟨1⟩2. LET:  $X = \prod_{\alpha \in J} X_\alpha$

- (1)3. PICK a well-ordering  $<$  on  $J$  such that  $J$  has a greatest element.  
 (1)4. For  $\alpha \in J$  and  $p = \{p_i \in X_i\}_{i \leq \alpha}$  a family of points,  
 LET:  $Y(p) = \{x \in X \mid \forall i \leq \alpha. x_i = p_i\}$   
 (1)5. If  $\alpha < \alpha'$  and  $p$  is an  $\alpha'$ -indexed family of points then  $Y(p) \subseteq Y(p \upharpoonright \alpha)$   
 PROOF: From definition.  
 (1)6. Given  $\beta \in J$  and  $p = \{p_i \in X_i\}_{i < \beta}$  a family of points,  
 LET:  $Z(p) = \bigcap_{\alpha < \beta} Y(p \upharpoonright \alpha)$   
 (1)7. Given  $\beta \in J$  and  $p = \{p_i \in X_i\}_{i < \beta}$  a family of points, if  $\mathcal{A}$  is a finite set of basic open spaces for  $X$  that covers  $Z(p)$ , then there exists  $\alpha < \beta$  such that  $\mathcal{A}$  covers  $Y(p \upharpoonright \alpha)$   
 (2)1. ASSUME: without loss of generality  $\beta$  has no immediate predecessor.  
 (2)2. For  $A \in \mathcal{A}$ ,  
 LET:  $J_A = \{i < \beta \mid \pi_i(A) \neq X_i\}$   
 (2)3. LET:  $\alpha = \max \bigcup_{A \in \mathcal{A}} J_A$   
 (2)4. LET:  $x \in Y(p \upharpoonright \alpha)$   
 (2)5. LET:  $y \in Z(p)$  be the point with  $y_i = p_i$  for  $i < \beta$  and  $y_i = x_i$  for  $i \geq \beta$   
 (2)6. PICK  $A \in \mathcal{A}$  such that  $y \in A$   
 PROOF: Since  $\mathcal{A}$  covers  $Z(p)$ .  
 (2)7. For  $i \in J_A$  we have  $x_i \in \pi_i(A)$   
 PROOF: Since  $i \leq \alpha$  so  $x_i = p_i$   
 (2)8. For  $i \in J \setminus J_A$  we have  $x_i \in \pi_i(A)$   
 PROOF: Since  $\pi_i(A) = X_i$   
 (2)9.  $x \in A$   
 (1)8. ASSUME: for a contraction  $\mathcal{A}$  is a set of basic open sets for  $X$  that covers  $X$  but such that no finite subset of  $\mathcal{A}$  covers  $X$   
 (1)9. PICK a set of points  $\{p_i\}_{i \in J}$  such that, for all  $\alpha \in J$ , we have  $Y(p \upharpoonright \alpha)$  is not finitely covered by  $\mathcal{A}$   
 (2)1. ASSUME: as transfinite induction hypothesis  $\alpha \in J$  and  $\{p_i\}_{i < \alpha}$  is a family of points such that, for all  $\alpha' < \alpha$ , we have  $Y(p \upharpoonright \alpha')$  is not finitely covered by  $\mathcal{A}$   
 (2)2.  $Z(p)$  is not finitely covered by  $\mathcal{A}$   
 PROOF: By (1)7.  
 (2)3. PICK  $p_\alpha \in X_\alpha$  such that  $Y(p)$  is not finitely covered by  $\mathcal{A}$   
 PROOF: By Lemma 9.322 since there is a homeomorphism  $\phi : Z(p) \cong X_\alpha \times \prod_{\alpha' > \alpha} X_{\alpha'}$  and, given  $p_\alpha$ , this homomorphism  $\phi$  restricts to a homeomorphism  $Y(p) \cong \{p_\alpha\} \times \prod_{\alpha' > \alpha} X_{\alpha'}$ .  
 (1)10. Q.E.D.  
 PROOF: If  $\omega$  is the greatest element of  $J$  then  $Y(p \upharpoonright \omega)$  is a singleton.

□

**Theorem 9.324.** *Every complete linearly ordered set in the order topology is compact.*

PROOF:

- (1)1. LET:  $X$  be a complete linearly ordered set with least element  $a$  and greatest element  $b$ .  
 (1)2. LET:  $\mathcal{A}$  be an open covering of  $X$ .

- ⟨1⟩3. For all  $x < b$ , there exists  $y > x$  such that  $[x, y]$  can be covered by at most two elements of  $\mathcal{A}$ .
- ⟨2⟩1. LET:  $x \in X$
- ⟨2⟩2. PICK  $A \in \mathcal{A}$  with  $x \in A$
- ⟨2⟩3. PICK  $y > x$  such that  $[x, y] \subseteq A$
- ⟨2⟩4. PICK  $B \in \mathcal{A}$  with  $y \in B$
- ⟨2⟩5.  $[x, y]$  is covered by  $A$  and  $B$
- ⟨1⟩4. LET:  $C = \{y \in X \mid [a, y] \text{ can be covered by finitely many elements of } \mathcal{A}\}$
- ⟨1⟩5. LET:  $c = \sup C$
- ⟨1⟩6.  $c > a$
- ⟨2⟩1. PICK  $x > a$  such that  $[a, x]$  can be covered by at most two elements of  $\mathcal{A}$ .
- PROOF: From ⟨1⟩3.
- ⟨2⟩2.  $x \in C$
- ⟨1⟩7.  $c \in C$
- ⟨2⟩1. PICK  $A \in \mathcal{A}$
- ⟨2⟩2. PICK  $x < c$  such that  $(x, c] \subseteq A$
- ⟨2⟩3. PICK  $y > x$  such that  $y \in C$
- ⟨2⟩4. PICK  $\mathcal{A}_0 \subseteq^{\text{fin}} \mathcal{A}$  that covers  $[a, y]$
- ⟨2⟩5.  $\mathcal{A}_0 \cup \{A\}$  covers  $[a, c]$
- ⟨1⟩8.  $c = b$
- ⟨2⟩1. ASSUME: for a contradiction  $c < b$
- ⟨2⟩2. PICK  $x > c$  such that  $[c, x]$  can be covered by at most two elements of  $\mathcal{A}$
- PROOF: From ⟨1⟩3.
- ⟨2⟩3.  $[a, x]$  can be finitely covered by  $\mathcal{A}$
- PROOF: From ⟨1⟩7.
- ⟨2⟩4. Q.E.D.
- PROOF: This contradicts the maximality of  $c$ .

□

**Corollary 9.324.1.** *Let  $X$  be a linearly ordered set with the least upper bound property. Then every closed interval in  $X$  is compact.*

**Corollary 9.324.2.** *Every closed interval in  $\mathbb{R}$  is compact.*

**Theorem 9.325** (Extreme Value Theorem). *Any linearly ordered set under the order topology that is compact has a greatest and a least element.*

PROOF:

- ⟨1⟩1. LET:  $X$  be a linearly ordered set under the order topology that is compact.
- ⟨1⟩2.  $X$  has a greatest element.
- ⟨2⟩1. ASSUME: for a contradiction  $X$  has no greatest element.
- ⟨2⟩2.  $\{(-\infty, a) \mid a \in X\}$  covers  $X$ .
- ⟨2⟩3. PICK a finite subcover  $\{(-\infty, a_1), \dots, (-\infty, a_n)\}$ , say.
- ⟨2⟩4. ASSUME: without loss of generality  $a_1 \leq \dots \leq a_n$
- ⟨2⟩5.  $X \subseteq (-\infty, a_n)$

- ⟨2⟩6.  $a_n < a_n$   
 ⟨1⟩3.  $X$  has a least element.

PROOF: Similar.

□

**Proposition 9.326.** *Every linearly ordered set in which every closed interval is compact satisfies the least upper bound property.*

PROOF:

- ⟨1⟩1. LET:  $X$  be a linearly ordered set in which every closed interval is compact.  
 ⟨1⟩2. LET:  $A \subseteq X$  be nonempty with upper bound  $u$   
 ⟨1⟩3. PICK  $a \in A$   
 ⟨1⟩4. The closed interval  $[a, u]$  is compact.  
 ⟨1⟩5. ASSUME: for a contradiction  $A$  has no supremum.  
 ⟨1⟩6.  $\{(-\infty, x) \mid x \in A\} \cup \{(x, +\infty) \mid x \text{ is an upper bound of } A\}$  covers  $[a, u]$ .  
 ⟨2⟩1. LET:  $x \in [a, u]$   
 ⟨2⟩2. ASSUME: for all  $y \in A$  we have  $x \notin (-\infty, y)$   
 ⟨2⟩3.  $x$  is an upper bound for  $A$   
 ⟨2⟩4. PICK an upper bound  $y$  for  $A$  with  $y < x$   
 ⟨2⟩5.  $x \in (y, +\infty)$   
 ⟨1⟩7. PICK a finite subcover  $\{(-\infty, x_1), \dots, (-\infty, x_m), (y_1, +\infty), \dots, (y_n, +\infty)\}$   
 ⟨1⟩8. ASSUME:  $x_m = \max(x_1, \dots, x_m)$  and  $y_1 = \min(y_1, \dots, y_n)$   
 ⟨1⟩9.  $x_m \notin (-\infty, x_i)$  for any  $i$   
 PROOF: Since  $x_i \leq x_m$   
 ⟨1⟩10.  $x_m \notin (y_i, +\infty)$  for any  $i$   
 PROOF: Since  $x_m \in A$  so  $x_m \leq y_i$   
 ⟨1⟩11.  $x_m \in [a, u]$   
 ⟨2⟩1.  $a \notin (y_i, +\infty)$  for any  $i$   
 PROOF: Since  $y_i$  is an upper bound for  $A$  and  $a \in A$ .  
 ⟨2⟩2.  $a \in (-\infty, x_i)$  for some  $i$   
 PROOF: From ⟨1⟩7.  
 ⟨2⟩3.  $a < x_m$   
 PROOF: Since  $x_i \leq x_m$   
 ⟨2⟩4.  $x_m \leq u$   
 PROOF: Since  $u$  is an upper bound for  $A$  and  $x_m \in A$ .  
 ⟨1⟩12. Q.E.D.  
 PROOF: This contradicts ⟨1⟩7.

□

**Example 9.327.** The set  $[0, 1]$  is not compact under the  $K$ -topology.

PROOF: For every  $n \geq 1$ , pick an open interval  $U_n$  such that  $U_n \cap K = \{1/n\}$ . Then the open cover  $\{[0, 1] - K\} \cup \{U_n \mid n \in \mathbb{Z}^+\}$  has no finite subcover. □

**Proposition 9.328 (AC).** *Let  $X$  be a compact Hausdorff space. Let  $\mathcal{A}$  be a countable set of closed sets in  $X$ . If every element of  $\mathcal{A}$  has empty interior, then  $\bigcup \mathcal{A}$  has empty interior.*

PROOF:

- <1>1. LET:  $X$  be a compact Hausdorff space.  
 <1>2. For every closed set  $A$  in  $X$  and open  $U$  in  $X$  with  $U \not\subseteq A$ , there exists a nonempty open set  $V$  such that  $\overline{V} \subseteq U - A$ .  
 <2>1. LET:  $A$  be a closed set in  $X$   
 <2>2. LET:  $U$  be an open set in  $X$  with  $U \not\subseteq A$   
 <2>3. PICK  $x \in U - A$   
 <2>4. PICK disjoint neighbourhoods  $W$  and  $V$  of  $A \cup (X - U)$  and  $x$  respectively.  
 PROOF: Proposition 9.311.  
 <2>5.  $\overline{V} \subseteq U - A$   
 PROOF:  

$$\begin{aligned}\overline{V} &\subseteq X - W && (\text{since } V \subseteq X - W) \\ &\subseteq X - (A \cup (X - U)) \\ &= (X - A) \cap U \\ &= U - A\end{aligned}$$
  
 <1>3. PICK an enumeration  $\{A_1, A_2, \dots\}$  of  $\mathcal{A}$   
 <1>4. LET:  $U_0$  be any nonempty open set  
 PROVE:  $U_0 \not\subseteq \bigcup \mathcal{A}$   
 <1>5. PICK a sequence of nonempty open sets  $U_1, U_2, \dots$  such that, for  $n \geq 1$ , we have  $\overline{U_n} \subseteq U_{n-1} - A_n$   
 <2>1. ASSUME: we have picked  $U_0, U_1, \dots, U_n$   
 <2>2.  $U_n \not\subseteq A_{n+1}$   
 PROOF: Since  $A_{n+1}$  has empty interior.  
 <2>3. PICK a nonempty open set  $U_{n+1}$  such that  $\overline{U_{n+1}} \subseteq U_n - A_{n+1}$   
 PROOF: By <1>2  
 <1>6. PICK  $a \in \bigcap_{n=0}^{\infty} \overline{U_n}$   
 PROOF: Corollary 9.307.1.  
 <1>7.  $a \in U_0$   
 PROOF: Since  $a \in \overline{U_1} \subseteq U_0$ .  
 <1>8.  $a \notin \bigcup \mathcal{A}$   
 PROOF: For all  $n$ , we have  $a \in \overline{U_n} \subseteq U_{n-1} - A_n$ .  
 $\square$

**Example 9.329.** The Cantor set is compact.

PROOF: It is a closed subset of the compact set  $[0, 1]$ .  $\square$

**Proposition 9.330.** *Every compact space is limit point compact.*

PROOF:

- <1>1. LET:  $X$  be a compact space.  
 <1>2. LET:  $A \subseteq X$  have no limit points.  
 PROVE:  $A$  is finite.  
 <1>3.  $A$  is closed.  
 PROOF: Corollary 9.49.1.  
 <1>4.  $A$  is compact.  
 PROOF: Proposition 9.304.

$\langle 1 \rangle 5.$   $\{U \mid U \text{ open}, |U \cap A| = 1\}$  covers  $A$ .

PROOF: From  $\langle 1 \rangle 2$ , for all  $a \in A$ , there is a neighbourhood  $U$  of  $a$  that intersects  $A$  in  $a$  only.

$\langle 1 \rangle 6.$  PICK a finite subcover  $\{U_1, \dots, U_n\}$

$\langle 1 \rangle 7.$  For  $i = 1, \dots, n$ ,

LET:  $U_i \cap A = \{x_i\}$ .

$\langle 1 \rangle 8.$   $A = \{x_1, \dots, x_n\}$

□

The following examples show that not every limit point compact space is compact.

**Example 9.331.** Let  $Y$  be a set with two elements under the indiscrete topology. Then  $\mathbb{Z}^+ \times Y$  is limit point compact, since every nonempty set has a limit point. It is not compact, since  $\{\{n\} \times Y \mid n \in \mathbb{Z}^+\}$  has no finite subcover.

**Example 9.332.** The space  $S_\Omega$  is limit point compact but not compact.

PROOF:

$\langle 1 \rangle 1.$   $S_\Omega$  is not compact.

PROOF: From the Extreme Value Theorem, since  $S_\Omega$  has no greatest element.

$\langle 1 \rangle 2.$  LET:  $A$  be an infinite subset of  $S_\Omega$ .

$\langle 1 \rangle 3.$  PICK  $B \subseteq A$  that is countably infinite.

PROOF: Proposition ??.

$\langle 1 \rangle 4.$  LET:  $b = \sup B$

$\langle 1 \rangle 5.$   $B \subseteq [0, b]$

$\langle 1 \rangle 6.$   $[0, b]$  is compact.

PROOF: Corollary 9.324.1.

$\langle 1 \rangle 7.$  PICK a limit point  $x$  of  $B$  in  $[0, b]$ .

PROOF: Proposition 9.330.

$\langle 1 \rangle 8.$   $x$  is a limit point of  $A$ .

PROOF: Lemma 9.51.

□

## 9.48 Perfect Maps

**Definition 9.333** (Perfect Map). Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Then  $f$  is a *perfect map* if and only if  $f$  is a closed map, continuous, surjective and  $p^{-1}(y)$  is compact for all  $y \in Y$ .

**Proposition 9.334.** Let  $X$  be a topological space,  $Y$  a compact space, and  $p : X \rightarrow Y$  a closed map such that, for all  $y \in Y$ , we have  $p^{-1}(y)$  is compact. Then  $X$  is compact.

PROOF:

$\langle 1 \rangle 1.$  LET:  $\mathcal{A}$  be a set of closed sets in  $X$  with the finite intersection property.

$\langle 1 \rangle 2.$   $\mathcal{B} = \{p(A_1 \cap \dots \cap A_n) \mid A_1, \dots, A_n \in \mathcal{A}\}$  is a set of closed sets in  $Y$  with the finite intersection property.

PROOF: Since  $p$  is a closed map.

⟨1⟩3. PICK  $y \in \bigcap \mathcal{B}$

PROOF: Theorem 9.307 since  $Y$  is compact.

⟨1⟩4.  $\{A \cap p^{-1}(y) \mid A \in \mathcal{A}\}$  is a set of closed sets in  $p^{-1}(y)$  with the finite intersection property.

⟨1⟩5. PICK  $x \in \bigcap_{A \in \mathcal{A}} (A \cap p^{-1}(y))$

PROOF: Theorem 9.307 since  $p^{-1}(y)$  is compact.

⟨1⟩6.  $x \in \bigcap \mathcal{A}$

⟨1⟩7. Q.E.D.

PROOF: Theorem 9.307.

□

## 9.49 Isolated Points

**Definition 9.335** (Isolated Point). Let  $X$  be a topological space and  $x \in X$ . Then  $x$  is an *isolated point* if and only if  $\{x\}$  is open.

**Theorem 9.336** (AC). A nonempty compact Hausdorff space with no isolated points is uncountable.

PROOF:

⟨1⟩1. LET:  $X$  be a nonempty compact Hausdorff space with no isolated points.

⟨1⟩2. For every nonempty open set  $U$  and every point  $x \in X$ , there exists a nonempty open set  $V \subseteq U$  such that  $x \notin \bar{V}$ .

⟨2⟩1. LET:  $U$  be a nonempty open set.

⟨2⟩2. LET:  $x \in X$

⟨2⟩3. PICK  $y \in U - \{x\}$

PROOF: This is possible because  $U$  cannot be  $\{x\}$ .

⟨2⟩4. PICK disjoint open neighbourhoods  $W_1$  of  $x$  and  $W_2$  of  $y$

⟨2⟩5. LET:  $V = W_2 \cap U$

⟨2⟩6.  $V$  is nonempty

PROOF: Since  $y \in V$

⟨2⟩7.  $V$  is open

PROOF: From ⟨2⟩1, ⟨2⟩4, ⟨2⟩5.

⟨2⟩8.  $V \subseteq U$

PROOF: From ⟨2⟩5

⟨2⟩9.  $x \notin V$

PROOF: From ⟨2⟩4 and ⟨2⟩5

⟨1⟩3. LET:  $(a_n)$  be any sequence of points in  $X$ .

PROVE: The set  $X - \{a_1, a_2, \dots\}$  is nonempty.

⟨1⟩4. PICK a sequence of nonempty open sets  $V_1, V_2, \dots$ , such that  $V_1 \supseteq V_2 \supseteq \dots$  and  $a_n \notin \bar{V}_n$  for all  $n$ .

PROOF: From ⟨1⟩2.

⟨1⟩5. PICK  $a \in \bigcap_{n=1}^{\infty} \bar{V}_n$

PROOF: Corollary 9.307.1.

⟨1⟩6.  $a \in X - \{a_1, a_2, \dots\}$

PROOF: We cannot have  $a = a_n$  because  $a \in \overline{V_n}$ .  
 $\square$

**Corollary 9.336.1.** *For all  $a, b \in \mathbb{R}$  with  $a < b$ , the closed interval  $[a, b]$  is uncountable.*

**Example 9.337.** The Cantor set has no isolated points, and is therefore uncountable.

PROOF:

$\langle 1 \rangle 1$ . LET:  $(A_n)$  be the sets in Definition 8.24.

$\langle 1 \rangle 2$ . LET:  $x \in C$

$\langle 1 \rangle 3$ . LET:  $A_n$  be the first set such that  $x$  is an endpoint of one of the intervals that make up  $A_n$

$\langle 1 \rangle 4$ . LET:  $(a_m)_{m \geq n}$  be the sequence of points defined by:  $a_m$  is the point such that either  $[a_m, x]$  or  $[x, a_m]$  is one of the intervals that make up  $A_m$ .

$\langle 1 \rangle 5$ .  $(a_m)$  is a sequence of points of  $C$  distinct from  $x$  that converges to  $x$ .

PROOF: Since  $|a_m - x| = 1/3^m$  for all  $m$ .

$\langle 1 \rangle 6$ .  $x$  is a limit point of  $C$ .

$\square$



## Chapter 10

# Topological Groups

**Definition 10.1** (Topological Group). A *topological group*  $G$  consists of a  $T_1$  space  $G$  and continuous maps  $\cdot : G^2 \rightarrow G$  and  $(\ )^{-1} : G \rightarrow G$  such that  $(G, \cdot, (\ )^{-1})$  is a group.

**Example 10.2.** 1. The integers  $\mathbb{Z}$  under addition are a topological group.

2. The real numbers  $\mathbb{R}$  under addition are a topological group.

3. The positive reals under multiplication are a topological group.

4. The set  $\{z \in \mathbb{C} \mid |z| = 1\}$  under multiplication and given the topology of  $S^1$  is a topological group.

5. For any  $n \geq 0$ , the general linear group  $GL_n(\mathbb{R})$  is a topological group under matrix multiplication, considered as a subspace of  $\mathbb{R}^{n^2}$ .

**Lemma 10.3.** Let  $G$  be a  $T_1$  space and  $\cdot : G^2 \rightarrow G$ ,  $(\ )^{-1} : G \rightarrow G$  be functions such that  $(G, \cdot, (\ )^{-1})$  is a group. Then  $G$  is a topological group if and only if the function  $f : G^2 \rightarrow G$  that maps  $(x, y)$  to  $xy^{-1}$  is continuous.

PROOF:

$\langle 1 \rangle 1$ . If  $G$  is a topological group then  $f$  is continuous.

PROOF: From Theorem 9.87.

$\langle 1 \rangle 2$ . If  $f$  is continuous then  $G$  is a topological group.

$\langle 2 \rangle 1$ . ASSUME:  $f$  is continuous.

$\langle 2 \rangle 2$ .  $(\ )^{-1}$  is continuous.

PROOF: Since  $x^{-1} = f(e, x)$ .

$\langle 2 \rangle 3$ .  $\cdot$  is continuous.

PROOF: Since  $xy = f(x, y^{-1})$ .

□

**Lemma 10.4.** Let  $G$  be a topological group and  $H$  a subgroup of  $G$ . Then  $H$  is a topological group under the subspace topology.

PROOF:

⟨1⟩1.  $H$  is  $T_1$ .

PROOF: From Proposition 9.157.

⟨1⟩2. multiplication and inverse on  $H$  are continuous.

PROOF: From Theorem 9.88.

□

**Lemma 10.5.** *Let  $G$  be a topological group and  $H$  a subgroup of  $G$ . Then  $\overline{H}$  is a subgroup of  $G$ .*

PROOF:

⟨1⟩1. LET:  $x, y \in \overline{H}$

PROVE:  $xy^{-1} \in \overline{H}$

⟨1⟩2. LET:  $U$  be any neighbourhood of  $xy^{-1}$

⟨1⟩3. LET:  $f : G^2 \rightarrow G$ ,  $f(a, b) = ab^{-1}$

⟨1⟩4.  $f^{-1}(U)$  is a neighbourhood of  $(x, y)$

⟨1⟩5. PICK neighbourhoods  $V, W$  of  $x$  and  $y$  respectively such that  $f(V \times W) \subseteq U$ .

⟨1⟩6. PICK  $a \in V \cap H$  and  $b \in W \cap H$

PROOF: Theorem 9.38.

⟨1⟩7.  $ab^{-1} \in U \cap H$

⟨1⟩8. Q.E.D.

PROOF: By Theorem 9.38.

□

**Proposition 10.6.** *Let  $G$  be a topological group and  $\alpha \in G$ . Then the maps  $l_\alpha, r_\alpha : G \rightarrow G$  defined by  $l_\alpha(x) = \alpha x$ ,  $r_\alpha(x) = x\alpha$  are homeomorphisms of  $G$  with itself.*

PROOF: They are continuous with continuous inverses  $l_{\alpha^{-1}}$  and  $r_{\alpha^{-1}}$ . □

**Corollary 10.6.1.** *Every topological group is homogeneous.*

PROOF: Given a topological group  $G$  and  $a, b \in G$ , we have  $l_{ba^{-1}}$  is a homeomorphism that maps  $a$  to  $b$ . □

**Proposition 10.7.** *Let  $G$  be a topological group and  $H$  a subgroup of  $G$ . Give  $G/H$  the quotient topology. For all  $\alpha \in G$ , the map  $\overline{f_\alpha}$  that sends  $xH$  to  $\alpha xH$  is a homeomorphism  $G/H \cong G/H$ .*

PROOF:

⟨1⟩1.  $\overline{f_\alpha}$  is well-defined.

PROOF: If  $xy^{-1} \in H$  then  $(\alpha x)(\alpha y)^{-1} = xy^{-1} \in H$ .

⟨1⟩2.  $\overline{f_\alpha}$  is continuous.

PROOF: Theorem 9.191 since  $\overline{f_\alpha} \circ p = p \circ f_\alpha$  is continuous, where  $p : G \rightarrow G/H$  is the canonical surjection.

⟨1⟩3.  $\overline{f_\alpha}^{-1}$  is continuous.

PROOF: Similar since  $\overline{f_\alpha}^{-1} = \overline{f_{\alpha^{-1}}}$ .

□

**Corollary 10.7.1.** *Let  $G$  be a topological group and  $H$  a subgroup of  $G$ . Give  $G/H$  the quotient topology. Then  $G/H$  is homogeneous.*

**Proposition 10.8.** *Let  $G$  be a topological group and  $H$  a subgroup of  $G$ . Give  $G/H$  the quotient topology. If  $H$  is closed in  $G$  then  $G/H$  is  $T_1$ .*

PROOF:

⟨1⟩1. LET:  $p : G \rightarrow G/H$  be the canonical surjection

⟨1⟩2. LET:  $x \in G$

⟨1⟩3.  $p^{-1}(xH) = f_x(H)$

⟨1⟩4.  $p^{-1}(xH)$  is closed in  $G$

PROOF: Since  $H$  is closed and  $f_x$  is a homomorphism of  $G$  with itself.

⟨1⟩5.  $\{xH\}$  is closed in  $G/H$

□

**Proposition 10.9.** *Let  $G$  be a topological group and  $H$  a subgroup of  $G$ . Give  $G/H$  the quotient topology. Then the canonical surjection  $p : G \rightarrow G/H$  is an open map.*

PROOF:

⟨1⟩1. LET:  $U \subseteq G$  be open.

⟨1⟩2.  $p^{-1}(p(U)) = \bigcup_{h \in H} r_h(U)$

⟨1⟩3.  $p^{-1}(p(U))$  is open.

⟨1⟩4.  $p(U)$  is open.

□

**Proposition 10.10.** *Let  $G$  be a topological group and  $H$  a closed normal subgroup of  $G$ . Then  $G/H$  is a topological group under the quotient topology.*

PROOF:

⟨1⟩1.  $G/H$  is  $T_1$

PROOF: Proposition 10.8.

⟨1⟩2. The map  $\bar{m} : (xH, yH) \mapsto xy^{-1}H$  is continuous.

⟨2⟩1.  $p^2 : G^2 \rightarrow (G/H)^2$  is a quotient map.

PROOF: Propositions 9.190, 10.9.

⟨2⟩2.  $\bar{m} \circ p^2$  is continuous.

PROOF: As it is  $p^2 \circ m$  where  $m : G^2 \rightarrow G$  with  $m(x, y) = xy^{-1}$

□

**Lemma 10.11.** *Let  $G$  be a topological group and  $A, B \subseteq G$ . If either  $A$  or  $B$  is open then  $AB$  is open.*

PROOF: If  $A$  is open we have  $AB = \bigcup_{b \in B} r_b(A)$ . Similarly if  $B$  is open. □

**Definition 10.12** (Symmetric Neighbourhood). Let  $G$  be a topological group. A neighbourhood  $V$  of  $e$  is *symmetric* if and only if  $V = V^{-1}$ .

**Lemma 10.13.** *Let  $G$  be a topological group. Let  $V$  be a neighbourhood of  $e$ . Then  $V$  is symmetric if and only if, for all  $x \in V$ , we have  $x^{-1} \in V$ .*

PROOF:

$\langle 1 \rangle 1$ . If  $V$  is symmetric then, for all  $x \in V$ , we have  $x^{-1} \in V$

PROOF: Immediate from definitions.

$\langle 1 \rangle 2$ . If, for all  $x \in V$ , we have  $x^{-1} \in V$ , then  $V$  is symmetric.

$\langle 2 \rangle 1$ . ASSUME: for all  $x \in V$  we have  $x^{-1} \in V$

$\langle 2 \rangle 2$ .  $V \subseteq V^{-1}$

PROOF: If  $x \in V$  then there exists  $y \in V$  such that  $x = y^{-1}$ , namely  $y = x^{-1}$

$\langle 2 \rangle 3$ .  $V^{-1} \subseteq V$

PROOF: Immediate from  $\langle 2 \rangle 1$ .

□

**Lemma 10.14.** *Let  $G$  be a topological group. For every neighbourhood  $U$  of  $e$ , there exists a symmetric neighbourhood  $V$  of  $e$  such that  $V^2 \subseteq U$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $U$  be a neighbourhood of  $e$ .

$\langle 1 \rangle 2$ . PICK a neighbourhood  $V'$  of  $e$  such that  $V'V' \subseteq U$

PROOF: Such a neighbourhood exists because multiplication in  $G$  is continuous.

$\langle 1 \rangle 3$ . PICK a neighbourhood  $W$  of  $e$  such that  $WW^{-1} \subseteq V'$

PROOF: Such a neighbourhood exists because the function that maps  $(x, y)$  to  $xy^{-1}$  is continuous.

$\langle 1 \rangle 4$ . LET:  $V = WW^{-1}$

$\langle 1 \rangle 5$ .  $V$  is a neighbourhood of  $e$

$\langle 2 \rangle 1$ .  $e \in V$

PROOF: Since  $e \in W$  so  $e = ee^{-1} \in V$ .

$\langle 2 \rangle 2$ .  $V$  is open

PROOF: Lemma 10.11.

$\langle 1 \rangle 6$ .  $V$  is symmetric

$\langle 2 \rangle 1$ . For all  $x \in V$  we have  $x^{-1} \in V$

$\langle 3 \rangle 1$ . LET:  $x \in V$

$\langle 3 \rangle 2$ . PICK  $y, z \in W$  such that  $x = yz^{-1}$

$\langle 3 \rangle 3$ .  $x^{-1} = zy^{-1}$

$\langle 3 \rangle 4$ .  $x^{-1} \in V$

$\langle 3 \rangle 5$ .  $x \in V^{-1}$

$\langle 2 \rangle 2$ . Q.E.D.

PROOF: Lemma 10.13

$\langle 1 \rangle 7$ .  $V^2 \subseteq U$

PROOF: We have  $V^2 \subseteq (V')^2 \subseteq U$

□

**Proposition 10.15.** *Every topological group is Hausdorff.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $G$  be a topological group.

$\langle 1 \rangle 2$ . LET:  $x, y \in G$  with  $x \neq y$

- ⟨1⟩3. LET:  $U = G \setminus \{x^{-1}y\}$
- ⟨1⟩4. PICK a symmetric neighbourhood  $V$  of  $e$  such that  $VV \subseteq U$ 
  - ⟨2⟩1.  $U$  is open  
PROOF: Since  $G$  is  $T_1$ .
  - ⟨2⟩2.  $e \in U$   
PROOF: Since  $x \neq y$
  - ⟨2⟩3. Q.E.D.  
PROOF: Lemma 10.14.
- ⟨1⟩5.  $Vx$  and  $Vy$  are disjoint neighbourhoods of  $x$  and  $y$  respectively.
  - ⟨2⟩1.  $Vx$  is open  
PROOF: Since  $Vx = r_x(V)$
  - ⟨2⟩2.  $Vy$  is open  
PROOF: Similar.
  - ⟨2⟩3.  $Vx \cap Vy = \emptyset$ 
    - ⟨3⟩1. ASSUME: for a contradiction  $z \in Vx \cap Vy$
    - ⟨3⟩2. PICK  $a, b \in V$  such that  $z = ax = by$
    - ⟨3⟩3.  $xy^{-1} \in VV$   
PROOF: Since  $xy^{-1} = a^{-1}b$
    - ⟨3⟩4.  $xy^{-1} \in U$
    - ⟨3⟩5. Q.E.D.  
PROOF: From ⟨1⟩3.

□

**Proposition 10.16.** *Every topological group is regular.*

PROOF:

- ⟨1⟩1. LET:  $G$  be a topological group.
- ⟨1⟩2. LET:  $A \subseteq G$  be a closed set and  $a \notin A$ .
- ⟨1⟩3. LET:  $U = G \setminus Aa^{-1}$
- ⟨1⟩4. PICK a symmetric neighbourhood  $V$  of  $e$  such that  $VV \subseteq U$ 
  - ⟨2⟩1.  $U$  is open  
PROOF: Since  $Aa^{-1} = r_{a^{-1}}(A)$  is closed.
  - ⟨2⟩2.  $e \in U$   
PROOF: Since  $a \notin A$ .
  - ⟨2⟩3. Q.E.D.  
PROOF: Lemma 10.14.
- ⟨1⟩5.  $VA$  and  $Va$  are disjoint open sets with  $A \subseteq VA$  and  $a \in Va$ 
  - ⟨2⟩1.  $VA$  is open  
PROOF: Lemma 10.11
  - ⟨2⟩2.  $Va$  is open  
PROOF: Lemma 10.11
  - ⟨2⟩3.  $VA \cap Va = \emptyset$ 
    - ⟨3⟩1. ASSUME: for a contradiction  $z \in VA \cap Va$
    - ⟨3⟩2. PICK  $b, c \in V$  and  $d \in A$  with  $z = bd = ca$
    - ⟨3⟩3.  $da^{-1} \in U$   
PROOF: Since  $da^{-1} = b^{-1}c \in VV \subseteq U$
    - ⟨3⟩4. Q.E.D.

PROOF: This contradicts  $\langle 1 \rangle 3$

□

**Proposition 10.17.** *Let  $G$  be a topological group and  $H$  a subgroup of  $G$ . Give  $G/H$  the quotient topology. If  $H$  is closed in  $G$  then  $G/H$  is regular.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $p : G \rightarrow G/H$  be the canonical surjection.
- $\langle 1 \rangle 2$ . LET:  $A$  be a closed set in  $G/H$  and  $aH \in (G/H) \setminus A$ .
- $\langle 1 \rangle 3$ . LET:  $B = p^{-1}(A)$
- $\langle 1 \rangle 4$ .  $B$  is a closed saturated set in  $G$ .
- $\langle 1 \rangle 5$ .  $B \cap aH = \emptyset$
- $\langle 1 \rangle 6$ .  $B = BH$
- $\langle 1 \rangle 7$ . PICK a symmetric neighbourhood  $V$  of  $e$  such that  $VB$  does not intersect  $Va$ 
  - $\langle 2 \rangle 1$ . LET:  $U = G \setminus Ba^{-1}$
  - $\langle 2 \rangle 2$ . PICK a symmetric neighbourhood  $V$  of  $e$  such that  $VV \subseteq U$ 
    - $\langle 3 \rangle 1$ .  $U$  is open
 

PROOF: Since  $Ba^{-1} = r_{a^{-1}}(B)$  is closed.
    - $\langle 3 \rangle 2$ .  $e \in U$ 

PROOF: If  $e \in Ba^{-1}$  then  $a \in B$
    - $\langle 3 \rangle 3$ . Q.E.D.
 

PROOF: Lemma 10.14
  - $\langle 2 \rangle 3$ .  $VB \cap Va = \emptyset$ 

PROOF: If  $vb = v'a$  for  $v, v' \in V$  and  $b \in B$  then we have  $ba^{-1} = v^{-1}v' \in Ba \cap VV \subseteq Ba \cap U$ .
- $\langle 1 \rangle 8$ .  $p(VB)$  and  $p(Va)$  are disjoint open sets
  - $\langle 2 \rangle 1$ .  $p(VB)$  and  $p(Va)$  are open.
 

PROOF: Proposition 10.9.
  - $\langle 2 \rangle 2$ .  $p(VB) \cap p(Va) = \emptyset$ 

PROOF: If  $vbH = v'aH$  for  $v, v' \in V$ ,  $b \in B$  then  $v'a = vbh$  for some  $h \in H$ . Hence  $v'a \in Va \cap VBH = Va \cap VB$ .
- $\langle 1 \rangle 9$ .  $A \subseteq p(VB)$
- $\langle 1 \rangle 10$ .  $aH \in p(Va)$

□

**Proposition 10.18.** *Let  $G$  be a topological group. The component of  $G$  that contains  $e$  is a normal subgroup of  $G$ .*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $C$  be the component of  $G$  that contains  $e$ .
- $\langle 1 \rangle 2$ . For all  $x \in G$ ,  $xC$  is the component of  $G$  that contains  $x$ .
  - $\langle 2 \rangle 1$ . LET:  $x \in G$
  - $\langle 2 \rangle 2$ . LET:  $D$  be the component of  $G$  that contains  $x$ .
  - $\langle 2 \rangle 3$ .  $xC \subseteq D$ 

PROOF: Since  $xC$  is connected by Theorem 9.215.
  - $\langle 2 \rangle 4$ .  $D \subseteq xC$

PROOF: Since  $x^{-1}D \subseteq C$  similarly.

$\langle 1 \rangle 3$ . For all  $x \in G$ ,  $Cx$  is the component of  $G$  that contains  $x$ .

PROOF: Similar.

$\langle 1 \rangle 4$ . For all  $x \in C$  we have  $xC = Cx = C$

$\langle 1 \rangle 5$ . For all  $x \in C$  we have  $x^{-1}C = C$

$\langle 1 \rangle 6$ . For all  $x \in C$  we have  $x^{-1} \in C$

$\langle 1 \rangle 7$ . For all  $x, y \in C$  we have  $xy \in C$

PROOF: Since  $xyC = xC = x$ .

$\langle 1 \rangle 8$ . For all  $x \in G$  we have  $xC = Cx$ .

PROOF: From  $\langle 1 \rangle 2$  and  $\langle 1 \rangle 3$ .

□

**Lemma 10.19.** *Let  $G$  be a topological group. Let  $A$  be a closed set in  $G$  and  $B$  a compact subspace of  $G$  such that  $A \cap B = \emptyset$ . Then there exists a symmetric neighbourhood  $U$  of  $e$  such that  $AU \cap BU = \emptyset$ .*

PROOF:

$\langle 1 \rangle 1$ . For all  $b \in B$  there exists a symmetric neighbourhood  $V$  of  $e$  such that  $bV^2 \cap A = \emptyset$

$\langle 2 \rangle 1$ . LET:  $b \in B$

$\langle 2 \rangle 2$ . LET:  $W = b^{-1}(G \setminus A)$

$\langle 2 \rangle 3$ .  $W$  is a neighbourhood of  $e$  and  $bW \cap A = \emptyset$

$\langle 2 \rangle 4$ . PICK a symmetric neighbourhood  $V$  of  $e$  such that  $V^2 \subseteq W$

$\langle 1 \rangle 2$ .  $\{bV^2 \mid b \in B, V \text{ is a symmetric neighbourhood of } e, bV^2 \cap A = \emptyset\}$  is an open cover of  $B$

$\langle 1 \rangle 3$ . PICK a finite subcover  $b_1V_1^2, \dots, b_nV_n^2$ , say.

$\langle 1 \rangle 4$ . LET:  $U = V_1 \cap \dots \cap V_n$

$\langle 1 \rangle 5$ .  $BU^2 \cap A = \emptyset$

$\langle 1 \rangle 6$ .  $AU \cap BU = \emptyset$

PROOF: If  $av \in BU$  where  $a \in A$  and  $v \in V$  then  $a = avv^{-1} \in BU^2 \cap A$ .

□

**Proposition 10.20 (AC).** *Let  $G$  be a topological group. Let  $A$  be a closed set in  $G$ , and  $B$  a compact subspace of  $G$ . Then  $AB$  is closed.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $x \in G \setminus AB$

$\langle 1 \rangle 2$ .  $A^{-1}x \cap B = \emptyset$

$\langle 1 \rangle 3$ .  $A^{-1}x$  is closed.

$\langle 1 \rangle 4$ . PICK a symmetric neighbourhood  $U$  of  $e$  such that  $A^{-1}xU \cap BU = \emptyset$

$\langle 1 \rangle 5$ .  $xU^2$  is open

PROOF: Lemma 10.11.

$\langle 1 \rangle 6$ .  $x \in xU^2 \subseteq G \setminus AB$

□

**Corollary 10.20.1.** *Let  $G$  be a topological group and  $H \leq G$ . Let  $p : G \rightarrow G/H$  be the quotient map. If  $H$  is compact then  $p$  is a closed map.*

PROOF: For  $A$  closed in  $G$ , we have  $p^{-1}(p(A)) = AH$  is closed, and so  $p(A)$  is closed.  $\square$

**Corollary 10.20.2.** *Let  $G$  be a topological group and  $H \leq G$ . If  $H$  and  $G/H$  are compact then  $G$  is compact.*

PROOF: From Proposition 9.334 since, for all  $aH \in G/H$ , we have  $p^{-1}(aH) = aH$  is compact because it is homomorphic to  $H$ .  $\square$



## 10.1 The Metric Topology

**Definition 10.21** (Metric). Let  $X$  be a set. A *metric* on  $X$  is a function  $d : X^2 \rightarrow \mathbb{R}$  such that:

1. For all  $x, y \in X$ ,  $d(x, y) \geq 0$
2. For all  $x, y \in X$ ,  $d(x, y) = 0$  if and only if  $x = y$
3. For all  $x, y \in X$ ,  $d(x, y) = d(y, x)$
4. (*Triangle Inequality*) For all  $x, y, z \in X$ ,  $d(x, z) \leq d(x, y) + d(y, z)$

We call  $d(x, y)$  the *distance* between  $x$  and  $y$ .

**Definition 10.22** (Open Ball). Let  $X$  be a metric space. Let  $a \in X$  and  $\epsilon > 0$ . The *open ball* with *centre*  $a$  and *radius*  $\epsilon$  is

$$B(a, \epsilon) = \{x \in X \mid d(a, x) < \epsilon\} .$$

**Definition 10.23** (Metric Topology). Let  $X$  be a metric space. The *metric topology* on  $X$  is the topology generated by the basis consisting of all the open balls.

We prove this is a basis for a topology.

PROOF:

$\langle 1 \rangle 1$ . For every point  $a$ , there exists a ball  $B$  such that  $a \in B$

PROOF: We have  $a \in B(a, 1)$ .

$\langle 1 \rangle 2$ . For any balls  $B_1, B_2$  and point  $a \in B_1 \cap B_2$ , there exists a ball  $B_3$  such that  $a \in B_3 \subseteq B_1 \cap B_2$

$\langle 2 \rangle 1$ . LET:  $B_1 = B(c_1, \epsilon_1)$  and  $B_2 = B(c_2, \epsilon_2)$

$\langle 2 \rangle 2$ . LET:  $\delta = \min(\epsilon_1 - d(c_1, a), \epsilon_2 - d(c_2, a))$

PROVE:  $B(a, \delta) \subseteq B_1 \cap B_2$

$\langle 2 \rangle 3$ . LET:  $x \in B(a, \delta)$

$\langle 2 \rangle 4$ .  $x \in B_1$

PROOF:

$$\begin{aligned} d(x, c_1) &= d(x, a) + d(a, c_1) \\ &< \delta + d(a, c_1) \\ &\leq \epsilon_1 \end{aligned}$$

$\langle 2 \rangle 5$ .  $x \in B_2$

PROOF: Similar.

□

**Proposition 10.24.** Let  $X$  be a metric space and  $U \subseteq X$ . Then  $U$  is open if and only if, for every  $x \in U$ , there exists  $\epsilon > 0$  such that  $B(x, \epsilon) \subseteq U$ .

PROOF:

$\langle 1 \rangle 1$ . If  $U$  is open then, for all  $x \in U$ , there exists  $\epsilon > 0$  such that  $B(x, \epsilon) \subseteq U$ .

$\langle 2 \rangle 1$ . ASSUME:  $U$  is open.

- (2)2. LET:  $x \in U$   
 (2)3. PICK  $a \in X$  and  $\delta > 0$  such that  $x \in B(a, \delta) \subseteq U$   
 (2)4. LET:  $\epsilon = \delta - d(a, x)$   
 PROVE:  $B(x, \epsilon) \subseteq U$   
 (2)5. LET:  $y \in B(x, \epsilon)$   
 (2)6.  $d(y, a) < \delta$   
 PROOF:

$$\begin{aligned}
 d(y, a) &\leq d(a, x) + d(x, y) \\
 &< \delta + d(x, y) \\
 &= \epsilon
 \end{aligned}$$

- (2)7.  $y \in U$   
 (1)2. If, for all  $x \in U$ , there exists  $\epsilon > 0$  such that  $B(x, \epsilon) \subseteq U$ , then  $U$  is open.  
 PROOF: Immediate from definitions.

□

**Definition 10.25** (Discrete Metric). Let  $X$  be a set. The *discrete metric* on  $X$  is defined by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

**Proposition 10.26.** *The discrete metric induces the discrete topology.*

PROOF: For any (open) set  $U$  and point  $a \in U$ , we have  $a \in B(a, 1) \subseteq U$ . □

**Definition 10.27** (Standard Metric on  $\mathbb{R}$ ). The *standard metric* on  $\mathbb{R}$  is defined by  $d(x, y) = |x - y|$ .

**Proposition 10.28.** *The standard metric on  $\mathbb{R}$  induces the standard topology on  $\mathbb{R}$ .*

PROOF:

- (1)1. Every open ball is open in the standard topology on  $\mathbb{R}$ .  
 PROOF:  $B(a, \epsilon) = (a - \epsilon, a + \epsilon)$   
 (1)2. For every open set  $U$  and point  $a \in U$ , there exists  $\epsilon > 0$  such that  
 $B(a, \epsilon) \subseteq U$   
 (2)1. LET:  $U$  be an open set and  $a \in U$   
 (2)2. PICK an open interval  $b, c$  such that  $a \in (b, c) \subseteq U$   
 (2)3. LET:  $\epsilon = \min(a - b, c - a)$   
 (2)4.  $B(a, \epsilon) \subseteq U$

□

**Definition 10.29** (Metrizability). A topological space  $X$  is *metrizable* if and only if there exists a metric on  $X$  that induces the topology.

**Definition 10.30** (Bounded). Let  $X$  be a metric space and  $A \subseteq X$ . Then  $A$  is *bounded* if and only if there exists  $M$  such that, for all  $x, y \in A$ , we have  $d(x, y) \leq M$ .

**Definition 10.31** (Diameter). Let  $X$  be a metric space and  $A \subseteq X$ . The *diameter* of  $A$  is

$$\text{diam } A = \sup_{x,y \in A} d(x,y) .$$

**Definition 10.32** (Standard Bounded Metric). Let  $d$  be a metric on  $X$ . The *standard bounded metric* corresponding to  $d$  is the metric  $\bar{d}$  defined by

$$\bar{d}(x,y) = \min(d(x,y), 1) .$$

We prove this is a metric.

PROOF:

$\langle 1 \rangle 1. \bar{d}(x,y) \geq 0$

PROOF: Since  $d(x,y) \geq 0$

$\langle 1 \rangle 2. \bar{d}(x,y) = 0$  if and only if  $x = y$

PROOF:  $\bar{d}(x,y) = 0$  if and only if  $d(x,y) = 0$  if and only if  $x = y$

$\langle 1 \rangle 3. \bar{d}(x,y) = \bar{d}(y,x)$

PROOF: Since  $d(x,y) = d(y,x)$

$\langle 1 \rangle 4. \bar{d}(x,z) \leq \bar{d}(x,y) + \bar{d}(y,z)$

PROOF:

$$\begin{aligned} \bar{d}(x,y) + \bar{d}(y,z) &= \min(d(x,y), 1) + \min(d(y,z), 1) \\ &= \min(d(x,y) + d(y,z), d(x,y) + 1, d(y,z) + 1, 2) \\ &\geq \min(d(x,z), 1) \\ &= \bar{d}(x,z) \end{aligned}$$

□

**Lemma 10.33.** In any metric space  $X$ , the set  $\mathcal{B} = \{B(a, \epsilon) \mid a \in X, \epsilon < 1\}$  is a basis for the metric topology.

PROOF:

$\langle 1 \rangle 1.$  Every element of  $\mathcal{B}$  is open.

PROOF: From Lemma 9.53.

$\langle 1 \rangle 2.$  For every open set  $U$  and point  $a \in U$ , there exists  $B \in \mathcal{B}$  such that  $a \in B \subseteq U$

$\langle 2 \rangle 1.$  LET:  $U$  be an open set and  $a \in U$

$\langle 2 \rangle 2.$  PICK  $\epsilon > 0$  such that  $B(a, \epsilon) \subseteq U$

$\langle 2 \rangle 3.$   $B(a, \min(\epsilon, 1/2)) \subseteq U$

$\langle 1 \rangle 3.$  Q.E.D.

PROOF: Lemma 9.54.

□

**Proposition 10.34.** Let  $d$  be a metric on the set  $X$ . Then the standard bounded metric  $\bar{d}$  induces the same metric as  $d$ .

PROOF: This follows from Lemma 10.33 since the open balls with radius  $< 1$  are the same under both metrics. □

**Lemma 10.35.** *Let  $d$  and  $d'$  be two metrics on the same set  $X$ . Let  $\mathcal{T}$  and  $\mathcal{T}'$  be the topologies they induce. Then  $\mathcal{T} \subseteq \mathcal{T}'$  if and only if, for all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that*

$$B_{d'}(x, \delta) \subseteq B_d(x, \epsilon) .$$

PROOF:

$\langle 1 \rangle 1$ . If  $\mathcal{T} \subseteq \mathcal{T}'$  then, for all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$

PROOF: From Proposition 10.24 since  $x \in B_d(x, \epsilon) \in \mathcal{T}'$ .

$\langle 1 \rangle 2$ . If, for all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$ , then  $\mathcal{T} \subseteq \mathcal{T}'$

$\langle 2 \rangle 1$ . ASSUME: For all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$

$\langle 2 \rangle 2$ . LET:  $U \in \mathcal{T}$

$\langle 2 \rangle 3$ . For all  $x \in U$  there exists  $\delta > 0$  such that  $B_{d'}(x, \delta) \subseteq U$ .

$\langle 3 \rangle 1$ . LET:  $x \in U$

$\langle 3 \rangle 2$ . PICK  $\epsilon > 0$  such that  $B_d(x, \epsilon) \subseteq U$

PROOF: Proposition 10.24

$\langle 3 \rangle 3$ . PICK  $\delta > 0$  such that  $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$

PROOF: By  $\langle 2 \rangle 1$

$\langle 3 \rangle 4$ .  $B_{d'}(x, \delta) \subseteq U$

$\langle 2 \rangle 4$ .  $U \in \mathcal{T}'$

PROOF: Proposition 10.24.

□

**Proposition 10.36.**  $\mathbb{R}^2$  under the dictionary order topology is metrizable.

PROOF: Define  $d : \mathbb{R}^2 \rightarrow \mathbb{R}$  by

$$d((x, y), (x, z)) = \max(|y - z|, 1)$$

$$d((x, y), (x', y')) = 1 \quad \text{if } x \neq x' \square$$

$\langle 1 \rangle 1$ .  $x \in \bigcap_{i=1}^N \pi_i^{-1}() \subseteq B_D(a, \epsilon)$

**Proposition 10.37.** *Let  $d : X^2 \rightarrow \mathbb{R}$  be a metric on  $X$ . Then the metric topology on  $X$  is the coarsest topology such that  $d$  is continuous.*

PROOF:

$\langle 1 \rangle 1$ .  $d$  is continuous.

$\langle 2 \rangle 1$ . LET:  $a, b \in X$

$\langle 2 \rangle 2$ . LET:  $\epsilon > 0$

$\langle 2 \rangle 3$ . LET:  $\delta = \epsilon/2$

$\langle 2 \rangle 4$ . LET:  $x, y \in X$

$\langle 2 \rangle 5$ . ASSUME:  $\rho((a, b), (x, y)) < \delta$

$\langle 2 \rangle 6$ .  $|d(a, b) - d(x, y)| < \epsilon$

$\langle 3 \rangle 1$ .  $d(a, b) - d(x, y) < \epsilon$

PROOF:

$$\begin{aligned}
 d(a, b) &\leq d(a, x) + d(x, y) + d(y, b) \\
 &\leq d(x, y) + 2\rho((a, b), (x, y)) \\
 &< d(x, y) + 2\delta \\
 &= d(x, y) + \epsilon
 \end{aligned}$$

$\langle 3 \rangle 2.$   $d(a, b) - d(x, y) > -\epsilon$

PROOF: Similar.

$\langle 2 \rangle 7.$  Q.E.D.

$\langle 1 \rangle 2.$  If  $\mathcal{T}$  is any topology under which  $d$  is continuous then  $\mathcal{T}$  is finer than the metric topology.

PROOF: Since  $B(a, \epsilon) = d_a^{-1}((-\infty, \epsilon))$

□

**Proposition 10.38.** *Let  $X$  be a metric space with metric  $d$  and  $A \subseteq X$ . The restriction of  $d$  to  $A$  is a metric on  $A$  that induces the subspace topology.*

PROOF:

$\langle 1 \rangle 1.$  The restriction of  $d$  to  $A$  is a metric on  $A$ .

$\langle 1 \rangle 2.$  Every open ball under  $d \upharpoonright A$  is open under the subspace topology.

PROOF:  $B_{d \upharpoonright A}(a, \epsilon) = B_d(a, \epsilon) \cap A$ .

$\langle 1 \rangle 3.$  If  $U$  is open in the subspace topology and  $x \in U$ , then there exists a  $d \upharpoonright A$ -ball  $B$  such that  $x \in B \subseteq U$ .

$\langle 2 \rangle 1.$  PICK  $V$  open in  $X$  such that  $U = V \cap A$

$\langle 2 \rangle 2.$  PICK  $\epsilon > 0$  such that  $B_d(x, \epsilon) \subseteq V$

$\langle 2 \rangle 3.$  Take  $B = B_{d \upharpoonright A}(x, \epsilon)$

□

**Corollary 10.38.1.** *A subspace of a metrizable space is metrizable.*

**Proposition 10.39.** *Every metrizable space is Hausdorff.*

PROOF:

$\langle 1 \rangle 1.$  LET:  $X$  be a metric space

$\langle 1 \rangle 2.$  LET:  $a, b \in X$  with  $a \neq b$

$\langle 1 \rangle 3.$  LET:  $\epsilon = d(a, b)/2$

$\langle 1 \rangle 4.$  LET:  $U = B(a, \epsilon)$  and  $V = B(b, \epsilon)$

$\langle 1 \rangle 5.$   $U$  and  $V$  are disjoint neighbourhoods of  $a$  and  $b$  respectively.

□

**Proposition 10.40 (CC).** *The product of a countable family of metrizable spaces is metrizable.*

PROOF:

$\langle 1 \rangle 1.$  LET:  $(X_n, d_n)$  be a sequence of metric spaces.

$\langle 1 \rangle 2.$  ASSUME: w.l.o.g. each  $d_n$  is bounded above by 1.

PROOF: By Proposition 10.34.

$\langle 1 \rangle 3.$  LET:  $D$  be the metric on  $\mathbb{R}^\omega$  defined by  $D(x, y) = \sup_i (d_i(x_i, y_i)/i)$ .

- (2)1.  $D(x, y) \geq 0$
- (2)2.  $D(x, y) = 0$  if and only if  $x = y$
- (2)3.  $D(x, y) = D(y, x)$
- (2)4.  $D(x, z) \leq D(x, y) + D(y, z)$

PROOF:

$$\begin{aligned}
 D(x, z) &= \sup_i \frac{d_i(x_i, z_i)}{i} \\
 &\leq \sup_i \frac{d_i(x_i, y_i) + d_i(y_i, z_i)}{i} \\
 &\leq \sup_i \frac{d_i(x_i, y_i)}{i} + \sup_i \frac{d_i(y_i, z_i)}{i} \\
 &= D(x, y) + D(y, z)
 \end{aligned}$$

- (1)4. Every open ball  $B_D(a, \epsilon)$  is open in the product topology.
- (2)1. PICK  $N$  such that  $1/\epsilon < N$
- (2)2.  $B_D(a, \epsilon) = \prod_{i=1}^{\infty} U_i$  where  $U_i = B(a_i, i\epsilon)$  if  $i \leq N$ , and  $U_i = X_i$  if  $i > N$
- (1)5. For any open set  $U$  and  $a \in U$ , there exists  $\epsilon > 0$  such that  $B_D(a, \epsilon) \subseteq U$ .
- (2)1. LET:  $n \geq 1$ ,  $V$  be an open set in  $\mathbb{R}$  and  $a \in \pi_n^{-1}(V)$
- (2)2. PICK  $\epsilon > 0$  such that  $B_{d_n}(a, \epsilon) \subseteq V$
- (2)3.  $B_D(a, \epsilon/n) \subseteq \pi_n^{-1}(V)$

□

**Theorem 10.41.** *Let  $X$  and  $Y$  be metric spaces and  $f : X \rightarrow Y$ . Then  $f$  is continuous if and only if, for all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that, for all  $y \in X$ , if  $d(x, y) < \delta$  then  $d(f(x), f(y)) < \epsilon$ .*

PROOF:

- (1)1. If  $f$  is continuous then, for all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that, for all  $y \in X$ , if  $d(x, y) < \delta$  then  $d(f(x), f(y)) < \epsilon$
- (2)1. ASSUME:  $f$  is continuous.
- (2)2. LET:  $x \in X$  and  $\epsilon > 0$
- (2)3. PICK a neighbourhood  $U$  of  $x$  such that  $f(U) \subseteq B(f(x), \epsilon)$
- PROOF: Theorem 9.84.
- (2)4. PICK  $\delta > 0$  such that  $B(x, \delta) \subseteq U$
- PROOF: Proposition 10.24.
- (2)5. For all  $y \in X$ , if  $d(x, y) < \delta$  then  $d(f(x), f(y)) < \epsilon$
- (1)2. If for all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that, for all  $y \in X$ , if  $d(x, y) < \delta$  then  $d(f(x), f(y)) < \epsilon$ , then  $f$  is continuous.
- (2)1. ASSUME: for all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that, for all  $y \in X$ , if  $d(x, y) < \delta$  then  $d(f(x), f(y)) < \epsilon$
- (2)2. LET:  $x \in X$  and  $V$  be a neighbourhood of  $f(x)$
- (2)3. PICK  $\epsilon > 0$  such that  $B(f(x), \epsilon) \subseteq V$
- PROOF: Proposition 10.24.
- (2)4. PICK  $\delta > 0$  such that, for all  $y \in X$ , if  $d(x, y) < \delta$  then  $d(f(x), f(y)) < \epsilon$
- PROOF: By (2)1
- (2)5. LET:  $U = B(x, \delta)$
- (2)6.  $U$  is a neighbourhood of  $x$  with  $f(U) \subseteq V$

⟨2⟩7. Q.E.D.

PROOF: Theorem 9.84.

□

**Proposition 10.42.** *Let  $X$  be a metric space. Let  $(a_n)$  be a sequence in  $X$  and  $l \in X$ . Then  $a_n \rightarrow l$  as  $n \rightarrow \infty$  if and only if, for all  $\epsilon > 0$ , there exists  $N$  such that, for all  $n \geq N$ , we have  $d(a_n, l) < \epsilon$ .*

PROOF: From Proposition 9.67. □

**Proposition 10.43.** *Every metrizable space is first countable.*

PROOF: In any metric space  $X$ , the open balls  $B(a, 1/n)$  for  $n \geq 1$  form a local basis at  $a$ .

**Example 10.44.**  $\mathbb{R}^\omega$  under the box topology is not metrizable.

**Example 10.45.** If  $J$  is uncountable then  $\mathbb{R}^J$  under the product topology is not metrizable.

**Proposition 10.46.** *A compact subspace of a metric space is bounded.*

PROOF:

⟨1⟩1. LET:  $X$  be a metric space and  $A \subseteq X$  be compact.

⟨1⟩2. PICK  $a \in A$

⟨1⟩3.  $\{B(a, n) \mid n \in \mathbb{Z}^+\}$  covers  $A$

⟨1⟩4. PICK a finite subcover  $\{B(a, n_1), \dots, B(a, n_k)\}$

⟨1⟩5. LET:  $N = \max(n_1, \dots, n_k)$

⟨1⟩6. For all  $x, y \in A$  we have  $d(x, y) < 2N$

PROOF:

$$\begin{aligned} d(x, y) &\leq d(x, a) + d(a, y) \\ &< N + N \end{aligned}$$

□

This example shows the converse does not hold:

**Example 10.47.** The space  $\mathbb{R}$  under the standard bounded metric is bounded but not compact.

**Proposition 10.48.** *A connected metric space with more than one point is uncountable.*

PROOF:

⟨1⟩1. LET:  $X$  be a connected metric space with more than one point.

⟨1⟩2. PICK  $a \in X$

⟨1⟩3.  $d(a, -) : X \rightarrow \mathbb{R}$  is continuous.

PROOF: Proposition 10.37.

⟨1⟩4.  $\{d(a, x) \mid x \in X\}$  is a connected subspace of  $\mathbb{R}$  that includes 0.

PROOF: Theorem 9.215.

⟨1⟩5.  $\{d(a, x) \mid x \in X\} \neq \{0\}$

PROOF: Since  $X$  has more than one point.  
 (1)6.  $\{d(a, x) \mid x \in X\}$  is uncountable.  
 PROOF: Since it includes a closed interval (Corollary 9.336.1).  
 $\square$

## 10.2 Real Linear Algebra

**Definition 10.49** (Square Metric). The *square metric*  $\rho$  on  $\mathbb{R}^n$  is defined by

$$\rho(\vec{x}, \vec{y}) = \max(|x_1 - y_1|, \dots, |x_n - y_n|)$$

We prove this is a metric.

PROOF:

(1)1.  $\rho(\vec{x}, \vec{y}) \geq 0$

PROOF: Immediate from definition.

(1)2.  $\rho(\vec{x}, \vec{y}) = 0$  if and only if  $\vec{x} = \vec{y}$

PROOF: Immediate from definition.

(1)3.  $\rho(\vec{x}, \vec{y}) = \rho(\vec{y}, \vec{x})$

PROOF: Immediate from definition.

(1)4.  $\rho(\vec{x}, \vec{z}) \leq \rho(\vec{x}, \vec{y}) + \rho(\vec{y}, \vec{z})$

PROOF: Since  $|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$ .

$\square$

**Proposition 10.50.** *The square metric induces the standard topology on  $\mathbb{R}^n$ .*

PROOF:

(1)1. For every  $a \in X$  and  $\epsilon > 0$ , we have  $B_\rho(a, \epsilon)$  is open in the standard product topology.

PROOF:

$$B_\rho(a, \epsilon) = (a_1 - \epsilon, a_1 + \epsilon) \times \dots \times (a_n - \epsilon, a_n + \epsilon)$$

(1)2. For any open sets  $U_1, \dots, U_n$  in  $\mathbb{R}$ , we have  $U_1 \times \dots \times U_n$  is open in the square metric topology.

(2)1. LET:  $\vec{a} \in U_1 \times \dots \times U_n$

(2)2. For  $i = 1, \dots, n$ , PICK  $\epsilon_i > 0$  such that  $(a_i - \epsilon_i, a_i + \epsilon_i) \subseteq U_i$

(2)3. LET:  $\epsilon = \min(\epsilon_1, \dots, \epsilon_n)$

(2)4.  $B_\rho(\vec{a}, \epsilon) \subseteq U$

$\square$

**Definition 10.51.** Given  $\vec{x}, \vec{y} \in \mathbb{R}^n$ , define the *sum*  $\vec{x} + \vec{y}$  by

$$(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n) .$$

**Definition 10.52.** Given  $\lambda \in \mathbb{R}$  and  $\vec{x} \in \mathbb{R}^n$ , define the *scalar product*  $\lambda\vec{x} \in \mathbb{R}^n$  by

$$\lambda(x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n)$$



**Definition 10.53** (Inner Product). Given  $\vec{x}, \vec{y} \in \mathbb{R}^n$ , define the *inner product*  $\vec{x} \cdot \vec{y} \in \mathbb{R}$  by

$$(x_1, \dots, x_n) \cdot (y_1, \dots, y_n) = x_1 y_1 + \dots + x_n y_n \quad .$$

We write  $\vec{x}^2$  for  $\vec{x} \cdot \vec{x}$ .

**Definition 10.54** (Norm). Let  $n \geq 1$ . The *norm* on  $\mathbb{R}^n$  is the function  $\| \cdot \| : \mathbb{R}^n \rightarrow \mathbb{R}$  defined by

$$\|(x_1, \dots, x_n)\| = \sqrt{x_1^2 + \dots + x_n^2}$$

**Lemma 10.55.**

$$\|\vec{x}\|^2 = \vec{x}^2$$

PROOF: Immediate from definitions.  $\square$

**Lemma 10.56.**

$$\vec{x} \cdot (\vec{y} + \vec{z}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z}$$

PROOF: Each is equal to  $(x_1 y_1 + x_1 z_1, \dots, x_n y_n + x_n z_n)$ .  $\square$

**Lemma 10.57.**

$$|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \|\vec{y}\|$$

PROOF:

$\langle 1 \rangle 1$ . ASSUME:  $\vec{x} \neq \vec{0} \neq \vec{y}$

PROOF: Otherwise both sides are 0.

$\langle 1 \rangle 2$ . LET:  $a = 1/\|\vec{x}\|$

$\langle 1 \rangle 3$ . LET:  $b = 1/\|\vec{y}\|$

$\langle 1 \rangle 4$ .  $(a\vec{x} + b\vec{y})^2 \geq 0$  and  $(a\vec{x} - b\vec{y})^2 \geq 0$

$\langle 1 \rangle 5$ .  $a^2 \|\vec{x}\|^2 + 2ab\vec{x} \cdot \vec{y} + b^2 \|\vec{y}\|^2 \geq 0$  and  $a^2 \|\vec{x}\|^2 - 2ab\vec{x} \cdot \vec{y} + b^2 \|\vec{y}\|^2 \geq 0$

$\langle 1 \rangle 6$ .  $2ab\vec{x} \cdot \vec{y} + 2 \geq 0$  and  $-2ab\vec{x} \cdot \vec{y} + 2 \geq 0$

$\langle 1 \rangle 7$ .  $\vec{x} \cdot \vec{y} \geq -1/ab$  and  $\vec{x} \cdot \vec{y} \leq 1/ab$

$\langle 1 \rangle 8$ .  $\vec{x} \cdot \vec{y} \geq -\|\vec{x}\| \|\vec{y}\|$  and  $\vec{x} \cdot \vec{y} \leq \|\vec{x}\| \|\vec{y}\|$

$\square$

**Lemma 10.58** (Triangle Inequality).

$$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$$

PROOF:

$$\begin{aligned} \|\vec{x} + \vec{y}\|^2 &= \|\vec{x}\|^2 + 2\vec{x} \cdot \vec{y} + \|\vec{y}\|^2 \\ &\leq \|\vec{x}\|^2 + 2\|\vec{x}\| \|\vec{y}\| + \|\vec{y}\|^2 && (\text{Lemma 10.57}) \\ &= (\|\vec{x}\| + \|\vec{y}\|)^2 && \square \end{aligned}$$

**Definition 10.59** (Euclidean Metric). Let  $n \geq 1$ . The *Euclidean metric* on  $\mathbb{R}^n$  is defined by

$$d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\| \quad .$$

We prove this is a metric.

(1)1.  $d(\vec{x}, \vec{y}) \geq 0$

PROOF: Immediate from definition.

(1)2.  $d(\vec{x}, \vec{y}) = 0$  if and only if  $\vec{x} = \vec{y}$

PROOF:  $d(\vec{x}, \vec{y}) = 0$  if and only if  $\vec{x} - \vec{y} = \vec{0}$ .

(1)3.  $d(\vec{x}, \vec{y}) = d(\vec{y}, \vec{x})$

PROOF: Immediate from definition.

(1)4.  $d(\vec{x}, \vec{z}) \leq d(\vec{x}, \vec{y}) + d(\vec{y}, \vec{z})$

PROOF:

$$\begin{aligned} \|\vec{x} - \vec{z}\| &= \|(\vec{x} - \vec{y}) + (\vec{y} - \vec{z})\| \\ &\leq \|\vec{x} - \vec{y}\| + \|\vec{y} - \vec{z}\| \end{aligned} \quad (\text{Lemma 10.58})$$

□

**Proposition 10.60.** *The Euclidean metric induces the standard topology on  $\mathbb{R}^n$ .*

PROOF:

(1)1. LET:  $\rho$  be the square metric.

(1)2. For all  $\vec{a} \in \mathbb{R}^n$  and  $\epsilon > 0$ , we have  $B_d(\vec{a}, \epsilon) \subseteq B_\rho(\vec{a}, \epsilon)$

(2)1. LET:  $\vec{x} \in B_d(\vec{a}, \epsilon)$

(2)2.  $\sqrt{(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2} < \epsilon$

(2)3.  $(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2 < \epsilon^2$

(2)4. For all  $i$  we have  $(x_i - a_i)^2 < \epsilon^2$

(2)5. For all  $i$  we have  $|x_i - a_i| < \epsilon$

(2)6.  $\rho(\vec{x}, \vec{a}) < \epsilon$

(1)3. For all  $\vec{a} \in \mathbb{R}^n$  and  $\epsilon > 0$ , we have  $B_\rho(\vec{a}, \epsilon/\sqrt{n}) \subseteq B_d(\vec{a}, \epsilon)$

(2)1. LET:  $\vec{x} \in B_\rho(\vec{a}, \epsilon/\sqrt{n})$

(2)2.  $\rho(\vec{x}, \vec{a}) < \epsilon/\sqrt{n}$

(2)3. For all  $i$  we have  $|x_i - a_i| < \epsilon/\sqrt{n}$

(2)4. For all  $i$  we have  $(x_i - a_i)^2 < \epsilon^2/n$

(2)5.  $d(\vec{x}, \vec{a}) < \epsilon$

(1)4. Q.E.D.

PROOF: By Lemma 10.35.

□

**Proposition 10.61.** *Let  $n \geq 0$ . For all  $c \in \mathbb{R}^n$  and  $\epsilon > 0$ , the open ball  $B(c, \epsilon)$  is path connected.*

PROOF:

(1)1. LET:  $a, b \in B(c, \epsilon)$

(1)2. LET:  $p : [0, 1] \rightarrow B(c, \epsilon)$  be the function  $p(t) = (1 - t)a + tb$

PROOF: We have  $p(t) \in B(c, \epsilon)$  for all  $t$  because

$$\begin{aligned} d(p(t), c) &= \|(1-t)a + tb - c\| \\ &= \|(1-t)(a-c) + t(b-c)\| \\ &\leq (1-t)\|a-c\| + t\|b-c\| \\ &< (1-t)\epsilon + t\epsilon \\ &= \epsilon \end{aligned}$$

$\langle 1 \rangle 3$ .  $p$  is a path from  $a$  to  $b$ .  
□

**Proposition 10.62.** *Let  $n \geq 0$ . For all  $c \in \mathbb{R}^n$  and  $\epsilon > 0$ , the closed ball  $\overline{B}(c, \epsilon)$  is path connected.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $a, b \in \overline{B}(c, \epsilon)$

$\langle 1 \rangle 2$ . LET:  $p : [0, 1] \rightarrow \overline{B}(c, \epsilon)$  be the function  $p(t) = (1-t)a + tb$

PROOF: We have  $p(t) \in \overline{B}(c, \epsilon)$  for all  $t$  because

$$\begin{aligned} d(p(t), c) &= \|(1-t)a + tb - c\| \\ &= \|(1-t)(a-c) + t(b-c)\| \\ &\leq (1-t)\|a-c\| + t\|b-c\| \\ &\leq (1-t)\epsilon + t\epsilon \\ &= \epsilon \end{aligned}$$

$\langle 1 \rangle 3$ .  $p$  is a path from  $a$  to  $b$ .  
□

**Lemma 10.63.** *If  $\sum_{i=0}^{\infty} x_i^2$  and  $\sum_{i=0}^{\infty} y_i^2$  converge then  $\sum_{i=0}^{\infty} |x_i y_i|$  converges.*

PROOF:

$\langle 1 \rangle 1$ . For all  $N \geq 0$  we have  $\sum_{i=0}^N |x_i y_i| \leq \sqrt{\sum_{i=0}^N |x_i|^2} \sqrt{\sum_{i=0}^N |y_i|^2}$

PROOF: By the Cauchy-Schwarz inequality

$\langle 1 \rangle 2$ . Q.E.D.

PROOF: Since  $\sum_{i=0}^N |x_i y_i|$  is an increasing sequence bounded above by  $(\sum_{i=0}^{\infty} x_i^2)(\sum_{i=0}^{\infty} y_i^2)$ .

□

**Corollary 10.63.1.** *If  $\sum_{i=0}^{\infty} x_i^2$  and  $\sum_{i=0}^{\infty} y_i^2$  converge then  $\sum_{i=0}^{\infty} (x_i + y_i)^2$  converges.*

PROOF: Since  $\sum_{i=0}^{\infty} x_i^2$ ,  $\sum_{i=0}^{\infty} y_i^2$  and  $2\sum_{i=0}^{\infty} x_i y_i$  all converge.

**Definition 10.64** ( $l^2$ -metric). The  $l^2$ -metric on

$$\left\{ (x_n) \in \mathbb{R}^{\omega} \mid \sum_{n=0}^{\infty} x_n^2 \text{ converges} \right\}$$

is defined by

$$d(x, y) = \left( \sum_{n=0}^{\infty} (x_n - y_n)^2 \right)^{1/2}$$

We prove this is a metric.

PROOF:

⟨1⟩1.  $d$  is well-defined.

PROOF: By Corollary 10.63.1.

⟨1⟩2.  $d(x, y) \geq 0$

⟨1⟩3.  $d(x, y) = 0$  if and only if  $x = y$

⟨1⟩4.  $d(x, y) = d(y, x)$

⟨1⟩5.  $d(x, z) \leq d(x, y) + d(y, z)$

PROOF: By Lemma 10.58.

□

**Theorem 10.65.** *Addition is a continuous function  $\mathbb{R}^2 \rightarrow \mathbb{R}$ .*

PROOF:

⟨1⟩1. LET:  $a, b \in \mathbb{R}$

⟨1⟩2. LET:  $\epsilon > 0$

⟨1⟩3. LET:  $\delta = \epsilon/2$

⟨1⟩4. LET:  $x, y \in \mathbb{R}$

⟨1⟩5. ASSUME:  $\rho((a, b), (x, y)) < \delta$

⟨1⟩6.  $|(a + b) - (x + y)| < \epsilon$

PROOF:

$$\begin{aligned} |(a + b) - (x + y)| &= |a - x| + |b - y| \\ &\leq 2\rho((a, b), (x, y)) \\ &< 2\delta \\ &= \epsilon \end{aligned}$$

⟨1⟩7. Q.E.D.

PROOF: Theorem 10.41

□

**Theorem 10.66.** *Multiplication is a continuous function  $\mathbb{R}^2 \rightarrow \mathbb{R}$ .*

PROOF:

⟨1⟩1. LET:  $a, b \in \mathbb{R}$

⟨1⟩2. LET:  $\epsilon > 0$

⟨1⟩3. LET:  $\delta = \min(\epsilon/(|a| + |b| + 1), 1)$

⟨1⟩4. LET:  $x, y \in \mathbb{R}$

⟨1⟩5. ASSUME:  $\rho((a, b), (x, y)) < \delta$

⟨1⟩6.  $|ab - xy| < \epsilon$

PROOF:

$$\begin{aligned} |ab - xy| &= |a(b - y) + (a - x)b - (a - x)(b - y)| \\ &\leq |a||b - y| + |b||a - x| + |a - x||b - y| \\ &< |a|\delta + |b|\delta + \delta^2 && ((1)5) \\ &\leq |a|\delta + |b|\delta + \delta && ((1)3) \\ &\leq \epsilon && ((1)3) \end{aligned}$$

⟨1⟩7. Q.E.D.

PROOF: Theorem 10.41

□

**Theorem 10.67.** *The function  $f : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$  defined by  $f(x) = x^{-1}$  is continuous.*

PROOF:

⟨1⟩1. For all  $a \in \mathbb{R}$  we have  $f^{-1}((a, +\infty))$  is open.

PROOF: The set is

$$(a^{-1}, +\infty) \text{ if } a > 0$$

$$(0, +\infty) \text{ if } a = 0$$

$$(-\infty, a^{-1}) \cup (0, +\infty) \text{ if } a < 0$$

⟨1⟩2. For all  $a \in \mathbb{R}$  we have  $f^{-1}((-\infty, a))$  is open.

PROOF: Similar.

⟨1⟩3. Q.E.D.

PROOF: By Proposition 9.81 and Lemma 9.104.

□

**Definition 10.68.** For  $n \geq 0$ , the *unit ball*  $B^n$  is the space  $\{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ .

**Proposition 10.69.** *For all  $n \geq 0$ , the unit ball  $B^n$  is path connected.*

PROOF:

⟨1⟩1. LET:  $a, b \in B^n$

⟨1⟩2. LET:  $p : [0, 1] \rightarrow B^n$  be the function  $p(t) = (1 - t)a + tb$

PROOF: We have  $p(t) \in B^n$  for all  $t$  because

$$\begin{aligned} \|(1 - t)a + tb\| &\leq (1 - t)\|a\| + t\|b\| \\ &\leq (1 - t) + t \\ &= 1 \end{aligned}$$

⟨1⟩3.  $p$  is a path from  $a$  to  $b$ .

□

**Definition 10.70** (Punctured Euclidean Space). For  $n \geq 0$ , defined *punctured Euclidean space* to be  $\mathbb{R}^n \setminus \{0\}$ .

**Proposition 10.71.** *For  $n > 1$ , punctured Euclidean space  $\mathbb{R}^n \setminus \{0\}$  is path connected.*

PROOF:

⟨1⟩1. LET:  $a, b \in \mathbb{R}^n \setminus \{0\}$

⟨1⟩2. CASE: 0 is on the line from  $a$  to  $b$

⟨2⟩1. PICK a point  $c$  not on the line from  $a$  to  $b$

⟨2⟩2. The path consisting of a straight line from  $a$  to  $c$  followed by a straight line from  $c$  to  $b$  is a path from  $a$  to  $b$ .

⟨1⟩3. CASE: 0 is not on the line from  $a$  to  $b$

PROOF: The straight line from  $a$  to  $b$  is a path from  $a$  to  $b$ .

**Corollary 10.71.1.** *For  $n > 1$ , the spaces  $\mathbb{R}$  and  $\mathbb{R}^n$  are not homeomorphic.*

PROOF: For any point  $a$ , the space  $\mathbb{R} \setminus \{a\}$  is disconnected.

**Definition 10.72** (Unit Sphere). For  $n \geq 1$ , the *unit sphere*  $S^{n-1}$  is the space

$$S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}.$$

**Proposition 10.73.** For  $n > 1$ , the unit sphere  $S^{n-1}$  is path connected.

PROOF: The map  $g : \mathbb{R}^n \setminus \{0\} \rightarrow S^{n-1}$  defined by  $g(x) = x/\|x\|$  is continuous and surjective. The result follows by Proposition 9.238.  $\square$

**Proposition 10.74.** Let  $f : S^1 \rightarrow \mathbb{R}$  be continuous. Then there exists  $x \in S^1$  such that  $f(x) = f(-x)$ .

PROOF:

(1)1. LET:  $g : S^1 \rightarrow \mathbb{R}$  be the function  $g(x) = f(x) - f(-x)$

PROVE: There exists  $x \in S^1$  such that  $g(x) = 0$

(1)2. ASSUME: without loss of generality  $g((1, 0)) > 0$

(1)3.  $g((-1, 0)) < 0$

(1)4. There exists  $x$  such that  $g(x) = 0$

PROOF: By the Intermediate Value Theorem.

$\square$

**Definition 10.75** (Topologist's Sine Curve). Let  $S = \{(x, \sin 1/x) \mid 0 < x \leq 1\}$ . The *topologist's sine curve* is the closure  $\overline{S}$  of  $S$ .

**Proposition 10.76.**

$$\overline{S} = S \cup (\{0\} \times [-1, 1])$$

**Proposition 10.77.** The topologist's sine curve is connected.

PROOF:

(1)1. LET:  $S = \{(x, \sin 1/x) \mid 0 < x \leq 1\}$

(1)2.  $S$  is connected.

PROOF: Theorem 9.215.

(1)3.  $\overline{S}$  is connected.

PROOF: Theorem 9.214.

$\square$

**Proposition 10.78** (CC). The topologist's sine curve is not path connected.

PROOF:

(1)1. ASSUME: for a contradiction  $p : [0, 1] \rightarrow \overline{S}$  is a path from  $(0, 0)$  to  $(1, \sin 1)$ .

(1)2.  $p^{-1}(\{0\} \times [0, 1])$  is closed.

(1)3. LET:  $b$  be the greatest element of  $p^{-1}(\{0\} \times [0, 1])$ .

(1)4.  $b < 1$

PROOF: Since  $p(1) = (1, \sin 1)$ .

(1)5. PICK a sequence  $(t_n)_{n \geq 1}$  in  $(b, 1]$  such that  $t_n \rightarrow b$  and  $\pi_2(p(t_n)) = (-1)^n$

(2)1. LET:  $n \geq 1$

(2)2. PICK  $u$  with  $0 < u < \pi_1(p(1/n))$  such that  $\sin(1/u) = (-1)^n$

(2)3. PICK  $t_n$  such that  $0 < t_n < 1/n$  and  $\pi_1(p(t_n)) = u$   
 PROOF: One exists by the Intermediate Value Theorem.  
 (1)6. Q.E.D.  
 PROOF: This contradicts 9.96.

□

**Theorem 10.79.** *Let  $A$  be a subspace of  $\mathbb{R}^n$ . Then the following are equivalent:*

1.  $A$  is compact.
2.  $A$  is closed and bounded under the Euclidean metric.
3.  $A$  is closed and bounded under the square metric.

PROOF:

(1)1.  $1 \Rightarrow 2$

PROOF: By Corollary 9.311.1 and Proposition 10.46.

(1)2.  $2 \Rightarrow 3$

PROOF: If  $d(x, y) \leq M$  for all  $x, y \in A$  then  $\rho(x, y) \leq M/\sqrt{2}$ .

(1)3.  $3 \Rightarrow 1$

(2)1. ASSUME:  $A$  is closed and  $\rho(x, y) \leq M$  for all  $x, y \in A$

(2)2. PICK  $a \in A$

PROOF: We may assume w.l.o.g.  $A$  is nonempty since the empty space is compact.

(2)3.  $A$  is a closed subspace of  $[a_1 - M, a_1 + M] \times \cdots \times [a_n - M, a_n + M]$

(2)4.  $A$  is compact

PROOF: Proposition 9.304.

□

**Corollary 10.79.1.** *The unit sphere  $S^{n-1}$  and the closed unit ball  $B^n$  are compact for any  $n$ .*

## 10.3 The Uniform Topology

**Definition 10.80** (Uniform Metric). Let  $J$  be a set. The *uniform metric*  $\bar{\rho}$  on  $\mathbb{R}^J$  is defined by

$$\bar{\rho}(a, b) = \sup_{j \in J} \bar{d}(a_j, b_j)$$

where  $\bar{d}$  is the standard bounded metric on  $\mathbb{R}$ .

The *uniform topology* on  $\mathbb{R}^J$  is the topology induced by the uniform metric.

We prove this is a metric.

PROOF:

(1)1.  $\bar{\rho}(a, b) \geq 0$

PROOF: Immediate from definitions.

(1)2.  $\bar{\rho}(a, b) = 0$  if and only if  $a = b$

PROOF: Immediate from definitions.

⟨1⟩3.  $\bar{\rho}(a, b) = \bar{\rho}(b, a)$

PROOF: Immediate from definitions.

⟨1⟩4.  $\bar{\rho}(a, c) \leq \bar{\rho}(a, b) + \bar{\rho}(b, c)$

PROOF:

$$\begin{aligned}\bar{\rho}(a, c) &= \sup_{j \in J} \bar{d}(a_j, c_j) \\ &\leq \sup_{j \in J} (\bar{d}(a_j, b_j) + \bar{d}(b_j, c_j)) \\ &\leq \sup_{j \in J} \bar{d}(a_j, b_j) + \sup_{j \in J} \bar{d}(b_j, c_j) \\ &= \bar{\rho}(a, b) + \bar{\rho}(b, c)\end{aligned}$$

□

**Proposition 10.81.** *The uniform topology on  $\mathbb{R}^J$  is finer than the product topology.*

PROOF:

⟨1⟩1. LET:  $j \in J$  and  $U$  be open in  $\mathbb{R}$

PROVE:  $\pi_j^{-1}(U)$  is open in the uniform topology.

⟨1⟩2. LET:  $a \in \pi_j^{-1}(U)$

⟨1⟩3. PICK  $\epsilon > 0$  such that  $(a_j - \epsilon, a_j + \epsilon) \subseteq U$

⟨1⟩4.  $B_{\bar{\rho}}(a, \epsilon) \subseteq \pi_j^{-1}(U)$

□

**Proposition 10.82.** *The uniform topology on  $\mathbb{R}^J$  is coarser than the box topology.*

PROOF:

⟨1⟩1. LET:  $a \in \mathbb{R}^J$  and  $\epsilon > 0$

PROVE:  $B(a, \epsilon)$  is open in the box topology.

⟨1⟩2. LET:  $b \in B(a, \epsilon)$

⟨1⟩3. For  $j \in J$  we have  $|a_j - b_j| < \epsilon$

⟨1⟩4. For  $j \in J$ ,

LET:  $\delta_j = (\epsilon - |a_j - b_j|)/2$

⟨1⟩5.  $\prod_{j \in J} (b_j - \delta_j, b_j + \delta_j) \subseteq B(a, \epsilon)$

□

**Proposition 10.83.** *The uniform topology on  $\mathbb{R}^J$  is strictly finer than the product topology if and only if  $J$  is infinite.*

PROOF:

⟨1⟩1. If  $J$  is finite then the uniform and product topologies coincide.

PROOF: The uniform, box and product topologies are all the same.

⟨1⟩2. If  $J$  is infinite then the uniform and product topologies are different.

PROOF: The set  $B(\vec{0}, 1)$  is open in the uniform topology but not the product topology.

□



**Proposition 10.84** (DC). *The uniform topology on  $\mathbb{R}^J$  is strictly coarser than the box topology if and only if  $J$  is infinite.*

PROOF:

$\langle 1 \rangle 1$ . If  $J$  is finite then the uniform and box topologies coincide.

PROOF: The uniform, box and product topologies are all the same.

$\langle 1 \rangle 2$ . If  $J$  is infinite then the uniform and box topologies are different.

PROOF: Pick an  $\omega$ -sequence  $(j_1, j_2, \dots)$  in  $J$ . Let  $U = \prod_{j \in J} U_j$  where  $U_{j_i} = (-1/i, 1/i)$  and  $U_j = (-1, 1)$  for all other  $j$ . Then  $\vec{0} \in U$  but there is no  $\epsilon > 0$  such that  $B(\vec{0}, \epsilon) \subseteq U$ .

□

**Proposition 10.85.** *The closure of  $\mathbb{R}^\infty$  in  $\mathbb{R}^\omega$  under the uniform topology is  $\mathbb{R}^\omega$ .*

PROOF: Given any open ball  $B(a, \epsilon)$ , pick an integer  $N$  such that  $1/\epsilon < N$ . Then  $B(a, \epsilon)$  includes sequences whose  $n$ th entry is 0 for all  $n \geq N$ . □

**Example 10.86.** The space  $\mathbb{R}^\omega$  is disconnected under the uniform topology. The set of bounded sequences and the set of unbounded sequences form a separation.

**Proposition 10.87.** *Give  $\mathbb{R}^\omega$  the uniform topology. Let  $x, y \in \mathbb{R}^\omega$ . Then  $x$  and  $y$  are in the same component if and only if  $x - y$  is bounded.*

PROOF:

$\langle 1 \rangle 1$ . The component containing 0 is the set of bounded sequences.

$\langle 2 \rangle 1$ . LET:  $B$  be the set of bounded sequences.

$\langle 2 \rangle 2$ .  $B$  is path-connected.

$\langle 3 \rangle 1$ . LET:  $x, y \in B$

$\langle 3 \rangle 2$ . PICK  $b > 0$  such that  $|x_j|, |y_j| \leq b$  for all  $j$

$\langle 3 \rangle 3$ . LET:  $p : [0, 1] \rightarrow B$  be the function  $p(t) = (1 - t)x + ty$

PROVE:  $p$  is continuous.

$\langle 3 \rangle 4$ . LET:  $t \in [0, 1]$  and  $\epsilon > 0$

$\langle 3 \rangle 5$ . LET:  $\delta = \epsilon/2b$

$\langle 3 \rangle 6$ . LET:  $s \in [0, 1]$  with  $|s - t| < \delta$

$\langle 3 \rangle 7$ .  $\bar{\rho}(p(s), p(t)) < \epsilon$

PROOF:

$$\begin{aligned} \bar{\rho}(p(s), p(t)) &= \sup_j \bar{d}((1-s)x_j + sy_j, (1-t)x_j + ty_j) \\ &\leq |(s-t)x_j + (t-s)y_j| \\ &\leq |s-t||x_j - y_j| \\ &< 2b\delta \\ &= \epsilon \end{aligned}$$

$\langle 2 \rangle 3$ .  $B$  is connected.

PROOF: Proposition 9.236.

$\langle 2 \rangle 4$ . If  $C$  is connected and  $B \subseteq C$  then  $B = C$ .

PROOF: Otherwise  $B \cap C$  and  $C \setminus B$  form a separation of  $C$ .  
 (1)2. Q.E.D.

PROOF: Since  $\lambda x.x - y$  is a Homeomorphism of  $\mathbb{R}^\omega$  with itself.  
 □

## 10.4 Uniform Convergence

**Definition 10.88** (Uniform Convergence). Let  $X$  be a set and  $Y$  a metric space. Let  $(f_n : X \rightarrow Y)$  be a sequence of functions and  $f : X \rightarrow Y$  be a function. Then  $f_n$  converges uniformly to  $f$  as  $n \rightarrow \infty$  if and only if, for all  $\epsilon > 0$ , there exists  $N$  such that, for all  $n \geq N$  and  $x \in X$ , we have  $d(f_n(x), f(x)) < \epsilon$ .

**Example 10.89.** Define  $f_n : [0, 1] \rightarrow \mathbb{R}$  by  $f_n(x) = x^n$  for  $n \geq 1$ , and  $f : [0, 1] \rightarrow \mathbb{R}$  by  $f(x) = 0$  if  $x < 1$ ,  $f(1) = 1$ . Then  $f_n$  converges to  $f$  pointwise but not uniformly.

**Theorem 10.90** (Uniform Limit Theorem). Let  $X$  be a topological space and  $Y$  a metric space. Let  $(f_n : X \rightarrow Y)$  be a sequence of continuous functions and  $f : X \rightarrow Y$  be a function. If  $f_n$  converges uniformly to  $f$  as  $n \rightarrow \infty$ , then  $f$  is continuous.

PROOF:

- (1)1. LET:  $x \in X$  and  $\epsilon > 0$   
 (1)2. PICK  $N$  such that, for all  $n \geq N$  and  $y \in X$ , we have  $d(f_n(y), f(y)) < \epsilon/3$   
 (1)3. PICK a neighbourhood  $U$  of  $x$  such that  $f_N(U) \subseteq B(f_N(x), \epsilon/3)$   
     PROVE:  $f(U) \subseteq B(f(x), \epsilon)$   
 (1)4. LET:  $y \in U$   
 (1)5.  $d(f(y), f(x)) < \epsilon$

PROOF:

$$\begin{aligned} d(f(y), f(x)) &\leq d(f(y), f_N(y)) + d(f_N(y), f_N(x)) + d(f_N(x), f(x)) \quad (\text{Triangle Inequality}) \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 \quad ((1)2, (1)3) \\ &= \epsilon \end{aligned}$$

□

**Proposition 10.91.** Let  $X$  be a topological space and  $Y$  a metric space. Let  $(f_n : X \rightarrow Y)$  be a sequence of continuous functions and  $f : X \rightarrow Y$  be a function. Let  $(a_n)$  be a sequence of points in  $X$  and  $a \in X$ . If  $f_n$  converges uniformly to  $f$  and  $a_n$  converges to  $a$  in  $X$  then  $f_n(a_n)$  converges to  $f(a)$  uniformly in  $Y$ .

PROOF:

- (1)1. LET:  $\epsilon > 0$   
 (1)2. PICK  $N_1$  such that, for all  $n \geq N_1$  and  $x \in X$ , we have  $d(f_n(x), f(x)) < \epsilon/2$   
 (1)3. PICK  $N_2$  such that, for all  $n \geq N_2$ , we have  $a_n \in f^{-1}(B(a, \epsilon/2))$

PROOF: Using the fact that  $f$  is continuous from the Uniform Limit Theorem.

⟨1⟩4. LET:  $N = \max(N_1, N_2)$

⟨1⟩5. LET:  $n \geq N$

⟨1⟩6.  $d(f_n(a_n), f(a)) < \epsilon$

PROOF:

$$\begin{aligned} d(f_n(a_n), f(a)) &\leq d(f_n(a_n), f(a_n)) + d(f(a_n), f(a)) \quad (\text{Triangle Inequality}) \\ &< \epsilon/2 + \epsilon/2 \quad (\langle 1 \rangle 2, \langle 1 \rangle 3) \\ &= \epsilon \end{aligned}$$

□

**Proposition 10.92.** *Let  $X$  be a set. Let  $(f_n : X \rightarrow \mathbb{R})$  be a sequence of functions and  $f : X \rightarrow \mathbb{R}$  be a function. Then  $f_n$  converges uniformly to  $f$  as  $n \rightarrow \infty$  if and only if  $f_n \rightarrow f$  as  $n \rightarrow \infty$  in  $\mathbb{R}^X$  under the uniform topology.*

PROOF:

⟨1⟩1. If  $f_n$  converges uniformly to  $f$  then  $f_n$  converges to  $f$  under the uniform topology.

⟨2⟩1. ASSUME:  $f_n$  converges uniformly to  $f$

⟨2⟩2. LET:  $\epsilon > 0$

⟨2⟩3. PICK  $N$  such that, for all  $n \geq N$  and  $x \in X$ , we have  $d(f_n(x), f(x)) < \epsilon/2$

⟨2⟩4. For all  $n \geq N$  we have  $\bar{\rho}(f_n, f) \leq \epsilon/2$

⟨2⟩5. For all  $n \geq N$  we have  $\bar{\rho}(f_n, f) < \epsilon$

⟨1⟩2. If  $f_n$  converges to  $f$  under the uniform topology then  $f_n$  converges uniformly to  $f$ .

⟨2⟩1. ASSUME:  $f_n$  converges to  $f$  under the uniform topology.

⟨2⟩2. LET:  $\epsilon > 0$

⟨2⟩3. PICK  $N$  such that, for all  $n \geq N$ , we have  $\bar{\rho}(f_n, f) < \min(\epsilon, 1/2)$

⟨2⟩4. LET:  $n \geq N$

⟨2⟩5. LET:  $x \in X$

⟨2⟩6.  $\bar{\rho}(f_n, f) < \min(\epsilon, 1/2)$

PROOF: From ⟨2⟩3.

⟨2⟩7.  $d(f_n(x), f(x)) < \min(\epsilon, 1/2)$

⟨2⟩8.  $d(f_n(x), f(x)) < \epsilon$

□

## 10.5 Isometric Imbeddings

**Definition 10.93.** Let  $X$  and  $Y$  be metric spaces. An *isometric imbedding*  $f : X \rightarrow Y$  is a function such that, for all  $x, y \in X$ , we have  $d(f(x), f(y)) = d(x, y)$ .

**Proposition 10.94.** *Every isometric imbedding is an imbedding.*

PROOF:

⟨1⟩1. LET:  $f : X \rightarrow Y$  be an isometric imbedding.

⟨1⟩2.  $f$  is injective.

PROOF: If  $f(x) = f(y)$  then  $d(f(x), f(y)) = 0$  hence  $d(x, y) = 0$  hence  $x = y$ .

⟨1⟩3.  $f$  is continuous.

PROOF: For all  $\epsilon > 0$ , if  $d(x, y) < \epsilon$  then  $d(f(x), f(y)) < \epsilon$ .

⟨1⟩4.  $f : X \rightarrow f(X)$  is an open map.

PROOF:  $f(B(a, \epsilon)) = B(f(a), \epsilon) \cap f(X)$ .

□

## 10.6 Distance to a Set

**Definition 10.95.** Let  $X$  be a metric space,  $x \in X$  and  $A \subseteq X$  be nonempty. The *distance* from  $x$  to  $A$  is defined as

$$d(x, A) = \inf_{a \in A} d(x, a) .$$

**Proposition 10.96.** Let  $X$  be a metric space and  $A \subseteq X$  be nonempty. Then the function  $d(-, A) : X \rightarrow \mathbb{R}$  is continuous.

PROOF:

⟨1⟩1. LET:  $X$  be a metric space.

⟨1⟩2. LET:  $A \subseteq X$  be nonempty.

⟨1⟩3. LET:  $x \in X$  and  $\epsilon > 0$

⟨1⟩4. LET:  $\delta = \epsilon$

⟨1⟩5. LET:  $y \in B(x, \delta)$

⟨1⟩6.  $|d(x, A) - d(y, A)| < \epsilon$

⟨2⟩1.  $d(x, A) - d(y, A) < \epsilon$

PROOF:

⟨3⟩1. For all  $a \in A$  we have  $d(x, A) \leq d(x, y) + d(y, a)$

PROOF:

$$d(x, A) \leq d(x, a) \quad (\text{definition of } d(x, A))$$

$$\leq d(x, y) + d(y, a) \quad (\text{Triangle Inequality})$$

⟨3⟩2.  $d(x, A) - d(x, y) \leq d(y, A)$

⟨2⟩2.  $d(y, A) - d(x, A) < \epsilon$

PROOF: Similar.

⟨1⟩7. Q.E.D.

PROOF: Theorem 10.41.

□

**Theorem 10.97.** Let  $X$  be a metric space,  $A \subseteq X$  be nonempty, and  $x \in X$ . Then  $d(x, A) = 0$  if and only if  $x \in \overline{A}$ .

PROOF:

⟨1⟩1. LET:  $X$  be a metric space.

⟨1⟩2. LET:  $A \subseteq X$  be nonempty.

⟨1⟩3. LET:  $x \in X$

⟨1⟩4. If  $d(x, A) = 0$  then  $x \in \overline{A}$

⟨2⟩1. ASSUME:  $d(x, A) = 0$

⟨2⟩2. LET:  $U$  be any neighbourhood of  $x$ .

(2)3. PICK  $\epsilon > 0$  such that  $B(x, \epsilon) \subseteq U$   
 PROOF: Proposition 10.24, (1)1, (2)2.  
 (2)4. PICK  $a \in A$  such that  $d(x, a) < \epsilon$   
 PROOF: From (2)1, (2)3.  
 (2)5.  $a \in A \cap U$   
 PROOF: From (2)3, (2)4.  
 (2)6. Q.E.D.  
 PROOF: Theorem 9.38.  
 (1)5. If  $x \in \bar{A}$  then  $d(x, A) = 0$   
 $\square$

**Theorem 10.98.** *Let  $X$  be a metric space. Let  $A \subseteq X$  be nonempty and compact. Let  $x \in X$ . Then there exists  $a \in A$  such that  $d(x, A) = d(x, a)$ .*

PROOF: By the Extreme Value Theorem, the function  $d(x, -) : A \rightarrow \mathbb{R}$  attains its minimum.  $\square$

## 10.7 Lebesgue Numbers

**Definition 10.99** (Lebesgue Number). Let  $X$  be a metric space. Let  $\mathcal{U}$  be an open covering of  $X$ . A *Lebesgue number* for  $\mathcal{U}$  is a real number  $\delta > 0$  such that, for every subset  $A \subseteq X$  with diameter  $\text{diam}(A) < \delta$ , there exists  $U \in \mathcal{U}$  such that  $A \subseteq U$ .

**Theorem 10.100** (Lebesgue Number Lemma). *Every open covering of a compact metric space has a Lebesgue number.*

PROOF:

(1)1. LET:  $X$  be a compact metric space.  
 (1)2. LET:  $\mathcal{U}$  be an open covering of  $X$ .  
 (1)3. PICK a finite subset  $\{U_1, \dots, U_n\}$  of  $\mathcal{U}$  that covers  $X$ .  
 (1)4. For  $i = 1, \dots, n$ ,  
 LET:  $C_i = X - U_i$   
 (1)5. LET:  $f : X \rightarrow \mathbb{R}$ ,

$$f(x) = 1/n \sum_{i=1}^n d(x, C_i)$$

(1)6. For all  $x \in X$  we have  $f(x) > 0$   
 (2)1. LET:  $x \in X$   
 (2)2. PICK  $i$  such that  $x \in U_i$   
 PROOF: From (1)3.  
 (2)3. PICK  $\epsilon > 0$  such that  $B(x, \epsilon) \subseteq U_i$   
 PROOF: Proposition 10.24.  
 (2)4.  $d(x, C_i) \geq \epsilon$   
 (2)5.  $f(x) \geq \epsilon/n$   
 (1)7.  $f$  is continuous.  
 PROOF: Proposition 10.96.

- (1)8. LET:  $\delta$  be the minimum value of  $f(X)$   
 PROOF: By the Extreme Value Theorem  
 (1)9.  $\delta > 0$   
 PROOF: From (1)6  
 (1)10. For every subset  $A \subseteq X$  with diameter  $< \delta$ , there exists  $U \in \mathcal{U}$  such that  
 $A \subseteq U$   
 (2)1. LET:  $A \subseteq X$  with  $\text{diam } A < \delta$   
 (2)2. PICK  $x_0 \in A$   
 (2)3.  $A \subseteq B(x_0, \delta)$   
 (2)4.  $f(x_0) \geq \delta$   
 (2)5. PICK  $m$  such that  $d(x_0, C_m)$  is the largest out of  $d(x_0, C_1), \dots, d(x_0, C_n)$   
 (2)6.  $d(x_0, C_m) \geq f(x_0)$   
 (2)7.  $B(x_0, \delta) \subseteq U_m$   
 (2)8.  $A \subseteq U_m$   
 (1)11.  $\delta$  is a Lebesgue number for  $\mathcal{U}$   
 $\square$

**Theorem 10.101 (AC).** *Every sequentially compact metric space is compact.*

PROOF:

- (1)1. LET:  $X$  be a sequentially compact metric space.  
 (1)2. Every open covering of  $X$  has a Lebesgue number.  
 (2)1. LET:  $\mathcal{A}$  be an open covering of  $X$ .  
 (2)2. ASSUME: for a contradiction  $\mathcal{A}$  has no Lebesgue number.  
 (2)3. For  $n \geq 1$ , PICK a set  $C_n$  with diameter  $< 1/n$  that is not included in any member of  $\mathcal{A}$ .  
 (2)4. For  $n \geq 1$ , PICK  $x_n \in C_n$ .  
 (2)5. PICK a convergent subsequence  $(C_{n_r})$  of  $(C_n)$  with limit  $a$ .  
 (2)6. PICK  $A \in \mathcal{A}$  such that  $a \in A$ .  
 (2)7. PICK  $\epsilon > 0$  such that  $B(a, \epsilon) \subseteq A$ .  
 (2)8. PICK  $r$  such that  $1/n_r < \epsilon/2$  and  $d(x_{n_r}, a) < \epsilon/2$   
 (2)9.  $C_{n_r} \subseteq B(a, \epsilon)$   
 (2)10.  $C_{n_r} \subseteq A$   
 (2)11. Q.E.D.  
 PROOF: This contradicts (2)3.  
 (1)3. For every  $\epsilon > 0$ , there exists a finite covering of  $X$  by  $\epsilon$ -balls.  
 (2)1. ASSUME: for a contradiction that there exists  $\epsilon > 0$  such that  $X$  cannot be finitely covered by  $\epsilon$ -balls.  
 (2)2. PICK a sequence of points  $(x_n)$  such that  $x_n \in X - (B(x_1, \epsilon) \cup \dots \cup B(x_{n-1}, \epsilon))$   
 (2)3.  $d(x_m, x_n) \geq \epsilon$  for all  $m, n$  distinct  
 (2)4.  $(x_n)$  has no convergent subsequence  
 (2)5. Q.E.D.  
 PROOF: This contradicts (1)1.  
 (1)4. LET:  $\mathcal{A}$  be an open covering of  $X$ .  
 (1)5. PICK a Lebesgue number  $\delta$  for  $\mathcal{A}$ .  
 PROOF: By (1)2.

- ⟨1⟩6. LET:  $\epsilon = \delta/3$   
 ⟨1⟩7. PICK a finite covering  $\{B_1, \dots, B_n\}$  of  $X$  be  $\epsilon$ -balls.  
 PROOF: By ⟨1⟩3.  
 ⟨1⟩8. For  $i = 1, \dots, n$ , PICK  $U_i \in \mathcal{A}$  such that  $B_i \subseteq A_i$   
 PROOF: By ⟨1⟩5 since  $\text{diam } B_i = 2\epsilon < \delta$ .  
 ⟨1⟩9.  $\{U_1, \dots, U_n\}$  covers  $X$ .  
 □

## 10.8 Uniform Continuity

**Definition 10.102** (Uniformly Continuous). Let  $X$  and  $Y$  be metric spaces. Let  $f : X \rightarrow Y$ . Then  $f$  is *uniformly continuous* if and only if, for every  $\epsilon > 0$ , there exists  $\delta > 0$  such that, for all  $x, y \in X$ , if  $d(x, y) < \delta$  then  $d(f(x), f(y)) < \epsilon$ .

**Theorem 10.103** (Uniform Continuity Theorem). *Every continuous function from a compact metric space to a metric space is uniformly continuous.*

PROOF:

- ⟨1⟩1. LET:  $X$  be a compact metric space.  
 ⟨1⟩2. LET:  $Y$  be a metric space.  
 ⟨1⟩3. LET:  $f : X \rightarrow Y$  be a continuous function.  
 ⟨1⟩4. LET:  $\epsilon > 0$   
 ⟨1⟩5. LET:  $\mathcal{U} = \{f^{-1}(B(y, \epsilon/2)) \mid y \in Y\}$   
 ⟨1⟩6. PICK a Lebesgue number  $\delta > 0$  for  $\mathcal{U}$ .  
 PROOF: By the Lebesgue Number Lemma.  
 ⟨1⟩7. LET:  $x, x' \in X$   
 ⟨1⟩8. ASSUME:  $d(x, x') < \delta$   
 ⟨1⟩9. PICK  $y \in Y$  such that  $\{x, x'\} \subseteq f^{-1}(B(y, \epsilon/2))$   
 PROOF: Since  $\text{diam}\{x, x'\} < \delta$ .  
 ⟨1⟩10.  $d(f(x), f(x')) < \epsilon$

PROOF:

$$\begin{aligned}
 d(f(x), f(x')) &\leq d(f(x), y) + d(y, f(x')) && \text{(Triangle Inequality)} \\
 &< \epsilon/2 + \epsilon/2 && (\langle 1 \rangle 9) \\
 &= \epsilon
 \end{aligned}$$

□

## 10.9 Epsilon-neighbourhoods

**Definition 10.104** ( $\epsilon$ -neighbourhood). Let  $X$  be a metric space. Let  $A \subseteq X$  be nonempty. Let  $\epsilon > 0$ . Then the  $\epsilon$ -neighbourhood of  $A$ ,  $U(A, \epsilon)$ , is the set

$$U(A, \epsilon) = \{x \in X \mid d(x, A) < \epsilon\}.$$

**Proposition 10.105.** *Let  $X$  be a metric space. Let  $A \subseteq X$  be nonempty. Let  $\epsilon > 0$ . Then  $U(A, \epsilon) = \bigcup_{a \in A} B(a, \epsilon)$ .*

PROOF:

- ⟨1⟩1. LET:  $X$  be a metric space.
- ⟨1⟩2. LET:  $A \subseteq X$  be nonempty.
- ⟨1⟩3. LET:  $\epsilon > 0$
- ⟨1⟩4.  $U(A, \epsilon) \subseteq \bigcup_{a \in A} B(a, \epsilon)$ 
  - ⟨2⟩1. LET:  $x \in U(A, \epsilon)$
  - ⟨2⟩2.  $d(x, A) < \epsilon$
  - ⟨2⟩3.  $\epsilon$  is not a lower bound for  $\{d(x, a) \mid a \in A\}$
  - ⟨2⟩4. PICK  $a \in A$  such that  $d(x, a) < \epsilon$
  - ⟨2⟩5.  $x \in B(a, \epsilon)$
- ⟨1⟩5.  $\bigcup_{a \in A} B(a, \epsilon) \subseteq U(A, \epsilon)$ 
  - ⟨2⟩1. LET:  $a \in A$  and  $x \in B(a, \epsilon)$
  - ⟨2⟩2.  $d(x, A) \leq d(x, a)$
  - ⟨2⟩3.  $d(x, A) < \epsilon$
  - ⟨2⟩4.  $x \in U(A, \epsilon)$

□

**Proposition 10.106.** *Let  $X$  be a metric space. Let  $A \subseteq X$  be nonempty and compact. Let  $U$  be an open set such that  $A \subseteq U$ . Then there exists  $\epsilon > 0$  such that  $U(A, \epsilon) \subseteq U$ .*

PROOF:

- ⟨1⟩1. LET:  $X$  be a metric space.
- ⟨1⟩2. LET:  $A \subseteq X$  be nonempty and compact.
- ⟨1⟩3. LET:  $U$  be an open set such that  $A \subseteq U$
- ⟨1⟩4.  $\{B(a, \epsilon) \mid a \in A, \epsilon > 0, B(a, 2\epsilon) \subseteq U\}$  covers  $A$ .  
PROOF: By Proposition 10.24.
- ⟨1⟩5. PICK a finite subcover  $\{B(a_1, \epsilon_1), \dots, B(a_n, \epsilon_n)\}$   
PROOF: Since  $A$  is compact (⟨1⟩2).
- ⟨1⟩6. LET:  $\epsilon = \min(\epsilon_1, \dots, \epsilon_n)$   
PROVE:  $U(A, \epsilon) \subseteq U$
- ⟨1⟩7. LET:  $x \in U(A, \epsilon)$
- ⟨1⟩8. PICK  $a \in A$  such that  $d(x, a) < \epsilon$   
PROOF: Proposition 10.105.
- ⟨1⟩9. PICK  $i$  such that  $a \in B(a_i, \epsilon_i)$   
PROOF: By ⟨1⟩5.
- ⟨1⟩10.  $d(x, a_i) < 2\epsilon$   
PROOF: By the Triangle Inequality.
- ⟨1⟩11.  $x \in U$   
PROOF: From ⟨1⟩4.

□

This example shows that we cannot weaken the hypothesis that  $A$  is compact to  $A$  being closed:

**Example 10.107.** Let  $X = \mathbb{R}^2$ . Let  $A = \{(x, 1/x) \mid x > 0\}$ . Let  $U = \{(x, y) \mid x > 0, y > 0\}$ . Then  $A$  is nonempty and closed (Proposition 9.315). The set  $U$  is open and  $A \subseteq U$ . But there is no  $\epsilon > 0$  such that  $U(A, \epsilon) \subseteq U$ .



PROOF:

$\langle 1 \rangle 1.$  LET:  $\epsilon > 0$

$\langle 1 \rangle 2.$   $(2/\epsilon, \epsilon/2) \in A$

$\langle 1 \rangle 3.$   $(2/\epsilon, 0) \in U(A, \epsilon)$

$\langle 1 \rangle 4.$   $(2/\epsilon, 0) \notin U$

□