# Topology

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# Part I

# Set Theory

# Chapter 1

# Set Theory

## 1.1 Membership

We take as undefined the binary relation of *membership*, $\in$. If $a \in A$ we say $a$ is a *member* or *element* of $A$. If this does not hold, we write $a \notin A$.

**Axiom 1.1** (Axiom of Extensionality)**.** *Two sets with exactly the same elements are equal.*

## 1.2 Subsets

**Definition 1.2** (Subset)**.** Let $A$ and $B$ be sets. We say $A$ is a *subset* of $B$, $A \subseteq B$, if and only if every member of $A$ is a member of $B$.

## 1.3 The Empty Set

**Axiom 1.3** (Empty Set Axiom)**.** *There exists a set with no members.*

**Proposition 1.4** (Extensionality, Empty Set Axiom)**.** *There exists a unique set with no members.*

PROOF: Existence follows from the Empty Set Axiom, and uniqueness from the Axiom of Extensionality. $\square$

**Definition 1.5** (Empty Set (Extensionality, Empty Set Axiom))**.** The *empty set* $\emptyset$ is the set with no members.

## 1.4 Pair Sets

**Axiom 1.6** (Pairing Axiom)**.** *For any sets $u$ and $v$, there exists a set having as members just $u$ and $v$.*

**Proposition 1.7** (Extensionality, Pairing Axiom)**.** *For any sets $u$ and $v$, there exists a unique set having as members just $u$ and $v$.*

Proof: Existence follows from the Pairing Axiom, and uniqueness from the Axiom of Extensionality. □

**Definition 1.8** (Pair Set (Extensionality, Pairing Axiom))**.** For any sets $u$ and $v$, the *pair set* $\{u, v\}$ is the set whose members are just $u$ and $v$.

## 1.5 Unions

**Axiom 1.9** (Union Axiom, Preliminary Form)**.** *For any sets $a$ and $b$, there exists a set whose members are those sets belonging to $a$ or to $b$ (or both).*

**Proposition 1.10** (Extensionality, Union Axiom Preliminary Form)**.** *For any sets $a$ and $b$, there exists a unique set whose members are those sets belonging to $a$ or to $b$ (or both).*

Proof: Existence follows from the Union Axiom Preliminary Form, and uniqueness from the Axiom of Extensionality. □

**Definition 1.11** (Union (Extensionality, Union Axiom Preliminary Form))**.** For any sets $a$ and $b$, the *union* $a \cup b$ is the unique set whose members are those sets belonging to $a$ or to $b$ (or both).

## 1.6 Power Set

**Axiom 1.12** (Power Set Axiom)**.** *For any set $a$, there is a set whose members are exactly the subsets of $a$.*

**Proposition 1.13** (Extensionality, Power Set)**.** *For any set $a$, there exists a unique set whose members are the subsets of $a$*

Proof: Existence follows from the Power Set Axiom, and uniqueness from the Axiom of Extensionality. □

**Definition 1.14** (Power Set (Extensionality, Power Set))**.** For any set $a$, the *power set* $\mathcal{P}a$ is the unique set whose members are those sets belonging to $a$ or to $b$ (or both).

## 1.7 Covers

**Definition 1.15** (Cover)**.** Let $X$ be a set and $\mathcal{A} \subseteq \mathcal{P}X$. Then $\mathcal{A}$ *covers* $X$, or is a *covering* of $X$, if and only if $\bigcup \mathcal{A} = X$.

## 1.8 The Finite Intersection Property

**Definition 1.16** (Finite Intersection Property)**.** Let $X$ be a set and $\mathcal{A} \subseteq \mathcal{P}X$. Then $\mathcal{A}$ satisfies the *finite intersection property* if and only if every nonempty finite subset of $\mathcal{A}$ has nonempty intersection.

**Lemma 1.17.** *Let $X$ be a set. Let $\mathcal{A} \subseteq \mathcal{P}X$ have the finite intersection property. Then there exists a maximal set $\mathcal{D}$ such that $\mathcal{A} \subseteq \mathcal{D} \subseteq \mathcal{P}X$ and $\mathcal{D}$ has the finite intersection property.*

PROOF:
⟨1⟩1. LET: $\mathbb{F} = \{\mathcal{D} \mid \mathcal{A} \subseteq \mathcal{D} \subseteq \mathcal{P}X, \mathcal{D}$ has the finite intersection property$\}$
⟨1⟩2. Every chain in $\mathbb{F}$ has an upper bound.
  ⟨2⟩1. LET: $\mathbb{C}$ be a chain in $\mathbb{F}$.
  ⟨2⟩2. ASSUME: without loss of generality $\mathbb{C} \neq \emptyset$
    PROVE: $\bigcup \mathbb{C} \in \mathbb{F}$
   PROOF: If $\mathbb{C} = \emptyset$ then $\mathcal{A}$ is an upper bound.
  ⟨2⟩3. $\mathcal{A} \subseteq \bigcup \mathbb{C} \subseteq \mathcal{P}X$
  ⟨2⟩4. LET: $C_1, \ldots, C_n \in \mathbb{C}$
    PROVE: $C_1 \cap \cdots \cap C_n \neq \emptyset$
  ⟨2⟩5. PICK $\mathcal{C}_1, \ldots, \mathcal{C}_n \in \mathbb{C}$ such that $C_i \in \mathcal{C}_i$ for all $i$.
  ⟨2⟩6. ASSUME: without loss of generality $\mathcal{C}_1 \subseteq \cdots \subseteq \mathcal{C}_n$
  ⟨2⟩7. $C_1, \ldots, C_n \in \mathcal{C}_n$
  ⟨2⟩8. $\mathcal{C}_n$ satisfies the finite intersection property.
  ⟨2⟩9. $C_1 \cap \cdots \cap C_n \neq \emptyset$
⟨1⟩3. Q.E.D.
  PROOF: By Zorn's Lemma.
□

**Lemma 1.18.** *Let $X$ be a set. Let $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the finite intersection property. Then any finite intersection of elements of $\mathcal{D}$ is an element of $\mathcal{D}$.*

PROOF:
⟨1⟩1. LET: $D_1, D_2 \in \mathcal{D}$
⟨1⟩2. $\mathcal{D} \cup \{D_1 \cap D_2\}$ has the finite intersection property.
  PROOF: Any finite intersection of members of $\mathcal{D} \cup \{D_1 \cap D_2\}$ is a finite intersection of members of $\mathcal{D}$.
⟨1⟩3. $\mathcal{D} = \mathcal{D} \cup \{D_1 \cap D_2\}$
  PROOF: By maximality of $\mathcal{D}$.
⟨1⟩4. $D_1 \cap D_2 \in \mathcal{D}$.
□

**Lemma 1.19.** *Let $X$ be a set. Let $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the finite intersection property. Let $A \subseteq X$. If $A$ intersects every member of $\mathcal{D}$ then $A \in \mathcal{D}$.*

PROOF:

⟨1⟩1. $\mathcal{D} \cup \{A\}$ has the finite intersection property.
⟨2⟩1. Let: $D_1, \ldots, D_n \in \mathcal{D}$
Prove: $D_1 \cap \cdots \cap D_n \cap A \neq \emptyset$
⟨2⟩2. $D_1 \cap \cdots \cap D_n \in \mathcal{D}$
Proof: Lemma 1.18.
⟨2⟩3. $D_1 \cap \cdots \cap D_n \cap A \neq \emptyset$
Proof: Since $A$ intersects every member of $\mathcal{D}$.
⟨1⟩2. Q.E.D.
Proof: By maximality of $\mathcal{D}$.
□

**Proposition 1.20.** *Let $X$ be a set. Let $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the finite intersection property. Let $A, D \in \mathcal{P}X$. If $D \in \mathcal{D}$ and $D \subseteq A$ then $A \in \mathcal{D}$.*

Proof:
⟨1⟩1. $\mathcal{D} \cup \{A\}$ satisfies the finite intersection property.
⟨2⟩1. Let: $D_1, \ldots, D_n \in \mathcal{D}$
⟨2⟩2. $D_1 \cap \cdots \cap D_n \cap D \neq \emptyset$
Proof: Since $\mathcal{D}$ satisfies the finite intersection property.
⟨2⟩3. $D_1 \cap \cdots \cap D_n \cap A \neq \emptyset$
⟨1⟩2. $\mathcal{D} = \mathcal{D} \cup \{A\}$
Proof: By the maximality of $\mathcal{D}$.
⟨1⟩3. $A \in \mathcal{D}$
□

**Definition 1.21** (Graph)**.** Let $f : A \to B$. The *graph* of $f$ is the set $\{(x, f(x)) \mid x \in A\} \subseteq A \times B$.

## 1.9 Countable Intersection Property

**Definition 1.22** (Countable Intersection Property)**.** Let $X$ be a set and $\mathcal{A} \subseteq \mathcal{P}X$. Then $\mathcal{A}$ satisfies the *countable intersection property* if and only if every countable subset of $\mathcal{A}$ has nonempty intersection.

**Lemma 1.23.** *Let $X$ be a set. Let $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the countable intersection property. Then any countable intersection of elements of $\mathcal{D}$ is an element of $\mathcal{D}$.*

Proof:
⟨1⟩1. Let: $\mathcal{D}_0 \subseteq \mathcal{D}$ be countable.
⟨1⟩2. $\mathcal{D} \cup \{\bigcap \mathcal{D}_0\}$ has the countable intersection property.
Proof: Any countable intersection of members of $\mathcal{D} \cup \{\bigcap \mathcal{D}_0\}$ is a finite intersection of members of $\mathcal{D}$.
⟨1⟩3. $\mathcal{D} = \mathcal{D} \cup \{\bigcap \mathcal{D}_0\}$
Proof: By maximality of $\mathcal{D}$.
⟨1⟩4. $\bigcap \mathcal{D}_0 \in \mathcal{D}$.

□

**Lemma 1.24.** *Let $X$ be a set. Let $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the countable intersection property. Let $A \subseteq X$. If $A$ intersects every member of $\mathcal{D}$ then $A \in \mathcal{D}$.*

PROOF:
⟨1⟩1. $\mathcal{D} \cup \{A\}$ has the countable intersection property.
  ⟨2⟩1. LET: $\mathcal{D}_0 \subseteq \mathcal{D}$ be countable.
      PROVE: $\bigcap \mathcal{D}_0 \cap A \neq \emptyset$
  ⟨2⟩2. $\bigcap \mathcal{D}_0 \in \mathcal{D}$
    PROOF: Lemma 1.23.
  ⟨2⟩3. $\bigcap \mathcal{D}_0 \cap A \neq \emptyset$
    PROOF: Since $A$ intersects every member of $\mathcal{D}$.
⟨1⟩2. Q.E.D.
  PROOF: By maximality of $\mathcal{D}$.
□

## 1.10 The Axiom of Choice

**Axiom 1.25** (Axiom of Choice)**.** *Let $\mathcal{A}$ be a set of disjoint nonempty sets. Then there exists a set $C$ consisting of exactly one element from each member of $\mathcal{A}$.*

## 1.11 Choice Functions

**Definition 1.26** (Choice Function)**.** Let $\mathcal{B}$ be a set of nonempty sets. A *choice function* for $\mathcal{B}$ is a function $c : \mathcal{B} \to \bigcup \mathcal{B}$ such that, for all $B \in \mathcal{B}$, we have $c(B) \in B$.

**Lemma 1.27** (Existence of a Choice Function (AC))**.** *Every set of nonempty sets has a choice function.*

PROOF:
⟨1⟩1. LET: $\mathcal{B}$ be a set of nonempty sets.
⟨1⟩2. For $B \in \mathcal{B}$,
    LET: $B' = \{B\} \times B$
⟨1⟩3. $\{B' \mid B \in \mathcal{B}\}$ is a set of disjoint nonempty sets.
⟨1⟩4. PICK a set $c$ consisting of exactly one element from each $B'$ for $B \in \mathcal{B}$.
⟨1⟩5. $c$ is a choice function for $\mathcal{B}$.
□

## 1.12 Order Theory

**Definition 1.28** (Preorder)**.** Let $X$ be a set. A *preorder* on $X$ is a binary relation $\leq$ on $X$ such that:

**Reflexivity** For all $x \in X$, we have $x \leq x$

**Transitivity** For all $x, y, z \in X$, if $x \leq y$ and $y \leq z$ then $x \leq z$.

**Definition 1.29** (Preordered Set)**.** A *preordered set* consists of a set $X$ and a preorder $\leq$ on $X$.

**Proposition 1.30.** *Let $X$ and $Y$ be linearly ordered sets. Let $f : X \twoheadrightarrow Y$ be strictly monotone and surjective. Then $f$ is a poset isomorphism.*

PROOF:
⟨1⟩1. $f$ is injective.
  ⟨2⟩1. LET: $x, y \in X$
  ⟨2⟩2. ASSUME: $f(x) = f(y)$
  ⟨2⟩3. $x \nless y$
    PROOF: By strong motonicity.
  ⟨2⟩4. $y \nless x$
    PROOF: By strong motonicity.
  ⟨2⟩5. $x = y$
    PROOF: By trichotomy.
⟨1⟩2. $f^{-1}$ is monotone.
  ⟨2⟩1. LET: $x, y \in X$
  ⟨2⟩2. ASSUME: $x \leq y$
  ⟨2⟩3. $f^{-1}(x) \ngtr f^{-1}(y)$
    PROOF: By strong motonicity.
  ⟨2⟩4. $f^{-1}(x) < f^{-1}(y)$
    PROOF: By trichotomy.
□

**Definition 1.31** (Interval)**.** Let $X$ be a preordered set and $Y \subseteq X$. Then $Y$ is an *interval* if and only if, for all $a, b \in Y$ and $c \in X$, if $a \leq c \leq b$ then $c \in Y$.

**Definition 1.32** (Linear Continuum)**.** A linearly ordered set $L$ is a *linear continuum* if and only if:

1. every nonempty subset of $L$ that is bounded above has a supremum
2. $L$ is dense

**Proposition 1.33.** *Every interval in a linear continuum is a linear continuum.*

PROOF:
⟨1⟩1. LET: $L$ be a linear continuum and $I$ an interval in $L$.
⟨1⟩2. Every nonempty subset of $I$ that is bounded above has a supremum in $I$.
  ⟨2⟩1. LET: $X \subseteq I$ be nonempty and bounded above by $b \in I$.

⟨2⟩2. LET: $s$ be the supremum of $X$ in $L$.
PROOF: Since $L$ is a linear continuum.
⟨2⟩3. $s \in I$
⟨3⟩1. PICK $a \in X$
PROOF: Since $X$ is nonempty (⟨2⟩1).
⟨3⟩2. $a \leq s \leq b$
⟨3⟩3. $s \in I$
PROOF: Since $I$ is an interval (⟨1⟩1).
⟨2⟩4. $s$ is the supremum of $X$ in $I$
⟨1⟩3. $I$ is dense.
⟨2⟩1. LET: $x, y \in I$ with $x < y$
⟨2⟩2. PICK $z \in L$ with $x < z < y$
PROOF: Since $L$ is dense.
⟨2⟩3. $z \in I$
PROOF: Since $I$ is an interval.
□

**Definition 1.34** (Ordered Square)**.** The *ordered square* $I_o^2$ is the set $[0,1]^2$ under the dictionary order.

**Proposition 1.35.** *The ordered square is a linear continuum.*

PROOF:
⟨1⟩1. Every nonempty subset of $I_o^2$ bounded above has a supremum.
⟨2⟩1. LET: $X \subseteq I_o^2$ be nonempty and bounded above by $(b, c)$
⟨2⟩2. LET: $s = \sup \pi_1(X)$
PROOF: The set $\pi_1(X)$ is nonempty and bounded above by $b$.
⟨2⟩3. CASE: $s \in \pi_1(X)$
⟨3⟩1. LET: $t = \sup\{y \in [0,1] \mid (s, y) \in X\}$
PROOF: This set is nonempty and bounded above by $c$.
⟨3⟩2. $(s, t)$ is the supremum of $X$.
⟨2⟩4. CASE: $s \notin \pi_1(X)$
PROOF: In this case $(s, 0)$ is the supremum of $X$.
⟨1⟩2. $I_o^2$ is dense.
⟨2⟩1. LET: $(x_1, y_1), (x_2, y_2) \in I_o^2$ with $(x_1, y_1) < (x_2, y_2)$
⟨2⟩2. CASE: $x_1 < x_2$
⟨3⟩1. PICK $x_3$ with $x_1 < x_3 < x_2$
⟨3⟩2. $(x_1, y_1) < (x_3, y_1) < (x_2, y_2)$
⟨2⟩3. CASE: $x_1 = x_2$ and $y_1 < y_2$
⟨3⟩1. PICK $y_3$ with $y_1 < y_3 < y_2$
⟨3⟩2. $(x_1, y_1) < (x_1, y_3) < (x_2, y_2)$
□

**Proposition 1.36.** *If $X$ is a well-ordered set then $X \times [0,1)$ under the dictionary order is a linear continuum.*

PROOF:
⟨1⟩1. Every nonempty set $A \subseteq X \times [0,1)$ bounded above has a supremum.

$\langle 2 \rangle 1$. LET: $A \subseteq X \times [0,1)$ be nonempty and bounded above
$\langle 2 \rangle 2$. LET: $x_0$ be the supremum of $\pi_1(A)$
$\langle 2 \rangle 3$. CASE: $x_0 \in \pi_1(A)$
$\langle 3 \rangle 1$. LET: $y_0$ be the supremum of $\{y \in [0,1) \mid (x_0, y) \in A\}$
$\langle 3 \rangle 2$. $(x_0, y_0)$ is the supremum of $A$.
$\langle 2 \rangle 4$. CASE: $x_0 \notin \pi_1(A)$
PROOF: In this case $(x_0, 0)$ is the supremum of $A$.
$\langle 1 \rangle 2$. $X \times [0,1)$ is dense.
$\langle 2 \rangle 1$. LET: $(x_1, y_1), (x_2, y_2) \in X \times [0,1)$ with $(x_1, y_1) < (x_2, y_2)$
$\langle 2 \rangle 2$. CASE: $x_1 < x_2$
$\langle 3 \rangle 1$. PICK $y_3$ such that $y_1 < y_3 < 1$
$\langle 3 \rangle 2$. $(x_1, y_1) < (x_1, y_3) < (x_2, y_2)$
$\langle 2 \rangle 3$. CASE: $x_1 = x_2$ and $y_1 < y_2$
$\langle 3 \rangle 1$. PICK $y_3$ such that $y_1 < y_3 < y_2$
$\langle 3 \rangle 2$. $(x_1, y_1) < (x_1, y_3) < (x_2, y_2)$
□

**Lemma 1.37.** *For all $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$, we have $[a, b) \cong [c, d)$*

PROOF: The map $\lambda t.c + (t - a)(d - c)/(b - a)$ is an order isomorphism.

**Proposition 1.38.** *Let $X$ be a linearly ordered set. Let $a < b < c$ in $X$. Then $[a, c) \cong [0, 1)$ if and only if $[a, b) \cong [b, c) \cong [0, 1)$.*

PROOF:
$\langle 1 \rangle 1$. If $[a, c) \cong [0,1)$ then $[a, b) \cong [b, c) \cong [0,1)$
$\langle 2 \rangle 1$. ASSUME: $f : [a, c) \cong [0, 1)$ is an order isomorphism
$\langle 2 \rangle 2$. $[a, b) \cong [0, 1)$
PROOF:

$$\begin{aligned} [a, b) &\cong [0, f(b)) && \text{(by the restriction of } f) \\ &\cong [0, 1) && \text{(Lemma 1.37)} \end{aligned}$$

$\langle 2 \rangle 3$. $[b, c) \cong [0, 1)$
PROOF: Similar.
$\langle 1 \rangle 2$. If $[a, b) \cong [b, c) \cong [0, 1)$ then $[a, c) \cong [0, 1)$
PROOF:

$$\begin{aligned} [a, c) &= [a, b) * [b, c) \\ &\cong [0, 1) * [0, 1) \\ &\cong [0, 1/2) * [1/2, 1) && \text{(Lemma 1.37)} \\ &= 1 \end{aligned}$$

□

**Proposition 1.39** (CC)**.** *Let $X$ be a linearly ordered set. Let $x_0 < x_1 < \cdots$ be a strictly increasing sequence in $X$ with supremum $b$. Then $[x_0, b) \cong [0, 1)$ if and only if $[x_i, x_{i+1}) \cong [0, 1)$ for all $i$.*

PROOF:
$\langle 1 \rangle 1$. If $[x_0, b) \cong [0, 1)$ then $[x_i, x_{i+1}) \cong [0, 1)$ for all $i$.

PROOF: By Lemma 1.37

⟨1⟩2. If $[x_i, x_{i+1}) \cong [0,1)$ for all $i$ then $[x_0, b) \cong [0,1)$

⟨2⟩1. ASSUME: $[x_i, x_{i+1}) \cong [0,1)$ for all $i$

⟨2⟩2. PICK an order isomorphism $f_i : [x_i, x_{i+1}) \cong [1/2^i, 2/2^{i+1})$ for each $i$.

PROOF: By Lemma 1.37

⟨2⟩3. The union of the $f_i$s is an order isomorphism $[x_0, b) \cong [0,1)$

□

## 1.13 Partially Ordered Sets

**Definition 1.40** (Partial Order)**.** A *partial order* on a set $X$ is a preorder $\leq$ that is *anti-symmetric*, i.e. whenever $x \leq y$ and $y \leq x$ then $x = y$.

**Definition 1.41** (Linear Order)**.** A *linear order* on a set $X$ is a partial order such that, for any $x, y \in X$, either $x \leq y$ or $y \leq x$.

**Definition 1.42** (Well-ordering)**.** A *well-order* on a set $X$ is a linear order such that every nonempty set has a least element.

**Definition 1.43** (Section)**.** Given a well-ordered set $X$ and $\alpha \in X$, the *section* of $X$ by $\alpha$ is $S_\alpha = \{x \in X \mid x < \alpha\}$.

**Theorem 1.44** (Transfinite Induction)**.** *Let $J$ be a well-ordered set and $J_0 \subseteq J$. Suppose that, for all $\alpha \in J$, if $S_\alpha \subseteq J_0$ then $\alpha \in J_0$. Then $J_0 = J$.*

PROOF:

⟨1⟩1. ASSUME: for a contradiction $J_0 \neq J$

⟨1⟩2. LET: $\alpha$ be the least element of $J \setminus J_0$

⟨1⟩3. $S_\alpha \subseteq J_0$

⟨1⟩4. $\alpha \in J_0$

⟨1⟩5. Q.E.D.

PROOF: This contradicts ⟨1⟩2.

□

**Theorem 1.45** (Transfinite Recursion)**.** *Let $J$ be a well-ordered set and $C$ a set. Let $\mathcal{F}$ be the set of all functions from a section of $J$ to $C$. Let $\rho : \mathcal{F} \to C$. Then there exists a unique function $h : J \to C$ such that, for all $\alpha \in J$, we have $h(\alpha) = \rho(h \restriction S_\alpha)$.*

PROOF:

⟨1⟩1. For every $\beta \in J$, there exists a unique $h_\beta : S_\beta \to J$ such that, for all $\alpha < \beta$, we have $h_\beta(\alpha) = \rho(h_\beta \restriction \alpha)$

⟨2⟩1. LET: $\beta \in J$

⟨2⟩2. ASSUME: for all $\gamma < \beta$ there exists a unique $h : S_\gamma \to J$ such that, for all $\alpha < \gamma$, we have $h(\alpha) = \rho(h \restriction S_\alpha)$

⟨2⟩3. For $\gamma < \beta$,

LET: $h_\gamma : S_\gamma \to J$ be the function such that, for all $\alpha < \gamma$, we have $h_\gamma(\alpha) = \rho(h_\gamma \restriction S_\alpha)$

⟨2⟩4. LET: $h : S_\beta \to J$ be the function $h(\gamma) = \rho(h_\gamma)$ for $\gamma < \beta$

⟨2⟩5. For $\gamma < \beta$ we have $h \restriction S_\gamma = h_\gamma$

⟨3⟩1. LET: $\gamma < \beta$

⟨3⟩2. ASSUME: For all $\alpha < \gamma$ we have $h \restriction S_\alpha = h_\alpha$

⟨3⟩3. For all $\alpha < \gamma$ we have $(h \restriction S_\gamma)(\alpha) = \rho((h \restriction S_\gamma) \restriction S_\alpha)$

PROOF:

$$\begin{aligned}(h \restriction S_\gamma)(\alpha) &= h(\alpha) \\ &= \rho(h_\alpha) && (\langle 2\rangle 4) \\ &= \rho(h \restriction S_\alpha) && (\langle 3\rangle 2) \\ &= \rho((h \restriction S_\gamma) \restriction S_\alpha)\end{aligned}$$

⟨3⟩4. $h \restriction S_\gamma = h_\gamma$

PROOF: From ⟨2⟩4

⟨3⟩5. Q.E.D.

PROOF: By transfinite induction.

⟨2⟩6. For $\alpha < \beta$ we have $h(\alpha) = \rho(h \restriction S_\alpha)$

⟨2⟩7. If $h' : S_\beta \to J$ and $h'(\alpha) = \rho(h' \restriction S_\alpha)$ for all $\alpha < \beta$, then $h' = h$)

⟨3⟩1. LET: $h' : S_\beta \to J$ and $h'(\alpha) = \rho(h' \restriction S_\alpha)$ for all $\alpha < \beta$

⟨3⟩2. For all $\gamma < \beta$ we have $h' \restriction S_\gamma = h_\gamma$

⟨4⟩1. For all $\alpha < \gamma$ we have $(h' \restriction S_\gamma)(\alpha) = \rho((h' \restriction S_\gamma) \restriction S_\alpha)$

PROOF:

$$\begin{aligned}(h' \restriction S_\gamma)(\alpha) &= h'(\alpha) \\ &= \rho(h' \restriction S_\alpha) && (\langle 3\rangle 1) \\ &= \rho((h' \restriction S_\gamma) \restriction S_\alpha)\end{aligned}$$

⟨4⟩2. Q.E.D.

PROOF: From ⟨2⟩4

⟨3⟩3. For all $\alpha < \beta$ we have $h'(\alpha) = \rho(h_\alpha)$

⟨1⟩2. There exists $h : J \to C$ such that, for all $\alpha \in J$, we have $h(\alpha) = \rho(h \restriction S_\alpha)$

⟨2⟩1. For $\alpha \in J$,

LET: $h(\alpha) = \rho(h_\alpha)$

⟨2⟩2. For $\alpha \in J$ we have $h \restriction S_\alpha = h_\alpha$

⟨3⟩1. LET: $\alpha \in J$

⟨3⟩2. ASSUME: For all $\beta < \alpha$ we have $h \restriction S_\beta = h_\beta$

⟨3⟩3. For all $\beta < \alpha$ we have $(h \restriction S_\alpha)(\beta) = \rho((h \restriction S_\alpha) \restriction S_\beta)$

PROOF:

$$\begin{aligned}(h \restriction S_\alpha)(\beta) &= h(\beta) \\ &= \rho(h_\beta) && (\langle 2\rangle 1) \\ &= \rho(h \restriction S_\beta) && (\langle 3\rangle 2) \\ &= \rho((h \restriction S_\alpha) \restriction S_\beta)\end{aligned}$$

⟨3⟩4. $h \restriction S_\alpha = h_\alpha$

PROOF: From ⟨1⟩1

⟨3⟩5. Q.E.D.

PROOF: By transfinite induction.

⟨2⟩3. For $\alpha \in J$ we have $h(\alpha) = \rho(h \restriction S_\alpha)$

⟨1⟩3. If $h, h' : J \to C$ and, for all $\alpha \in J$, we have $h(\alpha) = \rho(h \restriction S_\alpha)$ and

$h'(\alpha) = \rho(h' \upharpoonright S_\alpha)$, then $h = h'$

$\langle 2\rangle 1$. ASSUME: $h, h' : J \to C$ and, for all $\alpha \in J$, we have $h(\alpha) = \rho(h \upharpoonright S_\alpha)$ and $h'(\alpha) = \rho(h' \upharpoonright S_\alpha)$

$\langle 2\rangle 2$. LET: $\alpha \in J$

$\langle 2\rangle 3$. ASSUME: for all $\beta < \alpha$ we have $h(\beta) = h'(\beta)$

$\langle 2\rangle 4$. $h(\alpha) = h'(\alpha)$

PROOF:

$$\begin{aligned} h(\alpha) &= \rho(h \upharpoonright S_\alpha) \\ &= \rho(h' \upharpoonright S_\alpha) && (\langle 2\rangle 3) \\ &= h'(\alpha) \end{aligned}$$

$\langle 2\rangle 5$. Q.E.D.

PROOF: By transfinite induction.

□

**Theorem 1.46** (Well-Ordering Theorem (AC)). *Every set has a well-ordering.*

PROOF:

$\langle 1\rangle 1$. LET: $X$ be a set.

$\langle 1\rangle 2$. PICK a choice function for $\mathcal{P}X \setminus \{\emptyset\}$

PROOF: Lemma 1.27.

$\langle 1\rangle 3$. LET: a *tower* in $X$ be a pair $(T, <)$ where $T \subseteq X$, $<$ is a well-ordering of $T$, and $x = c(X \setminus \{y \in T \mid y < x\})$.

$\langle 1\rangle 4$. For any two towers $(T_1, <_1)$ and $(T_2, <_2)$, either these two posets are equal or one is a section of the other.

$\langle 2\rangle 1$.

$\langle 1\rangle 5$. For any tower $(T, <)$ in $X$ with $T \neq X$, there exists a tower in $X$ of which $(T, <)$ is a section.

$\langle 1\rangle 6$. LET: $T = \bigcup\{T' \subseteq X \mid \exists R.(T, R) \text{ is a tower in } X\}$

$\langle 1\rangle 7$. Define $<$ on $T$ by: $x < y$ iff there exists a tower $(T, R)$ in $X$ such that $x, y \in T$ and $xRy$.

$\langle 1\rangle 8$. $(T, <)$ is a tower in $X$.

$\langle 1\rangle 9$. $T = X$

$\langle 1\rangle 10$. $<$ is a well-ordering of $X$.

□

**Theorem 1.47** (Maximum Principle (AC)). *Every poset has a maximal chain.*

**Lemma 1.48** (Zorn's Lemma (AC)). *Let $A$ be a poset. If every chain in $A$ has an upper bound in $A$, then $A$ has a maximal element.*

## 1.14 Real Analysis

**Definition 1.49.** Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences $(a_n)$ such that $a_n = 0$ for all but finitely many $n$.

## 1.15 Group Theory

**Definition 1.50.** Given a group $G$ and sets $A, B \subseteq G$, let $AB = \{ab \mid a \in A, b \in B\}$.

**Definition 1.51.** Given a group $G$ and a set $A \subseteq G$, let $A^{-1} = \{a^{-1} \mid a \in A\}$.

## 1.16 Topological Spaces

**Definition 1.52** (Topology)**.** A *topology* on a set $X$ is a set $\mathcal{T} \subseteq \mathcal{P}X$ such that:

- $X \in \mathcal{T}$.
- For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$.
- For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$.

We call the elements of $X$ *points* and the elements of $\mathcal{T}$ *open sets*.

**Definition 1.53** (Topological Space)**.** A *topological space* $X$ consists of a set $X$ and a topology on $X$.

**Definition 1.54** (Discrete Space)**.** For any set $X$, the *discrete* topology on $X$ is $\mathcal{P}X$.

**Definition 1.55** (Indiscrete Space)**.** For any set $X$, the *indiscrete* or *trivial* topology on $X$ is $\{\emptyset, X\}$.

**Definition 1.56** (Finite Complement Topology)**.** For any set $X$, the *finite complement topology* on $X$ is $\{U \in \mathcal{P}X \mid X \setminus U \text{ is finite}\} \cup \{\emptyset\}$.

**Definition 1.57** (Countable Complement Topology)**.** For any set $X$, the *countable complement topology* on $X$ is $\{U \in \mathcal{P}X \mid X \setminus U \text{ is countable}\} \cup \{\emptyset\}$.

**Definition 1.58** (Finer, Coarser)**.** Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a given set $X$. If $\mathcal{T}' \supseteq \mathcal{T}$, we say that $\mathcal{T}'$ is *finer* than $\mathcal{T}$; if $\mathcal{T}'$ *properly* contains $\mathcal{T}$, we say that $\mathcal{T}'$ is *strictly* finer than $\mathcal{T}$. We also say that $\mathcal{T}$ is *coarser* than $\mathcal{T}'$, or *stricly* coarser, in these two respective situations. We say $\mathcal{T}$ is *comparable* with $\mathcal{T}'$ if either $\mathcal{T}' \supseteq \mathcal{T}$ or $\mathcal{T} \supseteq \mathcal{T}'$.

**Lemma 1.59.** *Let $X$ be a topological space and $U \subseteq X$. Then $U$ is open if and only if, for all $x \in U$, there exists an open set $V$ such that $x \in V \subseteq U$.*

PROOF:
$\langle 1 \rangle 1.$ $\Rightarrow$
  PROOF: Take $V = U$
$\langle 1 \rangle 2.$ $\Leftarrow$
  PROOF: We have $U = \bigcup \{V \text{ open in } X \mid V \subseteq U\}$.
□

**Lemma 1.60.** *Let $X$ be a set and $\mathcal{T}$ a nonempty set of topologies on $X$. Then $\bigcap \mathcal{T}$ is a topology on $X$, and is the finest topology that is coarser than every member of $\mathcal{T}$.*

PROOF:
$\langle 1 \rangle 1.$ $X \in \bigcap \mathcal{T}$
  PROOF: Since $X$ is in every member of $\mathcal{T}$.
$\langle 1 \rangle 2.$ $\bigcap \mathcal{T}$ is closed under union.

⟨2⟩1. LET: $\mathcal{U} \subseteq \bigcap \mathcal{T}$
⟨2⟩2. For all $T \in \mathcal{T}$ we have $\mathcal{U} \subseteq T$
⟨2⟩3. For all $T \in \mathcal{T}$ we have $\bigcup \mathcal{U} \in T$
⟨2⟩4. $\bigcup \mathcal{U} \in \bigcap \mathcal{T}$

⟨1⟩3. $\bigcap \mathcal{T}$ is closed under binary intersection.
⟨2⟩1. LET: $U, V \in \bigcap \mathcal{T}$
⟨2⟩2. For all $T \in \mathcal{T}$ we have $U, V \in T$
⟨2⟩3. For all $T \in \mathcal{T}$ we have $U \cap V \in T$
⟨2⟩4. $U \cap V \in \bigcap \mathcal{T}$

□

**Lemma 1.61.** *Let $X$ be a set and $\mathcal{T}$ a set of topologies on $X$. Then there exists a unique coarsest topology that is finer than every member of $\mathcal{T}$.*

PROOF: The required topology is given by
$\bigcap \{T \in \mathcal{PP}X \mid T \text{ is a topology on } X \text{ that is finer than every member of } \mathcal{T}\}$ ,
The set is nonempty since it contains the discrete topology. □

**Definition 1.62** (Neighbourhood)**.** A *neighbourhood* of a point $x$ is an open set that contains $x$.

## 1.17 Closed Set

**Definition 1.63** (Closed Set)**.** Let $X$ be a topological space and $A \subseteq X$. Then $A$ is *closed* if and only if $X \setminus A$ is open.

**Lemma 1.64.** *The empty set is closed.*

PROOF: Since the whole space $X$ is always open. □

**Lemma 1.65.** *The topological space $X$ is closed.*

PROOF: Since $\emptyset$ is open. □

**Lemma 1.66.** *The intersection of a nonempty set of closed sets is closed.*

PROOF: Let $\mathcal{C}$ be a nonempty set of closed sets. Then $X \setminus \bigcap \mathcal{C} = \bigcup \{X \setminus C \mid C \in \mathcal{C}\}$ is open. □

**Lemma 1.67.** *The union of two closed sets is closed.*

PROOF: Let $C$ and $D$ be closed. Then $X \setminus (C \cup D) = (X \setminus C) \cap (X \setminus D)$ is open. □

**Proposition 1.68.** *Let $X$ be a set and $\mathcal{C} \subseteq \mathcal{P}X$ a set such that:*

1. $\emptyset \in \mathcal{C}$
2. $X \in \mathcal{C}$
3. *For all $\mathcal{A} \subseteq \mathcal{C}$ nonempty we have $\bigcap \mathcal{A} \in \mathcal{C}$*

*4. For all $C, D \in \mathcal{C}$ we have $C \cup D \in \mathcal{C}$.*

*Then there exists a unique topology $\mathcal{T}$ such that $\mathcal{C}$ is the set of closed sets, namely*

$$\mathcal{T} = \{X \setminus C \mid C \in \mathcal{C}\}$$

PROOF:
$\langle 1 \rangle 1$. LET: $\mathcal{T} = \{X \setminus C \mid C \in \mathcal{C}\}$
$\langle 1 \rangle 2$. $\mathcal{T}$ is a topology
$\langle 2 \rangle 1$. $X \in \mathcal{T}$
PROOF: Since $\emptyset \in \mathcal{C}$
$\langle 2 \rangle 2$. For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$
$\langle 3 \rangle 1$. LET: $\mathcal{U} \subseteq \mathcal{T}$
$\langle 3 \rangle 2$. CASE: $\mathcal{U} = \emptyset$
PROOF: In this case $\bigcup \mathcal{U} = \emptyset \in \mathcal{T}$ since $X \in \mathcal{C}$
$\langle 3 \rangle 3$. CASE: $\mathcal{U} \neq \emptyset$
PROOF: In this case $X \setminus \bigcup \mathcal{U} = \bigcap \{X \setminus U \mid U \in \mathcal{U}\} \in \mathcal{C}$.
$\langle 2 \rangle 3$. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$
PROOF: Since $X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V) \in \mathcal{C}$.
$\langle 1 \rangle 3$. $\mathcal{C}$ is the set of all closed sets in $\mathcal{T}$
PROOF:

$$\begin{aligned} & C \text{ is closed in } \mathcal{T} \\ \Leftrightarrow & X \setminus C \in \mathcal{T} \\ \Leftrightarrow & C \in \mathcal{C} \end{aligned}$$

$\langle 1 \rangle 4$. If $\mathcal{T}'$ is a topology and $\mathcal{C}$ is the set of closed sets in $\mathcal{T}'$ then $\mathcal{T}' = \mathcal{T}$
PROOF: We have

$$\begin{aligned} & U \in \mathcal{T} \\ \Leftrightarrow & X \setminus U \in \mathcal{C} \\ \Leftrightarrow & X \setminus U \text{ is closed in } \mathcal{T}' \\ \Leftrightarrow & U \in \mathcal{T}' \end{aligned}$$

□

**Proposition 1.69.** *If $U$ is open and $A$ is closed then $U \setminus A$ is open.*

PROOF: $U \setminus A = U \cap (X \setminus A)$ is the intersection of two open sets. □

**Proposition 1.70.** *If $U$ is open and $A$ is closed then $A \setminus U$ is closed.*

PROOF: $A \setminus U = A \cap (X \setminus U)$ is the intersection of two closed sets. □

## 1.18 Interior

**Definition 1.71** (Interior)**.** Let $X$ be a topological space and $A \subseteq X$. The *interior* of $A$, $\operatorname{Int} A$, is the union of all the open subsets of $A$.

**Lemma 1.72.** *The interior of a set is open.*

PROOF: It is a union of open sets. □

**Lemma 1.73.**

$$\operatorname{Int} A \subseteq A$$

PROOF: Immediate from definition. □

**Lemma 1.74.** *If $U$ is open and $U \subseteq A$ then $U \subseteq \operatorname{Int} A$*

PROOF: Immediate from definition. □

**Lemma 1.75.** *A set $A$ is open if and only if $A = \operatorname{Int} A$.*

PROOF: If $A = \operatorname{Int} A$ then $A$ is open by Lemma 1.72. Conversely if $A$ is open then $A \subseteq \operatorname{Int} A$ by the definition of interior and so $A = \operatorname{Int} A$.

## 1.19 Closure

**Definition 1.76** (Closure)**.** Let $X$ be a topological space and $A \subseteq X$. The *closure* of $A$, $\overline{A}$, is the intersection of all the closed sets that include $A$.

This intersection exists since $X$ is a closed set that includes A (Lemma 1.65).

**Lemma 1.77.** *The closure of a set is closed.*

PROOF: Dual to Lemma 1.72. □

**Lemma 1.78.**

$$A \subseteq \overline{A}$$

PROOF: Immediate from definition. □

**Lemma 1.79.** *If $C$ is closed and $A \subseteq C$ then $\overline{A} \subseteq C$.*

PROOF: Immediate from definition. □

**Lemma 1.80.** *A set $A$ is closed if and only if $A = \overline{A}$.*

PROOF: Dual to Lemma 1.75. □

**Theorem 1.81.** *Let $X$ be a topological space, $A \subseteq X$ and $x \in X$. Then $x \in \overline{A}$ if and only if every neighbourhood of $x$ intersects $A$.*

PROOF: We have

$$\begin{aligned}
& x \in \overline{A} \\
\Leftrightarrow & \forall C. C \text{ closed } \wedge A \subseteq C \Rightarrow x \in C \\
\Leftrightarrow & \forall U. U \text{ open } \wedge A \cap U = \emptyset \Rightarrow x \notin U \\
\Leftrightarrow & \forall U. U \text{ open } \wedge x \in U \Rightarrow U \text{ intersects } A
\end{aligned}$$

□

**Proposition 1.82.** *If $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$.*

PROOF: This holds because $\overline{B}$ is a closed set that includes $A$. $\square$

**Proposition 1.83.**

$$\overline{A \cup B} = \overline{A} \cup \overline{B}$$

PROOF:
$\langle 1 \rangle 1.$ $\overline{A} \subseteq \overline{A \cup B}$
PROOF: By Proposition 1.82.
$\langle 1 \rangle 2.$ $\overline{B} \subseteq \overline{A \cup B}$
PROOF: By Proposition 1.82.
$\langle 1 \rangle 3.$ $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$
$\langle 2 \rangle 1.$ LET: $x \in \overline{A \cup B}$
$\langle 2 \rangle 2.$ ASSUME: $x \notin \overline{A}$
PROVE: $x \in \overline{B}$
$\langle 2 \rangle 3.$ PICK a neighbourhood $U$ of $x$ that does not intersect $A$
$\langle 2 \rangle 4.$ LET: $V$ be any neighbourhood of $x$
$\langle 2 \rangle 5.$ $U \cap V$ is a neighbourhood of $x$
$\langle 2 \rangle 6.$ $U \cap V$ intersects $A \cup B$
PROOF: From $\langle 2 \rangle 1$ and Theorem 1.81.
$\langle 2 \rangle 7.$ $U \cap V$ intersects $B$
PROOF: From $\langle 2 \rangle 3$
$\langle 2 \rangle 8.$ $V$ intersects $B$
$\langle 2 \rangle 9.$ Q.E.D.
PROOF: We have $x \in \overline{B}$ from Theorem 1.81.

$\square$

**Proposition 1.84.** *Let $X$ be a topological space. Let $\mathcal{D}$ be a set of subsets of $X$ that is maximal with respect to the finite intersection property. Let $x \in X$. Then the following are equivalent:*

1. *For all $D \in \mathcal{D}$ we have $x \in \overline{D}$*
2. *Every neighbourhood of $x$ is in $\mathcal{D}$.*

PROOF:
$\langle 1 \rangle 1.$ $1 \Rightarrow 2$
$\langle 2 \rangle 1.$ For all $D \in \mathcal{D}$ we have $x \in \overline{D}$
$\langle 2 \rangle 2.$ LET: $U$ be a neighbourhood of $x$
$\langle 2 \rangle 3.$ $\mathcal{D} \cup \{U\}$ satisfies the finite intersection property.
$\langle 3 \rangle 1.$ LET: $D_1, \ldots, D_n \in \mathcal{D}$
$\langle 3 \rangle 2.$ $D_1 \cap \cdots \cap D_n \in \mathcal{D}$
PROOF: Lemma 1.18.
$\langle 3 \rangle 3.$ $x \in \overline{D_1 \cap \cdots \cap D_n}$
PROOF: $\langle 2 \rangle 1$, $\langle 3 \rangle 2$
$\langle 3 \rangle 4.$ $D_1 \cap \cdots \cap D_n \cap U \neq \emptyset$
PROOF: Theorem 1.81, $\langle 2 \rangle 2$, $\langle 3 \rangle 3$.
$\langle 2 \rangle 4.$ $\mathcal{D} = \mathcal{D} \cup \{U\}$
PROOF: By the maximality of $\mathcal{D}$.

⟨2⟩5. $U \in \mathcal{D}$

⟨1⟩2. $2 \Rightarrow 1$

⟨2⟩1. Assume: Every neighbourhood of $x$ is in $\mathcal{D}$.

⟨2⟩2. Let: $D \in \mathcal{D}$

⟨2⟩3. Every neighbourhood of $x$ intersects $D$.

Proof: From ⟨2⟩1, ⟨2⟩2 and the fact that $\mathcal{D}$ satisfies the finite intersection property.

⟨2⟩4. $x \in \overline{D}$

Proof: Theorem 1.81, ⟨2⟩3.

∎

## 1.20 Boundary

**Definition 1.85** (Boundary)**.** The *boundary* of a set $A$ is the set $\partial A = \overline{A} \cap \overline{X \setminus A}$.

**Proposition 1.86.**

$$\operatorname{Int} A \cap \partial A = \emptyset$$

Proof: Since $\overline{X \setminus A} = X \setminus \operatorname{Int} A$. ∎

**Proposition 1.87.**

$$\overline{A} = \operatorname{Int} A \cup \partial A$$

Proof:

$$\begin{aligned} \operatorname{Int} A \cup \partial A &= \operatorname{Int} A \cup (\overline{A} \cap \overline{X \setminus A}) \\ &= (\operatorname{Int} A \cup \overline{A}) \cap (\operatorname{Int} A \cup \overline{X \setminus A}) \\ &= \overline{A} \cap X \\ &= \overline{A} \end{aligned}$$

**Proposition 1.88.** *$\partial A = \emptyset$ if and only if $A$ is open and closed.*

Proof: If $\partial A = \emptyset$ then $\overline{A} = \operatorname{Int} A$ by Proposition 1.87.

**Proposition 1.89.** *A set $U$ is open if and only if $\partial U = \overline{U} \setminus U$.*

Proof:

$$\begin{aligned} & \partial U = \overline{U} \setminus U \\ \Leftrightarrow & \overline{U} \setminus \operatorname{Int} U = \overline{U} \setminus U && \text{(Propositions 1.86, 1.87)} \\ \Leftrightarrow & \operatorname{Int} U = U \end{aligned}$$

∎

## 1.21 Limit Points

**Definition 1.90** (Limit Point)**.** Let $X$ be a topological space, $a \in X$ and $A \subseteq X$. Then $a$ is a *limit point*, *cluster point* or *point of accumulation* for $A$ if and only if every neighbourhood of $a$ intersects $A$ at a point other than $a$.

**Lemma 1.91.** *The point $a$ is an accumulation point for $A$ if and only if $a \in \overline{A \setminus \{a\}}$.*

PROOF: From Theorem 1.81. □

**Theorem 1.92.** *Let $X$ be a topological space and $A \subseteq X$. Let $A'$ be the set of all limit points of $A$. Then $\overline{A} = A \cup A'$.*

PROOF:
⟨1⟩1. For all $x \in \overline{A}$, if $x \notin A$ then $x \in A'$
  PROOF: From Theorem 1.81.
⟨1⟩2. $A \subseteq \overline{A}$
  PROOF: Lemma 1.78.
⟨1⟩3. $A' \subseteq \overline{A}$
  PROOF: From Theorem 1.81.
□

**Corollary 1.92.1.** *A set is closed if and only if it contains all its limit points.*

**Proposition 1.93.** *In an indiscrete topology, every point is a limit point of any set that has more than one point.*

PROOF: Let $X$ be an indiscrete space. Let $A$ be a set with more than one point and $x$ be a point. The only neighbourhood of $x$ is $X$, which must intersect $A$ at a point other than $x$. □

**Lemma 1.94.** *Let $X$ be a topological space and $A \subseteq B \subseteq X$. Then every limit point of $A$ is a limit point of $B$.*

PROOF: Immediate from definitions. □

## 1.22 Basis for a Topology

**Definition 1.95** (Basis)**.** If $X$ is a set, a *basis* for a topology on $X$ is a set $\mathcal{B} \subseteq \mathcal{P}X$ called *basis elements* such that

1. For all $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$.
2. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

If $\mathcal{B}$ satisfies these two conditions, then we define the topology *generated* by $\mathcal{B}$ to be $\mathcal{T} = \{U \in \mathcal{P}X \mid \forall x \in U.\exists B \in \mathcal{B}.x \in B \subseteq U\}$.

We prove this is a topology.

PROOF:
⟨1⟩1. $X \in \mathcal{T}$
  PROOF: For all $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B \subseteq X$ by condition 1.

$\langle 1\rangle 2.$ For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$
$\langle 2\rangle 1.$ LET: $\mathcal{U} \subseteq \mathcal{T}$
$\langle 2\rangle 2.$ LET: $x \in \bigcup \mathcal{U}$
$\langle 2\rangle 3.$ PICK $U \in \mathcal{U}$ such that $x \in U$
$\langle 2\rangle 4.$ PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$
PROOF: Since $U \in \mathcal{T}$ by $\langle 2\rangle 1$ and $\langle 2\rangle 3$.
$\langle 2\rangle 5.$ $x \in B \subseteq \bigcup \mathcal{U}$
$\langle 1\rangle 3.$ For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$
$\langle 2\rangle 1.$ LET: $U, V \in \mathcal{T}$
$\langle 2\rangle 2.$ LET: $x \in U \cap V$
$\langle 2\rangle 3.$ PICK $B_1 \in \mathcal{B}$ such that $x \in B_1 \subseteq U$
$\langle 2\rangle 4.$ PICK $B_2 \in \mathcal{B}$ such that $x \in B_2 \subseteq V$
$\langle 2\rangle 5.$ PICK $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$
PROOF: By condition 2.
$\langle 2\rangle 6.$ $x \in B_3 \subseteq U \cap V$
□

**Lemma 1.96.** *Let $X$ be a set. Let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on $X$. Then $\mathcal{T}$ is the set of all unions of subsets of $\mathcal{B}$.*

PROOF:
$\langle 1\rangle 1.$ For all $U \in \mathcal{T}$, there exists $\mathcal{A} \subseteq \mathcal{B}$ such that $U = \bigcup \mathcal{A}$
$\langle 2\rangle 1.$ LET: $U \in \mathcal{T}$
$\langle 2\rangle 2.$ LET: $\mathcal{A} = \{B \in \mathcal{B} \mid B \subseteq U\}$
$\langle 2\rangle 3.$ $U \subseteq \bigcup \mathcal{A}$
$\langle 3\rangle 1.$ LET: $x \in U$
$\langle 3\rangle 2.$ PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$
PROOF: Since $\mathcal{B}$ is a basis for $\mathcal{T}$.
$\langle 3\rangle 3.$ $x \in B \in \mathcal{A}$
$\langle 2\rangle 4.$ $\bigcup \mathcal{A} \subseteq U$
PROOF: From the definition of $\mathcal{A}$ ($\langle 2\rangle 2$).
$\langle 1\rangle 2.$ For all $\mathcal{A} \subseteq \mathcal{B}$ we have $\bigcup \mathcal{A} \in \mathcal{T}$
$\langle 2\rangle 1.$ $\mathcal{B} \subseteq \mathcal{T}$
PROOF: If $B \in \mathcal{B}$ and $x \in B$, then there exists $B' \in \mathcal{B}$ such that $x \in B' \subseteq B$, namely $B' = B$.
$\langle 2\rangle 2.$ Q.E.D.
PROOF: Since $\mathcal{T}$ is closed under union.
□

**Corollary 1.96.1.** *Let $X$ be a set. Let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on $X$. Then $\mathcal{T}$ is the coarsest topology that includes $\mathcal{B}$.*

PROOF: Since every topology that includes $\mathcal{B}$ includes all unions of subsets of $\mathcal{B}$. □

**Lemma 1.97.** *Let $X$ be a topological space. Suppose that $\mathcal{C}$ is a set of open sets such that, for every open set $U$ and every point $x \in U$, there exists $C \in \mathcal{C}$ such that $x \in C \subseteq U$. Then $\mathcal{C}$ is a basis for the topology on $X$.*

PROOF:
$\langle 1 \rangle 1$. For all $x \in X$, there exists $C \in \mathcal{C}$ such that $x \in C$
PROOF: Immediate from hypothesis.
$\langle 1 \rangle 2$. For all $C_1, C_2 \in \mathcal{C}$ and $x \in C_1 \cap C_2$, there exists $C_3 \in \mathcal{C}$ such that $x \in C_3 \subseteq C_1 \cap C_2$
PROOF: Since $C_1 \cap C_2$ is open.
$\langle 1 \rangle 3$. Every open set is open in the topology generated by $\mathcal{C}$
PROOF: Immediate from hypothesis.
$\langle 1 \rangle 4$. Every union of a subset of $\mathcal{C}$ is open.
PROOF: Since every member of $\mathcal{C}$ is open.
□

**Lemma 1.98.** *Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$ respectively on the set $X$. Then the following are equivalent.*

1. $\mathcal{T} \subseteq \mathcal{T}'$
2. *For all $B \in \mathcal{B}$ and $x \in B$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.*

PROOF:
$\langle 1 \rangle 1$. $1 \Rightarrow 2$
$\langle 2 \rangle 1$. ASSUME: $\mathcal{T} \subseteq \mathcal{T}'$
$\langle 2 \rangle 2$. LET: $B \in \mathcal{B}$ and $x \in B$
$\langle 2 \rangle 3$. $B \in \mathcal{T}$
PROOF: Corollary 1.96.1.
$\langle 2 \rangle 4$. $B \in \mathcal{T}'$
PROOF: By $\langle 2 \rangle 1$
$\langle 2 \rangle 5$. There exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$
PROOF: Since $\mathcal{B}'$ is a basis for $\mathcal{T}'$.
$\langle 1 \rangle 2$. $2 \Rightarrow 1$
$\langle 2 \rangle 1$. ASSUME: 2
$\langle 2 \rangle 2$. LET: $U \in \mathcal{T}$
PROVE: $U \in \mathcal{T}'$
$\langle 2 \rangle 3$. LET: $x \in U$
PROVE: There exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq U$
$\langle 2 \rangle 4$. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$
PROOF: Since $\mathcal{B}$ is a basis for $\mathcal{T}$.
$\langle 2 \rangle 5$. PICK $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$
PROOF: By $\langle 2 \rangle 1$.
$\langle 2 \rangle 6$. $x \in B' \subseteq U$
□

**Theorem 1.99.** *Let $X$ be a topological space, $A \subseteq X$ and $x \in X$. Let $\mathcal{B}$ be a basis for $X$. Then $x \in \overline{A}$ if and only if, for all $B \in \mathcal{B}$, if $x \in B$ then $B$ intersects $A$.*

PROOF:
$\langle 1 \rangle 1$. If $x \in \overline{A}$ then, for all $B \in \mathcal{B}$, if $x \in B$ then $B$ intersects $A$.

PROOF: This follows from Theorem 1.81 since every element of $\mathcal{B}$ is open (Corollary 1.96.1).

⟨1⟩2. Suppose that, for all $B \in \mathcal{B}$, if $x \in B$ then $B$ intersects $A$. Then $x \in \overline{A}$.

⟨2⟩1. ASSUME: For all $B \in \mathcal{B}$, if $x \in B$ then $B$ intersects $A$.

⟨2⟩2. LET: $U$ be an open set that contains $x$
PROVE: $U$ intersects $A$.

⟨2⟩3. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

⟨2⟩4. $B$ intersects $A$.
PROOF: From ⟨2⟩1.

⟨2⟩5. $U$ intersects $A$.

⟨2⟩6. Q.E.D.
PROOF: By Theorem 1.81.

□

**Definition 1.100** (Lower Limit Topology on the Real Line)**.** The *lower limit topology on the real line* is the topology on $\mathbb{R}$ generated by the basis consisting of all half-open intervals of the form $[a, b)$.

We write $\mathbb{R}_l$ for the topological space $\mathbb{R}$ under the lower limit topology.

We prove this is a basis for a topology.

PROOF:

⟨1⟩1. For all $x \in \mathbb{R}$ there exists an interval $[a, b)$ such that $x \in [a, b)$.
PROOF: Take $[a, b) = [x, x + 1)$.

⟨1⟩2. For any open intervals $[a, b)$, $[c, d)$ if $x \in [a, b) \cap [c, d)$, then there exists an interval $[e, f)$ such that $x \in [e, f) \subseteq [a, b) \cap [c, d)$
PROOF: Take $[e, f) = [\max(a, c), \min(b, d))$.

□

**Definition 1.101** ($K$-topology on the Real Line)**.** Let $K = \{1/n \mid n \in \mathbb{Z}^+\}$.

The *$K$-topology on the real line* is the topology on $\mathbb{R}$ generated by the basis consisting of all open intervals $(a, b)$ and all sets of the form $(a, b) \setminus K$.

We write $\mathbb{R}_K$ for the topological space $\mathbb{R}$ under the $K$-topology.

We prove this is a basis for a topology.

PROOF:

⟨1⟩1. For all $x \in \mathbb{R}$ there exists an open interval $(a, b)$ such that $x \in (a, b)$.
PROOF: Take $(a, b) = (x - 1, x + 1)$.

⟨1⟩2. For any basic open sets $B_1$, $B_2$ if $x \in B_1 \cap B_2$, then there exists a basic open set $B_3$ such that $x \in B_3 \subseteq B_1 \cap B_2$

⟨2⟩1. CASE: $B_1 = (a, b)$, $B_2 = (c, d)$
PROOF: Take $B_3 = (\max(a, c), \min(b, d))$.

⟨2⟩2. CASE: $B_1 = (a, b)$ or $(a, b) \setminus K$, $B_2 = (c, d)$ or $(c, d) \setminus K$, and they are not both open intervals.
PROOF: Take $B_3 = (\max(a, c), \min(b, d)) \setminus K$.

□

**Lemma 1.102.** *The lower limit topology and the $K$-topology are incomparable.*

Proof:
⟨1⟩1. The interval $[10, 11)$ is not open in the $K$-topology.
Proof: There is no open interval $(a, b)$ such that $10 \in (a, b) \subseteq [10, 11)$ or $10 \in (a, b) \setminus K \subseteq [10, 11)$.
⟨1⟩2. The set $(-1, 1) \setminus K$ is not open in the lower limit topology.
Proof: There is no half-open interval $[a, b)$ such that $0 \in [a, b) \subseteq (-1, 1) \setminus K$, since there must be a positive integer $n$ with $1/n \in [a, b)$.
□

**Definition 1.103** (Subbasis)**.** A *subbasis* $\mathcal{S}$ for a topology on $X$ is a set $\mathcal{S} \subseteq \mathcal{P}X$ such that $\bigcup \mathcal{S} = X$.

The topology *generated* by the subbasis $\mathcal{S}$ is the set of all unions of finite intersections of elements of $\mathcal{S}$.

We prove this is a topology.

Proof:
⟨1⟩1. The set $\mathcal{B}$ of all finite intersections of elements of $\mathcal{S}$ forms a basis for a topology on $X$.
⟨2⟩1. $\bigcup \mathcal{B} = X$
Proof: Since $\mathcal{S} \subseteq \mathcal{B}$.
⟨2⟩2. $\mathcal{B}$ is closed under binary intersection.
Proof: By definition.
⟨1⟩2. Q.E.D.
Proof: By Lemma 1.96.
□

We have simultaneously proved:

**Proposition 1.104.** *Let $\mathcal{S}$ be a subbasis for the topology on $X$. Then the set of all finite intersections of elements of $\mathcal{S}$ is a basis for the topology on $X$.*

**Proposition 1.105.** *Let $X$ be a set. Let $\mathcal{S}$ be a subbasis for a topology $\mathcal{T}$ on $X$. Then $\mathcal{T}$ is the coarsest topology that includes $\mathcal{S}$.*

Proof: Since every topology that includes $\mathcal{S}$ includes every union of finite intersections of elements of $\mathcal{S}$. □

## 1.23 Local Basis at a Point

**Definition 1.106** (Local Basis)**.** Let $X$ be a topological space and $a \in X$. A *(local) basis at $a$* is a set $\mathcal{B}$ of neighbourhoods of $a$ such that every neighbourhood of $a$ includes some member of $\mathcal{B}$.

**Lemma 1.107.** *If there exists a countable local basis at a point $a$, then there exists a countable local basis $\{B_n \mid n \geq 1\}$ such that $B_1 \supseteq B_2 \supseteq \cdots$.*

Proof: Given a countable local basis $\{C_n \mid n \geq 1\}$, take $B_n = C_1 \cap \cdots \cap C_n$. □

## 1.24 Convergence

**Definition 1.108** (Convergence)**.** Let $X$ be a topological space. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of points in $X$ and $l \in X$. Then the sequence $(a_n)_{n\in\mathbb{N}}$ *converges* to the *limit* $l$, $a_n \to l$ as $n \to \infty$, if and only if, for every neighbourhood $U$ of $l$, there exists $N$ such that, for all $n \geq N$, we have $a_n \in U$.

**Lemma 1.109.** *Let $X$ be a topological space. Let $A \subseteq X$ and $l \in X$. If there is a sequence of points in $A$ that converges to $l$ then $l \in \overline{A}$.*

PROOF:
⟨1⟩1. LET: $(a_n)$ be a sequence of points in $A$ that converges to $l$.
⟨1⟩2. LET: $U$ be a neighbourhood of $l$.
⟨1⟩3. PICK $N$ such that, for all $n \geq N$, we have $a_n \in U$.
⟨1⟩4. $a_N \in U \cap A$
⟨1⟩5. Q.E.D.
PROOF: Theorem 1.81.
□

**Proposition 1.110.** *Let $X$ be a topological space. Let $\mathcal{B}$ be a basis for the topology on $X$. Let $(a_n)$ be a sequence in $X$ and $l \in X$. Then $a_n \to l$ as $n \to \infty$ if and only if, for every $B \in \mathcal{B}$ with $l \in B$, there exists $N$ such that, for all $n \geq N$, we have $a_n \in B$.*

PROOF:
⟨1⟩1. If $a_n \to l$ as $n \to \infty$ then, for every $B \in \mathcal{B}$ with $l \in B$, there exists $N$ such that, for all $n \geq N$, we have $a_n \in B$.
PROOF: Since every element of $\mathcal{B}$ is open (Corollary 1.96.1).
⟨1⟩2. If, for every $B \in \mathcal{B}$ with $l \in B$, there exists $N$ such that, for all $n \geq N$, we have $a_n \in B$, then $a_n \to l$ as $n \to \infty$.
⟨2⟩1. ASSUME: for every $B \in \mathcal{B}$ with $l \in B$, there exists $N$ such that, for all $n \geq N$, we have $a_n \in B$
⟨2⟩2. LET: $U$ be a neighbourhood of $l$.
⟨2⟩3. PICK $B \in \mathcal{B}$ such that $l \in B \subseteq U$
⟨2⟩4. PICK $N$ such that, for all $n \geq N$, we have $a_n \in B$
PROOF: From ⟨2⟩1.
⟨2⟩5. For all $n \geq N$ we have $a_n \in U$
□

**Lemma 1.111.** *If a sequence $(a_n)$ is constant with $a_n = l$ for all $n$, then $a_n \to l$ as $n \to \infty$.*

PROOF: Immediate from definitions. □

**Theorem 1.112.** *Let $X$ be a linearly ordered set. Let $(s_n)$ be a monotone increasing sequence in $X$ with a supremum $s$. Then $s_n \to s$ as $n \to \infty$.*

PROOF:
⟨1⟩1. ASSUME: $s$ is not least in $X$.

PROOF: Otherwise $(s_n)$ is the constant sequence $s$ and the result follows from Lemma 1.111.
$\langle 1 \rangle 2$. LET: $U$ be a neighbourhood of $s$.
$\langle 1 \rangle 3$. PICK$a < s$ such that $(a, s] \subseteq U$
$\langle 1 \rangle 4$. PICK $N$ such that $a < a_N$.
$\langle 1 \rangle 5$. For all $n \geq N$ we have $a_n \in (a, s]$
$\langle 1 \rangle 6$. For all $n \geq N$ we have $a_n \in U$.
□

**Theorem 1.113.** *If $\sum_{i=0}^{\infty} a_i = s$ and $\sum_{i=0}^{\infty} b_i = t$ then $\sum_{i=0}^{\infty}(ca_i + b_i) = cs + t$.*

PROOF:$\sum_{i=0}^{N}(ca_i + b_i) = c\sum_{i=0}^{N} a_i + \sum_{i=0}^{N} b_i \to cs + t$ as $n \to \infty$. □

**Theorem 1.114** (Comparison Test)**.** *If $|a_i| \leq b_i$ for all $i$ and $\sum_{i=0}^{\infty} b_i$ converges then $\sum_{i=0}^{\infty} a_i$ converges.*

PROOF:
$\langle 1 \rangle 1$. $\sum_{i=0}^{\infty} |a_i|$ converges
PROOF: The partial sums $\sum_{i=0}^{N} |a_i|$ form an increasing sequence bounded above by $\sum_{i=0}^{\infty} b_i$.
$\langle 1 \rangle 2$. LET: $c_i = |a_i| + a_i$ for all $i$
$\langle 1 \rangle 3$. $\sum_{i=0}^{\infty} c_i$ converges
PROOF: Each $c_i$ is either $2|a_i|$ or 0. So the partial sums $\sum_{i=0}^{N} c_i$ form an increasing sequence bounded above by $2\sum_{i=0}^{\infty} b_i$.
$\langle 1 \rangle 4$. Q.E.D.
PROOF: Since $a_i = c_i - |a_i|$.
□

**Corollary 1.114.1.** *If $\sum_{i=0}^{\infty} |a_i|$ converges then $\sum_{i=0}^{\infty} a_i$ converges.*

**Theorem 1.115** (Weierstrass $M$-test)**.** *Let $X$ be a set and $(f_n : X \to \mathbb{R})$ be a sequence of functions. Let*

$$s_n(x) = \sum_{i=0}^{n} f_i(x)$$

*for all $n$, $x$. Suppose $|f_i(x)| \leq M_i$ for all $i \geq 0$ and $x \in X$. If the series $\sum_{i=0}^{\infty} M_i$ converges, then the sequence $(s_n)$ converges uniformly to*

$$s(x) = \sum_{i=0}^{\infty} f_i(x) \enspace .$$

PROOF:
$\langle 1 \rangle 1$. LET: $r_n = \sum_{i=n+1}^{\infty} M_i$ for all $n$
$\langle 1 \rangle 2$. Given $0 \leq n < k$, we have $|s_k(x) - s_n(x)| \leq r_n$

PROOF:

$$\begin{aligned}|s_k(x) - s_n(x)| &= |\sum_{i=n+1}^{k} f_i(x)| \\ &\leq \sum_{i=n+1}^{k} |f_i(x)| \\ &\leq \sum_{i=n+1}^{k} M_i \\ &\leq r_n\end{aligned}$$

⟨1⟩3. Given $n \geq 0$ we have $|s(x) - s_n(x)| \leq r_n$

PROOF: By taking the limit $k \to \infty$ in ⟨1⟩2.

⟨1⟩4. Q.E.D.

PROOF: Since $r_n \to 0$ as $n \to \infty$.

□

## 1.25 Locally Finite Sets

**Definition 1.116** (Locally Finite)**.** Let $X$ be a topological space and $\{A_\alpha\}$ a family of subsets of $X$. Then $\mathcal{A}$ is *locally finite* if and only if every point in $X$ has a neighbourhood that intersects $A_\alpha$ for only finitely many $\alpha$.

**Theorem 1.117** (Pasting Lemma)**.** *Let $X$ and $Y$ be topological spaces and $f : X \to Y$. Let $\{A_\alpha\}$ be a locally finite family of closed subsets of $X$ that cover $X$. Suppose $f \upharpoonright A_\alpha$ is continuous for all $\alpha$. Then $f$ is continuous.*

PROOF:

⟨1⟩1. Let $X$ and $Y$ be topological spaces and $f : X \to Y$. Let $A$ and $B$ be closed subsets of $X$ such that $X = A \cup B$. Suppose $f \upharpoonright A$ and $f \upharpoonright B$ are continuous. Then $f$ is continuous.

⟨2⟩1. LET: $C \subseteq Y$ be closed.

⟨2⟩2. $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$

⟨2⟩3. $f^{-1}(C)$ and $g^{-1}(C)$ are closed in $X$.

PROOF: Theorems 1.127 and 1.178.

⟨2⟩4. $h^{-1}(C)$ is closed in $X$.

PROOF: Lemma 1.67.

⟨2⟩5. Q.E.D.

PROOF: Theorem 1.127.

⟨1⟩2. Let $X$ and $Y$ be topological spaces and $f : X \to Y$. Let $\{A_\alpha\}$ be a finite family of closed subsets of $X$ that cover $X$. Suppose $f \upharpoonright A_\alpha$ is continuous for all $\alpha$. Then $f$ is continuous.

PROOF: From ⟨1⟩1 by induction.

⟨1⟩3. Let $X$ and $Y$ be topological spaces and $f : X \to Y$. Let $\{A_\alpha\}$ be a locally finite family of closed subsets of $X$ that cover $X$. Suppose $f \upharpoonright A_\alpha$ is continuous for all $\alpha$. Then $f$ is continuous.

$\langle 2\rangle 1$. LET: $x \in X$
PROVE: $f$ is continuous at $x$
$\langle 2\rangle 2$. PICK a neighbourhood $U$ of $x$ that intersects $A_\alpha$ for only finitely many $\alpha$.
$\langle 2\rangle 3$. $f \upharpoonright U$ is continuous
PROOF: By $\langle 1\rangle 2$.
$\langle 2\rangle 4$. Q.E.D.
PROOF: Lemma 1.137.
□

The following example shows that we cannot remove the assumption of local finiteness.

**Example 1.118.** Define $f : [-1,1] \to \mathbb{R}$ by: $f(x) = 1$ if $x < -1$, $f(x) = 0$ if $x > 1$. Let $C_n = [-1, -1/n]$ for $n \geq 1$, and $D = [0,1]$. Then $[-1,1] = \bigcup_{n=1}^{\infty} C_n \cup D$ and $f$ is continuous on each $C_n$ and each $D$, but $f$ is not continuous on $[-1,1]$.

## 1.26 Open Maps

**Definition 1.119** (Open Map)**.** Let $X$ and $Y$ be topological spaces and $f : X \to Y$. Then $f$ is an *open map* if and only if, for every open set $U$ in $X$, the set $f(U)$ is open in $Y$.

**Lemma 1.120.** *Let $X$ and $Y$ be topological spaces and $f : X \to Y$. Let $\mathcal{B}$ be a basis for the topology on $X$. If $f(B)$ is open in $Y$ for all $B \in \mathcal{B}$, then $f$ is an open map.*

PROOF: From Lemma 1.96. □

**Proposition 1.121.** *Let $X$ and $Y$ be topological spaces. Let $\mathcal{B}$ be a basis for the topology on $X$. Let $f : X \to Y$. Suppose that, for all $B \in \mathcal{B}$, we have $f(B)$ is open tn $Y$. Then $f$ is an open map.*

PROOF: For any $\mathcal{A} \subseteq \mathcal{B}$, we have $f(\bigcup \mathcal{A}) = \bigcup_{B \in \mathcal{B}} f(B)$ is open in $Y$. The result follows from Lemma 1.96. □

## 1.27 Continuous Functions

**Definition 1.122** (Continuous)**.** Let $X$ and $Y$ be topological spaces. A function $f : X \to Y$ is *continuous* if and only if, for every open set $V$ in $Y$, the set $f^{-1}(V)$ is open in $X$.

**Proposition 1.123.** *Let $X$ and $Y$ be topological spaces and $f : X \to Y$. Let $\mathcal{B}$ be a basis for $Y$. Then $f$ is continuous if and only if, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in $X$.*

PROOF:
$\langle 1\rangle 1$. If $f$ is continuous then, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in $X$.

PROOF: Since every element of $B$ is open (Lemma 1.96).

⟨1⟩2. Suppose that, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in $X$. Then $f$ is continuous.

⟨2⟩1. ASSUME: For all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in $X$.

⟨2⟩2. LET: $V$ be open in $Y$.

⟨2⟩3. PICK $\mathcal{A} \subseteq \mathcal{B}$ such that $V = \bigcup \mathcal{A}$

PROOF: By Lemma 1.96.

⟨2⟩4. $f^{-1}(V)$ is open in $X$.

PROOF:

$$\begin{aligned} f^{-1}(V) &= f^{-1}\left(\bigcup \mathcal{A}\right) \\ &= \bigcup_{B \in \mathcal{A}} f^{-1}(B) \end{aligned}$$

□

**Proposition 1.124.** *Let $X$ and $Y$ be topological spaces and $f : X \to Y$. Let $\mathcal{S}$ be a subbasis for $Y$. Then $f$ is continuous if and only if, for all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in $X$.*

PROOF:

⟨1⟩1. If $f$ is continuous then, for all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in $X$.

PROOF: Since every element of $S$ is open.

⟨1⟩2. Suppose that, for all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in $X$. Then $f$ is continuous.

⟨2⟩1. ASSUME: For all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in $X$.

⟨2⟩2. LET: $S_1, \ldots, S_n \in \mathcal{S}$

⟨2⟩3. $f^{-1}(S_1 \cap \cdots \cap S_n)$ is open in $A$

PROOF: Since $f^{-1}(S_1 \cap \cdots \cap S_n) = f^{-1}(S_1) \cap \cdots \cap f^{-1}(S_n)$.

⟨2⟩4. Q.E.D.

PROOF: By Propositions 1.123 and 1.104.

□

**Proposition 1.125.** *Let $X$ and $Y$ be topological spaces and $f : X \to Y$. Let $\mathcal{S}$ be a basis for $Y$. Then $f$ is continuous if and only if, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in $X$.*

PROOF:

⟨1⟩1. If $f$ is continuous then, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in $X$.

PROOF: Since every element of $\mathcal{S}$ is open.

⟨1⟩2. Suppose that, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in $X$. Then $f$ is continuous.

⟨2⟩1. ASSUME: For all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in $X$.

⟨2⟩2. For every set $B$ that is the finite intersection of elemets of $\mathcal{S}$, we have $f^{-1}(B)$ is open in $X$.

PROOF: Because $f^{-1}(V_1 \cap \cdots \cap V_n) = f^{-1}(V_1) \cap \cdots \cap f^{-1}(V_n)$.

⟨2⟩3. Q.E.D.

PROOF: From Propositions 1.104 and 1.123.

□

**Definition 1.126** (Continuous at a Point)**.** Let $X$ and $Y$ be topological spaces. Let $f : X \to Y$ and $x \in X$. Then $f$ is *continuous at* $x$ if and only if, for every neighbourhood $V$ of $f(x)$, there exists a neighbourhood $U$ of $x$ such that $f(U) \subseteq V$.

**Theorem 1.127.** *Let $X$ and $Y$ be topological spaces and $f : X \to Y$. Then the following are equivalent:*

1. *$f$ is continuous.*
2. *For all $A \subseteq X$, we have $f(\overline{A}) \subseteq \overline{f(A)}$*
3. *For all $B \subseteq Y$ closed, we have $f^{-1}(B)$ is closed in $X$.*
4. *$f$ is continuous at every point of $X$.*

PROOF:
⟨1⟩1. $1 \Rightarrow 2$
  ⟨2⟩1. ASSUME: $f$ is continuous.
  ⟨2⟩2. LET: $A \subseteq X$
  ⟨2⟩3. LET: $x \in \overline{A}$
    PROVE: $f(x) \in \overline{f(A)}$
  ⟨2⟩4. LET: $V$ be a neighbourhood of $f(x)$
  ⟨2⟩5. $f^{-1}(V)$ is a neighbourhood of $x$
  ⟨2⟩6. PICK $y \in A \cap f^{-1}(V)$
    PROOF: By Theorem 1.81.
  ⟨2⟩7. $f(y) \in V \cap f(A)$
  ⟨2⟩8. Q.E.D.
    PROOF: By Theorem 1.81.
⟨1⟩2. $2 \Rightarrow 3$
  ⟨2⟩1. ASSUME: 2
  ⟨2⟩2. LET: $B$ be closed in $Y$
  ⟨2⟩3. LET: $x \in \overline{f^{-1}(B)}$
    PROVE: $x \in f^{-1}(B)$
  ⟨2⟩4. $f(x) \in B$
    PROOF:

$$\begin{aligned} f(x) &\in f(\overline{f^{-1}(B)}) \\ &\subseteq \overline{f(f^{-1}(B))} && (\langle 2\rangle 1) \\ &\subseteq \overline{B} && (Proposition 1.82) \\ &= B \end{aligned}$$

⟨1⟩3. $3 \Rightarrow 1$
  ⟨2⟩1. ASSUME: 3
  ⟨2⟩2. LET: $V$ be open in $Y$
  ⟨2⟩3. $Y \setminus V$ is closed in $Y$
  ⟨2⟩4. $f^{-1}(Y \setminus V)$ is closed in $X$
  ⟨2⟩5. $X \setminus f^{-1}(V)$ is closed in $X$
  ⟨2⟩6. $f^{-1}(V)$ is open in $X$

$\langle 1\rangle 4.$ $1 \Rightarrow 4$
PROOF: For any neighbourhood $V$ of $f(x)$, the set $U = f^{-1}(V)$ is a neighbourhood of $x$ such that $f(U) \subseteq V$.
$\langle 1\rangle 5.$ $4 \Rightarrow 1$
$\langle 2\rangle 1.$ ASSUME: 4
$\langle 2\rangle 2.$ LET: $V$ be open in $Y$
$\langle 2\rangle 3.$ LET: $x \in f^{-1}(V)$
$\langle 2\rangle 4.$ $V$ is a neighbourhood of $f(x)$
$\langle 2\rangle 5.$ PICK a neighbourhood $U$ of $x$ such that $f(U) \subseteq V$
$\langle 2\rangle 6.$ $x \in U \subseteq f^{-1}(V)$
$\langle 2\rangle 7.$ Q.E.D.
PROOF: By Lemma 1.59.
□

**Theorem 1.128.** *A constant function is continuous.*

PROOF: Let $X$ and $Y$ be topological spaces. Let $b \in Y$, and let $f : X \to Y$ be the constant function with value $b$. For any open $V \subseteq Y$, the set $f^{-1}(V)$ is either $X$ (if $b \in V$) or $\emptyset$ (if $b \notin V$). □

**Theorem 1.129.** *If $A$ is a subspace of $X$ then the inclusion $j : A \to X$ is continuous.*

PROOF: For any $V$ open in $X$, we have $j^{-1}(V) = V \cap A$ is open in $A$. □

**Theorem 1.130.** *The composite of two continuous functions is continuous.*

PROOF: Let $f : X \to Y$ and $g : Y \to Z$ be continuous. For any $V$ open in $Z$, we have $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is open in $X$. □

**Theorem 1.131.** *Let $f : X \to Y$ be a continuous function and $A$ be a subspace of $X$. Then the restriction $f \restriction A : A \to Y$ is continuous.*

PROOF: Let $V$ be open in $Y$. Then $(f \restriction A)^{-1}(V) = f^{-1}(V) \cap A$ is open in $A$. □

**Theorem 1.132.** *Let $f : X \to Y$ be continuous. Let $Z$ be a subspace of $Y$ such that $f(X) \subseteq Z$. Then the corestriction $f : X \to Z$ is continuous.*

PROOF:
$\langle 1\rangle 1.$ LET: $V$ be open in $Z$.
$\langle 1\rangle 2.$ PICK $U$ open in $Y$ such that $V = U \cap Z$.
$\langle 1\rangle 3.$ $f^{-1}(V) = f^{-1}(U)$
$\langle 1\rangle 4.$ $f^{-1}(V)$ is open in $X$.
□

**Theorem 1.133.** *Let $f : X \to Y$ be continuous. Let $Z$ be a space such that $Y$ is a subspace of $Z$. Then the expansion $f : X \to Z$ is continuous.*

PROOF: Let $V$ be open in $Z$. Then $f^{-1}(V) = f^{-1}(V \cap Y)$ is open in $X$. □

**Theorem 1.134.** *Let $X$ and $Y$ be topological spaces. Let $f : X \to Y$. Suppose $\mathcal{U}$ is a set of open sets in $X$ such that $X = \bigcup \mathcal{U}$ and, for all $U \in \mathcal{U}$, we have $f \restriction U : U \to Y$ is continuous. Then $f$ is continuous.*

PROOF:
⟨1⟩1. LET: $V$ be open in $Y$
⟨1⟩2. $f^{-1}(V) = \bigcup_{U \in \mathcal{U}} (f \restriction U)^{-1}(V)$
⟨1⟩3. For all $U \in \mathcal{U}$, we have $(f \restriction U)^{-1}(V)$ is open in $U$.
⟨1⟩4. For all $U \in \mathcal{U}$, we have $(f \restriction U)^{-1}(V)$ is open in $X$.
  PROOF: Lemma 1.177.
□

**Proposition 1.135.** *Let $X$ and $X'$ be the same set $X$ under two topologies $\mathcal{T}$ and $\mathcal{T}'$. Let $i : X \to X'$ be the identity function. Then $i$ is continuous if and only if $\mathcal{T}' \subseteq \mathcal{T}$.*

PROOF: Immediate from definitions. □

**Proposition 1.136.** *Let $f : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$. Then $f$ is continuous on the right at $a$ if and only if $f$ is continuous at $a$ as a function $\mathbb{R}_l \to \mathbb{R}$.*

PROOF:
⟨1⟩1. If $f$ is continuous on the right at $a$ then $f$ is continuous at $a$ as a function $\mathbb{R}_l \to \mathbb{R}$.
  ⟨2⟩1. ASSUME: $f$ is continuous on the right at $a$.
  ⟨2⟩2. LET: $V$ be a neighbourhood of $f(a)$
  ⟨2⟩3. PICK $b$, $c$ such that $f(a) \in (b, c) \subseteq V$.
  ⟨2⟩4. LET: $\epsilon = \min(c - f(a), f(a) - b)$
  ⟨2⟩5. PICK $\delta > 0$ such that, for all $x$, if $a < x < a + \delta$ then $|f(x) - f(a)| < \epsilon$
  ⟨2⟩6. LET: $U = [a, a + \delta)$
  ⟨2⟩7. $f(U) \subseteq V$
⟨1⟩2. If $f$ is continuous at $a$ as a function $\mathbb{R}_l \to \mathbb{R}$ then $f$ is continuous on the right at $a$.
  ⟨2⟩1. ASSUME: $f$ is continuous at $a$ as a function $\mathbb{R}_l \to \mathbb{R}$
  ⟨2⟩2. LET: $\epsilon > 0$
  ⟨2⟩3. PICK a neighbourhood $U$ of $a$ such that $f(U) \subseteq (f(a) - \epsilon, f(a) + \epsilon)$
  ⟨2⟩4. PICK $b$, $c$ such that $a \in [b, c) \subset U$
  ⟨2⟩5. LET: $\delta = c - a$
  ⟨2⟩6. For all $x$, if $a < x < a + \delta$ then $|f(x) - f(a)| < \epsilon$
□

**Lemma 1.137.** *Let $f : X \to Y$. Let $Z$ be an open subspace of $X$ and $a \in Z$. If $f \restriction Z$ is continuous at $a$ then $f$ is continuous at $a$.*

PROOF:
⟨1⟩1. LET: $V$ be a neighbourhood of $f(x)$
⟨1⟩2. PICK a neighbourhood $W$ of $x$ in $Z$ such that $f(W) \subseteq V$
⟨1⟩3. $W$ is a neighbourhood of $x$ in $X$ such that $f(W) \subseteq V$

PROOF: Lemma 1.177.
∎

**Proposition 1.138.** *Let $f : A \to B$ and $g : C \to D$ be continuous. Define $f \times g : A \times C \to B \times D$ by*

$$(f \times g)(a, c) = (f(a), g(c)) \ .$$

*Then $f \times g$ is continuous.*

PROOF: $\pi_1 \circ (f \times g) = f \circ \pi_1$ and $\pi_2 \circ (f \times g) = g \circ \pi_2$ are continuous by Theorem 1.130. The result follows by Theorem 1.166.

**Proposition 1.139.** *Let $X$ and $Y$ be topological spaces and $f : X \to Y$ be continuous. If $a_n \to l$ as $n \to \infty$ in $X$ then $f(a_n) \to f(l)$ as $n \to \infty$.*

PROOF:
$\langle 1 \rangle 1$. LET: $V$ be a neighbourhood of $f(l)$
$\langle 1 \rangle 2$. PICK a neighbourhood $U$ of $l$ such that $f(U) \subseteq V$
$\langle 1 \rangle 3$. PICK $N$ such that, for all $n \geq N$, we have $a_n \in U$
$\langle 1 \rangle 4$. For all $n \geq N$ we have $f(n) \in V$
∎

## 1.28 Homeomorphisms

**Definition 1.140** (Homeomorphism)**.** Let $X$ and $Y$ be topological spaces. A *Homeomorphism* $f$ between $X$ and $Y$, $f : X \cong Y$, is a bijection $f : X \to Y$ such that both $f$ and $f^{-1}$ are continuous.

**Lemma 1.141.** *Let $X$ and $Y$ be topological spaces and $f : X \to Y$ a bijection. Then the following are equivalent:*

1. *$f$ is a homeomorphism.*
2. *$f$ is continuous and an open map.*
3. *$f$ is continuous and a closed map.*
4. *For any $U \subseteq X$, we have $U$ is open if and only if $f(U)$ is open.*

PROOF: Immediate from definitions. ∎

**Proposition 1.142.** *Let $X$ and $X'$ be the same set $X$ under two topologies $\mathcal{T}$ and $\mathcal{T}'$. Let $i : X \to X'$ be the identity function. Then $i$ is a homeomorphism if and only if $\mathcal{T} = \mathcal{T}'$.*

PROOF: Immediate from definitions. ∎

**Definition 1.143** (Topological Property)**.** Let $P$ be a property of topological spaces. Then $P$ is a *topological* property if and only if, for any spaces $X$ and $Y$, if $P$ holds of $X$ and $X \cong Y$ then $P$ holds of $Y$.

**Definition 1.144** (Topological Imbedding)**.** Let $X$ and $Y$ be topological spaces and $f : X \to Y$. Then $f$ is a *topological imbedding* if and only if the corestriction $f : X \to f(X)$ is a homeomorphism.

**Proposition 1.145.** *Let $X$ and $Y$ be topological spaces and $a \in X$. The function $i : Y \to X \times Y$ that maps $y$ to $(a, y)$ is an imbedding.*

PROOF:
$\langle 1 \rangle 1$. $i$ is injective
$\langle 1 \rangle 2$. $i$ is continuous.
PROOF: For $U$ open in $X$ and $V$ open in $Y$, we have $i^{-1}(U \times V)$ is $V$ if $a \in U$, and $\emptyset$ if $a \notin U$.
$\langle 1 \rangle 3$. $i : Y \to i(Y)$ is an open map.
PROOF: For $V$ open in $Y$ we have $i(V) = (X \times V) \cap i(Y)$.
□

## 1.29 The Order Topology

**Definition 1.146** (Order Topology)**.** Let $X$ be a linearly ordered set with at least two points. The *order topology* on $X$ is the topology generated by the basis $\mathcal{B}$ consisting of:

- all open intervals $(a, b)$;
- all intervals of the form $[\bot, b)$ where $\bot$ is least in $X$;
- all intervals of the form $(a, \top]$ where $\top$ is greatest in $X$.

We prove this is a basis for a topology.

PROOF:
$\langle 1 \rangle 1$. For all $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B$.
$\langle 2 \rangle 1$. LET: $x \in X$
$\langle 2 \rangle 2$. CASE: $x$ is greatest in $X$.
$\langle 3 \rangle 1$. PICK $y \in X$ with $y \neq x$
$\langle 3 \rangle 2$. $x \in (y, x] \in \mathcal{B}$
$\langle 2 \rangle 3$. CASE: $x$ is least in $X$.
$\langle 3 \rangle 1$. PICK $y \in X$ with $y \neq x$
$\langle 3 \rangle 2$. $x \in [x, y) \in \mathcal{B}$
$\langle 2 \rangle 4$. CASE: $x$ is neither greatest nor least in $X$.
$\langle 3 \rangle 1$. PICK $a, b \in X$ with $a < x$ and $x < b$
$\langle 3 \rangle 2$. $x \in (a, b) \in \mathcal{B}$
$\langle 1 \rangle 2$. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$
$\langle 2 \rangle 1$. LET: $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$
$\langle 2 \rangle 2$. CASE: $B_1 = (a, b)$, $B_2 = (c, d)$
PROOF: Take $B_3 = (\max(a, c), \min(b, d))$.
$\langle 2 \rangle 3$. CASE: $B_1 = (a, b)$, $B_2 = [\bot, d)$

PROOF: Take $B_3 = (a, \min(b, d))$.
⟨2⟩4. CASE: $B_1 = (a, b)$, $B_2 = (c, \top]$
PROOF: Take $B_3 = (\max(a, c), b)$.
⟨2⟩5. CASE: $B_1 = [\bot, b)$, $B_2 = [\bot, d)$
PROOF: Take $B_3 = [\bot, \min(b, d))$.
⟨2⟩6. CASE: $B_1 = [\bot, b)$, $B_2 = (c, \top]$
PROOF: Take $B_3 = (c, b)$.
□

**Lemma 1.147.** *Let $X$ be a linearly ordered set. Then the open rays form a subbasis for the order topology on $X$.*

PROOF:
⟨1⟩1. Every open ray is open.
⟨2⟩1. For all $a \in X$, the ray $(-\infty, a)$ is open.
⟨3⟩1. LET: $x \in (-\infty, a)$
⟨3⟩2. CASE: $x$ is least in $X$
PROOF: $xin[x, a) = (-\infty, a)$.
⟨3⟩3. CASE: $x$ is not least in $X$
⟨4⟩1. PICK $y < x$
⟨4⟩2. $x \in (y, a) \subseteq (-\infty, a)$
⟨2⟩2. For all $a \in X$, the ray $(a, +\infty)$ is open.
PROOF: Similar.
⟨1⟩2. Every basic open set is a finite intersection of open rays.
PROOF: We have $(a, b) = (a, +\infty) \cap (-\infty, b)$, $[\bot, b) = (-\infty, b)$ and $(a, \top] = (a, +\infty)$.
□

**Definition 1.148** (Standard Topology on the Real Line)**.** The *standard topology on the real line* is the order topology on $\mathbb{R}$ generated by the standard order.

**Lemma 1.149.** *The lower limit topology is strictly finer than the standard topology on $\mathbb{R}$.*

PROOF:
⟨1⟩1. Every open interval is open in the lower limit topology.
PROOF: If $x \in (a, b)$ then $x \in [x, b) \subseteq (a, b)$.
⟨1⟩2. The half-open interval $[0, 1)$ is not open in the standard topology.
PROOF: There is no open interval $(a, b)$ such that $0 \in (a, b) \subseteq [0, 1)$.
□

**Lemma 1.150.** *The $K$-topology is strictly finer than the standard topology on $\mathbb{R}$.*

PROOF:
⟨1⟩1. Every open interval is open in the $K$-topology.
PROOF: Corollary 1.96.1.
⟨1⟩2. The set $(-1, 1) \setminus K$ is not open in the standard topology.

PROOF: There is no open interval $(a, b)$ such that $0 \in (a, b) \subseteq (-1, 1) \setminus K$, since there must be a positive integer $n$ with $1/n \in (a, b)$.
□

**Lemma 1.151.** *Let $X$ be a topological space. Let $Y$ be a linearly ordered set in the order topology. Let $f, g : X \to Y$ be continuous. Then $C = \{x \in X \mid f(x) \leq g(x)\}$ is closed.*

PROOF:
⟨1⟩1. LET: $x \in X \setminus C$
⟨1⟩2. $f(x) > g(x)$
PROVE: There exists a neighbourhood $U$ of $x$ such that $U \subseteq X \setminus C$
⟨1⟩3. CASE: There exists $y$ such that $g(x) < y < f(x)$
PROOF: Take $U = g^{-1}((-\infty, y)) \cup f^{-1}(y, +\infty)$.
⟨1⟩4. CASE: There is no $y$ such that $g(x) < y < f(x)$
PROOF: Take $U = g^{-1}((-\infty, f(x))) \cup f^{-1}(g(x), +\infty)$.
□

**Proposition 1.152.** *Let $X$ be a topological space. Let $Y$ be a linearly ordered set in the order topology. Let $f, g : X \to Y$ be continuous. Define $h : X \to Y$ by $h(x) = \min(f(x), g(x))$. Then $h$ is continuous.*

PROOF: By the Pasting Lemma applied to $\{x \in X \mid f(x) \leq g(x)\}$ and $\{x \in X \mid g(x) \leq f(x)\}$, which are closed by Lemma 1.151.

**Proposition 1.153.** *Let $X$ and $Y$ be linearly ordered sets in the order topology. Let $f : X \to Y$ be strictly monotone and surjective. Then $f$ is a homeomorphism.*

PROOF:
⟨1⟩1. $f$ is bijective.
PROOF: Proposition 1.30.
⟨1⟩2. $f$ is continuous.
⟨2⟩1. For all $y \in Y$ we have $f^{-1}((y, +\infty))$ is open.
⟨3⟩1. LET: $y \in Y$
⟨3⟩2. PICK$x \in X$ such that $f(x) = y$
PROOF: Since $f$ is surjective.
⟨3⟩3. $f^{-1}((y, +\infty)) = (x, +\infty)$
PROOF: By strict monotoncity.
⟨2⟩2. For all $y \in Y$ we have $f^{-1}((-\infty, y))$ is open.
PROOF: Similar.
⟨1⟩3. $f^{-1}$ is continuous.
⟨2⟩1. For all $x \in X$ we have $f((x, +\infty))$ is open.
PROOF: $f((x, +\infty)) = (f(x), +\infty)$.
⟨2⟩2. For all $x \in X$ we have $f((-\infty, x))$ is open.
PROOF: $f((-\infty, x)) = (-\infty, f(x))$.
□

## 1.30 The $n$th Root Function

**Proposition 1.154.** *For all $n \geq 1$, the function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined by $f(x) = x^n$ is a homemorphism.*

PROOF:
⟨1⟩1. $f$ is strictly monotone.
⟨2⟩1. LET: $x, y \in \mathbb{R}$ with $0 \leq x < y$
⟨2⟩2. $x^n < y^n$

$$\begin{aligned} y^n - x^n &= (y-x)(y^{n-1} + y^{n-2}x + y^{n-3}x^2 + \cdots + x^{n-1}) \\ &> 0 \end{aligned}$$

⟨1⟩2. $f$ is surjective.
⟨2⟩1. LET: $y \in \mathbb{R}_{\geq 0}$
⟨2⟩2. PICK $x \in \mathbb{R}$ such that $y \leq x^n$
PROOF: If $y \leq 1$ take $x = 1$, otherwise take $x = y$.
⟨2⟩3. There exists $x' \in [0, x]$ such that $(x')^n = y$
PROOF: By the Intermediate Value Theorem.
⟨1⟩3. Q.E.D.
PROOF: Proposition 1.153.
□

**Definition 1.155.** For $n \geq 1$, the *nth root function* is the function $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ that is the inverse of $\lambda x.x^n$.

## 1.31 The Product Topology

**Definition 1.156** (Product Topology)**.** Let $\{A_i\}_{i \in I}$ be a family of topological spaces. The *product topology* on $\prod_{i \in I} A_i$ is the topology generated by the subbasis consisting of the sets of the form ${\pi_i}^{-1}(U)$ where $i \in I$ and $U$ is open in $A_i$.

**Proposition 1.157.** *The product topology on $\prod_{i \in I} A_i$ is generated by the basis consisting of all sets of the form $\prod_{i \in I} U_i$ where $\{U_i\}_{i \in I}$ is a family such that each $U_i$ is an open set in $A_i$ and $U_i = A_i$ for all but finitely many $i$.*

PROOF: From Proposition 1.104. □

**Proposition 1.158.** *If $A_i$ is closed in $X_i$ for all $i \in I$ then $\prod_{i \in I} A_i$ is closed in $\prod_{i \in I} X_i$.*

PROOF:

$$(\prod_{i \in I} X_i) \setminus (\prod_{i \in I} A_i) = \bigcup_{j \in I} \left( \prod_{i \in I} X_i \setminus {\pi_j}^{-1}(A_j) \right) \square$$

**Proposition 1.159.** *Let $\{A_i\}_{i \in I}$ be a family of topological spaces. For $i \in I$, let $\mathcal{B}_i$ be a basis for the topology on $A_i$. Then $\mathcal{B} = \{\prod_{i \in I} B_i \mid \forall i \in I . B_i \in \mathcal{B}_i, B_i = A_i$ for all but finitely many $i\}$ is a basis for the box topology on $\prod_{i \in I} A_i$.*

Proof:
⟨1⟩1. Every set in $\mathcal{B}$ is open.
⟨1⟩2. For every point $a \in \prod_{i \in I} A_i$ and every open set $U$ with $a \in U$, there exists $B \in \mathcal{B}$ such that $a \in B \subseteq U$.
⟨2⟩1. Let: $U$ be open and $a \in U$
⟨2⟩2. Pick a family $\{U_i\}_{i \in I}$ such that each $U_i$ is open in $A_i$, such that $U_i = A_i$ except for $i = i_1, \ldots, i_n$, and such that $a \in \prod_{i \in I} U_i \subseteq U$.
⟨2⟩3. For $j = 1, \ldots, n$, Pick $B_{i_j} \in \mathcal{B}_{i_j}$ such that $a_{i_j} \in B_{i_j} \subseteq U_{i_j}$
⟨2⟩4. Let: $B = \prod_{i \in I} B_i$ where $B_i = A_i$ for $i \neq i_1, \ldots, i_n$
⟨2⟩5. $B \in \mathcal{B}$
⟨2⟩6. $a \in B \subseteq U$
⟨1⟩3. Q.E.D.
Proof: Lemma 1.97.
□

**Proposition 1.160.** *Let $\{A_i\}_{i \in I}$ be a family of topological spaces. Then the projections $\pi_i : \prod_{i \in I} A_i \to A_i$ are open maps.*

Proof: From Lemma 1.120. □

**Example 1.161.** The projections are not always closed maps. For example, $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ maps the closed set $\{(x, 1/x) \mid x > 0\}$ to $(0, +\infty)$.

**Proposition 1.162.** *Let $\{X_i\}_{i \in I}$ be a family of sets. For $i \in I$, let $\mathcal{T}_i$ and $\mathcal{U}_i$ be topologies on $X_i$. Let $\mathcal{P}$ be the product topology on $\prod_{i \in I} X_i$ generated by the topologies $\mathcal{T}_i$, and $\mathcal{Q}$ the product topology on the same set generated by the topologies $\mathcal{U}_i$. Then $\mathcal{P} \subseteq \mathcal{Q}$ if and only if $\mathcal{T}_i \subseteq \mathcal{U}_i$ for all $i$.*

Proof:
⟨1⟩1. If $\mathcal{T}_i \subseteq \mathcal{U}_i$ for all $i$ then $\mathcal{P} \subseteq \mathcal{Q}$
Proof: By Corollary 1.96.1.
⟨1⟩2. If $\mathcal{P} \subseteq \mathcal{Q}$ then $\mathcal{T}_i \subseteq \mathcal{U}_i$ for all $i$
⟨2⟩1. Assume: $\mathcal{P} \subseteq \mathcal{Q}$
⟨2⟩2. Let: $i \in I$
⟨2⟩3. Let: $U \in \mathcal{T}_i$
⟨2⟩4. Let: $U_i = U$ and $U_j = X_j$ for $j \neq i$
⟨2⟩5. $\prod_{i \in I} U_i \in \mathcal{P}$
⟨2⟩6. $\prod_{i \in I} U_i \in \mathcal{Q}$
⟨2⟩7. $U \in \mathcal{U}_i$
Proof: From Proposition 1.160.
□

**Proposition 1.163** (AC)**.** *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let $A_i \subseteq X_i$ for all $i \in I$. Then*

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i} \ .$$

Proof:

⟨1⟩1. $\overline{\prod_{i \in I} A_i} \subseteq \prod_{i \in I} \overline{A_i}$

⟨2⟩1. For all $i \in I$ we have $A_i \subseteq \overline{A_i}$

PROOF: Lemma 1.78.

⟨2⟩2. $\prod_{i \in I} A_i \subseteq \prod_{i \in I} \overline{A_i}$

⟨2⟩3. Q.E.D.

PROOF: Since $\prod_{i \in I} A_i$ is closed by Proposition 1.158.

⟨1⟩2. $\prod_{i \in I} \overline{A_i} \subseteq \overline{\prod_{i \in I} A_i}$

⟨2⟩1. LET: $x \in \prod_{i \in I} \overline{A_i}$

⟨2⟩2. LET: $U$ be a neighbourhood of $x$

⟨2⟩3. PICK $V_i$ open in $X_i$ such that $x \in \prod_{i \in I} V_i \subseteq U$ with $V_i = X_i$ except for $i = i_1, \ldots, i_n$

⟨2⟩4. For $i \in I$, pick $a_i \in V_i \cap A_i$

PROOF: By Theorem 1.81 and ⟨2⟩1 using the Axiom of Choice.

⟨2⟩5. $U$ intersects $\prod_{i \in I} A_i$

⟨2⟩6. Q.E.D.

PROOF: $a \in U \cap \prod_{i \in I} A_i$

□

**Example 1.164.** The closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ is $\mathbb{R}^\omega$

PROOF:

⟨1⟩1. LET: $a \in \mathbb{R}^\omega$

⟨1⟩2. LET: $U$ be any neighbourhoods of $a$.

⟨1⟩3. PICK $U_n$ open in $\mathbb{R}$ for all $n$ such that $a \in \prod_{n \geq 0} U_n \subseteq U$ and $U_n = \mathbb{R}$ for all $n$ except $n_1, \ldots, n_k$

⟨1⟩4. LET: $b_n = a_n$ for $n = n_1, \ldots, n_k$ and $b_n = 0$ for all other $n$

⟨1⟩5. $b \in \mathbb{R}^\infty \cap U$

⟨1⟩6. Q.E.D.

PROOF: From Theorem 1.81.

□

**Proposition 1.165.** *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let $(a_n)$ be a sequence in $\prod_{i \in I} X_i$ and $l \in \prod_{i \in I} X_i$. Then $a_n \to l$ as $n \to \infty$ if and only if, for all $i \in I$, we have $\pi_i(a_n) \to \pi_i(l)$ as $n \to \infty$.*

PROOF:

⟨1⟩1. If $a_n \to l$ as $n \to \infty$ then, for all $i \in I$, we have $\pi_i(a_n) \to \pi_i(l)$ as $n \to \infty$

PROOF: Proposition 1.139.

⟨1⟩2. If, for all $i \in I$, we have $\pi_i(a_n) \to \pi_i(l)$ as $n \to \infty$, then $a_n \to l$ as $n \to \infty$

⟨2⟩1. ASSUME: For all $i \in I$, we have $\pi_i(a_n) \to \pi_i(l)$ as $n \to \infty$

⟨2⟩2. LET: $V$ be a neighbourhood of $l$

⟨2⟩3. PICK open sets $U_i$ in $X_i$ such that $l \in \prod_{i \in I} U_i \subseteq V$ and $U_i = X_i$ for all $i$ except $i = i_1, \ldots, i_k$

⟨2⟩4. For $j = 1, \ldots, k$, PICK $N_j$ such that, for all $n \geq N_j$, we have $\pi_{i_j}(a_n) \in U_{i_j}$

⟨2⟩5. LET: $N = \max(N_1, \ldots, N_k)$

⟨2⟩6. For all $n \geq N$ we have $a_n \in V$

□

**Theorem 1.166.** *Let $A$ be a topological space and $\{X_i\}_{i \in I}$ be a family of topological spaces. Let $f : A \to \prod_{i \in I} X_i$ be a function. If $\pi_i \circ f$ is continuous for all $i \in I$ then $f$ is continuous.*

PROOF:
$\langle 1 \rangle 1$. LET: $i \in I$ and $U$ be open in $X_i$
$\langle 1 \rangle 2$. $f^{-1}({\pi_i}^{-1}(U))$ is open in $A$
$\langle 1 \rangle 3$. Q.E.D.
PROOF: Proposition 1.124.
□

### 1.31.1 Continuous in Each Variable Separately

**Definition 1.167** (Continuous in Each Variable Separately)**.** Let $F : X \times Y \to Z$. Then $F$ is *continuous in each variable separately* if and only if:

- for every $a \in X$ the function $\lambda y \in Y.F(a, y)$ is continuous;
- for every $b \in Y$ the function $\lambda x \in X.F(x, b)$ is continuous.

**Proposition 1.168.** *Let $F : X \times Y \to Z$. If $F$ is continuous then $F$ is continuous in each variable separately.*

PROOF: For $a \in X$, the function $\lambda y \in Y.F(a, y)$ is $F \circ i$ where $i : Y \to X \times Y$ maps $y$ to $(a, y)$. We have $i$ is continuous by Proposition 1.145, hence $F \circ i$ is continuous by Theorem 1.130.
Similarly for $\lambda x \in X.F(x, b)$ for $b \in Y$. □

**Example 1.169.** Define $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$F(x, y) = \begin{cases} xy/(x^2 + y^2) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Then $F$ is continuous in each variable separately but not continuous.

**Proposition 1.170.** *Let $f : A \to C$ and $g : B \to D$ be open maps. Then $f \times g : A \times B \to C \times D$ is an open map.*

PROOF: Given $U$ open in $A$ and $V$ open in $B$. Then $(f \times g)(U \times V) = f(U) \times g(V)$ is open in $C \times D$. The result follows from Proposition 1.121. □

**Definition 1.171** (Sorgenfrey Plane)**.** The *Sorgenfrey plane* is $\mathbb{R}_l^2$.

## 1.32 The Subspace Topology

**Definition 1.172** (Subspace Topology)**.** Let $X$ be a topological space and $Y \subseteq X$. The *subspace topology* on $Y$ is $\mathcal{T} = \{U \cap Y \mid U \text{ is open in } X\}$.

We prove this is a topology.

PROOF:
⟨1⟩1. $Y \in \mathcal{T}$
  PROOF: Since $Y = X \cap Y$
⟨1⟩2. For all $\mathcal{U} \subseteq \mathcal{T}$, we have $\bigcup \mathcal{U} \in \mathcal{T}$
  ⟨2⟩1. LET: $\mathcal{U} \subseteq \mathcal{T}$
  ⟨2⟩2. LET: $\mathcal{V} = \{V \text{ open in } X \mid V \cap Y \in \mathcal{U}\}$
  ⟨2⟩3. $\bigcup \mathcal{U} = (\bigcup \mathcal{V}) \cap Y$
⟨1⟩3. For all $U, V \in \mathcal{T}$, we have $U \cap V \in \mathcal{T}$
  ⟨2⟩1. LET: $U, V \in \mathcal{T}$
  ⟨2⟩2. PICK $U'$, $V'$ open in $X$ such that $U = U' \cap Y$ and $V = V' \cap Y$
  ⟨2⟩3. $(U \cap V) = (U' \cap V') \cap Y$
□

**Theorem 1.173.** *Let $X$ be a topological space and $Y$ a subspace of $X$. Let $A \subseteq Y$. Then $A$ is closed in $Y$ if and only if there exists a closed set $C$ in $X$ such that $A = C \cap Y$.*

PROOF: We have

$$\begin{aligned} & A \text{ is closed in } Y \\ \Leftrightarrow & Y \setminus A \text{ is open in } Y \\ \Leftrightarrow & \exists U \text{ open in } X . Y \setminus A = Y \cap U \\ \Leftrightarrow & \exists C \text{ closed in } X . Y \setminus A = Y \cap (X \setminus U) \\ \Leftrightarrow & \exists C \text{ closed in } X . A = Y \cap U \end{aligned}$$

□

**Theorem 1.174.** *Let $Y$ be a subspace of $X$. Let $A \subseteq Y$. Let $\overline{A}$ be the closure of $A$ in $X$. Then the closure of $A$ in $Y$ is $\overline{A} \cap Y$.*

PROOF: The closure of $A$ in $Y$ is

$$\begin{aligned} & \bigcap \{C \text{ closed in } Y \mid A \subseteq C\} \\ = & \bigcap \{D \cap Y \mid D \text{ closed in } X, A \subseteq D \cap Y\} && \text{(Theorem 1.173)} \\ = & \bigcap \{D \mid D \text{ closed in } X, A \subseteq D\} \cap Y \\ = & \overline{A} \cap Y \end{aligned}$$

□

**Lemma 1.175.** *Let $X$ be a topological space and $Y \subseteq X$. Let $\mathcal{B}$ be a basis for the topology on $X$. Then $\mathcal{B}' = \{B \cap Y \mid B \in \mathcal{B}\}$ is a basis for the subspace topology on $Y$.*

PROOF:
⟨1⟩1. Every element in $\mathcal{B}'$ is open in $Y$
⟨1⟩2. For every open set $U$ in $Y$ and point $y \in U$, there exists $B' \in \mathcal{B}'$ such that $y \in B' \subseteq U$
  ⟨2⟩1. LET: $U$ be open in $Y$ and $y \in U$
  ⟨2⟩2. PICK $V$ open in $X$ such that $U = V \cap Y$
  ⟨2⟩3. PICK $B \in \mathcal{B}$ such that $y \in B \subseteq V$

⟨2⟩4. LET: $B' = B \cap Y$
⟨2⟩5. $B' \in \mathcal{B}'$
⟨2⟩6. $y \in B' \subseteq U$
⟨1⟩3. Q.E.D.
PROOF: By Lemma 1.97.
□

**Lemma 1.176.** *Let $X$ be a topological space and $Y \subseteq X$. Let $\mathcal{S}$ be a basis for the topology on $X$. Then $\mathcal{S}' = \{S \cap Y \mid S \in \mathcal{S}\}$ is a subbasis for the subspace topology on $Y$.*

PROOF: The set $\{B \cap Y \mid B$ is a finite intersection of elements of $\mathcal{S}\}$ is a basis for the subspace topology by Lemma 1.175, and this is the set of all finite intersections of elements of $\mathcal{S}'$. □

**Lemma 1.177.** *Let $Y$ be a subspace of $X$. If $U$ is open in $Y$ and $Y$ is open in $X$ then $U$ is open in $X$.*

PROOF:
⟨1⟩1. PICK $V$ open in $X$ such that $U = V \cap Y$
⟨1⟩2. $U$ is open in $X$
PROOF: Since it is the intersection of two open sets $V$ and $Y$.
□

**Theorem 1.178.** *Let $Y$ be a subspace of $X$ and $A \subseteq Y$. If $A$ is closed in $Y$ and $Y$ is closed in $X$ then $A$ is closed in $X$.*

PROOF: Pick a closed set $C$ in $X$ such that $A = C \cap Y$ (Theorem 1.173). Then $A$ is the intersection of two sets closed in $X$, hence $A$ is closed in $X$ (Lemma 1.66). □

**Theorem 1.179.** *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let $A_i$ be a subspace of $X_i$ for all $i \in I$. Then the product topology on $\prod_{i \in I} A_i$ is the same as the topology it inherits as a subspace of $\prod_{i \in I} X_i$.*

PROOF: The product topology is generated by the subbasis

$$\begin{aligned}
&\{{\pi_i}^{-1}(U) \mid i \in I, U \text{ open in } A_i\} \\
=&\{{\pi_i}^{-1}(V) \cap A_i \mid i \in I, V \text{ open in } X_i\} \\
=&\{{\pi_i}^{-1}(V) \mid i \in I, V \text{ open in } X_i\} \cap \prod_{i \in I} A_i
\end{aligned}$$

and this is a subbasis for the subspace topology by Lemma 1.176. □

**Theorem 1.180.** *Let $X$ be an ordered set in the order topology. Let $Y \subseteq X$ be an interval. Then the order topology on $Y$ is the same as the subspace topology on $Y$.*

PROOF:
⟨1⟩1. The order topology is finer than the subspace topology.

⟨2⟩1. For every open ray $R$ in $X$, the set $R \cap Y$ is open in the order topology.
⟨3⟩1. For all $a \in X$, we have $(-\infty, a) \cap Y$ is open in the order topology.
⟨4⟩1. CASE: For all $y \in Y$ we have $y < a$
PROOF: In this case $(-\infty, a) \cap Y = Y$.
⟨4⟩2. CASE: For all $y \in Y$ we have $a < y$
PROOF: In this case $(-\infty, a) \cap Y = \emptyset$.
⟨4⟩3. CASE: There exists $y \in Y$ such that $y \leq a$ and $y \in Y$ such that $a \leq y$
⟨5⟩1. $a \in Y$
PROOF: Because $Y$ is an interval.
⟨5⟩2. $(-\infty, a) \cap Y = \{y \in Y \mid y < a\}$
⟨3⟩2. For all $a \in X$, we have $(a, +\infty) \cap Y$ is open in the order topology.
PROOF: Similar.
⟨2⟩2. Q.E.D.
PROOF: By Lemmas 1.147 and 1.176 and Proposition 1.105.
⟨1⟩2. The subspace topology is finer than the order topology.
⟨2⟩1. Every open ray in $Y$ is open in the subspace topology.
PROOF: For any $a \in Y$ we have $(-\infty, a)_Y = (-\infty, a)_X \cap Y$ and $(a, +\infty)_Y = (a, +\infty)_X \cap Y$.
⟨2⟩2. Q.E.D.
PROOF: By Lemma 1.147 and Proposition 1.105
□

This example shows that we cannot remove the hypothesis that $Y$ is an interval:

**Example 1.181.** The order topology on $I_o^2$ is different from the subspace topology as a subspace of $\mathbb{R}^2$ under the dictionary order topology.

PROOF: The set $\{1/2\} \times (1/2, 1)$ is open in the subspace topology but not in the order topology. □

**Proposition 1.182.** *Let $X$ be a topological space, $Y$ a subspace of $X$, and $Z$ a subspace of $Y$. Then the subspace topology on $Z$ inherited from $X$ is the same as the subspace topology on $Z$ inherited from $Y$.*

PROOF: The subspace topology inherited from $Y$ is

$$\begin{aligned} &\{V \cap Z \mid V \text{ open in } Y\} \\ =&\{U \cap Y \cap Z \mid U \text{ open in } X\} \\ =&\{U \cap Z \mid U \text{ open in } X\} \end{aligned}$$

which is the subspace topology inherited from $X$. □

**Definition 1.183** (Unit Circle)**.** The *unit circle* $S^1$ is $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ as a subspace of $\mathbb{R}^2$.

**Definition 1.184** (Unit 2-sphere)**.** The *unit 2-sphere* is $S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$ as a subspace of $\mathbb{R}^3$.

**Proposition 1.185.** *Let $f : X \to Y$ be an open map and $A \subseteq X$ be open. Then the restriction $f \upharpoonright A : A \to f(A)$ is an open map.*

PROOF:
$\langle 1 \rangle 1$. LET: $U$ be open in $A$
$\langle 1 \rangle 2$. $U$ is open in $X$
  PROOF: Lemma 1.177.
$\langle 1 \rangle 3$. $f(U)$ is open in $Y$
$\langle 1 \rangle 4$. $f(U)$ is open in $f(A)$
  PROOF: Since $f(U) = f(U) \cap f(A)$.
$\square$

**Example 1.186.** This example shows that we cannot remove the hypothesis that $A$ is open.

Let $A = \{(x, y) \in \mathbb{R}^2 \mid (x > 0 \text{ and } y = 1/x) \text{ or } x = y = 0\}$. Then $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ is an open map, but $\pi_1 \upharpoonright A : A \to [0, +\infty)$ is not, because it maps the set $\{0, 0\}$ which is open in $A$ to $\{0\}$ which is not open in $[0, +\infty)$.

**Proposition 1.187.** *Let $Y$ be a subspace of $X$. Let $A \subseteq Y$ and $l \in Y$. Then $l$ is a limit point of $A$ in $Y$ if and only if $l$ is a limit point of $A$ in $X$.*

PROOF: Both are equivalent to the condition that any neighbourhood of $l$ in $X$ intersects $A$ in a point other than $l$. $\square$

## 1.33 The Box Topology

**Definition 1.188** (Box Topology)**.** Let $\{A_i\}_{i \in I}$ be a family of topological spaces. The *box topology* on $\prod_{i \in I} A_i$ is the topology generated by the set of all sets of the form $\prod_{i \in I} U_i$ where $\{U_i\}_{i \in I}$ is a family such that each $U_i$ is open in $A_i$.

This is a basis since it covers $\prod_{i \in I} A_i$ and is closed under intersection.

**Proposition 1.189.** *The box topology is finer than the product topology.*

PROOF: From Proposition 1.157. $\square$

**Corollary 1.189.1.** *If $A_i$ is closed in $X_i$ for all $i \in I$ then $\prod_{i \in I} A_i$ is closed in $\prod_{i \in I} X_i$ under the box topology.*

PROOF: From Proposition 1.158.

**Proposition 1.190** (AC)**.** *Let $\{A_i\}_{i \in I}$ be a family of topological spaces. For $i \in I$, let $\mathcal{B}_i$ be a basis for the topology on $A_i$. Then $\mathcal{B} = \{\prod_{i \in I} B_i \mid \forall i \in I. B_i \in \mathcal{B}_i\}$ is a basis for the box topology on $\prod_{i \in I} A_i$.*

PROOF:
$\langle 1 \rangle 1$. Every set of the form $\prod_{i \in I} B_i$ is open.
$\langle 1 \rangle 2$. For every point $a \in \prod_{i \in I} A_i$ and every open set $U$ with $a \in U$, there exists $B \in \mathcal{B}$ such that $a \in B \subseteq U$.

⟨2⟩1. LET: $U$ be open and $a \in U$

⟨2⟩2. PICK a family $\{U_i\}_{i \in I}$ such that each $U_i$ is open in $A_i$ and $a \in \prod_{i \in I} U_i \subseteq U$.

⟨2⟩3. For $i \in I$, PICK $B_i \in \mathcal{B}_i$ such that $a_i \in B_i \subseteq U_i$

PROOF: Using the Axiom of Choice.

⟨2⟩4. $a \in \prod_{i \in I} B_i \subseteq U$

⟨1⟩3. Q.E.D.

PROOF: Lemma 1.97.

□

**Theorem 1.191.** *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let $A_i$ be a subspace of $X_i$ for all $i \in I$. Give $\prod_{i \in I} X_i$ the box topology. Then the box topology on $\prod_{i \in I} A_i$ is the same as the topology it inherits as a subspace of $\prod_{i \in I} X_i$.*

PROOF: The box topology is generated by the basis

$$\begin{aligned} &\{\prod_{i \in I} U_i \mid \forall i \in I, U_i \text{ open in } A_i\} \\ =&\{\prod_{i \in I} (V_i \cap A_i) \mid \forall i \in I, V_i \text{ open in } X_i\} \\ =&\{\prod_{i \in I} V_i \mid \forall i \in I, V_i \text{ open in } X_i\} \cap \prod_{i \in I} A_i \end{aligned}$$

and this is a basis for the subspace topology by Lemma 1.175. □

**Proposition 1.192** (AC)**.** *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Give $\prod_{i \in I} X_i$ the box topology. Let $A_i \subseteq X_i$ for all $i \in I$. Then*

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i} \ .$$

PROOF:

⟨1⟩1. $\overline{\prod_{i \in I} A_i} \subseteq \prod_{i \in I} \overline{A_i}$

⟨2⟩1. For all $i \in I$ we have $A_i \subseteq \overline{A_i}$

PROOF: Lemma 1.78.

⟨2⟩2. $\prod_{i \in I} A_i \subseteq \prod_{i \in I} \overline{A_i}$

⟨2⟩3. Q.E.D.

PROOF: Since $\prod_{i \in I} A_i$ is closed by Corollary 1.189.1.

⟨1⟩2. $\prod_{i \in I} \overline{A_i} \subseteq \overline{\prod_{i \in I} A_i}$

⟨2⟩1. LET: $x \in \prod_{i \in I} \overline{A_i}$

⟨2⟩2. LET: $U$ be a neighbourhood of $x$

⟨2⟩3. PICK $V_i$ open in $X_i$ such that $x \in \prod_{i \in I} V_i \subseteq U$

⟨2⟩4. For $i \in I$, pick $a_i \in V_i \cap A_i$

PROOF: By Theorem 1.81 and ⟨2⟩1 using the Axiom of Choice.

⟨2⟩5. $U$ intersects $\prod_{i \in I} A_i$

⟨2⟩6. Q.E.D.

PROOF: $a \in U \cap \prod_{i \in I} A_i$.

□

The following example shows that Theorem 1.166 fails in the box topology.

**Example 1.193.** Define $f : \mathbb{R} \to \mathbb{R}^\omega$ by $f(t) = (t, t, \ldots)$. Then $\pi_n \circ f = \mathrm{id}_\mathbb{R}$ is continuous for all $n$. But $f$ is not continuous when $\mathbb{R}^\omega$ is given the box topology because the inverse image of

$$(-1, 1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times \cdots$$

is $\{0\}$ which is not open.

The following example shows that Proposition 1.165 fails in the box topology.

**Example 1.194.** Give $\mathbb{R}^\omega$ the box topology. Let $a_n = (1/n, 1/n, \ldots)$ for $n \geq 1$ and $l = (0, 0, \ldots)$. Then $\pi_i(a_n) \to \pi_i(l)$ as $n \to \infty$ for all $i$, but $a_n \not\to l$ as $n \to \infty$ since the open set

$$(-1, 1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times \cdots$$

contains $l$ but does not contain any $a_n$.

**Example 1.195.** The set $\mathbb{R}^\infty$ is closed in $\mathbb{R}^\omega$ under the box topology. For let $(a_n)$ be any sequence not in $\mathbb{R}^\infty$. Let $U_n$ be an open interval around $a_n$ that does not contain 0 if $a_n \neq 0$, and $U_n = \mathbb{R}$ if $a_n = 0$. Then $\prod_{n \geq 0} U_n$ is a neighbourhood of $(a_n)$ that does not intersect $\mathbb{R}^\infty$.

## 1.34 $T_1$ Spaces

**Definition 1.196** ($T_1$ Space)**.** A topological space is $T_1$ if and only if every singleton is closed.

**Lemma 1.197.** *A space is $T_1$ if and only if every finite set is closed.*

PROOF: From Lemma 1.67. □

**Theorem 1.198.** *In a $T_1$ space, a point $a$ is a limit point of a set $A$ if and only if every neighbourhood of $a$ contains infinitely many points of $A$.*

PROOF:
$\langle 1 \rangle 1$. If $a$ is a limit point of $A$ then every neighbourhood of $a$ contains infinitely many points of $A$.
  $\langle 2 \rangle 1$. ASSUME: $a$ is a limit point of $A$.
  $\langle 2 \rangle 2$. LET: $U$ be a neighbourhood of $a$.
  $\langle 2 \rangle 3$. ASSUME: for a contradiction $U$ contains only finitely many points of $A$.
  $\langle 2 \rangle 4$. $(U \cap A) \setminus \{a\}$ is closed.
    PROOF: By the $T_1$ axiom.
  $\langle 2 \rangle 5$. $(U \setminus A) \cup \{a\}$ is open.
    PROOF: It is $U \setminus ((U \cap A) \setminus \{a\})$.
  $\langle 2 \rangle 6$. $(U \setminus A) \cup \{a\}$ intersects $A$ in a point other than $a$.

PROOF: From ⟨2⟩1.

⟨2⟩7. Q.E.D.

□

⟨1⟩2. If every neighbourhood of $a$ contains infinitely many points of $A$ then $a$ is a limit point of $A$.

PROOF: Immediate from definitions.

□

(To see this does not hold in every space, see Proposition 1.93.)

**Proposition 1.199.** *A space is $T_1$ if and only if, for any two distinct points $x$ and $y$, there exist neighbourhoods $U$ of $x$ and $V$ of $y$ such that $x \notin V$ and $y \notin U$.*

PROOF:

⟨1⟩1. LET: $X$ be a topological space.

⟨1⟩2. If $X$ is $T_1$ then, for any two distinct points $x$ and $y$, there exist neighbourhoods $U$ of $x$ and $V$ of $y$ such that $x \notin V$ and $y \notin U$.

PROOF: This holds because $\{x\}$ and $\{y\}$ are closed.

⟨1⟩3. Suppose, for any two distinct points $x$ and $y$, there exist neighbourhoods $U$ of $x$ and $V$ of $y$ such that $x \notin V$ and $y \notin U$. Then $X$ is $T_1$.

⟨2⟩1. ASSUME: For any two distinct points $x$ and $y$, there exist neighbourhoods $U$ of $x$ and $V$ of $y$ such that $x \notin V$ and $y \notin U$.

⟨2⟩2. LET: $a \in X$

⟨2⟩3. $\{a\}$ is closed.

PROOF: For all $b \neq a$ there exists a neighbourhood $U$ of $b$ such that $U \subseteq X \setminus \{a\}$.

□

**Proposition 1.200.** *A subspace of a $T_1$ space is $T_1$.*

PROOF: From Proposition 1.178.

## 1.35 Hausdorff Spaces

**Definition 1.201** (Hausdorff Space)**.** A topological space is *Hausdorff* if and only if, for any points $x$, $y$ with $x \neq y$, there exist disjoint open sets $U$ and $V$ such that $x \in U$ and $y \in V$.

**Theorem 1.202.** *Every Hausdorff space is $T_1$.*

PROOF:

⟨1⟩1. LET: $X$ be a Hausdorff space.

⟨1⟩2. LET: $b \in X$

PROVE: $\overline{\{b\}} = \{b\}$

⟨1⟩3. ASSUME: $a \in \overline{\{b\}}$ and $a \neq b$

⟨1⟩4. PICK disjoint neighbourhoods $U$ of $a$ and $V$ of $b$.

⟨1⟩5. $U$ intersects $\{b\}$

PROOF: Theorem 1.81.

$\langle 1 \rangle 6.$ $b \in U$
$\langle 1 \rangle 7.$ Q.E.D.
PROOF: This contradicts the fact that $U$ and $V$ are disjoint ($\langle 1 \rangle 4$).
□

**Proposition 1.203.** *An infinite set under the finite complement topology is $T_1$ but not Hausdorff.*

PROOF:
$\langle 1 \rangle 1.$ LET: $X$ be an infinite set under the finite complement topology.
$\langle 1 \rangle 2.$ Every singleton is closed.
PROOF: By definition.
$\langle 1 \rangle 3.$ PICK$a, b \in X$ with $a \neq b$
$\langle 1 \rangle 4.$ There are no disjoint neighbourhoods $U$ of $a$ and $V$ of $b$.
$\langle 2 \rangle 1.$ LET: $U$ be a neighbourhood of $a$ and $V$ a neighbourhood of $b$.
$\langle 2 \rangle 2.$ $X \setminus U$ and $X \setminus V$ are finite.
$\langle 2 \rangle 3.$ PICK $c \in X$ that is not in $X \setminus U$ or $X \setminus V$.
$\langle 2 \rangle 4.$ $c \in U \cap V$
□

**Proposition 1.204.** *The product of a family of Hausdorff spaces is Hausdorff.*

PROOF:
$\langle 1 \rangle 1.$ LET: $\{X_i\}_{i \in I}$ be a family of Hausdorff spaces.
$\langle 1 \rangle 2.$ LET: $a, b \in \prod_{i \in I} X_i$ with $a \neq b$
$\langle 1 \rangle 3.$ PICK $i \in I$ such that $a_i \neq b_i$
$\langle 1 \rangle 4.$ PICK $U$, $V$ disjoint open sets in $X_i$ with $a_i \in U$ and $b_i \in V$
$\langle 1 \rangle 5.$ ${\pi_i}^{-1}(U)$ and ${\pi_i}^{-1}(V)$ are disjoint open sets in $\prod_{i \in I} X_i$ with $a \in {\pi_i}^{-1}(U)$ and $b \in {\pi_i}^{-1}(V)$
□

**Theorem 1.205.** *Every linearly ordered set under the order topology is Hausdorff.*

PROOF:
$\langle 1 \rangle 1.$ LET: $X$ be a linearly ordered set under the order topology.
$\langle 1 \rangle 2.$ LET: $a, b \in X$ with $a \neq b$
$\langle 1 \rangle 3.$ ASSUME: w.l.o.g. $a < b$
$\langle 1 \rangle 4.$ CASE: There exists $c$ such that $a < c < b$
PROOF: The sets $(-\infty, c)$ and $(c, +\infty)$ are disjoint neighbourhoods of $a$ and $b$ respectively.
$\langle 1 \rangle 5.$ CASE: There is no $c$ such that $a < c < b$
PROOF: The sets $(-\infty, b)$ and $(a, +\infty)$ are disjoint neighbourhoods of $a$ and $b$ respectively.
□

**Theorem 1.206.** *A subspace of a Hausdorff space is Hausdorff.*

PROOF:

⟨1⟩1. LET: $X$ be a Hausdorff space and $Y$ a subspace of $X$.
⟨1⟩2. LET: $x, y \in Y$ with $x \neq y$
⟨1⟩3. PICK disjoint neighbourhoods $U$ of $x$ and $V$ of $y$ in $X$.
⟨1⟩4. $U \cap Y$ and $V \cap Y$ are disjoint neighbourhoods of $x$ and $y$ respectively in $Y$.
□

**Proposition 1.207.** *A space $X$ is Hausdorff if and only if the diagonal $\Delta = \{(x, x) \mid x \in X\}$ is closed in $X^2$.*

PROOF:

$$\begin{aligned} & X \text{ is Hausdorff} \\ \Leftrightarrow & \forall x, y \in X . x \neq y \Rightarrow \exists V, W \text{ open}. x \in V \wedge y \in W \wedge V \cap W = \emptyset \\ \Leftrightarrow & \forall (x, y) \in X^2 \setminus \Delta . \exists V, W \text{ open}. (x, y) \subseteq V \times W \subseteq X^2 \setminus \Delta \\ \Leftrightarrow & \Delta \text{ is closed} \end{aligned}$$

□

**Theorem 1.208.** *In a Hausdorff space, a sequence has at most one limit.*

PROOF:
⟨1⟩1. LET: $X$ be a Hausdorff space.
⟨1⟩2. ASSUME: for a contradiction $a_n \to l$ as $n \to \infty$, $a_n \to m$ as $n \to \infty$, and $l \neq m$
⟨1⟩3. PICK disjoint neighbourhoods $U$ of $l$ and $V$ of $m$
PROOF: By the Hausdorff axiom.
⟨1⟩4. PICK $M$ and $N$ such that $a_n \in U$ for $n \geq M$ and $a_n \in V$ for $n \geq N$
⟨1⟩5. $a_{\max(M,N)} \in U \cap V$
⟨1⟩6. Q.E.D.
PROOF: This contradicts the fact that $U$ and $V$ are disjoint (⟨1⟩3).
□

To see this is not always true in spaces that are $T_1$ but not Hausdorff:

**Proposition 1.209.** *Let $X$ be an infinite set under the finite complement topology. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence with all points distinct. Then for every $l \in X$ we have $a_n \to l$ as $n \to \infty$.*

PROOF: Let $U$ be any neighbourhood of $l$. Since $X \setminus U$ is finite, there must exist $N$ such that, for all $n \geq N$, we have $a_n \in U$. □

**Proposition 1.210.** *Let $X$ be a topological space. Let $Y$ a Hausdorff space. Let $A \subseteq X$. Let $f, g : \overline{A} \to Y$ be continuous. If $f$ and $g$ agree on $A$ then $f = g$.*

PROOF:
⟨1⟩1. LET: $x \in \overline{A}$
⟨1⟩2. ASSUME: $f(x) \neq g(x)$
⟨1⟩3. PICK disjoint neighbourhoods $V$ of $f(x)$ and $W$ of $g(x)$.
⟨1⟩4. PICK $y \in f^{-1}(V) \cap g^{-1}(W) \cap A$
PROOF: Since $f^{-1}(V) \cap g^{-1}(W)$ is a neighbourhood of $x$ and hence intersects $A$.

⟨1⟩5. $f(y) = g(y) \in V \cap W$
⟨1⟩6. Q.E.D.
PROOF: This contradicts the fact that $V$ and $W$ are disjoint (⟨1⟩3).
□

**Proposition 1.211.** *Let $\{X_i\}_{i \in I}$ be a family of Hausdorff spaces. Then $\prod_{i \in I} X_i$ under the box topology is Hausdorff.*

PROOF:
⟨1⟩1. LET: $\{X_i\}_{i \in I}$ be a family of Hausdorff spaces.
⟨1⟩2. LET: $a, b \in \prod_{i \in I} X_i$ with $a \neq b$
⟨1⟩3. PICK $i \in I$ such that $a_i \neq b_i$
⟨1⟩4. PICK $U$, $V$ disjoint open sets in $X_i$ with $a_i \in U$ and $b_i \in V$
⟨1⟩5. ${\pi_i}^{-1}(U)$ and ${\pi_i}^{-1}(V)$ are disjoint open sets in $\prod_{i \in I} X_i$ with $a \in {\pi_i}^{-1}(U)$ and $b \in {\pi_i}^{-1}(V)$
□

**Proposition 1.212.** *Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on the same set $X$ with $\mathcal{T} \subseteq \mathcal{T}'$ If $\mathcal{T}$ is Haudorff then $\mathcal{T}'$ is Hausdorff.*

PROOF: Immediate from definitions.

**Proposition 1.213.** *Let $X$ be a Hausdorff space. Let $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the finite intersection property. Then $\bigcap_{D \in \mathcal{D}} \overline{D}$ contains at most one point.*

PROOF:
⟨1⟩1. LET: $x, y \in \bigcap_{D \in \mathcal{D}} \overline{D}$
⟨1⟩2. ASSUME: for a contradiction $x \neq y$
⟨1⟩3. PICK disjoint open subsets $U$ and $V$ of $x$ and $y$ respectively.
⟨1⟩4. $U, V \in \mathcal{D}$
PROOF: Proposition 1.84.
⟨1⟩5. Q.E.D.
PROOF: This contradicts the fact that $\mathcal{D}$ satisfies the finite intersection property.
□

## 1.36 The First Countability Axiom

**Definition 1.214** (First Countability Axiom)**.** A topological space $X$ satisfies the *first countability axiom*, or is *first countable*, if and only if every point has a countable local basis.

**Lemma 1.215** (Sequence Lemma (CC))**.** *Let $X$ be a first countable space. Let $A \subseteq X$ and $l \in \overline{A}$. Then there exists a sequence in $A$ that converges to $l$.*

PROOF:
⟨1⟩1. PICK a countable local basis $\{B_n \mid n \in \mathbb{Z}^+\}$ at $l$ such that $B_1 \supseteq B_2 \supseteq \cdots$.

PROOF: Lemma 1.107.

⟨1⟩2. For all $n \geq 1$, PICK $a_n \in A \cap B_n$.
PROVE: $a_n \to l$ as $n \to \infty$

⟨1⟩3. LET: $U$ be a neighbourhood of $A$

⟨1⟩4. PICK $N$ such that $B_N \subseteq U$

⟨1⟩5. For $n \geq N$ we have $a_n \in U$
PROOF: $a_n \in B_n \subseteq B_N \subseteq U$

□

**Theorem 1.216** (CC)**.** *Let $X$ be a first countable space and $Y$ a topological space. Let $f : X \to Y$. Suppose that, for every sequence $(x_n)$ in $X$ and $l \in X$, if $x_n \to l$ as $n \to \infty$, then $f(x_n) \to f(l)$ as $n \to \infty$. Then $f$ is continuous.*

PROOF:

⟨1⟩1. LET: $A \subseteq X$

⟨1⟩2. LET: $a \in A$
PROVE: $f(a) \in \overline{f(A)}$

⟨1⟩3. PICK a sequence $(x_n)$ in $A$ that converges to $a$.
PROOF: By the Sequence Lemma.

⟨1⟩4. $f(x_n) \to f(a)$

⟨1⟩5. $f(a) \in \overline{f(A)}$
PROOF: By Lemma 1.109.

⟨1⟩6. Q.E.D.
PROOF: By Theorem 1.127.

□

**Example 1.217** (CC)**.** The space $\mathbb{R}^\omega$ under the box product is not first countable.

PROOF: Let $\{B_n \mid n \geq 0\}$ be a countable set of neighbourhoods of $\vec{0}$. We will construct a neighbourhood of $\vec{0}$ that does not include any of these.
For $n \geq 0$, pick a neighbourhood $U_n$ of 0 such that $U_n \subset \pi_n(B_p)$. Then $\prod_{n=0}^{\infty} U_n$ is a neighbourhood of $\vec{0}$ that does not include any $B_n$. □

**Example 1.218.** If $J$ is an uncountable set then $\mathbb{R}^J$ is not first countable.

PROOF:

⟨1⟩1. LET: $\{B_n \mid n \geq 0\}$ be any countable set of neighbourhoods of $\vec{0}$.

⟨1⟩2. For $n \geq 0$, PICK a basis element $\prod_{\alpha \in J} U_{n\alpha}$ that contains $\vec{0}$ and is included in $B_n$.
PROOF: Using the Axiom of Countable Choice.

⟨1⟩3. For $n \geq 0$,
LET: $J_n = \{\alpha \in J \mid U_{n\alpha} \neq \mathbb{R}\}$

⟨1⟩4. PICK $\beta \in J$ such that $\beta \notin J_n$ for any $n$.
PROOF: Since each $J_n$ is finite so $\bigcup_n J_n$ is countable.

⟨1⟩5. $\pi_\beta((-1, 1))$ is a neighbourhood of $\vec{0}$ that does not include any $B_n$.

□

**Example 1.219.** The space $\mathbb{R}_l$ is first countable.

PROOF: For any $a \in \mathbb{R}$, the set $\{[a, a + 1/n) \mid n \geq 1\}$ is a countable local basis.

**Example 1.220.** The ordered square is first countable.

PROOF: For any $(a, b) \in I_o^2$ with $b \neq 0, 1$, the set $\{(\{a\} \times (b-1/n, b+1/n)) \cap I_o^2 \mid n \geq 1\}$ is a countable local basis.

## 1.37 Strong Continuity

**Definition 1.221** (Strongly Continuous)**.** Let $X$ and $Y$ be topological spaces and $f : X \to Y$ be a function. Then $f$ is *strongly continuous* if and only if, for every subset $U \subseteq Y$, we have $U$ is open in $Y$ if and only if $f^{-1}(U)$ is open in $X$.

**Proposition 1.222.** *Let $X$ and $Y$ be topological spaces and $f : X \to Y$ be a function. Then $f$ is* strongly continuous *if and only if, for every subset $C \subseteq Y$, we have $C$ is closed in $Y$ if and only if $f^{-1}(C)$ is closed in $X$.*

PROOF: Since $X \setminus f^{-1}(C) = f^{-1}(Y \setminus C)$. □

**Proposition 1.223.** *Let $X$, $Y$ and $Z$ be topological spaces. Let $f : X \to Y$ and $g : Y \to Z$. If $f$ and $g$ are strongly continuous then so is $g \circ f$.*

PROOF: Since $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$. □

**Proposition 1.224.** *Let $X$, $Y$ and $Z$ be topological spaces. Let $f : X \to Y$ and $g : Y \to Z$. If $g \circ f$ is continuous and $f$ is strongly continuous then $g$ is continuous.*

PROOF:
⟨1⟩1. LET: $V \subseteq Z$ be open.
⟨1⟩2. $f^{-1}(g^{-1}(V))$ is open in $X$.
  PROOF: Since $g \circ f$ is continuous.
⟨1⟩3. $f^{-1}(V)$ is open in $Y$.
  PROOF: Since $g$ is strongly continuous.
□

**Proposition 1.225.** *Let $X$, $Y$ and $Z$ be topological spaces. Let $f : X \to Y$ and $g : Y \to Z$. If $g \circ f$ is strongly continuous and $f$ is strongly continuous then $g$ is strongly continuous.*

PROOF: For $V \subseteq Z$, we have $V$ is open iff $f^{-1}(g^{-1}(V))$ is open iff $g^{-1}(V)$ is open.

## 1.38 Saturated Sets

**Definition 1.226.** Let $X$ and $Y$ be sets and $p : X \twoheadrightarrow Y$ a surjective function. Let $C \subseteq X$. Then $C$ is *saturated* with respect to $p$ if and only if, for all $x, y \in X$, if $x \in C$ and $p(x) = p(y)$ then $y \in C$.

**Proposition 1.227.** *Let $X$ and $Y$ be sets and $p : X \twoheadrightarrow Y$ a surjective function. Let $C \subseteq X$. Then the following are equivalent:*

1. *$C$ is saturated with respect to $p$.*
2. *There exists $D \subseteq Y$ such that $C = p^{-1}(D)$*
3. *$C = p^{-1}(p(C))$.*

PROOF:
$\langle 1 \rangle 1.$ $1 \Rightarrow 3$
  $\langle 2 \rangle 1.$ ASSUME: $C$ is saturated with respect to $p$.
  $\langle 2 \rangle 2.$ $C \subseteq p^{-1}(p(C))$
    PROOF: Trivial.
  $\langle 2 \rangle 3.$ $p^{-1}(p(C)) \subseteq C$
    $\langle 3 \rangle 1.$ LET: $x \in p^{-1}(p(C))$
    $\langle 3 \rangle 2.$ $p(x) \in p(C)$
    $\langle 3 \rangle 3.$ There exists $y \in C$ such that $p(x) = p(y)$
    $\langle 3 \rangle 4.$ $x \in C$
      PROOF: From $\langle 2 \rangle 1$.
$\langle 1 \rangle 2.$ $3 \Rightarrow 2$
  PROOF: Trivial.
$\langle 1 \rangle 3.$ $2 \Rightarrow 1$
  PROOF: This follows because if $p(x) \in D$ and $p(x) = p(y)$ then $p(y) \in D$.
$\square$

## 1.39 Quotient Maps

**Definition 1.228** (Quotient Map)**.** Let $X$ and $Y$ be topological spaces and $p : X \to Y$. Then $p$ is a *quotient map* if and only if $p$ is surjective and strongly continuous.

**Proposition 1.229.** *Let $X$ and $Y$ be topological spaces and $p : X \twoheadrightarrow Y$ be a surjective function. Then the following are equivalent.*

1. *$p$ is a quotient map.*
2. *$p$ is continuous and maps saturated open sets to open sets.*
3. *$p$ is continuous and maps saturated closed sets to closed sets.*

PROOF:
$\langle 1 \rangle 1.$ $1 \Rightarrow 2$
  $\langle 2 \rangle 1.$ ASSUME: $p$ is a quotient map.
  $\langle 2 \rangle 2.$ LET: $U$ be a saturated open set in $X$.
  $\langle 2 \rangle 3.$ $p^{-1}(p(U))$ is open in $X$.
    PROOF: Since $U = p^{-1}(p(U))$ be Proposition 1.227.
  $\langle 2 \rangle 4.$ $p(U)$ is open in $Y$.

PROOF: From $\langle 2\rangle 1$.

$\langle 1\rangle 2$. $1 \Rightarrow 3$

PROOF: Similar.

$\langle 1\rangle 3$. $2 \Rightarrow 1$

$\langle 2\rangle 1$. ASSUME: $p$ is continuous and maps saturated open sets to open sets.

$\langle 2\rangle 2$. LET: $U \subseteq Y$

$\langle 2\rangle 3$. ASSUME: $p^{-1}(U)$ is open in $X$

$\langle 2\rangle 4$. $p^{-1}(U)$ is saturated.

PROOF: Proposition 1.227.

$\langle 2\rangle 5$. $U$ is open in $Y$.

$\langle 1\rangle 4$. $3 \Rightarrow 1$

PROOF: Similar.

□

**Corollary 1.229.1.** *Every surjective continuous open map is a quotient map.*

**Corollary 1.229.2.** *Every surjective continuous closed map is a quotient map.*

**Example 1.230.** The converses of these corollaries do not hold.

Let $A = \{(x, y) \mid x \geq 0\} \cup \{(x, y) \mid y = 0\}$. Then $\pi_1 : A \to \mathbb{R}$ is a quotient map, but not an open map or a closed map.

We prove that $\pi_1$ maps saturated open sets to open sets:

$\langle 1\rangle 1$. LET: ${\pi_1}^{-1}(U)$ be a saturated open set in $A$

PROVE: $U$ is open in $\mathbb{R}$

$\langle 1\rangle 2$. LET: $x \in U$

$\langle 1\rangle 3$. $(x, 0) \in \pi_1(U)^{-1}$

$\langle 1\rangle 4$. PICK $W$, $V$ open in $\mathbb{R}$ such that $(x, 0) \subseteq W \times V \subseteq \pi_1(U)^{-1}$

$\langle 1\rangle 5$. $x \in W \subseteq U$

It is not an open map because it maps $((-1, 1) \times (1, 2)) \cap A$ to $[0, 1)$.

It is not a closed map because it maps $\{(x, 1/x) \mid x > 0\}$ to $(0, +\infty)$.

**Proposition 1.231.** *Let $p : X \twoheadrightarrow Y$ be a quotient map. Let $A \subseteq X$ be saturated with respect to $p$. Let $q : A \twoheadrightarrow p(A)$ be the restriction of $p$.*

1. *If $A$ is either open or closed in $X$ then $q$ is a quotient map.*

2. *If $p$ is either an open map or a closed map then $q$ is a quotient map.*

PROOF:

$\langle 1\rangle 1$. LET: $p : X \twoheadrightarrow Y$ be a quotient map.

$\langle 1\rangle 2$. LET: $A \subseteq X$ be saturated with respect to $p$.

$\langle 1\rangle 3$. LET: $q : A \twoheadrightarrow p(A)$ be the restriction of $p$.

$\langle 1\rangle 4$. $q$ is continuous.

PROOF: Theorem 1.131.

$\langle 1\rangle 5$. If $A$ is open in $X$ then $q$ is a quotient map.

$\langle 2\rangle 1$. ASSUME: $A$ is open in $X$.

$\langle 2\rangle 2$. $q$ maps saturated open sets to open sets.

$\langle 3\rangle 1$. LET: $U \subseteq A$ be saturated with respect to $q$ and open in $A$
$\langle 3\rangle 2$. $U$ is saturated with respect to $p$
$\langle 4\rangle 1$. LET: $x, y \in X$
$\langle 4\rangle 2$. ASSUME: $x \in U$
$\langle 4\rangle 3$. ASSUME: $p(x) = p(y)$
$\langle 4\rangle 4$. $x \in A$
PROOF: From $\langle 3\rangle 1$ and $\langle 4\rangle 2$.
$\langle 4\rangle 5$. $y \in A$
PROOF: From $\langle 1\rangle 2$ and $\langle 4\rangle 3$
$\langle 4\rangle 6$. $q(x) = x(y)$
PROOF: From $\langle 1\rangle 3$, $\langle 4\rangle 3$, $\langle 4\rangle 4$, $\langle 4\rangle 5$.
$\langle 4\rangle 7$. $y \in U$
PROOF: From $\langle 3\rangle 1$, $\langle 4\rangle 2$, $\langle 4\rangle 6$
$\langle 3\rangle 3$. $U$ is open in $X$
PROOF: Lemma 1.177, $\langle 2\rangle 1$, $\langle 3\rangle 1$.
$\langle 3\rangle 4$. $p(U)$ is open in $Y$
PROOF: Proposition 1.229, $\langle 1\rangle 1$, $\langle 3\rangle 2$, $\langle 3\rangle 3$
$\langle 3\rangle 5$. $q(U)$ is open in $p(A)$
PROOF: Since $q(U) = p(U) = p(U) \cap p(A)$.
$\langle 2\rangle 3$. Q.E.D.
PROOF: By Proposition 1.229.
$\langle 1\rangle 6$. If $A$ is closed in $X$ then $q$ is a quotient map.
PROOF: Similar.
$\langle 1\rangle 7$. If $p$ is an open map then $q$ is a quotient map.
$\langle 2\rangle 1$. ASSUME: $p$ is an open map
$\langle 2\rangle 2$. $q$ maps saturated open sets to open sets.
$\langle 3\rangle 1$. LET: $U$ be open in $A$ and saturated with respect to $q$
$\langle 3\rangle 2$. PICK $V$ open in $X$ such that $U = A \cap V$
$\langle 3\rangle 3$. $p(V)$ is open in $Y$
$\langle 3\rangle 4$. $q(U) = p(V) \cap p(A)$
$\langle 4\rangle 1$. $q(U) \subseteq p(V) \cap p(A)$
PROOF: From $\langle 3\rangle 2$.
$\langle 4\rangle 2$. $p(V) \cap p(A) \subseteq q(U)$
$\langle 5\rangle 1$. LET: $y \in p(V) \cap p(A)$
$\langle 5\rangle 2$. PICK $x \in V$ and $x' \in A$ such that $p(x) = p(x') = y$
$\langle 5\rangle 3$. $x \in A$
PROOF: By $\langle 1\rangle 2$.
$\langle 5\rangle 4$. $x \in U$
PROOF: From $\langle 3\rangle 2$
$\langle 2\rangle 3$. Q.E.D.
PROOF: By Proposition 1.229.
$\langle 1\rangle 8$. If $p$ is a closed map then $q$ is a quotient map.
PROOF: Similar.
□

**Example 1.232.** This example shows we cannot remove the hypotheses on $A$

and $p$.

Define $f : [0,1] \to [2,3] \to [0,2]$ by $f(x) = x$ if $x \leq 1$, $f(x) = x - 1$ if $x \geq 2$. Then $f$ is a quotient map but its restriction $f'$ to $[0,1) \cup [2,3]$ is not, because ${f'}^{-1}([1,2])$ is open but $[1,2]$ is not.

For a counterexample where $A$ is saturated, see Example 1.238.

**Proposition 1.233.** *Let $p : A \twoheadrightarrow C$ and $q : B \twoheadrightarrow D$ be open quotient maps. Then $p \times q : A \times B \to C \times D$ is an open quotient map.*

PROOF: From Corollary 1.229.1, Proposition 1.170 and Theorem 1.166. ∎

**Theorem 1.234.** *Let $p : X \twoheadrightarrow Y$ be a quotient map. Let $Z$ be a topological space and $f : Y \to Z$ be a function. Then*

1. *$f \circ p$ is continuous if and only if $f$ is continuous.*
2. *$f \circ p$ is a quotient map if and only if $f$ is a quotient map.*

PROOF:
$\langle 1 \rangle 1$. If $f \circ p$ is continuous then $f$ is continuous.
PROOF: Proposition 1.224.
$\langle 1 \rangle 2$. If $f$ is continuous then $f \circ p$ is continuous.
PROOF: Theorem 1.130.
$\langle 1 \rangle 3$. If $f \circ p$ is a quotient map then $f$ is a quotient map.
PROOF: Proposition 1.225.
$\langle 1 \rangle 4$. If $f$ is a quotient map then $f \circ p$ is a quotient map.
PROOF: From Proposition 1.223.
∎

**Proposition 1.235.** *Let $X$ and $Y$ be topological spaces. Let $p : X \to Y$ and $f : Y \to X$ be continuous maps such that $p \circ f = \mathrm{id}_Y$. Then $p$ is a quotient map.*

PROOF:
$\langle 1 \rangle 1$. LET: $V \subseteq Y$
$\langle 1 \rangle 2$. ASSUME: $p^{-1}(V)$ is open in $X$.
$\langle 1 \rangle 3$. $f^{-1}(p^{-1}(V))$ is open in $Y$.
PROOF: Because $f$ is continuous.
$\langle 1 \rangle 4$. $V$ is open in $Y$.
PROOF: Because $f^{-1}(p^{-1}(V)) = V$.
∎

## 1.40 Quotient Topology

**Definition 1.236** (Quotient Topology)**.** Let $X$ be a topological space, $Y$ a set and $p : X \twoheadrightarrow Y$ be a surjective function. Then the *quotient topology* on $Y$ is the unique topology on $Y$ with respect to which $p$ is a quotient map, namely $\mathcal{T} = \{U \in \mathcal{P}Y \mid p^{-1}(U) \text{ is open in } X\}$.

We prove this is a topology.

PROOF:
$\langle 1\rangle 1.$ $Y \in \mathcal{T}$
  PROOF: Since $p^{-1}(Y) = X$ by surjectivity.
$\langle 1\rangle 2.$ For all $\mathcal{A} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{A} \in \mathcal{T}$
  PROOF: Since $p^{-1}(\bigcup \mathcal{A}) = \bigcup_{U \in \mathcal{A}} p^{-1}(U)$
$\langle 1\rangle 3.$ For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$
  PROOF: Since $p^{-1}(U \cap V) = p^{-1}(U) \cap p^{-1}(V)$.
□

**Definition 1.237** (Quotient Space)**.** Let $X$ be a topological space and $\sim$ an equivalence relation on $X$. Let $p : X \twoheadrightarrow X/\sim$ be the canonical surjection. Then $X/\sim$ under the quotient topology is called a *quotient space*, *identification space* or *decomposition space* of $X$.

Here is a counterexample showing we cannot remove all the hypotheses of Proposition 1.231 except that $A$ is saturated.

**Example 1.238.** Let $X = (0, 1/2] \cup \{1\} \cup \{1 + 1/n : n \geq 2\}$ as a subspace of $\mathbb{R}$. Define $R$ to be the equivalence relation on $X$ where $xRy$ iff ($x = y$ or $|x - y| = 1$), so we identify $1/n$ with $1 + 1/n$ for all $n \geq 2$. Let $Y$ be the resulting quotient space $X/R$ in the quotient topology and $p : X \twoheadrightarrow Y$ the canonical surjection.

Let $A = \{1\} \cup (0, 1/2] \setminus \{1/n : n \geq 2\} \subseteq X$. Then $A$ is saturated under $p$ but the restriction $q$ of $p$ to $A$ is not a quotient map because it maps the saturated open set $\{1\}$ to $\{1\}$ which is not open in $p(A)$.

**Proposition 1.239.** *Let $X$, $Y$ and $Z$ be topological spaces. Let $f : X \to Y$ and $g : Y \to Z$. If $f$ and $g$ are quotient maps then so is $g \circ f$.*

PROOF: From Proposition 1.223. □

**Example 1.240.** The product of two quotient maps is not necessarily a quotient map.

Let $X = \mathbb{R}$ and $X^*$ the quotient space formed by identifying all positive integers. Let $p : X \twoheadrightarrow X^*$ be the canonical surjection.

We prove $p \times \mathrm{id}_{\mathbb{Q}} : X \times \mathbb{Q} \to X^* \times \mathbb{Q}$ is not a quotient map.

PROOF:
$\langle 1\rangle 1.$ For $n \geq 1$,
  LET: $c_n = \sqrt{2}/n$
$\langle 1\rangle 2.$ For $n \geq 1$,
  LET: $U_n = \{(x, y) \in X \times \mathbb{Q} \mid n - 1/4 < x < n + 1/4, (y + n > x + c_n \text{ and } y + n > -x + c_n) \text{ or } (y + n < x + c_n \text{ and } y + n < -x + c_n)\}$
$\langle 1\rangle 3.$ For $n \geq 1$, we have $U_n$ is open in $X \times \mathbb{Q}$
$\langle 1\rangle 4.$ For $n \geq 1$, we have $\{n\} \times \mathbb{Q} \subseteq U_n$
$\langle 1\rangle 5.$ LET: $U = \bigcup_{n=1}^{\infty} U_n$
$\langle 1\rangle 6.$ $U$ is open in $X \times \mathbb{Q}$
$\langle 1\rangle 7.$ $U$ is saturated with respect to $p \times \mathrm{id}_{\mathbb{Q}}$

$\langle 1\rangle 8.$ LET: $U' = (p \times \mathrm{id}_{\mathbb{Q}})(U)$
$\langle 1\rangle 9.$ ASSUME: for a contradiction $U'$ is open in $X^* \times \mathbb{Q}$
$\langle 1\rangle 10.$ $(1, 0) \in U'$
$\langle 1\rangle 11.$ PICK a neighbourhood $W$ of 1 in $X^*$ and $\delta > 0$ such that $W \times (-\delta, \delta) \subseteq U'$
$\langle 1\rangle 12.$ $p^{-1}(W) \times (-\delta, \delta) \subseteq U$
$\langle 1\rangle 13.$ PICK $n$ such that $c_n < \delta$
$\langle 1\rangle 14.$ $n \in p^{-1}(W)$
$\langle 1\rangle 15.$ PICK $\epsilon > 0$ such that $\epsilon < \delta - c_n$ and $\epsilon < 1/4$ and $(n - \epsilon, n + \epsilon) \subseteq p^{-1}(W)$
$\langle 1\rangle 16.$ $(n - \epsilon, n + \epsilon) \times (-\delta, \delta) \subseteq U$
$\langle 1\rangle 17.$ PICK a rational $y$ such that $c_n - \epsilon/2 < y < c_n + \epsilon/2$
$\langle 1\rangle 18.$ $(n + \epsilon/2, y) \notin U$
$\langle 1\rangle 19.$ Q.E.D.
PROOF: This contradicts $\langle 1\rangle 16$.
□

**Proposition 1.241.** *Let $X$ be a topological space and $\sim$ an equivalence relation on $X$. Then $X/\sim$ is $T_1$ if and only if every equivalence class is closed in $X$.*

PROOF: Immediate from definitions. □

## 1.41 Retractions

**Definition 1.242** (Retraction)**.** Let $X$ be a topological space and $A \subseteq X$. A *retraction* of $X$ onto $A$ is a continuous map $r : X \to A$ such that, for all $a \in A$, we have $r(a) = a$.

**Proposition 1.243.** *Every retraction is a quotient map.*

PROOF: Proposition 1.235 with $f$ the inclusion $A \hookrightarrow X$. □

## 1.42 Homogeneous Spaces

**Definition 1.244** (Homogeneous)**.** A topological space $X$ is *homogeneous* if and only if, for any points $a, b \in X$, there exists a homeomorphism $\phi : X \cong X$ such that $\phi(a) = b$.

## 1.43 Regular Spaces

**Definition 1.245** (Regular Space)**.** A topological space $X$ is *regular* if and only if, for any closed set $A$ and point $a \notin A$, there exist disjoint open sets $U$, $V$ such that $A \subseteq U$ and $a \in V$.

## 1.44 Connected Spaces

**Definition 1.246** (Separation)**.** A *separation* of a topological space $X$ is a pair of disjoint open sets $U$, $V$ such that $U \cup V = \emptyset$.

**Definition 1.247** (Connected)**.** A topological space is *connected* if and only if it has no separation; otherwise it is *disconnected*.

**Proposition 1.248.** *A topological space $X$ is connected if and only if the only sets that are both open and closed are $X$ and $\emptyset$.*

Immediate from defintions.

**Lemma 1.249.** *If $Y$ is a subspace of $X$, a separation of $Y$ is a pair of disjoint nonempty sets $A$ and $B$ whose union is $Y$, neither of which contains a limit point of the other.*

PROOF:
⟨1⟩1. LET: $A, B \subseteq Y$
⟨1⟩2. If $A$ and $B$ form a separation of $Y$ then $A$ and $B$ are disjoint and nonempty, $A \cup B = Y$, and neither of $A$ and $B$ contains a limit point of the other.
  ⟨2⟩1. ASSUME: $A$ and $B$ form a separation of $Y$
  ⟨2⟩2. $A$ and $B$ are disjoint and nonempty and $A \cup B = Y$
    PROOF: From ⟨2⟩1 and the definition of separation.
  ⟨2⟩3. $A$ does not contain a limit point of $B$
    ⟨3⟩1. ASSUME: for a contradiction $l \in A$ and $l$ is a limit point of $B$ in $X$.
    ⟨3⟩2. $l$ is a limit point of $B$ in $Y$
      PROOF: Proposition 1.187.
    ⟨3⟩3. $l \in B$
      ⟨4⟩1. $B$ is closed in $Y$
        PROOF: Since $A$ is open in $Y$ and $B = Y \setminus A$ from ⟨2⟩1.
      ⟨4⟩2. Q.E.D.
        PROOF: Corollary 1.92.1.
    ⟨3⟩4. Q.E.D.
      PROOF: This contradicts the fact that $A \cap B = \emptyset$ (⟨2⟩1).
  ⟨2⟩4. $B$ does not contain a limit point of $A$
    PROOF: Similar.
⟨1⟩3. If $A$ and $B$ are disjoint and nonempty, $A \cup B = Y$, and neither of $A$ and $B$ contains a limit point of the other, then $A$ and $B$ form a separation of $Y$.
  ⟨2⟩1. ASSUME: $A$ and $B$ are disjoint and nonempty, $A \cup B = Y$, and neither of $A$ and $B$ contains a limit point of the other.
  ⟨2⟩2. $A$ is open in $Y$
    ⟨3⟩1. $B$ is closed in $Y$
      ⟨4⟩1. LET: $l$ be a limit point of $B$ in $Y$
      ⟨4⟩2. $l$ is a limit point of $B$ in $X$
        PROOF: Proposition 1.187.
      ⟨4⟩3. $l \notin A$
        PROOF: By ⟨2⟩1
      ⟨4⟩4. $l \in B$
        PROOF: By ⟨2⟩1 since $A \cup B = Y$
      ⟨4⟩5. Q.E.D.

PROOF: Corollary 1.92.1.
⟨3⟩2. Q.E.D.
PROOF: Since $A = Y \setminus B$.
⟨2⟩3. $B$ is open in $Y$
PROOF: Similar.
□

**Example 1.250.** Every set under the indiscrete topology is connected.

**Example 1.251.** The discrete topology on a set $X$ is connected if and only if $|X| \leq 1$.

**Example 1.252.** The finite complement topology on a set $X$ is connected if and only if either $|X| \leq 1$ or $X$ is infinite.

**Example 1.253.** The countable complement topology on a set $X$ is connected if and only if either $|X| \leq 1$ or $X$ is uncountable.

**Example 1.254.** The rationals $\mathbb{Q}$ are disconnected. For any irrational $a$, the sets $(-\infty, a) \cap \mathbb{Q}$ and $(a, +\infty) \cap \mathbb{Q}$ form a separation of $\mathbb{Q}$.

**Lemma 1.255.** *Let $X$ be a topological space. If $C$ and $D$ form a separation of $X$, and $Y$ is a connected subspace of $X$, then either $Y \subseteq C$ or $Y \subseteq D$.*

PROOF: Otherwise $Y \cap C$ and $Y \cap D$ would form a separation of $Y$. □

**Theorem 1.256.** *The union of a set of connected subspaces of a space $X$ that have a point in common is connected.*

PROOF:
⟨1⟩1. LET: $\mathcal{A}$ be a set of connected subspaces of the space $X$ that have the point $a$ in common.
⟨1⟩2. ASSUME: for a contradiction $C$ and $D$ form a separation of $\bigcup \mathcal{A}$
⟨1⟩3. ASSUME: without loss of generality $a \in C$
⟨1⟩4. For all $A \in \mathcal{A}$ we have $A \subseteq C$
PROOF: Lemma 1.255.
⟨1⟩5. $D = \emptyset$
⟨1⟩6. Q.E.D.
PROOF: This contradicts ⟨1⟩2.
□

**Theorem 1.257.** *Let $X$ be a topological space and $A$ a connected subspace of $X$. If $A \subseteq B \subseteq \overline{A}$ then $B$ is connected.*

PROOF:
⟨1⟩1. ASSUME: for a contradiction $C$ and $D$ form a separation of $B$.
⟨1⟩2. ASSUME: without loss of generality $A \subseteq C$
PROOF: Lemma 1.255.
⟨1⟩3. $B \subseteq C$
⟨2⟩1. LET: $x \in B$

⟨2⟩2. $x \in \overline{A}$
⟨2⟩3. Either $x \in A$ or $x$ is a limit point of $A$.
PROOF: Theorem 1.92.
⟨2⟩4. Either $x \in A$ or $x$ is a limit point of $C$.
PROOF: Lemma 1.94, ⟨1⟩2.
⟨2⟩5. $x \in C$
PROOF: Lemma 1.249.
⟨1⟩4. $D = \emptyset$
⟨1⟩5. Q.E.D.
PROOF: This contradicts ⟨1⟩1.
□

**Theorem 1.258.** *The image of a connected space under a continuous map is connected.*

PROOF:
⟨1⟩1. LET: $f : X \twoheadrightarrow Y$ be a surjective continuous map where $X$ is connected.
⟨1⟩2. ASSUME: for a contradiction $C$ and $D$ form a separation of $Y$.
⟨1⟩3. $f^{-1}(C)$ and $f^{-1}(D)$ form a separation of $X$.
□

**Theorem 1.259.** *The product of a family of connected spaces is connected.*

PROOF:
⟨1⟩1. The product of two connected spaces is connected.
⟨2⟩1. LET: $X$ and $Y$ be connected spaces.
⟨2⟩2. PICK $a \in X$ and $b \in Y$
PROOF: We may assume $X$ and $Y$ are nonempty since otherwise $X \times Y = \emptyset$ which is connected.
⟨2⟩3. $X \times \{b\}$ is connected.
PROOF: It is homeomorphic to $X$.
⟨2⟩4. For all $x \in X$ we have $\{x\} \times Y$ is connected.
PROOF: It is homeomorphic to $Y$.
⟨2⟩5. For any $x \in X$
LET: $T_x = (X \times \{b\}) \cup (\{x\} \times Y)$
⟨2⟩6. For all $x \in X$, $T_x$ is connected.
PROOF: Theorem 1.256 since $(x, b) \in (X \times \{b\}) \cap (\{x\} \times Y)$.
⟨2⟩7. $X \times Y$ is connected.
PROOF: Theorem 1.256 since $X \times Y = \bigcup_{x \in X} T_x$ and $(a, b)$ is a point in every $T_x$.
⟨1⟩2. The product of a finite family of connected spaces is connected.
PROOF: From ⟨1⟩1 by induction.
⟨1⟩3. The product of any family of connected spaces is connected.
⟨2⟩1. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of connected spaces.
⟨2⟩2. LET: $X = \prod_{\alpha \in J} X_\alpha$
⟨2⟩3. PICK $a \in X$
PROOF: We may assume $X \neq \emptyset$ as the empty space is connected.

⟨2⟩4. For every finite subset $K$ of $J$,
LET: $X_K = \{x \in X \mid \forall \alpha \in J \setminus K. x_\alpha = a_\alpha\}$

⟨2⟩5. For every finite $K \subseteq J$, we have $X_K$ is connected.
PROOF: From ⟨1⟩2 since $X_K \cong \prod_{\alpha \in K} X_K$.

⟨2⟩6. LET: $Y = \bigcup_K X_K$

⟨2⟩7. $Y$ is connected
PROOF: Theorem 1.256 since $a$ is a common point.

⟨2⟩8. $X = \overline{Y}$

⟨3⟩1. LET: $x \in X$

⟨3⟩2. LET: $U = \prod_{\alpha \in J} U_\alpha$ be a basic neighbourhood of $x$ where $U_\alpha = X_\alpha$ for all $\alpha$ except $\alpha \in K$ for some finite $K \subseteq J$

⟨3⟩3. LET: $y \in X$ be the point with $y_\alpha = x_\alpha$ for $\alpha \in K$ and $y_\alpha = a_\alpha$ for all other $\alpha$

⟨3⟩4. $y \in U \cap X_K$

⟨3⟩5. $y \in U \cap Y$

⟨2⟩9. $X$ is connected.
PROOF: Theorem 1.257.

□

**Example 1.260.** The set $\mathbb{R}^\omega$ is disconnected under the box topology. The set of bounded sequences and the set of unbounded sequences form a separation.

**Proposition 1.261.** *Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on the same set $X$. If $\mathcal{T} \subseteq \mathcal{T}'$ and $(X, \mathcal{T}')$ is connected then $(X, \mathcal{T})$ is connected.*

PROOF: If $U$ and $V$ form a separation of $(X, \mathcal{T})$ then they form a separation of $(X, \mathcal{T}')$. □

**Proposition 1.262.** *Let $X$ be a topological space and $(A_n)$ a sequence of connected subspaces of $X$. If $A_n \cap A_{n+1} \neq \emptyset$ for all $n$ then $\bigcup_n A_n$ is connected.*

PROOF:

⟨1⟩1. ASSUME: for a contradiction $C$ and $D$ form a separation of $\bigcup_n A_n$

⟨1⟩2. ASSUME: without loss of generality $A_0 \subseteq C$
PROOF: Lemma 1.255.

⟨1⟩3. For all $n$ we gave $A_n \subseteq C$
PROOF:

⟨2⟩1. ASSUME: $A_n \subseteq C$

⟨2⟩2. PICK $x \in A_n \cap A_{n+1}$

⟨2⟩3. $x \in C$

⟨2⟩4. $A_{n+1} \subseteq C$
PROOF: Lemma 1.255.

⟨2⟩5. Q.E.D.
PROOF: The result follows by induction.

⟨1⟩4. $D = \emptyset$

⟨1⟩5. Q.E.D.
PROOF: This contradicts ⟨1⟩1.

□

**Proposition 1.263.** *Let $X$ be a topological space. Let $A, C \subseteq X$. If $C$ is connected and intersects both $A$ and $X \setminus A$ then $C$ intersects $\partial A$.*

PROOF: Otherwise $C \cap A^\circ$ and $C \setminus \overline{A}$ would form a separation of $C$. ∎

**Example 1.264.** The space $\mathbb{R}_l$ is disconnected. For any real $x$, the sets $(-\infty, x)$ and $[x, +\infty)$ form a separation.

**Proposition 1.265.** *Let $X$ and $Y$ be connected spaces. Let $A$ be a proper subset of $X$ and $B$ a proper subset of $Y$. Then $(X \times Y) \setminus (A \times B)$ is connected.*

PROOF:
$\langle 1 \rangle 1$. PICK $a \in X \setminus A$ and $b \in Y \setminus B$
$\langle 1 \rangle 2$. For $x \in X \setminus A$ we have $(X \times \{b\}) \cup (\{x\} \times Y)$ is connected.
PROOF: Theorem 1.256 since $(x, b)$ is a common point.
$\langle 1 \rangle 3$. For $y \in Y \setminus B$ we have $(X \times \{y\}) \cup (\{a\} \times Y)$ is connected.
PROOF: Theorem 1.256 since $(a, y)$ is a common point.
$\langle 1 \rangle 4$. $(X \times Y) \setminus (A \times B)$ is connected.
PROOF: Theorem 1.256 since it is the union of the sets in $\langle 1 \rangle 2$ and $\langle 1 \rangle 3$ with $(a, b)$ as a common point.
∎

**Proposition 1.266.** *Let $p : X \twoheadrightarrow Y$ be a quotient map. If $Y$ is connected and $p^{-1}(y)$ is connected for all $y \in Y$, then $X$ is connected.*

PROOF:
$\langle 1 \rangle 1$. ASSUME: for a contradiction $C$ and $D$ form a separation of $X$.
$\langle 1 \rangle 2$. $C$ is saturated.
$\langle 2 \rangle 1$. LET: $x \in C$, $y \in X$ with $p(x) = p(y) = a$, say
$\langle 2 \rangle 2$. $y \notin D$
PROOF: Otherwise $p^{-1}(a) \cap C$ and $p^{-1}(a) \cap D$ form a separation of $p^{-1}(a)$.
$\langle 2 \rangle 3$. $y \in C$
$\langle 1 \rangle 3$. $D$ is saturated.
PROOF: Similar.
$\langle 1 \rangle 4$. $p(C)$ and $p(D)$ form a separation of $Y$.
∎

**Proposition 1.267.** *Let $X$ be a connected space and $Y$ a connected subspace of $X$. Suppose $A$ and $B$ form a separation of $X \setminus Y$. Then $Y \cup A$ and $Y \cup B$ are both connected.*

PROOF:
$\langle 1 \rangle 1$. $Y \cup A$ is connected.
$\langle 2 \rangle 1$. ASSUME: for a contradiction $C$ and $D$ form a separation of $Y \cup A$
$\langle 2 \rangle 2$. ASSUME: without loss of generality $Y \subseteq C$
$\langle 2 \rangle 3$. PICK open sets $A_1$, $B_1$, $C_1$, $D_1$ in $X$ with

$$A = A_1 \setminus Y$$
$$B = B_1 \setminus Y$$
$$C = C_1 \cap (Y \cup A)$$
$$D = D_1 \cap (Y \cup A)$$

$\langle 2\rangle 4$. $B_1 \cup C_1$ and $A_1 \cap D_1$ form a separation of $X$
$\langle 1\rangle 2$. $Y \cup B$ is connected.
PROOF: Similar.
□

**Theorem 1.268.** *Let $L$ be a linearly ordered set under the order topology. Then $L$ is connected if and only if $L$ is a linear continuum.*

PROOF:
$\langle 1\rangle 1$. If $L$ is a linear continuum then $L$ is connected.
$\langle 2\rangle 1$. LET: $L$ be a linear continuum under the order topology.
$\langle 2\rangle 2$. ASSUME: for a contradiction $C$ and $D$ form a separation of $L$.
$\langle 2\rangle 3$. PICK $a \in C$ and $b \in D$.
$\langle 2\rangle 4$. ASSUME: without loss of generality $a < b$.
$\langle 2\rangle 5$. LET: $S = \{x \in L \mid a < x \text{ and } [a, x) \subseteq C\}$
$\langle 2\rangle 6$. $S$ is nonempty.
PROOF: Since $a \in C$ and $C$ is open.
$\langle 2\rangle 7$. $S$ is bounded above by $b$.
PROOF: Since $b \notin C$.
$\langle 2\rangle 8$. LET: $s = \sup S$
$\langle 2\rangle 9$. $s \in S$
$\langle 3\rangle 1$. LET: $y \in [a, s)$
PROVE: $y \in C$
$\langle 3\rangle 2$. PICK $z$ with $y < z \in S$
PROOF: By minimality of $s$.
$\langle 3\rangle 3$. $y \in [a, z) \subseteq C$
$\langle 2\rangle 10$. CASE: $s \in C$
$\langle 3\rangle 1$. PICK $x$ such that $s < x$ and $[s, x) \subseteq C$
PROOF: Since $C$ is open and $s$ is not greatest in $L$ because $s < b$.
$\langle 3\rangle 2$. $x \in S$
PROOF: Since $[a, x) = [a, s) \cup [s, x) \subseteq C$.
$\langle 3\rangle 3$. Q.E.D.
PROOF: This contradicts the fact that $s$ is an upper bound for $S$.
$\langle 2\rangle 11$. CASE: $s \in D$
$\langle 3\rangle 1$. PICK $x < s$ such that $(x, s] \subseteq D$
$\langle 3\rangle 2$. PICK $y$ with $x < y < s$
PROOF: Since $L$ is dense.
$\langle 3\rangle 3$. $y \in C$
PROOF: From $\langle 2\rangle 9$.
$\langle 3\rangle 4$. $y \in D$
PROOF: From $\langle 3\rangle 1$.
$\langle 3\rangle 5$. Q.E.D.
$\langle 3\rangle 6$. LET: $L$ be a linear continuum under the order topology.
$\langle 3\rangle 7$. ASSUME: for a contradiction $C$ and $D$ form a separation of $L$.
$\langle 3\rangle 8$. PICK $a \in C$ and $b \in D$.
$\langle 3\rangle 9$. ASSUME: without loss of generality $a < b$.
$\langle 3\rangle 10$. LET: $S = \{x \in L \mid a < x \text{ and } [a, x) \subseteq C\}$

$\langle 3\rangle 11$. $S$ is nonempty.
PROOF: Since $a \in C$ and $C$ is open.
$\langle 3\rangle 12$. $S$ is bounded above by $b$.
PROOF: Since $b \notin C$.
$\langle 3\rangle 13$. LET: $s = \sup S$
$\langle 3\rangle 14$. $s \in S$
$\langle 4\rangle 1$. LET: $y \in [a, s)$
PROVE: $y \in C$
$\langle 4\rangle 2$. PICK $z$ with $y < z \in S$
PROOF: By minimality of $s$.
$\langle 4\rangle 3$. $y \in [a, z) \subseteq C$
$\langle 3\rangle 15$. CASE: $s \in C$
$\langle 4\rangle 1$. PICK $x$ such that $s < x$ and $[s, x) \subseteq C$
PROOF: Since $C$ is open and $s$ is not greatest in $L$ because $s < b$.
$\langle 4\rangle 2$. $x \in S$
PROOF: Since $[a, x) = [a, s) \cup [s, x) \subseteq C$.
$\langle 4\rangle 3$. Q.E.D.
PROOF: This contradicts the fact that $s$ is an upper bound for $S$.
$\langle 3\rangle 16$. CASE: $s \in D$
$\langle 4\rangle 1$. PICK $x < s$ such that $(x, s] \subseteq D$
$\langle 4\rangle 2$. PICK $y$ with $x < y < s$
PROOF: Since $L$ is dense.
$\langle 4\rangle 3$. $y \in C$
PROOF: From $\langle 2\rangle 9$.
$\langle 4\rangle 4$. $y \in D$
PROOF: From $\langle 3\rangle 1$.
$\langle 4\rangle 5$. Q.E.D.
PROOF: This contradicts $\langle 2\rangle 2$.
$\langle 1\rangle 2$. If $L$ is connected then $L$ is a linear continuum.
$\langle 2\rangle 1$. ASSUME: $L$ is connected.
$\langle 2\rangle 2$. Every nonempty subset of $L$ that is bounded above has a supremum.
$\langle 3\rangle 1$. LET: $X$ be a nonempty subset of $L$ bounded above by $b$.
$\langle 3\rangle 2$. ASSUME: for a contradiction $X$ has no supremum.
$\langle 3\rangle 3$. LET: $U$ be the set of upper bounds of $X$,
$\langle 3\rangle 4$. $U$ is nonempty.
PROOF: Since $b \in U$.
$\langle 3\rangle 5$. $U$ is open.
$\langle 4\rangle 1$. LET: $x \in U$
$\langle 4\rangle 2$. PICK an upper bound $y$ for $X$ such that $y < x$
$\langle 4\rangle 3$. Either $x$ is greatest in $L$ and $(y, x] \subseteq U$, or there exists $z > x$ such that $(y, z) \subseteq U$
$\langle 3\rangle 6$. LET: $V$ be the set of lower bounds of $U$.
$\langle 3\rangle 7$. $V$ is nonempty.
PROOF: Since $X \subseteq V$
$\langle 3\rangle 8$. $V$ is open.
$\langle 4\rangle 1$. LET: $x \in V$

⟨4⟩2. PICK $y \in X$ with $x < y$
PROOF: $x$ cannot be an upper bound for $X$, because it would be the supremum of $X$.
⟨4⟩3. Either $x$ least in $L$ and $[x, y) \subseteq V$, or there exists $z < x$ such that $(z, y) \subseteq V$
⟨3⟩9. $L = U \cup V$
⟨4⟩1. LET: $x \in L \setminus U$
⟨4⟩2. PICK $y \in X$ such that $x < y$
⟨4⟩3. For all $u \in U$ we have $x < y \leq u$
⟨4⟩4. $x \in V$
⟨3⟩10. $U \cap V = \emptyset$
PROOF: Any element of $U \cap V$ would be a supremum of $X$.
⟨3⟩11. $U$ and $V$ form a separation of $L$.
⟨3⟩12. Q.E.D.
PROOF: This contradicts ⟨2⟩1.
⟨2⟩3. $L$ is dense.
⟨3⟩1. LET: $x, y \in L$ with $x < y$
⟨3⟩2. There exists $z \in L$ such that $x < z < y$
PROOF: Otherwise $(-\infty, y)$ and $(x, +\infty)$ would form a separation of $L$.
□

**Corollary 1.268.1.** *The real line $\mathbb{R}$ is connected.*

**Corollary 1.268.2.** *Every interval in $\mathbb{R}$ is connected.*

**Corollary 1.268.3.** *The ordered square is connected.*

**Theorem 1.269** (Intermediate Value Theorem)**.** *Let $X$ be a connected space. Let $Y$ be a linearly ordered set under the order topology. Let $f : X \to Y$ be continuous. Let $a, b \in X$ and $r \in Y$. Suppose $f(a) < r < f(b)$. Then there exists $c \in X$ such that $f(c) = r$.*

PROOF: Otherwise $f^{-1}((-\infty, r))$ and $f^{-1}((r, +\infty))$ would form a separation of $X$. □

**Proposition 1.270.** *Every function $f : [0, 1] \to [0, 1]$ has a fixed point.*

PROOF:
⟨1⟩1. LET: $g : [0, 1] \to [-1, 1]$ be the function $g(x) = f(x) - x$
PROVE: there exists $x \in [0, 1]$ such that $g(x) = 0$
⟨1⟩2. ASSUME: without loss of generality $g(0) \neq 0$ and $g(1) \neq 0$
⟨1⟩3. $g(0) > 0$
⟨1⟩4. $g(1) < 0$
⟨1⟩5. There exists $x \in (0, 1)$ such that $g(x) = 0$
PROOF: By the Intermediate Value Theorem.

**Proposition 1.271.** *Give $\mathbb{R}^\omega$ the box topology. Let $x, y \in \mathbb{R}^\omega$. Then $x$ and $y$ lie in the same comoponent if and only if $x - y$ is eventually zero, i.e. there exists $N$ such that, for all $n \geq N$, we have $x_n = y_n$.*

PROOF:
$\langle 1\rangle 1$. The component containing 0 is the set of sequences that are eventually zero.
$\langle 2\rangle 1$. LET: $B$ be the set of sequences that are eventually zero.
$\langle 2\rangle 2$. $B$ is path-connected.
$\langle 3\rangle 1$. LET: $x, y \in B$
$\langle 3\rangle 2$. PICK $N$ such that, for all $n \geq N$, we have $x_n = y_n = 0$
$\langle 3\rangle 3$. LET: $p : [0, 1] \to \mathbb{R}^\omega$, $p(t) = (1 - t)x + ty$
PROVE: $p$ is continuous.
$\langle 3\rangle 4$. LET: $t \in [0, 1]$ and $\prod_j U_j$ be a basic open neighbourhood of $p(t)$, where each $U_j$ is open in $\mathbb{R}$
$\langle 3\rangle 5$. PICK $\delta$ such that, for all $n < N$ and all $s \in [0, 1]$, if $|s - t| < \delta$ then $p(s)_n \in U_n$
$\langle 3\rangle 6$. For all $s \in [0, 1]$, if $|s - t| < \delta$ then $p(s) \in \prod_j U_j$
$\langle 2\rangle 3$. $B$ is connected.
PROOF: Proposition 1.277.
$\langle 2\rangle 4$. If $C$ is connected and $B \subseteq C$ then $B = C$.
$\langle 3\rangle 1$. ASSUME: $C$ is connected and $B \subseteq C$
$\langle 3\rangle 2$. ASSUME: for a contradiction $x \in C \setminus B$
$\langle 3\rangle 3$. For $n \geq 1$,
LET: $c_n = 1$ if $x_n = 0$, $c_n = n/x_n$ otherwise
$\langle 3\rangle 4$. LET: $h : \mathbb{R}^\omega \to \mathbb{R}^\omega$ be the function $h(x) = (c_n x_n)_{n \geq 1}$
$\langle 3\rangle 5$. $h$ is a homeomorphism of $\mathbb{R}^\omega$ with itself.
$\langle 3\rangle 6$. $h(x)$ is unbounded.
PROOF: For any $b > 0$, pick $N > b$ such that $x_N \neq 0$. Then $h(x)_N > b$.
$\langle 3\rangle 7$. $h^{-1}(\{\text{bounded sequences}\}) \cap C$ and $h^{-1}(\{\text{unbounded sequences}\}) \cap C$ form a separation of $C$
$\langle 3\rangle 8$. Q.E.D.
PROOF: This contradicts $\langle 3\rangle 1$.
$\langle 1\rangle 2$. Q.E.D.
PROOF: Since $\lambda x.x - y$ is a homeomorphism of $\mathbb{R}^\omega$ with itself.
□

## 1.45 Totally Disconnected Spaces

**Definition 1.272** (Totally Disconnected)**.** A topological space $X$ is *totally disconnected* if and only if the only connected subspaces are the singletons.

**Example 1.273.** Every discrete space is totally disconnected.

**Example 1.274.** The rationals $\mathbb{Q}$ are totally disconnected.

## 1.46 Paths and Path Connectedness

**Definition 1.275** (Path)**.** Let $X$ be a topological space and $a, b \in X$. A *path* from $a$ to $b$ is a continuous function $p : [0, 1] \to X$ such that $p(0) = a$ and

$p(1) = b$.

**Definition 1.276** (Path Connected)**.** A topological space is *path connected* if and only if there exists a path between any two points.

**Proposition 1.277.** *Every path connected space is connected.*

PROOF:
$\langle 1 \rangle 1$. LET: $X$ be a path connected space.
$\langle 1 \rangle 2$. ASSUME: for a contradiction $C$ and $D$ form a separation of $X$.
$\langle 1 \rangle 3$. PICK $a \in C$ and $b \in D$.
$\langle 1 \rangle 4$. PICK a path $p : [0, 1] \to X$ from $a$ to $b$.
$\langle 1 \rangle 5$. $p^{-1}(C)$ and $p^{-1}(D)$ form a separation of $[0, 1]$.
$\langle 1 \rangle 6$. Q.E.D.
PROOF: This contradicts Corollary 1.268.2.
□

An example that shows the converse does not hold:

**Example 1.278.** The ordered square is not path connected.

PROOF:
$\langle 1 \rangle 1$. ASSUME: for a contradiction $p : [0, 1] \to I_o^2$ is a path from $(0, 0)$ to $(1, 1)$.
$\langle 1 \rangle 2$. $p$ is surjective.
PROOF: By the Intermediate Value Theorem.
$\langle 1 \rangle 3$. For $x \in [0, 1]$, PICK a rational $q_x \in p^{-1}((x, 0), (x, 1))$
PROOF: Since $p^{-1}((x, 0), (x, 1))$ is open and nonempty by $\langle 1 \rangle 2$.
$\langle 1 \rangle 4$. For $x, y \in [0, 1]$, if $x \neq y$ then $q_x \neq q_y$
PROOF: We have $p(q_x) \neq p(q_y)$ because $((x, 0), (x, 1))$ and $((y, 0), (y, 1))$ are disjoint.
$\langle 1 \rangle 5$. $\{q_x \mid x \in [0, 1]\}$ is an uncountable set of rationals.
$\langle 1 \rangle 6$. Q.E.D.
PROOF: This contradicts the fact that the rationals are countable.
□

**Proposition 1.279.** *The continuous image of a path connected space is path connected.*

PROOF:
$\langle 1 \rangle 1$. LET: $X$ be a path connected space, $Y$ a topological space, and $f : X \twoheadrightarrow Y$ be continuous and surjective.
$\langle 1 \rangle 2$. LET: $a, b \in Y$
$\langle 1 \rangle 3$. PICK $c, d \in X$ with $f(c) = a$ and $f(d) = b$
$\langle 1 \rangle 4$. PICK a path $p : [0, 1] \to X$ from $c$ to $d$.
$\langle 1 \rangle 5$. $f \circ p$ is a path from $a$ to $b$ in $Y$.
□

**Proposition 1.280** (AC)**.** *The product of a family of path-connected spaces is path-connected.*

PROOF:
⟨1⟩1. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of path-connected spaces.
⟨1⟩2. LET: $a, b \in \prod_{\alpha \in J} X_\alpha$
⟨1⟩3. For $\alpha \in J$, PICK a path $p_\alpha : [0,1] \to X_\alpha$ from $a_\alpha$ to $b_\alpha$
PROOF: Using the Axiom of Choice.
⟨1⟩4. Define $p : [0.1] \to \prod_{\alpha \in J} X_\alpha$ by $p(t)_\alpha = p_\alpha(t)$
⟨1⟩5. $p$ is a path from $a$ to $b$.
PROOF: Theorem 1.166.
□

**Proposition 1.281.** *The continuous image of a path-connected space is path-connected.*

PROOF:
⟨1⟩1. LET: $f : X \twoheadrightarrow Y$ be continuous and surjective where $X$ is path-connected.
⟨1⟩2. LET: $a, b \in Y$
⟨1⟩3. PICK $a', b' \in X$ with $f(a') = a$ and $f(b') = b$.
⟨1⟩4. PICK a path $p : [0,1] \to X$ from $a'$ to $b'$.
⟨1⟩5. $f \circ p$ is a path from $a$ to $b$.
□

**Proposition 1.282.** *Let $X$ be a topological space. The union of a set of path-connected subspaces of $X$ that have a point in common is path-connected.*

PROOF:
⟨1⟩1. LET: $\mathcal{A}$ be a set of path-connected subspaces of $X$ with the point $a$ in common.
⟨1⟩2. LET: $b, c \in \bigcup \mathcal{A}$
⟨1⟩3. PICK $B, C \in \mathcal{A}$ with $b \in B$ and $c \in C$.
⟨1⟩4. PICK a path $p$ in $B$ from $b$ to $a$.
⟨1⟩5. PICK a path $q$ in $C$ from $a$ to $c$.
⟨1⟩6. The concatenation of $p$ and $q$ is a path from $b$ to $c$ in $\bigcup \mathcal{A}$.
□

**Proposition 1.283.** *Let $A \subseteq \mathbb{R}^2$ be countable. Then $\mathbb{R}^2 \setminus A$ is path-connected.*

PROOF:
⟨1⟩1. LET: $a, b \in \mathbb{R}^2 \setminus A$
⟨1⟩2. PICK a line $l$ in $\mathbb{R}^2$ with $a$ on one side and $b$ on the other.
⟨1⟩3. For every point $x$ on $l$,
LET: $p_x$ be the path in $\mathbb{R}^2$ consisting of a line from $a$ to $x$ then a line from $x$ to $b$
⟨1⟩4. For $x \neq y$ we have $p_x$ and $p_y$ have no points in common except $a$ and $b$
⟨1⟩5. There are only countably many $x$ such that a point of $A$ lies on $p_x$.
⟨1⟩6. There exists $x$ such that $p_x$ is a path from $a$ to $b$ in $\mathbb{R}^2 \setminus A$.
□

**Proposition 1.284.** *Every open connected subspace of $\mathbb{R}^2$ is path-connected.*

PROOF:
⟨1⟩1. LET: $U$ be an open connected subspace of $\mathbb{R}^2$.
⟨1⟩2. For all $x_0 \in U$,
LET: $PC(x_0) = \{y \in U \mid \text{there exists a path from } x \text{ to } y\}$
⟨1⟩3. For all $x_0 \in U$, the set $PC(x_0)$ is open and closed in $U$.
⟨2⟩1. LET: $x_0 \in U$
⟨2⟩2. $PC(x_0)$ is open in $U$
⟨3⟩1. LET: $y \in PC(x_0)$
⟨3⟩2. PICK $\epsilon > 0$ such that $B(y, \epsilon) \subseteq U$
PROOF: Since $U$ is open.
⟨3⟩3. $B(y, \epsilon) \subseteq PC(x_0)$
PROOF: For all $z \in B(y, \epsilon)$, pick a path from $x_0$ to $y$ then concatenate the straight line from $y$ to $z$.
⟨2⟩3. $PC(x_0)$ is closed in $U$
⟨3⟩1. LET: $y \in U$ be a limit point of $PC(x_0)$
⟨3⟩2. PICK $\epsilon > 0$ such that $B(y, \epsilon) \subseteq U$
⟨3⟩3. PICK $z \in PC(x_0) \cap B(y, \epsilon)$
⟨3⟩4. $y \in PC(x_0)$
PROOF: Pick a path from $x_0$ to $z$ then concatenate the straight line from $z$ to $y$.
⟨1⟩4. $PC(x_0) = U$
PROOF: Proposition 1.248.
□

**Example 1.285.** If $A$ is a connected subspace of $X$, then $A^\circ$ is not necessarily connected.

Take two closed circles in $\mathbb{R}^2$ that touch at one point. The interior of this space is two open circles, and these two circles form a separation.

**Example 1.286.** If $A$ is a connected subspace of $X$ then $\partial A$ is not necessarily connected.

We have $[0, 1]$ is connected but $\partial[0, 1] = \{0, 1\}$ is not.

**Example 1.287.** If $A$ is a subspace of $X$ and $A^\circ$ and $\partial A$ are connected, then $A$ is not necessarily connected.

We have $\mathbb{Q}^\circ = \emptyset$ and $\partial\mathbb{Q} = \mathbb{R}$ are connected but $\mathbb{Q}$ is not connected.

## 1.47 The Topologist's Sine Curve

**Definition 1.288** (The Topologist's Sine Curve)**.** Let $S = \{(x, \sin 1/x) \mid 0 < x \leq 1\}$, The *topologist's sine curve* is the closure $\overline{S}$ of $S$ in $\mathbb{R}^2$.

**Proposition 1.289.** *The topologist's sine curve is connected.*

PROOF:
⟨1⟩1. LET: $S = \{(x, \sin 1/x) \mid 0 < x \leq 1\}$
⟨1⟩2. $S$ is connected.

PROOF: Theorem 1.258.
⟨1⟩3. $\overline{S}$ is connected.
PROOF: Theorem 1.257.
□

**Proposition 1.290.** *The topologist's sine curve is* $\{(x, \sin 1/x) \mid 0 < x \leq 1\} \cup (\{0\} \times [-1.1])$.

PROOF: Sketch proof: Given a point $(0.y)$ with $-1 \leq y \leq 1$, pick $a$ such that $\sin a = y$. Then $((1/a, y), (1/(a+2\pi), y), (1/(a+4\pi), y), \ldots)$ is a sequence in $S$ that converges to $(0, y)$.
Conversely, let $(x, y)$ be any point not in $S \cup (\{0\} \times [-1, 1])$. If $x < 0$ or $y > 1$ or $y < -1$ then we can easily find a neighbourhood that does not intersect $S \cup (\{0\} \times [-1, 1])$. If $x > 0$ and $-1 \leq y \leq 1$, then we have $y \neq \sin 1/x$. Hence pick a neighbourhood that does not intersect $S$.

**Proposition 1.291.** *Every closed subset of* $\mathbb{R}$ *that is bounded above has a greatest element.*

PROOF: It has a supremum, which is a limit point of the set and hence an element. □

**Proposition 1.292** (CC)**.** *The topologist's sine curve is not path connected.*

PROOF:
⟨1⟩1. ASSUME: For a contradction $p : [0, 1] \to \overline{S}$ is a path from $(0, 0)$ to $(1, \sin 1)$.
⟨1⟩2. $\{t \in [0, 1] \mid p(t) \in \{0\} \times [-1, 1]\}$ is closed.
PROOF: Since $p$ is continuous and $\{0\} \times [-1, 1]$ is closed.
⟨1⟩3. LET: $b$ be the largest number in $[0, 1]$ such that $p(b) \in \{0\} \times [-1, 1]$.
PROOF: Proposition 1.291.
⟨1⟩4. LET: $x : [b, 1] \to \overline{S}$ be the function $\pi_1 \circ p$
⟨1⟩5. LET: $y : [b, 1] \to \overline{S}$ be the function $\pi_2 \circ p$
⟨1⟩6. PICK a sequence $t_n$ in $(b, 1]$ such that $t_n \to b$ and $y(t_n) = (-1)^n$ for all $n$
⟨2⟩1. LET: $n \geq 1$
⟨2⟩2. PICK $u$ with $0 < u < x(1/n)$ and $\sin(1/u) = (-1)^n$
⟨2⟩3. PICK $t_n$ with $b < t_n < 1/n$ and $x(t_n) = u$
PROOF: By the Intermediate Value Theorem
⟨1⟩7. Q.E.D.
PROOF: This contradicts Proposition 1.139 since $y$ is continuous and $y(t_n)$ does not converge.
□

**Corollary 1.292.1.** *The closure of a path-connected subspace of a space is not necessarily path-connected.*

## 1.48 The Long Line

**Definition 1.293** (The Long Line)**.** The *long line* is the space $\omega_1 \times [0, 1)$ in the dictionary order under the order topology, where $\omega_1$ is the first uncountable ordinal.

**Lemma 1.294.** *For any ordinal $\alpha$ with $0 < \alpha < \omega_1$ we have $[(0,0),(\alpha,0)) \cong [0,1)$*

$\langle 1 \rangle 1.$ $[(0,0),(1,0)) \cong [0,1)$
PROOF: The map $\pi_2$ is a homeomorphism.
$\langle 1 \rangle 2.$ If $[(0,0),(\alpha,0)) \cong [0,1)$ then $[(0,0),(\alpha+1,0)) \cong [0,1)$
PROOF: Proposition 1.38.
$\langle 1 \rangle 3.$ If $\lambda$ is a limit ordinal with $\lambda < \omega_1$ and $[(0,0),(\alpha,0)) \cong [0,1)$ for all $\alpha$ with $0 < \alpha < \lambda$ then $[(0,0),(\lambda,0)) \cong [0,1)$
$\langle 2 \rangle 1.$ LET: $\lambda$ be a limit ordinal $< \omega_1$
$\langle 2 \rangle 2.$ ASSUME: $[(0,0),(\alpha,0)) \cong [0,1)$ for all $\alpha$ with $0 < \alpha < \lambda$
$\langle 2 \rangle 3.$ PICK a sequence of ordinals $\alpha_0 < \alpha_1 < \cdots$ with limit $\lambda$
PROOF: Since $\lambda$ is countable.
$\langle 2 \rangle 4.$ $[(\alpha_i,0),(\alpha_{i+1},0)) \cong [0,1)$ for all $i$
PROOF: Lemma 1.37.
$\langle 2 \rangle 5.$ Q.E.D.
PROOF: By Proposition 1.39.
$\langle 1 \rangle 4.$ Q.E.D.
PROOF: By transfinite induction.

**Proposition 1.295** (CC)**.** *The long line is path-connected.*

PROOF:
$\langle 1 \rangle 1.$ LET: $(\alpha,i),(\beta,j) \in \omega_1 \times [0,1)$
$\langle 1 \rangle 2.$ ASSUME: without loss of generality $(\alpha,i) < (\beta,j)$
$\langle 1 \rangle 3.$ $[(0,0),(\beta+1,0)) \cong [0,1)$
PROOF: By Lemma 1.294
$\langle 1 \rangle 4.$ $[(\alpha,i),(\beta,j)) \cong [0,1)$
PROOF: Lemma 1.37.
$\langle 1 \rangle 5.$ PICK a homeomorphism $q : [0,1) \to [(\alpha,i),(\beta,j))$
$\langle 1 \rangle 6.$ $q \cup \{(1,(\beta,j))\}$ is a path from $(\alpha,i)$ to $(\beta,j)$
□

**Proposition 1.296.** *Every point in the long line has a neighbourhood homeomorphic to an interval in $\mathbb{R}$.*

PROOF: For any $(\alpha,i)$ in the long line, the neighbourhood $[(0,0),(\alpha+1,0))$ satisfies the condition by Lemma 1.294.

**Proposition 1.297.** *The long line $L$ is not second countable.*

PROOF:
$\langle 1 \rangle 1.$ LET: $\mathcal{B}$ be a basis for $L$.
$\langle 1 \rangle 2.$ For $\alpha < \omega_1$, PICK $B_\alpha \in \mathcal{B}$ such that $(\alpha,1/2) \in B_\alpha \subseteq ((\alpha,0),(\alpha+1,0))$
$\langle 1 \rangle 3.$ $\mathcal{B}$ is uncountable.
PROOF: The mapping $\alpha \mapsto B_\alpha$ is an injection $\omega_1 \to \mathcal{B}$.

**Corollary 1.297.1.** *The long line cannot be imbedded into $\mathbb{R}^n$ for any $n$.*

## 1.49 Components

**Proposition 1.298.** *Let $X$ be a topological space. Define the relation $\sim$ on $X$ by $a \sim b$ if and only if there exists a connected subspace $A \subseteq X$ with $a, b \in A$. Then $\sim$ is an equivalence relation on $X$.*

PROOF:
$\langle 1 \rangle 1$. $\sim$ is reflexive.
  PROOF: For any $a \in X$ we have $\{a\}$ is a connected subspace that contains $a$.
$\langle 1 \rangle 2$. $\sim$ is symmetric.
  PROOF: Trivial.
$\langle 1 \rangle 3$. $\sim$ is transitive.
  $\langle 2 \rangle 1$. LET: $a, b, c \in X$
  $\langle 2 \rangle 2$. ASSUME: $a \sim b$ and $b \sim c$
  $\langle 2 \rangle 3$. PICK connected subspaces $A$ and $B$ with $a, b \in A$ and $b, c \in B$
  $\langle 2 \rangle 4$. $A \cup B$ is a connected subspace that contains $a$ and $c$
    PROOF: Theorem 1.256.
□

**Definition 1.299** ((Connected) Component)**.** Let $X$ be a topological space. The *(connected) components* of $X$ are the equivalence classes under the above $\sim$.

**Lemma 1.300.** *Let $X$ be a topological space. If $A \subseteq X$ is connected and nonempty then there exists a unique component $C$ of $X$ such that $A \subseteq C$.*

PROOF:
$\langle 1 \rangle 1$. PICK $a \in A$
$\langle 1 \rangle 2$. LET: $C$ be the $\sim$-equivalence class of $a$.
$\langle 1 \rangle 3$. $A \subseteq C$
  PROOF: For all $x \in A$ we have $x \sim a$.
$\langle 1 \rangle 4$. If $C'$ is a component and $A \subseteq C'$ then $C = C'$
  PROOF: Since we have $a \in C'$.
□

**Theorem 1.301.** *Let $X$ be a topological space. The components of $X$ are connected disjoint subspaces of $X$ whose union is $X$ such that each nonempty connected subspace of $X$ intersects only one of them.*

PROOF:
$\langle 1 \rangle 1$. Every component of $X$ is connected.
  PROOF: For $a \in X$, the $\sim$-equivalence class of $a$ is $\bigcup \{A \subseteq X \mid A \text{ is connected}, a \in A\}$ which is connected by Theorem 1.256.
$\langle 1 \rangle 2$. The components form a partition of $X$.
  PROOF: Immediate from the definition.
$\langle 1 \rangle 3$. Every nonempty connected subspace of $X$ intersects a unique component of $X$.
  $\langle 2 \rangle 1$. LET: $A \subseteq X$ be connected and nonempty.

$\langle 2 \rangle 2$. LET: $C$ be the component such that $A \subseteq C$
PROOF: Lemma 1.300.
$\langle 2 \rangle 3$. $A$ intersects $C$
$\langle 2 \rangle 4$. If $A$ intersects the component $C'$ then $C' = C$
$\langle 3 \rangle 1$. LET: $C'$ be a component that intersects $A$
$\langle 3 \rangle 2$. PICK $b \in A \cap C'$
$\langle 3 \rangle 3$. $A \subseteq C'$
PROOF: For all $x \in A$ we have $x \sim b$.
$\langle 3 \rangle 4$. $C = C'$
PROOF: By uniqueness in $\langle 2 \rangle 2$.
□

**Proposition 1.302.** *Every component of a space is closed.*

PROOF:
$\langle 1 \rangle 1$. LET: $X$ be a topological space and $C$ a component of $X$.
$\langle 1 \rangle 2$. $\overline{C}$ is connected.
PROOF: Theorem 1.257.
$\langle 1 \rangle 3$. $C = \overline{C}$
PROOF: Lemma 1.255.
$\langle 1 \rangle 4$. $C$ is closed.
PROOF: Lemma 1.80.
□

**Proposition 1.303.** *If a topological space has finitely many components then every component is open.*

PROOF: Each component is the complement of a finite union of closed sets. □

## 1.50 Path Components

**Proposition 1.304.** *Let $X$ be a topological space. Define the relation $\sim$ on $X$ by: $a \sim b$ if and only if there exists a path in $X$ from $a$ to $b$. Then $\sim$ is an equivalence relation on $X$.*

PROOF:
$\langle 1 \rangle 1$. $\sim$ is reflexive.
PROOF: For $a \in X$, the constant function $[0,1] \to X$ with value $a$ is a path from $a$ to $a$.
$\langle 1 \rangle 2$. $\sim$ is symmetric.
PROOF: If $p : [0,1] \to X$ is a path from $a$ to $b$, then $\lambda t.p(1-t)$ is a path from $b$ to $a$.
$\langle 1 \rangle 3$. $\sim$ is transitive.
PROOF: Concatenate paths.
□

**Definition 1.305** (Path Component)**.** Let $X$ be a topological space. The *path components* of $X$ are the equivalence relations under $\sim$.

.

**Theorem 1.306.** *The path components of $X$ are path-connected disjoint subspaces of $X$ whose union is $X$ such that every nonempty path-connected subspace of $X$ intersects exactly one path component.*

PROOF:
⟨1⟩1. Every path component is path-connected.
PROOF: If $a$ and $b$ are in the same path component then $a \sim b$, i.e. there exists a path from $a$ to $b$.
⟨1⟩2. The path components are disjoint and their union is $X$.
PROOF: Immediate from the definition.
⟨1⟩3. Every non-empty path-cönnected subspace of $X$ intersects exactly one path component.
⟨2⟩1. LET: $A$ be a nonempty path-connected subspace of $X$.
⟨2⟩2. PICK $a \in A$
⟨2⟩3. $A$ intersects the $\sim$-equivalence class of $a$.
⟨2⟩4. LET: $C$ be any path component that intersects $A$.
⟨2⟩5. PICK $b \in A \cap C$
⟨2⟩6. $a \sim b$
PROOF: Since $A$ is path-connected.
⟨2⟩7. $C$ is the $\sim$-equivalence class of $a$.
□

**Proposition 1.307.** *Every path component is included in a component.*

PROOF:
⟨1⟩1. LET: $X$ be a topological space and $C$ a path component of $X$.
⟨1⟩2. $C$ is path-connected.
PROOF: Theorem 1.306.
⟨1⟩3. $C$ is connected.
PROOF: Proposition 1.277.
⟨1⟩4. $C$ is included in a component.
PROOF: Lemma 1.300.
□

## 1.51 Local Connectedness

**Definition 1.308** (Locally Connected)**.** Let $X$ be a topological space and $a \in X$. Then $X$ is *locally connected* at $a$ if and only if every neighbourhood of $a$ includes a connected neighbourhood of $a$.

The space $X$ is *locally connected* if and only if it is locally connected at every point.

**Example 1.309.** The real line is both connected and locally connected.

**Example 1.310.** The space $\mathbb{R} \setminus \{0\}$ is disconnected but locally connected.

**Example 1.311.** The topologist's sine curve is connected but not locally connected.

**Example 1.312.** The rationals $\mathbb{Q}$ are neither connected nor locally connected.

**Theorem 1.313.** *A topological space $X$ is locally connected if and only if, for every open set $U$ in $X$, every component of $U$ is open in $X$.*

PROOF:
$\langle 1 \rangle 1$. If $X$ is locally connected then, for every open set $U$ in $X$, every component of $U$ is open in $X$.
  $\langle 2 \rangle 1$. ASSUME: $X$ is locally connected.
  $\langle 2 \rangle 2$. LET: $U$ be open in $X$.
  $\langle 2 \rangle 3$. LET: $C$ be a component of $U$.
  $\langle 2 \rangle 4$. LET: $a \in C$
  $\langle 2 \rangle 5$. LET: $V$ be a connected neighbourhood of $a$ such that $V \subseteq U$
  $\langle 2 \rangle 6$. $V \subseteq C$
    PROOF: Lemma 1.300.
  $\langle 2 \rangle 7$. Q.E.D.
    PROOF: Lemma 1.59.
$\langle 1 \rangle 2$. If, for every open set $U$ in $X$, every component of $U$ is open in $X$, then $X$ is locally connected.
  $\langle 2 \rangle 1$. ASSUME: for every open set $U$ in $X$, every component of $U$ is open in $X$.
  $\langle 2 \rangle 2$. LET: $a \in X$
  $\langle 2 \rangle 3$. LET: $U$ be a neighbourhood of $a$
  $\langle 2 \rangle 4$. The component of $U$ that contains $a$ is a connected neighbourhood of $a$ included in $U$.

$\square$

**Example 1.314.** The ordered square is locally connected.

PROOF: Every neighbourhood of a point includes an interval around that point, which is connected by Theorem 1.268.

**Example 1.315.** Let $X$ be the set of all rational points on the line segment $[0,1] \times \{0\}$, and $Y$ the set of all rational points on the line segment $[0,1] \times \{1\}$. Let $A$ be the space consisting of all line segments joining the point $(0,1)$ to a point of $X$, and all line segments joining the point $(1,0)$ to a point of $Y$. Then $A$ is path-connected but is not locally connected at any point,

**Proposition 1.316.** *Let $X$ and $Y$ be topological spaces and $p : X \twoheadrightarrow Y$ be a quotient map. If $X$ is locally connected then so is $Y$.*

PROOF:
$\langle 1 \rangle 1$. LET: $U$ be an open set in $Y$.
$\langle 1 \rangle 2$. LET: $C$ be a component of $U$.
$\langle 1 \rangle 3$. $p^{-1}(C)$ is a union of components of $p^{-1}(U)$
  $\langle 2 \rangle 1$. LET: $x \in p^{-1}(C)$

$\langle 2 \rangle 2$. Let: $D$ be the component of $p^{-1}(U)$ that contains $x$.
$\langle 2 \rangle 3$. $p(D)$ is connected.
Proof: Theorem 1.258.
$\langle 2 \rangle 4$. $p(D) \subseteq C$.
Proof: From $\langle 1 \rangle 2$ since $p(x) \in p(D) \cap C$ ($\langle 2 \rangle 1$, $\langle 2 \rangle 2$).
$\langle 2 \rangle 5$. $D \subseteq p^{-1}(C)$
$\langle 1 \rangle 4$. $p^{-1}(C)$ is open in $p^{-1}(U)$
Proof: Theorem 1.313.
$\langle 1 \rangle 5$. $C$ is open in $U$
Proof: Since the restriction of $p$ to $p : p^{-1}(U) \twoheadrightarrow U$ is a quotient map by Proposition 1.231.
$\langle 1 \rangle 6$. Q.E.D.
Proof: Theorem 1.313.
□

## 1.52 Local Path Connectedness

**Definition 1.317** (Locally Path-Connected)**.** Let $X$ be a topological space and $a \in X$. Then $X$ is *locally path-connected* at $a$ if and only if every neighbourhood of $a$ includes a path-connected neighbourhood of $a$.

The space $X$ is *locally path-connected* if and only if it is locally path-connected at every point.

**Theorem 1.318.** *A topological space $X$ is locally path-connected if and only if, for every open set $U$ in $X$, every path component of $U$ is open in $X$.*

Proof:
$\langle 1 \rangle 1$. If $X$ is locally path-connected then, for every open set $U$ in $X$, every path component of $U$ is open in $X$.
$\langle 2 \rangle 1$. Assume: $X$ is locally path-connected.
$\langle 2 \rangle 2$. Let: $U$ be open in $X$.
$\langle 2 \rangle 3$. Let: $C$ be a path component of $U$.
$\langle 2 \rangle 4$. Let: $a \in C$
$\langle 2 \rangle 5$. Let: $V$ be a path-connected neighbourhood of $a$ such that $V \subseteq U$
$\langle 2 \rangle 6$. $V \subseteq C$
Proof: Lemma 1.300.
$\langle 2 \rangle 7$. Q.E.D.
Proof: Lemma 1.59.
$\langle 1 \rangle 2$. If, for every open set $U$ in $X$, every component of $U$ is open in $X$, then $X$ is locally connected.
$\langle 2 \rangle 1$. Assume: for every open set $U$ in $X$, every component of $U$ is open in $X$.
$\langle 2 \rangle 2$. Let: $a \in X$
$\langle 2 \rangle 3$. Let: $U$ be a neighbourhood of $a$
$\langle 2 \rangle 4$. The component of $U$ that contains $a$ is a connected neighbourhood of $a$ included in $U$.

□

**Theorem 1.319.** *If a space is locally path connected then its components and its path components are the same.*

PROOF:
⟨1⟩1. LET: $X$ be a locally path connected space.
⟨1⟩2. LET: $C$ be a component of $X$.
⟨1⟩3. LET: $x \in C$
⟨1⟩4. LET: $P$ be the path component of $x$
PROVE: $P = C$
⟨1⟩5. $P \subseteq C$
PROOF: Proposition 1.307.
⟨1⟩6. LET: $Q$ be the union of the other path components included in $C$
⟨1⟩7. $C = P \cup Q$
PROOF: Proposition 1.307.
⟨1⟩8. $P$ and $Q$ are open in $C$
⟨2⟩1. $C$ is open.
PROOF: Theorem 1.313.
⟨2⟩2. Q.E.D.
PROOF: Theorem 1.318.
⟨1⟩9. $Q = \emptyset$
PROOF: Otherwise $P$ and $Q$ would form a separation of $C$.
□

**Example 1.320.** The ordered square is not locally path connected, since it is connected but not path connected.

**Proposition 1.321.** *Let $X$ be a locally path-connected space. Then every connected open subspace of $X$ is path-connected.*

PROOF:
⟨1⟩1. LET: $U$ be a connected open subspace of $X$.
⟨1⟩2. LET: $P$ be a path component of $U$.
⟨1⟩3. LET: $Q$ be the union of the other path components of $U$.
⟨1⟩4. $P$ and $Q$ are open in $U$.
PROOF: Theorem 1.318.
⟨1⟩5. $Q = \emptyset$
PROOF: Otherwise $P$ and $Q$ form a separation of $U$.
□

## 1.53 Weak Local Connectedness

**Definition 1.322** (Weakly Locally Connected)**.** Let $X$ be a topological space and $a \in X$. Then $X$ is *weakly locally connected* at $a$ if and only if every neighbourhood of $a$ includes a connected subspace that includes a neighbourhood of $a$.

**Proposition 1.323.** *Let $X$ be a topological space. If $X$ is weakly locally connected at every point then $X$ is locally connected.*

PROOF:
$\langle 1\rangle 1.$ ASSUME: $X$ is weakly locally connected at every point.
$\langle 1\rangle 2.$ LET: $U$ be open in $X$.
$\langle 1\rangle 3.$ LET: $C$ be a component of $U$.
$\langle 1\rangle 4.$ $C$ is open in $X$.
  $\langle 2\rangle 1.$ LET: $x \in C$
  $\langle 2\rangle 2.$ PICK a connected subspace $D$ of $U$ that includes a neighbourhood $V$ of $x$.
  $\langle 2\rangle 3.$ $D \subseteq C$
    PROOF: Lemma 1.300.
  $\langle 2\rangle 4.$ $x \in V \subseteq C$
  $\langle 2\rangle 5.$ Q.E.D.
    PROOF: Lemma 1.59.
$\langle 1\rangle 5.$ Q.E.D.
  PROOF: Theorem 1.313.
□

**Example 1.324.** The 'infinite broom' (Munkres p.163) is an example of a space that is weakly locally connected at a point $p$ but not locally connected at $p$.

## 1.54 Quasicomponents

**Proposition 1.325.** *Let $X$ be a topological space. Define $\sim$ on $X$ by $x \sim y$ if and only if there exists no separation $U$ and $V$ of $X$ such that $x \in U$ and $y \in V$. Then $\sim$ is an equivalence relation on $X$.*

PROOF:
$\langle 1\rangle 1.$ $\sim$ is reflexive.
  PROOF: Since the two sets that make up a separation are disjoint.
$\langle 1\rangle 2.$ $\sim$ is symmetric.
  PROOF: Immediate from the defintion.
$\langle 1\rangle 3.$ $\sim$ is transitive.
  $\langle 2\rangle 1.$ ASSUME: $x \sim y$ and $y \sim z$
  $\langle 2\rangle 2.$ ASSUME: for a contradiction there is a separation $U$ and $V$ of $X$ with $x \in U$ and $z \in V$
  $\langle 2\rangle 3.$ $y \in U$ or $y \in V$
  $\langle 2\rangle 4.$ Q.E.D.
    PROOF: Either case contradicts $\langle 2\rangle 1$.
□

**Definition 1.326** (Quasicomponents)**.** For $X$ a topological space, the *quasicomponents* of $X$ are the equivalence classes under $\sim$.

**Proposition 1.327.** *Let $X$ be a topological space. Then every component of $X$ is included in a quasicomponent of $X$.*

PROOF:
$\langle 1\rangle 1$. LET: $C$ be a component of $X$.
$\langle 1\rangle 2$. LET: $x, y \in C$
PROVE: $x \sim y$
$\langle 1\rangle 3$. ASSUME: for a contradiction there exists a separation $U$ and $V$ of $X$ with $x \in U$ and $y \in V$
$\langle 1\rangle 4$. $C \cap U$ and $C \cap V$ form a separation of $C$.
$\langle 1\rangle 5$. Q.E.D.
PROOF: This contradicts $\langle 1\rangle 1$.

**Proposition 1.328.** *In a locally connected space, the components and the quasicomponents are the same.*

PROOF:
$\langle 1\rangle 1$. LET: $X$ be a locally connected space and $Q$ a quasicomponent of $X$.
$\langle 1\rangle 2$. PICK a component $C$ of $X$ such that $C \subseteq Q$
$\langle 1\rangle 3$. LET: $D$ be the union of the components of $X$
$\langle 1\rangle 4$. $C$ and $D$ are open in $X$.
PROOF: Theorem 1.313.
$\langle 1\rangle 5$. $D$ cannot contain any points of $Q$.
PROOF: If it did, then $C$ and $D$ would form a separation of $X$ and there would be points $x, y \in Q$ with $x \in C$ and $y \in D$.
$\langle 1\rangle 6$. $C = Q$
□

## 1.55 Open Coverings

**Definition 1.329** (Open Covering)**.** Let $X$ be a topological space. An *open covering* of $X$ is a covering of $X$ whose elements are all open sets.

## 1.56 Lindelöf Spaces

**Definition 1.330** (Lindelöf Space)**.** A topological space $X$ is *Lindelöf* if and only if every open covering has a countable subcovering.

**Proposition 1.331.** *Let $X$ be a topological space. Then $X$ is Lindelöf if and only if every set of closed sets that has the countable intersection property has nonempty intersection.*

PROOF: The following are equivalent.

1. $X$ is compact.
2. Every open covering of $X$ has a countable subcovering.
3. For any set $\mathcal{C}$ of closed sets, if $\{X \setminus C \mid C \in \mathcal{C}\}$ covers $X$ then there is a countable subset $\mathcal{C}_0$ such that $\{X \setminus C \mid C \in \mathcal{C}_0\}$ covers $X$

4. For any set $\mathcal{C}$ of closed sets, if $\bigcap \mathcal{C} = \emptyset$ then there is a countable subset $\mathcal{C}_0$ with empty intersection.

5. Any set of closed sets with the countable intersection property has nonempty intersection.

$\square$

**Proposition 1.332** (CC)**.** *Let $X$ be a topological space and $\mathcal{B}$ a basis for the topology on $X$. Then the following are equivalent.*

1. *$X$ is Lindelöf.*

2. *Every open covering of $X$ by elements of $\mathcal{B}$ has a countable subcovering.*

Proof:
$\langle 1 \rangle 1.$ $1 \Rightarrow 2$
  Proof: Immediate from definitions.
$\langle 1 \rangle 2.$ $2 \Rightarrow 1$
  $\langle 2 \rangle 1.$ Assume: Every open covering of $X$ by elements of $\mathcal{B}$ has a countable subcovering.
  $\langle 2 \rangle 2.$ Let: $\mathcal{U}$ be an open covering of $X$.
  $\langle 2 \rangle 3.$ $\{B \in \mathcal{B} \mid \exists U \in \mathcal{U}. B \subseteq U\}$ covers $X$.
  $\langle 2 \rangle 4.$ Pick a finite subcovering $\mathcal{B}_0$.
  $\langle 2 \rangle 5.$ For $B \in BB$, Pick $U_B \in \mathcal{U}$ such that $B \subseteq U_B$
  $\langle 2 \rangle 6.$ $\{U_B \mid B \in \mathcal{B}_0\}$ covers $X$.
$\square$

## 1.57 The Second Countability Axiom

**Definition 1.333** (Second Countability Axiom)**.** A topological space satisfies the *second countability axiom*, or is *second countable*, if and only if it has a countable basis.

**Example 1.334.** The space $\mathbb{R}$ is second countable.

Proof: The set $\{(a, b) \mid a, b \in \mathbb{Q}\}$ is a basis. $\square$

**Proposition 1.335.** *A subspace of a second countable space is second countable.*

Proof: If $\mathcal{B}$ is a countable basis for $X$ and $Y \subseteq X$ then $\{B \cap Y \mid B \in \mathcal{B}\}$ is a countable basis for $Y$. $\square$

**Proposition 1.336** (CC)**.** *Every second countable space is Lindelöf.*

Proof: From Proposition 1.332.

**Example 1.337** (CC)**.** The space $\mathbb{R}_l$ is Lindelöf.

$\langle 1 \rangle 1.$ Let: $\mathcal{A}$ be a covering of $\mathbb{R}_l$ by basic open sets of the form $[a, b)$
$\langle 1 \rangle 2.$ Let: $C = \bigcup \{(a, b) \mid [a, b) \in \mathcal{A}\}$

$\langle 1 \rangle 3.$ $\mathbb{R} \setminus C$ is countable.
$\langle 2 \rangle 1.$ For every $x \in \mathbb{R} \setminus C$, Pick a rational $q_x$ such that $(x, q_x) \subseteq C$
$\langle 3 \rangle 1.$ Let: $x \in \mathbb{R} \setminus C$
$\langle 3 \rangle 2.$ Pick $b$ such that $[x, b) \in \mathcal{A}$
$\langle 3 \rangle 3.$ Pick a rational $q$ such that $q \in (x, b)$
$\langle 2 \rangle 2.$ The mapping $x \mapsto q_x$ is an injection $\mathbb{R} \setminus C \to \mathbb{Q}$
$\langle 1 \rangle 4.$ Pick a countable $\mathcal{A}' \subseteq \mathcal{A}$ that covers $\mathbb{R} \setminus C$
$\langle 1 \rangle 5.$ Under the standard topology on $\mathbb{R}$, $C$ is second countable.
Proof: Proposition 1.335.
$\langle 1 \rangle 6.$ Pick a countable $\mathcal{A}'' \subseteq \mathcal{A}$ such that $\{(a, b) \mid [a, b) \in \mathcal{A}''\}$ covers $C$.
Proof: Proposition 1.332.
$\langle 1 \rangle 7.$ $\mathcal{A}' \cup \mathcal{A}''$ covers $\mathbb{R}_l$.
□

**Example 1.338.** The product of two Lindelöf spaces is not necessarily Lindelöf.
We prove that the Sorgenfrey plane is not Lindelöf.

Proof:
$\langle 1 \rangle 1.$ Let: $L = \{(x, -x) \mid x \in \mathbb{R}\}$
$\langle 1 \rangle 2.$ $L$ is closed in $\mathbb{R}_l^2$
$\langle 1 \rangle 3.$ Let: $\mathcal{U} = \{[a, b) \times [a, -d) \mid a, b, d \in \mathbb{R}\}$
$\langle 1 \rangle 4.$ $\mathcal{U} \cup \{\mathbb{R} \setminus L\}$ covers $\mathbb{R}_l^2$
$\langle 1 \rangle 5.$ Every element of $\mathcal{U}$ intersects $L$ at exactly one point.
$\langle 1 \rangle 6.$ No countable subset of $\mathcal{U}$ covers $\mathbb{R}_l^2$.
□

## 1.58 Compact Spaces

**Definition 1.339** (Compact)**.** A topological space is *compact* if and only if every open covering has a finite subcovering.

**Lemma 1.340.** *Let $X$ be a topological space and $Y$ a subspace of $X$. Then $Y$ is compact if and only if every covering of $Y$ by sets open in $X$ has a finite subcovering.*

Proof:
$\langle 1 \rangle 1.$ If $Y$ is compact then every covering of $Y$ by sets open in $X$ has a finite subcovering.
$\langle 2 \rangle 1.$ Assume: $Y$ is compact.
$\langle 2 \rangle 2.$ Let: $\mathcal{U}$ be a covering of $Y$ by sets open in $X$.
$\langle 2 \rangle 3.$ $\{U \cap Y \mid U \in \mathcal{U}\}$ is an open covering of $Y$.
$\langle 2 \rangle 4.$ Pick a finite subcovering $\{U_1 \cap Y, \ldots, U_n \cap Y\}$
$\langle 2 \rangle 5.$ $\{U_1, \ldots, U_n\}$ is a finite subcovering of $\mathcal{U}$.
$\langle 1 \rangle 2.$ If every covering of $Y$ by sets open in $X$ has a finite subcovering then $Y$ is compact.
$\langle 2 \rangle 1.$ Let: $\mathcal{U}$ be an open covering of $Y$.
$\langle 2 \rangle 2.$ Let: $\mathcal{V} = \{V \text{ open in } X \mid V \cap Y \in \mathcal{U}\}$.

⟨2⟩3. $\mathcal{V}$ is a covering of $Y$ by sets open in $X$.
⟨2⟩4. PICK a finite subcovering $\{V_1, \ldots, V_n\}$
⟨2⟩5. $\{V_1 \cap Y, \ldots, V_n \cap Y\}$ is a finite subcovering of $\mathcal{U}$.
□

**Proposition 1.341.** *Every closed subspace of a compact space is compact.*

PROOF:
⟨1⟩1. LET: $X$ be a compact space and $Y \subseteq X$ be closed.
⟨1⟩2. LET: $\mathcal{U}$ be a covering of $Y$ by sets open in $X$.
⟨1⟩3. $\mathcal{U} \cup \{X \setminus Y\}$ is an open covering of $X$.
⟨1⟩4. PICK a finite subcovering $\mathcal{U}_0$
⟨1⟩5. $\mathcal{U}_0 \cap \mathcal{U}$ is a finite subset of $\mathcal{U}$ that covers $Y$.
□

**Theorem 1.342.** *The continuous image of a compact space is compact.*

PROOF:
⟨1⟩1. LET: $f : X \twoheadrightarrow Y$ be continuous and surjective.
⟨1⟩2. LET: $\mathcal{V}$ be an open covering of $Y$
⟨1⟩3. $\{p^{-1}(V) \mid V \in \mathcal{V}\}$ is an open covering of $X$.
⟨1⟩4. PICK a finite subcovering $\{p^{-1}(V_1), \ldots, p^{-1}(V_n)\}$
⟨1⟩5. $\{V_1, \ldots, V_n\}$ covers $Y$.
□

**Theorem 1.343.** *Let $A$ and $B$ be compact subspaces of $X$ and $Y$ respectively. Let $N$ be an open set in $X \times Y$ that includes $A \times B$. Then there exist open sets $U$ and $V$ in $X$ and $Y$ respectively such that $A \subseteq U$, $B \subseteq V$ and $U \times V \subseteq N$.*

PROOF:
⟨1⟩1. For all $x \in A$, there exist neighbourhoods $U$ of $x$ and $V$ of $B$ such that $U \times V \subseteq N$.
⟨2⟩1. LET: $x \in A$
⟨2⟩2. For all $y \in B$, there exist neighbourhoods $U$ of $x$ and $V$ of $y$ such that $U \times V \subseteq N$
⟨2⟩3. $\{V \text{ open in } Y \mid \exists \text{ neighbourhood } U \text{ of } x, U \times V \subseteq N\}$ covers $B$.
⟨2⟩4. PICK a finite subcover $\{V_1, \ldots, V_n\}$
⟨2⟩5. For $i = 1, \ldots, n$, PICK a neighbourhood $U_i$ of $x$ such that $U_i \times V_i \subseteq N$
⟨2⟩6. LET: $U = U_1 \cap \cdots \cap U_n$
⟨2⟩7. LET: $V = V_1 \cup \cdots \cup V_n$
⟨2⟩8. $U$ is a neighbourhood of $x$.
⟨2⟩9. $V$ is a neighbourhood of $B$.
⟨2⟩10. $U \times V \subseteq N$
⟨1⟩2. $\{U \text{ open in } X \mid \exists \text{ neighbourhood } V \text{ of } B . U \times V \subseteq N\}$ covers $A$.
⟨1⟩3. PICK a finite subcover $\{U_1, \ldots, U_n\}$
⟨1⟩4. For $i = 1, \ldots, n$, PICK a neighbourhood $V_i$ of $B$ such that $U_i \times V_i \subseteq N$
⟨1⟩5. LET: $U = U_1 \cup \cdots \cup U_n$
⟨1⟩6. LET: $V = V_1 \cap \cdots \cap V_n$

⟨1⟩7. $U$ and $V$ are open.
⟨1⟩8. $A \subseteq U$
⟨1⟩9. $B \subseteq V$
⟨1⟩10. $U \times V \subseteq N$
□

**Corollary 1.343.1** (Tube Lemma)**.** *Let $X$ and $Y$ be topological spaces with $Y$ compact. Let $a \in X$ and $N$ be an open set in $X \times Y$ that includes $\{a\} \times Y$. Then there exists a neighbourhood $W$ of $a$ such that $N$ includes the* tube *$W \times Y$.*

**Theorem 1.344.** *Let $X$ be a topological space. Then $X$ is compact if and only if every set of closed sets that has the finite intersection property has nonempty intersection.*

PROOF: The following are equivalent.

1. $X$ is compact.
2. Every open covering of $X$ has a finite subcovering.
3. For any set $\mathcal{C}$ of closed sets, if $\{X \setminus C \mid C \in \mathcal{C}\}$ covers $X$ then there is a finite subset $\mathcal{C}_0$ such that $\{X \setminus C \mid C \in \mathcal{C}_0\}$ covers $X$
4. For any set $\mathcal{C}$ of closed sets, if $\bigcap \mathcal{C} = \emptyset$ then there is a finite subset $\mathcal{C}_0$ with empty intersection.
5. Any set of closed sets with the finite intersection property has nonempty intersection.

□

**Corollary 1.344.1.** *Let $X$ be a topological space and $C_1 \supseteq C_2 \supseteq \cdots$ a nested sequence of nonempty closed sets. Then $\bigcap_n C_n$ is nonempty.*

**Proposition 1.345.** *Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on the same set $X$ with $\mathcal{T} \subseteq \mathcal{T}'$. If $\mathcal{T}'$ is compact then $\mathcal{T}$ is compact.*

PROOF:
⟨1⟩1. LET: $\mathcal{U} \subseteq \mathcal{T}$ cover $X$
⟨1⟩2. $\mathcal{U} \subseteq \mathcal{T}'$
⟨1⟩3. A finite subset of $\mathcal{U}$ covers $X$.
□

**Corollary 1.345.1.** *If $\mathcal{T}$ and $\mathcal{T}'$ are two compact Hausdorff topologies on the same set $X$, then either $\mathcal{T} = \mathcal{T}'$ or $\mathcal{T}$ and $\mathcal{T}'$ are incomparable.*

PROOF: From the Proposition and Proposition 1.212. □

**Example 1.346.** Any set under the finite complement topology is compact.

**Proposition 1.347.** *Let $X$ be a topological space. A finite union of compact subspaces of $X$ is compact.*

PROOF:
⟨1⟩1. LET: $A$ and $B$ be compact subspaces of $X$.
⟨1⟩2. LET: $\mathcal{U}$ be a set of open sets in $X$ that covers $A \cup B$
⟨1⟩3. PICK a finite subset $\mathcal{U}_1$ that covers $A$.
PROOF: Lemma 1.340.
⟨1⟩4. PICK a finite subset $\mathcal{U}_2$ that covers $B$.
PROOF: Lemma 1.340.
⟨1⟩5. $\mathcal{U}_1 \cup \mathcal{U}_2$ is a finite subset that covers $A \cup B$.
⟨1⟩6. Q.E.D.
PROOF: Lemma 1.340.
□

**Proposition 1.348.** *Let $A$ and $B$ be disjoint compact subspaces of the Hausdorff space $X$. Then there exist disjoint open sets $U$ and $V$ that include $A$ and $B$ respectively.*

PROOF: From Theorem 1.343 with $N = X^2 \setminus \{(x, x) \mid x \in X\}$. □

**Corollary 1.348.1.** *Every compact subspace of a Hausdorff space is closed.*

**Theorem 1.349.** *Let $f : X \to Y$ be a bijective continuous function. If $X$ is compact and $Y$ is Hausdorff then $f$ is a homeomorphism.*

PROOF:
⟨1⟩1. LET: $C \subseteq X$ be closed.
⟨1⟩2. $C$ is compact.
PROOF: Proposition 1.341.
⟨1⟩3. $f(C)$ is compact.
PROOF: Theorem 1.342.
⟨1⟩4. $f(C)$ is closed.
PROOF: Corollary 1.348.1.
⟨1⟩5. Q.E.D.
PROOF: Lemma 1.141.
□

**Proposition 1.350.** *Let $X$ be a compact space, $Y$ a Hausdorff space, and $f : X \to Y$ a continuous map. Then $f$ is a closed map.*

PROOF:
⟨1⟩1. LET: $C \subseteq X$ be closed.
⟨1⟩2. $C$ is compact.
PROOF: Proposition 1.341.
⟨1⟩3. $f(C)$ is compact.
PROOF: Theorem 1.342.
⟨1⟩4. $f(C)$ is closed.
PROOF: Corollary 1.348.1.
□

**Proposition 1.351.** *If $Y$ is compact then the projection $\pi_1 : X \times Y \to X$ is a closed map.*

PROOF:
⟨1⟩1. LET: $A \subseteq X \times Y$ be closed.
⟨1⟩2. LET: $x \in X \setminus \pi_1(A)$
⟨1⟩3. PICK a neighbourhood $U$ of $x$ such that $U \times Y \subseteq (X \times Y) \setminus A$
PROOF: By the Tube Lemma.
⟨1⟩4. $x \in U \subseteq X \setminus \pi_1(A)$
⟨1⟩5. Q.E.D.
PROOF: So $X \setminus \pi_1(A)$ is open by Lemma 1.59.
□

**Theorem 1.352.** *Let $X$ be a topological space and $Y$ a compact Hausdorff space. Let $f : X \to Y$ be a function. Then $f$ is continuous if and only if the graph of $f$ is closed in $X \times Y$.*

PROOF:
⟨1⟩1. LET: $G_f$ be the graph of $f$.
⟨1⟩2. If $f$ is continuous then $G_f$ is closed.
⟨2⟩1. ASSUME: $f$ is continuous.
⟨2⟩2. LET: $(x, y) \in (X \times Y) \setminus G_f$
⟨2⟩3. PICK disjoint neighbourhoods $U$ and $V$ of $y$ and $f(x)$ respectively.
⟨2⟩4. $f^{-1}(V) \times U$ is a neighbourhood of $(x, y)$ disjoint from $G_f$.
⟨1⟩3. If $G_f$ is closed then $f$ is continuous.
⟨2⟩1. ASSUME: $G_f$ is closed.
⟨2⟩2. LET: $x \in X$ and $V$ be a neighbourhood of $f(x)$.
⟨2⟩3. $G_f \cap (X \times (Y \setminus V))$ is closed.
⟨2⟩4. $\pi_1(G_f \cap (X \times (Y \setminus V)))$ is closed.
PROOF: Proposition 1.351.
⟨2⟩5. LET: $U = X \setminus \pi_1(G_f \cap (X \times (Y \setminus V)))$
⟨2⟩6. $U$ is a neighbourhood of $x$
⟨2⟩7. $f(U) \subseteq V$
□

**Theorem 1.353.** *Let $X$ be a compact topological space. Let $(f_n : X \to \mathbb{R})$ be a monotone increasing sequence of continuous functions and $f : X \to \mathbb{R}$ a continuous function. If $(f_n)$ converges pointwise to $f$, then $(f_n)$ converges uniformly to $f$.*

PROOF:
⟨1⟩1. LET: $\epsilon > 0$
⟨1⟩2. For all $x \in X$, there exists $N$ such that, for all $n \geq N$, we have $|f_n(x) - f(x)| < \epsilon$
⟨1⟩3. For $n \geq 1$,
LET: $U_n = \{x \in X \mid |f_n(x) - f(x)| < \epsilon\}$
⟨1⟩4. For $n \geq 1$, we have $U_n$ is open in $X$.
⟨2⟩1. LET: $x \in X$
⟨2⟩2. LET: $\delta = \epsilon - |f_n(x) - f(x)|$
⟨2⟩3. PICK a neighbourhood $U$ of $x$ such that $f(U) \subseteq B(f(x), \delta/2)$

⟨2⟩4. PICK a neighbourhood $V$ of $x$ such that $f_n(V) \subseteq B(f_n(x), \delta/2)$

⟨2⟩5. $f(U \cap V) \subseteq U_n$

PROOF: For $y \in U \cap V$ we have

$$\begin{aligned} |f_n(y) - f(y)| &\leq |f_n(y) - f_n(x)| + |f_n(x) - f(x)| + |f(x) - y(y)| \\ &< \delta/2 + |f_n(x) - f(x)| + \delta/2 \\ &= \epsilon \end{aligned}$$

⟨1⟩5. $\{U_n \mid n \geq 1\}$ covers $X$

PROOF: From ⟨1⟩2

⟨1⟩6. PICK $N$ such that $X = U_N$

⟨2⟩1. PICK $n_1, \ldots, n_k$ such that $U_{n_1}, \ldots, U_{n_k}$ cover $X$.

⟨2⟩2. LET: $N = \max(n_1, \ldots, n_k)$

⟨2⟩3. For all $i$ we have $U_{n_i} \subseteq U_N$

PROOF: Since $(f_n)$ is monotone increasing.

⟨2⟩4. $X = U_N$

⟨1⟩7. For all $x \in X$ and $n \geq N$ we have $|f_n(x) - f(x)| < \epsilon$

□

An example to show that we cannot remove the hypothesis that $X$ is compact:

**Example 1.354.** Let $X = (0,1)$, $f_n(x) = -x^n$ and $f(x) = 0$ for $x \in X$ and $n \geq 1$. Then $f_n \to f$ pointwise and $(f_n)$ is monotone increasing but the convergence is not uniform since, for all $N \geq 1$, there exists $x \in (0,1)$ such that $-x^N < -1/2$.

An example to show that we cannot remove the hypothesis that $(f_n)$ is monotone increasing:

**Example 1.355.** Let $X = [0,1]$, $f_n(x) = 1/(n^3(x - 1/n)^2 + 1)$ and $f(x) = 0$ for $x \in X$ and $n \geq 1$. Then $X$ is compact and $f_n \to f$ pointwise but the convergence is not uniform since, for all $N \geq 1$, there exists $x \in [0,1]$ such that $f_N(x) = 1$, namely $x = 1/N$.

**Theorem 1.356.** *Let $X$ be a compact Hausdorff space. Let $\mathcal{A}$ be a chain of closed connected subsets of $X$. Then $\bigcap \mathcal{A}$ is connected.*

PROOF:

⟨1⟩1. ASSUME: for a contradiction $C$ and $D$ form a separation of $\bigcap \mathcal{A}$.

⟨1⟩2. PICK disjoint open sets $U$ and $V$ that include $C$ and $D$ respectively.

PROOF: Proposition 1.348.

⟨1⟩3. $\{A \setminus (U \cup V) \mid A \in \mathcal{A}\}$ is a set of closed sets with the finite intersection property.

⟨2⟩1. For all $A \in \mathcal{A}$ we have $A \setminus (U \cup V)$ is closed.

⟨2⟩2. For all $A_1, \ldots, A_n \in \mathcal{A}$ we have $(A_1 \cap \cdots \cap A_n) \setminus (U \cup V)$ is nonempty.

PROOF:

⟨3⟩1. LET: $A_1, \ldots, A_n \in \mathcal{A}$

⟨3⟩2. ASSUME: without loss of generality $A_1 \subseteq A_2, \ldots, A_n$

PROOF: Since $\mathcal{A}$ is a chain.

⟨3⟩3. $A_1 \setminus (U \cup V)$ is nonempty

PROOF: Otherwise $(A_1 \cap \cdots \cap A_n \cap U)$ and $(A_1 \cap \cdots \cap A_n \cap V)$ would form a separation of $A_n$.

⟨1⟩4. $\bigcap \mathcal{A} \setminus (U \cup V)$ is nonempty.

PROOF: Theorem 1.344.

⟨1⟩5. Q.E.D.

PROOF: This contradicts ⟨1⟩1 since $\bigcap AA \setminus (U \cup V) = \bigcap \mathcal{A} \setminus (C \cup D)$.

□

**Theorem 1.357** (Tychonoff Theorem (AC))**.** *The product of a family of compact spaces is compact.*

PROOF:

⟨1⟩1. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of compact spaces.

⟨1⟩2. LET: $X = \prod_{\alpha \in J} X_\alpha$

⟨1⟩3. For any $\mathcal{A} \subseteq \mathcal{P}X$, we have $\bigcap_{A \in \mathcal{A}} \overline{A} \neq \emptyset$

⟨2⟩1. LET: $\mathcal{A} \subseteq \mathcal{P}X$

⟨2⟩2. PICK $\mathcal{D} \supseteq \mathcal{A}$ that is maximal with respect to the finite intersection property.

PROVE: $\bigcap_{D \in \mathcal{D}} \overline{D} \neq \emptyset$

PROOF: Lemma 1.17.

⟨2⟩3. For $\alpha \in J$, PICK $x_\alpha \in X_\alpha$ such that $x_\alpha \in \bigcap_{D \in \mathcal{D}} \overline{\pi_\alpha(D)}$

PROOF: Theorem 1.344 since $\{\overline{\pi_\alpha(D)} \mid D \in \mathcal{D}\}$ is a set of closed sets in $X_\alpha$ with the finite intersection property.

⟨2⟩4. LET: $x = (x_\alpha)_{\alpha \in J}$

PROVE: $x \in \bigcap_{D \in \mathcal{D}} \overline{D}$

⟨2⟩5. For any $\beta \in J$ and neighbourhood $U$ of $x_\beta$ in $X_\beta$, we have ${\pi_\beta}^{-1}(U)$ intersects every element of $\mathcal{D}$

⟨3⟩1. LET: $\beta \in J$

⟨3⟩2. LET: $U$ be a neighbourhood of $x_\beta$ in $X_\beta$.

⟨3⟩3. LET: $D \in \mathcal{D}$

⟨3⟩4. $x_\beta \in \overline{\pi_\beta(D)}$

PROOF: From ⟨2⟩3

⟨3⟩5. $U$ intersects $\pi_\beta(D)$.

⟨3⟩6. ${\pi_\beta}^{-1}(U)$ intersects $D$.

⟨2⟩6. For any $\beta \in J$ and neighbourhood $U$ of $x_\beta$ in $X_\beta$, we have ${\pi_\beta}^{-1}(U) \in \mathcal{D}$

PROOF: Lemma 1.19.

⟨2⟩7. Every basic neighbourhood of $x$ is an element of $\mathcal{D}$

PROOF: Lemma 1.18.

⟨2⟩8. Every basic neighbourhood of $x$ intersects every element of $\mathcal{D}$

PROOF: Since $\mathcal{D}$ satisfies the finite intersection property.

⟨2⟩9. For all $D \in \mathcal{D}$ we have $x \in \overline{D}$

⟨1⟩4. Q.E.D.

PROOF: Theorem 1.344.

□

**Lemma 1.358.** *Let $X$ and $Y$ be topological spaces. Let $\mathcal{A}$ be a set of basis* *elements for the product topology on $X \times Y$ such that no finite subset of $\mathcal{A}$ covers $X \times Y$. If $X$ is compact, then there exists $x \in X$ such that no finite subset of $\mathcal{A}$ covers the slice $\{x\} \times Y$.*

PROOF:
⟨1⟩1. ASSUME: for every $x \in X$, there exists a finite subset of $\mathcal{A}$ that covers $\{x\} \times Y$
PROVE: A finite subset of $\mathcal{A}$ covers $X \times Y$
⟨1⟩2. $\{U \mid \exists U_1 \times V_1, \ldots, U_n \times V_n \in \mathcal{A}. U = U_1 \cap \cdots \cap U_n, V_1 \cup \cdots \cup V_n = Y\}$ covers $X$
⟨1⟩3. PICK a finite subcover $U_1, \ldots, U_m$
⟨1⟩4. PICK $U_{ij} \times V_{ij} \in \mathcal{A}$ such that, for every $i$, we have $U_i = \bigcap_j U_{ij}$ and $Y = \bigcup_j V_{ij}$
⟨1⟩5. The collection of all $U_{ij} \times V_{ij}$ covers $X \times Y$
□

**Theorem 1.359** (AC). *Let $X$ be a compact Hausdorff space. Then the quasicomponents and the components of $X$ are the same.*

PROOF:
⟨1⟩1. LET: $x, y \in X$
⟨1⟩2. ASSUME: $x$ and $y$ are in the same quasicomponent.
PROVE: $x$ and $y$ are in the same component.
⟨1⟩3. LET: $\mathcal{A}$ be the set of all closed subsets $A$ of $X$ such that $x$ and $y$ are in the same quasicomponent of $A$.
⟨1⟩4. For every chain $\mathcal{B} \subseteq \mathcal{A}$ we have $\bigcap \mathcal{B} \in \mathcal{A}$
⟨2⟩1. LET: $BB \subseteq \mathcal{A}$ be a chain.
⟨2⟩2. ASSUME: for a contradiction $U$ and $V$ form a separation of $\bigcap \mathcal{B}$ with $x \in U$ and $y \in V$
⟨2⟩3. PICK disjoint open sets $U'$, $V'$ in $X$ such that $U \subseteq U'$ and $V \subseteq V'$
⟨2⟩4. $\{B \setminus (U' \cup V') \mid B \in \mathcal{B}\}$ satisfies the finite intersection property.
⟨3⟩1. LET: $B_1, \ldots, B_n \in \mathcal{B}$
⟨3⟩2. ASSUME: without loss of generality $B_1 \subseteq \cdots \subseteq B_n$
PROOF: Since $\mathcal{B}$ is a chain.
⟨3⟩3. $\bigcap \{B_1 \setminus (U' \cup V'), \ldots, B_n \setminus (U' \cup V')\} = B_1 \setminus (U' \cup V')$
⟨3⟩4. $B_1 \setminus (U' \cup V')$ is nonempty
PROOF: Otherwise $B_1 \cap U'$ and $B_1 \cap V'$ would form a separation of $B_1$, contradicting the fact that $x$ and $y$ are in the same quasicomponent of $B_1$.
⟨2⟩5. $\bigcap \mathcal{B} \setminus (U \cup V)$ is nonempty
PROOF: Theorem 1.344.
⟨2⟩6. Q.E.D.
PROOF: This contradicts ⟨2⟩2.
⟨1⟩5. PICK a minimal element $D$ in $\mathcal{A}$.
PROVE: $D$ is connected.
PROOF: By Zorn's Lemma.
⟨1⟩6. ASSUME: for a contradiction $U$ and $V$ form a separation of $D$.

⟨1⟩7. ASSUME: without loss of generality $x, y \in U$

PROOF: We cannot have that one of $x$, $y$ is in $U$ and the other in $V$ sicnce $D \in \mathcal{A}$.

⟨1⟩8. $U \in \mathcal{A}$

PROOF: If $X$ and $Y$ form a separation of $U$ with $x \in X$ and $y \in Y$, then $X$ and $Y \cup V$ form a separation of $D$ with $x \in X$ and $y \in Y \cup V$.

⟨1⟩9. Q.E.D.

PROOF: There is a connected set $D$ that contains both $x$ and $y$.

$\square$

PROOF:

⟨1⟩1. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of compact spaces.

⟨1⟩2. LET: $X = \prod_{\alpha \in J} X_\alpha$

⟨1⟩3. PICK a well-ordering $<$ on $J$ such that $J$ has a greatest element.

⟨1⟩4. For $\alpha \in J$ and $p = \{p_i \in X_i\}_{i \leq \alpha}$ a family of points,
LET: $Y(p) = \{x \in X \mid \forall i \leq \alpha . x_i = p_i\}$

⟨1⟩5. If $\alpha < \alpha'$ and $p$ is an $\alpha'$-indexed family of points then $Y(p) \subseteq Y(p \upharpoonright \alpha)$

PROOF: From definition.

⟨1⟩6. Given $\beta \in J$ and $p = \{p_i \in X_i\}_{i < \beta}$ a family of points,
LET: $Z(p) = \bigcap_{\alpha < \beta} Y(p \upharpoonright \alpha)$

⟨1⟩7. Given $\beta \in J$ and $p = \{p_i \in X_i\}_{i < \beta}$ a family of points, if $\mathcal{A}$ is a finite set of basic open spaces for $X$ that covers $Z(p)$, then there exists $\alpha < \beta$ such that $\mathcal{A}$ covers $Y(p \upharpoonright \alpha)$

⟨2⟩1. ASSUME: without loss of generality $\beta$ has no immediate predecessor.

⟨2⟩2. For $A \in \mathcal{A}$,
LET: $J_A = \{i < \beta \mid \pi_i(A) \neq X_i\}$

⟨2⟩3. LET: $\alpha = \max \bigcup_{A \in \mathcal{A}} J_A$

⟨2⟩4. LET: $x \in Y(p \upharpoonright \alpha)$

⟨2⟩5. LET: $y \in Z(p)$ be the point with $y_i = p_i$ for $i < \beta$ and $y_i = x_i$ for $i \geq \beta$

⟨2⟩6. PICK $A \in \mathcal{A}$ such that $y \in A$

PROOF: Since $\mathcal{A}$ covers $Z(p)$.

⟨2⟩7. For $i \in J_A$ we have $x_i \in \pi_i(A)$

PROOF: Since $i \leq \alpha$ so $x_i = p_i$

⟨2⟩8. For $i \in J \setminus J_A$ we have $x_i \in \pi_i(A)$

PROOF: Since $\pi_i(A) = X_i$

⟨2⟩9. $x \in A$

⟨1⟩8. ASSUME: for a contraction $\mathcal{A}$ is a set of basic open sets for $X$ that covers $X$ but such that no finite subset of $\mathcal{A}$ covers $X$

⟨1⟩9. PICK a set of points $\{p_i\}_{i \in J}$ such that, for all $\alpha \in J$, we have $Y(p \upharpoonright \alpha)$ is not finitely covered by $\mathcal{A}$

⟨2⟩1. ASSUME: as transfinite induction hypothesis $\alpha \in J$ and $\{p_i\}_{i < \alpha}$ is a family of points such that, for all $\alpha' < \alpha$, we have $Y(p \upharpoonright \alpha')$ is not finitely covered by $\mathcal{A}$

⟨2⟩2. $Z(p)$ is not finitely covered by $\mathcal{A}$

PROOF: By ⟨1⟩7.

⟨2⟩3. PICK $p_\alpha \in X_\alpha$ such that $Y(p)$ is not finitely covered by $\mathcal{A}$

Proof: By Lemma 1.358 since there is a homeomorphism $\phi : Z(p) \cong X_\alpha \times \prod_{\alpha' > \alpha} X_{\alpha'}$ and, given $p_\alpha$, this homemorphism $\phi$ restricts to a homeomorphism $Y(p) \cong \{p_\alpha\} \times \prod_{\alpha' > \alpha} X_{\alpha'}$.

⟨1⟩10. Q.E.D.

Proof: If $\omega$ is the greatest element of $J$ then $Y(p \restriction \omega)$ is a singleton.

□

**Theorem 1.360.** *Every complete linearly ordered set in the order topology is compact.*

Proof:

⟨1⟩1. Let: $X$ be a complete linearly ordered set with least element $a$ and greatest element $b$.

⟨1⟩2. Let: $\mathcal{A}$ be an open covering of $X$.

⟨1⟩3. For all $x < b$, there exists $y > x$ such that $[x, y]$ can be covered by at most two elements of $\mathcal{A}$.

⟨2⟩1. Let: $x \in X$

⟨2⟩2. Pick $A \in \mathcal{A}$ with $x \in A$

⟨2⟩3. Pick $y > x$ such that $[x, y) \subseteq A$

⟨2⟩4. Pick $B \in \mathcal{A}$ with $y \in B$

⟨2⟩5. $[x, y]$ is covered by $A$ and $B$

⟨1⟩4. Let: $C = \{y \in X \mid [a, y] \text{ can be covered by finitely many elements of } \mathcal{A}\}$

⟨1⟩5. Let: $c = \sup C$

⟨1⟩6. $c > a$

⟨2⟩1. Pick $x > a$ such that $[a, x]$ can be covered by at most two elements of $\mathcal{A}$.

Proof: From ⟨1⟩3.

⟨2⟩2. $x \in C$

⟨1⟩7. $c \in C$

⟨2⟩1. Pick $A \in \mathcal{A}$

⟨2⟩2. Pick $x < c$ such that $(x, c] \subseteq A$

⟨2⟩3. Pick $y > x$ such that $y \in C$

⟨2⟩4. Pick $\mathcal{A}_0 \subseteq^{\text{fin}} \mathcal{A}$ that covers $[a, y]$

⟨2⟩5. $\mathcal{A}_0 \cup \{A\}$ covers $[a, c]$

⟨1⟩8. $c = b$

⟨2⟩1. Assume: for a contradiction $c < b$

⟨2⟩2. Pick $x > c$ such that $[c, x]$ can be covered by at most two elements of $\mathcal{A}$

Proof: From ⟨1⟩3.

⟨2⟩3. $[a, x]$ can be finitely covered by $\mathcal{A}$

Proof: From ⟨1⟩7.

⟨2⟩4. Q.E.D.

Proof: This contradicts the maximality of $c$.

□

**Corollary 1.360.1.** *Let $X$ be a linearly ordered set with the least upper bound property. Then every closed interval in $X$ is compact.*

**Corollary 1.360.2.** *Every closed interval in $\mathbb{R}$ is compact.*

**Theorem 1.361** (Extreme Value Theorem)**.** *Any linearly ordered set under the order topology that is compact has a greatest and a least element.*

PROOF:
$\langle 1 \rangle 1$. LET: $X$ be a linearly ordered set under the order topology that is compact.
$\langle 1 \rangle 2$. $X$ has a greatest element.
  $\langle 2 \rangle 1$. ASSUME: for a contradiction $X$ has no greatest element.
  $\langle 2 \rangle 2$. $\{(-\infty, a) \mid a \in X\}$ covers $X$.
  $\langle 2 \rangle 3$. PICK a finite subcover $\{(-\infty, a_1), \ldots, (-\infty, a_n)\}$, say.
  $\langle 2 \rangle 4$. ASSUME: without loss of generality $a_1 \leq \cdots \leq a_n$
  $\langle 2 \rangle 5$. $X \subseteq (-\infty, a_n)$
  $\langle 2 \rangle 6$. $a_n < a_n$
$\langle 1 \rangle 3$. $X$ has a least element.
  PROOF: Similar.
□

## 1.59 Perfect Maps

**Definition 1.362** (Perfect Map)**.** Let $X$ and $Y$ be topological spaces and $f : X \to Y$. Then $f$ is a *perfect map* if and only if $f$ is a closed map, continuous, surjective and $p^{-1}(y)$ is compact for all $y \in Y$.

**Proposition 1.363.** *Let $X$ be a topological space, $Y$ a compact space, and $p : X \to Y$ a closed map such that, for all $y \in Y$, we have $p^{-1}(y)$ is compact. Then $X$ is compact.*

PROOF:
$\langle 1 \rangle 1$. LET: $\mathcal{A}$ be a set of closed sets in $X$ with the finite intersection property.
$\langle 1 \rangle 2$. $\mathcal{B} = \{p(A_1 \cap \cdots \cap A_n) \mid A_1, \ldots, A_n \in \mathcal{A}\}$ is a set of closed sets in $Y$ with the finite intersection property.
  PROOF: Since $p$ is a closed map.
$\langle 1 \rangle 3$. PICK $y \in \bigcap \mathcal{B}$
  PROOF: Theorem 1.344 since $Y$ is compact.
$\langle 1 \rangle 4$. $\{A \cap p^{-1}(y) \mid A \in \mathcal{A}\}$ is a set of closed sets in $p^{-1}(y)$ with the finite intersection property.
$\langle 1 \rangle 5$. PICK $x \in \bigcap_{A \in \mathcal{A}} (A \cap p^{-1}(y))$
  PROOF: Theorem 1.344 since $p^{-1}(y)$ is compact.
$\langle 1 \rangle 6$. $x \in \bigcap \mathcal{A}$
$\langle 1 \rangle 7$. Q.E.D.
  PROOF: Theorem 1.344.
□

## 1.60 Topological Groups

**Definition 1.364** (Topological Group)**.** A *topological group* $G$ consists of a $T_1$ space $G$ and continuous maps $\cdot : G^2 \to G$ and $(\ )^{-1} : G \to G$ such that $(G, \cdot, (\ )^{-1})$ is a group.

**Example 1.365.**

1. The integers $\mathbb{Z}$ under addition are a topological group.
2. The real numbers $\mathbb{R}$ under addition are a topological group.
3. The positive reals under multiplication are a topological group.
4. The set $\{z \in \mathbb{C} \mid |z| = 1\}$ under multiplication and given the topology of $S^1$ is a topological group.
5. For any $n \geq 0$, the general linear group $GL_n(\mathbb{R})$ is a topological group under matrix multiplication, considered as a subspace of $\mathbb{R}^{n^2}$.

**Lemma 1.366.** *Let $G$ be a $T_1$ space and $\cdot : G^2 \to G$, $(\ )^{-1} : G \to G$ be functions such that $(G, \cdot, (\ )^{-1})$ is a group. Then $G$ is a topological group if and only if the function $f : G^2 \to G$ that maps $(x, y)$ to $xy^{-1}$ is continuous.*

Proof:
$\langle 1 \rangle 1$. If $G$ is a topological group then $f$ is continuous.
  Proof: From Theorem 1.130.
$\langle 1 \rangle 2$. If $f$ is continuous then $G$ is a topological group.
  $\langle 2 \rangle 1$. Assume: $f$ is continuous.
  $\langle 2 \rangle 2$. $(\ )^{-1}$ is continuous.
    Proof: Since $x^{-1} = f(e, x)$.
  $\langle 2 \rangle 3$. $\cdot$ is continuous.
    Proof: Since $xy = f(x, y^{-1})$.
∎

**Lemma 1.367.** *Let $G$ be a topological group and $H$ a subgroup of $G$. Then $H$ is a topological group under the subspace topology.*

Proof:
$\langle 1 \rangle 1$. $H$ is $T_1$.
  Proof: From Proposition 1.200.
$\langle 1 \rangle 2$. multiplication and inverse on $H$ are continuous.
  Proof: From Theorem 1.131.
∎

**Lemma 1.368.** *Let $G$ be a topological group and $H$ a subgroup of $G$. Then $\overline{H}$ is a subgroup of $G$.*

Proof:
$\langle 1 \rangle 1$. Let: $x, y \in \overline{H}$
    Prove: $xy^{-1} \in \overline{H}$

⟨1⟩2. LET: $U$ be any neighbourhood of $xy^{-1}$
⟨1⟩3. LET: $f : G^2 \to G$, $f(a,b) = ab^{-1}$
⟨1⟩4. $f^{-1}(U)$ is a neighbourhood of $(x,y)$
⟨1⟩5. PICK neighbourhoods $V$, $W$ of $x$ and $y$ respectively such that $f(V \times W) \subseteq U$.
⟨1⟩6. PICK $a \in V \cap H$ and $b \in W \cap H$
PROOF: Theorem 1.81.
⟨1⟩7. $ab^{-1} \in U \cap H$
⟨1⟩8. Q.E.D.
PROOF: By Theorem 1.81.
□

**Proposition 1.369.** *Let $G$ be a topological group and $\alpha \in G$. Then the maps $l_\alpha, r_\alpha : G \to G$ defined by $l_\alpha(x) = \alpha x$, $r_\alpha(x) = x\alpha$ are homeomorphisms of $G$ with itself.*

PROOF: They are continuous with continuous inverses $l_{\alpha^{-1}}$ and $r_{\alpha^{-1}}$. □

**Corollary 1.369.1.** *Every topological group is homogeneous.*

PROOF: Given a topological group $G$ and $a, b \in G$, we have $l_{ba^{-1}}$ is a homeomorphism that maps $a$ to $b$. □

**Proposition 1.370.** *Let $G$ be a topological group and $H$ a subgroup of $G$. Give $G/H$ the quotient topology. For all $\alpha \in G$, the map $\overline{f_\alpha}$ that sends $xH$ to $\alpha xH$ is a homeomorphism $G/H \cong G/H$.*

PROOF:
⟨1⟩1. $\overline{f_\alpha}$ is well-defined.
PROOF: If $xy^{-1} \in H$ then $(\alpha x)(\alpha y)^{-1} = xy^{-1} \in H$.
⟨1⟩2. $\overline{f_\alpha}$ is continuous.
PROOF: Theorem 1.234 since $\overline{f_\alpha} \circ p = p \circ f_\alpha$ is continuous, where $p : G \twoheadrightarrow G/H$ is the canonical surjection.
⟨1⟩3. $\overline{f_\alpha}^{-1}$ is continuous.
PROOF: Similar since $\overline{f_\alpha}^{-1} = \overline{f_{\alpha^{-1}}}$.
□

**Corollary 1.370.1.** *Let $G$ be a topological group and $H$ a subgroup of $G$. Give $G/H$ the quotient topology. Then $G/H$ is homogeneous.*

**Proposition 1.371.** *Let $G$ be a topological group and $H$ a subgroup of $G$. Give $G/H$ the quotient topology. If $H$ is closed in $G$ then $G/H$ is $T_1$.*

PROOF:
⟨1⟩1. LET: $p : G \twoheadrightarrow G/H$ be the canonical surjection
⟨1⟩2. LET: $x \in G$
⟨1⟩3. $p^{-1}(xH) = f_x(H)$
⟨1⟩4. $p^{-1}(xH)$ is closed in $G$
PROOF: Since $H$ is closed and $f_x$ is a homemorphism of $G$ with itself.

$\langle 1\rangle 5$. $\{xH\}$ is closed in $G/H$
□

**Proposition 1.372.** *Let $G$ be a topological group and $H$ a subgroup of $G$. Give $G/H$ the quotient topology. Then the canonical surjection $p : G \twoheadrightarrow G/H$ is an open map.*

PROOF:
$\langle 1\rangle 1$. LET: $U \subseteq G$ be open.
$\langle 1\rangle 2$. $p^{-1}(p(U)) = \bigcup_{h \in H} r_h(U)$
$\langle 1\rangle 3$. $p^{-1}(p(U))$ is open.
$\langle 1\rangle 4$. $p(U)$ is open.
□

**Proposition 1.373.** *Let $G$ be a topological group and $H$ a closed normal subgroup of $G$. Then $G/H$ is a topological group under the quotient topology.*

PROOF:
$\langle 1\rangle 1$. $G/H$ is $T_1$
  PROOF: Proposition 1.371.
$\langle 1\rangle 2$. The map $\overline{m} : (xH, yH) \mapsto xy^{-1}H$ is continuous.
  $\langle 2\rangle 1$. $p^2 : G^2 \to (G/H)^2$ is a quotient map.
    PROOF: Propositions 1.233, 1.372.
  $\langle 2\rangle 2$. $\overline{m} \circ p^2$ is continuous.
    PROOF: As it is $p^2 \circ m$ where $m : G^2 \to G$ with $m(x, y) = xy^{-1}$
□

**Lemma 1.374.** *Let $G$ be a topological group and $A, B \subseteq G$. If either $A$ or $B$ is open then $AB$ is open.*

PROOF: If $A$ is open we have $AB = \bigcup_{b \in B} r_b(A)$. Similarly if $B$ is open. □

**Definition 1.375** (Symmetric Neighbourhood)**.** Let $G$ be a topological group. A neighbourhood $V$ of $e$ is *symmetric* if and only if $V = V^{-1}$.

**Lemma 1.376.** *Let $G$ be a topological group. Let $V$ be a neighbourhood of $e$. Then $V$ is symmetric if and only if, for all $x \in V$, we have $x^{-1} \in V$.*

PROOF:
$\langle 1\rangle 1$. If $V$ is symmetric then, for all $x \in V$, we have $x^{-1} \in V$
  PROOF: Immediate from defintions.
$\langle 1\rangle 2$. If, for all $x \in V$, we have $x^{-1} \in V$, then $V$ is symmetric.
  $\langle 2\rangle 1$. ASSUME: for all $x \in V$ we have $x^{-1} \in V$
  $\langle 2\rangle 2$. $V \subseteq V^{-1}$
    PROOF: If $x \in V$ then there exists $y \in V$ such that $x = y^{-1}$, namely $y = x^{-1}$
  $\langle 2\rangle 3$. $V^{-1} \subseteq V$
    PROOF: Immediate from $\langle 2\rangle 1$.
□

**Lemma 1.377.** *Let $G$ be a topological group. For every neighbourhood $U$ of $e$, there exists a symmetric neighbourhood $V$ of $e$ such that $V^2 \subseteq U$.*

PROOF:
$\langle 1\rangle 1$. LET: $U$ be a neighbourhood of $e$.
$\langle 1\rangle 2$. PICK a neighbourhood $V'$ of $e$ such that $V'V' \subseteq U$
PROOF: Such a neighbourhood exists because multiplication in $G$ is continuous.
$\langle 1\rangle 3$. PICK a neighbourhood $W$ of $e$ such that $WW^{-1} \subseteq V'$
PROOF: Such a neighbourhood exists because the function that maps $(x, y)$ to $xy^{-1}$ is continuous.
$\langle 1\rangle 4$. LET: $V = WW^{-1}$
$\langle 1\rangle 5$. $V$ is a neighbourhood of $e$
$\langle 2\rangle 1$. $e \in V$
PROOF: Since $e \in W$ so $e = ee^{-1} \in V$.
$\langle 2\rangle 2$. $V$ is open
PROOF: Lemma 1.374.
$\langle 1\rangle 6$. $V$ is symmetric
$\langle 2\rangle 1$. For all $x \in V$ we have $x^{-1} \in V$
$\langle 3\rangle 1$. LET: $x \in V$
$\langle 3\rangle 2$. PICK$y, z \in W$ such that $x = yz^{-1}$
$\langle 3\rangle 3$. $x^{-1} = zy^{-1}$
$\langle 3\rangle 4$. $x^{-1} \in V$
$\langle 3\rangle 5$. $x \in V^{-1}$
$\langle 2\rangle 2$. Q.E.D.
PROOF: Lemma 1.376
$\langle 1\rangle 7$. $V^2 \subseteq U$
PROOF: We have $V^2 \subseteq (V')^2 \subseteq U$
□

**Proposition 1.378.** *Every topological group is Hausdorff.*

PROOF:
$\langle 1\rangle 1$. LET: $G$ be a topological group.
$\langle 1\rangle 2$. LET: $x, y \in G$ with $x \neq y$
$\langle 1\rangle 3$. LET: $U = G \setminus \{x[^{-1}y]\}$
$\langle 1\rangle 4$. PICK a symmetric neighbourhood $V$ of $e$ such that $VV \subseteq U$
$\langle 2\rangle 1$. $U$ is open
PROOF: Since $G$ is $T_1$.
$\langle 2\rangle 2$. $e \in U$
PROOF: Since $x \neq y$
$\langle 2\rangle 3$. Q.E.D.
PROOF: Lemma 1.377.
$\langle 1\rangle 5$. $Vx$ and $Vy$ are disjoint neighbourhoods of $x$ and $y$ respectively.
$\langle 2\rangle 1$. $Vx$ is open
PROOF: Since $Vx = r_x(V)$
$\langle 2\rangle 2$. $Vy$ is open

PROOF: Similar.
$\langle 2\rangle 3.$ $Vx \cap Vy = \emptyset$
$\langle 3\rangle 1.$ ASSUME: for a contradiction $z \in Vx \cap Vy$
$\langle 3\rangle 2.$ PICK $a, b \in V$ such that $z = ax = by$
$\langle 3\rangle 3.$ $xy^{-1} \in VV$
PROOF: Since $xy^{-1} = a^{-1}b$
$\langle 3\rangle 4.$ $xy^{-1} \in U$
$\langle 3\rangle 5.$ Q.E.D.
PROOF: From $\langle 1\rangle 3$.
□

**Proposition 1.379.** *Every topological group is regular.*

PROOF:
$\langle 1\rangle 1.$ LET: $G$ be a topological group.
$\langle 1\rangle 2.$ LET: $A \subseteq G$ be a closed set and $a \notin A$.
$\langle 1\rangle 3.$ LET: $U = G \setminus Aa^{-1}$
$\langle 1\rangle 4.$ PICK a symmetric neighbourhood $V$ of $e$ such that $VV \subseteq U$
$\langle 2\rangle 1.$ $U$ is open
PROOF: Since $Aa^{-1} = r_{a^{-1}}(A)$ is closed.
$\langle 2\rangle 2.$ $e \in U$
PROOF: Since $a \notin A$.
$\langle 2\rangle 3.$ Q.E.D.
PROOF: Lemma 1.377.
$\langle 1\rangle 5.$ $VA$ and $Va$ are disjoint open sets with $A \subseteq VA$ and $a \in Va$
$\langle 2\rangle 1.$ $VA$ is open
PROOF: Lemma 1.374
$\langle 2\rangle 2.$ $Va$ is open
PROOF: Lemma 1.374
$\langle 2\rangle 3.$ $VA \cap Va = \emptyset$
$\langle 3\rangle 1.$ ASSUME: for a contradiction $z \in VA \cap Va$
$\langle 3\rangle 2.$ PICK $b, c \in V$ and $d \in A$ with $z = bd = ca$
$\langle 3\rangle 3.$ $da^{-1} \in U$
PROOF: Since $da^{-1} = b^{-1}c \in VV \subseteq U$
$\langle 3\rangle 4.$ Q.E.D.
PROOF: This contradicts $\langle 1\rangle 3$
□

**Proposition 1.380.** *Let $G$ be a topological group and $H$ a subgroup of $G$. Give $G/H$ the quotient topology. If $H$ is closed in $G$ then $G/H$ is regular.*

PROOF:
$\langle 1\rangle 1.$ LET: $p : G \twoheadrightarrow G/H$ be the canonical surjection.
$\langle 1\rangle 2.$ LET: $A$ be a closed set in $G/H$ and $aH \in (G/H) \setminus A$.
$\langle 1\rangle 3.$ LET: $B = p^{-1}(A)$
$\langle 1\rangle 4.$ $B$ is a closed saturated set in $G$.
$\langle 1\rangle 5.$ $B \cap aH = \emptyset$

$\langle 1\rangle 6.$ $B = BH$
$\langle 1\rangle 7.$ Pick a symmetric neighbourhood $V$ of $e$ such that $VB$ does not intersect $Va$
$\langle 2\rangle 1.$ Let: $U = G \setminus Ba^{-1}$
$\langle 2\rangle 2.$ Pick a symmetric neighbourhood $V$ of $e$ such that $VV \subseteq U$
$\langle 3\rangle 1.$ $U$ is open
Proof: Since $Ba^{-1} = r_{a^{-1}}(B)$ is closed.
$\langle 3\rangle 2.$ $e \in U$
Proof: If $e \in Ba^{-1}$ then $a \in B$
$\langle 3\rangle 3.$ Q.E.D.
Proof: Lemma 1.377
$\langle 2\rangle 3.$ $VB \cap Va = \emptyset$
Proof: If $vb = v'a$ for $v, v' \in V$ and $b \in B$ then we have $ba^{-1} = v^{-1}v' \in Ba \cap VV \subseteq Ba \cap U$.
$\langle 1\rangle 8.$ $p(VB)$ and $p(Va)$ are disjoint open sets
$\langle 2\rangle 1.$ $p(VB)$ and $p(Va)$ are open.
Proof: Proposition 1.372.
$\langle 2\rangle 2.$ $p(VB) \cap p(Va) = \emptyset$
Proof: If $vbH = v'aH$ for $v, v' \in V$, $b \in B$ then $v'a = vbh$ for some $h \in H$. Hence $v'a \in Va \cap VBH = Va \cap VB$.
$\langle 1\rangle 9.$ $A \subseteq p(VB)$
$\langle 1\rangle 10.$ $aH \in p(Va)$
□

**Proposition 1.381.** *Let $G$ be a topological group. The component of $G$ that contains $e$ is a normal subgroup of $G$.*

Proof:
$\langle 1\rangle 1.$ Let: $C$ be the component of $G$ that contains $e$.
$\langle 1\rangle 2.$ For all $x \in G$, $xC$ is the component of $G$ that contains $x$.
$\langle 2\rangle 1.$ Let: $x \in G$
$\langle 2\rangle 2.$ Let: $D$ be the component of $G$ that contains $x$.
$\langle 2\rangle 3.$ $xC \subseteq D$
Proof: Since $xC$ is connected by Theorem 1.258.
$\langle 2\rangle 4.$ $D \subseteq xC$
Proof: Since $x^{-1}D \subseteq C$ similarly.
$\langle 1\rangle 3.$ For all $x \in G$, $Cx$ is the component of $G$ that contains $x$.
Proof: Similar.
$\langle 1\rangle 4.$ For all $x \in C$ we have $xC = Cx = C$
$\langle 1\rangle 5.$ For all $x \in C$ we have $x^{-1}C = C$
$\langle 1\rangle 6.$ For all $x \in C$ we have $x^{-1} \in C$
$\langle 1\rangle 7.$ For all $x, y \in C$ we have $xy \in C$
Proof: Since $xyC = xC = x$.
$\langle 1\rangle 8.$ For all $x \in G$ we have $xC = Cx$.
Proof: From $\langle 1\rangle 2$ and $\langle 1\rangle 3$.
□

**Lemma 1.382.** *Let $G$ be a topological group. Let $A$ be a closed set in $G$ and $B$*

*a compact subspace of $G$ such that $A \cap B = \emptyset$. Then there exists a symmetric neighbourhood $U$ of $e$ such that $AU \cap BU = \emptyset$.*

PROOF:
$\langle 1 \rangle 1$. For all $b \in B$ there exists a symmetric neighbourhood $V$ of $e$ such that $bV^2 \cap A = \emptyset$
  $\langle 2 \rangle 1$. LET: $b \in B$
  $\langle 2 \rangle 2$. LET: $W = b^{-1}(G \setminus A)$
  $\langle 2 \rangle 3$. $W$ is a neighbourhood of $e$ and $bW \cap A = \emptyset$
  $\langle 2 \rangle 4$. PICK a symmetric neighbourhood $V$ of $e$ such that $V^2 \subseteq W$
$\langle 1 \rangle 2$. $\{bV^2 \mid b \in B, V \text{ is a symmetric neighbourhood of } e, bV^2 \cap A = \emptyset\}$ is an open cover of $B$
$\langle 1 \rangle 3$. PICK a finite subcover $b_1 V_1^2, \ldots, b_n V_n^2$, say.
$\langle 1 \rangle 4$. LET: $U = V_1 \cap \cdots \cap V_n$
$\langle 1 \rangle 5$. $BU^2 \cap A = \emptyset$
$\langle 1 \rangle 6$. $AU \cap BU = \emptyset$
  PROOF: If $av \in BU$ where $a \in A$ and $v \in V$ then $a = avv^{-1} \in BU^2 \cap A$.
□

**Proposition 1.383** (AC)**.** *Let $G$ be a topological group. Let $A$ be a closed set in $G$, and $B$ a compact subspace of $G$. Then $AB$ is closed.*

PROOF:
$\langle 1 \rangle 1$. LET: $x \in G \setminus AB$
$\langle 1 \rangle 2$. $A^{-1}x \cap B = \emptyset$
$\langle 1 \rangle 3$. $A^{-1}x$ is closed.
$\langle 1 \rangle 4$. PICK a symmetric neighbourhood $U$ of $e$ such that $A^{-1}xU \cap BU = \emptyset$
$\langle 1 \rangle 5$. $xU^2$ is open
  PROOF: Lemma 1.374.
$\langle 1 \rangle 6$. $x \in xU^2 \subseteq G \setminus AB$
□

**Corollary 1.383.1.** *Let $G$ be a topological group and $H \leq G$. Let $p : G \twoheadrightarrow G/H$ be the quotient map. If $H$ is compact then $p$ is a closed map.*

PROOF: For $A$ closed in $G$, we have $p^{-1}(p(A)) = AH$ is closed, and so $p(A)$ is closed. □

**Corollary 1.383.2.** *Let $G$ be a topological group and $H \leq G$. If $H$ and $G/H$ are compact then $G$ is compact.*

PROOF: From Proposition 1.363 since, for all $aH \in G/H$, we have $p^{-1}(aH) = aH$ is compact because it is homemorphic to $H$. □

## 1.61 The Metric Topology

**Definition 1.384** (Metric)**.** Let $X$ be a set. A *metric* on $X$ is a function $d : X^2 \to \mathbb{R}$ such that:

1. For all $x, y \in X$, $d(x, y) \geq 0$
2. For all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$
3. For all $x, y \in X$, $d(x, y) = d(y, x)$
4. (*Triangle Inequality*) For all $x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$

We call $d(x, y)$ the *distance* between $x$ and $y$.

**Definition 1.385** (Open Ball)**.** Let $X$ be a metric space. Let $a \in X$ and $\epsilon > 0$. The *open ball* with *centre* $a$ and *radius* $\epsilon$ is

$$B(a, \epsilon) = \{x \in X \mid d(a, x) < \epsilon\} \enspace .$$

**Definition 1.386** (Metric Topology)**.** Let $X$ be a metric space. The *metric topology* on $X$ is the topology generated by the basis consisting of all the open balls.

We prove this is a basis for a topology.

Proof:
⟨1⟩1. For every point $a$, there exists a ball $B$ such that $a \in B$
Proof: We have $a \in B(a, 1)$.
⟨1⟩2. For any balls $B_1$, $B_2$ and point $a \in B_1 \cap B_2$, there exists a ball $B_3$ such that $a \in B_3 \subseteq B_1 \cap B_2$
⟨2⟩1. Let: $B_1 = B(c_1, \epsilon_1)$ and $B_2 = B(c_2, \epsilon_2)$
⟨2⟩2. Let: $\delta = \min(\epsilon_1 - d(c_1, a), \epsilon_2 - d(c_2, a))$
Prove: $B(a, \delta) \subseteq B_1 \cap B_2$
⟨2⟩3. Let: $x \in B(a, \delta)$
⟨2⟩4. $x \in B_1$
Proof:

$$\begin{aligned} d(x, c_1) &= d(x, a) + d(a, c_1) \\ &< \delta + d(a, c_1) \\ &\leq \epsilon_1 \end{aligned}$$

⟨2⟩5. $x \in B_2$
Proof: Similar.
□

**Proposition 1.387.** *Let $X$ be a metric space and $U \subseteq X$. Then $U$ is open if and only if, for every $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$.*

Proof:
⟨1⟩1. If $U$ is open then, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$.
⟨2⟩1. Assume: $U$ is open.

⟨2⟩2. LET: $x \in U$
⟨2⟩3. PICK $a \in X$ and $\delta > 0$ such that $x \in B(a, \delta) \subseteq U$
⟨2⟩4. LET: $\epsilon = \delta - d(a, x)$
PROVE: $B(x, \epsilon) \subseteq U$
⟨2⟩5. LET: $y \in B(x, \epsilon)$
⟨2⟩6. $d(y, a) < \delta$
PROOF:

$$\begin{aligned} d(y,a) &\leq d(a,x) + d(x,y) \\ &< \delta + d(x,y) \\ &= \epsilon \end{aligned}$$

⟨2⟩7. $y \in U$
⟨1⟩2. If, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$, then $U$ is open.
PROOF: Immediate from definitions.
□

**Definition 1.388** (Discrete Metric)**.** Let $X$ be a set. The *discrete metric* on $X$ is defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

**Proposition 1.389.** *The discrete metric induces the discrete topology.*

PROOF: For any (open) set $U$ and point $a \in U$, we have $a \in B(a, 1) \subseteq U$. □

**Definition 1.390** (Standard Metric on $\mathbb{R}$)**.** The *standard metric* on $\mathbb{R}$ is defined by $d(x, y) = |x - y|$.

**Proposition 1.391.** *The standard metric on $\mathbb{R}$ induces the standard topology on $\mathbb{R}$.*

PROOF:
⟨1⟩1. Every open ball is open in the standard topology on $\mathbb{R}$.
PROOF: $B(a, \epsilon) = (a - \epsilon, a + \epsilon)$
⟨1⟩2. For every open set $U$ and point $a \in U$, there exists $\epsilon > 0$ such that $B(a, \epsilon) \subseteq U$
⟨2⟩1. LET: $U$ be an open set and $a \in U$
⟨2⟩2. PICK an open interval $b$, $c$ such that $a \in (b, c) \subseteq U$
⟨2⟩3. LET: $\epsilon = \min(a - b, c - a)$
⟨2⟩4. $B(a, \epsilon) \subseteq U$
□

**Definition 1.392** (Metrizable)**.** A topological space $X$ is *metrizable* if and only if there exists a metric on $X$ that induces the topology.

**Definition 1.393** (Bounded)**.** Let $X$ be a metric space and $A \subseteq X$. Then $A$ is *bounded* if and only if there exists $M$ such that, for all $x, y \in A$, we have $d(x, y) \leq M$.

**Definition 1.394** (Diameter)**.** Let $X$ be a metric space and $A \subseteq X$. The *diameter* of $A$ is

$$\operatorname{diam} A = \sup_{x,y \in A} d(x,y) \ .$$

**Definition 1.395** (Standard Bounded Metric)**.** Let $d$ be a metric on $X$. The *standard bounded metric* corresponding to $d$ is the metric $\overline{d}$ defined by

$$\overline{d}(x,y) = \min(d(x,y), 1) \ .$$

We prove this is a metric.

PROOF:
⟨1⟩1. $\overline{d}(x,y) \geq 0$
  PROOF: Since $d(x,y) \geq 0$
⟨1⟩2. $\overline{d}(x,y) = 0$ if and only if $x = y$
  PROOF: $\overline{d}(x,y) = 0$ if and only if $d(x,y) = 0$ if and only if $x = y$
⟨1⟩3. $\overline{d}(x,y) = \overline{d}(y,x)$
  PROOF: Since $d(x,y) = d(y,x)$
⟨1⟩4. $\overline{d}(x,z) \leq \overline{d}(x,y) + \overline{d}(y,z)$
  PROOF:

$$\begin{aligned}
\overline{d}(x,y) + \overline{d}(y,z) &= \min(d(x,y),1) + \min(d(y,z),1) \\
&= \min(d(x,y) + d(y,z), d(x,y) + 1, d(y,z) + 1, 2) \\
&\geq \min(d(x,z), 1) \\
&= \overline{d}(x,z)
\end{aligned}$$

□

**Lemma 1.396.** *In any metric space $X$, the set $\mathcal{B} = \{B(a,\epsilon) \mid a \in X, \epsilon < 1\}$ is a basis for the metric topology.*

PROOF:
⟨1⟩1. Every element of $\mathcal{B}$ is open.
  PROOF: From Lemma 1.96.
⟨1⟩2. For every open set $U$ and point $a \in U$, there exists $B \in \mathcal{B}$ such that $a \in B \subseteq U$
  ⟨2⟩1. LET: $U$ be an open set and $a \in U$
  ⟨2⟩2. PICK$\epsilon > 0$ such that $B(a,\epsilon) \subseteq U$
  ⟨2⟩3. $B(a, \min(\epsilon, 1/2)) \subseteq U$
⟨1⟩3. Q.E.D.
  PROOF: Lemma 1.97.

□

**Proposition 1.397.** *Let $d$ be a metric on the set $X$. Then the standard bounded metric $\overline{d}$ induces the same metric as $d$.*

PROOF: This follows from Lemma 1.396 since the open balls with radius $< 1$ are the same under both metrics. □

**Lemma 1.398.** *Let $d$ and $d'$ be two metrics on the same set $X$. Let $\mathcal{T}$ and $\mathcal{T}'$ be the topologies they induce. Then $\mathcal{T} \subseteq \mathcal{T}'$ if and only if, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that*

$$B_{d'}(x, \delta) \subseteq B_d(x, \epsilon) \ .$$

PROOF:
⟨1⟩1. If $\mathcal{T} \subseteq \mathcal{T}'$ then, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$
  PROOF: From Proposition 1.387 since $x \in B_d(x, \epsilon) \in \mathcal{T}'$.
⟨1⟩2. If, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$, then $\mathcal{T} \subseteq \mathcal{T}'$
  ⟨2⟩1. ASSUME: For all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$
  ⟨2⟩2. LET: $U \in \mathcal{T}$
  ⟨2⟩3. For all $x \in U$ there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq U$.
    ⟨3⟩1. LET: $x \in U$
    ⟨3⟩2. PICK $\epsilon > 0$ such that $B_d(x, \epsilon) \subseteq U$
      PROOF: Proposition 1.387
    ⟨3⟩3. PICK $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$
      PROOF: By ⟨2⟩1
    ⟨3⟩4. $B_{d'}(x, \delta) \subseteq U$
  ⟨2⟩4. $U \in \mathcal{T}'$
    PROOF: Proposition 1.387.
□

**Proposition 1.399.** $\mathbb{R}^2$ *under the dictionary order topology is metrizable.*

PROOF: Define $d : \mathbb{R}^2 \to \mathbb{R}$ by

$$d((x, y), (x, z)) = \max(|y - z|, 1)$$

$$d((x, y), (x', y')) = 1 \qquad \text{if } x \neq x' \square$$

⟨1⟩1. $x \in \bigcap_{i=1}^{N} {\pi_i}^{-1}() \subseteq B_D(a, \epsilon)$

**Proposition 1.400.** *Let $d : X^2 \to \mathbb{R}$ be a metric on $X$. Then the metric topology on $X$ is the coarsest topology such that $d$ is continuous.*

PROOF:
⟨1⟩1. $d$ is continuous.
  ⟨2⟩1. LET: $a, b \in X$
  ⟨2⟩2. LET: $\epsilon > 0$
  ⟨2⟩3. LET: $\delta = \epsilon/2$
  ⟨2⟩4. LET: $x, y \in X$
  ⟨2⟩5. ASSUME: $\rho((a, b), (x, y)) < \delta$
  ⟨2⟩6. $|d(a, b) - d(x, y)| < \epsilon$
    ⟨3⟩1. $d(a, b) - d(x, y) < \epsilon$

PROOF:

$$\begin{aligned} d(a,b) &\leq d(a,x) + d(x,y) + d(y,b) \\ &\leq d(x,y) + 2\rho((a,b),(x,y)) \\ &< d(x,y) + 2\delta \\ &= d(x,y) + \epsilon \end{aligned}$$

⟨3⟩2. $d(a.b) - d(x,y) > -\epsilon$

PROOF: Similar.

⟨2⟩7. Q.E.D.

⟨1⟩2. If $\mathcal{T}$ is any topology under which $d$ is continuous then $\mathcal{T}$ is finer than the metric topology.

PROOF: Since $B(a,\epsilon) = {d_a}^{-1}((-\infty,\epsilon))$

□

**Proposition 1.401.** *Let $X$ be a metric space with metric $d$ and $A \subseteq X$. The restriction of $d$ to $A$ is a metric on $A$ that induces the subspace topology.*

PROOF:

⟨1⟩1. The restriction of $d$ to $A$ is a metric on $A$.

⟨1⟩2. Every open ball under $d \upharpoonright A$ is open under the subspace topology.

PROOF: $B_{d\upharpoonright A}(a,\epsilon) = B_d(a,\epsilon) \cap A$.

⟨1⟩3. If $U$ is open in the subspace topology and $x \in U$, then there exists a $d \upharpoonright A$-ball $B$ such that $x \in B \subseteq U$.

⟨2⟩1. PICK $V$ open in $X$ such that $U = V \cap A$

⟨2⟩2. PICK $\epsilon > 0$ such that $B_d(x,\epsilon) \subseteq V$

⟨2⟩3. Take $B = B_{d\upharpoonright A}(x,\epsilon)$

□

**Corollary 1.401.1.** *A subspace of a metrizable space is metrizable.*

**Proposition 1.402.** *Every metrizable space is Hausdorff.*

PROOF:

⟨1⟩1. LET: $X$ be a metric space

⟨1⟩2. LET: $a,b \in X$ with $a \neq b$

⟨1⟩3. LET: $\epsilon = d(a,b)/2$

⟨1⟩4. LET: $U = B(a,\epsilon)$ and $V = B(b,\epsilon)$

⟨1⟩5. $U$ and $V$ are disjoint neighbourhoods of $a$ and $b$ respectively.

□

**Proposition 1.403** (CC)**.** *The product of a countable family of metrizable spaces is metrizable.*

PROOF:

⟨1⟩1. LET: $(X_n, d_n)$ be a sequence of metric spaces.

⟨1⟩2. ASSUME: w.l.o.g. each $d_n$ is bounded above by 1.

PROOF: By Proposition 1.397.

⟨1⟩3. LET: $D$ be the metric on $\mathbb{R}^\omega$ defined by $D(x,y) = \sup_i(d_i(x_i,y_i)/i)$.

⟨2⟩1. $D(x,y) \geq 0$
⟨2⟩2. $D(x,y) = 0$ if and only if $x = y$
⟨2⟩3. $D(x,y) = D(y,x)$
⟨2⟩4. $D(x,z) \leq D(x,y) + D(y,z)$
PROOF:

$$\begin{aligned}
D(x,z) &= \sup_i \frac{d_i(x_i, z_i)}{i} \\
&\leq \sup_i \frac{d_i(x_i, y_i) + d_i(y_i, z_i)}{i} \\
&\leq \sup_i \frac{d_i(x_i, y_i)}{i} + \sup_i \frac{d_i(y_i, z_i)}{i} \\
&= D(x,y) + D(y,z)
\end{aligned}$$

⟨1⟩4. Every open ball $B_D(a, \epsilon)$ is open in the product topology.
⟨2⟩1. PICK $N$ such that $1/\epsilon < N$
⟨2⟩2. $B_D(a,\epsilon) = \prod_{i=1}^{\infty} U_i$ where $U_i = B(a_i, i\epsilon)$ if $i \leq N$, and $U_i = X_i$ if $i > N$
⟨1⟩5. For any open set $U$ and $a \in U$, there exists $\epsilon > 0$ such that $B_D(a,\epsilon) \subseteq U$.
⟨2⟩1. LET: $n \geq 1$, $V$ be an open set in $\mathbb{R}$ and $a \in {\pi_n}^{-1}(V)$
⟨2⟩2. PICK $\epsilon > 0$ such that $B_{d_n}(a, \epsilon) \subseteq V$
⟨2⟩3. $B_D(a, \epsilon/n) \subseteq {\pi_n}^{-1}(V)$
□

**Theorem 1.404.** *Let $X$ and $Y$ be metric spaces and $f : X \to Y$. Then $f$ is continuous if and only if, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x,y) < \delta$ then $d(f(x), f(y)) < \epsilon$.*

PROOF:
⟨1⟩1. If $f$ is continuous then, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x,y) < \delta$ then $d(f(x), f(y)) < \epsilon$
⟨2⟩1. ASSUME: $f$ is continuous.
⟨2⟩2. LET: $x \in X$ and $\epsilon > 0$
⟨2⟩3. PICK a neighbourhood $U$ of $x$ such that $f(U) \subseteq B(f(x), \epsilon)$
PROOF: Theorem 1.127.
⟨2⟩4. PICK $\delta > 0$ such that $B(x, \delta) \subseteq U$
PROOF: Proposition 1.387.
⟨2⟩5. For all $y \in X$, if $d(x,y) < \delta$ then $d(f(x), f(y)) < \epsilon$
⟨1⟩2. If for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x,y) < \delta$ then $d(f(x), f(y)) < \epsilon$, then $f$ is continuous.
⟨2⟩1. ASSUME: for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x,y) < \delta$ then $d(f(x), f(y)) < \epsilon$
⟨2⟩2. LET: $x \in X$ and $V$ be a neighbourhood of $f(x)$
⟨2⟩3. PICK $\epsilon > 0$ such that $B(f(x), \epsilon) \subseteq V$
PROOF: Proposition 1.387.
⟨2⟩4. PICK $\delta > 0$ such that, for all $y \in X$, if $d(x,y) < \delta$ then $d(f(x), f(y)) < \epsilon$
PROOF: By ⟨2⟩1
⟨2⟩5. LET: $U = B(x, \delta)$
⟨2⟩6. $U$ is a neighbourhood of $x$ with $f(U) \subseteq V$

⟨2⟩7. Q.E.D.
PROOF: Theorem 1.127.
□

**Proposition 1.405.** *Let $X$ be a metric space. Let $(a_n)$ be a sequence in $X$ and $l \in X$. Then $a_n \to l$ as $n \to \infty$ if and only if, for all $\epsilon > 0$, there exists $N$ such that, for all $n \geq N$, we have $d(a_n, l) < \epsilon$.*

PROOF: From Proposition 1.110. □

**Proposition 1.406.** *Every metrizable space is first countable.*

PROOF: In any metric space $X$, the open balls $B(a, 1/n)$ for $n \geq 1$ form a local basis at $a$.

**Example 1.407.** $\mathbb{R}^\omega$ under the box topology is not metrizable.

**Example 1.408.** If $J$ is uncountable then $\mathbb{R}^J$ under the product topology is not metrizable.

**Proposition 1.409.** *A compact subspace of a metric space is bounded.*

PROOF:
⟨1⟩1. LET: $X$ be a metric space and $A \subseteq X$ be compact.
⟨1⟩2. PICK $a \in A$
⟨1⟩3. $\{B(a,n) \mid n \in \mathbb{Z}^+\}$ covers $A$
⟨1⟩4. PICK a finite subcover $\{B(a, n_1), \ldots, B(a, n_k)\}$
⟨1⟩5. LET: $N = \max(n_1, \ldots, n_k)$
⟨1⟩6. For all $x, y \in A$ we have $d(x,y) < 2N$
PROOF:

$$\begin{aligned} d(x,y) &\leq d(x,a) + d(a,y) \\ &< N + N \end{aligned}$$

□

This example shows the converse does not hold:

**Example 1.410.** The space $\mathbb{R}$ under the standard bounded metric is bounded but not compact.

## 1.62 Real Linear Algebra

**Definition 1.411** (Square Metric)**.** The *square metric* $\rho$ on $\mathbb{R}^n$ is defined by

$$\rho(\vec{x}, \vec{y}) = \max(|x_1 - y_1|, \ldots, |x_n - y_n|)$$

We prove this is a metric.

PROOF:
⟨1⟩1. $\rho(\vec{x}, \vec{y}) \geq 0$
PROOF: Immediate from definition.

⟨1⟩2. $\rho(\vec{x}, \vec{y}) = 0$ if and only if $\vec{x} = \vec{y}$

PROOF: Immediate from definition.

⟨1⟩3. $\rho(\vec{x}, \vec{y}) = \rho(\vec{y}, \vec{x})$

PROOF: Immediate from definition.

⟨1⟩4. $\rho(\vec{x}, \vec{z}) \leq \rho(\vec{x}, \vec{y}) + \rho(\vec{y}, \vec{z})$

PROOF: Since $|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$.

□

**Proposition 1.412.** *The square metric induces the standard topology on $\mathbb{R}^n$.*

PROOF:

⟨1⟩1. For every $a \in X$ and $\epsilon > 0$, we have $B_\rho(a, \epsilon)$ is open in the standard product topology.

PROOF:

$$B_\rho(a, \epsilon) = (a_1 - \epsilon, a_1 + \epsilon) \times \cdots \times (a_n - \epsilon, a_n + \epsilon)$$

⟨1⟩2. For any open sets $U_1, \ldots, U_n$ in $\mathbb{R}$, we have $U_1 \times \cdots \times U_n$ is open in the square metric topology.

⟨2⟩1. LET: $\vec{a} \in U_1 \times \cdots \times U_n$

⟨2⟩2. For $i = 1, \ldots, n$, PICK $\epsilon_i > 0$ such that $(a_i - \epsilon_i, a_i + \epsilon_i) \subseteq U_i$

⟨2⟩3. LET: $\epsilon = \min(\epsilon_1, \ldots, \epsilon_n)$

⟨2⟩4. $B_\rho(\vec{a}, \epsilon) \subseteq U$

□

**Definition 1.413.** Given $\vec{x}, \vec{y} \in \mathbb{R}^n$, define the *sum* $\vec{x} + \vec{y}$ by

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n) \ .$$

**Definition 1.414.** Given $\lambda \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^n$, define the *scalar product* $\lambda\vec{x} \in \mathbb{R}^n$ by

$$\lambda(x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n)$$

**Definition 1.415** (Inner Product)**.** Given $\vec{x}, \vec{y} \in \mathbb{R}^n$, define the *inner product* $\vec{x} \cdot \vec{y} \in \mathbb{R}$ by

$$(x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) = x_1 y_1 + \cdots + x_n y_n \ .$$

We write $\vec{x}^2$ for $\vec{x} \cdot \vec{x}$.

**Definition 1.416** (Norm)**.** Let $n \geq 1$. The *norm* on $\mathbb{R}^n$ is the function $\| \ \| : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\|(x_1, \ldots, x_n)\| = \sqrt{x_1^2 + \cdots + x_n^2}$$

**Lemma 1.417.**

$$\|\vec{x}\|^2 = \vec{x}^2$$

PROOF: Immediate from definitions. □

**Lemma 1.418.**

$$\vec{x} \cdot (\vec{y} + \vec{z}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z}$$

PROOF: Each is equal to $(x_1y_1 + x_1z_1, \ldots, x_ny_n + x_nz_n)$. ∎

**Lemma 1.419.**

$$|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\|\|\vec{y}\|$$

PROOF:
⟨1⟩1. ASSUME: $\vec{x} \neq \vec{0} \neq \vec{y}$
PROOF: Otherwise both sides are 0.
⟨1⟩2. LET: $a = 1/\|\vec{x}\|$
⟨1⟩3. LET: $b = 1/\|\vec{y}\|$
⟨1⟩4. $(a\vec{x} + b\vec{y})^2 \geq 0$ and $(a\vec{x} - b\vec{y})^2 \geq 0$
⟨1⟩5. $a^2\|\vec{x}\|^2 + 2ab\vec{x} \cdot \vec{y} + b^2\|\vec{y}\|^2 \geq 0$ and $a^2\|\vec{x}\|^2 - 2ab\vec{x} \cdot \vec{y} + b^2\|\vec{y}\|^2 \geq 0$
⟨1⟩6. $2ab\vec{x} \cdot \vec{y} + 2 \geq 0$ and $-2ab\vec{x} \cdot \vec{y} + 2 \geq 0$
⟨1⟩7. $\vec{x} \cdot \vec{y} \geq -1/ab$ and $\vec{x} \cdot \vec{y} \leq 1/ab$
⟨1⟩8. $\vec{x} \cdot \vec{y} \geq -\|\vec{x}\|\|\vec{y}\|$ and $\vec{x} \cdot \vec{y} \leq \|\vec{x}\|\|\vec{y}\|$
∎

**Lemma 1.420** (Triangle Inequality)**.**

$$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$$

PROOF:

$$\begin{aligned}
\|\vec{x} + \vec{y}\|^2 &= \|\vec{x}\|^2 + 2\vec{x} \cdot \vec{y} + \|\vec{y}\|^2 \\
&\leq \|\vec{x}\|^2 + 2\|\vec{x}\|\|\vec{y}\| + \|\vec{y}\|^2 && \text{(Lemma 1.419)} \\
&= (\|\vec{x}\| + \|\vec{y}\|)^2
\end{aligned}$$

∎

**Definition 1.421** (Euclidean Metric)**.** Let $n \geq 1$. The *Euclidean metric* on $\mathbb{R}^n$ is defined by

$$d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\| \ .$$

We prove this is a metric.

⟨1⟩1. $d(\vec{x}, \vec{y}) \geq 0$
PROOF: Immediate from definition.
⟨1⟩2. $d(\vec{x}, \vec{y}) = 0$ if and only if $\vec{x} = \vec{y}$
PROOF: $d(\vec{x}, \vec{y}) = 0$ if and only if $\vec{x} - \vec{y} = \vec{0}$.
⟨1⟩3. $d(\vec{x}, \vec{y}) = d(\vec{y}, \vec{x})$
PROOF: Immediate from definition.
⟨1⟩4. $d(\vec{x}, \vec{z}) \leq d(\vec{x}, \vec{y}) + d(\vec{y}, \vec{z})$
PROOF:

$$\begin{aligned}
\|\vec{x} - \vec{z}\| &= \|(\vec{x} - \vec{y}) + (\vec{y} - \vec{z})\| \\
&\leq \|\vec{x} - \vec{y}\| + \|\vec{y} - \vec{z}\| && \text{(Lemma 1.420)}
\end{aligned}$$

∎

**Proposition 1.422.** *The Euclidean metric induces the standard topology on* $\mathbb{R}^n$*.*

PROOF:

⟨1⟩1. LET: $\rho$ be the square metric.
⟨1⟩2. For all $\vec{a} \in \mathbb{R}^n$ and $\epsilon > 0$, we have $B_d(\vec{a}, \epsilon) \subseteq B_\rho(\vec{a}, \epsilon)$
  ⟨2⟩1. LET: $\vec{x} \in B_d(\vec{a}, \epsilon)$
  ⟨2⟩2. $\sqrt{(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2} < \epsilon$
  ⟨2⟩3. $(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2 < \epsilon^2$
  ⟨2⟩4. For all $i$ we have $(x_i - a_i)^2 < \epsilon^2$
  ⟨2⟩5. For all $i$ we have $|x_i - a_i| < \epsilon$
  ⟨2⟩6. $\rho(\vec{x}, \vec{a}) < \epsilon$
⟨1⟩3. For all $\vec{a} \in \mathbb{R}^n$ and $\epsilon > 0$, we have $B_\rho(\vec{a}, \epsilon/\sqrt{n}) \subseteq B_d(\vec{a}, \epsilon)$
  ⟨2⟩1. LET: $\vec{x} \in B_\rho(\vec{a}, \epsilon/\sqrt{n})$
  ⟨2⟩2. $\rho(\vec{x}, \vec{a}) < \epsilon/\sqrt{n}$
  ⟨2⟩3. For all $i$ we have $|x_i - x_a| < \epsilon/\sqrt{n}$
  ⟨2⟩4. For all $i$ we have $(x_i - x_a)^2 < \epsilon^2/n$
  ⟨2⟩5. $d(\vec{x}, \vec{a}) < \epsilon$
⟨1⟩4. Q.E.D.
  PROOF: By Lemma 1.398.
□

**Proposition 1.423.** *Let $n \geq 0$. For all $c \in \mathbb{R}^n$ and $\epsilon > 0$, the open ball $B(c, \epsilon)$ is path connected.*

PROOF:
⟨1⟩1. LET: $a, b \in B(c, \epsilon)$
⟨1⟩2. LET: $p : [0, 1] \to B(c, \epsilon)$ be the function $p(t) = (1 - t)a + tb$
  PROOF: We have $p(t) \in B(c, \epsilon)$ for all $t$ because

$$\begin{aligned} d(p(t), c) &= \|(1 - t)a + tb - c\| \\ &= \|(1 - t)(a - c) + t(b - c)\| \\ &\leq (1 - t)\|a - c\| + t\|b - c\| \\ &< (1 - t)\epsilon + t\epsilon \\ &= \epsilon \end{aligned}$$

⟨1⟩3. $p$ is a path from $a$ to $b$.
□

**Proposition 1.424.** *Let $n \geq 0$. For all $c \in \mathbb{R}^n$ and $\epsilon > 0$, the closed ball $\overline{B(c, \epsilon)}$ is path connected.*

PROOF:
⟨1⟩1. LET: $a, b \in \overline{B(c, \epsilon)}$
⟨1⟩2. LET: $p : [0, 1] \to \overline{B(c, \epsilon)}$ be the function $p(t) = (1 - t)a + tb$
  PROOF: We have $p(t) \in \overline{B(c, \epsilon)}$ for all $t$ because

$$\begin{aligned} d(p(t), c) &= \|(1 - t)a + tb - c\| \\ &= \|(1 - t)(a - c) + t(b - c)\| \\ &\leq (1 - t)\|a - c\| + t\|b - c\| \\ &\leq (1 - t)\epsilon + t\epsilon \\ &= \epsilon \end{aligned}$$

⟨1⟩3. $p$ is a path from $a$ to $b$.

□

**Lemma 1.425.** *If $\sum_{i=0}^{\infty} x_i^2$ and $\sum_{i=0}^{\infty} y_i^2$ converge then $\sum_{i=0}^{\infty} |x_i y_i|$ converges.*

PROOF:
⟨1⟩1. For all $N \geq 0$ we have $\sum_{i=0}^{N} |x_i y_i| \leq \sqrt{\sum_{i=0}^{N} |x_i|^2} \sqrt{\sum_{i=0}^{N} |y_i|^2}$
PROOF: By the Cauchy-Schwarz inequality
⟨1⟩2. Q.E.D.
PROOF: Since $\sum_{i=0}^{N} |x_i y_i|$ is an increasing sequence bounded above by $(\sum_{i=0}^{\infty} x_i^2)(\sum_{i=0}^{\infty} y_i^2)$.

□

**Corollary 1.425.1.** *If $\sum_{i=0}^{\infty} x_i^2$ and $\sum_{i=0}^{\infty} y_i^2$ converge then $\sum_{i=0}^{\infty} (x_i + y_i)^2$ converges.*

PROOF: Since $\sum_{i=0}^{\infty} x_i^2$, $\sum_{i=0}^{\infty} y_i^2$ and $2 \sum_{i=0}^{\infty} x_i y_i$ all converge.

**Definition 1.426** ($l^2$-metric)**.** The $l^2$-*metric* on

$$\left\{ (x_n) \in \mathbb{R}^{\omega} \mid \sum_{n=0}^{\infty} x_n^2 \text{ converges} \right\}$$

is defined by

$$d(x, y) = \left( \sum_{n=0}^{\infty} (x_n - y_n)^2 \right)^{1/2}$$

We prove this is a metric.

PROOF:
⟨1⟩1. $d$ is well-defined.
PROOF: By Corollary 1.425.1.
⟨1⟩2. $d(x, y) \geq 0$
⟨1⟩3. $d(x, y) = 0$ if and only if $x = y$
⟨1⟩4. $d(x, y) = d(y, x)$
⟨1⟩5. $d(x, z) \leq d(x, y) + d(y, z)$
PROOF: By Lemma 1.420.

□

**Theorem 1.427.** *Addition is a continuous function $\mathbb{R}^2 \to \mathbb{R}$.*

PROOF:
⟨1⟩1. LET: $a, b \in \mathbb{R}$
⟨1⟩2. LET: $\epsilon > 0$
⟨1⟩3. LET: $\delta = \epsilon / 2$
⟨1⟩4. LET: $x, y \in \mathbb{R}$
⟨1⟩5. ASSUME: $\rho((a, b), (x, y)) < \delta$
⟨1⟩6. $|(a + b) - (x + y)| < \epsilon$

PROOF:

$$\begin{aligned}|(a+b)-(x+y)| &= |a-x|+|b-y| \\ &\leq 2\rho((a,b),(x,y)) \\ &< 2\delta \\ &= \epsilon\end{aligned}$$

$\langle 1\rangle$7. Q.E.D.
PROOF: Theorem 1.404
$\square$

**Theorem 1.428.** *Multiplication is a continuous function* $\mathbb{R}^2 \to \mathbb{R}$.

PROOF:
$\langle 1\rangle$1. LET: $a, b \in \mathbb{R}$
$\langle 1\rangle$2. LET: $\epsilon > 0$
$\langle 1\rangle$3. LET: $\delta = \min(\epsilon/(|a|+|b|+1), 1)$
$\langle 1\rangle$4. LET: $x, y \in \mathbb{R}$
$\langle 1\rangle$5. ASSUME: $\rho((a,b),(x,y)) < \delta$
$\langle 1\rangle$6. $|ab-xy| < \epsilon$
PROOF:

$$\begin{aligned}|ab-xy| &= |a(b-y)+(a-x)b-(a-x)(b-y)| \\ &\leq |a||b-y|+|b||a-x|+|a-x||b-y| \\ &< |a|\delta+|b|\delta+\delta^2 && (\langle 1\rangle 5) \\ &\leq |a|\delta+|b|\delta+\delta && (\langle 1\rangle 3) \\ &\leq \epsilon && (\langle 1\rangle 3)\end{aligned}$$

$\langle 1\rangle$7. Q.E.D.
PROOF: Theorem 1.404
$\square$

**Theorem 1.429.** *The function* $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ *defined by* $f(x) = x^{-1}$ *is continuous.*

PROOF:
$\langle 1\rangle$1. For all $a \in \mathbb{R}$ we have $f^{-1}((a,+\infty))$ is open.
PROOF: The set is

$$\begin{aligned}(a^{-1},+\infty) &\text{if } a > 0 \\ (0,+\infty) &\text{if } a = 0 \\ (-\infty, a^{-1}) \cup (0,+\infty) &\text{if } a < 0\end{aligned}$$

$\langle 1\rangle$2. For all $a \in \mathbb{R}$ we have $f^{-1}((-\infty,a))$ is open.
PROOF: Similar.
$\langle 1\rangle$3. Q.E.D.
PROOF: By Proposition 1.124 and Lemma 1.147.
$\square$

**Definition 1.430.** For $n \geq 0$, the *unit ball* $B^n$ is the space $\{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$.

**Proposition 1.431.** *For all* $n \geq 0$*, the unit ball* $B^n$ *is path connected.*

PROOF:
⟨1⟩1. LET: $a, b \in B^n$
⟨1⟩2. LET: $p : [0, 1] \to B^n$ be the function $p(t) = (1 - t)a + tb$
PROOF: We have $p(t) \in B^n$ for all $t$ because

$$\begin{aligned}\|(1-t)a + tb\| &\leq (1-t)\|a\| + t\|b\| \\ &\leq (1-t) + t \\ &= 1\end{aligned}$$

⟨1⟩3. $p$ is a path from $a$ to $b$.
□

**Definition 1.432** (Punctured Euclidean Space)**.** For $n \geq 0$, defined *punctured Euclidean space* to be $\mathbb{R}^n \setminus \{0\}$.

**Proposition 1.433.** *For $n > 1$, punctured Euclidean space $\mathbb{R}^n \setminus \{0\}$ is path connected.*

PROOF:
⟨1⟩1. LET: $a, b \in \mathbb{R}^n \setminus \{0\}$
⟨1⟩2. CASE: 0 is on the line from $a$ to $b$
⟨2⟩1. PICK a point $c$ not on the line from $a$ to $b$
⟨2⟩2. The path consisting of a straight line from $a$ to $c$ followed by a straight line from $c$ to $b$ is a path from $a$ to $b$.
⟨1⟩3. CASE: 0 is not on the line from $a$ to $b$
PROOF: The straight line from $a$ to $b$ is a path from $a$ to $b$.

**Corollary 1.433.1.** *For $n > 1$, the spaces $\mathbb{R}$ and $\mathbb{R}^n$ are not homeomorphic.*

PROOF: For any point $a$, the space $\mathbb{R} \setminus \{a\}$ is disconnected.

**Definition 1.434** (Unit Sphere)**.** For $n \geq 1$, the *unit sphere* $S^{n-1}$ is the space

$$S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\} \ .$$

**Proposition 1.435.** *For $n > 1$, the unit sphere $S^{n-1}$ is path connected.*

PROOF: The map $g : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$ defined by $g(x) = x/\|x\|$ is continuous and surjective. The result follows by Proposition 1.279. □

**Proposition 1.436.** *Let $f : S^1 \to \mathbb{R}$ be continuous. Then there exists $x \in S^1$ such that $f(x) = f(-x)$.*

PROOF:
⟨1⟩1. LET: $g : S^1 \to \mathbb{R}$ be the function $g(x) = f(x) - f(-x)$
PROVE: There exists $x \in S^1$ such that $g(x) = 0$
⟨1⟩2. ASSUME: without loss of generality $g((1, 0)) > 0$
⟨1⟩3. $g((-1, 0)) < 0$
⟨1⟩4. There exists $x$ such that $g(x) = 0$
PROOF: By the Intermediate Value Theorem.
□

**Definition 1.437** (Topologist's Sine Curve)**.** Let $S = \{(x, \sin 1/x) \mid 0 < x \leq 1\}$. The *topologist's sine curve* is the closure $\overline{S}$ of $S$.

**Proposition 1.438.**

$$\overline{S} = S \cup (\{0\} \times [-1, 1])$$

**Proposition 1.439.** *The topologist's sine curve is connected.*

PROOF:
⟨1⟩1. LET: $S = \{(x, \sin 1/x) \mid 0 < x \leq 1\}$
⟨1⟩2. $S$ is connected.
  PROOF: Theorem 1.258.
⟨1⟩3. $\overline{S}$ is connected.
  PROOF: Theorem 1.257.
□

**Proposition 1.440** (CC)**.** *The topologist's sine curve is not path connected.*

PROOF:
⟨1⟩1. ASSUME: for a contradiction $p : [0, 1] \to \overline{S}$ is a path from $(0, 0)$ to $(1, \sin 1)$.
⟨1⟩2. $p^{-1}(\{0\} \times [0, 1])$ is closed.
⟨1⟩3. LET: $b$ be the greatest element of $p^{-1}(\{0\} \times [0, 1])$.
⟨1⟩4. $b < 1$
  PROOF: Since $p(1) = (1, \sin 1)$.
⟨1⟩5. PICK a sequence $(t_n)_{n \geq 1}$ in $(b, 1]$ such that $t_n \to b$ and $\pi_2(p(t_n)) = (-1)^n$
  ⟨2⟩1. LET: $n \geq 1$
  ⟨2⟩2. PICK $u$ with $0 < u < \pi_1(p(1/n))$ such that $\sin(1/u) = (-1)^n$
  ⟨2⟩3. PICK $t_n$ such that $0 < t_n < 1/n$ and $\pi_1(p(t_n)) = u$
    PROOF: One exists by the Intermediate Value Theorem.
⟨1⟩6. Q.E.D.
  PROOF: This contradicts 1.139.
□

**Theorem 1.441.** *Let $A$ be a subspace of $\mathbb{R}^n$. Then the following are equivalent:*

1. *$A$ is compact.*
2. *$A$ is closed and bounded under the Euclidean metric.*
3. *$A$ is closed and bounded under the square metric.*

PROOF:
⟨1⟩1. $1 \Rightarrow 2$
  PROOF: By Corollary 1.348.1 and Proposition 1.409.
⟨1⟩2. $2 \Rightarrow 3$
  PROOF: If $d(x, y) \leq M$ for all $x, y \in A$ then $\rho(x, y) \leq M/\sqrt{2}$.
⟨1⟩3. $3 \Rightarrow 1$
  ⟨2⟩1. ASSUME: $A$ is closed and $\rho(x, y) \leq M$ for all $x, y \in A$
  ⟨2⟩2. PICK $a \in A$

PROOF: We may assume w.l.o.g. $A$ is nonempty since the empty space is compact.

$\langle 2\rangle 3.$ $A$ is a closed subspace of $[a_1 - M, a_1 + M] \times \cdots \times [a_n - M, a_n + M]$

$\langle 2\rangle 4.$ $A$ is compact

PROOF: Proposition 1.341.

$\square$

**Corollary 1.441.1.** *The unit sphere $S^{n-1}$ and the closed unit ball $B^n$ are compact for any $n$.*

## 1.63 The Uniform Topology

**Definition 1.442** (Uniform Metric)**.** Let $J$ be a set. The *uniform metric* $\overline{\rho}$ on $\mathbb{R}^J$ is defined by

$$\overline{\rho}(a, b) = \sup_{j \in J} \overline{d}(a_j, b_j)$$

where $\overline{d}$ is the standard bounded metric on $\mathbb{R}$.

The *uniform topology* on $\mathbb{R}^J$ is the topology induced by the uniform metric.

We prove this is a metric.

PROOF:

$\langle 1\rangle 1.$ $\overline{\rho}(a, b) \geq 0$

PROOF: Immediate from definitions.

$\langle 1\rangle 2.$ $\overline{\rho}(a, b) = 0$ if and only if $a = b$

PROOF: Immediate from definitions.

$\langle 1\rangle 3.$ $\overline{\rho}(a, b) = \overline{\rho}(b, a)$

PROOF: Immediate from definitions.

$\langle 1\rangle 4.$ $\overline{\rho}(a, c) \leq \overline{\rho}(a, b) + \overline{\rho}(b, c)$

PROOF:

$$\begin{aligned}
\overline{\rho}(a, c) &= \sup_{j \in J} \overline{d}(a_j, c_j) \\
&\leq \sup_{j \in J} (\overline{d}(a_j, b_j) + \overline{d}(b_j, c_j)) \\
&\leq \sup_{j \in J} \overline{d}(a_j, b_j) + \sup_{j \in J} \overline{d}(b_j, c_j) \\
&= \overline{\rho}(a, b) + \overline{\rho}(b, c)
\end{aligned}$$

$\square$

**Proposition 1.443.** *The uniform topology on $\mathbb{R}^J$ is finer than the product topology.*

PROOF:

$\langle 1\rangle 1.$ LET: $j \in J$ and $U$ be open in $\mathbb{R}$

PROVE: ${\pi_j}^{-1}(U)$ is open in the uniform topology.

$\langle 1\rangle 2.$ LET: $a \in {\pi_j}^{-1}(U)$

$\langle 1\rangle 3.$ PICK $\epsilon > 0$ such that $(a_j - \epsilon, a_j + \epsilon) \subseteq U$

⟨1⟩4. $B_{\overline{\rho}}(a, \epsilon) \subseteq {\pi_j}^{-1}(U)$
□

**Proposition 1.444.** *The uniform topology on* $\mathbb{R}^J$ *is coarser than the box topology.*

PROOF:
⟨1⟩1. LET: $a \in \mathbb{R}^J$ and $\epsilon > 0$
PROVE: $B(a, \epsilon)$ is open in the box topology.
⟨1⟩2. LET: $b \in B(a, \epsilon)$
⟨1⟩3. For $j \in J$ we have $|a_j - b_j| < \epsilon$
⟨1⟩4. For $j \in J$,
LET: $\delta_j = (\epsilon - |a_j - b_j|)/2$
⟨1⟩5. $\prod_{j \in J}(b_j - \delta_j, b_j + \delta_j) \subseteq B(a, \epsilon)$
□

**Proposition 1.445.** *The uniform topology on* $\mathbb{R}^J$ *is strictly finer than the product topology if and only if* $J$ *is infinite.*

PROOF:
⟨1⟩1. If $J$ is finite then the uniform and product topologies coincide.
PROOF: The uniform, box and product topologies are all the same.
⟨1⟩2. If $J$ is infinite then the uniform and product topologies are different.
PROOF: The set $B(\vec{0}, 1)$ is open in the uniform topology but not the product topology.
□

**Proposition 1.446** (DC)**.** *The uniform topology on* $\mathbb{R}^J$ *is strictly coarser than the box topology if and only if* $J$ *is infinite.*

PROOF:
⟨1⟩1. If $J$ is finite then the uniform and box topologies coincide.
PROOF: The uniform, box and product topologies are all the same.
⟨1⟩2. If $J$ is infinite then the uniform and box topologies are different.
PROOF: Pick an $\omega$-sequence $(j_1, j_2, \ldots)$ in $J$. Let $U = \prod_{j \in J} U_j$ where $U_{j_i} = (-1/i, 1/i)$ and $U_j = (-1, 1)$ for all other $j$. Then $\vec{0} \in U$ but there is no $\epsilon > 0$ such that $B(\vec{0}, \epsilon) \subseteq U$.
□

**Proposition 1.447.** *The closure of* $\mathbb{R}^\infty$ *in* $\mathbb{R}^\omega$ *under the uniform topology is* $\mathbb{R}^\omega$*.*

PROOF: Given any open ball $B(a, \epsilon)$, pick an integer $N$ such that $1/\epsilon < N$. Then $B(a, \epsilon)$ includes sequences whose $n$th entry is 0 for all $n \geq N$. □

**Example 1.448.** The space $\mathbb{R}^\omega$ is disconnected under the uniform topology. The set of bounded sequences and the set of unbounded sequences form a separation.

**Proposition 1.449.** *Give $\mathbb{R}^\omega$ the uniform topology. Let $x, y \in \mathbb{R}^\omega$. Then $x$ and $y$ are in the same component if and only if $x - y$ is bounded.*

PROOF:
⟨1⟩1. The component containing 0 is the set of bounded sequences.
  ⟨2⟩1. LET: $B$ be the set of bounded sequences.
  ⟨2⟩2. $B$ is path-connected.
    ⟨3⟩1. LET: $x.y \in B$
    ⟨3⟩2. PICK $b > 0$ such that $|x_j|, |y_j| \leq b$ for all $j$
    ⟨3⟩3. LET: $p : [0,1] \to B$ be the function $p(t) = (1-t)x + ty$
      PROVE: $p$ is continuous.
    ⟨3⟩4. LET: $t \in [0,1]$ and $\epsilon > 0$
    ⟨3⟩5. LET: $\delta = \epsilon/2b$
    ⟨3⟩6. LET: $s \in [0,1]$ with $|s - t| < \delta$
    ⟨3⟩7. $\overline{\rho}(p(s), p(t)) < \epsilon$
      PROOF:

$$\begin{aligned}
\overline{\rho}(p(s), p(t)) &= \sup_j \overline{d}((1-s)x_j + sy_j, (1-t)x_j + ty_j) \\
&\leq |(s-t)x_j + (t-s)y_j| \\
&\leq |s-t||x_j - y_j| \\
&< 2b\delta \\
&= \epsilon
\end{aligned}$$

  ⟨2⟩3. $B$ is connected.
    PROOF: Proposition 1.277.
  ⟨2⟩4. If $C$ is connected and $B \subseteq C$ then $B = C$.
    PROOF: Otherwise $B \cap C$ and $C \setminus B$ form a separation of $C$.
⟨1⟩2. Q.E.D.
  PROOF: Since $\lambda x.x - y$ is a Homeomorphism of $\mathbb{R}^\omega$ with itself.
□

## 1.64 Uniform Convergence

**Definition 1.450** (Uniform Convergence)**.** Let $X$ be a set and $Y$ a metric space. Let $(f_n : X \to Y)$ be a sequence of functions and $f : X \to Y$ be a function. Then $f_n$ *converges uniformly* to $f$ as $n \to \infty$ if and only if, for all $\epsilon > 0$, there exists $N$ such that, for all $n \geq N$ and $x \in X$, we have $d(f_n(x), f(x)) < \epsilon$.

**Example 1.451.** Define $f_n : [0,1] \to \mathbb{R}$ by $f_n(x) = x^n$ for $n \geq 1$, and $f : [0,1] \to \mathbb{R}$ by $f(x) = 0$ if $x < 1$, $f(1) = 1$. Then $f_n$ converges to $f$ pointwise but not uniformly.

**Theorem 1.452** (Uniform Limit Theorem)**.** *Let $X$ be a topological space and $Y$ a metric space. Let $(f_n : X \to Y)$ be a sequence of continuous functions and $f : X \to Y$ be a function. If $f_n$ converges uniformly to $f$ as $n \to \infty$, then $f$ is continuous.*

Proof:
$\langle 1\rangle 1$. Let: $x \in X$ and $\epsilon > 0$
$\langle 1\rangle 2$. Pick $N$ such that, for all $n \geq N$ and $y \in X$, we have $d(f_n(y), f(y)) < \epsilon/3$
$\langle 1\rangle 3$. Pick a neighbourhood $U$ of $x$ such that $f_N(U) \subseteq B(f_N(x), \epsilon/3)$
Prove: $f(U) \subseteq B(f(x), \epsilon)$
$\langle 1\rangle 4$. Let: $y \in U$
$\langle 1\rangle 5$. $d(f(y), f(x)) < \epsilon$
Proof:

$$\begin{aligned} d(f(y), f(x)) &\leq d(f(y), f_N(y)) + d(f_N(y), f_N(x)) + d(f_N(x), f(x)) && \text{(Triangle Inequality)} \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 && (\langle 1\rangle 2,\ \langle 1\rangle 3) \\ &= \epsilon \end{aligned}$$

□

**Proposition 1.453.** *Let $X$ be a topological space and $Y$ a metric space. Let $(f_n : X \to Y)$ be a sequence of continuous functions and $f : X \to Y$ be a function. Let $(a_n)$ be a sequence of points in $X$ and $a \in X$. If $f_n$ converges uniformly to $f$ and $a_n$ converges to $a$ in $X$ then $f_n(a_n)$ converges to $f(a)$ uniformly in $Y$.*

Proof:
$\langle 1\rangle 1$. Let: $\epsilon > 0$
$\langle 1\rangle 2$. Pick $N_1$ such that, for all $n \geq N_1$ and $x \in X$, we have $d(f_n(x), f(x)) < \epsilon/2$
$\langle 1\rangle 3$. Pick $N_2$ such that, for all $n \geq N_2$, we have $a_n \in f^{-1}(B(a, \epsilon/2))$
Proof: Using the fact that $f$ is continuous from the Uniform Limit Theorem.
$\langle 1\rangle 4$. Let: $N = \max(N_1, N_2)$
$\langle 1\rangle 5$. Let: $n \geq N$
$\langle 1\rangle 6$. $d(f_n(a_n), f(a)) < \epsilon$
Proof:

$$\begin{aligned} d(f_n(a_n), f(a)) &\leq d(f_n(a_n), f(a_n)) + d(f(a_n), f(a)) && \text{(Triangle Inequality)} \\ &< \epsilon/2 + \epsilon/2 && (\langle 1\rangle 2,\ \langle 1\rangle 3) \\ &= \epsilon \end{aligned}$$

□

**Proposition 1.454.** *Let $X$ be a set. Let $(f_n : X \to \mathbb{R})$ be a sequence of functions and $f : X \to \mathbb{R}$ be a function. Then $f_n$ converges unifomly to $f$ as $n \to \infty$ if and only if $f_n \to f$ as $n \to \infty$ in $\mathbb{R}^X$ under the uniform topology.*

Proof:
$\langle 1\rangle 1$. If $f_n$ converges uniformly to $f$ then $f_n$ converges to $f$ under the uniform topology.
$\langle 2\rangle 1$. Assume: $f_n$ converges uniformly to $f$
$\langle 2\rangle 2$. Let: $\epsilon > 0$
$\langle 2\rangle 3$. Pick $N$ such that, for all $n \geq N$ and $x \in X$, we have $d(f_n(x), f(x)) < \epsilon/2$
$\langle 2\rangle 4$. For all $n \geq N$ we have $\overline{\rho}(f_n, f) \leq \epsilon/2$

⟨2⟩5. For all $n \geq N$ we have $\overline{\rho}(f_n, f) < \epsilon$
⟨1⟩2. If $f_n$ converges to $f$ under the uniform topology then $f_n$ converges uniformly to $f$.
⟨2⟩1. ASSUME: $f_n$ converges to $f$ under the uniform topology.
⟨2⟩2. LET: $\epsilon > 0$
⟨2⟩3. PICK $N$ such that, for all $n \geq N$, we have $\overline{\rho}(f_n, f) < \min(\epsilon, 1/2)$
⟨2⟩4. LET: $n \geq N$
⟨2⟩5. LET: $x \in X$
⟨2⟩6. $\overline{\rho}(f_n, f) < \min(\epsilon, 1/2)$
PROOF: From ⟨2⟩3.
⟨2⟩7. $d(f_n(x), f(x)) < \min(\epsilon, 1/2)$
⟨2⟩8. $d(f_n(x), f(x)) < \epsilon$
□

## 1.65 Isometric Imbeddings

**Definition 1.455.** Let $X$ and $Y$ be metric spaces. An *isometric imbedding* $f : X \to Y$ is a function such that, for all $x, y \in X$, we have $d(f(x), f(y)) = d(x, y)$.

**Proposition 1.456.** *Every isometric imbedding is an imbedding.*

PROOF:
⟨1⟩1. LET: $f : X \to Y$ be an isometric imbedding.
⟨1⟩2. $f$ is injective.
PROOF: If $f(x) = f(y)$ then $d(f(x), f(y)) = 0$ hence $d(x, y) = 0$ hence $x = y$.
⟨1⟩3. $f$ is continuous.
PROOF: For all $\epsilon > 0$, if $d(x, y) < \epsilon$ then $d(f(x), f(y)) < \epsilon$.
⟨1⟩4. $f : X \to f(X)$ is an open map.
PROOF: $f(B(a, \epsilon)) = B(f(a), \epsilon) \cap f(X)$.
□

## 1.66 Distance to a Set

**Definition 1.457.** Let $X$ be a metric space, $x \in X$ and $A \subseteq X$ be nonempty. The *distance* from $x$ to $A$ is defined as

$$d(x, A) = \inf_{a \in A} d(x, a) \ .$$