

Topology

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Part I

Set Theory

Chapter 1

Classes

1.1 Classes

We speak informally of *classes*. A class is specified by a unary predicate. We write $\{x \mid P(x)\}$ for the class determined by the predicate $P(x)$.

Definition 1.1 (Membership). Let a be an object and \mathbf{A} a class. We define the proposition $a \in \mathbf{A}$ (a is a *member* or *element* of A) as follows:

The proposition $a \in \{x \mid P(x)\}$ is the proposition $P(a)$.

Definition 1.2 (Equality of Classes). Let \mathbf{A} and \mathbf{B} be classes. We say \mathbf{A} and \mathbf{B} are *equal*, $\mathbf{A} = \mathbf{B}$, if and only if they have exactly the same elements.

1.2 Subclasses

Definition 1.3 (Subclass). Let \mathbf{A} and \mathbf{B} be classes. We say \mathbf{A} is a *subclass* of \mathbf{B} , $\mathbf{A} \subseteq \mathbf{B}$, if and only if every member of \mathbf{A} is a member of \mathbf{B} .

We say \mathbf{A} is a *proper* subclass of \mathbf{B} , $\mathbf{A} \subset \mathbf{B}$, if and only if $\mathbf{A} \subseteq \mathbf{B}$ and $\mathbf{A} \neq \mathbf{B}$.

1.3 The Empty Class

Definition 1.4 (Empty Class). The *empty* class \emptyset is $\{x \mid \perp\}$.

1.4 Finite Classes

Definition 1.5. For any objects a_1, \dots, a_n , we write $\{a_1, \dots, a_n\}$ for the class $\{x \mid x = a_1 \vee \dots \vee x = a_n\}$.

1.5 Universal Class

Definition 1.6 (Universal Class). The *universal class* \mathbf{V} is the class $\{x \mid \top\}$.

1.6 Union

Definition 1.7 (Union). For any classes \mathbf{A} and \mathbf{B} , the *union* $\mathbf{A} \cup \mathbf{B}$ is the class $\{x \mid x \in \mathbf{A} \vee x \in \mathbf{B}\}$.

1.7 Intersection

Definition 1.8 (Intersection). For any classes \mathbf{A} and \mathbf{B} , the *intersection* $\mathbf{A} \cap \mathbf{B}$ is the class $\{x \mid x \in \mathbf{A} \wedge x \in \mathbf{B}\}$.

1.8 Relative Complement

Definition 1.9 (Relative Complement). For any classes \mathbf{A} and \mathbf{B} , the *relative complement* $\mathbf{A} - \mathbf{B}$ is $\{x \in \mathbf{A} \mid x \notin \mathbf{B}\}$.

Chapter 2

Sets

2.1 Membership

We take as undefined the notion of *set*.

We take as undefined the binary relation of *membership*, \in . If $a \in A$ we say a is a *member* or *element* of A . If this does not hold, we write $a \notin A$.

Axiom 2.1 (Axiom of Extensionality). *Two sets with exactly the same elements are equal.*

We may therefore identify the set A with the class $\{x \mid x \in A\}$.

We say a class \mathbf{A} is a *set* iff there exists a set A such that $A = \mathbf{A}$. That is, $\{x \mid P(x)\}$ is a set if and only if there exists a set A such that, for all x , we have $x \in A$ if and only if $P(x)$.

2.2 The Empty Set

Axiom 2.2 (Empty Set Axiom). *The empty class \emptyset is a set.*

2.3 Pair Sets

Axiom 2.3 (Pairing Axiom). *For any objects u and v , the class $\{u, v\}$ is a set.*

Theorem 2.4 (Pairing). *For any object a , the class $\{a\}$ is a set.*

PROOF: It is $\{a, a\}$. \square

2.4 Unions

Definition 2.5 (Union). For any class of sets \mathbf{A} , the *union* $\bigcup \mathbf{A}$ is the class $\{x \mid \exists A \in \mathbf{A}. x \in A\}$.

Axiom 2.6 (Union Axiom). *For any set A , the union $\bigcup A$ is a set.*

Theorem 2.7 (Union, Pairing). *For any sets A and B , the class $A \cup B$ is a set.*

PROOF: It is $\bigcup\{A, B\}$. \square

Theorem Schema 2.8 (Union, Pairing). *For any objects a_1, \dots, a_n , the class $\{a_1, \dots, a_n\}$ is a set.*

PROOF: We prove each theorem using the last since $\{a_1, \dots, a_n, a_{n+1}\} = \{a_1, \dots, a_n\} \cup \{a_{n+1}\}$. \square

2.5 Power Set

Definition 2.9 (Power Class). For any class \mathbf{A} , the *power class* $\mathcal{P}\mathbf{A}$ is the class of all subsets of \mathbf{A} .

Axiom 2.10 (Power Set Axiom). *For any set A , the power class $\mathcal{P}A$ is a set.*

2.6 Covers

Definition 2.11 (Cover). Let \mathbf{X} be a class and $\mathcal{A} \subseteq \mathcal{P}\mathbf{X}$. Then \mathcal{A} *covers* \mathbf{X} , or is a *covering* of \mathbf{X} , if and only if $\bigcup \mathcal{A} = \mathbf{X}$.

2.7 Subset Axioms

Axiom Schema 2.12 (Subset Axioms, Aussonderung Axioms). *For any classes \mathbf{A} and \mathbf{B} , if $\mathbf{A} \subseteq \mathbf{B}$ and \mathbf{B} is a set then \mathbf{A} is a set.*

Theorem 2.13 (Subset). *The universal class \mathbf{V} is not a set.*

PROOF:

$\langle 1 \rangle 1$. ASSUME: \mathbf{V} is a set.

$\langle 1 \rangle 2$. LET: $R = \{x \in \mathbf{V} \mid x \notin x\}$

$\langle 1 \rangle 3$. $R \in R$ if and only if $R \notin R$

$\langle 1 \rangle 4$. Q.E.D.

PROOF: This is a contradiction.

\square

Theorem 2.14 (Subset). *If A is a set and \mathbf{B} is a class then $A - \mathbf{B}$ is a set.*

PROOF: It is a subset of A . \square

2.8 Intersection

Definition 2.15 (Intersection). For any class \mathbf{A} of sets, the *intersection* $\bigcap \mathbf{A}$ is the class $\{x \mid \forall A \in \mathbf{A}. x \in A\}$.

Theorem 2.16 (Subset). *For any nonempty class \mathbf{A} of sets, we have $\bigcap \mathbf{A}$ is a set.*

PROOF:

$\langle 1 \rangle 1.$ PICK $A \in \mathbf{A}$

$\langle 1 \rangle 2.$ $\bigcap \mathbf{A} \subseteq A$

$\langle 1 \rangle 3.$ Q.E.D.

PROOF: By a Subset Axiom.

□

Theorem 2.17 (Subset). *For any sets A and B , the class $A \cap B$ is a set.*

PROOF: From a Subset Axiom since $A \cap B \subseteq A$. □

Chapter 3

Relations

3.1 Ordered Pairs

Definition 3.1 (Ordered Pair (Pairing)). For any sets x and y , the *ordered pair* (x, y) is defined to be $\{\{x\}, \{x, y\}\}$.

Theorem 3.2 (Pairing). For any sets u, v, x, y , we have $(u, v) = (x, y)$ if and only if $u = x$ and $v = y$

PROOF:

$\langle 1 \rangle 1$. ASSUME: $\{\{u\}, \{u, v\}\} = \{\{x\}, \{x, y\}\}$

$\langle 1 \rangle 2$. $\{u\} \in \{\{x\}, \{x, y\}\}$

$\langle 1 \rangle 3$. $\{u, v\} \in \{\{x\}, \{x, y\}\}$

$\langle 1 \rangle 4$. $\{u\} = \{x\}$ or $\{u\} = \{x, y\}$

$\langle 1 \rangle 5$. $\{u, v\} = \{x\}$ or $\{u, v\} = \{x, y\}$

$\langle 1 \rangle 6$. CASE: $\{u\} = \{x, y\}$

$\langle 2 \rangle 1$. $u = x = y$

$\langle 2 \rangle 2$. $u = v = x = y$

PROOF: From $\langle 1 \rangle 5$

$\langle 1 \rangle 7$. CASE: $\{u, v\} = \{x\}$

PROOF: Similar.

$\langle 1 \rangle 8$. CASE: $\{u\} = \{x\}$ and $\{u, v\} = \{x, y\}$

$\langle 2 \rangle 1$. $u = x$

$\langle 2 \rangle 2$. $u = y$ or $v = y$

$\langle 2 \rangle 3$. CASE: $u = y$

PROOF: This case is the case considered in $\langle 1 \rangle 6$.

$\langle 2 \rangle 4$. CASE: $v = y$

PROOF: We have $u = x$ and $v = y$ as required.

□

Lemma 3.3 (Pairing, Power Set). Let x, y and C be sets. If $x \in C$ and $y \in C$ then $(x, y) \in \mathcal{P}C$.

PROOF:

- $\langle 1 \rangle 1.$ LET: x, y and C be sets.
- $\langle 1 \rangle 2.$ ASSUME: $x \in C$
- $\langle 1 \rangle 3.$ ASSUME: $y \in C$
- $\langle 1 \rangle 4.$ $\{x\} \subseteq C$
- $\langle 1 \rangle 5.$ $\{x, y\} \subseteq C$
- $\langle 1 \rangle 6.$ $\{x\} \in \mathcal{P}C$
- $\langle 1 \rangle 7.$ $\{x, y\} \in \mathcal{P}C$
- $\langle 1 \rangle 8.$ $\{\{x\}, \{x, y\}\} \subseteq \mathcal{P}C$
- $\langle 1 \rangle 9.$ $\{\{x\}, \{x, y\}\} \in \mathcal{P}\mathcal{P}C$

□

Lemma 3.4 (Pairing, Union). *Let x, y and A be sets. If $(x, y) \in A$ then x and y belong to $\bigcup \bigcup A$.*

PROOF:

- $\langle 1 \rangle 1.$ LET: x, y and A be sets.
- $\langle 1 \rangle 2.$ ASSUME: $(x, y) \in A$
- $\langle 1 \rangle 3.$ $\{x, y\} \in \bigcup A$
- $\langle 1 \rangle 4.$ $x \in \bigcup \bigcup A$
- $\langle 1 \rangle 5.$ $y \in \bigcup \bigcup A$

□

3.2 Cartesian Product

Definition 3.5 (Cartesian Product (Pairing)). Let \mathbf{A} and \mathbf{B} be classes. The *Cartesian product* $\mathbf{A} \times \mathbf{B}$ is the class $\{(x, y) \mid x \in \mathbf{A}, y \in \mathbf{B}\}$.

Theorem 3.6 (Pairing, Union, Power Set, Subset). *For any sets A and B , the Cartesian product $A \times B$ is a set.*

PROOF: It is a subset of $\mathcal{P}\mathcal{P}(A \cup B)$ by Lemma 3.3. □

3.3 Relations

Definition 3.7 (Relation (Pairing)). A *relation* is a class of ordered pairs.

Given a relation \mathbf{R} , we write $x\mathbf{R}y$ for $(x, y) \in \mathbf{R}$.

A relation is *small* iff it is a set.

3.4 Domain

Definition 3.8 (Domain (Pairing)). Let \mathbf{R} be a class. The *domain* of \mathbf{R} is $\text{dom } \mathbf{R} = \{x \mid \exists y. x\mathbf{R}y\}$.

Theorem 3.9 (Pairing, Union, Subset). *For any set R , the domain $\text{dom } R$ is a set.*

PROOF: It is a subset of $\bigcup \bigcup R$ by Lemma 3.4. □

3.5 Range

Definition 3.10 (Domain (Pairing)). Let \mathbf{R} be a class. The *range* of \mathbf{R} is $\text{ran } \mathbf{R} = \{y \mid \exists x. x\mathbf{R}y\}$.

Theorem 3.11 (Pairing, Union, Subset). *For any set R , the range $\text{ran } R$ is a set.*

PROOF: It is a subset of $\bigcup \bigcup R$ by Lemma 3.4. \square

3.6 Field

Definition 3.12 (Field). Let \mathbf{R} be a class. The *field* of \mathbf{R} is $\text{fld } \mathbf{R} = \text{dom } \mathbf{R} \cup \text{ran } \mathbf{R}$.

Theorem 3.13 (Pairing, Union, Subset). *For any set R , the field $\text{fld } R$ is a set.*

PROOF: Theorems 2.7, 3.9 and 3.11. \square

3.7 Functions

Definition 3.14 (Class Term (Pairing)). A *class term* is a relation \mathbf{F} such that, for all x, y, y' , if $x\mathbf{F}y$ and $x\mathbf{F}y'$ then $y = y'$.

If \mathbf{F} is a class term and $x \in \text{dom } \mathbf{F}$, then we write $\mathbf{F}(x)$ for the unique y such that $x\mathbf{F}y$.

We write $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$ iff \mathbf{F} is a class term, $\text{dom } \mathbf{F} = \mathbf{A}$ and $\text{ran } \mathbf{F} \subseteq \mathbf{B}$.

A *function* is a class term that is a set.

Axiom 3.15 (Axiom of Choice, First Form (Pairing)). *For any relation R , there exists a function $H \subseteq R$ such that $\text{dom } H = \text{dom } R$.*

Theorem 3.16. *The following are equivalent.*

1. *The Axiom of Choice*
2. (**Multiplicative Axiom**) *For any function H with domain I such that $H(i)$ is nonempty for all $i \in I$, there exists a function f with domain I such that, for all $i \in I$, we have $f(i) \in H(i)$.*
3. *Every set has a choice function.*
4. *Let \mathcal{A} be a set of pairwise disjoint nonempty sets. Then there exists a set C containing exactly one element from each member of \mathcal{A} .*

3.8 Single-Rooted

Definition 3.17 (Single-Rooted (Pairing)). A class \mathbf{R} is *single-rooted* if and only if, for all x, x', y , if $x\mathbf{R}y$ and $x'\mathbf{R}y$ then $x = x'$.

We call a class term *one-to-one*, *injective* or an *injection* if and only if it is single-rooted.

3.9 Surjective

Definition 3.18 (Surjective (Pairing)). Let $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$. Then \mathbf{F} is *surjective* if and only if $\text{ran } \mathbf{F} = \mathbf{B}$.

3.10 Inverse

Definition 3.19 (Inverse (Pairing)). Let \mathbf{R} be a class. The *inverse* of \mathbf{R} is $\mathbf{R}^{-1} = \{(y, x) \mid x\mathbf{R}y\}$.

Theorem 3.20 (Pairing, Union, Power Set, Subset). *For any set R , the inverse R^{-1} is a set.*

PROOF: It is a subset of $\text{ran } R \times \text{dom } R$. \square

Theorem 3.21 (Pairing). *For any class \mathbf{F} , we have $\text{dom } \mathbf{F}^{-1} = \text{ran } \mathbf{F}$.*

PROOF: For any x , we have

$$\begin{aligned} x \in \text{dom } \mathbf{F}^{-1} &\Leftrightarrow \exists y.(x, y) \in \mathbf{F}^{-1} \\ &\Leftrightarrow \exists y.(y, x) \in \mathbf{F} \\ &\Leftrightarrow x \in \text{ran } \mathbf{F} \end{aligned} \quad \square$$

Theorem 3.22 (Pairing). *For any set \mathbf{F} , we have $\text{ran } \mathbf{F}^{-1} = \text{dom } \mathbf{F}$.*

PROOF: For any x , we have

$$\begin{aligned} x \in \text{ran } \mathbf{F}^{-1} &\Leftrightarrow \exists y.(y, x) \in \mathbf{F}^{-1} \\ &\Leftrightarrow \exists y.(x, y) \in \mathbf{F} \\ &\Leftrightarrow x \in \text{dom } \mathbf{F} \end{aligned} \quad \square$$

Theorem 3.23 (Pairing). *For any relation \mathbf{F} , we have $(\mathbf{F}^{-1})^{-1} = \mathbf{F}$.*

PROOF: For any z we have

$$\begin{aligned} z \in (\mathbf{F}^{-1})^{-1} &\Leftrightarrow \exists x, y. z = (x, y) \wedge (y, x) \in \mathbf{F}^{-1} \\ &\Leftrightarrow \exists x, y. z = (x, y) \wedge (x, y) \in \mathbf{F} \\ &\Leftrightarrow z \in \mathbf{F} \end{aligned} \quad (\mathbf{F} \text{ is a relation}) \square$$

Theorem 3.24 (Pairing). *For any class \mathbf{F} , we have \mathbf{F}^{-1} is a class term if and only if \mathbf{F} is single-rooted.*

PROOF: Immediate from definitions. \square

Theorem 3.25 (Pairing). *Let \mathbf{F} be a relation. Then \mathbf{F} is a class term if and only if \mathbf{F}^{-1} is single-rooted.*

PROOF: Immediate from definitions. \square

Theorem 3.26 (Pairing). *Let \mathbf{F} be a one-to-one class term and $x \in \text{dom } \mathbf{F}$. Then $\mathbf{F}^{-1}(\mathbf{F}(x)) = x$.*

PROOF: We have $(x, \mathbf{F}(x)) \in \mathbf{F}$ and so $(\mathbf{F}(x), x) \in \mathbf{F}^{-1}$. \square

Theorem 3.27 (Pairing). *Let \mathbf{F} be a one-to-one function and $y \in \text{ran } \mathbf{F}$. Then $\mathbf{F}(\mathbf{F}^{-1}(y)) = y$.*

PROOF: From Theorems 3.21, 3.23 and 3.26. \square

3.11 Composition

Definition 3.28 (Composition (Pairing)). Let \mathbf{R} and \mathbf{S} be relations. The *composition* of \mathbf{R} and \mathbf{S} is $\mathbf{S} \circ \mathbf{R} = \{(x, z) \mid \exists y. x\mathbf{R}y \wedge y\mathbf{S}z\}$.

Theorem 3.29 (Pairing, Union, Power Set, Subset). *If R and S are small relations then $S \circ R$ is small.*

PROOF: It is a subset of $\text{dom } R \times \text{ran } S$. \square

Theorem 3.30 (Pairing). *Let \mathbf{F} and \mathbf{G} be class terms. Then $\mathbf{G} \circ \mathbf{F}$ is a function, its domain is $\{x \in \text{dom } \mathbf{F} \mid \mathbf{F}(x) \in \text{dom } \mathbf{G}\}$, and for x in this domain, we have $(\mathbf{F} \circ \mathbf{G})(x) = \mathbf{F}(\mathbf{G}(x))$.*

PROOF:

$\langle 1 \rangle 1$. $\mathbf{G} \circ \mathbf{F}$ is a class term.

$\langle 2 \rangle 1$. LET: $x(\mathbf{G} \circ \mathbf{F})z$ and $x(\mathbf{G} \circ \mathbf{F})z'$

$\langle 2 \rangle 2$. PICK y, y' such that $x\mathbf{F}y, x\mathbf{F}y', y\mathbf{G}z$ and $y'\mathbf{G}z'$

$\langle 2 \rangle 3$. $y = y'$

PROOF: Since \mathbf{F} is a class term.

$\langle 2 \rangle 4$. $z = z'$

PROOF: Since \mathbf{G} is a class term.

$\langle 1 \rangle 2$. $\text{dom}(\mathbf{G} \circ \mathbf{F}) = \{x \in \text{dom } \mathbf{F} \mid \mathbf{F}(x) \in \text{dom } \mathbf{G}\}$

PROOF:

$$\begin{aligned} x \in \text{dom}(\mathbf{G} \circ \mathbf{F}) &\Leftrightarrow \exists z. x(\mathbf{G} \circ \mathbf{F})z \\ &\Leftrightarrow \exists y, z. x\mathbf{F}y \wedge y\mathbf{G}z \\ &\Leftrightarrow x \in \text{dom } \mathbf{F} \wedge \mathbf{F}(x) \in \text{dom } \mathbf{G} \end{aligned}$$

$\langle 1 \rangle 3$. For x in this domain, we have $(\mathbf{F} \circ \mathbf{G})(x) = \mathbf{F}(\mathbf{G}(x))$.

PROOF: Since $(x, \mathbf{F}(x)) \in \mathbf{F}$ and $(\mathbf{F}(x), \mathbf{G}(\mathbf{F}(x))) \in \mathbf{G}$.

\square

Theorem 3.31 (Pairing). *For any classes \mathbf{F} and \mathbf{G} , we have $(\mathbf{G} \circ \mathbf{F})^{-1} = \mathbf{F}^{-1} \circ \mathbf{G}^{-1}$.*

PROOF:

$$\begin{aligned} (x, z) \in (\mathbf{G} \circ \mathbf{F})^{-1} &\Leftrightarrow (z, x) \in \mathbf{G} \circ \mathbf{F} \\ &\Leftrightarrow \exists y. z\mathbf{F}y \wedge y\mathbf{G}x \\ &\Leftrightarrow \exists y. (y, z) \in \mathbf{F}^{-1} \wedge (x, y) \in \mathbf{G}^{-1} \\ &\Leftrightarrow (x, z) \in \mathbf{F}^{-1} \circ \mathbf{G}^{-1} \end{aligned} \quad \square$$

3.12 Identity Function

Definition 3.32 (Identity Class Term (Pairing)). Let \mathbf{A} be a set. The *identity class term* $\text{id}_{\mathbf{A}}$ on \mathbf{A} is $\{(x, x) \mid x \in \mathbf{A}\}$.

Theorem 3.33 (Pairing, Power Set, Subset). *For any set A , we have id_A is a function.*

PROOF: It is a subset of $\mathcal{P}\mathcal{P}A$. \square

Theorem 3.34 (Extensionality, Pairing, Union, Power Set, Subset). *Let $F : A \rightarrow B$ and A be nonempty. Then there exists a function $G : B \rightarrow A$ such that $G \circ F = \text{id}_A$ if and only if F is one-to-one.*

PROOF:

$\langle 1 \rangle 1$. LET: $F : A \rightarrow B$

$\langle 1 \rangle 2$. ASSUME: A is nonempty

$\langle 1 \rangle 3$. If there exists $G : B \rightarrow A$ such that $G \circ F = \text{id}_A$ then F is one-to-one.

$\langle 2 \rangle 1$. ASSUME: $G : B \rightarrow A$ and $G \circ F = \text{id}_A$

$\langle 2 \rangle 2$. LET: $x, y \in A$

$\langle 2 \rangle 3$. ASSUME: $F(x) = F(y)$

$\langle 2 \rangle 4$. $x = y$

PROOF: $x = G(F(x)) = G(F(y)) = y$.

$\langle 1 \rangle 4$. If F is one-to-one then there exists $G : B \rightarrow A$ such that $G \circ F = \text{id}_A$.

$\langle 2 \rangle 1$. ASSUME: F is one-to-one.

$\langle 2 \rangle 2$. PICK $a \in A$

$\langle 2 \rangle 3$. Define $G : B \rightarrow A$ by: $G(y)$ is the x such that $F(x) = y$ if $y \in \text{ran } F$, otherwise $G(y) = a$

$\langle 2 \rangle 4$. $G \circ F = \text{id}_A$

PROOF: For $x \in A$ we have $(G \circ F)(x) = G(F(x)) = x$ by Theorem 3.30.

\square

Theorem 3.35 (Extensionality, Pairing, Union, Power Set, Subset). *Let $F : A \rightarrow B$ and A be nonempty. If there exists a function $H : B \rightarrow A$ such that $F \circ H = \text{id}_B$ then F is surjective.*

PROOF:

$\langle 1 \rangle 1$. LET: $F : A \rightarrow B$

$\langle 1 \rangle 2$. ASSUME: A is nonempty.

$\langle 1 \rangle 3$. LET: $H : B \rightarrow A$ satisfy $F \circ H = \text{id}_B$

$\langle 1 \rangle 4$. LET: $y \in B$

$\langle 1 \rangle 5$. $F(H(y)) = y$.

\square

Theorem 3.36 (Extensionality, Pairing, Union, Power Set, Subset, Choice). *Let $F : A \rightarrow B$ and A be nonempty. If F is surjective then there exists a function $H : B \rightarrow A$ such that $F \circ H = \text{id}_B$.*

PROOF:

$\langle 1 \rangle 1.$ ASSUME: F is surjective.
 $\langle 1 \rangle 2.$ PICK a function $H \subseteq F^{-1}$ with $\text{dom } H = B$
 PROOF: By the Axiom of Choice.
 $\langle 1 \rangle 3.$ $H : B \rightarrow A$
 $\langle 1 \rangle 4.$ $F \circ H = \text{id}_B$
 $\langle 2 \rangle 1.$ LET: $y \in B$
 $\langle 2 \rangle 2.$ $(y, H(y)) \in F^{-1}$
 $\langle 2 \rangle 3.$ $(H(y), y) \in F$
 $\langle 2 \rangle 4.$ $F(H(y)) = y$
 \square

3.13 Restriction

Definition 3.37 (Restriction (Pairing)). Let \mathbf{R} be a relation and \mathbf{A} a class. The *restriction* of \mathbf{R} to \mathbf{A} is $\mathbf{R} \upharpoonright \mathbf{A} = \{(x, y) \mid x \in \mathbf{A} \wedge x \mathbf{R} y\}$.

Theorem 3.38 (Pairing, Subset). *If R is a small relation then $R \upharpoonright \mathbf{A}$ is small.*

PROOF: Since it is a subset of R . \square

3.14 Image

Definition 3.39 (Image (Pairing)). Let \mathbf{F} and \mathbf{A} be classes. The *image* of \mathbf{A} under \mathbf{F} is $\mathbf{F}(\mathbf{A}) = \{\mathbf{F}(x) \mid x \in \mathbf{A}\}$.

Theorem 3.40 (Pairing, Union, Subset). *If F is a set then $F(\mathbf{A})$ is a set.*

PROOF: Since it is a subset of $\text{ran } F$. \square

Theorem 3.41 (Pairing). *For any classes \mathbf{F} and \mathcal{A} we have*

$$\mathbf{F}\left(\bigcup \mathcal{A}\right) = \bigcup_{A \in \mathcal{A}} \mathbf{F}(A)$$

PROOF: Each is the class of all y such that $\exists x. \exists A. x \in A \in \mathcal{A} \wedge y = \mathbf{F}(x)$. \square

Theorem 3.42 (Pairing). *For any classes \mathbf{F} , $\mathbf{A}_1, \dots, \mathbf{A}_n$, we have*

$$\mathbf{F}(\mathbf{A}_1 \cup \dots \cup \mathbf{A}_n) = \mathbf{F}(\mathbf{A}_1) \cup \dots \cup \mathbf{F}(\mathbf{A}_n) .$$

PROOF: Similar. \square

Theorem 3.43 (Pairing). *For any classes \mathbf{F} and \mathcal{A} , we have*

$$\mathbf{F}\left(\bigcap \mathcal{A}\right) \subseteq \bigcap_{A \in \mathcal{A}} \mathbf{F}(A) .$$

Equality holds if \mathbf{F} is single-rooted and \mathcal{A} is nonempty.

PROOF:

- ⟨1⟩1. $\mathbf{F}(\bigcap \mathcal{A}) \subseteq \bigcap_{A \in \mathcal{A}} \mathbf{F}(A)$
- ⟨2⟩1. LET: $y \in \mathbf{F}(\bigcap \mathcal{A})$
- ⟨2⟩2. PICK $x \in \bigcap \mathcal{A}$ such that $y = \mathbf{F}(x)$
- ⟨2⟩3. LET: $A \in \mathcal{A}$
- ⟨2⟩4. $x \in A$
- ⟨2⟩5. $y \in \mathbf{F}(A)$
- ⟨1⟩2. If \mathbf{F} is single-rooted then $\mathbf{F}(\bigcap \mathcal{A}) = \bigcap_{A \in \mathcal{A}} \mathbf{F}(A)$
- ⟨2⟩1. ASSUME: \mathbf{F} is single-rooted and \mathcal{A} is nonempty.
- ⟨2⟩2. LET: $y \in \bigcap_{A \in \mathcal{A}} \mathbf{F}(A)$
- ⟨2⟩3. PICK $A \in \mathcal{A}$
- ⟨2⟩4. PICK $x \in A$ such that $y = \mathbf{F}(x)$
- ⟨2⟩5. $x \in \bigcap \mathcal{A}$
- ⟨3⟩1. LET: $A' \in \mathcal{A}$
- ⟨3⟩2. PICK $x' \in A'$ such that $y = \mathbf{F}(x')$
- ⟨3⟩3. $x = x'$
- PROOF: By ⟨2⟩1.
- ⟨3⟩4. $x \in A'$

□

Corollary 3.43.1 (Pairing). *For any class \mathbf{F} and nonempty class \mathcal{A} , we have*

$$\mathbf{F}^{-1}\left(\bigcap \mathcal{A}\right) = \bigcap_{A \in \mathcal{A}} \mathbf{F}^{-1}(A) .$$

Theorem 3.44 (Pairing). *For any classes \mathbf{F} , $\mathbf{A}_1, \dots, \mathbf{A}_n$, we have*

$$\mathbf{F}(\mathbf{A}_1 \cap \dots \cap \mathbf{A}_n) \subseteq \mathbf{F}(\mathbf{A}_1) \cap \dots \cap \mathbf{F}(\mathbf{A}_n) .$$

Equality holds if \mathbf{F} is single-rooted.

PROOF: Similar.

Corollary 3.44.1 (Pairing). *For any classes \mathbf{F} , $\mathbf{A}_1, \dots, \mathbf{A}_n$, we have*

$$\mathbf{F}^{-1}(\mathbf{A}_1 \cap \dots \cap \mathbf{A}_n) = \mathbf{F}^{-1}(\mathbf{A}_1) \cap \dots \cap \mathbf{F}^{-1}(\mathbf{A}_n) .$$

Theorem 3.45 (Pairing). *For any classes \mathbf{F} , \mathbf{A} and \mathbf{B} , we have*

$$\mathbf{F}(\mathbf{A}) - \mathbf{F}(\mathbf{B}) \subseteq \mathbf{F}(\mathbf{A} - \mathbf{B}) .$$

Equality holds if \mathbf{F} is single-rooted.

PROOF:

- ⟨1⟩1. LET: \mathbf{F} , \mathbf{A} and \mathbf{B} be sets.
- ⟨1⟩2. $\mathbf{F}(\mathbf{A}) - \mathbf{F}(\mathbf{B}) \subseteq \mathbf{F}(\mathbf{A} - \mathbf{B})$
- ⟨2⟩1. LET: $y \in \mathbf{F}(\mathbf{A}) - \mathbf{F}(\mathbf{B})$
- ⟨2⟩2. PICK $x \in \mathbf{A}$ such that $x\mathbf{F}y$
- ⟨2⟩3. $x \in \mathbf{A} - \mathbf{B}$

- ⟨1⟩3. If \mathbf{F} is single-rooted then $\mathbf{F}(\mathbf{A} - \mathbf{B}) = \mathbf{F}(\mathbf{A}) - \mathbf{F}(\mathbf{B})$.
- ⟨2⟩1. ASSUME: \mathbf{F} is single-rooted.
- ⟨2⟩2. LET: $y \in \mathbf{F}(\mathbf{A} - \mathbf{B})$
- ⟨2⟩3. PICK $x \in \mathbf{A} - \mathbf{B}$ such that $y = \mathbf{F}(x)$
- ⟨2⟩4. $y \in \mathbf{F}(\mathbf{A})$
- ⟨2⟩5. $y \notin \mathbf{F}(\mathbf{B})$
- ⟨3⟩1. ASSUME: for a contradiction $x' \in \mathbf{B}$ and $x'\mathbf{F}y$
- ⟨3⟩2. $x' = x$
PROOF: From ⟨2⟩1
- ⟨3⟩3. $x \in \mathbf{B}$
- ⟨3⟩4. Q.E.D.
PROOF: This contradicts ⟨2⟩3.

□

Corollary 3.45.1 (Pairing). *For any classes \mathbf{F} and sets \mathbf{A} and \mathbf{B} , we have*

$$\mathbf{F}^{-1}(\mathbf{A}) - \mathbf{F}^{-1}(\mathbf{B}) = \mathbf{F}^{-1}(\mathbf{A} - \mathbf{B}) .$$

3.15 Infinite Cartesian Product

Definition 3.46 (Infinite Cartesian Product (Pairing)). Let H be a function with domain I . The *Cartesian product* $\prod_{i \in I} H(i)$ is the class of all functions f with domain I such that, for all $i \in I$, we have $f(i) \in H(i)$.

Theorem 3.47 (Pairing, Union, Power Set, Subset). *If H is a function with domain I then $\prod_{i \in I} H(i)$ is a set.*

PROOF: It is a subset of $\mathcal{P}(I \times \bigcup \text{ran } H)$. □

Theorem 3.48 (Axiom of Choice, Second Version (Pairing, Union, Power Set, Subset)). *The Axiom of Choice is equivalent to the statement: for any function H with domain I , if $H(i)$ is nonempty for all $i \in I$, then $\prod_{i \in I} H(i)$ is nonempty.*

PROOF:

- ⟨1⟩1. If the Axiom of Choice is true then, for any function H with domain I , if $H(i)$ is nonempty for all $i \in I$, then $\prod_{i \in I} H(i)$ is nonempty.
- ⟨2⟩1. ASSUME: The Axiom of Choice
- ⟨2⟩2. LET: H be a function with domain I such that $H(i)$ is nonempty for all $i \in I$.
- ⟨2⟩3. PICK a function $f \subseteq \{(i, x) \mid x \in H(i)\}$
- ⟨2⟩4. $f \in \prod_{i \in I} H(i)$
- ⟨1⟩2. If, for any function H with domain I , if $H(i)$ is nonempty for all $i \in I$, then $\prod_{i \in I} H(i)$ is nonempty, then the Axiom of Choice is true.
- ⟨2⟩1. ASSUME: for any function H with domain I , if $H(i)$ is nonempty for all $i \in I$, then $\prod_{i \in I} H(i)$ is nonempty
- ⟨2⟩2. LET: R be a relation.
- ⟨2⟩3. LET: $I = \text{dom } R$

⟨2⟩4. LET: H be the function with domain I such that $H(i) = \{y \mid iRy\}$ for all i .

⟨2⟩5. PICK $f \in \prod_{i \in I} H(i)$

⟨2⟩6. $f \subseteq R$

□

3.16 Reflexive Relations

Definition 3.49 (Reflexive (Pairing)). Let \mathbf{R} be a relation on \mathbf{A} . Then \mathbf{R} is *reflexive* on A if and only if, for all $x \in \mathbf{A}$, we have $x\mathbf{R}x$.

3.17 Symmetric

Definition 3.50 (Symmetric (Pairing)). Let \mathbf{R} be a relation. Then \mathbf{R} is *symmetric* if and only if, whenever $x\mathbf{R}y$, then $y\mathbf{R}x$.

3.18 Transitivity

Definition 3.51 (Transitivity (Pairing)). Let \mathbf{R} be a relation. Then \mathbf{R} is *transitive* if and only if, whenever $x\mathbf{R}y$ and $y\mathbf{R}z$, then $x\mathbf{R}z$.

3.19 Equivalence Relations

Definition 3.52 (Equivalence Relation (Pairing)). Let \mathbf{R} be a relation on \mathbf{A} . Then \mathbf{R} is an *equivalence relation* on \mathbf{A} if and only if \mathbf{R} is reflexive on \mathbf{A} , symmetric and transitive.

Theorem 3.53 (Pairing). *If \mathbf{R} is a symmetric and transitive relation then \mathbf{R} is an equivalence relation on $\text{fld } \mathbf{R}$.*

PROOF:

⟨1⟩1. LET: \mathbf{R} be a symmetric and transitive relation.

⟨1⟩2. LET: $x \in \text{fld } \mathbf{R}$

⟨1⟩3. PICK y such that $x\mathbf{R}y$ or $y\mathbf{R}x$

⟨1⟩4. $x\mathbf{R}y$ and $y\mathbf{R}x$

PROOF: By symmetry.

⟨1⟩5. $x\mathbf{R}x$

PROOF: By transitivity.

□

3.20 Equivalence Class

Definition 3.54 (Equivalence Class (Pairing)). Let \mathbf{R} be an equivalence relation on \mathbf{A} and $a \in \mathbf{A}$. Then the *equivalence class of a modulo \mathbf{R}* is

$$[a]_{\mathbf{R}} = \{x \in \mathbf{A} \mid a\mathbf{R}x\} .$$

Lemma 3.55 (Extensionality, Pairing, Subset). *Let \mathbf{R} be an equivalence relation on \mathbf{A} and $x, y \in \mathbf{A}$. Then $[x]_{\mathbf{R}} = [y]_{\mathbf{R}}$ if and only if $x\mathbf{R}y$.*

PROOF:

$\langle 1 \rangle 1$. If $[x]_{\mathbf{R}} = [y]_{\mathbf{R}}$ then $x\mathbf{R}y$.

$\langle 2 \rangle 1$. ASSUME: $[x]_{\mathbf{R}} = [y]_{\mathbf{R}}$

$\langle 2 \rangle 2$. $y \in [y]_{\mathbf{R}}$

PROOF: Since $y\mathbf{R}y$ by reflexivity.

$\langle 2 \rangle 3$. $y \in [x]_{\mathbf{R}}$

$\langle 2 \rangle 4$. $x\mathbf{R}y$

$\langle 1 \rangle 2$. If $x\mathbf{R}y$ then $[x]_{\mathbf{R}} = [y]_{\mathbf{R}}$.

$\langle 2 \rangle 1$. ASSUME: $x\mathbf{R}y$

$\langle 2 \rangle 2$. $[y]_{\mathbf{R}} \subseteq [x]_{\mathbf{R}}$

PROOF: If $y\mathbf{R}z$ then $x\mathbf{R}z$ by transitivity.

$\langle 2 \rangle 3$. $[x]_{\mathbf{R}} \subseteq [y]_{\mathbf{R}}$

PROOF: Similar since $y\mathbf{R}x$ by symmetry.

□

3.21 Disjoint

Definition 3.56 (Disjoint). Two classes \mathbf{A} and \mathbf{B} are *disjoint* if and only if there is no x such that $x \in \mathbf{A}$ and $x \in \mathbf{B}$.

Axiom 3.57 (Regularity). *For any nonempty set A , there exists $m \in A$ such that m and A are disjoint.*

Theorem 3.58 (Regularity). *No set is a member of itself.*

Theorem 3.59 (Regularity). *There are no sets A and B such that $A \in B$ and $B \in A$.*

3.22 Partitions

Definition 3.60 (Partition). A *partition P* of a set A is a set of nonempty subsets of A such that:

1. For all $x \in A$ there exists $S \in P$ such that $x \in S$.
2. Any two distinct elements of P are disjoint.

3.23 Quotient Sets

Definition 3.61 (Quotient Set (Pairing, Power Set, Subset)). Let R be an equivalence relation on A . The *quotient set* A/R is the set of all equivalence classes modulo R .

This is a set because it is a subset of $\mathcal{P}A$.

Theorem 3.62 (Extensionality, Pairing, Power Set, Subset). *Let R be an equivalence relation on A . Then the quotient set A/R is a partition of A .*

PROOF:

$\langle 1 \rangle 1$. For all $x \in A$ there exists $y \in A$ such that $x \in [y]_R$

PROOF: Take $y = x$.

$\langle 1 \rangle 2$. Any two distinct equivalence classes are disjoint.

$\langle 2 \rangle 1$. ASSUME: $z \in [x]_R$ and $z \in [y]_R$

$\langle 2 \rangle 2$. xRz and yRz

$\langle 2 \rangle 3$. $[x]_R = [z]_R = [y]_R$

PROOF: Lemma 3.55.

□

Definition 3.63 (Canonical Map (Pairing, Power Set, Subset)). Let R be an equivalence relation on A . The *canonical map* $\phi : A \rightarrow A/R$ is the function defined by $\phi(a) = [a]_R$.

Theorem 3.64. *Let R be an equivalence relation on A and $F : A \rightarrow B$. Then the following are equivalent:*

1. For all $x, y \in A$, if xRy then $F(x) = F(y)$.
2. There exists $G : A/R \rightarrow B$ such that $F = G \circ \phi$, where $\phi : A \rightarrow A/R$ is the canonical map.

In this case, G is unique.

PROOF:

$\langle 1 \rangle 1$. $1 \Rightarrow 2$

$\langle 2 \rangle 1$. ASSUME: 1

$\langle 2 \rangle 2$. Let $G = \{([a]_R, b) \mid F(a) = b\}$

$\langle 2 \rangle 3$. G is a function.

$\langle 3 \rangle 1$. LET: $(c, b), (c, b') \in G$

$\langle 3 \rangle 2$. PICK $a, a' \in A$ such that $c = [a]_R = [a']_R$ with $F(a) = b$ and $F(a') = b'$

$\langle 3 \rangle 3$. aRa'

PROOF: Lemma 3.55.

$\langle 3 \rangle 4$. $F(a) = F(a')$

PROOF: From $\langle 2 \rangle 1$.

$\langle 3 \rangle 5$. $b = b'$

PROOF: From $\langle 3 \rangle 2$.

$\langle 2 \rangle 4$. $F = G \circ \phi$

PROOF: For $a \in A$ we have $G(\phi(a)) = G([a]) = F(a)$.

$\langle 1 \rangle 2$. $2 \Rightarrow 1$

$\langle 2 \rangle 1$. LET: $G : A/R \rightarrow B$ be such that $F = G \circ \phi$

$\langle 2 \rangle 2$. LET: $x, y \in A$

$\langle 2 \rangle 3$. ASSUME: xRy

$\langle 2 \rangle 4$. $G([x]) = G([y])$

PROOF: Lemma 3.55

$\langle 2 \rangle 5$. $F(x) = F(y)$

PROOF: From $\langle 2 \rangle 1$.

$\langle 1 \rangle 3$. If $G, G' : A/R \rightarrow B$ and $G \circ \phi = G' \circ \phi$ then $G = G'$

PROOF: For any $a \in A$ we have $G([a]) = G'([a])$.

□

3.24 The Finite Intersection Property

Definition 3.65 (Finite Intersection Property). Let X be a set and $\mathcal{A} \subseteq \mathcal{P}X$. Then \mathcal{A} satisfies the *finite intersection property* if and only if every nonempty finite subset of \mathcal{A} has nonempty intersection.

Lemma 3.66. Let X be a set. Let $\mathcal{A} \subseteq \mathcal{P}X$ have the finite intersection property. Then there exists a maximal set \mathcal{D} such that $\mathcal{A} \subseteq \mathcal{D} \subseteq \mathcal{P}X$ and \mathcal{D} has the finite intersection property.

PROOF:

$\langle 1 \rangle 1$. LET: $\mathbb{F} = \{\mathcal{D} \mid \mathcal{A} \subseteq \mathcal{D} \subseteq \mathcal{P}X, \mathcal{D} \text{ has the finite intersection property}\}$

$\langle 1 \rangle 2$. Every chain in \mathbb{F} has an upper bound.

$\langle 2 \rangle 1$. LET: \mathbb{C} be a chain in \mathbb{F} .

$\langle 2 \rangle 2$. ASSUME: without loss of generality $\mathbb{C} \neq \emptyset$

PROVE: $\bigcup \mathbb{C} \in \mathbb{F}$

PROOF: If $\mathbb{C} = \emptyset$ then \mathcal{A} is an upper bound.

$\langle 2 \rangle 3$. $\mathcal{A} \subseteq \bigcup \mathbb{C} \subseteq \mathcal{P}X$

$\langle 2 \rangle 4$. LET: $C_1, \dots, C_n \in \mathbb{C}$

PROVE: $C_1 \cap \dots \cap C_n \neq \emptyset$

$\langle 2 \rangle 5$. PICK $C_1, \dots, C_n \in \mathbb{C}$ such that $C_i \in \mathbb{C}_i$ for all i .

$\langle 2 \rangle 6$. ASSUME: without loss of generality $C_1 \subseteq \dots \subseteq C_n$

$\langle 2 \rangle 7$. $C_1, \dots, C_n \in \mathbb{C}_n$

$\langle 2 \rangle 8$. \mathbb{C}_n satisfies the finite intersection property.

$\langle 2 \rangle 9$. $C_1 \cap \dots \cap C_n \neq \emptyset$

$\langle 1 \rangle 3$. Q.E.D.

PROOF: By Zorn's Lemma.

□

Lemma 3.67. Let X be a set. Let $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the finite intersection property. Then any finite intersection of elements of \mathcal{D} is an element of \mathcal{D} .

PROOF:

$\langle 1 \rangle 1.$ LET: $D_1, D_2 \in \mathcal{D}$
 $\langle 1 \rangle 2.$ $\mathcal{D} \cup \{D_1 \cap D_2\}$ has the finite intersection property.
 PROOF: Any finite intersection of members of $\mathcal{D} \cup \{D_1 \cap D_2\}$ is a finite intersection of members of \mathcal{D} .
 $\langle 1 \rangle 3.$ $\mathcal{D} = \mathcal{D} \cup \{D_1 \cap D_2\}$
 PROOF: By maximality of \mathcal{D} .
 $\langle 1 \rangle 4.$ $D_1 \cap D_2 \in \mathcal{D}$.
 \square

Lemma 3.68. *Let X be a set. Let $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the finite intersection property. Let $A \subseteq X$. If A intersects every member of \mathcal{D} then $A \in \mathcal{D}$.*

PROOF:

$\langle 1 \rangle 1.$ $\mathcal{D} \cup \{A\}$ has the finite intersection property.
 $\langle 2 \rangle 1.$ LET: $D_1, \dots, D_n \in \mathcal{D}$
 PROVE: $D_1 \cap \dots \cap D_n \cap A \neq \emptyset$
 $\langle 2 \rangle 2.$ $D_1 \cap \dots \cap D_n \in \mathcal{D}$
 PROOF: Lemma 3.67.
 $\langle 2 \rangle 3.$ $D_1 \cap \dots \cap D_n \cap A \neq \emptyset$
 PROOF: Since A intersects every member of \mathcal{D} .
 $\langle 1 \rangle 2.$ Q.E.D.
 PROOF: By maximality of \mathcal{D} .
 \square

Proposition 3.69. *Let X be a set. Let $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the finite intersection property. Let $A, D \in \mathcal{P}X$. If $D \in \mathcal{D}$ and $D \subseteq A$ then $A \in \mathcal{D}$.*

PROOF:

$\langle 1 \rangle 1.$ $\mathcal{D} \cup \{A\}$ satisfies the finite intersection property.
 $\langle 2 \rangle 1.$ LET: $D_1, \dots, D_n \in \mathcal{D}$
 $\langle 2 \rangle 2.$ $D_1 \cap \dots \cap D_n \cap D \neq \emptyset$
 PROOF: Since \mathcal{D} satisfies the finite intersection property.
 $\langle 2 \rangle 3.$ $D_1 \cap \dots \cap D_n \cap A \neq \emptyset$
 $\langle 1 \rangle 2.$ $\mathcal{D} = \mathcal{D} \cup \{A\}$
 PROOF: By the maximality of \mathcal{D} .
 $\langle 1 \rangle 3.$ $A \in \mathcal{D}$
 \square

Definition 3.70 (Graph). Let $f : A \rightarrow B$. The *graph* of f is the set $\{(x, f(x)) \mid x \in A\} \subseteq A \times B$.

3.25 Countable Intersection Property

Definition 3.71 (Countable Intersection Property). Let X be a set and $\mathcal{A} \subseteq \mathcal{P}X$. Then \mathcal{A} satisfies the *countable intersection property* if and only if every countable subset of \mathcal{A} has nonempty intersection.

Lemma 3.72. *Let X be a set. Let $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the countable intersection property. Then any countable intersection of elements of \mathcal{D} is an element of \mathcal{D} .*

PROOF:

$\langle 1 \rangle 1.$ LET: $\mathcal{D}_0 \subseteq \mathcal{D}$ be countable.

$\langle 1 \rangle 2.$ $\mathcal{D} \cup \{\bigcap \mathcal{D}_0\}$ has the countable intersection property.

PROOF: Any countable intersection of members of $\mathcal{D} \cup \{\bigcap \mathcal{D}_0\}$ is a finite intersection of members of \mathcal{D} .

$\langle 1 \rangle 3.$ $\mathcal{D} = \mathcal{D} \cup \{\bigcap \mathcal{D}_0\}$

PROOF: By maximality of \mathcal{D} .

$\langle 1 \rangle 4.$ $\bigcap \mathcal{D}_0 \in \mathcal{D}$.

□

Lemma 3.73. *Let X be a set. Let $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the countable intersection property. Let $A \subseteq X$. If A intersects every member of \mathcal{D} then $A \in \mathcal{D}$.*

PROOF:

$\langle 1 \rangle 1.$ $\mathcal{D} \cup \{A\}$ has the countable intersection property.

$\langle 2 \rangle 1.$ LET: $\mathcal{D}_0 \subseteq \mathcal{D}$ be countable.

PROVE: $\bigcap \mathcal{D}_0 \cap A \neq \emptyset$

$\langle 2 \rangle 2.$ $\bigcap \mathcal{D}_0 \in \mathcal{D}$

PROOF: Lemma 3.72.

$\langle 2 \rangle 3.$ $\bigcap \mathcal{D}_0 \cap A \neq \emptyset$

PROOF: Since A intersects every member of \mathcal{D} .

$\langle 1 \rangle 2.$ Q.E.D.

PROOF: By maximality of \mathcal{D} .

□

3.26 The Axiom of Choice

Axiom 3.74 (Axiom of Choice). *Let \mathcal{A} be a set of disjoint nonempty sets. Then there exists a set C consisting of exactly one element from each member of \mathcal{A} .*

3.27 Choice Functions

Definition 3.75 (Choice Function). Let \mathcal{B} be a set of nonempty sets. A *choice function* for \mathcal{B} is a function $c : \mathcal{B} \rightarrow \bigcup \mathcal{B}$ such that, for all $B \in \mathcal{B}$, we have $c(B) \in B$.

Lemma 3.76 (Existence of a Choice Function (AC)). *Every set of nonempty sets has a choice function.*

PROOF:

$\langle 1 \rangle 1.$ LET: \mathcal{B} be a set of nonempty sets.

- $\langle 1 \rangle 2$. For $B \in \mathcal{B}$,
 LET: $B' = \{B\} \times B$
 $\langle 1 \rangle 3$. $\{B' \mid B \in \mathcal{B}\}$ is a set of disjoint nonempty sets.
 $\langle 1 \rangle 4$. PICK a set c consisting of exactly one element from each B' for $B \in \mathcal{B}$.
 $\langle 1 \rangle 5$. c is a choice function for \mathcal{B} .
 \square

3.28 Transitive

Definition 3.77 (Transitive Set). A set A is *transitive* if and only if, whenever $x \in y \in A$ then $x \in A$.

Theorem 3.78 (Union, Power Set). *Let A be a set. Then the following are equivalent.*

1. A is transitive.
2. $\bigcup A \subseteq A$
3. For all $a \in A$ we have $a \subseteq A$
4. $A \subseteq \mathcal{P}A$

PROOF: From definitions. \square

3.29 Minimal Elements

Definition 3.79 (Minimal). Let R be a binary relation and A a set. An element $a \in A$ is *minimal* w.r.t. R iff there is no $x \in A$ such that xRa .

3.30 Well-Founded Relations

Definition 3.80 (Well-Founded). Let R be a relation on A . Then R is *well-founded* iff every nonempty subset of A has an R -minimal element.

Theorem 3.81 (Transfinite Induction). *Let R be a well-founded relation on A and $B \subseteq A$. Assume that, for every $t \in A$, if $\{x \in A \mid xRt\} \subseteq B$ then $t \in B$. Then we have $B = A$.*

Theorem 3.82 (Transfinite Recursion). *Let R be a well-founded relation on a set C .*

Let \mathbf{A} be a class. Let \mathbf{B} be the class of all functions from a subset of C to \mathbf{A} . Let $\mathbf{F} : \mathbf{B} \times C \rightarrow \mathbf{A}$ be a class term.

Then there exists a unique function $f : C \rightarrow \mathbf{A}$ such that, for all $t \in C$, we have $f(t) = \mathbf{F}(f \upharpoonright \{x \in C \mid xRt\}, t)$.

3.31 Transitive Closure

Theorem 3.83. *Let R be a relation. Then there exists a unique relation R^t such that R^t is transitive, $R \subseteq R^t$, and for every transitive relation S with $R \subseteq S$ we have $R^t \subseteq S$.*

Definition 3.84 (Transitive Closure). The *transitive closure* of a relation R is this relation R^t .

Theorem 3.85. *If R is well-founded then R^t is well-founded.*

Chapter 4

Cardinal Numbers

Definition 4.1 (Equinumerous). Two sets A and B are *equinumerous* if and only if there exists a bijection between them.

Theorem 4.2. *Equinumerosity is an equivalence relation on the class of all sets.*

Theorem 4.3 (Cantor). *No set is equinumerous with its power set.*

Definition 4.4. We say a set A is *dominated* by B , $A \preceq B$, iff A is equinumerous with a subset of B .

Theorem 4.5. $A \preceq A$

Theorem 4.6. *If $A \preceq B \preceq C$ then $A \preceq C$.*

Theorem 4.7 (Schröder-Bernstein Theorem). *If $A \preceq B$ and $B \preceq A$ then $A \equiv B$.*

PROOF:

$\langle 1 \rangle 1$. LET: $f : A \rightarrow B$ and $g : B \rightarrow A$ be injections.

$\langle 1 \rangle 2$. Define a sequence of sets $C_n \subseteq A$ by

$$C_0 = A - \text{ran } g$$

$$C_{n+1} = g(f(C_n))$$

$\langle 1 \rangle 3$. Define $h : A \rightarrow B$ by

$$h(x) = \begin{cases} f(x) & \text{if } x \in \bigcup_n C_n \\ g^{-1}(x) & \text{if not} \end{cases}$$

$\langle 1 \rangle 4$. h is a bijection.

□

Theorem 4.8 (AC). *For any infinite set A we have $\mathbb{N} \preceq A$.*

PROOF: Given a choice function f for A , choose a sequence (a_n) in A by $a_n = f(A - \{a_0, \dots, a_{n-1}\})$. □

Corollary 4.8.1 (AC). *A set is infinite if and only if it is equinumerous with a proper subset.*

4.1 Countability

Definition 4.9 (Countable). A set A is *countable* iff $A \preccurlyeq \mathbb{N}$.

Theorem 4.10 (AC). *A countable union of countable sets is countable.*

4.2 Order Theory

Definition 4.11 (Preorder). Let X be a set. A *preorder* on X is a binary relation \leq on X such that:

Reflexivity For all $x \in X$, we have $x \leq x$

Transitivity For all $x, y, z \in X$, if $x \leq y$ and $y \leq z$ then $x \leq z$.

Definition 4.12 (Preordered Set). A *preordered set* consists of a set X and a preorder \leq on X .

Proposition 4.13. Let X and Y be linearly ordered sets. Let $f : X \rightarrow Y$ be strictly monotone and surjective. Then f is a poset isomorphism.

PROOF:

$\langle 1 \rangle 1.$ f is injective.

$\langle 2 \rangle 1.$ LET: $x, y \in X$

$\langle 2 \rangle 2.$ ASSUME: $f(x) = f(y)$

$\langle 2 \rangle 3.$ $x \not\leq y$

PROOF: By strong monotonicity.

$\langle 2 \rangle 4.$ $y \not\leq x$

PROOF: By strong monotonicity.

$\langle 2 \rangle 5.$ $x = y$

PROOF: By trichotomy.

$\langle 1 \rangle 2.$ f^{-1} is monotone.

$\langle 2 \rangle 1.$ LET: $x, y \in X$

$\langle 2 \rangle 2.$ ASSUME: $x \leq y$

$\langle 2 \rangle 3.$ $f^{-1}(x) \not\leq f^{-1}(y)$

PROOF: By strong monotonicity.

$\langle 2 \rangle 4.$ $f^{-1}(x) < f^{-1}(y)$

PROOF: By trichotomy.

□

Definition 4.14 (Interval). Let X be a preordered set and $Y \subseteq X$. Then Y is an *interval* if and only if, for all $a, b \in Y$ and $c \in X$, if $a \leq c \leq b$ then $c \in Y$.

Definition 4.15 (Linear Continuum). A linearly ordered set L is a *linear continuum* if and only if:

1. every nonempty subset of L that is bounded above has a supremum
2. L is dense

Proposition 4.16. Every interval in a linear continuum is a linear continuum.

PROOF:

$\langle 1 \rangle 1.$ LET: L be a linear continuum and I an interval in L .

$\langle 1 \rangle 2.$ Every nonempty subset of I that is bounded above has a supremum in I .

$\langle 2 \rangle 1.$ LET: $X \subseteq I$ be nonempty and bounded above by $b \in I$.

$\langle 2 \rangle 2$. LET: s be the supremum of X in L .
 PROOF: Since L is a linear continuum.
 $\langle 2 \rangle 3$. $s \in I$
 $\langle 3 \rangle 1$. PICK $a \in X$
 PROOF: Since X is nonempty ($\langle 2 \rangle 1$).
 $\langle 3 \rangle 2$. $a \leq s \leq b$
 $\langle 3 \rangle 3$. $s \in I$
 PROOF: Since I is an interval ($\langle 1 \rangle 1$).
 $\langle 2 \rangle 4$. s is the supremum of X in I
 $\langle 1 \rangle 3$. I is dense.
 $\langle 2 \rangle 1$. LET: $x, y \in I$ with $x < y$
 $\langle 2 \rangle 2$. PICK $z \in I$ with $x < z < y$
 PROOF: Since L is dense.
 $\langle 2 \rangle 3$. $z \in I$
 PROOF: Since I is an interval.

□

Definition 4.17 (Ordered Square). The *ordered square* I_o^2 is the set $[0, 1]^2$ under the dictionary order.

Proposition 4.18. *The ordered square is a linear continuum.*

PROOF:

$\langle 1 \rangle 1$. Every nonempty subset of I_o^2 bounded above has a supremum.
 $\langle 2 \rangle 1$. LET: $X \subseteq I_o^2$ be nonempty and bounded above by (b, c)
 $\langle 2 \rangle 2$. LET: $s = \sup \pi_1(X)$
 PROOF: The set $\pi_1(X)$ is nonempty and bounded above by b .
 $\langle 2 \rangle 3$. CASE: $s \in \pi_1(X)$
 $\langle 3 \rangle 1$. LET: $t = \sup \{y \in [0, 1] \mid (s, y) \in X\}$
 PROOF: This set is nonempty and bounded above by c .
 $\langle 3 \rangle 2$. (s, t) is the supremum of X .
 $\langle 2 \rangle 4$. CASE: $s \notin \pi_1(X)$
 PROOF: In this case $(s, 0)$ is the supremum of X .
 $\langle 1 \rangle 2$. I_o^2 is dense.
 $\langle 2 \rangle 1$. LET: $(x_1, y_1), (x_2, y_2) \in I_o^2$ with $(x_1, y_1) < (x_2, y_2)$
 $\langle 2 \rangle 2$. CASE: $x_1 < x_2$
 $\langle 3 \rangle 1$. PICK x_3 with $x_1 < x_3 < x_2$
 $\langle 3 \rangle 2$. $(x_1, y_1) < (x_3, y_1) < (x_2, y_2)$
 $\langle 2 \rangle 3$. CASE: $x_1 = x_2$ and $y_1 < y_2$
 $\langle 3 \rangle 1$. PICK y_3 with $y_1 < y_3 < y_2$
 $\langle 3 \rangle 2$. $(x_1, y_1) < (x_1, y_3) < (x_2, y_2)$

□

Proposition 4.19. *If X is a well-ordered set then $X \times [0, 1)$ under the dictionary order is a linear continuum.*

PROOF:

$\langle 1 \rangle 1$. Every nonempty set $A \subseteq X \times [0, 1)$ bounded above has a supremum.

- ⟨2⟩1. LET: $A \subseteq X \times [0, 1)$ be nonempty and bounded above
- ⟨2⟩2. LET: x_0 be the supremum of $\pi_1(A)$
- ⟨2⟩3. CASE: $x_0 \in \pi_1(A)$
 - ⟨3⟩1. LET: y_0 be the supremum of $\{y \in [0, 1) \mid (x_0, y) \in A\}$
 - ⟨3⟩2. (x_0, y_0) is the supremum of A .
- ⟨2⟩4. CASE: $x_0 \notin \pi_1(A)$
 - PROOF: In this case $(x_0, 0)$ is the supremum of A .
- ⟨1⟩2. $X \times [0, 1)$ is dense.
 - ⟨2⟩1. LET: $(x_1, y_1), (x_2, y_2) \in X \times [0, 1)$ with $(x_1, y_1) < (x_2, y_2)$
 - ⟨2⟩2. CASE: $x_1 < x_2$
 - ⟨3⟩1. PICK y_3 such that $y_1 < y_3 < 1$
 - ⟨3⟩2. $(x_1, y_1) < (x_1, y_3) < (x_2, y_2)$
 - ⟨2⟩3. CASE: $x_1 = x_2$ and $y_1 < y_2$
 - ⟨3⟩1. PICK y_3 such that $y_1 < y_3 < y_2$
 - ⟨3⟩2. $(x_1, y_1) < (x_1, y_3) < (x_2, y_2)$

□

Lemma 4.20. For all $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$, we have $[a, b) \cong [c, d)$

PROOF: The map $\lambda t.c + (t - a)(d - c)/(b - a)$ is an order isomorphism.

Proposition 4.21. Let X be a linearly ordered set. Let $a < b < c$ in X . Then $[a, c) \cong [0, 1)$ if and only if $[a, b) \cong [b, c) \cong [0, 1)$.

PROOF:

- ⟨1⟩1. If $[a, c) \cong [0, 1)$ then $[a, b) \cong [b, c) \cong [0, 1)$
- ⟨2⟩1. ASSUME: $f : [a, c) \cong [0, 1)$ is an order isomorphism
- ⟨2⟩2. $[a, b) \cong [0, 1)$
 - PROOF:
$$\begin{aligned} [a, b) &\cong [0, f(b)) && \text{(by the restriction of } f) \\ &\cong [0, 1) && \text{(Lemma 4.20)} \end{aligned}$$
- ⟨2⟩3. $[b, c) \cong [0, 1)$
 - PROOF: Similar.
- ⟨1⟩2. If $[a, b) \cong [b, c) \cong [0, 1)$ then $[a, c) \cong [0, 1)$
 - PROOF:
$$\begin{aligned} [a, c) &= [a, b) * [b, c) \\ &\cong [0, 1) * [0, 1) \\ &\cong [0, 1/2) * [1/2, 1) && \text{(Lemma 4.20)} \\ &= 1 \end{aligned}$$

□

Proposition 4.22 (CC). Let X be a linearly ordered set. Let $x_0 < x_1 < \dots$ be a strictly increasing sequence in X with supremum b . Then $[x_0, b) \cong [0, 1)$ if and only if $[x_i, x_{i+1}) \cong [0, 1)$ for all i .

PROOF:

- ⟨1⟩1. If $[x_0, b) \cong [0, 1)$ then $[x_i, x_{i+1}) \cong [0, 1)$ for all i .

PROOF: By Lemma 4.20

⟨1⟩2. If $[x_i, x_{i+1}] \cong [0, 1]$ for all i then $[x_0, b] \cong [0, 1]$

⟨2⟩1. ASSUME: $[x_i, x_{i+1}] \cong [0, 1]$ for all i

⟨2⟩2. PICK an order isomorphism $f_i : [x_i, x_{i+1}] \cong [1/2^i, 2/2^{i+1}]$ for each i .

PROOF: By Lemma 4.20

⟨2⟩3. The union of the f_i s is an order isomorphism $[x_0, b] \cong [0, 1]$

□

4.3 Partially Ordered Sets

Definition 4.23 (Partial Order). A *partial order* on a set X is a preorder \leq that is *anti-symmetric*, i.e. whenever $x \leq y$ and $y \leq x$ then $x = y$.

4.4 Strict Partial Order

Definition 4.24 (Strict Partial Order). A *strict partial order* on a set X is a relation on X that is transitive and irreflexive.

Proposition 4.25. If $<$ is a strict partial order on X and $x, y \in X$, then at most one of $x < y$, $y < x$, $x = y$ holds.

Proposition 4.26. If $<$ is a strict partial order then the relation \leq defined by: $x \leq y$ iff $x < y$ or $x = y$, is a partial order.

Theorem 4.27. If R is a well-founded relation then its transitive closure is a partial order.

Definition 4.28 (Linear Order). A *linear order* on a set X is a partial order such that, for any $x, y \in X$, either $x \leq y$ or $y \leq x$.

4.5 Strict Linear Orders

Definition 4.29 (Strict Linear Order (Extensionality, Pairing)). Let A be a set. A *strict linear order* on A is a binary relation R on A that is transitive and satisfies *trichotomy*: for any $x, y \in A$, exactly one of xRy , $x = y$, yRx holds.

Theorem 4.30. Let R be a strict linear order on A . Then there is no $x \in A$ such that xRx .

PROOF: Immediate from trichotomy.

4.6 Well Orderings

Definition 4.31 (Well-ordering). A *well-order* on a set X is a linear order such that every nonempty set has a least element.

Proposition 4.32. *Let \leq be a linear order on X . Then \leq is a well-order iff there is no function $f : \mathbb{N} \rightarrow X$ such that $f(n+1) < f(n)$ for all n .*

Definition 4.33 (Initial Segment). Given a well-ordered set X and $\alpha \in X$, the initial segment of X up to α is $\text{seg } \alpha = \{x \in X \mid x < \alpha\}$.

Theorem 4.34 (Transfinite Induction). *Let \leq be a linear order on J . Then the following are equivalent:*

1. \leq is a well-order on J .
2. For every subset $J_0 \subseteq J$, if the following condition holds:
 - For every $\alpha \in J$, if $\text{seg } \alpha \subseteq J_0$ then $\alpha \in J_0$.
then $J_0 = J$.

Axiom Schema 4.35 (Replacement). *Let \mathbf{H} be a class term. If $\text{dom } \mathbf{H}$ is a set then \mathbf{H} is a set.*

Theorem 4.36 (Transfinite Recursion). *Let J be a well-ordered set and C a set. Let \mathcal{F} be the set of all functions from a section of J to C . Let G be a function with domain \mathcal{F} . Then there exists a unique function h with domain J such that, for all $\alpha \in J$, we have $h(\alpha) = \rho(h \upharpoonright \text{seg } \alpha)$.*

PROOF:

- <1>1. If v is a function and $t \in J$, we say v is ρ -constructed up to t iff $\text{dom } v = \{x \in J \mid x \leq t\}$ and, for all $x \in \text{dom } v$, we have $v(x) = \rho(v \upharpoonright \text{seg } x)$
- <1>2. If $t_1 \leq t_2$, v_1 is ρ -constructed up to t_1 , and v_2 is ρ -constructed up to t_2 , then $v_1 = v_2 \upharpoonright \{x \in J \mid x \leq t_1\}$
- <1>3. LET: \mathcal{K} be the set of all functions that are ρ -constructed up to some $t \in J$
PROOF: \mathcal{K} is a set by a Replacement Axiom.
- <1>4. LET: $F = \bigcup \mathcal{K}$
- <1>5. F is a function
- <1>6. For all $x \in \text{dom } F$ we have $F(x) = \rho(F \upharpoonright \text{seg } x)$
- <1>7. $\text{dom } F = J$
- <1>8. F is unique

□

Theorem 4.37. *The following are equivalent.*

1. The Axiom of Choice
2. (Well-Ordering Theorem) Every set has a well-ordering.
3. (Zorn's Lemma) Let X be a poset. If every chain in X has an upper bound in X , then X has a maximal element.

PROOF:

- <1>1. $1 \Rightarrow 2$

PROOF:

- ⟨2⟩1. ASSUME: The Axiom of Choice
- ⟨2⟩2. LET: X be a set.
- ⟨2⟩3. PICK a choice function for $\mathcal{P}X \setminus \{\emptyset\}$
PROOF: Lemma 3.76.
- ⟨2⟩4. LET: a *tower* in X be a pair $(T, <)$ where $T \subseteq X$, $<$ is a well-ordering of T , and $x = c(X \setminus \{y \in T \mid y < x\})$.
- ⟨2⟩5. For any two towers $(T_1, <_1)$ and $(T_2, <_2)$, either these two posets are equal or one is a section of the other.
- ⟨3⟩1.
- ⟨2⟩6. For any tower $(T, <)$ in X with $T \neq X$, there exists a tower in X of which $(T, <)$ is a section.
- ⟨2⟩7. LET: $T = \bigcup \{T' \subseteq X \mid \exists R. (T, R) \text{ is a tower in } X\}$
- ⟨2⟩8. Define $<$ on T by: $x < y$ iff there exists a tower (T, R) in X such that $x, y \in T$ and xRy .
- ⟨2⟩9. $(T, <)$ is a tower in X .
- ⟨2⟩10. $T = X$
- ⟨2⟩11. $<$ is a well-ordering of X .
- ⟨1⟩2. $2 \Rightarrow 3$
- ⟨2⟩1. ASSUME: The Well-Ordering Theorem
- ⟨2⟩2. LET: X be a poset in which every chain has an upper bound.
- ⟨2⟩3. PICK a well-ordering R of X
- ⟨2⟩4. Define $F : X \rightarrow \{0, 1\}$ by transfinite R -recursion by:
$$F(a) = \begin{cases} 1 & \text{if } b < a \text{ for all } b \text{ such that } bRa \text{ and } f(b) = 1 \\ 0 & \text{otherwise} \end{cases}$$
- ⟨2⟩5. LET: $C = \{a \in X \mid f(a) = 1\}$
- ⟨2⟩6. C is a chain in X
 - ⟨3⟩1. LET: $x, y \in C$
 - ⟨3⟩2. ASSUME: without loss of generality xRy
 - ⟨3⟩3. $f(y) = 1$
 - ⟨3⟩4. for all z such that zRy and $f(z) = 1$ we have $z < y$
 - ⟨3⟩5. $x < y$
- ⟨2⟩7. PICK an upper bound u for C
- ⟨2⟩8. u is maximal in X
 - ⟨3⟩1. LET: $x \in X$ with $u \leq x$
 - ⟨3⟩2. for all b such that bRx and $f(b) = 1$ we have $b < x$
PROOF: Since $b \in C$ so $b \leq u \leq x$
 - ⟨3⟩3. $f(u) = 1$
 - ⟨3⟩4. $u \leq x$
 - ⟨3⟩5. $u = x$
- ⟨2⟩9. $3 \Rightarrow 1$
 - ⟨3⟩1. ASSUME: Zorn's Lemma
 - ⟨3⟩2. LET: R be a relation
 - ⟨3⟩3. LET: \mathcal{A} be the poset of functions that are subsets of R under \subseteq
 - ⟨3⟩4. Every chain in \mathcal{A} has an upper bound
 - ⟨4⟩1. LET: $\mathcal{C} \subseteq \mathcal{A}$ be a chain.

PROVE: $\bigcup \mathcal{C} \in \mathcal{A}$
 ⟨4⟩2. ASSUME: $(x, y), (x, z) \in \bigcup \mathcal{C}$
 ⟨4⟩3. PICK $f, g \in \mathcal{C}$ such that $f(x) = y$ and $g(x) = z$
 ⟨4⟩4. ASSUME: without loss of generality $f \subseteq g$
 ⟨4⟩5. $g(x) = y$
 ⟨4⟩6. $y = z$
 ⟨3⟩5. PICK F maximal in \mathcal{A}
 ⟨3⟩6. $\text{dom } F = \text{dom } R$
 ⟨4⟩1. ASSUME: for a contradiction $x \in \text{dom } R - \text{dom } F$
 ⟨4⟩2. PICK y such that xRy
 ⟨4⟩3. LET: $G = F \cup \{(x, y)\}$
 ⟨4⟩4. $G \in \mathcal{A}$
 ⟨4⟩5. $F \subset G$
 ⟨4⟩6. Q.E.D.
 PROOF: This contradicts the maximality of F .

□

Theorem 4.38 (Cardinal Comparability). *The Axiom of Choice is equivalent to the Cardinal Comparability Theorem: for any two sets A and B , either $A \preccurlyeq B$ or $B \preccurlyeq A$.*

PROOF:

⟨1⟩1. Zorn's Lemma implies Cardinal Comparability
 ⟨2⟩1. ASSUME: Zorn's Lemma
 ⟨2⟩2. LET: A and B be sets.
 ⟨2⟩3. LET: \mathcal{A} be the poset of all injective functions f such that $\text{dom } f \subseteq C$ and $\text{ran } f \subseteq D$ under \subseteq
 ⟨2⟩4. Every chain in \mathcal{A} has an upper bound.
 ⟨3⟩1. LET: $\mathcal{C} \subseteq \mathcal{A}$ be a chain.
 PROVE: $\bigcup \mathcal{C} \in \mathcal{A}$
 ⟨3⟩2. $\bigcup \mathcal{C}$ is a function.
 ⟨4⟩1. LET: $(x, y), (x, z) \in \bigcup \mathcal{C}$
 ⟨4⟩2. PICK $f, g \in \mathcal{C}$ such that $f(x) = y$ and $g(x) = z$
 ⟨4⟩3. ASSUME: without loss of generality $f \subseteq g$
 ⟨4⟩4. $g(x) = y$
 ⟨4⟩5. $y = z$
 ⟨3⟩3. $\bigcup \mathcal{C}$ is injective.
 PROOF: Similar.
 ⟨2⟩5. PICK \hat{f} maximal in \mathcal{A}
 PROOF: By Zorn's Lemma.
 ⟨2⟩6. Either $\text{dom } \hat{f} = C$ or $\text{ran } \hat{f} = D$
 ⟨3⟩1. ASSUME: for a contradiction $\text{dom } \hat{f} \subset C$ and $\text{ran } \hat{f} \subset D$
 ⟨3⟩2. PICK $x \in C - \text{dom } \hat{f}$ and $y \in D - \text{ran } \hat{f}$
 ⟨3⟩3. LET: $g = \hat{f} \cup \{(x, y)\}$
 ⟨3⟩4. $g \in \mathcal{A}$
 ⟨3⟩5. $\hat{f} \subset g$

⟨3⟩6. Q.E.D.

PROOF: This contradicts the maximality of \hat{f} .

⟨2⟩7. If $\text{dom } \hat{f} = C$ then $C \preccurlyeq D$

⟨2⟩8. If $\text{ran } \hat{f} = D$ then $D \preccurlyeq C$

⟨1⟩2. Cardinal Comparability implies the Well-Ordering Theorem

⟨2⟩1. ASSUME: Cardinal Comparability

⟨2⟩2. LET: A be a set

⟨2⟩3. PICK an ordinal α such that $\alpha \not\preccurlyeq A$

⟨2⟩4. $A \preccurlyeq \alpha$

PROOF: By Cardinal Comparability.

⟨2⟩5. PICK an injection $f : A \rightarrow \alpha$

⟨2⟩6. Define $<$ on A by $x < y$ iff $f(x) \in f(y)$

⟨2⟩7. $<$ is a well-ordering on A .

□

Theorem 4.39. *Given two well-ordered sets A and B , either $A \cong B$ or one of A, B is isomorphic to an initial segment of the other.*

4.7 Ordinal Numbers

Definition 4.40. Let (A, \leq) be a well-ordered set. The *ordinal number* of (A, \leq) is the range of E , where E is the unique function with domain A such that $E(t) = \text{ran}(E \upharpoonright \text{seg } t)$ for all $t \in A$.

Theorem 4.41. *Let (A, \leq) be a well-ordered set and $E : A \rightarrow \alpha$ be the canonical function onto the ordinal of A . Then:*

1. *For all $t \in A$ we have $E(t) \notin E(t)$.*
2. *E is a bijection.*
3. *For any $s, t \in A$, we have $s < t$ if and only if $E(s) \in E(t)$.*
4. *α is a transitive set.*
5. *α is well-ordered by \in*
6. *E is an order isomorphism between (A, \leq) and (α, \subseteq) .*

Theorem 4.42. *Two well-ordered sets are isomorphic if and only if they have the same ordinal number.*

Theorem 4.43. *A set is an ordinal number if and only if it is a transitive set well-ordered by \in .*

Theorem 4.44. *Every member of an ordinal number is an ordinal number.*

Theorem 4.45. *Any transitive set of ordinal numbers is an ordinal number.*

Theorem 4.46. *The empty set is an ordinal number.*

Theorem 4.47. *The successor of an ordinal number is an ordinal number.*

Theorem 4.48. *If A is a set of ordinal numbers then $\bigcup A$ is an ordinal number.*

Theorem 4.49. *Any nonempty set of ordinal numbers has a least element.*

Theorem 4.50 (Burali-Forti Paradox). *The class of ordinal numbers is a proper class.*

Theorem 4.51 (Hartogs' Theorem). *For any set A , there exists an ordinal that is not dominated by A .*

PROOF:

$\langle 1 \rangle 1$. LET: α be the class of all ordinals β such that $\beta \preccurlyeq A$

$\langle 1 \rangle 2$. α is a set.

$\langle 2 \rangle 1$. LET: W be the set of all pairs (B, \leq) such that $B \subseteq A$ and \leq is a well-ordering on B .

$\langle 2 \rangle 2$. Every member of α is the ordinal number of a member of W

$\langle 2 \rangle 3$. Q.E.D.

PROOF: By a Replacement Axiom.

$\langle 1 \rangle 3$. α is an ordinal.

$\langle 1 \rangle 4$. α is not dominated by A .

□

Definition 4.52. A class term $\mathbf{F} : \mathbf{On} \rightarrow \mathbf{On}$ is *continuous* iff, for every limit ordinal λ , we have $\mathbf{F}(\lambda) = \sup_{\alpha < \lambda} \mathbf{F}(\alpha)$.

Theorem 4.53. *Let $\mathbf{F} : \mathbf{On} \rightarrow \mathbf{On}$. If \mathbf{F} is continuous and $\mathbf{F}(\alpha) < \mathbf{F}(\alpha + 1)$ for every ordinal α , then \mathbf{F} is strictly monotone.*

Definition 4.54. A class term $\mathbf{F} : \mathbf{On} \rightarrow \mathbf{On}$ is *normal* iff it is strictly monotone and continuous.

Theorem 4.55. *Let $\mathbf{F} : \mathbf{On} \rightarrow \mathbf{On}$ be normal. For every ordinal $\beta \geq \mathbf{F}(0)$, there exists a greatest ordinal α such that $\mathbf{F}(\alpha) \leq \beta$.*

Theorem 4.56. *Let $\mathbf{F} : \mathbf{On} \rightarrow \mathbf{On}$ be normal. Let S be a set of ordinals. Then $\mathbf{F}(\sup S) = \sup_{\alpha \in S} \mathbf{F}(\alpha)$.*

Theorem 4.57 (Veblen Fixed-Point Theorem). *Let $\mathbf{F} : \mathbf{On} \rightarrow \mathbf{On}$ be normal. For every ordinal α , there exists $\beta \geq \alpha$ such that $\mathbf{F}(\beta) = \beta$.*

PROOF: Let β be the supremum of $\alpha, \mathbf{F}(\alpha), \mathbf{F}^2(\alpha), \dots$ □

Lemma 4.58. *Let α be an ordinal. Let $(f(\gamma))_{\gamma < \alpha}$ be an α -sequence of ordinals. Then there exists $\beta \leq \alpha$ and an increasing sequence of ordinals $(g(\gamma))_{\gamma < \beta}$ such that $\sup_{\gamma < \alpha} f(\gamma) = \sup_{\gamma < \beta} g(\gamma)$.*

4.8 Cardinal Numbers

Definition 4.59 (Cardinal Number (AC)). For any set A , the *cardinal number* of A , $\text{card } A$, is the least ordinal equinumerous with A .

There exists some ordinal equinumerous with A by the Well-Ordering Theorem.

Theorem 4.60. For any sets A and B , we have $A \equiv B$ if and only if $\text{card } A = \text{card } B$.

Theorem 4.61. A set A is finite if and only if $\text{card } A$ is a natural number.

Theorem 4.62. The supremum of a set of cardinal numbers is a cardinal number.

4.9 Cardinal Arithmetic

Definition 4.63. For cardinal numbers κ and λ , the *sum* $\kappa + \lambda$ is the cardinal number of $A \cup B$, where A and B are disjoint sets of cardinality κ and λ respectively.

Theorem 4.64. $\kappa + \lambda = \lambda + \kappa$

Theorem 4.65. $\kappa + (\lambda + \mu) = (\kappa + \lambda) + \mu$

Theorem 4.66. The definition of addition agrees with the definition on natural numbers.

Definition 4.67. For cardinal numbers κ and λ , the *product* $\kappa\lambda$ is the cardinality of $\kappa \times \lambda$.

Theorem 4.68. $\kappa\lambda = \lambda\kappa$

Theorem 4.69. $\kappa(\lambda\mu) = (\kappa\lambda)\mu$

Theorem 4.70. $\kappa(\lambda + \mu) = \kappa\lambda + \kappa\mu$

Theorem 4.71. The definition of multiplication agrees with the definition on natural numbers.

Theorem 4.72 (AC). For any infinite cardinal κ we have $\kappa\kappa = \kappa$.

PROOF:

- $\langle 1 \rangle 1$. LET: B be a set with cardinality κ
- $\langle 1 \rangle 2$. LET: $\mathcal{H} = \{\emptyset\} \cup \{f \mid \exists A \subseteq B. A \text{ is infinite and } f \text{ is a bijection between } A \times A \text{ and } A\}$
- $\langle 1 \rangle 3$. For every chain $\mathcal{C} \subseteq \mathcal{H}$ we have $\bigcup \mathcal{C} \in \mathcal{H}$
- $\langle 1 \rangle 4$. PICK a maximal f_0 in \mathcal{H}
- $\langle 1 \rangle 5$. $f_0 \neq \emptyset$

PROOF: B has a subset of cardinality \aleph_0 and $\aleph_0\aleph_0 = \aleph_0$.

- $\langle 1 \rangle 6$. LET: A_0 be the set such that f_0 is a bijection between $A_0 \times A_0$ and A_0

$\langle 1 \rangle 7$. LET: $\lambda = \text{card } A_0$

$\langle 1 \rangle 8$. $\text{card}(B - A_0) < \lambda$

$\langle 1 \rangle 9$. $\kappa = \lambda$

PROOF:

$$\begin{aligned}
 \kappa &= \text{card } A_0 + \text{card}(B - A_0) \\
 &\leq \lambda + \lambda \\
 &= 2\lambda \\
 &\leq \lambda\lambda \\
 &= \lambda && (\langle 1 \rangle 6) \\
 &\leq \kappa && \square
 \end{aligned}$$

Theorem 4.73 (Absorption Law). *Let κ and λ be cardinal numbers such that $0 < \kappa \leq \lambda$ and λ is infinite. Then*

$$\kappa + \lambda = \lambda .$$

Theorem 4.74 (Absorption Law). *Let κ and λ be cardinal numbers such that $0 < \kappa \leq \lambda$ and λ is infinite. Then*

$$\kappa\lambda = \lambda .$$

Definition 4.75. For cardinal numbers κ and λ , we write κ^λ for the cardinality of the set of functions from λ to κ .

Theorem 4.76. $\kappa^{\lambda+\mu} = \kappa^\lambda + \kappa^\mu$

Theorem 4.77. $(\kappa\lambda)^\mu = \kappa^\mu \lambda^\mu$

Theorem 4.78. $(\kappa^\lambda)^\mu = \kappa^{\lambda\mu}$

Theorem 4.79. *The definition of exponentiation agrees with the definition on natural numbers.*

Theorem 4.80. *Given sets A and B , we have $\text{card } A \leq \text{card } B$ if and only if $A \preccurlyeq B$.*

Definition 4.81. Let $\aleph_0 = \text{card } \mathbb{N}$.

Theorem 4.82 (AC). *For any infinite cardinal κ we have $\aleph_0 \leq \kappa$.*

Theorem 4.83 (Maximum Principle (AC)). *Every poset has a maximal chain.*

4.10 Rank of a Set

Definition 4.84 (Cumulative Hierarchy of Sets). For every ordinal α , define the rank V_α by transfinite recursion thus:

$$\begin{aligned}
 V_0 &= \emptyset \\
 V_{\alpha+1} &= \mathcal{P}V_\alpha \\
 V_\lambda &= \bigcup_{\alpha < \lambda} V_\alpha
 \end{aligned}$$

for λ a limit ordinal.

The *von Neumann universe* is the class $\mathbf{V} = \bigcup_{\alpha \in \mathbf{On}} V_\alpha$.

Theorem 4.85. *If λ is a limit ordinal and $\lambda > \omega$ then V_λ is a model of Zermelo set theory.*

Lemma 4.86 (AC). *There exists a well-ordered set in V_{ω_2} whose ordinal is not in V_{ω_2} .*

PROOF: Pick a well-ordering $<$ of \mathcal{PN} . Then $(\mathcal{PN}, <) \in V_{\omega_2}$ but its ordinal is not because its ordinal is uncountable. \square

Theorem 4.87. *The set V_{ω_2} is not a model of Zermelo-Fraenkel set theory.*

Thus, the Replacement Axioms cannot be proven from the other axioms.

Definition 4.88 (Well-Founded Set). A set A is *well-founded* iff $A \in V_\alpha$ for some $\alpha \in \mathbf{On}$.

Definition 4.89 (Rank). The *rank* of a well-founded set A , $\text{rank } A$, is the least ordinal α such that $A \in V_\alpha$.

Theorem 4.90. *If $A \in B$ and B is well-founded then A is well-founded and $\text{rank } A < \text{rank } B$.*

Theorem 4.91. *If A is a set and every member of A is well-founded then A is well-founded and $\text{rank } A = \sup_{B \in A} (\text{rank } B + 1)$.*

Theorem 4.92. *The Axiom of Regularity is equivalent to the statement that every set is well-founded.*

4.11 Transfinite Recursion Again

Theorem 4.93. *Let \mathbf{A} be a class. Let \mathbf{B} be the class of all functions $f : \alpha \rightarrow \mathbf{A}$ for some ordinal α . Let $\mathbf{F} : \mathbf{B} \rightarrow \mathbf{A}$ be a class term. Then there exists a unique class term $\mathbf{G} : \mathbf{On} \rightarrow \mathbf{A}$ such that, for all $\alpha \in \mathbf{On}$, we have $\mathbf{G}(\alpha) = \mathbf{F}(\mathbf{G} \upharpoonright \alpha)$.*

4.12 Alephs

Definition 4.94. Define the cardinal number \aleph_α for every ordinal α by transfinite recursion on α thus: \aleph_α is the least infinite cardinal different from \aleph_β for all $\beta < \alpha$.

Theorem 4.95. *If $\alpha < \beta$ then $\aleph_\alpha < \aleph_\beta$.*

Theorem 4.96. *Every infinite cardinal has the form \aleph_α for some ordinal α .*

4.13 Ordinal Arithmetic

Definition 4.97 (Sum). Let α and β be ordinals. The *sum* $\alpha + \beta$ is the ordinal of the concatenation of A followed by B , where A is a well-ordered set of ordinal α and B a well-ordered set of ordinal β .

Theorem 4.98. *Addition is associative.*

Theorem 4.99. $\alpha + 0 = \alpha$

Theorem 4.100. $0 + \alpha = \alpha$

Theorem 4.101. *For λ a limit ordinal we have $\alpha + \lambda = \sup_{\beta < \lambda} (\alpha + \beta)$*

Theorem 4.102. *For any α , the class term that maps β to $\alpha + \beta$ is normal.*

Theorem 4.103. $\beta < \gamma$ iff $\alpha + \beta < \alpha + \gamma$.

Theorem 4.104. *If $\beta \leq \gamma$ then $\beta + \alpha \leq \gamma + \alpha$.*

Theorem 4.105 (Subtraction Theorem). *If $\alpha < \beta$ then there exists a unique δ such that $\alpha + \delta < \beta$.*

Definition 4.106 (Product). Let α and β be ordinals. The *sum* $\alpha \cdot \beta$ is the ordinal of $A \times B$ ordered under the Hebrew lexicographic order, where A is a well-ordered set of ordinal α and B a well-ordered set of ordinal β .

Theorem 4.107. *Multiplication is associative.*

Theorem 4.108. *Multiplication distributes over addition on the left.*

Theorem 4.109. $\alpha \cdot 1 = \alpha$

Theorem 4.110. $1 \cdot \alpha = \alpha$

Theorem 4.111. $\alpha \cdot 0 = 0$

Theorem 4.112. $0 \cdot \alpha = 0$

Theorem 4.113. *For λ a limit ordinal, we have $\alpha \cdot \lambda = \sup_{\beta < \lambda} (\alpha \cdot \beta)$.*

Theorem 4.114. *For $\alpha > 0$, the class term that maps β to $\alpha \cdot \beta$ is normal.*

Theorem 4.115. *If $\alpha > 0$, then $\beta < \gamma$ iff $\alpha \cdot \beta < \alpha \cdot \gamma$.*

Theorem 4.116. *If $\beta \leq \gamma$ then $\beta \cdot \alpha \leq \gamma \cdot \alpha$.*

Theorem 4.117 (Division Theorem). *For any ordinals α and δ with $\delta \neq 0$, there exist unique ordinals β and γ with $\gamma < \delta$ and $\alpha = \delta \cdot \beta + \gamma$.*

Definition 4.118 (Exponentiation). For ordinals α and β , define the ordinal α^β by transfinite recursion on β by:

$$\begin{aligned}\alpha^0 &= 1 \\ \alpha^{\beta+1} &= \alpha^\beta + \alpha \\ \alpha^\lambda &= \sup_{\beta < \lambda} \alpha^\beta\end{aligned}$$

for λ a limit ordinal.

Theorem 4.119. For $\alpha > 1$, the class term that maps β to α^β is normal.

Theorem 4.120. If $\alpha > 1$, then $\beta < \gamma$ iff $\alpha^\beta < \alpha^\gamma$.

Theorem 4.121. If $\beta \leq \gamma$ then $\beta^\alpha \leq \gamma^\alpha$.

Theorem 4.122 (Logarithm Theorem). Let α and β be ordinals with $\alpha \neq 0$ and $\beta > 1$. Then there exist unique ordinals γ , δ and ρ such that $\delta \neq 0$, $\delta < \beta$, $\rho < \beta^\gamma$, and $\alpha = \beta^\gamma \delta + \rho$.

Theorem 4.123.

$$\alpha^{\beta+\gamma} = \alpha^\beta \alpha^\gamma$$

Theorem 4.124.

$$(\alpha^\beta)^\gamma = \alpha^{\beta\gamma}$$

4.14 Beth Cardinals

Definition 4.125. Define the cardinal \beth_α for every ordinal α by:

$$\begin{aligned}\beth_0 &= \aleph_0 \\ \beth_{\alpha+1} &= 2^{\beth_\alpha} \\ \beth_\lambda &= \sup_{\alpha < \lambda} \beth_\alpha\end{aligned}$$

for λ a limit ordinal.

Lemma 4.126. For any ordinal α we have $\text{card } V_{\omega+\alpha} = \beth_\alpha$.

4.15 Cofinality

Definition 4.127 (Cofinality). For λ a limit ordinal, the *cofinality* of λ , $\text{cf } \lambda$, is the least cardinal κ such that λ is the supremum of a set of κ smaller ordinals.

We extend cf to all the ordinals by setting $\text{cf } 0 = 0$ and $\text{cf}(\alpha + 1) = 1$.

Theorem 4.128. For any limit ordinal λ we have $\text{cf } \aleph_\lambda = \text{cf } \lambda$.

Lemma 4.129. Let λ be a limit ordinal. Then $\text{cf } \lambda$ is the least ordinal α such that there exists an increasing α -sequence of ordinals with limit λ .

Theorem 4.130. Let λ be an infinite cardinal. Then $\text{cf } \lambda$ is the least cardinal number κ such that λ can be partitioned into κ sets each of cardinality $< \lambda$.

Theorem 4.131 (König's Theorem). Let κ be an infinite cardinal. Then $\kappa < 2^{\text{cf } \kappa}$.

Corollary 4.131.1. $2^{\aleph_0} \neq \aleph_\omega$.

Definition 4.132 (Regular). A cardinal κ is *regular* iff $\text{cf } \kappa = \kappa$.

Theorem 4.133. For any ordinal λ , we have $\text{cf } \lambda$ is a regular cardinal.

Definition 4.134 (Singular). A cardinal κ is *singular* iff $\text{cf } \kappa < \kappa$.

Theorem 4.135. For any ordinal α we have $\aleph_{\alpha+1}$ is a regular cardinal.

4.16 Inaccessible Cardinals

Definition 4.136 (Inaccessible). A cardinal number κ is *inaccessible* iff:

- $\kappa > \aleph_0$
- For every cardinal $\lambda < \kappa$ we have $2^\lambda < \kappa$
- κ is regular.

Lemma 4.137. *If κ is inaccessible and $\alpha < \kappa$ then $\beth_\alpha < \kappa$.*

Lemma 4.138. *If κ is inaccessible and $A \in V_\kappa$ then $\text{card } A < \kappa$.*

Theorem 4.139. *If κ is inaccessible then V_κ is a model of ZF.*

Chapter 5

Natural Numbers

5.1 Successors

Definition 5.1 (Successor (Pairing, Union)). For any set a , its *Successor* a^+ is the set $a \cup \{a\}$

Theorem 5.2 (Pairing, Union). *If a is a transitive set then $\bigcup(a^+) = a$.*

PROOF:

$$\begin{aligned}\bigcup(a^+) &= \bigcup(a \cup \{a\}) \\ &= \bigcup a \cup \bigcup \{a\} \\ &= \bigcup a \cup a \\ &= a\end{aligned}\quad (\bigcup a \subseteq a) \square$$

Theorem 5.3. *If A is a transitive set then A^+ is transitive.*

PROOF: If A is transitive then $\bigcup(A^+) = A \subseteq A^+$. \square

5.2 Inductive Sets

Definition 5.4 (Inductive (Extensionality, Empty Set, Pairing, Union)). A set A is *inductive* iff $\emptyset \in A$ and, for every $a \in A$, we have $a^+ \in A$.

Axiom 5.5 (Axiom of Infinity (Extensionality, Empty Set, Pairing, Union)). *There exists an inductive set.*

5.3 Natural Numbers

Definition 5.6 (Natural Number (Extensionality, Empty Set, Pairing, Union)). A *natural number* is a set that belongs to every inductive set.

We write \mathbb{N} for the class of all natural numbers.

Theorem 5.7 (Extensionality, Empty Set, Pairing, Union, Infinity, Subset).
The class of natural numbers is a set.

PROOF:

⟨1⟩1. PICK an inductive set I .

PROOF: By the Axiom of Infinity.

⟨1⟩2. $\mathbb{N} \subseteq I$

⟨1⟩3. Q.E.D.

PROOF: By a Subset Axiom.

□

Theorem 5.8 (Extensionality, Empty Set, Pairing, Union, Infinity, Subset).
The set \mathbb{N} is inductive.

PROOF:

⟨1⟩1. $\emptyset \in \mathbb{N}$

PROOF: Since \emptyset is a member of every inductive set.

⟨1⟩2. For all $n \in \mathbb{N}$ we have $n^+ \in \mathbb{N}$

PROOF: If n is a member of every inductive set then so is n^+ .

□

Theorem 5.9 (Extensionality, Empty Set, Pairing, Union, Infinity, Subset).
The set \mathbb{N} is a subset of every inductive set.

PROOF: Immediate from definition. □

Corollary 5.9.1 (Proof by Induction (Extensionality, Empty Set, Pairing, Union, Infinity, Subset)). *If $A \subseteq \mathbb{N}$ and A is inductive then $A = \mathbb{N}$.*

Definition 5.10 (Zero (Empty Set)). The natural number *zero*, 0 , is defined to be \emptyset .

Theorem 5.11 (Extensionality, Empty Set, Pairing, Union, Infinity, Subset).
Every natural number except 0 is a successor of a natural number.

PROOF: The set $\{x \in \mathbb{N} \mid x = 0 \vee \exists y \in \mathbb{N}. x = y^+\}$ is inductive. □

Theorem 5.12 (Extensionality, Empty Set, Pairing, Union, Infinity, Subset).
Every natural number is transitive.

PROOF: By induction using Theorem 5.3. □

Theorem 5.13 (Extensionality, Empty Set, Pairing, Union, Infinity, Subset).
The set \mathbb{N} is transitive.

PROOF:

⟨1⟩1. For every natural number n and every $m \in n$ then m is a natural number.

⟨2⟩1. Every member of \emptyset is a natural number.

PROOF: Vacuous.

⟨2⟩2. If n is a natural number and a set of natural numbers then n^+ is a set of natural numbers.

PROOF: From the definition of n^+ .

⟨2⟩3. Q.E.D.

PROOF: By induction.

□

Theorem 5.14 (Extensionality, Empty Set, Pairing, Union, Infinity, Subset).
Let A be a set, $a \in A$, and $F : A \rightarrow A$. Then there exists a unique function $h : \mathbb{N} \rightarrow A$ such that $h(0) = a$ and, for all $n \in \mathbb{N}$, we have $h(n^+) = F(h(n))$.

PROOF:

⟨1⟩1. Call a function v *acceptable* iff $\text{dom } v \subseteq \mathbb{N}$, $\text{ran } v \subseteq A$, and:

1. If $0 \in \text{dom } v$ then $v(0) = a$.

2. For all $n \in \mathbb{N}$, if $n^+ \in \text{dom } v$ then $n \in \text{dom } v$ and $v(n^+) = F(v(n))$.

⟨1⟩2. LET: \mathcal{K} be the set of all acceptable functions.

⟨1⟩3. LET: $h = \bigcup \mathcal{K}$

⟨1⟩4. h is a function.

⟨2⟩1. If $(0, y) \in h$ and $(0, y') \in h$ then $y = y'$

PROOF: We have $y = y' = a$.

⟨2⟩2. For any natural number n , if there is at most one y such that $(n, y) \in h$, then there is at most one y such that $(n^+, y) \in h$

⟨3⟩1. LET: n be a natural number.

⟨3⟩2. ASSUME: there is at most one y such that $(n, y) \in h$

⟨3⟩3. ASSUME: (n^+, y) and (n^+, y') are in h

⟨3⟩4. PICK acceptable functions u and v such that $u(n^+) = y$ and $v(n^+) = y'$

⟨3⟩5. $n \in \text{dom } u$, $n \in \text{dom } v$ and $y = F(u(n))$, $y' = F(v(n))$

⟨3⟩6. $u(n) = v(n)$

PROOF: By the induction hypothesis ⟨3⟩2

⟨3⟩7. $y = y'$

⟨2⟩3. Q.E.D.

PROOF: By induction.

⟨1⟩5. h is acceptable.

⟨2⟩1. If $0 \in \text{dom } h$ then $h(0) = a$

⟨2⟩2. If $n^+ \in \text{dom } h$ then $n \in \text{dom } h$ and $h(n^+) = F(h(n))$

⟨3⟩1. ASSUME: $n^+ \in \text{dom } h$

⟨3⟩2. PICK an acceptable v such that $n^+ \in \text{dom } v$

⟨3⟩3. $v(n^+) = F(v(n))$

⟨3⟩4. $h(n^+) = F(h(n))$

⟨1⟩6. $\text{dom } h = \mathbb{N}$

⟨2⟩1. $0 \in \text{dom } h$

PROOF: Since $\{(0, a)\}$ is an acceptable function.

⟨2⟩2. For all $n \in \text{dom } h$ we have $n^+ \in \text{dom } h$

⟨3⟩1. ASSUME: $n \in \text{dom } h$

⟨3⟩2. LET: v be an acceptable function with $n \in \text{dom } v$

⟨3⟩3. ASSUME: without loss of generality $n^+ \notin \text{dom } v$

⟨3⟩4. $v \cup \{(n^+, F(v(n)))\}$ is acceptable

$\langle 3 \rangle 5. n^+ \in \text{dom } v$
 $\langle 1 \rangle 7. \text{ If } h' : \mathbb{N} \rightarrow A, h'(0) = a \text{ and, for all } n \in \mathbb{N}, \text{ we have } h'(n^+) = F(h'(n)),$
 $\text{ then } h' = h$
 PROOF: Prove $h(n) = h'(n)$ by induction on n .
 \square

5.4 Peano Systems

Definition 5.15 (Peano System). A *Peano system* consists of a set N , an element $z \in N$, and a function $S : N \rightarrow N$ such that:

- S is one-to-one
- $z \notin \text{ran } S$
- For any set $A \subseteq N$, if $z \in A$ and $S(A) \subseteq A$ then $A = N$.

Theorem 5.16. \mathbb{N} is a Peano system with zero 0 and successor $n \mapsto n^+$.

Theorem 5.17. For any Peano system (N, z, S) , there exists a unique bijection $h : \mathbb{N} \cong N$ such that $h(0) = z$ and $S(h(n)) = h(n^+)$ for all n .

5.5 Arithmetic

Definition 5.18 (Addition). Define *addition* $+: \mathbb{N}^2 \rightarrow \mathbb{N}$ recursively by

$$\begin{aligned}
 m + 0 &= m \\
 m + n^+ &= (m + n)^+
 \end{aligned}$$

for any $m, n \in \mathbb{N}$.

Theorem 5.19. Addition is associative.

Theorem 5.20. Addition is commutative

Definition 5.21 (Multiplication). Define *multiplication* $\cdot : \mathbb{N}^2 \rightarrow \mathbb{N}$ recursively by

$$\begin{aligned}
 m0 &= 0 \\
 mn^+ &= mn + m
 \end{aligned}$$

for any $m, n \in \mathbb{N}$

Theorem 5.22. Multiplication is associative.

Theorem 5.23. Multiplication is commutative.

Theorem 5.24. Multiplication distributes over addition.

Definition 5.25. For natural numbers m and n , we write $m < n$ iff $m \in n$. We write $m \leq n$ iff $m < n$ or $m = n$.

Theorem 5.26. We have $m < n$ iff $m^+ < n^+$.

Theorem 5.27. We never have $n < n$.

Theorem 5.28. The ordering on \mathbb{N} satisfies trichotomy; that is, for any m, n , exactly one of $m < n$, $m = n$, $n < m$ holds.

Theorem 5.29. For any natural numbers m and n , we have $m \leq n$ iff $m \subseteq n$.

Theorem 5.30. We have $m < n$ iff $m + p < n + p$.

Corollary 5.30.1. If $m + p = n + p$ then $m = n$.

Theorem 5.31. If $p \neq 0$ then $m < n$ iff $mp < np$.

Corollary 5.31.1. If $mp = np$ and $p \neq 0$ then $m = n$.

Theorem 5.32 (Well-Ordering of \mathbb{N}). Any nonempty set $A \subseteq \mathbb{N}$ has a least element.

Corollary 5.32.1. There is no function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that $f(n^+) < f(n)$ for all n .

Theorem 5.33 (Strong Induction). Let $A \subseteq \mathbb{N}$. Suppose that, for every natural number n , if $\forall m < n. m \in A$ then $n \in A$. Then $A = \mathbb{N}$.

Theorem 5.34 (Pigeonhole Principle). No natural number is equinumerous with a proper subset of itself.

PROOF: Prove by induction on n that if $f : n \rightarrow n$ is injective then it is surjective.
 \square

Chapter 6

Integers

Lemma 6.1. Define \sim on \mathbb{N}^2 by: $(m, n) \sim (p, q)$ iff $m + q = n + p$. Then \sim is an equivalence relation on \mathbb{N}^2 .

Definition 6.2 (Integers). The set \mathbb{Z} of *integers* is \mathbb{N}^2 / \sim .

Definition 6.3. Define *addition* $+: \mathbb{Z}^2 \rightarrow \mathbb{Z}$ by: $(m, n) + (p, q) = (m + p, n + q)$.
Prove this is well-defined.

Theorem 6.4. *Addition is associative and commutative.*

Definition 6.5 (Zero). The integer *zero* is $0 = (0, 0)$.

Theorem 6.6. *For any integer a , we have $a + 0 = a$.*

Theorem 6.7. *For any integer a , there exists a unique integer b such that $a + b = 0$.*

Definition 6.8 (Multiplication). Define multiplication on \mathbb{Z} by $(m, n)(p, q) = (mp + nq, mq + np)$.

Theorem 6.9. *Multiplication is associative, commutative and distributive over addition.*

Definition 6.10. The integer *one* is $1 = (1, 0)$.

Theorem 6.11. *For any integer a we have $a1 = a$.*

Theorem 6.12. $1 \neq 0$

Theorem 6.13. *Whenever $ab = 0$ then either $a = 0$ or $b = 0$.*

Definition 6.14. Define $<$ on \mathbb{Z} by: $(m, n) < (p, q)$ iff $m + q < n + p$.

Theorem 6.15. *The relation $<$ is a strict linear ordering on \mathbb{Z} .*

Theorem 6.16. *We have $a < b$ iff $+c < b + c$.*

Corollary 6.16.1. *If $a + c = b + c$ then $a = b$.*

Theorem 6.17. *If $0 < c$ then $a < b$ iff $ac < bc$.*

Corollary 6.17.1. *If $ac = bc$ and $c \neq 0$ then $a = b$.*

Definition 6.18. We identify any natural number n with the integer $(n, 0)$.

Theorem 6.19. *This embedding preserves 0, 1, addition, multiplication and the ordering.*

Chapter 7

Rational Numbers

Definition 7.1 (Rational Numbers). The set of *rational numbers* \mathbb{Q} is $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) / \sim$, where $(a, b) \sim (c, d)$ iff $ad = bc$.

Definition 7.2 (Addition). Define addition on \mathbb{Q} by: $(a, b) + (c, d) = (ad + bc, bd)$.

Theorem 7.3. *Addition is commutative and associative*

Definition 7.4. The rational number 0 is $(0, 1)$.

Theorem 7.5. *For any rational q we have $q + 0 = q$.*

Theorem 7.6. *For any rational q , there exists a unique rational r such that $q + r = 0$.*

Definition 7.7. Define multiplication on \mathbb{Q} by: $(a, b)(c, d) = (ac, bd)$.

Theorem 7.8. *Multiplication is commutative, associative and distributive over addition.*

Definition 7.9. The rational number 1 is $(1, 1)$.

Theorem 7.10. *For every nonzero rational r , there exists a nonzero rational q such that $rq = 1$.*

Corollary 7.10.1. *If $qr = 0$ then either $q = 0$ or $r = 0$.*

Definition 7.11. Define $<$ on \mathbb{Q} by: for b and d positive, $(a, b) < (c, d)$ iff $ad < bc$.

Theorem 7.12. *The relation $<$ is a strict linear ordering on \mathbb{Q} .*

Theorem 7.13. *We have $q < r$ iff $q + s < r + s$*

Corollary 7.13.1. *If $q + s = r + s$ then $q = r$.*

Theorem 7.14. *If $s > 0$ then we have $q < r$ iff $qs < rs$.*

Corollary 7.14.1. *If $qs = rs$ and $s \neq 0$ then $q = r$.*

Definition 7.15. We identify an integer n with the rational $(n, 1)$.

Theorem 7.16. *This embedding preserves zero, one, addition, multiplication and the ordering.*

Chapter 8

Real Numbers

Definition 8.1 (Dedekind Cut). A *Dedekind cut* is a subset $X \subseteq \mathbb{Q}$ such that:

- X is nonempty
- $X \neq \mathbb{Q}$
- X is closed downward
- X has no largest element.

Definition 8.2 (Real Numbers). The set of *real numbers* \mathbb{R} is the set of all Dedekind cuts.

Definition 8.3. Define $<$ on \mathbb{R} by: $x < y$ iff x is a proper subset of y .

Theorem 8.4. *The relation $<$ is a strict linear ordering on \mathbb{R} .*

Theorem 8.5. *Any bounded nonempty subset of \mathbb{R} has a least upper bound.*

Definition 8.6. Define addition on \mathbb{R} by: $x + y = \{q + r \mid q \in x, r \in y\}$.

Theorem 8.7. *Addition is associative and commutative.*

Definition 8.8. The zero real 0 is $\{q \in \mathbb{Q} \mid q < 0\}$.

Theorem 8.9. *For any $x \in \mathbb{R}$ we have $x + 0 = x$.*

Definition 8.10. Given a real x , define $-x = \{q \in \mathbb{Q} \mid \exists r > q. -r \notin x\}$.

Theorem 8.11. *For any real x we have $x + (-x) = 0$.*

Corollary 8.11.1. *If $x + z = y + z$ then $x = y$.*

Theorem 8.12. *We have $x < y$ iff $x + z < y + z$.*

Definition 8.13. Define the *absolute value* of a real x by $|x| = x \cup -x$.

Theorem 8.14. *For any real x we have $0 \leq |x|$.*

Definition 8.15. Define multiplication on \mathbb{R} by:

- If x and y are nonnegative then

$$xy = 0 \cup \{qr \mid 0 \leq q, 0 \leq r, q \in x, r \in y\}$$

- If x and y are both negative then $xy = |x||y|$
- If one of x and y is negative and the other not then $xy = -|x||y|$.

Theorem 8.16. *Multiplication is associative, commutative and distributive over addition.*

Definition 8.17. The real number 1 is $\{q \in \mathbb{Q} \mid q < 1\}$.

Theorem 8.18. $0 \neq 1$

Theorem 8.19. *For any real x we have $x1 = x$*

Theorem 8.20. *For any nonzero x , there exists a real y with $xy = 1$.*

Theorem 8.21. *If $0 < x$ then $y < z$ iff $xy < xz$.*

Definition 8.22. Identify a rational q with $\{r \in \mathbb{Q} \mid r < q\}$.

Theorem 8.23. *This embedding preserves zero, one, addition, multiplication and the ordering.*

Chapter 9

Finite Sets

Definition 9.1 (Finite). A set is *finite* iff it is equinumerous with a natural number; otherwise it is *infinite*.

Theorem 9.2. *No finite set is equinumerous with a proper subset of itself.*

PROOF: From the Pigeonhole Principle.

Corollary 9.2.1. *The set \mathbb{N} is infinite.*

Corollary 9.2.2. *A finite set is equinumerous with a unique natural number.*

Lemma 9.3. *If A is a proper subset of a natural number n then there exists $m < n$ such that $C \equiv m$.*

Corollary 9.3.1. *A subset of a finite set is finite.*

Theorem 9.4 (Regularity). *There is no function f with domain \mathbb{N} such that $f(n+1) \in f(n)$ for all n .*

PROOF:

$\langle 1 \rangle 1$. ASSUME: for a contradiction f is a function with domain \mathbb{N} such that $f(n+1) \in f(n)$ for all n .

$\langle 1 \rangle 2$. PICK $m \in \text{ran } f$ such that $m \cap \text{ran } f = \emptyset$

PROOF: By the Axiom of Regularity.

$\langle 1 \rangle 3$. PICK $n \in \mathbb{N}$ such that $f(n) = m$

$\langle 1 \rangle 4$. $f(n+1) \in m \cap \text{ran } f$

$\langle 1 \rangle 5$. Q.E.D.

PROOF: This is a contradiction.

□

Theorem 9.5. *A relation R is well-founded if and only if there is no function f with domain \mathbb{N} such that, for all $n \in \mathbb{N}$, we have $f(n+1)Rf(n)$.*

9.1 Real Analysis

Definition 9.6. Let \mathbb{R}^∞ be the subset of \mathbb{R}^ω consisting of all sequences (a_n) such that $a_n = 0$ for all but finitely many n .

9.2 Group Theory

Definition 9.7. Given a group G and sets $A, B \subseteq G$, let $AB = \{ab \mid a \in A, b \in B\}$.

Definition 9.8. Given a group G and a set $A \subseteq G$, let $A^{-1} = \{a^{-1} \mid a \in A\}$.

9.3 Topological Spaces

Definition 9.9 (Topology). A *topology* on a set X is a set $\mathcal{T} \subseteq \mathcal{P}X$ such that:

- $X \in \mathcal{T}$.
- For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$.
- For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$.

We call the elements of X *points* and the elements of \mathcal{T} *open sets*.

Definition 9.10 (Topological Space). A *topological space* X consists of a set X and a topology on X .

Definition 9.11 (Discrete Space). For any set X , the *discrete* topology on X is $\mathcal{P}X$.

Definition 9.12 (Indiscrete Space). For any set X , the *indiscrete* or *trivial* topology on X is $\{\emptyset, X\}$.

Definition 9.13 (Finite Complement Topology). For any set X , the *finite complement topology* on X is $\{U \in \mathcal{P}X \mid X \setminus U \text{ is finite}\} \cup \{\emptyset\}$.

Definition 9.14 (Countable Complement Topology). For any set X , the *countable complement topology* on X is $\{U \in \mathcal{P}X \mid X \setminus U \text{ is countable}\} \cup \{\emptyset\}$.

Definition 9.15 (Finer, Coarser). Suppose that \mathcal{T} and \mathcal{T}' are two topologies on a given set X . If $\mathcal{T}' \supseteq \mathcal{T}$, we say that \mathcal{T}' is *finer* than \mathcal{T} ; if \mathcal{T}' properly contains \mathcal{T} , we say that \mathcal{T}' is *strictly finer* than \mathcal{T} . We also say that \mathcal{T} is *coarser* than \mathcal{T}' , or *strictly coarser*, in these two respective situations. We say \mathcal{T} is *comparable* with \mathcal{T}' if either $\mathcal{T}' \supseteq \mathcal{T}$ or $\mathcal{T} \supseteq \mathcal{T}'$.

Lemma 9.16. Let X be a topological space and $U \subseteq X$. Then U is open if and only if, for all $x \in U$, there exists an open set V such that $x \in V \subseteq U$.

PROOF:

$\langle 1 \rangle 1. \Rightarrow$

PROOF: Take $V = U$

$\langle 1 \rangle 2. \Leftarrow$

PROOF: We have $U = \bigcup \{V \text{ open in } X \mid V \subseteq U\}$.

□

Lemma 9.17. Let X be a set and \mathcal{T} a nonempty set of topologies on X . Then $\bigcap \mathcal{T}$ is a topology on X , and is the finest topology that is coarser than every member of \mathcal{T} .

PROOF:

$\langle 1 \rangle 1. X \in \bigcap \mathcal{T}$

PROOF: Since X is in every member of \mathcal{T} .

$\langle 1 \rangle 2. \bigcap \mathcal{T}$ is closed under union.

- ⟨2⟩1. LET: $\mathcal{U} \subseteq \bigcap \mathcal{T}$
- ⟨2⟩2. For all $T \in \mathcal{T}$ we have $\mathcal{U} \subseteq T$
- ⟨2⟩3. For all $T \in \mathcal{T}$ we have $\bigcup \mathcal{U} \in T$
- ⟨2⟩4. $\bigcup \mathcal{U} \in \bigcap \mathcal{T}$
- ⟨1⟩3. $\bigcap \mathcal{T}$ is closed under binary intersection.
- ⟨2⟩1. LET: $U, V \in \bigcap \mathcal{T}$
- ⟨2⟩2. For all $T \in \mathcal{T}$ we have $U, V \in T$
- ⟨2⟩3. For all $T \in \mathcal{T}$ we have $U \cap V \in T$
- ⟨2⟩4. $U \cap V \in \bigcap \mathcal{T}$

□

Lemma 9.18. *Let X be a set and \mathcal{T} a set of topologies on X . Then there exists a unique coarsest topology that is finer than every member of \mathcal{T} .*

PROOF: The required topology is given by

$$\bigcap \{T \in \mathcal{P}X \mid T \text{ is a topology on } X \text{ that is finer than every member of } \mathcal{T}\},$$

The set is nonempty since it contains the discrete topology. □

Definition 9.19 (Neighbourhood). A *neighbourhood* of a point x is an open set that contains x .

9.4 Closed Set

Definition 9.20 (Closed Set). Let X be a topological space and $A \subseteq X$. Then A is *closed* if and only if $X \setminus A$ is open.

Lemma 9.21. *The empty set is closed.*

PROOF: Since the whole space X is always open. □

Lemma 9.22. *The topological space X is closed.*

PROOF: Since \emptyset is open. □

Lemma 9.23. *The intersection of a nonempty set of closed sets is closed.*

PROOF: Let \mathcal{C} be a nonempty set of closed sets. Then $X \setminus \bigcap \mathcal{C} = \bigcup \{X \setminus C \mid C \in \mathcal{C}\}$ is open. □

Lemma 9.24. *The union of two closed sets is closed.*

PROOF: Let C and D be closed. Then $X \setminus (C \cup D) = (X \setminus C) \cap (X \setminus D)$ is open. □

Proposition 9.25. *Let X be a set and $\mathcal{C} \subseteq \mathcal{P}X$ a set such that:*

1. $\emptyset \in \mathcal{C}$
2. $X \in \mathcal{C}$
3. For all $\mathcal{A} \subseteq \mathcal{C}$ nonempty we have $\bigcap \mathcal{A} \in \mathcal{C}$

4. For all $C, D \in \mathcal{C}$ we have $C \cup D \in \mathcal{C}$.

Then there exists a unique topology \mathcal{T} such that \mathcal{C} is the set of closed sets, namely

$$\mathcal{T} = \{X \setminus C \mid C \in \mathcal{C}\}$$

PROOF:

$\langle 1 \rangle 1$. LET: $\mathcal{T} = \{X \setminus C \mid C \in \mathcal{C}\}$

$\langle 1 \rangle 2$. \mathcal{T} is a topology

$\langle 2 \rangle 1$. $X \in \mathcal{T}$

PROOF: Since $\emptyset \in \mathcal{C}$

$\langle 2 \rangle 2$. For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$

$\langle 3 \rangle 1$. LET: $\mathcal{U} \subseteq \mathcal{T}$

$\langle 3 \rangle 2$. CASE: $\mathcal{U} = \emptyset$

PROOF: In this case $\bigcup \mathcal{U} = \emptyset \in \mathcal{T}$ since $X \in \mathcal{C}$

$\langle 3 \rangle 3$. CASE: $\mathcal{U} \neq \emptyset$

PROOF: In this case $X \setminus \bigcup \mathcal{U} = \bigcap \{X \setminus U \mid U \in \mathcal{U}\} \in \mathcal{C}$.

$\langle 2 \rangle 3$. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$

PROOF: Since $X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V) \in \mathcal{C}$.

$\langle 1 \rangle 3$. \mathcal{C} is the set of all closed sets in \mathcal{T}

PROOF:

C is closed in \mathcal{T}

$$\Leftrightarrow X \setminus C \in \mathcal{T}$$

$$\Leftrightarrow C \in \mathcal{C}$$

$\langle 1 \rangle 4$. If \mathcal{T}' is a topology and \mathcal{C} is the set of closed sets in \mathcal{T}' then $\mathcal{T}' = \mathcal{T}$

PROOF: We have

$$U \in \mathcal{T}$$

$$\Leftrightarrow X \setminus U \in \mathcal{C}$$

$$\Leftrightarrow X \setminus U \text{ is closed in } \mathcal{T}'$$

$$\Leftrightarrow U \in \mathcal{T}'$$

□

Proposition 9.26. If U is open and A is closed then $U \setminus A$ is open.

PROOF: $U \setminus A = U \cap (X \setminus A)$ is the intersection of two open sets. □

Proposition 9.27. If U is open and A is closed then $A \setminus U$ is closed.

PROOF: $A \setminus U = A \cap (X \setminus U)$ is the intersection of two closed sets. □

9.5 Interior

Definition 9.28 (Interior). Let X be a topological space and $A \subseteq X$. The *interior* of A , $\text{Int } A$, is the union of all the open subsets of A .

Lemma 9.29. The interior of a set is open.

PROOF: It is a union of open sets. \square

Lemma 9.30.

$$\text{Int } A \subseteq A$$

PROOF: Immediate from definition. \square

Lemma 9.31. *If U is open and $U \subseteq A$ then $U \subseteq \text{Int } A$*

PROOF: Immediate from definition. \square

Lemma 9.32. *A set A is open if and only if $A = \text{Int } A$.*

PROOF: If $A = \text{Int } A$ then A is open by Lemma 9.29. Conversely if A is open then $A \subseteq \text{Int } A$ by the definition of interior and so $A = \text{Int } A$.

9.6 Closure

Definition 9.33 (Closure). Let X be a topological space and $A \subseteq X$. The *closure* of A , \overline{A} , is the intersection of all the closed sets that include A .

This intersection exists since X is a closed set that includes A (Lemma 9.22).

Lemma 9.34. *The closure of a set is closed.*

PROOF: Dual to Lemma 9.29. \square

Lemma 9.35.

$$A \subseteq \overline{A}$$

PROOF: Immediate from definition. \square

Lemma 9.36. *If C is closed and $A \subseteq C$ then $\overline{A} \subseteq C$.*

PROOF: Immediate from definition. \square

Lemma 9.37. *A set A is closed if and only if $A = \overline{A}$.*

PROOF: Dual to Lemma 9.32. \square

Theorem 9.38. *Let X be a topological space, $A \subseteq X$ and $x \in X$. Then $x \in \overline{A}$ if and only if every neighbourhood of x intersects A .*

PROOF: We have

$$\begin{aligned} x \in \overline{A} \\ \Leftrightarrow \forall C. C \text{ closed} \wedge A \subseteq C \Rightarrow x \in C \\ \Leftrightarrow \forall U. U \text{ open} \wedge A \cap U = \emptyset \Rightarrow x \notin U \\ \Leftrightarrow \forall U. U \text{ open} \wedge x \in U \Rightarrow U \text{ intersects } A \end{aligned} \quad \square$$

Proposition 9.39. *If $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$.*

PROOF: This holds because \overline{B} is a closed set that includes A . \square

Proposition 9.40.

$$\overline{A \cup B} = \overline{A} \cup \overline{B}$$

PROOF:

$\langle 1 \rangle 1. \overline{A} \subseteq \overline{A \cup B}$

PROOF: By Proposition 9.39.

$\langle 1 \rangle 2. \overline{B} \subseteq \overline{A \cup B}$

PROOF: By Proposition 9.39.

$\langle 1 \rangle 3. \overline{A \cup B} \subseteq \overline{A \cup B}$

$\langle 2 \rangle 1. \text{ LET: } x \in \overline{A \cup B}$

$\langle 2 \rangle 2. \text{ ASSUME: } x \notin \overline{A}$

PROVE: $x \in \overline{B}$

$\langle 2 \rangle 3. \text{ PICK a neighbourhood } U \text{ of } x \text{ that does not intersect } A$

$\langle 2 \rangle 4. \text{ LET: } V \text{ be any neighbourhood of } x$

$\langle 2 \rangle 5. U \cap V \text{ is a neighbourhood of } x$

$\langle 2 \rangle 6. U \cap V \text{ intersects } A \cup B$

PROOF: From $\langle 2 \rangle 1$ and Theorem 9.38.

$\langle 2 \rangle 7. U \cap V \text{ intersects } B$

PROOF: From $\langle 2 \rangle 3$

$\langle 2 \rangle 8. V \text{ intersects } B$

$\langle 2 \rangle 9. \text{ Q.E.D.}$

PROOF: We have $x \in \overline{B}$ from Theorem 9.38.

\square

Proposition 9.41. *Let X be a topological space. Let \mathcal{D} be a set of subsets of X that is maximal with respect to the finite intersection property. Let $x \in X$. Then the following are equivalent:*

1. *For all $D \in \mathcal{D}$ we have $x \in \overline{D}$*
2. *Every neighbourhood of x is in \mathcal{D} .*

PROOF:

$\langle 1 \rangle 1. 1 \Rightarrow 2$

$\langle 2 \rangle 1. \text{ For all } D \in \mathcal{D} \text{ we have } x \in \overline{D}$

$\langle 2 \rangle 2. \text{ LET: } U \text{ be a neighbourhood of } x$

$\langle 2 \rangle 3. \mathcal{D} \cup \{U\} \text{ satisfies the finite intersection property.}$

$\langle 3 \rangle 1. \text{ LET: } D_1, \dots, D_n \in \mathcal{D}$

$\langle 3 \rangle 2. D_1 \cap \dots \cap D_n \in \mathcal{D}$

PROOF: Lemma 3.67.

$\langle 3 \rangle 3. x \in \overline{D_1 \cap \dots \cap D_n}$

PROOF: $\langle 2 \rangle 1, \langle 3 \rangle 2$

$\langle 3 \rangle 4. D_1 \cap \dots \cap D_n \cap U \neq \emptyset$

PROOF: Theorem 9.38, $\langle 2 \rangle 2, \langle 3 \rangle 3$.

$\langle 2 \rangle 4. \mathcal{D} = \mathcal{D} \cup \{U\}$

PROOF: By the maximality of \mathcal{D} .

$\langle 2 \rangle 5. U \in \mathcal{D}$
 $\langle 1 \rangle 2. 2 \Rightarrow 1$
 $\langle 2 \rangle 1.$ ASSUME: Every neighbourhood of x is in \mathcal{D} .
 $\langle 2 \rangle 2.$ LET: $D \in \mathcal{D}$
 $\langle 2 \rangle 3.$ Every neighbourhood of x intersects D .
 PROOF: From $\langle 2 \rangle 1$, $\langle 2 \rangle 2$ and the fact that \mathcal{D} satisfies the finite intersection property.
 $\langle 2 \rangle 4. x \in \overline{D}$
 PROOF: Theorem 9.38, $\langle 2 \rangle 3$.
 \square

9.7 Boundary

Definition 9.42 (Boundary). The *boundary* of a set A is the set $\partial A = \overline{A} \cap \overline{X \setminus A}$.

Proposition 9.43.

$$\text{Int } A \cap \partial A = \emptyset$$

PROOF: Since $\overline{X \setminus A} = X \setminus \text{Int } A$. \square

Proposition 9.44.

$$\overline{A} = \text{Int } A \cup \partial A$$

PROOF:

$$\begin{aligned}
 \text{Int } A \cup \partial A &= \text{Int } A \cup (\overline{A} \cap \overline{X \setminus A}) \\
 &= (\text{Int } A \cup \overline{A}) \cap (\text{Int } A \cup \overline{X \setminus A}) \\
 &= \overline{A} \cap X \\
 &= \overline{A}
 \end{aligned}$$

Proposition 9.45. $\partial A = \emptyset$ if and only if A is open and closed.

PROOF: If $\partial A = \emptyset$ then $\overline{A} = \text{Int } A$ by Proposition 9.44.

Proposition 9.46. A set U is open if and only if $\partial U = \overline{U} \setminus U$.

PROOF:

$$\begin{aligned}
 \partial U &= \overline{U} \setminus U \\
 \Leftrightarrow \overline{U} \setminus \text{Int } U &= \overline{U} \setminus U && (\text{Propositions 9.43, 9.44}) \\
 \Leftrightarrow \text{Int } U &= U && \square
 \end{aligned}$$

9.8 Limit Points

Definition 9.47 (Limit Point). Let X be a topological space, $a \in X$ and $A \subseteq X$. Then a is a *limit point*, *cluster point* or *point of accumulation* for A if and only if every neighbourhood of a intersects A at a point other than a .

Lemma 9.48. *The point a is an accumulation point for A if and only if $a \in \overline{A} \setminus \{a\}$.*

PROOF: From Theorem 9.38. \square

Theorem 9.49. *Let X be a topological space and $A \subseteq X$. Let A' be the set of all limit points of A . Then $\overline{A} = A \cup A'$.*

PROOF:

$\langle 1 \rangle 1.$ For all $x \in \overline{A}$, if $x \notin A$ then $x \in A'$

PROOF: From Theorem 9.38.

$\langle 1 \rangle 2.$ $A \subseteq \overline{A}$

PROOF: Lemma 9.35.

$\langle 1 \rangle 3.$ $A' \subseteq \overline{A}$

PROOF: From Theorem 9.38.

\square

Corollary 9.49.1. *A set is closed if and only if it contains all its limit points.*

Proposition 9.50. *In an indiscrete topology, every point is a limit point of any set that has more than one point.*

PROOF: Let X be an indiscrete space. Let A be a set with more than one point and x be a point. The only neighbourhood of x is X , which must intersect A at a point other than x . \square

Lemma 9.51. *Let X be a topological space and $A \subseteq B \subseteq X$. Then every limit point of A is a limit point of B .*

PROOF: Immediate from definitions. \square

9.9 Basis for a Topology

Definition 9.52 (Basis). If X is a set, a *basis* for a topology on X is a set $\mathcal{B} \subseteq \mathcal{P}X$ called *basis elements* such that

1. For all $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$.
2. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

If \mathcal{B} satisfies these two conditions, then we define the topology *generated* by \mathcal{B} to be $\mathcal{T} = \{U \in \mathcal{P}X \mid \forall x \in U. \exists B \in \mathcal{B}. x \in B \subseteq U\}$.

We prove this is a topology.

PROOF:

$\langle 1 \rangle 1.$ $X \in \mathcal{T}$

PROOF: For all $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B \subseteq X$ by condition 1.

- ⟨1⟩2. For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$
 - ⟨2⟩1. LET: $\mathcal{U} \subseteq \mathcal{T}$
 - ⟨2⟩2. LET: $x \in \bigcup \mathcal{U}$
 - ⟨2⟩3. PICK $U \in \mathcal{U}$ such that $x \in U$
 - ⟨2⟩4. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$
 - PROOF: Since $U \in \mathcal{T}$ by ⟨2⟩1 and ⟨2⟩3.
 - ⟨2⟩5. $x \in B \subseteq \bigcup \mathcal{U}$
- ⟨1⟩3. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$
 - ⟨2⟩1. LET: $U, V \in \mathcal{T}$
 - ⟨2⟩2. LET: $x \in U \cap V$
 - ⟨2⟩3. PICK $B_1 \in \mathcal{B}$ such that $x \in B_1 \subseteq U$
 - ⟨2⟩4. PICK $B_2 \in \mathcal{B}$ such that $x \in B_2 \subseteq V$
 - ⟨2⟩5. PICK $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$
 - PROOF: By condition 2.
 - ⟨2⟩6. $x \in B_3 \subseteq U \cap V$

□

Lemma 9.53. *Let X be a set. Let \mathcal{B} be a basis for a topology \mathcal{T} on X . Then \mathcal{T} is the set of all unions of subsets of \mathcal{B} .*

PROOF:

- ⟨1⟩1. For all $U \in \mathcal{T}$, there exists $\mathcal{A} \subseteq \mathcal{B}$ such that $U = \bigcup \mathcal{A}$
 - ⟨2⟩1. LET: $U \in \mathcal{T}$
 - ⟨2⟩2. LET: $\mathcal{A} = \{B \in \mathcal{B} \mid B \subseteq U\}$
 - ⟨2⟩3. $U \subseteq \bigcup \mathcal{A}$
 - ⟨3⟩1. LET: $x \in U$
 - ⟨3⟩2. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$
 - PROOF: Since \mathcal{B} is a basis for \mathcal{T} .
 - ⟨3⟩3. $x \in B \in \mathcal{A}$
 - ⟨2⟩4. $\bigcup \mathcal{A} \subseteq U$
 - PROOF: From the definition of \mathcal{A} (⟨2⟩2).
- ⟨1⟩2. For all $\mathcal{A} \subseteq \mathcal{B}$ we have $\bigcup \mathcal{A} \in \mathcal{T}$
 - ⟨2⟩1. $\mathcal{B} \subseteq \mathcal{T}$
 - PROOF: If $B \in \mathcal{B}$ and $x \in B$, then there exists $B' \in \mathcal{B}$ such that $x \in B' \subseteq B$, namely $B' = B$.
 - ⟨2⟩2. Q.E.D.
 - PROOF: Since \mathcal{T} is closed under union.

□

Corollary 9.53.1. *Let X be a set. Let \mathcal{B} be a basis for a topology \mathcal{T} on X . Then \mathcal{T} is the coarsest topology that includes \mathcal{B} .*

PROOF: Since every topology that includes \mathcal{B} includes all unions of subsets of \mathcal{B} . □

Lemma 9.54. *Let X be a topological space. Suppose that \mathcal{C} is a set of open sets such that, for every open set U and every point $x \in U$, there exists $C \in \mathcal{C}$ such that $x \in C \subseteq U$. Then \mathcal{C} is a basis for the topology on X .*

PROOF:

⟨1⟩1. For all $x \in X$, there exists $C \in \mathcal{C}$ such that $x \in C$

PROOF: Immediate from hypothesis.

⟨1⟩2. For all $C_1, C_2 \in \mathcal{C}$ and $x \in C_1 \cap C_2$, there exists $C_3 \in \mathcal{C}$ such that $x \in C_3 \subseteq C_1 \cap C_2$

PROOF: Since $C_1 \cap C_2$ is open.

⟨1⟩3. Every open set is open in the topology generated by \mathcal{C}

PROOF: Immediate from hypothesis.

⟨1⟩4. Every union of a subset of \mathcal{C} is open.

PROOF: Since every member of \mathcal{C} is open.

□

Lemma 9.55. *Let \mathcal{B} and \mathcal{B}' be bases for the topologies \mathcal{T} and \mathcal{T}' respectively on the set X . Then the following are equivalent.*

1. $\mathcal{T} \subseteq \mathcal{T}'$

2. For all $B \in \mathcal{B}$ and $x \in B$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.

PROOF:

⟨1⟩1. $1 \Rightarrow 2$

⟨2⟩1. ASSUME: $\mathcal{T} \subseteq \mathcal{T}'$

⟨2⟩2. LET: $B \in \mathcal{B}$ and $x \in B$

⟨2⟩3. $B \in \mathcal{T}$

PROOF: Corollary 9.53.1.

⟨2⟩4. $B \in \mathcal{T}'$

PROOF: By ⟨2⟩1

⟨2⟩5. There exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$

PROOF: Since \mathcal{B}' is a basis for \mathcal{T}' .

⟨1⟩2. $2 \Rightarrow 1$

⟨2⟩1. ASSUME: 2

⟨2⟩2. LET: $U \in \mathcal{T}$

PROVE: $U \in \mathcal{T}'$

⟨2⟩3. LET: $x \in U$

PROVE: There exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq U$

⟨2⟩4. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$

PROOF: Since \mathcal{B} is a basis for \mathcal{T} .

⟨2⟩5. PICK $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$

PROOF: By ⟨2⟩1.

⟨2⟩6. $x \in B' \subseteq U$

□

Theorem 9.56. *Let X be a topological space, $A \subseteq X$ and $x \in X$. Let \mathcal{B} be a basis for X . Then $x \in \overline{A}$ if and only if, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A .*

PROOF:

⟨1⟩1. If $x \in \overline{A}$ then, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A .

PROOF: This follows from Theorem 9.38 since every element of \mathcal{B} is open (Corollary 9.53.1).

$\langle 1 \rangle 2$. Suppose that, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A . Then $x \in \bar{A}$.

$\langle 2 \rangle 1$. ASSUME: For all $B \in \mathcal{B}$, if $x \in B$ then B intersects A .

$\langle 2 \rangle 2$. LET: U be an open set that contains x

PROVE: U intersects A .

$\langle 2 \rangle 3$. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

$\langle 2 \rangle 4$. B intersects A .

PROOF: From $\langle 2 \rangle 1$.

$\langle 2 \rangle 5$. U intersects A .

$\langle 2 \rangle 6$. Q.E.D.

PROOF: By Theorem 9.38.

□

Definition 9.57 (Lower Limit Topology on the Real Line). The *lower limit topology on the real line* is the topology on \mathbb{R} generated by the basis consisting of all half-open intervals of the form $[a, b)$.

We write \mathbb{R}_l for the topological space \mathbb{R} under the lower limit topology.

We prove this is a basis for a topology.

PROOF:

$\langle 1 \rangle 1$. For all $x \in \mathbb{R}$ there exists an interval $[a, b)$ such that $x \in [a, b)$.

PROOF: Take $[a, b) = [x, x + 1)$.

$\langle 1 \rangle 2$. For any open intervals $[a, b)$, $[c, d)$ if $x \in [a, b) \cap [c, d)$, then there exists an interval $[e, f)$ such that $x \in [e, f) \subseteq [a, b) \cap [c, d)$

PROOF: Take $[e, f) = [\max(a, c), \min(b, d))$.

□

Definition 9.58 (K -topology on the Real Line). Let $K = \{1/n \mid n \in \mathbb{Z}^+\}$.

The *K -topology on the real line* is the topology on \mathbb{R} generated by the basis consisting of all open intervals (a, b) and all sets of the form $(a, b) \setminus K$.

We write \mathbb{R}_K for the topological space \mathbb{R} under the K -topology.

We prove this is a basis for a topology.

PROOF:

$\langle 1 \rangle 1$. For all $x \in \mathbb{R}$ there exists an open interval (a, b) such that $x \in (a, b)$.

PROOF: Take $(a, b) = (x - 1, x + 1)$.

$\langle 1 \rangle 2$. For any basic open sets B_1, B_2 if $x \in B_1 \cap B_2$, then there exists a basic open set B_3 such that $x \in B_3 \subseteq B_1 \cap B_2$

$\langle 2 \rangle 1$. CASE: $B_1 = (a, b)$, $B_2 = (c, d)$

PROOF: Take $B_3 = (\max(a, c), \min(b, d))$.

$\langle 2 \rangle 2$. CASE: $B_1 = (a, b)$ or $(a, b) \setminus K$, $B_2 = (c, d)$ or $(c, d) \setminus K$, and they are not both open intervals.

PROOF: Take $B_3 = (\max(a, c), \min(b, d)) \setminus K$.

□

Lemma 9.59. *The lower limit topology and the K -topology are incomparable.*

PROOF:

$\langle 1 \rangle 1$. The interval $[10, 11)$ is not open in the K -topology.

PROOF: There is no open interval (a, b) such that $10 \in (a, b) \subseteq [10, 11)$ or $10 \in (a, b) \setminus K \subseteq [10, 11)$.

$\langle 1 \rangle 2$. The set $(-1, 1) \setminus K$ is not open in the lower limit topology.

PROOF: There is no half-open interval $[a, b)$ such that $0 \in [a, b) \subseteq (-1, 1) \setminus K$, since there must be a positive integer n with $1/n \in [a, b)$.

□

Definition 9.60 (Subbasis). A *subbasis* \mathcal{S} for a topology on X is a set $\mathcal{S} \subseteq \mathcal{P}X$ such that $\bigcup \mathcal{S} = X$.

The topology *generated* by the subbasis \mathcal{S} is the set of all unions of finite intersections of elements of \mathcal{S} .

We prove this is a topology.

PROOF:

$\langle 1 \rangle 1$. The set \mathcal{B} of all finite intersections of elements of \mathcal{S} forms a basis for a topology on X .

$\langle 2 \rangle 1$. $\bigcup \mathcal{B} = X$

PROOF: Since $\mathcal{S} \subseteq \mathcal{B}$.

$\langle 2 \rangle 2$. \mathcal{B} is closed under binary intersection.

PROOF: By definition.

$\langle 1 \rangle 2$. Q.E.D.

PROOF: By Lemma 9.53.

□

We have simultaneously proved:

Proposition 9.61. Let \mathcal{S} be a subbasis for the topology on X . Then the set of all finite intersections of elements of \mathcal{S} is a basis for the topology on X .

Proposition 9.62. Let X be a set. Let \mathcal{S} be a subbasis for a topology \mathcal{T} on X . Then \mathcal{T} is the coarsest topology that includes \mathcal{S} .

PROOF: Since every topology that includes \mathcal{S} includes every union of finite intersections of elements of \mathcal{S} . □

9.10 Local Basis at a Point

Definition 9.63 (Local Basis). Let X be a topological space and $a \in X$. A *(local) basis at a* is a set \mathcal{B} of neighbourhoods of a such that every neighbourhood of a includes some member of \mathcal{B} .

Lemma 9.64. If there exists a countable local basis at a point a , then there exists a countable local basis $\{B_n \mid n \geq 1\}$ such that $B_1 \supseteq B_2 \supseteq \dots$.

PROOF: Given a countable local basis $\{C_n \mid n \geq 1\}$, take $B_n = C_1 \cap \dots \cap C_n$. □

9.11 Convergence

Definition 9.65 (Convergence). Let X be a topological space. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of points in X and $l \in X$. Then the sequence $(a_n)_{n \in \mathbb{N}}$ *converges* to the *limit* l , $a_n \rightarrow l$ as $n \rightarrow \infty$, if and only if, for every neighbourhood U of l , there exists N such that, for all $n \geq N$, we have $a_n \in U$.

Lemma 9.66. *Let X be a topological space. Let $A \subseteq X$ and $l \in X$. If there is a sequence of points in A that converges to l then $l \in \bar{A}$.*

PROOF:

- $\langle 1 \rangle 1$. LET: (a_n) be a sequence of points in A that converges to l .
- $\langle 1 \rangle 2$. LET: U be a neighbourhood of l .
- $\langle 1 \rangle 3$. PICK N such that, for all $n \geq N$, we have $a_n \in U$.
- $\langle 1 \rangle 4$. $a_N \in U \cap A$
- $\langle 1 \rangle 5$. Q.E.D.

PROOF: Theorem 9.38.

□

Proposition 9.67. *Let X be a topological space. Let \mathcal{B} be a basis for the topology on X . Let (a_n) be a sequence in X and $l \in X$. Then $a_n \rightarrow l$ as $n \rightarrow \infty$ if and only if, for every $B \in \mathcal{B}$ with $l \in B$, there exists N such that, for all $n \geq N$, we have $a_n \in B$.*

PROOF:

- $\langle 1 \rangle 1$. If $a_n \rightarrow l$ as $n \rightarrow \infty$ then, for every $B \in \mathcal{B}$ with $l \in B$, there exists N such that, for all $n \geq N$, we have $a_n \in B$.

PROOF: Since every element of \mathcal{B} is open (Corollary 9.53.1).

- $\langle 1 \rangle 2$. If, for every $B \in \mathcal{B}$ with $l \in B$, there exists N such that, for all $n \geq N$, we have $a_n \in B$, then $a_n \rightarrow l$ as $n \rightarrow \infty$.
 - $\langle 2 \rangle 1$. ASSUME: for every $B \in \mathcal{B}$ with $l \in B$, there exists N such that, for all $n \geq N$, we have $a_n \in B$
 - $\langle 2 \rangle 2$. LET: U be a neighbourhood of l .
 - $\langle 2 \rangle 3$. PICK $B \in \mathcal{B}$ such that $l \in B \subseteq U$
 - $\langle 2 \rangle 4$. PICK N such that, for all $n \geq N$, we have $a_n \in B$

PROOF: From $\langle 2 \rangle 1$.

- $\langle 2 \rangle 5$. For all $n \geq N$ we have $a_n \in U$

□

Lemma 9.68. *If a sequence (a_n) is constant with $a_n = l$ for all n , then $a_n \rightarrow l$ as $n \rightarrow \infty$.*

PROOF: Immediate from definitions. □

Theorem 9.69. *Let X be a linearly ordered set. Let (s_n) be a monotone increasing sequence in X with a supremum s . Then $s_n \rightarrow s$ as $n \rightarrow \infty$.*

PROOF:

- $\langle 1 \rangle 1$. ASSUME: s is not least in X .

PROOF: Otherwise (s_n) is the constant sequence s and the result follows from Lemma 9.68.

- ⟨1⟩2. LET: U be a neighbourhood of s .
- ⟨1⟩3. PICK $a < s$ such that $(a, s] \subseteq U$
- ⟨1⟩4. PICK N such that $a < a_N$.
- ⟨1⟩5. For all $n \geq N$ we have $a_n \in (a, s]$
- ⟨1⟩6. For all $n \geq N$ we have $a_n \in U$.

□

Theorem 9.70. If $\sum_{i=0}^{\infty} a_i = s$ and $\sum_{i=0}^{\infty} b_i = t$ then $\sum_{i=0}^{\infty} (ca_i + b_i) = cs + t$.

PROOF: $\sum_{i=0}^N (ca_i + b_i) = c \sum_{i=0}^N a_i + \sum_{i=0}^N b_i \rightarrow cs + t$ as $n \rightarrow \infty$. □

Theorem 9.71 (Comparison Test). If $|a_i| \leq b_i$ for all i and $\sum_{i=0}^{\infty} b_i$ converges then $\sum_{i=0}^{\infty} a_i$ converges.

PROOF:

- ⟨1⟩1. $\sum_{i=0}^{\infty} |a_i|$ converges

PROOF: The partial sums $\sum_{i=0}^N |a_i|$ form an increasing sequence bounded above by $\sum_{i=0}^{\infty} b_i$.

- ⟨1⟩2. LET: $c_i = |a_i| + a_i$ for all i

- ⟨1⟩3. $\sum_{i=0}^{\infty} c_i$ converges

PROOF: Each c_i is either $2|a_i|$ or 0. So the partial sums $\sum_{i=0}^N c_i$ form an increasing sequence bounded above by $2 \sum_{i=0}^{\infty} b_i$.

- ⟨1⟩4. Q.E.D.

PROOF: Since $a_i = c_i - |a_i|$.

□

Corollary 9.71.1. If $\sum_{i=0}^{\infty} |a_i|$ converges then $\sum_{i=0}^{\infty} a_i$ converges.

Theorem 9.72 (Weierstrass M-test). Let X be a set and $(f_n : X \rightarrow \mathbb{R})$ be a sequence of functions. Let

$$s_n(x) = \sum_{i=0}^n f_i(x)$$

for all n, x . Suppose $|f_i(x)| \leq M_i$ for all $i \geq 0$ and $x \in X$. If the series $\sum_{i=0}^{\infty} M_i$ converges, then the sequence (s_n) converges uniformly to

$$s(x) = \sum_{i=0}^{\infty} f_i(x) .$$

PROOF:

- ⟨1⟩1. LET: $r_n = \sum_{i=n+1}^{\infty} M_i$ for all n

- ⟨1⟩2. Given $0 \leq n < k$, we have $|s_k(x) - s_n(x)| \leq r_n$

PROOF:

$$\begin{aligned}
|s_k(x) - s_n(x)| &= \left| \sum_{i=n+1}^k f_i(x) \right| \\
&\leq \sum_{i=n+1}^k |f_i(x)| \\
&\leq \sum_{i=n+1}^k M_i \\
&\leq r_n
\end{aligned}$$

⟨1⟩3. Given $n \geq 0$ we have $|s(x) - s_n(x)| \leq r_n$

PROOF: By taking the limit $k \rightarrow \infty$ in ⟨1⟩2.

⟨1⟩4. Q.E.D.

PROOF: Since $r_n \rightarrow 0$ as $n \rightarrow \infty$.

□

9.12 Locally Finite Sets

Definition 9.73 (Locally Finite). Let X be a topological space and $\{A_\alpha\}$ a family of subsets of X . Then \mathcal{A} is *locally finite* if and only if every point in X has a neighbourhood that intersects A_α for only finitely many α .

Theorem 9.74 (Pasting Lemma). *Let X and Y be topological spaces and $f : X \rightarrow Y$. Let $\{A_\alpha\}$ be a locally finite family of closed subsets of X that cover X . Suppose $f \upharpoonright A_\alpha$ is continuous for all α . Then f is continuous.*

PROOF:

⟨1⟩1. Let X and Y be topological spaces and $f : X \rightarrow Y$. Let A and B be closed subsets of X such that $X = A \cup B$. Suppose $f \upharpoonright A$ and $f \upharpoonright B$ are continuous. Then f is continuous.

⟨2⟩1. LET: $C \subseteq Y$ be closed.

⟨2⟩2. $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$

⟨2⟩3. $f^{-1}(C)$ and $g^{-1}(C)$ are closed in X .

PROOF: Theorems 9.84 and 9.135.

⟨2⟩4. $h^{-1}(C)$ is closed in X .

PROOF: Lemma 9.24.

⟨2⟩5. Q.E.D.

PROOF: Theorem 9.84.

⟨1⟩2. Let X and Y be topological spaces and $f : X \rightarrow Y$. Let $\{A_\alpha\}$ be a finite family of closed subsets of X that cover X . Suppose $f \upharpoonright A_\alpha$ is continuous for all α . Then f is continuous.

PROOF: From ⟨1⟩1 by induction.

⟨1⟩3. Let X and Y be topological spaces and $f : X \rightarrow Y$. Let $\{A_\alpha\}$ be a locally finite family of closed subsets of X that cover X . Suppose $f \upharpoonright A_\alpha$ is continuous for all α . Then f is continuous.

- (2)1. LET: $x \in X$
 PROVE: f is continuous at x
 (2)2. PICK a neighbourhood U of x that intersects A_α for only finitely many α .
 (2)3. $f \upharpoonright U$ is continuous
 PROOF: By (1)2.
 (2)4. Q.E.D.
 PROOF: Lemma 9.94.

□

The following example shows that we cannot remove the assumption of local finiteness.

Example 9.75. Define $f : [-1, 1] \rightarrow \mathbb{R}$ by: $f(x) = 1$ if $x < -1$, $f(x) = 0$ if $x > 1$. Let $C_n = [-1, -1/n]$ for $n \geq 1$, and $D = [0, 1]$. Then $[-1, 1] = \bigcup_{n=1}^{\infty} C_n \cup D$ and f is continuous on each C_n and each D , but f is not continuous on $[-1, 1]$.

9.13 Open Maps

Definition 9.76 (Open Map). Let X and Y be topological spaces and $f : X \rightarrow Y$. Then f is an *open map* if and only if, for every open set U in X , the set $f(U)$ is open in Y .

Lemma 9.77. Let X and Y be topological spaces and $f : X \rightarrow Y$. Let \mathcal{B} be a basis for the topology on X . If $f(B)$ is open in Y for all $B \in \mathcal{B}$, then f is an open map.

PROOF: From Lemma 9.53. □

Proposition 9.78. Let X and Y be topological spaces. Let \mathcal{B} be a basis for the topology on X . Let $f : X \rightarrow Y$. Suppose that, for all $B \in \mathcal{B}$, we have $f(B)$ is open in Y . Then f is an open map.

PROOF: For any $\mathcal{A} \subseteq \mathcal{B}$, we have $f(\bigcup \mathcal{A}) = \bigcup_{B \in \mathcal{A}} f(B)$ is open in Y . The result follows from Lemma 9.53. □

9.14 Continuous Functions

Definition 9.79 (Continuous). Let X and Y be topological spaces. A function $f : X \rightarrow Y$ is *continuous* if and only if, for every open set V in Y , the set $f^{-1}(V)$ is open in X .

Proposition 9.80. Let X and Y be topological spaces and $f : X \rightarrow Y$. Let \mathcal{B} be a basis for Y . Then f is continuous if and only if, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X .

PROOF:

- (1)1. If f is continuous then, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X .

PROOF: Since every element of \mathcal{B} is open (Lemma 9.53).

⟨1⟩2. Suppose that, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X . Then f is continuous.

⟨2⟩1. ASSUME: For all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X .

⟨2⟩2. LET: V be open in Y .

⟨2⟩3. PICK $\mathcal{A} \subseteq \mathcal{B}$ such that $V = \bigcup \mathcal{A}$

PROOF: By Lemma 9.53.

⟨2⟩4. $f^{-1}(V)$ is open in X .

PROOF:

$$\begin{aligned} f^{-1}(V) &= f^{-1}\left(\bigcup \mathcal{A}\right) \\ &= \bigcup_{B \in \mathcal{A}} f^{-1}(B) \end{aligned}$$

□

Proposition 9.81. *Let X and Y be topological spaces and $f : X \rightarrow Y$. Let \mathcal{S} be a subbasis for Y . Then f is continuous if and only if, for all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in X .*

PROOF:

⟨1⟩1. If f is continuous then, for all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in X .

PROOF: Since every element of \mathcal{S} is open.

⟨1⟩2. Suppose that, for all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in X . Then f is continuous.

⟨2⟩1. ASSUME: For all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in X .

⟨2⟩2. LET: $S_1, \dots, S_n \in \mathcal{S}$

⟨2⟩3. $f^{-1}(S_1 \cap \dots \cap S_n)$ is open in X

PROOF: Since $f^{-1}(S_1 \cap \dots \cap S_n) = f^{-1}(S_1) \cap \dots \cap f^{-1}(S_n)$.

⟨2⟩4. Q.E.D.

PROOF: By Propositions 9.80 and 9.61.

□

Proposition 9.82. *Let X and Y be topological spaces and $f : X \rightarrow Y$. Let \mathcal{S} be a basis for Y . Then f is continuous if and only if, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X .*

PROOF:

⟨1⟩1. If f is continuous then, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X .

PROOF: Since every element of \mathcal{S} is open.

⟨1⟩2. Suppose that, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X . Then f is continuous.

⟨2⟩1. ASSUME: For all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X .

⟨2⟩2. For every set B that is the finite intersection of elements of \mathcal{S} , we have $f^{-1}(B)$ is open in X .

PROOF: Because $f^{-1}(V_1 \cap \dots \cap V_n) = f^{-1}(V_1) \cap \dots \cap f^{-1}(V_n)$.

⟨2⟩3. Q.E.D.

PROOF: From Propositions 9.61 and 9.80.

□

Definition 9.83 (Continuous at a Point). Let X and Y be topological spaces. Let $f : X \rightarrow Y$ and $x \in X$. Then f is *continuous at x* if and only if, for every neighbourhood V of $f(x)$, there exists a neighbourhood U of x such that $f(U) \subseteq V$.

Theorem 9.84. Let X and Y be topological spaces and $f : X \rightarrow Y$. Then the following are equivalent:

1. f is continuous.
2. For all $A \subseteq X$, we have $f(\overline{A}) \subseteq \overline{f(A)}$
3. For all $B \subseteq Y$ closed, we have $f^{-1}(B)$ is closed in X .
4. f is continuous at every point of X .

PROOF:

$\langle 1 \rangle 1. 1 \Rightarrow 2$

$\langle 2 \rangle 1.$ ASSUME: f is continuous.

$\langle 2 \rangle 2.$ LET: $A \subseteq X$

$\langle 2 \rangle 3.$ LET: $x \in \overline{A}$

PROVE: $f(x) \in \overline{f(A)}$

$\langle 2 \rangle 4.$ LET: V be a neighbourhood of $f(x)$

$\langle 2 \rangle 5.$ $f^{-1}(V)$ is a neighbourhood of x

$\langle 2 \rangle 6.$ PICK $y \in A \cap f^{-1}(V)$

PROOF: By Theorem 9.38.

$\langle 2 \rangle 7.$ $f(y) \in V \cap f(A)$

$\langle 2 \rangle 8.$ Q.E.D.

PROOF: By Theorem 9.38.

$\langle 1 \rangle 2. 2 \Rightarrow 3$

$\langle 2 \rangle 1.$ ASSUME: 2

$\langle 2 \rangle 2.$ LET: B be closed in Y

$\langle 2 \rangle 3.$ LET: $x \in \overline{f^{-1}(B)}$

PROVE: $x \in f^{-1}(B)$

$\langle 2 \rangle 4.$ $f(x) \in B$

PROOF:

$$f(x) \in f(\overline{f^{-1}(B)})$$

$$\subseteq \overline{f(f^{-1}(B))}$$

$(\langle 2 \rangle 1)$

$$\subseteq \overline{B}$$

$(\text{Proposition 9.39})$

$$= B$$

$\langle 1 \rangle 3. 3 \Rightarrow 1$

$\langle 2 \rangle 1.$ ASSUME: 3

$\langle 2 \rangle 2.$ LET: V be open in Y

$\langle 2 \rangle 3.$ $Y \setminus V$ is closed in Y

$\langle 2 \rangle 4.$ $f^{-1}(Y \setminus V)$ is closed in X

$\langle 2 \rangle 5.$ $X \setminus f^{-1}(V)$ is closed in X

$\langle 2 \rangle 6.$ $f^{-1}(V)$ is open in X

⟨1⟩4. $1 \Rightarrow 4$

PROOF: For any neighbourhood V of $f(x)$, the set $U = f^{-1}(V)$ is a neighbourhood of x such that $f(U) \subseteq V$.

⟨1⟩5. $4 \Rightarrow 1$

⟨2⟩1. ASSUME: 4

⟨2⟩2. LET: V be open in Y

⟨2⟩3. LET: $x \in f^{-1}(V)$

⟨2⟩4. V is a neighbourhood of $f(x)$

⟨2⟩5. PICK a neighbourhood U of x such that $f(U) \subseteq V$

⟨2⟩6. $x \in U \subseteq f^{-1}(V)$

⟨2⟩7. Q.E.D.

PROOF: By Lemma 9.16.

□

Theorem 9.85. *A constant function is continuous.*

PROOF: Let X and Y be topological spaces. Let $b \in Y$, and let $f : X \rightarrow Y$ be the constant function with value b . For any open $V \subseteq Y$, the set $f^{-1}(V)$ is either X (if $b \in V$) or \emptyset (if $b \notin V$). □

Theorem 9.86. *If A is a subspace of X then the inclusion $j : A \rightarrow X$ is continuous.*

PROOF: For any V open in X , we have $j^{-1}(V) = V \cap A$ is open in A . □

Theorem 9.87. *The composite of two continuous functions is continuous.*

PROOF: Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be continuous. For any V open in Z , we have $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is open in X . □

Theorem 9.88. *Let $f : X \rightarrow Y$ be a continuous function and A be a subspace of X . Then the restriction $f \upharpoonright A : A \rightarrow Y$ is continuous.*

PROOF: Let V be open in Y . Then $(f \upharpoonright A)^{-1}(V) = f^{-1}(V) \cap A$ is open in A . □

Theorem 9.89. *Let $f : X \rightarrow Y$ be continuous. Let Z be a subspace of Y such that $f(X) \subseteq Z$. Then the corestriction $f : X \rightarrow Z$ is continuous.*

PROOF:

⟨1⟩1. LET: V be open in Z .

⟨1⟩2. PICK U open in Y such that $V = U \cap Z$.

⟨1⟩3. $f^{-1}(V) = f^{-1}(U)$

⟨1⟩4. $f^{-1}(V)$ is open in X .

□

Theorem 9.90. *Let $f : X \rightarrow Y$ be continuous. Let Z be a space such that Y is a subspace of Z . Then the expansion $f : X \rightarrow Z$ is continuous.*

PROOF: Let V be open in Z . Then $f^{-1}(V) = f^{-1}(V \cap Y)$ is open in X . □

Theorem 9.91. *Let X and Y be topological spaces. Let $f : X \rightarrow Y$. Suppose \mathcal{U} is a set of open sets in X such that $X = \bigcup \mathcal{U}$ and, for all $U \in \mathcal{U}$, we have $f \upharpoonright U : U \rightarrow Y$ is continuous. Then f is continuous.*

PROOF:

- $\langle 1 \rangle 1$. LET: V be open in Y
- $\langle 1 \rangle 2$. $f^{-1}(V) = \bigcup_{U \in \mathcal{U}} (f \upharpoonright U)^{-1}(V)$
- $\langle 1 \rangle 3$. For all $U \in \mathcal{U}$, we have $(f \upharpoonright U)^{-1}(V)$ is open in U .
- $\langle 1 \rangle 4$. For all $U \in \mathcal{U}$, we have $(f \upharpoonright U)^{-1}(V)$ is open in X .

PROOF: Lemma 9.134.

□

Proposition 9.92. *Let X and X' be the same set X under two topologies \mathcal{T} and \mathcal{T}' . Let $i : X \rightarrow X'$ be the identity function. Then i is continuous if and only if $\mathcal{T}' \subseteq \mathcal{T}$.*

PROOF: Immediate from definitions. □

Proposition 9.93. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ and $a \in \mathbb{R}$. Then f is continuous on the right at a if and only if f is continuous at a as a function $\mathbb{R}_l \rightarrow \mathbb{R}$.*

PROOF:

- $\langle 1 \rangle 1$. If f is continuous on the right at a then f is continuous at a as a function $\mathbb{R}_l \rightarrow \mathbb{R}$.
 - $\langle 2 \rangle 1$. ASSUME: f is continuous on the right at a .
 - $\langle 2 \rangle 2$. LET: V be a neighbourhood of $f(a)$
 - $\langle 2 \rangle 3$. PICK b, c such that $f(a) \in (b, c) \subseteq V$.
 - $\langle 2 \rangle 4$. LET: $\epsilon = \min(c - f(a), f(a) - b)$
 - $\langle 2 \rangle 5$. PICK $\delta > 0$ such that, for all x , if $a < x < a + \delta$ then $|f(x) - f(a)| < \epsilon$
 - $\langle 2 \rangle 6$. LET: $U = [a, a + \delta)$
 - $\langle 2 \rangle 7$. $f(U) \subseteq V$
- $\langle 1 \rangle 2$. If f is continuous at a as a function $\mathbb{R}_l \rightarrow \mathbb{R}$ then f is continuous on the right at a .
 - $\langle 2 \rangle 1$. ASSUME: f is continuous at a as a function $\mathbb{R}_l \rightarrow \mathbb{R}$
 - $\langle 2 \rangle 2$. LET: $\epsilon > 0$
 - $\langle 2 \rangle 3$. PICK a neighbourhood U of a such that $f(U) \subseteq (f(a) - \epsilon, f(a) + \epsilon)$
 - $\langle 2 \rangle 4$. PICK b, c such that $a \in [b, c) \subset U$
 - $\langle 2 \rangle 5$. LET: $\delta = c - a$
 - $\langle 2 \rangle 6$. For all x , if $a < x < a + \delta$ then $|f(x) - f(a)| < \epsilon$

□

Lemma 9.94. *Let $f : X \rightarrow Y$. Let Z be an open subspace of X and $a \in Z$. If $f \upharpoonright Z$ is continuous at a then f is continuous at a .*

PROOF:

- $\langle 1 \rangle 1$. LET: V be a neighbourhood of $f(a)$
- $\langle 1 \rangle 2$. PICK a neighbourhood W of a in Z such that $f(W) \subseteq V$
- $\langle 1 \rangle 3$. W is a neighbourhood of a in X such that $f(W) \subseteq V$

PROOF: Lemma 9.134.

□

Proposition 9.95. *Let $f : A \rightarrow B$ and $g : C \rightarrow D$ be continuous. Define $f \times g : A \times C \rightarrow B \times D$ by*

$$(f \times g)(a, c) = (f(a), g(c)) .$$

Then $f \times g$ is continuous.

PROOF: $\pi_1 \circ (f \times g) = f \circ \pi_1$ and $\pi_2 \circ (f \times g) = g \circ \pi_2$ are continuous by Theorem 9.87. The result follows by Theorem 9.123.

Proposition 9.96. *Let X and Y be topological spaces and $f : X \rightarrow Y$ be continuous. If $a_n \rightarrow l$ as $n \rightarrow \infty$ in X then $f(a_n) \rightarrow f(l)$ as $n \rightarrow \infty$.*

PROOF:

- ⟨1⟩1. LET: V be a neighbourhood of $f(l)$
- ⟨1⟩2. PICK a neighbourhood U of l such that $f(U) \subseteq V$
- ⟨1⟩3. PICK N such that, for all $n \geq N$, we have $a_n \in U$
- ⟨1⟩4. For all $n \geq N$ we have $f(a_n) \in V$

□

9.15 Homeomorphisms

Definition 9.97 (Homeomorphism). Let X and Y be topological spaces. A *Homeomorphism* f between X and Y , $f : X \cong Y$, is a bijection $f : X \rightarrow Y$ such that both f and f^{-1} are continuous.

Lemma 9.98. *Let X and Y be topological spaces and $f : X \rightarrow Y$ a bijection. Then the following are equivalent:*

1. f is a homeomorphism.
2. f is continuous and an open map.
3. f is continuous and a closed map.
4. For any $U \subseteq X$, we have U is open if and only if $f(U)$ is open.

PROOF: Immediate from definitions. □

Proposition 9.99. *Let X and X' be the same set X under two topologies \mathcal{T} and \mathcal{T}' . Let $i : X \rightarrow X'$ be the identity function. Then i is a homeomorphism if and only if $\mathcal{T} = \mathcal{T}'$.*

PROOF: Immediate from definitions. □

Definition 9.100 (Topological Property). Let P be a property of topological spaces. Then P is a *topological* property if and only if, for any spaces X and Y , if P holds of X and $X \cong Y$ then P holds of Y .

Definition 9.101 (Topological Imbedding). Let X and Y be topological spaces and $f : X \rightarrow Y$. Then f is a *topological imbedding* if and only if the corestriction $f : X \rightarrow f(X)$ is a homeomorphism.

Proposition 9.102. Let X and Y be topological spaces and $a \in X$. The function $i : Y \rightarrow X \times Y$ that maps y to (a, y) is an imbedding.

PROOF:

$\langle 1 \rangle 1$. i is injective

$\langle 1 \rangle 2$. i is continuous.

PROOF: For U open in X and V open in Y , we have $i^{-1}(U \times V)$ is V if $a \in U$, and \emptyset if $a \notin U$.

$\langle 1 \rangle 3$. $i : Y \rightarrow i(Y)$ is an open map.

PROOF: For V open in Y we have $i(V) = (X \times V) \cap i(Y)$.

□

9.16 The Order Topology

Definition 9.103 (Order Topology). Let X be a linearly ordered set with at least two points. The *order topology* on X is the topology generated by the basis \mathcal{B} consisting of:

- all open intervals (a, b) ;
- all intervals of the form $[\perp, b)$ where \perp is least in X ;
- all intervals of the form $(a, \top]$ where \top is greatest in X .

We prove this is a basis for a topology.

PROOF:

$\langle 1 \rangle 1$. For all $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B$.

$\langle 2 \rangle 1$. LET: $x \in X$

$\langle 2 \rangle 2$. CASE: x is greatest in X .

$\langle 3 \rangle 1$. PICK $y \in X$ with $y \neq x$

$\langle 3 \rangle 2$. $x \in (y, x] \in \mathcal{B}$

$\langle 2 \rangle 3$. CASE: x is least in X .

$\langle 3 \rangle 1$. PICK $y \in X$ with $y \neq x$

$\langle 3 \rangle 2$. $x \in [x, y) \in \mathcal{B}$

$\langle 2 \rangle 4$. CASE: x is neither greatest nor least in X .

$\langle 3 \rangle 1$. PICK $a, b \in X$ with $a < x$ and $x < b$

$\langle 3 \rangle 2$. $x \in (a, b) \in \mathcal{B}$

$\langle 1 \rangle 2$. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$

$\langle 2 \rangle 1$. LET: $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$

$\langle 2 \rangle 2$. CASE: $B_1 = (a, b)$, $B_2 = (c, d)$

PROOF: Take $B_3 = (\max(a, c), \min(b, d))$.

$\langle 2 \rangle 3$. CASE: $B_1 = (a, b)$, $B_2 = [\perp, d)$

PROOF: Take $B_3 = (a, \min(b, d))$.
 $\langle 2 \rangle 4$. CASE: $B_1 = (a, b)$, $B_2 = (c, \top]$
PROOF: Take $B_3 = (\max(a, c), b)$.
 $\langle 2 \rangle 5$. CASE: $B_1 = [\perp, b)$, $B_2 = [\perp, d)$
PROOF: Take $B_3 = [\perp, \min(b, d))$.
 $\langle 2 \rangle 6$. CASE: $B_1 = [\perp, b)$, $B_2 = (c, \top]$
PROOF: Take $B_3 = (c, b)$.

□

Lemma 9.104. *Let X be a linearly ordered set. Then the open rays form a subbasis for the order topology on X .*

PROOF:

$\langle 1 \rangle 1$. Every open ray is open.
 $\langle 2 \rangle 1$. For all $a \in X$, the ray $(-\infty, a)$ is open.
 $\langle 3 \rangle 1$. LET: $x \in (-\infty, a)$
 $\langle 3 \rangle 2$. CASE: x is least in X
PROOF: $x \text{ in } [x, a) = (-\infty, a)$.
 $\langle 3 \rangle 3$. CASE: x is not least in X
 $\langle 4 \rangle 1$. PICK $y < x$
 $\langle 4 \rangle 2$. $x \in (y, a) \subseteq (-\infty, a)$
 $\langle 2 \rangle 2$. For all $a \in X$, the ray $(a, +\infty)$ is open.
PROOF: Similar.
 $\langle 1 \rangle 2$. Every basic open set is a finite intersection of open rays.
PROOF: We have $(a, b) = (a, +\infty) \cap (-\infty, b)$, $[\perp, b) = (-\infty, b)$ and $(a, \top] = (a, +\infty)$.

□

Definition 9.105 (Standard Topology on the Real Line). The *standard topology on the real line* is the order topology on \mathbb{R} generated by the standard order.

Lemma 9.106. *The lower limit topology is strictly finer than the standard topology on \mathbb{R} .*

PROOF:

$\langle 1 \rangle 1$. Every open interval is open in the lower limit topology.
PROOF: If $x \in (a, b)$ then $x \in [x, b) \subseteq (a, b)$.
 $\langle 1 \rangle 2$. The half-open interval $[0, 1)$ is not open in the standard topology.
PROOF: There is no open interval (a, b) such that $0 \in (a, b) \subseteq [0, 1)$.

□

Lemma 9.107. *The K -topology is strictly finer than the standard topology on \mathbb{R} .*

PROOF:

$\langle 1 \rangle 1$. Every open interval is open in the K -topology.
PROOF: Corollary 9.53.1.
 $\langle 1 \rangle 2$. The set $(-1, 1) \setminus K$ is not open in the standard topology.

PROOF: There is no open interval (a, b) such that $0 \in (a, b) \subseteq (-1, 1) \setminus K$, since there must be a positive integer n with $1/n \in (a, b)$.

□

Lemma 9.108. *Let X be a topological space. Let Y be a linearly ordered set in the order topology. Let $f, g : X \rightarrow Y$ be continuous. Then $C = \{x \in X \mid f(x) \leq g(x)\}$ is closed.*

PROOF:

⟨1⟩1. LET: $x \in X \setminus C$

⟨1⟩2. $f(x) > g(x)$

PROVE: There exists a neighbourhood U of x such that $U \subseteq X \setminus C$

⟨1⟩3. CASE: There exists y such that $g(x) < y < f(x)$

PROOF: Take $U = g^{-1}((-\infty, y)) \cup f^{-1}(y, +\infty)$.

⟨1⟩4. CASE: There is no y such that $g(x) < y < f(x)$

PROOF: Take $U = g^{-1}((-\infty, f(x))) \cup f^{-1}(g(x), +\infty)$.

□

Proposition 9.109. *Let X be a topological space. Let Y be a linearly ordered set in the order topology. Let $f, g : X \rightarrow Y$ be continuous. Define $h : X \rightarrow Y$ by $h(x) = \min(f(x), g(x))$. Then h is continuous.*

PROOF: By the Pasting Lemma applied to $\{x \in X \mid f(x) \leq g(x)\}$ and $\{x \in X \mid g(x) \leq f(x)\}$, which are closed by Lemma 9.108.

Proposition 9.110. *Let X and Y be linearly ordered sets in the order topology. Let $f : X \rightarrow Y$ be strictly monotone and surjective. Then f is a homeomorphism.*

PROOF:

⟨1⟩1. f is bijective.

PROOF: Proposition 4.13.

⟨1⟩2. f is continuous.

⟨2⟩1. For all $y \in Y$ we have $f^{-1}((y, +\infty))$ is open.

⟨3⟩1. LET: $y \in Y$

⟨3⟩2. PICK $x \in X$ such that $f(x) = y$

PROOF: Since f is surjective.

⟨3⟩3. $f^{-1}((y, +\infty)) = (x, +\infty)$

PROOF: By strict monotonicity.

⟨2⟩2. For all $y \in Y$ we have $f^{-1}((-\infty, y))$ is open.

PROOF: Similar.

⟨1⟩3. f^{-1} is continuous.

⟨2⟩1. For all $x \in X$ we have $f((x, +\infty))$ is open.

PROOF: $f((x, +\infty)) = (f(x), +\infty)$.

⟨2⟩2. For all $x \in X$ we have $f((-\infty, x))$ is open.

PROOF: $f((-\infty, x)) = (-\infty, f(x))$.

□

9.17 The n th Root Function

Proposition 9.111. For all $n \geq 1$, the function $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ defined by $f(x) = x^n$ is a homeomorphism.

PROOF:

$\langle 1 \rangle 1$. f is strictly monotone.

$\langle 2 \rangle 1$. LET: $x, y \in \mathbb{R}$ with $0 \leq x < y$

$\langle 2 \rangle 2$. $x^n < y^n$

$$\begin{aligned} y^n - x^n &= (y - x)(y^{n-1} + y^{n-2}x + y^{n-3}x^2 + \cdots + x^{n-1}) \\ &> 0 \end{aligned}$$

$\langle 1 \rangle 2$. f is surjective.

$\langle 2 \rangle 1$. LET: $y \in \mathbb{R}_{\geq 0}$

$\langle 2 \rangle 2$. PICK $x \in \mathbb{R}$ such that $y \leq x^n$

PROOF: If $y \leq 1$ take $x = 1$, otherwise take $x = y$.

$\langle 2 \rangle 3$. There exists $x' \in [0, x]$ such that $(x')^n = y$

PROOF: By the Intermediate Value Theorem.

$\langle 1 \rangle 3$. Q.E.D.

PROOF: Proposition 9.110.

□

Definition 9.112. For $n \geq 1$, the n th root function is the function $\mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ that is the inverse of $\lambda x.x^n$.

9.18 The Product Topology

Definition 9.113 (Product Topology). Let $\{A_i\}_{i \in I}$ be a family of topological spaces. The *product topology* on $\prod_{i \in I} A_i$ is the topology generated by the sub-basis consisting of the sets of the form $\pi_i^{-1}(U)$ where $i \in I$ and U is open in A_i .

Proposition 9.114. The product topology on $\prod_{i \in I} A_i$ is generated by the basis consisting of all sets of the form $\prod_{i \in I} U_i$ where $\{U_i\}_{i \in I}$ is a family such that each U_i is an open set in A_i and $U_i = A_i$ for all but finitely many i .

PROOF: From Proposition 9.61. □

Proposition 9.115. If A_i is closed in X_i for all $i \in I$ then $\prod_{i \in I} A_i$ is closed in $\prod_{i \in I} X_i$.

PROOF:

$$\left(\prod_{i \in I} X_i \right) \setminus \left(\prod_{i \in I} A_i \right) = \bigcup_{j \in I} \left(\prod_{i \in I} X_i \setminus \pi_j^{-1}(A_j) \right) \square$$

Proposition 9.116. Let $\{A_i\}_{i \in I}$ be a family of topological spaces. For $i \in I$, let \mathcal{B}_i be a basis for the topology on A_i . Then $\mathcal{B} = \{ \prod_{i \in I} B_i \mid \forall i \in I. B_i \in \mathcal{B}_i, B_i = A_i \text{ for all but finitely many } i \}$ is a basis for the box topology on $\prod_{i \in I} A_i$.

PROOF:

- ⟨1⟩1. Every set in \mathcal{B} is open.
- ⟨1⟩2. For every point $a \in \prod_{i \in I} A_i$ and every open set U with $a \in U$, there exists $B \in \mathcal{B}$ such that $a \in B \subseteq U$.
 - ⟨2⟩1. LET: U be open and $a \in U$
 - ⟨2⟩2. PICK a family $\{U_i\}_{i \in I}$ such that each U_i is open in A_i , such that $U_i = A_i$ except for $i = i_1, \dots, i_n$, and such that $a \in \prod_{i \in I} U_i \subseteq U$.
 - ⟨2⟩3. For $j = 1, \dots, n$, PICK $B_{i_j} \in \mathcal{B}_{i_j}$ such that $a_{i_j} \in B_{i_j} \subseteq U_{i_j}$
 - ⟨2⟩4. LET: $B = \prod_{i \in I} B_i$ where $B_i = A_i$ for $i \neq i_1, \dots, i_n$
 - ⟨2⟩5. $B \in \mathcal{B}$
 - ⟨2⟩6. $a \in B \subseteq U$
- ⟨1⟩3. Q.E.D.

PROOF: Lemma 9.54.

□

Proposition 9.117. *Let $\{A_i\}_{i \in I}$ be a family of topological spaces. Then the projections $\pi_i : \prod_{i \in I} A_i \rightarrow A_i$ are open maps.*

PROOF: From Lemma 9.77. □

Example 9.118. The projections are not always closed maps. For example, $\pi_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$ maps the closed set $\{(x, 1/x) \mid x > 0\}$ to $(0, +\infty)$.

Proposition 9.119. *Let $\{X_i\}_{i \in I}$ be a family of sets. For $i \in I$, let \mathcal{T}_i and \mathcal{U}_i be topologies on X_i . Let \mathcal{P} be the product topology on $\prod_{i \in I} X_i$ generated by the topologies \mathcal{T}_i , and \mathcal{Q} the product topology on the same set generated by the topologies \mathcal{U}_i . Then $\mathcal{P} \subseteq \mathcal{Q}$ if and only if $\mathcal{T}_i \subseteq \mathcal{U}_i$ for all i .*

PROOF:

- ⟨1⟩1. If $\mathcal{T}_i \subseteq \mathcal{U}_i$ for all i then $\mathcal{P} \subseteq \mathcal{Q}$

PROOF: By Corollary 9.53.1.
- ⟨1⟩2. If $\mathcal{P} \subseteq \mathcal{Q}$ then $\mathcal{T}_i \subseteq \mathcal{U}_i$ for all i
 - ⟨2⟩1. ASSUME: $\mathcal{P} \subseteq \mathcal{Q}$
 - ⟨2⟩2. LET: $i \in I$
 - ⟨2⟩3. LET: $U \in \mathcal{T}_i$
 - ⟨2⟩4. LET: $U_i = U$ and $U_j = X_j$ for $j \neq i$
 - ⟨2⟩5. $\prod_{i \in I} U_i \in \mathcal{P}$
 - ⟨2⟩6. $\prod_{i \in I} U_i \in \mathcal{Q}$
 - ⟨2⟩7. $U \in \mathcal{U}_i$

PROOF: From Proposition 9.117.

□

Proposition 9.120 (AC). *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let $A_i \subseteq X_i$ for all $i \in I$. Then*

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i} .$$

PROOF:

- (1)1. $\overline{\prod_{i \in I} A_i} \subseteq \prod_{i \in I} \overline{A_i}$
 (2)1. For all $i \in I$ we have $A_i \subseteq \overline{A_i}$
 PROOF: Lemma 9.35.
 (2)2. $\prod_{i \in I} A_i \subseteq \prod_{i \in I} \overline{A_i}$
 (2)3. Q.E.D.
 PROOF: Since $\prod_{i \in I} A_i$ is closed by Proposition 9.115.
 (1)2. $\prod_{i \in I} \overline{A_i} \subseteq \overline{\prod_{i \in I} A_i}$
 (2)1. LET: $x \in \prod_{i \in I} \overline{A_i}$
 (2)2. LET: U be a neighbourhood of x
 (2)3. PICK V_i open in X_i such that $x \in \prod_{i \in I} V_i \subseteq U$ with $V_i = X_i$ except for
 $i = i_1, \dots, i_n$
 (2)4. For $i \in I$, pick $a_i \in V_i \cap A_i$
 PROOF: By Theorem 9.38 and (2)1 using the Axiom of Choice.
 (2)5. U intersects $\prod_{i \in I} A_i$
 (2)6. Q.E.D.
 PROOF: $a \in U \cap \prod_{i \in I} A_i$

□

Example 9.121. The closure of \mathbb{R}^∞ in \mathbb{R}^ω is \mathbb{R}^ω

PROOF:

- (1)1. LET: $a \in \mathbb{R}^\omega$
 (1)2. LET: U be any neighbourhoods of a .
 (1)3. PICK U_n open in \mathbb{R} for all n such that $a \in \prod_{n \geq 0} U_n \subseteq U$ and $U_n = \mathbb{R}$ for
 all n except n_1, \dots, n_k
 (1)4. LET: $b_n = a_n$ for $n = n_1, \dots, n_k$ and $b_n = 0$ for all other n
 (1)5. $b \in \mathbb{R}^\infty \cap U$
 (1)6. Q.E.D.

PROOF: From Theorem 9.38.

□

Proposition 9.122. Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let (a_n) be a sequence in $\prod_{i \in I} X_i$ and $l \in \prod_{i \in I} X_i$. Then $a_n \rightarrow l$ as $n \rightarrow \infty$ if and only if, for all $i \in I$, we have $\pi_i(a_n) \rightarrow \pi_i(l)$ as $n \rightarrow \infty$.

PROOF:

- (1)1. If $a_n \rightarrow l$ as $n \rightarrow \infty$ then, for all $i \in I$, we have $\pi_i(a_n) \rightarrow \pi_i(l)$ as $n \rightarrow \infty$
 PROOF: Proposition 9.96.
 (1)2. If, for all $i \in I$, we have $\pi_i(a_n) \rightarrow \pi_i(l)$ as $n \rightarrow \infty$, then $a_n \rightarrow l$ as $n \rightarrow \infty$
 (2)1. ASSUME: For all $i \in I$, we have $\pi_i(a_n) \rightarrow \pi_i(l)$ as $n \rightarrow \infty$
 (2)2. LET: V be a neighbourhood of l
 (2)3. PICK open sets U_i in X_i such that $l \in \prod_{i \in I} U_i \subseteq V$ and $U_i = X_i$ for all
 i except $i = i_1, \dots, i_k$
 (2)4. For $j = 1, \dots, k$, PICK N_j such that, for all $n \geq N_j$, we have $\pi_{i_j}(a_n) \in$
 U_{i_j}
 (2)5. LET: $N = \max(N_1, \dots, N_k)$
 (2)6. For all $n \geq N$ we have $a_n \in V$

□

Theorem 9.123. *Let A be a topological space and $\{X_i\}_{i \in I}$ be a family of topological spaces. Let $f : A \rightarrow \prod_{i \in I} X_i$ be a function. If $\pi_i \circ f$ is continuous for all $i \in I$ then f is continuous.*

PROOF:

⟨1⟩1. LET: $i \in I$ and U be open in X_i

⟨1⟩2. $f^{-1}(\pi_i^{-1}(U))$ is open in A

⟨1⟩3. Q.E.D.

PROOF: Proposition 9.81.

□

9.18.1 Continuous in Each Variable Separately

Definition 9.124 (Continuous in Each Variable Separately). Let $F : X \times Y \rightarrow Z$. Then F is *continuous in each variable separately* if and only if:

- for every $a \in X$ the function $\lambda y \in Y. F(a, y)$ is continuous;
- for every $b \in Y$ the function $\lambda x \in X. F(x, b)$ is continuous.

Proposition 9.125. *Let $F : X \times Y \rightarrow Z$. If F is continuous then F is continuous in each variable separately.*

PROOF: For $a \in X$, the function $\lambda y \in Y. F(a, y)$ is $F \circ i$ where $i : Y \rightarrow X \times Y$ maps y to (a, y) . We have i is continuous by Proposition 9.102, hence $F \circ i$ is continuous by Theorem 9.87.

Similarly for $\lambda x \in X. F(x, b)$ for $b \in Y$. □

Example 9.126. Define $F : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$F(x, y) = \begin{cases} xy/(x^2 + y^2) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Then F is continuous in each variable separately but not continuous.

Proposition 9.127. *Let $f : A \rightarrow C$ and $g : B \rightarrow D$ be open maps. Then $f \times g : A \times B \rightarrow C \times D$ is an open map.*

PROOF: Given U open in A and V open in B . Then $(f \times g)(U \times V) = f(U) \times g(V)$ is open in $C \times D$. The result follows from Proposition 9.78. □

Definition 9.128 (Sorgenfrey Plane). The *Sorgenfrey plane* is \mathbb{R}_l^2 .

9.19 The Subspace Topology

Definition 9.129 (Subspace Topology). Let X be a topological space and $Y \subseteq X$. The *subspace topology* on Y is $\mathcal{T} = \{U \cap Y \mid U \text{ is open in } X\}$.

We prove this is a topology.

PROOF:

$\langle 1 \rangle 1. Y \in \mathcal{T}$

PROOF: Since $Y = X \cap Y$

$\langle 1 \rangle 2. \text{ For all } \mathcal{U} \subseteq \mathcal{T}, \text{ we have } \bigcup \mathcal{U} \in \mathcal{T}$

$\langle 2 \rangle 1. \text{ LET: } \mathcal{U} \subseteq \mathcal{T}$

$\langle 2 \rangle 2. \text{ LET: } \mathcal{V} = \{V \text{ open in } X \mid V \cap Y \in \mathcal{U}\}$

$\langle 2 \rangle 3. \bigcup \mathcal{U} = (\bigcup \mathcal{V}) \cap Y$

$\langle 1 \rangle 3. \text{ For all } U, V \in \mathcal{T}, \text{ we have } U \cap V \in \mathcal{T}$

$\langle 2 \rangle 1. \text{ LET: } U, V \in \mathcal{T}$

$\langle 2 \rangle 2. \text{ PICK } U', V' \text{ open in } X \text{ such that } U = U' \cap Y \text{ and } V = V' \cap Y$

$\langle 2 \rangle 3. (U \cap V) = (U' \cap V') \cap Y$

□

Theorem 9.130. *Let X be a topological space and Y a subspace of X . Let $A \subseteq Y$. Then A is closed in Y if and only if there exists a closed set C in X such that $A = C \cap Y$.*

PROOF: We have

$$\begin{aligned} & A \text{ is closed in } Y \\ \Leftrightarrow & Y \setminus A \text{ is open in } Y \\ \Leftrightarrow & \exists U \text{ open in } X. Y \setminus A = Y \cap U \\ \Leftrightarrow & \exists C \text{ closed in } X. Y \setminus A = Y \cap (X \setminus U) \\ \Leftrightarrow & \exists C \text{ closed in } X. A = Y \cap U \end{aligned}$$

□

Theorem 9.131. *Let Y be a subspace of X . Let $A \subseteq Y$. Let \overline{A} be the closure of A in X . Then the closure of A in Y is $\overline{A} \cap Y$.*

PROOF: The closure of A in Y is

$$\begin{aligned} & \bigcap \{C \text{ closed in } Y \mid A \subseteq C\} \\ &= \bigcap \{D \cap Y \mid D \text{ closed in } X, A \subseteq D \cap Y\} \quad (\text{Theorem 9.130}) \\ &= \bigcap \{D \mid D \text{ closed in } X, A \subseteq D\} \cap Y \\ &= \overline{A} \cap Y \end{aligned}$$

□

Lemma 9.132. *Let X be a topological space and $Y \subseteq X$. Let \mathcal{B} be a basis for the topology on X . Then $\mathcal{B}' = \{B \cap Y \mid B \in \mathcal{B}\}$ is a basis for the subspace topology on Y .*

PROOF:

$\langle 1 \rangle 1. \text{ Every element in } \mathcal{B}' \text{ is open in } Y$

$\langle 1 \rangle 2. \text{ For every open set } U \text{ in } Y \text{ and point } y \in U, \text{ there exists } B' \in \mathcal{B}' \text{ such that } y \in B' \subseteq U$

$\langle 2 \rangle 1. \text{ LET: } U \text{ be open in } Y \text{ and } y \in U$

$\langle 2 \rangle 2. \text{ PICK } V \text{ open in } X \text{ such that } U = V \cap Y$

$\langle 2 \rangle 3. \text{ PICK } B \in \mathcal{B} \text{ such that } y \in B \subseteq V$

⟨2⟩4. LET: $B' = B \cap Y$

⟨2⟩5. $B' \in \mathcal{B}'$

⟨2⟩6. $y \in B' \subseteq U$

⟨1⟩3. Q.E.D.

PROOF: By Lemma 9.54.

□

Lemma 9.133. *Let X be a topological space and $Y \subseteq X$. Let \mathcal{S} be a basis for the topology on X . Then $\mathcal{S}' = \{S \cap Y \mid S \in \mathcal{S}\}$ is a subbasis for the subspace topology on Y .*

PROOF: The set $\{B \cap Y \mid B \text{ is a finite intersection of elements of } \mathcal{S}\}$ is a basis for the subspace topology by Lemma 9.132, and this is the set of all finite intersections of elements of \mathcal{S}' . □

Lemma 9.134. *Let Y be a subspace of X . If U is open in Y and Y is open in X then U is open in X .*

PROOF:

⟨1⟩1. PICK V open in X such that $U = V \cap Y$

⟨1⟩2. U is open in X

PROOF: Since it is the intersection of two open sets V and Y .

□

Theorem 9.135. *Let Y be a subspace of X and $A \subseteq Y$. If A is closed in Y and Y is closed in X then A is closed in X .*

PROOF: Pick a closed set C in X such that $A = C \cap Y$ (Theorem 9.130). Then A is the intersection of two sets closed in X , hence A is closed in X (Lemma 9.23). □

Theorem 9.136. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let A_i be a subspace of X_i for all $i \in I$. Then the product topology on $\prod_{i \in I} A_i$ is the same as the topology it inherits as a subspace of $\prod_{i \in I} X_i$.*

PROOF: The product topology is generated by the subbasis

$$\begin{aligned} & \{\pi_i^{-1}(U) \mid i \in I, U \text{ open in } A_i\} \\ &= \{\pi_i^{-1}(V) \cap A_i \mid i \in I, V \text{ open in } X_i\} \\ &= \{\pi_i^{-1}(V) \mid i \in I, V \text{ open in } X_i\} \cap \prod_{i \in I} A_i \end{aligned}$$

and this is a subbasis for the subspace topology by Lemma 9.133. □

Theorem 9.137. *Let X be an ordered set in the order topology. Let $Y \subseteq X$ be an interval. Then the order topology on Y is the same as the subspace topology on Y .*

PROOF:

⟨1⟩1. The order topology is finer than the subspace topology.

- ⟨2⟩1. For every open ray R in X , the set $R \cap Y$ is open in the order topology.
- ⟨3⟩1. For all $a \in X$, we have $(-\infty, a) \cap Y$ is open in the order topology.
 - ⟨4⟩1. CASE: For all $y \in Y$ we have $y < a$
 PROOF: In this case $(-\infty, a) \cap Y = Y$.
 - ⟨4⟩2. CASE: For all $y \in Y$ we have $a < y$
 PROOF: In this case $(-\infty, a) \cap Y = \emptyset$.
 - ⟨4⟩3. CASE: There exists $y \in Y$ such that $y \leq a$ and $y \in Y$ such that
 $a \leq y$
 - ⟨5⟩1. $a \in Y$
 PROOF: Because Y is an interval.
 - ⟨5⟩2. $(-\infty, a) \cap Y = \{y \in Y \mid y < a\}$
- ⟨3⟩2. For all $a \in X$, we have $(a, +\infty) \cap Y$ is open in the order topology.
 PROOF: Similar.
- ⟨2⟩2. Q.E.D.
 PROOF: By Lemmas 9.104 and 9.133 and Proposition 9.62.
- ⟨1⟩2. The subspace topology is finer than the order topology.
- ⟨2⟩1. Every open ray in Y is open in the subspace topology.
 PROOF: For any $a \in Y$ we have $(-\infty, a)_Y = (-\infty, a)_X \cap Y$ and $(a, +\infty)_Y = (a, +\infty)_X \cap Y$.
- ⟨2⟩2. Q.E.D.
 PROOF: By Lemma 9.104 and Proposition 9.62

□

This example shows that we cannot remove the hypothesis that Y is an interval:

Example 9.138. The order topology on I_o^2 is different from the subspace topology as a subspace of \mathbb{R}^2 under the dictionary order topology.

PROOF: The set $\{1/2\} \times (1/2, 1)$ is open in the subspace topology but not in the order topology. □

Proposition 9.139. *Let X be a topological space, Y a subspace of X , and Z a subspace of Y . Then the subspace topology on Z inherited from X is the same as the subspace topology on Z inherited from Y .*

PROOF: The subspace topology inherited from Y is

$$\begin{aligned}
 & \{V \cap Z \mid V \text{ open in } Y\} \\
 &= \{U \cap Y \cap Z \mid U \text{ open in } X\} \\
 &= \{U \cap Z \mid U \text{ open in } X\}
 \end{aligned}$$

which is the subspace topology inherited from X . □

Definition 9.140 (Unit Circle). The *unit circle* S^1 is $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ as a subspace of \mathbb{R}^2 .

Definition 9.141 (Unit 2-sphere). The *unit 2-sphere* is $S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$ as a subspace of \mathbb{R}^3 .

Proposition 9.142. *Let $f : X \rightarrow Y$ be an open map and $A \subseteq X$ be open. Then the restriction $f \upharpoonright A : A \rightarrow f(A)$ is an open map.*

PROOF:

$\langle 1 \rangle 1$. LET: U be open in A

$\langle 1 \rangle 2$. U is open in X

PROOF: Lemma 9.134.

$\langle 1 \rangle 3$. $f(U)$ is open in Y

$\langle 1 \rangle 4$. $f(U)$ is open in $f(A)$

PROOF: Since $f(U) = f(U) \cap f(A)$.

□

Example 9.143. This example shows that we cannot remove the hypothesis that A is open.

Let $A = \{(x, y) \in \mathbb{R}^2 \mid (x > 0 \text{ and } y = 1/x) \text{ or } x = y = 0\}$. Then $\pi_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$ is an open map, but $\pi_1 \upharpoonright A : A \rightarrow [0, +\infty)$ is not, because it maps the set $\{0, 0\}$ which is open in A to $\{0\}$ which is not open in $[0, +\infty)$.

Proposition 9.144. *Let Y be a subspace of X . Let $A \subseteq Y$ and $l \in Y$. Then l is a limit point of A in Y if and only if l is a limit point of A in X .*

PROOF: Both are equivalent to the condition that any neighbourhood of l in X intersects A in a point other than l . □

9.20 The Box Topology

Definition 9.145 (Box Topology). Let $\{A_i\}_{i \in I}$ be a family of topological spaces. The *box topology* on $\prod_{i \in I} A_i$ is the topology generated by the set of all sets of the form $\prod_{i \in I} U_i$ where $\{U_i\}_{i \in I}$ is a family such that each U_i is open in A_i .

This is a basis since it covers $\prod_{i \in I} A_i$ and is closed under intersection.

Proposition 9.146. *The box topology is finer than the product topology.*

PROOF: From Proposition 9.114. □

Corollary 9.146.1. *If A_i is closed in X_i for all $i \in I$ then $\prod_{i \in I} A_i$ is closed in $\prod_{i \in I} X_i$ under the box topology.*

PROOF: From Proposition 9.115.

Proposition 9.147 (AC). *Let $\{A_i\}_{i \in I}$ be a family of topological spaces. For $i \in I$, let \mathcal{B}_i be a basis for the topology on A_i . Then $\mathcal{B} = \{\prod_{i \in I} B_i \mid \forall i \in I. B_i \in \mathcal{B}_i\}$ is a basis for the box topology on $\prod_{i \in I} A_i$.*

PROOF:

$\langle 1 \rangle 1$. Every set of the form $\prod_{i \in I} B_i$ is open.

$\langle 1 \rangle 2$. For every point $a \in \prod_{i \in I} A_i$ and every open set U with $a \in U$, there exists $B \in \mathcal{B}$ such that $a \in B \subseteq U$.

- (2)1. LET: U be open and $a \in U$
 (2)2. PICK a family $\{U_i\}_{i \in I}$ such that each U_i is open in A_i and $a \in \prod_{i \in I} U_i \subseteq U$.
 (2)3. For $i \in I$, PICK $B_i \in \mathcal{B}_i$ such that $a_i \in B_i \subseteq U_i$
 PROOF: Using the Axiom of Choice.
 (2)4. $a \in \prod_{i \in I} B_i \subseteq U$
 (1)3. Q.E.D.
 PROOF: Lemma 9.54.

□

Theorem 9.148. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let A_i be a subspace of X_i for all $i \in I$. Give $\prod_{i \in I} X_i$ the box topology. Then the box topology on $\prod_{i \in I} A_i$ is the same as the topology it inherits as a subspace of $\prod_{i \in I} X_i$.*

PROOF: The box topology is generated by the basis

$$\begin{aligned}
 & \left\{ \prod_{i \in I} U_i \mid \forall i \in I, U_i \text{ open in } A_i \right\} \\
 &= \left\{ \prod_{i \in I} (V_i \cap A_i) \mid \forall i \in I, V_i \text{ open in } X_i \right\} \\
 &= \left\{ \prod_{i \in I} V_i \mid \forall i \in I, V_i \text{ open in } X_i \right\} \cap \prod_{i \in I} A_i
 \end{aligned}$$

and this is a basis for the subspace topology by Lemma 9.132. □

Proposition 9.149 (AC). *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Give $\prod_{i \in I} X_i$ the box topology. Let $A_i \subseteq X_i$ for all $i \in I$. Then*

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i}.$$

PROOF:

- (1)1. $\overline{\prod_{i \in I} A_i} \subseteq \prod_{i \in I} \overline{A_i}$
 (2)1. For all $i \in I$ we have $A_i \subseteq \overline{A_i}$
 PROOF: Lemma 9.35.
 (2)2. $\prod_{i \in I} A_i \subseteq \prod_{i \in I} \overline{A_i}$
 (2)3. Q.E.D.
 PROOF: Since $\prod_{i \in I} A_i$ is closed by Corollary 9.146.1.
 (1)2. $\prod_{i \in I} \overline{A_i} \subseteq \overline{\prod_{i \in I} A_i}$
 (2)1. LET: $x \in \prod_{i \in I} \overline{A_i}$
 (2)2. LET: U be a neighbourhood of x
 (2)3. PICK V_i open in X_i such that $x \in \prod_{i \in I} V_i \subseteq U$
 (2)4. For $i \in I$, pick $a_i \in V_i \cap A_i$
 PROOF: By Theorem 9.38 and (2)1 using the Axiom of Choice.
 (2)5. U intersects $\prod_{i \in I} A_i$
 (2)6. Q.E.D.
 PROOF: $a \in U \cap \prod_{i \in I} A_i$.

□

The following example shows that Theorem 9.123 fails in the box topology.

Example 9.150. Define $f : \mathbb{R} \rightarrow \mathbb{R}^\omega$ by $f(t) = (t, t, \dots)$. Then $\pi_n \circ f = \text{id}_{\mathbb{R}}$ is continuous for all n . But f is not continuous when \mathbb{R}^ω is given the box topology because the inverse image of

$$(-1, 1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times \dots$$

is $\{0\}$ which is not open.

The following example shows that Proposition 9.122 fails in the box topology.

Example 9.151. Give \mathbb{R}^ω the box topology. Let $a_n = (1/n, 1/n, \dots)$ for $n \geq 1$ and $l = (0, 0, \dots)$. Then $\pi_i(a_n) \rightarrow \pi_i(l)$ as $n \rightarrow \infty$ for all i , but $a_n \not\rightarrow l$ as $n \rightarrow \infty$ since the open set

$$(-1, 1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times \dots$$

contains l but does not contain any a_n .

Example 9.152. The set \mathbb{R}^∞ is closed in \mathbb{R}^ω under the box topology. For let (a_n) be any sequence not in \mathbb{R}^∞ . Let U_n be an open interval around a_n that does not contain 0 if $a_n \neq 0$, and $U_n = \mathbb{R}$ if $a_n = 0$. Then $\prod_{n \geq 0} U_n$ is a neighbourhood of (a_n) that does not intersect \mathbb{R}^∞ .

9.21 T_1 Spaces

Definition 9.153 (T_1 Space). A topological space is T_1 if and only if every singleton is closed.

Lemma 9.154. A space is T_1 if and only if every finite set is closed.

PROOF: From Lemma 9.24. □

Theorem 9.155. In a T_1 space, a point a is a limit point of a set A if and only if every neighbourhood of a contains infinitely many points of A .

PROOF:

⟨1⟩1. If a is a limit point of A then every neighbourhood of a contains infinitely many points of A .

⟨2⟩1. ASSUME: a is a limit point of A .

⟨2⟩2. LET: U be a neighbourhood of a .

⟨2⟩3. ASSUME: for a contradiction U contains only finitely many points of A .

⟨2⟩4. $(U \cap A) \setminus \{a\}$ is closed.

PROOF: By the T_1 axiom.

⟨2⟩5. $(U \setminus A) \cup \{a\}$ is open.

PROOF: It is $U \setminus ((U \cap A) \setminus \{a\})$.

⟨2⟩6. $(U \setminus A) \cup \{a\}$ intersects A in a point other than a .

PROOF: From $\langle 2 \rangle 1$.

$\langle 2 \rangle 7$. Q.E.D.

□

$\langle 1 \rangle 2$. If every neighbourhood of a contains infinitely many points of A then a is a limit point of A .

PROOF: Immediate from definitions.

□

(To see this does not hold in every space, see Proposition 9.50.)

Proposition 9.156. *A space is T_1 if and only if, for any two distinct points x and y , there exist neighbourhoods U of x and V of y such that $x \notin V$ and $y \notin U$.*

PROOF:

$\langle 1 \rangle 1$. LET: X be a topological space.

$\langle 1 \rangle 2$. If X is T_1 then, for any two distinct points x and y , there exist neighbourhoods U of x and V of y such that $x \notin V$ and $y \notin U$.

PROOF: This holds because $\{x\}$ and $\{y\}$ are closed.

$\langle 1 \rangle 3$. Suppose, for any two distinct points x and y , there exist neighbourhoods U of x and V of y such that $x \notin V$ and $y \notin U$. Then X is T_1 .

$\langle 2 \rangle 1$. ASSUME: For any two distinct points x and y , there exist neighbourhoods U of x and V of y such that $x \notin V$ and $y \notin U$.

$\langle 2 \rangle 2$. LET: $a \in X$

$\langle 2 \rangle 3$. $\{a\}$ is closed.

PROOF: For all $b \neq a$ there exists a neighbourhood U of b such that $U \subseteq X \setminus \{a\}$.

□

Proposition 9.157. *A subspace of a T_1 space is T_1 .*

PROOF: From Proposition 9.135.

9.22 Hausdorff Spaces

Definition 9.158 (Hausdorff Space). A topological space is *Hausdorff* if and only if, for any points x, y with $x \neq y$, there exist disjoint open sets U and V such that $x \in U$ and $y \in V$.

Theorem 9.159. *Every Hausdorff space is T_1 .*

PROOF:

$\langle 1 \rangle 1$. LET: X be a Hausdorff space.

$\langle 1 \rangle 2$. LET: $b \in X$

PROVE: $\overline{\{b\}} = \{b\}$

$\langle 1 \rangle 3$. ASSUME: $a \in \overline{\{b\}}$ and $a \neq b$

$\langle 1 \rangle 4$. PICK disjoint neighbourhoods U of a and V of b .

$\langle 1 \rangle 5$. U intersects $\{b\}$

PROOF: Theorem 9.38.

⟨1⟩6. $b \in U$

⟨1⟩7. Q.E.D.

PROOF: This contradicts the fact that U and V are disjoint (⟨1⟩4).

□

Proposition 9.160. *An infinite set under the finite complement topology is T_1 but not Hausdorff.*

PROOF:

⟨1⟩1. LET: X be an infinite set under the finite complement topology.

⟨1⟩2. Every singleton is closed.

PROOF: By definition.

⟨1⟩3. PICK $a, b \in X$ with $a \neq b$

⟨1⟩4. There are no disjoint neighbourhoods U of a and V of b .

⟨2⟩1. LET: U be a neighbourhood of a and V a neighbourhood of b .

⟨2⟩2. $X \setminus U$ and $X \setminus V$ are finite.

⟨2⟩3. PICK $c \in X$ that is not in $X \setminus U$ or $X \setminus V$.

⟨2⟩4. $c \in U \cap V$

□

Proposition 9.161. *The product of a family of Hausdorff spaces is Hausdorff.*

PROOF:

⟨1⟩1. LET: $\{X_i\}_{i \in I}$ be a family of Hausdorff spaces.

⟨1⟩2. LET: $a, b \in \prod_{i \in I} X_i$ with $a \neq b$

⟨1⟩3. PICK $i \in I$ such that $a_i \neq b_i$

⟨1⟩4. PICK U, V disjoint open sets in X_i with $a_i \in U$ and $b_i \in V$

⟨1⟩5. $\pi_i^{-1}(U)$ and $\pi_i^{-1}(V)$ are disjoint open sets in $\prod_{i \in I} X_i$ with $a \in \pi_i^{-1}(U)$ and $b \in \pi_i^{-1}(V)$

□

Theorem 9.162. *Every linearly ordered set under the order topology is Hausdorff.*

PROOF:

⟨1⟩1. LET: X be a linearly ordered set under the order topology.

⟨1⟩2. LET: $a, b \in X$ with $a \neq b$

⟨1⟩3. ASSUME: w.l.o.g. $a < b$

⟨1⟩4. CASE: There exists c such that $a < c < b$

PROOF: The sets $(-\infty, c)$ and $(c, +\infty)$ are disjoint neighbourhoods of a and b respectively.

⟨1⟩5. CASE: There is no c such that $a < c < b$

PROOF: The sets $(-\infty, b)$ and $(a, +\infty)$ are disjoint neighbourhoods of a and b respectively.

□

Theorem 9.163. *A subspace of a Hausdorff space is Hausdorff.*

PROOF:

- ⟨1⟩1. LET: X be a Hausdorff space and Y a subspace of X .
- ⟨1⟩2. LET: $x, y \in Y$ with $x \neq y$
- ⟨1⟩3. PICK disjoint neighbourhoods U of x and V of y in X .
- ⟨1⟩4. $U \cap Y$ and $V \cap Y$ are disjoint neighbourhoods of x and y respectively in Y .

□

Proposition 9.164. *A space X is Hausdorff if and only if the diagonal $\Delta = \{(x, x) \mid x \in X\}$ is closed in X^2 .*

PROOF:

X is Hausdorff

$$\Leftrightarrow \forall x, y \in X. x \neq y \Rightarrow \exists V, W \text{ open. } x \in V \wedge y \in W \wedge V \cap W = \emptyset$$

$$\Leftrightarrow \forall (x, y) \in X^2 \setminus \Delta. \exists V, W \text{ open. } (x, y) \subseteq V \times W \subseteq X^2 \setminus \Delta$$

$$\Leftrightarrow \Delta \text{ is closed}$$

□

Theorem 9.165. *In a Hausdorff space, a sequence has at most one limit.*

PROOF:

- ⟨1⟩1. LET: X be a Hausdorff space.
- ⟨1⟩2. ASSUME: for a contradiction $a_n \rightarrow l$ as $n \rightarrow \infty$, $a_n \rightarrow m$ as $n \rightarrow \infty$, and $l \neq m$
- ⟨1⟩3. PICK disjoint neighbourhoods U of l and V of m
- PROOF: By the Hausdorff axiom.
- ⟨1⟩4. PICK M and N such that $a_n \in U$ for $n \geq M$ and $a_n \in V$ for $n \geq N$
- ⟨1⟩5. $a_{\max(M, N)} \in U \cap V$
- ⟨1⟩6. Q.E.D.

PROOF: This contradicts the fact that U and V are disjoint (⟨1⟩3).

□

To see this is not always true in spaces that are T_1 but not Hausdorff:

Proposition 9.166. *Let X be an infinite set under the finite complement topology. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence with all points distinct. Then for every $l \in X$ we have $a_n \rightarrow l$ as $n \rightarrow \infty$.*

PROOF: Let U be any neighbourhood of l . Since $X \setminus U$ is finite, there must exist N such that, for all $n \geq N$, we have $a_n \in U$. □

Proposition 9.167. *Let X be a topological space. Let Y a Hausdorff space. Let $A \subseteq X$. Let $f, g : \overline{A} \rightarrow Y$ be continuous. If f and g agree on A then $f = g$.*

PROOF:

- ⟨1⟩1. LET: $x \in \overline{A}$
- ⟨1⟩2. ASSUME: $f(x) \neq g(x)$
- ⟨1⟩3. PICK disjoint neighbourhoods V of $f(x)$ and W of $g(x)$.
- ⟨1⟩4. PICK $y \in f^{-1}(V) \cap g^{-1}(W) \cap A$

PROOF: Since $f^{-1}(V) \cap g^{-1}(W)$ is a neighbourhood of x and hence intersects A .

⟨1⟩5. $f(y) = g(y) \in V \cap W$

⟨1⟩6. Q.E.D.

PROOF: This contradicts the fact that V and W are disjoint (⟨1⟩3).

□

Proposition 9.168. *Let $\{X_i\}_{i \in I}$ be a family of Hausdorff spaces. Then $\prod_{i \in I} X_i$ under the box topology is Hausdorff.*

PROOF:

⟨1⟩1. LET: $\{X_i\}_{i \in I}$ be a family of Hausdorff spaces.

⟨1⟩2. LET: $a, b \in \prod_{i \in I} X_i$ with $a \neq b$

⟨1⟩3. PICK $i \in I$ such that $a_i \neq b_i$

⟨1⟩4. PICK U, V disjoint open sets in X_i with $a_i \in U$ and $b_i \in V$

⟨1⟩5. $\pi_i^{-1}(U)$ and $\pi_i^{-1}(V)$ are disjoint open sets in $\prod_{i \in I} X_i$ with $a \in \pi_i^{-1}(U)$ and $b \in \pi_i^{-1}(V)$

□

Proposition 9.169. *Let \mathcal{T} and \mathcal{T}' be two topologies on the same set X with $\mathcal{T} \subseteq \mathcal{T}'$. If \mathcal{T} is Hausdorff then \mathcal{T}' is Hausdorff.*

PROOF: Immediate from definitions.

Proposition 9.170. *Let X be a Hausdorff space. Let $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the finite intersection property. Then $\bigcap_{D \in \mathcal{D}} \overline{D}$ contains at most one point.*

PROOF:

⟨1⟩1. LET: $x, y \in \bigcap_{D \in \mathcal{D}} \overline{D}$

⟨1⟩2. ASSUME: for a contradiction $x \neq y$

⟨1⟩3. PICK disjoint open subsets U and V of x and y respectively.

⟨1⟩4. $U, V \in \mathcal{D}$

PROOF: Proposition 9.41.

⟨1⟩5. Q.E.D.

PROOF: This contradicts the fact that \mathcal{D} satisfies the finite intersection property.

□

9.23 The First Countability Axiom

Definition 9.171 (First Countability Axiom). A topological space X satisfies the *first countability axiom*, or is *first countable*, if and only if every point has a countable local basis.

Lemma 9.172 (Sequence Lemma (CC)). *Let X be a first countable space. Let $A \subseteq X$ and $l \in \overline{A}$. Then there exists a sequence in A that converges to l .*

PROOF:

⟨1⟩1. PICK a countable local basis $\{B_n \mid n \in \mathbb{Z}^+\}$ at l such that $B_1 \supseteq B_2 \supseteq \dots$.

PROOF: Lemma 9.64.

⟨1⟩2. For all $n \geq 1$, PICK $a_n \in A \cap B_n$.

PROVE: $a_n \rightarrow l$ as $n \rightarrow \infty$

⟨1⟩3. LET: U be a neighbourhood of A

⟨1⟩4. PICK N such that $B_N \subseteq U$

⟨1⟩5. For $n \geq N$ we have $a_n \in U$

PROOF: $a_n \in B_n \subseteq B_N \subseteq U$

□

Theorem 9.173 (CC). *Let X be a first countable space and Y a topological space. Let $f : X \rightarrow Y$. Suppose that, for every sequence (x_n) in X and $l \in X$, if $x_n \rightarrow l$ as $n \rightarrow \infty$, then $f(x_n) \rightarrow f(l)$ as $n \rightarrow \infty$. Then f is continuous.*

PROOF:

⟨1⟩1. LET: $A \subseteq X$

⟨1⟩2. LET: $a \in A$

PROVE: $f(a) \in \overline{f(A)}$

⟨1⟩3. PICK a sequence (x_n) in A that converges to a .

PROOF: By the Sequence Lemma.

⟨1⟩4. $f(x_n) \rightarrow f(a)$

⟨1⟩5. $f(a) \in \overline{f(A)}$

PROOF: By Lemma 9.66.

⟨1⟩6. Q.E.D.

PROOF: By Theorem 9.84.

□

Example 9.174 (CC). The space \mathbb{R}^ω under the box product is not first countable.

PROOF: Let $\{B_n \mid n \geq 0\}$ be a countable set of neighbourhoods of $\vec{0}$. We will construct a neighbourhood of $\vec{0}$ that does not include any of these.

For $n \geq 0$, pick a neighbourhood U_n of 0 such that $U_n \subset \pi_n(B_p)$. Then $\prod_{n=0}^\infty U_n$ is a neighbourhood of $\vec{0}$ that does not include any B_n . □

Example 9.175. If J is an uncountable set then \mathbb{R}^J is not first countable.

PROOF:

⟨1⟩1. LET: $\{B_n \mid n \geq 0\}$ be any countable set of neighbourhoods of $\vec{0}$.

⟨1⟩2. For $n \geq 0$, PICK a basis element $\prod_{\alpha \in J} U_{n\alpha}$ that contains $\vec{0}$ and is included in B_n .

PROOF: Using the Axiom of Countable Choice.

⟨1⟩3. For $n \geq 0$,

LET: $J_n = \{\alpha \in J \mid U_{n\alpha} \neq \mathbb{R}\}$

⟨1⟩4. PICK $\beta \in J$ such that $\beta \notin J_n$ for any n .

PROOF: Since each J_n is finite so $\bigcup_n J_n$ is countable.

⟨1⟩5. $\pi_\beta((-1, 1))$ is a neighbourhood of $\vec{0}$ that does not include any B_n .

□

Example 9.176. The space \mathbb{R}_l is first countable.

PROOF: For any $a \in \mathbb{R}$, the set $\{[a, a + 1/n) \mid n \geq 1\}$ is a countable local basis.

Example 9.177. The ordered square is first countable.

PROOF: For any $(a, b) \in I_o^2$ with $b \neq 0, 1$, the set $\{(\{a\} \times (b - 1/n, b + 1/n)) \cap I_o^2 \mid n \geq 1\}$ is a countable local basis.

9.24 Strong Continuity

Definition 9.178 (Strongly Continuous). Let X and Y be topological spaces and $f : X \rightarrow Y$ be a function. Then f is *strongly continuous* if and only if, for every subset $U \subseteq Y$, we have U is open in Y if and only if $f^{-1}(U)$ is open in X .

Proposition 9.179. Let X and Y be topological spaces and $f : X \rightarrow Y$ be a function. Then f is strongly continuous if and only if, for every subset $C \subseteq Y$, we have C is closed in Y if and only if $f^{-1}(C)$ is closed in X .

PROOF: Since $X \setminus f^{-1}(C) = f^{-1}(Y \setminus C)$. \square

Proposition 9.180. Let X, Y and Z be topological spaces. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$. If f and g are strongly continuous then so is $g \circ f$.

PROOF: Since $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$. \square

Proposition 9.181. Let X, Y and Z be topological spaces. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$. If $g \circ f$ is continuous and f is strongly continuous then g is continuous.

PROOF:

$\langle 1 \rangle 1$. LET: $V \subseteq Z$ be open.

$\langle 1 \rangle 2$. $f^{-1}(g^{-1}(V))$ is open in X .

PROOF: Since $g \circ f$ is continuous.

$\langle 1 \rangle 3$. $f^{-1}(V)$ is open in Y .

PROOF: Since g is strongly continuous.

\square

Proposition 9.182. Let X, Y and Z be topological spaces. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$. If $g \circ f$ is strongly continuous and f is strongly continuous then g is strongly continuous.

PROOF: For $V \subseteq Z$, we have V is open iff $f^{-1}(g^{-1}(V))$ is open iff $g^{-1}(V)$ is open.

9.25 Saturated Sets

Definition 9.183. Let X and Y be sets and $p : X \rightarrow Y$ a surjective function. Let $C \subseteq X$. Then C is *saturated* with respect to p if and only if, for all $x, y \in X$, if $x \in C$ and $p(x) = p(y)$ then $y \in C$.

Proposition 9.184. *Let X and Y be sets and $p : X \rightarrow Y$ a surjective function. Let $C \subseteq X$. Then the following are equivalent:*

1. C is saturated with respect to p .
2. There exists $D \subseteq Y$ such that $C = p^{-1}(D)$
3. $C = p^{-1}(p(C))$.

PROOF:

$\langle 1 \rangle 1. 1 \Rightarrow 3$

$\langle 2 \rangle 1$. ASSUME: C is saturated with respect to p .

$\langle 2 \rangle 2. C \subseteq p^{-1}(p(C))$

PROOF: Trivial.

$\langle 2 \rangle 3. p^{-1}(p(C)) \subseteq C$

$\langle 3 \rangle 1$. LET: $x \in p^{-1}(p(C))$

$\langle 3 \rangle 2. p(x) \in p(C)$

$\langle 3 \rangle 3$. There exists $y \in C$ such that $p(x) = p(y)$

$\langle 3 \rangle 4. x \in C$

PROOF: From $\langle 2 \rangle 1$.

$\langle 1 \rangle 2. 3 \Rightarrow 2$

PROOF: Trivial.

$\langle 1 \rangle 3. 2 \Rightarrow 1$

PROOF: This follows because if $p(x) \in D$ and $p(x) = p(y)$ then $p(y) \in D$.

□

9.26 Quotient Maps

Definition 9.185 (Quotient Map). Let X and Y be topological spaces and $p : X \rightarrow Y$. Then p is a *quotient map* if and only if p is surjective and strongly continuous.

Proposition 9.186. *Let X and Y be topological spaces and $p : X \rightarrow Y$ be a surjective function. Then the following are equivalent.*

1. p is a quotient map.
2. p is continuous and maps saturated open sets to open sets.
3. p is continuous and maps saturated closed sets to closed sets.

PROOF:

$\langle 1 \rangle 1. 1 \Rightarrow 2$

$\langle 2 \rangle 1$. ASSUME: p is a quotient map.

$\langle 2 \rangle 2$. LET: U be a saturated open set in X .

$\langle 2 \rangle 3. p^{-1}(p(U))$ is open in X .

PROOF: Since $U = p^{-1}(p(U))$ by Proposition 9.184.

$\langle 2 \rangle 4. p(U)$ is open in Y .

PROOF: From $\langle 2 \rangle 1$.

$\langle 1 \rangle 2$. $1 \Rightarrow 3$

PROOF: Similar.

$\langle 1 \rangle 3$. $2 \Rightarrow 1$

$\langle 2 \rangle 1$. ASSUME: p is continuous and maps saturated open sets to open sets.

$\langle 2 \rangle 2$. LET: $U \subseteq Y$

$\langle 2 \rangle 3$. ASSUME: $p^{-1}(U)$ is open in X

$\langle 2 \rangle 4$. $p^{-1}(U)$ is saturated.

PROOF: Proposition 9.184.

$\langle 2 \rangle 5$. U is open in Y .

$\langle 1 \rangle 4$. $3 \Rightarrow 1$

PROOF: Similar.

□

Corollary 9.186.1. *Every surjective continuous open map is a quotient map.*

Corollary 9.186.2. *Every surjective continuous closed map is a quotient map.*

Example 9.187. The converses of these corollaries do not hold.

Let $A = \{(x, y) \mid x \geq 0\} \cup \{(x, y) \mid y = 0\}$. Then $\pi_1 : A \rightarrow \mathbb{R}$ is a quotient map, but not an open map or a closed map.

We prove that π_1 maps saturated open sets to open sets:

$\langle 1 \rangle 1$. LET: $\pi_1^{-1}(U)$ be a saturated open set in A

PROVE: U is open in \mathbb{R}

$\langle 1 \rangle 2$. LET: $x \in U$

$\langle 1 \rangle 3$. $(x, 0) \in \pi_1^{-1}(U)$

$\langle 1 \rangle 4$. PICK W, V open in \mathbb{R} such that $(x, 0) \in W \times V \subseteq \pi_1^{-1}(U)$

$\langle 1 \rangle 5$. $x \in W \subseteq U$

It is not an open map because it maps $((-1, 1) \times (1, 2)) \cap A$ to $[0, 1)$.

It is not a closed map because it maps $\{(x, 1/x) \mid x > 0\}$ to $(0, +\infty)$.

Proposition 9.188. *Let $p : X \twoheadrightarrow Y$ be a quotient map. Let $A \subseteq X$ be saturated with respect to p . Let $q : A \twoheadrightarrow p(A)$ be the restriction of p .*

1. *If A is either open or closed in X then q is a quotient map.*

2. *If p is either an open map or a closed map then q is a quotient map.*

PROOF:

$\langle 1 \rangle 1$. LET: $p : X \twoheadrightarrow Y$ be a quotient map.

$\langle 1 \rangle 2$. LET: $A \subseteq X$ be saturated with respect to p .

$\langle 1 \rangle 3$. LET: $q : A \twoheadrightarrow p(A)$ be the restriction of p .

$\langle 1 \rangle 4$. q is continuous.

PROOF: Theorem 9.88.

$\langle 1 \rangle 5$. If A is open in X then q is a quotient map.

$\langle 2 \rangle 1$. ASSUME: A is open in X .

$\langle 2 \rangle 2$. q maps saturated open sets to open sets.

$\langle 3 \rangle 1$. LET: $U \subseteq A$ be saturated with respect to q and open in A
 $\langle 3 \rangle 2$. U is saturated with respect to p
 $\langle 4 \rangle 1$. LET: $x, y \in X$
 $\langle 4 \rangle 2$. ASSUME: $x \in U$
 $\langle 4 \rangle 3$. ASSUME: $p(x) = p(y)$
 $\langle 4 \rangle 4$. $x \in A$
PROOF: From $\langle 3 \rangle 1$ and $\langle 4 \rangle 2$.
 $\langle 4 \rangle 5$. $y \in A$
PROOF: From $\langle 1 \rangle 2$ and $\langle 4 \rangle 3$
 $\langle 4 \rangle 6$. $q(x) = x(y)$
PROOF: From $\langle 1 \rangle 3$, $\langle 4 \rangle 3$, $\langle 4 \rangle 4$, $\langle 4 \rangle 5$.
 $\langle 4 \rangle 7$. $y \in U$
PROOF: From $\langle 3 \rangle 1$, $\langle 4 \rangle 2$, $\langle 4 \rangle 6$
 $\langle 3 \rangle 3$. U is open in X
PROOF: Lemma 9.134, $\langle 2 \rangle 1$, $\langle 3 \rangle 1$.
 $\langle 3 \rangle 4$. $p(U)$ is open in Y
PROOF: Proposition 9.186, $\langle 1 \rangle 1$, $\langle 3 \rangle 2$, $\langle 3 \rangle 3$
 $\langle 3 \rangle 5$. $q(U)$ is open in $p(A)$
PROOF: Since $q(U) = p(U) = p(U) \cap p(A)$.
 $\langle 2 \rangle 3$. Q.E.D.
PROOF: By Proposition 9.186.
 $\langle 1 \rangle 6$. If A is closed in X then q is a quotient map.
PROOF: Similar.
 $\langle 1 \rangle 7$. If p is an open map then q is a quotient map.
 $\langle 2 \rangle 1$. ASSUME: p is an open map
 $\langle 2 \rangle 2$. q maps saturated open sets to open sets.
 $\langle 3 \rangle 1$. LET: U be open in A and saturated with respect to q
 $\langle 3 \rangle 2$. PICK V open in X such that $U = A \cap V$
 $\langle 3 \rangle 3$. $p(V)$ is open in Y
 $\langle 3 \rangle 4$. $q(U) = p(V) \cap p(A)$
 $\langle 4 \rangle 1$. $q(U) \subseteq p(V) \cap p(A)$
PROOF: From $\langle 3 \rangle 2$.
 $\langle 4 \rangle 2$. $p(V) \cap p(A) \subseteq q(U)$
 $\langle 5 \rangle 1$. LET: $y \in p(V) \cap p(A)$
 $\langle 5 \rangle 2$. PICK $x \in V$ and $x' \in A$ such that $p(x) = p(x') = y$
 $\langle 5 \rangle 3$. $x \in A$
PROOF: By $\langle 1 \rangle 2$.
 $\langle 5 \rangle 4$. $x \in U$
PROOF: From $\langle 3 \rangle 2$
 $\langle 2 \rangle 3$. Q.E.D.
PROOF: By Proposition 9.186.
 $\langle 1 \rangle 8$. If p is a closed map then q is a quotient map.
PROOF: Similar.

Example 9.189. This example shows we cannot remove the hypotheses on A

and p .

Define $f : [0, 1] \rightarrow [2, 3] \rightarrow [0, 2]$ by $f(x) = x$ if $x \leq 1$, $f(x) = x - 1$ if $x \geq 2$. Then f is a quotient map but its restriction f' to $[0, 1] \cup [2, 3]$ is not, because $f'^{-1}([1, 2])$ is open but $[1, 2]$ is not.

For a counterexample where A is saturated, see Example 9.195.

Proposition 9.190. *Let $p : A \twoheadrightarrow C$ and $q : B \twoheadrightarrow D$ be open quotient maps. Then $p \times q : A \times B \rightarrow C \times D$ is an open quotient map.*

PROOF: From Corollary 9.186.1, Proposition 9.127 and Theorem 9.123. \square

Theorem 9.191. *Let $p : X \twoheadrightarrow Y$ be a quotient map. Let Z be a topological space and $f : Y \rightarrow Z$ be a function. Then*

1. $f \circ p$ is continuous if and only if f is continuous.
2. $f \circ p$ is a quotient map if and only if f is a quotient map.

PROOF:

$\langle 1 \rangle 1$. If $f \circ p$ is continuous then f is continuous.

PROOF: Proposition 9.181.

$\langle 1 \rangle 2$. If f is continuous then $f \circ p$ is continuous.

PROOF: Theorem 9.87.

$\langle 1 \rangle 3$. If $f \circ p$ is a quotient map then f is a quotient map.

PROOF: Proposition 9.182.

$\langle 1 \rangle 4$. If f is a quotient map then $f \circ p$ is a quotient map.

PROOF: From Proposition 9.180.

\square

Proposition 9.192. *Let X and Y be topological spaces. Let $p : X \rightarrow Y$ and $f : Y \rightarrow X$ be continuous maps such that $p \circ f = \text{id}_Y$. Then p is a quotient map.*

PROOF:

$\langle 1 \rangle 1$. LET: $V \subseteq Y$

$\langle 1 \rangle 2$. ASSUME: $p^{-1}(V)$ is open in X .

$\langle 1 \rangle 3$. $f^{-1}(p^{-1}(V))$ is open in Y .

PROOF: Because f is continuous.

$\langle 1 \rangle 4$. V is open in Y .

PROOF: Because $f^{-1}(p^{-1}(V)) = V$.

\square

9.27 Quotient Topology

Definition 9.193 (Quotient Topology). Let X be a topological space, Y a set and $p : X \twoheadrightarrow Y$ be a surjective function. Then the *quotient topology* on Y is the unique topology on Y with respect to which p is a quotient map, namely $\mathcal{T} = \{U \in \mathcal{P}Y \mid p^{-1}(U) \text{ is open in } X\}$.

We prove this is a topology.

PROOF:

$\langle 1 \rangle 1. Y \in \mathcal{T}$

PROOF: Since $p^{-1}(Y) = X$ by surjectivity.

$\langle 1 \rangle 2. \text{ For all } \mathcal{A} \subseteq \mathcal{T} \text{ we have } \bigcup \mathcal{A} \in \mathcal{T}$

PROOF: Since $p^{-1}(\bigcup \mathcal{A}) = \bigcup_{U \in \mathcal{A}} p^{-1}(U)$

$\langle 1 \rangle 3. \text{ For all } U, V \in \mathcal{T} \text{ we have } U \cap V \in \mathcal{T}$

PROOF: Since $p^{-1}(U \cap V) = p^{-1}(U) \cap p^{-1}(V)$.

□

Definition 9.194 (Quotient Space). Let X be a topological space and \sim an equivalence relation on X . Let $p : X \twoheadrightarrow X/\sim$ be the canonical surjection. Then X/\sim under the quotient topology is called a *quotient space*, *identification space* or *decomposition space* of X .

Here is a counterexample showing we cannot remove all the hypotheses of Proposition 9.188 except that A is saturated.

Example 9.195. Let $X = (0, 1/2] \cup \{1\} \cup \{1 + 1/n : n \geq 2\}$ as a subspace of \mathbb{R} . Define R to be the equivalence relation on X where xRy iff $(x = y \text{ or } |x - y| = 1)$, so we identify $1/n$ with $1 + 1/n$ for all $n \geq 2$. Let Y be the resulting quotient space X/R in the quotient topology and $p : X \twoheadrightarrow Y$ the canonical surjection.

Let $A = \{1\} \cup (0, 1/2] \setminus \{1/n : n \geq 2\} \subseteq X$. Then A is saturated under p but the restriction q of p to A is not a quotient map because it maps the saturated open set $\{1\}$ to $\{1\}$ which is not open in $p(A)$.

Proposition 9.196. Let X, Y and Z be topological spaces. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$. If f and g are quotient maps then so is $g \circ f$.

PROOF: From Proposition 9.180. □

Example 9.197. The product of two quotient maps is not necessarily a quotient map.

Let $X = \mathbb{R}$ and X^* the quotient space formed by identifying all positive integers. Let $p : X \twoheadrightarrow X^*$ be the canonical surjection.

We prove $p \times \text{id}_{\mathbb{Q}} : X \times \mathbb{Q} \rightarrow X^* \times \mathbb{Q}$ is not a quotient map.

PROOF:

$\langle 1 \rangle 1. \text{ For } n \geq 1,$

LET: $c_n = \sqrt{2}/n$

$\langle 1 \rangle 2. \text{ For } n \geq 1,$

LET: $U_n = \{(x, y) \in X \times \mathbb{Q} \mid n - 1/4 < x < n + 1/4, (y + n > x + c_n \text{ and } y + n > -x + c_n) \text{ or } (y + n < x + c_n \text{ and } y + n < -x + c_n)\}$

$\langle 1 \rangle 3. \text{ For } n \geq 1, \text{ we have } U_n \text{ is open in } X \times \mathbb{Q}$

$\langle 1 \rangle 4. \text{ For } n \geq 1, \text{ we have } \{n\} \times \mathbb{Q} \subseteq U_n$

$\langle 1 \rangle 5. \text{ LET: } U = \bigcup_{n=1}^{\infty} U_n$

$\langle 1 \rangle 6. U \text{ is open in } X \times \mathbb{Q}$

$\langle 1 \rangle 7. U \text{ is saturated with respect to } p \times \text{id}_{\mathbb{Q}}$

- ⟨1⟩8. LET: $U' = (p \times \text{id}_{\mathbb{Q}})(U)$
- ⟨1⟩9. ASSUME: for a contradiction U' is open in $X^* \times \mathbb{Q}$
- ⟨1⟩10. $(1, 0) \in U'$
- ⟨1⟩11. PICK a neighbourhood W of 1 in X^* and $\delta > 0$ such that $W \times (-\delta, \delta) \subseteq U'$
- ⟨1⟩12. $p^{-1}(W) \times (-\delta, \delta) \subseteq U$
- ⟨1⟩13. PICK n such that $c_n < \delta$
- ⟨1⟩14. $n \in p^{-1}(W)$
- ⟨1⟩15. PICK $\epsilon > 0$ such that $\epsilon < \delta - c_n$ and $\epsilon < 1/4$ and $(n - \epsilon, n + \epsilon) \subseteq p^{-1}(W)$
- ⟨1⟩16. $(n - \epsilon, n + \epsilon) \times (-\delta, \delta) \subseteq U$
- ⟨1⟩17. PICK a rational y such that $c_n - \epsilon/2 < y < c_n + \epsilon/2$
- ⟨1⟩18. $(n + \epsilon/2, y) \notin U$
- ⟨1⟩19. Q.E.D.

PROOF: This contradicts ⟨1⟩16.

□

Proposition 9.198. *Let X be a topological space and \sim an equivalence relation on X . Then X/\sim is T_1 if and only if every equivalence class is closed in X .*

PROOF: Immediate from definitions. □

9.28 Retractions

Definition 9.199 (Retraction). Let X be a topological space and $A \subseteq X$. A *retraction* of X onto A is a continuous map $r : X \rightarrow A$ such that, for all $a \in A$, we have $r(a) = a$.

Proposition 9.200. *Every retraction is a quotient map.*

PROOF: Proposition 9.192 with f the inclusion $A \hookrightarrow X$. □

9.29 Homogeneous Spaces

Definition 9.201 (Homogeneous). A topological space X is *homogeneous* if and only if, for any points $a, b \in X$, there exists a homeomorphism $\phi : X \cong X$ such that $\phi(a) = b$.

9.30 Regular Spaces

Definition 9.202 (Regular Space). A topological space X is *regular* if and only if, for any closed set A and point $a \notin A$, there exist disjoint open sets U, V such that $A \subseteq U$ and $a \in V$.

9.31 Connected Spaces

Definition 9.203 (Separation). A *separation* of a topological space X is a pair of disjoint open sets U, V such that $U \cup V = \emptyset$.

Definition 9.204 (Connected). A topological space is *connected* if and only if it has no separation; otherwise it is *disconnected*.

Proposition 9.205. *A topological space X is connected if and only if the only sets that are both open and closed are X and \emptyset .*

Immediate from definitions.

Lemma 9.206. *If Y is a subspace of X , a separation of Y is a pair of disjoint nonempty sets A and B whose union is Y , neither of which contains a limit point of the other.*

PROOF:

- $\langle 1 \rangle 1$. LET: $A, B \subseteq Y$
- $\langle 1 \rangle 2$. If A and B form a separation of Y then A and B are disjoint and nonempty, $A \cup B = Y$, and neither of A and B contains a limit point of the other.
 - $\langle 2 \rangle 1$. ASSUME: A and B form a separation of Y
 - $\langle 2 \rangle 2$. A and B are disjoint and nonempty and $A \cup B = Y$
 PROOF: From $\langle 2 \rangle 1$ and the definition of separation.
 - $\langle 2 \rangle 3$. A does not contain a limit point of B
 - $\langle 3 \rangle 1$. ASSUME: for a contradiction $l \in A$ and l is a limit point of B in X .
 - $\langle 3 \rangle 2$. l is a limit point of B in Y
 PROOF: Proposition 9.144.
 - $\langle 3 \rangle 3$. $l \in B$
 - $\langle 4 \rangle 1$. B is closed in Y
 PROOF: Since A is open in Y and $B = Y \setminus A$ from $\langle 2 \rangle 1$.
 - $\langle 4 \rangle 2$. Q.E.D.
 PROOF: Corollary 9.49.1.
 - $\langle 3 \rangle 4$. Q.E.D.
 PROOF: This contradicts the fact that $A \cap B = \emptyset$ ($\langle 2 \rangle 1$).
- $\langle 2 \rangle 4$. B does not contain a limit point of A
 PROOF: Similar.
- $\langle 1 \rangle 3$. If A and B are disjoint and nonempty, $A \cup B = Y$, and neither of A and B contains a limit point of the other, then A and B form a separation of Y .
 - $\langle 2 \rangle 1$. ASSUME: A and B are disjoint and nonempty, $A \cup B = Y$, and neither of A and B contains a limit point of the other.
 - $\langle 2 \rangle 2$. A is open in Y
 - $\langle 3 \rangle 1$. B is closed in Y
 - $\langle 4 \rangle 1$. LET: l be a limit point of B in Y
 - $\langle 4 \rangle 2$. l is a limit point of B in X
 PROOF: Proposition 9.144.
 - $\langle 4 \rangle 3$. $l \notin A$
 PROOF: By $\langle 2 \rangle 1$
 - $\langle 4 \rangle 4$. $l \in B$
 PROOF: By $\langle 2 \rangle 1$ since $A \cup B = Y$
 - $\langle 4 \rangle 5$. Q.E.D.

PROOF: Corollary 9.49.1.

⟨3⟩2. Q.E.D.

PROOF: Since $A = Y \setminus B$.

⟨2⟩3. B is open in Y

PROOF: Similar.

□

Example 9.207. Every set under the indiscrete topology is connected.

Example 9.208. The discrete topology on a set X is connected if and only if $|X| \leq 1$.

Example 9.209. The finite complement topology on a set X is connected if and only if either $|X| \leq 1$ or X is infinite.

Example 9.210. The countable complement topology on a set X is connected if and only if either $|X| \leq 1$ or X is uncountable.

Example 9.211. The rationals \mathbb{Q} are disconnected. For any irrational a , the sets $(-\infty, a) \cap \mathbb{Q}$ and $(a, +\infty) \cap \mathbb{Q}$ form a separation of \mathbb{Q} .

Lemma 9.212. *Let X be a topological space. If C and D form a separation of X , and Y is a connected subspace of X , then either $Y \subseteq C$ or $Y \subseteq D$.*

PROOF: Otherwise $Y \cap C$ and $Y \cap D$ would form a separation of Y . □

Theorem 9.213. *The union of a set of connected subspaces of a space X that have a point in common is connected.*

PROOF:

⟨1⟩1. LET: \mathcal{A} be a set of connected subspaces of the space X that have the point a in common.

⟨1⟩2. ASSUME: for a contradiction C and D form a separation of $\bigcup \mathcal{A}$

⟨1⟩3. ASSUME: without loss of generality $a \in C$

⟨1⟩4. For all $A \in \mathcal{A}$ we have $A \subseteq C$

PROOF: Lemma 9.212.

⟨1⟩5. $D = \emptyset$

⟨1⟩6. Q.E.D.

PROOF: This contradicts ⟨1⟩2.

□

Theorem 9.214. *Let X be a topological space and A a connected subspace of X . If $A \subseteq B \subseteq \bar{A}$ then B is connected.*

PROOF:

⟨1⟩1. ASSUME: for a contradiction C and D form a separation of B .

⟨1⟩2. ASSUME: without loss of generality $A \subseteq C$

PROOF: Lemma 9.212.

⟨1⟩3. $B \subseteq C$

⟨2⟩1. LET: $x \in B$

- ⟨2⟩2. $x \in \overline{A}$
- ⟨2⟩3. Either $x \in A$ or x is a limit point of A .
PROOF: Theorem 9.49.
- ⟨2⟩4. Either $x \in A$ or x is a limit point of C .
PROOF: Lemma 9.51, ⟨1⟩2.
- ⟨2⟩5. $x \in C$
PROOF: Lemma 9.206.
- ⟨1⟩4. $D = \emptyset$
- ⟨1⟩5. Q.E.D.
- PROOF: This contradicts ⟨1⟩1.

□

Theorem 9.215. *The image of a connected space under a continuous map is connected.*

PROOF:

- ⟨1⟩1. LET: $f : X \rightarrow Y$ be a surjective continuous map where X is connected.
- ⟨1⟩2. ASSUME: for a contradiction C and D form a separation of Y .
- ⟨1⟩3. $f^{-1}(C)$ and $f^{-1}(D)$ form a separation of X .

□

Theorem 9.216. *The product of a family of connected spaces is connected.*

PROOF:

- ⟨1⟩1. The product of two connected spaces is connected.
 - ⟨2⟩1. LET: X and Y be connected spaces.
 - ⟨2⟩2. PICK $a \in X$ and $b \in Y$
PROOF: We may assume X and Y are nonempty since otherwise $X \times Y = \emptyset$ which is connected.
 - ⟨2⟩3. $X \times \{b\}$ is connected.
PROOF: It is homeomorphic to X .
 - ⟨2⟩4. For all $x \in X$ we have $\{x\} \times Y$ is connected.
PROOF: It is homeomorphic to Y .
 - ⟨2⟩5. For any $x \in X$
LET: $T_x = (X \times \{b\}) \cup (\{x\} \times Y)$
 - ⟨2⟩6. For all $x \in X$, T_x is connected.
PROOF: Theorem 9.213 since $(x, b) \in (X \times \{b\}) \cap (\{x\} \times Y)$.
 - ⟨2⟩7. $X \times Y$ is connected.
PROOF: Theorem 9.213 since $X \times Y = \bigcup_{x \in X} T_x$ and (a, b) is a point in every T_x .
- ⟨1⟩2. The product of a finite family of connected spaces is connected.
PROOF: From ⟨1⟩1 by induction.
- ⟨1⟩3. The product of any family of connected spaces is connected.
 - ⟨2⟩1. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of connected spaces.
 - ⟨2⟩2. LET: $X = \prod_{\alpha \in J} X_\alpha$
 - ⟨2⟩3. PICK $a \in X$
PROOF: We may assume $X \neq \emptyset$ as the empty space is connected.

- (2)4. For every finite subset K of J ,
 LET: $X_K = \{x \in X \mid \forall \alpha \in J \setminus K. x_\alpha = a_\alpha\}$
 (2)5. For every finite $K \subseteq J$, we have X_K is connected.
 PROOF: From (1)2 since $X_K \cong \prod_{\alpha \in K} X_K$.
 (2)6. LET: $Y = \bigcup_K X_K$
 (2)7. Y is connected
 PROOF: Theorem 9.213 since a is a common point.
 (2)8. $X = \bar{Y}$
 (3)1. LET: $x \in X$
 (3)2. LET: $U = \prod_{\alpha \in J} U_\alpha$ be a basic neighbourhood of x where $U_\alpha = X_\alpha$
 for all α except $\alpha \in K$ for some finite $K \subseteq J$
 (3)3. LET: $y \in X$ be the point with $y_\alpha = x_\alpha$ for $\alpha \in K$ and $y_\alpha = a_\alpha$ for
 all other α
 (3)4. $y \in U \cap X_K$
 (3)5. $y \in U \cap Y$
 (2)9. X is connected.
 PROOF: Theorem 9.214.

□

Example 9.217. The set \mathbb{R}^ω is disconnected under the box topology. The set of bounded sequences and the set of unbounded sequences form a separation.

Proposition 9.218. Let \mathcal{T} and \mathcal{T}' be two topologies on the same set X . If $\mathcal{T} \subseteq \mathcal{T}'$ and (X, \mathcal{T}') is connected then (X, \mathcal{T}) is connected.

PROOF: If U and V form a separation of (X, \mathcal{T}) then they form a separation of (X, \mathcal{T}') . □

Proposition 9.219. Let X be a topological space and (A_n) a sequence of connected subspaces of X . If $A_n \cap A_{n+1} \neq \emptyset$ for all n then $\bigcup_n A_n$ is connected.

PROOF:

- (1)1. ASSUME: for a contradiction C and D form a separation of $\bigcup_n A_n$
 (1)2. ASSUME: without loss of generality $A_0 \subseteq C$
 PROOF: Lemma 9.212.
 (1)3. For all n we have $A_n \subseteq C$
 PROOF:
 (2)1. ASSUME: $A_n \subseteq C$
 (2)2. PICK $x \in A_n \cap A_{n+1}$
 (2)3. $x \in C$
 (2)4. $A_{n+1} \subseteq C$
 PROOF: Lemma 9.212.
 (2)5. Q.E.D.

PROOF: The result follows by induction.

- (1)4. $D = \emptyset$
 (1)5. Q.E.D.
 PROOF: This contradicts (1)1.

□

Proposition 9.220. *Let X be a topological space. Let $A, C \subseteq X$. If C is connected and intersects both A and $X \setminus A$ then C intersects ∂A .*

PROOF: Otherwise $C \cap A^\circ$ and $C \setminus \overline{A}$ would form a separation of C . \square

Example 9.221. The space \mathbb{R}_l is disconnected. For any real x , the sets $(-\infty, x)$ and $[x, +\infty)$ form a separation.

Proposition 9.222. *Let X and Y be connected spaces. Let A be a proper subset of X and B a proper subset of Y . Then $(X \times Y) \setminus (A \times B)$ is connected.*

PROOF:

$\langle 1 \rangle 1$. PICK $a \in X \setminus A$ and $b \in Y \setminus B$

$\langle 1 \rangle 2$. For $x \in X \setminus A$ we have $(X \times \{b\}) \cup (\{x\} \times Y)$ is connected.

PROOF: Theorem 9.213 since (x, b) is a common point.

$\langle 1 \rangle 3$. For $y \in Y \setminus B$ we have $(X \times \{y\}) \cup (\{a\} \times Y)$ is connected.

PROOF: Theorem 9.213 since (a, y) is a common point.

$\langle 1 \rangle 4$. $(X \times Y) \setminus (A \times B)$ is connected.

PROOF: Theorem 9.213 since it is the union of the sets in $\langle 1 \rangle 2$ and $\langle 1 \rangle 3$ with (a, b) as a common point.

\square

Proposition 9.223. *Let $p : X \rightarrow Y$ be a quotient map. If Y is connected and $p^{-1}(y)$ is connected for all $y \in Y$, then X is connected.*

PROOF:

$\langle 1 \rangle 1$. ASSUME: for a contradiction C and D form a separation of X .

$\langle 1 \rangle 2$. C is saturated.

$\langle 2 \rangle 1$. LET: $x \in C$, $y \in X$ with $p(x) = p(y) = a$, say

$\langle 2 \rangle 2$. $y \notin D$

PROOF: Otherwise $p^{-1}(a) \cap C$ and $p^{-1}(a) \cap D$ form a separation of $p^{-1}(a)$.

$\langle 2 \rangle 3$. $y \in C$

$\langle 1 \rangle 3$. D is saturated.

PROOF: Similar.

$\langle 1 \rangle 4$. $p(C)$ and $p(D)$ form a separation of Y .

\square

Proposition 9.224. *Let X be a connected space and Y a connected subspace of X . Suppose A and B form a separation of $X \setminus Y$. Then $Y \cup A$ and $Y \cup B$ are both connected.*

PROOF:

$\langle 1 \rangle 1$. $Y \cup A$ is connected.

$\langle 2 \rangle 1$. ASSUME: for a contradiction C and D form a separation of $Y \cup A$

$\langle 2 \rangle 2$. ASSUME: without loss of generality $Y \subseteq C$

$\langle 2 \rangle 3$. PICK open sets A_1, B_1, C_1, D_1 in X with

$$A = A_1 \setminus Y$$

$$B = B_1 \setminus Y$$

$$C = C_1 \cap (Y \cup A)$$

$$D = D_1 \cap (Y \cup A)$$

- ⟨2⟩4. $B_1 \cup C_1$ and $A_1 \cap D_1$ form a separation of X
- ⟨1⟩2. $Y \cup B$ is connected.

PROOF: Similar.

□

Theorem 9.225. *Let L be a linearly ordered set under the order topology. Then L is connected if and only if L is a linear continuum.*

PROOF:

- ⟨1⟩1. If L is a linear continuum then L is connected.
 - ⟨2⟩1. LET: L be a linear continuum under the order topology.
 - ⟨2⟩2. ASSUME: for a contradiction C and D form a separation of L .
 - ⟨2⟩3. PICK $a \in C$ and $b \in D$.
 - ⟨2⟩4. ASSUME: without loss of generality $a < b$.
 - ⟨2⟩5. LET: $S = \{x \in L \mid a < x \text{ and } [a, x) \subseteq C\}$
 - ⟨2⟩6. S is nonempty.
 - PROOF: Since $a \in C$ and C is open.
 - ⟨2⟩7. S is bounded above by b .
 - PROOF: Since $b \notin C$.
 - ⟨2⟩8. LET: $s = \sup S$
 - ⟨2⟩9. $s \in S$
 - ⟨3⟩1. LET: $y \in [a, s)$
 - PROVE: $y \in C$
 - ⟨3⟩2. PICK z with $y < z \in S$
 - PROOF: By minimality of s .
 - ⟨3⟩3. $y \in [a, z) \subseteq C$
 - ⟨2⟩10. CASE: $s \in C$
 - ⟨3⟩1. PICK x such that $s < x$ and $[s, x) \subseteq C$
 - PROOF: Since C is open and s is not greatest in L because $s < b$.
 - ⟨3⟩2. $x \in S$
 - PROOF: Since $[a, x) = [a, s) \cup [s, x) \subseteq C$.
 - ⟨3⟩3. Q.E.D.
 - PROOF: This contradicts the fact that s is an upper bound for S .
 - ⟨2⟩11. CASE: $s \in D$
 - ⟨3⟩1. PICK $x < s$ such that $(x, s] \subseteq D$
 - ⟨3⟩2. PICK y with $x < y < s$
 - PROOF: Since L is dense.
 - ⟨3⟩3. $y \in C$
 - PROOF: From ⟨2⟩9.
 - ⟨3⟩4. $y \in D$
 - PROOF: From ⟨3⟩1.
 - ⟨3⟩5. Q.E.D.
 - ⟨3⟩6. LET: L be a linear continuum under the order topology.
 - ⟨3⟩7. ASSUME: for a contradiction C and D form a separation of L .
 - ⟨3⟩8. PICK $a \in C$ and $b \in D$.
 - ⟨3⟩9. ASSUME: without loss of generality $a < b$.
 - ⟨3⟩10. LET: $S = \{x \in L \mid a < x \text{ and } [a, x) \subseteq C\}$

PROOF: Since $a \in C$ and C is open.

PROOF: Since $b \notin C$.

$$\langle 3 \rangle 14. \quad s \in S$$

PROVE: $y \in C$

PROOF: By minimality of s .

⟨3⟩15. CASE: $s \in C$

PROOF: Since C is open and s is not great

PROOF: S

PROOF: T

⟨4⟩1. PICK $x < s$

PROOF: Since L is dense.

PROOF: F

PROOF: From $\langle 3 \rangle 1$.

PROOF: T

2)1. ASSUME: L is connected.

(3)1. LET: X be a nonempty subset of L bounded above by b .

⟨3⟩3. LET: U be the set of upper bounds of X ,

PROOF: Since $b \in b$

⟨4⟩1. LET: $x \in U$

⟨4⟩3. Either x is greatest in L and $(y, x] \subset U$, or there exists $z > x$ such

⟨3⟩6. LET: V be the set of lower bounds of U .

PROOF: Since $X \subseteq V$

⟨4⟩1. LET: $x \in V$

(4)2. PICK $y \in X$ with $x < y$
 PROOF: x cannot be an upper bound for X , because it would be the supremum of X .
 (4)3. Either x least in L and $[x, y) \subseteq V$, or there exists $z < x$ such that $(z, y) \subseteq V$
 (3)9. $L = U \cup V$
 (4)1. LET: $x \in L \setminus U$
 (4)2. PICK $y \in X$ such that $x < y$
 (4)3. For all $u \in U$ we have $x < y \leq u$
 (4)4. $x \in V$
 (3)10. $U \cap V = \emptyset$
 PROOF: Any element of $U \cap V$ would be a supremum of X .
 (3)11. U and V form a separation of L .
 (3)12. Q.E.D.
 PROOF: This contradicts (2)1.
 (2)3. L is dense.
 (3)1. LET: $x, y \in L$ with $x < y$
 (3)2. There exists $z \in L$ such that $x < z < y$
 PROOF: Otherwise $(-\infty, y)$ and $(x, +\infty)$ would form a separation of L .
 □

Corollary 9.225.1. *The real line \mathbb{R} is connected.*

Corollary 9.225.2. *Every interval in \mathbb{R} is connected.*

Corollary 9.225.3. *The ordered square is connected.*

Theorem 9.226 (Intermediate Value Theorem). *Let X be a connected space. Let Y be a linearly ordered set under the order topology. Let $f : X \rightarrow Y$ be continuous. Let $a, b \in X$ and $r \in Y$. Suppose $f(a) < r < f(b)$. Then there exists $c \in X$ such that $f(c) = r$.*

PROOF: Otherwise $f^{-1}((-\infty, r))$ and $f^{-1}((r, +\infty))$ would form a separation of X . □

Proposition 9.227. *Every function $f : [0, 1] \rightarrow [0, 1]$ has a fixed point.*

PROOF:

(1)1. LET: $g : [0, 1] \rightarrow [-1, 1]$ be the function $g(x) = f(x) - x$
 PROVE: there exists $x \in [0, 1]$ such that $g(x) = 0$
 (1)2. ASSUME: without loss of generality $g(0) \neq 0$ and $g(1) \neq 0$
 (1)3. $g(0) > 0$
 (1)4. $g(1) < 0$
 (1)5. There exists $x \in (0, 1)$ such that $g(x) = 0$
 PROOF: By the Intermediate Value Theorem.

Proposition 9.228. *Give \mathbb{R}^ω the box topology. Let $x, y \in \mathbb{R}^\omega$. Then x and y lie in the same component if and only if $x - y$ is eventually zero, i.e. there exists N such that, for all $n \geq N$, we have $x_n = y_n$.*

PROOF:

- ⟨1⟩1. The component containing 0 is the set of sequences that are eventually zero.
- ⟨2⟩1. LET: B be the set of sequences that are eventually zero.
- ⟨2⟩2. B is path-connected.
- ⟨3⟩1. LET: $x, y \in B$
- ⟨3⟩2. PICK N such that, for all $n \geq N$, we have $x_n = y_n = 0$
- ⟨3⟩3. LET: $p : [0, 1] \rightarrow \mathbb{R}^\omega$, $p(t) = (1 - t)x + ty$
PROVE: p is continuous.
- ⟨3⟩4. LET: $t \in [0, 1]$ and $\prod_j U_j$ be a basic open neighbourhood of $p(t)$, where each U_j is open in \mathbb{R}
- ⟨3⟩5. PICK δ such that, for all $n < N$ and all $s \in [0, 1]$, if $|s - t| < \delta$ then $p(s)_n \in U_n$
- ⟨3⟩6. For all $s \in [0, 1]$, if $|s - t| < \delta$ then $p(s) \in \prod_j U_j$
- ⟨2⟩3. B is connected.
PROOF: Proposition 9.234.
- ⟨2⟩4. If C is connected and $B \subseteq C$ then $B = C$.
- ⟨3⟩1. ASSUME: C is connected and $B \subseteq C$
- ⟨3⟩2. ASSUME: for a contradiction $x \in C \setminus B$
- ⟨3⟩3. For $n \geq 1$,
LET: $c_n = 1$ if $x_n = 0$, $c_n = n/x_n$ otherwise
- ⟨3⟩4. LET: $h : \mathbb{R}^\omega \rightarrow \mathbb{R}^\omega$ be the function $h(x) = (c_n x_n)_{n \geq 1}$
- ⟨3⟩5. h is a homeomorphism of \mathbb{R}^ω with itself.
- ⟨3⟩6. $h(x)$ is unbounded.
PROOF: For any $b > 0$, pick $N > b$ such that $x_N \neq 0$. Then $h(x)_N > b$.
- ⟨3⟩7. $h^{-1}(\{\text{bounded sequences}\}) \cap C$ and $h^{-1}(\{\text{unbounded sequences}\}) \cap C$ form a separation of C
- ⟨3⟩8. Q.E.D.
PROOF: This contradicts ⟨3⟩1.
- ⟨1⟩2. Q.E.D.
PROOF: Since $\lambda x. x - y$ is a homeomorphism of \mathbb{R}^ω with itself.

□

9.32 Totally Disconnected Spaces

Definition 9.229 (Totally Disconnected). A topological space X is *totally disconnected* if and only if the only connected subspaces are the singletons.

Example 9.230. Every discrete space is totally disconnected.

Example 9.231. The rationals \mathbb{Q} are totally disconnected.

9.33 Paths and Path Connectedness

Definition 9.232 (Path). Let X be a topological space and $a, b \in X$. A *path* from a to b is a continuous function $p : [0, 1] \rightarrow X$ such that $p(0) = a$ and

$$p(1) = b.$$

Definition 9.233 (Path Connected). A topological space is *path connected* if and only if there exists a path between any two points.

Proposition 9.234. *Every path connected space is connected.*

PROOF:

- ⟨1⟩1. LET: X be a path connected space.
- ⟨1⟩2. ASSUME: for a contradiction C and D form a separation of X .
- ⟨1⟩3. PICK $a \in C$ and $b \in D$.
- ⟨1⟩4. PICK a path $p : [0, 1] \rightarrow X$ from a to b .
- ⟨1⟩5. $p^{-1}(C)$ and $p^{-1}(D)$ form a separation of $[0, 1]$.
- ⟨1⟩6. Q.E.D.

PROOF: This contradicts Corollary 9.225.2.

□

An example that shows the converse does not hold:

Example 9.235. The ordered square is not path connected.

PROOF:

- ⟨1⟩1. ASSUME: for a contradiction $p : [0, 1] \rightarrow I_o^2$ is a path from $(0, 0)$ to $(1, 1)$.
- ⟨1⟩2. p is surjective.
- PROOF: By the Intermediate Value Theorem.
- ⟨1⟩3. For $x \in [0, 1]$, PICK a rational $q_x \in p^{-1}((x, 0), (x, 1))$
- PROOF: Since $p^{-1}((x, 0), (x, 1))$ is open and nonempty by ⟨1⟩2.
- ⟨1⟩4. For $x, y \in [0, 1]$, if $x \neq y$ then $q_x \neq q_y$
- PROOF: We have $p(q_x) \neq p(q_y)$ because $((x, 0), (x, 1))$ and $((y, 0), (y, 1))$ are disjoint.
- ⟨1⟩5. $\{q_x \mid x \in [0, 1]\}$ is an uncountable set of rationals.
- ⟨1⟩6. Q.E.D.

PROOF: This contradicts the fact that the rationals are countable.

□

Proposition 9.236. *The continuous image of a path connected space is path connected.*

PROOF:

- ⟨1⟩1. LET: X be a path connected space, Y a topological space, and $f : X \rightarrow Y$ be continuous and surjective.
- ⟨1⟩2. LET: $a, b \in Y$
- ⟨1⟩3. PICK $c, d \in X$ with $f(c) = a$ and $f(d) = b$
- ⟨1⟩4. PICK a path $p : [0, 1] \rightarrow X$ from c to d .
- ⟨1⟩5. $f \circ p$ is a path from a to b in Y .

□

Proposition 9.237 (AC). *The product of a family of path-connected spaces is path-connected.*

PROOF:

- ⟨1⟩1. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of path-connected spaces.
- ⟨1⟩2. LET: $a, b \in \prod_{\alpha \in J} X_\alpha$
- ⟨1⟩3. For $\alpha \in J$, PICK a path $p_\alpha : [0, 1] \rightarrow X_\alpha$ from a_α to b_α
PROOF: Using the Axiom of Choice.
- ⟨1⟩4. Define $p : [0, 1] \rightarrow \prod_{\alpha \in J} X_\alpha$ by $p(t)_\alpha = p_\alpha(t)$
- ⟨1⟩5. p is a path from a to b .
PROOF: Theorem 9.123.

□

Proposition 9.238. *The continuous image of a path-connected space is path-connected.*

PROOF:

- ⟨1⟩1. LET: $f : X \rightarrow Y$ be continuous and surjective where X is path-connected.
- ⟨1⟩2. LET: $a, b \in Y$
- ⟨1⟩3. PICK $a', b' \in X$ with $f(a') = a$ and $f(b') = b$.
- ⟨1⟩4. PICK a path $p : [0, 1] \rightarrow X$ from a' to b' .
- ⟨1⟩5. $f \circ p$ is a path from a to b .

□

Proposition 9.239. *Let X be a topological space. The union of a set of path-connected subspaces of X that have a point in common is path-connected.*

PROOF:

- ⟨1⟩1. LET: \mathcal{A} be a set of path-connected subspaces of X with the point a in common.
- ⟨1⟩2. LET: $b, c \in \bigcup \mathcal{A}$
- ⟨1⟩3. PICK $B, C \in \mathcal{A}$ with $b \in B$ and $c \in C$.
- ⟨1⟩4. PICK a path p in B from b to a .
- ⟨1⟩5. PICK a path q in C from a to c .
- ⟨1⟩6. The concatenation of p and q is a path from b to c in $\bigcup \mathcal{A}$.

□

Proposition 9.240. *Let $A \subseteq \mathbb{R}^2$ be countable. Then $\mathbb{R}^2 \setminus A$ is path-connected.*

PROOF:

- ⟨1⟩1. LET: $a, b \in \mathbb{R}^2 \setminus A$
- ⟨1⟩2. PICK a line l in \mathbb{R}^2 with a on one side and b on the other.
- ⟨1⟩3. For every point x on l ,
LET: p_x be the path in \mathbb{R}^2 consisting of a line from a to x then a line from x to b
- ⟨1⟩4. For $x \neq y$ we have p_x and p_y have no points in common except a and b
- ⟨1⟩5. There are only countably many x such that a point of A lies on p_x .
- ⟨1⟩6. There exists x such that p_x is a path from a to b in $\mathbb{R}^2 \setminus A$.

□

Proposition 9.241. *Every open connected subspace of \mathbb{R}^2 is path-connected.*

PROOF:

$\langle 1 \rangle 1$. LET: U be an open connected subspace of \mathbb{R}^2 .

$\langle 1 \rangle 2$. For all $x_0 \in U$,

LET: $PC(x_0) = \{y \in U \mid \text{there exists a path from } x \text{ to } y\}$

$\langle 1 \rangle 3$. For all $x_0 \in U$, the set $PC(x_0)$ is open and closed in U .

$\langle 2 \rangle 1$. LET: $x_0 \in U$

$\langle 2 \rangle 2$. $PC(x_0)$ is open in U

$\langle 3 \rangle 1$. LET: $y \in PC(x_0)$

$\langle 3 \rangle 2$. PICK $\epsilon > 0$ such that $B(y, \epsilon) \subseteq U$

PROOF: Since U is open.

$\langle 3 \rangle 3$. $B(y, \epsilon) \subseteq PC(x_0)$

PROOF: For all $z \in B(y, \epsilon)$, pick a path from x_0 to y then concatenate the straight line from y to z .

$\langle 2 \rangle 3$. $PC(x_0)$ is closed in U

$\langle 3 \rangle 1$. LET: $y \in U$ be a limit point of $PC(x_0)$

$\langle 3 \rangle 2$. PICK $\epsilon > 0$ such that $B(y, \epsilon) \subseteq U$

$\langle 3 \rangle 3$. PICK $z \in PC(x_0) \cap B(y, \epsilon)$

$\langle 3 \rangle 4$. $y \in PC(x_0)$

PROOF: Pick a path from x_0 to z then concatenate the straight line from z to y .

$\langle 1 \rangle 4$. $PC(x_0) = U$

PROOF: Proposition 9.205.

□

Example 9.242. If A is a connected subspace of X , then A° is not necessarily connected.

Take two closed circles in \mathbb{R}^2 that touch at one point. The interior of this space is two open circles, and these two circles form a separation.

Example 9.243. If A is a connected subspace of X then ∂A is not necessarily connected.

We have $[0, 1]$ is connected but $\partial[0, 1] = \{0, 1\}$ is not.

Example 9.244. If A is a subspace of X and A° and ∂A are connected, then A is not necessarily connected.

We have $\mathbb{Q}^\circ = \emptyset$ and $\partial\mathbb{Q} = \mathbb{R}$ are connected but \mathbb{Q} is not connected.

9.34 The Topologist's Sine Curve

Definition 9.245 (The Topologist's Sine Curve). Let $S = \{(x, \sin 1/x) \mid 0 < x \leq 1\}$, The *topologist's sine curve* is the closure \overline{S} of S in \mathbb{R}^2 .

Proposition 9.246. The topologist's sine curve is connected.

PROOF:

$\langle 1 \rangle 1$. LET: $S = \{(x, \sin 1/x) \mid 0 < x \leq 1\}$

$\langle 1 \rangle 2$. S is connected.

PROOF: Theorem 9.215.

<1>3. \bar{S} is connected.

PROOF: Theorem 9.214.

□

Proposition 9.247. *The topologist's sine curve is $\{(x, \sin 1/x) \mid 0 < x \leq 1\} \cup (\{0\} \times [-1, 1])$.*

PROOF: Sketch proof: Given a point $(0, y)$ with $-1 \leq y \leq 1$, pick a such that $\sin a = y$. Then $((1/a, y), (1/(a + 2\pi), y), (1/(a + 4\pi), y), \dots)$ is a sequence in S that converges to $(0, y)$.

Conversely, let (x, y) be any point not in $S \cup (\{0\} \times [-1, 1])$. If $x < 0$ or $y > 1$ or $y < -1$ then we can easily find a neighbourhood that does not intersect $S \cup (\{0\} \times [-1, 1])$. If $x > 0$ and $-1 \leq y \leq 1$, then we have $y \neq \sin 1/x$. Hence pick a neighbourhood that does not intersect S .

Proposition 9.248. *Every closed subset of \mathbb{R} that is bounded above has a greatest element.*

PROOF: It has a supremum, which is a limit point of the set and hence an element. □

Proposition 9.249 (CC). *The topologist's sine curve is not path connected.*

PROOF:

<1>1. ASSUME: For a contradiction $p : [0, 1] \rightarrow \bar{S}$ is a path from $(0, 0)$ to $(1, \sin 1)$.

<1>2. $\{t \in [0, 1] \mid p(t) \in \{0\} \times [-1, 1]\}$ is closed.

PROOF: Since p is continuous and $\{0\} \times [-1, 1]$ is closed.

<1>3. LET: b be the largest number in $[0, 1]$ such that $p(b) \in \{0\} \times [-1, 1]$.

PROOF: Proposition 9.248.

<1>4. LET: $x : [b, 1] \rightarrow \bar{S}$ be the function $\pi_1 \circ p$

<1>5. LET: $y : [b, 1] \rightarrow \bar{S}$ be the function $\pi_2 \circ p$

<1>6. PICK a sequence t_n in $(b, 1]$ such that $t_n \rightarrow b$ and $y(t_n) = (-1)^n$ for all n

<2>1. LET: $n \geq 1$

<2>2. PICK u with $0 < u < x(1/n)$ and $\sin(1/u) = (-1)^n$

<2>3. PICK t_n with $b < t_n < 1/n$ and $x(t_n) = u$

PROOF: By the Intermediate Value Theorem

<1>7. Q.E.D.

PROOF: This contradicts Proposition 9.96 since y is continuous and $y(t_n)$ does not converge.

□

Corollary 9.249.1. *The closure of a path-connected subspace of a space is not necessarily path-connected.*

9.35 The Long Line

Definition 9.250 (The Long Line). The *long line* is the space $\omega_1 \times [0, 1]$ in the dictionary order under the order topology, where ω_1 is the first uncountable ordinal.

Lemma 9.251. *For any ordinal α with $0 < \alpha < \omega_1$ we have $[(0, 0), (\alpha, 0)) \cong [0, 1)$*

$\langle 1 \rangle 1.$ $[(0, 0), (1, 0)) \cong [0, 1)$

PROOF: The map π_2 is a homeomorphism.

$\langle 1 \rangle 2.$ If $[(0, 0), (\alpha, 0)) \cong [0, 1)$ then $[(0, 0), (\alpha + 1, 0)) \cong [0, 1)$

PROOF: Proposition 4.21.

$\langle 1 \rangle 3.$ If λ is a limit ordinal with $\lambda < \omega_1$ and $[(0, 0), (\alpha, 0)) \cong [0, 1)$ for all α with $0 < \alpha < \lambda$ then $[(0, 0), (\lambda, 0)) \cong [0, 1)$

$\langle 2 \rangle 1.$ LET: λ be a limit ordinal $< \omega_1$

$\langle 2 \rangle 2.$ ASSUME: $[(0, 0), (\alpha, 0)) \cong [0, 1)$ for all α with $0 < \alpha < \lambda$

$\langle 2 \rangle 3.$ PICK a sequence of ordinals $\alpha_0 < \alpha_1 < \dots$ with limit λ

PROOF: Since λ is countable.

$\langle 2 \rangle 4.$ $[(\alpha_i, 0), (\alpha_{i+1}, 0)) \cong [0, 1)$ for all i

PROOF: Lemma 4.20.

$\langle 2 \rangle 5.$ Q.E.D.

PROOF: By Proposition 4.22.

$\langle 1 \rangle 4.$ Q.E.D.

PROOF: By transfinite induction.

Proposition 9.252 (CC). *The long line is path-connected.*

PROOF:

$\langle 1 \rangle 1.$ LET: $(\alpha, i), (\beta, j) \in \omega_1 \times [0, 1)$

$\langle 1 \rangle 2.$ ASSUME: without loss of generality $(\alpha, i) < (\beta, j)$

$\langle 1 \rangle 3.$ $[(0, 0), (\beta + 1, 0)) \cong [0, 1)$

PROOF: By Lemma 9.251

$\langle 1 \rangle 4.$ $[(\alpha, i), (\beta, j)) \cong [0, 1)$

PROOF: Lemma 4.20.

$\langle 1 \rangle 5.$ PICK a homeomorphism $q : [0, 1) \rightarrow [(\alpha, i), (\beta, j))$

$\langle 1 \rangle 6.$ $q \cup \{(1, (\beta, j))\}$ is a path from (α, i) to (β, j)

□

Proposition 9.253. *Every point in the long line has a neighbourhood homeomorphic to an interval in \mathbb{R} .*

PROOF: For any (α, i) in the long line, the neighbourhood $[(0, 0), (\alpha + 1, 0))$ satisfies the condition by Lemma 9.251.

Proposition 9.254. *The long line L is not second countable.*

PROOF:

$\langle 1 \rangle 1.$ LET: \mathcal{B} be a basis for L .

$\langle 1 \rangle 2.$ For $\alpha < \omega_1$, PICK $B_\alpha \in \mathcal{B}$ such that $(\alpha, 1/2) \in B_\alpha \subseteq ((\alpha, 0), (\alpha + 1, 0))$

$\langle 1 \rangle 3.$ \mathcal{B} is uncountable.

PROOF: The mapping $\alpha \mapsto B_\alpha$ is an injection $\omega_1 \rightarrow \mathcal{B}$.

Corollary 9.254.1. *The long line cannot be imbedded into \mathbb{R}^n for any n .*

9.36 Components

Proposition 9.255. *Let X be a topological space. Define the relation \sim on X by $a \sim b$ if and only if there exists a connected subspace $A \subseteq X$ with $a, b \in A$. Then \sim is an equivalence relation on X .*

PROOF:

$\langle 1 \rangle 1.$ \sim is reflexive.

PROOF: For any $a \in X$ we have $\{a\}$ is a connected subspace that contains a .

$\langle 1 \rangle 2.$ \sim is symmetric.

PROOF: Trivial.

$\langle 1 \rangle 3.$ \sim is transitive.

$\langle 2 \rangle 1.$ LET: $a, b, c \in X$

$\langle 2 \rangle 2.$ ASSUME: $a \sim b$ and $b \sim c$

$\langle 2 \rangle 3.$ PICK connected subspaces A and B with $a, b \in A$ and $b, c \in B$

$\langle 2 \rangle 4.$ $A \cup B$ is a connected subspace that contains a and c

PROOF: Theorem 9.213.

□

Definition 9.256 ((Connected) Component). Let X be a topological space. The *(connected) components* of X are the equivalence classes under the above \sim .

Lemma 9.257. *Let X be a topological space. If $A \subseteq X$ is connected and nonempty then there exists a unique component C of X such that $A \subseteq C$.*

PROOF:

$\langle 1 \rangle 1.$ PICK $a \in A$

$\langle 1 \rangle 2.$ LET: C be the \sim -equivalence class of a .

$\langle 1 \rangle 3.$ $A \subseteq C$

PROOF: For all $x \in A$ we have $x \sim a$.

$\langle 1 \rangle 4.$ If C' is a component and $A \subseteq C'$ then $C = C'$

PROOF: Since we have $a \in C'$.

□

Theorem 9.258. *Let X be a topological space. The components of X are connected disjoint subspaces of X whose union is X such that each nonempty connected subspace of X intersects only one of them.*

PROOF:

$\langle 1 \rangle 1.$ Every component of X is connected.

PROOF: For $a \in X$, the \sim -equivalence class of a is $\bigcup \{A \subseteq X \mid A \text{ is connected, } a \in A\}$ which is connected by Theorem 9.213.

$\langle 1 \rangle 2.$ The components form a partition of X .

PROOF: Immediate from the definition.

$\langle 1 \rangle 3.$ Every nonempty connected subspace of X intersects a unique component of X .

$\langle 2 \rangle 1.$ LET: $A \subseteq X$ be connected and nonempty.

$\langle 2 \rangle 2$. LET: C be the component such that $A \subseteq C$
 PROOF: Lemma 9.257.
 $\langle 2 \rangle 3$. A intersects C
 $\langle 2 \rangle 4$. If A intersects the component C' then $C' = C$
 $\langle 3 \rangle 1$. LET: C' be a component that intersects A
 $\langle 3 \rangle 2$. PICK $b \in A \cap C'$
 $\langle 3 \rangle 3$. $A \subseteq C'$
 PROOF: For all $x \in A$ we have $x \sim b$.
 $\langle 3 \rangle 4$. $C = C'$
 PROOF: By uniqueness in $\langle 2 \rangle 2$.

□

Proposition 9.259. *Every component of a space is closed.*

PROOF:

$\langle 1 \rangle 1$. LET: X be a topological space and C a component of X .
 $\langle 1 \rangle 2$. \overline{C} is connected.
 PROOF: Theorem 9.214.
 $\langle 1 \rangle 3$. $C = \overline{C}$
 PROOF: Lemma 9.212.
 $\langle 1 \rangle 4$. C is closed.
 PROOF: Lemma 9.37.

□

Proposition 9.260. *If a topological space has finitely many components then every component is open.*

PROOF: Each component is the complement of a finite union of closed sets. □

9.37 Path Components

Proposition 9.261. *Let X be a topological space. Define the relation \sim on X by: $a \sim b$ if and only if there exists a path in X from a to b . Then \sim is an equivalence relation on X .*

PROOF:

$\langle 1 \rangle 1$. \sim is reflexive.
 PROOF: For $a \in X$, the constant function $[0, 1] \rightarrow X$ with value a is a path from a to a .
 $\langle 1 \rangle 2$. \sim is symmetric.
 PROOF: If $p : [0, 1] \rightarrow X$ is a path from a to b , then $\lambda t.p(1-t)$ is a path from b to a .
 $\langle 1 \rangle 3$. \sim is transitive.
 PROOF: Concatenate paths.

□

Definition 9.262 (Path Component). Let X be a topological space. The *path components* of X are the equivalence relations under \sim .

Theorem 9.263. *The path components of X are path-connected disjoint subspaces of X whose union is X such that every nonempty path-connected subspace of X intersects exactly one path component.*

PROOF:

⟨1⟩1. Every path component is path-connected.

PROOF: If a and b are in the same path component then $a \sim b$, i.e. there exists a path from a to b .

⟨1⟩2. The path components are disjoint and their union is X .

PROOF: Immediate from the definition.

⟨1⟩3. Every non-empty path-connected subspace of X intersects exactly one path component.

⟨2⟩1. LET: A be a nonempty path-connected subspace of X .

⟨2⟩2. PICK $a \in A$

⟨2⟩3. A intersects the \sim -equivalence class of a .

⟨2⟩4. LET: C be any path component that intersects A .

⟨2⟩5. PICK $b \in A \cap C$

⟨2⟩6. $a \sim b$

PROOF: Since A is path-connected.

⟨2⟩7. C is the \sim -equivalence class of a .

□

Proposition 9.264. *Every path component is included in a component.*

PROOF:

⟨1⟩1. LET: X be a topological space and C a path component of X .

⟨1⟩2. C is path-connected.

PROOF: Theorem 9.263.

⟨1⟩3. C is connected.

PROOF: Proposition 9.234.

⟨1⟩4. C is included in a component.

PROOF: Lemma 9.257.

□

9.38 Local Connectedness

Definition 9.265 (Locally Connected). Let X be a topological space and $a \in X$. Then X is *locally connected* at a if and only if every neighbourhood of a includes a connected neighbourhood of a .

The space X is *locally connected* if and only if it is locally connected at every point.

Example 9.266. The real line is both connected and locally connected.

Example 9.267. The space $\mathbb{R} \setminus \{0\}$ is disconnected but locally connected.

Example 9.268. The topologist's sine curve is connected but not locally connected.

Example 9.269. The rationals \mathbb{Q} are neither connected nor locally connected.

Theorem 9.270. *A topological space X is locally connected if and only if, for every open set U in X , every component of U is open in X .*

PROOF:

⟨1⟩1. If X is locally connected then, for every open set U in X , every component of U is open in X .

⟨2⟩1. ASSUME: X is locally connected.

⟨2⟩2. LET: U be open in X .

⟨2⟩3. LET: C be a component of U .

⟨2⟩4. LET: $a \in C$

⟨2⟩5. LET: V be a connected neighbourhood of a such that $V \subseteq U$

⟨2⟩6. $V \subseteq C$

PROOF: Lemma 9.257.

⟨2⟩7. Q.E.D.

PROOF: Lemma 9.16.

⟨1⟩2. If, for every open set U in X , every component of U is open in X , then X is locally connected.

⟨2⟩1. ASSUME: for every open set U in X , every component of U is open in X .

⟨2⟩2. LET: $a \in X$

⟨2⟩3. LET: U be a neighbourhood of a

⟨2⟩4. The component of U that contains a is a connected neighbourhood of a included in U .

□

Example 9.271. The ordered square is locally connected.

PROOF: Every neighbourhood of a point includes an interval around that point, which is connected by Theorem 9.225.

Example 9.272. Let X be the set of all rational points on the line segment $[0, 1] \times \{0\}$, and Y the set of all rational points on the line segment $[0, 1] \times \{1\}$. Let A be the space consisting of all line segments joining the point $(0, 1)$ to a point of X , and all line segments joining the point $(1, 0)$ to a point of Y . Then A is path-connected but is not locally connected at any point,

Proposition 9.273. *Let X and Y be topological spaces and $p : X \rightarrow Y$ be a quotient map. If X is locally connected then so is Y .*

PROOF:

⟨1⟩1. LET: U be an open set in Y .

⟨1⟩2. LET: C be a component of U .

⟨1⟩3. $p^{-1}(C)$ is a union of components of $p^{-1}(U)$

⟨2⟩1. LET: $x \in p^{-1}(C)$

$\langle 2 \rangle 2$. LET: D be the component of $p^{-1}(U)$ that contains x .
 $\langle 2 \rangle 3$. $p(D)$ is connected.
 PROOF: Theorem 9.215.
 $\langle 2 \rangle 4$. $p(D) \subseteq C$.
 PROOF: From $\langle 1 \rangle 2$ since $p(x) \in p(D) \cap C$ ($\langle 2 \rangle 1$, $\langle 2 \rangle 2$).
 $\langle 2 \rangle 5$. $D \subseteq p^{-1}(C)$
 $\langle 1 \rangle 4$. $p^{-1}(C)$ is open in $p^{-1}(U)$
 PROOF: Theorem 9.270.
 $\langle 1 \rangle 5$. C is open in U
 PROOF: Since the restriction of p to $p : p^{-1}(U) \rightarrow U$ is a quotient map by Proposition 9.188.
 $\langle 1 \rangle 6$. Q.E.D.
 PROOF: Theorem 9.270.
 \square

9.39 Local Path Connectedness

Definition 9.274 (Locally Path-Connected). Let X be a topological space and $a \in X$. Then X is *locally path-connected* at a if and only if every neighbourhood of a includes a path-connected neighbourhood of a .

The space X is *locally path-connected* if and only if it is locally path-connected at every point.

Theorem 9.275. *A topological space X is locally path-connected if and only if, for every open set U in X , every path component of U is open in X .*

PROOF:

$\langle 1 \rangle 1$. If X is locally path-connected then, for every open set U in X , every path component of U is open in X .
 $\langle 2 \rangle 1$. ASSUME: X is locally path-connected.
 $\langle 2 \rangle 2$. LET: U be open in X .
 $\langle 2 \rangle 3$. LET: C be a path component of U .
 $\langle 2 \rangle 4$. LET: $a \in C$
 $\langle 2 \rangle 5$. LET: V be a path-connected neighbourhood of a such that $V \subseteq U$
 $\langle 2 \rangle 6$. $V \subseteq C$
 PROOF: Lemma 9.257.
 $\langle 2 \rangle 7$. Q.E.D.
 PROOF: Lemma 9.16.
 $\langle 1 \rangle 2$. If, for every open set U in X , every component of U is open in X , then X is locally connected.
 $\langle 2 \rangle 1$. ASSUME: for every open set U in X , every component of U is open in X .
 $\langle 2 \rangle 2$. LET: $a \in X$
 $\langle 2 \rangle 3$. LET: U be a neighbourhood of a
 $\langle 2 \rangle 4$. The component of U that contains a is a connected neighbourhood of a included in U .

□

Theorem 9.276. *If a space is locally path connected then its components and its path components are the same.*

PROOF:

⟨1⟩1. LET: X be a locally path connected space.

⟨1⟩2. LET: C be a component of X .

⟨1⟩3. LET: $x \in C$

⟨1⟩4. LET: P be the path component of x

PROVE: $P = C$

⟨1⟩5. $P \subseteq C$

PROOF: Proposition 9.264.

⟨1⟩6. LET: Q be the union of the other path components included in C

⟨1⟩7. $C = P \cup Q$

PROOF: Proposition 9.264.

⟨1⟩8. P and Q are open in C

⟨2⟩1. C is open.

PROOF: Theorem 9.270.

⟨2⟩2. Q.E.D.

PROOF: Theorem 9.275.

⟨1⟩9. $Q = \emptyset$

PROOF: Otherwise P and Q would form a separation of C .

□

Example 9.277. The ordered square is not locally path connected, since it is connected but not path connected.

Proposition 9.278. *Let X be a locally path-connected space. Then every connected open subspace of X is path-connected.*

PROOF:

⟨1⟩1. LET: U be a connected open subspace of X .

⟨1⟩2. LET: P be a path component of U .

⟨1⟩3. LET: Q be the union of the other path components of U .

⟨1⟩4. P and Q are open in U .

PROOF: Theorem 9.275.

⟨1⟩5. $Q = \emptyset$

PROOF: Otherwise P and Q form a separation of U .

□

9.40 Weak Local Connectedness

Definition 9.279 (Weakly Locally Connected). Let X be a topological space and $a \in X$. Then X is *weakly locally connected* at a if and only if every neighbourhood of a includes a connected subspace that includes a neighbourhood of a .

Proposition 9.280. *Let X be a topological space. If X is weakly locally connected at every point then X is locally connected.*

PROOF:

⟨1⟩1. ASSUME: X is weakly locally connected at every point.

⟨1⟩2. LET: U be open in X .

⟨1⟩3. LET: C be a component of U .

⟨1⟩4. C is open in X .

⟨2⟩1. LET: $x \in C$

⟨2⟩2. PICK a connected subspace D of U that includes a neighbourhood V of x .

⟨2⟩3. $D \subseteq C$

PROOF: Lemma 9.257.

⟨2⟩4. $x \in V \subseteq C$

⟨2⟩5. Q.E.D.

PROOF: Lemma 9.16.

⟨1⟩5. Q.E.D.

PROOF: Theorem 9.270.

□

Example 9.281. The 'infinite broom' (Munkres p.163) is an example of a space that is weakly locally connected at a point p but not locally connected at p .

9.41 Quasicomponents

Proposition 9.282. *Let X be a topological space. Define \sim on X by $x \sim y$ if and only if there exists no separation U and V of X such that $x \in U$ and $y \in V$. Then \sim is an equivalence relation on X .*

PROOF:

⟨1⟩1. \sim is reflexive.

PROOF: Since the two sets that make up a separation are disjoint.

⟨1⟩2. \sim is symmetric.

PROOF: Immediate from the definition.

⟨1⟩3. \sim is transitive.

⟨2⟩1. ASSUME: $x \sim y$ and $y \sim z$

⟨2⟩2. ASSUME: for a contradiction there is a separation U and V of X with $x \in U$ and $z \in V$

⟨2⟩3. $y \in U$ or $y \in V$

⟨2⟩4. Q.E.D.

PROOF: Either case contradicts ⟨2⟩1.

□

Definition 9.283 (Quasicomponents). For X a topological space, the *quasi-components* of X are the equivalence classes under \sim .

Proposition 9.284. *Let X be a topological space. Then every component of X is included in a quasicomponent of X .*

PROOF:

$\langle 1 \rangle 1$. LET: C be a component of X .

$\langle 1 \rangle 2$. LET: $x, y \in C$

PROVE: $x \sim y$

$\langle 1 \rangle 3$. ASSUME: for a contradiction there exists a separation U and V of X with
 $x \in U$ and $y \in V$

$\langle 1 \rangle 4$. $C \cap U$ and $C \cap V$ form a separation of C .

$\langle 1 \rangle 5$. Q.E.D.

PROOF: This contradicts $\langle 1 \rangle 1$.

Proposition 9.285. *In a locally connected space, the components and the quasicomponents are the same.*

PROOF:

$\langle 1 \rangle 1$. LET: X be a locally connected space and Q a quasicomponent of X .

$\langle 1 \rangle 2$. PICK a component C of X such that $C \subseteq Q$

$\langle 1 \rangle 3$. LET: D be the union of the components of X

$\langle 1 \rangle 4$. C and D are open in X .

PROOF: Theorem 9.270.

$\langle 1 \rangle 5$. D cannot contain any points of Q .

PROOF: If it did, then C and D would form a separation of X and there would be points $x, y \in Q$ with $x \in C$ and $y \in D$.

$\langle 1 \rangle 6$. $C = Q$

□

9.42 Open Coverings

Definition 9.286 (Open Covering). Let X be a topological space. An *open covering* of X is a covering of X whose elements are all open sets.

9.43 Lindelöf Spaces

Definition 9.287 (Lindelöf Space). A topological space X is *Lindelöf* if and only if every open covering has a countable subcovering.

Proposition 9.288. *Let X be a topological space. Then X is Lindelöf if and only if every set of closed sets that has the countable intersection property has nonempty intersection.*

PROOF: The following are equivalent.

1. X is compact.
2. Every open covering of X has a countable subcovering.
3. For any set \mathcal{C} of closed sets, if $\{X \setminus C \mid C \in \mathcal{C}\}$ covers X then there is a countable subset \mathcal{C}_0 such that $\{X \setminus C \mid C \in \mathcal{C}_0\}$ covers X

4. For any set \mathcal{C} of closed sets, if $\bigcap \mathcal{C} = \emptyset$ then there is a countable subset \mathcal{C}_0 with empty intersection.
5. Any set of closed sets with the countable intersection property has nonempty intersection.

□

Proposition 9.289 (CC). *Let X be a topological space and \mathcal{B} a basis for the topology on X . Then the following are equivalent.*

1. X is Lindelöf.
2. Every open covering of X by elements of \mathcal{B} has a countable subcovering.

PROOF:

⟨1⟩1. $1 \Rightarrow 2$

PROOF: Immediate from definitions.

⟨1⟩2. $2 \Rightarrow 1$

⟨2⟩1. ASSUME: Every open covering of X by elements of \mathcal{B} has a countable subcovering.

⟨2⟩2. LET: \mathcal{U} be an open covering of X .

⟨2⟩3. $\{B \in \mathcal{B} \mid \exists U \in \mathcal{U}. B \subseteq U\}$ covers X .

⟨2⟩4. PICK a finite subcovering \mathcal{B}_0 .

⟨2⟩5. For $B \in \mathcal{B}_0$, PICK $U_B \in \mathcal{U}$ such that $B \subseteq U_B$

⟨2⟩6. $\{U_B \mid B \in \mathcal{B}_0\}$ covers X .

□

9.44 The Second Countability Axiom

Definition 9.290 (Second Countability Axiom). A topological space satisfies the *second countability axiom*, or is *second countable*, if and only if it has a countable basis.

Example 9.291. The space \mathbb{R} is second countable.

PROOF: The set $\{(a, b) \mid a, b \in \mathbb{Q}\}$ is a basis. □

Proposition 9.292. *A subspace of a second countable space is second countable.*

PROOF: If \mathcal{B} is a countable basis for X and $Y \subseteq X$ then $\{B \cap Y \mid B \in \mathcal{B}\}$ is a countable basis for Y . □

Proposition 9.293 (CC). *Every second countable space is Lindelöf.*

PROOF: From Proposition 9.289.

Example 9.294 (CC). The space \mathbb{R}_l is Lindelöf.

⟨1⟩1. LET: \mathcal{A} be a covering of \mathbb{R}_l by basic open sets of the form $[a, b)$

⟨1⟩2. LET: $C = \bigcup \{(a, b) \mid [a, b) \in \mathcal{A}\}$

- ⟨1⟩3. $\mathbb{R} \setminus C$ is countable.
 - ⟨2⟩1. For every $x \in \mathbb{R} \setminus C$, PICK a rational q_x such that $(x, q_x) \subseteq C$
 - ⟨3⟩1. LET: $x \in \mathbb{R} \setminus C$
 - ⟨3⟩2. PICK b such that $[x, b) \in \mathcal{A}$
 - ⟨3⟩3. PICK a rational q such that $q \in (x, b)$
 - ⟨2⟩2. The mapping $x \mapsto q_x$ is an injection $\mathbb{R} \setminus C \rightarrow \mathbb{Q}$
 - ⟨1⟩4. PICK a countable $\mathcal{A}' \subseteq \mathcal{A}$ that covers $\mathbb{R} \setminus C$
 - ⟨1⟩5. Under the standard topology on \mathbb{R} , C is second countable.
PROOF: Proposition 9.292.
 - ⟨1⟩6. PICK a countable $\mathcal{A}'' \subseteq \mathcal{A}$ such that $\{(a, b) \mid [a, b) \in \mathcal{A}''\}$ covers C .
PROOF: Proposition 9.289.
 - ⟨1⟩7. $\mathcal{A}' \cup \mathcal{A}''$ covers \mathbb{R}_l .
-

Example 9.295. The product of two Lindelöf spaces is not necessarily Lindelöf.
We prove that the Sorgenfrey plane is not Lindelöf.

PROOF:

- ⟨1⟩1. LET: $L = \{(x, -x) \mid x \in \mathbb{R}\}$
 - ⟨1⟩2. L is closed in \mathbb{R}_l^2
 - ⟨1⟩3. LET: $\mathcal{U} = \{[a, b) \times [a, -d) \mid a, b, d \in \mathbb{R}\}$
 - ⟨1⟩4. $\mathcal{U} \cup \{\mathbb{R} \setminus L\}$ covers \mathbb{R}_l^2
 - ⟨1⟩5. Every element of \mathcal{U} intersects L at exactly one point.
 - ⟨1⟩6. No countable subset of \mathcal{U} covers \mathbb{R}_l^2 .
-

9.45 Compact Spaces

Definition 9.296 (Compact). A topological space is *compact* if and only if every open covering has a finite subcovering.

Lemma 9.297. Let X be a topological space and Y a subspace of X . Then Y is compact if and only if every covering of Y by sets open in X has a finite subcovering.

PROOF:

- ⟨1⟩1. If Y is compact then every covering of Y by sets open in X has a finite subcovering.
- ⟨2⟩1. ASSUME: Y is compact.
- ⟨2⟩2. LET: \mathcal{U} be a covering of Y by sets open in X .
- ⟨2⟩3. $\{U \cap Y \mid U \in \mathcal{U}\}$ is an open covering of Y .
- ⟨2⟩4. PICK a finite subcovering $\{U_1 \cap Y, \dots, U_n \cap Y\}$
- ⟨2⟩5. $\{U_1, \dots, U_n\}$ is a finite subcovering of \mathcal{U} .
- ⟨1⟩2. If every covering of Y by sets open in X has a finite subcovering then Y is compact.
- ⟨2⟩1. LET: \mathcal{U} be an open covering of Y .
- ⟨2⟩2. LET: $\mathcal{V} = \{V \text{ open in } X \mid V \cap Y \in \mathcal{U}\}$.

- ⟨2⟩3. \mathcal{V} is a covering of Y by sets open in X .
- ⟨2⟩4. PICK a finite subcovering $\{V_1, \dots, V_n\}$
- ⟨2⟩5. $\{V_1 \cap Y, \dots, V_n \cap Y\}$ is a finite subcovering of \mathcal{U} .

□

Proposition 9.298. *Every closed subspace of a compact space is compact.*

PROOF:

- ⟨1⟩1. LET: X be a compact space and $Y \subseteq X$ be closed.
- ⟨1⟩2. LET: \mathcal{U} be a covering of Y by sets open in X .
- ⟨1⟩3. $\mathcal{U} \cup \{X \setminus Y\}$ is an open covering of X .
- ⟨1⟩4. PICK a finite subcovering \mathcal{U}_0
- ⟨1⟩5. $\mathcal{U}_0 \cap \mathcal{U}$ is a finite subset of \mathcal{U} that covers Y .

□

Theorem 9.299. *The continuous image of a compact space is compact.*

PROOF:

- ⟨1⟩1. LET: $f : X \rightarrow Y$ be continuous and surjective.
- ⟨1⟩2. LET: \mathcal{V} be an open covering of Y
- ⟨1⟩3. $\{p^{-1}(V) \mid V \in \mathcal{V}\}$ is an open covering of X .
- ⟨1⟩4. PICK a finite subcovering $\{p^{-1}(V_1), \dots, p^{-1}(V_n)\}$
- ⟨1⟩5. $\{V_1, \dots, V_n\}$ covers Y .

□

Theorem 9.300. *Let A and B be compact subspaces of X and Y respectively. Let N be an open set in $X \times Y$ that includes $A \times B$. Then there exist open sets U and V in X and Y respectively such that $A \subseteq U$, $B \subseteq V$ and $U \times V \subseteq N$.*

PROOF:

- ⟨1⟩1. For all $x \in A$, there exist neighbourhoods U of x and V of B such that $U \times V \subseteq N$.
- ⟨2⟩1. LET: $x \in A$
- ⟨2⟩2. For all $y \in B$, there exist neighbourhoods U of x and V of y such that $U \times V \subseteq N$
- ⟨2⟩3. $\{V \text{ open in } Y \mid \exists \text{ neighbourhood } U \text{ of } x, U \times V \subseteq N\}$ covers B .
- ⟨2⟩4. PICK a finite subcover $\{V_1, \dots, V_n\}$
- ⟨2⟩5. For $i = 1, \dots, n$, PICK a neighbourhood U_i of x such that $U_i \times V_i \subseteq N$
- ⟨2⟩6. LET: $U = U_1 \cap \dots \cap U_n$
- ⟨2⟩7. LET: $V = V_1 \cup \dots \cup V_n$
- ⟨2⟩8. U is a neighbourhood of x .
- ⟨2⟩9. V is a neighbourhood of B .
- ⟨2⟩10. $U \times V \subseteq N$
- ⟨1⟩2. $\{U \text{ open in } X \mid \exists \text{ neighbourhood } V \text{ of } B, U \times V \subseteq N\}$ covers A .
- ⟨1⟩3. PICK a finite subcover $\{U_1, \dots, U_n\}$
- ⟨1⟩4. For $i = 1, \dots, n$, PICK a neighbourhood V_i of B such that $U_i \times V_i \subseteq N$
- ⟨1⟩5. LET: $U = U_1 \cup \dots \cup U_n$
- ⟨1⟩6. LET: $V = V_1 \cap \dots \cap V_n$

⟨1⟩7. U and V are open.

⟨1⟩8. $A \subseteq U$

⟨1⟩9. $B \subseteq V$

⟨1⟩10. $U \times V \subseteq N$

□

Corollary 9.300.1 (Tube Lemma). *Let X and Y be topological spaces with Y compact. Let $a \in X$ and N be an open set in $X \times Y$ that includes $\{a\} \times Y$. Then there exists a neighbourhood W of a such that N includes the tube $W \times Y$.*

Theorem 9.301. *Let X be a topological space. Then X is compact if and only if every set of closed sets that has the finite intersection property has nonempty intersection.*

PROOF: The following are equivalent.

1. X is compact.
2. Every open covering of X has a finite subcovering.
3. For any set \mathcal{C} of closed sets, if $\{X \setminus C \mid C \in \mathcal{C}\}$ covers X then there is a finite subset \mathcal{C}_0 such that $\{X \setminus C \mid C \in \mathcal{C}_0\}$ covers X
4. For any set \mathcal{C} of closed sets, if $\bigcap \mathcal{C} = \emptyset$ then there is a finite subset \mathcal{C}_0 with empty intersection.
5. Any set of closed sets with the finite intersection property has nonempty intersection.

□

Corollary 9.301.1. *Let X be a topological space and $C_1 \supseteq C_2 \supseteq \cdots$ a nested sequence of nonempty closed sets. Then $\bigcap_n C_n$ is nonempty.*

Proposition 9.302. *Let \mathcal{T} and \mathcal{T}' be two topologies on the same set X with $\mathcal{T} \subseteq \mathcal{T}'$. If \mathcal{T}' is compact then \mathcal{T} is compact.*

PROOF:

⟨1⟩1. LET: $\mathcal{U} \subseteq \mathcal{T}$ cover X

⟨1⟩2. $\mathcal{U} \subseteq \mathcal{T}'$

⟨1⟩3. A finite subset of \mathcal{U} covers X .

□

Corollary 9.302.1. *If \mathcal{T} and \mathcal{T}' are two compact Hausdorff topologies on the same set X , then either $\mathcal{T} = \mathcal{T}'$ or \mathcal{T} and \mathcal{T}' are incomparable.*

PROOF: From the Proposition and Proposition 9.169. □

Example 9.303. Any set under the finite complement topology is compact.

Proposition 9.304. *Let X be a topological space. A finite union of compact subspaces of X is compact.*

PROOF:

- $\langle 1 \rangle 1$. LET: A and B be compact subspaces of X .
- $\langle 1 \rangle 2$. LET: \mathcal{U} be a set of open sets in X that covers $A \cup B$
- $\langle 1 \rangle 3$. PICK a finite subset \mathcal{U}_1 that covers A .
PROOF: Lemma 9.297.
- $\langle 1 \rangle 4$. PICK a finite subset \mathcal{U}_2 that covers B .
PROOF: Lemma 9.297.
- $\langle 1 \rangle 5$. $\mathcal{U}_1 \cup \mathcal{U}_2$ is a finite subset that covers $A \cup B$.
- $\langle 1 \rangle 6$. Q.E.D.
PROOF: Lemma 9.297.

□

Proposition 9.305. *Let A and B be disjoint compact subspaces of the Hausdorff space X . Then there exist disjoint open sets U and V that include A and B respectively.*

PROOF: From Theorem 9.300 with $N = X^2 \setminus \{(x, x) \mid x \in X\}$. □

Corollary 9.305.1. *Every compact subspace of a Hausdorff space is closed.*

Theorem 9.306. *Let $f : X \rightarrow Y$ be a bijective continuous function. If X is compact and Y is Hausdorff then f is a homeomorphism.*

PROOF:

- $\langle 1 \rangle 1$. LET: $C \subseteq X$ be closed.
- $\langle 1 \rangle 2$. C is compact.
PROOF: Proposition 9.298.
- $\langle 1 \rangle 3$. $f(C)$ is compact.
PROOF: Theorem 9.299.
- $\langle 1 \rangle 4$. $f(C)$ is closed.
PROOF: Corollary 9.305.1.
- $\langle 1 \rangle 5$. Q.E.D.
PROOF: Lemma 9.98.

□

Proposition 9.307. *Let X be a compact space, Y a Hausdorff space, and $f : X \rightarrow Y$ a continuous map. Then f is a closed map.*

PROOF:

- $\langle 1 \rangle 1$. LET: $C \subseteq X$ be closed.
- $\langle 1 \rangle 2$. C is compact.
PROOF: Proposition 9.298.
- $\langle 1 \rangle 3$. $f(C)$ is compact.
PROOF: Theorem 9.299.
- $\langle 1 \rangle 4$. $f(C)$ is closed.
PROOF: Corollary 9.305.1.

□

Proposition 9.308. *If Y is compact then the projection $\pi_1 : X \times Y \rightarrow X$ is a closed map.*

PROOF:

- ⟨1⟩1. LET: $A \subseteq X \times Y$ be closed.
- ⟨1⟩2. LET: $x \in X \setminus \pi_1(A)$
- ⟨1⟩3. PICK a neighbourhood U of x such that $U \times Y \subseteq (X \times Y) \setminus A$
PROOF: By the Tube Lemma.
- ⟨1⟩4. $x \in U \subseteq X \setminus \pi_1(A)$
- ⟨1⟩5. Q.E.D.

PROOF: So $X \setminus \pi_1(A)$ is open by Lemma 9.16.

□

Theorem 9.309. *Let X be a topological space and Y a compact Hausdorff space. Let $f : X \rightarrow Y$ be a function. Then f is continuous if and only if the graph of f is closed in $X \times Y$.*

PROOF:

- ⟨1⟩1. LET: G_f be the graph of f .
- ⟨1⟩2. If f is continuous then G_f is closed.
 - ⟨2⟩1. ASSUME: f is continuous.
 - ⟨2⟩2. LET: $(x, y) \in (X \times Y) \setminus G_f$
 - ⟨2⟩3. PICK disjoint neighbourhoods U and V of y and $f(x)$ respectively.
 - ⟨2⟩4. $f^{-1}(V) \times U$ is a neighbourhood of (x, y) disjoint from G_f .
- ⟨1⟩3. If G_f is closed then f is continuous.
 - ⟨2⟩1. ASSUME: G_f is closed.
 - ⟨2⟩2. LET: $x \in X$ and V be a neighbourhood of $f(x)$.
 - ⟨2⟩3. $G_f \cap (X \times (Y \setminus V))$ is closed.
 - ⟨2⟩4. $\pi_1(G_f \cap (X \times (Y \setminus V)))$ is closed.

PROOF: Proposition 9.308.

- ⟨2⟩5. LET: $U = X \setminus \pi_1(G_f \cap (X \times (Y \setminus V)))$
- ⟨2⟩6. U is a neighbourhood of x
- ⟨2⟩7. $f(U) \subseteq V$

□

Theorem 9.310. *Let X be a compact topological space. Let $(f_n : X \rightarrow \mathbb{R})$ be a monotone increasing sequence of continuous functions and $f : X \rightarrow \mathbb{R}$ a continuous function. If (f_n) converges pointwise to f , then (f_n) converges uniformly to f .*

PROOF:

- ⟨1⟩1. LET: $\epsilon > 0$
- ⟨1⟩2. For all $x \in X$, there exists N such that, for all $n \geq N$, we have $|f_n(x) - f(x)| < \epsilon$
- ⟨1⟩3. For $n \geq 1$,
LET: $U_n = \{x \in X \mid |f_n(x) - f(x)| < \epsilon\}$
- ⟨1⟩4. For $n \geq 1$, we have U_n is open in X .
 - ⟨2⟩1. LET: $x \in X$
 - ⟨2⟩2. LET: $\delta = \epsilon - |f_n(x) - f(x)|$
 - ⟨2⟩3. PICK a neighbourhood U of x such that $f(U) \subseteq B(f(x), \delta/2)$

⟨2⟩4. PICK a neighbourhood V of x such that $f_n(V) \subseteq B(f_n(x), \delta/2)$

⟨2⟩5. $f(U \cap V) \subseteq U_n$

PROOF: For $y \in U \cap V$ we have

$$\begin{aligned} |f_n(y) - f(y)| &\leq |f_n(y) - f_n(x)| + |f_n(x) - f(x)| + |f(x) - y(y)| \\ &< \delta/2 + |f_n(x) - f(x)| + \delta/2 \\ &= \epsilon \end{aligned}$$

⟨1⟩5. $\{U_n \mid n \geq 1\}$ covers X

PROOF: From ⟨1⟩2

⟨1⟩6. PICK N such that $X = U_N$

⟨2⟩1. PICK n_1, \dots, n_k such that U_{n_1}, \dots, U_{n_k} cover X .

⟨2⟩2. LET: $N = \max(n_1, \dots, n_k)$

⟨2⟩3. For all i we have $U_{n_i} \subseteq U_N$

PROOF: Since (f_n) is monotone increasing.

⟨2⟩4. $X = U_N$

⟨1⟩7. For all $x \in X$ and $n \geq N$ we have $|f_n(x) - f(x)| < \epsilon$

□

An example to show that we cannot remove the hypothesis that X is compact:

Example 9.311. Let $X = (0, 1)$, $f_n(x) = -x^n$ and $f(x) = 0$ for $x \in X$ and $n \geq 1$. Then $f_n \rightarrow f$ pointwise and (f_n) is monotone increasing but the convergence is not uniform since, for all $N \geq 1$, there exists $x \in (0, 1)$ such that $-x^N < -1/2$.

An example to show that we cannot remove the hypothesis that (f_n) is monotone increasing:

Example 9.312. Let $X = [0, 1]$, $f_n(x) = 1/(n^3(x - 1/n)^2 + 1)$ and $f(x) = 0$ for $x \in X$ and $n \geq 1$. Then X is compact and $f_n \rightarrow f$ pointwise but the convergence is not uniform since, for all $N \geq 1$, there exists $x \in [0, 1]$ such that $f_N(x) = 1$, namely $x = 1/N$.

Theorem 9.313. Let X be a compact Hausdorff space. Let \mathcal{A} be a chain of closed connected subsets of X . Then $\bigcap \mathcal{A}$ is connected.

PROOF:

⟨1⟩1. ASSUME: for a contradiction C and D form a separation of $\bigcap \mathcal{A}$.

⟨1⟩2. PICK disjoint open sets U and V that include C and D respectively.

PROOF: Proposition 9.305.

⟨1⟩3. $\{A \setminus (U \cup V) \mid A \in \mathcal{A}\}$ is a set of closed sets with the finite intersection property.

⟨2⟩1. For all $A \in \mathcal{A}$ we have $A \setminus (U \cup V)$ is closed.

⟨2⟩2. For all $A_1, \dots, A_n \in \mathcal{A}$ we have $(A_1 \cap \dots \cap A_n) \setminus (U \cup V)$ is nonempty.

PROOF:

⟨3⟩1. LET: $A_1, \dots, A_n \in \mathcal{A}$

⟨3⟩2. ASSUME: without loss of generality $A_1 \subseteq A_2, \dots, A_n$

PROOF: Since \mathcal{A} is a chain.

⟨3⟩3. $A_1 \setminus (U \cup V)$ is nonempty

PROOF: Otherwise $(A_1 \cap \cdots \cap A_n \cap U)$ and $(A_1 \cap \cdots \cap A_n \cap V)$ would form a separation of A_n .

⟨1⟩4. $\bigcap \mathcal{A} \setminus (U \cup V)$ is nonempty.

PROOF: Theorem 9.301.

⟨1⟩5. Q.E.D.

PROOF: This contradicts ⟨1⟩1 since $\bigcap \mathcal{A} \setminus (U \cup V) = \bigcap \mathcal{A} \setminus (C \cup D)$.

□

Theorem 9.314 (Tychonoff Theorem (AC)). *The product of a family of compact spaces is compact.*

PROOF:

⟨1⟩1. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of compact spaces.

⟨1⟩2. LET: $X = \prod_{\alpha \in J} X_\alpha$

⟨1⟩3. For any $\mathcal{A} \subseteq \mathcal{P}X$, we have $\bigcap_{A \in \mathcal{A}} \overline{A} \neq \emptyset$

⟨2⟩1. LET: $\mathcal{A} \subseteq \mathcal{P}X$

⟨2⟩2. PICK $\mathcal{D} \supseteq \mathcal{A}$ that is maximal with respect to the finite intersection property.

PROVE: $\bigcap_{D \in \mathcal{D}} \overline{D} \neq \emptyset$

PROOF: Lemma 3.66.

⟨2⟩3. For $\alpha \in J$, PICK $x_\alpha \in X_\alpha$ such that $x_\alpha \in \bigcap_{D \in \mathcal{D}} \overline{\pi_\alpha(D)}$

PROOF: Theorem 9.301 since $\{\overline{\pi_\alpha(D)} \mid D \in \mathcal{D}\}$ is a set of closed sets in X_α with the finite intersection property.

⟨2⟩4. LET: $x = (x_\alpha)_{\alpha \in J}$

PROVE: $x \in \bigcap_{D \in \mathcal{D}} \overline{D}$

⟨2⟩5. For any $\beta \in J$ and neighbourhood U of x_β in X_β , we have $\pi_\beta^{-1}(U)$ intersects every element of \mathcal{D}

⟨3⟩1. LET: $\beta \in J$

⟨3⟩2. LET: U be a neighbourhood of x_β in X_β .

⟨3⟩3. LET: $D \in \mathcal{D}$

⟨3⟩4. $x_\beta \in \overline{\pi_\beta(D)}$

PROOF: From ⟨2⟩3

⟨3⟩5. U intersects $\pi_\beta(D)$.

⟨3⟩6. $\pi_\beta^{-1}(U)$ intersects D .

⟨2⟩6. For any $\beta \in J$ and neighbourhood U of x_β in X_β , we have $\pi_\beta^{-1}(U) \in \mathcal{D}$

PROOF: Lemma 3.68.

⟨2⟩7. Every basic neighbourhood of x is an element of \mathcal{D}

PROOF: Lemma 3.67.

⟨2⟩8. Every basic neighbourhood of x intersects every element of \mathcal{D}

PROOF: Since \mathcal{D} satisfies the finite intersection property.

⟨2⟩9. For all $D \in \mathcal{D}$ we have $x \in \overline{D}$

⟨1⟩4. Q.E.D.

PROOF: Theorem 9.301.

□

Lemma 9.315. *Let X and Y be topological spaces. Let \mathcal{A} be a set of basis*

elements for the product topology on $X \times Y$ such that no finite subset of \mathcal{A} covers $X \times Y$. If X is compact, then there exists $x \in X$ such that no finite subset of \mathcal{A} covers the slice $\{x\} \times Y$.

PROOF:

- ⟨1⟩1. ASSUME: for every $x \in X$, there exists a finite subset of \mathcal{A} that covers $\{x\} \times Y$
PROVE: A finite subset of \mathcal{A} covers $X \times Y$
 - ⟨1⟩2. $\{U \mid \exists U_1 \times V_1, \dots, U_n \times V_n \in \mathcal{A}. U = U_1 \cap \dots \cap U_n, V_1 \cup \dots \cup V_n = Y\}$ covers X
 - ⟨1⟩3. PICK a finite subcover U_1, \dots, U_m
 - ⟨1⟩4. PICK $U_{ij} \times V_{ij} \in \mathcal{A}$ such that, for every i , we have $U_i = \bigcap_j U_{ij}$ and $Y = \bigcup_j V_{ij}$
 - ⟨1⟩5. The collection of all $U_{ij} \times V_{ij}$ covers $X \times Y$
-

Theorem 9.316 (AC). *Let X be a compact Hausdorff space. Then the quasicomponents and the components of X are the same.*

PROOF:

- ⟨1⟩1. LET: $x, y \in X$
- ⟨1⟩2. ASSUME: x and y are in the same quasicomponent.
PROVE: x and y are in the same component.
- ⟨1⟩3. LET: \mathcal{A} be the set of all closed subsets A of X such that x and y are in the same quasicomponent of A .
- ⟨1⟩4. For every chain $\mathcal{B} \subseteq \mathcal{A}$ we have $\bigcap \mathcal{B} \in \mathcal{A}$
 - ⟨2⟩1. LET: $\mathcal{B} \subseteq \mathcal{A}$ be a chain.
 - ⟨2⟩2. ASSUME: for a contradiction U and V form a separation of $\bigcap \mathcal{B}$ with $x \in U$ and $y \in V$
 - ⟨2⟩3. PICK disjoint open sets U', V' in X such that $U \subseteq U'$ and $V \subseteq V'$
 - ⟨2⟩4. $\{B \setminus (U' \cup V') \mid B \in \mathcal{B}\}$ satisfies the finite intersection property.
 - ⟨3⟩1. LET: $B_1, \dots, B_n \in \mathcal{B}$
 - ⟨3⟩2. ASSUME: without loss of generality $B_1 \subseteq \dots \subseteq B_n$
PROOF: Since \mathcal{B} is a chain.
 - ⟨3⟩3. $\bigcap \{B_1 \setminus (U' \cup V'), \dots, B_n \setminus (U' \cup V')\} = B_1 \setminus (U' \cup V')$
 - ⟨3⟩4. $B_1 \setminus (U' \cup V')$ is nonempty
PROOF: Otherwise $B_1 \cap U'$ and $B_1 \cap V'$ would form a separation of B_1 , contradicting the fact that x and y are in the same quasicomponent of B_1 .
 - ⟨2⟩5. $\bigcap \mathcal{B} \setminus (U \cup V)$ is nonempty
PROOF: Theorem 9.301.
 - ⟨2⟩6. Q.E.D.
PROOF: This contradicts ⟨2⟩2.
- ⟨1⟩5. PICK a minimal element D in \mathcal{A} .
PROVE: D is connected.
PROOF: By Zorn's Lemma.
- ⟨1⟩6. ASSUME: for a contradiction U and V form a separation of D .

- ⟨1⟩7. ASSUME: without loss of generality $x, y \in U$
 PROOF: We cannot have that one of x, y is in U and the other in V since $D \in \mathcal{A}$.
- ⟨1⟩8. $U \in \mathcal{A}$
 PROOF: If X and Y form a separation of U with $x \in X$ and $y \in Y$, then X and $Y \cup V$ form a separation of D with $x \in X$ and $y \in Y \cup V$.
- ⟨1⟩9. Q.E.D.
 PROOF: There is a connected set D that contains both x and y .
 \square
- PROOF:
- ⟨1⟩1. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of compact spaces.
 ⟨1⟩2. LET: $X = \prod_{\alpha \in J} X_\alpha$
 ⟨1⟩3. PICK a well-ordering $<$ on J such that J has a greatest element.
 ⟨1⟩4. For $\alpha \in J$ and $p = \{p_i \in X_i\}_{i \leq \alpha}$ a family of points,
 LET: $Y(p) = \{x \in X \mid \forall i \leq \alpha. x_i = p_i\}$
 ⟨1⟩5. If $\alpha < \alpha'$ and p is an α' -indexed family of points then $Y(p) \subseteq Y(p \restriction \alpha)$
 PROOF: From definition.
- ⟨1⟩6. Given $\beta \in J$ and $p = \{p_i \in X_i\}_{i < \beta}$ a family of points,
 LET: $Z(p) = \bigcap_{\alpha < \beta} Y(p \restriction \alpha)$
- ⟨1⟩7. Given $\beta \in J$ and $p = \{p_i \in X_i\}_{i < \beta}$ a family of points, if \mathcal{A} is a finite set of basic open spaces for X that covers $Z(p)$, then there exists $\alpha < \beta$ such that \mathcal{A} covers $Y(p \restriction \alpha)$
- ⟨2⟩1. ASSUME: without loss of generality β has no immediate predecessor.
 ⟨2⟩2. For $A \in \mathcal{A}$,
 LET: $J_A = \{i < \beta \mid \pi_i(A) \neq X_i\}$
 ⟨2⟩3. LET: $\alpha = \max \bigcup_{A \in \mathcal{A}} J_A$
 ⟨2⟩4. LET: $x \in Y(p \restriction \alpha)$
 ⟨2⟩5. LET: $y \in Z(p)$ be the point with $y_i = p_i$ for $i < \beta$ and $y_i = x_i$ for $i \geq \beta$
 ⟨2⟩6. PICK $A \in \mathcal{A}$ such that $y \in A$
 PROOF: Since \mathcal{A} covers $Z(p)$.
 ⟨2⟩7. For $i \in J_A$ we have $x_i \in \pi_i(A)$
 PROOF: Since $i \leq \alpha$ so $x_i = p_i$
 ⟨2⟩8. For $i \in J \setminus J_A$ we have $x_i \in \pi_i(A)$
 PROOF: Since $\pi_i(A) = X_i$
 ⟨2⟩9. $x \in A$
- ⟨1⟩8. ASSUME: for a contraction \mathcal{A} is a set of basic open sets for X that covers X but such that no finite subset of \mathcal{A} covers X
- ⟨1⟩9. PICK a set of points $\{p_i\}_{i \in J}$ such that, for all $\alpha \in J$, we have $Y(p \restriction \alpha)$ is not finitely covered by \mathcal{A}
- ⟨2⟩1. ASSUME: as transfinite induction hypothesis $\alpha \in J$ and $\{p_i\}_{i < \alpha}$ is a family of points such that, for all $\alpha' < \alpha$, we have $Y(p \restriction \alpha')$ is not finitely covered by \mathcal{A}
- ⟨2⟩2. $Z(p)$ is not finitely covered by \mathcal{A}
 PROOF: By ⟨1⟩7.
- ⟨2⟩3. PICK $p_\alpha \in X_\alpha$ such that $Y(p)$ is not finitely covered by \mathcal{A}

PROOF: By Lemma 9.315 since there is a homeomorphism $\phi : Z(p) \cong X_\alpha \times \prod_{\alpha' > \alpha} X_{\alpha'}$ and, given p_α , this homomorphism ϕ restricts to a homeomorphism $Y(p) \cong \{p_\alpha\} \times \prod_{\alpha' > \alpha} X_{\alpha'}$.

$\langle 1 \rangle 10$. Q.E.D.

PROOF: If ω is the greatest element of J then $Y(p \upharpoonright \omega)$ is a singleton.

□

Theorem 9.317. *Every complete linearly ordered set in the order topology is compact.*

PROOF:

$\langle 1 \rangle 1$. LET: X be a complete linearly ordered set with least element a and greatest element b .

$\langle 1 \rangle 2$. LET: \mathcal{A} be an open covering of X .

$\langle 1 \rangle 3$. For all $x < b$, there exists $y > x$ such that $[x, y]$ can be covered by at most two elements of \mathcal{A} .

$\langle 2 \rangle 1$. LET: $x \in X$

$\langle 2 \rangle 2$. PICK $A \in \mathcal{A}$ with $x \in A$

$\langle 2 \rangle 3$. PICK $y > x$ such that $[x, y] \subseteq A$

$\langle 2 \rangle 4$. PICK $B \in \mathcal{A}$ with $y \in B$

$\langle 2 \rangle 5$. $[x, y]$ is covered by A and B

$\langle 1 \rangle 4$. LET: $C = \{y \in X \mid [a, y] \text{ can be covered by finitely many elements of } \mathcal{A}\}$

$\langle 1 \rangle 5$. LET: $c = \sup C$

$\langle 1 \rangle 6$. $c > a$

$\langle 2 \rangle 1$. PICK $x > a$ such that $[a, x]$ can be covered by at most two elements of \mathcal{A} .

PROOF: From $\langle 1 \rangle 3$.

$\langle 2 \rangle 2$. $x \in C$

$\langle 1 \rangle 7$. $c \in C$

$\langle 2 \rangle 1$. PICK $A \in \mathcal{A}$

$\langle 2 \rangle 2$. PICK $x < c$ such that $(x, c] \subseteq A$

$\langle 2 \rangle 3$. PICK $y > x$ such that $y \in C$

$\langle 2 \rangle 4$. PICK $\mathcal{A}_0 \subseteq^{\text{fin}} \mathcal{A}$ that covers $[a, y]$

$\langle 2 \rangle 5$. $\mathcal{A}_0 \cup \{A\}$ covers $[a, c]$

$\langle 1 \rangle 8$. $c = b$

$\langle 2 \rangle 1$. ASSUME: for a contradiction $c < b$

$\langle 2 \rangle 2$. PICK $x > c$ such that $[c, x]$ can be covered by at most two elements of \mathcal{A}

PROOF: From $\langle 1 \rangle 3$.

$\langle 2 \rangle 3$. $[a, x]$ can be finitely covered by \mathcal{A}

PROOF: From $\langle 1 \rangle 7$.

$\langle 2 \rangle 4$. Q.E.D.

PROOF: This contradicts the maximality of c .

□

Corollary 9.317.1. *Let X be a linearly ordered set with the least upper bound property. Then every closed interval in X is compact.*

Corollary 9.317.2. *Every closed interval in \mathbb{R} is compact.*

Theorem 9.318 (Extreme Value Theorem). *Any linearly ordered set under the order topology that is compact has a greatest and a least element.*

PROOF:

$\langle 1 \rangle 1$. LET: X be a linearly ordered set under the order topology that is compact.

$\langle 1 \rangle 2$. X has a greatest element.

$\langle 2 \rangle 1$. ASSUME: for a contradiction X has no greatest element.

$\langle 2 \rangle 2$. $\{(-\infty, a) \mid a \in X\}$ covers X .

$\langle 2 \rangle 3$. PICK a finite subcover $\{(-\infty, a_1), \dots, (-\infty, a_n)\}$, say.

$\langle 2 \rangle 4$. ASSUME: without loss of generality $a_1 \leq \dots \leq a_n$

$\langle 2 \rangle 5$. $X \subseteq (-\infty, a_n)$

$\langle 2 \rangle 6$. $a_n < a_n$

$\langle 1 \rangle 3$. X has a least element.

PROOF: Similar.

□

9.46 Perfect Maps

Definition 9.319 (Perfect Map). Let X and Y be topological spaces and $f : X \rightarrow Y$. Then f is a *perfect map* if and only if f is a closed map, continuous, surjective and $p^{-1}(y)$ is compact for all $y \in Y$.

Proposition 9.320. *Let X be a topological space, Y a compact space, and $p : X \rightarrow Y$ a closed map such that, for all $y \in Y$, we have $p^{-1}(y)$ is compact. Then X is compact.*

PROOF:

$\langle 1 \rangle 1$. LET: \mathcal{A} be a set of closed sets in X with the finite intersection property.

$\langle 1 \rangle 2$. $\mathcal{B} = \{p(A_1 \cap \dots \cap A_n) \mid A_1, \dots, A_n \in \mathcal{A}\}$ is a set of closed sets in Y with the finite intersection property.

PROOF: Since p is a closed map.

$\langle 1 \rangle 3$. PICK $y \in \bigcap \mathcal{B}$

PROOF: Theorem 9.301 since Y is compact.

$\langle 1 \rangle 4$. $\{A \cap p^{-1}(y) \mid A \in \mathcal{A}\}$ is a set of closed sets in $p^{-1}(y)$ with the finite intersection property.

$\langle 1 \rangle 5$. PICK $x \in \bigcap_{A \in \mathcal{A}} (A \cap p^{-1}(y))$

PROOF: Theorem 9.301 since $p^{-1}(y)$ is compact.

$\langle 1 \rangle 6$. $x \in \bigcap \mathcal{A}$

$\langle 1 \rangle 7$. Q.E.D.

PROOF: Theorem 9.301.

□

9.47 Topological Groups

Definition 9.321 (Topological Group). A *topological group* G consists of a T_1 space G and continuous maps $\cdot : G^2 \rightarrow G$ and $(\)^{-1} : G \rightarrow G$ such that $(G, \cdot, (\)^{-1})$ is a group.

Example 9.322. 1. The integers \mathbb{Z} under addition are a topological group.

2. The real numbers \mathbb{R} under addition are a topological group.

3. The positive reals under multiplication are a topological group.

4. The set $\{z \in \mathbb{C} \mid |z| = 1\}$ under multiplication and given the topology of S^1 is a topological group.

5. For any $n \geq 0$, the general linear group $GL_n(\mathbb{R})$ is a topological group under matrix multiplication, considered as a subspace of \mathbb{R}^{n^2} .

Lemma 9.323. Let G be a T_1 space and $\cdot : G^2 \rightarrow G$, $(\)^{-1} : G \rightarrow G$ be functions such that $(G, \cdot, (\)^{-1})$ is a group. Then G is a topological group if and only if the function $f : G^2 \rightarrow G$ that maps (x, y) to xy^{-1} is continuous.

PROOF:

$\langle 1 \rangle 1$. If G is a topological group then f is continuous.

PROOF: From Theorem 9.87.

$\langle 1 \rangle 2$. If f is continuous then G is a topological group.

$\langle 2 \rangle 1$. ASSUME: f is continuous.

$\langle 2 \rangle 2$. $(\)^{-1}$ is continuous.

PROOF: Since $x^{-1} = f(e, x)$.

$\langle 2 \rangle 3$. \cdot is continuous.

PROOF: Since $xy = f(x, y^{-1})$.

□

Lemma 9.324. Let G be a topological group and H a subgroup of G . Then H is a topological group under the subspace topology.

PROOF:

$\langle 1 \rangle 1$. H is T_1 .

PROOF: From Proposition 9.157.

$\langle 1 \rangle 2$. multiplication and inverse on H are continuous.

PROOF: From Theorem 9.88.

□

Lemma 9.325. Let G be a topological group and H a subgroup of G . Then \overline{H} is a subgroup of G .

PROOF:

$\langle 1 \rangle 1$. LET: $x, y \in \overline{H}$

PROVE: $xy^{-1} \in \overline{H}$

- <1>2. LET: U be any neighbourhood of xy^{-1}
 <1>3. LET: $f : G^2 \rightarrow G$, $f(a, b) = ab^{-1}$
 <1>4. $f^{-1}(U)$ is a neighbourhood of (x, y)
 <1>5. PICK neighbourhoods V, W of x and y respectively such that $f(V \times W) \subseteq U$.
 <1>6. PICK $a \in V \cap H$ and $b \in W \cap H$
 PROOF: Theorem 9.38.
 <1>7. $ab^{-1} \in U \cap H$
 <1>8. Q.E.D.
 PROOF: By Theorem 9.38.

□

Proposition 9.326. *Let G be a topological group and $\alpha \in G$. Then the maps $l_\alpha, r_\alpha : G \rightarrow G$ defined by $l_\alpha(x) = \alpha x$, $r_\alpha(x) = x\alpha$ are homeomorphisms of G with itself.*

PROOF: They are continuous with continuous inverses $l_{\alpha^{-1}}$ and $r_{\alpha^{-1}}$. □

Corollary 9.326.1. *Every topological group is homogeneous.*

PROOF: Given a topological group G and $a, b \in G$, we have $l_{ba^{-1}}$ is a homeomorphism that maps a to b . □

Proposition 9.327. *Let G be a topological group and H a subgroup of G . Give G/H the quotient topology. For all $\alpha \in G$, the map \bar{f}_α that sends xH to αxH is a homeomorphism $G/H \cong G/H$.*

PROOF:

- <1>1. \bar{f}_α is well-defined.

PROOF: If $xy^{-1} \in H$ then $(\alpha x)(\alpha y)^{-1} = xy^{-1} \in H$.

- <1>2. \bar{f}_α is continuous.

PROOF: Theorem 9.191 since $\bar{f}_\alpha \circ p = p \circ f_\alpha$ is continuous, where $p : G \twoheadrightarrow G/H$ is the canonical surjection.

- <1>3. \bar{f}_α^{-1} is continuous.

PROOF: Similar since $\bar{f}_\alpha^{-1} = \overline{f_{\alpha^{-1}}}$.

□

Corollary 9.327.1. *Let G be a topological group and H a subgroup of G . Give G/H the quotient topology. Then G/H is homogeneous.*

Proposition 9.328. *Let G be a topological group and H a subgroup of G . Give G/H the quotient topology. If H is closed in G then G/H is T_1 .*

PROOF:

- <1>1. LET: $p : G \twoheadrightarrow G/H$ be the canonical surjection

- <1>2. LET: $x \in G$

- <1>3. $p^{-1}(xH) = f_x(H)$

- <1>4. $p^{-1}(xH)$ is closed in G

PROOF: Since H is closed and f_x is a homomorphism of G with itself.

⟨1⟩5. $\{xH\}$ is closed in G/H

□

Proposition 9.329. *Let G be a topological group and H a subgroup of G . Give G/H the quotient topology. Then the canonical surjection $p : G \rightarrow G/H$ is an open map.*

PROOF:

⟨1⟩1. LET: $U \subseteq G$ be open.

⟨1⟩2. $p^{-1}(p(U)) = \bigcup_{h \in H} r_h(U)$

⟨1⟩3. $p^{-1}(p(U))$ is open.

⟨1⟩4. $p(U)$ is open.

□

Proposition 9.330. *Let G be a topological group and H a closed normal subgroup of G . Then G/H is a topological group under the quotient topology.*

PROOF:

⟨1⟩1. G/H is T_1

PROOF: Proposition 9.328.

⟨1⟩2. The map $\overline{m} : (xH, yH) \mapsto xy^{-1}H$ is continuous.

⟨2⟩1. $p^2 : G^2 \rightarrow (G/H)^2$ is a quotient map.

PROOF: Propositions 9.190, 9.329.

⟨2⟩2. $\overline{m} \circ p^2$ is continuous.

PROOF: As it is $p^2 \circ m$ where $m : G^2 \rightarrow G$ with $m(x, y) = xy^{-1}$

□

Lemma 9.331. *Let G be a topological group and $A, B \subseteq G$. If either A or B is open then AB is open.*

PROOF: If A is open we have $AB = \bigcup_{b \in B} r_b(A)$. Similarly if B is open. □

Definition 9.332 (Symmetric Neighbourhood). Let G be a topological group. A neighbourhood V of e is *symmetric* if and only if $V = V^{-1}$.

Lemma 9.333. *Let G be a topological group. Let V be a neighbourhood of e . Then V is symmetric if and only if, for all $x \in V$, we have $x^{-1} \in V$.*

PROOF:

⟨1⟩1. If V is symmetric then, for all $x \in V$, we have $x^{-1} \in V$

PROOF: Immediate from definitions.

⟨1⟩2. If, for all $x \in V$, we have $x^{-1} \in V$, then V is symmetric.

⟨2⟩1. ASSUME: for all $x \in V$ we have $x^{-1} \in V$

⟨2⟩2. $V \subseteq V^{-1}$

PROOF: If $x \in V$ then there exists $y \in V$ such that $x = y^{-1}$, namely

$y = x^{-1}$

⟨2⟩3. $V^{-1} \subseteq V$

PROOF: Immediate from ⟨2⟩1.

□

Lemma 9.334. *Let G be a topological group. For every neighbourhood U of e , there exists a symmetric neighbourhood V of e such that $V^2 \subseteq U$.*

PROOF:

$\langle 1 \rangle 1$. LET: U be a neighbourhood of e .

$\langle 1 \rangle 2$. PICK a neighbourhood V' of e such that $V'V' \subseteq U$

PROOF: Such a neighbourhood exists because multiplication in G is continuous.

$\langle 1 \rangle 3$. PICK a neighbourhood W of e such that $WW^{-1} \subseteq V'$

PROOF: Such a neighbourhood exists because the function that maps (x, y) to xy^{-1} is continuous.

$\langle 1 \rangle 4$. LET: $V = WW^{-1}$

$\langle 1 \rangle 5$. V is a neighbourhood of e

$\langle 2 \rangle 1$. $e \in V$

PROOF: Since $e \in W$ so $e = ee^{-1} \in V$.

$\langle 2 \rangle 2$. V is open

PROOF: Lemma 9.331.

$\langle 1 \rangle 6$. V is symmetric

$\langle 2 \rangle 1$. For all $x \in V$ we have $x^{-1} \in V$

$\langle 3 \rangle 1$. LET: $x \in V$

$\langle 3 \rangle 2$. PICK $y, z \in W$ such that $x = yz^{-1}$

$\langle 3 \rangle 3$. $x^{-1} = zy^{-1}$

$\langle 3 \rangle 4$. $x^{-1} \in V$

$\langle 3 \rangle 5$. $x \in V^{-1}$

$\langle 2 \rangle 2$. Q.E.D.

PROOF: Lemma 9.333

$\langle 1 \rangle 7$. $V^2 \subseteq U$

PROOF: We have $V^2 \subseteq (V')^2 \subseteq U$

□

Proposition 9.335. *Every topological group is Hausdorff.*

PROOF:

$\langle 1 \rangle 1$. LET: G be a topological group.

$\langle 1 \rangle 2$. LET: $x, y \in G$ with $x \neq y$

$\langle 1 \rangle 3$. LET: $U = G \setminus \{x^{-1}y\}$

$\langle 1 \rangle 4$. PICK a symmetric neighbourhood V of e such that $VV \subseteq U$

$\langle 2 \rangle 1$. U is open

PROOF: Since G is T_1 .

$\langle 2 \rangle 2$. $e \in U$

PROOF: Since $x \neq y$

$\langle 2 \rangle 3$. Q.E.D.

PROOF: Lemma 9.334.

$\langle 1 \rangle 5$. Vx and Vy are disjoint neighbourhoods of x and y respectively.

$\langle 2 \rangle 1$. Vx is open

PROOF: Since $Vx = r_x(V)$

$\langle 2 \rangle 2$. Vy is open

PROOF: Similar.

$\langle 2 \rangle 3$. $Vx \cap Vy = \emptyset$

$\langle 3 \rangle 1$. ASSUME: for a contradiction $z \in Vx \cap Vy$

$\langle 3 \rangle 2$. PICK $a, b \in V$ such that $z = ax = by$

$\langle 3 \rangle 3$. $xy^{-1} \in VV$

PROOF: Since $xy^{-1} = a^{-1}b$

$\langle 3 \rangle 4$. $xy^{-1} \in U$

$\langle 3 \rangle 5$. Q.E.D.

PROOF: From $\langle 1 \rangle 3$.

□

Proposition 9.336. *Every topological group is regular.*

PROOF:

$\langle 1 \rangle 1$. LET: G be a topological group.

$\langle 1 \rangle 2$. LET: $A \subseteq G$ be a closed set and $a \notin A$.

$\langle 1 \rangle 3$. LET: $U = G \setminus Aa^{-1}$

$\langle 1 \rangle 4$. PICK a symmetric neighbourhood V of e such that $VV \subseteq U$

$\langle 2 \rangle 1$. U is open

PROOF: Since $Aa^{-1} = r_{a^{-1}}(A)$ is closed.

$\langle 2 \rangle 2$. $e \in U$

PROOF: Since $a \notin A$.

$\langle 2 \rangle 3$. Q.E.D.

PROOF: Lemma 9.334.

$\langle 1 \rangle 5$. VA and Va are disjoint open sets with $A \subseteq VA$ and $a \in Va$

$\langle 2 \rangle 1$. VA is open

PROOF: Lemma 9.331

$\langle 2 \rangle 2$. Va is open

PROOF: Lemma 9.331

$\langle 2 \rangle 3$. $VA \cap Va = \emptyset$

$\langle 3 \rangle 1$. ASSUME: for a contradiction $z \in VA \cap Va$

$\langle 3 \rangle 2$. PICK $b, c \in V$ and $d \in A$ with $z = bd = ca$

$\langle 3 \rangle 3$. $da^{-1} \in U$

PROOF: Since $da^{-1} = b^{-1}c \in VV \subseteq U$

$\langle 3 \rangle 4$. Q.E.D.

PROOF: This contradicts $\langle 1 \rangle 3$

□

Proposition 9.337. *Let G be a topological group and H a subgroup of G . Give G/H the quotient topology. If H is closed in G then G/H is regular.*

PROOF:

$\langle 1 \rangle 1$. LET: $p : G \twoheadrightarrow G/H$ be the canonical surjection.

$\langle 1 \rangle 2$. LET: A be a closed set in G/H and $aH \in (G/H) \setminus A$.

$\langle 1 \rangle 3$. LET: $B = p^{-1}(A)$

$\langle 1 \rangle 4$. B is a closed saturated set in G .

$\langle 1 \rangle 5$. $B \cap aH = \emptyset$

- ⟨1⟩6. $B = BH$
- ⟨1⟩7. PICK a symmetric neighbourhood V of e such that VB does not intersect Va
 - ⟨2⟩1. LET: $U = G \setminus Ba^{-1}$
 - ⟨2⟩2. PICK a symmetric neighbourhood V of e such that $VV \subseteq U$
 - ⟨3⟩1. U is open
PROOF: Since $Ba^{-1} = r_{a^{-1}}(B)$ is closed.
 - ⟨3⟩2. $e \in U$
PROOF: If $e \in Ba^{-1}$ then $a \in B$
 - ⟨3⟩3. Q.E.D.
PROOF: Lemma 9.334
 - ⟨2⟩3. $VB \cap Va = \emptyset$
PROOF: If $vb = v'a$ for $v, v' \in V$ and $b \in B$ then we have $ba^{-1} = v^{-1}v' \in Ba \cap VV \subseteq Ba \cap U$.
- ⟨1⟩8. $p(VB)$ and $p(Va)$ are disjoint open sets
 - ⟨2⟩1. $p(VB)$ and $p(Va)$ are open.
PROOF: Proposition 9.329.
 - ⟨2⟩2. $p(VB) \cap p(Va) = \emptyset$
PROOF: If $vbH = v'aH$ for $v, v' \in V, b \in B$ then $v'a = vbh$ for some $h \in H$. Hence $v'a \in Va \cap VBH = Va \cap VB$.
- ⟨1⟩9. $A \subseteq p(VB)$
- ⟨1⟩10. $aH \in p(Va)$

□

Proposition 9.338. *Let G be a topological group. The component of G that contains e is a normal subgroup of G .*

PROOF:

- ⟨1⟩1. LET: C be the component of G that contains e .
- ⟨1⟩2. For all $x \in G$, xC is the component of G that contains x .
 - ⟨2⟩1. LET: $x \in G$
 - ⟨2⟩2. LET: D be the component of G that contains x .
 - ⟨2⟩3. $xC \subseteq D$
PROOF: Since xC is connected by Theorem 9.215.
 - ⟨2⟩4. $D \subseteq xC$
PROOF: Since $x^{-1}D \subseteq C$ similarly.
- ⟨1⟩3. For all $x \in G$, Cx is the component of G that contains x .
PROOF: Similar.
- ⟨1⟩4. For all $x \in C$ we have $xC = Cx = C$
- ⟨1⟩5. For all $x \in C$ we have $x^{-1}C = C$
- ⟨1⟩6. For all $x \in C$ we have $x^{-1} \in C$
- ⟨1⟩7. For all $x, y \in C$ we have $xy \in C$
PROOF: Since $xyC = xC = x$.
- ⟨1⟩8. For all $x \in G$ we have $xC = Cx$.
PROOF: From ⟨1⟩2 and ⟨1⟩3.

□

Lemma 9.339. *Let G be a topological group. Let A be a closed set in G and B*

a compact subspace of G such that $A \cap B = \emptyset$. Then there exists a symmetric neighbourhood U of e such that $AU \cap BU = \emptyset$.

PROOF:

- $\langle 1 \rangle 1$. For all $b \in B$ there exists a symmetric neighbourhood V of e such that $bV^2 \cap A = \emptyset$
- $\langle 2 \rangle 1$. LET: $b \in B$
- $\langle 2 \rangle 2$. LET: $W = b^{-1}(G \setminus A)$
- $\langle 2 \rangle 3$. W is a neighbourhood of e and $bW \cap A = \emptyset$
- $\langle 2 \rangle 4$. PICK a symmetric neighbourhood V of e such that $V^2 \subseteq W$
- $\langle 1 \rangle 2$. $\{bV^2 \mid b \in B, V \text{ is a symmetric neighbourhood of } e, bV^2 \cap A = \emptyset\}$ is an open cover of B
- $\langle 1 \rangle 3$. PICK a finite subcover $b_1V_1^2, \dots, b_nV_n^2$, say.
- $\langle 1 \rangle 4$. LET: $U = V_1 \cap \dots \cap V_n$
- $\langle 1 \rangle 5$. $BU^2 \cap A = \emptyset$
- $\langle 1 \rangle 6$. $AU \cap BU = \emptyset$

PROOF: If $av \in BU$ where $a \in A$ and $v \in V$ then $a = avv^{-1} \in BU^2 \cap A$.

□

Proposition 9.340 (AC). Let G be a topological group. Let A be a closed set in G , and B a compact subspace of G . Then AB is closed.

PROOF:

- $\langle 1 \rangle 1$. LET: $x \in G \setminus AB$
- $\langle 1 \rangle 2$. $A^{-1}x \cap B = \emptyset$
- $\langle 1 \rangle 3$. $A^{-1}x$ is closed.
- $\langle 1 \rangle 4$. PICK a symmetric neighbourhood U of e such that $A^{-1}xU \cap BU = \emptyset$
- $\langle 1 \rangle 5$. xU^2 is open

PROOF: Lemma 9.331.

- $\langle 1 \rangle 6$. $x \in xU^2 \subseteq G \setminus AB$

□

Corollary 9.340.1. Let G be a topological group and $H \leq G$. Let $p : G \twoheadrightarrow G/H$ be the quotient map. If H is compact then p is a closed map.

PROOF: For A closed in G , we have $p^{-1}(p(A)) = AH$ is closed, and so $p(A)$ is closed. □

Corollary 9.340.2. Let G be a topological group and $H \leq G$. If H and G/H are compact then G is compact.

PROOF: From Proposition 9.320 since, for all $aH \in G/H$, we have $p^{-1}(aH) = aH$ is compact because it is homomorphic to H . □

9.48 The Metric Topology

Definition 9.341 (Metric). Let X be a set. A *metric* on X is a function $d : X^2 \rightarrow \mathbb{R}$ such that:

1. For all $x, y \in X$, $d(x, y) \geq 0$
2. For all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$
3. For all $x, y \in X$, $d(x, y) = d(y, x)$
4. (*Triangle Inequality*) For all $x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$

We call $d(x, y)$ the *distance* between x and y .

Definition 9.342 (Open Ball). Let X be a metric space. Let $a \in X$ and $\epsilon > 0$. The *open ball* with *centre* a and *radius* ϵ is

$$B(a, \epsilon) = \{x \in X \mid d(a, x) < \epsilon\} .$$

Definition 9.343 (Metric Topology). Let X be a metric space. The *metric topology* on X is the topology generated by the basis consisting of all the open balls.

We prove this is a basis for a topology.

PROOF:

$\langle 1 \rangle 1$. For every point a , there exists a ball B such that $a \in B$

PROOF: We have $a \in B(a, 1)$.

$\langle 1 \rangle 2$. For any balls B_1, B_2 and point $a \in B_1 \cap B_2$, there exists a ball B_3 such that $a \in B_3 \subseteq B_1 \cap B_2$

$\langle 2 \rangle 1$. LET: $B_1 = B(c_1, \epsilon_1)$ and $B_2 = B(c_2, \epsilon_2)$

$\langle 2 \rangle 2$. LET: $\delta = \min(\epsilon_1 - d(c_1, a), \epsilon_2 - d(c_2, a))$

PROVE: $B(a, \delta) \subseteq B_1 \cap B_2$

$\langle 2 \rangle 3$. LET: $x \in B(a, \delta)$

$\langle 2 \rangle 4$. $x \in B_1$

PROOF:

$$\begin{aligned} d(x, c_1) &= d(x, a) + d(a, c_1) \\ &< \delta + d(a, c_1) \\ &\leq \epsilon_1 \end{aligned}$$

$\langle 2 \rangle 5$. $x \in B_2$

PROOF: Similar.

□

Proposition 9.344. Let X be a metric space and $U \subseteq X$. Then U is open if and only if, for every $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$.

PROOF:

$\langle 1 \rangle 1$. If U is open then, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$.

$\langle 2 \rangle 1$. ASSUME: U is open.

- ⟨2⟩2. LET: $x \in U$
 ⟨2⟩3. PICK $a \in X$ and $\delta > 0$ such that $x \in B(a, \delta) \subseteq U$
 ⟨2⟩4. LET: $\epsilon = \delta - d(a, x)$
 PROVE: $B(x, \epsilon) \subseteq U$
 ⟨2⟩5. LET: $y \in B(x, \epsilon)$
 ⟨2⟩6. $d(y, a) < \delta$
 PROOF:

$$\begin{aligned}
 d(y, a) &\leq d(a, x) + d(x, y) \\
 &< \delta + d(x, y) \\
 &= \epsilon
 \end{aligned}$$

- ⟨2⟩7. $y \in U$
 ⟨1⟩2. If, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$, then U is open.
 PROOF: Immediate from definitions.

□

Definition 9.345 (Discrete Metric). Let X be a set. The *discrete metric* on X is defined by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Proposition 9.346. *The discrete metric induces the discrete topology.*

PROOF: For any (open) set U and point $a \in U$, we have $a \in B(a, 1) \subseteq U$. □

Definition 9.347 (Standard Metric on \mathbb{R}). The *standard metric* on \mathbb{R} is defined by $d(x, y) = |x - y|$.

Proposition 9.348. *The standard metric on \mathbb{R} induces the standard topology on \mathbb{R} .*

PROOF:

- ⟨1⟩1. Every open ball is open in the standard topology on \mathbb{R} .
 PROOF: $B(a, \epsilon) = (a - \epsilon, a + \epsilon)$
 ⟨1⟩2. For every open set U and point $a \in U$, there exists $\epsilon > 0$ such that $B(a, \epsilon) \subseteq U$
 ⟨2⟩1. LET: U be an open set and $a \in U$
 ⟨2⟩2. PICK an open interval b, c such that $a \in (b, c) \subseteq U$
 ⟨2⟩3. LET: $\epsilon = \min(a - b, c - a)$
 ⟨2⟩4. $B(a, \epsilon) \subseteq U$

□

Definition 9.349 (Metrizable). A topological space X is *metrizable* if and only if there exists a metric on X that induces the topology.

Definition 9.350 (Bounded). Let X be a metric space and $A \subseteq X$. Then A is *bounded* if and only if there exists M such that, for all $x, y \in A$, we have $d(x, y) \leq M$.

Definition 9.351 (Diameter). Let X be a metric space and $A \subseteq X$. The *diameter* of A is

$$\text{diam } A = \sup_{x,y \in A} d(x,y) .$$

Definition 9.352 (Standard Bounded Metric). Let d be a metric on X . The *standard bounded metric* corresponding to d is the metric \bar{d} defined by

$$\bar{d}(x,y) = \min(d(x,y), 1) .$$

We prove this is a metric.

PROOF:

$\langle 1 \rangle 1. \bar{d}(x,y) \geq 0$

PROOF: Since $d(x,y) \geq 0$

$\langle 1 \rangle 2. \bar{d}(x,y) = 0$ if and only if $x = y$

PROOF: $\bar{d}(x,y) = 0$ if and only if $d(x,y) = 0$ if and only if $x = y$

$\langle 1 \rangle 3. \bar{d}(x,y) = \bar{d}(y,x)$

PROOF: Since $d(x,y) = d(y,x)$

$\langle 1 \rangle 4. \bar{d}(x,z) \leq \bar{d}(x,y) + \bar{d}(y,z)$

PROOF:

$$\begin{aligned} \bar{d}(x,y) + \bar{d}(y,z) &= \min(d(x,y), 1) + \min(d(y,z), 1) \\ &= \min(d(x,y) + d(y,z), d(x,y) + 1, d(y,z) + 1, 2) \\ &\geq \min(d(x,z), 1) \\ &= \bar{d}(x,z) \end{aligned}$$

□

Lemma 9.353. In any metric space X , the set $\mathcal{B} = \{B(a, \epsilon) \mid a \in X, \epsilon < 1\}$ is a basis for the metric topology.

PROOF:

$\langle 1 \rangle 1.$ Every element of \mathcal{B} is open.

PROOF: From Lemma 9.53.

$\langle 1 \rangle 2.$ For every open set U and point $a \in U$, there exists $B \in \mathcal{B}$ such that $a \in B \subseteq U$

$\langle 2 \rangle 1.$ LET: U be an open set and $a \in U$

$\langle 2 \rangle 2.$ PICK $\epsilon > 0$ such that $B(a, \epsilon) \subseteq U$

$\langle 2 \rangle 3.$ $B(a, \min(\epsilon, 1/2)) \subseteq U$

$\langle 1 \rangle 3.$ Q.E.D.

PROOF: Lemma 9.54.

□

Proposition 9.354. Let d be a metric on the set X . Then the standard bounded metric \bar{d} induces the same metric as d .

PROOF: This follows from Lemma 9.353 since the open balls with radius < 1 are the same under both metrics. □

Lemma 9.355. *Let d and d' be two metrics on the same set X . Let \mathcal{T} and \mathcal{T}' be the topologies they induce. Then $\mathcal{T} \subseteq \mathcal{T}'$ if and only if, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that*

$$B_{d'}(x, \delta) \subseteq B_d(x, \epsilon) .$$

PROOF:

$\langle 1 \rangle 1$. If $\mathcal{T} \subseteq \mathcal{T}'$ then, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$

PROOF: From Proposition 9.344 since $x \in B_d(x, \epsilon) \in \mathcal{T}'$.

$\langle 1 \rangle 2$. If, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$, then $\mathcal{T} \subseteq \mathcal{T}'$

$\langle 2 \rangle 1$. ASSUME: For all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$

$\langle 2 \rangle 2$. LET: $U \in \mathcal{T}$

$\langle 2 \rangle 3$. For all $x \in U$ there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq U$.

$\langle 3 \rangle 1$. LET: $x \in U$

$\langle 3 \rangle 2$. PICK $\epsilon > 0$ such that $B_d(x, \epsilon) \subseteq U$

PROOF: Proposition 9.344

$\langle 3 \rangle 3$. PICK $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$

PROOF: By $\langle 2 \rangle 1$

$\langle 3 \rangle 4$. $B_{d'}(x, \delta) \subseteq U$

$\langle 2 \rangle 4$. $U \in \mathcal{T}'$

PROOF: Proposition 9.344.

□

Proposition 9.356. \mathbb{R}^2 under the dictionary order topology is metrizable.

PROOF: Define $d : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$d((x, y), (x, z)) = \max(|y - z|, 1)$$

$$d((x, y), (x', y')) = 1$$

$$\text{if } x \neq x' \square$$

$\langle 1 \rangle 1$. $x \in \bigcap_{i=1}^N \pi_i^{-1}() \subseteq B_D(a, \epsilon)$

Proposition 9.357. *Let $d : X^2 \rightarrow \mathbb{R}$ be a metric on X . Then the metric topology on X is the coarsest topology such that d is continuous.*

PROOF:

$\langle 1 \rangle 1$. d is continuous.

$\langle 2 \rangle 1$. LET: $a, b \in X$

$\langle 2 \rangle 2$. LET: $\epsilon > 0$

$\langle 2 \rangle 3$. LET: $\delta = \epsilon/2$

$\langle 2 \rangle 4$. LET: $x, y \in X$

$\langle 2 \rangle 5$. ASSUME: $\rho((a, b), (x, y)) < \delta$

$\langle 2 \rangle 6$. $|d(a, b) - d(x, y)| < \epsilon$

$\langle 3 \rangle 1$. $d(a, b) - d(x, y) < \epsilon$

PROOF:

$$\begin{aligned}
 d(a, b) &\leq d(a, x) + d(x, y) + d(y, b) \\
 &\leq d(x, y) + 2\rho((a, b), (x, y)) \\
 &< d(x, y) + 2\delta \\
 &= d(x, y) + \epsilon
 \end{aligned}$$

$$\langle 3 \rangle 2. \quad d(a, b) - d(x, y) > -\epsilon$$

PROOF: Similar.

$\langle 2 \rangle 7.$ Q.E.D.

$\langle 1 \rangle 2.$ If \mathcal{T} is any topology under which d is continuous then \mathcal{T} is finer than the metric topology.

PROOF: Since $B(a, \epsilon) = d_a^{-1}((-\infty, \epsilon))$

□

Proposition 9.358. *Let X be a metric space with metric d and $A \subseteq X$. The restriction of d to A is a metric on A that induces the subspace topology.*

PROOF:

$\langle 1 \rangle 1.$ The restriction of d to A is a metric on A .

$\langle 1 \rangle 2.$ Every open ball under $d \upharpoonright A$ is open under the subspace topology.

PROOF: $B_{d \upharpoonright A}(a, \epsilon) = B_d(a, \epsilon) \cap A$.

$\langle 1 \rangle 3.$ If U is open in the subspace topology and $x \in U$, then there exists a $d \upharpoonright A$ -ball B such that $x \in B \subseteq U$.

$\langle 2 \rangle 1.$ PICK V open in X such that $U = V \cap A$

$\langle 2 \rangle 2.$ PICK $\epsilon > 0$ such that $B_d(x, \epsilon) \subseteq V$

$\langle 2 \rangle 3.$ Take $B = B_{d \upharpoonright A}(x, \epsilon)$

□

Corollary 9.358.1. *A subspace of a metrizable space is metrizable.*

Proposition 9.359. *Every metrizable space is Hausdorff.*

PROOF:

$\langle 1 \rangle 1.$ LET: X be a metric space

$\langle 1 \rangle 2.$ LET: $a, b \in X$ with $a \neq b$

$\langle 1 \rangle 3.$ LET: $\epsilon = d(a, b)/2$

$\langle 1 \rangle 4.$ LET: $U = B(a, \epsilon)$ and $V = B(b, \epsilon)$

$\langle 1 \rangle 5.$ U and V are disjoint neighbourhoods of a and b respectively.

□

Proposition 9.360 (CC). *The product of a countable family of metrizable spaces is metrizable.*

PROOF:

$\langle 1 \rangle 1.$ LET: (X_n, d_n) be a sequence of metric spaces.

$\langle 1 \rangle 2.$ ASSUME: w.l.o.g. each d_n is bounded above by 1.

PROOF: By Proposition 9.354.

$\langle 1 \rangle 3.$ LET: D be the metric on \mathbb{R}^ω defined by $D(x, y) = \sup_i (d_i(x_i, y_i)/i)$.

- ⟨2⟩1. $D(x, y) \geq 0$
- ⟨2⟩2. $D(x, y) = 0$ if and only if $x = y$
- ⟨2⟩3. $D(x, y) = D(y, x)$
- ⟨2⟩4. $D(x, z) \leq D(x, y) + D(y, z)$

PROOF:

$$\begin{aligned}
 D(x, z) &= \sup_i \frac{d_i(x_i, z_i)}{i} \\
 &\leq \sup_i \frac{d_i(x_i, y_i) + d_i(y_i, z_i)}{i} \\
 &\leq \sup_i \frac{d_i(x_i, y_i)}{i} + \sup_i \frac{d_i(y_i, z_i)}{i} \\
 &= D(x, y) + D(y, z)
 \end{aligned}$$

- ⟨1⟩4. Every open ball $B_D(a, \epsilon)$ is open in the product topology.
- ⟨2⟩1. PICK N such that $1/\epsilon < N$
- ⟨2⟩2. $B_D(a, \epsilon) = \prod_{i=1}^{\infty} U_i$ where $U_i = B(a_i, i\epsilon)$ if $i \leq N$, and $U_i = X_i$ if $i > N$
- ⟨1⟩5. For any open set U and $a \in U$, there exists $\epsilon > 0$ such that $B_D(a, \epsilon) \subseteq U$.
- ⟨2⟩1. LET: $n \geq 1$, V be an open set in \mathbb{R} and $a \in \pi_n^{-1}(V)$
- ⟨2⟩2. PICK $\epsilon > 0$ such that $B_{d_n}(a, \epsilon) \subseteq V$
- ⟨2⟩3. $B_D(a, \epsilon/n) \subseteq \pi_n^{-1}(V)$

□

Theorem 9.361. *Let X and Y be metric spaces and $f : X \rightarrow Y$. Then f is continuous if and only if, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$.*

PROOF:

- ⟨1⟩1. If f is continuous then, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$
- ⟨2⟩1. ASSUME: f is continuous.
- ⟨2⟩2. LET: $x \in X$ and $\epsilon > 0$
- ⟨2⟩3. PICK a neighbourhood U of x such that $f(U) \subseteq B(f(x), \epsilon)$
- PROOF: Theorem 9.84.
- ⟨2⟩4. PICK $\delta > 0$ such that $B(x, \delta) \subseteq U$
- PROOF: Proposition 9.344.
- ⟨2⟩5. For all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$
- ⟨1⟩2. If for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$, then f is continuous.
- ⟨2⟩1. ASSUME: for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$
- ⟨2⟩2. LET: $x \in X$ and V be a neighbourhood of $f(x)$
- ⟨2⟩3. PICK $\epsilon > 0$ such that $B(f(x), \epsilon) \subseteq V$
- PROOF: Proposition 9.344.
- ⟨2⟩4. PICK $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$
- PROOF: By ⟨2⟩1
- ⟨2⟩5. LET: $U = B(x, \delta)$
- ⟨2⟩6. U is a neighbourhood of x with $f(U) \subseteq V$

⟨2⟩7. Q.E.D.

PROOF: Theorem 9.84.

□

Proposition 9.362. *Let X be a metric space. Let (a_n) be a sequence in X and $l \in X$. Then $a_n \rightarrow l$ as $n \rightarrow \infty$ if and only if, for all $\epsilon > 0$, there exists N such that, for all $n \geq N$, we have $d(a_n, l) < \epsilon$.*

PROOF: From Proposition 9.67. □

Proposition 9.363. *Every metrizable space is first countable.*

PROOF: In any metric space X , the open balls $B(a, 1/n)$ for $n \geq 1$ form a local basis at a .

Example 9.364. \mathbb{R}^ω under the box topology is not metrizable.

Example 9.365. If J is uncountable then \mathbb{R}^J under the product topology is not metrizable.

Proposition 9.366. *A compact subspace of a metric space is bounded.*

PROOF:

⟨1⟩1. LET: X be a metric space and $A \subseteq X$ be compact.

⟨1⟩2. PICK $a \in A$

⟨1⟩3. $\{B(a, n) \mid n \in \mathbb{Z}^+\}$ covers A

⟨1⟩4. PICK a finite subcover $\{B(a, n_1), \dots, B(a, n_k)\}$

⟨1⟩5. LET: $N = \max(n_1, \dots, n_k)$

⟨1⟩6. For all $x, y \in A$ we have $d(x, y) < 2N$

PROOF:

$$\begin{aligned} d(x, y) &\leq d(x, a) + d(a, y) \\ &< N + N \end{aligned}$$

□

This example shows the converse does not hold:

Example 9.367. The space \mathbb{R} under the standard bounded metric is bounded but not compact.

9.49 Real Linear Algebra

Definition 9.368 (Square Metric). The *square metric* ρ on \mathbb{R}^n is defined by

$$\rho(\vec{x}, \vec{y}) = \max(|x_1 - y_1|, \dots, |x_n - y_n|)$$

We prove this is a metric.

PROOF:

⟨1⟩1. $\rho(\vec{x}, \vec{y}) \geq 0$

PROOF: Immediate from definition.

- $\langle 1 \rangle 2.$ $\rho(\vec{x}, \vec{y}) = 0$ if and only if $\vec{x} = \vec{y}$
 PROOF: Immediate from definition.
 $\langle 1 \rangle 3.$ $\rho(\vec{x}, \vec{y}) = \rho(\vec{y}, \vec{x})$
 PROOF: Immediate from definition.
 $\langle 1 \rangle 4.$ $\rho(\vec{x}, \vec{z}) \leq \rho(\vec{x}, \vec{y}) + \rho(\vec{y}, \vec{z})$
 PROOF: Since $|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$.
 \square

Proposition 9.369. *The square metric induces the standard topology on \mathbb{R}^n .*

PROOF:

- $\langle 1 \rangle 1.$ For every $a \in X$ and $\epsilon > 0$, we have $B_\rho(a, \epsilon)$ is open in the standard product topology.

PROOF:

$$B_\rho(a, \epsilon) = (a_1 - \epsilon, a_1 + \epsilon) \times \cdots \times (a_n - \epsilon, a_n + \epsilon)$$

- $\langle 1 \rangle 2.$ For any open sets U_1, \dots, U_n in \mathbb{R} , we have $U_1 \times \cdots \times U_n$ is open in the square metric topology.
 $\langle 2 \rangle 1.$ LET: $\vec{a} \in U_1 \times \cdots \times U_n$
 $\langle 2 \rangle 2.$ For $i = 1, \dots, n$, PICK $\epsilon_i > 0$ such that $(a_i - \epsilon_i, a_i + \epsilon_i) \subseteq U_i$
 $\langle 2 \rangle 3.$ LET: $\epsilon = \min(\epsilon_1, \dots, \epsilon_n)$
 $\langle 2 \rangle 4.$ $B_\rho(\vec{a}, \epsilon) \subseteq U$
 \square

Definition 9.370. Given $\vec{x}, \vec{y} \in \mathbb{R}^n$, define the *sum* $\vec{x} + \vec{y}$ by

$$(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n) .$$

Definition 9.371. Given $\lambda \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^n$, define the *scalar product* $\lambda \vec{x} \in \mathbb{R}^n$ by

$$\lambda(x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n)$$

Definition 9.372 (Inner Product). Given $\vec{x}, \vec{y} \in \mathbb{R}^n$, define the *inner product* $\vec{x} \cdot \vec{y} \in \mathbb{R}$ by

$$(x_1, \dots, x_n) \cdot (y_1, \dots, y_n) = x_1 y_1 + \cdots + x_n y_n .$$

We write \vec{x}^2 for $\vec{x} \cdot \vec{x}$.

Definition 9.373 (Norm). Let $n \geq 1$. The *norm* on \mathbb{R}^n is the function $\| \cdot \| : \mathbb{R}^n \rightarrow \mathbb{R}$ defined by

$$\|(x_1, \dots, x_n)\| = \sqrt{x_1^2 + \cdots + x_n^2}$$

Lemma 9.374.

$$\|\vec{x}\|^2 = \vec{x}^2$$

PROOF: Immediate from definitions. \square

Lemma 9.375.

$$\vec{x} \cdot (\vec{y} + \vec{z}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z}$$

PROOF: Each is equal to $(x_1y_1 + x_1z_1, \dots, x_ny_n + x_nz_n)$. \square

Lemma 9.376.

$$|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \|\vec{y}\|$$

PROOF:

$\langle 1 \rangle 1$. ASSUME: $\vec{x} \neq \vec{0} \neq \vec{y}$

PROOF: Otherwise both sides are 0.

$\langle 1 \rangle 2$. LET: $a = 1/\|\vec{x}\|$

$\langle 1 \rangle 3$. LET: $b = 1/\|\vec{y}\|$

$\langle 1 \rangle 4$. $(a\vec{x} + b\vec{y})^2 \geq 0$ and $(a\vec{x} - b\vec{y})^2 \geq 0$

$\langle 1 \rangle 5$. $a^2\|\vec{x}\|^2 + 2ab\vec{x} \cdot \vec{y} + b^2\|\vec{y}\|^2 \geq 0$ and $a^2\|\vec{x}\|^2 - 2ab\vec{x} \cdot \vec{y} + b^2\|\vec{y}\|^2 \geq 0$

$\langle 1 \rangle 6$. $2ab\vec{x} \cdot \vec{y} + 2 \geq 0$ and $-2ab\vec{x} \cdot \vec{y} + 2 \geq 0$

$\langle 1 \rangle 7$. $\vec{x} \cdot \vec{y} \geq -1/ab$ and $\vec{x} \cdot \vec{y} \leq 1/ab$

$\langle 1 \rangle 8$. $\vec{x} \cdot \vec{y} \geq -\|\vec{x}\| \|\vec{y}\|$ and $\vec{x} \cdot \vec{y} \leq \|\vec{x}\| \|\vec{y}\|$

\square

Lemma 9.377 (Triangle Inequality).

$$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$$

PROOF:

$$\begin{aligned} \|\vec{x} + \vec{y}\|^2 &= \|\vec{x}\|^2 + 2\vec{x} \cdot \vec{y} + \|\vec{y}\|^2 \\ &\leq \|\vec{x}\|^2 + 2\|\vec{x}\| \|\vec{y}\| + \|\vec{y}\|^2 && \text{(Lemma 9.376)} \\ &= (\|\vec{x}\| + \|\vec{y}\|)^2 && \square \end{aligned}$$

Definition 9.378 (Euclidean Metric). Let $n \geq 1$. The *Euclidean metric* on \mathbb{R}^n is defined by

$$d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|.$$

We prove this is a metric.

$\langle 1 \rangle 1$. $d(\vec{x}, \vec{y}) \geq 0$

PROOF: Immediate from definition.

$\langle 1 \rangle 2$. $d(\vec{x}, \vec{y}) = 0$ if and only if $\vec{x} = \vec{y}$

PROOF: $d(\vec{x}, \vec{y}) = 0$ if and only if $\vec{x} - \vec{y} = \vec{0}$.

$\langle 1 \rangle 3$. $d(\vec{x}, \vec{y}) = d(\vec{y}, \vec{x})$

PROOF: Immediate from definition.

$\langle 1 \rangle 4$. $d(\vec{x}, \vec{z}) \leq d(\vec{x}, \vec{y}) + d(\vec{y}, \vec{z})$

PROOF:

$$\begin{aligned} \|\vec{x} - \vec{z}\| &= \|(\vec{x} - \vec{y}) + (\vec{y} - \vec{z})\| \\ &\leq \|\vec{x} - \vec{y}\| + \|\vec{y} - \vec{z}\| && \text{(Lemma 9.377)} \end{aligned}$$

\square

Proposition 9.379. The Euclidean metric induces the standard topology on \mathbb{R}^n .

PROOF:

- (1)1. LET: ρ be the square metric.
 (1)2. For all $\vec{a} \in \mathbb{R}^n$ and $\epsilon > 0$, we have $B_d(\vec{a}, \epsilon) \subseteq B_\rho(\vec{a}, \epsilon)$
 (2)1. LET: $\vec{x} \in B_d(\vec{a}, \epsilon)$
 (2)2. $\sqrt{(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2} < \epsilon$
 (2)3. $(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2 < \epsilon^2$
 (2)4. For all i we have $(x_i - a_i)^2 < \epsilon^2$
 (2)5. For all i we have $|x_i - a_i| < \epsilon$
 (2)6. $\rho(\vec{x}, \vec{a}) < \epsilon$
 (1)3. For all $\vec{a} \in \mathbb{R}^n$ and $\epsilon > 0$, we have $B_\rho(\vec{a}, \epsilon/\sqrt{n}) \subseteq B_d(\vec{a}, \epsilon)$
 (2)1. LET: $\vec{x} \in B_\rho(\vec{a}, \epsilon/\sqrt{n})$
 (2)2. $\rho(\vec{x}, \vec{a}) < \epsilon/\sqrt{n}$
 (2)3. For all i we have $|x_i - a_i| < \epsilon/\sqrt{n}$
 (2)4. For all i we have $(x_i - a_i)^2 < \epsilon^2/n$
 (2)5. $d(\vec{x}, \vec{a}) < \epsilon$
 (1)4. Q.E.D.

PROOF: By Lemma 9.355.

□

Proposition 9.380. *Let $n \geq 0$. For all $c \in \mathbb{R}^n$ and $\epsilon > 0$, the open ball $B(c, \epsilon)$ is path connected.*

PROOF:

- (1)1. LET: $a, b \in B(c, \epsilon)$
 (1)2. LET: $p : [0, 1] \rightarrow B(c, \epsilon)$ be the function $p(t) = (1 - t)a + tb$
 PROOF: We have $p(t) \in B(c, \epsilon)$ for all t because

$$\begin{aligned}
 d(p(t), c) &= \|(1 - t)a + tb - c\| \\
 &= \|(1 - t)(a - c) + t(b - c)\| \\
 &\leq (1 - t)\|a - c\| + t\|b - c\| \\
 &< (1 - t)\epsilon + t\epsilon \\
 &= \epsilon
 \end{aligned}$$

- (1)3. p is a path from a to b .

□

Proposition 9.381. *Let $n \geq 0$. For all $c \in \mathbb{R}^n$ and $\epsilon > 0$, the closed ball $\overline{B(c, \epsilon)}$ is path connected.*

PROOF:

- (1)1. LET: $a, b \in \overline{B(c, \epsilon)}$
 (1)2. LET: $p : [0, 1] \rightarrow \overline{B(c, \epsilon)}$ be the function $p(t) = (1 - t)a + tb$
 PROOF: We have $p(t) \in \overline{B(c, \epsilon)}$ for all t because

$$\begin{aligned}
 d(p(t), c) &= \|(1 - t)a + tb - c\| \\
 &= \|(1 - t)(a - c) + t(b - c)\| \\
 &\leq (1 - t)\|a - c\| + t\|b - c\| \\
 &\leq (1 - t)\epsilon + t\epsilon \\
 &= \epsilon
 \end{aligned}$$

⟨1⟩3. p is a path from a to b .

□

Lemma 9.382. *If $\sum_{i=0}^{\infty} x_i^2$ and $\sum_{i=0}^{\infty} y_i^2$ converge then $\sum_{i=0}^{\infty} |x_i y_i|$ converges.*

PROOF:

⟨1⟩1. For all $N \geq 0$ we have $\sum_{i=0}^N |x_i y_i| \leq \sqrt{\sum_{i=0}^N |x_i|^2} \sqrt{\sum_{i=0}^N |y_i|^2}$

PROOF: By the Cauchy-Schwarz inequality

⟨1⟩2. Q.E.D.

PROOF: Since $\sum_{i=0}^N |x_i y_i|$ is an increasing sequence bounded above by $(\sum_{i=0}^{\infty} x_i^2)(\sum_{i=0}^{\infty} y_i^2)$.

□

Corollary 9.382.1. *If $\sum_{i=0}^{\infty} x_i^2$ and $\sum_{i=0}^{\infty} y_i^2$ converge then $\sum_{i=0}^{\infty} (x_i + y_i)^2$ converges.*

PROOF: Since $\sum_{i=0}^{\infty} x_i^2$, $\sum_{i=0}^{\infty} y_i^2$ and $2 \sum_{i=0}^{\infty} x_i y_i$ all converge.

Definition 9.383 (l^2 -metric). The l^2 -metric on

$$\left\{ (x_n) \in \mathbb{R}^{\omega} \mid \sum_{n=0}^{\infty} x_n^2 \text{ converges} \right\}$$

is defined by

$$d(x, y) = \left(\sum_{n=0}^{\infty} (x_n - y_n)^2 \right)^{1/2}$$

We prove this is a metric.

PROOF:

⟨1⟩1. d is well-defined.

PROOF: By Corollary 9.382.1.

⟨1⟩2. $d(x, y) \geq 0$

⟨1⟩3. $d(x, y) = 0$ if and only if $x = y$

⟨1⟩4. $d(x, y) = d(y, x)$

⟨1⟩5. $d(x, z) \leq d(x, y) + d(y, z)$

PROOF: By Lemma 9.377.

□

Theorem 9.384. *Addition is a continuous function $\mathbb{R}^2 \rightarrow \mathbb{R}$.*

PROOF:

⟨1⟩1. LET: $a, b \in \mathbb{R}$

⟨1⟩2. LET: $\epsilon > 0$

⟨1⟩3. LET: $\delta = \epsilon/2$

⟨1⟩4. LET: $x, y \in \mathbb{R}$

⟨1⟩5. ASSUME: $\rho((a, b), (x, y)) < \delta$

⟨1⟩6. $|(a + b) - (x + y)| < \epsilon$

PROOF:

$$\begin{aligned}
|(a+b) - (x+y)| &= |a-x| + |b-y| \\
&\leq 2\rho((a,b), (x,y)) \\
&< 2\delta \\
&= \epsilon
\end{aligned}$$

⟨1⟩7. Q.E.D.

PROOF: Theorem 9.361

□

Theorem 9.385. *Multiplication is a continuous function $\mathbb{R}^2 \rightarrow \mathbb{R}$.*

PROOF:

⟨1⟩1. LET: $a, b \in \mathbb{R}$

⟨1⟩2. LET: $\epsilon > 0$

⟨1⟩3. LET: $\delta = \min(\epsilon/(|a| + |b| + 1), 1)$

⟨1⟩4. LET: $x, y \in \mathbb{R}$

⟨1⟩5. ASSUME: $\rho((a,b), (x,y)) < \delta$

⟨1⟩6. $|ab - xy| < \epsilon$

PROOF:

$$\begin{aligned}
|ab - xy| &= |a(b-y) + (a-x)b - (a-x)(b-y)| \\
&\leq |a||b-y| + |b||a-x| + |a-x||b-y| \\
&< |a|\delta + |b|\delta + \delta^2 && (\langle 1 \rangle 5) \\
&\leq |a|\delta + |b|\delta + \delta && (\langle 1 \rangle 3) \\
&\leq \epsilon && (\langle 1 \rangle 3)
\end{aligned}$$

⟨1⟩7. Q.E.D.

PROOF: Theorem 9.361

□

Theorem 9.386. *The function $f : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ defined by $f(x) = x^{-1}$ is continuous.*

PROOF:

⟨1⟩1. For all $a \in \mathbb{R}$ we have $f^{-1}((a, +\infty))$ is open.

PROOF: The set is

$$(a^{-1}, +\infty) \text{ if } a > 0$$

$$(0, +\infty) \text{ if } a = 0$$

$$(-\infty, a^{-1}) \cup (0, +\infty) \text{ if } a < 0$$

⟨1⟩2. For all $a \in \mathbb{R}$ we have $f^{-1}((-\infty, a))$ is open.

PROOF: Similar.

⟨1⟩3. Q.E.D.

PROOF: By Proposition 9.81 and Lemma 9.104.

□

Definition 9.387. For $n \geq 0$, the *unit ball* B^n is the space $\{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$.

Proposition 9.388. *For all $n \geq 0$, the unit ball B^n is path connected.*

PROOF:

⟨1⟩1. LET: $a, b \in B^n$

⟨1⟩2. LET: $p : [0, 1] \rightarrow B^n$ be the function $p(t) = (1 - t)a + tb$

PROOF: We have $p(t) \in B^n$ for all t because

$$\begin{aligned}\|(1 - t)a + tb\| &\leq (1 - t)\|a\| + t\|b\| \\ &\leq (1 - t) + t \\ &= 1\end{aligned}$$

⟨1⟩3. p is a path from a to b .

□

Definition 9.389 (Punctured Euclidean Space). For $n \geq 0$, defined *punctured Euclidean space* to be $\mathbb{R}^n \setminus \{0\}$.

Proposition 9.390. For $n > 1$, *punctured Euclidean space* $\mathbb{R}^n \setminus \{0\}$ is path connected.

PROOF:

⟨1⟩1. LET: $a, b \in \mathbb{R}^n \setminus \{0\}$

⟨1⟩2. CASE: 0 is on the line from a to b

⟨2⟩1. PICK a point c not on the line from a to b

⟨2⟩2. The path consisting of a straight line from a to c followed by a straight line from c to b is a path from a to b .

⟨1⟩3. CASE: 0 is not on the line from a to b

PROOF: The straight line from a to b is a path from a to b .

Corollary 9.390.1. For $n > 1$, the spaces \mathbb{R} and \mathbb{R}^n are not homeomorphic.

PROOF: For any point a , the space $\mathbb{R} \setminus \{a\}$ is disconnected.

Definition 9.391 (Unit Sphere). For $n \geq 1$, the *unit sphere* S^{n-1} is the space

$$S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\} .$$

Proposition 9.392. For $n > 1$, the unit sphere S^{n-1} is path connected.

PROOF: The map $g : \mathbb{R}^n \setminus \{0\} \rightarrow S^{n-1}$ defined by $g(x) = x/\|x\|$ is continuous and surjective. The result follows by Proposition 9.236. □

Proposition 9.393. Let $f : S^1 \rightarrow \mathbb{R}$ be continuous. Then there exists $x \in S^1$ such that $f(x) = f(-x)$.

PROOF:

⟨1⟩1. LET: $g : S^1 \rightarrow \mathbb{R}$ be the function $g(x) = f(x) - f(-x)$

PROVE: There exists $x \in S^1$ such that $g(x) = 0$

⟨1⟩2. ASSUME: without loss of generality $g((1, 0)) > 0$

⟨1⟩3. $g((-1, 0)) < 0$

⟨1⟩4. There exists x such that $g(x) = 0$

PROOF: By the Intermediate Value Theorem.

□

Definition 9.394 (Topologist's Sine Curve). Let $S = \{(x, \sin 1/x) \mid 0 < x \leq 1\}$. The *topologist's sine curve* is the closure \bar{S} of S .

Proposition 9.395.

$$\bar{S} = S \cup (\{0\} \times [-1, 1])$$

Proposition 9.396. *The topologist's sine curve is connected.*

PROOF:

$\langle 1 \rangle 1$. LET: $S = \{(x, \sin 1/x) \mid 0 < x \leq 1\}$

$\langle 1 \rangle 2$. S is connected.

PROOF: Theorem 9.215.

$\langle 1 \rangle 3$. \bar{S} is connected.

PROOF: Theorem 9.214.

□

Proposition 9.397 (CC). *The topologist's sine curve is not path connected.*

PROOF:

$\langle 1 \rangle 1$. ASSUME: for a contradiction $p : [0, 1] \rightarrow \bar{S}$ is a path from $(0, 0)$ to $(1, \sin 1)$.

$\langle 1 \rangle 2$. $p^{-1}(\{0\} \times [0, 1])$ is closed.

$\langle 1 \rangle 3$. LET: b be the greatest element of $p^{-1}(\{0\} \times [0, 1])$.

$\langle 1 \rangle 4$. $b < 1$

PROOF: Since $p(1) = (1, \sin 1)$.

$\langle 1 \rangle 5$. PICK a sequence $(t_n)_{n \geq 1}$ in $(b, 1]$ such that $t_n \rightarrow b$ and $\pi_2(p(t_n)) = (-1)^n$

$\langle 2 \rangle 1$. LET: $n \geq 1$

$\langle 2 \rangle 2$. PICK u with $0 < u < \pi_1(p(1/n))$ such that $\sin(1/u) = (-1)^n$

$\langle 2 \rangle 3$. PICK t_n such that $0 < t_n < 1/n$ and $\pi_1(p(t_n)) = u$

PROOF: One exists by the Intermediate Value Theorem.

$\langle 1 \rangle 6$. Q.E.D.

PROOF: This contradicts 9.96.

□

Theorem 9.398. *Let A be a subspace of \mathbb{R}^n . Then the following are equivalent:*

1. A is compact.
2. A is closed and bounded under the Euclidean metric.
3. A is closed and bounded under the square metric.

PROOF:

$\langle 1 \rangle 1$. $1 \Rightarrow 2$

PROOF: By Corollary 9.305.1 and Proposition 9.366.

$\langle 1 \rangle 2$. $2 \Rightarrow 3$

PROOF: If $d(x, y) \leq M$ for all $x, y \in A$ then $\rho(x, y) \leq M/\sqrt{2}$.

$\langle 1 \rangle 3$. $3 \Rightarrow 1$

$\langle 2 \rangle 1$. ASSUME: A is closed and $\rho(x, y) \leq M$ for all $x, y \in A$

$\langle 2 \rangle 2$. PICK $a \in A$

PROOF: We may assume w.l.o.g. A is nonempty since the empty space is compact.

⟨2⟩3. A is a closed subspace of $[a_1 - M, a_1 + M] \times \cdots \times [a_n - M, a_n + M]$

⟨2⟩4. A is compact

PROOF: Proposition 9.298.

□

Corollary 9.398.1. *The unit sphere S^{n-1} and the closed unit ball B^n are compact for any n .*

9.50 The Uniform Topology

Definition 9.399 (Uniform Metric). Let J be a set. The *uniform metric* $\bar{\rho}$ on \mathbb{R}^J is defined by

$$\bar{\rho}(a, b) = \sup_{j \in J} \bar{d}(a_j, b_j)$$

where \bar{d} is the standard bounded metric on \mathbb{R} .

The *uniform topology* on \mathbb{R}^J is the topology induced by the uniform metric.

We prove this is a metric.

PROOF:

⟨1⟩1. $\bar{\rho}(a, b) \geq 0$

PROOF: Immediate from definitions.

⟨1⟩2. $\bar{\rho}(a, b) = 0$ if and only if $a = b$

PROOF: Immediate from definitions.

⟨1⟩3. $\bar{\rho}(a, b) = \bar{\rho}(b, a)$

PROOF: Immediate from definitions.

⟨1⟩4. $\bar{\rho}(a, c) \leq \bar{\rho}(a, b) + \bar{\rho}(b, c)$

PROOF:

$$\begin{aligned} \bar{\rho}(a, c) &= \sup_{j \in J} \bar{d}(a_j, c_j) \\ &\leq \sup_{j \in J} (\bar{d}(a_j, b_j) + \bar{d}(b_j, c_j)) \\ &\leq \sup_{j \in J} \bar{d}(a_j, b_j) + \sup_{j \in J} \bar{d}(b_j, c_j) \\ &= \bar{\rho}(a, b) + \bar{\rho}(b, c) \end{aligned}$$

□

Proposition 9.400. *The uniform topology on \mathbb{R}^J is finer than the product topology.*

PROOF:

⟨1⟩1. LET: $j \in J$ and U be open in \mathbb{R}

PROVE: $\pi_j^{-1}(U)$ is open in the uniform topology.

⟨1⟩2. LET: $a \in \pi_j^{-1}(U)$

⟨1⟩3. PICK $\epsilon > 0$ such that $(a_j - \epsilon, a_j + \epsilon) \subseteq U$

⟨1⟩4. $B_{\bar{p}}(a, \epsilon) \subseteq \pi_j^{-1}(U)$
 \square

Proposition 9.401. *The uniform topology on \mathbb{R}^J is coarser than the box topology.*

PROOF:

⟨1⟩1. LET: $a \in \mathbb{R}^J$ and $\epsilon > 0$

PROVE: $B(a, \epsilon)$ is open in the box topology.

⟨1⟩2. LET: $b \in B(a, \epsilon)$

⟨1⟩3. For $j \in J$ we have $|a_j - b_j| < \epsilon$

⟨1⟩4. For $j \in J$,

LET: $\delta_j = (\epsilon - |a_j - b_j|)/2$

⟨1⟩5. $\prod_{j \in J} (b_j - \delta_j, b_j + \delta_j) \subseteq B(a, \epsilon)$

\square

Proposition 9.402. *The uniform topology on \mathbb{R}^J is strictly finer than the product topology if and only if J is infinite.*

PROOF:

⟨1⟩1. If J is finite then the uniform and product topologies coincide.

PROOF: The uniform, box and product topologies are all the same.

⟨1⟩2. If J is infinite then the uniform and product topologies are different.

PROOF: The set $B(\vec{0}, 1)$ is open in the uniform topology but not the product topology.

\square

Proposition 9.403 (DC). *The uniform topology on \mathbb{R}^J is strictly coarser than the box topology if and only if J is infinite.*

PROOF:

⟨1⟩1. If J is finite then the uniform and box topologies coincide.

PROOF: The uniform, box and product topologies are all the same.

⟨1⟩2. If J is infinite then the uniform and box topologies are different.

PROOF: Pick an ω -sequence (j_1, j_2, \dots) in J . Let $U = \prod_{j \in J} U_j$ where $U_{j_i} = (-1/i, 1/i)$ and $U_j = (-1, 1)$ for all other j . Then $\vec{0} \in U$ but there is no $\epsilon > 0$ such that $B(\vec{0}, \epsilon) \subseteq U$.

\square

Proposition 9.404. *The closure of \mathbb{R}^∞ in \mathbb{R}^ω under the uniform topology is \mathbb{R}^ω .*

PROOF: Given any open ball $B(a, \epsilon)$, pick an integer N such that $1/\epsilon < N$. Then $B(a, \epsilon)$ includes sequences whose n th entry is 0 for all $n \geq N$. \square

Example 9.405. The space \mathbb{R}^ω is disconnected under the uniform topology. The set of bounded sequences and the set of unbounded sequences form a separation.

Proposition 9.406. Give \mathbb{R}^ω the uniform topology. Let $x, y \in \mathbb{R}^\omega$. Then x and y are in the same component if and only if $x - y$ is bounded.

PROOF:

$\langle 1 \rangle 1$. The component containing 0 is the set of bounded sequences.

$\langle 2 \rangle 1$. LET: B be the set of bounded sequences.

$\langle 2 \rangle 2$. B is path-connected.

$\langle 3 \rangle 1$. LET: $x, y \in B$

$\langle 3 \rangle 2$. PICK $b > 0$ such that $|x_j|, |y_j| \leq b$ for all j

$\langle 3 \rangle 3$. LET: $p : [0, 1] \rightarrow B$ be the function $p(t) = (1 - t)x + ty$

PROVE: p is continuous.

$\langle 3 \rangle 4$. LET: $t \in [0, 1]$ and $\epsilon > 0$

$\langle 3 \rangle 5$. LET: $\delta = \epsilon/2b$

$\langle 3 \rangle 6$. LET: $s \in [0, 1]$ with $|s - t| < \delta$

$\langle 3 \rangle 7$. $\bar{\rho}(p(s), p(t)) < \epsilon$

PROOF:

$$\begin{aligned} \bar{\rho}(p(s), p(t)) &= \sup_j \bar{d}((1 - s)x_j + sy_j, (1 - t)x_j + ty_j) \\ &\leq |(s - t)x_j + (t - s)y_j| \\ &\leq |s - t||x_j - y_j| \\ &< 2b\delta \\ &= \epsilon \end{aligned}$$

$\langle 2 \rangle 3$. B is connected.

PROOF: Proposition 9.234.

$\langle 2 \rangle 4$. If C is connected and $B \subseteq C$ then $B = C$.

PROOF: Otherwise $B \cap C$ and $C \setminus B$ form a separation of C .

$\langle 1 \rangle 2$. Q.E.D.

PROOF: Since $\lambda x. x - y$ is a Homeomorphism of \mathbb{R}^ω with itself.

□

9.51 Uniform Convergence

Definition 9.407 (Uniform Convergence). Let X be a set and Y a metric space. Let $(f_n : X \rightarrow Y)$ be a sequence of functions and $f : X \rightarrow Y$ be a function. Then f_n converges uniformly to f as $n \rightarrow \infty$ if and only if, for all $\epsilon > 0$, there exists N such that, for all $n \geq N$ and $x \in X$, we have $d(f_n(x), f(x)) < \epsilon$.

Example 9.408. Define $f_n : [0, 1] \rightarrow \mathbb{R}$ by $f_n(x) = x^n$ for $n \geq 1$, and $f : [0, 1] \rightarrow \mathbb{R}$ by $f(x) = 0$ if $x < 1$, $f(1) = 1$. Then f_n converges to f pointwise but not uniformly.

Theorem 9.409 (Uniform Limit Theorem). Let X be a topological space and Y a metric space. Let $(f_n : X \rightarrow Y)$ be a sequence of continuous functions and $f : X \rightarrow Y$ be a function. If f_n converges uniformly to f as $n \rightarrow \infty$, then f is continuous.

PROOF:

⟨1⟩1. LET: $x \in X$ and $\epsilon > 0$

⟨1⟩2. PICK N such that, for all $n \geq N$ and $y \in X$, we have $d(f_n(y), f(y)) < \epsilon/3$

⟨1⟩3. PICK a neighbourhood U of x such that $f_N(U) \subseteq B(f_N(x), \epsilon/3)$

PROVE: $f(U) \subseteq B(f(x), \epsilon)$

⟨1⟩4. LET: $y \in U$

⟨1⟩5. $d(f(y), f(x)) < \epsilon$

PROOF:

$d(f(y), f(x)) \leq d(f(y), f_N(y)) + d(f_N(y), f_N(x)) + d(f_N(x), f(x))$ (Triangle Inequality)

$< \epsilon/3 + \epsilon/3 + \epsilon/3$

(⟨1⟩2, ⟨1⟩3)

$= \epsilon$

□

Proposition 9.410. *Let X be a topological space and Y a metric space. Let $(f_n : X \rightarrow Y)$ be a sequence of continuous functions and $f : X \rightarrow Y$ be a function. Let (a_n) be a sequence of points in X and $a \in X$. If f_n converges uniformly to f and a_n converges to a in X then $f_n(a_n)$ converges to $f(a)$ uniformly in Y .*

PROOF:

⟨1⟩1. LET: $\epsilon > 0$

⟨1⟩2. PICK N_1 such that, for all $n \geq N_1$ and $x \in X$, we have $d(f_n(x), f(x)) < \epsilon/2$

⟨1⟩3. PICK N_2 such that, for all $n \geq N_2$, we have $a_n \in f^{-1}(B(a, \epsilon/2))$

PROOF: Using the fact that f is continuous from the Uniform Limit Theorem.

⟨1⟩4. LET: $N = \max(N_1, N_2)$

⟨1⟩5. LET: $n \geq N$

⟨1⟩6. $d(f_n(a_n), f(a)) < \epsilon$

PROOF:

$d(f_n(a_n), f(a)) \leq d(f_n(a_n), f(a_n)) + d(f(a_n), f(a))$ (Triangle Inequality)

$< \epsilon/2 + \epsilon/2$

(⟨1⟩2, ⟨1⟩3)

$= \epsilon$

□

Proposition 9.411. *Let X be a set. Let $(f_n : X \rightarrow \mathbb{R})$ be a sequence of functions and $f : X \rightarrow \mathbb{R}$ be a function. Then f_n converges uniformly to f as $n \rightarrow \infty$ if and only if $f_n \rightarrow f$ as $n \rightarrow \infty$ in \mathbb{R}^X under the uniform topology.*

PROOF:

⟨1⟩1. If f_n converges uniformly to f then f_n converges to f under the uniform topology.

⟨2⟩1. ASSUME: f_n converges uniformly to f

⟨2⟩2. LET: $\epsilon > 0$

⟨2⟩3. PICK N such that, for all $n \geq N$ and $x \in X$, we have $d(f_n(x), f(x)) < \epsilon/2$

⟨2⟩4. For all $n \geq N$ we have $\bar{\rho}(f_n, f) \leq \epsilon/2$

- ⟨2⟩5. For all $n \geq N$ we have $\bar{\rho}(f_n, f) < \epsilon$
 ⟨1⟩2. If f_n converges to f under the uniform topology then f_n converges uniformly to f .
 ⟨2⟩1. ASSUME: f_n converges to f under the uniform topology.
 ⟨2⟩2. LET: $\epsilon > 0$
 ⟨2⟩3. PICK N such that, for all $n \geq N$, we have $\bar{\rho}(f_n, f) < \min(\epsilon, 1/2)$
 ⟨2⟩4. LET: $n \geq N$
 ⟨2⟩5. LET: $x \in X$
 ⟨2⟩6. $\bar{\rho}(f_n, f) < \min(\epsilon, 1/2)$
 PROOF: From ⟨2⟩3.
 ⟨2⟩7. $d(f_n(x), f(x)) < \min(\epsilon, 1/2)$
 ⟨2⟩8. $d(f_n(x), f(x)) < \epsilon$

□

9.52 Isometric Imbeddings

Definition 9.412. Let X and Y be metric spaces. An *isometric imbedding* $f : X \rightarrow Y$ is a function such that, for all $x, y \in X$, we have $d(f(x), f(y)) = d(x, y)$.

Proposition 9.413. Every isometric imbedding is an imbedding.

PROOF:

- ⟨1⟩1. LET: $f : X \rightarrow Y$ be an isometric imbedding.
 ⟨1⟩2. f is injective.
 PROOF: If $f(x) = f(y)$ then $d(f(x), f(y)) = 0$ hence $d(x, y) = 0$ hence $x = y$.
 ⟨1⟩3. f is continuous.
 PROOF: For all $\epsilon > 0$, if $d(x, y) < \epsilon$ then $d(f(x), f(y)) < \epsilon$.
 ⟨1⟩4. $f : X \rightarrow f(X)$ is an open map.
 PROOF: $f(B(a, \epsilon)) = B(f(a), \epsilon) \cap f(X)$.

□

9.53 Distance to a Set

Definition 9.414. Let X be a metric space, $x \in X$ and $A \subseteq X$ be nonempty. The *distance* from x to A is defined as

$$d(x, A) = \inf_{a \in A} d(x, a) .$$

Proposition 9.415. Let X be a metric space and $A \subseteq X$ be nonempty. Then the function $d(-, A) : X \rightarrow \mathbb{R}$ is continuous.

PROOF:

- ⟨1⟩1. LET: X be a metric space.
 ⟨1⟩2. LET: $A \subseteq X$ be nonempty.
 ⟨1⟩3. LET: $x \in X$ and $\epsilon > 0$
 ⟨1⟩4. LET: $\delta = \epsilon$

⟨1⟩5. LET: $y \in B(x, \delta)$
 ⟨1⟩6. $|d(x, A) - d(y, A)| < \epsilon$
 ⟨2⟩1. $d(x, A) - d(y, A) < \epsilon$
 PROOF:
 ⟨3⟩1. For all $a \in A$ we have $d(x, A) \leq d(x, y) + d(y, a)$
 PROOF:

$$d(x, A) \leq d(x, a) \quad (\text{definition of } d(x, A))$$

$$\leq d(x, y) + d(y, a) \quad (\text{Triangle Inequality})$$
 ⟨3⟩2. $d(x, A) - d(x, y) \leq d(y, A)$
 ⟨2⟩2. $d(y, A) - d(x, A) < \epsilon$
 PROOF: Similar.
 ⟨1⟩7. Q.E.D.
 PROOF: Theorem 9.361.
 □

9.54 Lebesgue Numbers

Definition 9.416 (Lebesgue Number). Let X be a metric space. Let \mathcal{U} be an open covering of X . A *Lebesgue number* for \mathcal{U} is a real number $\delta > 0$ such that, for every subset $A \subseteq X$ with diameter $\text{diam } A < \delta$, there exists $U \in \mathcal{U}$ such that $A \subseteq U$.

Theorem 9.417 (Lebesgue Number Lemma). *Every open covering of a compact metric space has a Lebesgue number.*

PROOF:

⟨1⟩1. LET: X be a compact metric space.
 ⟨1⟩2. LET: \mathcal{U} be an open covering of X .
 ⟨1⟩3. PICK a finite subset $\{U_1, \dots, U_n\}$ of \mathcal{U} that covers X .
 ⟨1⟩4. For $i = 1, \dots, n$,
 LET: $C_i = X - U_i$
 ⟨1⟩5. LET: $f : X \rightarrow \mathbb{R}$,

$$f(x) = 1/n \sum_{i=1}^n d(x, C_i)$$

⟨1⟩6. For all $x \in X$ we have $f(x) > 0$
 ⟨2⟩1. LET: $x \in X$
 ⟨2⟩2. PICK i such that $x \in U_i$
 PROOF: From ⟨1⟩3.
 ⟨2⟩3. PICK $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U_i$
 PROOF: Proposition 9.344.
 ⟨2⟩4. $d(x, C_i) \geq \epsilon$
 ⟨2⟩5. $f(x) \geq \epsilon/n$
 ⟨1⟩7. f is continuous.
 PROOF: Proposition 9.415.
 ⟨1⟩8. LET: δ be the minimum value of $f(X)$

PROOF: By the Extreme Value Theorem

⟨1⟩9. $\delta > 0$

PROOF: From ⟨1⟩6

⟨1⟩10. For every subset $A \subseteq X$ with diameter $< \delta$, there exists $U \in \mathcal{U}$ such that $A \subseteq U$

⟨2⟩1. LET: $A \subseteq X$ with $\text{diam } A < \delta$

⟨2⟩2. PICK $x_0 \in A$

⟨2⟩3. $A \subseteq B(x_0, \delta)$

⟨2⟩4. $f(x_0) \geq \delta$

⟨2⟩5. PICK m such that $d(x_0, C_m)$ is the largest out of $d(x_0, C_1), \dots, d(x_0, C_n)$

⟨2⟩6. $d(x_0, C_m) \geq f(x_0)$

⟨2⟩7. $B(x_0, \delta) \subseteq U_m$

⟨2⟩8. $A \subseteq U_m$

⟨1⟩11. δ is a Lebesgue number for \mathcal{U}

□

9.55 Uniform Continuity

Definition 9.418 (Uniformly Continuous). Let X and Y be metric spaces. Let $f : X \rightarrow Y$. Then f is *uniformly continuous* if and only if, for every $\epsilon > 0$, there exists $\delta > 0$ such that, for all $x, y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$.

Theorem 9.419 (Uniform Continuity Theorem). *Every continuous function from a compact metric space to a metric space is uniformly continuous.*

PROOF:

⟨1⟩1. LET: X be a compact metric space.

⟨1⟩2. LET: Y be a metric space.

⟨1⟩3. LET: $f : X \rightarrow Y$ be a continuous function.

⟨1⟩4. LET: $\epsilon > 0$

⟨1⟩5. LET: $\mathcal{U} = \{f^{-1}(B(y, \epsilon/2)) \mid y \in Y\}$

⟨1⟩6. PICK a Lebesgue number $\delta > 0$ for \mathcal{U} .

PROOF: By the Lebesgue Number Lemma.

⟨1⟩7. LET: $x, x' \in X$

⟨1⟩8. ASSUME: $d(x, x') < \delta$

⟨1⟩9. PICK $y \in Y$ such that $\{x, x'\} \subseteq f^{-1}(B(y, \epsilon/2))$

PROOF: Since $\text{diam}\{x, x'\} < \delta$.

⟨1⟩10. $d(f(x), f(x')) < \epsilon$

PROOF:

$$\begin{aligned} d(f(x), f(x')) &\leq d(f(x), y) + d(y, f(x')) && \text{(Triangle Inequality)} \\ &< \epsilon/2 + \epsilon/2 && (\langle 1 \rangle 9) \\ &= \epsilon \end{aligned}$$

□