# Topology

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# 1 Order Theory

**Definition 1** (Preorder)**.** Let $X$ be a set. A *preorder* on $X$ is a binary relation $\leq$ on $X$ such that:

**Reflexivity** For all $x \in X$, we have $x \leq x$

**Transitivity** For all $x, y, z \in X$, if $x \leq y$ and $y \leq z$ then $x \leq z$.

**Definition 2** (Preordered Set)**.** A *preordered set* consists of a set $X$ and a preorder $\leq$ on $X$.

**Definition 3** (Interval)**.** Let $X$ be a preordered set and $Y \subseteq X$. Then $Y$ is an *interval* if and only if, for all $a, b \in Y$ and $c \in X$, if $a \leq c \leq b$ then $c \in Y$.

# 2 Real Analysis

**Definition 4.** Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences $(a_n)$ such that $a_n = 0$ for all but finitely many $n$.

# 3 Topological Spaces

**Definition 5** (Topology)**.** A *topology* on a set $X$ is a set $\mathcal{T} \subseteq \mathcal{P}X$ such that:

- $X \in \mathcal{T}$.
- For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$.
- For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$.

We call the elements of $X$ *points* and the elements of $\mathcal{T}$ *open sets*.

**Definition 6** (Topological Space)**.** A *topological space* $X$ consists of a set $X$ and a topology on $X$.

**Definition 7** (Discrete Space)**.** For any set $X$, the *discrete* topology on $X$ is $\mathcal{P}X$.

**Definition 8** (Indiscrete Space)**.** For any set $X$, the *indiscrete* or *trivial* topology on $X$ is $\{\emptyset, X\}$.

**Definition 9** (Finite Complement Topology)**.** For any set $X$, the *finite complement topology* on $X$ is $\{U \in \mathcal{P}X \mid X \setminus U \text{ is finite}\} \cup \{\emptyset\}$.

**Definition 10** (Countable Complement Topology)**.** For any set $X$, the *countable complement topology* on $X$ is $\{U \in \mathcal{P}X \mid X \setminus U \text{ is countable}\} \cup \{\emptyset\}$.

**Definition 11** (Finer, Coarser)**.** Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a given set $X$. If $\mathcal{T}' \supseteq \mathcal{T}$, we say that $\mathcal{T}'$ is *finer* than $\mathcal{T}$; if $\mathcal{T}'$ *properly* contains $\mathcal{T}$, we say that $\mathcal{T}'$ is *strictly* finer than $\mathcal{T}$. We also say that $\mathcal{T}$ is *coarser* than $\mathcal{T}'$, or *stricly* coarser, in these two respective situations. We say $\mathcal{T}$ is *comparable* with $\mathcal{T}'$ if either $\mathcal{T}' \supseteq \mathcal{T}$ or $\mathcal{T} \supseteq \mathcal{T}'$.

**Lemma 12.** *Let $X$ be a topological space and $U \subseteq X$. Then $U$ is open if and only if, for all $x \in U$, there exists an open set $V$ such that $x \in V \subseteq U$.*

PROOF:
$\langle 1 \rangle 1.$ $\Rightarrow$
  PROOF: Take $V = U$
$\langle 1 \rangle 2.$ $\Leftarrow$
  PROOF: We have $U = \bigcup \{V \text{ open in } X \mid V \subseteq U\}$.
$\square$

**Lemma 13.** *Let $X$ be a set and $\mathcal{T}$ a nonempty set of topologies on $X$. Then $\bigcap \mathcal{T}$ is a topology on $X$, and is the finest topology that is coarser than every member of $\mathcal{T}$.*

PROOF:
$\langle 1 \rangle 1.$ $X \in \bigcap \mathcal{T}$
  PROOF: Since $X$ is in every member of $\mathcal{T}$.
$\langle 1 \rangle 2.$ $\bigcap \mathcal{T}$ is closed under union.

$\langle 2\rangle 1.$ LET: $\mathcal{U} \subseteq \bigcap \mathcal{T}$
$\langle 2\rangle 2.$ For all $T \in \mathcal{T}$ we have $\mathcal{U} \subseteq T$
$\langle 2\rangle 3.$ For all $T \in \mathcal{T}$ we have $\bigcup \mathcal{U} \in T$
$\langle 2\rangle 4.$ $\bigcup \mathcal{U} \in \bigcap \mathcal{T}$
$\langle 1\rangle 3.$ $\bigcap \mathcal{T}$ is closed under binary intersection.
$\langle 2\rangle 1.$ LET: $U, V \in \bigcap \mathcal{T}$
$\langle 2\rangle 2.$ For all $T \in \mathcal{T}$ we have $U, V \in T$
$\langle 2\rangle 3.$ For all $T \in \mathcal{T}$ we have $U \cap V \in T$
$\langle 2\rangle 4.$ $U \cap V \in \bigcap \mathcal{T}$
□

**Lemma 14.** *Let $X$ be a set and $\mathcal{T}$ a set of topologies on $X$. Then there exists a unique coarsest topology that is finer than every member of $\mathcal{T}$.*

PROOF: The required topology is given by
$\bigcap \{T \in \mathcal{PP}X \mid T \text{ is a topology on } X \text{ that is finer than every member of } \mathcal{T}\}$ ,
The set is nonempty since it contains the discrete topology. □

**Definition 15** (Neighbourhood)**.** A *neighbourhood* of a point $x$ is an open set that contains $x$.

# 4 Closed Set

**Definition 16** (Closed Set)**.** Let $X$ be a topological space and $A \subseteq X$. Then $A$ is *closed* if and only if $X \setminus A$ is open.

**Lemma 17.** *The empty set is closed.*

PROOF: Since the whole space $X$ is always open. □

**Lemma 18.** *The topological space $X$ is closed.*

PROOF: Since $\emptyset$ is open. □

**Lemma 19.** *The intersection of a nonempty set of closed sets is closed.*

PROOF: Let $\mathcal{C}$ be a nonempty set of closed sets. Then $X \setminus \bigcap \mathcal{C} = \bigcup \{X \setminus C \mid C \in \mathcal{C}\}$ is open. □

**Lemma 20.** *The union of two closed sets is closed.*

PROOF: Let $C$ and $D$ be closed. Then $X \setminus (C \cup D) = (X \setminus C) \cap (X \setminus D)$ is open. □

**Proposition 21.** *Let $X$ be a set and $\mathcal{C} \subseteq \mathcal{P}X$ a set such that:*

1. $\emptyset \in \mathcal{C}$
2. $X \in \mathcal{C}$
3. *For all $\mathcal{A} \subseteq \mathcal{C}$ nonempty we have $\bigcap \mathcal{A} \in \mathcal{C}$*

*4. For all $C, D \in \mathcal{C}$ we have $C \cup D \in \mathcal{C}$.*

*Then there exists a unique topology $\mathcal{T}$ such that $\mathcal{C}$ is the set of closed sets, namely*

$$\mathcal{T} = \{X \setminus C \mid C \in \mathcal{C}\}$$

PROOF:
$\langle 1\rangle 1$. LET: $\mathcal{T} = \{X \setminus C \mid C \in \mathcal{C}\}$
$\langle 1\rangle 2$. $\mathcal{T}$ is a topology
  $\langle 2\rangle 1$. $X \in \mathcal{T}$
    PROOF: Since $\emptyset \in \mathcal{C}$
  $\langle 2\rangle 2$. For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$
    $\langle 3\rangle 1$. LET: $\mathcal{U} \subseteq \mathcal{T}$
    $\langle 3\rangle 2$. CASE: $\mathcal{U} = \emptyset$
      PROOF: In this case $\bigcup \mathcal{U} = \emptyset \in \mathcal{T}$ since $X \in \mathcal{C}$
    $\langle 3\rangle 3$. CASE: $\mathcal{U} \neq \emptyset$
      PROOF: In this case $X \setminus \bigcup \mathcal{U} = \bigcap \{X \setminus U \mid U \in \mathcal{U}\} \in \mathcal{C}$.
  $\langle 2\rangle 3$. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$
    PROOF: Since $X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V) \in \mathcal{C}$.
$\langle 1\rangle 3$. $\mathcal{C}$ is the set of all closed sets in $\mathcal{T}$
  PROOF:

$$\begin{aligned} & C \text{ is closed in } \mathcal{T} \\ \Leftrightarrow & X \setminus C \in \mathcal{T} \\ \Leftrightarrow & C \in \mathcal{C} \end{aligned}$$

$\langle 1\rangle 4$. If $\mathcal{T}'$ is a topology and $\mathcal{C}$ is the set of closed sets in $\mathcal{T}'$ then $\mathcal{T}' = \mathcal{T}$
  PROOF: We have

$$\begin{aligned} & U \in \mathcal{T} \\ \Leftrightarrow & X \setminus U \in \mathcal{C} \\ \Leftrightarrow & X \setminus U \text{ is closed in } \mathcal{T}' \\ \Leftrightarrow & U \in \mathcal{T}' \end{aligned}$$

$\square$

**Proposition 22.** *If $U$ is open and $A$ is closed then $U \setminus A$ is open.*

PROOF: $U \setminus A = U \cap (X \setminus A)$ is the intersection of two open sets. $\square$

**Proposition 23.** *If $U$ is open and $A$ is closed then $A \setminus U$ is closed.*

PROOF: $A \setminus U = A \cap (X \setminus U)$ is the intersection of two closed sets. $\square$

# 5 Interior

**Definition 24** (Interior)**.** Let $X$ be a topological space and $A \subseteq X$. The *interior* of $A$, $\operatorname{Int} A$, is the union of all the open subsets of $A$.

**Lemma 25.** *The interior of a set is open.*

PROOF: It is a union of open sets. □

**Lemma 26.**

$$\operatorname{Int} A \subseteq A$$

PROOF: Immediate from definition. □

**Lemma 27.** *If $U$ is open and $U \subseteq A$ then $U \subseteq \operatorname{Int} A$*

PROOF: Immediate from definition. □

**Lemma 28.** *A set $A$ is open if and only if $A = \operatorname{Int} A$.*

PROOF: If $A = \operatorname{Int} A$ then $A$ is open by Lemma 25. Conversely if $A$ is open then $A \subseteq \operatorname{Int} A$ by the definition of interior and so $A = \operatorname{Int} A$.

# 6 Closure

**Definition 29** (Closure)**.** Let $X$ be a topological space and $A \subseteq X$. The *closure* of $A$, $\overline{A}$, is the intersection of all the closed sets that include $A$.

This intersection exists since $X$ is a closed set that includes A (Lemma 18).

**Lemma 30.** *The closure of a set is closed.*

PROOF: Dual to Lemma 25. □

**Lemma 31.**

$$A \subseteq \overline{A}$$

PROOF: Immediate from definition. □

**Lemma 32.** *If $C$ is closed and $A \subseteq C$ then $\overline{A} \subseteq C$.*

PROOF: Immediate from definition. □

**Lemma 33.** *A set $A$ is closed if and only if $A = \overline{A}$.*

PROOF: Dual to Lemma 28. □

**Theorem 34.** *Let $X$ be a topological space, $A \subseteq X$ and $x \in X$. Then $x \in \overline{A}$ if and only if every neighbourhood of $x$ intersects $A$.*

PROOF: We have

$$\begin{aligned}
& x \in \overline{A} \\
\Leftrightarrow & \forall C.C \text{ closed } \wedge A \subseteq C \Rightarrow x \in C \\
\Leftrightarrow & \forall U.U \text{ open } \wedge A \cap U = \emptyset \Rightarrow x \notin U \\
\Leftrightarrow & \forall U.U \text{ open } \wedge x \in U \Rightarrow U \text{ intersects } A
\end{aligned}$$

□

**Proposition 35.** *If $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$.*

PROOF: This holds because $\overline{B}$ is a closed set that includes $A$. ∎

**Proposition 36.**

$$\overline{A \cup B} = \overline{A} \cup \overline{B}$$

PROOF:
⟨1⟩1. $\overline{A} \subseteq \overline{A \cup B}$
  PROOF: By Proposition 35.
⟨1⟩2. $\overline{B} \subseteq \overline{A \cup B}$
  PROOF: By Proposition 35.
⟨1⟩3. $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$
  ⟨2⟩1. LET: $x \in \overline{A \cup B}$
  ⟨2⟩2. ASSUME: $x \notin \overline{A}$
    PROVE: $x \in \overline{B}$
  ⟨2⟩3. PICK a neighbourhood $U$ of $x$ that does not intersect $A$
  ⟨2⟩4. LET: $V$ be any neighbourhood of $x$
  ⟨2⟩5. $U \cap V$ is a neighbourhood of $x$
  ⟨2⟩6. $U \cap V$ intersects $A \cup B$
    PROOF: From ⟨2⟩1 and Theorem 34.
  ⟨2⟩7. $U \cap V$ intersects $B$
    PROOF: From ⟨2⟩3
  ⟨2⟩8. $V$ intersects $B$
  ⟨2⟩9. Q.E.D.
    PROOF: We have $x \in \overline{B}$ from Theorem 34.

∎

# 7 Boundary

**Definition 37** (Boundary)**.** The *boundary* of a set $A$ is the set $\partial A = \overline{A} \cap \overline{X \setminus A}$.

**Proposition 38.**

$$\mathrm{Int}\, A \cap \partial A = \emptyset$$

PROOF: Since $\overline{X \setminus A} = X \setminus \mathrm{Int}\, A$. ∎

**Proposition 39.**

$$\overline{A} = \mathrm{Int}\, A \cup \partial A$$

PROOF:

$$\begin{aligned} \mathrm{Int}\, A \cup \partial A &= \mathrm{Int}\, A \cup (\overline{A} \cap \overline{X \setminus A}) \\ &= (\mathrm{Int}\, A \cup \overline{A}) \cap (\mathrm{Int}\, A \cup \overline{X \setminus A}) \\ &= \overline{A} \cap X \\ &= \overline{A} \end{aligned}$$

**Proposition 40.** *$\partial A = \emptyset$ if and only if $A$ is open and closed.*

PROOF: If $\partial A = \emptyset$ then $\overline{A} = \mathrm{Int}\, A$ by Proposition 39.

**Proposition 41.** *A set $U$ is open if and only if $\partial U = \overline{U} \setminus U$.*

PROOF:

$$\begin{aligned} & \partial U = \overline{U} \setminus U \\ \Leftrightarrow & \overline{U} \setminus \operatorname{Int} U = \overline{U} \setminus U && \text{(Propositions 38, 39)} \\ \Leftrightarrow & \operatorname{Int} U = U && \square \end{aligned}$$

# 8 Limit Points

**Definition 42** (Limit Point)**.** Let $X$ be a topological space, $a \in X$ and $A \subseteq X$. Then $a$ is a *limit point*, *cluster point* or *point of accumulation* for $A$ if and only if every neighbourhood of $a$ intersects $A$ at a point other than $a$.

**Lemma 43.** *The point $a$ is an accumulation point for $A$ if and only if $a \in \overline{A \setminus \{a\}}$.*

PROOF: From Theorem 34. $\square$

**Theorem 44.** *Let $X$ be a topological space and $A \subseteq X$. Let $A'$ be the set of all limit points of $A$. Then $\overline{A} = A \cup A'$.*

PROOF:
$\langle 1 \rangle 1$. For all $x \in \overline{A}$, if $x \notin A$ then $x \in A'$
PROOF: From Theorem 34.
$\langle 1 \rangle 2$. $A \subseteq \overline{A}$
PROOF: Lemma 31.
$\langle 1 \rangle 3$. $A' \subseteq \overline{A}$
PROOF: From Theorem 34.
$\square$

**Corollary 44.1.** *A set is closed if and only if it contains all its limit points.*

**Proposition 45.** *In an indiscrete topology, every point is a limit point of any set that has more than one point.*

PROOF: Let $X$ be an indiscrete space. Let $A$ be a set with more than one point and $x$ be a point. The only neighbourhood of $x$ is $X$, which must intersect $A$ at a point other than $x$. $\square$

# 9 Basis for a Topology

**Definition 46** (Basis)**.** If $X$ is a set, a *basis* for a topology on $X$ is a set $\mathcal{B} \subseteq \mathcal{P}X$ called *basis elements* such that

1. For all $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$.
2. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

If $\mathcal{B}$ satisfies these two conditions, then we define the topology *generated* by $\mathcal{B}$ to be $\mathcal{T} = \{U \in \mathcal{P}X \mid \forall x \in U. \exists B \in \mathcal{B}. x \in B \subseteq U\}$.

We prove this is a topology.

PROOF:
$\langle 1 \rangle 1.$ $X \in \mathcal{T}$
  PROOF: For all $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B \subseteq X$ by condition 1.
$\langle 1 \rangle 2.$ For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$
  $\langle 2 \rangle 1.$ LET: $\mathcal{U} \subseteq \mathcal{T}$
  $\langle 2 \rangle 2.$ LET: $x \in \bigcup \mathcal{U}$
  $\langle 2 \rangle 3.$ PICK $U \in \mathcal{U}$ such that $x \in U$
  $\langle 2 \rangle 4.$ PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$
    PROOF: Since $U \in \mathcal{T}$ by $\langle 2 \rangle 1$ and $\langle 2 \rangle 3$.
  $\langle 2 \rangle 5.$ $x \in B \subseteq \bigcup \mathcal{U}$
$\langle 1 \rangle 3.$ For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$
  $\langle 2 \rangle 1.$ LET: $U, V \in \mathcal{T}$
  $\langle 2 \rangle 2.$ LET: $x \in U \cap V$
  $\langle 2 \rangle 3.$ PICK $B_1 \in \mathcal{B}$ such that $x \in B_1 \subseteq U$
  $\langle 2 \rangle 4.$ PICK $B_2 \in \mathcal{B}$ such that $x \in B_2 \subseteq V$
  $\langle 2 \rangle 5.$ PICK $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$
    PROOF: By condition 2.
  $\langle 2 \rangle 6.$ $x \in B_3 \subseteq U \cap V$
$\square$

**Lemma 47.** *Let $X$ be a set. Let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on $X$. Then $\mathcal{T}$ is the set of all unions of subsets of $\mathcal{B}$.*

PROOF:
$\langle 1 \rangle 1.$ For all $U \in \mathcal{T}$, there exists $\mathcal{A} \subseteq \mathcal{B}$ such that $U = \bigcup \mathcal{A}$
  $\langle 2 \rangle 1.$ LET: $U \in \mathcal{T}$
  $\langle 2 \rangle 2.$ LET: $\mathcal{A} = \{B \in \mathcal{B} \mid B \subseteq U\}$
  $\langle 2 \rangle 3.$ $U \subseteq \bigcup \mathcal{A}$
    $\langle 3 \rangle 1.$ LET: $x \in U$
    $\langle 3 \rangle 2.$ PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$
      PROOF: Since $\mathcal{B}$ is a basis for $\mathcal{T}$.
    $\langle 3 \rangle 3.$ $x \in B \in \mathcal{A}$
  $\langle 2 \rangle 4.$ $\bigcup \mathcal{A} \subseteq U$
    PROOF: From the definition of $\mathcal{A}$ ($\langle 2 \rangle 2$).
$\langle 1 \rangle 2.$ For all $\mathcal{A} \subseteq \mathcal{B}$ we have $\bigcup \mathcal{A} \in \mathcal{T}$
  $\langle 2 \rangle 1.$ $\mathcal{B} \subseteq \mathcal{T}$
    PROOF: If $B \in \mathcal{B}$ and $x \in B$, then there exists $B' \in \mathcal{B}$ such that $x \in B' \subseteq B$, namely $B' = B$.
  $\langle 2 \rangle 2.$ Q.E.D.
    PROOF: Since $\mathcal{T}$ is closed under union.
$\square$

**Corollary 47.1.** *Let $X$ be a set. Let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on $X$. Then $\mathcal{T}$ is the coarsest topology that includes $\mathcal{B}$.*

PROOF: Since every topology that includes $\mathcal{B}$ includes all unions of subsets of $\mathcal{B}$. ∎

**Lemma 48.** *Let $X$ be a topological space. Suppose that $\mathcal{C}$ is a set of open sets such that, for every open set $U$ and every point $x \in U$, there exists $C \in \mathcal{C}$ such that $x \in C \subseteq U$. Then $\mathcal{C}$ is a basis for the topology on $X$.*

PROOF:
$\langle 1 \rangle 1$. For all $x \in X$, there exists $C \in \mathcal{C}$ such that $x \in C$
  PROOF: Immediate from hypothesis.
$\langle 1 \rangle 2$. For all $C_1, C_2 \in \mathcal{C}$ and $x \in C_1 \cap C_2$, there exists $C_3 \in \mathcal{C}$ such that $x \in C_3 \subseteq C_1 \cap C_2$
  PROOF: Since $C_1 \cap C_2$ is open.
$\langle 1 \rangle 3$. Every open set is open in the topology generated by $\mathcal{C}$
  PROOF: Immediate from hypothesis.
$\langle 1 \rangle 4$. Every union of a subset of $\mathcal{C}$ is open.
  PROOF: Since every member of $\mathcal{C}$ is open.
∎

**Lemma 49.** *Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$ respectively on the set $X$. Then the following are equivalent.*

1. $\mathcal{T} \subseteq \mathcal{T}'$
2. *For all $B \in \mathcal{B}$ and $x \in B$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.*

PROOF:
$\langle 1 \rangle 1$. $1 \Rightarrow 2$
  $\langle 2 \rangle 1$. ASSUME: $\mathcal{T} \subseteq \mathcal{T}'$
  $\langle 2 \rangle 2$. LET: $B \in \mathcal{B}$ and $x \in B$
  $\langle 2 \rangle 3$. $B \in \mathcal{T}$
    PROOF: Corollary 47.1.
  $\langle 2 \rangle 4$. $B \in \mathcal{T}'$
    PROOF: By $\langle 2 \rangle 1$
  $\langle 2 \rangle 5$. There exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$
    PROOF: Since $\mathcal{B}'$ is a basis for $\mathcal{T}'$.
$\langle 1 \rangle 2$. $2 \Rightarrow 1$
  $\langle 2 \rangle 1$. ASSUME: 2
  $\langle 2 \rangle 2$. LET: $U \in \mathcal{T}$
    PROVE: $U \in \mathcal{T}'$
  $\langle 2 \rangle 3$. LET: $x \in U$
    PROVE: There exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq U$
  $\langle 2 \rangle 4$. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$
    PROOF: Since $\mathcal{B}$ is a basis for $\mathcal{T}$.
  $\langle 2 \rangle 5$. PICK $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$

PROOF: By $\langle 2\rangle 1$.
$\langle 2\rangle 6$. $x \in B' \subseteq U$
□

**Theorem 50.** *Let $X$ be a topological space, $A \subseteq X$ and $x \in X$. Let $\mathcal{B}$ be a basis for $X$. Then $x \in \overline{A}$ if and only if, for all $B \in \mathcal{B}$, if $x \in B$ then $B$ intersects $A$.*

PROOF:
$\langle 1\rangle 1$. If $x \in \overline{A}$ then, for all $B \in \mathcal{B}$, if $x \in B$ then $B$ intersects $A$.
PROOF: This follows from Theorem 34 since every element of $\mathcal{B}$ is open (Corollary 47.1).
$\langle 1\rangle 2$. Suppose that, for all $B \in \mathcal{B}$, if $x \in B$ then $B$ intersects $A$. Then $x \in \overline{A}$.
$\langle 2\rangle 1$. ASSUME: For all $B \in \mathcal{B}$, if $x \in B$ then $B$ intersects $A$.
$\langle 2\rangle 2$. LET: $U$ be an open set that contains $x$
PROVE: $U$ intersects $A$.
$\langle 2\rangle 3$. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$.
$\langle 2\rangle 4$. $B$ intersects $A$.
PROOF: From $\langle 2\rangle 1$.
$\langle 2\rangle 5$. $U$ intersects $A$.
$\langle 2\rangle 6$. Q.E.D.
PROOF: By Theorem 34.
□

**Definition 51** (Lower Limit Topology on the Real Line)**.** The *lower limit topology on the real line* is the topology on $\mathbb{R}$ generated by the basis consisting of all half-open intervals of the form $[a, b)$.

We write $\mathbb{R}_l$ for the topological space $\mathbb{R}$ under the lower limit topology.

We prove this is a basis for a topology.

PROOF:
$\langle 1\rangle 1$. For all $x \in \mathbb{R}$ there exists an interval $[a, b)$ such that $x \in [a, b)$.
PROOF: Take $[a, b) = [x, x + 1)$.
$\langle 1\rangle 2$. For any open intervals $[a, b)$, $[c, d)$ if $x \in [a, b) \cap [c, d)$, then there exists an interval $[e, f)$ such that $x \in [e, f) \subseteq [a, b) \cap [c, d)$
PROOF: Take $[e, f) = [\max(a, c), \min(b, d))$.
□

**Definition 52** ($K$-topology on the Real Line)**.** Let $K = \{1/n \mid n \in \mathbb{Z}^+\}$.

The *$K$-topology on the real line* is the topology on $\mathbb{R}$ generated by the basis consisting of all open intervals $(a, b)$ and all sets of the form $(a, b) \setminus K$.

We write $\mathbb{R}_K$ for the topological space $\mathbb{R}$ under the $K$-topology.

We prove this is a basis for a topology.

PROOF:
$\langle 1\rangle 1$. For all $x \in \mathbb{R}$ there exists an open interval $(a, b)$ such that $x \in (a, b)$.
PROOF: Take $(a, b) = (x - 1, x + 1)$.
$\langle 1\rangle 2$. For any basic open sets $B_1$, $B_2$ if $x \in B_1 \cap B_2$, then there exists a basic open set $B_3$ such that $x \in B_3 \subseteq B_1 \cap B_2$

⟨2⟩1. Case: $B_1 = (a, b)$, $B_2 = (c, d)$

Proof: Take $B_3 = (\max(a, c), \min(b, d))$.

⟨2⟩2. Case: $B_1 = (a, b)$ or $(a, b) \setminus K$, $B_2 = (c, d)$ or $(c, d) \setminus K$, and they are not both open intervals.

Proof: Take $B_3 = (\max(a, c), \min(b, d)) \setminus K$.

□

**Lemma 53.** *The lower limit topology and the $K$-topology are incomparable.*

Proof:

⟨1⟩1. The interval $[10, 11)$ is not open in the $K$-topology.

Proof: There is no open interval $(a, b)$ such that $10 \in (a, b) \subseteq [10, 11)$ or $10 \in (a, b) \setminus K \subseteq [10, 11)$.

⟨1⟩2. The set $(-1, 1) \setminus K$ is not open in the lower limit topology.

Proof: There is no half-open interval $[a, b)$ such that $0 \in [a, b) \subseteq (-1, 1) \setminus K$, since there must be a positive integer $n$ with $1/n \in [a, b)$.

□

**Definition 54** (Subbasis)**.** A *subbasis* $\mathcal{S}$ for a topology on $X$ is a set $\mathcal{S} \subseteq \mathcal{P}X$ such that $\bigcup \mathcal{S} = X$.

The topology *generated* by the subbasis $\mathcal{S}$ is the set of all unions of finite intersections of elements of $\mathcal{S}$.

We prove this is a topology.

Proof:

⟨1⟩1. The set $\mathcal{B}$ of all finite intersections of elements of $\mathcal{S}$ forms a basis for a topology on $X$.

⟨2⟩1. $\bigcup \mathcal{B} = X$

Proof: Since $\mathcal{S} \subseteq \mathcal{B}$.

⟨2⟩2. $\mathcal{B}$ is closed under binary intersection.

Proof: By definition.

⟨1⟩2. Q.E.D.

Proof: By Lemma 47.

□

We have simultaneously proved:

**Proposition 55.** *Let $\mathcal{S}$ be a subbasis for the topology on $X$. Then the set of all finite intersections of elements of $\mathcal{S}$ is a basis for the topology on $X$.*

**Proposition 56.** *Let $X$ be a set. Let $\mathcal{S}$ be a subbasis for a topology $\mathcal{T}$ on $X$. Then $\mathcal{T}$ is the coarsest topology that includes $\mathcal{S}$.*

Proof: Since every topology that includes $\mathcal{S}$ includes every union of finite intersections of elements of $\mathcal{S}$. □

# 10 Convergence

**Definition 57** (Convergence)**.** Let $X$ be a topological space. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of points in $X$ and $l \in X$. Then the sequence $(a_n)_{n \in \mathbb{N}}$ *converges* to the *limit* $l$, $a_n \to l$ as $n \to \infty$, if and only if, for every neighbourhood $U$ of $l$, there exists $N$ such that, for all $n \geq N$, we have $a_n \in U$.

**Theorem 58.** *In a Hausdorff space, a sequence has at most one limit.*

PROOF:
$\langle 1 \rangle 1$. LET: $X$ be a Hausdorff space.
$\langle 1 \rangle 2$. ASSUME: for a contradiction $a_n \to l$ as $n \to \infty$, $a_n \to m$ as $n \to \infty$, and $l \neq m$
$\langle 1 \rangle 3$. PICK disjoint neighbourhoods $U$ of $l$ and $V$ of $m$
PROOF: By the Hausdorff axiom.
$\langle 1 \rangle 4$. PICK $M$ and $N$ such that $a_n \in U$ for $n \geq M$ and $a_n \in V$ for $n \geq N$
$\langle 1 \rangle 5$. $a_{\max(M,N)} \in U \cap V$
$\langle 1 \rangle 6$. Q.E.D.
PROOF: This contradicts the fact that $U$ and $V$ are disjoint ($\langle 1 \rangle 3$).
□

To see this is not always true in spaces that are $T_1$ but not Hausdorff:

**Proposition 59.** *Let $X$ be an infinite set under the finite complement topology. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence with all points distinct. Then for every $l \in X$ we have $a_n \to l$ as $n \to \infty$.*

PROOF: Let $U$ be any neighbourhood of $l$. Since $X \setminus U$ is finite, there must exist $N$ such that, for all $n \geq N$, we have $a_n \in U$. □

**Lemma 60** (Sequence Lemma)**.** *Let $X$ be a topological space. Let $A \subseteq X$ and $l \in X$. If there is a sequence of points in $A$ that converges to $l$ then $l \in \overline{A}$.*

PROOF:
$\langle 1 \rangle 1$. LET: $(a_n)$ be a sequence of points in $A$ that converges to $l$.
$\langle 1 \rangle 2$. LET: $U$ be a neighbourhood of $l$.
$\langle 1 \rangle 3$. PICK $N$ such that, for all $n \geq N$, we have $a_n \in U$.
$\langle 1 \rangle 4$. $a_N \in U \cap A$
$\langle 1 \rangle 5$. Q.E.D.
PROOF: Theorem 34.
□

**Proposition 61.** *Let $X$ be a topological space. Let $\mathcal{B}$ be a basis for the topology on $X$. Let $(a_n)$ be a sequence in $X$ and $l \in X$. Then $a_n \to l$ as $n \to \infty$ if and only if, for every $B \in \mathcal{B}$ with $l \in B$, there exists $N$ such that, for all $n \geq N$, we have $a_n \in B$.*

PROOF:
$\langle 1 \rangle 1$. If $a_n \to l$ as $n \to \infty$ then, for every $B \in \mathcal{B}$ with $l \in B$, there exists $N$ such that, for all $n \geq N$, we have $a_n \in B$.
PROOF: Since every element of $\mathcal{B}$ is open (Corollary 47.1).
$\langle 1 \rangle 2$. If, for every $B \in \mathcal{B}$ with $l \in B$, there exists $N$ such that, for all $n \geq N$, we have $a_n \in B$, then $a_n \to l$ as $n \to \infty$.

⟨2⟩1. Assume: for every $B \in \mathcal{B}$ with $l \in B$, there exists $N$ such that, for all $n \geq N$, we have $a_n \in B$
⟨2⟩2. Let: $U$ be a neighbourhood of $l$.
⟨2⟩3. Pick $B \in \mathcal{B}$ such that $l \in B \subseteq U$
⟨2⟩4. Pick $N$ such that, for all $n \geq N$, we have $a_n \in B$
Proof: From ⟨2⟩1.
⟨2⟩5. For all $n \geq N$ we have $a_n \in U$
□

**Lemma 62.** *If a sequence $(a_n)$ is constant with $a_n = l$ for all $n$, then $a_n \to l$ as $n \to \infty$.*

Proof: Immediate from definitions. □

**Theorem 63.** *Let $X$ be a linearly ordered set. Let $(s_n)$ be a monotone increasing sequence in $X$ with a supremum $s$. Then $s_n \to s$ as $n \to \infty$.*

Proof:
⟨1⟩1. Assume: $s$ is not least in $X$.
Proof: Otherwise $(s_n)$ is the constant sequence $s$ and the result follows from Lemma 62.
⟨1⟩2. Let: $U$ be a neighbourhood of $s$.
⟨1⟩3. Pick$a < s$ such that $(a, s] \subseteq U$
⟨1⟩4. Pick $N$ such that $a < a_N$.
⟨1⟩5. For all $n \geq N$ we have $a_n \in (a, s]$
⟨1⟩6. For all $n \geq N$ we have $a_n \in U$.
□

**Theorem 64.** *If $\sum_{i=0}^{\infty} a_i = s$ and $\sum_{i=0}^{\infty} b_i = t$ then $\sum_{i=0}^{\infty}(ca_i + b_i) = cs + t$.*

Proof:$\sum_{i=0}^{N}(ca_i + b_i) = c\sum_{i=0}^{N} a_i + \sum_{i=0}^{N} b_i \to cs + t$ as $n \to \infty$. □

**Theorem 65** (Comparison Test)**.** *If $|a_i| \leq b_i$ for all $i$ and $\sum_{i=0}^{\infty} b_i$ converges then $\sum_{i=0}^{\infty} a_i$ converges.*

Proof:
⟨1⟩1. $\sum_{i=0}^{\infty} |a_i|$ converges
Proof: The partial sums $\sum_{i=0}^{N} |a_i|$ form an increasing sequence bounded above by $\sum_{i=0}^{\infty} b_i$.
⟨1⟩2. Let: $c_i = |a_i| + a_i$ for all $i$
⟨1⟩3. $\sum_{i=0}^{\infty} c_i$ converges
Proof: Each $c_i$ is either $2|a_i|$ or 0. So the partial sums $\sum_{i=0}^{N} c_i$ form an increasing sequence bounded above by $2\sum_{i=0}^{\infty} b_i$.
⟨1⟩4. Q.E.D.
Proof: Since $a_i = c_i - |a_i|$.
□

**Corollary 65.1.** *If $\sum_{i=0}^{\infty} |a_i|$ converges then $\sum_{i=0}^{\infty} a_i$ converges.*

**Theorem 66** (Weierstrass $M$-test)**.** *Let $X$ be a set and $(f_n : X \to \mathbb{R})$ be a sequence of functions. Let*

$$s_n(x) = \sum_{i=0}^{n} f_i(x)$$

*for all $n$, $x$. Suppose $|f_i(x)| \leq M_i$ for all $i \geq 0$ and $x \in X$. If the series $\sum_{i=0}^{\infty} M_i$ converges, then the sequence $(s_n)$ converges uniformly to*

$$s(x) = \sum_{i=0}^{\infty} f_i(x) \quad .$$

Proof:
$\langle 1 \rangle 1$. Let: $r_n = \sum_{i=n+1}^{\infty} M_i$ for all $n$
$\langle 1 \rangle 2$. Given $0 \leq n < k$, we have $|s_k(x) - s_n(x)| \leq r_n$
Proof:

$$\begin{aligned} |s_k(x) - s_n(x)| &= |\sum_{i=n+1}^{k} f_i(x)| \\ &\leq \sum_{i=n+1}^{k} |f_i(x)| \\ &\leq \sum_{i=n+1}^{k} M_i \\ &\leq r_n \end{aligned}$$

$\langle 1 \rangle 3$. Given $n \geq 0$ we have $|s(x) - s_n(x)| \leq r_n$
Proof: By taking the limit $k \to \infty$ in $\langle 1 \rangle 2$.
$\langle 1 \rangle 4$. Q.E.D.
Proof: Since $r_n \to 0$ as $n \to \infty$.
$\square$

## 11 Locally Finite Sets

**Definition 67** (Locally Finite)**.** Let $X$ be a topological space and $\{A_\alpha\}$ a family of subsets of $X$. Then $\mathcal{A}$ is *locally finite* if and only if every point in $X$ has a neighbourhood that intersects $A_\alpha$ for only finitely many $\alpha$.

**Theorem 68** (Pasting Lemma)**.** *Let $X$ and $Y$ be topological spaces and $f : X \to Y$. Let $\{A_\alpha\}$ be a locally finite family of closed subsets of $X$ that cover $X$. Suppose $f \restriction A_\alpha$ is continuous for all $\alpha$. Then $f$ is continuous.*

Proof:
$\langle 1 \rangle 1$. Let $X$ and $Y$ be topological spaces and $f : X \to Y$. Let $A$ and $B$ be closed subsets of $X$ such that $X = A \cup B$. Suppose $f \restriction A$ and $f \restriction B$ are continuous. Then $f$ is continuous.
$\langle 2 \rangle 1$. Let: $C \subseteq Y$ be closed.

⟨2⟩2. $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$
⟨2⟩3. $f^{-1}(C)$ and $g^{-1}(C)$ are closed in $X$.
PROOF: Theorems 78 and 124.
⟨2⟩4. $h^{-1}(C)$ is closed in $X$.
PROOF: Lemma 20.
⟨2⟩5. Q.E.D.
PROOF: Theorem 78.

⟨1⟩2. Let $X$ and $Y$ be topological spaces and $f : X \to Y$. Let $\{A_\alpha\}$ be a finite family of closed subsets of $X$ that cover $X$. Suppose $f \restriction A_\alpha$ is continuous for all $\alpha$. Then $f$ is continuous.
PROOF: From ⟨1⟩1 by induction.

⟨1⟩3. Let $X$ and $Y$ be topological spaces and $f : X \to Y$. Let $\{A_\alpha\}$ be a locally finite family of closed subsets of $X$ that cover $X$. Suppose $f \restriction A_\alpha$ is continuous for all $\alpha$. Then $f$ is continuous.
⟨2⟩1. LET: $x \in X$
PROVE: $f$ is continuous at $x$
⟨2⟩2. PICK a neighbourhood $U$ of $x$ that intersects $A_\alpha$ for only finitely many $\alpha$.
⟨2⟩3. $f \restriction U$ is continuous
PROOF: By ⟨1⟩2.
⟨2⟩4. Q.E.D.
PROOF: Lemma 90.

□

The following example shows that we cannot remove the assumption of local finiteness.

**Example 69.** Define $f : [-1, 1] \to \mathbb{R}$ by: $f(x) = 1$ if $x < -1$, $f(x) = 0$ if $x > 1$. Let $C_n = [-1, -1/n]$ for $n \geq 1$, and $D = [0, 1]$. Then $[-1, 1] = \bigcup_{n=1}^{\infty} C_n \cup D$ and $f$ is continuous on each $C_n$ and each $D$, but $f$ is not continuous on $[-1, 1]$.

**Proposition 70.** *Let $X$ be a topological space. Let $Y$ be a linearly ordered set in the order topology. Let $f, g : X \to Y$ be continuous. Define $h : X \to Y$ by $h(x) = \min(f(x), g(x))$. Then $h$ is continuous.*

PROOF: By the Pasting Lemma applied to $\{x \in X \mid f(x) \leq g(x)\}$ and $\{x \in X \mid g(x) \leq f(x)\}$, which are closed by Lemma 89.

# 12 Open Maps

**Definition 71** (Open Map)**.** Let $X$ and $Y$ be topological spaces and $f : X \to Y$. Then $f$ is an *open map* if and only if, for every open set $U$ in $X$, the set $f(U)$ is open in $Y$.

**Lemma 72.** *Let $X$ and $Y$ be topological spaces and $f : X \to Y$. Let $\mathcal{B}$ be a basis for the topology on $X$. If $f(B)$ is open in $Y$ for all $B \in \mathcal{B}$, then $f$ is an open map.*

PROOF: From Lemma 47. □

## 13 Continuous Functions

**Definition 73** (Continuous)**.** Let $X$ and $Y$ be topological spaces. A function $f : X \to Y$ is *continuous* if and only if, for every open set $V$ in $Y$, the set $f^{-1}(V)$ is open in $X$.

**Proposition 74.** *Let $X$ and $Y$ be topological spaces and $f : X \to Y$. Let $\mathcal{B}$ be a basis for $Y$. Then $f$ is continuous if and only if, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in $X$.*

PROOF:
$\langle 1 \rangle 1$. If $f$ is continuous then, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in $X$.
  PROOF: Since every element of $B$ is open (Lemma 47).
$\langle 1 \rangle 2$. Suppose that, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in $X$. Then $f$ is continuous.
  $\langle 2 \rangle 1$. ASSUME: For all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in $X$.
  $\langle 2 \rangle 2$. LET: $V$ be open in $Y$.
  $\langle 2 \rangle 3$. PICK $\mathcal{A} \subseteq \mathcal{B}$ such that $V = \bigcup \mathcal{A}$
    PROOF: By Lemma 47.
  $\langle 2 \rangle 4$. $f^{-1}(V)$ is open in $X$.
    PROOF:

$$\begin{aligned} f^{-1}(V) &= f^{-1}\left(\bigcup \mathcal{A}\right) \\ &= \bigcup_{B \in \mathcal{A}} f^{-1}(B) \end{aligned}$$

□

**Proposition 75.** *Let $X$ and $Y$ be topological spaces and $f : X \to Y$. Let $\mathcal{S}$ be a subbasis for $Y$. Then $f$ is continuous if and only if, for all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in $X$.*

PROOF:
$\langle 1 \rangle 1$. If $f$ is continuous then, for all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in $X$.
  PROOF: Since every element of $S$ is open.
$\langle 1 \rangle 2$. Suppose that, for all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in $X$. Then $f$ is continuous.
  $\langle 2 \rangle 1$. ASSUME: For all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in $X$.
  $\langle 2 \rangle 2$. LET: $S_1, \ldots, S_n \in \mathcal{S}$
  $\langle 2 \rangle 3$. $f^{-1}(S_1 \cap \cdots \cap S_n)$ is open in $A$
    PROOF: Since $f^{-1}(S_1 \cap \cdots \cap S_n) = f^{-1}(S_1) \cap \cdots \cap f^{-1}(S_n)$.
  $\langle 2 \rangle 4$. Q.E.D.
    PROOF: By Propositions 74 and 55.

□

**Proposition 76.** *Let $X$ and $Y$ be topological spaces and $f : X \to Y$. Let $\mathcal{S}$ be a basis for $Y$. Then $f$ is continuous if and only if, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in $X$.*

PROOF:

⟨1⟩1. If $f$ is continuous then, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in $X$.
PROOF: Since every element of $\mathcal{S}$ is open.
⟨1⟩2. Suppose that, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in $X$. Then $f$ is continuous.
⟨2⟩1. ASSUME: For all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in $X$.
⟨2⟩2. For every set $B$ that is the finite intersection of elemets of $\mathcal{S}$, we have $f^{-1}(B)$ is open in $X$.
PROOF: Because $f^{-1}(V_1 \cap \cdots \cap V_n) = f^{-1}(V_1) \cap \cdots \cap f^{-1}(V_n)$.
⟨2⟩3. Q.E.D.
PROOF: From Propositions 55 and 74.
□

**Definition 77** (Continuous at a Point)**.** Let $X$ and $Y$ be topological spaces. Let $f : X \to Y$ and $x \in X$. Then $f$ is *continuous at* $x$ if and only if, for every neighbourhood $V$ of $f(x)$, there exists a neighbourhood $U$ of $x$ such that $f(U) \subseteq V$.

**Theorem 78.** *Let $X$ and $Y$ be topological spaces and $f : X \to Y$. Then the following are equivalent:*

1. *$f$ is continuous.*
2. *For all $A \subseteq X$, we have $f(\overline{A}) \subseteq \overline{f(A)}$*
3. *For all $B \subseteq Y$ closed, we have $f^{-1}(B)$ is closed in $X$.*
4. *$f$ is continuous at every point of $X$.*

PROOF:
⟨1⟩1. $1 \Rightarrow 2$
⟨2⟩1. ASSUME: $f$ is continuous.
⟨2⟩2. LET: $A \subseteq X$
⟨2⟩3. LET: $x \in \overline{A}$
PROVE: $f(x) \in \overline{f(A)}$
⟨2⟩4. LET: $V$ be a neighbourhood of $f(x)$
⟨2⟩5. $f^{-1}(V)$ is a neighbourhood of $x$
⟨2⟩6. PICK $y \in A \cap f^{-1}(V)$
PROOF: By Theorem 34.
⟨2⟩7. $f(y) \in V \cap f(A)$
⟨2⟩8. Q.E.D.
PROOF: By Theorem 34.
⟨1⟩2. $2 \Rightarrow 3$
⟨2⟩1. ASSUME: 2
⟨2⟩2. LET: $B$ be closed in $Y$
⟨2⟩3. LET: $x \in \overline{f^{-1}(B)}$
PROVE: $x \in f^{-1}(B)$
⟨2⟩4. $f(x) \in B$

PROOF:

$$\begin{aligned} f(x) &\in f(\overline{f^{-1}(B)}) \\ &\subseteq \overline{f(f^{-1}(B))} && (\langle 2\rangle 1) \\ &\subseteq \overline{B} && (Proposition 35) \\ &= B \end{aligned}$$

$\langle 1\rangle 3.$ $3 \Rightarrow 1$

$\langle 2\rangle 1.$ ASSUME: 3

$\langle 2\rangle 2.$ LET: $V$ be open in $Y$

$\langle 2\rangle 3.$ $Y \setminus V$ is closed in $Y$

$\langle 2\rangle 4.$ $f^{-1}(Y \setminus V)$ is closed in $X$

$\langle 2\rangle 5.$ $X \setminus f^{-1}(V)$ is closed in $X$

$\langle 2\rangle 6.$ $f^{-1}(V)$ is open in $X$

$\langle 1\rangle 4.$ $1 \Rightarrow 4$

PROOF: For any neighbourhood $V$ of $f(x)$, the set $U = f^{-1}(V)$ is a neighbourhood of $x$ such that $f(U) \subseteq V$.

$\langle 1\rangle 5.$ $4 \Rightarrow 1$

$\langle 2\rangle 1.$ ASSUME: 4

$\langle 2\rangle 2.$ LET: $V$ be open in $Y$

$\langle 2\rangle 3.$ LET: $x \in f^{-1}(V)$

$\langle 2\rangle 4.$ $V$ is a neighbourhood of $f(x)$

$\langle 2\rangle 5.$ PICK a neighbourhood $U$ of $x$ such that $f(U) \subseteq V$

$\langle 2\rangle 6.$ $x \in U \subseteq f^{-1}(V)$

$\langle 2\rangle 7.$ Q.E.D.

PROOF: By Lemma 12.

□

**Theorem 79.** *A constant function is continuous.*

PROOF: Let $X$ and $Y$ be topological spaces. Let $b \in Y$, and let $f : X \to Y$ be the constant function with value $b$. For any open $V \subseteq Y$, the set $f^{-1}(V)$ is either $X$ (if $b \in V$) or $\emptyset$ (if $b \notin V$). □

**Theorem 80.** *If $A$ is a subspace of $X$ then the inclusion $j : A \to X$ is continuous.*

PROOF: For any $V$ open in $X$, we have $j^{-1}(V) = V \cap A$ is open in $A$. □

**Theorem 81.** *The composite of two continuous functions is continuous.*

PROOF: Let $f : X \to Y$ and $g : Y \to Z$ be continuous. For any $V$ open in $Z$, we have $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is open in $X$. □

**Theorem 82.** *Let $f : X \to Y$ be a continuous function and $A$ be a subspace of $X$. Then the restriction $f \upharpoonright A : A \to Y$ is continuous.*

PROOF: Let $V$ be open in $Y$. Then $(f \upharpoonright A)^{-1}(V) = f^{-1}(V) \cap A$ is open in $A$. □

**Theorem 83.** *Let $f : X \to Y$ be continuous. Let $Z$ be a subspace of $Y$ such that $f(X) \subseteq Z$. Then the corestriction $f : X \to Z$ is continuous.*

PROOF:
$\langle 1\rangle 1$. LET: $V$ be open in $Z$.
$\langle 1\rangle 2$. PICK $U$ open in $Y$ such that $V = U \cap Z$.
$\langle 1\rangle 3$. $f^{-1}(V) = f^{-1}(U)$
$\langle 1\rangle 4$. $f^{-1}(V)$ is open in $X$.
□

**Theorem 84.** *Let $f : X \to Y$ be continuous. Let $Z$ be a space such that $Y$ is a subspace of $Z$. Then the expansion $f : X \to Z$ is continuous.*

PROOF: Let $V$ be open in $Z$. Then $f^{-1}(V) = f^{-1}(V \cap Y)$ is open in $X$. □

**Theorem 85.** *Let $X$ and $Y$ be topological spaces. Let $f : X \to Y$. Suppose $\mathcal{U}$ is a set of open sets in $X$ such that $X = \bigcup \mathcal{U}$ and, for all $U \in \mathcal{U}$, we have $f \restriction U : U \to Y$ is continuous. Then $f$ is continuous.*

PROOF:
$\langle 1\rangle 1$. LET: $V$ be open in $Y$
$\langle 1\rangle 2$. $f^{-1}(V) = \bigcup_{U \in \mathcal{U}} (f \restriction U)^{-1}(V)$
$\langle 1\rangle 3$. For all $U \in \mathcal{U}$, we have $(f \restriction U)^{-1}(V)$ is open in $U$.
$\langle 1\rangle 4$. For all $U \in \mathcal{U}$, we have $(f \restriction U)^{-1}(V)$ is open in $X$.
  PROOF: Lemma 123.
□

**Theorem 86.** *Let $A$ be a topological space and $\{X_i\}_{i \in I}$ be a family of topological spaces. Let $f : A \to \prod_{i \in I} X_i$ be a function. If $\pi_i \circ f$ is continuous for all $i \in I$ then $f$ is continuous.*

PROOF:
$\langle 1\rangle 1$. LET: $i \in I$ and $U$ be open in $X_i$
$\langle 1\rangle 2$. $f^{-1}({\pi_i}^{-1}(U))$ is open in $A$
$\langle 1\rangle 3$. Q.E.D.
  PROOF: Proposition 75.
□

**Proposition 87.** *Let $X$ and $X'$ be the same set $X$ under two topologies $\mathcal{T}$ and $\mathcal{T}'$. Let $i : X \to X'$ be the identity function. Then $i$ is continuous if and only if $\mathcal{T}' \subseteq \mathcal{T}$.*

PROOF: Immediate from definitions. □

**Proposition 88.** *Let $f : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$. Then $f$ is continuous on the right at $a$ if and only if $f$ is continuous at $a$ as a function $\mathbb{R}_l \to \mathbb{R}$.*

PROOF:
$\langle 1\rangle 1$. If $f$ is continuous on the right at $a$ then $f$ is continuous at $a$ as a function $\mathbb{R}_l \to \mathbb{R}$.
  $\langle 2\rangle 1$. ASSUME: $f$ is continuous on the right at $a$.
  $\langle 2\rangle 2$. LET: $V$ be a neighbourhood of $f(a)$

⟨2⟩3. Pick $b$, $c$ such that $f(a) \in (b, c) \subseteq V$.
⟨2⟩4. Let: $\epsilon = \min(c - f(a), f(a) - b)$
⟨2⟩5. Pick $\delta > 0$ such that, for all $x$, if $a < x < a + \delta$ then $|f(x) - f(a)| < \epsilon$
⟨2⟩6. Let: $U = [a, a + \delta)$
⟨2⟩7. $f(U) \subseteq V$
⟨1⟩2. If $f$ is continuous at $a$ as a function $\mathbb{R}_l \to \mathbb{R}$ then $f$ is continuous on the right at $a$.
⟨2⟩1. Assume: $f$ is continuous at $a$ as a function $\mathbb{R}_l \to \mathbb{R}$
⟨2⟩2. Let: $\epsilon > 0$
⟨2⟩3. Pick a neighbourhood $U$ of $a$ such that $f(U) \subseteq (f(a) - \epsilon, f(a) + \epsilon)$
⟨2⟩4. Pick $b$, $c$ such that $a \in [b, c) \subset U$
⟨2⟩5. Let: $\delta = c - a$
⟨2⟩6. For all $x$, if $a < x < a + \delta$ then $|f(x) - f(a)| < \epsilon$
□

**Lemma 89.** *Let $X$ be a topological space. Let $Y$ be a linearly ordered set in the order topology. Let $f, g : X \to Y$ be continuous. Then $C = \{x \in X \mid f(x) \leq g(x)\}$ is closed.*

Proof:
⟨1⟩1. Let: $x \in X \setminus C$
⟨1⟩2. $f(x) > g(x)$
Prove: There exists a neighbourhood $U$ of $x$ such that $U \subseteq X \setminus C$
⟨1⟩3. Case: There exists $y$ such that $g(x) < y < f(x)$
Proof: Take $U = g^{-1}((-\infty, y)) \cup f^{-1}(y, +\infty)$.
⟨1⟩4. Case: There is no $y$ such that $g(x) < y < f(x)$
Proof: Take $U = g^{-1}((-\infty, f(x))) \cup f^{-1}(g(x), +\infty)$.
□

**Lemma 90.** *Let $f : X \to Y$. Let $Z$ be an open subspace of $X$ and $a \in Z$. If $f \restriction Z$ is continuous at $a$ then $f$ is continuous at $a$.*

Proof:
⟨1⟩1. Let: $V$ be a neighbourhood of $f(x)$
⟨1⟩2. Pick a neighbourhood $W$ of $x$ in $Z$ such that $f(W) \subseteq V$
⟨1⟩3. $W$ is a neighbourhood of $x$ in $X$ such that $f(W) \subseteq V$
Proof: Lemma 123.
□

**Proposition 91.** *Let $f : A \to B$ and $g : C \to D$ be continuous. Define $f \times g : A \times C \to B \times D$ by*

$$(f \times g)(a, c) = (f(a), g(c)) \ .$$

*Then $f \times g$ is continuous.*

Proof: $\pi_1 \circ (f \times g) = f \circ \pi_1$ and $\pi_2 \circ (f \times g) = g \circ \pi_2$ are continuous by Theorem 81. The result follows by Theorem 86.

**Proposition 92.** *Let $X$ be a topological space. Let $Y$ a Hausdorff space. Let $A \subseteq X$. Let $f, g : \overline{A} \to Y$ be continuous. If $f$ and $g$ agree on $A$ then $f = g$.*

Proof:
⟨1⟩1. Let: $x \in \overline{A}$
⟨1⟩2. Assume: $f(x) \neq g(x)$
⟨1⟩3. Pick disjoint neighbourhoods $V$ of $f(x)$ and $W$ of $g(x)$.
⟨1⟩4. Pick $y \in f^{-1}(V) \cap g^{-1}(W) \cap A$
Proof: Since $f^{-1}(V) \cap g^{-1}(W)$ is a neighbourhood of $x$ and hence intersects $A$.
⟨1⟩5. $f(y) = g(y) \in V \cap W$
⟨1⟩6. Q.E.D.
Proof: This contradicts the fact that $V$ and $W$ are disjoint (⟨1⟩3).
□

**Proposition 93.** *Let $X$ and $Y$ be topological spaces and $f : X \to Y$ be continuous. If $a_n \to l$ as $n \to \infty$ in $X$ then $f(a_n) \to f(l)$ as $n \to \infty$.*

Proof:
⟨1⟩1. Let: $V$ be a neighbourhood of $f(l)$
⟨1⟩2. Pick a neighbourhood $U$ of $l$ such that $f(U) \subseteq V$
⟨1⟩3. Pick $N$ such that, for all $n \geq N$, we have $a_n \in U$
⟨1⟩4. For all $n \geq N$ we have $f(n) \in V$
□

**Proposition 94.** *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let $(a_n)$ be a sequence in $\prod_{i \in I} X_i$ and $l \in \prod_{i \in I} X_i$. Then $a_n \to l$ as $n \to \infty$ if and only if, for all $i \in I$, we have $\pi_i(a_n) \to \pi_i(l)$ as $n \to \infty$.*

Proof:
⟨1⟩1. If $a_n \to l$ as $n \to \infty$ then, for all $i \in I$, we have $\pi_i(a_n) \to \pi_i(l)$ as $n \to \infty$
Proof: Proposition 93.
⟨1⟩2. If, for all $i \in I$, we have $\pi_i(a_n) \to \pi_i(l)$ as $n \to \infty$, then $a_n \to l$ as $n \to \infty$
⟨2⟩1. Assume: For all $i \in I$, we have $\pi_i(a_n) \to \pi_i(l)$ as $n \to \infty$
⟨2⟩2. Let: $V$ be a neighbourhood of $l$
⟨2⟩3. Pick open sets $U_i$ in $X_i$ such that $l \in \prod_{i \in I} U_i \subseteq V$ and $U_i = X_i$ for all $i$ except $i = i_1, \ldots, i_k$
⟨2⟩4. For $j = 1, \ldots, k$, Pick $N_j$ such that, for all $n \geq N_j$, we have $\pi_{i_j}(a_n) \in U_{i_j}$
⟨2⟩5. Let: $N = \max(N_1, \ldots, N_k)$
⟨2⟩6. For all $n \geq N$ we have $a_n \in V$
□

# 14 Homeomorphisms

**Definition 95** (Homeomorphism)**.** Let $X$ and $Y$ be topological spaces. A *Homeomorphism* $f$ between $X$ and $Y$, $f : X \cong Y$, is a bijection $f : X \to Y$ such that both $f$ and $f^{-1}$ are continuous.

**Lemma 96.** *Let $X$ and $Y$ be topological spaces and $f : X \to Y$ a bijection. Then the following are equivalent:*

1. *$f$ is a homeomorphism.*
2. *$f$ is continuous and an open map.*
3. *For any $U \subseteq X$, we have $U$ is open if and only if $f(U)$ is open.*

Proof: Immediate from definitions. $\square$

**Proposition 97.** *Let $X$ and $X'$ be the same set $X$ under two topologies $\mathcal{T}$ and $\mathcal{T}'$. Let $i : X \to X'$ be the identity function. Then $i$ is a homeomorphism if and only if $\mathcal{T} = \mathcal{T}'$.*

Proof: Immediate from definitions. $\square$

**Definition 98** (Topological Property)**.** Let $P$ be a property of topological spaces. Then $P$ is a *topological* property if and only if, for any spaces $X$ and $Y$, if $P$ holds of $X$ and $X \cong Y$ then $P$ holds of $Y$.

**Definition 99** (Topological Imbedding)**.** Let $X$ and $Y$ be topological spaces and $f : X \to Y$. Then $f$ is a *topological imbedding* if and only if the corestriction $f : X \to f(X)$ is a homeomorphism.

**Proposition 100.** *Let $X$ and $Y$ be topological spaces and $a \in X$. The function $i : Y \to X \times Y$ that maps $y$ to $(a, y)$ is an imbedding.*

Proof:
$\langle 1 \rangle 1.$ $i$ is injective
$\langle 1 \rangle 2.$ $i$ is continuous.
Proof: For $U$ open in $X$ and $V$ open in $Y$, we have $i^{-1}(U \times V)$ is $V$ if $a \in U$, and $\emptyset$ if $a \notin U$.
$\langle 1 \rangle 3.$ $i : Y \to i(Y)$ is an open map.
Proof: For $V$ open in $Y$ we have $i(V) = (X \times V) \cap i(Y)$.
$\square$

# 15 The Order Topology

**Definition 101** (Order Topology)**.** Let $X$ be a linearly ordered set with at least two points. The *order topology* on $X$ is the topology generated by the basis $\mathcal{B}$ consisting of:

- all open intervals $(a, b)$;
- all intervals of the form $[\bot, b)$ where $\bot$ is least in $X$;
- all intervals of the form $(a, \top]$ where $\top$ is greatest in $X$.

We prove this is a basis for a topology.

Proof:

$\langle 1 \rangle 1$. For all $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B$.
$\langle 2 \rangle 1$. LET: $x \in X$
$\langle 2 \rangle 2$. CASE: $x$ is greatest in $X$.
$\langle 3 \rangle 1$. PICK $y \in X$ with $y \neq x$
$\langle 3 \rangle 2$. $x \in (y, x] \in \mathcal{B}$
$\langle 2 \rangle 3$. CASE: $x$ is least in $X$.
$\langle 3 \rangle 1$. PICK $y \in X$ with $y \neq x$
$\langle 3 \rangle 2$. $x \in [x, y) \in \mathcal{B}$
$\langle 2 \rangle 4$. CASE: $x$ is neither greatest nor least in $X$.
$\langle 3 \rangle 1$. PICK $a, b \in X$ with $a < x$ and $x < b$
$\langle 3 \rangle 2$. $x \in (a, b) \in \mathcal{B}$
$\langle 1 \rangle 2$. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$
$\langle 2 \rangle 1$. LET: $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$
$\langle 2 \rangle 2$. CASE: $B_1 = (a, b)$, $B_2 = (c, d)$
PROOF: Take $B_3 = (\max(a, c), \min(b, d))$.
$\langle 2 \rangle 3$. CASE: $B_1 = (a, b)$, $B_2 = [\perp, d)$
PROOF: Take $B_3 = (a, \min(b, d))$.
$\langle 2 \rangle 4$. CASE: $B_1 = (a, b)$, $B_2 = (c, \top]$
PROOF: Take $B_3 = (\max(a, c), b)$.
$\langle 2 \rangle 5$. CASE: $B_1 = [\perp, b)$, $B_2 = [\perp, d)$
PROOF: Take $B_3 = [\perp, \min(b, d))$.
$\langle 2 \rangle 6$. CASE: $B_1 = [\perp, b)$, $B_2 = (c, \top]$
PROOF: Take $B_3 = (c, b)$.
□

**Lemma 102.** *Let $X$ be a linearly ordered set. Then the open rays form a subbasis for the order topology on $X$.*

PROOF:
$\langle 1 \rangle 1$. Every open ray is open.
$\langle 2 \rangle 1$. For all $a \in X$, the ray $(-\infty, a)$ is open.
$\langle 3 \rangle 1$. LET: $x \in (-\infty, a)$
$\langle 3 \rangle 2$. CASE: $x$ is least in $X$
PROOF: $xin[x, a) = (-\infty, a)$.
$\langle 3 \rangle 3$. CASE: $x$ is not least in $X$
$\langle 4 \rangle 1$. PICK $y < x$
$\langle 4 \rangle 2$. $x \in (y, a) \subseteq (-\infty, a)$
$\langle 2 \rangle 2$. For all $a \in X$, the ray $(a, +\infty)$ is open.
PROOF: Similar.
$\langle 1 \rangle 2$. Every basic open set is a finite intersection of open rays.
PROOF: We have $(a, b) = (a, +\infty) \cap (-\infty, b)$, $[\perp, b) = (-\infty, b)$ and $(a, \top] = (a, +\infty)$.
□

**Definition 103** (Standard Topology on the Real Line)**.** The *standard topology on the real line* is the order topology on $\mathbb{R}$ generated by the standard order.

**Lemma 104.** *The lower limit topology is strictly finer than the standard topology on $\mathbb{R}$.*

PROOF:
$\langle 1 \rangle 1$. Every open interval is open in the lower limit topology.
PROOF: If $x \in (a, b)$ then $x \in [x, b) \subseteq (a, b)$.
$\langle 1 \rangle 2$. The half-open interval $[0, 1)$ is not open in the standard topology.
PROOF: There is no open interval $(a, b)$ such that $0 \in (a, b) \subseteq [0, 1)$.
$\square$

**Lemma 105.** *The $K$-topology is strictly finer than the standard topology on $\mathbb{R}$.*

PROOF:
$\langle 1 \rangle 1$. Every open interval is open in the $K$-topology.
PROOF: Corollary 47.1.
$\langle 1 \rangle 2$. The set $(-1, 1) \setminus K$ is not open in the standard topology.
PROOF: There is no open interval $(a, b)$ such that $0 \in (a, b) \subseteq (-1, 1) \setminus K$, since there must be a positive integer $n$ with $1/n \in (a, b)$.
$\square$

**Definition 106** (Ordered Square)**.** The *ordered square* $I_o^2$ is the set $[0, 1]^2$ under the order topology generated by the dictionary order.

# 16 The Product Topology

**Definition 107** (Product Topology)**.** Let $\{A_i\}_{i \in I}$ be a family of topological spaces. The *product topology* on $\prod_{i \in I} A_i$ is the topology generated by the subbasis consisting of the sets of the form ${\pi_i}^{-1}(U)$ where $i \in I$ and $U$ is open in $A_i$.

**Proposition 108.** *The product topology on $\prod_{i \in I} A_i$ is generated by the basis consisting of all sets of the form $\prod_{i \in I} U_i$ where $\{U_i\}_{i \in I}$ is a family such that each $U_i$ is an open set in $A_i$ and $U_i = A_i$ for all but finitely many $i$.*

PROOF: From Proposition 55. $\square$

**Proposition 109.** *If $A_i$ is closed in $X_i$ for all $i \in I$ then $\prod_{i \in I} A_i$ is closed in $\prod_{i \in I} X_i$.*

PROOF:

$$(\prod_{i \in I} X_i) \setminus (\prod_{i \in I} A_i) = \bigcup_{j \in I} \left( \prod_{i \in I} X_i \setminus {\pi_j}^{-1}(A_j) \right) \square$$

**Proposition 110.** *Let $\{A_i\}_{i \in I}$ be a family of topological spaces. For $i \in I$, let $\mathcal{B}_i$ be a basis for the topology on $A_i$. Then $\mathcal{B} = \{\prod_{i \in I} B_i \mid \forall i \in I. B_i \in \mathcal{B}_i, B_i = A_i$ for all but finitely many $i\}$ is a basis for the box topology on $\prod_{i \in I} A_i$.*

PROOF:

⟨1⟩1. Every set in $\mathcal{B}$ is open.
⟨1⟩2. For every point $a \in \prod_{i \in I} A_i$ and every open set $U$ with $a \in U$, there exists $B \in \mathcal{B}$ such that $a \in B \subseteq U$.
⟨2⟩1. LET: $U$ be open and $a \in U$
⟨2⟩2. PICK a family $\{U_i\}_{i \in I}$ such that each $U_i$ is open in $A_i$, such that $U_i = A_i$ except for $i = i_1, \ldots, i_n$, and such that $a \in \prod_{i \in I} U_i \subseteq U$.
⟨2⟩3. For $j = 1, \ldots, n$, PICK $B_{i_j} \in \mathcal{B}_{i_j}$ such that $a_{i_j} \in B_{i_j} \subseteq U_{i_j}$
⟨2⟩4. LET: $B = \prod_{i \in I} B_i$ where $B_i = A_i$ for $i \neq i_1, \ldots, i_n$
⟨2⟩5. $B \in \mathcal{B}$
⟨2⟩6. $a \in B \subseteq U$
⟨1⟩3. Q.E.D.
PROOF: Lemma 48.
□

**Proposition 111.** *Let $\{A_i\}_{i \in I}$ be a family of topological spaces. Then the projections $\pi_i : \prod_{i \in I} A_i \to A_i$ are open maps.*

PROOF: From Lemma 72. □

**Proposition 112.** *Let $\{X_i\}_{i \in I}$ be a family of sets. For $i \in I$, let $\mathcal{T}_i$ and $\mathcal{U}_i$ be topologies on $X_i$. Let $\mathcal{P}$ be the product topology on $\prod_{i \in I} X_i$ generated by the topologies $\mathcal{T}_i$, and $\mathcal{Q}$ the product topology on the same set generated by the topologies $\mathcal{U}_i$. Then $\mathcal{P} \subseteq \mathcal{Q}$ if and only if $\mathcal{T}_i \subseteq \mathcal{U}_i$ for all $i$.*

PROOF:
⟨1⟩1. If $\mathcal{T}_i \subseteq \mathcal{U}_i$ for all $i$ then $\mathcal{P} \subseteq \mathcal{Q}$
PROOF: By Corollary 47.1.
⟨1⟩2. If $\mathcal{P} \subseteq \mathcal{Q}$ then $\mathcal{T}_i \subseteq \mathcal{U}_i$ for all $i$
⟨2⟩1. ASSUME: $\mathcal{P} \subseteq \mathcal{Q}$
⟨2⟩2. LET: $i \in I$
⟨2⟩3. LET: $U \in \mathcal{T}_i$
⟨2⟩4. LET: $U_i = U$ and $U_j = X_j$ for $j \neq i$
⟨2⟩5. $\prod_{i \in I} U_i \in \mathcal{P}$
⟨2⟩6. $\prod_{i \in I} U_i \in \mathcal{Q}$
⟨2⟩7. $U \in \mathcal{U}_i$
PROOF: From Proposition 111.
□

**Proposition 113** (AC)**.** *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let $A_i \subseteq X_i$ for all $i \in I$. Then*

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i} \ .$$

PROOF:
⟨1⟩1. $\overline{\prod_{i \in I} A_i} \subseteq \prod_{i \in I} \overline{A_i}$
⟨2⟩1. For all $i \in I$ we have $A_i \subseteq \overline{A_i}$
PROOF: Lemma 31.

$\langle 2\rangle 2.$ $\prod_{i \in I} A_i \subseteq \prod_{i \in I} \overline{A_i}$
$\langle 2\rangle 3.$ Q.E.D.
PROOF: Since $\prod_{i \in I} A_i$ is closed by Proposition 109.
$\langle 1\rangle 2.$ $\prod_{i \in I} \overline{A_i} \subseteq \overline{\prod_{i \in I} A_i}$
$\langle 2\rangle 1.$ LET: $x \in \prod_{i \in I} \overline{A_i}$
$\langle 2\rangle 2.$ LET: $U$ be a neighbourhood of $x$
$\langle 2\rangle 3.$ PICK $V_i$ open in $X_i$ such that $x \in \prod_{i \in I} V_i \subseteq U$ with $V_i = X_i$ except for $i = i_1, \ldots, i_n$
$\langle 2\rangle 4.$ For $i \in I$, pick $a_i \in V_i \cap A_i$
PROOF: By Theorem 34 and $\langle 2\rangle 1$ using the Axiom of Choice.
$\langle 2\rangle 5.$ $U$ intersects $\prod_{i \in I} A_i$
$\langle 2\rangle 6.$ Q.E.D.
PROOF: $a \in U \cap \prod_{i \in I} A_i$
$\square$

**Example 114.** The closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ is $\mathbb{R}^\omega$

PROOF:
$\langle 1\rangle 1.$ LET: $a \in \mathbb{R}^\omega$
$\langle 1\rangle 2.$ LET: $U$ be any neighbourhoods of $a$.
$\langle 1\rangle 3.$ PICK $U_n$ open in $\mathbb{R}$ for all $n$ such that $a \in \prod_{n \geq 0} U_n \subseteq U$ and $U_n = \mathbb{R}$ for all $n$ except $n_1, \ldots, n_k$
$\langle 1\rangle 4.$ LET: $b_n = a_n$ for $n = n_1, \ldots, n_k$ and $b_n = 0$ for all other $n$
$\langle 1\rangle 5.$ $b \in \mathbb{R}^\infty \cap U$
$\langle 1\rangle 6.$ Q.E.D.
PROOF: From Theorem 34.
$\square$

## 16.1 Continuous in Each Variable Separately

**Definition 115** (Continuous in Each Variable Separately)**.** Let $F : X \times Y \to Z$. Then $F$ is *continuous in each variable separately* if and only if:

- for every $a \in X$ the function $\lambda y \in Y.F(a, y)$ is continuous;
- for every $b \in Y$ the function $\lambda x \in X.F(x, b)$ is continuous.

**Proposition 116.** *Let $F : X \times Y \to Z$. If $F$ is continuous then $F$ is continuous in each variable separately.*

PROOF: For $a \in X$, the function $\lambda y \in Y.F(a, y)$ is $F \circ i$ where $i : Y \to X \times Y$ maps $y$ to $(a, y)$. We have $i$ is continuous by Proposition 100, hence $F \circ i$ is continuous by Theorem 81.
Similarly for $\lambda x \in X.F(x, b)$ for $b \in Y$. $\square$

**Example 117.** Define $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$F(x, y) = \begin{cases} xy/(x^2 + y^2) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Then $F$ is continuous in each variable separately but not continuous.

## 17 The Subspace Topology

**Definition 118** (Subspace Topology)**.** Let $X$ be a topological space and $Y \subseteq X$. The *subspace topology* on $Y$ is $\mathcal{T} = \{U \cap Y \mid U \text{ is open in } X\}$.

We prove this is a topology.

PROOF:
$\langle 1 \rangle$1. $Y \in \mathcal{T}$
PROOF: Since $Y = X \cap Y$
$\langle 1 \rangle$2. For all $\mathcal{U} \subseteq \mathcal{T}$, we have $\bigcup \mathcal{U} \in \mathcal{T}$
$\langle 2 \rangle$1. LET: $\mathcal{U} \subseteq \mathcal{T}$
$\langle 2 \rangle$2. LET: $\mathcal{V} = \{V \text{ open in } X \mid V \cap Y \in \mathcal{U}\}$
$\langle 2 \rangle$3. $\bigcup \mathcal{U} = (\bigcup \mathcal{V}) \cap Y$
$\langle 1 \rangle$3. For all $U, V \in \mathcal{T}$, we have $U \cap V \in \mathcal{T}$
$\langle 2 \rangle$1. LET: $U, V \in \mathcal{T}$
$\langle 2 \rangle$2. PICK $U'$, $V'$ open in $X$ such that $U = U' \cap Y$ and $V = V' \cap Y$
$\langle 2 \rangle$3. $(U \cap V) = (U' \cap V') \cap Y$
$\square$

**Theorem 119.** *Let $X$ be a topological space and $Y$ a subspace of $X$. Let $A \subseteq Y$. Then $A$ is closed in $Y$ if and only if there exists a closed set $C$ in $X$ such that $A = C \cap Y$.*

PROOF: We have

$$\begin{aligned}
& A \text{ is closed in } Y \\
\Leftrightarrow & Y \setminus A \text{ is open in } Y \\
\Leftrightarrow & \exists U \text{ open in } X . Y \setminus A = Y \cap U \\
\Leftrightarrow & \exists C \text{ closed in } X . Y \setminus A = Y \cap (X \setminus U) \\
\Leftrightarrow & \exists C \text{ closed in } X . A = Y \cap U
\end{aligned}$$
$\square$

**Theorem 120.** *Let $Y$ be a subspace of $X$. Let $A \subseteq Y$. Let $\overline{A}$ be the closure of $A$ in $X$. Then the closure of $A$ in $Y$ is $\overline{A} \cap Y$.*

PROOF: The closure of $A$ in $Y$ is

$$\begin{aligned}
& \bigcap \{C \text{ closed in } Y \mid A \subseteq C\} \\
= & \bigcap \{D \cap Y \mid D \text{ closed in } X, A \subseteq D \cap Y\} && \text{(Theorem 119)} \\
= & \bigcap \{D \mid D \text{ closed in } X, A \subseteq D\} \cap Y \\
= & \overline{A} \cap Y
\end{aligned}$$
$\square$

**Lemma 121.** *Let $X$ be a topological space and $Y \subseteq X$. Let $\mathcal{B}$ be a basis for the topology on $X$. Then $\mathcal{B}' = \{B \cap Y \mid B \in \mathcal{B}\}$ is a basis for the subspace topology on $Y$.*

PROOF:
$\langle 1 \rangle$1. Every element in $\mathcal{B}'$ is open in $Y$

⟨1⟩2. For every open set $U$ in $Y$ and point $y \in U$, there exists $B' \in \mathcal{B}'$ such that $y \in B' \subseteq U$

⟨2⟩1. LET: $U$ be open in $Y$ and $y \in U$

⟨2⟩2. PICK $V$ open in $X$ such that $U = V \cap Y$

⟨2⟩3. PICK $B \in \mathcal{B}$ such that $y \in B \subseteq V$

⟨2⟩4. LET: $B' = B \cap Y$

⟨2⟩5. $B' \in \mathcal{B}'$

⟨2⟩6. $y \in B' \subseteq U$

⟨1⟩3. Q.E.D.

PROOF: By Lemma 48.

□

**Lemma 122.** *Let $X$ be a topological space and $Y \subseteq X$. Let $\mathcal{S}$ be a basis for the topology on $X$. Then $\mathcal{S}' = \{S \cap Y \mid S \in \mathcal{S}\}$ is a subbasis for the subspace topology on $Y$.*

PROOF: The set $\{B \cap Y \mid B$ is a finite intersection of elements of $\mathcal{S}\}$ is a basis for the subspace topology by Lemma 121, and this is the set of all finite intersections of elements of $\mathcal{S}'$. □

**Lemma 123.** *Let $Y$ be a subspace of $X$. If $U$ is open in $Y$ and $Y$ is open in $X$ then $U$ is open in $X$.*

PROOF:

⟨1⟩1. PICK $V$ open in $X$ such that $U = V \cap Y$

⟨1⟩2. $U$ is open in $X$

PROOF: Since it is the intersection of two open sets $V$ and $Y$.

□

**Theorem 124.** *Let $Y$ be a subspace of $X$ and $A \subseteq Y$. If $A$ is closed in $Y$ and $Y$ is closed in $X$ then $A$ is closed in $X$.*

PROOF: Pick a closed set $C$ in $X$ such that $A = C \cap Y$ (Theorem 119). Then $A$ is the intersection of two sets closed in $X$, hence $A$ is closed in $X$ (Lemma 19). □

**Theorem 125.** *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let $A_i$ be a subspace of $X_i$ for all $i \in I$. Then the product topology on $\prod_{i \in I} A_i$ is the same as the topology it inherits as a subspace of $\prod_{i \in I} X_i$.*

PROOF: The product topology is generated by the subbasis

$$\begin{aligned} &\{{\pi_i}^{-1}(U) \mid i \in I, U \text{ open in } A_i\} \\ =&\{{\pi_i}^{-1}(V) \cap A_i \mid i \in I, V \text{ open in } X_i\} \\ =&\{{\pi_i}^{-1}(V) \mid i \in I, V \text{ open in } X_i\} \cap \prod_{i \in I} A_i \end{aligned}$$

and this is a subbasis for the subspace topology by Lemma 122. □

**Theorem 126.** *Let $X$ be an ordered set in the order topology. Let $Y \subseteq X$ be an interval. Then the order topology on $Y$ is the same as the subspace topology on $Y$.*

PROOF:
$\langle 1 \rangle 1$. The order topology is finer than the subspace topology.
$\langle 2 \rangle 1$. For every open ray $R$ in $X$, the set $R \cap Y$ is open in the order topology.
$\langle 3 \rangle 1$. For all $a \in X$, we have $(-\infty, a) \cap Y$ is open in the order topology.
$\langle 4 \rangle 1$. CASE: For all $y \in Y$ we have $y < a$
PROOF: In this case $(-\infty, a) \cap Y = Y$.
$\langle 4 \rangle 2$. CASE: For all $y \in Y$ we have $a < y$
PROOF: In this case $(-\infty, a) \cap Y = \emptyset$.
$\langle 4 \rangle 3$. CASE: There exists $y \in Y$ such that $y \leq a$ and $y \in Y$ such that $a \leq y$
$\langle 5 \rangle 1$. $a \in Y$
PROOF: Because $Y$ is an interval.
$\langle 5 \rangle 2$. $(-\infty, a) \cap Y = \{y \in Y \mid y < a\}$
$\langle 3 \rangle 2$. For all $a \in X$, we have $(a, +\infty) \cap Y$ is open in the order topology.
PROOF: Similar.
$\langle 2 \rangle 2$. Q.E.D.
PROOF: By Lemmas 102 and 122 and Proposition 56.
$\langle 1 \rangle 2$. The subspace topology is finer than the order topology.
$\langle 2 \rangle 1$. Every open ray in $Y$ is open in the subspace topology.
PROOF: For any $a \in Y$ we have $(-\infty, a)_Y = (-\infty, a)_X \cap Y$ and $(a, +\infty)_Y = (a, +\infty)_X \cap Y$.
$\langle 2 \rangle 2$. Q.E.D.
PROOF: By Lemma 102 and Proposition 56
$\square$

This example shows that we cannot remove the hypothesis that $Y$ is an interval:

**Example 127.** The order topology on $I_o^2$ is different from the subspace topology as a subspace of $\mathbb{R}^2$ under the dictionary order topology.

PROOF: The set $\{1/2\} \times (1/2, 1)$ is open in the subspace topology but not in the order topology. $\square$

**Proposition 128.** *Let $X$ be a topological space, $Y$ a subspace of $X$, and $Z$ a subspace of $Y$. Then the subspace topology on $Z$ inherited from $X$ is the same as the subspace topology on $Z$ inherited from $Y$.*

PROOF: The subspace topology inherited from $Y$ is

$$\begin{aligned} & \{V \cap Z \mid V \text{ open in } Y\} \\ = & \{U \cap Y \cap Z \mid U \text{ open in } X\} \\ = & \{U \cap Z \mid U \text{ open in } X\} \end{aligned}$$

which is the subspace topology inherited from $X$. $\square$

**Definition 129** (Unit Circle)**.** The *unit circle* $S^1$ is $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$

# 18 The Box Topology

**Definition 130** (Box Topology)**.** Let $\{A_i\}_{i \in I}$ be a family of topological spaces. The *box topology* on $\prod_{i \in I} A_i$ is the topology generated by the set of all sets of the form $\prod_{i \in I} U_i$ where $\{U_i\}_{i \in I}$ is a family such that each $U_i$ is open in $A_i$.

This is a basis since it covers $\prod_{i \in I} A_i$ and is closed under intersection.

**Proposition 131.** *The box topology is finer than the product topology.*

Proof: From Proposition 108. $\square$

**Corollary 131.1.** *If $A_i$ is closed in $X_i$ for all $i \in I$ then $\prod_{i \in I} A_i$ is closed in $\prod_{i \in I} X_i$ under the box topology.*

Proof: From Proposition 109.

**Proposition 132** (AC)**.** *Let $\{A_i\}_{i \in I}$ be a family of topological spaces. For $i \in I$, let $\mathcal{B}_i$ be a basis for the topology on $A_i$. Then $\mathcal{B} = \{\prod_{i \in I} B_i \mid \forall i \in I . B_i \in \mathcal{B}_i\}$ is a basis for the box topology on $\prod_{i \in I} A_i$.*

Proof:
⟨1⟩1. Every set of the form $\prod_{i \in I} B_i$ is open.
⟨1⟩2. For every point $a \in \prod_{i \in I} A_i$ and every open set $U$ with $a \in U$, there exists $B \in \mathcal{B}$ such that $a \in B \subseteq U$.
  ⟨2⟩1. Let: $U$ be open and $a \in U$
  ⟨2⟩2. Pick a family $\{U_i\}_{i \in I}$ such that each $U_i$ is open in $A_i$ and $a \in \prod_{i \in I} U_i \subseteq U$.
  ⟨2⟩3. For $i \in I$, Pick $B_i \in \mathcal{B}_i$ such that $a_i \in B_i \subseteq U_i$
    Proof: Using the Axiom of Choice.
  ⟨2⟩4. $a \in \prod_{i \in I} B_i \subseteq U$
⟨1⟩3. Q.E.D.
  Proof: Lemma 48.
$\square$

**Theorem 133.** *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let $A_i$ be a subspace of $X_i$ for all $i \in I$. Give $\prod_{i \in I} X_i$ the box topology. Then the box topology on $\prod_{i \in I} A_i$ is the same as the topology it inherits as a subspace of $\prod_{i \in I} X_i$.*

Proof: The box topology is generated by the basis

$$\begin{aligned} & \{\prod_{i \in I} U_i \mid \forall i \in I, U_i \text{ open in } A_i\} \\ = & \{\prod_{i \in I} (V_i \cap A_i) \mid \forall i \in I, V_i \text{ open in } X_i\} \\ = & \{\prod_{i \in I} V_i \mid \forall i \in I, V_i \text{ open in } X_i\} \cap \prod_{i \in I} A_i \end{aligned}$$

and this is a basis for the subspace topology by Lemma 121. $\square$

**Proposition 134.** *Let $\{X_i\}_{i \in I}$ be a family of Hausdorff spaces. Then $\prod_{i \in I} X_i$ under the box topology is Hausdorff.*

PROOF:
$\langle 1 \rangle 1$. LET: $\{X_i\}_{i \in I}$ be a family of Hausdorff spaces.
$\langle 1 \rangle 2$. LET: $a, b \in \prod_{i \in I} X_i$ with $a \neq b$
$\langle 1 \rangle 3$. PICK $i \in I$ such that $a_i \neq b_i$
$\langle 1 \rangle 4$. PICK $U$, $V$ disjoint open sets in $X_i$ with $a_i \in U$ and $b_i \in V$
$\langle 1 \rangle 5$. ${\pi_i}^{-1}(U)$ and ${\pi_i}^{-1}(V)$ are disjoint open sets in $\prod_{i \in I} X_i$ with $a \in {\pi_i}^{-1}(U)$ and $b \in {\pi_i}^{-1}(V)$
□

**Proposition 135** (AC)**.** *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Give $\prod_{i \in I} X_i$ the box topology. Let $A_i \subseteq X_i$ for all $i \in I$. Then*

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i} \enspace .$$

PROOF:
$\langle 1 \rangle 1$. $\overline{\prod_{i \in I} A_i} \subseteq \prod_{i \in I} \overline{A_i}$
  $\langle 2 \rangle 1$. For all $i \in I$ we have $A_i \subseteq \overline{A_i}$
    PROOF: Lemma 31.
  $\langle 2 \rangle 2$. $\prod_{i \in I} A_i \subseteq \prod_{i \in I} \overline{A_i}$
  $\langle 2 \rangle 3$. Q.E.D.
    PROOF: Since $\prod_{i \in I} A_i$ is closed by Corollary 131.1.
$\langle 1 \rangle 2$. $\prod_{i \in I} \overline{A_i} \subseteq \overline{\prod_{i \in I} A_i}$
  $\langle 2 \rangle 1$. LET: $x \in \prod_{i \in I} \overline{A_i}$
  $\langle 2 \rangle 2$. LET: $U$ be a neighbourhood of $x$
  $\langle 2 \rangle 3$. PICK $V_i$ open in $X_i$ such that $x \in \prod_{i \in I} V_i \subseteq U$
  $\langle 2 \rangle 4$. For $i \in I$, pick $a_i \in V_i \cap A_i$
    PROOF: By Theorem 34 and $\langle 2 \rangle 1$ using the Axiom of Choice.
  $\langle 2 \rangle 5$. $U$ intersects $\prod_{i \in I} A_i$
  $\langle 2 \rangle 6$. Q.E.D.
    PROOF: $a \in U \cap \prod_{i \in I} A_i$.
□

The following example shows that Theorem 86 fails in the box topology.

**Example 136.** Define $f : \mathbb{R} \to \mathbb{R}^\omega$ by $f(t) = (t, t, \ldots)$. Then $\pi_n \circ f = \mathrm{id}_{\mathbb{R}}$ is continuous for all $n$. But $f$ is not continuous when $\mathbb{R}^\omega$ is given the box topology because the inverse image of

$$(-1, 1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times \cdots$$

is $\{0\}$ which is not open.

The following example shows that Proposition 94 fails in the box topology.

**Example 137.** Give $\mathbb{R}^\omega$ the box topology. Let $a_n = (1/n, 1/n, \ldots)$ for $n \geq 1$ and $l = (0, 0, \ldots)$. Then $\pi_i(a_n) \to \pi_i(l)$ as $n \to \infty$ for all $i$, but $a_n \not\to l$ as $n \to \infty$ since the open set

$$(-1, 1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times \cdots$$

contains $l$ but does not contain any $a_n$.

**Example 138.** The set $\mathbb{R}^\infty$ is closed in $\mathbb{R}^\omega$ under the box topology. For let $(a_n)$ be any sequence not in $\mathbb{R}^\infty$. Let $U_n$ be an open interval around $a_n$ that does not contain 0 if $a_n \neq 0$, and $U_n = \mathbb{R}$ if $a_n = 0$. Then $\prod_{n \geq 0} U_n$ is a neighbourhood of $(a_n)$ that does not intersect $\mathbb{R}^\infty$.

# 19 $T_1$ Spaces

**Definition 139** ($T_1$ Space)**.** A topological space is $T_1$ if and only if every singleton is closed.

**Lemma 140.** *A space is $T_1$ if and only if every finite set is closed.*

Proof: From Lemma 20. □

**Theorem 141.** *In a $T_1$ space, a point $a$ is a limit point of a set $A$ if and only if every neighbourhood of $a$ contains infinitely many points of $A$.*

Proof:
$\langle 1 \rangle 1$. If $a$ is a limit point of $A$ then every neighbourhood of $a$ contains infinitely many points of $A$.
  $\langle 2 \rangle 1$. Assume: $a$ is a limit point of $A$.
  $\langle 2 \rangle 2$. Let: $U$ be a neighbourhood of $a$.
  $\langle 2 \rangle 3$. Assume: for a contradiction $U$ contains only finitely many points of $A$.
  $\langle 2 \rangle 4$. $(U \cap A) \setminus \{a\}$ is closed.
    Proof: By the $T_1$ axiom.
  $\langle 2 \rangle 5$. $(U \setminus A) \cup \{a\}$ is open.
    Proof: It is $U \setminus ((U \cap A) \setminus \{a\})$.
  $\langle 2 \rangle 6$. $(U \setminus A) \cup \{a\}$ intersects $A$ in a point other than $a$.
    Proof: From $\langle 2 \rangle 1$.
  $\langle 2 \rangle 7$. Q.E.D.
  □
$\langle 1 \rangle 2$. If every neighbourhood of $a$ contains infinitely many points of $A$ then $a$ is a limit point of $A$.
  Proof: Immediate from definitions.
□

(To see this does not hold in every space, see Proposition 45.)

**Proposition 142.** *A space is $T_1$ if and only if, for any two distinct points $x$ and $y$, there exist neighbourhoods $U$ of $x$ and $V$ of $y$ such that $x \notin V$ and $y \notin U$.*

PROOF:
⟨1⟩1. LET: $X$ be a topological space.
⟨1⟩2. If $X$ is $T_1$ then, for any two distinct points $x$ and $y$, there exist neighbourhoods $U$ of $x$ and $V$ of $y$ such that $x \notin V$ and $y \notin U$.
PROOF: This holds because $\{x\}$ and $\{y\}$ are closed.
⟨1⟩3. Suppose, for any two distinct points $x$ and $y$, there exist neighbourhoods $U$ of $x$ and $V$ of $y$ such that $x \notin V$ and $y \notin U$. Then $X$ is $T_1$.
⟨2⟩1. ASSUME: For any two distinct points $x$ and $y$, there exist neighbourhoods $U$ of $x$ and $V$ of $y$ such that $x \notin V$ and $y \notin U$.
⟨2⟩2. LET: $a \in X$
⟨2⟩3. $\{a\}$ is closed.
PROOF: For all $b \neq a$ there exists a neighbourhood $U$ of $b$ such that $U \subseteq X \setminus \{a\}$.
□

# 20 Hausdorff Spaces

**Definition 143** (Hausdorff Space)**.** A topological space is *Hausdorff* if and only if, for any points $x$, $y$ with $x \neq y$, there exist disjoint open sets $U$ and $V$ such that $x \in U$ and $y \in V$.

**Theorem 144.** *Every Hausdorff space is $T_1$.*

PROOF:
⟨1⟩1. LET: $X$ be a Hausdorff space.
⟨1⟩2. LET: $b \in X$
PROVE: $\overline{\{b\}} = \{b\}$
⟨1⟩3. ASSUME: $a \in \overline{\{b\}}$ and $a \neq b$
⟨1⟩4. PICK disjoint neighbourhoods $U$ of $a$ and $V$ of $b$.
⟨1⟩5. $U$ intersects $\{b\}$
PROOF: Theorem 34.
⟨1⟩6. $b \in U$
⟨1⟩7. Q.E.D.
PROOF: This contradicts the fact that $U$ and $V$ are disjoint (⟨1⟩4).
□

**Proposition 145.** *An infinite set under the finite complement topology is $T_1$ but not Hausdorff.*

PROOF:
⟨1⟩1. LET: $X$ be an infinite set under the finite complement topology.
⟨1⟩2. Every singleton is closed.
PROOF: By definition.
⟨1⟩3. PICK$a, b \in X$ with $a \neq b$
⟨1⟩4. There are no disjoint neighbourhoods $U$ of $a$ and $V$ of $b$.
⟨2⟩1. LET: $U$ be a neighbourhood of $a$ and $V$ a neighbourhood of $b$.
⟨2⟩2. $X \setminus U$ and $X \setminus V$ are finite.

$\langle 2\rangle 3.$ Pick $c \in X$ that is not in $X \setminus U$ or $X \setminus V$.
$\langle 2\rangle 4.$ $c \in U \cap V$
$\square$

**Proposition 146.** *The product of a family of Hausdorff spaces is Hausdorff.*

Proof:
$\langle 1\rangle 1.$ Let: $\{X_i\}_{i \in I}$ be a family of Hausdorff spaces.
$\langle 1\rangle 2.$ Let: $a, b \in \prod_{i \in I} X_i$ with $a \neq b$
$\langle 1\rangle 3.$ Pick $i \in I$ such that $a_i \neq b_i$
$\langle 1\rangle 4.$ Pick $U$, $V$ disjoint open sets in $X_i$ with $a_i \in U$ and $b_i \in V$
$\langle 1\rangle 5.$ ${\pi_i}^{-1}(U)$ and ${\pi_i}^{-1}(V)$ are disjoint open sets in $\prod_{i \in I} X_i$ with $a \in {\pi_i}^{-1}(U)$ and $b \in {\pi_i}^{-1}(V)$
$\square$

**Theorem 147.** *Every linearly ordered set under the order topology is Hausdorff.*

Proof:
$\langle 1\rangle 1.$ Let: $X$ be a linearly ordered set under the order topology.
$\langle 1\rangle 2.$ Let: $a, b \in X$ with $a \neq b$
$\langle 1\rangle 3.$ Assume: w.l.o.g. $a < b$
$\langle 1\rangle 4.$ Case: There exists $c$ such that $a < c < b$
Proof: The sets $(-\infty, c)$ and $(c, +\infty)$ are disjoint neighbourhoods of $a$ and $b$ respectively.
$\langle 1\rangle 5.$ Case: There is no $c$ such that $a < c < b$
Proof: The sets $(-\infty, b)$ and $(a, +\infty)$ are disjoint neighbourhoods of $a$ and $b$ respectively.
$\square$

**Theorem 148.** *A subspace of a Hausdorff space is Hausdorff.*

Proof:
$\langle 1\rangle 1.$ Let: $X$ be a Hausdorff space and $Y$ a subspace of $X$.
$\langle 1\rangle 2.$ Let: $x, y \in Y$ with $x \neq y$
$\langle 1\rangle 3.$ Pick disjoint neighbourhoods $U$ of $x$ and $V$ of $y$ in $X$.
$\langle 1\rangle 4.$ $U \cap Y$ and $V \cap Y$ are disjoint neighbourhoods of $x$ and $y$ respectively in $Y$.
$\square$

**Proposition 149.** *A space $X$ is Hausdorff if and only if the diagonal $\Delta = \{(x, x) \mid x \in X\}$ is closed in $X^2$.*

Proof:
$$\begin{aligned}
& X \text{ is Hausdorff} \\
\Leftrightarrow & \forall x, y \in X. x \neq y \Rightarrow \exists V, W \text{ open}. x \in V \wedge y \in W \wedge V \cap W = \emptyset \\
\Leftrightarrow & \forall (x, y) \in X^2 \setminus \Delta . \exists V, W \text{ open}. (x, y) \subseteq V \times W \subseteq X^2 \setminus \Delta \\
\Leftrightarrow & \Delta \text{ is closed}
\end{aligned}$$
$\square$

## 21 The Metric Topology

**Definition 150** (Metric)**.** Let $X$ be a set. A *metric* on $X$ is a function $d : X^2 \to \mathbb{R}$ such that:

1. For all $x, y \in X$, $d(x, y) \geq 0$
2. For all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$
3. For all $x, y \in X$, $d(x, y) = d(y, x)$
4. (*Triangle Inequality*) For all $x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$

We call $d(x, y)$ the *distance* between $x$ and $y$.

**Definition 151** (Open Ball)**.** Let $X$ be a metric space. Let $a \in X$ and $\epsilon > 0$. The *open ball* with *centre* $a$ and *radius* $\epsilon$ is

$$B(a, \epsilon) = \{x \in X \mid d(a, x) < \epsilon\} \ .$$

**Definition 152** (Metric Topology)**.** Let $X$ be a metric space. The *metric topology* on $X$ is the topology generated by the basis consisting of all the open balls.

We prove this is a basis for a topology.

PROOF:
⟨1⟩1. For every point $a$, there exists a ball $B$ such that $a \in B$
PROOF: We have $a \in B(a, 1)$.
⟨1⟩2. For any balls $B_1$, $B_2$ and point $a \in B_1 \cap B_2$, there exists a ball $B_3$ such that $a \in B_3 \subseteq B_1 \cap B_2$
⟨2⟩1. LET: $B_1 = B(c_1, \epsilon_1)$ and $B_2 = B(c_2, \epsilon_2)$
⟨2⟩2. LET: $\delta = \min(\epsilon_1 - d(c_1, a), \epsilon_2 - d(c_2, a))$
PROVE: $B(a, \delta) \subseteq B_1 \cap B_2$
⟨2⟩3. LET: $x \in B(a, \delta)$
⟨2⟩4. $x \in B_1$
PROOF:

$$\begin{aligned} d(x, c_1) &= d(x, a) + d(a, c_1) \\ &< \delta + d(a, c_1) \\ &\leq \epsilon_1 \end{aligned}$$

⟨2⟩5. $x \in B_2$
PROOF: Similar.
□

**Proposition 153.** *Let $X$ be a metric space and $U \subseteq X$. Then $U$ is open if and only if, for every $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$.*

PROOF:
⟨1⟩1. If $U$ is open then, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$.
⟨2⟩1. ASSUME: $U$ is open.

$\langle 2 \rangle 2$. LET: $x \in U$
$\langle 2 \rangle 3$. PICK $a \in X$ and $\delta > 0$ such that $x \in B(a, \delta) \subseteq U$
$\langle 2 \rangle 4$. LET: $\epsilon = \delta - d(a, x)$
PROVE: $B(x, \epsilon) \subseteq U$
$\langle 2 \rangle 5$. LET: $y \in B(x, \epsilon)$
$\langle 2 \rangle 6$. $d(y, a) < \delta$
PROOF:

$$\begin{aligned} d(y,a) &\leq d(a,x) + d(x,y) \\ &< \delta + d(x,y) \\ &= \epsilon \end{aligned}$$

$\langle 2 \rangle 7$. $y \in U$
$\langle 1 \rangle 2$. If, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$, then $U$ is open.
PROOF: Immediate from definitions.
□

**Definition 154** (Discrete Metric)**.** Let $X$ be a set. The *discrete metric* on $X$ is defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

**Proposition 155.** *The discrete metric induces the discrete topology.*

PROOF: For any (open) set $U$ and point $a \in U$, we have $a \in B(a, 1) \subseteq U$. □

**Definition 156** (Standard Metric on $\mathbb{R}$)**.** The *standard metric* on $\mathbb{R}$ is defined by $d(x, y) = |x - y|$.

**Proposition 157.** *The standard metric on $\mathbb{R}$ induces the standard topology on $\mathbb{R}$.*

PROOF:
$\langle 1 \rangle 1$. Every open ball is open in the standard topology on $\mathbb{R}$.
PROOF: $B(a, \epsilon) = (a - \epsilon, a + \epsilon)$
$\langle 1 \rangle 2$. For every open set $U$ and point $a \in U$, there exists $\epsilon > 0$ such that $B(a, \epsilon) \subseteq U$
$\langle 2 \rangle 1$. LET: $U$ be an open set and $a \in U$
$\langle 2 \rangle 2$. PICK an open interval $b$, $c$ such that $a \in (b, c) \subseteq U$
$\langle 2 \rangle 3$. LET: $\epsilon = \min(a - b, c - a)$
$\langle 2 \rangle 4$. $B(a, \epsilon) \subseteq U$
□

**Definition 158** (Metrizable)**.** A topological space $X$ is *metrizable* if and only if there exists a metric on $X$ that induces the topology.

**Definition 159** (Bounded)**.** Let $X$ be a metric space and $A \subseteq X$. Then $A$ is *bounded* if and only if there exists $M$ such that, for all $x, y \in A$, we have $d(x, y) \leq M$.

**Definition 160** (Diameter)**.** Let $X$ be a metric space and $A \subseteq X$. The *diameter* of $A$ is

$$\operatorname{diam} A = \sup_{x,y \in A} d(x,y) \ .$$

**Definition 161** (Standard Bounded Metric)**.** Let $d$ be a metric on $X$. The *standard bounded metric* corresponding to $d$ is the metric $\overline{d}$ defined by

$$\overline{d}(x,y) = \min(d(x,y), 1) \ .$$

We prove this is a metric.

Proof:
$\langle 1 \rangle 1.$ $\overline{d}(x,y) \geq 0$
  Proof: Since $d(x,y) \geq 0$
$\langle 1 \rangle 2.$ $\overline{d}(x,y) = 0$ if and only if $x = y$
  Proof: $\overline{d}(x,y) = 0$ if and only if $d(x,y) = 0$ if and only if $x = y$
$\langle 1 \rangle 3.$ $\overline{d}(x,y) = \overline{d}(y,x)$
  Proof: Since $d(x,y) = d(y,x)$
$\langle 1 \rangle 4.$ $\overline{d}(x,z) \leq \overline{d}(x,y) + \overline{d}(y,z)$
  Proof:

$$\begin{aligned}
\overline{d}(x,y) + \overline{d}(y,z) &= \min(d(x,y), 1) + \min(d(y,z), 1) \\
&= \min(d(x,y) + d(y,z), d(x,y) + 1, d(y,z) + 1, 2) \\
&\geq \min(d(x,z), 1) \\
&= \overline{d}(x,z)
\end{aligned}$$

□

**Lemma 162.** *In any metric space $X$, the set $\mathcal{B} = \{B(a,\epsilon) \mid a \in X, \epsilon < 1\}$ is a basis for the metric topology.*

Proof:
$\langle 1 \rangle 1.$ Every element of $\mathcal{B}$ is open.
  Proof: From Lemma 47.
$\langle 1 \rangle 2.$ For every open set $U$ and point $a \in U$, there exists $B \in \mathcal{B}$ such that $a \in B \subseteq U$
  $\langle 2 \rangle 1.$ Let: $U$ be an open set and $a \in U$
  $\langle 2 \rangle 2.$ Pick$\epsilon > 0$ such that $B(a,\epsilon) \subseteq U$
  $\langle 2 \rangle 3.$ $B(a, \min(\epsilon, 1/2)) \subseteq U$
$\langle 1 \rangle 3.$ Q.E.D.
  Proof: Lemma 48.

□

**Proposition 163.** *Let $d$ be a metric on the set $X$. Then the standard bounded metric $\overline{d}$ induces the same metric as $d$.*

Proof: This follows from Lemma 162 since the open balls with radius $< 1$ are the same under both metrics. □

**Lemma 164.** *Let $d$ and $d'$ be two metrics on the same set $X$. Let $\mathcal{T}$ and $\mathcal{T}'$ be the topologies they induce. Then $\mathcal{T} \subseteq \mathcal{T}'$ if and only if, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that*

$$B_{d'}(x, \delta) \subseteq B_d(x, \epsilon) \ .$$

PROOF:
⟨1⟩1. If $\mathcal{T} \subseteq \mathcal{T}'$ then, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$
  PROOF: From Proposition 153 since $x \in B_d(x, \epsilon) \in \mathcal{T}'$.
⟨1⟩2. If, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$, then $\mathcal{T} \subseteq \mathcal{T}'$
  ⟨2⟩1. ASSUME: For all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$
  ⟨2⟩2. LET: $U \in \mathcal{T}$
  ⟨2⟩3. For all $x \in U$ there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq U$.
    ⟨3⟩1. LET: $x \in U$
    ⟨3⟩2. PICK $\epsilon > 0$ such that $B_d(x, \epsilon) \subseteq U$
      PROOF: Proposition 153
    ⟨3⟩3. PICK $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$
      PROOF: By ⟨2⟩1
    ⟨3⟩4. $B_{d'}(x, \delta) \subseteq U$
  ⟨2⟩4. $U \in \mathcal{T}'$
    PROOF: Proposition 153.
□

**Proposition 165.** *$\mathbb{R}^2$ under the dictionary order topology is metrizable.*

PROOF: Define $d : \mathbb{R}^2 \to \mathbb{R}$ by

$$d((x, y), (x, z)) = \max(|y - z|, 1)$$

$$d((x, y), (x', y')) = 1 \qquad \text{if } x \neq x'$$ □

⟨1⟩1. $x \in \bigcap_{i=1}^N {\pi_i}^{-1}() \subseteq B_D(a, \epsilon)$

**Proposition 166.** *Let $d : X^2 \to \mathbb{R}$ be a metric on $X$. Then the metric topology on $X$ is the coarsest topology such that $d$ is continuous.*

PROOF:
⟨1⟩1. $d$ is continuous in each variable separately.
  ⟨2⟩1. LET: $a \in X$ and $d_a : X \to \mathbb{R}$ be the function $d(a, -)$
  ⟨2⟩2. LET: $b \in X$ and $\epsilon > 0$
  ⟨2⟩3. For all $x \in X$, if $d(b, x) < \epsilon$ then $|d(a, b) - d(a, x)| < \epsilon$
⟨1⟩2. If $\mathcal{T}$ is any topology under which $d$ is continuous then $\mathcal{T}$ is finer than the metric topology.
  PROOF: Since $B(a, \epsilon) = {d_a}^{-1}((-\infty, \epsilon))$
□

**Proposition 167.** *Let $X$ be a metric space with metric $d$ and $A \subseteq X$. The restriction of $d$ to $A$ is a metric on $A$ that induces the subspace topology.*

PROOF:
$\langle 1\rangle 1.$ The restriction of $d$ to $A$ is a metric on $A$.
$\langle 1\rangle 2.$ Every open ball under $d \upharpoonright A$ is open under the subspace topology.
PROOF: $B_{d \upharpoonright A}(a, \epsilon) = B_d(a, \epsilon) \cap A$.
$\langle 1\rangle 3.$ If $U$ is open in the subspace topology and $x \in U$, then there exists a $d \upharpoonright A$-ball $B$ such that $x \in B \subseteq U$.
$\langle 2\rangle 1.$ PICK $V$ open in $X$ such that $U = V \cap A$
$\langle 2\rangle 2.$ PICK $\epsilon > 0$ such that $B_d(x, \epsilon) \subseteq V$
$\langle 2\rangle 3.$ Take $B = B_{d \upharpoonright A}(x, \epsilon)$
$\square$

**Corollary 167.1.** *A subspace of a metrizable space is metrizable.*

**Proposition 168.** *Every metrizable space is Hausdorff.*

PROOF:
$\langle 1\rangle 1.$ LET: $X$ be a metric space
$\langle 1\rangle 2.$ LET: $a, b \in X$ with $a \neq b$
$\langle 1\rangle 3.$ LET: $\epsilon = d(a, b)/2$
$\langle 1\rangle 4.$ LET: $U = B(a, \epsilon)$ and $V = B(b, \epsilon)$
$\langle 1\rangle 5.$ $U$ and $V$ are disjoint neighbourhoods of $a$ and $b$ respectively.
$\square$

**Proposition 169** (CC)**.** *The product of a countable family of metrizable spaces is metrizable.*

PROOF:
$\langle 1\rangle 1.$ LET: $(X_n, d_n)$ be a sequence of metric spaces.
$\langle 1\rangle 2.$ ASSUME: w.l.o.g. each $d_n$ is bounded above by 1.
PROOF: By Proposition 163.
$\langle 1\rangle 3.$ LET: $D$ be the metric on $\mathbb{R}^\omega$ defined by $D(x, y) = \sup_i (d_i(x_i, y_i)/i)$.
$\langle 2\rangle 1.$ $D(x, y) \geq 0$
$\langle 2\rangle 2.$ $D(x, y) = 0$ if and only if $x = y$
$\langle 2\rangle 3.$ $D(x, y) = D(y, x)$
$\langle 2\rangle 4.$ $D(x, z) \leq D(x, y) + D(y, z)$
PROOF:
$$\begin{aligned} D(x, z) &= \sup_i \frac{d_i(x_i, z_i)}{i} \\ &\leq \sup_i \frac{d_i(x_i, y_i) + d_i(y_i, z_i)}{i} \\ &\leq \sup_i \frac{d_i(x_i, y_i)}{i} + \sup_i \frac{d_i(y_i, z_i)}{i} \\ &= D(x, y) + D(y, z) \end{aligned}$$
$\langle 1\rangle 4.$ Every open ball $B_D(a, \epsilon)$ is open in the product topology.
$\langle 2\rangle 1.$ PICK $N$ such that $1/\epsilon < N$
$\langle 2\rangle 2.$ $B_D(a, \epsilon) = \prod_{i=1}^\infty U_i$ where $U_i = B(a_i, i\epsilon)$ if $i \leq N$, and $U_i = X_i$ if $i > N$
$\langle 1\rangle 5.$ For any open set $U$ and $a \in U$, there exists $\epsilon > 0$ such that $B_D(a, \epsilon) \subseteq U$.

⟨2⟩1. LET: $n \geq 1$, $V$ be an open set in $\mathbb{R}$ and $a \in {\pi_n}^{-1}(V)$
⟨2⟩2. PICK $\epsilon > 0$ such that $B_{d_n}(a, \epsilon) \subseteq V$
⟨2⟩3. $B_D(a, \epsilon/n) \subseteq {\pi_n}^{-1}(V)$
□

**Theorem 170.** *Let $X$ and $Y$ be metric spaces and $f : X \to Y$. Then $f$ is continuous if and only if, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$.*

PROOF:
⟨1⟩1. If $f$ is continuous then, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$
⟨2⟩1. ASSUME: $f$ is continuous.
⟨2⟩2. LET: $x \in X$ and $\epsilon > 0$
⟨2⟩3. PICK a neighbourhood $U$ of $x$ such that $f(U) \subseteq B(f(x), \epsilon)$
PROOF: Theorem 78.
⟨2⟩4. PICK $\delta > 0$ such that $B(x, \delta) \subseteq U$
PROOF: Proposition 153.
⟨2⟩5. For all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$
⟨1⟩2. If for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$, then $f$ is continuous.
⟨2⟩1. ASSUME: for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$
⟨2⟩2. LET: $x \in X$ and $V$ be a neighbourhood of $f(x)$
⟨2⟩3. PICK $\epsilon > 0$ such that $B(f(x), \epsilon) \subseteq V$
PROOF: Proposition 153.
⟨2⟩4. PICK $\delta > 0$ such that, for all $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$
PROOF: By ⟨2⟩1
⟨2⟩5. LET: $U = B(x, \delta)$
⟨2⟩6. $U$ is a neighbourhood of $x$ with $f(U) \subseteq V$
⟨2⟩7. Q.E.D.
PROOF: Theorem 78.
□

**Proposition 171.** *Let $X$ be a metric space. Let $(a_n)$ be a sequence in $X$ and $l \in X$. Then $a_n \to l$ as $n \to \infty$ if and only if, for all $\epsilon > 0$, there exists $N$ such that, for all $n \geq N$, we have $d(a_n, l) < \epsilon$.*

PROOF: From Proposition 61. □

**Lemma 172** (Sequence Lemma (CC)). *Let $X$ be a metrizable space. Let $A \subseteq X$ and $l \in \overline{A}$. Then there exists a sequence in $A$ that converges to $l$.*

PROOF:
⟨1⟩1. For all $n \geq 1$, PICK $a_n \in A \cap B(l, 1/n)$
PROVE: $a_n \to l$ as $n \to \infty$
⟨1⟩2. LET: $\epsilon > 0$
⟨1⟩3. PICK $N$ such that $1/\epsilon < N$

$\langle 1\rangle 4.$ For $n \geq N$ we have $d(a_n, l) < \epsilon$
$\langle 1\rangle 5.$ Q.E.D.
PROOF: Proposition 171.
□

## 22 Real Linear Algebra

**Definition 173** (Square Metric)**.** The *square metric* $\rho$ on $\mathbb{R}^n$ is defined by

$$\rho(\vec{x}, \vec{y}) = \max(|x_1 - y_1|, \ldots, |x_n - y_n|)$$

We prove this is a metric.

PROOF:
$\langle 1\rangle 1.$ $\rho(\vec{x}, \vec{y}) \geq 0$
PROOF: Immediate from definition.
$\langle 1\rangle 2.$ $\rho(\vec{x}, \vec{y}) = 0$ if and only if $\vec{x} = \vec{y}$
PROOF: Immediate from definition.
$\langle 1\rangle 3.$ $\rho(\vec{x}, \vec{y}) = \rho(\vec{y}, \vec{x})$
PROOF: Immediate from definition.
$\langle 1\rangle 4.$ $\rho(\vec{x}, \vec{z}) \leq \rho(\vec{x}, \vec{y}) + \rho(\vec{y}, \vec{z})$
PROOF: Since $|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$.
□

**Proposition 174.** *The square metric induces the standard topology on $\mathbb{R}^n$.*

PROOF:
$\langle 1\rangle 1.$ For every $a \in X$ and $\epsilon > 0$, we have $B_\rho(a, \epsilon)$ is open in the standard product topology.
PROOF:
$$B_\rho(a, \epsilon) = (a_1 - \epsilon, a_1 + \epsilon) \times \cdots \times (a_n - \epsilon, a_n + \epsilon)$$
$\langle 1\rangle 2.$ For any open sets $U_1, \ldots, U_n$ in $\mathbb{R}$, we have $U_1 \times \cdots \times U_n$ is open in the square metric topology.
$\langle 2\rangle 1.$ LET: $\vec{a} \in U_1 \times \cdots \times U_n$
$\langle 2\rangle 2.$ For $i = 1, \ldots, n$, PICK $\epsilon_i > 0$ such that $(a_i - \epsilon_i, a_i + \epsilon_i) \subseteq U_i$
$\langle 2\rangle 3.$ LET: $\epsilon = \min(\epsilon_1, \ldots, \epsilon_n)$
$\langle 2\rangle 4.$ $B_\rho(\vec{a}, \epsilon) \subseteq U$
□

**Definition 175.** Given $\vec{x}, \vec{y} \in \mathbb{R}^n$, define the *sum* $\vec{x} + \vec{y}$ by

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n) \ .$$

**Definition 176.** Given $\lambda \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^n$, define the *scalar product* $\lambda\vec{x} \in \mathbb{R}^n$ by

$$\lambda(x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n)$$

**Definition 177** (Inner Product)**.** Given $\vec{x}, \vec{y} \in \mathbb{R}^n$, define the *inner product* $\vec{x} \cdot \vec{y} \in \mathbb{R}$ by

$$(x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) = x_1 y_1 + \cdots + x_n y_n \ .$$

We write $\vec{x}^2$ for $\vec{x} \cdot \vec{x}$.

**Definition 178** (Norm)**.** Let $n \geq 1$. The *norm* on $\mathbb{R}^n$ is the function $\| \ \| : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\|(x_1, \ldots, x_n)\| = \sqrt{x_1^2 + \cdots + x_n^2}$$

**Lemma 179.**

$$\|\vec{x}\|^2 = \vec{x}^2$$

Proof: Immediate from definitions. ∎

**Lemma 180.**

$$\vec{x} \cdot (\vec{y} + \vec{z}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z}$$

Proof: Each is equal to $(x_1 y_1 + x_1 z_1, \ldots, x_n y_n + x_n z_n)$. ∎

**Lemma 181.**

$$|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \|\vec{y}\|$$

Proof:
⟨1⟩1. Assume: $\vec{x} \neq \vec{0} \neq \vec{y}$
  Proof: Otherwise both sides are 0.
⟨1⟩2. Let: $a = 1/\|\vec{x}\|$
⟨1⟩3. Let: $b = 1/\|\vec{y}\|$
⟨1⟩4. $(a\vec{x} + b\vec{y})^2 \geq 0$ and $(a\vec{x} - b\vec{y})^2 \geq 0$
⟨1⟩5. $a^2\|\vec{x}\|^2 + 2ab\vec{x} \cdot \vec{y} + b^2\|\vec{y}\|^2 \geq 0$ and $a^2\|\vec{x}\|^2 - 2ab\vec{x} \cdot \vec{y} + b^2\|\vec{y}\|^2 \geq 0$
⟨1⟩6. $2ab\vec{x} \cdot \vec{y} + 2 \geq 0$ and $-2ab\vec{x} \cdot \vec{y} + 2 \geq 0$
⟨1⟩7. $\vec{x} \cdot \vec{y} \geq -1/ab$ and $\vec{x} \cdot \vec{y} \leq 1/ab$
⟨1⟩8. $\vec{x} \cdot \vec{y} \geq -\|\vec{x}\|\|\vec{y}\|$ and $\vec{x} \cdot \vec{y} \leq \|\vec{x}\|\|\vec{y}\|$
∎

**Lemma 182** (Triangle Inequality)**.**

$$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$$

Proof:

$$\begin{aligned} \|\vec{x} + \vec{y}\|^2 &= \|\vec{x}\|^2 + 2\vec{x} \cdot \vec{y} + \|\vec{y}\|^2 \\ &\leq \|\vec{x}\|^2 + 2\|\vec{x}\|\|\vec{y}\| + \|\vec{y}\|^2 && \text{(Lemma 181)} \\ &= (\|\vec{x}\| + \|\vec{y}\|)^2 && \square \end{aligned}$$

**Definition 183** (Euclidean Metric)**.** Let $n \geq 1$. The *Euclidean metric* on $\mathbb{R}^n$ is defined by

$$d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\| \ .$$

We prove this is a metric.

$\langle 1 \rangle 1.$ $d(\vec{x}, \vec{y}) \geq 0$

PROOF: Immediate from definition.

$\langle 1 \rangle 2.$ $d(\vec{x}, \vec{y}) = 0$ if and only if $\vec{x} = \vec{y}$

PROOF: $d(\vec{x}, \vec{y}) = 0$ if and only if $\vec{x} - \vec{y} = \vec{0}$.

$\langle 1 \rangle 3.$ $d(\vec{x}, \vec{y}) = d(\vec{y}, \vec{x})$

PROOF: Immediate from definition.

$\langle 1 \rangle 4.$ $d(\vec{x}, \vec{z}) \leq d(\vec{x}, \vec{y}) + d(\vec{y}, \vec{z})$

PROOF:

$$\begin{aligned} \|\vec{x} - \vec{z}\| &= \|(\vec{x} - \vec{y}) + (\vec{y} - \vec{z})\| \\ &\leq \|\vec{x} - \vec{y}\| + \|\vec{y} - \vec{z}\| && \text{(Lemma 182)} \end{aligned}$$

□

**Proposition 184.** *The Euclidean metric induces the standard topology on $\mathbb{R}^n$.*

PROOF:

$\langle 1 \rangle 1.$ LET: $\rho$ be the square metric.

$\langle 1 \rangle 2.$ For all $\vec{a} \in \mathbb{R}^n$ and $\epsilon > 0$, we have $B_d(\vec{a}, \epsilon) \subseteq B_\rho(\vec{a}, \epsilon)$

$\langle 2 \rangle 1.$ LET: $\vec{x} \in B_d(\vec{a}, \epsilon)$

$\langle 2 \rangle 2.$ $\sqrt{(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2} < \epsilon$

$\langle 2 \rangle 3.$ $(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2 < \epsilon^2$

$\langle 2 \rangle 4.$ For all $i$ we have $(x_i - a_i)^2 < \epsilon^2$

$\langle 2 \rangle 5.$ For all $i$ we have $|x_i - a_i| < \epsilon$

$\langle 2 \rangle 6.$ $\rho(\vec{x}, \vec{a}) < \epsilon$

$\langle 1 \rangle 3.$ For all $\vec{a} \in \mathbb{R}^n$ and $\epsilon > 0$, we have $B_\rho(\vec{a}, \epsilon/\sqrt{n}) \subseteq B_d(\vec{a}, \epsilon)$

$\langle 2 \rangle 1.$ LET: $\vec{x} \in B_\rho(\vec{a}, \epsilon/\sqrt{n})$

$\langle 2 \rangle 2.$ $\rho(\vec{x}, \vec{a}) < \epsilon/\sqrt{n}$

$\langle 2 \rangle 3.$ For all $i$ we have $|x_i - x_a| < \epsilon/\sqrt{n}$

$\langle 2 \rangle 4.$ For all $i$ we have $(x_i - x_a)^2 < \epsilon^2/n$

$\langle 2 \rangle 5.$ $d(\vec{x}, \vec{a}) < \epsilon$

$\langle 1 \rangle 4.$ Q.E.D.

PROOF: By Lemma 164.

□

**Lemma 185.** *If $\sum_{i=0}^\infty x_i^2$ and $\sum_{i=0}^\infty y_i^2$ converge then $\sum_{i=0}^\infty |x_i y_i|$ converges.*

PROOF:

$\langle 1 \rangle 1.$ For all $N \geq 0$ we have $\sum_{i=0}^N |x_i y_i| \leq \sqrt{\sum_{i=0}^N |x_i|^2} \sqrt{\sum_{i=0}^N |y_i|^2}$

PROOF: By the Cauchy-Schwarz inequality

$\langle 1 \rangle 2.$ Q.E.D.

PROOF: Since $\sum_{i=0}^N |x_i y_i|$ is an increasing sequence bounded above by $(\sum_{i=0}^\infty x_i^2)(\sum_{i=0}^\infty y_i^2)$.

□

**Corollary 185.1.** *If $\sum_{i=0}^\infty x_i^2$ and $\sum_{i=0}^\infty y_i^2$ converge then $\sum_{i=0}^\infty (x_i + y_i)^2$ converges.*

PROOF: Since $\sum_{i=0}^{\infty} x_i^2$, $\sum_{i=0}^{\infty} y_i^2$ and $2\sum_{i=0}^{\infty} x_i y_i$ all converge.

**Definition 186** ($l^2$-metric)**.** The *$l^2$-metric* on

$$\left\{ (x_n) \in \mathbb{R}^\omega \mid \sum_{n=0}^{\infty} x_n^2 \text{ converges} \right\}$$

is defined by

$$d(x, y) = \left( \sum_{n=0}^{\infty} (x_n - y_n)^2 \right)^{1/2}$$

We prove this is a metric.

PROOF:
$\langle 1 \rangle 1.$ $d$ is well-defined.
PROOF: By Corollary 185.1.
$\langle 1 \rangle 2.$ $d(x, y) \geq 0$
$\langle 1 \rangle 3.$ $d(x, y) = 0$ if and only if $x = y$
$\langle 1 \rangle 4.$ $d(x, y) = d(y, x)$
$\langle 1 \rangle 5.$ $d(x, z) \leq d(x, y) + d(y, z)$
PROOF: By Lemma 182.
□

# 23 The Uniform Topology

**Definition 187** (Uniform Metric)**.** Let $J$ be a set. The *uniform metric* $\overline{\rho}$ on $\mathbb{R}^J$ is defined by

$$\overline{\rho}(a, b) = \sup_{j \in J} \overline{d}(a_j, b_j)$$

where $\overline{d}$ is the standard bounded metric on $\mathbb{R}$.

The *uniform topology* on $\mathbb{R}^J$ is the topology induced by the uniform metric.

We prove this is a metric.

PROOF:
$\langle 1 \rangle 1.$ $\overline{\rho}(a, b) \geq 0$
PROOF: Immediate from definitions.
$\langle 1 \rangle 2.$ $\overline{\rho}(a, b) = 0$ if and only if $a = b$
PROOF: Immediate from definitions.
$\langle 1 \rangle 3.$ $\overline{\rho}(a, b) = \overline{\rho}(b, a)$
PROOF: Immediate from definitions.
$\langle 1 \rangle 4.$ $\overline{\rho}(a, c) \leq \overline{\rho}(a, b) + \overline{\rho}(b, c)$

PROOF:

$$\begin{aligned}\overline{\rho}(a,c) &= \sup_{j\in J} \overline{d}(a_j,c_j) \\ &\leq \sup_{j\in J}(\overline{d}(a_j,b_j)+\overline{d}(b_j,c_j)) \\ &\leq \sup_{j\in J}\overline{d}(a_j,b_j)+\sup_{j\in J}\overline{d}(b_j,c_j) \\ &= \overline{\rho}(a,b)+\overline{\rho}(b,c)\end{aligned}$$

□

**Proposition 188.** *The uniform topology on $\mathbb{R}^J$ is finer than the product topology.*

PROOF:
⟨1⟩1. LET: $j \in J$ and $U$ be open in $\mathbb{R}$
PROVE: ${\pi_j}^{-1}(U)$ is open in the uniform topology.
⟨1⟩2. LET: $a \in {\pi_j}^{-1}(U)$
⟨1⟩3. PICK $\epsilon > 0$ such that $(a_j - \epsilon, a_j + \epsilon) \subseteq U$
⟨1⟩4. $B_{\overline{\rho}}(a,\epsilon) \subseteq {\pi_j}^{-1}(U)$
□

**Proposition 189.** *The uniform topology on $\mathbb{R}^J$ is coarser than the box topology.*

PROOF:
⟨1⟩1. LET: $a \in \mathbb{R}^J$ and $\epsilon > 0$
PROVE: $B(a,\epsilon)$ is open in the box topology.
⟨1⟩2. LET: $b \in B(a,\epsilon)$
⟨1⟩3. For $j \in J$ we have $|a_j - b_j| < \epsilon$
⟨1⟩4. For $j \in J$,
LET: $\delta_j = (\epsilon - |a_j - b_j|)/2$
⟨1⟩5. $\prod_{j\in J}(b_j - \delta_j, b_j + \delta_j) \subseteq B(a,\epsilon)$
□

**Proposition 190.** *The uniform topology on $\mathbb{R}^J$ is strictly finer than the product topology if and only if $J$ is infinite.*

PROOF:
⟨1⟩1. If $J$ is finite then the uniform and product topologies coincide.
PROOF: The uniform, box and product topologies are all the same.
⟨1⟩2. If $J$ is infinite then the uniform and product topologies are different.
PROOF: The set $B(\vec{0},1)$ is open in the uniform topology but not the product topology.
□

**Proposition 191** (DC)**.** *The uniform topology on $\mathbb{R}^J$ is strictly coarser than the box topology if and only if $J$ is infinite.*

PROOF:
⟨1⟩1. If $J$ is finite then the uniform and box topologies coincide.

PROOF: The uniform, box and product topologies are all the same.
⟨1⟩2. If $J$ is infinite then the uniform and box topologies are different.
PROOF: Pick an $\omega$-sequence $(j_1, j_2, \ldots)$ in $J$. Let $U = \prod_{j \in J} U_j$ where $U_{j_i} = (-1/i, 1/i)$ and $U_j = (-1, 1)$ for all other $j$. Then $\vec{0} \in U$ but there is no $\epsilon > 0$ such that $B(\vec{0}, \epsilon) \subseteq U$.
□

**Proposition 192.** *The closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ under the uniform topology is $\mathbb{R}^\omega$.*

PROOF: Given any open ball $B(a, \epsilon)$, pick an integer $N$ such that $1/\epsilon < N$. Then $B(a, \epsilon)$ includes sequences whose $n$th entry is 0 for all $n \geq N$. □