

# Topology

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May 8, 2022

## 1 Real Numbers

**Definition 1.** Let  $\mathbb{R}^\infty$  be the subset of  $\mathbb{R}^\omega$  consisting of all sequences  $(a_n)$  such that  $a_n = 0$  for all but finitely many  $n$ .

## 2 Order Theory

**Definition 2** (Convex). Let  $X$  be a linearly ordered set and  $Y \subseteq X$ . Then  $Y$  is *convex* if and only if, for all  $a, b \in Y$  and  $c \in X$ , if  $a < c < b$  then  $c \in Y$ .

## 3 Topological Spaces

**Definition 3** (Topology). A *topology* on a set  $X$  is a set  $\mathcal{T} \subseteq \mathcal{P}X$  such that:

- $X \in \mathcal{T}$ .
- For all  $\mathcal{U} \subseteq \mathcal{T}$  we have  $\bigcup \mathcal{U} \in \mathcal{T}$ .
- For all  $U, V \in \mathcal{T}$  we have  $U \cap V \in \mathcal{T}$ .

We call the elements of  $X$  *points* and the elements of  $\mathcal{T}$  *open sets*.

**Definition 4** (Topological Space). A *topological space*  $X$  consists of a set  $X$  and a topology on  $X$ .

**Definition 5** (Discrete Space). For any set  $X$ , the *discrete* topology on  $X$  is  $\mathcal{P}X$ .

**Definition 6** (Indiscrete Space). For any set  $X$ , the *indiscrete* or *trivial* topology on  $X$  is  $\{\emptyset, X\}$ .

**Definition 7** (Finite Complement Topology). For any set  $X$ , the *finite complement topology* on  $X$  is  $\{U \in \mathcal{P}X \mid X \setminus U \text{ is finite}\} \cup \{\emptyset\}$ .

**Definition 8** (Countable Complement Topology). For any set  $X$ , the *countable complement topology* on  $X$  is  $\{U \in \mathcal{P}X \mid X \setminus U \text{ is countable}\} \cup \{\emptyset\}$ .

**Definition 9** (Finer, Coarser). Suppose that  $\mathcal{T}$  and  $\mathcal{T}'$  are two topologies on a given set  $X$ . If  $\mathcal{T}' \supseteq \mathcal{T}$ , we say that  $\mathcal{T}'$  is *finer* than  $\mathcal{T}$ ; if  $\mathcal{T}'$  *properly* contains  $\mathcal{T}$ , we say that  $\mathcal{T}'$  is *strictly finer* than  $\mathcal{T}$ . We also say that  $\mathcal{T}$  is *coarser* than  $\mathcal{T}'$ , or *strictly coarser*, in these two respective situations. We say  $\mathcal{T}$  is *comparable* with  $\mathcal{T}'$  if either  $\mathcal{T}' \supseteq \mathcal{T}$  or  $\mathcal{T} \supseteq \mathcal{T}'$ .

**Lemma 10.** *Let  $X$  be a topological space and  $U \subseteq X$ . Then  $U$  is open if and only if, for all  $x \in U$ , there exists an open set  $V$  such that  $x \in V \subseteq U$ .*

PROOF:

$\langle 1 \rangle 1. \Rightarrow$

PROOF: Take  $V = U$

$\langle 1 \rangle 2. \Leftarrow$

PROOF: We have  $U = \bigcup \{V \in \mathcal{P}X \mid V \subseteq U\}$ .

□

**Lemma 11.** *Let  $X$  be a set and  $\mathcal{T}$  a nonempty set of topologies on  $X$ . Then  $\bigcap \mathcal{T}$  is a topology on  $X$ , and is the finest topology that is coarser than every member of  $\mathcal{T}$ .*

PROOF:

$\langle 1 \rangle 1. X \in \bigcap \mathcal{T}$

PROOF: Since  $X$  is in every member of  $\mathcal{T}$ .

$\langle 1 \rangle 2. \bigcap \mathcal{T}$  is closed under union.

$\langle 2 \rangle 1. \text{ LET: } \mathcal{U} \subseteq \bigcap \mathcal{T}$

$\langle 2 \rangle 2. \text{ For all } T \in \mathcal{T} \text{ we have } \mathcal{U} \subseteq T$

$\langle 2 \rangle 3. \text{ For all } T \in \mathcal{T} \text{ we have } \bigcup \mathcal{U} \in T$

$\langle 2 \rangle 4. \bigcup \mathcal{U} \in \bigcap \mathcal{T}$

$\langle 1 \rangle 3. \bigcap \mathcal{T}$  is closed under binary intersection.

$\langle 2 \rangle 1. \text{ LET: } U, V \in \bigcap \mathcal{T}$

$\langle 2 \rangle 2. \text{ For all } T \in \mathcal{T} \text{ we have } U, V \in T$

$\langle 2 \rangle 3. \text{ For all } T \in \mathcal{T} \text{ we have } U \cap V \in T$

$\langle 2 \rangle 4. U \cap V \in \bigcap \mathcal{T}$

□

**Lemma 12.** *Let  $X$  be a set and  $\mathcal{T}$  a set of topologies on  $X$ . Then there exists a unique coarsest topology that is finer than every member of  $\mathcal{T}$ .*

PROOF: The required topology is given by

$$\bigcap \{T \in \mathcal{P}X \mid T \text{ is a topology on } X \text{ that is finer than every member of } \mathcal{T}\},$$

The set is nonempty since it contains the discrete topology. □

## 4 Basis for a Topology

**Definition 13** (Basis). If  $X$  is a set, a *basis* for a topology on  $X$  is a set  $\mathcal{B} \subseteq \mathcal{P}X$  called *basis elements* such that

1. For all  $x \in X$ , there exists  $B \in \mathcal{B}$  such that  $x \in B$ .

2. For all  $B_1, B_2 \in \mathcal{B}$  and  $x \in B_1 \cap B_2$ , there exists  $B_3 \in \mathcal{B}$  such that  $x \in B_3 \subseteq B_1 \cap B_2$ .

If  $\mathcal{B}$  satisfies these two conditions, then we define the topology *generated* by  $\mathcal{B}$  to be  $\mathcal{T} = \{U \in \mathcal{P}X \mid \forall x \in U. \exists B \in \mathcal{B}. x \in B \subseteq U\}$ .

We prove this is a topology.

PROOF:

$\langle 1 \rangle 1.$   $X \in \mathcal{T}$

PROOF: For all  $x \in X$  there exists  $B \in \mathcal{B}$  such that  $x \in B \subseteq X$  by condition 1.

$\langle 1 \rangle 2.$  For all  $\mathcal{U} \subseteq \mathcal{T}$  we have  $\bigcup \mathcal{U} \in \mathcal{T}$

$\langle 2 \rangle 1.$  LET:  $\mathcal{U} \subseteq \mathcal{T}$

$\langle 2 \rangle 2.$  LET:  $x \in \bigcup \mathcal{U}$

$\langle 2 \rangle 3.$  PICK  $U \in \mathcal{U}$  such that  $x \in U$

$\langle 2 \rangle 4.$  PICK  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$

PROOF: Since  $U \in \mathcal{T}$  by  $\langle 2 \rangle 1$  and  $\langle 2 \rangle 3$ .

$\langle 2 \rangle 5.$   $x \in B \subseteq \bigcup \mathcal{U}$

$\langle 1 \rangle 3.$  For all  $U, V \in \mathcal{T}$  we have  $U \cap V \in \mathcal{T}$

$\langle 2 \rangle 1.$  LET:  $U, V \in \mathcal{T}$

$\langle 2 \rangle 2.$  LET:  $x \in U \cap V$

$\langle 2 \rangle 3.$  PICK  $B_1 \in \mathcal{B}$  such that  $x \in B_1 \subseteq U$

$\langle 2 \rangle 4.$  PICK  $B_2 \in \mathcal{B}$  such that  $x \in B_2 \subseteq V$

$\langle 2 \rangle 5.$  PICK  $B_3 \in \mathcal{B}$  such that  $x \in B_3 \subseteq B_1 \cap B_2$

PROOF: By condition 2.

$\langle 2 \rangle 6.$   $x \in B_3 \subseteq U \cap V$

□

**Lemma 14.** *Let  $X$  be a set. Let  $\mathcal{B}$  be a basis for a topology  $\mathcal{T}$  on  $X$ . Then  $\mathcal{T}$  is the set of all unions of subsets of  $\mathcal{B}$ .*

PROOF:

$\langle 1 \rangle 1.$  For all  $U \in \mathcal{T}$ , there exists  $\mathcal{A} \subseteq \mathcal{B}$  such that  $U = \bigcup \mathcal{A}$

$\langle 2 \rangle 1.$  LET:  $U \in \mathcal{T}$

$\langle 2 \rangle 2.$  LET:  $\mathcal{A} = \{B \in \mathcal{B} \mid B \subseteq U\}$

$\langle 2 \rangle 3.$   $U \subseteq \bigcup \mathcal{A}$

$\langle 3 \rangle 1.$  LET:  $x \in U$

$\langle 3 \rangle 2.$  PICK  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$

PROOF: Since  $\mathcal{B}$  is a basis for  $\mathcal{T}$ .

$\langle 3 \rangle 3.$   $x \in B \in \mathcal{A}$

$\langle 2 \rangle 4.$   $\bigcup \mathcal{A} \subseteq U$

PROOF: From the definition of  $\mathcal{A}$  ( $\langle 2 \rangle 2$ ).

$\langle 1 \rangle 2.$  For all  $\mathcal{A} \subseteq \mathcal{B}$  we have  $\bigcup \mathcal{A} \in \mathcal{T}$

$\langle 2 \rangle 1.$   $\bigcup \mathcal{A} \in \mathcal{T}$

PROOF: If  $B \in \mathcal{B}$  and  $x \in B$ , then there exists  $B' \in \mathcal{B}$  such that  $x \in B' \subseteq B$ , namely  $B' = B$ .

$\langle 2 \rangle 2.$  Q.E.D.

PROOF: Since  $\mathcal{T}$  is closed under union.

□

**Corollary 14.1.** *Let  $X$  be a set. Let  $\mathcal{B}$  be a basis for a topology  $\mathcal{T}$  on  $X$ . Then  $\mathcal{T}$  is the coarsest topology that includes  $\mathcal{B}$ .*

PROOF: Since every topology that includes  $\mathcal{B}$  includes all unions of subsets of  $\mathcal{B}$ . □

**Lemma 15.** *Let  $X$  be a topological space. Suppose that  $\mathcal{C}$  is a set of open sets such that, for every open set  $U$  and every point  $x \in U$ , there exists  $C \in \mathcal{C}$  such that  $x \in C \subseteq U$ . Then  $\mathcal{C}$  is a basis for the topology on  $X$ .*

PROOF:

⟨1⟩1. For all  $x \in X$ , there exists  $C \in \mathcal{C}$  such that  $x \in C$

PROOF: Immediate from hypothesis.

⟨1⟩2. For all  $C_1, C_2 \in \mathcal{C}$  and  $x \in C_1 \cap C_2$ , there exists  $C_3 \in \mathcal{C}$  such that  $x \in C_3 \subseteq C_1 \cap C_2$

PROOF: Since  $C_1 \cap C_2$  is open.

⟨1⟩3. Every open set is open in the topology generated by  $\mathcal{C}$

PROOF: Immediate from hypothesis.

⟨1⟩4. Every union of a subset of  $\mathcal{C}$  is open.

PROOF: Since every member of  $\mathcal{C}$  is open.

□

**Lemma 16.** *Let  $\mathcal{B}$  and  $\mathcal{B}'$  be bases for the topologies  $\mathcal{T}$  and  $\mathcal{T}'$  respectively on the set  $X$ . Then the following are equivalent.*

1.  $\mathcal{T} \subseteq \mathcal{T}'$

2. For all  $B \in \mathcal{B}$  and  $x \in B$ , there exists  $B' \in \mathcal{B}'$  such that  $x \in B' \subseteq B$ .

PROOF:

⟨1⟩1.  $1 \Rightarrow 2$

⟨2⟩1. ASSUME:  $\mathcal{T} \subseteq \mathcal{T}'$

⟨2⟩2. LET:  $B \in \mathcal{B}$  and  $x \in B$

⟨2⟩3.  $B \in \mathcal{T}$

PROOF: Corollary 14.1.

⟨2⟩4.  $B \in \mathcal{T}'$

PROOF: By ⟨2⟩1

⟨2⟩5. There exists  $B' \in \mathcal{B}'$  such that  $x \in B' \subseteq B$

PROOF: Since  $\mathcal{B}'$  is a basis for  $\mathcal{T}'$ .

⟨1⟩2.  $2 \Rightarrow 1$

⟨2⟩1. ASSUME: 2

⟨2⟩2. LET:  $U \in \mathcal{T}$

PROVE:  $U \in \mathcal{T}'$

⟨2⟩3. LET:  $x \in U$

PROVE: There exists  $B' \in \mathcal{B}'$  such that  $x \in B' \subseteq U$

⟨2⟩4. PICK  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$

PROOF: Since  $\mathcal{B}$  is a basis for  $\mathcal{T}$ .

$\langle 2 \rangle 5$ . PICK  $B' \in \mathcal{B}'$  such that  $x \in B' \subseteq B$

PROOF: By  $\langle 2 \rangle 1$ .

$\langle 2 \rangle 6$ .  $x \in B' \subseteq U$

□

**Definition 17** (Lower Limit Topology on the Real Line). The *lower limit topology on the real line* is the topology on  $\mathbb{R}$  generated by the basis consisting of all half-open intervals of the form  $[a, b)$ .

We write  $\mathbb{R}_l$  for the topological space  $\mathbb{R}$  under the lower limit topology.

We prove this is a basis for a topology.

PROOF:

$\langle 1 \rangle 1$ . For all  $x \in \mathbb{R}$  there exists an interval  $[a, b)$  such that  $x \in [a, b)$ .

PROOF: Take  $[a, b) = [x, x + 1)$ .

$\langle 1 \rangle 2$ . For any open intervals  $[a, b)$ ,  $[c, d)$  if  $x \in [a, b) \cap [c, d)$ , then there exists an interval  $[e, f)$  such that  $x \in [e, f) \subseteq [a, b) \cap [c, d)$

PROOF: Take  $[e, f) = [\max(a, c), \min(b, d))$ .

□

**Definition 18** ( $K$ -topology on the Real Line). Let  $K = \{1/n \mid n \in \mathbb{Z}^+\}$ .

The  *$K$ -topology on the real line* is the topology on  $\mathbb{R}$  generated by the basis consisting of all open intervals  $(a, b)$  and all sets of the form  $(a, b) \setminus K$ .

We write  $\mathbb{R}_K$  for the topological space  $\mathbb{R}$  under the  $K$ -topology.

We prove this is a basis for a topology.

PROOF:

$\langle 1 \rangle 1$ . For all  $x \in \mathbb{R}$  there exists an open interval  $(a, b)$  such that  $x \in (a, b)$ .

PROOF: Take  $(a, b) = (x - 1, x + 1)$ .

$\langle 1 \rangle 2$ . For any basic open sets  $B_1, B_2$  if  $x \in B_1 \cap B_2$ , then there exists a basic open set  $B_3$  such that  $x \in B_3 \subseteq B_1 \cap B_2$

$\langle 2 \rangle 1$ . CASE:  $B_1 = (a, b)$ ,  $B_2 = (c, d)$

PROOF: Take  $B_3 = (\max(a, c), \min(b, d))$ .

$\langle 2 \rangle 2$ . CASE:  $B_1 = (a, b)$  or  $(a, b) \setminus K$ ,  $B_2 = (c, d)$  or  $(c, d) \setminus K$ , and they are not both open intervals.

PROOF: Take  $B_3 = (\max(a, c), \min(b, d)) \setminus K$ .

□

**Lemma 19.** *The lower limit topology and the  $K$ -topology are incomparable.*

PROOF:

$\langle 1 \rangle 1$ . The interval  $[10, 11)$  is not open in the  $K$ -topology.

PROOF: There is no open interval  $(a, b)$  such that  $10 \in (a, b) \subseteq [10, 11)$  or  $10 \in (a, b) \setminus K \subseteq [10, 11)$ .

$\langle 1 \rangle 2$ . The set  $(-1, 1) \setminus K$  is not open in the lower limit topology.

PROOF: There is no half-open interval  $[a, b)$  such that  $0 \in [a, b) \subseteq (-1, 1) \setminus K$ , since there must be a positive integer  $n$  with  $1/n \in [a, b)$ .

□

**Definition 20** (Subbasis). A *subbasis*  $\mathcal{S}$  for a topology on  $X$  is a set  $\mathcal{S} \subseteq \mathcal{P}X$  such that  $\bigcup \mathcal{S} = X$ .

The topology *generated* by the subbasis  $\mathcal{S}$  is the set of all unions of finite intersections of elements of  $\mathcal{S}$ .

We prove this is a topology.

PROOF:

⟨1⟩1. The set  $\mathcal{B}$  of all finite intersections of elements of  $\mathcal{S}$  forms a basis for a topology on  $X$ .

⟨2⟩1.  $\bigcup \mathcal{B} = X$

PROOF: Since  $\mathcal{S} \subseteq \mathcal{B}$ .

⟨2⟩2.  $\mathcal{B}$  is closed under binary intersection.

PROOF: By definition.

⟨1⟩2. Q.E.D.

PROOF: By Lemma 14.

□

We have simultaneously proved:

**Proposition 21.** *Let  $\mathcal{S}$  be a subbasis for the topology on  $X$ . Then the set of all finite intersections of elements of  $\mathcal{S}$  is a basis for the topology on  $X$ .*

**Proposition 22.** *Let  $X$  be a set. Let  $\mathcal{S}$  be a subbasis for a topology  $\mathcal{T}$  on  $X$ . Then  $\mathcal{T}$  is the coarsest topology that includes  $\mathcal{S}$ .*

PROOF: Since every topology that includes  $\mathcal{S}$  includes every union of finite intersections of elements of  $\mathcal{S}$ . □

## 5 Open Maps

**Definition 23** (Open Map). Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Then  $f$  is an *open map* if and only if, for every open set  $U$  in  $X$ , the set  $f(U)$  is open in  $Y$ .

**Lemma 24.** *Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Let  $\mathcal{B}$  be a basis for the topology on  $X$ . If  $f(B)$  is open in  $Y$  for all  $B \in \mathcal{B}$ , then  $f$  is an open map.*

PROOF: From Lemma 14. □

## 6 The Order Topology

**Definition 25** (Order Topology). Let  $X$  be a linearly ordered set with at least two points. The *order topology* on  $X$  is the topology generated by the basis  $\mathcal{B}$  consisting of:

- all open intervals  $(a, b)$ ;
- all intervals of the form  $[\perp, b)$  where  $\perp$  is least in  $X$ ;
- all intervals of the form  $(a, \top]$  where  $\top$  is greatest in  $X$ .

We prove this is a basis for a topology.

PROOF:

- ⟨1⟩1. For all  $x \in X$  there exists  $B \in \mathcal{B}$  such that  $x \in B$ .
  - ⟨2⟩1. LET:  $x \in X$
  - ⟨2⟩2. CASE:  $x$  is greatest in  $X$ .
    - ⟨3⟩1. PICK  $y \in X$  with  $y \neq x$
    - ⟨3⟩2.  $x \in (y, x] \in \mathcal{B}$
  - ⟨2⟩3. CASE:  $x$  is least in  $X$ .
    - ⟨3⟩1. PICK  $y \in X$  with  $y \neq x$
    - ⟨3⟩2.  $x \in [x, y) \in \mathcal{B}$
  - ⟨2⟩4. CASE:  $x$  is neither greatest nor least in  $X$ .
    - ⟨3⟩1. PICK  $a, b \in X$  with  $a < x$  and  $x < b$
    - ⟨3⟩2.  $x \in (a, b) \in \mathcal{B}$
- ⟨1⟩2. For all  $B_1, B_2 \in \mathcal{B}$  and  $x \in B_1 \cap B_2$ , there exists  $B_3 \in \mathcal{B}$  such that  $x \in B_3 \subseteq B_1 \cap B_2$ 
  - ⟨2⟩1. LET:  $B_1, B_2 \in \mathcal{B}$  and  $x \in B_1 \cap B_2$
  - ⟨2⟩2. CASE:  $B_1 = (a, b), B_2 = (c, d)$   
 PROOF: Take  $B_3 = (\max(a, c), \min(b, d))$ .
  - ⟨2⟩3. CASE:  $B_1 = (a, b), B_2 = [\perp, d)$   
 PROOF: Take  $B_3 = (a, \min(b, d))$ .
  - ⟨2⟩4. CASE:  $B_1 = (a, b), B_2 = (c, \top]$   
 PROOF: Take  $B_3 = (\max(a, c), b)$ .
  - ⟨2⟩5. CASE:  $B_1 = [\perp, b), B_2 = [\perp, d)$   
 PROOF: Take  $B_3 = [\perp, \min(b, d))$ .
  - ⟨2⟩6. CASE:  $B_1 = [\perp, b), B_2 = (c, \top]$   
 PROOF: Take  $B_3 = (c, b)$ .

□

**Lemma 26.** *Let  $X$  be a linearly ordered set. Then the open rays form a subbasis for the order topology on  $X$ .*

PROOF:

- ⟨1⟩1. Every open ray is open.
  - ⟨2⟩1. For all  $a \in X$ , the ray  $(-\infty, a)$  is open.
    - ⟨3⟩1. LET:  $x \in (-\infty, a)$
    - ⟨3⟩2. CASE:  $x$  is least in  $X$   
 PROOF:  $x \in [x, a) = (-\infty, a)$ .
    - ⟨3⟩3. CASE:  $x$  is not least in  $X$ 
      - ⟨4⟩1. PICK  $y < x$
      - ⟨4⟩2.  $x \in (y, a) \subseteq (-\infty, a)$
  - ⟨2⟩2. For all  $a \in X$ , the ray  $(a, +\infty)$  is open.

PROOF: Similar.

⟨1⟩2. Every basic open set is a finite intersection of open rays.

PROOF: We have  $(a, b) = (a, +\infty) \cap (-\infty, b)$ ,  $[\perp, b) = (-\infty, b)$  and  $(a, \top] = (a, +\infty)$ .

□

**Definition 27** (Standard Topology on the Real Line). The *standard topology on the real line* is the order topology on  $\mathbb{R}$  generated by the standard order.

**Lemma 28.** *The lower limit topology is strictly finer than the standard topology on  $\mathbb{R}$ .*

PROOF:

⟨1⟩1. Every open interval is open in the lower limit topology.

PROOF: If  $x \in (a, b)$  then  $x \in [x, b) \subseteq (a, b)$ .

⟨1⟩2. The half-open interval  $[0, 1)$  is not open in the standard topology.

PROOF: There is no open interval  $(a, b)$  such that  $0 \in (a, b) \subseteq [0, 1)$ .

□

**Lemma 29.** *The  $K$ -topology is strictly finer than the standard topology on  $\mathbb{R}$ .*

PROOF:

⟨1⟩1. Every open interval is open in the  $K$ -topology.

PROOF: Corollary 14.1.

⟨1⟩2. The set  $(-1, 1) \setminus K$  is not open in the standard topology.

PROOF: There is no open interval  $(a, b)$  such that  $0 \in (a, b) \subseteq (-1, 1) \setminus K$ , since there must be a positive integer  $n$  with  $1/n \in (a, b)$ .

□

**Definition 30** (Ordered Square). The *ordered square*  $I_o^2$  is the set  $[0, 1]^2$  under the order topology generated by the dictionary order.

## 7 The Product Topology

**Definition 31** (Product Topology). Let  $\{A_i\}_{i \in I}$  be a family of topological spaces. The *product topology* on  $\prod_{i \in I} A_i$  is the topology generated by the sub-basis consisting of the sets of the form  $\pi_i^{-1}(U)$  where  $i \in I$  and  $U$  is open in  $A_i$ .

**Proposition 32.** *The product topology on  $\prod_{i \in I} A_i$  is generated by the basis consisting of all sets of the form  $\prod_{i \in I} U_i$  where  $\{U_i\}_{i \in I}$  is a family such that each  $U_i$  is an open set in  $A_i$  and  $U_i = A_i$  for all but finitely many  $i$ .*

PROOF: From Proposition 21. □

**Proposition 33.** *Let  $\{A_i\}_{i \in I}$  be a family of topological spaces. For  $i \in I$ , let  $\mathcal{B}_i$  be a basis for the topology on  $A_i$ . Then  $\mathcal{B} = \{\prod_{i \in I} B_i \mid \forall i \in I, B_i \in \mathcal{B}_i, B_i = A_i \text{ for all but finitely many } i\}$  is a basis for the box topology on  $\prod_{i \in I} A_i$ .*



PROOF:

- ⟨1⟩1. Every set in  $\mathcal{B}$  is open.
- ⟨1⟩2. For every point  $a \in \prod_{i \in I} A_i$  and every open set  $U$  with  $a \in U$ , there exists  $B \in \mathcal{B}$  such that  $a \in B \subseteq U$ .
  - ⟨2⟩1. LET:  $U$  be open and  $a \in U$
  - ⟨2⟩2. PICK a family  $\{U_i\}_{i \in I}$  such that each  $U_i$  is open in  $A_i$ , such that  $U_i = A_i$  except for  $i = i_1, \dots, i_n$ , and such that  $a \in \prod_{i \in I} U_i \subseteq U$ .
  - ⟨2⟩3. For  $j = 1, \dots, n$ , PICK  $B_{i_j} \in \mathcal{B}_{i_j}$  such that  $a_{i_j} \in B_{i_j} \subseteq U_{i_j}$
  - ⟨2⟩4. LET:  $B = \prod_{i \in I} B_i$  where  $B_i = A_i$  for  $i \neq i_1, \dots, i_n$
  - ⟨2⟩5.  $B \in \mathcal{B}$
  - ⟨2⟩6.  $a \in B \subseteq U$
- ⟨1⟩3. Q.E.D.

PROOF: Lemma 15.

□

**Proposition 34.** *Let  $\{A_i\}_{i \in I}$  be a family of topological spaces. Then the projections  $\pi_i : \prod_{i \in I} A_i \rightarrow A_i$  are open maps.*

PROOF: From Lemma 24. □

**Proposition 35.** *Let  $\{X_i\}_{i \in I}$  be a family of sets. For  $i \in I$ , let  $\mathcal{T}_i$  and  $\mathcal{U}_i$  be topologies on  $X_i$ . Let  $\mathcal{P}$  be the product topology on  $\prod_{i \in I} X_i$  generated by the topologies  $\mathcal{T}_i$ , and  $\mathcal{Q}$  the product topology on the same set generated by the topologies  $\mathcal{U}_i$ . Then  $\mathcal{P} \subseteq \mathcal{Q}$  if and only if  $\mathcal{T}_i \subseteq \mathcal{U}_i$  for all  $i$ .*

PROOF:

- ⟨1⟩1. If  $\mathcal{T}_i \subseteq \mathcal{U}_i$  for all  $i$  then  $\mathcal{P} \subseteq \mathcal{Q}$   
 PROOF: By Corollary 14.1.
- ⟨1⟩2. If  $\mathcal{P} \subseteq \mathcal{Q}$  then  $\mathcal{T}_i \subseteq \mathcal{U}_i$  for all  $i$ 
  - ⟨2⟩1. ASSUME:  $\mathcal{P} \subseteq \mathcal{Q}$
  - ⟨2⟩2. LET:  $i \in I$
  - ⟨2⟩3. LET:  $U \in \mathcal{T}_i$
  - ⟨2⟩4. LET:  $U_i = U$  and  $U_j = X_j$  for  $j \neq i$
  - ⟨2⟩5.  $\prod_{i \in I} U_i \in \mathcal{P}$
  - ⟨2⟩6.  $\prod_{i \in I} U_i \in \mathcal{Q}$
  - ⟨2⟩7.  $U \in \mathcal{U}_i$

PROOF: From Proposition 34.

□

## 8 The Subspace Topology

**Definition 36** (Subspace Topology). Let  $X$  be a topological space and  $Y \subseteq X$ . The *subspace topology* on  $Y$  is  $\mathcal{T} = \{U \cap Y \mid U \text{ is open in } X\}$ .

We prove this is a topology.

PROOF:

- ⟨1⟩1.  $Y \in \mathcal{T}$

PROOF: Since  $Y = X \cap Y$

$\langle 1 \rangle 2$ . For all  $\mathcal{U} \subseteq \mathcal{T}$ , we have  $\bigcup \mathcal{U} \in \mathcal{T}$

$\langle 2 \rangle 1$ . LET:  $\mathcal{U} \subseteq \mathcal{T}$

$\langle 2 \rangle 2$ . LET:  $\mathcal{V} = \{V \text{ open in } X \mid V \cap Y \in \mathcal{U}\}$

$\langle 2 \rangle 3$ .  $\bigcup \mathcal{U} = (\bigcup \mathcal{V}) \cap Y$

$\langle 1 \rangle 3$ . For all  $U, V \in \mathcal{T}$ , we have  $U \cap V \in \mathcal{T}$

$\langle 2 \rangle 1$ . LET:  $U, V \in \mathcal{T}$

$\langle 2 \rangle 2$ . PICK  $U', V'$  open in  $X$  such that  $U = U' \cap Y$  and  $V = V' \cap Y$

$\langle 2 \rangle 3$ .  $(U \cap V) = (U' \cap V') \cap Y$

□

**Lemma 37.** *Let  $X$  be a topological space and  $Y \subseteq X$ . Let  $\mathcal{B}$  be a basis for the topology on  $X$ . Then  $\mathcal{B}' = \{B \cap Y \mid B \in \mathcal{B}\}$  is a basis for the subspace topology on  $Y$ .*

PROOF:

$\langle 1 \rangle 1$ . Every element in  $\mathcal{B}'$  is open in  $Y$

$\langle 1 \rangle 2$ . For every open set  $U$  in  $Y$  and point  $y \in U$ , there exists  $B' \in \mathcal{B}'$  such that  $y \in B' \subseteq U$

$\langle 2 \rangle 1$ . LET:  $U$  be open in  $Y$  and  $y \in U$

$\langle 2 \rangle 2$ . PICK  $V$  open in  $X$  such that  $U = V \cap Y$

$\langle 2 \rangle 3$ . PICK  $B \in \mathcal{B}$  such that  $y \in B \subseteq V$

$\langle 2 \rangle 4$ . LET:  $B' = B \cap Y$

$\langle 2 \rangle 5$ .  $B' \in \mathcal{B}'$

$\langle 2 \rangle 6$ .  $y \in B' \subseteq U$

$\langle 1 \rangle 3$ . Q.E.D.

PROOF: By Lemma 15.

□

**Lemma 38.** *Let  $X$  be a topological space and  $Y \subseteq X$ . Let  $\mathcal{S}$  be a basis for the topology on  $X$ . Then  $\mathcal{S}' = \{S \cap Y \mid S \in \mathcal{S}\}$  is a subbasis for the subspace topology on  $Y$ .*

PROOF: The set  $\{B \cap Y \mid B \text{ is a finite intersection of elements of } \mathcal{S}\}$  is a basis for the subspace topology by Lemma 37, and this is the set of all finite intersections of elements of  $\mathcal{S}'$ . □

**Lemma 39.** *Let  $Y$  be a subspace of  $X$ . If  $U$  is open in  $Y$  and  $Y$  is open in  $X$  then  $U$  is open in  $X$ .*

PROOF:

$\langle 1 \rangle 1$ . PICK  $V$  open in  $X$  such that  $U = V \cap Y$

$\langle 1 \rangle 2$ .  $U$  is open in  $X$

PROOF: Since it is the intersection of two open sets  $V$  and  $Y$ .

□

**Theorem 40.** *Let  $\{X_i\}_{i \in I}$  be a family of topological spaces. Let  $A_i$  be a subspace of  $X_i$  for all  $i \in I$ . Then the product topology on  $\prod_{i \in I} A_i$  is the same as the topology it inherits as a subspace of  $\prod_{i \in I} X_i$ .*

PROOF: The product topology is generated by the subbasis

$$\begin{aligned} & \{\pi_i^{-1}(U) \mid i \in I, U \text{ open in } A_i\} \\ &= \{\pi_i^{-1}(V) \cap A_i \mid i \in I, V \text{ open in } X_i\} \\ &= \{\pi_i^{-1}(V) \mid i \in I, V \text{ open in } X_i\} \cap \prod_{i \in I} A_i \end{aligned}$$

and this is a subbasis for the subspace topology by Lemma 38.  $\square$

**Theorem 41.** *Let  $X$  be an ordered set in the order topology. Let  $Y \subseteq X$  be convex. Then the order topology on  $Y$  is the same as the subspace topology on  $Y$ .*

PROOF:

$\langle 1 \rangle 1$ . The order topology is finer than the subspace topology.

$\langle 2 \rangle 1$ . For every open ray  $R$  in  $X$ , the set  $R \cap Y$  is open in the order topology.

$\langle 3 \rangle 1$ . For all  $a \in X$ , we have  $(-\infty, a) \cap Y$  is open in the order topology.

$\langle 4 \rangle 1$ . CASE: For all  $y \in Y$  we have  $y < a$

PROOF: In this case  $(-\infty, a) \cap Y = Y$ .

$\langle 4 \rangle 2$ . CASE: For all  $y \in Y$  we have  $a < y$

PROOF: In this case  $(-\infty, a) \cap Y = \emptyset$ .

$\langle 4 \rangle 3$ . CASE: There exists  $y \in Y$  such that  $y \leq a$  and  $y \in Y$  such that  $a \leq y$

$\langle 5 \rangle 1$ .  $a \in Y$

PROOF: By convexity.

$\langle 5 \rangle 2$ .  $(-\infty, a) \cap Y = \{y \in Y \mid y < a\}$

$\langle 3 \rangle 2$ . For all  $a \in X$ , we have  $(a, +\infty) \cap Y$  is open in the order topology.

PROOF: Similar.

$\langle 2 \rangle 2$ . Q.E.D.

PROOF: By Lemmas 26 and 38 and Proposition 22.

$\langle 1 \rangle 2$ . The subspace topology is finer than the order topology.

$\langle 2 \rangle 1$ . Every open ray in  $Y$  is open in the subspace topology.

PROOF: For any  $a \in Y$  we have  $(-\infty, a)_Y = (-\infty, a)_X \cap Y$  and  $(a, +\infty)_Y = (a, +\infty)_X \cap Y$ .

$\langle 2 \rangle 2$ . Q.E.D.

PROOF: By Lemma 26 and Proposition 22

$\square$

**Proposition 42.** *The order topology on  $I_o^2$  is different from the subspace topology as a subspace of  $\mathbb{R}^2$  under the dictionary order topology.*

PROOF: The set  $\{1/2\} \times (1/2, 1)$  is open in the subspace topology but not in the order topology.  $\square$

**Proposition 43.** *Let  $X$  be a topological space,  $Y$  a subspace of  $X$ , and  $Z$  a subspace of  $Y$ . Then the subspace topology on  $Z$  inherited from  $X$  is the same as the subspace topology on  $Z$  inherited from  $Y$ .*

PROOF: The subspace topology inherited from  $Y$  is

$$\begin{aligned} & \{V \cap Z \mid V \text{ open in } Y\} \\ &= \{U \cap Y \cap Z \mid U \text{ open in } X\} \\ &= \{U \cap Z \mid U \text{ open in } X\} \end{aligned}$$

which is the subspace topology inherited from  $X$ .  $\square$

**Definition 44** (Unit Circle). The *unit circle*  $S^1$  is  $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$

## 9 Closed Set

**Definition 45** (Closed Set). Let  $X$  be a topological space and  $A \subseteq X$ . Then  $A$  is *closed* if and only if  $X \setminus A$  is open.

**Lemma 46.** *The empty set is closed.*

PROOF: Since the whole space  $X$  is always open.  $\square$

**Lemma 47.** *The topological space  $X$  is closed.*

PROOF: Since  $\emptyset$  is open.  $\square$

**Lemma 48.** *The intersection of a nonempty set of closed sets is closed.*

PROOF: Let  $\mathcal{C}$  be a nonempty set of closed sets. Then  $X \setminus \bigcap \mathcal{C} = \bigcup \{X \setminus C \mid C \in \mathcal{C}\}$  is open.  $\square$

**Lemma 49.** *The union of two closed sets is closed.*

PROOF: Let  $C$  and  $D$  be closed. Then  $X \setminus (C \cup D) = (X \setminus C) \cap (X \setminus D)$  is open.  $\square$

**Theorem 50.** *Let  $X$  be a topological space and  $Y$  a subspace of  $X$ . Let  $A \subseteq Y$ . Then  $A$  is closed in  $Y$  if and only if there exists a closed set  $C$  in  $X$  such that  $A = C \cap Y$ .*

PROOF: We have

$$\begin{aligned} & A \text{ is closed in } Y \\ \Leftrightarrow & Y \setminus A \text{ is open in } Y \\ \Leftrightarrow & \exists U \text{ open in } X. Y \setminus A = Y \cap U \\ \Leftrightarrow & \exists C \text{ closed in } X. Y \setminus A = Y \cap (X \setminus U) \\ \Leftrightarrow & \exists C \text{ closed in } X. A = Y \cap C \end{aligned} \quad \square$$

**Theorem 51.** *Let  $Y$  be a subspace of  $X$  and  $A \subseteq Y$ . If  $A$  is closed in  $Y$  and  $Y$  is closed in  $X$  then  $A$  is closed in  $X$ .*

PROOF: Pick a closed set  $C$  in  $X$  such that  $A = C \cap Y$  (Theorem 50). Then  $A$  is the intersection of two sets closed in  $X$ , hence  $A$  is closed in  $X$  (Lemma 48).  $\square$

**Proposition 52.** Let  $X$  be a set and  $\mathcal{C} \subseteq \mathcal{P}X$  a set such that:

1.  $\emptyset \in \mathcal{C}$
2.  $X \in \mathcal{C}$
3. For all  $\mathcal{A} \subseteq \mathcal{C}$  nonempty we have  $\bigcap \mathcal{A} \in \mathcal{C}$
4. For all  $C, D \in \mathcal{C}$  we have  $C \cup D \in \mathcal{C}$ .

Then there exists a unique topology  $\mathcal{T}$  such that  $\mathcal{C}$  is the set of closed sets, namely

$$\mathcal{T} = \{X \setminus C \mid C \in \mathcal{C}\}$$

PROOF:

$\langle 1 \rangle 1.$  LET:  $\mathcal{T} = \{X \setminus C \mid C \in \mathcal{C}\}$

$\langle 1 \rangle 2.$   $\mathcal{T}$  is a topology

$\langle 2 \rangle 1.$   $X \in \mathcal{T}$

PROOF: Since  $\emptyset \in \mathcal{C}$

$\langle 2 \rangle 2.$  For all  $\mathcal{U} \subseteq \mathcal{T}$  we have  $\bigcup \mathcal{U} \in \mathcal{T}$

$\langle 3 \rangle 1.$  LET:  $\mathcal{U} \subseteq \mathcal{T}$

$\langle 3 \rangle 2.$  CASE:  $\mathcal{U} = \emptyset$

PROOF: In this case  $\bigcup \mathcal{U} = \emptyset \in \mathcal{T}$  since  $X \in \mathcal{C}$

$\langle 3 \rangle 3.$  CASE:  $\mathcal{U} \neq \emptyset$

PROOF: In this case  $X \setminus \bigcup \mathcal{U} = \bigcap \{X \setminus U \mid U \in \mathcal{U}\} \in \mathcal{C}$ .

$\langle 2 \rangle 3.$  For all  $U, V \in \mathcal{T}$  we have  $U \cap V \in \mathcal{T}$

PROOF: Since  $X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V) \in \mathcal{C}$ .

$\langle 1 \rangle 3.$   $\mathcal{C}$  is the set of all closed sets in  $\mathcal{T}$

PROOF:

$C$  is closed in  $\mathcal{T}$

$$\Leftrightarrow X \setminus C \in \mathcal{T}$$

$$\Leftrightarrow C \in \mathcal{C}$$

$\langle 1 \rangle 4.$  If  $\mathcal{T}'$  is a topology and  $\mathcal{C}$  is the set of closed sets in  $\mathcal{T}'$  then  $\mathcal{T}' = \mathcal{T}$

PROOF: We have

$$U \in \mathcal{T}$$

$$\Leftrightarrow X \setminus U \in \mathcal{C}$$

$$\Leftrightarrow X \setminus U \text{ is closed in } \mathcal{T}'$$

$$\Leftrightarrow U \in \mathcal{T}'$$

□

**Proposition 53.** If  $A_i$  is closed in  $X_i$  for all  $i \in I$  then  $\prod_{i \in I} A_i$  is closed in  $\prod_{i \in I} X_i$ .

PROOF:

$$\left( \prod_{i \in I} X_i \right) \setminus \left( \prod_{i \in I} A_i \right) = \bigcup_{j \in I} \left( \prod_{i \in I} X_i \setminus \pi_j^{-1}(A_j) \right) \square$$

**Proposition 54.** If  $U$  is open and  $A$  is closed then  $U \setminus A$  is open.

PROOF:  $U \setminus A = U \cap (X \setminus A)$  is the intersection of two open sets.  $\square$

**Proposition 55.** *If  $U$  is open and  $A$  is closed then  $A \setminus U$  is closed.*

PROOF:  $A \setminus U = A \cap (X \setminus U)$  is the intersection of two closed sets.  $\square$

## 10 Interior

**Definition 56** (Interior). Let  $X$  be a topological space and  $A \subseteq X$ . The *interior* of  $A$ ,  $\text{Int } A$ , is the union of all the open subsets of  $A$ .

**Lemma 57.** *The interior of a set is open.*

PROOF: It is a union of open sets.  $\square$

**Lemma 58.**

$$\text{Int } A \subseteq A$$

PROOF: Immediate from definition.  $\square$

**Lemma 59.** *A set  $A$  is open if and only if  $A = \text{Int } A$ .*

PROOF: If  $A = \text{Int } A$  then  $A$  is open by Lemma 57. Conversely if  $A$  is open then  $A \subseteq \text{Int } A$  by the definition of interior and so  $A = \text{Int } A$ .

## 11 Neighbourhood

**Definition 60** (Neighbourhood). A *neighbourhood* of a point  $x$  is an open set that contains  $x$ .

## 12 Closure

**Definition 61** (Closure). Let  $X$  be a topological space and  $A \subseteq X$ . The *closure* of  $A$ ,  $\overline{A}$ , is the intersection of all the closed sets that include  $A$ .

This intersection exists since  $X$  is a closed set that includes  $A$  (Lemma 47).

**Lemma 62.** *The closure of a set is closed.*

PROOF: Dual to Lemma 57.  $\square$

**Lemma 63.**

$$A \subseteq \overline{A}$$

PROOF: Immediate from definition.  $\square$

**Lemma 64.** *A set  $A$  is closed if and only if  $A = \overline{A}$ .*

PROOF: Dual to Lemma 59.  $\square$

**Theorem 65.** Let  $Y$  be a subspace of  $X$ . Let  $A \subseteq Y$ . Let  $\bar{A}$  be the closure of  $A$  in  $X$ . Then the closure of  $A$  in  $Y$  is  $\bar{A} \cap Y$ .

PROOF: The closure of  $A$  in  $Y$  is

$$\begin{aligned} & \bigcap \{C \text{ closed in } Y \mid A \subseteq C\} \\ &= \bigcap \{D \cap Y \mid D \text{ closed in } X, A \subseteq D \cap Y\} \quad (\text{Theorem 50}) \\ &= \bigcap \{D \mid D \text{ closed in } X, A \subseteq D\} \cap Y \\ &= \bar{A} \cap Y \quad \square \end{aligned}$$

**Theorem 66.** Let  $X$  be a topological space,  $A \subseteq X$  and  $x \in X$ . Then  $x \in \bar{A}$  if and only if every neighbourhood of  $x$  intersects  $A$ .

PROOF: We have

$$\begin{aligned} x \in \bar{A} & \Leftrightarrow \forall C. C \text{ closed} \wedge A \subseteq C \Rightarrow x \in C \\ & \Leftrightarrow \forall U. U \text{ open} \wedge A \cap U = \emptyset \Rightarrow x \notin U \\ & \Leftrightarrow \forall U. U \text{ open} \wedge x \in U \Rightarrow U \text{ intersects } A \quad \square \end{aligned}$$

**Theorem 67.** Let  $X$  be a topological space,  $A \subseteq X$  and  $x \in X$ . Let  $\mathcal{B}$  be a basis for  $X$ . Then  $x \in \bar{A}$  if and only if, for all  $B \in \mathcal{B}$ , if  $x \in B$  then  $B$  intersects  $A$ .

PROOF:

$\langle 1 \rangle 1$ . If  $x \in \bar{A}$  then, for all  $B \in \mathcal{B}$ , if  $x \in B$  then  $B$  intersects  $A$ .

PROOF: This follows from Theorem 66 since every element of  $\mathcal{B}$  is open (Corollary 14.1).

$\langle 1 \rangle 2$ . Suppose that, for all  $B \in \mathcal{B}$ , if  $x \in B$  then  $B$  intersects  $A$ . Then  $x \in \bar{A}$ .

$\langle 2 \rangle 1$ . ASSUME: For all  $B \in \mathcal{B}$ , if  $x \in B$  then  $B$  intersects  $A$ .

$\langle 2 \rangle 2$ . LET:  $U$  be an open set that contains  $x$

PROVE:  $U$  intersects  $A$ .

$\langle 2 \rangle 3$ . PICK  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$ .

$\langle 2 \rangle 4$ .  $B$  intersects  $A$ .

PROOF: From  $\langle 2 \rangle 1$ .

$\langle 2 \rangle 5$ .  $U$  intersects  $A$ .

$\langle 2 \rangle 6$ . Q.E.D.

PROOF: By Theorem 66.

$\square$

**Proposition 68.** If  $A \subseteq B$  then  $\bar{A} \subseteq \bar{B}$ .

PROOF: This holds because  $\bar{B}$  is a closed set that includes  $A$ .  $\square$

**Proposition 69.**

$$\overline{A \cup B} = \bar{A} \cup \bar{B}$$

PROOF:

$\langle 1 \rangle 1$ .  $\bar{A} \subseteq \overline{A \cup B}$

PROOF: By Proposition 68.

⟨1⟩2.  $\overline{B} \subseteq \overline{A \cup B}$

PROOF: By Proposition 68.

⟨1⟩3.  $\overline{A \cup B} \subseteq \overline{A \cup B}$

⟨2⟩1. LET:  $x \in \overline{A \cup B}$

⟨2⟩2. ASSUME:  $x \notin \overline{A}$

PROVE:  $x \in \overline{B}$

⟨2⟩3. PICK a neighbourhood  $U$  of  $x$  that does not intersect  $A$

⟨2⟩4. LET:  $V$  be any neighbourhood of  $x$

⟨2⟩5.  $U \cap V$  is a neighbourhood of  $x$

⟨2⟩6.  $U \cap V$  intersects  $A \cup B$

PROOF: From ⟨2⟩1 and Theorem 66.

⟨2⟩7.  $U \cap V$  intersects  $B$

PROOF: From ⟨2⟩3

⟨2⟩8.  $V$  intersects  $B$

⟨2⟩9. Q.E.D.

PROOF: We have  $x \in \overline{B}$  from Theorem 66.

□

**Proposition 70 (AC).** Let  $\{X_i\}_{i \in I}$  be a family of topological spaces. Let  $A_i \subseteq X_i$  for all  $i \in I$ . Then

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i} .$$

PROOF:

⟨1⟩1.  $\overline{\prod_{i \in I} A_i} \subseteq \prod_{i \in I} \overline{A_i}$

⟨2⟩1. For all  $i \in I$  we have  $A_i \subseteq \overline{A_i}$

PROOF: Lemma 63.

⟨2⟩2.  $\prod_{i \in I} A_i \subseteq \prod_{i \in I} \overline{A_i}$

⟨2⟩3. Q.E.D.

PROOF: Since  $\prod_{i \in I} A_i$  is closed by Proposition 53.

⟨1⟩2.  $\prod_{i \in I} \overline{A_i} \subseteq \overline{\prod_{i \in I} A_i}$

⟨2⟩1. LET:  $x \in \prod_{i \in I} \overline{A_i}$

⟨2⟩2. LET:  $U$  be a neighbourhood of  $x$

⟨2⟩3. PICK  $V_i$  open in  $X_i$  such that  $x \in \prod_{i \in I} V_i \subseteq U$  with  $V_i = X_i$  except for  $i = i_1, \dots, i_n$

⟨2⟩4. For  $i \in I$ , pick  $a_i \in V_i \cap A_i$

PROOF: By Theorem 66 and ⟨2⟩1 using the Axiom of Choice.

⟨2⟩5.  $U$  intersects  $\prod_{i \in I} A_i$

⟨2⟩6. Q.E.D.

PROOF:  $a \in U \cap \prod_{i \in I} A_i$

□

**Example 71.** The closure of  $\mathbb{R}^\infty$  in  $\mathbb{R}^\omega$  is  $\mathbb{R}^\omega$

PROOF:

⟨1⟩1. LET:  $a \in \mathbb{R}^\omega$



- $\langle 1 \rangle 2$ . LET:  $U$  be any neighbourhoods of  $a$ .  
 $\langle 1 \rangle 3$ . PICK  $U_n$  open in  $\mathbb{R}$  for all  $n$  such that  $a \in \prod_{n \geq 0} U_n \subseteq U$  and  $U_n = \mathbb{R}$  for all  $n$  except  $n_1, \dots, n_k$   
 $\langle 1 \rangle 4$ . LET:  $b_n = a_n$  for  $n = n_1, \dots, n_k$  and  $b_n = 0$  for all other  $n$   
 $\langle 1 \rangle 5$ .  $b \in \mathbb{R}^\infty \cap U$   
 $\langle 1 \rangle 6$ . Q.E.D.  
 PROOF: From Theorem 66.

□

## 13 Limit Points

**Definition 72** (Limit Point). Let  $X$  be a topological space,  $a \in X$  and  $A \subseteq X$ . Then  $a$  is a *limit point*, *cluster point* or *point of accumulation* for  $A$  if and only if every neighbourhood of  $a$  intersects  $A$  at a point other than  $a$ .

**Lemma 73.** *The point  $a$  is an accumulation point for  $A$  if and only if  $a \in \overline{A \setminus \{a\}}$ .*

PROOF: From Theorem 66. □

**Theorem 74.** *Let  $X$  be a topological space and  $A \subseteq X$ . Let  $A'$  be the set of all limit points of  $A$ . Then  $\overline{A} = A \cup A'$ .*

PROOF:

- $\langle 1 \rangle 1$ . For all  $x \in \overline{A}$ , if  $x \notin A$  then  $x \in A'$

PROOF: From Theorem 66.

- $\langle 1 \rangle 2$ .  $A \subseteq \overline{A}$

PROOF: Lemma 63.

- $\langle 1 \rangle 3$ .  $A' \subseteq \overline{A}$

PROOF: From Theorem 66.

□

**Corollary 74.1.** *A set is closed if and only if it contains all its limit points.*

**Proposition 75.** *In an indiscrete topology, every point is a limit point of any set that has more than one point.*

PROOF: Let  $X$  be an indiscrete space. Let  $A$  be a set with more than one point and  $x$  be a point. The only neighbourhood of  $x$  is  $X$ , which must intersect  $A$  at a point other than  $x$ . □

## 14 $T_1$ Spaces

**Definition 76** ( $T_1$  Space). A topological space is  $T_1$  if and only if every singleton is closed.

**Lemma 77.** *A space is  $T_1$  if and only if every finite set is closed.*

PROOF: From Lemma 49.  $\square$

**Theorem 78.** *In a  $T_1$  space, a point  $a$  is a limit point of a set  $A$  if and only if every neighbourhood of  $a$  contains infinitely many points of  $A$ .*

PROOF:

$\langle 1 \rangle 1$ . If  $a$  is a limit point of  $A$  then every neighbourhood of  $a$  contains infinitely many points of  $A$ .

$\langle 2 \rangle 1$ . ASSUME:  $a$  is a limit point of  $A$ .

$\langle 2 \rangle 2$ . LET:  $U$  be a neighbourhood of  $a$ .

$\langle 2 \rangle 3$ . ASSUME: for a contradiction  $U$  contains only finitely many points of  $A$ .

$\langle 2 \rangle 4$ .  $(U \cap A) \setminus \{a\}$  is closed.

PROOF: By the  $T_1$  axiom.

$\langle 2 \rangle 5$ .  $(U \setminus A) \cup \{a\}$  is open.

PROOF: It is  $U \setminus ((U \cap A) \setminus \{a\})$ .

$\langle 2 \rangle 6$ .  $(U \setminus A) \cup \{a\}$  intersects  $A$  in a point other than  $a$ .

PROOF: From  $\langle 2 \rangle 1$ .

$\langle 2 \rangle 7$ . Q.E.D.

$\square$

$\langle 1 \rangle 2$ . If every neighbourhood of  $a$  contains infinitely many points of  $A$  then  $a$  is a limit point of  $A$ .

PROOF: Immediate from definitions.

$\square$

(To see this does not hold in every space, see Proposition 75.)

**Proposition 79.** *A space is  $T_1$  if and only if, for any two distinct points  $x$  and  $y$ , there exist neighbourhoods  $U$  of  $x$  and  $V$  of  $y$  such that  $x \notin V$  and  $y \notin U$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $X$  be a topological space.

$\langle 1 \rangle 2$ . If  $X$  is  $T_1$  then, for any two distinct points  $x$  and  $y$ , there exist neighbourhoods  $U$  of  $x$  and  $V$  of  $y$  such that  $x \notin V$  and  $y \notin U$ .

PROOF: This holds because  $\{x\}$  and  $\{y\}$  are closed.

$\langle 1 \rangle 3$ . Suppose, for any two distinct points  $x$  and  $y$ , there exist neighbourhoods  $U$  of  $x$  and  $V$  of  $y$  such that  $x \notin V$  and  $y \notin U$ . Then  $X$  is  $T_1$ .

$\langle 2 \rangle 1$ . ASSUME: For any two distinct points  $x$  and  $y$ , there exist neighbourhoods  $U$  of  $x$  and  $V$  of  $y$  such that  $x \notin V$  and  $y \notin U$ .

$\langle 2 \rangle 2$ . LET:  $a \in X$

$\langle 2 \rangle 3$ .  $\{a\}$  is closed.

PROOF: For all  $b \neq a$  there exists a neighbourhood  $U$  of  $b$  such that  $U \subseteq X \setminus \{a\}$ .

$\square$

## 15 Hausdorff Spaces

**Definition 80** (Hausdorff Space). A topological space is *Hausdorff* if and only if, for any points  $x, y$  with  $x \neq y$ , there exist disjoint open sets  $U$  and  $V$  such

that  $x \in U$  and  $y \in V$ .

**Theorem 81.** *Every Hausdorff space is  $T_1$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $X$  be a Hausdorff space.

$\langle 1 \rangle 2$ . LET:  $b \in X$

PROVE:  $\overline{\{b\}} = \{b\}$

$\langle 1 \rangle 3$ . ASSUME:  $a \in \overline{\{b\}}$  and  $a \neq b$

$\langle 1 \rangle 4$ . PICK disjoint neighbourhoods  $U$  of  $a$  and  $V$  of  $b$ .

$\langle 1 \rangle 5$ .  $U$  intersects  $\{b\}$

PROOF: Theorem 66.

$\langle 1 \rangle 6$ .  $b \in U$

$\langle 1 \rangle 7$ . Q.E.D.

PROOF: This contradicts the fact that  $U$  and  $V$  are disjoint ( $\langle 1 \rangle 4$ ).

□

**Proposition 82.** *An infinite set under the finite complement topology is  $T_1$  but not Hausdorff.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $X$  be an infinite set under the finite complement topology.

$\langle 1 \rangle 2$ . Every singleton is closed.

PROOF: By definition.

$\langle 1 \rangle 3$ . PICK  $a, b \in X$  with  $a \neq b$

$\langle 1 \rangle 4$ . There are no disjoint neighbourhoods  $U$  of  $a$  and  $V$  of  $b$ .

$\langle 2 \rangle 1$ . LET:  $U$  be a neighbourhood of  $a$  and  $V$  a neighbourhood of  $b$ .

$\langle 2 \rangle 2$ .  $X \setminus U$  and  $X \setminus V$  are finite.

$\langle 2 \rangle 3$ . PICK  $c \in X$  that is not in  $X \setminus U$  or  $X \setminus V$ .

$\langle 2 \rangle 4$ .  $c \in U \cap V$

□

**Proposition 83.** *The product of a family of Hausdorff spaces is Hausdorff.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $\{X_i\}_{i \in I}$  be a family of Hausdorff spaces.

$\langle 1 \rangle 2$ . LET:  $a, b \in \prod_{i \in I} X_i$  with  $a \neq b$

$\langle 1 \rangle 3$ . PICK  $i \in I$  such that  $a_i \neq b_i$

$\langle 1 \rangle 4$ . PICK  $U, V$  disjoint open sets in  $X_i$  with  $a_i \in U$  and  $b_i \in V$

$\langle 1 \rangle 5$ .  $\pi_i^{-1}(U)$  and  $\pi_i^{-1}(V)$  are disjoint open sets in  $\prod_{i \in I} X_i$  with  $a \in \pi_i^{-1}(U)$  and  $b \in \pi_i^{-1}(V)$

□

**Theorem 84.** *Every linearly ordered set under the order topology is Hausdorff.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $X$  be a linearly ordered set under the order topology.

$\langle 1 \rangle 2$ . LET:  $a, b \in X$  with  $a \neq b$

$\langle 1 \rangle 3$ . ASSUME: w.l.o.g.  $a < b$

⟨1⟩4. CASE: There exists  $c$  such that  $a < c < b$

PROOF: The sets  $(-\infty, c)$  and  $(c, +\infty)$  are disjoint neighbourhoods of  $a$  and  $b$  respectively.

⟨1⟩5. CASE: There is no  $c$  such that  $a < c < b$

PROOF: The sets  $(-\infty, b)$  and  $(a, +\infty)$  are disjoint neighbourhoods of  $a$  and  $b$  respectively.

□

**Theorem 85.** *A subspace of a Hausdorff space is Hausdorff.*

PROOF:

⟨1⟩1. LET:  $X$  be a Hausdorff space and  $Y$  a subspace of  $X$ .

⟨1⟩2. LET:  $x, y \in Y$  with  $x \neq y$

⟨1⟩3. PICK disjoint neighbourhoods  $U$  of  $x$  and  $V$  of  $y$  in  $X$ .

⟨1⟩4.  $U \cap Y$  and  $V \cap Y$  are disjoint neighbourhoods of  $x$  and  $y$  respectively in  $Y$ .

□

**Proposition 86.** *A space  $X$  is Hausdorff if and only if the diagonal  $\Delta = \{(x, x) \mid x \in X\}$  is closed in  $X^2$ .*

PROOF:

$X$  is Hausdorff

$$\Leftrightarrow \forall x, y \in X. x \neq y \Rightarrow \exists V, W \text{ open. } x \in V \wedge y \in W \wedge V \cap W = \emptyset$$

$$\Leftrightarrow \forall (x, y) \in X^2 \setminus \Delta. \exists V, W \text{ open. } (x, y) \in V \times W \subseteq X^2 \setminus \Delta$$

$$\Leftrightarrow \Delta \text{ is closed}$$

□

## 16 Convergence

**Definition 87** (Convergence). Let  $X$  be a topological space. Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence of points in  $X$  and  $l \in X$ . Then the sequence  $(a_n)_{n \in \mathbb{N}}$  *converges* to the *limit*  $l$ ,  $a_n \rightarrow l$  as  $n \rightarrow \infty$ , if and only if, for every neighbourhood  $U$  of  $l$ , there exists  $N$  such that, for all  $n \geq N$ , we have  $a_n \in U$ .

**Theorem 88.** *In a Hausdorff space, a sequence has at most one limit.*

PROOF:

⟨1⟩1. LET:  $X$  be a Hausdorff space.

⟨1⟩2. ASSUME: for a contradiction  $a_n \rightarrow l$  as  $n \rightarrow \infty$ ,  $a_n \rightarrow m$  as  $n \rightarrow \infty$ , and  $l \neq m$

⟨1⟩3. PICK disjoint neighbourhoods  $U$  of  $l$  and  $V$  of  $m$

PROOF: By the Hausdorff axiom.

⟨1⟩4. PICK  $M$  and  $N$  such that  $a_n \in U$  for  $n \geq M$  and  $a_n \in V$  for  $n \geq N$

⟨1⟩5.  $a_{\max(M, N)} \in U \cap V$

⟨1⟩6. Q.E.D.

PROOF: This contradicts the fact that  $U$  and  $V$  are disjoint (⟨1⟩3).

□

To see this is not always true in spaces that are  $T_1$  but not Hausdorff:

**Proposition 89.** *Let  $X$  be an infinite set under the finite complement topology. Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence with all points distinct. Then for every  $l \in X$  we have  $a_n \rightarrow l$  as  $n \rightarrow \infty$ .*

PROOF: Let  $U$  be any neighbourhood of  $l$ . Since  $X \setminus U$  is finite, there must exist  $N$  such that, for all  $n \geq N$ , we have  $a_n \in U$ . □

## 17 Boundary

**Definition 90** (Boundary). The *boundary* of a set  $A$  is the set  $\partial A = \overline{A} \cap \overline{X \setminus A}$ .

**Proposition 91.**

$$\text{Int } A \cap \partial A = \emptyset$$

PROOF: Since  $\overline{X \setminus A} = X \setminus \text{Int } A$ . □

**Proposition 92.**

$$\overline{A} = \text{Int } A \cup \partial A$$

PROOF:

$$\begin{aligned} \text{Int } A \cup \partial A &= \text{Int } A \cup (\overline{A} \cap \overline{X \setminus A}) \\ &= (\text{Int } A \cup \overline{A}) \cap (\text{Int } A \cup \overline{X \setminus A}) \\ &= \overline{A} \cap X \\ &= \overline{A} \end{aligned}$$

**Proposition 93.**  $\partial A = \emptyset$  if and only if  $A$  is open and closed.

PROOF: If  $\partial A = \emptyset$  then  $\overline{A} = \text{Int } A$  by Proposition 92.

**Proposition 94.** A set  $U$  is open if and only if  $\partial U = \overline{U} \setminus U$ .

PROOF:

$$\begin{aligned} \partial U &= \overline{U} \setminus U \\ \Leftrightarrow \overline{U} \setminus \text{Int } U &= \overline{U} \setminus U && \text{(Propositions 91, 92)} \\ \Leftrightarrow \text{Int } U &= U && \square \end{aligned}$$

## 18 Continuous Functions

**Definition 95** (Continuous). Let  $X$  and  $Y$  be topological spaces. A function  $f : X \rightarrow Y$  is *continuous* if and only if, for every open set  $V$  in  $Y$ , the set  $f^{-1}(V)$  is open in  $X$ .

**Proposition 96.** *Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Let  $\mathcal{B}$  be a basis for  $Y$ . Then  $f$  is continuous if and only if, for all  $B \in \mathcal{B}$ , we have  $f^{-1}(B)$  is open in  $X$ .*

PROOF:

$\langle 1 \rangle 1$ . If  $f$  is continuous then, for all  $B \in \mathcal{B}$ , we have  $f^{-1}(B)$  is open in  $X$ .

PROOF: Since every element of  $\mathcal{B}$  is open (Lemma 14).

$\langle 1 \rangle 2$ . Suppose that, for all  $B \in \mathcal{B}$ , we have  $f^{-1}(B)$  is open in  $X$ . Then  $f$  is continuous.

$\langle 2 \rangle 1$ . ASSUME: For all  $B \in \mathcal{B}$ , we have  $f^{-1}(B)$  is open in  $X$ .

$\langle 2 \rangle 2$ . LET:  $V$  be open in  $Y$ .

$\langle 2 \rangle 3$ . PICK  $\mathcal{A} \subseteq \mathcal{B}$  such that  $V = \bigcup \mathcal{A}$

PROOF: By Lemma 14.

$\langle 2 \rangle 4$ .  $f^{-1}(V)$  is open in  $X$ .

PROOF:

$$\begin{aligned} f^{-1}(V) &= f^{-1}\left(\bigcup \mathcal{A}\right) \\ &= \bigcup_{B \in \mathcal{A}} f^{-1}(B) \end{aligned}$$

□

**Proposition 97.** *Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Let  $\mathcal{S}$  be a subbasis for  $Y$ . Then  $f$  is continuous if and only if, for all  $S \in \mathcal{S}$ , we have  $f^{-1}(S)$  is open in  $X$ .*

PROOF:

$\langle 1 \rangle 1$ . If  $f$  is continuous then, for all  $S \in \mathcal{S}$ , we have  $f^{-1}(S)$  is open in  $X$ .

PROOF: Since every element of  $\mathcal{S}$  is open.

$\langle 1 \rangle 2$ . Suppose that, for all  $S \in \mathcal{S}$ , we have  $f^{-1}(S)$  is open in  $X$ . Then  $f$  is continuous.

$\langle 2 \rangle 1$ . ASSUME: For all  $S \in \mathcal{S}$ , we have  $f^{-1}(S)$  is open in  $X$ .

$\langle 2 \rangle 2$ . LET:  $S_1, \dots, S_n \in \mathcal{S}$

$\langle 2 \rangle 3$ .  $f^{-1}(S_1 \cap \dots \cap S_n)$  is open in  $X$

PROOF: Since  $f^{-1}(S_1 \cap \dots \cap S_n) = f^{-1}(S_1) \cap \dots \cap f^{-1}(S_n)$ .

$\langle 2 \rangle 4$ . Q.E.D.

PROOF: By Propositions 96 and 21.

□

**Proposition 98.** *Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Let  $\mathcal{S}$  be a subbasis for  $Y$ . Then  $f$  is continuous if and only if, for all  $V \in \mathcal{S}$ , we have  $f^{-1}(V)$  is open in  $X$ .*

PROOF:

$\langle 1 \rangle 1$ . If  $f$  is continuous then, for all  $V \in \mathcal{S}$ , we have  $f^{-1}(V)$  is open in  $X$ .

PROOF: Since every element of  $\mathcal{S}$  is open.

$\langle 1 \rangle 2$ . Suppose that, for all  $V \in \mathcal{S}$ , we have  $f^{-1}(V)$  is open in  $X$ . Then  $f$  is continuous.

$\langle 2 \rangle 1$ . ASSUME: For all  $V \in \mathcal{S}$ , we have  $f^{-1}(V)$  is open in  $X$ .

⟨2⟩2. For every set  $B$  that is the finite intersection of elements of  $\mathcal{S}$ , we have  $f^{-1}(B)$  is open in  $X$ .

PROOF: Because  $f^{-1}(V_1 \cap \dots \cap V_n) = f^{-1}(V_1) \cap \dots \cap f^{-1}(V_n)$ .

⟨2⟩3. Q.E.D.

PROOF: From Propositions 21 and 96.

□

**Definition 99** (Continuous at a Point). Let  $X$  and  $Y$  be topological spaces. Let  $f : X \rightarrow Y$  and  $x \in X$ . Then  $f$  is *continuous at  $x$*  if and only if, for every neighbourhood  $V$  of  $f(x)$ , there exists a neighbourhood  $U$  of  $x$  such that  $f(U) \subseteq V$ .

**Theorem 100.** Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Then the following are equivalent:

1.  $f$  is continuous.
2. For all  $A \subseteq X$ , we have  $f(\overline{A}) \subseteq \overline{f(A)}$
3. For all  $B \subseteq Y$  closed, we have  $f^{-1}(B)$  is closed in  $X$ .
4.  $f$  is continuous at every point of  $X$ .

PROOF:

⟨1⟩1.  $1 \Rightarrow 2$

⟨2⟩1. ASSUME:  $f$  is continuous.

⟨2⟩2. LET:  $A \subseteq X$

⟨2⟩3. LET:  $x \in \overline{A}$

PROVE:  $f(x) \in \overline{f(A)}$

⟨2⟩4. LET:  $V$  be a neighbourhood of  $f(x)$

⟨2⟩5.  $f^{-1}(V)$  is a neighbourhood of  $x$

⟨2⟩6. PICK  $y \in A \cap f^{-1}(V)$

PROOF: By Theorem 66.

⟨2⟩7.  $f(y) \in V \cap f(A)$

⟨2⟩8. Q.E.D.

PROOF: By Theorem 66.

⟨1⟩2.  $2 \Rightarrow 3$

⟨2⟩1. ASSUME: 2

⟨2⟩2. LET:  $B$  be closed in  $Y$

⟨2⟩3. LET:  $x \in \overline{f^{-1}(B)}$

PROVE:  $x \in f^{-1}(B)$

⟨2⟩4.  $f(x) \in B$

PROOF:

$$\begin{aligned}
 f(x) &\in \overline{f(f^{-1}(B))} \\
 &\subseteq \overline{f(f^{-1}(B))} && (\langle 2 \rangle 1) \\
 &\subseteq \overline{B} && (Proposition 68) \\
 &= B
 \end{aligned}$$

⟨1⟩3.  $3 \Rightarrow 1$

⟨2⟩1. ASSUME: 3

⟨2⟩2. LET:  $V$  be open in  $Y$

⟨2⟩3.  $Y \setminus V$  is closed in  $Y$

⟨2⟩4.  $f^{-1}(Y \setminus V)$  is closed in  $X$

⟨2⟩5.  $X \setminus f^{-1}(V)$  is closed in  $X$

⟨2⟩6.  $f^{-1}(V)$  is open in  $X$

⟨1⟩4.  $1 \Rightarrow 4$

PROOF: For any neighbourhood  $V$  of  $f(x)$ , the set  $U = f^{-1}(V)$  is a neighbourhood of  $x$  such that  $f(U) \subseteq V$ .

⟨1⟩5.  $4 \Rightarrow 1$

⟨2⟩1. ASSUME: 4

⟨2⟩2. LET:  $V$  be open in  $Y$

⟨2⟩3. LET:  $x \in f^{-1}(V)$

⟨2⟩4.  $V$  is a neighbourhood of  $f(x)$

⟨2⟩5. PICK a neighbourhood  $U$  of  $x$  such that  $f(U) \subseteq V$

⟨2⟩6.  $x \in U \subseteq f^{-1}(V)$

⟨2⟩7. Q.E.D.

PROOF: By Lemma 10.

□

**Theorem 101.** *A constant function is continuous.*

PROOF: Let  $X$  and  $Y$  be topological spaces. Let  $b \in Y$ , and let  $f : X \rightarrow Y$  be the constant function with value  $b$ . For any open  $V \subseteq Y$ , the set  $f^{-1}(V)$  is either  $X$  (if  $b \in V$ ) or  $\emptyset$  (if  $b \notin V$ ). □

**Theorem 102.** *If  $A$  is a subspace of  $X$  then the inclusion  $j : A \rightarrow X$  is continuous.*

PROOF: For any  $V$  open in  $X$ , we have  $j^{-1}(V) = V \cap A$  is open in  $A$ . □

**Theorem 103.** *The composite of two continuous functions is continuous.*

PROOF: Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  be continuous. For any  $V$  open in  $Z$ , we have  $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$  is open in  $X$ . □

**Theorem 104.** *Let  $f : X \rightarrow Y$  be a continuous function and  $A$  be a subspace of  $X$ . Then the restriction  $f \upharpoonright A : A \rightarrow Y$  is continuous.*

PROOF: Let  $V$  be open in  $Y$ . Then  $(f \upharpoonright A)^{-1}(V) = f^{-1}(V) \cap A$  is open in  $A$ . □

**Theorem 105.** *Let  $f : X \rightarrow Y$  be continuous. Let  $Z$  be a subspace of  $Y$  such that  $f(X) \subseteq Z$ . Then the corestriction  $f : X \rightarrow Z$  is continuous.*

PROOF:

⟨1⟩1. LET:  $V$  be open in  $Z$ .

⟨1⟩2. PICK  $U$  open in  $Y$  such that  $V = U \cap Z$ .

⟨1⟩3.  $f^{-1}(V) = f^{-1}(U)$



$\langle 1 \rangle 4.$   $f^{-1}(V)$  is open in  $X$ .

□

**Theorem 106.** *Let  $f : X \rightarrow Y$  be continuous. Let  $Z$  be a space such that  $Y$  is a subspace of  $Z$ . Then the expansion  $f : X \rightarrow Z$  is continuous.*

PROOF: Let  $V$  be open in  $Z$ . Then  $f^{-1}(V) = f^{-1}(V \cap Y)$  is open in  $X$ . □

**Theorem 107.** *Let  $X$  and  $Y$  be topological spaces. Let  $f : X \rightarrow Y$ . Suppose  $\mathcal{U}$  is a set of open sets in  $X$  such that  $X = \bigcup \mathcal{U}$  and, for all  $U \in \mathcal{U}$ , we have  $f \upharpoonright U : U \rightarrow Y$  is continuous. Then  $f$  is continuous.*

PROOF:

$\langle 1 \rangle 1.$  LET:  $V$  be open in  $Y$

$\langle 1 \rangle 2.$   $f^{-1}(V) = \bigcup_{U \in \mathcal{U}} (f \upharpoonright U)^{-1}(V)$

$\langle 1 \rangle 3.$  For all  $U \in \mathcal{U}$ , we have  $(f \upharpoonright U)^{-1}(V)$  is open in  $U$ .

$\langle 1 \rangle 4.$  For all  $U \in \mathcal{U}$ , we have  $(f \upharpoonright U)^{-1}(V)$  is open in  $X$ .

PROOF: Lemma 39.

□

**Theorem 108.** *Let  $A$  be a topological space and  $\{X_i\}_{i \in I}$  be a family of topological spaces. Let  $f : A \rightarrow \prod_{i \in I} X_i$  be a function. If  $\pi_i \circ f$  is continuous for all  $i \in I$  then  $f$  is continuous.*

PROOF:

$\langle 1 \rangle 1.$  LET:  $i \in I$  and  $U$  be open in  $X_i$

$\langle 1 \rangle 2.$   $f^{-1}(\pi_i^{-1}(U))$  is open in  $A$

$\langle 1 \rangle 3.$  Q.E.D.

PROOF: Proposition 97.

□

**Proposition 109.** *Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  and  $x \in \mathbb{R}$ . Then  $f$  is continuous at  $x$  if and only if, for all  $\epsilon > 0$ , there exists  $\delta > 0$  such that, for all  $y \in \mathbb{R}$ , if  $|x - y| < \delta$  then  $|f(x) - f(y)| < \epsilon$ .*

PROOF:

$\langle 1 \rangle 1.$  If  $f$  is continuous at  $x$  then, for all  $\epsilon > 0$ , there exists  $\delta > 0$  such that, for all  $y \in \mathbb{R}$ , if  $|x - y| < \delta$  then  $|f(x) - f(y)| < \epsilon$

$\langle 2 \rangle 1.$  ASSUME:  $f$  is continuous.

$\langle 2 \rangle 2.$  LET:  $\epsilon > 0$

$\langle 2 \rangle 3.$   $f^{-1}((f(x) - \epsilon, f(x) + \epsilon))$  is open.

$\langle 2 \rangle 4.$  PICK  $a, b$  such that  $x \in (a, b) \subseteq f^{-1}((f(x) - \epsilon, f(x) + \epsilon))$

$\langle 2 \rangle 5.$  LET:  $\delta = \min(x - a, b - x)$

$\langle 2 \rangle 6.$  LET:  $y \in \mathbb{R}$  with  $|x - y| < \delta$

$\langle 2 \rangle 7.$   $y \in (a, b)$

$\langle 2 \rangle 8.$   $f(y) \in (f(x) - \epsilon, f(x) + \epsilon)$

$\langle 2 \rangle 9.$   $|f(x) - f(y)| < \epsilon$

$\langle 1 \rangle 2.$  Suppose, for all  $\epsilon > 0$ , there exists  $\delta > 0$  such that, for all  $y \in \mathbb{R}$ , if  $|x - y| < \delta$  then  $|f(x) - f(y)| < \epsilon$ . Then  $f$  is continuous at  $x$ .

- ⟨2⟩1. ASSUME: For all  $\epsilon > 0$ , there exists  $\delta > 0$  such that. for all  $y \in \mathbb{R}$ , if  $|x - y| < \delta$  then  $|f(x) - f(y)| < \epsilon$
- ⟨2⟩2. LET:  $V$  be a neighbourhood of  $f(x)$
- ⟨2⟩3. PICK  $a, b$  such that  $f(x) \in (a, b) \subseteq V$
- ⟨2⟩4. LET:  $\epsilon = \min(f(x) - a, b - f(x))$
- ⟨2⟩5. PICK  $\delta > 0$  such that, for all  $y \in \mathbb{R}$ , if  $|x - y| < \delta$  then  $|f(x) - f(y)| < \epsilon$
- ⟨2⟩6. LET:  $U = (x - \delta, x + \delta)$
- ⟨2⟩7.  $x \in U \subseteq f^{-1}(V)$

□

**Proposition 110.** *Let  $X$  and  $X'$  be the same set  $X$  under two topologies  $\mathcal{T}$  and  $\mathcal{T}'$ . Let  $i : X \rightarrow X'$  be the identity function. Then  $i$  is continuous if and only if  $\mathcal{T}' \subseteq \mathcal{T}$ .*

PROOF: Immediate from definitions. □

**Proposition 111.** *Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  and  $a \in \mathbb{R}$ . Then  $f$  is continuous on the right at  $a$  if and only if  $f$  is continuous at  $a$  as a function  $\mathbb{R}_l \rightarrow \mathbb{R}$ .*

PROOF:

- ⟨1⟩1. If  $f$  is continuous on the right at  $a$  then  $f$  is continuous at  $a$  as a function  $\mathbb{R}_l \rightarrow \mathbb{R}$ .
  - ⟨2⟩1. ASSUME:  $f$  is continuous on the right at  $a$ .
  - ⟨2⟩2. LET:  $V$  be a neighbourhood of  $f(a)$
  - ⟨2⟩3. PICK  $b, c$  such that  $f(a) \in (b, c) \subseteq V$ .
  - ⟨2⟩4. LET:  $\epsilon = \min(c - f(a), f(a) - b)$
  - ⟨2⟩5. PICK  $\delta > 0$  such that, for all  $x$ , if  $a < x < a + \delta$  then  $|f(x) - f(a)| < \epsilon$
  - ⟨2⟩6. LET:  $U = [a, a + \delta)$
  - ⟨2⟩7.  $f(U) \subseteq V$
- ⟨1⟩2. If  $f$  is continuous at  $a$  as a function  $\mathbb{R}_l \rightarrow \mathbb{R}$  then  $f$  is continuous on the right at  $a$ .
  - ⟨2⟩1. ASSUME:  $f$  is continuous at  $a$  as a function  $\mathbb{R}_l \rightarrow \mathbb{R}$
  - ⟨2⟩2. LET:  $\epsilon > 0$
  - ⟨2⟩3. PICK a neighbourhood  $U$  of  $a$  such that  $f(U) \subseteq (f(a) - \epsilon, f(a) + \epsilon)$
  - ⟨2⟩4. PICK  $b, c$  such that  $a \in [b, c) \subset U$
  - ⟨2⟩5. LET:  $\delta = c - a$
  - ⟨2⟩6. For all  $x$ , if  $a < x < a + \delta$  then  $|f(x) - f(a)| < \epsilon$

□

**Lemma 112.** *Let  $X$  be a topological space. Let  $Y$  be a linearly ordered set in the order topology. Let  $f, g : X \rightarrow Y$  be continuous. Then  $C = \{x \in X \mid f(x) \leq g(x)\}$  is closed.*

PROOF:

- ⟨1⟩1. LET:  $x \in X \setminus C$
- ⟨1⟩2.  $f(x) > g(x)$ 
  - PROVE: There exists a neighbourhood  $U$  of  $x$  such that  $U \subseteq X \setminus C$
- ⟨1⟩3. CASE: There exists  $y$  such that  $g(x) < y < f(x)$

PROOF: Take  $U = g^{-1}((-\infty, y)) \cup f^{-1}(y, +\infty)$ .  
 (1)4. CASE: There is no  $y$  such that  $g(x) < y < f(x)$   
 PROOF: Take  $U = g^{-1}((-\infty, f(x))) \cup f^{-1}(g(x), +\infty)$ .  
 $\square$

**Lemma 113.** *Let  $f : X \rightarrow Y$ . Let  $Z$  be an open subspace of  $X$  and  $a \in Z$ . If  $f \upharpoonright Z$  is continuous at  $a$  then  $f$  is continuous at  $a$ .*

PROOF:  
 (1)1. LET:  $V$  be a neighbourhood of  $f(x)$   
 (1)2. PICK a neighbourhood  $W$  of  $x$  in  $Z$  such that  $f(W) \subseteq V$   
 (1)3.  $W$  is a neighbourhood of  $x$  in  $X$  such that  $f(W) \subseteq V$   
 PROOF: Lemma 39.  
 $\square$

**Proposition 114.** *Let  $f : A \rightarrow B$  and  $g : C \rightarrow D$  be continuous. Define  $f \times g : A \times C \rightarrow B \times D$  by*

$$(f \times g)(a, c) = (f(a), g(c)) .$$

*Then  $f \times g$  is continuous.*

PROOF:  $\pi_1 \circ (f \times g) = f \circ \pi_1$  and  $\pi_2 \circ (f \times g) = g \circ \pi_2$  are continuous by Theorem 103. The result follows by Theorem 108.

**Proposition 115.** *Let  $X$  be a topological space. Let  $Y$  a Hausdorff space. Let  $A \subseteq X$ . Let  $f, g : \overline{A} \rightarrow Y$  be continuous. If  $f$  and  $g$  agree on  $A$  then  $f = g$ .*

PROOF:  
 (1)1. LET:  $x \in \overline{A}$   
 (1)2. ASSUME:  $f(x) \neq g(x)$   
 (1)3. PICK disjoint neighbourhoods  $V$  of  $f(x)$  and  $W$  of  $g(x)$ .  
 (1)4. PICK  $y \in f^{-1}(V) \cap g^{-1}(W) \cap A$   
 PROOF: Since  $f^{-1}(V) \cap g^{-1}(W)$  is a neighbourhood of  $x$  and hence intersects  $A$ .  
 (1)5.  $f(y) = g(y) \in V \cap W$   
 (1)6. Q.E.D.  
 PROOF: This contradicts the fact that  $V$  and  $W$  are disjoint ((1)3).  
 $\square$

**Proposition 116.** *Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$  be continuous. If  $a_n \rightarrow l$  as  $n \rightarrow \infty$  in  $X$  then  $f(a_n) \rightarrow f(l)$  as  $n \rightarrow \infty$ .*

PROOF:  
 (1)1. LET:  $V$  be a neighbourhood of  $f(l)$   
 (1)2. PICK a neighbourhood  $U$  of  $l$  such that  $f(U) \subseteq V$   
 (1)3. PICK  $N$  such that, for all  $n \geq N$ , we have  $a_n \in U$   
 (1)4. For all  $n \geq N$  we have  $f(a_n) \in V$   
 $\square$

**Proposition 117.** Let  $\{X_i\}_{i \in I}$  be a family of topological spaces. Let  $(a_n)$  be a sequence in  $\prod_{i \in I} X_i$  and  $l \in \prod_{i \in I} X_i$ . Then  $a_n \rightarrow l$  as  $n \rightarrow \infty$  if and only if, for all  $i \in I$ , we have  $\pi_i(a_n) \rightarrow \pi_i(l)$  as  $n \rightarrow \infty$ .

PROOF:

$\langle 1 \rangle 1$ . If  $a_n \rightarrow l$  as  $n \rightarrow \infty$  then, for all  $i \in I$ , we have  $\pi_i(a_n) \rightarrow \pi_i(l)$  as  $n \rightarrow \infty$

PROOF: Proposition 116.

$\langle 1 \rangle 2$ . If, for all  $i \in I$ , we have  $\pi_i(a_n) \rightarrow \pi_i(l)$  as  $n \rightarrow \infty$ , then  $a_n \rightarrow l$  as  $n \rightarrow \infty$

$\langle 2 \rangle 1$ . ASSUME: For all  $i \in I$ , we have  $\pi_i(a_n) \rightarrow \pi_i(l)$  as  $n \rightarrow \infty$

$\langle 2 \rangle 2$ . LET:  $V$  be a neighbourhood of  $l$

$\langle 2 \rangle 3$ . PICK open sets  $U_i$  in  $X_i$  such that  $l \in \prod_{i \in I} U_i \subseteq V$  and  $U_i = X_i$  for all  $i$  except  $i = i_1, \dots, i_k$

$\langle 2 \rangle 4$ . For  $j = 1, \dots, k$ , PICK  $N_j$  such that, for all  $n \geq N_j$ , we have  $\pi_{i_j}(a_n) \in U_{i_j}$

$\langle 2 \rangle 5$ . LET:  $N = \max(N_1, \dots, N_k)$

$\langle 2 \rangle 6$ . For all  $n \geq N$  we have  $a_n \in V$

□

## 19 Homeomorphisms

**Definition 118** (Homeomorphism). Let  $X$  and  $Y$  be topological spaces. A *Homeomorphism*  $f$  between  $X$  and  $Y$ ,  $f : X \cong Y$ , is a bijection  $f : X \rightarrow Y$  such that both  $f$  and  $f^{-1}$  are continuous.

**Lemma 119.** Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$  a bijection. Then the following are equivalent:

1.  $f$  is a homeomorphism.
2.  $f$  is continuous and an open map.
3. For any  $U \subseteq X$ , we have  $U$  is open if and only if  $f(U)$  is open.

PROOF: Immediate from definitions. □

**Proposition 120.** Let  $X$  and  $X'$  be the same set  $X$  under two topologies  $\mathcal{T}$  and  $\mathcal{T}'$ . Let  $i : X \rightarrow X'$  be the identity function. Then  $i$  is a homeomorphism if and only if  $\mathcal{T} = \mathcal{T}'$ .

PROOF: Immediate from definitions. □

**Definition 121** (Topological Property). Let  $P$  be a property of topological spaces. Then  $P$  is a *topological* property if and only if, for any spaces  $X$  and  $Y$ , if  $P$  holds of  $X$  and  $X \cong Y$  then  $P$  holds of  $Y$ .

**Definition 122** (Topological Imbedding). Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Then  $f$  is a *topological imbedding* if and only if the corestriction  $f : X \rightarrow f(X)$  is a homeomorphism.

**Proposition 123.** *Let  $X$  and  $Y$  be topological spaces and  $a \in X$ . The function  $i : Y \rightarrow X \times Y$  that maps  $y$  to  $(a, y)$  is an imbedding.*

PROOF:

$\langle 1 \rangle 1$ .  $i$  is injective

$\langle 1 \rangle 2$ .  $i$  is continuous.

PROOF: For  $U$  open in  $X$  and  $V$  open in  $Y$ , we have  $i^{-1}(U \times V)$  is  $V$  if  $a \in U$ , and  $\emptyset$  if  $a \notin U$ .

$\langle 1 \rangle 3$ .  $i : Y \rightarrow i(Y)$  is an open map.

PROOF: For  $V$  open in  $Y$  we have  $i(V) = (X \times V) \cap i(Y)$ .

□

## 20 Locally Finite Sets

**Definition 124** (Locally Finite). Let  $X$  be a topological space and  $\{A_\alpha\}$  a family of subsets of  $X$ . Then  $\mathcal{A}$  is *locally finite* if and only if every point in  $X$  has a neighbourhood that intersects  $A_\alpha$  for only finitely many  $\alpha$ .

**Theorem 125** (Pasting Lemma). *Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Let  $\{A_\alpha\}$  be a locally finite family of closed subsets of  $X$  that cover  $X$ . Suppose  $f \upharpoonright A_\alpha$  is continuous for all  $\alpha$ . Then  $f$  is continuous.*

PROOF:

$\langle 1 \rangle 1$ . Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Let  $A$  and  $B$  be closed subsets of  $X$  such that  $X = A \cup B$ . Suppose  $f \upharpoonright A$  and  $f \upharpoonright B$  are continuous. Then  $f$  is continuous.

$\langle 2 \rangle 1$ . LET:  $C \subseteq Y$  be closed.

$\langle 2 \rangle 2$ .  $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$

$\langle 2 \rangle 3$ .  $f^{-1}(C)$  and  $g^{-1}(C)$  are closed in  $X$ .

PROOF: Theorems 100 and 51.

$\langle 2 \rangle 4$ .  $h^{-1}(C)$  is closed in  $X$ .

PROOF: Lemma 49.

$\langle 2 \rangle 5$ . Q.E.D.

PROOF: Theorem 100.

$\langle 1 \rangle 2$ . Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Let  $\{A_\alpha\}$  be a finite family of closed subsets of  $X$  that cover  $X$ . Suppose  $f \upharpoonright A_\alpha$  is continuous for all  $\alpha$ . Then  $f$  is continuous.

PROOF: From  $\langle 1 \rangle 1$  by induction.

$\langle 1 \rangle 3$ . Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Let  $\{A_\alpha\}$  be a locally finite family of closed subsets of  $X$  that cover  $X$ . Suppose  $f \upharpoonright A_\alpha$  is continuous for all  $\alpha$ . Then  $f$  is continuous.

$\langle 2 \rangle 1$ . LET:  $x \in X$

PROVE:  $f$  is continuous at  $x$

$\langle 2 \rangle 2$ . PICK a neighbourhood  $U$  of  $x$  that intersects  $A_\alpha$  for only finitely many  $\alpha$ .

$\langle 2 \rangle 3$ .  $f \upharpoonright U$  is continuous

PROOF: By  $\langle 1 \rangle 2$ .  
 $\langle 2 \rangle 4$ . Q.E.D.  
 PROOF: Lemma 113.

□

The following example shows that we cannot remove the assumption of local finiteness.

**Example 126.** Define  $f : [-1, 1] \rightarrow \mathbb{R}$  by:  $f(x) = 1$  if  $x < -1$ ,  $f(x) = 0$  if  $x > 1$ . Let  $C_n = [-1, -1/n]$  for  $n \geq 1$ , and  $D = [0, 1]$ . Then  $[-1, 1] = \bigcup_{n=1}^{\infty} C_n \cup D$  and  $f$  is continuous on each  $C_n$  and each  $D$ , but  $f$  is not continuous on  $[-1, 1]$ .

**Proposition 127.** Let  $X$  be a topological space. Let  $Y$  be a linearly ordered set in the order topology. Let  $f, g : X \rightarrow Y$  be continuous. Define  $h : X \rightarrow Y$  by  $h(x) = \min(f(x), g(x))$ . Then  $h$  is continuous.

PROOF: By the Pasting Lemma applied to  $\{x \in X \mid f(x) \leq g(x)\}$  and  $\{x \in X \mid g(x) \leq f(x)\}$ , which are closed by Lemma 112.

## 21 Continuous in Each Variable Separately

**Definition 128** (Continuous in Each Variable Separately). Let  $F : X \times Y \rightarrow Z$ . Then  $F$  is *continuous in each variable separately* if and only if:

- for every  $a \in X$  the function  $\lambda y \in Y. F(a, y)$  is continuous;
- for every  $b \in Y$  the function  $\lambda x \in X. F(x, b)$  is continuous.

**Proposition 129.** Let  $F : X \times Y \rightarrow Z$ . If  $F$  is continuous then  $F$  is continuous in each variable separately.

PROOF: For  $a \in X$ , the function  $\lambda y \in Y. F(a, y)$  is  $F \circ i$  where  $i : Y \rightarrow X \times Y$  maps  $y$  to  $(a, y)$ . We have  $i$  is continuous by Proposition 123, hence  $F \circ i$  is continuous by Theorem 103.

Similarly for  $\lambda x \in X. F(x, b)$  for  $b \in Y$ . □

**Example 130.** Define  $F : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  by

$$F(x, y) = \begin{cases} xy/(x^2 + y^2) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Then  $F$  is continuous in each variable separately but not continuous.

## 22 The Box Topology

**Definition 131** (Box Topology). Let  $\{A_i\}_{i \in I}$  be a family of topological spaces. The *box topology* on  $\prod_{i \in I} A_i$  is the topology generated by the set of all sets of the form  $\prod_{i \in I} U_i$  where  $\{U_i\}_{i \in I}$  is a family such that each  $U_i$  is open in  $A_i$ .

This is a basis since it covers  $\prod_{i \in I} A_i$  and is closed under intersection.

**Proposition 132.** *The box topology is finer than the product topology.*

PROOF: From Proposition 32.  $\square$

**Corollary 132.1.** *If  $A_i$  is closed in  $X_i$  for all  $i \in I$  then  $\prod_{i \in I} A_i$  is closed in  $\prod_{i \in I} X_i$  under the box topology.*

PROOF: From Proposition 53.

**Proposition 133 (AC).** *Let  $\{A_i\}_{i \in I}$  be a family of topological spaces. For  $i \in I$ , let  $\mathcal{B}_i$  be a basis for the topology on  $A_i$ . Then  $\mathcal{B} = \{\prod_{i \in I} B_i \mid \forall i \in I, B_i \in \mathcal{B}_i\}$  is a basis for the box topology on  $\prod_{i \in I} A_i$ .*

PROOF:

- $\langle 1 \rangle 1$ . Every set of the form  $\prod_{i \in I} B_i$  is open.
- $\langle 1 \rangle 2$ . For every point  $a \in \prod_{i \in I} A_i$  and every open set  $U$  with  $a \in U$ , there exists  $B \in \mathcal{B}$  such that  $a \in B \subseteq U$ .
  - $\langle 2 \rangle 1$ . LET:  $U$  be open and  $a \in U$
  - $\langle 2 \rangle 2$ . PICK a family  $\{U_i\}_{i \in I}$  such that each  $U_i$  is open in  $A_i$  and  $a \in \prod_{i \in I} U_i \subseteq U$ .
  - $\langle 2 \rangle 3$ . For  $i \in I$ , PICK  $B_i \in \mathcal{B}_i$  such that  $a_i \in B_i \subseteq U_i$
- PROOF: Using the Axiom of Choice.
- $\langle 2 \rangle 4$ .  $a \in \prod_{i \in I} B_i \subseteq U$
- $\langle 1 \rangle 3$ . Q.E.D.

PROOF: Lemma 15.

$\square$

**Theorem 134.** *Let  $\{X_i\}_{i \in I}$  be a family of topological spaces. Let  $A_i$  be a subspace of  $X_i$  for all  $i \in I$ . Give  $\prod_{i \in I} X_i$  the box topology. Then the box topology on  $\prod_{i \in I} A_i$  is the same as the topology it inherits as a subspace of  $\prod_{i \in I} X_i$ .*

PROOF: The box topology is generated by the basis

$$\begin{aligned}
& \left\{ \prod_{i \in I} U_i \mid \forall i \in I, U_i \text{ open in } A_i \right\} \\
&= \left\{ \prod_{i \in I} (V_i \cap A_i) \mid \forall i \in I, V_i \text{ open in } X_i \right\} \\
&= \left\{ \prod_{i \in I} V_i \mid \forall i \in I, V_i \text{ open in } X_i \right\} \cap \prod_{i \in I} A_i
\end{aligned}$$

and this is a basis for the subspace topology by Lemma 37.  $\square$

**Proposition 135.** *Let  $\{X_i\}_{i \in I}$  be a family of Hausdorff spaces. Then  $\prod_{i \in I} X_i$  under the box topology is Hausdorff.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $\{X_i\}_{i \in I}$  be a family of Hausdorff spaces.
- $\langle 1 \rangle 2$ . LET:  $a, b \in \prod_{i \in I} X_i$  with  $a \neq b$

- ⟨1⟩3. PICK  $i \in I$  such that  $a_i \neq b_i$   
 ⟨1⟩4. PICK  $U, V$  disjoint open sets in  $X_i$  with  $a_i \in U$  and  $b_i \in V$   
 ⟨1⟩5.  $\pi_i^{-1}(U)$  and  $\pi_i^{-1}(V)$  are disjoint open sets in  $\prod_{i \in I} X_i$  with  $a \in \pi_i^{-1}(U)$   
 and  $b \in \pi_i^{-1}(V)$

□

**Proposition 136 (AC).** *Let  $\{X_i\}_{i \in I}$  be a family of topological spaces. Give  $\prod_{i \in I} X_i$  the box topology. Let  $A_i \subseteq X_i$  for all  $i \in I$ . Then*

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i} .$$

PROOF:

⟨1⟩1.  $\overline{\prod_{i \in I} A_i} \subseteq \prod_{i \in I} \overline{A_i}$

⟨2⟩1. For all  $i \in I$  we have  $A_i \subseteq \overline{A_i}$

PROOF: Lemma 63.

⟨2⟩2.  $\prod_{i \in I} A_i \subseteq \prod_{i \in I} \overline{A_i}$

⟨2⟩3. Q.E.D.

PROOF: Since  $\prod_{i \in I} \overline{A_i}$  is closed by Corollary 132.1.

⟨1⟩2.  $\prod_{i \in I} \overline{A_i} \subseteq \overline{\prod_{i \in I} A_i}$

⟨2⟩1. LET:  $x \in \prod_{i \in I} \overline{A_i}$

⟨2⟩2. LET:  $U$  be a neighbourhood of  $x$

⟨2⟩3. PICK  $V_i$  open in  $X_i$  such that  $x \in \prod_{i \in I} V_i \subseteq U$

⟨2⟩4. For  $i \in I$ , pick  $a_i \in V_i \cap A_i$

PROOF: By Theorem 66 and ⟨2⟩1 using the Axiom of Choice.

⟨2⟩5.  $U$  intersects  $\prod_{i \in I} A_i$

⟨2⟩6. Q.E.D.

PROOF:  $a \in U \cap \prod_{i \in I} A_i$ .

□

The following example shows that Theorem 108 fails in the box topology.

**Example 137.** Define  $f : \mathbb{R} \rightarrow \mathbb{R}^\omega$  by  $f(t) = (t, t, \dots)$ . Then  $\pi_n \circ f = \text{id}_{\mathbb{R}}$  is continuous for all  $n$ . But  $f$  is not continuous when  $\mathbb{R}^\omega$  is given the box topology because the inverse image of

$$(-1, 1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times \dots$$

is  $\{0\}$  which is not open.

The following example shows that Proposition 117 fails in the box topology.

**Example 138.** Give  $\mathbb{R}^\omega$  the box topology. Let  $a_n = (1/n, 1/n, \dots)$  for  $n \geq 1$  and  $l = (0, 0, \dots)$ . Then  $\pi_i(a_n) \rightarrow \pi_i(l)$  as  $n \rightarrow \infty$  for all  $i$ , but  $a_n \not\rightarrow l$  as  $n \rightarrow \infty$  since the open set

$$(-1, 1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times \dots$$

contains  $l$  but does not contain any  $a_n$ .