

Topology

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Part I

Set Theory

Chapter 1

Classes

1.1 Classes

We speak informally of *classes*. A class is specified by a unary predicate. We write $\{x \mid P(x)\}$ for the class determined by the predicate $P(x)$.

Definition 1.1.1 (Membership). Let a be an object and \mathbf{A} a class. We define the proposition $a \in \mathbf{A}$ (a is a *member* or *element* of \mathbf{A}) as follows:

The proposition $a \in \{x \mid P(x)\}$ is the proposition $P(a)$.

Definition 1.1.2 (Equality of Classes). Let \mathbf{A} and \mathbf{B} be classes. We say \mathbf{A} and \mathbf{B} are *equal*, $\mathbf{A} = \mathbf{B}$, if and only if they have exactly the same elements.

1.2 Subclasses

Definition 1.2.1 (Subclass). Let \mathbf{A} and \mathbf{B} be classes. We say \mathbf{A} is a *subclass* of \mathbf{B} , $\mathbf{A} \subseteq \mathbf{B}$, if and only if every member of \mathbf{A} is a member of \mathbf{B} .

We say \mathbf{A} is a *proper* subclass of \mathbf{B} , $\mathbf{A} \subset \mathbf{B}$, if and only if $\mathbf{A} \subseteq \mathbf{B}$ and $\mathbf{A} \neq \mathbf{B}$.

1.3 The Empty Class

Definition 1.3.1 (Empty Class). The *empty* class \emptyset is $\{x \mid \perp\}$.

1.4 Finite Classes

Definition 1.4.1. For any objects a_1, \dots, a_n , we write $\{a_1, \dots, a_n\}$ for the class $\{x \mid x = a_1 \vee \dots \vee x = a_n\}$.

1.5 Universal Class

Definition 1.5.1 (Universal Class). The *universal class* \mathbf{V} is the class $\{x \mid \top\}$.

1.6 Union

Definition 1.6.1 (Union). For any classes \mathbf{A} and \mathbf{B} , the *union* $\mathbf{A} \cup \mathbf{B}$ is the class $\{x \mid x \in \mathbf{A} \vee x \in \mathbf{B}\}$.

1.7 Intersection

Definition 1.7.1 (Intersection). For any classes \mathbf{A} and \mathbf{B} , the *intersection* $\mathbf{A} \cap \mathbf{B}$ is the class $\{x \mid x \in \mathbf{A} \wedge x \in \mathbf{B}\}$.

1.8 Relative Complement

Definition 1.8.1 (Relative Complement). For any classes \mathbf{A} and \mathbf{B} , the *relative complement* $\mathbf{A} - \mathbf{B}$ is $\{x \in \mathbf{A} \mid x \notin \mathbf{B}\}$.

Chapter 2

Sets

2.1 Membership

We take as undefined the notion of *set*.

We take as undefined the binary relation of *membership*, \in . If $a \in A$ we say a is a *member* or *element* of A . If this does not hold, we write $a \notin A$.

Axiom 2.1.1 (Axiom of Extensionality). *Two sets with exactly the same elements are equal.*

We may therefore identify the set A with the class $\{x \mid x \in A\}$.

We say a class \mathbf{A} is a *set* iff there exists a set A such that $A = \mathbf{A}$. That is, $\{x \mid P(x)\}$ is a set if and only if there exists a set A such that, for all x , we have $x \in A$ if and only if $P(x)$.

2.2 The Empty Set

Axiom 2.2.1 (Empty Set Axiom). *The empty class \emptyset is a set.*

2.3 Pair Sets

Axiom 2.3.1 (Pairing Axiom). *For any objects u and v , the class $\{u, v\}$ is a set.*

Theorem 2.3.2 (Pairing). *For any object a , the class $\{a\}$ is a set.*

PROOF: It is $\{a, a\}$. \square

2.4 Unions

Definition 2.4.1 (Union). For any class of sets \mathbf{A} , the *union* $\bigcup \mathbf{A}$ is the class $\{x \mid \exists A \in \mathbf{A}. x \in A\}$.

Axiom 2.4.2 (Union Axiom). *For any set A , the union $\bigcup A$ is a set.*

Theorem 2.4.3 (Union, Pairing). *For any sets A and B , the class $A \cup B$ is a set.*

PROOF: It is $\bigcup\{A, B\}$. \square

Theorem Schema 2.4.4 (Union, Pairing). *For any objects a_1, \dots, a_n , the class $\{a_1, \dots, a_n\}$ is a set.*

PROOF: We prove each theorem using the last since $\{a_1, \dots, a_n, a_{n+1}\} = \{a_1, \dots, a_n\} \cup \{a_{n+1}\}$. \square

2.5 Power Set

Definition 2.5.1 (Power Class). For any class \mathbf{A} , the *power class* $\mathcal{P}\mathbf{A}$ is the class of all subsets of \mathbf{A} .

Axiom 2.5.2 (Power Set Axiom). *For any set A , the power class $\mathcal{P}A$ is a set.*

2.6 Covers

Definition 2.6.1 (Cover). Let \mathbf{X} be a class and $\mathcal{A} \subseteq \mathcal{P}\mathbf{X}$. Then \mathcal{A} *covers* \mathbf{X} , or is a *covering* of \mathbf{X} , if and only if $\bigcup \mathcal{A} = \mathbf{X}$.

2.7 Subset Axioms

Axiom Schema 2.7.1 (Subset Axioms, Aussonderung Axioms). *For any classes \mathbf{A} and \mathbf{B} , if $\mathbf{A} \subseteq \mathbf{B}$ and \mathbf{B} is a set then \mathbf{A} is a set.*

Theorem 2.7.2 (Subset). *The universal class \mathbf{V} is not a set.*

PROOF:

$\langle 1 \rangle 1$. ASSUME: \mathbf{V} is a set.

$\langle 1 \rangle 2$. LET: $R = \{x \in \mathbf{V} \mid x \notin x\}$

$\langle 1 \rangle 3$. $R \in R$ if and only if $R \notin R$

$\langle 1 \rangle 4$. Q.E.D.

PROOF: This is a contradiction.

\square

Theorem 2.7.3 (Subset). *If A is a set and \mathbf{B} is a class then $A - \mathbf{B}$ is a set.*

PROOF: It is a subset of A . \square

2.8 Intersection

Definition 2.8.1 (Intersection). For any class \mathbf{A} of sets, the *intersection* $\bigcap \mathbf{A}$ is the class $\{x \mid \forall A \in \mathbf{A}. x \in A\}$.

Theorem 2.8.2 (Subset). *For any nonempty class \mathbf{A} of sets, we have $\bigcap \mathbf{A}$ is a set.*

PROOF:

$\langle 1 \rangle 1.$ PICK $A \in \mathbf{A}$

$\langle 1 \rangle 2.$ $\bigcap \mathbf{A} \subseteq A$

$\langle 1 \rangle 3.$ Q.E.D.

PROOF: By a Subset Axiom.

□

Theorem 2.8.3 (Subset). *For any sets A and B , the class $A \cap B$ is a set.*

PROOF: From a Subset Axiom since $A \cap B \subseteq A$. □

Chapter 3

Relations

3.1 Ordered Pairs

Definition 3.1.1 (Ordered Pair (Pairing)). For any sets x and y , the *ordered pair* (x, y) is defined to be $\{\{x\}, \{x, y\}\}$.

Theorem 3.1.2 (Pairing). For any sets u, v, x, y , we have $(u, v) = (x, y)$ if and only if $u = x$ and $v = y$

PROOF:

$\langle 1 \rangle 1.$ ASSUME: $\{\{u\}, \{u, v\}\} = \{\{x\}, \{x, y\}\}$

$\langle 1 \rangle 2.$ $\{u\} \in \{\{x\}, \{x, y\}\}$

$\langle 1 \rangle 3.$ $\{u, v\} \in \{\{x\}, \{x, y\}\}$

$\langle 1 \rangle 4.$ $\{u\} = \{x\}$ or $\{u\} = \{x, y\}$

$\langle 1 \rangle 5.$ $\{u, v\} = \{x\}$ or $\{u, v\} = \{x, y\}$

$\langle 1 \rangle 6.$ CASE: $\{u\} = \{x, y\}$

$\langle 2 \rangle 1.$ $u = x = y$

$\langle 2 \rangle 2.$ $u = v = x = y$

PROOF: From $\langle 1 \rangle 5$

$\langle 1 \rangle 7.$ CASE: $\{u, v\} = \{x\}$

PROOF: Similar.

$\langle 1 \rangle 8.$ CASE: $\{u\} = \{x\}$ and $\{u, v\} = \{x, y\}$

$\langle 2 \rangle 1.$ $u = x$

$\langle 2 \rangle 2.$ $u = y$ or $v = y$

$\langle 2 \rangle 3.$ CASE: $u = y$

PROOF: This case is the case considered in $\langle 1 \rangle 6$.

$\langle 2 \rangle 4.$ CASE: $v = y$

PROOF: We have $u = x$ and $v = y$ as required.

□

Lemma 3.1.3 (Pairing, Power Set). Let x, y and C be sets. If $x \in C$ and $y \in C$ then $(x, y) \in \mathcal{P}C$.

PROOF:

- $\langle 1 \rangle 1.$ LET: x, y and C be sets.
- $\langle 1 \rangle 2.$ ASSUME: $x \in C$
- $\langle 1 \rangle 3.$ ASSUME: $y \in C$
- $\langle 1 \rangle 4.$ $\{x\} \subseteq C$
- $\langle 1 \rangle 5.$ $\{x, y\} \subseteq C$
- $\langle 1 \rangle 6.$ $\{x\} \in \mathcal{P}C$
- $\langle 1 \rangle 7.$ $\{x, y\} \in \mathcal{P}C$
- $\langle 1 \rangle 8.$ $\{\{x\}, \{x, y\}\} \subseteq \mathcal{P}C$
- $\langle 1 \rangle 9.$ $\{\{x\}, \{x, y\}\} \in \mathcal{P}\mathcal{P}C$

□

Lemma 3.1.4 (Pairing, Union). *Let x, y and A be sets. If $(x, y) \in A$ then x and y belong to $\bigcup \bigcup A$.*

PROOF:

- $\langle 1 \rangle 1.$ LET: x, y and A be sets.
- $\langle 1 \rangle 2.$ ASSUME: $(x, y) \in A$
- $\langle 1 \rangle 3.$ $\{x, y\} \in \bigcup A$
- $\langle 1 \rangle 4.$ $x \in \bigcup \bigcup A$
- $\langle 1 \rangle 5.$ $y \in \bigcup \bigcup A$

□

3.2 Cartesian Product

Definition 3.2.1 (Cartesian Product (Pairing)). Let \mathbf{A} and \mathbf{B} be classes. The *Cartesian product* $\mathbf{A} \times \mathbf{B}$ is the class $\{(x, y) \mid x \in \mathbf{A}, y \in \mathbf{B}\}$.

Theorem 3.2.2 (Pairing, Union, Power Set, Subset). *For any sets A and B , the Cartesian product $A \times B$ is a set.*

PROOF: It is a subset of $\mathcal{P}\mathcal{P}(A \cup B)$ by Lemma 3.1.3. □

3.3 Relations

Definition 3.3.1 (Relation (Pairing)). A *relation* is a class of ordered pairs.

Given a relation \mathbf{R} , we write $x\mathbf{R}y$ for $(x, y) \in \mathbf{R}$.

A relation is *small* iff it is a set.

3.4 Domain

Definition 3.4.1 (Domain (Pairing)). Let \mathbf{R} be a class. The *domain* of \mathbf{R} is $\text{dom } \mathbf{R} = \{x \mid \exists y. x\mathbf{R}y\}$.

Theorem 3.4.2 (Pairing, Union, Subset). *For any set R , the domain $\text{dom } R$ is a set.*

PROOF: It is a subset of $\bigcup \bigcup R$ by Lemma 3.1.4. □

3.5 Range

Definition 3.5.1 (Domain (Pairing)). Let \mathbf{R} be a class. The *range* of \mathbf{R} is $\text{ran } \mathbf{R} = \{y \mid \exists x. x\mathbf{R}y\}$.

Theorem 3.5.2 (Pairing, Union, Subset). *For any set R , the range $\text{ran } R$ is a set.*

PROOF: It is a subset of $\bigcup \bigcup R$ by Lemma 3.1.4. \square

3.6 Field

Definition 3.6.1 (Field). Let \mathbf{R} be a class. The *field* of \mathbf{R} is $\text{fld } \mathbf{R} = \text{dom } \mathbf{R} \cup \text{ran } \mathbf{R}$.

Theorem 3.6.2 (Pairing, Union, Subset). *For any set R , the field $\text{fld } R$ is a set.*

PROOF: Theorems 2.4.3, 3.4.2 and 3.5.2. \square

3.7 Functions

Definition 3.7.1 (Class Term (Pairing)). A *class term* is a relation \mathbf{F} such that, for all x, y, y' , if $x\mathbf{F}y$ and $x\mathbf{F}y'$ then $y = y'$.

If \mathbf{F} is a class term and $x \in \text{dom } \mathbf{F}$, then we write $\mathbf{F}(x)$ for the unique y such that $x\mathbf{F}y$.

We write $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$ iff \mathbf{F} is a class term, $\text{dom } \mathbf{F} = \mathbf{A}$ and $\text{ran } \mathbf{F} \subseteq \mathbf{B}$.

A *function* is a class term that is a set.

Axiom 3.7.2 (Axiom of Choice, First Form (Pairing)). *For any relation R , there exists a function $H \subseteq R$ such that $\text{dom } H = \text{dom } R$.*

Theorem 3.7.3. *The following are equivalent.*

1. *The Axiom of Choice*
2. (**Multiplicative Axiom**) *For any function H with domain I such that $H(i)$ is nonempty for all $i \in I$, there exists a function f with domain I such that, for all $i \in I$, we have $f(i) \in H(i)$.*
3. *Every set has a choice function.*
4. *Let \mathcal{A} be a set of pairwise disjoint nonempty sets. Then there exists a set C containing exactly one element from each member of \mathcal{A} .*

3.8 Single-Rooted

Definition 3.8.1 (Single-Rooted (Pairing)). A class \mathbf{R} is *single-rooted* if and only if, for all x, x', y , if $x\mathbf{R}y$ and $x'\mathbf{R}y$ then $x = x'$.

We call a class term *one-to-one*, *injective* or an *injection* if and only if it is single-rooted.

3.9 Surjective

Definition 3.9.1 (Surjective (Pairing)). Let $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$. Then \mathbf{F} is *surjective* if and only if $\text{ran } \mathbf{F} = \mathbf{B}$.

3.10 Inverse

Definition 3.10.1 (Inverse (Pairing)). Let \mathbf{R} be a class. The *inverse* of \mathbf{R} is $\mathbf{R}^{-1} = \{(y, x) \mid x\mathbf{R}y\}$.

Theorem 3.10.2 (Pairing, Union, Power Set, Subset). *For any set R , the inverse R^{-1} is a set.*

PROOF: It is a subset of $\text{ran } R \times \text{dom } R$. \square

Theorem 3.10.3 (Pairing). *For any class \mathbf{F} , we have $\text{dom } \mathbf{F}^{-1} = \text{ran } \mathbf{F}$.*

PROOF: For any x , we have

$$\begin{aligned} x \in \text{dom } \mathbf{F}^{-1} &\Leftrightarrow \exists y.(x, y) \in \mathbf{F}^{-1} \\ &\Leftrightarrow \exists y.(y, x) \in \mathbf{F} \\ &\Leftrightarrow x \in \text{ran } \mathbf{F} \end{aligned} \quad \square$$

Theorem 3.10.4 (Pairing). *For any set \mathbf{F} , we have $\text{ran } \mathbf{F}^{-1} = \text{dom } \mathbf{F}$.*

PROOF: For any x , we have

$$\begin{aligned} x \in \text{ran } \mathbf{F}^{-1} &\Leftrightarrow \exists y.(y, x) \in \mathbf{F}^{-1} \\ &\Leftrightarrow \exists y.(x, y) \in \mathbf{F} \\ &\Leftrightarrow x \in \text{dom } \mathbf{F} \end{aligned} \quad \square$$

Theorem 3.10.5 (Pairing). *For any relation \mathbf{F} , we have $(\mathbf{F}^{-1})^{-1} = \mathbf{F}$.*

PROOF: For any z we have

$$\begin{aligned} z \in (\mathbf{F}^{-1})^{-1} &\Leftrightarrow \exists x, y. z = (x, y) \wedge (y, x) \in \mathbf{F}^{-1} \\ &\Leftrightarrow \exists x, y. z = (x, y) \wedge (x, y) \in \mathbf{F} \\ &\Leftrightarrow z \in \mathbf{F} \end{aligned} \quad (\mathbf{F} \text{ is a relation}) \square$$

Theorem 3.10.6 (Pairing). *For any class \mathbf{F} , we have \mathbf{F}^{-1} is a class term if and only if \mathbf{F} is single-rooted.*

PROOF: Immediate from definitions. \square

Theorem 3.10.7 (Pairing). *Let \mathbf{F} be a relation. Then \mathbf{F} is a class term if and only if \mathbf{F}^{-1} is single-rooted.*

PROOF: Immediate from definitions. \square

Theorem 3.10.8 (Pairing). *Let \mathbf{F} be a one-to-one class term and $x \in \text{dom } \mathbf{F}$. Then $\mathbf{F}^{-1}(\mathbf{F}(x)) = x$.*

PROOF: We have $(x, \mathbf{F}(x)) \in \mathbf{F}$ and so $(\mathbf{F}(x), x) \in \mathbf{F}^{-1}$. \square

Theorem 3.10.9 (Pairing). *Let \mathbf{F} be a one-to-one function and $y \in \text{ran } \mathbf{F}$. Then $\mathbf{F}(\mathbf{F}^{-1}(y)) = y$.*

PROOF: From Theorems 3.10.3, 3.10.5 and 3.10.8. \square

3.11 Composition

Definition 3.11.1 (Composition (Pairing)). Let \mathbf{R} and \mathbf{S} be relations. The *composition* of \mathbf{R} and \mathbf{S} is $\mathbf{S} \circ \mathbf{R} = \{(x, z) \mid \exists y. x\mathbf{R}y \wedge y\mathbf{S}z\}$.

Theorem 3.11.2 (Pairing, Union, Power Set, Subset). *If R and S are small relations then $S \circ R$ is small.*

PROOF: It is a subset of $\text{dom } R \times \text{ran } S$. \square

Theorem 3.11.3 (Pairing). *Let \mathbf{F} and \mathbf{G} be class terms. Then $\mathbf{G} \circ \mathbf{F}$ is a function, its domain is $\{x \in \text{dom } \mathbf{F} \mid \mathbf{F}(x) \in \text{dom } \mathbf{G}\}$, and for x in this domain, we have $(\mathbf{F} \circ \mathbf{G})(x) = \mathbf{F}(\mathbf{G}(x))$.*

PROOF:

$\langle 1 \rangle 1$. $\mathbf{G} \circ \mathbf{F}$ is a class term.

$\langle 2 \rangle 1$. LET: $x(\mathbf{G} \circ \mathbf{F})z$ and $x(\mathbf{G} \circ \mathbf{F})z'$

$\langle 2 \rangle 2$. PICK y, y' such that $x\mathbf{F}y, x\mathbf{F}y', y\mathbf{G}z$ and $y'\mathbf{G}z'$

$\langle 2 \rangle 3$. $y = y'$

PROOF: Since \mathbf{F} is a class term.

$\langle 2 \rangle 4$. $z = z'$

PROOF: Since \mathbf{G} is a class term.

$\langle 1 \rangle 2$. $\text{dom}(\mathbf{G} \circ \mathbf{F}) = \{x \in \text{dom } \mathbf{F} \mid \mathbf{F}(x) \in \text{dom } \mathbf{G}\}$

PROOF:

$$\begin{aligned} x \in \text{dom}(\mathbf{G} \circ \mathbf{F}) &\Leftrightarrow \exists z. x(\mathbf{G} \circ \mathbf{F})z \\ &\Leftrightarrow \exists y, z. x\mathbf{F}y \wedge y\mathbf{G}z \\ &\Leftrightarrow x \in \text{dom } \mathbf{F} \wedge \mathbf{F}(x) \in \text{dom } \mathbf{G} \end{aligned}$$

$\langle 1 \rangle 3$. For x in this domain, we have $(\mathbf{F} \circ \mathbf{G})(x) = \mathbf{F}(\mathbf{G}(x))$.

PROOF: Since $(x, \mathbf{F}(x)) \in \mathbf{F}$ and $(\mathbf{F}(x), \mathbf{G}(\mathbf{F}(x))) \in \mathbf{G}$.

\square

Theorem 3.11.4 (Pairing). *For any classes \mathbf{F} and \mathbf{G} , we have $(\mathbf{G} \circ \mathbf{F})^{-1} = \mathbf{F}^{-1} \circ \mathbf{G}^{-1}$.*

PROOF:

$$\begin{aligned}
(x, z) \in (\mathbf{G} \circ \mathbf{F})^{-1} &\Leftrightarrow (z, x) \in \mathbf{G} \circ \mathbf{F} \\
&\Leftrightarrow \exists y. z \mathbf{F} y \wedge y \mathbf{G} x \\
&\Leftrightarrow \exists y. (y, z) \in \mathbf{F}^{-1} \wedge (x, y) \in \mathbf{G}^{-1} \\
&\Leftrightarrow (x, z) \in \mathbf{F}^{-1} \circ \mathbf{G}^{-1} \quad \square
\end{aligned}$$

3.12 Identity Function

Definition 3.12.1 (Identity Class Term (Pairing)). Let \mathbf{A} be a set. The *identity class term* $\text{id}_{\mathbf{A}}$ on \mathbf{A} is $\{(x, x) \mid x \in \mathbf{A}\}$.

Theorem 3.12.2 (Pairing, Power Set, Subset). *For any set A , we have id_A is a function.*

PROOF: It is a subset of $\mathcal{P}\mathcal{P}A$. \square

Theorem 3.12.3 (Extensionality, Pairing, Union, Power Set, Subset). *Let $F : A \rightarrow B$ and A be nonempty. Then there exists a function $G : B \rightarrow A$ such that $G \circ F = \text{id}_A$ if and only if F is one-to-one.*

PROOF:

- $\langle 1 \rangle 1$. LET: $F : A \rightarrow B$
- $\langle 1 \rangle 2$. ASSUME: A is nonempty
- $\langle 1 \rangle 3$. If there exists $G : B \rightarrow A$ such that $G \circ F = \text{id}_A$ then F is one-to-one.
 - $\langle 2 \rangle 1$. ASSUME: $G : B \rightarrow A$ and $G \circ F = \text{id}_A$
 - $\langle 2 \rangle 2$. LET: $x, y \in A$
 - $\langle 2 \rangle 3$. ASSUME: $F(x) = F(y)$
 - $\langle 2 \rangle 4$. $x = y$
- PROOF: $x = G(F(x)) = G(F(y)) = y$.
- $\langle 1 \rangle 4$. If F is one-to-one then there exists $G : B \rightarrow A$ such that $G \circ F = \text{id}_A$.
 - $\langle 2 \rangle 1$. ASSUME: F is one-to-one.
 - $\langle 2 \rangle 2$. PICK $a \in A$
 - $\langle 2 \rangle 3$. Define $G : B \rightarrow A$ by: $G(y)$ is the x such that $F(x) = y$ if $y \in \text{ran } F$, otherwise $G(y) = a$
 - $\langle 2 \rangle 4$. $G \circ F = \text{id}_A$
- PROOF: For $x \in A$ we have $(G \circ F)(x) = G(F(x)) = x$ by Theorem 3.11.3.

\square

Theorem 3.12.4 (Extensionality, Pairing, Union, Power Set, Subset). *Let $F : A \rightarrow B$ and A be nonempty. If there exists a function $H : B \rightarrow A$ such that $F \circ H = \text{id}_B$ then F is surjective.*

PROOF:

- $\langle 1 \rangle 1$. LET: $F : A \rightarrow B$
- $\langle 1 \rangle 2$. ASSUME: A is nonempty.
- $\langle 1 \rangle 3$. LET: $H : B \rightarrow A$ satisfy $F \circ H = \text{id}_B$

⟨1⟩4. LET: $y \in B$
 ⟨1⟩5. $F(H(y)) = y$.
 □

Theorem 3.12.5 (Extensionality, Pairing, Union, Power Set, Subset, Choice).
Let $F : A \rightarrow B$ and A be nonempty. If F is surjective then there exists a function $H : B \rightarrow A$ such that $F \circ H = \text{id}_B$.

PROOF:

⟨1⟩1. ASSUME: F is surjective.
 ⟨1⟩2. PICK a function $H \subseteq F^{-1}$ with $\text{dom } H = B$

PROOF: By the Axiom of Choice.

⟨1⟩3. $H : B \rightarrow A$
 ⟨1⟩4. $F \circ H = \text{id}_B$
 ⟨2⟩1. LET: $y \in B$
 ⟨2⟩2. $(y, H(y)) \in F^{-1}$
 ⟨2⟩3. $(H(y), y) \in F$
 ⟨2⟩4. $F(H(y)) = y$

□

3.13 Restriction

Definition 3.13.1 (Restriction (Pairing)). Let \mathbf{R} be a relation and \mathbf{A} a class.
 The *restriction* of \mathbf{R} to \mathbf{A} is $\mathbf{R} \upharpoonright \mathbf{A} = \{(x, y) \mid x \in \mathbf{A} \wedge x\mathbf{R}y\}$.

Theorem 3.13.2 (Pairing, Subset). *If R is a small relation then $R \upharpoonright \mathbf{A}$ is small.*

PROOF: Since it is a subset of R . □

3.14 Image

Definition 3.14.1 (Image (Pairing)). Let \mathbf{F} and \mathbf{A} be classes. The *image* of \mathbf{A} under \mathbf{F} is $\mathbf{F}(\mathbf{A}) = \{\mathbf{F}(x) \mid x \in \mathbf{A}\}$.

Theorem 3.14.2 (Pairing, Union, Subset). *If F is a set then $F(\mathbf{A})$ is a set.*

PROOF: Since it is a subset of $\text{ran } F$. □

Theorem 3.14.3 (Pairing). *For any classes \mathbf{F} and \mathcal{A} we have*

$$\mathbf{F}\left(\bigcup \mathcal{A}\right) = \bigcup_{A \in \mathcal{A}} \mathbf{F}(A)$$

PROOF: Each is the class of all y such that $\exists x. \exists A. x \in A \in \mathcal{A} \wedge y = \mathbf{F}(x)$. □

Theorem 3.14.4 (Pairing). *For any classes \mathbf{F} , $\mathbf{A}_1, \dots, \mathbf{A}_n$, we have*

$$\mathbf{F}(\mathbf{A}_1 \cup \dots \cup \mathbf{A}_n) = \mathbf{F}(\mathbf{A}_1) \cup \dots \cup \mathbf{F}(\mathbf{A}_n) .$$

PROOF: Similar. \square

Theorem 3.14.5 (Pairing). *For any classes \mathbf{F} and \mathcal{A} , we have*

$$\mathbf{F}\left(\bigcap \mathcal{A}\right) \subseteq \bigcap_{A \in \mathcal{A}} \mathbf{F}(A) .$$

Equality holds if \mathbf{F} is single-rooted and \mathcal{A} is nonempty.

PROOF:

- $\langle 1 \rangle 1.$ $\mathbf{F}\left(\bigcap \mathcal{A}\right) \subseteq \bigcap_{A \in \mathcal{A}} \mathbf{F}(A)$
- $\langle 2 \rangle 1.$ LET: $y \in \mathbf{F}\left(\bigcap \mathcal{A}\right)$
- $\langle 2 \rangle 2.$ PICK $x \in \bigcap \mathcal{A}$ such that $y = \mathbf{F}(x)$
- $\langle 2 \rangle 3.$ LET: $A \in \mathcal{A}$
- $\langle 2 \rangle 4.$ $x \in A$
- $\langle 2 \rangle 5.$ $y \in \mathbf{F}(A)$
- $\langle 1 \rangle 2.$ If \mathbf{F} is single-rooted then $\mathbf{F}\left(\bigcap \mathcal{A}\right) = \bigcap_{A \in \mathcal{A}} \mathbf{F}(A)$
- $\langle 2 \rangle 1.$ ASSUME: \mathbf{F} is single-rooted and \mathcal{A} is nonempty.
- $\langle 2 \rangle 2.$ LET: $y \in \bigcap_{A \in \mathcal{A}} \mathbf{F}(A)$
- $\langle 2 \rangle 3.$ PICK $A \in \mathcal{A}$
- $\langle 2 \rangle 4.$ PICK $x \in A$ such that $y = \mathbf{F}(x)$
- $\langle 2 \rangle 5.$ $x \in \bigcap \mathcal{A}$
- $\langle 3 \rangle 1.$ LET: $A' \in \mathcal{A}$
- $\langle 3 \rangle 2.$ PICK $x' \in A'$ such that $y = \mathbf{F}(x')$
- $\langle 3 \rangle 3.$ $x = x'$
- PROOF: By $\langle 2 \rangle 1.$
- $\langle 3 \rangle 4.$ $x \in A'$

\square

Corollary 3.14.5.1 (Pairing). *For any class \mathbf{F} and nonempty class \mathcal{A} , we have*

$$\mathbf{F}^{-1}\left(\bigcap \mathcal{A}\right) = \bigcap_{A \in \mathcal{A}} \mathbf{F}^{-1}(A) .$$

Theorem 3.14.6 (Pairing). *For any classes \mathbf{F} , $\mathbf{A}_1, \dots, \mathbf{A}_n$, we have*

$$\mathbf{F}(\mathbf{A}_1 \cap \dots \cap \mathbf{A}_n) \subseteq \mathbf{F}(\mathbf{A}_1) \cap \dots \cap \mathbf{F}(\mathbf{A}_n) .$$

Equality holds if \mathbf{F} is single-rooted.

PROOF: Similar.

Corollary 3.14.6.1 (Pairing). *For any classes \mathbf{F} , $\mathbf{A}_1, \dots, \mathbf{A}_n$, we have*

$$\mathbf{F}^{-1}(\mathbf{A}_1 \cap \dots \cap \mathbf{A}_n) = \mathbf{F}^{-1}(\mathbf{A}_1) \cap \dots \cap \mathbf{F}^{-1}(\mathbf{A}_n) .$$

Theorem 3.14.7 (Pairing). *For any classes \mathbf{F} , \mathbf{A} and \mathbf{B} , we have*

$$\mathbf{F}(\mathbf{A}) - \mathbf{F}(\mathbf{B}) \subseteq \mathbf{F}(\mathbf{A} - \mathbf{B}) .$$

Equality holds if \mathbf{F} is single-rooted.

PROOF:

- ⟨1⟩1. LET: \mathbf{F} , \mathbf{A} and \mathbf{B} be sets.
- ⟨1⟩2. $\mathbf{F}(\mathbf{A}) - \mathbf{F}(\mathbf{B}) \subseteq \mathbf{F}(\mathbf{A} - \mathbf{B})$
 - ⟨2⟩1. LET: $y \in \mathbf{F}(\mathbf{A}) - \mathbf{F}(\mathbf{B})$
 - ⟨2⟩2. PICK $x \in \mathbf{A}$ such that $x\mathbf{F}y$
 - ⟨2⟩3. $x \in \mathbf{A} - \mathbf{B}$
- ⟨1⟩3. If \mathbf{F} is single-rooted then $\mathbf{F}(\mathbf{A} - \mathbf{B}) = \mathbf{F}(\mathbf{A}) - \mathbf{F}(\mathbf{B})$.
 - ⟨2⟩1. ASSUME: \mathbf{F} is single-rooted.
 - ⟨2⟩2. LET: $y \in \mathbf{F}(\mathbf{A} - \mathbf{B})$
 - ⟨2⟩3. PICK $x \in \mathbf{A} - \mathbf{B}$ such that $y = \mathbf{F}(x)$
 - ⟨2⟩4. $y \in \mathbf{F}(\mathbf{A})$
 - ⟨2⟩5. $y \notin \mathbf{F}(\mathbf{B})$
 - ⟨3⟩1. ASSUME: for a contradiction $x' \in \mathbf{B}$ and $x'\mathbf{F}y$
 - ⟨3⟩2. $x' = x$
 - PROOF: From ⟨2⟩1
 - ⟨3⟩3. $x \in \mathbf{B}$
 - ⟨3⟩4. Q.E.D.
 - PROOF: This contradicts ⟨2⟩3.

□

Corollary 3.14.7.1 (Pairing). *For any classes \mathbf{F} and sets \mathbf{A} and \mathbf{B} , we have*

$$\mathbf{F}^{-1}(\mathbf{A}) - \mathbf{F}^{-1}(\mathbf{B}) = \mathbf{F}^{-1}(\mathbf{A} - \mathbf{B}) .$$

3.15 Infinite Cartesian Product

Definition 3.15.1 (Infinite Cartesian Product (Pairing)). Let H be a function with domain I . The *Cartesian product* $\prod_{i \in I} H(i)$ is the class of all functions f with domain I such that, for all $i \in I$, we have $f(i) \in H(i)$.

Theorem 3.15.2 (Pairing, Union, Power Set, Subset). *If H is a function with domain I then $\prod_{i \in I} H(i)$ is a set.*

PROOF: It is a subset of $\mathcal{P}(I \times \bigcup \text{ran } H)$. □

Theorem 3.15.3 (Axiom of Choice, Second Version (Pairing, Union, Power Set, Subset)). *The Axiom of Choice is equivalent to the statement: for any function H with domain I , if $H(i)$ is nonempty for all $i \in I$, then $\prod_{i \in I} H(i)$ is nonempty.*

PROOF:

- ⟨1⟩1. If the Axiom of Choice is true then, for any function H with domain I , if $H(i)$ is nonempty for all $i \in I$, then $\prod_{i \in I} H(i)$ is nonempty.
- ⟨2⟩1. ASSUME: The Axiom of Choice
- ⟨2⟩2. LET: H be a function with domain I such that $H(i)$ is nonempty for all $i \in I$.
- ⟨2⟩3. PICK a function $f \subseteq \{(i, x) \mid x \in H(i)\}$

- (2)4. $f \in \prod_{i \in I} H(i)$
- (1)2. If, for any function H with domain I , if $H(i)$ is nonempty for all $i \in I$, then $\prod_{i \in I} H(i)$ is nonempty, then the Axiom of Choice is true.
- (2)1. ASSUME: for any function H with domain I , if $H(i)$ is nonempty for all $i \in I$, then $\prod_{i \in I} H(i)$ is nonempty
- (2)2. LET: R be a relation.
- (2)3. LET: $I = \text{dom } R$
- (2)4. LET: H be the function with domain I such that $H(i) = \{y \mid iRy\}$ for all i .
- (2)5. PICK $f \in \prod_{i \in I} H(i)$
- (2)6. $f \subseteq R$

□

3.16 Reflexive Relations

Definition 3.16.1 (Reflexive (Pairing)). Let \mathbf{R} be a relation on \mathbf{A} . Then \mathbf{R} is *reflexive* on A if and only if, for all $x \in \mathbf{A}$, we have $x\mathbf{R}x$.

3.17 Symmetric

Definition 3.17.1 (Symmetric (Pairing)). Let \mathbf{R} be a relation. Then \mathbf{R} is *symmetric* if and only if, whenever $x\mathbf{R}y$, then $y\mathbf{R}x$.

3.18 Transitivity

Definition 3.18.1 (Transitivity (Pairing)). Let \mathbf{R} be a relation. Then \mathbf{R} is *transitive* if and only if, whenever $x\mathbf{R}y$ and $y\mathbf{R}z$, then $x\mathbf{R}z$.

3.19 Equivalence Relations

Definition 3.19.1 (Equivalence Relation (Pairing)). Let \mathbf{R} be a relation on \mathbf{A} . Then \mathbf{R} is an *equivalence relation* on \mathbf{A} if and only if \mathbf{R} is reflexive on \mathbf{A} , symmetric and transitive.

Theorem 3.19.2 (Pairing). *If \mathbf{R} is a symmetric and transitive relation then \mathbf{R} is an equivalence relation on $\text{fld } \mathbf{R}$.*

PROOF:

- (1)1. LET: \mathbf{R} be a symmetric and transitive relation.
- (1)2. LET: $x \in \text{fld } \mathbf{R}$
- (1)3. PICK y such that $x\mathbf{R}y$ or $y\mathbf{R}x$
- (1)4. $x\mathbf{R}y$ and $y\mathbf{R}x$
- PROOF: By symmetry.
- (1)5. $x\mathbf{R}x$

PROOF: By transitivity.

□

3.20 Equivalence Class

Definition 3.20.1 (Equivalence Class (Pairing)). Let \mathbf{R} be an equivalence relation on \mathbf{A} and $a \in \mathbf{A}$. Then the *equivalence class* of a modulo \mathbf{R} is

$$[a]_{\mathbf{R}} = \{x \in \mathbf{A} \mid a\mathbf{R}x\} .$$

Lemma 3.20.2 (Extensionality, Pairing, Subset). *Let \mathbf{R} be an equivalence relation on \mathbf{A} and $x, y \in \mathbf{A}$. Then $[x]_{\mathbf{R}} = [y]_{\mathbf{R}}$ if and only if $x\mathbf{R}y$.*

PROOF:

⟨1⟩1. If $[x]_{\mathbf{R}} = [y]_{\mathbf{R}}$ then $x\mathbf{R}y$.

⟨2⟩1. ASSUME: $[x]_{\mathbf{R}} = [y]_{\mathbf{R}}$

⟨2⟩2. $y \in [y]_{\mathbf{R}}$

PROOF: Since $y\mathbf{R}y$ by reflexivity.

⟨2⟩3. $y \in [x]_{\mathbf{R}}$

⟨2⟩4. $x\mathbf{R}y$

⟨1⟩2. If $x\mathbf{R}y$ then $[x]_{\mathbf{R}} = [y]_{\mathbf{R}}$.

⟨2⟩1. ASSUME: $x\mathbf{R}y$

⟨2⟩2. $[y]_{\mathbf{R}} \subseteq [x]_{\mathbf{R}}$

PROOF: If $y\mathbf{R}z$ then $x\mathbf{R}z$ by transitivity.

⟨2⟩3. $[x]_{\mathbf{R}} \subseteq [y]_{\mathbf{R}}$

PROOF: Similar since $y\mathbf{R}x$ by symmetry.

□

3.21 Disjoint

Definition 3.21.1 (Disjoint). Two classes \mathbf{A} and \mathbf{B} are *disjoint* if and only if there is no x such that $x \in \mathbf{A}$ and $x \in \mathbf{B}$.

Axiom 3.21.2 (Regularity). *For any nonempty set A , there exists $m \in A$ such that m and A are disjoint.*

Theorem 3.21.3 (Regularity). *No set is a member of itself.*

Theorem 3.21.4 (Regularity). *There are no sets A and B such that $A \in B$ and $B \in A$.*

3.22 Partitions

Definition 3.22.1 (Partition). A *partition* P of a set A is a set of nonempty subsets of A such that:

1. For all $x \in A$ there exists $S \in P$ such that $x \in S$.
2. Any two distinct elements of P are disjoint.

3.23 Quotient Sets

Definition 3.23.1 (Quotient Set (Pairing, Power Set, Subset)). Let R be an equivalence relation on A . The *quotient set* A/R is the set of all equivalence classes modulo R .

This is a set because it is a subset of $\mathcal{P}A$.

Theorem 3.23.2 (Extensionality, Pairing, Power Set, Subset). *Let R be an equivalence relation on A . Then the quotient set A/R is a partition of A .*

PROOF:

$\langle 1 \rangle 1$. For all $x \in A$ there exists $y \in A$ such that $x \in [y]_R$

PROOF: Take $y = x$.

$\langle 1 \rangle 2$. Any two distinct equivalence classes are disjoint.

$\langle 2 \rangle 1$. ASSUME: $z \in [x]_R$ and $z \in [y]_R$

$\langle 2 \rangle 2$. xRz and yRz

$\langle 2 \rangle 3$. $[x]_R = [z]_R = [y]_R$

PROOF: Lemma 3.20.2.

□

Definition 3.23.3 (Canonical Map (Pairing, Power Set, Subset)). Let R be an equivalence relation on A . The *canonical map* $\phi : A \rightarrow A/R$ is the function defined by $\phi(a) = [a]_R$.

Theorem 3.23.4. *Let R be an equivalence relation on A and $F : A \rightarrow B$. Then the following are equivalent:*

1. For all $x, y \in A$, if xRy then $F(x) = F(y)$.

2. There exists $G : A/R \rightarrow B$ such that $F = G \circ \phi$, where $\phi : A \rightarrow A/R$ is the canonical map.

In this case, G is unique.

PROOF:

$\langle 1 \rangle 1$. $1 \Rightarrow 2$

$\langle 2 \rangle 1$. ASSUME: 1

$\langle 2 \rangle 2$. Let $G = \{([a]_R, b) \mid F(a) = b\}$

$\langle 2 \rangle 3$. G is a function.

$\langle 3 \rangle 1$. LET: $(c, b), (c, b') \in G$

$\langle 3 \rangle 2$. PICK $a, a' \in A$ such that $c = [a]_R = [a']_R$ with $F(a) = b$ and $F(a') = b'$

$\langle 3 \rangle 3$. aRa'

PROOF: Lemma 3.20.2.

$\langle 3 \rangle 4$. $F(a) = F(a')$

PROOF: From $\langle 2 \rangle 1$.

$\langle 3 \rangle 5$. $b = b'$

PROOF: From $\langle 3 \rangle 2$.

$\langle 2 \rangle 4$. $F = G \circ \phi$

PROOF: For $a \in A$ we have $G(\phi(a)) = G([a]) = F(a)$.

$\langle 1 \rangle 2$. $2 \Rightarrow 1$

$\langle 2 \rangle 1$. LET: $G : A/R \rightarrow B$ be such that $F = G \circ \phi$

$\langle 2 \rangle 2$. LET: $x, y \in A$

$\langle 2 \rangle 3$. ASSUME: xRy

$\langle 2 \rangle 4$. $G([x]) = G([y])$

PROOF: Lemma 3.20.2

$\langle 2 \rangle 5$. $F(x) = F(y)$

PROOF: From $\langle 2 \rangle 1$.

$\langle 1 \rangle 3$. If $G, G' : A/R \rightarrow B$ and $G \circ \phi = G' \circ \phi$ then $G = G'$

PROOF: For any $a \in A$ we have $G([a]) = G'([a])$.

□

3.24 The Finite Intersection Property

Definition 3.24.1 (Finite Intersection Property). Let X be a set and $\mathcal{A} \subseteq \mathcal{P}X$. Then \mathcal{A} satisfies the *finite intersection property* if and only if every nonempty finite subset of \mathcal{A} has nonempty intersection.

Lemma 3.24.2. Let X be a set. Let $\mathcal{A} \subseteq \mathcal{P}X$ have the finite intersection property. Then there exists a maximal set \mathcal{D} such that $\mathcal{A} \subseteq \mathcal{D} \subseteq \mathcal{P}X$ and \mathcal{D} has the finite intersection property.

PROOF:

$\langle 1 \rangle 1$. LET: $\mathbb{F} = \{\mathcal{D} \mid \mathcal{A} \subseteq \mathcal{D} \subseteq \mathcal{P}X, \mathcal{D} \text{ has the finite intersection property}\}$

$\langle 1 \rangle 2$. Every chain in \mathbb{F} has an upper bound.

$\langle 2 \rangle 1$. LET: \mathbb{C} be a chain in \mathbb{F} .

$\langle 2 \rangle 2$. ASSUME: without loss of generality $\mathbb{C} \neq \emptyset$

PROVE: $\bigcup \mathbb{C} \in \mathbb{F}$

PROOF: If $\mathbb{C} = \emptyset$ then \mathcal{A} is an upper bound.

$\langle 2 \rangle 3$. $\mathcal{A} \subseteq \bigcup \mathbb{C} \subseteq \mathcal{P}X$

$\langle 2 \rangle 4$. LET: $C_1, \dots, C_n \in \mathbb{C}$

PROVE: $C_1 \cap \dots \cap C_n \neq \emptyset$

$\langle 2 \rangle 5$. PICK $\mathcal{C}_1, \dots, \mathcal{C}_n \in \mathbb{C}$ such that $C_i \in \mathcal{C}_i$ for all i .

$\langle 2 \rangle 6$. ASSUME: without loss of generality $\mathcal{C}_1 \subseteq \dots \subseteq \mathcal{C}_n$

$\langle 2 \rangle 7$. $C_1, \dots, C_n \in \mathcal{C}_n$

$\langle 2 \rangle 8$. \mathcal{C}_n satisfies the finite intersection property.

$\langle 2 \rangle 9$. $C_1 \cap \dots \cap C_n \neq \emptyset$

$\langle 1 \rangle 3$. Q.E.D.

PROOF: By Zorn's Lemma.

□

Lemma 3.24.3. Let X be a set. Let $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the finite intersection property. Then any finite intersection of elements of \mathcal{D} is an element of \mathcal{D} .

PROOF:

- ⟨1⟩1. LET: $D_1, D_2 \in \mathcal{D}$
 ⟨1⟩2. $\mathcal{D} \cup \{D_1 \cap D_2\}$ has the finite intersection property.
 PROOF: Any finite intersection of members of $\mathcal{D} \cup \{D_1 \cap D_2\}$ is a finite intersection of members of \mathcal{D} .
 ⟨1⟩3. $\mathcal{D} = \mathcal{D} \cup \{D_1 \cap D_2\}$
 PROOF: By maximality of \mathcal{D} .
 ⟨1⟩4. $D_1 \cap D_2 \in \mathcal{D}$.
 □

Lemma 3.24.4. *Let X be a set. Let $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the finite intersection property. Let $A \subseteq X$. If A intersects every member of \mathcal{D} then $A \in \mathcal{D}$.*

- PROOF:
 ⟨1⟩1. $\mathcal{D} \cup \{A\}$ has the finite intersection property.
 ⟨2⟩1. LET: $D_1, \dots, D_n \in \mathcal{D}$
 PROVE: $D_1 \cap \dots \cap D_n \cap A \neq \emptyset$
 ⟨2⟩2. $D_1 \cap \dots \cap D_n \in \mathcal{D}$
 PROOF: Lemma 3.24.3.
 ⟨2⟩3. $D_1 \cap \dots \cap D_n \cap A \neq \emptyset$
 PROOF: Since A intersects every member of \mathcal{D} .
 ⟨1⟩2. Q.E.D.
 PROOF: By maximality of \mathcal{D} .
 □

Proposition 3.24.5. *Let X be a set. Let $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the finite intersection property. Let $A, D \in \mathcal{P}X$. If $D \in \mathcal{D}$ and $D \subseteq A$ then $A \in \mathcal{D}$.*

- PROOF:
 ⟨1⟩1. $\mathcal{D} \cup \{A\}$ satisfies the finite intersection property.
 ⟨2⟩1. LET: $D_1, \dots, D_n \in \mathcal{D}$
 ⟨2⟩2. $D_1 \cap \dots \cap D_n \cap D \neq \emptyset$
 PROOF: Since \mathcal{D} satisfies the finite intersection property.
 ⟨2⟩3. $D_1 \cap \dots \cap D_n \cap A \neq \emptyset$
 ⟨1⟩2. $\mathcal{D} = \mathcal{D} \cup \{A\}$
 PROOF: By the maximality of \mathcal{D} .
 ⟨1⟩3. $A \in \mathcal{D}$
 □

Definition 3.24.6 (Graph). Let $f : A \rightarrow B$. The *graph* of f is the set $\{(x, f(x)) \mid x \in A\} \subseteq A \times B$.

3.25 Countable Intersection Property

Definition 3.25.1 (Countable Intersection Property). Let X be a set and $\mathcal{A} \subseteq \mathcal{P}X$. Then \mathcal{A} satisfies the *countable intersection property* if and only if every countable subset of \mathcal{A} has nonempty intersection.

Lemma 3.25.2. *Let X be a set. Let $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the countable intersection property. Then any countable intersection of elements of \mathcal{D} is an element of \mathcal{D} .*

PROOF:

$\langle 1 \rangle 1$. LET: $\mathcal{D}_0 \subseteq \mathcal{D}$ be countable.

$\langle 1 \rangle 2$. $\mathcal{D} \cup \{\bigcap \mathcal{D}_0\}$ has the countable intersection property.

PROOF: Any countable intersection of members of $\mathcal{D} \cup \{\bigcap \mathcal{D}_0\}$ is a finite intersection of members of \mathcal{D} .

$\langle 1 \rangle 3$. $\mathcal{D} = \mathcal{D} \cup \{\bigcap \mathcal{D}_0\}$

PROOF: By maximality of \mathcal{D} .

$\langle 1 \rangle 4$. $\bigcap \mathcal{D}_0 \in \mathcal{D}$.

□

Lemma 3.25.3. *Let X be a set. Let $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the countable intersection property. Let $A \subseteq X$. If A intersects every member of \mathcal{D} then $A \in \mathcal{D}$.*

PROOF:

$\langle 1 \rangle 1$. $\mathcal{D} \cup \{A\}$ has the countable intersection property.

$\langle 2 \rangle 1$. LET: $\mathcal{D}_0 \subseteq \mathcal{D}$ be countable.

PROVE: $\bigcap \mathcal{D}_0 \cap A \neq \emptyset$

$\langle 2 \rangle 2$. $\bigcap \mathcal{D}_0 \in \mathcal{D}$

PROOF: Lemma 3.25.2.

$\langle 2 \rangle 3$. $\bigcap \mathcal{D}_0 \cap A \neq \emptyset$

PROOF: Since A intersects every member of \mathcal{D} .

$\langle 1 \rangle 2$. Q.E.D.

PROOF: By maximality of \mathcal{D} .

□

3.26 The Axiom of Choice

Axiom 3.26.1 (Axiom of Choice). *Let \mathcal{A} be a set of disjoint nonempty sets. Then there exists a set C consisting of exactly one element from each member of \mathcal{A} .*

3.27 Choice Functions

Definition 3.27.1 (Choice Function). Let \mathcal{B} be a set of nonempty sets. A *choice function* for \mathcal{B} is a function $c : \mathcal{B} \rightarrow \bigcup \mathcal{B}$ such that, for all $B \in \mathcal{B}$, we have $c(B) \in B$.

Lemma 3.27.2 (Existence of a Choice Function (AC)). *Every set of nonempty sets has a choice function.*

PROOF:

- ⟨1⟩1. LET: \mathcal{B} be a set of nonempty sets.
 - ⟨1⟩2. For $B \in \mathcal{B}$,
LET: $B' = \{B\} \times B$
 - ⟨1⟩3. $\{B' \mid B \in \mathcal{B}\}$ is a set of disjoint nonempty sets.
 - ⟨1⟩4. PICK a set c consisting of exactly one element from each B' for $B \in \mathcal{B}$.
 - ⟨1⟩5. c is a choice function for \mathcal{B} .
-

3.28 Transitive

Definition 3.28.1 (Transitive Set). A set A is *transitive* if and only if, whenever $x \in y \in A$ then $x \in A$.

Theorem 3.28.2 (Union, Power Set). *Let A be a set. Then the following are equivalent.*

- 1. A is transitive.
- 2. $\bigcup A \subseteq A$
- 3. For all $a \in A$ we have $a \subseteq A$
- 4. $A \subseteq \mathcal{P}A$

PROOF: From definitions. □

3.29 Minimal Elements

Definition 3.29.1 (Minimal). Let R be a binary relation and A a set. An element $a \in A$ is *minimal* w.r.t. R iff there is no $x \in A$ such that xRa .

3.30 Well-Founded Relations

Definition 3.30.1 (Well-Founded). Let R be a relation on A . Then R is *well-founded* iff every nonempty subset of A has an R -minimal element.

Theorem 3.30.2 (Transfinite Induction). *Let R be a well-founded relation on A and $B \subseteq A$. Assume that, for every $t \in A$, if $\{x \in A \mid xRt\} \subseteq B$ then $t \in B$. Then we have $B = A$.*

Theorem 3.30.3 (Transfinite Recursion). *Let R be a well-founded relation on a set C .*

Let \mathbf{A} be a class. Let \mathbf{B} be the class of all functions from a subset of C to \mathbf{A} . Let $\mathbf{F} : \mathbf{B} \times C \rightarrow \mathbf{A}$ be a class term.

Then there exists a unique function $f : C \rightarrow \mathbf{A}$ such that, for all $t \in C$, we have $f(t) = \mathbf{F}(f \upharpoonright \{x \in C \mid xRt\}, t)$.

3.31 Transitive Closure

Theorem 3.31.1. *Let R be a relation. Then there exists a unique relation R^t such that R^t is transitive, $R \subseteq R^t$, and for every transitive relation S with $R \subseteq S$ we have $R^t \subseteq S$.*

Definition 3.31.2 (Transitive Closure). The *transitive closure* of a relation R is this relation R^t .

Theorem 3.31.3. *If R is well-founded then R^t is well-founded.*

3.32 Fixed Points

Definition 3.32.1 (Fixed Point). Let X be a set. Let $f : X \rightarrow X$. Then a *fixed point* of f is an element $a \in X$ such that $f(a) = a$.

Chapter 4

Cardinal Numbers

Definition 4.0.1 (Equinumerous). Two sets A and B are *equinumerous* if and only if there exists a bijection between them.

Theorem 4.0.2. *Equinumerosity is an equivalence relation on the class of all sets.*

Theorem 4.0.3 (Cantor). *No set is equinumerous with its power set.*

Definition 4.0.4. We say a set A is *dominated* by B , $A \preccurlyeq B$, iff A is equinumerous with a subset of B .

Theorem 4.0.5. $A \preccurlyeq A$

Theorem 4.0.6. *If $A \preccurlyeq B \preccurlyeq C$ then $A \preccurlyeq C$.*

Theorem 4.0.7 (Schröder-Bernstein Theorem). *If $A \preccurlyeq B$ and $B \preccurlyeq A$ then $A \equiv B$.*

PROOF:

⟨1⟩1. LET: $f : A \rightarrow B$ and $g : B \rightarrow A$ be injections.

⟨1⟩2. Define a sequence of sets $C_n \subseteq A$ by

$$\begin{aligned} C_0 &= A - \text{ran } g \\ C_{n+1} &= g(f(C_n)) \end{aligned}$$

⟨1⟩3. Define $h : A \rightarrow B$ by

$$h(x) = \begin{cases} f(x) & \text{if } x \in \bigcup_n C_n \\ g^{-1}(x) & \text{if not} \end{cases}$$

⟨1⟩4. h is a bijection.

□

Theorem 4.0.8 (AC). *For any infinite set A we have $\mathbb{N} \preccurlyeq A$.*

PROOF: Given a choice function f for A , choose a sequence (a_n) in A by $a_n = f(A - \{a_0, \dots, a_{n-1}\})$. □

Corollary 4.0.8.1 (AC). *A set is infinite if and only if it is equinumerous with a proper subset.*

4.1 Countability

Definition 4.1.1 (Countable). A set A is *countable* iff $A \preccurlyeq \mathbb{N}$.

Theorem 4.1.2 (AC). *A countable union of countable sets is countable.*

Proposition 4.1.3 (AC). *Every infinite set has a countable subset.*

4.2 Order Theory

Definition 4.2.1 (Preorder). Let X be a set. A *preorder* on X is a binary relation \leq on X such that:

Reflexivity For all $x \in X$, we have $x \leq x$

Transitivity For all $x, y, z \in X$, if $x \leq y$ and $y \leq z$ then $x \leq z$.

Definition 4.2.2 (Preordered Set). A *preordered set* consists of a set X and a preorder \leq on X .

Proposition 4.2.3. Let X and Y be linearly ordered sets. Let $f : X \rightarrow Y$ be strictly monotone and surjective. Then f is a poset isomorphism.

PROOF:

$\langle 1 \rangle 1.$ f is injective.

$\langle 2 \rangle 1.$ LET: $x, y \in X$

$\langle 2 \rangle 2.$ ASSUME: $f(x) = f(y)$

$\langle 2 \rangle 3.$ $x \not\leq y$

PROOF: By strong monotonicity.

$\langle 2 \rangle 4.$ $y \not\leq x$

PROOF: By strong monotonicity.

$\langle 2 \rangle 5.$ $x = y$

PROOF: By trichotomy.

$\langle 1 \rangle 2.$ f^{-1} is monotone.

$\langle 2 \rangle 1.$ LET: $x, y \in X$

$\langle 2 \rangle 2.$ ASSUME: $x \leq y$

$\langle 2 \rangle 3.$ $f^{-1}(x) \not\leq f^{-1}(y)$

PROOF: By strong monotonicity.

$\langle 2 \rangle 4.$ $f^{-1}(x) < f^{-1}(y)$

PROOF: By trichotomy.

□

Definition 4.2.4 (Interval). Let X be a preordered set and $Y \subseteq X$. Then Y is an *interval* if and only if, for all $a, b \in Y$ and $c \in X$, if $a \leq c \leq b$ then $c \in Y$.

Definition 4.2.5 (Linear Continuum). A linearly ordered set L is a *linear continuum* if and only if:

1. every nonempty subset of L that is bounded above has a supremum
2. L is dense

Proposition 4.2.6. Every interval in a linear continuum is a linear continuum.

PROOF:

$\langle 1 \rangle 1.$ LET: L be a linear continuum and I an interval in L .

$\langle 1 \rangle 2.$ Every nonempty subset of I that is bounded above has a supremum in I .

$\langle 2 \rangle 1.$ LET: $X \subseteq I$ be nonempty and bounded above by $b \in I$.

(2)2. LET: s be the supremum of X in L .
 PROOF: Since L is a linear continuum.
 (2)3. $s \in I$
 (3)1. PICK $a \in X$
 PROOF: Since X is nonempty ((2)1).
 (3)2. $a \leq s \leq b$
 (3)3. $s \in I$
 PROOF: Since I is an interval ((1)1).
 (2)4. s is the supremum of X in I
 (1)3. I is dense.
 (2)1. LET: $x, y \in I$ with $x < y$
 (2)2. PICK $z \in L$ with $x < z < y$
 PROOF: Since L is dense.
 (2)3. $z \in I$
 PROOF: Since I is an interval.

□

Definition 4.2.7 (Ordered Square). The *ordered square* I_o^2 is the set $[0, 1]^2$ under the dictionary order.

Proposition 4.2.8. *The ordered square is a linear continuum.*

PROOF:

(1)1. Every nonempty subset of I_o^2 bounded above has a supremum.
 (2)1. LET: $X \subseteq I_o^2$ be nonempty and bounded above by (b, c)
 (2)2. LET: $s = \sup \pi_1(X)$
 PROOF: The set $\pi_1(X)$ is nonempty and bounded above by b .
 (2)3. CASE: $s \in \pi_1(X)$
 (3)1. LET: $t = \sup\{y \in [0, 1] \mid (s, y) \in X\}$
 PROOF: This set is nonempty and bounded above by c .
 (3)2. (s, t) is the supremum of X .
 (2)4. CASE: $s \notin \pi_1(X)$
 PROOF: In this case $(s, 0)$ is the supremum of X .
 (1)2. I_o^2 is dense.
 (2)1. LET: $(x_1, y_1), (x_2, y_2) \in I_o^2$ with $(x_1, y_1) < (x_2, y_2)$
 (2)2. CASE: $x_1 < x_2$
 (3)1. PICK x_3 with $x_1 < x_3 < x_2$
 (3)2. $(x_1, y_1) < (x_3, y_1) < (x_2, y_2)$
 (2)3. CASE: $x_1 = x_2$ and $y_1 < y_2$
 (3)1. PICK y_3 with $y_1 < y_3 < y_2$
 (3)2. $(x_1, y_1) < (x_1, y_3) < (x_2, y_2)$

□

Proposition 4.2.9. *If X is a well-ordered set then $X \times [0, 1)$ under the dictionary order is a linear continuum.*

PROOF:

(1)1. Every nonempty set $A \subseteq X \times [0, 1)$ bounded above has a supremum.

- ⟨2⟩1. LET: $A \subseteq X \times [0, 1]$ be nonempty and bounded above
- ⟨2⟩2. LET: x_0 be the supremum of $\pi_1(A)$
- ⟨2⟩3. CASE: $x_0 \in \pi_1(A)$
 - ⟨3⟩1. LET: y_0 be the supremum of $\{y \in [0, 1] \mid (x_0, y) \in A\}$
 - ⟨3⟩2. (x_0, y_0) is the supremum of A .
- ⟨2⟩4. CASE: $x_0 \notin \pi_1(A)$
 - PROOF: In this case $(x_0, 0)$ is the supremum of A .
- ⟨1⟩2. $X \times [0, 1]$ is dense.
 - ⟨2⟩1. LET: $(x_1, y_1), (x_2, y_2) \in X \times [0, 1]$ with $(x_1, y_1) < (x_2, y_2)$
 - ⟨2⟩2. CASE: $x_1 < x_2$
 - ⟨3⟩1. PICK y_3 such that $y_1 < y_3 < 1$
 - ⟨3⟩2. $(x_1, y_1) < (x_1, y_3) < (x_2, y_2)$
 - ⟨2⟩3. CASE: $x_1 = x_2$ and $y_1 < y_2$
 - ⟨3⟩1. PICK y_3 such that $y_1 < y_3 < y_2$
 - ⟨3⟩2. $(x_1, y_1) < (x_1, y_3) < (x_2, y_2)$

□

Lemma 4.2.10. *For all $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$, we have $[a, b) \cong [c, d)$*

PROOF: The map $\lambda t.c + (t - a)(d - c)/(b - a)$ is an order isomorphism.

Proposition 4.2.11. *Let X be a linearly ordered set. Let $a < b < c$ in X . Then $[a, c) \cong [0, 1)$ if and only if $[a, b) \cong [b, c) \cong [0, 1)$.*

PROOF:

- ⟨1⟩1. If $[a, c) \cong [0, 1)$ then $[a, b) \cong [b, c) \cong [0, 1)$
- ⟨2⟩1. ASSUME: $f : [a, c) \cong [0, 1)$ is an order isomorphism
- ⟨2⟩2. $[a, b) \cong [0, 1)$
 - PROOF:
$$\begin{aligned} [a, b) &\cong [0, f(b)) && \text{(by the restriction of } f) \\ &\cong [0, 1) && \text{(Lemma 4.2.10)} \end{aligned}$$
- ⟨2⟩3. $[b, c) \cong [0, 1)$
 - PROOF: Similar.

- ⟨1⟩2. If $[a, b) \cong [b, c) \cong [0, 1)$ then $[a, c) \cong [0, 1)$

PROOF:

$$\begin{aligned} [a, c) &= [a, b) * [b, c) \\ &\cong [0, 1) * [0, 1) \\ &\cong [0, 1/2) * [1/2, 1) && \text{(Lemma 4.2.10)} \\ &= 1 \end{aligned}$$

□

Proposition 4.2.12 (CC). *Let X be a linearly ordered set. Let $x_0 < x_1 < \dots$ be a strictly increasing sequence in X with supremum b . Then $[x_0, b) \cong [0, 1)$ if and only if $[x_i, x_{i+1}) \cong [0, 1)$ for all i .*

PROOF:

- (1)1. If $[x_0, b) \cong [0, 1)$ then $[x_i, x_{i+1}) \cong [0, 1)$ for all i .
 PROOF: By Lemma 4.2.10
 (1)2. If $[x_i, x_{i+1}) \cong [0, 1)$ for all i then $[x_0, b) \cong [0, 1)$
 (2)1. ASSUME: $[x_i, x_{i+1}) \cong [0, 1)$ for all i
 (2)2. PICK an order isomorphism $f_i : [x_i, x_{i+1}) \cong [1/2^i, 2/2^{i+1})$ for each i .
 PROOF: By Lemma 4.2.10
 (2)3. The union of the f_i s is an order isomorphism $[x_0, b) \cong [0, 1)$
 □

4.3 Partially Ordered Sets

Definition 4.3.1 (Partial Order). A *partial order* on a set X is a preorder \leq that is *anti-symmetric*, i.e. whenever $x \leq y$ and $y \leq x$ then $x = y$.

4.4 Strict Partial Order

Definition 4.4.1 (Strict Partial Order). A *strict partial order* on a set X is a relation on X that is transitive and irreflexive.

Proposition 4.4.2. If $<$ is a strict partial order on X and $x, y \in X$, then at most one of $x < y$, $y < x$, $x = y$ holds.

Proposition 4.4.3. If $<$ is a strict partial order then the relation \leq defined by: $x \leq y$ iff $x < y$ or $x = y$, is a partial order.

Theorem 4.4.4. If R is a well-founded relation then its transitive closure is a partial order.

Definition 4.4.5 (Linear Order). A *linear order* on a set X is a partial order such that, for any $x, y \in X$, either $x \leq y$ or $y \leq x$.

4.5 Strict Linear Orders

Definition 4.5.1 (Strict Linear Order (Extensionality, Pairing)). Let A be a set. A *strict linear order* on A is a binary relation R on A that is transitive and satisfies *trichotomy*: for any $x, y \in A$, exactly one of xRy , $x = y$, yRx holds.

Theorem 4.5.2. Let R be a strict linear order on A . Then there is no $x \in A$ such that xRx .

PROOF: Immediate from trichotomy.

4.6 Well Orderings

Definition 4.6.1 (Well-ordering). A *well-order* on a set X is a linear order such that every nonempty set has a least element.

Proposition 4.6.2. *Let \leq be a linear order on X . Then \leq is a well-order iff there is no function $f : \mathbb{N} \rightarrow X$ such that $f(n+1) < f(n)$ for all n .*

Definition 4.6.3 (Initial Segment). Given a well-ordered set X and $\alpha \in X$, the *initial segment* of X up to α is $\text{seg } \alpha = \{x \in X \mid x < \alpha\}$.

Theorem 4.6.4 (Transfinite Induction). *Let \leq be a linear order on J . Then the following are equivalent:*

1. \leq is a well-order on J .
2. For every subset $J_0 \subseteq J$, if the following condition holds:
 - For every $\alpha \in J$, if $\text{seg } \alpha \subseteq J_0$ then $\alpha \in J_0$.
then $J_0 = J$.

Axiom Schema 4.6.5 (Replacement). *Let \mathbf{H} be a class term. If $\text{dom } \mathbf{H}$ is a set then \mathbf{H} is a set.*

Theorem 4.6.6 (Transfinite Recursion). *Let J be a well-ordered set and C a set. Let \mathcal{F} be the set of all functions from a section of J to C . Let G be a function with domain \mathcal{F} . Then there exists a unique function h with domain J such that, for all $\alpha \in J$, we have $h(\alpha) = \rho(h \upharpoonright \text{seg } \alpha)$.*

PROOF:

- $\langle 1 \rangle 1$. If v is a function and $t \in J$, we say v is ρ -constructed up to t iff $\text{dom } v = \{x \in J \mid x \leq t\}$ and, for all $x \in \text{dom } v$, we have $v(x) = \rho(v \upharpoonright \text{seg } x)$
- $\langle 1 \rangle 2$. If $t_1 \leq t_2$, v_1 is ρ -constructed up to t_1 , and v_2 is ρ -constructed up to t_2 , then $v_1 = v_2 \upharpoonright \{x \in J \mid x \leq t_1\}$
- $\langle 1 \rangle 3$. LET: \mathcal{K} be the set of all functions that are ρ -constructed up to some $t \in J$
PROOF: \mathcal{K} is a set by a Replacement Axiom.
- $\langle 1 \rangle 4$. LET: $F = \bigcup \mathcal{K}$
- $\langle 1 \rangle 5$. F is a function
- $\langle 1 \rangle 6$. For all $x \in \text{dom } F$ we have $F(x) = \rho(F \upharpoonright \text{seg } x)$
- $\langle 1 \rangle 7$. $\text{dom } F = J$
- $\langle 1 \rangle 8$. F is unique

□

Theorem 4.6.7. *The following are equivalent.*

1. The Axiom of Choice
2. (Well-Ordering Theorem) Every set has a well-ordering.
3. (Zorn's Lemma) Let X be a poset. If every chain in X has an upper bound in X , then X has a maximal element.

PROOF:

- $\langle 1 \rangle 1$. $1 \Rightarrow 2$

PROOF:

- (2)1. ASSUME: The Axiom of Choice
- (2)2. LET: X be a set.
- (2)3. PICK a choice function for $\mathcal{P}X \setminus \{\emptyset\}$
PROOF: Lemma 3.27.2.
- (2)4. LET: a *tower* in X be a pair $(T, <)$ where $T \subseteq X$, $<$ is a well-ordering of T , and $x = c(X \setminus \{y \in T \mid y < x\})$.
- (2)5. For any two towers $(T_1, <_1)$ and $(T_2, <_2)$, either these two posets are equal or one is a section of the other.
- (3)1.
- (2)6. For any tower $(T, <)$ in X with $T \neq X$, there exists a tower in X of which $(T, <)$ is a section.
- (2)7. LET: $T = \bigcup \{T' \subseteq X \mid \exists R. (T, R) \text{ is a tower in } X\}$
- (2)8. Define $<$ on T by: $x < y$ iff there exists a tower (T, R) in X such that $x, y \in T$ and xRy .
- (2)9. $(T, <)$ is a tower in X .
- (2)10. $T = X$
- (2)11. $<$ is a well-ordering of X .
- (1)2. $2 \Rightarrow 3$
- (2)1. ASSUME: The Well-Ordering Theorem
- (2)2. LET: X be a poset in which every chain has an upper bound.
- (2)3. PICK a well-ordering R of X
- (2)4. Define $F : X \rightarrow \{0, 1\}$ by transfinite R -recursion by:

$$F(a) = \begin{cases} 1 & \text{if } b < a \text{ for all } b \text{ such that } bRa \text{ and } f(b) = 1 \\ 0 & \text{otherwise} \end{cases}$$
- (2)5. LET: $C = \{a \in X \mid f(a) = 1\}$
- (2)6. C is a chain in X
 - (3)1. LET: $x, y \in C$
 - (3)2. ASSUME: without loss of generality xRy
 - (3)3. $f(y) = 1$
 - (3)4. for all z such that zRy and $f(z) = 1$ we have $z < y$
 - (3)5. $x < y$
- (2)7. PICK an upper bound u for C
- (2)8. u is maximal in X
 - (3)1. LET: $x \in X$ with $u \leq x$
 - (3)2. for all b such that bRx and $f(b) = 1$ we have $b < x$
PROOF: Since $b \in C$ so $b \leq u \leq x$
 - (3)3. $f(u) = 1$
 - (3)4. $u \leq x$
 - (3)5. $u = x$
- (2)9. $3 \Rightarrow 1$
 - (3)1. ASSUME: Zorn's Lemma
 - (3)2. LET: R be a relation
 - (3)3. LET: \mathcal{A} be the poset of functions that are subsets of R under \subseteq
 - (3)4. Every chain in \mathcal{A} has an upper bound
 - (4)1. LET: $\mathcal{C} \subseteq \mathcal{A}$ be a chain.

- PROVE: $\bigcup \mathcal{C} \in \mathcal{A}$
- $\langle 4 \rangle 2$. ASSUME: $(x, y), (x, z) \in \bigcup \mathcal{C}$
 - $\langle 4 \rangle 3$. PICK $f, g \in \mathcal{C}$ such that $f(x) = y$ and $g(x) = z$
 - $\langle 4 \rangle 4$. ASSUME: without loss of generality $f \subseteq g$
 - $\langle 4 \rangle 5$. $g(x) = y$
 - $\langle 4 \rangle 6$. $y = z$
 - $\langle 3 \rangle 5$. PICK F maximal in \mathcal{A}
 - $\langle 3 \rangle 6$. $\text{dom } F = \text{dom } R$
 - $\langle 4 \rangle 1$. ASSUME: for a contradiction $x \in \text{dom } R - \text{dom } F$
 - $\langle 4 \rangle 2$. PICK y such that xRy
 - $\langle 4 \rangle 3$. LET: $G = F \cup \{(x, y)\}$
 - $\langle 4 \rangle 4$. $G \in \mathcal{A}$
 - $\langle 4 \rangle 5$. $F \subset G$
 - $\langle 4 \rangle 6$. Q.E.D.
- PROOF: This contradicts the maximality of F .

□

Theorem 4.6.8 (Cardinal Comparability). *The Axiom of Choice is equivalent to the Cardinal Comparability Theorem: for any two sets A and B , either $A \preceq B$ or $B \preceq A$.*

PROOF:

- $\langle 1 \rangle 1$. Zorn's Lemma implies Cardinal Comparability
- $\langle 2 \rangle 1$. ASSUME: Zorn's Lemma
- $\langle 2 \rangle 2$. LET: A and B be sets.
- $\langle 2 \rangle 3$. LET: \mathcal{A} be the poset of all injective functions f such that $\text{dom } f \subseteq C$ and $\text{ran } f \subseteq D$ under \subseteq
- $\langle 2 \rangle 4$. Every chain in \mathcal{A} has an upper bound.
- $\langle 3 \rangle 1$. LET: $\mathcal{C} \subseteq \mathcal{A}$ be a chain.
- PROVE: $\bigcup \mathcal{C} \in \mathcal{A}$
- $\langle 3 \rangle 2$. $\bigcup \mathcal{C}$ is a function.
- $\langle 4 \rangle 1$. LET: $(x, y), (x, z) \in \bigcup \mathcal{C}$
- $\langle 4 \rangle 2$. PICK $f, g \in \mathcal{C}$ such that $f(x) = y$ and $g(x) = z$
- $\langle 4 \rangle 3$. ASSUME: without loss of generality $f \subseteq g$
- $\langle 4 \rangle 4$. $g(x) = y$
- $\langle 4 \rangle 5$. $y = z$
- $\langle 3 \rangle 3$. $\bigcup \mathcal{C}$ is injective.
- PROOF: Similar.
- $\langle 2 \rangle 5$. PICK \hat{f} maximal in \mathcal{A}
- PROOF: By Zorn's Lemma.
- $\langle 2 \rangle 6$. Either $\text{dom } \hat{f} = C$ or $\text{ran } \hat{f} = D$
- $\langle 3 \rangle 1$. ASSUME: for a contradiction $\text{dom } \hat{f} \subset C$ and $\text{ran } \hat{f} \subset D$
- $\langle 3 \rangle 2$. PICK $x \in C - \text{dom } \hat{f}$ and $y \in D - \text{ran } \hat{f}$
- $\langle 3 \rangle 3$. LET: $g = \hat{f} \cup \{(x, y)\}$
- $\langle 3 \rangle 4$. $g \in \mathcal{A}$
- $\langle 3 \rangle 5$. $\hat{f} \subset g$

⟨3⟩6. Q.E.D.

PROOF: This contradicts the maximality of \hat{f} .

⟨2⟩7. If $\text{dom } \hat{f} = C$ then $C \preceq D$

⟨2⟩8. If $\text{ran } \hat{f} = D$ then $D \preceq C$

⟨1⟩2. Cardinal Comparability implies the Well-Ordering Theorem

⟨2⟩1. ASSUME: Cardinal Comparability

⟨2⟩2. LET: A be a set

⟨2⟩3. PICK an ordinal α such that $\alpha \not\preceq A$

⟨2⟩4. $A \preceq \alpha$

PROOF: By Cardinal Comparability.

⟨2⟩5. PICK an injection $f : A \rightarrow \alpha$

⟨2⟩6. Define $<$ on A by $x < y$ iff $f(x) \in f(y)$

⟨2⟩7. $<$ is a well-ordering on A .

□

Theorem 4.6.9. *Given two well-ordered sets A and B , either $A \cong B$ or one of A, B is isomorphic to an initial segment of the other.*

4.7 Ordinal Numbers

Definition 4.7.1. Let (A, \leq) be a well-ordered set. The *ordinal number* of (A, \leq) is the range of E , where E is the unique function with domain A such that $E(t) = \text{ran}(E \upharpoonright \text{seg } t)$ for all $t \in A$.

Theorem 4.7.2. *Let (A, \leq) be a well-ordered set and $E : A \rightarrow \alpha$ be the canonical function onto the ordinal of A . Then:*

1. *For all $t \in A$ we have $E(t) \notin E(t)$.*
2. *E is a bijection.*
3. *For any $s, t \in A$, we have $s < t$ if and only if $E(s) \in E(t)$.*
4. *α is a transitive set.*
5. *α is well-ordered by \in*
6. *E is an order isomorphism between (A, \leq) and (α, \subseteq) .*

Theorem 4.7.3. *Two well-ordered sets are isomorphic if and only if they have the same ordinal number.*

Theorem 4.7.4. *A set is an ordinal number if and only if it is a transitive set well-ordered by \in .*

Theorem 4.7.5. *Every member of an ordinal number is an ordinal number.*

Theorem 4.7.6. *Any transitive set of ordinal numbers is an ordinal number.*

Theorem 4.7.7. *The empty set is an ordinal number.*

Theorem 4.7.8. *The successor of an ordinal number is an ordinal number.*

Theorem 4.7.9. *If A is a set of ordinal numbers then $\bigcup A$ is an ordinal number.*

Theorem 4.7.10. *Any nonempty set of ordinal numbers has a least element.*

Theorem 4.7.11 (Burali-Forti Paradox). *The class of ordinal numbers is a proper class.*

Theorem 4.7.12 (Hartogs' Theorem). *For any set A , there exists an ordinal that is not dominated by A .*

PROOF:

$\langle 1 \rangle 1$. LET: α be the class of all ordinals β such that $\beta \preccurlyeq A$

$\langle 1 \rangle 2$. α is a set.

$\langle 2 \rangle 1$. LET: W be the set of all pairs (B, \leq) such that $B \subseteq A$ and \leq is a well-ordering on B .

$\langle 2 \rangle 2$. Every member of α is the ordinal number of a member of W

$\langle 2 \rangle 3$. Q.E.D.

PROOF: By a Replacement Axiom.

$\langle 1 \rangle 3$. α is an ordinal.

$\langle 1 \rangle 4$. α is not dominated by A .

□

Definition 4.7.13. A class term $\mathbf{F} : \mathbf{On} \rightarrow \mathbf{On}$ is *continuous* iff, for every limit ordinal λ , we have $\mathbf{F}(\lambda) = \sup_{\alpha < \lambda} \mathbf{F}(\alpha)$.

Theorem 4.7.14. *Let $\mathbf{F} : \mathbf{On} \rightarrow \mathbf{On}$. If \mathbf{F} is continuous and $\mathbf{F}(\alpha) < \mathbf{F}(\alpha + 1)$ for every ordinal α , then \mathbf{F} is strictly monotone.*

Definition 4.7.15. A class term $\mathbf{F} : \mathbf{On} \rightarrow \mathbf{On}$ is *normal* iff it is strictly monotone and continuous.

Theorem 4.7.16. *Let $\mathbf{F} : \mathbf{On} \rightarrow \mathbf{On}$ be normal. For every ordinal $\beta \geq \mathbf{F}(0)$, there exists a greatest ordinal α such that $\mathbf{F}(\alpha) \leq \beta$.*

Theorem 4.7.17. *Let $\mathbf{F} : \mathbf{On} \rightarrow \mathbf{On}$ be normal. Let S be a set of ordinals. Then $\mathbf{F}(\sup S) = \sup_{\alpha \in S} \mathbf{F}(\alpha)$.*

Theorem 4.7.18 (Veblen Fixed-Point Theorem). *Let $\mathbf{F} : \mathbf{On} \rightarrow \mathbf{On}$ be normal. For every ordinal α , there exists $\beta \geq \alpha$ such that $\mathbf{F}(\beta) = \beta$.*

PROOF: Let β be the supremum of $\alpha, \mathbf{F}(\alpha), \mathbf{F}^2(\alpha), \dots$ □

Lemma 4.7.19. *Let α be an ordinal. Let $(f(\gamma))_{\gamma < \alpha}$ be an α -sequence of ordinals. Then there exists $\beta \leq \alpha$ and an increasing sequence of ordinals $(g(\gamma))_{\gamma < \beta}$ such that $\sup_{\gamma < \alpha} f(\gamma) = \sup_{\gamma < \beta} g(\gamma)$.*

4.8 Cardinal Numbers

Definition 4.8.1 (Cardinal Number (AC)). For any set A , the *cardinal number* of A , $\text{card } A$, is the least ordinal equinumerous with A .

There exists some ordinal equinumerous with A by the Well-Ordering Theorem.

Theorem 4.8.2. For any sets A and B , we have $A \equiv B$ if and only if $\text{card } A = \text{card } B$.

Theorem 4.8.3. A set A is finite if and only if $\text{card } A$ is a natural number.

Theorem 4.8.4. The supremum of a set of cardinal numbers is a cardinal number.

4.9 Cardinal Arithmetic

Definition 4.9.1. For cardinal numbers κ and λ , the *sum* $\kappa + \lambda$ is the cardinal number of $A \cup B$, where A and B are disjoint sets of cardinality κ and λ respectively.

Theorem 4.9.2. $\kappa + \lambda = \lambda + \kappa$

Theorem 4.9.3. $\kappa + (\lambda + \mu) = (\kappa + \lambda) + \mu$

Theorem 4.9.4. The definition of addition agrees with the definition on natural numbers.

Definition 4.9.5. For cardinal numbers κ and λ , the *product* $\kappa\lambda$ is the cardinality of $\kappa \times \lambda$.

Theorem 4.9.6. $\kappa\lambda = \lambda\kappa$

Theorem 4.9.7. $\kappa(\lambda\mu) = (\kappa\lambda)\mu$

Theorem 4.9.8. $\kappa(\lambda + \mu) = \kappa\lambda + \kappa\mu$

Theorem 4.9.9. The definition of multiplication agrees with the definition on natural numbers.

Theorem 4.9.10 (AC). For any infinite cardinal κ we have $\kappa\kappa = \kappa$.

PROOF:

- $\langle 1 \rangle 1$. LET: B be a set with cardinality κ
- $\langle 1 \rangle 2$. LET: $\mathcal{H} = \{\emptyset\} \cup \{f \mid \exists A \subseteq B. A \text{ is infinite and } f \text{ is a bijection between } A \times A \text{ and } A\}$
- $\langle 1 \rangle 3$. For every chain $\mathcal{C} \subseteq \mathcal{H}$ we have $\bigcup \mathcal{C} \in \mathcal{H}$
- $\langle 1 \rangle 4$. PICK a maximal f_0 in \mathcal{H}
- $\langle 1 \rangle 5$. $f_0 \neq \emptyset$

PROOF: B has a subset of cardinality \aleph_0 and $\aleph_0\aleph_0 = \aleph_0$.

- $\langle 1 \rangle 6$. LET: A_0 be the set such that f_0 is a bijection between $A_0 \times A_0$ and A_0

$\langle 1 \rangle 7.$ LET: $\lambda = \text{card } A_0$

$\langle 1 \rangle 8.$ $\text{card}(B - A_0) < \lambda$

$\langle 1 \rangle 9.$ $\kappa = \lambda$

PROOF:

$$\begin{aligned}
 \kappa &= \text{card } A_0 + \text{card}(B - A_0) \\
 &\leq \lambda + \lambda \\
 &= 2\lambda \\
 &\leq \lambda\lambda \\
 &= \lambda && (\langle 1 \rangle 6) \\
 &\leq \kappa && \square
 \end{aligned}$$

Theorem 4.9.11 (Absorption Law). *Let κ and λ be cardinal numbers such that $0 < \kappa \leq \lambda$ and λ is infinite. Then*

$$\kappa + \lambda = \lambda .$$

Theorem 4.9.12 (Absorption Law). *Let κ and λ be cardinal numbers such that $0 < \kappa \leq \lambda$ and λ is infinite. Then*

$$\kappa\lambda = \lambda .$$

Definition 4.9.13. For cardinal numbers κ and λ , we write κ^λ for the cardinality of the set of functions from λ to κ .

Theorem 4.9.14. $\kappa^{\lambda+\mu} = \kappa^\lambda + \kappa^\mu$

Theorem 4.9.15. $(\kappa\lambda)^\mu = \kappa^\mu \lambda^\mu$

Theorem 4.9.16. $(\kappa^\lambda)^\mu = \kappa^{\lambda\mu}$

Theorem 4.9.17. *The definition of exponentiation agrees with the definition on natural numbers.*

Theorem 4.9.18. *Given sets A and B , we have $\text{card } A \leq \text{card } B$ if and only if $A \preceq B$.*

Definition 4.9.19. Let $\aleph_0 = \text{card } \mathbb{N}$.

Theorem 4.9.20 (AC). *For any infinite cardinal κ we have $\aleph_0 \leq \kappa$.*

Theorem 4.9.21 (Maximum Principle (AC)). *Every poset has a maximal chain.*

4.10 Rank of a Set

Definition 4.10.1 (Cumulative Hierarchy of Sets). For every ordinal α , define the rank V_α by transfinite recursion thus:

$$\begin{aligned}
 V_0 &= \emptyset \\
 V_{\alpha+1} &= \mathcal{P}V_\alpha \\
 V_\lambda &= \bigcup_{\alpha < \lambda} V_\alpha
 \end{aligned}$$

for λ a limit ordinal.

The *von Neumann universe* is the class $\mathbf{V} = \bigcup_{\alpha \in \mathbf{On}} V_\alpha$.

Theorem 4.10.2. *If λ is a limit ordinal and $\lambda > \omega$ then V_λ is a model of Zermelo set theory.*

Lemma 4.10.3 (AC). *There exists a well-ordered set in V_{ω_2} whose ordinal is not in V_{ω_2} .*

PROOF: Pick a well-ordering $<$ of \mathcal{PN} . Then $(\mathcal{PN}, <) \in V_{\omega_2}$ but its ordinal is not because its ordinal is uncountable. \square

Theorem 4.10.4. *The set V_{ω_2} is not a model of Zermelo-Fraenkel set theory.*

Thus, the Replacement Axioms cannot be proven from the other axioms.

Definition 4.10.5 (Well-Founded Set). A set A is *well-founded* iff $A \in V_\alpha$ for some $\alpha \in \mathbf{On}$.

Definition 4.10.6 (Rank). The *rank* of a well-founded set A , $\text{rank } A$, is the least ordinal α such that $A \in V_\alpha$.

Theorem 4.10.7. *If $A \in B$ and B is well-founded then A is well-founded and $\text{rank } A < \text{rank } B$.*

Theorem 4.10.8. *If A is a set and every member of A is well-founded then A is well-founded and $\text{rank } A = \sup_{B \in A} (\text{rank } B + 1)$.*

Theorem 4.10.9. *The Axiom of Regularity is equivalent to the statement that every set is well-founded.*

4.11 Transfinite Recursion Again

Theorem 4.11.1. *Let \mathbf{A} be a class. Let \mathbf{B} be the class of all functions $f : \alpha \rightarrow \mathbf{A}$ for some ordinal α . Let $\mathbf{F} : \mathbf{B} \rightarrow \mathbf{A}$ be a class term. Then there exists a unique class term $\mathbf{G} : \mathbf{On} \rightarrow \mathbf{A}$ such that, for all $\alpha \in \mathbf{On}$, we have $\mathbf{G}(\alpha) = \mathbf{F}(\mathbf{G} \upharpoonright \alpha)$.*

4.12 Alephs

Definition 4.12.1. Define the cardinal number \aleph_α for every ordinal α by transfinite recursion on α thus: \aleph_α is the least infinite cardinal different from \aleph_β for all $\beta < \alpha$.

Theorem 4.12.2. *If $\alpha < \beta$ then $\aleph_\alpha < \aleph_\beta$.*

Theorem 4.12.3. *Every infinite cardinal has the form \aleph_α for some ordinal α .*

4.13 Ordinal Arithmetic

Definition 4.13.1 (Sum). Let α and β be ordinals. The *sum* $\alpha + \beta$ is the ordinal of the concatenation of A followed by B , where A is a well-ordered set of ordinal α and B a well-ordered set of ordinal β .

Theorem 4.13.2. *Addition is associative.*

Theorem 4.13.3. $\alpha + 0 = \alpha$

Theorem 4.13.4. $0 + \alpha = \alpha$

Theorem 4.13.5. *For λ a limit ordinal we have $\alpha + \lambda = \sup_{\beta < \lambda} (\alpha + \beta)$*

Theorem 4.13.6. *For any α , the class term that maps β to $\alpha + \beta$ is normal.*

Theorem 4.13.7. $\beta < \gamma$ iff $\alpha + \beta < \alpha + \gamma$.

Theorem 4.13.8. *If $\beta \leq \gamma$ then $\beta + \alpha \leq \gamma + \alpha$.*

Theorem 4.13.9 (Subtraction Theorem). *If $\alpha < \beta$ then there exists a unique δ such that $\alpha + \delta < \beta$.*

Definition 4.13.10 (Product). Let α and β be ordinals. The *sum* $\alpha \times \beta$ is the ordinal of $A \times B$ ordered under the Hebrew lexicographic order, where A is a well-ordered set of ordinal α and B a well-ordered set of ordinal β .

Theorem 4.13.11. *Multiplication is associative.*

Theorem 4.13.12. *Multiplication distributes over addition on the left.*

Theorem 4.13.13. $\alpha 1 = \alpha$

Theorem 4.13.14. $1\alpha = \alpha$

Theorem 4.13.15. $\alpha 0 = 0$

Theorem 4.13.16. $0\alpha = 0$

Theorem 4.13.17. *For λ a limit ordinal, we have $\alpha\lambda = \sup_{\beta < \lambda} (\alpha\beta)$.*

Theorem 4.13.18. *For $\alpha > 0$, the class term that maps β to $\alpha\beta$ is normal.*

Theorem 4.13.19. *If $\alpha > 0$, then $\beta < \gamma$ iff $\alpha\beta < \alpha\gamma$.*

Theorem 4.13.20. *If $\beta \leq \gamma$ then $\beta\alpha \leq \gamma\alpha$.*

Theorem 4.13.21 (Division Theorem). *For any ordinals α and δ with $\delta \neq 0$, there exist unique ordinals β and γ with $\gamma < \delta$ and $\alpha = \delta\beta + \gamma$.*

Definition 4.13.22 (Exponentiation). For ordinals α and β , define the ordinal α^β by transfinite recursion on β by:

$$\begin{aligned}\alpha^0 &= 1 \\ \alpha^{\beta+1} &= \alpha^\beta + \alpha \\ \alpha^\lambda &= \sup_{\beta < \lambda} \alpha^\beta\end{aligned}$$

for λ a limit ordinal.

Theorem 4.13.23. For $\alpha > 1$, the class term that maps β to α^β is normal.

Theorem 4.13.24. If $\alpha > 1$, then $\beta < \gamma$ iff $\alpha^\beta < \alpha^\gamma$.

Theorem 4.13.25. If $\beta \leq \gamma$ then $\beta^\alpha \leq \gamma^\alpha$.

Theorem 4.13.26 (Logarithm Theorem). Let α and β be ordinals with $\alpha \neq 0$ and $\beta > 1$. Then there exist unique ordinals γ , δ and ρ such that $\delta \neq 0$, $\delta < \beta$, $\rho < \beta^\gamma$, and $\alpha = \beta^\gamma \delta + \rho$.

Theorem 4.13.27.

$$\alpha^{\beta+\gamma} = \alpha^\beta \alpha^\gamma$$

Theorem 4.13.28.

$$(\alpha^\beta)^\gamma = \alpha^{\beta\gamma}$$

4.14 Beth Cardinals

Definition 4.14.1. Define the cardinal \beth_α for every ordinal α by:

$$\begin{aligned}\beth_0 &= \aleph_0 \\ \beth_{\alpha+1} &= 2^{\beth_\alpha} \\ \beth_\lambda &= \sup_{\alpha < \lambda} \beth_\alpha\end{aligned}$$

for λ a limit ordinal.

Lemma 4.14.2. For any ordinal α we have $\text{card } V_{\omega+\alpha} = \beth_\alpha$.

4.15 Cofinality

Definition 4.15.1 (Cofinality). For λ a limit ordinal, the *cofinality* of λ , $\text{cf } \lambda$, is the least cardinal κ such that λ is the supremum of a set of κ smaller ordinals.

We extend cf to all the ordinals by setting $\text{cf } 0 = 0$ and $\text{cf}(\alpha + 1) = 1$.

Theorem 4.15.2. For any limit ordinal λ we have $\text{cf } \aleph_\lambda = \text{cf } \lambda$.

Lemma 4.15.3. Let λ be a limit ordinal. Then $\text{cf } \lambda$ is the least ordinal α such that there exists an increasing α -sequence of ordinals with limit λ .

Theorem 4.15.4. Let λ be an infinite cardinal. Then $\text{cf } \lambda$ is the least cardinal number κ such that λ can be partitioned into κ sets each of cardinality $< \lambda$.

Theorem 4.15.5 (König's Theorem). Let κ be an infinite cardinal. Then $\kappa < 2^{\text{cf } \kappa}$.

Corollary 4.15.5.1. $2^{\aleph_0} \neq \aleph_\omega$.

Definition 4.15.6 (Regular). A cardinal κ is *regular* iff $\text{cf } \kappa = \kappa$.

Theorem 4.15.7. For any ordinal λ , we have $\text{cf } \lambda$ is a regular cardinal.

Definition 4.15.8 (Singular). A cardinal κ is *singular* iff $\text{cf } \kappa < \kappa$.

Theorem 4.15.9. For any ordinal α we have $\aleph_{\alpha+1}$ is a regular cardinal.

4.16 Inaccessible Cardinals

Definition 4.16.1 (Inaccessible). A cardinal number κ is *inaccessible* iff:

- $\kappa > \aleph_0$
- For every cardinal $\lambda < \kappa$ we have $2^\lambda < \kappa$
- κ is regular.

Lemma 4.16.2. *If κ is inaccessible and $\alpha < \kappa$ then $\beth_\alpha < \kappa$.*

Lemma 4.16.3. *If κ is inaccessible and $A \in V_\kappa$ then $\text{card } A < \kappa$.*

Theorem 4.16.4. *If κ is inaccessible then V_κ is a model of ZF.*

Chapter 5

Natural Numbers

5.1 Successors

Definition 5.1.1 (Successor (Pairing, Union)). For any set a , its *Successor* a^+ is the set $a \cup \{a\}$

Theorem 5.1.2 (Pairing, Union). *If a is a transitive set then $\bigcup(a^+) = a$.*

PROOF:

$$\begin{aligned}\bigcup(a^+) &= \bigcup(a \cup \{a\}) \\ &= \bigcup a \cup \bigcup \{a\} \\ &= \bigcup a \cup a \\ &= a \qquad \qquad \qquad (\bigcup a \subseteq a) \square\end{aligned}$$

Theorem 5.1.3. *If A is a transitive set then A^+ is transitive.*

PROOF: If A is transitive then $\bigcup(A^+) = A \subseteq A^+$. \square

5.2 Inductive Sets

Definition 5.2.1 (Inductive (Extensionality, Empty Set, Pairing, Union)). A set A is *inductive* iff $\emptyset \in A$ and, for every $a \in A$, we have $a^+ \in A$.

Axiom 5.2.2 (Axiom of Infinity (Extensionality, Empty Set, Pairing, Union)). *There exists an inductive set.*

5.3 Natural Numbers

Definition 5.3.1 (Natural Number (Extensionality, Empty Set, Pairing, Union)). A *natural number* is a set that belongs to every inductive set.

We write \mathbb{N} for the class of all natural numbers.

Theorem 5.3.2 (Extensionality, Empty Set, Pairing, Union, Infinity, Subset).
The class of natural numbers is a set.

PROOF:

⟨1⟩1. PICK an inductive set I .

PROOF: By the Axiom of Infinity.

⟨1⟩2. $\mathbb{N} \subseteq I$

⟨1⟩3. Q.E.D.

PROOF: By a Subset Axiom.

□

Theorem 5.3.3 (Extensionality, Empty Set, Pairing, Union, Infinity, Subset).
The set \mathbb{N} is inductive.

PROOF:

⟨1⟩1. $\emptyset \in \mathbb{N}$

PROOF: Since \emptyset is a member of every inductive set.

⟨1⟩2. For all $n \in \mathbb{N}$ we have $n^+ \in \mathbb{N}$

PROOF: If n is a member of every inductive set then so is n^+ .

□

Theorem 5.3.4 (Extensionality, Empty Set, Pairing, Union, Infinity, Subset).
The set \mathbb{N} is a subset of every inductive set.

PROOF: Immediate from definition. □

Corollary 5.3.4.1 (Proof by Induction (Extensionality, Empty Set, Pairing, Union, Infinity, Subset)). *If $A \subseteq \mathbb{N}$ and A is inductive then $A = \mathbb{N}$.*

Definition 5.3.5 (Zero (Empty Set)). The natural number *zero*, 0 , is defined to be \emptyset .

Theorem 5.3.6 (Extensionality, Empty Set, Pairing, Union, Infinity, Subset).
Every natural number except 0 is a successor of a natural number.

PROOF: The set $\{x \in \mathbb{N} \mid x = 0 \vee \exists y \in \mathbb{N}. x = y^+\}$ is inductive. □

Theorem 5.3.7 (Extensionality, Empty Set, Pairing, Union, Infinity, Subset).
Every natural number is transitive.

PROOF: By induction using Theorem 5.1.3. □

Theorem 5.3.8 (Extensionality, Empty Set, Pairing, Union, Infinity, Subset).
The set \mathbb{N} is transitive.

PROOF:

⟨1⟩1. For every natural number n and every $m \in n$ then m is a natural number.

⟨2⟩1. Every member of \emptyset is a natural number.

PROOF: Vacuous.

⟨2⟩2. If n is a natural number and a set of natural numbers then n^+ is a set of natural numbers.

PROOF: From the definition of n^+ .

⟨2⟩3. Q.E.D.

PROOF: By induction.

□

Theorem 5.3.9 (Extensionality, Empty Set, Pairing, Union, Infinity, Subset).
Let A be a set, $a \in A$, and $F : A \rightarrow A$. Then there exists a unique function $h : \mathbb{N} \rightarrow A$ such that $h(0) = a$ and, for all $n \in \mathbb{N}$, we have $h(n^+) = F(h(n))$.

PROOF:

⟨1⟩1. Call a function v *acceptable* iff $\text{dom } v \subseteq \mathbb{N}$, $\text{ran } v \subseteq A$, and:

1. If $0 \in \text{dom } v$ then $v(0) = a$.

2. For all $n \in \mathbb{N}$, if $n^+ \in \text{dom } v$ then $n \in \text{dom } v$ and $v(n^+) = F(v(n))$.

⟨1⟩2. LET: \mathcal{K} be the set of all acceptable functions.

⟨1⟩3. LET: $h = \bigcup \mathcal{K}$

⟨1⟩4. h is a function.

⟨2⟩1. If $(0, y) \in h$ and $(0, y') \in h$ then $y = y'$

PROOF: We have $y = y' = a$.

⟨2⟩2. For any natural number n , if there is at most one y such that $(n, y) \in h$, then there is at most one y such that $(n^+, y) \in h$

⟨3⟩1. LET: n be a natural number.

⟨3⟩2. ASSUME: there is at most one y such that $(n, y) \in h$

⟨3⟩3. ASSUME: (n^+, y) and (n^+, y') are in h

⟨3⟩4. PICK acceptable functions u and v such that $u(n^+) = y$ and $v(n^+) = y'$

⟨3⟩5. $n \in \text{dom } u$, $n \in \text{dom } v$ and $y = F(u(n))$, $y' = F(v(n))$

⟨3⟩6. $u(n) = v(n)$

PROOF: By the induction hypothesis ⟨3⟩2

⟨3⟩7. $y = y'$

⟨2⟩3. Q.E.D.

PROOF: By induction.

⟨1⟩5. h is acceptable.

⟨2⟩1. If $0 \in \text{dom } h$ then $h(0) = a$

⟨2⟩2. If $n^+ \in \text{dom } h$ then $n \in \text{dom } h$ and $h(n^+) = F(h(n))$

⟨3⟩1. ASSUME: $n^+ \in \text{dom } h$

⟨3⟩2. PICK an acceptable v such that $n^+ \in \text{dom } v$

⟨3⟩3. $v(n^+) = F(v(n))$

⟨3⟩4. $h(n^+) = F(h(n))$

⟨1⟩6. $\text{dom } h = \mathbb{N}$

⟨2⟩1. $0 \in \text{dom } h$

PROOF: Since $\{(0, a)\}$ is an acceptable function.

⟨2⟩2. For all $n \in \text{dom } h$ we have $n^+ \in \text{dom } h$

⟨3⟩1. ASSUME: $n \in \text{dom } h$

⟨3⟩2. LET: v be an acceptable function with $n \in \text{dom } v$

⟨3⟩3. ASSUME: without loss of generality $n^+ \notin \text{dom } v$

⟨3⟩4. $v \cup \{(n^+, F(v(n)))\}$ is acceptable

$\langle 3 \rangle 5. \quad n^+ \in \text{dom } v$
 $\langle 1 \rangle 7. \quad \text{If } h' : \mathbb{N} \rightarrow A, h'(0) = a \text{ and, for all } n \in \mathbb{N}, \text{ we have } h'(n^+) = F(h'(n)),$
 $\text{then } h' = h$
 PROOF: Prove $h(n) = h'(n)$ by induction on n .
 \square

5.4 Peano Systems

Definition 5.4.1 (Peano System). A *Peano system* consists of a set N , an element $z \in N$, and a function $S : N \rightarrow N$ such that:

- S is one-to-one
- $z \notin \text{ran } S$
- For any set $A \subseteq N$, if $z \in A$ and $S(A) \subseteq A$ then $A = N$.

Theorem 5.4.2. \mathbb{N} is a Peano system with zero 0 and successor $n \mapsto n^+$.

Theorem 5.4.3. For any Peano system (N, z, S) , there exists a unique bijection $h : \mathbb{N} \cong N$ such that $h(0) = z$ and $S(h(n)) = h(n^+)$ for all n .

5.5 Arithmetic

Definition 5.5.1 (Addition). Define *addition* $+ : \mathbb{N}^2 \rightarrow \mathbb{N}$ recursively by

$$\begin{aligned}
 m + 0 &= m \\
 m + n^+ &= (m + n)^+
 \end{aligned}$$

for any $m, n \in \mathbb{N}$.

Theorem 5.5.2. *Addition is associative.*

Theorem 5.5.3. *Addition is commutative*

Definition 5.5.4 (Multiplication). Define *multiplication* $\cdot : \mathbb{N}^2 \rightarrow \mathbb{N}$ recursively by

$$\begin{aligned}
 m0 &= 0 \\
 mn^+ &= mn + m
 \end{aligned}$$

for any $m, n \in \mathbb{N}$

Theorem 5.5.5. *Multiplication is associative.*

Theorem 5.5.6. *Multiplication is commutative.*

Theorem 5.5.7. *Multiplication distributes over addition.*

Definition 5.5.8. For natural numbers m and n , we write $m < n$ iff $m \in n$. We write $m \leq n$ iff $m < n$ or $m = n$.

Theorem 5.5.9. We have $m < n$ iff $m^+ < n^+$.

Theorem 5.5.10. We never have $n < n$.

Theorem 5.5.11. The ordering on \mathbb{N} satisfies trichotomy; that is, for any m, n , exactly one of $m < n$, $m = n$, $n < m$ holds.

Theorem 5.5.12. For any natural numbers m and n , we have $m \leq n$ iff $m \subseteq n$.

Theorem 5.5.13. We have $m < n$ iff $m + p < n + p$.

Corollary 5.5.13.1. If $m + p = n + p$ then $m = n$.

Theorem 5.5.14. If $p \neq 0$ then $m < n$ iff $mp < np$.

Corollary 5.5.14.1. If $mp = np$ and $p \neq 0$ then $m = n$.

Theorem 5.5.15 (Well-Ordering of \mathbb{N}). Any nonempty set $A \subseteq \mathbb{N}$ has a least element.

Corollary 5.5.15.1. There is no function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that $f(n^+) < f(n)$ for all n .

Theorem 5.5.16 (Strong Induction). Let $A \subseteq \mathbb{N}$. Suppose that, for every natural number n , if $\forall m < n. m \in A$ then $n \in A$. Then $A = \mathbb{N}$.

Theorem 5.5.17 (Pigeonhole Principle). No natural number is equinumerous with a proper subset of itself.

PROOF: Prove by induction on n that if $f : n \rightarrow n$ is injective then it is surjective.
 \square

Chapter 6

Integers

Lemma 6.0.1. Define \sim on \mathbb{N}^2 by: $(m, n) \sim (p, q)$ iff $m + q = n + p$. Then \sim is an equivalence relation on \mathbb{N}^2 .

Definition 6.0.2 (Integers). The set \mathbb{Z} of *integers* is \mathbb{N}^2 / \sim .

Definition 6.0.3. Define *addition* $+: \mathbb{Z}^2 \rightarrow \mathbb{Z}$ by: $(m, n) + (p, q) = (m + p, n + q)$.

Prove this is well-defined.

Theorem 6.0.4. *Addition is associative and commutative.*

Definition 6.0.5 (Zero). The integer *zero* is $0 = (0, 0)$.

Theorem 6.0.6. *For any integer a , we have $a + 0 = a$.*

Theorem 6.0.7. *For any integer a , there exists a unique integer b such that $a + b = 0$.*

Definition 6.0.8 (Multiplication). Define multiplication on \mathbb{Z} by $(m, n)(p, q) = (mp + nq, mq + np)$.

Theorem 6.0.9. *Multiplication is associative, commutative and distributive over addition.*

Definition 6.0.10. The integer *one* is $1 = (1, 0)$.

Theorem 6.0.11. *For any integer a we have $a1 = a$.*

Theorem 6.0.12. $1 \neq 0$

Theorem 6.0.13. *Whenever $ab = 0$ then either $a = 0$ or $b = 0$.*

Definition 6.0.14. Define $<$ on \mathbb{Z} by: $(m, n) < (p, q)$ iff $m + q < n + p$.

Theorem 6.0.15. *The relation $<$ is a strict linear ordering on \mathbb{Z} .*

Theorem 6.0.16. *We have $a < b$ iff $+c < b + c$.*

Corollary 6.0.16.1. *If $a + c = b + c$ then $a = b$.*

Theorem 6.0.17. *If $0 < c$ then $a < b$ iff $ac < bc$.*

Corollary 6.0.17.1. *If $ac = bc$ and $c \neq 0$ then $a = b$.*

Definition 6.0.18. We identify any natural number n with the integer $(n, 0)$.

Theorem 6.0.19. *This embedding preserves 0, 1, addition, multiplication and the ordering.*

Chapter 7

Rational Numbers

Definition 7.0.1 (Rational Numbers). The set of *rational numbers* \mathbb{Q} is $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) / \sim$, where $(a, b) \sim (c, d)$ iff $ad = bc$.

Definition 7.0.2 (Addition). Define addition on \mathbb{Q} by: $(a, b) + (c, d) = (ad + bc, bd)$.

Theorem 7.0.3. *Addition is commutative and associative*

Definition 7.0.4. The rational number 0 is $(0, 1)$.

Theorem 7.0.5. *For any rational q we have $q + 0 = q$.*

Theorem 7.0.6. *For any rational q , there exists a unique rational r such that $q + r = 0$.*

Definition 7.0.7. Define multiplication on \mathbb{Q} by: $(a, b)(c, d) = (ac, bd)$.

Theorem 7.0.8. *Multiplication is commutative, associative and distributive over addition.*

Definition 7.0.9. The rational number 1 is $(1, 1)$.

Theorem 7.0.10. *For every nonzero rational r , there exists a nonzero rational q such that $rq = 1$.*

Corollary 7.0.10.1. *If $qr = 0$ then either $q = 0$ or $r = 0$.*

Definition 7.0.11. Define $<$ on \mathbb{Q} by: for b and d positive, $(a, b) < (c, d)$ iff $ad < bc$.

Theorem 7.0.12. *The relation $<$ is a strict linear ordering on \mathbb{Q} .*

Theorem 7.0.13. *We have $q < r$ iff $q + s < r + s$*

Corollary 7.0.13.1. *If $q + s = r + s$ then $q = r$.*

Theorem 7.0.14. *If $s > 0$ then we have $q < r$ iff $qs < rs$.*

Corollary 7.0.14.1. *If $qs = rs$ and $s \neq 0$ then $q = r$.*

Definition 7.0.15. We identify an integer n with the rational $(n, 1)$.

Theorem 7.0.16. *This embedding preserves zero, one, addition, multiplication and the ordering.*

Chapter 8

Real Numbers

Definition 8.0.1 (Dedekind Cut). A *Dedekind cut* is a subset $X \subseteq \mathbb{Q}$ such that:

- X is nonempty
- $X \neq \mathbb{Q}$
- X is closed downward
- X has no largest element.

Definition 8.0.2 (Real Numbers). The set of *real numbers* \mathbb{R} is the set of all Dedekind cuts.

Definition 8.0.3. Define $<$ on \mathbb{R} by: $x < y$ iff x is a proper subset of y .

Theorem 8.0.4. *The relation $<$ is a strict linear ordering on \mathbb{R} .*

Theorem 8.0.5. *Any bounded nonempty subset of \mathbb{R} has a least upper bound.*

Definition 8.0.6. Define addition on \mathbb{R} by: $x + y = \{q + r \mid q \in x, r \in y\}$.

Theorem 8.0.7. *Addition is associative and commutative.*

Definition 8.0.8. The zero real 0 is $\{q \in \mathbb{Q} \mid q < 0\}$.

Theorem 8.0.9. *For any $x \in \mathbb{R}$ we have $x + 0 = x$.*

Definition 8.0.10. Given a real x , define $-x = \{q \in \mathbb{Q} \mid \exists r > q. -r \notin x\}$.

Theorem 8.0.11. *For any real x we have $x + (-x) = 0$.*

Corollary 8.0.11.1. *If $x + z = y + z$ then $x = y$.*

Theorem 8.0.12. *We have $x < y$ iff $x + z < y + z$.*

Definition 8.0.13. Define the *absolute value* of a real x by $|x| = x \cup -x$.

Theorem 8.0.14. *For any real x we have $0 \leq |x|$.*

Definition 8.0.15. Define multiplication on \mathbb{R} by:

- If x and y are nonnegative then

$$xy = 0 \cup \{qr \mid 0 \leq q, 0 \leq r, q \in x, r \in y\}$$

- If x and y are both negative then $xy = |x||y|$
- If one of x and y is negative and the other not then $xy = -|x||y|$.

Theorem 8.0.16. *Multiplication is associative, commutative and distributive over addition.*

Definition 8.0.17. The real number 1 is $\{q \in \mathbb{Q} \mid q < 1\}$.

Theorem 8.0.18. $0 \neq 1$

Theorem 8.0.19. *For any real x we have $x1 = x$*

Theorem 8.0.20. *For any nonzero x , there exists a real y with $xy = 1$.*

Theorem 8.0.21. *If $0 < x$ then $y < z$ iff $xy < xz$.*

Definition 8.0.22. Identify a rational q with $\{r \in \mathbb{Q} \mid r < q\}$.

Theorem 8.0.23. *This embedding preserves zero, one, addition, multiplication and the ordering.*

8.1 The Cantor Set

Definition 8.1.1 (Cantor Set). Define the sequence of sets $A_n \subseteq \mathbb{R}$ by

$$A_0 = [0, 1]$$

$$A_n = A_{n-1} - \bigcup_{k=0}^{3^{n-1}-1} ((3k+1)/3^n, (3k+2)/3^n)$$

The Cantor set is $\bigcap_{n=0}^{\infty} A_n$.

Proposition 8.1.2. *The set A_n is a union of finitely many disjoint closed intervals of length $1/3^n$, and the endpoints of these intervals lie in C .*

PROOF: An easy induction on n . \square

Chapter 9

Finite Sets

Definition 9.0.1 (Finite). A set is *finite* iff it is equinumerous with a natural number; otherwise it is *infinite*.

Theorem 9.0.2. *No finite set is equinumerous with a proper subset of itself.*

PROOF: From the Pigeonhole Principle.

Corollary 9.0.2.1. *The set \mathbb{N} is infinite.*

Corollary 9.0.2.2. *A finite set is equinumerous with a unique natural number.*

Lemma 9.0.3. *If A is a proper subset of a natural number n then there exists $m < n$ such that $C \equiv m$.*

Corollary 9.0.3.1. *A subset of a finite set is finite.*

Theorem 9.0.4 (Regularity). *There is no function f with domain \mathbb{N} such that $f(n+1) \in f(n)$ for all n .*

PROOF:

$\langle 1 \rangle 1$. ASSUME: for a contradiction f is a function with domain \mathbb{N} such that $f(n+1) \in f(n)$ for all n .

$\langle 1 \rangle 2$. PICK $m \in \text{ran } f$ such that $m \cap \text{ran } f = \emptyset$

PROOF: By the Axiom of Regularity.

$\langle 1 \rangle 3$. PICK $n \in \mathbb{N}$ such that $f(n) = m$

$\langle 1 \rangle 4$. $f(n+1) \in m \cap \text{ran } f$

$\langle 1 \rangle 5$. Q.E.D.

PROOF: This is a contradiction.

□

Theorem 9.0.5. *A relation R is well-founded if and only if there is no function f with domain \mathbb{N} such that, for all $n \in \mathbb{N}$, we have $f(n+1)Rf(n)$.*

9.1 Real Analysis

Definition 9.1.1. Let \mathbb{R}^∞ be the subset of \mathbb{R}^ω consisting of all sequences (a_n) such that $a_n = 0$ for all but finitely many n .

9.2 Group Theory

Definition 9.2.1. Given a group G and sets $A, B \subseteq G$, let $AB = \{ab \mid a \in A, b \in B\}$.

Definition 9.2.2. Given a group G and a set $A \subseteq G$, let $A^{-1} = \{a^{-1} \mid a \in A\}$.

9.3 Topological Spaces

Definition 9.3.1 (Topology). A *topology* on a set X is a set $\mathcal{T} \subseteq \mathcal{P}X$ such that:

- $X \in \mathcal{T}$.
- For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$.
- For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$.

We call the elements of X *points* and the elements of \mathcal{T} *open sets*.

Definition 9.3.2 (Topological Space). A *topological space* X consists of a set X and a topology on X .

Definition 9.3.3 (Discrete Space). For any set X , the *discrete* topology on X is $\mathcal{P}X$.

Definition 9.3.4 (Indiscrete Space). For any set X , the *indiscrete* or *trivial* topology on X is $\{\emptyset, X\}$.

Definition 9.3.5 (Finite Complement Topology). For any set X , the *finite complement topology* on X is $\{U \in \mathcal{P}X \mid X \setminus U \text{ is finite}\} \cup \{\emptyset\}$.

Definition 9.3.6 (Countable Complement Topology). For any set X , the *countable complement topology* on X is $\{U \in \mathcal{P}X \mid X \setminus U \text{ is countable}\} \cup \{\emptyset\}$.

Definition 9.3.7 (Finer, Coarser). Suppose that \mathcal{T} and \mathcal{T}' are two topologies on a given set X . If $\mathcal{T}' \supseteq \mathcal{T}$, we say that \mathcal{T}' is *finer* than \mathcal{T} ; if \mathcal{T}' *properly* contains \mathcal{T} , we say that \mathcal{T}' is *strictly finer* than \mathcal{T} . We also say that \mathcal{T} is *coarser* than \mathcal{T}' , or *strictly coarser*, in these two respective situations. We say \mathcal{T} is *comparable* with \mathcal{T}' if either $\mathcal{T}' \supseteq \mathcal{T}$ or $\mathcal{T} \supseteq \mathcal{T}'$.

Lemma 9.3.8. *Let X be a topological space and $U \subseteq X$. Then U is open if and only if, for all $x \in U$, there exists an open set V such that $x \in V \subseteq U$.*

PROOF:

$\langle 1 \rangle 1. \Rightarrow$

PROOF: Take $V = U$

$\langle 1 \rangle 2. \Leftarrow$

PROOF: We have $U = \bigcup \{V \text{ open in } X \mid V \subseteq U\}$.

□

Lemma 9.3.9. *Let X be a set and \mathcal{T} a nonempty set of topologies on X . Then $\bigcap \mathcal{T}$ is a topology on X , and is the finest topology that is coarser than every member of \mathcal{T} .*

PROOF:

$\langle 1 \rangle 1. X \in \bigcap \mathcal{T}$

PROOF: Since X is in every member of \mathcal{T} .

- ⟨1⟩2. $\bigcap \mathcal{T}$ is closed under union.
 - ⟨2⟩1. LET: $\mathcal{U} \subseteq \bigcap \mathcal{T}$
 - ⟨2⟩2. For all $T \in \mathcal{T}$ we have $\mathcal{U} \subseteq T$
 - ⟨2⟩3. For all $T \in \mathcal{T}$ we have $\bigcup \mathcal{U} \in T$
 - ⟨2⟩4. $\bigcup \mathcal{U} \in \bigcap \mathcal{T}$
- ⟨1⟩3. $\bigcap \mathcal{T}$ is closed under binary intersection.
 - ⟨2⟩1. LET: $U, V \in \bigcap \mathcal{T}$
 - ⟨2⟩2. For all $T \in \mathcal{T}$ we have $U, V \in T$
 - ⟨2⟩3. For all $T \in \mathcal{T}$ we have $U \cap V \in T$
 - ⟨2⟩4. $U \cap V \in \bigcap \mathcal{T}$

□

Lemma 9.3.10. *Let X be a set and \mathcal{T} a set of topologies on X . Then there exists a unique coarsest topology that is finer than every member of \mathcal{T} .*

PROOF: The required topology is given by

$$\bigcap \{T \in \mathcal{P}X \mid T \text{ is a topology on } X \text{ that is finer than every member of } \mathcal{T}\},$$

The set is nonempty since it contains the discrete topology. □

Definition 9.3.11 (Neighbourhood). A *neighbourhood* of a point x is an open set that contains x .

9.4 Closed Set

Definition 9.4.1 (Closed Set). Let X be a topological space and $A \subseteq X$. Then A is *closed* if and only if $X \setminus A$ is open.

Lemma 9.4.2. *The empty set is closed.*

PROOF: Since the whole space X is always open. □

Lemma 9.4.3. *The topological space X is closed.*

PROOF: Since \emptyset is open. □

Lemma 9.4.4. *The intersection of a nonempty set of closed sets is closed.*

PROOF: Let \mathcal{C} be a nonempty set of closed sets. Then $X \setminus \bigcap \mathcal{C} = \bigcup \{X \setminus C \mid C \in \mathcal{C}\}$ is open. □

Lemma 9.4.5. *The union of two closed sets is closed.*

PROOF: Let C and D be closed. Then $X \setminus (C \cup D) = (X \setminus C) \cap (X \setminus D)$ is open. □

Proposition 9.4.6. *Let X be a set and $\mathcal{C} \subseteq \mathcal{P}X$ a set such that:*

1. $\emptyset \in \mathcal{C}$
2. $X \in \mathcal{C}$

3. For all $\mathcal{A} \subseteq \mathcal{C}$ nonempty we have $\bigcap \mathcal{A} \in \mathcal{C}$

4. For all $C, D \in \mathcal{C}$ we have $C \cup D \in \mathcal{C}$.

Then there exists a unique topology \mathcal{T} such that \mathcal{C} is the set of closed sets, namely

$$\mathcal{T} = \{X \setminus C \mid C \in \mathcal{C}\}$$

PROOF:

$\langle 1 \rangle 1$. LET: $\mathcal{T} = \{X \setminus C \mid C \in \mathcal{C}\}$

$\langle 1 \rangle 2$. \mathcal{T} is a topology

$\langle 2 \rangle 1$. $X \in \mathcal{T}$

PROOF: Since $\emptyset \in \mathcal{C}$

$\langle 2 \rangle 2$. For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$

$\langle 3 \rangle 1$. LET: $\mathcal{U} \subseteq \mathcal{T}$

$\langle 3 \rangle 2$. CASE: $\mathcal{U} = \emptyset$

PROOF: In this case $\bigcup \mathcal{U} = \emptyset \in \mathcal{T}$ since $X \in \mathcal{C}$

$\langle 3 \rangle 3$. CASE: $\mathcal{U} \neq \emptyset$

PROOF: In this case $X \setminus \bigcup \mathcal{U} = \bigcap \{X \setminus U \mid U \in \mathcal{U}\} \in \mathcal{C}$.

$\langle 2 \rangle 3$. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$

PROOF: Since $X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V) \in \mathcal{C}$.

$\langle 1 \rangle 3$. \mathcal{C} is the set of all closed sets in \mathcal{T}

PROOF:

C is closed in \mathcal{T}

$$\Leftrightarrow X \setminus C \in \mathcal{T}$$

$$\Leftrightarrow C \in \mathcal{C}$$

$\langle 1 \rangle 4$. If \mathcal{T}' is a topology and \mathcal{C} is the set of closed sets in \mathcal{T}' then $\mathcal{T}' = \mathcal{T}$

PROOF: We have

$$U \in \mathcal{T}$$

$$\Leftrightarrow X \setminus U \in \mathcal{C}$$

$$\Leftrightarrow X \setminus U \text{ is closed in } \mathcal{T}'$$

$$\Leftrightarrow U \in \mathcal{T}'$$

□

Proposition 9.4.7. If U is open and A is closed then $U \setminus A$ is open.

PROOF: $U \setminus A = U \cap (X \setminus A)$ is the intersection of two open sets. □

Proposition 9.4.8. If U is open and A is closed then $A \setminus U$ is closed.

PROOF: $A \setminus U = A \cap (X \setminus U)$ is the intersection of two closed sets. □

9.5 Interior

Definition 9.5.1 (Interior). Let X be a topological space and $A \subseteq X$. The *interior* of A , $\text{Int } A$, is the union of all the open subsets of A .

Lemma 9.5.2. *The interior of a set is open.*

PROOF: It is a union of open sets. \square

Lemma 9.5.3.

$$\text{Int } A \subseteq A$$

PROOF: Immediate from definition. \square

Lemma 9.5.4. *If U is open and $U \subseteq A$ then $U \subseteq \text{Int } A$*

PROOF: Immediate from definition. \square

Lemma 9.5.5. *A set A is open if and only if $A = \text{Int } A$.*

PROOF: If $A = \text{Int } A$ then A is open by Lemma 9.5.2. Conversely if A is open then $A \subseteq \text{Int } A$ by the definition of interior and so $A = \text{Int } A$.

9.6 Closure

Definition 9.6.1 (Closure). Let X be a topological space and $A \subseteq X$. The *closure* of A , \overline{A} , is the intersection of all the closed sets that include A .

This intersection exists since X is a closed set that includes A (Lemma 9.4.3).

Lemma 9.6.2. *The closure of a set is closed.*

PROOF: Dual to Lemma 9.5.2. \square

Lemma 9.6.3.

$$A \subseteq \overline{A}$$

PROOF: Immediate from definition. \square

Lemma 9.6.4. *If C is closed and $A \subseteq C$ then $\overline{A} \subseteq C$.*

PROOF: Immediate from definition. \square

Lemma 9.6.5. *A set A is closed if and only if $A = \overline{A}$.*

PROOF: Dual to Lemma 9.5.5. \square

Theorem 9.6.6. *Let X be a topological space, $A \subseteq X$ and $x \in X$. Then $x \in \overline{A}$ if and only if every neighbourhood of x intersects A .*

PROOF: We have

$$x \in \overline{A}$$

$$\Leftrightarrow \forall C. C \text{ closed} \wedge A \subseteq C \Rightarrow x \in C$$

$$\Leftrightarrow \forall U. U \text{ open} \wedge A \cap U = \emptyset \Rightarrow x \notin U$$

$$\Leftrightarrow \forall U. U \text{ open} \wedge x \in U \Rightarrow U \text{ intersects } A \quad \square$$

Proposition 9.6.7. *If $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$.*

PROOF: This holds because \overline{B} is a closed set that includes A . \square

Proposition 9.6.8.

$$\overline{A \cup B} = \overline{A} \cup \overline{B}$$

PROOF:

$\langle 1 \rangle 1.$ $\overline{A} \subseteq \overline{A \cup B}$

PROOF: By Proposition 9.6.7.

$\langle 1 \rangle 2.$ $\overline{B} \subseteq \overline{A \cup B}$

PROOF: By Proposition 9.6.7.

$\langle 1 \rangle 3.$ $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$

$\langle 2 \rangle 1.$ LET: $x \in \overline{A \cup B}$

$\langle 2 \rangle 2.$ ASSUME: $x \notin \overline{A}$

PROVE: $x \in \overline{B}$

$\langle 2 \rangle 3.$ PICK a neighbourhood U of x that does not intersect A

$\langle 2 \rangle 4.$ LET: V be any neighbourhood of x

$\langle 2 \rangle 5.$ $U \cap V$ is a neighbourhood of x

$\langle 2 \rangle 6.$ $U \cap V$ intersects $A \cup B$

PROOF: From $\langle 2 \rangle 1$ and Theorem 9.6.6.

$\langle 2 \rangle 7.$ $U \cap V$ intersects B

PROOF: From $\langle 2 \rangle 3$

$\langle 2 \rangle 8.$ V intersects B

$\langle 2 \rangle 9.$ Q.E.D.

PROOF: We have $x \in \overline{B}$ from Theorem 9.6.6.

\square

Proposition 9.6.9. *Let X be a topological space. Let \mathcal{D} be a set of subsets of X that is maximal with respect to the finite intersection property. Let $x \in X$. Then the following are equivalent:*

1. For all $D \in \mathcal{D}$ we have $x \in \overline{D}$
2. Every neighbourhood of x is in \mathcal{D} .

PROOF:

$\langle 1 \rangle 1.$ $1 \Rightarrow 2$

$\langle 2 \rangle 1.$ For all $D \in \mathcal{D}$ we have $x \in \overline{D}$

$\langle 2 \rangle 2.$ LET: U be a neighbourhood of x

$\langle 2 \rangle 3.$ $\mathcal{D} \cup \{U\}$ satisfies the finite intersection property.

$\langle 3 \rangle 1.$ LET: $D_1, \dots, D_n \in \mathcal{D}$

$\langle 3 \rangle 2.$ $D_1 \cap \dots \cap D_n \in \mathcal{D}$

PROOF: Lemma 3.24.3.

$\langle 3 \rangle 3.$ $x \in \overline{D_1 \cap \dots \cap D_n}$

PROOF: $\langle 2 \rangle 1, \langle 3 \rangle 2$

$\langle 3 \rangle 4.$ $D_1 \cap \dots \cap D_n \cap U \neq \emptyset$

PROOF: Theorem 9.6.6, $\langle 2 \rangle 2, \langle 3 \rangle 3.$

$\langle 2 \rangle 4.$ $\mathcal{D} = \mathcal{D} \cup \{U\}$
 PROOF: By the maximality of \mathcal{D} .
 $\langle 2 \rangle 5.$ $U \in \mathcal{D}$
 $\langle 1 \rangle 2.$ $2 \Rightarrow 1$
 $\langle 2 \rangle 1.$ ASSUME: Every neighbourhood of x is in \mathcal{D} .
 $\langle 2 \rangle 2.$ LET: $D \in \mathcal{D}$
 $\langle 2 \rangle 3.$ Every neighbourhood of x intersects D .
 PROOF: From $\langle 2 \rangle 1$, $\langle 2 \rangle 2$ and the fact that \mathcal{D} satisfies the finite intersection property.
 $\langle 2 \rangle 4.$ $x \in \overline{D}$
 PROOF: Theorem 9.6.6, $\langle 2 \rangle 3$.
 \square

9.7 Boundary

Definition 9.7.1 (Boundary). The *boundary* of a set A is the set $\partial A = \overline{A} \cap \overline{X \setminus A}$.

Proposition 9.7.2.

$$\text{Int } A \cap \partial A = \emptyset$$

PROOF: Since $\overline{X \setminus A} = X \setminus \text{Int } A$. \square

Proposition 9.7.3.

$$\overline{A} = \text{Int } A \cup \partial A$$

PROOF:

$$\begin{aligned}
 \text{Int } A \cup \partial A &= \text{Int } A \cup (\overline{A} \cap \overline{X \setminus A}) \\
 &= (\text{Int } A \cup \overline{A}) \cap (\text{Int } A \cup \overline{X \setminus A}) \\
 &= \overline{A} \cap X \\
 &= \overline{A}
 \end{aligned}$$

Proposition 9.7.4. $\partial A = \emptyset$ if and only if A is open and closed.

PROOF: If $\partial A = \emptyset$ then $\overline{A} = \text{Int } A$ by Proposition 9.7.3.

Proposition 9.7.5. A set U is open if and only if $\partial U = \overline{U} \setminus U$.

PROOF:

$$\begin{aligned}
 \partial U &= \overline{U} \setminus U \\
 \Leftrightarrow \overline{U} \setminus \text{Int } U &= \overline{U} \setminus U && (\text{Propositions 9.7.2, 9.7.3}) \\
 \Leftrightarrow \text{Int } U &= U && \square
 \end{aligned}$$

9.8 Limit Points

Definition 9.8.1 (Limit Point). Let X be a topological space, $a \in X$ and $A \subseteq X$. Then a is a *limit point*, *cluster point* or *point of accumulation* for A if and only if every neighbourhood of a intersects A at a point other than a .

Lemma 9.8.2. *The point a is an accumulation point for A if and only if $a \in \overline{A \setminus \{a\}}$.*

PROOF: From Theorem 9.6.6. \square

Theorem 9.8.3. *Let X be a topological space and $A \subseteq X$. Let A' be the set of all limit points of A . Then $\overline{A} = A \cup A'$.*

PROOF:

$\langle 1 \rangle 1$. For all $x \in \overline{A}$, if $x \notin A$ then $x \in A'$

PROOF: From Theorem 9.6.6.

$\langle 1 \rangle 2$. $A \subseteq \overline{A}$

PROOF: Lemma 9.6.3.

$\langle 1 \rangle 3$. $A' \subseteq \overline{A}$

PROOF: From Theorem 9.6.6.

\square

Corollary 9.8.3.1. *A set is closed if and only if it contains all its limit points.*

Proposition 9.8.4. *In an indiscrete topology, every point is a limit point of any set that has more than one point.*

PROOF: Let X be an indiscrete space. Let A be a set with more than one point and x be a point. The only neighbourhood of x is X , which must intersect A at a point other than x . \square

Lemma 9.8.5. *Let X be a topological space and $A \subseteq B \subseteq X$. Then every limit point of A is a limit point of B .*

PROOF: Immediate from definitions. \square

9.9 Basis for a Topology

Definition 9.9.1 (Basis). If X is a set, a *basis* for a topology on X is a set $\mathcal{B} \subseteq \mathcal{P}X$ called *basis elements* such that

1. For all $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$.
2. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

If \mathcal{B} satisfies these two conditions, then we define the topology *generated* by \mathcal{B} to be $\mathcal{T} = \{U \in \mathcal{P}X \mid \forall x \in U. \exists B \in \mathcal{B}. x \in B \subseteq U\}$.

We prove this is a topology.

PROOF:

$\langle 1 \rangle 1. X \in \mathcal{T}$

PROOF: For all $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B \subseteq X$ by condition 1.

$\langle 1 \rangle 2.$ For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$

$\langle 2 \rangle 1.$ LET: $\mathcal{U} \subseteq \mathcal{T}$

$\langle 2 \rangle 2.$ LET: $x \in \bigcup \mathcal{U}$

$\langle 2 \rangle 3.$ PICK $U \in \mathcal{U}$ such that $x \in U$

$\langle 2 \rangle 4.$ PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$

PROOF: Since $U \in \mathcal{T}$ by $\langle 2 \rangle 1$ and $\langle 2 \rangle 3$.

$\langle 2 \rangle 5.$ $x \in B \subseteq \bigcup \mathcal{U}$

$\langle 1 \rangle 3.$ For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$

$\langle 2 \rangle 1.$ LET: $U, V \in \mathcal{T}$

$\langle 2 \rangle 2.$ LET: $x \in U \cap V$

$\langle 2 \rangle 3.$ PICK $B_1 \in \mathcal{B}$ such that $x \in B_1 \subseteq U$

$\langle 2 \rangle 4.$ PICK $B_2 \in \mathcal{B}$ such that $x \in B_2 \subseteq V$

$\langle 2 \rangle 5.$ PICK $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$

PROOF: By condition 2.

$\langle 2 \rangle 6.$ $x \in B_3 \subseteq U \cap V$

□

Lemma 9.9.2. *Let X be a set. Let \mathcal{B} be a basis for a topology \mathcal{T} on X . Then \mathcal{T} is the set of all unions of subsets of \mathcal{B} .*

PROOF:

$\langle 1 \rangle 1.$ For all $U \in \mathcal{T}$, there exists $\mathcal{A} \subseteq \mathcal{B}$ such that $U = \bigcup \mathcal{A}$

$\langle 2 \rangle 1.$ LET: $U \in \mathcal{T}$

$\langle 2 \rangle 2.$ LET: $\mathcal{A} = \{B \in \mathcal{B} \mid B \subseteq U\}$

$\langle 2 \rangle 3.$ $U \subseteq \bigcup \mathcal{A}$

$\langle 3 \rangle 1.$ LET: $x \in U$

$\langle 3 \rangle 2.$ PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$

PROOF: Since \mathcal{B} is a basis for \mathcal{T} .

$\langle 3 \rangle 3.$ $x \in B \in \mathcal{A}$

$\langle 2 \rangle 4.$ $\bigcup \mathcal{A} \subseteq U$

PROOF: From the definition of \mathcal{A} ($\langle 2 \rangle 2$).

$\langle 1 \rangle 2.$ For all $\mathcal{A} \subseteq \mathcal{B}$ we have $\bigcup \mathcal{A} \in \mathcal{T}$

$\langle 2 \rangle 1.$ $\bigcup \mathcal{A} \subseteq \mathcal{T}$

PROOF: If $B \in \mathcal{B}$ and $x \in B$, then there exists $B' \in \mathcal{B}$ such that $x \in B' \subseteq B$, namely $B' = B$.

$\langle 2 \rangle 2.$ Q.E.D.

PROOF: Since \mathcal{T} is closed under union.

□

Corollary 9.9.2.1. *Let X be a set. Let \mathcal{B} be a basis for a topology \mathcal{T} on X . Then \mathcal{T} is the coarsest topology that includes \mathcal{B} .*

PROOF: Since every topology that includes \mathcal{B} includes all unions of subsets of \mathcal{B} . \square

Lemma 9.9.3. *Let X be a topological space. Suppose that \mathcal{C} is a set of open sets such that, for every open set U and every point $x \in U$, there exists $C \in \mathcal{C}$ such that $x \in C \subseteq U$. Then \mathcal{C} is a basis for the topology on X .*

PROOF:

$\langle 1 \rangle 1$. For all $x \in X$, there exists $C \in \mathcal{C}$ such that $x \in C$

PROOF: Immediate from hypothesis.

$\langle 1 \rangle 2$. For all $C_1, C_2 \in \mathcal{C}$ and $x \in C_1 \cap C_2$, there exists $C_3 \in \mathcal{C}$ such that $x \in C_3 \subseteq C_1 \cap C_2$

PROOF: Since $C_1 \cap C_2$ is open.

$\langle 1 \rangle 3$. Every open set is open in the topology generated by \mathcal{C}

PROOF: Immediate from hypothesis.

$\langle 1 \rangle 4$. Every union of a subset of \mathcal{C} is open.

PROOF: Since every member of \mathcal{C} is open.

\square

Lemma 9.9.4. *Let \mathcal{B} and \mathcal{B}' be bases for the topologies \mathcal{T} and \mathcal{T}' respectively on the set X . Then the following are equivalent.*

1. $\mathcal{T} \subseteq \mathcal{T}'$

2. For all $B \in \mathcal{B}$ and $x \in B$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.

PROOF:

$\langle 1 \rangle 1$. $1 \Rightarrow 2$

$\langle 2 \rangle 1$. ASSUME: $\mathcal{T} \subseteq \mathcal{T}'$

$\langle 2 \rangle 2$. LET: $B \in \mathcal{B}$ and $x \in B$

$\langle 2 \rangle 3$. $B \in \mathcal{T}$

PROOF: Corollary 9.9.2.1.

$\langle 2 \rangle 4$. $B \in \mathcal{T}'$

PROOF: By $\langle 2 \rangle 1$

$\langle 2 \rangle 5$. There exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$

PROOF: Since \mathcal{B}' is a basis for \mathcal{T}' .

$\langle 1 \rangle 2$. $2 \Rightarrow 1$

$\langle 2 \rangle 1$. ASSUME: 2

$\langle 2 \rangle 2$. LET: $U \in \mathcal{T}$

PROVE: $U \in \mathcal{T}'$

$\langle 2 \rangle 3$. LET: $x \in U$

PROVE: There exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq U$

$\langle 2 \rangle 4$. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$

PROOF: Since \mathcal{B} is a basis for \mathcal{T} .

$\langle 2 \rangle 5$. PICK $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$

PROOF: By $\langle 2 \rangle 1$.

$\langle 2 \rangle 6$. $x \in B' \subseteq U$

\square

Theorem 9.9.5. *Let X be a topological space, $A \subseteq X$ and $x \in X$. Let \mathcal{B} be a basis for X . Then $x \in \overline{A}$ if and only if, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A .*

PROOF:

$\langle 1 \rangle 1$. If $x \in \overline{A}$ then, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A .

PROOF: This follows from Theorem 9.6.6 since every element of \mathcal{B} is open (Corollary 9.9.2.1).

$\langle 1 \rangle 2$. Suppose that, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A . Then $x \in \overline{A}$.

$\langle 2 \rangle 1$. ASSUME: For all $B \in \mathcal{B}$, if $x \in B$ then B intersects A .

$\langle 2 \rangle 2$. LET: U be an open set that contains x

PROVE: U intersects A .

$\langle 2 \rangle 3$. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

$\langle 2 \rangle 4$. B intersects A .

PROOF: From $\langle 2 \rangle 1$.

$\langle 2 \rangle 5$. U intersects A .

$\langle 2 \rangle 6$. Q.E.D.

PROOF: By Theorem 9.6.6.

□

Definition 9.9.6 (Lower Limit Topology on the Real Line). The *lower limit topology on the real line* is the topology on \mathbb{R} generated by the basis consisting of all half-open intervals of the form $[a, b)$.

We write \mathbb{R}_l for the topological space \mathbb{R} under the lower limit topology.

We prove this is a basis for a topology.

PROOF:

$\langle 1 \rangle 1$. For all $x \in \mathbb{R}$ there exists an interval $[a, b)$ such that $x \in [a, b)$.

PROOF: Take $[a, b) = [x, x + 1)$.

$\langle 1 \rangle 2$. For any open intervals $[a, b)$, $[c, d)$ if $x \in [a, b) \cap [c, d)$, then there exists an interval $[e, f)$ such that $x \in [e, f) \subseteq [a, b) \cap [c, d)$

PROOF: Take $[e, f) = [\max(a, c), \min(b, d))$.

□

Definition 9.9.7 (K -topology on the Real Line). Let $K = \{1/n \mid n \in \mathbb{Z}^+\}$.

The *K -topology on the real line* is the topology on \mathbb{R} generated by the basis consisting of all open intervals (a, b) and all sets of the form $(a, b) \setminus K$.

We write \mathbb{R}_K for the topological space \mathbb{R} under the K -topology.

We prove this is a basis for a topology.

PROOF:

$\langle 1 \rangle 1$. For all $x \in \mathbb{R}$ there exists an open interval (a, b) such that $x \in (a, b)$.

PROOF: Take $(a, b) = (x - 1, x + 1)$.

$\langle 1 \rangle 2$. For any basic open sets B_1, B_2 if $x \in B_1 \cap B_2$, then there exists a basic open set B_3 such that $x \in B_3 \subseteq B_1 \cap B_2$

$\langle 2 \rangle 1$. CASE: $B_1 = (a, b)$, $B_2 = (c, d)$

PROOF: Take $B_3 = (\max(a, c), \min(b, d))$.

⟨2⟩2. CASE: $B_1 = (a, b)$ or $(a, b) \setminus K$, $B_2 = (c, d)$ or $(c, d) \setminus K$, and they are not both open intervals.

PROOF: Take $B_3 = (\max(a, c), \min(b, d)) \setminus K$.

□

Lemma 9.9.8. *The lower limit topology and the K -topology are incomparable.*

PROOF:

⟨1⟩1. The interval $[10, 11)$ is not open in the K -topology.

PROOF: There is no open interval (a, b) such that $10 \in (a, b) \subseteq [10, 11)$ or $10 \in (a, b) \setminus K \subseteq [10, 11)$.

⟨1⟩2. The set $(-1, 1) \setminus K$ is not open in the lower limit topology.

PROOF: There is no half-open interval $[a, b)$ such that $0 \in [a, b) \subseteq (-1, 1) \setminus K$, since there must be a positive integer n with $1/n \in [a, b)$.

□

Definition 9.9.9 (Subbasis). A *subbasis* \mathcal{S} for a topology on X is a set $\mathcal{S} \subseteq \mathcal{P}X$ such that $\bigcup \mathcal{S} = X$.

The topology *generated* by the subbasis \mathcal{S} is the set of all unions of finite intersections of elements of \mathcal{S} .

We prove this is a topology.

PROOF:

⟨1⟩1. The set \mathcal{B} of all finite intersections of elements of \mathcal{S} forms a basis for a topology on X .

⟨2⟩1. $\bigcup \mathcal{B} = X$

PROOF: Since $\mathcal{S} \subseteq \mathcal{B}$.

⟨2⟩2. \mathcal{B} is closed under binary intersection.

PROOF: By definition.

⟨1⟩2. Q.E.D.

PROOF: By Lemma 9.9.2.

□

We have simultaneously proved:

Proposition 9.9.10. *Let \mathcal{S} be a subbasis for the topology on X . Then the set of all finite intersections of elements of \mathcal{S} is a basis for the topology on X .*

Proposition 9.9.11. *Let X be a set. Let \mathcal{S} be a subbasis for a topology \mathcal{T} on X . Then \mathcal{T} is the coarsest topology that includes \mathcal{S} .*

PROOF: Since every topology that includes \mathcal{S} includes every union of finite intersections of elements of \mathcal{S} . □

9.10 Local Basis at a Point

Definition 9.10.1 (Local Basis). Let X be a topological space and $a \in X$. A *(local) basis at a* is a set \mathcal{B} of neighbourhoods of a such that every neighbourhood of a includes some member of \mathcal{B} .

Lemma 9.10.2. *If there exists a countable local basis at a point a , then there exists a countable local basis $\{B_n \mid n \geq 1\}$ such that $B_1 \supseteq B_2 \supseteq \cdots$.*

PROOF: Given a countable local basis $\{C_n \mid n \geq 1\}$, take $B_n = C_1 \cap \cdots \cap C_n$. \square

9.11 Convergence

Definition 9.11.1 (Convergence). Let X be a topological space. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of points in X and $l \in X$. Then the sequence $(a_n)_{n \in \mathbb{N}}$ *converges* to the *limit* l , $a_n \rightarrow l$ as $n \rightarrow \infty$, if and only if, for every neighbourhood U of l , there exists N such that, for all $n \geq N$, we have $a_n \in U$.

Lemma 9.11.2. *Let X be a topological space. Let $A \subseteq X$ and $l \in X$. If there is a sequence of points in A that converges to l then $l \in \overline{A}$.*

PROOF:

- $\langle 1 \rangle 1$. LET: (a_n) be a sequence of points in A that converges to l .
- $\langle 1 \rangle 2$. LET: U be a neighbourhood of l .
- $\langle 1 \rangle 3$. PICK N such that, for all $n \geq N$, we have $a_n \in U$.
- $\langle 1 \rangle 4$. $a_N \in U \cap A$
- $\langle 1 \rangle 5$. Q.E.D.

PROOF: Theorem 9.6.6.

\square

Proposition 9.11.3. *Let X be a topological space. Let \mathcal{B} be a basis for the topology on X . Let (a_n) be a sequence in X and $l \in X$. Then $a_n \rightarrow l$ as $n \rightarrow \infty$ if and only if, for every $B \in \mathcal{B}$ with $l \in B$, there exists N such that, for all $n \geq N$, we have $a_n \in B$.*

PROOF:

- $\langle 1 \rangle 1$. If $a_n \rightarrow l$ as $n \rightarrow \infty$ then, for every $B \in \mathcal{B}$ with $l \in B$, there exists N such that, for all $n \geq N$, we have $a_n \in B$.

PROOF: Since every element of \mathcal{B} is open (Corollary 9.9.2.1).

- $\langle 1 \rangle 2$. If, for every $B \in \mathcal{B}$ with $l \in B$, there exists N such that, for all $n \geq N$, we have $a_n \in B$, then $a_n \rightarrow l$ as $n \rightarrow \infty$.

- $\langle 2 \rangle 1$. ASSUME: for every $B \in \mathcal{B}$ with $l \in B$, there exists N such that, for all $n \geq N$, we have $a_n \in B$

- $\langle 2 \rangle 2$. LET: U be a neighbourhood of l .

- $\langle 2 \rangle 3$. PICK $B \in \mathcal{B}$ such that $l \in B \subseteq U$

- $\langle 2 \rangle 4$. PICK N such that, for all $n \geq N$, we have $a_n \in B$

PROOF: From $\langle 2 \rangle 1$.

- $\langle 2 \rangle 5$. For all $n \geq N$ we have $a_n \in U$

\square

Lemma 9.11.4. *If a sequence (a_n) is constant with $a_n = l$ for all n , then $a_n \rightarrow l$ as $n \rightarrow \infty$.*

PROOF: Immediate from definitions. \square

Theorem 9.11.5. *Let X be a linearly ordered set. Let (s_n) be a monotone increasing sequence in X with a supremum s . Then $s_n \rightarrow s$ as $n \rightarrow \infty$.*

PROOF:

$\langle 1 \rangle 1$. ASSUME: s is not least in X .

PROOF: Otherwise (s_n) is the constant sequence s and the result follows from Lemma 9.11.4.

$\langle 1 \rangle 2$. LET: U be a neighbourhood of s .

$\langle 1 \rangle 3$. PICK $a < s$ such that $(a, s] \subseteq U$

$\langle 1 \rangle 4$. PICK N such that $a < a_N$.

$\langle 1 \rangle 5$. For all $n \geq N$ we have $a_n \in (a, s]$

$\langle 1 \rangle 6$. For all $n \geq N$ we have $a_n \in U$.

□

Theorem 9.11.6. *If $\sum_{i=0}^{\infty} a_i = s$ and $\sum_{i=0}^{\infty} b_i = t$ then $\sum_{i=0}^{\infty} (ca_i + b_i) = cs + t$.*

PROOF: $\sum_{i=0}^N (ca_i + b_i) = c \sum_{i=0}^N a_i + \sum_{i=0}^N b_i \rightarrow cs + t$ as $n \rightarrow \infty$. □

Theorem 9.11.7 (Comparison Test). *If $|a_i| \leq b_i$ for all i and $\sum_{i=0}^{\infty} b_i$ converges then $\sum_{i=0}^{\infty} a_i$ converges.*

PROOF:

$\langle 1 \rangle 1$. $\sum_{i=0}^{\infty} |a_i|$ converges

PROOF: The partial sums $\sum_{i=0}^N |a_i|$ form an increasing sequence bounded above by $\sum_{i=0}^{\infty} b_i$.

$\langle 1 \rangle 2$. LET: $c_i = |a_i| + a_i$ for all i

$\langle 1 \rangle 3$. $\sum_{i=0}^{\infty} c_i$ converges

PROOF: Each c_i is either $2|a_i|$ or 0. So the partial sums $\sum_{i=0}^N c_i$ form an increasing sequence bounded above by $2 \sum_{i=0}^{\infty} b_i$.

$\langle 1 \rangle 4$. Q.E.D.

PROOF: Since $a_i = c_i - |a_i|$.

□

Corollary 9.11.7.1. *If $\sum_{i=0}^{\infty} |a_i|$ converges then $\sum_{i=0}^{\infty} a_i$ converges.*

Theorem 9.11.8 (Weierstrass M-test). *Let X be a set and $(f_n : X \rightarrow \mathbb{R})$ be a sequence of functions. Let*

$$s_n(x) = \sum_{i=0}^n f_i(x)$$

for all n, x . Suppose $|f_i(x)| \leq M_i$ for all $i \geq 0$ and $x \in X$. If the series $\sum_{i=0}^{\infty} M_i$ converges, then the sequence (s_n) converges uniformly to

$$s(x) = \sum_{i=0}^{\infty} f_i(x) .$$

PROOF:

$\langle 1 \rangle 1$. LET: $r_n = \sum_{i=n+1}^{\infty} M_i$ for all n

⟨1⟩2. Given $0 \leq n < k$, we have $|s_k(x) - s_n(x)| \leq r_n$

PROOF:

$$\begin{aligned} |s_k(x) - s_n(x)| &= \left| \sum_{i=n+1}^k f_i(x) \right| \\ &\leq \sum_{i=n+1}^k |f_i(x)| \\ &\leq \sum_{i=n+1}^k M_i \\ &\leq r_n \end{aligned}$$

⟨1⟩3. Given $n \geq 0$ we have $|s(x) - s_n(x)| \leq r_n$

PROOF: By taking the limit $k \rightarrow \infty$ in ⟨1⟩2.

⟨1⟩4. Q.E.D.

PROOF: Since $r_n \rightarrow 0$ as $n \rightarrow \infty$.

□

9.12 Locally Finite Sets

Definition 9.12.1 (Locally Finite). Let X be a topological space and $\{A_\alpha\}$ a family of subsets of X . Then \mathcal{A} is *locally finite* if and only if every point in X has a neighbourhood that intersects A_α for only finitely many α .

Theorem 9.12.2 (Pasting Lemma). *Let X and Y be topological spaces and $f : X \rightarrow Y$. Let $\{A_\alpha\}$ be a locally finite family of closed subsets of X that cover X . Suppose $f \upharpoonright A_\alpha$ is continuous for all α . Then f is continuous.*

PROOF:

⟨1⟩1. Let X and Y be topological spaces and $f : X \rightarrow Y$. Let A and B be closed subsets of X such that $X = A \cup B$. Suppose $f \upharpoonright A$ and $f \upharpoonright B$ are continuous. Then f is continuous.

⟨2⟩1. LET: $C \subseteq Y$ be closed.

⟨2⟩2. $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$

⟨2⟩3. $f^{-1}(C)$ and $g^{-1}(C)$ are closed in X .

PROOF: Theorems 9.14.6 and 9.19.7.

⟨2⟩4. $h^{-1}(C)$ is closed in X .

PROOF: Lemma 9.4.5.

⟨2⟩5. Q.E.D.

PROOF: Theorem 9.14.6.

⟨1⟩2. Let X and Y be topological spaces and $f : X \rightarrow Y$. Let $\{A_\alpha\}$ be a finite family of closed subsets of X that cover X . Suppose $f \upharpoonright A_\alpha$ is continuous for all α . Then f is continuous.

PROOF: From ⟨1⟩1 by induction.

⟨1⟩3. Let X and Y be topological spaces and $f : X \rightarrow Y$. Let $\{A_\alpha\}$ be a locally finite family of closed subsets of X that cover X . Suppose $f \upharpoonright A_\alpha$ is continuous for all α . Then f is continuous.

- (2)1. LET: $x \in X$
 PROVE: f is continuous at x
 (2)2. PICK a neighbourhood U of x that intersects A_α for only finitely many α .
 (2)3. $f \upharpoonright U$ is continuous
 PROOF: By (1)2.
 (2)4. Q.E.D.
 PROOF: Lemma 9.14.16.

□

The following example shows that we cannot remove the assumption of local finiteness.

Example 9.12.3. Define $f : [-1, 1] \rightarrow \mathbb{R}$ by: $f(x) = 1$ if $x < -1$, $f(x) = 0$ if $x > 1$. Let $C_n = [-1, -1/n]$ for $n \geq 1$, and $D = [0, 1]$. Then $[-1, 1] = \bigcup_{n=1}^{\infty} C_n \cup D$ and f is continuous on each C_n and each D , but f is not continuous on $[-1, 1]$.

9.13 Open Maps

Definition 9.13.1 (Open Map). Let X and Y be topological spaces and $f : X \rightarrow Y$. Then f is an *open map* if and only if, for every open set U in X , the set $f(U)$ is open in Y .

Lemma 9.13.2. Let X and Y be topological spaces and $f : X \rightarrow Y$. Let \mathcal{B} be a basis for the topology on X . If $f(B)$ is open in Y for all $B \in \mathcal{B}$, then f is an open map.

PROOF: From Lemma 9.9.2. □

Proposition 9.13.3. Let X and Y be topological spaces. Let \mathcal{B} be a basis for the topology on X . Let $f : X \rightarrow Y$. Suppose that, for all $B \in \mathcal{B}$, we have $f(B)$ is open in Y . Then f is an open map.

PROOF: For any $\mathcal{A} \subseteq \mathcal{B}$, we have $f(\bigcup \mathcal{A}) = \bigcup_{B \in \mathcal{A}} f(B)$ is open in Y . The result follows from Lemma 9.9.2. □

9.14 Continuous Functions

Definition 9.14.1 (Continuous). Let X and Y be topological spaces. A function $f : X \rightarrow Y$ is *continuous* if and only if, for every open set V in Y , the set $f^{-1}(V)$ is open in X .

Proposition 9.14.2. Let X and Y be topological spaces and $f : X \rightarrow Y$. Let \mathcal{B} be a basis for Y . Then f is continuous if and only if, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X .

PROOF:

- ⟨1⟩1. If f is continuous then, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X .
 PROOF: Since every element of \mathcal{B} is open (Lemma 9.9.2).
 ⟨1⟩2. Suppose that, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X . Then f is continuous.
 ⟨2⟩1. ASSUME: For all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X .
 ⟨2⟩2. LET: V be open in Y .
 ⟨2⟩3. PICK $\mathcal{A} \subseteq \mathcal{B}$ such that $V = \bigcup \mathcal{A}$
 PROOF: By Lemma 9.9.2.
 ⟨2⟩4. $f^{-1}(V)$ is open in X .
 PROOF:

$$\begin{aligned} f^{-1}(V) &= f^{-1}\left(\bigcup \mathcal{A}\right) \\ &= \bigcup_{B \in \mathcal{A}} f^{-1}(B) \end{aligned}$$

□

Proposition 9.14.3. *Let X and Y be topological spaces and $f : X \rightarrow Y$. Let \mathcal{S} be a subbasis for Y . Then f is continuous if and only if, for all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in X .*

PROOF:

- ⟨1⟩1. If f is continuous then, for all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in X .
 PROOF: Since every element of \mathcal{S} is open.
 ⟨1⟩2. Suppose that, for all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in X . Then f is continuous.
 ⟨2⟩1. ASSUME: For all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in X .
 ⟨2⟩2. LET: $S_1, \dots, S_n \in \mathcal{S}$
 ⟨2⟩3. $f^{-1}(S_1 \cap \dots \cap S_n)$ is open in X
 PROOF: Since $f^{-1}(S_1 \cap \dots \cap S_n) = f^{-1}(S_1) \cap \dots \cap f^{-1}(S_n)$.
 ⟨2⟩4. Q.E.D.
 PROOF: By Propositions 9.14.2 and 9.9.10.

□

Proposition 9.14.4. *Let X and Y be topological spaces and $f : X \rightarrow Y$. Let \mathcal{S} be a basis for Y . Then f is continuous if and only if, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X .*

PROOF:

- ⟨1⟩1. If f is continuous then, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X .
 PROOF: Since every element of \mathcal{S} is open.
 ⟨1⟩2. Suppose that, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X . Then f is continuous.
 ⟨2⟩1. ASSUME: For all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X .
 ⟨2⟩2. For every set B that is the finite intersection of elements of \mathcal{S} , we have $f^{-1}(B)$ is open in X .
 PROOF: Because $f^{-1}(V_1 \cap \dots \cap V_n) = f^{-1}(V_1) \cap \dots \cap f^{-1}(V_n)$.
 ⟨2⟩3. Q.E.D.
 PROOF: From Propositions 9.9.10 and 9.14.2.

□

Definition 9.14.5 (Continuous at a Point). Let X and Y be topological spaces. Let $f : X \rightarrow Y$ and $x \in X$. Then f is *continuous at x* if and only if, for every neighbourhood V of $f(x)$, there exists a neighbourhood U of x such that $f(U) \subseteq V$.

Theorem 9.14.6. Let X and Y be topological spaces and $f : X \rightarrow Y$. Then the following are equivalent:

1. f is continuous.
2. For all $A \subseteq X$, we have $f(\overline{A}) \subseteq \overline{f(A)}$
3. For all $B \subseteq Y$ closed, we have $f^{-1}(B)$ is closed in X .
4. f is continuous at every point of X .

PROOF:

⟨1⟩1. $1 \Rightarrow 2$

⟨2⟩1. ASSUME: f is continuous.

⟨2⟩2. LET: $A \subseteq X$

⟨2⟩3. LET: $x \in \overline{A}$

PROVE: $f(x) \in \overline{f(A)}$

⟨2⟩4. LET: V be a neighbourhood of $f(x)$

⟨2⟩5. $f^{-1}(V)$ is a neighbourhood of x

⟨2⟩6. PICK $y \in A \cap f^{-1}(V)$

PROOF: By Theorem 9.6.6.

⟨2⟩7. $f(y) \in V \cap f(A)$

⟨2⟩8. Q.E.D.

PROOF: By Theorem 9.6.6.

⟨1⟩2. $2 \Rightarrow 3$

⟨2⟩1. ASSUME: 2

⟨2⟩2. LET: B be closed in Y

⟨2⟩3. LET: $x \in \overline{f^{-1}(B)}$

PROVE: $x \in f^{-1}(B)$

⟨2⟩4. $f(x) \in B$

PROOF:

$$f(x) \in \overline{f(f^{-1}(B))}$$

$$\subseteq \overline{f(f^{-1}(B))}$$

(⟨2⟩1)

$$\subseteq \overline{B}$$

(Proposition 9.6.7)

$$= B$$

⟨1⟩3. $3 \Rightarrow 1$

⟨2⟩1. ASSUME: 3

⟨2⟩2. LET: V be open in Y

⟨2⟩3. $Y \setminus V$ is closed in Y

⟨2⟩4. $f^{-1}(Y \setminus V)$ is closed in X

(2)5. $X \setminus f^{-1}(V)$ is closed in X
 (2)6. $f^{-1}(V)$ is open in X
 (1)4. $1 \Rightarrow 4$
 PROOF: For any neighbourhood V of $f(x)$, the set $U = f^{-1}(V)$ is a neighbourhood of x such that $f(U) \subseteq V$.
 (1)5. $4 \Rightarrow 1$
 (2)1. ASSUME: 4
 (2)2. LET: V be open in Y
 (2)3. LET: $x \in f^{-1}(V)$
 (2)4. V is a neighbourhood of $f(x)$
 (2)5. PICK a neighbourhood U of x such that $f(U) \subseteq V$
 (2)6. $x \in U \subseteq f^{-1}(V)$
 (2)7. Q.E.D.
 PROOF: By Lemma 9.3.8.

□

Theorem 9.14.7. *A constant function is continuous.*

PROOF: Let X and Y be topological spaces. Let $b \in Y$, and let $f : X \rightarrow Y$ be the constant function with value b . For any open $V \subseteq Y$, the set $f^{-1}(V)$ is either X (if $b \in V$) or \emptyset (if $b \notin V$). □

Theorem 9.14.8. *If A is a subspace of X then the inclusion $j : A \rightarrow X$ is continuous.*

PROOF: For any V open in X , we have $j^{-1}(V) = V \cap A$ is open in A . □

Theorem 9.14.9. *The composite of two continuous functions is continuous.*

PROOF: Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be continuous. For any V open in Z , we have $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is open in X . □

Theorem 9.14.10. *Let $f : X \rightarrow Y$ be a continuous function and A be a subspace of X . Then the restriction $f \upharpoonright A : A \rightarrow Y$ is continuous.*

PROOF: Let V be open in Y . Then $(f \upharpoonright A)^{-1}(V) = f^{-1}(V) \cap A$ is open in A . □

Theorem 9.14.11. *Let $f : X \rightarrow Y$ be continuous. Let Z be a subspace of Y such that $f(X) \subseteq Z$. Then the corestriction $f : X \rightarrow Z$ is continuous.*

PROOF:

(1)1. LET: V be open in Z .
 (1)2. PICK U open in Y such that $V = U \cap Z$.
 (1)3. $f^{-1}(V) = f^{-1}(U)$
 (1)4. $f^{-1}(V)$ is open in X .

□

Theorem 9.14.12. *Let $f : X \rightarrow Y$ be continuous. Let Z be a space such that Y is a subspace of Z . Then the expansion $f : X \rightarrow Z$ is continuous.*

PROOF: Let V be open in Z . Then $f^{-1}(V) = f^{-1}(V \cap Y)$ is open in X . □

Theorem 9.14.13. *Let X and Y be topological spaces. Let $f : X \rightarrow Y$. Suppose \mathcal{U} is a set of open sets in X such that $X = \bigcup \mathcal{U}$ and, for all $U \in \mathcal{U}$, we have $f \upharpoonright U : U \rightarrow Y$ is continuous. Then f is continuous.*

PROOF:

- $\langle 1 \rangle 1$. LET: V be open in Y
- $\langle 1 \rangle 2$. $f^{-1}(V) = \bigcup_{U \in \mathcal{U}} (f \upharpoonright U)^{-1}(V)$
- $\langle 1 \rangle 3$. For all $U \in \mathcal{U}$, we have $(f \upharpoonright U)^{-1}(V)$ is open in U .
- $\langle 1 \rangle 4$. For all $U \in \mathcal{U}$, we have $(f \upharpoonright U)^{-1}(V)$ is open in X .

PROOF: Lemma 9.19.6.

□

Proposition 9.14.14. *Let X and X' be the same set X under two topologies \mathcal{T} and \mathcal{T}' . Let $i : X \rightarrow X'$ be the identity function. Then i is continuous if and only if $\mathcal{T}' \subseteq \mathcal{T}$.*

PROOF: Immediate from definitions. □

Proposition 9.14.15. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ and $a \in \mathbb{R}$. Then f is continuous on the right at a if and only if f is continuous at a as a function $\mathbb{R}_l \rightarrow \mathbb{R}$.*

PROOF:

- $\langle 1 \rangle 1$. If f is continuous on the right at a then f is continuous at a as a function $\mathbb{R}_l \rightarrow \mathbb{R}$.
 - $\langle 2 \rangle 1$. ASSUME: f is continuous on the right at a .
 - $\langle 2 \rangle 2$. LET: V be a neighbourhood of $f(a)$
 - $\langle 2 \rangle 3$. PICK b, c such that $f(a) \in (b, c) \subseteq V$.
 - $\langle 2 \rangle 4$. LET: $\epsilon = \min(c - f(a), f(a) - b)$
 - $\langle 2 \rangle 5$. PICK $\delta > 0$ such that, for all x , if $a < x < a + \delta$ then $|f(x) - f(a)| < \epsilon$
 - $\langle 2 \rangle 6$. LET: $U = [a, a + \delta)$
 - $\langle 2 \rangle 7$. $f(U) \subseteq V$
- $\langle 1 \rangle 2$. If f is continuous at a as a function $\mathbb{R}_l \rightarrow \mathbb{R}$ then f is continuous on the right at a .
 - $\langle 2 \rangle 1$. ASSUME: f is continuous at a as a function $\mathbb{R}_l \rightarrow \mathbb{R}$
 - $\langle 2 \rangle 2$. LET: $\epsilon > 0$
 - $\langle 2 \rangle 3$. PICK a neighbourhood U of a such that $f(U) \subseteq (f(a) - \epsilon, f(a) + \epsilon)$
 - $\langle 2 \rangle 4$. PICK b, c such that $a \in [b, c) \subset U$
 - $\langle 2 \rangle 5$. LET: $\delta = c - a$
 - $\langle 2 \rangle 6$. For all x , if $a < x < a + \delta$ then $|f(x) - f(a)| < \epsilon$

□

Lemma 9.14.16. *Let $f : X \rightarrow Y$. Let Z be an open subspace of X and $a \in Z$. If $f \upharpoonright Z$ is continuous at a then f is continuous at a .*

PROOF:

- $\langle 1 \rangle 1$. LET: V be a neighbourhood of $f(a)$
- $\langle 1 \rangle 2$. PICK a neighbourhood W of a in Z such that $f(W) \subseteq V$
- $\langle 1 \rangle 3$. W is a neighbourhood of a in X such that $f(W) \subseteq V$

PROOF: Lemma 9.19.6.

□

Proposition 9.14.17. *Let $f : A \rightarrow B$ and $g : C \rightarrow D$ be continuous. Define $f \times g : A \times C \rightarrow B \times D$ by*

$$(f \times g)(a, c) = (f(a), g(c)) .$$

Then $f \times g$ is continuous.

PROOF: $\pi_1 \circ (f \times g) = f \circ \pi_1$ and $\pi_2 \circ (f \times g) = g \circ \pi_2$ are continuous by Theorem 9.14.9. The result follows by Theorem 9.18.11.

Proposition 9.14.18. *Let X and Y be topological spaces and $f : X \rightarrow Y$ be continuous. If $a_n \rightarrow l$ as $n \rightarrow \infty$ in X then $f(a_n) \rightarrow f(l)$ as $n \rightarrow \infty$.*

PROOF:

- ⟨1⟩1. LET: V be a neighbourhood of $f(l)$
- ⟨1⟩2. PICK a neighbourhood U of l such that $f(U) \subseteq V$
- ⟨1⟩3. PICK N such that, for all $n \geq N$, we have $a_n \in U$
- ⟨1⟩4. For all $n \geq N$ we have $f(a_n) \in V$

□

9.15 Homeomorphisms

Definition 9.15.1 (Homeomorphism). Let X and Y be topological spaces. A *Homeomorphism* f between X and Y , $f : X \cong Y$, is a bijection $f : X \rightarrow Y$ such that both f and f^{-1} are continuous.

Lemma 9.15.2. *Let X and Y be topological spaces and $f : X \rightarrow Y$ a bijection. Then the following are equivalent:*

1. f is a homeomorphism.
2. f is continuous and an open map.
3. f is continuous and a closed map.
4. For any $U \subseteq X$, we have U is open if and only if $f(U)$ is open.

PROOF: Immediate from definitions. □

Proposition 9.15.3. *Let X and X' be the same set X under two topologies \mathcal{T} and \mathcal{T}' . Let $i : X \rightarrow X'$ be the identity function. Then i is a homeomorphism if and only if $\mathcal{T} = \mathcal{T}'$.*

PROOF: Immediate from definitions. □

Definition 9.15.4 (Topological Property). Let P be a property of topological spaces. Then P is a *topological* property if and only if, for any spaces X and Y , if P holds of X and $X \cong Y$ then P holds of Y .

Definition 9.15.5 (Topological Imbedding). Let X and Y be topological spaces and $f : X \rightarrow Y$. Then f is a *topological imbedding* if and only if the corestriction $f : X \rightarrow f(X)$ is a homeomorphism.

Proposition 9.15.6. Let X and Y be topological spaces and $a \in X$. The function $i : Y \rightarrow X \times Y$ that maps y to (a, y) is an imbedding.

PROOF:

$\langle 1 \rangle 1$. i is injective

$\langle 1 \rangle 2$. i is continuous.

PROOF: For U open in X and V open in Y , we have $i^{-1}(U \times V)$ is V if $a \in U$, and \emptyset if $a \notin U$.

$\langle 1 \rangle 3$. $i : Y \rightarrow i(Y)$ is an open map.

PROOF: For V open in Y we have $i(V) = (X \times V) \cap i(Y)$.

□

9.16 The Order Topology

Definition 9.16.1 (Order Topology). Let X be a linearly ordered set with at least two points. The *order topology* on X is the topology generated by the basis \mathcal{B} consisting of:

- all open intervals (a, b) ;
- all intervals of the form $[\perp, b)$ where \perp is least in X ;
- all intervals of the form $(a, \top]$ where \top is greatest in X .

We prove this is a basis for a topology.

PROOF:

$\langle 1 \rangle 1$. For all $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B$.

$\langle 2 \rangle 1$. LET: $x \in X$

$\langle 2 \rangle 2$. CASE: x is greatest in X .

$\langle 3 \rangle 1$. PICK $y \in X$ with $y \neq x$

$\langle 3 \rangle 2$. $x \in (y, x] \in \mathcal{B}$

$\langle 2 \rangle 3$. CASE: x is least in X .

$\langle 3 \rangle 1$. PICK $y \in X$ with $y \neq x$

$\langle 3 \rangle 2$. $x \in [x, y) \in \mathcal{B}$

$\langle 2 \rangle 4$. CASE: x is neither greatest nor least in X .

$\langle 3 \rangle 1$. PICK $a, b \in X$ with $a < x$ and $x < b$

$\langle 3 \rangle 2$. $x \in (a, b) \in \mathcal{B}$

$\langle 1 \rangle 2$. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$

$\langle 2 \rangle 1$. LET: $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$

$\langle 2 \rangle 2$. CASE: $B_1 = (a, b)$, $B_2 = (c, d)$

PROOF: Take $B_3 = (\max(a, c), \min(b, d))$.

$\langle 2 \rangle 3$. CASE: $B_1 = (a, b)$, $B_2 = [\perp, d)$

PROOF: Take $B_3 = (a, \min(b, d))$.

(2)4. CASE: $B_1 = (a, b)$, $B_2 = (c, \top]$

PROOF: Take $B_3 = (\max(a, c), b)$.

(2)5. CASE: $B_1 = [\perp, b)$, $B_2 = [\perp, d)$

PROOF: Take $B_3 = [\perp, \min(b, d))$.

(2)6. CASE: $B_1 = [\perp, b)$, $B_2 = (c, \top]$

PROOF: Take $B_3 = (c, b)$.

□

Lemma 9.16.2. *Let X be a linearly ordered set. Then the open rays form a subbasis for the order topology on X .*

PROOF:

(1)1. Every open ray is open.

(2)1. For all $a \in X$, the ray $(-\infty, a)$ is open.

(3)1. LET: $x \in (-\infty, a)$

(3)2. CASE: x is least in X

PROOF: $x \in \text{min}[x, a) = (-\infty, a)$.

(3)3. CASE: x is not least in X

(4)1. PICK $y < x$

(4)2. $x \in (y, a) \subseteq (-\infty, a)$

(2)2. For all $a \in X$, the ray $(a, +\infty)$ is open.

PROOF: Similar.

(1)2. Every basic open set is a finite intersection of open rays.

PROOF: We have $(a, b) = (a, +\infty) \cap (-\infty, b)$, $[\perp, b) = (-\infty, b)$ and $(a, \top] = (a, +\infty)$.

□

Definition 9.16.3 (Standard Topology on the Real Line). The *standard topology on the real line* is the order topology on \mathbb{R} generated by the standard order.

Lemma 9.16.4. *The lower limit topology is strictly finer than the standard topology on \mathbb{R} .*

PROOF:

(1)1. Every open interval is open in the lower limit topology.

PROOF: If $x \in (a, b)$ then $x \in [x, b) \subseteq (a, b)$.

(1)2. The half-open interval $[0, 1)$ is not open in the standard topology.

PROOF: There is no open interval (a, b) such that $0 \in (a, b) \subseteq [0, 1)$.

□

Lemma 9.16.5. *The K -topology is strictly finer than the standard topology on \mathbb{R} .*

PROOF:

(1)1. Every open interval is open in the K -topology.

PROOF: Corollary 9.9.2.1.

(1)2. The set $(-1, 1) \setminus K$ is not open in the standard topology.

PROOF: There is no open interval (a, b) such that $0 \in (a, b) \subseteq (-1, 1) \setminus K$, since there must be a positive integer n with $1/n \in (a, b)$.

□

Lemma 9.16.6. *Let X be a topological space. Let Y be a linearly ordered set in the order topology. Let $f, g : X \rightarrow Y$ be continuous. Then $C = \{x \in X \mid f(x) \leq g(x)\}$ is closed.*

PROOF:

⟨1⟩1. LET: $x \in X \setminus C$

⟨1⟩2. $f(x) > g(x)$

PROVE: There exists a neighbourhood U of x such that $U \subseteq X \setminus C$

⟨1⟩3. CASE: There exists y such that $g(x) < y < f(x)$

PROOF: Take $U = g^{-1}((-\infty, y)) \cup f^{-1}(y, +\infty)$.

⟨1⟩4. CASE: There is no y such that $g(x) < y < f(x)$

PROOF: Take $U = g^{-1}((-\infty, f(x))) \cup f^{-1}(g(x), +\infty)$.

□

Proposition 9.16.7. *Let X be a topological space. Let Y be a linearly ordered set in the order topology. Let $f, g : X \rightarrow Y$ be continuous. Define $h : X \rightarrow Y$ by $h(x) = \min(f(x), g(x))$. Then h is continuous.*

PROOF: By the Pasting Lemma applied to $\{x \in X \mid f(x) \leq g(x)\}$ and $\{x \in X \mid g(x) \leq f(x)\}$, which are closed by Lemma 9.16.6.

Proposition 9.16.8. *Let X and Y be linearly ordered sets in the order topology. Let $f : X \rightarrow Y$ be strictly monotone and surjective. Then f is a homeomorphism.*

PROOF:

⟨1⟩1. f is bijective.

PROOF: Proposition 4.2.3.

⟨1⟩2. f is continuous.

⟨2⟩1. For all $y \in Y$ we have $f^{-1}((y, +\infty))$ is open.

⟨3⟩1. LET: $y \in Y$

⟨3⟩2. PICK $x \in X$ such that $f(x) = y$

PROOF: Since f is surjective.

⟨3⟩3. $f^{-1}((y, +\infty)) = (x, +\infty)$

PROOF: By strict monotonicity.

⟨2⟩2. For all $y \in Y$ we have $f^{-1}((-\infty, y))$ is open.

PROOF: Similar.

⟨1⟩3. f^{-1} is continuous.

⟨2⟩1. For all $x \in X$ we have $f((x, +\infty))$ is open.

PROOF: $f((x, +\infty)) = (f(x), +\infty)$.

⟨2⟩2. For all $x \in X$ we have $f((-\infty, x))$ is open.

PROOF: $f((-\infty, x)) = (-\infty, f(x))$.

□

9.17 The nth Root Function

Proposition 9.17.1. *For all $n \geq 1$, the function $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ defined by $f(x) = x^n$ is a homomorphism.*

PROOF:

$\langle 1 \rangle 1$. f is strictly monotone.

$\langle 2 \rangle 1$. LET: $x, y \in \mathbb{R}$ with $0 \leq x < y$

$\langle 2 \rangle 2$. $x^n < y^n$

$$\begin{aligned} y^n - x^n &= (y - x)(y^{n-1} + y^{n-2}x + y^{n-3}x^2 + \cdots + x^{n-1}) \\ &> 0 \end{aligned}$$

$\langle 1 \rangle 2$. f is surjective.

$\langle 2 \rangle 1$. LET: $y \in \mathbb{R}_{\geq 0}$

$\langle 2 \rangle 2$. PICK $x \in \mathbb{R}$ such that $y \leq x^n$

PROOF: If $y \leq 1$ take $x = 1$, otherwise take $x = y$.

$\langle 2 \rangle 3$. There exists $x' \in [0, x]$ such that $(x')^n = y$

PROOF: By the Intermediate Value Theorem.

$\langle 1 \rangle 3$. Q.E.D.

PROOF: Proposition 9.16.8.

□

Definition 9.17.2. For $n \geq 1$, the n th root function is the function $\mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ that is the inverse of $\lambda x.x^n$.

9.18 The Product Topology

Definition 9.18.1 (Product Topology). Let $\{A_i\}_{i \in I}$ be a family of topological spaces. The *product topology* on $\prod_{i \in I} A_i$ is the topology generated by the sub-basis consisting of the sets of the form $\pi_i^{-1}(U)$ where $i \in I$ and U is open in A_i .

Proposition 9.18.2. *The product topology on $\prod_{i \in I} A_i$ is generated by the basis consisting of all sets of the form $\prod_{i \in I} U_i$ where $\{U_i\}_{i \in I}$ is a family such that each U_i is an open set in A_i and $U_i = A_i$ for all but finitely many i .*

PROOF: From Proposition 9.9.10. □

Proposition 9.18.3. *If A_i is closed in X_i for all $i \in I$ then $\prod_{i \in I} A_i$ is closed in $\prod_{i \in I} X_i$.*

PROOF:

$$\left(\prod_{i \in I} X_i \right) \setminus \left(\prod_{i \in I} A_i \right) = \bigcup_{j \in I} \left(\prod_{i \in I} X_i \setminus \pi_j^{-1}(A_j) \right) \square$$

Proposition 9.18.4. *Let $\{A_i\}_{i \in I}$ be a family of topological spaces. For $i \in I$, let \mathcal{B}_i be a basis for the topology on A_i . Then $\mathcal{B} = \{\prod_{i \in I} B_i \mid \forall i \in I. B_i \in \mathcal{B}_i, B_i = A_i \text{ for all but finitely many } i\}$ is a basis for the box topology on $\prod_{i \in I} A_i$.*

PROOF:

- ⟨1⟩1. Every set in \mathcal{B} is open.
- ⟨1⟩2. For every point $a \in \prod_{i \in I} A_i$ and every open set U with $a \in U$, there exists $B \in \mathcal{B}$ such that $a \in B \subseteq U$.
 - ⟨2⟩1. LET: U be open and $a \in U$
 - ⟨2⟩2. PICK a family $\{U_i\}_{i \in I}$ such that each U_i is open in A_i , such that $U_i = A_i$ except for $i = i_1, \dots, i_n$, and such that $a \in \prod_{i \in I} U_i \subseteq U$.
 - ⟨2⟩3. For $j = 1, \dots, n$, PICK $B_{i_j} \in \mathcal{B}_{i_j}$ such that $a_{i_j} \in B_{i_j} \subseteq U_{i_j}$
 - ⟨2⟩4. LET: $B = \prod_{i \in I} B_i$ where $B_i = A_i$ for $i \neq i_1, \dots, i_n$
 - ⟨2⟩5. $B \in \mathcal{B}$
 - ⟨2⟩6. $a \in B \subseteq U$
- ⟨1⟩3. Q.E.D.

PROOF: Lemma 9.9.3.

□

Proposition 9.18.5. *Let $\{A_i\}_{i \in I}$ be a family of topological spaces. Then the projections $\pi_i : \prod_{i \in I} A_i \rightarrow A_i$ are open maps.*

PROOF: From Lemma 9.13.2. □

Example 9.18.6. The projections are not always closed maps. For example, $\pi_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$ maps the closed set $\{(x, 1/x) \mid x > 0\}$ to $(0, +\infty)$.

Proposition 9.18.7. *Let $\{X_i\}_{i \in I}$ be a family of sets. For $i \in I$, let \mathcal{T}_i and \mathcal{U}_i be topologies on X_i . Let \mathcal{P} be the product topology on $\prod_{i \in I} X_i$ generated by the topologies \mathcal{T}_i , and \mathcal{Q} the product topology on the same set generated by the topologies \mathcal{U}_i . Then $\mathcal{P} \subseteq \mathcal{Q}$ if and only if $\mathcal{T}_i \subseteq \mathcal{U}_i$ for all i .*

PROOF:

- ⟨1⟩1. If $\mathcal{T}_i \subseteq \mathcal{U}_i$ for all i then $\mathcal{P} \subseteq \mathcal{Q}$

PROOF: By Corollary 9.9.2.1.
- ⟨1⟩2. If $\mathcal{P} \subseteq \mathcal{Q}$ then $\mathcal{T}_i \subseteq \mathcal{U}_i$ for all i
 - ⟨2⟩1. ASSUME: $\mathcal{P} \subseteq \mathcal{Q}$
 - ⟨2⟩2. LET: $i \in I$
 - ⟨2⟩3. LET: $U \in \mathcal{T}_i$
 - ⟨2⟩4. LET: $U_i = U$ and $U_j = X_j$ for $j \neq i$
 - ⟨2⟩5. $\prod_{i \in I} U_i \in \mathcal{P}$
 - ⟨2⟩6. $\prod_{i \in I} U_i \in \mathcal{Q}$
 - ⟨2⟩7. $U \in \mathcal{U}_i$

PROOF: From Proposition 9.18.5.

□

Proposition 9.18.8 (AC). *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let $A_i \subseteq X_i$ for all $i \in I$. Then*

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i} .$$

PROOF:

- (1)1. $\overline{\prod_{i \in I} A_i} \subseteq \prod_{i \in I} \overline{A_i}$
 (2)1. For all $i \in I$ we have $A_i \subseteq \overline{A_i}$
 PROOF: Lemma 9.6.3.
 (2)2. $\prod_{i \in I} A_i \subseteq \prod_{i \in I} \overline{A_i}$
 (2)3. Q.E.D.
 PROOF: Since $\prod_{i \in I} A_i$ is closed by Proposition 9.18.3.
 (1)2. $\prod_{i \in I} \overline{A_i} \subseteq \overline{\prod_{i \in I} A_i}$
 (2)1. LET: $x \in \prod_{i \in I} \overline{A_i}$
 (2)2. LET: U be a neighbourhood of x
 (2)3. PICK V_i open in X_i such that $x \in \prod_{i \in I} V_i \subseteq U$ with $V_i = X_i$ except for
 $i = i_1, \dots, i_n$
 (2)4. For $i \in I$, pick $a_i \in V_i \cap A_i$
 PROOF: By Theorem 9.6.6 and (2)1 using the Axiom of Choice.
 (2)5. U intersects $\prod_{i \in I} A_i$
 (2)6. Q.E.D.
 PROOF: $a \in U \cap \prod_{i \in I} A_i$

□

Example 9.18.9. The closure of \mathbb{R}^∞ in \mathbb{R}^ω is \mathbb{R}^ω

PROOF:

- (1)1. LET: $a \in \mathbb{R}^\omega$
 (1)2. LET: U be any neighbourhoods of a .
 (1)3. PICK U_n open in \mathbb{R} for all n such that $a \in \prod_{n \geq 0} U_n \subseteq U$ and $U_n = \mathbb{R}$ for
 all n except n_1, \dots, n_k
 (1)4. LET: $b_n = a_n$ for $n = n_1, \dots, n_k$ and $b_n = 0$ for all other n
 (1)5. $b \in \mathbb{R}^\infty \cap U$
 (1)6. Q.E.D.
 PROOF: From Theorem 9.6.6.

□

Proposition 9.18.10. Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let (a_n) be a sequence in $\prod_{i \in I} X_i$ and $l \in \prod_{i \in I} X_i$. Then $a_n \rightarrow l$ as $n \rightarrow \infty$ if and only if, for all $i \in I$, we have $\pi_i(a_n) \rightarrow \pi_i(l)$ as $n \rightarrow \infty$.

PROOF:

- (1)1. If $a_n \rightarrow l$ as $n \rightarrow \infty$ then, for all $i \in I$, we have $\pi_i(a_n) \rightarrow \pi_i(l)$ as $n \rightarrow \infty$
 PROOF: Proposition 9.14.18.
 (1)2. If, for all $i \in I$, we have $\pi_i(a_n) \rightarrow \pi_i(l)$ as $n \rightarrow \infty$, then $a_n \rightarrow l$ as $n \rightarrow \infty$
 (2)1. ASSUME: For all $i \in I$, we have $\pi_i(a_n) \rightarrow \pi_i(l)$ as $n \rightarrow \infty$
 (2)2. LET: V be a neighbourhood of l
 (2)3. PICK open sets U_i in X_i such that $l \in \prod_{i \in I} U_i \subseteq V$ and $U_i = X_i$ for all
 i except $i = i_1, \dots, i_k$
 (2)4. For $j = 1, \dots, k$, PICK N_j such that, for all $n \geq N_j$, we have $\pi_{i_j}(a_n) \in$
 U_{i_j}
 (2)5. LET: $N = \max(N_1, \dots, N_k)$
 (2)6. For all $n \geq N$ we have $a_n \in V$

□

Theorem 9.18.11. *Let A be a topological space and $\{X_i\}_{i \in I}$ be a family of topological spaces. Let $f : A \rightarrow \prod_{i \in I} X_i$ be a function. If $\pi_i \circ f$ is continuous for all $i \in I$ then f is continuous.*

PROOF:

⟨1⟩1. LET: $i \in I$ and U be open in X_i

⟨1⟩2. $f^{-1}(\pi_i^{-1}(U))$ is open in A

⟨1⟩3. Q.E.D.

PROOF: Proposition 9.14.3.

□

9.18.1 Continuous in Each Variable Separately

Definition 9.18.12 (Continuous in Each Variable Separately). Let $F : X \times Y \rightarrow Z$. Then F is *continuous in each variable separately* if and only if:

- for every $a \in X$ the function $\lambda y \in Y. F(a, y)$ is continuous;
- for every $b \in Y$ the function $\lambda x \in X. F(x, b)$ is continuous.

Proposition 9.18.13. *Let $F : X \times Y \rightarrow Z$. If F is continuous then F is continuous in each variable separately.*

PROOF: For $a \in X$, the function $\lambda y \in Y. F(a, y)$ is $F \circ i$ where $i : Y \rightarrow X \times Y$ maps y to (a, y) . We have i is continuous by Proposition 9.15.6, hence $F \circ i$ is continuous by Theorem 9.14.9.

Similarly for $\lambda x \in X. F(x, b)$ for $b \in Y$. □

Example 9.18.14. Define $F : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$F(x, y) = \begin{cases} xy/(x^2 + y^2) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Then F is continuous in each variable separately but not continuous.

Proposition 9.18.15. *Let $f : A \rightarrow C$ and $g : B \rightarrow D$ be open maps. Then $f \times g : A \times B \rightarrow C \times D$ is an open map.*

PROOF: Given U open in A and V open in B . Then $(f \times g)(U \times V) = f(U) \times g(V)$ is open in $C \times D$. The result follows from Proposition 9.13.3. □

Definition 9.18.16 (Sorgenfrey Plane). The *Sorgenfrey plane* is \mathbb{R}_l^2 .

9.19 The Subspace Topology

Definition 9.19.1 (Subspace Topology). Let X be a topological space and $Y \subseteq X$. The *subspace topology* on Y is $\mathcal{T} = \{U \cap Y \mid U \text{ is open in } X\}$.

We prove this is a topology.

PROOF:

(1)1. $Y \in \mathcal{T}$

PROOF: Since $Y = X \cap Y$

(1)2. For all $\mathcal{U} \subseteq \mathcal{T}$, we have $\bigcup \mathcal{U} \in \mathcal{T}$

(2)1. LET: $\mathcal{U} \subseteq \mathcal{T}$

(2)2. LET: $\mathcal{V} = \{V \text{ open in } X \mid V \cap Y \in \mathcal{U}\}$

(2)3. $\bigcup \mathcal{U} = (\bigcup \mathcal{V}) \cap Y$

(1)3. For all $U, V \in \mathcal{T}$, we have $U \cap V \in \mathcal{T}$

(2)1. LET: $U, V \in \mathcal{T}$

(2)2. PICK U', V' open in X such that $U = U' \cap Y$ and $V = V' \cap Y$

(2)3. $(U \cap V) = (U' \cap V') \cap Y$

□

Theorem 9.19.2. *Let X be a topological space and Y a subspace of X . Let $A \subseteq Y$. Then A is closed in Y if and only if there exists a closed set C in X such that $A = C \cap Y$.*

PROOF: We have

A is closed in Y

$\Leftrightarrow Y \setminus A$ is open in Y

$\Leftrightarrow \exists U$ open in $X. Y \setminus A = Y \cap U$

$\Leftrightarrow \exists C$ closed in $X. Y \setminus A = Y \cap (X \setminus U)$

$\Leftrightarrow \exists C$ closed in $X. A = Y \cap U$

□

Theorem 9.19.3. *Let Y be a subspace of X . Let $A \subseteq Y$. Let \bar{A} be the closure of A in X . Then the closure of A in Y is $\bar{A} \cap Y$.*

PROOF: The closure of A in Y is

$\bigcap \{C \text{ closed in } Y \mid A \subseteq C\}$

$= \bigcap \{D \cap Y \mid D \text{ closed in } X, A \subseteq D \cap Y\}$ (Theorem 9.19.2)

$= \bigcap \{D \mid D \text{ closed in } X, A \subseteq D\} \cap Y$

$= \bar{A} \cap Y$

□

Lemma 9.19.4. *Let X be a topological space and $Y \subseteq X$. Let \mathcal{B} be a basis for the topology on X . Then $\mathcal{B}' = \{B \cap Y \mid B \in \mathcal{B}\}$ is a basis for the subspace topology on Y .*

PROOF:

(1)1. Every element in \mathcal{B}' is open in Y

(1)2. For every open set U in Y and point $y \in U$, there exists $B' \in \mathcal{B}'$ such that $y \in B' \subseteq U$

(2)1. LET: U be open in Y and $y \in U$

(2)2. PICK V open in X such that $U = V \cap Y$

(2)3. PICK $B \in \mathcal{B}$ such that $y \in B \subseteq V$

- (2)4. LET: $B' = B \cap Y$
 (2)5. $B' \in \mathcal{B}'$
 (2)6. $y \in B' \subseteq U$
 (1)3. Q.E.D.

PROOF: By Lemma 9.9.3.

□

Lemma 9.19.5. *Let X be a topological space and $Y \subseteq X$. Let \mathcal{S} be a basis for the topology on X . Then $\mathcal{S}' = \{S \cap Y \mid S \in \mathcal{S}\}$ is a subbasis for the subspace topology on Y .*

PROOF: The set $\{B \cap Y \mid B \text{ is a finite intersection of elements of } \mathcal{S}\}$ is a basis for the subspace topology by Lemma 9.19.4, and this is the set of all finite intersections of elements of \mathcal{S}' . □

Lemma 9.19.6. *Let Y be a subspace of X . If U is open in Y and Y is open in X then U is open in X .*

PROOF:

- (1)1. PICK V open in X such that $U = V \cap Y$
 (1)2. U is open in X

PROOF: Since it is the intersection of two open sets V and Y .

□

Theorem 9.19.7. *Let Y be a subspace of X and $A \subseteq Y$. If A is closed in Y and Y is closed in X then A is closed in X .*

PROOF: Pick a closed set C in X such that $A = C \cap Y$ (Theorem 9.19.2). Then A is the intersection of two sets closed in X , hence A is closed in X (Lemma 9.4.4). □

Theorem 9.19.8. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let A_i be a subspace of X_i for all $i \in I$. Then the product topology on $\prod_{i \in I} A_i$ is the same as the topology it inherits as a subspace of $\prod_{i \in I} X_i$.*

PROOF: The product topology is generated by the subbasis

$$\begin{aligned}
 & \{\pi_i^{-1}(U) \mid i \in I, U \text{ open in } A_i\} \\
 &= \{\pi_i^{-1}(V) \cap A_i \mid i \in I, V \text{ open in } X_i\} \\
 &= \{\pi_i^{-1}(V) \mid i \in I, V \text{ open in } X_i\} \cap \prod_{i \in I} A_i
 \end{aligned}$$

and this is a subbasis for the subspace topology by Lemma 9.19.5. □

Theorem 9.19.9. *Let X be an ordered set in the order topology. Let $Y \subseteq X$ be an interval. Then the order topology on Y is the same as the subspace topology on Y .*

PROOF:

- (1)1. The order topology is finer than the subspace topology.

- (2)1. For every open ray R in X , the set $R \cap Y$ is open in the order topology.
 (3)1. For all $a \in X$, we have $(-\infty, a) \cap Y$ is open in the order topology.
 (4)1. CASE: For all $y \in Y$ we have $y < a$
 PROOF: In this case $(-\infty, a) \cap Y = Y$.
 (4)2. CASE: For all $y \in Y$ we have $a < y$
 PROOF: In this case $(-\infty, a) \cap Y = \emptyset$.
 (4)3. CASE: There exists $y \in Y$ such that $y \leq a$ and $y \in Y$ such that

$$a \leq y$$

 (5)1. $a \in Y$
 PROOF: Because Y is an interval.
 (5)2. $(-\infty, a) \cap Y = \{y \in Y \mid y < a\}$
 (3)2. For all $a \in X$, we have $(a, +\infty) \cap Y$ is open in the order topology.
 PROOF: Similar.
 (2)2. Q.E.D.
 PROOF: By Lemmas 9.16.2 and 9.19.5 and Proposition 9.9.11.
 (1)2. The subspace topology is finer than the order topology.
 (2)1. Every open ray in Y is open in the subspace topology.
 PROOF: For any $a \in Y$ we have $(-\infty, a)_Y = (-\infty, a)_X \cap Y$ and $(a, +\infty)_Y = (a, +\infty)_X \cap Y$.
 (2)2. Q.E.D.
 PROOF: By Lemma 9.16.2 and Proposition 9.9.11

□

This example shows that we cannot remove the hypothesis that Y is an interval:

Example 9.19.10. The order topology on I_o^2 is different from the subspace topology as a subspace of \mathbb{R}^2 under the dictionary order topology.

PROOF: The set $\{1/2\} \times (1/2, 1)$ is open in the subspace topology but not in the order topology. □

Proposition 9.19.11. *Let X be a topological space, Y a subspace of X , and Z a subspace of Y . Then the subspace topology on Z inherited from X is the same as the subspace topology on Z inherited from Y .*

PROOF: The subspace topology inherited from Y is

$$\begin{aligned}
 & \{V \cap Z \mid V \text{ open in } Y\} \\
 &= \{U \cap Y \cap Z \mid U \text{ open in } X\} \\
 &= \{U \cap Z \mid U \text{ open in } X\}
 \end{aligned}$$

which is the subspace topology inherited from X . □

Definition 9.19.12 (Unit Circle). The *unit circle* S^1 is $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ as a subspace of \mathbb{R}^2 .

Definition 9.19.13 (Unit 2-sphere). The *unit 2-sphere* is $S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$ as a subspace of \mathbb{R}^3 .

Proposition 9.19.14. *Let $f : X \rightarrow Y$ be an open map and $A \subseteq X$ be open. Then the restriction $f \upharpoonright A : A \rightarrow f(A)$ is an open map.*

PROOF:

$\langle 1 \rangle 1.$ LET: U be open in A

$\langle 1 \rangle 2.$ U is open in X

PROOF: Lemma 9.19.6.

$\langle 1 \rangle 3.$ $f(U)$ is open in Y

$\langle 1 \rangle 4.$ $f(U)$ is open in $f(A)$

PROOF: Since $f(U) = f(U) \cap f(A)$.

□

Example 9.19.15. This example shows that we cannot remove the hypothesis that A is open.

Let $A = \{(x, y) \in \mathbb{R}^2 \mid (x > 0 \text{ and } y = 1/x) \text{ or } x = y = 0\}$. Then $\pi_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$ is an open map, but $\pi_1 \upharpoonright A : A \rightarrow [0, +\infty)$ is not, because it maps the set $\{0, 0\}$ which is open in A to $\{0\}$ which is not open in $[0, +\infty)$.

Proposition 9.19.16. *Let Y be a subspace of X . Let $A \subseteq Y$ and $l \in Y$. Then l is a limit point of A in Y if and only if l is a limit point of A in X .*

PROOF: Both are equivalent to the condition that any neighbourhood of l in X intersects A in a point other than l . □

9.20 The Box Topology

Definition 9.20.1 (Box Topology). Let $\{A_i\}_{i \in I}$ be a family of topological spaces. The *box topology* on $\prod_{i \in I} A_i$ is the topology generated by the set of all sets of the form $\prod_{i \in I} U_i$ where $\{U_i\}_{i \in I}$ is a family such that each U_i is open in A_i .

This is a basis since it covers $\prod_{i \in I} A_i$ and is closed under intersection.

Proposition 9.20.2. *The box topology is finer than the product topology.*

PROOF: From Proposition 9.18.2. □

Corollary 9.20.2.1. *If A_i is closed in X_i for all $i \in I$ then $\prod_{i \in I} A_i$ is closed in $\prod_{i \in I} X_i$ under the box topology.*

PROOF: From Proposition 9.18.3.

Proposition 9.20.3 (AC). *Let $\{A_i\}_{i \in I}$ be a family of topological spaces. For $i \in I$, let \mathcal{B}_i be a basis for the topology on A_i . Then $\mathcal{B} = \{\prod_{i \in I} B_i \mid \forall i \in I, B_i \in \mathcal{B}_i\}$ is a basis for the box topology on $\prod_{i \in I} A_i$.*

PROOF:

$\langle 1 \rangle 1.$ Every set of the form $\prod_{i \in I} B_i$ is open.

$\langle 1 \rangle 2.$ For every point $a \in \prod_{i \in I} A_i$ and every open set U with $a \in U$, there exists $B \in \mathcal{B}$ such that $a \in B \subseteq U$.

- (2)1. LET: U be open and $a \in U$
 (2)2. PICK a family $\{U_i\}_{i \in I}$ such that each U_i is open in A_i and $a \in \prod_{i \in I} U_i \subseteq U$.
 (2)3. For $i \in I$, PICK $B_i \in \mathcal{B}_i$ such that $a_i \in B_i \subseteq U_i$
 PROOF: Using the Axiom of Choice.
 (2)4. $a \in \prod_{i \in I} B_i \subseteq U$
 (1)3. Q.E.D.
 PROOF: Lemma 9.9.3.
 \square

Theorem 9.20.4. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let A_i be a subspace of X_i for all $i \in I$. Give $\prod_{i \in I} X_i$ the box topology. Then the box topology on $\prod_{i \in I} A_i$ is the same as the topology it inherits as a subspace of $\prod_{i \in I} X_i$.*

PROOF: The box topology is generated by the basis

$$\begin{aligned}
 & \left\{ \prod_{i \in I} U_i \mid \forall i \in I, U_i \text{ open in } A_i \right\} \\
 &= \left\{ \prod_{i \in I} (V_i \cap A_i) \mid \forall i \in I, V_i \text{ open in } X_i \right\} \\
 &= \left\{ \prod_{i \in I} V_i \mid \forall i \in I, V_i \text{ open in } X_i \right\} \cap \prod_{i \in I} A_i
 \end{aligned}$$

and this is a basis for the subspace topology by Lemma 9.19.4. \square

Proposition 9.20.5 (AC). *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Give $\prod_{i \in I} X_i$ the box topology. Let $A_i \subseteq X_i$ for all $i \in I$. Then*

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i}.$$

PROOF:

- (1)1. $\overline{\prod_{i \in I} A_i} \subseteq \prod_{i \in I} \overline{A_i}$
 (2)1. For all $i \in I$ we have $A_i \subseteq \overline{A_i}$
 PROOF: Lemma 9.6.3.
 (2)2. $\prod_{i \in I} A_i \subseteq \prod_{i \in I} \overline{A_i}$
 (2)3. Q.E.D.
 PROOF: Since $\prod_{i \in I} A_i$ is closed by Corollary 9.20.2.1.
 (1)2. $\prod_{i \in I} \overline{A_i} \subseteq \overline{\prod_{i \in I} A_i}$
 (2)1. LET: $x \in \prod_{i \in I} \overline{A_i}$
 (2)2. LET: U be a neighbourhood of x
 (2)3. PICK V_i open in X_i such that $x \in \prod_{i \in I} V_i \subseteq U$
 (2)4. For $i \in I$, pick $a_i \in V_i \cap A_i$
 PROOF: By Theorem 9.6.6 and (2)1 using the Axiom of Choice.
 (2)5. U intersects $\prod_{i \in I} A_i$
 (2)6. Q.E.D.
 PROOF: $a \in U \cap \prod_{i \in I} A_i$.

□

The following example shows that Theorem 9.18.11 fails in the box topology.

Example 9.20.6. Define $f : \mathbb{R} \rightarrow \mathbb{R}^\omega$ by $f(t) = (t, t, \dots)$. Then $\pi_n \circ f = \text{id}_{\mathbb{R}}$ is continuous for all n . But f is not continuous when \mathbb{R}^ω is given the box topology because the inverse image of

$$(-1, 1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times \dots$$

is $\{0\}$ which is not open.

The following example shows that Proposition 9.18.10 fails in the box topology.

Example 9.20.7. Give \mathbb{R}^ω the box topology. Let $a_n = (1/n, 1/n, \dots)$ for $n \geq 1$ and $l = (0, 0, \dots)$. Then $\pi_i(a_n) \rightarrow \pi_i(l)$ as $n \rightarrow \infty$ for all i , but $a_n \not\rightarrow l$ as $n \rightarrow \infty$ since the open set

$$(-1, 1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times \dots$$

contains l but does not contain any a_n .

Example 9.20.8. The set \mathbb{R}^∞ is closed in \mathbb{R}^ω under the box topology. For let (a_n) be any sequence not in \mathbb{R}^∞ . Let U_n be an open interval around a_n that does not contain 0 if $a_n \neq 0$, and $U_n = \mathbb{R}$ if $a_n = 0$. Then $\prod_{n \geq 0} U_n$ is a neighbourhood of (a_n) that does not intersect \mathbb{R}^∞ .

9.21 T_1 Spaces

Definition 9.21.1 (T_1 Space). A topological space is T_1 if and only if every singleton is closed.

Lemma 9.21.2. A space is T_1 if and only if every finite set is closed.

PROOF: From Lemma 9.4.5. □

Theorem 9.21.3. In a T_1 space, a point a is a limit point of a set A if and only if every neighbourhood of a contains infinitely many points of A .

PROOF:

⟨1⟩1. If a is a limit point of A then every neighbourhood of a contains infinitely many points of A .

⟨2⟩1. ASSUME: a is a limit point of A .

⟨2⟩2. LET: U be a neighbourhood of a .

⟨2⟩3. ASSUME: for a contradiction U contains only finitely many points of A .

⟨2⟩4. $(U \cap A) \setminus \{a\}$ is closed.

PROOF: By the T_1 axiom.

⟨2⟩5. $(U \setminus A) \cup \{a\}$ is open.

PROOF: It is $U \setminus ((U \cap A) \setminus \{a\})$.

⟨2⟩6. $(U \setminus A) \cup \{a\}$ intersects A in a point other than a .

PROOF: From ⟨2⟩1.

⟨2⟩7. Q.E.D.

□

⟨1⟩2. If every neighbourhood of a contains infinitely many points of A then a is a limit point of A .

PROOF: Immediate from definitions.

□

(To see this does not hold in every space, see Proposition 9.8.4.)

Proposition 9.21.4. *A space is T_1 if and only if, for any two distinct points x and y , there exist neighbourhoods U of x and V of y such that $x \notin V$ and $y \notin U$.*

PROOF:

⟨1⟩1. LET: X be a topological space.

⟨1⟩2. If X is T_1 then, for any two distinct points x and y , there exist neighbourhoods U of x and V of y such that $x \notin V$ and $y \notin U$.

PROOF: This holds because $\{x\}$ and $\{y\}$ are closed.

⟨1⟩3. Suppose, for any two distinct points x and y , there exist neighbourhoods U of x and V of y such that $x \notin V$ and $y \notin U$. Then X is T_1 .

⟨2⟩1. ASSUME: For any two distinct points x and y , there exist neighbourhoods U of x and V of y such that $x \notin V$ and $y \notin U$.

⟨2⟩2. LET: $a \in X$

⟨2⟩3. $\{a\}$ is closed.

PROOF: For all $b \neq a$ there exists a neighbourhood U of b such that $U \subseteq X \setminus \{a\}$.

□

Proposition 9.21.5. *A subspace of a T_1 space is T_1 .*

PROOF: From Proposition 9.19.7.

9.22 Hausdorff Spaces

Definition 9.22.1 (Hausdorff Space). A topological space is *Hausdorff* if and only if, for any points x, y with $x \neq y$, there exist disjoint open sets U and V such that $x \in U$ and $y \in V$.

Theorem 9.22.2. *Every Hausdorff space is T_1 .*

PROOF:

⟨1⟩1. LET: X be a Hausdorff space.

⟨1⟩2. LET: $b \in X$

PROVE: $\overline{\{b\}} = \{b\}$

⟨1⟩3. ASSUME: $a \in \overline{\{b\}}$ and $a \neq b$

⟨1⟩4. PICK disjoint neighbourhoods U of a and V of b .

$\langle 1 \rangle 5$. U intersects $\{b\}$

PROOF: Theorem 9.6.6.

$\langle 1 \rangle 6$. $b \in U$

$\langle 1 \rangle 7$. Q.E.D.

PROOF: This contradicts the fact that U and V are disjoint ($\langle 1 \rangle 4$).

□

Proposition 9.22.3. *An infinite set under the finite complement topology is T_1 but not Hausdorff.*

PROOF:

$\langle 1 \rangle 1$. LET: X be an infinite set under the finite complement topology.

$\langle 1 \rangle 2$. Every singleton is closed.

PROOF: By definition.

$\langle 1 \rangle 3$. PICK $a, b \in X$ with $a \neq b$

$\langle 1 \rangle 4$. There are no disjoint neighbourhoods U of a and V of b .

$\langle 2 \rangle 1$. LET: U be a neighbourhood of a and V a neighbourhood of b .

$\langle 2 \rangle 2$. $X \setminus U$ and $X \setminus V$ are finite.

$\langle 2 \rangle 3$. PICK $c \in X$ that is not in $X \setminus U$ or $X \setminus V$.

$\langle 2 \rangle 4$. $c \in U \cap V$

□

Proposition 9.22.4. *The product of a family of Hausdorff spaces is Hausdorff.*

PROOF:

$\langle 1 \rangle 1$. LET: $\{X_i\}_{i \in I}$ be a family of Hausdorff spaces.

$\langle 1 \rangle 2$. LET: $a, b \in \prod_{i \in I} X_i$ with $a \neq b$

$\langle 1 \rangle 3$. PICK $i \in I$ such that $a_i \neq b_i$

$\langle 1 \rangle 4$. PICK U, V disjoint open sets in X_i with $a_i \in U$ and $b_i \in V$

$\langle 1 \rangle 5$. $\pi_i^{-1}(U)$ and $\pi_i^{-1}(V)$ are disjoint open sets in $\prod_{i \in I} X_i$ with $a \in \pi_i^{-1}(U)$ and $b \in \pi_i^{-1}(V)$

□

Theorem 9.22.5. *Every linearly ordered set under the order topology is Hausdorff.*

PROOF:

$\langle 1 \rangle 1$. LET: X be a linearly ordered set under the order topology.

$\langle 1 \rangle 2$. LET: $a, b \in X$ with $a \neq b$

$\langle 1 \rangle 3$. ASSUME: w.l.o.g. $a < b$

$\langle 1 \rangle 4$. CASE: There exists c such that $a < c < b$

PROOF: The sets $(-\infty, c)$ and $(c, +\infty)$ are disjoint neighbourhoods of a and b respectively.

$\langle 1 \rangle 5$. CASE: There is no c such that $a < c < b$

PROOF: The sets $(-\infty, b)$ and $(a, +\infty)$ are disjoint neighbourhoods of a and b respectively.

□

Theorem 9.22.6. *A subspace of a Hausdorff space is Hausdorff.*

PROOF:

- ⟨1⟩1. LET: X be a Hausdorff space and Y a subspace of X .
- ⟨1⟩2. LET: $x, y \in Y$ with $x \neq y$
- ⟨1⟩3. PICK disjoint neighbourhoods U of x and V of y in X .
- ⟨1⟩4. $U \cap Y$ and $V \cap Y$ are disjoint neighbourhoods of x and y respectively in Y .

□

Proposition 9.22.7. *A space X is Hausdorff if and only if the diagonal $\Delta = \{(x, x) \mid x \in X\}$ is closed in X^2 .*

PROOF:

X is Hausdorff

$$\Leftrightarrow \forall x, y \in X. x \neq y \Rightarrow \exists V, W \text{ open. } x \in V \wedge y \in W \wedge V \cap W = \emptyset$$

$$\Leftrightarrow \forall (x, y) \in X^2 \setminus \Delta. \exists V, W \text{ open. } (x, y) \in V \times W \subseteq X^2 \setminus \Delta$$

$$\Leftrightarrow \Delta \text{ is closed}$$

□

Theorem 9.22.8. *In a Hausdorff space, a sequence has at most one limit.*

PROOF:

- ⟨1⟩1. LET: X be a Hausdorff space.
- ⟨1⟩2. ASSUME: for a contradiction $a_n \rightarrow l$ as $n \rightarrow \infty$, $a_n \rightarrow m$ as $n \rightarrow \infty$, and $l \neq m$
- ⟨1⟩3. PICK disjoint neighbourhoods U of l and V of m
PROOF: By the Hausdorff axiom.
- ⟨1⟩4. PICK M and N such that $a_n \in U$ for $n \geq M$ and $a_n \in V$ for $n \geq N$
- ⟨1⟩5. $a_{\max(M, N)} \in U \cap V$
- ⟨1⟩6. Q.E.D.

PROOF: This contradicts the fact that U and V are disjoint ((1)3).

□

To see this is not always true in spaces that are T_1 but not Hausdorff:

Proposition 9.22.9. *Let X be an infinite set under the finite complement topology. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence with all points distinct. Then for every $l \in X$ we have $a_n \rightarrow l$ as $n \rightarrow \infty$.*

PROOF: Let U be any neighbourhood of l . Since $X \setminus U$ is finite, there must exist N such that, for all $n \geq N$, we have $a_n \in U$. □

Proposition 9.22.10. *Let X be a topological space. Let Y a Hausdorff space. Let $A \subseteq X$. Let $f, g : \overline{A} \rightarrow Y$ be continuous. If f and g agree on A then $f = g$.*

PROOF:

- ⟨1⟩1. LET: $x \in \overline{A}$
- ⟨1⟩2. ASSUME: $f(x) \neq g(x)$
- ⟨1⟩3. PICK disjoint neighbourhoods V of $f(x)$ and W of $g(x)$.
- ⟨1⟩4. PICK $y \in f^{-1}(V) \cap g^{-1}(W) \cap A$

PROOF: Since $f^{-1}(V) \cap g^{-1}(W)$ is a neighbourhood of x and hence intersects A .

$\langle 1 \rangle 5$. $f(y) = g(y) \in V \cap W$

$\langle 1 \rangle 6$. Q.E.D.

PROOF: This contradicts the fact that V and W are disjoint ($\langle 1 \rangle 3$).

□

Proposition 9.22.11. *Let $\{X_i\}_{i \in I}$ be a family of Hausdorff spaces. Then $\prod_{i \in I} X_i$ under the box topology is Hausdorff.*

PROOF:

$\langle 1 \rangle 1$. LET: $\{X_i\}_{i \in I}$ be a family of Hausdorff spaces.

$\langle 1 \rangle 2$. LET: $a, b \in \prod_{i \in I} X_i$ with $a \neq b$

$\langle 1 \rangle 3$. PICK $i \in I$ such that $a_i \neq b_i$

$\langle 1 \rangle 4$. PICK U, V disjoint open sets in X_i with $a_i \in U$ and $b_i \in V$

$\langle 1 \rangle 5$. $\pi_i^{-1}(U)$ and $\pi_i^{-1}(V)$ are disjoint open sets in $\prod_{i \in I} X_i$ with $a \in \pi_i^{-1}(U)$ and $b \in \pi_i^{-1}(V)$

□

Proposition 9.22.12. *Let \mathcal{T} and \mathcal{T}' be two topologies on the same set X with $\mathcal{T} \subseteq \mathcal{T}'$. If \mathcal{T} is Hausdorff then \mathcal{T}' is Hausdorff.*

PROOF: Immediate from definitions.

Proposition 9.22.13. *Let X be a Hausdorff space. Let $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the finite intersection property. Then $\bigcap_{D \in \mathcal{D}} \overline{D}$ contains at most one point.*

PROOF:

$\langle 1 \rangle 1$. LET: $x, y \in \bigcap_{D \in \mathcal{D}} \overline{D}$

$\langle 1 \rangle 2$. ASSUME: for a contradiction $x \neq y$

$\langle 1 \rangle 3$. PICK disjoint open subsets U and V of x and y respectively.

$\langle 1 \rangle 4$. $U, V \in \mathcal{D}$

PROOF: Proposition 9.6.9.

$\langle 1 \rangle 5$. Q.E.D.

PROOF: This contradicts the fact that \mathcal{D} satisfies the finite intersection property.

□

9.23 The First Countability Axiom

Definition 9.23.1 (First Countability Axiom). A topological space X satisfies the *first countability axiom*, or is *first countable*, if and only if every point has a countable local basis.

Example 9.23.2. The space S_Ω is first countable. For any $\alpha \in S_\Omega$, the set $\{(\beta, \alpha + 1) \mid \beta < \alpha\} \cup \{[0, \alpha + 1)\}$ is a local basis at α .

Lemma 9.23.3 (Sequence Lemma (CC)). *Let X be a first countable space. Let $A \subseteq X$ and $l \in \overline{A}$. Then there exists a sequence in A that converges to l .*

PROOF:

⟨1⟩1. PICK a countable local basis $\{B_n \mid n \in \mathbb{Z}^+\}$ at l such that $B_1 \supseteq B_2 \supseteq \dots$.

PROOF: Lemma 9.10.2.

⟨1⟩2. For all $n \geq 1$, PICK $a_n \in A \cap B_n$.

PROVE: $a_n \rightarrow l$ as $n \rightarrow \infty$

⟨1⟩3. LET: U be a neighbourhood of l

⟨1⟩4. PICK N such that $B_N \subseteq U$

⟨1⟩5. For $n \geq N$ we have $a_n \in U$

PROOF: $a_n \in B_n \subseteq B_N \subseteq U$

□

Example 9.23.4. The space $\overline{S_\Omega}$ is not first countable, since Ω is a limit point for S_Ω but there is no sequence of points in S_Ω that converges to Ω .

Theorem 9.23.5 (CC). *Let X be a first countable space and Y a topological space. Let $f : X \rightarrow Y$. Suppose that, for every sequence (x_n) in X and $l \in X$, if $x_n \rightarrow l$ as $n \rightarrow \infty$, then $f(x_n) \rightarrow f(l)$ as $n \rightarrow \infty$. Then f is continuous.*

PROOF:

⟨1⟩1. LET: $A \subseteq X$

⟨1⟩2. LET: $a \in A$

PROVE: $f(a) \in \overline{f(A)}$

⟨1⟩3. PICK a sequence (x_n) in A that converges to a .

PROOF: By the Sequence Lemma.

⟨1⟩4. $f(x_n) \rightarrow f(a)$

⟨1⟩5. $f(a) \in \overline{f(A)}$

PROOF: By Lemma 9.11.2.

⟨1⟩6. Q.E.D.

PROOF: By Theorem 9.14.6.

□

Example 9.23.6 (CC). The space \mathbb{R}^ω under the box product is not first countable.

PROOF: Let $\{B_n \mid n \geq 0\}$ be a countable set of neighbourhoods of $\vec{0}$. We will construct a neighbourhood of $\vec{0}$ that does not include any of these.

For $n \geq 0$, pick a neighbourhood U_n of 0 such that $U_n \subset \pi_n(B_p)$. Then $\prod_{n=0}^\infty U_n$ is a neighbourhood of $\vec{0}$ that does not include any B_n . □

Example 9.23.7. If J is an uncountable set then \mathbb{R}^J is not first countable.

PROOF:

⟨1⟩1. LET: $\{B_n \mid n \geq 0\}$ be any countable set of neighbourhoods of $\vec{0}$.

⟨1⟩2. For $n \geq 0$, PICK a basis element $\prod_{\alpha \in J} U_{n\alpha}$ that contains $\vec{0}$ and is included in B_n .

PROOF: Using the Axiom of Countable Choice.

⟨1⟩3. For $n \geq 0$,

LET: $J_n = \{\alpha \in J \mid U_{n\alpha} \neq \mathbb{R}\}$

⟨1⟩4. PICK $\beta \in J$ such that $\beta \notin J_n$ for any n .

PROOF: Since each J_n is finite so $\bigcup_n J_n$ is countable.

⟨1⟩5. $\pi_\beta((-1, 1))$ is a neighbourhood of $\vec{0}$ that does not include any B_n .

□

Example 9.23.8. The space \mathbb{R}_l is first countable.

PROOF: For any $a \in \mathbb{R}$, the set $\{[a, a + 1/n) \mid n \geq 1\}$ is a countable local basis.

Example 9.23.9. The ordered square is first countable.

PROOF: For any $(a, b) \in I_o^2$ with $b \neq 0, 1$, the set $\{(\{a\} \times (b - 1/n, b + 1/n)) \cap I_o^2 \mid n \geq 1\}$ is a countable local basis.

9.24 Strong Continuity

Definition 9.24.1 (Strongly Continuous). Let X and Y be topological spaces and $f : X \rightarrow Y$ be a function. Then f is *strongly continuous* if and only if, for every subset $U \subseteq Y$, we have U is open in Y if and only if $f^{-1}(U)$ is open in X .

Proposition 9.24.2. Let X and Y be topological spaces and $f : X \rightarrow Y$ be a function. Then f is strongly continuous if and only if, for every subset $C \subseteq Y$, we have C is closed in Y if and only if $f^{-1}(C)$ is closed in X .

PROOF: Since $X \setminus f^{-1}(C) = f^{-1}(Y \setminus C)$. □

Proposition 9.24.3. Let X, Y and Z be topological spaces. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$. If f and g are strongly continuous then so is $g \circ f$.

PROOF: Since $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$. □

Proposition 9.24.4. Let X, Y and Z be topological spaces. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$. If $g \circ f$ is continuous and f is strongly continuous then g is continuous.

PROOF:

⟨1⟩1. LET: $V \subseteq Z$ be open.

⟨1⟩2. $f^{-1}(g^{-1}(V))$ is open in X .

PROOF: Since $g \circ f$ is continuous.

⟨1⟩3. $f^{-1}(V)$ is open in Y .

PROOF: Since g is strongly continuous.

□

Proposition 9.24.5. Let X, Y and Z be topological spaces. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$. If $g \circ f$ is strongly continuous and f is strongly continuous then g is strongly continuous.

PROOF: For $V \subseteq Z$, we have V is open iff $f^{-1}(g^{-1}(V))$ is open iff $g^{-1}(V)$ is open.

9.25 Saturated Sets

Definition 9.25.1. Let X and Y be sets and $p : X \twoheadrightarrow Y$ a surjective function. Let $C \subseteq X$. Then C is *saturated* with respect to p if and only if, for all $x, y \in X$, if $x \in C$ and $p(x) = p(y)$ then $y \in C$.

Proposition 9.25.2. Let X and Y be sets and $p : X \twoheadrightarrow Y$ a surjective function. Let $C \subseteq X$. Then the following are equivalent:

1. C is saturated with respect to p .
2. There exists $D \subseteq Y$ such that $C = p^{-1}(D)$
3. $C = p^{-1}(p(C))$.

PROOF:

$\langle 1 \rangle 1. 1 \Rightarrow 3$

$\langle 2 \rangle 1.$ ASSUME: C is saturated with respect to p .

$\langle 2 \rangle 2. C \subseteq p^{-1}(p(C))$

PROOF: Trivial.

$\langle 2 \rangle 3. p^{-1}(p(C)) \subseteq C$

$\langle 3 \rangle 1.$ LET: $x \in p^{-1}(p(C))$

$\langle 3 \rangle 2. p(x) \in p(C)$

$\langle 3 \rangle 3.$ There exists $y \in C$ such that $p(x) = p(y)$

$\langle 3 \rangle 4. x \in C$

PROOF: From $\langle 2 \rangle 1$.

$\langle 1 \rangle 2. 3 \Rightarrow 2$

PROOF: Trivial.

$\langle 1 \rangle 3. 2 \Rightarrow 1$

PROOF: This follows because if $p(x) \in D$ and $p(x) = p(y)$ then $p(y) \in D$.

□

9.26 Quotient Maps

Definition 9.26.1 (Quotient Map). Let X and Y be topological spaces and $p : X \rightarrow Y$. Then p is a *quotient map* if and only if p is surjective and strongly continuous.

Proposition 9.26.2. Let X and Y be topological spaces and $p : X \twoheadrightarrow Y$ be a surjective function. Then the following are equivalent.

1. p is a quotient map.
2. p is continuous and maps saturated open sets to open sets.
3. p is continuous and maps saturated closed sets to closed sets.

PROOF:

$\langle 1 \rangle 1. 1 \Rightarrow 2$

- (2)1. ASSUME: p is a quotient map.
 (2)2. LET: U be a saturated open set in X .
 (2)3. $p^{-1}(p(U))$ is open in X .
 PROOF: Since $U = p^{-1}(p(U))$ be Proposition 9.25.2.
 (2)4. $p(U)$ is open in Y .
 PROOF: From (2)1.
 (1)2. $1 \Rightarrow 3$
 PROOF: Similar.
 (1)3. $2 \Rightarrow 1$
 (2)1. ASSUME: p is continuous and maps saturated open sets to open sets.
 (2)2. LET: $U \subseteq Y$
 (2)3. ASSUME: $p^{-1}(U)$ is open in X
 (2)4. $p^{-1}(U)$ is saturated.
 PROOF: Proposition 9.25.2.
 (2)5. U is open in Y .
 (1)4. $3 \Rightarrow 1$
 PROOF: Similar.
 □

Corollary 9.26.2.1. *Every surjective continuous open map is a quotient map.*

Corollary 9.26.2.2. *Every surjective continuous closed map is a quotient map.*

Example 9.26.3. The converses of these corollaries do not hold.

Let $A = \{(x, y) \mid x \geq 0\} \cup \{(x, y) \mid y = 0\}$. Then $\pi_1 : A \rightarrow \mathbb{R}$ is a quotient map, but not an open map or a closed map.

We prove that π_1 maps saturated open sets to open sets:

- (1)1. LET: $\pi_1^{-1}(U)$ be a saturated open set in A
 PROVE: U is open in \mathbb{R}
 (1)2. LET: $x \in U$
 (1)3. $(x, 0) \in \pi_1(U)^{-1}$
 (1)4. PICK W, V open in \mathbb{R} such that $(x, 0) \subseteq W \times V \subseteq \pi_1(U)^{-1}$
 (1)5. $x \in W \subseteq U$

It is not an open map because it maps $((-1, 1) \times (1, 2)) \cap A$ to $[0, 1)$.

It is not a closed map because it maps $\{(x, 1/x) \mid x > 0\}$ to $(0, +\infty)$.

Proposition 9.26.4. *Let $p : X \twoheadrightarrow Y$ be a quotient map. Let $A \subseteq X$ be saturated with respect to p . Let $q : A \twoheadrightarrow p(A)$ be the restriction of p .*

1. *If A is either open or closed in X then q is a quotient map.*
2. *If p is either an open map or a closed map then q is a quotient map.*

PROOF:

- (1)1. LET: $p : X \twoheadrightarrow Y$ be a quotient map.
 (1)2. LET: $A \subseteq X$ be saturated with respect to p .
 (1)3. LET: $q : A \twoheadrightarrow p(A)$ be the restriction of p .

$\langle 1 \rangle 4$. q is continuous.
 PROOF: Theorem 9.14.10.

$\langle 1 \rangle 5$. If A is open in X then q is a quotient map.

$\langle 2 \rangle 1$. ASSUME: A is open in X .

$\langle 2 \rangle 2$. q maps saturated open sets to open sets.

$\langle 3 \rangle 1$. LET: $U \subseteq A$ be saturated with respect to q and open in A

$\langle 3 \rangle 2$. U is saturated with respect to p

$\langle 4 \rangle 1$. LET: $x, y \in X$

$\langle 4 \rangle 2$. ASSUME: $x \in U$

$\langle 4 \rangle 3$. ASSUME: $p(x) = p(y)$

$\langle 4 \rangle 4$. $x \in A$

PROOF: From $\langle 3 \rangle 1$ and $\langle 4 \rangle 2$.

$\langle 4 \rangle 5$. $y \in A$

PROOF: From $\langle 1 \rangle 2$ and $\langle 4 \rangle 3$

$\langle 4 \rangle 6$. $q(x) = x(y)$

PROOF: From $\langle 1 \rangle 3$, $\langle 4 \rangle 3$, $\langle 4 \rangle 4$, $\langle 4 \rangle 5$.

$\langle 4 \rangle 7$. $y \in U$

PROOF: From $\langle 3 \rangle 1$, $\langle 4 \rangle 2$, $\langle 4 \rangle 6$

$\langle 3 \rangle 3$. U is open in X

PROOF: Lemma 9.19.6, $\langle 2 \rangle 1$, $\langle 3 \rangle 1$.

$\langle 3 \rangle 4$. $p(U)$ is open in Y

PROOF: Proposition 9.26.2, $\langle 1 \rangle 1$, $\langle 3 \rangle 2$, $\langle 3 \rangle 3$

$\langle 3 \rangle 5$. $q(U)$ is open in $p(A)$

PROOF: Since $q(U) = p(U) = p(U) \cap p(A)$.

$\langle 2 \rangle 3$. Q.E.D.

PROOF: By Proposition 9.26.2.

$\langle 1 \rangle 6$. If A is closed in X then q is a quotient map.

PROOF: Similar.

$\langle 1 \rangle 7$. If p is an open map then q is a quotient map.

$\langle 2 \rangle 1$. ASSUME: p is an open map

$\langle 2 \rangle 2$. q maps saturated open sets to open sets.

$\langle 3 \rangle 1$. LET: U be open in A and saturated with respect to q

$\langle 3 \rangle 2$. PICK V open in X such that $U = A \cap V$

$\langle 3 \rangle 3$. $p(V)$ is open in Y

$\langle 3 \rangle 4$. $q(U) = p(V) \cap p(A)$

$\langle 4 \rangle 1$. $q(U) \subseteq p(V) \cap p(A)$

PROOF: From $\langle 3 \rangle 2$.

$\langle 4 \rangle 2$. $p(V) \cap p(A) \subseteq q(U)$

$\langle 5 \rangle 1$. LET: $y \in p(V) \cap p(A)$

$\langle 5 \rangle 2$. PICK $x \in V$ and $x' \in A$ such that $p(x) = p(x') = y$

$\langle 5 \rangle 3$. $x \in A$

PROOF: By $\langle 1 \rangle 2$.

$\langle 5 \rangle 4$. $x \in U$

PROOF: From $\langle 3 \rangle 2$

$\langle 2 \rangle 3$. Q.E.D.

PROOF: By Proposition 9.26.2.

⟨1⟩8. If p is a closed map then q is a quotient map.

PROOF: Similar.

□

Example 9.26.5. This example shows we cannot remove the hypotheses on A and p .

Define $f : [0, 1] \rightarrow [2, 3] \rightarrow [0, 2]$ by $f(x) = x$ if $x \leq 1$, $f(x) = x - 1$ if $x \geq 2$. Then f is a quotient map but its restriction f' to $[0, 1] \cup [2, 3]$ is not, because $f'^{-1}([1, 2])$ is open but $[1, 2]$ is not.

For a counterexample where A is saturated, see Example 9.27.3.

Proposition 9.26.6. Let $p : A \twoheadrightarrow C$ and $q : B \twoheadrightarrow D$ be open quotient maps. Then $p \times q : A \times B \rightarrow C \times D$ is an open quotient map.

PROOF: From Corollary 9.26.2.1, Proposition 9.18.15 and Theorem 9.18.11. □

Theorem 9.26.7. Let $p : X \twoheadrightarrow Y$ be a quotient map. Let Z be a topological space and $f : Y \rightarrow Z$ be a function. Then

1. $f \circ p$ is continuous if and only if f is continuous.
2. $f \circ p$ is a quotient map if and only if f is a quotient map.

PROOF:

⟨1⟩1. If $f \circ p$ is continuous then f is continuous.

PROOF: Proposition 9.24.4.

⟨1⟩2. If f is continuous then $f \circ p$ is continuous.

PROOF: Theorem 9.14.9.

⟨1⟩3. If $f \circ p$ is a quotient map then f is a quotient map.

PROOF: Proposition 9.24.5.

⟨1⟩4. If f is a quotient map then $f \circ p$ is a quotient map.

PROOF: From Proposition 9.24.3.

□

Proposition 9.26.8. Let X and Y be topological spaces. Let $p : X \rightarrow Y$ and $f : Y \rightarrow X$ be continuous maps such that $p \circ f = \text{id}_Y$. Then p is a quotient map.

PROOF:

⟨1⟩1. LET: $V \subseteq Y$

⟨1⟩2. ASSUME: $p^{-1}(V)$ is open in X .

⟨1⟩3. $f^{-1}(p^{-1}(V))$ is open in Y .

PROOF: Because f is continuous.

⟨1⟩4. V is open in Y .

PROOF: Because $f^{-1}(p^{-1}(V)) = V$.

□

9.27 Quotient Topology

Definition 9.27.1 (Quotient Topology). Let X be a topological space, Y a set and $p : X \twoheadrightarrow Y$ be a surjective function. Then the *quotient topology* on Y is the unique topology on Y with respect to which p is a quotient map, namely $\mathcal{T} = \{U \in \mathcal{P}Y \mid p^{-1}(U) \text{ is open in } X\}$.

We prove this is a topology.

PROOF:

$\langle 1 \rangle 1. Y \in \mathcal{T}$

PROOF: Since $p^{-1}(Y) = X$ by surjectivity.

$\langle 1 \rangle 2. \text{ For all } \mathcal{A} \subseteq \mathcal{T} \text{ we have } \bigcup \mathcal{A} \in \mathcal{T}$

PROOF: Since $p^{-1}(\bigcup \mathcal{A}) = \bigcup_{U \in \mathcal{A}} p^{-1}(U)$

$\langle 1 \rangle 3. \text{ For all } U, V \in \mathcal{T} \text{ we have } U \cap V \in \mathcal{T}$

PROOF: Since $p^{-1}(U \cap V) = p^{-1}(U) \cap p^{-1}(V)$.

□

Definition 9.27.2 (Quotient Space). Let X be a topological space and \sim an equivalence relation on X . Let $p : X \twoheadrightarrow X/\sim$ be the canonical surjection. Then X/\sim under the quotient topology is called a *quotient space*, *identification space* or *decomposition space* of X .

Here is a counterexample showing we cannot remove all the hypotheses of Proposition 9.26.4 except that A is saturated.

Example 9.27.3. Let $X = (0, 1/2] \cup \{1\} \cup \{1 + 1/n : n \geq 2\}$ as a subspace of \mathbb{R} . Define R to be the equivalence relation on X where xRy iff $(x = y \text{ or } |x - y| = 1)$, so we identify $1/n$ with $1 + 1/n$ for all $n \geq 2$. Let Y be the resulting quotient space X/R in the quotient topology and $p : X \twoheadrightarrow Y$ the canonical surjection.

Let $A = \{1\} \cup (0, 1/2] \setminus \{1/n : n \geq 2\} \subseteq X$. Then A is saturated under p but the restriction q of p to A is not a quotient map because it maps the saturated open set $\{1\}$ to $\{1\}$ which is not open in $p(A)$.

Proposition 9.27.4. Let X, Y and Z be topological spaces. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$. If f and g are quotient maps then so is $g \circ f$.

PROOF: From Proposition 9.24.3. □

Example 9.27.5. The product of two quotient maps is not necessarily a quotient map.

Let $X = \mathbb{R}$ and X^* the quotient space formed by identifying all positive integers. Let $p : X \twoheadrightarrow X^*$ be the canonical surjection.

We prove $p \times \text{id}_{\mathbb{Q}} : X \times \mathbb{Q} \rightarrow X^* \times \mathbb{Q}$ is not a quotient map.

PROOF:

$\langle 1 \rangle 1. \text{ For } n \geq 1,$

LET: $c_n = \sqrt{2}/n$

$\langle 1 \rangle 2. \text{ For } n \geq 1,$

LET: $U_n = \{(x, y) \in X \times \mathbb{Q} \mid n - 1/4 < x < n + 1/4, (y + n > x + c_n \text{ and } y + n > -x + c_n) \text{ or } (y + n < x + c_n \text{ and } y + n < -x + c_n)\}$

⟨1⟩3. For $n \geq 1$, we have U_n is open in $X \times \mathbb{Q}$

⟨1⟩4. For $n \geq 1$, we have $\{n\} \times \mathbb{Q} \subseteq U_n$

⟨1⟩5. LET: $U = \bigcup_{n=1}^{\infty} U_n$

⟨1⟩6. U is open in $X \times \mathbb{Q}$

⟨1⟩7. U is saturated with respect to $p \times \text{id}_{\mathbb{Q}}$

⟨1⟩8. LET: $U' = (p \times \text{id}_{\mathbb{Q}})(U)$

⟨1⟩9. ASSUME: for a contradiction U' is open in $X^* \times \mathbb{Q}$

⟨1⟩10. $(1, 0) \in U'$

⟨1⟩11. PICK a neighbourhood W of 1 in X^* and $\delta > 0$ such that $W \times (-\delta, \delta) \subseteq U'$

⟨1⟩12. $p^{-1}(W) \times (-\delta, \delta) \subseteq U$

⟨1⟩13. PICK n such that $c_n < \delta$

⟨1⟩14. $n \in p^{-1}(W)$

⟨1⟩15. PICK $\epsilon > 0$ such that $\epsilon < \delta - c_n$ and $\epsilon < 1/4$ and $(n - \epsilon, n + \epsilon) \subseteq p^{-1}(W)$

⟨1⟩16. $(n - \epsilon, n + \epsilon) \times (-\delta, \delta) \subseteq U$

⟨1⟩17. PICK a rational y such that $c_n - \epsilon/2 < y < c_n + \epsilon/2$

⟨1⟩18. $(n + \epsilon/2, y) \notin U$

⟨1⟩19. Q.E.D.

PROOF: This contradicts ⟨1⟩16.

□

Proposition 9.27.6. *Let X be a topological space and \sim an equivalence relation on X . Then X/\sim is T_1 if and only if every equivalence class is closed in X .*

PROOF: Immediate from definitions. □

9.28 Retractions

Definition 9.28.1 (Retraction). Let X be a topological space and $A \subseteq X$. A *retraction* of X onto A is a continuous map $r : X \rightarrow A$ such that, for all $a \in A$, we have $r(a) = a$.

Proposition 9.28.2. *Every retraction is a quotient map.*

PROOF: Proposition 9.26.8 with f the inclusion $A \hookrightarrow X$. □

9.29 Homogeneous Spaces

Definition 9.29.1 (Homogeneous). A topological space X is *homogeneous* if and only if, for any points $a, b \in X$, there exists a homeomorphism $\phi : X \cong X$ such that $\phi(a) = b$.

9.30 Regular Spaces

Definition 9.30.1 (Regular Space). A topological space X is *regular* if and only if, for any closed set A and point $a \notin A$, there exist disjoint open sets U ,

V such that $A \subseteq U$ and $a \in V$.

9.31 Connected Spaces

Definition 9.31.1 (Separation). A *separation* of a topological space X is a pair of disjoint open sets U, V such that $U \cup V = \emptyset$.

Definition 9.31.2 (Connected). A topological space is *connected* if and only if it has no separation; otherwise it is *disconnected*.

Proposition 9.31.3. A topological space X is connected if and only if the only sets that are both open and closed are X and \emptyset .

Immediate from definitions.

Lemma 9.31.4. If Y is a subspace of X , a separation of Y is a pair of disjoint nonempty sets A and B whose union is Y , neither of which contains a limit point of the other.

PROOF:

- $\langle 1 \rangle 1$. LET: $A, B \subseteq Y$
- $\langle 1 \rangle 2$. If A and B form a separation of Y then A and B are disjoint and nonempty, $A \cup B = Y$, and neither of A and B contains a limit point of the other.
 - $\langle 2 \rangle 1$. ASSUME: A and B form a separation of Y
 - $\langle 2 \rangle 2$. A and B are disjoint and nonempty and $A \cup B = Y$
 - PROOF: From $\langle 2 \rangle 1$ and the definition of separation.
 - $\langle 2 \rangle 3$. A does not contain a limit point of B
 - $\langle 3 \rangle 1$. ASSUME: for a contradiction $l \in A$ and l is a limit point of B in X .
 - $\langle 3 \rangle 2$. l is a limit point of B in Y
 - PROOF: Proposition 9.19.16.
 - $\langle 3 \rangle 3$. $l \in B$
 - $\langle 4 \rangle 1$. B is closed in Y
 - PROOF: Since A is open in Y and $B = Y \setminus A$ from $\langle 2 \rangle 1$.
 - $\langle 4 \rangle 2$. Q.E.D.
 - PROOF: Corollary 9.8.3.1.
 - $\langle 3 \rangle 4$. Q.E.D.
 - PROOF: This contradicts the fact that $A \cap B = \emptyset$ ($\langle 2 \rangle 1$).
 - $\langle 2 \rangle 4$. B does not contain a limit point of A
 - PROOF: Similar.
- $\langle 1 \rangle 3$. If A and B are disjoint and nonempty, $A \cup B = Y$, and neither of A and B contains a limit point of the other, then A and B form a separation of Y .
 - $\langle 2 \rangle 1$. ASSUME: A and B are disjoint and nonempty, $A \cup B = Y$, and neither of A and B contains a limit point of the other.
 - $\langle 2 \rangle 2$. A is open in Y
 - $\langle 3 \rangle 1$. B is closed in Y
 - $\langle 4 \rangle 1$. LET: l be a limit point of B in Y

⟨4⟩2. l is a limit point of B in X

PROOF: Proposition 9.19.16.

⟨4⟩3. $l \notin A$

PROOF: By ⟨2⟩1

⟨4⟩4. $l \in B$

PROOF: By ⟨2⟩1 since $A \cup B = Y$

⟨4⟩5. Q.E.D.

PROOF: Corollary 9.8.3.1.

⟨3⟩2. Q.E.D.

PROOF: Since $A = Y \setminus B$.

⟨2⟩3. B is open in Y

PROOF: Similar.

□

Example 9.31.5. Every set under the indiscrete topology is connected.

Example 9.31.6. The discrete topology on a set X is connected if and only if $|X| \leq 1$.

Example 9.31.7. The finite complement topology on a set X is connected if and only if either $|X| \leq 1$ or X is infinite.

Example 9.31.8. The countable complement topology on a set X is connected if and only if either $|X| \leq 1$ or X is uncountable.

Example 9.31.9. The rationals \mathbb{Q} are disconnected. For any irrational a , the sets $(-\infty, a) \cap \mathbb{Q}$ and $(a, +\infty) \cap \mathbb{Q}$ form a separation of \mathbb{Q} .

Lemma 9.31.10. Let X be a topological space. If C and D form a separation of X , and Y is a connected subspace of X , then either $Y \subseteq C$ or $Y \subseteq D$.

PROOF: Otherwise $Y \cap C$ and $Y \cap D$ would form a separation of Y . □

Theorem 9.31.11. The union of a set of connected subspaces of a space X that have a point in common is connected.

PROOF:

⟨1⟩1. LET: \mathcal{A} be a set of connected subspaces of the space X that have the point a in common.

⟨1⟩2. ASSUME: for a contradiction C and D form a separation of $\bigcup \mathcal{A}$

⟨1⟩3. ASSUME: without loss of generality $a \in C$

⟨1⟩4. For all $A \in \mathcal{A}$ we have $A \subseteq C$

PROOF: Lemma 9.31.10.

⟨1⟩5. $D = \emptyset$

⟨1⟩6. Q.E.D.

PROOF: This contradicts ⟨1⟩2.

□

Theorem 9.31.12. Let X be a topological space and A a connected subspace of X . If $A \subseteq B \subseteq \overline{A}$ then B is connected.

PROOF:

⟨1⟩1. ASSUME: for a contradiction C and D form a separation of B .

⟨1⟩2. ASSUME: without loss of generality $A \subseteq C$

PROOF: Lemma 9.31.10.

⟨1⟩3. $B \subseteq C$

⟨2⟩1. LET: $x \in B$

⟨2⟩2. $x \in \overline{A}$

⟨2⟩3. Either $x \in A$ or x is a limit point of A .

PROOF: Theorem 9.8.3.

⟨2⟩4. Either $x \in A$ or x is a limit point of C .

PROOF: Lemma 9.8.5, ⟨1⟩2.

⟨2⟩5. $x \in C$

PROOF: Lemma 9.31.4.

⟨1⟩4. $D = \emptyset$

⟨1⟩5. Q.E.D.

PROOF: This contradicts ⟨1⟩1.

□

Theorem 9.31.13. *The image of a connected space under a continuous map is connected.*

PROOF:

⟨1⟩1. LET: $f : X \rightarrow Y$ be a surjective continuous map where X is connected.

⟨1⟩2. ASSUME: for a contradiction C and D form a separation of Y .

⟨1⟩3. $f^{-1}(C)$ and $f^{-1}(D)$ form a separation of X .

□

Theorem 9.31.14. *The product of a family of connected spaces is connected.*

PROOF:

⟨1⟩1. The product of two connected spaces is connected.

⟨2⟩1. LET: X and Y be connected spaces.

⟨2⟩2. PICK $a \in X$ and $b \in Y$

PROOF: We may assume X and Y are nonempty since otherwise $X \times Y = \emptyset$ which is connected.

⟨2⟩3. $X \times \{b\}$ is connected.

PROOF: It is homeomorphic to X .

⟨2⟩4. For all $x \in X$ we have $\{x\} \times Y$ is connected.

PROOF: It is homeomorphic to Y .

⟨2⟩5. For any $x \in X$

LET: $T_x = (X \times \{b\}) \cup (\{x\} \times Y)$

⟨2⟩6. For all $x \in X$, T_x is connected.

PROOF: Theorem 9.31.11 since $(x, b) \in (X \times \{b\}) \cap (\{x\} \times Y)$.

⟨2⟩7. $X \times Y$ is connected.

PROOF: Theorem 9.31.11 since $X \times Y = \bigcup_{x \in X} T_x$ and (a, b) is a point in every T_x .

⟨1⟩2. The product of a finite family of connected spaces is connected.

PROOF: From $\langle 1 \rangle 1$ by induction.

$\langle 1 \rangle 3$. The product of any family of connected spaces is connected.

$\langle 2 \rangle 1$. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of connected spaces.

$\langle 2 \rangle 2$. LET: $X = \prod_{\alpha \in J} X_\alpha$

$\langle 2 \rangle 3$. PICK $a \in X$

PROOF: We may assume $X \neq \emptyset$ as the empty space is connected.

$\langle 2 \rangle 4$. For every finite subset K of J ,

LET: $X_K = \{x \in X \mid \forall \alpha \in J \setminus K. x_\alpha = a_\alpha\}$

$\langle 2 \rangle 5$. For every finite $K \subseteq J$, we have X_K is connected.

PROOF: From $\langle 1 \rangle 2$ since $X_K \cong \prod_{\alpha \in K} X_\alpha$.

$\langle 2 \rangle 6$. LET: $Y = \bigcup_K X_K$

$\langle 2 \rangle 7$. Y is connected

PROOF: Theorem 9.31.11 since a is a common point.

$\langle 2 \rangle 8$. $X = \bar{Y}$

$\langle 3 \rangle 1$. LET: $x \in X$

$\langle 3 \rangle 2$. LET: $U = \prod_{\alpha \in J} U_\alpha$ be a basic neighbourhood of x where $U_\alpha = X_\alpha$ for all α except $\alpha \in K$ for some finite $K \subseteq J$

$\langle 3 \rangle 3$. LET: $y \in X$ be the point with $y_\alpha = x_\alpha$ for $\alpha \in K$ and $y_\alpha = a_\alpha$ for all other α

$\langle 3 \rangle 4$. $y \in U \cap X_K$

$\langle 3 \rangle 5$. $y \in U \cap Y$

$\langle 2 \rangle 9$. X is connected.

PROOF: Theorem 9.31.12.

□

Example 9.31.15. The set \mathbb{R}^ω is disconnected under the box topology. The set of bounded sequences and the set of unbounded sequences form a separation.

Proposition 9.31.16. Let \mathcal{T} and \mathcal{T}' be two topologies on the same set X . If $\mathcal{T} \subseteq \mathcal{T}'$ and (X, \mathcal{T}') is connected then (X, \mathcal{T}) is connected.

PROOF: If U and V form a separation of (X, \mathcal{T}) then they form a separation of (X, \mathcal{T}') . □

Proposition 9.31.17. Let X be a topological space and (A_n) a sequence of connected subspaces of X . If $A_n \cap A_{n+1} \neq \emptyset$ for all n then $\bigcup_n A_n$ is connected.

PROOF:

$\langle 1 \rangle 1$. ASSUME: for a contradiction C and D form a separation of $\bigcup_n A_n$

$\langle 1 \rangle 2$. ASSUME: without loss of generality $A_0 \subseteq C$

PROOF: Lemma 9.31.10.

$\langle 1 \rangle 3$. For all n we have $A_n \subseteq C$

PROOF:

$\langle 2 \rangle 1$. ASSUME: $A_n \subseteq C$

$\langle 2 \rangle 2$. PICK $x \in A_n \cap A_{n+1}$

$\langle 2 \rangle 3$. $x \in C$

$\langle 2 \rangle 4$. $A_{n+1} \subseteq C$

PROOF: Lemma 9.31.10.

⟨2⟩5. Q.E.D.

PROOF: The result follows by induction.

⟨1⟩4. $D = \emptyset$

⟨1⟩5. Q.E.D.

PROOF: This contradicts ⟨1⟩1.

□

Proposition 9.31.18. *Let X be a topological space. Let $A, C \subseteq X$. If C is connected and intersects both A and $X \setminus A$ then C intersects ∂A .*

PROOF: Otherwise $C \cap A^\circ$ and $C \setminus \bar{A}$ would form a separation of C . □

Example 9.31.19. The space \mathbb{R}_l is disconnected. For any real x , the sets $(-\infty, x)$ and $[x, +\infty)$ form a separation.

Proposition 9.31.20. *Let X and Y be connected spaces. Let A be a proper subset of X and B a proper subset of Y . Then $(X \times Y) \setminus (A \times B)$ is connected.*

PROOF:

⟨1⟩1. PICK $a \in X \setminus A$ and $b \in Y \setminus B$

⟨1⟩2. For $x \in X \setminus A$ we have $(X \times \{b\}) \cup (\{x\} \times Y)$ is connected.

PROOF: Theorem 9.31.11 since (x, b) is a common point.

⟨1⟩3. For $y \in Y \setminus B$ we have $(X \times \{y\}) \cup (\{a\} \times Y)$ is connected.

PROOF: Theorem 9.31.11 since (a, y) is a common point.

⟨1⟩4. $(X \times Y) \setminus (A \times B)$ is connected.

PROOF: Theorem 9.31.11 since it is the union of the sets in ⟨1⟩2 and ⟨1⟩3 with (a, b) as a common point.

□

Proposition 9.31.21. *Let $p : X \twoheadrightarrow Y$ be a quotient map. If Y is connected and $p^{-1}(y)$ is connected for all $y \in Y$, then X is connected.*

PROOF:

⟨1⟩1. ASSUME: for a contradiction C and D form a separation of X .

⟨1⟩2. C is saturated.

⟨2⟩1. LET: $x \in C, y \in X$ with $p(x) = p(y) = a$, say

⟨2⟩2. $y \notin D$

PROOF: Otherwise $p^{-1}(a) \cap C$ and $p^{-1}(a) \cap D$ form a separation of $p^{-1}(a)$.

⟨2⟩3. $y \in C$

⟨1⟩3. D is saturated.

PROOF: Similar.

⟨1⟩4. $p(C)$ and $p(D)$ form a separation of Y .

□

Proposition 9.31.22. *Let X be a connected space and Y a connected subspace of X . Suppose A and B form a separation of $X \setminus Y$. Then $Y \cup A$ and $Y \cup B$ are both connected.*

PROOF:

- ⟨1⟩1. $Y \cup A$ is connected.
- ⟨2⟩1. ASSUME: for a contradiction C and D form a separation of $Y \cup A$
- ⟨2⟩2. ASSUME: without loss of generality $Y \subseteq C$
- ⟨2⟩3. PICK open sets A_1, B_1, C_1, D_1 in X with

$$A = A_1 \setminus Y$$

$$B = B_1 \setminus Y$$

$$C = C_1 \cap (Y \cup A)$$

$$D = D_1 \cap (Y \cup A)$$
- ⟨2⟩4. $B_1 \cup C_1$ and $A_1 \cap D_1$ form a separation of X
- ⟨1⟩2. $Y \cup B$ is connected.
- PROOF: Similar.
-

Theorem 9.31.23. *Let L be a linearly ordered set under the order topology. Then L is connected if and only if L is a linear continuum.*

PROOF:

- ⟨1⟩1. If L is a linear continuum then L is connected.
- ⟨2⟩1. LET: L be a linear continuum under the order topology.
- ⟨2⟩2. ASSUME: for a contradiction C and D form a separation of L .
- ⟨2⟩3. PICK $a \in C$ and $b \in D$.
- ⟨2⟩4. ASSUME: without loss of generality $a < b$.
- ⟨2⟩5. LET: $S = \{x \in L \mid a < x \text{ and } [a, x) \subseteq C\}$
- ⟨2⟩6. S is nonempty.
 - PROOF: Since $a \in C$ and C is open.
- ⟨2⟩7. S is bounded above by b .
 - PROOF: Since $b \notin C$.
- ⟨2⟩8. LET: $s = \sup S$
- ⟨2⟩9. $s \in S$
 - ⟨3⟩1. LET: $y \in [a, s)$
 - PROVE: $y \in C$
 - ⟨3⟩2. PICK z with $y < z \in S$
 - PROOF: By minimality of s .
 - ⟨3⟩3. $y \in [a, z) \subseteq C$
- ⟨2⟩10. CASE: $s \in C$
 - ⟨3⟩1. PICK x such that $s < x$ and $[s, x) \subseteq C$
 - PROOF: Since C is open and s is not greatest in L because $s < b$.
 - ⟨3⟩2. $x \in S$
 - PROOF: Since $[a, x) = [a, s) \cup [s, x) \subseteq C$.
 - ⟨3⟩3. Q.E.D.
 - PROOF: This contradicts the fact that s is an upper bound for S .
- ⟨2⟩11. CASE: $s \in D$
 - ⟨3⟩1. PICK $x < s$ such that $(x, s] \subseteq D$
 - ⟨3⟩2. PICK y with $x < y < s$
 - PROOF: Since L is dense.
 - ⟨3⟩3. $y \in C$

PROOF: From $\langle 2 \rangle 9$.

$\langle 3 \rangle 4$. $y \in D$
PROOF: From $\langle 3 \rangle 1$.

$\langle 3 \rangle 5$. Q.E.D.

$\langle 3 \rangle 6$. LET: L be a linear continuum under the order topology.

$\langle 3 \rangle 7$. ASSUME: for a contradiction C and D form a separation of L .

$\langle 3 \rangle 8$. PICK $a \in C$ and $b \in D$.

$\langle 3 \rangle 9$. ASSUME: without loss of generality $a < b$.

$\langle 3 \rangle 10$. LET: $S = \{x \in L \mid a < x \text{ and } [a, x) \subseteq C\}$

$\langle 3 \rangle 11$. S is nonempty.
PROOF: Since $a \in C$ and C is open.

$\langle 3 \rangle 12$. S is bounded above by b .
PROOF: Since $b \notin C$.

$\langle 3 \rangle 13$. LET: $s = \sup S$

$\langle 3 \rangle 14$. $s \in S$
 $\langle 4 \rangle 1$. LET: $y \in [a, s)$
PROVE: $y \in C$

$\langle 4 \rangle 2$. PICK z with $y < z \in S$
PROOF: By minimality of s .

$\langle 4 \rangle 3$. $y \in [a, z) \subseteq C$

$\langle 3 \rangle 15$. CASE: $s \in C$
 $\langle 4 \rangle 1$. PICK x such that $s < x$ and $[s, x) \subseteq C$
PROOF: Since C is open and s is not greatest in L because $s < b$.

$\langle 4 \rangle 2$. $x \in S$
PROOF: Since $[a, x) = [a, s) \cup [s, x) \subseteq C$.

$\langle 4 \rangle 3$. Q.E.D.
PROOF: This contradicts the fact that s is an upper bound for S .

$\langle 3 \rangle 16$. CASE: $s \in D$
 $\langle 4 \rangle 1$. PICK $x < s$ such that $(x, s] \subseteq D$

$\langle 4 \rangle 2$. PICK y with $x < y < s$
PROOF: Since L is dense.

$\langle 4 \rangle 3$. $y \in C$
PROOF: From $\langle 2 \rangle 9$.

$\langle 4 \rangle 4$. $y \in D$
PROOF: From $\langle 3 \rangle 1$.

$\langle 4 \rangle 5$. Q.E.D.
PROOF: This contradicts $\langle 2 \rangle 2$.

$\langle 1 \rangle 2$. If L is connected then L is a linear continuum.

$\langle 2 \rangle 1$. ASSUME: L is connected.

$\langle 2 \rangle 2$. Every nonempty subset of L that is bounded above has a supremum.

$\langle 3 \rangle 1$. LET: X be a nonempty subset of L bounded above by b .

$\langle 3 \rangle 2$. ASSUME: for a contradiction X has no supremum.

$\langle 3 \rangle 3$. LET: U be the set of upper bounds of X ,

$\langle 3 \rangle 4$. U is nonempty.
PROOF: Since $b \in U$.

$\langle 3 \rangle 5$. U is open.

⟨4⟩1. LET: $x \in U$
 ⟨4⟩2. PICK an upper bound y for X such that $y < x$
 ⟨4⟩3. Either x is greatest in L and $(y, x] \subseteq U$, or there exists $z > x$ such that $(y, z) \subseteq U$
 ⟨3⟩6. LET: V be the set of lower bounds of U .
 ⟨3⟩7. V is nonempty.
 PROOF: Since $X \subseteq V$
 ⟨3⟩8. V is open.
 ⟨4⟩1. LET: $x \in V$
 ⟨4⟩2. PICK $y \in X$ with $x < y$
 PROOF: x cannot be an upper bound for X , because it would be the supremum of X .
 ⟨4⟩3. Either x least in L and $[x, y) \subseteq V$, or there exists $z < x$ such that $(z, y) \subseteq V$
 ⟨3⟩9. $L = U \cup V$
 ⟨4⟩1. LET: $x \in L \setminus U$
 ⟨4⟩2. PICK $y \in X$ such that $x < y$
 ⟨4⟩3. For all $u \in U$ we have $x < y \leq u$
 ⟨4⟩4. $x \in V$
 ⟨3⟩10. $U \cap V = \emptyset$
 PROOF: Any element of $U \cap V$ would be a supremum of X .
 ⟨3⟩11. U and V form a separation of L .
 ⟨3⟩12. Q.E.D.
 PROOF: This contradicts ⟨2⟩1.
 ⟨2⟩3. L is dense.
 ⟨3⟩1. LET: $x, y \in L$ with $x < y$
 ⟨3⟩2. There exists $z \in L$ such that $x < z < y$
 PROOF: Otherwise $(-\infty, y)$ and $(x, +\infty)$ would form a separation of L .

□

Corollary 9.31.23.1. *The real line \mathbb{R} is connected.*

Corollary 9.31.23.2. *Every interval in \mathbb{R} is connected.*

Corollary 9.31.23.3. *The ordered square is connected.*

Theorem 9.31.24 (Intermediate Value Theorem). *Let X be a connected space. Let Y be a linearly ordered set under the order topology. Let $f : X \rightarrow Y$ be continuous. Let $a, b \in X$ and $r \in Y$. Suppose $f(a) < r < f(b)$. Then there exists $c \in X$ such that $f(c) = r$.*

PROOF: Otherwise $f^{-1}((-\infty, r))$ and $f^{-1}((r, +\infty))$ would form a separation of X . □

Proposition 9.31.25. *Every function $f : [0, 1] \rightarrow [0, 1]$ has a fixed point.*

PROOF:

⟨1⟩1. LET: $g : [0, 1] \rightarrow [-1, 1]$ be the function $g(x) = f(x) - x$

- PROVE: there exists $x \in [0, 1]$ such that $g(x) = 0$
- ⟨1⟩2. ASSUME: without loss of generality $g(0) \neq 0$ and $g(1) \neq 0$
- ⟨1⟩3. $g(0) > 0$
- ⟨1⟩4. $g(1) < 0$
- ⟨1⟩5. There exists $x \in (0, 1)$ such that $g(x) = 0$
- PROOF: By the Intermediate Value Theorem.

Proposition 9.31.26. Give \mathbb{R}^ω the box topology. Let $x, y \in \mathbb{R}^\omega$. Then x and y lie in the same component if and only if $x - y$ is eventually zero, i.e. there exists N such that, for all $n \geq N$, we have $x_n = y_n$.

PROOF:

- ⟨1⟩1. The component containing 0 is the set of sequences that are eventually zero.
- ⟨2⟩1. LET: B be the set of sequences that are eventually zero.
- ⟨2⟩2. B is path-connected.
- ⟨3⟩1. LET: $x, y \in B$
- ⟨3⟩2. PICK N such that, for all $n \geq N$, we have $x_n = y_n = 0$
- ⟨3⟩3. LET: $p : [0, 1] \rightarrow \mathbb{R}^\omega$, $p(t) = (1 - t)x + ty$
- PROVE: p is continuous.
- ⟨3⟩4. LET: $t \in [0, 1]$ and $\prod_j U_j$ be a basic open neighbourhood of $p(t)$, where each U_j is open in \mathbb{R}
- ⟨3⟩5. PICK δ such that, for all $n < N$ and all $s \in [0, 1]$, if $|s - t| < \delta$ then $p(s)_n \in U_n$
- ⟨3⟩6. For all $s \in [0, 1]$, if $|s - t| < \delta$ then $p(s) \in \prod_j U_j$
- ⟨2⟩3. B is connected.
- PROOF: Proposition 9.33.3.
- ⟨2⟩4. If C is connected and $B \subseteq C$ then $B = C$.
- ⟨3⟩1. ASSUME: C is connected and $B \subseteq C$
- ⟨3⟩2. ASSUME: for a contradiction $x \in C \setminus B$
- ⟨3⟩3. For $n \geq 1$,
LET: $c_n = 1$ if $x_n = 0$, $c_n = n/x_n$ otherwise
- ⟨3⟩4. LET: $h : \mathbb{R}^\omega \rightarrow \mathbb{R}^\omega$ be the function $h(x) = (c_n x_n)_{n \geq 1}$
- ⟨3⟩5. h is a homeomorphism of \mathbb{R}^ω with itself.
- ⟨3⟩6. $h(x)$ is unbounded.
- PROOF: For any $b > 0$, pick $N > b$ such that $x_N \neq 0$. Then $h(x)_N > b$.
- ⟨3⟩7. $h^{-1}(\{\text{bounded sequences}\}) \cap C$ and $h^{-1}(\{\text{unbounded sequences}\}) \cap C$ form a separation of C
- ⟨3⟩8. Q.E.D.
- PROOF: This contradicts ⟨3⟩1.
- ⟨1⟩2. Q.E.D.
- PROOF: Since $\lambda x. x - y$ is a homeomorphism of \mathbb{R}^ω with itself.
-

Example 9.31.27. The space \mathbb{R}_K is connected.

PROOF:

⟨1⟩1. ASSUME: for a contradiction U and V form a separation of \mathbb{R}_K
 ⟨1⟩2. ASSUME: without loss of generality $0 \in U$
 ⟨1⟩3. There exists an open interval (a, b) such that $(a, b) - K \subseteq U$ and $(a, b) \not\subseteq U$
 PROOF: Otherwise U and V would form a separation of \mathbb{R} .
 ⟨1⟩4. PICK $1/n \in (a, b) - U$
 ⟨1⟩5. $1/n \in V$
 ⟨1⟩6. There exists an open interval (c, d) around $1/n$ that is included in V
 ⟨1⟩7. Q.E.D.
 PROOF: This is a contradiction since $(a, b) - K$ and (c, d) must intersect.

□

9.32 Totally Disconnected Spaces

Definition 9.32.1 (Totally Disconnected). A topological space X is *totally disconnected* if and only if the only connected subspaces are the singletons.

Example 9.32.2. Every discrete space is totally disconnected.

Example 9.32.3. The rationals \mathbb{Q} are totally disconnected.

Example 9.32.4. The Cantor set is totally disconnected.

PROOF:

⟨1⟩1. LET: (A_n) be the sequence of sets in Definition 8.1.1.
 ⟨1⟩2. LET: C be the Cantor set $\bigcap_n A_n$
 ⟨1⟩3. ASSUME:
 for a contradiction $D \subseteq C$ is connected and has more than one point.
 ⟨1⟩4. LET: $x, y \in D$ with $x < y$
 ⟨1⟩5. PICK n such that $|x - y| > 1/3^n$
 ⟨1⟩6. A_n is a sequence of disjoint intervals of length $1/3^n$
 ⟨1⟩7. x and y are in two different intervals out of the intervals that make up A_n
 ⟨1⟩8. There exists z with $x < z < y$ such that $z \notin A_n$
 ⟨1⟩9. $(-\infty, z) \cap D$ and $(z, +\infty) \cap D$ form a separation of D .

□

9.33 Paths and Path Connectedness

Definition 9.33.1 (Path). Let X be a topological space and $a, b \in X$. A *path* from a to b is a continuous function $p : [0, 1] \rightarrow X$ such that $p(0) = a$ and $p(1) = b$.

Definition 9.33.2 (Path Connected). A topological space is *path connected* if and only if there exists a path between any two points.

Proposition 9.33.3. *Every path connected space is connected.*

PROOF:

- ⟨1⟩1. LET: X be a path connected space.
- ⟨1⟩2. ASSUME: for a contradiction C and D form a separation of X .
- ⟨1⟩3. PICK $a \in C$ and $b \in D$.
- ⟨1⟩4. PICK a path $p : [0, 1] \rightarrow X$ from a to b .
- ⟨1⟩5. $p^{-1}(C)$ and $p^{-1}(D)$ form a separation of $[0, 1]$.
- ⟨1⟩6. Q.E.D.

PROOF: This contradicts Corollary 9.31.23.2.

□

An example that shows the converse does not hold:

Example 9.33.4. The ordered square is not path connected.

PROOF:

- ⟨1⟩1. ASSUME: for a contradiction $p : [0, 1] \rightarrow I_o^2$ is a path from $(0, 0)$ to $(1, 1)$.
- ⟨1⟩2. p is surjective.

PROOF: By the Intermediate Value Theorem.

- ⟨1⟩3. For $x \in [0, 1]$, PICK a rational $q_x \in p^{-1}((x, 0), (x, 1))$

PROOF: Since $p^{-1}((x, 0), (x, 1))$ is open and nonempty by ⟨1⟩2.

- ⟨1⟩4. For $x, y \in [0, 1]$, if $x \neq y$ then $q_x \neq q_y$

PROOF: We have $p(q_x) \neq p(q_y)$ because $((x, 0), (x, 1))$ and $((y, 0), (y, 1))$ are disjoint.

- ⟨1⟩5. $\{q_x \mid x \in [0, 1]\}$ is an uncountable set of rationals.

- ⟨1⟩6. Q.E.D.

PROOF: This contradicts the fact that the rationals are countable.

□

Proposition 9.33.5. *The continuous image of a path connected space is path connected.*

PROOF:

- ⟨1⟩1. LET: X be a path connected space, Y a topological space, and $f : X \rightarrow Y$ be continuous and surjective.
- ⟨1⟩2. LET: $a, b \in Y$
- ⟨1⟩3. PICK $c, d \in X$ with $f(c) = a$ and $f(d) = b$
- ⟨1⟩4. PICK a path $p : [0, 1] \rightarrow X$ from c to d .
- ⟨1⟩5. $f \circ p$ is a path from a to b in Y .

□

Proposition 9.33.6 (AC). *The product of a family of path-connected spaces is path-connected.*

PROOF:

- ⟨1⟩1. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of path-connected spaces.
 - ⟨1⟩2. LET: $a, b \in \prod_{\alpha \in J} X_\alpha$
 - ⟨1⟩3. For $\alpha \in J$, PICK a path $p_\alpha : [0, 1] \rightarrow X_\alpha$ from a_α to b_α
- PROOF: Using the Axiom of Choice.
- ⟨1⟩4. Define $p : [0, 1] \rightarrow \prod_{\alpha \in J} X_\alpha$ by $p(t)_\alpha = p_\alpha(t)$
 - ⟨1⟩5. p is a path from a to b .

PROOF: Theorem 9.18.11.

□

Proposition 9.33.7. *The continuous image of a path-connected space is path-connected.*

PROOF:

⟨1⟩1. LET: $f : X \rightarrow Y$ be continuous and surjective where X is path-connected.

⟨1⟩2. LET: $a, b \in Y$

⟨1⟩3. PICK $a', b' \in X$ with $f(a') = a$ and $f(b') = b$.

⟨1⟩4. PICK a path $p : [0, 1] \rightarrow X$ from a' to b' .

⟨1⟩5. $f \circ p$ is a path from a to b .

□

Proposition 9.33.8. *Let X be a topological space. The union of a set of path-connected subspaces of X that have a point in common is path-connected.*

PROOF:

⟨1⟩1. LET: \mathcal{A} be a set of path-connected subspaces of X with the point a in common.

⟨1⟩2. LET: $b, c \in \bigcup \mathcal{A}$

⟨1⟩3. PICK $B, C \in \mathcal{A}$ with $b \in B$ and $c \in C$.

⟨1⟩4. PICK a path p in B from b to a .

⟨1⟩5. PICK a path q in C from a to c .

⟨1⟩6. The concatenation of p and q is a path from b to c in $\bigcup \mathcal{A}$.

□

Proposition 9.33.9. *Let $A \subseteq \mathbb{R}^2$ be countable. Then $\mathbb{R}^2 \setminus A$ is path-connected.*

PROOF:

⟨1⟩1. LET: $a, b \in \mathbb{R}^2 \setminus A$

⟨1⟩2. PICK a line l in \mathbb{R}^2 with a on one side and b on the other.

⟨1⟩3. For every point x on l ,

LET: p_x be the path in \mathbb{R}^2 consisting of a line from a to x then a line from x to b

⟨1⟩4. For $x \neq y$ we have p_x and p_y have no points in common except a and b

⟨1⟩5. There are only countably many x such that a point of A lies on p_x .

⟨1⟩6. There exists x such that p_x is a path from a to b in $\mathbb{R}^2 \setminus A$.

□

Proposition 9.33.10. *Every open connected subspace of \mathbb{R}^2 is path-connected.*

PROOF:

⟨1⟩1. LET: U be an open connected subspace of \mathbb{R}^2 .

⟨1⟩2. For all $x_0 \in U$,

LET: $PC(x_0) = \{y \in U \mid \text{there exists a path from } x_0 \text{ to } y\}$

⟨1⟩3. For all $x_0 \in U$, the set $PC(x_0)$ is open and closed in U .

⟨2⟩1. LET: $x_0 \in U$

⟨2⟩2. $PC(x_0)$ is open in U

⟨3⟩1. LET: $y \in PC(x_0)$
 ⟨3⟩2. PICK $\epsilon > 0$ such that $B(y, \epsilon) \subseteq U$
 PROOF: Since U is open.
 ⟨3⟩3. $B(y, \epsilon) \subseteq PC(x_0)$
 PROOF: For all $z \in B(y, \epsilon)$, pick a path from x_0 to y then concatenate the straight line from y to z .
 ⟨2⟩3. $PC(x_0)$ is closed in U
 ⟨3⟩1. LET: $y \in U$ be a limit point of $PC(x_0)$
 ⟨3⟩2. PICK $\epsilon > 0$ such that $B(y, \epsilon) \subseteq U$
 ⟨3⟩3. PICK $z \in PC(x_0) \cap B(y, \epsilon)$
 ⟨3⟩4. $y \in PC(x_0)$
 PROOF: Pick a path from x_0 to z then concatenate the straight line from z to y .
 ⟨1⟩4. $PC(x_0) = U$
 PROOF: Proposition 9.31.3.

□

Example 9.33.11. If A is a connected subspace of X , then A° is not necessarily connected.

Take two closed circles in \mathbb{R}^2 that touch at one point. The interior of this space is two open circles, and these two circles form a separation.

Example 9.33.12. If A is a connected subspace of X then ∂A is not necessarily connected.

We have $[0, 1]$ is connected but $\partial[0, 1] = \{0, 1\}$ is not.

Example 9.33.13. If A is a subspace of X and A° and ∂A are connected, then A is not necessarily connected.

We have $\mathbb{Q}^\circ = \emptyset$ and $\partial\mathbb{Q} = \mathbb{R}$ are connected but \mathbb{Q} is not connected.

Example 9.33.14. The space \mathbb{R}_K is not path connected.

PROOF:

⟨1⟩1. ASSUME: for a contradiction $p : [0, 1] \rightarrow \mathbb{R}_K$ was a path from 0 to 1.
 ⟨1⟩2. $p([0, 1])$ as a subspace of \mathbb{R}_K is compact.
 PROOF: Theorem 9.48.4.
 ⟨1⟩3. $p([0, 1])$ as a subspace of \mathbb{R}_K is connected.
 PROOF: Theorem 9.31.13.
 ⟨1⟩4. $p([0, 1])$ is connected as a subspace of \mathbb{R} .
 PROOF: Theorem 9.31.13 as the identity map is continuous as a map $\mathbb{R}_K \rightarrow \mathbb{R}$.
 ⟨1⟩5. $p([0, 1])$ is convex.
 ⟨2⟩1. LET: $a, b \in p([0, 1])$ and $a < c < b$
 ⟨2⟩2. ASSUME: for a contradiction $c \notin p([0, 1])$
 ⟨2⟩3. $(-\infty, c) \cap p([0, 1])$ and $(c, +\infty) \cap p([0, 1])$ form a separation of $p([0, 1])$ as a subspace of \mathbb{R} .
 ⟨2⟩4. Q.E.D.
 PROOF: This contradicts ⟨1⟩4.
 ⟨1⟩6. $[0, 1] \subseteq p([0, 1])$

⟨1⟩7. $[0, 1]$ as a subspace of \mathbb{R}_K is compact.

PROOF: By Proposition 9.48.3 and ⟨1⟩2.

⟨1⟩8. Q.E.D.

PROOF: This contradicts Example 9.48.26.

□

9.34 The Topologist's Sine Curve

Definition 9.34.1 (The Topologist's Sine Curve). Let $S = \{(x, \sin 1/x) \mid 0 < x \leq 1\}$, The *topologist's sine curve* is the closure \bar{S} of S in \mathbb{R}^2 .

Proposition 9.34.2. *The topologist's sine curve is connected.*

PROOF:

⟨1⟩1. LET: $S = \{(x, \sin 1/x) \mid 0 < x \leq 1\}$

⟨1⟩2. S is connected.

PROOF: Theorem 9.31.13.

⟨1⟩3. \bar{S} is connected.

PROOF: Theorem 9.31.12.

□

Proposition 9.34.3. *The topologist's sine curve is $\{(x, \sin 1/x) \mid 0 < x \leq 1\} \cup (\{0\} \times [-1, 1])$.*

PROOF: Sketch proof: Given a point $(0, y)$ with $-1 \leq y \leq 1$, pick a such that $\sin a = y$. Then $((1/a, y), (1/(a + 2\pi), y), (1/(a + 4\pi), y), \dots)$ is a sequence in S that converges to $(0, y)$.

Conversely, let (x, y) be any point not in $S \cup (\{0\} \times [-1, 1])$. If $x < 0$ or $y > 1$ or $y < -1$ then we can easily find a neighbourhood that does not intersect $S \cup (\{0\} \times [-1, 1])$. If $x > 0$ and $-1 \leq y \leq 1$, then we have $y \neq \sin 1/x$. Hence pick a neighbourhood that does not intersect S .

Proposition 9.34.4. *Every closed subset of \mathbb{R} that is bounded above has a greatest element.*

PROOF: It has a supremum, which is a limit point of the set and hence an element. □

Proposition 9.34.5 (CC). *The topologist's sine curve is not path connected.*

PROOF:

⟨1⟩1. ASSUME: For a contradiction $p : [0, 1] \rightarrow \bar{S}$ is a path from $(0, 0)$ to $(1, \sin 1)$.

⟨1⟩2. $\{t \in [0, 1] \mid p(t) \in \{0\} \times [-1, 1]\}$ is closed.

PROOF: Since p is continuous and $\{0\} \times [-1, 1]$ is closed.

⟨1⟩3. LET: b be the largest number in $[0, 1]$ such that $p(b) \in \{0\} \times [-1, 1]$.

PROOF: Proposition 9.34.4.

⟨1⟩4. LET: $x : [b, 1] \rightarrow \bar{S}$ be the function $\pi_1 \circ p$

⟨1⟩5. LET: $y : [b, 1] \rightarrow \bar{S}$ be the function $\pi_2 \circ p$

- (1)6. PICK a sequence t_n in $(b, 1]$ such that $t_n \rightarrow b$ and $y(t_n) = (-1)^n$ for all n
 (2)1. LET: $n \geq 1$
 (2)2. PICK u with $0 < u < x(1/n)$ and $\sin(1/u) = (-1)^n$
 (2)3. PICK t_n with $b < t_n < 1/n$ and $x(t_n) = u$
 PROOF: By the Intermediate Value Theorem
 (1)7. Q.E.D.
 PROOF: This contradicts Proposition 9.14.18 since y is continuous and $y(t_n)$ does not converge.

□

Corollary 9.34.5.1. *The closure of a path-connected subspace of a space is not necessarily path-connected.*

9.35 The Long Line

Definition 9.35.1 (The Long Line). The *long line* is the space $\omega_1 \times [0, 1)$ in the dictionary order under the order topology, where ω_1 is the first uncountable ordinal.

Lemma 9.35.2. *For any ordinal α with $0 < \alpha < \omega_1$ we have $[(0, 0), (\alpha, 0)) \cong [0, 1)$*

- (1)1. $[(0, 0), (1, 0)) \cong [0, 1)$
 PROOF: The map π_2 is a homeomorphism.
 (1)2. If $[(0, 0), (\alpha, 0)) \cong [0, 1)$ then $[(0, 0), (\alpha + 1, 0)) \cong [0, 1)$
 PROOF: Proposition 4.2.11.
 (1)3. If λ is a limit ordinal with $\lambda < \omega_1$ and $[(0, 0), (\alpha, 0)) \cong [0, 1)$ for all α with $0 < \alpha < \lambda$ then $[(0, 0), (\lambda, 0)) \cong [0, 1)$
 (2)1. LET: λ be a limit ordinal $< \omega_1$
 (2)2. ASSUME: $[(0, 0), (\alpha, 0)) \cong [0, 1)$ for all α with $0 < \alpha < \lambda$
 (2)3. PICK a sequence of ordinals $\alpha_0 < \alpha_1 < \dots$ with limit λ
 PROOF: Since λ is countable.
 (2)4. $[(\alpha_i, 0), (\alpha_{i+1}, 0)) \cong [0, 1)$ for all i
 PROOF: Lemma 4.2.10.
 (2)5. Q.E.D.
 PROOF: By Proposition 4.2.12.
 (1)4. Q.E.D.
 PROOF: By transfinite induction.

Proposition 9.35.3 (CC). *The long line is path-connected.*

PROOF:

- (1)1. LET: $(\alpha, i), (\beta, j) \in \omega_1 \times [0, 1)$
 (1)2. ASSUME: without loss of generality $(\alpha, i) < (\beta, j)$
 (1)3. $[(0, 0), (\beta + 1, 0)) \cong [0, 1)$
 PROOF: By Lemma 9.35.2
 (1)4. $[(\alpha, i), (\beta, j)) \cong [0, 1)$

PROOF: Lemma 4.2.10.

⟨1⟩5. PICK a homeomorphism $q : [0, 1] \rightarrow [(\alpha, i), (\beta, j))$

⟨1⟩6. $q \cup \{(1, (\beta, j))\}$ is a path from (α, i) to (β, j)

□

Proposition 9.35.4. *Every point in the long line has a neighbourhood homeomorphic to an interval in \mathbb{R} .*

PROOF: For any (α, i) in the long line, the neighbourhood $[(0, 0), (\alpha + 1, 0))$ satisfies the condition by Lemma 9.35.2.

Proposition 9.35.5. *The long line L is not second countable.*

PROOF:

⟨1⟩1. LET: \mathcal{B} be a basis for L .

⟨1⟩2. For $\alpha < \omega_1$, PICK $B_\alpha \in \mathcal{B}$ such that $(\alpha, 1/2) \in B_\alpha \subseteq ((\alpha, 0), (\alpha + 1, 0))$

⟨1⟩3. \mathcal{B} is uncountable.

PROOF: The mapping $\alpha \mapsto B_\alpha$ is an injection $\omega_1 \rightarrow \mathcal{B}$.

Corollary 9.35.5.1. *The long line cannot be imbedded into \mathbb{R}^n for any n .*

9.36 Components

Proposition 9.36.1. *Let X be a topological space. Define the relation \sim on X by $a \sim b$ if and only if there exists a connected subspace $A \subseteq X$ with $a, b \in A$. Then \sim is an equivalence relation on X .*

PROOF:

⟨1⟩1. \sim is reflexive.

PROOF: For any $a \in X$ we have $\{a\}$ is a connected subspace that contains a .

⟨1⟩2. \sim is symmetric.

PROOF: Trivial.

⟨1⟩3. \sim is transitive.

⟨2⟩1. LET: $a, b, c \in X$

⟨2⟩2. ASSUME: $a \sim b$ and $b \sim c$

⟨2⟩3. PICK connected subspaces A and B with $a, b \in A$ and $b, c \in B$

⟨2⟩4. $A \cup B$ is a connected subspace that contains a and c

PROOF: Theorem 9.31.11.

□

Definition 9.36.2 ((Connected) Component). Let X be a topological space. The (*connected*) *components* of X are the equivalence classes under the above \sim .

Lemma 9.36.3. *Let X be a topological space. If $A \subseteq X$ is connected and nonempty then there exists a unique component C of X such that $A \subseteq C$.*

PROOF:

- ⟨1⟩1. PICK $a \in A$
- ⟨1⟩2. LET: C be the \sim -equivalence class of a .
- ⟨1⟩3. $A \subseteq C$
PROOF: For all $x \in A$ we have $x \sim a$.
- ⟨1⟩4. If C' is a component and $A \subseteq C'$ then $C = C'$
PROOF: Since we have $a \in C'$.

□

Theorem 9.36.4. *Let X be a topological space. The components of X are connected disjoint subspaces of X whose union is X such that each nonempty connected subspace of X intersects only one of them.*

PROOF:

- ⟨1⟩1. Every component of X is connected.
PROOF: For $a \in X$, the \sim -equivalence class of a is $\bigcup\{A \subseteq X \mid A \text{ is connected, } a \in A\}$ which is connected by Theorem 9.31.11.
- ⟨1⟩2. The components form a partition of X .
PROOF: Immediate from the definition.
- ⟨1⟩3. Every nonempty connected subspace of X intersects a unique component of X .
 - ⟨2⟩1. LET: $A \subseteq X$ be connected and nonempty.
 - ⟨2⟩2. LET: C be the component such that $A \subseteq C$
PROOF: Lemma 9.36.3.
 - ⟨2⟩3. A intersects C
 - ⟨2⟩4. If A intersects the component C' then $C' = C$
 - ⟨3⟩1. LET: C' be a component that intersects A
 - ⟨3⟩2. PICK $b \in A \cap C'$
 - ⟨3⟩3. $A \subseteq C'$
PROOF: For all $x \in A$ we have $x \sim b$.
 - ⟨3⟩4. $C = C'$
PROOF: By uniqueness in ⟨2⟩2.

□

Proposition 9.36.5. *Every component of a space is closed.*

PROOF:

- ⟨1⟩1. LET: X be a topological space and C a component of X .
- ⟨1⟩2. \overline{C} is connected.
PROOF: Theorem 9.31.12.
- ⟨1⟩3. $C = \overline{C}$
PROOF: Lemma 9.31.10.
- ⟨1⟩4. C is closed.
PROOF: Lemma 9.6.5.

□

Proposition 9.36.6. *If a topological space has finitely many components then every component is open.*

PROOF: Each component is the complement of a finite union of closed sets. □

9.37 Path Components

Proposition 9.37.1. *Let X be a topological space. Define the relation \sim on X by: $a \sim b$ if and only if there exists a path in X from a to b . Then \sim is an equivalence relation on X .*

PROOF:

(1)1. \sim is reflexive.

PROOF: For $a \in X$, the constant function $[0, 1] \rightarrow X$ with value a is a path from a to a .

(1)2. \sim is symmetric.

PROOF: If $p : [0, 1] \rightarrow X$ is a path from a to b , then $\lambda t.p(1 - t)$ is a path from b to a .

(1)3. \sim is transitive.

PROOF: Concatenate paths.

□

Definition 9.37.2 (Path Component). Let X be a topological space. The *path components* of X are the equivalence relations under \sim .

Theorem 9.37.3. *The path components of X are path-connected disjoint subspaces of X whose union is X such that every nonempty path-connected subspace of X intersects exactly one path component.*

PROOF:

(1)1. Every path component is path-connected.

PROOF: If a and b are in the same path component then $a \sim b$, i.e. there exists a path from a to b .

(1)2. The path components are disjoint and their union is X .

PROOF: Immediate from the definition.

(1)3. Every non-empty path-connected subspace of X intersects exactly one path component.

(2)1. LET: A be a nonempty path-connected subspace of X .

(2)2. PICK $a \in A$

(2)3. A intersects the \sim -equivalence class of a .

(2)4. LET: C be any path component that intersects A .

(2)5. PICK $b \in A \cap C$

(2)6. $a \sim b$

PROOF: Since A is path-connected.

(2)7. C is the \sim -equivalence class of a .

□

Proposition 9.37.4. *Every path component is included in a component.*

PROOF:

(1)1. LET: X be a topological space and C a path component of X .

(1)2. C is path-connected.

PROOF: Theorem 9.37.3.
 ⟨1⟩3. C is connected.
 PROOF: Proposition 9.33.3.
 ⟨1⟩4. C is included in a component.
 PROOF: Lemma 9.36.3.
 □

9.38 Local Connectedness

Definition 9.38.1 (Locally Connected). Let X be a topological space and $a \in X$. Then X is *locally connected* at a if and only if every neighbourhood of a includes a connected neighbourhood of a .

The space X is *locally connected* if and only if it is locally connected at every point.

Example 9.38.2. The real line is both connected and locally connected.

Example 9.38.3. The space $\mathbb{R} \setminus \{0\}$ is disconnected but locally connected.

Example 9.38.4. The topologist's sine curve is connected but not locally connected.

Example 9.38.5. The rationals \mathbb{Q} are neither connected nor locally connected.

Theorem 9.38.6. *A topological space X is locally connected if and only if, for every open set U in X , every component of U is open in X .*

PROOF:

- ⟨1⟩1. If X is locally connected then, for every open set U in X , every component of U is open in X .
 ⟨2⟩1. ASSUME: X is locally connected.
 ⟨2⟩2. LET: U be open in X .
 ⟨2⟩3. LET: C be a component of U .
 ⟨2⟩4. LET: $a \in C$
 ⟨2⟩5. LET: V be a connected neighbourhood of a such that $V \subseteq U$
 ⟨2⟩6. $V \subseteq C$
 PROOF: Lemma 9.36.3.
 ⟨2⟩7. Q.E.D.
 PROOF: Lemma 9.3.8.
 ⟨1⟩2. If, for every open set U in X , every component of U is open in X , then X is locally connected.
 ⟨2⟩1. ASSUME: for every open set U in X , every component of U is open in X .
 ⟨2⟩2. LET: $a \in X$
 ⟨2⟩3. LET: U be a neighbourhood of a
 ⟨2⟩4. The component of U that contains a is a connected neighbourhood of a included in U .

□

Example 9.38.7. The ordered square is locally connected.

PROOF: Every neighbourhood of a point includes an interval around that point, which is connected by Theorem 9.31.23.

Example 9.38.8. Let X be the set of all rational points on the line segment $[0, 1] \times \{0\}$, and Y the set of all rational points on the line segment $[0, 1] \times \{1\}$. Let A be the space consisting of all line segments joining the point $(0, 1)$ to a point of X , and all line segments joining the point $(1, 0)$ to a point of Y . Then A is path-connected but is not locally connected at any point,

Proposition 9.38.9. Let X and Y be topological spaces and $p : X \rightarrow Y$ be a quotient map. If X is locally connected then so is Y .

PROOF:

⟨1⟩1. LET: U be an open set in Y .

⟨1⟩2. LET: C be a component of U .

⟨1⟩3. $p^{-1}(C)$ is a union of components of $p^{-1}(U)$

⟨2⟩1. LET: $x \in p^{-1}(C)$

⟨2⟩2. LET: D be the component of $p^{-1}(U)$ that contains x .

⟨2⟩3. $p(D)$ is connected.

PROOF: Theorem 9.31.13.

⟨2⟩4. $p(D) \subseteq C$.

PROOF: From ⟨1⟩2 since $p(x) \in p(D) \cap C$ (⟨2⟩1, ⟨2⟩2).

⟨2⟩5. $D \subseteq p^{-1}(C)$

⟨1⟩4. $p^{-1}(C)$ is open in $p^{-1}(U)$

PROOF: Theorem 9.38.6.

⟨1⟩5. C is open in U

PROOF: Since the restriction of p to $p : p^{-1}(U) \rightarrow U$ is a quotient map by Proposition 9.26.4.

⟨1⟩6. Q.E.D.

PROOF: Theorem 9.38.6.

□

9.39 Local Path Connectedness

Definition 9.39.1 (Locally Path-Connected). Let X be a topological space and $a \in X$. Then X is *locally path-connected* at a if and only if every neighbourhood of a includes a path-connected neighbourhood of a .

The space X is *locally path-connected* if and only if it is locally path-connected at every point.

Theorem 9.39.2. A topological space X is locally path-connected if and only if, for every open set U in X , every path component of U is open in X .

PROOF:

- ⟨1⟩1. If X is locally path-connected then, for every open set U in X , every path component of U is open in X .
- ⟨2⟩1. ASSUME: X is locally path-connected.
- ⟨2⟩2. LET: U be open in X .
- ⟨2⟩3. LET: C be a path component of U .
- ⟨2⟩4. LET: $a \in C$
- ⟨2⟩5. LET: V be a path-connected neighbourhood of a such that $V \subseteq U$
- ⟨2⟩6. $V \subseteq C$
PROOF: Lemma 9.36.3.
- ⟨2⟩7. Q.E.D.
PROOF: Lemma 9.3.8.
- ⟨1⟩2. If, for every open set U in X , every component of U is open in X , then X is locally connected.
- ⟨2⟩1. ASSUME: for every open set U in X , every component of U is open in X .
- ⟨2⟩2. LET: $a \in X$
- ⟨2⟩3. LET: U be a neighbourhood of a
- ⟨2⟩4. The component of U that contains a is a connected neighbourhood of a included in U .

□

Theorem 9.39.3. *If a space is locally path connected then its components and its path components are the same.*

PROOF:

- ⟨1⟩1. LET: X be a locally path connected space.
- ⟨1⟩2. LET: C be a component of X .
- ⟨1⟩3. LET: $x \in C$
- ⟨1⟩4. LET: P be the path component of x
PROVE: $P = C$
- ⟨1⟩5. $P \subseteq C$
PROOF: Proposition 9.37.4.
- ⟨1⟩6. LET: Q be the union of the other path components included in C
- ⟨1⟩7. $C = P \cup Q$
PROOF: Proposition 9.37.4.
- ⟨1⟩8. P and Q are open in C
- ⟨2⟩1. C is open.
PROOF: Theorem 9.38.6.
- ⟨2⟩2. Q.E.D.
PROOF: Theorem 9.39.2.
- ⟨1⟩9. $Q = \emptyset$
PROOF: Otherwise P and Q would form a separation of C .

□

Example 9.39.4. The ordered square is not locally path connected, since it is connected but not path connected.

Proposition 9.39.5. *Let X be a locally path-connected space. Then every connected open subspace of X is path-connected.*

PROOF:

- ⟨1⟩1. LET: U be a connected open subspace of X .
- ⟨1⟩2. LET: P be a path component of U .
- ⟨1⟩3. LET: Q be the union of the other path components of U .
- ⟨1⟩4. P and Q are open in U .

PROOF: Theorem 9.39.2.

- ⟨1⟩5. $Q = \emptyset$

PROOF: Otherwise P and Q form a separation of U .

□

9.40 Weak Local Connectedness

Definition 9.40.1 (Weakly Locally Connected). Let X be a topological space and $a \in X$. Then X is *weakly locally connected* at a if and only if every neighbourhood of a includes a connected subspace that includes a neighbourhood of a .

Proposition 9.40.2. *Let X be a topological space. If X is weakly locally connected at every point then X is locally connected.*

PROOF:

- ⟨1⟩1. ASSUME: X is weakly locally connected at every point.
- ⟨1⟩2. LET: U be open in X .
- ⟨1⟩3. LET: C be a component of U .
- ⟨1⟩4. C is open in X .
- ⟨2⟩1. LET: $x \in C$
- ⟨2⟩2. PICK a connected subspace D of U that includes a neighbourhood V of x .
- ⟨2⟩3. $D \subseteq C$
- PROOF: Lemma 9.36.3.
- ⟨2⟩4. $x \in V \subseteq C$
- ⟨2⟩5. Q.E.D.

PROOF: Lemma 9.3.8.

- ⟨1⟩5. Q.E.D.

PROOF: Theorem 9.38.6.

□

Example 9.40.3. The 'infinite broom' (Munkres p.163) is an example of a space that is weakly locally connected at a point p but not locally connected at p .

9.41 Quasicomponents

Proposition 9.41.1. *Let X be a topological space. Define \sim on X by $x \sim y$ if and only if there exists no separation U and V of X such that $x \in U$ and $y \in V$. Then \sim is an equivalence relation on X .*

PROOF:

$\langle 1 \rangle 1.$ \sim is reflexive.

PROOF: Since the two sets that make up a separation are disjoint.

$\langle 1 \rangle 2.$ \sim is symmetric.

PROOF: Immediate from the definition.

$\langle 1 \rangle 3.$ \sim is transitive.

$\langle 2 \rangle 1.$ ASSUME: $x \sim y$ and $y \sim z$

$\langle 2 \rangle 2.$ ASSUME: for a contradiction there is a separation U and V of X with
 $x \in U$ and $z \in V$

$\langle 2 \rangle 3.$ $y \in U$ or $y \in V$

$\langle 2 \rangle 4.$ Q.E.D.

PROOF: Either case contradicts $\langle 2 \rangle 1.$

□

Definition 9.41.2 (Quasicomponents). For X a topological space, the *quasi-components* of X are the equivalence classes under \sim .

Proposition 9.41.3. *Let X be a topological space. Then every component of X is included in a quasicomponent of X .*

PROOF:

$\langle 1 \rangle 1.$ LET: C be a component of X .

$\langle 1 \rangle 2.$ LET: $x, y \in C$

PROVE: $x \sim y$

$\langle 1 \rangle 3.$ ASSUME: for a contradiction there exists a separation U and V of X with
 $x \in U$ and $y \in V$

$\langle 1 \rangle 4.$ $C \cap U$ and $C \cap V$ form a separation of C .

$\langle 1 \rangle 5.$ Q.E.D.

PROOF: This contradicts $\langle 1 \rangle 1.$

Proposition 9.41.4. *In a locally connected space, the components and the quasicomponents are the same.*

PROOF:

$\langle 1 \rangle 1.$ LET: X be a locally connected space and Q a quasicomponent of X .

$\langle 1 \rangle 2.$ PICK a component C of X such that $C \subseteq Q$

$\langle 1 \rangle 3.$ LET: D be the union of the components of X

$\langle 1 \rangle 4.$ C and D are open in X .

PROOF: Theorem 9.38.6.

$\langle 1 \rangle 5.$ D cannot contain any points of Q .

PROOF: If it did, then C and D would form a separation of X and there would be points $x, y \in Q$ with $x \in C$ and $y \in D$.

$\langle 1 \rangle 6.$ $C = Q$

□

9.42 Open Coverings

Definition 9.42.1 (Open Covering). Let X be a topological space. An *open covering* of X is a covering of X whose elements are all open sets.

9.43 Lindelöf Spaces

Definition 9.43.1 (Lindelöf Space). A topological space X is *Lindelöf* if and only if every open covering has a countable subcovering.

Proposition 9.43.2. *Let X be a topological space. Then X is Lindelöf if and only if every set of closed sets that has the countable intersection property has nonempty intersection.*

PROOF: The following are equivalent.

1. X is compact.
2. Every open covering of X has a countable subcovering.
3. For any set \mathcal{C} of closed sets, if $\{X \setminus C \mid C \in \mathcal{C}\}$ covers X then there is a countable subset \mathcal{C}_0 such that $\{X \setminus C \mid C \in \mathcal{C}_0\}$ covers X .
4. For any set \mathcal{C} of closed sets, if $\bigcap \mathcal{C} = \emptyset$ then there is a countable subset \mathcal{C}_0 with empty intersection.
5. Any set of closed sets with the countable intersection property has nonempty intersection.

□

Proposition 9.43.3 (CC). *Let X be a topological space and \mathcal{B} a basis for the topology on X . Then the following are equivalent.*

1. X is Lindelöf.
2. Every open covering of X by elements of \mathcal{B} has a countable subcovering.

PROOF:

(1)1. $1 \Rightarrow 2$

PROOF: Immediate from definitions.

(1)2. $2 \Rightarrow 1$

(2)1. ASSUME: Every open covering of X by elements of \mathcal{B} has a countable subcovering.

(2)2. LET: \mathcal{U} be an open covering of X .

(2)3. $\{B \in \mathcal{B} \mid \exists U \in \mathcal{U}. B \subseteq U\}$ covers X .

(2)4. PICK a finite subcovering \mathcal{B}_0 .

(2)5. For $B \in \mathcal{B}_0$, PICK $U_B \in \mathcal{U}$ such that $B \subseteq U_B$.

(2)6. $\{U_B \mid B \in \mathcal{B}_0\}$ covers X .

□

9.44 The Second Countability Axiom

Definition 9.44.1 (Second Countability Axiom). A topological space satisfies the *second countability axiom*, or is *second countable*, if and only if it has a countable basis.

Example 9.44.2. The space \mathbb{R} is second countable.

PROOF: The set $\{(a, b) \mid a, b \in \mathbb{Q}\}$ is a basis. \square

Proposition 9.44.3. A subspace of a second countable space is second countable.

PROOF: If \mathcal{B} is a countable basis for X and $Y \subseteq X$ then $\{B \cap Y \mid B \in \mathcal{B}\}$ is a countable basis for Y . \square

Proposition 9.44.4 (CC). Every second countable space is Lindelöf.

PROOF: From Proposition 9.43.3.

Example 9.44.5 (CC). The space \mathbb{R}_l is Lindelöf.

$\langle 1 \rangle 1$. LET: \mathcal{A} be a covering of \mathbb{R}_l by basic open sets of the form $[a, b)$

$\langle 1 \rangle 2$. LET: $C = \bigcup \{(a, b) \mid [a, b) \in \mathcal{A}\}$

$\langle 1 \rangle 3$. $\mathbb{R} \setminus C$ is countable.

$\langle 2 \rangle 1$. For every $x \in \mathbb{R} \setminus C$, PICK a rational q_x such that $(x, q_x) \subseteq C$

$\langle 3 \rangle 1$. LET: $x \in \mathbb{R} \setminus C$

$\langle 3 \rangle 2$. PICK b such that $[x, b) \in \mathcal{A}$

$\langle 3 \rangle 3$. PICK a rational q such that $q \in (x, b)$

$\langle 2 \rangle 2$. The mapping $x \mapsto q_x$ is an injection $\mathbb{R} \setminus C \rightarrow \mathbb{Q}$

$\langle 1 \rangle 4$. PICK a countable $\mathcal{A}' \subseteq \mathcal{A}$ that covers $\mathbb{R} \setminus C$

$\langle 1 \rangle 5$. Under the standard topology on \mathbb{R} , C is second countable.

PROOF: Proposition 9.44.3.

$\langle 1 \rangle 6$. PICK a countable $\mathcal{A}'' \subseteq \mathcal{A}$ such that $\{(a, b) \mid [a, b) \in \mathcal{A}''\}$ covers C .

PROOF: Proposition 9.43.3.

$\langle 1 \rangle 7$. $\mathcal{A}' \cup \mathcal{A}''$ covers \mathbb{R}_l .

\square

Example 9.44.6. The product of two Lindelöf spaces is not necessarily Lindelöf.

We prove that the Sorgenfrey plane is not Lindelöf.

PROOF:

$\langle 1 \rangle 1$. LET: $L = \{(x, -x) \mid x \in \mathbb{R}\}$

$\langle 1 \rangle 2$. L is closed in \mathbb{R}_l^2

$\langle 1 \rangle 3$. LET: $\mathcal{U} = \{[a, b) \times [a, -d) \mid a, b, d \in \mathbb{R}\}$

$\langle 1 \rangle 4$. $\mathcal{U} \cup \{\mathbb{R} \setminus L\}$ covers \mathbb{R}_l^2

$\langle 1 \rangle 5$. Every element of \mathcal{U} intersects L at exactly one point.

$\langle 1 \rangle 6$. No countable subset of \mathcal{U} covers \mathbb{R}_l^2 .

\square

9.45 Sequential Compactness

Definition 9.45.1 (Sequentially Compact). A topological space is *sequentially compact* if and only if every sequence has a convergent subsequence.

9.46 Limit Point Compactness

Definition 9.46.1 (Limit Point Compact Space). A topological space is *limit point compact* if and only if every infinite set has a limit point.

Proposition 9.46.2. *Every limit point compact T_1 space is sequentially compact.*

PROOF:

- ⟨1⟩1. LET: X be a limit point compact T_1 space.
- ⟨1⟩2. LET: (x_n) be a sequence in X .
- ⟨1⟩3. CASE: $\{x_n \mid n \geq 1\}$ is finite.
 - ⟨2⟩1. PICK n such that x_n occurs infinitely often in the sequence (x_n)
 - ⟨2⟩2. The subsequence consisting of all the terms equal to x_n is convergent.
- ⟨1⟩4. CASE: $\{x_n \mid n \geq 1\}$ is infinite.
 - ⟨2⟩1. PICK a limit point l for $\{x_n \mid n \geq 1\}$
 - ⟨2⟩2. PICK an increasing sequence n_r with $x_{n_r} \in B(x, 1/r)$ for all r
PROOF: This is always possible by Theorem 9.21.3.
 - ⟨2⟩3. (x_{n_r}) converges to l .

□

Corollary 9.46.2.1. *Every compact T_1 space is sequentially compact.*

Example 9.46.3. The space $[0, 1]^\omega$ under the uniform topology is not limit point compact.

The infinite set $\{0, 1\}^\omega$ has no limit point.

Example 9.46.4. The space $[0, 1]$ under the lower limit topology is not limit point compact.

The infinite set $A = \{1 - 1/n \mid n \geq 1\}$ has no limit point. 1 is not a limit point because the neighbourhood $\{1\}$ does not intersect A .

Proposition 9.46.5. *A closed subspace of a limit point compact space is limit point compact.*

PROOF:

- ⟨1⟩1. LET: X be a limit point compact space.
- ⟨1⟩2. LET: $A \subseteq X$ be closed.
- ⟨1⟩3. LET: $B \subseteq A$ be infinite.
- ⟨1⟩4. PICK a limit point l of B in X .
- ⟨1⟩5. $l \in A$
- ⟨1⟩6. l is a limit point of B in A .

□

Example 9.46.6. An open subspace of a limit point compact space is not necessarily limit point compact.

The space $[0, 1]$ is limit point compact but $(0, 1)$ is not.

Example 9.46.7. The continuous image of a limit point compact space is not necessarily limit point compact.

Let Y be the indiscrete space with two points. Then $\mathbb{Z}^+ \times Y$ is limit point compact but \mathbb{Z}^+ is not.

Example 9.46.8. A limit point compact subspace of a Hausdorff space is not necessarily closed.

The space S_Ω is limit point compact but is not closed in $\overline{S_\Omega}$.

For an example that shows that the product of two limit point compact spaces is not necessarily limit point compact, see L. A. Steen and J. A. Seebach Jr. *Counterexamples in Topology* Example 112.

9.47 Countable Compactness

Definition 9.47.1 (Countably Compact). A topological space is *countably compact* if and only if every countable open covering has a finite subcovering.

Proposition 9.47.2 (AC). *Every closed subspace of a countably compact space is countably compact.*

PROOF:

- <1>1. LET: X be a countably compact space.
- <1>2. LET: $A \subseteq X$ be closed.
- <1>3. LET: \mathcal{U} be a countable open cover of A .
- <1>4. For $U \in \mathcal{U}$, PICK an open set V_U in X such that $U = V_U \cap A$
- <1>5. $\{V_U \mid U \in \mathcal{U}\} \cup \{X - A\}$ is a countable open cover of X
- <1>6. PICK a finite subcover $\{V_{U_1}, \dots, V_{U_n}, X - A\}$
- <1>7. $\{U_1, \dots, U_n\}$ covers A .

□

Proposition 9.47.3 (AC). *Every countably compact space is limit point compact.*

PROOF:

- <1>1. ASSUME: X is countably compact.
- <1>2. LET: $A \subseteq X$ be infinite.
- <1>3. ASSUME: for a contradiction A has no limit point.
- <1>4. PICK a countably infinite $B \subseteq A$
- <1>5. B is discrete.

PROOF: For all $b \in B$, there exists U_b open in X such that $U_b \cap B = \{b\}$.

- <1>6. $\{\{b\} \mid b \in B\}$ is a countable cover of B that has no finite subcover.
- <1>7. B is not countably compact.
- <1>8. B is not closed in X

- ⟨1⟩9. B has a limit point.
 - ⟨1⟩10. A has a limit point.
 - ⟨1⟩11. Q.E.D.
- PROOF: This contradicts ⟨1⟩3.

□

Proposition 9.47.4 (AC). *Every limit point compact T_1 space is countably compact.*

PROOF:

- ⟨1⟩1. LET: X be a limit point compact T_1 space.
- ⟨1⟩2. LET: $\{U_n \mid n \in \mathbb{Z}^+\}$ be a countable open cover of X .
- ⟨1⟩3. For $n \in \mathbb{Z}^+$,
LET: $V_n = U_1 \cup \dots \cup U_n$
- ⟨1⟩4. ASSUME: for a contradiction none of the V_n covers X
- ⟨1⟩5. For $n \in \mathbb{Z}^+$, PICK $a_n \in X - V_n$
- ⟨1⟩6. PICK a limit point l for $\{a_n \mid n \in \mathbb{Z}^+\}$
- ⟨1⟩7. PICK n such that $l \in U_n$
- ⟨1⟩8. CASE: $l = a_m$ for some $m \leq n$
PROOF: $U_n - \{a_1, \dots, a_{m-1}, a_{m+1}, \dots, a_n\}$ is a neighbourhood of l that intersects $\{a_n \mid n \in \mathbb{Z}^+\}$ only at l , contradicting ⟨1⟩6.
- ⟨1⟩9. CASE: $l \neq a_m$ for any $m \leq n$
PROOF: $U_n - \{a_1, \dots, a_n\}$ is a neighbourhood of l that does not intersect $\{a_n \mid n \in \mathbb{Z}^+\}$, which contradicts ⟨1⟩6.

□

The following example shows we cannot remove the hypothesis that the space is T_1 .

Example 9.47.5. Let Y be the indiscrete space with two points. Then $\mathbb{Z}^+ \times Y$ is a limit point compact space that is not countably compact, since $\{\{n\} \times Y \mid n \in \mathbb{Z}^+\}$ is a countable open cover that has no finite subcover.

Proposition 9.47.6. *A topological space is countably compact if and only if every nested sequence $C_1 \supseteq C_2 \supseteq \dots$ of nonempty closed sets has nonempty intersection.*

PROOF:

- ⟨1⟩1. LET: X be a topological space.
- ⟨1⟩2. If X is countably compact then every nested sequence of nonempty closed sets has nonempty intersection.
- ⟨2⟩1. ASSUME: X is countably compact.
- ⟨2⟩2. LET: $C_1 \supseteq C_2 \supseteq \dots$ be a nested sequence of nonempty closed sets.
- ⟨2⟩3. ASSUME: for a contradiction $\bigcap_n C_n = \emptyset$
- ⟨2⟩4. $\{X - C_n \mid n \in \mathbb{Z}^+\}$ covers X
- ⟨2⟩5. PICK a finite subcover $\{X - C_{n_1}, \dots, X - C_{n_k}\}$ where $n_1 < \dots < n_k$
- ⟨2⟩6. $C_{n_k} = \emptyset$
- ⟨2⟩7. Q.E.D.

PROOF: This contradicts $\langle 2 \rangle 2$.

□

- $\langle 1 \rangle 3$. If every nested sequence of nonempty closed sets has nonempty intersection then X is countably compact.
- $\langle 2 \rangle 1$. ASSUME: Every nested sequence of nonempty closed sets has nonempty intersection.
- $\langle 2 \rangle 2$. LET: $\{U_n \mid n \geq 1\}$ is a countable open cover of X .
- $\langle 2 \rangle 3$. $X - U_1 \supseteq X - (U_1 \cup U_2) \supseteq \cdots$ is a nested sequence of closed sets with empty intersection.
- $\langle 2 \rangle 4$. PICK k such that $X - (U_1 \cup \cdots \cup U_k) = \emptyset$
- $\langle 2 \rangle 5$. $\{U_1, \dots, U_k\}$ covers X .

□

9.48 Compact Spaces

Definition 9.48.1 (Compact). A topological space is *compact* if and only if every open covering has a finite subcovering.

Lemma 9.48.2. *Let X be a topological space and Y a subspace of X . Then Y is compact if and only if every covering of Y by sets open in X has a finite subcovering.*

PROOF:

- $\langle 1 \rangle 1$. If Y is compact then every covering of Y by sets open in X has a finite subcovering.
 - $\langle 2 \rangle 1$. ASSUME: Y is compact.
 - $\langle 2 \rangle 2$. LET: \mathcal{U} be a covering of Y by sets open in X .
 - $\langle 2 \rangle 3$. $\{U \cap Y \mid U \in \mathcal{U}\}$ is an open covering of Y .
 - $\langle 2 \rangle 4$. PICK a finite subcovering $\{U_1 \cap Y, \dots, U_n \cap Y\}$
 - $\langle 2 \rangle 5$. $\{U_1, \dots, U_n\}$ is a finite subcovering of \mathcal{U} .
- $\langle 1 \rangle 2$. If every covering of Y by sets open in X has a finite subcovering then Y is compact.
 - $\langle 2 \rangle 1$. LET: \mathcal{U} be an open covering of Y .
 - $\langle 2 \rangle 2$. LET: $\mathcal{V} = \{V \text{ open in } X \mid V \cap Y \in \mathcal{U}\}$.
 - $\langle 2 \rangle 3$. \mathcal{V} is a covering of Y by sets open in X .
 - $\langle 2 \rangle 4$. PICK a finite subcovering $\{V_1, \dots, V_n\}$
 - $\langle 2 \rangle 5$. $\{V_1 \cap Y, \dots, V_n \cap Y\}$ is a finite subcovering of \mathcal{U} .

□

Proposition 9.48.3. *Every closed subspace of a compact space is compact.*

PROOF:

- $\langle 1 \rangle 1$. LET: X be a compact space and $Y \subseteq X$ be closed.
- $\langle 1 \rangle 2$. LET: \mathcal{U} be a covering of Y by sets open in X .
- $\langle 1 \rangle 3$. $\mathcal{U} \cup \{X \setminus Y\}$ is an open covering of X .
- $\langle 1 \rangle 4$. PICK a finite subcovering \mathcal{U}_0
- $\langle 1 \rangle 5$. $\mathcal{U}_0 \cap \mathcal{U}$ is a finite subset of \mathcal{U} that covers Y .

□

Theorem 9.48.4. *The continuous image of a compact space is compact.*

PROOF:

- ⟨1⟩1. LET: $f : X \rightarrow Y$ be continuous and surjective.
- ⟨1⟩2. LET: \mathcal{V} be an open covering of Y
- ⟨1⟩3. $\{p^{-1}(V) \mid V \in \mathcal{V}\}$ is an open covering of X .
- ⟨1⟩4. PICK a finite subcovering $\{p^{-1}(V_1), \dots, p^{-1}(V_n)\}$
- ⟨1⟩5. $\{V_1, \dots, V_n\}$ covers Y .

□

Theorem 9.48.5. *Let A and B be compact subspaces of X and Y respectively. Let N be an open set in $X \times Y$ that includes $A \times B$. Then there exist open sets U and V in X and Y respectively such that $A \subseteq U$, $B \subseteq V$ and $U \times V \subseteq N$.*

PROOF:

- ⟨1⟩1. For all $x \in A$, there exist neighbourhoods U of x and V of B such that $U \times V \subseteq N$.
- ⟨2⟩1. LET: $x \in A$
- ⟨2⟩2. For all $y \in B$, there exist neighbourhoods U of x and V of y such that $U \times V \subseteq N$
- ⟨2⟩3. $\{V \text{ open in } Y \mid \exists \text{ neighbourhood } U \text{ of } x, U \times V \subseteq N\}$ covers B .
- ⟨2⟩4. PICK a finite subcover $\{V_1, \dots, V_n\}$
- ⟨2⟩5. For $i = 1, \dots, n$, PICK a neighbourhood U_i of x such that $U_i \times V_i \subseteq N$
- ⟨2⟩6. LET: $U = U_1 \cap \dots \cap U_n$
- ⟨2⟩7. LET: $V = V_1 \cup \dots \cup V_n$
- ⟨2⟩8. U is a neighbourhood of x .
- ⟨2⟩9. V is a neighbourhood of B .
- ⟨2⟩10. $U \times V \subseteq N$
- ⟨1⟩2. $\{U \text{ open in } X \mid \exists \text{ neighbourhood } V \text{ of } B, U \times V \subseteq N\}$ covers A .
- ⟨1⟩3. PICK a finite subcover $\{U_1, \dots, U_n\}$
- ⟨1⟩4. For $i = 1, \dots, n$, PICK a neighbourhood V_i of B such that $U_i \times V_i \subseteq N$
- ⟨1⟩5. LET: $U = U_1 \cup \dots \cup U_n$
- ⟨1⟩6. LET: $V = V_1 \cap \dots \cap V_n$
- ⟨1⟩7. U and V are open.
- ⟨1⟩8. $A \subseteq U$
- ⟨1⟩9. $B \subseteq V$
- ⟨1⟩10. $U \times V \subseteq N$

□

Corollary 9.48.5.1 (Tube Lemma). *Let X and Y be topological spaces with Y compact. Let $a \in X$ and N be an open set in $X \times Y$ that includes $\{a\} \times Y$. Then there exists a neighbourhood W of a such that N includes the tube $W \times Y$.*

Theorem 9.48.6. *Let X be a topological space. Then X is compact if and only if every set of closed sets that has the finite intersection property has nonempty intersection.*

PROOF: The following are equivalent.

1. X is compact.
2. Every open covering of X has a finite subcovering.
3. For any set \mathcal{C} of closed sets, if $\{X \setminus C \mid C \in \mathcal{C}\}$ covers X then there is a finite subset \mathcal{C}_0 such that $\{X \setminus C \mid C \in \mathcal{C}_0\}$ covers X .
4. For any set \mathcal{C} of closed sets, if $\bigcap \mathcal{C} = \emptyset$ then there is a finite subset \mathcal{C}_0 with empty intersection.
5. Any set of closed sets with the finite intersection property has nonempty intersection.

□

Corollary 9.48.6.1. *Let X be a topological space and $C_1 \supseteq C_2 \supseteq \cdots$ a nested sequence of nonempty closed sets. Then $\bigcap_n C_n$ is nonempty.*

Proposition 9.48.7. *Let \mathcal{T} and \mathcal{T}' be two topologies on the same set X with $\mathcal{T} \subseteq \mathcal{T}'$. If \mathcal{T}' is compact then \mathcal{T} is compact.*

PROOF:

- ⟨1⟩1. LET: $\mathcal{U} \subseteq \mathcal{T}$ cover X
 ⟨1⟩2. $\mathcal{U} \subseteq \mathcal{T}'$
 ⟨1⟩3. A finite subset of \mathcal{U} covers X .

□

Corollary 9.48.7.1. *If \mathcal{T} and \mathcal{T}' are two compact Hausdorff topologies on the same set X , then either $\mathcal{T} = \mathcal{T}'$ or \mathcal{T} and \mathcal{T}' are incomparable.*

PROOF: From the Proposition and Proposition 9.22.12. □

Example 9.48.8. Any set under the finite complement topology is compact.

Proposition 9.48.9. *Let X be a topological space. A finite union of compact subspaces of X is compact.*

PROOF:

- ⟨1⟩1. LET: A and B be compact subspaces of X .
 ⟨1⟩2. LET: \mathcal{U} be a set of open sets in X that covers $A \cup B$
 ⟨1⟩3. PICK a finite subset \mathcal{U}_1 that covers A .
 PROOF: Lemma 9.48.2.
 ⟨1⟩4. PICK a finite subset \mathcal{U}_2 that covers B .
 PROOF: Lemma 9.48.2.
 ⟨1⟩5. $\mathcal{U}_1 \cup \mathcal{U}_2$ is a finite subset that covers $A \cup B$.
 ⟨1⟩6. Q.E.D.
 PROOF: Lemma 9.48.2.

□

Proposition 9.48.10. *Let A and B be disjoint compact subspaces of the Hausdorff space X . Then there exist disjoint open sets U and V that include A and B respectively.*

PROOF: From Theorem 9.48.5 with $N = X^2 \setminus \{(x, x) \mid x \in X\}$. \square

Corollary 9.48.10.1. *Every compact subspace of a Hausdorff space is closed.*

Theorem 9.48.11. *Let $f : X \rightarrow Y$ be a bijective continuous function. If X is compact and Y is Hausdorff then f is a homeomorphism.*

PROOF:

$\langle 1 \rangle 1$. LET: $C \subseteq X$ be closed.

$\langle 1 \rangle 2$. C is compact.

PROOF: Proposition 9.48.3.

$\langle 1 \rangle 3$. $f(C)$ is compact.

PROOF: Theorem 9.48.4.

$\langle 1 \rangle 4$. $f(C)$ is closed.

PROOF: Corollary 9.48.10.1.

$\langle 1 \rangle 5$. Q.E.D.

PROOF: Lemma 9.15.2.

\square

Proposition 9.48.12. *Let X be a compact space, Y a Hausdorff space, and $f : X \rightarrow Y$ a continuous map. Then f is a closed map.*

PROOF:

$\langle 1 \rangle 1$. LET: $C \subseteq X$ be closed.

$\langle 1 \rangle 2$. C is compact.

PROOF: Proposition 9.48.3.

$\langle 1 \rangle 3$. $f(C)$ is compact.

PROOF: Theorem 9.48.4.

$\langle 1 \rangle 4$. $f(C)$ is closed.

PROOF: Corollary 9.48.10.1.

\square

Proposition 9.48.13. *If Y is compact then the projection $\pi_1 : X \times Y \rightarrow X$ is a closed map.*

PROOF:

$\langle 1 \rangle 1$. LET: $A \subseteq X \times Y$ be closed.

$\langle 1 \rangle 2$. LET: $x \in X \setminus \pi_1(A)$

$\langle 1 \rangle 3$. PICK a neighbourhood U of x such that $U \times Y \subseteq (X \times Y) \setminus A$

PROOF: By the Tube Lemma.

$\langle 1 \rangle 4$. $x \in U \subseteq X \setminus \pi_1(A)$

$\langle 1 \rangle 5$. Q.E.D.

PROOF: So $X \setminus \pi_1(A)$ is open by Lemma 9.3.8.

\square

Proposition 9.48.14. *Let X be a topological space and Y a Hausdorff space. Let $f : X \rightarrow Y$ be continuous. Then the graph of f is closed in $X \times Y$.*

- $\langle 1 \rangle 1$. ASSUME: f is continuous.
- $\langle 1 \rangle 2$. LET: $(x, y) \in (X \times Y) \setminus G_f$
- $\langle 1 \rangle 3$. PICK disjoint neighbourhoods U and V of y and $f(x)$ respectively.
- $\langle 1 \rangle 4$. $f^{-1}(V) \times U$ is a neighbourhood of (x, y) disjoint from G_f .

□

Theorem 9.48.15. *Let X be a topological space and Y a compact space. Let $f : X \rightarrow Y$ be a function. If the graph of f is closed in $X \times Y$ then f is continuous.*

PROOF:

- $\langle 1 \rangle 1$. ASSUME: G_f is closed.
- $\langle 1 \rangle 2$. LET: $x \in X$ and V be a neighbourhood of $f(x)$.
- $\langle 1 \rangle 3$. $G_f \cap (X \times (Y \setminus V))$ is closed.
- $\langle 1 \rangle 4$. $\pi_1(G_f \cap (X \times (Y \setminus V)))$ is closed.

PROOF: Proposition 9.48.13.

- $\langle 1 \rangle 5$. LET: $U = X \setminus \pi_1(G_f \cap (X \times (Y \setminus V)))$
- $\langle 1 \rangle 6$. U is a neighbourhood of x
- $\langle 1 \rangle 7$. $f(U) \subseteq V$

□

Theorem 9.48.16. *Let X be a compact topological space. Let $(f_n : X \rightarrow \mathbb{R})$ be a monotone increasing sequence of continuous functions and $f : X \rightarrow \mathbb{R}$ a continuous function. If (f_n) converges pointwise to f , then (f_n) converges uniformly to f .*

PROOF:

- $\langle 1 \rangle 1$. LET: $\epsilon > 0$
- $\langle 1 \rangle 2$. For all $x \in X$, there exists N such that, for all $n \geq N$, we have $|f_n(x) - f(x)| < \epsilon$
- $\langle 1 \rangle 3$. For $n \geq 1$,
LET: $U_n = \{x \in X \mid |f_n(x) - f(x)| < \epsilon\}$
- $\langle 1 \rangle 4$. For $n \geq 1$, we have U_n is open in X .
 $\langle 2 \rangle 1$. LET: $x \in X$
 $\langle 2 \rangle 2$. LET: $\delta = \epsilon - |f_n(x) - f(x)|$
 $\langle 2 \rangle 3$. PICK a neighbourhood U of x such that $f(U) \subseteq B(f(x), \delta/2)$
 $\langle 2 \rangle 4$. PICK a neighbourhood V of x such that $f_n(V) \subseteq B(f_n(x), \delta/2)$
 $\langle 2 \rangle 5$. $f(U \cap V) \subseteq U_n$

PROOF: For $y \in U \cap V$ we have

$$\begin{aligned} |f_n(y) - f(y)| &\leq |f_n(y) - f_n(x)| + |f_n(x) - f(x)| + |f(x) - f(y)| \\ &< \delta/2 + |f_n(x) - f(x)| + \delta/2 \\ &= \epsilon \end{aligned}$$

- $\langle 1 \rangle 5$. $\{U_n \mid n \geq 1\}$ covers X

PROOF: From $\langle 1 \rangle 2$

- (1)6. PICK N such that $X = U_N$
 (2)1. PICK n_1, \dots, n_k such that U_{n_1}, \dots, U_{n_k} cover X .
 (2)2. LET: $N = \max(n_1, \dots, n_k)$
 (2)3. For all i we have $U_{n_i} \subseteq U_N$
 PROOF: Since (f_n) is monotone increasing.
 (2)4. $X = U_N$
 (1)7. For all $x \in X$ and $n \geq N$ we have $|f_n(x) - f(x)| < \epsilon$
 \square

An example to show that we cannot remove the hypothesis that X is compact:

Example 9.48.17. Let $X = (0, 1)$, $f_n(x) = -x^n$ and $f(x) = 0$ for $x \in X$ and $n \geq 1$. Then $f_n \rightarrow f$ pointwise and (f_n) is monotone increasing but the convergence is not uniform since, for all $N \geq 1$, there exists $x \in (0, 1)$ such that $-x^N < -1/2$.

An example to show that we cannot remove the hypothesis that (f_n) is monotone increasing:

Example 9.48.18. Let $X = [0, 1]$, $f_n(x) = 1/(n^3(x - 1/n)^2 + 1)$ and $f(x) = 0$ for $x \in X$ and $n \geq 1$. Then X is compact and $f_n \rightarrow f$ pointwise but the convergence is not uniform since, for all $N \geq 1$, there exists $x \in [0, 1]$ such that $f_N(x) = 1$, namely $x = 1/N$.

Theorem 9.48.19. Let X be a compact Hausdorff space. Let \mathcal{A} be a chain of closed connected subsets of X . Then $\bigcap \mathcal{A}$ is connected.

PROOF:

- (1)1. ASSUME: for a contradiction C and D form a separation of $\bigcap \mathcal{A}$.
 (1)2. PICK disjoint open sets U and V that include C and D respectively.
 PROOF: Proposition 9.48.10.
 (1)3. $\{A \setminus (U \cup V) \mid A \in \mathcal{A}\}$ is a set of closed sets with the finite intersection property.
 (2)1. For all $A \in \mathcal{A}$ we have $A \setminus (U \cup V)$ is closed.
 (2)2. For all $A_1, \dots, A_n \in \mathcal{A}$ we have $(A_1 \cap \dots \cap A_n) \setminus (U \cup V)$ is nonempty.
 PROOF:
 (3)1. LET: $A_1, \dots, A_n \in \mathcal{A}$
 (3)2. ASSUME: without loss of generality $A_1 \subseteq A_2, \dots, A_n$
 PROOF: Since \mathcal{A} is a chain.
 (3)3. $A_1 \setminus (U \cup V)$ is nonempty
 PROOF: Otherwise $(A_1 \cap \dots \cap A_n \cap U)$ and $(A_1 \cap \dots \cap A_n \cap V)$ would form a separation of A_n .
 (1)4. $\bigcap \mathcal{A} \setminus (U \cup V)$ is nonempty.
 PROOF: Theorem 9.48.6.
 (1)5. Q.E.D.
 PROOF: This contradicts (1)1 since $\bigcap \mathcal{A} \setminus (U \cup V) = \bigcap \mathcal{A} \setminus (C \cup D)$.
 \square

Theorem 9.48.20 (Tychonoff Theorem (AC)). *The product of a family of compact spaces is compact.*

PROOF:

- ⟨1⟩1. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of compact spaces.
- ⟨1⟩2. LET: $X = \prod_{\alpha \in J} X_\alpha$
- ⟨1⟩3. For any $\mathcal{A} \subseteq \mathcal{P}X$, we have $\bigcap_{A \in \mathcal{A}} \overline{A} \neq \emptyset$
 - ⟨2⟩1. LET: $\mathcal{A} \subseteq \mathcal{P}X$
 - ⟨2⟩2. PICK $\mathcal{D} \supseteq \mathcal{A}$ that is maximal with respect to the finite intersection property.
 - PROVE: $\bigcap_{D \in \mathcal{D}} \overline{D} \neq \emptyset$
 - PROOF: Lemma 3.24.2.
 - ⟨2⟩3. For $\alpha \in J$, PICK $x_\alpha \in X_\alpha$ such that $x_\alpha \in \bigcap_{D \in \mathcal{D}} \overline{\pi_\alpha(D)}$
 - PROOF: Theorem 9.48.6 since $\{\overline{\pi_\alpha(D)} \mid D \in \mathcal{D}\}$ is a set of closed sets in X_α with the finite intersection property.
 - ⟨2⟩4. LET: $x = (x_\alpha)_{\alpha \in J}$
 - PROVE: $x \in \bigcap_{D \in \mathcal{D}} \overline{D}$
 - ⟨2⟩5. For any $\beta \in J$ and neighbourhood U of x_β in X_β , we have $\pi_\beta^{-1}(U)$ intersects every element of \mathcal{D}
 - ⟨3⟩1. LET: $\beta \in J$
 - ⟨3⟩2. LET: U be a neighbourhood of x_β in X_β .
 - ⟨3⟩3. LET: $D \in \mathcal{D}$
 - ⟨3⟩4. $x_\beta \in \overline{\pi_\beta(D)}$
 - PROOF: From ⟨2⟩3
 - ⟨3⟩5. U intersects $\pi_\beta(D)$.
 - ⟨3⟩6. $\pi_\beta^{-1}(U)$ intersects D .
 - ⟨2⟩6. For any $\beta \in J$ and neighbourhood U of x_β in X_β , we have $\pi_\beta^{-1}(U) \in \mathcal{D}$
 - PROOF: Lemma 3.24.4.
 - ⟨2⟩7. Every basic neighbourhood of x is an element of \mathcal{D}
 - PROOF: Lemma 3.24.3.
 - ⟨2⟩8. Every basic neighbourhood of x intersects every element of \mathcal{D}
 - PROOF: Since \mathcal{D} satisfies the finite intersection property.
 - ⟨2⟩9. For all $D \in \mathcal{D}$ we have $x \in \overline{D}$
- ⟨1⟩4. Q.E.D.
 - PROOF: Theorem 9.48.6.

□

Lemma 9.48.21. *Let X and Y be topological spaces. Let \mathcal{A} be a set of basis elements for the product topology on $X \times Y$ such that no finite subset of \mathcal{A} covers $X \times Y$. If X is compact, then there exists $x \in X$ such that no finite subset of \mathcal{A} covers the slice $\{x\} \times Y$.*

PROOF:

- ⟨1⟩1. ASSUME: for every $x \in X$, there exists a finite subset of \mathcal{A} that covers $\{x\} \times Y$
 - PROVE: A finite subset of \mathcal{A} covers $X \times Y$

- ⟨1⟩2. $\{U \mid \exists U_1 \times V_1, \dots, U_n \times V_n \in \mathcal{A}. U = U_1 \cap \dots \cap U_n, V_1 \cup \dots \cup V_n = Y\}$
covers X
 - ⟨1⟩3. PICK a finite subcover U_1, \dots, U_m
 - ⟨1⟩4. PICK $U_{ij} \times V_{ij} \in \mathcal{A}$ such that, for every i , we have $U_i = \bigcap_j U_{ij}$ and
 $Y = \bigcup_j V_{ij}$
 - ⟨1⟩5. The collection of all $U_{ij} \times V_{ij}$ covers $X \times Y$
-

Theorem 9.48.22 (AC). *Let X be a compact Hausdorff space. Then the quasicomponents and the components of X are the same.*

PROOF:

- ⟨1⟩1. LET: $x, y \in X$
- ⟨1⟩2. ASSUME: x and y are in the same quasicomponent.
PROVE: x and y are in the same component.
- ⟨1⟩3. LET: \mathcal{A} be the set of all closed subsets A of X such that x and y are in the same quasicomponent of A .
- ⟨1⟩4. For every chain $\mathcal{B} \subseteq \mathcal{A}$ we have $\bigcap \mathcal{B} \in \mathcal{A}$
 - ⟨2⟩1. LET: $\mathcal{B} \subseteq \mathcal{A}$ be a chain.
 - ⟨2⟩2. ASSUME: for a contradiction U and V form a separation of $\bigcap \mathcal{B}$ with
 $x \in U$ and $y \in V$
 - ⟨2⟩3. PICK disjoint open sets U', V' in X such that $U \subseteq U'$ and $V \subseteq V'$
 - ⟨2⟩4. $\{B \setminus (U' \cup V') \mid B \in \mathcal{B}\}$ satisfies the finite intersection property.
 - ⟨3⟩1. LET: $B_1, \dots, B_n \in \mathcal{B}$
 - ⟨3⟩2. ASSUME: without loss of generality $B_1 \subseteq \dots \subseteq B_n$
PROOF: Since \mathcal{B} is a chain.
 - ⟨3⟩3. $\bigcap \{B_1 \setminus (U' \cup V'), \dots, B_n \setminus (U' \cup V')\} = B_1 \setminus (U' \cup V')$
 - ⟨3⟩4. $B_1 \setminus (U' \cup V')$ is nonempty
PROOF: Otherwise $B_1 \cap U'$ and $B_1 \cap V'$ would form a separation of B_1 ,
contradicting the fact that x and y are in the same quasicomponent of B_1 .
 - ⟨2⟩5. $\bigcap \mathcal{B} \setminus (U \cup V)$ is nonempty
PROOF: Theorem 9.48.6.
 - ⟨2⟩6. Q.E.D.
PROOF: This contradicts ⟨2⟩2.
- ⟨1⟩5. PICK a minimal element D in \mathcal{A} .
PROVE: D is connected.
PROOF: By Zorn's Lemma.
- ⟨1⟩6. ASSUME: for a contradiction U and V form a separation of D .
- ⟨1⟩7. ASSUME: without loss of generality $x, y \in U$
PROOF: We cannot have that one of x, y is in U and the other in V since
 $D \in \mathcal{A}$.
- ⟨1⟩8. $U \in \mathcal{A}$
PROOF: If X and Y form a separation of U with $x \in X$ and $y \in Y$, then X
and $Y \cup V$ form a separation of D with $x \in X$ and $y \in Y \cup V$.
- ⟨1⟩9. Q.E.D.
PROOF: There is a connected set D that contains both x and y .

□

PROOF:

- (1)1. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of compact spaces.
- (1)2. LET: $X = \prod_{\alpha \in J} X_\alpha$
- (1)3. PICK a well-ordering $<$ on J such that J has a greatest element.
- (1)4. For $\alpha \in J$ and $p = \{p_i \in X_i\}_{i \leq \alpha}$ a family of points,
LET: $Y(p) = \{x \in X \mid \forall i \leq \alpha. x_i = p_i\}$
- (1)5. If $\alpha < \alpha'$ and p is an α' -indexed family of points then $Y(p) \subseteq Y(p \upharpoonright \alpha)$
PROOF: From definition.
- (1)6. Given $\beta \in J$ and $p = \{p_i \in X_i\}_{i < \beta}$ a family of points,
LET: $Z(p) = \bigcap_{\alpha < \beta} Y(p \upharpoonright \alpha)$
- (1)7. Given $\beta \in J$ and $p = \{p_i \in X_i\}_{i < \beta}$ a family of points, if \mathcal{A} is a finite set of basic open spaces for X that covers $Z(p)$, then there exists $\alpha < \beta$ such that \mathcal{A} covers $Y(p \upharpoonright \alpha)$
- (2)1. ASSUME: without loss of generality β has no immediate predecessor.
- (2)2. For $A \in \mathcal{A}$,
LET: $J_A = \{i < \beta \mid \pi_i(A) \neq X_i\}$
- (2)3. LET: $\alpha = \max \bigcup_{A \in \mathcal{A}} J_A$
- (2)4. LET: $x \in Y(p \upharpoonright \alpha)$
- (2)5. LET: $y \in Z(p)$ be the point with $y_i = p_i$ for $i < \beta$ and $y_i = x_i$ for $i \geq \beta$
- (2)6. PICK $A \in \mathcal{A}$ such that $y \in A$
PROOF: Since \mathcal{A} covers $Z(p)$.
- (2)7. For $i \in J_A$ we have $x_i \in \pi_i(A)$
PROOF: Since $i \leq \alpha$ so $x_i = p_i$
- (2)8. For $i \in J \setminus J_A$ we have $x_i \in \pi_i(A)$
PROOF: Since $\pi_i(A) = X_i$
- (2)9. $x \in A$
- (1)8. ASSUME: for a contraction \mathcal{A} is a set of basic open sets for X that covers X but such that no finite subset of \mathcal{A} covers X
- (1)9. PICK a set of points $\{p_i\}_{i \in J}$ such that, for all $\alpha \in J$, we have $Y(p \upharpoonright \alpha)$ is not finitely covered by \mathcal{A}
- (2)1. ASSUME: as transfinite induction hypothesis $\alpha \in J$ and $\{p_i\}_{i < \alpha}$ is a family of points such that, for all $\alpha' < \alpha$, we have $Y(p \upharpoonright \alpha')$ is not finitely covered by \mathcal{A}
- (2)2. $Z(p)$ is not finitely covered by \mathcal{A}
PROOF: By (1)7.
- (2)3. PICK $p_\alpha \in X_\alpha$ such that $Y(p)$ is not finitely covered by \mathcal{A}
PROOF: By Lemma 9.48.21 since there is a homeomorphism $\phi : Z(p) \cong X_\alpha \times \prod_{\alpha' > \alpha} X_{\alpha'}$ and, given p_α , this homomorphism ϕ restricts to a homeomorphism $Y(p) \cong \{p_\alpha\} \times \prod_{\alpha' > \alpha} X_{\alpha'}$.
- (1)10. Q.E.D.
PROOF: If ω is the greatest element of J then $Y(p \upharpoonright \omega)$ is a singleton.

□

Theorem 9.48.23. *Every complete linearly ordered set in the order topology is compact.*

PROOF:

- ⟨1⟩1. LET: X be a complete linearly ordered set with least element a and greatest element b .
- ⟨1⟩2. LET: \mathcal{A} be an open covering of X .
- ⟨1⟩3. For all $x < b$, there exists $y > x$ such that $[x, y]$ can be covered by at most two elements of \mathcal{A} .
 - ⟨2⟩1. LET: $x \in X$
 - ⟨2⟩2. PICK $A \in \mathcal{A}$ with $x \in A$
 - ⟨2⟩3. PICK $y > x$ such that $[x, y] \subseteq A$
 - ⟨2⟩4. PICK $B \in \mathcal{A}$ with $y \in B$
 - ⟨2⟩5. $[x, y]$ is covered by A and B
- ⟨1⟩4. LET: $C = \{y \in X \mid [a, y] \text{ can be covered by finitely many elements of } \mathcal{A}\}$
- ⟨1⟩5. LET: $c = \sup C$
- ⟨1⟩6. $c > a$
 - ⟨2⟩1. PICK $x > a$ such that $[a, x]$ can be covered by at most two elements of \mathcal{A} .

PROOF: From ⟨1⟩3.
 - ⟨2⟩2. $x \in C$
- ⟨1⟩7. $c \in C$
 - ⟨2⟩1. PICK $A \in \mathcal{A}$
 - ⟨2⟩2. PICK $x < c$ such that $(x, c] \subseteq A$
 - ⟨2⟩3. PICK $y > x$ such that $y \in C$
 - ⟨2⟩4. PICK $\mathcal{A}_0 \subseteq^{\text{fin}} \mathcal{A}$ that covers $[a, y]$
 - ⟨2⟩5. $\mathcal{A}_0 \cup \{A\}$ covers $[a, c]$
- ⟨1⟩8. $c = b$
 - ⟨2⟩1. ASSUME: for a contradiction $c < b$
 - ⟨2⟩2. PICK $x > c$ such that $[c, x]$ can be covered by at most two elements of \mathcal{A}

PROOF: From ⟨1⟩3.
 - ⟨2⟩3. $[a, x]$ can be finitely covered by \mathcal{A}

PROOF: From ⟨1⟩7.
 - ⟨2⟩4. Q.E.D.

□

Corollary 9.48.23.1. *Let X be a linearly ordered set with the least upper bound property. Then every closed interval in X is compact.*

Corollary 9.48.23.2. *Every closed interval in \mathbb{R} is compact.*

Theorem 9.48.24 (Extreme Value Theorem). *Any linearly ordered set under the order topology that is compact has a greatest and a least element.*

PROOF:

- ⟨1⟩1. LET: X be a linearly ordered set under the order topology that is compact.
- ⟨1⟩2. X has a greatest element.
 - ⟨2⟩1. ASSUME: for a contradiction X has no greatest element.

- ⟨2⟩2. $\{(-\infty, a) \mid a \in X\}$ covers X .
 - ⟨2⟩3. PICK a finite subcover $\{(-\infty, a_1), \dots, (-\infty, a_n)\}$, say.
 - ⟨2⟩4. ASSUME: without loss of generality $a_1 \leq \dots \leq a_n$
 - ⟨2⟩5. $X \subseteq (-\infty, a_n)$
 - ⟨2⟩6. $a_n < a_n$
 - ⟨1⟩3. X has a least element.
- PROOF: Similar.

□

Proposition 9.48.25. *Every linearly ordered set in which every closed interval is compact satisfies the least upper bound property.*

PROOF:

- ⟨1⟩1. LET: X be a linearly ordered set in which every closed interval is compact.
- ⟨1⟩2. LET: $A \subseteq X$ be nonempty with upper bound u
- ⟨1⟩3. PICK $a \in A$
- ⟨1⟩4. The closed interval $[a, u]$ is compact.
- ⟨1⟩5. ASSUME: for a contradiction A has no supremum.
- ⟨1⟩6. $\{(-\infty, x) \mid x \in A\} \cup \{(x, +\infty) \mid x \text{ is an upper bound of } A\}$ covers $[a, u]$.
- ⟨2⟩1. LET: $x \in [a, u]$
- ⟨2⟩2. ASSUME: for all $y \in A$ we have $x \notin (-\infty, y)$
- ⟨2⟩3. x is an upper bound for A
- ⟨2⟩4. PICK an upper bound y for A with $y < x$
- ⟨2⟩5. $x \in (y, +\infty)$
- ⟨1⟩7. PICK a finite subcover $\{(-\infty, x_1), \dots, (-\infty, x_m), (y_1, +\infty), \dots, (y_n, +\infty)\}$
- ⟨1⟩8. ASSUME: $x_m = \max(x_1, \dots, x_m)$ and $y_1 = \min(y_1, \dots, y_n)$
- ⟨1⟩9. $x_m \notin (-\infty, x_i)$ for any i
- PROOF: Since $x_i \leq x_m$
- ⟨1⟩10. $x_m \notin (y_i, +\infty)$ for any i
- PROOF: Since $x_m \in A$ so $x_m \leq y_i$
- ⟨1⟩11. $x_m \in [a, u]$
- ⟨2⟩1. $a \notin (y_i, +\infty)$ for any i
- PROOF: Since y_i is an upper bound for A and $a \in A$.
- ⟨2⟩2. $a \in (-\infty, x_i)$ for some i
- PROOF: From ⟨1⟩7.
- ⟨2⟩3. $a < x_m$
- PROOF: Since $x_i \leq x_m$
- ⟨2⟩4. $x_m \leq u$
- PROOF: Since u is an upper bound for A and $x_m \in A$.
- ⟨1⟩12. Q.E.D.
- PROOF: This contradicts ⟨1⟩7.

□

Example 9.48.26. The set $[0, 1]$ is not compact under the K -topology.

PROOF: For every $n \geq 1$, pick an open interval U_n such that $U_n \cap K = \{1/n\}$. Then the open cover $\{[0, 1] - K\} \cup \{U_n \mid n \in \mathbb{Z}^+\}$ has no finite subcover. □

Proposition 9.48.27 (AC). *Let X be a compact Hausdorff space. Let \mathcal{A} be a countable set of closed sets in X . If every element of \mathcal{A} has empty interior, then $\bigcup \mathcal{A}$ has empty interior.*

PROOF:

- ⟨1⟩1. LET: X be a compact Hausdorff space.
- ⟨1⟩2. For every closed set A in X and open U in X with $U \not\subseteq A$, there exists a nonempty open set V such that $\overline{V} \subseteq U - A$.
- ⟨2⟩1. LET: A be a closed set in X
- ⟨2⟩2. LET: U be an open set in X with $U \not\subseteq A$
- ⟨2⟩3. PICK $x \in U - A$
- ⟨2⟩4. PICK disjoint neighbourhoods W and V of $A \cup (X - U)$ and x respectively.

PROOF: Proposition 9.48.10.

- ⟨2⟩5. $\overline{V} \subseteq U - A$

PROOF:

$$\begin{aligned} \overline{V} &\subseteq X - W && \text{(since } V \subseteq X - W) \\ &\subseteq X - (A \cup (X - U)) \\ &= (X - A) \cap U \\ &= U - A \end{aligned}$$

- ⟨1⟩3. PICK an enumeration $\{A_1, A_2, \dots\}$ of \mathcal{A}
- ⟨1⟩4. LET: U_0 be any nonempty open set
PROVE: $U_0 \not\subseteq \bigcup \mathcal{A}$
- ⟨1⟩5. PICK a sequence of nonempty open sets U_1, U_2, \dots such that, for $n \geq 1$, we have $\overline{U_n} \subseteq U_{n-1} - A_n$
 - ⟨2⟩1. ASSUME: we have picked U_0, U_1, \dots, U_n
 - ⟨2⟩2. $U_n \not\subseteq A_{n+1}$
PROOF: Since A_{n+1} has empty interior.
 - ⟨2⟩3. PICK a nonempty open set U_{n+1} such that $\overline{U_{n+1}} \subseteq U_n - A_{n+1}$
PROOF: By ⟨1⟩2
- ⟨1⟩6. PICK $a \in \bigcap_{n=0}^{\infty} \overline{U_n}$
PROOF: Corollary 9.48.6.1.
- ⟨1⟩7. $a \in U_0$
PROOF: Since $a \in \overline{U_1} \subseteq U_0$.
- ⟨1⟩8. $a \notin \bigcup \mathcal{A}$
PROOF: For all n , we have $a \in \overline{U_n} \subseteq U_{n-1} - A_n$.

□

Example 9.48.28. The Cantor set is compact.

PROOF: It is a closed subset of the compact set $[0, 1]$. □

Proposition 9.48.29. *Every compact space is limit point compact.*

PROOF:

- ⟨1⟩1. LET: X be a compact space.
- ⟨1⟩2. LET: $A \subseteq X$ have no limit points.

PROVE: A is finite.

⟨1⟩3. A is closed.
PROOF: Corollary 9.8.3.1.

⟨1⟩4. A is compact.
PROOF: Proposition 9.48.3.

⟨1⟩5. $\{U \mid U \text{ open}, |U \cap A| = 1\}$ covers A .
PROOF: From ⟨1⟩2, for all $a \in A$, there is a neighbourhood U of a that intersects A in a only.

⟨1⟩6. PICK a finite subcover $\{U_1, \dots, U_n\}$

⟨1⟩7. For $i = 1, \dots, n$,
LET: $U_i \cap A = \{x_i\}$.

⟨1⟩8. $A = \{x_1, \dots, x_n\}$
□

The following examples show that not every limit point compact space is compact.

Example 9.48.30. Let Y be a set with two elements under the indiscrete topology. Then $\mathbb{Z}^+ \times Y$ is limit point compact, since every nonempty set has a limit point. It is not compact, since $\{\{n\} \times Y \mid n \in \mathbb{Z}^+\}$ has no finite subcover.

Example 9.48.31. The space S_Ω is limit point compact but not compact.

PROOF:

⟨1⟩1. S_Ω is not compact.
PROOF: From the Extreme Value Theorem, since S_Ω has no greatest element.

⟨1⟩2. LET: A be an infinite subset of S_Ω .

⟨1⟩3. PICK $B \subseteq A$ that is countably infinite.
PROOF: Proposition ??.

⟨1⟩4. LET: $b = \sup B$

⟨1⟩5. $B \subseteq [0, b]$

⟨1⟩6. $[0, b]$ is compact.
PROOF: Corollary 9.48.23.1.

⟨1⟩7. PICK a limit point x of B in $[0, b]$.
PROOF: Proposition 9.48.29.

⟨1⟩8. x is a limit point of A .
PROOF: Lemma 9.8.5.
□

9.49 Perfect Maps

Definition 9.49.1 (Perfect Map). Let X and Y be topological spaces and $f : X \rightarrow Y$. Then f is a *perfect map* if and only if f is a closed map, continuous, surjective and $p^{-1}(y)$ is compact for all $y \in Y$.

Proposition 9.49.2. Let X be a topological space, Y a compact space, and $p : X \rightarrow Y$ a closed map such that, for all $y \in Y$, we have $p^{-1}(y)$ is compact. Then X is compact.

PROOF:

- ⟨1⟩1. LET: \mathcal{A} be a set of closed sets in X with the finite intersection property.
- ⟨1⟩2. $\mathcal{B} = \{p(A_1 \cap \cdots \cap A_n) \mid A_1, \dots, A_n \in \mathcal{A}\}$ is a set of closed sets in Y with the finite intersection property.

PROOF: Since p is a closed map.

- ⟨1⟩3. PICK $y \in \bigcap \mathcal{B}$

PROOF: Theorem 9.48.6 since Y is compact.

- ⟨1⟩4. $\{A \cap p^{-1}(y) \mid A \in \mathcal{A}\}$ is a set of closed sets in $p^{-1}(y)$ with the finite intersection property.

- ⟨1⟩5. PICK $x \in \bigcap_{A \in \mathcal{A}} (A \cap p^{-1}(y))$

PROOF: Theorem 9.48.6 since $p^{-1}(y)$ is compact.

- ⟨1⟩6. $x \in \bigcap \mathcal{A}$

- ⟨1⟩7. Q.E.D.

PROOF: Theorem 9.48.6.

□

9.50 Isolated Points

Definition 9.50.1 (Isolated Point). Let X be a topological space and $x \in X$. Then x is an *isolated point* if and only if $\{x\}$ is open.

Theorem 9.50.2 (AC). *A nonempty compact Hausdorff space with no isolated points is uncountable.*

PROOF:

- ⟨1⟩1. LET: X be a nonempty compact Hausdorff space with no isolated points.
- ⟨1⟩2. For every nonempty open set U and every point $x \in X$, there exists a nonempty open set $V \subseteq U$ such that $x \notin \overline{V}$.

- ⟨2⟩1. LET: U be a nonempty open set.

- ⟨2⟩2. LET: $x \in X$

- ⟨2⟩3. PICK $y \in U - \{x\}$

PROOF: This is possible because U cannot be $\{x\}$.

- ⟨2⟩4. PICK disjoint open neighbourhoods W_1 of x and W_2 of y

- ⟨2⟩5. LET: $V = W_2 \cap U$

- ⟨2⟩6. V is nonempty

PROOF: Since $y \in V$

- ⟨2⟩7. V is open

PROOF: From ⟨2⟩1, ⟨2⟩4, ⟨2⟩5.

- ⟨2⟩8. $V \subseteq U$

PROOF: From ⟨2⟩5

- ⟨2⟩9. $x \notin V$

PROOF: From ⟨2⟩4 and ⟨2⟩5

- ⟨1⟩3. LET: (a_n) be any sequence of points in X .

PROVE: The set $X - \{a_1, a_2, \dots\}$ is nonempty.

- ⟨1⟩4. PICK a sequence of nonempty open sets V_1, V_2, \dots , such that $V_1 \supseteq V_2 \supseteq \dots$ and $a_n \notin \overline{V_n}$ for all n .

PROOF: From $\langle 1 \rangle 2$.
 $\langle 1 \rangle 5$. PICK $a \in \bigcap_{n=1}^{\infty} \overline{V_n}$
 PROOF: Corollary 9.48.6.1.
 $\langle 1 \rangle 6$. $a \in X - \{a_1, a_2, \dots\}$
 PROOF: We cannot have $a = a_n$ because $a \in \overline{V_n}$.
 \square

Corollary 9.50.2.1. For all $a, b \in \mathbb{R}$ with $a < b$, the closed interval $[a, b]$ is uncountable.

Example 9.50.3. The Cantor set has no isolated points, and is therefore uncountable.

PROOF:

$\langle 1 \rangle 1$. LET: (A_n) be the sets in Definition 8.1.1.
 $\langle 1 \rangle 2$. LET: $x \in C$
 $\langle 1 \rangle 3$. LET: A_n be the first set such that x is an endpoint of one of the intervals that make up A_n
 $\langle 1 \rangle 4$. LET: $(a_m)_{m \geq n}$ be the sequence of points defined by: a_m is the point such that either $[a_m, x]$ or $[x, a_m]$ is one of the intervals that make up A_m .
 $\langle 1 \rangle 5$. (a_m) is a sequence of points of C distinct from x that converges to x .
 PROOF: Since $|a_m - x| = 1/3^m$ for all m .
 $\langle 1 \rangle 6$. x is a limit point of C .
 \square

9.51 Local Compactness

Definition 9.51.1 (Locally Compact). Let X be a topological space and $x \in X$. Then X is *locally compact* at x if and only if there exists a compact subspace of X that includes a neighbourhood of x .

A space is *locally compact* if and only if it is locally compact at every point.

Example 9.51.2. The real line is locally compact, because for every real number x we have $x \in (x - 1, x + 1) \subseteq [x - 1, x + 1]$.

Example 9.51.3. For all $n \geq 1$, we have \mathbb{R}^n is locally compact. For any point $x = (x_1, \dots, x_n)$, we have $x \in (x_1 - 1, x_1 + 1) \times \dots \times (x_n - 1, x_n + 1) \subseteq [x_1 - 1, x_1 + 1] \times \dots \times [x_n - 1, x_n + 1]$.

The following example shows that a countable product of locally compact spaces is not necessarily locally compact.

Example 9.51.4. The space \mathbb{R}^ω is not locally compact.

PROOF:

$\langle 1 \rangle 1$. ASSUME: for a contradiction $0 \in U \subseteq C$ where U is open and C is compact.

- (1)2. PICK a basic open set $B = (a_1, b_1) \times \cdots \times (a_n, b_n) \times \mathbb{R} \times \mathbb{R} \times \cdots$ such that $0 \in B \subseteq U$
 (1)3. $\overline{B} = [a_1, b_1] \times \cdots \times [a_n, b_n] \times \mathbb{R} \times \mathbb{R} \times \cdots$ is compact.
 PROOF: Proposition 9.48.3.
 (1)4. Q.E.D.
 PROOF: This is a contradiction.
 □

Example 9.51.5. Every linearly ordered set X with the least upper bound property is locally compact under the order topology.

For any point x , pick a basic open set B such that $x \in B$. Then $x \in B \subseteq \overline{B}$ and \overline{B} is a closed interval, hence compact (Corollary 9.48.23.1).

Proposition 9.51.6. *Any closed subspace of a locally compact space is locally compact.*

PROOF:

- (1)1. LET: X be a locally compact space and $Y \subseteq X$ be closed.
 (1)2. LET: $y \in Y$.
 (1)3. PICK a compact subspace C of X and neighbourhood U of y in X such that $U \subseteq C$
 (1)4. $y \in U \cap Y \subseteq C \cap Y$
 (1)5. $C \cap Y$ is compact.
 PROOF: Proposition 9.48.3.
 □

Proposition 9.51.7. *Let X be a Hausdorff space. Let $x \in X$. Then X is locally compact at x if and only if, for every neighbourhood U of x , there exists a neighbourhood V of x such that \overline{V} is compact and $\overline{V} \subseteq U$.*

Corollary 9.51.7.1. *Every open subspace of a locally compact Hausdorff space is locally compact Hausdorff.*

This example shows that a subspace of a locally compact Hausdorff space is not necessarily locally compact Hausdorff.

Example 9.51.8. The rationals \mathbb{Q} are not locally compact.

Assume for a contradiction $C \subseteq \mathbb{Q}$ is compact and includes $(-\epsilon, \epsilon) \cap \mathbb{Q}$. Pick an irrational $\xi \in (-\epsilon, \epsilon)$. Then $\{(-\infty, q) \cap \mathbb{Q} \mid q \in \mathbb{Q}, q < \xi\} \cup \{(q, +\infty) \cap \mathbb{Q} \mid q \in \mathbb{Q}, q > \xi\}$ covers C but no finite subcover does.

Proposition 9.51.9 (AC). *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of nonempty spaces. If $\prod_{\alpha \in J} X_\alpha$ is locally compact then each X_α is locally compact, and X_α is compact for all but finitely many $\alpha \in J$.*

PROOF:

- (1)1. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of nonempty spaces.
 (1)2. ASSUME: $\prod_{\alpha \in J} X_\alpha$ is locally compact.
 (1)3. For all $\alpha \in J$ we have X_α is locally compact.

- (2)1. LET: $\alpha \in J$
- (2)2. LET: $x_\alpha \in X_\alpha$
- (2)3. Extend x_α to a family $(x_\beta)_{\beta \in J} \in \prod_{\beta \in J} X_\beta$
- (2)4. PICK a compact $C \subseteq \prod_{\beta \in J} X_\beta$ that includes a basic open neighbourhood $\prod_{\beta \in J} U_\beta$ of (x_β) such that each U_β is open in X_β .
- (2)5. $x_\alpha \in U_\alpha \subseteq \pi_\alpha(C)$
- (2)6. $\pi_\alpha(C)$ is compact.
- PROOF: Theorem 9.48.4.
- (1)4. X_α is compact for all but finitely many $\alpha \in J$.
- (2)1. PICK $x_\alpha \in X_\alpha$ for all α .
- (2)2. PICK a compact $C \subseteq \prod_{\alpha \in J} X_\alpha$ that includes a basic open neighbourhood $\prod_{\alpha \in J} U_\alpha$ of (x_α) such that each U_α is open in X_α , and $U_\alpha = X_\alpha$ for all but finitely many α .
- (2)3. For all but finitely many $\alpha \in J$, we have $X_\alpha = \pi_\alpha(C)$
- (2)4. For all but finitely many $\alpha \in J$, we have X_α is compact.
- PROOF: Theorem 9.48.4.

□

9.52 Compactifications

Definition 9.52.1 (Compactification). Let X be a topological space. A *compactification* of X consists of a compact Hausdorff space Y and an imbedding $X \rightarrow Y$.

Definition 9.52.2 (One-Point Compactification). Let X be a topological space. A *one-point compactification* of X is a compactification $i : X \rightarrow Y$ such that $Y - i(X)$ consists of a single point.

Theorem 9.52.3. *Let X be a topological space. Then X is locally compact Hausdorff if and only if there exists a one-point compactification $i : X \rightarrow Y$. In this case, Y is unique up to unique homeomorphism that commutes with i .*

PROOF:

- (1)1. For any compact Hausdorff space Y and point $a \in Y$, the space $Y - \{a\}$ is locally compact Hausdorff.
- (2)1. LET: Y be a compact Hausdorff space.
- (2)2. LET: $a \in Y$
- (2)3. $Y - \{a\}$ is closed.
- (2)4. $Y - \{a\}$ is locally compact.
- PROOF: Proposition 9.51.6.
- (2)5. $Y - \{a\}$ is Hausdorff.
- PROOF: Theorem 9.22.6.
- (1)2. For any locally compact Hausdorff space X , there exists a compact Hausdorff space Y and imbedding $i : X \rightarrow Y$ such that $Y - i(X)$ is a single point.
- (2)1. LET: X be a locally compact Hausdorff space.

- (2)2. LET: $Y = X \cup \{\infty\}$
 (2)3. Define a topology on Y by: $U \subseteq Y$ is open if and only if U is an open set in X or $U = Y - C$ where C is a compact subspace of X .
 (3)1. Y is open.
 PROOF: Since $Y = Y - \emptyset$ and \emptyset is a compact subspace of X .
 (3)2. For any set of open sets \mathcal{U} we have $\bigcup \mathcal{U}$ is open.
 PROOF: We have $\bigcup \mathcal{U} = Y - (\bigcap \{C \subseteq X \mid C \text{ is compact, } Y - C \in \mathcal{U}\} - \bigcup \{U \subseteq X \mid U \text{ is open in } X, U \in \mathcal{U}\})$, where we take the empty intersection to be Y .
 (3)3. For any open sets U and V we have $U \cap V$ is open.
 (4)1. LET: U and V be open sets.
 (4)2. CASE: U and V are open sets in X .
 PROOF: In this case $U \cap V$ is open in X .
 (4)3. CASE: C_1 and C_2 are compact subspaces of X and $U = X - C_1$, $V = X - C_2$
 PROOF: In this case $C_1 \cup C_2$ is compact and $U \cap V = X - (C_1 \cup C_2)$.
 (4)4. CASE: U is open in X , C is a compact subspace of X and $V = X - C$
 PROOF: In this case $U \cap V = U - C$ which is open since C is closed.
 (2)4. Y is compact.
 (3)1. LET: \mathcal{A} be an open cover of Y .
 (3)2. PICK C compact in X such that $Y - C \in \mathcal{A}$
 PROOF: There must be at least one such member of \mathcal{A} since $\infty \in \bigcup \mathcal{A}$.
 (3)3. $\{U \cap X \mid U \in \mathcal{A} - \{Y - C\}\}$ is a set of open sets in X that covers C .
 (3)4. PICK a finite subcover $\{U_1 \cap X, \dots, U_n \cap X\}$
 (3)5. $\{U_1 \cap X, \dots, U_n \cap X, Y - C\}$ covers Y .
 (2)5. Y is Hausdorff.
 (3)1. LET: $x, y \in Y$ with $x \neq y$
 (3)2. CASE: $x, y \in X$
 PROOF: There are disjoint open sets U, V in X such that $x \in U, y \in V$.
 (3)3. CASE: $x \in X, y = \infty$
 (4)1. PICK a compact C that includes a neighbourhood U of x
 PROOF: Since X is locally compact.
 (4)2. U and $Y - C$ are disjoint open sets in Y with $x \in U$ and $\infty \in Y - C$
 (2)6. Let $i : X \rightarrow Y$ be the inclusion.
 (2)7. i is an imbedding.
 (3)1. i is continuous
 (3)2. i is an open map.
 (2)8. $Y - i(X) = \{\infty\}$
 (1)3. If X is locally compact Hausdorff, Y and Y' are compact Hausdorff, and $i : X \rightarrow Y, i' : X \rightarrow Y'$ are imbeddings such that $Y - i(X)$ and $Y' - i'(X)$ each have just one point, then there exists a unique homeomorphism $\theta : Y \cong Y'$ such that $\theta \circ i = i'$.
 (2)1. LET: $Y - i(X) = \{a\}$ and $Y' - i'(X) = \{b\}$
 (2)2. LET: $\theta : Y \rightarrow Y'$ be the function with $\theta(a) = b$ and $\theta(i(x)) = i'(x)$
 (2)3. θ is a bijection

$\langle 2 \rangle 4$. θ is continuous.

$\langle 3 \rangle 1$. LET: $U \subseteq Y'$ be open.

PROVE: $\theta^{-1}(U)$ is open.

$\langle 3 \rangle 2$. CASE: $b \in U$

$\langle 4 \rangle 1$. $Y' - U$ is compact

$\langle 4 \rangle 2.$ $i(i'^{-1}(Y' - U))$ is compact.

$\langle 4 \rangle 3$. $i(i'^{-1}(Y' - U))$ is closed.

$$\langle 4 \rangle 4. \quad \theta^{-1}(U) = X - i(i'^{-1}(Y' - U))$$

3. CASE: $b \notin U$

PROOF: $U = i'(V)$ for some V open in X and $\theta^{-1}(U) = i(V)$.

⟨2⟩5. θ is an open map.

PROOF: Similar.

$\langle 2 \rangle 6$. θ is unique.



Example 9.52.4. S^1 is the one-point compactification of \mathbb{R} .

Example 9.52.5. S^2 is the one-point compactification of \mathbb{R}^2 .

Definition 9.52.6 (Riemann Sphere). The *Riemann sphere* or *extended complex plane* is $\mathcal{C} \cup \{\infty\}$ topologized as the one-point compactification of \mathcal{C} . It is homeomorphic to S^2 .

Chapter 10

Topological Groups

Definition 10.0.1 (Topological Group). A *topological group* G consists of a T_1 space G and continuous maps $\cdot : G^2 \rightarrow G$ and $(\)^{-1} : G \rightarrow G$ such that $(G, \cdot, (\)^{-1})$ is a group.

Example 10.0.2. 1. The integers \mathbb{Z} under addition are a topological group.

2. The real numbers \mathbb{R} under addition are a topological group.

3. The positive reals under multiplication are a topological group.

4. The set $\{z \in \mathbb{C} \mid |z| = 1\}$ under multiplication and given the topology of S^1 is a topological group.

5. For any $n \geq 0$, the general linear group $GL_n(\mathbb{R})$ is a topological group under matrix multiplication, considered as a subspace of \mathbb{R}^{n^2} .

Lemma 10.0.3. Let G be a T_1 space and $\cdot : G^2 \rightarrow G$, $(\)^{-1} : G \rightarrow G$ be functions such that $(G, \cdot, (\)^{-1})$ is a group. Then G is a topological group if and only if the function $f : G^2 \rightarrow G$ that maps (x, y) to xy^{-1} is continuous.

PROOF:

$\langle 1 \rangle 1$. If G is a topological group then f is continuous.

PROOF: From Theorem 9.14.9.

$\langle 1 \rangle 2$. If f is continuous then G is a topological group.

$\langle 2 \rangle 1$. ASSUME: f is continuous.

$\langle 2 \rangle 2$. $(\)^{-1}$ is continuous.

PROOF: Since $x^{-1} = f(e, x)$.

$\langle 2 \rangle 3$. \cdot is continuous.

PROOF: Since $xy = f(x, y^{-1})$.

□

Lemma 10.0.4. Let G be a topological group and H a subgroup of G . Then H is a topological group under the subspace topology.

PROOF:

⟨1⟩1. H is T_1 .

PROOF: From Proposition 9.21.5.

⟨1⟩2. multiplication and inverse on H are continuous.

PROOF: From Theorem 9.14.10.

□

Lemma 10.0.5. *Let G be a topological group and H a subgroup of G . Then \overline{H} is a subgroup of G .*

PROOF:

⟨1⟩1. LET: $x, y \in \overline{H}$

PROVE: $xy^{-1} \in \overline{H}$

⟨1⟩2. LET: U be any neighbourhood of xy^{-1}

⟨1⟩3. LET: $f : G^2 \rightarrow G$, $f(a, b) = ab^{-1}$

⟨1⟩4. $f^{-1}(U)$ is a neighbourhood of (x, y)

⟨1⟩5. PICK neighbourhoods V, W of x and y respectively such that $f(V \times W) \subseteq U$.

⟨1⟩6. PICK $a \in V \cap H$ and $b \in W \cap H$

PROOF: Theorem 9.6.6.

⟨1⟩7. $ab^{-1} \in U \cap H$

⟨1⟩8. Q.E.D.

PROOF: By Theorem 9.6.6.

□

Proposition 10.0.6. *Let G be a topological group and $\alpha \in G$. Then the maps $l_\alpha, r_\alpha : G \rightarrow G$ defined by $l_\alpha(x) = \alpha x$, $r_\alpha(x) = x\alpha$ are homeomorphisms of G with itself.*

PROOF: They are continuous with continuous inverses $l_{\alpha^{-1}}$ and $r_{\alpha^{-1}}$. □

Corollary 10.0.6.1. *Every topological group is homogeneous.*

PROOF: Given a topological group G and $a, b \in G$, we have $l_{ba^{-1}}$ is a homeomorphism that maps a to b . □

Proposition 10.0.7. *Let G be a topological group and H a subgroup of G . Give G/H the quotient topology. For all $\alpha \in G$, the map $\overline{f_\alpha}$ that sends xH to αxH is a homeomorphism $G/H \cong G/H$.*

PROOF:

⟨1⟩1. $\overline{f_\alpha}$ is well-defined.

PROOF: If $xy^{-1} \in H$ then $(\alpha x)(\alpha y)^{-1} = xy^{-1} \in H$.

⟨1⟩2. $\overline{f_\alpha}$ is continuous.

PROOF: Theorem 9.26.7 since $\overline{f_\alpha} \circ p = p \circ f_\alpha$ is continuous, where $p : G \rightarrow G/H$ is the canonical surjection.

⟨1⟩3. $\overline{f_\alpha}^{-1}$ is continuous.

PROOF: Similar since $\overline{f_\alpha}^{-1} = \overline{f_{\alpha^{-1}}}$.

□

Corollary 10.0.7.1. *Let G be a topological group and H a subgroup of G . Give G/H the quotient topology. Then G/H is homogeneous.*

Proposition 10.0.8. *Let G be a topological group and H a subgroup of G . Give G/H the quotient topology. If H is closed in G then G/H is T_1 .*

PROOF:

⟨1⟩1. LET: $p : G \rightarrow G/H$ be the canonical surjection

⟨1⟩2. LET: $x \in G$

⟨1⟩3. $p^{-1}(xH) = f_x(H)$

⟨1⟩4. $p^{-1}(xH)$ is closed in G

PROOF: Since H is closed and f_x is a homomorphism of G with itself.

⟨1⟩5. $\{xH\}$ is closed in G/H

□

Proposition 10.0.9. *Let G be a topological group and H a subgroup of G . Give G/H the quotient topology. Then the canonical surjection $p : G \rightarrow G/H$ is an open map.*

PROOF:

⟨1⟩1. LET: $U \subseteq G$ be open.

⟨1⟩2. $p^{-1}(p(U)) = \bigcup_{h \in H} r_h(U)$

⟨1⟩3. $p^{-1}(p(U))$ is open.

⟨1⟩4. $p(U)$ is open.

□

Proposition 10.0.10. *Let G be a topological group and H a closed normal subgroup of G . Then G/H is a topological group under the quotient topology.*

PROOF:

⟨1⟩1. G/H is T_1

PROOF: Proposition 10.0.8.

⟨1⟩2. The map $\bar{m} : (xH, yH) \mapsto xy^{-1}H$ is continuous.

⟨2⟩1. $p^2 : G^2 \rightarrow (G/H)^2$ is a quotient map.

PROOF: Propositions 9.26.6, 10.0.9.

⟨2⟩2. $\bar{m} \circ p^2$ is continuous.

PROOF: As it is $p^2 \circ m$ where $m : G^2 \rightarrow G$ with $m(x, y) = xy^{-1}$

□

Lemma 10.0.11. *Let G be a topological group and $A, B \subseteq G$. If either A or B is open then AB is open.*

PROOF: If A is open we have $AB = \bigcup_{b \in B} r_b(A)$. Similarly if B is open. □

Definition 10.0.12 (Symmetric Neighbourhood). Let G be a topological group. A neighbourhood V of e is *symmetric* if and only if $V = V^{-1}$.

Lemma 10.0.13. *Let G be a topological group. Let V be a neighbourhood of e . Then V is symmetric if and only if, for all $x \in V$, we have $x^{-1} \in V$.*

PROOF:

⟨1⟩1. If V is symmetric then, for all $x \in V$, we have $x^{-1} \in V$

PROOF: Immediate from definitions.

⟨1⟩2. If, for all $x \in V$, we have $x^{-1} \in V$, then V is symmetric.

⟨2⟩1. ASSUME: for all $x \in V$ we have $x^{-1} \in V$

⟨2⟩2. $V \subseteq V^{-1}$

PROOF: If $x \in V$ then there exists $y \in V$ such that $x = y^{-1}$, namely $y = x^{-1}$

⟨2⟩3. $V^{-1} \subseteq V$

PROOF: Immediate from ⟨2⟩1.

□

Lemma 10.0.14. *Let G be a topological group. For every neighbourhood U of e , there exists a symmetric neighbourhood V of e such that $V^2 \subseteq U$.*

PROOF:

⟨1⟩1. LET: U be a neighbourhood of e .

⟨1⟩2. PICK a neighbourhood V' of e such that $V'V' \subseteq U$

PROOF: Such a neighbourhood exists because multiplication in G is continuous.

⟨1⟩3. PICK a neighbourhood W of e such that $WW^{-1} \subseteq V'$

PROOF: Such a neighbourhood exists because the function that maps (x, y) to xy^{-1} is continuous.

⟨1⟩4. LET: $V = WW^{-1}$

⟨1⟩5. V is a neighbourhood of e

⟨2⟩1. $e \in V$

PROOF: Since $e \in W$ so $e = ee^{-1} \in V$.

⟨2⟩2. V is open

PROOF: Lemma 10.0.11.

⟨1⟩6. V is symmetric

⟨2⟩1. For all $x \in V$ we have $x^{-1} \in V$

⟨3⟩1. LET: $x \in V$

⟨3⟩2. PICK $y, z \in W$ such that $x = yz^{-1}$

⟨3⟩3. $x^{-1} = zy^{-1}$

⟨3⟩4. $x^{-1} \in V$

⟨3⟩5. $x \in V^{-1}$

⟨2⟩2. Q.E.D.

PROOF: Lemma 10.0.13

⟨1⟩7. $V^2 \subseteq U$

PROOF: We have $V^2 \subseteq (V')^2 \subseteq U$

□

Proposition 10.0.15. *Every topological group is Hausdorff.*

PROOF:

⟨1⟩1. LET: G be a topological group.

⟨1⟩2. LET: $x, y \in G$ with $x \neq y$

- ⟨1⟩3. LET: $U = G \setminus \{x^{-1}y\}$
- ⟨1⟩4. PICK a symmetric neighbourhood V of e such that $VV \subseteq U$
 - ⟨2⟩1. U is open
PROOF: Since G is T_1 .
 - ⟨2⟩2. $e \in U$
PROOF: Since $x \neq y$
 - ⟨2⟩3. Q.E.D.
PROOF: Lemma 10.0.14.
- ⟨1⟩5. Vx and Vy are disjoint neighbourhoods of x and y respectively.
 - ⟨2⟩1. Vx is open
PROOF: Since $Vx = r_x(V)$
 - ⟨2⟩2. Vy is open
PROOF: Similar.
 - ⟨2⟩3. $Vx \cap Vy = \emptyset$
 - ⟨3⟩1. ASSUME: for a contradiction $z \in Vx \cap Vy$
 - ⟨3⟩2. PICK $a, b \in V$ such that $z = ax = by$
 - ⟨3⟩3. $xy^{-1} \in VV$
PROOF: Since $xy^{-1} = a^{-1}b$
 - ⟨3⟩4. $xy^{-1} \in U$
 - ⟨3⟩5. Q.E.D.
PROOF: From ⟨1⟩3.

□

Proposition 10.0.16. *Every topological group is regular.*

PROOF:

- ⟨1⟩1. LET: G be a topological group.
- ⟨1⟩2. LET: $A \subseteq G$ be a closed set and $a \notin A$.
- ⟨1⟩3. LET: $U = G \setminus Aa^{-1}$
- ⟨1⟩4. PICK a symmetric neighbourhood V of e such that $VV \subseteq U$
 - ⟨2⟩1. U is open
PROOF: Since $Aa^{-1} = r_{a^{-1}}(A)$ is closed.
 - ⟨2⟩2. $e \in U$
PROOF: Since $a \notin A$.
 - ⟨2⟩3. Q.E.D.
PROOF: Lemma 10.0.14.
- ⟨1⟩5. VA and Va are disjoint open sets with $A \subseteq VA$ and $a \in Va$
 - ⟨2⟩1. VA is open
PROOF: Lemma 10.0.11
 - ⟨2⟩2. Va is open
PROOF: Lemma 10.0.11
 - ⟨2⟩3. $VA \cap Va = \emptyset$
 - ⟨3⟩1. ASSUME: for a contradiction $z \in VA \cap Va$
 - ⟨3⟩2. PICK $b, c \in V$ and $d \in A$ with $z = bd = ca$
 - ⟨3⟩3. $da^{-1} \in U$
PROOF: Since $da^{-1} = b^{-1}c \in VV \subseteq U$
 - ⟨3⟩4. Q.E.D.

PROOF: This contradicts $\langle 1 \rangle 3$

□

Proposition 10.0.17. *Let G be a topological group and H a subgroup of G . Give G/H the quotient topology. If H is closed in G then G/H is regular.*

PROOF:

- $\langle 1 \rangle 1$. LET: $p : G \rightarrow G/H$ be the canonical surjection.
- $\langle 1 \rangle 2$. LET: A be a closed set in G/H and $aH \in (G/H) \setminus A$.
- $\langle 1 \rangle 3$. LET: $B = p^{-1}(A)$
- $\langle 1 \rangle 4$. B is a closed saturated set in G .
- $\langle 1 \rangle 5$. $B \cap aH = \emptyset$
- $\langle 1 \rangle 6$. $B = BH$
- $\langle 1 \rangle 7$. PICK a symmetric neighbourhood V of e such that VB does not intersect Va
 - $\langle 2 \rangle 1$. LET: $U = G \setminus Ba^{-1}$
 - $\langle 2 \rangle 2$. PICK a symmetric neighbourhood V of e such that $VV \subseteq U$
 - $\langle 3 \rangle 1$. U is open

PROOF: Since $Ba^{-1} = r_{a^{-1}}(B)$ is closed.
 - $\langle 3 \rangle 2$. $e \in U$

PROOF: If $e \in Ba^{-1}$ then $a \in B$
 - $\langle 3 \rangle 3$. Q.E.D.

PROOF: Lemma 10.0.14
 - $\langle 2 \rangle 3$. $VB \cap Va = \emptyset$

PROOF: If $vb = v'a$ for $v, v' \in V$ and $b \in B$ then we have $ba^{-1} = v^{-1}v' \in Ba \cap VV \subseteq Ba \cap U$.
- $\langle 1 \rangle 8$. $p(VB)$ and $p(Va)$ are disjoint open sets
 - $\langle 2 \rangle 1$. $p(VB)$ and $p(Va)$ are open.

PROOF: Proposition 10.0.9.
 - $\langle 2 \rangle 2$. $p(VB) \cap p(Va) = \emptyset$

PROOF: If $vbH = v'aH$ for $v, v' \in V$, $b \in B$ then $v'a = vbh$ for some $h \in H$. Hence $v'a \in Va \cap VBH = Va \cap VB$.
- $\langle 1 \rangle 9$. $A \subseteq p(VB)$
- $\langle 1 \rangle 10$. $aH \in p(Va)$

□

Proposition 10.0.18. *Let G be a topological group. The component of G that contains e is a normal subgroup of G .*

PROOF:

- $\langle 1 \rangle 1$. LET: C be the component of G that contains e .
- $\langle 1 \rangle 2$. For all $x \in G$, xC is the component of G that contains x .
 - $\langle 2 \rangle 1$. LET: $x \in G$
 - $\langle 2 \rangle 2$. LET: D be the component of G that contains x .
 - $\langle 2 \rangle 3$. $xC \subseteq D$

PROOF: Since xC is connected by Theorem 9.31.13.
 - $\langle 2 \rangle 4$. $D \subseteq xC$

PROOF: Since $x^{-1}D \subseteq C$ similarly.

$\langle 1 \rangle 3$. For all $x \in G$, Cx is the component of G that contains x .

PROOF: Similar.

$\langle 1 \rangle 4$. For all $x \in C$ we have $xC = Cx = C$

$\langle 1 \rangle 5$. For all $x \in C$ we have $x^{-1}C = C$

$\langle 1 \rangle 6$. For all $x \in C$ we have $x^{-1} \in C$

$\langle 1 \rangle 7$. For all $x, y \in C$ we have $xy \in C$

PROOF: Since $xyC = xC = x$.

$\langle 1 \rangle 8$. For all $x \in G$ we have $xC = Cx$.

PROOF: From $\langle 1 \rangle 2$ and $\langle 1 \rangle 3$.

□

Lemma 10.0.19. *Let G be a topological group. Let A be a closed set in G and B a compact subspace of G such that $A \cap B = \emptyset$. Then there exists a symmetric neighbourhood U of e such that $AU \cap BU = \emptyset$.*

PROOF:

$\langle 1 \rangle 1$. For all $b \in B$ there exists a symmetric neighbourhood V of e such that $bV^2 \cap A = \emptyset$

$\langle 2 \rangle 1$. LET: $b \in B$

$\langle 2 \rangle 2$. LET: $W = b^{-1}(G \setminus A)$

$\langle 2 \rangle 3$. W is a neighbourhood of e and $bW \cap A = \emptyset$

$\langle 2 \rangle 4$. PICK a symmetric neighbourhood V of e such that $V^2 \subseteq W$

$\langle 1 \rangle 2$. $\{bV^2 \mid b \in B, V \text{ is a symmetric neighbourhood of } e, bV^2 \cap A = \emptyset\}$ is an open cover of B

$\langle 1 \rangle 3$. PICK a finite subcover $b_1V_1^2, \dots, b_nV_n^2$, say.

$\langle 1 \rangle 4$. LET: $U = V_1 \cap \dots \cap V_n$

$\langle 1 \rangle 5$. $BU^2 \cap A = \emptyset$

$\langle 1 \rangle 6$. $AU \cap BU = \emptyset$

PROOF: If $av \in BU$ where $a \in A$ and $v \in V$ then $a = avv^{-1} \in BU^2 \cap A$.

□

Proposition 10.0.20 (AC). *Let G be a topological group. Let A be a closed set in G , and B a compact subspace of G . Then AB is closed.*

PROOF:

$\langle 1 \rangle 1$. LET: $x \in G \setminus AB$

$\langle 1 \rangle 2$. $A^{-1}x \cap B = \emptyset$

$\langle 1 \rangle 3$. $A^{-1}x$ is closed.

$\langle 1 \rangle 4$. PICK a symmetric neighbourhood U of e such that $A^{-1}xU \cap BU = \emptyset$

$\langle 1 \rangle 5$. xU^2 is open

PROOF: Lemma 10.0.11.

$\langle 1 \rangle 6$. $x \in xU^2 \subseteq G \setminus AB$

□

Corollary 10.0.20.1. *Let G be a topological group and $H \leq G$. Let $p : G \twoheadrightarrow G/H$ be the quotient map. If H is compact then p is a closed map.*

PROOF: For A closed in G , we have $p^{-1}(p(A)) = AH$ is closed, and so $p(A)$ is closed. \square

Corollary 10.0.20.2. *Let G be a topological group and $H \leq G$. If H and G/H are compact then G is compact.*

PROOF: From Proposition 9.49.2 since, for all $aH \in G/H$, we have $p^{-1}(aH) = aH$ is compact because it is homomorphic to H . \square

10.1 The Metric Topology

Definition 10.1.1 (Metric). Let X be a set. A *metric* on X is a function $d : X^2 \rightarrow \mathbb{R}$ such that:

1. For all $x, y \in X$, $d(x, y) \geq 0$
2. For all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$
3. For all $x, y \in X$, $d(x, y) = d(y, x)$
4. (*Triangle Inequality*) For all $x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$

We call $d(x, y)$ the *distance* between x and y .

Definition 10.1.2 (Open Ball). Let X be a metric space. Let $a \in X$ and $\epsilon > 0$. The *open ball* with *centre* a and *radius* ϵ is

$$B(a, \epsilon) = \{x \in X \mid d(a, x) < \epsilon\} .$$

Definition 10.1.3 (Metric Topology). Let X be a metric space. The *metric topology* on X is the topology generated by the basis consisting of all the open balls.

We prove this is a basis for a topology.

PROOF:

$\langle 1 \rangle 1$. For every point a , there exists a ball B such that $a \in B$

PROOF: We have $a \in B(a, 1)$.

$\langle 1 \rangle 2$. For any balls B_1, B_2 and point $a \in B_1 \cap B_2$, there exists a ball B_3 such that $a \in B_3 \subseteq B_1 \cap B_2$

$\langle 2 \rangle 1$. LET: $B_1 = B(c_1, \epsilon_1)$ and $B_2 = B(c_2, \epsilon_2)$

$\langle 2 \rangle 2$. LET: $\delta = \min(\epsilon_1 - d(c_1, a), \epsilon_2 - d(c_2, a))$

PROVE: $B(a, \delta) \subseteq B_1 \cap B_2$

$\langle 2 \rangle 3$. LET: $x \in B(a, \delta)$

$\langle 2 \rangle 4$. $x \in B_1$

PROOF:

$$\begin{aligned} d(x, c_1) &= d(x, a) + d(a, c_1) \\ &< \delta + d(a, c_1) \\ &\leq \epsilon_1 \end{aligned}$$

$\langle 2 \rangle 5$. $x \in B_2$

PROOF: Similar.

□

Proposition 10.1.4. Let X be a metric space and $U \subseteq X$. Then U is open if and only if, for every $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$.

PROOF:

$\langle 1 \rangle 1$. If U is open then, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$.

$\langle 2 \rangle 1$. ASSUME: U is open.

- (2)2. LET: $x \in U$
 (2)3. PICK $a \in X$ and $\delta > 0$ such that $x \in B(a, \delta) \subseteq U$
 (2)4. LET: $\epsilon = \delta - d(a, x)$
 PROVE: $B(x, \epsilon) \subseteq U$
 (2)5. LET: $y \in B(x, \epsilon)$
 (2)6. $d(y, a) < \delta$
 PROOF:

$$\begin{aligned}
 d(y, a) &\leq d(a, x) + d(x, y) \\
 &< \delta + d(x, y) \\
 &= \epsilon
 \end{aligned}$$

- (2)7. $y \in U$
 (1)2. If, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$, then U is open.
 PROOF: Immediate from definitions.

□

Definition 10.1.5 (Discrete Metric). Let X be a set. The *discrete metric* on X is defined by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Proposition 10.1.6. The discrete metric induces the discrete topology.

PROOF: For any (open) set U and point $a \in U$, we have $a \in B(a, 1) \subseteq U$. □

Definition 10.1.7 (Standard Metric on \mathbb{R}). The *standard metric* on \mathbb{R} is defined by $d(x, y) = |x - y|$.

Proposition 10.1.8. The standard metric on \mathbb{R} induces the standard topology on \mathbb{R} .

PROOF:

- (1)1. Every open ball is open in the standard topology on \mathbb{R} .
 PROOF: $B(a, \epsilon) = (a - \epsilon, a + \epsilon)$
 (1)2. For every open set U and point $a \in U$, there exists $\epsilon > 0$ such that
 $B(a, \epsilon) \subseteq U$
 (2)1. LET: U be an open set and $a \in U$
 (2)2. PICK an open interval b, c such that $a \in (b, c) \subseteq U$
 (2)3. LET: $\epsilon = \min(a - b, c - a)$
 (2)4. $B(a, \epsilon) \subseteq U$

□

Definition 10.1.9 (Metrizable). A topological space X is *metrizable* if and only if there exists a metric on X that induces the topology.

Definition 10.1.10 (Bounded). Let X be a metric space and $A \subseteq X$. Then A is *bounded* if and only if there exists M such that, for all $x, y \in A$, we have $d(x, y) \leq M$.

Definition 10.1.11 (Diameter). Let X be a metric space and $A \subseteq X$. The *diameter* of A is

$$\text{diam } A = \sup_{x,y \in A} d(x,y) .$$

Definition 10.1.12 (Standard Bounded Metric). Let d be a metric on X . The *standard bounded metric* corresponding to d is the metric \bar{d} defined by

$$\bar{d}(x,y) = \min(d(x,y), 1) .$$

We prove this is a metric.

PROOF:

$\langle 1 \rangle 1. \bar{d}(x,y) \geq 0$

PROOF: Since $d(x,y) \geq 0$

$\langle 1 \rangle 2. \bar{d}(x,y) = 0$ if and only if $x = y$

PROOF: $\bar{d}(x,y) = 0$ if and only if $d(x,y) = 0$ if and only if $x = y$

$\langle 1 \rangle 3. \bar{d}(x,y) = \bar{d}(y,x)$

PROOF: Since $d(x,y) = d(y,x)$

$\langle 1 \rangle 4. \bar{d}(x,z) \leq \bar{d}(x,y) + \bar{d}(y,z)$

PROOF:

$$\begin{aligned} \bar{d}(x,y) + \bar{d}(y,z) &= \min(d(x,y), 1) + \min(d(y,z), 1) \\ &= \min(d(x,y) + d(y,z), d(x,y) + 1, d(y,z) + 1, 2) \\ &\geq \min(d(x,z), 1) \\ &= \bar{d}(x,z) \end{aligned}$$

□

Lemma 10.1.13. In any metric space X , the set $\mathcal{B} = \{B(a, \epsilon) \mid a \in X, \epsilon < 1\}$ is a basis for the metric topology.

PROOF:

$\langle 1 \rangle 1.$ Every element of \mathcal{B} is open.

PROOF: From Lemma 9.9.2.

$\langle 1 \rangle 2.$ For every open set U and point $a \in U$, there exists $B \in \mathcal{B}$ such that $a \in B \subseteq U$

$\langle 2 \rangle 1.$ LET: U be an open set and $a \in U$

$\langle 2 \rangle 2.$ PICK $\epsilon > 0$ such that $B(a, \epsilon) \subseteq U$

$\langle 2 \rangle 3.$ $B(a, \min(\epsilon, 1/2)) \subseteq U$

$\langle 1 \rangle 3.$ Q.E.D.

PROOF: Lemma 9.9.3.

□

Proposition 10.1.14. Let d be a metric on the set X . Then the standard bounded metric \bar{d} induces the same metric as d .

PROOF: This follows from Lemma 10.1.13 since the open balls with radius < 1 are the same under both metrics. □

Lemma 10.1.15. *Let d and d' be two metrics on the same set X . Let \mathcal{T} and \mathcal{T}' be the topologies they induce. Then $\mathcal{T} \subseteq \mathcal{T}'$ if and only if, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that*

$$B_{d'}(x, \delta) \subseteq B_d(x, \epsilon) .$$

PROOF:

$\langle 1 \rangle 1$. If $\mathcal{T} \subseteq \mathcal{T}'$ then, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$

PROOF: From Proposition 10.1.4 since $x \in B_d(x, \epsilon) \in \mathcal{T}'$.

$\langle 1 \rangle 2$. If, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$, then $\mathcal{T} \subseteq \mathcal{T}'$

$\langle 2 \rangle 1$. ASSUME: For all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$

$\langle 2 \rangle 2$. LET: $U \in \mathcal{T}$

$\langle 2 \rangle 3$. For all $x \in U$ there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq U$.

$\langle 3 \rangle 1$. LET: $x \in U$

$\langle 3 \rangle 2$. PICK $\epsilon > 0$ such that $B_d(x, \epsilon) \subseteq U$

PROOF: Proposition 10.1.4

$\langle 3 \rangle 3$. PICK $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$

PROOF: By $\langle 2 \rangle 1$

$\langle 3 \rangle 4$. $B_{d'}(x, \delta) \subseteq U$

$\langle 2 \rangle 4$. $U \in \mathcal{T}'$

PROOF: Proposition 10.1.4.

□

Proposition 10.1.16. \mathbb{R}^2 under the dictionary order topology is metrizable.

PROOF: Define $d : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$d((x, y), (x, z)) = \max(|y - z|, 1)$$

$$d((x, y), (x', y')) = 1 \quad \text{if } x \neq x' \quad \square$$

$\langle 1 \rangle 1$. $x \in \bigcap_{i=1}^N \pi_i^{-1}() \subseteq B_D(a, \epsilon)$

Proposition 10.1.17. *Let $d : X^2 \rightarrow \mathbb{R}$ be a metric on X . Then the metric topology on X is the coarsest topology such that d is continuous.*

PROOF:

$\langle 1 \rangle 1$. d is continuous.

$\langle 2 \rangle 1$. LET: $a, b \in X$

$\langle 2 \rangle 2$. LET: $\epsilon > 0$

$\langle 2 \rangle 3$. LET: $\delta = \epsilon/2$

$\langle 2 \rangle 4$. LET: $x, y \in X$

$\langle 2 \rangle 5$. ASSUME: $\rho((a, b), (x, y)) < \delta$

$\langle 2 \rangle 6$. $|d(a, b) - d(x, y)| < \epsilon$

$\langle 3 \rangle 1$. $d(a, b) - d(x, y) < \epsilon$

PROOF:

$$\begin{aligned}
 d(a, b) &\leq d(a, x) + d(x, y) + d(y, b) \\
 &\leq d(x, y) + 2\rho((a, b), (x, y)) \\
 &< d(x, y) + 2\delta \\
 &= d(x, y) + \epsilon
 \end{aligned}$$

$\langle 3 \rangle 2. d(a, b) - d(x, y) > -\epsilon$

PROOF: Similar.

$\langle 2 \rangle 7. \text{ Q.E.D.}$

$\langle 1 \rangle 2. \text{ If } \mathcal{T} \text{ is any topology under which } d \text{ is continuous then } \mathcal{T} \text{ is finer than the metric topology.}$

PROOF: Since $B(a, \epsilon) = d_a^{-1}((-\infty, \epsilon))$

□

Proposition 10.1.18. *Let X be a metric space with metric d and $A \subseteq X$. The restriction of d to A is a metric on A that induces the subspace topology.*

PROOF:

$\langle 1 \rangle 1. \text{ The restriction of } d \text{ to } A \text{ is a metric on } A.$

$\langle 1 \rangle 2. \text{ Every open ball under } d \upharpoonright A \text{ is open under the subspace topology.}$

PROOF: $B_{d \upharpoonright A}(a, \epsilon) = B_d(a, \epsilon) \cap A.$

$\langle 1 \rangle 3. \text{ If } U \text{ is open in the subspace topology and } x \in U, \text{ then there exists a } d \upharpoonright A\text{-ball } B \text{ such that } x \in B \subseteq U.$

$\langle 2 \rangle 1. \text{ PICK } V \text{ open in } X \text{ such that } U = V \cap A$

$\langle 2 \rangle 2. \text{ PICK } \epsilon > 0 \text{ such that } B_d(x, \epsilon) \subseteq V$

$\langle 2 \rangle 3. \text{ Take } B = B_{d \upharpoonright A}(x, \epsilon)$

□

Corollary 10.1.18.1. *A subspace of a metrizable space is metrizable.*

Proposition 10.1.19. *Every metrizable space is Hausdorff.*

PROOF:

$\langle 1 \rangle 1. \text{ LET: } X \text{ be a metric space}$

$\langle 1 \rangle 2. \text{ LET: } a, b \in X \text{ with } a \neq b$

$\langle 1 \rangle 3. \text{ LET: } \epsilon = d(a, b)/2$

$\langle 1 \rangle 4. \text{ LET: } U = B(a, \epsilon) \text{ and } V = B(b, \epsilon)$

$\langle 1 \rangle 5. U \text{ and } V \text{ are disjoint neighbourhoods of } a \text{ and } b \text{ respectively.}$

□

Corollary 10.1.19.1. *Every metrizable space is T_1 .*

Proposition 10.1.20 (CC). *The product of a countable family of metrizable spaces is metrizable.*

PROOF:

$\langle 1 \rangle 1. \text{ LET: } (X_n, d_n) \text{ be a sequence of metric spaces.}$

$\langle 1 \rangle 2. \text{ ASSUME: w.l.o.g. each } d_n \text{ is bounded above by 1.}$

PROOF: By Proposition 10.1.14.

(1)3. LET: D be the metric on \mathbb{R}^ω defined by $D(x, y) = \sup_i (d_i(x_i, y_i)/i)$.

(2)1. $D(x, y) \geq 0$

(2)2. $D(x, y) = 0$ if and only if $x = y$

(2)3. $D(x, y) = D(y, x)$

(2)4. $D(x, z) \leq D(x, y) + D(y, z)$

PROOF:

$$\begin{aligned} D(x, z) &= \sup_i \frac{d_i(x_i, z_i)}{i} \\ &\leq \sup_i \frac{d_i(x_i, y_i) + d_i(y_i, z_i)}{i} \\ &\leq \sup_i \frac{d_i(x_i, y_i)}{i} + \sup_i \frac{d_i(y_i, z_i)}{i} \\ &= D(x, y) + D(y, z) \end{aligned}$$

(1)4. Every open ball $B_D(a, \epsilon)$ is open in the product topology.

(2)1. PICK N such that $1/\epsilon < N$

(2)2. $B_D(a, \epsilon) = \prod_{i=1}^\infty U_i$ where $U_i = B(a_i, i\epsilon)$ if $i \leq N$, and $U_i = X_i$ if $i > N$

(1)5. For any open set U and $a \in U$, there exists $\epsilon > 0$ such that $B_D(a, \epsilon) \subseteq U$.

(2)1. LET: $n \geq 1$, V be an open set in \mathbb{R} and $a \in \pi_n^{-1}(V)$

(2)2. PICK $\epsilon > 0$ such that $B_{d_n}(a, \epsilon) \subseteq V$

(2)3. $B_D(a, \epsilon/n) \subseteq \pi_n^{-1}(V)$

□

Theorem 10.1.21. *Let X and Y be metric spaces and $f : X \rightarrow Y$. Then f is continuous if and only if, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$.*

PROOF:

(1)1. If f is continuous then, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$

(2)1. ASSUME: f is continuous.

(2)2. LET: $x \in X$ and $\epsilon > 0$

(2)3. PICK a neighbourhood U of x such that $f(U) \subseteq B(f(x), \epsilon)$

PROOF: Theorem 9.14.6.

(2)4. PICK $\delta > 0$ such that $B(x, \delta) \subseteq U$

PROOF: Proposition 10.1.4.

(2)5. For all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$

(1)2. If for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$, then f is continuous.

(2)1. ASSUME: for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$

(2)2. LET: $x \in X$ and V be a neighbourhood of $f(x)$

(2)3. PICK $\epsilon > 0$ such that $B(f(x), \epsilon) \subseteq V$

PROOF: Proposition 10.1.4.

(2)4. PICK $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$

PROOF: By (2)1

(2)5. LET: $U = B(x, \delta)$

⟨2⟩6. U is a neighbourhood of x with $f(U) \subseteq V$

⟨2⟩7. Q.E.D.

PROOF: Theorem 9.14.6.

□

Proposition 10.1.22. *Let X be a metric space. Let (a_n) be a sequence in X and $l \in X$. Then $a_n \rightarrow l$ as $n \rightarrow \infty$ if and only if, for all $\epsilon > 0$, there exists N such that, for all $n \geq N$, we have $d(a_n, l) < \epsilon$.*

PROOF: From Proposition 9.11.3. □

Proposition 10.1.23. *Every metrizable space is first countable.*

PROOF: In any metric space X , the open balls $B(a, 1/n)$ for $n \geq 1$ form a local basis at a .

Example 10.1.24. \mathbb{R}^ω under the box topology is not metrizable.

Example 10.1.25. If J is uncountable then \mathbb{R}^J under the product topology is not metrizable.

Example 10.1.26. The space $\overline{S_\Omega}$ is not metrizable by Example 9.23.4.

Proposition 10.1.27. *A compact subspace of a metric space is bounded.*

PROOF:

⟨1⟩1. LET: X be a metric space and $A \subseteq X$ be compact.

⟨1⟩2. PICK $a \in A$

⟨1⟩3. $\{B(a, n) \mid n \in \mathbb{Z}^+\}$ covers A

⟨1⟩4. PICK a finite subcover $\{B(a, n_1), \dots, B(a, n_k)\}$

⟨1⟩5. LET: $N = \max(n_1, \dots, n_k)$

⟨1⟩6. For all $x, y \in A$ we have $d(x, y) < 2N$

PROOF:

$$\begin{aligned} d(x, y) &\leq d(x, a) + d(a, y) \\ &< N + N \end{aligned}$$

□

This example shows the converse does not hold:

Example 10.1.28. The space \mathbb{R} under the standard bounded metric is bounded but not compact.

Proposition 10.1.29. *A connected metric space with more than one point is uncountable.*

PROOF:

⟨1⟩1. LET: X be a connected metric space with more than one point.

⟨1⟩2. PICK $a \in X$

⟨1⟩3. $d(a, -) : X \rightarrow \mathbb{R}$ is continuous.

PROOF: Proposition 10.1.17.

⟨1⟩4. $\{d(a, x) \mid x \in X\}$ is a connected subspace of \mathbb{R} that includes 0.

PROOF: Theorem 9.31.13.

⟨1⟩5. $\{d(a, x) \mid x \in X\} \neq \{0\}$

PROOF: Since X has more than one point.

⟨1⟩6. $\{d(a, x) \mid x \in X\}$ is uncountable.

PROOF: Since it includes a closed interval (Corollary 9.50.2.1).

□

10.2 Real Linear Algebra

Definition 10.2.1 (Square Metric). The *square metric* ρ on \mathbb{R}^n is defined by

$$\rho(\vec{x}, \vec{y}) = \max(|x_1 - y_1|, \dots, |x_n - y_n|)$$

We prove this is a metric.

PROOF:

⟨1⟩1. $\rho(\vec{x}, \vec{y}) \geq 0$

PROOF: Immediate from definition.

⟨1⟩2. $\rho(\vec{x}, \vec{y}) = 0$ if and only if $\vec{x} = \vec{y}$

PROOF: Immediate from definition.

⟨1⟩3. $\rho(\vec{x}, \vec{y}) = \rho(\vec{y}, \vec{x})$

PROOF: Immediate from definition.

⟨1⟩4. $\rho(\vec{x}, \vec{z}) \leq \rho(\vec{x}, \vec{y}) + \rho(\vec{y}, \vec{z})$

PROOF: Since $|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$.

□

Proposition 10.2.2. The square metric induces the standard topology on \mathbb{R}^n .

PROOF:

⟨1⟩1. For every $a \in X$ and $\epsilon > 0$, we have $B_\rho(a, \epsilon)$ is open in the standard product topology.

PROOF:

$$B_\rho(a, \epsilon) = (a_1 - \epsilon, a_1 + \epsilon) \times \cdots \times (a_n - \epsilon, a_n + \epsilon)$$

⟨1⟩2. For any open sets U_1, \dots, U_n in \mathbb{R} , we have $U_1 \times \cdots \times U_n$ is open in the square metric topology.

⟨2⟩1. LET: $\vec{a} \in U_1 \times \cdots \times U_n$

⟨2⟩2. For $i = 1, \dots, n$, PICK $\epsilon_i > 0$ such that $(a_i - \epsilon_i, a_i + \epsilon_i) \subseteq U_i$

⟨2⟩3. LET: $\epsilon = \min(\epsilon_1, \dots, \epsilon_n)$

⟨2⟩4. $B_\rho(\vec{a}, \epsilon) \subseteq U$

□

Definition 10.2.3. Given $\vec{x}, \vec{y} \in \mathbb{R}^n$, define the *sum* $\vec{x} + \vec{y}$ by

$$(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n) .$$

Definition 10.2.4. Given $\lambda \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^n$, define the *scalar product* $\lambda\vec{x} \in \mathbb{R}^n$ by

$$\lambda(x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n)$$

Definition 10.2.5 (Inner Product). Given $\vec{x}, \vec{y} \in \mathbb{R}^n$, define the *inner product* $\vec{x} \cdot \vec{y} \in \mathbb{R}$ by

$$(x_1, \dots, x_n) \cdot (y_1, \dots, y_n) = x_1 y_1 + \dots + x_n y_n \ .$$

We write \vec{x}^2 for $\vec{x} \cdot \vec{x}$.

Definition 10.2.6 (Norm). Let $n \geq 1$. The *norm* on \mathbb{R}^n is the function $\| \cdot \| : \mathbb{R}^n \rightarrow \mathbb{R}$ defined by

$$\|(x_1, \dots, x_n)\| = \sqrt{x_1^2 + \dots + x_n^2}$$

Lemma 10.2.7.

$$\|\vec{x}\|^2 = \vec{x}^2$$

PROOF: Immediate from definitions. \square

Lemma 10.2.8.

$$\vec{x} \cdot (\vec{y} + \vec{z}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z}$$

PROOF: Each is equal to $(x_1 y_1 + x_1 z_1, \dots, x_n y_n + x_n z_n)$. \square

Lemma 10.2.9.

$$|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \|\vec{y}\|$$

PROOF:

$\langle 1 \rangle 1$. ASSUME: $\vec{x} \neq \vec{0} \neq \vec{y}$

PROOF: Otherwise both sides are 0.

$\langle 1 \rangle 2$. LET: $a = 1/\|\vec{x}\|$

$\langle 1 \rangle 3$. LET: $b = 1/\|\vec{y}\|$

$\langle 1 \rangle 4$. $(a\vec{x} + b\vec{y})^2 \geq 0$ and $(a\vec{x} - b\vec{y})^2 \geq 0$

$\langle 1 \rangle 5$. $a^2 \|\vec{x}\|^2 + 2ab\vec{x} \cdot \vec{y} + b^2 \|\vec{y}\|^2 \geq 0$ and $a^2 \|\vec{x}\|^2 - 2ab\vec{x} \cdot \vec{y} + b^2 \|\vec{y}\|^2 \geq 0$

$\langle 1 \rangle 6$. $2ab\vec{x} \cdot \vec{y} + 2 \geq 0$ and $-2ab\vec{x} \cdot \vec{y} + 2 \geq 0$

$\langle 1 \rangle 7$. $\vec{x} \cdot \vec{y} \geq -1/ab$ and $\vec{x} \cdot \vec{y} \leq 1/ab$

$\langle 1 \rangle 8$. $\vec{x} \cdot \vec{y} \geq -\|\vec{x}\| \|\vec{y}\|$ and $\vec{x} \cdot \vec{y} \leq \|\vec{x}\| \|\vec{y}\|$

\square

Lemma 10.2.10 (Triangle Inequality).

$$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$$

PROOF:

$$\begin{aligned} \|\vec{x} + \vec{y}\|^2 &= \|\vec{x}\|^2 + 2\vec{x} \cdot \vec{y} + \|\vec{y}\|^2 \\ &\leq \|\vec{x}\|^2 + 2\|\vec{x}\| \|\vec{y}\| + \|\vec{y}\|^2 && \text{(Lemma 10.2.9)} \\ &= (\|\vec{x}\| + \|\vec{y}\|)^2 && \square \end{aligned}$$

Definition 10.2.11 (Euclidean Metric). Let $n \geq 1$. The *Euclidean metric* on \mathbb{R}^n is defined by

$$d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\| \ .$$

We prove this is a metric.

(1)1. $d(\vec{x}, \vec{y}) \geq 0$

PROOF: Immediate from definition.

(1)2. $d(\vec{x}, \vec{y}) = 0$ if and only if $\vec{x} = \vec{y}$

PROOF: $d(\vec{x}, \vec{y}) = 0$ if and only if $\vec{x} - \vec{y} = \vec{0}$.

(1)3. $d(\vec{x}, \vec{y}) = d(\vec{y}, \vec{x})$

PROOF: Immediate from definition.

(1)4. $d(\vec{x}, \vec{z}) \leq d(\vec{x}, \vec{y}) + d(\vec{y}, \vec{z})$

PROOF:

$$\begin{aligned} \|\vec{x} - \vec{z}\| &= \|(\vec{x} - \vec{y}) + (\vec{y} - \vec{z})\| \\ &\leq \|\vec{x} - \vec{y}\| + \|\vec{y} - \vec{z}\| \end{aligned} \quad (\text{Lemma 10.2.10})$$

□

Proposition 10.2.12. *The Euclidean metric induces the standard topology on \mathbb{R}^n .*

PROOF:

(1)1. LET: ρ be the square metric.

(1)2. For all $\vec{a} \in \mathbb{R}^n$ and $\epsilon > 0$, we have $B_d(\vec{a}, \epsilon) \subseteq B_\rho(\vec{a}, \epsilon)$

(2)1. LET: $\vec{x} \in B_d(\vec{a}, \epsilon)$

(2)2. $\sqrt{(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2} < \epsilon$

(2)3. $(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2 < \epsilon^2$

(2)4. For all i we have $(x_i - a_i)^2 < \epsilon^2$

(2)5. For all i we have $|x_i - a_i| < \epsilon$

(2)6. $\rho(\vec{x}, \vec{a}) < \epsilon$

(1)3. For all $\vec{a} \in \mathbb{R}^n$ and $\epsilon > 0$, we have $B_\rho(\vec{a}, \epsilon/\sqrt{n}) \subseteq B_d(\vec{a}, \epsilon)$

(2)1. LET: $\vec{x} \in B_\rho(\vec{a}, \epsilon/\sqrt{n})$

(2)2. $\rho(\vec{x}, \vec{a}) < \epsilon/\sqrt{n}$

(2)3. For all i we have $|x_i - a_i| < \epsilon/\sqrt{n}$

(2)4. For all i we have $(x_i - a_i)^2 < \epsilon^2/n$

(2)5. $d(\vec{x}, \vec{a}) < \epsilon$

(1)4. Q.E.D.

PROOF: By Lemma 10.1.15.

□

Proposition 10.2.13. *Let $n \geq 0$. For all $c \in \mathbb{R}^n$ and $\epsilon > 0$, the open ball $B(c, \epsilon)$ is path connected.*

PROOF:

(1)1. LET: $a, b \in B(c, \epsilon)$

(1)2. LET: $p : [0, 1] \rightarrow B(c, \epsilon)$ be the function $p(t) = (1 - t)a + tb$

PROOF: We have $p(t) \in B(c, \epsilon)$ for all t because

$$\begin{aligned} d(p(t), c) &= \|(1-t)a + tb - c\| \\ &= \|(1-t)(a-c) + t(b-c)\| \\ &\leq (1-t)\|a-c\| + t\|b-c\| \\ &< (1-t)\epsilon + t\epsilon \\ &= \epsilon \end{aligned}$$

$\langle 1 \rangle 3$. p is a path from a to b .

□

Proposition 10.2.14. *Let $n \geq 0$. For all $c \in \mathbb{R}^n$ and $\epsilon > 0$, the closed ball $\overline{B}(c, \epsilon)$ is path connected.*

PROOF:

$\langle 1 \rangle 1$. LET: $a, b \in \overline{B}(c, \epsilon)$

$\langle 1 \rangle 2$. LET: $p : [0, 1] \rightarrow \overline{B}(c, \epsilon)$ be the function $p(t) = (1-t)a + tb$

PROOF: We have $p(t) \in \overline{B}(c, \epsilon)$ for all t because

$$\begin{aligned} d(p(t), c) &= \|(1-t)a + tb - c\| \\ &= \|(1-t)(a-c) + t(b-c)\| \\ &\leq (1-t)\|a-c\| + t\|b-c\| \\ &\leq (1-t)\epsilon + t\epsilon \\ &= \epsilon \end{aligned}$$

$\langle 1 \rangle 3$. p is a path from a to b .

□

Lemma 10.2.15. *If $\sum_{i=0}^{\infty} x_i^2$ and $\sum_{i=0}^{\infty} y_i^2$ converge then $\sum_{i=0}^{\infty} |x_i y_i|$ converges.*

PROOF:

$\langle 1 \rangle 1$. For all $N \geq 0$ we have $\sum_{i=0}^N |x_i y_i| \leq \sqrt{\sum_{i=0}^N |x_i|^2} \sqrt{\sum_{i=0}^N |y_i|^2}$

PROOF: By the Cauchy-Schwarz inequality

$\langle 1 \rangle 2$. Q.E.D.

PROOF: Since $\sum_{i=0}^N |x_i y_i|$ is an increasing sequence bounded above by $(\sum_{i=0}^{\infty} x_i^2)(\sum_{i=0}^{\infty} y_i^2)$.

□

Corollary 10.2.15.1. *If $\sum_{i=0}^{\infty} x_i^2$ and $\sum_{i=0}^{\infty} y_i^2$ converge then $\sum_{i=0}^{\infty} (x_i + y_i)^2$ converges.*

PROOF: Since $\sum_{i=0}^{\infty} x_i^2$, $\sum_{i=0}^{\infty} y_i^2$ and $2\sum_{i=0}^{\infty} x_i y_i$ all converge.

Definition 10.2.16 (l^2 -metric). The l^2 -metric on

$$\left\{ (x_n) \in \mathbb{R}^{\omega} \mid \sum_{n=0}^{\infty} x_n^2 \text{ converges} \right\}$$

is defined by

$$d(x, y) = \left(\sum_{n=0}^{\infty} (x_n - y_n)^2 \right)^{1/2}$$

We prove this is a metric.

PROOF:

⟨1⟩1. d is well-defined.

PROOF: By Corollary 10.2.15.1.

⟨1⟩2. $d(x, y) \geq 0$

⟨1⟩3. $d(x, y) = 0$ if and only if $x = y$

⟨1⟩4. $d(x, y) = d(y, x)$

⟨1⟩5. $d(x, z) \leq d(x, y) + d(y, z)$

PROOF: By Lemma 10.2.10.

□

Theorem 10.2.17. *Addition is a continuous function $\mathbb{R}^2 \rightarrow \mathbb{R}$.*

PROOF:

⟨1⟩1. LET: $a, b \in \mathbb{R}$

⟨1⟩2. LET: $\epsilon > 0$

⟨1⟩3. LET: $\delta = \epsilon/2$

⟨1⟩4. LET: $x, y \in \mathbb{R}$

⟨1⟩5. ASSUME: $\rho((a, b), (x, y)) < \delta$

⟨1⟩6. $|(a + b) - (x + y)| < \epsilon$

PROOF:

$$\begin{aligned} |(a + b) - (x + y)| &= |a - x| + |b - y| \\ &\leq 2\rho((a, b), (x, y)) \\ &< 2\delta \\ &= \epsilon \end{aligned}$$

⟨1⟩7. Q.E.D.

PROOF: Theorem 10.1.21

□

Theorem 10.2.18. *Multiplication is a continuous function $\mathbb{R}^2 \rightarrow \mathbb{R}$.*

PROOF:

⟨1⟩1. LET: $a, b \in \mathbb{R}$

⟨1⟩2. LET: $\epsilon > 0$

⟨1⟩3. LET: $\delta = \min(\epsilon/(|a| + |b| + 1), 1)$

⟨1⟩4. LET: $x, y \in \mathbb{R}$

⟨1⟩5. ASSUME: $\rho((a, b), (x, y)) < \delta$

⟨1⟩6. $|ab - xy| < \epsilon$

PROOF:

$$\begin{aligned} |ab - xy| &= |a(b - y) + (a - x)b - (a - x)(b - y)| \\ &\leq |a||b - y| + |b||a - x| + |a - x||b - y| \\ &< |a|\delta + |b|\delta + \delta^2 && ((1)5) \\ &\leq |a|\delta + |b|\delta + \delta && ((1)3) \\ &\leq \epsilon && ((1)3) \end{aligned}$$

⟨1⟩7. Q.E.D.

PROOF: Theorem 10.1.21

□

Theorem 10.2.19. *The function $f : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ defined by $f(x) = x^{-1}$ is continuous.*

PROOF:

⟨1⟩1. For all $a \in \mathbb{R}$ we have $f^{-1}((a, +\infty))$ is open.

PROOF: The set is

$$(a^{-1}, +\infty) \text{ if } a > 0$$

$$(0, +\infty) \text{ if } a = 0$$

$$(-\infty, a^{-1}) \cup (0, +\infty) \text{ if } a < 0$$

⟨1⟩2. For all $a \in \mathbb{R}$ we have $f^{-1}((-\infty, a))$ is open.

PROOF: Similar.

⟨1⟩3. Q.E.D.

PROOF: By Proposition 9.14.3 and Lemma 9.16.2.

□

Definition 10.2.20. For $n \geq 0$, the *unit ball* B^n is the space $\{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$.

Proposition 10.2.21. *For all $n \geq 0$, the unit ball B^n is path connected.*

PROOF:

⟨1⟩1. LET: $a, b \in B^n$

⟨1⟩2. LET: $p : [0, 1] \rightarrow B^n$ be the function $p(t) = (1 - t)a + tb$

PROOF: We have $p(t) \in B^n$ for all t because

$$\|(1 - t)a + tb\| \leq (1 - t)\|a\| + t\|b\|$$

$$\leq (1 - t) + t$$

$$= 1$$

⟨1⟩3. p is a path from a to b .

□

Definition 10.2.22 (Punctured Euclidean Space). For $n \geq 0$, defined *punctured Euclidean space* to be $\mathbb{R}^n \setminus \{0\}$.

Proposition 10.2.23. *For $n > 1$, punctured Euclidean space $\mathbb{R}^n \setminus \{0\}$ is path connected.*

PROOF:

⟨1⟩1. LET: $a, b \in \mathbb{R}^n \setminus \{0\}$

⟨1⟩2. CASE: 0 is on the line from a to b

⟨2⟩1. PICK a point c not on the line from a to b

⟨2⟩2. The path consisting of a straight line from a to c followed by a straight line from c to b is a path from a to b .

⟨1⟩3. CASE: 0 is not on the line from a to b

PROOF: The straight line from a to b is a path from a to b .

Corollary 10.2.23.1. For $n > 1$, the spaces \mathbb{R} and \mathbb{R}^n are not homeomorphic.

PROOF: For any point a , the space $\mathbb{R} \setminus \{a\}$ is disconnected.

Definition 10.2.24 (Unit Sphere). For $n \geq 1$, the *unit sphere* S^{n-1} is the space

$$S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}.$$

Proposition 10.2.25. For $n > 1$, the unit sphere S^{n-1} is path connected.

PROOF: The map $g : \mathbb{R}^n \setminus \{0\} \rightarrow S^{n-1}$ defined by $g(x) = x/\|x\|$ is continuous and surjective. The result follows by Proposition 9.33.5. \square

Proposition 10.2.26. Let $f : S^1 \rightarrow \mathbb{R}$ be continuous. Then there exists $x \in S^1$ such that $f(x) = f(-x)$.

PROOF:

$\langle 1 \rangle 1$. LET: $g : S^1 \rightarrow \mathbb{R}$ be the function $g(x) = f(x) - f(-x)$

PROVE: There exists $x \in S^1$ such that $g(x) = 0$

$\langle 1 \rangle 2$. ASSUME: without loss of generality $g((1, 0)) > 0$

$\langle 1 \rangle 3$. $g((-1, 0)) < 0$

$\langle 1 \rangle 4$. There exists x such that $g(x) = 0$

PROOF: By the Intermediate Value Theorem.

\square

Definition 10.2.27 (Topologist's Sine Curve). Let $S = \{(x, \sin 1/x) \mid 0 < x \leq 1\}$. The *topologist's sine curve* is the closure \bar{S} of S .

Proposition 10.2.28.

$$\bar{S} = S \cup (\{0\} \times [-1, 1])$$

Proposition 10.2.29. The topologist's sine curve is connected.

PROOF:

$\langle 1 \rangle 1$. LET: $S = \{(x, \sin 1/x) \mid 0 < x \leq 1\}$

$\langle 1 \rangle 2$. S is connected.

PROOF: Theorem 9.31.13.

$\langle 1 \rangle 3$. \bar{S} is connected.

PROOF: Theorem 9.31.12.

\square

Proposition 10.2.30 (CC). The topologist's sine curve is not path connected.

PROOF:

$\langle 1 \rangle 1$. ASSUME: for a contradiction $p : [0, 1] \rightarrow \bar{S}$ is a path from $(0, 0)$ to $(1, \sin 1)$.

$\langle 1 \rangle 2$. $p^{-1}(\{0\} \times [0, 1])$ is closed.

$\langle 1 \rangle 3$. LET: b be the greatest element of $p^{-1}(\{0\} \times [0, 1])$.

$\langle 1 \rangle 4$. $b < 1$

PROOF: Since $p(1) = (1, \sin 1)$.

$\langle 1 \rangle 5$. PICK a sequence $(t_n)_{n \geq 1}$ in $(b, 1]$ such that $t_n \rightarrow b$ and $\pi_2(p(t_n)) = (-1)^n$

(2)1. LET: $n \geq 1$
 (2)2. PICK u with $0 < u < \pi_1(p(1/n))$ such that $\sin(1/u) = (-1)^n$
 (2)3. PICK t_n such that $0 < t_n < 1/n$ and $\pi_1(p(t_n)) = u$
 PROOF: One exists by the Intermediate Value Theorem.
 (1)6. Q.E.D.
 PROOF: This contradicts 9.14.18.
 □

Theorem 10.2.31. *Let A be a subspace of \mathbb{R}^n . Then the following are equivalent:*

1. A is compact.
2. A is closed and bounded under the Euclidean metric.
3. A is closed and bounded under the square metric.

PROOF:
 (1)1. $1 \Rightarrow 2$
 PROOF: By Corollary 9.48.10.1 and Proposition 10.1.27.
 (1)2. $2 \Rightarrow 3$
 PROOF: If $d(x, y) \leq M$ for all $x, y \in A$ then $\rho(x, y) \leq M/\sqrt{2}$.
 (1)3. $3 \Rightarrow 1$
 (2)1. ASSUME: A is closed and $\rho(x, y) \leq M$ for all $x, y \in A$
 (2)2. PICK $a \in A$
 PROOF: We may assume w.l.o.g. A is nonempty since the empty space is compact.
 (2)3. A is a closed subspace of $[a_1 - M, a_1 + M] \times \cdots \times [a_n - M, a_n + M]$
 (2)4. A is compact
 PROOF: Proposition 9.48.3.
 □

Corollary 10.2.31.1. *The unit sphere S^{n-1} and the closed unit ball B^n are compact for any n .*

10.3 The Uniform Topology

Definition 10.3.1 (Uniform Metric). Let J be a set. The *uniform metric* $\bar{\rho}$ on \mathbb{R}^J is defined by

$$\bar{\rho}(a, b) = \sup_{j \in J} \bar{d}(a_j, b_j)$$

where \bar{d} is the standard bounded metric on \mathbb{R} .

The *uniform topology* on \mathbb{R}^J is the topology induced by the uniform metric.

We prove this is a metric.

PROOF:

(1)1. $\bar{\rho}(a, b) \geq 0$

PROOF: Immediate from definitions.

⟨1⟩2. $\bar{\rho}(a, b) = 0$ if and only if $a = b$

PROOF: Immediate from definitions.

⟨1⟩3. $\bar{\rho}(a, b) = \bar{\rho}(b, a)$

PROOF: Immediate from definitions.

⟨1⟩4. $\bar{\rho}(a, c) \leq \bar{\rho}(a, b) + \bar{\rho}(b, c)$

PROOF:

$$\begin{aligned}\bar{\rho}(a, c) &= \sup_{j \in J} \bar{d}(a_j, c_j) \\ &\leq \sup_{j \in J} (\bar{d}(a_j, b_j) + \bar{d}(b_j, c_j)) \\ &\leq \sup_{j \in J} \bar{d}(a_j, b_j) + \sup_{j \in J} \bar{d}(b_j, c_j) \\ &= \bar{\rho}(a, b) + \bar{\rho}(b, c)\end{aligned}$$

□

Proposition 10.3.2. *The uniform topology on \mathbb{R}^J is finer than the product topology.*

PROOF:

⟨1⟩1. LET: $j \in J$ and U be open in \mathbb{R}

PROVE: $\pi_j^{-1}(U)$ is open in the uniform topology.

⟨1⟩2. LET: $a \in \pi_j^{-1}(U)$

⟨1⟩3. PICK $\epsilon > 0$ such that $(a_j - \epsilon, a_j + \epsilon) \subseteq U$

⟨1⟩4. $B_{\bar{\rho}}(a, \epsilon) \subseteq \pi_j^{-1}(U)$

□

Proposition 10.3.3. *The uniform topology on \mathbb{R}^J is coarser than the box topology.*

PROOF:

⟨1⟩1. LET: $a \in \mathbb{R}^J$ and $\epsilon > 0$

PROVE: $B(a, \epsilon)$ is open in the box topology.

⟨1⟩2. LET: $b \in B(a, \epsilon)$

⟨1⟩3. For $j \in J$ we have $|a_j - b_j| < \epsilon$

⟨1⟩4. For $j \in J$,

LET: $\delta_j = (\epsilon - |a_j - b_j|)/2$

⟨1⟩5. $\prod_{j \in J} (b_j - \delta_j, b_j + \delta_j) \subseteq B(a, \epsilon)$

□

Proposition 10.3.4. *The uniform topology on \mathbb{R}^J is strictly finer than the product topology if and only if J is infinite.*

PROOF:

⟨1⟩1. If J is finite then the uniform and product topologies coincide.

PROOF: The uniform, box and product topologies are all the same.

⟨1⟩2. If J is infinite then the uniform and product topologies are different.

PROOF: The set $B(\vec{0}, 1)$ is open in the uniform topology but not the product topology.

□

Proposition 10.3.5 (DC). *The uniform topology on \mathbb{R}^J is strictly coarser than the box topology if and only if J is infinite.*

PROOF:

⟨1⟩1. If J is finite then the uniform and box topologies coincide.

PROOF: The uniform, box and product topologies are all the same.

⟨1⟩2. If J is infinite then the uniform and box topologies are different.

PROOF: Pick an ω -sequence (j_1, j_2, \dots) in J . Let $U = \prod_{j \in J} U_j$ where $U_{j_i} = (-1/i, 1/i)$ and $U_j = (-1, 1)$ for all other j . Then $\vec{0} \in U$ but there is no $\epsilon > 0$ such that $B(\vec{0}, \epsilon) \subseteq U$.

□

Proposition 10.3.6. *The closure of \mathbb{R}^∞ in \mathbb{R}^ω under the uniform topology is \mathbb{R}^ω .*

PROOF: Given any open ball $B(a, \epsilon)$, pick an integer N such that $1/\epsilon < N$. Then $B(a, \epsilon)$ includes sequences whose n th entry is 0 for all $n \geq N$. □

Example 10.3.7. The space \mathbb{R}^ω is disconnected under the uniform topology. The set of bounded sequences and the set of unbounded sequences form a separation.

Proposition 10.3.8. *Give \mathbb{R}^ω the uniform topology. Let $x, y \in \mathbb{R}^\omega$. Then x and y are in the same component if and only if $x - y$ is bounded.*

PROOF:

⟨1⟩1. The component containing 0 is the set of bounded sequences.

⟨2⟩1. LET: B be the set of bounded sequences.

⟨2⟩2. B is path-connected.

⟨3⟩1. LET: $x, y \in B$

⟨3⟩2. PICK $b > 0$ such that $|x_j|, |y_j| \leq b$ for all j

⟨3⟩3. LET: $p : [0, 1] \rightarrow B$ be the function $p(t) = (1 - t)x + ty$

PROVE: p is continuous.

⟨3⟩4. LET: $t \in [0, 1]$ and $\epsilon > 0$

⟨3⟩5. LET: $\delta = \epsilon/2b$

⟨3⟩6. LET: $s \in [0, 1]$ with $|s - t| < \delta$

⟨3⟩7. $\bar{\rho}(p(s), p(t)) < \epsilon$

PROOF:

$$\begin{aligned} \bar{\rho}(p(s), p(t)) &= \sup_j \bar{d}((1 - s)x_j + sy_j, (1 - t)x_j + ty_j) \\ &\leq |(s - t)x_j + (t - s)y_j| \\ &\leq |s - t||x_j - y_j| \\ &< 2b\delta \\ &= \epsilon \end{aligned}$$

(2)3. B is connected.

PROOF: Proposition 9.33.3.

(2)4. If C is connected and $B \subseteq C$ then $B = C$.

PROOF: Otherwise $B \cap C$ and $C \setminus B$ form a separation of C .

(1)2. Q.E.D.

PROOF: Since $\lambda x.x - y$ is a Homeomorphism of \mathbb{R}^ω with itself.

□

10.4 Uniform Convergence

Definition 10.4.1 (Uniform Convergence). Let X be a set and Y a metric space. Let $(f_n : X \rightarrow Y)$ be a sequence of functions and $f : X \rightarrow Y$ be a function. Then f_n converges uniformly to f as $n \rightarrow \infty$ if and only if, for all $\epsilon > 0$, there exists N such that, for all $n \geq N$ and $x \in X$, we have $d(f_n(x), f(x)) < \epsilon$.

Example 10.4.2. Define $f_n : [0, 1] \rightarrow \mathbb{R}$ by $f_n(x) = x^n$ for $n \geq 1$, and $f : [0, 1] \rightarrow \mathbb{R}$ by $f(x) = 0$ if $x < 1$, $f(1) = 1$. Then f_n converges to f pointwise but not uniformly.

Theorem 10.4.3 (Uniform Limit Theorem). Let X be a topological space and Y a metric space. Let $(f_n : X \rightarrow Y)$ be a sequence of continuous functions and $f : X \rightarrow Y$ be a function. If f_n converges uniformly to f as $n \rightarrow \infty$, then f is continuous.

PROOF:

(1)1. LET: $x \in X$ and $\epsilon > 0$

(1)2. PICK N such that, for all $n \geq N$ and $y \in X$, we have $d(f_n(y), f(y)) < \epsilon/3$

(1)3. PICK a neighbourhood U of x such that $f_N(U) \subseteq B(f_N(x), \epsilon/3)$

PROVE: $f(U) \subseteq B(f(x), \epsilon)$

(1)4. LET: $y \in U$

(1)5. $d(f(y), f(x)) < \epsilon$

PROOF:

$$d(f(y), f(x)) \leq d(f(y), f_N(y)) + d(f_N(y), f_N(x)) + d(f_N(x), f(x)) \quad (\text{Triangle Inequality})$$

$$< \epsilon/3 + \epsilon/3 + \epsilon/3 \quad ((1)2, (1)3)$$

$$= \epsilon$$

□

Proposition 10.4.4. Let X be a topological space and Y a metric space. Let $(f_n : X \rightarrow Y)$ be a sequence of continuous functions and $f : X \rightarrow Y$ be a function. Let (a_n) be a sequence of points in X and $a \in X$. If f_n converges uniformly to f and a_n converges to a in X then $f_n(a_n)$ converges to $f(a)$ uniformly in Y .

PROOF:

(1)1. LET: $\epsilon > 0$

(1)2. PICK N_1 such that, for all $n \geq N_1$ and $x \in X$, we have $d(f_n(x), f(x)) < \epsilon/2$

⟨1⟩3. PICK N_2 such that, for all $n \geq N_2$, we have $a_n \in f^{-1}(B(a, \epsilon/2))$

PROOF: Using the fact that f is continuous from the Uniform Limit Theorem.

⟨1⟩4. LET: $N = \max(N_1, N_2)$

⟨1⟩5. LET: $n \geq N$

⟨1⟩6. $d(f_n(a_n), f(a)) < \epsilon$

PROOF:

$$\begin{aligned} d(f_n(a_n), f(a)) &\leq d(f_n(a_n), f(a_n)) + d(f(a_n), f(a)) \quad (\text{Triangle Inequality}) \\ &< \epsilon/2 + \epsilon/2 \quad (\langle 1 \rangle 2, \langle 1 \rangle 3) \\ &= \epsilon \end{aligned}$$

□

Proposition 10.4.5. *Let X be a set. Let $(f_n : X \rightarrow \mathbb{R})$ be a sequence of functions and $f : X \rightarrow \mathbb{R}$ be a function. Then f_n converges uniformly to f as $n \rightarrow \infty$ if and only if $f_n \rightarrow f$ as $n \rightarrow \infty$ in \mathbb{R}^X under the uniform topology.*

PROOF:

⟨1⟩1. If f_n converges uniformly to f then f_n converges to f under the uniform topology.

⟨2⟩1. ASSUME: f_n converges uniformly to f

⟨2⟩2. LET: $\epsilon > 0$

⟨2⟩3. PICK N such that, for all $n \geq N$ and $x \in X$, we have $d(f_n(x), f(x)) < \epsilon/2$

⟨2⟩4. For all $n \geq N$ we have $\bar{\rho}(f_n, f) \leq \epsilon/2$

⟨2⟩5. For all $n \geq N$ we have $\bar{\rho}(f_n, f) < \epsilon$

⟨1⟩2. If f_n converges to f under the uniform topology then f_n converges uniformly to f .

⟨2⟩1. ASSUME: f_n converges to f under the uniform topology.

⟨2⟩2. LET: $\epsilon > 0$

⟨2⟩3. PICK N such that, for all $n \geq N$, we have $\bar{\rho}(f_n, f) < \min(\epsilon, 1/2)$

⟨2⟩4. LET: $n \geq N$

⟨2⟩5. LET: $x \in X$

⟨2⟩6. $\bar{\rho}(f_n, f) < \min(\epsilon, 1/2)$

PROOF: From ⟨2⟩3.

⟨2⟩7. $d(f_n(x), f(x)) < \min(\epsilon, 1/2)$

⟨2⟩8. $d(f_n(x), f(x)) < \epsilon$

□

10.5 Isometric Imbeddings

Definition 10.5.1. Let X and Y be metric spaces. An *isometric imbedding* $f : X \rightarrow Y$ is a function such that, for all $x, y \in X$, we have $d(f(x), f(y)) = d(x, y)$.

Proposition 10.5.2. *Every isometric imbedding is an imbedding.*

PROOF:

- ⟨1⟩1. LET: $f : X \rightarrow Y$ be an isometric imbedding.
 ⟨1⟩2. f is injective.
 PROOF: If $f(x) = f(y)$ then $d(f(x), f(y)) = 0$ hence $d(x, y) = 0$ hence $x = y$.
 ⟨1⟩3. f is continuous.
 PROOF: For all $\epsilon > 0$, if $d(x, y) < \epsilon$ then $d(f(x), f(y)) < \epsilon$.
 ⟨1⟩4. $f : X \rightarrow f(X)$ is an open map.
 PROOF: $f(B(a, \epsilon)) = B(f(a), \epsilon) \cap f(X)$.
 □

10.6 Distance to a Set

Definition 10.6.1. Let X be a metric space, $x \in X$ and $A \subseteq X$ be nonempty. The *distance* from x to A is defined as

$$d(x, A) = \inf_{a \in A} d(x, a) .$$

Proposition 10.6.2. Let X be a metric space and $A \subseteq X$ be nonempty. Then the function $d(-, A) : X \rightarrow \mathbb{R}$ is continuous.

PROOF:

- ⟨1⟩1. LET: X be a metric space.
 ⟨1⟩2. LET: $A \subseteq X$ be nonempty.
 ⟨1⟩3. LET: $x \in X$ and $\epsilon > 0$
 ⟨1⟩4. LET: $\delta = \epsilon$
 ⟨1⟩5. LET: $y \in B(x, \delta)$
 ⟨1⟩6. $|d(x, A) - d(y, A)| < \epsilon$
 ⟨2⟩1. $d(x, A) - d(y, A) < \epsilon$

PROOF:

- ⟨3⟩1. For all $a \in A$ we have $d(x, A) \leq d(x, y) + d(y, a)$

PROOF:

$$\begin{aligned}
 d(x, A) &\leq d(x, a) && \text{(definition of } d(x, A)) \\
 &\leq d(x, y) + d(y, a) && \text{(Triangle Inequality)}
 \end{aligned}$$

- ⟨3⟩2. $d(x, A) - d(x, y) \leq d(y, A)$

- ⟨2⟩2. $d(y, A) - d(x, A) < \epsilon$

PROOF: Similar.

- ⟨1⟩7. Q.E.D.

PROOF: Theorem 10.1.21.

□

Theorem 10.6.3. Let X be a metric space, $A \subseteq X$ be nonempty, and $x \in X$. Then $d(x, A) = 0$ if and only if $x \in \bar{A}$.

PROOF:

- ⟨1⟩1. LET: X be a metric space.
 ⟨1⟩2. LET: $A \subseteq X$ be nonempty.
 ⟨1⟩3. LET: $x \in X$

- ⟨1⟩4. If $d(x, A) = 0$ then $x \in \overline{A}$
 ⟨2⟩1. ASSUME: $d(x, A) = 0$
 ⟨2⟩2. LET: U be any neighbourhood of x .
 ⟨2⟩3. PICK $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$
 PROOF: Proposition 10.1.4, ⟨1⟩1, ⟨2⟩2.
 ⟨2⟩4. PICK $a \in A$ such that $d(x, a) < \epsilon$
 PROOF: From ⟨2⟩1, ⟨2⟩3.
 ⟨2⟩5. $a \in A \cap U$
 PROOF: From ⟨2⟩3, ⟨2⟩4.
 ⟨2⟩6. Q.E.D.
 PROOF: Theorem 9.6.6.
 ⟨1⟩5. If $x \in \overline{A}$ then $d(x, A) = 0$
 □

Theorem 10.6.4. *Let X be a metric space. Let $A \subseteq X$ be nonempty and compact. Let $x \in X$. Then there exists $a \in A$ such that $d(x, A) = d(x, a)$.*

PROOF: By the Extreme Value Theorem, the function $d(x, -) : A \rightarrow \mathbb{R}$ attains its minimum. □

10.7 Lebesgue Numbers

Definition 10.7.1 (Lebesgue Number). Let X be a metric space. Let \mathcal{U} be an open covering of X . A *Lebesgue number* for \mathcal{U} is a real number $\delta > 0$ such that, for every subset $A \subseteq X$ with diameter $\text{diam}(A) < \delta$, there exists $U \in \mathcal{U}$ such that $A \subseteq U$.

Theorem 10.7.2 (Lebesgue Number Lemma). *Every open covering of a compact metric space has a Lebesgue number.*

PROOF:

- ⟨1⟩1. LET: X be a compact metric space.
 ⟨1⟩2. LET: \mathcal{U} be an open covering of X .
 ⟨1⟩3. PICK a finite subset $\{U_1, \dots, U_n\}$ of \mathcal{U} that covers X .
 ⟨1⟩4. For $i = 1, \dots, n$,
 LET: $C_i = X - U_i$
 ⟨1⟩5. LET: $f : X \rightarrow \mathbb{R}$,

$$f(x) = 1/n \sum_{i=1}^n d(x, C_i)$$

- ⟨1⟩6. For all $x \in X$ we have $f(x) > 0$
 ⟨2⟩1. LET: $x \in X$
 ⟨2⟩2. PICK i such that $x \in U_i$
 PROOF: From ⟨1⟩3.
 ⟨2⟩3. PICK $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U_i$
 PROOF: Proposition 10.1.4.
 ⟨2⟩4. $d(x, C_i) \geq \epsilon$

- (2)5. $f(x) \geq \epsilon/n$
- (1)7. f is continuous.
PROOF: Proposition 10.6.2.
- (1)8. LET: δ be the minimum value of $f(X)$
PROOF: By the Extreme Value Theorem
- (1)9. $\delta > 0$
PROOF: From (1)6
- (1)10. For every subset $A \subseteq X$ with diameter $< \delta$, there exists $U \in \mathcal{U}$ such that
 $A \subseteq U$
 - (2)1. LET: $A \subseteq X$ with $\text{diam } A < \delta$
 - (2)2. PICK $x_0 \in A$
 - (2)3. $A \subseteq B(x_0, \delta)$
 - (2)4. $f(x_0) \geq \delta$
 - (2)5. PICK m such that $d(x_0, C_m)$ is the largest out of $d(x_0, C_1), \dots, d(x_0, C_n)$
 - (2)6. $d(x_0, C_m) \geq f(x_0)$
 - (2)7. $B(x_0, \delta) \subseteq U_m$
 - (2)8. $A \subseteq U_m$
- (1)11. δ is a Lebesgue number for \mathcal{U}

□

Theorem 10.7.3 (AC). *Every sequentially compact metric space is compact.*

PROOF:

- (1)1. LET: X be a sequentially compact metric space.
- (1)2. Every open covering of X has a Lebesgue number.
 - (2)1. LET: \mathcal{A} be an open covering of X .
 - (2)2. ASSUME: for a contradiction \mathcal{A} has no Lebesgue number.
 - (2)3. For $n \geq 1$, PICK a set C_n with diameter $< 1/n$ that is not included in any member of \mathcal{A} .
 - (2)4. For $n \geq 1$, PICK $x_n \in C_n$.
 - (2)5. PICK a convergent subsequence (C_{n_r}) of (C_n) with limit a .
 - (2)6. PICK $A \in \mathcal{A}$ such that $a \in A$.
 - (2)7. PICK $\epsilon > 0$ such that $B(a, \epsilon) \subseteq A$.
 - (2)8. PICK r such that $1/n_r < \epsilon/2$ and $d(x_{n_r}, a) < \epsilon/2$
 - (2)9. $C_{n_r} \subseteq B(a, \epsilon)$
 - (2)10. $C_{n_r} \subseteq A$
 - (2)11. Q.E.D.
- PROOF: This contradicts (2)3.
- (1)3. For every $\epsilon > 0$, there exists a finite covering of X by ϵ -balls.
 - (2)1. ASSUME: for a contradiction that there exists $\epsilon > 0$ such that X cannot be finitely covered by ϵ -balls.
 - (2)2. PICK a sequence of points (x_n) such that $x_n \in X - (B(x_1, \epsilon) \cup \dots \cup B(x_{n-1}, \epsilon))$
 - (2)3. $d(x_m, x_n) \geq \epsilon$ for all m, n distinct
 - (2)4. (x_n) has no convergent subsequence
 - (2)5. Q.E.D.
- PROOF: This contradicts (1)1.

- ⟨1⟩4. LET: \mathcal{A} be an open covering of X .
 ⟨1⟩5. PICK a Lebesgue number δ for \mathcal{A} .
 PROOF: By ⟨1⟩2.
 ⟨1⟩6. LET: $\epsilon = \delta/3$
 ⟨1⟩7. PICK a finite covering $\{B_1, \dots, B_n\}$ of X be ϵ -balls.
 PROOF: By ⟨1⟩3.
 ⟨1⟩8. For $i = 1, \dots, n$, PICK $U_i \in \mathcal{A}$ such that $B_i \subseteq U_i$
 PROOF: By ⟨1⟩5 since $\text{diam } B_i = 2\epsilon < \delta$.
 ⟨1⟩9. $\{U_1, \dots, U_n\}$ covers X .

□

Example 10.7.4. The space S_Ω is not metrizable, because it is sequentially compact but not compact.

10.8 Uniform Continuity

Definition 10.8.1 (Uniformly Continuous). Let X and Y be metric spaces. Let $f : X \rightarrow Y$. Then f is *uniformly continuous* if and only if, for every $\epsilon > 0$, there exists $\delta > 0$ such that, for all $x, y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$.

Theorem 10.8.2 (Uniform Continuity Theorem). *Every continuous function from a compact metric space to a metric space is uniformly continuous.*

PROOF:

- ⟨1⟩1. LET: X be a compact metric space.
 ⟨1⟩2. LET: Y be a metric space.
 ⟨1⟩3. LET: $f : X \rightarrow Y$ be a continuous function.
 ⟨1⟩4. LET: $\epsilon > 0$
 ⟨1⟩5. LET: $\mathcal{U} = \{f^{-1}(B(y, \epsilon/2)) \mid y \in Y\}$
 ⟨1⟩6. PICK a Lebesgue number $\delta > 0$ for \mathcal{U} .
 PROOF: By the Lebesgue Number Lemma.
 ⟨1⟩7. LET: $x, x' \in X$
 ⟨1⟩8. ASSUME: $d(x, x') < \delta$
 ⟨1⟩9. PICK $y \in Y$ such that $\{x, x'\} \subseteq f^{-1}(B(y, \epsilon/2))$
 PROOF: Since $\text{diam}\{x, x'\} < \delta$.
 ⟨1⟩10. $d(f(x), f(x')) < \epsilon$

PROOF:

$$\begin{aligned}
 d(f(x), f(x')) &\leq d(f(x), y) + d(y, f(x')) && \text{(Triangle Inequality)} \\
 &< \epsilon/2 + \epsilon/2 && (\langle 1 \rangle 9) \\
 &= \epsilon
 \end{aligned}$$

□

10.9 Epsilon-neighbourhoods

Definition 10.9.1 (ϵ -neighbourhood). Let X be a metric space. Let $A \subseteq X$ be nonempty. Let $\epsilon > 0$. Then the ϵ -neighbourhood of A , $U(A, \epsilon)$, is the set

$$U(A, \epsilon) = \{x \in X \mid d(x, A) < \epsilon\}.$$

Proposition 10.9.2. Let X be a metric space. Let $A \subseteq X$ be nonempty. Let $\epsilon > 0$. Then $U(A, \epsilon) = \bigcup_{a \in A} B(a, \epsilon)$.

PROOF:

- $\langle 1 \rangle 1$. LET: X be a metric space.
- $\langle 1 \rangle 2$. LET: $A \subseteq X$ be nonempty.
- $\langle 1 \rangle 3$. LET: $\epsilon > 0$
- $\langle 1 \rangle 4$. $U(A, \epsilon) \subseteq \bigcup_{a \in A} B(a, \epsilon)$
 - $\langle 2 \rangle 1$. LET: $x \in U(A, \epsilon)$
 - $\langle 2 \rangle 2$. $d(x, A) < \epsilon$
 - $\langle 2 \rangle 3$. ϵ is not a lower bound for $\{d(x, a) \mid a \in A\}$
 - $\langle 2 \rangle 4$. PICK $a \in A$ such that $d(x, a) < \epsilon$
 - $\langle 2 \rangle 5$. $x \in B(a, \epsilon)$
- $\langle 1 \rangle 5$. $\bigcup_{a \in A} B(a, \epsilon) \subseteq U(A, \epsilon)$
 - $\langle 2 \rangle 1$. LET: $a \in A$ and $x \in B(a, \epsilon)$
 - $\langle 2 \rangle 2$. $d(x, A) \leq d(x, a)$
 - $\langle 2 \rangle 3$. $d(x, A) < \epsilon$
 - $\langle 2 \rangle 4$. $x \in U(A, \epsilon)$

□

Proposition 10.9.3. Let X be a metric space. Let $A \subseteq X$ be nonempty and compact. Let U be an open set such that $A \subseteq U$. Then there exists $\epsilon > 0$ such that $U(A, \epsilon) \subseteq U$.

PROOF:

- $\langle 1 \rangle 1$. LET: X be a metric space.
- $\langle 1 \rangle 2$. LET: $A \subseteq X$ be nonempty and compact.
- $\langle 1 \rangle 3$. LET: U be an open set such that $A \subseteq U$
- $\langle 1 \rangle 4$. $\{B(a, \epsilon) \mid a \in A, \epsilon > 0, B(a, 2\epsilon) \subseteq U\}$ covers A .
PROOF: By Proposition 10.1.4.
- $\langle 1 \rangle 5$. PICK a finite subcover $\{B(a_1, \epsilon_1), \dots, B(a_n, \epsilon_n)\}$
PROOF: Since A is compact ($\langle 1 \rangle 2$).
- $\langle 1 \rangle 6$. LET: $\epsilon = \min(\epsilon_1, \dots, \epsilon_n)$
PROVE: $U(A, \epsilon) \subseteq U$
- $\langle 1 \rangle 7$. LET: $x \in U(A, \epsilon)$
- $\langle 1 \rangle 8$. PICK $a \in A$ such that $d(x, a) < \epsilon$
PROOF: Proposition 10.9.2.
- $\langle 1 \rangle 9$. PICK i such that $a \in B(a_i, \epsilon_i)$
PROOF: By $\langle 1 \rangle 5$.
- $\langle 1 \rangle 10$. $d(x, a_i) < 2\epsilon$
PROOF: By the Triangle Inequality.

⟨1⟩11. $x \in U$

PROOF: From ⟨1⟩4.

□

This example shows that we cannot weaken the hypothesis that A is compact to A being closed:

Example 10.9.4. Let $X = \mathbb{R}^2$. Let $A = \{(x, 1/x) \mid x > 0\}$. Let $U = \{(x, y) \mid x > 0, y > 0\}$. Then A is nonempty and closed (Proposition 9.48.14). The set U is open and $A \subseteq U$. But there is no $\epsilon > 0$ such that $U(A, \epsilon) \subseteq U$.

PROOF:

⟨1⟩1. LET: $\epsilon > 0$

⟨1⟩2. $(2/\epsilon, \epsilon/2) \in A$

⟨1⟩3. $(2/\epsilon, 0) \in U(A, \epsilon)$

⟨1⟩4. $(2/\epsilon, 0) \notin U$

□

10.10 Isometry

Definition 10.10.1 (Isometry). Let X be a metric space. An *isometry* of X is a function $f : X \rightarrow X$ such that, for all $x, y \in X$, we have $d(x, y) = d(f(x), f(y))$.

Proposition 10.10.2. *An isometry on a compact metric space is a homeomorphism.*

PROOF:

⟨1⟩1. LET: X be a compact metric space.

⟨1⟩2. LET: $f : X \rightarrow X$ be an isometry.

⟨1⟩3. f is an imbedding

PROOF: Proposition 10.5.2.

⟨1⟩4. f is surjective.

⟨2⟩1. ASSUME: for a contradiction $a \notin f(X)$

⟨2⟩2. $f(X)$ is closed

PROOF: Proposition 9.48.12.

⟨2⟩3. PICK $\epsilon > 0$ such that $B(a, \epsilon) \cap f(X) = \emptyset$

⟨2⟩4. For $m, n \in \mathbb{N}$ with $m \neq n$, we have $d(f^m(a), f^n(a)) \geq \epsilon$

⟨3⟩1. ASSUME: without loss of generality $m < n$

⟨3⟩2. $d(a, f^{n-m}(a)) \geq \epsilon$

PROOF: ⟨2⟩3

⟨3⟩3. $d(f^m(a), f^n(a)) \geq \epsilon$

PROOF: ⟨1⟩2

⟨2⟩5. The sequence $(f^n(a))$ has a convergent subsequence.

PROOF: Corollary 9.46.2.1, ⟨1⟩1, Corollary 10.1.19.1.

⟨2⟩6. Q.E.D.

PROOF: ⟨2⟩4 and ⟨2⟩5 form a contradiction.

□

□

10.11 Shrinking Maps

Definition 10.11.1 (Shrinking Map). Let X be a metric space. Let $f : X \rightarrow X$. Then f is a *shrinking map* if and only if, for all $x, y \in X$ with $x \neq y$, we have $d(f(x), f(y)) < d(x, y)$.

Proposition 10.11.2. Let X be a compact metric space. Let $f : X \rightarrow X$ be a contraction. Then f has a unique fixed point.

PROOF:

$\langle 1 \rangle 1$. LET: $A_n = f^n(X)$ for $n \geq 1$

$\langle 1 \rangle 2$. For all $n \geq 1$ we have A_n is closed.

PROOF: Proposition 9.48.12.

$\langle 1 \rangle 3$. LET: $A = \bigcap_{n=1}^{\infty} A_n$

$\langle 1 \rangle 4$. PICK $a \in A$

PROOF: Proposition 9.47.6.

$\langle 1 \rangle 5$. $f(A) = A$

$\langle 2 \rangle 1$. $f(A) \subseteq A$

$\langle 2 \rangle 2$. $A \subseteq f(A)$

$\langle 3 \rangle 1$. LET: $x \in A$

$\langle 3 \rangle 2$. For $n \geq 1$, PICK x_n such that $x = f^n(x_n)$

$\langle 3 \rangle 3$. PICK a convergent subsequence $(f^{n_r-1}(x_{n_r}))$ of $(f^{n-1}(x_n))$ with limit l

PROOF: Corollary 9.46.2.1.

$\langle 3 \rangle 4$. $f(l) = x$

PROOF: Both are the limit of $f(f^{n_r-1}(a_{n_r})) = f^{n_r}(a_{n_r})$.

$\langle 3 \rangle 5$. $l \in A$

$\langle 4 \rangle 1$. ASSUME: for a contradiction $l \notin A$

$\langle 4 \rangle 2$. PICK N such that $l \notin A_N$

$\langle 4 \rangle 3$. PICK R such that $n_R > N$

$\langle 4 \rangle 4$. For $r \geq R$ we have $f^{n_r-1}(a_{n_r}) \in A_N$

$\langle 4 \rangle 5$. Q.E.D.

PROOF: This is a contradiction.

$\langle 1 \rangle 6$. $\text{diam } A = 0$

$\langle 2 \rangle 1$. PICK $x, y \in A$ such that $d(x, y) = \text{diam } A$

PROOF: By the Extreme Value Theorem.

$\langle 2 \rangle 2$. PICK $x', y' \in A$ such that $x = f(x')$ and $y = f(y')$

PROOF: By $\langle 1 \rangle 5$.

$\langle 2 \rangle 3$. $x' = y'$

PROOF: If $x' \neq y'$ then $\text{diam } A = d(x, y) < d(x', y')$ which is a contradiction.

$\langle 2 \rangle 4$. $x = y$

$\langle 1 \rangle 7$. $f(a) = a$

PROOF: Since $a, f(a) \in A$

$\langle 1 \rangle 8$. If $f(b) = b$ then $b = a$

PROOF: If $f(b) = b$ then $b \in A$.

□

The following example shows that we cannot weaken the hypothesis from 'X is a compact metric space' to 'X is a complete metric space'.

Example 10.11.3. The function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = [x + (x^2 + 1)^{1/2}]/2$ is a shrinking map with no fixed point.

10.12 Contractions

Definition 10.12.1 (Contraction). Let X be a metric space. Let $f : X \rightarrow X$. Then f is a *contraction* if and only if there exists $\alpha < 1$ such that, for all $x, y \in X$, we have $d(f(x), f(y)) \leq \alpha d(x, y)$.