

# Topology

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July 9, 2022

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Part I

Set Theory

# Chapter 1

## Classes

### 1.1 Classes

We speak informally of *classes*. A class is specified by a unary predicate. We write  $\{x \mid P(x)\}$  for the class determined by the predicate  $P(x)$ .

**Definition 1.1.1** (Membership). Let  $a$  be an object and  $\mathbf{A}$  a class. We define the proposition  $a \in \mathbf{A}$  ( $a$  is a *member* or *element* of  $\mathbf{A}$ ) as follows:

The proposition  $a \in \{x \mid P(x)\}$  is the proposition  $P(a)$ .

**Definition 1.1.2** (Equality of Classes). Let  $\mathbf{A}$  and  $\mathbf{B}$  be classes. We say  $\mathbf{A}$  and  $\mathbf{B}$  are *equal*,  $\mathbf{A} = \mathbf{B}$ , if and only if they have exactly the same elements.

### 1.2 Subclasses

**Definition 1.2.1** (Subclass). Let  $\mathbf{A}$  and  $\mathbf{B}$  be classes. We say  $\mathbf{A}$  is a *subclass* of  $\mathbf{B}$ ,  $\mathbf{A} \subseteq \mathbf{B}$ , if and only if every member of  $\mathbf{A}$  is a member of  $\mathbf{B}$ .

We say  $\mathbf{A}$  is a *proper* subclass of  $\mathbf{B}$ ,  $\mathbf{A} \subset \mathbf{B}$ , if and only if  $\mathbf{A} \subseteq \mathbf{B}$  and  $\mathbf{A} \neq \mathbf{B}$ .

### 1.3 The Empty Class

**Definition 1.3.1** (Empty Class). The *empty* class  $\emptyset$  is  $\{x \mid \perp\}$ .

### 1.4 Finite Classes

**Definition 1.4.1.** For any objects  $a_1, \dots, a_n$ , we write  $\{a_1, \dots, a_n\}$  for the class  $\{x \mid x = a_1 \vee \dots \vee x = a_n\}$ .



## 1.5 Universal Class

**Definition 1.5.1** (Universal Class). The *universal class*  $\mathbf{V}$  is the class  $\{x \mid \top\}$ .

## 1.6 Union

**Definition 1.6.1** (Union). For any classes  $\mathbf{A}$  and  $\mathbf{B}$ , the *union*  $\mathbf{A} \cup \mathbf{B}$  is the class  $\{x \mid x \in \mathbf{A} \vee x \in \mathbf{B}\}$ .

## 1.7 Intersection

**Definition 1.7.1** (Intersection). For any classes  $\mathbf{A}$  and  $\mathbf{B}$ , the *intersection*  $\mathbf{A} \cap \mathbf{B}$  is the class  $\{x \mid x \in \mathbf{A} \wedge x \in \mathbf{B}\}$ .

## 1.8 Disjoint Classes

**Definition 1.8.1** (Disjoint). Classes  $\mathbf{A}$  and  $\mathbf{B}$  are *disjoint* if and only if  $\mathbf{A} \cap \mathbf{B} = \emptyset$ .

## 1.9 Relative Complement

**Definition 1.9.1** (Relative Complement). For any classes  $\mathbf{A}$  and  $\mathbf{B}$ , the *relative complement*  $\mathbf{A} - \mathbf{B}$  is  $\{x \in \mathbf{A} \mid x \notin \mathbf{B}\}$ .

# Chapter 2

## Sets

### 2.1 Membership

We take as undefined the notion of *set*.

We take as undefined the binary relation of *membership*,  $\in$ . If  $a \in A$  we say  $a$  is a *member* or *element* of  $A$ . If this does not hold, we write  $a \notin A$ .

**Axiom 2.1.1** (Axiom of Extensionality). *Two sets with exactly the same elements are equal.*

We may therefore identify the set  $A$  with the class  $\{x \mid x \in A\}$ .

We say a class  $\mathbf{A}$  *is a set* iff there exists a set  $A$  such that  $A = \mathbf{A}$ . That is,  $\{x \mid P(x)\}$  is a set if and only if there exists a set  $A$  such that, for all  $x$ , we have  $x \in A$  if and only if  $P(x)$ .

### 2.2 The Empty Set

**Axiom 2.2.1** (Empty Set Axiom). *The empty class  $\emptyset$  is a set.*

### 2.3 Pair Sets

**Axiom 2.3.1** (Pairing Axiom). *For any objects  $u$  and  $v$ , the class  $\{u, v\}$  is a set.*

**Theorem 2.3.2.** *For any object  $a$ , the class  $\{a\}$  is a set.*

PROOF: It is  $\{a, a\}$ .  $\square$

### 2.4 Unions

**Definition 2.4.1** (Union). For any class of sets  $\mathbf{A}$ , the *union*  $\bigcup \mathbf{A}$  is the class  $\{x \mid \exists A \in \mathbf{A}. x \in A\}$ .

**Axiom 2.4.2** (Union Axiom). *For any set  $A$ , the union  $\bigcup A$  is a set.*

**Theorem 2.4.3.** *For any sets  $A$  and  $B$ , the class  $A \cup B$  is a set.*

PROOF: It is  $\bigcup\{A, B\}$ .  $\square$

**Theorem Schema 2.4.4.** *For any objects  $a_1, \dots, a_n$ , the class  $\{a_1, \dots, a_n\}$  is a set.*

PROOF: It is  $\{a_1\} \cup \dots \cup \{a_n\}$ .  $\square$

## 2.5 Power Set

**Definition 2.5.1** (Power Class). For any class  $\mathbf{A}$ , the *power class*  $\mathcal{P}\mathbf{A}$  is the class of all subsets of  $\mathbf{A}$ .

**Axiom 2.5.2** (Power Set Axiom). *For any set  $A$ , the power class  $\mathcal{P}A$  is a set.*

## 2.6 Covers

**Definition 2.6.1** (Cover). Let  $\mathbf{X}$  be a class and  $\mathcal{A} \subseteq \mathcal{P}\mathbf{X}$ . Then  $\mathcal{A}$  *covers*  $\mathbf{X}$ , or is a *covering* of  $\mathbf{X}$ , if and only if  $\bigcup \mathcal{A} = \mathbf{X}$ .

## 2.7 Subset Axioms

**Axiom Schema 2.7.1** (Subset Axioms, Aussonderung Axioms). *For any classes  $\mathbf{A}$  and set  $B$ , if  $\mathbf{A} \subseteq B$  then  $\mathbf{A}$  is a set.*

**Theorem 2.7.2.** *The universal class  $\mathbf{V}$  is not a set.*

PROOF:

$\langle 1 \rangle 1$ . ASSUME:  $\mathbf{V}$  is a set.

$\langle 1 \rangle 2$ . LET:  $R = \{x \in \mathbf{V} \mid x \notin x\}$

$\langle 1 \rangle 3$ .  $R \in R$  if and only if  $R \notin R$

$\langle 1 \rangle 4$ . Q.E.D.

PROOF: This is a contradiction.

$\square$

**Theorem 2.7.3.** *If  $A$  is a set and  $\mathbf{B}$  is a class then  $A - \mathbf{B}$  is a set.*

PROOF: It is a subset of  $A$ .  $\square$

**Theorem 2.7.4.** *For any set  $A$  and class  $\mathbf{B}$ , the class  $A \cap \mathbf{B}$  is a set.*

PROOF: It is a subset of  $A$ .

## 2.8 Intersection

**Definition 2.8.1** (Intersection). For any class  $\mathbf{A}$  of sets, the *intersection*  $\bigcap \mathbf{A}$  is the class  $\{x \mid \forall A \in \mathbf{A}. x \in A\}$ .

**Theorem 2.8.2.** For any nonempty class  $\mathbf{A}$  of sets, we have  $\bigcap \mathbf{A}$  is a set.

PROOF:

$\langle 1 \rangle 1$ . PICK  $A \in \mathbf{A}$

$\langle 1 \rangle 2$ .  $\bigcap \mathbf{A} \subseteq A$

$\langle 1 \rangle 3$ . Q.E.D.

PROOF: By a Subset Axiom.

□

## 2.9 Pairwise Disjoint Classes

**Definition 2.9.1** (Pairwise Disjoint). Let  $\mathbf{A}$  be a class of sets. Then  $\mathbf{A}$  is *pairwise disjoint* iff any two distinct elements of  $\mathbf{A}$  are disjoint.

## 2.10 Axiom of Choice

**Axiom 2.10.1** (Axiom of Choice). Let  $\mathcal{A}$  be a set of pairwise disjoint nonempty sets. Then there exists a set  $C$  containing exactly one element from each member of  $\mathcal{A}$ .

## 2.11 Axiom of Regularity

**Axiom 2.11.1** (Regularity). For any nonempty set  $A$ , there exists  $m \in A$  such that  $m$  and  $A$  are disjoint.

**Theorem 2.11.2.** No set is a member of itself.

PROOF: From the Axiom of Regularity, for any set  $A$ , we have  $A$  and  $\{A\}$  are disjoint, i.e.  $A \notin A$ . □

**Theorem 2.11.3.** There are no sets  $A$  and  $B$  such that  $A \in B$  and  $B \in A$ .

PROOF:

$\langle 1 \rangle 1$ . LET:  $A$  and  $B$  be sets.

$\langle 1 \rangle 2$ . PICK  $m \in \{A, B\}$  such that  $m \cap \{A, B\} = \emptyset$

$\langle 1 \rangle 3$ . CASE:  $m = A$

PROOF: In this case  $B \notin A$ .

$\langle 1 \rangle 4$ . CASE:  $m = B$

PROOF: In this case  $A \notin B$ .

□

## 2.12 Transitive Sets

**Definition 2.12.1** (Transitive Set). A set  $A$  is *transitive* if and only if, whenever  $x \in y \in A$  then  $x \in A$ .

**Theorem 2.12.2.** *Let  $A$  be a set. Then the following are equivalent.*

1.  $A$  is transitive.
2.  $\bigcup A \subseteq A$
3. For all  $a \in A$  we have  $a \subseteq A$
4.  $A \subseteq \mathcal{P}A$

PROOF: From definitions.  $\square$

## 2.13 Partitions

**Definition 2.13.1** (Partition). A *partition*  $P$  of a set  $A$  is a set of nonempty subsets of  $A$  such that:

1. For all  $x \in A$  there exists  $S \in P$  such that  $x \in S$ .
2. Any two distinct elements of  $P$  are disjoint.

# Chapter 3

## Relations

### 3.1 Ordered Pairs

**Definition 3.1.1** (Ordered Pair). For any sets  $x$  and  $y$ , the *ordered pair*  $(x, y)$  is defined to be  $\{\{x\}, \{x, y\}\}$ .

**Theorem 3.1.2.** For any sets  $u, v, x, y$ , we have  $(u, v) = (x, y)$  if and only if  $u = x$  and  $v = y$

PROOF:

$\langle 1 \rangle 1$ . ASSUME:  $\{\{u\}, \{u, v\}\} = \{\{x\}, \{x, y\}\}$

$\langle 1 \rangle 2$ .  $\{u\} \in \{\{x\}, \{x, y\}\}$

$\langle 1 \rangle 3$ .  $\{u, v\} \in \{\{x\}, \{x, y\}\}$

$\langle 1 \rangle 4$ .  $\{u\} = \{x\}$  or  $\{u\} = \{x, y\}$

$\langle 1 \rangle 5$ .  $\{u, v\} = \{x\}$  or  $\{u, v\} = \{x, y\}$

$\langle 1 \rangle 6$ . CASE:  $\{u\} = \{x, y\}$

$\langle 2 \rangle 1$ .  $u = x = y$

$\langle 2 \rangle 2$ .  $u = v = x = y$

PROOF: From  $\langle 1 \rangle 5$

$\langle 1 \rangle 7$ . CASE:  $\{u, v\} = \{x\}$

PROOF: Similar.

$\langle 1 \rangle 8$ . CASE:  $\{u\} = \{x\}$  and  $\{u, v\} = \{x, y\}$

$\langle 2 \rangle 1$ .  $u = x$

$\langle 2 \rangle 2$ .  $u = y$  or  $v = y$

$\langle 2 \rangle 3$ . CASE:  $u = y$

PROOF: This case is the case considered in  $\langle 1 \rangle 6$ .

$\langle 2 \rangle 4$ . CASE:  $v = y$

PROOF: We have  $u = x$  and  $v = y$  as required.

□

**Lemma 3.1.3.** Let  $x, y$  and  $C$  be sets. If  $x \in C$  and  $y \in C$  then  $(x, y) \in \mathcal{PPC}$ .

PROOF:

- $\langle 1 \rangle 1.$  LET:  $x, y$  and  $C$  be sets.
  - $\langle 1 \rangle 2.$  ASSUME:  $x \in C$
  - $\langle 1 \rangle 3.$  ASSUME:  $y \in C$
  - $\langle 1 \rangle 4.$   $\{x\} \in \mathcal{P}C$
  - $\langle 1 \rangle 5.$   $\{x, y\} \in \mathcal{P}C$
  - $\langle 1 \rangle 6.$   $\{\{x\}, \{x, y\}\} \in \mathcal{P}\mathcal{P}C$
- 

**Lemma 3.1.4.** *Let  $x, y$  and  $A$  be sets. If  $(x, y) \in A$  then  $x$  and  $y$  belong to  $\bigcup \bigcup A$ .*

PROOF:

- $\langle 1 \rangle 1.$  LET:  $x, y$  and  $A$  be sets.
  - $\langle 1 \rangle 2.$  ASSUME:  $(x, y) \in A$
  - $\langle 1 \rangle 3.$   $\{x, y\} \in \bigcup A$
  - $\langle 1 \rangle 4.$   $x \in \bigcup \bigcup A$
  - $\langle 1 \rangle 5.$   $y \in \bigcup \bigcup A$
- 

## 3.2 Cartesian Product

**Definition 3.2.1** (Cartesian Product). Let  $\mathbf{A}$  and  $\mathbf{B}$  be classes. The *Cartesian product*  $\mathbf{A} \times \mathbf{B}$  is the class  $\{(x, y) \mid x \in \mathbf{A}, y \in \mathbf{B}\}$ .

**Theorem 3.2.2.** *For any sets  $A$  and  $B$ , the Cartesian product  $A \times B$  is a set.*

PROOF: It is a subset of  $\mathcal{P}\mathcal{P}(A \cup B)$  by Lemma 3.1.3. □

## 3.3 Relations

**Definition 3.3.1** (Relation). A *relation* is a class of ordered pairs.

Given a relation  $\mathbf{R}$ , we write  $x\mathbf{R}y$  for  $(x, y) \in \mathbf{R}$ .

A relation is *small* iff it is a set.

## 3.4 Domain

**Definition 3.4.1** (Domain). Let  $\mathbf{R}$  be a relation. The *domain* of  $\mathbf{R}$  is  $\text{dom } \mathbf{R} = \{x \mid \exists y. x\mathbf{R}y\}$ .

**Theorem 3.4.2.** *For any set  $R$ , the domain  $\text{dom } R$  is a set.*

PROOF: It is a subset of  $\bigcup \bigcup R$  by Lemma 3.1.4. □

### 3.5 Range

**Definition 3.5.1** (Range). Let  $\mathbf{R}$  be a relation. The *range* of  $\mathbf{R}$  is  $\text{ran } \mathbf{R} = \{y \mid \exists x. x\mathbf{R}y\}$ .

**Theorem 3.5.2.** For any set  $R$ , the range  $\text{ran } R$  is a set.

PROOF: It is a subset of  $\bigcup \bigcup R$  by Lemma 3.1.4.  $\square$

### 3.6 Single-Rooted

**Definition 3.6.1** (Single-Rooted). A relation  $\mathbf{R}$  is *single-rooted* if and only if, for all  $x, x', y$ , if  $x\mathbf{R}y$  and  $x'\mathbf{R}y$  then  $x = x'$ .

### 3.7 Inverse

**Definition 3.7.1** (Inverse). Let  $\mathbf{R}$  be a class. The *inverse* of  $\mathbf{R}$  is  $\mathbf{R}^{-1} = \{(y, x) \mid x\mathbf{R}y\}$ .

**Theorem 3.7.2.** For any small relation  $R$ , the inverse  $R^{-1}$  is small.

PROOF: It is a subset of  $\text{ran } R \times \text{dom } R$ .  $\square$

**Theorem 3.7.3.** For any relation  $\mathbf{F}$ , we have  $\text{dom } \mathbf{F}^{-1} = \text{ran } \mathbf{F}$ .

PROOF: For any  $x$ , we have

$$\begin{aligned} x \in \text{dom } \mathbf{F}^{-1} &\Leftrightarrow \exists y. (x, y) \in \mathbf{F}^{-1} \\ &\Leftrightarrow \exists y. (y, x) \in \mathbf{F} \\ &\Leftrightarrow x \in \text{ran } \mathbf{F} \end{aligned} \quad \square$$

**Theorem 3.7.4.** For any relation  $\mathbf{F}$ , we have  $\text{ran } \mathbf{F}^{-1} = \text{dom } \mathbf{F}$ .

PROOF: For any  $x$ , we have

$$\begin{aligned} x \in \text{ran } \mathbf{F}^{-1} &\Leftrightarrow \exists y. (y, x) \in \mathbf{F}^{-1} \\ &\Leftrightarrow \exists y. (x, y) \in \mathbf{F} \\ &\Leftrightarrow x \in \text{dom } \mathbf{F} \end{aligned} \quad \square$$

**Theorem 3.7.5.** For any relation  $\mathbf{F}$ , we have  $(\mathbf{F}^{-1})^{-1} = \mathbf{F}$ .

PROOF: For any  $z$  we have

$$\begin{aligned} z \in (\mathbf{F}^{-1})^{-1} &\Leftrightarrow \exists x, y. z = (x, y) \wedge (y, x) \in \mathbf{F}^{-1} \\ &\Leftrightarrow \exists x, y. z = (x, y) \wedge (x, y) \in \mathbf{F} \\ &\Leftrightarrow z \in \mathbf{F} \end{aligned} \quad (\mathbf{F} \text{ is a relation}) \square$$



### 3.8 Composition

**Definition 3.8.1** (Composition). Let  $\mathbf{R}$  and  $\mathbf{S}$  be relations. The *composition* of  $\mathbf{R}$  and  $\mathbf{S}$  is  $\mathbf{S} \circ \mathbf{R} = \{(x, z) \mid \exists y. x\mathbf{R}y \wedge y\mathbf{S}z\}$ .

**Theorem 3.8.2.** *If  $R$  and  $S$  are small relations then  $S \circ R$  is small.*

PROOF: It is a subset of  $\text{dom } R \times \text{ran } S$ .  $\square$

**Theorem 3.8.3.** *For any relations  $\mathbf{F}$  and  $\mathbf{G}$ , we have  $(\mathbf{G} \circ \mathbf{F})^{-1} = \mathbf{F}^{-1} \circ \mathbf{G}^{-1}$ .*

PROOF:

$$\begin{aligned}
 (x, z) \in (\mathbf{G} \circ \mathbf{F})^{-1} &\Leftrightarrow (z, x) \in \mathbf{G} \circ \mathbf{F} \\
 &\Leftrightarrow \exists y. z\mathbf{F}y \wedge y\mathbf{G}x \\
 &\Leftrightarrow \exists y. (y, z) \in \mathbf{F}^{-1} \wedge (x, y) \in \mathbf{G}^{-1} \\
 &\Leftrightarrow (x, z) \in \mathbf{F}^{-1} \circ \mathbf{G}^{-1} \quad \square
 \end{aligned}$$

### 3.9 Restriction

**Definition 3.9.1** (Restriction). Let  $\mathbf{R}$  be a relation and  $\mathbf{A}$  a class. The *restriction* of  $\mathbf{R}$  to  $\mathbf{A}$  is  $\mathbf{R} \upharpoonright \mathbf{A} = \{(x, y) \mid x \in \mathbf{A} \wedge x\mathbf{R}y\}$ .

**Theorem 3.9.2.** *If  $R$  is a small relation then  $R \upharpoonright \mathbf{A}$  is small.*

PROOF: Since it is a subset of  $R$ .  $\square$

### 3.10 Image

**Definition 3.10.1** (Image). Let  $\mathbf{F}$  be a relation and  $\mathbf{A}$  a class. The *image* of  $\mathbf{A}$  under  $\mathbf{F}$  is  $\mathbf{F}(\mathbf{A}) = \{\mathbf{F}(x) \mid x \in \mathbf{A}\}$ .

**Theorem 3.10.2.** *If  $F$  is small then  $F(\mathbf{A})$  is a set.*

PROOF: Since it is a subset of  $\text{ran } F$ .  $\square$

**Theorem 3.10.3.** *For any relation  $\mathbf{F}$  and class of sets  $\mathcal{A}$  we have*

$$\mathbf{F}\left(\bigcup \mathcal{A}\right) = \bigcup_{A \in \mathcal{A}} \mathbf{F}(A)$$

PROOF: Each is the class of all  $y$  such that  $\exists x. \exists A. x \in A \in \mathcal{A} \wedge y = \mathbf{F}(x)$ .  $\square$

**Theorem 3.10.4.** *For any relation  $\mathbf{F}$  and classes  $\mathbf{A}_1, \dots, \mathbf{A}_n$ , we have*

$$\mathbf{F}(\mathbf{A}_1 \cup \dots \cup \mathbf{A}_n) = \mathbf{F}(\mathbf{A}_1) \cup \dots \cup \mathbf{F}(\mathbf{A}_n) .$$

PROOF: Similar.  $\square$

**Theorem 3.10.5.** *For any relation  $\mathbf{F}$  and class of sets  $\mathcal{A}$ , we have*

$$\mathbf{F}\left(\bigcap \mathcal{A}\right) \subseteq \bigcap_{A \in \mathcal{A}} \mathbf{F}(A) \ .$$

*Equality holds if  $\mathbf{F}$  is single-rooted and  $\mathcal{A}$  is nonempty.*

PROOF:

- $\langle 1 \rangle 1.$   $\mathbf{F}(\bigcap \mathcal{A}) \subseteq \bigcap_{A \in \mathcal{A}} \mathbf{F}(A)$
- $\langle 2 \rangle 1.$  LET:  $y \in \mathbf{F}(\bigcap \mathcal{A})$
- $\langle 2 \rangle 2.$  PICK  $x \in \bigcap \mathcal{A}$  such that  $y = \mathbf{F}(x)$
- $\langle 2 \rangle 3.$  LET:  $A \in \mathcal{A}$
- $\langle 2 \rangle 4.$   $x \in A$
- $\langle 2 \rangle 5.$   $y \in \mathbf{F}(A)$
- $\langle 1 \rangle 2.$  If  $\mathbf{F}$  is single-rooted then  $\mathbf{F}(\bigcap \mathcal{A}) = \bigcap_{A \in \mathcal{A}} \mathbf{F}(A)$
- $\langle 2 \rangle 1.$  ASSUME:  $\mathbf{F}$  is single-rooted and  $\mathcal{A}$  is nonempty.
- $\langle 2 \rangle 2.$  LET:  $y \in \bigcap_{A \in \mathcal{A}} \mathbf{F}(A)$
- $\langle 2 \rangle 3.$  PICK  $A \in \mathcal{A}$
- $\langle 2 \rangle 4.$  PICK  $x \in A$  such that  $y = \mathbf{F}(x)$
- $\langle 2 \rangle 5.$   $x \in \bigcap \mathcal{A}$
- $\langle 3 \rangle 1.$  LET:  $A' \in \mathcal{A}$
- $\langle 3 \rangle 2.$  PICK  $x' \in A'$  such that  $y = \mathbf{F}(x')$
- $\langle 3 \rangle 3.$   $x = x'$
- PROOF: BY  $\langle 2 \rangle 1.$
- $\langle 3 \rangle 4.$   $x \in A'$

□

**Theorem 3.10.6.** *For any relation  $\mathbf{F}$  and classes  $\mathbf{A}_1, \dots, \mathbf{A}_n$ , we have*

$$\mathbf{F}(\mathbf{A}_1 \cap \dots \cap \mathbf{A}_n) \subseteq \mathbf{F}(\mathbf{A}_1) \cap \dots \cap \mathbf{F}(\mathbf{A}_n) \ .$$

*Equality holds if  $\mathbf{F}$  is single-rooted.*

PROOF: Similar.

**Theorem 3.10.7.** *For any relation  $\mathbf{F}$  and classes  $\mathbf{A}$  and  $\mathbf{B}$ , we have*

$$\mathbf{F}(\mathbf{A}) - \mathbf{F}(\mathbf{B}) \subseteq \mathbf{F}(\mathbf{A} - \mathbf{B}) \ .$$

*Equality holds if  $\mathbf{F}$  is single-rooted.*

PROOF:

- $\langle 1 \rangle 1.$  LET:  $\mathbf{F}$ ,  $\mathbf{A}$  and  $\mathbf{B}$  be sets.
- $\langle 1 \rangle 2.$   $\mathbf{F}(\mathbf{A}) - \mathbf{F}(\mathbf{B}) \subseteq \mathbf{F}(\mathbf{A} - \mathbf{B})$
- $\langle 2 \rangle 1.$  LET:  $y \in \mathbf{F}(\mathbf{A}) - \mathbf{F}(\mathbf{B})$
- $\langle 2 \rangle 2.$  PICK  $x \in \mathbf{A}$  such that  $x\mathbf{F}y$
- $\langle 2 \rangle 3.$   $x \in \mathbf{A} - \mathbf{B}$
- $\langle 1 \rangle 3.$  If  $\mathbf{F}$  is single-rooted then  $\mathbf{F}(\mathbf{A} - \mathbf{B}) = \mathbf{F}(\mathbf{A}) - \mathbf{F}(\mathbf{B})$ .
- $\langle 2 \rangle 1.$  ASSUME:  $\mathbf{F}$  is single-rooted.

⟨2⟩2. LET:  $y \in \mathbf{F}(\mathbf{A} - \mathbf{B})$   
 ⟨2⟩3. PICK  $x \in \mathbf{A} - \mathbf{B}$  such that  $y = \mathbf{F}(x)$   
 ⟨2⟩4.  $y \in \mathbf{F}(\mathbf{A})$   
 ⟨2⟩5.  $y \notin \mathbf{F}(\mathbf{B})$   
 ⟨3⟩1. ASSUME: for a contradiction  $x' \in \mathbf{B}$  and  $x' \mathbf{F} y$   
 ⟨3⟩2.  $x' = x$   
 PROOF: From ⟨2⟩1  
 ⟨3⟩3.  $x \in \mathbf{B}$   
 ⟨3⟩4. Q.E.D.  
 PROOF: This contradicts ⟨2⟩3.  
 □

### 3.11 Reflexive Relations

**Definition 3.11.1** (Reflexive). Let  $\mathbf{R}$  be a relation on  $\mathbf{A}$ . Then  $\mathbf{R}$  is *reflexive* on  $A$  if and only if, for all  $x \in \mathbf{A}$ , we have  $x\mathbf{R}x$ .

### 3.12 Symmetric

**Definition 3.12.1** (Symmetric (Pairing)). Let  $\mathbf{R}$  be a relation. Then  $\mathbf{R}$  is *symmetric* if and only if, whenever  $x\mathbf{R}y$ , then  $y\mathbf{R}x$ .

### 3.13 Transitivity

**Definition 3.13.1** (Transitivity (Pairing)). Let  $\mathbf{R}$  be a relation. Then  $\mathbf{R}$  is *transitive* if and only if, whenever  $x\mathbf{R}y$  and  $y\mathbf{R}z$ , then  $x\mathbf{R}z$ .

### 3.14 Equivalence Relations

**Definition 3.14.1** (Equivalence Relation (Pairing)). Let  $\mathbf{R}$  be a relation on  $\mathbf{A}$ . Then  $\mathbf{R}$  is an *equivalence relation* on  $\mathbf{A}$  if and only if  $\mathbf{R}$  is reflexive on  $\mathbf{A}$ , symmetric and transitive.

**Theorem 3.14.2.** *If  $\mathbf{R}$  is a symmetric and transitive relation then  $\mathbf{R}$  is an equivalence relation on  $\text{fld } \mathbf{R}$ .*

PROOF:

⟨1⟩1. LET:  $\mathbf{R}$  be a symmetric and transitive relation.  
 ⟨1⟩2. LET:  $x \in \text{fld } \mathbf{R}$   
 ⟨1⟩3. PICK  $y$  such that  $x\mathbf{R}y$  or  $y\mathbf{R}x$   
 ⟨1⟩4.  $x\mathbf{R}y$  and  $y\mathbf{R}x$   
 PROOF: By symmetry.  
 ⟨1⟩5.  $x\mathbf{R}x$   
 PROOF: By transitivity.  
 □

### 3.15 Equivalence Class

**Definition 3.15.1** (Equivalence Class). Let  $\mathbf{R}$  be an equivalence relation on  $\mathbf{A}$  and  $a \in \mathbf{A}$ . Then the *equivalence class* of  $a$  modulo  $\mathbf{R}$  is

$$[a]_{\mathbf{R}} = \{x \in \mathbf{A} \mid a\mathbf{R}x\} .$$

**Lemma 3.15.2.** Let  $\mathbf{R}$  be an equivalence relation on  $\mathbf{A}$  and  $x, y \in \mathbf{A}$ . Then  $[x]_{\mathbf{R}} = [y]_{\mathbf{R}}$  if and only if  $x\mathbf{R}y$ .

PROOF:

$\langle 1 \rangle 1$ . If  $[x]_{\mathbf{R}} = [y]_{\mathbf{R}}$  then  $x\mathbf{R}y$ .

$\langle 2 \rangle 1$ . ASSUME:  $[x]_{\mathbf{R}} = [y]_{\mathbf{R}}$

$\langle 2 \rangle 2$ .  $y \in [y]_{\mathbf{R}}$

PROOF: Since  $y\mathbf{R}y$  by reflexivity.

$\langle 2 \rangle 3$ .  $y \in [x]_{\mathbf{R}}$

$\langle 2 \rangle 4$ .  $x\mathbf{R}y$

$\langle 1 \rangle 2$ . If  $x\mathbf{R}y$  then  $[x]_{\mathbf{R}} = [y]_{\mathbf{R}}$ .

$\langle 2 \rangle 1$ . ASSUME:  $x\mathbf{R}y$

$\langle 2 \rangle 2$ .  $[y]_{\mathbf{R}} \subseteq [x]_{\mathbf{R}}$

PROOF: If  $y\mathbf{R}z$  then  $x\mathbf{R}z$  by transitivity.

$\langle 2 \rangle 3$ .  $[x]_{\mathbf{R}} \subseteq [y]_{\mathbf{R}}$

PROOF: Similar since  $y\mathbf{R}x$  by symmetry.

□

### 3.16 Quotient Sets

**Definition 3.16.1** (Quotient Set). Let  $R$  be an equivalence relation on  $A$ . The *quotient set*  $A/R$  is the set of all equivalence classes modulo  $R$ .

This is a set because it is a subset of  $\mathcal{P}A$ .

**Theorem 3.16.2.** Let  $R$  be an equivalence relation on  $A$ . Then the quotient set  $A/R$  is a partition of  $A$ .

PROOF:

$\langle 1 \rangle 1$ . For all  $x \in A$  there exists  $y \in A$  such that  $x \in [y]_R$

PROOF: Take  $y = x$ .

$\langle 1 \rangle 2$ . Any two distinct equivalence classes are disjoint.

$\langle 2 \rangle 1$ . ASSUME:  $z \in [x]_R$  and  $z \in [y]_R$

$\langle 2 \rangle 2$ .  $xRz$  and  $yRz$

$\langle 2 \rangle 3$ .  $[x]_R = [z]_R = [y]_R$

PROOF: Lemma 3.15.2.

□

### 3.17 Minimal Elements

**Definition 3.17.1** (Minimal). Let  $R$  be a binary relation and  $A$  a set. An element  $a \in A$  is *minimal* w.r.t.  $R$  iff there is no  $x \in A$  such that  $xRa$ .

### 3.18 Well-Founded Relations

**Definition 3.18.1** (Well-Founded). Let  $R$  be a relation on  $A$ . Then  $R$  is *well-founded* iff every nonempty subset of  $A$  has an  $R$ -minimal element.

**Theorem 3.18.2** (Transfinite Induction). Let  $R$  be a well-founded relation on  $A$  and  $B \subseteq A$ . Assume that, for every  $t \in A$ , if  $\{x \in A \mid xRt\} \subseteq B$  then  $t \in B$ . Then we have  $B = A$ .

PROOF:

- $\langle 1 \rangle 1$ . ASSUME: for a contradiction  $B \neq A$
- $\langle 1 \rangle 2$ . PICK an  $R$ -minimal element  $t$  of  $A - B$
- $\langle 1 \rangle 3$ . For all  $x \in A$ , if  $xRt$  then  $x \in B$
- $\langle 1 \rangle 4$ .  $t \in B$
- $\langle 1 \rangle 5$ . Q.E.D.

PROOF: This contradicts  $\langle 1 \rangle 2$ .

□

### 3.19 Transitive Closure

**Theorem 3.19.1.** Let  $R$  be a relation. Then there exists a unique relation  $R^t$  such that  $R^t$  is transitive,  $R \subseteq R^t$ , and for every transitive relation  $S$  with  $R \subseteq S$  we have  $R^t \subseteq S$ .

PROOF:

- $\langle 1 \rangle 1$ . LET:  $R^t = \bigcap \{S \mid R \subseteq S, S \text{ is a transitive relation}\}$
- $\langle 1 \rangle 2$ .  $R^t$  is transitive
  - $\langle 2 \rangle 1$ . LET:  $(x, y), (y, z) \in R^t$   
PROVE:  $(x, z) \in R^t$
  - $\langle 2 \rangle 2$ . LET:  $S$  be a transitive relation with  $R \subseteq S$
  - $\langle 2 \rangle 3$ .  $xSy$  and  $ySz$
  - $\langle 2 \rangle 4$ .  $xSz$
- $\langle 1 \rangle 3$ .  $R \subseteq R^t$
- $\langle 1 \rangle 4$ . For any transitive relation  $S$  with  $R \subseteq S$  we have  $R^t \subseteq S$
- $\langle 1 \rangle 5$ .  $R^t$  is unique.

PROOF: If  $S$  satisfies the same properties then  $R^t \subseteq S$  and  $S \subseteq R^t$ .

□

**Definition 3.19.2** (Transitive Closure). The *transitive closure* of a relation  $R$  is this relation  $R^t$ .

**Theorem 3.19.3.** If  $R$  is well-founded then  $R^t$  is well-founded.

PROOF:

- $\langle 1 \rangle 1$ . LET:  $R$  be a well-founded relation on  $A$
- $\langle 1 \rangle 2$ . For all  $x, y \in A$ , if  $xR^t y$  then there exists  $z$  such that  $zR^t y$ 
  - $\langle 2 \rangle 1$ . LET:  $S = \{(x, y) \mid \exists z. zRy\}$   
PROVE:  $R^t \subseteq S$

$\langle 2 \rangle 2$ .  $S$  is transitive  
 $\langle 3 \rangle 1$ . ASSUME:  $xSy$  and  $ySz$   
 $\langle 3 \rangle 2$ . There exists  $t$  such that  $tRz$   
 $\langle 3 \rangle 3$ .  $xSz$   
 $\langle 2 \rangle 3$ .  $R \subseteq S$   
 $\langle 1 \rangle 3$ . LET:  $B \subseteq A$  be nonempty.  
 $\langle 1 \rangle 4$ . PICK an  $R$ -minimal element  $b$  of  $B$   
 $\langle 1 \rangle 5$ .  $b$  is  $R^t$ -minimal.  
 PROOF: From  $\langle 1 \rangle 2$ .  
 $\square$

# Chapter 4

## Functions

### 4.1 Functions

**Definition 4.1.1** (Class Function). A *class function* is a relation  $\mathbf{F}$  such that, for all  $x, y, y'$ , if  $x\mathbf{F}y$  and  $x\mathbf{F}y'$  then  $y = y'$ .

If  $\mathbf{F}$  is a class function and  $x \in \text{dom } \mathbf{F}$ , then we write  $\mathbf{F}(x)$  for the unique  $y$  such that  $x\mathbf{F}y$ .

We write  $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$  iff  $\mathbf{F}$  is a class function,  $\text{dom } \mathbf{F} = \mathbf{A}$  and  $\text{ran } \mathbf{F} \subseteq \mathbf{B}$ .

A *function* is a class function that is a set.

**Theorem 4.1.2.** *The Axiom of Choice is equivalent to the following statement:*

*For any small relation  $R$ , there exists a function  $H \subseteq R$  such that  $\text{dom } H = \text{dom } R$ .*

PROOF:

$\langle 1 \rangle 1$ . If, for any small relation  $R$ , there exists a function  $H \subseteq R$  such that  $\text{dom } H = \text{dom } R$ , then the Axiom of Choice is true.

$\langle 2 \rangle 1$ . ASSUME: for any small relation  $R$ , there exists a function  $H \subseteq R$  such that  $\text{dom } H = \text{dom } R$

$\langle 2 \rangle 2$ . LET:  $\mathcal{A}$  be a set of pairwise disjoint nonempty sets.

$\langle 2 \rangle 3$ . LET:  $R = \{(A, a) \mid A \in \mathcal{A}, a \in A\}$

$\langle 2 \rangle 4$ . PICK a function  $H \subseteq R$  such that  $\text{dom } H = \text{dom } R$

PROOF: By  $\langle 2 \rangle 1$ .

$\langle 2 \rangle 5$ . LET:  $C = \text{ran } H$

$\langle 2 \rangle 6$ .  $C$  contains exactly one element from each  $A \in \mathcal{A}$ , namely  $H(A)$ .

$\langle 1 \rangle 2$ . If the Axiom of Choice is true then, for any small relation  $R$ , there exists a function  $H \subseteq R$  such that  $\text{dom } H = \text{dom } R$ .

$\langle 2 \rangle 1$ . ASSUME: the Axiom of Choice

$\langle 2 \rangle 2$ . LET:  $R$  be a small relation.

$\langle 2 \rangle 3$ . For  $a \in \text{dom } R$ ,

LET:  $R_a = \{(a, b) \mid aRb\}$

$\langle 2 \rangle 4$ . LET:  $\mathcal{A} = \{R_a \mid a \in \text{dom } R\}$

(2)5. PICK a set  $H$  that contains exactly one element from each  $R_a$ .

PROOF: By the Axiom of Choice ( $\langle 2 \rangle 1$ ).

(2)6.  $H$  is a function,  $H \subseteq R$  and  $\text{dom } H = \text{dom } R$ .

□

**Theorem 4.1.3.** *For any relation  $\mathbf{F}$ , we have  $\mathbf{F}^{-1}$  is a class function if and only if  $\mathbf{F}$  is single-rooted.*

PROOF: Immediate from definitions. □

**Theorem 4.1.4.** *Let  $\mathbf{F}$  be a relation. Then  $\mathbf{F}$  is a class function if and only if  $\mathbf{F}^{-1}$  is single-rooted.*

PROOF: Immediate from definitions. □

**Theorem 4.1.5.** *Let  $\mathbf{F}$  and  $\mathbf{G}$  be class functions. Then  $\mathbf{G} \circ \mathbf{F}$  is a function, its domain is  $\{x \in \text{dom } \mathbf{F} \mid \mathbf{F}(x) \in \text{dom } \mathbf{G}\}$ , and for  $x$  in this domain, we have  $(\mathbf{F} \circ \mathbf{G})(x) = \mathbf{F}(\mathbf{G}(x))$ .*

PROOF:

(1)1.  $\mathbf{G} \circ \mathbf{F}$  is a class function.

(2)1. LET:  $x(\mathbf{G} \circ \mathbf{F})z$  and  $x(\mathbf{G} \circ \mathbf{F})z'$

(2)2. PICK  $y, y'$  such that  $x\mathbf{F}y, x\mathbf{F}y', y\mathbf{G}z$  and  $y'\mathbf{G}z'$

(2)3.  $y = y'$

PROOF: Since  $\mathbf{F}$  is a class function.

(2)4.  $z = z'$

PROOF: Since  $\mathbf{G}$  is a class function.

(1)2.  $\text{dom}(\mathbf{G} \circ \mathbf{F}) = \{x \in \text{dom } \mathbf{F} \mid \mathbf{F}(x) \in \text{dom } \mathbf{G}\}$

PROOF:

$$\begin{aligned} x \in \text{dom}(\mathbf{G} \circ \mathbf{F}) &\Leftrightarrow \exists z. x(\mathbf{G} \circ \mathbf{F})z \\ &\Leftrightarrow \exists y, z. x\mathbf{F}y \wedge y\mathbf{G}z \\ &\Leftrightarrow x \in \text{dom } \mathbf{F} \wedge \mathbf{F}(x) \in \text{dom } \mathbf{G} \end{aligned}$$

(1)3. For  $x$  in this domain, we have  $(\mathbf{F} \circ \mathbf{G})(x) = \mathbf{F}(\mathbf{G}(x))$ .

PROOF: Since  $(x, \mathbf{F}(x)) \in \mathbf{F}$  and  $(\mathbf{F}(x), \mathbf{G}(\mathbf{F}(x))) \in \mathbf{G}$ .

□

**Axiom Schema 4.1.6** (Replacement). *Let  $\mathbf{H}$  be a class function. If  $\text{dom } \mathbf{H}$  is a set then  $\mathbf{H}$  is a set.*

## 4.2 Choice Functions

**Definition 4.2.1** (Choice Function). Let  $\mathcal{B}$  be a set of nonempty sets. A *choice function* for  $\mathcal{B}$  is a function  $c : \mathcal{B} \rightarrow \bigcup \mathcal{B}$  such that, for all  $B \in \mathcal{B}$ , we have  $c(B) \in B$ .

**Theorem 4.2.2.** *The Axiom of Choice is equivalent to the statement: every set of nonempty sets has a choice function.*



PROOF:

- ⟨1⟩1. If the Axiom of Choice is true then every set of nonempty sets has a choice function.
- ⟨2⟩1. ASSUME: The Axiom of Choice
- ⟨2⟩2. LET:  $\mathcal{B}$  be a set of nonempty sets.
- ⟨2⟩3. LET:  $R = \{(A, a) \mid A \in \mathcal{B}, a \in A\}$
- ⟨2⟩4.  $R$  is a set.  
PROOF: It is a subset of  $\mathcal{B} \times \bigcup \mathcal{B}$ .
- ⟨2⟩5. PICK a function  $c \subseteq R$  with  $\text{dom } c = \text{dom } R$   
PROOF: Theorem 4.1.2.
- ⟨2⟩6.  $\text{dom } c = \mathcal{B}$ 
  - ⟨3⟩1. LET:  $A \in \mathcal{B}$
  - ⟨3⟩2. PICK  $a \in A$   
PROOF:  $A$  is nonempty (⟨2⟩2)
  - ⟨3⟩3.  $ARa$   
PROOF: By ⟨2⟩3.
  - ⟨3⟩4.  $A \in \text{dom } R$
  - ⟨3⟩5.  $A \in \text{dom } c$   
PROOF: By ⟨2⟩5.
- ⟨2⟩7. For all  $A \in \mathcal{B}$  we have  $c(A) \in A$   
PROOF: From ⟨2⟩5.
- ⟨1⟩2. If every set of nonempty sets has a choice function then the Axiom of Choice is true.
  - ⟨2⟩1. ASSUME: Every set of nonempty sets has a choice function.
  - ⟨2⟩2. LET:  $\mathcal{A}$  be a set of pairwise disjoint nonempty sets.
  - ⟨2⟩3. PICK a choice function  $c$  for  $\mathcal{A}$
  - ⟨2⟩4. LET:  $C = \text{ran } c$
  - ⟨2⟩5.  $C$  contains exactly one element from each  $A \in \mathcal{A}$ , namely  $c(A)$

□

### 4.3 Injective Functions

**Definition 4.3.1** (Injective). We call a class function *one-to-one*, *injective* or an *injection* if and only if it is single-rooted.

**Theorem 4.3.2.** Let  $\mathbf{F}$  be a one-to-one class function and  $x \in \text{dom } \mathbf{F}$ . Then  $\mathbf{F}^{-1}(\mathbf{F}(x)) = x$ .

PROOF: We have  $(x, \mathbf{F}(x)) \in \mathbf{F}$  and so  $(\mathbf{F}(x), x) \in \mathbf{F}^{-1}$ . □

**Theorem 4.3.3.** Let  $\mathbf{F}$  be a one-to-one function and  $y \in \text{ran } \mathbf{F}$ . Then  $\mathbf{F}(\mathbf{F}^{-1}(y)) = y$ .

PROOF: From Theorems 3.7.3, 3.7.5 and 4.3.2. □

## 4.4 Surjective Functions

**Definition 4.4.1** (Surjective). Let  $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$ . Then  $\mathbf{F}$  is *surjective* if and only if  $\text{ran } \mathbf{F} = \mathbf{B}$ .

## 4.5 Bijective Functions

**Definition 4.5.1** (Bijective). A class function  $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$  is *bijective* or a *bijection* if and only if it is injective and surjective.

## 4.6 Identity Function

**Definition 4.6.1** (Identity class function). Let  $\mathbf{A}$  be a class. The *identity class function*  $\text{id}_{\mathbf{A}}$  on  $\mathbf{A}$  is  $\{(x, x) \mid x \in \mathbf{A}\}$ .

**Theorem 4.6.2.** For any set  $A$ , we have  $\text{id}_A$  is a function.

PROOF: It is a subset of  $A \times A$ .  $\square$

**Theorem 4.6.3.** Let  $F : A \rightarrow B$  and  $A$  be nonempty. Then there exists a function  $G : B \rightarrow A$  such that  $G \circ F = \text{id}_A$  if and only if  $F$  is one-to-one.

PROOF:

- $\langle 1 \rangle 1$ . LET:  $F : A \rightarrow B$
- $\langle 1 \rangle 2$ . ASSUME:  $A$  is nonempty
- $\langle 1 \rangle 3$ . If there exists  $G : B \rightarrow A$  such that  $G \circ F = \text{id}_A$  then  $F$  is one-to-one.
  - $\langle 2 \rangle 1$ . ASSUME:  $G : B \rightarrow A$  and  $G \circ F = \text{id}_A$
  - $\langle 2 \rangle 2$ . LET:  $x, y \in A$
  - $\langle 2 \rangle 3$ . ASSUME:  $F(x) = F(y)$
  - $\langle 2 \rangle 4$ .  $x = y$   
PROOF:  $x = G(F(x)) = G(F(y)) = y$ .
- $\langle 1 \rangle 4$ . If  $F$  is one-to-one then there exists  $G : B \rightarrow A$  such that  $G \circ F = \text{id}_A$ .
  - $\langle 2 \rangle 1$ . ASSUME:  $F$  is one-to-one.
  - $\langle 2 \rangle 2$ . PICK  $a \in A$
  - $\langle 2 \rangle 3$ . Define  $G : B \rightarrow A$  by:  $G(y)$  is the  $x$  such that  $F(x) = y$  if  $y \in \text{ran } F$ ,  
otherwise  $G(y) = a$
  - $\langle 2 \rangle 4$ .  $G \circ F = \text{id}_A$   
PROOF: For  $x \in A$  we have  $(G \circ F)(x) = G(F(x)) = x$  by Theorem 4.1.5.

$\square$

**Theorem 4.6.4.** Let  $F : A \rightarrow B$  and  $A$  be nonempty. If there exists a function  $H : B \rightarrow A$  such that  $F \circ H = \text{id}_B$  then  $F$  is surjective.

PROOF:

- $\langle 1 \rangle 1$ . LET:  $F : A \rightarrow B$
- $\langle 1 \rangle 2$ . ASSUME:  $A$  is nonempty.
- $\langle 1 \rangle 3$ . LET:  $H : B \rightarrow A$  satisfy  $F \circ H = \text{id}_B$

- ⟨1⟩4. LET:  $y \in B$
- ⟨1⟩5.  $F(H(y)) = y$ .

□

**Theorem 4.6.5** (Choice). *Let  $F : A \rightarrow B$  and  $A$  be nonempty. If  $F$  is surjective then there exists a function  $H : B \rightarrow A$  such that  $F \circ H = \text{id}_B$ .*

PROOF:

- ⟨1⟩1. ASSUME:  $F$  is surjective.
- ⟨1⟩2. PICK a function  $H \subseteq F^{-1}$  with  $\text{dom } H = B$

PROOF: By the Axiom of Choice.

- ⟨1⟩3.  $H : B \rightarrow A$
- ⟨1⟩4.  $F \circ H = \text{id}_B$ 
  - ⟨2⟩1. LET:  $y \in B$
  - ⟨2⟩2.  $(y, H(y)) \in F^{-1}$
  - ⟨2⟩3.  $(H(y), y) \in F$
  - ⟨2⟩4.  $F(H(y)) = y$

□

## 4.7 Infinite Cartesian Product

**Definition 4.7.1** (Infinite Cartesian Product). Let  $H$  be a function with domain  $I$  such that, for all  $i \in I$ ,  $H(i)$  is a set. The *Cartesian product*  $\prod_{i \in I} H(i)$  is the class of all functions  $f$  with domain  $I$  such that, for all  $i \in I$ , we have  $f(i) \in H(i)$ .

**Theorem 4.7.2.** *If  $H$  is a function with domain  $I$  and  $H(i)$  is a set for all  $i \in I$ , then  $\prod_{i \in I} H(i)$  is a set.*

PROOF: It is a subset of  $\mathcal{P}(I \times \bigcup \text{ran } H)$ . □

**Theorem 4.7.3** (Multiplicative Axiom). *The Axiom of Choice is equivalent to the Multiplicative Axiom: for any function  $H$  with domain  $I$ , if  $H(i)$  is nonempty for all  $i \in I$ , then  $\prod_{i \in I} H(i)$  is nonempty.*

PROOF:

- ⟨1⟩1. If the Axiom of Choice is true then the Multiplicative Axiom is true.
  - ⟨2⟩1. ASSUME: The Axiom of Choice
  - ⟨2⟩2. LET:  $H$  be a function with domain  $I$  such that  $H(i)$  is nonempty for all  $i \in I$ .
  - ⟨2⟩3. PICK a function  $f \subseteq \{(i, x) \mid x \in H(i)\}$
  - ⟨2⟩4.  $f \in \prod_{i \in I} H(i)$
- ⟨1⟩2. If the Multiplicative Axiom is true then the Axiom of Choice is true.
  - ⟨2⟩1. ASSUME: for any function  $H$  with domain  $I$ , if  $H(i)$  is nonempty for all  $i \in I$ , then  $\prod_{i \in I} H(i)$  is nonempty
  - ⟨2⟩2. LET:  $R$  be a relation.
  - ⟨2⟩3. LET:  $I = \text{dom } R$

- (2)4. LET:  $H$  be the function with domain  $I$  such that  $H(i) = \{y \mid iRy\}$  for all  $i$ .  
 (2)5. PICK  $f \in \prod_{i \in I} H(i)$   
 (2)6.  $f \subseteq R$   
 $\square$

## 4.8 Quotient Sets

**Definition 4.8.1** (Canonical Map). Let  $R$  be an equivalence relation on  $A$ . The *canonical map*  $\phi : A \rightarrow A/R$  is the function defined by  $\phi(a) = [a]_R$ .

**Theorem 4.8.2.** Let  $R$  be an equivalence relation on  $A$  and  $F : A \rightarrow B$ . Then the following are equivalent:

1. For all  $x, y \in A$ , if  $xRy$  then  $F(x) = F(y)$ .
2. There exists  $G : A/R \rightarrow B$  such that  $F = G \circ \phi$ , where  $\phi : A \rightarrow A/R$  is the canonical map.

In this case,  $G$  is unique.

PROOF:

- (1)1.  $1 \Rightarrow 2$   
 (2)1. ASSUME: 1  
 (2)2. Let  $G = \{([a]_R, b) \mid F(a) = b\}$   
 (2)3.  $G$  is a function.  
 (3)1. LET:  $(c, b), (c, b') \in G$   
 (3)2. PICK  $a, a' \in A$  such that  $c = [a]_R = [a']_R$  with  $F(a) = b$  and  $F(a') = b'$   
 (3)3.  $aRa'$   
 PROOF: Lemma 3.15.2.  
 (3)4.  $F(a) = F(a')$   
 PROOF: From (2)1.  
 (3)5.  $b = b'$   
 PROOF: From (3)2.  
 (2)4.  $F = G \circ \phi$   
 PROOF: For  $a \in A$  we have  $G(\phi(a)) = G([a]) = F(a)$ .  
 (1)2.  $2 \Rightarrow 1$   
 (2)1. LET:  $G : A/R \rightarrow B$  be such that  $F = G \circ \phi$   
 (2)2. LET:  $x, y \in A$   
 (2)3. ASSUME:  $xRy$   
 (2)4.  $G([x]) = G([y])$   
 PROOF: Lemma 3.15.2  
 (2)5.  $F(x) = F(y)$   
 PROOF: From (2)1.  
 (1)3. If  $G, G' : A/R \rightarrow B$  and  $G \circ \phi = G' \circ \phi$  then  $G = G'$   
 PROOF: For any  $a \in A$  we have  $G([a]) = G'([a])$ .  
 $\square$

## 4.9 Transfinite Recursion

**Theorem 4.9.1** (Transfinite Recursion). *Let  $R$  be a well-founded relation on a set  $C$ .*

*Let  $\mathbf{A}$  be a class. Let  $\mathbf{B}$  be the class of all functions from a subset of  $C$  to  $\mathbf{A}$ . Let  $\mathbf{F} : \mathbf{B} \times C \rightarrow \mathbf{A}$  be a class function.*

*Then there exists a unique function  $f : C \rightarrow \mathbf{A}$  such that, for all  $t \in C$ , we have  $f(t) = \mathbf{F}(f \upharpoonright \{x \in C \mid xRt\}, t)$ .*

PROOF:

$\langle 1 \rangle 1$ . Let us say a function  $v$  is *acceptable* if and only if  $\text{dom } v \subseteq C$ ,  $\text{ran } v \subseteq \mathbf{A}$  and, for all  $t \in \text{dom } v$ , we have  $\{x \in C \mid xRt\} \subseteq \text{dom } v$  and  $v(t) = \mathbf{F}(v \upharpoonright \{x \in C \mid xRt\})$

$\langle 1 \rangle 2$ . LET:  $\mathcal{K}$  be the set of all acceptable functions.

PROOF: This is a set by an Axiom of Replacement.

$\langle 1 \rangle 3$ . LET:  $h = \bigcup \mathcal{K}$

$\langle 1 \rangle 4$ .  $h$  is a function.

$\langle 2 \rangle 1$ . LET:  $x \in C$

$\langle 2 \rangle 2$ . ASSUME: as induction hypothesis, for all  $yRx$  we have that, whenever  $(y, a)$  and  $(y, b)$  in  $h$  then  $a = b$

$\langle 2 \rangle 3$ . ASSUME:  $(x, a)$  and  $(x, b)$  are in  $h$

$\langle 2 \rangle 4$ . PICK acceptable  $u$  and  $v$  such that  $u(x) = a$  and  $v(x) = b$

$\langle 2 \rangle 5$ . For all  $yRx$  we have  $u(y) = v(y)$

PROOF: From  $\langle 2 \rangle 2$  since  $(y, u(y))$  and  $(y, v(y))$  are in  $h$ .

$\langle 2 \rangle 6$ .  $a = b$

PROOF:

$$a = u(x) \quad (\langle 2 \rangle 4)$$

$$= \mathbf{F}(u \upharpoonright \{y \in C \mid yRx\})$$

$$= \mathbf{F}(v \upharpoonright \{y \in C \mid yRx\}) \quad (\langle 2 \rangle 5)$$

$$= v(x)$$

$$= b \quad (\langle 2 \rangle 4)$$

$\langle 2 \rangle 7$ . Q.E.D.

PROOF: By transfinite induction, for all  $x \in C$ , if  $(x, a)$  and  $(x, b)$  are in  $h$  then  $a = b$ .

$\langle 1 \rangle 5$ .  $h$  is acceptable.

$\langle 2 \rangle 1$ . LET:  $t \in \text{dom } h$

$\langle 2 \rangle 2$ . PICK  $v$  acceptable such that  $(t, h(t)) \in v$

$\langle 2 \rangle 3$ .  $\{x \in C \mid xRt\} \subseteq \text{dom } v$  and  $v(t) = \mathbf{F}(v \upharpoonright \{x \in C \mid xRt\})$

$\langle 2 \rangle 4$ .  $v \upharpoonright \{x \in C \mid xRt\} = h \upharpoonright \{x \in C \mid xRt\}$

PROOF: By  $\langle 1 \rangle 4$ .

$\langle 2 \rangle 5$ .  $h(t) = \mathbf{F}(h \upharpoonright \{x \in C \mid xRt\})$

$\langle 1 \rangle 6$ .  $\text{dom } h = C$

$\langle 2 \rangle 1$ . LET:  $t \in C$

$\langle 2 \rangle 2$ . ASSUME: as induction hypothesis, for all  $xRt$ , we have  $x \in \text{dom } h$

$\langle 2 \rangle 3$ . ASSUME: for a contradiction  $t \notin \text{dom } h$

- (2)4.  $h \cup (t, \mathbf{F}(h \upharpoonright \{x \in C \mid xRt\}))$  is acceptable)
- (2)5.  $h \cup (t, \mathbf{F}(h \upharpoonright \{x \in C \mid xRt\})) \subseteq h$  is acceptable)
- (2)6. Q.E.D.

PROOF: This contradicts (2)3. Thus, by transfinite induction, for all  $t \in C$  we have  $t \in \text{dom } h$ .

- (1)7. If  $h' : C \rightarrow \mathbf{A}$  is acceptable then  $h' = h$ .

(2)1. LET:  $t \in C$

(2)2. ASSUME: as induction hypothesis, for all  $xRt$ , we have  $h'(x) = h(x)$

(2)3.  $h'(t) = h(t)$

PROOF:

$$\begin{aligned}
 h'(t) &= \mathbf{F}(h' \upharpoonright \{x \in C \mid xRt\}) \\
 &= \mathbf{F}(h \upharpoonright \{x \in C \mid xRt\}) & ((2)2) \\
 &= h(t)
 \end{aligned}$$

(2)4. Q.E.D.

PROOF: By transfinite induction, for all  $t \in C$ , we have  $h'(t) = h(t)$ .

□

## 4.10 Fixed Points

**Definition 4.10.1** (Fixed Point). Let  $X$  be a set. Let  $f : X \rightarrow X$ . Then a *fixed point* of  $f$  is an element  $a \in X$  such that  $f(a) = a$ .

## Chapter 5

# Cardinal Numbers

### 5.1 Equinumerosity

**Definition 5.1.1** (Equinumerous). Two sets  $A$  and  $B$  are *equinumerous* if and only if there exists a bijection between them.

**Theorem 5.1.2.** *Equinumerosity is an equivalence relation on the class of all sets.*

**Theorem 5.1.3** (Cantor). *No set is equinumerous with its power set.*

**Definition 5.1.4.** We say a set  $A$  is *dominated* by  $B$ ,  $A \preccurlyeq B$ , iff  $A$  is equinumerous with a subset of  $B$ .

**Theorem 5.1.5.**  $A \preccurlyeq A$

**Theorem 5.1.6.** *If  $A \preccurlyeq B \preccurlyeq C$  then  $A \preccurlyeq C$ .*

**Theorem 5.1.7** (Schröder-Bernstein Theorem). *If  $A \preccurlyeq B$  and  $B \preccurlyeq A$  then  $A \equiv B$ .*

PROOF:

⟨1⟩1. LET:  $f : A \rightarrow B$  and  $g : B \rightarrow A$  be injections.

⟨1⟩2. Define a sequence of sets  $C_n \subseteq A$  by

$$C_0 = A - \text{ran } g$$

$$C_{n+1} = g(f(C_n))$$

⟨1⟩3. Define  $h : A \rightarrow B$  by

$$h(x) = \begin{cases} f(x) & \text{if } x \in \bigcup_n C_n \\ g^{-1}(x) & \text{if not} \end{cases}$$

⟨1⟩4.  $h$  is a bijection.

□

**Theorem 5.1.8** (AC). *For any infinite set  $A$  we have  $\mathbb{N} \preccurlyeq A$ .*

PROOF: Given a choice function  $f$  for  $A$ , choose a sequence  $(a_n)$  in  $A$  by  $a_n = f(A - \{a_0, \dots, a_{n-1}\})$ .  $\square$

**Corollary 5.1.8.1** (AC). *A set is infinite if and only if it is equinumerous with a proper subset.*

## 5.2 Countability

**Definition 5.2.1** (Countable). A set  $A$  is *countable* iff  $A \preceq \mathbb{N}$ .

**Theorem 5.2.2** (AC). *A countable union of countable sets is countable.*

**Proposition 5.2.3** (AC). *Every infinite set has a countable subset.*



## 5.3 Order Theory

**Definition 5.3.1** (Preorder). Let  $X$  be a set. A *preorder* on  $X$  is a binary relation  $\leq$  on  $X$  such that:

**Reflexivity** For all  $x \in X$ , we have  $x \leq x$

**Transitivity** For all  $x, y, z \in X$ , if  $x \leq y$  and  $y \leq z$  then  $x \leq z$ .

**Definition 5.3.2** (Preordered Set). A *preordered set* consists of a set  $X$  and a preorder  $\leq$  on  $X$ .

**Proposition 5.3.3.** Let  $X$  and  $Y$  be linearly ordered sets. Let  $f : X \rightarrow Y$  be strictly monotone and surjective. Then  $f$  is a poset isomorphism.

PROOF:

$\langle 1 \rangle 1.$   $f$  is injective.

$\langle 2 \rangle 1.$  LET:  $x, y \in X$

$\langle 2 \rangle 2.$  ASSUME:  $f(x) = f(y)$

$\langle 2 \rangle 3.$   $x \not\leq y$

PROOF: By strong monotonicity.

$\langle 2 \rangle 4.$   $y \not\leq x$

PROOF: By strong monotonicity.

$\langle 2 \rangle 5.$   $x = y$

PROOF: By trichotomy.

$\langle 1 \rangle 2.$   $f^{-1}$  is monotone.

$\langle 2 \rangle 1.$  LET:  $x, y \in X$

$\langle 2 \rangle 2.$  ASSUME:  $x \leq y$

$\langle 2 \rangle 3.$   $f^{-1}(x) \not\leq f^{-1}(y)$

PROOF: By strong monotonicity.

$\langle 2 \rangle 4.$   $f^{-1}(x) < f^{-1}(y)$

PROOF: By trichotomy.

□

**Definition 5.3.4** (Interval). Let  $X$  be a preordered set and  $Y \subseteq X$ . Then  $Y$  is an *interval* if and only if, for all  $a, b \in Y$  and  $c \in X$ , if  $a \leq c \leq b$  then  $c \in Y$ .

**Definition 5.3.5** (Linear Continuum). A linearly ordered set  $L$  is a *linear continuum* if and only if:

1. every nonempty subset of  $L$  that is bounded above has a supremum
2.  $L$  is dense

**Proposition 5.3.6.** Every interval in a linear continuum is a linear continuum.

PROOF:

$\langle 1 \rangle 1.$  LET:  $L$  be a linear continuum and  $I$  an interval in  $L$ .

$\langle 1 \rangle 2.$  Every nonempty subset of  $I$  that is bounded above has a supremum in  $I$ .

$\langle 2 \rangle 1.$  LET:  $X \subseteq I$  be nonempty and bounded above by  $b \in I$ .

(2)2. LET:  $s$  be the supremum of  $X$  in  $L$ .  
 PROOF: Since  $L$  is a linear continuum.  
 (2)3.  $s \in I$   
 (3)1. PICK  $a \in X$   
 PROOF: Since  $X$  is nonempty ((2)1).  
 (3)2.  $a \leq s \leq b$   
 (3)3.  $s \in I$   
 PROOF: Since  $I$  is an interval ((1)1).  
 (2)4.  $s$  is the supremum of  $X$  in  $I$   
 (1)3.  $I$  is dense.  
 (2)1. LET:  $x, y \in I$  with  $x < y$   
 (2)2. PICK  $z \in L$  with  $x < z < y$   
 PROOF: Since  $L$  is dense.  
 (2)3.  $z \in I$   
 PROOF: Since  $I$  is an interval.

□

**Definition 5.3.7** (Ordered Square). The *ordered square*  $I_o^2$  is the set  $[0, 1]^2$  under the dictionary order.

**Proposition 5.3.8.** *The ordered square is a linear continuum.*

PROOF:

(1)1. Every nonempty subset of  $I_o^2$  bounded above has a supremum.  
 (2)1. LET:  $X \subseteq I_o^2$  be nonempty and bounded above by  $(b, c)$   
 (2)2. LET:  $s = \sup \pi_1(X)$   
 PROOF: The set  $\pi_1(X)$  is nonempty and bounded above by  $b$ .  
 (2)3. CASE:  $s \in \pi_1(X)$   
 (3)1. LET:  $t = \sup\{y \in [0, 1] \mid (s, y) \in X\}$   
 PROOF: This set is nonempty and bounded above by  $c$ .  
 (3)2.  $(s, t)$  is the supremum of  $X$ .  
 (2)4. CASE:  $s \notin \pi_1(X)$   
 PROOF: In this case  $(s, 0)$  is the supremum of  $X$ .  
 (1)2.  $I_o^2$  is dense.  
 (2)1. LET:  $(x_1, y_1), (x_2, y_2) \in I_o^2$  with  $(x_1, y_1) < (x_2, y_2)$   
 (2)2. CASE:  $x_1 < x_2$   
 (3)1. PICK  $x_3$  with  $x_1 < x_3 < x_2$   
 (3)2.  $(x_1, y_1) < (x_3, y_1) < (x_2, y_2)$   
 (2)3. CASE:  $x_1 = x_2$  and  $y_1 < y_2$   
 (3)1. PICK  $y_3$  with  $y_1 < y_3 < y_2$   
 (3)2.  $(x_1, y_1) < (x_1, y_3) < (x_2, y_2)$

□

**Proposition 5.3.9.** *If  $X$  is a well-ordered set then  $X \times [0, 1)$  under the dictionary order is a linear continuum.*

PROOF:

(1)1. Every nonempty set  $A \subseteq X \times [0, 1)$  bounded above has a supremum.

- (2)1. LET:  $A \subseteq X \times [0, 1]$  be nonempty and bounded above
- (2)2. LET:  $x_0$  be the supremum of  $\pi_1(A)$
- (2)3. CASE:  $x_0 \in \pi_1(A)$ 
  - (3)1. LET:  $y_0$  be the supremum of  $\{y \in [0, 1] \mid (x_0, y) \in A\}$
  - (3)2.  $(x_0, y_0)$  is the supremum of  $A$ .
- (2)4. CASE:  $x_0 \notin \pi_1(A)$ 
  - PROOF: In this case  $(x_0, 0)$  is the supremum of  $A$ .
- (1)2.  $X \times [0, 1]$  is dense.
  - (2)1. LET:  $(x_1, y_1), (x_2, y_2) \in X \times [0, 1]$  with  $(x_1, y_1) < (x_2, y_2)$
  - (2)2. CASE:  $x_1 < x_2$ 
    - (3)1. PICK  $y_3$  such that  $y_1 < y_3 < 1$
    - (3)2.  $(x_1, y_1) < (x_1, y_3) < (x_2, y_2)$
  - (2)3. CASE:  $x_1 = x_2$  and  $y_1 < y_2$ 
    - (3)1. PICK  $y_3$  such that  $y_1 < y_3 < y_2$
    - (3)2.  $(x_1, y_1) < (x_1, y_3) < (x_2, y_2)$

□

**Lemma 5.3.10.** *For all  $a, b, c, d \in \mathbb{R}$  with  $a < b$  and  $c < d$ , we have  $[a, b) \cong [c, d)$*

PROOF: The map  $\lambda t.c + (t - a)(d - c)/(b - a)$  is an order isomorphism.

**Proposition 5.3.11.** *Let  $X$  be a linearly ordered set. Let  $a < b < c$  in  $X$ . Then  $[a, c) \cong [0, 1)$  if and only if  $[a, b) \cong [b, c) \cong [0, 1)$ .*

PROOF:

- (1)1. If  $[a, c) \cong [0, 1)$  then  $[a, b) \cong [b, c) \cong [0, 1)$
- (2)1. ASSUME:  $f : [a, c) \cong [0, 1)$  is an order isomorphism
- (2)2.  $[a, b) \cong [0, 1)$

PROOF:

$$\begin{aligned} [a, b) &\cong [0, f(b)) && \text{(by the restriction of } f) \\ &\cong [0, 1) && \text{(Lemma 5.3.10)} \end{aligned}$$

- (2)3.  $[b, c) \cong [0, 1)$

PROOF: Similar.

- (1)2. If  $[a, b) \cong [b, c) \cong [0, 1)$  then  $[a, c) \cong [0, 1)$

PROOF:

$$\begin{aligned} [a, c) &= [a, b) * [b, c) \\ &\cong [0, 1) * [0, 1) \\ &\cong [0, 1/2) * [1/2, 1) && \text{(Lemma 5.3.10)} \\ &= 1 \end{aligned}$$

□

**Proposition 5.3.12 (CC).** *Let  $X$  be a linearly ordered set. Let  $x_0 < x_1 < \dots$  be a strictly increasing sequence in  $X$  with supremum  $b$ . Then  $[x_0, b) \cong [0, 1)$  if and only if  $[x_i, x_{i+1}) \cong [0, 1)$  for all  $i$ .*

PROOF:

- (1)1. If  $[x_0, b) \cong [0, 1)$  then  $[x_i, x_{i+1}) \cong [0, 1)$  for all  $i$ .  
 PROOF: By Lemma 5.3.10  
 (1)2. If  $[x_i, x_{i+1}) \cong [0, 1)$  for all  $i$  then  $[x_0, b) \cong [0, 1)$   
 (2)1. ASSUME:  $[x_i, x_{i+1}) \cong [0, 1)$  for all  $i$   
 (2)2. PICK an order isomorphism  $f_i : [x_i, x_{i+1}) \cong [1/2^i, 2/2^{i+1})$  for each  $i$ .  
 PROOF: By Lemma 5.3.10  
 (2)3. The union of the  $f_i$ s is an order isomorphism  $[x_0, b) \cong [0, 1)$   
 □

## 5.4 Partially Ordered Sets

**Definition 5.4.1** (Partial Order). A *partial order* on a set  $X$  is a preorder  $\leq$  that is *anti-symmetric*, i.e. whenever  $x \leq y$  and  $y \leq x$  then  $x = y$ .

## 5.5 Strict Partial Order

**Definition 5.5.1** (Strict Partial Order). A *strict partial order* on a set  $X$  is a relation on  $X$  that is transitive and irreflexive.

**Proposition 5.5.2.** If  $<$  is a strict partial order on  $X$  and  $x, y \in X$ , then at most one of  $x < y$ ,  $y < x$ ,  $x = y$  holds.

**Proposition 5.5.3.** If  $<$  is a strict partial order then the relation  $\leq$  defined by:  $x \leq y$  iff  $x < y$  or  $x = y$ , is a partial order.

**Theorem 5.5.4.** If  $R$  is a well-founded relation then its transitive closure is a partial order.

**Definition 5.5.5** (Linear Order). A *linear order* on a set  $X$  is a partial order such that, for any  $x, y \in X$ , either  $x \leq y$  or  $y \leq x$ .

## 5.6 Strict Linear Orders

**Definition 5.6.1** (Strict Linear Order (Extensionality, Pairing)). Let  $A$  be a set. A *strict linear order* on  $A$  is a binary relation  $R$  on  $A$  that is transitive and satisfies *trichotomy*: for any  $x, y \in A$ , exactly one of  $xRy$ ,  $x = y$ ,  $yRx$  holds.

**Theorem 5.6.2.** Let  $R$  be a strict linear order on  $A$ . Then there is no  $x \in A$  such that  $xRx$ .

PROOF: Immediate from trichotomy.

## 5.7 Well Orderings

**Definition 5.7.1** (Well-ordering). A *well-order* on a set  $X$  is a linear order such that every nonempty set has a least element.

**Proposition 5.7.2.** *Let  $\leq$  be a linear order on  $X$ . Then  $\leq$  is a well-order iff there is no function  $f : \mathbb{N} \rightarrow X$  such that  $f(n+1) < f(n)$  for all  $n$ .*

**Definition 5.7.3** (Initial Segment). Given a well-ordered set  $X$  and  $\alpha \in X$ , the *initial segment* of  $X$  up to  $\alpha$  is  $\text{seg } \alpha = \{x \in X \mid x < \alpha\}$ .

**Theorem 5.7.4** (Transfinite Induction). *Let  $\leq$  be a linear order on  $J$ . Then the following are equivalent:*

1.  $\leq$  is a well-order on  $J$ .
2. For every subset  $J_0 \subseteq J$ , if the following condition holds:
  - For every  $\alpha \in J$ , if  $\text{seg } \alpha \subseteq J_0$  then  $\alpha \in J_0$ .
then  $J_0 = J$ .

**Theorem 5.7.5** (Transfinite Recursion). *Let  $J$  be a well-ordered set and  $C$  a set. Let  $\mathcal{F}$  be the set of all functions from a section of  $J$  to  $C$ . Let  $G$  be a function with domain  $\mathcal{F}$ . Then there exists a unique function  $h$  with domain  $J$  such that, for all  $\alpha \in J$ , we have  $h(\alpha) = \rho(h \upharpoonright \text{seg } \alpha)$ .*

PROOF:

- $\langle 1 \rangle 1$ . If  $v$  is a function and  $t \in J$ , we say  $v$  is  $\rho$ -constructed up to  $t$  iff  $\text{dom } v = \{x \in J \mid x \leq t\}$  and, for all  $x \in \text{dom } v$ , we have  $v(x) = \rho(v \upharpoonright \text{seg } x)$
- $\langle 1 \rangle 2$ . If  $t_1 \leq t_2$ ,  $v_1$  is  $\rho$ -constructed up to  $t_1$ , and  $v_2$  is  $\rho$ -constructed up to  $t_2$ , then  $v_1 = v_2 \upharpoonright \{x \in J \mid x \leq t_1\}$
- $\langle 1 \rangle 3$ . LET:  $\mathcal{K}$  be the set of all functions that are  $\rho$ -constructed up to some  $t \in J$   
PROOF:  $\mathcal{K}$  is a set by a Replacement Axiom.
- $\langle 1 \rangle 4$ . LET:  $F = \bigcup \mathcal{K}$
- $\langle 1 \rangle 5$ .  $F$  is a function
- $\langle 1 \rangle 6$ . For all  $x \in \text{dom } F$  we have  $F(x) = \rho(F \upharpoonright \text{seg } x)$
- $\langle 1 \rangle 7$ .  $\text{dom } F = J$
- $\langle 1 \rangle 8$ .  $F$  is unique

□

**Theorem 5.7.6.** *The following are equivalent.*

1. The Axiom of Choice
2. (Well-Ordering Theorem) Every set has a well-ordering.
3. (Zorn's Lemma) Let  $X$  be a poset. If every chain in  $X$  has an upper bound in  $X$ , then  $X$  has a maximal element.

PROOF:

- $\langle 1 \rangle 1$ .  $1 \Rightarrow 2$

PROOF:

- $\langle 2 \rangle 1$ . ASSUME: The Axiom of Choice
- $\langle 2 \rangle 2$ . LET:  $X$  be a set.

- ⟨2⟩3. PICK a choice function for  $\mathcal{P}X \setminus \{\emptyset\}$   
PROOF: Lemma ??.
- ⟨2⟩4. LET: a *tower* in  $X$  be a pair  $(T, <)$  where  $T \subseteq X$ ,  $<$  is a well-ordering of  $T$ , and  $x = c(X \setminus \{y \in T \mid y < x\})$ .
- ⟨2⟩5. For any two towers  $(T_1, <_1)$  and  $(T_2, <_2)$ , either these two posets are equal or one is a section of the other.
- ⟨3⟩1.
- ⟨2⟩6. For any tower  $(T, <)$  in  $X$  with  $T \neq X$ , there exists a tower in  $X$  of which  $(T, <)$  is a section.
- ⟨2⟩7. LET:  $T = \bigcup \{T' \subseteq X \mid \exists R. (T, R) \text{ is a tower in } X\}$
- ⟨2⟩8. Define  $<$  on  $T$  by:  $x < y$  iff there exists a tower  $(T, R)$  in  $X$  such that  $x, y \in T$  and  $xRy$ .
- ⟨2⟩9.  $(T, <)$  is a tower in  $X$ .
- ⟨2⟩10.  $T = X$
- ⟨2⟩11.  $<$  is a well-ordering of  $X$ .
- ⟨1⟩2.  $2 \Rightarrow 3$
- ⟨2⟩1. ASSUME: The Well-Ordering Theorem
- ⟨2⟩2. LET:  $X$  be a poset in which every chain has an upper bound.
- ⟨2⟩3. PICK a well-ordering  $R$  of  $X$
- ⟨2⟩4. Define  $F : X \rightarrow \{0, 1\}$  by transfinite  $R$ -recursion by:
 
$$F(a) = \begin{cases} 1 & \text{if } b < a \text{ for all } b \text{ such that } bRa \text{ and } f(b) = 1 \\ 0 & \text{otherwise} \end{cases}$$
- ⟨2⟩5. LET:  $C = \{a \in X \mid f(a) = 1\}$
- ⟨2⟩6.  $C$  is a chain in  $X$
- ⟨3⟩1. LET:  $x, y \in C$
- ⟨3⟩2. ASSUME: without loss of generality  $xRy$
- ⟨3⟩3.  $f(y) = 1$
- ⟨3⟩4. for all  $z$  such that  $zRy$  and  $f(z) = 1$  we have  $z < y$
- ⟨3⟩5.  $x < y$
- ⟨2⟩7. PICK an upper bound  $u$  for  $C$
- ⟨2⟩8.  $u$  is maximal in  $X$
- ⟨3⟩1. LET:  $x \in X$  with  $u \leq x$
- ⟨3⟩2. for all  $b$  such that  $bRx$  and  $f(b) = 1$  we have  $b < x$   
PROOF: Since  $b \in C$  so  $b \leq u \leq x$
- ⟨3⟩3.  $f(u) = 1$
- ⟨3⟩4.  $u \leq x$
- ⟨3⟩5.  $u = x$
- ⟨2⟩9.  $3 \Rightarrow 1$
- ⟨3⟩1. ASSUME: Zorn's Lemma
- ⟨3⟩2. LET:  $R$  be a relation
- ⟨3⟩3. LET:  $\mathcal{A}$  be the poset of functions that are subsets of  $R$  under  $\subseteq$
- ⟨3⟩4. Every chain in  $\mathcal{A}$  has an upper bound
- ⟨4⟩1. LET:  $\mathcal{C} \subseteq \mathcal{A}$  be a chain.  
PROVE:  $\bigcup \mathcal{C} \in \mathcal{A}$
- ⟨4⟩2. ASSUME:  $(x, y), (x, z) \in \bigcup \mathcal{C}$

- ⟨4⟩3. PICK  $f, g \in \mathcal{C}$  such that  $f(x) = y$  and  $g(x) = z$
- ⟨4⟩4. ASSUME: without loss of generality  $f \subseteq g$
- ⟨4⟩5.  $g(x) = y$
- ⟨4⟩6.  $y = z$
- ⟨3⟩5. PICK  $F$  maximal in  $\mathcal{A}$
- ⟨3⟩6.  $\text{dom } F = \text{dom } R$
- ⟨4⟩1. ASSUME: for a contradiction  $x \in \text{dom } R - \text{dom } F$
- ⟨4⟩2. PICK  $y$  such that  $xRy$
- ⟨4⟩3. LET:  $G = F \cup \{(x, y)\}$
- ⟨4⟩4.  $G \in \mathcal{A}$
- ⟨4⟩5.  $F \subset G$
- ⟨4⟩6. Q.E.D.

PROOF: This contradicts the maximality of  $F$ .

□

**Lemma 5.7.7** (Choice). *Let  $X$  be a set. Let  $\mathcal{A} \subseteq \mathcal{P}X$  have the finite intersection property. Then there exists a maximal set  $\mathcal{D}$  such that  $\mathcal{A} \subseteq \mathcal{D} \subseteq \mathcal{P}X$  and  $\mathcal{D}$  has the finite intersection property.*

PROOF:

- ⟨1⟩1. LET:  $\mathbb{F} = \{\mathcal{D} \mid \mathcal{A} \subseteq \mathcal{D} \subseteq \mathcal{P}X, \mathcal{D} \text{ has the finite intersection property}\}$
- ⟨1⟩2. Every chain in  $\mathbb{F}$  has an upper bound.
  - ⟨2⟩1. LET:  $\mathbb{C}$  be a chain in  $\mathbb{F}$ .
  - ⟨2⟩2. ASSUME: without loss of generality  $\mathbb{C} \neq \emptyset$
  - PROVE:  $\bigcup \mathbb{C} \in \mathbb{F}$
  - PROOF: If  $\mathbb{C} = \emptyset$  then  $\mathcal{A}$  is an upper bound.
  - ⟨2⟩3.  $\mathcal{A} \subseteq \bigcup \mathbb{C} \subseteq \mathcal{P}X$
  - ⟨2⟩4. LET:  $C_1, \dots, C_n \in \mathbb{C}$
  - PROVE:  $C_1 \cap \dots \cap C_n \neq \emptyset$
  - ⟨2⟩5. PICK  $C_1, \dots, C_n \in \mathbb{C}$  such that  $C_i \in \mathcal{C}_i$  for all  $i$ .
  - ⟨2⟩6. ASSUME: without loss of generality  $C_1 \subseteq \dots \subseteq C_n$
  - ⟨2⟩7.  $C_1, \dots, C_n \in \mathcal{C}_n$
  - ⟨2⟩8.  $\mathcal{C}_n$  satisfies the finite intersection property.
  - ⟨2⟩9.  $C_1 \cap \dots \cap C_n \neq \emptyset$
- ⟨1⟩3. Q.E.D.

PROOF: By Zorn's Lemma.

□

**Theorem 5.7.8** (Cardinal Comparability). *The Axiom of Choice is equivalent to the Cardinal Comparability Theorem: for any two sets  $A$  and  $B$ , either  $A \preccurlyeq B$  or  $B \preccurlyeq A$ .*

PROOF:

- ⟨1⟩1. Zorn's Lemma implies Cardinal Comparability
- ⟨2⟩1. ASSUME: Zorn's Lemma
- ⟨2⟩2. LET:  $A$  and  $B$  be sets.
- ⟨2⟩3. LET:  $\mathcal{A}$  be the poset of all injective functions  $f$  such that  $\text{dom } f \subseteq C$  and  $\text{ran } f \subseteq D$  under  $\subseteq$

- ⟨2⟩4. Every chain in  $\mathcal{A}$  has an upper bound.
- ⟨3⟩1. LET:  $\mathcal{C} \subseteq \mathcal{A}$  be a chain.  
PROVE:  $\bigcup \mathcal{C} \in \mathcal{A}$
- ⟨3⟩2.  $\bigcup \mathcal{C}$  is a function.
  - ⟨4⟩1. LET:  $(x, y), (x, z) \in \bigcup \mathcal{C}$
  - ⟨4⟩2. PICK  $f, g \in \mathcal{C}$  such that  $f(x) = y$  and  $g(x) = z$
  - ⟨4⟩3. ASSUME: without loss of generality  $f \subseteq g$
  - ⟨4⟩4.  $g(x) = y$
  - ⟨4⟩5.  $y = z$
- ⟨3⟩3.  $\bigcup \mathcal{C}$  is injective.  
PROOF: Similar.
- ⟨2⟩5. PICK  $\hat{f}$  maximal in  $\mathcal{A}$   
PROOF: By Zorn's Lemma.
- ⟨2⟩6. Either  $\text{dom } \hat{f} = C$  or  $\text{ran } \hat{f} = D$ 
  - ⟨3⟩1. ASSUME: for a contradiction  $\text{dom } \hat{f} \subset C$  and  $\text{ran } \hat{f} \subset D$
  - ⟨3⟩2. PICK  $x \in C - \text{dom } \hat{f}$  and  $y \in D - \text{ran } \hat{f}$
  - ⟨3⟩3. LET:  $g = \hat{f} \cup \{(x, y)\}$
  - ⟨3⟩4.  $g \in \mathcal{A}$
  - ⟨3⟩5.  $\hat{f} \subset g$
  - ⟨3⟩6. Q.E.D.
- PROOF: This contradicts the maximality of  $\hat{f}$ .
- ⟨2⟩7. If  $\text{dom } \hat{f} = C$  then  $C \preceq D$
- ⟨2⟩8. If  $\text{ran } \hat{f} = D$  then  $D \preceq C$
- ⟨1⟩2. Cardinal Comparability implies the Well-Ordering Theorem
  - ⟨2⟩1. ASSUME: Cardinal Comparability
  - ⟨2⟩2. LET:  $A$  be a set
  - ⟨2⟩3. PICK an ordinal  $\alpha$  such that  $\alpha \not\preceq A$
  - ⟨2⟩4.  $A \preceq \alpha$   
PROOF: By Cardinal Comparability.
  - ⟨2⟩5. PICK an injection  $f : A \rightarrow \alpha$
  - ⟨2⟩6. Define  $<$  on  $A$  by  $x < y$  iff  $f(x) \in f(y)$
  - ⟨2⟩7.  $<$  is a well-ordering on  $A$ .

□

**Theorem 5.7.9.** *Given two well-ordered sets  $A$  and  $B$ , either  $A \cong B$  or one of  $A, B$  is isomorphic to an initial segment of the other.*

## 5.8 Ordinal Numbers

**Definition 5.8.1.** Let  $(A, \leq)$  be a well-ordered set. The *ordinal number* of  $(A, \leq)$  is the range of  $E$ , where  $E$  is the unique function with domain  $A$  such that  $E(t) = \text{ran}(E \upharpoonright \text{seg } t)$  for all  $t \in A$ .

**Theorem 5.8.2.** *Let  $(A, \leq)$  be a well-ordered set and  $E : A \rightarrow \alpha$  be the canonical function onto the ordinal of  $A$ . Then:*



1. For all  $t \in A$  we have  $E(t) \notin E(t)$ .
2.  $E$  is a bijection.
3. For any  $s, t \in A$ , we have  $s < t$  if and only if  $E(s) \in E(t)$ .
4.  $\alpha$  is a transitive set.
5.  $\alpha$  is well-ordered by  $\in$
6.  $E$  is an order isomorphism between  $(A, \leq)$  and  $(\alpha, \subseteq)$ .

**Theorem 5.8.3.** *Two well-ordered sets are isomorphic if and only if they have the same ordinal number.*

**Theorem 5.8.4.** *A set is an ordinal number if and only if it is a transitive set well-ordered by  $\in$ .*

**Theorem 5.8.5.** *Every member of an ordinal number is an ordinal number.*

**Theorem 5.8.6.** *Any transitive set of ordinal numbers is an ordinal number.*

**Theorem 5.8.7.** *The empty set is an ordinal number.*

**Theorem 5.8.8.** *The successor of an ordinal number is an ordinal number.*

**Theorem 5.8.9.** *If  $A$  is a set of ordinal numbers then  $\bigcup A$  is an ordinal number.*

**Theorem 5.8.10.** *Any nonempty set of ordinal numbers has a least element.*

**Theorem 5.8.11** (Burali-Forti Paradox). *The class of ordinal numbers is a proper class.*

**Theorem 5.8.12** (Hartogs' Theorem). *For any set  $A$ , there exists an ordinal that is not dominated by  $A$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $\alpha$  be the class of all ordinals  $\beta$  such that  $\beta \preccurlyeq A$

$\langle 1 \rangle 2$ .  $\alpha$  is a set.

$\langle 2 \rangle 1$ . LET:  $W$  be the set of all pairs  $(B, \leq)$  such that  $B \subseteq A$  and  $\leq$  is a well-ordering on  $B$ .

$\langle 2 \rangle 2$ . Every member of  $\alpha$  is the ordinal number of a member of  $W$

$\langle 2 \rangle 3$ . Q.E.D.

PROOF: By a Replacement Axiom.

$\langle 1 \rangle 3$ .  $\alpha$  is an ordinal.

$\langle 1 \rangle 4$ .  $\alpha$  is not dominated by  $A$ .

□

**Definition 5.8.13.** A class term  $\mathbf{F} : \mathbf{On} \rightarrow \mathbf{On}$  is *continuous* iff, for every limit ordinal  $\lambda$ , we have  $\mathbf{F}(\lambda) = \sup_{\alpha < \lambda} \mathbf{F}(\alpha)$ .

**Theorem 5.8.14.** *Let  $\mathbf{F} : \mathbf{On} \rightarrow \mathbf{On}$ . If  $\mathbf{F}$  is continuous and  $\mathbf{F}(\alpha) < \mathbf{F}(\alpha + 1)$  for every ordinal  $\alpha$ , then  $\mathbf{F}$  is strictly monotone.*

**Definition 5.8.15.** A class term  $\mathbf{F} : \mathbf{On} \rightarrow \mathbf{On}$  is *normal* iff it is strictly monotone and continuous.

**Theorem 5.8.16.** Let  $\mathbf{F} : \mathbf{On} \rightarrow \mathbf{On}$  be normal. For every ordinal  $\beta \geq \mathbf{F}(0)$ , there exists a greatest ordinal  $\alpha$  such that  $\mathbf{F}(\alpha) \leq \beta$ .

**Theorem 5.8.17.** Let  $\mathbf{F} : \mathbf{On} \rightarrow \mathbf{On}$  be normal. Let  $S$  be a set of ordinals. Then  $\mathbf{F}(\sup S) = \sup_{\alpha \in S} \mathbf{F}(\alpha)$ .

**Theorem 5.8.18** (Veblen Fixed-Point Theorem). Let  $\mathbf{F} : \mathbf{On} \rightarrow \mathbf{On}$  be normal. For every ordinal  $\alpha$ , there exists  $\beta \geq \alpha$  such that  $\mathbf{F}(\beta) = \beta$ .

PROOF: Let  $\beta$  be the supremum of  $\alpha, \mathbf{F}(\alpha), \mathbf{F}^2(\alpha), \dots$ .  $\square$

**Lemma 5.8.19.** Let  $\alpha$  be an ordinal. Let  $(f(\gamma))_{\gamma < \alpha}$  be an  $\alpha$ -sequence of ordinals. Then there exists  $\beta \leq \alpha$  and an increasing sequence of ordinals  $(g(\gamma))_{\gamma < \beta}$  such that  $\sup_{\gamma < \alpha} f(\gamma) = \sup_{\gamma < \beta} g(\gamma)$ .

## 5.9 Cardinal Numbers

**Definition 5.9.1** (Cardinal Number (AC)). For any set  $A$ , the *cardinal number* of  $A$ ,  $\text{card } A$ , is the least ordinal equinumerous with  $A$ .

There exists some ordinal equinumerous with  $A$  by the Well-Ordering Theorem.

**Theorem 5.9.2.** For any sets  $A$  and  $B$ , we have  $A \equiv B$  if and only if  $\text{card } A = \text{card } B$ .

**Theorem 5.9.3.** A set  $A$  is finite if and only if  $\text{card } A$  is a natural number.

**Theorem 5.9.4.** The supremum of a set of cardinal numbers is a cardinal number.

## 5.10 Cardinal Arithmetic

**Definition 5.10.1.** For cardinal numbers  $\kappa$  and  $\lambda$ , the *sum*  $\kappa + \lambda$  is the cardinal number of  $A \cup B$ , where  $A$  and  $B$  are disjoint sets of cardinality  $\kappa$  and  $\lambda$  respectively.

**Theorem 5.10.2.**  $\kappa + \lambda = \lambda + \kappa$

**Theorem 5.10.3.**  $\kappa + (\lambda + \mu) = (\kappa + \lambda) + \mu$

**Theorem 5.10.4.** The definition of addition agrees with the definition on natural numbers.

**Definition 5.10.5.** For cardinal numbers  $\kappa$  and  $\lambda$ , the *product*  $\kappa\lambda$  is the cardinality of  $\kappa \times \lambda$ .

**Theorem 5.10.6.**  $\kappa\lambda = \lambda\kappa$

**Theorem 5.10.7.**  $\kappa(\lambda\mu) = (\kappa\lambda)\mu$

**Theorem 5.10.8.**  $\kappa(\lambda + \mu) = \kappa\lambda + \kappa\mu$

**Theorem 5.10.9.** *The definition of multiplication agrees with the definition on natural numbers.*

**Theorem 5.10.10 (AC).** *For any infinite cardinal  $\kappa$  we have  $\kappa\kappa = \kappa$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $B$  be a set with cardinality  $\kappa$

$\langle 1 \rangle 2$ . LET:  $\mathcal{H} = \{\emptyset\} \cup \{f \mid \exists A \subseteq B. A \text{ is infinite and } f \text{ is a bijection between } A \times A \text{ and } A\}$

$\langle 1 \rangle 3$ . For every chain  $\mathcal{C} \subseteq \mathcal{H}$  we have  $\bigcup \mathcal{C} \in \mathcal{H}$

$\langle 1 \rangle 4$ . PICK a maximal  $f_0$  in  $\mathcal{H}$

$\langle 1 \rangle 5$ .  $f_0 \neq \emptyset$

PROOF:  $B$  has a subset of cardinality  $\aleph_0$  and  $\aleph_0\aleph_0 = \aleph_0$ .

$\langle 1 \rangle 6$ . LET:  $A_0$  be the set such that  $f_0$  is a bijection between  $A_0 \times A_0$  and  $A_0$

$\langle 1 \rangle 7$ . LET:  $\lambda = \text{card } A_0$

$\langle 1 \rangle 8$ .  $\text{card}(B - A_0) < \lambda$

$\langle 1 \rangle 9$ .  $\kappa = \lambda$

PROOF:

$$\begin{aligned} \kappa &= \text{card } A_0 + \text{card}(B - A_0) \\ &\leq \lambda + \lambda \\ &= 2\lambda \\ &\leq \lambda\lambda \\ &= \lambda && (\langle 1 \rangle 6) \\ &\leq \kappa && \square \end{aligned}$$

**Theorem 5.10.11 (Absorption Law).** *Let  $\kappa$  and  $\lambda$  be cardinal numbers such that  $0 < \kappa \leq \lambda$  and  $\lambda$  is infinite. Then*

$$\kappa + \lambda = \lambda .$$

**Theorem 5.10.12 (Absorption Law).** *Let  $\kappa$  and  $\lambda$  be cardinal numbers such that  $0 < \kappa \leq \lambda$  and  $\lambda$  is infinite. Then*

$$\kappa\lambda = \lambda .$$

**Definition 5.10.13.** For cardinal numbers  $\kappa$  and  $\lambda$ , we write  $\kappa^\lambda$  for the cardinality of the set of functions from  $\lambda$  to  $\kappa$ .

**Theorem 5.10.14.**  $\kappa^{\lambda+\mu} = \kappa^\lambda + \kappa^\mu$

**Theorem 5.10.15.**  $(\kappa\lambda)^\mu = \kappa^\mu\lambda^\mu$

**Theorem 5.10.16.**  $(\kappa^\lambda)^\mu = \kappa^{\lambda\mu}$

**Theorem 5.10.17.** *The definition of exponentiation agrees with the definition on natural numbers.*

**Theorem 5.10.18.** *Given sets  $A$  and  $B$ , we have  $\text{card } A \leq \text{card } B$  if and only if  $A \preceq B$ .*

**Definition 5.10.19.** Let  $\aleph_0 = \text{card } \mathbb{N}$ .

**Theorem 5.10.20 (AC).** *For any infinite cardinal  $\kappa$  we have  $\aleph_0 \leq \kappa$ .*

**Theorem 5.10.21 (Maximum Principle (AC)).** *Every poset has a maximal chain.*

## 5.11 Rank of a Set

**Definition 5.11.1 (Cumulative Hierarchy of Sets).** For every ordinal  $\alpha$ , define the rank  $V_\alpha$  by transfinite recursion thus:

$$\begin{aligned} V_0 &= \emptyset \\ V_{\alpha+1} &= \mathcal{P}V_\alpha \\ V_\lambda &= \bigcup_{\alpha < \lambda} V_\alpha \end{aligned}$$

for  $\lambda$  a limit ordinal.

The *von Neumann universe* is the class  $\mathbf{V} = \bigcup_{\alpha \in \mathbf{On}} V_\alpha$ .

**Theorem 5.11.2.** *If  $\lambda$  is a limit ordinal and  $\lambda > \omega$  then  $V_\lambda$  is a model of Zermelo set theory.*

**Lemma 5.11.3 (AC).** *There exists a well-ordered set in  $V_{\omega^2}$  whose ordinal is not in  $V_{\omega^2}$ .*

PROOF: Pick a well-ordering  $<$  of  $\mathcal{P}\mathbb{N}$ . Then  $(\mathcal{P}\mathbb{N}, <) \in V_{\omega^2}$  but its ordinal is not because its ordinal is uncountable.  $\square$

**Theorem 5.11.4.** *The set  $V_{\omega^2}$  is not a model of Zermelo-Fraenkel set theory.*

Thus, the Replacement Axioms cannot be proven from the other axioms.

**Definition 5.11.5 (Well-Founded Set).** A set  $A$  is *well-founded* iff  $A \in V_\alpha$  for some  $\alpha \in \mathbf{On}$ .

**Definition 5.11.6 (Rank).** The *rank* of a well-founded set  $A$ ,  $\text{rank } A$ , is the least ordinal  $\alpha$  such that  $A \in V_\alpha$ .

**Theorem 5.11.7.** *If  $A \in B$  and  $B$  is well-founded then  $A$  is well-founded and  $\text{rank } A < \text{rank } B$ .*

**Theorem 5.11.8.** *If  $A$  is a set and every member of  $A$  is well-founded then  $A$  is well-founded and  $\text{rank } A = \sup_{B \in A} (\text{rank } B + 1)$ .*

**Theorem 5.11.9.** *The Axiom of Regularity is equivalent to the statement that every set is well-founded.*

## 5.12 Transfinite Recursion Again

**Theorem 5.12.1.** *Let  $\mathbf{A}$  be a class. Let  $\mathbf{B}$  be the class of all functions  $f : \alpha \rightarrow \mathbf{A}$  for some ordinal  $\alpha$ . Let  $\mathbf{F} : \mathbf{B} \rightarrow \mathbf{A}$  be a class term. Then there exists a unique class term  $\mathbf{G} : \mathbf{On} \rightarrow \mathbf{A}$  such that, for all  $\alpha \in \mathbf{On}$ , we have  $\mathbf{G}(\alpha) = \mathbf{F}(\mathbf{G} \upharpoonright \alpha)$ .*

## 5.13 Alephs

**Definition 5.13.1.** Define the cardinal number  $\aleph_\alpha$  for every ordinal  $\alpha$  by transfinite recursion on  $\alpha$  thus:  $\aleph_\alpha$  is the least infinite cardinal different from  $\aleph_\beta$  for all  $\beta < \alpha$ .

**Theorem 5.13.2.** *If  $\alpha < \beta$  then  $\aleph_\alpha < \aleph_\beta$ .*

**Theorem 5.13.3.** *Every infinite cardinal has the form  $\aleph_\alpha$  for some ordinal  $\alpha$ .*

## 5.14 Ordinal Arithmetic

**Definition 5.14.1** (Sum). Let  $\alpha$  and  $\beta$  be ordinals. The *sum*  $\alpha + \beta$  is the ordinal of the concatenation of  $A$  followed by  $B$ , where  $A$  is a well-ordered set of ordinal  $\alpha$  and  $B$  a well-ordered set of ordinal  $\beta$ .

**Theorem 5.14.2.** *Addition is associative.*

**Theorem 5.14.3.**  $\alpha + 0 = \alpha$

**Theorem 5.14.4.**  $0 + \alpha = \alpha$

**Theorem 5.14.5.** *For  $\lambda$  a limit ordinal we have  $\alpha + \lambda = \sup_{\beta < \lambda} (\alpha + \beta)$*

**Theorem 5.14.6.** *For any  $\alpha$ , the class term that maps  $\beta$  to  $\alpha + \beta$  is normal.*

**Theorem 5.14.7.**  $\beta < \gamma$  iff  $\alpha + \beta < \alpha + \gamma$ .

**Theorem 5.14.8.** *If  $\beta \leq \gamma$  then  $\beta + \alpha \leq \gamma + \alpha$ .*

**Theorem 5.14.9** (Subtraction Theorem). *If  $\alpha < \beta$  then there exists a unique  $\delta$  such that  $\alpha + \delta < \beta$ .*

**Definition 5.14.10** (Product). Let  $\alpha$  and  $\beta$  be ordinals. The *sum*  $\alpha + \beta$  is the ordinal of  $A \times B$  ordered under the Hebrew lexicographic order, where  $A$  is a well-ordered set of ordinal  $\alpha$  and  $B$  a well-ordered set of ordinal  $\beta$ .

**Theorem 5.14.11.** *Multiplication is associative.*

**Theorem 5.14.12.** *Multiplication distributes over addition on the left.*

**Theorem 5.14.13.**  $\alpha 1 = \alpha$

**Theorem 5.14.14.**  $1\alpha = \alpha$

**Theorem 5.14.15.**  $\alpha 0 = 0$

**Theorem 5.14.16.**  $0\alpha = 0$

**Theorem 5.14.17.** For  $\lambda$  a limit ordinal, we have  $\alpha\lambda = \sup_{\beta < \lambda}(\alpha\beta)$ .

**Theorem 5.14.18.** For  $\alpha > 0$ , the class term that maps  $\beta$  to  $\alpha\beta$  is normal.

**Theorem 5.14.19.** If  $\alpha > 0$ , then  $\beta < \gamma$  iff  $\alpha\beta < \alpha\gamma$ .

**Theorem 5.14.20.** If  $\beta \leq \gamma$  then  $\beta\alpha \leq \gamma\alpha$ .

**Theorem 5.14.21** (Division Theorem). For any ordinals  $\alpha$  and  $\delta$  with  $\delta \neq 0$ , there exist unique ordinals  $\beta$  and  $\gamma$  with  $\gamma < \delta$  and  $\alpha = \delta\beta + \gamma$ .

**Definition 5.14.22** (Exponentiation). For ordinals  $\alpha$  and  $\beta$ , define the ordinal  $\alpha^\beta$  by transfinite recursion on  $\beta$  by:

$$\begin{aligned}\alpha^0 &= 1 \\ \alpha^{\beta+1} &= \alpha^\beta + \alpha \\ \alpha^\lambda &= \sup_{\beta < \lambda} \alpha^\beta\end{aligned}$$

for  $\lambda$  a limit ordinal.

**Theorem 5.14.23.** For  $\alpha > 1$ , the class term that maps  $\beta$  to  $\alpha^\beta$  is normal.

**Theorem 5.14.24.** If  $\alpha > 1$ , then  $\beta < \gamma$  iff  $\alpha^\beta < \alpha^\gamma$ .

**Theorem 5.14.25.** If  $\beta \leq \gamma$  then  $\beta^\alpha \leq \gamma^\alpha$ .

**Theorem 5.14.26** (Logarithm Theorem). Let  $\alpha$  and  $\beta$  be ordinals with  $\alpha \neq 0$  and  $\beta > 1$ . Then there exist unique ordinals  $\gamma$ ,  $\delta$  and  $\rho$  such that  $\delta \neq 0$ ,  $\delta < \beta$ ,  $\rho < \beta^\gamma$ , and  $\alpha = \beta^\gamma\delta + \rho$ .

**Theorem 5.14.27.**

$$\alpha^{\beta+\gamma} = \alpha^\beta \alpha^\gamma$$

**Theorem 5.14.28.**

$$(\alpha^\beta)^\gamma = \alpha^{\beta\gamma}$$

## 5.15 Beth Cardinals

**Definition 5.15.1.** Define the cardinal  $\beth_\alpha$  for every ordinal  $\alpha$  by:

$$\begin{aligned}\beth_0 &= \aleph_0 \\ \beth_{\alpha+1} &= 2^{\beth_\alpha} \\ \beth_\lambda &= \sup_{\alpha < \lambda} \beth_\alpha\end{aligned}$$

for  $\lambda$  a limit ordinal.

**Lemma 5.15.2.** For any ordinal  $\alpha$  we have  $\text{card } V_{\omega+\alpha} = \beth_\alpha$ .

## 5.16 Cofinality

**Definition 5.16.1** (Cofinality). For  $\lambda$  a limit ordinal, the *cofinality* of  $\lambda$ ,  $\text{cf } \lambda$ , is the least cardinal  $\kappa$  such that  $\lambda$  is the supremum of a set of  $\kappa$  smaller ordinals.

We extend  $\text{cf}$  to all the ordinals by setting  $\text{cf } 0 = 0$  and  $\text{cf}(\alpha + 1) = 1$ .

**Theorem 5.16.2.** For any limit ordinal  $\lambda$  we have  $\text{cf } \aleph_\lambda = \aleph_\lambda$ .

**Lemma 5.16.3.** Let  $\lambda$  be a limit ordinal. Then  $\text{cf } \lambda$  is the least ordinal  $\alpha$  such that there exists an increasing  $\alpha$ -sequence of ordinals with limit  $\lambda$ .

**Theorem 5.16.4.** Let  $\lambda$  be an infinite cardinal. Then  $\text{cf } \lambda$  is the least cardinal number  $\kappa$  such that  $\lambda$  can be partitioned into  $\kappa$  sets each of cardinality  $< \lambda$ .

**Theorem 5.16.5** (König's Theorem). Let  $\kappa$  be an infinite cardinal. Then  $\kappa < 2^{\text{cf } \kappa}$ .

**Corollary 5.16.5.1.**  $2^{\aleph_0} \neq \aleph_\omega$ .

**Definition 5.16.6** (Regular). A cardinal  $\kappa$  is *regular* iff  $\text{cf } \kappa = \kappa$ .

**Theorem 5.16.7.** For any ordinal  $\lambda$ , we have  $\text{cf } \lambda$  is a regular cardinal.

**Definition 5.16.8** (Singular). A cardinal  $\kappa$  is *singular* iff  $\text{cf } \kappa < \kappa$ .

**Theorem 5.16.9.** For any ordinal  $\alpha$  we have  $\aleph_{\alpha+1}$  is a regular cardinal.

## 5.17 Inaccessible Cardinals

**Definition 5.17.1** (Inaccessible). A cardinal number  $\kappa$  is *inaccessible* iff:

- $\kappa > \aleph_0$
- For every cardinal  $\lambda < \kappa$  we have  $2^\lambda < \kappa$
- $\kappa$  is regular.

**Lemma 5.17.2.** If  $\kappa$  is inaccessible and  $\alpha < \kappa$  then  $\beth_\alpha < \kappa$ .

**Lemma 5.17.3.** If  $\kappa$  is inaccessible and  $A \in V_\kappa$  then  $\text{card } A < \kappa$ .

**Theorem 5.17.4.** If  $\kappa$  is inaccessible then  $V_\kappa$  is a model of ZF.

## 5.18 Directed Set

**Definition 5.18.1** (Directed Set). A preordered set  $P$  is *directed* iff, for all  $a, b \in P$ , there exists  $c \in P$  such that  $a \leq c$  and  $b \leq c$ .

**Proposition 5.18.2.** Every linearly ordered set is directed.

**Proposition 5.18.3.** For any set  $A$ , the  $\mathcal{P}A$  under  $\subseteq$  is directed.

## 5.19 Cofinal Set

**Definition 5.19.1** (Cofinal). Let  $A$  be a preordered set and  $B \subseteq A$ . Then  $B$  is *cofinal* if and only if, for every  $x \in A$ , there exists  $y \in B$  such that  $x \leq y$ .

**Proposition 5.19.2.** *If  $A$  is a directed preordered set and  $B \subseteq A$  is cofinal then  $B$  is directed.*

PROOF:

- $\langle 1 \rangle$ 1. LET:  $x, y \in B$
- $\langle 1 \rangle$ 2. PICK  $z \in A$  such that  $x \leq z$  and  $y \leq z$
- $\langle 1 \rangle$ 3. PICK  $z' \in B$  such that  $z \leq z'$
- $\langle 1 \rangle$ 4.  $x \leq z'$  and  $y \leq z'$

□



## Chapter 6

# Natural Numbers

### 6.1 Successors

**Definition 6.1.1** (Successor (Pairing, Union)). For any set  $a$ , its *Successor*  $a^+$  is the set  $a \cup \{a\}$

**Theorem 6.1.2** (Pairing, Union). *If  $a$  is a transitive set then  $\bigcup(a^+) = a$ .*

PROOF:

$$\begin{aligned}\bigcup(a^+) &= \bigcup(a \cup \{a\}) \\ &= \bigcup a \cup \bigcup \{a\} \\ &= \bigcup a \cup a \\ &= a \qquad \qquad \qquad (\bigcup a \subseteq a) \square\end{aligned}$$

**Theorem 6.1.3.** *If  $A$  is a transitive set then  $A^+$  is transitive.*

PROOF: If  $A$  is transitive then  $\bigcup(A^+) = A \subseteq A^+$ .  $\square$

### 6.2 Inductive Sets

**Definition 6.2.1** (Inductive (Extensionality, Empty Set, Pairing, Union)). A set  $A$  is *inductive* iff  $\emptyset \in A$  and, for every  $a \in A$ , we have  $a^+ \in A$ .

**Axiom 6.2.2** (Axiom of Infinity (Extensionality, Empty Set, Pairing, Union)). *There exists an inductive set.*

### 6.3 Natural Numbers

**Definition 6.3.1** (Natural Number (Extensionality, Empty Set, Pairing, Union)). A *natural number* is a set that belongs to every inductive set.

We write  $\mathbb{N}$  for the class of all natural numbers.

**Theorem 6.3.2** (Extensionality, Empty Set, Pairing, Union, Infinity, Subset).  
*The class of natural numbers is a set.*

PROOF:

⟨1⟩1. PICK an inductive set  $I$ .

PROOF: By the Axiom of Infinity.

⟨1⟩2.  $\mathbb{N} \subseteq I$

⟨1⟩3. Q.E.D.

PROOF: By a Subset Axiom.

□

**Theorem 6.3.3** (Extensionality, Empty Set, Pairing, Union, Infinity, Subset).  
*The set  $\mathbb{N}$  is inductive.*

PROOF:

⟨1⟩1.  $\emptyset \in \mathbb{N}$

PROOF: Since  $\emptyset$  is a member of every inductive set.

⟨1⟩2. For all  $n \in \mathbb{N}$  we have  $n^+ \in \mathbb{N}$

PROOF: If  $n$  is a member of every inductive set then so is  $n^+$ .

□

**Theorem 6.3.4** (Extensionality, Empty Set, Pairing, Union, Infinity, Subset).  
*The set  $\mathbb{N}$  is a subset of every inductive set.*

PROOF: Immediate from definition. □

**Corollary 6.3.4.1** (Proof by Induction (Extensionality, Empty Set, Pairing, Union, Infinity, Subset)). *If  $A \subseteq \mathbb{N}$  and  $A$  is inductive then  $A = \mathbb{N}$ .*

**Definition 6.3.5** (Zero (Empty Set)). The natural number *zero*,  $0$ , is defined to be  $\emptyset$ .

**Theorem 6.3.6** (Extensionality, Empty Set, Pairing, Union, Infinity, Subset).  
*Every natural number except  $0$  is a successor of a natural number.*

PROOF: The set  $\{x \in \mathbb{N} \mid x = 0 \vee \exists y \in \mathbb{N}. x = y^+\}$  is inductive. □

**Theorem 6.3.7** (Extensionality, Empty Set, Pairing, Union, Infinity, Subset).  
*Every natural number is transitive.*

PROOF: By induction using Theorem 6.1.3. □

**Theorem 6.3.8** (Extensionality, Empty Set, Pairing, Union, Infinity, Subset).  
*The set  $\mathbb{N}$  is transitive.*

PROOF:

⟨1⟩1. For every natural number  $n$  and every  $m \in n$  then  $m$  is a natural number.

⟨2⟩1. Every member of  $\emptyset$  is a natural number.

PROOF: Vacuous.

⟨2⟩2. If  $n$  is a natural number and a set of natural numbers then  $n^+$  is a set of natural numbers.

PROOF: From the definition of  $n^+$ .

⟨2⟩3. Q.E.D.

PROOF: By induction.

□

**Theorem 6.3.9** (Extensionality, Empty Set, Pairing, Union, Infinity, Subset).  
*Let  $A$  be a set,  $a \in A$ , and  $F : A \rightarrow A$ . Then there exists a unique function  $h : \mathbb{N} \rightarrow A$  such that  $h(0) = a$  and, for all  $n \in \mathbb{N}$ , we have  $h(n^+) = F(h(n))$ .*

PROOF:

⟨1⟩1. Call a function  $v$  *acceptable* iff  $\text{dom } v \subseteq \mathbb{N}$ ,  $\text{ran } v \subseteq A$ , and:

1. If  $0 \in \text{dom } v$  then  $v(0) = a$ .

2. For all  $n \in \mathbb{N}$ , if  $n^+ \in \text{dom } v$  then  $n \in \text{dom } v$  and  $v(n^+) = F(v(n))$ .

⟨1⟩2. LET:  $\mathcal{K}$  be the set of all acceptable functions.

⟨1⟩3. LET:  $h = \bigcup \mathcal{K}$

⟨1⟩4.  $h$  is a function.

⟨2⟩1. If  $(0, y) \in h$  and  $(0, y') \in h$  then  $y = y'$

PROOF: We have  $y = y' = a$ .

⟨2⟩2. For any natural number  $n$ , if there is at most one  $y$  such that  $(n, y) \in h$ , then there is at most one  $y$  such that  $(n^+, y) \in h$

⟨3⟩1. LET:  $n$  be a natural number.

⟨3⟩2. ASSUME: there is at most one  $y$  such that  $(n, y) \in h$

⟨3⟩3. ASSUME:  $(n^+, y)$  and  $(n^+, y')$  are in  $h$

⟨3⟩4. PICK acceptable functions  $u$  and  $v$  such that  $u(n^+) = y$  and  $v(n^+) = y'$

⟨3⟩5.  $n \in \text{dom } u$ ,  $n \in \text{dom } v$  and  $y = F(u(n))$ ,  $y' = F(v(n))$

⟨3⟩6.  $u(n) = v(n)$

PROOF: By the induction hypothesis ⟨3⟩2

⟨3⟩7.  $y = y'$

⟨2⟩3. Q.E.D.

PROOF: By induction.

⟨1⟩5.  $h$  is acceptable.

⟨2⟩1. If  $0 \in \text{dom } h$  then  $h(0) = a$

⟨2⟩2. If  $n^+ \in \text{dom } h$  then  $n \in \text{dom } h$  and  $h(n^+) = F(h(n))$

⟨3⟩1. ASSUME:  $n^+ \in \text{dom } h$

⟨3⟩2. PICK an acceptable  $v$  such that  $n^+ \in \text{dom } v$

⟨3⟩3.  $v(n^+) = F(v(n))$

⟨3⟩4.  $h(n^+) = F(h(n))$

⟨1⟩6.  $\text{dom } h = \mathbb{N}$

⟨2⟩1.  $0 \in \text{dom } h$

PROOF: Since  $\{(0, a)\}$  is an acceptable function.

⟨2⟩2. For all  $n \in \text{dom } h$  we have  $n^+ \in \text{dom } h$

⟨3⟩1. ASSUME:  $n \in \text{dom } h$

⟨3⟩2. LET:  $v$  be an acceptable function with  $n \in \text{dom } v$

⟨3⟩3. ASSUME: without loss of generality  $n^+ \notin \text{dom } v$

⟨3⟩4.  $v \cup \{(n^+, F(v(n)))\}$  is acceptable

$\langle 3 \rangle 5. n^+ \in \text{dom } v$   
 $\langle 1 \rangle 7. \text{ If } h' : \mathbb{N} \rightarrow A, h'(0) = a \text{ and, for all } n \in \mathbb{N}, \text{ we have } h'(n^+) = F(h'(n)),$   
 $\text{then } h' = h$   
 PROOF: Prove  $h(n) = h'(n)$  by induction on  $n$ .  
 $\square$

## 6.4 Peano Systems

**Definition 6.4.1** (Peano System). A *Peano system* consists of a set  $N$ , an element  $z \in N$ , and a function  $S : N \rightarrow N$  such that:

- $S$  is one-to-one
- $z \notin \text{ran } S$
- For any set  $A \subseteq N$ , if  $z \in A$  and  $S(A) \subseteq A$  then  $A = N$ .

**Theorem 6.4.2.**  $\mathbb{N}$  is a Peano system with zero 0 and successor  $n \mapsto n^+$ .

**Theorem 6.4.3.** For any Peano system  $(N, z, S)$ , there exists a unique bijection  $h : \mathbb{N} \cong N$  such that  $h(0) = z$  and  $S(h(n)) = h(n^+)$  for all  $n$ .

## 6.5 Arithmetic

**Definition 6.5.1** (Addition). Define *addition*  $+ : \mathbb{N}^2 \rightarrow \mathbb{N}$  recursively by

$$\begin{aligned}
 m + 0 &= m \\
 m + n^+ &= (m + n)^+
 \end{aligned}$$

for any  $m, n \in \mathbb{N}$ .

**Theorem 6.5.2.** *Addition is associative.*

**Theorem 6.5.3.** *Addition is commutative*

**Definition 6.5.4** (Multiplication). Define *multiplication*  $\cdot : \mathbb{N}^2 \rightarrow \mathbb{N}$  recursively by

$$\begin{aligned}
 m0 &= 0 \\
 mn^+ &= mn + m
 \end{aligned}$$

for any  $m, n \in \mathbb{N}$

**Theorem 6.5.5.** *Multiplication is associative.*

**Theorem 6.5.6.** *Multiplication is commutative.*

**Theorem 6.5.7.** *Multiplication distributes over addition.*

**Definition 6.5.8.** For natural numbers  $m$  and  $n$ , we write  $m < n$  iff  $m \in n$ . We write  $m \leq n$  iff  $m < n$  or  $m = n$ .

**Theorem 6.5.9.** We have  $m < n$  iff  $m^+ < n^+$ .

**Theorem 6.5.10.** We never have  $n < n$ .

**Theorem 6.5.11.** The ordering on  $\mathbb{N}$  satisfies trichotomy; that is, for any  $m, n$ , exactly one of  $m < n$ ,  $m = n$ ,  $n < m$  holds.

**Theorem 6.5.12.** For any natural numbers  $m$  and  $n$ , we have  $m \leq n$  iff  $m \subseteq n$ .

**Theorem 6.5.13.** We have  $m < n$  iff  $m + p < n + p$ .

**Corollary 6.5.13.1.** If  $m + p = n + p$  then  $m = n$ .

**Theorem 6.5.14.** If  $p \neq 0$  then  $m < n$  iff  $mp < np$ .

**Corollary 6.5.14.1.** If  $mp = np$  and  $p \neq 0$  then  $m = n$ .

**Theorem 6.5.15** (Well-Ordering of  $\mathbb{N}$ ). Any nonempty set  $A \subseteq \mathbb{N}$  has a least element.

**Corollary 6.5.15.1.** There is no function  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that  $f(n^+) < f(n)$  for all  $n$ .

**Theorem 6.5.16** (Strong Induction). Let  $A \subseteq \mathbb{N}$ . Suppose that, for every natural number  $n$ , if  $\forall m < n. m \in A$  then  $n \in A$ . Then  $A = \mathbb{N}$ .

**Theorem 6.5.17** (Pigeonhole Principle). No natural number is equinumerous with a proper subset of itself.

PROOF: Prove by induction on  $n$  that if  $f : n \rightarrow n$  is injective then it is surjective.  
 $\square$

## Chapter 7

# Integers

**Lemma 7.0.1.** Define  $\sim$  on  $\mathbb{N}^2$  by:  $(m, n) \sim (p, q)$  iff  $m + q = n + p$ . Then  $\sim$  is an equivalence relation on  $\mathbb{N}^2$ .

**Definition 7.0.2** (Integers). The set  $\mathbb{Z}$  of *integers* is  $\mathbb{N}^2 / \sim$ .

**Definition 7.0.3.** Define *addition*  $+: \mathbb{Z}^2 \rightarrow \mathbb{Z}$  by:  $(m, n) + (p, q) = (m + p, n + q)$ .

Prove this is well-defined.

**Theorem 7.0.4.** *Addition is associative and commutative.*

**Definition 7.0.5** (Zero). The integer *zero* is  $0 = (0, 0)$ .

**Theorem 7.0.6.** *For any integer  $a$ , we have  $a + 0 = a$ .*

**Theorem 7.0.7.** *For any integer  $a$ , there exists a unique integer  $b$  such that  $a + b = 0$ .*

**Definition 7.0.8** (Multiplication). Define multiplication on  $\mathbb{Z}$  by  $(m, n)(p, q) = (mp + nq, mq + np)$ .

**Theorem 7.0.9.** *Multiplication is associative, commutative and distributive over addition.*

**Definition 7.0.10.** The integer *one* is  $1 = (1, 0)$ .

**Theorem 7.0.11.** *For any integer  $a$  we have  $a1 = a$ .*

**Theorem 7.0.12.**  $1 \neq 0$

**Theorem 7.0.13.** *Whenever  $ab = 0$  then either  $a = 0$  or  $b = 0$ .*

**Definition 7.0.14.** Define  $<$  on  $\mathbb{Z}$  by:  $(m, n) < (p, q)$  iff  $m + q < n + p$ .

**Theorem 7.0.15.** *The relation  $<$  is a strict linear ordering on  $\mathbb{Z}$ .*

**Theorem 7.0.16.** *We have  $a < b$  iff  $+c < b + c$ .*

**Corollary 7.0.16.1.** *If  $a + c = b + c$  then  $a = b$ .*

**Theorem 7.0.17.** *If  $0 < c$  then  $a < b$  iff  $ac < bc$ .*

**Corollary 7.0.17.1.** *If  $ac = bc$  and  $c \neq 0$  then  $a = b$ .*

**Definition 7.0.18.** We identify any natural number  $n$  with the integer  $(n, 0)$ .

**Theorem 7.0.19.** *This embedding preserves 0, 1, addition, multiplication and the ordering.*

## Chapter 8

# Rational Numbers

**Definition 8.0.1** (Rational Numbers). The set of *rational numbers*  $\mathbb{Q}$  is  $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) / \sim$ , where  $(a, b) \sim (c, d)$  iff  $ad = bc$ .

**Definition 8.0.2** (Addition). Define addition on  $\mathbb{Q}$  by:  $(a, b) + (c, d) = (ad + bc, bd)$ .

**Theorem 8.0.3.** *Addition is commutative and associative*

**Definition 8.0.4.** The rational number 0 is  $(0, 1)$ .

**Theorem 8.0.5.** *For any rational  $q$  we have  $q + 0 = q$ .*

**Theorem 8.0.6.** *For any rational  $q$ , there exists a unique rational  $r$  such that  $q + r = 0$ .*

**Definition 8.0.7.** Define multiplication on  $\mathbb{Q}$  by:  $(a, b)(c, d) = (ac, bd)$ .

**Theorem 8.0.8.** *Multiplication is commutative, associative and distributive over addition.*

**Definition 8.0.9.** The rational number 1 is  $(1, 1)$ .

**Theorem 8.0.10.** *For every nonzero rational  $r$ , there exists a nonzero rational  $q$  such that  $rq = 1$ .*

**Corollary 8.0.10.1.** *If  $qr = 0$  then either  $q = 0$  or  $r = 0$ .*

**Definition 8.0.11.** Define  $<$  on  $\mathbb{Q}$  by: for  $b$  and  $d$  positive,  $(a, b) < (c, d)$  iff  $ad < bc$ .

**Theorem 8.0.12.** *The relation  $<$  is a strict linear ordering on  $\mathbb{Q}$ .*

**Theorem 8.0.13.** *We have  $q < r$  iff  $q + s < r + s$*

**Corollary 8.0.13.1.** *If  $q + s = r + s$  then  $q = r$ .*

**Theorem 8.0.14.** *If  $s > 0$  then we have  $q < r$  iff  $qs < rs$ .*



**Corollary 8.0.14.1.** *If  $qs = rs$  and  $s \neq 0$  then  $q = r$ .*

**Definition 8.0.15.** We identify an integer  $n$  with the rational  $(n, 1)$ .

**Theorem 8.0.16.** *This embedding preserves zero, one, addition, multiplication and the ordering.*

## Chapter 9

# Real Numbers

**Definition 9.0.1** (Dedekind Cut). A *Dedekind cut* is a subset  $X \subseteq \mathbb{Q}$  such that:

- $X$  is nonempty
- $X \neq \mathbb{Q}$
- $X$  is closed downward
- $X$  has no largest element.

**Definition 9.0.2** (Real Numbers). The set of *real numbers*  $\mathbb{R}$  is the set of all Dedekind cuts.

**Definition 9.0.3.** Define  $<$  on  $\mathbb{R}$  by:  $x < y$  iff  $x$  is a proper subset of  $y$ .

**Theorem 9.0.4.** *The relation  $<$  is a strict linear ordering on  $\mathbb{R}$ .*

**Theorem 9.0.5.** *Any bounded nonempty subset of  $\mathbb{R}$  has a least upper bound.*

**Definition 9.0.6.** Define addition on  $\mathbb{R}$  by:  $x + y = \{q + r \mid q \in x, r \in y\}$ .

**Theorem 9.0.7.** *Addition is associative and commutative.*

**Definition 9.0.8.** The zero real  $0$  is  $\{q \in \mathbb{Q} \mid q < 0\}$ .

**Theorem 9.0.9.** *For any  $x \in \mathbb{R}$  we have  $x + 0 = x$ .*

**Definition 9.0.10.** Given a real  $x$ , define  $-x = \{q \in \mathbb{Q} \mid \exists r > q. -r \notin x\}$ .

**Theorem 9.0.11.** *For any real  $x$  we have  $x + (-x) = 0$ .*

**Corollary 9.0.11.1.** *If  $x + z = y + z$  then  $x = y$ .*

**Theorem 9.0.12.** *We have  $x < y$  iff  $x + z < y + z$ .*

**Definition 9.0.13.** Define the *absolute value* of a real  $x$  by  $|x| = x \cup -x$ .

**Theorem 9.0.14.** For any real  $x$  we have  $0 \leq |x|$ .

**Definition 9.0.15.** Define multiplication on  $\mathbb{R}$  by:

- If  $x$  and  $y$  are nonnegative then

$$xy = 0 \cup \{qr \mid 0 \leq q, 0 \leq r, q \in x, r \in y\}$$

- If  $x$  and  $y$  are both negative then  $xy = |x||y|$
- If one of  $x$  and  $y$  is negative and the other not then  $xy = -|x||y|$ .

**Theorem 9.0.16.** Multiplication is associative, commutative and distributive over addition.

**Definition 9.0.17.** The real number 1 is  $\{q \in \mathbb{Q} \mid q < 1\}$ .

**Theorem 9.0.18.**  $0 \neq 1$

**Theorem 9.0.19.** For any real  $x$  we have  $x1 = x$

**Theorem 9.0.20.** For any nonzero  $x$ , there exists a real  $y$  with  $xy = 1$ .

**Theorem 9.0.21.** If  $0 < x$  then  $y < z$  iff  $xy < xz$ .

**Definition 9.0.22.** Identify a rational  $q$  with  $\{r \in \mathbb{Q} \mid r < q\}$ .

**Theorem 9.0.23.** This embedding preserves zero, one, addition, multiplication and the ordering.

## 9.1 The Cantor Set

**Definition 9.1.1** (Cantor Set). Define the sequence of sets  $A_n \subseteq \mathbb{R}$  by

$$A_0 = [0, 1]$$

$$A_n = A_{n-1} - \bigcup_{k=0}^{3^{n-1}-1} ((3k+1)/3^n, (3k+2)/3^n)$$

The Cantor set is  $\bigcap_{n=0}^{\infty} A_n$ .

**Proposition 9.1.2.** The set  $A_n$  is a union of finitely many disjoint closed intervals of length  $1/3^n$ , and the endpoints of these intervals lie in  $C$ .

PROOF: An easy induction on  $n$ .  $\square$

# Chapter 10

## Finite Sets

**Definition 10.0.1** (Finite). A set is *finite* iff it is equinumerous with a natural number; otherwise it is *infinite*.

**Theorem 10.0.2.** *No finite set is equinumerous with a proper subset of itself.*

PROOF: From the Pigeonhole Principle.

**Corollary 10.0.2.1.** *The set  $\mathbb{N}$  is infinite.*

**Corollary 10.0.2.2.** *A finite set is equinumerous with a unique natural number.*

**Lemma 10.0.3.** *If  $A$  is a proper subset of a natural number  $n$  then there exists  $m < n$  such that  $C \equiv m$ .*

**Corollary 10.0.3.1.** *A subset of a finite set is finite.*

**Theorem 10.0.4** (Regularity). *There is no function  $f$  with domain  $\mathbb{N}$  such that  $f(n+1) \in f(n)$  for all  $n$ .*

PROOF:

$\langle 1 \rangle 1$ . ASSUME: for a contradiction  $f$  is a function with domain  $\mathbb{N}$  such that  $f(n+1) \in f(n)$  for all  $n$ .

$\langle 1 \rangle 2$ . PICK  $m \in \text{ran } f$  such that  $m \cap \text{ran } f = \emptyset$

PROOF: By the Axiom of Regularity.

$\langle 1 \rangle 3$ . PICK  $n \in \mathbb{N}$  such that  $f(n) = m$

$\langle 1 \rangle 4$ .  $f(n+1) \in m \cap \text{ran } f$

$\langle 1 \rangle 5$ . Q.E.D.

PROOF: This is a contradiction.

□

**Theorem 10.0.5.** *A relation  $R$  is well-founded if and only if there is no function  $f$  with domain  $\mathbb{N}$  such that, for all  $n \in \mathbb{N}$ , we have  $f(n+1) R f(n)$ .*

## 10.1 The Finite Intersection Property

**Definition 10.1.1** (Finite Intersection Property). Let  $X$  be a set and  $\mathcal{A} \subseteq \mathcal{P}X$ . Then  $\mathcal{A}$  satisfies the *finite intersection property* if and only if every nonempty finite subset of  $\mathcal{A}$  has nonempty intersection.

**Lemma 10.1.2.** Let  $X$  be a set. Let  $\mathcal{D} \subseteq \mathcal{P}X$  be maximal with respect to the finite intersection property. Then any finite intersection of elements of  $\mathcal{D}$  is an element of  $\mathcal{D}$ .

PROOF:

$\langle 1 \rangle 1.$  LET:  $D_1, D_2 \in \mathcal{D}$

$\langle 1 \rangle 2.$   $\mathcal{D} \cup \{D_1 \cap D_2\}$  has the finite intersection property.

PROOF: Any finite intersection of members of  $\mathcal{D} \cup \{D_1 \cap D_2\}$  is a finite intersection of members of  $\mathcal{D}$ .

$\langle 1 \rangle 3.$   $\mathcal{D} = \mathcal{D} \cup \{D_1 \cap D_2\}$

PROOF: By maximality of  $\mathcal{D}$ .

$\langle 1 \rangle 4.$   $D_1 \cap D_2 \in \mathcal{D}$ .

□

**Lemma 10.1.3.** Let  $X$  be a set. Let  $\mathcal{D} \subseteq \mathcal{P}X$  be maximal with respect to the finite intersection property. Let  $A \subseteq X$ . If  $A$  intersects every member of  $\mathcal{D}$  then  $A \in \mathcal{D}$ .

PROOF:

$\langle 1 \rangle 1.$   $\mathcal{D} \cup \{A\}$  has the finite intersection property.

$\langle 2 \rangle 1.$  LET:  $D_1, \dots, D_n \in \mathcal{D}$

PROVE:  $D_1 \cap \dots \cap D_n \cap A \neq \emptyset$

$\langle 2 \rangle 2.$   $D_1 \cap \dots \cap D_n \in \mathcal{D}$

PROOF: Lemma 10.1.2.

$\langle 2 \rangle 3.$   $D_1 \cap \dots \cap D_n \cap A \neq \emptyset$

PROOF: Since  $A$  intersects every member of  $\mathcal{D}$ .

$\langle 1 \rangle 2.$  Q.E.D.

PROOF: By maximality of  $\mathcal{D}$ .

□

**Proposition 10.1.4.** Let  $X$  be a set. Let  $\mathcal{D} \subseteq \mathcal{P}X$  be maximal with respect to the finite intersection property. Let  $A, D \in \mathcal{P}X$ . If  $D \in \mathcal{D}$  and  $D \subseteq A$  then  $A \in \mathcal{D}$ .

PROOF:

$\langle 1 \rangle 1.$   $\mathcal{D} \cup \{A\}$  satisfies the finite intersection property.

$\langle 2 \rangle 1.$  LET:  $D_1, \dots, D_n \in \mathcal{D}$

$\langle 2 \rangle 2.$   $D_1 \cap \dots \cap D_n \cap D \neq \emptyset$

PROOF: Since  $\mathcal{D}$  satisfies the finite intersection property.

$\langle 2 \rangle 3.$   $D_1 \cap \dots \cap D_n \cap A \neq \emptyset$

$\langle 1 \rangle 2.$   $\mathcal{D} = \mathcal{D} \cup \{A\}$

PROOF: By the maximality of  $\mathcal{D}$ .

$$\langle 1 \rangle 3. \ A \in \mathcal{D}$$

□

## 10.2 Real Analysis

**Definition 10.2.1.** Let  $\mathbb{R}^\infty$  be the subset of  $\mathbb{R}^\omega$  consisting of all sequences  $(a_n)$  such that  $a_n = 0$  for all but finitely many  $n$ .

## 10.3 Group Theory

**Definition 10.3.1.** Given a group  $G$  and sets  $A, B \subseteq G$ , let  $AB = \{ab \mid a \in A, b \in B\}$ .

**Definition 10.3.2.** Given a group  $G$  and a set  $A \subseteq G$ , let  $A^{-1} = \{a^{-1} \mid a \in A\}$ .



# Chapter 11

## Topological Spaces

### 11.1 Topologies

**Definition 11.1.1** (Topology). A *topology* on a set  $X$  is a set  $\mathcal{T} \subseteq \mathcal{P}X$  such that:

- $X \in \mathcal{T}$ .
- For all  $\mathcal{U} \subseteq \mathcal{T}$  we have  $\bigcup \mathcal{U} \in \mathcal{T}$ .
- For all  $U, V \in \mathcal{T}$  we have  $U \cap V \in \mathcal{T}$ .

We call the elements of  $X$  *points* and the elements of  $\mathcal{T}$  *open sets*.

**Definition 11.1.2** (Topological Space). A *topological space*  $X$  consists of a set  $X$  and a topology on  $X$ .

**Definition 11.1.3** (Discrete Space). For any set  $X$ , the *discrete* topology on  $X$  is  $\mathcal{P}X$ .

**Definition 11.1.4** (Indiscrete Space). For any set  $X$ , the *indiscrete* or *trivial* topology on  $X$  is  $\{\emptyset, X\}$ .

**Definition 11.1.5** (Finite Complement Topology). For any set  $X$ , the *finite complement topology* on  $X$  is  $\{U \in \mathcal{P}X \mid X \setminus U \text{ is finite}\} \cup \{\emptyset\}$ .

**Definition 11.1.6** (Countable Complement Topology). For any set  $X$ , the *countable complement topology* on  $X$  is  $\{U \in \mathcal{P}X \mid X \setminus U \text{ is countable}\} \cup \{\emptyset\}$ .

**Definition 11.1.7** (Finer, Coarser). Suppose that  $\mathcal{T}$  and  $\mathcal{T}'$  are two topologies on a given set  $X$ . If  $\mathcal{T}' \supseteq \mathcal{T}$ , we say that  $\mathcal{T}'$  is *finer* than  $\mathcal{T}$ ; if  $\mathcal{T}'$  properly contains  $\mathcal{T}$ , we say that  $\mathcal{T}'$  is *strictly finer* than  $\mathcal{T}$ . We also say that  $\mathcal{T}$  is *coarser* than  $\mathcal{T}'$ , or *strictly coarser*, in these two respective situations. We say  $\mathcal{T}$  is *comparable* with  $\mathcal{T}'$  if either  $\mathcal{T}' \supseteq \mathcal{T}$  or  $\mathcal{T} \supseteq \mathcal{T}'$ .

**Lemma 11.1.8.** Let  $X$  be a topological space and  $U \subseteq X$ . Then  $U$  is open if and only if, for all  $x \in U$ , there exists an open set  $V$  such that  $x \in V \subseteq U$ .

PROOF:

$\langle 1 \rangle 1. \Rightarrow$

PROOF: Take  $V = U$

$\langle 1 \rangle 2. \Leftarrow$

PROOF: We have  $U = \bigcup \{V \text{ open in } X \mid V \subseteq U\}$ .

□

**Lemma 11.1.9.** *Let  $X$  be a set and  $\mathcal{T}$  a nonempty set of topologies on  $X$ . Then  $\bigcap \mathcal{T}$  is a topology on  $X$ , and is the finest topology that is coarser than every member of  $\mathcal{T}$ .*

PROOF:

$\langle 1 \rangle 1. X \in \bigcap \mathcal{T}$

PROOF: Since  $X$  is in every member of  $\mathcal{T}$ .

$\langle 1 \rangle 2. \bigcap \mathcal{T}$  is closed under union.

$\langle 2 \rangle 1. \text{ LET: } \mathcal{U} \subseteq \bigcap \mathcal{T}$

$\langle 2 \rangle 2. \text{ For all } T \in \mathcal{T} \text{ we have } \mathcal{U} \subseteq T$

$\langle 2 \rangle 3. \text{ For all } T \in \mathcal{T} \text{ we have } \bigcup \mathcal{U} \in T$

$\langle 2 \rangle 4. \bigcup \mathcal{U} \in \bigcap \mathcal{T}$

$\langle 1 \rangle 3. \bigcap \mathcal{T}$  is closed under binary intersection.

$\langle 2 \rangle 1. \text{ LET: } U, V \in \bigcap \mathcal{T}$

$\langle 2 \rangle 2. \text{ For all } T \in \mathcal{T} \text{ we have } U, V \in T$

$\langle 2 \rangle 3. \text{ For all } T \in \mathcal{T} \text{ we have } U \cap V \in T$

$\langle 2 \rangle 4. U \cap V \in \bigcap \mathcal{T}$

□

**Lemma 11.1.10.** *Let  $X$  be a set and  $\mathcal{T}$  a set of topologies on  $X$ . Then there exists a unique coarsest topology that is finer than every member of  $\mathcal{T}$ .*

PROOF: The required topology is given by

$\bigcap \{T \in \mathcal{P}P X \mid T \text{ is a topology on } X \text{ that is finer than every member of } \mathcal{T}\}$  ,

The set is nonempty since it contains the discrete topology. □

**Definition 11.1.11** (Neighbourhood). A *neighbourhood* of a point  $x$  is an open set that contains  $x$ .

## 11.2 Closed Set

**Definition 11.2.1** (Closed Set). Let  $X$  be a topological space and  $A \subseteq X$ . Then  $A$  is *closed* if and only if  $X \setminus A$  is open.

**Lemma 11.2.2.** *The empty set is closed.*

PROOF: Since the whole space  $X$  is always open. □

**Lemma 11.2.3.** *The topological space  $X$  is closed.*

PROOF: Since  $\emptyset$  is open. □

**Lemma 11.2.4.** *The intersection of a nonempty set of closed sets is closed.*

PROOF: Let  $\mathcal{C}$  be a nonempty set of closed sets. Then  $X \setminus \bigcap \mathcal{C} = \bigcup \{X \setminus C \mid C \in \mathcal{C}\}$  is open.  $\square$

**Lemma 11.2.5.** *The union of two closed sets is closed.*

PROOF: Let  $C$  and  $D$  be closed. Then  $X \setminus (C \cup D) = (X \setminus C) \cap (X \setminus D)$  is open.  $\square$

**Proposition 11.2.6.** *Let  $X$  be a set and  $\mathcal{C} \subseteq \mathcal{P}X$  a set such that:*

1.  $\emptyset \in \mathcal{C}$
2.  $X \in \mathcal{C}$
3. For all  $\mathcal{A} \subseteq \mathcal{C}$  nonempty we have  $\bigcap \mathcal{A} \in \mathcal{C}$
4. For all  $C, D \in \mathcal{C}$  we have  $C \cup D \in \mathcal{C}$ .

*Then there exists a unique topology  $\mathcal{T}$  such that  $\mathcal{C}$  is the set of closed sets, namely*

$$\mathcal{T} = \{X \setminus C \mid C \in \mathcal{C}\}$$

PROOF:

$\langle 1 \rangle 1$ . LET:  $\mathcal{T} = \{X \setminus C \mid C \in \mathcal{C}\}$

$\langle 1 \rangle 2$ .  $\mathcal{T}$  is a topology

$\langle 2 \rangle 1$ .  $X \in \mathcal{T}$

PROOF: Since  $\emptyset \in \mathcal{C}$

$\langle 2 \rangle 2$ . For all  $\mathcal{U} \subseteq \mathcal{T}$  we have  $\bigcup \mathcal{U} \in \mathcal{T}$

$\langle 3 \rangle 1$ . LET:  $\mathcal{U} \subseteq \mathcal{T}$

$\langle 3 \rangle 2$ . CASE:  $\mathcal{U} = \emptyset$

PROOF: In this case  $\bigcup \mathcal{U} = \emptyset \in \mathcal{T}$  since  $X \in \mathcal{C}$

$\langle 3 \rangle 3$ . CASE:  $\mathcal{U} \neq \emptyset$

PROOF: In this case  $X \setminus \bigcup \mathcal{U} = \bigcap \{X \setminus U \mid U \in \mathcal{U}\} \in \mathcal{C}$ .

$\langle 2 \rangle 3$ . For all  $U, V \in \mathcal{T}$  we have  $U \cap V \in \mathcal{T}$

PROOF: Since  $X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V) \in \mathcal{C}$ .

$\langle 1 \rangle 3$ .  $\mathcal{C}$  is the set of all closed sets in  $\mathcal{T}$

PROOF:

$C$  is closed in  $\mathcal{T}$

$$\Leftrightarrow X \setminus C \in \mathcal{T}$$

$$\Leftrightarrow C \in \mathcal{C}$$

$\langle 1 \rangle 4$ . If  $\mathcal{T}'$  is a topology and  $\mathcal{C}$  is the set of closed sets in  $\mathcal{T}'$  then  $\mathcal{T}' = \mathcal{T}$

PROOF: We have

$$U \in \mathcal{T}$$

$$\Leftrightarrow X \setminus U \in \mathcal{C}$$

$$\Leftrightarrow X \setminus U \text{ is closed in } \mathcal{T}'$$

$$\Leftrightarrow U \in \mathcal{T}'$$

□

**Proposition 11.2.7.** *If  $U$  is open and  $A$  is closed then  $U \setminus A$  is open.*

PROOF:  $U \setminus A = U \cap (X \setminus A)$  is the intersection of two open sets. □

**Proposition 11.2.8.** *If  $U$  is open and  $A$  is closed then  $A \setminus U$  is closed.*

PROOF:  $A \setminus U = A \cap (X \setminus U)$  is the intersection of two closed sets. □

## 11.3 Interior

**Definition 11.3.1** (Interior). Let  $X$  be a topological space and  $A \subseteq X$ . The *interior* of  $A$ ,  $\text{Int } A$ , is the union of all the open subsets of  $A$ .

**Lemma 11.3.2.** *The interior of a set is open.*

PROOF: It is a union of open sets. □

**Lemma 11.3.3.**

$$\text{Int } A \subseteq A$$

PROOF: Immediate from definition. □

**Lemma 11.3.4.** *If  $U$  is open and  $U \subseteq A$  then  $U \subseteq \text{Int } A$*

PROOF: Immediate from definition. □

**Lemma 11.3.5.** *A set  $A$  is open if and only if  $A = \text{Int } A$ .*

PROOF: If  $A = \text{Int } A$  then  $A$  is open by Lemma 11.3.2. Conversely if  $A$  is open then  $A \subseteq \text{Int } A$  by the definition of interior and so  $A = \text{Int } A$ .

## 11.4 Closure

**Definition 11.4.1** (Closure). Let  $X$  be a topological space and  $A \subseteq X$ . The *closure* of  $A$ ,  $\bar{A}$ , is the intersection of all the closed sets that include  $A$ .

This intersection exists since  $X$  is a closed set that includes  $A$  (Lemma 11.2.3).

**Lemma 11.4.2.** *The closure of a set is closed.*

PROOF: Dual to Lemma 11.3.2. □

**Lemma 11.4.3.**

$$A \subseteq \bar{A}$$

PROOF: Immediate from definition. □

**Lemma 11.4.4.** *If  $C$  is closed and  $A \subseteq C$  then  $\bar{A} \subseteq C$ .*

PROOF: Immediate from definition.  $\square$

**Lemma 11.4.5.** *A set  $A$  is closed if and only if  $A = \overline{A}$ .*

PROOF: Dual to Lemma 11.3.5.  $\square$

**Theorem 11.4.6.** *Let  $X$  be a topological space,  $A \subseteq X$  and  $x \in X$ . Then  $x \in \overline{A}$  if and only if every neighbourhood of  $x$  intersects  $A$ .*

PROOF: We have

$$\begin{aligned} x \in \overline{A} \\ \Leftrightarrow \forall C. C \text{ closed} \wedge A \subseteq C \Rightarrow x \in C \\ \Leftrightarrow \forall U. U \text{ open} \wedge A \cap U = \emptyset \Rightarrow x \notin U \\ \Leftrightarrow \forall U. U \text{ open} \wedge x \in U \Rightarrow U \text{ intersects } A \end{aligned} \quad \square$$

**Proposition 11.4.7.** *If  $A \subseteq B$  then  $\overline{A} \subseteq \overline{B}$ .*

PROOF: This holds because  $\overline{B}$  is a closed set that includes  $A$ .  $\square$

**Proposition 11.4.8.**

$$\overline{A \cup B} = \overline{A} \cup \overline{B}$$

PROOF:

$\langle 1 \rangle 1. \overline{A} \subseteq \overline{A \cup B}$

PROOF: By Proposition 11.4.7.

$\langle 1 \rangle 2. \overline{B} \subseteq \overline{A \cup B}$

PROOF: By Proposition 11.4.7.

$\langle 1 \rangle 3. \overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$

$\langle 2 \rangle 1. \text{ LET: } x \in \overline{A \cup B}$

$\langle 2 \rangle 2. \text{ ASSUME: } x \notin \overline{A}$

PROVE:  $x \in \overline{B}$

$\langle 2 \rangle 3. \text{ PICK a neighbourhood } U \text{ of } x \text{ that does not intersect } A$

$\langle 2 \rangle 4. \text{ LET: } V \text{ be any neighbourhood of } x$

$\langle 2 \rangle 5. U \cap V \text{ is a neighbourhood of } x$

$\langle 2 \rangle 6. U \cap V \text{ intersects } A \cup B$

PROOF: From  $\langle 2 \rangle 1$  and Theorem 11.4.6.

$\langle 2 \rangle 7. U \cap V \text{ intersects } B$

PROOF: From  $\langle 2 \rangle 3$

$\langle 2 \rangle 8. V \text{ intersects } B$

$\langle 2 \rangle 9. \text{ Q.E.D.}$

PROOF: We have  $x \in \overline{B}$  from Theorem 11.4.6.

$\square$

**Proposition 11.4.9.** *Let  $X$  be a topological space. Let  $\mathcal{D}$  be a set of subsets of  $X$  that is maximal with respect to the finite intersection property. Let  $x \in X$ . Then the following are equivalent:*

1. For all  $D \in \mathcal{D}$  we have  $x \in \overline{D}$

2. Every neighbourhood of  $x$  is in  $\mathcal{D}$ .

PROOF:

$\langle 1 \rangle 1. 1 \Rightarrow 2$

$\langle 2 \rangle 1.$  For all  $D \in \mathcal{D}$  we have  $x \in \overline{D}$

$\langle 2 \rangle 2.$  LET:  $U$  be a neighbourhood of  $x$

$\langle 2 \rangle 3.$   $\mathcal{D} \cup \{U\}$  satisfies the finite intersection property.

$\langle 3 \rangle 1.$  LET:  $D_1, \dots, D_n \in \mathcal{D}$

$\langle 3 \rangle 2.$   $D_1 \cap \dots \cap D_n \in \mathcal{D}$

PROOF: Lemma 10.1.2.

$\langle 3 \rangle 3.$   $x \in \overline{D_1 \cap \dots \cap D_n}$

PROOF:  $\langle 2 \rangle 1, \langle 3 \rangle 2$

$\langle 3 \rangle 4.$   $D_1 \cap \dots \cap D_n \cap U \neq \emptyset$

PROOF: Theorem 11.4.6,  $\langle 2 \rangle 2, \langle 3 \rangle 3.$

$\langle 2 \rangle 4.$   $\mathcal{D} = \mathcal{D} \cup \{U\}$

PROOF: By the maximality of  $\mathcal{D}$ .

$\langle 2 \rangle 5.$   $U \in \mathcal{D}$

$\langle 1 \rangle 2. 2 \Rightarrow 1$

$\langle 2 \rangle 1.$  ASSUME: Every neighbourhood of  $x$  is in  $\mathcal{D}$ .

$\langle 2 \rangle 2.$  LET:  $D \in \mathcal{D}$

$\langle 2 \rangle 3.$  Every neighbourhood of  $x$  intersects  $D$ .

PROOF: From  $\langle 2 \rangle 1, \langle 2 \rangle 2$  and the fact that  $\mathcal{D}$  satisfies the finite intersection property.

$\langle 2 \rangle 4.$   $x \in \overline{D}$

PROOF: Theorem 11.4.6,  $\langle 2 \rangle 3.$

□

## 11.5 Boundary

**Definition 11.5.1** (Boundary). The *boundary* of a set  $A$  is the set  $\partial A = \overline{A} \cap \overline{X \setminus A}$ .

**Proposition 11.5.2.**

$$\text{Int } A \cap \partial A = \emptyset$$

PROOF: Since  $\overline{X \setminus A} = X \setminus \text{Int } A$ . □

**Proposition 11.5.3.**

$$\overline{A} = \text{Int } A \cup \partial A$$

PROOF:

$$\begin{aligned} \text{Int } A \cup \partial A &= \text{Int } A \cup (\overline{A} \cap \overline{X \setminus A}) \\ &= (\text{Int } A \cup \overline{A}) \cap (\text{Int } A \cup \overline{X \setminus A}) \\ &= \overline{A} \cap X \\ &= \overline{A} \end{aligned}$$

**Proposition 11.5.4.**  $\partial A = \emptyset$  if and only if  $A$  is open and closed.

PROOF: If  $\partial A = \emptyset$  then  $\overline{A} = \text{Int } A$  by Proposition 11.5.3.

**Proposition 11.5.5.** *A set  $U$  is open if and only if  $\partial U = \overline{U} \setminus U$ .*

PROOF:

$$\begin{aligned} \partial U &= \overline{U} \setminus U \\ \Leftrightarrow \overline{U} \setminus \text{Int } U &= \overline{U} \setminus U && (\text{Propositions 11.5.2, 11.5.3}) \\ \Leftrightarrow \text{Int } U &= U && \square \end{aligned}$$

## 11.6 Limit Points

**Definition 11.6.1** (Limit Point). Let  $X$  be a topological space,  $a \in X$  and  $A \subseteq X$ . Then  $a$  is a *limit point*, *cluster point* or *point of accumulation* for  $A$  if and only if every neighbourhood of  $a$  intersects  $A$  at a point other than  $a$ .

**Lemma 11.6.2.** *The point  $a$  is an accumulation point for  $A$  if and only if  $a \in \overline{A \setminus \{a\}}$ .*

PROOF: From Theorem 11.4.6.  $\square$

**Theorem 11.6.3.** *Let  $X$  be a topological space and  $A \subseteq X$ . Let  $A'$  be the set of all limit points of  $A$ . Then  $\overline{A} = A \cup A'$ .*

PROOF:

$\langle 1 \rangle 1$ . For all  $x \in \overline{A}$ , if  $x \notin A$  then  $x \in A'$

PROOF: From Theorem 11.4.6.

$\langle 1 \rangle 2$ .  $A \subseteq \overline{A}$

PROOF: Lemma 11.4.3.

$\langle 1 \rangle 3$ .  $A' \subseteq \overline{A}$

PROOF: From Theorem 11.4.6.

$\square$

**Corollary 11.6.3.1.** *A set is closed if and only if it contains all its limit points.*

**Proposition 11.6.4.** *In an indiscrete topology, every point is a limit point of any set that has more than one point.*

PROOF: Let  $X$  be an indiscrete space. Let  $A$  be a set with more than one point and  $x$  be a point. The only neighbourhood of  $x$  is  $X$ , which must intersect  $A$  at a point other than  $x$ .  $\square$

**Lemma 11.6.5.** *Let  $X$  be a topological space and  $A \subseteq B \subseteq X$ . Then every limit point of  $A$  is a limit point of  $B$ .*

PROOF: Immediate from definitions.  $\square$

## 11.7 Basis for a Topology

**Definition 11.7.1** (Basis). If  $X$  is a set, a *basis* for a topology on  $X$  is a set  $\mathcal{B} \subseteq \mathcal{P}X$  called *basis elements* such that

1. For all  $x \in X$ , there exists  $B \in \mathcal{B}$  such that  $x \in B$ .
2. For all  $B_1, B_2 \in \mathcal{B}$  and  $x \in B_1 \cap B_2$ , there exists  $B_3 \in \mathcal{B}$  such that  $x \in B_3 \subseteq B_1 \cap B_2$ .

If  $\mathcal{B}$  satisfies these two conditions, then we define the topology *generated* by  $\mathcal{B}$  to be  $\mathcal{T} = \{U \in \mathcal{P}X \mid \forall x \in U. \exists B \in \mathcal{B}. x \in B \subseteq U\}$ .

We prove this is a topology.

PROOF:

$\langle 1 \rangle 1.$   $X \in \mathcal{T}$

PROOF: For all  $x \in X$  there exists  $B \in \mathcal{B}$  such that  $x \in B \subseteq X$  by condition 1.

$\langle 1 \rangle 2.$  For all  $\mathcal{U} \subseteq \mathcal{T}$  we have  $\bigcup \mathcal{U} \in \mathcal{T}$

$\langle 2 \rangle 1.$  LET:  $\mathcal{U} \subseteq \mathcal{T}$

$\langle 2 \rangle 2.$  LET:  $x \in \bigcup \mathcal{U}$

$\langle 2 \rangle 3.$  PICK  $U \in \mathcal{U}$  such that  $x \in U$

$\langle 2 \rangle 4.$  PICK  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$

PROOF: Since  $U \in \mathcal{T}$  by  $\langle 2 \rangle 1$  and  $\langle 2 \rangle 3$ .

$\langle 2 \rangle 5.$   $x \in B \subseteq \bigcup \mathcal{U}$

$\langle 1 \rangle 3.$  For all  $U, V \in \mathcal{T}$  we have  $U \cap V \in \mathcal{T}$

$\langle 2 \rangle 1.$  LET:  $U, V \in \mathcal{T}$

$\langle 2 \rangle 2.$  LET:  $x \in U \cap V$

$\langle 2 \rangle 3.$  PICK  $B_1 \in \mathcal{B}$  such that  $x \in B_1 \subseteq U$

$\langle 2 \rangle 4.$  PICK  $B_2 \in \mathcal{B}$  such that  $x \in B_2 \subseteq V$

$\langle 2 \rangle 5.$  PICK  $B_3 \in \mathcal{B}$  such that  $x \in B_3 \subseteq B_1 \cap B_2$

PROOF: By condition 2.

$\langle 2 \rangle 6.$   $x \in B_3 \subseteq U \cap V$

□

**Lemma 11.7.2.** Let  $X$  be a set. Let  $\mathcal{B}$  be a basis for a topology  $\mathcal{T}$  on  $X$ . Then  $\mathcal{T}$  is the set of all unions of subsets of  $\mathcal{B}$ .

PROOF:

$\langle 1 \rangle 1.$  For all  $U \in \mathcal{T}$ , there exists  $\mathcal{A} \subseteq \mathcal{B}$  such that  $U = \bigcup \mathcal{A}$

$\langle 2 \rangle 1.$  LET:  $U \in \mathcal{T}$

$\langle 2 \rangle 2.$  LET:  $\mathcal{A} = \{B \in \mathcal{B} \mid B \subseteq U\}$

$\langle 2 \rangle 3.$   $U \subseteq \bigcup \mathcal{A}$

$\langle 3 \rangle 1.$  LET:  $x \in U$

$\langle 3 \rangle 2.$  PICK  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$

PROOF: Since  $\mathcal{B}$  is a basis for  $\mathcal{T}$ .

$\langle 3 \rangle 3.$   $x \in B \in \mathcal{A}$

$\langle 2 \rangle 4.$   $\bigcup \mathcal{A} \subseteq U$



PROOF: From the definition of  $\mathcal{A}$  ( $\langle 2 \rangle 2$ ).

$\langle 1 \rangle 2$ . For all  $\mathcal{A} \subseteq \mathcal{B}$  we have  $\bigcup \mathcal{A} \in \mathcal{T}$

$\langle 2 \rangle 1$ .  $\mathcal{B} \subseteq \mathcal{T}$

PROOF: If  $B \in \mathcal{B}$  and  $x \in B$ , then there exists  $B' \in \mathcal{B}$  such that  $x \in B' \subseteq B$ , namely  $B' = B$ .

$\langle 2 \rangle 2$ . Q.E.D.

PROOF: Since  $\mathcal{T}$  is closed under union.

□

**Corollary 11.7.2.1.** *Let  $X$  be a set. Let  $\mathcal{B}$  be a basis for a topology  $\mathcal{T}$  on  $X$ . Then  $\mathcal{T}$  is the coarsest topology that includes  $\mathcal{B}$ .*

PROOF: Since every topology that includes  $\mathcal{B}$  includes all unions of subsets of  $\mathcal{B}$ . □

**Lemma 11.7.3.** *Let  $X$  be a topological space. Suppose that  $\mathcal{C}$  is a set of open sets such that, for every open set  $U$  and every point  $x \in U$ , there exists  $C \in \mathcal{C}$  such that  $x \in C \subseteq U$ . Then  $\mathcal{C}$  is a basis for the topology on  $X$ .*

PROOF:

$\langle 1 \rangle 1$ . For all  $x \in X$ , there exists  $C \in \mathcal{C}$  such that  $x \in C$

PROOF: Immediate from hypothesis.

$\langle 1 \rangle 2$ . For all  $C_1, C_2 \in \mathcal{C}$  and  $x \in C_1 \cap C_2$ , there exists  $C_3 \in \mathcal{C}$  such that  $x \in C_3 \subseteq C_1 \cap C_2$

PROOF: Since  $C_1 \cap C_2$  is open.

$\langle 1 \rangle 3$ . Every open set is open in the topology generated by  $\mathcal{C}$

PROOF: Immediate from hypothesis.

$\langle 1 \rangle 4$ . Every union of a subset of  $\mathcal{C}$  is open.

PROOF: Since every member of  $\mathcal{C}$  is open.

□

**Lemma 11.7.4.** *Let  $\mathcal{B}$  and  $\mathcal{B}'$  be bases for the topologies  $\mathcal{T}$  and  $\mathcal{T}'$  respectively on the set  $X$ . Then the following are equivalent.*

1.  $\mathcal{T} \subseteq \mathcal{T}'$
2. For all  $B \in \mathcal{B}$  and  $x \in B$ , there exists  $B' \in \mathcal{B}'$  such that  $x \in B' \subseteq B$ .

PROOF:

$\langle 1 \rangle 1$ .  $1 \Rightarrow 2$

$\langle 2 \rangle 1$ . ASSUME:  $\mathcal{T} \subseteq \mathcal{T}'$

$\langle 2 \rangle 2$ . LET:  $B \in \mathcal{B}$  and  $x \in B$

$\langle 2 \rangle 3$ .  $B \in \mathcal{T}$

PROOF: Corollary 11.7.2.1.

$\langle 2 \rangle 4$ .  $B \in \mathcal{T}'$

PROOF: By  $\langle 2 \rangle 1$

$\langle 2 \rangle 5$ . There exists  $B' \in \mathcal{B}'$  such that  $x \in B' \subseteq B$

PROOF: Since  $\mathcal{B}'$  is a basis for  $\mathcal{T}'$ .

$\langle 1 \rangle 2$ .  $2 \Rightarrow 1$

- ⟨2⟩1. ASSUME: 2
  - ⟨2⟩2. LET:  $U \in \mathcal{T}$   
PROVE:  $U \in \mathcal{T}'$
  - ⟨2⟩3. LET:  $x \in U$   
PROVE: There exists  $B' \in \mathcal{B}'$  such that  $x \in B' \subseteq U$
  - ⟨2⟩4. PICK  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$   
PROOF: Since  $\mathcal{B}$  is a basis for  $\mathcal{T}$ .
  - ⟨2⟩5. PICK  $B' \in \mathcal{B}'$  such that  $x \in B' \subseteq B$   
PROOF: By ⟨2⟩1.
  - ⟨2⟩6.  $x \in B' \subseteq U$
- 

**Theorem 11.7.5.** *Let  $X$  be a topological space,  $A \subseteq X$  and  $x \in X$ . Let  $\mathcal{B}$  be a basis for  $X$ . Then  $x \in \overline{A}$  if and only if, for all  $B \in \mathcal{B}$ , if  $x \in B$  then  $B$  intersects  $A$ .*

PROOF:

- ⟨1⟩1. If  $x \in \overline{A}$  then, for all  $B \in \mathcal{B}$ , if  $x \in B$  then  $B$  intersects  $A$ .  
PROOF: This follows from Theorem 11.4.6 since every element of  $\mathcal{B}$  is open (Corollary 11.7.2.1).
  - ⟨1⟩2. Suppose that, for all  $B \in \mathcal{B}$ , if  $x \in B$  then  $B$  intersects  $A$ . Then  $x \in \overline{A}$ .  
    - ⟨2⟩1. ASSUME: For all  $B \in \mathcal{B}$ , if  $x \in B$  then  $B$  intersects  $A$ .
    - ⟨2⟩2. LET:  $U$  be an open set that contains  $x$   
PROVE:  $U$  intersects  $A$ .
    - ⟨2⟩3. PICK  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$ .
    - ⟨2⟩4.  $B$  intersects  $A$ .  
PROOF: From ⟨2⟩1.
    - ⟨2⟩5.  $U$  intersects  $A$ .
    - ⟨2⟩6. Q.E.D.
- PROOF: By Theorem 11.4.6.
- 

**Definition 11.7.6** (Lower Limit Topology on the Real Line). The *lower limit topology on the real line* is the topology on  $\mathbb{R}$  generated by the basis consisting of all half-open intervals of the form  $[a, b)$ .

We write  $\mathbb{R}_l$  for the topological space  $\mathbb{R}$  under the lower limit topology.

We prove this is a basis for a topology.

PROOF:

- ⟨1⟩1. For all  $x \in \mathbb{R}$  there exists an interval  $[a, b)$  such that  $x \in [a, b)$ .  
PROOF: Take  $[a, b) = [x, x + 1)$ .
  - ⟨1⟩2. For any open intervals  $[a, b)$ ,  $[c, d)$  if  $x \in [a, b) \cap [c, d)$ , then there exists an interval  $[e, f)$  such that  $x \in [e, f) \subseteq [a, b) \cap [c, d)$   
PROOF: Take  $[e, f) = [\max(a, c), \min(b, d))$ .
- 

**Definition 11.7.7** ( $K$ -topology on the Real Line). Let  $K = \{1/n \mid n \in \mathbb{Z}^+\}$ .

The  $K$ -topology on the real line is the topology on  $\mathbb{R}$  generated by the basis consisting of all open intervals  $(a, b)$  and all sets of the form  $(a, b) \setminus K$ .

We write  $\mathbb{R}_K$  for the topological space  $\mathbb{R}$  under the  $K$ -topology.

We prove this is a basis for a topology.

PROOF:

(1)1. For all  $x \in \mathbb{R}$  there exists an open interval  $(a, b)$  such that  $x \in (a, b)$ .

PROOF: Take  $(a, b) = (x - 1, x + 1)$ .

(1)2. For any basic open sets  $B_1, B_2$  if  $x \in B_1 \cap B_2$ , then there exists a basic open set  $B_3$  such that  $x \in B_3 \subseteq B_1 \cap B_2$

(2)1. CASE:  $B_1 = (a, b), B_2 = (c, d)$

PROOF: Take  $B_3 = (\max(a, c), \min(b, d))$ .

(2)2. CASE:  $B_1 = (a, b)$  or  $(a, b) \setminus K, B_2 = (c, d)$  or  $(c, d) \setminus K$ , and they are not both open intervals.

PROOF: Take  $B_3 = (\max(a, c), \min(b, d)) \setminus K$ .

□

**Lemma 11.7.8.** *The lower limit topology and the  $K$ -topology are incomparable.*

PROOF:

(1)1. The interval  $[10, 11)$  is not open in the  $K$ -topology.

PROOF: There is no open interval  $(a, b)$  such that  $10 \in (a, b) \subseteq [10, 11)$  or  $10 \in (a, b) \setminus K \subseteq [10, 11)$ .

(1)2. The set  $(-1, 1) \setminus K$  is not open in the lower limit topology.

PROOF: There is no half-open interval  $[a, b)$  such that  $0 \in [a, b) \subseteq (-1, 1) \setminus K$ , since there must be a positive integer  $n$  with  $1/n \in [a, b)$ .

□

**Definition 11.7.9** (Subbasis). A *subbasis*  $\mathcal{S}$  for a topology on  $X$  is a set  $\mathcal{S} \subseteq \mathcal{P}X$  such that  $\bigcup \mathcal{S} = X$ .

The topology *generated* by the subbasis  $\mathcal{S}$  is the set of all unions of finite intersections of elements of  $\mathcal{S}$ .

We prove this is a topology.

PROOF:

(1)1. The set  $\mathcal{B}$  of all finite intersections of elements of  $\mathcal{S}$  forms a basis for a topology on  $X$ .

(2)1.  $\bigcup \mathcal{B} = X$

PROOF: Since  $\mathcal{S} \subseteq \mathcal{B}$ .

(2)2.  $\mathcal{B}$  is closed under binary intersection.

PROOF: By definition.

(1)2. Q.E.D.

PROOF: By Lemma 11.7.2.

□

We have simultaneously proved:

**Proposition 11.7.10.** *Let  $\mathcal{S}$  be a subbasis for the topology on  $X$ . Then the set of all finite intersections of elements of  $\mathcal{S}$  is a basis for the topology on  $X$ .*

**Proposition 11.7.11.** *Let  $X$  be a set. Let  $\mathcal{S}$  be a subbasis for a topology  $\mathcal{T}$  on  $X$ . Then  $\mathcal{T}$  is the coarsest topology that includes  $\mathcal{S}$ .*

PROOF: Since every topology that includes  $\mathcal{S}$  includes every union of finite intersections of elements of  $\mathcal{S}$ .  $\square$

## 11.8 Local Basis at a Point

**Definition 11.8.1** (Local Basis). Let  $X$  be a topological space and  $a \in X$ . A (local) basis at  $a$  is a set  $\mathcal{B}$  of neighbourhoods of  $a$  such that every neighbourhood of  $a$  includes some member of  $\mathcal{B}$ .

**Lemma 11.8.2.** *If there exists a countable local basis at a point  $a$ , then there exists a countable local basis  $\{B_n \mid n \geq 1\}$  such that  $B_1 \supseteq B_2 \supseteq \dots$ .*

PROOF: Given a countable local basis  $\{C_n \mid n \geq 1\}$ , take  $B_n = C_1 \cap \dots \cap C_n$ .  $\square$

## 11.9 Nets

**Definition 11.9.1** (Net). Let  $X$  be a topological space. A net in  $X$  consists of a directed poset  $J$  and a family  $(x_\alpha)_{\alpha \in J}$  of points of  $X$  indexed by  $J$ .

**Definition 11.9.2** (Convergence). Let  $X$  be a topological space. Let  $(x_\alpha)_{\alpha \in J}$  be a net in  $X$  and  $l \in X$ . Then  $(x_\alpha)$  converges to the limit  $l$  iff, for every limit  $U$  of  $l$ , there exists  $\alpha \in J$  such that, for all  $\beta \geq \alpha$ , we have  $x_\beta \in U$ .

**Lemma 11.9.3.** *Let  $X$  be a topological space. Let  $A \subseteq X$  and  $l \in X$ . Then  $l \in \bar{A}$  if and only if there exists a net of points in  $A$  that converges to  $l$ .*

PROOF:

$\langle 1 \rangle 1$ . If  $l \in \bar{A}$  then there exists a net of points in  $A$  that converges to  $l$ .

$\langle 2 \rangle 1$ . ASSUME:  $l \in \bar{A}$

$\langle 2 \rangle 2$ . LET:  $J$  be the set of neighbourhoods of  $l$  under  $\supseteq$

$\langle 2 \rangle 3$ . For  $U \in J$ , PICK  $a_U \in U \cap A$

PROVE:  $a_U \rightarrow l$  as  $U \rightarrow \infty$

PROOF: Theorem 11.4.6.

$\langle 2 \rangle 4$ . LET:  $U$  be a neighbourhood of  $l$ .

$\langle 2 \rangle 5$ . For any  $V \subseteq U$  we have  $a_V \in V$ .

$\langle 1 \rangle 2$ . If there exists a net of points in  $A$  that converges to  $l$ , then  $l \in \bar{A}$ .

$\langle 2 \rangle 1$ . LET:  $(a_\alpha)_{\alpha \in J}$  be a sequence of points in  $A$  that converges to  $l$ .

$\langle 2 \rangle 2$ . LET:  $U$  be a neighbourhood of  $l$ .

$\langle 2 \rangle 3$ . PICK  $\alpha \in J$  such that, for all  $\beta \geq \alpha$ , we have  $a_\beta \in U$ .

$\langle 2 \rangle 4$ .  $a_\alpha \in U \cap A$

$\langle 2 \rangle 5$ . Q.E.D.

PROOF: Theorem 11.4.6.

$\square$

**Proposition 11.9.4.** *Let  $X$  be a topological space. Let  $\mathcal{B}$  be a basis for the topology on  $X$ . Let  $(a_n)$  be a sequence in  $X$  and  $l \in X$ . Then  $a_n \rightarrow l$  as  $n \rightarrow \infty$  if and only if, for every  $B \in \mathcal{B}$  with  $l \in B$ , there exists  $N$  such that, for all  $n \geq N$ , we have  $a_n \in B$ .*

PROOF:

$\langle 1 \rangle 1$ . If  $a_n \rightarrow l$  as  $n \rightarrow \infty$  then, for every  $B \in \mathcal{B}$  with  $l \in B$ , there exists  $N$  such that, for all  $n \geq N$ , we have  $a_n \in B$ .

PROOF: Since every element of  $\mathcal{B}$  is open (Corollary 11.7.2.1).

$\langle 1 \rangle 2$ . If, for every  $B \in \mathcal{B}$  with  $l \in B$ , there exists  $N$  such that, for all  $n \geq N$ , we have  $a_n \in B$ , then  $a_n \rightarrow l$  as  $n \rightarrow \infty$ .

$\langle 2 \rangle 1$ . ASSUME: for every  $B \in \mathcal{B}$  with  $l \in B$ , there exists  $N$  such that, for all  $n \geq N$ , we have  $a_n \in B$

$\langle 2 \rangle 2$ . LET:  $U$  be a neighbourhood of  $l$ .

$\langle 2 \rangle 3$ . PICK  $B \in \mathcal{B}$  such that  $l \in B \subseteq U$

$\langle 2 \rangle 4$ . PICK  $N$  such that, for all  $n \geq N$ , we have  $a_n \in B$

PROOF: From  $\langle 2 \rangle 1$ .

$\langle 2 \rangle 5$ . For all  $n \geq N$  we have  $a_n \in U$

□

**Lemma 11.9.5.** *If a sequence  $(a_n)$  is constant with  $a_n = l$  for all  $n$ , then  $a_n \rightarrow l$  as  $n \rightarrow \infty$ .*

PROOF: Immediate from definitions. □

**Theorem 11.9.6.** *Let  $X$  be a linearly ordered set. Let  $(s_n)$  be a monotone increasing sequence in  $X$  with a supremum  $s$ . Then  $s_n \rightarrow s$  as  $n \rightarrow \infty$ .*

PROOF:

$\langle 1 \rangle 1$ . ASSUME:  $s$  is not least in  $X$ .

PROOF: Otherwise  $(s_n)$  is the constant sequence  $s$  and the result follows from Lemma 11.9.5.

$\langle 1 \rangle 2$ . LET:  $U$  be a neighbourhood of  $s$ .

$\langle 1 \rangle 3$ . PICK  $a < s$  such that  $(a, s] \subseteq U$

$\langle 1 \rangle 4$ . PICK  $N$  such that  $a < a_N$ .

$\langle 1 \rangle 5$ . For all  $n \geq N$  we have  $a_n \in (a, s]$

$\langle 1 \rangle 6$ . For all  $n \geq N$  we have  $a_n \in U$ .

□

**Theorem 11.9.7.** *If  $\sum_{i=0}^{\infty} a_i = s$  and  $\sum_{i=0}^{\infty} b_i = t$  then  $\sum_{i=0}^{\infty} (ca_i + b_i) = cs + t$ .*

PROOF:  $\sum_{i=0}^N (ca_i + b_i) = c \sum_{i=0}^N a_i + \sum_{i=0}^N b_i \rightarrow cs + t$  as  $n \rightarrow \infty$ . □

**Theorem 11.9.8** (Comparison Test). *If  $|a_i| \leq b_i$  for all  $i$  and  $\sum_{i=0}^{\infty} b_i$  converges then  $\sum_{i=0}^{\infty} a_i$  converges.*

PROOF:

$\langle 1 \rangle 1$ .  $\sum_{i=0}^{\infty} |a_i|$  converges

PROOF: The partial sums  $\sum_{i=0}^N |a_i|$  form an increasing sequence bounded above by  $\sum_{i=0}^{\infty} b_i$ .  
 $\langle 1 \rangle 2$ . LET:  $c_i = |a_i| + a_i$  for all  $i$   
 $\langle 1 \rangle 3$ .  $\sum_{i=0}^{\infty} c_i$  converges  
 PROOF: Each  $c_i$  is either  $2|a_i|$  or 0. So the partial sums  $\sum_{i=0}^N c_i$  form an increasing sequence bounded above by  $2 \sum_{i=0}^{\infty} b_i$ .  
 $\langle 1 \rangle 4$ . Q.E.D.  
 PROOF: Since  $a_i = c_i - |a_i|$ .  
 $\square$

**Corollary 11.9.8.1.** *If  $\sum_{i=0}^{\infty} |a_i|$  converges then  $\sum_{i=0}^{\infty} a_i$  converges.*

**Theorem 11.9.9** (Weierstrass M-test). *Let  $X$  be a set and  $(f_n : X \rightarrow \mathbb{R})$  be a sequence of functions. Let*

$$s_n(x) = \sum_{i=0}^n f_i(x)$$

*for all  $n, x$ . Suppose  $|f_i(x)| \leq M_i$  for all  $i \geq 0$  and  $x \in X$ . If the series  $\sum_{i=0}^{\infty} M_i$  converges, then the sequence  $(s_n)$  converges uniformly to*

$$s(x) = \sum_{i=0}^{\infty} f_i(x) .$$

PROOF:

$\langle 1 \rangle 1$ . LET:  $r_n = \sum_{i=n+1}^{\infty} M_i$  for all  $n$   
 $\langle 1 \rangle 2$ . Given  $0 \leq n < k$ , we have  $|s_k(x) - s_n(x)| \leq r_n$   
 PROOF:

$$\begin{aligned} |s_k(x) - s_n(x)| &= \left| \sum_{i=n+1}^k f_i(x) \right| \\ &\leq \sum_{i=n+1}^k |f_i(x)| \\ &\leq \sum_{i=n+1}^k M_i \\ &\leq r_n \end{aligned}$$

$\langle 1 \rangle 3$ . Given  $n \geq 0$  we have  $|s(x) - s_n(x)| \leq r_n$

PROOF: By taking the limit  $k \rightarrow \infty$  in  $\langle 1 \rangle 2$ .

$\langle 1 \rangle 4$ . Q.E.D.

PROOF: Since  $r_n \rightarrow 0$  as  $n \rightarrow \infty$ .

$\square$

## 11.10 Locally Finite Sets

**Definition 11.10.1** (Locally Finite). Let  $X$  be a topological space and  $\{A_\alpha\}$  a family of subsets of  $X$ . Then  $\mathcal{A}$  is *locally finite* if and only if every point in  $X$

has a neighbourhood that intersects  $A_\alpha$  for only finitely many  $\alpha$ .

The following example shows that we cannot remove the assumption of local finiteness.

**Example 11.10.2.** Define  $f : [-1, 1] \rightarrow \mathbb{R}$  by:  $f(x) = 1$  if  $x < -1$ ,  $f(x) = 0$  if  $x > 1$ . Let  $C_n = [-1, -1/n]$  for  $n \geq 1$ , and  $D = [0, 1]$ . Then  $[-1, 1] = \bigcup_{n=1}^{\infty} C_n \cup D$  and  $f$  is continuous on each  $C_n$  and each  $D$ , but  $f$  is not continuous on  $[-1, 1]$ .

## 11.11 Open Maps

**Definition 11.11.1** (Open Map). Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Then  $f$  is an *open map* if and only if, for every open set  $U$  in  $X$ , the set  $f(U)$  is open in  $Y$ .

**Lemma 11.11.2.** Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Let  $\mathcal{B}$  be a basis for the topology on  $X$ . If  $f(B)$  is open in  $Y$  for all  $B \in \mathcal{B}$ , then  $f$  is an open map.

PROOF: From Lemma 11.7.2.  $\square$

**Proposition 11.11.3.** Let  $X$  and  $Y$  be topological spaces. Let  $\mathcal{B}$  be a basis for the topology on  $X$ . Let  $f : X \rightarrow Y$ . Suppose that, for all  $B \in \mathcal{B}$ , we have  $f(B)$  is open in  $Y$ . Then  $f$  is an open map.

PROOF: For any  $\mathcal{A} \subseteq \mathcal{B}$ , we have  $f(\bigcup \mathcal{A}) = \bigcup_{B \in \mathcal{A}} f(B)$  is open in  $Y$ . The result follows from Lemma 11.7.2.  $\square$

## 11.12 Continuous Functions

**Definition 11.12.1** (Continuous). Let  $X$  and  $Y$  be topological spaces. A function  $f : X \rightarrow Y$  is *continuous* if and only if, for every open set  $V$  in  $Y$ , the set  $f^{-1}(V)$  is open in  $X$ .

**Proposition 11.12.2.** Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Let  $\mathcal{B}$  be a basis for  $Y$ . Then  $f$  is continuous if and only if, for all  $B \in \mathcal{B}$ , we have  $f^{-1}(B)$  is open in  $X$ .

PROOF:

$\langle 1 \rangle 1$ . If  $f$  is continuous then, for all  $B \in \mathcal{B}$ , we have  $f^{-1}(B)$  is open in  $X$ .

PROOF: Since every element of  $\mathcal{B}$  is open (Lemma 11.7.2).

$\langle 1 \rangle 2$ . Suppose that, for all  $B \in \mathcal{B}$ , we have  $f^{-1}(B)$  is open in  $X$ . Then  $f$  is continuous.

$\langle 2 \rangle 1$ . ASSUME: For all  $B \in \mathcal{B}$ , we have  $f^{-1}(B)$  is open in  $X$ .

$\langle 2 \rangle 2$ . LET:  $V$  be open in  $Y$ .

$\langle 2 \rangle 3$ . PICK  $\mathcal{A} \subseteq \mathcal{B}$  such that  $V = \bigcup \mathcal{A}$

PROOF: By Lemma 11.7.2.

(2)4.  $f^{-1}(V)$  is open in  $X$ .

PROOF:

$$\begin{aligned} f^{-1}(V) &= f^{-1}\left(\bigcup \mathcal{A}\right) \\ &= \bigcup_{B \in \mathcal{A}} f^{-1}(B) \end{aligned}$$

□

**Proposition 11.12.3.** *Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Let  $\mathcal{S}$  be a subbasis for  $Y$ . Then  $f$  is continuous if and only if, for all  $S \in \mathcal{S}$ , we have  $f^{-1}(S)$  is open in  $X$ .*

PROOF:

(1)1. If  $f$  is continuous then, for all  $S \in \mathcal{S}$ , we have  $f^{-1}(S)$  is open in  $X$ .

PROOF: Since every element of  $\mathcal{S}$  is open.

(1)2. Suppose that, for all  $S \in \mathcal{S}$ , we have  $f^{-1}(S)$  is open in  $X$ . Then  $f$  is continuous.

(2)1. ASSUME: For all  $S \in \mathcal{S}$ , we have  $f^{-1}(S)$  is open in  $X$ .

(2)2. LET:  $S_1, \dots, S_n \in \mathcal{S}$

(2)3.  $f^{-1}(S_1 \cap \dots \cap S_n)$  is open in  $X$

PROOF: Since  $f^{-1}(S_1 \cap \dots \cap S_n) = f^{-1}(S_1) \cap \dots \cap f^{-1}(S_n)$ .

(2)4. Q.E.D.

PROOF: By Propositions 11.12.2 and 11.7.10.

□

**Proposition 11.12.4.** *Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Let  $\mathcal{S}$  be a basis for  $Y$ . Then  $f$  is continuous if and only if, for all  $V \in \mathcal{S}$ , we have  $f^{-1}(V)$  is open in  $X$ .*

PROOF:

(1)1. If  $f$  is continuous then, for all  $V \in \mathcal{S}$ , we have  $f^{-1}(V)$  is open in  $X$ .

PROOF: Since every element of  $\mathcal{S}$  is open.

(1)2. Suppose that, for all  $V \in \mathcal{S}$ , we have  $f^{-1}(V)$  is open in  $X$ . Then  $f$  is continuous.

(2)1. ASSUME: For all  $V \in \mathcal{S}$ , we have  $f^{-1}(V)$  is open in  $X$ .

(2)2. For every set  $B$  that is the finite intersection of elements of  $\mathcal{S}$ , we have  $f^{-1}(B)$  is open in  $X$ .

PROOF: Because  $f^{-1}(V_1 \cap \dots \cap V_n) = f^{-1}(V_1) \cap \dots \cap f^{-1}(V_n)$ .

(2)3. Q.E.D.

PROOF: From Propositions 11.7.10 and 11.12.2.

□

**Definition 11.12.5** (Continuous at a Point). Let  $X$  and  $Y$  be topological spaces. Let  $f : X \rightarrow Y$  and  $x \in X$ . Then  $f$  is *continuous at  $x$*  if and only if, for every neighbourhood  $V$  of  $f(x)$ , there exists a neighbourhood  $U$  of  $x$  such that  $f(U) \subseteq V$ .



**Theorem 11.12.6.** *Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Then the following are equivalent:*

1.  $f$  is continuous.
2. For all  $A \subseteq X$ , we have  $f(\overline{A}) \subseteq \overline{f(A)}$
3. For all  $B \subseteq Y$  closed, we have  $f^{-1}(B)$  is closed in  $X$ .
4.  $f$  is continuous at every point of  $X$ .

PROOF:

$\langle 1 \rangle 1. 1 \Rightarrow 2$

$\langle 2 \rangle 1.$  ASSUME:  $f$  is continuous.

$\langle 2 \rangle 2.$  LET:  $A \subseteq X$

$\langle 2 \rangle 3.$  LET:  $x \in \overline{A}$

PROVE:  $f(x) \in \overline{f(A)}$

$\langle 2 \rangle 4.$  LET:  $V$  be a neighbourhood of  $f(x)$

$\langle 2 \rangle 5.$   $f^{-1}(V)$  is a neighbourhood of  $x$

$\langle 2 \rangle 6.$  PICK  $y \in A \cap f^{-1}(V)$

PROOF: By Theorem 11.4.6.

$\langle 2 \rangle 7.$   $f(y) \in V \cap f(A)$

$\langle 2 \rangle 8.$  Q.E.D.

PROOF: By Theorem 11.4.6.

$\langle 1 \rangle 2. 2 \Rightarrow 3$

$\langle 2 \rangle 1.$  ASSUME: 2

$\langle 2 \rangle 2.$  LET:  $B$  be closed in  $Y$

$\langle 2 \rangle 3.$  LET:  $x \in \overline{f^{-1}(B)}$

PROVE:  $x \in f^{-1}(B)$

$\langle 2 \rangle 4.$   $f(x) \in B$

PROOF:

$$f(x) \in \overline{f(f^{-1}(B))}$$

$$\subseteq \overline{f(f^{-1}(B))}$$

$(\langle 2 \rangle 1)$

$$\subseteq \overline{B}$$

$(\text{Proposition 11.4.7})$

$$= B$$

$\langle 1 \rangle 3. 3 \Rightarrow 1$

$\langle 2 \rangle 1.$  ASSUME: 3

$\langle 2 \rangle 2.$  LET:  $V$  be open in  $Y$

$\langle 2 \rangle 3.$   $Y \setminus V$  is closed in  $Y$

$\langle 2 \rangle 4.$   $f^{-1}(Y \setminus V)$  is closed in  $X$

$\langle 2 \rangle 5.$   $X \setminus f^{-1}(V)$  is closed in  $X$

$\langle 2 \rangle 6.$   $f^{-1}(V)$  is open in  $X$

$\langle 1 \rangle 4. 1 \Rightarrow 4$

PROOF: For any neighbourhood  $V$  of  $f(x)$ , the set  $U = f^{-1}(V)$  is a neighbourhood of  $x$  such that  $f(U) \subseteq V$ .

$\langle 1 \rangle 5. 4 \Rightarrow 1$

- ⟨2⟩1. ASSUME: 4
- ⟨2⟩2. LET:  $V$  be open in  $Y$
- ⟨2⟩3. LET:  $x \in f^{-1}(V)$
- ⟨2⟩4.  $V$  is a neighbourhood of  $f(x)$
- ⟨2⟩5. PICK a neighbourhood  $U$  of  $x$  such that  $f(U) \subseteq V$
- ⟨2⟩6.  $x \in U \subseteq f^{-1}(V)$
- ⟨2⟩7. Q.E.D.

PROOF: By Lemma 11.1.8.

□

**Theorem 11.12.7.** *A constant function is continuous.*

PROOF: Let  $X$  and  $Y$  be topological spaces. Let  $b \in Y$ , and let  $f : X \rightarrow Y$  be the constant function with value  $b$ . For any open  $V \subseteq Y$ , the set  $f^{-1}(V)$  is either  $X$  (if  $b \in V$ ) or  $\emptyset$  (if  $b \notin V$ ). □

**Theorem 11.12.8.** *The composite of two continuous functions is continuous.*

PROOF: Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  be continuous. For any  $V$  open in  $Z$ , we have  $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$  is open in  $X$ . □

**Theorem 11.12.9.** *Let  $f : X \rightarrow Y$  be a continuous function and  $A$  be a subspace of  $X$ . Then the restriction  $f \upharpoonright A : A \rightarrow Y$  is continuous.*

PROOF: Let  $V$  be open in  $Y$ . Then  $(f \upharpoonright A)^{-1}(V) = f^{-1}(V) \cap A$  is open in  $A$ . □

**Theorem 11.12.10.** *Let  $f : X \rightarrow Y$  be continuous. Let  $Z$  be a subspace of  $Y$  such that  $f(X) \subseteq Z$ . Then the corestriction  $f : X \rightarrow Z$  is continuous.*

PROOF:

- ⟨1⟩1. LET:  $V$  be open in  $Z$ .
- ⟨1⟩2. PICK  $U$  open in  $Y$  such that  $V = U \cap Z$ .
- ⟨1⟩3.  $f^{-1}(V) = f^{-1}(U)$
- ⟨1⟩4.  $f^{-1}(V)$  is open in  $X$ .

□

**Theorem 11.12.11.** *Let  $f : X \rightarrow Y$  be continuous. Let  $Z$  be a space such that  $Y$  is a subspace of  $Z$ . Then the expansion  $f : X \rightarrow Z$  is continuous.*

PROOF: Let  $V$  be open in  $Z$ . Then  $f^{-1}(V) = f^{-1}(V \cap Y)$  is open in  $X$ . □

**Theorem 11.12.12.** *Let  $X$  and  $Y$  be topological spaces. Let  $f : X \rightarrow Y$ . Suppose  $\mathcal{U}$  is a set of open sets in  $X$  such that  $X = \bigcup \mathcal{U}$  and, for all  $U \in \mathcal{U}$ , we have  $f \upharpoonright U : U \rightarrow Y$  is continuous. Then  $f$  is continuous.*

PROOF:

- ⟨1⟩1. LET:  $V$  be open in  $Y$
- ⟨1⟩2.  $f^{-1}(V) = \bigcup_{U \in \mathcal{U}} (f \upharpoonright U)^{-1}(V)$
- ⟨1⟩3. For all  $U \in \mathcal{U}$ , we have  $(f \upharpoonright U)^{-1}(V)$  is open in  $U$ .
- ⟨1⟩4. For all  $U \in \mathcal{U}$ , we have  $(f \upharpoonright U)^{-1}(V)$  is open in  $X$ .

PROOF: Lemma 11.18.6.

□

**Proposition 11.12.13.** *Let  $X$  and  $X'$  be the same set  $X$  under two topologies  $\mathcal{T}$  and  $\mathcal{T}'$ . Let  $i : X \rightarrow X'$  be the identity function. Then  $i$  is continuous if and only if  $\mathcal{T}' \subseteq \mathcal{T}$ .*

PROOF: Immediate from definitions. □

**Proposition 11.12.14.** *Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  and  $a \in \mathbb{R}$ . Then  $f$  is continuous on the right at  $a$  if and only if  $f$  is continuous at  $a$  as a function  $\mathbb{R}_l \rightarrow \mathbb{R}$ .*

PROOF:

⟨1⟩1. If  $f$  is continuous on the right at  $a$  then  $f$  is continuous at  $a$  as a function  $\mathbb{R}_l \rightarrow \mathbb{R}$ .

⟨2⟩1. ASSUME:  $f$  is continuous on the right at  $a$ .

⟨2⟩2. LET:  $V$  be a neighbourhood of  $f(a)$

⟨2⟩3. PICK  $b, c$  such that  $f(a) \in (b, c) \subseteq V$ .

⟨2⟩4. LET:  $\epsilon = \min(c - f(a), f(a) - b)$

⟨2⟩5. PICK  $\delta > 0$  such that, for all  $x$ , if  $a < x < a + \delta$  then  $|f(x) - f(a)| < \epsilon$

⟨2⟩6. LET:  $U = [a, a + \delta)$

⟨2⟩7.  $f(U) \subseteq V$

⟨1⟩2. If  $f$  is continuous at  $a$  as a function  $\mathbb{R}_l \rightarrow \mathbb{R}$  then  $f$  is continuous on the right at  $a$ .

⟨2⟩1. ASSUME:  $f$  is continuous at  $a$  as a function  $\mathbb{R}_l \rightarrow \mathbb{R}$

⟨2⟩2. LET:  $\epsilon > 0$

⟨2⟩3. PICK a neighbourhood  $U$  of  $a$  such that  $f(U) \subseteq (f(a) - \epsilon, f(a) + \epsilon)$

⟨2⟩4. PICK  $b, c$  such that  $a \in [b, c) \subset U$

⟨2⟩5. LET:  $\delta = c - a$

⟨2⟩6. For all  $x$ , if  $a < x < a + \delta$  then  $|f(x) - f(a)| < \epsilon$

□

**Lemma 11.12.15.** *Let  $f : X \rightarrow Y$ . Let  $Z$  be an open subspace of  $X$  and  $a \in Z$ . If  $f \upharpoonright Z$  is continuous at  $a$  then  $f$  is continuous at  $a$ .*

PROOF:

⟨1⟩1. LET:  $V$  be a neighbourhood of  $f(x)$

⟨1⟩2. PICK a neighbourhood  $W$  of  $x$  in  $Z$  such that  $f(W) \subseteq V$

⟨1⟩3.  $W$  is a neighbourhood of  $x$  in  $X$  such that  $f(W) \subseteq V$

PROOF: Lemma 11.18.6.

□

**Proposition 11.12.16.** *Let  $f : A \rightarrow B$  and  $g : C \rightarrow D$  be continuous. Define  $f \times g : A \times C \rightarrow B \times D$  by*

$$(f \times g)(a, c) = (f(a), g(c)) .$$

*Then  $f \times g$  is continuous.*

PROOF:  $\pi_1 \circ (f \times g) = f \circ \pi_1$  and  $\pi_2 \circ (f \times g) = g \circ \pi_2$  are continuous by Theorem 11.12.8. The result follows by Theorem 11.17.12.

**Proposition 11.12.17.** *Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Then  $f$  is continuous if and only if, for any net  $(a_\alpha)_{\alpha \in J}$  in  $X$  and  $l \in X$ , if  $a_\alpha \rightarrow l$  as  $\alpha \rightarrow \infty$  in  $X$  then  $f(a_\alpha) \rightarrow f(l)$  as  $\alpha \rightarrow \infty$ .*

PROOF:

- (1)1. If  $f$  is continuous then, for every net  $(a_\alpha)_{\alpha \in J}$  in  $X$  and  $l \in X$ , if  $a_\alpha \rightarrow l$  as  $\alpha \rightarrow \infty$  then  $f(a_\alpha) \rightarrow f(l)$  as  $\alpha \rightarrow \infty$
- (2)1. ASSUME:  $f$  is continuous.
- (2)2. LET:  $(a_\alpha)_{\alpha \in J}$  be a net in  $X$
- (2)3. LET:  $l \in X$
- (2)4. ASSUME:  $a_\alpha \rightarrow l$  as  $\alpha \rightarrow \infty$
- (2)5. LET:  $V$  be a neighbourhood of  $f(l)$
- (2)6. PICK a neighbourhood  $U$  of  $l$  such that  $f(U) \subseteq V$
- (2)7. PICK  $\alpha \in J$  such that, for all  $\beta \geq \alpha$ , we have  $a_\beta \in U$
- (2)8. For all  $\beta \geq \alpha$  we have  $f(a_\beta) \in V$
- (1)2. If, for every net  $(a_\alpha)_{\alpha \in J}$  in  $X$  and  $l \in X$ , if  $a_\alpha \rightarrow l$  as  $\alpha \rightarrow \infty$  then  $f(a_\alpha) \rightarrow f(l)$  as  $\alpha \rightarrow \infty$ , then  $f$  is continuous.
- (2)1. ASSUME: for every net  $(a_\alpha)_{\alpha \in J}$  in  $X$  and  $l \in X$ , if  $a_\alpha \rightarrow l$  as  $\alpha \rightarrow \infty$  then  $f(a_\alpha) \rightarrow f(l)$  as  $\alpha \rightarrow \infty$
- PROVE: For every  $A \subseteq X$ , we have  $f(\overline{A}) \subseteq \overline{f(A)}$
- (2)2. LET:  $A \subseteq X$
- (2)3. LET:  $x \in \overline{A}$
- PROVE:  $f(x) \in \overline{f(A)}$
- (2)4. PICK a net  $(a_\alpha)_{\alpha \in J}$  of points in  $A$  that converges to  $x$
- PROOF: Lemma 11.9.3.
- (2)5.  $(f(a_\alpha))_{\alpha \in J}$  is a net of points in  $f(A)$  that converges to  $f(x)$
- PROOF: From (2)1.
- (2)6.  $f(x) \in \overline{f(A)}$
- PROOF: Lemma 11.9.3.
- (2)7. Q.E.D.
- PROOF: Theorem 11.12.6.

□

## 11.13 Closed Maps

**Definition 11.13.1** (Closed Map). Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Then  $f$  is a *closed map* if and only if, for every closed set  $A$  in  $X$ , we have  $f(A)$  is closed in  $Y$ .

**Lemma 11.13.2.** *Let  $X$  and  $Y$  be topological spaces. Let  $f : X \rightarrow Y$  be a closed map. Let  $B \subseteq Y$ . Let  $U$  be an open neighbourhood of  $p^{-1}(B)$ . Then there exists an open neighbourhood  $V$  of  $B$  such that  $p^{-1}(V) \subseteq U$ .*

PROOF:

(1)1. LET:  $V = Y - p(X - U)$   
 (1)2.  $V$  is open  
 PROOF: Since  $p(X - U)$  is closed.  
 (1)3.  $B \subseteq V$   
 PROOF: If  $p(x) = b \in B$  then  $x \in U$ .  
 (1)4.  $p^{-1}(V) \subseteq U$   
 PROOF: If  $p(x) \in V$  then  $x \in U$ .  
 $\square$

## 11.14 Homeomorphisms

**Definition 11.14.1** (Homeomorphism). Let  $X$  and  $Y$  be topological spaces. A *Homeomorphism*  $f$  between  $X$  and  $Y$ ,  $f : X \cong Y$ , is a bijection  $f : X \rightarrow Y$  such that both  $f$  and  $f^{-1}$  are continuous.

**Lemma 11.14.2.** *Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$  a bijection. Then the following are equivalent:*

1.  $f$  is a homeomorphism.
2.  $f$  is continuous and an open map.
3.  $f$  is continuous and a closed map.
4. For any  $U \subseteq X$ , we have  $U$  is open if and only if  $f(U)$  is open.

PROOF: Immediate from definitions.  $\square$

**Proposition 11.14.3.** *Let  $X$  and  $X'$  be the same set  $X$  under two topologies  $\mathcal{T}$  and  $\mathcal{T}'$ . Let  $i : X \rightarrow X'$  be the identity function. Then  $i$  is a homeomorphism if and only if  $\mathcal{T} = \mathcal{T}'$ .*

PROOF: Immediate from definitions.  $\square$

**Definition 11.14.4** (Topological Property). Let  $P$  be a property of topological spaces. Then  $P$  is a *topological* property if and only if, for any spaces  $X$  and  $Y$ , if  $P$  holds of  $X$  and  $X \cong Y$  then  $P$  holds of  $Y$ .

**Definition 11.14.5** (Topological Imbedding). Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Then  $f$  is a *topological imbedding* if and only if the corestriction  $f : X \rightarrow f(X)$  is a homeomorphism.

**Proposition 11.14.6.** *Let  $X$  and  $Y$  be topological spaces and  $a \in X$ . The function  $i : Y \rightarrow X \times Y$  that maps  $y$  to  $(a, y)$  is an imbedding.*

PROOF:

- (1)1.  $i$  is injective  
 (1)2.  $i$  is continuous.

PROOF: For  $U$  open in  $X$  and  $V$  open in  $Y$ , we have  $i^{-1}(U \times V)$  is  $V$  if  $a \in U$ , and  $\emptyset$  if  $a \notin U$ .

$\langle 1 \rangle 3$ .  $i : Y \rightarrow i(Y)$  is an open map.

PROOF: For  $V$  open in  $Y$  we have  $i(V) = (X \times V) \cap i(Y)$ .

□

## 11.15 The Order Topology

**Definition 11.15.1** (Order Topology). Let  $X$  be a linearly ordered set with at least two points. The *order topology* on  $X$  is the topology generated by the basis  $\mathcal{B}$  consisting of:

- all open intervals  $(a, b)$ ;
- all intervals of the form  $[\perp, b)$  where  $\perp$  is least in  $X$ ;
- all intervals of the form  $(a, \top]$  where  $\top$  is greatest in  $X$ .

We prove this is a basis for a topology.

PROOF:

$\langle 1 \rangle 1$ . For all  $x \in X$  there exists  $B \in \mathcal{B}$  such that  $x \in B$ .

$\langle 2 \rangle 1$ . LET:  $x \in X$

$\langle 2 \rangle 2$ . CASE:  $x$  is greatest in  $X$ .

$\langle 3 \rangle 1$ . PICK  $y \in X$  with  $y \neq x$

$\langle 3 \rangle 2$ .  $x \in (y, x] \in \mathcal{B}$

$\langle 2 \rangle 3$ . CASE:  $x$  is least in  $X$ .

$\langle 3 \rangle 1$ . PICK  $y \in X$  with  $y \neq x$

$\langle 3 \rangle 2$ .  $x \in [x, y) \in \mathcal{B}$

$\langle 2 \rangle 4$ . CASE:  $x$  is neither greatest nor least in  $X$ .

$\langle 3 \rangle 1$ . PICK  $a, b \in X$  with  $a < x$  and  $x < b$

$\langle 3 \rangle 2$ .  $x \in (a, b) \in \mathcal{B}$

$\langle 1 \rangle 2$ . For all  $B_1, B_2 \in \mathcal{B}$  and  $x \in B_1 \cap B_2$ , there exists  $B_3 \in \mathcal{B}$  such that  $x \in B_3 \subseteq B_1 \cap B_2$

$\langle 2 \rangle 1$ . LET:  $B_1, B_2 \in \mathcal{B}$  and  $x \in B_1 \cap B_2$

$\langle 2 \rangle 2$ . CASE:  $B_1 = (a, b)$ ,  $B_2 = (c, d)$

PROOF: Take  $B_3 = (\max(a, c), \min(b, d))$ .

$\langle 2 \rangle 3$ . CASE:  $B_1 = (a, b)$ ,  $B_2 = [\perp, d)$

PROOF: Take  $B_3 = (a, \min(b, d))$ .

$\langle 2 \rangle 4$ . CASE:  $B_1 = (a, b)$ ,  $B_2 = (c, \top]$

PROOF: Take  $B_3 = (\max(a, c), b)$ .

$\langle 2 \rangle 5$ . CASE:  $B_1 = [\perp, b)$ ,  $B_2 = [\perp, d)$

PROOF: Take  $B_3 = [\perp, \min(b, d))$ .

$\langle 2 \rangle 6$ . CASE:  $B_1 = [\perp, b)$ ,  $B_2 = (c, \top]$

PROOF: Take  $B_3 = (c, b)$ .

□

**Lemma 11.15.2.** Let  $X$  be a linearly ordered set. Then the open rays form a subbasis for the order topology on  $X$ .

PROOF:

⟨1⟩1. Every open ray is open.

⟨2⟩1. For all  $a \in X$ , the ray  $(-\infty, a)$  is open.

⟨3⟩1. LET:  $x \in (-\infty, a)$

⟨3⟩2. CASE:  $x$  is least in  $X$

PROOF:  $x \text{ in } [x, a) = (-\infty, a)$ .

⟨3⟩3. CASE:  $x$  is not least in  $X$

⟨4⟩1. PICK  $y < x$

⟨4⟩2.  $x \in (y, a) \subseteq (-\infty, a)$

⟨2⟩2. For all  $a \in X$ , the ray  $(a, +\infty)$  is open.

PROOF: Similar.

⟨1⟩2. Every basic open set is a finite intersection of open rays.

PROOF: We have  $(a, b) = (a, +\infty) \cap (-\infty, b)$ ,  $[\perp, b) = (-\infty, b)$  and  $(a, \top] = (a, +\infty)$ .

□

**Definition 11.15.3** (Standard Topology on the Real Line). The *standard topology on the real line* is the order topology on  $\mathbb{R}$  generated by the standard order.

**Lemma 11.15.4.** *The lower limit topology is strictly finer than the standard topology on  $\mathbb{R}$ .*

PROOF:

⟨1⟩1. Every open interval is open in the lower limit topology.

PROOF: If  $x \in (a, b)$  then  $x \in [x, b) \subseteq (a, b)$ .

⟨1⟩2. The half-open interval  $[0, 1)$  is not open in the standard topology.

PROOF: There is no open interval  $(a, b)$  such that  $0 \in (a, b) \subseteq [0, 1)$ .

□

**Lemma 11.15.5.** *The  $K$ -topology is strictly finer than the standard topology on  $\mathbb{R}$ .*

PROOF:

⟨1⟩1. Every open interval is open in the  $K$ -topology.

PROOF: Corollary 11.7.2.1.

⟨1⟩2. The set  $(-1, 1) \setminus K$  is not open in the standard topology.

PROOF: There is no open interval  $(a, b)$  such that  $0 \in (a, b) \subseteq (-1, 1) \setminus K$ , since there must be a positive integer  $n$  with  $1/n \in (a, b)$ .

□

**Lemma 11.15.6.** *Let  $X$  be a topological space. Let  $Y$  be a linearly ordered set in the order topology. Let  $f, g : X \rightarrow Y$  be continuous. Then  $C = \{x \in X \mid f(x) \leq g(x)\}$  is closed.*

PROOF:

⟨1⟩1. LET:  $x \in X \setminus C$

⟨1⟩2.  $f(x) > g(x)$

PROVE: There exists a neighbourhood  $U$  of  $x$  such that  $U \subseteq X \setminus C$

⟨1⟩3. CASE: There exists  $y$  such that  $g(x) < y < f(x)$

PROOF: Take  $U = g^{-1}((-\infty, y)) \cup f^{-1}(y, +\infty)$ .

⟨1⟩4. CASE: There is no  $y$  such that  $g(x) < y < f(x)$

PROOF: Take  $U = g^{-1}((-\infty, f(x))) \cup f^{-1}(g(x), +\infty)$ .

□

**Proposition 11.15.7.** *Let  $X$  be a topological space. Let  $Y$  be a linearly ordered set in the order topology. Let  $f, g : X \rightarrow Y$  be continuous. Define  $h : X \rightarrow Y$  by  $h(x) = \min(f(x), g(x))$ . Then  $h$  is continuous.*

PROOF: By the Pasting Lemma applied to  $\{x \in X \mid f(x) \leq g(x)\}$  and  $\{x \in X \mid g(x) \leq f(x)\}$ , which are closed by Lemma 11.15.6.

**Proposition 11.15.8.** *Let  $X$  and  $Y$  be linearly ordered sets in the order topology. Let  $f : X \rightarrow Y$  be strictly monotone and surjective. Then  $f$  is a homeomorphism.*

PROOF:

⟨1⟩1.  $f$  is bijective.

PROOF: Proposition 5.3.3.

⟨1⟩2.  $f$  is continuous.

⟨2⟩1. For all  $y \in Y$  we have  $f^{-1}((y, +\infty))$  is open.

⟨3⟩1. LET:  $y \in Y$

⟨3⟩2. PICK  $x \in X$  such that  $f(x) = y$

PROOF: Since  $f$  is surjective.

⟨3⟩3.  $f^{-1}((y, +\infty)) = (x, +\infty)$

PROOF: By strict monotonicity.

⟨2⟩2. For all  $y \in Y$  we have  $f^{-1}((-\infty, y))$  is open.

PROOF: Similar.

⟨1⟩3.  $f^{-1}$  is continuous.

⟨2⟩1. For all  $x \in X$  we have  $f((x, +\infty))$  is open.

PROOF:  $f((x, +\infty)) = (f(x), +\infty)$ .

⟨2⟩2. For all  $x \in X$  we have  $f((-\infty, x))$  is open.

PROOF:  $f((-\infty, x)) = (-\infty, f(x))$ .

□

## 11.16 The $n$ th Root Function

**Proposition 11.16.1.** *For all  $n \geq 1$ , the function  $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  defined by  $f(x) = x^n$  is a homeomorphism.*

PROOF:

⟨1⟩1.  $f$  is strictly monotone.

⟨2⟩1. LET:  $x, y \in \mathbb{R}$  with  $0 \leq x < y$

⟨2⟩2.  $x^n < y^n$

$$\begin{aligned} y^n - x^n &= (y - x)(y^{n-1} + y^{n-2}x + y^{n-3}x^2 + \cdots + x^{n-1}) \\ &> 0 \end{aligned}$$



⟨1⟩2.  $f$  is surjective.

⟨2⟩1. LET:  $y \in \mathbb{R}_{\geq 0}$

⟨2⟩2. PICK  $x \in \mathbb{R}$  such that  $y \leq x^n$

PROOF: If  $y \leq 1$  take  $x = 1$ , otherwise take  $x = y$ .

⟨2⟩3. There exists  $x' \in [0, x]$  such that  $(x')^n = y$

PROOF: By the Intermediate Value Theorem.

⟨1⟩3. Q.E.D.

PROOF: Proposition 11.15.8.

□

**Definition 11.16.2.** For  $n \geq 1$ , the  $n$ th root function is the function  $\mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  that is the inverse of  $\lambda x.x^n$ .

## 11.17 The Product Topology

**Definition 11.17.1** (Product Topology). Let  $\{A_i\}_{i \in I}$  be a family of topological spaces. The *product topology* on  $\prod_{i \in I} A_i$  is the topology generated by the subbasis consisting of the sets of the form  $\pi_i^{-1}(U)$  where  $i \in I$  and  $U$  is open in  $A_i$ .

**Proposition 11.17.2.** The product topology on  $\prod_{i \in I} A_i$  is generated by the basis consisting of all sets of the form  $\prod_{i \in I} U_i$  where  $\{U_i\}_{i \in I}$  is a family such that each  $U_i$  is an open set in  $A_i$  and  $U_i = A_i$  for all but finitely many  $i$ .

PROOF: From Proposition 11.7.10. □

**Proposition 11.17.3.** If  $A_i$  is closed in  $X_i$  for all  $i \in I$  then  $\prod_{i \in I} A_i$  is closed in  $\prod_{i \in I} X_i$ .

PROOF:

$$\left( \prod_{i \in I} X_i \right) \setminus \left( \prod_{i \in I} A_i \right) = \bigcup_{j \in I} \left( \prod_{i \in I} X_i \setminus \pi_j^{-1}(A_j) \right) \square$$

**Proposition 11.17.4.** Let  $\{A_i\}_{i \in I}$  be a family of topological spaces. For  $i \in I$ , let  $\mathcal{B}_i$  be a basis for the topology on  $A_i$ . Then  $\mathcal{B} = \{\prod_{i \in I} B_i \mid \forall i \in I. B_i \in \mathcal{B}_i, B_i = A_i \text{ for all but finitely many } i\}$  is a basis for the box topology on  $\prod_{i \in I} A_i$ .

PROOF:

⟨1⟩1. Every set in  $\mathcal{B}$  is open.

⟨1⟩2. For every point  $a \in \prod_{i \in I} A_i$  and every open set  $U$  with  $a \in U$ , there exists  $B \in \mathcal{B}$  such that  $a \in B \subseteq U$ .

⟨2⟩1. LET:  $U$  be open and  $a \in U$

⟨2⟩2. PICK a family  $\{U_i\}_{i \in I}$  such that each  $U_i$  is open in  $A_i$ , such that  $U_i = A_i$  except for  $i = i_1, \dots, i_n$ , and such that  $a \in \prod_{i \in I} U_i \subseteq U$ .

⟨2⟩3. For  $j = 1, \dots, n$ , PICK  $B_{i_j} \in \mathcal{B}_{i_j}$  such that  $a_{i_j} \in B_{i_j} \subseteq U_{i_j}$

⟨2⟩4. LET:  $B = \prod_{i \in I} B_i$  where  $B_i = A_i$  for  $i \neq i_1, \dots, i_n$

- (2)5.  $B \in \mathcal{B}$   
 (2)6.  $a \in B \subseteq U$   
 (1)3. Q.E.D.

PROOF: Lemma 11.7.3.

□

**Proposition 11.17.5.** *Let  $\{A_i\}_{i \in I}$  be a family of topological spaces. Then the projections  $\pi_i : \prod_{i \in I} A_i \rightarrow A_i$  are open maps.*

PROOF: From Lemma 11.11.2. □

**Example 11.17.6.** The projections are not always closed maps. For example,  $\pi_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$  maps the closed set  $\{(x, 1/x) \mid x > 0\}$  to  $(0, +\infty)$ .

**Proposition 11.17.7.** *Let  $\{X_i\}_{i \in I}$  be a family of sets. For  $i \in I$ , let  $\mathcal{T}_i$  and  $\mathcal{U}_i$  be topologies on  $X_i$ . Let  $\mathcal{P}$  be the product topology on  $\prod_{i \in I} X_i$  generated by the topologies  $\mathcal{T}_i$ , and  $\mathcal{Q}$  the product topology on the same set generated by the topologies  $\mathcal{U}_i$ . Then  $\mathcal{P} \subseteq \mathcal{Q}$  if and only if  $\mathcal{T}_i \subseteq \mathcal{U}_i$  for all  $i$ .*

PROOF:

- (1)1. If  $\mathcal{T}_i \subseteq \mathcal{U}_i$  for all  $i$  then  $\mathcal{P} \subseteq \mathcal{Q}$

PROOF: By Corollary 11.7.2.1.

- (1)2. If  $\mathcal{P} \subseteq \mathcal{Q}$  then  $\mathcal{T}_i \subseteq \mathcal{U}_i$  for all  $i$

- (2)1. ASSUME:  $\mathcal{P} \subseteq \mathcal{Q}$

- (2)2. LET:  $i \in I$

- (2)3. LET:  $U \in \mathcal{T}_i$

- (2)4. LET:  $U_i = U$  and  $U_j = X_j$  for  $j \neq i$

- (2)5.  $\prod_{i \in I} U_i \in \mathcal{P}$

- (2)6.  $\prod_{i \in I} U_i \in \mathcal{Q}$

- (2)7.  $U \in \mathcal{U}_i$

PROOF: From Proposition 11.17.5.

□

**Proposition 11.17.8 (Choice).** *Let  $\{X_i\}_{i \in I}$  be a family of topological spaces. Let  $A_i \subseteq X_i$  for all  $i \in I$ . Then*

$$\overline{\prod_{i \in I} A_i} \subseteq \prod_{i \in I} \overline{A_i} .$$

- (1)1. For all  $i \in I$  we have  $A_i \subseteq \overline{A_i}$

PROOF: Lemma 11.4.3.

- (1)2.  $\prod_{i \in I} A_i \subseteq \prod_{i \in I} \overline{A_i}$

- (1)3. Q.E.D.

PROOF: Since  $\prod_{i \in I} A_i$  is closed by Proposition 11.17.3.

**Proposition 11.17.9 (Choice).** *Let  $\{X_i\}_{i \in I}$  be a family of topological spaces. Let  $A_i \subseteq X_i$  for all  $i \in I$ . Then*

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i} .$$

PROOF:

$\langle 1 \rangle 1. \overline{\prod_{i \in I} A_i} \subseteq \prod_{i \in I} \overline{A_i}$

PROOF: Proposition 11.17.8.

$\langle 1 \rangle 2. \prod_{i \in I} \overline{A_i} \subseteq \overline{\prod_{i \in I} A_i}$

$\langle 2 \rangle 1. \text{ LET: } x \in \prod_{i \in I} \overline{A_i}$

$\langle 2 \rangle 2. \text{ LET: } U \text{ be a neighbourhood of } x$

$\langle 2 \rangle 3. \text{ PICK } V_i \text{ open in } X_i \text{ such that } x \in \prod_{i \in I} V_i \subseteq U \text{ with } V_i = X_i \text{ except for } i = i_1, \dots, i_n$

$\langle 2 \rangle 4. \text{ For } i \in I, \text{ pick } a_i \in V_i \cap A_i$

PROOF: By Theorem 11.4.6 and  $\langle 2 \rangle 1$  using the Axiom of Choice.

$\langle 2 \rangle 5. U \text{ intersects } \prod_{i \in I} A_i$

$\langle 2 \rangle 6. \text{ Q.E.D.}$

PROOF:  $a \in U \cap \prod_{i \in I} A_i$

□

**Example 11.17.10.** The closure of  $\mathbb{R}^\infty$  in  $\mathbb{R}^\omega$  is  $\mathbb{R}^\omega$

PROOF:

$\langle 1 \rangle 1. \text{ LET: } a \in \mathbb{R}^\omega$

$\langle 1 \rangle 2. \text{ LET: } U \text{ be any neighbourhoods of } a.$

$\langle 1 \rangle 3. \text{ PICK } U_n \text{ open in } \mathbb{R} \text{ for all } n \text{ such that } a \in \prod_{n \geq 0} U_n \subseteq U \text{ and } U_n = \mathbb{R} \text{ for all } n \text{ except } n_1, \dots, n_k$

$\langle 1 \rangle 4. \text{ LET: } b_n = a_n \text{ for } n = n_1, \dots, n_k \text{ and } b_n = 0 \text{ for all other } n$

$\langle 1 \rangle 5. b \in \mathbb{R}^\infty \cap U$

$\langle 1 \rangle 6. \text{ Q.E.D.}$

PROOF: From Theorem 11.4.6.

□

**Proposition 11.17.11.** Let  $\{X_i\}_{i \in I}$  be a family of topological spaces. Let  $(a_\alpha)_{\alpha \in J}$  be a net in  $\prod_{i \in I} X_i$  and  $l \in \prod_{i \in I} X_i$ . Then  $a_\alpha \rightarrow l$  as  $\alpha \rightarrow \infty$  if and only if, for all  $i \in I$ , we have  $\pi_i(a_\alpha) \rightarrow \pi_i(l)$  as  $\alpha \rightarrow \infty$ .

PROOF:

$\langle 1 \rangle 1. \text{ If } a_\alpha \rightarrow l \text{ as } \alpha \rightarrow \infty \text{ then, for all } i \in I, \text{ we have } \pi_i(a_\alpha) \rightarrow \pi_i(l) \text{ as } \alpha \rightarrow \infty$

PROOF: Proposition 11.12.17.

$\langle 1 \rangle 2. \text{ If, for all } i \in I, \text{ we have } \pi_i(a_\alpha) \rightarrow \pi_i(l) \text{ as } \alpha \rightarrow \infty, \text{ then } a_\alpha \rightarrow l \text{ as } \alpha \rightarrow \infty$

$\langle 2 \rangle 1. \text{ ASSUME: For all } i \in I, \text{ we have } \pi_i(a_\alpha) \rightarrow \pi_i(l) \text{ as } \alpha \rightarrow \infty$

$\langle 2 \rangle 2. \text{ LET: } V \text{ be a neighbourhood of } l$

$\langle 2 \rangle 3. \text{ PICK open sets } U_i \text{ in } X_i \text{ such that } l \in \prod_{i \in I} U_i \subseteq V \text{ and } U_i = X_i \text{ for all } i \text{ except } i = i_1, \dots, i_k$

$\langle 2 \rangle 4. \text{ For } j = 1, \dots, k, \text{ PICK } \alpha_j \text{ such that, for all } \beta \geq \alpha_j, \text{ we have } \pi_{i_j}(a_\beta) \in U_{i_j}$

$\langle 2 \rangle 5. \text{ PICK } \alpha \in J \text{ such that } \alpha_1, \dots, \alpha_k \leq \alpha$

$\langle 2 \rangle 6. \text{ For all } \beta \geq \alpha \text{ we have } a_\beta \in V$

□

**Theorem 11.17.12.** Let  $A$  be a topological space and  $\{X_i\}_{i \in I}$  be a family of topological spaces. Let  $f : A \rightarrow \prod_{i \in I} X_i$  be a function. If  $\pi_i \circ f$  is continuous for all  $i \in I$  then  $f$  is continuous.

PROOF:

- ⟨1⟩1. LET:  $i \in I$  and  $U$  be open in  $X_i$
- ⟨1⟩2.  $f^{-1}(\pi_i^{-1}(U))$  is open in  $A$
- ⟨1⟩3. Q.E.D.

PROOF: Proposition 11.12.3.

□

### 11.17.1 Continuous in Each Variable Separately

**Definition 11.17.13** (Continuous in Each Variable Separately). Let  $F : X \times Y \rightarrow Z$ . Then  $F$  is *continuous in each variable separately* if and only if:

- for every  $a \in X$  the function  $\lambda y \in Y. F(a, y)$  is continuous;
- for every  $b \in Y$  the function  $\lambda x \in X. F(x, b)$  is continuous.

**Proposition 11.17.14.** Let  $F : X \times Y \rightarrow Z$ . If  $F$  is continuous then  $F$  is continuous in each variable separately.

PROOF: For  $a \in X$ , the function  $\lambda y \in Y. F(a, y)$  is  $F \circ i$  where  $i : Y \rightarrow X \times Y$  maps  $y$  to  $(a, y)$ . We have  $i$  is continuous by Proposition 11.14.6, hence  $F \circ i$  is continuous by Theorem 11.12.8.

Similarly for  $\lambda x \in X. F(x, b)$  for  $b \in Y$ . □

**Example 11.17.15.** Define  $F : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  by

$$F(x, y) = \begin{cases} xy/(x^2 + y^2) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Then  $F$  is continuous in each variable separately but not continuous.

**Proposition 11.17.16.** Let  $f : A \rightarrow C$  and  $g : B \rightarrow D$  be open maps. Then  $f \times g : A \times B \rightarrow C \times D$  is an open map.

PROOF: Given  $U$  open in  $A$  and  $V$  open in  $B$ . Then  $(f \times g)(U \times V) = f(U) \times g(V)$  is open in  $C \times D$ . The result follows from Proposition 11.11.3. □

**Definition 11.17.17** (Sorgenfrey Plane). The *Sorgenfrey plane* is  $\mathbb{R}_l^2$ .

## 11.18 The Subspace Topology

**Definition 11.18.1** (Subspace Topology). Let  $X$  be a topological space and  $Y \subseteq X$ . The *subspace topology* on  $Y$  is  $\mathcal{T} = \{U \cap Y \mid U \text{ is open in } X\}$ .

We prove this is a topology.

PROOF:

- ⟨1⟩1.  $Y \in \mathcal{T}$

PROOF: Since  $Y = X \cap Y$

- ⟨1⟩2. For all  $\mathcal{U} \subseteq \mathcal{T}$ , we have  $\bigcup \mathcal{U} \in \mathcal{T}$
- ⟨2⟩1. LET:  $\mathcal{U} \subseteq \mathcal{T}$
- ⟨2⟩2. LET:  $\mathcal{V} = \{V \text{ open in } X \mid V \cap Y \in \mathcal{U}\}$
- ⟨2⟩3.  $\bigcup \mathcal{U} = (\bigcup \mathcal{V}) \cap Y$
- ⟨1⟩3. For all  $U, V \in \mathcal{T}$ , we have  $U \cap V \in \mathcal{T}$
- ⟨2⟩1. LET:  $U, V \in \mathcal{T}$
- ⟨2⟩2. PICK  $U', V'$  open in  $X$  such that  $U = U' \cap Y$  and  $V = V' \cap Y$
- ⟨2⟩3.  $(U \cap V) = (U' \cap V') \cap Y$

□

**Theorem 11.18.2.** *Let  $X$  be a topological space and  $Y$  a subspace of  $X$ . Let  $A \subseteq Y$ . Then  $A$  is closed in  $Y$  if and only if there exists a closed set  $C$  in  $X$  such that  $A = C \cap Y$ .*

PROOF: We have

$$\begin{aligned}
& A \text{ is closed in } Y \\
& \Leftrightarrow Y \setminus A \text{ is open in } Y \\
& \Leftrightarrow \exists U \text{ open in } X. Y \setminus A = Y \cap U \\
& \Leftrightarrow \exists C \text{ closed in } X. Y \setminus A = Y \cap (X \setminus U) \\
& \Leftrightarrow \exists C \text{ closed in } X. A = Y \cap U
\end{aligned}$$

□

**Theorem 11.18.3.** *Let  $Y$  be a subspace of  $X$ . Let  $A \subseteq Y$ . Let  $\bar{A}$  be the closure of  $A$  in  $X$ . Then the closure of  $A$  in  $Y$  is  $\bar{A} \cap Y$ .*

PROOF: The closure of  $A$  in  $Y$  is

$$\begin{aligned}
& \bigcap \{C \text{ closed in } Y \mid A \subseteq C\} \\
& = \bigcap \{D \cap Y \mid D \text{ closed in } X, A \subseteq D \cap Y\} \quad (\text{Theorem 11.18.2}) \\
& = \bigcap \{D \mid D \text{ closed in } X, A \subseteq D\} \cap Y \\
& = \bar{A} \cap Y
\end{aligned}$$

□

**Lemma 11.18.4.** *Let  $X$  be a topological space and  $Y \subseteq X$ . Let  $\mathcal{B}$  be a basis for the topology on  $X$ . Then  $\mathcal{B}' = \{B \cap Y \mid B \in \mathcal{B}\}$  is a basis for the subspace topology on  $Y$ .*

PROOF:

- ⟨1⟩1. Every element in  $\mathcal{B}'$  is open in  $Y$
- ⟨1⟩2. For every open set  $U$  in  $Y$  and point  $y \in U$ , there exists  $B' \in \mathcal{B}'$  such that  $y \in B' \subseteq U$
- ⟨2⟩1. LET:  $U$  be open in  $Y$  and  $y \in U$
- ⟨2⟩2. PICK  $V$  open in  $X$  such that  $U = V \cap Y$
- ⟨2⟩3. PICK  $B \in \mathcal{B}$  such that  $y \in B \subseteq V$
- ⟨2⟩4. LET:  $B' = B \cap Y$
- ⟨2⟩5.  $B' \in \mathcal{B}'$
- ⟨2⟩6.  $y \in B' \subseteq U$
- ⟨1⟩3. Q.E.D.

PROOF: By Lemma 11.7.3.

□

**Lemma 11.18.5.** *Let  $X$  be a topological space and  $Y \subseteq X$ . Let  $\mathcal{S}$  be a basis for the topology on  $X$ . Then  $\mathcal{S}' = \{S \cap Y \mid S \in \mathcal{S}\}$  is a subbasis for the subspace topology on  $Y$ .*

PROOF: The set  $\{B \cap Y \mid B \text{ is a finite intersection of elements of } \mathcal{S}\}$  is a basis for the subspace topology by Lemma 11.18.4, and this is the set of all finite intersections of elements of  $\mathcal{S}'$ . □

**Lemma 11.18.6.** *Let  $Y$  be a subspace of  $X$ . If  $U$  is open in  $Y$  and  $Y$  is open in  $X$  then  $U$  is open in  $X$ .*

PROOF:

⟨1⟩1. PICK  $V$  open in  $X$  such that  $U = V \cap Y$

⟨1⟩2.  $U$  is open in  $X$

PROOF: Since it is the intersection of two open sets  $V$  and  $Y$ .

□

**Theorem 11.18.7.** *Let  $Y$  be a subspace of  $X$  and  $A \subseteq Y$ . If  $A$  is closed in  $Y$  and  $Y$  is closed in  $X$  then  $A$  is closed in  $X$ .*

PROOF: Pick a closed set  $C$  in  $X$  such that  $A = C \cap Y$  (Theorem 11.18.2). Then  $A$  is the intersection of two sets closed in  $X$ , hence  $A$  is closed in  $X$  (Lemma 11.2.4). □

**Theorem 11.18.8** (Pasting Lemma). *Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Let  $\{A_\alpha\}$  be a locally finite family of closed subsets of  $X$  that cover  $X$ . Suppose  $f \upharpoonright A_\alpha$  is continuous for all  $\alpha$ . Then  $f$  is continuous.*

PROOF:

⟨1⟩1. Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Let  $A$  and  $B$  be closed subsets of  $X$  such that  $X = A \cup B$ . Suppose  $f \upharpoonright A$  and  $f \upharpoonright B$  are continuous. Then  $f$  is continuous.

⟨2⟩1. LET:  $C \subseteq Y$  be closed.

⟨2⟩2.  $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$

⟨2⟩3.  $f^{-1}(C)$  and  $g^{-1}(C)$  are closed in  $X$ .

PROOF: Theorems 11.12.6 and 11.18.7.

⟨2⟩4.  $h^{-1}(C)$  is closed in  $X$ .

PROOF: Lemma 11.2.5.

⟨2⟩5. Q.E.D.

PROOF: Theorem 11.12.6.

⟨1⟩2. Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Let  $\{A_\alpha\}$  be a finite family of closed subsets of  $X$  that cover  $X$ . Suppose  $f \upharpoonright A_\alpha$  is continuous for all  $\alpha$ . Then  $f$  is continuous.

PROOF: From ⟨1⟩1 by induction.

⟨1⟩3. Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Let  $\{A_\alpha\}$  be a locally finite family of closed subsets of  $X$  that cover  $X$ . Suppose  $f \upharpoonright A_\alpha$  is continuous for all  $\alpha$ . Then  $f$  is continuous.

- (2)1. LET:  $x \in X$   
 PROVE:  $f$  is continuous at  $x$   
 (2)2. PICK a neighbourhood  $U$  of  $x$  that intersects  $A_\alpha$  for only finitely many  $\alpha$ .  
 (2)3.  $f \upharpoonright U$  is continuous  
 PROOF: By (1)2.  
 (2)4. Q.E.D.  
 PROOF: Lemma 11.12.15.

□

**Theorem 11.18.9.** *Let  $\{X_i\}_{i \in I}$  be a family of topological spaces. Let  $A_i$  be a subspace of  $X_i$  for all  $i \in I$ . Then the product topology on  $\prod_{i \in I} A_i$  is the same as the topology it inherits as a subspace of  $\prod_{i \in I} X_i$ .*

PROOF: The product topology is generated by the subbasis

$$\begin{aligned}
 & \{\pi_i^{-1}(U) \mid i \in I, U \text{ open in } A_i\} \\
 &= \{\pi_i^{-1}(V) \cap A_i \mid i \in I, V \text{ open in } X_i\} \\
 &= \{\pi_i^{-1}(V) \mid i \in I, V \text{ open in } X_i\} \cap \prod_{i \in I} A_i
 \end{aligned}$$

and this is a subbasis for the subspace topology by Lemma 11.18.5. □

**Theorem 11.18.10.** *Let  $X$  be an ordered set in the order topology. Let  $Y \subseteq X$  be an interval. Then the order topology on  $Y$  is the same as the subspace topology on  $Y$ .*

PROOF:

- (1)1. The order topology is finer than the subspace topology.  
 (2)1. For every open ray  $R$  in  $X$ , the set  $R \cap Y$  is open in the order topology.  
 (3)1. For all  $a \in X$ , we have  $(-\infty, a) \cap Y$  is open in the order topology.  
 (4)1. CASE: For all  $y \in Y$  we have  $y < a$   
 PROOF: In this case  $(-\infty, a) \cap Y = Y$ .  
 (4)2. CASE: For all  $y \in Y$  we have  $a < y$   
 PROOF: In this case  $(-\infty, a) \cap Y = \emptyset$ .  
 (4)3. CASE: There exists  $y \in Y$  such that  $y \leq a$  and  $y \in Y$  such that  $a \leq y$   
 (5)1.  $a \in Y$   
 PROOF: Because  $Y$  is an interval.  
 (5)2.  $(-\infty, a) \cap Y = \{y \in Y \mid y < a\}$   
 (3)2. For all  $a \in X$ , we have  $(a, +\infty) \cap Y$  is open in the order topology.  
 PROOF: Similar.  
 (2)2. Q.E.D.  
 PROOF: By Lemmas 11.15.2 and 11.18.5 and Proposition 11.7.11.  
 (1)2. The subspace topology is finer than the order topology.  
 (2)1. Every open ray in  $Y$  is open in the subspace topology.  
 PROOF: For any  $a \in Y$  we have  $(-\infty, a)_Y = (-\infty, a)_X \cap Y$  and  $(a, +\infty)_Y = (a, +\infty)_X \cap Y$ .

⟨2⟩2. Q.E.D.

PROOF: By Lemma 11.15.2 and Proposition 11.7.11

□

This example shows that we cannot remove the hypothesis that  $Y$  is an interval:

**Example 11.18.11.** The order topology on  $I_o^2$  is different from the subspace topology as a subspace of  $\mathbb{R}^2$  under the dictionary order topology.

PROOF: The set  $\{1/2\} \times (1/2, 1)$  is open in the subspace topology but not in the order topology. □

**Proposition 11.18.12.** *Let  $X$  be a topological space,  $Y$  a subspace of  $X$ , and  $Z$  a subspace of  $Y$ . Then the subspace topology on  $Z$  inherited from  $X$  is the same as the subspace topology on  $Z$  inherited from  $Y$ .*

PROOF: The subspace topology inherited from  $Y$  is

$$\begin{aligned} & \{V \cap Z \mid V \text{ open in } Y\} \\ &= \{U \cap Y \cap Z \mid U \text{ open in } X\} \\ &= \{U \cap Z \mid U \text{ open in } X\} \end{aligned}$$

which is the subspace topology inherited from  $X$ . □

**Definition 11.18.13** (Unit Circle). The *unit circle*  $S^1$  is  $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$  as a subspace of  $\mathbb{R}^2$ .

**Definition 11.18.14** (Unit 2-sphere). The *unit 2-sphere* is  $S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$  as a subspace of  $\mathbb{R}^3$ .

**Proposition 11.18.15.** *Let  $f : X \rightarrow Y$  be an open map and  $A \subseteq X$  be open. Then the restriction  $f \upharpoonright A : A \rightarrow f(A)$  is an open map.*

PROOF:

⟨1⟩1. LET:  $U$  be open in  $A$

⟨1⟩2.  $U$  is open in  $X$

PROOF: Lemma 11.18.6.

⟨1⟩3.  $f(U)$  is open in  $Y$

⟨1⟩4.  $f(U)$  is open in  $f(A)$

PROOF: Since  $f(U) = f(U) \cap f(A)$ .

□

**Example 11.18.16.** This example shows that we cannot remove the hypothesis that  $A$  is open.

Let  $A = \{(x, y) \in \mathbb{R}^2 \mid (x > 0 \text{ and } y = 1/x) \text{ or } x = y = 0\}$ . Then  $\pi_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$  is an open map, but  $\pi_1 \upharpoonright A : A \rightarrow [0, +\infty)$  is not, because it maps the set  $\{0, 0\}$  which is open in  $A$  to  $\{0\}$  which is not open in  $[0, +\infty)$ .

**Proposition 11.18.17.** *Let  $Y$  be a subspace of  $X$ . Let  $A \subseteq Y$  and  $l \in Y$ . Then  $l$  is a limit point of  $A$  in  $Y$  if and only if  $l$  is a limit point of  $A$  in  $X$ .*



PROOF: Both are equivalent to the condition that any neighbourhood of  $l$  in  $X$  intersects  $A$  in a point other than  $l$ .  $\square$

**Theorem 11.18.18.** *If  $A$  is a subspace of  $X$  then the inclusion  $j : A \rightarrow X$  is continuous.*

PROOF: For any  $V$  open in  $X$ , we have  $j^{-1}(V) = V \cap A$  is open in  $A$ .  $\square$

## 11.19 The Box Topology

**Definition 11.19.1** (Box Topology). Let  $\{A_i\}_{i \in I}$  be a family of topological spaces. The *box topology* on  $\prod_{i \in I} A_i$  is the topology generated by the set of all sets of the form  $\prod_{i \in I} U_i$  where  $\{U_i\}_{i \in I}$  is a family such that each  $U_i$  is open in  $A_i$ .

This is a basis since it covers  $\prod_{i \in I} A_i$  and is closed under intersection.

**Proposition 11.19.2.** *The box topology is finer than the product topology.*

PROOF: From Proposition 11.17.2.  $\square$

**Corollary 11.19.2.1.** *If  $A_i$  is closed in  $X_i$  for all  $i \in I$  then  $\prod_{i \in I} A_i$  is closed in  $\prod_{i \in I} X_i$  under the box topology.*

PROOF: From Proposition 11.17.3.

**Proposition 11.19.3** (AC). *Let  $\{A_i\}_{i \in I}$  be a family of topological spaces. For  $i \in I$ , let  $\mathcal{B}_i$  be a basis for the topology on  $A_i$ . Then  $\mathcal{B} = \{\prod_{i \in I} B_i \mid \forall i \in I, B_i \in \mathcal{B}_i\}$  is a basis for the box topology on  $\prod_{i \in I} A_i$ .*

PROOF:

- $\langle 1 \rangle 1$ . Every set of the form  $\prod_{i \in I} B_i$  is open.
- $\langle 1 \rangle 2$ . For every point  $a \in \prod_{i \in I} A_i$  and every open set  $U$  with  $a \in U$ , there exists  $B \in \mathcal{B}$  such that  $a \in B \subseteq U$ .
- $\langle 2 \rangle 1$ . LET:  $U$  be open and  $a \in U$
- $\langle 2 \rangle 2$ . PICK a family  $\{U_i\}_{i \in I}$  such that each  $U_i$  is open in  $A_i$  and  $a \in \prod_{i \in I} U_i \subseteq U$ .
- $\langle 2 \rangle 3$ . For  $i \in I$ , PICK  $B_i \in \mathcal{B}_i$  such that  $a_i \in B_i \subseteq U_i$   
PROOF: Using the Axiom of Choice.
- $\langle 2 \rangle 4$ .  $a \in \prod_{i \in I} B_i \subseteq U$
- $\langle 1 \rangle 3$ . Q.E.D.

PROOF: Lemma 11.7.3.

$\square$

**Theorem 11.19.4.** *Let  $\{X_i\}_{i \in I}$  be a family of topological spaces. Let  $A_i$  be a subspace of  $X_i$  for all  $i \in I$ . Give  $\prod_{i \in I} X_i$  the box topology. Then the box topology on  $\prod_{i \in I} A_i$  is the same as the topology it inherits as a subspace of  $\prod_{i \in I} X_i$ .*

PROOF: The box topology is generated by the basis

$$\begin{aligned} & \left\{ \prod_{i \in I} U_i \mid \forall i \in I, U_i \text{ open in } A_i \right\} \\ &= \left\{ \prod_{i \in I} (V_i \cap A_i) \mid \forall i \in I, V_i \text{ open in } X_i \right\} \\ &= \left\{ \prod_{i \in I} V_i \mid \forall i \in I, V_i \text{ open in } X_i \right\} \cap \prod_{i \in I} A_i \end{aligned}$$

and this is a basis for the subspace topology by Lemma 11.18.4.  $\square$

**Proposition 11.19.5 (AC).** *Let  $\{X_i\}_{i \in I}$  be a family of topological spaces. Give  $\prod_{i \in I} X_i$  the box topology. Let  $A_i \subseteq X_i$  for all  $i \in I$ . Then*

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i}.$$

PROOF:

$$\langle 1 \rangle 1. \overline{\prod_{i \in I} A_i} \subseteq \prod_{i \in I} \overline{A_i}$$

$$\langle 2 \rangle 1. \text{ For all } i \in I \text{ we have } A_i \subseteq \overline{A_i}$$

PROOF: Lemma 11.4.3.

$$\langle 2 \rangle 2. \prod_{i \in I} A_i \subseteq \prod_{i \in I} \overline{A_i}$$

$$\langle 2 \rangle 3. \text{ Q.E.D.}$$

PROOF: Since  $\prod_{i \in I} A_i$  is closed by Corollary 11.19.2.1.

$$\langle 1 \rangle 2. \prod_{i \in I} \overline{A_i} \subseteq \overline{\prod_{i \in I} A_i}$$

$$\langle 2 \rangle 1. \text{ LET: } x \in \prod_{i \in I} \overline{A_i}$$

$$\langle 2 \rangle 2. \text{ LET: } U \text{ be a neighbourhood of } x$$

$$\langle 2 \rangle 3. \text{ PICK } V_i \text{ open in } X_i \text{ such that } x \in \prod_{i \in I} V_i \subseteq U$$

$$\langle 2 \rangle 4. \text{ For } i \in I, \text{ pick } a_i \in V_i \cap A_i$$

PROOF: By Theorem 11.4.6 and  $\langle 2 \rangle 1$  using the Axiom of Choice.

$$\langle 2 \rangle 5. U \text{ intersects } \prod_{i \in I} A_i$$

$$\langle 2 \rangle 6. \text{ Q.E.D.}$$

PROOF:  $a \in U \cap \prod_{i \in I} A_i$ .

$\square$

The following example shows that Theorem 11.17.12 fails in the box topology.

**Example 11.19.6.** Define  $f : \mathbb{R} \rightarrow \mathbb{R}^\omega$  by  $f(t) = (t, t, \dots)$ . Then  $\pi_n \circ f = \text{id}_{\mathbb{R}}$  is continuous for all  $n$ . But  $f$  is not continuous when  $\mathbb{R}^\omega$  is given the box topology because the inverse image of

$$(-1, 1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times \dots$$

is  $\{0\}$  which is not open.

The following example shows that Proposition 11.17.11 fails in the box topology.

**Example 11.19.7.** Give  $\mathbb{R}^\omega$  the box topology. Let  $a_n = (1/n, 1/n, \dots)$  for  $n \geq 1$  and  $l = (0, 0, \dots)$ . Then  $\pi_i(a_n) \rightarrow \pi_i(l)$  as  $n \rightarrow \infty$  for all  $i$ , but  $a_n \not\rightarrow l$  as

$n \rightarrow \infty$  since the open set

$$(-1, 1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times \cdots$$

contains  $l$  but does not contain any  $a_n$ .

**Example 11.19.8.** The set  $\mathbb{R}^\infty$  is closed in  $\mathbb{R}^\omega$  under the box topology. For let  $(a_n)$  be any sequence not in  $\mathbb{R}^\infty$ . Let  $U_n$  be an open interval around  $a_n$  that does not contain 0 if  $a_n \neq 0$ , and  $U_n = \mathbb{R}$  if  $a_n = 0$ . Then  $\prod_{n \geq 0} U_n$  is a neighbourhood of  $(a_n)$  that does not intersect  $\mathbb{R}^\infty$ .

## 11.20 $T_1$ Spaces

**Definition 11.20.1** ( $T_1$  Space). A topological space is  $T_1$  if and only if every singleton is closed.

**Lemma 11.20.2.** *A space is  $T_1$  if and only if every finite set is closed.*

PROOF: From Lemma 11.2.5.  $\square$

**Theorem 11.20.3.** *In a  $T_1$  space, a point  $a$  is a limit point of a set  $A$  if and only if every neighbourhood of  $a$  contains infinitely many points of  $A$ .*

PROOF:

$\langle 1 \rangle 1$ . If  $a$  is a limit point of  $A$  then every neighbourhood of  $a$  contains infinitely many points of  $A$ .

$\langle 2 \rangle 1$ . ASSUME:  $a$  is a limit point of  $A$ .

$\langle 2 \rangle 2$ . LET:  $U$  be a neighbourhood of  $a$ .

$\langle 2 \rangle 3$ . ASSUME: for a contradiction  $U$  contains only finitely many points of  $A$ .

$\langle 2 \rangle 4$ .  $(U \cap A) \setminus \{a\}$  is closed.

PROOF: By the  $T_1$  axiom.

$\langle 2 \rangle 5$ .  $(U \setminus A) \cup \{a\}$  is open.

PROOF: It is  $U \setminus ((U \cap A) \setminus \{a\})$ .

$\langle 2 \rangle 6$ .  $(U \setminus A) \cup \{a\}$  intersects  $A$  in a point other than  $a$ .

PROOF: From  $\langle 2 \rangle 1$ .

$\langle 2 \rangle 7$ . Q.E.D.

$\square$

$\langle 1 \rangle 2$ . If every neighbourhood of  $a$  contains infinitely many points of  $A$  then  $a$  is a limit point of  $A$ .

PROOF: Immediate from definitions.

$\square$

(To see this does not hold in every space, see Proposition 11.6.4.)

**Proposition 11.20.4.** *A space is  $T_1$  if and only if, for any two distinct points  $x$  and  $y$ , there exist neighbourhoods  $U$  of  $x$  and  $V$  of  $y$  such that  $x \notin V$  and  $y \notin U$ .*

PROOF:

(1)1. LET:  $X$  be a topological space.  
 (1)2. If  $X$  is  $T_1$  then, for any two distinct points  $x$  and  $y$ , there exist neighbourhoods  $U$  of  $x$  and  $V$  of  $y$  such that  $x \notin V$  and  $y \notin U$ .  
 PROOF: This holds because  $\{x\}$  and  $\{y\}$  are closed.  
 (1)3. Suppose, for any two distinct points  $x$  and  $y$ , there exist neighbourhoods  $U$  of  $x$  and  $V$  of  $y$  such that  $x \notin V$  and  $y \notin U$ . Then  $X$  is  $T_1$ .  
 (2)1. ASSUME: For any two distinct points  $x$  and  $y$ , there exist neighbourhoods  $U$  of  $x$  and  $V$  of  $y$  such that  $x \notin V$  and  $y \notin U$ .  
 (2)2. LET:  $a \in X$   
 (2)3.  $\{a\}$  is closed.  
 PROOF: For all  $b \neq a$  there exists a neighbourhood  $U$  of  $b$  such that  $U \subseteq X \setminus \{a\}$ .  
 □

**Proposition 11.20.5.** *A subspace of a  $T_1$  space is  $T_1$ .*

PROOF: From Proposition 11.18.7.

## 11.21 Hausdorff Spaces

**Definition 11.21.1** (Hausdorff Space). A topological space is *Hausdorff* if and only if, for any points  $x, y$  with  $x \neq y$ , there exist disjoint open sets  $U$  and  $V$  such that  $x \in U$  and  $y \in V$ .

**Theorem 11.21.2.** *Every Hausdorff space is  $T_1$ .*

PROOF:

(1)1. LET:  $X$  be a Hausdorff space.  
 (1)2. LET:  $b \in X$   
 PROVE:  $\overline{\{b\}} = \{b\}$   
 (1)3. ASSUME:  $a \in \overline{\{b\}}$  and  $a \neq b$   
 (1)4. PICK disjoint neighbourhoods  $U$  of  $a$  and  $V$  of  $b$ .  
 (1)5.  $U$  intersects  $\{b\}$   
 PROOF: Theorem 11.4.6.  
 (1)6.  $b \in U$   
 (1)7. Q.E.D.  
 PROOF: This contradicts the fact that  $U$  and  $V$  are disjoint ((1)4).  
 □

**Proposition 11.21.3.** *An infinite set under the finite complement topology is  $T_1$  but not Hausdorff.*

PROOF:

(1)1. LET:  $X$  be an infinite set under the finite complement topology.  
 (1)2. Every singleton is closed.  
 PROOF: By definition.  
 (1)3. PICK  $a, b \in X$  with  $a \neq b$

- ⟨1⟩4. There are no disjoint neighbourhoods  $U$  of  $a$  and  $V$  of  $b$ .
- ⟨2⟩1. LET:  $U$  be a neighbourhood of  $a$  and  $V$  a neighbourhood of  $b$ .
- ⟨2⟩2.  $X \setminus U$  and  $X \setminus V$  are finite.
- ⟨2⟩3. PICK  $c \in X$  that is not in  $X \setminus U$  or  $X \setminus V$ .
- ⟨2⟩4.  $c \in U \cap V$

□

**Proposition 11.21.4.** *The product of a family of Hausdorff spaces is Hausdorff.*

PROOF:

- ⟨1⟩1. LET:  $\{X_i\}_{i \in I}$  be a family of Hausdorff spaces.
- ⟨1⟩2. LET:  $a, b \in \prod_{i \in I} X_i$  with  $a \neq b$
- ⟨1⟩3. PICK  $i \in I$  such that  $a_i \neq b_i$
- ⟨1⟩4. PICK  $U, V$  disjoint open sets in  $X_i$  with  $a_i \in U$  and  $b_i \in V$
- ⟨1⟩5.  $\pi_i^{-1}(U)$  and  $\pi_i^{-1}(V)$  are disjoint open sets in  $\prod_{i \in I} X_i$  with  $a \in \pi_i^{-1}(U)$  and  $b \in \pi_i^{-1}(V)$

□

**Theorem 11.21.5.** *Every linearly ordered set under the order topology is Hausdorff.*

PROOF:

- ⟨1⟩1. LET:  $X$  be a linearly ordered set under the order topology.
- ⟨1⟩2. LET:  $a, b \in X$  with  $a \neq b$
- ⟨1⟩3. ASSUME: w.l.o.g.  $a < b$
- ⟨1⟩4. CASE: There exists  $c$  such that  $a < c < b$   
PROOF: The sets  $(-\infty, c)$  and  $(c, +\infty)$  are disjoint neighbourhoods of  $a$  and  $b$  respectively.
- ⟨1⟩5. CASE: There is no  $c$  such that  $a < c < b$   
PROOF: The sets  $(-\infty, b)$  and  $(a, +\infty)$  are disjoint neighbourhoods of  $a$  and  $b$  respectively.

□

**Theorem 11.21.6.** *A subspace of a Hausdorff space is Hausdorff.*

PROOF:

- ⟨1⟩1. LET:  $X$  be a Hausdorff space and  $Y$  a subspace of  $X$ .
- ⟨1⟩2. LET:  $x, y \in Y$  with  $x \neq y$
- ⟨1⟩3. PICK disjoint neighbourhoods  $U$  of  $x$  and  $V$  of  $y$  in  $X$ .
- ⟨1⟩4.  $U \cap Y$  and  $V \cap Y$  are disjoint neighbourhoods of  $x$  and  $y$  respectively in  $Y$ .

□

**Corollary 11.21.6.1.** *For any family of nonempty spaces  $\{X_\alpha\}_{\alpha \in J}$ , if  $\prod_{\alpha \in J} X_\alpha$  is Hausdorff then each  $X_\alpha$ .*

**Proposition 11.21.7.** *A space  $X$  is Hausdorff if and only if the diagonal  $\Delta = \{(x, x) \mid x \in X\}$  is closed in  $X^2$ .*

PROOF:

$X$  is Hausdorff

$$\Leftrightarrow \forall x, y \in X. x \neq y \Rightarrow \exists V, W \text{ open. } x \in V \wedge y \in W \wedge V \cap W = \emptyset$$

$$\Leftrightarrow \forall (x, y) \in X^2 \setminus \Delta. \exists V, W \text{ open. } (x, y) \subseteq V \times W \subseteq X^2 \setminus \Delta$$

$$\Leftrightarrow \Delta \text{ is closed}$$

□

**Theorem 11.21.8.** *In a Hausdorff space, a net has at most one limit.*

PROOF:

<1>1. LET:  $X$  be a Hausdorff space.

<1>2. ASSUME: for a contradiction  $(a_\alpha)_{\alpha \in J}$  is a net with limits  $l$  and  $m$ .

<1>3. PICK disjoint neighbourhoods  $U$  of  $l$  and  $V$  of  $m$

PROOF: By the Hausdorff axiom.

<1>4. PICK  $\alpha$  and  $\beta$  such that  $a_\gamma \in U$  for  $\gamma \geq \alpha$  and  $a_\gamma \in V$  for  $\gamma \geq \beta$

<1>5. PICK  $\gamma \in J$  with  $\gamma \geq \alpha$  and  $\gamma \geq \beta$

<1>6.  $a_\gamma \in U \cap V$

<1>7. Q.E.D.

PROOF: This contradicts the fact that  $U$  and  $V$  are disjoint (<1>3).

□

To see this is not always true in spaces that are  $T_1$  but not Hausdorff:

**Proposition 11.21.9.** *Let  $X$  be an infinite set under the finite complement topology. Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence with all points distinct. Then for every  $l \in X$  we have  $a_n \rightarrow l$  as  $n \rightarrow \infty$ .*

PROOF: Let  $U$  be any neighbourhood of  $l$ . Since  $X \setminus U$  is finite, there must exist  $N$  such that, for all  $n \geq N$ , we have  $a_n \in U$ . □

**Proposition 11.21.10.** *Let  $X$  be a topological space. Let  $Y$  a Hausdorff space. Let  $A \subseteq X$ . Let  $f, g : \overline{A} \rightarrow Y$  be continuous. If  $f$  and  $g$  agree on  $A$  then  $f = g$ .*

PROOF:

<1>1. LET:  $x \in \overline{A}$

<1>2. ASSUME:  $f(x) \neq g(x)$

<1>3. PICK disjoint neighbourhoods  $V$  of  $f(x)$  and  $W$  of  $g(x)$ .

<1>4. PICK  $y \in f^{-1}(V) \cap g^{-1}(W) \cap A$

PROOF: Since  $f^{-1}(V) \cap g^{-1}(W)$  is a neighbourhood of  $x$  and hence intersects  $A$ .

<1>5.  $f(y) = g(y) \in V \cap W$

<1>6. Q.E.D.

PROOF: This contradicts the fact that  $V$  and  $W$  are disjoint (<1>3).

□

**Proposition 11.21.11.** *Let  $\{X_i\}_{i \in I}$  be a family of Hausdorff spaces. Then  $\prod_{i \in I} X_i$  under the box topology is Hausdorff.*

PROOF:

- ⟨1⟩1. LET:  $\{X_i\}_{i \in I}$  be a family of Hausdorff spaces.
- ⟨1⟩2. LET:  $a, b \in \prod_{i \in I} X_i$  with  $a \neq b$
- ⟨1⟩3. PICK  $i \in I$  such that  $a_i \neq b_i$
- ⟨1⟩4. PICK  $U, V$  disjoint open sets in  $X_i$  with  $a_i \in U$  and  $b_i \in V$
- ⟨1⟩5.  $\pi_i^{-1}(U)$  and  $\pi_i^{-1}(V)$  are disjoint open sets in  $\prod_{i \in I} X_i$  with  $a \in \pi_i^{-1}(U)$  and  $b \in \pi_i^{-1}(V)$

□

**Proposition 11.21.12.** *Let  $\mathcal{T}$  and  $\mathcal{T}'$  be two topologies on the same set  $X$  with  $\mathcal{T} \subseteq \mathcal{T}'$ . If  $\mathcal{T}$  is Hausdorff then  $\mathcal{T}'$  is Hausdorff.*

PROOF: Immediate from definitions.

**Proposition 11.21.13.** *Let  $X$  be a Hausdorff space. Let  $\mathcal{D} \subseteq \mathcal{P}X$  be maximal with respect to the finite intersection property. Then  $\bigcap_{D \in \mathcal{D}} \overline{D}$  contains at most one point.*

PROOF:

- ⟨1⟩1. LET:  $x, y \in \bigcap_{D \in \mathcal{D}} \overline{D}$
- ⟨1⟩2. ASSUME: for a contradiction  $x \neq y$
- ⟨1⟩3. PICK disjoint open subsets  $U$  and  $V$  of  $x$  and  $y$  respectively.
- ⟨1⟩4.  $U, V \in \mathcal{D}$

PROOF: Proposition 11.4.9.

- ⟨1⟩5. Q.E.D.

PROOF: This contradicts the fact that  $\mathcal{D}$  satisfies the finite intersection property.

□

**Proposition 11.21.14.** *Let  $X$  be a topological space. Let  $Y$  be a Hausdorff space. Let  $f : X \rightarrow Y$  be continuous. Then  $\{x \in X \mid f(x) = g(x)\}$  is closed in  $X$ .*

PROOF:

- ⟨1⟩1. LET:  $A = \{x \in X \mid f(x) = g(x)\}$
- ⟨1⟩2. LET:  $x \in X - A$
- ⟨1⟩3. PICK disjoint open neighbourhoods  $U$  and  $V$  of  $f(x)$  and  $g(x)$  in  $Y$
- ⟨1⟩4.  $x \in f^{-1}(U) \cap g^{-1}(V) \subseteq X - A$

□

## 11.22 The First Countability Axiom

**Definition 11.22.1** (First Countability Axiom). A topological space  $X$  satisfies the *first countability axiom*, or is *first countable*, if and only if every point has a countable local basis.

**Example 11.22.2.** The space  $S_\Omega$  is first countable. For any  $\alpha \in S_\Omega$ , the set  $\{(\beta, \alpha + 1) \mid \beta < \alpha\} \cup \{[0, \alpha + 1)\}$  is a local basis at  $\alpha$ .

**Lemma 11.22.3** (Sequence Lemma (CC)). *Let  $X$  be a first countable space. Let  $A \subseteq X$  and  $l \in \overline{A}$ . Then there exists a sequence in  $A$  that converges to  $l$ .*

PROOF:

⟨1⟩1. PICK a countable local basis  $\{B_n \mid n \in \mathbb{Z}^+\}$  at  $l$  such that  $B_1 \supseteq B_2 \supseteq \dots$ .

PROOF: Lemma 11.8.2.

⟨1⟩2. For all  $n \geq 1$ , PICK  $a_n \in A \cap B_n$ .

PROVE:  $a_n \rightarrow l$  as  $n \rightarrow \infty$

⟨1⟩3. LET:  $U$  be a neighbourhood of  $l$

⟨1⟩4. PICK  $N$  such that  $B_N \subseteq U$

⟨1⟩5. For  $n \geq N$  we have  $a_n \in U$

PROOF:  $a_n \in B_n \subseteq B_N \subseteq U$

□

**Example 11.22.4.** The space  $\overline{S_\Omega}$  is not first countable, since  $\Omega$  is a limit point for  $S_\Omega$  but there is no sequence of points in  $S_\Omega$  that converges to  $\Omega$ .

**Theorem 11.22.5** (CC). *Let  $X$  be a first countable space and  $Y$  a topological space. Let  $f : X \rightarrow Y$ . Suppose that, for every sequence  $(x_n)$  in  $X$  and  $l \in X$ , if  $x_n \rightarrow l$  as  $n \rightarrow \infty$ , then  $f(x_n) \rightarrow f(l)$  as  $n \rightarrow \infty$ . Then  $f$  is continuous.*

PROOF:

⟨1⟩1. LET:  $A \subseteq X$

⟨1⟩2. LET:  $a \in A$

PROVE:  $f(a) \in \overline{f(A)}$

⟨1⟩3. PICK a sequence  $(x_n)$  in  $A$  that converges to  $a$ .

PROOF: By the Sequence Lemma.

⟨1⟩4.  $f(x_n) \rightarrow f(a)$

⟨1⟩5.  $f(a) \in \overline{f(A)}$

PROOF: By Lemma 11.9.3.

⟨1⟩6. Q.E.D.

PROOF: By Theorem 11.12.6.

□

**Example 11.22.6** (CC). The space  $\mathbb{R}^\omega$  under the box product is not first countable.

PROOF: Let  $\{B_n \mid n \geq 0\}$  be a countable set of neighbourhoods of  $\vec{0}$ . We will construct a neighbourhood of  $\vec{0}$  that does not include any of these.

For  $n \geq 0$ , pick a neighbourhood  $U_n$  of  $0$  such that  $U_n \subset \pi_n(B_p)$ . Then  $\prod_{n=0}^\infty U_n$  is a neighbourhood of  $\vec{0}$  that does not include any  $B_n$ . □

**Example 11.22.7.** If  $J$  is an uncountable set then  $\mathbb{R}^J$  is not first countable.

PROOF:

⟨1⟩1. LET:  $\{B_n \mid n \geq 0\}$  be any countable set of neighbourhoods of  $\vec{0}$ .

⟨1⟩2. For  $n \geq 0$ , PICK a basis element  $\prod_{\alpha \in J} U_{n\alpha}$  that contains  $\vec{0}$  and is included in  $B_n$ .

PROOF: Using the Axiom of Countable Choice.



- (1)3. For  $n \geq 0$ ,  
 LET:  $J_n = \{\alpha \in J \mid U_{n\alpha} \neq \mathbb{R}\}$   
 (1)4. PICK  $\beta \in J$  such that  $\beta \notin J_n$  for any  $n$ .  
 PROOF: Since each  $J_n$  is finite so  $\bigcup_n J_n$  is countable.  
 (1)5.  $\pi_\beta((-1, 1))$  is a neighbourhood of  $\vec{0}$  that does not include any  $B_n$ .  
 $\square$

**Example 11.22.8.** The space  $\mathbb{R}_l$  is first countable.

PROOF: For any real number  $x$ , the set  $\{[x, q) \mid q \in \mathbb{Q}, q > x\}$  is a countable local basis at  $x$ .  $\square$

PROOF: For any  $a \in \mathbb{R}$ , the set  $\{[a, a + 1/n) \mid n \geq 1\}$  is a countable local basis.

**Example 11.22.9.** The ordered square is first countable.

PROOF: For any  $(a, b) \in I_o^2$  with  $b \neq 0, 1$ , the set  $\{(\{a\} \times (b - 1/n, b + 1/n)) \cap I_o^2 \mid n \geq 1\}$  is a countable local basis.

**Proposition 11.22.10.** *A subspace of a first countable space is first countable.*

PROOF:

- (1)1. LET:  $X$  be a first countable space.  
 (1)2. LET:  $Y \subseteq X$   
 (1)3. LET:  $y \in Y$   
 (1)4. PICK a countable local basis  $\mathcal{B}$  for  $y$  in  $X$   
 (1)5.  $\{B \cap Y \mid B \in \mathcal{B}\}$  is a countable local basis for  $y$  in  $Y$   
 $\square$

**Proposition 11.22.11 (AC).** *A countable product of first countable spaces is first countable.*

PROOF:

- (1)1. LET:  $(X_n)$  be a sequence of first countable spaces.  
 (1)2. LET:  $(x_n) \in \prod_n X_n$   
 (1)3. For all  $n$ , PICK a countable local basis  $\mathcal{B}_n$  for  $x_n$  in  $X_n$   
 (1)4. LET:  $\mathcal{B}$  be the set of all sets of the form  $\prod_n U_n$  where  $(U_n)$  is a family such that  $U_n \in \mathcal{B}_n$  for finitely many  $n$  and  $U_n = X_n$  for all other  $n$   
 (1)5.  $\mathcal{B}$  is a countable local basis for  $(x_n)$   
 $\square$

**Example 11.22.12.** The space  $S_\Omega$  is first countable. For any  $x \in S_\Omega$ , the set  $\{[0, x + 1)\} \cup \{(y, x + 1) \mid y < x\}$  is a countable local basis at  $x$ .

**Example 11.22.13.** The space  $\overline{S_\Omega}$  is not first countable.

PROOF:

- (1)1. ASSUME: for a contradiction  $\mathcal{B}$  is a countable local basis at  $\Omega$ .  
 (1)2. For  $B \in \mathcal{B}$ ,  
 LET:  $a_B$  be least such that  $(a_B, \Omega] \subseteq B$ .  
 (1)3. LET:  $b = \sup_{B \in \mathcal{B}} a_B$

- (1)4.  $b < \Omega$   
 (1)5. There is no  $B \in \mathcal{B}$  such that  $\Omega \in B \subseteq (b+1, \Omega]$   
 $\square$

**Proposition 11.22.14.** *The image of a first countable space under a continuous open map is first countable.*

PROOF:

- (1)1. LET:  $X$  be a first countable space.  
 (1)2. LET:  $Y$  be a topological space.  
 (1)3. LET:  $f : X \rightarrow Y$  be a surjective, continuous open map.  
 (1)4. LET:  $y \in Y$   
 (1)5. PICK  $x \in X$  such that  $f(x) = y$   
 (1)6. PICK a countable local basis  $\mathcal{B}$  at  $x$ .  
 PROVE:  $\{f(B) \mid B \in \mathcal{B}\}$  is a local basis at  $y$ .  
 (1)7. LET:  $V$  be a neighbourhood of  $y$   
 (1)8. PICK  $B \in \mathcal{B}$  such that  $x \in B \subseteq f^{-1}(V)$   
 (1)9.  $y \in f(B) \subseteq V$   
 $\square$

**Example 11.22.15** (Choice). Let  $\mathbb{Q}^\infty$  be the set of all sequences of rationals that end in an infinite sequence of 0s. Then  $\mathbb{Q}^\infty$  as a subspace of  $\mathbb{R}^\omega$  under the box topology is not first countable.

PROOF:

- (1)1. ASSUME: for a contradiction  $\{B_n \mid n \in \mathbb{Z}^+\}$  is a countable local basis at 0.  
 (1)2. For  $n \in \mathbb{Z}^+$ , PICK a sequence of open neighbourhoods  $(U_{nm})$  of 0 in  $\mathbb{R}$  such that  $\prod_m U_{nm} \cap \mathbb{Q}^\infty \subseteq B_n$   
 (1)3. For  $n \in \mathbb{Z}^+$ , PICK a nonempty open proper subset  $V_n$  of  $U_{nn}$  that contains 0.  
 (1)4.  $\prod_n V_n$  is a nonempty open neighbourhood of 0 but there is no  $n$  such that  $B_n \subseteq \prod_n V_n$   
 $\square$

## 11.23 Strong Continuity

**Definition 11.23.1** (Strongly Continuous). Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$  be a function. Then  $f$  is *strongly continuous* if and only if, for every subset  $U \subseteq Y$ , we have  $U$  is open in  $Y$  if and only if  $f^{-1}(U)$  is open in  $X$ .

**Proposition 11.23.2.** *Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$  be a function. Then  $f$  is strongly continuous if and only if, for every subset  $C \subseteq Y$ , we have  $C$  is closed in  $Y$  if and only if  $f^{-1}(C)$  is closed in  $X$ .*

PROOF: Since  $X \setminus f^{-1}(C) = f^{-1}(Y \setminus C)$ .  $\square$

**Proposition 11.23.3.** *Let  $X$ ,  $Y$  and  $Z$  be topological spaces. Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$ . If  $f$  and  $g$  are strongly continuous then so is  $g \circ f$ .*

PROOF: Since  $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ .  $\square$

**Proposition 11.23.4.** *Let  $X, Y$  and  $Z$  be topological spaces. Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$ . If  $g \circ f$  is continuous and  $f$  is strongly continuous then  $g$  is continuous.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $V \subseteq Z$  be open.

$\langle 1 \rangle 2$ .  $f^{-1}(g^{-1}(V))$  is open in  $X$ .

PROOF: Since  $g \circ f$  is continuous.

$\langle 1 \rangle 3$ .  $f^{-1}(V)$  is open in  $Y$ .

PROOF: Since  $g$  is strongly continuous.

$\square$

**Proposition 11.23.5.** *Let  $X, Y$  and  $Z$  be topological spaces. Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$ . If  $g \circ f$  is strongly continuous and  $f$  is strongly continuous then  $g$  is strongly continuous.*

PROOF: For  $V \subseteq Z$ , we have  $V$  is open iff  $f^{-1}(g^{-1}(V))$  is open iff  $g^{-1}(V)$  is open.

## 11.24 Saturated Sets

**Definition 11.24.1.** Let  $X$  and  $Y$  be sets and  $p : X \rightarrow Y$  a surjective function. Let  $C \subseteq X$ . Then  $C$  is *saturated* with respect to  $p$  if and only if, for all  $x, y \in X$ , if  $x \in C$  and  $p(x) = p(y)$  then  $y \in C$ .

**Proposition 11.24.2.** *Let  $X$  and  $Y$  be sets and  $p : X \rightarrow Y$  a surjective function. Let  $C \subseteq X$ . Then the following are equivalent:*

1.  $C$  is saturated with respect to  $p$ .
2. There exists  $D \subseteq Y$  such that  $C = p^{-1}(D)$
3.  $C = p^{-1}(p(C))$ .

PROOF:

$\langle 1 \rangle 1$ .  $1 \Rightarrow 3$

$\langle 2 \rangle 1$ . ASSUME:  $C$  is saturated with respect to  $p$ .

$\langle 2 \rangle 2$ .  $C \subseteq p^{-1}(p(C))$

PROOF: Trivial.

$\langle 2 \rangle 3$ .  $p^{-1}(p(C)) \subseteq C$

$\langle 3 \rangle 1$ . LET:  $x \in p^{-1}(p(C))$

$\langle 3 \rangle 2$ .  $p(x) \in p(C)$

$\langle 3 \rangle 3$ . There exists  $y \in C$  such that  $p(x) = p(y)$

$\langle 3 \rangle 4$ .  $x \in C$

PROOF: From  $\langle 2 \rangle 1$ .

$\langle 1 \rangle 2$ .  $3 \Rightarrow 2$

PROOF: Trivial.

⟨1⟩3.  $2 \Rightarrow 1$

PROOF: This follows because if  $p(x) \in D$  and  $p(x) = p(y)$  then  $p(y) \in D$ .

□

## 11.25 Quotient Maps

**Definition 11.25.1** (Quotient Map). Let  $X$  and  $Y$  be topological spaces and  $p : X \rightarrow Y$ . Then  $p$  is a *quotient map* if and only if  $p$  is surjective and strongly continuous.

**Proposition 11.25.2.** *Let  $X$  and  $Y$  be topological spaces and  $p : X \rightarrow Y$  be a surjective function. Then the following are equivalent.*

1.  $p$  is a quotient map.
2.  $p$  is continuous and maps saturated open sets to open sets.
3.  $p$  is continuous and maps saturated closed sets to closed sets.

PROOF:

⟨1⟩1.  $1 \Rightarrow 2$

⟨2⟩1. ASSUME:  $p$  is a quotient map.

⟨2⟩2. LET:  $U$  be a saturated open set in  $X$ .

⟨2⟩3.  $p^{-1}(p(U))$  is open in  $X$ .

PROOF: Since  $U = p^{-1}(p(U))$  by Proposition 11.24.2.

⟨2⟩4.  $p(U)$  is open in  $Y$ .

PROOF: From ⟨2⟩1.

⟨1⟩2.  $1 \Rightarrow 3$

PROOF: Similar.

⟨1⟩3.  $2 \Rightarrow 1$

⟨2⟩1. ASSUME:  $p$  is continuous and maps saturated open sets to open sets.

⟨2⟩2. LET:  $U \subseteq Y$

⟨2⟩3. ASSUME:  $p^{-1}(U)$  is open in  $X$

⟨2⟩4.  $p^{-1}(U)$  is saturated.

PROOF: Proposition 11.24.2.

⟨2⟩5.  $U$  is open in  $Y$ .

⟨1⟩4.  $3 \Rightarrow 1$

PROOF: Similar.

□

**Corollary 11.25.2.1.** *Every surjective continuous open map is a quotient map.*

**Corollary 11.25.2.2.** *Every surjective continuous closed map is a quotient map.*

**Example 11.25.3.** The converses of these corollaries do not hold.

Let  $A = \{(x, y) \mid x \geq 0\} \cup \{(x, y) \mid y = 0\}$ . Then  $\pi_1 : A \rightarrow \mathbb{R}$  is a quotient map, but not an open map or a closed map.

We prove that  $\pi_1$  maps saturated open sets to open sets:

- ⟨1⟩1. LET:  $\pi_1^{-1}(U)$  be a saturated open set in  $A$   
PROVE:  $U$  is open in  $\mathbb{R}$
- ⟨1⟩2. LET:  $x \in U$
- ⟨1⟩3.  $(x, 0) \in \pi_1(U)^{-1}$
- ⟨1⟩4. PICK  $W, V$  open in  $\mathbb{R}$  such that  $(x, 0) \subseteq W \times V \subseteq \pi_1(U)^{-1}$
- ⟨1⟩5.  $x \in W \subseteq U$

It is not an open map because it maps  $((-1, 1) \times (1, 2)) \cap A$  to  $[0, 1)$ .

It is not a closed map because it maps  $\{(x, 1/x) \mid x > 0\}$  to  $(0, +\infty)$ .

**Proposition 11.25.4.** *Let  $p : X \twoheadrightarrow Y$  be a quotient map. Let  $A \subseteq X$  be saturated with respect to  $p$ . Let  $q : A \twoheadrightarrow p(A)$  be the restriction of  $p$ .*

- 1. *If  $A$  is either open or closed in  $X$  then  $q$  is a quotient map.*
- 2. *If  $p$  is either an open map or a closed map then  $q$  is a quotient map.*

PROOF:

- ⟨1⟩1. LET:  $p : X \twoheadrightarrow Y$  be a quotient map.
- ⟨1⟩2. LET:  $A \subseteq X$  be saturated with respect to  $p$ .
- ⟨1⟩3. LET:  $q : A \twoheadrightarrow p(A)$  be the restriction of  $p$ .
- ⟨1⟩4.  $q$  is continuous.
- PROOF: Theorem 11.12.9.
- ⟨1⟩5. If  $A$  is open in  $X$  then  $q$  is a quotient map.
- ⟨2⟩1. ASSUME:  $A$  is open in  $X$ .
- ⟨2⟩2.  $q$  maps saturated open sets to open sets.
- ⟨3⟩1. LET:  $U \subseteq A$  be saturated with respect to  $q$  and open in  $A$
- ⟨3⟩2.  $U$  is saturated with respect to  $p$ 
  - ⟨4⟩1. LET:  $x, y \in X$
  - ⟨4⟩2. ASSUME:  $x \in U$
  - ⟨4⟩3. ASSUME:  $p(x) = p(y)$
  - ⟨4⟩4.  $x \in A$
  - PROOF: From ⟨3⟩1 and ⟨4⟩2.
  - ⟨4⟩5.  $y \in A$
  - PROOF: From ⟨1⟩2 and ⟨4⟩3
  - ⟨4⟩6.  $q(x) = q(y)$
  - PROOF: From ⟨1⟩3, ⟨4⟩3, ⟨4⟩4, ⟨4⟩5.
  - ⟨4⟩7.  $y \in U$
  - PROOF: From ⟨3⟩1, ⟨4⟩2, ⟨4⟩6
- ⟨3⟩3.  $U$  is open in  $X$
- PROOF: Lemma 11.18.6, ⟨2⟩1, ⟨3⟩1.
- ⟨3⟩4.  $p(U)$  is open in  $Y$
- PROOF: Proposition 11.25.2, ⟨1⟩1, ⟨3⟩2, ⟨3⟩3
- ⟨3⟩5.  $q(U)$  is open in  $p(A)$
- PROOF: Since  $q(U) = p(U) = p(U) \cap p(A)$ .
- ⟨2⟩3. Q.E.D.
- PROOF: By Proposition 11.25.2.
- ⟨1⟩6. If  $A$  is closed in  $X$  then  $q$  is a quotient map.

PROOF: Similar.

- ⟨1⟩7. If  $p$  is an open map then  $q$  is a quotient map.
- ⟨2⟩1. ASSUME:  $p$  is an open map
- ⟨2⟩2.  $q$  maps saturated open sets to open sets.
  - ⟨3⟩1. LET:  $U$  be open in  $A$  and saturated with respect to  $q$
  - ⟨3⟩2. PICK  $V$  open in  $X$  such that  $U = A \cap V$
  - ⟨3⟩3.  $p(V)$  is open in  $Y$
  - ⟨3⟩4.  $q(U) = p(V) \cap p(A)$ 
    - ⟨4⟩1.  $q(U) \subseteq p(V) \cap p(A)$ 
      - PROOF: From ⟨3⟩2.
    - ⟨4⟩2.  $p(V) \cap p(A) \subseteq q(U)$ 
      - ⟨5⟩1. LET:  $y \in p(V) \cap p(A)$
      - ⟨5⟩2. PICK  $x \in V$  and  $x' \in A$  such that  $p(x) = p(x') = y$
      - ⟨5⟩3.  $x \in A$ 
        - PROOF: By ⟨1⟩2.
      - ⟨5⟩4.  $x \in U$ 
        - PROOF: From ⟨3⟩2
- ⟨2⟩3. Q.E.D.
- PROOF: By Proposition 11.25.2.
- ⟨1⟩8. If  $p$  is a closed map then  $q$  is a quotient map.
- PROOF: Similar.

□

**Example 11.25.5.** This example shows we cannot remove the hypotheses on  $A$  and  $p$ .

Define  $f : [0, 1] \rightarrow [2, 3] \rightarrow [0, 2]$  by  $f(x) = x$  if  $x \leq 1$ ,  $f(x) = x - 1$  if  $x \geq 2$ . Then  $f$  is a quotient map but its restriction  $f'$  to  $[0, 1] \cup [2, 3]$  is not, because  $f'^{-1}([1, 2])$  is open but  $[1, 2]$  is not.

For a counterexample where  $A$  is saturated, see Example 11.26.3.

**Proposition 11.25.6.** Let  $p : A \twoheadrightarrow C$  and  $q : B \twoheadrightarrow D$  be open quotient maps. Then  $p \times q : A \times B \rightarrow C \times D$  is an open quotient map.

PROOF: From Corollary 11.25.2.1, Proposition 11.17.16 and Theorem 11.17.12.

□

**Theorem 11.25.7.** Let  $p : X \twoheadrightarrow Y$  be a quotient map. Let  $Z$  be a topological space and  $f : Y \rightarrow Z$  be a function. Then

- 1.  $f \circ p$  is continuous if and only if  $f$  is continuous.
- 2.  $f \circ p$  is a quotient map if and only if  $f$  is a quotient map.

PROOF:

- ⟨1⟩1. If  $f \circ p$  is continuous then  $f$  is continuous.
  - PROOF: Proposition 11.23.4.
- ⟨1⟩2. If  $f$  is continuous then  $f \circ p$  is continuous.
  - PROOF: Theorem 11.12.8.

⟨1⟩3. If  $f \circ p$  is a quotient map then  $f$  is a quotient map.

PROOF: Proposition 11.23.5.

⟨1⟩4. If  $f$  is a quotient map then  $f \circ p$  is a quotient map.

PROOF: From Proposition 11.23.3.

□

**Proposition 11.25.8.** *Let  $X$  and  $Y$  be topological spaces. Let  $p : X \rightarrow Y$  and  $f : Y \rightarrow X$  be continuous maps such that  $p \circ f = \text{id}_Y$ . Then  $p$  is a quotient map.*

PROOF:

⟨1⟩1. LET:  $V \subseteq Y$

⟨1⟩2. ASSUME:  $p^{-1}(V)$  is open in  $X$ .

⟨1⟩3.  $f^{-1}(p^{-1}(V))$  is open in  $Y$ .

PROOF: Because  $f$  is continuous.

⟨1⟩4.  $V$  is open in  $Y$ .

PROOF: Because  $f^{-1}(p^{-1}(V)) = V$ .

□

## 11.26 Quotient Topology

**Definition 11.26.1** (Quotient Topology). Let  $X$  be a topological space,  $Y$  a set and  $p : X \rightarrow Y$  be a surjective function. Then the *quotient topology* on  $Y$  is the unique topology on  $Y$  with respect to which  $p$  is a quotient map, namely  $\mathcal{T} = \{U \in \mathcal{P}Y \mid p^{-1}(U) \text{ is open in } X\}$ .

We prove this is a topology.

PROOF:

⟨1⟩1.  $Y \in \mathcal{T}$

PROOF: Since  $p^{-1}(Y) = X$  by surjectivity.

⟨1⟩2. For all  $\mathcal{A} \subseteq \mathcal{T}$  we have  $\bigcup \mathcal{A} \in \mathcal{T}$

PROOF: Since  $p^{-1}(\bigcup \mathcal{A}) = \bigcup_{U \in \mathcal{A}} p^{-1}(U)$

⟨1⟩3. For all  $U, V \in \mathcal{T}$  we have  $U \cap V \in \mathcal{T}$

PROOF: Since  $p^{-1}(U \cap V) = p^{-1}(U) \cap p^{-1}(V)$ .

□

**Definition 11.26.2** (Quotient Space). Let  $X$  be a topological space and  $\sim$  an equivalence relation on  $X$ . Let  $p : X \rightarrow X/\sim$  be the canonical surjection. Then  $X/\sim$  under the quotient topology is called a *quotient space*, *identification space* or *decomposition space* of  $X$ .

Here is a counterexample showing we cannot remove all the hypotheses of Proposition 11.25.4 except that  $A$  is saturated.

**Example 11.26.3.** Let  $X = (0, 1/2] \cup \{1\} \cup \{1 + 1/n : n \geq 2\}$  as a subspace of  $\mathbb{R}$ . Define  $R$  to be the equivalence relation on  $X$  where  $xRy$  iff  $(x = y \text{ or } |x - y| = 1)$ , so we identify  $1/n$  with  $1 + 1/n$  for all  $n \geq 2$ . Let  $Y$  be the

resulting quotient space  $X/R$  in the quotient topology and  $p : X \twoheadrightarrow Y$  the canonical surjection.

Let  $A = \{1\} \cup (0, 1/2] \setminus \{1/n : n \geq 2\} \subseteq X$ . Then  $A$  is saturated under  $p$  but the restriction  $q$  of  $p$  to  $A$  is not a quotient map because it maps the saturated open set  $\{1\}$  to  $\{1\}$  which is not open in  $p(A)$ .

**Proposition 11.26.4.** *Let  $X, Y$  and  $Z$  be topological spaces. Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$ . If  $f$  and  $g$  are quotient maps then so is  $g \circ f$ .*

PROOF: From Proposition 11.23.3.  $\square$

**Example 11.26.5.** The product of two quotient maps is not necessarily a quotient map.

Let  $X = \mathbb{R}$  and  $X^*$  the quotient space formed by identifying all positive integers. Let  $p : X \twoheadrightarrow X^*$  be the canonical surjection.

We prove  $p \times \text{id}_{\mathbb{Q}} : X \times \mathbb{Q} \rightarrow X^* \times \mathbb{Q}$  is not a quotient map.

PROOF:

- $\langle 1 \rangle 1$ . For  $n \geq 1$ ,  
LET:  $c_n = \sqrt{2}/n$
- $\langle 1 \rangle 2$ . For  $n \geq 1$ ,  
LET:  $U_n = \{(x, y) \in X \times \mathbb{Q} \mid n - 1/4 < x < n + 1/4, (y + n > x + c_n \text{ and } y + n > -x + c_n) \text{ or } (y + n < x + c_n \text{ and } y + n < -x + c_n)\}$
- $\langle 1 \rangle 3$ . For  $n \geq 1$ , we have  $U_n$  is open in  $X \times \mathbb{Q}$
- $\langle 1 \rangle 4$ . For  $n \geq 1$ , we have  $\{n\} \times \mathbb{Q} \subseteq U_n$
- $\langle 1 \rangle 5$ . LET:  $U = \bigcup_{n=1}^{\infty} U_n$
- $\langle 1 \rangle 6$ .  $U$  is open in  $X \times \mathbb{Q}$
- $\langle 1 \rangle 7$ .  $U$  is saturated with respect to  $p \times \text{id}_{\mathbb{Q}}$
- $\langle 1 \rangle 8$ . LET:  $U' = (p \times \text{id}_{\mathbb{Q}})(U)$
- $\langle 1 \rangle 9$ . ASSUME: for a contradiction  $U'$  is open in  $X^* \times \mathbb{Q}$
- $\langle 1 \rangle 10$ .  $(1, 0) \in U'$
- $\langle 1 \rangle 11$ . PICK a neighbourhood  $W$  of 1 in  $X^*$  and  $\delta > 0$  such that  $W \times (-\delta, \delta) \subseteq U'$
- $\langle 1 \rangle 12$ .  $p^{-1}(W) \times (-\delta, \delta) \subseteq U$
- $\langle 1 \rangle 13$ . PICK  $n$  such that  $c_n < \delta$
- $\langle 1 \rangle 14$ .  $n \in p^{-1}(W)$
- $\langle 1 \rangle 15$ . PICK  $\epsilon > 0$  such that  $\epsilon < \delta - c_n$  and  $\epsilon < 1/4$  and  $(n - \epsilon, n + \epsilon) \subseteq p^{-1}(W)$
- $\langle 1 \rangle 16$ .  $(n - \epsilon, n + \epsilon) \times (-\delta, \delta) \subseteq U$
- $\langle 1 \rangle 17$ . PICK a rational  $y$  such that  $c_n - \epsilon/2 < y < c_n + \epsilon/2$
- $\langle 1 \rangle 18$ .  $(n + \epsilon/2, y) \notin U$
- $\langle 1 \rangle 19$ . Q.E.D.

PROOF: This contradicts  $\langle 1 \rangle 16$ .

$\square$

**Proposition 11.26.6.** *Let  $X$  be a topological space and  $\sim$  an equivalence relation on  $X$ . Then  $X/\sim$  is  $T_1$  if and only if every equivalence class is closed in  $X$ .*

PROOF: Immediate from definitions.  $\square$



## 11.27 Retractions

**Definition 11.27.1** (Retraction). Let  $X$  be a topological space and  $A \subseteq X$ . A *retraction* of  $X$  onto  $A$  is a continuous map  $r : X \rightarrow A$  such that, for all  $a \in A$ , we have  $r(a) = a$ .

**Proposition 11.27.2.** *Every retraction is a quotient map.*

PROOF: Proposition 11.25.8 with  $f$  the inclusion  $A \hookrightarrow X$ .  $\square$

## 11.28 Homogeneous Spaces

**Definition 11.28.1** (Homogeneous). A topological space  $X$  is *homogeneous* if and only if, for any points  $a, b \in X$ , there exists a homeomorphism  $\phi : X \cong X$  such that  $\phi(a) = b$ .

## 11.29 Regular Spaces

**Definition 11.29.1** (Regular Space). A topological space  $X$  is *regular* if and only if it is  $T_1$  and, for any closed set  $A$  and point  $a \notin A$ , there exist disjoint open sets  $U, V$  such that  $A \subseteq U$  and  $a \in V$ .

**Proposition 11.29.2.** *Every regular space is Hausdorff.*

PROOF: Immediate from definitions.  $\square$

**Proposition 11.29.3.** *Let  $X$  be a  $T_1$  space. Then  $X$  is regular if and only if, for any point  $x \in X$  and open neighbourhood  $U$  of  $x$ , there exists a neighbourhood  $V$  of  $x$  with  $\overline{V} \subseteq U$ .*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $X$  be a  $T_1$  space.
- $\langle 1 \rangle 2$ . If  $X$  is regular then, for any point  $x \in X$  and open neighbourhood  $U$  of  $x$ , there exists a neighbourhood  $V$  of  $x$  with  $\overline{V} \subseteq U$ .
- $\langle 2 \rangle 1$ . ASSUME:  $X$  is regular.
- $\langle 2 \rangle 2$ . LET:  $x \in X$
- $\langle 2 \rangle 3$ . LET:  $U$  be an open neighbourhood of  $x$ .
- $\langle 2 \rangle 4$ . PICK disjoint open sets  $V$  and  $W$  such that  $X \setminus U \subseteq W$  and  $x \in V$
- $\langle 2 \rangle 5$ .  $\overline{V} \subseteq U$

PROOF: Since  $V \subseteq X - W \subseteq U$ .

- $\langle 1 \rangle 3$ . If, for any point  $x \in X$  and open neighbourhood  $U$  of  $x$ , there exists a neighbourhood  $V$  of  $x$  with  $\overline{V} \subseteq U$ , then  $X$  is regular.
- $\langle 2 \rangle 1$ . ASSUME: for any point  $x \in X$  and open neighbourhood  $U$  of  $x$ , there exists a neighbourhood  $V$  of  $x$  with  $\overline{V} \subseteq U$ .
- $\langle 2 \rangle 2$ . LET:  $A$  be a closed set with  $a \notin A$
- $\langle 2 \rangle 3$ . PICK a neighbourhood  $V$  of  $a$  with  $\overline{V} \subseteq X - A$
- $\langle 2 \rangle 4$ . LET:  $U = X - \overline{V}$

⟨2⟩5.  $U$  and  $V$  are disjoint open sets with  $A \subseteq U$  and  $a \in V$   
 $\square$

**Proposition 11.29.4.** *A subspace of a regular space is regular.*

PROOF:

⟨1⟩1. LET:  $X$  be a regular space.

⟨1⟩2. LET:  $Y \subseteq X$

⟨1⟩3.  $Y$  is  $T_1$

PROOF: Proposition 11.20.5.

⟨1⟩4. LET:  $A$  be a closed set in  $Y$  and  $a \in Y - A$

⟨1⟩5. PICK a closed set  $B$  in  $X$  such that  $A = Y \cap B$

PROOF: Theorem 11.18.7.

⟨1⟩6. PICK disjoint open sets  $U$  and  $V$  in  $X$  such that  $B \subseteq U$  and  $a \in V$

⟨1⟩7.  $U \cap Y$  and  $V \cap Y$  are disjoint open sets in  $Y$  such that  $A \subseteq U \cap Y$  and  $a \in V \cap Y$ .  
 $\square$

**Corollary 11.29.4.1.** *For any family of nonempty spaces  $\{X_\alpha\}_{\alpha \in J}$ , if  $\prod_{\alpha \in J} X_\alpha$  is regular, then each  $X_\alpha$  is regular.*

**Proposition 11.29.5.** *A product of regular spaces is regular.*

PROOF:

⟨1⟩1. LET:  $\{X_\alpha\}_{\alpha \in J}$  be a family of regular spaces.

⟨1⟩2. LET:  $X = \prod_{\alpha \in J} X_\alpha$

⟨1⟩3. LET:  $x \in X$  and  $U$  be an open neighbourhood of  $x$ .

⟨1⟩4. PICK  $U_\alpha$  open in  $X_\alpha$  for all  $\alpha$  such that  $x \in \prod_{\alpha \in J} U_\alpha \subseteq U$  and  $U_\alpha = X_\alpha$  for all  $\alpha$  except  $\alpha_1, \dots, \alpha_n$ .

⟨1⟩5. For  $1 \leq i \leq n$  PICK an open neighbourhood  $V_{\alpha_i}$  of  $\pi_i(x)$  in  $X_{\alpha_i}$  such that  $\bar{V}_{\alpha_i} \subseteq U_{\alpha_i}$

⟨1⟩6. For  $\alpha \neq \alpha_1, \dots, \alpha_n$ ,

LET:  $V_\alpha = X_\alpha$

⟨1⟩7. LET:  $V = \prod_{\alpha} V_\alpha$

⟨1⟩8.  $V$  is an open neighbourhood of  $x$ .

⟨1⟩9.  $\bar{V} \subseteq U$

PROOF: Proposition 11.17.8.

$\square$

**Example 11.29.6.** The space  $\mathbb{R}_K$  is not regular.

PROOF:

⟨1⟩1.  $K$  is closed in  $\mathbb{R}_K$

⟨1⟩2. ASSUME: for a contradiction there exist disjoint open sets  $U$  and  $V$  with  $0 \in U$  and  $K \subseteq V$

⟨1⟩3. PICK an open interval  $(a, b)$  such that  $0 \in (a, b)$  and  $(a, b) - K \subseteq U$

⟨1⟩4. PICK  $n \in \mathbb{Z}^+$  such that  $1/n \in (a, b)$

⟨1⟩5. PICK an open interval  $(c, d)$  such that  $1/n \in (c, d) \subseteq V$

⟨1⟩6. PICK  $z \in (\max(c, 1/(n+1)), 1/n)$

$\langle 1 \rangle 7. z \in U$

PROOF:

$\langle 2 \rangle 1. a < z$

PROOF:  $a < 0 < 1/(n+1) < z$  ( $\langle 1 \rangle 3$ ,  $\langle 1 \rangle 6$ )

$\langle 2 \rangle 2. z < b$

PROOF:  $z < 1/n < b$  ( $\langle 1 \rangle 4$ ,  $\langle 1 \rangle 6$ )

$\langle 2 \rangle 3. z \in (a, b) - K \subseteq U$

PROOF:  $\langle 1 \rangle 3$

$\langle 1 \rangle 8. z \in V$

PROOF:

$\langle 2 \rangle 1. c < z$

PROOF: From  $\langle 1 \rangle 6$ .

$\langle 2 \rangle 2. z < d$

PROOF:  $z < 1/n < d$  ( $\langle 1 \rangle 5$ ,  $\langle 1 \rangle 6$ )

$\langle 2 \rangle 3. z \in (c, d) \subseteq V$

PROOF:  $\langle 1 \rangle 5$

$\langle 1 \rangle 9. \text{Q.E.D.}$

PROOF: This contradicts the fact that  $U$  and  $V$  are disjoint ( $\langle 1 \rangle 2$ ).

□

**Proposition 11.29.7.** *In a regular space, every pair of distinct points have neighbourhoods whose closures are disjoint.*

PROOF:

$\langle 1 \rangle 1. \text{LET: } X \text{ be a regular space.}$

$\langle 1 \rangle 2. \text{LET: } a, b \in X \text{ with } a \neq b.$

$\langle 1 \rangle 3. \text{PICK a neighbourhood } U \text{ of } a \text{ such that } \overline{U} \subseteq X - \{b\}.$

PROOF: Proposition 11.29.3.

$\langle 1 \rangle 4. \text{PICK a neighbourhood } V \text{ of } b \text{ such that } \overline{V} \subseteq X - \overline{U}.$

PROOF: Proposition 11.29.3.

$\langle 1 \rangle 5. U \text{ and } V \text{ are disjoint neighbourhoods of } a \text{ and } b \text{ respectively with disjoint closures.}$

□

**Proposition 11.29.8.** *Every linearly ordered set under the order topology is regular.*

PROOF:

$\langle 1 \rangle 1. \text{LET: } X \text{ be a linearly ordered set in the order topology.}$

$\langle 1 \rangle 2. X \text{ is } T_1.$

$\langle 1 \rangle 3. \text{LET: } x \in X$

$\langle 1 \rangle 4. \text{LET: } U \text{ be an open neighbourhood of } x.$

$\langle 1 \rangle 5. \text{PICK a basic open set } B \text{ such that } x \in B \subseteq U$

PROVE: There exists an open neighbourhood  $V$  of  $x$  such that  $\overline{V} \subseteq B$

$\langle 1 \rangle 6. \text{CASE: } B = (a, b)$

$\langle 2 \rangle 1. \text{CASE: There exists } c, d \text{ with } a < c < x < d < b$

$\langle 3 \rangle 1. \text{LET: } V = (c, d)$



**Definition 11.31.2** (Connected). A topological space is *connected* if and only if it has no separation; otherwise it is *disconnected*.

**Proposition 11.31.3.** A topological space  $X$  is connected if and only if the only sets that are both open and closed are  $X$  and  $\emptyset$ .

Immediate from definitions.

**Lemma 11.31.4.** If  $Y$  is a subspace of  $X$ , a separation of  $Y$  is a pair of disjoint nonempty sets  $A$  and  $B$  whose union is  $Y$ , neither of which contains a limit point of the other.

PROOF:

- ⟨1⟩1. LET:  $A, B \subseteq Y$
- ⟨1⟩2. If  $A$  and  $B$  form a separation of  $Y$  then  $A$  and  $B$  are disjoint and nonempty,  $A \cup B = Y$ , and neither of  $A$  and  $B$  contains a limit point of the other.
  - ⟨2⟩1. ASSUME:  $A$  and  $B$  form a separation of  $Y$
  - ⟨2⟩2.  $A$  and  $B$  are disjoint and nonempty and  $A \cup B = Y$ 
    - PROOF: From ⟨2⟩1 and the definition of separation.
  - ⟨2⟩3.  $A$  does not contain a limit point of  $B$ 
    - ⟨3⟩1. ASSUME: for a contradiction  $l \in A$  and  $l$  is a limit point of  $B$  in  $X$ .
    - ⟨3⟩2.  $l$  is a limit point of  $B$  in  $Y$ 
      - PROOF: Proposition 11.18.17.
    - ⟨3⟩3.  $l \in B$ 
      - ⟨4⟩1.  $B$  is closed in  $Y$ 
        - PROOF: Since  $A$  is open in  $Y$  and  $B = Y \setminus A$  from ⟨2⟩1.
      - ⟨4⟩2. Q.E.D.
        - PROOF: Corollary 11.6.3.1.
    - ⟨3⟩4. Q.E.D.
      - PROOF: This contradicts the fact that  $A \cap B = \emptyset$  (⟨2⟩1).
  - ⟨2⟩4.  $B$  does not contain a limit point of  $A$ 
    - PROOF: Similar.
- ⟨1⟩3. If  $A$  and  $B$  are disjoint and nonempty,  $A \cup B = Y$ , and neither of  $A$  and  $B$  contains a limit point of the other, then  $A$  and  $B$  form a separation of  $Y$ .
  - ⟨2⟩1. ASSUME:  $A$  and  $B$  are disjoint and nonempty,  $A \cup B = Y$ , and neither of  $A$  and  $B$  contains a limit point of the other.
  - ⟨2⟩2.  $A$  is open in  $Y$ 
    - ⟨3⟩1.  $B$  is closed in  $Y$ 
      - ⟨4⟩1. LET:  $l$  be a limit point of  $B$  in  $Y$
      - ⟨4⟩2.  $l$  is a limit point of  $B$  in  $X$ 
        - PROOF: Proposition 11.18.17.
      - ⟨4⟩3.  $l \notin A$ 
        - PROOF: By ⟨2⟩1
      - ⟨4⟩4.  $l \in B$ 
        - PROOF: By ⟨2⟩1 since  $A \cup B = Y$
      - ⟨4⟩5. Q.E.D.

PROOF: Corollary 11.6.3.1.

⟨3⟩2. Q.E.D.

PROOF: Since  $A = Y \setminus B$ .

⟨2⟩3.  $B$  is open in  $Y$

PROOF: Similar.

□

**Example 11.31.5.** Every set under the indiscrete topology is connected.

**Example 11.31.6.** The discrete topology on a set  $X$  is connected if and only if  $|X| \leq 1$ .

**Example 11.31.7.** The finite complement topology on a set  $X$  is connected if and only if either  $|X| \leq 1$  or  $X$  is infinite.

**Example 11.31.8.** The countable complement topology on a set  $X$  is connected if and only if either  $|X| \leq 1$  or  $X$  is uncountable.

**Example 11.31.9.** The rationals  $\mathbb{Q}$  are disconnected. For any irrational  $a$ , the sets  $(-\infty, a) \cap \mathbb{Q}$  and  $(a, +\infty) \cap \mathbb{Q}$  form a separation of  $\mathbb{Q}$ .

**Lemma 11.31.10.** *Let  $X$  be a topological space. If  $C$  and  $D$  form a separation of  $X$ , and  $Y$  is a connected subspace of  $X$ , then either  $Y \subseteq C$  or  $Y \subseteq D$ .*

PROOF: Otherwise  $Y \cap C$  and  $Y \cap D$  would form a separation of  $Y$ . □

**Theorem 11.31.11.** *The union of a set of connected subspaces of a space  $X$  that have a point in common is connected.*

PROOF:

⟨1⟩1. LET:  $\mathcal{A}$  be a set of connected subspaces of the space  $X$  that have the point  $a$  in common.

⟨1⟩2. ASSUME: for a contradiction  $C$  and  $D$  form a separation of  $\bigcup \mathcal{A}$

⟨1⟩3. ASSUME: without loss of generality  $a \in C$

⟨1⟩4. For all  $A \in \mathcal{A}$  we have  $A \subseteq C$

PROOF: Lemma 11.31.10.

⟨1⟩5.  $D = \emptyset$

⟨1⟩6. Q.E.D.

PROOF: This contradicts ⟨1⟩2.

□

**Theorem 11.31.12.** *Let  $X$  be a topological space and  $A$  a connected subspace of  $X$ . If  $A \subseteq B \subseteq \overline{A}$  then  $B$  is connected.*

PROOF:

⟨1⟩1. ASSUME: for a contradiction  $C$  and  $D$  form a separation of  $B$ .

⟨1⟩2. ASSUME: without loss of generality  $A \subseteq C$

PROOF: Lemma 11.31.10.

⟨1⟩3.  $B \subseteq C$

⟨2⟩1. LET:  $x \in B$

- (2)2.  $x \in \overline{A}$
- (2)3. Either  $x \in A$  or  $x$  is a limit point of  $A$ .  
PROOF: Theorem 11.6.3.
- (2)4. Either  $x \in A$  or  $x$  is a limit point of  $C$ .  
PROOF: Lemma 11.6.5, (1)2.
- (2)5.  $x \in C$   
PROOF: Lemma 11.31.4.
- (1)4.  $D = \emptyset$
- (1)5. Q.E.D.

PROOF: This contradicts (1)1.  
□

**Theorem 11.31.13.** *The image of a connected space under a continuous map is connected.*

PROOF:

- (1)1. LET:  $f : X \rightarrow Y$  be a surjective continuous map where  $X$  is connected.
- (1)2. ASSUME: for a contradiction  $C$  and  $D$  form a separation of  $Y$ .
- (1)3.  $f^{-1}(C)$  and  $f^{-1}(D)$  form a separation of  $X$ .

□

**Theorem 11.31.14.** *The product of a family of connected spaces is connected.*

PROOF:

- (1)1. The product of two connected spaces is connected.
  - (2)1. LET:  $X$  and  $Y$  be connected spaces.
  - (2)2. PICK  $a \in X$  and  $b \in Y$   
PROOF: We may assume  $X$  and  $Y$  are nonempty since otherwise  $X \times Y = \emptyset$  which is connected.
  - (2)3.  $X \times \{b\}$  is connected.  
PROOF: It is homeomorphic to  $X$ .
  - (2)4. For all  $x \in X$  we have  $\{x\} \times Y$  is connected.  
PROOF: It is homeomorphic to  $Y$ .
  - (2)5. For any  $x \in X$   
LET:  $T_x = (X \times \{b\}) \cup (\{x\} \times Y)$
  - (2)6. For all  $x \in X$ ,  $T_x$  is connected.  
PROOF: Theorem 11.31.11 since  $(x, b) \in (X \times \{b\}) \cap (\{x\} \times Y)$ .
  - (2)7.  $X \times Y$  is connected.  
PROOF: Theorem 11.31.11 since  $X \times Y = \bigcup_{x \in X} T_x$  and  $(a, b)$  is a point in every  $T_x$ .
- (1)2. The product of a finite family of connected spaces is connected.  
PROOF: From (1)1 by induction.
- (1)3. The product of any family of connected spaces is connected.
  - (2)1. LET:  $\{X_\alpha\}_{\alpha \in J}$  be a family of connected spaces.
  - (2)2. LET:  $X = \prod_{\alpha \in J} X_\alpha$
  - (2)3. PICK  $a \in X$   
PROOF: We may assume  $X \neq \emptyset$  as the empty space is connected.

- (2)4. For every finite subset  $K$  of  $J$ ,  
 LET:  $X_K = \{x \in X \mid \forall \alpha \in J \setminus K. x_\alpha = a_\alpha\}$   
 (2)5. For every finite  $K \subseteq J$ , we have  $X_K$  is connected.  
 PROOF: From (1)2 since  $X_K \cong \prod_{\alpha \in K} X_K$ .  
 (2)6. LET:  $Y = \bigcup_K X_K$   
 (2)7.  $Y$  is connected  
 PROOF: Theorem 11.31.11 since  $a$  is a common point.  
 (2)8.  $X = \bar{Y}$   
 (3)1. LET:  $x \in X$   
 (3)2. LET:  $U = \prod_{\alpha \in J} U_\alpha$  be a basic neighbourhood of  $x$  where  $U_\alpha = X_\alpha$   
 for all  $\alpha$  except  $\alpha \in K$  for some finite  $K \subseteq J$   
 (3)3. LET:  $y \in X$  be the point with  $y_\alpha = x_\alpha$  for  $\alpha \in K$  and  $y_\alpha = a_\alpha$  for  
 all other  $\alpha$   
 (3)4.  $y \in U \cap X_K$   
 (3)5.  $y \in U \cap Y$   
 (2)9.  $X$  is connected.  
 PROOF: Theorem 11.31.12.

□

**Example 11.31.15.** The set  $\mathbb{R}^\omega$  is disconnected under the box topology. The set of bounded sequences and the set of unbounded sequences form a separation.

**Proposition 11.31.16.** Let  $\mathcal{T}$  and  $\mathcal{T}'$  be two topologies on the same set  $X$ . If  $\mathcal{T} \subseteq \mathcal{T}'$  and  $(X, \mathcal{T}')$  is connected then  $(X, \mathcal{T})$  is connected.

PROOF: If  $U$  and  $V$  form a separation of  $(X, \mathcal{T})$  then they form a separation of  $(X, \mathcal{T}')$ . □

**Proposition 11.31.17.** Let  $X$  be a topological space and  $(A_n)$  a sequence of connected subspaces of  $X$ . If  $A_n \cap A_{n+1} \neq \emptyset$  for all  $n$  then  $\bigcup_n A_n$  is connected.

PROOF:

- (1)1. ASSUME: for a contradiction  $C$  and  $D$  form a separation of  $\bigcup_n A_n$   
 (1)2. ASSUME: without loss of generality  $A_0 \subseteq C$   
 PROOF: Lemma 11.31.10.  
 (1)3. For all  $n$  we have  $A_n \subseteq C$   
 PROOF:  
 (2)1. ASSUME:  $A_n \subseteq C$   
 (2)2. PICK  $x \in A_n \cap A_{n+1}$   
 (2)3.  $x \in C$   
 (2)4.  $A_{n+1} \subseteq C$   
 PROOF: Lemma 11.31.10.  
 (2)5. Q.E.D.  
 PROOF: The result follows by induction.  
 (1)4.  $D = \emptyset$   
 (1)5. Q.E.D.  
 PROOF: This contradicts (1)1.

□



**Proposition 11.31.18.** *Let  $X$  be a topological space. Let  $A, C \subseteq X$ . If  $C$  is connected and intersects both  $A$  and  $X \setminus A$  then  $C$  intersects  $\partial A$ .*

PROOF: Otherwise  $C \cap A^\circ$  and  $C \setminus \bar{A}$  would form a separation of  $C$ .  $\square$

**Example 11.31.19.** The space  $\mathbb{R}_l$  is disconnected. For any real  $x$ , the sets  $(-\infty, x)$  and  $[x, +\infty)$  form a separation.

**Proposition 11.31.20.** *Let  $X$  and  $Y$  be connected spaces. Let  $A$  be a proper subset of  $X$  and  $B$  a proper subset of  $Y$ . Then  $(X \times Y) \setminus (A \times B)$  is connected.*

PROOF:

$\langle 1 \rangle 1$ . PICK  $a \in X \setminus A$  and  $b \in Y \setminus B$

$\langle 1 \rangle 2$ . For  $x \in X \setminus A$  we have  $(X \times \{b\}) \cup (\{x\} \times Y)$  is connected.

PROOF: Theorem 11.31.11 since  $(x, b)$  is a common point.

$\langle 1 \rangle 3$ . For  $y \in Y \setminus B$  we have  $(X \times \{y\}) \cup (\{a\} \times Y)$  is connected.

PROOF: Theorem 11.31.11 since  $(a, y)$  is a common point.

$\langle 1 \rangle 4$ .  $(X \times Y) \setminus (A \times B)$  is connected.

PROOF: Theorem 11.31.11 since it is the union of the sets in  $\langle 1 \rangle 2$  and  $\langle 1 \rangle 3$  with  $(a, b)$  as a common point.

$\square$

**Proposition 11.31.21.** *Let  $p : X \twoheadrightarrow Y$  be a quotient map. If  $Y$  is connected and  $p^{-1}(y)$  is connected for all  $y \in Y$ , then  $X$  is connected.*

PROOF:

$\langle 1 \rangle 1$ . ASSUME: for a contradiction  $C$  and  $D$  form a separation of  $X$ .

$\langle 1 \rangle 2$ .  $C$  is saturated.

$\langle 2 \rangle 1$ . LET:  $x \in C, y \in X$  with  $p(x) = p(y) = a$ , say

$\langle 2 \rangle 2$ .  $y \notin D$

PROOF: Otherwise  $p^{-1}(a) \cap C$  and  $p^{-1}(a) \cap D$  form a separation of  $p^{-1}(a)$ .

$\langle 2 \rangle 3$ .  $y \in C$

$\langle 1 \rangle 3$ .  $D$  is saturated.

PROOF: Similar.

$\langle 1 \rangle 4$ .  $p(C)$  and  $p(D)$  form a separation of  $Y$ .

$\square$

**Proposition 11.31.22.** *Let  $X$  be a connected space and  $Y$  a connected subspace of  $X$ . Suppose  $A$  and  $B$  form a separation of  $X \setminus Y$ . Then  $Y \cup A$  and  $Y \cup B$  are both connected.*

PROOF:

$\langle 1 \rangle 1$ .  $Y \cup A$  is connected.

$\langle 2 \rangle 1$ . ASSUME: for a contradiction  $C$  and  $D$  form a separation of  $Y \cup A$

$\langle 2 \rangle 2$ . ASSUME: without loss of generality  $Y \subseteq C$

$\langle 2 \rangle 3$ . PICK open sets  $A_1, B_1, C_1, D_1$  in  $X$  with

$$A = A_1 \setminus Y$$

$$B = B_1 \setminus Y$$

$$C = C_1 \cap (Y \cup A)$$

$$D = D_1 \cap (Y \cup A)$$

- ⟨2⟩4.  $B_1 \cup C_1$  and  $A_1 \cap D_1$  form a separation of  $X$   
 ⟨1⟩2.  $Y \cup B$  is connected.

PROOF: Similar.

□

**Theorem 11.31.23.** *Let  $L$  be a linearly ordered set under the order topology. Then  $L$  is connected if and only if  $L$  is a linear continuum.*

PROOF:

- ⟨1⟩1. If  $L$  is a linear continuum then  $L$  is connected.  
 ⟨2⟩1. LET:  $L$  be a linear continuum under the order topology.  
 ⟨2⟩2. ASSUME: for a contradiction  $C$  and  $D$  form a separation of  $L$ .  
 ⟨2⟩3. PICK  $a \in C$  and  $b \in D$ .  
 ⟨2⟩4. ASSUME: without loss of generality  $a < b$ .  
 ⟨2⟩5. LET:  $S = \{x \in L \mid a < x \text{ and } [a, x) \subseteq C\}$   
 ⟨2⟩6.  $S$  is nonempty.  
     PROOF: Since  $a \in C$  and  $C$  is open.  
 ⟨2⟩7.  $S$  is bounded above by  $b$ .  
     PROOF: Since  $b \notin C$ .  
 ⟨2⟩8. LET:  $s = \sup S$   
 ⟨2⟩9.  $s \in S$   
     ⟨3⟩1. LET:  $y \in [a, s)$   
         PROVE:  $y \in C$   
     ⟨3⟩2. PICK  $z$  with  $y < z \in S$   
         PROOF: By minimality of  $s$ .  
     ⟨3⟩3.  $y \in [a, z) \subseteq C$   
 ⟨2⟩10. CASE:  $s \in C$   
     ⟨3⟩1. PICK  $x$  such that  $s < x$  and  $[s, x) \subseteq C$   
         PROOF: Since  $C$  is open and  $s$  is not greatest in  $L$  because  $s < b$ .  
     ⟨3⟩2.  $x \in S$   
         PROOF: Since  $[a, x) = [a, s) \cup [s, x) \subseteq C$ .  
     ⟨3⟩3. Q.E.D.  
     PROOF: This contradicts the fact that  $s$  is an upper bound for  $S$ .  
 ⟨2⟩11. CASE:  $s \in D$   
     ⟨3⟩1. PICK  $x < s$  such that  $(x, s] \subseteq D$   
     ⟨3⟩2. PICK  $y$  with  $x < y < s$   
         PROOF: Since  $L$  is dense.  
     ⟨3⟩3.  $y \in C$   
         PROOF: From ⟨2⟩9.  
     ⟨3⟩4.  $y \in D$   
         PROOF: From ⟨3⟩1.  
     ⟨3⟩5. Q.E.D.  
     ⟨3⟩6. LET:  $L$  be a linear continuum under the order topology.  
     ⟨3⟩7. ASSUME: for a contradiction  $C$  and  $D$  form a separation of  $L$ .  
     ⟨3⟩8. PICK  $a \in C$  and  $b \in D$ .  
     ⟨3⟩9. ASSUME: without loss of generality  $a < b$ .  
     ⟨3⟩10. LET:  $S = \{x \in L \mid a < x \text{ and } [a, x) \subseteq C\}$

PROOF: Since  $a \in C$  and  $C$  is open.

PROOF: Since  $b \notin C$ .

$\langle 3 \rangle 14. s \in S$

PROVE:  $y \in C$

PROOF: By minimality of  $s$ .

⟨3⟩15. CASE:  $s \in C$

PROOF: Since  $C$  is open and  $s$  is not greatest in  $L$  because  $s < b$ .

PROOF: Since  $[a, x) = [a, s) \cup [s, x) \subseteq C$ .

PROOF: This contradicts the fact that  $s$  is an upper bound for  $S$ .

⟨4⟩1. PICK  $x < s$  such that  $(x, s] \subseteq D$

PROOF: Since  $L$  is dense.

PROOF: From  $\langle 2 \rangle 9$ .

PROOF: From  $\langle 3 \rangle 1$ .

PROOF: This contradicts  $\langle 2 \rangle 2$ .

$\langle 2 \rangle 1$ . ASSUME:  $L$  is connected.

⟨3⟩1. LET:  $X$  be a nonempty subset of  $L$  bounded above by  $b$ .

3. LET:  $U$  be the set of upper bounds of  $X$ ,

PROOF: Since  $b \in U$ .

$\langle 4 \rangle 1$ . LET:  $x \in U$

(4)3. Either  $x$  is greatest in  $L$  and  $(y, x] \subset U$ , or there exists  $z > x$  such

⟨3⟩6. LET:  $V$  be the set of lower bounds of  $U$ .

PROOF: Since  $X \subset V$

1. LET:  $x \in V$

⟨4⟩2. PICK  $y \in X$  with  $x < y$

PROOF:  $x$  cannot be an upper bound for  $X$ , because it would be the supremum of  $X$ .

⟨4⟩3. Either  $x$  least in  $L$  and  $[x, y) \subseteq V$ , or there exists  $z < x$  such that  $(z, y) \subseteq V$

⟨3⟩9.  $L = U \cup V$

⟨4⟩1. LET:  $x \in L \setminus U$

⟨4⟩2. PICK  $y \in X$  such that  $x < y$

⟨4⟩3. For all  $u \in U$  we have  $x < y \leq u$

⟨4⟩4.  $x \in V$

⟨3⟩10.  $U \cap V = \emptyset$

PROOF: Any element of  $U \cap V$  would be a supremum of  $X$ .

⟨3⟩11.  $U$  and  $V$  form a separation of  $L$ .

⟨3⟩12. Q.E.D.

PROOF: This contradicts ⟨2⟩1.

⟨2⟩3.  $L$  is dense.

⟨3⟩1. LET:  $x, y \in L$  with  $x < y$

⟨3⟩2. There exists  $z \in L$  such that  $x < z < y$

PROOF: Otherwise  $(-\infty, y)$  and  $(x, +\infty)$  would form a separation of  $L$ .

□

**Corollary 11.31.23.1.** *The real line  $\mathbb{R}$  is connected.*

**Corollary 11.31.23.2.** *Every interval in  $\mathbb{R}$  is connected.*

**Corollary 11.31.23.3.** *The ordered square is connected.*

**Theorem 11.31.24** (Intermediate Value Theorem). *Let  $X$  be a connected space. Let  $Y$  be a linearly ordered set under the order topology. Let  $f : X \rightarrow Y$  be continuous. Let  $a, b \in X$  and  $r \in Y$ . Suppose  $f(a) < r < f(b)$ . Then there exists  $c \in X$  such that  $f(c) = r$ .*

PROOF: Otherwise  $f^{-1}((-\infty, r))$  and  $f^{-1}((r, +\infty))$  would form a separation of  $X$ . □

**Proposition 11.31.25.** *Every function  $f : [0, 1] \rightarrow [0, 1]$  has a fixed point.*

PROOF:

⟨1⟩1. LET:  $g : [0, 1] \rightarrow [-1, 1]$  be the function  $g(x) = f(x) - x$

PROVE: there exists  $x \in [0, 1]$  such that  $g(x) = 0$

⟨1⟩2. ASSUME: without loss of generality  $g(0) \neq 0$  and  $g(1) \neq 0$

⟨1⟩3.  $g(0) > 0$

⟨1⟩4.  $g(1) < 0$

⟨1⟩5. There exists  $x \in (0, 1)$  such that  $g(x) = 0$

PROOF: By the Intermediate Value Theorem.

**Proposition 11.31.26.** *Give  $\mathbb{R}^\omega$  the box topology. Let  $x, y \in \mathbb{R}^\omega$ . Then  $x$  and  $y$  lie in the same component if and only if  $x - y$  is eventually zero, i.e. there exists  $N$  such that, for all  $n \geq N$ , we have  $x_n = y_n$ .*

PROOF:

- ⟨1⟩1. The component containing 0 is the set of sequences that are eventually zero.
- ⟨2⟩1. LET:  $B$  be the set of sequences that are eventually zero.
- ⟨2⟩2.  $B$  is path-connected.
  - ⟨3⟩1. LET:  $x, y \in B$
  - ⟨3⟩2. PICK  $N$  such that, for all  $n \geq N$ , we have  $x_n = y_n = 0$
  - ⟨3⟩3. LET:  $p : [0, 1] \rightarrow \mathbb{R}^\omega$ ,  $p(t) = (1 - t)x + ty$   
 PROVE:  $p$  is continuous.
  - ⟨3⟩4. LET:  $t \in [0, 1]$  and  $\prod_j U_j$  be a basic open neighbourhood of  $p(t)$ ,  
 where each  $U_j$  is open in  $\mathbb{R}$
  - ⟨3⟩5. PICK  $\delta$  such that, for all  $n < N$  and all  $s \in [0, 1]$ , if  $|s - t| < \delta$  then  
 $p(s)_n \in U_n$
  - ⟨3⟩6. For all  $s \in [0, 1]$ , if  $|s - t| < \delta$  then  $p(s) \in \prod_j U_j$
- ⟨2⟩3.  $B$  is connected.  
 PROOF: Proposition 11.33.3.
- ⟨2⟩4. If  $C$  is connected and  $B \subseteq C$  then  $B = C$ .
  - ⟨3⟩1. ASSUME:  $C$  is connected and  $B \subseteq C$
  - ⟨3⟩2. ASSUME: for a contradiction  $x \in C \setminus B$
  - ⟨3⟩3. For  $n \geq 1$ ,  
 LET:  $c_n = 1$  if  $x_n = 0$ ,  $c_n = n/x_n$  otherwise
  - ⟨3⟩4. LET:  $h : \mathbb{R}^\omega \rightarrow \mathbb{R}^\omega$  be the function  $h(x) = (c_n x_n)_{n \geq 1}$
  - ⟨3⟩5.  $h$  is a homeomorphism of  $\mathbb{R}^\omega$  with itself.
  - ⟨3⟩6.  $h(x)$  is unbounded.  
 PROOF: For any  $b > 0$ , pick  $N > b$  such that  $x_N \neq 0$ . Then  $h(x)_N > b$ .
  - ⟨3⟩7.  $h^{-1}(\{\text{bounded sequences}\}) \cap C$  and  $h^{-1}(\{\text{unbounded sequences}\}) \cap C$   
 form a separation of  $C$
  - ⟨3⟩8. Q.E.D.  
 PROOF: This contradicts ⟨3⟩1.
- ⟨1⟩2. Q.E.D.  
 PROOF: Since  $\lambda x.x - y$  is a homeomorphism of  $\mathbb{R}^\omega$  with itself.

□

**Example 11.31.27.** The space  $\mathbb{R}_K$  is connected.

PROOF:

- ⟨1⟩1. ASSUME: for a contradiction  $U$  and  $V$  form a separation of  $\mathbb{R}_K$
- ⟨1⟩2. ASSUME: without loss of generality  $0 \in U$
- ⟨1⟩3. There exists an open interval  $(a, b)$  such that  $(a, b) - K \subseteq U$  and  $(a, b) \not\subseteq U$   
 PROOF: Otherwise  $U$  and  $V$  would form a separation of  $\mathbb{R}$ .
- ⟨1⟩4. PICK  $1/n \in (a, b) - U$
- ⟨1⟩5.  $1/n \in V$
- ⟨1⟩6. There exists an open interval  $(c, d)$  around  $1/n$  that is included in  $V$
- ⟨1⟩7. Q.E.D.  
 PROOF: This is a contradiction since  $(a, b) - K$  and  $(c, d)$  must intersect.

□

## 11.32 Totally Disconnected Spaces

**Definition 11.32.1** (Totally Disconnected). A topological space  $X$  is *totally disconnected* if and only if the only connected subspaces are the singletons.

**Example 11.32.2.** Every discrete space is totally disconnected.

**Example 11.32.3.** The rationals  $\mathbb{Q}$  are totally disconnected.

**Example 11.32.4.** The Cantor set is totally disconnected.

PROOF:

- ⟨1⟩1. LET:  $(A_n)$  be the sequence of sets in Definition 9.1.1.
- ⟨1⟩2. LET:  $C$  be the Cantor set  $\bigcap_n A_n$
- ⟨1⟩3. ASSUME:  
for a contradiction  $D \subseteq C$  is connected and has more than one point.
- ⟨1⟩4. LET:  $x, y \in D$  with  $x < y$
- ⟨1⟩5. PICK  $n$  such that  $|x - y| > 1/3^n$
- ⟨1⟩6.  $A_n$  is a sequence of disjoint intervals of length  $1/3^n$
- ⟨1⟩7.  $x$  and  $y$  are in two different intervals out of the intervals that make up  $A_n$
- ⟨1⟩8. There exists  $z$  with  $x < z < y$  such that  $z \notin A_n$
- ⟨1⟩9.  $(-\infty, z) \cap D$  and  $(z, +\infty) \cap D$  form a separation of  $D$ .

□

## 11.33 Paths and Path Connectedness

**Definition 11.33.1** (Path). Let  $X$  be a topological space and  $a, b \in X$ . A *path* from  $a$  to  $b$  is a continuous function  $p : [0, 1] \rightarrow X$  such that  $p(0) = a$  and  $p(1) = b$ .

**Definition 11.33.2** (Path Connected). A topological space is *path connected* if and only if there exists a path between any two points.

**Proposition 11.33.3.** *Every path connected space is connected.*

PROOF:

- ⟨1⟩1. LET:  $X$  be a path connected space.
- ⟨1⟩2. ASSUME: for a contradiction  $C$  and  $D$  form a separation of  $X$ .
- ⟨1⟩3. PICK  $a \in C$  and  $b \in D$ .
- ⟨1⟩4. PICK a path  $p : [0, 1] \rightarrow X$  from  $a$  to  $b$ .
- ⟨1⟩5.  $p^{-1}(C)$  and  $p^{-1}(D)$  form a separation of  $[0, 1]$ .
- ⟨1⟩6. Q.E.D.

PROOF: This contradicts Corollary 11.31.23.2.

□

An example that shows the converse does not hold:

**Example 11.33.4.** The ordered square is not path connected.

PROOF:

- (1)1. ASSUME: for a contradiction  $p : [0, 1] \rightarrow I_o^2$  is a path from  $(0, 0)$  to  $(1, 1)$ .  
 (1)2.  $p$  is surjective.  
 PROOF: By the Intermediate Value Theorem.  
 (1)3. For  $x \in [0, 1]$ , PICK a rational  $q_x \in p^{-1}((x, 0), (x, 1))$   
 PROOF: Since  $p^{-1}((x, 0), (x, 1))$  is open and nonempty by (1)2.  
 (1)4. For  $x, y \in [0, 1]$ , if  $x \neq y$  then  $q_x \neq q_y$   
 PROOF: We have  $p(q_x) \neq p(q_y)$  because  $((x, 0), (x, 1))$  and  $((y, 0), (y, 1))$  are disjoint.  
 (1)5.  $\{q_x \mid x \in [0, 1]\}$  is an uncountable set of rationals.  
 (1)6. Q.E.D.  
 PROOF: This contradicts the fact that the rationals are countable.

□

**Proposition 11.33.5.** *The continuous image of a path connected space is path connected.*

PROOF:

- (1)1. LET:  $X$  be a path connected space,  $Y$  a topological space, and  $f : X \rightarrow Y$  be continuous and surjective.  
 (1)2. LET:  $a, b \in Y$   
 (1)3. PICK  $c, d \in X$  with  $f(c) = a$  and  $f(d) = b$   
 (1)4. PICK a path  $p : [0, 1] \rightarrow X$  from  $c$  to  $d$ .  
 (1)5.  $f \circ p$  is a path from  $a$  to  $b$  in  $Y$ .

□

**Proposition 11.33.6 (AC).** *The product of a family of path-connected spaces is path-connected.*

PROOF:

- (1)1. LET:  $\{X_\alpha\}_{\alpha \in J}$  be a family of path-connected spaces.  
 (1)2. LET:  $a, b \in \prod_{\alpha \in J} X_\alpha$   
 (1)3. For  $\alpha \in J$ , PICK a path  $p_\alpha : [0, 1] \rightarrow X_\alpha$  from  $a_\alpha$  to  $b_\alpha$   
 PROOF: Using the Axiom of Choice.  
 (1)4. Define  $p : [0, 1] \rightarrow \prod_{\alpha \in J} X_\alpha$  by  $p(t)_\alpha = p_\alpha(t)$   
 (1)5.  $p$  is a path from  $a$  to  $b$ .

PROOF: Theorem 11.17.12.

□

**Proposition 11.33.7.** *The continuous image of a path-connected space is path-connected.*

PROOF:

- (1)1. LET:  $f : X \rightarrow Y$  be continuous and surjective where  $X$  is path-connected.  
 (1)2. LET:  $a, b \in Y$   
 (1)3. PICK  $a', b' \in X$  with  $f(a') = a$  and  $f(b') = b$ .  
 (1)4. PICK a path  $p : [0, 1] \rightarrow X$  from  $a'$  to  $b'$ .  
 (1)5.  $f \circ p$  is a path from  $a$  to  $b$ .

□

**Proposition 11.33.8.** *Let  $X$  be a topological space. The union of a set of path-connected subspaces of  $X$  that have a point in common is path-connected.*

PROOF:

- ⟨1⟩1. LET:  $\mathcal{A}$  be a set of path-connected subspaces of  $X$  with the point  $a$  in common.
  - ⟨1⟩2. LET:  $b, c \in \bigcup \mathcal{A}$
  - ⟨1⟩3. PICK  $B, C \in \mathcal{A}$  with  $b \in B$  and  $c \in C$ .
  - ⟨1⟩4. PICK a path  $p$  in  $B$  from  $b$  to  $a$ .
  - ⟨1⟩5. PICK a path  $q$  in  $C$  from  $a$  to  $c$ .
  - ⟨1⟩6. The concatenation of  $p$  and  $q$  is a path from  $b$  to  $c$  in  $\bigcup \mathcal{A}$ .
- 

**Proposition 11.33.9.** *Let  $A \subseteq \mathbb{R}^2$  be countable. Then  $\mathbb{R}^2 \setminus A$  is path-connected.*

PROOF:

- ⟨1⟩1. LET:  $a, b \in \mathbb{R}^2 \setminus A$
  - ⟨1⟩2. PICK a line  $l$  in  $\mathbb{R}^2$  with  $a$  on one side and  $b$  on the other.
  - ⟨1⟩3. For every point  $x$  on  $l$ ,  
LET:  $p_x$  be the path in  $\mathbb{R}^2$  consisting of a line from  $a$  to  $x$  then a line from  $x$  to  $b$
  - ⟨1⟩4. For  $x \neq y$  we have  $p_x$  and  $p_y$  have no points in common except  $a$  and  $b$
  - ⟨1⟩5. There are only countably many  $x$  such that a point of  $A$  lies on  $p_x$ .
  - ⟨1⟩6. There exists  $x$  such that  $p_x$  is a path from  $a$  to  $b$  in  $\mathbb{R}^2 \setminus A$ .
- 

**Proposition 11.33.10.** *Every open connected subspace of  $\mathbb{R}^2$  is path-connected.*

PROOF:

- ⟨1⟩1. LET:  $U$  be an open connected subspace of  $\mathbb{R}^2$ .
- ⟨1⟩2. For all  $x_0 \in U$ ,  
LET:  $PC(x_0) = \{y \in U \mid \text{there exists a path from } x_0 \text{ to } y\}$
- ⟨1⟩3. For all  $x_0 \in U$ , the set  $PC(x_0)$  is open and closed in  $U$ .
  - ⟨2⟩1. LET:  $x_0 \in U$
  - ⟨2⟩2.  $PC(x_0)$  is open in  $U$ 
    - ⟨3⟩1. LET:  $y \in PC(x_0)$
    - ⟨3⟩2. PICK  $\epsilon > 0$  such that  $B(y, \epsilon) \subseteq U$   
PROOF: Since  $U$  is open.
    - ⟨3⟩3.  $B(y, \epsilon) \subseteq PC(x_0)$   
PROOF: For all  $z \in B(y, \epsilon)$ , pick a path from  $x_0$  to  $y$  then concatenate the straight line from  $y$  to  $z$ .
- ⟨2⟩3.  $PC(x_0)$  is closed in  $U$ 
  - ⟨3⟩1. LET:  $y \in U$  be a limit point of  $PC(x_0)$
  - ⟨3⟩2. PICK  $\epsilon > 0$  such that  $B(y, \epsilon) \subseteq U$
  - ⟨3⟩3. PICK  $z \in PC(x_0) \cap B(y, \epsilon)$
  - ⟨3⟩4.  $y \in PC(x_0)$   
PROOF: Pick a path from  $x_0$  to  $z$  then concatenate the straight line from  $z$  to  $y$ .



⟨1⟩4.  $PC(x_0) = U$

PROOF: Proposition 11.31.3.

□

**Example 11.33.11.** If  $A$  is a connected subspace of  $X$ , then  $A^\circ$  is not necessarily connected.

Take two closed circles in  $\mathbb{R}^2$  that touch at one point. The interior of this space is two open circles, and these two circles form a separation.

**Example 11.33.12.** If  $A$  is a connected subspace of  $X$  then  $\partial A$  is not necessarily connected.

We have  $[0, 1]$  is connected but  $\partial[0, 1] = \{0, 1\}$  is not.

**Example 11.33.13.** If  $A$  is a subspace of  $X$  and  $A^\circ$  and  $\partial A$  are connected, then  $A$  is not necessarily connected.

We have  $\mathbb{Q}^\circ = \emptyset$  and  $\partial\mathbb{Q} = \mathbb{R}$  are connected but  $\mathbb{Q}$  is not connected.

## 11.34 The Topologist's Sine Curve

**Definition 11.34.1** (The Topologist's Sine Curve). Let  $S = \{(x, \sin 1/x) \mid 0 < x \leq 1\}$ , The *topologist's sine curve* is the closure  $\bar{S}$  of  $S$  in  $\mathbb{R}^2$ .

**Proposition 11.34.2.** *The topologist's sine curve is connected.*

PROOF:

⟨1⟩1. LET:  $S = \{(x, \sin 1/x) \mid 0 < x \leq 1\}$

⟨1⟩2.  $S$  is connected.

PROOF: Theorem 11.31.13.

⟨1⟩3.  $\bar{S}$  is connected.

PROOF: Theorem 11.31.12.

□

**Proposition 11.34.3.** *The topologist's sine curve is  $\{(x, \sin 1/x) \mid 0 < x \leq 1\} \cup (\{0\} \times [-1, 1])$ .*

PROOF: Sketch proof: Given a point  $(0, y)$  with  $-1 \leq y \leq 1$ , pick  $a$  such that  $\sin a = y$ . Then  $((1/a, y), (1/(a + 2\pi), y), (1/(a + 4\pi), y), \dots)$  is a sequence in  $S$  that converges to  $(0, y)$ .

Conversely, let  $(x, y)$  be any point not in  $S \cup (\{0\} \times [-1, 1])$ . If  $x < 0$  or  $y > 1$  or  $y < -1$  then we can easily find a neighbourhood that does not intersect  $S \cup (\{0\} \times [-1, 1])$ . If  $x > 0$  and  $-1 \leq y \leq 1$ , then we have  $y \neq \sin 1/x$ . Hence pick a neighbourhood that does not intersect  $S$ .

**Proposition 11.34.4.** *Every closed subset of  $\mathbb{R}$  that is bounded above has a greatest element.*

PROOF: It has a supremum, which is a limit point of the set and hence an element. □

**Proposition 11.34.5 (CC).** *The topologist's sine curve is not path connected.*

PROOF:

(1)1. ASSUME: For a contradiction  $p : [0, 1] \rightarrow \bar{S}$  is a path from  $(0, 0)$  to  $(1, \sin 1)$ .

(1)2.  $\{t \in [0, 1] \mid p(t) \in \{0\} \times [-1, 1]\}$  is closed.

PROOF: Since  $p$  is continuous and  $\{0\} \times [-1, 1]$  is closed.

(1)3. LET:  $b$  be the largest number in  $[0, 1]$  such that  $p(b) \in \{0\} \times [-1, 1]$ .

PROOF: Proposition 11.34.4.

(1)4. LET:  $x : [b, 1] \rightarrow \bar{S}$  be the function  $\pi_1 \circ p$

(1)5. LET:  $y : [b, 1] \rightarrow \bar{S}$  be the function  $\pi_2 \circ p$

(1)6. PICK a sequence  $t_n$  in  $(b, 1]$  such that  $t_n \rightarrow b$  and  $y(t_n) = (-1)^n$  for all  $n$

(2)1. LET:  $n \geq 1$

(2)2. PICK  $u$  with  $0 < u < x(1/n)$  and  $\sin(1/u) = (-1)^n$

(2)3. PICK  $t_n$  with  $b < t_n < 1/n$  and  $x(t_n) = u$

PROOF: By the Intermediate Value Theorem

(1)7. Q.E.D.

PROOF: This contradicts Proposition 11.12.17 since  $y$  is continuous and  $y(t_n)$  does not converge.

□

**Corollary 11.34.5.1.** *The closure of a path-connected subspace of a space is not necessarily path-connected.*

## 11.35 The Long Line

**Definition 11.35.1** (The Long Line). The *long line* is the space  $\omega_1 \times [0, 1)$  in the dictionary order under the order topology, where  $\omega_1$  is the first uncountable ordinal.

**Lemma 11.35.2.** *For any ordinal  $\alpha$  with  $0 < \alpha < \omega_1$  we have  $[(0, 0), (\alpha, 0)) \cong [0, 1)$*

(1)1.  $[(0, 0), (1, 0)) \cong [0, 1)$

PROOF: The map  $\pi_2$  is a homeomorphism.

(1)2. If  $[(0, 0), (\alpha, 0)) \cong [0, 1)$  then  $[(0, 0), (\alpha + 1, 0)) \cong [0, 1)$

PROOF: Proposition 5.3.11.

(1)3. If  $\lambda$  is a limit ordinal with  $\lambda < \omega_1$  and  $[(0, 0), (\alpha, 0)) \cong [0, 1)$  for all  $\alpha$  with  $0 < \alpha < \lambda$  then  $[(0, 0), (\lambda, 0)) \cong [0, 1)$

(2)1. LET:  $\lambda$  be a limit ordinal  $< \omega_1$

(2)2. ASSUME:  $[(0, 0), (\alpha, 0)) \cong [0, 1)$  for all  $\alpha$  with  $0 < \alpha < \lambda$

(2)3. PICK a sequence of ordinals  $\alpha_0 < \alpha_1 < \dots$  with limit  $\lambda$

PROOF: Since  $\lambda$  is countable.

(2)4.  $[(\alpha_i, 0), (\alpha_{i+1}, 0)) \cong [0, 1)$  for all  $i$

PROOF: Lemma 5.3.10.

(2)5. Q.E.D.

PROOF: By Proposition 5.3.12.

(1)4. Q.E.D.

PROOF: By transfinite induction.

**Proposition 11.35.3 (CC).** *The long line is path-connected.*

PROOF:

⟨1⟩1. LET:  $(\alpha, i), (\beta, j) \in \omega_1 \times [0, 1)$

⟨1⟩2. ASSUME: without loss of generality  $(\alpha, i) < (\beta, j)$

⟨1⟩3.  $[(0, 0), (\beta + 1, 0)) \cong [0, 1)$

PROOF: By Lemma 11.35.2

⟨1⟩4.  $[(\alpha, i), (\beta, j)) \cong [0, 1)$

PROOF: Lemma 5.3.10.

⟨1⟩5. PICK a homeomorphism  $q : [0, 1) \rightarrow [(\alpha, i), (\beta, j))$

⟨1⟩6.  $q \cup \{(1, (\beta, j))\}$  is a path from  $(\alpha, i)$  to  $(\beta, j)$

□

**Proposition 11.35.4.** *Every point in the long line has a neighbourhood homeomorphic to an interval in  $\mathbb{R}$ .*

PROOF: For any  $(\alpha, i)$  in the long line, the neighbourhood  $[(0, 0), (\alpha + 1, 0))$  satisfies the condition by Lemma 11.35.2.

## 11.36 Components

**Proposition 11.36.1.** *Let  $X$  be a topological space. Define the relation  $\sim$  on  $X$  by  $a \sim b$  if and only if there exists a connected subspace  $A \subseteq X$  with  $a, b \in A$ . Then  $\sim$  is an equivalence relation on  $X$ .*

PROOF:

⟨1⟩1.  $\sim$  is reflexive.

PROOF: For any  $a \in X$  we have  $\{a\}$  is a connected subspace that contains  $a$ .

⟨1⟩2.  $\sim$  is symmetric.

PROOF: Trivial.

⟨1⟩3.  $\sim$  is transitive.

⟨2⟩1. LET:  $a, b, c \in X$

⟨2⟩2. ASSUME:  $a \sim b$  and  $b \sim c$

⟨2⟩3. PICK connected subspaces  $A$  and  $B$  with  $a, b \in A$  and  $b, c \in B$

⟨2⟩4.  $A \cup B$  is a connected subspace that contains  $a$  and  $c$

PROOF: Theorem 11.31.11.

□

**Definition 11.36.2** ((Connected) Component). Let  $X$  be a topological space. The *(connected) components* of  $X$  are the equivalence classes under the above  $\sim$ .

**Lemma 11.36.3.** *Let  $X$  be a topological space. If  $A \subseteq X$  is connected and nonempty then there exists a unique component  $C$  of  $X$  such that  $A \subseteq C$ .*

PROOF:

- ⟨1⟩1. PICK  $a \in A$
- ⟨1⟩2. LET:  $C$  be the  $\sim$ -equivalence class of  $a$ .
- ⟨1⟩3.  $A \subseteq C$   
PROOF: For all  $x \in A$  we have  $x \sim a$ .
- ⟨1⟩4. If  $C'$  is a component and  $A \subseteq C'$  then  $C = C'$   
PROOF: Since we have  $a \in C'$ .

□

**Theorem 11.36.4.** *Let  $X$  be a topological space. The components of  $X$  are connected disjoint subspaces of  $X$  whose union is  $X$  such that each nonempty connected subspace of  $X$  intersects only one of them.*

PROOF:

- ⟨1⟩1. Every component of  $X$  is connected.  
PROOF: For  $a \in X$ , the  $\sim$ -equivalence class of  $a$  is  $\bigcup\{A \subseteq X \mid A \text{ is connected, } a \in A\}$  which is connected by Theorem 11.31.11.
- ⟨1⟩2. The components form a partition of  $X$ .  
PROOF: Immediate from the definition.
- ⟨1⟩3. Every nonempty connected subspace of  $X$  intersects a unique component of  $X$ .  
  - ⟨2⟩1. LET:  $A \subseteq X$  be connected and nonempty.
  - ⟨2⟩2. LET:  $C$  be the component such that  $A \subseteq C$   
PROOF: Lemma 11.36.3.
  - ⟨2⟩3.  $A$  intersects  $C$
  - ⟨2⟩4. If  $A$  intersects the component  $C'$  then  $C' = C$ 
    - ⟨3⟩1. LET:  $C'$  be a component that intersects  $A$
    - ⟨3⟩2. PICK  $b \in A \cap C'$
    - ⟨3⟩3.  $A \subseteq C'$   
PROOF: For all  $x \in A$  we have  $x \sim b$ .
    - ⟨3⟩4.  $C = C'$   
PROOF: By uniqueness in ⟨2⟩2.

□

**Proposition 11.36.5.** *Every component of a space is closed.*

PROOF:

- ⟨1⟩1. LET:  $X$  be a topological space and  $C$  a component of  $X$ .
- ⟨1⟩2.  $\overline{C}$  is connected.  
PROOF: Theorem 11.31.12.
- ⟨1⟩3.  $C = \overline{C}$   
PROOF: Lemma 11.31.10.
- ⟨1⟩4.  $C$  is closed.  
PROOF: Lemma 11.4.5.

□

**Proposition 11.36.6.** *If a topological space has finitely many components then every component is open.*

PROOF: Each component is the complement of a finite union of closed sets. □

## 11.37 Path Components

**Proposition 11.37.1.** *Let  $X$  be a topological space. Define the relation  $\sim$  on  $X$  by:  $a \sim b$  if and only if there exists a path in  $X$  from  $a$  to  $b$ . Then  $\sim$  is an equivalence relation on  $X$ .*

PROOF:

(1)1.  $\sim$  is reflexive.

PROOF: For  $a \in X$ , the constant function  $[0, 1] \rightarrow X$  with value  $a$  is a path from  $a$  to  $a$ .

(1)2.  $\sim$  is symmetric.

PROOF: If  $p : [0, 1] \rightarrow X$  is a path from  $a$  to  $b$ , then  $\lambda t.p(1 - t)$  is a path from  $b$  to  $a$ .

(1)3.  $\sim$  is transitive.

PROOF: Concatenate paths.

□

**Definition 11.37.2** (Path Component). Let  $X$  be a topological space. The *path components* of  $X$  are the equivalence relations under  $\sim$ .

**Theorem 11.37.3.** *The path components of  $X$  are path-connected disjoint subspaces of  $X$  whose union is  $X$  such that every nonempty path-connected subspace of  $X$  intersects exactly one path component.*

PROOF:

(1)1. Every path component is path-connected.

PROOF: If  $a$  and  $b$  are in the same path component then  $a \sim b$ , i.e. there exists a path from  $a$  to  $b$ .

(1)2. The path components are disjoint and their union is  $X$ .

PROOF: Immediate from the definition.

(1)3. Every non-empty path-connected subspace of  $X$  intersects exactly one path component.

(2)1. LET:  $A$  be a nonempty path-connected subspace of  $X$ .

(2)2. PICK  $a \in A$

(2)3.  $A$  intersects the  $\sim$ -equivalence class of  $a$ .

(2)4. LET:  $C$  be any path component that intersects  $A$ .

(2)5. PICK  $b \in A \cap C$

(2)6.  $a \sim b$

PROOF: Since  $A$  is path-connected.

(2)7.  $C$  is the  $\sim$ -equivalence class of  $a$ .

□

**Proposition 11.37.4.** *Every path component is included in a component.*

PROOF:

(1)1. LET:  $X$  be a topological space and  $C$  a path component of  $X$ .

(1)2.  $C$  is path-connected.

PROOF: Theorem 11.37.3.  
 ⟨1⟩3.  $C$  is connected.  
 PROOF: Proposition 11.33.3.  
 ⟨1⟩4.  $C$  is included in a component.  
 PROOF: Lemma 11.36.3.  
 □

## 11.38 Local Connectedness

**Definition 11.38.1** (Locally Connected). Let  $X$  be a topological space and  $a \in X$ . Then  $X$  is *locally connected* at  $a$  if and only if every neighbourhood of  $a$  includes a connected neighbourhood of  $a$ .

The space  $X$  is *locally connected* if and only if it is locally connected at every point.

**Example 11.38.2.** The real line is both connected and locally connected.

**Example 11.38.3.** The space  $\mathbb{R} \setminus \{0\}$  is disconnected but locally connected.

**Example 11.38.4.** The topologist's sine curve is connected but not locally connected.

**Example 11.38.5.** The rationals  $\mathbb{Q}$  are neither connected nor locally connected.

**Theorem 11.38.6.** *A topological space  $X$  is locally connected if and only if, for every open set  $U$  in  $X$ , every component of  $U$  is open in  $X$ .*

PROOF:

- ⟨1⟩1. If  $X$  is locally connected then, for every open set  $U$  in  $X$ , every component of  $U$  is open in  $X$ .
- ⟨2⟩1. ASSUME:  $X$  is locally connected.
- ⟨2⟩2. LET:  $U$  be open in  $X$ .
- ⟨2⟩3. LET:  $C$  be a component of  $U$ .
- ⟨2⟩4. LET:  $a \in C$
- ⟨2⟩5. LET:  $V$  be a connected neighbourhood of  $a$  such that  $V \subseteq U$
- ⟨2⟩6.  $V \subseteq C$   
 PROOF: Lemma 11.36.3.
- ⟨2⟩7. Q.E.D.  
 PROOF: Lemma 11.1.8.
- ⟨1⟩2. If, for every open set  $U$  in  $X$ , every component of  $U$  is open in  $X$ , then  $X$  is locally connected.
- ⟨2⟩1. ASSUME: for every open set  $U$  in  $X$ , every component of  $U$  is open in  $X$ .
- ⟨2⟩2. LET:  $a \in X$
- ⟨2⟩3. LET:  $U$  be a neighbourhood of  $a$
- ⟨2⟩4. The component of  $U$  that contains  $a$  is a connected neighbourhood of  $a$  included in  $U$ .

□

**Example 11.38.7.** The ordered square is locally connected.

PROOF: Every neighbourhood of a point includes an interval around that point, which is connected by Theorem 11.31.23.

**Example 11.38.8.** Let  $X$  be the set of all rational points on the line segment  $[0, 1] \times \{0\}$ , and  $Y$  the set of all rational points on the line segment  $[0, 1] \times \{1\}$ . Let  $A$  be the space consisting of all line segments joining the point  $(0, 1)$  to a point of  $X$ , and all line segments joining the point  $(1, 0)$  to a point of  $Y$ . Then  $A$  is path-connected but is not locally connected at any point,

**Proposition 11.38.9.** Let  $X$  and  $Y$  be topological spaces and  $p : X \rightarrow Y$  be a quotient map. If  $X$  is locally connected then so is  $Y$ .

PROOF:

⟨1⟩1. LET:  $U$  be an open set in  $Y$ .

⟨1⟩2. LET:  $C$  be a component of  $U$ .

⟨1⟩3.  $p^{-1}(C)$  is a union of components of  $p^{-1}(U)$

⟨2⟩1. LET:  $x \in p^{-1}(C)$

⟨2⟩2. LET:  $D$  be the component of  $p^{-1}(U)$  that contains  $x$ .

⟨2⟩3.  $p(D)$  is connected.

PROOF: Theorem 11.31.13.

⟨2⟩4.  $p(D) \subseteq C$ .

PROOF: From ⟨1⟩2 since  $p(x) \in p(D) \cap C$  (⟨2⟩1, ⟨2⟩2).

⟨2⟩5.  $D \subseteq p^{-1}(C)$

⟨1⟩4.  $p^{-1}(C)$  is open in  $p^{-1}(U)$

PROOF: Theorem 11.38.6.

⟨1⟩5.  $C$  is open in  $U$

PROOF: Since the restriction of  $p$  to  $p : p^{-1}(U) \rightarrow U$  is a quotient map by Proposition 11.25.4.

⟨1⟩6. Q.E.D.

PROOF: Theorem 11.38.6.

□

## 11.39 Local Path Connectedness

**Definition 11.39.1** (Locally Path-Connected). Let  $X$  be a topological space and  $a \in X$ . Then  $X$  is *locally path-connected* at  $a$  if and only if every neighbourhood of  $a$  includes a path-connected neighbourhood of  $a$ .

The space  $X$  is *locally path-connected* if and only if it is locally path-connected at every point.

**Theorem 11.39.2.** A topological space  $X$  is locally path-connected if and only if, for every open set  $U$  in  $X$ , every path component of  $U$  is open in  $X$ .

PROOF:

- ⟨1⟩1. If  $X$  is locally path-connected then, for every open set  $U$  in  $X$ , every path component of  $U$  is open in  $X$ .
- ⟨2⟩1. ASSUME:  $X$  is locally path-connected.
- ⟨2⟩2. LET:  $U$  be open in  $X$ .
- ⟨2⟩3. LET:  $C$  be a path component of  $U$ .
- ⟨2⟩4. LET:  $a \in C$
- ⟨2⟩5. LET:  $V$  be a path-connected neighbourhood of  $a$  such that  $V \subseteq U$
- ⟨2⟩6.  $V \subseteq C$   
PROOF: Lemma 11.36.3.
- ⟨2⟩7. Q.E.D.  
PROOF: Lemma 11.1.8.
- ⟨1⟩2. If, for every open set  $U$  in  $X$ , every component of  $U$  is open in  $X$ , then  $X$  is locally connected.
- ⟨2⟩1. ASSUME: for every open set  $U$  in  $X$ , every component of  $U$  is open in  $X$ .
- ⟨2⟩2. LET:  $a \in X$
- ⟨2⟩3. LET:  $U$  be a neighbourhood of  $a$
- ⟨2⟩4. The component of  $U$  that contains  $a$  is a connected neighbourhood of  $a$  included in  $U$ .

□

**Theorem 11.39.3.** *If a space is locally path connected then its components and its path components are the same.*

PROOF:

- ⟨1⟩1. LET:  $X$  be a locally path connected space.
- ⟨1⟩2. LET:  $C$  be a component of  $X$ .
- ⟨1⟩3. LET:  $x \in C$
- ⟨1⟩4. LET:  $P$  be the path component of  $x$   
PROVE:  $P = C$
- ⟨1⟩5.  $P \subseteq C$   
PROOF: Proposition 11.37.4.
- ⟨1⟩6. LET:  $Q$  be the union of the other path components included in  $C$
- ⟨1⟩7.  $C = P \cup Q$   
PROOF: Proposition 11.37.4.
- ⟨1⟩8.  $P$  and  $Q$  are open in  $C$
- ⟨2⟩1.  $C$  is open.  
PROOF: Theorem 11.38.6.
- ⟨2⟩2. Q.E.D.  
PROOF: Theorem 11.39.2.
- ⟨1⟩9.  $Q = \emptyset$   
PROOF: Otherwise  $P$  and  $Q$  would form a separation of  $C$ .

□

**Example 11.39.4.** The ordered square is not locally path connected, since it is connected but not path connected.



**Proposition 11.39.5.** *Let  $X$  be a locally path-connected space. Then every connected open subspace of  $X$  is path-connected.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $U$  be a connected open subspace of  $X$ .
- $\langle 1 \rangle 2$ . LET:  $P$  be a path component of  $U$ .
- $\langle 1 \rangle 3$ . LET:  $Q$  be the union of the other path components of  $U$ .
- $\langle 1 \rangle 4$ .  $P$  and  $Q$  are open in  $U$ .

PROOF: Theorem 11.39.2.

- $\langle 1 \rangle 5$ .  $Q = \emptyset$

PROOF: Otherwise  $P$  and  $Q$  form a separation of  $U$ .

□

## 11.40 Weak Local Connectedness

**Definition 11.40.1** (Weakly Locally Connected). Let  $X$  be a topological space and  $a \in X$ . Then  $X$  is *weakly locally connected* at  $a$  if and only if every neighbourhood of  $a$  includes a connected subspace that includes a neighbourhood of  $a$ .

**Proposition 11.40.2.** *Let  $X$  be a topological space. If  $X$  is weakly locally connected at every point then  $X$  is locally connected.*

PROOF:

- $\langle 1 \rangle 1$ . ASSUME:  $X$  is weakly locally connected at every point.
- $\langle 1 \rangle 2$ . LET:  $U$  be open in  $X$ .
- $\langle 1 \rangle 3$ . LET:  $C$  be a component of  $U$ .
- $\langle 1 \rangle 4$ .  $C$  is open in  $X$ .
- $\langle 2 \rangle 1$ . LET:  $x \in C$
- $\langle 2 \rangle 2$ . PICK a connected subspace  $D$  of  $U$  that includes a neighbourhood  $V$  of  $x$ .
- $\langle 2 \rangle 3$ .  $D \subseteq C$
- PROOF: Lemma 11.36.3.
- $\langle 2 \rangle 4$ .  $x \in V \subseteq C$
- $\langle 2 \rangle 5$ . Q.E.D.

PROOF: Lemma 11.1.8.

- $\langle 1 \rangle 5$ . Q.E.D.

PROOF: Theorem 11.38.6.

□

**Example 11.40.3.** The 'infinite broom' (Munkres p.163) is an example of a space that is weakly locally connected at a point  $p$  but not locally connected at  $p$ .

## 11.41 Quasicomponents

**Proposition 11.41.1.** *Let  $X$  be a topological space. Define  $\sim$  on  $X$  by  $x \sim y$  if and only if there exists no separation  $U$  and  $V$  of  $X$  such that  $x \in U$  and  $y \in V$ . Then  $\sim$  is an equivalence relation on  $X$ .*

PROOF:

$\langle 1 \rangle 1.$   $\sim$  is reflexive.

PROOF: Since the two sets that make up a separation are disjoint.

$\langle 1 \rangle 2.$   $\sim$  is symmetric.

PROOF: Immediate from the definition.

$\langle 1 \rangle 3.$   $\sim$  is transitive.

$\langle 2 \rangle 1.$  ASSUME:  $x \sim y$  and  $y \sim z$

$\langle 2 \rangle 2.$  ASSUME: for a contradiction there is a separation  $U$  and  $V$  of  $X$  with  
 $x \in U$  and  $z \in V$

$\langle 2 \rangle 3.$   $y \in U$  or  $y \in V$

$\langle 2 \rangle 4.$  Q.E.D.

PROOF: Either case contradicts  $\langle 2 \rangle 1.$

□

**Definition 11.41.2** (Quasicomponents). For  $X$  a topological space, the *quasi-components* of  $X$  are the equivalence classes under  $\sim$ .

**Proposition 11.41.3.** *Let  $X$  be a topological space. Then every component of  $X$  is included in a quasicomponent of  $X$ .*

PROOF:

$\langle 1 \rangle 1.$  LET:  $C$  be a component of  $X$ .

$\langle 1 \rangle 2.$  LET:  $x, y \in C$

PROVE:  $x \sim y$

$\langle 1 \rangle 3.$  ASSUME: for a contradiction there exists a separation  $U$  and  $V$  of  $X$  with  
 $x \in U$  and  $y \in V$

$\langle 1 \rangle 4.$   $C \cap U$  and  $C \cap V$  form a separation of  $C$ .

$\langle 1 \rangle 5.$  Q.E.D.

PROOF: This contradicts  $\langle 1 \rangle 1.$

**Proposition 11.41.4.** *In a locally connected space, the components and the quasicomponents are the same.*

PROOF:

$\langle 1 \rangle 1.$  LET:  $X$  be a locally connected space and  $Q$  a quasicomponent of  $X$ .

$\langle 1 \rangle 2.$  PICK a component  $C$  of  $X$  such that  $C \subseteq Q$

$\langle 1 \rangle 3.$  LET:  $D$  be the union of the components of  $X$

$\langle 1 \rangle 4.$   $C$  and  $D$  are open in  $X$ .

PROOF: Theorem 11.38.6.

$\langle 1 \rangle 5.$   $D$  cannot contain any points of  $Q$ .

PROOF: If it did, then  $C$  and  $D$  would form a separation of  $X$  and there would be points  $x, y \in Q$  with  $x \in C$  and  $y \in D$ .

$\langle 1 \rangle 6.$   $C = Q$

□

## 11.42 Open Coverings

**Definition 11.42.1** (Open Covering). Let  $X$  be a topological space. An *open covering* of  $X$  is a covering of  $X$  whose elements are all open sets.

## 11.43 Lindelöf Spaces

**Definition 11.43.1** (Lindelöf Space). A topological space  $X$  is *Lindelöf* if and only if every open covering has a countable subcovering.

**Proposition 11.43.2.** *Let  $X$  be a topological space. Then  $X$  is Lindelöf if and only if every set of closed sets that has the countable intersection property has nonempty intersection.*

PROOF: The following are equivalent.

1.  $X$  is compact.
2. Every open covering of  $X$  has a countable subcovering.
3. For any set  $\mathcal{C}$  of closed sets, if  $\{X \setminus C \mid C \in \mathcal{C}\}$  covers  $X$  then there is a countable subset  $\mathcal{C}_0$  such that  $\{X \setminus C \mid C \in \mathcal{C}_0\}$  covers  $X$ .
4. For any set  $\mathcal{C}$  of closed sets, if  $\bigcap \mathcal{C} = \emptyset$  then there is a countable subset  $\mathcal{C}_0$  with empty intersection.
5. Any set of closed sets with the countable intersection property has nonempty intersection.

□

**Proposition 11.43.3** (CC). *Let  $X$  be a topological space and  $\mathcal{B}$  a basis for the topology on  $X$ . Then the following are equivalent.*

1.  $X$  is Lindelöf.
2. Every open covering of  $X$  by elements of  $\mathcal{B}$  has a countable subcovering.

PROOF:

(1)1.  $1 \Rightarrow 2$

PROOF: Immediate from definitions.

(1)2.  $2 \Rightarrow 1$

(2)1. ASSUME: Every open covering of  $X$  by elements of  $\mathcal{B}$  has a countable subcovering.

(2)2. LET:  $\mathcal{U}$  be an open covering of  $X$ .

(2)3.  $\{B \in \mathcal{B} \mid \exists U \in \mathcal{U}. B \subseteq U\}$  covers  $X$ .

(2)4. PICK a finite subcovering  $\mathcal{B}_0$ .

(2)5. For  $B \in \mathcal{B}_0$ , PICK  $U_B \in \mathcal{U}$  such that  $B \subseteq U_B$ .

(2)6.  $\{U_B \mid B \in \mathcal{B}_0\}$  covers  $X$ .

□

**Example 11.43.4** (AC). The space  $\overline{S_\Omega}$  is Lindelöf.

PROOF:

- ⟨1⟩1. LET:  $\mathcal{A}$  be any open cover of  $\overline{S_\Omega}$
- ⟨1⟩2. PICK  $U \in \mathcal{A}$  such that  $\Omega \in U$
- ⟨1⟩3. PICK  $\alpha < \Omega$  such that  $(\alpha, \Omega] \subseteq U$
- ⟨1⟩4. For  $\beta < \alpha$ , PICK  $U_\beta \in \mathcal{A}$  such that  $\beta \in U_\beta$
- ⟨1⟩5.  $\{U_\beta \mid \beta < \alpha\} \cup \{U\}$  covers  $\overline{S_\Omega}$

□

**Proposition 11.43.5.** Every closed subspace of a Lindelöf space is Lindelöf.

PROOF:

- ⟨1⟩1. LET:  $X$  be a Lindelöf space.
- ⟨1⟩2. LET:  $Y \subseteq X$  be closed.
- ⟨1⟩3. LET:  $\mathcal{A}$  be an open covering of  $Y$ .
- ⟨1⟩4. LET:  $\mathcal{B} = \{U \text{ open in } X \mid U \cap Y \in \mathcal{A}\} \cup \{X - Y\}$
- ⟨1⟩5.  $\mathcal{B}$  is an open covering of  $X$ .
- ⟨1⟩6. PICK a countable subcovering  $\mathcal{B}_0$
- ⟨1⟩7.  $\{U \cap Y \mid U \in \mathcal{B}_0\} - \{\emptyset\}$  is a countable subset of  $\mathcal{A}$  that covers  $Y$ .

□

The following examples show that an open subspace of a Lindelöf space is not necessarily Lindelöf.

**Example 11.43.6.** The space  $S_\Omega$  is not Lindelöf, because the open cover  $\{[0, x) \mid x \in S_\Omega\}$  has no countable subcover.

**Example 11.43.7.** The set  $[0, 1] \times (0, 1)$  as a subspace of the ordered square is not Lindelöf.

The open cover  $\{\{x\} \times (0, 1) \mid x \in [0, 1]\}$  has no countable subcover.

**Proposition 11.43.8** (Choice). The continuous image of a Lindelöf space is Lindelöf.

PROOF:

- ⟨1⟩1. LET:  $X$  be a Lindelöf space.
- ⟨1⟩2. LET:  $Y$  be a topological space.
- ⟨1⟩3. LET:  $f : X \rightarrow Y$  be continuous and surjective.
- ⟨1⟩4. LET:  $\mathcal{A}$  be an open cover of  $Y$ .
- ⟨1⟩5.  $\{f^{-1}(V) \mid V \in \mathcal{A}\}$  is an open cover of  $X$ .
- ⟨1⟩6. PICK a countable subcover  $\mathcal{B}$
- ⟨1⟩7. For  $U \in \mathcal{B}$ , PICK  $V_U \in \mathcal{A}$  such that  $U = f^{-1}(V_U)$
- ⟨1⟩8.  $\{V_U \mid U \in \mathcal{B}\}$  covers  $Y$ .

□

The following is an example of a Lindelöf space that is not first countable.

**Example 11.43.9** (Choice). Let  $\mathbb{Q}^\infty$  be the set of all sequences of rationals that end in an infinite sequence of 0s. Then  $\mathbb{Q}^\infty$  as a subspace of  $\mathbb{R}^\omega$  under the box topology is Lindelöf, because it is countable.

## 11.44 Separable Spaces

**Definition 11.44.1** (Separable). A topological space is *separable* if and only if it has a countable dense subset.

**Proposition 11.44.2** (AC). *A countable product of separable spaces is separable.*

PROOF:

- ⟨1⟩1. LET:  $(X_n)$  be a sequence of separable spaces.
- ⟨1⟩2. For each  $n$ , PICK a countable dense set  $D_n$  in  $X_n$   
       PROVE:  $\prod_n D_n$  is dense in  $\prod_n X_n$
- ⟨1⟩3. LET:  $\prod_n U_n$  be a nonempty basic open set where each  $U_n$  is open in  $X_n$ .
- ⟨1⟩4. For each  $n$ , PICK  $a_n \in D_n \cap U_n$
- ⟨1⟩5.  $(a_n) \in \prod_n D_n \cap \prod_n U_n$

□

**Example 11.44.3.** The space  $\mathbb{R}_l$  is separable. The set  $\mathbb{Q}$  is dense.

The following example shows that a closed subspace of a separable space is not necessarily separable.

**Example 11.44.4** (AC). The space  $\mathbb{R}_l^2$  is separable, but  $\{(x, -x) \mid x \in \mathbb{R}\}$  as a subspace is uncountable and discrete, and hence not separable.

**Example 11.44.5.** The space  $S_\Omega$  is not separable. For any countable  $D \subseteq S_\Omega$ , we have  $\sup D + 1 \notin \overline{D}$ .

**Example 11.44.6.** The space  $\overline{S_\Omega}$  is not separable. For any countable  $D \subseteq \overline{S_\Omega}$ , we have  $\sup(D - \{\Omega\}) + 1 \notin \overline{D}$ .

**Proposition 11.44.7.** *The continuous image of a separable space is separable.*

PROOF:

- ⟨1⟩1. LET:  $X$  be a separable space.
- ⟨1⟩2. LET:  $Y$  be a topological space.
- ⟨1⟩3. LET:  $f : X \rightarrow Y$  be continuous and separable.
- ⟨1⟩4. PICK a countable dense  $D$  in  $X$   
       PROVE:  $f(D)$  is dense in  $Y$ .
- ⟨1⟩5. LET:  $V$  be open in  $Y$  and nonempty.
- ⟨1⟩6. PICK  $a \in f^{-1}(V) \cap D$
- ⟨1⟩7.  $f(a) \in V \cap f(D)$

□

**Proposition 11.44.8** (Choice). *In a separable space, every set of disjoint open sets is countable.*

PROOF:

- ⟨1⟩1. LET:  $X$  be a separable space.
- ⟨1⟩2. PICK a countable dense set  $D$ .

- ⟨1⟩3. LET:  $\mathcal{A}$  be a set of disjoint open sets.
  - ⟨1⟩4. For  $U \in \mathcal{A}$  nonempty, PICK  $a_U \in U \cap D$
  - ⟨1⟩5. The mapping  $\mathcal{A} \rightarrow D$  that maps  $U$  to  $a_U$  is injective.
  - ⟨1⟩6.  $\mathcal{A}$  is countable.
- 

**Proposition 11.44.9.** *The product space  $\mathbb{R}^I$  is separable.*

PROOF:

- ⟨1⟩1. LET:  $D$  be the set of all step functions with rational values and rational endpoints of the intervals.
- ⟨1⟩2. LET:  $U = \prod_{x \in I} U_x$  be a basic open set in  $\mathbb{R}^I$ , where  $U_x = \mathbb{R}$  except for  $x = x_1, \dots, x_n$  with  $0 \leq x_1 < \dots < x_n \leq 1$
- ⟨1⟩3. ASSUME: w.l.o.g.  $x_1 = 0$  and  $x_n = 1$
- ⟨1⟩4. PICK rationals  $q_1, \dots, q_{n-1}$  such that  $x_1 < q_1 < x_2 < q_2 < \dots < x_{n-1} < q_{n-1} < x_n$
- ⟨1⟩5. For  $1 \leq i \leq n$ , PICK a rational  $r_i \in U_{x_i}$
- ⟨1⟩6. LET:  $g$  be the step function with

$$g(x) = \begin{cases} r_1 & \text{if } 0 \leq x < q_1 \\ r_{i+1} & \text{if } q_i \leq x < q_{i+1}, 1 \leq i \leq n-2 \\ r_n & \text{if } q_{n-1} \leq x \leq 1 \end{cases}$$

- ⟨1⟩7.  $g \in D \cap U$
- 

**Proposition 11.44.10.** *If  $J$  is a set and  $|J| > 2^{\aleph_0}$  then  $\mathbb{R}^J$  is not separable.*

PROOF:

- ⟨1⟩1. LET:  $J$  be a set.
  - ⟨1⟩2. LET:  $D$  be dense in  $\mathbb{R}^J$ .
  - ⟨1⟩3. Define  $f : J \rightarrow \mathcal{P}D$  by:  $f(\alpha) = D \cap \pi_\alpha^{-1}(I)$  where  $I = (0, 1)$ .
  - ⟨1⟩4.  $f$  is injective.
    - ⟨2⟩1. LET:  $\alpha, \beta \in J$
    - ⟨2⟩2. ASSUME:  $\alpha \neq \beta$
    - ⟨2⟩3. PICK  $x \in D \cap \pi_\alpha^{-1}(I) \cap \pi_\beta^{-1}((1, 2))$
    - ⟨2⟩4.  $x \in f(\alpha)$  and  $x \notin f(\beta)$
  - ⟨1⟩5.  $|J| \leq 2^{|D|}$
  - ⟨1⟩6. If  $|J| > 2^{\aleph_0}$  then  $D$  is uncountable.
- 

The following is an example of a separable space that is not first countable.

**Example 11.44.11** (Choice). Let  $\mathbb{Q}^\infty$  be the set of all sequences of rationals that end in an infinite sequence of 0s. Then  $\mathbb{Q}^\infty$  as a subspace of  $\mathbb{R}^\omega$  under the box topology is separable, because it is countable.

## 11.45 The Second Countability Axiom

**Definition 11.45.1** (Second Countability Axiom). A topological space satisfies the *second countability axiom*, or is *second countable*, if and only if it has a countable basis.

**Example 11.45.2.** The space  $\mathbb{R}$  is second countable.

The set  $\{(a, b) \mid a, b \in \mathbb{Q}\}$  is a basis.

**Proposition 11.45.3.** *A subspace of a second countable space is second countable.*

PROOF: If  $\mathcal{B}$  is a countable basis for  $X$  and  $Y \subseteq X$  then  $\{B \cap Y \mid B \in \mathcal{B}\}$  is a countable basis for  $Y$ .  $\square$

**Proposition 11.45.4** (CC). *Every second countable space is Lindelöf.*

PROOF: From Proposition 11.43.3.

**Example 11.45.5.** The space  $S_\Omega$  is not second countable, because it is not Lindelöf.

The following example shows that the product of two Lindelöf spaces is not necessarily Lindelöf.

**Example 11.45.6.** The Sorgenfrey plane is not Lindelöf.

PROOF:

- $\langle 1 \rangle 1.$  LET:  $L = \{(x, -x) \mid x \in \mathbb{R}\}$
  - $\langle 1 \rangle 2.$   $L$  is closed in  $\mathbb{R}_l^2$
  - $\langle 1 \rangle 3.$  LET:  $\mathcal{U} = \{[a, b) \times [a, -d) \mid a, b, d \in \mathbb{R}\}$
  - $\langle 1 \rangle 4.$   $\mathcal{U} \cup \{\mathbb{R} \setminus L\}$  covers  $\mathbb{R}_l^2$
  - $\langle 1 \rangle 5.$  Every element of  $\mathcal{U}$  intersects  $L$  at exactly one point.
  - $\langle 1 \rangle 6.$  No countable subset of  $\mathcal{U}$  covers  $\mathbb{R}_l^2$ .
- $\square$

**Proposition 11.45.7.** *The long line  $L$  is not second countable.*

PROOF:

- $\langle 1 \rangle 1.$  LET:  $\mathcal{B}$  be a basis for  $L$ .
- $\langle 1 \rangle 2.$  For  $\alpha < \omega_1$ , PICK  $B_\alpha \in \mathcal{B}$  such that  $(\alpha, 1/2) \in B_\alpha \subseteq ((\alpha, 0), (\alpha + 1, 0))$
- $\langle 1 \rangle 3.$   $\mathcal{B}$  is uncountable.

PROOF: The mapping  $\alpha \mapsto B_\alpha$  is an injection  $\omega_1 \rightarrow \mathcal{B}$ .

**Corollary 11.45.7.1.** *The long line cannot be imbedded into  $\mathbb{R}^n$  for any  $n$ .*

**Proposition 11.45.8.** *Every second countable space is first countable.*

PROOF:

- $\langle 1 \rangle 1.$  LET:  $X$  be a second countable space.
- $\langle 1 \rangle 2.$  PICK a countable bases  $\mathcal{B}$  for  $X$ .

- ⟨1⟩3. LET:  $x \in X$   
 ⟨1⟩4.  $\{B \in \mathcal{B} \mid x \in B\}$  is a countable local basis at  $x$ .  
 □

**Proposition 11.45.9 (AC).** *A countable product of second countable spaces is second countable.*

PROOF:

- ⟨1⟩1. LET:  $(X_n)$  be a sequence of second countable spaces.  
 ⟨1⟩2. For each  $n$ , PICK a countable basis  $\mathcal{B}_n$  of  $X_n$   
 ⟨1⟩3. LET:  $\mathcal{B} = \{\prod_i U_i \mid U_i \in \mathcal{B}_i \text{ for finitely many } i, U_i = X_i \text{ for all other } i\}$   
 ⟨1⟩4.  $\mathcal{B}$  is a countable basis for  $\prod_n X_n$   
 □

**Proposition 11.45.10 (AC).** *Any discrete subspace of a second countable space is countable.*

PROOF:

- ⟨1⟩1. LET:  $X$  be a second countable space.  
 ⟨1⟩2. LET:  $A \subseteq X$  be discrete.  
 ⟨1⟩3. PICK a countable basis  $\mathcal{B}$  for  $X$ .  
 ⟨1⟩4. For all  $a \in A$ , PICK  $B_a \in \mathcal{B}$  such that  $B_a \cap A = \{a\}$   
     ⟨2⟩1. LET:  $a \in A$   
     ⟨2⟩2. PICK  $U$  open in  $X$  such that  $U \cap A = \{a\}$   
     ⟨2⟩3. PICK  $B \in \mathcal{B}$  such that  $a \in B \subseteq U$   
 ⟨1⟩5. The mapping  $A \rightarrow \mathcal{B}$  that maps  $a$  to  $B_a$  is injective.  
 ⟨1⟩6.  $A$  is countable.  
 □

**Proposition 11.45.11 (AC).** *Every second countable space is separable.*

PROOF:

- ⟨1⟩1. LET:  $X$  be a second countable space.  
 ⟨1⟩2. PICK a countable basis  $\mathcal{B}$  for  $X$ .  
 ⟨1⟩3. For all  $B \in \mathcal{B}$  nonempty PICK  $a_B \in B$ .  
 ⟨1⟩4. LET:  $A = \{a_B \mid B \in \mathcal{B}, B \neq \emptyset\}$   
     PROVE:  $A$  is dense  
 ⟨1⟩5. LET:  $x \in X$   
     PROVE:  $x \in \overline{A}$   
 ⟨1⟩6. LET:  $U$  be a neighbourhood of  $x$   
     PROVE:  $U$  intersects  $A$   
 ⟨1⟩7. PICK  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$   
 ⟨1⟩8.  $a_B \in U \cap A$   
 □

**Example 11.45.12 (AC).** The space  $\mathbb{R}_l$  is not second countable.

PROOF:

- ⟨1⟩1. LET:  $\mathcal{B}$  be any basis for  $\mathbb{R}_l$



$\langle 1 \rangle 2$ . For  $x \in \mathbb{R}$ , PICK  $B_x \in \mathcal{B}$  such that  $x \in B_x \subseteq [x, x+1)$

$\langle 1 \rangle 3$ . The mapping  $\mathbb{R} \rightarrow \mathcal{B}$  that maps  $x$  to  $B_x$  is injective.

PROOF: If  $x < y$  then  $x \in B_x$  and  $x \notin B_y$ .

$\langle 1 \rangle 4$ .  $\mathcal{B}$  is uncountable.

□

**Example 11.45.13 (CC).** The space  $\mathbb{R}_l$  is Lindelöf.

$\langle 1 \rangle 1$ . LET:  $\mathcal{A}$  be a covering of  $\mathbb{R}_l$  by basic open sets of the form  $[a, b)$

$\langle 1 \rangle 2$ . LET:  $C = \bigcup \{(a, b) \mid [a, b) \in \mathcal{A}\}$

$\langle 1 \rangle 3$ .  $\mathbb{R} \setminus C$  is countable.

$\langle 2 \rangle 1$ . For every  $x \in \mathbb{R} \setminus C$ , PICK a rational  $q_x$  such that  $(x, q_x) \subseteq C$

$\langle 3 \rangle 1$ . LET:  $x \in \mathbb{R} \setminus C$

$\langle 3 \rangle 2$ . PICK  $b$  such that  $[x, b) \in \mathcal{A}$

$\langle 3 \rangle 3$ . PICK a rational  $q$  such that  $q \in (x, b)$

$\langle 2 \rangle 2$ . The mapping  $x \mapsto q_x$  is an injection  $\mathbb{R} \setminus C \rightarrow \mathbb{Q}$

$\langle 1 \rangle 4$ . PICK a countable  $\mathcal{A}' \subseteq \mathcal{A}$  that covers  $\mathbb{R} \setminus C$

$\langle 1 \rangle 5$ . Under the standard topology on  $\mathbb{R}$ ,  $C$  is second countable.

PROOF: Proposition 11.45.3.

$\langle 1 \rangle 6$ . PICK a countable  $\mathcal{A}'' \subseteq \mathcal{A}$  such that  $\{(a, b) \mid [a, b) \in \mathcal{A}''\}$  covers  $C$ .

PROOF: Proposition 11.43.3.

$\langle 1 \rangle 7$ .  $\mathcal{A}' \cup \mathcal{A}''$  covers  $\mathbb{R}_l$ .

□

**Proposition 11.45.14 (AC).** *A topological space is second countable if and only if every basis includes a countable basis.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $X$  be a topological space.

$\langle 1 \rangle 2$ . If  $X$  is second countable then every basis includes a countable basis.

$\langle 2 \rangle 1$ . ASSUME:  $X$  is second countable.

$\langle 2 \rangle 2$ . LET:  $\mathcal{B}$  be a basis.

$\langle 2 \rangle 3$ . PICK a countable basis  $\mathcal{C}$ .

$\langle 2 \rangle 4$ . For every pair of basis elements  $C, C' \in \mathcal{C}$  such that there exists  $B \in \mathcal{B}$  with  $C \subseteq B \subseteq C'$ , PICK  $B_{CC'} \in \mathcal{B}$  such that  $C \subseteq B_{CC'} \subseteq C'$

PROVE: The set of all  $B_{CC'}$  form a basis for  $X$ .

$\langle 2 \rangle 5$ . LET:  $x \in X$

$\langle 2 \rangle 6$ . LET:  $U$  be a neighbourhood of  $x$ .

$\langle 2 \rangle 7$ . PICK  $C' \in \mathcal{C}$  such that  $x \in C' \subseteq U$

$\langle 2 \rangle 8$ . PICK  $B \in \mathcal{B}$  such that  $x \in B \subseteq C'$

$\langle 2 \rangle 9$ . PICK  $C \in \mathcal{C}$  such that  $x \in C \subseteq B$

$\langle 2 \rangle 10$ .  $x \in B_{CC'} \subseteq U$

$\langle 1 \rangle 3$ . If every basis includes a countable basis then  $X$  is second countable.

PROOF: The set of all open sets is a basis and so includes a countable basis.

□

**Proposition 11.45.15 (AC).** *Let  $X$  be a second countable space. Let  $A \subseteq X$  be uncountable. Then  $A$  contains uncountably many of its own limit points.*

PROOF:

- ⟨1⟩1. LET:  $X$  be a second countable space.
- ⟨1⟩2. PICK a countable basis  $\mathcal{B}$  for  $X$ .
- ⟨1⟩3. LET:  $A \subseteq X$
- ⟨1⟩4. ASSUME: only countably many points of  $A$  are limit points of  $A$ .
- ⟨1⟩5. For every point  $x$  of  $A$  that is not a limit point of  $A$ , PICK  $B_x \in \mathcal{B}$  such that  $B_x \cap A = \{x\}$ .
- ⟨1⟩6. The mapping  $A - A' \rightarrow \mathcal{B}$  that maps  $x$  to  $B_x$  is injective.
- ⟨1⟩7.  $A$  is countable.

□

**Example 11.45.16.** The space  $\overline{S_\Omega}$  is not second countable because it is neither first countable nor separable.

**Proposition 11.45.17.** *The image of a first countable space under a continuous open map is first countable.*

PROOF:

- ⟨1⟩1. LET:  $X$  be a first countable space.
- ⟨1⟩2. LET:  $Y$  be a topological space.
- ⟨1⟩3. LET:  $f : X \rightarrow Y$  be a surjective, continuous open map.
- ⟨1⟩4. LET:  $y \in Y$
- ⟨1⟩5. PICK  $x \in X$  such that  $f(x) = y$
- ⟨1⟩6. PICK a countable local basis  $\mathcal{B}$  at  $x$ .  
PROVE:  $\{f(B) \mid B \in \mathcal{B}\}$  is a local basis at  $y$ .
- ⟨1⟩7. LET:  $V$  be a neighbourhood of  $y$
- ⟨1⟩8. PICK  $B \in \mathcal{B}$  such that  $x \in B \subseteq f^{-1}(V)$
- ⟨1⟩9.  $y \in f(B) \subseteq V$

□

**Proposition 11.45.18.** *The image of a second countable space under a continuous open map is second countable.*

PROOF:

- ⟨1⟩1. LET:  $X$  be a second countable space.
- ⟨1⟩2. LET:  $Y$  be a topological space.
- ⟨1⟩3. LET:  $f : X \rightarrow Y$  be a surjective, continuous open map.
- ⟨1⟩4. PICK a countable basis  $\mathcal{B}$ .  
PROVE:  $\{f(B) \mid B \in \mathcal{B}\}$  is a basis.
- ⟨1⟩5. LET:  $V$  be a neighbourhood of  $y$
- ⟨1⟩6. PICK  $x \in X$  such that  $f(x) = y$
- ⟨1⟩7. PICK  $B \in \mathcal{B}$  such that  $x \in B \subseteq f^{-1}(V)$
- ⟨1⟩8.  $y \in f(B) \subseteq V$

□

## 11.46 Sequential Compactness

**Definition 11.46.1** (Sequentially Compact). A topological space is *sequentially compact* if and only if every sequence has a convergent subsequence.

## 11.47 Limit Point Compactness

**Definition 11.47.1** (Limit Point Compact Space). A topological space is *limit point compact* if and only if every infinite set has a limit point.

**Proposition 11.47.2.** *Every limit point compact  $T_1$  space is sequentially compact.*

PROOF:

- ⟨1⟩1. LET:  $X$  be a limit point compact  $T_1$  space.
- ⟨1⟩2. LET:  $(x_n)$  be a sequence in  $X$ .
- ⟨1⟩3. CASE:  $\{x_n \mid n \geq 1\}$  is finite.
  - ⟨2⟩1. PICK  $n$  such that  $x_n$  occurs infinitely often in the sequence  $(x_n)$
  - ⟨2⟩2. The subsequence consisting of all the terms equal to  $x_n$  is convergent.
- ⟨1⟩4. CASE:  $\{x_n \mid n \geq 1\}$  is infinite.
  - ⟨2⟩1. PICK a limit point  $l$  for  $\{x_n \mid n \geq 1\}$
  - ⟨2⟩2. PICK an increasing sequence  $n_r$  with  $x_{n_r} \in B(x, 1/r)$  for all  $r$

PROOF: This is always possible by Theorem 11.20.3.

- ⟨2⟩3.  $(x_{n_r})$  converges to  $l$ .

□

**Corollary 11.47.2.1.** *Every compact  $T_1$  space is sequentially compact.*

**Example 11.47.3.** The space  $[0, 1]^\omega$  under the uniform topology is not limit point compact.

The infinite set  $\{0, 1\}^\omega$  has no limit point.

**Example 11.47.4.** The space  $[0, 1]$  under the lower limit topology is not limit point compact.

The infinite set  $A = \{1 - 1/n \mid n \geq 1\}$  has no limit point. 1 is not a limit point because the neighbourhood  $\{1\}$  does not intersect  $A$ .

**Proposition 11.47.5.** *A closed subspace of a limit point compact space is limit point compact.*

PROOF:

- ⟨1⟩1. LET:  $X$  be a limit point compact space.
- ⟨1⟩2. LET:  $A \subseteq X$  be closed.
- ⟨1⟩3. LET:  $B \subseteq A$  be infinite.
- ⟨1⟩4. PICK a limit point  $l$  of  $B$  in  $X$ .
- ⟨1⟩5.  $l \in A$
- ⟨1⟩6.  $l$  is a limit point of  $B$  in  $A$ .

□

**Example 11.47.6.** An open subspace of a limit point compact space is not necessarily limit point compact.

The space  $[0, 1]$  is limit point compact but  $(0, 1)$  is not.

**Example 11.47.7.** The continuous image of a limit point compact space is not necessarily limit point compact.

Let  $Y$  be the indiscrete space with two points. Then  $\mathbb{Z}^+ \times Y$  is limit point compact but  $\mathbb{Z}^+$  is not.

**Example 11.47.8.** A limit point compact subspace of a Hausdorff space is not necessarily closed.

The space  $S_\Omega$  is limit point compact but is not closed in  $\overline{S_\Omega}$ .

For an example that shows that the product of two limit point compact spaces is not necessarily limit point compact, see L. A. Steen and J. A. Seebach Jr. *Counterexamples in Topology* Example 112.

## 11.48 Countable Compactness

**Definition 11.48.1** (Countably Compact). A topological space is *countably compact* if and only if every countable open covering has a finite subcovering.

**Proposition 11.48.2** (AC). *Every closed subspace of a countably compact space is countably compact.*

PROOF:

- <1>1. LET:  $X$  be a countably compact space.
- <1>2. LET:  $A \subseteq X$  be closed.
- <1>3. LET:  $\mathcal{U}$  be a countable open cover of  $A$ .
- <1>4. For  $U \in \mathcal{U}$ , PICK an open set  $V_U$  in  $X$  such that  $U = V_U \cap A$
- <1>5.  $\{V_U \mid U \in \mathcal{U}\} \cup \{X - A\}$  is a countable open cover of  $X$
- <1>6. PICK a finite subcover  $\{V_{U_1}, \dots, V_{U_n}, X - A\}$
- <1>7.  $\{U_1, \dots, U_n\}$  covers  $A$ .

□

**Proposition 11.48.3** (AC). *Every countably compact space is limit point compact.*

PROOF:

- <1>1. ASSUME:  $X$  is countably compact.
- <1>2. LET:  $A \subseteq X$  be infinite.
- <1>3. ASSUME: for a contradiction  $A$  has no limit point.
- <1>4. PICK a countably infinite  $B \subseteq A$
- <1>5.  $B$  is discrete.

PROOF: For all  $b \in B$ , there exists  $U_b$  open in  $X$  such that  $U_b \cap B = \{b\}$ .

- <1>6.  $\{\{b\} \mid b \in B\}$  is a countable cover of  $B$  that has no finite subcover.
- <1>7.  $B$  is not countably compact.
- <1>8.  $B$  is not closed in  $X$

⟨1⟩9.  $B$  has a limit point.

⟨1⟩10.  $A$  has a limit point.

⟨1⟩11. Q.E.D.

PROOF: This contradicts ⟨1⟩3.

□

**Proposition 11.48.4 (AC).** *Every limit point compact  $T_1$  space is countably compact.*

PROOF:

⟨1⟩1. LET:  $X$  be a limit point compact  $T_1$  space.

⟨1⟩2. LET:  $\{U_n \mid n \in \mathbb{Z}^+\}$  be a countable open cover of  $X$ .

⟨1⟩3. For  $n \in \mathbb{Z}^+$ ,

LET:  $V_n = U_1 \cup \cdots \cup U_n$

⟨1⟩4. ASSUME: for a contradiction none of the  $V_n$  covers  $X$

⟨1⟩5. For  $n \in \mathbb{Z}^+$ , PICK  $a_n \in X - V_n$

⟨1⟩6. PICK a limit point  $l$  for  $\{a_n \mid n \in \mathbb{Z}^+\}$

⟨1⟩7. PICK  $n$  such that  $l \in U_n$

⟨1⟩8. CASE:  $l = a_m$  for some  $m \leq n$

PROOF:  $U_n - \{a_1, \dots, a_{m-1}, a_{m+1}, \dots, a_n\}$  is a neighbourhood of  $l$  that intersects  $\{a_n \mid n \in \mathbb{Z}^+\}$  only at  $l$ , contradicting ⟨1⟩6.

⟨1⟩9. CASE:  $l \neq a_m$  for any  $m \leq n$

PROOF:  $U_n - \{a_1, \dots, a_n\}$  is a neighbourhood of  $l$  that does not intersect  $\{a_n \mid n \in \mathbb{Z}^+\}$ , which contradicts ⟨1⟩6.

□

The following example shows we cannot remove the hypothesis that the space is  $T_1$ .

**Example 11.48.5.** Let  $Y$  be the indiscrete space with two points. Then  $\mathbb{Z}^+ \times Y$  is a limit point compact space that is not countably compact, since  $\{\{n\} \times Y \mid n \in \mathbb{Z}^+\}$  is a countable open cover that has no finite subcover.

**Proposition 11.48.6.** *A topological space is countably compact if and only if every nested sequence  $C_1 \supseteq C_2 \supseteq \cdots$  of nonempty closed sets has nonempty intersection.*

PROOF:

⟨1⟩1. LET:  $X$  be a topological space.

⟨1⟩2. If  $X$  is countably compact then every nested sequence of nonempty closed sets has nonempty intersection.

⟨2⟩1. ASSUME:  $X$  is countably compact.

⟨2⟩2. LET:  $C_1 \supseteq C_2 \supseteq \cdots$  be a nested sequence of nonempty closed sets.

⟨2⟩3. ASSUME: for a contradiction  $\bigcap_n C_n = \emptyset$

⟨2⟩4.  $\{X - C_n \mid n \in \mathbb{Z}^+\}$  covers  $X$

⟨2⟩5. PICK a finite subcover  $\{X - C_{n_1}, \dots, X - C_{n_k}\}$  where  $n_1 < \cdots < n_k$

⟨2⟩6.  $C_{n_k} = \emptyset$

⟨2⟩7. Q.E.D.

PROOF: This contradicts  $\langle 2 \rangle 2$ .

□

$\langle 1 \rangle 3$ . If every nested sequence of nonempty closed sets has nonempty intersection then  $X$  is countably compact.

$\langle 2 \rangle 1$ . ASSUME: Every nested sequence of nonempty closed sets has nonempty intersection.

$\langle 2 \rangle 2$ . LET:  $\{U_n \mid n \geq 1\}$  is a countable open cover of  $X$ .

$\langle 2 \rangle 3$ .  $X - U_1 \supseteq X - (U_1 \cup U_2) \supseteq \cdots$  is a nested sequence of closed sets with empty intersection.

$\langle 2 \rangle 4$ . PICK  $k$  such that  $X - (U_1 \cup \cdots \cup U_k) = \emptyset$

$\langle 2 \rangle 5$ .  $\{U_1, \dots, U_k\}$  covers  $X$ .

□

## 11.49 Subnets

**Definition 11.49.1** (Subnet). Let  $X$  be a topological space. Let  $(a_\alpha)_{\alpha \in J}$  be a net in  $X$ . A *subnet* of  $(a_\alpha)_{\alpha \in J}$  is a net of the form  $(a_{g(\beta)})_{\beta \in K}$  where  $K$  is a directed set,  $g : K \rightarrow J$  is monotone, and  $g(K)$  is cofinal in  $J$ .

**Proposition 11.49.2.** Let  $X$  be a topological space. Let  $(a_\alpha)_{\alpha \in J}$  be a net in  $X$ . Let  $l \in X$ . If  $(a_\alpha)$  converges to  $l$  then any subnet of  $(a_\alpha)$  converges to  $l$ .

PROOF:

$\langle 1 \rangle 1$ . LET:  $X$  be a topological space.

$\langle 1 \rangle 2$ . LET:  $(a_\alpha)_{\alpha \in J}$  be a net in  $X$ .

$\langle 1 \rangle 3$ . LET:  $l \in X$

$\langle 1 \rangle 4$ . ASSUME:  $a_\alpha \rightarrow l$  as  $\alpha \rightarrow \infty$

$\langle 1 \rangle 5$ . LET:  $(a_{g(\beta)})_{\beta \in K}$  be a subnet of  $(a_\alpha)_{\alpha \in J}$

$\langle 1 \rangle 6$ . LET:  $U$  be a neighbourhood of  $l$ .

$\langle 1 \rangle 7$ . PICK  $\alpha \in J$  be such that, for all  $\alpha' \geq \alpha$ , we have  $a_{\alpha'} \in U$

$\langle 1 \rangle 8$ . PICK  $\beta \in K$  such that  $g(\beta) \geq \alpha$ .

$\langle 1 \rangle 9$ . For all  $\beta' \geq \beta$  we have  $a_{g(\beta')} \in U$ .

□

## 11.50 Accumulation Points

**Definition 11.50.1** (Accumulation Point). Let  $X$  be a topological space. Let  $(a_\alpha)_{\alpha \in J}$  be a net in  $X$ . Let  $l \in X$ . Then  $l$  is an *accumulation point* of  $(a_\alpha)_{\alpha \in J}$  if and only if, for every neighbourhood  $U$  of  $l$ , the set  $\{\alpha \in J \mid a_\alpha \in U\}$  is cofinal in  $J$ .

**Lemma 11.50.2.** Let  $X$  be a topological space. Let  $(a_\alpha)_{\alpha \in J}$  be a net in  $X$ . Let  $l \in X$ . Then  $l$  is an accumulation point of  $(a_\alpha)_{\alpha \in J}$  if and only if there exists a subnet of  $(a_\alpha)_{\alpha \in J}$  that converges to  $l$ .

PROOF:

- ⟨1⟩1. LET:  $X$  be a topological space.
- ⟨1⟩2. LET:  $(a_\alpha)_{\alpha \in J}$  be a net in  $X$ .
- ⟨1⟩3. LET:  $l \in X$
- ⟨1⟩4. If  $l$  is an accumulation point of  $(a_\alpha)_{\alpha \in J}$  then there exists a subnet of  $(a_\alpha)_{\alpha \in J}$  that converges to  $l$ .
- ⟨2⟩1. ASSUME:  $l$  is an accumulation point of  $(a_\alpha)_{\alpha \in J}$ .
- ⟨2⟩2. LET:  $K = \{(\alpha, U) \mid \alpha \in J, U \text{ is a neighbourhood of } l, a_\alpha \in U\}$  with  $(\alpha, U) \leq (\beta, V)$  if and only if  $\alpha \leq \beta$  and  $V \subseteq U$
- ⟨2⟩3.  $K$  is a directed set
  - ⟨3⟩1.  $\leq$  is reflexive on  $K$ .
  - ⟨3⟩2.  $\leq$  is transitive on  $K$ .
  - ⟨3⟩3.  $\leq$  is antisymmetric on  $K$ .
  - ⟨3⟩4. For all  $(\alpha, U), (\beta, V) \in K$ , there exists  $(\gamma, W)$  such that  $(\alpha, U) \leq (\gamma, W)$  and  $(\beta, V) \leq (\gamma, W)$
- ⟨4⟩1. LET:  $(\alpha, U), (\beta, V) \in K$
- ⟨4⟩2. PICK  $\gamma \in J$  with  $\alpha \leq \gamma$  and  $\beta \leq \gamma$
- ⟨4⟩3. PICK  $\delta \in J$  with  $\gamma \leq \delta$  and  $a_\delta \in U \cap V$
- ⟨4⟩4.  $(\alpha, U) \leq (\delta, U \cap V)$  and  $(\beta, V) \leq (\delta, U \cap V)$
- ⟨2⟩4. LET:  $g : K \rightarrow J, g(\alpha, U) = \alpha$
- ⟨2⟩5.  $g$  is monotone
- ⟨2⟩6.  $g(K)$  is cofinal in  $J$
- PROOF: For all  $\alpha \in J$  we have  $\alpha = g(\alpha, X)$ .
- ⟨2⟩7.  $(a_{g(\alpha, U)})_{(\alpha, U) \in K}$  converges to  $l$ .
- ⟨3⟩1. LET:  $U$  be a neighbourhood of  $l$
- ⟨3⟩2. PICK  $\alpha \in J$  such that  $a_\alpha \in U$
- ⟨3⟩3. For all  $(\beta, V) \geq (\alpha, U)$  we have  $a_\beta \in U$
- PROOF: Since  $a_\beta \in V \subseteq U$
- ⟨1⟩5. If there exists a subnet of  $(a_\alpha)_{\alpha \in J}$  that converges to  $l$  then  $l$  is an accumulation point of  $(a_\alpha)_{\alpha \in J}$ .
- ⟨2⟩1. ASSUME:  $(a_{g(\beta)})_{\beta \in K}$  converges to  $l$
- ⟨2⟩2. LET:  $U$  be a neighbourhood of  $l$
- ⟨2⟩3. LET:  $\alpha \in J$
- ⟨2⟩4. PICK  $\beta \in K$  such that, for all  $\beta' \geq \beta$ , we have  $a_{g(\beta')} \in U$
- ⟨2⟩5. PICK  $\gamma \in K$  such that  $g(\gamma) \geq \alpha$
- ⟨2⟩6. PICK  $\delta \in K$  with  $\beta \leq \delta$  and  $\gamma \leq \delta$
- ⟨2⟩7.  $\alpha \leq g(\delta)$
- ⟨2⟩8.  $a_{g(\delta)} \in U$

□

## 11.51 Compact Spaces

**Definition 11.51.1** (Compact). A topological space is *compact* if and only if every open covering has a finite subcovering.

**Lemma 11.51.2.** Let  $X$  be a topological space and  $Y$  a subspace of  $X$ . Then

*$Y$  is compact if and only if every covering of  $Y$  by sets open in  $X$  has a finite subcovering.*

PROOF:

- ⟨1⟩1. If  $Y$  is compact then every covering of  $Y$  by sets open in  $X$  has a finite subcovering.
- ⟨2⟩1. ASSUME:  $Y$  is compact.
- ⟨2⟩2. LET:  $\mathcal{U}$  be a covering of  $Y$  by sets open in  $X$ .
- ⟨2⟩3.  $\{U \cap Y \mid U \in \mathcal{U}\}$  is an open covering of  $Y$ .
- ⟨2⟩4. PICK a finite subcovering  $\{U_1 \cap Y, \dots, U_n \cap Y\}$
- ⟨2⟩5.  $\{U_1, \dots, U_n\}$  is a finite subcovering of  $\mathcal{U}$ .
- ⟨1⟩2. If every covering of  $Y$  by sets open in  $X$  has a finite subcovering then  $Y$  is compact.
- ⟨2⟩1. LET:  $\mathcal{U}$  be an open covering of  $Y$ .
- ⟨2⟩2. LET:  $\mathcal{V} = \{V \text{ open in } X \mid V \cap Y \in \mathcal{U}\}$ .
- ⟨2⟩3.  $\mathcal{V}$  is a covering of  $Y$  by sets open in  $X$ .
- ⟨2⟩4. PICK a finite subcovering  $\{V_1, \dots, V_n\}$
- ⟨2⟩5.  $\{V_1 \cap Y, \dots, V_n \cap Y\}$  is a finite subcovering of  $\mathcal{U}$ .

□

**Proposition 11.51.3.** *Every closed subspace of a compact space is compact.*

PROOF:

- ⟨1⟩1. LET:  $X$  be a compact space and  $Y \subseteq X$  be closed.
- ⟨1⟩2. LET:  $\mathcal{U}$  be a covering of  $Y$  by sets open in  $X$ .
- ⟨1⟩3.  $\mathcal{U} \cup \{X \setminus Y\}$  is an open covering of  $X$ .
- ⟨1⟩4. PICK a finite subcovering  $\mathcal{U}_0$
- ⟨1⟩5.  $\mathcal{U}_0 \cap \mathcal{U}$  is a finite subset of  $\mathcal{U}$  that covers  $Y$ .

□

**Theorem 11.51.4.** *The continuous image of a compact space is compact.*

PROOF:

- ⟨1⟩1. LET:  $f : X \rightarrow Y$  be continuous and surjective.
- ⟨1⟩2. LET:  $\mathcal{V}$  be an open covering of  $Y$
- ⟨1⟩3.  $\{p^{-1}(V) \mid V \in \mathcal{V}\}$  is an open covering of  $X$ .
- ⟨1⟩4. PICK a finite subcovering  $\{p^{-1}(V_1), \dots, p^{-1}(V_n)\}$
- ⟨1⟩5.  $\{V_1, \dots, V_n\}$  covers  $Y$ .

□

**Theorem 11.51.5.** *Let  $A$  and  $B$  be compact subspaces of  $X$  and  $Y$  respectively. Let  $N$  be an open set in  $X \times Y$  that includes  $A \times B$ . Then there exist open sets  $U$  and  $V$  in  $X$  and  $Y$  respectively such that  $A \subseteq U$ ,  $B \subseteq V$  and  $U \times V \subseteq N$ .*

PROOF:

- ⟨1⟩1. For all  $x \in A$ , there exist neighbourhoods  $U$  of  $x$  and  $V$  of  $B$  such that  $U \times V \subseteq N$ .
- ⟨2⟩1. LET:  $x \in A$



- (2)2. For all  $y \in B$ , there exist neighbourhoods  $U$  of  $x$  and  $V$  of  $y$  such that  $U \times V \subseteq N$
- (2)3.  $\{V \text{ open in } Y \mid \exists \text{ neighbourhood } U \text{ of } x, U \times V \subseteq N\}$  covers  $B$ .
- (2)4. PICK a finite subcover  $\{V_1, \dots, V_n\}$
- (2)5. For  $i = 1, \dots, n$ , PICK a neighbourhood  $U_i$  of  $x$  such that  $U_i \times V_i \subseteq N$
- (2)6. LET:  $U = U_1 \cap \dots \cap U_n$
- (2)7. LET:  $V = V_1 \cup \dots \cup V_n$
- (2)8.  $U$  is a neighbourhood of  $x$ .
- (2)9.  $V$  is a neighbourhood of  $B$ .
- (2)10.  $U \times V \subseteq N$
- (1)2.  $\{U \text{ open in } X \mid \exists \text{ neighbourhood } V \text{ of } B, U \times V \subseteq N\}$  covers  $A$ .
- (1)3. PICK a finite subcover  $\{U_1, \dots, U_n\}$
- (1)4. For  $i = 1, \dots, n$ , PICK a neighbourhood  $V_i$  of  $B$  such that  $U_i \times V_i \subseteq N$
- (1)5. LET:  $U = U_1 \cup \dots \cup U_n$
- (1)6. LET:  $V = V_1 \cap \dots \cap V_n$
- (1)7.  $U$  and  $V$  are open.
- (1)8.  $A \subseteq U$
- (1)9.  $B \subseteq V$
- (1)10.  $U \times V \subseteq N$

□

**Corollary 11.51.5.1** (Tube Lemma). *Let  $X$  and  $Y$  be topological spaces with  $Y$  compact. Let  $a \in X$  and  $N$  be an open set in  $X \times Y$  that includes  $\{a\} \times Y$ . Then there exists a neighbourhood  $W$  of  $a$  such that  $N$  includes the tube  $W \times Y$ .*

**Theorem 11.51.6.** *Let  $X$  be a topological space. Then  $X$  is compact if and only if every set of closed sets that has the finite intersection property has nonempty intersection.*

PROOF: The following are equivalent.

1.  $X$  is compact.
2. Every open covering of  $X$  has a finite subcovering.
3. For any set  $\mathcal{C}$  of closed sets, if  $\{X \setminus C \mid C \in \mathcal{C}\}$  covers  $X$  then there is a finite subset  $\mathcal{C}_0$  such that  $\{X \setminus C \mid C \in \mathcal{C}_0\}$  covers  $X$
4. For any set  $\mathcal{C}$  of closed sets, if  $\bigcap \mathcal{C} = \emptyset$  then there is a finite subset  $\mathcal{C}_0$  with empty intersection.
5. Any set of closed sets with the finite intersection property has nonempty intersection.

□

**Corollary 11.51.6.1.** *Let  $X$  be a topological space and  $C_1 \supseteq C_2 \supseteq \dots$  a nested sequence of nonempty closed sets. Then  $\bigcap_n C_n$  is nonempty.*

**Proposition 11.51.7.** *Let  $\mathcal{T}$  and  $\mathcal{T}'$  be two topologies on the same set  $X$  with  $\mathcal{T} \subseteq \mathcal{T}'$ . If  $\mathcal{T}'$  is compact then  $\mathcal{T}$  is compact.*

PROOF:

- ⟨1⟩1. LET:  $\mathcal{U} \subseteq \mathcal{T}$  cover  $X$
  - ⟨1⟩2.  $\mathcal{U} \subseteq \mathcal{T}'$
  - ⟨1⟩3. A finite subset of  $\mathcal{U}$  covers  $X$ .
- 

**Corollary 11.51.7.1.** *If  $\mathcal{T}$  and  $\mathcal{T}'$  are two compact Hausdorff topologies on the same set  $X$ , then either  $\mathcal{T} = \mathcal{T}'$  or  $\mathcal{T}$  and  $\mathcal{T}'$  are incomparable.*

PROOF: From the Proposition and Proposition 11.21.12. □

**Example 11.51.8.** Any set under the finite complement topology is compact.

**Proposition 11.51.9.** *Let  $X$  be a topological space. A finite union of compact subspaces of  $X$  is compact.*

PROOF:

- ⟨1⟩1. LET:  $A$  and  $B$  be compact subspaces of  $X$ .
  - ⟨1⟩2. LET:  $\mathcal{U}$  be a set of open sets in  $X$  that covers  $A \cup B$
  - ⟨1⟩3. PICK a finite subset  $\mathcal{U}_1$  that covers  $A$ .  
PROOF: Lemma 11.51.2.
  - ⟨1⟩4. PICK a finite subset  $\mathcal{U}_2$  that covers  $B$ .  
PROOF: Lemma 11.51.2.
  - ⟨1⟩5.  $\mathcal{U}_1 \cup \mathcal{U}_2$  is a finite subset that covers  $A \cup B$ .
  - ⟨1⟩6. Q.E.D.  
PROOF: Lemma 11.51.2.
- 

**Proposition 11.51.10.** *Let  $A$  and  $B$  be disjoint compact subspaces of the Hausdorff space  $X$ . Then there exist disjoint open sets  $U$  and  $V$  that include  $A$  and  $B$  respectively.*

PROOF: From Theorem 11.51.5 with  $N = X^2 \setminus \{(x, x) \mid x \in X\}$ . □

**Corollary 11.51.10.1.** *Every compact subspace of a Hausdorff space is closed.*

**Theorem 11.51.11.** *Let  $f : X \rightarrow Y$  be a bijective continuous function. If  $X$  is compact and  $Y$  is Hausdorff then  $f$  is a homeomorphism.*

PROOF:

- ⟨1⟩1. LET:  $C \subseteq X$  be closed.
- ⟨1⟩2.  $C$  is compact.  
PROOF: Proposition 11.51.3.
- ⟨1⟩3.  $f(C)$  is compact.  
PROOF: Theorem 11.51.4.
- ⟨1⟩4.  $f(C)$  is closed.  
PROOF: Corollary 11.51.10.1.
- ⟨1⟩5. Q.E.D.  
PROOF: Lemma 11.14.2.

□

**Proposition 11.51.12.** *Let  $X$  be a compact space,  $Y$  a Hausdorff space, and  $f : X \rightarrow Y$  a continuous map. Then  $f$  is a closed map.*

PROOF:

⟨1⟩1. LET:  $C \subseteq X$  be closed.

⟨1⟩2.  $C$  is compact.

PROOF: Proposition 11.51.3.

⟨1⟩3.  $f(C)$  is compact.

PROOF: Theorem 11.51.4.

⟨1⟩4.  $f(C)$  is closed.

PROOF: Corollary 11.51.10.1.

□

**Proposition 11.51.13.** *If  $Y$  is compact then the projection  $\pi_1 : X \times Y \rightarrow X$  is a closed map.*

PROOF:

⟨1⟩1. LET:  $A \subseteq X \times Y$  be closed.

⟨1⟩2. LET:  $x \in X \setminus \pi_1(A)$

⟨1⟩3. PICK a neighbourhood  $U$  of  $x$  such that  $U \times Y \subseteq (X \times Y) \setminus A$

PROOF: By the Tube Lemma.

⟨1⟩4.  $x \in U \subseteq X \setminus \pi_1(A)$

⟨1⟩5. Q.E.D.

PROOF: So  $X \setminus \pi_1(A)$  is open by Lemma 11.1.8.

□

**Proposition 11.51.14.** *Let  $X$  be a topological space and  $Y$  a Hausdorff space. Let  $f : X \rightarrow Y$  be continuous. Then the graph of  $f$  is closed in  $X \times Y$ .*

⟨1⟩1. ASSUME:  $f$  is continuous.

⟨1⟩2. LET:  $(x, y) \in (X \times Y) \setminus G_f$

⟨1⟩3. PICK disjoint neighbourhoods  $U$  and  $V$  of  $y$  and  $f(x)$  respectively.

⟨1⟩4.  $f^{-1}(V) \times U$  is a neighbourhood of  $(x, y)$  disjoint from  $G_f$ .

□

**Theorem 11.51.15.** *Let  $X$  be a topological space and  $Y$  a compact space. Let  $f : X \rightarrow Y$  be a function. If the graph of  $f$  is closed in  $X \times Y$  then  $f$  is continuous.*

PROOF:

⟨1⟩1. ASSUME:  $G_f$  is closed.

⟨1⟩2. LET:  $x \in X$  and  $V$  be a neighbourhood of  $f(x)$ .

⟨1⟩3.  $G_f \cap (X \times (Y \setminus V))$  is closed.

⟨1⟩4.  $\pi_1(G_f \cap (X \times (Y \setminus V)))$  is closed.

PROOF: Proposition 11.51.13.

⟨1⟩5. LET:  $U = X \setminus \pi_1(G_f \cap (X \times (Y \setminus V)))$

⟨1⟩6.  $U$  is a neighbourhood of  $x$

⟨1⟩7.  $f(U) \subseteq V$

□

**Theorem 11.51.16.** *Let  $X$  be a compact topological space. Let  $(f_n : X \rightarrow \mathbb{R})$  be a monotone increasing sequence of continuous functions and  $f : X \rightarrow \mathbb{R}$  a continuous function. If  $(f_n)$  converges pointwise to  $f$ , then  $(f_n)$  converges uniformly to  $f$ .*

PROOF:

⟨1⟩1. LET:  $\epsilon > 0$

⟨1⟩2. For all  $x \in X$ , there exists  $N$  such that, for all  $n \geq N$ , we have  $|f_n(x) - f(x)| < \epsilon$

⟨1⟩3. For  $n \geq 1$ ,

LET:  $U_n = \{x \in X \mid |f_n(x) - f(x)| < \epsilon\}$

⟨1⟩4. For  $n \geq 1$ , we have  $U_n$  is open in  $X$ .

⟨2⟩1. LET:  $x \in X$

⟨2⟩2. LET:  $\delta = \epsilon - |f_n(x) - f(x)|$

⟨2⟩3. PICK a neighbourhood  $U$  of  $x$  such that  $f(U) \subseteq B(f(x), \delta/2)$

⟨2⟩4. PICK a neighbourhood  $V$  of  $x$  such that  $f_n(V) \subseteq B(f_n(x), \delta/2)$

⟨2⟩5.  $f(U \cap V) \subseteq U_n$

PROOF: For  $y \in U \cap V$  we have

$$\begin{aligned} |f_n(y) - f(y)| &\leq |f_n(y) - f_n(x)| + |f_n(x) - f(x)| + |f(x) - f(y)| \\ &< \delta/2 + |f_n(x) - f(x)| + \delta/2 \\ &= \epsilon \end{aligned}$$

⟨1⟩5.  $\{U_n \mid n \geq 1\}$  covers  $X$

PROOF: From ⟨1⟩2

⟨1⟩6. PICK  $N$  such that  $X = U_N$

⟨2⟩1. PICK  $n_1, \dots, n_k$  such that  $U_{n_1}, \dots, U_{n_k}$  cover  $X$ .

⟨2⟩2. LET:  $N = \max(n_1, \dots, n_k)$

⟨2⟩3. For all  $i$  we have  $U_{n_i} \subseteq U_N$

PROOF: Since  $(f_n)$  is monotone increasing.

⟨2⟩4.  $X = U_N$

⟨1⟩7. For all  $x \in X$  and  $n \geq N$  we have  $|f_n(x) - f(x)| < \epsilon$

□

An example to show that we cannot remove the hypothesis that  $X$  is compact:

**Example 11.51.17.** Let  $X = (0, 1)$ ,  $f_n(x) = -x^n$  and  $f(x) = 0$  for  $x \in X$  and  $n \geq 1$ . Then  $f_n \rightarrow f$  pointwise and  $(f_n)$  is monotone increasing but the convergence is not uniform since, for all  $N \geq 1$ , there exists  $x \in (0, 1)$  such that  $-x^N < -1/2$ .

An example to show that we cannot remove the hypothesis that  $(f_n)$  is monotone increasing:

**Example 11.51.18.** Let  $X = [0, 1]$ ,  $f_n(x) = 1/(n^3(x - 1/n)^2 + 1)$  and  $f(x) = 0$  for  $x \in X$  and  $n \geq 1$ . Then  $X$  is compact and  $f_n \rightarrow f$  pointwise but the

convergence is not uniform since, for all  $N \geq 1$ , there exists  $x \in [0, 1]$  such that  $f_N(x) = 1$ , namely  $x = 1/N$ .

**Theorem 11.51.19.** *Let  $X$  be a compact Hausdorff space. Let  $\mathcal{A}$  be a chain of closed connected subsets of  $X$ . Then  $\bigcap \mathcal{A}$  is connected.*

PROOF:

$\langle 1 \rangle 1$ . ASSUME: for a contradiction  $C$  and  $D$  form a separation of  $\bigcap \mathcal{A}$ .

$\langle 1 \rangle 2$ . PICK disjoint open sets  $U$  and  $V$  that include  $C$  and  $D$  respectively.

PROOF: Proposition 11.51.10.

$\langle 1 \rangle 3$ .  $\{A \setminus (U \cup V) \mid A \in \mathcal{A}\}$  is a set of closed sets with the finite intersection property.

$\langle 2 \rangle 1$ . For all  $A \in \mathcal{A}$  we have  $A \setminus (U \cup V)$  is closed.

$\langle 2 \rangle 2$ . For all  $A_1, \dots, A_n \in \mathcal{A}$  we have  $(A_1 \cap \dots \cap A_n) \setminus (U \cup V)$  is nonempty.

PROOF:

$\langle 3 \rangle 1$ . LET:  $A_1, \dots, A_n \in \mathcal{A}$

$\langle 3 \rangle 2$ . ASSUME: without loss of generality  $A_1 \subseteq A_2, \dots, A_n$

PROOF: Since  $\mathcal{A}$  is a chain.

$\langle 3 \rangle 3$ .  $A_1 \setminus (U \cup V)$  is nonempty

PROOF: Otherwise  $(A_1 \cap \dots \cap A_n \cap U)$  and  $(A_1 \cap \dots \cap A_n \cap V)$  would form a separation of  $A_n$ .

$\langle 1 \rangle 4$ .  $\bigcap \mathcal{A} \setminus (U \cup V)$  is nonempty.

PROOF: Theorem 11.51.6.

$\langle 1 \rangle 5$ . Q.E.D.

PROOF: This contradicts  $\langle 1 \rangle 1$  since  $\bigcap \mathcal{A} \setminus (U \cup V) = \bigcap \mathcal{A} \setminus (C \cup D)$ .

□

**Theorem 11.51.20** (Tychonoff Theorem (AC)). *The product of a family of compact spaces is compact.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $\{X_\alpha\}_{\alpha \in J}$  be a family of compact spaces.

$\langle 1 \rangle 2$ . LET:  $X = \prod_{\alpha \in J} X_\alpha$

$\langle 1 \rangle 3$ . For any  $\mathcal{A} \subseteq \mathcal{P}X$ , we have  $\bigcap_{A \in \mathcal{A}} \overline{A} \neq \emptyset$

$\langle 2 \rangle 1$ . LET:  $\mathcal{A} \subseteq \mathcal{P}X$

$\langle 2 \rangle 2$ . PICK  $\mathcal{D} \supseteq \mathcal{A}$  that is maximal with respect to the finite intersection property.

PROVE:  $\bigcap_{D \in \mathcal{D}} \overline{D} \neq \emptyset$

PROOF: Lemma 5.7.7.

$\langle 2 \rangle 3$ . For  $\alpha \in J$ , PICK  $x_\alpha \in X_\alpha$  such that  $x_\alpha \in \bigcap_{D \in \mathcal{D}} \overline{\pi_\alpha(D)}$

PROOF: Theorem 11.51.6 since  $\{\overline{\pi_\alpha(D)} \mid D \in \mathcal{D}\}$  is a set of closed sets in  $X_\alpha$  with the finite intersection property.

$\langle 2 \rangle 4$ . LET:  $x = (x_\alpha)_{\alpha \in J}$

PROVE:  $x \in \bigcap_{D \in \mathcal{D}} \overline{D}$

$\langle 2 \rangle 5$ . For any  $\beta \in J$  and neighbourhood  $U$  of  $x_\beta$  in  $X_\beta$ , we have  $\pi_\beta^{-1}(U)$  intersects every element of  $\mathcal{D}$

$\langle 3 \rangle 1$ . LET:  $\beta \in J$

- ⟨3⟩2. LET:  $U$  be a neighbourhood of  $x_\beta$  in  $X_\beta$ .  
 ⟨3⟩3. LET:  $D \in \mathcal{D}$   
 ⟨3⟩4.  $x_\beta \in \pi_\beta(\overline{D})$   
 PROOF: From ⟨2⟩3  
 ⟨3⟩5.  $U$  intersects  $\pi_\beta(D)$ .  
 ⟨3⟩6.  $\pi_\beta^{-1}(U)$  intersects  $D$ .  
 ⟨2⟩6. For any  $\beta \in J$  and neighbourhood  $U$  of  $x_\beta$  in  $X_\beta$ , we have  $\pi_\beta^{-1}(U) \in \mathcal{D}$   
 PROOF: Lemma 10.1.3.  
 ⟨2⟩7. Every basic neighbourhood of  $x$  is an element of  $\mathcal{D}$   
 PROOF: Lemma 10.1.2.  
 ⟨2⟩8. Every basic neighbourhood of  $x$  intersects every element of  $\mathcal{D}$   
 PROOF: Since  $\mathcal{D}$  satisfies the finite intersection property.  
 ⟨2⟩9. For all  $D \in \mathcal{D}$  we have  $x \in \overline{D}$   
 ⟨1⟩4. Q.E.D.  
 PROOF: Theorem 11.51.6.

□

**Lemma 11.51.21.** *Let  $X$  and  $Y$  be topological spaces. Let  $\mathcal{A}$  be a set of basis elements for the product topology on  $X \times Y$  such that no finite subset of  $\mathcal{A}$  covers  $X \times Y$ . If  $X$  is compact, then there exists  $x \in X$  such that no finite subset of  $\mathcal{A}$  covers the slice  $\{x\} \times Y$ .*

PROOF:

- ⟨1⟩1. ASSUME: for every  $x \in X$ , there exists a finite subset of  $\mathcal{A}$  that covers  $\{x\} \times Y$   
 PROVE: A finite subset of  $\mathcal{A}$  covers  $X \times Y$   
 ⟨1⟩2.  $\{U \mid \exists U_1 \times V_1, \dots, U_n \times V_n \in \mathcal{A}. U = U_1 \cap \dots \cap U_n, V_1 \cup \dots \cup V_n = Y\}$  covers  $X$   
 ⟨1⟩3. PICK a finite subcover  $U_1, \dots, U_m$   
 ⟨1⟩4. PICK  $U_{ij} \times V_{ij} \in \mathcal{A}$  such that, for every  $i$ , we have  $U_i = \bigcap_j U_{ij}$  and  $Y = \bigcup_j V_{ij}$   
 ⟨1⟩5. The collection of all  $U_{ij} \times V_{ij}$  covers  $X \times Y$

□

**Theorem 11.51.22 (AC).** *Let  $X$  be a compact Hausdorff space. Then the quasicomponents and the components of  $X$  are the same.*

PROOF:

- ⟨1⟩1. LET:  $x, y \in X$   
 ⟨1⟩2. ASSUME:  $x$  and  $y$  are in the same quasicomponent.  
 PROVE:  $x$  and  $y$  are in the same component.  
 ⟨1⟩3. LET:  $\mathcal{A}$  be the set of all closed subsets  $A$  of  $X$  such that  $x$  and  $y$  are in the same quasicomponent of  $A$ .  
 ⟨1⟩4. For every chain  $\mathcal{B} \subseteq \mathcal{A}$  we have  $\bigcap \mathcal{B} \in \mathcal{A}$   
 ⟨2⟩1. LET:  $\mathcal{B} \subseteq \mathcal{A}$  be a chain.  
 ⟨2⟩2. ASSUME: for a contradiction  $U$  and  $V$  form a separation of  $\bigcap \mathcal{B}$  with  $x \in U$  and  $y \in V$

- (2)3. PICK disjoint open sets  $U', V'$  in  $X$  such that  $U \subseteq U'$  and  $V \subseteq V'$   
 (2)4.  $\{B \setminus (U' \cup V') \mid B \in \mathcal{B}\}$  satisfies the finite intersection property.  
 (3)1. LET:  $B_1, \dots, B_n \in \mathcal{B}$   
 (3)2. ASSUME: without loss of generality  $B_1 \subseteq \dots \subseteq B_n$   
 PROOF: Since  $\mathcal{B}$  is a chain.  
 (3)3.  $\bigcap \{B_1 \setminus (U' \cup V'), \dots, B_n \setminus (U' \cup V')\} = B_1 \setminus (U' \cup V')$   
 (3)4.  $B_1 \setminus (U' \cup V')$  is nonempty  
 PROOF: Otherwise  $B_1 \cap U'$  and  $B_1 \cap V'$  would form a separation of  $B_1$ , contradicting the fact that  $x$  and  $y$  are in the same quasicomponent of  $B_1$ .  
 (2)5.  $\bigcap \mathcal{B} \setminus (U \cup V)$  is nonempty  
 PROOF: Theorem 11.51.6.  
 (2)6. Q.E.D.  
 PROOF: This contradicts (2)2.  
 (1)5. PICK a minimal element  $D$  in  $\mathcal{A}$ .  
 PROVE:  $D$  is connected.  
 PROOF: By Zorn's Lemma.  
 (1)6. ASSUME: for a contradiction  $U$  and  $V$  form a separation of  $D$ .  
 (1)7. ASSUME: without loss of generality  $x, y \in U$   
 PROOF: We cannot have that one of  $x, y$  is in  $U$  and the other in  $V$  since  $D \in \mathcal{A}$ .  
 (1)8.  $U \in \mathcal{A}$   
 PROOF: If  $X$  and  $Y$  form a separation of  $U$  with  $x \in X$  and  $y \in Y$ , then  $X$  and  $Y \cup V$  form a separation of  $D$  with  $x \in X$  and  $y \in Y \cup V$ .  
 (1)9. Q.E.D.  
 PROOF: There is a connected set  $D$  that contains both  $x$  and  $y$ .

□

PROOF:

- (1)1. LET:  $\{X_\alpha\}_{\alpha \in J}$  be a family of compact spaces.  
 (1)2. LET:  $X = \prod_{\alpha \in J} X_\alpha$   
 (1)3. PICK a well-ordering  $<$  on  $J$  such that  $J$  has a greatest element.  
 (1)4. For  $\alpha \in J$  and  $p = \{p_i \in X_i\}_{i \leq \alpha}$  a family of points,  
 LET:  $Y(p) = \{x \in X \mid \forall i \leq \alpha. x_i = p_i\}$   
 (1)5. If  $\alpha < \alpha'$  and  $p$  is an  $\alpha'$ -indexed family of points then  $Y(p) \subseteq Y(p \upharpoonright \alpha)$   
 PROOF: From definition.  
 (1)6. Given  $\beta \in J$  and  $p = \{p_i \in X_i\}_{i < \beta}$  a family of points,  
 LET:  $Z(p) = \bigcap_{\alpha < \beta} Y(p \upharpoonright \alpha)$   
 (1)7. Given  $\beta \in J$  and  $p = \{p_i \in X_i\}_{i < \beta}$  a family of points, if  $\mathcal{A}$  is a finite set of basic open spaces for  $X$  that covers  $Z(p)$ , then there exists  $\alpha < \beta$  such that  $\mathcal{A}$  covers  $Y(p \upharpoonright \alpha)$   
 (2)1. ASSUME: without loss of generality  $\beta$  has no immediate predecessor.  
 (2)2. For  $A \in \mathcal{A}$ ,  
 LET:  $J_A = \{i < \beta \mid \pi_i(A) \neq X_i\}$   
 (2)3. LET:  $\alpha = \max \bigcup_{A \in \mathcal{A}} J_A$   
 (2)4. LET:  $x \in Y(p \upharpoonright \alpha)$

- (2)5. LET:  $y \in Z(p)$  be the point with  $y_i = p_i$  for  $i < \beta$  and  $y_i = x_i$  for  $i \geq \beta$
  - (2)6. PICK  $A \in \mathcal{A}$  such that  $y \in A$   
PROOF: Since  $\mathcal{A}$  covers  $Z(p)$ .
  - (2)7. For  $i \in J_A$  we have  $x_i \in \pi_i(A)$   
PROOF: Since  $i \leq \alpha$  so  $x_i = p_i$
  - (2)8. For  $i \in J \setminus J_A$  we have  $x_i \in \pi_i(A)$   
PROOF: Since  $\pi_i(A) = X_i$
  - (2)9.  $x \in A$
  - (1)8. ASSUME: for a contraction  $\mathcal{A}$  is a set of basic open sets for  $X$  that covers  $X$  but such that no finite subset of  $\mathcal{A}$  covers  $X$
  - (1)9. PICK a set of points  $\{p_i\}_{i \in J}$  such that, for all  $\alpha \in J$ , we have  $Y(p \upharpoonright \alpha)$  is not finitely covered by  $\mathcal{A}$
  - (2)1. ASSUME: as transfinite induction hypothesis  $\alpha \in J$  and  $\{p_i\}_{i < \alpha}$  is a family of points such that, for all  $\alpha' < \alpha$ , we have  $Y(p \upharpoonright \alpha')$  is not finitely covered by  $\mathcal{A}$
  - (2)2.  $Z(p)$  is not finitely covered by  $\mathcal{A}$   
PROOF: By (1)7.
  - (2)3. PICK  $p_\alpha \in X_\alpha$  such that  $Y(p)$  is not finitely covered by  $\mathcal{A}$   
PROOF: By Lemma 11.51.21 since there is a homeomorphism  $\phi : Z(p) \cong X_\alpha \times \prod_{\alpha' > \alpha} X_{\alpha'}$  and, given  $p_\alpha$ , this homomorphism  $\phi$  restricts to a homeomorphism  $Y(p) \cong \{p_\alpha\} \times \prod_{\alpha' > \alpha} X_{\alpha'}$ .
  - (1)10. Q.E.D.  
PROOF: If  $\omega$  is the greatest element of  $J$  then  $Y(p \upharpoonright \omega)$  is a singleton.
- 

**Theorem 11.51.23.** *Every complete linearly ordered set in the order topology is compact.*

PROOF:

- (1)1. LET:  $X$  be a complete linearly ordered set with least element  $a$  and greatest element  $b$ .
- (1)2. LET:  $\mathcal{A}$  be an open covering of  $X$ .
- (1)3. For all  $x < b$ , there exists  $y > x$  such that  $[x, y]$  can be covered by at most two elements of  $\mathcal{A}$ .
- (2)1. LET:  $x \in X$
- (2)2. PICK  $A \in \mathcal{A}$  with  $x \in A$
- (2)3. PICK  $y > x$  such that  $[x, y] \subseteq A$
- (2)4. PICK  $B \in \mathcal{A}$  with  $y \in B$
- (2)5.  $[x, y]$  is covered by  $A$  and  $B$
- (1)4. LET:  $C = \{y \in X \mid [a, y] \text{ can be covered by finitely many elements of } \mathcal{A}\}$
- (1)5. LET:  $c = \sup C$
- (1)6.  $c > a$   
(2)1. PICK  $x > a$  such that  $[a, x]$  can be covered by at most two elements of  $\mathcal{A}$ .  
PROOF: From (1)3.
- (2)2.  $x \in C$
- (1)7.  $c \in C$



- ⟨2⟩1. PICK  $A \in \mathcal{A}$
- ⟨2⟩2. PICK  $x < c$  such that  $(x, c] \subseteq A$
- ⟨2⟩3. PICK  $y > x$  such that  $y \in C$
- ⟨2⟩4. PICK  $\mathcal{A}_0 \subseteq^{\text{fin}} \mathcal{A}$  that covers  $[a, y]$
- ⟨2⟩5.  $\mathcal{A}_0 \cup \{A\}$  covers  $[a, c]$
- ⟨1⟩8.  $c = b$
- ⟨2⟩1. ASSUME: for a contradiction  $c < b$
- ⟨2⟩2. PICK  $x > c$  such that  $[c, x]$  can be covered by at most two elements of  $\mathcal{A}$
- PROOF: From ⟨1⟩3.
- ⟨2⟩3.  $[a, x]$  can be finitely covered by  $\mathcal{A}$
- PROOF: From ⟨1⟩7.
- ⟨2⟩4. Q.E.D.
- PROOF: This contradicts the maximality of  $c$ .

□

**Corollary 11.51.23.1.** *Let  $X$  be a linearly ordered set with the least upper bound property. Then every closed interval in  $X$  is compact.*

**Corollary 11.51.23.2.** *Every closed interval in  $\mathbb{R}$  is compact.*

**Theorem 11.51.24** (Extreme Value Theorem). *Any linearly ordered set under the order topology that is compact has a greatest and a least element.*

PROOF:

- ⟨1⟩1. LET:  $X$  be a linearly ordered set under the order topology that is compact.
- ⟨1⟩2.  $X$  has a greatest element.
- ⟨2⟩1. ASSUME: for a contradiction  $X$  has no greatest element.
- ⟨2⟩2.  $\{(-\infty, a) \mid a \in X\}$  covers  $X$ .
- ⟨2⟩3. PICK a finite subcover  $\{(-\infty, a_1), \dots, (-\infty, a_n)\}$ , say.
- ⟨2⟩4. ASSUME: without loss of generality  $a_1 \leq \dots \leq a_n$
- ⟨2⟩5.  $X \subseteq (-\infty, a_n)$
- ⟨2⟩6.  $a_n < a_n$
- ⟨1⟩3.  $X$  has a least element.
- PROOF: Similar.

□

**Proposition 11.51.25.** *Every linearly ordered set in which every closed interval is compact satisfies the least upper bound property.*

PROOF:

- ⟨1⟩1. LET:  $X$  be a linearly ordered set in which every closed interval is compact.
- ⟨1⟩2. LET:  $A \subseteq X$  be nonempty with upper bound  $u$
- ⟨1⟩3. PICK  $a \in A$
- ⟨1⟩4. The closed interval  $[a, u]$  is compact.
- ⟨1⟩5. ASSUME: for a contradiction  $A$  has no supremum.
- ⟨1⟩6.  $\{(-\infty, x) \mid x \in A\} \cup \{(x, +\infty) \mid x \text{ is an upper bound of } A\}$  covers  $[a, u]$ .
- ⟨2⟩1. LET:  $x \in [a, u]$

- (2)2. ASSUME: for all  $y \in A$  we have  $x \notin (-\infty, y)$
  - (2)3.  $x$  is an upper bound for  $A$
  - (2)4. PICK an upper bound  $y$  for  $A$  with  $y < x$
  - (2)5.  $x \in (y, +\infty)$
  - (1)7. PICK a finite subcover  $\{(-\infty, x_1), \dots, (-\infty, x_m), (y_1, +\infty), \dots, (y_n, +\infty)\}$
  - (1)8. ASSUME:  $x_m = \max(x_1, \dots, x_m)$  and  $y_1 = \min(y_1, \dots, y_n)$
  - (1)9.  $x_m \notin (-\infty, x_i)$  for any  $i$   
PROOF: Since  $x_i \leq x_m$
  - (1)10.  $x_m \notin (y_i, +\infty)$  for any  $i$   
PROOF: Since  $x_m \in A$  so  $x_m \leq y_i$
  - (1)11.  $x_m \in [a, u]$
  - (2)1.  $a \notin (y_i, +\infty)$  for any  $i$   
PROOF: Since  $y_i$  is an upper bound for  $A$  and  $a \in A$ .
  - (2)2.  $a \in (-\infty, x_i)$  for some  $i$   
PROOF: From (1)7.
  - (2)3.  $a < x_m$   
PROOF: Since  $x_i \leq x_m$
  - (2)4.  $x_m \leq u$   
PROOF: Since  $u$  is an upper bound for  $A$  and  $x_m \in A$ .
  - (1)12. Q.E.D.  
PROOF: This contradicts (1)7.
- 

**Example 11.51.26.** The set  $[0, 1]$  is not compact under the  $K$ -topology.

PROOF: For every  $n \geq 1$ , pick an open interval  $U_n$  such that  $U_n \cap K = \{1/n\}$ . Then the open cover  $\{[0, 1] - K\} \cup \{U_n \mid n \in \mathbb{Z}^+\}$  has no finite subcover. □

**Proposition 11.51.27 (AC).** *Let  $X$  be a compact Hausdorff space. Let  $\mathcal{A}$  be a countable set of closed sets in  $X$ . If every element of  $\mathcal{A}$  has empty interior, then  $\bigcup \mathcal{A}$  has empty interior.*

PROOF:

- (1)1. LET:  $X$  be a compact Hausdorff space.
- (1)2. For every closed set  $A$  in  $X$  and open  $U$  in  $X$  with  $U \not\subseteq A$ , there exists a nonempty open set  $V$  such that  $\bar{V} \subseteq U - A$ .
- (2)1. LET:  $A$  be a closed set in  $X$
- (2)2. LET:  $U$  be an open set in  $X$  with  $U \not\subseteq A$
- (2)3. PICK  $x \in U - A$
- (2)4. PICK disjoint neighbourhoods  $W$  and  $V$  of  $A \cup (X - U)$  and  $x$  respectively.  
PROOF: Proposition 11.51.10.
- (2)5.  $\bar{V} \subseteq U - A$

PROOF:

$$\begin{aligned}
 \overline{V} &\subseteq X - W && (\text{since } V \subseteq X - W) \\
 &\subseteq X - (A \cup (X - U)) \\
 &= (X - A) \cap U \\
 &= U - A
 \end{aligned}$$

$\langle 1 \rangle 3$ . PICK an enumeration  $\{A_1, A_2, \dots\}$  of  $\mathcal{A}$

$\langle 1 \rangle 4$ . LET:  $U_0$  be any nonempty open set

PROVE:  $U_0 \not\subseteq \bigcup \mathcal{A}$

$\langle 1 \rangle 5$ . PICK a sequence of nonempty open sets  $U_1, U_2, \dots$  such that, for  $n \geq 1$ , we have  $\overline{U_n} \subseteq U_{n-1} - A_n$

$\langle 2 \rangle 1$ . ASSUME: we have picked  $U_0, U_1, \dots, U_n$

$\langle 2 \rangle 2$ .  $U_n \not\subseteq A_{n+1}$

PROOF: Since  $A_{n+1}$  has empty interior.

$\langle 2 \rangle 3$ . PICK a nonempty open set  $U_{n+1}$  such that  $\overline{U_{n+1}} \subseteq U_n - A_{n+1}$

PROOF: By  $\langle 1 \rangle 2$

$\langle 1 \rangle 6$ . PICK  $a \in \bigcap_{n=0}^{\infty} \overline{U_n}$

PROOF: Corollary 11.51.6.1.

$\langle 1 \rangle 7$ .  $a \in U_0$

PROOF: Since  $a \in \overline{U_1} \subseteq U_0$ .

$\langle 1 \rangle 8$ .  $a \notin \bigcup \mathcal{A}$

PROOF: For all  $n$ , we have  $a \in \overline{U_n} \subseteq U_{n-1} - A_n$ .

□

**Example 11.51.28.** The Cantor set is compact.

PROOF: It is a closed subset of the compact set  $[0, 1]$ . □

**Proposition 11.51.29.** Every compact space is limit point compact.

PROOF:

$\langle 1 \rangle 1$ . LET:  $X$  be a compact space.

$\langle 1 \rangle 2$ . LET:  $A \subseteq X$  have no limit points.

PROVE:  $A$  is finite.

$\langle 1 \rangle 3$ .  $A$  is closed.

PROOF: Corollary 11.6.3.1.

$\langle 1 \rangle 4$ .  $A$  is compact.

PROOF: Proposition 11.51.3.

$\langle 1 \rangle 5$ .  $\{U \mid U \text{ open, } |U \cap A| = 1\}$  covers  $A$ .

PROOF: From  $\langle 1 \rangle 2$ , for all  $a \in A$ , there is a neighbourhood  $U$  of  $a$  that intersects  $A$  in  $a$  only.

$\langle 1 \rangle 6$ . PICK a finite subcover  $\{U_1, \dots, U_n\}$

$\langle 1 \rangle 7$ . For  $i = 1, \dots, n$ ,

LET:  $U_i \cap A = \{x_i\}$ .

$\langle 1 \rangle 8$ .  $A = \{x_1, \dots, x_n\}$

□

The following examples show that not every limit point compact space is compact.

**Example 11.51.30.** Let  $Y$  be a set with two elements under the indiscrete topology. Then  $\mathbb{Z}^+ \times Y$  is limit point compact, since every nonempty set has a limit point. It is not compact, since  $\{\{n\} \times Y \mid n \in \mathbb{Z}^+\}$  has no finite subcover.

**Example 11.51.31.** The space  $S_\Omega$  is limit point compact but not compact.

PROOF:

(1)1.  $S_\Omega$  is not compact.

PROOF: From the Extreme Value Theorem, since  $S_\Omega$  has no greatest element.

(1)2. LET:  $A$  be an infinite subset of  $S_\Omega$ .

(1)3. PICK  $B \subseteq A$  that is countably infinite.

PROOF: Proposition ??.

(1)4. LET:  $b = \sup B$

(1)5.  $B \subseteq [0, b]$

(1)6.  $[0, b]$  is compact.

PROOF: Corollary 11.51.23.1.

(1)7. PICK a limit point  $x$  of  $B$  in  $[0, b]$ .

PROOF: Proposition 11.51.29.

(1)8.  $x$  is a limit point of  $A$ .

PROOF: Lemma 11.6.5.

□

**Proposition 11.51.32 (AC).** *A topological space is compact if and only if every net has a convergent subnet.*

PROOF:

(1)1. LET:  $X$  be a topological space.

(1)2. If  $X$  is compact then every net has a convergent subnet.

(2)1. ASSUME:  $X$  is compact.

(2)2. LET:  $(a_\alpha)_{\alpha \in J}$  be a net in  $X$ .

(2)3. For  $\alpha \in J$ ,

LET:  $B_\alpha = \{a_\beta \mid \alpha \leq \beta\}$

(2)4.  $\{B_\alpha \mid \alpha \in J\}$  has the finite intersection property.

(2)5. PICK  $x \in \bigcap_{\alpha \in J} \overline{B_\alpha}$

(2)6.  $x$  is an accumulation point of  $(a_\alpha)_{\alpha \in J}$

(3)1. LET:  $U$  be a neighbourhood of  $x$ .

(3)2. LET:  $\alpha \in J$

(3)3.  $x \in \overline{B_\alpha}$

(3)4. There exists  $\beta \geq \alpha$  such that  $a_\beta \in U$

(2)7. Q.E.D.

PROOF: Lemma 11.50.2.

(1)3. If every net in  $X$  has a convergent subnet then  $X$  is compact.

(2)1. ASSUME: Every net in  $X$  has a convergent subnet.

(2)2. LET:  $\mathcal{A}$  be a set of closed sets with the finite intersection property.

(2)3. LET:  $\mathcal{B}$  be the set of all finite intersections of elements of  $\mathcal{A}$  under  $\supseteq$ .

(2)4. For  $B \in \mathcal{B}$ , PICK  $a_B \in B$

(2)5. PICK a convergent subnet  $(a_{g(\alpha)})_{\alpha \in K}$  with limit  $l$ .

PROVE:  $l \in \bigcap \mathcal{A}$

PROOF: From  $\langle 2 \rangle 1$ .

$\langle 2 \rangle 6$ . LET:  $A \in \mathcal{A}$

$\langle 2 \rangle 7$ . ASSUME: for a contradiction  $l \notin A$

$\langle 2 \rangle 8$ . PICK  $\alpha \in K$  such that, for all  $\beta \geq \alpha$ , we have  $a_{g(\beta)} \in X - A$

$\langle 2 \rangle 9$ . PICK  $\beta \in K$  such that  $g(\beta) \geq A$

$\langle 2 \rangle 10$ . PICK  $\gamma \in K$  with  $\alpha \leq \gamma$  and  $\beta \leq \gamma$

$\langle 2 \rangle 11$ .  $a_{g(\gamma)} \in A$  and  $a_{g(\gamma)} \in X - A$

$\langle 2 \rangle 12$ . Q.E.D.

PROOF: This is a contradiction.

□

**Example 11.51.33.** The space  $\mathbb{R}_K$  is not path connected.

PROOF:

$\langle 1 \rangle 1$ . ASSUME: for a contradiction  $p : [0, 1] \rightarrow \mathbb{R}_K$  was a path from 0 to 1.

$\langle 1 \rangle 2$ .  $p([0, 1])$  as a subspace of  $\mathbb{R}_K$  is compact.

PROOF: Theorem 11.51.4.

$\langle 1 \rangle 3$ .  $p([0, 1])$  as a subspace of  $\mathbb{R}_K$  is connected.

PROOF: Theorem 11.31.13.

$\langle 1 \rangle 4$ .  $p([0, 1])$  is connected as a subspace of  $\mathbb{R}$ .

PROOF: Theorem 11.31.13 as the identity map is continuous as a map  $\mathbb{R}_K \rightarrow \mathbb{R}$ .

$\langle 1 \rangle 5$ .  $p([0, 1])$  is convex.

$\langle 2 \rangle 1$ . LET:  $a, b \in p([0, 1])$  and  $a < c < b$

$\langle 2 \rangle 2$ . ASSUME: for a contradiction  $c \notin p([0, 1])$

$\langle 2 \rangle 3$ .  $(-\infty, c) \cap p([0, 1])$  and  $(c, +\infty) \cap p([0, 1])$  form a separation of  $p([0, 1])$  as a subspace of  $\mathbb{R}$ .

$\langle 2 \rangle 4$ . Q.E.D.

PROOF: This contradicts  $\langle 1 \rangle 4$ .

$\langle 1 \rangle 6$ .  $[0, 1] \subseteq p([0, 1])$

$\langle 1 \rangle 7$ .  $[0, 1]$  as a subspace of  $\mathbb{R}_K$  is compact.

PROOF: By Proposition 11.51.3 and  $\langle 1 \rangle 2$ .

$\langle 1 \rangle 8$ . Q.E.D.

PROOF: This contradicts Example 11.51.26.

□

**Proposition 11.51.34** (Choice). *The product of a Lindelöf and a compact space is Lindelöf.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $X$  be a Lindelöf space.

$\langle 1 \rangle 2$ . LET:  $Y$  be a compact space.

$\langle 1 \rangle 3$ . LET:  $\mathcal{A}$  be an open cover of  $X \times Y$ .

$\langle 1 \rangle 4$ .  $\{W \text{ open in } X \mid W \times Y \text{ can be covered by finitely many elements of } \mathcal{A}\}$  is an open cover of  $X$ .

$\langle 2 \rangle 1$ . LET:  $x \in X$

$\langle 2 \rangle 2$ .  $\{V \text{ open in } Y \mid \exists U \text{ open in } X. \exists A \in \mathcal{A}. x \in U \text{ and } U \times V \subseteq A\}$  covers  $Y$

- (2)3. PICK a finite subcover  $V_1, \dots, V_n$
  - (2)4.  $\{x\} \times Y \subseteq V_1 \cup \dots \cup V_n$
  - (2)5. PICK a neighbourhood  $W$  of  $x$  such that  $W \times Y \subseteq V_1 \cup \dots \cup V_n$
  - PROOF: By the Tube Lemma.
  - (1)5. PICK a countable subcover  $\mathcal{B}$
  - (1)6. For  $W \in \mathcal{B}$ , PICK  $U_{W1}, \dots, U_{Wn_W} \in \mathcal{A}$  such that  $W \times Y \subseteq U_{W1} \cup \dots \cup U_{Wn_W}$
  - (1)7.  $\{U_{Wi} \mid W \in \mathcal{B}, 1 \leq i \leq n_W\}$  covers  $X \times Y$
- 

## 11.52 Perfect Maps

**Definition 11.52.1** (Perfect Map). Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Then  $f$  is a *perfect map* if and only if  $f$  is a closed map, continuous, surjective and  $p^{-1}(y)$  is compact for all  $y \in Y$ .

**Proposition 11.52.2.** Let  $X$  be a topological space,  $Y$  a compact space, and  $p : X \rightarrow Y$  a closed map such that, for all  $y \in Y$ , we have  $p^{-1}(y)$  is compact. Then  $X$  is compact.

PROOF:

- (1)1. LET:  $\mathcal{A}$  be a set of closed sets in  $X$  with the finite intersection property.
- (1)2.  $\mathcal{B} = \{p(A_1 \cap \dots \cap A_n) \mid A_1, \dots, A_n \in \mathcal{A}\}$  is a set of closed sets in  $Y$  with the finite intersection property.

PROOF: Since  $p$  is a closed map.

- (1)3. PICK  $y \in \bigcap \mathcal{B}$

PROOF: Theorem 11.51.6 since  $Y$  is compact.

- (1)4.  $\{A \cap p^{-1}(y) \mid A \in \mathcal{A}\}$  is a set of closed sets in  $p^{-1}(y)$  with the finite intersection property.

- (1)5. PICK  $x \in \bigcap_{A \in \mathcal{A}} (A \cap p^{-1}(y))$

PROOF: Theorem 11.51.6 since  $p^{-1}(y)$  is compact.

- (1)6.  $x \in \bigcap \mathcal{A}$

- (1)7. Q.E.D.

PROOF: Theorem 11.51.6.

□

**Proposition 11.52.3.** The image of a Hausdorff space under a perfect map is Hausdorff.

PROOF:

- (1)1. LET:  $X$  be a Hausdorff space.
- (1)2. LET:  $Y$  be a topological space.
- (1)3. LET:  $p : X \rightarrow Y$  be a perfect map.
- (1)4. LET:  $y, y' \in Y$  with  $y \neq y'$
- (1)5. PICK disjoint open neighbourhoods  $U_1$  and  $U_2$  of  $p^{-1}(y)$  and  $p^{-1}(y')$

PROOF: Proposition 11.51.10.

⟨1⟩6. PICK neighbourhoods  $V_1$  and  $V_2$  of  $y$  and  $y'$  respectively such that  $p^{-1}(V_1) \subseteq U_1$  and  $p^{-1}(V_2) \subseteq U_2$

PROOF: Lemma 11.13.2,

⟨1⟩7.  $V_1 \cap V_2 = \emptyset$

⟨2⟩1. ASSUME: for a contradiction  $z \in V_1 \cap V_2$

⟨2⟩2. PICK  $x \in X$  such that  $p(x) = z$

⟨2⟩3.  $x \in U_1 \cap U_2$

PROOF: From ⟨1⟩6.

⟨2⟩4. Q.E.D.

PROOF: This contradicts the fact that  $U_1$  and  $U_2$  are disjoint (⟨1⟩5).

□

**Proposition 11.52.4.** *The image of a regular space under a perfect map is regular.*

PROOF:

⟨1⟩1. LET:  $X$  be a regular space.

⟨1⟩2. LET:  $Y$  be a topological space.

⟨1⟩3. LET:  $p : X \rightarrow Y$  be a perfect map.

⟨1⟩4.  $Y$  is  $T_1$ .

PROOF: Proposition 11.52.3.

⟨1⟩5. LET:  $A$  be a closed set in  $Y$  and  $a \in Y - A$ .

⟨1⟩6.  $p^{-1}(A)$  is closed in  $X$  and  $p^{-1}(a)$  is disjoint from  $p^{-1}(A)$

⟨1⟩7. PICK disjoint open neighbourhoods  $U_1$  and  $U_2$  of  $p^{-1}(A)$  and  $p^{-1}(a)$  respectively.

PROOF: Proposition 11.29.9.

⟨1⟩8. PICK open neighbourhoods  $V_1$  and  $V_2$  of  $A$  and  $a$  respectively such that  $p^{-1}(V_1) \subseteq U_1$  and  $p^{-1}(V_2) \subseteq U_2$ .

PROOF: Lemma 11.13.2.

⟨1⟩9.  $V_1 \cap V_2 = \emptyset$

⟨2⟩1. ASSUME: for a contradiction  $y \in V_1 \cap V_2$

⟨2⟩2. PICK  $x \in X$  such that  $p(x) = y$

⟨2⟩3.  $x \in U_1 \cap U_2$

PROOF: From ⟨1⟩8

⟨2⟩4. Q.E.D.

PROOF: This contradicts the fact that  $U_1$  and  $U_2$  are disjoint (⟨1⟩7).

□

**Proposition 11.52.5.** *The image of a second countable space under a perfect map is second countable.*

PROOF:

⟨1⟩1. LET:  $X$  be a second countable space.

⟨1⟩2. LET:  $Y$  be a topological space.

⟨1⟩3. LET:  $p : X \rightarrow Y$  be a perfect map.

⟨1⟩4. PICK a countable basis  $\mathcal{B}$  for  $X$ .

⟨1⟩5. For  $J \subseteq^{\text{fin}} \mathcal{B}$ ,

LET:  $U_J = \bigcup \{W \mid W \text{ open in } Y, p^{-1}(W) \subseteq \bigcup J\}$   
 PROVE:  $\{U_J \mid J \subseteq^{\text{fin}} \mathcal{B}\}$  is a basis for  $Y$ .  
 (1)6. LET:  $y \in Y$   
 (1)7. LET:  $V$  be a neighbourhood of  $y$ .  
 (1)8. For all  $x \in p^{-1}(y)$ , there exists  $B \in \mathcal{B}$  such that  $x \in B \subseteq p^{-1}(V)$   
 (1)9. PICK finitely many sets  $B_1, \dots, B_n \in \mathcal{B}$  that cover  $p^{-1}(y)$  such that  $B_i \subseteq p^{-1}(V)$  for all  $i$ .  
 (1)10. LET:  $J = \{B_1, \dots, B_n\}$   
 PROVE:  $y \in U_J \subseteq V$   
 (1)11.  $y \in U_J$   
 PROOF: There exists an open neighbourhood  $W$  of  $y$  such that  $p^{-1}(W) \subseteq B_1 \cup \dots \cup B_n$  by Lemma 11.13.2.  
 (1)12.  $U_J \subseteq V$   
 PROOF: Since  $\bigcup J \subseteq V$  by (1)9.  
 □

## 11.53 Isolated Points

**Definition 11.53.1** (Isolated Point). Let  $X$  be a topological space and  $x \in X$ . Then  $x$  is an *isolated point* if and only if  $\{x\}$  is open.

**Theorem 11.53.2** (AC). A nonempty compact Hausdorff space with no isolated points is uncountable.

PROOF:

(1)1. LET:  $X$  be a nonempty compact Hausdorff space with no isolated points.  
 (1)2. For every nonempty open set  $U$  and every point  $x \in X$ , there exists a nonempty open set  $V \subseteq U$  such that  $x \notin \overline{V}$ .  
 (2)1. LET:  $U$  be a nonempty open set.  
 (2)2. LET:  $x \in X$   
 (2)3. PICK  $y \in U - \{x\}$   
 PROOF: This is possible because  $U$  cannot be  $\{x\}$ .  
 (2)4. PICK disjoint open neighbourhoods  $W_1$  of  $x$  and  $W_2$  of  $y$   
 (2)5. LET:  $V = W_2 \cap U$   
 (2)6.  $V$  is nonempty  
 PROOF: Since  $y \in V$   
 (2)7.  $V$  is open  
 PROOF: From (2)1, (2)4, (2)5.  
 (2)8.  $V \subseteq U$   
 PROOF: From (2)5  
 (2)9.  $x \notin \overline{V}$   
 PROOF: From (2)4 and (2)5  
 (1)3. LET:  $(a_n)$  be any sequence of points in  $X$ .  
 PROVE: The set  $X - \{a_1, a_2, \dots\}$  is nonempty.  
 (1)4. PICK a sequence of nonempty open sets  $V_1, V_2, \dots$ , such that  $V_1 \supseteq V_2 \supseteq \dots$  and  $a_n \notin \overline{V_n}$  for all  $n$ .



PROOF: From  $\langle 1 \rangle 2$ .  
 $\langle 1 \rangle 5$ . PICK  $a \in \bigcap_{n=1}^{\infty} \overline{V_n}$   
 PROOF: Corollary 11.51.6.1.  
 $\langle 1 \rangle 6$ .  $a \in X - \{a_1, a_2, \dots\}$   
 PROOF: We cannot have  $a = a_n$  because  $a \in \overline{V_n}$ .  
 $\square$

**Corollary 11.53.2.1.** *For all  $a, b \in \mathbb{R}$  with  $a < b$ , the closed interval  $[a, b]$  is uncountable.*

**Example 11.53.3.** The Cantor set has no isolated points, and is therefore uncountable.

PROOF:

$\langle 1 \rangle 1$ . LET:  $(A_n)$  be the sets in Definition 9.1.1.  
 $\langle 1 \rangle 2$ . LET:  $x \in C$   
 $\langle 1 \rangle 3$ . LET:  $A_n$  be the first set such that  $x$  is an endpoint of one of the intervals that make up  $A_n$   
 $\langle 1 \rangle 4$ . LET:  $(a_m)_{m \geq n}$  be the sequence of points defined by:  $a_m$  is the point such that either  $[a_m, x]$  or  $[x, a_m]$  is one of the intervals that make up  $A_m$ .  
 $\langle 1 \rangle 5$ .  $(a_m)$  is a sequence of points of  $C$  distinct from  $x$  that converges to  $x$ .  
 PROOF: Since  $|a_m - x| = 1/3^m$  for all  $m$ .  
 $\langle 1 \rangle 6$ .  $x$  is a limit point of  $C$ .  
 $\square$

## 11.54 Local Compactness

**Definition 11.54.1** (Locally Compact). Let  $X$  be a topological space and  $x \in X$ . Then  $X$  is *locally compact* at  $x$  if and only if there exists a compact subspace of  $X$  that includes a neighbourhood of  $x$ .

A space is *locally compact* if and only if it is locally compact at every point.

**Example 11.54.2.** The real line is locally compact, because for every real number  $x$  we have  $x \in (x-1, x+1) \subseteq [x-1, x+1]$ .

**Example 11.54.3.** For all  $n \geq 1$ , we have  $\mathbb{R}^n$  is locally compact. For any point  $x = (x_1, \dots, x_n)$ , we have  $x \in (x_1-1, x_1+1) \times \dots \times (x_n-1, x_n+1) \subseteq [x_1-1, x_1+1] \times \dots \times [x_n-1, x_n+1]$ .

The following example shows that a countable product of locally compact spaces is not necessarily locally compact.

**Example 11.54.4.** The space  $\mathbb{R}^{\omega}$  is not locally compact.

PROOF:

$\langle 1 \rangle 1$ . ASSUME: for a contradiction  $0 \in U \subseteq C$  where  $U$  is open and  $C$  is compact.

- (1)2. PICK a basic open set  $B = (a_1, b_1) \times \cdots \times (a_n, b_n) \times \mathbb{R} \times \mathbb{R} \times \cdots$  such that  $0 \in B \subseteq U$   
 (1)3.  $\overline{B} = [a_1, b_1] \times \cdots \times [a_n, b_n] \times \mathbb{R} \times \mathbb{R} \times \cdots$  is compact.  
 PROOF: Proposition 11.51.3.  
 (1)4. Q.E.D.  
 PROOF: This is a contradiction.  
 □

**Example 11.54.5.** Every linearly ordered set  $X$  with the least upper bound property is locally compact under the order topology.

For any point  $x$ , pick a basic open set  $B$  such that  $x \in B$ . Then  $x \in B \subseteq \overline{B}$  and  $\overline{B}$  is a closed interval, hence compact (Corollary 11.51.23.1).

**Proposition 11.54.6.** *Any closed subspace of a locally compact space is locally compact.*

PROOF:

- (1)1. LET:  $X$  be a locally compact space and  $Y \subseteq X$  be closed.  
 (1)2. LET:  $y \in Y$ .  
 (1)3. PICK a compact subspace  $C$  of  $X$  and neighbourhood  $U$  of  $y$  in  $X$  such that  $U \subseteq C$   
 (1)4.  $y \in U \cap Y \subseteq C \cap Y$   
 (1)5.  $C \cap Y$  is compact.  
 PROOF: Proposition 11.51.3.  
 □

**Proposition 11.54.7.** *Let  $X$  be a Hausdorff space. Let  $x \in X$ . Then  $X$  is locally compact at  $x$  if and only if, for every neighbourhood  $U$  of  $x$ , there exists a neighbourhood  $V$  of  $x$  such that  $\overline{V}$  is compact and  $\overline{V} \subseteq U$ .*

**Corollary 11.54.7.1.** *Every open subspace of a locally compact Hausdorff space is locally compact Hausdorff.*

This example shows that a subspace of a locally compact Hausdorff space is not necessarily locally compact Hausdorff.

**Example 11.54.8.** The rationals  $\mathbb{Q}$  are not locally compact.

Assume for a contradiction  $C \subseteq \mathbb{Q}$  is compact and includes  $(-\epsilon, \epsilon) \cap \mathbb{Q}$ . Pick an irrational  $\xi \in (-\epsilon, \epsilon)$ . Then  $\{(-\infty, q) \cap \mathbb{Q} \mid q \in \mathbb{Q}, q < \xi\} \cup \{(q, +\infty) \cap \mathbb{Q} \mid q \in \mathbb{Q}, q > \xi\}$  covers  $C$  but no finite subcover does.

**Proposition 11.54.9.** *Let  $\{X_\alpha\}_{\alpha \in J}$  be a family of nonempty spaces. If  $\prod_{\alpha \in J} X_\alpha$  is locally compact under the box topology then each  $X_\alpha$  is locally compact.*

PROOF:

- (1)1. LET:  $\alpha \in J$   
 (1)2. LET:  $x_\alpha \in X_\alpha$   
 (1)3. Extend  $x_\alpha$  to a family  $(x_\beta)_{\beta \in J} \in \prod_{\beta \in J} X_\beta$

- ⟨1⟩4. PICK a compact  $C \subseteq \prod_{\beta \in J} X_\beta$  that includes a basic open neighbourhood  $\prod_{\beta \in J} U_\beta$  of  $(x_\beta)$  such that each  $U_\beta$  is open in  $X_\beta$ .
- ⟨1⟩5.  $x_\alpha \in U_\alpha \subseteq \pi_\alpha(C)$
- ⟨1⟩6.  $\pi_\alpha(C)$  is compact.

PROOF: Theorem 11.51.4.

□

**Proposition 11.54.10 (AC).** *Let  $\{X_\alpha\}_{\alpha \in J}$  be a family of nonempty spaces. Then  $\prod_{\alpha \in J} X_\alpha$  is locally compact if and only if each  $X_\alpha$  is locally compact, and  $X_\alpha$  is compact for all but finitely many  $\alpha \in J$ .*

PROOF:

- ⟨1⟩1. LET:  $\{X_\alpha\}_{\alpha \in J}$  be a family of nonempty spaces.
- ⟨1⟩2. If  $\prod_{\alpha \in J} X_\alpha$  is locally compact then each  $X_\alpha$  is locally compact.
  - ⟨2⟩1. ASSUME:  $\prod_{\alpha \in J} X_\alpha$  is locally compact.
  - ⟨2⟩2. For all  $\alpha \in J$  we have  $X_\alpha$  is locally compact.
    - ⟨3⟩1. LET:  $\alpha \in J$
    - ⟨3⟩2. LET:  $x_\alpha \in X_\alpha$
    - ⟨3⟩3. Extend  $x_\alpha$  to a family  $(x_\beta)_{\beta \in J} \in \prod_{\beta \in J} X_\beta$
    - ⟨3⟩4. PICK a compact  $C \subseteq \prod_{\beta \in J} X_\beta$  that includes a basic open neighbourhood  $\prod_{\beta \in J} U_\beta$  of  $(x_\beta)$  such that each  $U_\beta$  is open in  $X_\beta$ .
    - ⟨3⟩5.  $x_\alpha \in U_\alpha \subseteq \pi_\alpha(C)$
    - ⟨3⟩6.  $\pi_\alpha(C)$  is compact.
- PROOF: Theorem 11.51.4.
- ⟨1⟩3. If  $\prod_{\alpha \in J} X_\alpha$  is locally compact then  $X_\alpha$  is compact for all but finitely many  $\alpha \in J$ .
  - ⟨2⟩1. ASSUME:  $\prod_{\alpha \in J} X_\alpha$  is locally compact.
  - ⟨2⟩2. PICK  $x_\alpha \in X_\alpha$  for all  $\alpha$ .
  - ⟨2⟩3. PICK a compact  $C \subseteq \prod_{\alpha \in J} X_\alpha$  that includes a basic open neighbourhood  $\prod_{\alpha \in J} U_\alpha$  of  $(x_\alpha)$  such that each  $U_\alpha$  is open in  $X_\alpha$ , and  $U_\alpha = X_\alpha$  for all but finitely many  $\alpha$ .
  - ⟨2⟩4. For all but finitely many  $\alpha \in J$ , we have  $X_\alpha = \pi_\alpha(C)$
  - ⟨2⟩5. For all but finitely many  $\alpha \in J$ , we have  $X_\alpha$  is compact.

PROOF: Theorem 11.51.4.

- ⟨1⟩4. If each  $X_\alpha$  is locally compact and  $X_\alpha$  is compact for all but finitely many  $\alpha \in J$  then  $\prod_{\alpha \in J} X_\alpha$  is locally compact.
  - ⟨2⟩1. ASSUME:  $X_\alpha$  is compact for all  $\alpha$  except  $\alpha_1, \dots, \alpha_n$
  - ⟨2⟩2. ASSUME:  $X_{\alpha_1}, \dots, X_{\alpha_n}$  are locally compact.
  - ⟨2⟩3. LET:  $(x_\alpha) \in \prod X_\alpha$
  - ⟨2⟩4. For  $i = 1, \dots, n$ , PICK a compact  $C_{\alpha_i} \subseteq X_{\alpha_i}$  that includes the neighbourhood  $U_{\alpha_i}$  of  $x_{\alpha_i}$ .
  - ⟨2⟩5. For  $\alpha \neq \alpha_1, \dots, \alpha_n$ ,  
LET:  $C_\alpha = U_\alpha = X_\alpha$
  - ⟨2⟩6.  $\prod_{\alpha \in J} C_\alpha$  is compact.
  - PROOF: Tychonoff's Theorem.
  - ⟨2⟩7.  $(x_\alpha) \in \prod U_\alpha \subseteq \prod C_\alpha$

□

The following example shows that the continuous image of a locally compact space is not necessarily locally compact.

**Example 11.54.11.** Pick an enumeration  $\{q_1, q_2, \dots\}$  of  $\mathbb{Q}$ . Let  $X = \bigcup_{n \in \mathbb{Z}^+} (n, n+1)$ . Define  $f : X \rightarrow \mathbb{Q}$  by  $f(x) = q_n$  if  $x \in (n, n+1)$ . Then  $f$  is continuous,  $X$  is locally compact, but  $f(X) = \mathbb{Q}$  is not locally compact.

**Proposition 11.54.12.** *The image of a locally compact space under a continuous open map is locally compact.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $X$  be locally compact and  $f : X \rightarrow Y$  be a surjective continuous open map.
  - $\langle 1 \rangle 2$ . LET:  $y \in Y$
  - $\langle 1 \rangle 3$ . PICK  $x \in X$  such that  $f(x) = y$
  - $\langle 1 \rangle 4$ . PICK a compact  $C \subseteq X$  that includes a neighbourhood  $U$  of  $x$
  - $\langle 1 \rangle 5$ .  $y \in f(U) \subseteq f(C)$  and  $f(U)$  is open,  $f(C)$  is compact.
- 

**Lemma 11.54.13.** *Let  $X, Y$  and  $Z$  be topological spaces and  $p : X \rightarrow Y$ . If  $p$  is a quotient map and  $Z$  is locally compact Hausdorff, then  $p \times \text{id}_Z : X \times Z \rightarrow Y \times Z$  is a quotient map.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $X, Y$  and  $Z$  be topological spaces and  $p : X \rightarrow Y$ .
  - $\langle 1 \rangle 2$ . ASSUME:  $p$  is a quotient map and  $Z$  is locally compact Hausdorff.
  - $\langle 1 \rangle 3$ . LET:  $\pi = p \times \text{id}_Z$
  - $\langle 1 \rangle 4$ .  $\pi$  is surjective.
  - $\langle 1 \rangle 5$ .  $\pi$  is continuous.
  - $\langle 1 \rangle 6$ .  $\pi$  is strongly continuous.
  - $\langle 2 \rangle 1$ . LET:  $A \subseteq Y \times Z$
  - $\langle 2 \rangle 2$ . ASSUME:  $\pi^{-1}(A)$  is open.
  - $\langle 2 \rangle 3$ . LET:  $(y, z) \in A$
  - $\langle 2 \rangle 4$ . PICK  $x \in X$  such that  $p(x) = y$
  - $\langle 2 \rangle 5$ . PICK open sets  $U_1$  in  $X$  and  $V$  in  $Z$  such that  $x \in U_1$ ,  $z \in V$ ,  $\bar{V}$  is compact, and  $U_1 \times \bar{V} \subseteq \pi^{-1}(A)$
  - $\langle 3 \rangle 1$ . PICK open sets  $U_1$  in  $X$  and  $V'$  in  $Z$  such that  $x \in U_1$ ,  $z \in V'$  and  $U' \times V' \subseteq \pi^{-1}(A)$
  - $\langle 3 \rangle 2$ . PICK  $V$  open in  $Z$  such that  $z \in V$ ,  $\bar{V}$  is compact and  $\bar{V} \subseteq V'$
- PROOF: Proposition 11.54.7.
- $\langle 2 \rangle 6$ . LET:  $U = \bigcup \{U' \text{ open in } X \mid U' \times \bar{V} \subseteq \pi^{-1}(A)\}$
  - $\langle 2 \rangle 7$ .  $U$  is saturated
  - $\langle 3 \rangle 1$ . LET:  $a \in U$ ,  $b \in X$  with  $p(a) = p(b)$
  - $\langle 3 \rangle 2$ .  $\{b\} \times \bar{V} \subseteq \pi^{-1}(A)$
  - $\langle 3 \rangle 3$ . PICK  $U'$  open in  $X$  such that  $b \in U'$  and  $U' \times \bar{V} \subseteq \pi^{-1}(A)$
- PROOF: By the Tube Lemma.
- $\langle 3 \rangle 4$ .  $b \in U' \subseteq U$
  - $\langle 2 \rangle 8$ .  $\pi(U \times V)$  is open

PROOF: Since  $\pi(U \times V) = p(U) \times V$ .

$\langle 2 \rangle 9. (y, z) \in \pi(U \times V)$

$\langle 2 \rangle 10. \pi(U \times V) \subseteq A$

□

**Theorem 11.54.14.** *Let  $A, B, C$  and  $D$  be topological spaces with  $B$  and  $C$  locally compact Hausdorff. Let  $p : A \twoheadrightarrow B$  and  $q : C \twoheadrightarrow D$  be quotient maps. Then  $p \times q : A \times C \twoheadrightarrow B \times D$ .*

PROOF: By Lemma 11.54.13 since  $p \times q = (\text{id}_B \times q) \circ (p \times \text{id}_C)$ . □

**Proposition 11.54.15.** *The image of a locally compact space under a perfect map is locally compact.*

PROOF:

$\langle 1 \rangle 1.$  Let  $X$  be a locally compact space and  $p : X \twoheadrightarrow Y$  be a perfect map.

$\langle 1 \rangle 2.$  LET:  $y \in Y$

$\langle 1 \rangle 3.$  For all  $x \in p^{-1}(y)$ , there exists a neighbourhood of  $x$  that is included in a compact subspace of  $X$ .

$\langle 1 \rangle 4.$   $\{U \text{ open in } X \mid \exists C \subseteq X. C \text{ is compact and } U \subseteq C\}$  covers  $p^{-1}(y)$

$\langle 1 \rangle 5.$  PICK a finite subcover  $\{U_1, \dots, U_n\}$

$\langle 1 \rangle 6.$  For  $1 \leq i \leq n$ , PICK  $C_i \subseteq X$  such that  $C_i$  is compact

$\langle 1 \rangle 7.$  LET:  $C = C_1 \cup \dots \cup C_n$

$\langle 1 \rangle 8.$   $C$  is compact

PROOF: Proposition 11.51.9.

$\langle 1 \rangle 9.$  PICK an open neighbourhood  $V$  of  $y$  such that  $p^{-1}(V) \subseteq U_1 \cup \dots \cup U_n$

PROOF: Lemma 11.13.2.

$\langle 1 \rangle 10.$   $V \subseteq p(C)$

$\langle 2 \rangle 1.$  LET:  $y \in V$

$\langle 2 \rangle 2.$  PICK  $x \in X$  such that  $p(x) = y$

$\langle 2 \rangle 3.$   $x \in C$

$\langle 2 \rangle 4.$   $y \in p(C)$

$\langle 1 \rangle 11.$   $p(C)$  is compact.

PROOF: Theorem 11.51.4.

□

## 11.55 Compactifications

**Definition 11.55.1** (Compactification). Let  $X$  be a topological space. A *compactification* of  $X$  consists of a compact Hausdorff space  $Y$  and an imbedding  $X \rightarrow Y$ .

**Definition 11.55.2** (One-Point Compactification). Let  $X$  be a topological space. A *one-point compactification* of  $X$  is a compactification  $i : X \rightarrow Y$  such that  $Y - i(x)$  consists of a single point.

**Theorem 11.55.3.** *Let  $X$  be a topological space. Then  $X$  is locally compact Hausdorff if and only if there exists a one-point compactification  $i : X \rightarrow Y$ . In this case,  $Y$  is unique up to unique homeomorphism that commutes with  $i$ .*

PROOF:

- ⟨1⟩1. For any compact Hausdorff space  $Y$  and point  $a \in Y$ , the space  $Y - \{a\}$  is locally compact Hausdorff.
- ⟨2⟩1. LET:  $Y$  be a compact Hausdorff space.
- ⟨2⟩2. LET:  $a \in Y$
- ⟨2⟩3.  $Y - \{a\}$  is closed.
- ⟨2⟩4.  $Y - \{a\}$  is locally compact.
- PROOF: Proposition 11.54.6.
- ⟨2⟩5.  $Y - \{a\}$  is Hausdorff.
- PROOF: Theorem 11.21.6.
- ⟨1⟩2. For any locally compact Hausdorff space  $X$ , there exists a compact Hausdorff space  $Y$  and imbedding  $i : X \rightarrow Y$  such that  $Y - i(X)$  is a single point.
- ⟨2⟩1. LET:  $X$  be a locally compact Hausdorff space.
- ⟨2⟩2. LET:  $Y = X \cup \{\infty\}$
- ⟨2⟩3. Define a topology on  $Y$  by:  $U \subseteq Y$  is open if and only if  $U$  is an open set in  $X$  or  $U = Y - C$  where  $C$  is a compact subspace of  $X$ .
- ⟨3⟩1.  $Y$  is open.
- PROOF: Since  $Y = Y - \emptyset$  and  $\emptyset$  is a compact subspace of  $X$ .
- ⟨3⟩2. For any set of open sets  $\mathcal{U}$  we have  $\bigcup \mathcal{U}$  is open.
- PROOF: We have  $\bigcup \mathcal{U} = Y - (\bigcap \{C \subseteq X \mid C \text{ is compact, } Y - C \in \mathcal{U}\} - \bigcup \{U \subseteq X \mid U \text{ is open in } X, U \in \mathcal{U}\})$ , where we take the empty intersection to be  $Y$ .
- ⟨3⟩3. For any open sets  $U$  and  $V$  we have  $U \cap V$  is open.
- ⟨4⟩1. LET:  $U$  and  $V$  be open sets.
- ⟨4⟩2. CASE:  $U$  and  $V$  are open sets in  $X$ .
- PROOF: In this case  $U \cap V$  is open in  $X$ .
- ⟨4⟩3. CASE:  $C_1$  and  $C_2$  are compact subspaces of  $X$  and  $U = X - C_1$ ,  $V = X - C_2$
- PROOF: In this case  $C_1 \cup C_2$  is compact and  $U \cap V = X - (C_1 \cup C_2)$ .
- ⟨4⟩4. CASE:  $U$  is open in  $X$ ,  $C$  is a compact subspace of  $X$  and  $V = X - C$
- PROOF: In this case  $U \cap V = U - C$  which is open since  $C$  is closed.
- ⟨2⟩4.  $Y$  is compact.
- ⟨3⟩1. LET:  $\mathcal{A}$  be an open cover of  $Y$ .
- ⟨3⟩2. PICK  $C$  compact in  $X$  such that  $Y - C \in \mathcal{A}$
- PROOF: There must be at least one such member of  $\mathcal{A}$  since  $\infty \in \bigcup \mathcal{A}$ .
- ⟨3⟩3.  $\{U \cap X \mid U \in \mathcal{A} - \{Y - C\}\}$  is a set of open sets in  $X$  that covers  $C$ .
- ⟨3⟩4. PICK a finite subcover  $\{U_1 \cap X, \dots, U_n \cap X\}$
- ⟨3⟩5.  $\{U_1 \cap X, \dots, U_n \cap X, Y - C\}$  covers  $Y$ .
- ⟨2⟩5.  $Y$  is Hausdorff.
- ⟨3⟩1. LET:  $x, y \in Y$  with  $x \neq y$
- ⟨3⟩2. CASE:  $x, y \in X$
- PROOF: There are disjoint open sets  $U, V$  in  $X$  such that  $x \in U, y \in V$ .
- ⟨3⟩3. CASE:  $x \in X, y = \infty$
- ⟨4⟩1. PICK a compact  $C$  that includes a neighbourhood  $U$  of  $x$

- PROOF: Since  $X$  is locally compact.
- ⟨4⟩2.  $U$  and  $Y - C$  are disjoint open sets in  $Y$  with  $x \in U$  and  $\infty \in Y - C$
  - ⟨2⟩6. Let  $i : X \rightarrow Y$  be the inclusion.
  - ⟨2⟩7.  $i$  is an imbedding.
    - ⟨3⟩1.  $i$  is continuous
    - ⟨3⟩2.  $i$  is an open map.
  - ⟨2⟩8.  $Y - i(X) = \{\infty\}$
  - ⟨1⟩3. If  $X$  is locally compact Hausdorff,  $Y$  and  $Y'$  are compact Hausdorff, and  $i : X \rightarrow Y, i' : X \rightarrow Y'$  are imbeddings such that  $Y - i(X)$  and  $Y' - i'(X)$  each have just one point, then there exists a unique homeomorphism  $\theta : Y \cong Y'$  such that  $\theta \circ i = i'$ .
    - ⟨2⟩1. LET:  $Y - i(X) = \{a\}$  and  $Y' - i'(X) = \{b\}$
    - ⟨2⟩2. LET:  $\theta : Y \rightarrow Y'$  be the function with  $\theta(a) = b$  and  $\theta(i(x)) = i'(x)$
    - ⟨2⟩3.  $\theta$  is a bijection
    - ⟨2⟩4.  $\theta$  is continuous.
      - ⟨3⟩1. LET:  $U \subseteq Y'$  be open.
 

PROVE:  $\theta^{-1}(U)$  is open.
      - ⟨3⟩2. CASE:  $b \in U$ 
        - ⟨4⟩1.  $Y' - U$  is compact
        - ⟨4⟩2.  $i(i'^{-1}(Y' - U))$  is compact.
        - ⟨4⟩3.  $i(i'^{-1}(Y' - U))$  is closed.
        - ⟨4⟩4.  $\theta^{-1}(U) = X - i(i'^{-1}(Y' - U))$
      - ⟨3⟩3. CASE:  $b \notin U$ 

PROOF:  $U = i'(V)$  for some  $V$  open in  $X$  and  $\theta^{-1}(U) = i(V)$ .
    - ⟨2⟩5.  $\theta$  is an open map.
 

PROOF: Similar.
    - ⟨2⟩6.  $\theta$  is unique.

□

**Example 11.55.4.**  $S^1$  is the one-point compactification of  $\mathbb{R}$ .

**Example 11.55.5.**  $S^2$  is the one-point compactification of  $\mathbb{R}^2$ .

**Definition 11.55.6** (Riemann Sphere). The *Riemann sphere* or *extended complex plane* is  $\mathcal{C} \cup \{\infty\}$  topologized as the one-point compactification of  $\mathcal{C}$ . It is homeomorphic to  $S^2$ .

**Example 11.55.7.** The one-point compactification of  $\mathbb{Z}^+$  is  $\{0\} \cup \{1/n \mid n \in \mathbb{Z}^+\}$ .

**Proposition 11.55.8.** Every locally compact Hausdorff space is regular.

PROOF: It is a subspace of its one-point compactification, which is regular since it is compact Hausdorff. □

## 11.56 $G_\delta$ Sets

**Definition 11.56.1** ( $G_\delta$  Set). Let  $X$  be a topological space and  $A \subseteq X$ . Then  $A$  is  $G_\delta$  if and only if it is the intersection of a countable set of open sets.

**Proposition 11.56.2.** *In a first countable  $T_1$  space, every singleton is  $G_\delta$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $X$  be a first countable  $T_1$  space.

$\langle 1 \rangle 2$ . LET:  $a \in X$

$\langle 1 \rangle 3$ . PICK a countable local basis  $\mathcal{B}$  at  $a$ .

$\langle 1 \rangle 4$ .  $\bigcap \mathcal{B} = \{a\}$

$\langle 2 \rangle 1$ . LET:  $b \in X - \{a\}$

PROVE:  $b \notin \bigcap \mathcal{B}$

$\langle 2 \rangle 2$ . PICK  $B \in \mathcal{B}$  with  $a \in B \subseteq X - \{b\}$

$\langle 2 \rangle 3$ .  $b \notin B$

□

**Example 11.56.3.** In the space  $\mathbb{R}^\omega$  under the box topology, every singleton is  $G_\delta$ . However,  $\mathbb{R}^\omega$  is not first countable.

## 11.57 Separated by a Continuous Function

**Definition 11.57.1** (Separated by a Continuous Function). Let  $X$  be a topological space. Let  $A, B \subseteq X$ . Then  $A$  and  $B$  can be *separated by a continuous function* if and only if there exists a continuous  $f : X \rightarrow [0, 1]$  such that  $f(A) = \{0\}$  and  $f(B) = \{1\}$ .

## 11.58 Completely Regular Spaces

**Definition 11.58.1** (Completely Regular). Let  $X$  be a topological space. Then  $X$  is *completely regular* if and only if it is  $T_1$  and, for every point  $x_0 \in X$  and closed set  $A$  not containing  $x_0$ , we have  $\{x_0\}$  and  $A$  can be separated by a continuous function.

**Proposition 11.58.2.** *Every completely regular space is regular.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $X$  be a completely regular space.

$\langle 1 \rangle 2$ . LET:  $x_0 \in X$  and  $A$  be a closed space not containing  $x_0$ .

$\langle 1 \rangle 3$ . PICK a continuous  $f : X \rightarrow [0, 1]$  with  $f(x_0) = \{0\}$  and  $f(A) = \{1\}$ .

$\langle 1 \rangle 4$ .  $f^{-1}((-\infty, 1/2))$  and  $f^{-1}((1/2, +\infty))$  are disjoint open neighbourhoods of  $x_0$  and  $A$  respectively.

□

**Proposition 11.58.3.** *A subspace of a completely regular space is completely regular.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $X$  be a completely regular space.

$\langle 1 \rangle 2$ . LET:  $Y \subseteq X$

$\langle 1 \rangle 3$ .  $Y$  is  $T_1$



PROOF: Proposition 11.20.5.

- ⟨1⟩4. LET:  $y_0 \in Y$
- ⟨1⟩5. LET:  $A$  be closed in  $Y$ .
- ⟨1⟩6. ASSUME:  $y_0 \notin A$
- ⟨1⟩7. PICK  $B$  closed in  $X$  such that  $A = B \cap Y$
- ⟨1⟩8. PICK  $f : X \rightarrow [0, 1]$  continuous such that  $f(y_0) = 0$  and  $f(B) = \{1\}$ .
- ⟨1⟩9.  $f \upharpoonright Y : Y \rightarrow [0, 1]$  is continuous and separates  $y_0$  from  $A$ .

□

**Proposition 11.58.4.** *The product of a family of completely regular spaces is completely regular.*

PROOF:

- ⟨1⟩1. LET:  $\{X_\alpha\}_{\alpha \in J}$  be any family of completely regular spaces.
- ⟨1⟩2. LET:  $X = \prod_{\alpha \in J} X_\alpha$
- ⟨1⟩3. LET:  $b \in X$
- ⟨1⟩4. LET:  $A$  be closed in  $X$ .
- ⟨1⟩5. ASSUME:  $b \notin A$
- ⟨1⟩6. PICK a basic open neighbourhood  $U = \prod_{\alpha \in J} U_\alpha$  of  $b$  that is disjoint from  $A$ , where for all  $\alpha$  we have  $U_\alpha$  is open in  $X_\alpha$ , and  $U_\alpha = X_\alpha$  for all  $\alpha$  except  $\alpha_1, \dots, \alpha_n$ .
- ⟨1⟩7. For  $1 \leq i \leq n$ , PICK a continuous function  $f_i : X_i \rightarrow [0, 1]$  such that  $f_i(b_{\alpha_i}) = 1$  and  $f(X_i - U_{\alpha_i}) = \{0\}$
- ⟨1⟩8. Define  $f : X \rightarrow [0, 1]$  by  $f(x) = f_1(x_{\alpha_1}) \cdots f_n(x_{\alpha_n})$
- ⟨1⟩9.  $f$  is continuous.
- ⟨1⟩10.  $f(b) = 1$
- ⟨1⟩11. For all  $x \in A$  we have  $f(x) = 0$

□

**Example 11.58.5** (Choice). Not every regular space is completely regular.

For  $m \in \mathbb{Z}$ , let  $L_m = \{m\} \times [-1, 0]$ .

For each odd integer  $n$  and each integer  $k \geq 2$ , let

$$C_{nk} = (\{n+1-1/k\} \times [-1, 0]) \cup (\{n-1+1/k\} \times [-1, 0]) \cup \{(x, y) \mid (x-n)^2 + y^2 = (1-1/k)^2 \text{ and } y \geq 0\}$$

and

$$p_{nk} = (n, 1 - 1/k)$$

Let

$$X = \bigcup_{m \in \mathbb{Z}} L_m \cup \bigcup_{n, k} C_{nk} \cup \{a, b\}$$

Let  $\mathcal{B}$  be the set consisting of all sets of one of the following four types:

- The intersection of  $X$  with a horizontal open line segment that contains none of the points  $p_{nk}$ ;
- A set formed from one of the  $C_{nk}$  by deleting finitely many points;

- For each even integer  $m$ , the union of  $\{a\}$  and the set of points  $(x, y) \in X$  such that  $x < m$ ;
- For each even integer  $m$ , the union of  $\{b\}$  and the set of points  $(x, y) \in X$  such that  $x > m$ .

Then  $\mathcal{B}$  is a basis for a topology on  $X$  that is completely regular but not normal.

PROOF:

- (1)1.  $\mathcal{B}$  forms a basis for a topology on  $X$ .
- (2)1. For all  $x \in X$ , there exists  $B \in \mathcal{B}$  such that  $x \in B$
- (3)1. Every  $x \in \mathbb{R}^2$  is in a set of type 3
- (3)2.  $a$  is in a set of type 3
- (3)3.  $b$  is in a set of type 4
- (2)2. For all  $B_1, B_2 \in \mathcal{B}$  and  $p \in B_1 \cap B_2$ , there exists  $B_3 \in \mathcal{B}$  such that  $p \in B_3 \subseteq B_1 \cap B_2$
- (3)1. CASE:  $B_1$  and  $B_2$  are of type 1  
PROOF: Take  $B_3 = B_1 \cap B_2$  which is of type 1.
- (3)2. CASE:  $B_1$  is of type 1 and  $B_2$  is of type 2  
PROOF: Take  $B_3 = B_1 \cap B_2$  which is of type 1.
- (3)3. CASE:  $B_1$  is of type 1 and  $B_2$  is of type 3  
PROOF: Take  $B_3 = B_1 \cap B_2$  which is of type 1.
- (3)4. CASE:  $B_1$  is of type 1 and  $B_2$  is of type 4  
PROOF: Take  $B_3 = B_1 \cap B_2$  which is of type 1.
- (3)5. CASE:  $B_1$  and  $B_2$  are of type 2  
PROOF: Take  $B_3 = B_1 \cap B_2$  which is of type 2.
- (3)6. CASE:  $B_1$  is of type 2 and  $B_2$  is of type 3  
PROOF: Take  $B_3 = B_1 \cap B_2$ .
- (3)7. CASE:  $B_1$  is of type 2 and  $B_2$  is of type 4  
PROOF: Take  $B_3 = B_1 \cap B_2$ .
- (3)8. CASE:  $B_1$  and  $B_2$  are of type 3  
PROOF: Take  $B_3 = B_1 \cap B_2$  which is equal to either  $B_1$  or  $B_2$ .
- (3)9. CASE:  $B_1$  is of type 3 and  $B_2$  is of type 4
- (4)1. LET:  $B_1 = \{(x, y) \in X \mid x < m\} \cup \{a\}$
- (4)2. LET:  $B_2 = \{(x, y) \in X \mid x > n\} \cup \{b\}$
- (4)3. LET:  $p = (x, y)$
- (4)4.  $n < x < m$
- (4)5. LET:  $B_3 = X \cap ((n, m) \times \{y\})$
- (4)6.  $B_3$  is of type 1.
- (4)7.  $p \in B_3 \subseteq B_1 \cap B_2$
- (3)10. CASE:  $B_1$  and  $B_2$  are of type 4  
PROOF: Take  $B_3 = B_1 \cap B_2$  which is equal to either  $B_1$  or  $B_2$ .
- (1)2. For any continuous  $f : X \rightarrow \mathbb{R}$  we have  $f(a) = f(b)$
- (2)1. LET:  $f : X \rightarrow \mathbb{R}$  be continuous.
- (2)2. For all  $c \in \mathbb{R}$ , we have  $f^{-1}(c)$  is  $G_\delta$  in  $X$   
PROOF:  $f^{-1}(c) = \bigcap_{q \in \mathbb{Q}^+} f^{-1}((c - q, c + q))$

- (2)3. For  $n$  an odd integer and  $k \geq 2$  an integer,  
 LET:  $S_{nk} = \{p \in C_{nk} \mid f(p) \neq f(p_{nk})\}$
- (2)4. Each  $S_{nk}$  is countable.
- (3)1. LET:  $n$  be an odd integer and  $k \geq 2$  an integer.
- (3)2.  $f^{-1}(f(p_{nk}))$  is  $G_\delta$
- PROOF: By (2)2
- (3)3. PICK a countable set  $\mathcal{U}$  of open sets such that  $f^{-1}(f(p_{nk})) = \bigcap \mathcal{U}$
- (3)4.  $S_{nk} = \bigcup \{A \subseteq^{\text{fin}} C_{nk} \mid C_{nk} - A \in \mathcal{U}\}$
- (2)5. PICK  $d \in [-1, 0]$  such that  $\mathbb{R} \times \{d\}$  intersects none of the sets  $S_{nk}$
- PROOF: There exists such a  $d$  because  $[-1, 0]$  is uncountable and  $\bigcup_{n,k} S_{nk}$  is countable.
- (2)6. For  $n$  odd,
 
$$f(n-1, d) = \lim_{k \rightarrow \infty} f(p_{nk}) = f(n+1, d)$$
- (3)1. LET:  $n$  be an odd integer.
- (3)2.  $f(p_{nk}) \rightarrow f(n-1, d)$  as  $k \rightarrow \infty$
- (4)1. For every  $k \geq 1$  we have  $f(p_{nk}) = f(n-1+1/k, d)$
- PROOF: Since  $(n-1+1/k, d) \notin S_{nk}$ .
- (4)2.  $(n-1+1/k, d) \rightarrow (n-1, d)$  as  $k \rightarrow \infty$
- (5)1. LET:  $U$  be an open neighbourhood of  $(n-1, d)$
- PROVE: There exists  $K$  such that, for all  $k \geq K$ ,  $(n-1+1/k, d) \in U$
- (5)2. CASE:  $U$  is of type 1
- (6)1. PICK  $\epsilon > 0$  such that  $(n-1-\epsilon, n-1+\epsilon) \times \{d\} \subseteq U$
- (6)2. PICK  $K$  such that  $1/K < \epsilon$
- (6)3. For  $k \geq K$  we have  $(n-1+1/k, d) \in K$
- (5)3. CASE:  $U$  is of type 3 or 4
- PROOF: For all  $k$  we have  $(n-1+1/k, d) \in U$
- (4)3.  $f(n-1+1/k, d) \rightarrow f(n, d)$  as  $k \rightarrow \infty$
- (3)3.  $f(p_{nk}) \rightarrow f(n+1, d)$  as  $k \rightarrow \infty$
- PROOF: Similar.
- (2)7.  $f(a) = f(b)$
- PROOF: Let  $c$  be the number such that  $f(2n) = c$  for every integer  $n$ . Then  $f(a) = f(b) = c$  since  $2n \rightarrow b$  as  $n \rightarrow \infty$  and  $-2n \rightarrow a$  as  $n \rightarrow \infty$ .
- (1)3.  $X$  is regular.
- (2)1.  $X$  is  $T_1$
- (3)1. For  $x \in \mathbb{R}^2$ ,  $X - \{x\}$  is open.
- PROOF: It is a union of one set of type 3, one set of type 4, and a set of sets of type 1.
- (3)2.  $X - \{a\}$  is open.
- PROOF: It is the union of all the sets of type 4.
- (3)3.  $X - \{b\}$  is open.
- PROOF: It is the union of all the sets of type 3.
- (2)2. For all  $x \in X$  and  $U$  an open neighbourhood of  $x$ , there exists a neighbourhood  $V$  of  $x$  such that  $\overline{V} \subseteq U$ .
- (3)1. LET:  $x \in X$

- ⟨3⟩2. LET:  $U$  be an open neighbourhood of  $x$   
 ⟨3⟩3. CASE:  $U$  is of type 1  
 PROOF: Let  $x = (s, t)$  be in an open line segment with  $x$ -coordinates the values in  $(a, b)$ . Pick  $c, d$  with  $a < c < s < d < b$ . Let  $V$  be the set of type 1 with  $y$ -coordinate  $t$  and  $x$ -coordinates in  $(c, d)$ . Then  $x \in V$  and  $\overline{V} \subseteq U$ .  
 ⟨3⟩4. CASE:  $U$  is of type 2  
 PROOF: Take  $V = U$ .  
 ⟨3⟩5. CASE:  $U$  is of type 3  
 PROOF: If  $U$  is  $a$  plus the set of all points with  $x$ -coordinate  $< m$ , take  $V$  to be  $a$  plus the set of all points with  $x$ -coordinate  $< m - 1$ .  
 ⟨3⟩6. CASE:  $U$  is of type 4  
 PROOF: Similar.  
 ⟨1⟩4.  $X$  is not completely regular.  
 PROOF: There is no continuous  $f : X \rightarrow [0, 1]$  with  $f(a) = 0$  and  $f(b) = 1$ .

□

**Proposition 11.58.6.** *Let  $X$  be completely regular. Let  $A$  be a compact subspace of  $X$  and  $B$  a closed set in  $X$  disjoint from  $A$ . Then  $A$  and  $B$  can be separated by a continuous function.*

PROOF:

- ⟨1⟩1. For all  $a \in A$ , there exists a continuous function  $f : X \rightarrow [0, 1]$  such that  $f(a) = 0$  and  $f(B) = \{1\}$ .  
 ⟨1⟩2.  $\{f^{-1}(0) \mid f : X \rightarrow [0, 1] \text{ continuous}, f(B) = \{1\}\}$  covers  $A$ .  
 ⟨1⟩3. PICK a finite subcover  $\{f_1^{-1}(0), \dots, f_n^{-1}(0)\}$   
 ⟨1⟩4. Define  $f : X \rightarrow [0, 1]$  by  $f(x) = f_1(x) \cdots f_n(x)$   
 ⟨1⟩5.  $f$  is continuous.  
 ⟨1⟩6.  $f(A) = \{0\}$   
 ⟨1⟩7.  $f(B) = \{1\}$

□

## 11.59 Normal Spaces

**Definition 11.59.1** (Normal). A topological space  $X$  is normal if and only if it is  $T_1$  and, for any disjoint closed sets  $A$  and  $B$ , there exist disjoint open sets  $U$  and  $V$  with  $A \subseteq U$  and  $B \subseteq V$ .

**Proposition 11.59.2.** *Every normal space is regular.*

PROOF: Immediate from definitions. □

**Proposition 11.59.3.** *Let  $X$  be a  $T_1$  space. Then  $X$  is normal if and only if, for any closed set  $A$  and open neighbourhood  $U$  of  $A$ , there exists a neighbourhood  $V$  of  $A$  with  $\overline{V} \subseteq U$ .*

PROOF:

- ⟨1⟩1. LET:  $X$  be a  $T_1$  space.
- ⟨1⟩2. If  $X$  is regular then, for any closed set  $A$  and open neighbourhood  $U$  of  $A$ , there exists a neighbourhood  $V$  of  $A$  with  $\bar{V} \subseteq U$ .
  - ⟨2⟩1. ASSUME:  $X$  is regular.
  - ⟨2⟩2. LET:  $A$  be closed in  $X$
  - ⟨2⟩3. LET:  $U$  be an open neighbourhood of  $A$ .
  - ⟨2⟩4. PICK disjoint open sets  $V$  and  $W$  such that  $X \setminus U \subseteq W$  and  $A \subseteq V$
  - ⟨2⟩5.  $\bar{V} \subseteq U$
- PROOF: Since  $V \subseteq X - W \subseteq U$ .
- ⟨1⟩3. If, for any closed set  $A$  and open neighbourhood  $U$  of  $A$ , there exists a neighbourhood  $V$  of  $A$  with  $\bar{V} \subseteq U$ , then  $X$  is regular.
  - ⟨2⟩1. ASSUME: for any closed set  $A$  and open neighbourhood  $U$  of  $A$ , there exists a neighbourhood  $V$  of  $A$  with  $\bar{V} \subseteq U$ .
  - ⟨2⟩2. LET:  $A$  and  $B$  be disjoint closed sets.
  - ⟨2⟩3. PICK a neighbourhood  $V$  of  $B$  with  $\bar{V} \subseteq X - A$
  - ⟨2⟩4. LET:  $U = X - \bar{V}$
  - ⟨2⟩5.  $U$  and  $V$  are disjoint open sets with  $A \subseteq U$  and  $B \subseteq V$

□

**Example 11.59.4.** The space  $\mathbb{R}_l$  is normal.

PROOF:

- ⟨1⟩1.  $\mathbb{R}_l$  is  $T_1$
- PROOF: Since the topology of  $\mathbb{R}_l$  is finer than  $\mathbb{R}$ .
- ⟨1⟩2. LET:  $A$  and  $B$  be disjoint closed sets in  $\mathbb{R}_l$
- ⟨1⟩3. LET:  $U$  be the union of all the basic open sets  $[a, x)$  with  $a \in A$  such that  $[a, x)$  does not intersect  $B$ .
- ⟨1⟩4. LET:  $V$  be the union of all the basic open sets  $[b, x)$  with  $b \in B$  such that  $[b, x)$  does not intersect  $A$ .
- ⟨1⟩5.  $U$  and  $V$  are disjoint.
  - ⟨2⟩1. ASSUME: for a contradiction  $x \in U \cap V$
  - ⟨2⟩2. PICK  $[a, y)$  such that  $x \in [a, y)$ ,  $a \in A$  and  $[a, y)$  does not intersect  $B$ .
  - ⟨2⟩3. PICK  $[b, z)$  such that  $x \in [b, z)$ ,  $b \in B$  and  $[b, z)$  does not intersect  $A$ .
  - ⟨2⟩4. ASSUME: w.l.o.g.  $a < b$
  - ⟨2⟩5.  $b \in [a, y)$
- PROOF: Since  $a < b \leq x < y$
- ⟨2⟩6. Q.E.D.
- PROOF: This contradicts the fact that  $[a, y)$  does not intersect  $B$  (⟨2⟩2).
- ⟨1⟩6.  $A \subseteq U$ 
  - ⟨2⟩1. LET:  $a \in A$
  - ⟨2⟩2.  $a \in X - B$
  - ⟨2⟩3. PICK  $x$  such that  $[a, x) \subseteq X - B$
  - ⟨2⟩4.  $a \in [a, x) \subseteq U$
- ⟨1⟩7.  $B \subseteq V$
- PROOF: Similar.

□

The following example shows that:

- a completely regular space is not necessarily normal;
- the product of two normal spaces is not necessarily normal.

**Example 11.59.5** (Choice). The Sorgenfrey plane is not normal.

PROOF:

- ⟨1⟩1. ASSUME: for a contradiction  $\mathbb{R}_l^2$  is normal.  
 ⟨1⟩2. LET:  $L = \{(x, -x) \mid x \in \mathbb{R}\}$   
 ⟨1⟩3.  $L$  is closed in  $\mathbb{R}_l^2$ .  
 ⟨1⟩4.  $L$  is discrete as a subspace of  $\mathbb{R}_l^2$ .  
 ⟨1⟩5. For every  $A \subseteq L$ , PICK disjoint open sets  $U$  and  $V$  in  $\mathbb{R}_l^2$  such that  $A \subseteq U$  and  $L - A \subseteq V$ .

PROOF:  $A$  and  $L - A$  are closed in  $\mathbb{R}_l^2$  by 11.18.7.

- ⟨1⟩6. LET:  $D = \mathbb{Q}^2$   
 ⟨1⟩7.  $D$  is dense in  $\mathbb{R}_l^2$   
 ⟨1⟩8. Define  $\theta : \mathcal{P}L \rightarrow \mathcal{P}D$  by  

$$\theta(A) = D \cap U_A \quad (\emptyset \subset A \subset L)$$
  

$$\theta(\emptyset) = \emptyset$$
  

$$\theta(L) = D$$

- ⟨1⟩9.  $\theta$  is injective.  
 ⟨2⟩1. LET:  $A, B$  be nonempty proper subsets of  $L$  with  $A \neq B$   
 ⟨2⟩2. ASSUME: w.l.o.g. there exists  $x \in A$  such that  $x \notin B$   
 ⟨2⟩3.  $U_A \cap V_B$  is nonempty  
 ⟨2⟩4. PICK  $d \in D \cap U_A \cap V_B$   
 ⟨2⟩5.  $d \in \theta(A)$  and  $d \notin \theta(B)$   
 ⟨2⟩6.  $\theta(A) \neq \theta(B)$

- ⟨1⟩10. Q.E.D.

PROOF: This contradicts Cantor's Theorem, since  $L$  is uncountable and  $D$  is countable.

□

PROOF:

- ⟨1⟩1. LET:  $A = \{(x, -x) \mid x \in \mathbb{Q}\}$   
 ⟨1⟩2. LET:  $B = \{(x, -x) \mid x \in \mathbb{R} - \mathbb{Q}\}$   
 ⟨1⟩3.  $A$  and  $B$  are closed in  $\mathbb{R}_l^2$   
 ⟨1⟩4. LET:  $V$  be any open set in  $\mathbb{R}_l^2$  that includes  $B$   
 PROVE: There is no open set  $U$  that includes  $A$  and is disjoint from  $B$ .  
 ⟨1⟩5. For  $n \in \mathbb{Z}^+$ ,  
 LET:  $K_n = \{x \in [0, 1] - \mathbb{Q} \mid [x, x + 1/n) \times [-x, -x + 1/n) \subseteq V\}$   
 ⟨1⟩6.  $[0, 1] - \mathbb{Q} = \bigcup_n K_n$  is countable.  
 ⟨2⟩1. LET:  $x \in [0, 1] - \mathbb{Q}$   
 PROVE: There exists  $n$  such that  $x \in K_n$   
 ⟨2⟩2. PICK  $\epsilon, \delta$  such that  $[x, x + \epsilon) \times [-x, -x + \delta) \subseteq V$   
 ⟨2⟩3. PICK  $n$  such that  $1/n \leq \epsilon$  and  $1/n \leq \delta$   
 ⟨2⟩4.  $x \in K_n$

- ⟨1⟩7. PICK  $n, a, b$  such that  $(a, b) \subseteq \overline{K_n}$   
 PROOF: By Proposition 11.51.27 it cannot be that every  $\overline{K_n}$  has empty interior.
- ⟨1⟩8. For all  $x \in (a, b)$  and  $\epsilon \in (0, 1/n)$  we have  $(x, -x + \epsilon) \in V$
- ⟨2⟩1. LET:  $x \in (a, b)$
- ⟨2⟩2. LET:  $\epsilon \in (0, 1/n)$
- ⟨2⟩3. PICK  $y \in K_n \cap (x - \epsilon, x)$   
 PROOF: Pick  $z \in (a, b) \cap (x - \epsilon, x)$ . Then  $z \in \overline{K_n}$ .
- ⟨2⟩4.  $(x, -x + \epsilon) \in [y, y + 1/n] \times [-y, -y + 1/n] \subseteq V$
- ⟨1⟩9. For every rational  $q \in (a, b)$ , we have  $(q, -q)$  is a limit point of  $V$ .
- ⟨1⟩10. There is no open set  $U$  that includes  $A$  and is disjoint from  $B$ .  
 $\square$

**Proposition 11.59.6.** *In a regular space, every pair of disjoint closed sets have open neighbourhoods whose closures are disjoint.*

PROOF:

- ⟨1⟩1. LET:  $X$  be a regular space.
- ⟨1⟩2. LET:  $A$  and  $B$  be disjoint closed sets in  $X$ .
- ⟨1⟩3. PICK a neighbourhood  $U$  of  $A$  such that  $\overline{U} \subseteq X - B$ .  
 PROOF: Proposition 11.59.3.
- ⟨1⟩4. PICK a neighbourhood  $V$  of  $B$  such that  $\overline{V} \subseteq X - \overline{U}$ .  
 PROOF: Proposition 11.59.3.
- ⟨1⟩5.  $U$  and  $V$  are disjoint open neighbourhoods of  $A$  and  $B$  respectively with disjoint closures.

$\square$

**Proposition 11.59.7.** *The image of a normal space under a closed continuous map is normal.*

PROOF:

- ⟨1⟩1. LET:  $X$  be a normal space.
- ⟨1⟩2. LET:  $Y$  be a topological space.
- ⟨1⟩3. LET:  $f : X \rightarrow Y$  be a closed continuous surjective map.
- ⟨1⟩4. For every  $y \in Y$  and every open neighbourhood  $U$  of  $p^{-1}(y)$ , there exists an open neighbourhood  $W$  of  $y$  such that  $p^{-1}(W) \subseteq U$
- ⟨2⟩1. LET:  $y \in Y$
- ⟨2⟩2. LET:  $U$  be an open neighbourhood of  $p^{-1}(y)$
- ⟨2⟩3. LET:  $W = Y - p(X - U)$
- ⟨2⟩4.  $W$  is open  
 PROOF: Since  $p$  is a closed map.
- ⟨2⟩5.  $y \in W$   
 PROOF: Since  $p^{-1}(y) \subseteq U$ .
- ⟨2⟩6.  $p^{-1}(W) \subseteq U$   
 PROOF: If  $p(x) \in W$  then  $p(x) \notin p(X - U)$  so  $x \in U$ .
- ⟨1⟩5. LET:  $A$  be a closed set in  $Y$  and  $U$  be an open neighbourhood of  $A$ .
- ⟨1⟩6. PICK an open neighbourhood  $V$  of  $p^{-1}(A)$  such that  $\overline{V} \subseteq p^{-1}(U)$

⟨1⟩7. LET:  $W = \bigcup \{W' \text{ open in } Y \mid p^{-1}(W') \subseteq V\}$   
 ⟨1⟩8.  $A \subseteq W$   
 ⟨2⟩1. LET:  $y \in A$   
 ⟨2⟩2.  $p^{-1}(y) \subseteq V$   
 ⟨2⟩3. PICK an open neighbourhood  $W'$  of  $y$  such that  $p^{-1}(W') \subseteq V$   
 PROOF: By ⟨1⟩4.  
 ⟨2⟩4.  $y \in W' \subseteq W$   
 ⟨1⟩9.  $\overline{W} \subseteq U$   
 ⟨2⟩1.  $W \subseteq p(\overline{V})$   
 ⟨3⟩1. LET:  $y \in W$   
 ⟨3⟩2. PICK  $W'$  open in  $Y$  such that  $y \in W'$  and  $p^{-1}(W') \subseteq V$   
 ⟨3⟩3. PICK  $x \in X$  such that  $p(x) = y$   
 ⟨3⟩4.  $x \in V$   
 ⟨3⟩5.  $x \in \overline{V}$   
 ⟨3⟩6.  $y \in p(\overline{V})$   
 ⟨2⟩2.  $\overline{W} \subseteq p(\overline{V})$   
 PROOF:  $p(\overline{V})$  is closed since  $p$  is a closed map.  
 ⟨2⟩3.  $p(\overline{V}) \subseteq U$

□

**Theorem 11.59.8.** *If the Continuum Hypothesis is true then  $\mathbb{R}^\omega$  under the box topology is normal.*

PROOF: See M. E. Rudin. The box product of countably many compact metric spaces. *General Topology and its Applications*, 2:293–298, 1972. □

**Theorem 11.59.9.** *Every compact Hausdorff space is normal.*

PROOF: From Propositions 11.51.10 and 11.51.3. □

**Corollary 11.59.9.1.** *Every locally compact Hausdorff space is completely regular.*

**Theorem 11.59.10.** *Every linearly ordered set under the order topology is normal.*

PROOF: See L. A. Steen and J. A. Seebach Jr. *Counterexamples in Topology* Example 39. □

**Proposition 11.59.11.** *Every closed subspace of a normal space is normal.*

PROOF:

⟨1⟩1. LET:  $X$  be a normal space.  
 ⟨1⟩2. LET:  $Y$  be a closed subspace of  $X$ .  
 ⟨1⟩3. LET:  $A$  and  $B$  be closed in  $Y$ .  
 ⟨1⟩4.  $A$  and  $B$  are closed in  $X$ .  
 ⟨1⟩5. PICK  $U'$  and  $V'$  open in  $X$  and disjoint such that  $A \subseteq U'$  and  $B \subseteq V'$   
 ⟨1⟩6.  $U' \cap Y$  and  $V' \cap Y$  are disjoint open neighbourhoods of  $A$  and  $B$  respectively in  $Y$ .



□

**Corollary 11.59.11.1.** *Let  $\{X_\alpha\}_{\alpha \in J}$  be a family of nonempty spaces. If  $\prod_{\alpha \in J} X_\alpha$  is normal then each  $X_\alpha$  is normal.*

The following example shows:

- A completely regular space is not necessarily normal.
- A subspace of a normal space is not necessarily normal. (Since  $\mathbb{R}^J \cong (0, 1)^J \subseteq [0, 1]^J$  and  $[0, 1]^J$  is compact Hausdorff hence normal.)
- A product of a family of normal spaces is not necessarily normal.

**Example 11.59.12** (Choice). If  $J$  is an uncountable set then  $\mathbb{R}^J$  is not normal.

PROOF:

⟨1⟩1. LET:  $J$  be an uncountable set.

⟨1⟩2. LET:  $X = (\mathbb{Z}^+)^J$

⟨1⟩3.  $X$  is not normal.

⟨2⟩1. For  $x \in X$  and  $B \subseteq^{\text{fin}} J$ ,

LET:  $U(x, B) = \{y \in X \mid \forall \alpha \in B. y(\alpha) = x(\alpha)\}$

⟨2⟩2.  $\{U(x, B) \mid x \in X, B \subseteq^{\text{fin}} J\}$  is a basis for  $X$

⟨3⟩1. LET:  $x \in X$  and  $U = \prod_{\alpha \in J} U_\alpha$  be a basic open neighbourhood of  $x$ , where  $U_\alpha = \mathbb{Z}^+$  for all  $\alpha$  except  $\alpha \in B$ , where  $B$  is some finite subset of  $J$

⟨3⟩2.  $x \in U(x, B) \subseteq U$

⟨2⟩3. For  $n \in \mathbb{Z}^+$ ,

LET:  $P_n = \{x \in X \mid x \text{ is injective on } J - x^{-1}(n)\}$

⟨2⟩4. For every  $n$ ,  $P_n$  is closed.

⟨3⟩1. LET:  $x \in X - P_n$

⟨3⟩2. PICK  $\alpha, \beta \in J$  such that  $x(\alpha) = x(\beta) \neq n$

⟨3⟩3.  $x \in B(x, \{\alpha, \beta\}) \subseteq X - P_n$

⟨2⟩5.  $P_1$  and  $P_2$  are disjoint.

PROOF: If  $x \in X$  was in  $P_1$  and  $P_2$  then it would be an injective function  $J \rightarrow \mathbb{Z}^+$ , contradicting ⟨1⟩1.

⟨2⟩6. ASSUME: for a contradiction  $U$  and  $V$  are disjoint open neighbourhoods of  $P_1$  and  $P_2$  respectively.

⟨2⟩7. For every sequence  $(\alpha_n)$  in  $J$ , sequence  $0 < n_1 < n_2 < \dots$  of integers, and  $i \geq 1$ ,

LET:  $B_i((\alpha_n), (n_k)) = \{\alpha_1, \alpha_2, \dots, \alpha_{n_i}\}$  and  $x_i((\alpha_n), (n_k)) \in X$  be the point

$$x_i((\alpha_n), (n_k))(\alpha_j) = j \quad (1 \leq j \leq n_{i-1})$$

$$x_i((\alpha_n), (n_k))(\alpha) = 1 \quad (\text{for all other } \alpha)$$

⟨2⟩8. PICK sequences  $(\alpha_n)$  and  $(n_k)$  such that, for all  $i$ , we have  $U(x_i((\alpha_n), (n_k)), B_i((\alpha_n), (n_k))) \subseteq U$

⟨3⟩1.  $x_1((\alpha_n), (n_k))(\alpha) = 1$  for all  $\alpha$

⟨3⟩2. PICK  $B_1$  such that  $U(x_1, B_1) \subseteq U$

⟨3⟩3. LET:  $p_1 = |B_1|$  and  $B_1 = \{\alpha_1, \alpha_2, \dots, \alpha_{n_1}\}$   
 ⟨3⟩4. ASSUME:  
     we have picked  $n_1, \dots, n_k$  and  $\alpha_1, \dots, \alpha_{n_k}$  satisfying the conditions.  
 ⟨3⟩5.  $x_{k+1}((\alpha_n), (n_k)) \in P_1$   
 ⟨3⟩6. PICK  $B_{k+1} \supset B_k$  such that  $U(x_{k+1}, B_{k+1}) \subseteq U$   
 ⟨3⟩7. LET:  $n_{k+1} = |B_{k+1}|$  and  $B_{k+1} - B_k = \{\alpha_{n_k+1}, \alpha_{n_k+2}, \dots, \alpha_{n_{k+1}}\}$   
 ⟨2⟩9. LET:  $A = \{\alpha_1, \alpha_2, \dots\}$   
 ⟨2⟩10. Define  $y : J \rightarrow \mathbb{Z}^+$  by  
      $y(\alpha_j) = j$  ( $j \in \mathbb{Z}^+$ )  
      $y(\alpha) = 2$  (for all other  $j$ )  
 ⟨2⟩11. PICK  $B$  such that  $U(y, B) \subseteq V$   
 ⟨2⟩12. LET:  $i$  be least such that  $B \cap A \subseteq B_i$   
 ⟨2⟩13.  $U(x_{i+1}((\alpha_n), (n_k)), B_{i+1}((\alpha_n), (n_k))) \cap U(y, B) \neq \emptyset$   
     PROOF:  $x_{i+1}$  is in both.  
 ⟨2⟩14. Q.E.D.  
     PROOF: This is a contradiction since  $U(x_{i+1}((\alpha_n), (n_k)), B_{i+1}((\alpha_n), (n_k))) \subseteq U$  and  $U(y, B) \subseteq V$ .  
 ⟨1⟩4. Q.E.D.  
     PROOF: By Propositionm 11.59.11 since  $(\mathbb{Z}^+)^J$  is a closed subspace of  $\mathbb{R}^J$ .  
 □

- A regular space is not necessarily normal.
- A subspace of a normal space is not necessarily normal.
- A product of two normal spaces is not necessarily normal.

**Example 11.59.13.** The space  $S_\Omega \times \overline{S_\Omega}$  is not normal.

PROOF:

⟨1⟩1. LET:  $A = \{(x, x) \mid x \in S_\Omega\}$   
 ⟨1⟩2. LET:  $B = \{(x, \Omega) \mid x \in S_\Omega\}$   
 ⟨1⟩3.  $A$  and  $B$  are disjoint closed subsets in  $S_\Omega \times \overline{S_\Omega}$   
 ⟨1⟩4. ASSUME: for a contradiction  $U$  and  $V$  are disjoint open neighbourhoods of  $A$  and  $B$  respectively.  
 ⟨1⟩5. For all  $x \in S_\Omega$ , there exists  $\beta$  with  $x < \beta < \Omega$  such that  $(x, \beta) \notin U$   
     ⟨2⟩1. LET:  $x \in S_\Omega$   
     ⟨2⟩2. PICK  $U_1$  open in  $S_\Omega$  and  $U_2$  open in  $\overline{S_\Omega}$  such that  $(x, \Omega) \in U_1 \times U_2 \subseteq V$   
     ⟨2⟩3. PICK  $\gamma$  such that  $(\gamma, \Omega] \subseteq U_2$   
     ⟨2⟩4. LET:  $\beta = \gamma + 1$   
     ⟨2⟩5.  $(x, \beta) \in V$   
     ⟨2⟩6.  $(x, \beta) \notin U$   
 ⟨1⟩6. For  $x \in S_\Omega$ ,  
     LET:  $\beta(x)$  be the least element such that  $x < \beta(x)$  and  $(x, \beta(x)) \notin U$   
 ⟨1⟩7. LET:  $b = \lim_{n \rightarrow \infty} \beta^n(0)$   
 ⟨1⟩8.  $(\beta^n(0), \beta^{n+1}(0)) \rightarrow (b, b)$  as  $n \rightarrow \infty$   
 ⟨1⟩9.  $(b, b) \in A \subseteq U$

⟨1⟩10. For all  $n$  we have  $(\beta^n(0), \beta^{n+1}(0)) \notin U$

⟨1⟩11. Q.E.D.

□

**Proposition 11.59.14.** *Every regular Lindelöf space is normal.*

PROOF:

⟨1⟩1. LET:  $X$  be a regular Lindelöf space.

⟨1⟩2. LET:  $A$  and  $B$  be disjoint closed sets in  $X$ .

⟨1⟩3. PICK a sequence  $(U_n)$  of open sets disjoint from  $B$  that cover  $A$

⟨1⟩4. PICK a sequence  $(V_n)$  of open sets disjoint from  $A$  that cover  $B$

⟨1⟩5. For  $n \in \mathbb{Z}^+$ ,

LET:  $U'_n = U_n - \bigcup_{i=1}^n \overline{V_i}$  and  $V'_n = V_n - \bigcup_{i=1}^n \overline{U_i}$

⟨1⟩6. LET:  $U' = \bigcup_n U'_n$  and  $V' = \bigcup_n V'_n$

⟨1⟩7.  $U'$  and  $V'$  are disjoint open neighbourhoods of  $A$  and  $B$  respectively.

□

**Corollary 11.59.14.1.** *The space  $\mathbb{R}^\omega$  is normal.*

**Theorem 11.59.15** (Urysohn Lemma). *In a normal space, any two disjoint closed sets can be separated by a continuous function.*

PROOF:

⟨1⟩1. LET:  $X$  be a normal space.

⟨1⟩2. LET:  $A$  and  $B$  be disjoint closed sets in  $X$ .

⟨1⟩3. LET:  $P = [0, 1] \cap \mathbb{Q}$

⟨1⟩4. PICK a family of open sets  $\{U_p\}_{p \in P}$  in  $X$  such that  $A \subseteq U_0$ ,  $U_1 = X - B$  and, for all  $p, q \in P$ , if  $p < q$  then  $\overline{U_p} \subseteq U_q$ .

⟨2⟩1. PICK an enumeration  $\{p_1, p_2, p_3, \dots\}$  of  $P$  such that  $p_1 = 1$  and  $p_2 = 0$ .

⟨2⟩2. LET:  $U_1 = X - B$

⟨2⟩3. PICK  $U_0$  open in  $X$  such that  $A \subseteq U_0$  and  $\overline{U_0} \subseteq X$

PROOF: Proposition 11.59.3.

⟨2⟩4. For all  $n$ , given open sets  $U_{p_1}, \dots, U_{p_n}$  that satisfies the condition, there exists  $U_{p_{n+1}}$  that satisfies the condition.

⟨3⟩1. Let  $q$  be the greatest element out of  $\{p_1, \dots, p_n\}$  such that  $q < p_{n+1}$  and  $r$  the least element such that  $p_{n+1} < r$

⟨3⟩2. PICK  $U_{p_{n+1}}$  open in  $X$  such that  $\overline{U_1} \subseteq U_{p_{n+1}}$  and  $U_{p_{n+1}} \subseteq U_r$

PROOF: Proposition 11.59.3.

⟨1⟩5. Extend the family to  $\{U_p\}_{p \in \mathbb{Q}}$  by defining  $U_p = \emptyset$  if  $p < 0$ , and  $U_p = X$  for  $p > 1$ .

⟨1⟩6. For  $x \in X$ ,

LET:  $\mathbb{Q}(x) = \{p \in \mathbb{Q} \mid x \in U_p\}$

⟨1⟩7. Define  $f : X \rightarrow [0, 1]$  by  $f(x) = \inf \mathbb{Q}(x)$

⟨2⟩1. For all  $x \in X$ , 0 is a lower bound for  $\mathbb{Q}(x)$

⟨2⟩2. For all  $x \in X$ , we have  $1 \in \mathbb{Q}(x)$

⟨1⟩8.  $f$  is continuous.

⟨2⟩1. For all  $r \in P$  and  $x \in \overline{U_r}$  we have  $f(x) \leq r$

⟨3⟩1. LET:  $r \in P$

- ⟨3⟩2. LET:  $x \in \overline{U_r}$
  - ⟨3⟩3. For all  $s > r$  we have  $x \in U_s$
  - ⟨3⟩4. For all  $s > r$  we have  $s \in \mathbb{Q}(x)$
  - ⟨3⟩5. For all  $s > r$  we have  $f(x) \leq s$
  - ⟨3⟩6.  $f(x) \leq r$
- ⟨2⟩2. For all  $r \in P$  and  $x \in U_r$  we have  $f(x) \geq r$ 
  - ⟨3⟩1. LET:  $r \in P$
  - ⟨3⟩2. LET:  $x \in X - U_r$
  - ⟨3⟩3. For all  $s < r$  we have  $x \notin U_s$
  - ⟨3⟩4.  $r$  is a lower bound for  $\mathbb{Q}(x)$
  - ⟨3⟩5.  $r \leq f(x)$
- ⟨2⟩3. LET:  $x_0 \in X$
- ⟨2⟩4. LET:  $(c, d)$  be an open interval that contains  $f(x_0)$ 
  - PROVE: There exists an open neighbourhood  $U$  of  $x_0$  such that  $f(U) \subseteq (c, d)$
- ⟨2⟩5. PICK rationals  $q, r$  with  $c < q < f(x_0) < r < d$
- ⟨2⟩6. LET:  $U = U_r - \overline{U_q}$
- ⟨2⟩7.  $U$  is open
- ⟨2⟩8.  $x_0 \in U$ 
  - ⟨3⟩1.  $x_0 \in U_r$
  - PROOF: From ⟨2⟩2.
  - ⟨3⟩2.  $x_0 \notin \overline{U_q}$
  - PROOF: From ⟨2⟩1.
- ⟨2⟩9.  $f(U) \subseteq (c, d)$ 
  - ⟨3⟩1. LET:  $x \in U$
  - ⟨3⟩2.  $r \leq f(x) \leq s$
  - PROOF: From ⟨2⟩1 and ⟨2⟩2.
  - ⟨3⟩3.  $c < f(x) < d$
- ⟨1⟩9.  $f(x) = 0$  for  $x \in A$ .
  - PROOF: Since for  $x \in A$  we have  $\mathbb{Q}(x) = \{q \in \mathbb{Q} \mid q \geq 0\}$ .
- ⟨1⟩10.  $f(x) = 1$  for  $x \in B$ .
  - PROOF: Since for  $x \in B$  we have  $\mathbb{Q}(x) = \{q \in \mathbb{Q} \mid q > 1\}$ .

□

**Corollary 11.59.15.1.** *Every normal space is completely regular.*

**Proposition 11.59.16.** *A connected normal space with more than one point is uncountable.*

PROOF:

- ⟨1⟩1. LET:  $X$  be a connected normal space with more than one point.
- ⟨1⟩2. PICK two distinct points  $a$  and  $b$  in  $X$
- ⟨1⟩3. PICK a continuous  $f : X \rightarrow [0, 1]$  with  $f(a) = 0$  and  $f(b) = 1$
- ⟨1⟩4.  $f$  is surjective.
  - PROOF: Since  $f(X)$  is connected therefore  $f(X) = [0, 1]$ .
- ⟨1⟩5.  $X$  is uncountable.

□

**Corollary 11.59.16.1.** *A connected regular space with more than one point is uncountable.*

PROOF: Since a countable regular space is Lindelöf, hence normal.  $\square$

**Example 11.59.17.** There exists a connected Hausdorff space that is countably infinite.

See L. A. Steen and J. A. Seerbach Jr. *Counterexamples in Topology* Example 75.

## 11.60 Completely Normal Spaces

**Definition 11.60.1** (Completely Normal). A topological space is *completely normal* if and only if every subspace is normal.

**Proposition 11.60.2.** *Every subspace of a completely normal space is completely normal.*

PROOF: From Proposition 11.18.12.  $\square$

## 11.61 Separated Sets

**Definition 11.61.1** (Separated). Let  $X$  be a topological space and  $A, B \subseteq X$ . Then  $A$  and  $B$  are *separated* if and only if  $\bar{A} \cap B = A \cap \bar{B} = \emptyset$ .

**Proposition 11.61.2.** *Let  $X$  be a topological space. Then  $X$  is completely normal if and only if, for all  $A, B \subseteq X$ , if  $A$  and  $B$  are separated then there exist disjoint open neighbourhoods  $U$  and  $V$  of  $A$  and  $B$  respectively.*

PROOF:

- (1)1. LET:  $X$  be a topological space.
- (1)2. If  $X$  is completely normal then every pair of separated sets in  $X$  have disjoint open neighbourhoods.
  - (2)1. ASSUME:  $X$  is completely normal.
  - (2)2. LET:  $A$  and  $B$  be separated sets in  $X$ .
  - (2)3.  $A$  and  $B$  are disjoint closed sets in  $X - (\bar{A} \cap \bar{B})$ .
  - (2)4. PICK disjoint open neighbourhoods  $U$  and  $V$  of  $A$  and  $B$  in  $X - (\bar{A} \cap \bar{B})$ .
  - (2)5.  $U$  and  $V$  are disjoint open neighbourhoods of  $A$  and  $B$  in  $X$ .
- (1)3. If every pair of separated sets in  $X$  have disjoint open neighbourhoods then  $X$  is completely normal.
  - (2)1. ASSUME: Every pair of separated sets in  $X$  have disjoint open neighbourhoods.
  - (2)2. LET:  $Y \subseteq X$
  - (2)3. LET:  $A$  and  $B$  be disjoint closed sets in  $Y$ .
  - (2)4.  $A$  and  $B$  are separated in  $X$ .
  - (2)5. PICK disjoint open neighbourhoods  $U$  and  $V$  of  $A$  and  $B$  in  $X$ .
  - (2)6.  $U \cap Y$  and  $V \cap Y$  are disjoint open neighbourhoods of  $A$  and  $B$  in  $Y$ .

□

**Example 11.61.3.** The space  $\mathbb{R}_l$  is completely normal.

PROOF:

- ⟨1⟩1. LET:  $A$  and  $B$  be separated sets in  $\mathbb{R}_l$ .
- ⟨1⟩2. For all  $a \in A$ , PICK  $x_a \in X$  such that  $[a, x_a) \cap B = \emptyset$
- ⟨1⟩3. For all  $b \in B$ , PICK  $y_b \in X$  such that  $[b, y_b) \cap A = \emptyset$
- ⟨1⟩4. LET:  $U = \bigcup_{a \in A} [a, x_a)$
- ⟨1⟩5. LET:  $V = \bigcup_{b \in B} [b, y_b)$
- ⟨1⟩6.  $U$  and  $V$  are open neighbourhoods of  $A$  and  $B$  respectively.
- ⟨1⟩7.  $U \cap V = \emptyset$ 
  - ⟨2⟩1. ASSUME: for a contradiction  $x \in U \cap V$
  - ⟨2⟩2. PICK  $a \in A$  and  $b \in B$  such that  $x \in [a, x_a)$  and  $x \in [b, y_b)$
  - ⟨2⟩3. ASSUME: w.l.o.g.  $a < b$
  - ⟨2⟩4.  $b \in [a, x_a)$ 
    - PROOF: Since  $a < b \leq x < x_a$ .
  - ⟨2⟩5. Q.E.D.
  - PROOF: This contradicts the fact that  $[a, x_a) \cap B = \emptyset$  (⟨1⟩2).
- ⟨1⟩8. Q.E.D.
- PROOF: Proposition 11.61.2.

□

**Corollary 11.61.3.1.** *The product of two completely normal spaces is not necessarily completely normal.*

PROOF: Since  $\mathbb{R}_l^2$  is not normal. □

**Proposition 11.61.4.** *Every linearly ordered set in the order topology is completely normal.*

PROOF: ??? □

**Example 11.61.5.** Not every compact Hausdorff space is completely normal.

PROOF: The space  $\overline{S_\Omega}^2$  is compact Hausdorff, but its subspace  $S_\Omega \times \overline{S_\Omega}$  is not normal. □

**Proposition 11.61.6.** *Every second countable regular space is completely normal.*

PROOF: Since a subspace of a second countable regular space is second countable and regular, hence normal. □

## 11.62 Vanish Precisely

**Definition 11.62.1** (Vanish Precisely). Let  $X$  be a set and  $f : X \rightarrow [0, 1]$ . Let  $A \subseteq X$ . Then  $f$  *vanishes precisely* on  $A$  if and only if  $A = f^{-1}(0)$ .

**Theorem 11.62.2.** *Let  $X$  be a normal space. Let  $A \subseteq X$ . Then there exists a continuous function  $f : X \rightarrow [0, 1]$  that vanishes precisely on  $A$  if and only if  $A$  is closed and  $G_\delta$ .*

PROOF:

(1)1. If there exists a continuous  $f : X \rightarrow [0, 1]$  that vanishes precisely on  $A$  then  $A$  is closed.

PROOF: Since  $A = f^{-1}(\{0\})$ .  $\square$

(1)2. If there exists a continuous  $f : X \rightarrow [0, 1]$  that vanishes precisely on  $A$  then  $A$  is  $G_\delta$ .

PROOF: Since  $A = \bigcap_{n \in \mathbb{Z}^+} f^{-1}((-1/n, 1/n))$ .

(1)3. If  $A$  is closed and  $G_\delta$  then there exists a continuous  $f : X \rightarrow [0, 1]$  that vanishes precisely on  $A$ .

(2)1. ASSUME:  $A$  is closed and  $G_\delta$ .

(2)2. PICK a sequence  $U_n$  of open sets such that  $A = \bigcap_n U_n$

(2)3. For  $n \in \mathbb{Z}^+$  PICK a continuous  $\phi_n : X \rightarrow [0, 1]$  such that  $\phi_n(x) = 0$  for  $x \in A$  and  $\phi_n(x) = 1$  for  $x \in X - U_n$

PROOF: By the Urysohn Lemma.

(2)4. Define  $f : X \rightarrow [0, 1]$  by  $f(x) = \sum_{n=1}^{\infty} 1/2^n \phi_n(x)$

(2)5.  $f$  is continuous.

PROOF: By the Weierstrass  $M$ -test.

(2)6.  $f$  vanishes precisely on  $A$ .

$\square$

**Theorem 11.62.3** (Strong Form of the Urysohn Lemma). *Let  $X$  be a normal space. Let  $A, B \subseteq X$ . Then there exists a continuous function  $f : X \rightarrow [0, 1]$  such that  $A = f^{-1}(0)$  and  $B = f^{-1}(1)$  if and only if  $A$  and  $B$  are disjoint closed  $G_\delta$  sets.*

PROOF:

(1)1. For any continuous  $f : X \rightarrow [0, 1]$ , we have  $f^{-1}(0)$  and  $f^{-1}(1)$  are disjoint closed  $G_\delta$  sets.

PROOF: By Theorem 11.62.2.

(1)2. If  $A$  and  $B$  are disjoint closed  $G_\delta$  sets, then there exists a continuous  $f : X \rightarrow [0, 1]$  such that  $A = f^{-1}(0)$  and  $B = f^{-1}(1)$ .

(2)1. ASSUME:  $A$  and  $B$  are disjoint closed  $G_\delta$  sets.

(2)2. PICK a continuous function  $g : X \rightarrow [0, 1]$  that vanishes precisely on  $A$ .

PROOF: By Theorem 11.62.2.

(2)3. PICK a continuous function  $h : X \rightarrow [0, 1]$  that vanishes precisely on  $B$ .

PROOF: By Theorem 11.62.2.

(2)4. LET:  $f : X \rightarrow [0, 1]$  by  $f(x) = g(x)/(g(x) + h(x))$

(2)5.  $f$  is continuous

(2)6.  $A = f^{-1}(0)$

(2)7.  $B = f^{-1}(1)$

$\square$

## 11.63 Perfectly Normal Spaces

**Definition 11.63.1** (Perfectly Normal). A topological space is *perfectly normal* if and only if it is normal and every closed set is  $G_\delta$ .

**Proposition 11.63.2.** *Every perfectly normal space is completely normal.*

PROOF:

⟨1⟩1. LET:  $X$  be a perfectly normal space.

⟨1⟩2.  $A$  and  $B$  be separated sets in  $X$ .

⟨1⟩3. PICK  $f, g : X \rightarrow [0, 1]$  such that  $f$  vanishes precisely on  $\overline{A}$  and  $g$  vanishes precisely on  $\overline{B}$

PROOF: Theorem 11.62.2.

⟨1⟩4. LET:  $h = f - g : X \rightarrow [-1, 1]$

⟨1⟩5.  $h^{-1}([-1, 0))$  and  $h^{-1}((0, 1])$  are disjoint open neighbourhoods of  $B$  and  $A$  respectively.

⟨1⟩6. Q.E.D.

PROOF: Proposition 11.61.2.

□

**Example 11.63.3.** The space  $\overline{S_\Omega}$  is completely normal but not perfectly normal, because  $\{\Omega\}$  is not  $G_\delta$ .



## Chapter 12

# Topological Groups

**Definition 12.0.1** (Topological Group). A *topological group*  $G$  consists of a  $T_1$  space  $G$  and continuous maps  $\cdot : G^2 \rightarrow G$  and  $(\ )^{-1} : G \rightarrow G$  such that  $(G, \cdot, (\ )^{-1})$  is a group.

**Example 12.0.2.** 1. The integers  $\mathbb{Z}$  under addition are a topological group.

2. The real numbers  $\mathbb{R}$  under addition are a topological group.

3. The positive reals under multiplication are a topological group.

4. The set  $\{z \in \mathbb{C} \mid |z| = 1\}$  under multiplication and given the topology of  $S^1$  is a topological group.

5. For any  $n \geq 0$ , the general linear group  $GL_n(\mathbb{R})$  is a topological group under matrix multiplication, considered as a subspace of  $\mathbb{R}^{n^2}$ .

**Lemma 12.0.3.** Let  $G$  be a  $T_1$  space and  $\cdot : G^2 \rightarrow G$ ,  $(\ )^{-1} : G \rightarrow G$  be functions such that  $(G, \cdot, (\ )^{-1})$  is a group. Then  $G$  is a topological group if and only if the function  $f : G^2 \rightarrow G$  that maps  $(x, y)$  to  $xy^{-1}$  is continuous.

PROOF:

$\langle 1 \rangle 1$ . If  $G$  is a topological group then  $f$  is continuous.

PROOF: From Theorem 11.12.8.

$\langle 1 \rangle 2$ . If  $f$  is continuous then  $G$  is a topological group.

$\langle 2 \rangle 1$ . ASSUME:  $f$  is continuous.

$\langle 2 \rangle 2$ .  $(\ )^{-1}$  is continuous.

PROOF: Since  $x^{-1} = f(e, x)$ .

$\langle 2 \rangle 3$ .  $\cdot$  is continuous.

PROOF: Since  $xy = f(x, y^{-1})$ .

□

**Lemma 12.0.4.** Let  $G$  be a topological group and  $H$  a subgroup of  $G$ . Then  $H$  is a topological group under the subspace topology.

PROOF:

⟨1⟩1.  $H$  is  $T_1$ .

PROOF: From Proposition 11.20.5.

⟨1⟩2. multiplication and inverse on  $H$  are continuous.

PROOF: From Theorem 11.12.9.

□

**Lemma 12.0.5.** *Let  $G$  be a topological group and  $H$  a subgroup of  $G$ . Then  $\overline{H}$  is a subgroup of  $G$ .*

PROOF:

⟨1⟩1. LET:  $x, y \in \overline{H}$

PROVE:  $xy^{-1} \in \overline{H}$

⟨1⟩2. LET:  $U$  be any neighbourhood of  $xy^{-1}$

⟨1⟩3. LET:  $f : G^2 \rightarrow G$ ,  $f(a, b) = ab^{-1}$

⟨1⟩4.  $f^{-1}(U)$  is a neighbourhood of  $(x, y)$

⟨1⟩5. PICK neighbourhoods  $V, W$  of  $x$  and  $y$  respectively such that  $f(V \times W) \subseteq U$ .

⟨1⟩6. PICK  $a \in V \cap H$  and  $b \in W \cap H$

PROOF: Theorem 11.4.6.

⟨1⟩7.  $ab^{-1} \in U \cap H$

⟨1⟩8. Q.E.D.

PROOF: By Theorem 11.4.6.

□

**Proposition 12.0.6.** *Let  $G$  be a topological group and  $\alpha \in G$ . Then the maps  $l_\alpha, r_\alpha : G \rightarrow G$  defined by  $l_\alpha(x) = \alpha x$ ,  $r_\alpha(x) = x\alpha$  are homeomorphisms of  $G$  with itself.*

PROOF: They are continuous with continuous inverses  $l_{\alpha^{-1}}$  and  $r_{\alpha^{-1}}$ . □

**Corollary 12.0.6.1.** *Every topological group is homogeneous.*

PROOF: Given a topological group  $G$  and  $a, b \in G$ , we have  $l_{ba^{-1}}$  is a homeomorphism that maps  $a$  to  $b$ . □

**Proposition 12.0.7.** *Let  $G$  be a topological group and  $H$  a subgroup of  $G$ . Give  $G/H$  the quotient topology. For all  $\alpha \in G$ , the map  $\overline{f_\alpha}$  that sends  $xH$  to  $\alpha xH$  is a homeomorphism  $G/H \cong G/H$ .*

PROOF:

⟨1⟩1.  $\overline{f_\alpha}$  is well-defined.

PROOF: If  $xy^{-1} \in H$  then  $(\alpha x)(\alpha y)^{-1} = xy^{-1} \in H$ .

⟨1⟩2.  $\overline{f_\alpha}$  is continuous.

PROOF: Theorem 11.25.7 since  $\overline{f_\alpha} \circ p = p \circ f_\alpha$  is continuous, where  $p : G \rightarrow G/H$  is the canonical surjection.

⟨1⟩3.  $\overline{f_\alpha}^{-1}$  is continuous.

PROOF: Similar since  $\overline{f_\alpha}^{-1} = \overline{f_{\alpha^{-1}}}$ .

□

**Corollary 12.0.7.1.** *Let  $G$  be a topological group and  $H$  a subgroup of  $G$ . Give  $G/H$  the quotient topology. Then  $G/H$  is homogeneous.*

**Proposition 12.0.8.** *Let  $G$  be a topological group and  $H$  a subgroup of  $G$ . Give  $G/H$  the quotient topology. If  $H$  is closed in  $G$  then  $G/H$  is  $T_1$ .*

PROOF:

⟨1⟩1. LET:  $p : G \rightarrow G/H$  be the canonical surjection

⟨1⟩2. LET:  $x \in G$

⟨1⟩3.  $p^{-1}(xH) = f_x(H)$

⟨1⟩4.  $p^{-1}(xH)$  is closed in  $G$

PROOF: Since  $H$  is closed and  $f_x$  is a homomorphism of  $G$  with itself.

⟨1⟩5.  $\{xH\}$  is closed in  $G/H$

□

**Proposition 12.0.9.** *Let  $G$  be a topological group and  $H$  a subgroup of  $G$ . Give  $G/H$  the quotient topology. Then the canonical surjection  $p : G \rightarrow G/H$  is an open map.*

PROOF:

⟨1⟩1. LET:  $U \subseteq G$  be open.

⟨1⟩2.  $p^{-1}(p(U)) = \bigcup_{h \in H} r_h(U)$

⟨1⟩3.  $p^{-1}(p(U))$  is open.

⟨1⟩4.  $p(U)$  is open.

□

**Proposition 12.0.10.** *Let  $G$  be a topological group and  $H$  a closed normal subgroup of  $G$ . Then  $G/H$  is a topological group under the quotient topology.*

PROOF:

⟨1⟩1.  $G/H$  is  $T_1$

PROOF: Proposition 12.0.8.

⟨1⟩2. The map  $\bar{m} : (xH, yH) \mapsto xy^{-1}H$  is continuous.

⟨2⟩1.  $p^2 : G^2 \rightarrow (G/H)^2$  is a quotient map.

PROOF: Propositions 11.25.6, 12.0.9.

⟨2⟩2.  $\bar{m} \circ p^2$  is continuous.

PROOF: As it is  $p^2 \circ m$  where  $m : G^2 \rightarrow G$  with  $m(x, y) = xy^{-1}$

□

**Lemma 12.0.11.** *Let  $G$  be a topological group and  $A, B \subseteq G$ . If either  $A$  or  $B$  is open then  $AB$  is open.*

PROOF: If  $A$  is open we have  $AB = \bigcup_{b \in B} r_b(A)$ . Similarly if  $B$  is open. □

**Definition 12.0.12** (Symmetric Neighbourhood). Let  $G$  be a topological group. A neighbourhood  $V$  of  $e$  is *symmetric* if and only if  $V = V^{-1}$ .

**Lemma 12.0.13.** *Let  $G$  be a topological group. Let  $V$  be a neighbourhood of  $e$ . Then  $V$  is symmetric if and only if, for all  $x \in V$ , we have  $x^{-1} \in V$ .*

PROOF:

⟨1⟩1. If  $V$  is symmetric then, for all  $x \in V$ , we have  $x^{-1} \in V$

PROOF: Immediate from definitions.

⟨1⟩2. If, for all  $x \in V$ , we have  $x^{-1} \in V$ , then  $V$  is symmetric.

⟨2⟩1. ASSUME: for all  $x \in V$  we have  $x^{-1} \in V$

⟨2⟩2.  $V \subseteq V^{-1}$

PROOF: If  $x \in V$  then there exists  $y \in V$  such that  $x = y^{-1}$ , namely  $y = x^{-1}$

⟨2⟩3.  $V^{-1} \subseteq V$

PROOF: Immediate from ⟨2⟩1.

□

**Lemma 12.0.14.** *Let  $G$  be a topological group. For every neighbourhood  $U$  of  $e$ , there exists a symmetric neighbourhood  $V$  of  $e$  such that  $V^2 \subseteq U$ .*

PROOF:

⟨1⟩1. LET:  $U$  be a neighbourhood of  $e$ .

⟨1⟩2. PICK a neighbourhood  $V'$  of  $e$  such that  $V'V' \subseteq U$

PROOF: Such a neighbourhood exists because multiplication in  $G$  is continuous.

⟨1⟩3. PICK a neighbourhood  $W$  of  $e$  such that  $WW^{-1} \subseteq V'$

PROOF: Such a neighbourhood exists because the function that maps  $(x, y)$  to  $xy^{-1}$  is continuous.

⟨1⟩4. LET:  $V = WW^{-1}$

⟨1⟩5.  $V$  is a neighbourhood of  $e$

⟨2⟩1.  $e \in V$

PROOF: Since  $e \in W$  so  $e = ee^{-1} \in V$ .

⟨2⟩2.  $V$  is open

PROOF: Lemma 12.0.11.

⟨1⟩6.  $V$  is symmetric

⟨2⟩1. For all  $x \in V$  we have  $x^{-1} \in V$

⟨3⟩1. LET:  $x \in V$

⟨3⟩2. PICK  $y, z \in W$  such that  $x = yz^{-1}$

⟨3⟩3.  $x^{-1} = zy^{-1}$

⟨3⟩4.  $x^{-1} \in V$

⟨3⟩5.  $x \in V^{-1}$

⟨2⟩2. Q.E.D.

PROOF: Lemma 12.0.13

⟨1⟩7.  $V^2 \subseteq U$

PROOF: We have  $V^2 \subseteq (V')^2 \subseteq U$

□

**Proposition 12.0.15.** *Every topological group is Hausdorff.*

PROOF:

⟨1⟩1. LET:  $G$  be a topological group.

⟨1⟩2. LET:  $x, y \in G$  with  $x \neq y$

- ⟨1⟩3. LET:  $U = G \setminus \{x^{-1}y\}$
- ⟨1⟩4. PICK a symmetric neighbourhood  $V$  of  $e$  such that  $VV \subseteq U$ 
  - ⟨2⟩1.  $U$  is open  
PROOF: Since  $G$  is  $T_1$ .
  - ⟨2⟩2.  $e \in U$   
PROOF: Since  $x \neq y$
  - ⟨2⟩3. Q.E.D.  
PROOF: Lemma 12.0.14.
- ⟨1⟩5.  $Vx$  and  $Vy$  are disjoint neighbourhoods of  $x$  and  $y$  respectively.
  - ⟨2⟩1.  $Vx$  is open  
PROOF: Since  $Vx = r_x(V)$
  - ⟨2⟩2.  $Vy$  is open  
PROOF: Similar.
  - ⟨2⟩3.  $Vx \cap Vy = \emptyset$ 
    - ⟨3⟩1. ASSUME: for a contradiction  $z \in Vx \cap Vy$
    - ⟨3⟩2. PICK  $a, b \in V$  such that  $z = ax = by$
    - ⟨3⟩3.  $xy^{-1} \in VV$   
PROOF: Since  $xy^{-1} = a^{-1}b$
    - ⟨3⟩4.  $xy^{-1} \in U$
    - ⟨3⟩5. Q.E.D.  
PROOF: From ⟨1⟩3.

□

**Proposition 12.0.16.** *Every topological group is regular.*

PROOF:

- ⟨1⟩1. LET:  $G$  be a topological group.
- ⟨1⟩2. LET:  $A \subseteq G$  be a closed set and  $a \notin A$ .
- ⟨1⟩3. LET:  $U = G \setminus Aa^{-1}$
- ⟨1⟩4. PICK a symmetric neighbourhood  $V$  of  $e$  such that  $VV \subseteq U$ 
  - ⟨2⟩1.  $U$  is open  
PROOF: Since  $Aa^{-1} = r_{a^{-1}}(A)$  is closed.
  - ⟨2⟩2.  $e \in U$   
PROOF: Since  $a \notin A$ .
  - ⟨2⟩3. Q.E.D.  
PROOF: Lemma 12.0.14.
- ⟨1⟩5.  $VA$  and  $Va$  are disjoint open sets with  $A \subseteq VA$  and  $a \in Va$ 
  - ⟨2⟩1.  $VA$  is open  
PROOF: Lemma 12.0.11
  - ⟨2⟩2.  $Va$  is open  
PROOF: Lemma 12.0.11
  - ⟨2⟩3.  $VA \cap Va = \emptyset$ 
    - ⟨3⟩1. ASSUME: for a contradiction  $z \in VA \cap Va$
    - ⟨3⟩2. PICK  $b, c \in V$  and  $d \in A$  with  $z = bd = ca$
    - ⟨3⟩3.  $da^{-1} \in U$   
PROOF: Since  $da^{-1} = b^{-1}c \in VV \subseteq U$
    - ⟨3⟩4. Q.E.D.

PROOF: This contradicts  $\langle 1 \rangle 3$

□

**Example 12.0.17.** Not every topological group is normal.

PROOF: For  $J$  an uncountable set, we have  $\mathbb{R}^J$  is a topological group under pointwise addition, but is not normal. □

**Proposition 12.0.18.** *Let  $G$  be a topological group and  $H$  a subgroup of  $G$ . Give  $G/H$  the quotient topology. If  $H$  is closed in  $G$  then  $G/H$  is regular.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $p : G \rightarrow G/H$  be the canonical surjection.
- $\langle 1 \rangle 2$ . LET:  $A$  be a closed set in  $G/H$  and  $aH \in (G/H) \setminus A$ .
- $\langle 1 \rangle 3$ . LET:  $B = p^{-1}(A)$
- $\langle 1 \rangle 4$ .  $B$  is a closed saturated set in  $G$ .
- $\langle 1 \rangle 5$ .  $B \cap aH = \emptyset$
- $\langle 1 \rangle 6$ .  $B = BH$
- $\langle 1 \rangle 7$ . PICK a symmetric neighbourhood  $V$  of  $e$  such that  $VB$  does not intersect  $Va$ 
  - $\langle 2 \rangle 1$ . LET:  $U = G \setminus Ba^{-1}$
  - $\langle 2 \rangle 2$ . PICK a symmetric neighbourhood  $V$  of  $e$  such that  $VV \subseteq U$ 
    - $\langle 3 \rangle 1$ .  $U$  is open
 

PROOF: Since  $Ba^{-1} = r_{a^{-1}}(B)$  is closed.
    - $\langle 3 \rangle 2$ .  $e \in U$ 

PROOF: If  $e \in Ba^{-1}$  then  $a \in B$
    - $\langle 3 \rangle 3$ . Q.E.D.
 

PROOF: Lemma 12.0.14
  - $\langle 2 \rangle 3$ .  $VB \cap Va = \emptyset$ 

PROOF: If  $vb = v'a$  for  $v, v' \in V$  and  $b \in B$  then we have  $ba^{-1} = v^{-1}v' \in Ba \cap VV \subseteq Ba \cap U$ .
- $\langle 1 \rangle 8$ .  $p(VB)$  and  $p(Va)$  are disjoint open sets
  - $\langle 2 \rangle 1$ .  $p(VB)$  and  $p(Va)$  are open.
 

PROOF: Proposition 12.0.9.
  - $\langle 2 \rangle 2$ .  $p(VB) \cap p(Va) = \emptyset$ 

PROOF: If  $vbH = v'aH$  for  $v, v' \in V$ ,  $b \in B$  then  $v'a = vbh$  for some  $h \in H$ . Hence  $v'a \in Va \cap VBH = Va \cap VB$ .
- $\langle 1 \rangle 9$ .  $A \subseteq p(VB)$
- $\langle 1 \rangle 10$ .  $aH \in p(Va)$

□

**Proposition 12.0.19.** *Let  $G$  be a topological group. The component of  $G$  that contains  $e$  is a normal subgroup of  $G$ .*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $C$  be the component of  $G$  that contains  $e$ .
- $\langle 1 \rangle 2$ . For all  $x \in G$ ,  $xC$  is the component of  $G$  that contains  $x$ .
  - $\langle 2 \rangle 1$ . LET:  $x \in G$

- (2)2. LET:  $D$  be the component of  $G$  that contains  $x$ .  
 (2)3.  $xC \subseteq D$   
 PROOF: Since  $xC$  is connected by Theorem 11.31.13.  
 (2)4.  $D \subseteq xC$   
 PROOF: Since  $x^{-1}D \subseteq C$  similarly.  
 (1)3. For all  $x \in G$ ,  $Cx$  is the component of  $G$  that contains  $x$ .  
 PROOF: Similar.  
 (1)4. For all  $x \in C$  we have  $xC = Cx = C$   
 (1)5. For all  $x \in C$  we have  $x^{-1}C = C$   
 (1)6. For all  $x \in C$  we have  $x^{-1} \in C$   
 (1)7. For all  $x, y \in C$  we have  $xy \in C$   
 PROOF: Since  $xyC = xC = x$ .  
 (1)8. For all  $x \in G$  we have  $xC = Cx$ .  
 PROOF: From (1)2 and (1)3.

□

**Lemma 12.0.20.** *Let  $G$  be a topological group. Let  $A$  be a closed set in  $G$  and  $B$  a compact subspace of  $G$  such that  $A \cap B = \emptyset$ . Then there exists a symmetric neighbourhood  $U$  of  $e$  such that  $AU \cap BU = \emptyset$ .*

PROOF:

- (1)1. For all  $b \in B$  there exists a symmetric neighbourhood  $V$  of  $e$  such that  $bV^2 \cap A = \emptyset$   
 (2)1. LET:  $b \in B$   
 (2)2. LET:  $W = b^{-1}(G \setminus A)$   
 (2)3.  $W$  is a neighbourhood of  $e$  and  $bW \cap A = \emptyset$   
 (2)4. PICK a symmetric neighbourhood  $V$  of  $e$  such that  $V^2 \subseteq W$   
 (1)2.  $\{bV^2 \mid b \in B, V \text{ is a symmetric neighbourhood of } e, bV^2 \cap A = \emptyset\}$  is an open cover of  $B$   
 (1)3. PICK a finite subcover  $b_1V_1^2, \dots, b_nV_n^2$ , say.  
 (1)4. LET:  $U = V_1 \cap \dots \cap V_n$   
 (1)5.  $BU^2 \cap A = \emptyset$   
 (1)6.  $AU \cap BU = \emptyset$

PROOF: If  $av \in BU$  where  $a \in A$  and  $v \in V$  then  $a = avv^{-1} \in BU^2 \cap A$ .

□

**Proposition 12.0.21 (AC).** *Let  $G$  be a topological group. Let  $A$  be a closed set in  $G$ , and  $B$  a compact subspace of  $G$ . Then  $AB$  is closed.*

PROOF:

- (1)1. LET:  $x \in G \setminus AB$   
 (1)2.  $A^{-1}x \cap B = \emptyset$   
 (1)3.  $A^{-1}x$  is closed.  
 (1)4. PICK a symmetric neighbourhood  $U$  of  $e$  such that  $A^{-1}xU \cap BU = \emptyset$   
 (1)5.  $xU^2$  is open  
 PROOF: Lemma 12.0.11.  
 (1)6.  $x \in xU^2 \subseteq G \setminus AB$

□

PROOF:

⟨1⟩1. LET:  $x \in \overline{AB}$

PROVE:  $x \in AB$

⟨1⟩2. PICK a net  $(a_\alpha b_\alpha)_{\alpha \in J}$  in  $AB$  that converges to  $x$ .

⟨1⟩3. PICK a convergent subnet  $(b_{g(\beta)})_{\beta \in K}$  of  $(b_\alpha)_{\alpha \in J}$  with limit  $l$ .

⟨1⟩4.  $a_{g(\beta)} \rightarrow xl^{-1}$  as  $\beta \rightarrow \infty$

PROOF:

$$\begin{aligned} a_{g(\beta)} &= a_{g(\beta)} b_{g(\beta)} b_{g(\beta)}^{-1} \\ &\rightarrow xl^{-1} \end{aligned}$$

⟨1⟩5.  $xl^{-1} \in A$

⟨1⟩6.  $l \in B$

PROOF:  $B$  is closed because it is compact.

⟨1⟩7.  $x \in AB$

□

**Corollary 12.0.21.1.** *Let  $G$  be a topological group and  $H \leq G$ . Let  $p : G \rightarrow G/H$  be the quotient map. If  $H$  is compact then  $p$  is a closed map.*

PROOF: For  $A$  closed in  $G$ , we have  $p^{-1}(p(A)) = AH$  is closed, and so  $p(A)$  is closed. □

**Corollary 12.0.21.2.** *Let  $G$  be a topological group and  $H \leq G$ . If  $H$  and  $G/H$  are compact then  $G$  is compact.*

PROOF: From Proposition 11.52.2 since, for all  $aH \in G/H$ , we have  $p^{-1}(aH) = aH$  is compact because it is homomorphic to  $H$ . □

**Proposition 12.0.22.** *Let  $G$  be a locally compact topological group. Let  $H \leq G$ . Then  $G/H$  is locally compact.*

PROOF: From Propositions 11.54.12 and 12.0.9. □

**Proposition 12.0.23.** *Every Lindelöf first countable topological group is second countable.*

PROOF:

⟨1⟩1. LET:  $G$  be a topological group that is Lindelöf and first countable.

⟨1⟩2. PICK a countable local basis  $\{B_n \mid n \in \mathbb{Z}^+\}$  at  $e$ .

⟨1⟩3. For  $n \in \mathbb{Z}^+$ , PICK a countable set  $C_n$  such that  $\{cB_n \mid c \in C\}$  covers  $G$ .

PROVE:  $\{cB_n \mid n \in \mathbb{Z}^+, c \in C_n\}$  is a basis for  $G$ .

⟨1⟩4. LET:  $x \in G$  and  $U$  be an open neighbourhood of  $x$

⟨1⟩5. PICK a symmetric open neighbourhood  $V$  of  $e$  such that  $VV \subseteq x^{-1}U$

⟨1⟩6. PICK  $n$  such that  $B_n \subseteq V$

⟨1⟩7. PICK  $c \in C_n$  such that  $x \in cB_n$

⟨1⟩8.  $cB_n \subseteq U$

⟨2⟩1. LET:  $b \in B_n$

PROVE:  $cb \in U$



- (2)2. PICK  $b' \in B_n$  such that  $x = cb'$   
 (2)3.  $cb = xb'^{-1}b$   
 (2)4.  $cb \in U$   
 PROOF: Since  $b'^{-1}b \in VV \subseteq x^{-1}U$ .

□

**Proposition 12.0.24.** *Every separable first countable topological group is second countable.*

PROOF:

- (1)1. LET:  $G$  be a topological group that is separable and first countable.  
 (1)2. PICK a countable local basis  $\{B_n \mid n \in \mathbb{Z}^+\}$  at  $e$ .  
 (1)3. PICK a countable dense set  $D$  in  $G$ .  
 PROVE:  $\{dB_n \mid n \in \mathbb{Z}^+, d \in D\}$  is a basis for  $G$ .  
 (1)4. LET:  $x \in G$  and  $U$  be an open neighbourhood of  $x$   
 (1)5. PICK a symmetric open neighbourhood  $V$  of  $e$  such that  $VV \subseteq x^{-1}U$   
 (1)6. PICK  $n$  such that  $B_n \subseteq V$   
 (1)7. PICK  $d \in D \cap xB_n^{-1}$   
 (1)8.  $x \in dB_n$   
 (1)9.  $dB_n \subseteq U$   
 (2)1. LET:  $b \in B_n$   
 PROVE:  $db \in U$   
 (2)2. PICK  $b' \in B_n$  such that  $d = xb'^{-1}$   
 (2)3.  $db = xb'^{-1}b$   
 (2)4.  $db \in U$   
 PROOF: Since  $b'^{-1}b \in VV \subseteq x^{-1}U$ .

□

## 12.1 Actions

**Definition 12.1.1** (Action). Let  $X$  be a topological space. Let  $G$  be a topological group. An *action* of  $G$  on  $X$  is a continuous map  $\cdot : G \times X \rightarrow X$  such that:

- For all  $x \in X$ ,  $ex = x$
- For all  $g, h \in G$  and  $x \in X$ ,  $g(hx) = (gh)x$

**Definition 12.1.2** (Orbit Space). Let  $X$  be a topological space. Let  $G$  be a topological group. Let  $\cdot$  be an action of  $G$  on  $X$ . The *orbit space* of the action,  $X/G$ , is the quotient space of  $X$  by  $\sim$ , where  $x \sim y$  iff there exists  $g \in G$  such that  $y = gx$ .

**Proposition 12.1.3.** *Let  $G$  be a compact topological group. Let  $X$  be a topological space. Let  $\cdot$  be an action of  $G$  on  $X$ . The canonical map  $\pi : X \rightarrow X/G$  is a perfect map.*

PROOF:

- (1)1.  $\pi$  is a closed map.  
 (2)1. LET:  $A \subseteq X$  be closed.  
 PROVE:  $\pi^{-1}(\pi(A)) = GA$  is closed.  
 (2)2. LET:  $x \in X - GA$   
 (2)3. For all  $g \in G$  we have  $gx \notin A$   
 (2)4. For all  $g \in G$ , we have  $(g, x) \in \alpha^{-1}(X - A)$ , where  $\alpha = \cdot : G \times X \rightarrow X$   
 (2)5. For all  $g \in G$ , there exist open neighbourhoods  $U$  of  $g$  and  $V$  of  $x$  such that  $U \times V \subseteq \alpha^{-1}(X - A)$   
 (2)6.  $\{U \text{ open in } G \mid \exists V \text{ open in } X. x \in V \text{ and } U \times V \subseteq \alpha^{-1}(X - A)\}$  covers  $G$   
 (2)7. PICK a finite subcover  $\{U_1, \dots, U_n\}$   
 (2)8. For  $1 \leq i \leq n$ , PICK  $V_i$  open in  $X$  such that  $x \in V_i$  and  $U_i \times V_i \subseteq \alpha^{-1}(X - A)$   
 (2)9. LET:  $V = V_1 \cap \dots \cap V_n$   
 (2)10.  $x \in V \subseteq X - GA$   
 (1)2.  $\pi$  is continuous.  
 PROOF: By construction it is a quotient map.  
 (1)3. For all  $y \in X/G$ , we have  $\pi^{-1}(y)$  is compact.  
 PROOF: By Theorem 11.51.4 since it is the image of the map  $G \rightarrow X$  that maps  $g$  to  $gy$ .

□

**Corollary 12.1.3.1.** *Let  $G$  be a compact topological group. Let  $X$  be a Hausdorff space. Let  $\cdot$  be an action of  $G$  on  $X$ . Then  $X/G$  is Hausdorff.*

PROOF: Proposition 11.52.3. □

**Corollary 12.1.3.2.** *Let  $G$  be a compact topological group. Let  $X$  be a regular space. Let  $\cdot$  be an action of  $G$  on  $X$ . Then  $X/G$  is regular.*

PROOF: Proposition 11.52.4. □

**Corollary 12.1.3.3.** *Let  $G$  be a compact topological group. Let  $X$  be a normal space. Let  $\cdot$  be an action of  $G$  on  $X$ . Then  $X/G$  is normal.*

PROOF: Proposition 11.59.7. □

**Corollary 12.1.3.4.** *Let  $G$  be a compact topological group. Let  $X$  be a locally compact space. Let  $\cdot$  be an action of  $G$  on  $X$ . Then  $X/G$  is locally compact.*

PROOF: Proposition 11.54.15. □

**Corollary 12.1.3.5.** *Let  $G$  be a compact topological group. Let  $X$  be a second countable space. Let  $\cdot$  be an action of  $G$  on  $X$ . Then  $X/G$  is second countable.*

PROOF: Proposition 11.52.5. □

**Proposition 12.1.4.** *Every topological group is completely regular.*

PROOF:

- (1)1. LET:  $G$  be a topological group.  
 (1)2. LET:  $A$  be a closed set that does not contain  $e$ .  
 (1)3. LET:  $V_0 = G - A$   
 (1)4. PICK a sequence  $(V_n)_{n \geq 1}$  of symmetric neighbourhoods of  $e$  such that, for all  $n \geq 1$ ,  $V_n V_n \subseteq V_{n-1}$ .  
 (1)5. Define an open set  $U(p)$  for every dyadic rational  $p$  by:

$$\begin{aligned}
 U(1/2^n) &= V_n & (n \geq 0) \\
 U((2k+1)/2^n) &= V_n U(k/2^{n-1}) & (n \geq 0, 0 < k < 2^n) \\
 U(p) &= \emptyset & (p \leq 0) \\
 U(p) &= G & (p > 1)
 \end{aligned}$$

- (1)6. For all  $n \geq 0$  and  $0 < k < 2^n$ , we have

$$V_n U(k/2^n) \subseteq U((k+1)/2^n)$$

PROOF: The proof is by induction on  $n$ .

- (2)1. ASSUME: For all  $0 < k < 2^n$ , we have  $V_n U(k/2^n) \subseteq U((k+1)/2^n)$

- (2)2.  $V_{n+1} U(1/2^{n+1}) \subseteq U(1/2^n)$

PROOF:

$$\begin{aligned}
 V_{n+1} U(1/2^{n+1}) &= V_{n+1} V_{n+1} & (\langle 1 \rangle 5) \\
 &\subseteq V_n & (\langle 1 \rangle 4) \\
 &= U(1/2^n) & (\langle 1 \rangle 5)
 \end{aligned}$$

- (2)3. For  $0 < k < 2^{n+1}$  even we have  $V_{n+1} U(k/2^{n+1}) \subseteq U((k+1)/2^{n+1})$

- (3)1. LET:  $0 < k < 2^{n+1}$  be even

- (3)2. LET:  $k = 2l$

- (3)3.  $V_{n+1} U(k/2^{n+1}) \subseteq U((k+1)/2^{n+1})$

PROOF:

$$\begin{aligned}
 V_{n+1} U(k/2^{n+1}) &= V_{n+1} U(l/2^n) \\
 &= U((2l+1)/2^{n+1}) & (\langle 1 \rangle 5) \\
 &= U((k+1)/2^{n+1})
 \end{aligned}$$

- (2)4. For  $0 < k < 2^{n+1}$  odd we have  $V_{n+1} U(k/2^{n+1}) \subseteq U((k+1)/2^{n+1})$

PROOF:

- (3)1. LET:  $0 < k < 2^{n+1}$  be odd

- (3)2. LET:  $k = 2l + 1$

- (3)3.  $V_{n+1} U(k/2^{n+1}) \subseteq U((k+1)/2^{n+1})$

PROOF:

$$\begin{aligned}
 V_{n+1} U(k/2^{n+1}) &= V_{n+1} U(2l+1/2^{n+1}) \\
 &= V_{n+1} V_{n+1} U(l/2^n) & (\langle 1 \rangle 5) \\
 &\subseteq V_n U(l/2^n) & (\langle 1 \rangle 4) \\
 &\subseteq U(l+1/2^n) & (\langle 2 \rangle 1) \\
 &= U((k+1)/2^{n+1})
 \end{aligned}$$

- (1)7. For any dyadic rationals  $p, q$ , if  $p < q$  then  $\overline{U(p)} \subseteq U(q)$

- (2)1. LET:  $p$  and  $q$  be dyadic rationals such that  $p < q$

- (2)2. PICK  $n > 0, k, l$  such that  $p = k/2^n$  and  $q = l/2^n$  and  $k < l$

- (2)3.  $V_n U(p) \subseteq U(q)$

- (2)4.  $\overline{U(p)} \subseteq V_n U(p)$ 
  - (3)1. LET:  $x \in \overline{U(p)}$
  - (3)2.  $V_n x$  is an open neighbourhood of  $x$
  - (3)3. PICK  $y \in U(p) \cap V_n x$
  - (3)4. PICK  $v \in V_n$  such that  $y = vx$
  - (3)5.  $x = v^{-1}y \in V_n U(p)$
- PROOF: Since  $V_n$  is symmetric.
- (1)8. For  $x \in G$ ,
  - LET:  $\mathbb{Q}(x) = \{p \mid x \in U(p)\}$
- (1)9. Define  $f : G \rightarrow [0, 1]$  by  $f(x) = \inf \mathbb{Q}(x)$
- (1)10.  $f$  is continuous.
- (1)11.  $f$  is continuous.
  - (2)1. For all  $r \in P$  and  $x \in \overline{U_r}$  we have  $f(x) \leq r$ 
    - (3)1. LET:  $r \in P$
    - (3)2. LET:  $x \in \overline{U_r}$
    - (3)3. For all  $s > r$  we have  $x \in U_s$
    - (3)4. For all  $s > r$  we have  $s \in \mathbb{Q}(x)$
    - (3)5. For all  $s > r$  we have  $f(x) \leq s$
    - (3)6.  $f(x) \leq r$
  - (2)2. For all  $r \in P$  and  $x \in U_r$  we have  $f(x) \geq r$ 
    - (3)1. LET:  $r \in P$
    - (3)2. LET:  $x \in X - U_r$
    - (3)3. For all  $s < r$  we have  $x \notin U_s$
    - (3)4.  $r$  is a lower bound for  $\mathbb{Q}(x)$
    - (3)5.  $r \leq f(x)$
- (2)3. LET:  $x_0 \in X$
- (2)4. LET:  $(c, d)$  be an open interval that contains  $f(x_0)$ 
  - PROVE: There exists an open neighbourhood  $U$  of  $x_0$  such that  $f(U) \subseteq (c, d)$
- (2)5. PICK rationals  $q, r$  with  $c < q < f(x_0) < r < d$
- (2)6. LET:  $U = U_r - \overline{U_q}$
- (2)7.  $U$  is open
- (2)8.  $x_0 \in U$ 
  - (3)1.  $x_0 \in U_r$
  - PROOF: From (2)2.
  - (3)2.  $x_0 \notin \overline{U_q}$
  - PROOF: From (2)1.
- (2)9.  $f(U) \subseteq (c, d)$ 
  - (3)1. LET:  $x \in U$
  - (3)2.  $r \leq f(x) \leq s$
  - PROOF: From (2)1 and (2)2.
  - (3)3.  $c < f(x) < d$
- (1)12.  $f(e) = 0$
- (1)13.  $f(A) = \{1\}$
- (1)14. Q.E.D.

PROOF: The result follows for any  $a \in G$  by taking the homeomorphism that

$\square$  maps  $a$  to  $e$ .

## 12.2 The Metric Topology

**Definition 12.2.1** (Metric). Let  $X$  be a set. A *metric* on  $X$  is a function  $d : X^2 \rightarrow \mathbb{R}$  such that:

1. For all  $x, y \in X$ ,  $d(x, y) \geq 0$
2. For all  $x, y \in X$ ,  $d(x, y) = 0$  if and only if  $x = y$
3. For all  $x, y \in X$ ,  $d(x, y) = d(y, x)$
4. (*Triangle Inequality*) For all  $x, y, z \in X$ ,  $d(x, z) \leq d(x, y) + d(y, z)$

We call  $d(x, y)$  the *distance* between  $x$  and  $y$ .

**Definition 12.2.2** (Open Ball). Let  $X$  be a metric space. Let  $a \in X$  and  $\epsilon > 0$ . The *open ball* with *centre*  $a$  and *radius*  $\epsilon$  is

$$B(a, \epsilon) = \{x \in X \mid d(a, x) < \epsilon\} .$$

**Definition 12.2.3** (Metric Topology). Let  $X$  be a metric space. The *metric topology* on  $X$  is the topology generated by the basis consisting of all the open balls.

We prove this is a basis for a topology.

PROOF:

$\langle 1 \rangle 1$ . For every point  $a$ , there exists a ball  $B$  such that  $a \in B$

PROOF: We have  $a \in B(a, 1)$ .

$\langle 1 \rangle 2$ . For any balls  $B_1, B_2$  and point  $a \in B_1 \cap B_2$ , there exists a ball  $B_3$  such that  $a \in B_3 \subseteq B_1 \cap B_2$

$\langle 2 \rangle 1$ . LET:  $B_1 = B(c_1, \epsilon_1)$  and  $B_2 = B(c_2, \epsilon_2)$

$\langle 2 \rangle 2$ . LET:  $\delta = \min(\epsilon_1 - d(c_1, a), \epsilon_2 - d(c_2, a))$

PROVE:  $B(a, \delta) \subseteq B_1 \cap B_2$

$\langle 2 \rangle 3$ . LET:  $x \in B(a, \delta)$

$\langle 2 \rangle 4$ .  $x \in B_1$

PROOF:

$$\begin{aligned} d(x, c_1) &= d(x, a) + d(a, c_1) \\ &< \delta + d(a, c_1) \\ &\leq \epsilon_1 \end{aligned}$$

$\langle 2 \rangle 5$ .  $x \in B_2$

PROOF: Similar.

□

**Proposition 12.2.4.** Let  $X$  be a metric space and  $U \subseteq X$ . Then  $U$  is open if and only if, for every  $x \in U$ , there exists  $\epsilon > 0$  such that  $B(x, \epsilon) \subseteq U$ .

PROOF:

$\langle 1 \rangle 1$ . If  $U$  is open then, for all  $x \in U$ , there exists  $\epsilon > 0$  such that  $B(x, \epsilon) \subseteq U$ .

$\langle 2 \rangle 1$ . ASSUME:  $U$  is open.

- (2)2. LET:  $x \in U$   
 (2)3. PICK  $a \in X$  and  $\delta > 0$  such that  $x \in B(a, \delta) \subseteq U$   
 (2)4. LET:  $\epsilon = \delta - d(a, x)$   
 PROVE:  $B(x, \epsilon) \subseteq U$   
 (2)5. LET:  $y \in B(x, \epsilon)$   
 (2)6.  $d(y, a) < \delta$   
 PROOF:

$$\begin{aligned}
 d(y, a) &\leq d(a, x) + d(x, y) \\
 &< \delta + d(x, y) \\
 &= \epsilon
 \end{aligned}$$

- (2)7.  $y \in U$   
 (1)2. If, for all  $x \in U$ , there exists  $\epsilon > 0$  such that  $B(x, \epsilon) \subseteq U$ , then  $U$  is open.  
 PROOF: Immediate from definitions.

□

**Definition 12.2.5** (Discrete Metric). Let  $X$  be a set. The *discrete metric* on  $X$  is defined by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

**Proposition 12.2.6.** *The discrete metric induces the discrete topology.*

PROOF: For any (open) set  $U$  and point  $a \in U$ , we have  $a \in B(a, 1) \subseteq U$ . □

**Definition 12.2.7** (Standard Metric on  $\mathbb{R}$ ). The *standard metric* on  $\mathbb{R}$  is defined by  $d(x, y) = |x - y|$ .

**Proposition 12.2.8.** *The standard metric on  $\mathbb{R}$  induces the standard topology on  $\mathbb{R}$ .*

PROOF:

- (1)1. Every open ball is open in the standard topology on  $\mathbb{R}$ .  
 PROOF:  $B(a, \epsilon) = (a - \epsilon, a + \epsilon)$   
 (1)2. For every open set  $U$  and point  $a \in U$ , there exists  $\epsilon > 0$  such that  
 $B(a, \epsilon) \subseteq U$   
 (2)1. LET:  $U$  be an open set and  $a \in U$   
 (2)2. PICK an open interval  $b, c$  such that  $a \in (b, c) \subseteq U$   
 (2)3. LET:  $\epsilon = \min(a - b, c - a)$   
 (2)4.  $B(a, \epsilon) \subseteq U$

□

**Definition 12.2.9** (Metrizable). A topological space  $X$  is *metrizable* if and only if there exists a metric on  $X$  that induces the topology.

**Definition 12.2.10** (Bounded). Let  $X$  be a metric space and  $A \subseteq X$ . Then  $A$  is *bounded* if and only if there exists  $M$  such that, for all  $x, y \in A$ , we have  $d(x, y) \leq M$ .

**Definition 12.2.11** (Diameter). Let  $X$  be a metric space and  $A \subseteq X$ . The *diameter* of  $A$  is

$$\text{diam } A = \sup_{x,y \in A} d(x,y) .$$

**Definition 12.2.12** (Standard Bounded Metric). Let  $d$  be a metric on  $X$ . The *standard bounded metric* corresponding to  $d$  is the metric  $\bar{d}$  defined by

$$\bar{d}(x,y) = \min(d(x,y), 1) .$$

We prove this is a metric.

PROOF:

$\langle 1 \rangle 1. \bar{d}(x,y) \geq 0$

PROOF: Since  $d(x,y) \geq 0$

$\langle 1 \rangle 2. \bar{d}(x,y) = 0$  if and only if  $x = y$

PROOF:  $\bar{d}(x,y) = 0$  if and only if  $d(x,y) = 0$  if and only if  $x = y$

$\langle 1 \rangle 3. \bar{d}(x,y) = \bar{d}(y,x)$

PROOF: Since  $d(x,y) = d(y,x)$

$\langle 1 \rangle 4. \bar{d}(x,z) \leq \bar{d}(x,y) + \bar{d}(y,z)$

PROOF:

$$\begin{aligned} \bar{d}(x,y) + \bar{d}(y,z) &= \min(d(x,y), 1) + \min(d(y,z), 1) \\ &= \min(d(x,y) + d(y,z), d(x,y) + 1, d(y,z) + 1, 2) \\ &\geq \min(d(x,z), 1) \\ &= \bar{d}(x,z) \end{aligned}$$

□

**Lemma 12.2.13.** In any metric space  $X$ , the set  $\mathcal{B} = \{B(a, \epsilon) \mid a \in X, \epsilon < 1\}$  is a basis for the metric topology.

PROOF:

$\langle 1 \rangle 1.$  Every element of  $\mathcal{B}$  is open.

PROOF: From Lemma 11.7.2.

$\langle 1 \rangle 2.$  For every open set  $U$  and point  $a \in U$ , there exists  $B \in \mathcal{B}$  such that  $a \in B \subseteq U$

$\langle 2 \rangle 1.$  LET:  $U$  be an open set and  $a \in U$

$\langle 2 \rangle 2.$  PICK  $\epsilon > 0$  such that  $B(a, \epsilon) \subseteq U$

$\langle 2 \rangle 3.$   $B(a, \min(\epsilon, 1/2)) \subseteq U$

$\langle 1 \rangle 3.$  Q.E.D.

PROOF: Lemma 11.7.3.

□

**Proposition 12.2.14.** Let  $d$  be a metric on the set  $X$ . Then the standard bounded metric  $\bar{d}$  induces the same metric as  $d$ .

PROOF: This follows from Lemma 12.2.13 since the open balls with radius  $< 1$  are the same under both metrics. □



**Lemma 12.2.15.** *Let  $d$  and  $d'$  be two metrics on the same set  $X$ . Let  $\mathcal{T}$  and  $\mathcal{T}'$  be the topologies they induce. Then  $\mathcal{T} \subseteq \mathcal{T}'$  if and only if, for all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that*

$$B_{d'}(x, \delta) \subseteq B_d(x, \epsilon) .$$

PROOF:

$\langle 1 \rangle 1$ . If  $\mathcal{T} \subseteq \mathcal{T}'$  then, for all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$

PROOF: From Proposition 12.2.4 since  $x \in B_d(x, \epsilon) \in \mathcal{T}'$ .

$\langle 1 \rangle 2$ . If, for all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$ , then  $\mathcal{T} \subseteq \mathcal{T}'$

$\langle 2 \rangle 1$ . ASSUME: For all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$

$\langle 2 \rangle 2$ . LET:  $U \in \mathcal{T}$

$\langle 2 \rangle 3$ . For all  $x \in U$  there exists  $\delta > 0$  such that  $B_{d'}(x, \delta) \subseteq U$ .

$\langle 3 \rangle 1$ . LET:  $x \in U$

$\langle 3 \rangle 2$ . PICK  $\epsilon > 0$  such that  $B_d(x, \epsilon) \subseteq U$

PROOF: Proposition 12.2.4

$\langle 3 \rangle 3$ . PICK  $\delta > 0$  such that  $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$

PROOF: By  $\langle 2 \rangle 1$

$\langle 3 \rangle 4$ .  $B_{d'}(x, \delta) \subseteq U$

$\langle 2 \rangle 4$ .  $U \in \mathcal{T}'$

PROOF: Proposition 12.2.4.

□

**Proposition 12.2.16.**  $\mathbb{R}^2$  under the dictionary order topology is metrizable.

PROOF: Define  $d : \mathbb{R}^2 \rightarrow \mathbb{R}$  by

$$d((x, y), (x, z)) = \max(|y - z|, 1)$$

$$d((x, y), (x', y')) = 1 \quad \text{if } x \neq x' \quad \square$$

$\langle 1 \rangle 1$ .  $x \in \bigcap_{i=1}^N \pi_i^{-1}() \subseteq B_D(a, \epsilon)$

**Proposition 12.2.17.** *Let  $d : X^2 \rightarrow \mathbb{R}$  be a metric on  $X$ . Then the metric topology on  $X$  is the coarsest topology such that  $d$  is continuous.*

PROOF:

$\langle 1 \rangle 1$ .  $d$  is continuous.

$\langle 2 \rangle 1$ . LET:  $a, b \in X$

$\langle 2 \rangle 2$ . LET:  $\epsilon > 0$

$\langle 2 \rangle 3$ . LET:  $\delta = \epsilon/2$

$\langle 2 \rangle 4$ . LET:  $x, y \in X$

$\langle 2 \rangle 5$ . ASSUME:  $\rho((a, b), (x, y)) < \delta$

$\langle 2 \rangle 6$ .  $|d(a, b) - d(x, y)| < \epsilon$

$\langle 3 \rangle 1$ .  $d(a, b) - d(x, y) < \epsilon$

PROOF:

$$\begin{aligned}
 d(a, b) &\leq d(a, x) + d(x, y) + d(y, b) \\
 &\leq d(x, y) + 2\rho((a, b), (x, y)) \\
 &< d(x, y) + 2\delta \\
 &= d(x, y) + \epsilon
 \end{aligned}$$

$\langle 3 \rangle 2.$   $d(a, b) - d(x, y) > -\epsilon$

PROOF: Similar.

$\langle 2 \rangle 7.$  Q.E.D.

$\langle 1 \rangle 2.$  If  $\mathcal{T}$  is any topology under which  $d$  is continuous then  $\mathcal{T}$  is finer than the metric topology.

PROOF: Since  $B(a, \epsilon) = d_a^{-1}((-\infty, \epsilon))$

□

**Proposition 12.2.18.** *Let  $X$  be a metric space with metric  $d$  and  $A \subseteq X$ . The restriction of  $d$  to  $A$  is a metric on  $A$  that induces the subspace topology.*

PROOF:

$\langle 1 \rangle 1.$  The restriction of  $d$  to  $A$  is a metric on  $A$ .

$\langle 1 \rangle 2.$  Every open ball under  $d \upharpoonright A$  is open under the subspace topology.

PROOF:  $B_{d \upharpoonright A}(a, \epsilon) = B_d(a, \epsilon) \cap A$ .

$\langle 1 \rangle 3.$  If  $U$  is open in the subspace topology and  $x \in U$ , then there exists a  $d \upharpoonright A$ -ball  $B$  such that  $x \in B \subseteq U$ .

$\langle 2 \rangle 1.$  PICK  $V$  open in  $X$  such that  $U = V \cap A$

$\langle 2 \rangle 2.$  PICK  $\epsilon > 0$  such that  $B_d(x, \epsilon) \subseteq V$

$\langle 2 \rangle 3.$  Take  $B = B_{d \upharpoonright A}(x, \epsilon)$

□

**Corollary 12.2.18.1.** *A subspace of a metrizable space is metrizable.*

**Proposition 12.2.19.** *Every metrizable space is Hausdorff.*

PROOF:

$\langle 1 \rangle 1.$  LET:  $X$  be a metric space

$\langle 1 \rangle 2.$  LET:  $a, b \in X$  with  $a \neq b$

$\langle 1 \rangle 3.$  LET:  $\epsilon = d(a, b)/2$

$\langle 1 \rangle 4.$  LET:  $U = B(a, \epsilon)$  and  $V = B(b, \epsilon)$

$\langle 1 \rangle 5.$   $U$  and  $V$  are disjoint neighbourhoods of  $a$  and  $b$  respectively.

□

**Corollary 12.2.19.1.** *Every metrizable space is  $T_1$ .*

**Proposition 12.2.20.** *Every metrizable space is normal.*

PROOF:

$\langle 1 \rangle 1.$  LET:  $X$  be a metric space.

$\langle 1 \rangle 2.$   $X$  is  $T_1$ .

PROOF: Corollary 12.2.19.1.

$\langle 1 \rangle 3.$  LET:  $A$  and  $B$  be disjoint closed sets in  $X$ .

- (1)4. LET:  $U = \bigcup \{B(a, \epsilon/2) \mid a \in A, B(a, \epsilon) \cap B = \emptyset\}$   
 (1)5. LET:  $V = \bigcup \{B(b, \epsilon/2) \mid b \in B, B(b, \epsilon) \cap A = \emptyset\}$   
 (1)6.  $U$  and  $V$  are disjoint open neighbourhoods of  $A$  and  $B$  respectively.  
 $\square$

**Corollary 12.2.20.1.** *Every metrizable space is completely normal.*

PROOF: Since a subspace of a metrizable space is metrizable.  $\square$

**Proposition 12.2.21 (CC).** *The product of a countable family of metrizable spaces is metrizable.*

PROOF:

- (1)1. LET:  $(X_n, d_n)$  be a sequence of metric spaces.  
 (1)2. ASSUME: w.l.o.g. each  $d_n$  is bounded above by 1.  
 PROOF: By Proposition 12.2.14.  
 (1)3. LET:  $D$  be the metric on  $\mathbb{R}^\omega$  defined by  $D(x, y) = \sup_i (d_i(x_i, y_i)/i)$ .  
 (2)1.  $D(x, y) \geq 0$   
 (2)2.  $D(x, y) = 0$  if and only if  $x = y$   
 (2)3.  $D(x, y) = D(y, x)$   
 (2)4.  $D(x, z) \leq D(x, y) + D(y, z)$

PROOF:

$$\begin{aligned}
 D(x, z) &= \sup_i \frac{d_i(x_i, z_i)}{i} \\
 &\leq \sup_i \frac{d_i(x_i, y_i) + d_i(y_i, z_i)}{i} \\
 &\leq \sup_i \frac{d_i(x_i, y_i)}{i} + \sup_i \frac{d_i(y_i, z_i)}{i} \\
 &= D(x, y) + D(y, z)
 \end{aligned}$$

- (1)4. Every open ball  $B_D(a, \epsilon)$  is open in the product topology.  
 (2)1. PICK  $N$  such that  $1/\epsilon < N$   
 (2)2.  $B_D(a, \epsilon) = \prod_{i=1}^\infty U_i$  where  $U_i = B(a_i, i\epsilon)$  if  $i \leq N$ , and  $U_i = X_i$  if  $i > N$   
 (1)5. For any open set  $U$  and  $a \in U$ , there exists  $\epsilon > 0$  such that  $B_D(a, \epsilon) \subseteq U$ .  
 (2)1. LET:  $n \geq 1$ ,  $V$  be an open set in  $\mathbb{R}$  and  $a \in \pi_n^{-1}(V)$   
 (2)2. PICK  $\epsilon > 0$  such that  $B_{d_n}(a, \epsilon) \subseteq V$   
 (2)3.  $B_D(a, \epsilon/n) \subseteq \pi_n^{-1}(V)$   
 $\square$

**Theorem 12.2.22.** *Let  $X$  and  $Y$  be metric spaces and  $f : X \rightarrow Y$ . Then  $f$  is continuous if and only if, for all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that, for all  $y \in X$ , if  $d(x, y) < \delta$  then  $d(f(x), f(y)) < \epsilon$ .*

PROOF:

- (1)1. If  $f$  is continuous then, for all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that, for all  $y \in X$ , if  $d(x, y) < \delta$  then  $d(f(x), f(y)) < \epsilon$   
 (2)1. ASSUME:  $f$  is continuous.  
 (2)2. LET:  $x \in X$  and  $\epsilon > 0$

- (2)3. PICK a neighbourhood  $U$  of  $x$  such that  $f(U) \subseteq B(f(x), \epsilon)$   
PROOF: Theorem 11.12.6.
- (2)4. PICK  $\delta > 0$  such that  $B(x, \delta) \subseteq U$   
PROOF: Proposition 12.2.4.
- (2)5. For all  $y \in X$ , if  $d(x, y) < \delta$  then  $d(f(x), f(y)) < \epsilon$
- (1)2. If for all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that, for all  $y \in X$ , if  $d(x, y) < \delta$  then  $d(f(x), f(y)) < \epsilon$ , then  $f$  is continuous.
- (2)1. ASSUME: for all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that, for all  $y \in X$ , if  $d(x, y) < \delta$  then  $d(f(x), f(y)) < \epsilon$
- (2)2. LET:  $x \in X$  and  $V$  be a neighbourhood of  $f(x)$
- (2)3. PICK  $\epsilon > 0$  such that  $B(f(x), \epsilon) \subseteq V$   
PROOF: Proposition 12.2.4.
- (2)4. PICK  $\delta > 0$  such that, for all  $y \in X$ , if  $d(x, y) < \delta$  then  $d(f(x), f(y)) < \epsilon$   
PROOF: By (2)1
- (2)5. LET:  $U = B(x, \delta)$
- (2)6.  $U$  is a neighbourhood of  $x$  with  $f(U) \subseteq V$
- (2)7. Q.E.D.  
PROOF: Theorem 11.12.6.

□

**Proposition 12.2.23.** *Let  $X$  be a metric space. Let  $(a_n)$  be a sequence in  $X$  and  $l \in X$ . Then  $a_n \rightarrow l$  as  $n \rightarrow \infty$  if and only if, for all  $\epsilon > 0$ , there exists  $N$  such that, for all  $n \geq N$ , we have  $d(a_n, l) < \epsilon$ .*

PROOF: From Proposition 11.9.4. □

**Proposition 12.2.24.** *Every metrizable space is first countable.*

PROOF: In any metric space  $X$ , the open balls  $B(a, 1/n)$  for  $n \geq 1$  form a local basis at  $a$ .

**Example 12.2.25.**  $\mathbb{R}^\omega$  under the box topology is not metrizable.

**Example 12.2.26.** If  $J$  is uncountable then  $\mathbb{R}^J$  under the product topology is not metrizable.

**Example 12.2.27.** The space  $\overline{S_\Omega}$  is not metrizable by Example 11.22.4.

**Proposition 12.2.28.** *A compact subspace of a metric space is bounded.*

PROOF:

- (1)1. LET:  $X$  be a metric space and  $A \subseteq X$  be compact.
- (1)2. PICK  $a \in A$
- (1)3.  $\{B(a, n) \mid n \in \mathbb{Z}^+\}$  covers  $A$
- (1)4. PICK a finite subcover  $\{B(a, n_1), \dots, B(a, n_k)\}$
- (1)5. LET:  $N = \max(n_1, \dots, n_k)$
- (1)6. For all  $x, y \in A$  we have  $d(x, y) < 2N$

PROOF:

$$\begin{aligned} d(x, y) &\leq d(x, a) + d(a, y) \\ &< N + N \end{aligned}$$

□

This example shows the converse does not hold:

**Example 12.2.29.** The space  $\mathbb{R}$  under the standard bounded metric is bounded but not compact.

**Proposition 12.2.30.** *A connected metric space with more than one point is uncountable.*

PROOF:

⟨1⟩1. LET:  $X$  be a connected metric space with more than one point.

⟨1⟩2. PICK  $a \in X$

⟨1⟩3.  $d(a, -) : X \rightarrow \mathbb{R}$  is continuous.

PROOF: Proposition 12.2.17.

⟨1⟩4.  $\{d(a, x) \mid x \in X\}$  is a connected subspace of  $\mathbb{R}$  that includes 0.

PROOF: Theorem 11.31.13.

⟨1⟩5.  $\{d(a, x) \mid x \in X\} \neq \{0\}$

PROOF: Since  $X$  has more than one point.

⟨1⟩6.  $\{d(a, x) \mid x \in X\}$  is uncountable.

PROOF: Since it includes a closed interval (Corollary 11.53.2.1).

□

## 12.3 Real Linear Algebra

**Definition 12.3.1** (Square Metric). The *square metric*  $\rho$  on  $\mathbb{R}^n$  is defined by

$$\rho(\vec{x}, \vec{y}) = \max(|x_1 - y_1|, \dots, |x_n - y_n|)$$

We prove this is a metric.

PROOF:

⟨1⟩1.  $\rho(\vec{x}, \vec{y}) \geq 0$

PROOF: Immediate from definition.

⟨1⟩2.  $\rho(\vec{x}, \vec{y}) = 0$  if and only if  $\vec{x} = \vec{y}$

PROOF: Immediate from definition.

⟨1⟩3.  $\rho(\vec{x}, \vec{y}) = \rho(\vec{y}, \vec{x})$

PROOF: Immediate from definition.

⟨1⟩4.  $\rho(\vec{x}, \vec{z}) \leq \rho(\vec{x}, \vec{y}) + \rho(\vec{y}, \vec{z})$

PROOF: Since  $|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$ .

□

**Proposition 12.3.2.** *The square metric induces the standard topology on  $\mathbb{R}^n$ .*

PROOF:

⟨1⟩1. For every  $a \in X$  and  $\epsilon > 0$ , we have  $B_\rho(a, \epsilon)$  is open in the standard product topology.

PROOF:

$$B_\rho(a, \epsilon) = (a_1 - \epsilon, a_1 + \epsilon) \times \cdots \times (a_n - \epsilon, a_n + \epsilon)$$

- ⟨1⟩2. For any open sets  $U_1, \dots, U_n$  in  $\mathbb{R}$ , we have  $U_1 \times \dots \times U_n$  is open in the square metric topology.
- ⟨2⟩1. LET:  $\vec{a} \in U_1 \times \dots \times U_n$
- ⟨2⟩2. For  $i = 1, \dots, n$ , PICK  $\epsilon_i > 0$  such that  $(a_i - \epsilon_i, a_i + \epsilon_i) \subseteq U_i$
- ⟨2⟩3. LET:  $\epsilon = \min(\epsilon_1, \dots, \epsilon_n)$
- ⟨2⟩4.  $B_\rho(\vec{a}, \epsilon) \subseteq U$

□

**Definition 12.3.3.** Given  $\vec{x}, \vec{y} \in \mathbb{R}^n$ , define the *sum*  $\vec{x} + \vec{y}$  by

$$(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n) .$$

**Definition 12.3.4.** Given  $\lambda \in \mathbb{R}$  and  $\vec{x} \in \mathbb{R}^n$ , define the *scalar product*  $\lambda\vec{x} \in \mathbb{R}^n$  by

$$\lambda(x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n)$$

**Definition 12.3.5** (Inner Product). Given  $\vec{x}, \vec{y} \in \mathbb{R}^n$ , define the *inner product*  $\vec{x} \cdot \vec{y} \in \mathbb{R}$  by

$$(x_1, \dots, x_n) \cdot (y_1, \dots, y_n) = x_1 y_1 + \dots + x_n y_n .$$

We write  $\vec{x}^2$  for  $\vec{x} \cdot \vec{x}$ .

**Definition 12.3.6** (Norm). Let  $n \geq 1$ . The *norm* on  $\mathbb{R}^n$  is the function  $\| \cdot \| : \mathbb{R}^n \rightarrow \mathbb{R}$  defined by

$$\|(x_1, \dots, x_n)\| = \sqrt{x_1^2 + \dots + x_n^2}$$

**Lemma 12.3.7.**

$$\|\vec{x}\|^2 = \vec{x}^2$$

PROOF: Immediate from definitions. □

**Lemma 12.3.8.**

$$\vec{x} \cdot (\vec{y} + \vec{z}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z}$$

PROOF: Each is equal to  $(x_1 y_1 + x_1 z_1, \dots, x_n y_n + x_n z_n)$ . □

**Lemma 12.3.9.**

$$|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \|\vec{y}\|$$

PROOF:

⟨1⟩1. ASSUME:  $\vec{x} \neq \vec{0} \neq \vec{y}$

PROOF: Otherwise both sides are 0.

⟨1⟩2. LET:  $a = 1/\|\vec{x}\|$

⟨1⟩3. LET:  $b = 1/\|\vec{y}\|$

⟨1⟩4.  $(a\vec{x} + b\vec{y})^2 \geq 0$  and  $(a\vec{x} - b\vec{y})^2 \geq 0$

⟨1⟩5.  $a^2\|\vec{x}\|^2 + 2ab\vec{x} \cdot \vec{y} + b^2\|\vec{y}\|^2 \geq 0$  and  $a^2\|\vec{x}\|^2 - 2ab\vec{x} \cdot \vec{y} + b^2\|\vec{y}\|^2 \geq 0$

⟨1⟩6.  $2ab\vec{x} \cdot \vec{y} + 2 \geq 0$  and  $-2ab\vec{x} \cdot \vec{y} + 2 \geq 0$

⟨1⟩7.  $\vec{x} \cdot \vec{y} \geq -1/ab$  and  $\vec{x} \cdot \vec{y} \leq 1/ab$

⟨1⟩8.  $\vec{x} \cdot \vec{y} \geq -\|\vec{x}\|\|\vec{y}\|$  and  $\vec{x} \cdot \vec{y} \leq \|\vec{x}\|\|\vec{y}\|$

□

**Lemma 12.3.10** (Triangle Inequality).

$$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$$

PROOF:

$$\begin{aligned} \|\vec{x} + \vec{y}\|^2 &= \|\vec{x}\|^2 + 2\vec{x} \cdot \vec{y} + \|\vec{y}\|^2 \\ &\leq \|\vec{x}\|^2 + 2\|\vec{x}\|\|\vec{y}\| + \|\vec{y}\|^2 && \text{(Lemma 12.3.9)} \\ &= (\|\vec{x}\| + \|\vec{y}\|)^2 && \square \end{aligned}$$

**Definition 12.3.11** (Euclidean Metric). Let  $n \geq 1$ . The *Euclidean metric* on  $\mathbb{R}^n$  is defined by

$$d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|.$$

We prove this is a metric.

⟨1⟩1.  $d(\vec{x}, \vec{y}) \geq 0$

PROOF: Immediate from definition.

⟨1⟩2.  $d(\vec{x}, \vec{y}) = 0$  if and only if  $\vec{x} = \vec{y}$

PROOF:  $d(\vec{x}, \vec{y}) = 0$  if and only if  $\vec{x} - \vec{y} = \vec{0}$ .

⟨1⟩3.  $d(\vec{x}, \vec{y}) = d(\vec{y}, \vec{x})$

PROOF: Immediate from definition.

⟨1⟩4.  $d(\vec{x}, \vec{z}) \leq d(\vec{x}, \vec{y}) + d(\vec{y}, \vec{z})$

PROOF:

$$\begin{aligned} \|\vec{x} - \vec{z}\| &= \|(\vec{x} - \vec{y}) + (\vec{y} - \vec{z})\| \\ &\leq \|\vec{x} - \vec{y}\| + \|\vec{y} - \vec{z}\| && \text{(Lemma 12.3.10)} \end{aligned}$$

□

**Proposition 12.3.12.** The Euclidean metric induces the standard topology on  $\mathbb{R}^n$ .

PROOF:

⟨1⟩1. LET:  $\rho$  be the square metric.

⟨1⟩2. For all  $\vec{a} \in \mathbb{R}^n$  and  $\epsilon > 0$ , we have  $B_d(\vec{a}, \epsilon) \subseteq B_\rho(\vec{a}, \epsilon)$

⟨2⟩1. LET:  $\vec{x} \in B_d(\vec{a}, \epsilon)$

⟨2⟩2.  $\sqrt{(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2} < \epsilon$

⟨2⟩3.  $(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2 < \epsilon^2$

⟨2⟩4. For all  $i$  we have  $(x_i - a_i)^2 < \epsilon^2$

⟨2⟩5. For all  $i$  we have  $|x_i - a_i| < \epsilon$

⟨2⟩6.  $\rho(\vec{x}, \vec{a}) < \epsilon$

⟨1⟩3. For all  $\vec{a} \in \mathbb{R}^n$  and  $\epsilon > 0$ , we have  $B_\rho(\vec{a}, \epsilon/\sqrt{n}) \subseteq B_d(\vec{a}, \epsilon)$

⟨2⟩1. LET:  $\vec{x} \in B_\rho(\vec{a}, \epsilon/\sqrt{n})$

⟨2⟩2.  $\rho(\vec{x}, \vec{a}) < \epsilon/\sqrt{n}$

⟨2⟩3. For all  $i$  we have  $|x_i - a_i| < \epsilon/\sqrt{n}$

⟨2⟩4. For all  $i$  we have  $(x_i - a_i)^2 < \epsilon^2/n$

⟨2⟩5.  $d(\vec{x}, \vec{a}) < \epsilon$

⟨1⟩4. Q.E.D.

PROOF: By Lemma 12.2.15.

□

**Proposition 12.3.13.** *Let  $n \geq 0$ . For all  $c \in \mathbb{R}^n$  and  $\epsilon > 0$ , the open ball  $B(c, \epsilon)$  is path connected.*

PROOF:

⟨1⟩1. LET:  $a, b \in B(c, \epsilon)$

⟨1⟩2. LET:  $p : [0, 1] \rightarrow B(c, \epsilon)$  be the function  $p(t) = (1 - t)a + tb$

PROOF: We have  $p(t) \in B(c, \epsilon)$  for all  $t$  because

$$\begin{aligned} d(p(t), c) &= \|(1 - t)a + tb - c\| \\ &= \|(1 - t)(a - c) + t(b - c)\| \\ &\leq (1 - t)\|a - c\| + t\|b - c\| \\ &< (1 - t)\epsilon + t\epsilon \\ &= \epsilon \end{aligned}$$

⟨1⟩3.  $p$  is a path from  $a$  to  $b$ .

□

**Proposition 12.3.14.** *Let  $n \geq 0$ . For all  $c \in \mathbb{R}^n$  and  $\epsilon > 0$ , the closed ball  $\overline{B}(c, \epsilon)$  is path connected.*

PROOF:

⟨1⟩1. LET:  $a, b \in \overline{B}(c, \epsilon)$

⟨1⟩2. LET:  $p : [0, 1] \rightarrow \overline{B}(c, \epsilon)$  be the function  $p(t) = (1 - t)a + tb$

PROOF: We have  $p(t) \in \overline{B}(c, \epsilon)$  for all  $t$  because

$$\begin{aligned} d(p(t), c) &= \|(1 - t)a + tb - c\| \\ &= \|(1 - t)(a - c) + t(b - c)\| \\ &\leq (1 - t)\|a - c\| + t\|b - c\| \\ &\leq (1 - t)\epsilon + t\epsilon \\ &= \epsilon \end{aligned}$$

⟨1⟩3.  $p$  is a path from  $a$  to  $b$ .

□

**Lemma 12.3.15.** *If  $\sum_{i=0}^{\infty} x_i^2$  and  $\sum_{i=0}^{\infty} y_i^2$  converge then  $\sum_{i=0}^{\infty} |x_i y_i|$  converges.*

PROOF:

⟨1⟩1. For all  $N \geq 0$  we have  $\sum_{i=0}^N |x_i y_i| \leq \sqrt{\sum_{i=0}^N |x_i|^2} \sqrt{\sum_{i=0}^N |y_i|^2}$

PROOF: By the Cauchy-Schwarz inequality

⟨1⟩2. Q.E.D.

PROOF: Since  $\sum_{i=0}^N |x_i y_i|$  is an increasing sequence bounded above by  $(\sum_{i=0}^{\infty} x_i^2)(\sum_{i=0}^{\infty} y_i^2)$ .

□

**Corollary 12.3.15.1.** *If  $\sum_{i=0}^{\infty} x_i^2$  and  $\sum_{i=0}^{\infty} y_i^2$  converge then  $\sum_{i=0}^{\infty} (x_i + y_i)^2$  converges.*



PROOF: Since  $\sum_{i=0}^{\infty} x_i^2$ ,  $\sum_{i=0}^{\infty} y_i^2$  and  $2\sum_{i=0}^{\infty} x_i y_i$  all converge.

**Definition 12.3.16** ( $l^2$ -metric). The  $l^2$ -metric on

$$\left\{ (x_n) \in \mathbb{R}^{\omega} \mid \sum_{n=0}^{\infty} x_n^2 \text{ converges} \right\}$$

is defined by

$$d(x, y) = \left( \sum_{n=0}^{\infty} (x_n - y_n)^2 \right)^{1/2}$$

We prove this is a metric.

PROOF:

$\langle 1 \rangle 1$ .  $d$  is well-defined.

PROOF: By Corollary 12.3.15.1.

$\langle 1 \rangle 2$ .  $d(x, y) \geq 0$

$\langle 1 \rangle 3$ .  $d(x, y) = 0$  if and only if  $x = y$

$\langle 1 \rangle 4$ .  $d(x, y) = d(y, x)$

$\langle 1 \rangle 5$ .  $d(x, z) \leq d(x, y) + d(y, z)$

PROOF: By Lemma 12.3.10.

□

**Theorem 12.3.17.** Addition is a continuous function  $\mathbb{R}^2 \rightarrow \mathbb{R}$ .

PROOF:

$\langle 1 \rangle 1$ . LET:  $a, b \in \mathbb{R}$

$\langle 1 \rangle 2$ . LET:  $\epsilon > 0$

$\langle 1 \rangle 3$ . LET:  $\delta = \epsilon/2$

$\langle 1 \rangle 4$ . LET:  $x, y \in \mathbb{R}$

$\langle 1 \rangle 5$ . ASSUME:  $\rho((a, b), (x, y)) < \delta$

$\langle 1 \rangle 6$ .  $|(a + b) - (x + y)| < \epsilon$

PROOF:

$$\begin{aligned} |(a + b) - (x + y)| &= |a - x| + |b - y| \\ &\leq 2\rho((a, b), (x, y)) \\ &< 2\delta \\ &= \epsilon \end{aligned}$$

$\langle 1 \rangle 7$ . Q.E.D.

PROOF: Theorem 12.2.22

□

**Theorem 12.3.18.** Multiplication is a continuous function  $\mathbb{R}^2 \rightarrow \mathbb{R}$ .

PROOF:

$\langle 1 \rangle 1$ . LET:  $a, b \in \mathbb{R}$

$\langle 1 \rangle 2$ . LET:  $\epsilon > 0$

$\langle 1 \rangle 3$ . LET:  $\delta = \min(\epsilon/(|a| + |b| + 1), 1)$

⟨1⟩4. LET:  $x, y \in \mathbb{R}$

⟨1⟩5. ASSUME:  $\rho((a, b), (x, y)) < \delta$

⟨1⟩6.  $|ab - xy| < \epsilon$

PROOF:

$$\begin{aligned}
 |ab - xy| &= |a(b - y) + (a - x)b - (a - x)(b - y)| \\
 &\leq |a||b - y| + |b||a - x| + |a - x||b - y| \\
 &< |a|\delta + |b|\delta + \delta^2 && (\langle 1 \rangle 5) \\
 &\leq |a|\delta + |b|\delta + \delta && (\langle 1 \rangle 3) \\
 &\leq \epsilon && (\langle 1 \rangle 3)
 \end{aligned}$$

⟨1⟩7. Q.E.D.

PROOF: Theorem 12.2.22

□

**Theorem 12.3.19.** *The function  $f : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$  defined by  $f(x) = x^{-1}$  is continuous.*

PROOF:

⟨1⟩1. For all  $a \in \mathbb{R}$  we have  $f^{-1}((a, +\infty))$  is open.

PROOF: The set is

$$\begin{aligned}
 &(a^{-1}, +\infty) \text{ if } a > 0 \\
 &(0, +\infty) \text{ if } a = 0 \\
 &(-\infty, a^{-1}) \cup (0, +\infty) \text{ if } a < 0
 \end{aligned}$$

⟨1⟩2. For all  $a \in \mathbb{R}$  we have  $f^{-1}((-\infty, a))$  is open.

PROOF: Similar.

⟨1⟩3. Q.E.D.

PROOF: By Proposition 11.12.3 and Lemma 11.15.2.

□

**Definition 12.3.20.** For  $n \geq 0$ , the *unit ball*  $B^n$  is the space  $\{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ .

**Proposition 12.3.21.** *For all  $n \geq 0$ , the unit ball  $B^n$  is path connected.*

PROOF:

⟨1⟩1. LET:  $a, b \in B^n$

⟨1⟩2. LET:  $p : [0, 1] \rightarrow B^n$  be the function  $p(t) = (1 - t)a + tb$

PROOF: We have  $p(t) \in B^n$  for all  $t$  because

$$\begin{aligned}
 \|(1 - t)a + tb\| &\leq (1 - t)\|a\| + t\|b\| \\
 &\leq (1 - t) + t \\
 &= 1
 \end{aligned}$$

⟨1⟩3.  $p$  is a path from  $a$  to  $b$ .

□

**Definition 12.3.22** (Punctured Euclidean Space). For  $n \geq 0$ , defined *punctured Euclidean space* to be  $\mathbb{R}^n \setminus \{0\}$ .

**Proposition 12.3.23.** For  $n > 1$ , punctured Euclidean space  $\mathbb{R}^n \setminus \{0\}$  is path connected.

PROOF:

- ⟨1⟩1. LET:  $a, b \in \mathbb{R}^n \setminus \{0\}$
- ⟨1⟩2. CASE: 0 is on the line from  $a$  to  $b$ 
  - ⟨2⟩1. PICK a point  $c$  not on the line from  $a$  to  $b$
  - ⟨2⟩2. The path consisting of a straight line from  $a$  to  $c$  followed by a straight line from  $c$  to  $b$  is a path from  $a$  to  $b$ .
- ⟨1⟩3. CASE: 0 is not on the line from  $a$  to  $b$ 
  - PROOF: The straight line from  $a$  to  $b$  is a path from  $a$  to  $b$ .

**Corollary 12.3.23.1.** For  $n > 1$ , the spaces  $\mathbb{R}$  and  $\mathbb{R}^n$  are not homeomorphic.

PROOF: For any point  $a$ , the space  $\mathbb{R} \setminus \{a\}$  is disconnected.

**Definition 12.3.24** (Unit Sphere). For  $n \geq 1$ , the *unit sphere*  $S^{n-1}$  is the space

$$S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}.$$

**Proposition 12.3.25.** For  $n > 1$ , the unit sphere  $S^{n-1}$  is path connected.

PROOF: The map  $g : \mathbb{R}^n \setminus \{0\} \rightarrow S^{n-1}$  defined by  $g(x) = x/\|x\|$  is continuous and surjective. The result follows by Proposition 11.33.5.  $\square$

**Proposition 12.3.26.** Let  $f : S^1 \rightarrow \mathbb{R}$  be continuous. Then there exists  $x \in S^1$  such that  $f(x) = f(-x)$ .

PROOF:

- ⟨1⟩1. LET:  $g : S^1 \rightarrow \mathbb{R}$  be the function  $g(x) = f(x) - f(-x)$ 
  - PROVE: There exists  $x \in S^1$  such that  $g(x) = 0$
- ⟨1⟩2. ASSUME: without loss of generality  $g((1, 0)) > 0$
- ⟨1⟩3.  $g((-1, 0)) < 0$
- ⟨1⟩4. There exists  $x$  such that  $g(x) = 0$ 
  - PROOF: By the Intermediate Value Theorem.

$\square$

**Definition 12.3.27** (Topologist's Sine Curve). Let  $S = \{(x, \sin 1/x) \mid 0 < x \leq 1\}$ . The *topologist's sine curve* is the closure  $\overline{S}$  of  $S$ .

**Proposition 12.3.28.**

$$\overline{S} = S \cup (\{0\} \times [-1, 1])$$

**Proposition 12.3.29.** The topologist's sine curve is connected.

PROOF:

- ⟨1⟩1. LET:  $S = \{(x, \sin 1/x) \mid 0 < x \leq 1\}$
- ⟨1⟩2.  $S$  is connected.
  - PROOF: Theorem 11.31.13.
- ⟨1⟩3.  $\overline{S}$  is connected.

PROOF: Theorem 11.31.12.

□

**Proposition 12.3.30 (CC).** *The topologist's sine curve is not path connected.*

PROOF:

⟨1⟩1. ASSUME: for a contradiction  $p : [0, 1] \rightarrow \overline{S}$  is a path from  $(0, 0)$  to  $(1, \sin 1)$ .

⟨1⟩2.  $p^{-1}(\{0\} \times [0, 1])$  is closed.

⟨1⟩3. LET:  $b$  be the greatest element of  $p^{-1}(\{0\} \times [0, 1])$ .

⟨1⟩4.  $b < 1$

PROOF: Since  $p(1) = (1, \sin 1)$ .

⟨1⟩5. PICK a sequence  $(t_n)_{n \geq 1}$  in  $(b, 1]$  such that  $t_n \rightarrow b$  and  $\pi_2(p(t_n)) = (-1)^n$

⟨2⟩1. LET:  $n \geq 1$

⟨2⟩2. PICK  $u$  with  $0 < u < \pi_1(p(1/n))$  such that  $\sin(1/u) = (-1)^n$

⟨2⟩3. PICK  $t_n$  such that  $0 < t_n < 1/n$  and  $\pi_1(p(t_n)) = u$

PROOF: One exists by the Intermediate Value Theorem.

⟨1⟩6. Q.E.D.

PROOF: This contradicts 11.12.17.

□

**Theorem 12.3.31.** *Let  $A$  be a subspace of  $\mathbb{R}^n$ . Then the following are equivalent:*

1.  $A$  is compact.

2.  $A$  is closed and bounded under the Euclidean metric.

3.  $A$  is closed and bounded under the square metric.

PROOF:

⟨1⟩1.  $1 \Rightarrow 2$

PROOF: By Corollary 11.51.10.1 and Proposition 12.2.28.

⟨1⟩2.  $2 \Rightarrow 3$

PROOF: If  $d(x, y) \leq M$  for all  $x, y \in A$  then  $\rho(x, y) \leq M/\sqrt{2}$ .

⟨1⟩3.  $3 \Rightarrow 1$

⟨2⟩1. ASSUME:  $A$  is closed and  $\rho(x, y) \leq M$  for all  $x, y \in A$

⟨2⟩2. PICK  $a \in A$

PROOF: We may assume w.l.o.g.  $A$  is nonempty since the empty space is compact.

⟨2⟩3.  $A$  is a closed subspace of  $[a_1 - M, a_1 + M] \times \cdots \times [a_n - M, a_n + M]$

⟨2⟩4.  $A$  is compact

PROOF: Proposition 11.51.3.

□

**Corollary 12.3.31.1.** *The unit sphere  $S^{n-1}$  and the closed unit ball  $B^n$  are compact for any  $n$ .*

## 12.4 The Uniform Topology

**Definition 12.4.1** (Uniform Metric). Let  $J$  be a set. The *uniform metric*  $\bar{\rho}$  on  $\mathbb{R}^J$  is defined by

$$\bar{\rho}(a, b) = \sup_{j \in J} \bar{d}(a_j, b_j)$$

where  $\bar{d}$  is the standard bounded metric on  $\mathbb{R}$ .

The *uniform topology* on  $\mathbb{R}^J$  is the topology induced by the uniform metric.

We prove this is a metric.

PROOF:

$\langle 1 \rangle 1. \bar{\rho}(a, b) \geq 0$

PROOF: Immediate from definitions.

$\langle 1 \rangle 2. \bar{\rho}(a, b) = 0$  if and only if  $a = b$

PROOF: Immediate from definitions.

$\langle 1 \rangle 3. \bar{\rho}(a, b) = \bar{\rho}(b, a)$

PROOF: Immediate from definitions.

$\langle 1 \rangle 4. \bar{\rho}(a, c) \leq \bar{\rho}(a, b) + \bar{\rho}(b, c)$

PROOF:

$$\begin{aligned} \bar{\rho}(a, c) &= \sup_{j \in J} \bar{d}(a_j, c_j) \\ &\leq \sup_{j \in J} (\bar{d}(a_j, b_j) + \bar{d}(b_j, c_j)) \\ &\leq \sup_{j \in J} \bar{d}(a_j, b_j) + \sup_{j \in J} \bar{d}(b_j, c_j) \\ &= \bar{\rho}(a, b) + \bar{\rho}(b, c) \end{aligned}$$

□

**Proposition 12.4.2.** *The uniform topology on  $\mathbb{R}^J$  is finer than the product topology.*

PROOF:

$\langle 1 \rangle 1.$  LET:  $j \in J$  and  $U$  be open in  $\mathbb{R}$

PROVE:  $\pi_j^{-1}(U)$  is open in the uniform topology.

$\langle 1 \rangle 2.$  LET:  $a \in \pi_j^{-1}(U)$

$\langle 1 \rangle 3.$  PICK  $\epsilon > 0$  such that  $(a_j - \epsilon, a_j + \epsilon) \subseteq U$

$\langle 1 \rangle 4. B_{\bar{\rho}}(a, \epsilon) \subseteq \pi_j^{-1}(U)$

□

**Proposition 12.4.3.** *The uniform topology on  $\mathbb{R}^J$  is coarser than the box topology.*

PROOF:

$\langle 1 \rangle 1.$  LET:  $a \in \mathbb{R}^J$  and  $\epsilon > 0$

PROVE:  $B(a, \epsilon)$  is open in the box topology.

$\langle 1 \rangle 2.$  LET:  $b \in B(a, \epsilon)$

$\langle 1 \rangle 3.$  For  $j \in J$  we have  $|a_j - b_j| < \epsilon$

- (1)4. For  $j \in J$ ,  
       LET:  $\delta_j = (\epsilon - |a_j - b_j|)/2$   
 (1)5.  $\prod_{j \in J} (b_j - \delta_j, b_j + \delta_j) \subseteq B(a, \epsilon)$   
 $\square$

**Proposition 12.4.4.** *The uniform topology on  $\mathbb{R}^J$  is strictly finer than the product topology if and only if  $J$  is infinite.*

PROOF:

- (1)1. If  $J$  is finite then the uniform and product topologies coincide.  
       PROOF: The uniform, box and product topologies are all the same.  
 (1)2. If  $J$  is infinite then the uniform and product topologies are different.  
       PROOF: The set  $B(\vec{0}, 1)$  is open in the uniform topology but not the product topology.  
 $\square$

**Proposition 12.4.5 (DC).** *The uniform topology on  $\mathbb{R}^J$  is strictly coarser than the box topology if and only if  $J$  is infinite.*

PROOF:

- (1)1. If  $J$  is finite then the uniform and box topologies coincide.  
       PROOF: The uniform, box and product topologies are all the same.  
 (1)2. If  $J$  is infinite then the uniform and box topologies are different.  
       PROOF: Pick an  $\omega$ -sequence  $(j_1, j_2, \dots)$  in  $J$ . Let  $U = \prod_{j \in J} U_j$  where  $U_{j_i} = (-1/i, 1/i)$  and  $U_j = (-1, 1)$  for all other  $j$ . Then  $\vec{0} \in U$  but there is no  $\epsilon > 0$  such that  $B(\vec{0}, \epsilon) \subseteq U$ .  
 $\square$

**Proposition 12.4.6.** *The closure of  $\mathbb{R}^\infty$  in  $\mathbb{R}^\omega$  under the uniform topology is  $\mathbb{R}^\omega$ .*

PROOF: Given any open ball  $B(a, \epsilon)$ , pick an integer  $N$  such that  $1/\epsilon < N$ . Then  $B(a, \epsilon)$  includes sequences whose  $n$ th entry is 0 for all  $n \geq N$ .  $\square$

**Example 12.4.7.** The space  $\mathbb{R}^\omega$  is disconnected under the uniform topology. The set of bounded sequences and the set of unbounded sequences form a separation.

**Proposition 12.4.8.** *Give  $\mathbb{R}^\omega$  the uniform topology. Let  $x, y \in \mathbb{R}^\omega$ . Then  $x$  and  $y$  are in the same component if and only if  $x - y$  is bounded.*

PROOF:

- (1)1. The component containing 0 is the set of bounded sequences.  
 (2)1. LET:  $B$  be the set of bounded sequences.  
 (2)2.  $B$  is path-connected.  
       (3)1. LET:  $x, y \in B$   
       (3)2. PICK  $b > 0$  such that  $|x_j|, |y_j| \leq b$  for all  $j$   
       (3)3. LET:  $p : [0, 1] \rightarrow B$  be the function  $p(t) = (1 - t)x + ty$   
           PROVE:  $p$  is continuous.

- ⟨3⟩4. LET:  $t \in [0, 1]$  and  $\epsilon > 0$   
 ⟨3⟩5. LET:  $\delta = \epsilon/2b$   
 ⟨3⟩6. LET:  $s \in [0, 1]$  with  $|s - t| < \delta$   
 ⟨3⟩7.  $\bar{\rho}(p(s), p(t)) < \epsilon$

PROOF:

$$\begin{aligned}
 \bar{\rho}(p(s), p(t)) &= \sup_j \bar{d}((1-s)x_j + sy_j, (1-t)x_j + ty_j) \\
 &\leq |(s-t)x_j + (t-s)y_j| \\
 &\leq |s-t||x_j - y_j| \\
 &< 2b\delta \\
 &= \epsilon
 \end{aligned}$$

- ⟨2⟩3.  $B$  is connected.

PROOF: Proposition 11.33.3.

- ⟨2⟩4. If  $C$  is connected and  $B \subseteq C$  then  $B = C$ .

PROOF: Otherwise  $B \cap C$  and  $C \setminus B$  form a separation of  $C$ .

- ⟨1⟩2. Q.E.D.

PROOF: Since  $\lambda x.x - y$  is a Homeomorphism of  $\mathbb{R}^\omega$  with itself.

□

**Example 12.4.9.** The space  $[0, 1]^\omega$  under the uniform topology is not locally compact.

It is not compact because the set  $\{0, 1\}^\omega$  has no limit point.

Now, assume for a contradiction  $[0, 1]^\omega$  is locally compact. Pick  $\epsilon > 0$  such that  $B(0, \epsilon)$  is included in a compact subspace. Then  $\overline{B(0, \epsilon)}$  is compact. But  $\overline{B(0, \epsilon)} = [0, 1]^\omega$  if  $\epsilon \geq 1$ , or  $[0, \epsilon]^\omega$  if  $\epsilon < 1$ . In either case  $\overline{B(0, \epsilon)} \cong [0, 1]^\epsilon$  which is not compact.

**Example 12.4.10.** The space  $\mathbb{R}^\omega$  under the uniform topology is not second countable.

PROOF: The set  $\{0, 1\}^\omega$  is an uncountable discrete subspace. □

**Proposition 12.4.11.** *Every separable metrizable space is second countable.*

PROOF:

- ⟨1⟩1. LET:  $X$  be a separable metrizable space.

- ⟨1⟩2. PICK a countable dense subset  $D$ .

PROVE:  $\{B(x, 1/n) \mid x \in D, n \in \mathbb{Z}^+\}$  is a basis for  $X$

- ⟨1⟩3. LET:  $x \in X$

- ⟨1⟩4. LET:  $U$  be a neighbourhood of  $x$ .

- ⟨1⟩5. PICK  $n$  such that  $B(x, 1/n) \subseteq U$ .

- ⟨1⟩6. PICK  $d \in D \cap B(x, 1/2n)$

- ⟨1⟩7.  $x \in B(d, 1/2n) \subseteq U$ .

□

**Proposition 12.4.12 (AC).** *Every Lindelöf metrizable space is second countable.*

PROOF:

- ⟨1⟩1. LET:  $X$  be a Lindelöf metrizable space.
- ⟨1⟩2. For  $n \in \mathbb{Z}^+$ , PICK a countable set  $\mathcal{A}_n$  of open balls of radius  $1/n$  that covers  $X$ .
- ⟨1⟩3. LET:  $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{A}_n$   
PROVE:  $\mathcal{B}$  is a basis for  $X$ .
- ⟨1⟩4. LET:  $x \in X$
- ⟨1⟩5. LET:  $U$  be a neighbourhood of  $x$ .
- ⟨1⟩6. PICK  $n$  such that  $B(x, 1/n) \subseteq U$
- ⟨1⟩7. PICK  $B \in \mathcal{A}_{2n}$  such that  $x \in B$
- ⟨1⟩8.  $B \subseteq U$

PROOF: Since  $\text{diam } B \leq n$ .

□

**Example 12.4.13.** The space  $\mathbb{R}_l$  is not metrizable, because it is Lindelöf but not second countable.

**Example 12.4.14.** The ordered square is not metrizable, because it is compact but not separable.

**Proposition 12.4.15.** For any set  $J$ , the space  $\mathbb{R}^J$  under the box topology is completely regular.

PROOF:

- ⟨1⟩1. LET:  $J$  be any set.
- ⟨1⟩2. For any closed set  $A \subseteq \mathbb{R}^J$  disjoint from  $(-1, 1)^J$ ,  $A$  and  $\{0\}$  can be separated by a continuous function.
  - ⟨2⟩1. LET:  $A$  be a closed set disjoint from  $(-1, 1)^J$
  - ⟨2⟩2. PICK a function  $f : \mathbb{R}^J \rightarrow [0, 1]$  continuous with respect to the uniform topology that separates  $\{0\}$  from  $A$ .
  - ⟨2⟩3.  $f$  is continuous with respect to the box topology.
- ⟨1⟩3. LET:  $A$  be any closed set.
- ⟨1⟩4. LET:  $a$  be any point not in  $A$ .
- ⟨1⟩5. PICK a basic open neighbourhood  $U$  of  $a$  disjoint from  $A$ .
- ⟨1⟩6. PICK a homomorphism  $\phi : \mathbb{R}^J \cong \mathbb{R}^J$  that maps  $a$  to 0 and  $U$  to  $(-1, 1)^J$
- ⟨1⟩7. PICK a continuous function  $f : \mathbb{R}^J \rightarrow [0, 1]$  such that  $f(0) = 0$  and  $f(\phi(A)) = \{1\}$
- ⟨1⟩8. LET:  $g = f \circ \phi$
- ⟨1⟩9.  $g(a) = 0$  and  $g(A) = \{1\}$

## 12.5 Uniform Convergence

**Definition 12.5.1** (Uniform Convergence). Let  $X$  be a set and  $Y$  a metric space. Let  $(f_n : X \rightarrow Y)$  be a sequence of functions and  $f : X \rightarrow Y$  be a function. Then  $f_n$  converges uniformly to  $f$  as  $n \rightarrow \infty$  if and only if, for all  $\epsilon > 0$ , there exists  $N$  such that, for all  $n \geq N$  and  $x \in X$ , we have  $d(f_n(x), f(x)) < \epsilon$ .



**Example 12.5.2.** Define  $f_n : [0, 1] \rightarrow \mathbb{R}$  by  $f_n(x) = x^n$  for  $n \geq 1$ , and  $f : [0, 1] \rightarrow \mathbb{R}$  by  $f(x) = 0$  if  $x < 1$ ,  $f(1) = 1$ . Then  $f_n$  converges to  $f$  pointwise but not uniformly.

**Theorem 12.5.3** (Uniform Limit Theorem). *Let  $X$  be a topological space and  $Y$  a metric space. Let  $(f_n : X \rightarrow Y)$  be a sequence of continuous functions and  $f : X \rightarrow Y$  be a function. If  $f_n$  converges uniformly to  $f$  as  $n \rightarrow \infty$ , then  $f$  is continuous.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $x \in X$  and  $\epsilon > 0$

$\langle 1 \rangle 2$ . PICK  $N$  such that, for all  $n \geq N$  and  $y \in X$ , we have  $d(f_n(y), f(y)) < \epsilon/3$

$\langle 1 \rangle 3$ . PICK a neighbourhood  $U$  of  $x$  such that  $f_N(U) \subseteq B(f_N(x), \epsilon/3)$

PROVE:  $f(U) \subseteq B(f(x), \epsilon)$

$\langle 1 \rangle 4$ . LET:  $y \in U$

$\langle 1 \rangle 5$ .  $d(f(y), f(x)) < \epsilon$

PROOF:

$$\begin{aligned} d(f(y), f(x)) &\leq d(f(y), f_N(y)) + d(f_N(y), f_N(x)) + d(f_N(x), f(x)) \quad (\text{Triangle Inequality}) \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 \quad (\langle 1 \rangle 2, \langle 1 \rangle 3) \\ &= \epsilon \end{aligned}$$

□

**Proposition 12.5.4.** *Let  $X$  be a topological space and  $Y$  a metric space. Let  $(f_n : X \rightarrow Y)$  be a sequence of continuous functions and  $f : X \rightarrow Y$  be a function. Let  $(a_n)$  be a sequence of points in  $X$  and  $a \in X$ . If  $f_n$  converges uniformly to  $f$  and  $a_n$  converges to  $a$  in  $X$  then  $f_n(a_n)$  converges to  $f(a)$  uniformly in  $Y$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $\epsilon > 0$

$\langle 1 \rangle 2$ . PICK  $N_1$  such that, for all  $n \geq N_1$  and  $x \in X$ , we have  $d(f_n(x), f(x)) < \epsilon/2$

$\langle 1 \rangle 3$ . PICK  $N_2$  such that, for all  $n \geq N_2$ , we have  $a_n \in f^{-1}(B(a, \epsilon/2))$

PROOF: Using the fact that  $f$  is continuous from the Uniform Limit Theorem.

$\langle 1 \rangle 4$ . LET:  $N = \max(N_1, N_2)$

$\langle 1 \rangle 5$ . LET:  $n \geq N$

$\langle 1 \rangle 6$ .  $d(f_n(a_n), f(a)) < \epsilon$

PROOF:

$$\begin{aligned} d(f_n(a_n), f(a)) &\leq d(f_n(a_n), f(a_n)) + d(f(a_n), f(a)) \quad (\text{Triangle Inequality}) \\ &< \epsilon/2 + \epsilon/2 \quad (\langle 1 \rangle 2, \langle 1 \rangle 3) \\ &= \epsilon \end{aligned}$$

□

**Proposition 12.5.5.** *Let  $X$  be a set. Let  $(f_n : X \rightarrow \mathbb{R})$  be a sequence of functions and  $f : X \rightarrow \mathbb{R}$  be a function. Then  $f_n$  converges uniformly to  $f$  as  $n \rightarrow \infty$  if and only if  $f_n \rightarrow f$  as  $n \rightarrow \infty$  in  $\mathbb{R}^X$  under the uniform topology.*

PROOF:

- ⟨1⟩1. If  $f_n$  converges uniformly to  $f$  then  $f_n$  converges to  $f$  under the uniform topology.
- ⟨2⟩1. ASSUME:  $f_n$  converges uniformly to  $f$
- ⟨2⟩2. LET:  $\epsilon > 0$
- ⟨2⟩3. PICK  $N$  such that, for all  $n \geq N$  and  $x \in X$ , we have  $d(f_n(x), f(x)) < \epsilon/2$
- ⟨2⟩4. For all  $n \geq N$  we have  $\bar{\rho}(f_n, f) \leq \epsilon/2$
- ⟨2⟩5. For all  $n \geq N$  we have  $\bar{\rho}(f_n, f) < \epsilon$
- ⟨1⟩2. If  $f_n$  converges to  $f$  under the uniform topology then  $f_n$  converges uniformly to  $f$ .
- ⟨2⟩1. ASSUME:  $f_n$  converges to  $f$  under the uniform topology.
- ⟨2⟩2. LET:  $\epsilon > 0$
- ⟨2⟩3. PICK  $N$  such that, for all  $n \geq N$ , we have  $\bar{\rho}(f_n, f) < \min(\epsilon, 1/2)$
- ⟨2⟩4. LET:  $n \geq N$
- ⟨2⟩5. LET:  $x \in X$
- ⟨2⟩6.  $\bar{\rho}(f_n, f) < \min(\epsilon, 1/2)$
- PROOF: From ⟨2⟩3.
- ⟨2⟩7.  $d(f_n(x), f(x)) < \min(\epsilon, 1/2)$
- ⟨2⟩8.  $d(f_n(x), f(x)) < \epsilon$

□

## 12.6 Isometric Imbeddings

**Definition 12.6.1.** Let  $X$  and  $Y$  be metric spaces. An *isometric imbedding*  $f : X \rightarrow Y$  is a function such that, for all  $x, y \in X$ , we have  $d(f(x), f(y)) = d(x, y)$ .

**Proposition 12.6.2.** Every isometric imbedding is an imbedding.

PROOF:

- ⟨1⟩1. LET:  $f : X \rightarrow Y$  be an isometric imbedding.
- ⟨1⟩2.  $f$  is injective.  
PROOF: If  $f(x) = f(y)$  then  $d(f(x), f(y)) = 0$  hence  $d(x, y) = 0$  hence  $x = y$ .
- ⟨1⟩3.  $f$  is continuous.  
PROOF: For all  $\epsilon > 0$ , if  $d(x, y) < \epsilon$  then  $d(f(x), f(y)) < \epsilon$ .
- ⟨1⟩4.  $f : X \rightarrow f(X)$  is an open map.  
PROOF:  $f(B(a, \epsilon)) = B(f(a), \epsilon) \cap f(X)$ .

□

## 12.7 Distance to a Set

**Definition 12.7.1.** Let  $X$  be a metric space,  $x \in X$  and  $A \subseteq X$  be nonempty. The *distance* from  $x$  to  $A$  is defined as

$$d(x, A) = \inf_{a \in A} d(x, a) .$$

**Proposition 12.7.2.** *Let  $X$  be a metric space and  $A \subseteq X$  be nonempty. Then the function  $d(-, A) : X \rightarrow \mathbb{R}$  is continuous.*

PROOF:

$\langle 1 \rangle 1.$  LET:  $X$  be a metric space.

$\langle 1 \rangle 2.$  LET:  $A \subseteq X$  be nonempty.

$\langle 1 \rangle 3.$  LET:  $x \in X$  and  $\epsilon > 0$

$\langle 1 \rangle 4.$  LET:  $\delta = \epsilon$

$\langle 1 \rangle 5.$  LET:  $y \in B(x, \delta)$

$\langle 1 \rangle 6.$   $|d(x, A) - d(y, A)| < \epsilon$

$\langle 2 \rangle 1.$   $d(x, A) - d(y, A) < \epsilon$

PROOF:

$\langle 3 \rangle 1.$  For all  $a \in A$  we have  $d(x, A) \leq d(x, y) + d(y, a)$

PROOF:

$$d(x, A) \leq d(x, a) \quad (\text{definition of } d(x, A))$$

$$\leq d(x, y) + d(y, a) \quad (\text{Triangle Inequality})$$

$\langle 3 \rangle 2.$   $d(x, A) - d(x, y) \leq d(y, A)$

$\langle 2 \rangle 2.$   $d(y, A) - d(x, A) < \epsilon$

PROOF: Similar.

$\langle 1 \rangle 7.$  Q.E.D.

PROOF: Theorem 12.2.22.

□

**Theorem 12.7.3.** *Let  $X$  be a metric space,  $A \subseteq X$  be nonempty, and  $x \in X$ . Then  $d(x, A) = 0$  if and only if  $x \in \overline{A}$ .*

PROOF:

$\langle 1 \rangle 1.$  LET:  $X$  be a metric space.

$\langle 1 \rangle 2.$  LET:  $A \subseteq X$  be nonempty.

$\langle 1 \rangle 3.$  LET:  $x \in X$

$\langle 1 \rangle 4.$  If  $d(x, A) = 0$  then  $x \in \overline{A}$

$\langle 2 \rangle 1.$  ASSUME:  $d(x, A) = 0$

$\langle 2 \rangle 2.$  LET:  $U$  be any neighbourhood of  $x$ .

$\langle 2 \rangle 3.$  PICK  $\epsilon > 0$  such that  $B(x, \epsilon) \subseteq U$

PROOF: Proposition 12.2.4,  $\langle 1 \rangle 1$ ,  $\langle 2 \rangle 2$ .

$\langle 2 \rangle 4.$  PICK  $a \in A$  such that  $d(x, a) < \epsilon$

PROOF: From  $\langle 2 \rangle 1$ ,  $\langle 2 \rangle 3$ .

$\langle 2 \rangle 5.$   $a \in A \cap U$

PROOF: From  $\langle 2 \rangle 3$ ,  $\langle 2 \rangle 4$ .

$\langle 2 \rangle 6.$  Q.E.D.

PROOF: Theorem 11.4.6.

$\langle 1 \rangle 5.$  If  $x \in \overline{A}$  then  $d(x, A) = 0$

□

**Theorem 12.7.4.** *Let  $X$  be a metric space. Let  $A \subseteq X$  be nonempty and compact. Let  $x \in X$ . Then there exists  $a \in A$  such that  $d(x, A) = d(x, a)$ .*

PROOF: By the Extreme Value Theorem, the function  $d(x, -) : A \rightarrow \mathbb{R}$  attains its minimum. □

## 12.8 Lebesgue Numbers

**Definition 12.8.1** (Lebesgue Number). Let  $X$  be a metric space. Let  $\mathcal{U}$  be an open covering of  $X$ . A *Lebesgue number* for  $\mathcal{U}$  is a real number  $\delta > 0$  such that, for every subset  $A \subseteq X$  with  $\text{diam } A < \delta$ , there exists  $U \in \mathcal{U}$  such that  $A \subseteq U$ .

**Theorem 12.8.2** (Lebesgue Number Lemma). *Every open covering of a compact metric space has a Lebesgue number.*

PROOF:

- ⟨1⟩1. LET:  $X$  be a compact metric space.
- ⟨1⟩2. LET:  $\mathcal{U}$  be an open covering of  $X$ .
- ⟨1⟩3. PICK a finite subset  $\{U_1, \dots, U_n\}$  of  $\mathcal{U}$  that covers  $X$ .
- ⟨1⟩4. For  $i = 1, \dots, n$ ,  
LET:  $C_i = X - U_i$
- ⟨1⟩5. LET:  $f : X \rightarrow \mathbb{R}$ ,

$$f(x) = 1/n \sum_{i=1}^n d(x, C_i)$$

- ⟨1⟩6. For all  $x \in X$  we have  $f(x) > 0$ 
  - ⟨2⟩1. LET:  $x \in X$
  - ⟨2⟩2. PICK  $i$  such that  $x \in U_i$   
PROOF: From ⟨1⟩3.
  - ⟨2⟩3. PICK  $\epsilon > 0$  such that  $B(x, \epsilon) \subseteq U_i$   
PROOF: Proposition 12.2.4.
  - ⟨2⟩4.  $d(x, C_i) \geq \epsilon$
  - ⟨2⟩5.  $f(x) \geq \epsilon/n$
- ⟨1⟩7.  $f$  is continuous.  
PROOF: Proposition 12.7.2.
- ⟨1⟩8. LET:  $\delta$  be the minimum value of  $f(X)$   
PROOF: By the Extreme Value Theorem
- ⟨1⟩9.  $\delta > 0$   
PROOF: From ⟨1⟩6
- ⟨1⟩10. For every subset  $A \subseteq X$  with  $\text{diam } A < \delta$ , there exists  $U \in \mathcal{U}$  such that  $A \subseteq U$ 
  - ⟨2⟩1. LET:  $A \subseteq X$  with  $\text{diam } A < \delta$
  - ⟨2⟩2. PICK  $x_0 \in A$
  - ⟨2⟩3.  $A \subseteq B(x_0, \delta)$
  - ⟨2⟩4.  $f(x_0) \geq \delta$
  - ⟨2⟩5. PICK  $m$  such that  $d(x_0, C_m)$  is the largest out of  $d(x_0, C_1), \dots, d(x_0, C_n)$
  - ⟨2⟩6.  $d(x_0, C_m) \geq f(x_0)$
  - ⟨2⟩7.  $B(x_0, \delta) \subseteq U_m$
  - ⟨2⟩8.  $A \subseteq U_m$
- ⟨1⟩11.  $\delta$  is a Lebesgue number for  $\mathcal{U}$

□

**Theorem 12.8.3** (AC). *Every sequentially compact metric space is compact.*

PROOF:

- ⟨1⟩1. LET:  $X$  be a sequentially compact metric space.
- ⟨1⟩2. Every open covering of  $X$  has a Lebesgue number.
- ⟨2⟩1. LET:  $\mathcal{A}$  be an open covering of  $X$ .
- ⟨2⟩2. ASSUME: for a contradiction  $\mathcal{A}$  has no Lebesgue number.
- ⟨2⟩3. For  $n \geq 1$ , PICK a set  $C_n$  with diameter  $< 1/n$  that is not included in any member of  $\mathcal{A}$ .
- ⟨2⟩4. For  $n \geq 1$ , PICK  $x_n \in C_n$ .
- ⟨2⟩5. PICK a convergent subsequence  $(C_{n_r})$  of  $(C_n)$  with limit  $a$ .
- ⟨2⟩6. PICK  $A \in \mathcal{A}$  such that  $a \in A$ .
- ⟨2⟩7. PICK  $\epsilon > 0$  such that  $B(a, \epsilon) \subseteq A$ .
- ⟨2⟩8. PICK  $r$  such that  $1/n_r < \epsilon/2$  and  $d(x_{n_r}, a) < \epsilon/2$ .
- ⟨2⟩9.  $C_{n_r} \subseteq B(a, \epsilon)$
- ⟨2⟩10.  $C_{n_r} \subseteq A$
- ⟨2⟩11. Q.E.D.

PROOF: This contradicts ⟨2⟩3.

- ⟨1⟩3. For every  $\epsilon > 0$ , there exists a finite covering of  $X$  by  $\epsilon$ -balls.
- ⟨2⟩1. ASSUME: for a contradiction that there exists  $\epsilon > 0$  such that  $X$  cannot be finitely covered by  $\epsilon$ -balls.
- ⟨2⟩2. PICK a sequence of points  $(x_n)$  such that  $x_n \in X - (B(x_1, \epsilon) \cup \dots \cup B(x_{n-1}, \epsilon))$
- ⟨2⟩3.  $d(x_m, x_n) \geq \epsilon$  for all  $m, n$  distinct
- ⟨2⟩4.  $(x_n)$  has no convergent subsequence
- ⟨2⟩5. Q.E.D.

PROOF: This contradicts ⟨1⟩1.

- ⟨1⟩4. LET:  $\mathcal{A}$  be an open covering of  $X$ .
- ⟨1⟩5. PICK a Lebesgue number  $\delta$  for  $\mathcal{A}$ .
- PROOF: By ⟨1⟩2.
- ⟨1⟩6. LET:  $\epsilon = \delta/3$
- ⟨1⟩7. PICK a finite covering  $\{B_1, \dots, B_n\}$  of  $X$  by  $\epsilon$ -balls.
- PROOF: By ⟨1⟩3.
- ⟨1⟩8. For  $i = 1, \dots, n$ , PICK  $U_i \in \mathcal{A}$  such that  $B_i \subseteq U_i$
- PROOF: By ⟨1⟩5 since  $\text{diam } B_i = 2\epsilon < \delta$ .
- ⟨1⟩9.  $\{U_1, \dots, U_n\}$  covers  $X$ .

□

**Example 12.8.4.** The space  $S_\Omega$  is not metrizable, because it is sequentially compact but not compact.

## 12.9 Uniform Continuity

**Definition 12.9.1** (Uniformly Continuous). Let  $X$  and  $Y$  be metric spaces. Let  $f : X \rightarrow Y$ . Then  $f$  is *uniformly continuous* if and only if, for every  $\epsilon > 0$ , there exists  $\delta > 0$  such that, for all  $x, y \in X$ , if  $d(x, y) < \delta$  then  $d(f(x), f(y)) < \epsilon$ .

**Theorem 12.9.2** (Uniform Continuity Theorem). *Every continuous function*

from a compact metric space to a metric space is uniformly continuous.

PROOF:

- <1>1. LET:  $X$  be a compact metric space.
- <1>2. LET:  $Y$  be a metric space.
- <1>3. LET:  $f : X \rightarrow Y$  be a continuous function.
- <1>4. LET:  $\epsilon > 0$
- <1>5. LET:  $\mathcal{U} = \{f^{-1}(B(y, \epsilon/2)) \mid y \in Y\}$
- <1>6. PICK a Lebesgue number  $\delta > 0$  for  $\mathcal{U}$ .

PROOF: By the Lebesgue Number Lemma.

- <1>7. LET:  $x, x' \in X$
- <1>8. ASSUME:  $d(x, x') < \delta$
- <1>9. PICK  $y \in Y$  such that  $\{x, x'\} \subseteq f^{-1}(B(y, \epsilon/2))$

PROOF: Since  $\text{diam}\{x, x'\} < \delta$ .

- <1>10.  $d(f(x), f(x')) < \epsilon$

PROOF:

$$\begin{aligned} d(f(x), f(x')) &\leq d(f(x), y) + d(y, f(x')) && \text{(Triangle Inequality)} \\ &< \epsilon/2 + \epsilon/2 && \text{(<1>9)} \\ &= \epsilon \end{aligned}$$

□

**Example 12.9.3.** The space  $\mathcal{C}(I, \mathbb{R})$  of continuous functions  $I \rightarrow \mathbb{R}$ , as a subspace of  $\mathbb{R}^I$  under the uniform topology, is second countable.

PROOF:

- <1>1. LET:  $D$  be the set of continuous functions whose graphs consist of finitely many line segments with rational end points.

PROVE:  $D$  is dense.

- <1>2. LET:  $f \in \mathcal{C}(I, \mathbb{R})$  and  $\epsilon > 0$

PROVE:  $B(f, \epsilon)$  intersects  $D$ .

- <1>3.  $f$  is uniformly continuous.

PROOF: Uniform Continuity Theorem.

- <1>4. PICK  $\delta > 0$  such that, for all  $x, y \in I$ , if  $|x - y| < \delta$  then  $|f(x) - f(y)| < \epsilon/4$

- <1>5. PICK  $N \in \mathbb{Z}^+$  such that  $1/N < \delta$

- <1>6. For  $0 \leq k \leq N$ , PICK a rational number  $q_k$  such that  $|q_k - f(k/N)| < \epsilon/6$

- <1>7. LET:  $q \in D$  be the function whose graph consists of the line segments with endpoints  $(0, q_0), (1/N, q_1), \dots, (1, q_N)$

PROVE:  $q \in B(f, \epsilon)$

- <1>8. For  $0 \leq k < N$ , we have  $|q_{k+1} - q_k| < 7\epsilon/12$

PROOF:

$$\begin{aligned} |q_{k+1} - q_k| &\leq |q_{k+1} - f((k+1)/N)| + |f((k+1)/N) - f(k/N)| + |f(k/N) - q_k| \\ &< \epsilon/6 + \epsilon/4 + \epsilon/6 \\ &= 7\epsilon/12 \end{aligned}$$

- <1>9. LET:  $x \in I$

- <1>10. PICK  $k$  such that  $k/N \leq x \leq (k+1)/N$

- <1>11.  $|f(x) - q(x)| < \epsilon$

PROOF:

$$\begin{aligned}
|f(x) - q(x)| &\leq |f(x) - f(k/N)| + |f(k/N) - q_k| + |q_k - q(x)| \\
&\leq |f(x) - f(k/N)| + |f(k/N) - q_k| + |q_k - q_{k+1}| \\
&< \epsilon/4 + \epsilon/6 + 7\epsilon/12 \\
&= \epsilon
\end{aligned}$$

□

## 12.10 Epsilon-neighbourhoods

**Definition 12.10.1** ( $\epsilon$ -neighbourhood). Let  $X$  be a metric space. Let  $A \subseteq X$  be nonempty. Let  $\epsilon > 0$ . Then the  $\epsilon$ -neighbourhood of  $A$ ,  $U(A, \epsilon)$ , is the set

$$U(A, \epsilon) = \{x \in X \mid d(x, A) < \epsilon\}.$$

**Proposition 12.10.2.** Let  $X$  be a metric space. Let  $A \subseteq X$  be nonempty. Let  $\epsilon > 0$ . Then  $U(A, \epsilon) = \bigcup_{a \in A} B(a, \epsilon)$ .

PROOF:

- ⟨1⟩1. LET:  $X$  be a metric space.
- ⟨1⟩2. LET:  $A \subseteq X$  be nonempty.
- ⟨1⟩3. LET:  $\epsilon > 0$
- ⟨1⟩4.  $U(A, \epsilon) \subseteq \bigcup_{a \in A} B(a, \epsilon)$ 
  - ⟨2⟩1. LET:  $x \in U(A, \epsilon)$
  - ⟨2⟩2.  $d(x, A) < \epsilon$
  - ⟨2⟩3.  $\epsilon$  is not a lower bound for  $\{d(x, a) \mid a \in A\}$
  - ⟨2⟩4. PICK  $a \in A$  such that  $d(x, a) < \epsilon$
  - ⟨2⟩5.  $x \in B(a, \epsilon)$
- ⟨1⟩5.  $\bigcup_{a \in A} B(a, \epsilon) \subseteq U(A, \epsilon)$ 
  - ⟨2⟩1. LET:  $a \in A$  and  $x \in B(a, \epsilon)$
  - ⟨2⟩2.  $d(x, A) \leq d(x, a)$
  - ⟨2⟩3.  $d(x, A) < \epsilon$
  - ⟨2⟩4.  $x \in U(A, \epsilon)$

□

**Proposition 12.10.3.** Let  $X$  be a metric space. Let  $A \subseteq X$  be nonempty and compact. Let  $U$  be an open set such that  $A \subseteq U$ . Then there exists  $\epsilon > 0$  such that  $U(A, \epsilon) \subseteq U$ .

PROOF:

- ⟨1⟩1. LET:  $X$  be a metric space.
- ⟨1⟩2. LET:  $A \subseteq X$  be nonempty and compact.
- ⟨1⟩3. LET:  $U$  be an open set such that  $A \subseteq U$
- ⟨1⟩4.  $\{B(a, \epsilon) \mid a \in A, \epsilon > 0, B(a, 2\epsilon) \subseteq U\}$  covers  $A$ .

PROOF: By Proposition 12.2.4.

- ⟨1⟩5. PICK a finite subcover  $\{B(a_1, \epsilon_1), \dots, B(a_n, \epsilon_n)\}$
- PROOF: Since  $A$  is compact (⟨1⟩2).

⟨1⟩6. LET:  $\epsilon = \min(\epsilon_1, \dots, \epsilon_n)$   
 PROVE:  $U(A, \epsilon) \subseteq U$   
 ⟨1⟩7. LET:  $x \in U(A, \epsilon)$   
 ⟨1⟩8. PICK  $a \in A$  such that  $d(x, a) < \epsilon$   
 PROOF: Proposition 12.10.2.  
 ⟨1⟩9. PICK  $i$  such that  $a \in B(a_i, \epsilon_i)$   
 PROOF: By ⟨1⟩5.  
 ⟨1⟩10.  $d(x, a_i) < 2\epsilon$   
 PROOF: By the Triangle Inequality.  
 ⟨1⟩11.  $x \in U$   
 PROOF: From ⟨1⟩4.

□

This example shows that we cannot weaken the hypothesis that  $A$  is compact to  $A$  being closed:

**Example 12.10.4.** Let  $X = \mathbb{R}^2$ . Let  $A = \{(x, 1/x) \mid x > 0\}$ . Let  $U = \{(x, y) \mid x > 0, y > 0\}$ . Then  $A$  is nonempty and closed (Proposition 11.51.14). The set  $U$  is open and  $A \subseteq U$ . But there is no  $\epsilon > 0$  such that  $U(A, \epsilon) \subseteq U$ .

PROOF:

⟨1⟩1. LET:  $\epsilon > 0$   
 ⟨1⟩2.  $(2/\epsilon, \epsilon/2) \in A$   
 ⟨1⟩3.  $(2/\epsilon, 0) \in U(A, \epsilon)$   
 ⟨1⟩4.  $(2/\epsilon, 0) \notin U$

□

**Proposition 12.10.5.** *Every metrizable space is perfectly normal.*

PROOF:

⟨1⟩1. LET:  $X$  be a metric space.  
 ⟨1⟩2.  $X$  is normal.

PROOF: Proposition 12.2.20.

⟨1⟩3. Every closed set is  $G_\delta$ .

PROOF: For any closed set  $A$  we have  $A = \bigcap_{n \in \mathbb{Z}^+} B(A, 1/n)$ .

□

## 12.11 Isometry

**Definition 12.11.1** (Isometry). Let  $X$  be a metric space. An *isometry* of  $X$  is a function  $f : X \rightarrow X$  such that, for all  $x, y \in X$ , we have  $d(x, y) = d(f(x), f(y))$ .

**Proposition 12.11.2.** *An isometry on a compact metric space is a homeomorphism.*

PROOF:

⟨1⟩1. LET:  $X$  be a compact metric space.  
 ⟨1⟩2. LET:  $f : X \rightarrow X$  be an isometry.



- ⟨1⟩3.  $f$  is an imbedding  
 PROOF: Proposition 12.6.2.  
 ⟨1⟩4.  $f$  is surjective.  
 ⟨2⟩1. ASSUME: for a contradiction  $a \notin f(X)$   
 ⟨2⟩2.  $f(X)$  is closed  
 PROOF: Proposition 11.51.12.  
 ⟨2⟩3. PICK  $\epsilon > 0$  such that  $B(a, \epsilon) \cap f(X) = \emptyset$   
 ⟨2⟩4. For  $m, n \in \mathbb{N}$  with  $m \neq n$ , we have  $d(f^m(a), f^n(a)) \geq \epsilon$   
 ⟨3⟩1. ASSUME: without loss of generality  $m < n$   
 ⟨3⟩2.  $d(a, f^{n-m}(a)) \geq \epsilon$   
 PROOF: ⟨2⟩3  
 ⟨3⟩3.  $d(f^m(a), f^n(a)) \geq \epsilon$   
 PROOF: ⟨1⟩2  
 ⟨2⟩5. The sequence  $(f^n(a))$  has a convergent subsequence.  
 PROOF: Corollary 11.47.2.1, ⟨1⟩1, Corollary 12.2.19.1.  
 ⟨2⟩6. Q.E.D.  
 PROOF: ⟨2⟩4 and ⟨2⟩5 form a contradiction.

□  
□

## 12.12 Shrinking Maps

**Definition 12.12.1** (Shrinking Map). Let  $X$  be a metric space. Let  $f : X \rightarrow X$ . Then  $f$  is a *shrinking map* if and only if, for all  $x, y \in X$  with  $x \neq y$ , we have  $d(f(x), f(y)) < d(x, y)$ .

**Proposition 12.12.2.** Let  $X$  be a compact metric space. Let  $f : X \rightarrow X$  be a contraction. Then  $f$  has a unique fixed point.

PROOF:

- ⟨1⟩1. LET:  $A_n = f^n(X)$  for  $n \geq 1$   
 ⟨1⟩2. For all  $n \geq 1$  we have  $A_n$  is closed.  
 PROOF: Proposition 11.51.12.  
 ⟨1⟩3. LET:  $A = \bigcap_{n=1}^{\infty} A_n$   
 ⟨1⟩4. PICK  $a \in A$   
 PROOF: Proposition 11.48.6.  
 ⟨1⟩5.  $f(A) = A$   
 ⟨2⟩1.  $f(A) \subseteq A$   
 ⟨2⟩2.  $A \subseteq f(A)$   
 ⟨3⟩1. LET:  $x \in A$   
 ⟨3⟩2. For  $n \geq 1$ , PICK  $x_n$  such that  $x = f^n(x_n)$   
 ⟨3⟩3. PICK a convergent subsequence  $(f^{n_r-1}(x_{n_r}))$  of  $(f^{n-1}(x_n))$  with limit  $l$   
 PROOF: Corollary 11.47.2.1.  
 ⟨3⟩4.  $f(l) = x$   
 PROOF: Both are the limit of  $f(f^{n_r-1}(a_{n_r})) = f^{n_r}(a_{n_r})$ .

$\langle 3 \rangle 5. \quad l \in A$

$\langle 4 \rangle 1$ . ASSUME: for a contradiction  $l \notin A$

$\langle 4 \rangle 2$ . PICK  $N$  such that  $l \notin A_N$

⟨4⟩3. PICK  $R$  such that  $n_R > N$

$\langle 4 \rangle 4$ . For  $r \geq R$  we have  $f^{n_r-1}(a_{n_r}) \in A_N$

⟨4⟩5. Q.E.D.

PROOF: This is a contradiction.

⟨1⟩6.  $\text{diam } A = A$

$\langle 2 \rangle 1$ . PICK  $x, y \in A$  such that  $d(x, y) = \text{diam } A$

PROOF: By the Extreme Value Theorem.

⟨2⟩2. PICK  $x', y' \in A$  such that  $x = f(x')$  and  $y = f(y')$

PROOF: By  $\langle 1 \rangle 5$ .

$\langle 2 \rangle 3. \ x' = y'$

PROOF: If  $x' \neq y'$  then  $\text{diam } A = d(x, y) < d(x', y')$  which is a contradiction.

$\langle 2 \rangle 4. \ x = y$

$\langle 1 \rangle 7. f(a) = a$

PROOF: Since  $a, f(a) \in A$

$\langle 1 \rangle 8$ . If  $f(b) = b$  then  $b = a$

PROOF: If  $f(b) = b$  then  $b \in A$ .



The following example shows that we cannot weaken the hypothesis from ' $X$  is a compact metric space' to ' $X$  is a complete metric space'.

**Example 12.12.3.** The function  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f(x) = [x + (x^2 + 1)^{1/2}] / 2$  is a shrinking map with no fixed point.

## 12.13 Contractions

**Definition 12.13.1** (Contraction). Let  $X$  be a metric space. Let  $f : X \rightarrow X$ . Then  $f$  is a *contraction* if and only if there exists  $\alpha < 1$  such that, for all  $x, y \in X$ , we have  $d(f(x), f(y)) \leq \alpha d(x, y)$ .