Topology

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Contents

Part I

Set Theory

Chapter 1

Classes

1.1 Classes

We speak informally of *classes*. A class is specified by a unary predicate. We write $\{x \mid P(x)\}$ for the class determined by the predicate $P(x)$.

Definition 1.1.1 (Membership)**.** Let a be an object and $\mathbf{A}$ a class. We define the proposition $a \in \mathbf{A}$ (a is a *member* or *element* of A) as follows:

The proposition $a \in \{x \mid P(x)\}$ is the proposition $P(a)$.

Definition 1.1.2 (Equality of Classes)**.** Let $\mathbf{A}$ and $\mathbf{B}$ be classes. We say $\mathbf{A}$ and $\mathbf{B}$ are *equal*, $\mathbf{A} = \mathbf{B}$, if and only if they have exactly the same elements.

1.2 Subclasses

Definition 1.2.1 (Subclass)**.** Let $\mathbf{A}$ and $\mathbf{B}$ be classes. We say $\mathbf{A}$ is a *subclass* of $\mathbf{B}$, $\mathbf{A} \subseteq \mathbf{B}$, if and only if every member of $\mathbf{A}$ is a member of $\mathbf{B}$.

We say $\mathbf{A}$ is a *proper* subclass of $\mathbf{B}$, $\mathbf{A} \subset \mathbf{B}$, if and only if $\mathbf{A} \subseteq \mathbf{B}$ and $\mathbf{A} \neq \mathbf{B}$.

1.3 The Empty Class

Definition 1.3.1 (Empty Class)**.** The *empty* class $\emptyset$ is $\{x \mid \bot\}$.

1.4 Finite Classes

Definition 1.4.1. For any objects $a_1, \ldots, a_n$, we write $\{a_1, \ldots, a_n\}$ for the class $\{x \mid x = a_1 \vee \cdots \vee x = a_n\}$.

1.5 Universal Class

Definition 1.5.1 (Universal Class)**.** The *universal class* $\mathbf{V}$ is the class $\{x \mid \top\}$.

1.6 Union

Definition 1.6.1 (Union)**.** For any classes $\mathbf{A}$ and $\mathbf{B}$, the *union* $\mathbf{A} \cup \mathbf{B}$ is the class $\{x \mid x \in \mathbf{A} \vee x \in \mathbf{B}\}$.

1.7 Intersection

Definition 1.7.1 (Intersection)**.** For any classes $\mathbf{A}$ and $\mathbf{B}$, the *intersection* $\mathbf{A} \cap \mathbf{B}$ is the class $\{x \mid x \in \mathbf{A} \wedge x \in \mathbf{B}\}$.

1.8 Relative Complement

Definition 1.8.1 (Relative Complement)**.** For any classes $\mathbf{A}$ and $\mathbf{B}$, the *relative complement* $\mathbf{A} - \mathbf{B}$ is $\{x \in \mathbf{A} \mid x \notin \mathbf{B}\}$.

Chapter 2

Sets

2.1 Membership

We take as undefined the notion of *set*.

We take as undefined the binary relation of *membership*, $\in$. If $a \in A$ we say a is a *member* or *element* of A. If this does not hold, we write $a \notin A$.

Axiom 2.1.1 (Axiom of Extensionality)**.** *Two sets with exactly the same elements are equal.*

We may therefore identify the set A with the class $\{x \mid x \in A\}$.

We say a class $\mathbf{A}$ *is a set* iff there exists a set A such that $A = \mathbf{A}$. That is, $\{x \mid P(x)\}$ is a set if and only if there exists a set A such that, for all x, we have $x \in A$ if and only if $P(x)$.

2.2 The Empty Set

Axiom 2.2.1 (Empty Set Axiom)**.** *The empty class $\emptyset$ is a set.*

2.3 Pair Sets

Axiom 2.3.1 (Pairing Axiom)**.** *For any objects u and v, the class $\{u, v\}$ is a set.*

Theorem 2.3.2 (Pairing)**.** *For any object a, the class $\{a\}$ is a set.*

Proof: It is $\{a, a\}$. □

2.4 Unions

Definition 2.4.1 (Union)**.** For any class of sets $\mathbf{A}$, the *union* $\bigcup \mathbf{A}$ is the class $\{x \mid \exists A \in \mathbf{A}. x \in A\}$.

Axiom 2.4.2 (Union Axiom)**.** *For any set A, the union $\bigcup A$ is a set.*

Theorem 2.4.3 (Union, Pairing)**.** *For any sets A and B, the class $A \cup B$ is a set.*

PROOF: It is $\bigcup\{A, B\}$. □

Theorem Schema 2.4.4 (Union, Pairing)**.** *For any objects a_1, …, a_n, the class $\{a_1, \ldots, a_n\}$ is a set.*

PROOF: We prove each theorem using the last since $\{a_1, \ldots, a_n, a_{n+1}\} = \{a_1, \ldots, a_n\} \cup \{a_{n+1}\}$. □

2.5 Power Set

Definition 2.5.1 (Power Class)**.** For any class $\mathbf{A}$, the *power* class $\mathcal{P}\mathbf{A}$ is the class of all subsets of $\mathbf{A}$.

Axiom 2.5.2 (Power Set Axiom)**.** *For any set A, the power class $\mathcal{P}A$ is a set.*

2.6 Covers

Definition 2.6.1 (Cover)**.** Let $\mathbf{X}$ be a class and $\mathcal{A} \subseteq \mathcal{P}\mathbf{X}$. Then $\mathcal{A}$ *covers* $\mathbf{X}$, or is a *covering* of $\mathbf{X}$, if and only if $\bigcup \mathcal{A} = \mathbf{X}$.

2.7 Subset Axioms

Axiom Schema 2.7.1 (Subset Axioms, Aussonderung Axioms)**.** *For any classes $\mathbf{A}$ and $\mathbf{B}$, if $\mathbf{A} \subseteq \mathbf{B}$ and $\mathbf{B}$ is a set then $\mathbf{A}$ is a set.*

Theorem 2.7.2 (Subset)**.** *The universal class $\mathbf{V}$ is not a set.*

PROOF:
$\langle 1 \rangle 1$. ASSUME: $\mathbf{V}$ is a set.
$\langle 1 \rangle 2$. LET: $R = \{x \in \mathbf{V} \mid x \notin x\}$
$\langle 1 \rangle 3$. $R \in R$ if and only if $R \notin R$
$\langle 1 \rangle 4$. Q.E.D.
PROOF: This is a contradiction.
□

Theorem 2.7.3 (Subset)**.** *If A is a set and $\mathbf{B}$ is a class then $A - \mathbf{B}$ is a set.*

PROOF: It is a subset of A. □

2.8 Intersection

Definition 2.8.1 (Intersection)**.** For any class $\mathbf{A}$ of sets, the *intersection* $\bigcap \mathbf{A}$ is the class $\{x \mid \forall A \in \mathbf{A}.x \in A\}$.

Theorem 2.8.2 (Subset)**.** *For any nonempty class* $\mathbf{A}$ *of sets, we have* $\bigcap \mathbf{A}$ *is a set.*

PROOF:
$\langle 1 \rangle 1.$ PICK $A \in \mathbf{A}$
$\langle 1 \rangle 2.$ $\bigcap \mathbf{A} \subseteq A$
$\langle 1 \rangle 3.$ Q.E.D.
  PROOF: By a Subset Axiom.
□

Theorem 2.8.3 (Subset)**.** *For any sets* A *and* B*, the class* $A \cap B$ *is a set.*

PROOF: From a Subset Axiom since $A \cap B \subseteq A$. □

Chapter 3

Relations

3.1 Ordered Pairs

Definition 3.1.1 (Ordered Pair (Pairing))**.** For any sets x and y, the *ordered pair* (x, y) is defined to be $\{\{x\}, \{x, y\}\}$.

Theorem 3.1.2 (Pairing)**.** *For any sets u, v, x, y, we have $(u, v) = (x, y)$ if and only if $u = x$ and $v = y$*

PROOF:
⟨1⟩1. ASSUME: $\{\{u\}, \{u, v\}\} = \{\{x\}, \{x, y\}\}$
⟨1⟩2. $\{u\} \in \{\{x\}, \{x, y\}\}$
⟨1⟩3. $\{u, v\} \in \{\{x\}, \{x, y\}\}$
⟨1⟩4. $\{u\} = \{x\}$ or $\{u\} = \{x, y\}$
⟨1⟩5. $\{u, v\} = \{x\}$ or $\{u, v\} = \{x, y\}$
⟨1⟩6. CASE: $\{u\} = \{x, y\}$
  ⟨2⟩1. $u = x = y$
  ⟨2⟩2. $u = v = x = y$
    PROOF: From ⟨1⟩5
⟨1⟩7. CASE: $\{u, v\} = \{x\}$
  PROOF: Similar.
⟨1⟩8. CASE: $\{u\} = \{x\}$ and $\{u, v\} = \{x, y\}$
  ⟨2⟩1. $u = x$
  ⟨2⟩2. $u = y$ or $v = y$
  ⟨2⟩3. CASE: $u = y$
    PROOF: This case is the case considered in ⟨1⟩6.
  ⟨2⟩4. CASE: $v = y$
    PROOF: We have $u = x$ and $v = y$ as required.
□

Lemma 3.1.3 (Pairing, Power Set)**.** *Let x, y and C be sets. If $x \in C$ and $y \in C$ then $(x, y) \in \mathcal{PP}C$.*

PROOF:

$\langle 1 \rangle 1$. LET: x, y and C be sets.
$\langle 1 \rangle 2$. ASSUME: $x \in C$
$\langle 1 \rangle 3$. ASSUME: $y \in C$
$\langle 1 \rangle 4$. $\{x\} \subseteq C$
$\langle 1 \rangle 5$. $\{x, y\} \subseteq C$
$\langle 1 \rangle 6$. $\{x\} \in \mathcal{P}C$
$\langle 1 \rangle 7$. $\{x, y\} \in \mathcal{P}C$
$\langle 1 \rangle 8$. $\{\{x\}, \{x, y\}\} \subseteq \mathcal{P}C$
$\langle 1 \rangle 9$. $\{\{x\}, \{x, y\}\} \in \mathcal{P}\mathcal{P}C$
$\square$

Lemma 3.1.4 (Pairing, Union)**.** *Let x, y and A be sets. If $(x, y) \in A$ then x and y belong to $\bigcup\bigcup A$.*

PROOF:
$\langle 1 \rangle 1$. LET: x, y and A be sets.
$\langle 1 \rangle 2$. ASSUME: $(x, y) \in A$
$\langle 1 \rangle 3$. $\{x, y\} \in \bigcup A$
$\langle 1 \rangle 4$. $x \in \bigcup\bigcup A$
$\langle 1 \rangle 5$. $y \in \bigcup\bigcup A$
$\square$

3.2 Cartesian Product

Definition 3.2.1 (Cartesian Product (Pairing))**.** Let $\mathbf{A}$ and $\mathbf{B}$ be classes. The *Cartesian product* $\mathbf{A} \times \mathbf{B}$ is the class $\{(x, y) \mid x \in \mathbf{A}, y \in \mathbf{B}\}$.

Theorem 3.2.2 (Pairing, Union, Power Set, Subset)**.** *For any sets A and B, the Cartesian product $A \times B$ is a set.*

PROOF: It is a subset of $\mathcal{P}\mathcal{P}(A \cup B)$ by Lemma 3.1.3. $\square$

3.3 Relations

Definition 3.3.1 (Relation (Pairing))**.** A *relation* is a class of ordered pairs.

Given a relation $\mathbf{R}$, we write $x\mathbf{R}y$ for $(x, y) \in \mathbf{R}$.

A relation is *small* iff it is a set.

3.4 Domain

Definition 3.4.1 (Domain (Pairing))**.** Let $\mathbf{R}$ be a class. The *domain* of $\mathbf{R}$ is $\operatorname{dom} \mathbf{R} = \{x \mid \exists y . x\mathbf{R}y\}$.

Theorem 3.4.2 (Pairing, Union, Subset)**.** *For any set R, the domain* $\operatorname{dom} R$ *is a set.*

PROOF: It is a subset of $\bigcup\bigcup R$ by Lemma 3.1.4. $\square$

3.5 Range

Definition 3.5.1 (Domain (Pairing))**.** Let $\mathbf{R}$ be a class. The *range* of $\mathbf{R}$ is $\operatorname{ran}\mathbf{R} = \{y \mid \exists x.x\mathbf{R}y\}$.

Theorem 3.5.2 (Pairing, Union, Subset)**.** *For any set R, the range $\operatorname{ran} R$ is a set.*

Proof: It is a subset of $\bigcup\bigcup R$ by Lemma 3.1.4. □

3.6 Field

Definition 3.6.1 (Field)**.** Let $\mathbf{R}$ be a class. The *field* of $\mathbf{R}$ is $\operatorname{fld}\mathbf{R} = \operatorname{dom}\mathbf{R} \cup \operatorname{ran}\mathbf{R}$.

Theorem 3.6.2 (Pairing, Union, Subset)**.** *For any set R, the field $\operatorname{fld} R$ is a set.*

Proof: Theorems 2.4.3, 3.4.2 and 3.5.2. □

3.7 Functions

Definition 3.7.1 (Class Term (Pairing))**.** A *class term* is a relation $\mathbf{F}$ such that, for all x, y, y', if $x\mathbf{F}y$ and $x\mathbf{F}y'$ then $y = y'$.

If $\mathbf{F}$ is a class term and $x \in \operatorname{dom}\mathbf{F}$, then we write $\mathbf{F}(x)$ for the unique y such that $x\mathbf{F}y$.

We write $\mathbf{F} : \mathbf{A} \to \mathbf{B}$ iff $\mathbf{F}$ is a class term, $\operatorname{dom}\mathbf{F} = \mathbf{A}$ and $\operatorname{ran}\mathbf{F} \subseteq \mathbf{B}$.

A *function* is a class term that is a set.

Axiom 3.7.2 (Axiom of Choice, First Form (Pairing))**.** *For any relation R, there exists a function $H \subseteq R$ such that $\operatorname{dom} H = \operatorname{dom} R$.*

Theorem 3.7.3. *The following are equivalent.*

1. *The Axiom of Choice*
2. *(**Multiplicative Axiom**) For any function H with domain I such that $H(i)$ is nonempty for all $i \in I$, there exists a function f with domain I such that, for all $i \in I$, we have $f(i) \in H(i)$.*
3. *Every set has a choice function.*
4. *Let $\mathcal{A}$ be a set of pairwise disjoint nonempty sets. Then there exists a set C containing exactly one element from each member of $\mathcal{A}$.*

3.8 Single-Rooted

Definition 3.8.1 (Single-Rooted (Pairing)). A class $\mathbf{R}$ is *single-rooted* if and only if, for all x, x', y, if $x\mathbf{R}y$ and $x'\mathbf{R}y$ then $x = x'$.

We call a class term *one-to-one*, *injective* or an *injection* if and only if it is single-rooted.

3.9 Surjective

Definition 3.9.1 (Surjective (Pairing)). Let $\mathbf{F} : \mathbf{A} \to \mathbf{B}$. Then $\mathbf{F}$ is *surjective* if and only if $\operatorname{ran} \mathbf{F} = \mathbf{B}$.

3.10 Inverse

Definition 3.10.1 (Inverse (Pairing)). Let $\mathbf{R}$ be a class. The *inverse* of $\mathbf{R}$ is $\mathbf{R}^{-1} = \{(y, x) \mid x\mathbf{R}y\}$.

Theorem 3.10.2 (Pairing, Union, Power Set, Subset). *For any set R, the inverse R^{-1} is a set.*

Proof: It is a subset of $\operatorname{ran} R \times \operatorname{dom} R$. ∎

Theorem 3.10.3 (Pairing). *For any class $\mathbf{F}$, we have $\operatorname{dom} \mathbf{F}^{-1} = \operatorname{ran} \mathbf{F}$.*

Proof: For any x, we have

$$\begin{aligned} x \in \operatorname{dom} \mathbf{F}^{-1} &\Leftrightarrow \exists y.(x, y) \in \mathbf{F}^{-1} \\ &\Leftrightarrow \exists y.(y, x) \in \mathbf{F} \\ &\Leftrightarrow x \in \operatorname{ran} \mathbf{F} \qquad \square \end{aligned}$$

Theorem 3.10.4 (Pairing). *For any set $\mathbf{F}$, we have $\operatorname{ran} \mathbf{F}^{-1} = \operatorname{dom} \mathbf{F}$.*

Proof: For any x, we have

$$\begin{aligned} x \in \operatorname{ran} \mathbf{F}^{-1} &\Leftrightarrow \exists y.(y, x) \in \mathbf{F}^{-1} \\ &\Leftrightarrow \exists y.(x, y) \in \mathbf{F} \\ &\Leftrightarrow x \in \operatorname{dom} \mathbf{F} \qquad \square \end{aligned}$$

Theorem 3.10.5 (Pairing). *For any relation $\mathbf{F}$, we have $(\mathbf{F}^{-1})^{-1} = \mathbf{F}$.*

Proof: For any z we have

$$\begin{aligned} z \in (\mathbf{F}^{-1})^{-1} &\Leftrightarrow \exists x, y.z = (x, y) \wedge (y, x) \in \mathbf{F}^{-1} \\ &\Leftrightarrow \exists x, y.z = (x, y) \wedge (x, y) \in \mathbf{F} \\ &\Leftrightarrow z \in \mathbf{F} \qquad (\mathbf{F} \text{ is a relation}) \square \end{aligned}$$

Theorem 3.10.6 (Pairing). *For any class $\mathbf{F}$, we have $\mathbf{F}^{-1}$ is a class term if and only if $\mathbf{F}$ is single-rooted.*

Proof: Immediate from definitions. $\square$

Theorem 3.10.7 (Pairing)**.** *Let* $\mathbf{F}$ *be a relation. Then* $\mathbf{F}$ *is a class term if and only if* $\mathbf{F}^{-1}$ *is single-rooted.*

Proof: Immediate from definitions. $\square$

Theorem 3.10.8 (Pairing)**.** *Let* $\mathbf{F}$ *be a one-to-one class term and* $x \in \operatorname{dom} \mathbf{F}$*. Then* $\mathbf{F}^{-1}(\mathbf{F}(x)) = x$*.*

Proof: We have $(x, \mathbf{F}(x)) \in \mathbf{F}$ and so $(\mathbf{F}(x), x) \in \mathbf{F}^{-1}$. $\square$

Theorem 3.10.9 (Pairing)**.** *Let* $\mathbf{F}$ *be a one-to-one function and* $y \in \operatorname{ran} \mathbf{F}$*. Then* $\mathbf{F}(\mathbf{F}^{-1}(y)) = y$*.*

Proof: From Theorems 3.10.3, 3.10.5 and 3.10.8. $\square$

3.11 Composition

Definition 3.11.1 (Composition (Pairming))**.** Let $\mathbf{R}$ and $\mathbf{S}$ be relations. The *composition* of $\mathbf{R}$ and $\mathbf{S}$ is $\mathbf{S} \circ \mathbf{R} = \{(x, z) \mid \exists y. x\mathbf{R}y \wedge y\mathbf{S}z\}$.

Theorem 3.11.2 (Pairing, Union, Power Set, Subset)**.** *If* R *and* S *are small relations then* $S \circ R$ *is small.*

Proof: It is a subset of $\operatorname{dom} R \times \operatorname{ran} S$. $\square$

Theorem 3.11.3 (Pairing)**.** *Let* $\mathbf{F}$ *and* $\mathbf{G}$ *be class terms. Then* $\mathbf{G} \circ \mathbf{F}$ *is a function, its domain is* $\{x \in \operatorname{dom} \mathbf{F} \mid \mathbf{F}(x) \in \operatorname{dom} \mathbf{G}\}$*, and for* x *in this domain, we have* $(\mathbf{F} \circ \mathbf{G})(x) = \mathbf{F}(\mathbf{G}(x))$*.*

Proof:
⟨1⟩1. $\mathbf{G} \circ \mathbf{F}$ is a class term.
  ⟨2⟩1. Let: $x(\mathbf{G} \circ \mathbf{F})z$ and $x(\mathbf{G} \circ \mathbf{F})z'$
  ⟨2⟩2. Pick y, y' such that $x\mathbf{F}y$, $x\mathbf{F}y'$, $y\mathbf{G}z$ and $y'\mathbf{G}z'$
  ⟨2⟩3. $y = y'$
    Proof: Since $\mathbf{F}$ is a class term.
  ⟨2⟩4. $z = z'$
    Proof: Since $\mathbf{G}$ is a class term.
⟨1⟩2. $\operatorname{dom}(\mathbf{G} \circ \mathbf{F}) = \{x \in \operatorname{dom} \mathbf{F} \mid \mathbf{F}(x) \in \operatorname{dom} \mathbf{G}\}$
  Proof:

$$\begin{aligned} x \in \operatorname{dom}(\mathbf{G} \circ \mathbf{F}) &\Leftrightarrow \exists z. x(\mathbf{G} \circ \mathbf{F})z \\ &\Leftrightarrow \exists y, z. x\mathbf{F}y \wedge y\mathbf{G}z \\ &\Leftrightarrow x \in \operatorname{dom} \mathbf{F} \wedge \mathbf{F}(x) \in \operatorname{dom} \mathbf{G} \end{aligned}$$

⟨1⟩3. For x in this domain, we have $(\mathbf{F} \circ \mathbf{G})(x) = \mathbf{F}(\mathbf{G}(x))$.
  Proof: Since $(x, \mathbf{F}(x)) \in \mathbf{F}$ and $(\mathbf{F}(x), \mathbf{G}(\mathbf{F}(x))) \in \mathbf{G}$.
$\square$

Theorem 3.11.4 (Pairing)**.** *For any classes* $\mathbf{F}$ *and* $\mathbf{G}$*, we have* $(\mathbf{G} \circ \mathbf{F})^{-1} = \mathbf{F}^{-1} \circ \mathbf{G}^{-1}$.

PROOF:

$$\begin{aligned}(x,z) \in (\mathbf{G} \circ \mathbf{F})^{-1} &\Leftrightarrow (z,x) \in \mathbf{G} \circ \mathbf{F} \\ &\Leftrightarrow \exists y.z\mathbf{F}y \wedge y\mathbf{G}x \\ &\Leftrightarrow \exists y.(y,z) \in \mathbf{F}^{-1} \wedge (x,y) \in \mathbf{G}^{-1} \\ &\Leftrightarrow (x,z) \in \mathbf{F}^{-1} \circ \mathbf{G}^{-1} \qquad \square\end{aligned}$$

3.12 Identity Function

Definition 3.12.1 (Identity Class Term (Pairing))**.** Let $\mathbf{A}$ be a set. The *identity class term* $\mathrm{id}_{\mathbf{A}}$ on $\mathbf{A}$ is $\{(x,x) \mid x \in \mathbf{A}\}$.

Theorem 3.12.2 (Pairing, Power Set, Subset)**.** *For any set A, we have id_A is a function.*

PROOF: It is a subset of $\mathcal{PP}A$. $\square$

Theorem 3.12.3 (Extensionality, Pairing, Union, Power Set, Subset)**.** *Let $F : A \to B$ and A be nonempty. Then there exists a function $G : B \to A$ such that $G \circ F = \mathrm{id}_A$ if and only if F is one-to-one.*

PROOF:
⟨1⟩1. LET: $F : A \to B$
⟨1⟩2. ASSUME: A is nonempty
⟨1⟩3. If there exists $G : B \to A$ such that $G \circ F = \mathrm{id}_A$ then F is one-to-one.
  ⟨2⟩1. ASSUME: $G : B \to A$ and $G \circ F = \mathrm{id}_A$
  ⟨2⟩2. LET: $x, y \in A$
  ⟨2⟩3. ASSUME: $F(x) = F(y)$
  ⟨2⟩4. $x = y$
    PROOF: $x = G(F(x)) = G(F(y)) = y$.
⟨1⟩4. If F is one-to-one then there exists $G : B \to A$ such that $G \circ F = \mathrm{id}_A$.
  ⟨2⟩1. ASSUME: F is one-to-one.
  ⟨2⟩2. PICK $a \in A$
  ⟨2⟩3. Define $G : B \to A$ by: $G(y)$ is the x such that $F(x) = y$ if $y \in \operatorname{ran} F$, otherwise $G(y) = a$
  ⟨2⟩4. $G \circ F = \mathrm{id}_A$
    PROOF: For $x \in A$ we have $(G \circ F)(x) = G(F(x)) = x$ by Theorem 3.11.3.
$\square$

Theorem 3.12.4 (Extensionality, Pairing, Union, Power Set, Subset)**.** *Let $F : A \to B$ and A be nonempty. If there exists a function $H : B \to A$ such that $F \circ H = \mathrm{id}_B$ then F is surjective.*

PROOF:
⟨1⟩1. LET: $F : A \to B$
⟨1⟩2. ASSUME: A is nonempty.
⟨1⟩3. LET: $H : B \to A$ satisfy $F \circ H = \mathrm{id}_B$

$\langle 1 \rangle 4$. LET: $y \in B$
$\langle 1 \rangle 5$. $F(H(y)) = y$.
$\square$

Theorem 3.12.5 (Extensionality, Pairing, Union, Power Set, Subset, Choice)**.** *Let $F : A \to B$ and A be nonempty. If F is surjective then there exists a function $H : B \to A$ such that $F \circ H = \mathrm{id}_B$.*

PROOF:
$\langle 1 \rangle 1$. ASSUME: F is surjective.
$\langle 1 \rangle 2$. PICK a function $H \subseteq F^{-1}$ with $\mathrm{dom}\, H = B$
  PROOF: By the Axiom of Choice.
$\langle 1 \rangle 3$. $H : B \to A$
$\langle 1 \rangle 4$. $F \circ H = \mathrm{id}_B$
  $\langle 2 \rangle 1$. LET: $y \in B$
  $\langle 2 \rangle 2$. $(y, H(y)) \in F^{-1}$
  $\langle 2 \rangle 3$. $(H(y), y) \in F$
  $\langle 2 \rangle 4$. $F(H(y)) = y$
$\square$

3.13 Restriction

Definition 3.13.1 (Restriction (Pairing))**.** Let $\mathbf{R}$ be a relation and $\mathbf{A}$ a class. The *restriction* of $\mathbf{R}$ to $\mathbf{A}$ is $\mathbf{R} \restriction \mathbf{A} = \{(x, y) \mid x \in \mathbf{A} \wedge x\mathbf{R}y\}$.

Theorem 3.13.2 (Pairing, Subset)**.** *If R is a small relation then $R \restriction \mathbf{A}$ is small.*

PROOF: Since it is a subset of R. $\square$

3.14 Image

Definition 3.14.1 (Image (Pairing))**.** Let $\mathbf{F}$ and $\mathbf{A}$ be classes. The *image* of $\mathbf{A}$ under $\mathbf{F}$ is $\mathbf{F}(\mathbf{A}) = \{\mathbf{F}(x) \mid x \in \mathbf{A}\}$.

Theorem 3.14.2 (Pairing, Union, Subset)**.** *If F is a set then $F(\mathbf{A})$ is a set.*

PROOF: Since it is a subset of $\mathrm{ran}\, F$. $\square$

Theorem 3.14.3 (Pairing)**.** *For any classes $\mathbf{F}$ and $\mathcal{A}$ we have*

$$\mathbf{F}\left(\bigcup \mathcal{A}\right) = \bigcup_{A \in \mathcal{A}} \mathbf{F}(A)$$

PROOF: Each is the class of all y such that $\exists x. \exists A. x \in A \in \mathcal{A} \wedge y = \mathbf{F}(x)$. $\square$

Theorem 3.14.4 (Pairing)**.** *For any classes $\mathbf{F}$, $\mathbf{A_1}$, ..., $\mathbf{A_n}$, we have*

$$\mathbf{F}(\mathbf{A_1} \cup \cdots \cup \mathbf{A_n}) = \mathbf{F}(\mathbf{A_1}) \cup \cdots \cup \mathbf{F}(\mathbf{A_n}) \ .$$

PROOF: Similar. ∎

Theorem 3.14.5 (Pairing)**.** *For any classes* $\mathbf{F}$ *and* $\mathcal{A}$*, we have*

$$\mathbf{F}\left(\bigcap \mathcal{A}\right) \subseteq \bigcap_{A \in \mathcal{A}} \mathbf{F}(A) \enspace .$$

Equality holds if $\mathbf{F}$ *is single-rooted and* $\mathcal{A}$ *is nonempty.*

PROOF:
⟨1⟩1. $\mathbf{F}\left(\bigcap \mathcal{A}\right) \subseteq \bigcap_{A \in \mathcal{A}} \mathbf{F}(A)$
  ⟨2⟩1. LET: $y \in \mathbf{F}\left(\bigcap \mathcal{A}\right)$
  ⟨2⟩2. PICK $x \in \bigcap \mathcal{A}$ such that $y = \mathbf{F}(x)$
  ⟨2⟩3. LET: $A \in \mathcal{A}$
  ⟨2⟩4. $x \in A$
  ⟨2⟩5. $y \in \mathbf{F}(A)$
⟨1⟩2. If $\mathbf{F}$ is single-rooted then $\mathbf{F}\left(\bigcap \mathcal{A}\right) = \bigcap_{A \in \mathcal{A}} \mathbf{F}(A)$
  ⟨2⟩1. ASSUME: $\mathbf{F}$ is single-rooted and $\mathcal{A}$ is nonempty.
  ⟨2⟩2. LET: $y \in \bigcap_{A \in \mathcal{A}} \mathbf{F}(A)$
  ⟨2⟩3. PICK $A \in \mathcal{A}$
  ⟨2⟩4. PICK $x \in A$ such that $y = \mathbf{F}(x)$
  ⟨2⟩5. $x \in \bigcap \mathcal{A}$
    ⟨3⟩1. LET: $A' \in \mathcal{A}$
    ⟨3⟩2. PICK $x' \in A'$ such that $y = \mathbf{F}(x')$
    ⟨3⟩3. $x = x'$
      PROOF: By ⟨2⟩1.
    ⟨3⟩4. $x \in A'$
∎

Corollary 3.14.5.1 (Pairing)**.** *For any class* $\mathbf{F}$ *and nonempty class* $\mathcal{A}$*, we have*

$$\mathbf{F}^{-1}\left(\bigcap \mathcal{A}\right) = \bigcap_{A \in \mathcal{A}} \mathbf{F}^{-1}(A) \enspace .$$

Theorem 3.14.6 (Pairing)**.** *For any classes* $\mathbf{F}$*,* $\mathbf{A_1}$*, ...,* $\mathbf{A_n}$*, we have*

$$\mathbf{F}(\mathbf{A_1} \cap \cdots \cap \mathbf{A_n}) \subseteq \mathbf{F}(\mathbf{A_1}) \cap \cdots \cap \mathbf{F}(\mathbf{A_n}) \enspace .$$

Equality holds if $\mathbf{F}$ *is single-rooted.*

PROOF: Similar.

Corollary 3.14.6.1 (Pairing)**.** *For any classes* $\mathbf{F}$*,* $\mathbf{A_1}$*, ...,* $\mathbf{A_n}$*, we have*

$$\mathbf{F}^{-1}(\mathbf{A_1} \cap \cdots \cap \mathbf{A_n}) = \mathbf{F}^{-1}(\mathbf{A_1}) \cap \cdots \cap \mathbf{F}^{-1}(\mathbf{A_n}) \enspace .$$

Theorem 3.14.7 (Pairing)**.** *For any classes* $\mathbf{F}$*,* $\mathbf{A}$ *and* $\mathbf{B}$*, we have*

$$\mathbf{F}(\mathbf{A}) - \mathbf{F}(\mathbf{B}) \subseteq \mathbf{F}(\mathbf{A} - \mathbf{B}) \enspace .$$

Equality holds if $\mathbf{F}$ *is single-rooted.*

PROOF:
⟨1⟩1. LET: $\mathbf{F}$, $\mathbf{A}$ and $\mathbf{B}$ be sets.
⟨1⟩2. $\mathbf{F}(\mathbf{A}) - \mathbf{F}(\mathbf{B}) \subseteq \mathbf{F}(\mathbf{A} - \mathbf{B})$
  ⟨2⟩1. LET: $y \in \mathbf{F}(\mathbf{A}) - \mathbf{F}(\mathbf{B})$
  ⟨2⟩2. PICK $x \in \mathbf{A}$ such that $x\mathbf{F}y$
  ⟨2⟩3. $x \in \mathbf{A} - \mathbf{B}$
⟨1⟩3. If $\mathbf{F}$ is single-rooted then $\mathbf{F}(\mathbf{A} - \mathbf{B}) = \mathbf{F}(\mathbf{A}) - \mathbf{F}(\mathbf{B})$.
  ⟨2⟩1. ASSUME: $\mathbf{F}$ is single-rooted.
  ⟨2⟩2. LET: $y \in \mathbf{F}(\mathbf{A} - \mathbf{B})$
  ⟨2⟩3. PICK $x \in \mathbf{A} - \mathbf{B}$ such that $y = \mathbf{F}(x)$
  ⟨2⟩4. $y \in \mathbf{F}(\mathbf{A})$
  ⟨2⟩5. $y \notin \mathbf{F}(\mathbf{B})$
    ⟨3⟩1. ASSUME: for a contradiction $x' \in \mathbf{B}$ and $x'\mathbf{F}y$
    ⟨3⟩2. $x' = x$
      PROOF: From ⟨2⟩1
    ⟨3⟩3. $x \in \mathbf{B}$
    ⟨3⟩4. Q.E.D.
      PROOF: This contradicts ⟨2⟩3.
□

Corollary 3.14.7.1 (Pairing)**.** *For any classes* $\mathbf{F}$ *and sets* $\mathbf{A}$ *and* $\mathbf{B}$*, we have*

$$\mathbf{F}^{-1}(\mathbf{A}) - \mathbf{F}^{-1}(\mathbf{B}) = \mathbf{F}^{-1}(\mathbf{A} - \mathbf{B}) \ .$$

3.15 Infinite Cartesian Product

Definition 3.15.1 (Infinite Cartesian Product (Pairing))**.** Let H be a function with domain I. The *Cartesian product* $\prod_{i \in I} H(i)$ is the class of all functions f with domain I such that, for all $i \in I$, we have $f(i) \in H(i)$.

Theorem 3.15.2 (Pairing, Union, Power Set, Subset)**.** *If* H *is a function with domain* I *then* $\prod_{i \in I} H(i)$ *is a set.*

PROOF: It is a subset of $\mathcal{P}(I \times \bigcup \operatorname{ran} H)$. □

Theorem 3.15.3 (Axiom of Choice, Second Version (Pairing, Union, Power Set, Subset))**.** *The Axiom of Choice is equivalent to the statement: for any function* H *with domain* I*, if* $H(i)$ *is nonempty for all* $i \in I$*, then* $\prod_{i \in I} H(i)$ *is nonempty.*

PROOF:
⟨1⟩1. If the Axiom of Choice is true then, for any function H with domain I, if $H(i)$ is nonempty for all $i \in I$, then $\prod_{i \in I} H(i)$ is nonempty.
  ⟨2⟩1. ASSUME: The Axiom of Choice
  ⟨2⟩2. LET: H be a function with domain I such that $H(i)$ is nonempty for all $i \in I$.
  ⟨2⟩3. PICK a function $f \subseteq \{(i, x) \mid x \in H(i)\}$

⟨2⟩4. $f \in \prod_{i \in I} H(i)$

⟨1⟩2. If, for any function H with domain I, if $H(i)$ is nonempty for all $i \in I$, then $\prod_{i \in I} H(i)$ is nonempty, then the Axiom of Choice is true.

⟨2⟩1. ASSUME: for any function H with domain I, if $H(i)$ is nonempty for all $i \in I$, then $\prod_{i \in I} H(i)$ is nonempty

⟨2⟩2. LET: R be a relation.

⟨2⟩3. LET: $I = \operatorname{dom} R$

⟨2⟩4. LET: H be the function with domain I such that $H(i) = \{y \mid iRy\}$ for all i.

⟨2⟩5. PICK $f \in \prod_{i \in I} H(i)$

⟨2⟩6. $f \subseteq R$

□

3.16 Reflexive Relations

Definition 3.16.1 (Reflexive (Pairing))**.** Let $\mathbf{R}$ be a relation on $\mathbf{A}$. Then $\mathbf{R}$ is *reflexive* on A if and only if, for all $x \in \mathbf{A}$, we have $x\mathbf{R}x$.

3.17 Symmetric

Definition 3.17.1 (Symmetric (Pairing))**.** Let $\mathbf{R}$ be a relation. Then $\mathbf{R}$ is *symmetric* if and only if, whenever $x\mathbf{R}y$, then $y\mathbf{R}x$.

3.18 Transitivity

Definition 3.18.1 (Transitivity (Pairing))**.** Let $\mathbf{R}$ be a relation. Then $\mathbf{R}$ is *transitive* if and only if, whenever $x\mathbf{R}y$ and $y\mathbf{R}z$, then $x\mathbf{R}z$.

3.19 Equivalence Relations

Definition 3.19.1 (Equivalence Relation (Pairing))**.** Let $\mathbf{R}$ be a relation on $\mathbf{A}$. Then $\mathbf{R}$ is an *equivalence relation* on $\mathbf{A}$ if and only if $\mathbf{R}$ is reflexive on $\mathbf{A}$, symmetric and transitive.

Theorem 3.19.2 (Pairing)**.** *If* $\mathbf{R}$ *is a symmetric and transitive relation then* $\mathbf{R}$ *is an equivalence relation on* $\operatorname{fld} \mathbf{R}$*.*

PROOF:

⟨1⟩1. LET: $\mathbf{R}$ be a symmetric and transitive relation.

⟨1⟩2. LET: $x \in \operatorname{fld} \mathbf{R}$

⟨1⟩3. PICK y such that $x\mathbf{R}y$ or $y\mathbf{R}x$

⟨1⟩4. $x\mathbf{R}y$ and $y\mathbf{R}x$

PROOF: By symmetry.

⟨1⟩5. $x\mathbf{R}x$

PROOF: By transitivity.
□

3.20 Equivalence Class

Definition 3.20.1 (Equivalence Class (Pairing))**.** Let $\mathbf{R}$ be an equivalence relation on $\mathbf{A}$ and $a \in \mathbf{A}$. Then the *equivalence class* of a *modulo* $\mathbf{R}$ is

$$[a]_{\mathbf{R}} = \{x \in \mathbf{A} \mid a\mathbf{R}x\} \enspace .$$

Lemma 3.20.2 (Extensionality, Pairing, Subset)**.** *Let* $\mathbf{R}$ *be an equivalence relation on* $\mathbf{A}$ *and* $x, y \in \mathbf{A}$*. Then* $[x]_{\mathbf{R}} = [y]_{\mathbf{R}}$ *if and only if* $x\mathbf{R}y$*.*

PROOF:
⟨1⟩1. If $[x]_{\mathbf{R}} = [y]_{\mathbf{R}}$ then $x\mathbf{R}y$.
  ⟨2⟩1. ASSUME: $[x]_{\mathbf{R}} = [y]_{\mathbf{R}}$
  ⟨2⟩2. $y \in [y]_{\mathbf{R}}$
    PROOF: Since $y\mathbf{R}y$ by reflexivity.
  ⟨2⟩3. $y \in [x]_{\mathbf{R}}$
  ⟨2⟩4. $x\mathbf{R}y$
⟨1⟩2. If $x\mathbf{R}y$ then $[x]_{\mathbf{R}} = [y]_{\mathbf{R}}$.
  ⟨2⟩1. ASSUME: $x\mathbf{R}y$
  ⟨2⟩2. $[y]_{\mathbf{R}} \subseteq [x]_{\mathbf{R}}$
    PROOF: If $y\mathbf{R}z$ then $x\mathbf{R}z$ by transitivity.
  ⟨2⟩3. $[x]_{\mathbf{R}} \subseteq [y]_{\mathbf{R}}$
    PROOF: Similar since $y\mathbf{R}x$ by symmetry.
□

3.21 Disjoint

Definition 3.21.1 (Disjoint)**.** Two classes $\mathbf{A}$ and $\mathbf{B}$ are *disjoint* if and only if there is no x such that $x \in \mathbf{A}$ and $x \in \mathbf{B}$.

Axiom 3.21.2 (Regularity)**.** *For any nonempty set A, there exists $m \in A$ such that m and A are disjoint.*

Theorem 3.21.3 (Regularity)**.** *No set is a member of itself.*

Theorem 3.21.4 (Regularity)**.** *There are no sets A and B such that $A \in B$ and $B \in A$.*

3.22 Partitions

Definition 3.22.1 (Partition)**.** A *partition* P of a set A is a set of nonempty subsets of A such that:

1. For all $x \in A$ there exists $S \in P$ such that $x \in S$.
2. Any two distinct elements of P are disjoint.

3.23 Quotient Sets

Definition 3.23.1 (Quotient Set (Pairing, Power Set, Subset))**.** Let R be an equivalence relation on A. The *quotient set* A/R is the set of all equivalence classes modulo R.

This is a set because it is a subset of $\mathcal{P}A$.

Theorem 3.23.2 (Extensionality, Pairing, Power Set, Subset)**.** *Let R be an equivalence relation on A. Then the quotient set A/R is a partition of A.*

PROOF:
$\langle 1 \rangle 1$. For all $x \in A$ there exists $y \in A$ such that $x \in [y]_R$
PROOF: Take $y = x$.
$\langle 1 \rangle 2$. Any two distinct equivalence classes are disjoint.
$\langle 2 \rangle 1$. ASSUME: $z \in [x]_R$ and $z \in [y]_R$
$\langle 2 \rangle 2$. xRz and yRz
$\langle 2 \rangle 3$. $[x]_R = [z]_R = [y]_R$
PROOF: Lemma 3.20.2.
□

Definition 3.23.3 (Canonical Map (Pairing, Power Set, Subset))**.** Let R be an equivalence relation on A. The *canonical map* $\phi : A \to A/R$ is the function defined by $\phi(a) = [a]_R$.

Theorem 3.23.4. *Let R be an equivalence relation on A and $F : A \to B$. Then the following are equivalent:*

1. *For all $x, y \in A$, if xRy then $F(x) = F(y)$.*
2. *There exists $G : A/R \to B$ such that $F = G \circ \phi$, where $\phi : A \to A/R$ is the canonical map.*

In this case, G is unique.

PROOF:
$\langle 1 \rangle 1$. $1 \Rightarrow 2$
$\langle 2 \rangle 1$. ASSUME: 1
$\langle 2 \rangle 2$. Let $G = \{([a]_R, b) \mid F(a) = b\}$
$\langle 2 \rangle 3$. G is a function.
$\langle 3 \rangle 1$. LET: $(c, b), (c, b') \in G$
$\langle 3 \rangle 2$. PICK $a, a' \in A$ such that $c = [a]_R = [a']_R$ with $F(a) = b$ and $F(a') = b'$
$\langle 3 \rangle 3$. aRa'
PROOF: Lemma 3.20.2.
$\langle 3 \rangle 4$. $F(a) = F(a')$
PROOF: From $\langle 2 \rangle 1$.
$\langle 3 \rangle 5$. $b = b'$
PROOF: From $\langle 3 \rangle 2$.
$\langle 2 \rangle 4$. $F = G \circ \phi$

PROOF: For $a \in A$ we have $G(\phi(a)) = G([a]) = F(a)$.
$\langle 1 \rangle 2$. $2 \Rightarrow 1$
$\langle 2 \rangle 1$. LET: $G : A/R \to B$ be such that $F = G \circ \phi$
$\langle 2 \rangle 2$. LET: $x, y \in A$
$\langle 2 \rangle 3$. ASSUME: xRy
$\langle 2 \rangle 4$. $G([x]) = G([y])$
PROOF: Lemma 3.20.2
$\langle 2 \rangle 5$. $F(x) = F(y)$
PROOF: From $\langle 2 \rangle 1$.
$\langle 1 \rangle 3$. If $G, G' : A/R \to B$ and $G \circ \phi = G' \circ \phi$ then $G = G'$
PROOF: For any $a \in A$ we have $G([a]) = G'([a])$.
□

3.24 The Finite Intersection Property

Definition 3.24.1 (Finite Intersection Property)**.** Let X be a set and $\mathcal{A} \subseteq \mathcal{P}X$. Then $\mathcal{A}$ satisfies the *finite intersection property* if and only if every nonempty finite subset of $\mathcal{A}$ has nonempty intersection.

Lemma 3.24.2. *Let X be a set. Let $\mathcal{A} \subseteq \mathcal{P}X$ have the finite intersection property. Then there exists a maximal set $\mathcal{D}$ such that $\mathcal{A} \subseteq \mathcal{D} \subseteq \mathcal{P}X$ and $\mathcal{D}$ has the finite intersection property.*

PROOF:
$\langle 1 \rangle 1$. LET: $\mathbb{F} = \{\mathcal{D} \mid \mathcal{A} \subseteq \mathcal{D} \subseteq \mathcal{P}X, \mathcal{D}$ has the finite intersection property$\}$
$\langle 1 \rangle 2$. Every chain in $\mathbb{F}$ has an upper bound.
$\langle 2 \rangle 1$. LET: $\mathbb{C}$ be a chain in $\mathbb{F}$.
$\langle 2 \rangle 2$. ASSUME: without loss of generality $\mathbb{C} \neq \emptyset$
PROVE: $\bigcup \mathbb{C} \in \mathbb{F}$
PROOF: If $\mathbb{C} = \emptyset$ then $\mathcal{A}$ is an upper bound.
$\langle 2 \rangle 3$. $\mathcal{A} \subseteq \bigcup \mathbb{C} \subseteq \mathcal{P}X$
$\langle 2 \rangle 4$. LET: $C_1, \ldots, C_n \in \mathbb{C}$
PROVE: $C_1 \cap \cdots \cap C_n \neq \emptyset$
$\langle 2 \rangle 5$. PICK $\mathcal{C}_1, \ldots, \mathcal{C}_n \in \mathbb{C}$ such that $C_i \in \mathcal{C}_i$ for all i.
$\langle 2 \rangle 6$. ASSUME: without loss of generality $\mathcal{C}_1 \subseteq \cdots \subseteq \mathcal{C}_n$
$\langle 2 \rangle 7$. $C_1, \ldots, C_n \in \mathcal{C}_n$
$\langle 2 \rangle 8$. $\mathcal{C}_n$ satisfies the finite intersection property.
$\langle 2 \rangle 9$. $C_1 \cap \cdots \cap C_n \neq \emptyset$
$\langle 1 \rangle 3$. Q.E.D.
PROOF: By Zorn's Lemma.
□

Lemma 3.24.3. *Let X be a set. Let $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the finite intersection property. Then any finite intersection of elements of $\mathcal{D}$ is an element of $\mathcal{D}$.*

PROOF:

$\langle 1\rangle 1.$ LET: $D_1, D_2 \in \mathcal{D}$
$\langle 1\rangle 2.$ $\mathcal{D} \cup \{D_1 \cap D_2\}$ has the finite intersection property.
PROOF: Any finite intersection of members of $\mathcal{D} \cup \{D_1 \cap D_2\}$ is a finite intersection of members of $\mathcal{D}$.
$\langle 1\rangle 3.$ $\mathcal{D} = \mathcal{D} \cup \{D_1 \cap D_2\}$
PROOF: By maximality of $\mathcal{D}$.
$\langle 1\rangle 4.$ $D_1 \cap D_2 \in \mathcal{D}$.
$\square$

Lemma 3.24.4. *Let X be a set. Let $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the finite intersection property. Let $A \subseteq X$. If A intersects every member of $\mathcal{D}$ then $A \in \mathcal{D}$.*

PROOF:
$\langle 1\rangle 1.$ $\mathcal{D} \cup \{A\}$ has the finite intersection property.
$\langle 2\rangle 1.$ LET: $D_1, \ldots, D_n \in \mathcal{D}$
PROVE: $D_1 \cap \cdots \cap D_n \cap A \neq \emptyset$
$\langle 2\rangle 2.$ $D_1 \cap \cdots \cap D_n \in \mathcal{D}$
PROOF: Lemma 3.24.3.
$\langle 2\rangle 3.$ $D_1 \cap \cdots \cap D_n \cap A \neq \emptyset$
PROOF: Since A intersects every member of $\mathcal{D}$.
$\langle 1\rangle 2.$ Q.E.D.
PROOF: By maximality of $\mathcal{D}$.
$\square$

Proposition 3.24.5. *Let X be a set. Let $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the finite intersection property. Let $A, D \in \mathcal{P}X$. If $D \in \mathcal{D}$ and $D \subseteq A$ then $A \in \mathcal{D}$.*

PROOF:
$\langle 1\rangle 1.$ $\mathcal{D} \cup \{A\}$ satisfies the finite intersection property.
$\langle 2\rangle 1.$ LET: $D_1, \ldots, D_n \in \mathcal{D}$
$\langle 2\rangle 2.$ $D_1 \cap \cdots \cap D_n \cap D \neq \emptyset$
PROOF: Since $\mathcal{D}$ satisfies the finite intersection property.
$\langle 2\rangle 3.$ $D_1 \cap \cdots \cap D_n \cap A \neq \emptyset$
$\langle 1\rangle 2.$ $\mathcal{D} = \mathcal{D} \cup \{A\}$
PROOF: By the maximality of $\mathcal{D}$.
$\langle 1\rangle 3.$ $A \in \mathcal{D}$
$\square$

Definition 3.24.6 (Graph)**.** Let $f : A \to B$. The *graph* of f is the set $\{(x, f(x)) \mid x \in A\} \subseteq A \times B$.

3.25 Countable Intersection Property

Definition 3.25.1 (Countable Intersection Property)**.** Let X be a set and $\mathcal{A} \subseteq \mathcal{P}X$. Then $\mathcal{A}$ satisfies the *countable intersection property* if and only if every countable subset of $\mathcal{A}$ has nonempty intersection.

Lemma 3.25.2. *Let X be a set. Let $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the countable intersection property. Then any countable intersection of elements of $\mathcal{D}$ is an element of $\mathcal{D}$.*

PROOF:
⟨1⟩1. LET: $\mathcal{D}_0 \subseteq \mathcal{D}$ be countable.
⟨1⟩2. $\mathcal{D} \cup \{\bigcap \mathcal{D}_0\}$ has the countable intersection property.
PROOF: Any countable intersection of members of $\mathcal{D} \cup \{\bigcap \mathcal{D}_0\}$ is a finite intersection of members of $\mathcal{D}$.
⟨1⟩3. $\mathcal{D} = \mathcal{D} \cup \{\bigcap \mathcal{D}_0\}$
PROOF: By maximality of $\mathcal{D}$.
⟨1⟩4. $\bigcap \mathcal{D}_0 \in \mathcal{D}$.
□

Lemma 3.25.3. *Let X be a set. Let $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the countable intersection property. Let $A \subseteq X$. If A intersects every member of $\mathcal{D}$ then $A \in \mathcal{D}$.*

PROOF:
⟨1⟩1. $\mathcal{D} \cup \{A\}$ has the countable intersection property.
⟨2⟩1. LET: $\mathcal{D}_0 \subseteq \mathcal{D}$ be countable.
PROVE: $\bigcap \mathcal{D}_0 \cap A \neq \emptyset$
⟨2⟩2. $\bigcap \mathcal{D}_0 \in \mathcal{D}$
PROOF: Lemma 3.25.2.
⟨2⟩3. $\bigcap \mathcal{D}_0 \cap A \neq \emptyset$
PROOF: Since A intersects every member of $\mathcal{D}$.
⟨1⟩2. Q.E.D.
PROOF: By maximality of $\mathcal{D}$.
□

3.26 The Axiom of Choice

Axiom 3.26.1 (Axiom of Choice)**.** *Let $\mathcal{A}$ be a set of disjoint nonempty sets. Then there exists a set C consisting of exactly one element from each member of $\mathcal{A}$.*

3.27 Choice Functions

Definition 3.27.1 (Choice Function)**.** Let $\mathcal{B}$ be a set of nonempty sets. A *choice function* for $\mathcal{B}$ is a function $c : \mathcal{B} \to \bigcup \mathcal{B}$ such that, for all $B \in \mathcal{B}$, we have $c(B) \in B$.

Lemma 3.27.2 (Existence of a Choice Function (AC))**.** *Every set of nonempty sets has a choice function.*

PROOF:

$\langle 1 \rangle 1$. Let: $\mathcal{B}$ be a set of nonempty sets.
$\langle 1 \rangle 2$. For $B \in \mathcal{B}$,
Let: $B' = \{B\} \times B$
$\langle 1 \rangle 3$. $\{B' \mid B \in \mathcal{B}\}$ is a set of disjoint nonempty sets.
$\langle 1 \rangle 4$. Pick a set c consisting of exactly one element from each B' for $B \in \mathcal{B}$.
$\langle 1 \rangle 5$. c is a choice function for $\mathcal{B}$.
$\square$

3.28 Transitive

Definition 3.28.1 (Transitive Set)**.** A set A is *transitive* if and only if, whenever $x \in y \in A$ then $x \in A$.

Theorem 3.28.2 (Union, Power Set)**.** *Let A be a set. Then the following are equivalent.*

1. *A is transitive.*
2. $\bigcup A \subseteq A$
3. *For all $a \in A$ we have $a \subseteq A$*
4. $A \subseteq \mathcal{P}A$

Proof: From definitions. $\square$

3.29 Minimal Elements

Definition 3.29.1 (Minimal)**.** Let R be a binary relation and A a set. An element $a \in A$ is *minimal* w.r.t. R iff there is no $x \in A$ such that xRa.

3.30 Well-Founded Relations

Definition 3.30.1 (Well-Founded)**.** Let R be a relation on A. Then R is *well-founded* iff every nonempty subset of A has an R-minimal element.

Theorem 3.30.2 (Transfinite Induction)**.** *Let R be a well-founded relation on A and $B \subseteq A$. Assume that, for every $t \in A$, if $\{x \in A \mid xRt\} \subseteq B$ then $t \in B$. Then we have $B = A$.*

Theorem 3.30.3 (Transfinite Recursion)**.** *Let R be a well-founded relation on a set C.*

Let $\mathbf{A}$ be a class. Let $\mathbf{B}$ be the class of all functions from a subset of C to $\mathbf{A}$. Let $\mathbf{F} : \mathbf{B} \times C \to \mathbf{A}$ be a class term.

Then there exists a unique function $f : C \to \mathbf{A}$ such that, for all $t \in C$, we have $f(t) = \mathbf{F}(f \upharpoonright \{x \in C \mid xRt\}, t)$.

3.31 Transitive Closure

Theorem 3.31.1. *Let R be a relation. Then there exists a unique relation R^t such that R^t is transitive, $R \subseteq R^t$, and for every transitive relation S with $R \subseteq S$ we have $R^t \subseteq S$.*

Definition 3.31.2 (Transitive Closure)**.** The *transitive closure* of a relation R is this relation R^t.

Theorem 3.31.3. *If R is well-founded then R^t is well-founded.*

3.32 Fixed Points

Definition 3.32.1 (Fixed Point)**.** Let X be a set. Let $f : X \to X$. Then a *fixed point* of f is an element $a \in X$ such that $f(a) = a$.

Chapter 4

Cardinal Numbers

Definition 4.0.1 (Equinumerous)**.** Two sets A and B are *equinumerous* if and only if there exists a bijection between them.

Theorem 4.0.2. *Equinumerosity is an equivalence relation on the class of all sets.*

Theorem 4.0.3 (Cantor)**.** *No set is equinumerous with its power set.*

Definition 4.0.4. We say a set A is *dominated* by B, $A \preccurlyeq B$, iff A is equinumerous with a subset of B.

Theorem 4.0.5. $A \preccurlyeq A$

Theorem 4.0.6. *If* $A \preccurlyeq B \preccurlyeq C$ *then* $A \preccurlyeq C$.

Theorem 4.0.7 (Schröder-Bernstein Theorem)**.** *If* $A \preccurlyeq B$ *and* $B \preccurlyeq A$ *then* $A \equiv B$.

Proof:
$\langle 1 \rangle 1.$ Let: $f : A \to B$ and $g : B \to A$ be injections.
$\langle 1 \rangle 2.$ Define a sequence of sets $C_n \subseteq A$ by

$$C_0 = A - \operatorname{ran} g$$
$$C_{n+1} = g(f(C_n))$$

$\langle 1 \rangle 3.$ Define $h : A \to B$ by

$$h(x) = \begin{cases} f(x) & \text{if } x \in \bigcup_n C_n \\ g^{-1}(x) & \text{if not} \end{cases}$$

$\langle 1 \rangle 4.$ h is a bijection.
$\square$

Theorem 4.0.8 (AC)**.** *For any infinite set A we have* $\mathbb{N} \preccurlyeq A$.

Proof: Given a choice funtion f for A, choose a sequence (a_n) in A by $a_n = f(A - \{a_0, \ldots, a_{n-1}\})$. $\square$

Corollary 4.0.8.1 (AC)**.** *A set is infinite if and only if it is equinumerous with a proper subset.*

4.1 Countability

Definition 4.1.1 (Countable)**.** A set A is *countable* iff $A \preccurlyeq \mathbb{N}$.

Theorem 4.1.2 (AC)**.** *A countable union of countable sets is countable.*

Proposition 4.1.3 (AC)**.** *Every infinite set has a countable subset.*

4.2 Order Theory

Definition 4.2.1 (Preorder)**.** Let X be a set. A *preorder* on X is a binary relation $\leq$ on X such that:

Reflexivity For all $x \in X$, we have $x \leq x$

Transitivity For all $x, y, z \in X$, if $x \leq y$ and $y \leq z$ then $x \leq z$.

Definition 4.2.2 (Preordered Set)**.** A *preordered set* consists of a set X and a preorder $\leq$ on X.

Proposition 4.2.3. *Let X and Y be linearly ordered sets. Let $f : X \twoheadrightarrow Y$ be strictly monotone and surjective. Then f is a poset isomorphism.*

PROOF:
⟨1⟩1. f is injective.
 ⟨2⟩1. LET: $x, y \in X$
 ⟨2⟩2. ASSUME: $f(x) = f(y)$
 ⟨2⟩3. $x \nless y$
  PROOF: By strong motonicity.
 ⟨2⟩4. $y \nless x$
  PROOF: By strong motonicity.
 ⟨2⟩5. $x = y$
  PROOF: By trichotomy.
⟨1⟩2. f^{-1} is monotone.
 ⟨2⟩1. LET: $x, y \in X$
 ⟨2⟩2. ASSUME: $x \leq y$
 ⟨2⟩3. $f^{-1}(x) \ngtr f^{-1}(y)$
  PROOF: By strong motonicity.
 ⟨2⟩4. $f^{-1}(x) < f^{-1}(y)$
  PROOF: By trichotomy.
□

Definition 4.2.4 (Interval)**.** Let X be a preordered set and $Y \subseteq X$. Then Y is an *interval* if and only if, for all $a, b \in Y$ and $c \in X$, if $a \leq c \leq b$ then $c \in Y$.

Definition 4.2.5 (Linear Continuum)**.** A linearly ordered set L is a *linear continuum* if and only if:

1. every nonempty subset of L that is bounded above has a supremum
2. L is dense

Proposition 4.2.6. *Every interval in a linear continuum is a linear continuum.*

PROOF:
⟨1⟩1. LET: L be a linear continuum and I an interval in L.
⟨1⟩2. Every nonempty subset of I that is bounded above has a supremum in I.
 ⟨2⟩1. LET: $X \subseteq I$ be nonempty and bounded above by $b \in I$.

⟨2⟩2. Let: s be the supremum of X in L.
Proof: Since L is a linear continuum.
⟨2⟩3. $s \in I$
⟨3⟩1. Pick $a \in X$
Proof: Since X is nonempty (⟨2⟩1).
⟨3⟩2. $a \leq s \leq b$
⟨3⟩3. $s \in I$
Proof: Since I is an interval (⟨1⟩1).
⟨2⟩4. s is the supremum of X in I
⟨1⟩3. I is dense.
⟨2⟩1. Let: $x, y \in I$ with $x < y$
⟨2⟩2. Pick $z \in L$ with $x < z < y$
Proof: Since L is dense.
⟨2⟩3. $z \in I$
Proof: Since I is an interval.
□

Definition 4.2.7 (Ordered Square)**.** The *ordered square* I_o^2 is the set $[0,1]^2$ under the dictionary order.

Proposition 4.2.8. *The ordered square is a linear continuum.*

Proof:
⟨1⟩1. Every nonempty subset of I_o^2 bounded above has a supremum.
⟨2⟩1. Let: $X \subseteq I_o^2$ be nonempty and bounded above by (b, c)
⟨2⟩2. Let: $s = \sup \pi_1(X)$
Proof: The set $\pi_1(X)$ is nonempty and bounded above by b.
⟨2⟩3. Case: $s \in \pi_1(X)$
⟨3⟩1. Let: $t = \sup\{y \in [0,1] \mid (s, y) \in X\}$
Proof: This set is nonempty and bounded above by c.
⟨3⟩2. (s, t) is the supremum of X.
⟨2⟩4. Case: $s \notin \pi_1(X)$
Proof: In this case $(s, 0)$ is the supremum of X.
⟨1⟩2. I_o^2 is dense.
⟨2⟩1. Let: $(x_1, y_1), (x_2, y_2) \in I_o^2$ with $(x_1, y_1) < (x_2, y_2)$
⟨2⟩2. Case: $x_1 < x_2$
⟨3⟩1. Pick x_3 with $x_1 < x_3 < x_2$
⟨3⟩2. $(x_1, y_1) < (x_3, y_1) < (x_2, y_2)$
⟨2⟩3. Case: $x_1 = x_2$ and $y_1 < y_2$
⟨3⟩1. Pick y_3 with $y_1 < y_3 < y_2$
⟨3⟩2. $(x_1, y_1) < (x_1, y_3) < (x_2, y_2)$
□

Proposition 4.2.9. *If X is a well-ordered set then $X \times [0, 1)$ under the dictionary order is a linear continuum.*

Proof:
⟨1⟩1. Every nonempty set $A \subseteq X \times [0, 1)$ bounded above has a supremum.

⟨2⟩1. LET: $A \subseteq X \times [0,1)$ be nonempty and bounded above
⟨2⟩2. LET: x_0 be the supremum of $\pi_1(A)$
⟨2⟩3. CASE: $x_0 \in \pi_1(A)$
⟨3⟩1. LET: y_0 be the supremum of $\{y \in [0,1) \mid (x_0, y) \in A\}$
⟨3⟩2. (x_0, y_0) is the supremum of A.
⟨2⟩4. CASE: $x_0 \notin \pi_1(A)$
PROOF: In this case $(x_0, 0)$ is the supremum of A.
⟨1⟩2. $X \times [0,1)$ is dense.
⟨2⟩1. LET: $(x_1, y_1), (x_2, y_2) \in X \times [0,1)$ with $(x_1, y_1) < (x_2, y_2)$
⟨2⟩2. CASE: $x_1 < x_2$
⟨3⟩1. PICK y_3 such that $y_1 < y_3 < 1$
⟨3⟩2. $(x_1, y_1) < (x_1, y_3) < (x_2, y_2)$
⟨2⟩3. CASE: $x_1 = x_2$ and $y_1 < y_2$
⟨3⟩1. PICK y_3 such that $y_1 < y_3 < y_2$
⟨3⟩2. $(x_1, y_1) < (x_1, y_3) < (x_2, y_2)$
□

Lemma 4.2.10. *For all $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$, we have $[a, b) \cong [c, d)$*

PROOF: The map $\lambda t.c + (t - a)(d - c)/(b - a)$ is an order isomorphism.

Proposition 4.2.11. *Let X be a linearly ordered set. Let $a < b < c$ in X. Then $[a, c) \cong [0, 1)$ if and only if $[a, b) \cong [b, c) \cong [0, 1)$.*

PROOF:
⟨1⟩1. If $[a, c) \cong [0, 1)$ then $[a, b) \cong [b, c) \cong [0, 1)$
⟨2⟩1. ASSUME: $f : [a, c) \cong [0, 1)$ is an order isomorphism
⟨2⟩2. $[a, b) \cong [0, 1)$
PROOF:
$$\begin{aligned} [a, b) &\cong [0, f(b)) && \text{(by the restriction of } f\text{)} \\ &\cong [0, 1) && \text{(Lemma 4.2.10)} \end{aligned}$$
⟨2⟩3. $[b, c) \cong [0, 1)$
PROOF: Similar.
⟨1⟩2. If $[a, b) \cong [b, c) \cong [0, 1)$ then $[a, c) \cong [0, 1)$
PROOF:
$$\begin{aligned} [a, c) &= [a, b) * [b, c) \\ &\cong [0, 1) * [0, 1) \\ &\cong [0, 1/2) * [1/2, 1) && \text{(Lemma 4.2.10)} \\ &= 1 \end{aligned}$$
□

Proposition 4.2.12 (CC)**.** *Let X be a linearly ordered set. Let $x_0 < x_1 < \cdots$ be a strictly increasing sequence in X with supremum b. Then $[x_0, b) \cong [0, 1)$ if and only if $[x_i, x_{i+1}) \cong [0, 1)$ for all i.*

PROOF:

$\langle 1 \rangle$1. If $[x_0, b) \cong [0, 1)$ then $[x_i, x_{i+1}) \cong [0, 1)$ for all i.
PROOF: By Lemma 4.2.10
$\langle 1 \rangle$2. If $[x_i, x_{i+1}) \cong [0, 1)$ for all i then $[x_0, b) \cong [0, 1)$
$\langle 2 \rangle$1. ASSUME: $[x_i, x_{i+1}) \cong [0, 1)$ for all i
$\langle 2 \rangle$2. PICK an order isomorphism $f_i : [x_i, x_{i+1}) \cong [1/2^i, 2/2^{i+1})$ for each i.
PROOF: By Lemma 4.2.10
$\langle 2 \rangle$3. The union of the f_is is an order isomorphism $[x_0, b) \cong [0, 1)$
□

4.3 Partially Ordered Sets

Definition 4.3.1 (Partial Order)**.** A *partial order* on a set X is a preorder $\leq$ that is *anti-symmetric*, i.e. whenever $x \leq y$ and $y \leq x$ then $x = y$.

4.4 Strict Partial Order

Definition 4.4.1 (Strict Partial Order)**.** A *strict partial order* on a set X is a relation on X that is transitive and irreflexive.

Proposition 4.4.2. *If $<$ is a strict partial order on X and $x, y \in X$, then at most one of $x < y$, $y < x$, $x = y$ holds.*

Proposition 4.4.3. *If $<$ is a strict partial order then the relation $\leq$ defined by: $x \leq y$ iff $x < y$ or $x = y$, is a partial order.*

Theorem 4.4.4. *If R is a well-founded relation then its transitive closure is a partial order.*

Definition 4.4.5 (Linear Order)**.** A *linear order* on a set X is a partial order such that, for any $x, y \in X$, either $x \leq y$ or $y \leq x$.

4.5 Strict Linear Orders

Definition 4.5.1 (Strict Linear Order (Extensionality, Pairing))**.** Let A be a set. A *strict linear order* on A is a binary relation R on A that is transitive and satisfies *trichotomy*: for any $x, y \in A$, exactly one of xRy, $x = y$, yRx holds.

Theorem 4.5.2. *Let R be a strict linear order on A. Then there is no $x \in A$ such that xRx.*

PROOF: Immediate from trichotomy.

4.6 Well Orderings

Definition 4.6.1 (Well-ordering)**.** A *well-order* on a set X is a linear order such that every nonempty set has a least element.

Proposition 4.6.2. *Let $\leq$ be a linear order on X. Then $\leq$ is a well-order iff there is no function $f : \mathbb{N} \to X$ such that $f(n+1) < f(n)$ for all n.*

Definition 4.6.3 (Initial Segment)**.** Given a well-ordered set X and $\alpha \in X$, the *initial segment* of X up to α is $\operatorname{seg} \alpha = \{x \in X \mid x < \alpha\}$.

Theorem 4.6.4 (Transfinite Induction)**.** *Let $\leq$ be a linear order on J. Then the following are equivalent:*

1. *$\leq$ is a well-order on J.*
2. *For every subset $J_0 \subseteq J$, if the following condition holds:*
   - *For every $\alpha \in J$, if $\operatorname{seg} \alpha \subseteq J_0$ then $\alpha \in J$.*

   then $J_0 = J$.

Axiom Schema 4.6.5 (Replacement)**.** *Let* $\mathbf{H}$ *be a class term. If* $\operatorname{dom} \mathbf{H}$ *is a set then* $\mathbf{H}$ *is a set.*

Theorem 4.6.6 (Transfinite Recursion)**.** *Let J be a well-ordered set and C a set. Let $\mathcal{F}$ be the set of all functions from a section of J to C. Let G be a function with domain $\mathcal{F}$. Then there exists a unique function h with domain J such that, for all $\alpha \in J$, we have $h(\alpha) = \rho(h \restriction \operatorname{seg} \alpha)$.*

PROOF:
$\langle 1 \rangle 1.$ If v is a function and $t \in J$, we say v is *ρ-constructed up to t* iff $\operatorname{dom} v = \{x \in J \mid x \leq t\}$ and, for all $x \in \operatorname{dom} v$, we have $v(x) = \rho(v \restriction \operatorname{seg} x)$
$\langle 1 \rangle 2.$ If $t_1 \leq t_2$, v_1 is ρ-constructed up to t_1, and v_2 is ρ-constructed up to t_2, then $v_1 = v_2 \restriction \{x \in J \mid x \leq t_1\}$
$\langle 1 \rangle 3.$ LET: $\mathcal{K}$ be the set of all functions that are ρ-constructed up to some $t \in J$
PROOF: $\mathcal{K}$ is a set by a Replacement Axiom.
$\langle 1 \rangle 4.$ LET: $F = \bigcup \mathcal{K}$
$\langle 1 \rangle 5.$ F is a function
$\langle 1 \rangle 6.$ For all $x \in \operatorname{dom} F$ we have $F(x) = \rho(F \restriction \operatorname{seg} x)$
$\langle 1 \rangle 7.$ $\operatorname{dom} F = J$
$\langle 1 \rangle 8.$ F is unique
$\square$

Theorem 4.6.7. *The following are equivalent.*

1. *The Axiom of Choice*
2. *(*Well-Ordering Theorem*) Every set has a well-ordering.*
3. *(*Zorn's Lemma*) Let X be a poset. If every chain in X has an upper bound in X, then X has a maximal element.*

PROOF:
$\langle 1 \rangle 1.$ $1 \Rightarrow 2$
PROOF:

⟨2⟩1. ASSUME: The Axiom of Choice
⟨2⟩2. LET: X be a set.
⟨2⟩3. PICK a choice function for $\mathcal{P}X \setminus \{\emptyset\}$
PROOF: Lemma 3.27.2.
⟨2⟩4. LET: a *tower* in X be a pair $(T, <)$ where $T \subseteq X$, $<$ is a well-ordering of T, and $x = c(X \setminus \{y \in T \mid y < x\})$.
⟨2⟩5. For any two towers $(T_1, <_1)$ and $(T_2, <_2)$, either these two posets are equal or one is a section of the other.
⟨3⟩1.
⟨2⟩6. For any tower $(T, <)$ in X with $T \neq X$, there exists a tower in X of which $(T, <)$ is a section.
⟨2⟩7. LET: $T = \bigcup\{T' \subseteq X \mid \exists R.(T, R) \text{ is a tower in } X\}$
⟨2⟩8. Define $<$ on T by: $x < y$ iff there exists a tower (T, R) in X such that $x, y \in T$ and xRy.
⟨2⟩9. $(T, <)$ is a tower in X.
⟨2⟩10. $T = X$
⟨2⟩11. $<$ is a well-ordering of X.
⟨1⟩2. $2 \Rightarrow 3$
⟨2⟩1. ASSUME: The Well-Ordering Theorem
⟨2⟩2. LET: X be a poset in which every chain has an upper bound.
⟨2⟩3. PICK a well-ordering R of X
⟨2⟩4. Define $F : X \to \{0, 1\}$ by transfinite R-recursion by:

$$F(a) = \begin{cases} 1 & \text{if } b < a \text{ for all } b \text{ such that } bRa \text{ and } f(b) = 1 \\ 0 & \text{otherwise} \end{cases}$$

⟨2⟩5. LET: $C = \{a \in X \mid f(a) = 1\}$
⟨2⟩6. C is a chain in X
⟨3⟩1. LET: $x, y \in C$
⟨3⟩2. ASSUME: without loss of generality xRy
⟨3⟩3. $f(y) = 1$
⟨3⟩4. for all z such that zRy and $f(z) = 1$ we have $z < y$
⟨3⟩5. $x < y$
⟨2⟩7. PICK an upper bound u for C
⟨2⟩8. u is maximal in X
⟨3⟩1. LET: $x \in X$ with $u \leq x$
⟨3⟩2. for all b such that bRx and $f(b) = 1$ we have $b < x$
PROOF: Since $b \in C$ so $b \leq u \leq x$
⟨3⟩3. $f(u) = 1$
⟨3⟩4. $u \leq x$
⟨3⟩5. $u = x$
⟨2⟩9. $3 \Rightarrow 1$
⟨3⟩1. ASSUME: Zorn's Lemma
⟨3⟩2. LET: R be a relation
⟨3⟩3. LET: $\mathcal{A}$ be the poset of functions that are subsets of R under $\subseteq$
⟨3⟩4. Every chain in $\mathcal{A}$ has an upper bound
⟨4⟩1. LET: $\mathcal{C} \subseteq \mathcal{A}$ be a chain.

PROVE: $\bigcup \mathcal{C} \in \mathcal{A}$
⟨4⟩2. ASSUME: $(x, y), (x, z) \in \bigcup \mathcal{C}$
⟨4⟩3. PICK $f, g \in \mathcal{C}$ such that $f(x) = y$ and $g(x) = z$
⟨4⟩4. ASSUME: without loss of generality $f \subseteq g$
⟨4⟩5. $g(x) = y$
⟨4⟩6. $y = z$
⟨3⟩5. PICK F maximal in $\mathcal{A}$
⟨3⟩6. $\operatorname{dom} F = \operatorname{dom} R$
⟨4⟩1. ASSUME: for a contradiction $x \in \operatorname{dom} R - \operatorname{dom} F$
⟨4⟩2. PICK y such that xRy
⟨4⟩3. LET: $G = F \cup \{(x, y)\}$
⟨4⟩4. $G \in \mathcal{A}$
⟨4⟩5. $F \subset G$
⟨4⟩6. Q.E.D.
PROOF: This contradicts the maximality of F.
□

Theorem 4.6.8 (Cardinal Comparability)**.** *The Axiom of Choice is equivalent to the* Cardinal Comparability Theorem*: for any two sets A and B, either $A \preccurlyeq B$ or $B \preccurlyeq A$.*

PROOF:
⟨1⟩1. Zorn's Lemma implies Cardinal Comparability
⟨2⟩1. ASSUME: Zorn's Lemma
⟨2⟩2. LET: A and B be sets.
⟨2⟩3. LET: $\mathcal{A}$ be the poset of all injective functions f such that $\operatorname{dom} f \subseteq C$ and $\operatorname{ran} f \subseteq D$ under $\subseteq$
⟨2⟩4. Every chain in $\mathcal{A}$ has an upper bound.
⟨3⟩1. LET: $\mathcal{C} \subseteq \mathcal{A}$ be a chain.
PROVE: $\bigcup \mathcal{C} \in \mathcal{A}$
⟨3⟩2. $\bigcup \mathcal{C}$ is a function.
⟨4⟩1. LET: $(x, y), (x, z) \in \bigcup \mathcal{C}$
⟨4⟩2. PICK $f, g \in \mathcal{C}$ such that $f(x) = y$ and $g(x) = z$
⟨4⟩3. ASSUME: without loss of generality $f \subseteq g$
⟨4⟩4. $g(x) = y$
⟨4⟩5. $y = z$
⟨3⟩3. $\bigcup \mathcal{C}$ is injective.
PROOF: Similar.
⟨2⟩5. PICK $\hat{f}$ maximal in $\mathcal{A}$
PROOF: By Zorn's Lemma.
⟨2⟩6. Either $\operatorname{dom} \hat{f} = C$ or $\operatorname{ran} \hat{f} = D$
⟨3⟩1. ASSUME: for a contradiction $\operatorname{dom} \hat{f} \subset C$ and $\operatorname{ran} \hat{f} \subset D$
⟨3⟩2. PICK $x \in C - \operatorname{dom} \hat{f}$ and $y \in D - \operatorname{ran} \hat{f}$
⟨3⟩3. LET: $g = \hat{f} \cup \{(x, y)\}$
⟨3⟩4. $g \in \mathcal{A}$
⟨3⟩5. $\hat{f} \subset g$

⟨3⟩6. Q.E.D.

PROOF: This contradicts the maximality of $\hat{f}$.

⟨2⟩7. If $\mathrm{dom}\,\hat{f} = C$ then $C \preccurlyeq D$

⟨2⟩8. If $\mathrm{ran}\,\hat{f} = D$ then $D \preccurlyeq C$

⟨1⟩2. Cardinal Comparability implies the Well-Ordering Theorem

⟨2⟩1. ASSUME: Cardinal Comparability

⟨2⟩2. LET: A be a set

⟨2⟩3. PICK an ordinal α such that $\alpha \not\preccurlyeq A$

⟨2⟩4. $A \preccurlyeq \alpha$

PROOF: By Cardinal Comparability.

⟨2⟩5. PICK an injection $f : A \to \alpha$

⟨2⟩6. Define $<$ on A by $x < y$ iff $f(x) \in f(y)$

⟨2⟩7. $<$ is a well-ordering on A.

□

Theorem 4.6.9. *Given two well-ordered sets A and B, either $A \cong B$ or one of A, B is isomorphic to an initial segment of the other.*

4.7 Ordinal Numbers

Definition 4.7.1. Let $(A, \leq)$ be a well-ordered set. The *ordinal number* of $(A, \leq)$ is the range of E, where E is the unique function with domain A such that $E(t) = \mathrm{ran}(E \restriction \mathrm{seg}\, t)$ for all $t \in A$.

Theorem 4.7.2. *Let $(A, \leq)$ be a well-ordered set and $E : A \to \alpha$ be the canonical function onto the ordinal of A. Then:*

1. *For all $t \in A$ we have $E(t) \notin E(t)$.*
2. *E is a bijection.*
3. *For any $s, t \in A$, we have $s < t$ if and only if $E(s) \in E(t)$.*
4. *α is a transitive set.*
5. *α is well-ordered by $\in$*
6. *E is an order isomorphism between $(A, \leq)$ and $(\alpha, \underline{\in})$.*

Theorem 4.7.3. *Two well-ordered sets are isomorphic if and only if they have the same ordinal number.*

Theorem 4.7.4. *A set is an ordinal number if and only if it is a transitive set well-ordered by $\in$.*

Theorem 4.7.5. *Every member of an ordinal number is an ordinal number.*

Theorem 4.7.6. *Any transitive set of ordinal numbers is an ordinal number.*

Theorem 4.7.7. *The empty set is an ordinal number.*

Theorem 4.7.8. *The successor of an ordinal number is an ordinal number.*

Theorem 4.7.9. *If A is a set of ordinal numbers then $\bigcup A$ is an ordinal number.*

Theorem 4.7.10. *Any nonempty set of ordinal numbers has a least element.*

Theorem 4.7.11 (Burali-Forti Paradox)**.** *The class of ordinal numbers is a proper class.*

Theorem 4.7.12 (Hartogs' Theorem)**.** *For any set A, there exists an ordinal that is not dominated by A.*

Proof:
$\langle 1\rangle 1.$ Let: α be the class of all ordinals β such that $\beta \preccurlyeq A$
$\langle 1\rangle 2.$ α is a set.
  $\langle 2\rangle 1.$ Let: W be the set of all pairs $(B, \leq)$ such that $B \subseteq A$ and $\leq$ is a well-ordering on B.
  $\langle 2\rangle 2.$ Every member of α is the ordinal number of a member of W
  $\langle 2\rangle 3.$ Q.E.D.
    Proof: By a Replacement Axiom.
$\langle 1\rangle 3.$ α is an ordinal.
$\langle 1\rangle 4.$ α is not dominated by A.
$\square$

Definition 4.7.13. A class term $\mathbf{F} : \mathbf{On} \to \mathbf{On}$ is *continuous* iff, for every limit ordinal λ, we have $\mathbf{F}(\lambda) = \sup_{\alpha<\lambda} F(\alpha)$.

Theorem 4.7.14. *Let $\mathbf{F} : \mathbf{On} \to \mathbf{On}$. If $\mathbf{F}$ is continuous and $\mathbf{F}(\alpha) < \mathbf{F}(\alpha+1)$ for every ordinal α, then $\mathbf{F}$ is strictly monotone.*

Definition 4.7.15. A class term $\mathbf{F} : \mathbf{On} \to \mathbf{On}$ is *normal* iff it is strictly monotone and continuous.

Theorem 4.7.16. *Let $\mathbf{F} : \mathbf{On} \to \mathbf{On}$ be normal. For every ordinal $\beta \geq \mathbf{F}(0)$, there exists a greatest ordinal α such that $\mathbf{F}(\alpha) \leq \beta$.*

Theorem 4.7.17. *Let $\mathbf{F} : \mathbf{On} \to \mathbf{On}$ be normal. Let S be a set of ordinals. Then $\mathbf{F}(\sup S) = \sup_{\alpha\in S} \mathbf{F}(\alpha)$.*

Theorem 4.7.18 (Veblen Fixed-Point Theorem)**.** *Let $\mathbf{F} : \mathbf{On} \to \mathbf{On}$ be normal. For every ordinal α, there exists $\beta \geq \alpha$ such that $\mathbf{F}(\beta) = \beta$.*

Proof: Let β be the supremum of α, $\mathbf{F}(\alpha)$, $\mathbf{F}^2(\alpha)$, ... $\square$

Lemma 4.7.19. *Let α be an ordinal. Let $(f(\gamma))_{\gamma<\alpha}$ be an α-sequence of ordinals. Then there exists $\beta \leq \alpha$ and an increasing sequence of ordinals $(g(\gamma))_{\gamma<\beta}$ such that $\sup_{\gamma<\alpha} f(\gamma) = \sup_{\gamma<\beta} g(\gamma)$.*

4.8 Cardinal Numbers

Definition 4.8.1 (Cardinal Number (AC))**.** For any set A, the *cardinal number* of A, $\operatorname{card} A$, is the least ordinal equinumerous with A.

There exists some ordinal equinumerous with A by the Well-Ordering Theorem.

Theorem 4.8.2. *For any sets A and B, we have $A \equiv B$ if and only if* $\operatorname{card} A = \operatorname{card} B$*.*

Theorem 4.8.3. *A set A is finite if and only if* $\operatorname{card} A$ *is a natural number.*

Theorem 4.8.4. *The supremum of a set of cardinal numbers is a cardinal number.*

4.9 Cardinal Arithmetic

Definition 4.9.1. For cardinal numbers κ and λ, the *sum* $\kappa + \lambda$ is the cardinal number of $A \cup B$, where A and B are disjoint sets of cardinality κ and λ respectively.

Theorem 4.9.2. $\kappa + \lambda = \lambda + \kappa$

Theorem 4.9.3. $\kappa + (\lambda + \mu) = (\kappa + \lambda) + \mu$

Theorem 4.9.4. *The definition of addition agrees with the definition on natural numbers.*

Definition 4.9.5. For cardinal numbers κ and λ, the *product* $\kappa\lambda$ is the cardinality of $\kappa \times \lambda$.

Theorem 4.9.6. $\kappa\lambda = \lambda\kappa$

Theorem 4.9.7. $\kappa(\lambda\mu) = (\kappa\lambda)\mu$

Theorem 4.9.8. $\kappa(\lambda + \mu) = \kappa\lambda + \kappa\mu$

Theorem 4.9.9. *The definition of multiplication agrees with the definition on natural numbers.*

Theorem 4.9.10 (AC)**.** *For any infinite cardinal κ we have $\kappa\kappa = \kappa$.*

Proof:
$\langle 1\rangle 1.$ Let: B be a set with cardinality κ
$\langle 1\rangle 2.$ Let: $\mathcal{H} = \{\emptyset\} \cup \{f \mid \exists A \subseteq B. A$ is infinite and f is a bijection between $A \times A$ and $A\}$
$\langle 1\rangle 3.$ For every chain $\mathcal{C} \subseteq \mathcal{H}$ we have $\bigcup \mathcal{C} \in \mathcal{H}$
$\langle 1\rangle 4.$ Pick a maximal f_0 in $\mathcal{H}$
$\langle 1\rangle 5.$ $f_0 \neq \emptyset$
Proof: B has a subset of cardinality $\aleph_0$ and $\aleph_0\aleph_0 = \aleph_0$.
$\langle 1\rangle 6.$ Let: A_0 be the set such that f_0 is a bijection between $A_0 \times A_0$ and A_0

$\langle 1 \rangle 7$. LET: $\lambda = \operatorname{card} A_0$
$\langle 1 \rangle 8$. $\operatorname{card}(B - A_0) < \lambda$
$\langle 1 \rangle 9$. $\kappa = \lambda$
PROOF:

$$\begin{aligned} \kappa &= \operatorname{card} A_0 + \operatorname{card}(B - A_0) \\ &\leq \lambda + \lambda \\ &= 2\lambda \\ &\leq \lambda\lambda \\ &= \lambda && (\langle 1 \rangle 6) \\ &\leq \kappa && \square \end{aligned}$$

Theorem 4.9.11 (Absorption Law)**.** *Let κ and λ be cardinal numbers such that $0 < \kappa \leq \lambda$ and λ is infinite. Then*

$$\kappa + \lambda = \lambda \ .$$

Theorem 4.9.12 (Absorption Law)**.** *Let κ and λ be cardinal numbers such that $0 < \kappa \leq \lambda$ and λ is infinite. Then*

$$\kappa\lambda = \lambda \ .$$

Definition 4.9.13. For cardinal numbers κ and λ, we write κ^λ for the cardinality of the set of functions from λ to κ.

Theorem 4.9.14. $\kappa^{\lambda+\mu} = \kappa^\lambda + \kappa^\mu$

Theorem 4.9.15. $(\kappa\lambda)^\mu = \kappa^\mu \lambda^\mu$

Theorem 4.9.16. $(\kappa^\lambda)^\mu = \kappa^{\lambda\mu}$

Theorem 4.9.17. *The definition of exponentiation agrees with the definition on natural numbers.*

Theorem 4.9.18. *Given sets A and B, we have* $\operatorname{card} A \leq \operatorname{card} B$ *if and only if $A \preccurlyeq B$.*

Definition 4.9.19. Let $\aleph_0 = \operatorname{card} \mathbb{N}$.

Theorem 4.9.20 (AC)**.** *For any infinite cardinal κ we have $\aleph_0 \leq \kappa$.*

Theorem 4.9.21 (Maximum Principle (AC))**.** *Every poset has a maximal chain.*

4.10 Rank of a Set

Definition 4.10.1 (Cumulative Hierarchy of Sets)**.** For every ordinal α, define the *rank* V_α by transfinite recursion thus:

$$\begin{aligned} V_0 &= \emptyset \\ V_{\alpha+1} &= \mathcal{P} V_\alpha \\ V_\lambda &= \bigcup_{\alpha < \lambda} V_\alpha \end{aligned}$$

for λ a limit ordinal.

The *von Neumann universe* is the class $\mathbf{V} = \bigcup_{\alpha \in \mathbf{On}} V_\alpha$.

Theorem 4.10.2. *If λ is a limit ordinal and $\lambda > \omega$ then V_λ is a model of Zermelo set theory.*

Lemma 4.10.3 (AC)**.** *There exists a well-ordered set in $V_{\omega 2}$ whose ordinal is not in $V_{\omega 2}$.*

Proof: Pick a well-ordering $<$ of $\mathcal{P}\mathbb{N}$. Then $(\mathcal{P}\mathbb{N}, <) \in V_{\omega 2}$ but its ordinal is not because its ordinal is uncountable. □

Theorem 4.10.4. *The set $V_{\omega 2}$ is not a model of Zermelo-Fraenkel set theory.*

Thus, the Replacement Axioms cannot be proven from the other axioms.

Definition 4.10.5 (Well-Founded Set)**.** A set A is *well-founded* iff $A \in V_\alpha$ for some $\alpha \in \mathbf{On}$.

Definition 4.10.6 (Rank)**.** The *rank* of a well-founded set A, $\operatorname{rank} A$, is the least ordinal α such that $A \in V_\alpha$.

Theorem 4.10.7. *If $A \in B$ and B is well-founded then A is well-founded and* $\operatorname{rank} A < \operatorname{rank} B$.

Theorem 4.10.8. *If A is a set and every member of A is well-founded then A is well-founded and* $\operatorname{rank} A = \sup_{B \in A}(\operatorname{rank} B + 1)$.

Theorem 4.10.9. *The Axiom of Regularity is equivalent to the statement that every set is well-founded.*

4.11 Transfinite Recursion Again

Theorem 4.11.1. *Let $\mathbf{A}$ be a class. Let $\mathbf{B}$ be the class of all functions $f : \alpha \to \mathbf{A}$ for some ordinal α. Let $\mathbf{F} : \mathbf{B} \to \mathbf{A}$ be a class term. Then there exists a unique class term $\mathbf{G} : \mathbf{On} \to \mathbf{A}$ such that, for all $\alpha \in \mathbf{On}$, we have $\mathbf{G}(\alpha) = \mathbf{F}(\mathbf{G} \restriction \alpha)$.*

4.12 Alephs

Definition 4.12.1. Define the cardinal number $\aleph_\alpha$ for every ordinal α by transfinite recursion on α thus: $\aleph_\alpha$ is the least infinite cardinal different from $\aleph_\beta$ for all $\beta < \alpha$.

Theorem 4.12.2. *If $\alpha < \beta$ then $\aleph_\alpha < \aleph_\beta$.*

Theorem 4.12.3. *Every infinite cardinal has the form $\aleph_\alpha$ for some ordinal α.*

4.13 Ordinal Arithmetic

Definition 4.13.1 (Sum)**.** Let α and β be ordinals. The *sum* $\alpha + \beta$ is the ordinal of the concatenation of A followed by B, where A is a well-ordered set of ordinal α and B a well-ordered set of ordinal β.

Theorem 4.13.2. *Addition is associative.*

Theorem 4.13.3. $\alpha + 0 = \alpha$

Theorem 4.13.4. $0 + \alpha = \alpha$

Theorem 4.13.5. *For λ a limit ordinal we have* $\alpha + \lambda = \sup_{\beta<\lambda}(\alpha + \beta)$

Theorem 4.13.6. *For any α, the class term that maps β to $\alpha + \beta$ is normal.*

Theorem 4.13.7. $\beta < \gamma$ *iff* $\alpha + \beta < \alpha + \gamma$.

Theorem 4.13.8. *If $\beta \leq \gamma$ then $\beta + \alpha \leq \gamma + \alpha$.*

Theorem 4.13.9 (Subtraction Theorem)**.** *If $\alpha < \beta$ then there exists a unique δ such that $\alpha + \delta < \beta$.*

Definition 4.13.10 (Product)**.** Let α and β be ordinals. The *sum* $\alpha + \beta$ is the ordinal of $A \times B$ ordered under the Hebrew lexicographic order, where A is a well-ordered set of ordinal α and B a well-ordered set of ordinal β.

Theorem 4.13.11. *Multiplication is associative.*

Theorem 4.13.12. *Multiplication distributes over addition on the left.*

Theorem 4.13.13. $\alpha 1 = \alpha$

Theorem 4.13.14. $1\alpha = \alpha$

Theorem 4.13.15. $\alpha 0 = 0$

Theorem 4.13.16. $0\alpha = 0$

Theorem 4.13.17. *For λ a limit ordinal, we have* $\alpha\lambda = \sup_{\beta<\lambda}(\alpha\beta)$.

Theorem 4.13.18. *For $\alpha > 0$, the class term that maps β to $\alpha\beta$ is normal.*

Theorem 4.13.19. *If $\alpha > 0$, then $\beta < \gamma$ iff $\alpha\beta < \alpha\gamma$.*

Theorem 4.13.20. *If $\beta \leq \gamma$ then $\beta\alpha \leq \gamma\alpha$.*

Theorem 4.13.21 (Division Theorem)**.** *For any ordinals α and δ with $\delta \neq 0$, there exist unique ordinals β and γ with $\gamma < \delta$ and $\alpha = \delta\beta + \gamma$.*

Definition 4.13.22 (Exponentiation)**.** For ordinals α and β, define the ordinal α^β by transfinite recursion on β by:

$$\begin{aligned} \alpha^0 &= 1 \\ \alpha^{\beta+1} &= \alpha^\beta + \alpha \\ \alpha^\lambda &= \sup_{\beta<\lambda} \alpha^\beta \end{aligned}$$

for λ a limit ordinal.

Theorem 4.13.23. *For $\alpha > 1$, the class term that maps β to α^β is normal.*

Theorem 4.13.24. *If $\alpha > 1$, then $\beta < \gamma$ iff $\alpha^\beta < \alpha^\gamma$.*

Theorem 4.13.25. *If $\beta \leq \gamma$ then $\beta^\alpha \leq \gamma^\alpha$.*

Theorem 4.13.26 (Logarithm Theorem)**.** *Let α and β be ordinals with $\alpha \neq 0$ and $\beta > 1$. Then there exist unique ordinals γ, δ and ρ such that $\delta \neq 0$, $\delta < \beta$, $\rho < \beta^\gamma$, and $\alpha = \beta^\gamma \delta + \rho$.*

Theorem 4.13.27.

$$\alpha^{\beta+\gamma} = \alpha^\beta \alpha^\gamma$$

Theorem 4.13.28.

$$(\alpha^\beta)^\gamma = \alpha^{\beta\gamma}$$

4.14 Beth Cardinals

Definition 4.14.1. Define the cardinal $\beth_\alpha$ for every ordinal α by:

$$\begin{aligned} \beth_0 &= \aleph_0 \\ \beth_{\alpha+1} &= 2^{\beth_\alpha} \\ \beth_\lambda &= \sup_{\alpha<\lambda} \beth_\alpha \end{aligned}$$

for λ a limit ordinal.

Lemma 4.14.2. *For any ordinal α we have* $\operatorname{card} V_{\omega+\alpha} = \beth_\alpha$.

4.15 Cofinality

Definition 4.15.1 (Cofinality)**.** For λ a limit ordinal, the *cofinality* of λ, $\operatorname{cf}\lambda$, is the least cardinal κ such that λ is the supremum of a set of κ smaller ordinals.

We extend cf to all the ordinals by setting $\operatorname{cf} 0 = 0$ and $\operatorname{cf}(\alpha+1) = 1$.

Theorem 4.15.2. *For any limit ordinal λ we have* $\operatorname{cf}\aleph_\lambda = \operatorname{cf}\lambda$.

Lemma 4.15.3. *Let λ be a limit ordinal. Then* $\operatorname{cf}\lambda$ *is the least ordinal α such that there exists an increasing α-sequence of ordinals with limit λ.*

Theorem 4.15.4. *Let λ be an infinite cardinal. Then* $\operatorname{cf}\lambda$ *is the least cardinal number κ such that λ can be partitioned into κ sets each of cardinality $< \lambda$.*

Theorem 4.15.5 (König's Theorem)**.** *Let κ be an infinite cardinal. Then $\kappa < 2^{\operatorname{cf}\kappa}$.*

Corollary 4.15.5.1. $2^{\aleph_0} \neq \aleph_\omega$.

Definition 4.15.6 (Regular)**.** A cardinal κ is *regular* iff $\operatorname{cf}\kappa = \kappa$.

Theorem 4.15.7. *For any ordinal λ, we have* $\operatorname{cf}\lambda$ *is a regular cardinal.*

Definition 4.15.8 (Singular)**.** A cardinal κ is *singular* iff $\operatorname{cf}\kappa < \kappa$.

Theorem 4.15.9. *For any ordinal α we have $\aleph_{\alpha+1}$ is a regular cardinal.*

4.16 Inaccessible Cardinals

Definition 4.16.1 (Inaccessible)**.** A cardinal number κ is *inaccessible* iff:

- $\kappa > \aleph_0$
- For every cardinal $\lambda < \kappa$ we have $2^\lambda < \kappa$
- κ is regular.

Lemma 4.16.2. *If κ is inaccessible and $\alpha < \kappa$ then $\beth_\alpha < \kappa$.*

Lemma 4.16.3. *If κ is inaccessible and $A \in V_\kappa$ then* $\operatorname{card} A < \kappa$*.*

Theorem 4.16.4. *If κ is inaccessible then V_κ is a model of ZF.*

4.17 Directed Set

Definition 4.17.1 (Directed Set)**.** A preodered set P is *directed* iff, for all $a, b \in P$, there exists $c \in P$ such that $a \leq c$ and $b \leq c$.

Proposition 4.17.2. *Every linearly ordered set is directed.*

Proposition 4.17.3. *For any set A, the $\mathcal{P}A$ under $\subseteq$ is directed.*

4.18 Cofinal Set

Definition 4.18.1 (Cofinal)**.** Let A be a preordered set and $B \subseteq A$. Then B is *cofinal* if and only if, for every $x \in A$, there exists $y \in B$ such that $x \leq y$.

Proposition 4.18.2. *If A is a directed preordered set and $B \subseteq A$ is cofinal then B is directed.*

PROOF:
$\langle 1 \rangle 1.$ LET: $x, y \in B$
$\langle 1 \rangle 2.$ PICK $z \in A$ such that $x \leq z$ and $y \leq z$
$\langle 1 \rangle 3.$ PICK $z' \in B$ such that $z \leq z'$
$\langle 1 \rangle 4.$ $x \leq z'$ and $y \leq z'$
□

Chapter 5

Natural Numbers

5.1 Successors

Definition 5.1.1 (Successor (Pairing, Union))**.** For any set a, its *Successor* a^+ is the set $a \cup \{a\}$

Theorem 5.1.2 (Pairing, Union)**.** *If a is a transitive set then $\bigcup(a^+) = a$.*

Proof:

$$\begin{aligned}
\bigcup(a^+) &= \bigcup(a \cup \{a\}) \\
&= \bigcup a \cup \bigcup \{a\} \\
&= \bigcup a \cup a \\
&= a && (\bigcup a \subseteq a) \square
\end{aligned}$$

Theorem 5.1.3. *If A is a transitive set then A^+ is transitive.*

Proof: If A is transitive then $\bigcup(A^+) = A \subseteq A^+$. $\square$

5.2 Inductive Sets

Definition 5.2.1 (Inductive (Extensionality, Empty Set, Pairing, Union))**.** A set A is *inductive* iff $\emptyset \in A$ and, for every $a \in A$, we have $a^+ \in A$.

Axiom 5.2.2 (Axiom of Infinity (Extensionality, Empty Set, Pairing, Union))**.** *There exists an inductive set.*

5.3 Natural Numbers

Definition 5.3.1 (Natural Number (Extensionality, Empty Set, Pairing, Union))**.** A *natural number* is a set that belongs to every inductive set.

We write $\mathbb{N}$ for the class of all natural numbers.

Theorem 5.3.2 (Extensionality, Empty Set, Pairing, Union, Infinity, Subset)**.** *The class of natural numbers is a set.*

PROOF:
⟨1⟩1. PICK an inductive set I.
  PROOF: By the Axiom of Infinity.
⟨1⟩2. $\mathbb{N} \subseteq I$
⟨1⟩3. Q.E.D.
  PROOF: By a Subset Axiom.
□

Theorem 5.3.3 (Extensionality, Empty Set, Pairing, Union, Infinity, Subset)**.** *The set* $\mathbb{N}$ *is inductive.*

PROOF:
⟨1⟩1. $\emptyset \in \mathbb{N}$
  PROOF: Since $\emptyset$ is a member of every inductive set.
⟨1⟩2. For all $n \in \mathbb{N}$ we have $n^+ \in \mathbb{N}$
  PROOF: If n is a member of every inductive set then so is n^+.
□

Theorem 5.3.4 (Extensionality, Empty Set, Pairing, Union, Infinity, Subset)**.** *The set* $\mathbb{N}$ *is a subset of every inductive set.*

PROOF: Immediate from definition. □

Corollary 5.3.4.1 (Proof by Induction (Extensionality, Empty Set, Pairing, Union, Infinity, Subset))**.** *If* $A \subseteq \mathbb{N}$ *and* A *is inductive then* $A = \mathbb{N}$*.*

Definition 5.3.5 (Zero (Empty Set))**.** The natural number *zero*, 0, is defined to be $\emptyset$.

Theorem 5.3.6 (Extensionality, Empty Set, Pairing, Union, Infinity, Subset)**.** *Every natural number except 0 is a successor of a natural number.*

PROOF: The set $\{x \in \mathbb{N} \mid x = 0 \vee \exists y \in \mathbb{N}. x = y^+\}$ is inductive. □

Theorem 5.3.7 (Extensionality, Empty Set, Pairing, Union, Infinity, Subset)**.** *Every natural number is transitive.*

PROOF: By induction using Theorem 5.1.3. □

Theorem 5.3.8 (Extensionality, Empty Set, Pairing, Union, Infinity, Subset)**.** *The set* $\mathbb{N}$ *is transitive.*

PROOF:
⟨1⟩1. For every natural number n and every $m \in n$ then m is a natural number.
  ⟨2⟩1. Every member of $\emptyset$ is a natural number.
    PROOF: Vacuous.
  ⟨2⟩2. If n is a natural number and a set of natural numbers then n^+ is a set of natural numbers.

PROOF: From the definition of n^+.

$\langle 2\rangle 3$. Q.E.D.

PROOF: By induction.

□

Theorem 5.3.9 (Extensionality, Empty Set, Pairing, Union, Infinity, Subset)**.** *Let A be a set, $a \in A$, and $F : A \to A$. Then there exists a unique function $h : \mathbb{N} \to A$ such that $h(0) = a$ and, for all $n \in \mathbb{N}$, we have $h(n^+) = F(h(n))$.*

PROOF:

$\langle 1\rangle 1$. Call a function v *acceptable* iff $\mathrm{dom}\, v \subseteq \mathbb{N}$, $\mathrm{ran}\, v \subseteq A$, and:

1. If $0 \in \mathrm{dom}\, v$ then $v(0) = a$.
2. For all $n \in \mathbb{N}$, if $n^+ \in \mathrm{dom}\, v$ then $n \in \mathrm{dom}\, v$ and $v(n^+) = F(v(n))$.

$\langle 1\rangle 2$. LET: $\mathcal{K}$ be the set of all acceptable functions.

$\langle 1\rangle 3$. LET: $h = \bigcup \mathcal{K}$

$\langle 1\rangle 4$. h is a function.

$\langle 2\rangle 1$. If $(0, y) \in h$ and $(0, y') \in h$ then $y = y'$

PROOF: We have $y = y' = a$.

$\langle 2\rangle 2$. For any natural number n, if there is at most one y such that $(n, y) \in h$, then there is at most one y such that $(n^+, y) \in h$

$\langle 3\rangle 1$. LET: n be a natural number.

$\langle 3\rangle 2$. ASSUME: there is at most one y such that $(n, y) \in h$

$\langle 3\rangle 3$. ASSUME: (n^+, y) and (n^+, y') are in h)

$\langle 3\rangle 4$. PICK acceptable functions u and v such that $u(n^+) = y$ and $v(n^+) = y'$

$\langle 3\rangle 5$. $n \in \mathrm{dom}\, u$, $n \in \mathrm{dom}\, v$ and $y = F(u(n))$, $y' = F(v(n))$

$\langle 3\rangle 6$. $u(n) = v(n)$

PROOF: By the induction hypothesis $\langle 3\rangle 2$

$\langle 3\rangle 7$. $y = y'$

$\langle 2\rangle 3$. Q.E.D.

PROOF: By induction.

$\langle 1\rangle 5$. h is acceptable.

$\langle 2\rangle 1$. If $0 \in \mathrm{dom}\, h$ then $h(0) = a$

$\langle 2\rangle 2$. If $n^+ \in \mathrm{dom}\, h$ then $n \in \mathrm{dom}\, h$ and $h(n^+) = F(h(n))$

$\langle 3\rangle 1$. ASSUME: $n^+ \in \mathrm{dom}\, h$

$\langle 3\rangle 2$. PICK an acceptable v such that $n^+ \in \mathrm{dom}\, v$

$\langle 3\rangle 3$. $v(n^+) = F(v(n))$

$\langle 3\rangle 4$. $h(n^+) = F(h(n))$

$\langle 1\rangle 6$. $\mathrm{dom}\, h = \mathbb{N}$

$\langle 2\rangle 1$. $0 \in \mathrm{dom}\, h$

PROOF: Since $\{(0, a)\}$ is an acceptable function.

$\langle 2\rangle 2$. For all $n \in \mathrm{dom}\, h$ we have $n^+ \in \mathrm{dom}\, h$

$\langle 3\rangle 1$. ASSUME: $n \in \mathrm{dom}\, h$

$\langle 3\rangle 2$. LET: v be an acceptable function with $n \in \mathrm{dom}\, v$

$\langle 3\rangle 3$. ASSUME: without loss of generality $n^+ \notin \mathrm{dom}\, v$

$\langle 3\rangle 4$. $v \cup \{(n^+, F(v(n)))\}$ is acceptable

⟨3⟩5. $n^+ \in \operatorname{dom} v$

⟨1⟩7. If $h' : \mathbb{N} \to A$, $h'(0) = a$ and, for all $n \in \mathbb{N}$, we have $h'(n^+) = F(h'(n))$, then $h' = h$

PROOF: Prove $h(n) = h'(n)$ by induction on n.

□

5.4 Peano Systems

Definition 5.4.1 (Peano System)**.** A *Peano system* consists of a set N, an element $z \in N$, and a function $S : N \to N$ such that:

- S is one-to-one
- $z \notin \operatorname{ran} S$
- For any set $A \subseteq N$, if $z \in A$ and $S(A) \subseteq A$ then $A = N$.

Theorem 5.4.2. *$\mathbb{N}$ is a Peano system with zero 0 and successor $n \mapsto n^+$.*

Theorem 5.4.3. *For any Peano system (N, z, S), there exists a unique bijection $h : \mathbb{N} \cong N$ such that $h(0) = z$ and $S(h(n)) = h(n^+)$ for all n.*

5.5 Arithmetic

Definition 5.5.1 (Addition)**.** Define *addition* $+ : \mathbb{N}^2 \to \mathbb{N}$ recursively by

$$m + 0 = m$$
$$m + n^+ = (m + n)^+$$

for any $m, n \in \mathbb{N}$.

Theorem 5.5.2. *Addition is associative.*

Theorem 5.5.3. *Addition is commutative*

Definition 5.5.4 (Multiplication)**.** Define *multiplication* $\cdot : \mathbb{N}^2 \to \mathbb{N}$ recursively by

$$m0 = 0$$
$$mn^+ = mn + m$$

for any $m, n \in \mathbb{N}$

Theorem 5.5.5. *Multiplication is associative.*

Theorem 5.5.6. *Multiplication is commutative.*

Theorem 5.5.7. *Multiplication distributes over addition.*

Definition 5.5.8. For natural numbers m and n, we write $m < n$ iff $m \in n$. We write $m \leq n$ iff $m < n$ or $m = n$.

Theorem 5.5.9. *We have $m < n$ iff $m^+ < n^+$.*

Theorem 5.5.10. *We never have $n < n$.*

Theorem 5.5.11. *The ordering on $\mathbb{N}$ satisfies trichotomy; that is, for any m, n, exactly one of $m < n$, $m = n$, $n < m$ holds.*

Theorem 5.5.12. *For any natural numbers m and n, we have $m \leq n$ iff $m \subseteq n$.*

Theorem 5.5.13. *We have $m < n$ iff $m + p < n + p$.*

Corollary 5.5.13.1. *If $m + p = n + p$ then $m = n$.*

Theorem 5.5.14. *If $p \neq 0$ then $m < n$ iff $mp < np$.*

Corollary 5.5.14.1. *If $mp = np$ and $p \neq 0$ then $m = n$.*

Theorem 5.5.15 (Well-Ordering of $\mathbb{N}$)**.** *Any nonempty set $A \subseteq \mathbb{N}$ has a least element.*

Corollary 5.5.15.1. *There is no function $f : \mathbb{N} \to \mathbb{N}$ such that $f(n^+) < f(n)$ for all n.*

Theorem 5.5.16 (Strong Induction)**.** *Let $A \subseteq \mathbb{N}$. Suppose that, for every natural number n, if $\forall m < n . m \in A$ then $n \in A$. Then $A = \mathbb{N}$.*

Theorem 5.5.17 (Pigeonhole Principle)**.** *No natural number is equinumerous with a proper subset of itself.*

Proof: Prove by induction on n that if $f : n \to n$ is injective then it is surjective.
□

Chapter 6

Integers

Lemma 6.0.1. *Define* $\sim$ *on* $\mathbb{N}^2$ *by:* $(m, n) \sim (p, q)$ *iff* $m + q = n + p$. *Then* $\sim$ *is an equivalence relation on* $\mathbb{N}^2$.

Definition 6.0.2 (Integers)**.** The set $\mathbb{Z}$ of *integers* is $\mathbb{N}^2/\sim$.

Definition 6.0.3. Define *addition* $+ : \mathbb{Z}^2 \to \mathbb{Z}$ by: $(m, n)+(p, q) = (m+p, n+q)$.

Prove this is well-defined.

Theorem 6.0.4. *Addition is associative and commutative.*

Definition 6.0.5 (Zero)**.** The integer *zero* is $0 = (0, 0)$.

Theorem 6.0.6. *For any integer* a*, we have* $a + 0 = a$.

Theorem 6.0.7. *For any integer* a*, there exists a unique integer* b *such that* $a + b = 0$.

Definition 6.0.8 (Multiplication)**.** Define multiplication on $\mathbb{Z}$ by $(m, n)(p, q) = (mp + nq, mq + np)$.

Theorem 6.0.9. *Multiplication is associative, commutative and distributive over addition.*

Definition 6.0.10. The integer *one* is $1 = (1, 0)$.

Theorem 6.0.11. *For any integer* a *we have* $a1 = a$.

Theorem 6.0.12. $1 \neq 0$

Theorem 6.0.13. *Whenever* $ab = 0$ *then either* $a = 0$ *or* $b = 0$.

Definition 6.0.14. Define $<$ on $\mathbb{Z}$ by: $(m, n) < (p, q)$ iff $m + q < n + p$.

Theorem 6.0.15. *The relation* $<$ *is a strict linear ordering on* $\mathbb{Z}$.

Theorem 6.0.16. *We have* $a < b$ *iff* $< +c < b + c$.

Corollary 6.0.16.1. *If $a + c = b + c$ then $a = b$.*

Theorem 6.0.17. *If $0 < c$ then $a < b$ iff $ac < bc$.*

Corollary 6.0.17.1. *If $ac = bc$ and $c \neq 0$ then $a = b$.*

Definition 6.0.18. We identify any natural number n with the integer $(n, 0)$.

Theorem 6.0.19. *This embedding preserves 0, 1, addition, multiplication and the ordering.*

Chapter 7

Rational Numbers

Definition 7.0.1 (Rational Numbers)**.** The set of *rationals* $\mathbb{Q}$ is $\mathbb{Z}\times(\mathbb{Z}\setminus\{0\})/\sim$, where $(a,b) \sim (c,d)$ iff $ad = bc$.

Definition 7.0.2 (Addition)**.** Define addition on $\mathbb{Q}$ by: $(a,b) + (c,d) = (ad + bc, bd)$.

Theorem 7.0.3. *Addition is commutative and associative*

Definition 7.0.4. The rational number 0 is $(0,1)$.

Theorem 7.0.5. *For any rational q we have $q + 0 = q$.*

Theorem 7.0.6. *For any rational q, there exists a unique rational r such that $q + r = 0$.*

Definition 7.0.7. Define multiplication on $\mathbb{Q}$ by: $(a,b)(c,d) = (ac, bd)$.

Theorem 7.0.8. *Multiplication is commutative, associative and distributive over addition.*

Definition 7.0.9. The rational number 1 is $(1,1)$.

Theorem 7.0.10. *For every nonzero rational r, there exists a nonzero rational q such that $rq = 1$.*

Corollary 7.0.10.1. *If $qr = 0$ then either $q = 0$ or $r = 0$.*

Definition 7.0.11. Define $<$ on $\mathbb{Q}$ by: for b and d positive, $(a,b) < (c,d)$ iff $ad < bc$.

Theorem 7.0.12. *The relation $<$ is a strict linear ordering on $\mathbb{Q}$.*

Theorem 7.0.13. *We have $q < r$ iff $q + s < r + s$*

Corollary 7.0.13.1. *If $q + s = r + s$ then $q = r$.*

Theorem 7.0.14. *If $s > 0$ then we have $q < r$ iff $qs < rs$.*

Corollary 7.0.14.1. *If $qs = rs$ and $s \neq 0$ then $q = r$.*

Definition 7.0.15. We identify an integer n with the rational $(n, 1)$.

Theorem 7.0.16. *This embedding preserves zero, one, addition, multiplication and the ordering.*

Chapter 8

Real Numbers

Definition 8.0.1 (Dedekind Cut)**.** A *Dedekind cut* is a subset $X \subseteq \mathbb{Q}$ such that:

- X is nonempty
- $X \neq \mathbb{Q}$
- X is closed downward
- X has no largest element.

Definition 8.0.2 (Real Numbers)**.** The set of *real numbers* $\mathbb{R}$ is the set of all Dedekind cuts.

Definition 8.0.3. Define $<$ on $\mathbb{R}$ by: $x < y$ iff x is a proper subset of y.

Theorem 8.0.4. *The relation $<$ is a strict linear ordering on $\mathbb{R}$.*

Theorem 8.0.5. *Any bounded nonempty subset of $\mathbb{R}$ has a least upper bound.*

Definition 8.0.6. Define addition on $\mathbb{R}$ by: $x + y = \{q + r \mid q \in x, r \in y\}$.

Theorem 8.0.7. *Addition is associative and commutative.*

Definition 8.0.8. The zero real 0 is $\{q \in \mathbb{Q} \mid q < 0\}$.

Theorem 8.0.9. *For any $x \in \mathbb{R}$ we have $x + 0 = x$.*

Definition 8.0.10. Given a real x, define $-x = \{q \in \mathbb{Q} \mid \exists r > q. -r \notin x\}$.

Theorem 8.0.11. *For any real x we have $x + (-x) = 0$.*

Corollary 8.0.11.1. *If $x + z = y + z$ then $x = y$.*

Theorem 8.0.12. *We have $x < y$ iff $x + z < y + z$.*

Definition 8.0.13. Define the *absolute value* of a real x by $|x| = x \cup -x$.

Theorem 8.0.14. *For any real x we have $0 \leq |x|$.*

Definition 8.0.15. Define multiplication on $\mathbb{R}$ by:

- If x and y are nonnegative then

$$xy = 0 \cup \{qr \mid 0 \leq q, 0 \leq r, q \in x, r \in y\}$$

- If x and y are both negative then $xy = |x||y|$
- If one of x and y is negaive and the other not then $xy = -|x||y|$.

Theorem 8.0.16. *Multiplication is associative, commutative and distributive over addition.*

Definition 8.0.17. The real number 1 is $\{q \in \mathbb{Q} \mid q < 1\}$.

Theorem 8.0.18. $0 \neq 1$

Theorem 8.0.19. *For any real x we have $x1 = x$*

Theorem 8.0.20. *For any nonzero x, there exists a real y with $xy = 1$.*

Theorem 8.0.21. *If $0 < x$ then $y < z$ iff $xy < xz$.*

Definition 8.0.22. Identify a rational q with $\{r \in \mathbb{Q} \mid r < q\}$.

Theorem 8.0.23. *This embedding preserves zero, one, addition, multiplication and the ordering.*

8.1 The Cantor Set

Definition 8.1.1 (Cantor Set)**.** Define the sequence of sets $A_n \subseteq \mathbb{R}$ by

$$\begin{aligned} A_0 &= [0, 1] \\ A_n &= A_{n-1} - \bigcup_{k=0}^{3^{n-1}-1} ((3k+1)/3^n, (3k+2)/3^n) \end{aligned}$$

The *Cantor set* is $\bigcap_{n=0}^{\infty} A_n$.

Proposition 8.1.2. *The set A_n is a union of finitely many disjoint closed intervals of length $1/3^n$, and the endpoints of these intervals lie in C.*

Proof: An easy induction on n. □

Chapter 9

Finite Sets

Definition 9.0.1 (Finite)**.** A set is *finite* iff it is equinumerous with a natural number; otherwise it is *infinite*.

Theorem 9.0.2. *No finite set is equinumerous with a proper subset of itself.*

PROOF: From the Pigeonhole Principle.

Corollary 9.0.2.1. *The set* $\mathbb{N}$ *is infinite.*

Corollary 9.0.2.2. *A finite set is equinumerous with a unique natural number.*

Lemma 9.0.3. *If* A *is a proper subset of a natural number* n *then there exists* $m < n$ *such that* $C \equiv m$*.*

Corollary 9.0.3.1. *A subset of a finite set is finite.*

Theorem 9.0.4 (Regularity)**.** *There is no function* f *with domain* $\mathbb{N}$ *such that* $f(n+1) \in f(n)$ *for all* n*.*

PROOF:
$\langle 1 \rangle 1$. ASSUME: for a contradiction f is a function with domain $\mathbb{N}$ such that $f(n+1) \in f(n)$ for all n.
$\langle 1 \rangle 2$. PICK $m \in \operatorname{ran} f$ such that $m \cap \operatorname{ran} f = \emptyset$
PROOF: By the Axiom of Regularity.
$\langle 1 \rangle 3$. PICK $n \in \mathbb{N}$ such that $f(n) = m$
$\langle 1 \rangle 4$. $f(n+1) \in m \cap \operatorname{ran} f$
$\langle 1 \rangle 5$. Q.E.D.
PROOF: This is a contradiction.
□

Theorem 9.0.5. *A relation* R *is well-founded if and only if there is no function* f *with domain* $\mathbb{N}$ *such that, for all* $n \in \mathbb{N}$*, we have* $f(n+1) R f(n)$*.*

9.1 Real Analysis

Definition 9.1.1. Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences (a_n) such that $a_n = 0$ for all but finitely many n.

9.2 Group Theory

Definition 9.2.1. Given a group G and sets $A, B \subseteq G$, let $AB = \{ab \mid a \in A, b \in B\}$.

Definition 9.2.2. Given a group G and a set $A \subseteq G$, let $A^{-1} = \{a^{-1} \mid a \in A\}$.

9.3 Topological Spaces

Definition 9.3.1 (Topology)**.** A *topology* on a set X is a set $\mathcal{T} \subseteq \mathcal{P}X$ such that:

- $X \in \mathcal{T}$.
- For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$.
- For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$.

We call the elements of X *points* and the elements of $\mathcal{T}$ *open sets*.

Definition 9.3.2 (Topological Space)**.** A *topological space* X consists of a set X and a topology on X.

Definition 9.3.3 (Discrete Space)**.** For any set X, the *discrete* topology on X is $\mathcal{P}X$.

Definition 9.3.4 (Indiscrete Space)**.** For any set X, the *indiscrete* or *trivial* topology on X is $\{\emptyset, X\}$.

Definition 9.3.5 (Finite Complement Topology)**.** For any set X, the *finite complement topology* on X is $\{U \in \mathcal{P}X \mid X \setminus U \text{ is finite}\} \cup \{\emptyset\}$.

Definition 9.3.6 (Countable Complement Topology)**.** For any set X, the *countable complement topology* on X is $\{U \in \mathcal{P}X \mid X \setminus U \text{ is countable}\} \cup \{\emptyset\}$.

Definition 9.3.7 (Finer, Coarser)**.** Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a given set X. If $\mathcal{T}' \supseteq \mathcal{T}$, we say that $\mathcal{T}'$ is *finer* than $\mathcal{T}$; if $\mathcal{T}'$ *properly* contains $\mathcal{T}$, we say that $\mathcal{T}'$ is *strictly* finer than $\mathcal{T}$. We also say that $\mathcal{T}$ is *coarser* than $\mathcal{T}'$, or *stricly* coarser, in these two respective situations. We say $\mathcal{T}$ is *comparable* with $\mathcal{T}'$ if either $\mathcal{T}' \supseteq \mathcal{T}$ or $\mathcal{T} \supseteq \mathcal{T}'$.

Lemma 9.3.8. *Let X be a topological space and $U \subseteq X$. Then U is open if and only if, for all $x \in U$, there exists an open set V such that $x \in V \subseteq U$.*

Proof:
$\langle 1 \rangle 1.$ $\Rightarrow$
 Proof: Take $V = U$
$\langle 1 \rangle 2.$ $\Leftarrow$
 Proof: We have $U = \bigcup \{V \text{ open in } X \mid V \subseteq U\}$.
$\square$

Lemma 9.3.9. *Let X be a set and $\mathcal{T}$ a nonempty set of topologies on X. Then $\bigcap \mathcal{T}$ is a topology on X, and is the finest topology that is coarser than every member of $\mathcal{T}$.*

Proof:
$\langle 1 \rangle 1.$ $X \in \bigcap \mathcal{T}$
 Proof: Since X is in every member of $\mathcal{T}$.

$\langle 1\rangle 2.$ $\bigcap \mathcal{T}$ is closed under union.
  $\langle 2\rangle 1.$ LET: $\mathcal{U} \subseteq \bigcap \mathcal{T}$
  $\langle 2\rangle 2.$ For all $T \in \mathcal{T}$ we have $\mathcal{U} \subseteq T$
  $\langle 2\rangle 3.$ For all $T \in \mathcal{T}$ we have $\bigcup \mathcal{U} \in T$
  $\langle 2\rangle 4.$ $\bigcup \mathcal{U} \in \bigcap \mathcal{T}$
$\langle 1\rangle 3.$ $\bigcap \mathcal{T}$ is closed under binary intersection.
  $\langle 2\rangle 1.$ LET: $U, V \in \bigcap \mathcal{T}$
  $\langle 2\rangle 2.$ For all $T \in \mathcal{T}$ we have $U, V \in T$
  $\langle 2\rangle 3.$ For all $T \in \mathcal{T}$ we have $U \cap V \in T$
  $\langle 2\rangle 4.$ $U \cap V \in \bigcap \mathcal{T}$
□

Lemma 9.3.10. *Let X be a set and $\mathcal{T}$ a set of topologies on X. Then there exists a unique coarsest topology that is finer than every member of $\mathcal{T}$.*

PROOF: The required topology is given by
$\bigcap \{T \in \mathcal{PP}X \mid T \text{ is a topology on } X \text{ that is finer than every member of } \mathcal{T}\}$,
The set is nonempty since it contains the discrete topology. □

Definition 9.3.11 (Neighbourhood)**.** A *neighbourhood* of a point x is an open set that contains x.

9.4 Closed Set

Definition 9.4.1 (Closed Set)**.** Let X be a topological space and $A \subseteq X$. Then A is *closed* if and only if $X \setminus A$ is open.

Lemma 9.4.2. *The empty set is closed.*

PROOF: Since the whole space X is always open. □

Lemma 9.4.3. *The topological space X is closed.*

PROOF: Since $\emptyset$ is open. □

Lemma 9.4.4. *The intersection of a nonempty set of closed sets is closed.*

PROOF: Let $\mathcal{C}$ be a nonempty set of closed sets. Then $X \setminus \bigcap \mathcal{C} = \bigcup \{X \setminus C \mid C \in \mathcal{C}\}$ is open. □

Lemma 9.4.5. *The union of two closed sets is closed.*

PROOF: Let C and D be closed. Then $X \setminus (C \cup D) = (X \setminus C) \cap (X \setminus D)$ is open. □

Proposition 9.4.6. *Let X be a set and $\mathcal{C} \subseteq \mathcal{P}X$ a set such that:*

1. $\emptyset \in \mathcal{C}$
2. $X \in \mathcal{C}$

3. *For all $\mathcal{A} \subseteq \mathcal{C}$ nonempty we have $\bigcap \mathcal{A} \in \mathcal{C}$*

4. *For all $C, D \in \mathcal{C}$ we have $C \cup D \in \mathcal{C}$.*

Then there exists a unique topology $\mathcal{T}$ such that $\mathcal{C}$ is the set of closed sets, namely

$$\mathcal{T} = \{X \setminus C \mid C \in \mathcal{C}\}$$

PROOF:
⟨1⟩1. LET: $\mathcal{T} = \{X \setminus C \mid C \in \mathcal{C}\}$
⟨1⟩2. $\mathcal{T}$ is a topology
⟨2⟩1. $X \in \mathcal{T}$
PROOF: Since $\emptyset \in \mathcal{C}$
⟨2⟩2. For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$
⟨3⟩1. LET: $\mathcal{U} \subseteq \mathcal{T}$
⟨3⟩2. CASE: $\mathcal{U} = \emptyset$
PROOF: In this case $\bigcup \mathcal{U} = \emptyset \in \mathcal{T}$ since $X \in \mathcal{C}$
⟨3⟩3. CASE: $\mathcal{U} \neq \emptyset$
PROOF: In this case $X \setminus \bigcup \mathcal{U} = \bigcap \{X \setminus U \mid U \in \mathcal{U}\} \in \mathcal{C}$.
⟨2⟩3. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$
PROOF: Since $X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V) \in \mathcal{C}$.
⟨1⟩3. $\mathcal{C}$ is the set of all closed sets in $\mathcal{T}$
PROOF:

$$\begin{aligned} & C \text{ is closed in } \mathcal{T} \\ \Leftrightarrow & X \setminus C \in \mathcal{T} \\ \Leftrightarrow & C \in \mathcal{C} \end{aligned}$$

⟨1⟩4. If $\mathcal{T}'$ is a topology and $\mathcal{C}$ is the set of closed sets in $\mathcal{T}'$ then $\mathcal{T}' = \mathcal{T}$
PROOF: We have

$$\begin{aligned} & U \in \mathcal{T} \\ \Leftrightarrow & X \setminus U \in \mathcal{C} \\ \Leftrightarrow & X \setminus U \text{ is closed in } \mathcal{T}' \\ \Leftrightarrow & U \in \mathcal{T}' \end{aligned}$$

□

Proposition 9.4.7. *If U is open and A is closed then $U \setminus A$ is open.*

PROOF: $U \setminus A = U \cap (X \setminus A)$ is the intersection of two open sets. □

Proposition 9.4.8. *If U is open and A is closed then $A \setminus U$ is closed.*

PROOF: $A \setminus U = A \cap (X \setminus U)$ is the intersection of two closed sets. □

9.5 Interior

Definition 9.5.1 (Interior)**.** Let X be a topological space and $A \subseteq X$. The *interior* of A, $\operatorname{Int} A$, is the union of all the open subsets of A.

Lemma 9.5.2. *The interior of a set is open.*

Proof: It is a union of open sets. □

Lemma 9.5.3.

$$\text{Int}\, A \subseteq A$$

Proof: Immediate from definition. □

Lemma 9.5.4. *If U is open and $U \subseteq A$ then $U \subseteq \text{Int}\, A$*

Proof: Immediate from definition. □

Lemma 9.5.5. *A set A is open if and only if $A = \text{Int}\, A$.*

Proof: If $A = \text{Int}\, A$ then A is open by Lemma 9.5.2. Conversely if A is open then $A \subseteq \text{Int}\, A$ by the definition of interior and so $A = \text{Int}\, A$.

9.6 Closure

Definition 9.6.1 (Closure)**.** Let X be a topological space and $A \subseteq X$. The *closure* of A, $\overline{A}$, is the intersection of all the closed sets that include A.

This intersection exists since X is a closed set that includes A (Lemma 9.4.3).

Lemma 9.6.2. *The closure of a set is closed.*

Proof: Dual to Lemma 9.5.2. □

Lemma 9.6.3.

$$A \subseteq \overline{A}$$

Proof: Immediate from definition. □

Lemma 9.6.4. *If C is closed and $A \subseteq C$ then $\overline{A} \subseteq C$.*

Proof: Immediate from definition. □

Lemma 9.6.5. *A set A is closed if and only if $A = \overline{A}$.*

Proof: Dual to Lemma 9.5.5. □

Theorem 9.6.6. *Let X be a topological space, $A \subseteq X$ and $x \in X$. Then $x \in \overline{A}$ if and only if every neighbourhood of x intersects A.*

Proof: We have

$$\begin{aligned}
& x \in \overline{A} \\
\Leftrightarrow & \forall C. C \text{ closed } \wedge A \subseteq C \Rightarrow x \in C \\
\Leftrightarrow & \forall U. U \text{ open } \wedge A \cap U = \emptyset \Rightarrow x \notin U \\
\Leftrightarrow & \forall U. U \text{ open } \wedge x \in U \Rightarrow U \text{ intersects } A
\end{aligned}$$

□

Proposition 9.6.7. *If $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$.*

PROOF: This holds because $\overline{B}$ is a closed set that includes A. ∎

Proposition 9.6.8.

$$\overline{A \cup B} = \overline{A} \cup \overline{B}$$

PROOF:
⟨1⟩1. $\overline{A} \subseteq \overline{A \cup B}$
  PROOF: By Proposition 9.6.7.
⟨1⟩2. $\overline{B} \subseteq \overline{A \cup B}$
  PROOF: By Proposition 9.6.7.
⟨1⟩3. $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$
  ⟨2⟩1. LET: $x \in \overline{A \cup B}$
  ⟨2⟩2. ASSUME: $x \notin \overline{A}$
    PROVE: $x \in \overline{B}$
  ⟨2⟩3. PICK a neighbourhood U of x that does not intersect A
  ⟨2⟩4. LET: V be any neighbourhood of x
  ⟨2⟩5. $U \cap V$ is a neighbourhood of x
  ⟨2⟩6. $U \cap V$ intersects $A \cup B$
    PROOF: From ⟨2⟩1 and Theorem 9.6.6.
  ⟨2⟩7. $U \cap V$ intersects B
    PROOF: From ⟨2⟩3
  ⟨2⟩8. V intersects B
  ⟨2⟩9. Q.E.D.
    PROOF: We have $x \in \overline{B}$ from Theorem 9.6.6.
∎

Proposition 9.6.9. *Let X be a topological space. Let $\mathcal{D}$ be a set of subsets of X that is maximal with respect to the finite intersection property. Let $x \in X$. Then the following are equivalent:*

1. *For all $D \in \mathcal{D}$ we have $x \in \overline{D}$*
2. *Every neighbourhood of x is in $\mathcal{D}$.*

PROOF:
⟨1⟩1. $1 \Rightarrow 2$
  ⟨2⟩1. For all $D \in \mathcal{D}$ we have $x \in \overline{D}$
  ⟨2⟩2. LET: U be a neighbourhood of x
  ⟨2⟩3. $\mathcal{D} \cup \{U\}$ satisfies the finite intersection property.
    ⟨3⟩1. LET: $D_1, \ldots, D_n \in \mathcal{D}$
    ⟨3⟩2. $D_1 \cap \cdots \cap D_n \in \mathcal{D}$
      PROOF: Lemma 3.24.3.
    ⟨3⟩3. $x \in \overline{D_1 \cap \cdots \cap D_n}$
      PROOF: ⟨2⟩1, ⟨3⟩2
    ⟨3⟩4. $D_1 \cap \cdots \cap D_n \cap U \neq \emptyset$
      PROOF: Theorem 9.6.6, ⟨2⟩2, ⟨3⟩3.

⟨2⟩4. $\mathcal{D} = \mathcal{D} \cup \{U\}$

PROOF: By the maximality of $\mathcal{D}$.

⟨2⟩5. $U \in \mathcal{D}$

⟨1⟩2. $2 \Rightarrow 1$

⟨2⟩1. ASSUME: Every neighbourhood of x is in $\mathcal{D}$.

⟨2⟩2. LET: $D \in \mathcal{D}$

⟨2⟩3. Every neighbourhood of x intersects D.

PROOF: From ⟨2⟩1, ⟨2⟩2 and the fact that $\mathcal{D}$ satisfies the finite intersection property.

⟨2⟩4. $x \in \overline{D}$

PROOF: Theorem 9.6.6, ⟨2⟩3.

□

9.7 Boundary

Definition 9.7.1 (Boundary)**.** The *boundary* of a set A is the set $\partial A = \overline{A} \cap \overline{X \setminus A}$.

Proposition 9.7.2.

$$\operatorname{Int} A \cap \partial A = \emptyset$$

PROOF: Since $\overline{X \setminus A} = X \setminus \operatorname{Int} A$. □

Proposition 9.7.3.

$$\overline{A} = \operatorname{Int} A \cup \partial A$$

PROOF:

$$\begin{aligned}
\operatorname{Int} A \cup \partial A &= \operatorname{Int} A \cup (\overline{A} \cap \overline{X \setminus A}) \\
&= (\operatorname{Int} A \cup \overline{A}) \cap (\operatorname{Int} A \cup \overline{X \setminus A}) \\
&= \overline{A} \cap X \\
&= \overline{A}
\end{aligned}$$

Proposition 9.7.4. *$\partial A = \emptyset$ if and only if A is open and closed.*

PROOF: If $\partial A = \emptyset$ then $\overline{A} = \operatorname{Int} A$ by Proposition 9.7.3.

Proposition 9.7.5. *A set U is open if and only if $\partial U = \overline{U} \setminus U$.*

PROOF:

$$\begin{aligned}
&\partial U = \overline{U} \setminus U \\
\Leftrightarrow &\overline{U} \setminus \operatorname{Int} U = \overline{U} \setminus U && \text{(Propositions 9.7.2, 9.7.3)} \\
\Leftrightarrow &\operatorname{Int} U = U && \square
\end{aligned}$$

9.8 Limit Points

Definition 9.8.1 (Limit Point)**.** Let X be a topological space, $a \in X$ and $A \subseteq X$. Then a is a *limit point*, *cluster point* or *point of accumulation* for A if and only if every neighbourhood of a intersects A at a point other than a.

Lemma 9.8.2. *The point a is an accumulation point for A if and only if $a \in \overline{A \setminus \{a\}}$.*

PROOF: From Theorem 9.6.6. ∎

Theorem 9.8.3. *Let X be a topological space and $A \subseteq X$. Let A' be the set of all limit points of A. Then $\overline{A} = A \cup A'$.*

PROOF:
$\langle 1 \rangle 1$. For all $x \in \overline{A}$, if $x \notin A$ then $x \in A'$
PROOF: From Theorem 9.6.6.
$\langle 1 \rangle 2$. $A \subseteq \overline{A}$
PROOF: Lemma 9.6.3.
$\langle 1 \rangle 3$. $A' \subseteq \overline{A}$
PROOF: From Theorem 9.6.6.
∎

Corollary 9.8.3.1. *A set is closed if and only if it contains all its limit points.*

Proposition 9.8.4. *In an indiscrete topology, every point is a limit point of any set that has more than one point.*

PROOF: Let X be an indiscrete space. Let A be a set with more than one point and x be a point. The only neighbourhood of x is X, which must intersect A at a point other than x. ∎

Lemma 9.8.5. *Let X be a topological space and $A \subseteq B \subseteq X$. Then every limit point of A is a limit point of B.*

PROOF: Immediate from definitions. ∎

9.9 Basis for a Topology

Definition 9.9.1 (Basis)**.** If X is a set, a *basis* for a topology on X is a set $\mathcal{B} \subseteq \mathcal{P}X$ called *basis elements* such that

1. For all $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$.
2. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

If $\mathcal{B}$ satisfies these two conditions, then we define the topology *generated* by $\mathcal{B}$ to be $\mathcal{T} = \{U \in \mathcal{P}X \mid \forall x \in U. \exists B \in \mathcal{B}. x \in B \subseteq U\}$.

We prove this is a topology.

PROOF:
⟨1⟩1. $X \in \mathcal{T}$
PROOF: For all $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B \subseteq X$ by condition 1.
⟨1⟩2. For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$
⟨2⟩1. LET: $\mathcal{U} \subseteq \mathcal{T}$
⟨2⟩2. LET: $x \in \bigcup \mathcal{U}$
⟨2⟩3. PICK $U \in \mathcal{U}$ such that $x \in U$
⟨2⟩4. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$
PROOF: Since $U \in \mathcal{T}$ by ⟨2⟩1 and ⟨2⟩3.
⟨2⟩5. $x \in B \subseteq \bigcup \mathcal{U}$
⟨1⟩3. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$
⟨2⟩1. LET: $U, V \in \mathcal{T}$
⟨2⟩2. LET: $x \in U \cap V$
⟨2⟩3. PICK $B_1 \in \mathcal{B}$ such that $x \in B_1 \subseteq U$
⟨2⟩4. PICK $B_2 \in \mathcal{B}$ such that $x \in B_2 \subseteq V$
⟨2⟩5. PICK $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$
PROOF: By condition 2.
⟨2⟩6. $x \in B_3 \subseteq U \cap V$
□

Lemma 9.9.2. *Let X be a set. Let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ is the set of all unions of subsets of $\mathcal{B}$.*

PROOF:
⟨1⟩1. For all $U \in \mathcal{T}$, there exists $\mathcal{A} \subseteq \mathcal{B}$ such that $U = \bigcup \mathcal{A}$
⟨2⟩1. LET: $U \in \mathcal{T}$
⟨2⟩2. LET: $\mathcal{A} = \{B \in \mathcal{B} \mid B \subseteq U\}$
⟨2⟩3. $U \subseteq \bigcup \mathcal{A}$
⟨3⟩1. LET: $x \in U$
⟨3⟩2. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$
PROOF: Since $\mathcal{B}$ is a basis for $\mathcal{T}$.
⟨3⟩3. $x \in B \in \mathcal{A}$
⟨2⟩4. $\bigcup \mathcal{A} \subseteq U$
PROOF: From the definition of $\mathcal{A}$ (⟨2⟩2).
⟨1⟩2. For all $\mathcal{A} \subseteq \mathcal{B}$ we have $\bigcup \mathcal{A} \in \mathcal{T}$
⟨2⟩1. $\mathcal{B} \subseteq \mathcal{T}$
PROOF: If $B \in \mathcal{B}$ and $x \in B$, then there exists $B' \in \mathcal{B}$ such that $x \in B' \subseteq B$, namely $B' = B$.
⟨2⟩2. Q.E.D.
PROOF: Since $\mathcal{T}$ is closed under union.
□

Corollary 9.9.2.1. *Let X be a set. Let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ is the coarsest topology that includes $\mathcal{B}$.*

Proof: Since every topology that includes $\mathcal{B}$ includes all unions of subsets of $\mathcal{B}$. □

Lemma 9.9.3. *Let X be a topological space. Suppose that $\mathcal{C}$ is a set of open sets such that, for every open set U and every point $x \in U$, there exists $C \in \mathcal{C}$ such that $x \in C \subseteq U$. Then $\mathcal{C}$ is a basis for the topology on X.*

Proof:
⟨1⟩1. For all $x \in X$, there exists $C \in \mathcal{C}$ such that $x \in C$
  Proof: Immediate from hypothesis.
⟨1⟩2. For all $C_1, C_2 \in \mathcal{C}$ and $x \in C_1 \cap C_2$, there exists $C_3 \in \mathcal{C}$ such that $x \in C_3 \subseteq C_1 \cap C_2$
  Proof: Since $C_1 \cap C_2$ is open.
⟨1⟩3. Every open set is open in the topology generated by $\mathcal{C}$
  Proof: Immediate from hypothesis.
⟨1⟩4. Every union of a subset of $\mathcal{C}$ is open.
  Proof: Since every member of $\mathcal{C}$ is open.
□

Lemma 9.9.4. *Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$ respectively on the set X. Then the following are equivalent.*

1. *$\mathcal{T} \subseteq \mathcal{T}'$*
2. *For all $B \in \mathcal{B}$ and $x \in B$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.*

Proof:
⟨1⟩1. $1 \Rightarrow 2$
  ⟨2⟩1. Assume: $\mathcal{T} \subseteq \mathcal{T}'$
  ⟨2⟩2. Let: $B \in \mathcal{B}$ and $x \in B$
  ⟨2⟩3. $B \in \mathcal{T}$
    Proof: Corollary 9.9.2.1.
  ⟨2⟩4. $B \in \mathcal{T}'$
    Proof: By ⟨2⟩1
  ⟨2⟩5. There exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$
    Proof: Since $\mathcal{B}'$ is a basis for $\mathcal{T}'$.
⟨1⟩2. $2 \Rightarrow 1$
  ⟨2⟩1. Assume: 2
  ⟨2⟩2. Let: $U \in \mathcal{T}$
    Prove: $U \in \mathcal{T}'$
  ⟨2⟩3. Let: $x \in U$
    Prove: There exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq U$
  ⟨2⟩4. Pick $B \in \mathcal{B}$ such that $x \in B \subseteq U$
    Proof: Since $\mathcal{B}$ is a basis for $\mathcal{T}$.
  ⟨2⟩5. Pick $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$
    Proof: By ⟨2⟩1.
  ⟨2⟩6. $x \in B' \subseteq U$
□

Theorem 9.9.5. *Let X be a topological space, $A \subseteq X$ and $x \in X$. Let $\mathcal{B}$ be a basis for X. Then $x \in \overline{A}$ if and only if, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A.*

PROOF:
$\langle 1 \rangle 1$. If $x \in \overline{A}$ then, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A.
PROOF: This follows from Theorem 9.6.6 since every element of $\mathcal{B}$ is open (Corollary 9.9.2.1).
$\langle 1 \rangle 2$. Suppose that, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A. Then $x \in \overline{A}$.
$\langle 2 \rangle 1$. ASSUME: For all $B \in \mathcal{B}$, if $x \in B$ then B intersects A.
$\langle 2 \rangle 2$. LET: U be an open set that contains x
PROVE: U intersects A.
$\langle 2 \rangle 3$. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$.
$\langle 2 \rangle 4$. B intersects A.
PROOF: From $\langle 2 \rangle 1$.
$\langle 2 \rangle 5$. U intersects A.
$\langle 2 \rangle 6$. Q.E.D.
PROOF: By Theorem 9.6.6.
□

Definition 9.9.6 (Lower Limit Topology on the Real Line)**.** The *lower limit topology on the real line* is the topology on $\mathbb{R}$ generated by the basis consisting of all half-open intervals of the form $[a, b)$.

We write $\mathbb{R}_l$ for the topological space $\mathbb{R}$ under the lower limit topology.

We prove this is a basis for a topology.

PROOF:
$\langle 1 \rangle 1$. For all $x \in \mathbb{R}$ there exists an interval $[a, b)$ such that $x \in [a, b)$.
PROOF: Take $[a, b) = [x, x + 1)$.
$\langle 1 \rangle 2$. For any open intervals $[a, b)$, $[c, d)$ if $x \in [a, b) \cap [c, d)$, then there exists an interval $[e, f)$ such that $x \in [e, f) \subseteq [a, b) \cap [c, d)$
PROOF: Take $[e, f) = [\max(a, c), \min(b, d))$.
□

Definition 9.9.7 (K-topology on the Real Line)**.** Let $K = \{1/n \mid n \in \mathbb{Z}^+\}$.

The *K-topology on the real line* is the topology on $\mathbb{R}$ generated by the basis consisting of all open intervals (a, b) and all sets of the form $(a, b) \setminus K$.

We write $\mathbb{R}_K$ for the topological space $\mathbb{R}$ under the K-topology.

We prove this is a basis for a topology.

PROOF:
$\langle 1 \rangle 1$. For all $x \in \mathbb{R}$ there exists an open interval (a, b) such that $x \in (a, b)$.
PROOF: Take $(a, b) = (x - 1, x + 1)$.
$\langle 1 \rangle 2$. For any basic open sets B_1, B_2 if $x \in B_1 \cap B_2$, then there exists a basic open set B_3 such that $x \in B_3 \subseteq B_1 \cap B_2$
$\langle 2 \rangle 1$. CASE: $B_1 = (a, b)$, $B_2 = (c, d)$
PROOF: Take $B_3 = (\max(a, c), \min(b, d))$.

$\langle 2\rangle 2$. Case: $B_1 = (a, b)$ or $(a, b) \setminus K$, $B_2 = (c, d)$ or $(c, d) \setminus K$, and they are not both open intervals.

Proof: Take $B_3 = (\max(a, c), \min(b, d)) \setminus K$.

□

Lemma 9.9.8. *The lower limit topology and the K-topology are incomparable.*

Proof:

$\langle 1\rangle 1$. The interval $[10, 11)$ is not open in the K-topology.

Proof: There is no open interval (a, b) such that $10 \in (a, b) \subseteq [10, 11)$ or $10 \in (a, b) \setminus K \subseteq [10, 11)$.

$\langle 1\rangle 2$. The set $(-1, 1) \setminus K$ is not open in the lower limit topology.

Proof: There is no half-open interval $[a, b)$ such that $0 \in [a, b) \subseteq (-1, 1) \setminus K$, since there must be a positive integer n with $1/n \in [a, b)$.

□

Definition 9.9.9 (Subbasis)**.** A *subbasis* $\mathcal{S}$ for a topology on X is a set $\mathcal{S} \subseteq \mathcal{P}X$ such that $\bigcup \mathcal{S} = X$.

The topology *generated* by the subbasis $\mathcal{S}$ is the set of all unions of finite intersections of elements of $\mathcal{S}$.

We prove this is a topology.

Proof:

$\langle 1\rangle 1$. The set $\mathcal{B}$ of all finite intersections of elements of $\mathcal{S}$ forms a basis for a topology on X.

$\langle 2\rangle 1$. $\bigcup \mathcal{B} = X$

Proof: Since $\mathcal{S} \subseteq \mathcal{B}$.

$\langle 2\rangle 2$. $\mathcal{B}$ is closed under binary intersection.

Proof: By definition.

$\langle 1\rangle 2$. Q.E.D.

Proof: By Lemma 9.9.2.

□

We have simultaneously proved:

Proposition 9.9.10. *Let $\mathcal{S}$ be a subbasis for the topology on X. Then the set of all finite intersections of elements of $\mathcal{S}$ is a basis for the topology on X.*

Proposition 9.9.11. *Let X be a set. Let $\mathcal{S}$ be a subbasis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ is the coarsest topology that includes $\mathcal{S}$.*

Proof: Since every topology that includes $\mathcal{S}$ includes every union of finite intersections of elements of $\mathcal{S}$. □

9.10 Local Basis at a Point

Definition 9.10.1 (Local Basis)**.** Let X be a topological space and $a \in X$. A *(local) basis at a* is a set $\mathcal{B}$ of neighbourhoods of a such that every neighbourhood of a includes some member of $\mathcal{B}$.

Lemma 9.10.2. *If there exists a countable local basis at a point a, then there exists a countable local basis $\{B_n \mid n \geq 1\}$ such that $B_1 \supseteq B_2 \supseteq \cdots$.*

PROOF: Given a countable local basis $\{C_n \mid n \geq 1\}$, take $B_n = C_1 \cap \cdots \cap C_n$. ∎

9.11 Nets

Definition 9.11.1 (Net)**.** Let X be a topological space. A *net* in X consists of a directed poset J and a family $(x_\alpha)_{\alpha \in J}$ of points of X indexed by J.

Definition 9.11.2 (Convergence)**.** Let X be a topological space. Let $(x_\alpha)_{\alpha \in J}$ be a net in X and $l \in X$. Then (x_α) *converges* to the *limit* l iff, for every limit U of l, there exists $\alpha \in J$ such that, for all $\beta \geq \alpha$, we have $x_\beta \in U$.

Lemma 9.11.3. *Let X be a topological space. Let $A \subseteq X$ and $l \in X$. If there is a sequence of points in A that converges to l then $l \in \overline{A}$.*

PROOF:
⟨1⟩1. LET: (a_n) be a sequence of points in A that converges to l.
⟨1⟩2. LET: U be a neighbourhood of l.
⟨1⟩3. PICK N such that, for all $n \geq N$, we have $a_n \in U$.
⟨1⟩4. $a_N \in U \cap A$
⟨1⟩5. Q.E.D.
  PROOF: Theorem 9.6.6.
∎

Proposition 9.11.4. *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Let (a_n) be a sequence in X and $l \in X$. Then $a_n \to l$ as $n \to \infty$ if and only if, for every $B \in \mathcal{B}$ with $l \in B$, there exists N such that, for all $n \geq N$, we have $a_n \in B$.*

PROOF:
⟨1⟩1. If $a_n \to l$ as $n \to \infty$ then, for every $B \in \mathcal{B}$ with $l \in B$, there exists N such that, for all $n \geq N$, we have $a_n \in B$.
  PROOF: Since every element of $\mathcal{B}$ is open (Corollary 9.9.2.1).
⟨1⟩2. If, for every $B \in \mathcal{B}$ with $l \in B$, there exists N such that, for all $n \geq N$, we have $a_n \in B$, then $a_n \to l$ as $n \to \infty$.
  ⟨2⟩1. ASSUME: for every $B \in \mathcal{B}$ with $l \in B$, there exists N such that, for all $n \geq N$, we have $a_n \in B$
  ⟨2⟩2. LET: U be a neighbourhood of l.
  ⟨2⟩3. PICK $B \in \mathcal{B}$ such that $l \in B \subseteq U$
  ⟨2⟩4. PICK N such that, for all $n \geq N$, we have $a_n \in B$
    PROOF: From ⟨2⟩1.
  ⟨2⟩5. For all $n \geq N$ we have $a_n \in U$
∎

Lemma 9.11.5. *If a sequence (a_n) is constant with $a_n = l$ for all n, then $a_n \to l$ as $n \to \infty$.*

PROOF: Immediate from definitions. □

Theorem 9.11.6. *Let X be a linearly ordered set. Let (s_n) be a monotone increasing sequence in X with a supremum s. Then $s_n \to s$ as $n \to \infty$.*

PROOF:
⟨1⟩1. ASSUME: s is not least in X.
  PROOF: Otherwise (s_n) is the constant sequence s and the result follows from Lemma 9.11.5.
⟨1⟩2. LET: U be a neighbourhood of s.
⟨1⟩3. PICK$a < s$ such that $(a, s] \subseteq U$
⟨1⟩4. PICK N such that $a < a_N$.
⟨1⟩5. For all $n \geq N$ we have $a_n \in (a, s]$
⟨1⟩6. For all $n \geq N$ we have $a_n \in U$.
□

Theorem 9.11.7. *If $\sum_{i=0}^{\infty} a_i = s$ and $\sum_{i=0}^{\infty} b_i = t$ then $\sum_{i=0}^{\infty}(ca_i + b_i) = cs + t$.*

PROOF:$\sum_{i=0}^{N}(ca_i + b_i) = c\sum_{i=0}^{N} a_i + \sum_{i=0}^{N} b_i \to cs + t$ as $n \to \infty$. □

Theorem 9.11.8 (Comparison Test)**.** *If $|a_i| \leq b_i$ for all i and $\sum_{i=0}^{\infty} b_i$ converges then $\sum_{i=0}^{\infty} a_i$ converges.*

PROOF:
⟨1⟩1. $\sum_{i=0}^{\infty} |a_i|$ converges
  PROOF: The partial sums $\sum_{i=0}^{N} |a_i|$ form an increasing sequence bounded above by $\sum_{i=0}^{\infty} b_i$.
⟨1⟩2. LET: $c_i = |a_i| + a_i$ for all i
⟨1⟩3. $\sum_{i=0}^{\infty} c_i$ converges
  PROOF: Each c_i is either $2|a_i|$ or 0. So the partial sums $\sum_{i=0}^{N} c_i$ form an increasing sequence bounded above by $2\sum_{i=0}^{\infty} b_i$.
⟨1⟩4. Q.E.D.
  PROOF: Since $a_i = c_i - |a_i|$.
□

Corollary 9.11.8.1. *If $\sum_{i=0}^{\infty} |a_i|$ converges then $\sum_{i=0}^{\infty} a_i$ converges.*

Theorem 9.11.9 (Weierstrass M-test)**.** *Let X be a set and $(f_n : X \to \mathbb{R})$ be a sequence of functions. Let*

$$s_n(x) = \sum_{i=0}^{n} f_i(x)$$

for all n, x. Suppose $|f_i(x)| \leq M_i$ for all $i \geq 0$ and $x \in X$. If the series $\sum_{i=0}^{\infty} M_i$ converges, then the sequence (s_n) converges uniformly to

$$s(x) = \sum_{i=0}^{\infty} f_i(x) \enspace .$$

PROOF:

⟨1⟩1. LET: $r_n = \sum_{i=n+1}^{\infty} M_i$ for all n

⟨1⟩2. Given $0 \leq n < k$, we have $|s_k(x) - s_n(x)| \leq r_n$

PROOF:

$$\begin{aligned} |s_k(x) - s_n(x)| &= |\sum_{i=n+1}^{k} f_i(x)| \\ &\leq \sum_{i=n+1}^{k} |f_i(x)| \\ &\leq \sum_{i=n+1}^{k} M_i \\ &\leq r_n \end{aligned}$$

⟨1⟩3. Given $n \geq 0$ we have $|s(x) - s_n(x)| \leq r_n$

PROOF: By taking the limit $k \to \infty$ in ⟨1⟩2.

⟨1⟩4. Q.E.D.

PROOF: Since $r_n \to 0$ as $n \to \infty$.

□

9.12 Locally Finite Sets

Definition 9.12.1 (Locally Finite)**.** Let X be a topological space and $\{A_\alpha\}$ a family of subsets of X. Then $\mathcal{A}$ is *locally finite* if and only if every point in X has a neighbourhood that intersects A_α for only finitely many α.

Theorem 9.12.2 (Pasting Lemma)**.** *Let X and Y be topological spaces and $f : X \to Y$. Let $\{A_\alpha\}$ be a locally finite family of closed subsets of X that cover X. Suppose $f \upharpoonright A_\alpha$ is continuous for all α. Then f is continuous.*

PROOF:

⟨1⟩1. Let X and Y be topological spaces and $f : X \to Y$. Let A and B be closed subsets of X such that $X = A \cup B$. Suppose $f \upharpoonright A$ and $f \upharpoonright B$ are continuous. Then f is continuous.

⟨2⟩1. LET: $C \subseteq Y$ be closed.

⟨2⟩2. $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$

⟨2⟩3. $f^{-1}(C)$ and $g^{-1}(C)$ are closed in X.

PROOF: Theorems 9.14.6 and 9.19.7.

⟨2⟩4. $h^{-1}(C)$ is closed in X.

PROOF: Lemma 9.4.5.

⟨2⟩5. Q.E.D.

PROOF: Theorem 9.14.6.

⟨1⟩2. Let X and Y be topological spaces and $f : X \to Y$. Let $\{A_\alpha\}$ be a finite family of closed subsets of X that cover X. Suppose $f \upharpoonright A_\alpha$ is continuous for all α. Then f is continuous.

PROOF: From ⟨1⟩1 by induction.

⟨1⟩3. Let X and Y be topological spaces and $f : X \to Y$. Let $\{A_\alpha\}$ be a locally finite family of closed subsets of X that cover X. Suppose $f \restriction A_\alpha$ is continuous for all α. Then f is continuous.

⟨2⟩1. LET: $x \in X$
PROVE: f is continuous at x

⟨2⟩2. PICK a neighbourhood U of x that intersects A_α for only finitely many α.

⟨2⟩3. $f \restriction U$ is continuous
PROOF: By ⟨1⟩2.

⟨2⟩4. Q.E.D.
PROOF: Lemma 9.14.16.

□

The following example shows that we cannot remove the assumption of local finiteness.

Example 9.12.3. Define $f : [-1, 1] \to \mathbb{R}$ by: $f(x) = 1$ if $x < -1$, $f(x) = 0$ if $x > 1$. Let $C_n = [-1, -1/n]$ for $n \geq 1$, and $D = [0, 1]$. Then $[-1, 1] = \bigcup_{n=1}^{\infty} C_n \cup D$ and f is continuous on each C_n and each D, but f is not continuous on $[-1, 1]$.

9.13 Open Maps

Definition 9.13.1 (Open Map)**.** Let X and Y be topological spaces and $f : X \to Y$. Then f is an *open map* if and only if, for every open set U in X, the set $f(U)$ is open in Y.

Lemma 9.13.2. *Let X and Y be topological spaces and $f : X \to Y$. Let $\mathcal{B}$ be a basis for the topology on X. If $f(B)$ is open in Y for all $B \in \mathcal{B}$, then f is an open map.*

PROOF: From Lemma 9.9.2. □

Proposition 9.13.3. *Let X and Y be topological spaces. Let $\mathcal{B}$ be a basis for the topology on X. Let $f : X \to Y$. Suppose that, for all $B \in \mathcal{B}$, we have $f(B)$ is open tn Y. Then f is an open map.*

PROOF: For any $\mathcal{A} \subseteq \mathcal{B}$, we have $f(\bigcup \mathcal{A}) = \bigcup_{B \in \mathcal{B}} f(B)$ is open in Y. The result follows from Lemma 9.9.2. □

9.14 Continuous Functions

Definition 9.14.1 (Continuous)**.** Let X and Y be topological spaces. A function $f : X \to Y$ is *continuous* if and only if, for every open set V in Y, the set $f^{-1}(V)$ is open in X.

Proposition 9.14.2. *Let X and Y be topological spaces and $f : X \to Y$. Let $\mathcal{B}$ be a basis for Y. Then f is continuous if and only if, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X.*

PROOF:
⟨1⟩1. If f is continuous then, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X.
PROOF: Since every element of B is open (Lemma 9.9.2).
⟨1⟩2. Suppose that, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X. Then f is continuous.
⟨2⟩1. ASSUME: For all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X.
⟨2⟩2. LET: V be open in Y.
⟨2⟩3. PICK $\mathcal{A} \subseteq \mathcal{B}$ such that $V = \bigcup \mathcal{A}$
PROOF: By Lemma 9.9.2.
⟨2⟩4. $f^{-1}(V)$ is open in X.
PROOF:

$$\begin{aligned} f^{-1}(V) &= f^{-1}\left(\bigcup \mathcal{A}\right) \\ &= \bigcup_{B \in \mathcal{A}} f^{-1}(B) \end{aligned}$$

□

Proposition 9.14.3. *Let X and Y be topological spaces and $f : X \to Y$. Let $\mathcal{S}$ be a subbasis for Y. Then f is continuous if and only if, for all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in X.*

PROOF:
⟨1⟩1. If f is continuous then, for all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in X.
PROOF: Since every element of S is open.
⟨1⟩2. Suppose that, for all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in X. Then f is continuous.
⟨2⟩1. ASSUME: For all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in X.
⟨2⟩2. LET: $S_1, \ldots, S_n \in \mathcal{S}$
⟨2⟩3. $f^{-1}(S_1 \cap \cdots \cap S_n)$ is open in A
PROOF: Since $f^{-1}(S_1 \cap \cdots \cap S_n) = f^{-1}(S_1) \cap \cdots \cap f^{-1}(S_n)$.
⟨2⟩4. Q.E.D.
PROOF: By Propositions 9.14.2 and 9.9.10.

□

Proposition 9.14.4. *Let X and Y be topological spaces and $f : X \to Y$. Let $\mathcal{S}$ be a basis for Y. Then f is continuous if and only if, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X.*

PROOF:
⟨1⟩1. If f is continuous then, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X.
PROOF: Since every element of $\mathcal{S}$ is open.
⟨1⟩2. Suppose that, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X. Then f is continuous.
⟨2⟩1. ASSUME: For all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X.
⟨2⟩2. For every set B that is the finite intersection of elemets of $\mathcal{S}$, we have $f^{-1}(B)$ is open in X.
PROOF: Because $f^{-1}(V_1 \cap \cdots \cap V_n) = f^{-1}(V_1) \cap \cdots \cap f^{-1}(V_n)$.
⟨2⟩3. Q.E.D.

PROOF: From Propositions 9.9.10 and 9.14.2.
□

Definition 9.14.5 (Continuous at a Point)**.** Let X and Y be topological spaces. Let $f : X \to Y$ and $x \in X$. Then f is *continuous at* x if and only if, for every neighbourhood V of $f(x)$, there exists a neighbourhood U of x such that $f(U) \subseteq V$.

Theorem 9.14.6. *Let X and Y be topological spaces and $f : X \to Y$. Then the following are equivalent:*

1. *f is continuous.*
2. *For all $A \subseteq X$, we have $f(\overline{A}) \subseteq \overline{f(A)}$*
3. *For all $B \subseteq Y$ closed, we have $f^{-1}(B)$ is closed in X.*
4. *f is continuous at every point of X.*

PROOF:
⟨1⟩1. $1 \Rightarrow 2$
⟨2⟩1. ASSUME: f is continuous.
⟨2⟩2. LET: $A \subseteq X$
⟨2⟩3. LET: $x \in \overline{A}$
PROVE: $f(x) \in \overline{f(A)}$
⟨2⟩4. LET: V be a neighbourhood of $f(x)$
⟨2⟩5. $f^{-1}(V)$ is a neighbourhood of x
⟨2⟩6. PICK $y \in A \cap f^{-1}(V)$
PROOF: By Theorem 9.6.6.
⟨2⟩7. $f(y) \in V \cap f(A)$
⟨2⟩8. Q.E.D.
PROOF: By Theorem 9.6.6.
⟨1⟩2. $2 \Rightarrow 3$
⟨2⟩1. ASSUME: 2
⟨2⟩2. LET: B be closed in Y
⟨2⟩3. LET: $x \in \overline{f^{-1}(B)}$
PROVE: $x \in f^{-1}(B)$
⟨2⟩4. $f(x) \in B$
PROOF:

$$\begin{aligned} f(x) &\in f(\overline{f^{-1}(B)}) \\ &\subseteq \overline{f(f^{-1}(B))} && (\langle 2\rangle 1) \\ &\subseteq \overline{B} && (Proposition 9.6.7) \\ &= B \end{aligned}$$

⟨1⟩3. $3 \Rightarrow 1$
⟨2⟩1. ASSUME: 3
⟨2⟩2. LET: V be open in Y
⟨2⟩3. $Y \setminus V$ is closed in Y

⟨2⟩4. $f^{-1}(Y \setminus V)$ is closed in X
⟨2⟩5. $X \setminus f^{-1}(V)$ is closed in X
⟨2⟩6. $f^{-1}(V)$ is open in X
⟨1⟩4. $1 \Rightarrow 4$
PROOF: For any neighbourhood V of $f(x)$, the set $U = f^{-1}(V)$ is a neighbourhood of x such that $f(U) \subseteq V$.
⟨1⟩5. $4 \Rightarrow 1$
⟨2⟩1. ASSUME: 4
⟨2⟩2. LET: V be open in Y
⟨2⟩3. LET: $x \in f^{-1}(V)$
⟨2⟩4. V is a neighbourhood of $f(x)$
⟨2⟩5. PICK a neighbourhood U of x such that $f(U) \subseteq V$
⟨2⟩6. $x \in U \subseteq f^{-1}(V)$
⟨2⟩7. Q.E.D.
PROOF: By Lemma 9.3.8.
□

Theorem 9.14.7. *A constant function is continuous.*

PROOF: Let X and Y be topological spaces. Let $b \in Y$, and let $f : X \to Y$ be the constant function with value b. For any open $V \subseteq Y$, the set $f^{-1}(V)$ is either X (if $b \in V$) or $\emptyset$ (if $b \notin V$). □

Theorem 9.14.8. *If A is a subspace of X then the inclusion $j : A \to X$ is continuous.*

PROOF: For any V open in X, we have $j^{-1}(V) = V \cap A$ is open in A. □

Theorem 9.14.9. *The composite of two continuous functions is continuous.*

PROOF: Let $f : X \to Y$ and $g : Y \to Z$ be continuous. For any V open in Z, we have $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is open in X. □

Theorem 9.14.10. *Let $f : X \to Y$ be a continuous function and A be a subspace of X. Then the restriction $f \upharpoonright A : A \to Y$ is continuous.*

PROOF: Let V be open in Y. Then $(f \upharpoonright A)^{-1}(V) = f^{-1}(V) \cap A$ is open in A. □

Theorem 9.14.11. *Let $f : X \to Y$ be continuous. Let Z be a subspace of Y such that $f(X) \subseteq Z$. Then the corestriction $f : X \to Z$ is continuous.*

PROOF:
⟨1⟩1. LET: V be open in Z.
⟨1⟩2. PICK U open in Y such that $V = U \cap Z$.
⟨1⟩3. $f^{-1}(V) = f^{-1}(U)$
⟨1⟩4. $f^{-1}(V)$ is open in X.
□

Theorem 9.14.12. *Let $f : X \to Y$ be continuous. Let Z be a space such that Y is a subspace of Z. Then the expansion $f : X \to Z$ is continuous.*

Proof: Let V be open in Z. Then $f^{-1}(V) = f^{-1}(V \cap Y)$ is open in X. □

Theorem 9.14.13. *Let X and Y be topological spaces. Let $f : X \to Y$. Suppose $\mathcal{U}$ is a set of open sets in X such that $X = \bigcup \mathcal{U}$ and, for all $U \in \mathcal{U}$, we have $f \restriction U : U \to Y$ is continuous. Then f is continuous.*

Proof:
⟨1⟩1. Let: V be open in Y
⟨1⟩2. $f^{-1}(V) = \bigcup_{U \in \mathcal{U}} (f \restriction U)^{-1}(V)$
⟨1⟩3. For all $U \in \mathcal{U}$, we have $(f \restriction U)^{-1}(V)$ is open in U.
⟨1⟩4. For all $U \in \mathcal{U}$, we have $(f \restriction U)^{-1}(V)$ is open in X.
  Proof: Lemma 9.19.6.
□

Proposition 9.14.14. *Let X and X' be the same set X under two topologies $\mathcal{T}$ and $\mathcal{T}'$. Let $i : X \to X'$ be the identity function. Then i is continuous if and only if $\mathcal{T}' \subseteq \mathcal{T}$.*

Proof: Immediate from definitions. □

Proposition 9.14.15. *Let $f : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$. Then f is continuous on the right at a if and only if f is continuous at a as a function $\mathbb{R}_l \to \mathbb{R}$.*

Proof:
⟨1⟩1. If f is continuous on the right at a then f is continuous at a as a function $\mathbb{R}_l \to \mathbb{R}$.
  ⟨2⟩1. Assume: f is continuous on the right at a.
  ⟨2⟩2. Let: V be a neighbourhood of $f(a)$
  ⟨2⟩3. Pick b, c such that $f(a) \in (b, c) \subseteq V$.
  ⟨2⟩4. Let: $\epsilon = \min(c - f(a), f(a) - b)$
  ⟨2⟩5. Pick $\delta > 0$ such that, for all x, if $a < x < a + \delta$ then $|f(x) - f(a)| < \epsilon$
  ⟨2⟩6. Let: $U = [a, a + \delta)$
  ⟨2⟩7. $f(U) \subseteq V$
⟨1⟩2. If f is continuous at a as a function $\mathbb{R}_l \to \mathbb{R}$ then f is continuous on the right at a.
  ⟨2⟩1. Assume: f is continuous at a as a function $\mathbb{R}_l \to \mathbb{R}$
  ⟨2⟩2. Let: $\epsilon > 0$
  ⟨2⟩3. Pick a neighbourhood U of a such that $f(U) \subseteq (f(a) - \epsilon, f(a) + \epsilon)$
  ⟨2⟩4. Pick b, c such that $a \in [b, c) \subset U$
  ⟨2⟩5. Let: $\delta = c - a$
  ⟨2⟩6. For all x, if $a < x < a + \delta$ then $|f(x) - f(a)| < \epsilon$
□

Lemma 9.14.16. *Let $f : X \to Y$. Let Z be an open subspace of X and $a \in Z$. If $f \restriction Z$ is continuous at a then f is continuous at a.*

Proof:
⟨1⟩1. Let: V be a neighbourhood of $f(x)$

$\langle 1\rangle 2$. PICK a neighbourhood W of x in Z such that $f(W) \subseteq V$
$\langle 1\rangle 3$. W is a neighbourhood of x in X such that $f(W) \subseteq V$
PROOF: Lemma 9.19.6.
□

Proposition 9.14.17. *Let $f : A \to B$ and $g : C \to D$ be continuous. Define $f \times g : A \times C \to B \times D$ by*

$$(f \times g)(a, c) = (f(a), g(c)) \ .$$

Then $f \times g$ is continuous.

PROOF: $\pi_1 \circ (f \times g) = f \circ \pi_1$ and $\pi_2 \circ (f \times g) = g \circ \pi_2$ are continuous by Theorem 9.14.9. The result follows by Theorem 9.18.11.

Proposition 9.14.18. *Let X and Y be topological spaces and $f : X \to Y$ be continuous. Let $(a_\alpha)_{\alpha \in J}$ be a net in X and $l \in X$. If $a_\alpha \to l$ as $\alpha \to \infty$ in X then $f(a_\alpha) \to f(l)$ as $\alpha \to \infty$.*

PROOF:
$\langle 1\rangle 1$. LET: V be a neighbourhood of $f(l)$
$\langle 1\rangle 2$. PICK a neighbourhood U of l such that $f(U) \subseteq V$
$\langle 1\rangle 3$. PICK $\alpha \in J$ such that, for all $\beta \geq \alpha$, we have $a_\beta \in U$
$\langle 1\rangle 4$. For all $\beta \geq \alpha$ we have $f(a_\beta) \in V$
□

9.15 Homeomorphisms

Definition 9.15.1 (Homeomorphism)**.** Let X and Y be topological spaces. A *Homeomorphism* f between X and Y, $f : X \cong Y$, is a bijection $f : X \to Y$ such that both f and f^{-1} are continuous.

Lemma 9.15.2. *Let X and Y be topological spaces and $f : X \to Y$ a bijection. Then the following are equivalent:*

1. *f is a homeomorphism.*
2. *f is continuous and an open map.*
3. *f is continuous and a closed map.*
4. *For any $U \subseteq X$, we have U is open if and only if $f(U)$ is open.*

PROOF: Immediate from definitions. □

Proposition 9.15.3. *Let X and X' be the same set X under two topologies $\mathcal{T}$ and $\mathcal{T}'$. Let $i : X \to X'$ be the identity function. Then i is a homeomorphism if and only if $\mathcal{T} = \mathcal{T}'$.*

PROOF: Immediate from definitions. □

Definition 9.15.4 (Topological Property)**.** Let P be a property of topological spaces. Then P is a *topological* property if and only if, for any spaces X and Y, if P holds of X and $X \cong Y$ then P holds of Y.

Definition 9.15.5 (Topological Imbedding)**.** Let X and Y be topological spaces and $f : X \to Y$. Then f is a *topological imbedding* if and only if the corestriction $f : X \to f(X)$ is a homeomorphism.

Proposition 9.15.6. *Let X and Y be topological spaces and $a \in X$. The function $i : Y \to X \times Y$ that maps y to (a, y) is an imbedding.*

PROOF:
$\langle 1 \rangle 1$. i is injective
$\langle 1 \rangle 2$. i is continuous.
  PROOF: For U open in X and V open in Y, we have $i^{-1}(U \times V)$ is V if $a \in U$, and $\emptyset$ if $a \notin U$.
$\langle 1 \rangle 3$. $i : Y \to i(Y)$ is an open map.
  PROOF: For V open in Y we have $i(V) = (X \times V) \cap i(Y)$.
□

9.16 The Order Topology

Definition 9.16.1 (Order Topology)**.** Let X be a linearly ordered set with at least two points. The *order topology* on X is the topology generated by the basis $\mathcal{B}$ consisting of:

- all open intervals (a, b);
- all intervals of the form $[\bot, b)$ where $\bot$ is least in X;
- all intervals of the form $(a, \top]$ where $\top$ is greatest in X.

We prove this is a basis for a topology.

PROOF:
$\langle 1 \rangle 1$. For all $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B$.
  $\langle 2 \rangle 1$. LET: $x \in X$
  $\langle 2 \rangle 2$. CASE: x is greatest in X.
    $\langle 3 \rangle 1$. PICK $y \in X$ with $y \neq x$
    $\langle 3 \rangle 2$. $x \in (y, x] \in \mathcal{B}$
  $\langle 2 \rangle 3$. CASE: x is least in X.
    $\langle 3 \rangle 1$. PICK $y \in X$ with $y \neq x$
    $\langle 3 \rangle 2$. $x \in [x, y) \in \mathcal{B}$
  $\langle 2 \rangle 4$. CASE: x is neither greatest nor least in X.
    $\langle 3 \rangle 1$. PICK $a, b \in X$ with $a < x$ and $x < b$
    $\langle 3 \rangle 2$. $x \in (a, b) \in \mathcal{B}$
$\langle 1 \rangle 2$. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$

⟨2⟩1. LET: $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$
⟨2⟩2. CASE: $B_1 = (a, b)$, $B_2 = (c, d)$
PROOF: Take $B_3 = (\max(a, c), \min(b, d))$.
⟨2⟩3. CASE: $B_1 = (a, b)$, $B_2 = [\perp, d)$
PROOF: Take $B_3 = (a, \min(b, d))$.
⟨2⟩4. CASE: $B_1 = (a, b)$, $B_2 = (c, \top]$
PROOF: Take $B_3 = (\max(a, c), b)$.
⟨2⟩5. CASE: $B_1 = [\perp, b)$, $B_2 = [\perp, d)$
PROOF: Take $B_3 = [\perp, \min(b, d))$.
⟨2⟩6. CASE: $B_1 = [\perp, b)$, $B_2 = (c, \top]$
PROOF: Take $B_3 = (c, b)$.
□

Lemma 9.16.2. *Let X be a linearly ordered set. Then the open rays form a subbasis for the order topology on X.*

PROOF:
⟨1⟩1. Every open ray is open.
⟨2⟩1. For all $a \in X$, the ray $(-\infty, a)$ is open.
⟨3⟩1. LET: $x \in (-\infty, a)$
⟨3⟩2. CASE: x is least in X
PROOF: $xin[x, a) = (-\infty, a)$.
⟨3⟩3. CASE: x is not least in X
⟨4⟩1. PICK $y < x$
⟨4⟩2. $x \in (y, a) \subseteq (-\infty, a)$
⟨2⟩2. For all $a \in X$, the ray $(a, +\infty)$ is open.
PROOF: Similar.
⟨1⟩2. Every basic open set is a finite intersection of open rays.
PROOF: We have $(a, b) = (a, +\infty) \cap (-\infty, b)$, $[\perp, b) = (-\infty, b)$ and $(a, \top] = (a, +\infty)$.
□

Definition 9.16.3 (Standard Topology on the Real Line)**.** The *standard topology on the real line* is the order topology on $\mathbb{R}$ generated by the standard order.

Lemma 9.16.4. *The lower limit topology is strictly finer than the standard topology on $\mathbb{R}$.*

PROOF:
⟨1⟩1. Every open interval is open in the lower limit topology.
PROOF: If $x \in (a, b)$ then $x \in [x, b) \subseteq (a, b)$.
⟨1⟩2. The half-open interval $[0, 1)$ is not open in the standard topology.
PROOF: There is no open interval (a, b) such that $0 \in (a, b) \subseteq [0, 1)$.
□

Lemma 9.16.5. *The K-topology is strictly finer than the standard topology on $\mathbb{R}$.*

PROOF:

⟨1⟩1. Every open interval is open in the K-topology.
PROOF: Corollary 9.9.2.1.
⟨1⟩2. The set $(-1,1) \setminus K$ is not open in the standard topology.
PROOF: There is no open interval (a,b) such that $0 \in (a,b) \subseteq (-1,1) \setminus K$, since there must be a positive integer n with $1/n \in (a,b)$.
□

Lemma 9.16.6. *Let X be a topological space. Let Y be a linearly ordered set in the order topology. Let $f, g : X \to Y$ be continuous. Then $C = \{x \in X \mid f(x) \leq g(x)\}$ is closed.*

PROOF:
⟨1⟩1. LET: $x \in X \setminus C$
⟨1⟩2. $f(x) > g(x)$
PROVE: There exists a neighbourhood U of x such that $U \subseteq X \setminus C$
⟨1⟩3. CASE: There exists y such that $g(x) < y < f(x)$
PROOF: Take $U = g^{-1}((-\infty, y)) \cup f^{-1}(y, +\infty)$.
⟨1⟩4. CASE: There is no y such that $g(x) < y < f(x)$
PROOF: Take $U = g^{-1}((-\infty, f(x))) \cup f^{-1}(g(x), +\infty)$.
□

Proposition 9.16.7. *Let X be a topological space. Let Y be a linearly ordered set in the order topology. Let $f, g : X \to Y$ be continuous. Define $h : X \to Y$ by $h(x) = \min(f(x), g(x))$. Then h is continuous.*

PROOF: By the Pasting Lemma applied to $\{x \in X \mid f(x) \leq g(x)\}$ and $\{x \in X \mid g(x) \leq f(x)\}$, which are closed by Lemma 9.16.6.

Proposition 9.16.8. *Let X and Y be linearly ordered sets in the order topology. Let $f : X \to Y$ be strictly monotone and surjective. Then f is a homeomorphism.*

PROOF:
⟨1⟩1. f is bijective.
PROOF: Proposition 4.2.3.
⟨1⟩2. f is continuous.
⟨2⟩1. For all $y \in Y$ we have $f^{-1}((y, +\infty))$ is open.
⟨3⟩1. LET: $y \in Y$
⟨3⟩2. PICK$x \in X$ such that $f(x) = y$
PROOF: Since f is surjective.
⟨3⟩3. $f^{-1}((y, +\infty)) = (x, +\infty)$
PROOF: By strict monotoncity.
⟨2⟩2. For all $y \in Y$ we have $f^{-1}((-\infty, y))$ is open.
PROOF: Similar.
⟨1⟩3. f^{-1} is continuous.
⟨2⟩1. For all $x \in X$ we have $f((x, +\infty))$ is open.
PROOF: $f((x, +\infty)) = (f(x), +\infty)$.
⟨2⟩2. For all $x \in X$ we have $f((-\infty, x))$ is open.
PROOF: $f((-\infty, x)) = (-\infty, f(x))$.
□

9.17 The nth Root Function

Proposition 9.17.1. *For all $n \geq 1$, the function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined by $f(x) = x^n$ is a homemorphism.*

PROOF:
$\langle 1 \rangle 1$. f is strictly monotone.
$\langle 2 \rangle 1$. LET: $x, y \in \mathbb{R}$ with $0 \leq x < y$
$\langle 2 \rangle 2$. $x^n < y^n$

$$\begin{aligned} y^n - x^n &= (y - x)(y^{n-1} + y^{n-2}x + y^{n-3}x^2 + \cdots + x^{n-1}) \\ &> 0 \end{aligned}$$

$\langle 1 \rangle 2$. f is surjective.
$\langle 2 \rangle 1$. LET: $y \in \mathbb{R}_{\geq 0}$
$\langle 2 \rangle 2$. PICK $x \in \mathbb{R}$ such that $y \leq x^n$
PROOF: If $y \leq 1$ take $x = 1$, otherwise take $x = y$.
$\langle 2 \rangle 3$. There exists $x' \in [0, x]$ such that $(x')^n = y$
PROOF: By the Intermediate Value Theorem.
$\langle 1 \rangle 3$. Q.E.D.
PROOF: Proposition 9.16.8.
$\square$

Definition 9.17.2. For $n \geq 1$, the *nth root function* is the function $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ that is the inverse of $\lambda x.x^n$.

9.18 The Product Topology

Definition 9.18.1 (Product Topology)**.** Let $\{A_i\}_{i \in I}$ be a family of topological spaces. The *product topology* on $\prod_{i \in I} A_i$ is the topology generated by the subbasis consisting of the sets of the form ${\pi_i}^{-1}(U)$ where $i \in I$ and U is open in A_i.

Proposition 9.18.2. *The product topology on $\prod_{i \in I} A_i$ is generated by the basis consisting of all sets of the form $\prod_{i \in I} U_i$ where $\{U_i\}_{i \in I}$ is a family such that each U_i is an open set in A_i and $U_i = A_i$ for all but finitely many i.*

PROOF: From Proposition 9.9.10. $\square$

Proposition 9.18.3. *If A_i is closed in X_i for all $i \in I$ then $\prod_{i \in I} A_i$ is closed in $\prod_{i \in I} X_i$.*

PROOF:

$$(\prod_{i \in I} X_i) \setminus (\prod_{i \in I} A_i) = \bigcup_{j \in I} \left(\prod_{i \in I} X_i \setminus {\pi_j}^{-1}(A_j) \right) \square$$

Proposition 9.18.4. *Let $\{A_i\}_{i \in I}$ be a family of topological spaces. For $i \in I$, let $\mathcal{B}_i$ be a basis for the topology on A_i. Then $\mathcal{B} = \{\prod_{i \in I} B_i \mid \forall i \in I. B_i \in \mathcal{B}_i, B_i = A_i$ for all but finitely many $i\}$ is a basis for the box topology on $\prod_{i \in I} A_i$.*

PROOF:
⟨1⟩1. Every set in $\mathcal{B}$ is open.
⟨1⟩2. For every point $a \in \prod_{i \in I} A_i$ and every open set U with $a \in U$, there exists $B \in \mathcal{B}$ such that $a \in B \subseteq U$.
⟨2⟩1. LET: U be open and $a \in U$
⟨2⟩2. PICK a family $\{U_i\}_{i \in I}$ such that each U_i is open in A_i, such that $U_i = A_i$ except for $i = i_1, \ldots, i_n$, and such that $a \in \prod_{i \in I} U_i \subseteq U$.
⟨2⟩3. For $j = 1, \ldots, n$, PICK $B_{i_j} \in \mathcal{B}_{i_j}$ such that $a_{i_j} \in B_{i_j} \subseteq U_{i_j}$
⟨2⟩4. LET: $B = \prod_{i \in I} B_i$ where $B_i = A_i$ for $i \neq i_1, \ldots, i_n$
⟨2⟩5. $B \in \mathcal{B}$
⟨2⟩6. $a \in B \subseteq U$
⟨1⟩3. Q.E.D.
PROOF: Lemma 9.9.3.
□

Proposition 9.18.5. *Let $\{A_i\}_{i \in I}$ be a family of topological spaces. Then the projections $\pi_i : \prod_{i \in I} A_i \to A_i$ are open maps.*

PROOF: From Lemma 9.13.2. □

Example 9.18.6. The projections are not always closed maps. For example, $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ maps the closed set $\{(x, 1/x) \mid x > 0\}$ to $(0, +\infty)$.

Proposition 9.18.7. *Let $\{X_i\}_{i \in I}$ be a family of sets. For $i \in I$, let $\mathcal{T}_i$ and $\mathcal{U}_i$ be topologies on X_i. Let $\mathcal{P}$ be the product topology on $\prod_{i \in I} X_i$ generated by the topologies $\mathcal{T}_i$, and $\mathcal{Q}$ the product topology on the same set generated by the topologies $\mathcal{U}_i$. Then $\mathcal{P} \subseteq \mathcal{Q}$ if and only if $\mathcal{T}_i \subseteq \mathcal{U}_i$ for all i.*

PROOF:
⟨1⟩1. If $\mathcal{T}_i \subseteq \mathcal{U}_i$ for all i then $\mathcal{P} \subseteq \mathcal{Q}$
PROOF: By Corollary 9.9.2.1.
⟨1⟩2. If $\mathcal{P} \subseteq \mathcal{Q}$ then $\mathcal{T}_i \subseteq \mathcal{U}_i$ for all i
⟨2⟩1. ASSUME: $\mathcal{P} \subseteq \mathcal{Q}$
⟨2⟩2. LET: $i \in I$
⟨2⟩3. LET: $U \in \mathcal{T}_i$
⟨2⟩4. LET: $U_i = U$ and $U_j = X_j$ for $j \neq i$
⟨2⟩5. $\prod_{i \in I} U_i \in \mathcal{P}$
⟨2⟩6. $\prod_{i \in I} U_i \in \mathcal{Q}$
⟨2⟩7. $U \in \mathcal{U}_i$
PROOF: From Proposition 9.18.5.
□

Proposition 9.18.8 (AC)**.** *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let $A_i \subseteq X_i$ for all $i \in I$. Then*

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i} \enspace .$$

PROOF:

⟨1⟩1. $\overline{\prod_{i \in I} A_i} \subseteq \prod_{i \in I} \overline{A_i}$

⟨2⟩1. For all $i \in I$ we have $A_i \subseteq \overline{A_i}$

Proof: Lemma 9.6.3.

⟨2⟩2. $\prod_{i \in I} A_i \subseteq \prod_{i \in I} \overline{A_i}$

⟨2⟩3. Q.E.D.

Proof: Since $\prod_{i \in I} A_i$ is closed by Proposition 9.18.3.

⟨1⟩2. $\prod_{i \in I} \overline{A_i} \subseteq \overline{\prod_{i \in I} A_i}$

⟨2⟩1. Let: $x \in \prod_{i \in I} \overline{A_i}$

⟨2⟩2. Let: U be a neighbourhood of x

⟨2⟩3. Pick V_i open in X_i such that $x \in \prod_{i \in I} V_i \subseteq U$ with $V_i = X_i$ except for $i = i_1, \ldots, i_n$

⟨2⟩4. For $i \in I$, pick $a_i \in V_i \cap A_i$

Proof: By Theorem 9.6.6 and ⟨2⟩1 using the Axiom of Choice.

⟨2⟩5. U intersects $\prod_{i \in I} A_i$

⟨2⟩6. Q.E.D.

Proof: $a \in U \cap \prod_{i \in I} A_i$

□

Example 9.18.9. The closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ is $\mathbb{R}^\omega$

Proof:

⟨1⟩1. Let: $a \in \mathbb{R}^\omega$

⟨1⟩2. Let: U be any neighbourhoods of a.

⟨1⟩3. Pick U_n open in $\mathbb{R}$ for all n such that $a \in \prod_{n \geq 0} U_n \subseteq U$ and $U_n = \mathbb{R}$ for all n except $n_1, \ldots, n_k$

⟨1⟩4. Let: $b_n = a_n$ for $n = n_1, \ldots, n_k$ and $b_n = 0$ for all other n

⟨1⟩5. $b \in \mathbb{R}^\infty \cap U$

⟨1⟩6. Q.E.D.

Proof: From Theorem 9.6.6.

□

Proposition 9.18.10. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let $(a_\alpha)_{\alpha \in J}$ be a net in $\prod_{i \in I} X_i$ and $l \in \prod_{i \in I} X_i$. Then $a_\alpha \to l$ as $\alpha \to \infty$ if and only if, for all $i \in I$, we have $\pi_i(a_\alpha) \to \pi_i(l)$ as $\alpha \to \infty$.*

Proof:

⟨1⟩1. If $a_\alpha \to l$ as $\alpha \to \infty$ then, for all $i \in I$, we have $\pi_i(a_\alpha) \to \pi_i(l)$ as $n \to \infty$

Proof: Proposition 9.14.18.

⟨1⟩2. If, for all $i \in I$, we have $\pi_i(a_\alpha) \to \pi_i(l)$ as $\alpha \to \infty$, then $a_\alpha \to l$ as $\alpha \to \infty$

⟨2⟩1. Assume: For all $i \in I$, we have $\pi_i(a_\alpha) \to \pi_i(l)$ as $\alpha \to \infty$

⟨2⟩2. Let: V be a neighbourhood of l

⟨2⟩3. Pick open sets U_i in X_i such that $l \in \prod_{i \in I} U_i \subseteq V$ and $U_i = X_i$ for all i except $i = i_1, \ldots, i_k$

⟨2⟩4. For $j = 1, \ldots, k$, Pick α_j such that, for all $\beta \geq \alpha_j$, we have $\pi_{i_j}(a_\beta) \in U_{i_j}$

⟨2⟩5. Pick $\alpha \in J$ such that $\alpha_1, \ldots, \alpha_k \leq \alpha$

⟨2⟩6. For all $\beta \geq \alpha$ we have $a_\beta \in V$

□

Theorem 9.18.11. *Let A be a topological space and $\{X_i\}_{i \in I}$ be a family of topological spaces. Let $f : A \to \prod_{i \in I} X_i$ be a function. If $\pi_i \circ f$ is continuous for all $i \in I$ then f is continuous.*

PROOF:
$\langle 1 \rangle 1$. LET: $i \in I$ and U be open in X_i
$\langle 1 \rangle 2$. $f^{-1}({\pi_i}^{-1}(U))$ is open in A
$\langle 1 \rangle 3$. Q.E.D.
PROOF: Proposition 9.14.3.
□

9.18.1 Continuous in Each Variable Separately

Definition 9.18.12 (Continuous in Each Variable Separately)**.** Let $F : X \times Y \to Z$. Then F is *continuous in each variable separately* if and only if:

- for every $a \in X$ the function $\lambda y \in Y.F(a, y)$ is continuous;
- for every $b \in Y$ the function $\lambda x \in X.F(x, b)$ is continuous.

Proposition 9.18.13. *Let $F : X \times Y \to Z$. If F is continuous then F is continuous in each variable separately.*

PROOF: For $a \in X$, the function $\lambda y \in Y.F(a, y)$ is $F \circ i$ where $i : Y \to X \times Y$ maps y to (a, y). We have i is continuous by Proposition 9.15.6, hence $F \circ i$ is continuous by Theorem 9.14.9.
Similarly for $\lambda x \in X.F(x, b)$ for $b \in Y$. □

Example 9.18.14. Define $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$F(x, y) = \begin{cases} xy/(x^2 + y^2) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Then F is continuous in each variable separately but not continuous.

Proposition 9.18.15. *Let $f : A \to C$ and $g : B \to D$ be open maps. Then $f \times g : A \times B \to C \times D$ is an open map.*

PROOF: Given U open in A and V open in B. Then $(f \times g)(U \times V) = f(U) \times g(V)$ is open in $C \times D$. The result follows from Proposition 9.13.3. □

Definition 9.18.16 (Sorgenfrey Plane)**.** The *Sorgenfrey plane* is $\mathbb{R}_l^2$.

9.19 The Subspace Topology

Definition 9.19.1 (Subspace Topology)**.** Let X be a topological space and $Y \subseteq X$. The *subspace topology* on Y is $\mathcal{T} = \{U \cap Y \mid U \text{ is open in } X\}$.

We prove this is a topology.

PROOF:

⟨1⟩1. $Y \in \mathcal{T}$

PROOF: Since $Y = X \cap Y$

⟨1⟩2. For all $\mathcal{U} \subseteq \mathcal{T}$, we have $\bigcup \mathcal{U} \in \mathcal{T}$

⟨2⟩1. LET: $\mathcal{U} \subseteq \mathcal{T}$

⟨2⟩2. LET: $\mathcal{V} = \{V \text{ open in } X \mid V \cap Y \in \mathcal{U}\}$

⟨2⟩3. $\bigcup \mathcal{U} = (\bigcup \mathcal{V}) \cap Y$

⟨1⟩3. For all $U, V \in \mathcal{T}$, we have $U \cap V \in \mathcal{T}$

⟨2⟩1. LET: $U, V \in \mathcal{T}$

⟨2⟩2. PICK U', V' open in X such that $U = U' \cap Y$ and $V = V' \cap Y$

⟨2⟩3. $(U \cap V) = (U' \cap V') \cap Y$

∎

Theorem 9.19.2. *Let X be a topological space and Y a subspace of X. Let $A \subseteq Y$. Then A is closed in Y if and only if there exists a closed set C in X such that $A = C \cap Y$.*

PROOF: We have

$$\begin{aligned}
& A \text{ is closed in } Y \\
\Leftrightarrow & Y \setminus A \text{ is open in } Y \\
\Leftrightarrow & \exists U \text{ open in } X . Y \setminus A = Y \cap U \\
\Leftrightarrow & \exists C \text{ closed in } X . Y \setminus A = Y \cap (X \setminus U) \\
\Leftrightarrow & \exists C \text{ closed in } X . A = Y \cap U
\end{aligned}$$

∎

Theorem 9.19.3. *Let Y be a subspace of X. Let $A \subseteq Y$. Let $\overline{A}$ be the closure of A in X. Then the closure of A in Y is $\overline{A} \cap Y$.*

PROOF: The closure of A in Y is

$$\begin{aligned}
& \bigcap \{C \text{ closed in } Y \mid A \subseteq C\} \\
= & \bigcap \{D \cap Y \mid D \text{ closed in } X, A \subseteq D \cap Y\} && \text{(Theorem 9.19.2)} \\
= & \bigcap \{D \mid D \text{ closed in } X, A \subseteq D\} \cap Y \\
= & \overline{A} \cap Y
\end{aligned}$$

∎

Lemma 9.19.4. *Let X be a topological space and $Y \subseteq X$. Let $\mathcal{B}$ be a basis for the topology on X. Then $\mathcal{B}' = \{B \cap Y \mid B \in \mathcal{B}\}$ is a basis for the subspace topology on Y.*

PROOF:

⟨1⟩1. Every element in $\mathcal{B}'$ is open in Y

⟨1⟩2. For every open set U in Y and point $y \in U$, there exists $B' \in \mathcal{B}'$ such that $y \in B' \subseteq U$

⟨2⟩1. LET: U be open in Y and $y \in U$

⟨2⟩2. PICK V open in X such that $U = V \cap Y$

⟨2⟩3. PICK $B \in \mathcal{B}$ such that $y \in B \subseteq V$

⟨2⟩4. LET: $B' = B \cap Y$
⟨2⟩5. $B' \in \mathcal{B}'$
⟨2⟩6. $y \in B' \subseteq U$
⟨1⟩3. Q.E.D.
PROOF: By Lemma 9.9.3.
□

Lemma 9.19.5. *Let X be a topological space and $Y \subseteq X$. Let $\mathcal{S}$ be a basis for the topology on X. Then $\mathcal{S}' = \{S \cap Y \mid S \in \mathcal{S}\}$ is a subbasis for the subspace topology on Y.*

PROOF: The set $\{B \cap Y \mid B$ is a finite intersection of elements of $\mathcal{S}\}$ is a basis for the subspace topology by Lemma 9.19.4, and this is the set of all finite intersections of elements of $\mathcal{S}'$. □

Lemma 9.19.6. *Let Y be a subspace of X. If U is open in Y and Y is open in X then U is open in X.*

PROOF:
⟨1⟩1. PICK V open in X such that $U = V \cap Y$
⟨1⟩2. U is open in X
PROOF: Since it is the intersection of two open sets V and Y.
□

Theorem 9.19.7. *Let Y be a subspace of X and $A \subseteq Y$. If A is closed in Y and Y is closed in X then A is closed in X.*

PROOF: Pick a closed set C in X such that $A = C \cap Y$ (Theorem 9.19.2). Then A is the intersection of two sets closed in X, hence A is closed in X (Lemma 9.4.4). □

Theorem 9.19.8. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let A_i be a subspace of X_i for all $i \in I$. Then the product topology on $\prod_{i \in I} A_i$ is the same as the topology it inherits as a subspace of $\prod_{i \in I} X_i$.*

PROOF: The product topology is generated by the subbasis

$$\begin{aligned} &\{{\pi_i}^{-1}(U) \mid i \in I, U \text{ open in } A_i\} \\ =&\{{\pi_i}^{-1}(V) \cap A_i \mid i \in I, V \text{ open in } X_i\} \\ =&\{{\pi_i}^{-1}(V) \mid i \in I, V \text{ open in } X_i\} \cap \prod_{i \in I} A_i \end{aligned}$$

and this is a subbasis for the subspace topology by Lemma 9.19.5. □

Theorem 9.19.9. *Let X be an ordered set in the order topology. Let $Y \subseteq X$ be an interval. Then the order topology on Y is the same as the subspace topology on Y.*

PROOF:
⟨1⟩1. The order topology is finer than the subspace topology.

$\langle 2\rangle 1$. For every open ray R in X, the set $R \cap Y$ is open in the order topology.
$\langle 3\rangle 1$. For all $a \in X$, we have $(-\infty, a) \cap Y$ is open in the order topology.
$\langle 4\rangle 1$. CASE: For all $y \in Y$ we have $y < a$
PROOF: In this case $(-\infty, a) \cap Y = Y$.
$\langle 4\rangle 2$. CASE: For all $y \in Y$ we have $a < y$
PROOF: In this case $(-\infty, a) \cap Y = \emptyset$.
$\langle 4\rangle 3$. CASE: There exists $y \in Y$ such that $y \leq a$ and $y \in Y$ such that $a \leq y$
$\langle 5\rangle 1$. $a \in Y$
PROOF: Because Y is an interval.
$\langle 5\rangle 2$. $(-\infty, a) \cap Y = \{y \in Y \mid y < a\}$
$\langle 3\rangle 2$. For all $a \in X$, we have $(a, +\infty) \cap Y$ is open in the order topology.
PROOF: Similar.
$\langle 2\rangle 2$. Q.E.D.
PROOF: By Lemmas 9.16.2 and 9.19.5 and Proposition 9.9.11.
$\langle 1\rangle 2$. The subspace topology is finer than the order topology.
$\langle 2\rangle 1$. Every open ray in Y is open in the subspace topology.
PROOF: For any $a \in Y$ we have $(-\infty, a)_Y = (-\infty, a)_X \cap Y$ and $(a, +\infty)_Y = (a, +\infty)_X \cap Y$.
$\langle 2\rangle 2$. Q.E.D.
PROOF: By Lemma 9.16.2 and Proposition 9.9.11
□

This example shows that we cannot remove the hypothesis that Y is an interval:

Example 9.19.10. The order topology on I_o^2 is different from the subspace topology as a subspace of $\mathbb{R}^2$ under the dictionary order topology.

PROOF: The set $\{1/2\} \times (1/2, 1)$ is open in the subspace topology but not in the order topology. □

Proposition 9.19.11. *Let X be a topological space, Y a subspace of X, and Z a subspace of Y. Then the subspace topology on Z inherited from X is the same as the subspace topology on Z inherited from Y.*

PROOF: The subspace topology inherited from Y is

$$\begin{aligned} &\{V \cap Z \mid V \text{ open in } Y\} \\ =&\{U \cap Y \cap Z \mid U \text{ open in } X\} \\ =&\{U \cap Z \mid U \text{ open in } X\} \end{aligned}$$

which is the subspace topology inherited from X. □

Definition 9.19.12 (Unit Circle)**.** The *unit circle* S^1 is $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ as a subspace of $\mathbb{R}^2$.

Definition 9.19.13 (Unit 2-sphere)**.** The *unit 2-sphere* is $S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$ as a subspace of $\mathbb{R}^3$.

Proposition 9.19.14. *Let $f : X \to Y$ be an open map and $A \subseteq X$ be open. Then the restriction $f \upharpoonright A : A \to f(A)$ is an open map.*

PROOF:
⟨1⟩1. LET: U be open in A
⟨1⟩2. U is open in X
  PROOF: Lemma 9.19.6.
⟨1⟩3. $f(U)$ is open in Y
⟨1⟩4. $f(U)$ is open in $f(A)$
  PROOF: Since $f(U) = f(U) \cap f(A)$.
□

Example 9.19.15. This example shows that we cannot remove the hypothesis that A is open.

Let $A = \{(x, y) \in \mathbb{R}^2 \mid (x > 0 \text{ and } y = 1/x) \text{ or } x = y = 0\}$. Then $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ is an open map, but $\pi_1 \upharpoonright A : A \to [0, +\infty)$ is not, because it maps the set $\{0, 0\}$ which is open in A to $\{0\}$ which is not open in $[0, +\infty)$.

Proposition 9.19.16. *Let Y be a subspace of X. Let $A \subseteq Y$ and $l \in Y$. Then l is a limit point of A in Y if and only if l is a limit point of A in X.*

PROOF: Both are equivalent to the condition that any neighbourhood of l in X intersects A in a point other than l. □

9.20 The Box Topology

Definition 9.20.1 (Box Topology)**.** Let $\{A_i\}_{i \in I}$ be a family of topological spaces. The *box topology* on $\prod_{i \in I} A_i$ is the topology generated by the set of all sets of the form $\prod_{i \in I} U_i$ where $\{U_i\}_{i \in I}$ is a family such that each U_i is open in A_i.

This is a basis since it covers $\prod_{i \in I} A_i$ and is closed under intersection.

Proposition 9.20.2. *The box topology is finer than the product topology.*

PROOF: From Proposition 9.18.2. □

Corollary 9.20.2.1. *If A_i is closed in X_i for all $i \in I$ then $\prod_{i \in I} A_i$ is closed in $\prod_{i \in I} X_i$ under the box topology.*

PROOF: From Proposition 9.18.3.

Proposition 9.20.3 (AC)**.** *Let $\{A_i\}_{i \in I}$ be a family of topological spaces. For $i \in I$, let $\mathcal{B}_i$ be a basis for the topology on A_i. Then $\mathcal{B} = \{\prod_{i \in I} B_i \mid \forall i \in I.B_i \in \mathcal{B}_i\}$ is a basis for the box topology on $\prod_{i \in I} A_i$.*

PROOF:
⟨1⟩1. Every set of the form $\prod_{i \in I} B_i$ is open.
⟨1⟩2. For every point $a \in \prod_{i \in I} A_i$ and every open set U with $a \in U$, there exists $B \in \mathcal{B}$ such that $a \in B \subseteq U$.

⟨2⟩1. LET: U be open and $a \in U$

⟨2⟩2. PICK a family $\{U_i\}_{i \in I}$ such that each U_i is open in A_i and $a \in \prod_{i \in I} U_i \subseteq U$.

⟨2⟩3. For $i \in I$, PICK $B_i \in \mathcal{B}_i$ such that $a_i \in B_i \subseteq U_i$

PROOF: Using the Axiom of Choice.

⟨2⟩4. $a \in \prod_{i \in I} B_i \subseteq U$

⟨1⟩3. Q.E.D.

PROOF: Lemma 9.9.3.

□

Theorem 9.20.4. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let A_i be a subspace of X_i for all $i \in I$. Give $\prod_{i \in I} X_i$ the box topology. Then the box topology on $\prod_{i \in I} A_i$ is the same as the topology it inherits as a subspace of $\prod_{i \in I} X_i$.*

PROOF: The box topology is generated by the basis

$$\begin{aligned}&\{\prod_{i \in I} U_i \mid \forall i \in I, U_i \text{ open in } A_i\}\\ =&\{\prod_{i \in I}(V_i \cap A_i) \mid \forall i \in I, V_i \text{ open in } X_i\}\\ =&\{\prod_{i \in I} V_i \mid \forall i \in I, V_i \text{ open in } X_i\} \cap \prod_{i \in I} A_i\end{aligned}$$

and this is a basis for the subspace topology by Lemma 9.19.4. □

Proposition 9.20.5 (AC). *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Give $\prod_{i \in I} X_i$ the box topology. Let $A_i \subseteq X_i$ for all $i \in I$. Then*

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i} \enspace .$$

PROOF:

⟨1⟩1. $\overline{\prod_{i \in I} A_i} \subseteq \prod_{i \in I} \overline{A_i}$

⟨2⟩1. For all $i \in I$ we have $A_i \subseteq \overline{A_i}$

PROOF: Lemma 9.6.3.

⟨2⟩2. $\prod_{i \in I} A_i \subseteq \prod_{i \in I} \overline{A_i}$

⟨2⟩3. Q.E.D.

PROOF: Since $\prod_{i \in I} A_i$ is closed by Corollary 9.20.2.1.

⟨1⟩2. $\prod_{i \in I} \overline{A_i} \subseteq \overline{\prod_{i \in I} A_i}$

⟨2⟩1. LET: $x \in \prod_{i \in I} \overline{A_i}$

⟨2⟩2. LET: U be a neighbourhood of x

⟨2⟩3. PICK V_i open in X_i such that $x \in \prod_{i \in I} V_i \subseteq U$

⟨2⟩4. For $i \in I$, pick $a_i \in V_i \cap A_i$

PROOF: By Theorem 9.6.6 and ⟨2⟩1 using the Axiom of Choice.

⟨2⟩5. U intersects $\prod_{i \in I} A_i$

⟨2⟩6. Q.E.D.

PROOF: $a \in U \cap \prod_{i \in I} A_i$.

∎

The following example shows that Theorem 9.18.11 fails in the box topology.

Example 9.20.6. Define $f : \mathbb{R} \to \mathbb{R}^\omega$ by $f(t) = (t, t, \ldots)$. Then $\pi_n \circ f = \mathrm{id}_\mathbb{R}$ is continuous for all n. But f is not continuous when $\mathbb{R}^\omega$ is given the box topology because the inverse image of

$$(-1, 1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times \cdots$$

is $\{0\}$ which is not open.

The following example shows that Proposition 9.18.10 fails in the box topology.

Example 9.20.7. Give $\mathbb{R}^\omega$ the box topology. Let $a_n = (1/n, 1/n, \ldots)$ for $n \geq 1$ and $l = (0, 0, \ldots)$. Then $\pi_i(a_n) \to \pi_i(l)$ as $n \to \infty$ for all i, but $a_n \not\to l$ as $n \to \infty$ since the open set

$$(-1, 1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times \cdots$$

contains l but does not contain any a_n.

Example 9.20.8. The set $\mathbb{R}^\infty$ is closed in $\mathbb{R}^\omega$ under the box topology. For let (a_n) be any sequence not in $\mathbb{R}^\infty$. Let U_n be an open interval around a_n that does not contain 0 if $a_n \neq 0$, and $U_n = \mathbb{R}$ if $a_n = 0$. Then $\prod_{n \geq 0} U_n$ is a neighbourhood of (a_n) that does not intersect $\mathbb{R}^\infty$.

9.21 T_1 Spaces

Definition 9.21.1 (T_1 Space)**.** A topological space is T_1 if and only if every singleton is closed.

Lemma 9.21.2. *A space is T_1 if and only if every finite set is closed.*

Proof: From Lemma 9.4.5. ∎

Theorem 9.21.3. *In a T_1 space, a point a is a limit point of a set A if and only if every neighbourhood of a contains infinitely many points of A.*

Proof:
⟨1⟩1. If a is a limit point of A then every neighbourhood of a contains infinitely many points of A.
⟨2⟩1. Assume: a is a limit point of A.
⟨2⟩2. Let: U be a neighbourhood of a.
⟨2⟩3. Assume: for a contradiction U contains only finitely many points of A.
⟨2⟩4. $(U \cap A) \setminus \{a\}$ is closed.
Proof: By the T_1 axiom.
⟨2⟩5. $(U \setminus A) \cup \{a\}$ is open.
Proof: It is $U \setminus ((U \cap A) \setminus \{a\})$.

$\langle 2\rangle 6.$ $(U \setminus A) \cup \{a\}$ intersects A in a point other than a.
PROOF: From $\langle 2\rangle 1$.
$\langle 2\rangle 7.$ Q.E.D.
□
$\langle 1\rangle 2.$ If every neighbourhood of a contains infinitely many points of A then a is a limit point of A.
PROOF: Immediate from definitions.
□

(To see this does not hold in every space, see Proposition 9.8.4.)

Proposition 9.21.4. *A space is T_1 if and only if, for any two distinct points x and y, there exist neighbourhoods U of x and V of y such that $x \notin V$ and $y \notin U$.*

PROOF:
$\langle 1\rangle 1.$ LET: X be a topological space.
$\langle 1\rangle 2.$ If X is T_1 then, for any two distinct points x and y, there exist neighbourhoods U of x and V of y such that $x \notin V$ and $y \notin U$.
PROOF: This holds because $\{x\}$ and $\{y\}$ are closed.
$\langle 1\rangle 3.$ Suppose, for any two distinct points x and y, there exist neighbourhoods U of x and V of y such that $x \notin V$ and $y \notin U$. Then X is T_1.
$\langle 2\rangle 1.$ ASSUME: For any two distinct points x and y, there exist neighbourhoods U of x and V of y such that $x \notin V$ and $y \notin U$.
$\langle 2\rangle 2.$ LET: $a \in X$
$\langle 2\rangle 3.$ $\{a\}$ is closed.
PROOF: For all $b \neq a$ there exists a neighbourhood U of b such that $U \subseteq X \setminus \{a\}$.
□

Proposition 9.21.5. *A subspace of a T_1 space is T_1.*

PROOF: From Proposition 9.19.7.

9.22 Hausdorff Spaces

Definition 9.22.1 (Hausdorff Space)**.** A topological space is *Hausdorff* if and only if, for any points x, y with $x \neq y$, there exist disjoint open sets U and V such that $x \in U$ and $y \in V$.

Theorem 9.22.2. *Every Hausdorff space is T_1.*

PROOF:
$\langle 1\rangle 1.$ LET: X be a Hausdorff space.
$\langle 1\rangle 2.$ LET: $b \in X$
PROVE: $\overline{\{b\}} = \{b\}$
$\langle 1\rangle 3.$ ASSUME: $a \in \overline{\{b\}}$ and $a \neq b$
$\langle 1\rangle 4.$ PICK disjoint neighbourhoods U of a and V of b.

⟨1⟩5. U intersects $\{b\}$
PROOF: Theorem 9.6.6.
⟨1⟩6. $b \in U$
⟨1⟩7. Q.E.D.
PROOF: This contradicts the fact that U and V are disjoint (⟨1⟩4).
∎

Proposition 9.22.3. *An infinite set under the finite complement topology is T_1 but not Hausdorff.*

PROOF:
⟨1⟩1. LET: X be an infinite set under the finite complement topology.
⟨1⟩2. Every singleton is closed.
PROOF: By definition.
⟨1⟩3. PICK$a, b \in X$ with $a \neq b$
⟨1⟩4. There are no disjoint neighbourhoods U of a and V of b.
⟨2⟩1. LET: U be a neighbourhood of a and V a neighbourhood of b.
⟨2⟩2. $X \setminus U$ and $X \setminus V$ are finite.
⟨2⟩3. PICK $c \in X$ that is not in $X \setminus U$ or $X \setminus V$.
⟨2⟩4. $c \in U \cap V$
∎

Proposition 9.22.4. *The product of a family of Hausdorff spaces is Hausdorff.*

PROOF:
⟨1⟩1. LET: $\{X_i\}_{i \in I}$ be a family of Hausdorff spaces.
⟨1⟩2. LET: $a, b \in \prod_{i \in I} X_i$ with $a \neq b$
⟨1⟩3. PICK $i \in I$ such that $a_i \neq b_i$
⟨1⟩4. PICK U, V disjoint open sets in X_i with $a_i \in U$ and $b_i \in V$
⟨1⟩5. ${\pi_i}^{-1}(U)$ and ${\pi_i}^{-1}(V)$ are disjoint open sets in $\prod_{i \in I} X_i$ with $a \in {\pi_i}^{-1}(U)$ and $b \in {\pi_i}^{-1}(V)$
∎

Theorem 9.22.5. *Every linearly ordered set under the order topology is Hausdorff.*

PROOF:
⟨1⟩1. LET: X be a linearly ordered set under the order topology.
⟨1⟩2. LET: $a, b \in X$ with $a \neq b$
⟨1⟩3. ASSUME: w.l.o.g. $a < b$
⟨1⟩4. CASE: There exists c such that $a < c < b$
PROOF: The sets $(-\infty, c)$ and $(c, +\infty)$ are disjoint neighbourhoods of a and b respectively.
⟨1⟩5. CASE: There is no c such that $a < c < b$
PROOF: The sets $(-\infty, b)$ and $(a, +\infty)$ are disjoint neighbourhoods of a and b respectively.
∎

Theorem 9.22.6. *A subspace of a Hausdorff space is Hausdorff.*

Proof:
$\langle 1 \rangle 1$. Let: X be a Hausdorff space and Y a subspace of X.
$\langle 1 \rangle 2$. Let: $x, y \in Y$ with $x \neq y$
$\langle 1 \rangle 3$. Pick disjoint neighbourhoods U of x and V of y in X.
$\langle 1 \rangle 4$. $U \cap Y$ and $V \cap Y$ are disjoint neighbourhoods of x and y respectively in Y.
□

Proposition 9.22.7. *A space X is Hausdorff if and only if the diagonal $\Delta = \{(x, x) \mid x \in X\}$ is closed in X^2.*

Proof:

$$
\begin{aligned}
& X \text{ is Hausdorff} \\
\Leftrightarrow & \forall x, y \in X . x \neq y \Rightarrow \exists V, W \text{ open} . x \in V \wedge y \in W \wedge V \cap W = \emptyset \\
\Leftrightarrow & \forall (x, y) \in X^2 \setminus \Delta . \exists V, W \text{ open} . (x, y) \subseteq V \times W \subseteq X^2 \setminus \Delta \\
\Leftrightarrow & \Delta \text{ is closed}
\end{aligned}
$$
□

Theorem 9.22.8. *In a Hausdorff space, a net has at most one limit.*

Proof:
$\langle 1 \rangle 1$. Let: X be a Hausdorff space.
$\langle 1 \rangle 2$. Assume: for a contradiction $(a_\alpha)_{\alpha \in J}$ is a net with limits l and m.
$\langle 1 \rangle 3$. Pick disjoint neighbourhoods U of l and V of m
Proof: By the Hausdorff axiom.
$\langle 1 \rangle 4$. Pick α and β such that $a_\gamma \in U$ for $\gamma \geq \alpha$ and $a_\gamma \in V$ for $\gamma \geq \beta$
$\langle 1 \rangle 5$. Pick $\gamma \in J$ with $\gamma \geq \alpha$ and $\gamma \geq \beta$
$\langle 1 \rangle 6$. $a_\gamma \in U \cap V$
$\langle 1 \rangle 7$. Q.E.D.
Proof: This contradicts the fact that U and V are disjoint ($\langle 1 \rangle 3$).
□

To see this is not always true in spaces that are T_1 but not Hausdorff:

Proposition 9.22.9. *Let X be an infinite set under the finite complement topology. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence with all points distinct. Then for every $l \in X$ we have $a_n \to l$ as $n \to \infty$.*

Proof: Let U be any neighbourhood of l. Since $X \setminus U$ is finite, there must exist N such that, for all $n \geq N$, we have $a_n \in U$. □

Proposition 9.22.10. *Let X be a topological space. Let Y a Hausdorff space. Let $A \subseteq X$. Let $f, g : \overline{A} \to Y$ be continuous. If f and g agree on A then $f = g$.*

Proof:
$\langle 1 \rangle 1$. Let: $x \in \overline{A}$
$\langle 1 \rangle 2$. Assume: $f(x) \neq g(x)$
$\langle 1 \rangle 3$. Pick disjoint neighbourhoods V of $f(x)$ and W of $g(x)$.
$\langle 1 \rangle 4$. Pick $y \in f^{-1}(V) \cap g^{-1}(W) \cap A$

PROOF: Since $f^{-1}(V) \cap g^{-1}(W)$ is a neighbourhood of x and hence intersects A.

⟨1⟩5. $f(y) = g(y) \in V \cap W$

⟨1⟩6. Q.E.D.

PROOF: This contradicts the fact that V and W are disjoint (⟨1⟩3).

□

Proposition 9.22.11. *Let $\{X_i\}_{i \in I}$ be a family of Hausdorff spaces. Then $\prod_{i \in I} X_i$ under the box topology is Hausdorff.*

PROOF:

⟨1⟩1. LET: $\{X_i\}_{i \in I}$ be a family of Hausdorff spaces.

⟨1⟩2. LET: $a, b \in \prod_{i \in I} X_i$ with $a \neq b$

⟨1⟩3. PICK $i \in I$ such that $a_i \neq b_i$

⟨1⟩4. PICK U, V disjoint open sets in X_i with $a_i \in U$ and $b_i \in V$

⟨1⟩5. ${\pi_i}^{-1}(U)$ and ${\pi_i}^{-1}(V)$ are disjoint open sets in $\prod_{i \in I} X_i$ with $a \in {\pi_i}^{-1}(U)$ and $b \in {\pi_i}^{-1}(V)$

□

Proposition 9.22.12. *Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on the same set X with $\mathcal{T} \subseteq \mathcal{T}'$ If $\mathcal{T}$ is Haudorff then $\mathcal{T}'$ is Hausdorff.*

PROOF: Immediate from definitions.

Proposition 9.22.13. *Let X be a Hausdorff space. Let $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the finite intersection property. Then $\bigcap_{D \in \mathcal{D}} \overline{D}$ contains at most one point.*

PROOF:

⟨1⟩1. LET: $x, y \in \bigcap_{D \in \mathcal{D}} \overline{D}$

⟨1⟩2. ASSUME: for a contradiction $x \neq y$

⟨1⟩3. PICK disjoint open subsets U and V of x and y respectively.

⟨1⟩4. $U, V \in \mathcal{D}$

PROOF: Proposition 9.6.9.

⟨1⟩5. Q.E.D.

PROOF: This contradicts the fact that $\mathcal{D}$ satisfies the finite intersection property.

□

9.23 The First Countability Axiom

Definition 9.23.1 (First Countability Axiom)**.** A topological space X satisfies the *first countability axiom*, or is *first countable*, if and only if every point has a countable local basis.

Example 9.23.2. The space S_Ω is first countable. For any $\alpha \in S_\Omega$, the set $\{(\beta, \alpha + 1) \mid \beta < \alpha\} \cup \{[0, \alpha + 1)\}$ is a local basis at α.

Lemma 9.23.3 (Sequence Lemma (CC)). *Let X be a first countable space. Let $A \subseteq X$ and $l \in \overline{A}$. Then there exists a sequence in A that converges to l.*

PROOF:
⟨1⟩1. PICK a countable local basis $\{B_n \mid n \in \mathbb{Z}^+\}$ at l such that $B_1 \supseteq B_2 \supseteq \cdots$.
  PROOF: Lemma 9.10.2.
⟨1⟩2. For all $n \geq 1$, PICK $a_n \in A \cap B_n$.
    PROVE: $a_n \to l$ as $n \to \infty$
⟨1⟩3. LET: U be a neighbourhood of A
⟨1⟩4. PICK N such that $B_N \subseteq U$
⟨1⟩5. For $n \geq N$ we have $a_n \in U$
  PROOF: $a_n \in B_n \subseteq B_N \subseteq U$
□

Example 9.23.4. The space $\overline{S_\Omega}$ is not first countable, since Ω is a limit point for S_Ω but there is no sequence of points in S_Ω that converges to Ω.

Theorem 9.23.5 (CC). *Let X be a first countable space and Y a topological space. Let $f : X \to Y$. Suppose that, for every sequence (x_n) in X and $l \in X$, if $x_n \to l$ as $n \to \infty$, then $f(x_n) \to f(l)$ as $n \to \infty$. Then f is continuous.*

PROOF:
⟨1⟩1. LET: $A \subseteq X$
⟨1⟩2. LET: $a \in A$
    PROVE: $f(a) \in \overline{f(A)}$
⟨1⟩3. PICK a sequence (x_n) in A that converges to a.
  PROOF: By the Sequence Lemma.
⟨1⟩4. $f(x_n) \to f(a)$
⟨1⟩5. $f(a) \in \overline{f(A)}$
  PROOF: By Lemma 9.11.3.
⟨1⟩6. Q.E.D.
  PROOF: By Theorem 9.14.6.
□

Example 9.23.6 (CC). The space $\mathbb{R}^\omega$ under the box product is not first countable.

PROOF: Let $\{B_n \mid n \geq 0\}$ be a countable set of neighbourhoods of $\vec{0}$. We will construct a neighbourhood of $\vec{0}$ that does not include any of these.
For $n \geq 0$, pick a neighbourhood U_n of 0 such that $U_n \subset \pi_n(B_p)$. Then $\prod_{n=0}^{\infty} U_n$ is a neighbourhood of $\vec{0}$ that does not include any B_n. □

Example 9.23.7. If J is an uncountable set then $\mathbb{R}^J$ is not first countable.

PROOF:
⟨1⟩1. LET: $\{B_n \mid n \geq 0\}$ be any countable set of neighbourhoods of $\vec{0}$.
⟨1⟩2. For $n \geq 0$, PICK a basis element $\prod_{\alpha \in J} U_{n\alpha}$ that contains $\vec{0}$ and is included in B_n.
  PROOF: Using the Axiom of Countable Choice.

⟨1⟩3. For $n \geq 0$,
  Let: $J_n = \{\alpha \in J \mid U_{n\alpha} \neq \mathbb{R}\}$
⟨1⟩4. Pick $\beta \in J$ such that $\beta \notin J_n$ for any n.
  Proof: Since each J_n is finite so $\bigcup_n J_n$ is countable.
⟨1⟩5. $\pi_\beta((-1,1))$ is a neighbourhood of $\vec{0}$ that does not include any B_n.
□

Example 9.23.8. The space $\mathbb{R}_l$ is first countable.

Proof: For any $a \in \mathbb{R}$, the set $\{[a, a+1/n) \mid n \geq 1\}$ is a countable local basis.

Example 9.23.9. The ordered square is first countable.

Proof: For any $(a,b) \in I_o^2$ with $b \neq 0, 1$, the set $\{(\{a\} \times (b-1/n, b+1/n)) \cap I_o^2 \mid n \geq 1\}$ is a countable local basis.

9.24 Strong Continuity

Definition 9.24.1 (Strongly Continuous)**.** Let X and Y be topological spaces and $f : X \to Y$ be a function. Then f is *strongly continuous* if and only if, for every subset $U \subseteq Y$, we have U is open in Y if and only if $f^{-1}(U)$ is open in X.

Proposition 9.24.2. *Let X and Y be topological spaces and $f : X \to Y$ be a function. Then f is* strongly continuous *if and only if, for every subset $C \subseteq Y$, we have C is closed in Y if and only if $f^{-1}(C)$ is closed in X.*

Proof: Since $X \setminus f^{-1}(C) = f^{-1}(Y \setminus C)$. □

Proposition 9.24.3. *Let X, Y and Z be topological spaces. Let $f : X \to Y$ and $g : Y \to Z$. If f and g are strongly continuous then so is $g \circ f$.*

Proof: Since $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$. □

Proposition 9.24.4. *Let X, Y and Z be topological spaces. Let $f : X \to Y$ and $g : Y \to Z$. If $g \circ f$ is continuous and f is strongly continuous then g is continuous.*

Proof:
⟨1⟩1. Let: $V \subseteq Z$ be open.
⟨1⟩2. $f^{-1}(g^{-1}(V))$ is open in X.
  Proof: Since $g \circ f$ is continuous.
⟨1⟩3. $f^{-1}(V)$ is open in Y.
  Proof: Since g is strongly continuous.
□

Proposition 9.24.5. *Let X, Y and Z be topological spaces. Let $f : X \to Y$ and $g : Y \to Z$. If $g \circ f$ is strongly continuous and f is strongly continuous then g is strongly continuous.*

Proof: For $V \subseteq Z$, we have V is open iff $f^{-1}(g^{-1}(V))$ is open iff $g^{-1}(V)$ is open.

9.25 Saturated Sets

Definition 9.25.1. Let X and Y be sets and $p : X \twoheadrightarrow Y$ a surjective function. Let $C \subseteq X$. Then C is *saturated* with respect to p if and only if, for all $x, y \in X$, if $x \in C$ and $p(x) = p(y)$ then $y \in C$.

Proposition 9.25.2. *Let X and Y be sets and $p : X \twoheadrightarrow Y$ a surjective function. Let $C \subseteq X$. Then the following are equivalent:*

1. *C is saturated with respect to p.*
2. *There exists $D \subseteq Y$ such that $C = p^{-1}(D)$*
3. *$C = p^{-1}(p(C))$.*

PROOF:
$\langle 1 \rangle 1$. $1 \Rightarrow 3$
  $\langle 2 \rangle 1$. ASSUME: C is saturated with respect to p.
  $\langle 2 \rangle 2$. $C \subseteq p^{-1}(p(C))$
    PROOF: Trivial.
  $\langle 2 \rangle 3$. $p^{-1}(p(C)) \subseteq C$
    $\langle 3 \rangle 1$. LET: $x \in p^{-1}(p(C))$
    $\langle 3 \rangle 2$. $p(x) \in p(C)$
    $\langle 3 \rangle 3$. There exists $y \in C$ such that $p(x) = p(y)$
    $\langle 3 \rangle 4$. $x \in C$
      PROOF: From $\langle 2 \rangle 1$.
$\langle 1 \rangle 2$. $3 \Rightarrow 2$
  PROOF: Trivial.
$\langle 1 \rangle 3$. $2 \Rightarrow 1$
  PROOF: This follows because if $p(x) \in D$ and $p(x) = p(y)$ then $p(y) \in D$.
$\square$

9.26 Quotient Maps

Definition 9.26.1 (Quotient Map)**.** Let X and Y be topological spaces and $p : X \to Y$. Then p is a *quotient map* if and only if p is surjective and strongly continuous.

Proposition 9.26.2. *Let X and Y be topological spaces and $p : X \twoheadrightarrow Y$ be a surjective function. Then the following are equivalent.*

1. *p is a quotient map.*
2. *p is continuous and maps saturated open sets to open sets.*
3. *p is continuous and maps saturated closed sets to closed sets.*

PROOF:
$\langle 1 \rangle 1$. $1 \Rightarrow 2$

⟨2⟩1. Assume: p is a quotient map.
⟨2⟩2. Let: U be a saturated open set in X.
⟨2⟩3. $p^{-1}(p(U))$ is open in X.
Proof: Since $U = p^{-1}(p(U))$ be Proposition 9.25.2.
⟨2⟩4. $p(U)$ is open in Y.
Proof: From ⟨2⟩1.
⟨1⟩2. $1 \Rightarrow 3$
Proof: Similar.
⟨1⟩3. $2 \Rightarrow 1$
⟨2⟩1. Assume: p is continuous and maps saturated open sets to open sets.
⟨2⟩2. Let: $U \subseteq Y$
⟨2⟩3. Assume: $p^{-1}(U)$ is open in X
⟨2⟩4. $p^{-1}(U)$ is saturated.
Proof: Proposition 9.25.2.
⟨2⟩5. U is open in Y.
⟨1⟩4. $3 \Rightarrow 1$
Proof: Similar.
□

Corollary 9.26.2.1. *Every surjective continuous open map is a quotient map.*

Corollary 9.26.2.2. *Every surjective continuous closed map is a quotient map.*

Example 9.26.3. The converses of these corollaries do not hold.

Let $A = \{(x, y) \mid x \geq 0\} \cup \{(x, y) \mid y = 0\}$. Then $\pi_1 : A \to \mathbb{R}$ is a quotient map, but not an open map or a closed map.

We prove that π_1 maps saturated open sets to open sets:

⟨1⟩1. Let: ${\pi_1}^{-1}(U)$ be a saturated open set in A
Prove: U is open in $\mathbb{R}$
⟨1⟩2. Let: $x \in U$
⟨1⟩3. $(x, 0) \in \pi_1(U)^{-1}$
⟨1⟩4. Pick W, V open in $\mathbb{R}$ such that $(x, 0) \subseteq W \times V \subseteq \pi_1(U)^{-1}$
⟨1⟩5. $x \in W \subseteq U$

It is not an open map because it maps $((-1, 1) \times (1, 2)) \cap A$ to $[0, 1)$.

It is not a closed map because it maps $\{(x, 1/x) \mid x > 0\}$ to $(0, +\infty)$.

Proposition 9.26.4. *Let $p : X \twoheadrightarrow Y$ be a quotient map. Let $A \subseteq X$ be saturated with respect to p. Let $q : A \twoheadrightarrow p(A)$ be the restriction of p.*

1. *If A is either open or closed in X then q is a quotient map.*

2. *If p is either an open map or a closed map then q is a quotient map.*

Proof:
⟨1⟩1. Let: $p : X \twoheadrightarrow Y$ be a quotient map.
⟨1⟩2. Let: $A \subseteq X$ be saturated with respect to p.
⟨1⟩3. Let: $q : A \twoheadrightarrow p(A)$ be the restriction of p.

$\langle 1\rangle 4.$ q is continuous.
Proof: Theorem 9.14.10.
$\langle 1\rangle 5.$ If A is open in X then q is a quotient map.
$\langle 2\rangle 1.$ Assume: A is open in X.
$\langle 2\rangle 2.$ q maps saturated open sets to open sets.
$\langle 3\rangle 1.$ Let: $U \subseteq A$ be saturated with respect to q and open in A
$\langle 3\rangle 2.$ U is saturated with respect to p
$\langle 4\rangle 1.$ Let: $x, y \in X$
$\langle 4\rangle 2.$ Assume: $x \in U$
$\langle 4\rangle 3.$ Assume: $p(x) = p(y)$
$\langle 4\rangle 4.$ $x \in A$
Proof: From $\langle 3\rangle 1$ and $\langle 4\rangle 2$.
$\langle 4\rangle 5.$ $y \in A$
Proof: From $\langle 1\rangle 2$ and $\langle 4\rangle 3$
$\langle 4\rangle 6.$ $q(x) = x(y)$
Proof: From $\langle 1\rangle 3$, $\langle 4\rangle 3$, $\langle 4\rangle 4$, $\langle 4\rangle 5$.
$\langle 4\rangle 7.$ $y \in U$
Proof: From $\langle 3\rangle 1$, $\langle 4\rangle 2$, $\langle 4\rangle 6$
$\langle 3\rangle 3.$ U is open in X
Proof: Lemma 9.19.6, $\langle 2\rangle 1$, $\langle 3\rangle 1$.
$\langle 3\rangle 4.$ $p(U)$ is open in Y
Proof: Proposition 9.26.2, $\langle 1\rangle 1$, $\langle 3\rangle 2$, $\langle 3\rangle 3$
$\langle 3\rangle 5.$ $q(U)$ is open in $p(A)$
Proof: Since $q(U) = p(U) = p(U) \cap p(A)$.
$\langle 2\rangle 3.$ Q.E.D.
Proof: By Proposition 9.26.2.
$\langle 1\rangle 6.$ If A is closed in X then q is a quotient map.
Proof: Similar.
$\langle 1\rangle 7.$ If p is an open map then q is a quotient map.
$\langle 2\rangle 1.$ Assume: p is an open map
$\langle 2\rangle 2.$ q maps saturated open sets to open sets.
$\langle 3\rangle 1.$ Let: U be open in A and saturated with respect to q
$\langle 3\rangle 2.$ Pick V open in X such that $U = A \cap V$
$\langle 3\rangle 3.$ $p(V)$ is open in Y
$\langle 3\rangle 4.$ $q(U) = p(V) \cap p(A)$
$\langle 4\rangle 1.$ $q(U) \subseteq p(V) \cap p(A)$
Proof: From $\langle 3\rangle 2$.
$\langle 4\rangle 2.$ $p(V) \cap p(A) \subseteq q(U)$
$\langle 5\rangle 1.$ Let: $y \in p(V) \cap p(A)$
$\langle 5\rangle 2.$ Pick $x \in V$ and $x' \in A$ such that $p(x) = p(x') = y$
$\langle 5\rangle 3.$ $x \in A$
Proof: By $\langle 1\rangle 2$.
$\langle 5\rangle 4.$ $x \in U$
Proof: From $\langle 3\rangle 2$
$\langle 2\rangle 3.$ Q.E.D.
Proof: By Proposition 9.26.2.

$\langle 1 \rangle 8$. If p is a closed map then q is a quotient map.
PROOF: Similar.
□

Example 9.26.5. This example shows we cannot remove the hypotheses on A and p.

Define $f : [0, 1] \to [2, 3] \to [0, 2]$ by $f(x) = x$ if $x \leq 1$, $f(x) = x - 1$ if $x \geq 2$. Then f is a quotient map but its restriction f' to $[0, 1) \cup [2, 3]$ is not, because ${f'}^{-1}([1, 2])$ is open but $[1, 2]$ is not.

For a counterexample where A is saturated, see Example 9.27.3.

Proposition 9.26.6. *Let $p : A \twoheadrightarrow C$ and $q : B \twoheadrightarrow D$ be open quotient maps. Then $p \times q : A \times B \to C \times D$ is an open quotient map.*

PROOF: From Corollary 9.26.2.1, Proposition 9.18.15 and Theorem 9.18.11. □

Theorem 9.26.7. *Let $p : X \twoheadrightarrow Y$ be a quotient map. Let Z be a topological space and $f : Y \to Z$ be a function. Then*

1. *$f \circ p$ is continuous if and only if f is continuous.*
2. *$f \circ p$ is a quotient map if and only if f is a quotient map.*

PROOF:
$\langle 1 \rangle 1$. If $f \circ p$ is continuous then f is continuous.
PROOF: Proposition 9.24.4.
$\langle 1 \rangle 2$. If f is continuous then $f \circ p$ is continuous.
PROOF: Theorem 9.14.9.
$\langle 1 \rangle 3$. If $f \circ p$ is a quotient map then f is a quotient map.
PROOF: Proposition 9.24.5.
$\langle 1 \rangle 4$. If f is a quotient map then $f \circ p$ is a quotient map.
PROOF: From Proposition 9.24.3.
□

Proposition 9.26.8. *Let X and Y be topological spaces. Let $p : X \to Y$ and $f : Y \to X$ be continuous maps such that $p \circ f = \mathrm{id}_Y$. Then p is a quotient map.*

PROOF:
$\langle 1 \rangle 1$. LET: $V \subseteq Y$
$\langle 1 \rangle 2$. ASSUME: $p^{-1}(V)$ is open in X.
$\langle 1 \rangle 3$. $f^{-1}(p^{-1}(V))$ is open in Y.
PROOF: Because f is continuous.
$\langle 1 \rangle 4$. V is open in Y.
PROOF: Because $f^{-1}(p^{-1}(V)) = V$.
□

9.27 Quotient Topology

Definition 9.27.1 (Quotient Topology)**.** Let X be a topological space, Y a set and $p : X \twoheadrightarrow Y$ be a surjective function. Then the *quotient topology* on Y is the unique topology on Y with respect to which p is a quotient map, namely $\mathcal{T} = \{U \in \mathcal{P}Y \mid p^{-1}(U) \text{ is open in } X\}$.

We prove this is a topology.

PROOF:
$\langle 1 \rangle 1$. $Y \in \mathcal{T}$
PROOF: Since $p^{-1}(Y) = X$ by surjectivity.
$\langle 1 \rangle 2$. For all $\mathcal{A} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{A} \in \mathcal{T}$
PROOF: Since $p^{-1}(\bigcup \mathcal{A}) = \bigcup_{U \in \mathcal{A}} p^{-1}(U)$
$\langle 1 \rangle 3$. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$
PROOF: Since $p^{-1}(U \cap V) = p^{-1}(U) \cap p^{-1}(V)$.
□

Definition 9.27.2 (Quotient Space)**.** Let X be a topological space and $\sim$ an equivalence relation on X. Let $p : X \twoheadrightarrow X/\sim$ be the canonical surjection. Then $X/\sim$ under the quotient topology is called a *quotient space*, *identification space* or *decomposition space* of X.

Here is a counterexample showing we cannot remove all the hypotheses of Proposition 9.26.4 except that A is saturated.

Example 9.27.3. Let $X = (0, 1/2] \cup \{1\} \cup \{1 + 1/n : n \geq 2\}$ as a subspace of $\mathbb{R}$. Define R to be the equivalence relation on X where xRy iff ($x = y$ or $|x - y| = 1$), so we identify $1/n$ with $1 + 1/n$ for all $n \geq 2$. Let Y be the resulting quotient space X/R in the quotient topology and $p : X \twoheadrightarrow Y$ the canonical surjection.

Let $A = \{1\} \cup (0, 1/2] \setminus \{1/n : n \geq 2\} \subseteq X$. Then A is saturated under p but the restriction q of p to A is not a quotient map because it maps the saturated open set $\{1\}$ to $\{1\}$ which is not open in $p(A)$.

Proposition 9.27.4. *Let X, Y and Z be topological spaces. Let $f : X \to Y$ and $g : Y \to Z$. If f and g are quotient maps then so is $g \circ f$.*

PROOF: From Proposition 9.24.3. □

Example 9.27.5. The product of two quotient maps is not necessarily a quotient map.

Let $X = \mathbb{R}$ and X^* the quotient space formed by identifying all positive integers. Let $p : X \twoheadrightarrow X^*$ be the canonical surjection.

We prove $p \times \mathrm{id}_{\mathbb{Q}} : X \times \mathbb{Q} \to X^* \times \mathbb{Q}$ is not a quotient map.

PROOF:
$\langle 1 \rangle 1$. For $n \geq 1$,
LET: $c_n = \sqrt{2}/n$
$\langle 1 \rangle 2$. For $n \geq 1$,

LET: $U_n = \{(x, y) \in X \times \mathbb{Q} \mid n - 1/4 < x < n + 1/4, (y + n > x + c_n \text{ and } y + n > -x + c_n) \text{ or } (y + n < x + c_n \text{ and } y + n < -x + c_n)\}$

$\langle 1 \rangle 3.$ For $n \geq 1$, we have U_n is open in $X \times \mathbb{Q}$

$\langle 1 \rangle 4.$ For $n \geq 1$, we have $\{n\} \times \mathbb{Q} \subseteq U_n$

$\langle 1 \rangle 5.$ LET: $U = \bigcup_{n=1}^{\infty} U_n$

$\langle 1 \rangle 6.$ U is open in $X \times \mathbb{Q}$

$\langle 1 \rangle 7.$ U is saturated with respect to $p \times \mathrm{id}_{\mathbb{Q}}$

$\langle 1 \rangle 8.$ LET: $U' = (p \times \mathrm{id}_{\mathbb{Q}})(U)$

$\langle 1 \rangle 9.$ ASSUME: for a contradiction U' is open in $X^* \times \mathbb{Q}$

$\langle 1 \rangle 10.$ $(1, 0) \in U'$

$\langle 1 \rangle 11.$ PICK a neighbourhood W of 1 in X^* and $\delta > 0$ such that $W \times (-\delta, \delta) \subseteq U'$

$\langle 1 \rangle 12.$ $p^{-1}(W) \times (-\delta, \delta) \subseteq U$

$\langle 1 \rangle 13.$ PICK n such that $c_n < \delta$

$\langle 1 \rangle 14.$ $n \in p^{-1}(W)$

$\langle 1 \rangle 15.$ PICK $\epsilon > 0$ such that $\epsilon < \delta - c_n$ and $\epsilon < 1/4$ and $(n - \epsilon, n + \epsilon) \subseteq p^{-1}(W)$

$\langle 1 \rangle 16.$ $(n - \epsilon, n + \epsilon) \times (-\delta, \delta) \subseteq U$

$\langle 1 \rangle 17.$ PICK a rational y such that $c_n - \epsilon/2 < y < c_n + \epsilon/2$

$\langle 1 \rangle 18.$ $(n + \epsilon/2, y) \notin U$

$\langle 1 \rangle 19.$ Q.E.D.

PROOF: This contradicts $\langle 1 \rangle 16$.

□

Proposition 9.27.6. *Let X be a topological space and $\sim$ an equivalence relation on X. Then $X/\sim$ is T_1 if and only if every equivalence class is closed in X.*

PROOF: Immediate from definitions. □

9.28 Retractions

Definition 9.28.1 (Retraction)**.** Let X be a topological space and $A \subseteq X$. A *retraction* of X onto A is a continuous map $r : X \to A$ such that, for all $a \in A$, we have $r(a) = a$.

Proposition 9.28.2. *Every retraction is a quotient map.*

PROOF: Proposition 9.26.8 with f the inclusion $A \hookrightarrow X$. □

9.29 Homogeneous Spaces

Definition 9.29.1 (Homogeneous)**.** A topological space X is *homogeneous* if and only if, for any points $a, b \in X$, there exists a homeomorphism $\phi : X \cong X$ such that $\phi(a) = b$.

9.30 Regular Spaces

Definition 9.30.1 (Regular Space)**.** A topological space X is *regular* if and only if, for any closed set A and point $a \notin A$, there exist disjoint open sets U,

V such that $A \subseteq U$ and $a \in V$.

9.31 Connected Spaces

Definition 9.31.1 (Separation)**.** A *separation* of a topological space X is a pair of disjoint open sets U, V such that $U \cup V = \emptyset$.

Definition 9.31.2 (Connected)**.** A topological space is *connected* if and only if it has no separation; otherwise it is *disconnected*.

Proposition 9.31.3. *A topological space X is connected if and only if the only sets that are both open and closed are X and $\emptyset$.*

Immediate from defintions.

Lemma 9.31.4. *If Y is a subspace of X, a separation of Y is a pair of disjoint nonempty sets A and B whose union is Y, neither of which contains a limit point of the other.*

PROOF:
$\langle 1 \rangle 1$. LET: $A, B \subseteq Y$
$\langle 1 \rangle 2$. If A and B form a separation of Y then A and B are disjoint and nonempty, $A \cup B = Y$, and neither of A and B contains a limit point of the other.
  $\langle 2 \rangle 1$. ASSUME: A and B form a separation of Y
  $\langle 2 \rangle 2$. A and B are disjoint and nonempty and $A \cup B = Y$
    PROOF: From $\langle 2 \rangle 1$ and the definition of separation.
  $\langle 2 \rangle 3$. A does not contain a limit point of B
    $\langle 3 \rangle 1$. ASSUME: for a contradiction $l \in A$ and l is a limit point of B in X.
    $\langle 3 \rangle 2$. l is a limit point of B in Y
      PROOF: Proposition 9.19.16.
    $\langle 3 \rangle 3$. $l \in B$
      $\langle 4 \rangle 1$. B is closed in Y
        PROOF: Since A is open in Y and $B = Y \setminus A$ from $\langle 2 \rangle 1$.
      $\langle 4 \rangle 2$. Q.E.D.
        PROOF: Corollary 9.8.3.1.
    $\langle 3 \rangle 4$. Q.E.D.
      PROOF: This contradicts the fact that $A \cap B = \emptyset$ ($\langle 2 \rangle 1$).
  $\langle 2 \rangle 4$. B does not contain a limit point of A
    PROOF: Similar.
$\langle 1 \rangle 3$. If A and B are disjoint and nonempty, $A \cup B = Y$, and neither of A and B contains a limit point of the other, then A and B form a separation of Y.
  $\langle 2 \rangle 1$. ASSUME: A and B are disjoint and nonempty, $A \cup B = Y$, and neither of A and B contains a limit point of the other.
  $\langle 2 \rangle 2$. A is open in Y
    $\langle 3 \rangle 1$. B is closed in Y
      $\langle 4 \rangle 1$. LET: l be a limit point of B in Y

$\langle 4\rangle 2$. l is a limit point of B in X
PROOF: Proposition 9.19.16.
$\langle 4\rangle 3$. $l \notin A$
PROOF: By $\langle 2\rangle 1$
$\langle 4\rangle 4$. $l \in B$
PROOF: By $\langle 2\rangle 1$ since $A \cup B = Y$
$\langle 4\rangle 5$. Q.E.D.
PROOF: Corollary 9.8.3.1.
$\langle 3\rangle 2$. Q.E.D.
PROOF: Since $A = Y \setminus B$.
$\langle 2\rangle 3$. B is open in Y
PROOF: Similar.
□

Example 9.31.5. Every set under the indiscrete topology is connected.

Example 9.31.6. The discrete topology on a set X is connected if and only if $|X| \leq 1$.

Example 9.31.7. The finite complement topology on a set X is connected if and only if either $|X| \leq 1$ or X is infinite.

Example 9.31.8. The countable complement topology on a set X is connected if and only if either $|X| \leq 1$ or X is uncountable.

Example 9.31.9. The rationals $\mathbb{Q}$ are disconnected. For any irrational a, the sets $(-\infty, a) \cap \mathbb{Q}$ and $(a, +\infty) \cap \mathbb{Q}$ form a separation of $\mathbb{Q}$.

Lemma 9.31.10. *Let X be a topological space. If C and D form a separation of X, and Y is a connected subspace of X, then either $Y \subseteq C$ or $Y \subseteq D$.*

PROOF: Otherwise $Y \cap C$ and $Y \cap D$ would form a separation of Y. □

Theorem 9.31.11. *The union of a set of connected subspaces of a space X that have a point in common is connected.*

PROOF:
$\langle 1\rangle 1$. LET: $\mathcal{A}$ be a set of connected subspaces of the space X that have the point a in common.
$\langle 1\rangle 2$. ASSUME: for a contradiction C and D form a separation of $\bigcup \mathcal{A}$
$\langle 1\rangle 3$. ASSUME: without loss of generality $a \in C$
$\langle 1\rangle 4$. For all $A \in \mathcal{A}$ we have $A \subseteq C$
PROOF: Lemma 9.31.10.
$\langle 1\rangle 5$. $D = \emptyset$
$\langle 1\rangle 6$. Q.E.D.
PROOF: This contradicts $\langle 1\rangle 2$.
□

Theorem 9.31.12. *Let X be a topological space and A a connected subspace of X. If $A \subseteq B \subseteq \overline{A}$ then B is connected.*

PROOF:
$\langle 1 \rangle 1$. ASSUME: for a contradiction C and D form a separation of B.
$\langle 1 \rangle 2$. ASSUME: without loss of generality $A \subseteq C$
PROOF: Lemma 9.31.10.
$\langle 1 \rangle 3$. $B \subseteq C$
$\langle 2 \rangle 1$. LET: $x \in B$
$\langle 2 \rangle 2$. $x \in \overline{A}$
$\langle 2 \rangle 3$. Either $x \in A$ or x is a limit point of A.
PROOF: Theorem 9.8.3.
$\langle 2 \rangle 4$. Either $x \in A$ or x is a limit point of C.
PROOF: Lemma 9.8.5, $\langle 1 \rangle 2$.
$\langle 2 \rangle 5$. $x \in C$
PROOF: Lemma 9.31.4.
$\langle 1 \rangle 4$. $D = \emptyset$
$\langle 1 \rangle 5$. Q.E.D.
PROOF: This contradicts $\langle 1 \rangle 1$.
□

Theorem 9.31.13. *The image of a connected space under a continuous map is connected.*

PROOF:
$\langle 1 \rangle 1$. LET: $f : X \twoheadrightarrow Y$ be a surjective continuous map where X is connected.
$\langle 1 \rangle 2$. ASSUME: for a contradiction C and D form a separation of Y.
$\langle 1 \rangle 3$. $f^{-1}(C)$ and $f^{-1}(D)$ form a separation of X.
□

Theorem 9.31.14. *The product of a family of connected spaces is connected.*

PROOF:
$\langle 1 \rangle 1$. The product of two connected spaces is connected.
$\langle 2 \rangle 1$. LET: X and Y be connected spaces.
$\langle 2 \rangle 2$. PICK $a \in X$ and $b \in Y$
PROOF: We may assume X and Y are nonempty since otherwise $X \times Y = \emptyset$ which is connected.
$\langle 2 \rangle 3$. $X \times \{b\}$ is connected.
PROOF: It is homeomorphic to X.
$\langle 2 \rangle 4$. For all $x \in X$ we have $\{x\} \times Y$ is connected.
PROOF: It is homeomorphic to Y.
$\langle 2 \rangle 5$. For any $x \in X$
LET: $T_x = (X \times \{b\}) \cup (\{x\} \times Y)$
$\langle 2 \rangle 6$. For all $x \in X$, T_x is connected.
PROOF: Theorem 9.31.11 since $(x, b) \in (X \times \{b\}) \cap (\{x\} \times Y)$.
$\langle 2 \rangle 7$. $X \times Y$ is connected.
PROOF: Theorem 9.31.11 since $X \times Y = \bigcup_{x \in X} T_x$ and (a, b) is a point in every T_x.
$\langle 1 \rangle 2$. The product of a finite family of connected spaces is connected.

PROOF: From $\langle 1\rangle 1$ by induction.
$\langle 1\rangle 3$. The product of any family of connected spaces is connected.
$\langle 2\rangle 1$. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of connected spaces.
$\langle 2\rangle 2$. LET: $X = \prod_{\alpha \in J} X_\alpha$
$\langle 2\rangle 3$. PICK $a \in X$
PROOF: We may assume $X \neq \emptyset$ as the empty space is connected.
$\langle 2\rangle 4$. For every finite subset K of J,
LET: $X_K = \{x \in X \mid \forall \alpha \in J \setminus K. x_\alpha = a_\alpha\}$
$\langle 2\rangle 5$. For every finite $K \subseteq J$, we have X_K is connected.
PROOF: From $\langle 1\rangle 2$ since $X_K \cong \prod_{\alpha \in K} X_K$.
$\langle 2\rangle 6$. LET: $Y = \bigcup_K X_K$
$\langle 2\rangle 7$. Y is connected
PROOF: Theorem 9.31.11 since a is a common point.
$\langle 2\rangle 8$. $X = \overline{Y}$
$\langle 3\rangle 1$. LET: $x \in X$
$\langle 3\rangle 2$. LET: $U = \prod_{\alpha \in J} U_\alpha$ be a basic neighbourhood of x where $U_\alpha = X_\alpha$ for all α except $\alpha \in K$ for some finite $K \subseteq J$
$\langle 3\rangle 3$. LET: $y \in X$ be the point with $y_\alpha = x_\alpha$ for $\alpha \in K$ and $y_\alpha = a_\alpha$ for all other α
$\langle 3\rangle 4$. $y \in U \cap X_K$
$\langle 3\rangle 5$. $y \in U \cap Y$
$\langle 2\rangle 9$. X is connected.
PROOF: Theorem 9.31.12.
□

Example 9.31.15. The set $\mathbb{R}^\omega$ is disconnected under the box topology. The set of bounded sequences and the set of unbounded sequences form a separation.

Proposition 9.31.16. *Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on the same set X. If $\mathcal{T} \subseteq \mathcal{T}'$ and $(X, \mathcal{T}')$ is connected then $(X, \mathcal{T})$ is connected.*

PROOF: If U and V form a separation of $(X, \mathcal{T})$ then they form a separation of $(X, \mathcal{T}')$. □

Proposition 9.31.17. *Let X be a topological space and (A_n) a sequence of connected subspaces of X. If $A_n \cap A_{n+1} \neq \emptyset$ for all n then $\bigcup_n A_n$ is connected.*

PROOF:
$\langle 1\rangle 1$. ASSUME: for a contradiction C and D form a separation of $\bigcup_n A_n$
$\langle 1\rangle 2$. ASSUME: without loss of generality $A_0 \subseteq C$
PROOF: Lemma 9.31.10.
$\langle 1\rangle 3$. For all n we gave $A_n \subseteq C$
PROOF:
$\langle 2\rangle 1$. ASSUME: $A_n \subseteq C$
$\langle 2\rangle 2$. PICK $x \in A_n \cap A_{n+1}$
$\langle 2\rangle 3$. $x \in C$
$\langle 2\rangle 4$. $A_{n+1} \subseteq C$
PROOF: Lemma 9.31.10.

$\langle 2\rangle 5$. Q.E.D.

PROOF: The result follows by induction.

$\langle 1\rangle 4$. $D = \emptyset$

$\langle 1\rangle 5$. Q.E.D.

PROOF: This contradicts $\langle 1\rangle 1$.

□

Proposition 9.31.18. *Let X be a topological space. Let $A, C \subseteq X$. If C is connected and intersects both A and $X \setminus A$ then C intersects ∂A.*

PROOF: Otherwise $C \cap A^\circ$ and $C \setminus \overline{A}$ would form a separation of C. □

Example 9.31.19. The space $\mathbb{R}_l$ is disconnected. For any real x, the sets $(-\infty, x)$ and $[x, +\infty)$ form a separation.

Proposition 9.31.20. *Let X and Y be connected spaces. Let A be a proper subset of X and B a proper subset of Y. Then $(X \times Y) \setminus (A \times B)$ is connected.*

PROOF:

$\langle 1\rangle 1$. PICK $a \in X \setminus A$ and $b \in Y \setminus B$

$\langle 1\rangle 2$. For $x \in X \setminus A$ we have $(X \times \{b\}) \cup (\{x\} \times Y)$ is connected.

PROOF: Theorem 9.31.11 since (x, b) is a common point.

$\langle 1\rangle 3$. For $y \in Y \setminus B$ we have $(X \times \{y\}) \cup (\{a\} \times Y)$ is connected.

PROOF: Theorem 9.31.11 since (a, y) is a common point.

$\langle 1\rangle 4$. $(X \times Y) \setminus (A \times B)$ is connected.

PROOF: Theorem 9.31.11 since it is the union of the sets in $\langle 1\rangle 2$ and $\langle 1\rangle 3$ with (a, b) as a common point.

□

Proposition 9.31.21. *Let $p : X \twoheadrightarrow Y$ be a quotient map. If Y is connected and $p^{-1}(y)$ is connected for all $y \in Y$, then X is connected.*

PROOF:

$\langle 1\rangle 1$. ASSUME: for a contradiction C and D form a separation of X.

$\langle 1\rangle 2$. C is saturated.

$\langle 2\rangle 1$. LET: $x \in C$, $y \in X$ with $p(x) = p(y) = a$, say

$\langle 2\rangle 2$. $y \notin D$

PROOF: Otherwise $p^{-1}(a) \cap C$ and $p^{-1}(a) \cap D$ form a separation of $p^{-1}(a)$.

$\langle 2\rangle 3$. $y \in C$

$\langle 1\rangle 3$. D is saturated.

PROOF: Similar.

$\langle 1\rangle 4$. $p(C)$ and $p(D)$ form a separation of Y.

□

Proposition 9.31.22. *Let X be a connected space and Y a connected subspace of X. Suppose A and B form a separation of $X \setminus Y$. Then $Y \cup A$ and $Y \cup B$ are both connected.*

PROOF:

⟨1⟩1. $Y \cup A$ is connected.

⟨2⟩1. Assume: for a contradiction C and D form a separation of $Y \cup A$

⟨2⟩2. Assume: without loss of generality $Y \subseteq C$

⟨2⟩3. Pick open sets A_1, B_1, C_1, D_1 in X with

$$A = A_1 \setminus Y$$
$$B = B_1 \setminus Y$$
$$C = C_1 \cap (Y \cup A)$$
$$D = D_1 \cap (Y \cup A)$$

⟨2⟩4. $B_1 \cup C_1$ and $A_1 \cap D_1$ form a separation of X

⟨1⟩2. $Y \cup B$ is connected.

Proof: Similar.

□

Theorem 9.31.23. *Let L be a linearly ordered set under the order topology. Then L is connected if and only if L is a linear continuum.*

Proof:

⟨1⟩1. If L is a linear continuum then L is connected.

⟨2⟩1. Let: L be a linear continuum under the order topology.

⟨2⟩2. Assume: for a contradiction C and D form a separation of L.

⟨2⟩3. Pick $a \in C$ and $b \in D$.

⟨2⟩4. Assume: without loss of generality $a < b$.

⟨2⟩5. Let: $S = \{x \in L \mid a < x \text{ and } [a, x) \subseteq C\}$

⟨2⟩6. S is nonempty.

Proof: Since $a \in C$ and C is open.

⟨2⟩7. S is bounded above by b.

Proof: Since $b \notin C$.

⟨2⟩8. Let: $s = \sup S$

⟨2⟩9. $s \in S$

⟨3⟩1. Let: $y \in [a, s)$

Prove: $y \in C$

⟨3⟩2. Pick z with $y < z \in S$

Proof: By minimality of s.

⟨3⟩3. $y \in [a, z) \subseteq C$

⟨2⟩10. Case: $s \in C$

⟨3⟩1. Pick x such that $s < x$ and $[s, x) \subseteq C$

Proof: Since C is open and s is not greatest in L because $s < b$.

⟨3⟩2. $x \in S$

Proof: Since $[a, x) = [a, s) \cup [s, x) \subseteq C$.

⟨3⟩3. Q.E.D.

Proof: This contradicts the fact that s is an upper bound for S.

⟨2⟩11. Case: $s \in D$

⟨3⟩1. Pick $x < s$ such that $(x, s] \subseteq D$

⟨3⟩2. Pick y with $x < y < s$

Proof: Since L is dense.

⟨3⟩3. $y \in C$

PROOF: From $\langle 2 \rangle 9$.

$\langle 3 \rangle 4$. $y \in D$

PROOF: From $\langle 3 \rangle 1$.

$\langle 3 \rangle 5$. Q.E.D.

$\langle 3 \rangle 6$. LET: L be a linear continuum under the order topology.

$\langle 3 \rangle 7$. ASSUME: for a contradiction C and D form a separation of L.

$\langle 3 \rangle 8$. PICK $a \in C$ and $b \in D$.

$\langle 3 \rangle 9$. ASSUME: without loss of generality $a < b$.

$\langle 3 \rangle 10$. LET: $S = \{x \in L \mid a < x \text{ and } [a, x) \subseteq C\}$

$\langle 3 \rangle 11$. S is nonempty.

PROOF: Since $a \in C$ and C is open.

$\langle 3 \rangle 12$. S is bounded above by b.

PROOF: Since $b \notin C$.

$\langle 3 \rangle 13$. LET: $s = \sup S$

$\langle 3 \rangle 14$. $s \in S$

$\langle 4 \rangle 1$. LET: $y \in [a, s)$

PROVE: $y \in C$

$\langle 4 \rangle 2$. PICK z with $y < z \in S$

PROOF: By minimality of s.

$\langle 4 \rangle 3$. $y \in [a, z) \subseteq C$

$\langle 3 \rangle 15$. CASE: $s \in C$

$\langle 4 \rangle 1$. PICK x such that $s < x$ and $[s, x) \subseteq C$

PROOF: Since C is open and s is not greatest in L because $s < b$.

$\langle 4 \rangle 2$. $x \in S$

PROOF: Since $[a, x) = [a, s) \cup [s, x) \subseteq C$.

$\langle 4 \rangle 3$. Q.E.D.

PROOF: This contradicts the fact that s is an upper bound for S.

$\langle 3 \rangle 16$. CASE: $s \in D$

$\langle 4 \rangle 1$. PICK $x < s$ such that $(x, s] \subseteq D$

$\langle 4 \rangle 2$. PICK y with $x < y < s$

PROOF: Since L is dense.

$\langle 4 \rangle 3$. $y \in C$

PROOF: From $\langle 2 \rangle 9$.

$\langle 4 \rangle 4$. $y \in D$

PROOF: From $\langle 3 \rangle 1$.

$\langle 4 \rangle 5$. Q.E.D.

PROOF: This contradicts $\langle 2 \rangle 2$.

$\langle 1 \rangle 2$. If L is connected then L is a linear continuum.

$\langle 2 \rangle 1$. ASSUME: L is connected.

$\langle 2 \rangle 2$. Every nonempty subset of L that is bounded above has a supremum.

$\langle 3 \rangle 1$. LET: X be a nonempty subset of L bounded above by b.

$\langle 3 \rangle 2$. ASSUME: for a contradiction X has no supremum.

$\langle 3 \rangle 3$. LET: U be the set of upper bounds of X,

$\langle 3 \rangle 4$. U is nonempty.

PROOF: Since $b \in U$.

$\langle 3 \rangle 5$. U is open.

⟨4⟩1. LET: $x \in U$
⟨4⟩2. PICK an upper bound y for X such that $y < x$
⟨4⟩3. Either x is greatest in L and $(y, x] \subseteq U$, or there exists $z > x$ such that $(y, z) \subseteq U$
⟨3⟩6. LET: V be the set of lower bounds of U.
⟨3⟩7. V is nonempty.
PROOF: Since $X \subseteq V$
⟨3⟩8. V is open.
⟨4⟩1. LET: $x \in V$
⟨4⟩2. PICK $y \in X$ with $x < y$
PROOF: x cannot be an upper bound for X, because it would be the supremum of X.
⟨4⟩3. Either x least in L and $[x, y) \subseteq V$, or there exists $z < x$ such that $(z, y) \subseteq V$
⟨3⟩9. $L = U \cup V$
⟨4⟩1. LET: $x \in L \setminus U$
⟨4⟩2. PICK $y \in X$ such that $x < y$
⟨4⟩3. For all $u \in U$ we have $x < y \leq u$
⟨4⟩4. $x \in V$
⟨3⟩10. $U \cap V = \emptyset$
PROOF: Any element of $U \cap V$ would be a supremum of X.
⟨3⟩11. U and V form a separation of L.
⟨3⟩12. Q.E.D.
PROOF: This contradicts ⟨2⟩1.
⟨2⟩3. L is dense.
⟨3⟩1. LET: $x, y \in L$ with $x < y$
⟨3⟩2. There exists $z \in L$ such that $x < z < y$
PROOF: Otherwise $(-\infty, y)$ and $(x, +\infty)$ would form a separation of L.
□

Corollary 9.31.23.1. *The real line* $\mathbb{R}$ *is connected.*

Corollary 9.31.23.2. *Every interval in* $\mathbb{R}$ *is connected.*

Corollary 9.31.23.3. *The ordered square is connected.*

Theorem 9.31.24 (Intermediate Value Theorem)**.** *Let X be a connected space. Let Y be a linearly ordered set under the order topology. Let $f : X \to Y$ be continuous. Let $a, b \in X$ and $r \in Y$. Suppose $f(a) < r < f(b)$. Then there exists $c \in X$ such that $f(c) = r$.*

PROOF: Otherwise $f^{-1}((-\infty, r))$ and $f^{-1}((r, +\infty))$ would form a separation of X. □

Proposition 9.31.25. *Every function* $f : [0, 1] \to [0, 1]$ *has a fixed point.*

PROOF:
⟨1⟩1. LET: $g : [0, 1] \to [-1, 1]$ be the function $g(x) = f(x) - x$

PROVE: there exists $x \in [0,1]$ such that $g(x) = 0$

$\langle 1 \rangle 2$. ASSUME: without loss of generality $g(0) \neq 0$ and $g(1) \neq 0$

$\langle 1 \rangle 3$. $g(0) > 0$

$\langle 1 \rangle 4$. $g(1) < 0$

$\langle 1 \rangle 5$. There exists $x \in (0,1)$ such that $g(x) = 0$

PROOF: By the Intermediate Value Theorem.

Proposition 9.31.26. *Give $\mathbb{R}^\omega$ the box topology. Let $x, y \in \mathbb{R}^\omega$. Then x and y lie in the same comoponent if and only if $x - y$ is eventually zero, i.e. there exists N such that, for all $n \geq N$, we have $x_n = y_n$.*

PROOF:

$\langle 1 \rangle 1$. The component containing 0 is the set of sequences that are eventually zero.

$\langle 2 \rangle 1$. LET: B be the set of sequences that are eventually zero.

$\langle 2 \rangle 2$. B is path-connected.

$\langle 3 \rangle 1$. LET: $x, y \in B$

$\langle 3 \rangle 2$. PICK N such that, for all $n \geq N$, we have $x_n = y_n = 0$

$\langle 3 \rangle 3$. LET: $p : [0,1] \to \mathbb{R}^\omega$, $p(t) = (1-t)x + ty$

PROVE: p is continuous.

$\langle 3 \rangle 4$. LET: $t \in [0,1]$ and $\prod_j U_j$ be a basic open neighbourhood of $p(t)$, where each U_j is open in $\mathbb{R}$

$\langle 3 \rangle 5$. PICK δ such that, for all $n < N$ and all $s \in [0,1]$, if $|s - t| < \delta$ then $p(s)_n \in U_n$

$\langle 3 \rangle 6$. For all $s \in [0,1]$, if $|s - t| < \delta$ then $p(s) \in \prod_j U_j$

$\langle 2 \rangle 3$. B is connected.

PROOF: Proposition 9.33.3.

$\langle 2 \rangle 4$. If C is connected and $B \subseteq C$ then $B = C$.

$\langle 3 \rangle 1$. ASSUME: C is connected and $B \subseteq C$

$\langle 3 \rangle 2$. ASSUME: for a contradiction $x \in C \setminus B$

$\langle 3 \rangle 3$. For $n \geq 1$,

LET: $c_n = 1$ if $x_n = 0$, $c_n = n/x_n$ otherwise

$\langle 3 \rangle 4$. LET: $h : \mathbb{R}^\omega \to \mathbb{R}^\omega$ be the function $h(x) = (c_n x_n)_{n \geq 1}$

$\langle 3 \rangle 5$. h is a homeomorphism of $\mathbb{R}^\omega$ with itself.

$\langle 3 \rangle 6$. $h(x)$ is unbounded.

PROOF: For any $b > 0$, pick $N > b$ such that $x_N \neq 0$. Then $h(x)_N > b$.

$\langle 3 \rangle 7$. $h^{-1}(\{\text{bounded sequences}\}) \cap C$ and $h^{-1}(\{\text{unbounded sequences}\}) \cap C$ form a separation of C

$\langle 3 \rangle 8$. Q.E.D.

PROOF: This contradicts $\langle 3 \rangle 1$.

$\langle 1 \rangle 2$. Q.E.D.

PROOF: Since $\lambda x.x - y$ is a homeomorphism of $\mathbb{R}^\omega$ with itself.

□

Example 9.31.27. The space $\mathbb{R}_K$ is connected.

PROOF:

$\langle 1 \rangle 1$. ASSUME: for a contradiction U and V form a separation of $\mathbb{R}_K$
$\langle 1 \rangle 2$. ASSUME: without loss of generality $0 \in U$
$\langle 1 \rangle 3$. There exists an open interval (a, b) such that $(a, b) - K \subseteq U$ and $(a, b) \nsubseteq U$
PROOF: Otherwise U and V would form a separation of $\mathbb{R}$.
$\langle 1 \rangle 4$. PICK $1/n \in (a, b) - U$
$\langle 1 \rangle 5$. $1/n \in V$
$\langle 1 \rangle 6$. There exists an open interval (c, d) around $1/n$ that is included in V
$\langle 1 \rangle 7$. Q.E.D.
PROOF: This is a contradiction since $(a, b) - K$ and (c, d) must intersect.
□

9.32 Totally Disconnected Spaces

Definition 9.32.1 (Totally Disconnected)**.** A topological space X is *totally disconnected* if and only if the only connected subspaces are the singletons.

Example 9.32.2. Every discrete space is totally disconnected.

Example 9.32.3. The rationals $\mathbb{Q}$ are totally disconnected.

Example 9.32.4. The Cantor set is totally disconnected.

PROOF:
$\langle 1 \rangle 1$. LET: (A_n) be the sequence of sets in Definition 8.1.1.
$\langle 1 \rangle 2$. LET: C be the Cantor set $\bigcap_n A_n$
$\langle 1 \rangle 3$. ASSUME:
for a contradiction $D \subseteq C$ is connected and has more than one point.
$\langle 1 \rangle 4$. LET: $x, y \in D$ with $x < y$
$\langle 1 \rangle 5$. PICK n such that $|x - y| > 1/3^n$
$\langle 1 \rangle 6$. A_n is a sequence of disjoint intervals of length $1/3^n$
$\langle 1 \rangle 7$. x and y are in two different intervals out of the intervals that make up A_n
$\langle 1 \rangle 8$. There exists z with $x < z < y$ such that $z \notin A_n$
$\langle 1 \rangle 9$. $(-\infty, z) \cap D$ and $(z, +\infty) \cap D$ form a separation of D.
□

9.33 Paths and Path Connectedness

Definition 9.33.1 (Path)**.** Let X be a topological space and $a, b \in X$. A *path* from a to b is a continuous function $p : [0, 1] \to X$ such that $p(0) = a$ and $p(1) = b$.

Definition 9.33.2 (Path Connected)**.** A topological space is *path connected* if and only if there exists a path between any two points.

Proposition 9.33.3. *Every path connected space is connected.*

PROOF:

$\langle 1 \rangle 1$. LET: X be a path connected space.
$\langle 1 \rangle 2$. ASSUME: for a contradiction C and D form a separation of X.
$\langle 1 \rangle 3$. PICK $a \in C$ and $b \in D$.
$\langle 1 \rangle 4$. PICK a path $p : [0,1] \to X$ from a to b.
$\langle 1 \rangle 5$. $p^{-1}(C)$ and $p^{-1}(D)$ form a separation of $[0,1]$.
$\langle 1 \rangle 6$. Q.E.D.
PROOF: This contradicts Corollary 9.31.23.2.
□

An example that shows the converse does not hold:

Example 9.33.4. The ordered square is not path connected.

PROOF:
$\langle 1 \rangle 1$. ASSUME: for a contradiction $p : [0,1] \to I_o^2$ is a path from $(0,0)$ to $(1,1)$.
$\langle 1 \rangle 2$. p is surjective.
PROOF: By the Intermediate Value Theorem.
$\langle 1 \rangle 3$. For $x \in [0,1]$, PICK a rational $q_x \in p^{-1}((x,0),(x,1))$
PROOF: Since $p^{-1}((x,0),(x,1))$ is open and nonempty by $\langle 1 \rangle 2$.
$\langle 1 \rangle 4$. For $x, y \in [0,1]$, if $x \neq y$ then $q_x \neq q_y$
PROOF: We have $p(q_x) \neq p(q_y)$ because $((x,0),(x,1))$ and $((y,0),(y,1))$ are disjoint.
$\langle 1 \rangle 5$. $\{q_x \mid x \in [0,1]\}$ is an uncountable set of rationals.
$\langle 1 \rangle 6$. Q.E.D.
PROOF: This contradicts the fact that the rationals are countable.
□

Proposition 9.33.5. *The continuous image of a path connected space is path connected.*

PROOF:
$\langle 1 \rangle 1$. LET: X be a path connected space, Y a topological space, and $f : X \twoheadrightarrow Y$ be continuous and surjective.
$\langle 1 \rangle 2$. LET: $a, b \in Y$
$\langle 1 \rangle 3$. PICK $c, d \in X$ with $f(c) = a$ and $f(d) = b$
$\langle 1 \rangle 4$. PICK a path $p : [0,1] \to X$ from c to d.
$\langle 1 \rangle 5$. $f \circ p$ is a path from a to b in Y.
□

Proposition 9.33.6 (AC)**.** *The product of a family of path-connected spaces is path-connected.*

PROOF:
$\langle 1 \rangle 1$. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of path-connected spaces.
$\langle 1 \rangle 2$. LET: $a, b \in \prod_{\alpha \in J} X_\alpha$
$\langle 1 \rangle 3$. For $\alpha \in J$, PICK a path $p_\alpha : [0,1] \to X_\alpha$ from a_α to b_α
PROOF: Using the Axiom of Choice.
$\langle 1 \rangle 4$. Define $p : [0.1] \to \prod_{\alpha \in J} X_\alpha$ by $p(t)_\alpha = p_\alpha(t)$
$\langle 1 \rangle 5$. p is a path from a to b.

PROOF: Theorem 9.18.11.
∎

Proposition 9.33.7. *The continuous image of a path-connected space is path-connected.*

PROOF:
⟨1⟩1. LET: $f : X \twoheadrightarrow Y$ be continuous and surjective where X is path-connected.
⟨1⟩2. LET: $a, b \in Y$
⟨1⟩3. PICK $a', b' \in X$ with $f(a') = a$ and $f(b') = b$.
⟨1⟩4. PICK a path $p : [0, 1] \to X$ from a' to b'.
⟨1⟩5. $f \circ p$ is a path from a to b.
∎

Proposition 9.33.8. *Let X be a topological space. The union of a set of path-connected subspaces of X that have a point in common is path-connected.*

PROOF:
⟨1⟩1. LET: $\mathcal{A}$ be a set of path-connected subspaces of X with the point a in common.
⟨1⟩2. LET: $b, c \in \bigcup \mathcal{A}$
⟨1⟩3. PICK $B, C \in \mathcal{A}$ with $b \in B$ and $c \in C$.
⟨1⟩4. PICK a path p in B from b to a.
⟨1⟩5. PICK a path q in C from a to c.
⟨1⟩6. The concatenation of p and q is a path from b to c in $\bigcup \mathcal{A}$.
∎

Proposition 9.33.9. *Let $A \subseteq \mathbb{R}^2$ be countable. Then $\mathbb{R}^2 \setminus A$ is path-connected.*

PROOF:
⟨1⟩1. LET: $a, b \in \mathbb{R}^2 \setminus A$
⟨1⟩2. PICK a line l in $\mathbb{R}^2$ with a on one side and b on the other.
⟨1⟩3. For every point x on l,
LET: p_x be the path in $\mathbb{R}^2$ consisting of a line from a to x then a line from x to b
⟨1⟩4. For $x \neq y$ we have p_x and p_y have no points in common except a and b
⟨1⟩5. There are only countably many x such that a point of A lies on p_x.
⟨1⟩6. There exists x such that p_x is a path from a to b in $\mathbb{R}^2 \setminus A$.
∎

Proposition 9.33.10. *Every open connected subspace of $\mathbb{R}^2$ is path-connected.*

PROOF:
⟨1⟩1. LET: U be an open connected subspace of $\mathbb{R}^2$.
⟨1⟩2. For all $x_0 \in U$,
LET: $PC(x_0) = \{y \in U \mid \text{there exists a path from } x \text{ to } y\}$
⟨1⟩3. For all $x_0 \in U$, the set $PC(x_0)$ is open and closed in U.
⟨2⟩1. LET: $x_0 \in U$
⟨2⟩2. $PC(x_0)$ is open in U

⟨3⟩1. LET: $y \in PC(x_0)$
⟨3⟩2. PICK $\epsilon > 0$ such that $B(y, \epsilon) \subseteq U$
PROOF: Since U is open.
⟨3⟩3. $B(y, \epsilon) \subseteq PC(x_0)$
PROOF: For all $z \in B(y, \epsilon)$, pick a path from x_0 to y then concatenate the straight line from y to z.
⟨2⟩3. $PC(x_0)$ is closed in U
⟨3⟩1. LET: $y \in U$ be a limit point of $PC(x_0)$
⟨3⟩2. PICK $\epsilon > 0$ such that $B(y, \epsilon) \subseteq U$
⟨3⟩3. PICK $z \in PC(x_0) \cap B(y, \epsilon)$
⟨3⟩4. $y \in PC(x_0)$
PROOF: Pick a path from x_0 to z then concatenate the straight line from z to y.
⟨1⟩4. $PC(x_0) = U$
PROOF: Proposition 9.31.3.
□

Example 9.33.11. If A is a connected subspace of X, then A° is not necessarily connected.

Take two closed circles in $\mathbb{R}^2$ that touch at one point. The interior of this space is two open circles, and these two circles form a separation.

Example 9.33.12. If A is a connected subspace of X then ∂A is not necessarily connected.

We have $[0, 1]$ is connected but $\partial[0, 1] = \{0, 1\}$ is not.

Example 9.33.13. If A is a subspace of X and A° and ∂A are connected, then A is not necessarily connected.

We have $\mathbb{Q}^\circ = \emptyset$ and $\partial\mathbb{Q} = \mathbb{R}$ are connected but $\mathbb{Q}$ is not connected.

Example 9.33.14. The space $\mathbb{R}_K$ is not path connected.

PROOF:
⟨1⟩1. ASSUME: for a contradiction $p : [0, 1] \to \mathbb{R}_K$ was a path from 0 to 1.
⟨1⟩2. $p([0, 1])$ as a subspace of $\mathbb{R}_K$ is compact.
PROOF: Theorem 9.48.4.
⟨1⟩3. $p([0, 1])$ as a subspace of $\mathbb{R}_K$ is connected.
PROOF: Theorem 9.31.13.
⟨1⟩4. $p([0, 1])$ is connected as a subspace of $\mathbb{R}$.
PROOF: Theorem 9.31.13 as the identity map is continuous as a map $\mathbb{R}_K \to \mathbb{R}$.
⟨1⟩5. $p([0, 1])$ is convex.
⟨2⟩1. LET: $a, b \in p([0, 1])$ and $a < c < b$
⟨2⟩2. ASSUME: for a contradiction $c \notin p([0, 1])$
⟨2⟩3. $(-\infty, c) \cap p([0, 1])$ and $(c, +\infty) \cap p([0, 1])$ form a separation of $p([0, 1])$ as a subspace of $\mathbb{R}$.
⟨2⟩4. Q.E.D.
PROOF: This contradicts ⟨1⟩4.
⟨1⟩6. $[0, 1] \subseteq p([0, 1])$

$\langle 1 \rangle 7$. $[0,1]$ as a subspace of $\mathbb{R}_K$ is compact.
PROOF: By Proposition 9.48.3 and $\langle 1 \rangle 2$.
$\langle 1 \rangle 8$. Q.E.D.
PROOF: This contradicts Example 9.48.26.
□

9.34 The Topologist's Sine Curve

Definition 9.34.1 (The Topologist's Sine Curve)**.** Let $S = \{(x, \sin 1/x) \mid 0 < x \leq 1\}$, The *topologist's sine curve* is the closure $\overline{S}$ of S in $\mathbb{R}^2$.

Proposition 9.34.2. *The topologist's sine curve is connected.*

PROOF:
$\langle 1 \rangle 1$. LET: $S = \{(x, \sin 1/x) \mid 0 < x \leq 1\}$
$\langle 1 \rangle 2$. S is connected.
PROOF: Theorem 9.31.13.
$\langle 1 \rangle 3$. $\overline{S}$ is connected.
PROOF: Theorem 9.31.12.
□

Proposition 9.34.3. *The topologist's sine curve is* $\{(x, \sin 1/x) \mid 0 < x \leq 1\} \cup (\{0\} \times [-1.1])$.

PROOF: Sketch proof: Given a point $(0.y)$ with $-1 \leq y \leq 1$, pick a such that $\sin a = y$. Then $((1/a, y), (1/(a + 2\pi), y), (1/(a + 4\pi), y), \ldots)$ is a sequence in S that converges to $(0, y)$.
Conversely, let (x, y) be any point not in $S \cup (\{0\} \times [-1, 1])$. If $x < 0$ or $y > 1$ or $y < -1$ then we can easily find a neighbourhood that does not intersect $S \cup (\{0\} \times [-1, 1])$. If $x > 0$ and $-1 \leq y \leq 1$, then we have $y \neq \sin 1/x$. Hence pick a neighbourhood that does not intersect S.

Proposition 9.34.4. *Every closed subset of $\mathbb{R}$ that is bounded above has a greatest element.*

PROOF: It has a supremum, which is a limit point of the set and hence an element. □

Proposition 9.34.5 (CC)**.** *The topologist's sine curve is not path connected.*

PROOF:
$\langle 1 \rangle 1$. ASSUME: For a contradction $p : [0, 1] \to \overline{S}$ is a path from $(0, 0)$ to $(1, \sin 1)$.
$\langle 1 \rangle 2$. $\{t \in [0, 1] \mid p(t) \in \{0\} \times [-1, 1]\}$ is closed.
PROOF: Since p is continuous and $\{0\} \times [-1, 1]$ is closed.
$\langle 1 \rangle 3$. LET: b be the largest number in $[0, 1]$ such that $p(b) \in \{0\} \times [-1, 1]$.
PROOF: Proposition 9.34.4.
$\langle 1 \rangle 4$. LET: $x : [b, 1] \to \overline{S}$ be the function $\pi_1 \circ p$
$\langle 1 \rangle 5$. LET: $y : [b, 1] \to \overline{S}$ be the function $\pi_2 \circ p$

⟨1⟩6. Pick a sequence t_n in $(b, 1]$ such that $t_n \to b$ and $y(t_n) = (-1)^n$ for all n
⟨2⟩1. Let: $n \geq 1$
⟨2⟩2. Pick u with $0 < u < x(1/n)$ and $\sin(1/u) = (-1)^n$
⟨2⟩3. Pick t_n with $b < t_n < 1/n$ and $x(t_n) = u$
Proof: By the Intermediate Value Theorem
⟨1⟩7. Q.E.D.
Proof: This contradicts Proposition 9.14.18 since y is continuous and $y(t_n)$ does not converge.
□

Corollary 9.34.5.1. *The closure of a path-connected subspace of a space is not necessarily path-connected.*

9.35 The Long Line

Definition 9.35.1 (The Long Line)**.** The *long line* is the space $\omega_1 \times [0, 1)$ in the dictionary order under the order topology, where ω_1 is the first uncountable ordinal.

Lemma 9.35.2. *For any ordinal α with $0 < \alpha < \omega_1$ we have $[(0, 0), (\alpha, 0)) \cong [0, 1)$*

⟨1⟩1. $[(0, 0), (1, 0)) \cong [0, 1)$
Proof: The map π_2 is a homeomorphism.
⟨1⟩2. If $[(0, 0), (\alpha, 0)) \cong [0, 1)$ then $[(0, 0), (\alpha + 1, 0)) \cong [0, 1)$
Proof: Proposition 4.2.11.
⟨1⟩3. If λ is a limit ordinal with $\lambda < \omega_1$ and $[(0, 0), (\alpha, 0)) \cong [0, 1)$ for all α with $0 < \alpha < \lambda$ then $[(0, 0), (\lambda, 0)) \cong [0, 1)$
⟨2⟩1. Let: λ be a limit ordinal $< \omega_1$
⟨2⟩2. Assume: $[(0, 0), (\alpha, 0)) \cong [0, 1)$ for all α with $0 < \alpha < \lambda$
⟨2⟩3. Pick a sequence of ordinals $\alpha_0 < \alpha_1 < \cdots$ with limit λ
Proof: Since λ is countable.
⟨2⟩4. $[(\alpha_i, 0), (\alpha_{i+1}, 0)) \cong [0, 1)$ for all i
Proof: Lemma 4.2.10.
⟨2⟩5. Q.E.D.
Proof: By Proposition 4.2.12.
⟨1⟩4. Q.E.D.
Proof: By transfinite induction.

Proposition 9.35.3 (CC)**.** *The long line is path-connected.*

Proof:
⟨1⟩1. Let: $(\alpha, i), (\beta, j) \in \omega_1 \times [0, 1)$
⟨1⟩2. Assume: without loss of generality $(\alpha, i) < (\beta, j)$
⟨1⟩3. $[(0, 0), (\beta + 1, 0)) \cong [0, 1)$
Proof: By Lemma 9.35.2
⟨1⟩4. $[(\alpha, i), (\beta, j)) \cong [0, 1)$

PROOF: Lemma 4.2.10.
⟨1⟩5. PICK a homeomorphism $q : [0,1) \to [(\alpha, i), (\beta, j))$
⟨1⟩6. $q \cup \{(1, (\beta, j))\}$ is a path from (α, i) to (β, j)
□

Proposition 9.35.4. *Every point in the long line has a neighbourhood homeomorphic to an interval in* $\mathbb{R}$.

PROOF: For any (α, i) in the long line, the neighbourhood $[(0,0),(\alpha+1,0))$ satisfies the condition by Lemma 9.35.2.

Proposition 9.35.5. *The long line L is not second countable.*

PROOF:
⟨1⟩1. LET: $\mathcal{B}$ be a basis for L.
⟨1⟩2. For $\alpha < \omega_1$, PICK $B_\alpha \in \mathcal{B}$ such that $(\alpha, 1/2) \in B_\alpha \subseteq ((\alpha, 0), (\alpha+1, 0))$
⟨1⟩3. $\mathcal{B}$ is uncountable.
PROOF: The mapping $\alpha \mapsto B_\alpha$ is an injection $\omega_1 \to \mathcal{B}$.

Corollary 9.35.5.1. *The long line cannot be imbedded into $\mathbb{R}^n$ for any n.*

9.36 Components

Proposition 9.36.1. *Let X be a topological space. Define the relation $\sim$ on X by $a \sim b$ if and only if there exists a connected subspace $A \subseteq X$ with $a, b \in A$. Then $\sim$ is an equivalence relation on X.*

PROOF:
⟨1⟩1. $\sim$ is reflexive.
PROOF: For any $a \in X$ we have $\{a\}$ is a connected subspace that contains a.
⟨1⟩2. $\sim$ is symmetric.
PROOF: Trivial.
⟨1⟩3. $\sim$ is transitive.
⟨2⟩1. LET: $a, b, c \in X$
⟨2⟩2. ASSUME: $a \sim b$ and $b \sim c$
⟨2⟩3. PICK connected subspaces A and B with $a, b \in A$ and $b, c \in B$
⟨2⟩4. $A \cup B$ is a connected subspace that contains a and c
PROOF: Theorem 9.31.11.
□

Definition 9.36.2 ((Connected) Component)**.** Let X be a topological space. The *(connected) components* of X are the equivalence classes under the above $\sim$.

Lemma 9.36.3. *Let X be a topological space. If $A \subseteq X$ is connected and nonempty then there exists a unique component C of X such that $A \subseteq C$.*

PROOF:

$\langle 1\rangle 1.$ Pick $a \in A$
$\langle 1\rangle 2.$ Let: C be the $\sim$-equivalence class of a.
$\langle 1\rangle 3.$ $A \subseteq C$
Proof: For all $x \in A$ we have $x \sim a$.
$\langle 1\rangle 4.$ If C' is a component and $A \subseteq C'$ then $C = C'$
Proof: Since we have $a \in C'$.
$\square$

Theorem 9.36.4. *Let X be a topological space. The components of X are connected disjoint subspaces of X whose union is X such that each nonempty connected subspace of X intersects only one of them.*

Proof:
$\langle 1\rangle 1.$ Every component of X is connected.
Proof: For $a \in X$, the $\sim$-equivalence class of a is $\bigcup\{A \subseteq X \mid A \text{ is connected}, a \in A\}$ which is connected by Theorem 9.31.11.
$\langle 1\rangle 2.$ The components form a partition of X.
Proof: Immediate from the definition.
$\langle 1\rangle 3.$ Every nonempty connected subspace of X intersects a unique component of X.
$\langle 2\rangle 1.$ Let: $A \subseteq X$ be connected and nonempty.
$\langle 2\rangle 2.$ Let: C be the component such that $A \subseteq C$
Proof: Lemma 9.36.3.
$\langle 2\rangle 3.$ A intersects C
$\langle 2\rangle 4.$ If A intersects the component C' then $C' = C$
$\langle 3\rangle 1.$ Let: C' be a component that intersects A
$\langle 3\rangle 2.$ Pick $b \in A \cap C'$
$\langle 3\rangle 3.$ $A \subseteq C'$
Proof: For all $x \in A$ we have $x \sim b$.
$\langle 3\rangle 4.$ $C = C'$
Proof: By uniqueness in $\langle 2\rangle 2$.
$\square$

Proposition 9.36.5. *Every component of a space is closed.*

Proof:
$\langle 1\rangle 1.$ Let: X be a topological space and C a component of X.
$\langle 1\rangle 2.$ $\overline{C}$ is connected.
Proof: Theorem 9.31.12.
$\langle 1\rangle 3.$ $C = \overline{C}$
Proof: Lemma 9.31.10.
$\langle 1\rangle 4.$ C is closed.
Proof: Lemma 9.6.5.
$\square$

Proposition 9.36.6. *If a topological space has finitely many components then every component is open.*

Proof: Each component is the complement of a finite union of closed sets. $\square$

9.37 Path Components

Proposition 9.37.1. *Let X be a topological space. Define the relation $\sim$ on X by: $a \sim b$ if and only if there exists a path in X from a to b. Then $\sim$ is an equivalence relation on X.*

PROOF:
⟨1⟩1. $\sim$ is reflexive.
 PROOF: For $a \in X$, the constant function $[0,1] \to X$ with value a is a path from a to a.
⟨1⟩2. $\sim$ is symmetric.
 PROOF: If $p : [0,1] \to X$ is a path from a to b, then $\lambda t.p(1-t)$ is a path from b to a.
⟨1⟩3. $\sim$ is transitive.
 PROOF: Concatenate paths.
□

Definition 9.37.2 (Path Component)**.** Let X be a topological space. The *path components* of X are the equivalence relations under $\sim$.

.

Theorem 9.37.3. *The path components of X are path-connected disjoint subspaces of X whose union is X such that every nonempty path-connected subspace of X intersects exactly one path component.*

PROOF:
⟨1⟩1. Every path component is path-connected.
 PROOF: If a and b are in the same path component then $a \sim b$, i.e. there exists a path from a to b.
⟨1⟩2. The path components are disjoint and their union is X.
 PROOF: Immediate from the definition.
⟨1⟩3. Every non-empty path-cönnected subspace of X intersects exactly one path component.
 ⟨2⟩1. LET: A be a nonempty path-connected subspace of X.
 ⟨2⟩2. PICK $a \in A$
 ⟨2⟩3. A intersects the $\sim$-equivalence class of a.
 ⟨2⟩4. LET: C be any path component that intersects A.
 ⟨2⟩5. PICK $b \in A \cap C$
 ⟨2⟩6. $a \sim b$
  PROOF: Since A is path-connected.
 ⟨2⟩7. C is the $\sim$-equivalence class of a.
□

Proposition 9.37.4. *Every path component is included in a component.*

PROOF:
⟨1⟩1. LET: X be a topological space and C a path component of X.
⟨1⟩2. C is path-connected.

PROOF: Theorem 9.37.3.
⟨1⟩3. C is connected.
PROOF: Proposition 9.33.3.
⟨1⟩4. C is included in a component.
PROOF: Lemma 9.36.3.
□

9.38 Local Connectedness

Definition 9.38.1 (Locally Connected)**.** Let X be a topological space and $a \in X$. Then X is *locally connected* at a if and only if every neighbourhood of a includes a connected neighbourhood of a.

The space X is *locally connected* if and only if it is locally connected at every point.

Example 9.38.2. The real line is both connected and locally connected.

Example 9.38.3. The space $\mathbb{R} \setminus \{0\}$ is disconnected but locally connected.

Example 9.38.4. The topologist's sine curve is connected but not locally connected.

Example 9.38.5. The rationals $\mathbb{Q}$ are neither connected nor locally connected.

Theorem 9.38.6. *A topological space X is locally connected if and only if, for every open set U in X, every component of U is open in X.*

PROOF:
⟨1⟩1. If X is locally connected then, for every open set U in X, every component of U is open in X.
⟨2⟩1. ASSUME: X is locally connected.
⟨2⟩2. LET: U be open in X.
⟨2⟩3. LET: C be a component of U.
⟨2⟩4. LET: $a \in C$
⟨2⟩5. LET: V be a connected neighbourhood of a such that $V \subseteq U$
⟨2⟩6. $V \subseteq C$
PROOF: Lemma 9.36.3.
⟨2⟩7. Q.E.D.
PROOF: Lemma 9.3.8.
⟨1⟩2. If, for every open set U in X, every component of U is open in X, then X is locally connected.
⟨2⟩1. ASSUME: for every open set U in X, every component of U is open in X.
⟨2⟩2. LET: $a \in X$
⟨2⟩3. LET: U be a neighbourhood of a
⟨2⟩4. The component of U that contains a is a connected neighbourhood of a included in U.

∎

Example 9.38.7. The ordered square is locally connected.

PROOF: Every neighbourhood of a point includes an interval around that point, which is connected by Theorem 9.31.23.

Example 9.38.8. Let X be the set of all rational points on the line segment $[0,1] \times \{0\}$, and Y the set of all rational points on the line segment $[0,1] \times \{1\}$. Let A be the space consisting of all line segments joining the point $(0,1)$ to a point of X, and all line segments joining the point $(1,0)$ to a point of Y. Then A is path-connected but is not locally connected at any point,

Proposition 9.38.9. *Let X and Y be topological spaces and $p : X \twoheadrightarrow Y$ be a quotient map. If X is locally connected then so is Y.*

PROOF:
$\langle 1 \rangle 1$. LET: U be an open set in Y.
$\langle 1 \rangle 2$. LET: C be a component of U.
$\langle 1 \rangle 3$. $p^{-1}(C)$ is a union of components of $p^{-1}(U)$
  $\langle 2 \rangle 1$. LET: $x \in p^{-1}(C)$
  $\langle 2 \rangle 2$. LET: D be the component of $p^{-1}(U)$ that contains x.
  $\langle 2 \rangle 3$. $p(D)$ is connected.
    PROOF: Theorem 9.31.13.
  $\langle 2 \rangle 4$. $p(D) \subseteq C$.
    PROOF: From $\langle 1 \rangle 2$ since $p(x) \in p(D) \cap C$ ($\langle 2 \rangle 1$, $\langle 2 \rangle 2$).
  $\langle 2 \rangle 5$. $D \subseteq p^{-1}(C)$
$\langle 1 \rangle 4$. $p^{-1}(C)$ is open in $p^{-1}(U)$
  PROOF: Theorem 9.38.6.
$\langle 1 \rangle 5$. C is open in U
  PROOF: Since the restriction of p to $p : p^{-1}(U) \twoheadrightarrow U$ is a quotient map by Proposition 9.26.4.
$\langle 1 \rangle 6$. Q.E.D.
  PROOF: Theorem 9.38.6.
∎

9.39 Local Path Connectedness

Definition 9.39.1 (Locally Path-Connected)**.** Let X be a topological space and $a \in X$. Then X is *locally path-connected* at a if and only if every neighbourhood of a includes a path-connected neighbourhood of a.

The space X is *locally path-connected* if and only if it is locally path-connected at every point.

Theorem 9.39.2. *A topological space X is locally path-connected if and only if, for every open set U in X, every path component of U is open in X.*

PROOF:

⟨1⟩1. If X is locally path-connected then, for every open set U in X, every path component of U is open in X.
⟨2⟩1. Assume: X is locally path-connected.
⟨2⟩2. Let: U be open in X.
⟨2⟩3. Let: C be a path component of U.
⟨2⟩4. Let: $a \in C$
⟨2⟩5. Let: V be a path-connected neighbourhood of a such that $V \subseteq U$
⟨2⟩6. $V \subseteq C$
Proof: Lemma 9.36.3.
⟨2⟩7. Q.E.D.
Proof: Lemma 9.3.8.
⟨1⟩2. If, for every open set U in X, every component of U is open in X, then X is locally connected.
⟨2⟩1. Assume: for every open set U in X, every component of U is open in X.
⟨2⟩2. Let: $a \in X$
⟨2⟩3. Let: U be a neighbourhood of a
⟨2⟩4. The component of U that contains a is a connected neighbourhood of a included in U.
□

Theorem 9.39.3. *If a space is locally path connected then its components and its path components are the same.*

Proof:
⟨1⟩1. Let: X be a locally path connected space.
⟨1⟩2. Let: C be a component of X.
⟨1⟩3. Let: $x \in C$
⟨1⟩4. Let: P be the path component of x
Prove: $P = C$
⟨1⟩5. $P \subseteq C$
Proof: Proposition 9.37.4.
⟨1⟩6. Let: Q be the union of the other path components included in C
⟨1⟩7. $C = P \cup Q$
Proof: Proposition 9.37.4.
⟨1⟩8. P and Q are open in C
⟨2⟩1. C is open.
Proof: Theorem 9.38.6.
⟨2⟩2. Q.E.D.
Proof: Theorem 9.39.2.
⟨1⟩9. $Q = \emptyset$
Proof: Otherwise P and Q would form a separation of C.
□

Example 9.39.4. The ordered square is not locally path connected, since it is connected but not path connected.

Proposition 9.39.5. *Let X be a locally path-connected space. Then every connected open subspace of X is path-connected.*

Proof:
$\langle 1 \rangle 1$. Let: U be a connected open subspace of X.
$\langle 1 \rangle 2$. Let: P be a path component of U.
$\langle 1 \rangle 3$. Let: Q be the union of the other path components of U.
$\langle 1 \rangle 4$. P and Q are open in U.
  Proof: Theorem 9.39.2.
$\langle 1 \rangle 5$. $Q = \emptyset$
  Proof: Otherwise P and Q form a separation of U.
□

9.40 Weak Local Connectedness

Definition 9.40.1 (Weakly Locally Connected)**.** Let X be a topological space and $a \in X$. Then X is *weakly locally connected* at a if and only if every neighbourhood of a includes a connected subspace that includes a neighbourhood of a.

Proposition 9.40.2. *Let X be a topological space. If X is weakly locally connected at every point then X is locally connected.*

Proof:
$\langle 1 \rangle 1$. Assume: X is weakly locally connected at every point.
$\langle 1 \rangle 2$. Let: U be open in X.
$\langle 1 \rangle 3$. Let: C be a component of U.
$\langle 1 \rangle 4$. C is open in X.
  $\langle 2 \rangle 1$. Let: $x \in C$
  $\langle 2 \rangle 2$. Pick a connected subspace D of U that includes a neighbourhood V of x.
  $\langle 2 \rangle 3$. $D \subseteq C$
    Proof: Lemma 9.36.3.
  $\langle 2 \rangle 4$. $x \in V \subseteq C$
  $\langle 2 \rangle 5$. Q.E.D.
    Proof: Lemma 9.3.8.
$\langle 1 \rangle 5$. Q.E.D.
  Proof: Theorem 9.38.6.
□

Example 9.40.3. The 'infinite broom' (Munkres p.163) is an example of a space that is weakly locally connected at a point p but not locally connected at p.

9.41 Quasicomponents

Proposition 9.41.1. *Let X be a topological space. Define $\sim$ on X by $x \sim y$ if and only if there exists no separation U and V of X such that $x \in U$ and $y \in V$. Then $\sim$ is an equivalence relation on X.*

PROOF:
$\langle 1 \rangle 1.$ $\sim$ is reflexive.
PROOF: Since the two sets that make up a separation are disjoint.
$\langle 1 \rangle 2.$ $\sim$ is symmetric.
PROOF: Immediate from the defintion.
$\langle 1 \rangle 3.$ $\sim$ is transitive.
$\langle 2 \rangle 1.$ ASSUME: $x \sim y$ and $y \sim z$
$\langle 2 \rangle 2.$ ASSUME: for a contradiction there is a separation U and V of X with $x \in U$ and $z \in V$
$\langle 2 \rangle 3.$ $y \in U$ or $y \in V$
$\langle 2 \rangle 4.$ Q.E.D.
PROOF: Either case contradicts $\langle 2 \rangle 1$.
□

Definition 9.41.2 (Quasicomponents)**.** For X a topological space, the *quasicomponents* of X are the equivalence classes under $\sim$.

Proposition 9.41.3. *Let X be a topological space. Then every component of X is included in a quasicomponent of X.*

PROOF:
$\langle 1 \rangle 1.$ LET: C be a component of X.
$\langle 1 \rangle 2.$ LET: $x, y \in C$
PROVE: $x \sim y$
$\langle 1 \rangle 3.$ ASSUME: for a contradiction there exists a separation U and V of X with $x \in U$ and $y \in V$
$\langle 1 \rangle 4.$ $C \cap U$ and $C \cap V$ form a separation of C.
$\langle 1 \rangle 5.$ Q.E.D.
PROOF: This contradicts $\langle 1 \rangle 1$.

Proposition 9.41.4. *In a locally connected space, the components and the quasicomponents are the same.*

PROOF:
$\langle 1 \rangle 1.$ LET: X be a locally connected space and Q a quasicomponent of X.
$\langle 1 \rangle 2.$ PICK a component C of X such that $C \subseteq Q$
$\langle 1 \rangle 3.$ LET: D be the union of the components of X
$\langle 1 \rangle 4.$ C and D are open in X.
PROOF: Theorem 9.38.6.
$\langle 1 \rangle 5.$ D cannot contain any points of Q.
PROOF: If it did, then C and D would form a separation of X and there would be points $x, y \in Q$ with $x \in C$ and $y \in D$.
$\langle 1 \rangle 6.$ $C = Q$
□

9.42 Open Coverings

Definition 9.42.1 (Open Covering)**.** Let X be a topological space. An *open covering* of X is a covering of X whose elements are all open sets.

9.43 Lindelöf Spaces

Definition 9.43.1 (Lindelöf Space)**.** A topological space X is *Lindelöf* if and only if every open covering has a countable subcovering.

Proposition 9.43.2. *Let X be a topological space. Then X is Lindelöf if and only if every set of closed sets that has the countable intersection property has nonempty intersection.*

Proof: The following are equivalent.

1. X is compact.
2. Every open covering of X has a countable subcovering.
3. For any set $\mathcal{C}$ of closed sets, if $\{X \setminus C \mid C \in \mathcal{C}\}$ covers X then there is a countable subset $\mathcal{C}_0$ such that $\{X \setminus C \mid C \in \mathcal{C}_0\}$ covers X
4. For any set $\mathcal{C}$ of closed sets, if $\bigcap \mathcal{C} = \emptyset$ then there is a countable subset $\mathcal{C}_0$ with empty intersection.
5. Any set of closed sets with the countable intersection property has nonempty intersection.

□

Proposition 9.43.3 (CC)**.** *Let X be a topological space and $\mathcal{B}$ a basis for the topology on X. Then the following are equivalent.*

1. *X is Lindelöf.*
2. *Every open covering of X by elements of $\mathcal{B}$ has a countable subcovering.*

Proof:
⟨1⟩1. $1 \Rightarrow 2$
  Proof: Immediate from definitions.
⟨1⟩2. $2 \Rightarrow 1$
  ⟨2⟩1. Assume: Every open covering of X by elements of $\mathcal{B}$ has a countable subcovering.
  ⟨2⟩2. Let: $\mathcal{U}$ be an open covering of X.
  ⟨2⟩3. $\{B \in \mathcal{B} \mid \exists U \in \mathcal{U}.B \subseteq U\}$ covers X.
  ⟨2⟩4. Pick a finite subcovering $\mathcal{B}_0$.
  ⟨2⟩5. For $B \in BB$, Pick $U_B \in \mathcal{U}$ such that $B \subseteq U_B$
  ⟨2⟩6. $\{U_B \mid B \in \mathcal{B}_0\}$ covers X.
□

9.44 The Second Countability Axiom

Definition 9.44.1 (Second Countability Axiom)**.** A topological space satisfies the *second countability axiom*, or is *second countable*, if and only if it has a countable basis.

Example 9.44.2. The space $\mathbb{R}$ is second countable.

PROOF: The set $\{(a,b) \mid a,b \in \mathbb{Q}\}$ is a basis. ∎

Proposition 9.44.3. *A subspace of a second countable space is second countable.*

PROOF: If $\mathcal{B}$ is a countable basis for X and $Y \subseteq X$ then $\{B \cap Y \mid B \in \mathcal{B}\}$ is a countable basis for Y. ∎

Proposition 9.44.4 (CC)**.** *Every second countable space is Lindelöf.*

PROOF: From Proposition 9.43.3.

Example 9.44.5 (CC)**.** The space $\mathbb{R}_l$ is Lindelöf.

$\langle 1 \rangle 1$. LET: $\mathcal{A}$ be a covering of $\mathbb{R}_l$ by basic open sets of the form $[a,b)$
$\langle 1 \rangle 2$. LET: $C = \bigcup\{(a,b) \mid [a,b) \in \mathcal{A}\}$
$\langle 1 \rangle 3$. $\mathbb{R} \setminus C$ is countable.
  $\langle 2 \rangle 1$. For every $x \in \mathbb{R} \setminus C$, PICK a rational q_x such that $(x, q_x) \subseteq C$
    $\langle 3 \rangle 1$. LET: $x \in \mathbb{R} \setminus C$
    $\langle 3 \rangle 2$. PICK b such that $[x,b) \in \mathcal{A}$
    $\langle 3 \rangle 3$. PICK a rational q such that $q \in (x,b)$
  $\langle 2 \rangle 2$. The mapping $x \mapsto q_x$ is an injection $\mathbb{R} \setminus C \to \mathbb{Q}$
$\langle 1 \rangle 4$. PICK a countable $\mathcal{A}' \subseteq \mathcal{A}$ that covers $\mathbb{R} \setminus C$
$\langle 1 \rangle 5$. Under the standard topology on $\mathbb{R}$, C is second countable.
  PROOF: Proposition 9.44.3.
$\langle 1 \rangle 6$. PICK a countable $\mathcal{A}'' \subseteq \mathcal{A}$ such that $\{(a,b) \mid [a,b) \in \mathcal{A}''\}$ covers C.
  PROOF: Proposition 9.43.3.
$\langle 1 \rangle 7$. $\mathcal{A}' \cup \mathcal{A}''$ covers $\mathbb{R}_l$.
∎

Example 9.44.6. The product of two Lindelöf spaces is not necessarily Lindelöf.

We prove that the Sorgenfrey plane is not Lindelöf.

PROOF:
$\langle 1 \rangle 1$. LET: $L = \{(x, -x) \mid x \in \mathbb{R}\}$
$\langle 1 \rangle 2$. L is closed in $\mathbb{R}_l^2$
$\langle 1 \rangle 3$. LET: $\mathcal{U} = \{[a,b) \times [a,-d) \mid a,b,d \in \mathbb{R}\}$
$\langle 1 \rangle 4$. $\mathcal{U} \cup \{\mathbb{R} \setminus L\}$ covers $\mathbb{R}_l^2$
$\langle 1 \rangle 5$. Every element of $\mathcal{U}$ intersects L at exactly one point.
$\langle 1 \rangle 6$. No countable subset of $\mathcal{U}$ covers $\mathbb{R}_l^2$.
∎

9.45 Sequential Compactness

Definition 9.45.1 (Sequentially Compact)**.** A topological space is *sequentially compact* if and only if every sequence has a convergent subsequence.

9.46 Limit Point Compactness

Definition 9.46.1 (Limit Point Compact Space)**.** A topological space is *limit point compact* if and only if every infinite set has a limit point.

Proposition 9.46.2. *Every limit point compact T_1 space is sequentially compact.*

PROOF:
⟨1⟩1. LET: X be a limit point compact T_1 space.
⟨1⟩2. LET: (x_n) be a sequence in X.
⟨1⟩3. CASE: $\{x_n \mid n \geq 1\}$ is finite.
  ⟨2⟩1. PICK n such that x_n occurs infintely often in the sequence (x_n)
  ⟨2⟩2. The subsequence consisting of all the terms equal to x_n is convergent.
⟨1⟩4. CASE: $\{x_n \mid n \geq 1\}$ is infinite.
  ⟨2⟩1. PICK a limit point l for $\{x_n \mid n \geq 1\}$
  ⟨2⟩2. PICK an increasing sequence n_r with $x_{n_r} \in B(x, 1/r)$ for all r
    PROOF: This is always possible by Theorem 9.21.3.
  ⟨2⟩3. (x_{n_r}) converges to l.
□

Corollary 9.46.2.1. *Every compact T_1 spact is sequentially compact.*

Example 9.46.3. The space $[0,1]^\omega$ under the uniform topology is not limit point compact.

The infinite set $\{0,1\}^\omega$ has no limit point.

Example 9.46.4. The space $[0,1]$ under the lower limit topology is not limit point compact.

The infinite set $A = \{1 - 1/n \mid n \geq 1\}$ has no limit point. 1 is not a limit point because the neighbourhood $\{1\}$ does not intersect A.

Proposition 9.46.5. *A closed subspace of a limit point compact space is limit point compact.*

PROOF:
⟨1⟩1. LET: X be a limit point compact space.
⟨1⟩2. LET: $A \subseteq X$ be closed.
⟨1⟩3. LET: $B \subseteq A$ be infinite.
⟨1⟩4. PICK a limit point l of B in X.
⟨1⟩5. $l \in A$
⟨1⟩6. l is a limit point of B in A.
□

Example 9.46.6. An open subspace of a limit point compact space is not necessarily limit point compact.

The space $[0,1]$ is limit point compact but $(0,1)$ is not.

Example 9.46.7. The continuous image of a limit point compact space is not necessarily limit point compact.

Let Y be the indiscrete space with two points. Then $\mathbb{Z}^+ \times Y$ is limit point compact but $\mathbb{Z}^+$ is not.

Example 9.46.8. A limit point compact subspace of a Hausdorff space is not necessarily closed.

The space S_Ω is limit point compact but is not closed in $\overline{S_\Omega}$.

For an example that shows that the product of two limit point compact spaces is not necessarily limit point compact, see L. A. Steen and J. A. Seerbach Jr. *Counterexamples in Topology* Example 112.

9.47 Countable Compactness

Definition 9.47.1 (Countably Compact)**.** A topological space is *countably compact* if and only if every countable open covering has a finite subcovering.

Proposition 9.47.2 (AC)**.** *Every closed subspace of a countably compact space is countably compact.*

PROOF:
$\langle 1\rangle 1$. LET: X be a countably compact space.
$\langle 1\rangle 2$. LET: $A \subseteq X$ be closed.
$\langle 1\rangle 3$. LET: $\mathcal{U}$ be a countable open cover of A.
$\langle 1\rangle 4$. For $U \in \mathcal{U}$, PICK an open set V_U is X such that $U = V_U \cap A$
$\langle 1\rangle 5$. $\{V_U \mid U \in \mathcal{U}\} \cup \{X - A\}$ is a countable open cover of X
$\langle 1\rangle 6$. PICK a finite subcover $\{V_{U_1}, \ldots, V_{U_n}, X - A\}$
$\langle 1\rangle 7$. $\{U_1, \ldots, U_n\}$ covers A.
□

Proposition 9.47.3 (AC)**.** *Every countably compact space is limit point compact.*

PROOF:
$\langle 1\rangle 1$. ASSUME: X is countably compact.
$\langle 1\rangle 2$. LET: $A \subseteq X$ be infinite.
$\langle 1\rangle 3$. ASSUME: for a contradiction A has no limit point.
$\langle 1\rangle 4$. PICK a countably infinite $B \subseteq A$
$\langle 1\rangle 5$. B is discrete.
PROOF: For all $b \in B$, there exists U_b open in X such that $U_b \cap B = \{b\}$.
$\langle 1\rangle 6$. $\{\{b\} \mid b \in B\}$ is a countable cover of B that has no finite subcover.
$\langle 1\rangle 7$. B is not countably compact.
$\langle 1\rangle 8$. B is not closed in X

⟨1⟩9. B has a limit point.
⟨1⟩10. A has a limit point.
⟨1⟩11. Q.E.D.
PROOF: This contradicts ⟨1⟩3.
□

Proposition 9.47.4 (AC). *Every limit point compact T_1 space is countably compact.*

PROOF:
⟨1⟩1. LET: X be a limit point compact T_1 space.
⟨1⟩2. LET: $\{U_n \mid n \in \mathbb{Z}^+\}$ be a countable open cover of X.
⟨1⟩3. For $n \in \mathbb{Z}^+$,
LET: $V_n = U_1 \cup \cdots \cup V_n$
⟨1⟩4. ASSUME: for a contradiction none of the V_n covers X
⟨1⟩5. For $n \in \mathbb{Z}^+$, PICK $a_n \in X - V_n$
⟨1⟩6. PICK a limit point l for $\{a_n \mid n \in \mathbb{Z}^+\}$
⟨1⟩7. PICK n such that $l \in U_n$
⟨1⟩8. CASE: $l = a_m$ for some $m \leq n$
PROOF: $U_n - \{a_1, \ldots, a_{m-1}, a_{m+1}, \ldots, a_n\}$ is a neighbourhood of l that intersects $\{a_n \mid n \in \mathbb{Z}^+\}$ only at l, contradicting ⟨1⟩6.
⟨1⟩9. CASE: $l \neq a_m$ for any $m \leq n$
PROOF: $U_n - \{a_1, \ldots, a_n\}$ is a neighbourhood of l that does not intersect $\{a_n \mid n \in \mathbb{Z}^+\}$, which contradicts ⟨1⟩6.
□

The following example shows we cannot remove the hypothesis that the space is T_1.

Example 9.47.5. Let Y be the indiscrete space with two points. Then $\mathbb{Z}^+ \times Y$ is a limit point compact space that is not countably compact, since $\{\{n\} \times Y \mid n \in \mathbb{Z}^+\}$ is a countable open cover that has no finite subcover.

Proposition 9.47.6. *A topological space is countably compact if and only if every nested sequence $C_1 \supseteq C_2 \supseteq \cdots$ of nonempty closed sets has nonempty intersection.*

PROOF:
⟨1⟩1. LET: X be a topological space.
⟨1⟩2. If X is countably compact then every nested sequence of nonempty closed sets has nonempty intersection.
⟨2⟩1. ASSUME: X is countably compact.
⟨2⟩2. LET: $C_1 \supseteq C_2 \supseteq \cdots$ be a nested sequence of nonempty closed sets.
⟨2⟩3. ASSUME: for a contradiction $\bigcap_n C_n = \emptyset$
⟨2⟩4. $\{X - C_n \mid n \in \mathbb{Z}^+\}$ covers X
⟨2⟩5. PICK a finite subcover $\{X - C_{n_1}, \ldots, X - C_{n_k}\}$ where $n_1 < \cdots < n_k$
⟨2⟩6. $C_{n_k} = \emptyset$
⟨2⟩7. Q.E.D.

PROOF: This contradicts $\langle 2\rangle 2$.

∎

$\langle 1\rangle 3$. If every nested sequence of nonempty closed sets has nonempty intersection then X is countably compact.

$\langle 2\rangle 1$. ASSUME: Every nested sequence of nonempty closed sets has nonempty intersection.

$\langle 2\rangle 2$. LET: $\{U_n \mid n \geq 1\}$ is a countable open cover of X.

$\langle 2\rangle 3$. $X - U_1 \supseteq X - (U_1 \cup U_2) \supseteq \cdots$ is a nested sequence of closed sets with empty intersection.

$\langle 2\rangle 4$. PICK k such that $X - (U_1 \cup \cdots \cup U_k) = \emptyset$

$\langle 2\rangle 5$. $\{U_1, \ldots, U_k\}$ covers X.

∎

9.48 Compact Spaces

Definition 9.48.1 (Compact)**.** A topological space is *compact* if and only if every open covering has a finite subcovering.

Lemma 9.48.2. *Let X be a topological space and Y a subspace of X. Then Y is compact if and only if every covering of Y by sets open in X has a finite subcovering.*

PROOF:

$\langle 1\rangle 1$. If Y is compact then every covering of Y by sets open in X has a finite subcovering.

$\langle 2\rangle 1$. ASSUME: Y is compact.

$\langle 2\rangle 2$. LET: $\mathcal{U}$ be a covering of Y by sets open in X.

$\langle 2\rangle 3$. $\{U \cap Y \mid U \in \mathcal{U}\}$ is an open covering of Y.

$\langle 2\rangle 4$. PICK a finite subcovering $\{U_1 \cap Y, \ldots, U_n \cap Y\}$

$\langle 2\rangle 5$. $\{U_1, \ldots, U_n\}$ is a finite subcovering of $\mathcal{U}$.

$\langle 1\rangle 2$. If every covering of Y by sets open in X has a finite subcovering then Y is compact.

$\langle 2\rangle 1$. LET: $\mathcal{U}$ be an open covering of Y.

$\langle 2\rangle 2$. LET: $\mathcal{V} = \{V \text{ open in } X \mid V \cap Y \in \mathcal{U}\}$.

$\langle 2\rangle 3$. $\mathcal{V}$ is a covering of Y by sets open in X.

$\langle 2\rangle 4$. PICK a finite subcovering $\{V_1, \ldots, V_n\}$

$\langle 2\rangle 5$. $\{V_1 \cap Y, \ldots, V_n \cap Y\}$ is a finite subcovering of $\mathcal{U}$.

∎

Proposition 9.48.3. *Every closed subspace of a compact space is compact.*

PROOF:

$\langle 1\rangle 1$. LET: X be a compact space and $Y \subseteq X$ be closed.

$\langle 1\rangle 2$. LET: $\mathcal{U}$ be a covering of Y by sets open in X.

$\langle 1\rangle 3$. $\mathcal{U} \cup \{X \setminus Y\}$ is an open covering of X.

$\langle 1\rangle 4$. PICK a finite subcovering $\mathcal{U}_0$

$\langle 1\rangle 5$. $\mathcal{U}_0 \cap \mathcal{U}$ is a finite subset of $\mathcal{U}$ that covers Y.

∎

Theorem 9.48.4. *The continuous image of a compact space is compact.*

Proof:
⟨1⟩1. Let: $f : X \twoheadrightarrow Y$ be continuous and surjective.
⟨1⟩2. Let: $\mathcal{V}$ be an open covering of Y
⟨1⟩3. $\{p^{-1}(V) \mid V \in \mathcal{V}\}$ is an open covering of X.
⟨1⟩4. Pick a finite subcovering $\{p^{-1}(V_1), \ldots, p^{-1}(V_n)\}$
⟨1⟩5. $\{V_1, \ldots, V_n\}$ covers Y.
∎

Theorem 9.48.5. *Let A and B be compact subspaces of X and Y respectively. Let N be an open set in $X \times Y$ that includes $A \times B$. Then there exist open sets U and V in X and Y respectively such that $A \subseteq U$, $B \subseteq V$ and $U \times V \subseteq N$.*

Proof:
⟨1⟩1. For all $x \in A$, there exist neighbourhoods U of x and V of B such that $U \times V \subseteq N$.
  ⟨2⟩1. Let: $x \in A$
  ⟨2⟩2. For all $y \in B$, there exist neighbourhoods U of x and V of y such that $U \times V \subseteq N$
  ⟨2⟩3. $\{V \text{ open in } Y \mid \exists \text{ neighbourhood } U \text{ of } x, U \times V \subseteq N\}$ covers B.
  ⟨2⟩4. Pick a finite subcover $\{V_1, \ldots, V_n\}$
  ⟨2⟩5. For $i = 1, \ldots, n$, Pick a neighbourhood U_i of x such that $U_i \times V_i \subseteq N$
  ⟨2⟩6. Let: $U = U_1 \cap \cdots \cap U_n$
  ⟨2⟩7. Let: $V = V_1 \cup \cdots \cup V_n$
  ⟨2⟩8. U is a neighbourhood of x.
  ⟨2⟩9. V is a neighbourhood of B.
  ⟨2⟩10. $U \times V \subseteq N$
⟨1⟩2. $\{U \text{ open in } X \mid \exists \text{ neighbourhood } V \text{ of } B . U \times V \subseteq N\}$ covers A.
⟨1⟩3. Pick a finite subcover $\{U_1, \ldots, U_n\}$
⟨1⟩4. For $i = 1, \ldots, n$, Pick a neighbourhood V_i of B such that $U_i \times V_i \subseteq N$
⟨1⟩5. Let: $U = U_1 \cup \cdots \cup U_n$
⟨1⟩6. Let: $V = V_1 \cap \cdots \cap V_n$
⟨1⟩7. U and V are open.
⟨1⟩8. $A \subseteq U$
⟨1⟩9. $B \subseteq V$
⟨1⟩10. $U \times V \subseteq N$
∎

Corollary 9.48.5.1 (Tube Lemma)**.** *Let X and Y be topological spaces with Y compact. Let $a \in X$ and N be an open set in $X \times Y$ that includes $\{a\} \times Y$. Then there exists a neighbourhood W of a such that N includes the* tube *$W \times Y$.*

Theorem 9.48.6. *Let X be a topological space. Then X is compact if and only if every set of closed sets that has the finite intersection property has nonempty intersection.*

Proof: The following are equivalent.

1. X is compact.
2. Every open covering of X has a finite subcovering.
3. For any set $\mathcal{C}$ of closed sets, if $\{X \setminus C \mid C \in \mathcal{C}\}$ covers X then there is a finite subset $\mathcal{C}_0$ such that $\{X \setminus C \mid C \in \mathcal{C}_0\}$ covers X
4. For any set $\mathcal{C}$ of closed sets, if $\bigcap \mathcal{C} = \emptyset$ then there is a finite subset $\mathcal{C}_0$ with empty intersection.
5. Any set of closed sets with the finite intersection property has nonempty intersection.

□

Corollary 9.48.6.1. *Let X be a topological space and $C_1 \supseteq C_2 \supseteq \cdots$ a nested sequence of nonempty closed sets. Then $\bigcap_n C_n$ is nonempty.*

Proposition 9.48.7. *Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on the same set X with $\mathcal{T} \subseteq \mathcal{T}'$. If $\mathcal{T}'$ is compact then $\mathcal{T}$ is compact.*

Proof:
$\langle 1 \rangle 1$. Let: $\mathcal{U} \subseteq \mathcal{T}$ cover X
$\langle 1 \rangle 2$. $\mathcal{U} \subseteq \mathcal{T}'$
$\langle 1 \rangle 3$. A finite subset of $\mathcal{U}$ covers X.
□

Corollary 9.48.7.1. *If $\mathcal{T}$ and $\mathcal{T}'$ are two compact Hausdorff topologies on the same set X, then either $\mathcal{T} = \mathcal{T}'$ or $\mathcal{T}$ and $\mathcal{T}'$ are incomparable.*

Proof: From the Proposition and Proposition 9.22.12. □

Example 9.48.8. Any set under the finite complement topology is compact.

Proposition 9.48.9. *Let X be a topological space. A finite union of compact subspaces of X is compact.*

Proof:
$\langle 1 \rangle 1$. Let: A and B be compact subspaces of X.
$\langle 1 \rangle 2$. Let: $\mathcal{U}$ be a set of open sets in X that covers $A \cup B$
$\langle 1 \rangle 3$. Pick a finite subset $\mathcal{U}_1$ that covers A.
Proof: Lemma 9.48.2.
$\langle 1 \rangle 4$. Pick a finite subset $\mathcal{U}_2$ that covers B.
Proof: Lemma 9.48.2.
$\langle 1 \rangle 5$. $\mathcal{U}_1 \cup \mathcal{U}_2$ is a finite subset that covers $A \cup B$.
$\langle 1 \rangle 6$. Q.E.D.
Proof: Lemma 9.48.2.
□

Proposition 9.48.10. *Let A and B be disjoint compact subspaces of the Hausdorff space X. Then there exist disjoint open sets U and V that include A and B respectively.*

PROOF: From Theorem 9.48.5 with $N = X^2 \setminus \{(x, x) \mid x \in X\}$. ∎

Corollary 9.48.10.1. *Every compact subspace of a Hausdorff space is closed.*

Theorem 9.48.11. *Let $f : X \to Y$ be a bijective continuous function. If X is compact and Y is Hausdorff then f is a homeomorphism.*

PROOF:
⟨1⟩1. LET: $C \subseteq X$ be closed.
⟨1⟩2. C is compact.
  PROOF: Proposition 9.48.3.
⟨1⟩3. $f(C)$ is compact.
  PROOF: Theorem 9.48.4.
⟨1⟩4. $f(C)$ is closed.
  PROOF: Corollary 9.48.10.1.
⟨1⟩5. Q.E.D.
  PROOF: Lemma 9.15.2.
∎

Proposition 9.48.12. *Let X be a compact space, Y a Hausdorff space, and $f : X \to Y$ a continuous map. Then f is a closed map.*

PROOF:
⟨1⟩1. LET: $C \subseteq X$ be closed.
⟨1⟩2. C is compact.
  PROOF: Proposition 9.48.3.
⟨1⟩3. $f(C)$ is compact.
  PROOF: Theorem 9.48.4.
⟨1⟩4. $f(C)$ is closed.
  PROOF: Corollary 9.48.10.1.
∎

Proposition 9.48.13. *If Y is compact then the projection $\pi_1 : X \times Y \to X$ is a closed map.*

PROOF:
⟨1⟩1. LET: $A \subseteq X \times Y$ be closed.
⟨1⟩2. LET: $x \in X \setminus \pi_1(A)$
⟨1⟩3. PICK a neighbourhood U of x such that $U \times Y \subseteq (X \times Y) \setminus A$
  PROOF: By the Tube Lemma.
⟨1⟩4. $x \in U \subseteq X \setminus \pi_1(A)$
⟨1⟩5. Q.E.D.
  PROOF: So $X \setminus \pi_1(A)$ is open by Lemma 9.3.8.
∎

Proposition 9.48.14. *Let X be a topological space and Y a Hausdorff space. Let $f : X \to Y$ be continuous. Then the graph of f is closed in $X \times Y$.*

$\langle 1 \rangle 1$. Assume: f is continuous.
$\langle 1 \rangle 2$. Let: $(x, y) \in (X \times Y) \setminus G_f$
$\langle 1 \rangle 3$. Pick disjoint neighbourhoods U and V of y and $f(x)$ respectively.
$\langle 1 \rangle 4$. $f^{-1}(V) \times U$ is a neighbourhood of (x, y) disjoint from G_f.
$\square$

Theorem 9.48.15. *Let X be a topological space and Y a compact space. Let $f : X \to Y$ be a function. If the granh of f is closed in $X \times Y$ then f is continuous.*

Proof:
$\langle 1 \rangle 1$. Assume: G_f is closed.
$\langle 1 \rangle 2$. Let: $x \in X$ and V be a neighbourhood of $f(x)$.
$\langle 1 \rangle 3$. $G_f \cap (X \times (Y \setminus V))$ is closed.
$\langle 1 \rangle 4$. $\pi_1(G_f \cap (X \times (Y \setminus V)))$ is closed.
Proof: Proposition 9.48.13.
$\langle 1 \rangle 5$. Let: $U = X \setminus \pi_1(G_f \cap (X \times (Y \setminus V)))$
$\langle 1 \rangle 6$. U is a neighbourhood of x
$\langle 1 \rangle 7$. $f(U) \subseteq V$
$\square$

Theorem 9.48.16. *Let X be a compact topological space. Let $(f_n : X \to \mathbb{R})$ be a monotone increasing sequence of continuous functions and $f : X \to \mathbb{R}$ a continuous function. If (f_n) converges pointwise to f, then (f_n) converges uniformly to f.*

Proof:
$\langle 1 \rangle 1$. Let: $\epsilon > 0$
$\langle 1 \rangle 2$. For all $x \in X$, there exists N such that, for all $n \geq N$, we have $|f_n(x) - f(x)| < \epsilon$
$\langle 1 \rangle 3$. For $n \geq 1$,
Let: $U_n = \{x \in X \mid |f_n(x) - f(x)| < \epsilon\}$
$\langle 1 \rangle 4$. For $n \geq 1$, we have U_n is open in X.
$\langle 2 \rangle 1$. Let: $x \in X$
$\langle 2 \rangle 2$. Let: $\delta = \epsilon - |f_n(x) - f(x)|$
$\langle 2 \rangle 3$. Pick a neighbourhood U of x such that $f(U) \subseteq B(f(x), \delta/2)$
$\langle 2 \rangle 4$. Pick a neighbourhood V of x such that $f_n(V) \subseteq B(f_n(x), \delta/2)$
$\langle 2 \rangle 5$. $f(U \cap V) \subseteq U_n$
Proof: For $y \in U \cap V$ we have

$$\begin{aligned} |f_n(y) - f(y)| &\leq |f_n(y) - f_n(x)| + |f_n(x) - f(x)| + |f(x) - y(y)| \\ &< \delta/2 + |f_n(x) - f(x)| + \delta/2 \\ &= \epsilon \end{aligned}$$

$\langle 1 \rangle 5$. $\{U_n \mid n \geq 1\}$ covers X
Proof: From $\langle 1 \rangle 2$

⟨1⟩6. PICK N such that $X = U_N$

⟨2⟩1. PICK $n_1, \ldots, n_k$ such that $U_{n_1}, \ldots, U_{n_k}$ cover X.

⟨2⟩2. LET: $N = \max(n_1, \ldots, n_k)$

⟨2⟩3. For all i we have $U_{n_i} \subseteq U_N$

PROOF: Since (f_n) is monotone increasing.

⟨2⟩4. $X = U_N$

⟨1⟩7. For all $x \in X$ and $n \geq N$ we have $|f_n(x) - f(x)| < \epsilon$

□

An example to show that we cannot remove the hypothesis that X is compact:

Example 9.48.17. Let $X = (0,1)$, $f_n(x) = -x^n$ and $f(x) = 0$ for $x \in X$ and $n \geq 1$. Then $f_n \to f$ pointwise and (f_n) is monotone increasing but the convergence is not uniform since, for all $N \geq 1$, there exists $x \in (0,1)$ such that $-x^N < -1/2$.

An example to show that we cannot remove the hypothesis that (f_n) is monotone increasing:

Example 9.48.18. Let $X = [0,1]$, $f_n(x) = 1/(n^3(x - 1/n)^2 + 1)$ and $f(x) = 0$ for $x \in X$ and $n \geq 1$. Then X is compact and $f_n \to f$ pointwise but the convergence is not uniform since, for all $N \geq 1$, there exists $x \in [0,1]$ such that $f_N(x) = 1$, namely $x = 1/N$.

Theorem 9.48.19. *Let X be a compact Hausdorff space. Let $\mathcal{A}$ be a chain of closed connected subsets of X. Then $\bigcap \mathcal{A}$ is connected.*

PROOF:

⟨1⟩1. ASSUME: for a contradiction C and D form a separation of $\bigcap \mathcal{A}$.

⟨1⟩2. PICK disjoint open sets U and V that include C and D respectively.

PROOF: Proposition 9.48.10.

⟨1⟩3. $\{A \setminus (U \cup V) \mid A \in \mathcal{A}\}$ is a set of closed sets with the finite intersection property.

⟨2⟩1. For all $A \in \mathcal{A}$ we have $A \setminus (U \cup V)$ is closed.

⟨2⟩2. For all $A_1, \ldots, A_n \in \mathcal{A}$ we have $(A_1 \cap \cdots \cap A_n) \setminus (U \cup V)$ is nonempty.

PROOF:

⟨3⟩1. LET: $A_1, \ldots, A_n \in \mathcal{A}$

⟨3⟩2. ASSUME: without loss of generality $A_1 \subseteq A_2, \ldots, A_n$

PROOF: Since $\mathcal{A}$ is a chain.

⟨3⟩3. $A_1 \setminus (U \cup V)$ is nonempty

PROOF: Otherwise $(A_1 \cap \cdots \cap A_n \cap U)$ and $(A_1 \cap \cdots \cap A_n \cap V)$ would form a separation of A_n.

⟨1⟩4. $\bigcap \mathcal{A} \setminus (U \cup V)$ is nonempty.

PROOF: Theorem 9.48.6.

⟨1⟩5. Q.E.D.

PROOF: This contradicts ⟨1⟩1 since $\bigcap AA \setminus (U \cup V) = \bigcap \mathcal{A} \setminus (C \cup D)$.

□

Theorem 9.48.20 (Tychonoff Theorem (AC))**.** *The product of a family of compact spaces is compact.*

PROOF:
⟨1⟩1. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of compact spaces.
⟨1⟩2. LET: $X = \prod_{\alpha \in J} X_\alpha$
⟨1⟩3. For any $\mathcal{A} \subseteq \mathcal{P}X$, we have $\bigcap_{A \in \mathcal{A}} \overline{A} \neq \emptyset$
⟨2⟩1. LET: $\mathcal{A} \subseteq \mathcal{P}X$
⟨2⟩2. PICK $\mathcal{D} \supseteq \mathcal{A}$ that is maximal with respect to the finite intersection property.
PROVE: $\bigcap_{D \in \mathcal{D}} \overline{D} \neq \emptyset$
PROOF: Lemma 3.24.2.
⟨2⟩3. For $\alpha \in J$, PICK $x_\alpha \in X_\alpha$ such that $x_\alpha \in \bigcap_{D \in \mathcal{D}} \overline{\pi_\alpha(D)}$
PROOF: Theorem 9.48.6 since $\{\overline{\pi_\alpha(D)} \mid D \in \mathcal{D}\}$ is a set of closed sets in X_α with the finite intersection property.
⟨2⟩4. LET: $x = (x_\alpha)_{\alpha \in J}$
PROVE: $x \in \bigcap_{D \in \mathcal{D}} \overline{D}$
⟨2⟩5. For any $\beta \in J$ and neighbourhood U of x_β in X_β, we have ${\pi_\beta}^{-1}(U)$ intersects every element of $\mathcal{D}$
⟨3⟩1. LET: $\beta \in J$
⟨3⟩2. LET: U be a neighbourhood of x_β in X_β.
⟨3⟩3. LET: $D \in \mathcal{D}$
⟨3⟩4. $x_\beta \in \overline{\pi_\beta(D)}$
PROOF: From ⟨2⟩3
⟨3⟩5. U intersects $\pi_\beta(D)$.
⟨3⟩6. ${\pi_\beta}^{-1}(U)$ intersects D.
⟨2⟩6. For any $\beta \in J$ and neighbourhood U of x_β in X_β, we have ${\pi_\beta}^{-1}(U) \in \mathcal{D}$
PROOF: Lemma 3.24.4.
⟨2⟩7. Every basic neighbourhood of x is an element of $\mathcal{D}$
PROOF: Lemma 3.24.3.
⟨2⟩8. Every basic neighbourhood of x intersects every element of $\mathcal{D}$
PROOF: Since $\mathcal{D}$ satisfies the finite intersection property.
⟨2⟩9. For all $D \in \mathcal{D}$ we have $x \in \overline{D}$
⟨1⟩4. Q.E.D.
PROOF: Theorem 9.48.6.
□

Lemma 9.48.21. *Let X and Y be topological spaces. Let $\mathcal{A}$ be a set of basis elements for the product topology on $X \times Y$ such that no finite subset of $\mathcal{A}$ covers $X \times Y$. If X is compact, then there exists $x \in X$ such that no finite subset of $\mathcal{A}$ covers the slice $\{x\} \times Y$.*

PROOF:
⟨1⟩1. ASSUME: for every $x \in X$, there exists a finite subset of $\mathcal{A}$ that covers $\{x\} \times Y$
PROVE: A finite subset of $\mathcal{A}$ covers $X \times Y$

⟨1⟩2. $\{U \mid \exists U_1 \times V_1, \ldots, U_n \times V_n \in \mathcal{A}. U = U_1 \cap \cdots \cap U_n, V_1 \cup \cdots \cup V_n = Y\}$ covers X

⟨1⟩3. Pick a finite subcover $U_1, \ldots, U_m$

⟨1⟩4. Pick $U_{ij} \times V_{ij} \in \mathcal{A}$ such that, for every i, we have $U_i = \bigcap_j U_{ij}$ and $Y = \bigcup_j V_{ij}$

⟨1⟩5. The collection of all $U_{ij} \times V_{ij}$ covers $X \times Y$

□

Theorem 9.48.22 (AC)**.** *Let X be a compact Hausdorff space. Then the quasicomponents and the components of X are the same.*

Proof:

⟨1⟩1. Let: $x, y \in X$

⟨1⟩2. Assume: x and y are in the same quasicomponent.
Prove: x and y are in the same component.

⟨1⟩3. Let: $\mathcal{A}$ be the set of all closed subsets A of X such that x and y are in the same quasicomponent of A.

⟨1⟩4. For every chain $\mathcal{B} \subseteq \mathcal{A}$ we have $\bigcap \mathcal{B} \in \mathcal{A}$

⟨2⟩1. Let: $BB \subseteq \mathcal{A}$ be a chain.

⟨2⟩2. Assume: for a contradiction U and V form a separation of $\bigcap \mathcal{B}$ with $x \in U$ and $y \in V$

⟨2⟩3. Pick disjoint open sets U', V' in X such that $U \subseteq U'$ and $V \subseteq V'$

⟨2⟩4. $\{B \setminus (U' \cup V') \mid B \in \mathcal{B}\}$ satisfies the finite intersection property.

⟨3⟩1. Let: $B_1, \ldots, B_n \in \mathcal{B}$

⟨3⟩2. Assume: without loss of generality $B_1 \subseteq \cdots \subseteq B_n$
Proof: Since $\mathcal{B}$ is a chain.

⟨3⟩3. $\bigcap\{B_1 \setminus (U' \cup V'), \ldots, B_n \setminus (U' \cup V')\} = B_1 \setminus (U' \cup V')$

⟨3⟩4. $B_1 \setminus (U' \cup V')$ is nonempty
Proof: Otherwise $B_1 \cap U'$ and $B_1 \cap V'$ would form a separation of B_1, contradicting the fact that x and y are in the same quasicomponent of B_1.

⟨2⟩5. $\bigcap \mathcal{B} \setminus (U \cup V)$ is nonempty
Proof: Theorem 9.48.6.

⟨2⟩6. Q.E.D.
Proof: This contradicts ⟨2⟩2.

⟨1⟩5. Pick a minimal element D in $\mathcal{A}$.
Prove: D is connected.
Proof: By Zorn's Lemma.

⟨1⟩6. Assume: for a contradiction U and V form a separation of D.

⟨1⟩7. Assume: without loss of generality $x, y \in U$
Proof: We cannot have that one of x, y is in U and the other in V sicnce $D \in \mathcal{A}$.

⟨1⟩8. $U \in \mathcal{A}$
Proof: If X and Y form a separation of U with $x \in X$ and $y \in Y$, then X and $Y \cup V$ form a separation of D with $x \in X$ and $y \in Y \cup V$.

⟨1⟩9. Q.E.D.
Proof: There is a connected set D that contains both x and y.

□

Proof:

⟨1⟩1. Let: $\{X_\alpha\}_{\alpha \in J}$ be a family of compact spaces.

⟨1⟩2. Let: $X = \prod_{\alpha \in J} X_\alpha$

⟨1⟩3. Pick a well-ordering $<$ on J such that J has a greatest element.

⟨1⟩4. For $\alpha \in J$ and $p = \{p_i \in X_i\}_{i \leq \alpha}$ a family of points,
Let: $Y(p) = \{x \in X \mid \forall i \leq \alpha . x_i = p_i\}$

⟨1⟩5. If $\alpha < \alpha'$ and p is an α'-indexed family of points then $Y(p) \subseteq Y(p \upharpoonright \alpha)$
Proof: From definition.

⟨1⟩6. Given $\beta \in J$ and $p = \{p_i \in X_i\}_{i < \beta}$ a family of points,
Let: $Z(p) = \bigcap_{\alpha < \beta} Y(p \upharpoonright \alpha)$

⟨1⟩7. Given $\beta \in J$ and $p = \{p_i \in X_i\}_{i < \beta}$ a family of points, if $\mathcal{A}$ is a finite set of basic open spaces for X that covers $Z(p)$, then there exists $\alpha < \beta$ such that $\mathcal{A}$ covers $Y(p \upharpoonright \alpha)$

⟨2⟩1. Assume: without loss of generality β has no immediate predecessor.

⟨2⟩2. For $A \in \mathcal{A}$,
Let: $J_A = \{i < \beta \mid \pi_i(A) \neq X_i\}$

⟨2⟩3. Let: $\alpha = \max \bigcup_{A \in \mathcal{A}} J_A$

⟨2⟩4. Let: $x \in Y(p \upharpoonright \alpha)$

⟨2⟩5. Let: $y \in Z(p)$ be the point with $y_i = p_i$ for $i < \beta$ and $y_i = x_i$ for $i \geq \beta$

⟨2⟩6. Pick $A \in \mathcal{A}$ such that $y \in A$
Proof: Since $\mathcal{A}$ covers $Z(p)$.

⟨2⟩7. For $i \in J_A$ we have $x_i \in \pi_i(A)$
Proof: Since $i \leq \alpha$ so $x_i = p_i$

⟨2⟩8. For $i \in J \setminus J_A$ we have $x_i \in \pi_i(A)$
Proof: Since $\pi_i(A) = X_i$

⟨2⟩9. $x \in A$

⟨1⟩8. Assume: for a contraction $\mathcal{A}$ is a set of basic open sets for X that covers X but such that no finite subset of $\mathcal{A}$ covers X

⟨1⟩9. Pick a set of points $\{p_i\}_{i \in J}$ such that, for all $\alpha \in J$, we have $Y(p \upharpoonright \alpha)$ is not finitely covered by $\mathcal{A}$

⟨2⟩1. Assume: as transfinite induction hypothesis $\alpha \in J$ and $\{p_i\}_{i < \alpha}$ is a family of points such that, for all $\alpha' < \alpha$, we have $Y(p \upharpoonright \alpha')$ is not finitely covered by $\mathcal{A}$

⟨2⟩2. $Z(p)$ is not finitely covered by $\mathcal{A}$
Proof: By ⟨1⟩7.

⟨2⟩3. Pick $p_\alpha \in X_\alpha$ such that $Y(p)$ is not finitely covered by $\mathcal{A}$
Proof: By Lemma 9.48.21 since there is a homeomorphism $\phi : Z(p) \cong X_\alpha \times \prod_{\alpha' > \alpha} X_{\alpha'}$ and, given p_α, this homemorphism ϕ restricts to a homeomorphism $Y(p) \cong \{p_\alpha\} \times \prod_{\alpha' > \alpha} X_{\alpha'}$.

⟨1⟩10. Q.E.D.
Proof: If ω is the greatest element of J then $Y(p \upharpoonright \omega)$ is a singleton.

□

Theorem 9.48.23. *Every complete linearly ordered set in the order topology is compact.*

PROOF:
⟨1⟩1. LET: X be a complete linearly ordered set with least element a and greatest element b.
⟨1⟩2. LET: $\mathcal{A}$ be an open covering of X.
⟨1⟩3. For all $x < b$, there exists $y > x$ such that $[x, y]$ can be covered by at most two elements of $\mathcal{A}$.
⟨2⟩1. LET: $x \in X$
⟨2⟩2. PICK $A \in \mathcal{A}$ with $x \in A$
⟨2⟩3. PICK $y > x$ such that $[x, y) \subseteq A$
⟨2⟩4. PICK $B \in \mathcal{A}$ with $y \in B$
⟨2⟩5. $[x, y]$ is covered by A and B
⟨1⟩4. LET: $C = \{y \in X \mid [a, y]$ can be covered by finitely many elements of $\mathcal{A}\}$
⟨1⟩5. LET: $c = \sup C$
⟨1⟩6. $c > a$
⟨2⟩1. PICK $x > a$ such that $[a, x]$ can be covered by at most two elements of $\mathcal{A}$.
PROOF: From ⟨1⟩3.
⟨2⟩2. $x \in C$
⟨1⟩7. $c \in C$
⟨2⟩1. PICK $A \in \mathcal{A}$
⟨2⟩2. PICK $x < c$ such that $(x, c] \subseteq A$
⟨2⟩3. PICK $y > x$ such that $y \in C$
⟨2⟩4. PICK $\mathcal{A}_0 \subseteq^{\text{fin}} \mathcal{A}$ that covers $[a, y]$
⟨2⟩5. $\mathcal{A}_0 \cup \{A\}$ covers $[a, c]$
⟨1⟩8. $c = b$
⟨2⟩1. ASSUME: for a contradiction $c < b$
⟨2⟩2. PICK $x > c$ such that $[c, x]$ can be covered by at most two elements of $\mathcal{A}$
PROOF: From ⟨1⟩3.
⟨2⟩3. $[a, x]$ can be finitely covered by $\mathcal{A}$
PROOF: From ⟨1⟩7.
⟨2⟩4. Q.E.D.
PROOF: This contradicts the maximality of c.
□

Corollary 9.48.23.1. *Let X be a linearly ordered set with the least upper bound property. Then every closed interval in X is compact.*

Corollary 9.48.23.2. *Every closed interval in $\mathbb{R}$ is compact.*

Theorem 9.48.24 (Extreme Value Theorem)**.** *Any linearly ordered set under the order topology that is compact has a greatest and a least element.*

PROOF:
⟨1⟩1. LET: X be a linearly ordered set under the order topology that is compact.
⟨1⟩2. X has a greatest element.
⟨2⟩1. ASSUME: for a contradiction X has no greatest element.

⟨2⟩2. $\{(-\infty, a) \mid a \in X\}$ covers X.
⟨2⟩3. Pick a finite subcover $\{(-\infty, a_1), \ldots, (-\infty, a_n)\}$, say.
⟨2⟩4. Assume: without loss of generality $a_1 \leq \cdots \leq a_n$
⟨2⟩5. $X \subseteq (-\infty, a_n)$
⟨2⟩6. $a_n < a_n$
⟨1⟩3. X has a least element.
Proof: Similar.
□

Proposition 9.48.25. *Every linearly ordered set in which every closed interval is compact satisfies the least upper bound property.*

Proof:
⟨1⟩1. Let: X be a linearly ordered set in which every closed interval is compact.
⟨1⟩2. Let: $A \subseteq X$ be nonempty with upper bound u
⟨1⟩3. Pick $a \in A$
⟨1⟩4. The closed interval $[a, u]$ is compact.
⟨1⟩5. Assume: for a contradiction A has no supremum.
⟨1⟩6. $\{(-\infty, x) \mid x \in A\} \cup \{(x, +\infty) \mid x$ is an upper bound of $A\}$ covers $[a, u]$.
⟨2⟩1. Let: $x \in [a, u]$
⟨2⟩2. Assume: for all $y \in A$ we have $x \notin (-\infty, y)$
⟨2⟩3. x is an upper bound for A
⟨2⟩4. Pick an upper bound y for A with $y < x$
⟨2⟩5. $x \in (y, +\infty)$
⟨1⟩7. Pick a finite subcover $\{(-\infty, x_1), \ldots, (-\infty, x_m), (y_1, +\infty), \ldots, (y_n, +\infty)\}$
⟨1⟩8. Assume: $x_m = \max(x_1, \ldots, x_m)$ and $y_1 = \min(y_1, \ldots, y_n)$
⟨1⟩9. $x_m \notin (-\infty, x_i)$ for any i
Proof: Since $x_i \leq x_m$
⟨1⟩10. $x_m \notin (y_i, +\infty)$ for any i
Proof: Since $x_m \in A$ so $x_m \leq y_i$
⟨1⟩11. $x_m \in [a, u]$
⟨2⟩1. $a \notin (y_i, +\infty)$ for any i
Proof: Since y_i is an upper bound for A and $a \in A$.
⟨2⟩2. $a \in (-\infty, x_i)$ for some i
Proof: From ⟨1⟩7.
⟨2⟩3. $a < x_m$
Proof: Since $x_i \leq x_m$
⟨2⟩4. $x_m \leq u$
Proof: Since u is an upper bound for A and $x_m \in A$.
⟨1⟩12. Q.E.D.
Proof: This contradicts ⟨1⟩7.
□

Example 9.48.26. The set $[0, 1]$ is not compact under the K-topology.

Proof: For every $n \geq 1$, pick an open interval U_n such that $U_n \cap K = \{1/n\}$. Then the open cover $\{[0, 1] - K\} \cup \{U_n \mid n \in \mathbb{Z}^+\}$ has no finite subcover. □

Proposition 9.48.27 (AC)**.** *Let X be a compact Hausdorff space. Let $\mathcal{A}$ be a countable set of closed sets in X. If every element of $\mathcal{A}$ has empty interior, then $\bigcup \mathcal{A}$ has empty interior.*

PROOF:
$\langle 1 \rangle 1$. LET: X be a compact Hausdorff space.
$\langle 1 \rangle 2$. For every closed set A in X and open U in X with $U \nsubseteq A$, there exists a nonempty open set V such that $\overline{V} \subseteq U - A$.
  $\langle 2 \rangle 1$. LET: A be a closed set in X
  $\langle 2 \rangle 2$. LET: U be an open set in X with $U \nsubseteq A$
  $\langle 2 \rangle 3$. PICK $x \in U - A$
  $\langle 2 \rangle 4$. PICK disjoint neighbourhoods W and V of $A \cup (X - U)$ and x respectively.
   PROOF: Proposition 9.48.10.
  $\langle 2 \rangle 5$. $\overline{V} \subseteq U - A$
   PROOF:

$$\begin{aligned} \overline{V} &\subseteq X - W && \text{(since } V \subseteq X - W\text{)} \\ &\subseteq X - (A \cup (X - U)) \\ &= (x - A) \cap U \\ &= U - A \end{aligned}$$

$\langle 1 \rangle 3$. PICK an enumeration $\{A_1, A_2, \ldots\}$ of $\mathcal{A}$
$\langle 1 \rangle 4$. LET: U_0 be any nonempty open set
  PROVE: $U_0 \nsubseteq \bigcup \mathcal{A}$
$\langle 1 \rangle 5$. PICK a sequence of nonempty open sets U_1, U_2, … such that, for $n \geq 1$, we have $\overline{U_n} \subseteq U_{n-1} - A_n$
  $\langle 2 \rangle 1$. ASSUME: we have picked U_0, U_1, …, U_n
  $\langle 2 \rangle 2$. $U_n \nsubseteq A_{n+1}$
   PROOF: Since A_{n+1} has empty interior.
  $\langle 2 \rangle 3$. PICK a nonempty open set U_{n+1} such that $\overline{U_{n+1}} \subseteq U_n - A_{n+1}$
   PROOF: By $\langle 1 \rangle 2$
$\langle 1 \rangle 6$. PICK $a \in \bigcap_{n=0}^{\infty} \overline{U_n}$
  PROOF: Corollary 9.48.6.1.
$\langle 1 \rangle 7$. $a \in U_0$
  PROOF: Since $a \in \overline{U_1} \subseteq U_0$.
$\langle 1 \rangle 8$. $a \notin \bigcup \mathcal{A}$
  PROOF: For all n, we have $a \in \overline{U_n} \subseteq U_{n-1} - A_n$.
□

Example 9.48.28. The Cantor set is compact.

PROOF: It is a closed subset of the compact set $[0, 1]$. □

Proposition 9.48.29. *Every compact space is limit point compact.*

PROOF:
$\langle 1 \rangle 1$. LET: X be a compact space.
$\langle 1 \rangle 2$. LET: $A \subseteq X$ have no limit points.

PROVE: A is finite.
$\langle 1\rangle 3$. A is closed.
PROOF: Corollary 9.8.3.1.
$\langle 1\rangle 4$. A is compact.
PROOF: Proposition 9.48.3.
$\langle 1\rangle 5$. $\{U \mid U \text{ open }, |U \cap A| = 1\}$ covers A.
PROOF: From $\langle 1\rangle 2$, for all $a \in A$, there is a neighbourhood U of a that intersects A in a only.
$\langle 1\rangle 6$. PICK a finite subcover $\{U_1, \ldots, U_n\}$
$\langle 1\rangle 7$. For $i = 1, \ldots, n$,
LET: $U_i \cap A = \{x_i\}$.
$\langle 1\rangle 8$. $A = \{x_1, \ldots, x_n\}$
□

The following examples show that not every limit point compact space is compact.

Example 9.48.30. Let Y be a set with two elements under the indiscrete topology. Then $\mathbb{Z}^+ \times Y$ is limit point compact, since every nonempty set has a limit point. It is not compact, since $\{\{n\} \times Y \mid n \in \mathbb{Z}^+\}$ has no finite subcover.

Example 9.48.31. The space S_Ω is limit point compact but not compact.

PROOF:
$\langle 1\rangle 1$. S_Ω is not compact.
PROOF: From the Extreme Value Theorem, since S_Ω has no greatest element.
$\langle 1\rangle 2$. LET: A be an infinite subset of S_Ω.
$\langle 1\rangle 3$. PICK $B \subseteq A$ that is countably infinite.
PROOF: Proposition **??**.
$\langle 1\rangle 4$. LET: $b = \sup B$
$\langle 1\rangle 5$. $B \subseteq [0, b]$
$\langle 1\rangle 6$. $[0, b]$ is compact.
PROOF: Corollary 9.48.23.1.
$\langle 1\rangle 7$. PICK a limit point x of B in $[0, b]$.
PROOF: Proposition 9.48.29.
$\langle 1\rangle 8$. x is a limit point of A.
PROOF: Lemma 9.8.5.
□

9.49 Perfect Maps

Definition 9.49.1 (Perfect Map)**.** Let X and Y be topological spaces and $f : X \to Y$. Then f is a *perfect map* if and only if f is a closed map, continuous, surjective and $p^{-1}(y)$ is compact for all $y \in Y$.

Proposition 9.49.2. *Let X be a topological space, Y a compact space, and $p : X \to Y$ a closed map such that, for all $y \in Y$, we have $p^{-1}(y)$ is compact. Then X is compact.*

Proof:
⟨1⟩1. Let: $\mathcal{A}$ be a set of closed sets in X with the finite intersection property.
⟨1⟩2. $\mathcal{B} = \{p(A_1 \cap \cdots \cap A_n) \mid A_1, \ldots, A_n \in \mathcal{A}\}$ is a set of closed sets in Y with the finite intersection property.
Proof: Since p is a closed map.
⟨1⟩3. Pick $y \in \bigcap \mathcal{B}$
Proof: Theorem 9.48.6 since Y is compact.
⟨1⟩4. $\{A \cap p^{-1}(y) \mid A \in \mathcal{A}\}$ is a set of closed sets in $p^{-1}(y)$ with the finite intersection property.
⟨1⟩5. Pick $x \in \bigcap_{A \in \mathcal{A}}(A \cap p^{-1}(y))$
Proof: Theorem 9.48.6 since $p^{-1}(y)$ is compact.
⟨1⟩6. $x \in \bigcap \mathcal{A}$
⟨1⟩7. Q.E.D.
Proof: Theorem 9.48.6.
□

9.50 Isolated Points

Definition 9.50.1 (Isolated Point). Let X be a topolgical space and $x \in X$. Then x is an *isolated point* if and only if $\{x\}$ is open.

Theorem 9.50.2 (AC). *A nonempty compact Hausdorff space with no isolated points is uncountable.*

Proof:
⟨1⟩1. Let: X be a nonempty compact Hausdorff space with no isolated points.
⟨1⟩2. For every nonempty open set U and every point $x \in X$, there exists a nonempty open set $V \subseteq U$ such that $x \notin \overline{V}$.
⟨2⟩1. Let: U be a nonempty open set.
⟨2⟩2. Let: $x \in X$
⟨2⟩3. Pick $y \in U - \{x\}$
Proof: This is possible because U cannot be $\{x\}$.
⟨2⟩4. Pick disjoint open neighbourhoods W_1 of x and W_2 of y
⟨2⟩5. Let: $V = W_2 \cap U$
⟨2⟩6. V is nonempty
Proof: Since $y \in V$
⟨2⟩7. V is open
Proof: From ⟨2⟩1, ⟨2⟩4, ⟨2⟩5.
⟨2⟩8. $V \subseteq U$
Proof: From ⟨2⟩5
⟨2⟩9. $x \notin V$
Proof: From ⟨2⟩4 and ⟨2⟩5
⟨1⟩3. Let: (a_n) be any sequence of points in X.
Prove: The set $X - \{a_1, a_2, \ldots\}$ is nonempty.
⟨1⟩4. Pick a sequence of nonempty open sets V_1, V_2, …, such that $V_1 \supseteq V_2 \supseteq \cdots$ and $a_n \notin \overline{V_n}$ for all n.

PROOF: From ⟨1⟩2.
⟨1⟩5. PICK $a \in \bigcap_{n=1}^{\infty} \overline{V_n}$
PROOF: Corollary 9.48.6.1.
⟨1⟩6. $a \in X - \{a_1, a_2, \ldots\}$
PROOF: We cannot have $a = a_n$ because $a \in \overline{V_n}$.
□

Corollary 9.50.2.1. *For all $a, b \in \mathbb{R}$ with $a < b$, the closed interval $[a, b]$ is uncountable.*

Example 9.50.3. The Cantor set has no isolated points, and is therefore uncountable.

PROOF:
⟨1⟩1. LET: (A_n) be the sets in Definition 8.1.1.
⟨1⟩2. LET: $x \in C$
⟨1⟩3. LET: A_n be the first set such that x is an endpoint of one of the intervals that make up A_n
⟨1⟩4. LET: $(a_m)_{m \geq n}$ be the sequence of points defined by: a_m is the point such that either $[a_m, x]$ or $[x, a_m]$ is one of the intervals that make up A_m.
⟨1⟩5. (a_m) is a sequence of points of C distinct from x that converges to x.
PROOF: Since $|a_m - x| = 1/3^m$ for all m.
⟨1⟩6. x is a limit point of C.
□

9.51 Local Compactness

Definition 9.51.1 (Locally Compact)**.** Let X be a topological space and $x \in X$. Then X is *locally compact* at x if and only if there exists a compact subspace of X that includes a neighbourhood of x.

A space is *locally compact* if and only if it is locally compact at every point.

Example 9.51.2. The real line is locally compact, because for every real number x we have $x \in (x-1, x+1) \subseteq [x-1, x+1]$.

Example 9.51.3. For all $n \geq 1$, we have $\mathbb{R}^n$ is locally compact. For any point $x = (x_1, \ldots, x_n)$, we have $x \in (x_1 - 1, x_1 + 1) \times \cdots \times (x_n - 1, x_n + 1) \subseteq [x_1 - 1, x_1 + 1] \times \cdots \times [x_n - 1, x_n + 1]$.

The following example shows that a countable product of locally compact spaces is not necessarily locally compact.

Example 9.51.4. The space $\mathbb{R}^\omega$ is not locally compact.

PROOF:
⟨1⟩1. ASSUME: for a contradiction $0 \in U \subseteq C$ where U is open and C is compact.

⟨1⟩2. Pick a basic open set $B = (a_1, b_1) \times \cdots \times (a_n, b_n) \times \mathbb{R} \times \mathbb{R} \times \cdots$ such that $0 \in B \subseteq U$

⟨1⟩3. $\overline{B} = [a_1, b_1] \times \cdots \times [a_n, b_n] \times \mathbb{R} \times \mathbb{R} \times \cdots$ is compact.

Proof:Proposition 9.48.3.

⟨1⟩4. Q.E.D.

Proof: This is a contradiction.

□

Example 9.51.5. Every linearly ordered set X with the least upper bound property is locally compact under the order topology.

For any point x, pick a basic open set B such that $x \in B$. Then $x \in B \subseteq \overline{B}$ and $\overline{B}$ is a closed interval, hence compact (Corollary 9.48.23.1).

Proposition 9.51.6. *Any closed subspace of a locally compact space is locally compact.*

Proof:

⟨1⟩1. Let: X be a locally compact space and $Y \subseteq X$ be closed.

⟨1⟩2. Let: $y \in Y$.

⟨1⟩3. Pick a compact subspace C of X and neighbourhood U of y in X such that $U \subseteq C$

⟨1⟩4. $y \in U \cap Y \subseteq C \cap Y$

⟨1⟩5. $C \cap Y$ is compact.

Proof:Proposition 9.48.3.

□

Proposition 9.51.7. *Let X be a Hausdorff space. Let $x \in X$. Then X is locally compact at x if and only if, for every neighbourhood U of x, there exists a neighbourhood V of x such that $\overline{V}$ is compact and $\overline{V} \subseteq U$.*

Corollary 9.51.7.1. *Every open subspace of a locally compact Hausdorff space is locally compact Hausdorff.*

This example shows that a subspace of a locally compact Hausdorff space is not necessarily locally compact Hausdorff.

Example 9.51.8. The rationals $\mathbb{Q}$ are not locally compact.

Assume for a contradiction $C \subseteq \mathbb{Q}$ is compact and includes $(-\epsilon, \epsilon) \cap \mathbb{Q}$. Pick an irrational $\xi \in (-\epsilon, \epsilon)$. Then $\{(-\infty, q) \cap \mathbb{Q} \mid q \in \mathbb{Q}, q < \xi\} \cup \{(q, +\infty) \cap \mathbb{Q} \mid q \in \mathbb{Q}, q > \xi\}$ covers C but no finite subcover does.

Proposition 9.51.9. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of nonempty spaces. If $\prod_{\alpha \in J} X_\alpha$ is locally compact under the box topology then each X_α is locally compact.*

Proof:

⟨1⟩1. Let: $\alpha \in J$

⟨1⟩2. Let: $x_\alpha \in X_\alpha$

⟨1⟩3. Extend x_α to a family $(x_\beta)_{\beta \in J} \in \prod_{\beta \in J} X_\beta$

⟨1⟩4. Pick a compact $C \subseteq \prod_{\beta \in J} X_\beta$ that includes a basic open neighbourhood $\prod_{\beta \in J} U_\beta$ of (x_β) such that each U_β is open in X_β.
⟨1⟩5. $x_\alpha \in U_\alpha \subseteq \pi_\alpha(C)$
⟨1⟩6. $\pi_\alpha(C)$ is compact.
Proof: Theorem 9.48.4.
□

Proposition 9.51.10 (AC)**.** *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of nonempty spaces. Then $\prod_{\alpha \in J} X_\alpha$ is locally compact if and only if each X_α is locally compact, and X_α is compact for all but finitely many $\alpha \in J$.*

Proof:
⟨1⟩1. Let: $\{X_\alpha\}_{\alpha \in J}$ be a family of nonempty spaces.
⟨1⟩2. If $\prod_{\alpha \in J} X_\alpha$ is locally compact then each X_α is locally compact.
⟨2⟩1. Assume: $\prod_{\alpha \in J} X_\alpha$ is locally compact.
⟨2⟩2. For all $\alpha \in J$ we have X_α is locally compact.
⟨3⟩1. Let: $\alpha \in J$
⟨3⟩2. Let: $x_\alpha \in X_\alpha$
⟨3⟩3. Extend x_α to a family $(x_\beta)_{\beta \in J} \in \prod_{\beta \in J} X_\beta$
⟨3⟩4. Pick a compact $C \subseteq \prod_{\beta \in J} X_\beta$ that includes a basic open neighbourhood $\prod_{\beta \in J} U_\beta$ of (x_β) such that each U_β is open in X_β.
⟨3⟩5. $x_\alpha \in U_\alpha \subseteq \pi_\alpha(C)$
⟨3⟩6. $\pi_\alpha(C)$ is compact.
Proof: Theorem 9.48.4.
⟨1⟩3. If $\prod_{\alpha \in J} X_\alpha$ is locally compact then X_α is compact for all but finitely many $\alpha \in J$.
⟨2⟩1. Assume: $\prod_{\alpha \in J} X_\alpha$ is locally compact.
⟨2⟩2. Pick $x_\alpha \in X_\alpha$ for all α.
⟨2⟩3. Pick a compact $C \subseteq \prod_{\alpha \in J} X_\alpha$ that includes a basic open neighbourhood $\prod_{\alpha \in J} U_\alpha$ of (x_α) such that each U_α is open in X_α, and $U_\alpha = X_\alpha$ for all but finitely many α.
⟨2⟩4. For all but finitely many $\alpha \in J$, we have $X_\alpha = \pi_\alpha(C)$
⟨2⟩5. For all but finitely many $\alpha \in J$, we have X_α is compact.
Proof: Theorem 9.48.4.
⟨1⟩4. If each X_α is locally compact and X_α is compact for all but finitely many $\alpha \in J$ then $\prod_{\alpha \in J} X_\alpha$ is locally compact.
⟨2⟩1. Assume: X_α is compact for all α except $\alpha_1, \ldots, \alpha_n$
⟨2⟩2. Assume: $X_{\alpha_1}, \ldots, X_{\alpha_n}$ are locally compact.
⟨2⟩3. Let: $(x_\alpha) \in \prod X_\alpha$
⟨2⟩4. For $i = 1, \ldots, n$, Pick a compact $C_{\alpha_i} \subseteq X_{\alpha_i}$ that includes the neighbourhood U_{α_i} of x_{α_i}.
⟨2⟩5. For $\alpha \neq \alpha_1, \ldots, \alpha_n$,
Let: $C_\alpha = U_\alpha = X_\alpha$
⟨2⟩6. $\prod_{\alpha \in J} C_\alpha$ is compact.
Proof: Tychonoff's Theorem.
⟨2⟩7. $(x_\alpha) \in \prod U_\alpha \subseteq \prod C_\alpha$
□

The following example shows that the continuous image of a locally compact space is not necessarily locally compact.

Example 9.51.11. Pick an enumeration $\{q_1, q_2, \ldots\}$ of $\mathbb{Q}$. Let $X = \bigcup_{n \in \mathbb{Z}^+} (n, n+1)$. Define $f : X \to \mathbb{Q}$ by $f(x) = q_n$ if $x \in (n, n+1)$. Then f is continuous, X is locally compact, but $f(X) = \mathbb{Q}$ is not locally compact.

Proposition 9.51.12. *The image of a locally compact space under a continuous open map is locally compact.*

Proof:
$\langle 1 \rangle 1$. Let: X be locally compact and $f : X \twoheadrightarrow Y$ be a surjective continuous open map.
$\langle 1 \rangle 2$. Let: $y \in Y$
$\langle 1 \rangle 3$. Pick $x \in X$ such that $f(x) = y$
$\langle 1 \rangle 4$. Pick a compact $C \subseteq X$ that includes a neighbourhood U of x
$\langle 1 \rangle 5$. $y \in f(U) \subseteq f(C)$ and $f(U)$ is open, $f(C)$ is compact.
$\square$

Lemma 9.51.13. *Let X, Y and Z be topological spaces and $p : X \to Y$. If p is a quotient map and Z is locally compact Hausdorff, then $p \times \mathrm{id}_Z : X \times Z \to Y \times Z$ is a quotient map.*

Proof:
$\langle 1 \rangle 1$. Let: X, Y and Z be topological spaces and $p : X \to Y$.
$\langle 1 \rangle 2$. Assume: p is a quotient map and Z is locally compact Hausdorff.
$\langle 1 \rangle 3$. Let: $\pi = p \times \mathrm{id}_Z$
$\langle 1 \rangle 4$. π is sujective.
$\langle 1 \rangle 5$. π is continuous.
$\langle 1 \rangle 6$. π is strongly continuous.
$\langle 2 \rangle 1$. Let: $A \subseteq Y \times Z$
$\langle 2 \rangle 2$. Assume: $\pi^{-1}(A)$ is open.
$\langle 2 \rangle 3$. Let: $(y, z) \in A$
$\langle 2 \rangle 4$. Pick $x \in X$ such that $p(x) = y$
$\langle 2 \rangle 5$. Pick open sets U_1 in X and V in Z such that $x \in U_1$, $z \in V$, $\overline{V}$ is compact, and $U_1 \times \overline{V} \subseteq \pi^{-1}(A)$
$\langle 3 \rangle 1$. Pick open sets U_1 in X and V' in Z such that $x \in U_1$, $z \in V'$ and $U' \times V' \subseteq \pi^{-1}(A)$
$\langle 3 \rangle 2$. Pick V open in Z such that $z \in V$, $\overline{V}$ is compact and $\overline{V} \subseteq V'$
Proof: Proposition 9.51.7.
$\langle 2 \rangle 6$. Let: $U = \bigcup \{U' \text{ open in } X \mid U' \times \overline{V} \subseteq \pi^{-1}(A)\}$
$\langle 2 \rangle 7$. U is saturated
$\langle 3 \rangle 1$. Let: $a \in U$, $b \in X$ with $p(a) = p(b)$
$\langle 3 \rangle 2$. $\{b\} \times \overline{V} \subseteq \pi^{-1}(A)$
$\langle 3 \rangle 3$. Pick U' open in X such that $b \in U'$ and $U' \times \overline{V} \subseteq \pi^{-1}(A)$
Proof: By the Tube Lemma.
$\langle 3 \rangle 4$. $b \in U' \subseteq U$
$\langle 2 \rangle 8$. $\pi(U \times V)$ is open

PROOF: Since $\pi(U \times V) = p(U) \times V$.

⟨2⟩9. $(y, z) \in \pi(U \times V)$

⟨2⟩10. $\pi(U \times V) \subseteq A$

□

Theorem 9.51.14. *Let A, B, C and D be topological spaces with B and C locally compact Hausdorff. Let $p : A \twoheadrightarrow B$ and $q : C \twoheadrightarrow D$ be quotient maps. Then $p \times q : A \times C \twoheadrightarrow B \times D$.*

PROOF: By Lemma 9.51.13 since $p \times q = (\mathrm{id}_B \times q) \circ (p \times \mathrm{id}_C)$. □

9.52 Compactifications

Definition 9.52.1 (Compactification)**.** Let X be a topological space. A *compactification* of X consists of a compact Hausdorff space Y and an imbedding $X \to Y$.

Definition 9.52.2 (One-Point Compactification)**.** Let X be a topological space. A *one-point compactification* of X is a compactification $i : X \to Y$ such that $Y - i(x)$ consists of a single point.

Theorem 9.52.3. *Let X be a topological space. Then X is locally compact Hausdorff if and only if there exists a one-point compactification $i : X \to Y$. In this case, Y is unique up to unique homeomorphism that commutes with i.*

PROOF:

⟨1⟩1. For any compact Hausdorff space Y and point $a \in Y$, the space $Y - \{a\}$ is locally compact Hausdorff.

⟨2⟩1. LET: Y be a compact Hausdorff space.

⟨2⟩2. LET: $a \in Y$

⟨2⟩3. $Y - \{a\}$ is closed.

⟨2⟩4. $Y - \{a\}$ is locally compact.

PROOF: Proposition 9.51.6.

⟨2⟩5. $Y - \{a\}$ is Hausdorff.

PROOF: Theorem 9.22.6.

⟨1⟩2. For any locally compact Hausdorff space X, there exists a compact Hausdorff space Y and imbedding $i : X \to Y$ such that $Y - i(X)$ is a single point.

⟨2⟩1. LET: X be a locally compact Hausdorff space.

⟨2⟩2. LET: $Y = X \cup \{\infty\}$

⟨2⟩3. Define a topology on Y by: $U \subseteq Y$ is open if and only if U is an open set in X or $U = Y - C$ where C is a compact subspace of X.

⟨3⟩1. Y is open.

PROOF: Since $Y = Y - \emptyset$ and $\emptyset$ is a compact subspace of X.

⟨3⟩2. For any set of open sets $\mathcal{U}$ we have $\bigcup \mathcal{U}$ is open.

PROOF: We have $\bigcup \mathcal{U} = Y - (\bigcap \{C \subseteq X \mid C \text{ is compact}, Y - C \in \mathcal{U}\} - \bigcup \{U \subseteq X \mid U \text{ is open in } X, U \in \mathcal{U}\})$, where we take the empty intersection to be Y.

$\langle 3 \rangle 3$. For any open sets U and V we have $U \cap V$ is open.
$\langle 4 \rangle 1$. LET: U and V be open sets.
$\langle 4 \rangle 2$. CASE: U and V are open sets in X.
PROOF: In this case $U \cap V$ is open in X.
$\langle 4 \rangle 3$. CASE: C_1 and C_2 are compact subspaces of X and $U = X - C_1$, $V = X - C_2$
PROOF: In this case $C_1 \cup C_2$ is compact and $U \cap V = X - (C_1 \cup C_2)$.
$\langle 4 \rangle 4$. CASE: U is open in X, C is a compact subspace of X and $V = X - C$
PROOF: In this case $U \cap V = U - C$ which is open since C is closed.
$\langle 2 \rangle 4$. Y is compact.
$\langle 3 \rangle 1$. LET: $\mathcal{A}$ be an open cover of Y.
$\langle 3 \rangle 2$. PICK C compact in X such that $Y - C \in \mathcal{A}$
PROOF: There must be at least one such member of $\mathcal{A}$ since $\infty \in \bigcup \mathcal{A}$.
$\langle 3 \rangle 3$. $\{U \cap X \mid U \in \mathcal{A} - \{Y - C\}\}$ is a set of open sets in X that covers C.
$\langle 3 \rangle 4$. PICK a finite subcover $\{U_1 \cap X, \ldots, U_n \cap X\}$
$\langle 3 \rangle 5$. $\{U_1 \cap X, \ldots, U_n \cap X, Y - C\}$ covers Y.
$\langle 2 \rangle 5$. Y is Hausdorff.
$\langle 3 \rangle 1$. LET: $x, y \in Y$ with $x \neq y$
$\langle 3 \rangle 2$. CASE: $x, y \in X$
PROOF: There are disjoint open sets U, V in X such that $x \in U$, $y \in V$.
$\langle 3 \rangle 3$. CASE: $x \in X$, $y = \infty$
$\langle 4 \rangle 1$. PICK a compact C that includes a neighbourhood U of x
PROOF: Since X is locally compact.
$\langle 4 \rangle 2$. U and $Y - C$ are disjoint open sets in Y with $x \in U$ and $\infty \in Y - C$
$\langle 2 \rangle 6$. Let $i : X \to Y$ be the inclusion.
$\langle 2 \rangle 7$. i is an imbedding.
$\langle 3 \rangle 1$. i is continuous
$\langle 3 \rangle 2$. i is an open map.
$\langle 2 \rangle 8$. $Y - i(X) = \{\infty\}$
$\langle 1 \rangle 3$. If X is locally compact Hausdorff, Y and Y' are compact Hausdorff, and $i : X \to Y$, $i' : \to Y'$ are imbeddings such that $Y - i(X)$ and $Y' - i'(X)$ each have just one point, then there exists a unique homeomorphism $\theta : Y \cong Y'$ such that $\theta \circ i = i'$.
$\langle 2 \rangle 1$. LET: $Y - i(X) = \{a\}$ and $Y' - i'(X) = \{b\}$
$\langle 2 \rangle 2$. LET: $\theta : Y \to Y'$ be the function with $\theta(a) = b$ and $\theta(i(x)) = i'(x)$
$\langle 2 \rangle 3$. θ is a bijection
$\langle 2 \rangle 4$. θ is continuous.
$\langle 3 \rangle 1$. LET: $U \subseteq Y'$ be open.
PROVE: $\theta^{-1}(U)$ is open.
$\langle 3 \rangle 2$. CASE: $b \in U$
$\langle 4 \rangle 1$. $Y' - U$ is compact
$\langle 4 \rangle 2$. $i({i'}^{-1}(Y' - U))$ is compact.
$\langle 4 \rangle 3$. $i({i'}^{-1}(Y' - U))$ is closed.
$\langle 4 \rangle 4$. $\theta^{-1}(U) = X - i({i'}^{-1}(Y' - U))$

⟨3⟩3. Case: $b \notin U$

Proof: $U = i'(V)$ for some V open in X and $\theta^{-1}(U) = i(V)$.

⟨2⟩5. θ is an open map.

Proof: Similar.

⟨2⟩6. θ is unique.

$\square$

Example 9.52.4. S^1 is the one-point compactification of $\mathbb{R}$.

Example 9.52.5. S^2 is the one-point compactification of $\mathbb{R}^2$.

Definition 9.52.6 (Riemann Sphere)**.** The *Riemann sphere* or *extended complex plane* is $\mathcal{C} \cup \{\infty\}$ topologized as the one-point compactification of $\mathcal{C}$. It is homeomorphic to S^2.

Example 9.52.7. The one-point compactification of $\mathbb{Z}^+$ is $\{0\} \cup \{1/n \mid n \in \mathbb{Z}^+\}$.

Chapter 10

Topological Groups

Definition 10.0.1 (Topological Group)**.** A *topological group* G consists of a T_1 space G and continuous maps $\cdot : G^2 \to G$ and $(\ )^{-1} : G \to G$ such that $(G, \cdot, (\ )^{-1})$ is a group.

Example 10.0.2. 1. The integers $\mathbb{Z}$ under addition are a topological group.

2. The real numbers $\mathbb{R}$ under addition are a topological group.
3. The positive reals under multiplication are a topological group.
4. The set $\{z \in \mathbb{C} \mid |z| = 1\}$ under multiplication and given the topology of S^1 is a topological group.
5. For any $n \geq 0$, the general linear group $GL_n(\mathbb{R})$ is a topological group under matrix multiplication, considered as a subspace of $\mathbb{R}^{n^2}$.

Lemma 10.0.3. *Let G be a T_1 space and $\cdot : G^2 \to G$, $(\ )^{-1} : G \to G$ be functions such that $(G, \cdot, (\ )^{-1})$ is a group. Then G is a topological group if and only if the function $f : G^2 \to G$ that maps (x, y) to xy^{-1} is continuous.*

PROOF:
$\langle 1 \rangle 1$. If G is a topological group then f is continuous.
PROOF: From Theorem 9.14.9.
$\langle 1 \rangle 2$. If f is continuous then G is a topological group.
$\langle 2 \rangle 1$. ASSUME: f is continuous.
$\langle 2 \rangle 2$. $(\ )^{-1}$ is continuous.
PROOF: Since $x^{-1} = f(e, x)$.
$\langle 2 \rangle 3$. $\cdot$ is continuous.
PROOF: Since $xy = f(x, y^{-1})$.
$\square$

Lemma 10.0.4. *Let G be a topological group and H a subgroup of G. Then H is a topological group under the subspace topology.*

PROOF:
⟨1⟩1. H is T_1.
PROOF: From Proposition 9.21.5.
⟨1⟩2. multiplication and inverse on H are continuous.
PROOF: From Theorem 9.14.10.
□

Lemma 10.0.5. *Let G be a topological group and H a subgroup of G. Then $\overline{H}$ is a subgroup of G.*

PROOF:
⟨1⟩1. LET: $x, y \in \overline{H}$
PROVE: $xy^{-1} \in \overline{H}$
⟨1⟩2. LET: U be any neighbourhood of xy^{-1}
⟨1⟩3. LET: $f : G^2 \to G$, $f(a, b) = ab^{-1}$
⟨1⟩4. $f^{-1}(U)$ is a neighbourhood of (x, y)
⟨1⟩5. PICK neighbourhoods V, W of x and y respectively such that $f(V \times W) \subseteq U$.
⟨1⟩6. PICK $a \in V \cap H$ and $b \in W \cap H$
PROOF: Theorem 9.6.6.
⟨1⟩7. $ab^{-1} \in U \cap H$
⟨1⟩8. Q.E.D.
PROOF: By Theorem 9.6.6.
□

Proposition 10.0.6. *Let G be a topological group and $\alpha \in G$. Then the maps $l_\alpha, r_\alpha : G \to G$ defined by $l_\alpha(x) = \alpha x$, $r_\alpha(x) = x\alpha$ are homeomorphisms of G with itself.*

PROOF: They are continuous with continuous inverses $l_{\alpha^{-1}}$ and $r_{\alpha^{-1}}$. □

Corollary 10.0.6.1. *Every topological group is homogeneous.*

PROOF: Given a topological group G and $a, b \in G$, we have $l_{ba^{-1}}$ is a homeomorphism that maps a to b. □

Proposition 10.0.7. *Let G be a topological group and H a subgroup of G. Give G/H the quotient topology. For all $\alpha \in G$, the map $\overline{f_\alpha}$ that sends xH to αxH is a homeomorphism $G/H \cong G/H$.*

PROOF:
⟨1⟩1. $\overline{f_\alpha}$ is well-defined.
PROOF: If $xy^{-1} \in H$ then $(\alpha x)(\alpha y)^{-1} = xy^{-1} \in H$.
⟨1⟩2. $\overline{f_\alpha}$ is continuous.
PROOF: Theorem 9.26.7 since $\overline{f_\alpha} \circ p = p \circ f_\alpha$ is continuous, where $p : G \twoheadrightarrow G/H$ is the canonical surjection.
⟨1⟩3. $\overline{f_\alpha}^{-1}$ is continuous.
PROOF: Similar since $\overline{f_\alpha}^{-1} = \overline{f_{\alpha^{-1}}}$.

□

Corollary 10.0.7.1. *Let G be a topological group and H a subgroup of G. Give G/H the quotient topology. Then G/H is homogeneous.*

Proposition 10.0.8. *Let G be a topological group and H a subgroup of G. Give G/H the quotient topology. If H is closed in G then G/H is T_1.*

PROOF:
$\langle 1\rangle 1$. LET: $p : G \twoheadrightarrow G/H$ be the canonical surjection
$\langle 1\rangle 2$. LET: $x \in G$
$\langle 1\rangle 3$. $p^{-1}(xH) = f_x(H)$
$\langle 1\rangle 4$. $p^{-1}(xH)$ is closed in G
PROOF: Since H is closed and f_x is a homemorphism of G with itself.
$\langle 1\rangle 5$. $\{xH\}$ is closed in G/H
□

Proposition 10.0.9. *Let G be a topological group and H a subgroup of G. Give G/H the quotient topology. Then the canonical surjection $p : G \twoheadrightarrow G/H$ is an open map.*

PROOF:
$\langle 1\rangle 1$. LET: $U \subseteq G$ be open.
$\langle 1\rangle 2$. $p^{-1}(p(U)) = \bigcup_{h \in H} r_h(U)$
$\langle 1\rangle 3$. $p^{-1}(p(U))$ is open.
$\langle 1\rangle 4$. $p(U)$ is open.
□

Proposition 10.0.10. *Let G be a topological group and H a closed normal subgroup of G. Then G/H is a topological group under the quotient topology.*

PROOF:
$\langle 1\rangle 1$. G/H is T_1
PROOF: Proposition 10.0.8.
$\langle 1\rangle 2$. The map $\overline{m} : (xH, yH) \mapsto xy^{-1}H$ is continuous.
$\langle 2\rangle 1$. $p^2 : G^2 \to (G/H)^2$ is a quotient map.
PROOF: Propositions 9.26.6, 10.0.9.
$\langle 2\rangle 2$. $\overline{m} \circ p^2$ is continuous.
PROOF: As it is $p^2 \circ m$ where $m : G^2 \to G$ with $m(x, y) = xy^{-1}$
□

Lemma 10.0.11. *Let G be a topological group and $A, B \subseteq G$. If either A or B is open then AB is open.*

PROOF: If A is open we have $AB = \bigcup_{b \in B} r_b(A)$. Similarly if B is open. □

Definition 10.0.12 (Symmetric Neighbourhood)**.** Let G be a topological group. A neighbourhood V of e is *symmetric* if and only if $V = V^{-1}$.

Lemma 10.0.13. *Let G be a topological group. Let V be a neighbourhood of e. Then V is symmetric if and only if, for all $x \in V$, we have $x^{-1} \in V$.*

PROOF:
⟨1⟩1. If V is symmetric then, for all $x \in V$, we have $x^{-1} \in V$
PROOF: Immediate from defintions.
⟨1⟩2. If, for all $x \in V$, we have $x^{-1} \in V$, then V is symmetric.
⟨2⟩1. ASSUME: for all $x \in V$ we have $x^{-1} \in V$
⟨2⟩2. $V \subseteq V^{-1}$
PROOF: If $x \in V$ then there exists $y \in V$ such that $x = y^{-1}$, namely $y = x^{-1}$
⟨2⟩3. $V^{-1} \subseteq V$
PROOF: Immediate from ⟨2⟩1.
□

Lemma 10.0.14. *Let G be a topological group. For every neighbourhood U of e, there exists a symmetric neighbourhood V of e such that $V^2 \subseteq U$.*

PROOF:
⟨1⟩1. LET: U be a neighbourhood of e.
⟨1⟩2. PICK a neighbourhood V' of e such that $V'V' \subseteq U$
PROOF: Such a neighbourhood exists because multiplication in G is continuous.
⟨1⟩3. PICK a neighbourhood W of e such that $WW^{-1} \subseteq V'$
PROOF: Such a neighbourhood exists because the function that maps (x, y) to xy^{-1} is continuous.
⟨1⟩4. LET: $V = WW^{-1}$
⟨1⟩5. V is a neighbourhood of e
⟨2⟩1. $e \in V$
PROOF: Since $e \in W$ so $e = ee^{-1} \in V$.
⟨2⟩2. V is open
PROOF: Lemma 10.0.11.
⟨1⟩6. V is symmetric
⟨2⟩1. For all $x \in V$ we have $x^{-1} \in V$
⟨3⟩1. LET: $x \in V$
⟨3⟩2. PICK$y, z \in W$ such that $x = yz^{-1}$
⟨3⟩3. $x^{-1} = zy^{-1}$
⟨3⟩4. $x^{-1} \in V$
⟨3⟩5. $x \in V^{-1}$
⟨2⟩2. Q.E.D.
PROOF: Lemma 10.0.13
⟨1⟩7. $V^2 \subseteq U$
PROOF: We have $V^2 \subseteq (V')^2 \subseteq U$
□

Proposition 10.0.15. *Every topological group is Hausdorff.*

PROOF:
⟨1⟩1. LET: G be a topological group.
⟨1⟩2. LET: $x, y \in G$ with $x \neq y$

⟨1⟩3. LET: $U = G \setminus \{x[^{-1}y]\}$
⟨1⟩4. PICK a symmetric neighbourhood V of e such that $VV \subseteq U$
⟨2⟩1. U is open
PROOF: Since G is T_1.
⟨2⟩2. $e \in U$
PROOF: Since $x \neq y$
⟨2⟩3. Q.E.D.
PROOF: Lemma 10.0.14.
⟨1⟩5. Vx and Vy are disjoint neighbourhoods of x and y respectively.
⟨2⟩1. Vx is open
PROOF: Since $Vx = r_x(V)$
⟨2⟩2. Vy is open
PROOF: Similar.
⟨2⟩3. $Vx \cap Vy = \emptyset$
⟨3⟩1. ASSUME: for a contradiction $z \in Vx \cap Vy$
⟨3⟩2. PICK $a, b \in V$ such that $z = ax = by$
⟨3⟩3. $xy^{-1} \in VV$
PROOF: Since $xy^{-1} = a^{-1}b$
⟨3⟩4. $xy^{-1} \in U$
⟨3⟩5. Q.E.D.
PROOF: From ⟨1⟩3.

□

Proposition 10.0.16. *Every topological group is regular.*

PROOF:
⟨1⟩1. LET: G be a topological group.
⟨1⟩2. LET: $A \subseteq G$ be a closed set and $a \notin A$.
⟨1⟩3. LET: $U = G \setminus Aa^{-1}$
⟨1⟩4. PICK a symmetric neighbourhood V of e such that $VV \subseteq U$
⟨2⟩1. U is open
PROOF: Since $Aa^{-1} = r_{a^{-1}}(A)$ is closed.
⟨2⟩2. $e \in U$
PROOF: Since $a \notin A$.
⟨2⟩3. Q.E.D.
PROOF: Lemma 10.0.14.
⟨1⟩5. VA and Va are disjoint open sets with $A \subseteq VA$ and $a \in Va$
⟨2⟩1. VA is open
PROOF: Lemma 10.0.11
⟨2⟩2. Va is open
PROOF: Lemma 10.0.11
⟨2⟩3. $VA \cap Va = \emptyset$
⟨3⟩1. ASSUME: for a contradiction $z \in VA \cap Va$
⟨3⟩2. PICK $b, c \in V$ and $d \in A$ with $z = bd = ca$
⟨3⟩3. $da^{-1} \in U$
PROOF: Since $da^{-1} = b^{-1}c \in VV \subseteq U$
⟨3⟩4. Q.E.D.

PROOF: This contradicts ⟨1⟩3

∎

Proposition 10.0.17. *Let G be a topological group and H a subgroup of G. Give G/H the quotient topology. If H is closed in G then G/H is regular.*

PROOF:
⟨1⟩1. LET: $p : G \twoheadrightarrow G/H$ be the canonical surjection.
⟨1⟩2. LET: A be a closed set in G/H and $aH \in (G/H) \setminus A$.
⟨1⟩3. LET: $B = p^{-1}(A)$
⟨1⟩4. B is a closed saturated set in G.
⟨1⟩5. $B \cap aH = \emptyset$
⟨1⟩6. $B = BH$
⟨1⟩7. PICK a symmetric neighbourhood V of e such that VB does not intersect Va
⟨2⟩1. LET: $U = G \setminus Ba^{-1}$
⟨2⟩2. PICK a symmetric neighbourhood V of e such that $VV \subseteq U$
⟨3⟩1. U is open
PROOF: Since $Ba^{-1} = r_{a^{-1}}(B)$ is closed.
⟨3⟩2. $e \in U$
PROOF: If $e \in Ba^{-1}$ then $a \in B$
⟨3⟩3. Q.E.D.
PROOF: Lemma 10.0.14
⟨2⟩3. $VB \cap Va = \emptyset$
PROOF: If $vb = v'a$ for $v, v' \in V$ and $b \in B$ then we have $ba^{-1} = v^{-1}v' \in Ba \cap VV \subseteq Ba \cap U$.
⟨1⟩8. $p(VB)$ and $p(Va)$ are disjoint open sets
⟨2⟩1. $p(VB)$ and $p(Va)$ are open.
PROOF: Proposition 10.0.9.
⟨2⟩2. $p(VB) \cap p(Va) = \emptyset$
PROOF: If $vbH = v'aH$ for $v, v' \in V$, $b \in B$ then $v'a = vbh$ for some $h \in H$. Hence $v'a \in Va \cap VBH = Va \cap VB$.
⟨1⟩9. $A \subseteq p(VB)$
⟨1⟩10. $aH \in p(Va)$

∎

Proposition 10.0.18. *Let G be a topological group. The component of G that contains e is a normal subgroup of G.*

PROOF:
⟨1⟩1. LET: C be the component of G that contains e.
⟨1⟩2. For all $x \in G$, xC is the component of G that contains x.
⟨2⟩1. LET: $x \in G$
⟨2⟩2. LET: D be the component of G that contains x.
⟨2⟩3. $xC \subseteq D$
PROOF: Since xC is connected by Theorem 9.31.13.
⟨2⟩4. $D \subseteq xC$

PROOF: Since $x^{-1}D \subseteq C$ similarly.
$\langle 1\rangle 3$. For all $x \in G$, Cx is the component of G that contains x.
PROOF: Similar.
$\langle 1\rangle 4$. For all $x \in C$ we have $xC = Cx = C$
$\langle 1\rangle 5$. For all $x \in C$ we have $x^{-1}C = C$
$\langle 1\rangle 6$. For all $x \in C$ we have $x^{-1} \in C$
$\langle 1\rangle 7$. For all $x, y \in C$ we have $xy \in C$
PROOF: Since $xyC = xC = x$.
$\langle 1\rangle 8$. For all $x \in G$ we have $xC = Cx$.
PROOF: From $\langle 1\rangle 2$ and $\langle 1\rangle 3$.
□

Lemma 10.0.19. *Let G be a topological group. Let A be a closed set in G and B a compact subspace of G such that $A \cap B = \emptyset$. Then there exists a symmetric neighbourhood U of e such that $AU \cap BU = \emptyset$.*

PROOF:
$\langle 1\rangle 1$. For all $b \in B$ there exists a symmetric neighbourhood V of e such that $bV^2 \cap A = \emptyset$
$\langle 2\rangle 1$. LET: $b \in B$
$\langle 2\rangle 2$. LET: $W = b^{-1}(G \setminus A)$
$\langle 2\rangle 3$. W is a neighbourhood of e and $bW \cap A = \emptyset$
$\langle 2\rangle 4$. PICK a symmetric neighbourhood V of e such that $V^2 \subseteq W$
$\langle 1\rangle 2$. $\{bV^2 \mid b \in B, V$ is a symmetric neighbourhood of $e, bV^2 \cap A = \emptyset\}$ is an open cover of B
$\langle 1\rangle 3$. PICK a finite subcover $b_1 V_1^2, \ldots, b_n V_n^2$, say.
$\langle 1\rangle 4$. LET: $U = V_1 \cap \cdots \cap V_n$
$\langle 1\rangle 5$. $BU^2 \cap A = \emptyset$
$\langle 1\rangle 6$. $AU \cap BU = \emptyset$
PROOF: If $av \in BU$ where $a \in A$ and $v \in V$ then $a = avv^{-1} \in BU^2 \cap A$.
□

Proposition 10.0.20 (AC)**.** *Let G be a topological group. Let A be a closed set in G, and B a compact subspace of G. Then AB is closed.*

PROOF:
$\langle 1\rangle 1$. LET: $x \in G \setminus AB$
$\langle 1\rangle 2$. $A^{-1}x \cap B = \emptyset$
$\langle 1\rangle 3$. $A^{-1}x$ is closed.
$\langle 1\rangle 4$. PICK a symmetric neighbourhood U of e such that $A^{-1}xU \cap BU = \emptyset$
$\langle 1\rangle 5$. xU^2 is open
PROOF: Lemma 10.0.11.
$\langle 1\rangle 6$. $x \in xU^2 \subseteq G \setminus AB$
□

Corollary 10.0.20.1. *Let G be a topological group and $H \leq G$. Let $p : G \twoheadrightarrow G/H$ be the quotient map. If H is compact then p is a closed map.*

Proof: For A closed in G, we have $p^{-1}(p(A)) = AH$ is closed, and so $p(A)$ is closed. □

Corollary 10.0.20.2. *Let G be a topological group and $H \leq G$. If H and G/H are compact then G is compact.*

Proof: From Proposition 9.49.2 since, for all $aH \in G/H$, we have $p^{-1}(aH) = aH$ is compact because it is homemorphic to H. □

Proposition 10.0.21. *Let G be a locally compact topological group. Let $H \leq G$. Then G/H is locally compact.*

Proof: From Propositions 9.51.12 and 10.0.9. □

10.1 The Metric Topology

Definition 10.1.1 (Metric)**.** Let X be a set. A *metric* on X is a function $d : X^2 \to \mathbb{R}$ such that:

1. For all $x, y \in X$, $d(x, y) \geq 0$
2. For all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$
3. For all $x, y \in X$, $d(x, y) = d(y, x)$
4. (*Triangle Inequality*) For all $x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$

We call $d(x, y)$ the *distance* between x and y.

Definition 10.1.2 (Open Ball)**.** Let X be a metric space. Let $a \in X$ and $\epsilon > 0$. The *open ball* with *centre* a and *radius* ϵ is

$$B(a, \epsilon) = \{x \in X \mid d(a, x) < \epsilon\} \ .$$

Definition 10.1.3 (Metric Topology)**.** Let X be a metric space. The *metric topology* on X is the topology generated by the basis consisting of all the open balls.

We prove this is a basis for a topology.

PROOF:
$\langle 1 \rangle 1$. For every point a, there exists a ball B such that $a \in B$
PROOF: We have $a \in B(a, 1)$.
$\langle 1 \rangle 2$. For any balls B_1, B_2 and point $a \in B_1 \cap B_2$, there exists a ball B_3 such that $a \in B_3 \subseteq B_1 \cap B_2$
$\langle 2 \rangle 1$. LET: $B_1 = B(c_1, \epsilon_1)$ and $B_2 = B(c_2, \epsilon_2)$
$\langle 2 \rangle 2$. LET: $\delta = \min(\epsilon_1 - d(c_1, a), \epsilon_2 - d(c_2, a))$
PROVE: $B(a, \delta) \subseteq B_1 \cap B_2$
$\langle 2 \rangle 3$. LET: $x \in B(a, \delta)$
$\langle 2 \rangle 4$. $x \in B_1$
PROOF:

$$\begin{aligned} d(x, c_1) &= d(x, a) + d(a, c_1) \\ &< \delta + d(a, c_1) \\ &\leq \epsilon_1 \end{aligned}$$

$\langle 2 \rangle 5$. $x \in B_2$
PROOF: Similar.
□

Proposition 10.1.4. *Let X be a metric space and $U \subseteq X$. Then U is open if and only if, for every $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$.*

PROOF:
$\langle 1 \rangle 1$. If U is open then, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$.
$\langle 2 \rangle 1$. ASSUME: U is open.

⟨2⟩2. LET: $x \in U$
⟨2⟩3. PICK $a \in X$ and $\delta > 0$ such that $x \in B(a, \delta) \subseteq U$
⟨2⟩4. LET: $\epsilon = \delta - d(a, x)$
PROVE: $B(x, \epsilon) \subseteq U$
⟨2⟩5. LET: $y \in B(x, \epsilon)$
⟨2⟩6. $d(y, a) < \delta$
PROOF:

$$\begin{aligned} d(y, a) &\leq d(a, x) + d(x, y) \\ &< \delta + d(x, y) \\ &= \epsilon \end{aligned}$$

⟨2⟩7. $y \in U$
⟨1⟩2. If, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$, then U is open.
PROOF: Immediate from definitions.
□

Definition 10.1.5 (Discrete Metric)**.** Let X be a set. The *discrete metric* on X is defined by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Proposition 10.1.6. *The discrete metric induces the discrete topology.*

PROOF: For any (open) set U and point $a \in U$, we have $a \in B(a, 1) \subseteq U$. □

Definition 10.1.7 (Standard Metric on $\mathbb{R}$)**.** The *standard metric* on $\mathbb{R}$ is defined by $d(x, y) = |x - y|$.

Proposition 10.1.8. *The standard metric on* $\mathbb{R}$ *induces the standard topology on* $\mathbb{R}$*.*

PROOF:
⟨1⟩1. Every open ball is open in the standard topology on $\mathbb{R}$.
PROOF: $B(a, \epsilon) = (a - \epsilon, a + \epsilon)$
⟨1⟩2. For every open set U and point $a \in U$, there exists $\epsilon > 0$ such that $B(a, \epsilon) \subseteq U$
⟨2⟩1. LET: U be an open set and $a \in U$
⟨2⟩2. PICK an open interval b, c such that $a \in (b, c) \subseteq U$
⟨2⟩3. LET: $\epsilon = \min(a - b, c - a)$
⟨2⟩4. $B(a, \epsilon) \subseteq U$
□

Definition 10.1.9 (Metrizable)**.** A topological space X is *metrizable* if and only if there exists a metric on X that induces the topology.

Definition 10.1.10 (Bounded)**.** Let X be a metric space and $A \subseteq X$. Then A is *bounded* if and only if there exists M such that, for all $x, y \in A$, we have $d(x, y) \leq M$.

Definition 10.1.11 (Diameter)**.** Let X be a metric space and $A \subseteq X$. The *diameter* of A is

$$\operatorname{diam} A = \sup_{x,y \in A} d(x,y) \ .$$

Definition 10.1.12 (Standard Bounded Metric)**.** Let d be a metric on X. The *standard bounded metric* corresponding to d is the metric $\overline{d}$ defined by

$$\overline{d}(x,y) = \min(d(x,y), 1) \ .$$

We prove this is a metric.

Proof:
⟨1⟩1. $\overline{d}(x,y) \geq 0$
  Proof: Since $d(x,y) \geq 0$
⟨1⟩2. $\overline{d}(x,y) = 0$ if and only if $x = y$
  Proof: $\overline{d}(x,y) = 0$ if and only if $d(x,y) = 0$ if and only if $x = y$
⟨1⟩3. $\overline{d}(x,y) = \overline{d}(y,x)$
  Proof: Since $d(x,y) = d(y,x)$
⟨1⟩4. $\overline{d}(x,z) \leq \overline{d}(x,y) + \overline{d}(y,z)$
  Proof:

$$\begin{aligned}
\overline{d}(x,y) + \overline{d}(y,z) &= \min(d(x,y),1) + \min(d(y,z),1) \\
&= \min(d(x,y) + d(y,z), d(x,y) + 1, d(y,z) + 1, 2) \\
&\geq \min(d(x,z), 1) \\
&= \overline{d}(x,z)
\end{aligned}$$

□

Lemma 10.1.13. *In any metric space X, the set $\mathcal{B} = \{B(a,\epsilon) \mid a \in X, \epsilon < 1\}$ is a basis for the metric topology.*

Proof:
⟨1⟩1. Every element of $\mathcal{B}$ is open.
  Proof: From Lemma 9.9.2.
⟨1⟩2. For every open set U and point $a \in U$, there exists $B \in \mathcal{B}$ such that $a \in B \subseteq U$
  ⟨2⟩1. Let: U be an open set and $a \in U$
  ⟨2⟩2. Pick$\epsilon > 0$ such that $B(a,\epsilon) \subseteq U$
  ⟨2⟩3. $B(a, \min(\epsilon, 1/2)) \subseteq U$
⟨1⟩3. Q.E.D.
  Proof: Lemma 9.9.3.

□

Proposition 10.1.14. *Let d be a metric on the set X. Then the standard bounded metric $\overline{d}$ induces the same metric as d.*

Proof: This follows from Lemma 10.1.13 since the open balls with radius < 1 are the same under both metrics. □

Lemma 10.1.15. *Let d and d' be two metrics on the same set X. Let $\mathcal{T}$ and $\mathcal{T}'$ be the topologies they induce. Then $\mathcal{T} \subseteq \mathcal{T}'$ if and only if, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that*

$$B_{d'}(x, \delta) \subseteq B_d(x, \epsilon) \enspace .$$

PROOF:
⟨1⟩1. If $\mathcal{T} \subseteq \mathcal{T}'$ then, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$
  PROOF: From Proposition 10.1.4 since $x \in B_d(x, \epsilon) \in \mathcal{T}'$.
⟨1⟩2. If, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$, then $\mathcal{T} \subseteq \mathcal{T}'$
  ⟨2⟩1. ASSUME: For all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$
  ⟨2⟩2. LET: $U \in \mathcal{T}$
  ⟨2⟩3. For all $x \in U$ there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq U$.
    ⟨3⟩1. LET: $x \in U$
    ⟨3⟩2. PICK $\epsilon > 0$ such that $B_d(x, \epsilon) \subseteq U$
      PROOF: Proposition 10.1.4
    ⟨3⟩3. PICK $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$
      PROOF: By ⟨2⟩1
    ⟨3⟩4. $B_{d'}(x, \delta) \subseteq U$
  ⟨2⟩4. $U \in \mathcal{T}'$
    PROOF: Proposition 10.1.4.
□

Proposition 10.1.16. *$\mathbb{R}^2$ under the dictionary order topology is metrizable.*

PROOF: Define $d : \mathbb{R}^2 \to \mathbb{R}$ by

$$d((x, y), (x, z)) = \max(|y - z|, 1)$$
$$d((x, y), (x', y')) = 1 \qquad \text{if } x \neq x' \square$$

⟨1⟩1. $x \in \bigcap_{i=1}^{N} \pi_i{}^{-1}() \subseteq B_D(a, \epsilon)$

Proposition 10.1.17. *Let $d : X^2 \to \mathbb{R}$ be a metric on X. Then the metric topology on X is the coarsest topology such that d is continuous.*

PROOF:
⟨1⟩1. d is continuous.
  ⟨2⟩1. LET: $a, b \in X$
  ⟨2⟩2. LET: $\epsilon > 0$
  ⟨2⟩3. LET: $\delta = \epsilon/2$
  ⟨2⟩4. LET: $x, y \in X$
  ⟨2⟩5. ASSUME: $\rho((a, b), (x, y)) < \delta$
  ⟨2⟩6. $|d(a, b) - d(x, y)| < \epsilon$
    ⟨3⟩1. $d(a, b) - d(x, y) < \epsilon$

PROOF:

$$\begin{aligned}d(a,b) &\leq d(a,x)+d(x,y)+d(y,b)\\ &\leq d(x,y)+2\rho((a,b),(x,y))\\ &< d(x,y)+2\delta\\ &= d(x,y)+\epsilon\end{aligned}$$

⟨3⟩2. $d(a.b)-d(x,y) > -\epsilon$
PROOF: Similar.

⟨2⟩7. Q.E.D.

⟨1⟩2. If $\mathcal{T}$ is any topology under which d is continuous then $\mathcal{T}$ is finer than the metric topology.
PROOF: Since $B(a,\epsilon) = {d_a}^{-1}((-\infty,\epsilon))$

□

Proposition 10.1.18. *Let X be a metric space with metric d and $A \subseteq X$. The restriction of d to A is a metric on A that induces the subspace topology.*

PROOF:
⟨1⟩1. The restriction of d to A is a metric on A.
⟨1⟩2. Every open ball under $d \upharpoonright A$ is open under the subspace topology.
PROOF: $B_{d\upharpoonright A}(a,\epsilon) = B_d(a,\epsilon) \cap A$.
⟨1⟩3. If U is open in the subspace topology and $x \in U$, then there exists a $d \upharpoonright A$-ball B such that $x \in B \subseteq U$.
⟨2⟩1. PICK V open in X such that $U = V \cap A$
⟨2⟩2. PICK $\epsilon > 0$ such that $B_d(x,\epsilon) \subseteq V$
⟨2⟩3. Take $B = B_{d\upharpoonright A}(x,\epsilon)$

□

Corollary 10.1.18.1. *A subspace of a metrizable space is metrizable.*

Proposition 10.1.19. *Every metrizable space is Hausdorff.*

PROOF:
⟨1⟩1. LET: X be a metric space
⟨1⟩2. LET: $a, b \in X$ with $a \neq b$
⟨1⟩3. LET: $\epsilon = d(a,b)/2$
⟨1⟩4. LET: $U = B(a,\epsilon)$ and $V = B(b,\epsilon)$
⟨1⟩5. U and V are disjoint neighbourhoods of a and b respectively.

□

Corollary 10.1.19.1. *Every metrizable space is T_1.*

Proposition 10.1.20 (CC)**.** *The product of a countable family of metrizable spaces is metrizable.*

PROOF:
⟨1⟩1. LET: (X_n, d_n) be a sequence of metric spaces.
⟨1⟩2. ASSUME: w.l.o.g. each d_n is bounded above by 1.
PROOF: By Proposition 10.1.14.

⟨1⟩3. LET: D be the metric on $\mathbb{R}^\omega$ defined by $D(x,y) = \sup_i(d_i(x_i,y_i)/i)$.
⟨2⟩1. $D(x,y) \geq 0$
⟨2⟩2. $D(x,y) = 0$ if and only if $x = y$
⟨2⟩3. $D(x,y) = D(y,x)$
⟨2⟩4. $D(x,z) \leq D(x,y) + D(y,z)$
PROOF:
$$\begin{aligned} D(x,z) &= \sup_i \frac{d_i(x_i,z_i)}{i} \\ &\leq \sup_i \frac{d_i(x_i,y_i) + d_i(y_i,z_i)}{i} \\ &\leq \sup_i \frac{d_i(x_i,y_i)}{i} + \sup_i \frac{d_i(y_i,z_i)}{i} \\ &= D(x,y) + D(y,z) \end{aligned}$$
⟨1⟩4. Every open ball $B_D(a,\epsilon)$ is open in the product topology.
⟨2⟩1. PICK N such that $1/\epsilon < N$
⟨2⟩2. $B_D(a,\epsilon) = \prod_{i=1}^\infty U_i$ where $U_i = B(a_i, i\epsilon)$ if $i \leq N$, and $U_i = X_i$ if $i > N$
⟨1⟩5. For any open set U and $a \in U$, there exists $\epsilon > 0$ such that $B_D(a,\epsilon) \subseteq U$.
⟨2⟩1. LET: $n \geq 1$, V be an open set in $\mathbb{R}$ and $a \in {\pi_n}^{-1}(V)$
⟨2⟩2. PICK $\epsilon > 0$ such that $B_{d_n}(a,\epsilon) \subseteq V$
⟨2⟩3. $B_D(a,\epsilon/n) \subseteq {\pi_n}^{-1}(V)$
□

Theorem 10.1.21. *Let X and Y be metric spaces and $f : X \to Y$. Then f is continuous if and only if, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x,y) < \delta$ then $d(f(x), f(y)) < \epsilon$.*

PROOF:
⟨1⟩1. If f is continuous then, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x,y) < \delta$ then $d(f(x), f(y)) < \epsilon$
⟨2⟩1. ASSUME: f is continuous.
⟨2⟩2. LET: $x \in X$ and $\epsilon > 0$
⟨2⟩3. PICK a neighbourhood U of x such that $f(U) \subseteq B(f(x),\epsilon)$
PROOF: Theorem 9.14.6.
⟨2⟩4. PICK $\delta > 0$ such that $B(x,\delta) \subseteq U$
PROOF: Proposition 10.1.4.
⟨2⟩5. For all $y \in X$, if $d(x,y) < \delta$ then $d(f(x), f(y)) < \epsilon$
⟨1⟩2. If for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x,y) < \delta$ then $d(f(x), f(y)) < \epsilon$, then f is continuous.
⟨2⟩1. ASSUME: for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d(x,y) < \delta$ then $d(f(x), f(y)) < \epsilon$
⟨2⟩2. LET: $x \in X$ and V be a neighbourhood of $f(x)$
⟨2⟩3. PICK $\epsilon > 0$ such that $B(f(x),\epsilon) \subseteq V$
PROOF: Proposition 10.1.4.
⟨2⟩4. PICK $\delta > 0$ such that, for all $y \in X$, if $d(x,y) < \delta$ then $d(f(x), f(y)) < \epsilon$
PROOF: By ⟨2⟩1
⟨2⟩5. LET: $U = B(x,\delta)$

⟨2⟩6. U is a neighbourhood of x with $f(U) \subseteq V$
⟨2⟩7. Q.E.D.
PROOF: Theorem 9.14.6.
□

Proposition 10.1.22. *Let X be a metric space. Let (a_n) be a sequence in X and $l \in X$. Then $a_n \to l$ as $n \to \infty$ if and only if, for all $\epsilon > 0$, there exists N such that, for all $n \geq N$, we have $d(a_n, l) < \epsilon$.*

PROOF: From Proposition 9.11.4. □

Proposition 10.1.23. *Every metrizable space is first countable.*

PROOF: In any metric space X, the open balls $B(a, 1/n)$ for $n \geq 1$ form a local basis at a.

Example 10.1.24. $\mathbb{R}^\omega$ under the box topology is not metrizable.

Example 10.1.25. If J is uncountable then $\mathbb{R}^J$ under the product topology is not metrizable.

Example 10.1.26. The space $\overline{S_\Omega}$ is not metrizable by Example 9.23.4.

Proposition 10.1.27. *A compact subspace of a metric space is bounded.*

PROOF:
⟨1⟩1. LET: X be a metric space and $A \subseteq X$ be compact.
⟨1⟩2. PICK $a \in A$
⟨1⟩3. $\{B(a, n) \mid n \in \mathbb{Z}^+\}$ covers A
⟨1⟩4. PICK a finite subcover $\{B(a, n_1), \ldots, B(a, n_k)\}$
⟨1⟩5. LET: $N = \max(n_1, \ldots, n_k)$
⟨1⟩6. For all $x, y \in A$ we have $d(x, y) < 2N$
PROOF:
$$\begin{aligned} d(x,y) &\leq d(x,a) + d(a,y) \\ &< N + N \end{aligned}$$
□

This example shows the converse does not hold:

Example 10.1.28. The space $\mathbb{R}$ under the standard bounded metric is bounded but not compact.

Proposition 10.1.29. *A connected metric space with more than one point is uncountable.*

PROOF:
⟨1⟩1. LET: X be a connected metric space with more than one point.
⟨1⟩2. PICK $a \in X$
⟨1⟩3. $d(a, -) : X \to \mathbb{R}$ is continuous.
PROOF: Proposition 10.1.17.

⟨1⟩4. $\{d(a,x) \mid x \in X\}$ is a connected subspace of $\mathbb{R}$ that includes 0.
PROOF: Theorem 9.31.13.
⟨1⟩5. $\{d(a,x) \mid x \in X\} \neq \{0\}$
PROOF: Since X has more than one point.
⟨1⟩6. $\{d(a,x) \mid x \in X\}$ is uncountable.
PROOF: Since it includes a closed interval (Corollary 9.50.2.1).
□

10.2 Real Linear Algebra

Definition 10.2.1 (Square Metric)**.** The *square metric* ρ on $\mathbb{R}^n$ is defined by

$$\rho(\vec{x}, \vec{y}) = \max(|x_1 - y_1|, \ldots, |x_n - y_n|)$$

We prove this is a metric.

PROOF:
⟨1⟩1. $\rho(\vec{x}, \vec{y}) \geq 0$
PROOF: Immediate from definition.
⟨1⟩2. $\rho(\vec{x}, \vec{y}) = 0$ if and only if $\vec{x} = \vec{y}$
PROOF: Immediate from definition.
⟨1⟩3. $\rho(\vec{x}, \vec{y}) = \rho(\vec{y}, \vec{x})$
PROOF: Immediate from definition.
⟨1⟩4. $\rho(\vec{x}, \vec{z}) \leq \rho(\vec{x}, \vec{y}) + \rho(\vec{y}, \vec{z})$
PROOF: Since $|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$.
□

Proposition 10.2.2. *The square metric induces the standard topology on $\mathbb{R}^n$.*

PROOF:
⟨1⟩1. For every $a \in X$ and $\epsilon > 0$, we have $B_\rho(a, \epsilon)$ is open in the standard product topology.
PROOF:
$$B_\rho(a, \epsilon) = (a_1 - \epsilon, a_1 + \epsilon) \times \cdots \times (a_n - \epsilon, a_n + \epsilon)$$
⟨1⟩2. For any open sets $U_1, \ldots, U_n$ in $\mathbb{R}$, we have $U_1 \times \cdots \times U_n$ is open in the square metric topology.
⟨2⟩1. LET: $\vec{a} \in U_1 \times \cdots \times U_n$
⟨2⟩2. For $i = 1, \ldots, n$, PICK $\epsilon_i > 0$ such that $(a_i - \epsilon_i, a_i + \epsilon_i) \subseteq U_i$
⟨2⟩3. LET: $\epsilon = \min(\epsilon_1, \ldots, \epsilon_n)$
⟨2⟩4. $B_\rho(\vec{a}, \epsilon) \subseteq U$
□

Definition 10.2.3. Given $\vec{x}, \vec{y} \in \mathbb{R}^n$, define the *sum* $\vec{x} + \vec{y}$ by

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n) \ .$$

Definition 10.2.4. Given $\lambda \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^n$, define the *scalar product* $\lambda\vec{x} \in \mathbb{R}^n$ by

$$\lambda(x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n)$$

Definition 10.2.5 (Inner Product)**.** Given $\vec{x}, \vec{y} \in \mathbb{R}^n$, define the *inner product* $\vec{x} \cdot \vec{y} \in \mathbb{R}$ by

$$(x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) = x_1 y_1 + \cdots + x_n y_n \ .$$

We write $\vec{x}^2$ for $\vec{x} \cdot \vec{x}$.

Definition 10.2.6 (Norm)**.** Let $n \geq 1$. The *norm* on $\mathbb{R}^n$ is the function $\| \ \| : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\|(x_1, \ldots, x_n)\| = \sqrt{x_1^2 + \cdots + x_n^2}$$

Lemma 10.2.7.

$$\|\vec{x}\|^2 = \vec{x}^2$$

Proof: Immediate from definitions. □

Lemma 10.2.8.

$$\vec{x} \cdot (\vec{y} + \vec{z}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z}$$

Proof: Each is equal to $(x_1 y_1 + x_1 z_1, \ldots, x_n y_n + x_n z_n)$. □

Lemma 10.2.9.

$$|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \|\vec{y}\|$$

Proof:
⟨1⟩1. Assume: $\vec{x} \neq \vec{0} \neq \vec{y}$
 Proof: Otherwise both sides are 0.
⟨1⟩2. Let: $a = 1/\|\vec{x}\|$
⟨1⟩3. Let: $b = 1/\|\vec{y}\|$
⟨1⟩4. $(a\vec{x} + b\vec{y})^2 \geq 0$ and $(a\vec{x} - b\vec{y})^2 \geq 0$
⟨1⟩5. $a^2\|\vec{x}\|^2 + 2ab\vec{x} \cdot \vec{y} + b^2\|\vec{y}\|^2 \geq 0$ and $a^2\|\vec{x}\|^2 - 2ab\vec{x} \cdot \vec{y} + b^2\|\vec{y}\|^2 \geq 0$
⟨1⟩6. $2ab\vec{x} \cdot \vec{y} + 2 \geq 0$ and $-2ab\vec{x} \cdot \vec{y} + 2 \geq 0$
⟨1⟩7. $\vec{x} \cdot \vec{y} \geq -1/ab$ and $\vec{x} \cdot \vec{y} \leq 1/ab$
⟨1⟩8. $\vec{x} \cdot \vec{y} \geq -\|\vec{x}\|\|\vec{y}\|$ and $\vec{x} \cdot \vec{y} \leq \|\vec{x}\|\|\vec{y}\|$
□

Lemma 10.2.10 (Triangle Inequality)**.**

$$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$$

Proof:

$$\begin{aligned} \|\vec{x} + \vec{y}\|^2 &= \|\vec{x}\|^2 + 2\vec{x} \cdot \vec{y} + \|\vec{y}\|^2 \\ &\leq \|\vec{x}\|^2 + 2\|\vec{x}\|\|\vec{y}\| + \|\vec{y}\|^2 && \text{(Lemma 10.2.9)} \\ &= (\|\vec{x}\| + \|\vec{y}\|)^2 && \square \end{aligned}$$

Definition 10.2.11 (Euclidean Metric)**.** Let $n \geq 1$. The *Euclidean metric* on $\mathbb{R}^n$ is defined by

$$d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\| \ .$$

We prove this is a metric.

$\langle 1\rangle 1.$ $d(\vec{x}, \vec{y}) \geq 0$

PROOF: Immediate from definition.

$\langle 1\rangle 2.$ $d(\vec{x}, \vec{y}) = 0$ if and only if $\vec{x} = \vec{y}$

PROOF: $d(\vec{x}, \vec{y}) = 0$ if and only if $\vec{x} - \vec{y} = \vec{0}$.

$\langle 1\rangle 3.$ $d(\vec{x}, \vec{y}) = d(\vec{y}, \vec{x})$

PROOF: Immediate from definition.

$\langle 1\rangle 4.$ $d(\vec{x}, \vec{z}) \leq d(\vec{x}, \vec{y}) + d(\vec{y}, \vec{z})$

PROOF:

$$\begin{aligned} \|\vec{x} - \vec{z}\| &= \|(\vec{x} - \vec{y}) + (\vec{y} - \vec{z})\| \\ &\leq \|\vec{x} - \vec{y}\| + \|\vec{y} - \vec{z}\| && \text{(Lemma 10.2.10)} \end{aligned}$$

□

Proposition 10.2.12. *The Euclidean metric induces the standard topology on $\mathbb{R}^n$.*

PROOF:

$\langle 1\rangle 1.$ LET: ρ be the square metric.

$\langle 1\rangle 2.$ For all $\vec{a} \in \mathbb{R}^n$ and $\epsilon > 0$, we have $B_d(\vec{a}, \epsilon) \subseteq B_\rho(\vec{a}, \epsilon)$

$\langle 2\rangle 1.$ LET: $\vec{x} \in B_d(\vec{a}, \epsilon)$

$\langle 2\rangle 2.$ $\sqrt{(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2} < \epsilon$

$\langle 2\rangle 3.$ $(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2 < \epsilon^2$

$\langle 2\rangle 4.$ For all i we have $(x_i - a_i)^2 < \epsilon^2$

$\langle 2\rangle 5.$ For all i we have $|x_i - a_i| < \epsilon$

$\langle 2\rangle 6.$ $\rho(\vec{x}, \vec{a}) < \epsilon$

$\langle 1\rangle 3.$ For all $\vec{a} \in \mathbb{R}^n$ and $\epsilon > 0$, we have $B_\rho(\vec{a}, \epsilon/\sqrt{n}) \subseteq B_d(\vec{a}, \epsilon)$

$\langle 2\rangle 1.$ LET: $\vec{x} \in B_\rho(\vec{a}, \epsilon/\sqrt{n})$

$\langle 2\rangle 2.$ $\rho(\vec{x}, \vec{a}) < \epsilon/\sqrt{n}$

$\langle 2\rangle 3.$ For all i we have $|x_i - x_a| < \epsilon/\sqrt{n}$

$\langle 2\rangle 4.$ For all i we have $(x_i - x_a)^2 < \epsilon^2/n$

$\langle 2\rangle 5.$ $d(\vec{x}, \vec{a}) < \epsilon$

$\langle 1\rangle 4.$ Q.E.D.

PROOF: By Lemma 10.1.15.

□

Proposition 10.2.13. *Let $n \geq 0$. For all $c \in \mathbb{R}^n$ and $\epsilon > 0$, the open ball $B(c, \epsilon)$ is path connected.*

PROOF:

$\langle 1\rangle 1.$ LET: $a, b \in B(c, \epsilon)$

$\langle 1\rangle 2.$ LET: $p : [0, 1] \to B(c, \epsilon)$ be the function $p(t) = (1 - t)a + tb$

PROOF: We have $p(t) \in B(c, \epsilon)$ for all t because

$$\begin{aligned} d(p(t), c) &= \|(1-t)a + tb - c\| \\ &= \|(1-t)(a-c) + t(b-c)\| \\ &\leq (1-t)\|a-c\| + t\|b-c\| \\ &< (1-t)\epsilon + t\epsilon \\ &= \epsilon \end{aligned}$$

⟨1⟩3. p is a path from a to b.

□

Proposition 10.2.14. *Let $n \geq 0$. For all $c \in \mathbb{R}^n$ and $\epsilon > 0$, the closed ball $\overline{B(c, \epsilon)}$ is path connected.*

PROOF:

⟨1⟩1. LET: $a, b \in \overline{B(c, \epsilon)}$

⟨1⟩2. LET: $p : [0, 1] \to \overline{B(c, \epsilon)}$ be the function $p(t) = (1-t)a + tb$

PROOF: We have $p(t) \in \overline{B(c, \epsilon)}$ for all t because

$$\begin{aligned} d(p(t), c) &= \|(1-t)a + tb - c\| \\ &= \|(1-t)(a-c) + t(b-c)\| \\ &\leq (1-t)\|a-c\| + t\|b-c\| \\ &\leq (1-t)\epsilon + t\epsilon \\ &= \epsilon \end{aligned}$$

⟨1⟩3. p is a path from a to b.

□

Lemma 10.2.15. *If $\sum_{i=0}^{\infty} x_i^2$ and $\sum_{i=0}^{\infty} y_i^2$ converge then $\sum_{i=0}^{\infty} |x_i y_i|$ converges.*

PROOF:

⟨1⟩1. For all $N \geq 0$ we have $\sum_{i=0}^{N} |x_i y_i| \leq \sqrt{\sum_{i=0}^{N} |x_i|^2} \sqrt{\sum_{i=0}^{N} |y_i|^2}$

PROOF: By the Cauchy-Schwarz inequality

⟨1⟩2. Q.E.D.

PROOF: Since $\sum_{i=0}^{N} |x_i y_i|$ is an increasing sequence bounded above by $(\sum_{i=0}^{\infty} x_i^2)(\sum_{i=0}^{\infty} y_i^2)$.

□

Corollary 10.2.15.1. *If $\sum_{i=0}^{\infty} x_i^2$ and $\sum_{i=0}^{\infty} y_i^2$ converge then $\sum_{i=0}^{\infty} (x_i + y_i)^2$ converges.*

PROOF: Since $\sum_{i=0}^{\infty} x_i^2$, $\sum_{i=0}^{\infty} y_i^2$ and $2 \sum_{i=0}^{\infty} x_i y_i$ all converge.

Definition 10.2.16 (l^2-metric)**.** The *l^2-metric* on

$$\left\{ (x_n) \in \mathbb{R}^{\omega} \mid \sum_{n=0}^{\infty} x_n^2 \text{ converges} \right\}$$

is defined by

$$d(x, y) = \left(\sum_{n=0}^{\infty} (x_n - y_n)^2 \right)^{1/2}$$

We prove this is a metric.

Proof:
$\langle 1\rangle 1.$ d is well-defined.
Proof: By Corollary 10.2.15.1.
$\langle 1\rangle 2.$ $d(x,y) \geq 0$
$\langle 1\rangle 3.$ $d(x,y) = 0$ if and only if $x = y$
$\langle 1\rangle 4.$ $d(x,y) = d(y,x)$
$\langle 1\rangle 5.$ $d(x,z) \leq d(x,y) + d(y,z)$
Proof: By Lemma 10.2.10.
□

Theorem 10.2.17. *Addition is a continuous function $\mathbb{R}^2 \to \mathbb{R}$.*

Proof:
$\langle 1\rangle 1.$ Let: $a, b \in \mathbb{R}$
$\langle 1\rangle 2.$ Let: $\epsilon > 0$
$\langle 1\rangle 3.$ Let: $\delta = \epsilon/2$
$\langle 1\rangle 4.$ Let: $x, y \in \mathbb{R}$
$\langle 1\rangle 5.$ Assume: $\rho((a,b),(x,y)) < \delta$
$\langle 1\rangle 6.$ $|(a+b) - (x+y)| < \epsilon$
Proof:

$$\begin{aligned} |(a+b)-(x+y)| &= |a-x| + |b-y| \\ &\leq 2\rho((a,b),(x,y)) \\ &< 2\delta \\ &= \epsilon \end{aligned}$$

$\langle 1\rangle 7.$ Q.E.D.
Proof: Theorem 10.1.21
□

Theorem 10.2.18. *Multiplication is a continuous function $\mathbb{R}^2 \to \mathbb{R}$.*

Proof:
$\langle 1\rangle 1.$ Let: $a, b \in \mathbb{R}$
$\langle 1\rangle 2.$ Let: $\epsilon > 0$
$\langle 1\rangle 3.$ Let: $\delta = \min(\epsilon/(|a| + |b| + 1), 1)$
$\langle 1\rangle 4.$ Let: $x, y \in \mathbb{R}$
$\langle 1\rangle 5.$ Assume: $\rho((a,b),(x,y)) < \delta$
$\langle 1\rangle 6.$ $|ab - xy| < \epsilon$
Proof:

$$\begin{aligned} |ab - xy| &= |a(b-y) + (a-x)b - (a-x)(b-y)| \\ &\leq |a||b-y| + |b||a-x| + |a-x||b-y| \\ &< |a|\delta + |b|\delta + \delta^2 && (\langle 1\rangle 5) \\ &\leq |a|\delta + |b|\delta + \delta && (\langle 1\rangle 3) \\ &\leq \epsilon && (\langle 1\rangle 3) \end{aligned}$$

$\langle 1\rangle 7.$ Q.E.D.

PROOF: Theorem 10.1.21
□

Theorem 10.2.19. *The function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by $f(x) = x^{-1}$ is continuous.*

PROOF:
⟨1⟩1. For all $a \in \mathbb{R}$ we have $f^{-1}((a, +\infty))$ is open.
PROOF: The set is

$$\begin{aligned}(a^{-1}, +\infty) & \text{if } a > 0 \\ (0, +\infty) & \text{if } a = 0 \\ (-\infty, a^{-1}) \cup (0, +\infty) & \text{if } a < 0\end{aligned}$$

⟨1⟩2. For all $a \in \mathbb{R}$ we have $f^{-1}((-\infty, a))$ is open.
PROOF: Similar.
⟨1⟩3. Q.E.D.
PROOF: By Proposition 9.14.3 and Lemma 9.16.2.
□

Definition 10.2.20. For $n \geq 0$, the *unit ball* B^n is the space $\{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$.

Proposition 10.2.21. *For all $n \geq 0$, the unit ball B^n is path connected.*

PROOF:
⟨1⟩1. LET: $a, b \in B^n$
⟨1⟩2. LET: $p : [0, 1] \to B^n$ be the function $p(t) = (1 - t)a + tb$
PROOF: We have $p(t) \in B^n$ for all t because

$$\begin{aligned}\|(1-t)a + tb\| &\leq (1-t)\|a\| + t\|b\| \\ &\leq (1-t) + t \\ &= 1\end{aligned}$$

⟨1⟩3. p is a path from a to b.
□

Definition 10.2.22 (Punctured Euclidean Space)**.** For $n \geq 0$, defined *punctured Euclidean space* to be $\mathbb{R}^n \setminus \{0\}$.

Proposition 10.2.23. *For $n > 1$, punctured Euclidean space $\mathbb{R}^n \setminus \{0\}$ is path connected.*

PROOF:
⟨1⟩1. LET: $a, b \in \mathbb{R}^n \setminus \{0\}$
⟨1⟩2. CASE: 0 is on the line from a to b
⟨2⟩1. PICK a point c not on the line from a to b
⟨2⟩2. The path consisting of a straight line from a to c followed by a straight line from c to b is a path from a to b.
⟨1⟩3. CASE: 0 is not on the line from a to b
PROOF: The straight line from a to b is a path from a to b.

Corollary 10.2.23.1. *For $n > 1$, the spaces $\mathbb{R}$ and $\mathbb{R}^n$ are not homeomorphic.*

PROOF: For any point a, the space $\mathbb{R} \setminus \{a\}$ is disconnected.

Definition 10.2.24 (Unit Sphere)**.** For $n \geq 1$, the *unit sphere* S^{n-1} is the space

$$S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\} \ .$$

Proposition 10.2.25. *For $n > 1$, the unit sphere S^{n-1} is path connected.*

PROOF: The map $g : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$ defined by $g(x) = x/\|x\|$ is continuous and surjective. The result follows by Proposition 9.33.5. □

Proposition 10.2.26. *Let $f : S^1 \to \mathbb{R}$ be continuous. Then there exists $x \in S^1$ such that $f(x) = f(-x)$.*

PROOF:
$\langle 1 \rangle 1$. LET: $g : S^1 \to \mathbb{R}$ be the function $g(x) = f(x) - f(-x)$
PROVE: There exists $x \in S^1$ such that $g(x) = 0$
$\langle 1 \rangle 2$. ASSUME: without loss of generality $g((1, 0)) > 0$
$\langle 1 \rangle 3$. $g((-1, 0)) < 0$
$\langle 1 \rangle 4$. There exists x such that $g(x) = 0$
PROOF: By the Intermediate Value Theorem.
□

Definition 10.2.27 (Topologist's Sine Curve)**.** Let $S = \{(x, \sin 1/x) \mid 0 < x \leq 1\}$. The *topologist's sine curve* is the closure $\overline{S}$ of S.

Proposition 10.2.28.

$$\overline{S} = S \cup (\{0\} \times [-1, 1])$$

Proposition 10.2.29. *The topologist's sine curve is connected.*

PROOF:
$\langle 1 \rangle 1$. LET: $S = \{(x, \sin 1/x) \mid 0 < x \leq 1\}$
$\langle 1 \rangle 2$. S is connected.
PROOF: Theorem 9.31.13.
$\langle 1 \rangle 3$. $\overline{S}$ is connected.
PROOF: Theorem 9.31.12.
□

Proposition 10.2.30 (CC)**.** *The topologist's sine curve is not path connected.*

PROOF:
$\langle 1 \rangle 1$. ASSUME: for a contradiction $p : [0, 1] \to \overline{S}$ is a path from $(0, 0)$ to $(1, \sin 1)$.
$\langle 1 \rangle 2$. $p^{-1}(\{0\} \times [0, 1])$ is closed.
$\langle 1 \rangle 3$. LET: b be the greatest element of $p^{-1}(\{0\} \times [0, 1])$.
$\langle 1 \rangle 4$. $b < 1$
PROOF: Since $p(1) = (1, \sin 1)$.
$\langle 1 \rangle 5$. PICK a sequence $(t_n)_{n \geq 1}$ in $(b, 1]$ such that $t_n \to b$ and $\pi_2(p(t_n)) = (-1)^n$

⟨2⟩1. LET: $n \geq 1$
⟨2⟩2. PICK u with $0 < u < \pi_1(p(1/n))$ such that $\sin(1/u) = (-1)^n$
⟨2⟩3. PICK t_n such that $0 < t_n < 1/n$ and $\pi_1(p(t_n)) = u$
PROOF: One exists by the Intermediate Value Theorem.
⟨1⟩6. Q.E.D.
PROOF: This contradicts 9.14.18.
□

Theorem 10.2.31. *Let A be a subspace of $\mathbb{R}^n$. Then the following are equivalent:*

1. *A is compact.*
2. *A is closed and bounded under the Euclidean metric.*
3. *A is closed and bounded under the square metric.*

PROOF:
⟨1⟩1. $1 \Rightarrow 2$
PROOF: By Corollary 9.48.10.1 and Proposition 10.1.27.
⟨1⟩2. $2 \Rightarrow 3$
PROOF: If $d(x, y) \leq M$ for all $x, y \in A$ then $\rho(x, y) \leq M/\sqrt{2}$.
⟨1⟩3. $3 \Rightarrow 1$
⟨2⟩1. ASSUME: A is closed and $\rho(x, y) \leq M$ for all $x, y \in A$
⟨2⟩2. PICK $a \in A$
PROOF: We may assume w.l.o.g. A is nonempty since the empty space is compact.
⟨2⟩3. A is a closed subspace of $[a_1 - M, a_1 + M] \times \cdots \times [a_n - M, a_n + M]$
⟨2⟩4. A is compact
PROOF: Proposition 9.48.3.
□

Corollary 10.2.31.1. *The unit sphere S^{n-1} and the closed unit ball B^n are compact for any n.*

10.3 The Uniform Topology

Definition 10.3.1 (Uniform Metric)**.** Let J be a set. The *uniform metric* $\overline{\rho}$ on $\mathbb{R}^J$ is defined by

$$\overline{\rho}(a, b) = \sup_{j \in J} \overline{d}(a_j, b_j)$$

where $\overline{d}$ is the standard bounded metric on $\mathbb{R}$.

The *uniform topology* on $\mathbb{R}^J$ is the topology induced by the uniform metric.

We prove this is a metric.

PROOF:
⟨1⟩1. $\overline{\rho}(a, b) \geq 0$

PROOF: Immediate from definitions.
⟨1⟩2. $\overline{\rho}(a, b) = 0$ if and only if $a = b$
PROOF: Immediate from definitions.
⟨1⟩3. $\overline{\rho}(a, b) = \overline{\rho}(b, a)$
PROOF: Immediate from definitions.
⟨1⟩4. $\overline{\rho}(a, c) \leq \overline{\rho}(a, b) + \overline{\rho}(b, c)$
PROOF:

$$\begin{aligned}\overline{\rho}(a, c) &= \sup_{j \in J} \overline{d}(a_j, c_j) \\ &\leq \sup_{j \in J}(\overline{d}(a_j, b_j) + \overline{d}(b_j, c_j)) \\ &\leq \sup_{j \in J} \overline{d}(a_j, b_j) + \sup_{j \in J} \overline{d}(b_j, c_j) \\ &= \overline{\rho}(a, b) + \overline{\rho}(b, c)\end{aligned}$$

□

Proposition 10.3.2. *The uniform topology on $\mathbb{R}^J$ is finer than the product topology.*

PROOF:
⟨1⟩1. LET: $j \in J$ and U be open in $\mathbb{R}$
PROVE: ${\pi_j}^{-1}(U)$ is open in the uniform topology.
⟨1⟩2. LET: $a \in {\pi_j}^{-1}(U)$
⟨1⟩3. PICK $\epsilon > 0$ such that $(a_j - \epsilon, a_j + \epsilon) \subseteq U$
⟨1⟩4. $B_{\overline{\rho}}(a, \epsilon) \subseteq {\pi_j}^{-1}(U)$
□

Proposition 10.3.3. *The uniform topology on $\mathbb{R}^J$ is coarser than the box topology.*

PROOF:
⟨1⟩1. LET: $a \in \mathbb{R}^J$ and $\epsilon > 0$
PROVE: $B(a, \epsilon)$ is open in the box topology.
⟨1⟩2. LET: $b \in B(a, \epsilon)$
⟨1⟩3. For $j \in J$ we have $|a_j - b_j| < \epsilon$
⟨1⟩4. For $j \in J$,
LET: $\delta_j = (\epsilon - |a_j - b_j|)/2$
⟨1⟩5. $\prod_{j \in J}(b_j - \delta_j, b_j + \delta_j) \subseteq B(a, \epsilon)$
□

Proposition 10.3.4. *The uniform topology on $\mathbb{R}^J$ is strictly finer than the product topology if and only if J is infinite.*

PROOF:
⟨1⟩1. If J is finite then the uniform and product topologies coincide.
PROOF: The uniform, box and product topologies are all the same.
⟨1⟩2. If J is infinite then the uniform and product topologies are different.

PROOF: The set $B(\vec{0}, 1)$ is open in the uniform topology but not the product topology.

□

Proposition 10.3.5 (DC)**.** *The uniform topology on $\mathbb{R}^J$ is strictly coarser than the box topology if and only if J is infinite.*

PROOF:

$\langle 1 \rangle 1$. If J is finite then the uniform and box topologies coincide.

PROOF: The uniform, box and product topologies are all the same.

$\langle 1 \rangle 2$. If J is infinite then the uniform and box topologies are different.

PROOF: Pick an ω-sequence $(j_1, j_2, \ldots)$ in J. Let $U = \prod_{j \in J} U_j$ where $U_{j_i} = (-1/i, 1/i)$ and $U_j = (-1, 1)$ for all other j. Then $\vec{0} \in U$ but there is no $\epsilon > 0$ such that $B(\vec{0}, \epsilon) \subseteq U$.

□

Proposition 10.3.6. *The closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ under the uniform topology is $\mathbb{R}^\omega$.*

PROOF: Given any open ball $B(a, \epsilon)$, pick an integer N such that $1/\epsilon < N$. Then $B(a, \epsilon)$ includes sequences whose nth entry is 0 for all $n \geq N$. □

Example 10.3.7. The space $\mathbb{R}^\omega$ is disconnected under the uniform topology. The set of bounded sequences and the set of unbounded sequences form a separation.

Proposition 10.3.8. *Give $\mathbb{R}^\omega$ the uniform topology. Let $x, y \in \mathbb{R}^\omega$. Then x and y are in the same component if and only if $x - y$ is bounded.*

PROOF:

$\langle 1 \rangle 1$. The component containing 0 is the set of bounded sequences.

$\langle 2 \rangle 1$. LET: B be the set of bounded sequences.

$\langle 2 \rangle 2$. B is path-connected.

$\langle 3 \rangle 1$. LET: $x.y \in B$

$\langle 3 \rangle 2$. PICK $b > 0$ such that $|x_j|, |y_j| \leq b$ for all j

$\langle 3 \rangle 3$. LET: $p : [0, 1] \to B$ be the function $p(t) = (1 - t)x + ty$

PROVE: p is continuous.

$\langle 3 \rangle 4$. LET: $t \in [0, 1]$ and $\epsilon > 0$

$\langle 3 \rangle 5$. LET: $\delta = \epsilon / 2b$

$\langle 3 \rangle 6$. LET: $s \in [0, 1]$ with $|s - t| < \delta$

$\langle 3 \rangle 7$. $\overline{\rho}(p(s), p(t)) < \epsilon$

PROOF:

$$\begin{aligned}
\overline{\rho}(p(s), p(t)) &= \sup_j \overline{d}((1 - s)x_j + sy_j, (1 - t)x_j + ty_j) \\
&\leq |(s - t)x_j + (t - s)y_j| \\
&\leq |s - t||x_j - y_j| \\
&< 2b\delta \\
&= \epsilon
\end{aligned}$$

⟨2⟩3. B is connected.

PROOF: Proposition 9.33.3.

⟨2⟩4. If C is connected and $B \subseteq C$ then $B = C$.

PROOF: Otherwise $B \cap C$ and $C \setminus B$ form a separation of C.

⟨1⟩2. Q.E.D.

PROOF: Since $\lambda x.x - y$ is a Homeomorphism of $\mathbb{R}^\omega$ with itself.

□

Example 10.3.9. The space $[0,1]^\omega$ under the uniform topology is not locally compact.

It is not compact because the set $\{0,1\}^\omega$ has no limit point.

Now, assume for a contradiction $[0,1]^\omega$ is locally compact. Pick $\epsilon > 0$ such that $B(0,\epsilon)$ is included in a compact subspace. Then $\overline{B(0,\epsilon)}$ is compact. But $\overline{B(0,\epsilon)} = [0,1]^\omega$ if $\epsilon \geq 1$, or $[0,\epsilon]^\omega$ if $\epsilon < 1$. In either case $\overline{B(0,\epsilon)} \cong [0,1]^\epsilon$ which is not compact.

10.4 Uniform Convergence

Definition 10.4.1 (Uniform Convergence)**.** Let X be a set and Y a metric space. Let $(f_n : X \to Y)$ be a sequence of functions and $f : X \to Y$ be a function. Then f_n *converges uniformly* to f as $n \to \infty$ if and only if, for all $\epsilon > 0$, there exists N such that, for all $n \geq N$ and $x \in X$, we have $d(f_n(x), f(x)) < \epsilon$.

Example 10.4.2. Define $f_n : [0,1] \to \mathbb{R}$ by $f_n(x) = x^n$ for $n \geq 1$, and $f : [0,1] \to \mathbb{R}$ by $f(x) = 0$ if $x < 1$, $f(1) = 1$. Then f_n converges to f pointwise but not uniformly.

Theorem 10.4.3 (Uniform Limit Theorem)**.** *Let X be a topological space and Y a metric space. Let $(f_n : X \to Y)$ be a sequence of continuous functions and $f : X \to Y$ be a function. If f_n converges uniformly to f as $n \to \infty$, then f is continuous.*

PROOF:

⟨1⟩1. LET: $x \in X$ and $\epsilon > 0$

⟨1⟩2. PICK N such that, for all $n \geq N$ and $y \in X$, we have $d(f_n(y), f(y)) < \epsilon/3$

⟨1⟩3. PICK a neighbourhood U of x such that $f_N(U) \subseteq B(f_N(x), \epsilon/3)$

PROVE: $f(U) \subseteq B(f(x), \epsilon)$

⟨1⟩4. LET: $y \in U$

⟨1⟩5. $d(f(y), f(x)) < \epsilon$

PROOF:

$$\begin{aligned} d(f(y), f(x)) &\leq d(f(y), f_N(y)) + d(f_N(y), f_N(x)) + d(f_N(x), f(x)) && \text{(Triangle Inequality)} \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 && (\langle 1\rangle 2,\ \langle 1\rangle 3) \\ &= \epsilon \end{aligned}$$

□

Proposition 10.4.4. *Let X be a topological space and Y a metric space. Let $(f_n : X \to Y)$ be a sequence of continuous functions and $f : X \to Y$ be a* *function. Let (a_n) be a sequence of points in X and $a \in X$. If f_n converges uniformly to f and a_n converges to a in X then $f_n(a_n)$ converges to $f(a)$ uniformly in Y.*

PROOF:
⟨1⟩1. LET: $\epsilon > 0$
⟨1⟩2. PICK N_1 such that, for all $n \geq N_1$ and $x \in X$, we have $d(f_n(x), f(x)) < \epsilon/2$
⟨1⟩3. PICK N_2 such that, for all $n \geq N_2$, we have $a_n \in f^{-1}(B(a, \epsilon/2))$
PROOF: Using the fact that f is continuous from the Uniform Limit Theorem.
⟨1⟩4. LET: $N = \max(N_1, N_2)$
⟨1⟩5. LET: $n \geq N$
⟨1⟩6. $d(f_n(a_n), f(a)) < \epsilon$
PROOF:

$$\begin{aligned} d(f_n(a_n), f(a)) &\leq d(f_n(a_n), f(a_n)) + d(f(a_n), f(a)) && \text{(Triangle Inequality)} \\ &< \epsilon/2 + \epsilon/2 && (\langle 1\rangle 2,\ \langle 1\rangle 3) \\ &= \epsilon \end{aligned}$$

□

Proposition 10.4.5. *Let X be a set. Let $(f_n : X \to \mathbb{R})$ be a sequence of functions and $f : X \to \mathbb{R}$ be a function. Then f_n converges unifomly to f as $n \to \infty$ if and only if $f_n \to f$ as $n \to \infty$ in $\mathbb{R}^X$ under the uniform topology.*

PROOF:
⟨1⟩1. If f_n converges uniformly to f then f_n converges to f under the uniform topology.
⟨2⟩1. ASSUME: f_n converges uniformly to f
⟨2⟩2. LET: $\epsilon > 0$
⟨2⟩3. PICK N such that, for all $n \geq N$ and $x \in X$, we have $d(f_n(x), f(x)) < \epsilon/2$
⟨2⟩4. For all $n \geq N$ we have $\overline{\rho}(f_n, f) \leq \epsilon/2$
⟨2⟩5. For all $n \geq N$ we have $\overline{\rho}(f_n, f) < \epsilon$
⟨1⟩2. If f_n converges to f under the uniform topology then f_n converges uniformly to f.
⟨2⟩1. ASSUME: f_n converges to f under the uniform topology.
⟨2⟩2. LET: $\epsilon > 0$
⟨2⟩3. PICK N such that, for all $n \geq N$, we have $\overline{\rho}(f_n, f) < \min(\epsilon, 1/2)$
⟨2⟩4. LET: $n \geq N$
⟨2⟩5. LET: $x \in X$
⟨2⟩6. $\overline{\rho}(f_n, f) < \min(\epsilon, 1/2)$
PROOF: From ⟨2⟩3.
⟨2⟩7. $d(f_n(x), f(x)) < \min(\epsilon, 1/2)$
⟨2⟩8. $d(f_n(x), f(x)) < \epsilon$

□

10.5 Isometric Imbeddings

Definition 10.5.1. Let X and Y be metric spaces. An *isometric imbedding* $f : X \to Y$ is a function such that, for all $x, y \in X$, we have $d(f(x), f(y)) = d(x, y)$.

Proposition 10.5.2. *Every isometric imbedding is an imbedding.*

PROOF:
⟨1⟩1. LET: $f : X \to Y$ be an isometric imbedding.
⟨1⟩2. f is injective.
PROOF: If $f(x) = f(y)$ then $d(f(x), f(y)) = 0$ hence $d(x, y) = 0$ hence $x = y$.
⟨1⟩3. f is continuous.
PROOF: For all $\epsilon > 0$, if $d(x, y) < \epsilon$ then $d(f(x), f(y)) < \epsilon$.
⟨1⟩4. $f : X \to f(X)$ is an open map.
PROOF: $f(B(a, \epsilon)) = B(f(a), \epsilon) \cap f(X)$.
□

10.6 Distance to a Set

Definition 10.6.1. Let X be a metric space, $x \in X$ and $A \subseteq X$ be nonempty. The *distance* from x to A is defined as

$$d(x, A) = \inf_{a \in A} d(x, a) \ .$$

Proposition 10.6.2. *Let X be a metric space and $A \subseteq X$ be nonempty. Then the function $d(-, A) : X \to \mathbb{R}$ is continuous.*

PROOF:
⟨1⟩1. LET: X be a metric space.
⟨1⟩2. LET: $A \subseteq X$ be nonempty.
⟨1⟩3. LET: $x \in X$ and $\epsilon > 0$
⟨1⟩4. LET: $\delta = \epsilon$
⟨1⟩5. LET: $y \in B(x, \delta)$
⟨1⟩6. $|d(x, A) - d(y, A)| < \epsilon$
⟨2⟩1. $d(x, A) - d(y, A) < \epsilon$
PROOF:
⟨3⟩1. For all $a \in A$ we have $d(x, A) \leq d(x, y) + d(y, a)$
PROOF:

$$\begin{aligned} d(x, A) &\leq d(x, a) && \text{(definition of } d(x, A)\text{)} \\ &\leq d(x, y) + d(y, a) && \text{(Triangle Inequality)} \end{aligned}$$

⟨3⟩2. $d(x, A) - d(x, y) \leq d(y, A)$
⟨2⟩2. $d(y, A) - d(x, A) < \epsilon$
PROOF: Similar.
⟨1⟩7. Q.E.D.
PROOF: Theorem 10.1.21.

□

Theorem 10.6.3. *Let X be a metric space, $A \subseteq X$ be nonempty, and $x \in X$. Then $d(x, A) = 0$ if and only if $x \in \overline{A}$.*

PROOF:
⟨1⟩1. LET: X be a metric space.
⟨1⟩2. LET: $A \subseteq X$ be nonempty.
⟨1⟩3. LET: $x \in X$
⟨1⟩4. If $d(x, A) = 0$ then $x \in \overline{A}$
  ⟨2⟩1. ASSUME: $d(x, A) = 0$
  ⟨2⟩2. LET: U be any neighbourhood of x.
  ⟨2⟩3. PICK $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$
    PROOF: Proposition 10.1.4, ⟨1⟩1, ⟨2⟩2.
  ⟨2⟩4. PICK $a \in A$ such that $d(x, a) < \epsilon$
    PROOF: From ⟨2⟩1, ⟨2⟩3.
  ⟨2⟩5. $a \in A \cap U$
    PROOF: From ⟨2⟩3, ⟨2⟩4.
  ⟨2⟩6. Q.E.D.
    PROOF: Theorem 9.6.6.
⟨1⟩5. If $x \in \overline{A}$ then $d(x, A) = 0$
□

Theorem 10.6.4. *Let X be a metric space. Let $A \subseteq X$ be nonempty and compact. Let $x \in X$. Then there exists $a \in A$ such that $d(x, A) = d(x, a)$.*

PROOF: By the Extreme Value Theorem, the function $d(x, -) : A \to \mathbb{R}$ attains its minimum. □

10.7 Lebesgue Numbers

Definition 10.7.1 (Lebesgue Number)**.** Let X be a metric space. Let $\mathcal{U}$ be an open covering of X. A *Lebesgue number* for $\mathcal{U}$ is a real number $\delta > 0$ such that, for every subset $A \subseteq X$ with diameter diameter $< \delta$, there exists $U \in \mathcal{U}$ such that $A \subseteq U$.

Theorem 10.7.2 (Lebesgue Number Lemma)**.** *Every open covering of a compact metric space has a Lebesgue number.*

PROOF:
⟨1⟩1. LET: X be a compact metric space.
⟨1⟩2. LET: $\mathcal{U}$ be an open covering of X.
⟨1⟩3. PICK a finite subset $\{U_1, \ldots, U_n\}$ of $\mathcal{U}$ that covers X.
⟨1⟩4. For $i = 1, \ldots, n$,
    LET: $C_i = X - U_i$
⟨1⟩5. LET: $f : X \to \mathbb{R}$,

$$f(x) = 1/n \sum_{i=1}^{n} d(x, C_i)$$

⟨1⟩6. For all $x \in X$ we have $f(x) > 0$
  ⟨2⟩1. LET: $x \in X$
  ⟨2⟩2. PICK i such that $x \in U_i$
    PROOF: From ⟨1⟩3.
  ⟨2⟩3. PICK $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U_i$
    PROOF: Proposition 10.1.4.
  ⟨2⟩4. $d(x, C_i) \geq \epsilon$
  ⟨2⟩5. $f(x) \geq \epsilon/n$
⟨1⟩7. f is continuous.
  PROOF: Proposition 10.6.2.
⟨1⟩8. LET: δ be the minimum value of $f(X)$
  PROOF: By the Extreme Value Theorem
⟨1⟩9. $\delta > 0$
  PROOF: From ⟨1⟩6
⟨1⟩10. For every subset $A \subseteq X$ with diameter $< \delta$, there exists $U \in \mathcal{U}$ such that $A \subseteq U$
  ⟨2⟩1. LET: $A \subseteq X$ with $\operatorname{diam} A < \delta$
  ⟨2⟩2. PICK $x_0 \in A$
  ⟨2⟩3. $A \subseteq B(x_0, \delta)$
  ⟨2⟩4. $f(x_0) \geq \delta$
  ⟨2⟩5. PICK m such that $d(x_0, C_m)$ is the largest out of $d(x_0, C_1), \ldots, d(x_0, C_n)$
  ⟨2⟩6. $d(x_0, C_m) \geq f(x_0)$
  ⟨2⟩7. $B(x_0, \delta) \subseteq U_m$
  ⟨2⟩8. $A \subseteq U_m$
⟨1⟩11. δ is a Lebesgue number for $\mathcal{U}$
□

Theorem 10.7.3 (AC). *Every sequentially compact metric space is compact.*

PROOF:
⟨1⟩1. LET: X be a sequentially comapct metric space.
⟨1⟩2. Every open covering of X has a Lebesgue number.
  ⟨2⟩1. LET: $\mathcal{A}$ be an open covering of X.
  ⟨2⟩2. ASSUME: for a contradiction $\mathcal{A}$ has no Lebesgue number.
  ⟨2⟩3. For $n \geq 1$, PICK a set C_n with diameter $< 1/n$ that is not included in any member of $\mathcal{A}$.
  ⟨2⟩4. For $n \geq 1$, PICK $x_n \in C_n$.
  ⟨2⟩5. PICK a convergent subsequence (C_{n_r}) of (C_n) with limit a.
  ⟨2⟩6. PICK $A \in \mathcal{A}$ such that $a \in A$
  ⟨2⟩7. PICK $\epsilon > 0$ such that $B(a, \epsilon) \subseteq A$.
  ⟨2⟩8. PICK r such that $1/n_r < \epsilon/2$ and $d(x_{n_r}, a) < \epsilon/2$
  ⟨2⟩9. $C_{n_r} \subseteq B(a, \epsilon)$
  ⟨2⟩10. $C_{n_r} \subseteq A$
  ⟨2⟩11. Q.E.D.
    PROOF: This contradicts ⟨2⟩3.
⟨1⟩3. For every $\epsilon > 0$, there exists a finite covering of X by ϵ-balls.

⟨2⟩1. ASSUME: for a contradiction that there exists $\epsilon > 0$ such that X cannot be finitely covered by ϵ-balls.

⟨2⟩2. PICK a sequence of points (x_n) such that $x_n \in X - (B(x_1, \epsilon) \cup \cdots \cup B(x_{n-1}, \epsilon))$

⟨2⟩3. $d(x_m, x_n) \geq \epsilon$ for all m, n distinct

⟨2⟩4. (x_n) has no convergent subsequence

⟨2⟩5. Q.E.D.

PROOF: This contradicts ⟨1⟩1.

⟨1⟩4. LET: $\mathcal{A}$ be an open covering of X.

⟨1⟩5. PICK a Lebesgue number δ for $\mathcal{A}$.

PROOF: By ⟨1⟩2.

⟨1⟩6. LET: $\epsilon = \delta/3$

⟨1⟩7. PICK a finite covering $\{B_1, \ldots, B_n\}$ of X be ϵ-balls.

PROOF: By ⟨1⟩3.

⟨1⟩8. For $i = 1, \ldots, n$, PICK $U_i \in \mathcal{A}$ such that $B_i \subseteq A_i$

PROOF: By ⟨1⟩5 since $\operatorname{diam} B_i = 2\epsilon < \delta$.

⟨1⟩9. $\{U_1, \ldots, U_n\}$ covers X.

□

Example 10.7.4. The space S_Ω is not metrizable, because it is sequentially compact but not compact.

10.8 Uniform Continuity

Definition 10.8.1 (Uniformly Continuous)**.** Let X and Y be metric spaces. Let $f : X \to Y$. Then f is *uniformly continuous* if and only if, for every $\epsilon > 0$, there exists $\delta > 0$ such that, for all $x, y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$.

Theorem 10.8.2 (Uniform Continuity Theorem)**.** *Every continuous function from a compact metric space to a metric space is uniformly continuous.*

PROOF:

⟨1⟩1. LET: X be a compact metric space.

⟨1⟩2. LET: Y be a metric space.

⟨1⟩3. LET: $f : X \to Y$ be a continuous function.

⟨1⟩4. LET: $\epsilon > 0$

⟨1⟩5. LET: $\mathcal{U} = \{f^{-1}(B(y, \epsilon/2)) \mid y \in Y\}$

⟨1⟩6. PICK a Lebesgue number $\delta > 0$ for $\mathcal{U}$.

PROOF: By the Lebesgue Number Lemma.

⟨1⟩7. LET: $x, x' \in X$

⟨1⟩8. ASSUME: $d(x, x') < \delta$

⟨1⟩9. PICK $y \in Y$ such that $\{x, x'\} \subseteq f^{-1}(B(y, \epsilon/2))$

PROOF: Since $\operatorname{diam}\{x, x'\} < \delta$.

⟨1⟩10. $d(f(x), f(x')) < \epsilon$

PROOF:

$$\begin{aligned} d(f(x), f(x')) &\leq d(f(x), y) + d(y, f(x')) && \text{(Triangle Inequality)} \\ &< \epsilon/2 + \epsilon/2 && (\langle 1\rangle 9) \\ &= \epsilon \end{aligned}$$

□

10.9 Epsilon-neighbourhoods

Definition 10.9.1 (ϵ-neighbourhood)**.** Let X be a metric space. Let $A \subseteq X$ be nonempty. Let $\epsilon > 0$. Then the *ϵ-neighbourhood* of A, $U(A, \epsilon)$, is the set

$$U(A, \epsilon) = \{x \in X \mid d(x, A) < \epsilon\} \enspace .$$

Proposition 10.9.2. *Let X be a metric space. Let $A \subseteq X$ be nonempty. Let $\epsilon > 0$. Then $U(A, \epsilon) = \bigcup_{a \in A} B(a, \epsilon)$.*

PROOF:
$\langle 1\rangle 1$. LET: X be a metric space.
$\langle 1\rangle 2$. LET: $A \subseteq X$ be nonempty.
$\langle 1\rangle 3$. LET: $\epsilon > 0$
$\langle 1\rangle 4$. $U(A, \epsilon) \subseteq \bigcup_{a \in A} B(a, \epsilon)$
  $\langle 2\rangle 1$. LET: $x \in U(A, \epsilon)$
  $\langle 2\rangle 2$. $d(x, A) < \epsilon$
  $\langle 2\rangle 3$. ϵ is not a lower bound for $\{d(x, a) \mid a \in A\}$
  $\langle 2\rangle 4$. PICK $a \in A$ such that $d(x, a) < \epsilon$
  $\langle 2\rangle 5$. $x \in B(a, \epsilon)$
$\langle 1\rangle 5$. $\bigcup_{a \in A} B(a, \epsilon) \subseteq U(A, \epsilon)$
  $\langle 2\rangle 1$. LET: $a \in A$ and $x \in B(a, \epsilon)$
  $\langle 2\rangle 2$. $d(x, A) \leq d(x, a)$
  $\langle 2\rangle 3$. $d(x, A) < \epsilon$
  $\langle 2\rangle 4$. $x \in U(A, \epsilon)$

□

Proposition 10.9.3. *Let X be a metric space. Let $A \subseteq X$ be nonempty and compact. Let U be an open set such that $A \subseteq U$. Then there exists $\epsilon > 0$ such that $U(A, \epsilon) \subseteq U$.*

PROOF:
$\langle 1\rangle 1$. LET: X be a metric space.
$\langle 1\rangle 2$. LET: $A \subseteq X$ be nonempty and compact.
$\langle 1\rangle 3$. LET: U be an open set such that $A \subseteq U$
$\langle 1\rangle 4$. $\{B(a, \epsilon) \mid a \in A, \epsilon > 0, B(a, 2\epsilon) \subseteq U\}$ covers A.
  PROOF: By Proposition 10.1.4.
$\langle 1\rangle 5$. PICK a finite subcover $\{B(a_1, \epsilon_1), \ldots, B(a_n, \epsilon_n)\}$
  PROOF: Since A is compact ($\langle 1\rangle 2$).
$\langle 1\rangle 6$. LET: $\epsilon = \min(\epsilon_1, \ldots, \epsilon_n)$

PROVE: $U(A, \epsilon) \subseteq U$
$\langle 1 \rangle 7$. LET: $x \in U(A, \epsilon)$
$\langle 1 \rangle 8$. PICK $a \in A$ such that $d(x, a) < \epsilon$
PROOF: Proposition 10.9.2.
$\langle 1 \rangle 9$. PICK i such that $a \in B(a_i, \epsilon_i)$
PROOF: By $\langle 1 \rangle 5$.
$\langle 1 \rangle 10$. $d(x, a_i) < 2\epsilon$
PROOF: By the Triangle Inequality.
$\langle 1 \rangle 11$. $x \in U$
PROOF: From $\langle 1 \rangle 4$.
□

This example shows that we cannot weaken the hypothesis that A is compact to A being closed:

Example 10.9.4. Let $X = \mathbb{R}^2$. Let $A = \{(x, 1/x) \mid x > 0\}$. Let $U = \{(x, y) \mid x > 0, y > 0\}$. Then A is nonempty and closed (Proposition 9.48.14). The set U is open and $A \subseteq U$. But there is no $\epsilon > 0$ such that $U(A, \epsilon) \subseteq U$.

PROOF:
$\langle 1 \rangle 1$. LET: $\epsilon > 0$
$\langle 1 \rangle 2$. $(2/\epsilon, \epsilon/2) \in A$
$\langle 1 \rangle 3$. $(2/\epsilon, 0) \in U(A, \epsilon)$
$\langle 1 \rangle 4$. $(2/\epsilon, 0) \notin U$
□

10.10 Isometry

Definition 10.10.1 (Isometry)**.** Let X be a metric space. An *isometry* of X is a function $f : X \to X$ such that, for all $x, y \in X$, we have $d(x, y) = d(f(x), f(y))$.

Proposition 10.10.2. *An isometry on a compact metric space is a homeomorphism.*

PROOF:
$\langle 1 \rangle 1$. LET: X be a compact metric space.
$\langle 1 \rangle 2$. LET: $f : X \to X$ be an isometry.
$\langle 1 \rangle 3$. f is an imbedding
PROOF: Proposition 10.5.2.
$\langle 1 \rangle 4$. f is surjective.
$\langle 2 \rangle 1$. ASSUME: for a contradiction $a \notin f(X)$
$\langle 2 \rangle 2$. $f(X)$ is closed
PROOF: Proposition 9.48.12.
$\langle 2 \rangle 3$. PICK $\epsilon > 0$ such that $B(a, \epsilon) \cap f(X) = \emptyset$
$\langle 2 \rangle 4$. For $m, n \in \mathbb{N}$ with $m \neq n$, we have $d(f^m(a), f^n(a)) \geq \epsilon$
$\langle 3 \rangle 1$. ASSUME: without loss of generality $m < n$
$\langle 3 \rangle 2$. $d(a, f^{n-m}(a)) \geq \epsilon$
PROOF: $\langle 2 \rangle 3$

⟨3⟩3. $d(f^m(a), f^n(a)) \geq \epsilon$

PROOF: ⟨1⟩2

⟨2⟩5. The sequence $(f^n(a))$ has a convergent subsequence.

PROOF: Corollary 9.46.2.1, ⟨1⟩1, Corollary 10.1.19.1.

⟨2⟩6. Q.E.D.

PROOF: ⟨2⟩4 and ⟨2⟩5 form a contradiction.

□

□

10.11 Shrinking Maps

Definition 10.11.1 (Shrinking Map)**.** Let X be a metric space. Let $f : X \to X$. Then f is a *shrinking map* if and only if, for all $x, y \in X$ with $x \neq y$, we have $d(f(x), f(y)) < d(x, y)$.

Proposition 10.11.2. *Let X be a compact metric space. Let $f : X \to X$ be a contraction. Then f has a unique fixed point.*

PROOF:

⟨1⟩1. LET: $A_n = f^n(X)$ for $n \geq 1$

⟨1⟩2. For all $n \geq 1$ we have A_n is closed.

PROOF: Proposition 9.48.12.

⟨1⟩3. LET: $A = \bigcap_{n=1}^{\infty} A_n$

⟨1⟩4. PICK $a \in A$

PROOF: Proposition 9.47.6.

⟨1⟩5. $f(A) = A$

⟨2⟩1. $f(A) \subseteq A$

⟨2⟩2. $A \subseteq f(A)$

⟨3⟩1. LET: $x \in A$

⟨3⟩2. For $n \geq 1$, PICK x_n such that $x = f^n(x_n)$

⟨3⟩3. PICK a convergent subsequence $(f^{n_r-1}(x_{n_r}))$ of $(f^{n-1}(x_n))$ with limit l

PROOF: Corollary 9.46.2.1.

⟨3⟩4. $f(l) = x$

PROOF: Both are the limit of $f(f^{n_r-1}(a_{n_r})) = f^{n_r}(a_{n_r})$.

⟨3⟩5. $l \in A$

⟨4⟩1. ASSUME: for a contradiction $l \notin A$

⟨4⟩2. PICK N such that $l \notin A_N$

⟨4⟩3. PICK R such that $n_R > N$

⟨4⟩4. For $r \geq R$ we have $f^{n_r-1}(a_{n_r}) \in A_N$

⟨4⟩5. Q.E.D.

PROOF: This is a contradiction.

⟨1⟩6. $\operatorname{diam} A = A$

⟨2⟩1. PICK $x, y \in A$ such that $d(x, y) = \operatorname{diam} A$

PROOF: By the Extreme Value Theorem.

⟨2⟩2. PICK $x', y' \in A$ such that $x = f(x')$ and $y = f(y')$

PROOF: By ⟨1⟩5.

⟨2⟩3. $x' = y'$

PROOF: If $x' \neq y'$ then $\operatorname{diam} A = d(x, y) < d(x', y')$ which is a contradiction.

⟨2⟩4. $x = y$

⟨1⟩7. $f(a) = a$

PROOF: Since $a, f(a) \in A$

⟨1⟩8. If $f(b) = b$ then $b = a$

PROOF: If $f(b) = b$ then $b \in A$.

□

The following example shows that we cannot weaken the hypothesis from 'X is a compact metric space' to 'X is a complete metric space'.

Example 10.11.3. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = [x+(x^2+1)^{1/2}]/2$ is a shrinking map with no fixed point.

10.12 Contractions

Definition 10.12.1 (Contraction)**.** Let X be a metric space. Let $f : X \to X$. Then f is a *contraction* if and only if there exists $\alpha < 1$ such that, for all $x, y \in X$, we have $d(f(x), f(y)) \leq \alpha d(x, y)$.