Topology

Robin Adams

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# 1 Order Theory

**Definition 1** (Preorder). Let X be a set. A *preorder* on X is a binary relation  $\leq$  on X such that:

**Reflexivity** For all  $x \in X$ , we have  $x \leq x$ 

**Transitivity** For all  $x, y, z \in X$ , if  $x \le y$  and  $y \le z$  then  $x \le z$ .

**Definition 2** (Preordered Set). A preordered set consists of a set X and a preorder  $\leq$  on X.

**Proposition 3.** Let X and Y be linearly ordered sets. Let  $f: X \rightarrow Y$  be strictly monotone and surjective. Then f is a poset isomorphism.

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Proof:
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\langle 1 \rangle 1. f is injective.
   \langle 2 \rangle 1. Let: x, y \in X
   \langle 2 \rangle 2. Assume: f(x) = f(y)
   \langle 2 \rangle 3. \ x \not< y
      PROOF: By strong motonicity.
   \langle 2 \rangle 4. \ y \not < x
      PROOF: By strong motonicity.
   \langle 2 \rangle 5. \ x = y
      PROOF: By trichotomy.
\langle 1 \rangle 2. f^{-1} is monotone.
   \langle 2 \rangle 1. Let: x, y \in X
   \langle 2 \rangle 2. Assume: x \leq y
   \langle 2 \rangle 3. \ f^{-1}(x) \not> f^{-1}(y)
      PROOF: By strong motonicity.
   \langle 2 \rangle 4. \ f^{-1}(x) < f^{-1}(y)
      PROOF: By trichotomy.
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**Definition 4** (Interval). Let X be a preordered set and  $Y \subseteq X$ . Then Y is an interval if and only if, for all  $a, b \in Y$  and  $c \in X$ , if  $a \le c \le b$  then  $c \in Y$ .

**Definition 5** (Linear Continuum). A linearly ordered set L is a *linear continuum* if and only if:

- 1. every nonempty subset of L that is bounded above has a supremum
- 2. L is dense

**Proposition 6.** Every interval in a linear continuum is a linear continuum.

- $\langle 1 \rangle 1$ . Let: L be a linear continuum and I an interval in L.
- $\langle 1 \rangle 2$ . Every nonempty subset of I that is bounded above has a supremum in I.
  - $\langle 2 \rangle 1$ . Let:  $X \subseteq I$  be nonempty and bounded above by  $b \in I$ .

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\langle 2 \rangle 2. Let: s be the supremum of X in L. Proof: Since L is a linear continuum. \langle 2 \rangle 3. s \in I \langle 3 \rangle 1. Pick a \in X Proof: Since X is nonempty (\langle 2 \rangle 1). \langle 3 \rangle 2. a \leq s \leq b \langle 3 \rangle 3. s \in I Proof: Since I is an interval (\langle 1 \rangle 1). \langle 2 \rangle 4. s is the supremum of X in I \langle 1 \rangle 3. I is dense. \langle 2 \rangle 1. Let: x, y \in I with x < y \langle 2 \rangle 2. Pick z \in L with x < z < y Proof: Since L is dense. \langle 2 \rangle 3. z \in I Proof: Since L is an interval. \Box
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**Definition 7** (Ordered Square). The ordered square  $I_o^2$  is the set  $[0,1]^2$  under the dictionary order.

**Proposition 8.** The ordered square is a linear continuum.

#### Proof:

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\langle 1 \rangle 1. Every nonempty subset of I_o^2 bounded above has a supremum.
   \langle 2 \rangle 1. Let: X \subseteq I_o^2 be nonempty and bounded above by (b,c)
   \langle 2 \rangle 2. Let: s = \sup \pi_1(X)
      PROOF: The set \pi_1(X) is nonempty and bounded above by b.
   \langle 2 \rangle 3. Case: s \in \pi_1(X)
       \langle 3 \rangle 1. Let: t = \sup\{y \in [0,1] \mid (s,y) \in X\}
          Proof: This set is nonempty and bounded above by c.
       \langle 3 \rangle 2. (s,t) is the supremum of X.
   \langle 2 \rangle 4. Case: s \notin \pi_1(X)
      PROOF: In this case (s,0) is the supremum of X.
\langle 1 \rangle 2. I_o^2 is dense.
   \langle 2 \rangle 1. Let: (x_1, y_1), (x_2, y_2) \in I_o^2 with (x_1, y_1) < (x_2, y_2)
   \langle 2 \rangle 2. Case: x_1 < x_2
       \langle 3 \rangle 1. PICK x_3 with x_1 < x_3 < x_2
       \langle 3 \rangle 2. \ (x_1, y_1) < (x_3, y_1) < (x_2, y_2)
   \langle 2 \rangle 3. Case: x_1 = x_2 and y_1 < y_2
      \langle 3 \rangle 1. Pick y_3 with y_1 < y_3 < y_2
       \langle 3 \rangle 2. \ (x_1, y_1) < (x_1, y_3) < (x_2, y_2)
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**Proposition 9.** If X is a well-ordered set then  $X \times [0,1)$  under the dictionary order is a linear continuum.

### Proof:

 $\langle 1 \rangle 1$ . Every nonempty set  $A \subseteq X \times [0,1)$  bounded above has a supremum.

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\langle 2 \rangle 1. Let: A \subseteq X \times [0,1) be nonempty and bounded above
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 $\langle 2 \rangle 2$ . Let:  $x_0$  be the supremum of  $\pi_1(A)$ 

 $\langle 2 \rangle 3$ . Case:  $x_0 \in \pi_1(A)$ 

 $\langle 3 \rangle 1$ . Let:  $y_0$  be the supremum of  $\{ y \in [0,1) \mid (x_0,y) \in A \}$ 

 $\langle 3 \rangle 2$ .  $(x_0, y_0)$  is the supremum of A.

 $\langle 2 \rangle 4$ . Case:  $x_0 \notin \pi_1(A)$ 

PROOF: In this case  $(x_0, 0)$  is the supremum of A.

 $\langle 1 \rangle 2$ .  $X \times [0,1)$  is dense.

$$\langle 2 \rangle 1$$
. Let:  $(x_1, y_1), (x_2, y_2) \in X \times [0, 1)$  with  $(x_1, y_1) < (x_2, y_2)$ 

 $\langle 2 \rangle 2$ . Case:  $x_1 < x_2$ 

 $\langle 3 \rangle 1$ . PICK  $y_3$  such that  $y_1 < y_3 < 1$ 

$$\langle 3 \rangle 2. \ (x_1, y_1) < (x_1, y_3) < (x_2, y_2)$$

 $\langle 2 \rangle 3$ . Case:  $x_1 = x_2$  and  $y_1 < y_2$ 

 $\langle 3 \rangle 1$ . PICK  $y_3$  such that  $y_1 < y_3 < y_2$ 

 $\langle 3 \rangle 2. \ (x_1, y_1) < (x_1, y_3) < (x_2, y_2)$ 

**Lemma 10.** For all  $a, b, c, d \in \mathbb{R}$  with a < b and c < d, we have  $[a, b) \cong [c, d)$ 

PROOF: The map  $\lambda t.c + (t-a)(d-c)/(b-a)$  is an order isomorphism.

**Proposition 11.** Let X be a linearly ordered set. Let a < b < c in X. Then  $[a, c) \cong [0, 1)$  if and only if  $[a, b) \cong [b, c) \cong [0, 1)$ .

PROOF

 $\langle 1 \rangle 1$ . If  $[a, c) \cong [0, 1)$  then  $[a, b) \cong [b, c) \cong [0, 1)$ 

 $\langle 2 \rangle 1$ . Assume:  $f: [a,c) \cong [0,1)$  is an order isomorphism

$$\langle 2 \rangle 2$$
.  $[a,b) \cong [0,1)$ 

Proof:

$$[a,b) \cong [0,f(b))$$
 (by the restriction of  $f$ )  
 $\cong [0,1)$  (Lemma 10)

 $\langle 2 \rangle 3. \ [b,c) \cong [0,1)$ 

PROOF: Similar.

 $\langle 1 \rangle 2$ . If  $[a,b) \cong [b,c) \cong [0,1)$  then  $[a,c) \cong [0,1)$ 

Proof:

$$[a,c) = [a,b) * [b,c)$$
  
 $\cong [0,1) * [0,1)$   
 $\cong [0,1/2) * [1/2,1)$  (Lemma 10)  
 $= 1$ 

**Proposition 12** (CC). Let X be a linearly ordered set. Let  $x_0 < x_1 < \cdots$  be a strictly increasing sequence in X with supremum b. Then  $[x_0, b) \cong [0, 1)$  if and only if  $[x_i, x_{i+1}) \cong [0, 1)$  for all i.

### PROOF:

 $\langle 1 \rangle 1$ . If  $[x_0, b) \cong [0, 1)$  then  $[x_i, x_{i+1}) \cong [0, 1)$  for all i.

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PROOF: By Lemma 10 \langle 1 \rangle 2. If [x_i, x_{i+1}) \cong [0, 1) for all i then [x_0, b) \cong [0, 1) \langle 2 \rangle 1. Assume: [x_i, x_{i+1}) \cong [0, 1) for all i \langle 2 \rangle 2. PICK an order isomorphism f_i : [x_i, x_{i+1}) \cong [1/2^i, 2/2^{i+1}) for each i. PROOF: By Lemma 10 \langle 2 \rangle 3. The union of the f_is is an order isomorphism [x_0, b) \cong [0, 1)
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# 2 Real Analysis

**Definition 13.** Let  $\mathbb{R}^{\infty}$  be the subset of  $\mathbb{R}^{\omega}$  consisting of all sequences  $(a_n)$  such that  $a_n = 0$  for all but finitely many n.

# 3 Group Theory

**Definition 14.** Given a group G and sets  $A, B \subseteq G$ , let  $AB = \{ab \mid a \in A, b \in B\}$ .

**Definition 15.** Given a group G and a set  $A \subseteq G$ , let  $A^{-1} = \{a^{-1} \mid a \in A\}$ .

# 4 Topological Spaces

**Definition 16** (Topology). A topology on a set X is a set  $\mathcal{T} \subseteq \mathcal{P}X$  such that:

- $X \in \mathcal{T}$ .
- For all  $\mathcal{U} \subseteq \mathcal{T}$  we have  $\bigcup \mathcal{U} \in \mathcal{T}$ .
- For all  $U, V \in \mathcal{T}$  we have  $U \cap V \in \mathcal{T}$ .

We call the elements of X points and the elements of  $\mathcal{T}$  open sets.

**Definition 17** (Topological Space). A topological space X consists of a set X and a topology on X.

**Definition 18** (Discrete Space). For any set X, the discrete topology on X is  $\mathcal{P}X$ .

**Definition 19** (Indiscrete Space). For any set X, the *indiscrete* or *trivial* topology on X is  $\{\emptyset, X\}$ .

**Definition 20** (Finite Complement Topology). For any set X, the *finite complement topology* on X is  $\{U \in \mathcal{P}X \mid X \setminus U \text{ is finite}\} \cup \{\emptyset\}$ .

**Definition 21** (Countable Complement Topology). For any set X, the *countable complement topology* on X is  $\{U \in \mathcal{P}X \mid X \setminus U \text{ is countable}\} \cup \{\emptyset\}$ .

**Definition 22** (Finer, Coarser). Suppose that  $\mathcal{T}$  and  $\mathcal{T}'$  are two topologies on a given set X. If  $\mathcal{T}' \supseteq \mathcal{T}$ , we say that  $\mathcal{T}'$  is *finer* than  $\mathcal{T}$ ; if  $\mathcal{T}'$  properly contains  $\mathcal{T}$ , we say that  $\mathcal{T}'$  is *strictly* finer than  $\mathcal{T}$ . We also say that  $\mathcal{T}$  is *coarser* than  $\mathcal{T}'$ , or *strictly* coarser, in these two respective situations. We say  $\mathcal{T}$  is *comparable* with  $\mathcal{T}'$  if either  $\mathcal{T}' \supseteq \mathcal{T}$  or  $\mathcal{T} \supseteq \mathcal{T}'$ .

**Lemma 23.** Let X be a topological space and  $U \subseteq X$ . Then U is open if and only if, for all  $x \in U$ , there exists an open set V such that  $x \in V \subseteq U$ .

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Proof:  \begin{array}{l} \langle 1 \rangle 1. \ \Rightarrow \\ \text{Proof: Take } V = U \\ \langle 1 \rangle 2. \ \Leftarrow \\ \text{Proof: We have } U = \bigcup \{V \text{ open in } X \mid V \subseteq U\}. \end{array}
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**Lemma 24.** Let X be a set and  $\mathcal{T}$  a nonempty set of topologies on X. Then  $\bigcap \mathcal{T}$  is a topology on X, and is the finest topology that is coarser than every member of  $\mathcal{T}$ .

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Proof:
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\langle 1 \rangle 1. \ X \in \bigcap \mathcal{T}
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PROOF: Since X is in every member of  $\mathcal{T}$ .

 $\langle 1 \rangle 2$ .  $\bigcap \mathcal{T}$  is closed under union.

- $\langle 2 \rangle 1$ . Let:  $\mathcal{U} \subseteq \bigcap \mathcal{T}$
- $\langle 2 \rangle 2$ . For all  $T \in \mathcal{T}$  we have  $\mathcal{U} \subseteq T$
- $\langle 2 \rangle 3$ . For all  $T \in \mathcal{T}$  we have  $\bigcup \mathcal{U} \in T$
- $\langle 2 \rangle 4. \bigcup \mathcal{U} \in \bigcap \mathcal{T}$
- $\langle 1 \rangle 3$ .  $\bigcap \mathcal{T}$  is closed under binary intersection.
  - $\langle 2 \rangle 1$ . Let:  $U, V \in \bigcap \mathcal{T}$
  - $\langle 2 \rangle 2$ . For all  $T \in \mathcal{T}$  we have  $U, V \in T$
  - $\langle 2 \rangle 3$ . For all  $T \in \mathcal{T}$  we have  $U \cap V \in T$
- $\sqrt{\langle 2 \rangle} 4. \ U \cap V \in \bigcap \mathcal{T}$

**Lemma 25.** Let X be a set and  $\mathcal{T}$  a set of topologies on X. Then there exists a unique coarsest topology that is finer than every member of  $\mathcal{T}$ .

PROOF: The required topology is given by

 $\bigcap \{T \in \mathcal{PP}X \mid T \text{ is a topology on } X \text{ that is finer than every member of } \mathcal{T}\} ,$ 

The set is nonempty since it contains the discrete topology.  $\square$ 

**Definition 26** (Neighbourhood). A *neighbourhood* of a point x is an open set that contains x.

### 5 Closed Set

**Definition 27** (Closed Set). Let X be a topological space and  $A \subseteq X$ . Then A is *closed* if and only if  $X \setminus A$  is open.

Lemma 28. The empty set is closed.

PROOF: Since the whole space X is always open.  $\square$ 

**Lemma 29.** The topological space X is closed.

PROOF: Since  $\emptyset$  is open.  $\square$ 

Lemma 30. The intersection of a nonempty set of closed sets is closed.

PROOF: Let  $\mathcal{C}$  be a nonempty set of closed sets. Then  $X \setminus \bigcap \mathcal{C} = \bigcup \{X \setminus C \mid C \in \mathcal{C}\}$  is open.  $\square$ 

Lemma 31. The union of two closed sets is closed.

PROOF: Let C and D be closed. Then  $X \setminus (C \cup D) = (X \setminus C) \cap (X \setminus D)$  is open.  $\sqcap$ 

**Proposition 32.** Let X be a set and  $C \subseteq PX$  a set such that:

- 1.  $\emptyset \in \mathcal{C}$
- 2.  $X \in \mathcal{C}$
- 3. For all  $A \subseteq C$  nonempty we have  $\bigcap A \in C$

4. For all  $C, D \in \mathcal{C}$  we have  $C \cup D \in \mathcal{C}$ .

Then there exists a unique topology  $\mathcal{T}$  such that  $\mathcal{C}$  is the set of closed sets, namely

$$\mathcal{T} = \{ X \setminus C \mid C \in \mathcal{C} \}$$

Proof:

- $\langle 1 \rangle 1$ . Let:  $\mathcal{T} = \{ X \setminus C \mid C \in \mathcal{C} \}$
- $\langle 1 \rangle 2$ .  $\mathcal{T}$  is a topology
  - $\langle 2 \rangle 1. \ X \in \mathcal{T}$

Proof: Since  $\emptyset \in \mathcal{C}$ 

- $\langle 2 \rangle 2$ . For all  $\mathcal{U} \subseteq \mathcal{T}$  we have  $\bigcup \mathcal{U} \in \mathcal{T}$ 
  - $\langle 3 \rangle 1$ . Let:  $\mathcal{U} \subseteq \mathcal{T}$
  - $\langle 3 \rangle 2$ . Case:  $\mathcal{U} = \emptyset$

Proof: In this case  $\bigcup \mathcal{U} = \emptyset \in \mathcal{T}$  since  $X \in \mathcal{C}$ 

 $\langle 3 \rangle 3$ . Case:  $\mathcal{U} \neq \emptyset$ 

PROOF: In this case  $X \setminus \bigcup \mathcal{U} = \bigcap \{X \setminus U \mid U \in \mathcal{U}\} \in \mathcal{C}$ .

 $\langle 2 \rangle 3$ . For all  $U, V \in \mathcal{T}$  we have  $U \cap V \in \mathcal{T}$ 

PROOF: Since  $X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V) \in \mathcal{C}$ .

 $\langle 1 \rangle 3$ . C is the set of all closed sets in T

Proof:

$$C$$
 is closed in  $\mathcal{T}$   
 $\Leftrightarrow X \setminus C \in \mathcal{T}$ 

$$\Leftrightarrow C \in \mathcal{C}$$

 $\langle 1 \rangle 4$ . If  $\mathcal{T}'$  is a topology and  $\mathcal{C}$  is the set of closed sets in  $\mathcal{T}'$  then  $\mathcal{T}' = \mathcal{T}$ 

PROOF: We have

$$U \in \mathcal{T}$$

$$\Leftrightarrow X \setminus U \in \mathcal{C}$$

$$\Leftrightarrow X \setminus U \text{ is closed in } \mathcal{T}'$$

$$\Leftrightarrow U \in \mathcal{T}'$$

**Proposition 33.** If U is open and A is closed then  $U \setminus A$  is open.

PROOF:  $U \setminus A = U \cap (X \setminus A)$  is the intersection of two open sets.  $\square$ 

**Proposition 34.** If U is open and A is closed then  $A \setminus U$  is closed.

PROOF:  $A \setminus U = A \cap (X \setminus U)$  is the intersection of two closed sets.  $\square$ 

## 6 Interior

**Definition 35** (Interior). Let X be a topological space and  $A \subseteq X$ . The *interior* of A, Int A, is the union of all the open subsets of A.

**Lemma 36.** The interior of a set is open.

PROOF: It is a union of open sets.  $\square$ Lemma 37.  $\operatorname{Int} A \subseteq A$ PROOF: Immediate from definition.  $\Box$ **Lemma 38.** If U is open and  $U \subseteq A$  then  $U \subseteq \operatorname{Int} A$ PROOF: Immediate from definition. **Lemma 39.** A set A is open if and only if A = Int A. PROOF: If A = Int A then A is open by Lemma 36. Conversely if A is open then  $A \subseteq \operatorname{Int} A$  by the definition of interior and so  $A = \operatorname{Int} A$ . 7 Closure **Definition 40** (Closure). Let X be a topological space and  $A \subseteq X$ . The *closure* of A,  $\overline{A}$ , is the intersection of all the closed sets that include A. This intersection exists since X is a closed set that includes A (Lemma 29). Lemma 41. The closure of a set is closed. PROOF: Dual to Lemma 36. Lemma 42.  $A\subseteq \overline{A}$ PROOF: Immediate from definition. **Lemma 43.** If C is closed and  $A \subseteq C$  then  $\overline{A} \subseteq C$ . PROOF: Immediate from definition. **Lemma 44.** A set A is closed if and only if  $A = \overline{A}$ . PROOF: Dual to Lemma 39. **Theorem 45.** Let X be a topological space,  $A \subseteq X$  and  $x \in X$ . Then  $x \in \overline{A}$  if and only if every neighbourhood of x intersects A. PROOF: We have  $x \in \overline{A}$  $\Leftrightarrow \forall C. C \text{ closed } \land A \subseteq C \Rightarrow x \in C$  $\Leftrightarrow \forall U.U \text{ open } \wedge A \cap U = \emptyset \Rightarrow x \not\in U$ 

**Proposition 46.** If  $A \subseteq B$  then  $\overline{A} \subseteq \overline{B}$ .

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 $\Leftrightarrow \forall U.U \text{ open } \land x \in U \Rightarrow U \text{ intersects } A$ 

PROOF: This holds because  $\overline{B}$  is a closed set that includes A.  $\square$ 

### Proposition 47.

$$\overline{A \cup B} = \overline{A} \cup \overline{B}$$

Proof:

 $\langle 1 \rangle 1. \ \overline{A} \subseteq \overline{A \cup B}$ 

PROOF: By Proposition 46.

 $\langle 1 \rangle 2$ .  $\overline{B} \subseteq \overline{A \cup B}$ 

PROOF: By Proposition 46.

 $\langle 1 \rangle 3. \ \overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$ 

- $\langle 2 \rangle 1$ . Let:  $x \in \overline{A \cup B}$
- $\langle 2 \rangle 2$ . Assume:  $x \notin \overline{A}$ PROVE:  $x \in \overline{B}$
- $\langle 2 \rangle 3$ . PICK a neighbourhood U of x that does not intersect A
- $\langle 2 \rangle 4$ . Let: V be any neighbourhood of x
- $\langle 2 \rangle$ 5.  $U \cap V$  is a neighbourhood of x
- $\langle 2 \rangle 6. \ U \cap V \text{ intersects } A \cup B$

PROOF: From  $\langle 2 \rangle 1$  and Theorem 45.

 $\langle 2 \rangle 7$ .  $U \cap V$  intersects B

PROOF: From  $\langle 2 \rangle 3$ 

- $\langle 2 \rangle 8$ . V intersects B
- $\langle 2 \rangle 9$ . Q.E.D.

PROOF: We have  $x \in \overline{B}$  from Theorem 45.

# 8 Boundary

**Definition 48** (Boundary). The *boundary* of a set A is the set  $\partial A = \overline{A} \cap \overline{X \setminus A}$ .

Proposition 49.

Int 
$$A \cap \partial A = \emptyset$$

PROOF: Since  $\overline{X \setminus A} = X \setminus \text{Int } A$ .  $\square$ 

Proposition 50.

$$\overline{A}=\operatorname{Int} A\cup\partial A$$

Proof:

$$\begin{split} \operatorname{Int} A \cup \partial A &= \operatorname{Int} A \cup \left( \overline{A} \cap \overline{X \setminus A} \right) \\ &= \left( \operatorname{Int} A \cup \overline{A} \right) \cap \left( \operatorname{Int} A \cup \overline{X \setminus A} \right) \\ &= \overline{A} \cap X \\ &= \overline{A} \end{split}$$

**Proposition 51.**  $\partial A = \emptyset$  if and only if A is open and closed.

PROOF: If  $\partial A = \emptyset$  then  $\overline{A} = \text{Int } A$  by Proposition 50.

**Proposition 52.** A set U is open if and only if  $\partial U = \overline{U} \setminus U$ .

Proof:

$$\partial U = \overline{U} \setminus U$$

$$\Leftrightarrow \overline{U} \setminus \text{Int } U = \overline{U} \setminus U \qquad (Propositions 49, 50)$$

$$\Leftrightarrow \text{Int } U = U$$

### 9 Limit Points

**Definition 53** (Limit Point). Let X be a topological space,  $a \in X$  and  $A \subseteq X$ . Then a is a *limit point*, *cluster point* or *point of accumulation* for A if and only if every neighbourhood of a intersects A at a point other than a.

**Lemma 54.** The point a is an accumulation point for A if and only if  $a \in \overline{A \setminus \{a\}}$ .

PROOF: From Theorem 45.  $\square$ 

**Theorem 55.** Let X be a topological space and  $A \subseteq X$ . Let A' be the set of all limit points of A. Then  $\overline{A} = A \cup A'$ .

Proof:

 $\langle 1 \rangle 1$ . For all  $x \in \overline{A}$ , if  $x \notin A$  then  $x \in A'$  PROOF: From Theorem 45.

 $\langle 1 \rangle 2$ .  $A \subseteq \overline{A}$ 

Proof: Lemma 42.

 $\langle 1 \rangle 3. \ A' \subseteq \overline{A}$ 

PROOF: From Theorem 45.

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Corollary 55.1. A set is closed if and only if it contains all its limit points.

**Proposition 56.** In an indiscrete topology, every point is a limit point of any set that has more than one point.

PROOF: Let X be an indiscrete space. Let A be a set with more than one point and x be a point. The only neighbourhood of x is X, which must intersect A at a point other than x.  $\square$ 

**Lemma 57.** Let X be a topological space and  $A \subseteq B \subseteq X$ . Then every limit point of A is a limit point of B.

PROOF: Immediate from definitions.

# 10 Basis for a Topology

**Definition 58** (Basis). If X is a set, a *basis* for a topology on X is a set  $\mathcal{B} \subseteq \mathcal{P}X$  called *basis elements* such that

- 1. For all  $x \in X$ , there exists  $B \in \mathcal{B}$  such that  $x \in B$ .
- 2. For all  $B_1, B_2 \in \mathcal{B}$  and  $x \in B_1 \cap B_2$ , there exists  $B_3 \in \mathcal{B}$  such that  $x \in B_3 \subseteq B_1 \cap B_2$ .

If  $\mathcal{B}$  satisfies these two conditions, then we define the topology *generated* by  $\mathcal{B}$  to be  $\mathcal{T} = \{U \in \mathcal{P}X \mid \forall x \in U. \exists B \in \mathcal{B}. x \in B \subseteq U\}.$ 

We prove this is a topology.

### Proof:

 $\langle 1 \rangle 1. \ X \in \mathcal{T}$ 

PROOF: For all  $x \in X$  there exists  $B \in \mathcal{B}$  such that  $x \in B \subseteq X$  by condition

- $\langle 1 \rangle 2$ . For all  $\mathcal{U} \subseteq \mathcal{T}$  we have  $\bigcup \mathcal{U} \in \mathcal{T}$ 
  - $\langle 2 \rangle 1$ . Let:  $\mathcal{U} \subseteq \mathcal{T}$
  - $\langle 2 \rangle 2$ . Let:  $x \in \bigcup \mathcal{U}$
  - $\langle 2 \rangle 3$ . Pick  $U \in \mathcal{U}$  such that  $x \in U$
  - $\langle 2 \rangle 4$ . PICK  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$ PROOF: Since  $U \in \mathcal{T}$  by  $\langle 2 \rangle 1$  and  $\langle 2 \rangle 3$ .
  - $\langle 2 \rangle 5. \ x \in B \subseteq \bigcup \mathcal{U}$
- $\langle 1 \rangle 3$ . For all  $U, V \in \mathcal{T}$  we have  $U \cap V \in \mathcal{T}$ 
  - $\langle 2 \rangle 1$ . Let:  $U, V \in \mathcal{T}$
  - $\langle 2 \rangle 2$ . Let:  $x \in U \cap V$
  - $\langle 2 \rangle 3$ . PICK  $B_1 \in \mathcal{B}$  such that  $x \in B_1 \subseteq U$
  - $\langle 2 \rangle 4$ . PICK  $B_2 \in \mathcal{B}$  such that  $x \in B_2 \subseteq V$
  - $\langle 2 \rangle$ 5. PICK  $B_3 \in \mathcal{B}$  such that  $x \in B_3 \subseteq B_1 \cap B_2$ PROOF: By condition 2.

 $\langle 2 \rangle 6. \ x \in B_3 \subseteq U \cap V$ 

**Lemma 59.** Let X be a set. Let  $\mathcal{B}$  be a basis for a topology  $\mathcal{T}$  on X. Then  $\mathcal{T}$  is the set of all unions of subsets of  $\mathcal{B}$ .

### Proof:

- $\langle 1 \rangle 1$ . For all  $U \in \mathcal{T}$ , there exists  $\mathcal{A} \subseteq \mathcal{B}$  such that  $U = \bigcup \mathcal{A}$ 
  - $\langle 2 \rangle 1$ . Let:  $U \in \mathcal{T}$
  - $\langle 2 \rangle 2$ . Let:  $\mathcal{A} = \{ B \in \mathcal{B} \mid B \subseteq U \}$
  - $\langle 2 \rangle 3. \ U \subseteq \bigcup \mathcal{A}$ 
    - $\langle 3 \rangle 1$ . Let:  $x \in U$
    - $\langle 3 \rangle 2$ . Pick  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$

PROOF: Since  $\mathcal{B}$  is a basis for  $\mathcal{T}$ .

- $\langle 3 \rangle 3. \ x \in B \in \mathcal{A}$
- $\langle 2 \rangle 4. \bigcup \mathcal{A} \subseteq U$

PROOF: From the definition of  $\mathcal{A}$  ( $\langle 2 \rangle 2$ ).

- $\langle 1 \rangle 2$ . For all  $\mathcal{A} \subseteq \mathcal{B}$  we have  $\bigcup \mathcal{A} \in \mathcal{T}$ 
  - $\langle 2 \rangle 1. \ \mathcal{B} \subseteq \mathcal{T}$

PROOF: If  $B \in \mathcal{B}$  and  $x \in B$ , then there exists  $B' \in \mathcal{B}$  such that  $x \in B' \subseteq B$ , namely B' = B.

 $\langle 2 \rangle 2$ . Q.E.D.

Proof: Since  $\mathcal{T}$  is closed under union.

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**Corollary 59.1.** Let X be a set. Let  $\mathcal{B}$  be a basis for a topology  $\mathcal{T}$  on X. Then  $\mathcal{T}$  is the coarsest topology that includes  $\mathcal{B}$ .

PROOF: Since every topology that includes  $\mathcal B$  includes all unions of subsets of  $\mathcal B$ .  $\square$ 

**Lemma 60.** Let X be a topological space. Suppose that C is a set of open sets such that, for every open set U and every point  $x \in U$ , there exists  $C \in C$  such that  $x \in C \subseteq U$ . Then C is a basis for the topology on X.

### PROOF:

 $\langle 1 \rangle 1$ . For all  $x \in X$ , there exists  $C \in \mathcal{C}$  such that  $x \in C$ 

PROOF: Immediate from hypothesis.

 $\langle 1 \rangle 2$ . For all  $C_1, C_2 \in \mathcal{C}$  and  $x \in C_1 \cap C_2$ , there exists  $C_3 \in \mathcal{C}$  such that  $x \in C_3 \subseteq C_1 \cap C_2$ 

PROOF: Since  $C_1 \cap C_2$  is open.

 $\langle 1 \rangle 3$ . Every open set is open in the topology generated by  $\mathcal{C}$ 

PROOF: Immediate from hypothesis.

 $\langle 1 \rangle 4$ . Every union of a subset of  $\mathcal{C}$  is open.

Proof: Since every member of  $\mathcal{C}$  is open.

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**Lemma 61.** Let  $\mathcal{B}$  and  $\mathcal{B}'$  be bases for the topologies  $\mathcal{T}$  and  $\mathcal{T}'$  respectively on the set X. Then the following are equivalent.

- 1.  $\mathcal{T} \subseteq \mathcal{T}'$
- 2. For all  $B \in \mathcal{B}$  and  $x \in B$ , there exists  $B' \in \mathcal{B}'$  such that  $x \in B' \subseteq B$ .

### Proof:

- $\langle 1 \rangle 1. \ 1 \Rightarrow 2$ 
  - $\langle 2 \rangle 1$ . Assume:  $\mathcal{T} \subseteq \mathcal{T}'$
  - $\langle 2 \rangle 2$ . Let:  $B \in \mathcal{B}$  and  $x \in B$
  - $\langle 2 \rangle 3. \ B \in \mathcal{T}$

Proof: Corollary 59.1.

 $\langle 2 \rangle 4. \ B \in \mathcal{T}'$ 

Proof: By  $\langle 2 \rangle 1$ 

 $\langle 2 \rangle$ 5. There exists  $B' \in \mathcal{B}'$  such that  $x \in B' \subseteq B$ 

PROOF: Since  $\mathcal{B}'$  is a basis for  $\mathcal{T}'$ .

- $\langle 1 \rangle 2$ .  $2 \Rightarrow 1$ 
  - $\langle 2 \rangle 1$ . Assume: 2
  - $\langle 2 \rangle 2$ . Let:  $U \in \mathcal{T}$

Prove:  $U \in \mathcal{T}'$ 

 $\langle 2 \rangle 3$ . Let:  $x \in U$ 

PROVE: There exists  $B' \in \mathcal{B}'$  such that  $x \in B' \subseteq U$ 

```
\langle 2 \rangle4. PICK B \in \mathcal{B} such that x \in B \subseteq U
PROOF: Since \mathcal{B} is a basis for \mathcal{T}.
\langle 2 \rangle5. PICK B' \in \mathcal{B}' such that x \in B' \subseteq B
PROOF: By \langle 2 \rangle1.
\langle 2 \rangle6. x \in B' \subseteq U
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**Theorem 62.** Let X be a topological space,  $A \subseteq X$  and  $x \in X$ . Let  $\mathcal{B}$  be a basis for X. Then  $x \in \overline{A}$  if and only if, for all  $B \in \mathcal{B}$ , if  $x \in B$  then B intersects A.

#### PROOF

 $\langle 1 \rangle 1$ . If  $x \in \overline{A}$  then, for all  $B \in \mathcal{B}$ , if  $x \in B$  then B intersects A. PROOF: This follows from Theorem 45 since every element of  $\mathcal{B}$  is open (Corollary 59.1).

 $\langle 1 \rangle 2$ . Suppose that, for all  $B \in \mathcal{B}$ , if  $x \in B$  then B intersects A. Then  $x \in \overline{A}$ .

 $\langle 2 \rangle 1$ . Assume: For all  $B \in \mathcal{B}$ , if  $x \in B$  then B intersects A.

 $\langle 2 \rangle 2$ . Let: U be an open set that contains x Prove: U intersects A.

 $\langle 2 \rangle 3$ . Pick  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$ .

 $\langle 2 \rangle 4$ . B intersects A.

PROOF: From  $\langle 2 \rangle 1$ .

 $\langle 2 \rangle$ 5. U intersects A.

 $\langle 2 \rangle 6$ . Q.E.D.

PROOF: By Theorem 45.

**Definition 63** (Lower Limit Topology on the Real Line). The *lower limit topology on the real line* is the topology on  $\mathbb{R}$  generated by the basis consisting of all half-open intervals of the form [a,b).

We write  $\mathbb{R}_l$  for the topological space  $\mathbb{R}$  under the lower limit topology.

We prove this is a basis for a topology.

### Proof:

 $\langle 1 \rangle 1$ . For all  $x \in \mathbb{R}$  there exists an interval [a,b) such that  $x \in [a,b)$ . PROOF: Take [a,b) = [x,x+1).

 $\langle 1 \rangle 2$ . For any open intervals [a,b), [c,d) if  $x \in [a,b) \cap [c,d)$ , then there exists an interval [e,f) such that  $x \in [e,f) \subseteq [a,b) \cap [c,d)$ 

PROOF: Take  $[e, f) = [\max(a, c), \min(b, d)).$ 

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**Definition 64** (K-topology on the Real Line). Let  $K = \{1/n \mid n \in \mathbb{Z}^+\}$ .

The K-topology on the real line is the topology on  $\mathbb{R}$  generated by the basis consisting of all open intervals (a,b) and all sets of the form  $(a,b) \setminus K$ .

We write  $\mathbb{R}_K$  for the topological space  $\mathbb{R}$  under the K-topology.

We prove this is a basis for a topology.

- $\langle 1 \rangle 1$ . For all  $x \in \mathbb{R}$  there exists an open interval (a,b) such that  $x \in (a,b)$ . PROOF: Take (a,b) = (x-1,x+1).
- $\langle 1 \rangle$ 2. For any basic open sets  $B_1$ ,  $B_2$  if  $x \in B_1 \cap B_2$ , then there exists a basic open set  $B_3$  such that  $x \in B_3 \subseteq B_1 \cap B_2$ 
  - $\langle 2 \rangle 1$ . Case:  $B_1 = (a, b), B_2 = (c, d)$

PROOF: Take  $B_3 = (\max(a, c), \min(b, d))$ .

 $\langle 2 \rangle 2$ . CASE:  $B_1 = (a, b)$  or  $(a, b) \setminus K$ ,  $B_2 = (c, d)$  or  $(c, d) \setminus K$ , and they are not both open intervals.

PROOF: Take  $B_3 = (\max(a, c), \min(b, d)) \setminus K$ .

**Lemma 65.** The lower limit topology and the K-topology are incomparable.

### Proof:

 $\langle 1 \rangle 1$ . The interval [10, 11) is not open in the K-topology.

PROOF: There is no open interval (a,b) such that  $10 \in (a,b) \subseteq [10,11)$  or  $10 \in (a,b) \setminus K \subseteq [10,11)$ .

 $\langle 1 \rangle 2$ . The set  $(-1,1) \setminus K$  is not open in the lower limit topology.

PROOF: There is no half-open interval [a,b) such that  $0 \in [a,b) \subseteq (-1,1) \setminus K$ , since there must be a positive integer n with  $1/n \in [a,b)$ .

**Definition 66** (Subbasis). A *subbasis* S for a topology on X is a set  $S \subseteq PX$  such that  $\bigcup S = X$ .

The topology generated by the subbasis S is the set of all unions of finite intersections of elements of S.

We prove this is a topology.

### Proof:

 $\langle 1 \rangle 1$ . The set  $\mathcal B$  of all finite intersections of elements of  $\mathcal S$  forms a basis for a topology on X.

 $\langle 2 \rangle 1$ .  $| \mathcal{B} = X$ 

PROOF: Since  $S \subseteq B$ .

 $\langle 2 \rangle 2$ .  $\mathcal{B}$  is closed under binary intersection.

PROOF: By definition.

 $\langle 1 \rangle 2$ . Q.E.D.

PROOF: By Lemma 59.

We have simultaneously proved:

**Proposition 67.** Let S be a subbasis for the topology on X. Then the set of all finite intersections of elements of S is a basis for the topology on X.

**Proposition 68.** Let X be a set. Let S be a subbasis for a topology T on X. Then T is the coarsest topology that includes S.

PROOF: Since every topology that includes  $\mathcal S$  includes every union of finite intersections of elements of  $\mathcal S$ .  $\square$ 

## 11 Local Basis at a Point

**Definition 69** (Local Basis). Let X be a topological space and  $a \in X$ . A (local) basis at a is a set  $\mathcal{B}$  of neighbourhoods of a such that every neighbourhood of a includes some member of  $\mathcal{B}$ .

**Lemma 70.** If there exists a countable local basis at a point a, then there exists a countable local basis  $\{B_n \mid n \geq 1\}$  such that  $B_1 \supseteq B_2 \supseteq \cdots$ .

PROOF: Given a countable local basis  $\{C_n \mid n \geq 1\}$ , take  $B_n = C_1 \cap \cdots \cap C_n$ .  $\square$ 

# 12 Convergence

**Definition 71** (Convergence). Let X be a topological space. Let  $(a_n)_{n\in\mathbb{N}}$  be a sequence of points in X and  $l\in X$ . Then the sequence  $(a_n)_{n\in\mathbb{N}}$  converges to the limit l,  $a_n\to l$  as  $n\to\infty$ , if and only if, for every neighbourhood U of l, there exists N such that, for all  $n\geq N$ , we have  $a_n\in U$ .

**Theorem 72.** In a Hausdorff space, a sequence has at most one limit.

### Proof:

- $\langle 1 \rangle 1$ . Let: X be a Hausdorff space.
- $\langle 1 \rangle 2$ . Assume: for a contradiction  $a_n \to l$  as  $n \to \infty$ ,  $a_n \to m$  as  $n \to \infty$ , and  $l \neq m$
- $\langle 1 \rangle 3$ . PICK disjoint neighbourhoods U of l and V of m

Proof: By the Hausdorff axiom.

- $\langle 1 \rangle 4$ . PICK M and N such that  $a_n \in U$  for  $n \geq M$  and  $a_n \in V$  for  $n \geq N$
- $\langle 1 \rangle 5. \ a_{\max(M,N)} \in U \cap V$
- $\langle 1 \rangle 6$ . Q.E.D.

PROOF: This contradicts the fact that U and V are disjoint ( $\langle 1 \rangle 3$ ).

To see this is not always true in spaces that are  $T_1$  but not Hausdorff:

**Proposition 73.** Let X be an infinite set under the finite complement topology. Let  $(a_n)_{n\in\mathbb{N}}$  be a sequence with all points distinct. Then for every  $l\in X$  we have  $a_n\to l$  as  $n\to\infty$ .

PROOF: Let U be any neighbourhood of l. Since  $X \setminus U$  is finite, there must exist N such that, for all  $n \geq N$ , we have  $a_n \in U$ .  $\square$ 

**Lemma 74.** Let X be a topological space. Let  $A \subseteq X$  and  $l \in X$ . If there is a sequence of points in A that converges to l then  $l \in \overline{A}$ .

- $\langle 1 \rangle 1$ . Let:  $(a_n)$  be a sequence of points in A that converges to l.
- $\langle 1 \rangle 2$ . Let: *U* be a neighbourhood of *l*.
- $\langle 1 \rangle 3$ . Pick N such that, for all  $n \geq N$ , we have  $a_n \in U$ .
- $\langle 1 \rangle 4. \ a_N \in U \cap A$

 $\langle 1 \rangle$ 5. Q.E.D.

Proof: Theorem 45.

**Proposition 75.** Let X be a topological space. Let  $\mathcal{B}$  be a basis for the topology on X. Let  $(a_n)$  be a sequence in X and  $l \in X$ . Then  $a_n \to l$  as  $n \to \infty$  if and only if, for every  $B \in \mathcal{B}$  with  $l \in B$ , there exists N such that, for all  $n \geq N$ , we have  $a_n \in B$ .

#### Proof:

 $\langle 1 \rangle 1$ . If  $a_n \to l$  as  $n \to \infty$  then, for every  $B \in \mathcal{B}$  with  $l \in B$ , there exists N such that, for all  $n \ge N$ , we have  $a_n \in B$ .

PROOF: Since every element of  $\mathcal{B}$  is open (Corollary 59.1).

- $\langle 1 \rangle 2$ . If, for every  $B \in \mathcal{B}$  with  $l \in B$ , there exists N such that, for all  $n \geq N$ , we have  $a_n \in B$ , then  $a_n \to l$  as  $n \to \infty$ .
  - $\langle 2 \rangle 1$ . Assume: for every  $B \in \mathcal{B}$  with  $l \in B$ , there exists N such that, for all  $n \geq N$ , we have  $a_n \in B$
  - $\langle 2 \rangle 2$ . Let: *U* be a neighbourhood of *l*.
  - $\langle 2 \rangle 3$ . Pick  $B \in \mathcal{B}$  such that  $l \in B \subseteq U$
  - $\langle 2 \rangle 4$ . PICK N such that, for all  $n \geq N$ , we have  $a_n \in B$  PROOF: From  $\langle 2 \rangle 1$ .
- $\langle 2 \rangle 5$ . For all  $n \geq N$  we have  $a_n \in U$

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**Lemma 76.** If a sequence  $(a_n)$  is constant with  $a_n = l$  for all n, then  $a_n \to l$  as  $n \to \infty$ .

PROOF: Immediate from definitions.

**Theorem 77.** Let X be a linearly ordered set. Let  $(s_n)$  be a monotone increasing sequence in X with a supremum s. Then  $s_n \to s$  as  $n \to \infty$ .

### Proof:

 $\langle 1 \rangle 1$ . Assume: s is not least in X.

PROOF: Otherwise  $(s_n)$  is the constant sequence s and the result follows from Lemma 76.

- $\langle 1 \rangle 2$ . Let: U be a neighbourhood of s.
- $\langle 1 \rangle 3$ . Picka < s such that  $(a, s] \subseteq U$
- $\langle 1 \rangle 4$ . PICK N such that  $a < a_N$ .
- $\langle 1 \rangle 5$ . For all  $n \geq N$  we have  $a_n \in (a, s]$
- $\langle 1 \rangle 6$ . For all  $n \geq N$  we have  $a_n \in U$ .

**Theorem 78.** If  $\sum_{i=0}^{\infty} a_i = s$  and  $\sum_{i=0}^{\infty} b_i = t$  then  $\sum_{i=0}^{\infty} (ca_i + b_i) = cs + t$ .

PROOF: 
$$\sum_{i=0}^{N} (ca_i + b_i) = c \sum_{i=0}^{N} a_i + \sum_{i=0}^{N} b_i \to cs + t \text{ as } n \to \infty.$$

**Theorem 79** (Comparison Test). If  $|a_i| \leq b_i$  for all i and  $\sum_{i=0}^{\infty} b_i$  converges then  $\sum_{i=0}^{\infty} a_i$  converges.

Proof:

 $\langle 1 \rangle 1$ .  $\sum_{i=0}^{\infty} |a_i|$  converges

PROOF: The partial sums  $\sum_{i=0}^{N} |a_i|$  form an increasing sequence bounded above by  $\sum_{i=0}^{\infty} b_i$ .

- $\langle 1 \rangle 2$ . Let:  $c_i = |a_i| + a_i$  for all  $i \langle 1 \rangle 3$ .  $\sum_{i=0}^{\infty} c_i$  converges

PROOF: Each  $c_i$  is either  $2|a_i|$  or 0. So the partial sums  $\sum_{i=0}^{N} c_i$  form an increasing sequence bounded above by  $2\sum_{i=0}^{\infty} b_i$ .

 $\langle 1 \rangle 4$ . Q.E.D.

PROOF: Since  $a_i = c_i - |a_i|$ .

Corollary 79.1. If  $\sum_{i=0}^{\infty} |a_i|$  converges then  $\sum_{i=0}^{\infty} a_i$  converges.

**Theorem 80** (Weierstrass M-test). Let X be a set and  $(f_n: X \to \mathbb{R})$  be a sequence of functions. Let

$$s_n(x) = \sum_{i=0}^n f_i(x)$$

for all n, x. Suppose  $|f_i(x)| \leq M_i$  for all  $i \geq 0$  and  $x \in X$ . If the series  $\sum_{i=0}^{\infty} M_i$  converges, then the sequence  $(s_n)$  converges uniformly to

$$s(x) = \sum_{i=0}^{\infty} f_i(x) .$$

 $\langle 1 \rangle 1$ . Let:  $r_n = \sum_{i=n+1}^{\infty} M_i$  for all  $n \langle 1 \rangle 2$ . Given  $0 \le n < k$ , we have  $|s_k(x) - s_n(x)| \le r_n$ 

$$|s_k(x) - s_n(x)| = |\sum_{i=n+1}^k f_i(x)|$$

$$\leq \sum_{i=n+1}^k |f_i(x)|$$

$$\leq \sum_{i=n+1}^k M_i$$

 $\langle 1 \rangle 3$ . Given  $n \geq 0$  we have  $|s(x) - s_n(x)| \leq r_n$ PROOF: By taking the limit  $k \to \infty$  in  $\langle 1 \rangle 2$ .

 $\langle 1 \rangle 4$ . Q.E.D.

PROOF: Since  $r_n \to 0$  as  $n \to \infty$ .

# 13 Locally Finite Sets

**Definition 81** (Locally Finite). Let X be a topological space and  $\{A_{\alpha}\}$  a family of subsets of X. Then  $\mathcal{A}$  is *locally finite* if and only if every point in X has a neighbourhood that intersects  $A_{\alpha}$  for only finitely many  $\alpha$ .

**Theorem 82** (Pasting Lemma). Let X and Y be topological spaces and  $f: X \to Y$ . Let  $\{A_{\alpha}\}$  be a locally finite family of closed subsets of X that cover X. Suppose  $f \upharpoonright A_{\alpha}$  is continuous for all  $\alpha$ . Then f is continuous.

#### **PROOF**

- $\langle 1 \rangle 1$ . Let X and Y be topological spaces and  $f: X \to Y$ . Let A and B be closed subsets of X such that  $X = A \cup B$ . Suppose  $f \upharpoonright A$  and  $f \upharpoonright B$  are continuous. Then f is continuous.
  - $\langle 2 \rangle 1$ . Let:  $C \subseteq Y$  be closed.
  - $\langle 2 \rangle 2$ .  $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$
  - $\langle 2 \rangle 3$ .  $f^{-1}(C)$  and  $g^{-1}(C)$  are closed in X.

PROOF: Theorems 92 and 143.

 $\langle 2 \rangle 4$ .  $h^{-1}(C)$  is closed in X.

Proof: Lemma 31.

 $\langle 2 \rangle 5$ . Q.E.D.

PROOF: Theorem 92.

 $\langle 1 \rangle$ 2. Let X and Y be topological spaces and  $f: X \to Y$ . Let  $\{A_{\alpha}\}$  be a finite family of closed subsets of X that cover X. Suppose  $f \upharpoonright A_{\alpha}$  is continuous for all  $\alpha$ . Then f is continuous.

PROOF: From  $\langle 1 \rangle 1$  by induction.

- $\langle 1 \rangle 3$ . Let X and Y be topological spaces and  $f: X \to Y$ . Let  $\{A_{\alpha}\}$  be a locally finite family of closed subsets of X that cover X. Suppose  $f \upharpoonright A_{\alpha}$  is continuous for all  $\alpha$ . Then f is continuous.
  - $\langle 2 \rangle 1$ . Let:  $x \in X$

Prove: f is continuous at x

- $\langle 2 \rangle 2$ . Pick a neighbourhood U of x that intersects  $A_{\alpha}$  for only finitely many  $\alpha$ .
- $\langle 2 \rangle 3$ .  $f \upharpoonright U$  is continuous

Proof: By  $\langle 1 \rangle 2$ .

 $\langle 2 \rangle 4$ . Q.E.D.

Proof: Lemma 102.

The following example shows that we cannot remove the assumption of local finiteness.

**Example 83.** Define  $f:[-1,1] \to \mathbb{R}$  by: f(x)=1 if x<-1, f(x)=0 if x>1. Let  $C_n=[-1,-1/n]$  for  $n\geq 1$ , and D=[0,1]. Then  $[-1,1]=\bigcup_{n=1}^{\infty}C_n\cup D$  and f is continuous on each  $C_n$  and each D, but f is not continuous on [-1,1].

# 14 Open Maps

**Definition 84** (Open Map). Let X and Y be topological spaces and  $f: X \to Y$ . Then f is an *open map* if and only if, for every open set U in X, the set f(U) is open in Y.

**Lemma 85.** Let X and Y be topological spaces and  $f: X \to Y$ . Let  $\mathcal{B}$  be a basis for the topology on X. If f(B) is open in Y for all  $B \in \mathcal{B}$ , then f is an open map.

PROOF: From Lemma 59.

**Proposition 86.** Let X and Y be topological spaces. Let  $\mathcal{B}$  be a basis for the topology on X. Let  $f: X \to Y$ . Suppose that, for all  $B \in \mathcal{B}$ , we have f(B) is open to Y. Then f is an open map.

PROOF: For any  $\mathcal{A} \subseteq \mathcal{B}$ , we have  $f(\bigcup \mathcal{A}) = \bigcup_{B \in \mathcal{B}} f(B)$  is open in Y. The result follows from Lemma 59.  $\square$ 

## 15 Continuous Functions

**Definition 87** (Continuous). Let X and Y be topological spaces. A function  $f: X \to Y$  is *continuous* if and only if, for every open set V in Y, the set  $f^{-1}(V)$  is open in X.

**Proposition 88.** Let X and Y be topological spaces and  $f: X \to Y$ . Let  $\mathcal{B}$  be a basis for Y. Then f is continuous if and only if, for all  $B \in \mathcal{B}$ , we have  $f^{-1}(B)$  is open in X.

Proof:

- $\langle 1 \rangle 1$ . If f is continuous then, for all  $B \in \mathcal{B}$ , we have  $f^{-1}(B)$  is open in X. PROOF: Since every element of B is open (Lemma 59).
- $\langle 1 \rangle 2$ . Suppose that, for all  $B \in \mathcal{B}$ , we have  $f^{-1}(B)$  is open in X. Then f is continuous.
  - $\langle 2 \rangle 1$ . Assume: For all  $B \in \mathcal{B}$ , we have  $f^{-1}(B)$  is open in X.
  - $\langle 2 \rangle 2$ . Let: V be open in Y.
  - $\langle 2 \rangle 3$ . Pick  $\mathcal{A} \subseteq \mathcal{B}$  such that  $V = \bigcup \mathcal{A}$

PROOF: By Lemma 59.

 $\langle 2 \rangle 4$ .  $f^{-1}(V)$  is open in X.

Proof:

$$\begin{split} f^{-1}(V) &= f^{-1}\left(\bigcup \mathcal{A}\right) \\ &= \bigcup_{B \in \mathcal{A}} f^{-1}(B) \end{split}$$

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**Proposition 89.** Let X and Y be topological spaces and  $f: X \to Y$ . Let S be a subbasis for Y. Then f is continuous if and only if, for all  $S \in S$ , we have  $f^{-1}(S)$  is open in X.

### PROOF:

- $\langle 1 \rangle 1$ . If f is continuous then, for all  $S \in \mathcal{S}$ , we have  $f^{-1}(S)$  is open in X. PROOF: Since every element of S is open.
- $\langle 1 \rangle 2$ . Suppose that, for all  $S \in \mathcal{S}$ , we have  $f^{-1}(S)$  is open in X. Then f is continuous.
  - $\langle 2 \rangle 1$ . Assume: For all  $S \in \mathcal{S}$ , we have  $f^{-1}(S)$  is open in X.
  - $\langle 2 \rangle 2$ . Let:  $S_1, \ldots, S_n \in \mathcal{S}$
  - $\langle 2 \rangle 3.$   $f^{-1}(S_1 \cap \cdots \cap S_n)$  is open in A

PROOF: Since  $f^{-1}(S_1 \cap \cdots \cap S_n) = f^{-1}(S_1) \cap \cdots \cap f^{-1}(S_n)$ .

 $\langle 2 \rangle 4$ . Q.E.D.

Proof: By Propositions 88 and 67.

**Proposition 90.** Let X and Y be topological spaces and  $f: X \to Y$ . Let S be a basis for Y. Then f is continuous if and only if, for all  $V \in S$ , we have  $f^{-1}(V)$  is open in X.

### Proof:

- $\langle 1 \rangle 1$ . If f is continuous then, for all  $V \in \mathcal{S}$ , we have  $f^{-1}(V)$  is open in X. PROOF: Since every element of  $\mathcal{S}$  is open.
- $\langle 1 \rangle 2$ . Suppose that, for all  $V \in \mathcal{S}$ , we have  $f^{-1}(V)$  is open in X. Then f is continuous.
  - $\langle 2 \rangle 1$ . Assume: For all  $V \in \mathcal{S}$ , we have  $f^{-1}(V)$  is open in X.
  - $\langle 2 \rangle 2$ . For every set B that is the finite intersection of elemets of  $\mathcal{S}$ , we have  $f^{-1}(B)$  is open in X.

PROOF: Because  $f^{-1}(V_1 \cap \cdots \cap V_n) = f^{-1}(V_1) \cap \cdots \cap f^{-1}(V_n)$ .

 $\langle 2 \rangle 3$ . Q.E.D.

Proof: From Propositions 67 and 88.

**Definition 91** (Continuous at a Point). Let X and Y be topological spaces. Let  $f: X \to Y$  and  $x \in X$ . Then f is *continuous at* x if and only if, for every neighbourhood V of f(x), there exists a neighbourhood U of x such that  $f(U) \subseteq V$ .

**Theorem 92.** Let X and Y be topological spaces and  $f: X \to Y$ . Then the following are equivalent:

- 1. f is continuous.
- 2. For all  $A \subseteq X$ , we have  $f(\overline{A}) \subseteq \overline{f(A)}$
- 3. For all  $B \subseteq Y$  closed, we have  $f^{-1}(B)$  is closed in X.
- 4. f is continuous at every point of X.

- $\langle 1 \rangle 1. \ 1 \Rightarrow 2$ 
  - $\langle 2 \rangle 1$ . Assume: f is continuous.

```
\langle 2 \rangle 2. Let: A \subseteq X
   \langle 2 \rangle 3. Let: x \in \overline{A}
             Prove: f(x) \in \overline{f(A)}
   \langle 2 \rangle 4. Let: V be a neighbourhood of f(x)
   \langle 2 \rangle 5. f^{-1}(V) is a neighbourhood of x
   \langle 2 \rangle 6. Pick y \in A \cap f^{-1}(V)
       PROOF: By Theorem 45.
   \langle 2 \rangle 7. \ f(y) \in V \cap f(A)
   \langle 2 \rangle 8. Q.E.D.
       PROOF: By Theorem 45.
\langle 1 \rangle 2. \ 2 \Rightarrow 3
    \langle 2 \rangle 1. Assume: 2
   \langle 2 \rangle 2. Let: B be closed in Y
   \langle 2 \rangle 3. Let: x \in \overline{f^{-1}(B)}
                      PROVE: x \in f^{-1}(B)
   \langle 2 \rangle 4. \ f(x) \in B
       Proof:
                             f(x) \in f(\overline{f^{-1}(B)})
                                     \subseteq \overline{f(f^{-1}(B))}
                                                                                                  (\langle 2 \rangle 1)
                                     \subseteq \overline{B}
                                                                                 (Proposition 46)
                                     = B
\langle 1 \rangle 3. \ 3 \Rightarrow 1
   \langle 2 \rangle 1. Assume: 3
   \langle 2 \rangle 2. Let: V be open in Y
   \langle 2 \rangle 3. Y \setminus V is closed in Y
   \langle 2 \rangle 4. f^{-1}(Y \setminus V) is closed in X
   \langle 2 \rangle 5. X \setminus f^{-1}(V) is closed in X
   \langle 2 \rangle 6. f^{-1}(V) is open in X
\langle 1 \rangle 4. \ 1 \Rightarrow 4
   PROOF: For any neighbourhood V of f(x), the set U = f^{-1}(V) is a neigh-
   bourhood of x such that f(U) \subseteq V.
\langle 1 \rangle 5. \ 4 \Rightarrow 1
   \langle 2 \rangle 1. Assume: 4
   \langle 2 \rangle 2. Let: V be open in Y
   \langle 2 \rangle 3. Let: x \in f^{-1}(V)
   \langle 2 \rangle 4. V is a neighbourhood of f(x)
   \langle 2 \rangle5. PICK a neighbourhood U of x such that f(U) \subseteq V
   \langle 2 \rangle 6. \ x \in U \subseteq f^{-1}(V)
   \langle 2 \rangle 7. Q.E.D.
       PROOF: By Lemma 23.
```

**Theorem 93.** A constant function is continuous.

PROOF: Let X and Y be topological spaces. Let  $b \in Y$ , and let  $f: X \to Y$  be the constant function with value b. For any open  $V \subseteq Y$ , the set  $f^{-1}(V)$  is

either X (if  $b \in V$ ) or  $\emptyset$  (if  $b \notin V$ ).  $\square$ 

**Theorem 94.** If A is a subspace of X then the inclusion  $j: A \to X$  is continuous

PROOF: For any V open in X, we have  $j^{-1}(V) = V \cap A$  is open in A.  $\square$ 

**Theorem 95.** The composite of two continuous functions is continuous.

PROOF: Let  $f: X \to Y$  and  $g: Y \to Z$  be continuous. For any V open in Z, we have  $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$  is open in X.  $\square$ 

**Theorem 96.** Let  $f: X \to Y$  be a continuous function and A be a subspace of X. Then the restriction  $f \upharpoonright A: A \to Y$  is continuous.

PROOF: Let V be open in Y. Then  $(f \upharpoonright A)^{-1}(V) = f^{-1}(V) \cap A$  is open in A.  $\square$ 

**Theorem 97.** Let  $f: X \to Y$  be continuous. Let Z be a subspace of Y such that  $f(X) \subseteq Z$ . Then the corestriction  $f: X \to Z$  is continuous.

### Proof:

- $\langle 1 \rangle 1$ . Let: V be open in Z.
- $\langle 1 \rangle 2$ . PICK U open in Y such that  $V = U \cap Z$ .
- $\langle 1 \rangle 3. \ f^{-1}(V) = f^{-1}(U)$
- $\langle 1 \rangle 4$ .  $f^{-1}(V)$  is open in X.

**Theorem 98.** Let  $f: X \to Y$  be continuous. Let Z be a space such that Y is a subspace of Z. Then the expansion  $f: X \to Z$  is continuous.

PROOF: Let V be open in Z. Then  $f^{-1}(V) = f^{-1}(V \cap Y)$  is open in X.  $\square$ 

**Theorem 99.** Let X and Y be topological spaces. Let  $f: X \to Y$ . Suppose  $\mathcal{U}$  is a set of open sets in X such that  $X = \bigcup \mathcal{U}$  and, for all  $U \in \mathcal{U}$ , we have  $f \upharpoonright U : U \to Y$  is continuous. Then f is continuous.

### Proof:

- $\langle 1 \rangle 1$ . Let: V be open in Y
- $\langle 1 \rangle 2. \ f^{-1}(V) = \bigcup_{U \in \mathcal{U}} (f \restriction U)^{-1}(V)$
- $\langle 1 \rangle 3$ . For all  $U \in \mathcal{U}$ , we have  $(f \upharpoonright U)^{-1}(V)$  is open in U.
- $\langle 1 \rangle 4$ . For all  $U \in \mathcal{U}$ , we have  $(f \upharpoonright U)^{-1}(V)$  is open in X. PROOF: Lemma 142.

П

**Proposition 100.** Let X and X' be the same set X under two topologies  $\mathcal{T}$  and  $\mathcal{T}'$ . Let  $i: X \to X'$  be the identity function. Then i is continuous if and only if  $\mathcal{T}' \subset \mathcal{T}$ .

PROOF: Immediate from definitions.

**Proposition 101.** Let  $f : \mathbb{R} \to \mathbb{R}$  and  $a \in \mathbb{R}$ . Then f is continuous on the right at a if and only if f is continuous at a as a function  $\mathbb{R}_l \to \mathbb{R}$ .

### Proof:

- $\langle 1 \rangle 1$ . If f is continuous on the right at a then f is continuous at a as a function  $\mathbb{R}_l \to \mathbb{R}$ .
  - $\langle 2 \rangle 1$ . Assume: f is continuous on the right at a.
  - $\langle 2 \rangle 2$ . Let: V be a neighbourhood of f(a)
  - $\langle 2 \rangle 3$ . Pick b, c such that  $f(a) \in (b,c) \subseteq V$ .
  - $\langle 2 \rangle 4$ . Let:  $\epsilon = \min(c f(a), f(a) b)$
  - $\langle 2 \rangle$ 5. PICK  $\delta > 0$  such that, for all x, if  $a < x < a + \delta$  then  $|f(x) f(a)| < \epsilon$
  - $\langle 2 \rangle 6$ . Let:  $U = [a, a + \delta)$
  - $\langle 2 \rangle 7. \ f(U) \subseteq V$
- $\langle 1 \rangle 2$ . If f is continuous at a as a function  $\mathbb{R}_l \to \mathbb{R}$  then f is continuous on the right at a.
  - $\langle 2 \rangle 1$ . Assume: f is continuous at a as a function  $\mathbb{R}_l \to \mathbb{R}$
  - $\langle 2 \rangle 2$ . Let:  $\epsilon > 0$
  - $\langle 2 \rangle 3$ . PICK a neighbourhood U of a such that  $f(U) \subseteq (f(a) \epsilon, f(a) + \epsilon)$
  - $\langle 2 \rangle 4$ . Pick b, c such that  $a \in [b, c) \subset U$
  - $\langle 2 \rangle$ 5. Let:  $\delta = c a$
- $\langle 2 \rangle 6$ . For all x, if  $a < x < a + \delta$  then  $|f(x) f(a)| < \epsilon$

**Lemma 102.** Let  $f: X \to Y$ . Let Z be an open subspace of X and  $a \in Z$ . If  $f \upharpoonright Z$  is continuous at a then f is continuous at a.

#### Proof

- $\langle 1 \rangle 1$ . Let: V be a neighbourhood of f(x)
- $\langle 1 \rangle 2$ . PICK a neighbourhood W of x in Z such that  $f(W) \subseteq V$
- $\langle 1 \rangle 3.$  W is a neighbourhood of x in X such that  $f(W) \subseteq V$  PROOF: Lemma 142.

**Proposition 103.** Let  $f: A \to B$  and  $g: C \to D$  be continuous. Define  $f \times g: A \times C \to B \times D$  by

$$(f \times g)(a,c) = (f(a), g(c)) .$$

Then  $f \times g$  is continuous.

PROOF:  $\pi_1 \circ (f \times g) = f \circ \pi_1$  and  $\pi_2 \circ (f \times g) = g \circ \pi_2$  are continuous by Theorem 95. The result follows by Theorem 132.

**Proposition 104.** Let X be a topological space. Let Y a Hausdorff space. Let  $A \subseteq X$ . Let  $f, g : \overline{A} \to Y$  be continuous. If f and g agree on A then f = g.

- $\langle 1 \rangle 1$ . Let:  $x \in \overline{A}$
- $\langle 1 \rangle 2$ . Assume:  $f(x) \neq g(x)$
- $\langle 1 \rangle 3$ . PICK disjoint neighbourhoods V of f(x) and W of g(x).
- $\langle 1 \rangle 4$ . Pick  $y \in f^{-1}(V) \cap g^{-1}(W) \cap A$

PROOF: Since  $f^{-1}(V) \cap g^{-1}(W)$  is a neighbourhood of x and hence intersects A.

- $\langle 1 \rangle 5. \ f(y) = g(y) \in V \cap W$
- $\langle 1 \rangle 6$ . Q.E.D.

PROOF: This contradicts the fact that V and W are disjoint  $(\langle 1 \rangle 3)$ .

**Proposition 105.** Let X and Y be topological spaces and  $f: X \to Y$  be continuous. If  $a_n \to l$  as  $n \to \infty$  in X then  $f(a_n) \to f(l)$  as  $n \to \infty$ .

#### Proof:

- $\langle 1 \rangle 1$ . Let: V be a neighbourhood of f(l)
- $\langle 1 \rangle 2$ . PICK a neighbourhood U of l such that  $f(U) \subseteq V$
- $\langle 1 \rangle 3$ . Pick N such that, for all  $n \geq N$ , we have  $a_n \in U$
- $\langle 1 \rangle 4$ . For all  $n \geq N$  we have  $f(n) \in V$

# 16 Homeomorphisms

**Definition 106** (Homeomorphism). Let X and Y be topological spaces. A Homeomorphism f between X and Y,  $f: X \cong Y$ , is a bijection  $f: X \to Y$  such that both f and  $f^{-1}$  are continuous.

**Lemma 107.** Let X and Y be topological spaces and  $f: X \to Y$  a bijection. Then the following are equivalent:

- 1. f is a homeomorphism.
- 2. f is continuous and an open map.
- 3. For any  $U \subseteq X$ , we have U is open if and only if f(U) is open.

PROOF: Immediate from definitions.  $\square$ 

**Proposition 108.** Let X and X' be the same set X under two topologies  $\mathcal{T}$  and  $\mathcal{T}'$ . Let  $i: X \to X'$  be the identity function. Then i is a homeomorphism if and only if  $\mathcal{T} = \mathcal{T}'$ .

PROOF: Immediate from definitions.

**Definition 109** (Topological Property). Let P be a property of topological spaces. Then P is a *topological* property if and only if, for any spaces X and Y, if P holds of X and  $X \cong Y$  then P holds of Y.

**Definition 110** (Topological Imbedding). Let X and Y be topological spaces and  $f: X \to Y$ . Then f is a topological imbedding if and only if the corestriction  $f: X \to f(X)$  is a homeomorphism.

**Proposition 111.** Let X and Y be topological spaces and  $a \in X$ . The function  $i: Y \to X \times Y$  that maps y to (a, y) is an imbedding.

```
Proof:
```

- $\langle 1 \rangle 1$ . *i* is injective
- $\langle 1 \rangle 2$ . *i* is continuous.

PROOF: For U open in X and V open in Y, we have  $i^{-1}(U \times V)$  is V if  $a \in U$ , and  $\emptyset$  if  $a \notin U$ .

 $\langle 1 \rangle 3.$   $i: Y \to i(Y)$  is an open map.

PROOF: For V open in Y we have  $i(V) = (X \times V) \cap i(Y)$ .

# 17 The Order Topology

**Definition 112** (Order Topology). Let X be a linearly ordered set with at least two points. The *order topology* on X is the topology generated by the basis  $\mathcal{B}$  consisting of:

- all open intervals (a, b);
- all intervals of the form  $[\bot, b)$  where  $\bot$  is least in X;
- all intervals of the form  $(a, \top]$  where  $\top$  is greatest in X.

We prove this is a basis for a topology.

### Proof:

- $\langle 1 \rangle 1$ . For all  $x \in X$  there exists  $B \in \mathcal{B}$  such that  $x \in B$ .
  - $\langle 2 \rangle 1$ . Let:  $x \in X$
  - $\langle 2 \rangle 2$ . Case: x is greatest in X.
    - $\langle 3 \rangle 1$ . Pick  $y \in X$  with  $y \neq x$
    - $\langle 3 \rangle 2. \ x \in (y, x] \in \mathcal{B}$
  - $\langle 2 \rangle 3$ . Case: x is least in X.
    - $\langle 3 \rangle 1$ . Pick  $y \in X$  with  $y \neq x$
    - $\langle 3 \rangle 2. \ x \in [x,y) \in \mathcal{B}$
  - $\langle 2 \rangle 4$ . Case: x is neither greatest nor least in X.
    - $\langle 3 \rangle 1$ . Pick  $a, b \in X$  with a < x and x < b
    - $\langle 3 \rangle 2. \ x \in (a,b) \in \mathcal{B}$
- $\langle 1 \rangle 2$ . For all  $B_1, B_2 \in \mathcal{B}$  and  $x \in B_1 \cap B_2$ , there exists  $B_3 \in \mathcal{B}$  such that  $x \in B_3 \subseteq B_1 \cap B_2$ 
  - $\langle 2 \rangle 1$ . Let:  $B_1, B_2 \in \mathcal{B}$  and  $x \in B_1 \cap B_2$
  - $\langle 2 \rangle 2$ . Case:  $B_1 = (a, b), B_2 = (c, d)$

PROOF: Take  $B_3 = (\max(a, c), \min(b, d))$ .

- $\langle 2 \rangle 3$ . Case:  $B_1 = (a, b), B_2 = [\bot, d)$ 
  - PROOF: Take  $B_3 = (a, \min(b, d))$ .
- $\langle 2 \rangle 4$ . Case:  $B_1 = (a, b), B_2 = (c, \top]$

PROOF: Take  $B_3 = (\max(a, c), b)$ .

 $\langle 2 \rangle 5$ . Case:  $B_1 = [\bot, b), B_2 = [\bot, d)$ 

PROOF: Take  $B_3 = [\bot, \min(b, d))$ .

 $\langle 2 \rangle 6$ . Case:  $B_1 = [\bot, b), B_2 = (c, \top]$ 

```
PROOF: Take B_3 = (c, b).
Lemma 113. Let X be a linearly ordered set. Then the open rays form a
subbasis for the order topology on X.
Proof:
\langle 1 \rangle 1. Every open ray is open.
   \langle 2 \rangle 1. For all a \in X, the ray (-\infty, a) is open.
      \langle 3 \rangle 1. Let: x \in (-\infty, a)
     \langle 3 \rangle 2. Case: x is least in X
        PROOF: xin[x, a) = (-\infty, a).
      \langle 3 \rangle 3. Case: x is not least in X
        \langle 4 \rangle 1. Pick y < x
        \langle 4 \rangle 2. \ x \in (y,a) \subseteq (-\infty,a)
   \langle 2 \rangle 2. For all a \in X, the ray (a, +\infty) is open.
     PROOF: Similar.
\langle 1 \rangle 2. Every basic open set is a finite intersection of open rays.
  PROOF: We have (a,b) = (a,+\infty) \cap (-\infty,b), [\bot,b] = (-\infty,b) and (a,\top] =
  (a, +\infty).
Definition 114 (Standard Topology on the Real Line). The standard topology
on the real line is the order topology on \mathbb{R} generated by the standard order.
Lemma 115. The lower limit topology is strictly finer than the standard topol-
ogy on \mathbb{R}.
Proof:
\langle 1 \rangle 1. Every open interval is open in the lower limit topology.
  PROOF: If x \in (a, b) then x \in [x, b) \subseteq (a, b).
\langle 1 \rangle 2. The half-open interval [0,1) is not open in the standard topology.
   PROOF: There is no open interval (a, b) such that 0 \in (a, b) \subseteq [0, 1).
Lemma 116. The K-topology is strictly finer than the standard topology on \mathbb{R}.
Proof:
\langle 1 \rangle 1. Every open interval is open in the K-topology.
  Proof: Corollary 59.1.
\langle 1 \rangle 2. The set (-1,1) \setminus K is not open in the standard topology.
```

**Lemma 117.** Let X be a topological space. Let Y be a linearly ordered set in the order topology. Let  $f, g: X \to Y$  be continuous. Then  $C = \{x \in X \mid f(x) \leq g(x)\}$  is closed.

PROOF: There is no open interval (a,b) such that  $0 \in (a,b) \subseteq (-1,1) \setminus K$ ,

since there must be a positive integer n with  $1/n \in (a, b)$ .

Proof:

**Proposition 118.** Let X be a topological space. Let Y be a linearly ordered set in the order topology. Let  $f, g: X \to Y$  be continuous. Define  $h: X \to Y$  by  $h(x) = \min(f(x), g(x))$ . Then h is continuous.

PROOF: By the Pasting Lemma applied to  $\{x \in X \mid f(x) \leq g(x)\}$  and  $\{x \in X \mid g(x) \leq f(x)\}$ , which are closed by Lemma 117.

**Proposition 119.** Let X and Y be linearly ordered sets in the order topology. Let  $f: X \to Y$  be strictly monotone and surjective. Then f is a homeomorphism.

```
Proof:
```

 $\langle 1 \rangle 1$ . f is bijective.

Proof: Proposition 3.

- $\langle 1 \rangle 2$ . f is continuous.
  - $\langle 2 \rangle 1$ . For all  $y \in Y$  we have  $f^{-1}((y, +\infty))$  is open.
    - $\langle 3 \rangle 1$ . Let:  $y \in Y$
    - $\langle 3 \rangle 2$ . PICK $x \in X$  such that f(x) = y

Proof: Since f is surjective.

$$\langle 3 \rangle 3. \ f^{-1}((y, +\infty)) = (x, +\infty)$$

PROOF: By strict monotoncity.

 $\langle 2 \rangle 2$ . For all  $y \in Y$  we have  $f^{-1}((-\infty, y))$  is open.

Proof: Similar.

- $\langle 1 \rangle 3$ .  $f^{-1}$  is continuous.
  - $\langle 2 \rangle 1$ . For all  $x \in X$  we have  $f((x, +\infty))$  is open.

PROOF:  $f((x, +\infty)) = (f(x), +\infty)$ .

 $\langle 2 \rangle 2$ . For all  $x \in X$  we have  $f((-\infty, x))$  is open.

PROOF:  $f((-\infty, x)) = (-\infty, f(x))$ .

П

### 18 The nth Root Function

**Proposition 120.** For all  $n \geq 1$ , the function  $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$  defined by  $f(x) = x^n$  is a homemorphism.

- $\langle 1 \rangle 1$ . f is strictly monotone.
  - $\langle 2 \rangle 1$ . Let:  $x, y \in \mathbb{R}$  with  $0 \le x < y$

$$\langle 2 \rangle 2$$
.  $x^n < y^n$ 

$$y^{n} - x^{n} = (y - x)(y^{n-1} + y^{n-2}x + y^{n-3}x^{2} + \dots + x^{n-1})$$
  
> 0

- $\langle 1 \rangle 2$ . f is surjective.
  - $\langle 2 \rangle 1$ . Let:  $y \in \mathbb{R}_{>0}$
  - $\langle 2 \rangle 2$ . PICK  $x \in \mathbb{R}$  such that  $y \leq x^n$

PROOF: If  $y \le 1$  take x = 1, otherwise take x = y.

 $\langle 2 \rangle 3$ . There exists  $x' \in [0, x]$  such that  $(x')^n = y$ 

PROOF: By the Intermediate Value Theorem.

 $\langle 1 \rangle 3$ . Q.E.D.

PROOF: Proposition 119.

**Definition 121.** For  $n \geq 1$ , the *nth root function* is the function  $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$  that is the inverse of  $\lambda x.x^n$ .

# 19 The Product Topology

**Definition 122** (Product Topology). Let  $\{A_i\}_{i\in I}$  be a family of topological spaces. The *product topology* on  $\prod_{i\in I} A_i$  is the topology generated by the subbasis consisting of the sets of the form  $\pi_i^{-1}(U)$  where  $i\in I$  and U is open in  $A_i$ .

**Proposition 123.** The product topology on  $\prod_{i \in I} A_i$  is generated by the basis consisting of all sets of the form  $\prod_{i \in I} U_i$  where  $\{U_i\}_{i \in I}$  is a family such that each  $U_i$  is an open set in  $A_i$  and  $U_i = A_i$  for all but finitely many i.

Proof: From Proposition 67.  $\square$ 

**Proposition 124.** If  $A_i$  is closed in  $X_i$  for all  $i \in I$  then  $\prod_{i \in I} A_i$  is closed in  $\prod_{i \in I} X_i$ .

Proof:

$$\left(\prod_{i\in I} X_i\right) \setminus \left(\prod_{i\in I} A_i\right) = \bigcup_{j\in I} \left(\prod_{i\in I} X_i \setminus \pi_j^{-1}(A_j)\right) \square$$

**Proposition 125.** Let  $\{A_i\}_{i\in I}$  be a family of topological spaces. For  $i\in I$ , let  $\mathcal{B}_i$  be a basis for the topology on  $A_i$ . Then  $\mathcal{B}=\{\prod_{i\in I}B_i\mid \forall i\in I.B_i\in \mathcal{B}_i, B_i=A_i \text{ for all but finitely many } i\}$  is a basis for the box topology on  $\prod_{i\in I}A_i$ .

- $\langle 1 \rangle 1$ . Every set in  $\mathcal{B}$  is open.
- $\langle 1 \rangle 2$ . For every point  $a \in \prod_{i \in I} A_i$  and every open set U with  $a \in U$ , there exists  $B \in \mathcal{B}$  such that  $a \in B \subseteq U$ .
  - $\langle 2 \rangle 1$ . Let: U be open and  $a \in U$

- $\langle 2 \rangle 2$ . PICK a family  $\{U_i\}_{i \in I}$  such that each  $U_i$  is open in  $A_i$ , such that  $U_i = A_i$  except for  $i = i_1, \ldots, i_n$ , and such that  $a \in \prod_{i \in I} U_i \subseteq U$ .
- $\langle 2 \rangle 3$ . For  $j = 1, \ldots, n$ , PICK  $B_{i_j} \in \mathcal{B}_{i_j}$  such that  $a_{i_j} \in B_{i_j} \subseteq U_{i_j}$
- $\langle 2 \rangle 4$ . Let:  $B = \prod_{i \in I} B_i$  where  $B_i = A_i$  for  $i \neq i_1, \ldots, i_n$
- $\langle 2 \rangle 5. \ B \in \mathcal{B}$
- $\langle 2 \rangle 6. \ a \in B \subseteq U$
- $\langle 1 \rangle 3$ . Q.E.D.

Proof: Lemma 60.

**Proposition 126.** Let  $\{A_i\}_{i\in I}$  be a family of topological spaces. Then the projections  $\pi_i:\prod_{i\in I}A_i\to A_i$  are open maps.

Proof: From Lemma 85.  $\square$ 

**Example 127.** The projections are not always closed maps. For example,  $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$  maps the closed set  $\{(x, 1/x) \mid x > 0\}$  to  $(0, +\infty)$ .

**Proposition 128.** Let  $\{X_i\}_{i\in I}$  be a family of sets. For  $i\in I$ , let  $\mathcal{T}_i$  and  $\mathcal{U}_i$  be topologies on  $X_i$ . Let  $\mathcal{P}$  be the product topology on  $\prod_{i\in I}X_i$  generated by the topologies  $\mathcal{T}_i$ , and  $\mathcal{Q}$  the product topology on the same set generated by the topologies  $\mathcal{U}_i$ . Then  $\mathcal{P}\subseteq\mathcal{Q}$  if and only if  $\mathcal{T}_i\subseteq\mathcal{U}_i$  for all i.

### Proof:

 $\langle 1 \rangle 1$ . If  $\mathcal{T}_i \subseteq \mathcal{U}_i$  for all i then  $\mathcal{P} \subseteq \mathcal{Q}$ 

PROOF: By Corollary 59.1.

- $\langle 1 \rangle 2$ . If  $\mathcal{P} \subseteq \mathcal{Q}$  then  $\mathcal{T}_i \subseteq \mathcal{U}_i$  for all i
  - $\langle 2 \rangle 1$ . Assume:  $\mathcal{P} \subseteq \mathcal{Q}$
  - $\langle 2 \rangle 2$ . Let:  $i \in I$
  - $\langle 2 \rangle 3$ . Let:  $U \in \mathcal{T}_i$
  - $\langle 2 \rangle 4$ . Let:  $U_i = U$  and  $U_j = X_j$  for  $j \neq i$
  - $\langle 2 \rangle 5. \prod_{i \in I} U_i \in \mathcal{P}$
  - $\langle 2 \rangle 6. \prod_{i \in I} U_i \in \mathcal{Q}$
  - $\langle 2 \rangle 7. \ U \in \mathcal{U}_i$

PROOF: From Proposition 126.

**Proposition 129** (AC). Let  $\{X_i\}_{i\in I}$  be a family of topological spaces. Let  $A_i \subseteq X_i$  for all  $i \in I$ . Then

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i} .$$

Proof:

 $\langle 1 \rangle 1. \ \overline{\prod_{i \in I} A_i} \subseteq \prod_{i \in I} \overline{A_i}$ 

 $\langle 2 \rangle 1$ . For all  $i \in I$  we have  $A_i \subseteq \overline{A_i}$ 

Proof: Lemma 42.

 $\langle 2 \rangle 2$ .  $\prod_{i \in I} A_i \subseteq \prod_{i \in I} \overline{A_i}$ 

 $\langle 2 \rangle 3$ . Q.E.D.

PROOF: Since  $\prod_{i \in I} A_i$  is closed by Proposition 124.

- $\langle 1 \rangle 2$ .  $\prod_{i \in I} \overline{A_i} \subseteq \overline{\prod_{i \in I} A_i}$ 
  - $\langle 2 \rangle 1$ . Let:  $x \in \prod_{i \in I} \overline{A_i}$
  - $\langle 2 \rangle 2$ . Let: U be a neighbourhood of x
  - $\langle 2 \rangle 3$ . PICK  $V_i$  open in  $X_i$  such that  $x \in \prod_{i \in I} V_i \subseteq U$  with  $V_i = X_i$  except for  $i = i_1, \ldots, i_n$
  - $\langle 2 \rangle 4$ . For  $i \in I$ , pick  $a_i \in V_i \cap A_i$

PROOF: By Theorem 45 and  $\langle 2 \rangle 1$  using the Axiom of Choice.

- $\langle 2 \rangle 5$ . *U* intersects  $\prod_{i \in I} A_i$
- $\langle 2 \rangle 6$ . Q.E.D.

PROOF:  $a \in U \cap \prod_{i \in I} A_i$ 

**Example 130.** The closure of  $\mathbb{R}^{\infty}$  in  $\mathbb{R}^{\omega}$  is  $\mathbb{R}^{\omega}$ 

### Proof:

- $\langle 1 \rangle 1$ . Let:  $a \in \mathbb{R}^{\omega}$
- $\langle 1 \rangle 2$ . Let: U be any neighbourhoods of a.
- $\langle 1 \rangle 3$ . PICK  $U_n$  open in  $\mathbb{R}$  for all n such that  $a \in \prod_{n \geq 0} U_n \subseteq U$  and  $U_n = \mathbb{R}$  for all n except  $n_1, \ldots, n_k$
- $\langle 1 \rangle 4$ . Let:  $b_n = a_n$  for  $n = n_1, \dots, n_k$  and  $b_n = 0$  for all other n
- $\langle 1 \rangle 5. \ b \in \mathbb{R}^{\infty} \cap U$
- $\langle 1 \rangle 6$ . Q.E.D.

PROOF: From Theorem 45.

**Proposition 131.** Let  $\{X_i\}_{i\in I}$  be a family of topological spaces. Let  $(a_n)$  be a sequence in  $\prod_{i\in I} X_i$  and  $l\in \prod_{i\in I} X_i$ . Then  $a_n\to l$  as  $n\to\infty$  if and only if, for all  $i\in I$ , we have  $\pi_i(a_n)\to\pi_i(l)$  as  $n\to\infty$ .

### Proof:

- $\langle 1 \rangle 1$ . If  $a_n \to l$  as  $n \to \infty$  then, for all  $i \in I$ , we have  $\pi_i(a_n) \to \pi_i(l)$  as  $n \to \infty$  PROOF: Proposition 105.
- $\langle 1 \rangle 2$ . If, for all  $i \in I$ , we have  $\pi_i(a_n) \to \pi_i(l)$  as  $n \to \infty$ , then  $a_n \to l$  as  $n \to \infty$ 
  - $\langle 2 \rangle 1$ . Assume: For all  $i \in I$ , we have  $\pi_i(a_n) \to \pi_i(l)$  as  $n \to \infty$
  - $\langle 2 \rangle 2$ . Let: V be a neighbourhood of l
  - $\langle 2 \rangle$ 3. PICK open sets  $U_i$  in  $X_i$  such that  $l \in \prod_{i \in I} U_i \subseteq V$  and  $U_i = X_i$  for all i except  $i = i_1, \ldots, i_k$
  - $\langle 2 \rangle 4$ . For j = 1, ..., k, PICK  $N_j$  such that, for all  $n \geq N_j$ , we have  $\pi_{i_j}(a_n) \in U_{i_j}$
  - $\langle 2 \rangle 5$ . Let:  $N = \max(N_1, \ldots, N_k)$
  - $\langle 2 \rangle 6$ . For all  $n \geq N$  we have  $a_n \in V$

**Theorem 132.** Let A be a topological space and  $\{X_i\}_{i\in I}$  be a family of topological spaces. Let  $f: A \to \prod_{i\in I} X_i$  be a function. If  $\pi_i \circ f$  is continuous for all  $i\in I$  then f is continuous.

Proof:

 $\langle 1 \rangle 1$ . Let:  $i \in I$  and U be open in  $X_i$ 

 $\langle 1 \rangle 2$ .  $f^{-1}(\pi_i^{-1}(U))$  is open in A

 $\langle 1 \rangle 3$ . Q.E.D.

PROOF: Proposition 89.

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### 19.1 Continuous in Each Variable Separately

**Definition 133** (Continuous in Each Variable Separately). Let  $F: X \times Y \to Z$ . Then F is continuous in each variable separately if and only if:

- for every  $a \in X$  the function  $\lambda y \in Y.F(a,y)$  is continuous;
- for every  $b \in Y$  the function  $\lambda x \in X.F(x,b)$  is continuous.

**Proposition 134.** Let  $F: X \times Y \to Z$ . If F is continuous then F is continuous in each variable separately.

PROOF: For  $a \in X$ , the function  $\lambda y \in Y.F(a,y)$  is  $F \circ i$  where  $i: Y \to X \times Y$  maps y to (a,y). We have i is continuous by Proposition 111, hence  $F \circ i$  is continuous by Theorem 95.

Similarly for  $\lambda x \in X.F(x,b)$  for  $b \in Y$ .  $\square$ 

**Example 135.** Define  $F: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$  by

$$F(x,y) = \begin{cases} xy/(x^2 + y^2) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Then F is continuous in each variable separately but not continuous.

**Proposition 136.** Let  $f: A \to C$  and  $g: B \to D$  be open maps. Then  $f \times g: A \times B \to C \times D$  is an open map.

PROOF: Given U open in A and V open in B. Then  $(f \times g)(U \times V) = f(U) \times g(V)$  is open in  $C \times D$ . The result follows from Proposition 86.  $\square$ 

# 20 The Subspace Topology

**Definition 137** (Subspace Topology). Let X be a topological space and  $Y \subseteq X$ . The *subspace topology* on Y is  $\mathcal{T} = \{U \cap Y \mid U \text{ is open in } X\}$ .

We prove this is a topology.

Proof:

 $\langle 1 \rangle 1. \ Y \in \mathcal{T}$ 

PROOF: Since  $Y = X \cap Y$ 

 $\langle 1 \rangle 2$ . For all  $\mathcal{U} \subseteq \mathcal{T}$ , we have  $\bigcup \mathcal{U} \in \mathcal{T}$ 

 $\langle 2 \rangle 1$ . Let:  $\mathcal{U} \subseteq \mathcal{T}$ 

```
 \begin{array}{l} \langle 2 \rangle \text{2. Let: } \mathcal{V} = \{ V \text{ open in } X \mid V \cap Y \in \mathcal{U} \} \\ \langle 2 \rangle \text{3. } \bigcup \mathcal{U} = (\bigcup \mathcal{V}) \cap Y \\ \langle 1 \rangle \text{3. For all } U, V \in \mathcal{T}, \text{ we have } U \cap V \in \mathcal{T} \\ \langle 2 \rangle \text{1. Let: } U, V \in \mathcal{T} \\ \langle 2 \rangle \text{2. Pick } U', \ V' \text{ open in } X \text{ such that } U = U' \cap Y \text{ and } V = V' \cap Y \\ \langle 2 \rangle \text{3. } (U \cap V) = (U' \cap V') \cap Y \\ & \\ \square \end{array}
```

**Theorem 138.** Let X be a topological space and Y a subspace of X. Let  $A \subseteq Y$ . Then A is closed in Y if and only if there exists a closed set C in X such that  $A = C \cap Y$ .

PROOF: We have

$$\begin{array}{l} A \text{ is closed in } Y \\ \Leftrightarrow Y \setminus A \text{ is open in } Y \\ \Leftrightarrow \exists U \text{ open in } X.Y \setminus A = Y \cap U \\ \Leftrightarrow \exists C \text{ closed in } X.Y \setminus A = Y \cap (X \setminus U) \\ \Leftrightarrow \exists C \text{ closed in } X.A = Y \cap U \end{array}$$

**Theorem 139.** Let Y be a subspace of X. Let  $A \subseteq Y$ . Let  $\overline{A}$  be the closure of A in X. Then the closure of A in Y is  $\overline{A} \cap Y$ .

PROOF: The closure of 
$$A$$
 in  $Y$  is 
$$\bigcap \{C \text{ closed in } Y \mid A \subseteq C\}$$
 
$$= \bigcap \{D \cap Y \mid D \text{ closed in } X, A \subseteq D \cap Y\}$$
 (Theorem 138) 
$$= \bigcap \{D \mid D \text{ closed in } X, A \subseteq D\} \cap Y$$
 
$$= \overline{A} \cap Y$$

**Lemma 140.** Let X be a topological space and  $Y \subseteq X$ . Let  $\mathcal{B}$  be a basis for the topology on X. Then  $\mathcal{B}' = \{B \cap Y \mid B \in \mathcal{B}\}$  is a basis for the subspace topology on Y.

### Proof:

- $\langle 1 \rangle 1$ . Every element in  $\mathcal{B}'$  is open in Y
- $\langle 1 \rangle 2$ . For every open set U in Y and point  $y \in U$ , there exists  $B' \in \mathcal{B}'$  such that  $y \in B' \subseteq U$ 
  - $\langle 2 \rangle 1$ . Let: U be open in Y and  $y \in U$
  - $\langle 2 \rangle 2$ . Pick V open in X such that  $U = V \cap Y$
  - $\langle 2 \rangle 3$ . Pick  $B \in \mathcal{B}$  such that  $y \in B \subseteq V$
  - $\langle 2 \rangle 4$ . Let:  $B' = B \cap Y$
  - $\langle 2 \rangle 5. \ B' \in \mathcal{B}'$
  - $\langle 2 \rangle 6. \ y \in B' \subseteq U$
- $\langle 1 \rangle 3$ . Q.E.D.

Proof: By Lemma 60.

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**Lemma 141.** Let X be a topological space and  $Y \subseteq X$ . Let S be a basis for the topology on X. Then  $S' = \{S \cap Y \mid S \in S\}$  is a subbasis for the subspace topology on Y.

PROOF: The set  $\{B \cap Y \mid B \text{ is a finite intersection of elements of } \mathcal{S}\}$  is a basis for the subspace topology by Lemma 140, and this is the set of all finite intersections of elements of  $\mathcal{S}'$ .  $\square$ 

**Lemma 142.** Let Y be a subspace of X. If U is open in Y and Y is open in X then U is open in X.

### Proof:

 $\langle 1 \rangle 1$ . PICK V open in X such that  $U = V \cap Y$ 

 $\langle 1 \rangle 2$ . *U* is open in *X* 

Proof: Since it is the intersection of two open sets V and Y.

**Theorem 143.** Let Y be a subspace of X and  $A \subseteq Y$ . If A is closed in Y and Y is closed in X then A is closed in X.

PROOF: Pick a closed set C in X such that  $A = C \cap Y$  (Theorem 138). Then A is the intersection of two sets closed in X, hence A is closed in X (Lemma 30).  $\square$ 

**Theorem 144.** Let  $\{X_i\}_{i\in I}$  be a family of topological spaces. Let  $A_i$  be a subspace of  $X_i$  for all  $i\in I$ . Then the product topology on  $\prod_{i\in I}A_i$  is the same as the topology it inherits as a subspace of  $\prod_{i\in I}X_i$ .

PROOF: The product topology is generated by the subbasis

$$\{\pi_{i}^{-1}(U) \mid i \in I, U \text{ open in } A_{i}\}\$$

$$= \{\pi_{i}^{-1}(V) \cap A_{i} \mid i \in I, V \text{ open in } X_{i}\}\$$

$$= \{\pi_{i}^{-1}(V) \mid i \in I, V \text{ open in } X_{i}\} \cap \prod_{i \in I} A_{i}\$$

and this is a subbasis for the subspace topology by Lemma 141.  $\square$ 

**Theorem 145.** Let X be an ordered set in the order topology. Let  $Y \subseteq X$  be an interval. Then the order topology on Y is the same as the subspace topology on Y.

### Proof:

- $\langle 1 \rangle 1$ . The order topology is finer than the subspace topology.
  - $\langle 2 \rangle 1$ . For every open ray R in X, the set  $R \cap Y$  is open in the order topology.
    - $\langle 3 \rangle 1$ . For all  $a \in X$ , we have  $(-\infty, a) \cap Y$  is open in the order topology.
      - $\langle 4 \rangle 1$ . Case: For all  $y \in Y$  we have y < a

PROOF: In this case  $(-\infty, a) \cap Y = Y$ .

 $\langle 4 \rangle 2$ . CASE: For all  $y \in Y$  we have a < y PROOF: In this case  $(-\infty, a) \cap Y = \emptyset$ .

 $\langle 4 \rangle 3$ . Case: There exists  $y \in Y$  such that  $y \leq a$  and  $y \in Y$  such that  $a \leq y$ 

 $\langle 5 \rangle 1. \ a \in Y$ 

PROOF: Because Y is an interval.

$$\langle 5 \rangle 2$$
.  $(-\infty, a) \cap Y = \{ y \in Y \mid y < a \}$ 

- $\langle 3 \rangle 2$ . For all  $a \in X$ , we have  $(a, +\infty) \cap Y$  is open in the order topology. PROOF: Similar.
- $\langle 2 \rangle 2$ . Q.E.D.

PROOF: By Lemmas 113 and 141 and Proposition 68.

- $\langle 1 \rangle 2$ . The subspace topology is finer than the order topology.
  - $\langle 2 \rangle 1$ . Every open ray in Y is open in the subspace topology.

PROOF: For any  $a \in Y$  we have  $(-\infty, a)_Y = (-\infty, a)_X \cap Y$  and  $(a, +\infty)_Y = (a, +\infty)_X \cap Y$ .

 $\langle 2 \rangle 2$ . Q.E.D.

PROOF: By Lemma 113 and Proposition 68

This example shows that we cannot remove the hypothesis that Y is an interval:

**Example 146.** The order topology on  $I_o^2$  is different from the subspace topology as a subspace of  $\mathbb{R}^2$  under the dictionary order topology.

PROOF: The set  $\{1/2\} \times (1/2,1)$  is open in the subspace topology but not in the order topology.  $\square$ 

**Proposition 147.** Let X be a topological space, Y a subspace of X, and Z a subspace of Y. Then the subspace topology on Z inherited from X is the same as the subspace topology on Z inherited from Y.

PROOF: The subspace topology inherited from Y is

$$\{V \cap Z \mid V \text{ open in } Y\}$$

$$=\{U \cap Y \cap Z \mid U \text{ open in } X\}$$

$$=\{U \cap Z \mid U \text{ open in } X\}$$

which is the subspace topology inherited from X.  $\square$ 

**Definition 148** (Unit Circle). The unit circle  $S^1$  is  $\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$  as a subspace of  $\mathbb{R}^2$ .

**Definition 149** (Unit 2-sphere). The unit 2-sphere is  $S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 \le 1\}$  as a subspace of  $\mathbb{R}^3$ .

**Proposition 150.** Let  $f: X \to Y$  be an open map and  $A \subseteq X$  be open. Then the restriction  $f \upharpoonright A: A \to f(A)$  is an open map.

Proof:

 $\langle 1 \rangle 1$ . Let: U be open in A

 $\langle 1 \rangle 2$ . U is open in X

PROOF: Lemma 142.

- $\langle 1 \rangle 3$ . f(U) is open in Y
- $\langle 1 \rangle 4$ . f(U) is open in f(A)

```
PROOF: Since f(U) = f(U) \cap f(A).
```

**Example 151.** This example shows that we cannot remove the hypothesis that A is open.

Let  $A = \{(x,y) \in \mathbb{R}^2 \mid (x > 0 \text{ and } y = 1/x) \text{ or } x = y = 0\}$ . Then  $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$  is an open map, but  $\pi_1 \upharpoonright A : A \to [0, +\infty)$  is not, because it maps the set  $\{0,0\}$  which is open in A to  $\{0\}$  which is not open in  $[0, +\infty)$ .

**Proposition 152.** Let Y be a subspace of X. Let  $A \subseteq Y$  and  $l \in Y$ . Then l is a limit point of A in Y if and only if l is a limit point of A in X.

PROOF: Both are equivalent to the condition that any neighbourhood of l in X intersects A in a point other than l.  $\square$ 

### 21 The Box Topology

**Definition 153** (Box Topology). Let  $\{A_i\}_{i\in I}$  be a family of topological spaces. The box topology on  $\prod_{i\in I} A_i$  is the topology generated by the set of all sets of the form  $\prod_{i\in I} U_i$  where  $\{U_i\}_{i\in I}$  is a family such that each  $U_i$  is open in  $A_i$ .

This is a basis since it covers  $\prod_{i \in I} A_i$  and is closed under intersection.

**Proposition 154.** The box topology is finer than the product topology.

PROOF: From Proposition 123.  $\square$ 

**Corollary 154.1.** If  $A_i$  is closed in  $X_i$  for all  $i \in I$  then  $\prod_{i \in I} A_i$  is closed in  $\prod_{i \in I} X_i$  under the box topology.

Proof: From Proposition 124.

**Proposition 155** (AC). Let  $\{A_i\}_{i\in I}$  be a family of topological spaces. For  $i \in I$ , let  $\mathcal{B}_i$  be a basis for the topology on  $A_i$ . Then  $\mathcal{B} = \{\prod_{i\in I} B_i \mid \forall i \in I.B_i \in \mathcal{B}_i\}$  is a basis for the box topology on  $\prod_{i\in I} A_i$ .

#### Proof:

- $\langle 1 \rangle 1$ . Every set of the form  $\prod_{i \in I} B_i$  is open.
- $\langle 1 \rangle 2$ . For every point  $a \in \prod_{i \in I} A_i$  and every open set U with  $a \in U$ , there exists  $B \in \mathcal{B}$  such that  $a \in B \subseteq U$ .
  - $\langle 2 \rangle 1$ . Let: *U* be open and  $a \in U$
  - $\langle 2 \rangle 2$ . PICK a family  $\{U_i\}_{i \in I}$  such that each  $U_i$  is open in  $A_i$  and  $a \in \prod_{i \in I} U_i \subseteq U$ .
  - $\langle 2 \rangle 3$ . For  $i \in I$ , PICK  $B_i \in \mathcal{B}_i$  such that  $a_i \in B_i \subseteq U_i$  PROOF: Using the Axiom of Choice.
  - $\langle 2 \rangle 4. \ a \in \prod_{i \in I} B_i \subseteq U$
- $\langle 1 \rangle 3$ . Q.E.D.

Proof: Lemma 60.

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**Theorem 156.** Let  $\{X_i\}_{i\in I}$  be a family of topological spaces. Let  $A_i$  be a subspace of  $X_i$  for all  $i\in I$ . Give  $\prod_{i\in I}X_i$  the box topology. Then the box topology on  $\prod_{i\in I}A_i$  is the same as the topology it inherits as a subspace of  $\prod_{i\in I}X_i$ .

PROOF: The box topology is generated by the basis

opology is generated by the basis 
$$\{\prod_{i \in I} U_i \mid \forall i \in I, U_i \text{ open in } A_i\}$$
$$=\{\prod_{i \in I} (V_i \cap A_i) \mid \forall i \in I, V_i \text{ open in } X_i\}$$
$$=\{\prod_{i \in I} V_i \mid \forall i \in I, V_i \text{ open in } X_i\} \cap \prod_{i \in I} A_i$$

and this is a basis for the subspace topology by Lemma 140.  $\Box$ 

**Proposition 157.** Let  $\{X_i\}_{i\in I}$  be a family of Hausdorff spaces. Then  $\prod_{i\in I} X_i$  under the box topology is Hausdorff.

Proof:

 $\langle 1 \rangle 1$ . Let:  $\{X_i\}_{i \in I}$  be a family of Hausdorff spaces.

 $\langle 1 \rangle 2$ . Let:  $a, b \in \prod_{i \in I} X_i$  with  $a \neq b$ 

 $\langle 1 \rangle 3$ . PICK  $i \in I$  such that  $a_i \neq b_i$ 

 $\langle 1 \rangle 4$ . PICK U, V disjoint open sets in  $X_i$  with  $a_i \in U$  and  $b_i \in V$ 

 $\langle 1 \rangle$ 5.  $\pi_i^{-1}(U)$  and  $\pi_i^{-1}(V)$  are disjoint open sets in  $\prod_{i \in I} X_i$  with  $a \in \pi_i^{-1}(U)$  and  $b \in \pi_i^{-1}(V)$ 

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**Proposition 158** (AC). Let  $\{X_i\}_{i\in I}$  be a family of topological spaces. Give  $\prod_{i\in I}X_i$  the box topology. Let  $A_i\subseteq X_i$  for all  $i\in I$ . Then

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i} .$$

Proof:

 $\langle 1 \rangle 1. \ \overline{\prod_{i \in I} A_i} \subseteq \prod_{i \in I} \overline{A_i}$ 

 $\langle 2 \rangle 1$ . For all  $i \in I$  we have  $A_i \subseteq \overline{A_i}$ 

Proof: Lemma 42.

 $\langle 2 \rangle 2$ .  $\prod_{i \in I} A_i \subseteq \prod_{i \in I} \overline{A_i}$ 

 $\langle 2 \rangle 3$ . Q.E.D.

PROOF: Since  $\prod_{i \in I} A_i$  is closed by Corollary 154.1.

 $\langle 1 \rangle 2$ .  $\prod_{i \in I} \overline{A_i} \subseteq \overline{\prod_{i \in I} A_i}$ 

 $\langle 2 \rangle 1$ . Let:  $x \in \prod_{i \in I} \overline{A_i}$ 

 $\langle 2 \rangle 2$ . Let: U be a neighbourhood of x

 $\langle 2 \rangle 3$ . PICK  $V_i$  open in  $X_i$  such that  $x \in \prod_{i \in I} V_i \subseteq U$ 

 $\langle 2 \rangle 4$ . For  $i \in I$ , pick  $a_i \in V_i \cap A_i$ 

PROOF: By Theorem 45 and  $\langle 2 \rangle 1$  using the Axiom of Choice.

 $\langle 2 \rangle 5$ . *U* intersects  $\prod_{i \in I} A_i$ 

$$\langle 2 \rangle$$
6. Q.E.D.  
PROOF:  $a \in U \cap \prod_{i \in I} A_i$ .

The following example shows that Theorem 132 fails in the box topology.

**Example 159.** Define  $f: \mathbb{R} \to \mathbb{R}^{\omega}$  by f(t) = (t, t, ...). Then  $\pi_n \circ f = \mathrm{id}_{\mathbb{R}}$  is continuous for all n. But f is not continuous when  $\mathbb{R}^{\omega}$  is given the box topology because the inverse image of

$$(-1,1) \times (-1/2,1/2) \times (-1/3,1/3) \times \cdots$$

is  $\{0\}$  which is not open.

The following example shows that Proposition 131 fails in the box topology.

**Example 160.** Give  $\mathbb{R}^{\omega}$  the box topology. Let  $a_n = (1/n, 1/n, \ldots)$  for  $n \geq 1$  and  $l = (0, 0, \ldots)$ . Then  $\pi_i(a_n) \to \pi_i(l)$  as  $n \to \infty$  for all i, but  $a_n \not\to l$  as  $n \to \infty$  since the open set

$$(-1,1) \times (-1/2,1/2) \times (-1/3,1/3) \times \cdots$$

contains l but does not contain any  $a_n$ .

**Example 161.** The set  $\mathbb{R}^{\infty}$  is closed in  $\mathbb{R}^{\omega}$  under the box topology. For let  $(a_n)$  be any sequence not in  $\mathbb{R}^{\infty}$ . Let  $U_n$  be an open interval around  $a_n$  that does not contain 0 if  $a_n \neq 0$ , and  $U_n = \mathbb{R}$  if  $a_n = 0$ . Then  $\prod_{n \geq 0} U_n$  is a neighbourhood of  $(a_n)$  that does not intersect  $\mathbb{R}^{\infty}$ .

# 22 $T_1$ Spaces

**Definition 162** ( $T_1$  Space). A topological space is  $T_1$  if and only if every singleton is closed.

**Lemma 163.** A space is  $T_1$  if and only if every finite set is closed.

PROOF: From Lemma 31.

**Theorem 164.** In a  $T_1$  space, a point a is a limit point of a set A if and only if every neighbourhood of a contains infinitely many points of A.

#### Proof:

- $\langle 1 \rangle 1$ . If a is a limit point of A then every neighbourhood of a contains infinitely many points of A.
  - $\langle 2 \rangle 1$ . Assume: a is a limit point of A.
  - $\langle 2 \rangle 2$ . Let: U be a neighbourhood of a.
  - $\langle 2 \rangle 3$ . Assume: for a contradiction U contains only finitely many points of A.
  - $\langle 2 \rangle 4$ .  $(U \cap A) \setminus \{a\}$  is closed.

PROOF: By the  $T_1$  axiom.

 $\langle 2 \rangle 5$ .  $(U \setminus A) \cup \{a\}$  is open.

```
PROOF: It is U \setminus ((U \cap A) \setminus \{a\}). \langle 2 \rangle 6. (U \setminus A) \cup \{a\} intersects A in a point other than a. PROOF: From \langle 2 \rangle 1. \langle 2 \rangle 7. Q.E.D.
```

 $\langle 1 \rangle 2$ . If every neighbourhood of a contains infinitely many points of A then a is a limit point of A.

PROOF: Immediate from definitions.

(To see this does not hold in every space, see Proposition 56.)

**Proposition 165.** A space is  $T_1$  if and only if, for any two distinct points x and y, there exist neighbourhoods U of x and y of y such that  $x \notin V$  and  $y \notin U$ .

#### PROOF:

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- $\langle 1 \rangle 1.$  Let: X be a topological space.
- $\langle 1 \rangle 2$ . If X is  $T_1$  then, for any two distinct points x and y, there exist neighbourhoods U of x and V of y such that  $x \notin V$  and  $y \notin U$ .

PROOF: This holds because  $\{x\}$  and  $\{y\}$  are closed.

- $\langle 1 \rangle 3$ . Suppose, for any two distinct points x and y, there exist neighbourhoods U of x and V of y such that  $x \notin V$  and  $y \notin U$ . Then X is  $T_1$ .
  - $\langle 2 \rangle 1$ . Assume: For any two distinct points x and y, there exist neighbourhoods U of x and V of y such that  $x \notin V$  and  $y \notin U$ .
  - $\langle 2 \rangle 2$ . Let:  $a \in X$
  - $\langle 2 \rangle 3$ .  $\{a\}$  is closed.

PROOF: For all  $b \neq a$  there exists a neighbourhood U of b such that  $U \subseteq X \setminus \{a\}$ .

**Proposition 166.** A subspace of a  $T_1$  space is  $T_1$ .

PROOF: From Proposition 143.

# 23 Hausdorff Spaces

**Definition 167** (Hausdorff Space). A topological space is *Hausdorff* if and only if, for any points x, y with  $x \neq y$ , there exist disjoint open sets U and V such that  $x \in U$  and  $y \in V$ .

**Theorem 168.** Every Hausdorff space is  $T_1$ .

#### Proof:

- $\langle 1 \rangle 1$ . Let: X be a Hausdorff space.
- $\langle 1 \rangle 2$ . Let:  $b \in X$

PROVE:  $\overline{\{b\}} = \{b\}$ 

- $\langle 1 \rangle 3$ . Assume:  $a \in \overline{\{b\}}$  and  $a \neq b$
- $\langle 1 \rangle 4$ . PICK disjoint neighbourhoods U of a and V of b.

```
\langle 1 \rangle 6. \ b \in U
\langle 1 \rangle 7. Q.E.D.
  PROOF: This contradicts the fact that U and V are disjoint (\langle 1 \rangle 4).
Proposition 169. An infinite set under the finite complement topology is T_1
but not Hausdorff.
Proof:
\langle 1 \rangle 1. Let: X be an infinite set under the finite complement topology.
\langle 1 \rangle 2. Every singleton is closed.
  PROOF: By definition.
\langle 1 \rangle 3. PICKa, b \in X with a \neq b
\langle 1 \rangle 4. There are no disjoint neighbourhoods U of a and V of b.
   \langle 2 \rangle 1. Let: U be a neighbourhood of a and V a neighbourhood of b.
   \langle 2 \rangle 2. X \setminus U and X \setminus V are finite.
   \langle 2 \rangle 3. Pick c \in X that is not in X \setminus U or X \setminus V.
   \langle 2 \rangle 4. \ c \in U \cap V
П
Proposition 170. The product of a family of Hausdorff spaces is Hausdorff.
Proof:
\langle 1 \rangle 1. Let: \{X_i\}_{i \in I} be a family of Hausdorff spaces.
\langle 1 \rangle 2. Let: a, b \in \prod_{i \in I} X_i with a \neq b
\langle 1 \rangle 3. Pick i \in I such that a_i \neq b_i
\langle 1 \rangle 4. PICK U, V disjoint open sets in X_i with a_i \in U and b_i \in V
\langle 1 \rangle 5. \pi_i^{-1}(U) and \pi_i^{-1}(V) are disjoint open sets in \prod_{i \in I} X_i with a \in \pi_i^{-1}(U)
       and b \in \pi_i^{-1}(V)
Theorem 171. Every linearly ordered set under the order topology is Hausdorff.
Proof:
\langle 1 \rangle 1. Let: X be a linearly ordered set under the order topology.
\langle 1 \rangle 2. Let: a, b \in X with a \neq b
\langle 1 \rangle 3. Assume: w.l.o.g. a < b
\langle 1 \rangle 4. Case: There exists c such that a < c < b
  PROOF: The sets (-\infty, c) and (c, +\infty) are disjoint neighbourhoods of a and
  b respectively.
\langle 1 \rangle5. Case: There is no c such that a < c < b
   PROOF: The sets (-\infty, b) and (a, +\infty) are disjoint neighbourhoods of a and
  b respectively.
```

 $\langle 1 \rangle$ 5. *U* intersects  $\{b\}$  PROOF: Theorem 45.

**Theorem 172.** A subspace of a Hausdorff space is Hausdorff.

#### PROOF

- $\langle 1 \rangle 1$ . Let: X be a Hausdorff space and Y a subspace of X.
- $\langle 1 \rangle 2$ . Let:  $x, y \in Y$  with  $x \neq y$
- $\langle 1 \rangle 3$ . PICK disjoint neighbourhoods U of x and V of y in X.
- $\langle 1 \rangle 4.$   $U \cap Y$  and  $V \cap Y$  are disjoint neighbourhoods of x and y respectively in Y.

**Proposition 173.** A space X is Hausdorff if and only if the diagonal  $\Delta = \{(x,x) \mid x \in X\}$  is closed in  $X^2$ .

Proof:

X is Hausdorff

$$\Leftrightarrow \forall x,y \in X. x \neq y \Rightarrow \exists V, W \text{ open.} x \in V \land y \in W \land V \cap W = \emptyset$$
 
$$\Leftrightarrow \forall (x,y) \in X^2 \setminus \Delta. \exists V, W \text{ open.} (x,y) \subseteq V \times W \subseteq X^2 \setminus \Delta$$
 
$$\Leftrightarrow \Delta \text{ is closed}$$

## 24 The First Countability Axiom

**Definition 174** (First Countability Axiom). A topological space X satisfies the *first countability axiom*, or is *first countable*, if and only if every point has a countable local basis.

**Lemma 175** (Sequence Lemma (CC)). Let X be a first countable space. Let  $A \subseteq X$  and  $l \in \overline{A}$ . Then there exists a sequence in A that converges to l.

#### Proof:

- $\langle 1 \rangle 1$ . PICK a countable local basis  $\{B_n \mid n \in \mathbb{Z}^+\}$  at l such that  $B_1 \supseteq B_2 \supseteq \cdots$ . PROOF: Lemma 70.
- $\langle 1 \rangle 2$ . For all  $n \geq 1$ , PICK  $a_n \in A \cap B_n$ . PROVE:  $a_n \to l$  as  $n \to \infty$
- $\langle 1 \rangle 3$ . Let: U be a neighbourhood of A
- $\langle 1 \rangle 4$ . PICK N such that  $B_N \subseteq U$
- $\langle 1 \rangle$ 5. For  $n \geq N$  we have  $a_n \in U$ PROOF:  $a_n \in B_n \subseteq B_N \subseteq U$

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**Theorem 176** (CC). Let X be a first countable space and Y a topological space. Let  $f: X \to Y$ . Suppose that, for every sequence  $(x_n)$  in X and  $l \in X$ , if  $x_n \to l$  as  $n \to \infty$ , then  $f(x_n) \to f(l)$  as  $n \to \infty$ . Then f is continuous.

### Proof:

- $\langle 1 \rangle 1$ . Let:  $A \subseteq X$
- $\langle 1 \rangle 2$ . Let:  $a \in A$

Prove:  $f(a) \in \overline{f(A)}$ 

 $\langle 1 \rangle 3$ . PICK a sequence  $(x_n)$  in A that converges to a.

PROOF: By the Sequence Lemma.

- $\langle 1 \rangle 4. \ f(x_n) \to f(a)$
- $\langle 1 \rangle 5. \ f(a) \in \overline{f(A)}$

PROOF: By Lemma 74.

 $\langle 1 \rangle 6$ . Q.E.D.

PROOF: By Theorem 92.

**Example 177** (CC). The space  $\mathbb{R}^{\omega}$  under the box product is not first countable.

PROOF: Let  $\{B_n \mid n \geq 0\}$  be a countable set of neighbourhoods of  $\vec{0}$ . We will construct a neighbourhood of  $\vec{0}$  that does not include any of these.

For  $n \geq 0$ , pick a neighbourhood  $U_n$  of 0 such that  $U_n \subset \pi_n(B_p)$ . Then  $\prod_{n=0}^{\infty} U_n$  is a neighbourhood of  $\vec{0}$  that does not include any  $B_n$ .  $\square$ 

**Example 178.** If J is an uncountable set then  $\mathbb{R}^J$  is not first countable.

#### PROOF:

- $\langle 1 \rangle 1$ . Let:  $\{B_n \mid n \geq 0\}$  be any countable set of neighbourhoods of  $\vec{0}$ .
- $\langle 1 \rangle 2$ . For  $n \geq 0$ , PICK a basis element  $\prod_{\alpha \in J} U_{n\alpha}$  that contains  $\vec{0}$  and is included in  $B_n$ .

PROOF: Using the Axiom of Countable Choice.

- $\langle 1 \rangle 3$ . For  $n \geq 0$ ,
  - Let:  $J_n = \{ \alpha \in J \mid U_{n\alpha} \neq \mathbb{R} \}$
- $\langle 1 \rangle 4$ . PICK  $\beta \in J$  such that  $\beta \notin J_n$  for any n.

PROOF: Since each  $J_n$  is finite so  $\bigcup_n J_n$  is countable.

 $\langle 1 \rangle$ 5.  $\pi_{\beta}((-1,1))$  is a neighbourhood of  $\vec{0}$  that does not include any  $B_n$ .

**Example 179.** The space  $\mathbb{R}_l$  is first countable.

PROOF: For any  $a \in \mathbb{R}$ , the set  $\{[a, a+1/n) \mid n \geq 1\}$  is a countable local basis.

Example 180. The ordered square is first countable.

PROOF: For any  $(a,b) \in I_o^2$  with  $b \neq 0,1$ , the set  $\{(\{a\} \times (b-1/n,b+1/n)) \cap I_o^2 \mid n \geq 1\}$  is a countable local basis.

# 25 Strong Continuity

**Definition 181** (Strongly Continuous). Let X and Y be topological spaces and  $f: X \to Y$  be a function. Then f is *strongly continuous* if and only if, for every subset  $U \subseteq Y$ , we have U is open in Y if and only if  $f^{-1}(U)$  is open in X.

**Proposition 182.** Let X and Y be topological spaces and  $f: X \to Y$  be a function. Then f is strongly continuous if and only if, for every subset  $C \subseteq Y$ , we have C is closed in Y if and only if  $f^{-1}(C)$  is closed in X.

PROOF: Since  $X \setminus f^{-1}(C) = f^{-1}(Y \setminus C)$ .  $\square$ 

**Proposition 183.** Let X, Y and Z be topological spaces. Let  $f: X \to Y$  and  $g: Y \to Z$ . If f and g are strongly continuous then so is  $g \circ f$ .

PROOF: Since  $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ .  $\Box$ 

**Proposition 184.** Let X, Y and Z be topological spaces. Let  $f: X \to Y$  and  $g: Y \to Z$ . If  $g \circ f$  is continuous and f is strongly continuous then g is continuous.

#### Proof:

- $\langle 1 \rangle 1$ . Let:  $V \subseteq Z$  be open.
- $\langle 1 \rangle 2$ .  $f^{-1}(g^{-1}(V))$  is open in X.

PROOF: Since  $g \circ f$  is continuous.

 $\langle 1 \rangle 3.$   $f^{-1}(V)$  is open in Y.

PROOF: Since g is strongly continuous.

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**Proposition 185.** Let X, Y and Z be topological spaces. Let  $f: X \to Y$  and  $g: Y \to Z$ . If  $g \circ f$  is strongly continuous and f is strongly continuous then g is strongly continuous.

PROOF: For  $V \subseteq Z$ , we have V is open iff  $f^{-1}(g^{-1}(V))$  is open iff  $g^{-1}(V)$  is open.

### 26 Saturated Sets

**Definition 186.** Let X and Y be sets and p: X woheadrightarrow Y a surjective function. Let  $C \subseteq X$ . Then C is *saturated* with respect to p if and only if, for all  $x, y \in X$ , if  $x \in C$  and p(x) = p(y) then  $y \in C$ .

**Proposition 187.** Let X and Y be sets and  $p: X \rightarrow Y$  a surjective function. Let  $C \subseteq X$ . Then the following are equivalent:

- 1. C is saturated with respect to p.
- 2. There exists  $D \subseteq Y$  such that  $C = p^{-1}(D)$
- 3.  $C = p^{-1}(p(C))$ .

#### Proof:

- $\langle 1 \rangle 1. \ 1 \Rightarrow 3$ 
  - $\langle 2 \rangle$ 1. Assume: C is saturated with respect to p.
  - $\langle 2 \rangle 2$ .  $C \subseteq p^{-1}(p(C))$

PROOF: Trivial.

- $\langle 2 \rangle 3. \ p^{-1}(p(C)) \subseteq C$ 
  - $\langle 3 \rangle 1$ . Let:  $x \in p^{-1}(p(C))$
  - $\langle 3 \rangle 2. \ p(x) \in p(C)$
  - $\langle 3 \rangle 3$ . There exists  $y \in C$  such that p(x) = p(y)
  - $\langle 3 \rangle 4. \ x \in C$

```
PROOF: From \langle 2 \rangle 1. \langle 1 \rangle 2. 3 \Rightarrow 2
PROOF: Trivial. \langle 1 \rangle 3. 2 \Rightarrow 1
PROOF: This follows because if p(x) \in D and p(x) = p(y) then p(y) \in D.
```

### 27 Quotient Maps

**Definition 188** (Quotient Map). Let X and Y be topological spaces and  $p: X \to Y$ . Then p is a *quotient map* if and only if p is surjective and strongly continuous.

**Proposition 189.** Let X and Y be topological spaces and  $p: X \rightarrow Y$  be a surjective function. Then the following are equivalent.

```
1. p is a quotient map.
```

- 2. p is continuous and maps saturated open sets to open sets.
- 3. p is continuous and maps saturated closed sets to closed sets.

```
Proof:
\langle 1 \rangle 1. \ 1 \Rightarrow 2
   \langle 2 \rangle 1. Assume: p is a quotient map.
   \langle 2 \rangle 2. Let: U be a saturated open set in X.
   \langle 2 \rangle 3. p^{-1}(p(U)) is open in X.
       PROOF: Since U = p^{-1}(p(U)) be Proposition 187.
   \langle 2 \rangle 4. p(U) is open in Y.
       PROOF: From \langle 2 \rangle 1.
\langle 1 \rangle 2. \ 1 \Rightarrow 3
   PROOF: Similar.
\langle 1 \rangle 3. \ 2 \Rightarrow 1
   \langle 2 \rangle 1. Assume: p is continuous and maps saturated open sets to open sets.
   \langle 2 \rangle 2. Let: U \subseteq Y
   \langle 2 \rangle 3. Assume: p^{-1}(U) is open in X
   \langle 2 \rangle 4. p^{-1}(U) is saturated.
       Proof: Proposition 187.
   \langle 2 \rangle 5. U is open in Y.
\langle 1 \rangle 4. \ 3 \Rightarrow 1
   PROOF: Similar.
```

 ${\bf Corollary\ 189.1.}\ {\it Every\ surjective\ continuous\ open\ map\ is\ a\ quotient\ map.}$ 

Corollary 189.2. Every surjective continuous closed map is a quotient map.

Example 190. The converses of these corollaries do not hold.

Let  $A = \{(x,y) \mid x \geq 0\} \cup \{(x,y) \mid y = 0\}$ . Then  $\pi_1 : A \to \mathbb{R}$  is a quotient map, but not an open map or a closed map.

We prove that  $\pi_1$  maps saturated open sets to open sets:

- $\langle 1 \rangle 1$ . Let:  $\pi_1^{-1}(U)$  be a saturated open set in A Prove: U is open in  $\mathbb R$
- $\langle 1 \rangle 2$ . Let:  $x \in U$
- $\langle 1 \rangle 3. \ (x,0) \in \pi_1(U)^{-1}$
- $\langle 1 \rangle 4$ . PICK W, V open in  $\mathbb{R}$  such that  $(x,0) \subseteq W \times V \subseteq \pi_1(U)^{-1}$
- $\langle 1 \rangle 5. \ x \in W \subseteq U$

It is not an open map because it maps  $((-1,1)\times(1,2))\cap A$  to [0,1).

It is not a closed map because it maps  $\{(x, 1/x) \mid x > 0\}$  to  $(0, +\infty)$ .

**Proposition 191.** Let  $p: X \to Y$  be a quotient map. Let  $A \subseteq X$  be saturated with respect to p. Let  $q: A \to p(A)$  be the restriction of p.

- 1. If A is either open or closed in X then q is a quotient map.
- 2. If p is either an open map or a closed map then q is a quotient map.

#### Proof:

- $\langle 1 \rangle 1$ . Let:  $p: X \to Y$  be a quotient map.
- $\langle 1 \rangle 2$ . Let:  $A \subseteq X$  be saturated with respect to p.
- $\langle 1 \rangle 3$ . Let:  $q: A \rightarrow p(A)$  be the restriction of p.
- $\langle 1 \rangle 4$ . q is continuous.

PROOF: Theorem 96.

- $\langle 1 \rangle 5$ . If A is open in X then q is a quotient map.
  - $\langle 2 \rangle 1$ . Assume: A is open in X.
  - $\langle 2 \rangle 2$ . q maps saturated open sets to open sets.
    - $\langle 3 \rangle 1$ . Let:  $U \subseteq A$  be saturated with respect to q and open in A
    - $\langle 3 \rangle 2$ . U is saturated with respect to p
      - $\langle 4 \rangle 1$ . Let:  $x, y \in X$
      - $\langle 4 \rangle 2$ . Assume:  $x \in U$
      - $\langle 4 \rangle 3$ . Assume: p(x) = p(y)
      - $\langle 4 \rangle 4. \ x \in A$

PROOF: From  $\langle 3 \rangle 1$  and  $\langle 4 \rangle 2$ .

 $\langle 4 \rangle 5. \ y \in A$ 

PROOF: From  $\langle 1 \rangle 2$  and  $\langle 4 \rangle 3$ 

 $\langle 4 \rangle 6. \ q(x) = x(y)$ 

PROOF: From  $\langle 1 \rangle 3$ ,  $\langle 4 \rangle 3$ ,  $\langle 4 \rangle 4$ ,  $\langle 4 \rangle 5$ .

 $\langle 4 \rangle 7. \ y \in U$ 

PROOF: From  $\langle 3 \rangle 1$ ,  $\langle 4 \rangle 2$ ,  $\langle 4 \rangle 6$ 

 $\langle 3 \rangle 3$ . U is open in X

Proof: Lemma 142,  $\langle 2 \rangle 1$ ,  $\langle 3 \rangle 1$ .

 $\langle 3 \rangle 4$ . p(U) is open in Y

Proof: Proposition 189,  $\langle 1 \rangle 1$ ,  $\langle 3 \rangle 2$ ,  $\langle 3 \rangle 3$ 

```
\langle 3 \rangle 5. q(U) is open in p(A)
         PROOF: Since q(U) = p(U) = p(U) \cap p(A).
   \langle 2 \rangle 3. Q.E.D.
      Proof: By Proposition 189.
\langle 1 \rangle 6. If A is closed in X then q is a quotient map.
   PROOF: Similar.
\langle 1 \rangle 7. If p is an open map then q is a quotient map.
   \langle 2 \rangle 1. Assume: p is an open map
   \langle 2 \rangle 2. q maps saturated open sets to open sets.
      \langle 3 \rangle 1. Let: U be open in A and saturated with respect to q
      \langle 3 \rangle 2. PICK V open in X such that U = A \cap V
      \langle 3 \rangle 3. p(V) is open in Y
      \langle 3 \rangle 4. \ \ q(U) = p(V) \cap p(A)
         \langle 4 \rangle 1. \ q(U) \subseteq p(V) \cap p(A)
            PROOF: From \langle 3 \rangle 2.
         \langle 4 \rangle 2. \ p(V) \cap p(A) \subseteq q(U)
             \langle 5 \rangle 1. Let: y \in p(V) \cap p(A)
             \langle 5 \rangle 2. Pick x \in V and x' \in A such that p(x) = p(x') = y
             \langle 5 \rangle 3. \ x \in A
                Proof: By \langle 1 \rangle 2.
             \langle 5 \rangle 4. \ x \in U
                PROOF: From \langle 3 \rangle 2
   \langle 2 \rangle 3. Q.E.D.
      Proof: By Proposition 189.
\langle 1 \rangle 8. If p is a closed map then q is a quotient map.
   PROOF: Similar.
```

**Example 192.** This example shows we cannot remove the hypotheses on A and p.

Define  $f:[0,1] \to [2,3] \to [0,2]$  by f(x)=x if  $x \le 1$ , f(x)=x-1 if  $x \ge 2$ . Then f is a quotient map but its restriction f' to  $[0,1) \cup [2,3]$  is not, because  ${f'}^{-1}([1,2])$  is open but [1,2] is not.

For a counterexample where A is saturated, see Example 198.

**Proposition 193.** Let  $p: A \twoheadrightarrow C$  and  $q: B \twoheadrightarrow D$  be open quotient maps. Then  $p \times q: A \times B \rightarrow C \times D$  is an open quotient map.

PROOF: From Corollary 189.1, Proposition 136 and Theorem 132.

**Theorem 194.** Let  $p: X \to Y$  be a quotient map. Let Z be a topological space and  $f: Y \to Z$  be a function. Then

- 1.  $f \circ p$  is continuous if and only if f is continuous.
- 2.  $f \circ p$  is a quotient map if and only if f is a quotient map.

Proof:

```
\langle 1 \rangle 1. If f \circ p is continuous then f is continuous. PROOF: Proposition 184. \langle 1 \rangle 2. If f is continuous then f \circ p is continuous. PROOF: Theorem 95. \langle 1 \rangle 3. If f \circ p is a quotient map then f is a quotient map. PROOF: Proposition 185. \langle 1 \rangle 4. If f is a quotient map then f \circ p is a quotient map. PROOF: From Proposition 183.
```

**Proposition 195.** Let X and Y be topological spaces. Let  $p: X \to Y$  and  $f: Y \to X$  be continuous maps such that  $p \circ f = \mathrm{id}_Y$ . Then p is a quotient map.

```
Proof:  \begin{array}{l} \langle 1 \rangle 1. \ \ \mathrm{Lett} \colon V \subseteq Y \\ \langle 1 \rangle 2. \ \ \mathrm{Assume} \colon \ p^{-1}(V) \ \mathrm{is} \ \mathrm{open} \ \mathrm{in} \ X. \\ \langle 1 \rangle 3. \ \ f^{-1}(p^{-1}(V)) \ \mathrm{is} \ \mathrm{open} \ \mathrm{in} \ Y. \\ \mathrm{Proof:} \ \ \mathrm{Because} \ f \ \mathrm{is} \ \mathrm{continuous}. \\ \langle 1 \rangle 4. \ \ V \ \mathrm{is} \ \mathrm{open} \ \mathrm{in} \ Y. \\ \mathrm{Proof:} \ \ \mathrm{Because} \ f^{-1}(p^{-1}(V)) = V. \\ \square \end{array}
```

## 28 Quotient Topology

**Definition 196** (Quotient Topology). Let X be a topological space, Y a set and  $p: X \to Y$  be a surjective function. Then the *quotient topology* on Y is the unique topology on Y with respect to which p is a quotient map, namely  $\mathcal{T} = \{U \in \mathcal{P}Y \mid p^{-1}(U) \text{ is open in } X\}.$ 

We prove this is a topology.

```
PROOF:  \langle 1 \rangle 1. \ Y \in \mathcal{T}  PROOF: Since p^{-1}(Y) = X by surjectivity.  \langle 1 \rangle 2. \ \text{For all} \ \mathcal{A} \subseteq \mathcal{T} \ \text{we have} \ \bigcup \mathcal{A} \in \mathcal{T}  PROOF: Since p^{-1}(\bigcup \mathcal{A}) = \bigcup_{U \in \mathcal{A}} p^{-1}(U)  \langle 1 \rangle 3. \ \text{For all} \ U, V \in \mathcal{T} \ \text{we have} \ U \cap V \in \mathcal{T}  PROOF: Since p^{-1}(U \cap V) = p^{-1}(U) \cap p^{-1}(V). \square
```

**Definition 197** (Quotient Space). Let X be a topological space and  $\sim$  an equivalence relation on X. Let  $p:X \twoheadrightarrow X/\sim$  be the canonical surjection. Then  $X/\sim$  under the quotient topology is called a *quotient space*, *identification space* or *decomposition space* of X.

Here is a counterexample showing we cannot remove all the hypotheses of Proposition 191 except that A is saturated.

**Example 198.** Let  $X = (0, 1/2] \cup \{1\} \cup \{1 + 1/n : n \ge 2\}$  as a subspace of  $\mathbb{R}$ . Define R to be the equivalence relation on X where xRy iff (x = y or |x-y| = 1), so we identify 1/n with 1+1/n for all  $n \geq 2$ . Let Y be the resulting quotient space X/R in the quotient topology and  $p:X \to Y$  the canonical surjection.

Let  $A = \{1\} \cup (0, 1/2] \setminus \{1/n : n \ge 2\} \subseteq X$ . Then A is saturated under p but the restriction q of p to A is not a quotient map because it maps the saturated open set  $\{1\}$  to  $\{1\}$  which is not open in p(A).

**Proposition 199.** Let X, Y and Z be topological spaces. Let  $f: X \to Y$  and  $g: Y \to Z$ . If f and g are quotient maps then so is  $g \circ f$ .

Proof: From Proposition 183.

**Example 200.** The product of two quotient maps is not necessarily a quotient

Let  $X = \mathbb{R}$  and  $X^*$  the quotient space formed by identifying all positive integers. Let  $p: X \to X^*$  be the canonical surjection.

We prove  $p \times \mathrm{id}_{\mathbb{Q}} : X \times \mathbb{Q} \to X^* \times \mathbb{Q}$  is not a quotient map.

```
Proof:
\langle 1 \rangle 1. For n \geq 1,
          Let: c_n = \sqrt{2}/n
\langle 1 \rangle 2. For n \geq 1,
          Let: U_n = \{(x,y) \in X \times \mathbb{Q} \mid n-1/4 < x < n+1/4, (y+n > x + y) \}
                     c_n \text{ and } y + n > -x + c_n \text{ or } (y + n < x + c_n \text{ and } y + n < -x + c_n)
\langle 1 \rangle 3. For n \geq 1, we have U_n is open in X \times \mathbb{Q}
\langle 1 \rangle 4. For n \geq 1, we have \{n\} \times \mathbb{Q} \subseteq U_n
\langle 1 \rangle5. Let: U = \bigcup_{n=1}^{\infty} U_n
\langle 1 \rangle6. U is open in X \times \mathbb{Q}
\langle 1 \rangle 7. U is saturated with respect to p \times id_{\mathbb{O}}
\langle 1 \rangle 8. Let: U' = (p \times id_{\mathbb{Q}})(U)
\langle 1 \rangle 9. Assume: for a contradiction U' is open in X^* \times \mathbb{Q}
\langle 1 \rangle 10. \ (1,0) \in U'
\langle 1 \rangle 11. PICK a neighbourhood W of 1 in X^* and \delta > 0 such that W \times (-\delta, \delta) \subseteq U'
\langle 1 \rangle 12. \ p^{-1}(W) \times (-\delta, \delta) \subseteq U
\langle 1 \rangle 13. PICK n such that c_n < \delta
\langle 1 \rangle 14. \ n \in p^{-1}(W)
(1)15. PICK \epsilon > 0 such that \epsilon < \delta - c_n and \epsilon < 1/4 and (n - \epsilon, n + \epsilon) \subseteq p^{-1}(W)
\langle 1 \rangle 16. \ (n - \epsilon, n + \epsilon) \times (-\delta, \delta) \subseteq U
```

**Proposition 201.** Let X be a topological space and  $\sim$  an equivalence relation on X. Then  $X/\sim is\ T_1$  if and only if every equivalence class is closed in X.

 $\langle 1 \rangle 17$ . PICK a rational y such that  $c_n - \epsilon/2 < y < c_n + \epsilon/2$ 

Proof: Immediate from definitions.  $\square$ 

Proof: This contradicts  $\langle 1 \rangle 16$ .

 $\langle 1 \rangle 18. \ (n + \epsilon/2, y) \notin U$ 

 $\langle 1 \rangle 19$ . Q.E.D.

### 29 Retractions

**Definition 202** (Retraction). Let X be a topological space and  $A \subseteq X$ . A retraction of X onto A is a continuous map  $r: X \to A$  such that, for all  $a \in A$ , we have r(a) = a.

**Proposition 203.** Every retraction is a quotient map.

PROOF: Proposition 195 with f the inclusion  $A \hookrightarrow X$ .  $\square$ 

### 30 Homogeneous Spaces

**Definition 204** (Homogeneous). A topological space X is homogeneous if and only if, for any points  $a, b \in X$ , there exists a homeomorphism  $\phi : X \cong X$  such that  $\phi(a) = b$ .

### 31 Regular Spaces

**Definition 205** (Regular Space). A topological space X is *regular* if and only if, for any closed set A and point  $a \notin A$ , there exist disjoint open sets U, V such that  $A \subseteq U$  and  $a \in V$ .

### 32 Connected Spaces

**Definition 206** (Separation). A *separation* of a topological space X is a pair of disjoint open sets U, V such that  $U \cup V = \emptyset$ .

**Definition 207** (Connected). A topological space is *connected* if and only if it has no separation; otherwise it is *disconnected*.

**Proposition 208.** A topological space X is connected if and only if the only sets that are both open and closed are X and  $\emptyset$ .

Immediate from defintions.

**Lemma 209.** If Y is a subspace of X, a separation of Y is a pair of disjoint nonempty sets A and B whose union is Y, neither of which contains a limit point of the other.

#### Proof:

- $\langle 1 \rangle 1$ . Let:  $A, B \subseteq Y$
- $\langle 1 \rangle 2$ . If A and B form a separation of Y then A and B are disjoint and nonempty,  $A \cup B = Y$ , and neither of A and B contains a limit point of the other.
  - $\langle 2 \rangle 1$ . Assume: A and B form a separation of Y
  - $\langle 2 \rangle 2$ . A and B are disjoint and nonempty and  $A \cup B = Y$ PROOF: From  $\langle 2 \rangle 1$  and the definition of separation.
  - $\langle 2 \rangle 3$ . A does not contain a limit point of B

```
\langle 3 \rangle 1. Assume: for a contradiction l \in A and l is a limit point of B in X.
      \langle 3 \rangle 2. l is a limit point of B in Y
        Proof: Proposition 152.
      \langle 3 \rangle 3. \ l \in B
        \langle 4 \rangle 1. B is closed in Y
            PROOF: Since A is open in Y and B = Y \setminus A from \langle 2 \rangle 1.
         \langle 4 \rangle 2. Q.E.D.
            Proof: Corollary 55.1.
      \langle 3 \rangle 4. Q.E.D.
         PROOF: This contradicts the fact that A \cap B = \emptyset (\langle 2 \rangle 1).
  \langle 2 \rangle 4. B does not contain a limit point of A
     Proof: Similar.
\langle 1 \rangle3. If A and B are disjoint and nonempty, A \cup B = Y, and neither of A and
       B contains a limit point of the other, then A and B form a separation of
       Y.
   \langle 2 \rangle 1. Assume: A and B are disjoint and nonempty, A \cup B = Y, and neither
                        of A and B contains a limit point of the other.
  \langle 2 \rangle 2. A is open in Y
     \langle 3 \rangle 1. B is closed in Y
         \langle 4 \rangle 1. Let: l be a limit point of B in Y
         \langle 4 \rangle 2. l is a limit point of B in X
            Proof: Proposition 152.
         \langle 4 \rangle 3. \ l \notin A
            Proof: By \langle 2 \rangle 1
         \langle 4 \rangle 4. \ l \in B
            PROOF: By \langle 2 \rangle 1 since A \cup B = Y
         \langle 4 \rangle5. Q.E.D.
            PROOF: Corollary 55.1.
      \langle 3 \rangle 2. Q.E.D.
        PROOF: Since A = Y \setminus B.
   \langle 2 \rangle 3. B is open in Y
     PROOF: Similar.
```

Example 210. Every set under the indiscrete topology is connected.

**Example 211.** The discrete topology on a set X is connected if and only if  $|X| \leq 1$ .

**Example 212.** The finite complement topology on a set X is connected if and only if either  $|X| \le 1$  or X is infinite.

**Example 213.** The countable complement topology on a set X is connected if and only if either  $|X| \leq 1$  or X is uncountable.

**Example 214.** The rationals  $\mathbb{Q}$  are disconnected. For any irrational a, the sets  $(-\infty, a) \cap \mathbb{Q}$  and  $(a, +\infty) \cap \mathbb{Q}$  form a separation of  $\mathbb{Q}$ .

**Lemma 215.** Let X be a topological space. If C and D form a separation of X, and Y is a connected subspace of X, then either  $Y \subseteq C$  or  $Y \subseteq D$ .

PROOF: Otherwise  $Y \cap C$  and  $Y \cap D$  would form a separation of Y.  $\square$ 

**Theorem 216.** The union of a set of connected subspaces of a space X that have a point in common is connected.

#### Proof:

- $\langle 1 \rangle 1$ . Let:  $\mathcal{A}$  be a set of connected subspaces of the space X that have the point a in common.
- $\langle 1 \rangle 2$ . Assume: for a contradiction C and D form a separation of  $\bigcup \mathcal{A}$
- $\langle 1 \rangle 3$ . Assume: without loss of generality  $a \in C$
- $\langle 1 \rangle 4$ . For all  $A \in \mathcal{A}$  we have  $A \subseteq C$

Proof: Lemma 215.

- $\langle 1 \rangle 5. \ D = \emptyset$
- $\langle 1 \rangle 6$ . Q.E.D.

PROOF: This contradicts  $\langle 1 \rangle 2$ .

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**Theorem 217.** Let X be a topological space and A a connected subspace of X. If  $A \subseteq B \subseteq \overline{A}$  then B is connected.

#### Proof:

- $\langle 1 \rangle 1$ . Assume: for a contradiction C and D form a separation of B.
- $\langle 1 \rangle 2$ . Assume: without loss of generality  $A \subseteq C$

Proof: Lemma 215.

- $\langle 1 \rangle 3. \ B \subset C$ 
  - $\langle 2 \rangle 1$ . Let:  $x \in B$
  - $\langle 2 \rangle 2. \ x \in \overline{A}$
  - $\langle 2 \rangle 3$ . Either  $x \in A$  or x is a limit point of A.

PROOF: Theorem 55.

 $\langle 2 \rangle 4$ . Either  $x \in A$  or x is a limit point of C.

PROOF: Lemma 57,  $\langle 1 \rangle 2$ .

 $\langle 2 \rangle 5. \ x \in C$ 

Proof: Lemma 209.

- $\langle 1 \rangle 4. \ D = \emptyset$
- $\langle 1 \rangle$ 5. Q.E.D.

PROOF: This contradicts  $\langle 1 \rangle 1$ .

**Theorem 218.** The image of a connected space under a continuous map is connected.

#### Proof:

- $\langle 1 \rangle 1$ . Let:  $f: X \to Y$  be a surjective continuous map where X is connected.
- $\langle 1 \rangle 2$ . Assume: for a contradiction C and D form a separation of Y.
- $\langle 1 \rangle 3$ .  $f^{-1}(C)$  and  $f^{-1}(D)$  form a separation of X.

**Theorem 219.** The product of a family of connected spaces is connected.

Proof:

- $\langle 1 \rangle 1$ . The product of two connected spaces is connected.
  - $\langle 2 \rangle 1$ . Let: X and Y be connected spaces.
  - $\langle 2 \rangle 2$ . Pick  $a \in X$  and  $b \in Y$

PROOF: We may assume X and Y are nonempty since otherwise  $X \times Y = \emptyset$  which is connected.

 $\langle 2 \rangle 3$ .  $X \times \{b\}$  is connected.

PROOF: It is homeomorphic to X.

 $\langle 2 \rangle 4$ . For all  $x \in X$  we have  $\{x\} \times Y$  is connected.

PROOF: It is homeomorphic to Y.

 $\langle 2 \rangle 5$ . For any  $x \in X$ 

Let:  $T_x = (X \times \{b\}) \cup (\{x\} \times Y)$ 

 $\langle 2 \rangle 6$ . For all  $x \in X$ ,  $T_x$  is connected.

PROOF: Theorem 216 since  $(x,b) \in (X \times \{b\}) \cap (\{x\} \times Y)$ .

 $\langle 2 \rangle 7$ .  $X \times Y$  is connected.

PROOF: Theorem 216 since  $X \times Y = \bigcup_{x \in X} T_x$  and (a, b) is a point in every  $T_x$ .

(1)2. The product of a finite family of connected spaces is connected.

PROOF: From  $\langle 1 \rangle 1$  by induction.

- $\langle 1 \rangle 3$ . The product of any family of connected spaces is connected.
  - $\langle 2 \rangle 1$ . Let:  $\{X_{\alpha}\}_{{\alpha} \in J}$  be a family of connected spaces.
  - $\langle 2 \rangle 2$ . Let:  $X = \prod_{\alpha \in J} X_{\alpha}$
  - $\langle 2 \rangle 3$ . Pick  $a \in X$

Proof: We may assume  $X \neq \emptyset$  as the empty space is connected.

 $\langle 2 \rangle 4$ . For every finite subset K of J,

Let:  $X_K = \{x \in X \mid \forall \alpha \in J \setminus K.x_\alpha = a_\alpha\}$ 

 $\langle 2 \rangle$ 5. For every finite  $K \subseteq J$ , we have  $X_K$  is connected.

PROOF: From  $\langle 1 \rangle 2$  since  $X_K \cong \prod_{\alpha \in K} X_K$ .

- $\langle 2 \rangle 6$ . Let:  $Y = \bigcup_K X_K$
- $\langle 2 \rangle 7$ . Y is connected

Proof: Theorem 216 since a is a common point.

- $\langle 2 \rangle 8. \ X = \overline{Y}$ 
  - $\langle 3 \rangle 1$ . Let:  $x \in X$
  - $\langle 3 \rangle 2$ . Let:  $U = \prod_{\alpha \in J} U_{\alpha}$  be a basic neighbourhood of x where  $U_{\alpha} = X_{\alpha}$  for all  $\alpha$  except  $\alpha \in K$  for some finite  $K \subseteq J$
  - $\langle 3 \rangle 3$ . Let:  $y \in X$  be the point with  $y_{\alpha} = x_{\alpha}$  for  $\alpha \in K$  and  $y_{\alpha} = a_{\alpha}$  for all other  $\alpha$
  - $\langle 3 \rangle 4. \ y \in U \cap X_K$
  - $\langle 3 \rangle 5. \ y \in U \cap Y$
- $\langle 2 \rangle 9$ . X is connected.

PROOF: Theorem 217.

**Example 220.** The set  $\mathbb{R}^{\omega}$  is disconnected under the uniform and box topolo-

gies. Under either topology, the set of bounded sequences and the set of unbounded sequences form a separation.

**Proposition 221.** Let  $\mathcal{T}$  and  $\mathcal{T}'$  be two topologies on the same set X. If  $\mathcal{T} \subseteq \mathcal{T}'$  and  $(X, \mathcal{T}')$  is connected then  $(X, \mathcal{T})$  is connected.

PROOF: If U and V form a separation of  $(X, \mathcal{T})$  then they form a separation of  $(X, \mathcal{T}')$ .  $\square$ 

**Proposition 222.** Let X be a topological space and  $(A_n)$  a sequence of connected subspaces of X. If  $A_n \cap A_{n+1} \neq \emptyset$  for all n then  $\bigcup_n A_n$  is connected.

#### Proof:

- $\langle 1 \rangle 1$ . Assume: for a contradiction C and D form a separation of  $\bigcup_n A_n$
- $\langle 1 \rangle 2$ . Assume: without loss of generality  $A_0 \subseteq C$

PROOF: Lemma 215.

 $\langle 1 \rangle 3$ . For all n we gave  $A_n \subseteq C$ 

#### Proof:

- $\langle 2 \rangle 1$ . Assume:  $A_n \subseteq C$
- $\langle 2 \rangle 2$ . Pick  $x \in A_n \cap A_{n+1}$
- $\langle 2 \rangle 3. \ x \in C$
- $\langle 2 \rangle 4$ .  $A_{n+1} \subseteq C$

Proof: Lemma 215.

 $\langle 2 \rangle$ 5. Q.E.D.

PROOF: The result follows by induction.

- $\langle 1 \rangle 4$ .  $D = \emptyset$
- $\langle 1 \rangle$ 5. Q.E.D.

PROOF: This contradicts  $\langle 1 \rangle 1$ .

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**Proposition 223.** Let X be a topological space. Let  $A, C \subseteq X$ . If C is connected and intersects both A and  $X \setminus A$  then C intersects  $\partial A$ .

PROOF: Otherwise  $C \cap A^{\circ}$  and  $C \setminus \overline{A}$  would form a separation of C.  $\square$ 

**Example 224.** The space  $\mathbb{R}_l$  is disconnected. For any real x, the sets  $(-\infty, x)$  and  $[x, +\infty)$  form a separation.

**Proposition 225.** Let X and Y be connected spaces. Let A be a proper subset of X and B a proper subset of Y. Then  $(X \times Y) \setminus (A \times B)$  is connected.

#### Proof:

- $\langle 1 \rangle 1$ . Pick  $a \in X \setminus A$  and  $b \in Y \setminus B$
- $\langle 1 \rangle 2$ . For  $x \in X \setminus A$  we have  $(X \times \{b\}) \cup (\{x\} \times Y)$  is connected.

PROOF: Theorem 216 since (x, b) is a common point.

 $\langle 1 \rangle 3$ . For  $y \in Y \setminus B$  we have  $(X \times \{y\}) \cup (\{a\} \times Y)$  is connected. PROOF: Theorem 216 since (a, y) is a common point.

 $\langle 1 \rangle 4$ .  $(X \times Y) \setminus (A \times B)$  is connected.

PROOF: Theorem 216 since it is the union of the sets in  $\langle 1 \rangle 2$  and  $\langle 1 \rangle 3$  with (a,b) as a common point.

**Proposition 226.** Let  $p: X \to Y$  be a quotient map. If Y is connected and  $p^{-1}(y)$  is connected for all  $y \in Y$ , then X is connected.

#### Proof:

- $\langle 1 \rangle 1$ . Assume: for a contradiction C and D form a separation of X.
- $\langle 1 \rangle 2$ . C is saturated.
  - $\langle 2 \rangle 1$ . Let:  $x \in C$ ,  $y \in X$  with p(x) = p(y) = a, say
  - $\langle 2 \rangle 2. \ y \notin D$

PROOF: Otherwise  $p^{-1}(a) \cap C$  and  $p^{-1}(a) \cap D$  form a separation of  $p^{-1}(a)$ .

- $\langle 2 \rangle 3. \ y \in C$
- $\langle 1 \rangle 3$ . D is saturated.

PROOF: Similar.

 $\langle 1 \rangle 4$ . p(C) and p(D) form a separation of Y.

**Proposition 227.** Let X be a connected space and Y a connected subspace of X. Suppose A and B form a separation of  $X \setminus Y$ . Then  $Y \cup A$  and  $Y \cup B$  are both connected.

#### Proof:

- $\langle 1 \rangle 1$ .  $Y \cup A$  is connected.
  - $\langle 2 \rangle 1$ . Assume: for a contradiction C and D form a separation of  $Y \cup A$
  - $\langle 2 \rangle 2$ . Assume: without loss of generality  $Y \subseteq C$
  - $\langle 2 \rangle 3$ . PICK open sets  $A_1, B_1, C_1, D_1$  in X with

$$A = A_1 \setminus Y$$

$$B = B_1 \setminus Y$$

$$C = C_1 \cap (Y \cup A)$$

$$D = D_1 \cap (Y \cup A)$$

- $\langle 2 \rangle 4$ .  $B_1 \cup C_1$  and  $A_1 \cap D_1$  form a separation of X
- $\langle 1 \rangle 2$ .  $Y \cup B$  is connected.

PROOF: Similar.

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**Theorem 228.** Let L be a linearly ordered set under the order topology. Then L is connected if and only if L is a linear continuum.

#### Proof:

- $\langle 1 \rangle 1$ . If L is a linear continuum then L is connected.
  - $\langle 2 \rangle 1$ . Let: L be a linear continuum under the order topology.
  - $\langle 2 \rangle 2$ . Assume: for a contradiction C and D form a separation of L.
  - $\langle 2 \rangle 3$ . Pick  $a \in C$  and  $b \in D$ .
  - $\langle 2 \rangle 4$ . Assume: without loss of generality a < b.
  - $\langle 2 \rangle$ 5. Let:  $S = \{x \in L \mid a < x \text{ and } [a, x) \subseteq C\}$
  - $\langle 2 \rangle 6$ . S is nonempty.

PROOF: Since  $a \in C$  and C is open.

```
PROOF: Since b \notin C.
\langle 2 \rangle 8. Let: s = \sup S
\langle 2 \rangle 9. \ s \in S
   \langle 3 \rangle 1. Let: y \in [a, s)
           Prove: y \in C
   \langle 3 \rangle 2. Pick z with y < z \in S
      PROOF: By minimality of s.
   \langle 3 \rangle 3. \ y \in [a,z) \subseteq C
\langle 2 \rangle 10. Case: s \in C
   \langle 3 \rangle 1. Pick x such that s < x and [s, x) \subseteq C
      PROOF: Since C is open and s is not greatest in L because s < b.
   \langle 3 \rangle 2. \ x \in S
      PROOF: Since [a, x) = [a, s) \cup [s, x) \subseteq C.
   \langle 3 \rangle 3. Q.E.D.
      PROOF: This contradicts the fact that s is an upper bound for S.
\langle 2 \rangle 11. Case: s \in D
   \langle 3 \rangle 1. Pick x < s such that (x, s] \subseteq D
   \langle 3 \rangle 2. Pick y with x < y < s
      Proof: Since L is dense.
   \langle 3 \rangle 3. \ y \in C
      Proof: From \langle 2 \rangle 9.
   \langle 3 \rangle 4. \ y \in D
      PROOF: From \langle 3 \rangle 1.
   \langle 3 \rangle 5. Q.E.D.
   \langle 3 \rangle 6. Let: L be a linear continuum under the order topology.
   \langle 3 \rangle7. Assume: for a contradiction C and D form a separation of L.
   \langle 3 \rangle 8. Pick a \in C and b \in D.
   \langle 3 \rangle 9. Assume: without loss of generality a < b.
   \langle 3 \rangle 10. Let: S = \{ x \in L \mid a < x \text{ and } [a, x) \subseteq C \}
   \langle 3 \rangle 11. S is nonempty.
      PROOF: Since a \in C and C is open.
   \langle 3 \rangle 12. S is bounded above by b.
      PROOF: Since b \notin C.
   \langle 3 \rangle 13. Let: s = \sup S
   \langle 3 \rangle 14. \ s \in S
      \langle 4 \rangle 1. Let: y \in [a, s)
               Prove: y \in C
      \langle 4 \rangle 2. Pick z with y < z \in S
         PROOF: By minimality of s.
      \langle 4 \rangle 3. \ y \in [a, z) \subseteq C
   \langle 3 \rangle 15. Case: s \in C
      \langle 4 \rangle 1. Pick x such that s < x and [s, x) \subseteq C
```

 $\langle 2 \rangle$ 7. S is bounded above by b.

PROOF: Since  $[a, x) = [a, s) \cup [s, x) \subseteq C$ .

 $\langle 4 \rangle 2. \ x \in S$ 

PROOF: Since C is open and s is not greatest in L because s < b.

 $\langle 4 \rangle 3$ . Q.E.D.

PROOF: This contradicts the fact that s is an upper bound for S.

- $\langle 3 \rangle 16$ . Case:  $s \in D$ 
  - $\langle 4 \rangle 1$ . PICK x < s such that  $(x, s] \subseteq D$
  - $\langle 4 \rangle 2$ . Pick y with x < y < s

Proof: Since L is dense.

 $\langle 4 \rangle 3. \ y \in C$ 

PROOF: From  $\langle 2 \rangle 9$ .

 $\langle 4 \rangle 4. \ y \in D$ 

PROOF: From  $\langle 3 \rangle 1$ .

 $\langle 4 \rangle$ 5. Q.E.D.

Proof: This contradicts  $\langle 2 \rangle 2$ .

- $\langle 1 \rangle 2$ . If L is connected then L is a linear continuum.
  - $\langle 2 \rangle 1$ . Assume: L is connected.
  - $\langle 2 \rangle 2$ . Every nonempty subset of L that is bounded above has a supremum.
    - $\langle 3 \rangle 1$ . Let: X be a nonempty subset of L bounded above by b.
    - $\langle 3 \rangle 2$ . Assume: for a contradiction X has no supremum.
    - $\langle 3 \rangle 3$ . Let: U be the set of upper bounds of X,
    - $\langle 3 \rangle 4$ . *U* is nonempty.

PROOF: Since  $b \in U$ .

- $\langle 3 \rangle 5$ . *U* is open.
  - $\langle 4 \rangle 1$ . Let:  $x \in U$
  - $\langle 4 \rangle 2$ . PICK an upper bound y for X such that y < x
  - $\langle 4 \rangle 3$ . Either x is greatest in L and  $(y, x] \subseteq U$ , or there exists z > x such that  $(y, z) \subseteq U$
- $\langle 3 \rangle 6$ . Let: V be the set of lower bounds of U.
- $\langle 3 \rangle$ 7. V is nonempty.

PROOF: Since  $X \subseteq V$ 

- $\langle 3 \rangle 8$ . V is open.
  - $\langle 4 \rangle 1$ . Let:  $x \in V$
  - $\langle 4 \rangle 2$ . Pick  $y \in X$  with x < y

PROOF: x cannot be an upper bound for X, because it would be the supremum of X.

- $\langle 4 \rangle 3$ . Either x least in L and  $[x,y) \subseteq V$ , or there exists z < x such that  $(z,y) \subseteq V$
- $\langle 3 \rangle 9. \ L = U \cup V$ 
  - $\langle 4 \rangle 1$ . Let:  $x \in L \setminus U$
  - $\langle 4 \rangle 2$ . PICK  $y \in X$  such that x < y
  - $\langle 4 \rangle 3$ . For all  $u \in U$  we have  $x < y \le u$
  - $\langle 4 \rangle 4. \ x \in V$
- $\langle 3 \rangle 10. \ U \cap V = \emptyset$

PROOF: Any element of  $U \cap V$  would be a supremum of X.

- $\langle 3 \rangle 11$ . *U* and *V* form a separation of *L*.
- $\langle 3 \rangle 12$ . Q.E.D.

PROOF: This contradicts  $\langle 2 \rangle 1$ .

 $\langle 2 \rangle 3$ . L is dense.

```
\langle 3 \rangle 1. Let: x, y \in L with x < y
```

 $\langle 3 \rangle 2$ . There exists  $z \in L$  such that x < z < y

PROOF: Otherwise  $(-\infty, y)$  and  $(x, +\infty)$  would form a separation of L.

Corollary 228.1. The real line  $\mathbb{R}$  is connected.

Corollary 228.2. Every interval in  $\mathbb{R}$  is connected.

Corollary 228.3. The ordered square is connected.

**Theorem 229** (Intermediate Value Theorem). Let X be a connected space. Let Y be a linearly ordered set under the order topology. Let  $f: X \to Y$  be continuous. Let  $a, b \in X$  and  $r \in Y$ . Suppose f(a) < r < f(b). Then there exists  $c \in X$  such that f(c) = r.

PROOF: Otherwise  $f^{-1}((-\infty,r))$  and  $f^{-1}((r,+\infty))$  would form a separation of X.  $\square$ 

**Proposition 230.** Every function  $f:[0,1] \to [0,1]$  has a fixed point.

#### Proof:

- $\langle 1 \rangle 1$ . Let:  $g: [0,1] \to [-1,1]$  be the function g(x) = f(x) xProve: there exists  $x \in [0,1]$  such that g(x) = 0
- $\langle 1 \rangle 2$ . Assume: without loss of generality  $g(0) \neq 0$  and  $g(1) \neq 0$
- $\langle 1 \rangle 3. \ g(0) > 0$
- $\langle 1 \rangle 4. \ g(1) < 0$
- $\langle 1 \rangle$ 5. There exists  $x \in (0,1)$  such that g(x) = 0 PROOF: By the Intermediate Value Theorem.

## 33 Totally Disconnected Spaces

**Definition 231** (Totally Disconnected). A topological space X is *totally disconnected* if and only if the only connected subspaces are the singletons.

Example 232. Every discrete space is totally disconnected.

**Example 233.** The rationals  $\mathbb{Q}$  are totally disconnected.

### 34 Paths and Path Connectedness

**Definition 234** (Path). Let X be a topological space and  $a, b \in X$ . A path from a to b is a continuous function  $p : [0,1] \to X$  such that p(0) = a and p(1) = b.

**Definition 235** (Path Connected). A topological space is *path connected* if and only if there exists a path between any two points.

**Proposition 236.** Every path connected space is connected.

#### PROOF:

- $\langle 1 \rangle 1$ . Let: X be a path connected space.
- $\langle 1 \rangle 2$ . Assume: for a contradiction C and D form a separation of X.
- $\langle 1 \rangle 3$ . Pick  $a \in C$  and  $b \in D$ .
- $\langle 1 \rangle 4$ . PICK a path  $p : [0,1] \to X$  from a to b.
- $\langle 1 \rangle 5$ .  $p^{-1}(C)$  and  $p^{-1}(D)$  form a separation of [0,1].
- $\langle 1 \rangle 6$ . Q.E.D.

Proof: This contradicts Corollary 228.2.

An example that shows the converse does not hold:

Example 237. The ordered square is not path connected.

#### Proof:

- $\langle 1 \rangle 1$ . Assume: for a contradiction  $p:[0,1] \to I_q^2$  is a path from (0,0) to (1,1).
- $\langle 1 \rangle 2$ . p is surjective.

PROOF: By the Intermediate Value Theorem.

 $\langle 1 \rangle 3$ . For  $x \in [0,1]$ , PICK a rational  $q_x \in p^{-1}((x,0),(x,1))$ 

PROOF: Since  $p^{-1}((x,0),(x,1))$  is open and nonempty by  $\langle 1 \rangle 2$ .

- $\langle 1 \rangle$ 4. For  $x, y \in [0, 1]$ , if  $x \neq y$  then  $q_x \neq q_y$ PROOF: We have  $p(q_x) \neq p(q_y)$  because ((x, 0), (x, 1)) and ((y, 0), (y, 1)) are disjoint.
- $\langle 1 \rangle 5$ .  $\{q_x \mid x \in [0,1]\}$  is an uncountable set of rationals.
- $\langle 1 \rangle 6$ . Q.E.D.

PROOF: This contradicts the fact that the rationals are countable.

**Proposition 238.** The continuous image of a path connected space is path connected.

#### Proof:

- $\langle 1 \rangle 1$ . Let: X be a path connected space, Y a topological space, and  $f: X \twoheadrightarrow Y$  be continuous and surjective.
- $\langle 1 \rangle 2$ . Let:  $a, b \in Y$
- $\langle 1 \rangle 3$ . Pick  $c, d \in X$  with f(c) = a and f(d) = b
- $\langle 1 \rangle 4$ . PICK a path  $p:[0,1] \to X$  from c to d.
- $\langle 1 \rangle 5$ .  $f \circ p$  is a path from a to b in Y.

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**Proposition 239** (AC). The product of a family of path-connected spaces is path-connected.

#### Proof:

- $\langle 1 \rangle 1$ . Let:  $\{X_{\alpha}\}_{{\alpha} \in J}$  be a family of path-connected spaces.
- $\langle 1 \rangle 2$ . Let:  $a, b \in \prod_{\alpha \in J} X_{\alpha}$
- $\langle 1 \rangle 3$ . For  $\alpha \in J$ , PICK a path  $p_{\alpha} : [0,1] \to X_{\alpha}$  from  $a_{\alpha}$  to  $b_{\alpha}$  PROOF: Using the Axiom of Choice.
- $\langle 1 \rangle 4$ . Define  $p: [0.1] \to \prod_{\alpha \in J} X_{\alpha}$  by  $p(t)_{\alpha} = p_{\alpha}(t)$

 $\langle 1 \rangle$ 5. p is a path from a to b. PROOF: Theorem 132.

**Proposition 240.** The continuous image of a path-connected space is path-connected.

#### PROOF:

- $\langle 1 \rangle 1$ . Let:  $f: X \to Y$  be continuous and surjective where X is path-connected.
- $\langle 1 \rangle 2$ . Let:  $a, b \in Y$
- $\langle 1 \rangle 3$ . PICK  $a', b' \in X$  with f(a') = a and f(b') = b.
- $\langle 1 \rangle 4$ . PICK a path  $p : [0,1] \to X$  from a' to b'.
- $\langle 1 \rangle 5$ .  $f \circ p$  is a path from a to b.

**Proposition 241.** Let X be a topological space. The union of a set of path-connected subspaces of X that have a point in common is path-connected.

#### PROOF:

- $\langle 1 \rangle 1$ . Let:  $\mathcal{A}$  be a set of path-connected subspaces of X with the point a in common.
- $\langle 1 \rangle 2$ . Let:  $b, c \in \bigcup \mathcal{A}$
- $\langle 1 \rangle 3$ . Pick  $B, C \in \mathcal{A}$  with  $b \in B$  and  $c \in C$ .
- $\langle 1 \rangle 4$ . PICK a path p in B from b to a.
- $\langle 1 \rangle$ 5. PICK a path q in C from a to c.
- $\langle 1 \rangle$ 6. The concatenation of p and q is a path from b to c in  $\bigcup \mathcal{A}$ .

**Proposition 242.** Let  $A \subseteq \mathbb{R}^2$  be countable. Then  $\mathbb{R}^2 \setminus A$  is path-connected.

#### Proof:

- $\langle 1 \rangle 1$ . Let:  $a, b \in \mathbb{R}^2 \setminus A$
- $\langle 1 \rangle 2$ . PICK a line l in  $\mathbb{R}^2$  with a on one side and b on the other.
- $\langle 1 \rangle 3$ . For every point x on l, Let:  $p_x$  be the path in  $\mathbb{R}^2$  consisting of a line from a to x then a line from x to b
- $\langle 1 \rangle 4$ . For  $x \neq y$  we have  $p_x$  and  $p_y$  have no points in common except a and b
- $\langle 1 \rangle 5$ . There are only countably many x such that a point of A lies on  $p_x$ .
- $\langle 1 \rangle$ 6. There exists x such that  $p_x$  is a path from a to b in  $\mathbb{R}^2 \setminus A$ .

**Proposition 243.** Every open connected subspace of  $\mathbb{R}^2$  is path-connected.

#### Proof:

- $\langle 1 \rangle 1$ . Let: U be an open connected subspace of  $\mathbb{R}^2$ .
- $\langle 1 \rangle 2$ . For all  $x_0 \in U$ ,

Let:  $PC(x_0) = \{ y \in U \mid \text{there exists a path from } x \text{ to } y \}$ 

- $\langle 1 \rangle 3$ . For all  $x_0 \in U$ , the set  $PC(x_0)$  is open and closed in U.
  - $\langle 2 \rangle 1$ . Let:  $x_0 \in U$

```
\langle 2 \rangle 2. PC(x_0) is open in U
       \langle 3 \rangle 1. Let: y \in PC(x_0)
      \langle 3 \rangle 2. Pick \epsilon > 0 such that B(y, \epsilon) \subseteq U
          Proof: Since U is open.
      \langle 3 \rangle 3. \ B(y, \epsilon) \subseteq PC(x_0)
         PROOF: For all z \in B(y, \epsilon), pick a path from x_0 to y then concatenate
         the straight line from y to z.
   \langle 2 \rangle 3. PC(x_0) is closed in U
      \langle 3 \rangle 1. Let: y \in U be a limit point of PC(x_0)
      \langle 3 \rangle 2. Pick \epsilon > 0 such that B(y, \epsilon) \subseteq U
       \langle 3 \rangle 3. Pick z \in PC(x_0) \cap B(y, \epsilon)
      \langle 3 \rangle 4. \ y \in PC(x_0)
         PROOF: Pick a path from x_0 to z then concatenate the straight line from
         z to y.
\langle 1 \rangle 4. PC(x_0) = U
   Proof: Proposition 208.
```

**Example 244.** If A is a connected subspace of X, then  $A^{\circ}$  is not necessarily connected.

Take two closed circles in  $\mathbb{R}^2$  that touch at one point. The interior of this space is two open circles, and these two circles form a separation.

**Example 245.** If A is a connected subspace of X then  $\partial A$  is not necessarily connected.

We have [0,1] is connected but  $\partial[0,1] = \{0,1\}$  is not.

**Example 246.** If A is a subspace of X and  $A^{\circ}$  and  $\partial A$  are connected, then A is not necessarily connected.

We have  $\mathbb{Q}^{\circ} = \emptyset$  and  $\partial \mathbb{Q} = \mathbb{R}$  are connected but  $\mathbb{Q}$  is not connected.

# 35 The Topologist's Sine Curve

**Definition 247** (The Topologist's Sine Curve). Let  $S = \{(x, \sin 1/x) \mid 0 < x \le 1\}$ , The topologist's sine curve is the closure  $\overline{S}$  of S in  $\mathbb{R}^2$ .

Proposition 248. The topologist's sine curve is connected.

```
Proof:  \langle 1 \rangle 1. \text{ Let: } S = \{(x, \sin 1/x) \mid 0 < x \leq 1\}   \langle 1 \rangle 2. \text{ $S$ is connected.}  Proof: Theorem 218.  \langle 1 \rangle 3. \overline{S} \text{ is connected.}  Proof: Theorem 217.  \Box
```

**Proposition 249.** The topologist's sine curve is  $\{(x, \sin 1/x) \mid 0 < x \le 1\} \cup (\{0\} \times [-1.1]).$ 

PROOF: Sketch proof: Given a point (0.y) with  $-1 \le y \le 1$ , pick a such that  $\sin a = y$ . Then  $((1/a, y), (1/(a + 2\pi), y), (1/(a + 4\pi), y), ...)$  is a sequence in S that converges to (0, y).

Conversely, let (x,y) be any point not in  $S \cup (\{0\} \times [-1,1])$ . If x < 0 or y > 1or y < -1 then we can easily find a neighbourhood that does not intersect  $S \cup (\{0\} \times [-1,1])$ . If x > 0 and  $-1 \le y \le 1$ , then we have  $y \ne \sin 1/x$ . Hence pick a neighbourhood that does not intersect S.

**Proposition 250.** Every closed subset of  $\mathbb{R}$  that is bounded above has a greatest element.

PROOF: It has a supremum, which is a limit point of the set and hence an element.  $\Box$ 

**Proposition 251** (CC). The topologist's sine curve is not path connected.

#### Proof:

- $\langle 1 \rangle 1$ . Assume: For a contradction  $p:[0,1] \to \overline{S}$  is a path from (0,0) to  $(1,\sin 1)$ .
- $\langle 1 \rangle 2$ .  $\{ t \in [0,1] \mid p(t) \in \{0\} \times [-1,1] \}$  is closed.

PROOF: Since p is continuous and  $\{0\} \times [-1, 1]$  is closed.

- $\langle 1 \rangle 3$ . Let: b be the largest number in [0,1] such that  $p(b) \in \{0\} \times [-1,1]$ . Proof: Proposition 250.
- $\langle 1 \rangle 4$ . Let:  $x:[b,1] \to \overline{S}$  be the function  $\pi_1 \circ p$   $\langle 1 \rangle 5$ . Let:  $y:[b,1] \to \overline{S}$  be the function  $\pi_2 \circ p$
- (1)6. PICK a sequence  $t_n$  in (b,1] such that  $t_n \to b$  and  $y(t_n) = (-1)^n$  for all n $\langle 2 \rangle 1$ . Let:  $n \geq 1$ 
  - $\langle 2 \rangle 2$ . PICK *u* with 0 < u < x(1/n) and  $\sin(1/u) = (-1)^n$
  - $\langle 2 \rangle 3$ . PICK  $t_n$  with  $b < t_n < 1/n$  and  $x(t_n) = u$

PROOF: By the Intermediate Value Theorem

 $\langle 1 \rangle 7$ . Q.E.D.

PROOF: This contradicts Proposition 105 since y is continuous and  $y(t_n)$  does not converge.

Corollary 251.1. The closure of a path-connected subspace of a space is not necessarily path-connected.

#### 36 The Long Line

**Definition 252** (The Long Line). The long line is the space  $S_{\Omega} \times [0,1)$  in the dictionary order under the order topology.

**Proposition 253.** The long line is path-connected.

#### 37 **Topological Groups**

**Definition 254** (Topological Group). A topological group G consists of a  $T_1$ space G and continuous maps  $\cdot: G^2 \to G$  and  $()^{-1}: G \to G$  such that  $(G,\cdot,()^{-1})$  is a group.

**Example 255.** 1. The integers  $\mathbb{Z}$  under addition are a topological group.

- 2. The real numbers  $\mathbb{R}$  under addition are a topological group.
- 3. The positive reals under multiplication are a topological group.
- 4. The set  $\{z \in \mathbb{C} \mid |z|=1\}$  under multiplication and given the topology of  $S^1$  is a topological group.
- 5. For any  $n \geq 0$ , the general linear group  $GL_n(\mathbb{R})$  is a topological group under matrix multiplication, considered as a subspace of  $\mathbb{R}^{n^2}$ .

**Lemma 256.** Let G be a  $T_1$  space and  $\cdot: G^2 \to G$ ,  $(\ )^{-1}: G \to G$  be functions such that  $(G,\cdot,(\ )^{-1})$  is a group. Then G is a topological group if and only if the function  $f: G^2 \to G$  that maps (x,y) to  $xy^{-1}$  is continuous.

Proof:

 $\langle 1 \rangle 1$ . If G is a topological group then f is continuous.

PROOF: From Theorem 95.

- $\langle 1 \rangle 2$ . If f is continuous then G is a topological group.
  - $\langle 2 \rangle 1$ . Assume: f is continuous.
  - $\langle 2 \rangle 2$ . ()<sup>-1</sup> is continuous.

PROOF: Since  $x^{-1} = f(e, x)$ .

 $\langle 2 \rangle 3$ . · is continuous.

PROOF: Since  $xy = f(x, y^{-1})$ .

**Lemma 257.** Let G be a topological group and H a subgroup of G. Then H is a topological group under the subspace topology.

Proof:

 $\langle 1 \rangle 1$ . H is  $T_1$ .

PROOF: From Proposition 166.

 $\langle 1 \rangle 2$ . multiplication and inverse on H are continuous.

PROOF: From Theorem 96.

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**Lemma 258.** Let G be a topological group and H a subgroup of G. Then  $\overline{H}$  is a subgroup of G.

Proof:

 $\begin{array}{ll} \langle 1 \rangle 1. \ \ \mathrm{Let:} \ \ x,y \in \overline{H} \\ \ \ \mathrm{Prove:} \ \ xy^{-1} \in \overline{H} \end{array}$ 

- $\langle 1 \rangle 2$ . Let: U be any neighbourhood of  $xy^{-1}$
- $\langle 1 \rangle 3$ . Let:  $f: G^2 \to G$ ,  $f(a,b) = ab^{-1}$
- $\langle 1 \rangle 4$ .  $f^{-1}(U)$  is a neighbourhood of (x,y)
- $\langle 1 \rangle 5.$  Pick neighbourhoods  $V,\,W$  of x and y respectively such that  $f(V \times W) \subseteq U.$

```
\langle 1 \rangle 6. Pick a \in V \cap H and b \in W \cap H
   PROOF: Theorem 45.
\langle 1 \rangle 7. \ ab^{-1} \in U \cap H
\langle 1 \rangle 8. Q.E.D.
   PROOF: By Theorem 45.
```

**Proposition 259.** Let G be a topological group and  $\alpha \in G$ . Then the maps  $l_{\alpha}, r_{\alpha}: G \to G$  defined by  $l_{\alpha}(x) = \alpha x$ ,  $r_{\alpha}(x) = x\alpha$  are homeomorphisms of G with itself.

PROOF: They are continuous with continuous inverses  $l_{\alpha^{-1}}$  and  $r_{\alpha^{-1}}$ .  $\sqcup$ 

Corollary 259.1. Every topological group is homogeneous.

PROOF: Given a topological group G and  $a, b \in G$ , we have  $l_{ba^{-1}}$  is a homeomorphism that maps a to b.

**Proposition 260.** Let G be a topological group and H a subgroup of G. Give G/H the quotient topology. For all  $\alpha \in G$ , the map  $\overline{f_{\alpha}}$  that sends xH to  $\alpha xH$ is a homeomorphism  $G/H \cong G/H$ .

#### Proof:

 $\langle 1 \rangle 1$ .  $\overline{f_{\alpha}}$  is well-defined.

PROOF: If  $xy^{-1} \in H$  then  $(\alpha x)(\alpha y)^{-1} = xy^{-1} \in H$ .

 $\langle 1 \rangle 2$ .  $\overline{f_{\alpha}}$  is continuous.

PROOF: Theorem 194 since  $\overline{f_{\alpha}} \circ p = p \circ f_{\alpha}$  is continuous, where  $p: G \to G/H$ is the canonical surjection.  $\langle 1 \rangle 3$ .  $\overline{f_{\alpha}}^{-1}$  is continuous.

PROOF: Similar since  $\overline{f_{\alpha}}^{-1} = \overline{f_{\alpha^{-1}}}$ .

Corollary 260.1. Let G be a topological group and H a subgroup of G. Give G/H the quotient topology. Then G/H is homogeneous.

**Proposition 261.** Let G be a topological group and H a subgroup of G. Give G/H the quotient topology. If H is closed in G then G/H is  $T_1$ .

#### Proof:

- $\langle 1 \rangle 1$ . Let:  $p: G \rightarrow G/H$  be the canonical surjection
- $\langle 1 \rangle 2$ . Let:  $x \in G$
- $\langle 1 \rangle 3. \ p^{-1}(xH) = f_x(H)$
- $\langle 1 \rangle 4$ .  $p^{-1}(xH)$  is closed in G

PROOF: Since H is closed and  $f_x$  is a homemorphism of G with itself.

 $\langle 1 \rangle 5$ .  $\{xH\}$  is closed in G/H

**Proposition 262.** Let G be a topological group and H a subgroup of G. Give G/H the quotient topology. Then the canonical surjection p:G woheadrightarrow G/H is an open map.

```
Proof:
```

```
\langle 1 \rangle 1. Let: U \subseteq G be open.

\langle 1 \rangle 2. p^{-1}(p(U)) = \bigcup_{h \in H} r_h(U)

\langle 1 \rangle 3. p^{-1}(p(U)) is open.
```

 $\langle 1 \rangle 4$ . p(U) is open.

**Proposition 263.** Let G be a topological group and H a closed normal subgroup of G. Then G/H is a topological group under the quotient topology.

#### Proof:

 $\langle 1 \rangle 1$ . G/H is  $T_1$ 

Proof: Proposition 261.

 $\langle 1 \rangle 2$ . The map  $\overline{m}: (xH, yH) \mapsto xy^{-1}H$  is continuous.

 $\langle 2 \rangle 1.$   $p^2: G^2 \to (G/H)^2$  is a quotient map.

Proof: Propositions 193, 262.

 $\langle 2 \rangle 2$ .  $\overline{m} \circ p^2$  is continuous.

PROOF: As it is  $p^2 \circ m$  where  $m: G^2 \to G$  with  $m(x,y) = xy^{-1}$ 

**Lemma 264.** Let G be a topological group and  $A, B \subseteq G$ . If either A or B is open then AB is open.

PROOF: If A is open we have  $AB = \bigcup_{b \in B} r_b(A)$ . Similarly if B is open.  $\square$ 

**Definition 265** (Symmetric Neighbourhood). Let G be a topological group. A neighbourhood V of e is *symmetric* if and only if  $V = V^{-1}$ .

**Lemma 266.** Let G be a topological group. Let V be a neighbourhood of e. Then V is symmetric if and only if, for all  $x \in V$ , we have  $x^{-1} \in V$ .

#### Proof:

 $\langle 1 \rangle 1$ . If V is symmetric then, for all  $x \in V$ , we have  $x^{-1} \in V$  PROOF: Immediate from defintions.

 $\langle 1 \rangle 2$ . If, for all  $x \in V$ , we have  $x^{-1} \in V$ , then V is symmetric.

 $\langle 2 \rangle 1$ . Assume: for all  $x \in V$  we have  $x^{-1} \in V$ 

 $\langle 2 \rangle 2$ .  $V \subset V^{-1}$ 

PROOF: If  $x \in V$  then there exists  $y \in V$  such that  $x = y^{-1}$ , namely  $y = x^{-1}$ 

 $\langle 2 \rangle 3. \ V^{-1} \subseteq V$ 

PROOF: Immediate from  $\langle 2 \rangle 1$ .

**Lemma 267.** Let G be a topological group. For every neighbourhood U of e, there exists a symmetric neighbourhood V of e such that  $V^2 \subseteq U$ .

#### PROOF

- $\langle 1 \rangle 1$ . Let: U be a neighbourhood of e.
- $\langle 1 \rangle 2$ . PICK a neighbourhood V' of e such that  $V'V' \subseteq U$

```
Proof: Such a neighbourhood exists because multiplication in G is continuous
```

- $\langle 1 \rangle 3$ . PICK a neighbourhood W of e such that  $WW^{-1} \subseteq V'$ 
  - PROOF: Such a neighbourhood exists because the function that maps (x, y) to  $xy^{-1}$  is continuous.
- $\langle 1 \rangle 4$ . Let:  $V = WW^{-1}$
- $\langle 1 \rangle 5$ . V is a neighbourhood of e
  - $\langle 2 \rangle 1. \ e \in V$

PROOF: Since  $e \in W$  so  $e = ee^{-1} \in V$ .

 $\langle 2 \rangle 2$ . V is open

Proof: Lemma 264.

- $\langle 1 \rangle 6$ . V is symmetric
  - $\langle 2 \rangle 1$ . For all  $x \in V$  we have  $x^{-1} \in V$ 
    - $\langle 3 \rangle 1$ . Let:  $x \in V$
    - $\langle 3 \rangle 2$ . PICK $y, z \in W$  such that  $x = yz^{-1}$
    - $\langle 3 \rangle 3. \ x^{-1} = zy^{-1}$
    - $(3)4. \ x^{-1} \in V$
    - $\langle 3 \rangle 5. \ x \in V^{-1}$
  - $\langle 2 \rangle 2$ . Q.E.D.

Proof: Lemma 266

 $\langle 1 \rangle 7. \ V^2 \subseteq U$ 

PROOF: We have  $V^2 \subseteq (V')^2 \subseteq U$ 

#### **Proposition 268.** Every topological group is Hausdorff.

#### Proof:

- $\langle 1 \rangle 1$ . Let: G be a topological group.
- $\langle 1 \rangle 2$ . Let:  $x, y \in G$  with  $x \neq y$
- $\langle 1 \rangle 3$ . Let:  $U = G \setminus \{x[^{-1}y]\}$
- $\langle 1 \rangle 4$ . Pick a symmetric neighbourhood V of e such that  $VV \subseteq U$ 
  - $\langle 2 \rangle 1$ . U is open

PROOF: Since G is  $T_1$ .

 $\langle 2 \rangle 2. \ e \in U$ 

PROOF: Since  $x \neq y$ 

 $\langle 2 \rangle 3$ . Q.E.D.

Proof: Lemma 267.

- $\langle 1 \rangle 5$ . Vx and Vy are disjoint neighbourhoods of x and y respectively.
  - $\langle 2 \rangle 1$ . Vx is open

PROOF: Since  $Vx = r_x(V)$ 

 $\langle 2 \rangle 2$ . Vy is open

PROOF: Similar.

- $\langle 2 \rangle 3. \ Vx \cap Vy = \emptyset$ 
  - $\langle 3 \rangle 1$ . Assume: for a contradiction  $z \in Vx \cap Vy$
  - $\langle 3 \rangle 2$ . Pick  $a, b \in V$  such that z = ax = by
  - $\langle 3 \rangle 3. \ xy^{-1} \in VV$

PROOF: Since  $xy^{-1} = a^{-1}b$ 

```
\langle 3 \rangle 4. \ xy^{-1} \in U
\langle 3 \rangle 5. \ Q.E.D.
PROOF: From \langle 1 \rangle 3.
```

Proposition 269. Every topological group is regular.

```
Proof:
```

- $\langle 1 \rangle 1$ . Let: G be a topological group.
- $\langle 1 \rangle 2$ . Let:  $A \subseteq G$  be a closed set and  $a \notin A$ .
- $\langle 1 \rangle 3$ . Let:  $U = G \setminus Aa^{-1}$
- $\langle 1 \rangle 4$ . PICK a symmetric neighbourhood V of e such that  $VV \subseteq U$ 
  - $\langle 2 \rangle 1$ . *U* is open

PROOF: Since  $Aa^{-1} = r_{a^{-1}}(A)$  is closed.

 $\langle 2 \rangle 2. \ e \in U$ 

PROOF: Since  $a \notin A$ .

 $\langle 2 \rangle 3$ . Q.E.D.

Proof: Lemma 267.

- $\langle 1 \rangle 5$ . VA and Va are disjoint open sets with  $A \subseteq VA$  and  $a \in Va$ 
  - $\langle 2 \rangle 1$ . VA is open

Proof: Lemma 264

 $\langle 2 \rangle 2$ . Va is open

Proof: Lemma 264

- $\langle 2 \rangle 3. \ VA \cap Va = \emptyset$ 
  - $\langle 3 \rangle 1$ . Assume: for a contradiction  $z \in VA \cap Va$
  - $\langle 3 \rangle 2$ . Pick  $b, c \in V$  and  $d \in A$  with z = bd = ca
  - $\langle 3 \rangle 3. \ da^{-1} \in U$

PROOF: Since  $da^{-1} = b^{-1}c \in VV \subseteq U$ 

 $\langle 3 \rangle 4$ . Q.E.D.

Proof: This contradicts  $\langle 1 \rangle 3$ 

**Proposition 270.** Let G be a topological group and H a subgroup of G. Give G/H the quotient topology. If H is closed in G then G/H is regular.

#### Proof

- $\langle 1 \rangle 1$ . Let:  $p: G \rightarrow G/H$  be the canonical surjection.
- $\langle 1 \rangle 2$ . Let: A be a closed set in G/H and  $aH \in (G/H) \setminus A$ .
- $\langle 1 \rangle 3$ . Let:  $B = p^{-1}(A)$
- $\langle 1 \rangle 4$ . B is a closed saturated set in G.
- $\langle 1 \rangle 5. \ B \cap aH = \emptyset$
- $\langle 1 \rangle 6. \ B = BH$
- $\langle 1 \rangle$ 7. Pick a symmetric neighbourhood V of e such that VB does not intersect Va
  - $\langle 2 \rangle 1$ . Let:  $U = G \setminus Ba^{-1}$
  - $\langle 2 \rangle 2.$  PICK a symmetric neighbourhood V of e such that  $VV \subseteq U$ 
    - $\langle 3 \rangle 1$ . *U* is open

```
PROOF: Since Ba^{-1} = r_{a^{-1}}(B) is closed.
      \langle 3 \rangle 2. \ e \in U
         PROOF: If e \in Ba^{-1} then a \in B
      \langle 3 \rangle 3. Q.E.D.
         Proof: Lemma 267
   \langle 2 \rangle 3. \ VB \cap Va = \emptyset
      PROOF: If vb = v'a for v, v' \in V and b \in B then we have ba^{-1} = v^{-1}v' \in V
      Ba \cap VV \subseteq Ba \cap U.
\langle 1 \rangle 8. p(VB) and p(Va) are disjoint open sets
   \langle 2 \rangle 1. p(VB) and p(Va) are open.
      PROOF: Proposition 262.
   \langle 2 \rangle 2. p(VB) \cap p(Va) = \emptyset
      PROOF: If vbH = v'aH for v, v' \in V, b \in B then v'a = vbh for some h \in H.
      Hence v'a \in Va \cap VBH = Va \cap VB.
\langle 1 \rangle 9. \ A \subseteq p(VB)
\langle 1 \rangle 10. \ aH \in p(Va)
```

## 38 The Metric Topology

**Definition 271** (Metric). Let X be a set. A *metric* on X is a function  $d: X^2 \to \mathbb{R}$  such that:

- 1. For all  $x, y \in X$ ,  $d(x, y) \ge 0$
- 2. For all  $x, y \in X$ , d(x, y) = 0 if and only if x = y
- 3. For all  $x, y \in X$ , d(x, y) = d(y, x)
- 4. (Triangle Inequality) For all  $x, y, z \in X$ ,  $d(x, z) \leq d(x, y) + d(y, z)$

We call d(x, y) the distance between x and y.

**Definition 272** (Open Ball). Let X be a metric space. Let  $a \in X$  and  $\epsilon > 0$ . The *open ball* with *centre a* and *radius*  $\epsilon$  is

$$B(a,\epsilon) = \{x \in X \mid d(a,x) < \epsilon\} .$$

**Definition 273** (Metric Topology). Let X be a metric space. The *metric topology* on X is the topology generated by the basis consisting of all the open balls.

We prove this is a basis for a topology.

Proof

 $\langle 1 \rangle 1$ . For every point a, there exists a ball B such that  $a \in B$  PROOF: We have  $a \in B(a,1)$ .

- $\langle 1 \rangle 2$ . For any balls  $B_1$ ,  $B_2$  and point  $a \in B_1 \cap B_2$ , there exists a ball  $B_3$  such that  $a \in B_3 \subseteq B_1 \cap B_2$ 
  - $\langle 2 \rangle 1$ . Let:  $B_1 = B(c_1, \epsilon_1)$  and  $B_2 = B(c_2, \epsilon_2)$
  - $\langle 2 \rangle 2$ . Let:  $\delta = \min(\epsilon_1 d(c_1, a), \epsilon_2 d(c_2, a))$ Prove:  $B(a, \delta) \subseteq B_1 \cap B_2$
  - $\langle 2 \rangle 3$ . Let:  $x \in B(a, \delta)$
  - $\langle 2 \rangle 4. \ x \in B_1$

Proof:

$$d(x, c_1) = d(x, a) + d(a, c_1)$$

$$< \delta + d(a, c_1)$$

$$\le \epsilon_1$$

 $\langle 2 \rangle 5. \ x \in B_2$ 

PROOF: Similar.

**Proposition 274.** Let X be a metric space and  $U \subseteq X$ . Then U is open if and only if, for every  $x \in U$ , there exists  $\epsilon > 0$  such that  $B(x, \epsilon) \subseteq U$ .

PROOF

 $\langle 1 \rangle 1$ . If U is open then, for all  $x \in U$ , there exists  $\epsilon > 0$  such that  $B(x, \epsilon) \subseteq U$ .  $\langle 2 \rangle 1$ . Assume: U is open.

- $\langle 2 \rangle 2$ . Let:  $x \in U$
- $\langle 2 \rangle 3$ . Pick  $a \in X$  and  $\delta > 0$  such that  $x \in B(a, \delta) \subseteq U$
- $\langle 2 \rangle 4$ . Let:  $\epsilon = \delta d(a, x)$ Prove:  $B(x, \epsilon) \subseteq U$
- $\langle 2 \rangle 5$ . Let:  $y \in B(x, \epsilon)$
- $\langle 2 \rangle 6.$   $d(y,a) < \delta$

Proof:

$$d(y,a) \le d(a,x) + d(x,y)$$
$$< \delta + d(x,y)$$
$$= \epsilon$$

 $\langle 2 \rangle 7. \ y \in U$ 

 $\langle 1 \rangle 2$ . If, for all  $x \in U$ , there exists  $\epsilon > 0$  such that  $B(x, \epsilon) \subseteq U$ , then U is open. PROOF: Immediate from definitions.

**Definition 275** (Discrete Metric). Let X be a set. The *discrete metric* on X is defined by

 $d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$ 

Proposition 276. The discrete metric induces the discrete topology.

PROOF: For any (open) set U and point  $a \in U$ , we have  $a \in B(a,1) \subseteq U$ .  $\square$ 

**Definition 277** (Standard Metric on  $\mathbb{R}$ ). The *standard metric* on  $\mathbb{R}$  is defined by d(x,y) = |x-y|.

**Proposition 278.** The standard metric on  $\mathbb{R}$  induces the standard topology on  $\mathbb{R}$ .

Proof:

 $\langle 1 \rangle 1$ . Every open ball is open in the standard topology on  $\mathbb{R}$ .

PROOF:  $B(a, \epsilon) = (a - \epsilon, a + \epsilon)$ 

- $\langle 1 \rangle 2$ . For every open set U and point  $a \in U$ , there exists  $\epsilon > 0$  such that  $B(a,\epsilon) \subseteq U$ 
  - $\langle 2 \rangle 1$ . Let: U be an open set and  $a \in U$
  - $\langle 2 \rangle 2$ . PICK an open interval b, c such that  $a \in (b,c) \subseteq U$
  - $\langle 2 \rangle 3$ . Let:  $\epsilon = \min(a b, c a)$
  - $\langle 2 \rangle 4. \ B(a, \epsilon) \subseteq U$

**Definition 279** (Metrizable). A topological space X is metrizable if and only if there exists a metric on X that induces the topology.

**Definition 280** (Bounded). Let X be a metric space and  $A \subseteq X$ . Then A is *bounded* if and only if there exists M such that, for all  $x, y \in A$ , we have  $d(x, y) \leq M$ .

**Definition 281** (Diameter). Let X be a metric space and  $A \subseteq X$ . The diameter of A is

$$\operatorname{diam} A = \sup_{x,y \in A} d(x,y) .$$

**Definition 282** (Standard Bounded Metric). Let d be a metric on X. The standard bounded metric corresponding to d is the metric  $\overline{d}$  defined by

$$\overline{d}(x,y) = \min(d(x,y),1) .$$

We prove this is a metric.

Proof:

```
\langle 1 \rangle 1. \overline{d}(x,y) \geq 0
   PROOF: Since d(x, y) \ge 0
\langle 1 \rangle 2. \overline{d}(x,y) = 0 if and only if x = y
   PROOF: \overline{d}(x,y) = 0 if and only if d(x,y) = 0 if and only if x = y
\langle 1 \rangle 3. \ \overline{d}(x,y) = \overline{d}(y,x)
   PROOF: Since d(x, y) = d(y, x)
\langle 1 \rangle 4. \ \overline{d}(x,z) \leq \overline{d}(x,y) + \overline{d}(y,z)
   Proof:
            \overline{d}(x,y) + \overline{d}(y,z) = \min(d(x,y),1) + \min(d(y,z),1)
                                     = \min(d(x,y) + d(y,z), d(x,y) + 1, d(y,z) + 1, 2)
                                     \geq \min(d(x,z),1)
                                     = \overline{d}(x,z)
```

**Lemma 283.** In any metric space X, the set  $\mathcal{B} = \{B(a, \epsilon) \mid a \in X, \epsilon < 1\}$  is a basis for the metric topology.

Proof:

 $\langle 1 \rangle 1$ . Every element of  $\mathcal{B}$  is open.

PROOF: From Lemma 59.

- $\langle 1 \rangle 2$ . For every open set U and point  $a \in U$ , there exists  $B \in \mathcal{B}$  such that  $a \in B \subseteq U$ 
  - $\langle 2 \rangle 1$ . Let: U be an open set and  $a \in U$
  - $\langle 2 \rangle 2$ . PICK $\epsilon > 0$  such that  $B(a, \epsilon) \subseteq U$
  - $\langle 2 \rangle 3$ .  $B(a, \min(\epsilon, 1/2)) \subseteq U$
- $\langle 1 \rangle 3$ . Q.E.D.

Proof: Lemma 60.

**Proposition 284.** Let d be a metric on the set X. Then the standard bounded metric d induces the same metric as d.

PROOF: This follows from Lemma 283 since the open balls with radius < 1 are the same under both metrics.  $\square$ 

**Lemma 285.** Let d and d' be two metrics on the same set X. Let  $\mathcal{T}$  and  $\mathcal{T}'$  be the topologies they induce. Then  $\mathcal{T} \subseteq \mathcal{T}'$  if and only if, for all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that

$$B_{d'}(x,\delta) \subseteq B_d(x,\epsilon)$$
.

Proof:

 $\langle 1 \rangle 1$ . If  $\mathcal{T} \subseteq \mathcal{T}'$  then, for all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $B_{d'}(x,\delta) \subseteq B_d(x,\epsilon)$ 

PROOF: From Proposition 274 since  $x \in B_d(x, \epsilon) \in \mathcal{T}'$ .

- $\langle 1 \rangle 2$ . If, for all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$ , then  $\mathcal{T} \subseteq \mathcal{T}'$ 
  - $\langle 2 \rangle 1$ . Assume: For all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $B_{d'}(x,\delta) \subseteq B_d(x,\epsilon)$
  - $\langle 2 \rangle 2$ . Let:  $U \in \mathcal{T}$
  - $\langle 2 \rangle 3$ . For all  $x \in U$  there exists  $\delta > 0$  such that  $B_{d'}(x, \delta) \subseteq U$ .
    - $\langle 3 \rangle 1$ . Let:  $x \in U$
    - $\langle 3 \rangle 2$ . PICK  $\epsilon > 0$  such that  $B_d(x, \epsilon) \subseteq U$

Proof: Proposition 274

 $\langle 3 \rangle 3$ . PICK  $\delta > 0$  such that  $B_{d'}(x,\delta) \subseteq B_d(x,\epsilon)$ 

Proof: By  $\langle 2 \rangle 1$ 

 $\langle 3 \rangle 4$ .  $B_{d'}(x,\delta) \subseteq U$ 

 $\langle 2 \rangle 4. \ U \in \mathcal{T}'$ 

Proof: Proposition 274.

**Proposition 286.**  $\mathbb{R}^2$  under the dictionary order topology is metrizable.

PROOF: Define  $d: \mathbb{R}^2 \to \mathbb{R}$  by

$$d((x,y),(x,z)) = \max(|y-z|,1)$$

$$d((x,y),(x',y')) = 1 if x \neq x' \square$$

$$\langle 1 \rangle 1. \ x \in \bigcap_{i=1}^{N} \pi_i^{-1}() \subseteq B_D(a,\epsilon)$$

**Proposition 287.** Let  $d: X^2 \to \mathbb{R}$  be a metric on X. Then the metric topology on X is the coarsest topology such that d is continuous.

- $\langle 1 \rangle 1$ . d is continuous.
  - $\langle 2 \rangle 1$ . Let:  $a, b \in X$
  - $\langle 2 \rangle 2$ . Let:  $\epsilon > 0$
  - $\langle 2 \rangle 3$ . Let:  $\delta = \epsilon/2$
  - $\langle 2 \rangle 4$ . Let:  $x, y \in X$
  - $\langle 2 \rangle$ 5. Assume:  $\rho((a,b),(x,y)) < \delta$
  - $\langle 2 \rangle 6$ .  $|d(a,b) d(x,y)| < \epsilon$ 
    - $\langle 3 \rangle 1. \ d(a,b) d(x,y) < \epsilon$

Proof:

$$d(a,b) \le d(a,x) + d(x,y) + d(y,b)$$

$$\le d(x,y) + 2\rho((a,b),(x,y))$$

$$< d(x,y) + 2\delta$$

$$= d(x,y) + \epsilon$$

 $\langle 3 \rangle 2. \ d(a.b) - d(x,y) > -\epsilon$ 

Proof: Similar.

 $\langle 2 \rangle$ 7. Q.E.D.

 $\langle 1 \rangle 2$ . If  $\mathcal{T}$  is any topology under which d is continuous then  $\mathcal{T}$  is finer than the metric topology.

Proof: Since  $B(a, \epsilon) = d_a^{-1}((-\infty, \epsilon))$ 

**Proposition 288.** Let X be a metric space with metric d and  $A \subseteq X$ . The restriction of d to A is a metric on A that induces the subspace topology.

#### Proof:

- $\langle 1 \rangle 1$ . The restriction of d to A is a metric on A.
- $\langle 1 \rangle 2.$  Every open ball under  $d \upharpoonright A$  is open under the subspace topology.

PROOF:  $B_{d \upharpoonright A}(a, \epsilon) = B_d(a, \epsilon) \cap A$ .

- $\langle 1 \rangle 3$ . If U is open in the subspace topology and  $x \in U$ , then there exists a  $d \upharpoonright A$ -ball B such that  $x \in B \subseteq U$ .
  - $\langle 2 \rangle 1$ . PICK V open in X such that  $U = V \cap A$
  - $\langle 2 \rangle 2$ . Pick  $\epsilon > 0$  such that  $B_d(x, \epsilon) \subseteq V$
  - $\langle 2 \rangle 3$ . Take  $B = B_{d \uparrow A}(x, \epsilon)$

Corollary 288.1. A subspace of a metrizable space is metrizable.

Proposition 289. Every metrizable space is Hausdorff.

#### Proof:

- $\langle 1 \rangle 1$ . Let: X be a metric space
- $\langle 1 \rangle 2$ . Let:  $a, b \in X$  with  $a \neq b$
- $\langle 1 \rangle 3$ . Let:  $\epsilon = d(a,b)/2$
- $\langle 1 \rangle 4$ . Let:  $U = B(a, \epsilon)$  and  $V = B(b, \epsilon)$
- $\langle 1 \rangle$ 5. U and V are disjoint neighbourhoods of a and b respectively.

**Proposition 290** (CC). The product of a countable family of metrizable spaces is metrizable.

### Proof:

- $\langle 1 \rangle 1$ . Let:  $(X_n, d_n)$  be a sequence of metric spaces.
- $\langle 1 \rangle 2$ . Assume: w.l.o.g. each  $d_n$  is bounded above by 1.

Proof: By Proposition 284.

(1)3. Let: D be the metric on  $\mathbb{R}^{\omega}$  defined by  $D(x,y) = \sup_{i} (d_i(x_i,y_i)/i)$ .

- $\langle 2 \rangle 1$ .  $D(x,y) \geq 0$
- $\langle 2 \rangle 2$ . D(x,y) = 0 if and only if x = y
- $\langle 2 \rangle 3. \ D(x,y) = D(y,x)$
- $\langle 2 \rangle 4. \ D(x,z) \le D(x,y) + D(y,z)$

Proof:

$$D(x, z) = \sup_{i} \frac{d_i(x_i, z_i)}{i}$$

$$\leq \sup_{i} \frac{d_i(x_i, y_i) + d_i(y_i, z_i)}{i}$$

$$\leq \sup_{i} \frac{d_i(x_i, y_i)}{i} + \sup_{i} \frac{d_i(y_i, z_i)}{i}$$

$$= D(x, y) + D(y, z)$$

- $\langle 1 \rangle 4$ . Every open ball  $B_D(a, \epsilon)$  is open in the product topology.
  - $\langle 2 \rangle 1$ . PICK N such that  $1/\epsilon < N$
- $\langle 2 \rangle 2$ .  $B_D(a, \epsilon) = \prod_{i=1}^{\infty} U_i$  where  $U_i = B(a_i, i\epsilon)$  if  $i \leq N$ , and  $U_i = X_i$  if i > N
- $\langle 1 \rangle 5$ . For any open set U and  $a \in U$ , there exists  $\epsilon > 0$  such that  $B_D(a, \epsilon) \subseteq U$ .
  - $\langle 2 \rangle 1$ . Let:  $n \geq 1$ , V be an open set in  $\mathbb{R}$  and  $a \in \pi_n^{-1}(V)$
  - $\langle 2 \rangle 2$ . PICK  $\epsilon > 0$  such that  $B_{d_n}(a, \epsilon) \subseteq V$
- $\langle 2 \rangle 3. \ B_D(a, \epsilon/n) \subseteq \pi_n^{-1}(V)$

**Theorem 291.** Let X and Y be metric spaces and  $f: X \to Y$ . Then f is continuous if and only if, for all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that, for all  $y \in X$ , if  $d(x, y) < \delta$  then  $d(f(x), f(y)) < \epsilon$ .

# Proof:

- $\langle 1 \rangle 1$ . If f is continuous then, for all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that, for all  $y \in X$ , if  $d(x,y) < \delta$  then  $d(f(x),f(y)) < \epsilon$ 
  - $\langle 2 \rangle 1$ . Assume: f is continuous.
  - $\langle 2 \rangle 2$ . Let:  $x \in X$  and  $\epsilon > 0$
  - $\langle 2 \rangle$ 3. PICK a neighbourhood U of x such that  $f(U) \subseteq B(f(x), \epsilon)$  PROOF: Theorem 92.
  - $\langle 2 \rangle 4$ . PICK  $\delta > 0$  such that  $B(x, \delta) \subseteq U$ PROOF: Proposition 274.
  - $\langle 2 \rangle$ 5. For all  $y \in X$ , if  $d(x,y) < \delta$  then  $d(f(x),f(y)) < \epsilon$
- $\langle 1 \rangle$ 2. If for all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that, for all  $y \in X$ , if  $d(x,y) < \delta$  then  $d(f(x),f(y)) < \epsilon$ , then f is continuous.
  - $\langle 2 \rangle$ 1. Assume: for all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that, for all  $y \in X$ , if  $d(x,y) < \delta$  then  $d(f(x),f(y)) < \epsilon$
  - $\langle 2 \rangle 2$ . Let:  $x \in X$  and V be a neighbourhood of f(x)
  - $\langle 2 \rangle$ 3. PICK  $\epsilon > 0$  such that  $B(f(x), \epsilon) \subseteq V$

Proof: Proposition 274.

- $\langle 2 \rangle$ 4. Pick  $\delta > 0$  such that, for all  $y \in X$ , if  $d(x,y) < \delta$  then  $d(f(x),f(y)) < \epsilon$  Proof: By  $\langle 2 \rangle$ 1
- $\langle 2 \rangle$ 5. Let:  $U = B(x, \delta)$
- $\langle 2 \rangle 6$ . U is a neighbourhood of x with  $f(U) \subseteq V$

 $\langle 2 \rangle 7$ . Q.E.D.

PROOF: Theorem 92.

Г

**Proposition 292.** Let X be a metric space. Let  $(a_n)$  be a sequence in X and  $l \in X$ . Then  $a_n \to l$  as  $n \to \infty$  if and only if, for all  $\epsilon > 0$ , there exists N such that, for all  $n \ge N$ , we have  $d(a_n, l) < \epsilon$ .

Proof: From Proposition 75.  $\square$ 

Proposition 293. Every metrizable space is first countable.

PROOF: In any metric space X, the open balls B(a,1/n) for  $n \ge 1$  form a local basis at a.

**Example 294.**  $\mathbb{R}^{\omega}$  under the box topology is not metrizable.

**Example 295.** If J is uncountable then  $\mathbb{R}^J$  under the product topology is not metrizable.

# 39 Real Linear Algebra

**Definition 296** (Square Metric). The square metric  $\rho$  on  $\mathbb{R}^n$  is defined by

$$\rho(\vec{x}, \vec{y}) = \max(|x_1 - y_1|, \dots, |x_n - y_n|)$$

We prove this is a metric.

Proof:

 $\langle 1 \rangle 1. \ \rho(\vec{x}, \vec{y}) \geq 0$ 

PROOF: Immediate from definition.

 $\langle 1 \rangle 2$ .  $\rho(\vec{x}, \vec{y}) = 0$  if and only if  $\vec{x} = \vec{y}$ 

PROOF: Immediate from definition.

 $\langle 1 \rangle 3. \ \rho(\vec{x}, \vec{y}) = \rho(\vec{y}, \vec{x})$ 

PROOF: Immediate from definition.

 $\langle 1 \rangle 4. \ \rho(\vec{x}, \vec{z}) \leq \rho(\vec{x}, \vec{y}) + \rho(\vec{y}, \vec{z})$ 

PROOF: Since  $|x_i - z_i| \le |x_i - y_i| + |y_i - z_i|$ .

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**Proposition 297.** The square metric induces the standard topology on  $\mathbb{R}^n$ .

PROOF

 $\langle 1 \rangle 1$ . For every  $a \in X$  and  $\epsilon > 0$ , we have  $B_{\rho}(a, \epsilon)$  is open in the standard product topology.

$$B_{\rho}(a,\epsilon) = (a_1 - \epsilon, a_1 + \epsilon) \times \cdots \times (a_n - \epsilon, a_n + \epsilon)$$

- $\langle 1 \rangle 2$ . For any open sets  $U_1, \ldots, U_n$  in  $\mathbb{R}$ , we have  $U_1 \times \cdots \times U_n$  is open in the square metric topology.
  - $\langle 2 \rangle 1$ . Let:  $\vec{a} \in U_1 \times \cdots \times U_n$

- $\langle 2 \rangle 2$ . For i = 1, ..., n, PICK  $\epsilon_i > 0$  such that  $(a_i \epsilon_i, a_i + \epsilon_i) \subseteq U_i$
- $\langle 2 \rangle 3$ . Let:  $\epsilon = \min(\epsilon_1, \dots, \epsilon_n)$
- $\langle 2 \rangle 4. \ B_{\rho}(\vec{a}, \epsilon) \subseteq U$

**Definition 298.** Given  $\vec{x}, \vec{y} \in \mathbb{R}^n$ , define the sum  $\vec{x} + \vec{y}$  by

$$(x_1,\ldots,x_n)+(y_1,\ldots,y_n)=(x_1+y_1,\ldots,x_n+y_n)$$
.

**Definition 299.** Given  $\lambda \in \mathbb{R}$  and  $\vec{x} \in \mathbb{R}^n$ , define the scalar product  $\lambda \vec{x} \in \mathbb{R}^n$  by

$$\lambda(x_1,\ldots,x_n)=(\lambda x_1,\ldots,\lambda x_n)$$

**Definition 300** (Inner Product). Given  $\vec{x}, \vec{y} \in \mathbb{R}^n$ , define the *inner product*  $\vec{x} \cdot \vec{y} \in \mathbb{R}$  by

$$(x_1,\ldots,x_n)\cdot(y_1,\ldots,y_n)=x_1y_1+\cdots+x_ny_n.$$

We write  $\vec{x}^2$  for  $\vec{x} \cdot \vec{x}$ .

**Definition 301** (Norm). Let  $n \geq 1$ . The *norm* on  $\mathbb{R}^n$  is the function  $\| \| : \mathbb{R}^n \to \mathbb{R}$  defined by

$$||(x_1,\ldots,x_n)|| = \sqrt{x_1^2 + \cdots + x_n^2}$$

Lemma 302.

$$\|\vec{x}\|^2 = \vec{x}^2$$

Proof: Immediate from definitions.  $\square$ 

Lemma 303.

$$\vec{x} \cdot (\vec{y} + \vec{z}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z}$$

PROOF: Each is equal to  $(x_1y_1 + x_1z_1, \dots, x_ny_n + x_nz_n)$ .

Lemma 304.

$$|\vec{x} \cdot \vec{y}| \le ||\vec{x}|| ||\vec{y}||$$

Proof:

 $\langle 1 \rangle 1$ . Assume:  $\vec{x} \neq \vec{0} \neq \vec{y}$ 

PROOF: Otherwise both sides are 0.

- $\langle 1 \rangle 2$ . Let:  $a = 1/||\vec{x}||$
- $\langle 1 \rangle 3$ . Let:  $b = 1/\|\vec{y}\|$
- (1)4.  $(a\vec{x} + b\vec{y})^2 \ge 0$  and  $(a\vec{x} b\vec{y})^2 \ge 0$
- $\langle 1 \rangle 5$ .  $\hat{a}^2 \|\vec{x}\|^2 + 2ab\vec{x} \cdot \vec{y} + \hat{b}^2 \|\vec{y}\|^2 \ge 0$  and  $\hat{a}^2 \|\vec{x}\|^2 2ab\vec{x} \cdot \vec{y} + \hat{b}^2 \|\vec{y}\|^2 \ge 0$
- $\langle 1 \rangle 6$ .  $2ab\vec{x} \cdot \vec{y} + 2 \ge 0$  and  $-2ab\vec{x} \cdot \vec{y} + 2 \ge 0$
- $\langle 1 \rangle 7$ .  $\vec{x} \cdot \vec{y} \ge -1/ab$  and  $\vec{x} \cdot \vec{y} \le 1/ab$
- $\langle 1 \rangle 8. \ \vec{x} \cdot \vec{y} \ge ||\vec{x}|| ||\vec{y}|| \text{ and } \vec{x} \cdot \vec{y} \le ||\vec{x}|| ||\vec{y}||$

Lemma 305 (Triangle Inequality).

$$\|\vec{x} + \vec{y}\| \le \|\vec{x}\| + \|\vec{y}\|$$

Proof:

$$\begin{split} \|\vec{x} + \vec{y}\|^2 &= \|\vec{x}\|^2 + 2\vec{x} \cdot \vec{y} + \|\vec{y}\|^2 \\ &\leq \|\vec{x}\|^2 + 2\|\vec{x}\| \|\vec{y}\| + \|\vec{y}\|^2 \\ &= (\|\vec{x}\| + \|\vec{y}\|)^2 \end{split}$$
 (Lemma 304)

**Definition 306** (Euclidean Metric). Let  $n \geq 1$ . The Euclidean metric on  $\mathbb{R}^n$ is defined by

$$d(\vec{x}, \vec{y}) = ||\vec{x} - \vec{y}||.$$

We prove this is a metric.

 $\langle 1 \rangle 1. \ d(\vec{x}, \vec{y}) \ge 0$ 

PROOF: Immediate from definition.

 $\langle 1 \rangle 2$ .  $d(\vec{x}, \vec{y}) = 0$  if and only if  $\vec{x} = \vec{y}$ 

PROOF:  $d(\vec{x}, \vec{y}) = 0$  if and only if  $\vec{x} - \vec{y} = \vec{0}$ .

 $\langle 1 \rangle 3. \ d(\vec{x}, \vec{y}) = d(\vec{y}, \vec{x})$ 

PROOF: Immediate from definition.

 $\langle 1 \rangle 4. \ d(\vec{x}, \vec{z}) \le d(\vec{x}, \vec{y}) + d(\vec{y}, \vec{z})$ 

Proof:

$$\|\vec{x} - \vec{z}\| = \|(\vec{x} - \vec{y}) + (\vec{y} - \vec{z})\|$$

$$\leq \|\vec{x} - \vec{y}\| + \|\vec{y} - \vec{z}\| \qquad \text{(Lemma 305)}$$

П

**Proposition 307.** The Euclidean metric induces the standard topology on  $\mathbb{R}^n$ .

PROOF:

- $\langle 1 \rangle 1$ . Let:  $\rho$  be the square metric.
- $\langle 1 \rangle 2$ . For all  $\vec{a} \in \mathbb{R}^n$  and  $\epsilon > 0$ , we have  $B_d(\vec{a}, \epsilon) \subseteq B_\rho(\vec{a}, \epsilon)$ 
  - $\langle 2 \rangle 1$ . Let:  $\vec{x} \in B_d(\vec{a}, \epsilon)$
  - $\langle 2 \rangle 2$ .  $\sqrt{(x_1 a_1)^2 + \dots + (x_n a_n)^2} < \epsilon$   $\langle 2 \rangle 3$ .  $(x_1 a_1)^2 + \dots + (x_n a_n)^2 < \epsilon^2$   $\langle 2 \rangle 4$ . For all i we have  $(x_i a_i)^2 < \epsilon^2$

  - $\langle 2 \rangle$ 5. For all i we have  $|x_i a_i| < \epsilon$
  - $\langle 2 \rangle 6. \ \rho(\vec{x}, \vec{a}) < \epsilon$
- $\langle 1 \rangle 3$ . For all  $\vec{a} \in \mathbb{R}^n$  and  $\epsilon > 0$ , we have  $B_{\rho}(\vec{a}, \epsilon / \sqrt{n}) \subseteq B_d(\vec{a}, \epsilon)$ 
  - $\langle 2 \rangle 1$ . Let:  $\vec{x} \in B_{\rho}(\vec{a}, \epsilon/\sqrt{n})$
  - $\langle 2 \rangle 2$ .  $\rho(\vec{x}, \vec{a}) < \epsilon / \sqrt{n}$
  - $\langle 2 \rangle 3$ . For all i we have  $|x_i x_a| < \epsilon / \sqrt{n}$
  - $\langle 2 \rangle 4$ . For all *i* we have  $(x_i x_a)^2 < \epsilon^2/n$
  - $\langle 2 \rangle 5. \ d(\vec{x}, \vec{a}) < \epsilon$
- $\langle 1 \rangle 4$ . Q.E.D.

PROOF: By Lemma 285.

**Proposition 308.** Let  $n \geq 0$ . For all  $c \in \mathbb{R}^n$  and  $\epsilon > 0$ , the open ball  $B(c, \epsilon)$  is path connected.

Proof:

 $\langle 1 \rangle 1$ . Let:  $a, b \in B(c, \epsilon)$ 

 $\langle 1 \rangle 2$ . Let:  $p:[0,1] \to B(c,\epsilon)$  be the function p(t)=(1-t)a+tb

PROOF: We have  $p(t) \in B(c, \epsilon)$  for all t because

$$d(p(t), c) = \|(1 - t)a + tb - c\|$$

$$= \|(1 - t)(a - c) + t(b - c)\|$$

$$\leq (1 - t)\|a - c\| + t\|b - c\|$$

$$< (1 - t)\epsilon + t\epsilon$$

 $\langle 1 \rangle 3$ . p is a path from a to b.

**Proposition 309.** Let  $n \geq 0$ . For all  $c \in \mathbb{R}^n$  and  $\epsilon > 0$ , the closed ball  $\overline{B(c, \epsilon)}$  is path connected.

Proof:

 $\langle 1 \rangle 1$ . Let:  $a, b \in \overline{B(c, \epsilon)}$ 

 $\langle 1 \rangle 2$ . Let:  $p:[0,1] \to B(c,\epsilon)$  be the function p(t)=(1-t)a+tb

PROOF: We have  $p(t) \in \overline{B(c, \epsilon)}$  for all t because

$$d(p(t), c) = \|(1 - t)a + tb - c\|$$

$$= \|(1 - t)(a - c) + t(b - c)\|$$

$$\leq (1 - t)\|a - c\| + t\|b - c\|$$

$$\leq (1 - t)\epsilon + t\epsilon$$

$$= \epsilon$$

 $\langle 1 \rangle 3$ . p is a path from a to b.

**Lemma 310.** If  $\sum_{i=0}^{\infty} x_i^2$  and  $\sum_{i=0}^{\infty} y_i^2$  converge then  $\sum_{i=0}^{\infty} |x_i y_i|$  converges.

Proof:

 $\langle 1 \rangle 1$ . For all  $N \geq 0$  we have  $\sum_{i=0}^{N} |x_i y_i| \leq \sqrt{\sum_{i=0}^{N} |x_i|^2} \sqrt{\sum_{i=0}^{N} |y_i|^2}$ PROOF: By the Cauchy-Schwarz inequality

 $\langle 1 \rangle 2$ . Q.E.D.

PROOF: Since  $\sum_{i=0}^{N} |x_i y_i|$  is an increasing sequence bounded above by  $(\sum_{i=0}^{\infty} x_i^2)(\sum_{i=0}^{\infty} y_i^2)$ .

Corollary 310.1. If  $\sum_{i=0}^{\infty} x_i^2$  and  $\sum_{i=0}^{\infty} y_i^2$  converge then  $\sum_{i=0}^{\infty} (x_i + y_i)^2$  converges.

PROOF: Since  $\sum_{i=0}^{\infty} x_i^2$ ,  $\sum_{i=0}^{\infty} y_i^2$  and  $2\sum_{i=0}^{\infty} x_i y_i$  all converge.

**Definition 311** ( $l^2$ -metric). The  $l^2$ -metric on

$$\left\{ (x_n) \in \mathbb{R}^{\omega} \mid \sum_{n=0}^{\infty} x_n^2 \text{ converges} \right\}$$

is defined by

$$d(x,y) = \left(\sum_{n=0}^{\infty} (x_n - y_n)^2\right)^{1/2}$$

We prove this is a metric.

Proof:

 $\langle 1 \rangle 1$ . d is well-defined.

PROOF: By Corollary 310.1.

- $\langle 1 \rangle 2. \ d(x,y) \ge 0$
- $\langle 1 \rangle 3$ . d(x,y) = 0 if and only if x = y
- $\langle 1 \rangle 4$ . d(x,y) = d(y,x)
- $\langle 1 \rangle 5. \ d(x,z) \le d(x,y) + d(y,z)$

PROOF: By Lemma 305.

**Theorem 312.** Addition is a continuous function  $\mathbb{R}^2 \to \mathbb{R}$ .

Proof:

- $\langle 1 \rangle 1$ . Let:  $a, b \in \mathbb{R}$
- $\langle 1 \rangle 2$ . Let:  $\epsilon > 0$
- $\langle 1 \rangle 3$ . Let:  $\delta = \epsilon/2$
- $\langle 1 \rangle 4$ . Let:  $x, y \in \mathbb{R}$
- $\langle 1 \rangle 5$ . Assume:  $\rho((a,b),(x,y)) < \delta$
- $\langle 1 \rangle 6$ .  $|(a+b)-(x+y)| < \epsilon$

PROOF:

$$\begin{aligned} |(a+b)-(x+y)| &= |a-x|+|b-y| \\ &\leq 2\rho((a,b),(x,y)) \\ &< 2\delta \\ &= \epsilon \end{aligned}$$

 $\langle 1 \rangle 7$ . Q.E.D.

PROOF: Theorem 291

**Theorem 313.** Multiplication is a continuous function  $\mathbb{R}^2 \to \mathbb{R}$ .

- $\langle 1 \rangle 1$ . Let:  $a, b \in \mathbb{R}$
- $\langle 1 \rangle 2$ . Let:  $\epsilon > 0$
- $\langle 1 \rangle 3$ . Let:  $\delta = \min(\epsilon/(|a|+|b|+1),1)$
- $\langle 1 \rangle 4$ . Let:  $x, y \in \mathbb{R}$
- $\langle 1 \rangle 5$ . Assume:  $\rho((a,b),(x,y)) < \delta$

 $\langle 1 \rangle 6$ .  $|ab - xy| < \epsilon$ 

Proof:

$$\begin{split} |ab-xy| &= |a(b-y) + (a-x)b - (a-x)(b-y)| \\ &\leq |a||b-y| + |b||a-x| + |a-x||b-y| \\ &< |a|\delta + |b|\delta + \delta^2 \\ &\leq |a|\delta + |b|\delta + \delta \qquad (\langle 1 \rangle 3) \\ &\leq \epsilon \qquad (\langle 1 \rangle 3) \end{split}$$

 $\langle 1 \rangle 7$ . Q.E.D.

PROOF: Theorem 291

**Theorem 314.** The function  $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$  defined by  $f(x) = x^{-1}$  is continuous.

Proof:

 $\langle 1 \rangle 1$ . For all  $a \in \mathbb{R}$  we have  $f^{-1}((a, +\infty))$  is open.

PROOF: The set is

$$(a^{-1}, +\infty) \text{if } a > 0$$
$$(0, +\infty) \text{if } a = 0$$
$$(-\infty, a^{-1}) \cup (0, +\infty) \text{if } a < 0$$

 $(-\infty,a^{-1})\cup(0,+\infty) \text{if }a<0$   $\langle 1\rangle 2.$  For all  $a\in\mathbb{R}$  we have  $f^{-1}((-\infty,a))$  is open.

Proof: Similar.

 $\langle 1 \rangle 3$ . Q.E.D.

PROOF: By Proposition 89 and Lemma 113.

**Definition 315.** For  $n \geq 0$ , the unit ball  $B^n$  is the space  $\{x \in \mathbb{R}^n \mid ||x|| \leq 1\}$ .

**Proposition 316.** For all  $n \geq 0$ , the unit ball  $B^n$  is path connected.

Proof:

 $\langle 1 \rangle 1$ . Let:  $a, b \in B^n$ 

 $\langle 1 \rangle 2$ . Let:  $p:[0,1] \to B^n$  be the function p(t)=(1-t)a+tb

PROOF: We have  $p(t) \in B^n$  for all t because

$$||(1-t)a + tb|| \le (1-t)||a|| + t||b||$$

$$\le (1-t) + t$$

$$= 1$$

 $\langle 1 \rangle 3$ . p is a path from a to b.

**Definition 317** (Punctured Euclidean Space). For  $n \geq 0$ , defined punctured Euclidean space to be  $\mathbb{R}^n \setminus \{0\}$ .

**Proposition 318.** For n > 1, punctured Euclidean space  $\mathbb{R}^n \setminus \{0\}$  is path connected.

- $\langle 1 \rangle 1$ . Let:  $a, b \in \mathbb{R}^n \setminus \{0\}$
- $\langle 1 \rangle 2$ . Case: 0 is on the line from a to b
  - $\langle 2 \rangle 1$ . PICK a point c not on the line from a to b
  - $\langle 2 \rangle 2$ . The path consisting of a straight line from a to c followed by a straight line from c to b is a path from a to b.
- $\langle 1 \rangle 3$ . Case: 0 is not on the line from a to b

PROOF: The straight line from a to b is a path from a to b.

**Corollary 318.1.** For n > 1, the spaces  $\mathbb{R}$  and  $\mathbb{R}^n$  are not homeomorphic.

PROOF: For any point a, the space  $\mathbb{R} \setminus \{a\}$  is disconnected.

**Definition 319** (Unit Sphere). For  $n \ge 1$ , the unit sphere  $S^{n-1}$  is the space

$$S^{n-1} = \{ x \in \mathbb{R}^n \mid ||x|| = 1 \} .$$

**Proposition 320.** For n > 1, the unit sphere  $S^{n-1}$  is path connected.

PROOF: The map  $g: \mathbb{R}^n \setminus \{0\} \to S^{n-1}$  defined by  $g(x) = x/\|x\|$  is continuous and surjective. The result follows by Proposition 238.  $\square$ 

**Proposition 321.** Let  $f: S^1 \to \mathbb{R}$  be continuous. Then there exists  $x \in S^1$  such that f(x) = f(-x).

PROOF:

- $\langle 1 \rangle 1$ . Let:  $g: S^1 \to \mathbb{R}$  be the function g(x) = f(x) f(-x)Prove: There exists  $x \in S^1$  such that g(x) = 0
- $\langle 1 \rangle 2$ . Assume: without loss of generality g((1,0)) > 0
- $\langle 1 \rangle 3. \ g((-1,0)) < 0$
- $\langle 1 \rangle 4$ . There exists x such that g(x) = 0

PROOF: By the Intermediate Value Theorem.

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**Definition 322** (Topologist's Sine Curve). Let  $S = \{(x, \sin 1/x) \mid 0 < x \le 1\}$ . The *topologist's sine curve* is the closure  $\overline{S}$  of S.

Proposition 323.

$$\overline{S} = S \cup (\{0\} \times [-1, 1])$$

**Proposition 324.** The topologist's sine curve is connected.

PROOF

- $\langle 1 \rangle 1$ . Let:  $S = \{(x, \sin 1/x) \mid 0 < x \le 1\}$
- $\langle 1 \rangle 2$ . S is connected.

PROOF: Theorem 218.

 $\langle 1 \rangle 3$ .  $\overline{S}$  is connected.

PROOF: Theorem 217.

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Proposition 325 (CC). The topologist's sine curve is not path connected.

**PROOF** 

- $\langle 1 \rangle 1$ . Assume: for a contradiction  $p:[0,1] \to \overline{S}$  is a path from (0,0) to  $(1,\sin 1)$ .
- $\langle 1 \rangle 2. \ p^{-1}(\{0\} \times [0,1])$  is closed.
- $\langle 1 \rangle 3$ . Let: b be the greatest element of  $p^{-1}(\{0\} \times [0,1])$ .
- $\langle 1 \rangle 4. \ b < 1$

PROOF: Since  $p(1) = (1, \sin 1)$ .

- $\langle 1 \rangle$ 5. Pick a sequence  $(t_n)_{n \geq 1}$  in (b,1] such that  $t_n \to b$  and  $\pi_2(p(t_n)) = (-1)^n$ 
  - $\langle 2 \rangle 1$ . Let:  $n \geq 1$
  - $\langle 2 \rangle 2$ . PICK u with  $0 < u < \pi_1(p(1/n))$  such that  $\sin(1/u) = (-1)^n$
  - $\langle 2 \rangle 3$ . Pick  $t_n$  such that  $0 < t_n < 1/n$  and  $\pi_1(p(t_n)) = u$

PROOF: One exists by the Intermediate Value Theorem.

 $\langle 1 \rangle 6$ . Q.E.D.

Proof: This contradicts 105.

# 40 The Uniform Topology

**Definition 326** (Uniform Metric). Let J be a set. The *uniform metric*  $\overline{\rho}$  on  $\mathbb{R}^J$  is defined by

$$\overline{\rho}(a,b) = \sup_{j \in J} \overline{d}(a_j, b_j)$$

where  $\overline{d}$  is the standard bounded metric on  $\mathbb{R}$ .

The uniform topology on  $\mathbb{R}^J$  is the topology induced by the uniform metric.

We prove this is a metric.

Proof:

 $\langle 1 \rangle 1$ .  $\overline{\rho}(a,b) \geq 0$ 

PROOF: Immediate from definitions.

 $\langle 1 \rangle 2$ .  $\overline{\rho}(a,b) = 0$  if and only if a = b

PROOF: Immediate from definitions.

 $\langle 1 \rangle 3. \ \overline{\rho}(a,b) = \overline{\rho}(b,a)$ 

PROOF: Immediate from definitions.

 $\langle 1 \rangle 4. \ \overline{\rho}(a,c) \le \overline{\rho}(a,b) + \overline{\rho}(b,c)$ 

Proof:

$$\begin{split} \overline{\rho}(a,c) &= \sup_{j \in J} \overline{d}(a_j,c_j) \\ &\leq \sup_{j \in J} (\overline{d}(a_j,b_j) + \overline{d}(b_j,c_j)) \\ &\leq \sup_{j \in J} \overline{d}(a_j,b_j) + \sup_{j \in J} \overline{d}(b_j,c_j) \\ &= \overline{\rho}(a,b) + \overline{\rho}(b,c) \end{split}$$

**Proposition 327.** The uniform topology on  $\mathbb{R}^J$  is finer than the product topology.

#### Proof:

- $\langle 1 \rangle 1$ . Let:  $j \in J$  and U be open in  $\mathbb{R}$ PROVE:  $\pi_j^{-1}(U)$  is open in the uniform topology.  $\langle 1 \rangle 2$ . Let:  $a \in \pi_j^{-1}(U)$
- $\langle 1 \rangle$ 3. PICK  $\epsilon > 0$  such that  $(a_j \epsilon, a_j + \epsilon) \subseteq U$  $\langle 1 \rangle$ 4.  $B_{\overline{\rho}}(a, \epsilon) \subseteq \pi_j^{-1}(U)$

**Proposition 328.** The uniform topology on  $\mathbb{R}^J$  is coarser than the box topology.

### Proof:

- $\langle 1 \rangle 1$ . Let:  $a \in \mathbb{R}^J$  and  $\epsilon > 0$ PROVE:  $B(a, \epsilon)$  is open in the box topology.
- $\langle 1 \rangle 2$ . Let:  $b \in B(a, \epsilon)$
- $\langle 1 \rangle 3$ . For  $j \in J$  we have  $|a_j b_j| < \epsilon$
- $\langle 1 \rangle 4$ . For  $j \in J$ , Let:  $\delta_j = (\epsilon - |a_j - b_j|)/2$  $\langle 1 \rangle 5. \prod_{j \in J} (b_j - \delta_j, b_j + \delta_j) \subseteq B(a, \epsilon)$

**Proposition 329.** The uniform topology on  $\mathbb{R}^J$  is strictly finer than the product topology if and only if J is infinite.

#### PROOF:

- $\langle 1 \rangle 1$ . If J is finite then the uniform and product topologies coincide.
  - PROOF: The uniform, box and product topologies are all the same.
- $\langle 1 \rangle 2$ . If J is infinite then the uniform and product topologies are different. PROOF: The set  $B(\vec{0},1)$  is open in the uniform topology but not the product topology.

**Proposition 330** (DC). The uniform topology on  $\mathbb{R}^J$  is strictly coarser than the box topology if and only if J is infinite.

#### Proof:

- $\langle 1 \rangle 1$ . If J is finite then the uniform and box topologies coincide.
  - PROOF: The uniform, box and product topologies are all the same.
- $\langle 1 \rangle 2$ . If J is infinite then the uniform and box topologies are different.

PROOF: Pick an  $\omega$ -sequence  $(j_1, j_2, ...)$  in J. Let  $U = \prod_{i \in J} U_i$  where  $U_{j_i} =$ (-1/i,1/i) and  $U_j=(-1,1)$  for all other j. Then  $\vec{0}\in U$  but there is no  $\epsilon>0$ such that  $B(\vec{0}, \epsilon) \subseteq U$ .

**Proposition 331.** The closure of  $\mathbb{R}^{\infty}$  in  $\mathbb{R}^{\omega}$  under the uniform topology is  $\mathbb{R}^{\omega}$ .

PROOF: Given any open ball  $B(a,\epsilon)$ , pick an integer N such that  $1/\epsilon < N$ . Then  $B(a, \epsilon)$  includes sequences whose nth entry is 0 for all  $n \geq N$ .  $\square$ 

# 41 Uniform Convergence

**Definition 332** (Uniform Convergence). Let X be a set and Y a metric space. Let  $(f_n: X \to Y)$  be a sequence of functions and  $f: X \to Y$  be a function. Then  $f_n$  converges uniformly to f as  $n \to \infty$  if and only if, for all  $\epsilon > 0$ , there exists N such that, for all  $n \ge N$  and  $x \in X$ , we have  $d(f_n(x), f(x)) < \epsilon$ .

**Example 333.** Define  $f_n : [0,1] \to \mathbb{R}$  by  $f_n(x) = x^n$  for  $n \ge 1$ , and  $f : [0,1] \to \mathbb{R}$  by f(x) = 0 if x < 1, f(1) = 1. Then  $f_n$  converges to f pointwise but not uniformly.

**Theorem 334** (Uniform Limit Theorem). Let X be a topological space and Y a metric space. Let  $(f_n : X \to Y)$  be a sequence of continuous functions and  $f : X \to Y$  be a function. If  $f_n$  converges uniformly to f as  $n \to \infty$ , then f is continuous.

### PROOF:

- $\langle 1 \rangle 1$ . Let:  $x \in X$  and  $\epsilon > 0$
- (1)2. PICK N such that, for all  $n \geq N$  and  $y \in X$ , we have  $d(f_n(y), f(y)) < \epsilon/3$
- $\langle 1 \rangle 3$ . PICK a neighbourhood U of x such that  $f_N(U) \subseteq B(f_N(x), \epsilon/3)$ PROVE:  $f(U) \subseteq B(f(x), \epsilon)$
- $\langle 1 \rangle 4$ . Let:  $y \in U$
- $\langle 1 \rangle 5. \ d(f(y), f(x)) < \epsilon$

Proof:

$$d(f(y), f(x)) \leq d(f(y), f_N(y)) + d(f_N(y), f_N(x)) + d(f_N(x), f(x)) \quad \text{(Triangle Inequality)}$$

$$< \epsilon/3 + \epsilon/3 + \epsilon/3 \qquad (\langle 1 \rangle 2, \langle 1 \rangle 3)$$

$$= \epsilon$$

**Proposition 335.** Let X be a topological space and Y a metric space. Let  $(f_n : X \to Y)$  be a sequence of continuous functions and  $f : X \to Y$  be a function. Let  $(a_n)$  be a sequence of points in X and  $a \in X$ . If  $f_n$  converges uniformly to f and  $a_n$  converges to a in X then  $f_n(a_n)$  converges to f(a) uniformly in Y.

#### Proof:

- $\langle 1 \rangle 1$ . Let:  $\epsilon > 0$
- $\langle 1 \rangle 2$ . PICK  $N_1$  such that, for all  $n \geq N_1$  and  $x \in X$ , we have  $d(f_n(x), f(x)) < \epsilon/2$
- $\langle 1 \rangle 3$ . PICK  $N_2$  such that, for all  $n \geq N_2$ , we have  $a_n \in f^{-1}(B(a, \epsilon/2))$

PROOF: Using the fact that f is continuous from the Uniform Limit Theorem.

- $\langle 1 \rangle 4$ . Let:  $N = \max(N_1, N_2)$
- $\langle 1 \rangle 5$ . Let:  $n \geq N$
- $\langle 1 \rangle 6. \ d(f_n(a_n), f(a)) < \epsilon$

$$d(f_n(a_n), f(a)) \le d(f_n(a_n), f(a_n)) + d(f(a_n), f(a)) \quad \text{(Triangle Inequality)}$$
$$< \epsilon/2 + \epsilon/2$$
$$= \epsilon \quad (\langle 1 \rangle 2, \langle 1 \rangle 3)$$

**Proposition 336.** Let X be a set. Let  $(f_n : X \to \mathbb{R})$  be a sequence of functions and  $f : X \to \mathbb{R}$  be a function. Then  $f_n$  converges unifomly to f as  $n \to \infty$  if and only if  $f_n \to f$  as  $n \to \infty$  in  $\mathbb{R}^X$  under the uniform topology.

#### Proof:

- $\langle 1 \rangle 1$ . If  $f_n$  converges uniformly to f then  $f_n$  converges to f under the uniform topology.
  - $\langle 2 \rangle 1$ . Assume:  $f_n$  converges uniformly to f
  - $\langle 2 \rangle 2$ . Let:  $\epsilon > 0$
  - $\langle 2 \rangle$ 3. PICK N such that, for all  $n \geq N$  and  $x \in X$ , we have  $d(f_n(x), f(x)) < \epsilon/2$
  - $\langle 2 \rangle 4$ . For all  $n \geq N$  we have  $\overline{\rho}(f_n, f) \leq \epsilon/2$
  - $\langle 2 \rangle 5$ . For all  $n \geq N$  we have  $\overline{\rho}(f_n, f) < \epsilon$
- $\langle 1 \rangle 2$ . If  $f_n$  converges to f under the uniform topology then  $f_n$  converges uniformly to f.
  - $\langle 2 \rangle 1$ . Assume:  $f_n$  converges to f under the uniform topology.
  - $\langle 2 \rangle 2$ . Let:  $\epsilon > 0$
  - $\langle 2 \rangle 3$ . PICK N such that, for all  $n \geq N$ , we have  $\overline{\rho}(f_n, f) < \min(\epsilon, 1/2)$
  - $\langle 2 \rangle 4$ . Let:  $n \geq N$
  - $\langle 2 \rangle 5$ . Let:  $x \in X$
  - $\langle 2 \rangle 6. \ \overline{\rho}(f_n, f) < \min(\epsilon, 1/2)$

PROOF: From  $\langle 2 \rangle 3$ .

- $\langle 2 \rangle 7. \ d(f_n(x), f(x)) < \min(\epsilon, 1/2)$
- $\langle 2 \rangle 8. \ d(f_n(x), f(x)) < \epsilon$

# 42 Isometric Imbeddings

**Definition 337.** Let X and Y be metric spaces. An isometric imbedding  $f: X \to Y$  is a function such that, for all  $x, y \in X$ , we have d(f(x), f(y)) = d(x, y).

Proposition 338. Every isometric imbedding is an imbedding.

## Proof:

- $\langle 1 \rangle 1$ . Let:  $f: X \to Y$  be an isometric imbedding.
- $\langle 1 \rangle 2$ . f is injective.

PROOF: If f(x) = f(y) then d(f(x), f(y)) = 0 hence d(x, y) = 0 hence x = y.  $\langle 1 \rangle 3$ . f is continuous.

PROOF: For all  $\epsilon > 0$ , if  $d(x,y) < \epsilon$  then  $d(f(x),f(y)) < \epsilon$ .

 $\langle 1 \rangle 4. \ f: X \to f(X)$  is an open map.

PROOF:  $f(B(a, \epsilon)) = B(f(a), \epsilon) \cap f(X)$ .