

# Topology

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## 1 Set Theory

**Definition 1** (Cover). Let  $X$  be a set and  $\mathcal{A} \subseteq \mathcal{P}X$ . Then  $\mathcal{A}$  *covers*  $X$ , or is a *covering* of  $X$ , if and only if  $\bigcup \mathcal{A} = X$ .

**Definition 2** (Finite Intersection Property). Let  $X$  be a set and  $\mathcal{A} \subseteq \mathcal{P}X$ . Then  $\mathcal{A}$  satisfies the *finite intersection property* if and only if every nonempty finite subset of  $\mathcal{A}$  has nonempty intersection.

## 2 Order Theory

**Definition 3** (Preorder). Let  $X$  be a set. A *preorder* on  $X$  is a binary relation  $\leq$  on  $X$  such that:

**Reflexivity** For all  $x \in X$ , we have  $x \leq x$

**Transitivity** For all  $x, y, z \in X$ , if  $x \leq y$  and  $y \leq z$  then  $x \leq z$ .

**Definition 4** (Preordered Set). A *preordered set* consists of a set  $X$  and a preorder  $\leq$  on  $X$ .

**Proposition 5.** Let  $X$  and  $Y$  be linearly ordered sets. Let  $f : X \rightarrow Y$  be strictly monotone and surjective. Then  $f$  is a poset isomorphism.

PROOF:

$\langle 1 \rangle 1.$   $f$  is injective.

$\langle 2 \rangle 1.$  LET:  $x, y \in X$

$\langle 2 \rangle 2.$  ASSUME:  $f(x) = f(y)$

$\langle 2 \rangle 3.$   $x \not\leq y$

PROOF: By strong monotonicity.

$\langle 2 \rangle 4.$   $y \not\leq x$

PROOF: By strong monotonicity.

$\langle 2 \rangle 5.$   $x = y$

PROOF: By trichotomy.

$\langle 1 \rangle 2.$   $f^{-1}$  is monotone.

$\langle 2 \rangle 1.$  LET:  $x, y \in X$

$\langle 2 \rangle 2.$  ASSUME:  $x \leq y$

$\langle 2 \rangle 3.$   $f^{-1}(x) \not\leq f^{-1}(y)$

PROOF: By strong monotonicity.

$\langle 2 \rangle 4.$   $f^{-1}(x) < f^{-1}(y)$

PROOF: By trichotomy.

□

**Definition 6** (Interval). Let  $X$  be a preordered set and  $Y \subseteq X$ . Then  $Y$  is an *interval* if and only if, for all  $a, b \in Y$  and  $c \in X$ , if  $a \leq c \leq b$  then  $c \in Y$ .

**Definition 7** (Linear Continuum). A linearly ordered set  $L$  is a *linear continuum* if and only if:

1. every nonempty subset of  $L$  that is bounded above has a supremum
2.  $L$  is dense

**Proposition 8.** Every interval in a linear continuum is a linear continuum.

PROOF:

$\langle 1 \rangle 1.$  LET:  $L$  be a linear continuum and  $I$  an interval in  $L$ .

$\langle 1 \rangle 2.$  Every nonempty subset of  $I$  that is bounded above has a supremum in  $I$ .

$\langle 2 \rangle 1.$  LET:  $X \subseteq I$  be nonempty and bounded above by  $b \in I$ .

$\langle 2 \rangle 2$ . LET:  $s$  be the supremum of  $X$  in  $L$ .  
 PROOF: Since  $L$  is a linear continuum.  
 $\langle 2 \rangle 3$ .  $s \in I$   
 $\langle 3 \rangle 1$ . PICK  $a \in X$   
 PROOF: Since  $X$  is nonempty ( $\langle 2 \rangle 1$ ).  
 $\langle 3 \rangle 2$ .  $a \leq s \leq b$   
 $\langle 3 \rangle 3$ .  $s \in I$   
 PROOF: Since  $I$  is an interval ( $\langle 1 \rangle 1$ ).  
 $\langle 2 \rangle 4$ .  $s$  is the supremum of  $X$  in  $I$   
 $\langle 1 \rangle 3$ .  $I$  is dense.  
 $\langle 2 \rangle 1$ . LET:  $x, y \in I$  with  $x < y$   
 $\langle 2 \rangle 2$ . PICK  $z \in I$  with  $x < z < y$   
 PROOF: Since  $L$  is dense.  
 $\langle 2 \rangle 3$ .  $z \in I$   
 PROOF: Since  $I$  is an interval.

□

**Definition 9** (Ordered Square). The *ordered square*  $I_o^2$  is the set  $[0, 1]^2$  under the dictionary order.

**Proposition 10.** *The ordered square is a linear continuum.*

PROOF:

$\langle 1 \rangle 1$ . Every nonempty subset of  $I_o^2$  bounded above has a supremum.  
 $\langle 2 \rangle 1$ . LET:  $X \subseteq I_o^2$  be nonempty and bounded above by  $(b, c)$   
 $\langle 2 \rangle 2$ . LET:  $s = \sup \pi_1(X)$   
 PROOF: The set  $\pi_1(X)$  is nonempty and bounded above by  $b$ .  
 $\langle 2 \rangle 3$ . CASE:  $s \in \pi_1(X)$   
 $\langle 3 \rangle 1$ . LET:  $t = \sup \{y \in [0, 1] \mid (s, y) \in X\}$   
 PROOF: This set is nonempty and bounded above by  $c$ .  
 $\langle 3 \rangle 2$ .  $(s, t)$  is the supremum of  $X$ .  
 $\langle 2 \rangle 4$ . CASE:  $s \notin \pi_1(X)$   
 PROOF: In this case  $(s, 0)$  is the supremum of  $X$ .  
 $\langle 1 \rangle 2$ .  $I_o^2$  is dense.  
 $\langle 2 \rangle 1$ . LET:  $(x_1, y_1), (x_2, y_2) \in I_o^2$  with  $(x_1, y_1) < (x_2, y_2)$   
 $\langle 2 \rangle 2$ . CASE:  $x_1 < x_2$   
 $\langle 3 \rangle 1$ . PICK  $x_3$  with  $x_1 < x_3 < x_2$   
 $\langle 3 \rangle 2$ .  $(x_1, y_1) < (x_3, y_1) < (x_2, y_2)$   
 $\langle 2 \rangle 3$ . CASE:  $x_1 = x_2$  and  $y_1 < y_2$   
 $\langle 3 \rangle 1$ . PICK  $y_3$  with  $y_1 < y_3 < y_2$   
 $\langle 3 \rangle 2$ .  $(x_1, y_1) < (x_1, y_3) < (x_2, y_2)$

□

**Proposition 11.** *If  $X$  is a well-ordered set then  $X \times [0, 1)$  under the dictionary order is a linear continuum.*

PROOF:

$\langle 1 \rangle 1$ . Every nonempty set  $A \subseteq X \times [0, 1)$  bounded above has a supremum.

- ⟨2⟩1. LET:  $A \subseteq X \times [0, 1)$  be nonempty and bounded above
- ⟨2⟩2. LET:  $x_0$  be the supremum of  $\pi_1(A)$
- ⟨2⟩3. CASE:  $x_0 \in \pi_1(A)$ 
  - ⟨3⟩1. LET:  $y_0$  be the supremum of  $\{y \in [0, 1) \mid (x_0, y) \in A\}$
  - ⟨3⟩2.  $(x_0, y_0)$  is the supremum of  $A$ .
- ⟨2⟩4. CASE:  $x_0 \notin \pi_1(A)$ 
  - PROOF: In this case  $(x_0, 0)$  is the supremum of  $A$ .
- ⟨1⟩2.  $X \times [0, 1)$  is dense.
  - ⟨2⟩1. LET:  $(x_1, y_1), (x_2, y_2) \in X \times [0, 1)$  with  $(x_1, y_1) < (x_2, y_2)$
  - ⟨2⟩2. CASE:  $x_1 < x_2$ 
    - ⟨3⟩1. PICK  $y_3$  such that  $y_1 < y_3 < 1$
    - ⟨3⟩2.  $(x_1, y_1) < (x_1, y_3) < (x_2, y_2)$
  - ⟨2⟩3. CASE:  $x_1 = x_2$  and  $y_1 < y_2$ 
    - ⟨3⟩1. PICK  $y_3$  such that  $y_1 < y_3 < y_2$
    - ⟨3⟩2.  $(x_1, y_1) < (x_1, y_3) < (x_2, y_2)$

□

**Lemma 12.** For all  $a, b, c, d \in \mathbb{R}$  with  $a < b$  and  $c < d$ , we have  $[a, b) \cong [c, d)$

PROOF: The map  $\lambda t.c + (t - a)(d - c)/(b - a)$  is an order isomorphism.

**Proposition 13.** Let  $X$  be a linearly ordered set. Let  $a < b < c$  in  $X$ . Then  $[a, c) \cong [0, 1)$  if and only if  $[a, b) \cong [b, c) \cong [0, 1)$ .

PROOF:

- ⟨1⟩1. If  $[a, c) \cong [0, 1)$  then  $[a, b) \cong [b, c) \cong [0, 1)$
- ⟨2⟩1. ASSUME:  $f : [a, c) \cong [0, 1)$  is an order isomorphism
- ⟨2⟩2.  $[a, b) \cong [0, 1)$ 
  - PROOF:
$$\begin{aligned} [a, b) &\cong [0, f(b)) && \text{(by the restriction of } f) \\ &\cong [0, 1) && \text{(Lemma 12)} \end{aligned}$$
- ⟨2⟩3.  $[b, c) \cong [0, 1)$ 
  - PROOF: Similar.
- ⟨1⟩2. If  $[a, b) \cong [b, c) \cong [0, 1)$  then  $[a, c) \cong [0, 1)$ 
  - PROOF:
$$\begin{aligned} [a, c) &= [a, b) * [b, c) \\ &\cong [0, 1) * [0, 1) \\ &\cong [0, 1/2) * [1/2, 1) && \text{(Lemma 12)} \\ &= 1 \end{aligned}$$

□

**Proposition 14 (CC).** Let  $X$  be a linearly ordered set. Let  $x_0 < x_1 < \dots$  be a strictly increasing sequence in  $X$  with supremum  $b$ . Then  $[x_0, b) \cong [0, 1)$  if and only if  $[x_i, x_{i+1}) \cong [0, 1)$  for all  $i$ .

PROOF:

- ⟨1⟩1. If  $[x_0, b) \cong [0, 1)$  then  $[x_i, x_{i+1}) \cong [0, 1)$  for all  $i$ .

PROOF: By Lemma 12

$\langle 1 \rangle 2$ . If  $[x_i, x_{i+1}) \cong [0, 1)$  for all  $i$  then  $[x_0, b) \cong [0, 1)$

$\langle 2 \rangle 1$ . ASSUME:  $[x_i, x_{i+1}) \cong [0, 1)$  for all  $i$

$\langle 2 \rangle 2$ . PICK an order isomorphism  $f_i : [x_i, x_{i+1}) \cong [1/2^i, 2/2^{i+1})$  for each  $i$ .

PROOF: By Lemma 12

$\langle 2 \rangle 3$ . The union of the  $f_i$ s is an order isomorphism  $[x_0, b) \cong [0, 1)$

□

### 3 Real Analysis

**Definition 15.** Let  $\mathbb{R}^\infty$  be the subset of  $\mathbb{R}^\omega$  consisting of all sequences  $(a_n)$  such that  $a_n = 0$  for all but finitely many  $n$ .

## 4 Group Theory

**Definition 16.** Given a group  $G$  and sets  $A, B \subseteq G$ , let  $AB = \{ab \mid a \in A, b \in B\}$ .

**Definition 17.** Given a group  $G$  and a set  $A \subseteq G$ , let  $A^{-1} = \{a^{-1} \mid a \in A\}$ .



## 5 Topological Spaces

**Definition 18** (Topology). A *topology* on a set  $X$  is a set  $\mathcal{T} \subseteq \mathcal{P}X$  such that:

- $X \in \mathcal{T}$ .
- For all  $\mathcal{U} \subseteq \mathcal{T}$  we have  $\bigcup \mathcal{U} \in \mathcal{T}$ .
- For all  $U, V \in \mathcal{T}$  we have  $U \cap V \in \mathcal{T}$ .

We call the elements of  $X$  *points* and the elements of  $\mathcal{T}$  *open sets*.

**Definition 19** (Topological Space). A *topological space*  $X$  consists of a set  $X$  and a topology on  $X$ .

**Definition 20** (Discrete Space). For any set  $X$ , the *discrete* topology on  $X$  is  $\mathcal{P}X$ .

**Definition 21** (Indiscrete Space). For any set  $X$ , the *indiscrete* or *trivial* topology on  $X$  is  $\{\emptyset, X\}$ .

**Definition 22** (Finite Complement Topology). For any set  $X$ , the *finite complement topology* on  $X$  is  $\{U \in \mathcal{P}X \mid X \setminus U \text{ is finite}\} \cup \{\emptyset\}$ .

**Definition 23** (Countable Complement Topology). For any set  $X$ , the *countable complement topology* on  $X$  is  $\{U \in \mathcal{P}X \mid X \setminus U \text{ is countable}\} \cup \{\emptyset\}$ .

**Definition 24** (Finer, Coarser). Suppose that  $\mathcal{T}$  and  $\mathcal{T}'$  are two topologies on a given set  $X$ . If  $\mathcal{T}' \supseteq \mathcal{T}$ , we say that  $\mathcal{T}'$  is *finer* than  $\mathcal{T}$ ; if  $\mathcal{T}'$  *properly* contains  $\mathcal{T}$ , we say that  $\mathcal{T}'$  is *strictly finer* than  $\mathcal{T}$ . We also say that  $\mathcal{T}$  is *coarser* than  $\mathcal{T}'$ , or *strictly coarser*, in these two respective situations. We say  $\mathcal{T}$  is *comparable* with  $\mathcal{T}'$  if either  $\mathcal{T}' \supseteq \mathcal{T}$  or  $\mathcal{T} \supseteq \mathcal{T}'$ .

**Lemma 25.** Let  $X$  be a topological space and  $U \subseteq X$ . Then  $U$  is open if and only if, for all  $x \in U$ , there exists an open set  $V$  such that  $x \in V \subseteq U$ .

PROOF:

$\langle 1 \rangle 1. \Rightarrow$

PROOF: Take  $V = U$

$\langle 1 \rangle 2. \Leftarrow$

PROOF: We have  $U = \bigcup \{V \text{ open in } X \mid V \subseteq U\}$ .

□

**Lemma 26.** Let  $X$  be a set and  $\mathcal{T}$  a nonempty set of topologies on  $X$ . Then  $\bigcap \mathcal{T}$  is a topology on  $X$ , and is the finest topology that is coarser than every member of  $\mathcal{T}$ .

PROOF:

$\langle 1 \rangle 1. X \in \bigcap \mathcal{T}$

PROOF: Since  $X$  is in every member of  $\mathcal{T}$ .

$\langle 1 \rangle 2. \bigcap \mathcal{T}$  is closed under union.

- ⟨2⟩1. LET:  $\mathcal{U} \subseteq \bigcap \mathcal{T}$
- ⟨2⟩2. For all  $T \in \mathcal{T}$  we have  $\mathcal{U} \subseteq T$
- ⟨2⟩3. For all  $T \in \mathcal{T}$  we have  $\bigcup \mathcal{U} \in T$
- ⟨2⟩4.  $\bigcup \mathcal{U} \in \bigcap \mathcal{T}$
- ⟨1⟩3.  $\bigcap \mathcal{T}$  is closed under binary intersection.
- ⟨2⟩1. LET:  $U, V \in \bigcap \mathcal{T}$
- ⟨2⟩2. For all  $T \in \mathcal{T}$  we have  $U, V \in T$
- ⟨2⟩3. For all  $T \in \mathcal{T}$  we have  $U \cap V \in T$
- ⟨2⟩4.  $U \cap V \in \bigcap \mathcal{T}$

□

**Lemma 27.** *Let  $X$  be a set and  $\mathcal{T}$  a set of topologies on  $X$ . Then there exists a unique coarsest topology that is finer than every member of  $\mathcal{T}$ .*

PROOF: The required topology is given by

$$\bigcap \{T \in \mathcal{P}X \mid T \text{ is a topology on } X \text{ that is finer than every member of } \mathcal{T}\},$$

The set is nonempty since it contains the discrete topology. □

**Definition 28** (Neighbourhood). A *neighbourhood* of a point  $x$  is an open set that contains  $x$ .

## 6 Closed Set

**Definition 29** (Closed Set). Let  $X$  be a topological space and  $A \subseteq X$ . Then  $A$  is *closed* if and only if  $X \setminus A$  is open.

**Lemma 30.** *The empty set is closed.*

PROOF: Since the whole space  $X$  is always open. □

**Lemma 31.** *The topological space  $X$  is closed.*

PROOF: Since  $\emptyset$  is open. □

**Lemma 32.** *The intersection of a nonempty set of closed sets is closed.*

PROOF: Let  $\mathcal{C}$  be a nonempty set of closed sets. Then  $X \setminus \bigcap \mathcal{C} = \bigcup \{X \setminus C \mid C \in \mathcal{C}\}$  is open. □

**Lemma 33.** *The union of two closed sets is closed.*

PROOF: Let  $C$  and  $D$  be closed. Then  $X \setminus (C \cup D) = (X \setminus C) \cap (X \setminus D)$  is open. □

**Proposition 34.** *Let  $X$  be a set and  $\mathcal{C} \subseteq \mathcal{P}X$  a set such that:*

1.  $\emptyset \in \mathcal{C}$
2.  $X \in \mathcal{C}$
3. For all  $\mathcal{A} \subseteq \mathcal{C}$  nonempty we have  $\bigcap \mathcal{A} \in \mathcal{C}$

4. For all  $C, D \in \mathcal{C}$  we have  $C \cup D \in \mathcal{C}$ .

Then there exists a unique topology  $\mathcal{T}$  such that  $\mathcal{C}$  is the set of closed sets, namely

$$\mathcal{T} = \{X \setminus C \mid C \in \mathcal{C}\}$$

PROOF:

$\langle 1 \rangle 1$ . LET:  $\mathcal{T} = \{X \setminus C \mid C \in \mathcal{C}\}$

$\langle 1 \rangle 2$ .  $\mathcal{T}$  is a topology

$\langle 2 \rangle 1$ .  $X \in \mathcal{T}$

PROOF: Since  $\emptyset \in \mathcal{C}$

$\langle 2 \rangle 2$ . For all  $\mathcal{U} \subseteq \mathcal{T}$  we have  $\bigcup \mathcal{U} \in \mathcal{T}$

$\langle 3 \rangle 1$ . LET:  $\mathcal{U} \subseteq \mathcal{T}$

$\langle 3 \rangle 2$ . CASE:  $\mathcal{U} = \emptyset$

PROOF: In this case  $\bigcup \mathcal{U} = \emptyset \in \mathcal{T}$  since  $X \in \mathcal{C}$

$\langle 3 \rangle 3$ . CASE:  $\mathcal{U} \neq \emptyset$

PROOF: In this case  $X \setminus \bigcup \mathcal{U} = \bigcap \{X \setminus U \mid U \in \mathcal{U}\} \in \mathcal{C}$ .

$\langle 2 \rangle 3$ . For all  $U, V \in \mathcal{T}$  we have  $U \cap V \in \mathcal{T}$

PROOF: Since  $X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V) \in \mathcal{C}$ .

$\langle 1 \rangle 3$ .  $\mathcal{C}$  is the set of all closed sets in  $\mathcal{T}$

PROOF:

$C$  is closed in  $\mathcal{T}$

$$\Leftrightarrow X \setminus C \in \mathcal{T}$$

$$\Leftrightarrow C \in \mathcal{C}$$

$\langle 1 \rangle 4$ . If  $\mathcal{T}'$  is a topology and  $\mathcal{C}$  is the set of closed sets in  $\mathcal{T}'$  then  $\mathcal{T}' = \mathcal{T}$

PROOF: We have

$$U \in \mathcal{T}$$

$$\Leftrightarrow X \setminus U \in \mathcal{C}$$

$$\Leftrightarrow X \setminus U \text{ is closed in } \mathcal{T}'$$

$$\Leftrightarrow U \in \mathcal{T}'$$

□

**Proposition 35.** If  $U$  is open and  $A$  is closed then  $U \setminus A$  is open.

PROOF:  $U \setminus A = U \cap (X \setminus A)$  is the intersection of two open sets. □

**Proposition 36.** If  $U$  is open and  $A$  is closed then  $A \setminus U$  is closed.

PROOF:  $A \setminus U = A \cap (X \setminus U)$  is the intersection of two closed sets. □

## 7 Interior

**Definition 37** (Interior). Let  $X$  be a topological space and  $A \subseteq X$ . The *interior* of  $A$ ,  $\text{Int } A$ , is the union of all the open subsets of  $A$ .

**Lemma 38.** The interior of a set is open.

PROOF: It is a union of open sets.  $\square$

**Lemma 39.**

$$\text{Int } A \subseteq A$$

PROOF: Immediate from definition.  $\square$

**Lemma 40.** *If  $U$  is open and  $U \subseteq A$  then  $U \subseteq \text{Int } A$*

PROOF: Immediate from definition.  $\square$

**Lemma 41.** *A set  $A$  is open if and only if  $A = \text{Int } A$ .*

PROOF: If  $A = \text{Int } A$  then  $A$  is open by Lemma 38. Conversely if  $A$  is open then  $A \subseteq \text{Int } A$  by the definition of interior and so  $A = \text{Int } A$ .

## 8 Closure

**Definition 42** (Closure). Let  $X$  be a topological space and  $A \subseteq X$ . The *closure* of  $A$ ,  $\overline{A}$ , is the intersection of all the closed sets that include  $A$ .

This intersection exists since  $X$  is a closed set that includes  $A$  (Lemma 31).

**Lemma 43.** *The closure of a set is closed.*

PROOF: Dual to Lemma 38.  $\square$

**Lemma 44.**

$$A \subseteq \overline{A}$$

PROOF: Immediate from definition.  $\square$

**Lemma 45.** *If  $C$  is closed and  $A \subseteq C$  then  $\overline{A} \subseteq C$ .*

PROOF: Immediate from definition.  $\square$

**Lemma 46.** *A set  $A$  is closed if and only if  $A = \overline{A}$ .*

PROOF: Dual to Lemma 41.  $\square$

**Theorem 47.** *Let  $X$  be a topological space,  $A \subseteq X$  and  $x \in X$ . Then  $x \in \overline{A}$  if and only if every neighbourhood of  $x$  intersects  $A$ .*

PROOF: We have

$$\begin{aligned} x \in \overline{A} \\ \Leftrightarrow \forall C. C \text{ closed} \wedge A \subseteq C \Rightarrow x \in C \\ \Leftrightarrow \forall U. U \text{ open} \wedge A \cap U = \emptyset \Rightarrow x \notin U \\ \Leftrightarrow \forall U. U \text{ open} \wedge x \in U \Rightarrow U \text{ intersects } A \end{aligned} \quad \square$$

**Proposition 48.** *If  $A \subseteq B$  then  $\overline{A} \subseteq \overline{B}$ .*

PROOF: This holds because  $\overline{B}$  is a closed set that includes  $A$ .  $\square$

**Proposition 49.**

$$\overline{A \cup B} = \overline{A} \cup \overline{B}$$

PROOF:

$\langle 1 \rangle 1. \overline{A} \subseteq \overline{A \cup B}$

PROOF: By Proposition 48.

$\langle 1 \rangle 2. \overline{B} \subseteq \overline{A \cup B}$

PROOF: By Proposition 48.

$\langle 1 \rangle 3. \overline{A \cup B} \subseteq \overline{A \cup B}$

$\langle 2 \rangle 1. \text{ LET: } x \in \overline{A \cup B}$

$\langle 2 \rangle 2. \text{ ASSUME: } x \notin \overline{A}$

PROVE:  $x \in \overline{B}$

$\langle 2 \rangle 3. \text{ PICK a neighbourhood } U \text{ of } x \text{ that does not intersect } A$

$\langle 2 \rangle 4. \text{ LET: } V \text{ be any neighbourhood of } x$

$\langle 2 \rangle 5. U \cap V \text{ is a neighbourhood of } x$

$\langle 2 \rangle 6. U \cap V \text{ intersects } A \cup B$

PROOF: From  $\langle 2 \rangle 1$  and Theorem 47.

$\langle 2 \rangle 7. U \cap V \text{ intersects } B$

PROOF: From  $\langle 2 \rangle 3$

$\langle 2 \rangle 8. V \text{ intersects } B$

$\langle 2 \rangle 9. \text{ Q.E.D.}$

PROOF: We have  $x \in \overline{B}$  from Theorem 47.

$\square$

## 9 Boundary

**Definition 50** (Boundary). The *boundary* of a set  $A$  is the set  $\partial A = \overline{A} \cap \overline{X \setminus A}$ .

**Proposition 51.**

$$\text{Int } A \cap \partial A = \emptyset$$

PROOF: Since  $\overline{X \setminus A} = X \setminus \text{Int } A$ .  $\square$

**Proposition 52.**

$$\overline{A} = \text{Int } A \cup \partial A$$

PROOF:

$$\begin{aligned} \text{Int } A \cup \partial A &= \text{Int } A \cup (\overline{A} \cap \overline{X \setminus A}) \\ &= (\text{Int } A \cup \overline{A}) \cap (\text{Int } A \cup \overline{X \setminus A}) \\ &= \overline{A} \cap X \\ &= \overline{A} \end{aligned}$$

**Proposition 53.**  $\partial A = \emptyset$  if and only if  $A$  is open and closed.

PROOF: If  $\partial A = \emptyset$  then  $\overline{A} = \text{Int } A$  by Proposition 52.

**Proposition 54.** *A set  $U$  is open if and only if  $\partial U = \overline{U} \setminus U$ .*

PROOF:

$$\begin{aligned} \partial U &= \overline{U} \setminus U \\ \Leftrightarrow \overline{U} \setminus \text{Int } U &= \overline{U} \setminus U && (\text{Propositions 51, 52}) \\ \Leftrightarrow \text{Int } U &= U && \square \end{aligned}$$

## 10 Limit Points

**Definition 55** (Limit Point). Let  $X$  be a topological space,  $a \in X$  and  $A \subseteq X$ . Then  $a$  is a *limit point*, *cluster point* or *point of accumulation* for  $A$  if and only if every neighbourhood of  $a$  intersects  $A$  at a point other than  $a$ .

**Lemma 56.** *The point  $a$  is an accumulation point for  $A$  if and only if  $a \in \overline{A \setminus \{a\}}$ .*

PROOF: From Theorem 47.  $\square$

**Theorem 57.** *Let  $X$  be a topological space and  $A \subseteq X$ . Let  $A'$  be the set of all limit points of  $A$ . Then  $\overline{A} = A \cup A'$ .*

PROOF:

$\langle 1 \rangle 1$ . For all  $x \in \overline{A}$ , if  $x \notin A$  then  $x \in A'$

PROOF: From Theorem 47.

$\langle 1 \rangle 2$ .  $A \subseteq \overline{A}$

PROOF: Lemma 44.

$\langle 1 \rangle 3$ .  $A' \subseteq \overline{A}$

PROOF: From Theorem 47.

$\square$

**Corollary 57.1.** *A set is closed if and only if it contains all its limit points.*

**Proposition 58.** *In an indiscrete topology, every point is a limit point of any set that has more than one point.*

PROOF: Let  $X$  be an indiscrete space. Let  $A$  be a set with more than one point and  $x$  be a point. The only neighbourhood of  $x$  is  $X$ , which must intersect  $A$  at a point other than  $x$ .  $\square$

**Lemma 59.** *Let  $X$  be a topological space and  $A \subseteq B \subseteq X$ . Then every limit point of  $A$  is a limit point of  $B$ .*

PROOF: Immediate from definitions.  $\square$

## 11 Basis for a Topology

**Definition 60** (Basis). If  $X$  is a set, a *basis* for a topology on  $X$  is a set  $\mathcal{B} \subseteq \mathcal{P}X$  called *basis elements* such that

1. For all  $x \in X$ , there exists  $B \in \mathcal{B}$  such that  $x \in B$ .
2. For all  $B_1, B_2 \in \mathcal{B}$  and  $x \in B_1 \cap B_2$ , there exists  $B_3 \in \mathcal{B}$  such that  $x \in B_3 \subseteq B_1 \cap B_2$ .

If  $\mathcal{B}$  satisfies these two conditions, then we define the topology *generated* by  $\mathcal{B}$  to be  $\mathcal{T} = \{U \in \mathcal{P}X \mid \forall x \in U. \exists B \in \mathcal{B}. x \in B \subseteq U\}$ .

We prove this is a topology.

PROOF:

$\langle 1 \rangle 1.$   $X \in \mathcal{T}$

PROOF: For all  $x \in X$  there exists  $B \in \mathcal{B}$  such that  $x \in B \subseteq X$  by condition 1.

$\langle 1 \rangle 2.$  For all  $\mathcal{U} \subseteq \mathcal{T}$  we have  $\bigcup \mathcal{U} \in \mathcal{T}$

$\langle 2 \rangle 1.$  LET:  $\mathcal{U} \subseteq \mathcal{T}$

$\langle 2 \rangle 2.$  LET:  $x \in \bigcup \mathcal{U}$

$\langle 2 \rangle 3.$  PICK  $U \in \mathcal{U}$  such that  $x \in U$

$\langle 2 \rangle 4.$  PICK  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$

PROOF: Since  $U \in \mathcal{T}$  by  $\langle 2 \rangle 1$  and  $\langle 2 \rangle 3$ .

$\langle 2 \rangle 5.$   $x \in B \subseteq \bigcup \mathcal{U}$

$\langle 1 \rangle 3.$  For all  $U, V \in \mathcal{T}$  we have  $U \cap V \in \mathcal{T}$

$\langle 2 \rangle 1.$  LET:  $U, V \in \mathcal{T}$

$\langle 2 \rangle 2.$  LET:  $x \in U \cap V$

$\langle 2 \rangle 3.$  PICK  $B_1 \in \mathcal{B}$  such that  $x \in B_1 \subseteq U$

$\langle 2 \rangle 4.$  PICK  $B_2 \in \mathcal{B}$  such that  $x \in B_2 \subseteq V$

$\langle 2 \rangle 5.$  PICK  $B_3 \in \mathcal{B}$  such that  $x \in B_3 \subseteq B_1 \cap B_2$

PROOF: By condition 2.

$\langle 2 \rangle 6.$   $x \in B_3 \subseteq U \cap V$

□

**Lemma 61.** *Let  $X$  be a set. Let  $\mathcal{B}$  be a basis for a topology  $\mathcal{T}$  on  $X$ . Then  $\mathcal{T}$  is the set of all unions of subsets of  $\mathcal{B}$ .*

PROOF:

$\langle 1 \rangle 1.$  For all  $U \in \mathcal{T}$ , there exists  $\mathcal{A} \subseteq \mathcal{B}$  such that  $U = \bigcup \mathcal{A}$

$\langle 2 \rangle 1.$  LET:  $U \in \mathcal{T}$

$\langle 2 \rangle 2.$  LET:  $\mathcal{A} = \{B \in \mathcal{B} \mid B \subseteq U\}$

$\langle 2 \rangle 3.$   $U \subseteq \bigcup \mathcal{A}$

$\langle 3 \rangle 1.$  LET:  $x \in U$

$\langle 3 \rangle 2.$  PICK  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$

PROOF: Since  $\mathcal{B}$  is a basis for  $\mathcal{T}$ .

$\langle 3 \rangle 3.$   $x \in B \in \mathcal{A}$

$\langle 2 \rangle 4.$   $\bigcup \mathcal{A} \subseteq U$

PROOF: From the definition of  $\mathcal{A}$  ( $\langle 2 \rangle 2$ ).

$\langle 1 \rangle 2.$  For all  $\mathcal{A} \subseteq \mathcal{B}$  we have  $\bigcup \mathcal{A} \in \mathcal{T}$

$\langle 2 \rangle 1.$   $\bigcup \mathcal{A} \in \mathcal{T}$

PROOF: If  $B \in \mathcal{B}$  and  $x \in B$ , then there exists  $B' \in \mathcal{B}$  such that  $x \in B' \subseteq B$ , namely  $B' = B$ .

⟨2⟩2. Q.E.D.

PROOF: Since  $\mathcal{T}$  is closed under union.

□

**Corollary 61.1.** *Let  $X$  be a set. Let  $\mathcal{B}$  be a basis for a topology  $\mathcal{T}$  on  $X$ . Then  $\mathcal{T}$  is the coarsest topology that includes  $\mathcal{B}$ .*

PROOF: Since every topology that includes  $\mathcal{B}$  includes all unions of subsets of  $\mathcal{B}$ . □

**Lemma 62.** *Let  $X$  be a topological space. Suppose that  $\mathcal{C}$  is a set of open sets such that, for every open set  $U$  and every point  $x \in U$ , there exists  $C \in \mathcal{C}$  such that  $x \in C \subseteq U$ . Then  $\mathcal{C}$  is a basis for the topology on  $X$ .*

PROOF:

⟨1⟩1. For all  $x \in X$ , there exists  $C \in \mathcal{C}$  such that  $x \in C$

PROOF: Immediate from hypothesis.

⟨1⟩2. For all  $C_1, C_2 \in \mathcal{C}$  and  $x \in C_1 \cap C_2$ , there exists  $C_3 \in \mathcal{C}$  such that  $x \in C_3 \subseteq C_1 \cap C_2$

PROOF: Since  $C_1 \cap C_2$  is open.

⟨1⟩3. Every open set is open in the topology generated by  $\mathcal{C}$

PROOF: Immediate from hypothesis.

⟨1⟩4. Every union of a subset of  $\mathcal{C}$  is open.

PROOF: Since every member of  $\mathcal{C}$  is open.

□

**Lemma 63.** *Let  $\mathcal{B}$  and  $\mathcal{B}'$  be bases for the topologies  $\mathcal{T}$  and  $\mathcal{T}'$  respectively on the set  $X$ . Then the following are equivalent.*

1.  $\mathcal{T} \subseteq \mathcal{T}'$

2. For all  $B \in \mathcal{B}$  and  $x \in B$ , there exists  $B' \in \mathcal{B}'$  such that  $x \in B' \subseteq B$ .

PROOF:

⟨1⟩1.  $1 \Rightarrow 2$

⟨2⟩1. ASSUME:  $\mathcal{T} \subseteq \mathcal{T}'$

⟨2⟩2. LET:  $B \in \mathcal{B}$  and  $x \in B$

⟨2⟩3.  $B \in \mathcal{T}$

PROOF: Corollary 61.1.

⟨2⟩4.  $B \in \mathcal{T}'$

PROOF: By ⟨2⟩1

⟨2⟩5. There exists  $B' \in \mathcal{B}'$  such that  $x \in B' \subseteq B$

PROOF: Since  $\mathcal{B}'$  is a basis for  $\mathcal{T}'$ .

⟨1⟩2.  $2 \Rightarrow 1$

⟨2⟩1. ASSUME: 2

⟨2⟩2. LET:  $U \in \mathcal{T}$

PROVE:  $U \in \mathcal{T}'$

⟨2⟩3. LET:  $x \in U$

PROVE: There exists  $B' \in \mathcal{B}'$  such that  $x \in B' \subseteq U$



- $\langle 2 \rangle 4$ . PICK  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$   
 PROOF: Since  $\mathcal{B}$  is a basis for  $\mathcal{T}$ .  
 $\langle 2 \rangle 5$ . PICK  $B' \in \mathcal{B}'$  such that  $x \in B' \subseteq B$   
 PROOF: By  $\langle 2 \rangle 1$ .  
 $\langle 2 \rangle 6$ .  $x \in B' \subseteq U$

□

**Theorem 64.** Let  $X$  be a topological space,  $A \subseteq X$  and  $x \in X$ . Let  $\mathcal{B}$  be a basis for  $X$ . Then  $x \in \overline{A}$  if and only if, for all  $B \in \mathcal{B}$ , if  $x \in B$  then  $B$  intersects  $A$ .

PROOF:

- $\langle 1 \rangle 1$ . If  $x \in \overline{A}$  then, for all  $B \in \mathcal{B}$ , if  $x \in B$  then  $B$  intersects  $A$ .  
 PROOF: This follows from Theorem 47 since every element of  $\mathcal{B}$  is open (Corollary 61.1).  
 $\langle 1 \rangle 2$ . Suppose that, for all  $B \in \mathcal{B}$ , if  $x \in B$  then  $B$  intersects  $A$ . Then  $x \in \overline{A}$ .  
 $\langle 2 \rangle 1$ . ASSUME: For all  $B \in \mathcal{B}$ , if  $x \in B$  then  $B$  intersects  $A$ .  
 $\langle 2 \rangle 2$ . LET:  $U$  be an open set that contains  $x$   
 PROVE:  $U$  intersects  $A$ .  
 $\langle 2 \rangle 3$ . PICK  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$ .  
 $\langle 2 \rangle 4$ .  $B$  intersects  $A$ .  
 PROOF: From  $\langle 2 \rangle 1$ .  
 $\langle 2 \rangle 5$ .  $U$  intersects  $A$ .  
 $\langle 2 \rangle 6$ . Q.E.D.  
 PROOF: By Theorem 47.

□

**Definition 65** (Lower Limit Topology on the Real Line). The *lower limit topology on the real line* is the topology on  $\mathbb{R}$  generated by the basis consisting of all half-open intervals of the form  $[a, b)$ .

We write  $\mathbb{R}_l$  for the topological space  $\mathbb{R}$  under the lower limit topology.

We prove this is a basis for a topology.

PROOF:

- $\langle 1 \rangle 1$ . For all  $x \in \mathbb{R}$  there exists an interval  $[a, b)$  such that  $x \in [a, b)$ .  
 PROOF: Take  $[a, b) = [x, x + 1)$ .  
 $\langle 1 \rangle 2$ . For any open intervals  $[a, b)$ ,  $[c, d)$  if  $x \in [a, b) \cap [c, d)$ , then there exists an interval  $[e, f)$  such that  $x \in [e, f) \subseteq [a, b) \cap [c, d)$   
 PROOF: Take  $[e, f) = [\max(a, c), \min(b, d))$ .

□

**Definition 66** ( $K$ -topology on the Real Line). Let  $K = \{1/n \mid n \in \mathbb{Z}^+\}$ .

The  *$K$ -topology on the real line* is the topology on  $\mathbb{R}$  generated by the basis consisting of all open intervals  $(a, b)$  and all sets of the form  $(a, b) \setminus K$ .

We write  $\mathbb{R}_K$  for the topological space  $\mathbb{R}$  under the  $K$ -topology.

We prove this is a basis for a topology.

PROOF:

- ⟨1⟩1. For all  $x \in \mathbb{R}$  there exists an open interval  $(a, b)$  such that  $x \in (a, b)$ .  
PROOF: Take  $(a, b) = (x - 1, x + 1)$ .
- ⟨1⟩2. For any basic open sets  $B_1, B_2$  if  $x \in B_1 \cap B_2$ , then there exists a basic open set  $B_3$  such that  $x \in B_3 \subseteq B_1 \cap B_2$
- ⟨2⟩1. CASE:  $B_1 = (a, b), B_2 = (c, d)$   
PROOF: Take  $B_3 = (\max(a, c), \min(b, d))$ .
- ⟨2⟩2. CASE:  $B_1 = (a, b)$  or  $(a, b) \setminus K, B_2 = (c, d)$  or  $(c, d) \setminus K$ , and they are not both open intervals.  
PROOF: Take  $B_3 = (\max(a, c), \min(b, d)) \setminus K$ .

□

**Lemma 67.** *The lower limit topology and the  $K$ -topology are incomparable.*

PROOF:

- ⟨1⟩1. The interval  $[10, 11)$  is not open in the  $K$ -topology.  
PROOF: There is no open interval  $(a, b)$  such that  $10 \in (a, b) \subseteq [10, 11)$  or  $10 \in (a, b) \setminus K \subseteq [10, 11)$ .
- ⟨1⟩2. The set  $(-1, 1) \setminus K$  is not open in the lower limit topology.  
PROOF: There is no half-open interval  $[a, b)$  such that  $0 \in [a, b) \subseteq (-1, 1) \setminus K$ , since there must be a positive integer  $n$  with  $1/n \in [a, b)$ .

□

**Definition 68** (Subbasis). A *subbasis*  $\mathcal{S}$  for a topology on  $X$  is a set  $\mathcal{S} \subseteq \mathcal{P}X$  such that  $\bigcup \mathcal{S} = X$ .

The topology *generated* by the subbasis  $\mathcal{S}$  is the set of all unions of finite intersections of elements of  $\mathcal{S}$ .

We prove this is a topology.

PROOF:

- ⟨1⟩1. The set  $\mathcal{B}$  of all finite intersections of elements of  $\mathcal{S}$  forms a basis for a topology on  $X$ .
- ⟨2⟩1.  $\bigcup \mathcal{B} = X$   
PROOF: Since  $\mathcal{S} \subseteq \mathcal{B}$ .
- ⟨2⟩2.  $\mathcal{B}$  is closed under binary intersection.  
PROOF: By definition.
- ⟨1⟩2. Q.E.D.  
PROOF: By Lemma 61.

□

We have simultaneously proved:

**Proposition 69.** *Let  $\mathcal{S}$  be a subbasis for the topology on  $X$ . Then the set of all finite intersections of elements of  $\mathcal{S}$  is a basis for the topology on  $X$ .*

**Proposition 70.** *Let  $X$  be a set. Let  $\mathcal{S}$  be a subbasis for a topology  $\mathcal{T}$  on  $X$ . Then  $\mathcal{T}$  is the coarsest topology that includes  $\mathcal{S}$ .*

PROOF: Since every topology that includes  $\mathcal{S}$  includes every union of finite intersections of elements of  $\mathcal{S}$ . □

## 12 Local Basis at a Point

**Definition 71** (Local Basis). Let  $X$  be a topological space and  $a \in X$ . A *(local) basis at  $a$*  is a set  $\mathcal{B}$  of neighbourhoods of  $a$  such that every neighbourhood of  $a$  includes some member of  $\mathcal{B}$ .

**Lemma 72.** *If there exists a countable local basis at a point  $a$ , then there exists a countable local basis  $\{B_n \mid n \geq 1\}$  such that  $B_1 \supseteq B_2 \supseteq \dots$ .*

PROOF: Given a countable local basis  $\{C_n \mid n \geq 1\}$ , take  $B_n = C_1 \cap \dots \cap C_n$ .  $\square$

## 13 Convergence

**Definition 73** (Convergence). Let  $X$  be a topological space. Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence of points in  $X$  and  $l \in X$ . Then the sequence  $(a_n)_{n \in \mathbb{N}}$  *converges* to the *limit*  $l$ ,  $a_n \rightarrow l$  as  $n \rightarrow \infty$ , if and only if, for every neighbourhood  $U$  of  $l$ , there exists  $N$  such that, for all  $n \geq N$ , we have  $a_n \in U$ .

**Lemma 74.** *Let  $X$  be a topological space. Let  $A \subseteq X$  and  $l \in X$ . If there is a sequence of points in  $A$  that converges to  $l$  then  $l \in \overline{A}$ .*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $(a_n)$  be a sequence of points in  $A$  that converges to  $l$ .
- $\langle 1 \rangle 2$ . LET:  $U$  be a neighbourhood of  $l$ .
- $\langle 1 \rangle 3$ . PICK  $N$  such that, for all  $n \geq N$ , we have  $a_n \in U$ .
- $\langle 1 \rangle 4$ .  $a_N \in U \cap A$
- $\langle 1 \rangle 5$ . Q.E.D.

PROOF: Theorem 47.

$\square$

**Proposition 75.** *Let  $X$  be a topological space. Let  $\mathcal{B}$  be a basis for the topology on  $X$ . Let  $(a_n)$  be a sequence in  $X$  and  $l \in X$ . Then  $a_n \rightarrow l$  as  $n \rightarrow \infty$  if and only if, for every  $B \in \mathcal{B}$  with  $l \in B$ , there exists  $N$  such that, for all  $n \geq N$ , we have  $a_n \in B$ .*

PROOF:

- $\langle 1 \rangle 1$ . If  $a_n \rightarrow l$  as  $n \rightarrow \infty$  then, for every  $B \in \mathcal{B}$  with  $l \in B$ , there exists  $N$  such that, for all  $n \geq N$ , we have  $a_n \in B$ .

PROOF: Since every element of  $\mathcal{B}$  is open (Corollary 61.1).

- $\langle 1 \rangle 2$ . If, for every  $B \in \mathcal{B}$  with  $l \in B$ , there exists  $N$  such that, for all  $n \geq N$ , we have  $a_n \in B$ , then  $a_n \rightarrow l$  as  $n \rightarrow \infty$ .

- $\langle 2 \rangle 1$ . ASSUME: for every  $B \in \mathcal{B}$  with  $l \in B$ , there exists  $N$  such that, for all  $n \geq N$ , we have  $a_n \in B$

- $\langle 2 \rangle 2$ . LET:  $U$  be a neighbourhood of  $l$ .

- $\langle 2 \rangle 3$ . PICK  $B \in \mathcal{B}$  such that  $l \in B \subseteq U$

- $\langle 2 \rangle 4$ . PICK  $N$  such that, for all  $n \geq N$ , we have  $a_n \in B$

PROOF: From  $\langle 2 \rangle 1$ .

- $\langle 2 \rangle 5$ . For all  $n \geq N$  we have  $a_n \in U$

□

**Lemma 76.** *If a sequence  $(a_n)$  is constant with  $a_n = l$  for all  $n$ , then  $a_n \rightarrow l$  as  $n \rightarrow \infty$ .*

PROOF: Immediate from definitions. □

**Theorem 77.** *Let  $X$  be a linearly ordered set. Let  $(s_n)$  be a monotone increasing sequence in  $X$  with a supremum  $s$ . Then  $s_n \rightarrow s$  as  $n \rightarrow \infty$ .*

PROOF:

⟨1⟩1. ASSUME:  $s$  is not least in  $X$ .

PROOF: Otherwise  $(s_n)$  is the constant sequence  $s$  and the result follows from Lemma 76.

⟨1⟩2. LET:  $U$  be a neighbourhood of  $s$ .

⟨1⟩3. PICK  $a < s$  such that  $(a, s] \subseteq U$

⟨1⟩4. PICK  $N$  such that  $a < a_N$ .

⟨1⟩5. For all  $n \geq N$  we have  $a_n \in (a, s]$

⟨1⟩6. For all  $n \geq N$  we have  $a_n \in U$ .

□

**Theorem 78.** *If  $\sum_{i=0}^{\infty} a_i = s$  and  $\sum_{i=0}^{\infty} b_i = t$  then  $\sum_{i=0}^{\infty} (ca_i + b_i) = cs + t$ .*

PROOF:  $\sum_{i=0}^N (ca_i + b_i) = c \sum_{i=0}^N a_i + \sum_{i=0}^N b_i \rightarrow cs + t$  as  $n \rightarrow \infty$ . □

**Theorem 79** (Comparison Test). *If  $|a_i| \leq b_i$  for all  $i$  and  $\sum_{i=0}^{\infty} b_i$  converges then  $\sum_{i=0}^{\infty} a_i$  converges.*

PROOF:

⟨1⟩1.  $\sum_{i=0}^{\infty} |a_i|$  converges

PROOF: The partial sums  $\sum_{i=0}^N |a_i|$  form an increasing sequence bounded above by  $\sum_{i=0}^{\infty} b_i$ .

⟨1⟩2. LET:  $c_i = |a_i| + a_i$  for all  $i$

⟨1⟩3.  $\sum_{i=0}^{\infty} c_i$  converges

PROOF: Each  $c_i$  is either  $2|a_i|$  or 0. So the partial sums  $\sum_{i=0}^N c_i$  form an increasing sequence bounded above by  $2 \sum_{i=0}^{\infty} b_i$ .

⟨1⟩4. Q.E.D.

PROOF: Since  $a_i = c_i - |a_i|$ .

□

**Corollary 79.1.** *If  $\sum_{i=0}^{\infty} |a_i|$  converges then  $\sum_{i=0}^{\infty} a_i$  converges.*

**Theorem 80** (Weierstrass M-test). *Let  $X$  be a set and  $(f_n : X \rightarrow \mathbb{R})$  be a sequence of functions. Let*

$$s_n(x) = \sum_{i=0}^n f_i(x)$$

*for all  $n, x$ . Suppose  $|f_i(x)| \leq M_i$  for all  $i \geq 0$  and  $x \in X$ . If the series  $\sum_{i=0}^{\infty} M_i$  converges, then the sequence  $(s_n)$  converges uniformly to*

$$s(x) = \sum_{i=0}^{\infty} f_i(x) .$$

PROOF:

- ⟨1⟩1. LET:  $r_n = \sum_{i=n+1}^{\infty} M_i$  for all  $n$   
 ⟨1⟩2. Given  $0 \leq n < k$ , we have  $|s_k(x) - s_n(x)| \leq r_n$

PROOF:

$$\begin{aligned} |s_k(x) - s_n(x)| &= \left| \sum_{i=n+1}^k f_i(x) \right| \\ &\leq \sum_{i=n+1}^k |f_i(x)| \\ &\leq \sum_{i=n+1}^k M_i \\ &\leq r_n \end{aligned}$$

- ⟨1⟩3. Given  $n \geq 0$  we have  $|s(x) - s_n(x)| \leq r_n$

PROOF: By taking the limit  $k \rightarrow \infty$  in ⟨1⟩2.

- ⟨1⟩4. Q.E.D.

PROOF: Since  $r_n \rightarrow 0$  as  $n \rightarrow \infty$ .

□

## 14 Locally Finite Sets

**Definition 81** (Locally Finite). Let  $X$  be a topological space and  $\{A_\alpha\}$  a family of subsets of  $X$ . Then  $\mathcal{A}$  is *locally finite* if and only if every point in  $X$  has a neighbourhood that intersects  $A_\alpha$  for only finitely many  $\alpha$ .

**Theorem 82** (Pasting Lemma). Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Let  $\{A_\alpha\}$  be a locally finite family of closed subsets of  $X$  that cover  $X$ . Suppose  $f \upharpoonright A_\alpha$  is continuous for all  $\alpha$ . Then  $f$  is continuous.

PROOF:

- ⟨1⟩1. Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Let  $A$  and  $B$  be closed subsets of  $X$  such that  $X = A \cup B$ . Suppose  $f \upharpoonright A$  and  $f \upharpoonright B$  are continuous. Then  $f$  is continuous.

- ⟨2⟩1. LET:  $C \subseteq Y$  be closed.

- ⟨2⟩2.  $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$

- ⟨2⟩3.  $f^{-1}(C)$  and  $g^{-1}(C)$  are closed in  $X$ .

PROOF: Theorems 92 and 142.

- ⟨2⟩4.  $h^{-1}(C)$  is closed in  $X$ .

PROOF: Lemma 33.

- ⟨2⟩5. Q.E.D.

PROOF: Theorem 92.

- ⟨1⟩2. Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Let  $\{A_\alpha\}$  be a finite family of closed subsets of  $X$  that cover  $X$ . Suppose  $f \upharpoonright A_\alpha$  is continuous for all  $\alpha$ . Then  $f$  is continuous.

PROOF: From ⟨1⟩1 by induction.

- ⟨1⟩3. Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Let  $\{A_\alpha\}$  be a locally finite family of closed subsets of  $X$  that cover  $X$ . Suppose  $f \upharpoonright A_\alpha$  is continuous for all  $\alpha$ . Then  $f$  is continuous.
- ⟨2⟩1. LET:  $x \in X$   
PROVE:  $f$  is continuous at  $x$
- ⟨2⟩2. PICK a neighbourhood  $U$  of  $x$  that intersects  $A_\alpha$  for only finitely many  $\alpha$ .
- ⟨2⟩3.  $f \upharpoonright U$  is continuous  
PROOF: By ⟨1⟩2.
- ⟨2⟩4. Q.E.D.  
PROOF: Lemma 102.

□

The following example shows that we cannot remove the assumption of local finiteness.

**Example 83.** Define  $f : [-1, 1] \rightarrow \mathbb{R}$  by:  $f(x) = 1$  if  $x < -1$ ,  $f(x) = 0$  if  $x > 1$ . Let  $C_n = [-1, -1/n]$  for  $n \geq 1$ , and  $D = [0, 1]$ . Then  $[-1, 1] = \bigcup_{n=1}^{\infty} C_n \cup D$  and  $f$  is continuous on each  $C_n$  and each  $D$ , but  $f$  is not continuous on  $[-1, 1]$ .

## 15 Open Maps

**Definition 84** (Open Map). Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Then  $f$  is an *open map* if and only if, for every open set  $U$  in  $X$ , the set  $f(U)$  is open in  $Y$ .

**Lemma 85.** Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Let  $\mathcal{B}$  be a basis for the topology on  $X$ . If  $f(B)$  is open in  $Y$  for all  $B \in \mathcal{B}$ , then  $f$  is an open map.

PROOF: From Lemma 61. □

**Proposition 86.** Let  $X$  and  $Y$  be topological spaces. Let  $\mathcal{B}$  be a basis for the topology on  $X$ . Let  $f : X \rightarrow Y$ . Suppose that, for all  $B \in \mathcal{B}$ , we have  $f(B)$  is open in  $Y$ . Then  $f$  is an open map.

PROOF: For any  $\mathcal{A} \subseteq \mathcal{B}$ , we have  $f(\bigcup \mathcal{A}) = \bigcup_{B \in \mathcal{A}} f(B)$  is open in  $Y$ . The result follows from Lemma 61. □

## 16 Continuous Functions

**Definition 87** (Continuous). Let  $X$  and  $Y$  be topological spaces. A function  $f : X \rightarrow Y$  is *continuous* if and only if, for every open set  $V$  in  $Y$ , the set  $f^{-1}(V)$  is open in  $X$ .

**Proposition 88.** Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Let  $\mathcal{B}$  be a basis for  $Y$ . Then  $f$  is continuous if and only if, for all  $B \in \mathcal{B}$ , we have  $f^{-1}(B)$  is open in  $X$ .

PROOF:

⟨1⟩1. If  $f$  is continuous then, for all  $B \in \mathcal{B}$ , we have  $f^{-1}(B)$  is open in  $X$ .

PROOF: Since every element of  $\mathcal{B}$  is open (Lemma 61).

⟨1⟩2. Suppose that, for all  $B \in \mathcal{B}$ , we have  $f^{-1}(B)$  is open in  $X$ . Then  $f$  is continuous.

⟨2⟩1. ASSUME: For all  $B \in \mathcal{B}$ , we have  $f^{-1}(B)$  is open in  $X$ .

⟨2⟩2. LET:  $V$  be open in  $Y$ .

⟨2⟩3. PICK  $\mathcal{A} \subseteq \mathcal{B}$  such that  $V = \bigcup \mathcal{A}$

PROOF: By Lemma 61.

⟨2⟩4.  $f^{-1}(V)$  is open in  $X$ .

PROOF:

$$\begin{aligned} f^{-1}(V) &= f^{-1}\left(\bigcup \mathcal{A}\right) \\ &= \bigcup_{B \in \mathcal{A}} f^{-1}(B) \end{aligned}$$

□

**Proposition 89.** *Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Let  $\mathcal{S}$  be a subbasis for  $Y$ . Then  $f$  is continuous if and only if, for all  $S \in \mathcal{S}$ , we have  $f^{-1}(S)$  is open in  $X$ .*

PROOF:

⟨1⟩1. If  $f$  is continuous then, for all  $S \in \mathcal{S}$ , we have  $f^{-1}(S)$  is open in  $X$ .

PROOF: Since every element of  $\mathcal{S}$  is open.

⟨1⟩2. Suppose that, for all  $S \in \mathcal{S}$ , we have  $f^{-1}(S)$  is open in  $X$ . Then  $f$  is continuous.

⟨2⟩1. ASSUME: For all  $S \in \mathcal{S}$ , we have  $f^{-1}(S)$  is open in  $X$ .

⟨2⟩2. LET:  $S_1, \dots, S_n \in \mathcal{S}$

⟨2⟩3.  $f^{-1}(S_1 \cap \dots \cap S_n)$  is open in  $X$

PROOF: Since  $f^{-1}(S_1 \cap \dots \cap S_n) = f^{-1}(S_1) \cap \dots \cap f^{-1}(S_n)$ .

⟨2⟩4. Q.E.D.

PROOF: By Propositions 88 and 69.

□

**Proposition 90.** *Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Let  $\mathcal{S}$  be a basis for  $Y$ . Then  $f$  is continuous if and only if, for all  $V \in \mathcal{S}$ , we have  $f^{-1}(V)$  is open in  $X$ .*

PROOF:

⟨1⟩1. If  $f$  is continuous then, for all  $V \in \mathcal{S}$ , we have  $f^{-1}(V)$  is open in  $X$ .

PROOF: Since every element of  $\mathcal{S}$  is open.

⟨1⟩2. Suppose that, for all  $V \in \mathcal{S}$ , we have  $f^{-1}(V)$  is open in  $X$ . Then  $f$  is continuous.

⟨2⟩1. ASSUME: For all  $V \in \mathcal{S}$ , we have  $f^{-1}(V)$  is open in  $X$ .

⟨2⟩2. For every set  $B$  that is the finite intersection of elements of  $\mathcal{S}$ , we have  $f^{-1}(B)$  is open in  $X$ .

PROOF: Because  $f^{-1}(V_1 \cap \dots \cap V_n) = f^{-1}(V_1) \cap \dots \cap f^{-1}(V_n)$ .

⟨2⟩3. Q.E.D.

PROOF: From Propositions 69 and 88.

□

**Definition 91** (Continuous at a Point). Let  $X$  and  $Y$  be topological spaces. Let  $f : X \rightarrow Y$  and  $x \in X$ . Then  $f$  is *continuous at  $x$*  if and only if, for every neighbourhood  $V$  of  $f(x)$ , there exists a neighbourhood  $U$  of  $x$  such that  $f(U) \subseteq V$ .

**Theorem 92.** Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Then the following are equivalent:

1.  $f$  is continuous.
2. For all  $A \subseteq X$ , we have  $f(\overline{A}) \subseteq \overline{f(A)}$
3. For all  $B \subseteq Y$  closed, we have  $f^{-1}(B)$  is closed in  $X$ .
4.  $f$  is continuous at every point of  $X$ .

PROOF:

⟨1⟩1.  $1 \Rightarrow 2$

⟨2⟩1. ASSUME:  $f$  is continuous.

⟨2⟩2. LET:  $A \subseteq X$

⟨2⟩3. LET:  $x \in \overline{A}$

PROVE:  $f(x) \in \overline{f(A)}$

⟨2⟩4. LET:  $V$  be a neighbourhood of  $f(x)$

⟨2⟩5.  $f^{-1}(V)$  is a neighbourhood of  $x$

⟨2⟩6. PICK  $y \in A \cap f^{-1}(V)$

PROOF: By Theorem 47.

⟨2⟩7.  $f(y) \in V \cap f(A)$

⟨2⟩8. Q.E.D.

PROOF: By Theorem 47.

⟨1⟩2.  $2 \Rightarrow 3$

⟨2⟩1. ASSUME: 2

⟨2⟩2. LET:  $B$  be closed in  $Y$

⟨2⟩3. LET:  $x \in \overline{f^{-1}(B)}$

PROVE:  $x \in f^{-1}(B)$

⟨2⟩4.  $f(x) \in B$

PROOF:

$$f(x) \in \overline{f(f^{-1}(B))}$$

$$\subseteq \overline{f(f^{-1}(B))}$$

(⟨2⟩1)

$$\subseteq \overline{B}$$

(Proposition 48)

$$= B$$

⟨1⟩3.  $3 \Rightarrow 1$

⟨2⟩1. ASSUME: 3

⟨2⟩2. LET:  $V$  be open in  $Y$

⟨2⟩3.  $Y \setminus V$  is closed in  $Y$



$\langle 2 \rangle 4$ .  $f^{-1}(Y \setminus V)$  is closed in  $X$   
 $\langle 2 \rangle 5$ .  $X \setminus f^{-1}(V)$  is closed in  $X$   
 $\langle 2 \rangle 6$ .  $f^{-1}(V)$  is open in  $X$   
 $\langle 1 \rangle 4$ .  $1 \Rightarrow 4$   
 PROOF: For any neighbourhood  $V$  of  $f(x)$ , the set  $U = f^{-1}(V)$  is a neighbourhood of  $x$  such that  $f(U) \subseteq V$ .  
 $\langle 1 \rangle 5$ .  $4 \Rightarrow 1$   
 $\langle 2 \rangle 1$ . ASSUME: 4  
 $\langle 2 \rangle 2$ . LET:  $V$  be open in  $Y$   
 $\langle 2 \rangle 3$ . LET:  $x \in f^{-1}(V)$   
 $\langle 2 \rangle 4$ .  $V$  is a neighbourhood of  $f(x)$   
 $\langle 2 \rangle 5$ . PICK a neighbourhood  $U$  of  $x$  such that  $f(U) \subseteq V$   
 $\langle 2 \rangle 6$ .  $x \in U \subseteq f^{-1}(V)$   
 $\langle 2 \rangle 7$ . Q.E.D.  
 PROOF: By Lemma 25.

□

**Theorem 93.** *A constant function is continuous.*

PROOF: Let  $X$  and  $Y$  be topological spaces. Let  $b \in Y$ , and let  $f : X \rightarrow Y$  be the constant function with value  $b$ . For any open  $V \subseteq Y$ , the set  $f^{-1}(V)$  is either  $X$  (if  $b \in V$ ) or  $\emptyset$  (if  $b \notin V$ ). □

**Theorem 94.** *If  $A$  is a subspace of  $X$  then the inclusion  $j : A \rightarrow X$  is continuous.*

PROOF: For any  $V$  open in  $X$ , we have  $j^{-1}(V) = V \cap A$  is open in  $A$ . □

**Theorem 95.** *The composite of two continuous functions is continuous.*

PROOF: Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  be continuous. For any  $V$  open in  $Z$ , we have  $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$  is open in  $X$ . □

**Theorem 96.** *Let  $f : X \rightarrow Y$  be a continuous function and  $A$  be a subspace of  $X$ . Then the restriction  $f \upharpoonright A : A \rightarrow Y$  is continuous.*

PROOF: Let  $V$  be open in  $Y$ . Then  $(f \upharpoonright A)^{-1}(V) = f^{-1}(V) \cap A$  is open in  $A$ . □

**Theorem 97.** *Let  $f : X \rightarrow Y$  be continuous. Let  $Z$  be a subspace of  $Y$  such that  $f(X) \subseteq Z$ . Then the corestriction  $f : X \rightarrow Z$  is continuous.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $V$  be open in  $Z$ .  
 $\langle 1 \rangle 2$ . PICK  $U$  open in  $Y$  such that  $V = U \cap Z$ .  
 $\langle 1 \rangle 3$ .  $f^{-1}(V) = f^{-1}(U)$   
 $\langle 1 \rangle 4$ .  $f^{-1}(V)$  is open in  $X$ .  
 □

**Theorem 98.** *Let  $f : X \rightarrow Y$  be continuous. Let  $Z$  be a space such that  $Y$  is a subspace of  $Z$ . Then the expansion  $f : X \rightarrow Z$  is continuous.*

PROOF: Let  $V$  be open in  $Z$ . Then  $f^{-1}(V) = f^{-1}(V \cap Y)$  is open in  $X$ .  $\square$

**Theorem 99.** *Let  $X$  and  $Y$  be topological spaces. Let  $f : X \rightarrow Y$ . Suppose  $\mathcal{U}$  is a set of open sets in  $X$  such that  $X = \bigcup \mathcal{U}$  and, for all  $U \in \mathcal{U}$ , we have  $f \upharpoonright U : U \rightarrow Y$  is continuous. Then  $f$  is continuous.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $V$  be open in  $Y$
- $\langle 1 \rangle 2$ .  $f^{-1}(V) = \bigcup_{U \in \mathcal{U}} (f \upharpoonright U)^{-1}(V)$
- $\langle 1 \rangle 3$ . For all  $U \in \mathcal{U}$ , we have  $(f \upharpoonright U)^{-1}(V)$  is open in  $U$ .
- $\langle 1 \rangle 4$ . For all  $U \in \mathcal{U}$ , we have  $(f \upharpoonright U)^{-1}(V)$  is open in  $X$ .

PROOF: Lemma 141.

$\square$

**Proposition 100.** *Let  $X$  and  $X'$  be the same set  $X$  under two topologies  $\mathcal{T}$  and  $\mathcal{T}'$ . Let  $i : X \rightarrow X'$  be the identity function. Then  $i$  is continuous if and only if  $\mathcal{T}' \subseteq \mathcal{T}$ .*

PROOF: Immediate from definitions.  $\square$

**Proposition 101.** *Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  and  $a \in \mathbb{R}$ . Then  $f$  is continuous on the right at  $a$  if and only if  $f$  is continuous at  $a$  as a function  $\mathbb{R}_l \rightarrow \mathbb{R}$ .*

PROOF:

- $\langle 1 \rangle 1$ . If  $f$  is continuous on the right at  $a$  then  $f$  is continuous at  $a$  as a function  $\mathbb{R}_l \rightarrow \mathbb{R}$ .
  - $\langle 2 \rangle 1$ . ASSUME:  $f$  is continuous on the right at  $a$ .
  - $\langle 2 \rangle 2$ . LET:  $V$  be a neighbourhood of  $f(a)$
  - $\langle 2 \rangle 3$ . PICK  $b, c$  such that  $f(a) \in (b, c) \subseteq V$ .
  - $\langle 2 \rangle 4$ . LET:  $\epsilon = \min(c - f(a), f(a) - b)$
  - $\langle 2 \rangle 5$ . PICK  $\delta > 0$  such that, for all  $x$ , if  $a < x < a + \delta$  then  $|f(x) - f(a)| < \epsilon$
  - $\langle 2 \rangle 6$ . LET:  $U = [a, a + \delta)$
  - $\langle 2 \rangle 7$ .  $f(U) \subseteq V$
- $\langle 1 \rangle 2$ . If  $f$  is continuous at  $a$  as a function  $\mathbb{R}_l \rightarrow \mathbb{R}$  then  $f$  is continuous on the right at  $a$ .
  - $\langle 2 \rangle 1$ . ASSUME:  $f$  is continuous at  $a$  as a function  $\mathbb{R}_l \rightarrow \mathbb{R}$
  - $\langle 2 \rangle 2$ . LET:  $\epsilon > 0$
  - $\langle 2 \rangle 3$ . PICK a neighbourhood  $U$  of  $a$  such that  $f(U) \subseteq (f(a) - \epsilon, f(a) + \epsilon)$
  - $\langle 2 \rangle 4$ . PICK  $b, c$  such that  $a \in [b, c) \subset U$
  - $\langle 2 \rangle 5$ . LET:  $\delta = c - a$
  - $\langle 2 \rangle 6$ . For all  $x$ , if  $a < x < a + \delta$  then  $|f(x) - f(a)| < \epsilon$

$\square$

**Lemma 102.** *Let  $f : X \rightarrow Y$ . Let  $Z$  be an open subspace of  $X$  and  $a \in Z$ . If  $f \upharpoonright Z$  is continuous at  $a$  then  $f$  is continuous at  $a$ .*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $V$  be a neighbourhood of  $f(a)$

⟨1⟩2. PICK a neighbourhood  $W$  of  $x$  in  $Z$  such that  $f(W) \subseteq V$

⟨1⟩3.  $W$  is a neighbourhood of  $x$  in  $X$  such that  $f(W) \subseteq V$

PROOF: Lemma 141.

□

**Proposition 103.** *Let  $f : A \rightarrow B$  and  $g : C \rightarrow D$  be continuous. Define  $f \times g : A \times C \rightarrow B \times D$  by*

$$(f \times g)(a, c) = (f(a), g(c)) .$$

*Then  $f \times g$  is continuous.*

PROOF:  $\pi_1 \circ (f \times g) = f \circ \pi_1$  and  $\pi_2 \circ (f \times g) = g \circ \pi_2$  are continuous by Theorem 95. The result follows by Theorem 131.

**Proposition 104.** *Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$  be continuous. If  $a_n \rightarrow l$  as  $n \rightarrow \infty$  in  $X$  then  $f(a_n) \rightarrow f(l)$  as  $n \rightarrow \infty$ .*

PROOF:

⟨1⟩1. LET:  $V$  be a neighbourhood of  $f(l)$

⟨1⟩2. PICK a neighbourhood  $U$  of  $l$  such that  $f(U) \subseteq V$

⟨1⟩3. PICK  $N$  such that, for all  $n \geq N$ , we have  $a_n \in U$

⟨1⟩4. For all  $n \geq N$  we have  $f(a_n) \in V$

□

## 17 Homeomorphisms

**Definition 105** (Homeomorphism). Let  $X$  and  $Y$  be topological spaces. A *Homeomorphism*  $f$  between  $X$  and  $Y$ ,  $f : X \cong Y$ , is a bijection  $f : X \rightarrow Y$  such that both  $f$  and  $f^{-1}$  are continuous.

**Lemma 106.** *Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$  a bijection. Then the following are equivalent:*

1.  $f$  is a homeomorphism.
2.  $f$  is continuous and an open map.
3.  $f$  is continuous and a closed map.
4. For any  $U \subseteq X$ , we have  $U$  is open if and only if  $f(U)$  is open.

PROOF: Immediate from definitions. □

**Proposition 107.** *Let  $X$  and  $X'$  be the same set  $X$  under two topologies  $\mathcal{T}$  and  $\mathcal{T}'$ . Let  $i : X \rightarrow X'$  be the identity function. Then  $i$  is a homeomorphism if and only if  $\mathcal{T} = \mathcal{T}'$ .*

PROOF: Immediate from definitions. □

**Definition 108** (Topological Property). Let  $P$  be a property of topological spaces. Then  $P$  is a *topological* property if and only if, for any spaces  $X$  and  $Y$ , if  $P$  holds of  $X$  and  $X \cong Y$  then  $P$  holds of  $Y$ .

**Definition 109** (Topological Imbedding). Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Then  $f$  is a *topological imbedding* if and only if the corestriction  $f : X \rightarrow f(X)$  is a homeomorphism.

**Proposition 110.** *Let  $X$  and  $Y$  be topological spaces and  $a \in X$ . The function  $i : Y \rightarrow X \times Y$  that maps  $y$  to  $(a, y)$  is an imbedding.*

PROOF:

$\langle 1 \rangle 1$ .  $i$  is injective

$\langle 1 \rangle 2$ .  $i$  is continuous.

PROOF: For  $U$  open in  $X$  and  $V$  open in  $Y$ , we have  $i^{-1}(U \times V)$  is  $V$  if  $a \in U$ , and  $\emptyset$  if  $a \notin U$ .

$\langle 1 \rangle 3$ .  $i : Y \rightarrow i(Y)$  is an open map.

PROOF: For  $V$  open in  $Y$  we have  $i(V) = (X \times V) \cap i(Y)$ .

□

## 18 The Order Topology

**Definition 111** (Order Topology). Let  $X$  be a linearly ordered set with at least two points. The *order topology* on  $X$  is the topology generated by the basis  $\mathcal{B}$  consisting of:

- all open intervals  $(a, b)$ ;
- all intervals of the form  $[\perp, b)$  where  $\perp$  is least in  $X$ ;
- all intervals of the form  $(a, \top]$  where  $\top$  is greatest in  $X$ .

We prove this is a basis for a topology.

PROOF:

$\langle 1 \rangle 1$ . For all  $x \in X$  there exists  $B \in \mathcal{B}$  such that  $x \in B$ .

$\langle 2 \rangle 1$ . LET:  $x \in X$

$\langle 2 \rangle 2$ . CASE:  $x$  is greatest in  $X$ .

$\langle 3 \rangle 1$ . PICK  $y \in X$  with  $y \neq x$

$\langle 3 \rangle 2$ .  $x \in (y, x] \in \mathcal{B}$

$\langle 2 \rangle 3$ . CASE:  $x$  is least in  $X$ .

$\langle 3 \rangle 1$ . PICK  $y \in X$  with  $y \neq x$

$\langle 3 \rangle 2$ .  $x \in [x, y) \in \mathcal{B}$

$\langle 2 \rangle 4$ . CASE:  $x$  is neither greatest nor least in  $X$ .

$\langle 3 \rangle 1$ . PICK  $a, b \in X$  with  $a < x$  and  $x < b$

$\langle 3 \rangle 2$ .  $x \in (a, b) \in \mathcal{B}$

$\langle 1 \rangle 2$ . For all  $B_1, B_2 \in \mathcal{B}$  and  $x \in B_1 \cap B_2$ , there exists  $B_3 \in \mathcal{B}$  such that  $x \in B_3 \subseteq B_1 \cap B_2$

- ⟨2⟩1. LET:  $B_1, B_2 \in \mathcal{B}$  and  $x \in B_1 \cap B_2$
- ⟨2⟩2. CASE:  $B_1 = (a, b)$ ,  $B_2 = (c, d)$   
PROOF: Take  $B_3 = (\max(a, c), \min(b, d))$ .
- ⟨2⟩3. CASE:  $B_1 = (a, b)$ ,  $B_2 = [\perp, d]$   
PROOF: Take  $B_3 = (a, \min(b, d))$ .
- ⟨2⟩4. CASE:  $B_1 = (a, b)$ ,  $B_2 = (c, \top]$   
PROOF: Take  $B_3 = (\max(a, c), b)$ .
- ⟨2⟩5. CASE:  $B_1 = [\perp, b)$ ,  $B_2 = [\perp, d]$   
PROOF: Take  $B_3 = [\perp, \min(b, d))$ .
- ⟨2⟩6. CASE:  $B_1 = [\perp, b)$ ,  $B_2 = (c, \top]$   
PROOF: Take  $B_3 = (c, b)$ .

□

**Lemma 112.** *Let  $X$  be a linearly ordered set. Then the open rays form a subbasis for the order topology on  $X$ .*

PROOF:

- ⟨1⟩1. Every open ray is open.
- ⟨2⟩1. For all  $a \in X$ , the ray  $(-\infty, a)$  is open.
  - ⟨3⟩1. LET:  $x \in (-\infty, a)$
  - ⟨3⟩2. CASE:  $x$  is least in  $X$   
PROOF:  $x \in [x, a) = (-\infty, a)$ .
  - ⟨3⟩3. CASE:  $x$  is not least in  $X$ 
    - ⟨4⟩1. PICK  $y < x$
    - ⟨4⟩2.  $x \in (y, a) \subseteq (-\infty, a)$
- ⟨2⟩2. For all  $a \in X$ , the ray  $(a, +\infty)$  is open.  
PROOF: Similar.
- ⟨1⟩2. Every basic open set is a finite intersection of open rays.  
PROOF: We have  $(a, b) = (a, +\infty) \cap (-\infty, b)$ ,  $[\perp, b) = (-\infty, b)$  and  $(a, \top] = (a, +\infty)$ .

□

**Definition 113** (Standard Topology on the Real Line). The *standard topology on the real line* is the order topology on  $\mathbb{R}$  generated by the standard order.

**Lemma 114.** *The lower limit topology is strictly finer than the standard topology on  $\mathbb{R}$ .*

PROOF:

- ⟨1⟩1. Every open interval is open in the lower limit topology.  
PROOF: If  $x \in (a, b)$  then  $x \in [x, b) \subseteq (a, b)$ .
- ⟨1⟩2. The half-open interval  $[0, 1)$  is not open in the standard topology.  
PROOF: There is no open interval  $(a, b)$  such that  $0 \in (a, b) \subseteq [0, 1)$ .

□

**Lemma 115.** *The  $K$ -topology is strictly finer than the standard topology on  $\mathbb{R}$ .*

PROOF:

⟨1⟩1. Every open interval is open in the  $K$ -topology.

PROOF: Corollary 61.1.

⟨1⟩2. The set  $(-1, 1) \setminus K$  is not open in the standard topology.

PROOF: There is no open interval  $(a, b)$  such that  $0 \in (a, b) \subseteq (-1, 1) \setminus K$ , since there must be a positive integer  $n$  with  $1/n \in (a, b)$ .

□

**Lemma 116.** *Let  $X$  be a topological space. Let  $Y$  be a linearly ordered set in the order topology. Let  $f, g : X \rightarrow Y$  be continuous. Then  $C = \{x \in X \mid f(x) \leq g(x)\}$  is closed.*

PROOF:

⟨1⟩1. LET:  $x \in X \setminus C$

⟨1⟩2.  $f(x) > g(x)$

PROVE: There exists a neighbourhood  $U$  of  $x$  such that  $U \subseteq X \setminus C$

⟨1⟩3. CASE: There exists  $y$  such that  $g(x) < y < f(x)$

PROOF: Take  $U = g^{-1}((-\infty, y)) \cup f^{-1}(y, +\infty)$ .

⟨1⟩4. CASE: There is no  $y$  such that  $g(x) < y < f(x)$

PROOF: Take  $U = g^{-1}((-\infty, f(x))) \cup f^{-1}(g(x), +\infty)$ .

□

**Proposition 117.** *Let  $X$  be a topological space. Let  $Y$  be a linearly ordered set in the order topology. Let  $f, g : X \rightarrow Y$  be continuous. Define  $h : X \rightarrow Y$  by  $h(x) = \min(f(x), g(x))$ . Then  $h$  is continuous.*

PROOF: By the Pasting Lemma applied to  $\{x \in X \mid f(x) \leq g(x)\}$  and  $\{x \in X \mid g(x) \leq f(x)\}$ , which are closed by Lemma 116.

**Proposition 118.** *Let  $X$  and  $Y$  be linearly ordered sets in the order topology. Let  $f : X \rightarrow Y$  be strictly monotone and surjective. Then  $f$  is a homeomorphism.*

PROOF:

⟨1⟩1.  $f$  is bijective.

PROOF: Proposition 5.

⟨1⟩2.  $f$  is continuous.

⟨2⟩1. For all  $y \in Y$  we have  $f^{-1}((y, +\infty))$  is open.

⟨3⟩1. LET:  $y \in Y$

⟨3⟩2. PICK  $x \in X$  such that  $f(x) = y$

PROOF: Since  $f$  is surjective.

⟨3⟩3.  $f^{-1}((y, +\infty)) = (x, +\infty)$

PROOF: By strict monotonicity.

⟨2⟩2. For all  $y \in Y$  we have  $f^{-1}((-\infty, y))$  is open.

PROOF: Similar.

⟨1⟩3.  $f^{-1}$  is continuous.

⟨2⟩1. For all  $x \in X$  we have  $f((x, +\infty))$  is open.

PROOF:  $f((x, +\infty)) = (f(x), +\infty)$ .

⟨2⟩2. For all  $x \in X$  we have  $f((-\infty, x))$  is open.

PROOF:  $f((-\infty, x)) = (-\infty, f(x))$ .

□

## 19 The $n$ th Root Function

**Proposition 119.** For all  $n \geq 1$ , the function  $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  defined by  $f(x) = x^n$  is a homeomorphism.

PROOF:

$\langle 1 \rangle 1$ .  $f$  is strictly monotone.

$\langle 2 \rangle 1$ . LET:  $x, y \in \mathbb{R}$  with  $0 \leq x < y$

$\langle 2 \rangle 2$ .  $x^n < y^n$

$$y^n - x^n = (y - x)(y^{n-1} + y^{n-2}x + y^{n-3}x^2 + \cdots + x^{n-1}) > 0$$

$\langle 1 \rangle 2$ .  $f$  is surjective.

$\langle 2 \rangle 1$ . LET:  $y \in \mathbb{R}_{\geq 0}$

$\langle 2 \rangle 2$ . PICK  $x \in \mathbb{R}$  such that  $y \leq x^n$

PROOF: If  $y \leq 1$  take  $x = 1$ , otherwise take  $x = y$ .

$\langle 2 \rangle 3$ . There exists  $x' \in [0, x]$  such that  $(x')^n = y$

PROOF: By the Intermediate Value Theorem.

$\langle 1 \rangle 3$ . Q.E.D.

PROOF: Proposition 118.

□

**Definition 120.** For  $n \geq 1$ , the  $n$ th root function is the function  $\mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  that is the inverse of  $\lambda x.x^n$ .

## 20 The Product Topology

**Definition 121** (Product Topology). Let  $\{A_i\}_{i \in I}$  be a family of topological spaces. The *product topology* on  $\prod_{i \in I} A_i$  is the topology generated by the subbasis consisting of the sets of the form  $\pi_i^{-1}(U)$  where  $i \in I$  and  $U$  is open in  $A_i$ .

**Proposition 122.** The product topology on  $\prod_{i \in I} A_i$  is generated by the basis consisting of all sets of the form  $\prod_{i \in I} U_i$  where  $\{U_i\}_{i \in I}$  is a family such that each  $U_i$  is an open set in  $A_i$  and  $U_i = A_i$  for all but finitely many  $i$ .

PROOF: From Proposition 69. □

**Proposition 123.** If  $A_i$  is closed in  $X_i$  for all  $i \in I$  then  $\prod_{i \in I} A_i$  is closed in  $\prod_{i \in I} X_i$ .

PROOF:

$$\left( \prod_{i \in I} X_i \right) \setminus \left( \prod_{i \in I} A_i \right) = \bigcup_{j \in I} \left( \prod_{i \in I} X_i \setminus \pi_j^{-1}(A_j) \right) \square$$

**Proposition 124.** Let  $\{A_i\}_{i \in I}$  be a family of topological spaces. For  $i \in I$ , let  $\mathcal{B}_i$  be a basis for the topology on  $A_i$ . Then  $\mathcal{B} = \{\prod_{i \in I} B_i \mid \forall i \in I. B_i \in \mathcal{B}_i, B_i = A_i \text{ for all but finitely many } i\}$  is a basis for the box topology on  $\prod_{i \in I} A_i$ .

PROOF:

- ⟨1⟩1. Every set in  $\mathcal{B}$  is open.
- ⟨1⟩2. For every point  $a \in \prod_{i \in I} A_i$  and every open set  $U$  with  $a \in U$ , there exists  $B \in \mathcal{B}$  such that  $a \in B \subseteq U$ .
  - ⟨2⟩1. LET:  $U$  be open and  $a \in U$
  - ⟨2⟩2. PICK a family  $\{U_i\}_{i \in I}$  such that each  $U_i$  is open in  $A_i$ , such that  $U_i = A_i$  except for  $i = i_1, \dots, i_n$ , and such that  $a \in \prod_{i \in I} U_i \subseteq U$ .
  - ⟨2⟩3. For  $j = 1, \dots, n$ , PICK  $B_{i_j} \in \mathcal{B}_{i_j}$  such that  $a_{i_j} \in B_{i_j} \subseteq U_{i_j}$
  - ⟨2⟩4. LET:  $B = \prod_{i \in I} B_i$  where  $B_i = A_i$  for  $i \neq i_1, \dots, i_n$
  - ⟨2⟩5.  $B \in \mathcal{B}$
  - ⟨2⟩6.  $a \in B \subseteq U$
- ⟨1⟩3. Q.E.D.

PROOF: Lemma 62.

□

**Proposition 125.** *Let  $\{A_i\}_{i \in I}$  be a family of topological spaces. Then the projections  $\pi_i : \prod_{i \in I} A_i \rightarrow A_i$  are open maps.*

PROOF: From Lemma 85. □

**Example 126.** The projections are not always closed maps. For example,  $\pi_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$  maps the closed set  $\{(x, 1/x) \mid x > 0\}$  to  $(0, +\infty)$ .

**Proposition 127.** *Let  $\{X_i\}_{i \in I}$  be a family of sets. For  $i \in I$ , let  $\mathcal{T}_i$  and  $\mathcal{U}_i$  be topologies on  $X_i$ . Let  $\mathcal{P}$  be the product topology on  $\prod_{i \in I} X_i$  generated by the topologies  $\mathcal{T}_i$ , and  $\mathcal{Q}$  the product topology on the same set generated by the topologies  $\mathcal{U}_i$ . Then  $\mathcal{P} \subseteq \mathcal{Q}$  if and only if  $\mathcal{T}_i \subseteq \mathcal{U}_i$  for all  $i$ .*

PROOF:

- ⟨1⟩1. If  $\mathcal{T}_i \subseteq \mathcal{U}_i$  for all  $i$  then  $\mathcal{P} \subseteq \mathcal{Q}$ 

PROOF: By Corollary 61.1.
- ⟨1⟩2. If  $\mathcal{P} \subseteq \mathcal{Q}$  then  $\mathcal{T}_i \subseteq \mathcal{U}_i$  for all  $i$ 
  - ⟨2⟩1. ASSUME:  $\mathcal{P} \subseteq \mathcal{Q}$
  - ⟨2⟩2. LET:  $i \in I$
  - ⟨2⟩3. LET:  $U \in \mathcal{T}_i$
  - ⟨2⟩4. LET:  $U_i = U$  and  $U_j = X_j$  for  $j \neq i$
  - ⟨2⟩5.  $\prod_{i \in I} U_i \in \mathcal{P}$
  - ⟨2⟩6.  $\prod_{i \in I} U_i \in \mathcal{Q}$
  - ⟨2⟩7.  $U \in \mathcal{U}_i$

PROOF: From Proposition 125.

□

**Proposition 128 (AC).** *Let  $\{X_i\}_{i \in I}$  be a family of topological spaces. Let  $A_i \subseteq X_i$  for all  $i \in I$ . Then*

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i} .$$

PROOF:



- (1)1.  $\overline{\prod_{i \in I} A_i} \subseteq \prod_{i \in I} \overline{A_i}$   
 (2)1. For all  $i \in I$  we have  $A_i \subseteq \overline{A_i}$   
 PROOF: Lemma 44.  
 (2)2.  $\prod_{i \in I} A_i \subseteq \prod_{i \in I} \overline{A_i}$   
 (2)3. Q.E.D.  
 PROOF: Since  $\prod_{i \in I} A_i$  is closed by Proposition 123.  
 (1)2.  $\prod_{i \in I} \overline{A_i} \subseteq \overline{\prod_{i \in I} A_i}$   
 (2)1. LET:  $x \in \prod_{i \in I} \overline{A_i}$   
 (2)2. LET:  $U$  be a neighbourhood of  $x$   
 (2)3. PICK  $V_i$  open in  $X_i$  such that  $x \in \prod_{i \in I} V_i \subseteq U$  with  $V_i = X_i$  except for  
 $i = i_1, \dots, i_n$   
 (2)4. For  $i \in I$ , pick  $a_i \in V_i \cap A_i$   
 PROOF: By Theorem 47 and (2)1 using the Axiom of Choice.  
 (2)5.  $U$  intersects  $\prod_{i \in I} A_i$   
 (2)6. Q.E.D.  
 PROOF:  $a \in U \cap \prod_{i \in I} A_i$

□

**Example 129.** The closure of  $\mathbb{R}^\infty$  in  $\mathbb{R}^\omega$  is  $\mathbb{R}^\omega$

PROOF:

- (1)1. LET:  $a \in \mathbb{R}^\omega$   
 (1)2. LET:  $U$  be any neighbourhoods of  $a$ .  
 (1)3. PICK  $U_n$  open in  $\mathbb{R}$  for all  $n$  such that  $a \in \prod_{n \geq 0} U_n \subseteq U$  and  $U_n = \mathbb{R}$  for  
 all  $n$  except  $n_1, \dots, n_k$   
 (1)4. LET:  $b_n = a_n$  for  $n = n_1, \dots, n_k$  and  $b_n = 0$  for all other  $n$   
 (1)5.  $b \in \mathbb{R}^\infty \cap U$   
 (1)6. Q.E.D.

PROOF: From Theorem 47.

□

**Proposition 130.** Let  $\{X_i\}_{i \in I}$  be a family of topological spaces. Let  $(a_n)$  be a sequence in  $\prod_{i \in I} X_i$  and  $l \in \prod_{i \in I} X_i$ . Then  $a_n \rightarrow l$  as  $n \rightarrow \infty$  if and only if, for all  $i \in I$ , we have  $\pi_i(a_n) \rightarrow \pi_i(l)$  as  $n \rightarrow \infty$ .

PROOF:

- (1)1. If  $a_n \rightarrow l$  as  $n \rightarrow \infty$  then, for all  $i \in I$ , we have  $\pi_i(a_n) \rightarrow \pi_i(l)$  as  $n \rightarrow \infty$   
 PROOF: Proposition 104.  
 (1)2. If, for all  $i \in I$ , we have  $\pi_i(a_n) \rightarrow \pi_i(l)$  as  $n \rightarrow \infty$ , then  $a_n \rightarrow l$  as  $n \rightarrow \infty$   
 (2)1. ASSUME: For all  $i \in I$ , we have  $\pi_i(a_n) \rightarrow \pi_i(l)$  as  $n \rightarrow \infty$   
 (2)2. LET:  $V$  be a neighbourhood of  $l$   
 (2)3. PICK open sets  $U_i$  in  $X_i$  such that  $l \in \prod_{i \in I} U_i \subseteq V$  and  $U_i = X_i$  for all  
 $i$  except  $i = i_1, \dots, i_k$   
 (2)4. For  $j = 1, \dots, k$ , PICK  $N_j$  such that, for all  $n \geq N_j$ , we have  $\pi_{i_j}(a_n) \in$   
 $U_{i_j}$   
 (2)5. LET:  $N = \max(N_1, \dots, N_k)$   
 (2)6. For all  $n \geq N$  we have  $a_n \in V$

□

**Theorem 131.** *Let  $A$  be a topological space and  $\{X_i\}_{i \in I}$  be a family of topological spaces. Let  $f : A \rightarrow \prod_{i \in I} X_i$  be a function. If  $\pi_i \circ f$  is continuous for all  $i \in I$  then  $f$  is continuous.*

PROOF:

⟨1⟩1. LET:  $i \in I$  and  $U$  be open in  $X_i$

⟨1⟩2.  $f^{-1}(\pi_i^{-1}(U))$  is open in  $A$

⟨1⟩3. Q.E.D.

PROOF: Proposition 89.

□

## 20.1 Continuous in Each Variable Separately

**Definition 132** (Continuous in Each Variable Separately). Let  $F : X \times Y \rightarrow Z$ . Then  $F$  is *continuous in each variable separately* if and only if:

- for every  $a \in X$  the function  $\lambda y \in Y. F(a, y)$  is continuous;
- for every  $b \in Y$  the function  $\lambda x \in X. F(x, b)$  is continuous.

**Proposition 133.** *Let  $F : X \times Y \rightarrow Z$ . If  $F$  is continuous then  $F$  is continuous in each variable separately.*

PROOF: For  $a \in X$ , the function  $\lambda y \in Y. F(a, y)$  is  $F \circ i$  where  $i : Y \rightarrow X \times Y$  maps  $y$  to  $(a, y)$ . We have  $i$  is continuous by Proposition 110, hence  $F \circ i$  is continuous by Theorem 95.

Similarly for  $\lambda x \in X. F(x, b)$  for  $b \in Y$ . □

**Example 134.** Define  $F : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  by

$$F(x, y) = \begin{cases} xy/(x^2 + y^2) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Then  $F$  is continuous in each variable separately but not continuous.

**Proposition 135.** *Let  $f : A \rightarrow C$  and  $g : B \rightarrow D$  be open maps. Then  $f \times g : A \times B \rightarrow C \times D$  is an open map.*

PROOF: Given  $U$  open in  $A$  and  $V$  open in  $B$ . Then  $(f \times g)(U \times V) = f(U) \times g(V)$  is open in  $C \times D$ . The result follows from Proposition 86. □

## 21 The Subspace Topology

**Definition 136** (Subspace Topology). Let  $X$  be a topological space and  $Y \subseteq X$ . The *subspace topology* on  $Y$  is  $\mathcal{T} = \{U \cap Y \mid U \text{ is open in } X\}$ .

We prove this is a topology.

PROOF:

⟨1⟩1.  $Y \in \mathcal{T}$

PROOF: Since  $Y = X \cap Y$

⟨1⟩2. For all  $\mathcal{U} \subseteq \mathcal{T}$ , we have  $\bigcup \mathcal{U} \in \mathcal{T}$

⟨2⟩1. LET:  $\mathcal{U} \subseteq \mathcal{T}$

⟨2⟩2. LET:  $\mathcal{V} = \{V \text{ open in } X \mid V \cap Y \in \mathcal{U}\}$

⟨2⟩3.  $\bigcup \mathcal{U} = (\bigcup \mathcal{V}) \cap Y$

⟨1⟩3. For all  $U, V \in \mathcal{T}$ , we have  $U \cap V \in \mathcal{T}$

⟨2⟩1. LET:  $U, V \in \mathcal{T}$

⟨2⟩2. PICK  $U', V'$  open in  $X$  such that  $U = U' \cap Y$  and  $V = V' \cap Y$

⟨2⟩3.  $(U \cap V) = (U' \cap V') \cap Y$

□

**Theorem 137.** *Let  $X$  be a topological space and  $Y$  a subspace of  $X$ . Let  $A \subseteq Y$ . Then  $A$  is closed in  $Y$  if and only if there exists a closed set  $C$  in  $X$  such that  $A = C \cap Y$ .*

PROOF: We have

$$\begin{aligned} & A \text{ is closed in } Y \\ \Leftrightarrow & Y \setminus A \text{ is open in } Y \\ \Leftrightarrow & \exists U \text{ open in } X. Y \setminus A = Y \cap U \\ \Leftrightarrow & \exists C \text{ closed in } X. Y \setminus A = Y \cap (X \setminus U) \\ \Leftrightarrow & \exists C \text{ closed in } X. A = Y \cap C \end{aligned} \quad \square$$

**Theorem 138.** *Let  $Y$  be a subspace of  $X$ . Let  $A \subseteq Y$ . Let  $\bar{A}$  be the closure of  $A$  in  $X$ . Then the closure of  $A$  in  $Y$  is  $\bar{A} \cap Y$ .*

PROOF: The closure of  $A$  in  $Y$  is

$$\begin{aligned} & \bigcap \{C \text{ closed in } Y \mid A \subseteq C\} \\ &= \bigcap \{D \cap Y \mid D \text{ closed in } X, A \subseteq D \cap Y\} \quad (\text{Theorem 137}) \\ &= \bigcap \{D \mid D \text{ closed in } X, A \subseteq D\} \cap Y \\ &= \bar{A} \cap Y \end{aligned} \quad \square$$

**Lemma 139.** *Let  $X$  be a topological space and  $Y \subseteq X$ . Let  $\mathcal{B}$  be a basis for the topology on  $X$ . Then  $\mathcal{B}' = \{B \cap Y \mid B \in \mathcal{B}\}$  is a basis for the subspace topology on  $Y$ .*

PROOF:

⟨1⟩1. Every element in  $\mathcal{B}'$  is open in  $Y$

⟨1⟩2. For every open set  $U$  in  $Y$  and point  $y \in U$ , there exists  $B' \in \mathcal{B}'$  such that  $y \in B' \subseteq U$

⟨2⟩1. LET:  $U$  be open in  $Y$  and  $y \in U$

⟨2⟩2. PICK  $V$  open in  $X$  such that  $U = V \cap Y$

⟨2⟩3. PICK  $B \in \mathcal{B}$  such that  $y \in B \subseteq V$

⟨2⟩4. LET:  $B' = B \cap Y$

⟨2⟩5.  $B' \in \mathcal{B}'$

⟨2⟩6.  $y \in B' \subseteq U$

⟨1⟩3. Q.E.D.

PROOF: By Lemma 62.

□

**Lemma 140.** *Let  $X$  be a topological space and  $Y \subseteq X$ . Let  $\mathcal{S}$  be a basis for the topology on  $X$ . Then  $\mathcal{S}' = \{S \cap Y \mid S \in \mathcal{S}\}$  is a subbasis for the subspace topology on  $Y$ .*

PROOF: The set  $\{B \cap Y \mid B \text{ is a finite intersection of elements of } \mathcal{S}\}$  is a basis for the subspace topology by Lemma 139, and this is the set of all finite intersections of elements of  $\mathcal{S}'$ . □

**Lemma 141.** *Let  $Y$  be a subspace of  $X$ . If  $U$  is open in  $Y$  and  $Y$  is open in  $X$  then  $U$  is open in  $X$ .*

PROOF:

⟨1⟩1. PICK  $V$  open in  $X$  such that  $U = V \cap Y$

⟨1⟩2.  $U$  is open in  $X$

PROOF: Since it is the intersection of two open sets  $V$  and  $Y$ .

□

**Theorem 142.** *Let  $Y$  be a subspace of  $X$  and  $A \subseteq Y$ . If  $A$  is closed in  $Y$  and  $Y$  is closed in  $X$  then  $A$  is closed in  $X$ .*

PROOF: Pick a closed set  $C$  in  $X$  such that  $A = C \cap Y$  (Theorem 137). Then  $A$  is the intersection of two sets closed in  $X$ , hence  $A$  is closed in  $X$  (Lemma 32).

□

**Theorem 143.** *Let  $\{X_i\}_{i \in I}$  be a family of topological spaces. Let  $A_i$  be a subspace of  $X_i$  for all  $i \in I$ . Then the product topology on  $\prod_{i \in I} A_i$  is the same as the topology it inherits as a subspace of  $\prod_{i \in I} X_i$ .*

PROOF: The product topology is generated by the subbasis

$$\begin{aligned} & \{\pi_i^{-1}(U) \mid i \in I, U \text{ open in } A_i\} \\ &= \{\pi_i^{-1}(V) \cap A_i \mid i \in I, V \text{ open in } X_i\} \\ &= \{\pi_i^{-1}(V) \mid i \in I, V \text{ open in } X_i\} \cap \prod_{i \in I} A_i \end{aligned}$$

and this is a subbasis for the subspace topology by Lemma 140. □

**Theorem 144.** *Let  $X$  be an ordered set in the order topology. Let  $Y \subseteq X$  be an interval. Then the order topology on  $Y$  is the same as the subspace topology on  $Y$ .*

PROOF:

⟨1⟩1. The order topology is finer than the subspace topology.

⟨2⟩1. For every open ray  $R$  in  $X$ , the set  $R \cap Y$  is open in the order topology.

⟨3⟩1. For all  $a \in X$ , we have  $(-\infty, a) \cap Y$  is open in the order topology.

⟨4⟩1. CASE: For all  $y \in Y$  we have  $y < a$   
 PROOF: In this case  $(-\infty, a) \cap Y = Y$ .  
 ⟨4⟩2. CASE: For all  $y \in Y$  we have  $a < y$   
 PROOF: In this case  $(-\infty, a) \cap Y = \emptyset$ .  
 ⟨4⟩3. CASE: There exists  $y \in Y$  such that  $y \leq a$  and  $y \in Y$  such that  
 $a \leq y$   
 ⟨5⟩1.  $a \in Y$   
 PROOF: Because  $Y$  is an interval.  
 ⟨5⟩2.  $(-\infty, a) \cap Y = \{y \in Y \mid y < a\}$   
 ⟨3⟩2. For all  $a \in X$ , we have  $(a, +\infty) \cap Y$  is open in the order topology.  
 PROOF: Similar.  
 ⟨2⟩2. Q.E.D.  
 PROOF: By Lemmas 112 and 140 and Proposition 70.  
 ⟨1⟩2. The subspace topology is finer than the order topology.  
 ⟨2⟩1. Every open ray in  $Y$  is open in the subspace topology.  
 PROOF: For any  $a \in Y$  we have  $(-\infty, a)_Y = (-\infty, a)_X \cap Y$  and  $(a, +\infty)_Y = (a, +\infty)_X \cap Y$ .  
 ⟨2⟩2. Q.E.D.  
 PROOF: By Lemma 112 and Proposition 70  
 □

This example shows that we cannot remove the hypothesis that  $Y$  is an interval:

**Example 145.** The order topology on  $I_o^2$  is different from the subspace topology as a subspace of  $\mathbb{R}^2$  under the dictionary order topology.

PROOF: The set  $\{1/2\} \times (1/2, 1)$  is open in the subspace topology but not in the order topology. □

**Proposition 146.** Let  $X$  be a topological space,  $Y$  a subspace of  $X$ , and  $Z$  a subspace of  $Y$ . Then the subspace topology on  $Z$  inherited from  $X$  is the same as the subspace topology on  $Z$  inherited from  $Y$ .

PROOF: The subspace topology inherited from  $Y$  is

$$\begin{aligned}
 & \{V \cap Z \mid V \text{ open in } Y\} \\
 &= \{U \cap Y \cap Z \mid U \text{ open in } X\} \\
 &= \{U \cap Z \mid U \text{ open in } X\}
 \end{aligned}$$

which is the subspace topology inherited from  $X$ . □

**Definition 147** (Unit Circle). The *unit circle*  $S^1$  is  $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$  as a subspace of  $\mathbb{R}^2$ .

**Definition 148** (Unit 2-sphere). The *unit 2-sphere* is  $S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$  as a subspace of  $\mathbb{R}^3$ .

**Proposition 149.** Let  $f : X \rightarrow Y$  be an open map and  $A \subseteq X$  be open. Then the restriction  $f \upharpoonright A : A \rightarrow f(A)$  is an open map.

PROOF:

$\langle 1 \rangle 1$ . LET:  $U$  be open in  $A$

$\langle 1 \rangle 2$ .  $U$  is open in  $X$

PROOF: Lemma 141.

$\langle 1 \rangle 3$ .  $f(U)$  is open in  $Y$

$\langle 1 \rangle 4$ .  $f(U)$  is open in  $f(A)$

PROOF: Since  $f(U) = f(U) \cap f(A)$ .

□

**Example 150.** This example shows that we cannot remove the hypothesis that  $A$  is open.

Let  $A = \{(x, y) \in \mathbb{R}^2 \mid (x > 0 \text{ and } y = 1/x) \text{ or } x = y = 0\}$ . Then  $\pi_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$  is an open map, but  $\pi_1 \upharpoonright A : A \rightarrow [0, +\infty)$  is not, because it maps the set  $\{0, 0\}$  which is open in  $A$  to  $\{0\}$  which is not open in  $[0, +\infty)$ .

**Proposition 151.** *Let  $Y$  be a subspace of  $X$ . Let  $A \subseteq Y$  and  $l \in Y$ . Then  $l$  is a limit point of  $A$  in  $Y$  if and only if  $l$  is a limit point of  $A$  in  $X$ .*

PROOF: Both are equivalent to the condition that any neighbourhood of  $l$  in  $X$  intersects  $A$  in a point other than  $l$ . □

## 22 The Box Topology

**Definition 152** (Box Topology). Let  $\{A_i\}_{i \in I}$  be a family of topological spaces. The *box topology* on  $\prod_{i \in I} A_i$  is the topology generated by the set of all sets of the form  $\prod_{i \in I} U_i$  where  $\{U_i\}_{i \in I}$  is a family such that each  $U_i$  is open in  $A_i$ .

This is a basis since it covers  $\prod_{i \in I} A_i$  and is closed under intersection.

**Proposition 153.** *The box topology is finer than the product topology.*

PROOF: From Proposition 122. □

**Corollary 153.1.** *If  $A_i$  is closed in  $X_i$  for all  $i \in I$  then  $\prod_{i \in I} A_i$  is closed in  $\prod_{i \in I} X_i$  under the box topology.*

PROOF: From Proposition 123.

**Proposition 154** (AC). *Let  $\{A_i\}_{i \in I}$  be a family of topological spaces. For  $i \in I$ , let  $\mathcal{B}_i$  be a basis for the topology on  $A_i$ . Then  $\mathcal{B} = \{\prod_{i \in I} B_i \mid \forall i \in I, B_i \in \mathcal{B}_i\}$  is a basis for the box topology on  $\prod_{i \in I} A_i$ .*

PROOF:

$\langle 1 \rangle 1$ . Every set of the form  $\prod_{i \in I} B_i$  is open.

$\langle 1 \rangle 2$ . For every point  $a \in \prod_{i \in I} A_i$  and every open set  $U$  with  $a \in U$ , there exists  $B \in \mathcal{B}$  such that  $a \in B \subseteq U$ .

$\langle 2 \rangle 1$ . LET:  $U$  be open and  $a \in U$

$\langle 2 \rangle 2$ . PICK a family  $\{U_i\}_{i \in I}$  such that each  $U_i$  is open in  $A_i$  and  $a \in \prod_{i \in I} U_i \subseteq U$ .

⟨2⟩3. For  $i \in I$ , PICK  $B_i \in \mathcal{B}_i$  such that  $a_i \in B_i \subseteq U_i$

PROOF: Using the Axiom of Choice.

⟨2⟩4.  $a \in \prod_{i \in I} B_i \subseteq U$

⟨1⟩3. Q.E.D.

PROOF: Lemma 62.

□

**Theorem 155.** Let  $\{X_i\}_{i \in I}$  be a family of topological spaces. Let  $A_i$  be a subspace of  $X_i$  for all  $i \in I$ . Give  $\prod_{i \in I} X_i$  the box topology. Then the box topology on  $\prod_{i \in I} A_i$  is the same as the topology it inherits as a subspace of  $\prod_{i \in I} X_i$ .

PROOF: The box topology is generated by the basis

$$\begin{aligned} & \left\{ \prod_{i \in I} U_i \mid \forall i \in I, U_i \text{ open in } A_i \right\} \\ &= \left\{ \prod_{i \in I} (V_i \cap A_i) \mid \forall i \in I, V_i \text{ open in } X_i \right\} \\ &= \left\{ \prod_{i \in I} V_i \mid \forall i \in I, V_i \text{ open in } X_i \right\} \cap \prod_{i \in I} A_i \end{aligned}$$

and this is a basis for the subspace topology by Lemma 139. □

**Proposition 156 (AC).** Let  $\{X_i\}_{i \in I}$  be a family of topological spaces. Give  $\prod_{i \in I} X_i$  the box topology. Let  $A_i \subseteq X_i$  for all  $i \in I$ . Then

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i}.$$

PROOF:

⟨1⟩1.  $\overline{\prod_{i \in I} A_i} \subseteq \prod_{i \in I} \overline{A_i}$

⟨2⟩1. For all  $i \in I$  we have  $A_i \subseteq \overline{A_i}$

PROOF: Lemma 44.

⟨2⟩2.  $\prod_{i \in I} A_i \subseteq \prod_{i \in I} \overline{A_i}$

⟨2⟩3. Q.E.D.

PROOF: Since  $\prod_{i \in I} A_i$  is closed by Corollary 153.1.

⟨1⟩2.  $\prod_{i \in I} \overline{A_i} \subseteq \overline{\prod_{i \in I} A_i}$

⟨2⟩1. LET:  $x \in \prod_{i \in I} \overline{A_i}$

⟨2⟩2. LET:  $U$  be a neighbourhood of  $x$

⟨2⟩3. PICK  $V_i$  open in  $X_i$  such that  $x \in \prod_{i \in I} V_i \subseteq U$

⟨2⟩4. For  $i \in I$ , pick  $a_i \in V_i \cap A_i$

PROOF: By Theorem 47 and ⟨2⟩1 using the Axiom of Choice.

⟨2⟩5.  $U$  intersects  $\prod_{i \in I} A_i$

⟨2⟩6. Q.E.D.

PROOF:  $a \in U \cap \prod_{i \in I} A_i$ .

□

The following example shows that Theorem 131 fails in the box topology.

**Example 157.** Define  $f : \mathbb{R} \rightarrow \mathbb{R}^\omega$  by  $f(t) = (t, t, \dots)$ . Then  $\pi_n \circ f = \text{id}_{\mathbb{R}}$  is continuous for all  $n$ . But  $f$  is not continuous when  $\mathbb{R}^\omega$  is given the box topology because the inverse image of

$$(-1, 1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times \dots$$

is  $\{0\}$  which is not open.

The following example shows that Proposition 130 fails in the box topology.

**Example 158.** Give  $\mathbb{R}^\omega$  the box topology. Let  $a_n = (1/n, 1/n, \dots)$  for  $n \geq 1$  and  $l = (0, 0, \dots)$ . Then  $\pi_i(a_n) \rightarrow \pi_i(l)$  as  $n \rightarrow \infty$  for all  $i$ , but  $a_n \not\rightarrow l$  as  $n \rightarrow \infty$  since the open set

$$(-1, 1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times \dots$$

contains  $l$  but does not contain any  $a_n$ .

**Example 159.** The set  $\mathbb{R}^\infty$  is closed in  $\mathbb{R}^\omega$  under the box topology. For let  $(a_n)$  be any sequence not in  $\mathbb{R}^\infty$ . Let  $U_n$  be an open interval around  $a_n$  that does not contain 0 if  $a_n \neq 0$ , and  $U_n = \mathbb{R}$  if  $a_n = 0$ . Then  $\prod_{n \geq 0} U_n$  is a neighbourhood of  $(a_n)$  that does not intersect  $\mathbb{R}^\infty$ .

## 23 $T_1$ Spaces

**Definition 160** ( $T_1$  Space). A topological space is  $T_1$  if and only if every singleton is closed.

**Lemma 161.** A space is  $T_1$  if and only if every finite set is closed.

PROOF: From Lemma 33.  $\square$

**Theorem 162.** In a  $T_1$  space, a point  $a$  is a limit point of a set  $A$  if and only if every neighbourhood of  $a$  contains infinitely many points of  $A$ .

PROOF:

$\langle 1 \rangle 1$ . If  $a$  is a limit point of  $A$  then every neighbourhood of  $a$  contains infinitely many points of  $A$ .

$\langle 2 \rangle 1$ . ASSUME:  $a$  is a limit point of  $A$ .

$\langle 2 \rangle 2$ . LET:  $U$  be a neighbourhood of  $a$ .

$\langle 2 \rangle 3$ . ASSUME: for a contradiction  $U$  contains only finitely many points of  $A$ .

$\langle 2 \rangle 4$ .  $(U \cap A) \setminus \{a\}$  is closed.

PROOF: By the  $T_1$  axiom.

$\langle 2 \rangle 5$ .  $(U \setminus A) \cup \{a\}$  is open.

PROOF: It is  $U \setminus ((U \cap A) \setminus \{a\})$ .

$\langle 2 \rangle 6$ .  $(U \setminus A) \cup \{a\}$  intersects  $A$  in a point other than  $a$ .

PROOF: From  $\langle 2 \rangle 1$ .

$\langle 2 \rangle 7$ . Q.E.D.

$\square$



⟨1⟩2. If every neighbourhood of  $a$  contains infinitely many points of  $A$  then  $a$  is a limit point of  $A$ .

PROOF: Immediate from definitions.

□

(To see this does not hold in every space, see Proposition 58.)

**Proposition 163.** *A space is  $T_1$  if and only if, for any two distinct points  $x$  and  $y$ , there exist neighbourhoods  $U$  of  $x$  and  $V$  of  $y$  such that  $x \notin V$  and  $y \notin U$ .*

PROOF:

⟨1⟩1. LET:  $X$  be a topological space.

⟨1⟩2. If  $X$  is  $T_1$  then, for any two distinct points  $x$  and  $y$ , there exist neighbourhoods  $U$  of  $x$  and  $V$  of  $y$  such that  $x \notin V$  and  $y \notin U$ .

PROOF: This holds because  $\{x\}$  and  $\{y\}$  are closed.

⟨1⟩3. Suppose, for any two distinct points  $x$  and  $y$ , there exist neighbourhoods  $U$  of  $x$  and  $V$  of  $y$  such that  $x \notin V$  and  $y \notin U$ . Then  $X$  is  $T_1$ .

⟨2⟩1. ASSUME: For any two distinct points  $x$  and  $y$ , there exist neighbourhoods  $U$  of  $x$  and  $V$  of  $y$  such that  $x \notin V$  and  $y \notin U$ .

⟨2⟩2. LET:  $a \in X$

⟨2⟩3.  $\{a\}$  is closed.

PROOF: For all  $b \neq a$  there exists a neighbourhood  $U$  of  $b$  such that  $U \subseteq X \setminus \{a\}$ .

□

**Proposition 164.** *A subspace of a  $T_1$  space is  $T_1$ .*

PROOF: From Proposition 142.

## 24 Hausdorff Spaces

**Definition 165** (Hausdorff Space). A topological space is *Hausdorff* if and only if, for any points  $x, y$  with  $x \neq y$ , there exist disjoint open sets  $U$  and  $V$  such that  $x \in U$  and  $y \in V$ .

**Theorem 166.** *Every Hausdorff space is  $T_1$ .*

PROOF:

⟨1⟩1. LET:  $X$  be a Hausdorff space.

⟨1⟩2. LET:  $b \in X$

PROVE:  $\overline{\{b\}} = \{b\}$

⟨1⟩3. ASSUME:  $a \in \overline{\{b\}}$  and  $a \neq b$

⟨1⟩4. PICK disjoint neighbourhoods  $U$  of  $a$  and  $V$  of  $b$ .

⟨1⟩5.  $U$  intersects  $\{b\}$

PROOF: Theorem 47.

⟨1⟩6.  $b \in U$

⟨1⟩7. Q.E.D.

PROOF: This contradicts the fact that  $U$  and  $V$  are disjoint (⟨1⟩4).

□

**Proposition 167.** *An infinite set under the finite complement topology is  $T_1$  but not Hausdorff.*

PROOF:

⟨1⟩1. LET:  $X$  be an infinite set under the finite complement topology.

⟨1⟩2. Every singleton is closed.

PROOF: By definition.

⟨1⟩3. PICK  $a, b \in X$  with  $a \neq b$

⟨1⟩4. There are no disjoint neighbourhoods  $U$  of  $a$  and  $V$  of  $b$ .

⟨2⟩1. LET:  $U$  be a neighbourhood of  $a$  and  $V$  a neighbourhood of  $b$ .

⟨2⟩2.  $X \setminus U$  and  $X \setminus V$  are finite.

⟨2⟩3. PICK  $c \in X$  that is not in  $X \setminus U$  or  $X \setminus V$ .

⟨2⟩4.  $c \in U \cap V$

□

**Proposition 168.** *The product of a family of Hausdorff spaces is Hausdorff.*

PROOF:

⟨1⟩1. LET:  $\{X_i\}_{i \in I}$  be a family of Hausdorff spaces.

⟨1⟩2. LET:  $a, b \in \prod_{i \in I} X_i$  with  $a \neq b$

⟨1⟩3. PICK  $i \in I$  such that  $a_i \neq b_i$

⟨1⟩4. PICK  $U, V$  disjoint open sets in  $X_i$  with  $a_i \in U$  and  $b_i \in V$

⟨1⟩5.  $\pi_i^{-1}(U)$  and  $\pi_i^{-1}(V)$  are disjoint open sets in  $\prod_{i \in I} X_i$  with  $a \in \pi_i^{-1}(U)$  and  $b \in \pi_i^{-1}(V)$

□

**Theorem 169.** *Every linearly ordered set under the order topology is Hausdorff.*

PROOF:

⟨1⟩1. LET:  $X$  be a linearly ordered set under the order topology.

⟨1⟩2. LET:  $a, b \in X$  with  $a \neq b$

⟨1⟩3. ASSUME: w.l.o.g.  $a < b$

⟨1⟩4. CASE: There exists  $c$  such that  $a < c < b$

PROOF: The sets  $(-\infty, c)$  and  $(c, +\infty)$  are disjoint neighbourhoods of  $a$  and  $b$  respectively.

⟨1⟩5. CASE: There is no  $c$  such that  $a < c < b$

PROOF: The sets  $(-\infty, b)$  and  $(a, +\infty)$  are disjoint neighbourhoods of  $a$  and  $b$  respectively.

□

**Theorem 170.** *A subspace of a Hausdorff space is Hausdorff.*

PROOF:

⟨1⟩1. LET:  $X$  be a Hausdorff space and  $Y$  a subspace of  $X$ .

⟨1⟩2. LET:  $x, y \in Y$  with  $x \neq y$

⟨1⟩3. PICK disjoint neighbourhoods  $U$  of  $x$  and  $V$  of  $y$  in  $X$ .

⟨1⟩4.  $U \cap Y$  and  $V \cap Y$  are disjoint neighbourhoods of  $x$  and  $y$  respectively in  $Y$ .

□

**Proposition 171.** *A space  $X$  is Hausdorff if and only if the diagonal  $\Delta = \{(x, x) \mid x \in X\}$  is closed in  $X^2$ .*

PROOF:

$X$  is Hausdorff

$$\Leftrightarrow \forall x, y \in X. x \neq y \Rightarrow \exists V, W \text{ open. } x \in V \wedge y \in W \wedge V \cap W = \emptyset$$

$$\Leftrightarrow \forall (x, y) \in X^2 \setminus \Delta. \exists V, W \text{ open. } (x, y) \subseteq V \times W \subseteq X^2 \setminus \Delta$$

$$\Leftrightarrow \Delta \text{ is closed}$$

□

**Theorem 172.** *In a Hausdorff space, a sequence has at most one limit.*

PROOF:

⟨1⟩1. LET:  $X$  be a Hausdorff space.

⟨1⟩2. ASSUME: for a contradiction  $a_n \rightarrow l$  as  $n \rightarrow \infty$ ,  $a_n \rightarrow m$  as  $n \rightarrow \infty$ , and  $l \neq m$

⟨1⟩3. PICK disjoint neighbourhoods  $U$  of  $l$  and  $V$  of  $m$

PROOF: By the Hausdorff axiom.

⟨1⟩4. PICK  $M$  and  $N$  such that  $a_n \in U$  for  $n \geq M$  and  $a_n \in V$  for  $n \geq N$

⟨1⟩5.  $a_{\max(M, N)} \in U \cap V$

⟨1⟩6. Q.E.D.

PROOF: This contradicts the fact that  $U$  and  $V$  are disjoint (⟨1⟩3).

□

To see this is not always true in spaces that are  $T_1$  but not Hausdorff:

**Proposition 173.** *Let  $X$  be an infinite set under the finite complement topology. Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence with all points distinct. Then for every  $l \in X$  we have  $a_n \rightarrow l$  as  $n \rightarrow \infty$ .*

PROOF: Let  $U$  be any neighbourhood of  $l$ . Since  $X \setminus U$  is finite, there must exist  $N$  such that, for all  $n \geq N$ , we have  $a_n \in U$ . □

**Proposition 174.** *Let  $X$  be a topological space. Let  $Y$  a Hausdorff space. Let  $A \subseteq X$ . Let  $f, g : \overline{A} \rightarrow Y$  be continuous. If  $f$  and  $g$  agree on  $A$  then  $f = g$ .*

PROOF:

⟨1⟩1. LET:  $x \in \overline{A}$

⟨1⟩2. ASSUME:  $f(x) \neq g(x)$

⟨1⟩3. PICK disjoint neighbourhoods  $V$  of  $f(x)$  and  $W$  of  $g(x)$ .

⟨1⟩4. PICK  $y \in f^{-1}(V) \cap g^{-1}(W) \cap A$

PROOF: Since  $f^{-1}(V) \cap g^{-1}(W)$  is a neighbourhood of  $x$  and hence intersects  $A$ .

⟨1⟩5.  $f(y) = g(y) \in V \cap W$

⟨1⟩6. Q.E.D.

PROOF: This contradicts the fact that  $V$  and  $W$  are disjoint ( $\langle 1 \rangle 3$ ).

□

**Proposition 175.** *Let  $\{X_i\}_{i \in I}$  be a family of Hausdorff spaces. Then  $\prod_{i \in I} X_i$  under the box topology is Hausdorff.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $\{X_i\}_{i \in I}$  be a family of Hausdorff spaces.
- $\langle 1 \rangle 2$ . LET:  $a, b \in \prod_{i \in I} X_i$  with  $a \neq b$
- $\langle 1 \rangle 3$ . PICK  $i \in I$  such that  $a_i \neq b_i$
- $\langle 1 \rangle 4$ . PICK  $U, V$  disjoint open sets in  $X_i$  with  $a_i \in U$  and  $b_i \in V$
- $\langle 1 \rangle 5$ .  $\pi_i^{-1}(U)$  and  $\pi_i^{-1}(V)$  are disjoint open sets in  $\prod_{i \in I} X_i$  with  $a \in \pi_i^{-1}(U)$  and  $b \in \pi_i^{-1}(V)$

□

**Proposition 176.** *Let  $\mathcal{T}$  and  $\mathcal{T}'$  be two topologies on the same set  $X$  with  $\mathcal{T} \subseteq \mathcal{T}'$ . If  $\mathcal{T}$  is Hausdorff then  $\mathcal{T}'$  is Hausdorff.*

PROOF: Immediate from definitions.

## 25 The First Countability Axiom

**Definition 177** (First Countability Axiom). A topological space  $X$  satisfies the *first countability axiom*, or is *first countable*, if and only if every point has a countable local basis.

**Lemma 178** (Sequence Lemma (CC)). *Let  $X$  be a first countable space. Let  $A \subseteq X$  and  $l \in \bar{A}$ . Then there exists a sequence in  $A$  that converges to  $l$ .*

PROOF:

- $\langle 1 \rangle 1$ . PICK a countable local basis  $\{B_n \mid n \in \mathbb{Z}^+\}$  at  $l$  such that  $B_1 \supseteq B_2 \supseteq \dots$ .

PROOF: Lemma 72.

- $\langle 1 \rangle 2$ . For all  $n \geq 1$ , PICK  $a_n \in A \cap B_n$ .

PROVE:  $a_n \rightarrow l$  as  $n \rightarrow \infty$

- $\langle 1 \rangle 3$ . LET:  $U$  be a neighbourhood of  $l$

- $\langle 1 \rangle 4$ . PICK  $N$  such that  $B_N \subseteq U$

- $\langle 1 \rangle 5$ . For  $n \geq N$  we have  $a_n \in U$

PROOF:  $a_n \in B_n \subseteq B_N \subseteq U$

□

**Theorem 179** (CC). *Let  $X$  be a first countable space and  $Y$  a topological space. Let  $f : X \rightarrow Y$ . Suppose that, for every sequence  $(x_n)$  in  $X$  and  $l \in X$ , if  $x_n \rightarrow l$  as  $n \rightarrow \infty$ , then  $f(x_n) \rightarrow f(l)$  as  $n \rightarrow \infty$ . Then  $f$  is continuous.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $A \subseteq X$

- $\langle 1 \rangle 2$ . LET:  $a \in A$

PROVE:  $f(a) \in \overline{f(A)}$

⟨1⟩3. PICK a sequence  $(x_n)$  in  $A$  that converges to  $a$ .

PROOF: By the Sequence Lemma.

⟨1⟩4.  $f(x_n) \rightarrow f(a)$

⟨1⟩5.  $f(a) \in f(A)$

PROOF: By Lemma 74.

⟨1⟩6. Q.E.D.

PROOF: By Theorem 92.

□

**Example 180 (CC).** The space  $\mathbb{R}^\omega$  under the box product is not first countable.

PROOF: Let  $\{B_n \mid n \geq 0\}$  be a countable set of neighbourhoods of  $\vec{0}$ . We will construct a neighbourhood of  $\vec{0}$  that does not include any of these.

For  $n \geq 0$ , pick a neighbourhood  $U_n$  of 0 such that  $U_n \subset \pi_n(B_p)$ . Then  $\prod_{n=0}^\infty U_n$  is a neighbourhood of  $\vec{0}$  that does not include any  $B_n$ . □

**Example 181.** If  $J$  is an uncountable set then  $\mathbb{R}^J$  is not first countable.

PROOF:

⟨1⟩1. LET:  $\{B_n \mid n \geq 0\}$  be any countable set of neighbourhoods of  $\vec{0}$ .

⟨1⟩2. For  $n \geq 0$ , PICK a basis element  $\prod_{\alpha \in J} U_{n\alpha}$  that contains  $\vec{0}$  and is included in  $B_n$ .

PROOF: Using the Axiom of Countable Choice.

⟨1⟩3. For  $n \geq 0$ ,

LET:  $J_n = \{\alpha \in J \mid U_{n\alpha} \neq \mathbb{R}\}$

⟨1⟩4. PICK  $\beta \in J$  such that  $\beta \notin J_n$  for any  $n$ .

PROOF: Since each  $J_n$  is finite so  $\bigcup_n J_n$  is countable.

⟨1⟩5.  $\pi_\beta((-1, 1))$  is a neighbourhood of  $\vec{0}$  that does not include any  $B_n$ .

□

**Example 182.** The space  $\mathbb{R}_l$  is first countable.

PROOF: For any  $a \in \mathbb{R}$ , the set  $\{[a, a + 1/n) \mid n \geq 1\}$  is a countable local basis.

**Example 183.** The ordered square is first countable.

PROOF: For any  $(a, b) \in I_o^2$  with  $b \neq 0, 1$ , the set  $\{(\{a\} \times (b - 1/n, b + 1/n)) \cap I_o^2 \mid n \geq 1\}$  is a countable local basis.

## 26 Strong Continuity

**Definition 184** (Strongly Continuous). Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$  be a function. Then  $f$  is *strongly continuous* if and only if, for every subset  $U \subseteq Y$ , we have  $U$  is open in  $Y$  if and only if  $f^{-1}(U)$  is open in  $X$ .

**Proposition 185.** Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$  be a function. Then  $f$  is strongly continuous if and only if, for every subset  $C \subseteq Y$ , we have  $C$  is closed in  $Y$  if and only if  $f^{-1}(C)$  is closed in  $X$ .

PROOF: Since  $X \setminus f^{-1}(C) = f^{-1}(Y \setminus C)$ .  $\square$

**Proposition 186.** *Let  $X, Y$  and  $Z$  be topological spaces. Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$ . If  $f$  and  $g$  are strongly continuous then so is  $g \circ f$ .*

PROOF: Since  $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ .  $\square$

**Proposition 187.** *Let  $X, Y$  and  $Z$  be topological spaces. Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$ . If  $g \circ f$  is continuous and  $f$  is strongly continuous then  $g$  is continuous.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $V \subseteq Z$  be open.

$\langle 1 \rangle 2$ .  $f^{-1}(g^{-1}(V))$  is open in  $X$ .

PROOF: Since  $g \circ f$  is continuous.

$\langle 1 \rangle 3$ .  $f^{-1}(V)$  is open in  $Y$ .

PROOF: Since  $g$  is strongly continuous.

$\square$

**Proposition 188.** *Let  $X, Y$  and  $Z$  be topological spaces. Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$ . If  $g \circ f$  is strongly continuous and  $f$  is strongly continuous then  $g$  is strongly continuous.*

PROOF: For  $V \subseteq Z$ , we have  $V$  is open iff  $f^{-1}(g^{-1}(V))$  is open iff  $g^{-1}(V)$  is open.

## 27 Saturated Sets

**Definition 189.** Let  $X$  and  $Y$  be sets and  $p : X \rightarrow Y$  a surjective function. Let  $C \subseteq X$ . Then  $C$  is *saturated* with respect to  $p$  if and only if, for all  $x, y \in X$ , if  $x \in C$  and  $p(x) = p(y)$  then  $y \in C$ .

**Proposition 190.** *Let  $X$  and  $Y$  be sets and  $p : X \rightarrow Y$  a surjective function. Let  $C \subseteq X$ . Then the following are equivalent:*

1.  $C$  is saturated with respect to  $p$ .
2. There exists  $D \subseteq Y$  such that  $C = p^{-1}(D)$
3.  $C = p^{-1}(p(C))$ .

PROOF:

$\langle 1 \rangle 1$ .  $1 \Rightarrow 3$

$\langle 2 \rangle 1$ . ASSUME:  $C$  is saturated with respect to  $p$ .

$\langle 2 \rangle 2$ .  $C \subseteq p^{-1}(p(C))$

PROOF: Trivial.

$\langle 2 \rangle 3$ .  $p^{-1}(p(C)) \subseteq C$

$\langle 3 \rangle 1$ . LET:  $x \in p^{-1}(p(C))$

$\langle 3 \rangle 2$ .  $p(x) \in p(C)$

$\langle 3 \rangle 3$ . There exists  $y \in C$  such that  $p(x) = p(y)$

$\langle 3 \rangle 4$ .  $x \in C$

PROOF: From  $\langle 2 \rangle 1$ .

$\langle 1 \rangle 2$ .  $3 \Rightarrow 2$

PROOF: Trivial.

$\langle 1 \rangle 3$ .  $2 \Rightarrow 1$

PROOF: This follows because if  $p(x) \in D$  and  $p(x) = p(y)$  then  $p(y) \in D$ .

□

## 28 Quotient Maps

**Definition 191** (Quotient Map). Let  $X$  and  $Y$  be topological spaces and  $p : X \rightarrow Y$ . Then  $p$  is a *quotient map* if and only if  $p$  is surjective and strongly continuous.

**Proposition 192.** Let  $X$  and  $Y$  be topological spaces and  $p : X \rightarrow Y$  be a surjective function. Then the following are equivalent.

1.  $p$  is a quotient map.
2.  $p$  is continuous and maps saturated open sets to open sets.
3.  $p$  is continuous and maps saturated closed sets to closed sets.

PROOF:

$\langle 1 \rangle 1$ .  $1 \Rightarrow 2$

$\langle 2 \rangle 1$ . ASSUME:  $p$  is a quotient map.

$\langle 2 \rangle 2$ . LET:  $U$  be a saturated open set in  $X$ .

$\langle 2 \rangle 3$ .  $p^{-1}(p(U))$  is open in  $X$ .

PROOF: Since  $U = p^{-1}(p(U))$  by Proposition 190.

$\langle 2 \rangle 4$ .  $p(U)$  is open in  $Y$ .

PROOF: From  $\langle 2 \rangle 1$ .

$\langle 1 \rangle 2$ .  $1 \Rightarrow 3$

PROOF: Similar.

$\langle 1 \rangle 3$ .  $2 \Rightarrow 1$

$\langle 2 \rangle 1$ . ASSUME:  $p$  is continuous and maps saturated open sets to open sets.

$\langle 2 \rangle 2$ . LET:  $U \subseteq Y$

$\langle 2 \rangle 3$ . ASSUME:  $p^{-1}(U)$  is open in  $X$

$\langle 2 \rangle 4$ .  $p^{-1}(U)$  is saturated.

PROOF: Proposition 190.

$\langle 2 \rangle 5$ .  $U$  is open in  $Y$ .

$\langle 1 \rangle 4$ .  $3 \Rightarrow 1$

PROOF: Similar.

□

**Corollary 192.1.** Every surjective continuous open map is a quotient map.

**Corollary 192.2.** Every surjective continuous closed map is a quotient map.

**Example 193.** The converses of these corollaries do not hold.

Let  $A = \{(x, y) \mid x \geq 0\} \cup \{(x, y) \mid y = 0\}$ . Then  $\pi_1 : A \rightarrow \mathbb{R}$  is a quotient map, but not an open map or a closed map.

We prove that  $\pi_1$  maps saturated open sets to open sets:

- $\langle 1 \rangle 1$ . LET:  $\pi_1^{-1}(U)$  be a saturated open set in  $A$   
PROVE:  $U$  is open in  $\mathbb{R}$
- $\langle 1 \rangle 2$ . LET:  $x \in U$
- $\langle 1 \rangle 3$ .  $(x, 0) \in \pi_1(U)^{-1}$
- $\langle 1 \rangle 4$ . PICK  $W, V$  open in  $\mathbb{R}$  such that  $(x, 0) \subseteq W \times V \subseteq \pi_1(U)^{-1}$
- $\langle 1 \rangle 5$ .  $x \in W \subseteq U$

It is not an open map because it maps  $((-1, 1) \times (1, 2)) \cap A$  to  $[0, 1)$ .

It is not a closed map because it maps  $\{(x, 1/x) \mid x > 0\}$  to  $(0, +\infty)$ .

**Proposition 194.** Let  $p : X \twoheadrightarrow Y$  be a quotient map. Let  $A \subseteq X$  be saturated with respect to  $p$ . Let  $q : A \twoheadrightarrow p(A)$  be the restriction of  $p$ .

- 1. If  $A$  is either open or closed in  $X$  then  $q$  is a quotient map.
- 2. If  $p$  is either an open map or a closed map then  $q$  is a quotient map.

PROOF:

- $\langle 1 \rangle 1$ . LET:  $p : X \twoheadrightarrow Y$  be a quotient map.
- $\langle 1 \rangle 2$ . LET:  $A \subseteq X$  be saturated with respect to  $p$ .
- $\langle 1 \rangle 3$ . LET:  $q : A \twoheadrightarrow p(A)$  be the restriction of  $p$ .
- $\langle 1 \rangle 4$ .  $q$  is continuous.  
PROOF: Theorem 96.
- $\langle 1 \rangle 5$ . If  $A$  is open in  $X$  then  $q$  is a quotient map.
  - $\langle 2 \rangle 1$ . ASSUME:  $A$  is open in  $X$ .
  - $\langle 2 \rangle 2$ .  $q$  maps saturated open sets to open sets.
    - $\langle 3 \rangle 1$ . LET:  $U \subseteq A$  be saturated with respect to  $q$  and open in  $A$
    - $\langle 3 \rangle 2$ .  $U$  is saturated with respect to  $p$ 
      - $\langle 4 \rangle 1$ . LET:  $x, y \in X$
      - $\langle 4 \rangle 2$ . ASSUME:  $x \in U$
      - $\langle 4 \rangle 3$ . ASSUME:  $p(x) = p(y)$
      - $\langle 4 \rangle 4$ .  $x \in A$   
PROOF: From  $\langle 3 \rangle 1$  and  $\langle 4 \rangle 2$ .
      - $\langle 4 \rangle 5$ .  $y \in A$   
PROOF: From  $\langle 1 \rangle 2$  and  $\langle 4 \rangle 3$
      - $\langle 4 \rangle 6$ .  $q(x) = x(y)$   
PROOF: From  $\langle 1 \rangle 3$ ,  $\langle 4 \rangle 3$ ,  $\langle 4 \rangle 4$ ,  $\langle 4 \rangle 5$ .
      - $\langle 4 \rangle 7$ .  $y \in U$   
PROOF: From  $\langle 3 \rangle 1$ ,  $\langle 4 \rangle 2$ ,  $\langle 4 \rangle 6$
  - $\langle 3 \rangle 3$ .  $U$  is open in  $X$   
PROOF: Lemma 141,  $\langle 2 \rangle 1$ ,  $\langle 3 \rangle 1$ .
  - $\langle 3 \rangle 4$ .  $p(U)$  is open in  $Y$   
PROOF: Proposition 192,  $\langle 1 \rangle 1$ ,  $\langle 3 \rangle 2$ ,  $\langle 3 \rangle 3$



⟨3⟩5.  $q(U)$  is open in  $p(A)$

PROOF: Since  $q(U) = p(U) = p(U) \cap p(A)$ .

⟨2⟩3. Q.E.D.

PROOF: By Proposition 192.

⟨1⟩6. If  $A$  is closed in  $X$  then  $q$  is a quotient map.

PROOF: Similar.

⟨1⟩7. If  $p$  is an open map then  $q$  is a quotient map.

⟨2⟩1. ASSUME:  $p$  is an open map

⟨2⟩2.  $q$  maps saturated open sets to open sets.

⟨3⟩1. LET:  $U$  be open in  $A$  and saturated with respect to  $q$

⟨3⟩2. PICK  $V$  open in  $X$  such that  $U = A \cap V$

⟨3⟩3.  $p(V)$  is open in  $Y$

⟨3⟩4.  $q(U) = p(V) \cap p(A)$

⟨4⟩1.  $q(U) \subseteq p(V) \cap p(A)$

PROOF: From ⟨3⟩2.

⟨4⟩2.  $p(V) \cap p(A) \subseteq q(U)$

⟨5⟩1. LET:  $y \in p(V) \cap p(A)$

⟨5⟩2. PICK  $x \in V$  and  $x' \in A$  such that  $p(x) = p(x') = y$

⟨5⟩3.  $x \in A$

PROOF: By ⟨1⟩2.

⟨5⟩4.  $x \in U$

PROOF: From ⟨3⟩2

⟨2⟩3. Q.E.D.

PROOF: By Proposition 192.

⟨1⟩8. If  $p$  is a closed map then  $q$  is a quotient map.

PROOF: Similar.

□

**Example 195.** This example shows we cannot remove the hypotheses on  $A$  and  $p$ .

Define  $f : [0, 1] \rightarrow [2, 3] \rightarrow [0, 2]$  by  $f(x) = x$  if  $x \leq 1$ ,  $f(x) = x - 1$  if  $x \geq 2$ . Then  $f$  is a quotient map but its restriction  $f'$  to  $[0, 1] \cup [2, 3]$  is not, because  $f'^{-1}([1, 2])$  is open but  $[1, 2]$  is not.

For a counterexample where  $A$  is saturated, see Example 201.

**Proposition 196.** Let  $p : A \rightarrow C$  and  $q : B \rightarrow D$  be open quotient maps. Then  $p \times q : A \times B \rightarrow C \times D$  is an open quotient map.

PROOF: From Corollary 192.1, Proposition 135 and Theorem 131. □

**Theorem 197.** Let  $p : X \rightarrow Y$  be a quotient map. Let  $Z$  be a topological space and  $f : Y \rightarrow Z$  be a function. Then

1.  $f \circ p$  is continuous if and only if  $f$  is continuous.

2.  $f \circ p$  is a quotient map if and only if  $f$  is a quotient map.

PROOF:

- $\langle 1 \rangle 1$ . If  $f \circ p$  is continuous then  $f$  is continuous.  
 PROOF: Proposition 187.  
 $\langle 1 \rangle 2$ . If  $f$  is continuous then  $f \circ p$  is continuous.  
 PROOF: Theorem 95.  
 $\langle 1 \rangle 3$ . If  $f \circ p$  is a quotient map then  $f$  is a quotient map.  
 PROOF: Proposition 188.  
 $\langle 1 \rangle 4$ . If  $f$  is a quotient map then  $f \circ p$  is a quotient map.  
 PROOF: From Proposition 186.

□

**Proposition 198.** *Let  $X$  and  $Y$  be topological spaces. Let  $p : X \rightarrow Y$  and  $f : Y \rightarrow X$  be continuous maps such that  $p \circ f = \text{id}_Y$ . Then  $p$  is a quotient map.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $V \subseteq Y$   
 $\langle 1 \rangle 2$ . ASSUME:  $p^{-1}(V)$  is open in  $X$ .  
 $\langle 1 \rangle 3$ .  $f^{-1}(p^{-1}(V))$  is open in  $Y$ .  
 PROOF: Because  $f$  is continuous.  
 $\langle 1 \rangle 4$ .  $V$  is open in  $Y$ .  
 PROOF: Because  $f^{-1}(p^{-1}(V)) = V$ .

□

## 29 Quotient Topology

**Definition 199** (Quotient Topology). Let  $X$  be a topological space,  $Y$  a set and  $p : X \rightarrow Y$  be a surjective function. Then the *quotient topology* on  $Y$  is the unique topology on  $Y$  with respect to which  $p$  is a quotient map, namely  $\mathcal{T} = \{U \in \mathcal{P}Y \mid p^{-1}(U) \text{ is open in } X\}$ .

We prove this is a topology.

PROOF:

- $\langle 1 \rangle 1$ .  $Y \in \mathcal{T}$   
 PROOF: Since  $p^{-1}(Y) = X$  by surjectivity.  
 $\langle 1 \rangle 2$ . For all  $\mathcal{A} \subseteq \mathcal{T}$  we have  $\bigcup \mathcal{A} \in \mathcal{T}$   
 PROOF: Since  $p^{-1}(\bigcup \mathcal{A}) = \bigcup_{U \in \mathcal{A}} p^{-1}(U)$   
 $\langle 1 \rangle 3$ . For all  $U, V \in \mathcal{T}$  we have  $U \cap V \in \mathcal{T}$   
 PROOF: Since  $p^{-1}(U \cap V) = p^{-1}(U) \cap p^{-1}(V)$ .

□

**Definition 200** (Quotient Space). Let  $X$  be a topological space and  $\sim$  an equivalence relation on  $X$ . Let  $p : X \rightarrow X/\sim$  be the canonical surjection. Then  $X/\sim$  under the quotient topology is called a *quotient space*, *identification space* or *decomposition space* of  $X$ .

Here is a counterexample showing we cannot remove all the hypotheses of Proposition 194 except that  $A$  is saturated.

**Example 201.** Let  $X = (0, 1/2] \cup \{1\} \cup \{1 + 1/n : n \geq 2\}$  as a subspace of  $\mathbb{R}$ . Define  $R$  to be the equivalence relation on  $X$  where  $xRy$  iff  $(x = y \text{ or } |x - y| = 1)$ , so we identify  $1/n$  with  $1 + 1/n$  for all  $n \geq 2$ . Let  $Y$  be the resulting quotient space  $X/R$  in the quotient topology and  $p : X \twoheadrightarrow Y$  the canonical surjection.

Let  $A = \{1\} \cup (0, 1/2] \setminus \{1/n : n \geq 2\} \subseteq X$ . Then  $A$  is saturated under  $p$  but the restriction  $q$  of  $p$  to  $A$  is not a quotient map because it maps the saturated open set  $\{1\}$  to  $\{1\}$  which is not open in  $p(A)$ .

**Proposition 202.** Let  $X, Y$  and  $Z$  be topological spaces. Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$ . If  $f$  and  $g$  are quotient maps then so is  $g \circ f$ .

PROOF: From Proposition 186.  $\square$

**Example 203.** The product of two quotient maps is not necessarily a quotient map.

Let  $X = \mathbb{R}$  and  $X^*$  the quotient space formed by identifying all positive integers. Let  $p : X \twoheadrightarrow X^*$  be the canonical surjection.

We prove  $p \times \text{id}_{\mathbb{Q}} : X \times \mathbb{Q} \rightarrow X^* \times \mathbb{Q}$  is not a quotient map.

PROOF:

$\langle 1 \rangle 1$ . For  $n \geq 1$ ,

LET:  $c_n = \sqrt{2}/n$

$\langle 1 \rangle 2$ . For  $n \geq 1$ ,

LET:  $U_n = \{(x, y) \in X \times \mathbb{Q} \mid n - 1/4 < x < n + 1/4, (y + n > x + c_n \text{ and } y + n > -x + c_n) \text{ or } (y + n < x + c_n \text{ and } y + n < -x + c_n)\}$

$\langle 1 \rangle 3$ . For  $n \geq 1$ , we have  $U_n$  is open in  $X \times \mathbb{Q}$

$\langle 1 \rangle 4$ . For  $n \geq 1$ , we have  $\{n\} \times \mathbb{Q} \subseteq U_n$

$\langle 1 \rangle 5$ . LET:  $U = \bigcup_{n=1}^{\infty} U_n$

$\langle 1 \rangle 6$ .  $U$  is open in  $X \times \mathbb{Q}$

$\langle 1 \rangle 7$ .  $U$  is saturated with respect to  $p \times \text{id}_{\mathbb{Q}}$

$\langle 1 \rangle 8$ . LET:  $U' = (p \times \text{id}_{\mathbb{Q}})(U)$

$\langle 1 \rangle 9$ . ASSUME: for a contradiction  $U'$  is open in  $X^* \times \mathbb{Q}$

$\langle 1 \rangle 10$ .  $(1, 0) \in U'$

$\langle 1 \rangle 11$ . PICK a neighbourhood  $W$  of 1 in  $X^*$  and  $\delta > 0$  such that  $W \times (-\delta, \delta) \subseteq U'$

$\langle 1 \rangle 12$ .  $p^{-1}(W) \times (-\delta, \delta) \subseteq U$

$\langle 1 \rangle 13$ . PICK  $n$  such that  $c_n < \delta$

$\langle 1 \rangle 14$ .  $n \in p^{-1}(W)$

$\langle 1 \rangle 15$ . PICK  $\epsilon > 0$  such that  $\epsilon < \delta - c_n$  and  $\epsilon < 1/4$  and  $(n - \epsilon, n + \epsilon) \subseteq p^{-1}(W)$

$\langle 1 \rangle 16$ .  $(n - \epsilon, n + \epsilon) \times (-\delta, \delta) \subseteq U$

$\langle 1 \rangle 17$ . PICK a rational  $y$  such that  $c_n - \epsilon/2 < y < c_n + \epsilon/2$

$\langle 1 \rangle 18$ .  $(n + \epsilon/2, y) \notin U$

$\langle 1 \rangle 19$ . Q.E.D.

PROOF: This contradicts  $\langle 1 \rangle 16$ .

$\square$

**Proposition 204.** Let  $X$  be a topological space and  $\sim$  an equivalence relation on  $X$ . Then  $X/\sim$  is  $T_1$  if and only if every equivalence class is closed in  $X$ .

PROOF: Immediate from definitions.  $\square$

## 30 Retractions

**Definition 205** (Retraction). Let  $X$  be a topological space and  $A \subseteq X$ . A *retraction* of  $X$  onto  $A$  is a continuous map  $r : X \rightarrow A$  such that, for all  $a \in A$ , we have  $r(a) = a$ .

**Proposition 206.** *Every retraction is a quotient map.*

PROOF: Proposition 198 with  $f$  the inclusion  $A \hookrightarrow X$ .  $\square$

## 31 Homogeneous Spaces

**Definition 207** (Homogeneous). A topological space  $X$  is *homogeneous* if and only if, for any points  $a, b \in X$ , there exists a homeomorphism  $\phi : X \cong X$  such that  $\phi(a) = b$ .

## 32 Regular Spaces

**Definition 208** (Regular Space). A topological space  $X$  is *regular* if and only if, for any closed set  $A$  and point  $a \notin A$ , there exist disjoint open sets  $U, V$  such that  $A \subseteq U$  and  $a \in V$ .

## 33 Connected Spaces

**Definition 209** (Separation). A *separation* of a topological space  $X$  is a pair of disjoint open sets  $U, V$  such that  $U \cup V = \emptyset$ .

**Definition 210** (Connected). A topological space is *connected* if and only if it has no separation; otherwise it is *disconnected*.

**Proposition 211.** *A topological space  $X$  is connected if and only if the only sets that are both open and closed are  $X$  and  $\emptyset$ .*

Immediate from definitions.

**Lemma 212.** *If  $Y$  is a subspace of  $X$ , a separation of  $Y$  is a pair of disjoint nonempty sets  $A$  and  $B$  whose union is  $Y$ , neither of which contains a limit point of the other.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $A, B \subseteq Y$

$\langle 1 \rangle 2$ . If  $A$  and  $B$  form a separation of  $Y$  then  $A$  and  $B$  are disjoint and nonempty,  $A \cup B = Y$ , and neither of  $A$  and  $B$  contains a limit point of the other.

$\langle 2 \rangle 1$ . ASSUME:  $A$  and  $B$  form a separation of  $Y$

$\langle 2 \rangle 2$ .  $A$  and  $B$  are disjoint and nonempty and  $A \cup B = Y$

PROOF: From  $\langle 2 \rangle 1$  and the definition of separation.

$\langle 2 \rangle 3$ .  $A$  does not contain a limit point of  $B$

⟨3⟩1. ASSUME: for a contradiction  $l \in A$  and  $l$  is a limit point of  $B$  in  $X$ .  
 ⟨3⟩2.  $l$  is a limit point of  $B$  in  $Y$   
 PROOF: Proposition 151.  
 ⟨3⟩3.  $l \in B$   
 ⟨4⟩1.  $B$  is closed in  $Y$   
 PROOF: Since  $A$  is open in  $Y$  and  $B = Y \setminus A$  from ⟨2⟩1.  
 ⟨4⟩2. Q.E.D.  
 PROOF: Corollary 57.1.  
 ⟨3⟩4. Q.E.D.  
 PROOF: This contradicts the fact that  $A \cap B = \emptyset$  (⟨2⟩1).  
 ⟨2⟩4.  $B$  does not contain a limit point of  $A$   
 PROOF: Similar.  
 ⟨1⟩3. If  $A$  and  $B$  are disjoint and nonempty,  $A \cup B = Y$ , and neither of  $A$  and  $B$  contains a limit point of the other, then  $A$  and  $B$  form a separation of  $Y$ .  
 ⟨2⟩1. ASSUME:  $A$  and  $B$  are disjoint and nonempty,  $A \cup B = Y$ , and neither of  $A$  and  $B$  contains a limit point of the other.  
 ⟨2⟩2.  $A$  is open in  $Y$   
 ⟨3⟩1.  $B$  is closed in  $Y$   
 ⟨4⟩1. LET:  $l$  be a limit point of  $B$  in  $Y$   
 ⟨4⟩2.  $l$  is a limit point of  $B$  in  $X$   
 PROOF: Proposition 151.  
 ⟨4⟩3.  $l \notin A$   
 PROOF: By ⟨2⟩1  
 ⟨4⟩4.  $l \in B$   
 PROOF: By ⟨2⟩1 since  $A \cup B = Y$   
 ⟨4⟩5. Q.E.D.  
 PROOF: Corollary 57.1.  
 ⟨3⟩2. Q.E.D.  
 PROOF: Since  $A = Y \setminus B$ .  
 ⟨2⟩3.  $B$  is open in  $Y$   
 PROOF: Similar.

□

**Example 213.** Every set under the indiscrete topology is connected.

**Example 214.** The discrete topology on a set  $X$  is connected if and only if  $|X| \leq 1$ .

**Example 215.** The finite complement topology on a set  $X$  is connected if and only if either  $|X| \leq 1$  or  $X$  is infinite.

**Example 216.** The countable complement topology on a set  $X$  is connected if and only if either  $|X| \leq 1$  or  $X$  is uncountable.

**Example 217.** The rationals  $\mathbb{Q}$  are disconnected. For any irrational  $a$ , the sets  $(-\infty, a) \cap \mathbb{Q}$  and  $(a, +\infty) \cap \mathbb{Q}$  form a separation of  $\mathbb{Q}$ .

**Lemma 218.** *Let  $X$  be a topological space. If  $C$  and  $D$  form a separation of  $X$ , and  $Y$  is a connected subspace of  $X$ , then either  $Y \subseteq C$  or  $Y \subseteq D$ .*

PROOF: Otherwise  $Y \cap C$  and  $Y \cap D$  would form a separation of  $Y$ .  $\square$

**Theorem 219.** *The union of a set of connected subspaces of a space  $X$  that have a point in common is connected.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $\mathcal{A}$  be a set of connected subspaces of the space  $X$  that have the point  $a$  in common.

$\langle 1 \rangle 2$ . ASSUME: for a contradiction  $C$  and  $D$  form a separation of  $\bigcup \mathcal{A}$

$\langle 1 \rangle 3$ . ASSUME: without loss of generality  $a \in C$

$\langle 1 \rangle 4$ . For all  $A \in \mathcal{A}$  we have  $A \subseteq C$

PROOF: Lemma 218.

$\langle 1 \rangle 5$ .  $D = \emptyset$

$\langle 1 \rangle 6$ . Q.E.D.

PROOF: This contradicts  $\langle 1 \rangle 2$ .

$\square$

**Theorem 220.** *Let  $X$  be a topological space and  $A$  a connected subspace of  $X$ . If  $A \subseteq B \subseteq \overline{A}$  then  $B$  is connected.*

PROOF:

$\langle 1 \rangle 1$ . ASSUME: for a contradiction  $C$  and  $D$  form a separation of  $B$ .

$\langle 1 \rangle 2$ . ASSUME: without loss of generality  $A \subseteq C$

PROOF: Lemma 218.

$\langle 1 \rangle 3$ .  $B \subseteq C$

$\langle 2 \rangle 1$ . LET:  $x \in B$

$\langle 2 \rangle 2$ .  $x \in \overline{A}$

$\langle 2 \rangle 3$ . Either  $x \in A$  or  $x$  is a limit point of  $A$ .

PROOF: Theorem 57.

$\langle 2 \rangle 4$ . Either  $x \in A$  or  $x$  is a limit point of  $C$ .

PROOF: Lemma 59,  $\langle 1 \rangle 2$ .

$\langle 2 \rangle 5$ .  $x \in C$

PROOF: Lemma 212.

$\langle 1 \rangle 4$ .  $D = \emptyset$

$\langle 1 \rangle 5$ . Q.E.D.

PROOF: This contradicts  $\langle 1 \rangle 1$ .

$\square$

**Theorem 221.** *The image of a connected space under a continuous map is connected.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $f : X \rightarrow Y$  be a surjective continuous map where  $X$  is connected.

$\langle 1 \rangle 2$ . ASSUME: for a contradiction  $C$  and  $D$  form a separation of  $Y$ .

$\langle 1 \rangle 3$ .  $f^{-1}(C)$  and  $f^{-1}(D)$  form a separation of  $X$ .

□

**Theorem 222.** *The product of a family of connected spaces is connected.*

PROOF:

⟨1⟩1. The product of two connected spaces is connected.

⟨2⟩1. LET:  $X$  and  $Y$  be connected spaces.

⟨2⟩2. PICK  $a \in X$  and  $b \in Y$

PROOF: We may assume  $X$  and  $Y$  are nonempty since otherwise  $X \times Y = \emptyset$  which is connected.

⟨2⟩3.  $X \times \{b\}$  is connected.

PROOF: It is homeomorphic to  $X$ .

⟨2⟩4. For all  $x \in X$  we have  $\{x\} \times Y$  is connected.

PROOF: It is homeomorphic to  $Y$ .

⟨2⟩5. For any  $x \in X$

LET:  $T_x = (X \times \{b\}) \cup (\{x\} \times Y)$

⟨2⟩6. For all  $x \in X$ ,  $T_x$  is connected.

PROOF: Theorem 219 since  $(x, b) \in (X \times \{b\}) \cap (\{x\} \times Y)$ .

⟨2⟩7.  $X \times Y$  is connected.

PROOF: Theorem 219 since  $X \times Y = \bigcup_{x \in X} T_x$  and  $(a, b)$  is a point in every  $T_x$ .

⟨1⟩2. The product of a finite family of connected spaces is connected.

PROOF: From ⟨1⟩1 by induction.

⟨1⟩3. The product of any family of connected spaces is connected.

⟨2⟩1. LET:  $\{X_\alpha\}_{\alpha \in J}$  be a family of connected spaces.

⟨2⟩2. LET:  $X = \prod_{\alpha \in J} X_\alpha$

⟨2⟩3. PICK  $a \in X$

PROOF: We may assume  $X \neq \emptyset$  as the empty space is connected.

⟨2⟩4. For every finite subset  $K$  of  $J$ ,

LET:  $X_K = \{x \in X \mid \forall \alpha \in J \setminus K, x_\alpha = a_\alpha\}$

⟨2⟩5. For every finite  $K \subseteq J$ , we have  $X_K$  is connected.

PROOF: From ⟨1⟩2 since  $X_K \cong \prod_{\alpha \in K} X_\alpha$ .

⟨2⟩6. LET:  $Y = \bigcup_K X_K$

⟨2⟩7.  $Y$  is connected

PROOF: Theorem 219 since  $a$  is a common point.

⟨2⟩8.  $X = \bar{Y}$

⟨3⟩1. LET:  $x \in X$

⟨3⟩2. LET:  $U = \prod_{\alpha \in J} U_\alpha$  be a basic neighbourhood of  $x$  where  $U_\alpha = X_\alpha$  for all  $\alpha$  except  $\alpha \in K$  for some finite  $K \subseteq J$

⟨3⟩3. LET:  $y \in X$  be the point with  $y_\alpha = x_\alpha$  for  $\alpha \in K$  and  $y_\alpha = a_\alpha$  for all other  $\alpha$

⟨3⟩4.  $y \in U \cap X_K$

⟨3⟩5.  $y \in U \cap Y$

⟨2⟩9.  $X$  is connected.

PROOF: Theorem 220.

□

**Example 223.** The set  $\mathbb{R}^\omega$  is disconnected under the box topology. The set of bounded sequences and the set of unbounded sequences form a separation.

**Proposition 224.** Let  $\mathcal{T}$  and  $\mathcal{T}'$  be two topologies on the same set  $X$ . If  $\mathcal{T} \subseteq \mathcal{T}'$  and  $(X, \mathcal{T}')$  is connected then  $(X, \mathcal{T})$  is connected.

PROOF: If  $U$  and  $V$  form a separation of  $(X, \mathcal{T})$  then they form a separation of  $(X, \mathcal{T}')$ .  $\square$

**Proposition 225.** Let  $X$  be a topological space and  $(A_n)$  a sequence of connected subspaces of  $X$ . If  $A_n \cap A_{n+1} \neq \emptyset$  for all  $n$  then  $\bigcup_n A_n$  is connected.

PROOF:

$\langle 1 \rangle 1$ . ASSUME: for a contradiction  $C$  and  $D$  form a separation of  $\bigcup_n A_n$

$\langle 1 \rangle 2$ . ASSUME: without loss of generality  $A_0 \subseteq C$

PROOF: Lemma 218.

$\langle 1 \rangle 3$ . For all  $n$  we have  $A_n \subseteq C$

PROOF:

$\langle 2 \rangle 1$ . ASSUME:  $A_n \subseteq C$

$\langle 2 \rangle 2$ . PICK  $x \in A_n \cap A_{n+1}$

$\langle 2 \rangle 3$ .  $x \in C$

$\langle 2 \rangle 4$ .  $A_{n+1} \subseteq C$

PROOF: Lemma 218.

$\langle 2 \rangle 5$ . Q.E.D.

PROOF: The result follows by induction.

$\langle 1 \rangle 4$ .  $D = \emptyset$

$\langle 1 \rangle 5$ . Q.E.D.

PROOF: This contradicts  $\langle 1 \rangle 1$ .

$\square$

**Proposition 226.** Let  $X$  be a topological space. Let  $A, C \subseteq X$ . If  $C$  is connected and intersects both  $A$  and  $X \setminus A$  then  $C$  intersects  $\partial A$ .

PROOF: Otherwise  $C \cap A^\circ$  and  $C \setminus \overline{A}$  would form a separation of  $C$ .  $\square$

**Example 227.** The space  $\mathbb{R}_l$  is disconnected. For any real  $x$ , the sets  $(-\infty, x)$  and  $[x, +\infty)$  form a separation.

**Proposition 228.** Let  $X$  and  $Y$  be connected spaces. Let  $A$  be a proper subset of  $X$  and  $B$  a proper subset of  $Y$ . Then  $(X \times Y) \setminus (A \times B)$  is connected.

PROOF:

$\langle 1 \rangle 1$ . PICK  $a \in X \setminus A$  and  $b \in Y \setminus B$

$\langle 1 \rangle 2$ . For  $x \in X \setminus A$  we have  $(X \times \{b\}) \cup (\{x\} \times Y)$  is connected.

PROOF: Theorem 219 since  $(x, b)$  is a common point.

$\langle 1 \rangle 3$ . For  $y \in Y \setminus B$  we have  $(X \times \{y\}) \cup (\{a\} \times Y)$  is connected.

PROOF: Theorem 219 since  $(a, y)$  is a common point.

$\langle 1 \rangle 4$ .  $(X \times Y) \setminus (A \times B)$  is connected.

PROOF: Theorem 219 since it is the union of the sets in  $\langle 1 \rangle 2$  and  $\langle 1 \rangle 3$  with  $(a, b)$  as a common point.



□

**Proposition 229.** *Let  $p : X \rightarrow Y$  be a quotient map. If  $Y$  is connected and  $p^{-1}(y)$  is connected for all  $y \in Y$ , then  $X$  is connected.*

PROOF:

⟨1⟩1. ASSUME: for a contradiction  $C$  and  $D$  form a separation of  $X$ .

⟨1⟩2.  $C$  is saturated.

⟨2⟩1. LET:  $x \in C, y \in X$  with  $p(x) = p(y) = a$ , say

⟨2⟩2.  $y \notin D$

PROOF: Otherwise  $p^{-1}(a) \cap C$  and  $p^{-1}(a) \cap D$  form a separation of  $p^{-1}(a)$ .

⟨2⟩3.  $y \in C$

⟨1⟩3.  $D$  is saturated.

PROOF: Similar.

⟨1⟩4.  $p(C)$  and  $p(D)$  form a separation of  $Y$ .

□

**Proposition 230.** *Let  $X$  be a connected space and  $Y$  a connected subspace of  $X$ . Suppose  $A$  and  $B$  form a separation of  $X \setminus Y$ . Then  $Y \cup A$  and  $Y \cup B$  are both connected.*

PROOF:

⟨1⟩1.  $Y \cup A$  is connected.

⟨2⟩1. ASSUME: for a contradiction  $C$  and  $D$  form a separation of  $Y \cup A$

⟨2⟩2. ASSUME: without loss of generality  $Y \subseteq C$

⟨2⟩3. PICK open sets  $A_1, B_1, C_1, D_1$  in  $X$  with

$$A = A_1 \setminus Y$$

$$B = B_1 \setminus Y$$

$$C = C_1 \cap (Y \cup A)$$

$$D = D_1 \cap (Y \cup A)$$

⟨2⟩4.  $B_1 \cup C_1$  and  $A_1 \cap D_1$  form a separation of  $X$

⟨1⟩2.  $Y \cup B$  is connected.

PROOF: Similar.

□

**Theorem 231.** *Let  $L$  be a linearly ordered set under the order topology. Then  $L$  is connected if and only if  $L$  is a linear continuum.*

PROOF:

⟨1⟩1. If  $L$  is a linear continuum then  $L$  is connected.

⟨2⟩1. LET:  $L$  be a linear continuum under the order topology.

⟨2⟩2. ASSUME: for a contradiction  $C$  and  $D$  form a separation of  $L$ .

⟨2⟩3. PICK  $a \in C$  and  $b \in D$ .

⟨2⟩4. ASSUME: without loss of generality  $a < b$ .

⟨2⟩5. LET:  $S = \{x \in L \mid a < x \text{ and } [a, x) \subseteq C\}$

⟨2⟩6.  $S$  is nonempty.

PROOF: Since  $a \in C$  and  $C$  is open.

(2)7.  $S$  is bounded above by  $b$ .  
 PROOF: Since  $b \notin C$ .  
 (2)8. LET:  $s = \sup S$   
 (2)9.  $s \in S$   
 (3)1. LET:  $y \in [a, s)$   
 PROVE:  $y \in C$   
 (3)2. PICK  $z$  with  $y < z \in S$   
 PROOF: By minimality of  $s$ .  
 (3)3.  $y \in [a, z) \subseteq C$   
 (2)10. CASE:  $s \in C$   
 (3)1. PICK  $x$  such that  $s < x$  and  $[s, x) \subseteq C$   
 PROOF: Since  $C$  is open and  $s$  is not greatest in  $L$  because  $s < b$ .  
 (3)2.  $x \in S$   
 PROOF: Since  $[a, x) = [a, s) \cup [s, x) \subseteq C$ .  
 (3)3. Q.E.D.  
 PROOF: This contradicts the fact that  $s$  is an upper bound for  $S$ .  
 (2)11. CASE:  $s \in D$   
 (3)1. PICK  $x < s$  such that  $(x, s] \subseteq D$   
 (3)2. PICK  $y$  with  $x < y < s$   
 PROOF: Since  $L$  is dense.  
 (3)3.  $y \in C$   
 PROOF: From (2)9.  
 (3)4.  $y \in D$   
 PROOF: From (3)1.  
 (3)5. Q.E.D.  
 (3)6. LET:  $L$  be a linear continuum under the order topology.  
 (3)7. ASSUME: for a contradiction  $C$  and  $D$  form a separation of  $L$ .  
 (3)8. PICK  $a \in C$  and  $b \in D$ .  
 (3)9. ASSUME: without loss of generality  $a < b$ .  
 (3)10. LET:  $S = \{x \in L \mid a < x \text{ and } [a, x) \subseteq C\}$   
 (3)11.  $S$  is nonempty.  
 PROOF: Since  $a \in C$  and  $C$  is open.  
 (3)12.  $S$  is bounded above by  $b$ .  
 PROOF: Since  $b \notin C$ .  
 (3)13. LET:  $s = \sup S$   
 (3)14.  $s \in S$   
 (4)1. LET:  $y \in [a, s)$   
 PROVE:  $y \in C$   
 (4)2. PICK  $z$  with  $y < z \in S$   
 PROOF: By minimality of  $s$ .  
 (4)3.  $y \in [a, z) \subseteq C$   
 (3)15. CASE:  $s \in C$   
 (4)1. PICK  $x$  such that  $s < x$  and  $[s, x) \subseteq C$   
 PROOF: Since  $C$  is open and  $s$  is not greatest in  $L$  because  $s < b$ .  
 (4)2.  $x \in S$   
 PROOF: Since  $[a, x) = [a, s) \cup [s, x) \subseteq C$ .

⟨4⟩3. Q.E.D.

PROOF: This contradicts the fact that  $s$  is an upper bound for  $S$ .

$\langle 3 \rangle 16$ . CASE:  $s \in D$

⟨4⟩1. PICK  $x < s$  such that  $(x, s] \subseteq D$

⟨4⟩2. PICK  $y$  with  $x < y < s$

PROOF: Since  $L$  is dense.

$\langle 4 \rangle 3. \ y \in C$

PROOF: From  $\langle 2 \rangle 9$ .

$\langle 4 \rangle 4.$   $y \in D$

PROOF: From  $\langle 3 \rangle 1$ .

⟨4⟩5. Q.E.D.

PROOF: This contradicts  $\langle 2 \rangle 2$ .

$\langle 1 \rangle 2$ . If  $L$  is connected then  $L$  is a linear continuum.

$\langle 2 \rangle 1$ . ASSUME:  $L$  is connected.

(2)2. Every nonempty subset of  $L$  that is bounded above has a supremum.

**(3)1. LET:**  $X$  be a nonempty subset of  $L$  bounded above by  $b$ .

⟨3⟩2. ASSUME: for a contradiction  $X$  has no supremum.

**3. LET:**  $U$  be the set of upper bounds of  $X$ ,

$\langle 3 \rangle 4$ .  $U$  is nonempty.

PROOF: Since  $b \in U$ .

$\langle 3 \rangle 5$ .  $U$  is open.

$\langle 4 \rangle 1$ . LET:  $x \in U$

4)2. PICK an upper bound  $y$  for  $X$  such that  $y < x$

⟨4⟩3. Either  $x$  is greatest in  $L$  and  $(y, x] \subseteq U$ , or there exists  $z > x$  such that  $(y, z) \subseteq U$

**⟨3⟩6. LET:**  $V$  be the set of lower bounds of  $U$ .

$\langle 3 \rangle 7$ .  $V$  is nonempty.

PROOF: Since  $X \subseteq V$

$\langle 3 \rangle 8$ .  $V$  is open.

$\langle 4 \rangle 1$ . LET:  $x \in V$

⟨4⟩2. PICK  $y \in X$  with  $x < y$

PROOF:  $x$  cannot be an upper bound for  $X$ , because it would be the supremum of  $X$ .

(4)3. Either  $x$  least in  $L$  and  $[x, y) \subseteq V$ , or there exists  $z < x$  such that  $(z, y) \subseteq V$

⟨3⟩9.  $L = U \cup V$

⟨4⟩1. LET:  $x \in L \setminus U$

⟨4⟩2. PICK  $y \in X$  such that  $x < y$

$\langle 4 \rangle 3$ . For all  $u \in U$  we have  $x < y \leq u$

$$\langle 4 \rangle 4. \quad x \in V$$

⟨3⟩10.  $U \cap V = \emptyset$

PROOF: Any element of  $U \cap V$  would be a supremum of  $X$ .

⟨3⟩11.  $U$  and  $V$  form a separation of  $L$ .

$\langle 3 \rangle$ 12. Q.E.D.

PROOF: This contradicts  $\langle 2 \rangle 1$ .

$\langle 2 \rangle 3$ .  $L$  is dense.

⟨3⟩1. LET:  $x, y \in L$  with  $x < y$

⟨3⟩2. There exists  $z \in L$  such that  $x < z < y$

PROOF: Otherwise  $(-\infty, y)$  and  $(x, +\infty)$  would form a separation of  $L$ .

□

**Corollary 231.1.** *The real line  $\mathbb{R}$  is connected.*

**Corollary 231.2.** *Every interval in  $\mathbb{R}$  is connected.*

**Corollary 231.3.** *The ordered square is connected.*

**Theorem 232** (Intermediate Value Theorem). *Let  $X$  be a connected space. Let  $Y$  be a linearly ordered set under the order topology. Let  $f : X \rightarrow Y$  be continuous. Let  $a, b \in X$  and  $r \in Y$ . Suppose  $f(a) < r < f(b)$ . Then there exists  $c \in X$  such that  $f(c) = r$ .*

PROOF: Otherwise  $f^{-1}((-\infty, r))$  and  $f^{-1}((r, +\infty))$  would form a separation of  $X$ . □

**Proposition 233.** *Every function  $f : [0, 1] \rightarrow [0, 1]$  has a fixed point.*

PROOF:

⟨1⟩1. LET:  $g : [0, 1] \rightarrow [-1, 1]$  be the function  $g(x) = f(x) - x$

PROVE: there exists  $x \in [0, 1]$  such that  $g(x) = 0$

⟨1⟩2. ASSUME: without loss of generality  $g(0) \neq 0$  and  $g(1) \neq 0$

⟨1⟩3.  $g(0) > 0$

⟨1⟩4.  $g(1) < 0$

⟨1⟩5. There exists  $x \in (0, 1)$  such that  $g(x) = 0$

PROOF: By the Intermediate Value Theorem.

**Proposition 234.** *Give  $\mathbb{R}^\omega$  the box topology. Let  $x, y \in \mathbb{R}^\omega$ . Then  $x$  and  $y$  lie in the same component if and only if  $x - y$  is eventually zero, i.e. there exists  $N$  such that, for all  $n \geq N$ , we have  $x_n = y_n$ .*

PROOF:

⟨1⟩1. The component containing 0 is the set of sequences that are eventually zero.

⟨2⟩1. LET:  $B$  be the set of sequences that are eventually zero.

⟨2⟩2.  $B$  is path-connected.

⟨3⟩1. LET:  $x, y \in B$

⟨3⟩2. PICK  $N$  such that, for all  $n \geq N$ , we have  $x_n = y_n = 0$

⟨3⟩3. LET:  $p : [0, 1] \rightarrow \mathbb{R}^\omega$ ,  $p(t) = (1 - t)x + ty$

PROVE:  $p$  is continuous.

⟨3⟩4. LET:  $t \in [0, 1]$  and  $\prod_j U_j$  be a basic open neighbourhood of  $p(t)$ , where each  $U_j$  is open in  $\mathbb{R}$

⟨3⟩5. PICK  $\delta$  such that, for all  $n < N$  and all  $s \in [0, 1]$ , if  $|s - t| < \delta$  then  $p(s)_n \in U_n$

⟨3⟩6. For all  $s \in [0, 1]$ , if  $|s - t| < \delta$  then  $p(s) \in \prod_j U_j$

⟨2⟩3.  $B$  is connected.

PROOF: Proposition 240.

⟨2⟩4. If  $C$  is connected and  $B \subseteq C$  then  $B = C$ .

⟨3⟩1. ASSUME:  $C$  is connected and  $B \subseteq C$

⟨3⟩2. ASSUME: for a contradiction  $x \in C \setminus B$

⟨3⟩3. For  $n \geq 1$ ,

LET:  $c_n = 1$  if  $x_n = 0$ ,  $c_n = n/x_n$  otherwise

⟨3⟩4. LET:  $h : \mathbb{R}^\omega \rightarrow \mathbb{R}^\omega$  be the function  $h(x) = (c_n x_n)_{n \geq 1}$

⟨3⟩5.  $h$  is a homeomorphism of  $\mathbb{R}^\omega$  with itself.

⟨3⟩6.  $h(x)$  is unbounded.

PROOF: For any  $b > 0$ , pick  $N > b$  such that  $x_N \neq 0$ . Then  $h(x)_N > b$ .

⟨3⟩7.  $h^{-1}(\{\text{bounded sequences}\}) \cap C$  and  $h^{-1}(\{\text{unbounded sequences}\}) \cap C$  form a separation of  $C$

⟨3⟩8. Q.E.D.

PROOF: This contradicts ⟨3⟩1.

⟨1⟩2. Q.E.D.

PROOF: Since  $\lambda x.x - y$  is a homeomorphism of  $\mathbb{R}^\omega$  with itself.

□

## 34 Totally Disconnected Spaces

**Definition 235** (Totally Disconnected). A topological space  $X$  is *totally disconnected* if and only if the only connected subspaces are the singletons.

**Example 236.** Every discrete space is totally disconnected.

**Example 237.** The rationals  $\mathbb{Q}$  are totally disconnected.

## 35 Paths and Path Connectedness

**Definition 238** (Path). Let  $X$  be a topological space and  $a, b \in X$ . A *path* from  $a$  to  $b$  is a continuous function  $p : [0, 1] \rightarrow X$  such that  $p(0) = a$  and  $p(1) = b$ .

**Definition 239** (Path Connected). A topological space is *path connected* if and only if there exists a path between any two points.

**Proposition 240.** *Every path connected space is connected.*

PROOF:

⟨1⟩1. LET:  $X$  be a path connected space.

⟨1⟩2. ASSUME: for a contradiction  $C$  and  $D$  form a separation of  $X$ .

⟨1⟩3. PICK  $a \in C$  and  $b \in D$ .

⟨1⟩4. PICK a path  $p : [0, 1] \rightarrow X$  from  $a$  to  $b$ .

⟨1⟩5.  $p^{-1}(C)$  and  $p^{-1}(D)$  form a separation of  $[0, 1]$ .

⟨1⟩6. Q.E.D.

PROOF: This contradicts Corollary 231.2.

□

An example that shows the converse does not hold:

**Example 241.** The ordered square is not path connected.

PROOF:

⟨1⟩1. ASSUME: for a contradiction  $p : [0, 1] \rightarrow I_o^2$  is a path from  $(0, 0)$  to  $(1, 1)$ .

⟨1⟩2.  $p$  is surjective.

PROOF: By the Intermediate Value Theorem.

⟨1⟩3. For  $x \in [0, 1]$ , PICK a rational  $q_x \in p^{-1}((x, 0), (x, 1))$

PROOF: Since  $p^{-1}((x, 0), (x, 1))$  is open and nonempty by ⟨1⟩2.

⟨1⟩4. For  $x, y \in [0, 1]$ , if  $x \neq y$  then  $q_x \neq q_y$

PROOF: We have  $p(q_x) \neq p(q_y)$  because  $((x, 0), (x, 1))$  and  $((y, 0), (y, 1))$  are disjoint.

⟨1⟩5.  $\{q_x \mid x \in [0, 1]\}$  is an uncountable set of rationals.

⟨1⟩6. Q.E.D.

PROOF: This contradicts the fact that the rationals are countable.

□

**Proposition 242.** *The continuous image of a path connected space is path connected.*

PROOF:

⟨1⟩1. LET:  $X$  be a path connected space,  $Y$  a topological space, and  $f : X \rightarrow Y$  be continuous and surjective.

⟨1⟩2. LET:  $a, b \in Y$

⟨1⟩3. PICK  $c, d \in X$  with  $f(c) = a$  and  $f(d) = b$

⟨1⟩4. PICK a path  $p : [0, 1] \rightarrow X$  from  $c$  to  $d$ .

⟨1⟩5.  $f \circ p$  is a path from  $a$  to  $b$  in  $Y$ .

□

**Proposition 243 (AC).** *The product of a family of path-connected spaces is path-connected.*

PROOF:

⟨1⟩1. LET:  $\{X_\alpha\}_{\alpha \in J}$  be a family of path-connected spaces.

⟨1⟩2. LET:  $a, b \in \prod_{\alpha \in J} X_\alpha$

⟨1⟩3. For  $\alpha \in J$ , PICK a path  $p_\alpha : [0, 1] \rightarrow X_\alpha$  from  $a_\alpha$  to  $b_\alpha$

PROOF: Using the Axiom of Choice.

⟨1⟩4. Define  $p : [0, 1] \rightarrow \prod_{\alpha \in J} X_\alpha$  by  $p(t)_\alpha = p_\alpha(t)$

⟨1⟩5.  $p$  is a path from  $a$  to  $b$ .

PROOF: Theorem 131.

□

**Proposition 244.** *The continuous image of a path-connected space is path-connected.*

PROOF:

⟨1⟩1. LET:  $f : X \rightarrow Y$  be continuous and surjective where  $X$  is path-connected.

- ⟨1⟩2. LET:  $a, b \in Y$
- ⟨1⟩3. PICK  $a', b' \in X$  with  $f(a') = a$  and  $f(b') = b$ .
- ⟨1⟩4. PICK a path  $p : [0, 1] \rightarrow X$  from  $a'$  to  $b'$ .
- ⟨1⟩5.  $f \circ p$  is a path from  $a$  to  $b$ .

□

**Proposition 245.** *Let  $X$  be a topological space. The union of a set of path-connected subspaces of  $X$  that have a point in common is path-connected.*

PROOF:

- ⟨1⟩1. LET:  $\mathcal{A}$  be a set of path-connected subspaces of  $X$  with the point  $a$  in common.
- ⟨1⟩2. LET:  $b, c \in \bigcup \mathcal{A}$
- ⟨1⟩3. PICK  $B, C \in \mathcal{A}$  with  $b \in B$  and  $c \in C$ .
- ⟨1⟩4. PICK a path  $p$  in  $B$  from  $b$  to  $a$ .
- ⟨1⟩5. PICK a path  $q$  in  $C$  from  $a$  to  $c$ .
- ⟨1⟩6. The concatenation of  $p$  and  $q$  is a path from  $b$  to  $c$  in  $\bigcup \mathcal{A}$ .

□

**Proposition 246.** *Let  $A \subseteq \mathbb{R}^2$  be countable. Then  $\mathbb{R}^2 \setminus A$  is path-connected.*

PROOF:

- ⟨1⟩1. LET:  $a, b \in \mathbb{R}^2 \setminus A$
- ⟨1⟩2. PICK a line  $l$  in  $\mathbb{R}^2$  with  $a$  on one side and  $b$  on the other.
- ⟨1⟩3. For every point  $x$  on  $l$ ,  
LET:  $p_x$  be the path in  $\mathbb{R}^2$  consisting of a line from  $a$  to  $x$  then a line from  $x$  to  $b$
- ⟨1⟩4. For  $x \neq y$  we have  $p_x$  and  $p_y$  have no points in common except  $a$  and  $b$
- ⟨1⟩5. There are only countably many  $x$  such that a point of  $A$  lies on  $p_x$ .
- ⟨1⟩6. There exists  $x$  such that  $p_x$  is a path from  $a$  to  $b$  in  $\mathbb{R}^2 \setminus A$ .

□

**Proposition 247.** *Every open connected subspace of  $\mathbb{R}^2$  is path-connected.*

PROOF:

- ⟨1⟩1. LET:  $U$  be an open connected subspace of  $\mathbb{R}^2$ .
- ⟨1⟩2. For all  $x_0 \in U$ ,  
LET:  $PC(x_0) = \{y \in U \mid \text{there exists a path from } x \text{ to } y\}$
- ⟨1⟩3. For all  $x_0 \in U$ , the set  $PC(x_0)$  is open and closed in  $U$ .
  - ⟨2⟩1. LET:  $x_0 \in U$
  - ⟨2⟩2.  $PC(x_0)$  is open in  $U$ 
    - ⟨3⟩1. LET:  $y \in PC(x_0)$
    - ⟨3⟩2. PICK  $\epsilon > 0$  such that  $B(y, \epsilon) \subseteq U$   
PROOF: Since  $U$  is open.
    - ⟨3⟩3.  $B(y, \epsilon) \subseteq PC(x_0)$   
PROOF: For all  $z \in B(y, \epsilon)$ , pick a path from  $x_0$  to  $y$  then concatenate the straight line from  $y$  to  $z$ .
  - ⟨2⟩3.  $PC(x_0)$  is closed in  $U$

$\langle 3 \rangle 1$ . LET:  $y \in U$  be a limit point of  $PC(x_0)$

$\langle 3 \rangle 2$ . PICK  $\epsilon > 0$  such that  $B(y, \epsilon) \subseteq U$

$\langle 3 \rangle 3$ . PICK  $z \in PC(x_0) \cap B(y, \epsilon)$

$\langle 3 \rangle 4$ .  $y \in PC(x_0)$

PROOF: Pick a path from  $x_0$  to  $z$  then concatenate the straight line from  $z$  to  $y$ .

$\langle 1 \rangle 4$ .  $PC(x_0) = U$

PROOF: Proposition 211.

□

**Example 248.** If  $A$  is a connected subspace of  $X$ , then  $A^\circ$  is not necessarily connected.

Take two closed circles in  $\mathbb{R}^2$  that touch at one point. The interior of this space is two open circles, and these two circles form a separation.

**Example 249.** If  $A$  is a connected subspace of  $X$  then  $\partial A$  is not necessarily connected.

We have  $[0, 1]$  is connected but  $\partial[0, 1] = \{0, 1\}$  is not.

**Example 250.** If  $A$  is a subspace of  $X$  and  $A^\circ$  and  $\partial A$  are connected, then  $A$  is not necessarily connected.

We have  $\mathbb{Q}^\circ = \emptyset$  and  $\partial\mathbb{Q} = \mathbb{R}$  are connected but  $\mathbb{Q}$  is not connected.

## 36 The Topologist's Sine Curve

**Definition 251** (The Topologist's Sine Curve). Let  $S = \{(x, \sin 1/x) \mid 0 < x \leq 1\}$ , The *topologist's sine curve* is the closure  $\bar{S}$  of  $S$  in  $\mathbb{R}^2$ .

**Proposition 252.** The topologist's sine curve is connected.

PROOF:

$\langle 1 \rangle 1$ . LET:  $S = \{(x, \sin 1/x) \mid 0 < x \leq 1\}$

$\langle 1 \rangle 2$ .  $S$  is connected.

PROOF: Theorem 221.

$\langle 1 \rangle 3$ .  $\bar{S}$  is connected.

PROOF: Theorem 220.

□

**Proposition 253.** The topologist's sine curve is  $\{(x, \sin 1/x) \mid 0 < x \leq 1\} \cup (\{0\} \times [-1, 1])$ .

PROOF: Sketch proof: Given a point  $(0, y)$  with  $-1 \leq y \leq 1$ , pick  $a$  such that  $\sin a = y$ . Then  $((1/a, y), (1/(a + 2\pi), y), (1/(a + 4\pi), y), \dots)$  is a sequence in  $S$  that converges to  $(0, y)$ .

Conversely, let  $(x, y)$  be any point not in  $S \cup (\{0\} \times [-1, 1])$ . If  $x < 0$  or  $y > 1$  or  $y < -1$  then we can easily find a neighbourhood that does not intersect  $S \cup (\{0\} \times [-1, 1])$ . If  $x > 0$  and  $-1 \leq y \leq 1$ , then we have  $y \neq \sin 1/x$ . Hence pick a neighbourhood that does not intersect  $S$ .



**Proposition 254.** *Every closed subset of  $\mathbb{R}$  that is bounded above has a greatest element.*

PROOF: It has a supremum, which is a limit point of the set and hence an element.  $\square$

**Proposition 255 (CC).** *The topologist's sine curve is not path connected.*

PROOF:

$\langle 1 \rangle 1$ . ASSUME: For a contradiction  $p : [0, 1] \rightarrow \bar{S}$  is a path from  $(0, 0)$  to  $(1, \sin 1)$ .

$\langle 1 \rangle 2$ .  $\{t \in [0, 1] \mid p(t) \in \{0\} \times [-1, 1]\}$  is closed.

PROOF: Since  $p$  is continuous and  $\{0\} \times [-1, 1]$  is closed.

$\langle 1 \rangle 3$ . LET:  $b$  be the largest number in  $[0, 1]$  such that  $p(b) \in \{0\} \times [-1, 1]$ .

PROOF: Proposition 254.

$\langle 1 \rangle 4$ . LET:  $x : [b, 1] \rightarrow \bar{S}$  be the function  $\pi_1 \circ p$

$\langle 1 \rangle 5$ . LET:  $y : [b, 1] \rightarrow \bar{S}$  be the function  $\pi_2 \circ p$

$\langle 1 \rangle 6$ . PICK a sequence  $t_n$  in  $(b, 1]$  such that  $t_n \rightarrow b$  and  $y(t_n) = (-1)^n$  for all  $n$

$\langle 2 \rangle 1$ . LET:  $n \geq 1$

$\langle 2 \rangle 2$ . PICK  $u$  with  $0 < u < x(1/n)$  and  $\sin(1/u) = (-1)^n$

$\langle 2 \rangle 3$ . PICK  $t_n$  with  $b < t_n < 1/n$  and  $x(t_n) = u$

PROOF: By the Intermediate Value Theorem

$\langle 1 \rangle 7$ . Q.E.D.

PROOF: This contradicts Proposition 104 since  $y$  is continuous and  $y(t_n)$  does not converge.

$\square$

**Corollary 255.1.** *The closure of a path-connected subspace of a space is not necessarily path-connected.*

## 37 The Long Line

**Definition 256 (The Long Line).** The *long line* is the space  $\omega_1 \times [0, 1]$  in the dictionary order under the order topology, where  $\omega_1$  is the first uncountable ordinal.

**Lemma 257.** *For any ordinal  $\alpha$  with  $0 < \alpha < \omega_1$  we have  $[(0, 0), (\alpha, 0)) \cong [0, 1]$*

$\langle 1 \rangle 1$ .  $[(0, 0), (1, 0)) \cong [0, 1]$

PROOF: The map  $\pi_2$  is a homeomorphism.

$\langle 1 \rangle 2$ . If  $[(0, 0), (\alpha, 0)) \cong [0, 1]$  then  $[(0, 0), (\alpha + 1, 0)) \cong [0, 1]$

PROOF: Proposition 13.

$\langle 1 \rangle 3$ . If  $\lambda$  is a limit ordinal with  $\lambda < \omega_1$  and  $[(0, 0), (\alpha, 0)) \cong [0, 1]$  for all  $\alpha$  with  $0 < \alpha < \lambda$  then  $[(0, 0), (\lambda, 0)) \cong [0, 1]$

$\langle 2 \rangle 1$ . LET:  $\lambda$  be a limit ordinal  $< \omega_1$

$\langle 2 \rangle 2$ . ASSUME:  $[(0, 0), (\alpha, 0)) \cong [0, 1]$  for all  $\alpha$  with  $0 < \alpha < \lambda$

$\langle 2 \rangle 3$ . PICK a sequence of ordinals  $\alpha_0 < \alpha_1 < \dots$  with limit  $\lambda$

PROOF: Since  $\lambda$  is countable.

$\langle 2 \rangle 4.$   $[(\alpha_i, 0), (\alpha_{i+1}, 0)) \cong [0, 1]$  for all  $i$

PROOF: Lemma 12.

$\langle 2 \rangle 5.$  Q.E.D.

PROOF: By Proposition 14.

$\langle 1 \rangle 4.$  Q.E.D.

PROOF: By transfinite induction.

**Proposition 258 (CC).** *The long line is path-connected.*

PROOF:

$\langle 1 \rangle 1.$  LET:  $(\alpha, i), (\beta, j) \in \omega_1 \times [0, 1]$

$\langle 1 \rangle 2.$  ASSUME: without loss of generality  $(\alpha, i) < (\beta, j)$

$\langle 1 \rangle 3.$   $[(0, 0), (\beta + 1, 0)) \cong [0, 1]$

PROOF: By Lemma 257

$\langle 1 \rangle 4.$   $[(\alpha, i), (\beta, j)) \cong [0, 1]$

PROOF: Lemma 12.

$\langle 1 \rangle 5.$  PICK a homeomorphism  $q : [0, 1] \rightarrow [(\alpha, i), (\beta, j))$

$\langle 1 \rangle 6.$   $q \cup \{(1, (\beta, j))\}$  is a path from  $(\alpha, i)$  to  $(\beta, j)$

□

**Proposition 259.** *Every point in the long line has a neighbourhood homeomorphic to an interval in  $\mathbb{R}$ .*

PROOF: For any  $(\alpha, i)$  in the long line, the neighbourhood  $[(0, 0), (\alpha + 1, 0))$  satisfies the condition by Lemma 257.

**Proposition 260.** *The long line  $L$  is not second countable.*

PROOF:

$\langle 1 \rangle 1.$  LET:  $\mathcal{B}$  be a basis for  $L$ .

$\langle 1 \rangle 2.$  For  $\alpha < \omega_1$ , PICK  $B_\alpha \in \mathcal{B}$  such that  $(\alpha, 1/2) \in B_\alpha \subseteq ((\alpha, 0), (\alpha + 1, 0))$

$\langle 1 \rangle 3.$   $\mathcal{B}$  is uncountable.

PROOF: The mapping  $\alpha \mapsto B_\alpha$  is an injection  $\omega_1 \rightarrow \mathcal{B}$ .

**Corollary 260.1.** *The long line cannot be imbedded into  $\mathbb{R}^n$  for any  $n$ .*

## 38 Components

**Proposition 261.** *Let  $X$  be a topological space. Define the relation  $\sim$  on  $X$  by  $a \sim b$  if and only if there exists a connected subspace  $A \subseteq X$  with  $a, b \in A$ . Then  $\sim$  is an equivalence relation on  $X$ .*

PROOF:

$\langle 1 \rangle 1.$   $\sim$  is reflexive.

PROOF: For any  $a \in X$  we have  $\{a\}$  is a connected subspace that contains  $a$ .

$\langle 1 \rangle 2.$   $\sim$  is symmetric.

PROOF: Trivial.

$\langle 1 \rangle 3.$   $\sim$  is transitive.

- ⟨2⟩1. LET:  $a, b, c \in X$
  - ⟨2⟩2. ASSUME:  $a \sim b$  and  $b \sim c$
  - ⟨2⟩3. PICK connected subspaces  $A$  and  $B$  with  $a, b \in A$  and  $b, c \in B$
  - ⟨2⟩4.  $A \cup B$  is a connected subspace that contains  $a$  and  $c$
- PROOF: Theorem 219.

□

**Definition 262** ((Connected) Component). Let  $X$  be a topological space. The *(connected) components* of  $X$  are the equivalence classes under the above  $\sim$ .

**Lemma 263.** *Let  $X$  be a topological space. If  $A \subseteq X$  is connected and nonempty then there exists a unique component  $C$  of  $X$  such that  $A \subseteq C$ .*

PROOF:

- ⟨1⟩1. PICK  $a \in A$
  - ⟨1⟩2. LET:  $C$  be the  $\sim$ -equivalence class of  $a$ .
  - ⟨1⟩3.  $A \subseteq C$
- PROOF: For all  $x \in A$  we have  $x \sim a$ .
- ⟨1⟩4. If  $C'$  is a component and  $A \subseteq C'$  then  $C = C'$
- PROOF: Since we have  $a \in C'$ .

□

**Theorem 264.** *Let  $X$  be a topological space. The components of  $X$  are connected disjoint subspaces of  $X$  whose union is  $X$  such that each nonempty connected subspace of  $X$  intersects only one of them.*

PROOF:

- ⟨1⟩1. Every component of  $X$  is connected.
- PROOF: For  $a \in X$ , the  $\sim$ -equivalence class of  $a$  is  $\bigcup \{A \subseteq X \mid A \text{ is connected, } a \in A\}$  which is connected by Theorem 219.
- ⟨1⟩2. The components form a partition of  $X$ .
- PROOF: Immediate from the definition.
- ⟨1⟩3. Every nonempty connected subspace of  $X$  intersects a unique component of  $X$ .
- ⟨2⟩1. LET:  $A \subseteq X$  be connected and nonempty.
  - ⟨2⟩2. LET:  $C$  be the component such that  $A \subseteq C$
- PROOF: Lemma 263.
- ⟨2⟩3.  $A$  intersects  $C$
  - ⟨2⟩4. If  $A$  intersects the component  $C'$  then  $C' = C$
- ⟨3⟩1. LET:  $C'$  be a component that intersects  $A$
  - ⟨3⟩2. PICK  $b \in A \cap C'$
  - ⟨3⟩3.  $A \subseteq C'$
- PROOF: For all  $x \in A$  we have  $x \sim b$ .
- ⟨3⟩4.  $C = C'$
- PROOF: By uniqueness in ⟨2⟩2.

□

**Proposition 265.** *Every component of a space is closed.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $X$  be a topological space and  $C$  a component of  $X$ .

$\langle 1 \rangle 2$ .  $\overline{C}$  is connected.

PROOF: Theorem 220.

$\langle 1 \rangle 3$ .  $C = \overline{C}$

PROOF: Lemma 218.

$\langle 1 \rangle 4$ .  $C$  is closed.

PROOF: Lemma 46.

□

**Proposition 266.** *If a topological space has finitely many components then every component is open.*

PROOF: Each component is the complement of a finite union of closed sets. □

## 39 Path Components

**Proposition 267.** *Let  $X$  be a topological space. Define the relation  $\sim$  on  $X$  by:  $a \sim b$  if and only if there exists a path in  $X$  from  $a$  to  $b$ . Then  $\sim$  is an equivalence relation on  $X$ .*

PROOF:

$\langle 1 \rangle 1$ .  $\sim$  is reflexive.

PROOF: For  $a \in X$ , the constant function  $[0, 1] \rightarrow X$  with value  $a$  is a path from  $a$  to  $a$ .

$\langle 1 \rangle 2$ .  $\sim$  is symmetric.

PROOF: If  $p : [0, 1] \rightarrow X$  is a path from  $a$  to  $b$ , then  $\lambda t.p(1-t)$  is a path from  $b$  to  $a$ .

$\langle 1 \rangle 3$ .  $\sim$  is transitive.

PROOF: Concatenate paths.

□

**Definition 268** (Path Component). Let  $X$  be a topological space. The *path components* of  $X$  are the equivalence relations under  $\sim$ .

**Theorem 269.** *The path components of  $X$  are path-connected disjoint subspaces of  $X$  whose union is  $X$  such that every nonempty path-connected subspace of  $X$  intersects exactly one path component.*

PROOF:

$\langle 1 \rangle 1$ . Every path component is path-connected.

PROOF: If  $a$  and  $b$  are in the same path component then  $a \sim b$ , i.e. there exists a path from  $a$  to  $b$ .

$\langle 1 \rangle 2$ . The path components are disjoint and their union is  $X$ .

PROOF: Immediate from the definition.

- ⟨1⟩3. Every non-empty path-connected subspace of  $X$  intersects exactly one path component.  
 ⟨2⟩1. LET:  $A$  be a nonempty path-connected subspace of  $X$ .  
 ⟨2⟩2. PICK  $a \in A$   
 ⟨2⟩3.  $A$  intersects the  $\sim$ -equivalence class of  $a$ .  
 ⟨2⟩4. LET:  $C$  be any path component that intersects  $A$ .  
 ⟨2⟩5. PICK  $b \in A \cap C$   
 ⟨2⟩6.  $a \sim b$   
 PROOF: Since  $A$  is path-connected.  
 ⟨2⟩7.  $C$  is the  $\sim$ -equivalence class of  $a$ .

□

**Proposition 270.** *Every path component is included in a component.*

PROOF:

- ⟨1⟩1. LET:  $X$  be a topological space and  $C$  a path component of  $X$ .  
 ⟨1⟩2.  $C$  is path-connected.  
 PROOF: Theorem 269.  
 ⟨1⟩3.  $C$  is connected.  
 PROOF: Proposition 240.  
 ⟨1⟩4.  $C$  is included in a component.  
 PROOF: Lemma 263.

□

## 40 Local Connectedness

**Definition 271** (Locally Connected). Let  $X$  be a topological space and  $a \in X$ . Then  $X$  is *locally connected* at  $a$  if and only if every neighbourhood of  $a$  includes a connected neighbourhood of  $a$ .

The space  $X$  is *locally connected* if and only if it is locally connected at every point.

**Example 272.** The real line is both connected and locally connected.

**Example 273.** The space  $\mathbb{R} \setminus \{0\}$  is disconnected but locally connected.

**Example 274.** The topologist's sine curve is connected but not locally connected.

**Example 275.** The rationals  $\mathbb{Q}$  are neither connected nor locally connected.

**Theorem 276.** *A topological space  $X$  is locally connected if and only if, for every open set  $U$  in  $X$ , every component of  $U$  is open in  $X$ .*

PROOF:

- ⟨1⟩1. If  $X$  is locally connected then, for every open set  $U$  in  $X$ , every component of  $U$  is open in  $X$ .  
 ⟨2⟩1. ASSUME:  $X$  is locally connected.

- ⟨2⟩2. LET:  $U$  be open in  $X$ .
- ⟨2⟩3. LET:  $C$  be a component of  $U$ .
- ⟨2⟩4. LET:  $a \in C$
- ⟨2⟩5. LET:  $V$  be a connected neighbourhood of  $a$  such that  $V \subseteq U$
- ⟨2⟩6.  $V \subseteq C$

PROOF: Lemma 263.

- ⟨2⟩7. Q.E.D.

PROOF: Lemma 25.

- ⟨1⟩2. If, for every open set  $U$  in  $X$ , every component of  $U$  is open in  $X$ , then  $X$  is locally connected.
- ⟨2⟩1. ASSUME: for every open set  $U$  in  $X$ , every component of  $U$  is open in  $X$ .
- ⟨2⟩2. LET:  $a \in X$
- ⟨2⟩3. LET:  $U$  be a neighbourhood of  $a$
- ⟨2⟩4. The component of  $U$  that contains  $a$  is a connected neighbourhood of  $a$  included in  $U$ .

□

**Example 277.** The ordered square is locally connected.

PROOF: Every neighbourhood of a point includes an interval around that point, which is connected by Theorem 231.

**Example 278.** Let  $X$  be the set of all rational points on the line segment  $[0, 1] \times \{0\}$ , and  $Y$  the set of all rational points on the line segment  $[0, 1] \times \{1\}$ . Let  $A$  be the space consisting of all line segments joining the point  $(0, 1)$  to a point of  $X$ , and all line segments joining the point  $(1, 0)$  to a point of  $Y$ . Then  $A$  is path-connected but is not locally connected at any point,

**Proposition 279.** Let  $X$  and  $Y$  be topological spaces and  $p : X \twoheadrightarrow Y$  be a quotient map. If  $X$  is locally connected then so is  $Y$ .

PROOF:

- ⟨1⟩1. LET:  $U$  be an open set in  $Y$ .
- ⟨1⟩2. LET:  $C$  be a component of  $U$ .
- ⟨1⟩3.  $p^{-1}(C)$  is a union of components of  $p^{-1}(U)$ 
  - ⟨2⟩1. LET:  $x \in p^{-1}(C)$
  - ⟨2⟩2. LET:  $D$  be the component of  $p^{-1}(U)$  that contains  $x$ .
  - ⟨2⟩3.  $p(D)$  is connected.

PROOF: Theorem 221.

- ⟨2⟩4.  $p(D) \subseteq C$ .

PROOF: From ⟨1⟩2 since  $p(x) \in p(D) \cap C$  (⟨2⟩1, ⟨2⟩2).

- ⟨2⟩5.  $D \subseteq p^{-1}(C)$

- ⟨1⟩4.  $p^{-1}(C)$  is open in  $p^{-1}(U)$

PROOF: Theorem 276.

- ⟨1⟩5.  $C$  is open in  $U$

PROOF: Since the restriction of  $p$  to  $p : p^{-1}(U) \twoheadrightarrow U$  is a quotient map by Proposition 194.

⟨1⟩6. Q.E.D.

PROOF: Theorem 276.

□

## 41 Local Path Connectedness

**Definition 280** (Locally Path-Connected). Let  $X$  be a topological space and  $a \in X$ . Then  $X$  is *locally path-connected* at  $a$  if and only if every neighbourhood of  $a$  includes a path-connected neighbourhood of  $a$ .

The space  $X$  is *locally path-connected* if and only if it is locally path-connected at every point.

**Theorem 281.** *A topological space  $X$  is locally path-connected if and only if, for every open set  $U$  in  $X$ , every path component of  $U$  is open in  $X$ .*

PROOF:

⟨1⟩1. If  $X$  is locally path-connected then, for every open set  $U$  in  $X$ , every path component of  $U$  is open in  $X$ .

⟨2⟩1. ASSUME:  $X$  is locally path-connected.

⟨2⟩2. LET:  $U$  be open in  $X$ .

⟨2⟩3. LET:  $C$  be a path component of  $U$ .

⟨2⟩4. LET:  $a \in C$

⟨2⟩5. LET:  $V$  be a path-connected neighbourhood of  $a$  such that  $V \subseteq U$

⟨2⟩6.  $V \subseteq C$

PROOF: Lemma 263.

⟨2⟩7. Q.E.D.

PROOF: Lemma 25.

⟨1⟩2. If, for every open set  $U$  in  $X$ , every component of  $U$  is open in  $X$ , then  $X$  is locally connected.

⟨2⟩1. ASSUME: for every open set  $U$  in  $X$ , every component of  $U$  is open in  $X$ .

⟨2⟩2. LET:  $a \in X$

⟨2⟩3. LET:  $U$  be a neighbourhood of  $a$

⟨2⟩4. The component of  $U$  that contains  $a$  is a connected neighbourhood of  $a$  included in  $U$ .

□

**Theorem 282.** *If a space is locally path connected then its components and its path components are the same.*

PROOF:

⟨1⟩1. LET:  $X$  be a locally path connected space.

⟨1⟩2. LET:  $C$  be a component of  $X$ .

⟨1⟩3. LET:  $x \in C$

⟨1⟩4. LET:  $P$  be the path component of  $x$

PROVE:  $P = C$

⟨1⟩5.  $P \subseteq C$

PROOF: Proposition 270.

⟨1⟩6. LET:  $Q$  be the union of the other path components included in  $C$

⟨1⟩7.  $C = P \cup Q$

PROOF: Proposition 270.

⟨1⟩8.  $P$  and  $Q$  are open in  $C$

⟨2⟩1.  $C$  is open.

PROOF: Theorem 276.

⟨2⟩2. Q.E.D.

PROOF: Theorem 281.

⟨1⟩9.  $Q = \emptyset$

PROOF: Otherwise  $P$  and  $Q$  would form a separation of  $C$ .

□

**Example 283.** The ordered square is not locally path connected, since it is connected but not path connected.

**Proposition 284.** *Let  $X$  be a locally path-connected space. Then every connected open subspace of  $X$  is path-connected.*

PROOF:

⟨1⟩1. LET:  $U$  be a connected open subspace of  $X$ .

⟨1⟩2. LET:  $P$  be a path component of  $U$ .

⟨1⟩3. LET:  $Q$  be the union of the other path components of  $U$ .

⟨1⟩4.  $P$  and  $Q$  are open in  $U$ .

PROOF: Theorem 281.

⟨1⟩5.  $Q = \emptyset$

PROOF: Otherwise  $P$  and  $Q$  form a separation of  $U$ .

□

## 42 Weak Local Connectedness

**Definition 285** (Weakly Locally Connected). Let  $X$  be a topological space and  $a \in X$ . Then  $X$  is *weakly locally connected* at  $a$  if and only if every neighbourhood of  $a$  includes a connected subspace that includes a neighbourhood of  $a$ .

**Proposition 286.** *Let  $X$  be a topological space. If  $X$  is weakly locally connected at every point then  $X$  is locally connected.*

PROOF:

⟨1⟩1. ASSUME:  $X$  is weakly locally connected at every point.

⟨1⟩2. LET:  $U$  be open in  $X$ .

⟨1⟩3. LET:  $C$  be a component of  $U$ .

⟨1⟩4.  $C$  is open in  $X$ .

⟨2⟩1. LET:  $x \in C$

⟨2⟩2. PICK a connected subspace  $D$  of  $U$  that includes a neighbourhood  $V$  of  $x$ .



⟨2⟩3.  $D \subseteq C$

PROOF: Lemma 263.

⟨2⟩4.  $x \in V \subseteq C$

⟨2⟩5. Q.E.D.

PROOF: Lemma 25.

⟨1⟩5. Q.E.D.

PROOF: Theorem 276.

□

**Example 287.** The 'infinite broom' (Munkres p.163) is an example of a space that is weakly locally connected at a point  $p$  but not locally connected at  $p$ .

## 43 Quasicomponents

**Proposition 288.** Let  $X$  be a topological space. Define  $\sim$  on  $X$  by  $x \sim y$  if and only if there exists no separation  $U$  and  $V$  of  $X$  such that  $x \in U$  and  $y \in V$ . Then  $\sim$  is an equivalence relation on  $X$ .

PROOF:

⟨1⟩1.  $\sim$  is reflexive.

PROOF: Since the two sets that make up a separation are disjoint.

⟨1⟩2.  $\sim$  is symmetric.

PROOF: Immediate from the definition.

⟨1⟩3.  $\sim$  is transitive.

⟨2⟩1. ASSUME:  $x \sim y$  and  $y \sim z$

⟨2⟩2. ASSUME: for a contradiction there is a separation  $U$  and  $V$  of  $X$  with  
 $x \in U$  and  $z \in V$

⟨2⟩3.  $y \in U$  or  $y \in V$

⟨2⟩4. Q.E.D.

PROOF: Either case contradicts ⟨2⟩1.

□

**Definition 289** (Quasicomponents). For  $X$  a topological space, the *quasicomponents* of  $X$  are the equivalence classes under  $\sim$ .

**Proposition 290.** Let  $X$  be a topological space. Then every component of  $X$  is included in a quasicomponent of  $X$ .

PROOF:

⟨1⟩1. LET:  $C$  be a component of  $X$ .

⟨1⟩2. LET:  $x, y \in C$

PROVE:  $x \sim y$

⟨1⟩3. ASSUME: for a contradiction there exists a separation  $U$  and  $V$  of  $X$  with  
 $x \in U$  and  $y \in V$

⟨1⟩4.  $C \cap U$  and  $C \cap V$  form a separation of  $C$ .

⟨1⟩5. Q.E.D.

PROOF: This contradicts ⟨1⟩1.

**Proposition 291.** *In a locally connected space, the components and the quasi-components are the same.*

PROOF:

⟨1⟩1. LET:  $X$  be a locally connected space and  $Q$  a quasicomponent of  $X$ .

⟨1⟩2. PICK a component  $C$  of  $X$  such that  $C \subseteq Q$

⟨1⟩3. LET:  $D$  be the union of the components of  $X$

⟨1⟩4.  $C$  and  $D$  are open in  $X$ .

PROOF: Theorem 276.

⟨1⟩5.  $D$  cannot contain any points of  $Q$ .

PROOF: If it did, then  $C$  and  $D$  would form a separation of  $X$  and there would be points  $x, y \in Q$  with  $x \in C$  and  $y \in D$ .

⟨1⟩6.  $C = Q$

□

## 44 Open Coverings

**Definition 292** (Open Covering). Let  $X$  be a topological space. An *open covering* of  $X$  is a covering of  $X$  whose elements are all open sets.

## 45 Compact Spaces

**Definition 293** (Compact). A topological space is *compact* if and only if every open covering has a finite subcovering.

**Lemma 294.** *Let  $X$  be a topological space and  $Y$  a subspace of  $X$ . Then  $Y$  is compact if and only if every covering of  $Y$  by sets open in  $X$  has a finite subcovering.*

PROOF:

⟨1⟩1. If  $Y$  is compact then every covering of  $Y$  by sets open in  $X$  has a finite subcovering.

⟨2⟩1. ASSUME:  $Y$  is compact.

⟨2⟩2. LET:  $\mathcal{U}$  be a covering of  $Y$  by sets open in  $X$ .

⟨2⟩3.  $\{U \cap Y \mid U \in \mathcal{U}\}$  is an open covering of  $Y$ .

⟨2⟩4. PICK a finite subcovering  $\{U_1 \cap Y, \dots, U_n \cap Y\}$

⟨2⟩5.  $\{U_1, \dots, U_n\}$  is a finite subcovering of  $\mathcal{U}$ .

⟨1⟩2. If every covering of  $Y$  by sets open in  $X$  has a finite subcovering then  $Y$  is compact.

⟨2⟩1. LET:  $\mathcal{U}$  be an open covering of  $Y$ .

⟨2⟩2. LET:  $\mathcal{V} = \{V \text{ open in } X \mid V \cap Y \in \mathcal{U}\}$ .

⟨2⟩3.  $\mathcal{V}$  is a covering of  $Y$  by sets open in  $X$ .

⟨2⟩4. PICK a finite subcovering  $\{V_1, \dots, V_n\}$

⟨2⟩5.  $\{V_1 \cap Y, \dots, V_n \cap Y\}$  is a finite subcovering of  $\mathcal{U}$ .

□

**Proposition 295.** *Every closed subspace of a compact space is compact.*

PROOF:

- ⟨1⟩1. LET:  $X$  be a compact space and  $Y \subseteq X$  be closed.
- ⟨1⟩2. LET:  $\mathcal{U}$  be a covering of  $Y$  by sets open in  $X$ .
- ⟨1⟩3.  $\mathcal{U} \cup \{X \setminus Y\}$  is an open covering of  $X$ .
- ⟨1⟩4. PICK a finite subcovering  $\mathcal{U}_0$
- ⟨1⟩5.  $\mathcal{U}_0 \cap \mathcal{U}$  is a finite subset of  $\mathcal{U}$  that covers  $Y$ .

□

**Lemma 296.** *If  $Y$  is a compact subspace of a Hausdorff space  $X$  and  $x_0 \in X \setminus Y$  then there exist disjoint open sets  $U$  and  $V$  with  $x_0 \in U$  and  $Y \subseteq V$ .*

PROOF:

- ⟨1⟩1. For all  $y \in Y$  there exist disjoint open sets  $U', V'$  with  $x_0 \in U'$  and  $y \in V'$
- ⟨1⟩2.  $\{V' \text{ open in } X \mid \text{there exists } U' \text{ open in } X \text{ with } x_0 \in U' \text{ and } U' \cap V' = \emptyset\}$  covers  $Y$
- ⟨1⟩3. PICK a finite subcover  $\{V_1, \dots, V_n\}$
- PROOF: Lemma 294.
- ⟨1⟩4. For  $i = 1, \dots, n$ , PICK  $U_i$  open in  $X$  with  $x_0 \in U_i$  and  $U_i \cap V_i = \emptyset$
- ⟨1⟩5. Take  $U = U_1 \cap \dots \cap U_n$  and  $V = V_1 \cup \dots \cup V_n$

□

**Corollary 296.1.** *Every compact subspace of a Hausdorff space is closed.*

**Theorem 297.** *The continuous image of a compact space is compact.*

PROOF:

- ⟨1⟩1. LET:  $f : X \rightarrow Y$  be continuous and surjective.
- ⟨1⟩2. LET:  $\mathcal{V}$  be an open covering of  $Y$
- ⟨1⟩3.  $\{p^{-1}(V) \mid V \in \mathcal{V}\}$  is an open covering of  $X$ .
- ⟨1⟩4. PICK a finite subcovering  $\{p^{-1}(V_1), \dots, p^{-1}(V_n)\}$
- ⟨1⟩5.  $\{V_1, \dots, V_n\}$  covers  $Y$ .

□

**Theorem 298.** *Let  $f : X \rightarrow Y$  be a bijective continuous function. If  $X$  is compact and  $Y$  is Hausdorff then  $f$  is a homeomorphism.*

PROOF:

- ⟨1⟩1. LET:  $C \subseteq X$  be closed.
- ⟨1⟩2.  $C$  is compact.
- PROOF: Proposition 295.
- ⟨1⟩3.  $f(C)$  is compact.
- PROOF: Theorem 297.
- ⟨1⟩4.  $f(C)$  is closed.
- PROOF: Corollary 296.1.
- ⟨1⟩5. Q.E.D.
- PROOF: Lemma 106.

□

**Lemma 299** (Tube Lemma). *Let  $X$  and  $Y$  be topological spaces with  $Y$  compact. Let  $a \in X$  and  $N$  be an open set in  $X \times Y$  that includes  $\{a\} \times Y$ . Then there exists a neighbourhood  $W$  of  $a$  such that  $N$  includes the tube  $W \times Y$ .*

PROOF:

- $\langle 1 \rangle 1$ . For every  $y \in Y$ , there exist open sets  $U$  in  $X$  and  $V$  in  $Y$  such that  $a \in U$ ,  $y \in V$  and  $U \times V \subseteq N$ .
- $\langle 1 \rangle 2$ .  $\{V \text{ open in } Y \mid \text{there exists } U \text{ open in } X \text{ such that } a \in U \text{ and } U \times V \subseteq N\}$  covers  $Y$ .
- $\langle 1 \rangle 3$ . PICK a finite subcovering  $\{V_1, \dots, V_n\}$
- $\langle 1 \rangle 4$ . For  $i = 1, \dots, n$ , PICK a neighbourhood  $U_i$  of  $a$  such that  $U_i \times V_i \subseteq N$
- $\langle 1 \rangle 5$ . LET:  $W = U_1 \cap \dots \cap U_n$
- $\langle 1 \rangle 6$ .  $W \times Y \subseteq N$

□

**Theorem 300.** *Let  $X$  be a topological space. Then  $X$  is compact if and only if every set of closed sets that has the finite intersection property has nonempty intersection.*

PROOF: The following are equivalent.

1.  $X$  is compact.
2. Every open covering of  $X$  has a finite subcovering.
3. For any set  $\mathcal{C}$  of closed sets, if  $\{X \setminus C \mid C \in \mathcal{C}\}$  covers  $X$  then there is a finite subset  $\mathcal{C}_0$  such that  $\{X \setminus C \mid C \in \mathcal{C}_0\}$  covers  $X$
4. For any set  $\mathcal{C}$  of closed sets, if  $\bigcap \mathcal{C} = \emptyset$  then there is a finite subset  $\mathcal{C}_0$  with empty intersection.
5. Any set of closed sets with the finite intersection property has nonempty intersection.

□

**Corollary 300.1.** *Let  $X$  be a topological space and  $C_1 \supseteq C_2 \supseteq \dots$  a nested sequence of nonempty closed sets. Then  $\bigcap_n C_n$  is nonempty.*

**Proposition 301.** *Let  $\mathcal{T}$  and  $\mathcal{T}'$  be two topologies on the same set  $X$  with  $\mathcal{T} \subseteq \mathcal{T}'$ . If  $\mathcal{T}'$  is compact then  $\mathcal{T}$  is compact.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $\mathcal{U} \subseteq \mathcal{T}$  cover  $X$
- $\langle 1 \rangle 2$ .  $\mathcal{U} \subseteq \mathcal{T}'$
- $\langle 1 \rangle 3$ . A finite subset of  $\mathcal{U}$  covers  $X$ .

□

**Corollary 301.1.** *If  $\mathcal{T}$  and  $\mathcal{T}'$  are two compact Hausdorff topologies on the same set  $X$ , then either  $\mathcal{T} = \mathcal{T}'$  or  $\mathcal{T}$  and  $\mathcal{T}'$  are incomparable.*

PROOF: From the Proposition and Proposition 176.  $\square$

**Example 302.** Any set under the finite complement topology is compact.

**Proposition 303.** *Let  $X$  be a topological space. A finite union of compact subspaces of  $X$  is compact.*

PROOF:

- $\langle 1 \rangle 1.$  LET:  $A$  and  $B$  be compact subspaces of  $X$ .
- $\langle 1 \rangle 2.$  LET:  $\mathcal{U}$  be a set of open sets in  $X$  that covers  $A \cup B$
- $\langle 1 \rangle 3.$  PICK a finite subset  $\mathcal{U}_1$  that covers  $A$ .
- PROOF: Lemma 294.
- $\langle 1 \rangle 4.$  PICK a finite subset  $\mathcal{U}_2$  that covers  $B$ .
- PROOF: Lemma 294.
- $\langle 1 \rangle 5.$   $\mathcal{U}_1 \cup \mathcal{U}_2$  is a finite subset that covers  $A \cup B$ .
- $\langle 1 \rangle 6.$  Q.E.D.

PROOF: Lemma 294.

$\square$

## 46 Topological Groups

**Definition 304** (Topological Group). A *topological group*  $G$  consists of a  $T_1$  space  $G$  and continuous maps  $\cdot : G^2 \rightarrow G$  and  $(\ )^{-1} : G \rightarrow G$  such that  $(G, \cdot, (\ )^{-1})$  is a group.

- Example 305.**
- 1. The integers  $\mathbb{Z}$  under addition are a topological group.
  - 2. The real numbers  $\mathbb{R}$  under addition are a topological group.
  - 3. The positive reals under multiplication are a topological group.
  - 4. The set  $\{z \in \mathbb{C} \mid |z| = 1\}$  under multiplication and given the topology of  $S^1$  is a topological group.
  - 5. For any  $n \geq 0$ , the general linear group  $GL_n(\mathbb{R})$  is a topological group under matrix multiplication, considered as a subspace of  $\mathbb{R}^{n^2}$ .

**Lemma 306.** *Let  $G$  be a  $T_1$  space and  $\cdot : G^2 \rightarrow G$ ,  $(\ )^{-1} : G \rightarrow G$  be functions such that  $(G, \cdot, (\ )^{-1})$  is a group. Then  $G$  is a topological group if and only if the function  $f : G^2 \rightarrow G$  that maps  $(x, y)$  to  $xy^{-1}$  is continuous.*

PROOF:

- $\langle 1 \rangle 1.$  If  $G$  is a topological group then  $f$  is continuous.  
PROOF: From Theorem 95.
- $\langle 1 \rangle 2.$  If  $f$  is continuous then  $G$  is a topological group.
  - $\langle 2 \rangle 1.$  ASSUME:  $f$  is continuous.
  - $\langle 2 \rangle 2.$   $(\ )^{-1}$  is continuous.  
PROOF: Since  $x^{-1} = f(e, x)$ .

⟨2⟩3.  $\cdot$  is continuous.

PROOF: Since  $xy = f(x, y^{-1})$ .

□

**Lemma 307.** *Let  $G$  be a topological group and  $H$  a subgroup of  $G$ . Then  $H$  is a topological group under the subspace topology.*

PROOF:

⟨1⟩1.  $H$  is  $T_1$ .

PROOF: From Proposition 164.

⟨1⟩2. multiplication and inverse on  $H$  are continuous.

PROOF: From Theorem 96.

□

**Lemma 308.** *Let  $G$  be a topological group and  $H$  a subgroup of  $G$ . Then  $\overline{H}$  is a subgroup of  $G$ .*

PROOF:

⟨1⟩1. LET:  $x, y \in \overline{H}$

PROVE:  $xy^{-1} \in \overline{H}$

⟨1⟩2. LET:  $U$  be any neighbourhood of  $xy^{-1}$

⟨1⟩3. LET:  $f : G^2 \rightarrow G$ ,  $f(a, b) = ab^{-1}$

⟨1⟩4.  $f^{-1}(U)$  is a neighbourhood of  $(x, y)$

⟨1⟩5. PICK neighbourhoods  $V, W$  of  $x$  and  $y$  respectively such that  $f(V \times W) \subseteq U$ .

⟨1⟩6. PICK  $a \in V \cap H$  and  $b \in W \cap H$

PROOF: Theorem 47.

⟨1⟩7.  $ab^{-1} \in U \cap H$

⟨1⟩8. Q.E.D.

PROOF: By Theorem 47.

□

**Proposition 309.** *Let  $G$  be a topological group and  $\alpha \in G$ . Then the maps  $l_\alpha, r_\alpha : G \rightarrow G$  defined by  $l_\alpha(x) = \alpha x$ ,  $r_\alpha(x) = x\alpha$  are homeomorphisms of  $G$  with itself.*

PROOF: They are continuous with continuous inverses  $l_{\alpha^{-1}}$  and  $r_{\alpha^{-1}}$ . □

**Corollary 309.1.** *Every topological group is homogeneous.*

PROOF: Given a topological group  $G$  and  $a, b \in G$ , we have  $l_{ba^{-1}}$  is a homeomorphism that maps  $a$  to  $b$ . □

**Proposition 310.** *Let  $G$  be a topological group and  $H$  a subgroup of  $G$ . Give  $G/H$  the quotient topology. For all  $\alpha \in G$ , the map  $\overline{f_\alpha}$  that sends  $xH$  to  $\alpha xH$  is a homeomorphism  $G/H \cong G/H$ .*

PROOF:

⟨1⟩1.  $\overline{f_\alpha}$  is well-defined.

PROOF: If  $xy^{-1} \in H$  then  $(\alpha x)(\alpha y)^{-1} = xy^{-1} \in H$ .

$\langle 1 \rangle 2$ .  $\overline{f_\alpha}$  is continuous.

PROOF: Theorem 197 since  $\overline{f_\alpha} \circ p = p \circ f_\alpha$  is continuous, where  $p : G \twoheadrightarrow G/H$  is the canonical surjection.

$\langle 1 \rangle 3$ .  $\overline{f_\alpha}^{-1}$  is continuous.

PROOF: Similar since  $\overline{f_\alpha}^{-1} = \overline{f_{\alpha^{-1}}}$ .

□

**Corollary 310.1.** *Let  $G$  be a topological group and  $H$  a subgroup of  $G$ . Give  $G/H$  the quotient topology. Then  $G/H$  is homogeneous.*

**Proposition 311.** *Let  $G$  be a topological group and  $H$  a subgroup of  $G$ . Give  $G/H$  the quotient topology. If  $H$  is closed in  $G$  then  $G/H$  is  $T_1$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $p : G \twoheadrightarrow G/H$  be the canonical surjection

$\langle 1 \rangle 2$ . LET:  $x \in G$

$\langle 1 \rangle 3$ .  $p^{-1}(xH) = f_x(H)$

$\langle 1 \rangle 4$ .  $p^{-1}(xH)$  is closed in  $G$

PROOF: Since  $H$  is closed and  $f_x$  is a homomorphism of  $G$  with itself.

$\langle 1 \rangle 5$ .  $\{xH\}$  is closed in  $G/H$

□

**Proposition 312.** *Let  $G$  be a topological group and  $H$  a subgroup of  $G$ . Give  $G/H$  the quotient topology. Then the canonical surjection  $p : G \twoheadrightarrow G/H$  is an open map.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $U \subseteq G$  be open.

$\langle 1 \rangle 2$ .  $p^{-1}(p(U)) = \bigcup_{h \in H} r_h(U)$

$\langle 1 \rangle 3$ .  $p^{-1}(p(U))$  is open.

$\langle 1 \rangle 4$ .  $p(U)$  is open.

□

**Proposition 313.** *Let  $G$  be a topological group and  $H$  a closed normal subgroup of  $G$ . Then  $G/H$  is a topological group under the quotient topology.*

PROOF:

$\langle 1 \rangle 1$ .  $G/H$  is  $T_1$

PROOF: Proposition 311.

$\langle 1 \rangle 2$ . The map  $\overline{m} : (xH, yH) \mapsto xy^{-1}H$  is continuous.

$\langle 2 \rangle 1$ .  $p^2 : G^2 \rightarrow (G/H)^2$  is a quotient map.

PROOF: Propositions 196, 312.

$\langle 2 \rangle 2$ .  $\overline{m} \circ p^2$  is continuous.

PROOF: As it is  $p^2 \circ m$  where  $m : G^2 \rightarrow G$  with  $m(x, y) = xy^{-1}$

□

**Lemma 314.** *Let  $G$  be a topological group and  $A, B \subseteq G$ . If either  $A$  or  $B$  is open then  $AB$  is open.*

PROOF: If  $A$  is open we have  $AB = \bigcup_{b \in B} r_b(A)$ . Similarly if  $B$  is open.  $\square$

**Definition 315** (Symmetric Neighbourhood). Let  $G$  be a topological group. A neighbourhood  $V$  of  $e$  is *symmetric* if and only if  $V = V^{-1}$ .

**Lemma 316.** Let  $G$  be a topological group. Let  $V$  be a neighbourhood of  $e$ . Then  $V$  is symmetric if and only if, for all  $x \in V$ , we have  $x^{-1} \in V$ .

PROOF:

$\langle 1 \rangle 1$ . If  $V$  is symmetric then, for all  $x \in V$ , we have  $x^{-1} \in V$

PROOF: Immediate from definitions.

$\langle 1 \rangle 2$ . If, for all  $x \in V$ , we have  $x^{-1} \in V$ , then  $V$  is symmetric.

$\langle 2 \rangle 1$ . ASSUME: for all  $x \in V$  we have  $x^{-1} \in V$

$\langle 2 \rangle 2$ .  $V \subseteq V^{-1}$

PROOF: If  $x \in V$  then there exists  $y \in V$  such that  $x = y^{-1}$ , namely  $y = x^{-1}$

$\langle 2 \rangle 3$ .  $V^{-1} \subseteq V$

PROOF: Immediate from  $\langle 2 \rangle 1$ .

$\square$

**Lemma 317.** Let  $G$  be a topological group. For every neighbourhood  $U$  of  $e$ , there exists a symmetric neighbourhood  $V$  of  $e$  such that  $V^2 \subseteq U$ .

PROOF:

$\langle 1 \rangle 1$ . LET:  $U$  be a neighbourhood of  $e$ .

$\langle 1 \rangle 2$ . PICK a neighbourhood  $V'$  of  $e$  such that  $V'V' \subseteq U$

PROOF: Such a neighbourhood exists because multiplication in  $G$  is continuous.

$\langle 1 \rangle 3$ . PICK a neighbourhood  $W$  of  $e$  such that  $WW^{-1} \subseteq V'$

PROOF: Such a neighbourhood exists because the function that maps  $(x, y)$  to  $xy^{-1}$  is continuous.

$\langle 1 \rangle 4$ . LET:  $V = WW^{-1}$

$\langle 1 \rangle 5$ .  $V$  is a neighbourhood of  $e$

$\langle 2 \rangle 1$ .  $e \in V$

PROOF: Since  $e \in W$  so  $e = ee^{-1} \in V$ .

$\langle 2 \rangle 2$ .  $V$  is open

PROOF: Lemma 314.

$\langle 1 \rangle 6$ .  $V$  is symmetric

$\langle 2 \rangle 1$ . For all  $x \in V$  we have  $x^{-1} \in V$

$\langle 3 \rangle 1$ . LET:  $x \in V$

$\langle 3 \rangle 2$ . PICK  $y, z \in W$  such that  $x = yz^{-1}$

$\langle 3 \rangle 3$ .  $x^{-1} = zy^{-1}$

$\langle 3 \rangle 4$ .  $x^{-1} \in V$

$\langle 3 \rangle 5$ .  $x \in V^{-1}$

$\langle 2 \rangle 2$ . Q.E.D.

PROOF: Lemma 316

$\langle 1 \rangle 7$ .  $V^2 \subseteq U$

PROOF: We have  $V^2 \subseteq (V')^2 \subseteq U$



□

**Proposition 318.** *Every topological group is Hausdorff.*

PROOF:

- ⟨1⟩1. LET:  $G$  be a topological group.
- ⟨1⟩2. LET:  $x, y \in G$  with  $x \neq y$
- ⟨1⟩3. LET:  $U = G \setminus \{x[{}^{-1}y]\}$
- ⟨1⟩4. PICK a symmetric neighbourhood  $V$  of  $e$  such that  $VV \subseteq U$ 
  - ⟨2⟩1.  $U$  is open  
PROOF: Since  $G$  is  $T_1$ .
  - ⟨2⟩2.  $e \in U$   
PROOF: Since  $x \neq y$
  - ⟨2⟩3. Q.E.D.  
PROOF: Lemma 317.
- ⟨1⟩5.  $Vx$  and  $Vy$  are disjoint neighbourhoods of  $x$  and  $y$  respectively.
  - ⟨2⟩1.  $Vx$  is open  
PROOF: Since  $Vx = r_x(V)$
  - ⟨2⟩2.  $Vy$  is open  
PROOF: Similar.
  - ⟨2⟩3.  $Vx \cap Vy = \emptyset$ 
    - ⟨3⟩1. ASSUME: for a contradiction  $z \in Vx \cap Vy$
    - ⟨3⟩2. PICK  $a, b \in V$  such that  $z = ax = by$
    - ⟨3⟩3.  $xy^{-1} \in VV$   
PROOF: Since  $xy^{-1} = a^{-1}b$
    - ⟨3⟩4.  $xy^{-1} \in U$
    - ⟨3⟩5. Q.E.D.  
PROOF: From ⟨1⟩3.

□

**Proposition 319.** *Every topological group is regular.*

PROOF:

- ⟨1⟩1. LET:  $G$  be a topological group.
- ⟨1⟩2. LET:  $A \subseteq G$  be a closed set and  $a \notin A$ .
- ⟨1⟩3. LET:  $U = G \setminus Aa^{-1}$
- ⟨1⟩4. PICK a symmetric neighbourhood  $V$  of  $e$  such that  $VV \subseteq U$ 
  - ⟨2⟩1.  $U$  is open  
PROOF: Since  $Aa^{-1} = r_{a^{-1}}(A)$  is closed.
  - ⟨2⟩2.  $e \in U$   
PROOF: Since  $a \notin A$ .
  - ⟨2⟩3. Q.E.D.  
PROOF: Lemma 317.
- ⟨1⟩5.  $VA$  and  $Va$  are disjoint open sets with  $A \subseteq VA$  and  $a \in Va$ 
  - ⟨2⟩1.  $VA$  is open  
PROOF: Lemma 314
  - ⟨2⟩2.  $Va$  is open

PROOF: Lemma 314

$\langle 2 \rangle 3$ .  $VA \cap Va = \emptyset$

$\langle 3 \rangle 1$ . ASSUME: for a contradiction  $z \in VA \cap Va$

$\langle 3 \rangle 2$ . PICK  $b, c \in V$  and  $d \in A$  with  $z = bd = ca$

$\langle 3 \rangle 3$ .  $da^{-1} \in U$

PROOF: Since  $da^{-1} = b^{-1}c \in VV \subseteq U$

$\langle 3 \rangle 4$ . Q.E.D.

PROOF: This contradicts  $\langle 1 \rangle 3$

□

**Proposition 320.** *Let  $G$  be a topological group and  $H$  a subgroup of  $G$ . Give  $G/H$  the quotient topology. If  $H$  is closed in  $G$  then  $G/H$  is regular.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $p : G \twoheadrightarrow G/H$  be the canonical surjection.

$\langle 1 \rangle 2$ . LET:  $A$  be a closed set in  $G/H$  and  $aH \in (G/H) \setminus A$ .

$\langle 1 \rangle 3$ . LET:  $B = p^{-1}(A)$

$\langle 1 \rangle 4$ .  $B$  is a closed saturated set in  $G$ .

$\langle 1 \rangle 5$ .  $B \cap aH = \emptyset$

$\langle 1 \rangle 6$ .  $B = BH$

$\langle 1 \rangle 7$ . PICK a symmetric neighbourhood  $V$  of  $e$  such that  $VB$  does not intersect  $Va$

$\langle 2 \rangle 1$ . LET:  $U = G \setminus Ba^{-1}$

$\langle 2 \rangle 2$ . PICK a symmetric neighbourhood  $V$  of  $e$  such that  $VV \subseteq U$

$\langle 3 \rangle 1$ .  $U$  is open

PROOF: Since  $Ba^{-1} = r_{a^{-1}}(B)$  is closed.

$\langle 3 \rangle 2$ .  $e \in U$

PROOF: If  $e \in Ba^{-1}$  then  $a \in B$

$\langle 3 \rangle 3$ . Q.E.D.

PROOF: Lemma 317

$\langle 2 \rangle 3$ .  $VB \cap Va = \emptyset$

PROOF: If  $vb = v'a$  for  $v, v' \in V$  and  $b \in B$  then we have  $ba^{-1} = v^{-1}v' \in Ba \cap VV \subseteq Ba \cap U$ .

$\langle 1 \rangle 8$ .  $p(VB)$  and  $p(Va)$  are disjoint open sets

$\langle 2 \rangle 1$ .  $p(VB)$  and  $p(Va)$  are open.

PROOF: Proposition 312.

$\langle 2 \rangle 2$ .  $p(VB) \cap p(Va) = \emptyset$

PROOF: If  $vbh = v'aH$  for  $v, v' \in V$ ,  $b \in B$  then  $v'a = vbh$  for some  $h \in H$ . Hence  $v'a \in Va \cap VBH = Va \cap VB$ .

$\langle 1 \rangle 9$ .  $A \subseteq p(VB)$

$\langle 1 \rangle 10$ .  $aH \in p(Va)$

□

**Proposition 321.** *Let  $G$  be a topological group. The component of  $G$  that contains  $e$  is a normal subgroup of  $G$ .*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $C$  be the component of  $G$  that contains  $e$ .
- $\langle 1 \rangle 2$ . For all  $x \in G$ ,  $xC$  is the component of  $G$  that contains  $x$ .
  - $\langle 2 \rangle 1$ . LET:  $x \in G$
  - $\langle 2 \rangle 2$ . LET:  $D$  be the component of  $G$  that contains  $x$ .
  - $\langle 2 \rangle 3$ .  $xC \subseteq D$ 
    - PROOF: Since  $xC$  is connected by Theorem 221.
  - $\langle 2 \rangle 4$ .  $D \subseteq xC$ 
    - PROOF: Since  $x^{-1}D \subseteq C$  similarly.
- $\langle 1 \rangle 3$ . For all  $x \in G$ ,  $Cx$  is the component of  $G$  that contains  $x$ .
  - PROOF: Similar.
- $\langle 1 \rangle 4$ . For all  $x \in C$  we have  $xC = Cx = C$
- $\langle 1 \rangle 5$ . For all  $x \in C$  we have  $x^{-1}C = C$
- $\langle 1 \rangle 6$ . For all  $x \in C$  we have  $x^{-1} \in C$
- $\langle 1 \rangle 7$ . For all  $x, y \in C$  we have  $xy \in C$ 
  - PROOF: Since  $xyC = xC = C$ .
- $\langle 1 \rangle 8$ . For all  $x \in G$  we have  $xC = Cx$ .
  - PROOF: From  $\langle 1 \rangle 2$  and  $\langle 1 \rangle 3$ .

□

## 47 The Metric Topology

**Definition 322** (Metric). Let  $X$  be a set. A *metric* on  $X$  is a function  $d : X^2 \rightarrow \mathbb{R}$  such that:

1. For all  $x, y \in X$ ,  $d(x, y) \geq 0$
2. For all  $x, y \in X$ ,  $d(x, y) = 0$  if and only if  $x = y$
3. For all  $x, y \in X$ ,  $d(x, y) = d(y, x)$
4. (*Triangle Inequality*) For all  $x, y, z \in X$ ,  $d(x, z) \leq d(x, y) + d(y, z)$

We call  $d(x, y)$  the *distance* between  $x$  and  $y$ .

**Definition 323** (Open Ball). Let  $X$  be a metric space. Let  $a \in X$  and  $\epsilon > 0$ . The *open ball* with *centre*  $a$  and *radius*  $\epsilon$  is

$$B(a, \epsilon) = \{x \in X \mid d(a, x) < \epsilon\} .$$

**Definition 324** (Metric Topology). Let  $X$  be a metric space. The *metric topology* on  $X$  is the topology generated by the basis consisting of all the open balls.

We prove this is a basis for a topology.

PROOF:

$\langle 1 \rangle 1$ . For every point  $a$ , there exists a ball  $B$  such that  $a \in B$

PROOF: We have  $a \in B(a, 1)$ .

$\langle 1 \rangle 2$ . For any balls  $B_1, B_2$  and point  $a \in B_1 \cap B_2$ , there exists a ball  $B_3$  such that  $a \in B_3 \subseteq B_1 \cap B_2$

$\langle 2 \rangle 1$ . LET:  $B_1 = B(c_1, \epsilon_1)$  and  $B_2 = B(c_2, \epsilon_2)$

$\langle 2 \rangle 2$ . LET:  $\delta = \min(\epsilon_1 - d(c_1, a), \epsilon_2 - d(c_2, a))$

PROVE:  $B(a, \delta) \subseteq B_1 \cap B_2$

$\langle 2 \rangle 3$ . LET:  $x \in B(a, \delta)$

$\langle 2 \rangle 4$ .  $x \in B_1$

PROOF:

$$\begin{aligned} d(x, c_1) &= d(x, a) + d(a, c_1) \\ &< \delta + d(a, c_1) \\ &\leq \epsilon_1 \end{aligned}$$

$\langle 2 \rangle 5$ .  $x \in B_2$

PROOF: Similar.

□

**Proposition 325.** Let  $X$  be a metric space and  $U \subseteq X$ . Then  $U$  is open if and only if, for every  $x \in U$ , there exists  $\epsilon > 0$  such that  $B(x, \epsilon) \subseteq U$ .

PROOF:

$\langle 1 \rangle 1$ . If  $U$  is open then, for all  $x \in U$ , there exists  $\epsilon > 0$  such that  $B(x, \epsilon) \subseteq U$ .

$\langle 2 \rangle 1$ . ASSUME:  $U$  is open.

- ⟨2⟩2. LET:  $x \in U$   
 ⟨2⟩3. PICK  $a \in X$  and  $\delta > 0$  such that  $x \in B(a, \delta) \subseteq U$   
 ⟨2⟩4. LET:  $\epsilon = \delta - d(a, x)$   
         PROVE:  $B(x, \epsilon) \subseteq U$   
 ⟨2⟩5. LET:  $y \in B(x, \epsilon)$   
 ⟨2⟩6.  $d(y, a) < \delta$   
         PROOF:

$$\begin{aligned}
 d(y, a) &\leq d(a, x) + d(x, y) \\
 &< \delta + d(x, y) \\
 &= \epsilon
 \end{aligned}$$

- ⟨2⟩7.  $y \in U$   
 ⟨1⟩2. If, for all  $x \in U$ , there exists  $\epsilon > 0$  such that  $B(x, \epsilon) \subseteq U$ , then  $U$  is open.  
 PROOF: Immediate from definitions.

□

**Definition 326** (Discrete Metric). Let  $X$  be a set. The *discrete metric* on  $X$  is defined by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

**Proposition 327.** *The discrete metric induces the discrete topology.*

PROOF: For any (open) set  $U$  and point  $a \in U$ , we have  $a \in B(a, 1) \subseteq U$ . □

**Definition 328** (Standard Metric on  $\mathbb{R}$ ). The *standard metric* on  $\mathbb{R}$  is defined by  $d(x, y) = |x - y|$ .

**Proposition 329.** *The standard metric on  $\mathbb{R}$  induces the standard topology on  $\mathbb{R}$ .*

PROOF:

- ⟨1⟩1. Every open ball is open in the standard topology on  $\mathbb{R}$ .  
         PROOF:  $B(a, \epsilon) = (a - \epsilon, a + \epsilon)$   
 ⟨1⟩2. For every open set  $U$  and point  $a \in U$ , there exists  $\epsilon > 0$  such that  
          $B(a, \epsilon) \subseteq U$   
         ⟨2⟩1. LET:  $U$  be an open set and  $a \in U$   
         ⟨2⟩2. PICK an open interval  $b, c$  such that  $a \in (b, c) \subseteq U$   
         ⟨2⟩3. LET:  $\epsilon = \min(a - b, c - a)$   
         ⟨2⟩4.  $B(a, \epsilon) \subseteq U$

□

**Definition 330** (Metriizable). A topological space  $X$  is *metrizable* if and only if there exists a metric on  $X$  that induces the topology.

**Definition 331** (Bounded). Let  $X$  be a metric space and  $A \subseteq X$ . Then  $A$  is *bounded* if and only if there exists  $M$  such that, for all  $x, y \in A$ , we have  $d(x, y) \leq M$ .

**Definition 332** (Diameter). Let  $X$  be a metric space and  $A \subseteq X$ . The *diameter* of  $A$  is

$$\text{diam } A = \sup_{x,y \in A} d(x,y) .$$

**Definition 333** (Standard Bounded Metric). Let  $d$  be a metric on  $X$ . The *standard bounded metric* corresponding to  $d$  is the metric  $\bar{d}$  defined by

$$\bar{d}(x,y) = \min(d(x,y), 1) .$$

We prove this is a metric.

PROOF:

$\langle 1 \rangle 1. \bar{d}(x,y) \geq 0$

PROOF: Since  $d(x,y) \geq 0$

$\langle 1 \rangle 2. \bar{d}(x,y) = 0$  if and only if  $x = y$

PROOF:  $\bar{d}(x,y) = 0$  if and only if  $d(x,y) = 0$  if and only if  $x = y$

$\langle 1 \rangle 3. \bar{d}(x,y) = \bar{d}(y,x)$

PROOF: Since  $d(x,y) = d(y,x)$

$\langle 1 \rangle 4. \bar{d}(x,z) \leq \bar{d}(x,y) + \bar{d}(y,z)$

PROOF:

$$\begin{aligned} \bar{d}(x,y) + \bar{d}(y,z) &= \min(d(x,y), 1) + \min(d(y,z), 1) \\ &= \min(d(x,y) + d(y,z), d(x,y) + 1, d(y,z) + 1, 2) \\ &\geq \min(d(x,z), 1) \\ &= \bar{d}(x,z) \end{aligned}$$

□

**Lemma 334.** In any metric space  $X$ , the set  $\mathcal{B} = \{B(a, \epsilon) \mid a \in X, \epsilon < 1\}$  is a basis for the metric topology.

PROOF:

$\langle 1 \rangle 1.$  Every element of  $\mathcal{B}$  is open.

PROOF: From Lemma 61.

$\langle 1 \rangle 2.$  For every open set  $U$  and point  $a \in U$ , there exists  $B \in \mathcal{B}$  such that  $a \in B \subseteq U$

$\langle 2 \rangle 1.$  LET:  $U$  be an open set and  $a \in U$

$\langle 2 \rangle 2.$  PICK  $\epsilon > 0$  such that  $B(a, \epsilon) \subseteq U$

$\langle 2 \rangle 3.$   $B(a, \min(\epsilon, 1/2)) \subseteq U$

$\langle 1 \rangle 3.$  Q.E.D.

PROOF: Lemma 62.

□

**Proposition 335.** Let  $d$  be a metric on the set  $X$ . Then the standard bounded metric  $\bar{d}$  induces the same metric as  $d$ .

PROOF: This follows from Lemma 334 since the open balls with radius  $< 1$  are the same under both metrics. □

**Lemma 336.** *Let  $d$  and  $d'$  be two metrics on the same set  $X$ . Let  $\mathcal{T}$  and  $\mathcal{T}'$  be the topologies they induce. Then  $\mathcal{T} \subseteq \mathcal{T}'$  if and only if, for all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that*

$$B_{d'}(x, \delta) \subseteq B_d(x, \epsilon) .$$

PROOF:

$\langle 1 \rangle 1$ . If  $\mathcal{T} \subseteq \mathcal{T}'$  then, for all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$

PROOF: From Proposition 325 since  $x \in B_d(x, \epsilon) \in \mathcal{T}'$ .

$\langle 1 \rangle 2$ . If, for all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$ , then  $\mathcal{T} \subseteq \mathcal{T}'$

$\langle 2 \rangle 1$ . ASSUME: For all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$

$\langle 2 \rangle 2$ . LET:  $U \in \mathcal{T}$

$\langle 2 \rangle 3$ . For all  $x \in U$  there exists  $\delta > 0$  such that  $B_{d'}(x, \delta) \subseteq U$ .

$\langle 3 \rangle 1$ . LET:  $x \in U$

$\langle 3 \rangle 2$ . PICK  $\epsilon > 0$  such that  $B_d(x, \epsilon) \subseteq U$

PROOF: Proposition 325

$\langle 3 \rangle 3$ . PICK  $\delta > 0$  such that  $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$

PROOF: By  $\langle 2 \rangle 1$

$\langle 3 \rangle 4$ .  $B_{d'}(x, \delta) \subseteq U$

$\langle 2 \rangle 4$ .  $U \in \mathcal{T}'$

PROOF: Proposition 325.

□

**Proposition 337.**  $\mathbb{R}^2$  under the dictionary order topology is metrizable.

PROOF: Define  $d : \mathbb{R}^2 \rightarrow \mathbb{R}$  by

$$d((x, y), (x, z)) = \max(|y - z|, 1)$$

$$d((x, y), (x', y')) = 1$$

$$\text{if } x \neq x' \square$$

$\langle 1 \rangle 1$ .  $x \in \bigcap_{i=1}^N \pi_i^{-1}() \subseteq B_D(a, \epsilon)$

**Proposition 338.** *Let  $d : X^2 \rightarrow \mathbb{R}$  be a metric on  $X$ . Then the metric topology on  $X$  is the coarsest topology such that  $d$  is continuous.*

PROOF:

$\langle 1 \rangle 1$ .  $d$  is continuous.

$\langle 2 \rangle 1$ . LET:  $a, b \in X$

$\langle 2 \rangle 2$ . LET:  $\epsilon > 0$

$\langle 2 \rangle 3$ . LET:  $\delta = \epsilon/2$

$\langle 2 \rangle 4$ . LET:  $x, y \in X$

$\langle 2 \rangle 5$ . ASSUME:  $\rho((a, b), (x, y)) < \delta$

$\langle 2 \rangle 6$ .  $|d(a, b) - d(x, y)| < \epsilon$

$\langle 3 \rangle 1$ .  $d(a, b) - d(x, y) < \epsilon$

PROOF:

$$\begin{aligned}
 d(a, b) &\leq d(a, x) + d(x, y) + d(y, b) \\
 &\leq d(x, y) + 2\rho((a, b), (x, y)) \\
 &< d(x, y) + 2\delta \\
 &= d(x, y) + \epsilon
 \end{aligned}$$

$$\langle 3 \rangle 2. \ d(a, b) - d(x, y) > -\epsilon$$

PROOF: Similar.

$\langle 2 \rangle 7.$  Q.E.D.

$\langle 1 \rangle 2.$  If  $\mathcal{T}$  is any topology under which  $d$  is continuous then  $\mathcal{T}$  is finer than the metric topology.

PROOF: Since  $B(a, \epsilon) = d_a^{-1}((-\infty, \epsilon))$

□

**Proposition 339.** *Let  $X$  be a metric space with metric  $d$  and  $A \subseteq X$ . The restriction of  $d$  to  $A$  is a metric on  $A$  that induces the subspace topology.*

PROOF:

$\langle 1 \rangle 1.$  The restriction of  $d$  to  $A$  is a metric on  $A$ .

$\langle 1 \rangle 2.$  Every open ball under  $d \upharpoonright A$  is open under the subspace topology.

PROOF:  $B_{d \upharpoonright A}(a, \epsilon) = B_d(a, \epsilon) \cap A$ .

$\langle 1 \rangle 3.$  If  $U$  is open in the subspace topology and  $x \in U$ , then there exists a  $d \upharpoonright A$ -ball  $B$  such that  $x \in B \subseteq U$ .

$\langle 2 \rangle 1.$  PICK  $V$  open in  $X$  such that  $U = V \cap A$

$\langle 2 \rangle 2.$  PICK  $\epsilon > 0$  such that  $B_d(x, \epsilon) \subseteq V$

$\langle 2 \rangle 3.$  Take  $B = B_{d \upharpoonright A}(x, \epsilon)$

□

**Corollary 339.1.** *A subspace of a metrizable space is metrizable.*

**Proposition 340.** *Every metrizable space is Hausdorff.*

PROOF:

$\langle 1 \rangle 1.$  LET:  $X$  be a metric space

$\langle 1 \rangle 2.$  LET:  $a, b \in X$  with  $a \neq b$

$\langle 1 \rangle 3.$  LET:  $\epsilon = d(a, b)/2$

$\langle 1 \rangle 4.$  LET:  $U = B(a, \epsilon)$  and  $V = B(b, \epsilon)$

$\langle 1 \rangle 5.$   $U$  and  $V$  are disjoint neighbourhoods of  $a$  and  $b$  respectively.

□

**Proposition 341 (CC).** *The product of a countable family of metrizable spaces is metrizable.*

PROOF:

$\langle 1 \rangle 1.$  LET:  $(X_n, d_n)$  be a sequence of metric spaces.

$\langle 1 \rangle 2.$  ASSUME: w.l.o.g. each  $d_n$  is bounded above by 1.

PROOF: By Proposition 335.

$\langle 1 \rangle 3.$  LET:  $D$  be the metric on  $\mathbb{R}^\omega$  defined by  $D(x, y) = \sup_i (d_i(x_i, y_i)/i)$ .



- ⟨2⟩1.  $D(x, y) \geq 0$
- ⟨2⟩2.  $D(x, y) = 0$  if and only if  $x = y$
- ⟨2⟩3.  $D(x, y) = D(y, x)$
- ⟨2⟩4.  $D(x, z) \leq D(x, y) + D(y, z)$

PROOF:

$$\begin{aligned}
 D(x, z) &= \sup_i \frac{d_i(x_i, z_i)}{i} \\
 &\leq \sup_i \frac{d_i(x_i, y_i) + d_i(y_i, z_i)}{i} \\
 &\leq \sup_i \frac{d_i(x_i, y_i)}{i} + \sup_i \frac{d_i(y_i, z_i)}{i} \\
 &= D(x, y) + D(y, z)
 \end{aligned}$$

- ⟨1⟩4. Every open ball  $B_D(a, \epsilon)$  is open in the product topology.
- ⟨2⟩1. PICK  $N$  such that  $1/\epsilon < N$
- ⟨2⟩2.  $B_D(a, \epsilon) = \prod_{i=1}^{\infty} U_i$  where  $U_i = B(a_i, i\epsilon)$  if  $i \leq N$ , and  $U_i = X_i$  if  $i > N$
- ⟨1⟩5. For any open set  $U$  and  $a \in U$ , there exists  $\epsilon > 0$  such that  $B_D(a, \epsilon) \subseteq U$ .
- ⟨2⟩1. LET:  $n \geq 1$ ,  $V$  be an open set in  $\mathbb{R}$  and  $a \in \pi_n^{-1}(V)$
- ⟨2⟩2. PICK  $\epsilon > 0$  such that  $B_{d_n}(a, \epsilon) \subseteq V$
- ⟨2⟩3.  $B_D(a, \epsilon/n) \subseteq \pi_n^{-1}(V)$

□

**Theorem 342.** *Let  $X$  and  $Y$  be metric spaces and  $f : X \rightarrow Y$ . Then  $f$  is continuous if and only if, for all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that, for all  $y \in X$ , if  $d(x, y) < \delta$  then  $d(f(x), f(y)) < \epsilon$ .*

PROOF:

- ⟨1⟩1. If  $f$  is continuous then, for all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that, for all  $y \in X$ , if  $d(x, y) < \delta$  then  $d(f(x), f(y)) < \epsilon$
- ⟨2⟩1. ASSUME:  $f$  is continuous.
- ⟨2⟩2. LET:  $x \in X$  and  $\epsilon > 0$
- ⟨2⟩3. PICK a neighbourhood  $U$  of  $x$  such that  $f(U) \subseteq B(f(x), \epsilon)$
- PROOF: Theorem 92.
- ⟨2⟩4. PICK  $\delta > 0$  such that  $B(x, \delta) \subseteq U$
- PROOF: Proposition 325.
- ⟨2⟩5. For all  $y \in X$ , if  $d(x, y) < \delta$  then  $d(f(x), f(y)) < \epsilon$
- ⟨1⟩2. If for all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that, for all  $y \in X$ , if  $d(x, y) < \delta$  then  $d(f(x), f(y)) < \epsilon$ , then  $f$  is continuous.
- ⟨2⟩1. ASSUME: for all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that, for all  $y \in X$ , if  $d(x, y) < \delta$  then  $d(f(x), f(y)) < \epsilon$
- ⟨2⟩2. LET:  $x \in X$  and  $V$  be a neighbourhood of  $f(x)$
- ⟨2⟩3. PICK  $\epsilon > 0$  such that  $B(f(x), \epsilon) \subseteq V$
- PROOF: Proposition 325.
- ⟨2⟩4. PICK  $\delta > 0$  such that, for all  $y \in X$ , if  $d(x, y) < \delta$  then  $d(f(x), f(y)) < \epsilon$
- PROOF: By ⟨2⟩1
- ⟨2⟩5. LET:  $U = B(x, \delta)$
- ⟨2⟩6.  $U$  is a neighbourhood of  $x$  with  $f(U) \subseteq V$

⟨2⟩7. Q.E.D.

PROOF: Theorem 92.

□

**Proposition 343.** *Let  $X$  be a metric space. Let  $(a_n)$  be a sequence in  $X$  and  $l \in X$ . Then  $a_n \rightarrow l$  as  $n \rightarrow \infty$  if and only if, for all  $\epsilon > 0$ , there exists  $N$  such that, for all  $n \geq N$ , we have  $d(a_n, l) < \epsilon$ .*

PROOF: From Proposition 75. □

**Proposition 344.** *Every metrizable space is first countable.*

PROOF: In any metric space  $X$ , the open balls  $B(a, 1/n)$  for  $n \geq 1$  form a local basis at  $a$ .

**Example 345.**  $\mathbb{R}^\omega$  under the box topology is not metrizable.

**Example 346.** If  $J$  is uncountable then  $\mathbb{R}^J$  under the product topology is not metrizable.

**Proposition 347.** *A compact subspace of a metric space is bounded.*

PROOF:

⟨1⟩1. LET:  $X$  be a metric space and  $A \subseteq X$  be compact.

⟨1⟩2. PICK  $a \in A$

⟨1⟩3.  $\{B(a, n) \mid n \in \mathbb{Z}^+\}$  covers  $A$

⟨1⟩4. PICK a finite subcover  $\{B(a, n_1), \dots, B(a, n_k)\}$

⟨1⟩5. LET:  $N = \max(n_1, \dots, n_k)$

⟨1⟩6. For all  $x, y \in A$  we have  $d(x, y) < 2N$

PROOF:

$$\begin{aligned} d(x, y) &\leq d(x, a) + d(a, y) \\ &< N + N \end{aligned}$$

□

## 48 Real Linear Algebra

**Definition 348** (Square Metric). The *square metric*  $\rho$  on  $\mathbb{R}^n$  is defined by

$$\rho(\vec{x}, \vec{y}) = \max(|x_1 - y_1|, \dots, |x_n - y_n|)$$

We prove this is a metric.

PROOF:

⟨1⟩1.  $\rho(\vec{x}, \vec{y}) \geq 0$

PROOF: Immediate from definition.

⟨1⟩2.  $\rho(\vec{x}, \vec{y}) = 0$  if and only if  $\vec{x} = \vec{y}$

PROOF: Immediate from definition.

⟨1⟩3.  $\rho(\vec{x}, \vec{y}) = \rho(\vec{y}, \vec{x})$

PROOF: Immediate from definition.

⟨1⟩4.  $\rho(\vec{x}, \vec{z}) \leq \rho(\vec{x}, \vec{y}) + \rho(\vec{y}, \vec{z})$

PROOF: Since  $|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$ .

□

**Proposition 349.** *The square metric induces the standard topology on  $\mathbb{R}^n$ .*

PROOF:

⟨1⟩1. For every  $a \in X$  and  $\epsilon > 0$ , we have  $B_\rho(a, \epsilon)$  is open in the standard product topology.

PROOF:

$$B_\rho(a, \epsilon) = (a_1 - \epsilon, a_1 + \epsilon) \times \cdots \times (a_n - \epsilon, a_n + \epsilon)$$

⟨1⟩2. For any open sets  $U_1, \dots, U_n$  in  $\mathbb{R}$ , we have  $U_1 \times \cdots \times U_n$  is open in the square metric topology.

⟨2⟩1. LET:  $\vec{a} \in U_1 \times \cdots \times U_n$

⟨2⟩2. For  $i = 1, \dots, n$ , PICK  $\epsilon_i > 0$  such that  $(a_i - \epsilon_i, a_i + \epsilon_i) \subseteq U_i$

⟨2⟩3. LET:  $\epsilon = \min(\epsilon_1, \dots, \epsilon_n)$

⟨2⟩4.  $B_\rho(\vec{a}, \epsilon) \subseteq U$

□

**Definition 350.** Given  $\vec{x}, \vec{y} \in \mathbb{R}^n$ , define the *sum*  $\vec{x} + \vec{y}$  by

$$(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n) .$$

**Definition 351.** Given  $\lambda \in \mathbb{R}$  and  $\vec{x} \in \mathbb{R}^n$ , define the *scalar product*  $\lambda \vec{x} \in \mathbb{R}^n$  by

$$\lambda(x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n)$$

**Definition 352** (Inner Product). Given  $\vec{x}, \vec{y} \in \mathbb{R}^n$ , define the *inner product*  $\vec{x} \cdot \vec{y} \in \mathbb{R}$  by

$$(x_1, \dots, x_n) \cdot (y_1, \dots, y_n) = x_1 y_1 + \cdots + x_n y_n .$$

We write  $\vec{x}^2$  for  $\vec{x} \cdot \vec{x}$ .

**Definition 353** (Norm). Let  $n \geq 1$ . The *norm* on  $\mathbb{R}^n$  is the function  $\| \cdot \| : \mathbb{R}^n \rightarrow \mathbb{R}$  defined by

$$\|(x_1, \dots, x_n)\| = \sqrt{x_1^2 + \cdots + x_n^2}$$

**Lemma 354.**

$$\|\vec{x}\|^2 = \vec{x}^2$$

PROOF: Immediate from definitions. □

**Lemma 355.**

$$\vec{x} \cdot (\vec{y} + \vec{z}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z}$$

PROOF: Each is equal to  $(x_1 y_1 + x_1 z_1, \dots, x_n y_n + x_n z_n)$ . □

**Lemma 356.**

$$|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \|\vec{y}\|$$

PROOF:

⟨1⟩1. ASSUME:  $\vec{x} \neq \vec{0} \neq \vec{y}$

PROOF: Otherwise both sides are 0.

⟨1⟩2. LET:  $a = 1/\|\vec{x}\|$

⟨1⟩3. LET:  $b = 1/\|\vec{y}\|$

⟨1⟩4.  $(a\vec{x} + b\vec{y})^2 \geq 0$  and  $(a\vec{x} - b\vec{y})^2 \geq 0$

⟨1⟩5.  $a^2\|\vec{x}\|^2 + 2ab\vec{x} \cdot \vec{y} + b^2\|\vec{y}\|^2 \geq 0$  and  $a^2\|\vec{x}\|^2 - 2ab\vec{x} \cdot \vec{y} + b^2\|\vec{y}\|^2 \geq 0$

⟨1⟩6.  $2ab\vec{x} \cdot \vec{y} + 2 \geq 0$  and  $-2ab\vec{x} \cdot \vec{y} + 2 \geq 0$

⟨1⟩7.  $\vec{x} \cdot \vec{y} \geq -1/ab$  and  $\vec{x} \cdot \vec{y} \leq 1/ab$

⟨1⟩8.  $\vec{x} \cdot \vec{y} \geq -\|\vec{x}\|\|\vec{y}\|$  and  $\vec{x} \cdot \vec{y} \leq \|\vec{x}\|\|\vec{y}\|$

□

**Lemma 357** (Triangle Inequality).

$$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$$

PROOF:

$$\begin{aligned} \|\vec{x} + \vec{y}\|^2 &= \|\vec{x}\|^2 + 2\vec{x} \cdot \vec{y} + \|\vec{y}\|^2 \\ &\leq \|\vec{x}\|^2 + 2\|\vec{x}\|\|\vec{y}\| + \|\vec{y}\|^2 && \text{(Lemma 356)} \\ &= (\|\vec{x}\| + \|\vec{y}\|)^2 \end{aligned}$$

□

**Definition 358** (Euclidean Metric). Let  $n \geq 1$ . The *Euclidean metric* on  $\mathbb{R}^n$  is defined by

$$d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\| .$$

We prove this is a metric.

⟨1⟩1.  $d(\vec{x}, \vec{y}) \geq 0$

PROOF: Immediate from definition.

⟨1⟩2.  $d(\vec{x}, \vec{y}) = 0$  if and only if  $\vec{x} = \vec{y}$

PROOF:  $d(\vec{x}, \vec{y}) = 0$  if and only if  $\vec{x} - \vec{y} = \vec{0}$ .

⟨1⟩3.  $d(\vec{x}, \vec{y}) = d(\vec{y}, \vec{x})$

PROOF: Immediate from definition.

⟨1⟩4.  $d(\vec{x}, \vec{z}) \leq d(\vec{x}, \vec{y}) + d(\vec{y}, \vec{z})$

PROOF:

$$\begin{aligned} \|\vec{x} - \vec{z}\| &= \|(\vec{x} - \vec{y}) + (\vec{y} - \vec{z})\| \\ &\leq \|\vec{x} - \vec{y}\| + \|\vec{y} - \vec{z}\| \end{aligned}$$

(Lemma 357)

□

**Proposition 359.** The Euclidean metric induces the standard topology on  $\mathbb{R}^n$ .

PROOF:

⟨1⟩1. LET:  $\rho$  be the square metric.

⟨1⟩2. For all  $\vec{a} \in \mathbb{R}^n$  and  $\epsilon > 0$ , we have  $B_d(\vec{a}, \epsilon) \subseteq B_\rho(\vec{a}, \epsilon)$

⟨2⟩1. LET:  $\vec{x} \in B_d(\vec{a}, \epsilon)$

⟨2⟩2.  $\sqrt{(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2} < \epsilon$

⟨2⟩3.  $(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2 < \epsilon^2$

- $\langle 2 \rangle 4$ . For all  $i$  we have  $(x_i - a_i)^2 < \epsilon^2$   
 $\langle 2 \rangle 5$ . For all  $i$  we have  $|x_i - a_i| < \epsilon$   
 $\langle 2 \rangle 6$ .  $\rho(\vec{x}, \vec{a}) < \epsilon$   
 $\langle 1 \rangle 3$ . For all  $\vec{a} \in \mathbb{R}^n$  and  $\epsilon > 0$ , we have  $B_\rho(\vec{a}, \epsilon/\sqrt{n}) \subseteq B_d(\vec{a}, \epsilon)$   
 $\langle 2 \rangle 1$ . LET:  $\vec{x} \in B_\rho(\vec{a}, \epsilon/\sqrt{n})$   
 $\langle 2 \rangle 2$ .  $\rho(\vec{x}, \vec{a}) < \epsilon/\sqrt{n}$   
 $\langle 2 \rangle 3$ . For all  $i$  we have  $|x_i - a_i| < \epsilon/\sqrt{n}$   
 $\langle 2 \rangle 4$ . For all  $i$  we have  $(x_i - a_i)^2 < \epsilon^2/n$   
 $\langle 2 \rangle 5$ .  $d(\vec{x}, \vec{a}) < \epsilon$   
 $\langle 1 \rangle 4$ . Q.E.D.  
 PROOF: By Lemma 336.

□

**Proposition 360.** *Let  $n \geq 0$ . For all  $c \in \mathbb{R}^n$  and  $\epsilon > 0$ , the open ball  $B(c, \epsilon)$  is path connected.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $a, b \in B(c, \epsilon)$   
 $\langle 1 \rangle 2$ . LET:  $p : [0, 1] \rightarrow B(c, \epsilon)$  be the function  $p(t) = (1 - t)a + tb$   
 PROOF: We have  $p(t) \in B(c, \epsilon)$  for all  $t$  because
 
$$\begin{aligned}
 d(p(t), c) &= \|(1 - t)a + tb - c\| \\
 &= \|(1 - t)(a - c) + t(b - c)\| \\
 &\leq (1 - t)\|a - c\| + t\|b - c\| \\
 &< (1 - t)\epsilon + t\epsilon \\
 &= \epsilon
 \end{aligned}$$

- $\langle 1 \rangle 3$ .  $p$  is a path from  $a$  to  $b$ .

□

**Proposition 361.** *Let  $n \geq 0$ . For all  $c \in \mathbb{R}^n$  and  $\epsilon > 0$ , the closed ball  $\overline{B(c, \epsilon)}$  is path connected.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $a, b \in \overline{B(c, \epsilon)}$   
 $\langle 1 \rangle 2$ . LET:  $p : [0, 1] \rightarrow \overline{B(c, \epsilon)}$  be the function  $p(t) = (1 - t)a + tb$   
 PROOF: We have  $p(t) \in \overline{B(c, \epsilon)}$  for all  $t$  because
 
$$\begin{aligned}
 d(p(t), c) &= \|(1 - t)a + tb - c\| \\
 &= \|(1 - t)(a - c) + t(b - c)\| \\
 &\leq (1 - t)\|a - c\| + t\|b - c\| \\
 &\leq (1 - t)\epsilon + t\epsilon \\
 &= \epsilon
 \end{aligned}$$

- $\langle 1 \rangle 3$ .  $p$  is a path from  $a$  to  $b$ .

□

**Lemma 362.** *If  $\sum_{i=0}^{\infty} x_i^2$  and  $\sum_{i=0}^{\infty} y_i^2$  converge then  $\sum_{i=0}^{\infty} |x_i y_i|$  converges.*

PROOF:

⟨1⟩1. For all  $N \geq 0$  we have  $\sum_{i=0}^N |x_i y_i| \leq \sqrt{\sum_{i=0}^N |x_i|^2} \sqrt{\sum_{i=0}^N |y_i|^2}$

PROOF: By the Cauchy-Schwarz inequality

⟨1⟩2. Q.E.D.

PROOF: Since  $\sum_{i=0}^N |x_i y_i|$  is an increasing sequence bounded above by  $(\sum_{i=0}^{\infty} x_i^2)(\sum_{i=0}^{\infty} y_i^2)$ .

□

**Corollary 362.1.** *If  $\sum_{i=0}^{\infty} x_i^2$  and  $\sum_{i=0}^{\infty} y_i^2$  converge then  $\sum_{i=0}^{\infty} (x_i + y_i)^2$  converges.*

PROOF: Since  $\sum_{i=0}^{\infty} x_i^2$ ,  $\sum_{i=0}^{\infty} y_i^2$  and  $2 \sum_{i=0}^{\infty} x_i y_i$  all converge.

**Definition 363** ( $l^2$ -metric). The  $l^2$ -metric on

$$\left\{ (x_n) \in \mathbb{R}^{\omega} \mid \sum_{n=0}^{\infty} x_n^2 \text{ converges} \right\}$$

is defined by

$$d(x, y) = \left( \sum_{n=0}^{\infty} (x_n - y_n)^2 \right)^{1/2}$$

We prove this is a metric.

PROOF:

⟨1⟩1.  $d$  is well-defined.

PROOF: By Corollary 362.1.

⟨1⟩2.  $d(x, y) \geq 0$

⟨1⟩3.  $d(x, y) = 0$  if and only if  $x = y$

⟨1⟩4.  $d(x, y) = d(y, x)$

⟨1⟩5.  $d(x, z) \leq d(x, y) + d(y, z)$

PROOF: By Lemma 357.

□

**Theorem 364.** *Addition is a continuous function  $\mathbb{R}^2 \rightarrow \mathbb{R}$ .*

PROOF:

⟨1⟩1. LET:  $a, b \in \mathbb{R}$

⟨1⟩2. LET:  $\epsilon > 0$

⟨1⟩3. LET:  $\delta = \epsilon/2$

⟨1⟩4. LET:  $x, y \in \mathbb{R}$

⟨1⟩5. ASSUME:  $\rho((a, b), (x, y)) < \delta$

⟨1⟩6.  $|(a + b) - (x + y)| < \epsilon$

PROOF:

$$\begin{aligned} |(a + b) - (x + y)| &= |a - x| + |b - y| \\ &\leq 2\rho((a, b), (x, y)) \\ &< 2\delta \\ &= \epsilon \end{aligned}$$

⟨1⟩7. Q.E.D.

PROOF: Theorem 342

□

**Theorem 365.** *Multiplication is a continuous function  $\mathbb{R}^2 \rightarrow \mathbb{R}$ .*

PROOF:

⟨1⟩1. LET:  $a, b \in \mathbb{R}$

⟨1⟩2. LET:  $\epsilon > 0$

⟨1⟩3. LET:  $\delta = \min(\epsilon/(|a| + |b| + 1), 1)$

⟨1⟩4. LET:  $x, y \in \mathbb{R}$

⟨1⟩5. ASSUME:  $\rho((a, b), (x, y)) < \delta$

⟨1⟩6.  $|ab - xy| < \epsilon$

PROOF:

$$\begin{aligned} |ab - xy| &= |a(b - y) + (a - x)b - (a - x)(b - y)| \\ &\leq |a||b - y| + |b||a - x| + |a - x||b - y| \\ &< |a|\delta + |b|\delta + \delta^2 && (\langle 1 \rangle 5) \\ &\leq |a|\delta + |b|\delta + \delta && (\langle 1 \rangle 3) \\ &\leq \epsilon && (\langle 1 \rangle 3) \end{aligned}$$

⟨1⟩7. Q.E.D.

PROOF: Theorem 342

□

**Theorem 366.** *The function  $f : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$  defined by  $f(x) = x^{-1}$  is continuous.*

PROOF:

⟨1⟩1. For all  $a \in \mathbb{R}$  we have  $f^{-1}((a, +\infty))$  is open.

PROOF: The set is

$$(a^{-1}, +\infty) \text{ if } a > 0$$

$$(0, +\infty) \text{ if } a = 0$$

$$(-\infty, a^{-1}) \cup (0, +\infty) \text{ if } a < 0$$

⟨1⟩2. For all  $a \in \mathbb{R}$  we have  $f^{-1}((-\infty, a))$  is open.

PROOF: Similar.

⟨1⟩3. Q.E.D.

PROOF: By Proposition 89 and Lemma 112.

□

**Definition 367.** For  $n \geq 0$ , the *unit ball*  $B^n$  is the space  $\{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ .

**Proposition 368.** *For all  $n \geq 0$ , the unit ball  $B^n$  is path connected.*

PROOF:

⟨1⟩1. LET:  $a, b \in B^n$

⟨1⟩2. LET:  $p : [0, 1] \rightarrow B^n$  be the function  $p(t) = (1 - t)a + tb$

PROOF: We have  $p(t) \in B^n$  for all  $t$  because

$$\begin{aligned}\|(1-t)a + tb\| &\leq (1-t)\|a\| + t\|b\| \\ &\leq (1-t) + t \\ &= 1\end{aligned}$$

$\langle 1 \rangle 3$ .  $p$  is a path from  $a$  to  $b$ .

□

**Definition 369** (Punctured Euclidean Space). For  $n \geq 0$ , defined *punctured Euclidean space* to be  $\mathbb{R}^n \setminus \{0\}$ .

**Proposition 370.** For  $n > 1$ , *punctured Euclidean space*  $\mathbb{R}^n \setminus \{0\}$  is path connected.

PROOF:

$\langle 1 \rangle 1$ . LET:  $a, b \in \mathbb{R}^n \setminus \{0\}$

$\langle 1 \rangle 2$ . CASE: 0 is on the line from  $a$  to  $b$

$\langle 2 \rangle 1$ . PICK a point  $c$  not on the line from  $a$  to  $b$

$\langle 2 \rangle 2$ . The path consisting of a straight line from  $a$  to  $c$  followed by a straight line from  $c$  to  $b$  is a path from  $a$  to  $b$ .

$\langle 1 \rangle 3$ . CASE: 0 is not on the line from  $a$  to  $b$

PROOF: The straight line from  $a$  to  $b$  is a path from  $a$  to  $b$ .

**Corollary 370.1.** For  $n > 1$ , the spaces  $\mathbb{R}$  and  $\mathbb{R}^n$  are not homeomorphic.

PROOF: For any point  $a$ , the space  $\mathbb{R} \setminus \{a\}$  is disconnected.

**Definition 371** (Unit Sphere). For  $n \geq 1$ , the *unit sphere*  $S^{n-1}$  is the space

$$S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}.$$

**Proposition 372.** For  $n > 1$ , the unit sphere  $S^{n-1}$  is path connected.

PROOF: The map  $g : \mathbb{R}^n \setminus \{0\} \rightarrow S^{n-1}$  defined by  $g(x) = x/\|x\|$  is continuous and surjective. The result follows by Proposition 242. □

**Proposition 373.** Let  $f : S^1 \rightarrow \mathbb{R}$  be continuous. Then there exists  $x \in S^1$  such that  $f(x) = f(-x)$ .

PROOF:

$\langle 1 \rangle 1$ . LET:  $g : S^1 \rightarrow \mathbb{R}$  be the function  $g(x) = f(x) - f(-x)$

PROVE: There exists  $x \in S^1$  such that  $g(x) = 0$

$\langle 1 \rangle 2$ . ASSUME: without loss of generality  $g((1, 0)) > 0$

$\langle 1 \rangle 3$ .  $g((-1, 0)) < 0$

$\langle 1 \rangle 4$ . There exists  $x$  such that  $g(x) = 0$

PROOF: By the Intermediate Value Theorem.

□

**Definition 374** (Topologist's Sine Curve). Let  $S = \{(x, \sin 1/x) \mid 0 < x \leq 1\}$ . The *topologist's sine curve* is the closure  $\bar{S}$  of  $S$ .



**Proposition 375.**

$$\overline{S} = S \cup (\{0\} \times [-1, 1])$$

**Proposition 376.** *The topologist's sine curve is connected.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $S = \{(x, \sin 1/x) \mid 0 < x \leq 1\}$

$\langle 1 \rangle 2$ .  $S$  is connected.

PROOF: Theorem 221.

$\langle 1 \rangle 3$ .  $\overline{S}$  is connected.

PROOF: Theorem 220.

□

**Proposition 377 (CC).** *The topologist's sine curve is not path connected.*

PROOF:

$\langle 1 \rangle 1$ . ASSUME: for a contradiction  $p : [0, 1] \rightarrow \overline{S}$  is a path from  $(0, 0)$  to  $(1, \sin 1)$ .

$\langle 1 \rangle 2$ .  $p^{-1}(\{0\} \times [0, 1])$  is closed.

$\langle 1 \rangle 3$ . LET:  $b$  be the greatest element of  $p^{-1}(\{0\} \times [0, 1])$ .

$\langle 1 \rangle 4$ .  $b < 1$

PROOF: Since  $p(1) = (1, \sin 1)$ .

$\langle 1 \rangle 5$ . PICK a sequence  $(t_n)_{n \geq 1}$  in  $(b, 1]$  such that  $t_n \rightarrow b$  and  $\pi_2(p(t_n)) = (-1)^n$

$\langle 2 \rangle 1$ . LET:  $n \geq 1$

$\langle 2 \rangle 2$ . PICK  $u$  with  $0 < u < \pi_1(p(1/n))$  such that  $\sin(1/u) = (-1)^n$

$\langle 2 \rangle 3$ . PICK  $t_n$  such that  $0 < t_n < 1/n$  and  $\pi_1(p(t_n)) = u$

PROOF: One exists by the Intermediate Value Theorem.

$\langle 1 \rangle 6$ . Q.E.D.

PROOF: This contradicts 104.

□

## 49 The Uniform Topology

**Definition 378** (Uniform Metric). Let  $J$  be a set. The *uniform metric*  $\bar{\rho}$  on  $\mathbb{R}^J$  is defined by

$$\bar{\rho}(a, b) = \sup_{j \in J} \bar{d}(a_j, b_j)$$

where  $\bar{d}$  is the standard bounded metric on  $\mathbb{R}$ .

The *uniform topology* on  $\mathbb{R}^J$  is the topology induced by the uniform metric.

We prove this is a metric.

PROOF:

$\langle 1 \rangle 1$ .  $\bar{\rho}(a, b) \geq 0$

PROOF: Immediate from definitions.

$\langle 1 \rangle 2$ .  $\bar{\rho}(a, b) = 0$  if and only if  $a = b$

PROOF: Immediate from definitions.

$\langle 1 \rangle 3$ .  $\bar{\rho}(a, b) = \bar{\rho}(b, a)$

PROOF: Immediate from definitions.

⟨1⟩4.  $\bar{\rho}(a, c) \leq \bar{\rho}(a, b) + \bar{\rho}(b, c)$

PROOF:

$$\begin{aligned}\bar{\rho}(a, c) &= \sup_{j \in J} \bar{d}(a_j, c_j) \\ &\leq \sup_{j \in J} (\bar{d}(a_j, b_j) + \bar{d}(b_j, c_j)) \\ &\leq \sup_{j \in J} \bar{d}(a_j, b_j) + \sup_{j \in J} \bar{d}(b_j, c_j) \\ &= \bar{\rho}(a, b) + \bar{\rho}(b, c)\end{aligned}$$

□

**Proposition 379.** *The uniform topology on  $\mathbb{R}^J$  is finer than the product topology.*

PROOF:

⟨1⟩1. LET:  $j \in J$  and  $U$  be open in  $\mathbb{R}$

PROVE:  $\pi_j^{-1}(U)$  is open in the uniform topology.

⟨1⟩2. LET:  $a \in \pi_j^{-1}(U)$

⟨1⟩3. PICK  $\epsilon > 0$  such that  $(a_j - \epsilon, a_j + \epsilon) \subseteq U$

⟨1⟩4.  $B_{\bar{\rho}}(a, \epsilon) \subseteq \pi_j^{-1}(U)$

□

**Proposition 380.** *The uniform topology on  $\mathbb{R}^J$  is coarser than the box topology.*

PROOF:

⟨1⟩1. LET:  $a \in \mathbb{R}^J$  and  $\epsilon > 0$

PROVE:  $B(a, \epsilon)$  is open in the box topology.

⟨1⟩2. LET:  $b \in B(a, \epsilon)$

⟨1⟩3. For  $j \in J$  we have  $|a_j - b_j| < \epsilon$

⟨1⟩4. For  $j \in J$ ,

LET:  $\delta_j = (\epsilon - |a_j - b_j|)/2$

⟨1⟩5.  $\prod_{j \in J} (b_j - \delta_j, b_j + \delta_j) \subseteq B(a, \epsilon)$

□

**Proposition 381.** *The uniform topology on  $\mathbb{R}^J$  is strictly finer than the product topology if and only if  $J$  is infinite.*

PROOF:

⟨1⟩1. If  $J$  is finite then the uniform and product topologies coincide.

PROOF: The uniform, box and product topologies are all the same.

⟨1⟩2. If  $J$  is infinite then the uniform and product topologies are different.

PROOF: The set  $B(\vec{0}, 1)$  is open in the uniform topology but not the product topology.

□

**Proposition 382 (DC).** *The uniform topology on  $\mathbb{R}^J$  is strictly coarser than the box topology if and only if  $J$  is infinite.*

PROOF:

⟨1⟩1. If  $J$  is finite then the uniform and box topologies coincide.

PROOF: The uniform, box and product topologies are all the same.

⟨1⟩2. If  $J$  is infinite then the uniform and box topologies are different.

PROOF: Pick an  $\omega$ -sequence  $(j_1, j_2, \dots)$  in  $J$ . Let  $U = \prod_{j \in J} U_j$  where  $U_{j_i} = (-1/i, 1/i)$  and  $U_j = (-1, 1)$  for all other  $j$ . Then  $\vec{0} \in U$  but there is no  $\epsilon > 0$  such that  $B(\vec{0}, \epsilon) \subseteq U$ .

□

**Proposition 383.** *The closure of  $\mathbb{R}^\infty$  in  $\mathbb{R}^\omega$  under the uniform topology is  $\mathbb{R}^\omega$ .*

PROOF: Given any open ball  $B(a, \epsilon)$ , pick an integer  $N$  such that  $1/\epsilon < N$ . Then  $B(a, \epsilon)$  includes sequences whose  $n$ th entry is 0 for all  $n \geq N$ . □

**Example 384.** The space  $\mathbb{R}^\omega$  is disconnected under the uniform topology. The set of bounded sequences and the set of unbounded sequences form a separation.

**Proposition 385.** *Give  $\mathbb{R}^\omega$  the uniform topology. Let  $x, y \in \mathbb{R}^\omega$ . Then  $x$  and  $y$  are in the same component if and only if  $x - y$  is bounded.*

PROOF:

⟨1⟩1. The component containing 0 is the set of bounded sequences.

⟨2⟩1. LET:  $B$  be the set of bounded sequences.

⟨2⟩2.  $B$  is path-connected.

⟨3⟩1. LET:  $x, y \in B$

⟨3⟩2. PICK  $b > 0$  such that  $|x_j|, |y_j| \leq b$  for all  $j$

⟨3⟩3. LET:  $p : [0, 1] \rightarrow B$  be the function  $p(t) = (1 - t)x + ty$

PROVE:  $p$  is continuous.

⟨3⟩4. LET:  $t \in [0, 1]$  and  $\epsilon > 0$

⟨3⟩5. LET:  $\delta = \epsilon/2b$

⟨3⟩6. LET:  $s \in [0, 1]$  with  $|s - t| < \delta$

⟨3⟩7.  $\bar{\rho}(p(s), p(t)) < \epsilon$

PROOF:

$$\begin{aligned} \bar{\rho}(p(s), p(t)) &= \sup_j \bar{d}((1 - s)x_j + sy_j, (1 - t)x_j + ty_j) \\ &\leq |(s - t)x_j + (t - s)y_j| \\ &\leq |s - t||x_j - y_j| \\ &< 2b\delta \\ &= \epsilon \end{aligned}$$

⟨2⟩3.  $B$  is connected.

PROOF: Proposition 240.

⟨2⟩4. If  $C$  is connected and  $B \subseteq C$  then  $B = C$ .

PROOF: Otherwise  $B \cap C$  and  $C \setminus B$  form a separation of  $C$ .

⟨1⟩2. Q.E.D.

PROOF: Since  $\lambda x. x - y$  is a Homeomorphism of  $\mathbb{R}^\omega$  with itself.

□

## 50 Uniform Convergence

**Definition 386** (Uniform Convergence). Let  $X$  be a set and  $Y$  a metric space. Let  $(f_n : X \rightarrow Y)$  be a sequence of functions and  $f : X \rightarrow Y$  be a function. Then  $f_n$  converges uniformly to  $f$  as  $n \rightarrow \infty$  if and only if, for all  $\epsilon > 0$ , there exists  $N$  such that, for all  $n \geq N$  and  $x \in X$ , we have  $d(f_n(x), f(x)) < \epsilon$ .

**Example 387.** Define  $f_n : [0, 1] \rightarrow \mathbb{R}$  by  $f_n(x) = x^n$  for  $n \geq 1$ , and  $f : [0, 1] \rightarrow \mathbb{R}$  by  $f(x) = 0$  if  $x < 1$ ,  $f(1) = 1$ . Then  $f_n$  converges to  $f$  pointwise but not uniformly.

**Theorem 388** (Uniform Limit Theorem). Let  $X$  be a topological space and  $Y$  a metric space. Let  $(f_n : X \rightarrow Y)$  be a sequence of continuous functions and  $f : X \rightarrow Y$  be a function. If  $f_n$  converges uniformly to  $f$  as  $n \rightarrow \infty$ , then  $f$  is continuous.

PROOF:

$\langle 1 \rangle 1$ . LET:  $x \in X$  and  $\epsilon > 0$

$\langle 1 \rangle 2$ . PICK  $N$  such that, for all  $n \geq N$  and  $y \in X$ , we have  $d(f_n(y), f(y)) < \epsilon/3$

$\langle 1 \rangle 3$ . PICK a neighbourhood  $U$  of  $x$  such that  $f_N(U) \subseteq B(f_N(x), \epsilon/3)$

PROVE:  $f(U) \subseteq B(f(x), \epsilon)$

$\langle 1 \rangle 4$ . LET:  $y \in U$

$\langle 1 \rangle 5$ .  $d(f(y), f(x)) < \epsilon$

PROOF:

$$\begin{aligned} d(f(y), f(x)) &\leq d(f(y), f_N(y)) + d(f_N(y), f_N(x)) + d(f_N(x), f(x)) \quad (\text{Triangle Inequality}) \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 \quad (\langle 1 \rangle 2, \langle 1 \rangle 3) \\ &= \epsilon \end{aligned}$$

□

**Proposition 389.** Let  $X$  be a topological space and  $Y$  a metric space. Let  $(f_n : X \rightarrow Y)$  be a sequence of continuous functions and  $f : X \rightarrow Y$  be a function. Let  $(a_n)$  be a sequence of points in  $X$  and  $a \in X$ . If  $f_n$  converges uniformly to  $f$  and  $a_n$  converges to  $a$  in  $X$  then  $f_n(a_n)$  converges to  $f(a)$  uniformly in  $Y$ .

PROOF:

$\langle 1 \rangle 1$ . LET:  $\epsilon > 0$

$\langle 1 \rangle 2$ . PICK  $N_1$  such that, for all  $n \geq N_1$  and  $x \in X$ , we have  $d(f_n(x), f(x)) < \epsilon/2$

$\langle 1 \rangle 3$ . PICK  $N_2$  such that, for all  $n \geq N_2$ , we have  $a_n \in f^{-1}(B(a, \epsilon/2))$

PROOF: Using the fact that  $f$  is continuous from the Uniform Limit Theorem.

$\langle 1 \rangle 4$ . LET:  $N = \max(N_1, N_2)$

$\langle 1 \rangle 5$ . LET:  $n \geq N$

$\langle 1 \rangle 6$ .  $d(f_n(a_n), f(a)) < \epsilon$

PROOF:

$$\begin{aligned} d(f_n(a_n), f(a)) &\leq d(f_n(a_n), f(a_n)) + d(f(a_n), f(a)) \quad (\text{Triangle Inequality}) \\ &< \epsilon/2 + \epsilon/2 \quad (\langle 1 \rangle 2, \langle 1 \rangle 3) \\ &= \epsilon \end{aligned}$$

□

**Proposition 390.** *Let  $X$  be a set. Let  $(f_n : X \rightarrow \mathbb{R})$  be a sequence of functions and  $f : X \rightarrow \mathbb{R}$  be a function. Then  $f_n$  converges uniformly to  $f$  as  $n \rightarrow \infty$  if and only if  $f_n \rightarrow f$  as  $n \rightarrow \infty$  in  $\mathbb{R}^X$  under the uniform topology.*

PROOF:

- ⟨1⟩1. If  $f_n$  converges uniformly to  $f$  then  $f_n$  converges to  $f$  under the uniform topology.
- ⟨2⟩1. ASSUME:  $f_n$  converges uniformly to  $f$
- ⟨2⟩2. LET:  $\epsilon > 0$
- ⟨2⟩3. PICK  $N$  such that, for all  $n \geq N$  and  $x \in X$ , we have  $d(f_n(x), f(x)) < \epsilon/2$
- ⟨2⟩4. For all  $n \geq N$  we have  $\bar{\rho}(f_n, f) \leq \epsilon/2$
- ⟨2⟩5. For all  $n \geq N$  we have  $\bar{\rho}(f_n, f) < \epsilon$
- ⟨1⟩2. If  $f_n$  converges to  $f$  under the uniform topology then  $f_n$  converges uniformly to  $f$ .
- ⟨2⟩1. ASSUME:  $f_n$  converges to  $f$  under the uniform topology.
- ⟨2⟩2. LET:  $\epsilon > 0$
- ⟨2⟩3. PICK  $N$  such that, for all  $n \geq N$ , we have  $\bar{\rho}(f_n, f) < \min(\epsilon, 1/2)$
- ⟨2⟩4. LET:  $n \geq N$
- ⟨2⟩5. LET:  $x \in X$
- ⟨2⟩6.  $\bar{\rho}(f_n, f) < \min(\epsilon, 1/2)$
- PROOF: From ⟨2⟩3.
- ⟨2⟩7.  $d(f_n(x), f(x)) < \min(\epsilon, 1/2)$
- ⟨2⟩8.  $d(f_n(x), f(x)) < \epsilon$

□

## 51 Isometric Imbeddings

**Definition 391.** Let  $X$  and  $Y$  be metric spaces. An *isometric imbedding*  $f : X \rightarrow Y$  is a function such that, for all  $x, y \in X$ , we have  $d(f(x), f(y)) = d(x, y)$ .

**Proposition 392.** *Every isometric imbedding is an imbedding.*

PROOF:

- ⟨1⟩1. LET:  $f : X \rightarrow Y$  be an isometric imbedding.
- ⟨1⟩2.  $f$  is injective.
- PROOF: If  $f(x) = f(y)$  then  $d(f(x), f(y)) = 0$  hence  $d(x, y) = 0$  hence  $x = y$ .
- ⟨1⟩3.  $f$  is continuous.
- PROOF: For all  $\epsilon > 0$ , if  $d(x, y) < \epsilon$  then  $d(f(x), f(y)) < \epsilon$ .
- ⟨1⟩4.  $f : X \rightarrow f(X)$  is an open map.
- PROOF:  $f(B(a, \epsilon)) = B(f(a), \epsilon) \cap f(X)$ .

□