Topology

Robin Adams

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1 Real Numbers

Definition 1. Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences (a_n) such that $a_n = 0$ for all but finitely many n.

2 Order Theory

Definition 2 (Convex)**.** Let X be a linearly ordered set and $Y \subseteq X$. Then Y is *convex* if and only if, for all $a, b \in Y$ and $c \in X$, if $a < c < b$ then $c \in Y$.

3 Topological Spaces

Definition 3 (Topology)**.** A *topology* on a set X is a set $\mathcal{T} \subseteq \mathcal{P}X$ such that:

- $X \in \mathcal{T}$.
- For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$.
- For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$.

We call the elements of X *points* and the elements of $\mathcal{T}$ *open sets*.

Definition 4 (Topological Space)**.** A *topological space* X consists of a set X and a topology on X.

Definition 5 (Discrete Space)**.** For any set X, the *discrete* topology on X is $\mathcal{P}X$.

Definition 6 (Indiscrete Space)**.** For any set X, the *indiscrete* or *trivial* topology on X is $\{\emptyset, X\}$.

Definition 7 (Finite Complement Topology)**.** For any set X, the *finite complement topology* on X is $\{U \in \mathcal{P}X \mid X \setminus U \text{ is finite}\} \cup \{\emptyset\}$.

Definition 8 (Countable Complement Topology)**.** For any set X, the *countable complement topology* on X is $\{U \in \mathcal{P}X \mid X \setminus U \text{ is countable}\} \cup \{\emptyset\}$.

Definition 9 (Finer, Coarser)**.** Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a given set X. If $\mathcal{T}' \supseteq \mathcal{T}$, we say that $\mathcal{T}'$ is *finer* than $\mathcal{T}$; if $\mathcal{T}'$ *properly* contains $\mathcal{T}$, we say that $\mathcal{T}'$ is *strictly* finer than $\mathcal{T}$. We also say that $\mathcal{T}$ is *coarser* than $\mathcal{T}'$, or *stricly* coarser, in these two respective situations. We say $\mathcal{T}$ is *comparable* with $\mathcal{T}'$ if either $\mathcal{T}' \supseteq \mathcal{T}$ or $\mathcal{T} \supseteq \mathcal{T}'$.

Lemma 10. *Let X be a topological space and $U \subseteq X$. Then U is open if and only if, for all $x \in U$, there exists an open set V such that $x \in V \subseteq U$.*

PROOF:
$\langle 1 \rangle 1.$ $\Rightarrow$
  PROOF: Take $V = U$
$\langle 1 \rangle 2.$ $\Leftarrow$
  PROOF: We have $U = \bigcup \{V \in \mathcal{P}X \mid V \subseteq U\}$.
□

Lemma 11. *Let X be a set and $\mathcal{T}$ a nonempty set of topologies on X. Then $\bigcap \mathcal{T}$ is a topology on X, and is the finest topology that is coarser than every member of $\mathcal{T}$.*

PROOF:
$\langle 1 \rangle 1.$ $X \in \bigcap \mathcal{T}$
  PROOF: Since X is in every member of $\mathcal{T}$.
$\langle 1 \rangle 2.$ $\bigcap \mathcal{T}$ is closed under union.
  $\langle 2 \rangle 1.$ LET: $\mathcal{U} \subseteq \bigcap \mathcal{T}$
  $\langle 2 \rangle 2.$ For all $T \in \mathcal{T}$ we have $\mathcal{U} \subseteq T$
  $\langle 2 \rangle 3.$ For all $T \in \mathcal{T}$ we have $\bigcup \mathcal{U} \in T$
  $\langle 2 \rangle 4.$ $\bigcup \mathcal{U} \in \bigcap \mathcal{T}$
$\langle 1 \rangle 3.$ $\bigcap \mathcal{T}$ is closed under binary intersection.
  $\langle 2 \rangle 1.$ LET: $U, V \in \bigcap \mathcal{T}$
  $\langle 2 \rangle 2.$ For all $T \in \mathcal{T}$ we have $U, V \in T$
  $\langle 2 \rangle 3.$ For all $T \in \mathcal{T}$ we have $U \cap V \in T$
  $\langle 2 \rangle 4.$ $U \cap V \in \bigcap \mathcal{T}$
□

Lemma 12. *Let X be a set and $\mathcal{T}$ a set of topologies on X. Then there exists a unique coarsest topology that is finer than every member of $\mathcal{T}$.*

PROOF: The required topology is given by
$\bigcap \{T \in \mathcal{P}\mathcal{P}X \mid T \textit{ is a topology on } X \textit{ that is finer than every member of } \mathcal{T}\}$,
The set is nonempty since it contains the discrete topology. □

4 Basis for a Topology

Definition 13 (Basis)**.** If X is a set, a *basis* for a topology on X is a set $\mathcal{B} \subseteq \mathcal{P}X$ called *basis elements* such that

1. For all $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$.

2. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

If $\mathcal{B}$ satisfies these two conditions, then we define the topology *generated* by $\mathcal{B}$ to be $\mathcal{T} = \{U \in \mathcal{P}X \mid \forall x \in U.\exists B \in \mathcal{B}.x \in B \subseteq U\}$.

We prove this is a topology.

PROOF:
⟨1⟩1. $X \in \mathcal{T}$
PROOF: For all $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B \subseteq X$ by condition 1.
⟨1⟩2. For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$
⟨2⟩1. LET: $\mathcal{U} \subseteq \mathcal{T}$
⟨2⟩2. LET: $x \in \bigcup \mathcal{U}$
⟨2⟩3. PICK $U \in \mathcal{U}$ such that $x \in U$
⟨2⟩4. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$
PROOF: Since $U \in \mathcal{T}$ by ⟨2⟩1 and ⟨2⟩3.
⟨2⟩5. $x \in B \subseteq \bigcup \mathcal{U}$
⟨1⟩3. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$
⟨2⟩1. LET: $U, V \in \mathcal{T}$
⟨2⟩2. LET: $x \in U \cap V$
⟨2⟩3. PICK $B_1 \in \mathcal{B}$ such that $x \in B_1 \subseteq U$
⟨2⟩4. PICK $B_2 \in \mathcal{B}$ such that $x \in B_2 \subseteq V$
⟨2⟩5. PICK $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$
PROOF: By condition 2.
⟨2⟩6. $x \in B_3 \subseteq U \cap V$
□

Lemma 14. *Let X be a set. Let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ is the set of all unions of subsets of $\mathcal{B}$.*

PROOF:
⟨1⟩1. For all $U \in \mathcal{T}$, there exists $\mathcal{A} \subseteq \mathcal{B}$ such that $U = \bigcup \mathcal{A}$
⟨2⟩1. LET: $U \in \mathcal{T}$
⟨2⟩2. LET: $\mathcal{A} = \{B \in \mathcal{B} \mid B \subseteq U\}$
⟨2⟩3. $U \subseteq \bigcup \mathcal{A}$
⟨3⟩1. LET: $x \in U$
⟨3⟩2. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$
PROOF: Since $\mathcal{B}$ is a basis for $\mathcal{T}$.
⟨3⟩3. $x \in B \in \mathcal{A}$
⟨2⟩4. $\bigcup \mathcal{A} \subseteq U$
PROOF: From the definition of $\mathcal{A}$ (⟨2⟩2).
⟨1⟩2. For all $\mathcal{A} \subseteq \mathcal{B}$ we have $\bigcup \mathcal{A} \in \mathcal{T}$
⟨2⟩1. $\mathcal{B} \subseteq \mathcal{T}$
PROOF: If $B \in \mathcal{B}$ and $x \in B$, then there exists $B' \in \mathcal{B}$ such that $x \in B' \subseteq B$, namely $B' = B$.
⟨2⟩2. Q.E.D.

PROOF: Since $\mathcal{T}$ is closed under union.
□

Corollary 14.1. *Let X be a set. Let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ is the coarsest topology that includes $\mathcal{B}$.*

PROOF: Since every topology that includes $\mathcal{B}$ includes all unions of subsets of $\mathcal{B}$. □

Lemma 15. *Let X be a topological space. Suppose that $\mathcal{C}$ is a set of open sets such that, for every open set U and every point $x \in U$, there exists $C \in \mathcal{C}$ such that $x \in C \subseteq U$. Then $\mathcal{C}$ is a basis for the topology on X.*

PROOF:
⟨1⟩1. For all $x \in X$, there exists $C \in \mathcal{C}$ such that $x \in C$
  PROOF: Immediate from hypothesis.
⟨1⟩2. For all $C_1, C_2 \in \mathcal{C}$ and $x \in C_1 \cap C_2$, there exists $C_3 \in \mathcal{C}$ such that $x \in C_3 \subseteq C_1 \cap C_2$
  PROOF: Since $C_1 \cap C_2$ is open.
⟨1⟩3. Every open set is open in the topology generated by $\mathcal{C}$
  PROOF: Immediate from hypothesis.
⟨1⟩4. Every union of a subset of $\mathcal{C}$ is open.
  PROOF: Since every member of $\mathcal{C}$ is open.
□

Lemma 16. *Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$ respectively on the set X. Then the following are equivalent.*

1. $\mathcal{T} \subseteq \mathcal{T}'$
2. *For all $B \in \mathcal{B}$ and $x \in B$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.*

PROOF:
⟨1⟩1. $1 \Rightarrow 2$
  ⟨2⟩1. ASSUME: $\mathcal{T} \subseteq \mathcal{T}'$
  ⟨2⟩2. LET: $B \in \mathcal{B}$ and $x \in B$
  ⟨2⟩3. $B \in \mathcal{T}$
    PROOF: Corollary 14.1.
  ⟨2⟩4. $B \in \mathcal{T}'$
    PROOF: By ⟨2⟩1
  ⟨2⟩5. There exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$
    PROOF: Since $\mathcal{B}'$ is a basis for $\mathcal{T}'$.
⟨1⟩2. $2 \Rightarrow 1$
  ⟨2⟩1. ASSUME: 2
  ⟨2⟩2. LET: $U \in \mathcal{T}$
    PROVE: $U \in \mathcal{T}'$
  ⟨2⟩3. LET: $x \in U$
    PROVE: There exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq U$
  ⟨2⟩4. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$

PROOF: Since $\mathcal{B}$ is a basis for $\mathcal{T}$.
⟨2⟩5. PICK $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$
PROOF: By ⟨2⟩1.
⟨2⟩6. $x \in B' \subseteq U$
□

Definition 17 (Lower Limit Topology on the Real Line)**.** The *lower limit topology on the real line* is the topology on $\mathbb{R}$ generated by the basis consisting of all half-open intervals of the form $[a, b)$.

We write $\mathbb{R}_l$ for the topological space $\mathbb{R}$ under the lower limit topology.

We prove this is a basis for a topology.

PROOF:
⟨1⟩1. For all $x \in \mathbb{R}$ there exists an interval $[a, b)$ such that $x \in [a, b)$.
PROOF: Take $[a, b) = [x, x + 1)$.
⟨1⟩2. For any open intervals $[a, b)$, $[c, d)$ if $x \in [a, b) \cap [c, d)$, then there exists an interval $[e, f)$ such that $x \in [e, f) \subseteq [a, b) \cap [c, d)$
PROOF: Take $[e, f) = [\max(a, c), \min(b, d))$.
□

Definition 18 (K-topology on the Real Line)**.** Let $K = \{1/n \mid n \in \mathbb{Z}^+\}$.

The *K-topology on the real line* is the topology on $\mathbb{R}$ generated by the basis consisting of all open intervals (a, b) and all sets of the form $(a, b) \setminus K$.

We write $\mathbb{R}_K$ for the topological space $\mathbb{R}$ under the K-topology.

We prove this is a basis for a topology.

PROOF:
⟨1⟩1. For all $x \in \mathbb{R}$ there exists an open interval (a, b) such that $x \in (a, b)$.
PROOF: Take $(a, b) = (x - 1, x + 1)$.
⟨1⟩2. For any basic open sets B_1, B_2 if $x \in B_1 \cap B_2$, then there exists a basic open set B_3 such that $x \in B_3 \subseteq B_1 \cap B_2$
⟨2⟩1. CASE: $B_1 = (a, b)$, $B_2 = (c, d)$
PROOF: Take $B_3 = (\max(a, c), \min(b, d))$.
⟨2⟩2. CASE: $B_1 = (a, b)$ or $(a, b) \setminus K$, $B_2 = (c, d)$ or $(c, d) \setminus K$, and they are not both open intervals.
PROOF: Take $B_3 = (\max(a, c), \min(b, d)) \setminus K$.
□

Lemma 19. *The lower limit topology and the K-topology are incomparable.*

PROOF:
⟨1⟩1. The interval $[10, 11)$ is not open in the K-topology.
PROOF: There is no open interval (a, b) such that $10 \in (a, b) \subseteq [10, 11)$ or $10 \in (a, b) \setminus K \subseteq [10, 11)$.
⟨1⟩2. The set $(-1, 1) \setminus K$ is not open in the lower limit topology.
PROOF: There is no half-open interval $[a, b)$ such that $0 \in [a, b) \subseteq (-1, 1) \setminus K$, since there must be a positive integer n with $1/n \in [a, b)$.

□

Definition 20 (Subbasis)**.** A *subbasis* $\mathcal{S}$ for a topology on X is a set $\mathcal{S} \subseteq \mathcal{P}X$ such that $\bigcup \mathcal{S} = X$.

The topology *generated* by the subbasis $\mathcal{S}$ is the set of all unions of finite intersections of elements of $\mathcal{S}$.

We prove this is a topology.

PROOF:
⟨1⟩1. The set $\mathcal{B}$ of all finite intersections of elements of $\mathcal{S}$ forms a basis for a topology on X.
  ⟨2⟩1. $\bigcup \mathcal{B} = X$
    PROOF: Since $\mathcal{S} \subseteq \mathcal{B}$.
  ⟨2⟩2. $\mathcal{B}$ is closed under binary intersection.
    PROOF: By definition.
⟨1⟩2. Q.E.D.
  PROOF: By Lemma 14.
□

We have simultaneously proved:

Proposition 21. *Let $\mathcal{S}$ be a subbasis for the topology on X. Then the set of all finite intersections of elements of $\mathcal{S}$ is a basis for the topology on X.*

Proposition 22. *Let X be a set. Let $\mathcal{S}$ be a subbasis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ is the coarsest topology that includes $\mathcal{S}$.*

PROOF: Since every topology that includes $\mathcal{S}$ includes every union of finite intersections of elements of $\mathcal{S}$. □

5 Open Maps

Definition 23 (Open Map)**.** Let X and Y be topological spaces and $f : X \to Y$. Then f is an *open map* if and only if, for every open set U in X, the set $f(U)$ is open in Y.

Lemma 24. *Let X and Y be topological spaces and $f : X \to Y$. Let $\mathcal{B}$ be a basis for the topology on X. If $f(B)$ is open in Y for all $B \in \mathcal{B}$, then f is an open map.*

PROOF: From Lemma 14. □

6 The Order Topology

Definition 25 (Order Topology)**.** Let X be a linearly ordered set with at least two points. The *order topology* on X is the topology generated by the basis $\mathcal{B}$ consisting of:

- all open intervals (a, b);
- all intervals of the form $[\bot, b)$ where $\bot$ is least in X;
- all intervals of the form $(a, \top]$ where $\top$ is greatest in X.

We prove this is a basis for a topology.

PROOF:
$\langle 1 \rangle 1$. For all $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B$.
$\langle 2 \rangle 1$. LET: $x \in X$
$\langle 2 \rangle 2$. CASE: x is greatest in X.
$\langle 3 \rangle 1$. PICK $y \in X$ with $y \neq x$
$\langle 3 \rangle 2$. $x \in (y, x] \in \mathcal{B}$
$\langle 2 \rangle 3$. CASE: x is least in X.
$\langle 3 \rangle 1$. PICK $y \in X$ with $y \neq x$
$\langle 3 \rangle 2$. $x \in [x, y) \in \mathcal{B}$
$\langle 2 \rangle 4$. CASE: x is neither greatest nor least in X.
$\langle 3 \rangle 1$. PICK $a, b \in X$ with $a < x$ and $x < b$
$\langle 3 \rangle 2$. $x \in (a, b) \in \mathcal{B}$
$\langle 1 \rangle 2$. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$
$\langle 2 \rangle 1$. LET: $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$
$\langle 2 \rangle 2$. CASE: $B_1 = (a, b)$, $B_2 = (c, d)$
PROOF: Take $B_3 = (\max(a, c), \min(b, d))$.
$\langle 2 \rangle 3$. CASE: $B_1 = (a, b)$, $B_2 = [\bot, d)$
PROOF: Take $B_3 = (a, \min(b, d))$.
$\langle 2 \rangle 4$. CASE: $B_1 = (a, b)$, $B_2 = (c, \top]$
PROOF: Take $B_3 = (\max(a, c), b)$.
$\langle 2 \rangle 5$. CASE: $B_1 = [\bot, b)$, $B_2 = [\bot, d)$
PROOF: Take $B_3 = [\bot, \min(b, d))$.
$\langle 2 \rangle 6$. CASE: $B_1 = [\bot, b)$, $B_2 = (c, \top]$
PROOF: Take $B_3 = (c, b)$.
□

Lemma 26. *Let X be a linearly ordered set. Then the open rays form a subbasis for the order topology on X.*

PROOF:
$\langle 1 \rangle 1$. Every open ray is open.
$\langle 2 \rangle 1$. For all $a \in X$, the ray $(-\infty, a)$ is open.
$\langle 3 \rangle 1$. LET: $x \in (-\infty, a)$
$\langle 3 \rangle 2$. CASE: x is least in X
PROOF: $xin[x, a) = (-\infty, a)$.
$\langle 3 \rangle 3$. CASE: x is not least in X
$\langle 4 \rangle 1$. PICK $y < x$
$\langle 4 \rangle 2$. $x \in (y, a) \subseteq (-\infty, a)$
$\langle 2 \rangle 2$. For all $a \in X$, the ray $(a, +\infty)$ is open.

PROOF: Similar.

⟨1⟩2. Every basic open set is a finite intersection of open rays.

PROOF: We have $(a,b) = (a,+\infty) \cap (-\infty,b)$, $[\bot,b) = (-\infty,b)$ and $(a,\top] = (a,+\infty)$.

□

Definition 27 (Standard Topology on the Real Line)**.** The *standard topology on the real line* is the order topology on $\mathbb{R}$ generated by the standard order.

Lemma 28. *The lower limit topology is strictly finer than the standard topology on* $\mathbb{R}$*.*

PROOF:

⟨1⟩1. Every open interval is open in the lower limit topology.

PROOF: If $x \in (a,b)$ then $x \in [x,b) \subseteq (a,b)$.

⟨1⟩2. The half-open interval $[0,1)$ is not open in the standard topology.

PROOF: There is no open interval (a,b) such that $0 \in (a,b) \subseteq [0,1)$.

□

Lemma 29. *The K-topology is strictly finer than the standard topology on* $\mathbb{R}$*.*

PROOF:

⟨1⟩1. Every open interval is open in the K-topology.

PROOF: Corollary 14.1.

⟨1⟩2. The set $(-1,1) \setminus K$ is not open in the standard topology.

PROOF: There is no open interval (a,b) such that $0 \in (a,b) \subseteq (-1,1) \setminus K$, since there must be a positive integer n with $1/n \in (a,b)$.

□

Definition 30 (Ordered Square)**.** The *ordered square* I_o^2 is the set $[0,1]^2$ under the order topology generated by the dictionary order.

7 The Product Topology

Definition 31 (Product Topology)**.** Let $\{A_i\}_{i \in I}$ be a family of topological spaces. The *product topology* on $\prod_{i \in I} A_i$ is the topology generated by the subbasis consisting of the sets of the form ${\pi_i}^{-1}(U)$ where $i \in I$ and U is open in A_i.

Proposition 32. *The product topology on $\prod_{i \in I} A_i$ is generated by the basis consisting of all sets of the form $\prod_{i \in I} U_i$ where $\{U_i\}_{i \in I}$ is a family such that each U_i is an open set in A_i and $U_i = A_i$ for all but finitely many i.*

PROOF: From Proposition 21. □

Proposition 33. *Let $\{A_i\}_{i \in I}$ be a family of topological spaces. For $i \in I$, let $\mathcal{B}_i$ be a basis for the topology on A_i. Then $\mathcal{B} = \{\prod_{i \in I} B_i \mid \forall i \in I . B_i \in \mathcal{B}_i, B_i = A_i$ for all but finitely many $i\}$ is a basis for the box topology on $\prod_{i \in I} A_i$.*

PROOF:
$\langle 1\rangle 1$. Every set in $\mathcal{B}$ is open.
$\langle 1\rangle 2$. For every point $a \in \prod_{i \in I} A_i$ and every open set U with $a \in U$, there exists $B \in \mathcal{B}$ such that $a \in B \subseteq U$.
$\langle 2\rangle 1$. LET: U be open and $a \in U$
$\langle 2\rangle 2$. PICK a family $\{U_i\}_{i \in I}$ such that each U_i is open in A_i, such that $U_i = A_i$ except for $i = i_1, \ldots, i_n$, and such that $a \in \prod_{i \in I} U_i \subseteq U$.
$\langle 2\rangle 3$. For $j = 1, \ldots, n$, PICK $B_{i_j} \in \mathcal{B}_{i_j}$ such that $a_{i_j} \in B_{i_j} \subseteq U_{i_j}$
$\langle 2\rangle 4$. LET: $B = \prod_{i \in I} B_i$ where $B_i = A_i$ for $i \neq i_1, \ldots, i_n$
$\langle 2\rangle 5$. $B \in \mathcal{B}$
$\langle 2\rangle 6$. $a \in B \subseteq U$
$\langle 1\rangle 3$. Q.E.D.
PROOF: Lemma 15.
□

Proposition 34. *Let $\{A_i\}_{i \in I}$ be a family of topological spaces. Then the projections $\pi_i : \prod_{i \in I} A_i \to A_i$ are open maps.*

PROOF: From Lemma 24. □

Proposition 35. *Let $\{X_i\}_{i \in I}$ be a family of sets. For $i \in I$, let $\mathcal{T}_i$ and $\mathcal{U}_i$ be topologies on X_i. Let $\mathcal{P}$ be the product topology on $\prod_{i \in I} X_i$ generated by the topologies $\mathcal{T}_i$, and $\mathcal{Q}$ the product topology on the same set generated by the topologies $\mathcal{U}_i$. Then $\mathcal{P} \subseteq \mathcal{Q}$ if and only if $\mathcal{T}_i \subseteq \mathcal{U}_i$ for all i.*

PROOF:
$\langle 1\rangle 1$. If $\mathcal{T}_i \subseteq \mathcal{U}_i$ for all i then $\mathcal{P} \subseteq \mathcal{Q}$
PROOF: By Corollary 14.1.
$\langle 1\rangle 2$. If $\mathcal{P} \subseteq \mathcal{Q}$ then $\mathcal{T}_i \subseteq \mathcal{U}_i$ for all i
$\langle 2\rangle 1$. ASSUME: $\mathcal{P} \subseteq \mathcal{Q}$
$\langle 2\rangle 2$. LET: $i \in I$
$\langle 2\rangle 3$. LET: $U \in \mathcal{T}_i$
$\langle 2\rangle 4$. LET: $U_i = U$ and $U_j = X_j$ for $j \neq i$
$\langle 2\rangle 5$. $\prod_{i \in I} U_i \in \mathcal{P}$
$\langle 2\rangle 6$. $\prod_{i \in I} U_i \in \mathcal{Q}$
$\langle 2\rangle 7$. $U \in \mathcal{U}_i$
PROOF: From Proposition 34.
□

8 The Subspace Topology

Definition 36 (Subspace Topology)**.** Let X be a topological space and $Y \subseteq X$. The *subspace topology* on Y is $\mathcal{T} = \{U \cap Y \mid U \text{ is open in } X\}$.

We prove this is a topology.

PROOF:
$\langle 1\rangle 1$. $Y \in \mathcal{T}$

PROOF: Since $Y = X \cap Y$

$\langle 1 \rangle 2$. For all $\mathcal{U} \subseteq \mathcal{T}$, we have $\bigcup \mathcal{U} \in \mathcal{T}$

$\langle 2 \rangle 1$. LET: $\mathcal{U} \subseteq \mathcal{T}$

$\langle 2 \rangle 2$. LET: $\mathcal{V} = \{V \text{ open in } X \mid V \cap Y \in \mathcal{U}\}$

$\langle 2 \rangle 3$. $\bigcup \mathcal{U} = (\bigcup \mathcal{V}) \cap Y$

$\langle 1 \rangle 3$. For all $U, V \in \mathcal{T}$, we have $U \cap V \in \mathcal{T}$

$\langle 2 \rangle 1$. LET: $U, V \in \mathcal{T}$

$\langle 2 \rangle 2$. PICK U', V' open in X such that $U = U' \cap Y$ and $V = V' \cap Y$

$\langle 2 \rangle 3$. $(U \cap V) = (U' \cap V') \cap Y$

□

Lemma 37. *Let X be a topological space and $Y \subseteq X$. Let $\mathcal{B}$ be a basis for the topology on X. Then $\mathcal{B}' = \{B \cap Y \mid B \in \mathcal{B}\}$ is a basis for the subspace topology on Y.*

PROOF:

$\langle 1 \rangle 1$. Every element in $\mathcal{B}'$ is open in Y

$\langle 1 \rangle 2$. For every open set U in Y and point $y \in U$, there exists $B' \in \mathcal{B}'$ such that $y \in B' \subseteq U$

$\langle 2 \rangle 1$. LET: U be open in Y and $y \in U$

$\langle 2 \rangle 2$. PICK V open in X such that $U = V \cap Y$

$\langle 2 \rangle 3$. PICK $B \in \mathcal{B}$ such that $y \in B \subseteq V$

$\langle 2 \rangle 4$. LET: $B' = B \cap Y$

$\langle 2 \rangle 5$. $B' \in \mathcal{B}'$

$\langle 2 \rangle 6$. $y \in B' \subseteq U$

$\langle 1 \rangle 3$. Q.E.D.

PROOF: By Lemma 15.

□

Lemma 38. *Let X be a topological space and $Y \subseteq X$. Let $\mathcal{S}$ be a basis for the topology on X. Then $\mathcal{S}' = \{S \cap Y \mid S \in \mathcal{S}\}$ is a subbasis for the subspace topology on Y.*

PROOF: The set $\{B \cap Y \mid B \text{ is a finite intersection of elements of } \mathcal{S}\}$ is a basis for the subspace topology by Lemma 37, and this is the set of all finite intersections of elements of $\mathcal{S}'$. □

Lemma 39. *Let Y be a subspace of X. If U is open in Y and Y is open in X then U is open in X.*

PROOF:

$\langle 1 \rangle 1$. PICK V open in X such that $U = V \cap Y$

$\langle 1 \rangle 2$. U is open in X

PROOF: Since it is the intersection of two open sets V and Y.

□

Theorem 40. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let A_i be a subspace of X_i for all $i \in I$. Then the product topology on $\prod_{i \in I} A_i$ is the same as the topology it inherits as a subspace of $\prod_{i \in I} X_i$.*

PROOF: The product topology is generated by the subbasis

$$\begin{aligned}&\{{\pi_i}^{-1}(U) \mid i \in I, U \text{ open in } A_i\}\\ =&\{{\pi_i}^{-1}(V) \cap A_i \mid i \in I, V \text{ open in } X_i\}\\ =&\{{\pi_i}^{-1}(V) \mid i \in I, V \text{ open in } X_i\} \cap \prod_{i \in I} A_i\end{aligned}$$

and this is a subbasis for the subspace topology by Lemma 38. □

Theorem 41. *Let X be an ordered set in the order topology. Let $Y \subseteq X$ be convex. Then the order topology on Y is the same as the subspace topology on Y.*

PROOF:
⟨1⟩1. The order topology is finer than the subspace topology.
  ⟨2⟩1. For every open ray R in X, the set $R \cap Y$ is open in the order topology.
    ⟨3⟩1. For all $a \in X$, we have $(-\infty, a) \cap Y$ is open in the order topology.
      ⟨4⟩1. CASE: For all $y \in Y$ we have $y < a$
        PROOF: In this case $(-\infty, a) \cap Y = Y$.
      ⟨4⟩2. CASE: For all $y \in Y$ we have $a < y$
        PROOF: In this case $(-\infty, a) \cap Y = \emptyset$.
      ⟨4⟩3. CASE: There exists $y \in Y$ such that $y \leq a$ and $y \in Y$ such that $a \leq y$
        ⟨5⟩1. $a \in Y$
          PROOF: By convexity.
        ⟨5⟩2. $(-\infty, a) \cap Y = \{y \in Y \mid y < a\}$
    ⟨3⟩2. For all $a \in X$, we have $(a, +\infty) \cap Y$ is open in the order topology.
      PROOF: Similar.
  ⟨2⟩2. Q.E.D.
    PROOF: By Lemmas 26 and 38 and Proposition 22.
⟨1⟩2. The subspace topology is finer than the order topology.
  ⟨2⟩1. Every open ray in Y is open in the subspace topology.
    PROOF: For any $a \in Y$ we have $(-\infty, a)_Y = (-\infty, a)_X \cap Y$ and $(a, +\infty)_Y = (a, +\infty)_X \cap Y$.
  ⟨2⟩2. Q.E.D.
    PROOF: By Lemma 26 and Proposition 22
□

Proposition 42. *The order topology on I_o^2 is different from the subspace topology as a subspace of $\mathbb{R}^2$ under the dictionary order topology.*

PROOF: The set $\{1/2\} \times (1/2, 1)$ is open in the subspace topology but not in the order topology. □

Proposition 43. *Let X be a topological space, Y a subspace of X, and Z a subspace of Y. Then the subspace topology on Z inherited from X is the same as the subspace topology on Z inherited from Y.*

PROOF: The subspace topology inherited from Y is

$$\begin{aligned}&\{V \cap Z \mid V \text{ open in } Y\}\\ =&\{U \cap Y \cap Z \mid U \text{ open in } X\}\\ =&\{U \cap Z \mid U \text{ open in } X\}\end{aligned}$$

which is the subspace topology inherited from X. ☐

Definition 44 (Unit Circle)**.** The *unit circle* S^1 is $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$

9 Closed Set

Definition 45 (Closed Set)**.** Let X be a topological space and $A \subseteq X$. Then A is *closed* if and only if $X \setminus A$ is open.

Lemma 46. *The empty set is closed.*

PROOF: Since the whole space X is always open. ☐

Lemma 47. *The topological space X is closed.*

PROOF: Since $\emptyset$ is open. ☐

Lemma 48. *The intersection of a nonempty set of closed sets is closed.*

PROOF: Let $\mathcal{C}$ be a nonempty set of closed sets. Then $X \setminus \bigcap \mathcal{C} = \bigcup \{X \setminus C \mid C \in \mathcal{C}\}$ is open. ☐

Lemma 49. *The union of two closed sets is closed.*

PROOF: Let C and D be closed. Then $X \setminus (C \cup D) = (X \setminus C) \cap (X \setminus D)$ is open. ☐

Theorem 50. *Let X be a topological space and Y a subspace of X. Let $A \subseteq Y$. Then A is closed in Y if and only if there exists a closed set C in X such that $A = C \cap Y$.*

PROOF: We have

$$\begin{aligned}&A \text{ is closed in } Y\\ \Leftrightarrow &Y \setminus A \text{ is open in } Y\\ \Leftrightarrow &\exists U \text{ open in } X . Y \setminus A = Y \cap U\\ \Leftrightarrow &\exists C \text{ closed in } X . Y \setminus A = Y \cap (X \setminus U)\\ \Leftrightarrow &\exists C \text{ closed in } X . A = Y \cap U\end{aligned}$$

☐

Theorem 51. *Let Y be a subspace of X and $A \subseteq Y$. If A is closed in Y and Y is closed in X then A is closed in X.*

PROOF: Pick a closed set C in X such that $A = C \cap Y$ (Theorem 50). Then A is the intersection of two sets closed in X, hence A is closed in X (Lemma 48). ☐

Proposition 52. *Let X be a set and $\mathcal{C} \subseteq \mathcal{P}X$ a set such that:*

1. $\emptyset \in \mathcal{C}$
2. $X \in \mathcal{C}$
3. *For all* $\mathcal{A} \subseteq \mathcal{C}$ *nonempty we have* $\bigcap \mathcal{A} \in \mathcal{C}$
4. *For all* $C, D \in \mathcal{C}$ *we have* $C \cup D \in \mathcal{C}$.

Then there exists a unique topology $\mathcal{T}$ such that $\mathcal{C}$ is the set of closed sets, namely

$$\mathcal{T} = \{X \setminus C \mid C \in \mathcal{C}\}$$

PROOF:
$\langle 1 \rangle 1$. LET: $\mathcal{T} = \{X \setminus C \mid C \in \mathcal{C}\}$
$\langle 1 \rangle 2$. $\mathcal{T}$ is a topology
  $\langle 2 \rangle 1$. $X \in \mathcal{T}$
    PROOF: Since $\emptyset \in \mathcal{C}$
  $\langle 2 \rangle 2$. For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$
    $\langle 3 \rangle 1$. LET: $\mathcal{U} \subseteq \mathcal{T}$
    $\langle 3 \rangle 2$. CASE: $\mathcal{U} = \emptyset$
      PROOF: In this case $\bigcup \mathcal{U} = \emptyset \in \mathcal{T}$ since $X \in \mathcal{C}$
    $\langle 3 \rangle 3$. CASE: $\mathcal{U} \neq \emptyset$
      PROOF: In this case $X \setminus \bigcup \mathcal{U} = \bigcap \{X \setminus U \mid U \in \mathcal{U}\} \in \mathcal{C}$.
  $\langle 2 \rangle 3$. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$
    PROOF: Since $X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V) \in \mathcal{C}$.
$\langle 1 \rangle 3$. $\mathcal{C}$ is the set of all closed sets in $\mathcal{T}$
  PROOF:

$$\begin{aligned} & C \text{ is closed in } \mathcal{T} \\ \Leftrightarrow & X \setminus C \in \mathcal{T} \\ \Leftrightarrow & C \in \mathcal{C} \end{aligned}$$

$\langle 1 \rangle 4$. If $\mathcal{T}'$ is a topology and $\mathcal{C}$ is the set of closed sets in $\mathcal{T}'$ then $\mathcal{T}' = \mathcal{T}$
  PROOF: We have

$$\begin{aligned} & U \in \mathcal{T} \\ \Leftrightarrow & X \setminus U \in \mathcal{C} \\ \Leftrightarrow & X \setminus U \text{ is closed in } \mathcal{T}' \\ \Leftrightarrow & U \in \mathcal{T}' \end{aligned}$$

$\square$

Proposition 53. *If A_i is closed in X_i for all $i \in I$ then $\prod_{i \in I} A_i$ is closed in $\prod_{i \in I} X_i$.*

PROOF:

$$(\prod_{i \in I} X_i) \setminus (\prod_{i \in I} A_i) = \bigcup_{j \in I} \left(\prod_{i \in I} X_i \setminus \pi_j{}^{-1}(A_j) \right) \square$$

Proposition 54. *If U is open and A is closed then $U \setminus A$ is open.*

Proof: $U \setminus A = U \cap (X \setminus A)$ is the intersection of two open sets. □

Proposition 55. *If U is open and A is closed then $A \setminus U$ is closed.*

Proof: $A \setminus U = A \cap (X \setminus U)$ is the intersection of two closed sets. □

10 Interior

Definition 56 (Interior)**.** Let X be a topological space and $A \subseteq X$. The *interior* of A, $\operatorname{Int} A$, is the union of all the open subsets of A.

Lemma 57. *The interior of a set is open.*

Proof: It is a union of open sets. □

Lemma 58.

$$\operatorname{Int} A \subseteq A$$

Proof: Immediate from definition. □

Lemma 59. *A set A is open if and only if $A = \operatorname{Int} A$.*

Proof: If $A = \operatorname{Int} A$ then A is open by Lemma 57. Conversely if A is open then $A \subseteq \operatorname{Int} A$ by the definition of interior and so $A = \operatorname{Int} A$.

11 Neighbourhood

Definition 60 (Neighbourhood)**.** A *neighbourhood* of a point x is an open set that contains x.

12 Closure

Definition 61 (Closure)**.** Let X be a topological space and $A \subseteq X$. The *closure* of A, $\overline{A}$, is the intersection of all the closed sets that include A.

This intersection exists since X is a closed set that includes A (Lemma 47).

Lemma 62. *The closure of a set is closed.*

Proof: Dual to Lemma 57. □

Lemma 63.

$$A \subseteq \overline{A}$$

Proof: Immediate from definition. □

Lemma 64. *A set A is closed if and only if $A = \overline{A}$.*

Proof: Dual to Lemma 59. □

Theorem 65. *Let Y be a subspace of X. Let $A \subseteq Y$. Let $\overline{A}$ be the closure of A in X. Then the closure of A in Y is $\overline{A} \cap Y$.*

Proof: The closure of A in Y is

$$\begin{aligned} & \bigcap \{C \text{ closed in } Y \mid A \subseteq C\} \\ = & \bigcap \{D \cap Y \mid D \text{ closed in } X, A \subseteq D \cap Y\} && \text{(Theorem 50)} \\ = & \bigcap \{D \mid D \text{ closed in } X, A \subseteq D\} \cap Y \\ = & \overline{A} \cap Y \end{aligned}$$

∎

Theorem 66. *Let X be a topological space, $A \subseteq X$ and $x \in X$. Then $x \in \overline{A}$ if and only if every neighbourhood of x intersects A.*

Proof: We have

$$\begin{aligned} & x \in \overline{A} \\ \Leftrightarrow & \forall C. C \text{ closed } \wedge A \subseteq C \Rightarrow x \in C \\ \Leftrightarrow & \forall U. U \text{ open } \wedge A \cap U = \emptyset \Rightarrow x \notin U \\ \Leftrightarrow & \forall U. U \text{ open } \wedge x \in U \Rightarrow U \text{ intersects } A \end{aligned}$$

∎

Theorem 67. *Let X be a topological space, $A \subseteq X$ and $x \in X$. Let $\mathcal{B}$ be a basis for X. Then $x \in \overline{A}$ if and only if, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A.*

Proof:
⟨1⟩1. If $x \in \overline{A}$ then, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A.
  Proof: This follows from Theorem 66 since every element of $\mathcal{B}$ is open (Corollary 14.1).
⟨1⟩2. Suppose that, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A. Then $x \in \overline{A}$.
  ⟨2⟩1. Assume: For all $B \in \mathcal{B}$, if $x \in B$ then B intersects A.
  ⟨2⟩2. Let: U be an open set that contains x
      Prove: U intersects A.
  ⟨2⟩3. Pick $B \in \mathcal{B}$ such that $x \in B \subseteq U$.
  ⟨2⟩4. B intersects A.
    Proof: From ⟨2⟩1.
  ⟨2⟩5. U intersects A.
  ⟨2⟩6. Q.E.D.
    Proof: By Theorem 66.

∎

Proposition 68. *If $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$.*

Proof: This holds because $\overline{B}$ is a closed set that includes A. ∎

Proposition 69.

$$\overline{A \cup B} = \overline{A} \cup \overline{B}$$

Proof:
⟨1⟩1. $\overline{A} \subseteq \overline{A \cup B}$

PROOF: By Proposition 68.

$\langle 1\rangle 2.$ $\overline{B} \subseteq \overline{A \cup B}$

PROOF: By Proposition 68.

$\langle 1\rangle 3.$ $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$

$\langle 2\rangle 1.$ LET: $x \in \overline{A \cup B}$

$\langle 2\rangle 2.$ ASSUME: $x \notin \overline{A}$

PROVE: $x \in \overline{B}$

$\langle 2\rangle 3.$ PICK a neighbourhood U of x that does not intersect A

$\langle 2\rangle 4.$ LET: V be any neighbourhood of x

$\langle 2\rangle 5.$ $U \cap V$ is a neighbourhood of x

$\langle 2\rangle 6.$ $U \cap V$ intersects $A \cup B$

PROOF: From $\langle 2\rangle 1$ and Theorem 66.

$\langle 2\rangle 7.$ $U \cap V$ intersects B

PROOF: From $\langle 2\rangle 3$

$\langle 2\rangle 8.$ V intersects B

$\langle 2\rangle 9.$ Q.E.D.

PROOF: We have $x \in \overline{B}$ from Theorem 66.

□

Proposition 70 (AC)**.** *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let $A_i \subseteq X_i$ for all $i \in I$. Then*

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i} \enspace .$$

PROOF:

$\langle 1\rangle 1.$ $\overline{\prod_{i \in I} A_i} \subseteq \prod_{i \in I} \overline{A_i}$

$\langle 2\rangle 1.$ For all $i \in I$ we have $A_i \subseteq \overline{A_i}$

PROOF: Lemma 63.

$\langle 2\rangle 2.$ $\prod_{i \in I} A_i \subseteq \prod_{i \in I} \overline{A_i}$

$\langle 2\rangle 3.$ Q.E.D.

PROOF: Since $\prod_{i \in I} A_i$ is closed by Proposition 53.

$\langle 1\rangle 2.$ $\prod_{i \in I} \overline{A_i} \subseteq \overline{\prod_{i \in I} A_i}$

$\langle 2\rangle 1.$ LET: $x \in \prod_{i \in I} \overline{A_i}$

$\langle 2\rangle 2.$ LET: U be a neighbourhood of x

$\langle 2\rangle 3.$ PICK V_i open in X_i such that $x \in \prod_{i \in I} V_i \subseteq U$ with $V_i = X_i$ except for $i = i_1, \ldots, i_n$

$\langle 2\rangle 4.$ For $i \in I$, pick $a_i \in V_i \cap A_i$

PROOF: By Theorem 66 and $\langle 2\rangle 1$ using the Axiom of Choice.

$\langle 2\rangle 5.$ U intersects $\prod_{i \in I} A_i$

$\langle 2\rangle 6.$ Q.E.D.

PROOF: $a \in U \cap \prod_{i \in I} A_i$

□

Example 71. The closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ is $\mathbb{R}^\omega$

PROOF:

$\langle 1\rangle 1.$ LET: $a \in \mathbb{R}^\omega$

$\langle 1 \rangle 2.$ LET: U be any neighbourhoods of a.
$\langle 1 \rangle 3.$ PICK U_n open in $\mathbb{R}$ for all n such that $a \in \prod_{n \geq 0} U_n \subseteq U$ and $U_n = \mathbb{R}$ for all n except $n_1, \ldots, n_k$
$\langle 1 \rangle 4.$ LET: $b_n = a_n$ for $n = n_1, \ldots, n_k$ and $b_n = 0$ for all other n
$\langle 1 \rangle 5.$ $b \in \mathbb{R}^\infty \cap U$
$\langle 1 \rangle 6.$ Q.E.D.
PROOF: From Theorem 66.
□

13 Limit Points

Definition 72 (Limit Point)**.** Let X be a topological space, $a \in X$ and $A \subseteq X$. Then a is a *limit point*, *cluster point* or *point of accumulation* for A if and only if every neighbourhood of a intersects A at a point other than a.

Lemma 73. *The point a is an accumulation point for A if and only if $a \in \overline{A \setminus \{a\}}$.*

PROOF: From Theorem 66. □

Theorem 74. *Let X be a topological space and $A \subseteq X$. Let A' be the set of all limit points of A. Then $\overline{A} = A \cup A'$.*

PROOF:
$\langle 1 \rangle 1.$ For all $x \in \overline{A}$, if $x \notin A$ then $x \in A'$
PROOF: From Theorem 66.
$\langle 1 \rangle 2.$ $A \subseteq \overline{A}$
PROOF: Lemma 63.
$\langle 1 \rangle 3.$ $A' \subseteq \overline{A}$
PROOF: From Theorem 66.
□

Corollary 74.1. *A set is closed if and only if it contains all its limit points.*

Proposition 75. *In an indiscrete topology, every point is a limit point of any set that has more than one point.*

PROOF: Let X be an indiscrete space. Let A be a set with more than one point and x be a point. The only neighbourhood of x is X, which must intersect A at a point other than x. □

14 T_1 Spaces

Definition 76 (T_1 Space)**.** A topological space is T_1 if and only if every singleton is closed.

Lemma 77. *A space is T_1 if and only if every finite set is closed.*

PROOF: From Lemma 49. □

Theorem 78. *In a T_1 space, a point a is a limit point of a set A if and only if every neighbourhood of a contains infinitely many points of A.*

PROOF:
$\langle 1\rangle 1$. If a is a limit point of A then every neighbourhood of a contains infinitely many points of A.
  $\langle 2\rangle 1$. ASSUME: a is a limit point of A.
  $\langle 2\rangle 2$. LET: U be a neighbourhood of a.
  $\langle 2\rangle 3$. ASSUME: for a contradiction U contains only finitely many points of A.
  $\langle 2\rangle 4$. $(U \cap A) \setminus \{a\}$ is closed.
    PROOF: By the T_1 axiom.
  $\langle 2\rangle 5$. $(U \setminus A) \cup \{a\}$ is open.
    PROOF: It is $U \setminus ((U \cap A) \setminus \{a\})$.
  $\langle 2\rangle 6$. $(U \setminus A) \cup \{a\}$ intersects A in a point other than a.
    PROOF: From $\langle 2\rangle 1$.
  $\langle 2\rangle 7$. Q.E.D.
  □
$\langle 1\rangle 2$. If every neighbourhood of a contains infinitely many points of A then a is a limit point of A.
  PROOF: Immediate from definitions.
□

(To see this does not hold in every space, see Proposition 75.)

Proposition 79. *A space is T_1 if and only if, for any two distinct points x and y, there exist neighbourhoods U of x and V of y such that $x \notin V$ and $y \notin U$.*

PROOF:
$\langle 1\rangle 1$. LET: X be a topological space.
$\langle 1\rangle 2$. If X is T_1 then, for any two distinct points x and y, there exist neighbourhoods U of x and V of y such that $x \notin V$ and $y \notin U$.
  PROOF: This holds because $\{x\}$ and $\{y\}$ are closed.
$\langle 1\rangle 3$. Suppose, for any two distinct points x and y, there exist neighbourhoods U of x and V of y such that $x \notin V$ and $y \notin U$. Then X is T_1.
  $\langle 2\rangle 1$. ASSUME: For any two distinct points x and y, there exist neighbourhoods U of x and V of y such that $x \notin V$ and $y \notin U$.
  $\langle 2\rangle 2$. LET: $a \in X$
  $\langle 2\rangle 3$. $\{a\}$ is closed.
    PROOF: For all $b \neq a$ there exists a neighbourhood U of b such that $U \subseteq X \setminus \{a\}$.
□

15 Hausdorff Spaces

Definition 80 (Hausdorff Space)**.** A topological space is *Hausdorff* if and only if, for any points x, y with $x \neq y$, there exist disjoint open sets U and V such that $x \in U$ and $y \in V$.

Theorem 81. *Every Hausdorff space is T_1.*

PROOF:
$\langle 1\rangle 1$. LET: X be a Hausdorff space.
$\langle 1\rangle 2$. LET: $b \in X$
PROVE: $\overline{\{b\}} = \{b\}$
$\langle 1\rangle 3$. ASSUME: $a \in \overline{\{b\}}$ and $a \neq b$
$\langle 1\rangle 4$. PICK disjoint neighbourhoods U of a and V of b.
$\langle 1\rangle 5$. U intersects $\{b\}$
PROOF: Theorem 66.
$\langle 1\rangle 6$. $b \in U$
$\langle 1\rangle 7$. Q.E.D.
PROOF: This contradicts the fact that U and V are disjoint ($\langle 1\rangle 4$).
□

Proposition 82. *An infinite set under the finite complement topology is T_1 but not Hausdorff.*

PROOF:
$\langle 1\rangle 1$. LET: X be an infinite set under the finite complement topology.
$\langle 1\rangle 2$. Every singleton is closed.
PROOF: By definition.
$\langle 1\rangle 3$. PICK$a, b \in X$ with $a \neq b$
$\langle 1\rangle 4$. There are no disjoint neighbourhoods U of a and V of b.
$\langle 2\rangle 1$. LET: U be a neighbourhood of a and V a neighbourhood of b.
$\langle 2\rangle 2$. $X \setminus U$ and $X \setminus V$ are finite.
$\langle 2\rangle 3$. PICK $c \in X$ that is not in $X \setminus U$ or $X \setminus V$.
$\langle 2\rangle 4$. $c \in U \cap V$
□

Proposition 83. *The product of a family of Hausdorff spaces is Hausdorff.*

PROOF:
$\langle 1\rangle 1$. LET: $\{X_i\}_{i \in I}$ be a family of Hausdorff spaces.
$\langle 1\rangle 2$. LET: $a, b \in \prod_{i \in I} X_i$ with $a \neq b$
$\langle 1\rangle 3$. PICK $i \in I$ such that $a_i \neq b_i$
$\langle 1\rangle 4$. PICK U, V disjoint open sets in X_i with $a_i \in U$ and $b_i \in V$
$\langle 1\rangle 5$. ${\pi_i}^{-1}(U)$ and ${\pi_i}^{-1}(V)$ are disjoint open sets in $\prod_{i \in I} X_i$ with $a \in {\pi_i}^{-1}(U)$ and $b \in {\pi_i}^{-1}(V)$
□

Theorem 84. *Every linearly ordered set under the order topology is Hausdorff.*

PROOF:
$\langle 1\rangle 1$. LET: X be a linearly ordered set under the order topology.
$\langle 1\rangle 2$. LET: $a, b \in X$ with $a \neq b$
$\langle 1\rangle 3$. ASSUME: w.l.o.g. $a < b$

⟨1⟩4. CASE: There exists c such that $a < c < b$
PROOF: The sets $(-\infty, c)$ and $(c, +\infty)$ are disjoint neighbourhoods of a and b respectively.
⟨1⟩5. CASE: There is no c such that $a < c < b$
PROOF: The sets $(-\infty, b)$ and $(a, +\infty)$ are disjoint neighbourhoods of a and b respectively.
□

Theorem 85. *A subspace of a Hausdorff space is Hausdorff.*

PROOF:
⟨1⟩1. LET: X be a Hausdorff space and Y a subspace of X.
⟨1⟩2. LET: $x, y \in Y$ with $x \neq y$
⟨1⟩3. PICK disjoint neighbourhoods U of x and V of y in X.
⟨1⟩4. $U \cap Y$ and $V \cap Y$ are disjoint neighbourhoods of x and y respectively in Y.
□

Proposition 86. *A space X is Hausdorff if and only if the diagonal $\Delta = \{(x, x) \mid x \in X\}$ is closed in X^2.*

PROOF:

$$
\begin{aligned}
& X \text{ is Hausdorff} \\
\Leftrightarrow & \forall x, y \in X . x \neq y \Rightarrow \exists V, W \text{ open}. x \in V \wedge y \in W \wedge V \cap W = \emptyset \\
\Leftrightarrow & \forall (x, y) \in X^2 \setminus \Delta . \exists V, W \text{ open}. (x, y) \subseteq V \times W \subseteq X^2 \setminus \Delta \\
\Leftrightarrow & \Delta \text{ is closed}
\end{aligned}
$$
□

16 Convergence

Definition 87 (Convergence)**.** Let X be a topological space. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of points in X and $l \in X$. Then the sequence $(a_n)_{n \in \mathbb{N}}$ *converges* to the *limit* l, $a_n \to l$ as $n \to \infty$, if and only if, for every neighbourhood U of l, there exists N such that, for all $n \geq N$, we have $a_n \in U$.

Theorem 88. *In a Hausdorff space, a sequence has at most one limit.*

PROOF:
⟨1⟩1. LET: X be a Hausdorff space.
⟨1⟩2. ASSUME: for a contradiction $a_n \to l$ as $n \to \infty$, $a_n \to m$ as $n \to \infty$, and $l \neq m$
⟨1⟩3. PICK disjoint neighbourhoods U of l and V of m
PROOF: By the Hausdorff axiom.
⟨1⟩4. PICK M and N such that $a_n \in U$ for $n \geq M$ and $a_n \in V$ for $n \geq N$
⟨1⟩5. $a_{\max(M,N)} \in U \cap V$
⟨1⟩6. Q.E.D.
PROOF: This contradicts the fact that U and V are disjoint (⟨1⟩3).

$\square$

To see this is not always true in spaces that are T_1 but not Hausdorff:

Proposition 89. *Let X be an infinite set under the finite complement topology. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence with all points distinct. Then for every $l \in X$ we have $a_n \to l$ as $n \to \infty$.*

Proof: Let U be any neighbourhood of l. Since $X \setminus U$ is finite, there must exist N such that, for all $n \geq N$, we have $a_n \in U$. $\square$

17 Boundary

Definition 90 (Boundary)**.** The *boundary* of a set A is the set $\partial A = \overline{A} \cap \overline{X \setminus A}$.

Proposition 91.

$$\operatorname{Int} A \cap \partial A = \emptyset$$

Proof: Since $\overline{X \setminus A} = X \setminus \operatorname{Int} A$. $\square$

Proposition 92.

$$\overline{A} = \operatorname{Int} A \cup \partial A$$

Proof:

$$\begin{aligned}
\operatorname{Int} A \cup \partial A &= \operatorname{Int} A \cup (\overline{A} \cap \overline{X \setminus A}) \\
&= (\operatorname{Int} A \cup \overline{A}) \cap (\operatorname{Int} A \cup \overline{X \setminus A}) \\
&= \overline{A} \cap X \\
&= \overline{A}
\end{aligned}$$

Proposition 93. *$\partial A = \emptyset$ if and only if A is open and closed.*

Proof: If $\partial A = \emptyset$ then $\overline{A} = \operatorname{Int} A$ by Proposition 92.

Proposition 94. *A set U is open if and only if $\partial U = \overline{U} \setminus U$.*

Proof:

$$\begin{aligned}
&\partial U = \overline{U} \setminus U \\
\Leftrightarrow &\overline{U} \setminus \operatorname{Int} U = \overline{U} \setminus U && \text{(Propositions 91, 92)} \\
\Leftrightarrow &\operatorname{Int} U = U && \square
\end{aligned}$$

18 Continuous Functions

Definition 95 (Continuous)**.** Let X and Y be topological spaces. A function $f : X \to Y$ is *continuous* if and only if, for every open set V in Y, the set $f^{-1}(V)$ is open in X.

Proposition 96. *Let X and Y be topological spaces and $f : X \to Y$. Let $\mathcal{B}$ be a basis for Y. Then f is continuous if and only if, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X.*

PROOF:
⟨1⟩1. If f is continuous then, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X.
 PROOF: Since every element of B is open (Lemma 14).
⟨1⟩2. Suppose that, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X. Then f is continuous.
 ⟨2⟩1. ASSUME: For all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X.
 ⟨2⟩2. LET: V be open in Y.
 ⟨2⟩3. PICK $\mathcal{A} \subseteq \mathcal{B}$ such that $V = \bigcup \mathcal{A}$
  PROOF: By Lemma 14.
 ⟨2⟩4. $f^{-1}(V)$ is open in X.
  PROOF:

$$\begin{aligned} f^{-1}(V) &= f^{-1}\left(\bigcup \mathcal{A}\right) \\ &= \bigcup_{B \in \mathcal{A}} f^{-1}(B) \end{aligned}$$

□

Proposition 97. *Let X and Y be topological spaces and $f : X \to Y$. Let $\mathcal{S}$ be a subbasis for Y. Then f is continuous if and only if, for all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in X.*

PROOF:
⟨1⟩1. If f is continuous then, for all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in X.
 PROOF: Since every element of S is open.
⟨1⟩2. Suppose that, for all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in X. Then f is continuous.
 ⟨2⟩1. ASSUME: For all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in X.
 ⟨2⟩2. LET: $S_1, \ldots, S_n \in \mathcal{S}$
 ⟨2⟩3. $f^{-1}(S_1 \cap \cdots \cap S_n)$ is open in A
  PROOF: Since $f^{-1}(S_1 \cap \cdots \cap S_n) = f^{-1}(S_1) \cap \cdots \cap f^{-1}(S_n)$.
 ⟨2⟩4. Q.E.D.
  PROOF: By Propositions 96 and 21.

□

Proposition 98. *Let X and Y be topological spaces and $f : X \to Y$. Let $\mathcal{S}$ be a basis for Y. Then f is continuous if and only if, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X.*

PROOF:
⟨1⟩1. If f is continuous then, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X.
 PROOF: Since every element of $\mathcal{S}$ is open.
⟨1⟩2. Suppose that, for all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X. Then f is continuous.
 ⟨2⟩1. ASSUME: For all $V \in \mathcal{S}$, we have $f^{-1}(V)$ is open in X.

⟨2⟩2. For every set B that is the finite intersection of elemets of $\mathcal{S}$, we have $f^{-1}(B)$ is open in X.

PROOF: Because $f^{-1}(V_1 \cap \cdots \cap V_n) = f^{-1}(V_1) \cap \cdots \cap f^{-1}(V_n)$.

⟨2⟩3. Q.E.D.

PROOF: From Propositions 21 and 96.

□

Definition 99 (Continuous at a Point)**.** Let X and Y be topological spaces. Let $f : X \to Y$ and $x \in X$. Then f is *continuous at* x if and only if, for every neighbourhood V of $f(x)$, there exists a neighbourhood U of x such that $f(U) \subseteq V$.

Theorem 100. *Let X and Y be topological spaces and $f : X \to Y$. Then the following are equivalent:*

1. *f is continuous.*
2. *For all $A \subseteq X$, we have $f(\overline{A}) \subseteq \overline{f(A)}$*
3. *For all $B \subseteq Y$ closed, we have $f^{-1}(B)$ is closed in X.*
4. *f is continuous at every point of X.*

PROOF:

⟨1⟩1. $1 \Rightarrow 2$

⟨2⟩1. ASSUME: f is continuous.

⟨2⟩2. LET: $A \subseteq X$

⟨2⟩3. LET: $x \in \overline{A}$

PROVE: $f(x) \in \overline{f(A)}$

⟨2⟩4. LET: V be a neighbourhood of $f(x)$

⟨2⟩5. $f^{-1}(V)$ is a neighbourhood of x

⟨2⟩6. PICK $y \in A \cap f^{-1}(V)$

PROOF: By Theorem 66.

⟨2⟩7. $f(y) \in V \cap f(A)$

⟨2⟩8. Q.E.D.

PROOF: By Theorem 66.

⟨1⟩2. $2 \Rightarrow 3$

⟨2⟩1. ASSUME: 2

⟨2⟩2. LET: B be closed in Y

⟨2⟩3. LET: $x \in \overline{f^{-1}(B)}$

PROVE: $x \in f^{-1}(B)$

⟨2⟩4. $f(x) \in B$

PROOF:

$$\begin{aligned} f(x) &\in f(\overline{f^{-1}(B)}) \\ &\subseteq \overline{f(f^{-1}(B))} && (\langle 2\rangle 1) \\ &\subseteq \overline{B} && (Proposition 68) \\ &= B \end{aligned}$$

$\langle 1\rangle 3.$ $3 \Rightarrow 1$
$\langle 2\rangle 1.$ Assume: 3
$\langle 2\rangle 2.$ Let: V be open in Y
$\langle 2\rangle 3.$ $Y \setminus V$ is closed in Y
$\langle 2\rangle 4.$ $f^{-1}(Y \setminus V)$ is closed in X
$\langle 2\rangle 5.$ $X \setminus f^{-1}(V)$ is closed in X
$\langle 2\rangle 6.$ $f^{-1}(V)$ is open in X
$\langle 1\rangle 4.$ $1 \Rightarrow 4$
Proof: For any neighbourhood V of $f(x)$, the set $U = f^{-1}(V)$ is a neighbourhood of x such that $f(U) \subseteq V$.
$\langle 1\rangle 5.$ $4 \Rightarrow 1$
$\langle 2\rangle 1.$ Assume: 4
$\langle 2\rangle 2.$ Let: V be open in Y
$\langle 2\rangle 3.$ Let: $x \in f^{-1}(V)$
$\langle 2\rangle 4.$ V is a neighbourhood of $f(x)$
$\langle 2\rangle 5.$ Pick a neighbourhood U of x such that $f(U) \subseteq V$
$\langle 2\rangle 6.$ $x \in U \subseteq f^{-1}(V)$
$\langle 2\rangle 7.$ Q.E.D.
Proof: By Lemma 10.
□

Theorem 101. *A constant function is continuous.*

Proof: Let X and Y be topological spaces. Let $b \in Y$, and let $f : X \to Y$ be the constant function with value b. For any open $V \subseteq Y$, the set $f^{-1}(V)$ is either X (if $b \in V$) or $\emptyset$ (if $b \notin V$). □

Theorem 102. *If A is a subspace of X then the inclusion $j : A \to X$ is continuous.*

Proof: For any V open in X, we have $j^{-1}(V) = V \cap A$ is open in A. □

Theorem 103. *The composite of two continuous functions is continuous.*

Proof: Let $f : X \to Y$ and $g : Y \to Z$ be continuous. For any V open in Z, we have $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is open in X. □

Theorem 104. *Let $f : X \to Y$ be a continuous function and A be a subspace of X. Then the restriction $f \restriction A : A \to Y$ is continuous.*

Proof: Let V be open in Y. Then $(f \restriction A)^{-1}(V) = f^{-1}(V) \cap A$ is open in A. □

Theorem 105. *Let $f : X \to Y$ be continuous. Let Z be a subspace of Y such that $f(X) \subseteq Z$. Then the corestriction $f : X \to Z$ is continuous.*

Proof:
$\langle 1\rangle 1.$ Let: V be open in Z.
$\langle 1\rangle 2.$ Pick U open in Y such that $V = U \cap Z$.
$\langle 1\rangle 3.$ $f^{-1}(V) = f^{-1}(U)$

⟨1⟩4. $f^{-1}(V)$ is open in X.
□

Theorem 106. *Let $f : X \to Y$ be continuous. Let Z be a space such that Y is a subspace of Z. Then the expansion $f : X \to Z$ is continuous.*

Proof: Let V be open in Z. Then $f^{-1}(V) = f^{-1}(V \cap Y)$ is open in X. □

Theorem 107. *Let X and Y be topological spaces. Let $f : X \to Y$. Suppose $\mathcal{U}$ is a set of open sets in X such that $X = \bigcup \mathcal{U}$ and, for all $U \in \mathcal{U}$, we have $f \restriction U : U \to Y$ is continuous. Then f is continuous.*

Proof:
⟨1⟩1. Let: V be open in Y
⟨1⟩2. $f^{-1}(V) = \bigcup_{U \in \mathcal{U}} (f \restriction U)^{-1}(V)$
⟨1⟩3. For all $U \in \mathcal{U}$, we have $(f \restriction U)^{-1}(V)$ is open in U.
⟨1⟩4. For all $U \in \mathcal{U}$, we have $(f \restriction U)^{-1}(V)$ is open in X.
Proof: Lemma 39.
□

Theorem 108. *Let A be a topological space and $\{X_i\}_{i \in I}$ be a family of topological spaces. Let $f : A \to \prod_{i \in I} X_i$ be a function. If $\pi_i \circ f$ is continuous for all $i \in I$ then f is continuous.*

Proof:
⟨1⟩1. Let: $i \in I$ and U be open in X_i
⟨1⟩2. $f^{-1}({\pi_i}^{-1}(U))$ is open in A
⟨1⟩3. Q.E.D.
Proof: Proposition 97.
□

Proposition 109. *Let $f : \mathbb{R} \to \mathbb{R}$ and $x \in \mathbb{R}$. Then f is continuous at x if and only if, for all $\epsilon > 0$, there exists $\delta > 0$ such that. for all $y \in \mathbb{R}$, if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$.*

Proof:
⟨1⟩1. If f is continuous at x then, for all $\epsilon > 0$, there exists $\delta > 0$ such that. for all $y \in \mathbb{R}$, if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$
⟨2⟩1. Assume: f is continuous.
⟨2⟩2. Let: $\epsilon > 0$
⟨2⟩3. $f^{-1}((f(x) - \epsilon, f(x) + \epsilon))$ is open.
⟨2⟩4. Pick a, b such that $x \in (a, b) \subseteq f^{-1}((f(x) - \epsilon, f(x) + \epsilon))$
⟨2⟩5. Let: $\delta = \min(x - a, b - x)$
⟨2⟩6. Let: $y \in \mathbb{R}$ with $|x - y| < \delta$
⟨2⟩7. $y \in (a, b)$
⟨2⟩8. $f(y) \in (f(x) - \epsilon, f(x) + \epsilon)$
⟨2⟩9. $|f(x) - f(y)| < \epsilon$
⟨1⟩2. Suppose, for all $\epsilon > 0$, there exists $\delta > 0$ such that. for all $y \in \mathbb{R}$, if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$. Then f is continuous at x.

⟨2⟩1. ASSUME: For all $\epsilon > 0$, there exists $\delta > 0$ such that. for all $y \in \mathbb{R}$, if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$
⟨2⟩2. LET: V be a neighbourhood of $f(x)$
⟨2⟩3. PICK a, b such that $f(x) \in (a, b) \subseteq V$
⟨2⟩4. LET: $\epsilon = \min(f(x) - a, b - f(x))$
⟨2⟩5. PICK $\delta > 0$ such that, for all $y \in \mathbb{R}$, if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$
⟨2⟩6. LET: $U = (x - \delta, x + \delta)$
⟨2⟩7. $x \in U \subseteq f^{-1}(V)$
□

Proposition 110. *Let X and X' be the same set X under two topologies $\mathcal{T}$ and $\mathcal{T}'$. Let $i : X \to X'$ be the identity function. Then i is continuous if and only if $\mathcal{T}' \subseteq \mathcal{T}$.*

PROOF: Immediate from definitions. □

Proposition 111. *Let $f : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$. Then f is continuous on the right at a if and only if f is continuous at a as a function $\mathbb{R}_l \to \mathbb{R}$.*

PROOF:
⟨1⟩1. If f is continuous on the right at a then f is continuous at a as a function $\mathbb{R}_l \to \mathbb{R}$.
⟨2⟩1. ASSUME: f is continuous on the right at a.
⟨2⟩2. LET: V be a neighbourhood of $f(a)$
⟨2⟩3. PICK b, c such that $f(a) \in (b, c) \subseteq V$.
⟨2⟩4. LET: $\epsilon = \min(c - f(a), f(a) - b)$
⟨2⟩5. PICK $\delta > 0$ such that, for all x, if $a < x < a + \delta$ then $|f(x) - f(a)| < \epsilon$
⟨2⟩6. LET: $U = [a, a + \delta)$
⟨2⟩7. $f(U) \subseteq V$
⟨1⟩2. If f is continuous at a as a function $\mathbb{R}_l \to \mathbb{R}$ then f is continuous on the right at a.
⟨2⟩1. ASSUME: f is continuous at a as a function $\mathbb{R}_l \to \mathbb{R}$
⟨2⟩2. LET: $\epsilon > 0$
⟨2⟩3. PICK a neighbourhood U of a such that $f(U) \subseteq (f(a) - \epsilon, f(a) + \epsilon)$
⟨2⟩4. PICK b, c such that $a \in [b, c) \subset U$
⟨2⟩5. LET: $\delta = c - a$
⟨2⟩6. For all x, if $a < x < a + \delta$ then $|f(x) - f(a)| < \epsilon$
□

Lemma 112. *Let X be a topological space. Let Y be a linearly ordered set in the order topology. Let $f, g : X \to Y$ be continuous. Then $C = \{x \in X \mid f(x) \leq g(x)\}$ is closed.*

PROOF:
⟨1⟩1. LET: $x \in X \setminus C$
⟨1⟩2. $f(x) > g(x)$
PROVE: There exists a neighbourhood U of x such that $U \subseteq X \setminus C$
⟨1⟩3. CASE: There exists y such that $g(x) < y < f(x)$

PROOF: Take $U = g^{-1}((-\infty, y)) \cup f^{-1}(y, +\infty)$.
$\langle 1 \rangle 4$. CASE: There is no y such that $g(x) < y < f(x)$
PROOF: Take $U = g^{-1}((-\infty, f(x))) \cup f^{-1}(g(x), +\infty)$.
□

Lemma 113. *Let $f : X \to Y$. Let Z be an open subspace of X and $a \in Z$. If $f \upharpoonright Z$ is continuous at a then f is continuous at a.*

PROOF:
$\langle 1 \rangle 1$. LET: V be a neighbourhood of $f(x)$
$\langle 1 \rangle 2$. PICK a neighbourhood W of x in Z such that $f(W) \subseteq V$
$\langle 1 \rangle 3$. W is a neighbourhood of x in X such that $f(W) \subseteq V$
PROOF: Lemma 39.
□

Proposition 114. *Let $f : A \to B$ and $g : C \to D$ be continuous. Define $f \times g : A \times C \to B \times D$ by*

$$(f \times g)(a, c) = (f(a), g(c)) \ .$$

Then $f \times g$ is continuous.

PROOF: $\pi_1 \circ (f \times g) = f \circ \pi_1$ and $\pi_2 \circ (f \times g) = g \circ \pi_2$ are continuous by Theorem 103. The result follows by Theorem 108.

Proposition 115. *Let X be a topological space. Let Y a Hausdorff space. Let $A \subseteq X$. Let $f, g : \overline{A} \to Y$ be continuous. If f and g agree on A then $f = g$.*

PROOF:
$\langle 1 \rangle 1$. LET: $x \in \overline{A}$
$\langle 1 \rangle 2$. ASSUME: $f(x) \neq g(x)$
$\langle 1 \rangle 3$. PICK disjoint neighbourhoods V of $f(x)$ and W of $g(x)$.
$\langle 1 \rangle 4$. PICK $y \in f^{-1}(V) \cap g^{-1}(W) \cap A$
PROOF: Since $f^{-1}(V) \cap g^{-1}(W)$ is a neighbourhood of x and hence intersects A.
$\langle 1 \rangle 5$. $f(y) = g(y) \in V \cap W$
$\langle 1 \rangle 6$. Q.E.D.
PROOF: This contradicts the fact that V and W are disjoint ($\langle 1 \rangle 3$).
□

Proposition 116. *Let X and Y be topological spaces and $f : X \to Y$ be continuous. If $a_n \to l$ as $n \to \infty$ in X then $f(a_n) \to f(l)$ as $n \to \infty$.*

PROOF:
$\langle 1 \rangle 1$. LET: V be a neighbourhood of $f(l)$
$\langle 1 \rangle 2$. PICK a neighbourhood U of l such that $f(U) \subseteq V$
$\langle 1 \rangle 3$. PICK N such that, for all $n \geq N$, we have $a_n \in U$
$\langle 1 \rangle 4$. For all $n \geq N$ we have $f(n) \in V$
□

Proposition 117. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let (a_n) be a sequence in $\prod_{i \in I} X_i$ and $l \in \prod_{i \in I} X_i$. Then $a_n \to l$ as $n \to \infty$ if and only if, for all $i \in I$, we have $\pi_i(a_n) \to \pi_i(l)$ as $n \to \infty$.*

PROOF:
$\langle 1 \rangle 1$. If $a_n \to l$ as $n \to \infty$ then, for all $i \in I$, we have $\pi_i(a_n) \to \pi_i(l)$ as $n \to \infty$
  PROOF: Proposition 116.
$\langle 1 \rangle 2$. If, for all $i \in I$, we have $\pi_i(a_n) \to \pi_i(l)$ as $n \to \infty$, then $a_n \to l$ as $n \to \infty$
  $\langle 2 \rangle 1$. ASSUME: For all $i \in I$, we have $\pi_i(a_n) \to \pi_i(l)$ as $n \to \infty$
  $\langle 2 \rangle 2$. LET: V be a neighbourhood of l
  $\langle 2 \rangle 3$. PICK open sets U_i in X_i such that $l \in \prod_{i \in I} U_i \subseteq V$ and $U_i = X_i$ for all i except $i = i_1, \ldots, i_k$
  $\langle 2 \rangle 4$. For $j = 1, \ldots, k$, PICK N_j such that, for all $n \geq N_j$, we have $\pi_{i_j}(a_n) \in U_{i_j}$
  $\langle 2 \rangle 5$. LET: $N = \max(N_1, \ldots, N_k)$
  $\langle 2 \rangle 6$. For all $n \geq N$ we have $a_n \in V$
$\square$

19 Homeomorphisms

Definition 118 (Homeomorphism)**.** Let X and Y be topological spaces. A *Homeomorphism* f between X and Y, $f : X \cong Y$, is a bijection $f : X \to Y$ such that both f and f^{-1} are continuous.

Lemma 119. *Let X and Y be topological spaces and $f : X \to Y$ a bijection. Then the following are equivalent:*

1. *f is a homeomorphism.*
2. *f is continuous and an open map.*
3. *For any $U \subseteq X$, we have U is open if and only if $f(U)$ is open.*

PROOF: Immediate from definitions. $\square$

Proposition 120. *Let X and X' be the same set X under two topologies $\mathcal{T}$ and $\mathcal{T}'$. Let $i : X \to X'$ be the identity function. Then i is a homeomorphism if and only if $\mathcal{T} = \mathcal{T}'$.*

PROOF: Immediate from definitions. $\square$

Definition 121 (Topological Property)**.** Let P be a property of topological spaces. Then P is a *topological* property if and only if, for any spaces X and Y, if P holds of X and $X \cong Y$ then P holds of Y.

Definition 122 (Topological Imbedding)**.** Let X and Y be topological spaces and $f : X \to Y$. Then f is a *topological imbedding* if and only if the corestriction $f : X \to f(X)$ is a homeomorphism.

Proposition 123. *Let X and Y be topological spaces and $a \in X$. The function $i : Y \to X \times Y$ that maps y to (a, y) is an imbedding.*

PROOF:
$\langle 1 \rangle 1.$ i is injective
$\langle 1 \rangle 2.$ i is continuous.
  PROOF: For U open in X and V open in Y, we have $i^{-1}(U \times V)$ is V if $a \in U$, and $\emptyset$ if $a \notin U$.
$\langle 1 \rangle 3.$ $i : Y \to i(Y)$ is an open map.
  PROOF: For V open in Y we have $i(V) = (X \times V) \cap i(Y)$.
$\square$

20 Locally Finite Sets

Definition 124 (Locally Finite)**.** Let X be a topological space and $\{A_\alpha\}$ a family of subsets of X. Then $\mathcal{A}$ is *locally finite* if and only if every point in X has a neighbourhood that intersects A_α for only finitely many α.

Theorem 125 (Pasting Lemma)**.** *Let X and Y be topological spaces and $f : X \to Y$. Let $\{A_\alpha\}$ be a locally finite family of closed subsets of X that cover X. Suppose $f \restriction A_\alpha$ is continuous for all α. Then f is continuous.*

PROOF:
$\langle 1 \rangle 1.$ Let X and Y be topological spaces and $f : X \to Y$. Let A and B be closed subsets of X such that $X = A \cup B$. Suppose $f \restriction A$ and $f \restriction B$ are continuous. Then f is continuous.
  $\langle 2 \rangle 1.$ LET: $C \subseteq Y$ be closed.
  $\langle 2 \rangle 2.$ $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$
  $\langle 2 \rangle 3.$ $f^{-1}(C)$ and $g^{-1}(C)$ are closed in X.
    PROOF: Theorems 100 and 51.
  $\langle 2 \rangle 4.$ $h^{-1}(C)$ is closed in X.
    PROOF: Lemma 49.
  $\langle 2 \rangle 5.$ Q.E.D.
    PROOF: Theorem 100.
$\langle 1 \rangle 2.$ Let X and Y be topological spaces and $f : X \to Y$. Let $\{A_\alpha\}$ be a finite family of closed subsets of X that cover X. Suppose $f \restriction A_\alpha$ is continuous for all α. Then f is continuous.
  PROOF: From $\langle 1 \rangle 1$ by induction.
$\langle 1 \rangle 3.$ Let X and Y be topological spaces and $f : X \to Y$. Let $\{A_\alpha\}$ be a locally finite family of closed subsets of X that cover X. Suppose $f \restriction A_\alpha$ is continuous for all α. Then f is continuous.
  $\langle 2 \rangle 1.$ LET: $x \in X$
    PROVE: f is continuous at x
  $\langle 2 \rangle 2.$ PICK a neighbourhood U of x that intersects A_α for only finitely many α.
  $\langle 2 \rangle 3.$ $f \restriction U$ is continuous

Proof: By ⟨1⟩2.

⟨2⟩4. Q.E.D.

Proof: Lemma 113.

□

The following example shows that we cannot remove the assumption of local finiteness.

Example 126. Define $f : [-1, 1] \to \mathbb{R}$ by: $f(x) = 1$ if $x < -1$, $f(x) = 0$ if $x > 1$. Let $C_n = [-1, -1/n]$ for $n \geq 1$, and $D = [0, 1]$. Then $[-1, 1] = \bigcup_{n=1}^{\infty} C_n \cup D$ and f is continuous on each C_n and each D, but f is not continuous on $[-1, 1]$.

Proposition 127. *Let X be a topological space. Let Y be a linearly ordered set in the order topology. Let $f, g : X \to Y$ be continuous. Define $h : X \to Y$ by $h(x) = \min(f(x), g(x))$. Then h is continuous.*

Proof: By the Pasting Lemma applied to $\{x \in X \mid f(x) \leq g(x)\}$ and $\{x \in X \mid g(x) \leq f(x)\}$, which are closed by Lemma 112.

21 Continuous in Each Variable Separately

Definition 128 (Continuous in Each Variable Separately)**.** Let $F : X \times Y \to Z$. Then F is *continuous in each variable separately* if and only if:

- for every $a \in X$ the function $\lambda y \in Y.F(a, y)$ is continuous;
- for every $b \in Y$ the function $\lambda x \in X.F(x, b)$ is continuous.

Proposition 129. *Let $F : X \times Y \to Z$. If F is continuous then F is continuous in each variable separately.*

Proof: For $a \in X$, the function $\lambda y \in Y.F(a, y)$ is $F \circ i$ where $i : Y \to X \times Y$ maps y to (a, y). We have i is continuous by Proposition 123, hence $F \circ i$ is continuous by Theorem 103.
Similarly for $\lambda x \in X.F(x, b)$ for $b \in Y$. □

Example 130. Define $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$F(x, y) = \begin{cases} xy/(x^2 + y^2) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Then F is continuous in each variable separately but not continuous.

22 The Box Topology

Definition 131 (Box Topology)**.** Let $\{A_i\}_{i \in I}$ be a family of topological spaces. The *box topology* on $\prod_{i \in I} A_i$ is the topology generated by the set of all sets of the form $\prod_{i \in I} U_i$ where $\{U_i\}_{i \in I}$ is a family such that each U_i is open in A_i.

This is a basis since it covers $\prod_{i \in I} A_i$ and is closed under intersection.

Proposition 132. *The box topology is finer than the product topology.*

PROOF: From Proposition 32. $\square$

Corollary 132.1. *If A_i is closed in X_i for all $i \in I$ then $\prod_{i \in I} A_i$ is closed in $\prod_{i \in I} X_i$ under the box topology.*

PROOF: From Proposition 53.

Proposition 133 (AC)**.** *Let $\{A_i\}_{i \in I}$ be a family of topological spaces. For $i \in I$, let $\mathcal{B}_i$ be a basis for the topology on A_i. Then $\mathcal{B} = \{\prod_{i \in I} B_i \mid \forall i \in I.B_i \in \mathcal{B}_i\}$ is a basis for the box topology on $\prod_{i \in I} A_i$.*

PROOF:
⟨1⟩1. Every set of the form $\prod_{i \in I} B_i$ is open.
⟨1⟩2. For every point $a \in \prod_{i \in I} A_i$ and every open set U with $a \in U$, there exists $B \in \mathcal{B}$ such that $a \in B \subseteq U$.
  ⟨2⟩1. LET: U be open and $a \in U$
  ⟨2⟩2. PICK a family $\{U_i\}_{i \in I}$ such that each U_i is open in A_i and $a \in \prod_{i \in I} U_i \subseteq U$.
  ⟨2⟩3. For $i \in I$, PICK $B_i \in \mathcal{B}_i$ such that $a_i \in B_i \subseteq U_i$
    PROOF: Using the Axiom of Choice.
  ⟨2⟩4. $a \in \prod_{i \in I} B_i \subseteq U$
⟨1⟩3. Q.E.D.
  PROOF: Lemma 15.
$\square$

Theorem 134. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let A_i be a subspace of X_i for all $i \in I$. Give $\prod_{i \in I} X_i$ the box topology. Then the box topology on $\prod_{i \in I} A_i$ is the same as the topology it inherits as a subspace of $\prod_{i \in I} X_i$.*

PROOF: The box topology is generated by the basis

$$\begin{aligned} &\{\prod_{i \in I} U_i \mid \forall i \in I, U_i \text{ open in } A_i\} \\ =&\{\prod_{i \in I} (V_i \cap A_i) \mid \forall i \in I, V_i \text{ open in } X_i\} \\ =&\{\prod_{i \in I} V_i \mid \forall i \in I, V_i \text{ open in } X_i\} \cap \prod_{i \in I} A_i \end{aligned}$$

and this is a basis for the subspace topology by Lemma 37. $\square$

Proposition 135. *Let $\{X_i\}_{i \in I}$ be a family of Hausdorff spaces. Then $\prod_{i \in I} X_i$ under the box topology is Hausdorff.*

PROOF:
⟨1⟩1. LET: $\{X_i\}_{i \in I}$ be a family of Hausdorff spaces.
⟨1⟩2. LET: $a, b \in \prod_{i \in I} X_i$ with $a \neq b$

⟨1⟩3. PICK $i \in I$ such that $a_i \neq b_i$
⟨1⟩4. PICK U, V disjoint open sets in X_i with $a_i \in U$ and $b_i \in V$
⟨1⟩5. ${\pi_i}^{-1}(U)$ and ${\pi_i}^{-1}(V)$ are disjoint open sets in $\prod_{i \in I} X_i$ with $a \in {\pi_i}^{-1}(U)$ and $b \in {\pi_i}^{-1}(V)$
□

Proposition 136 (AC)**.** *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Give $\prod_{i \in I} X_i$ the box topology. Let $A_i \subseteq X_i$ for all $i \in I$. Then*

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i} \enspace .$$

PROOF:
⟨1⟩1. $\overline{\prod_{i \in I} A_i} \subseteq \prod_{i \in I} \overline{A_i}$
⟨2⟩1. For all $i \in I$ we have $A_i \subseteq \overline{A_i}$
PROOF: Lemma 63.
⟨2⟩2. $\prod_{i \in I} A_i \subseteq \prod_{i \in I} \overline{A_i}$
⟨2⟩3. Q.E.D.
PROOF: Since $\prod_{i \in I} A_i$ is closed by Corollary 132.1.
⟨1⟩2. $\prod_{i \in I} \overline{A_i} \subseteq \overline{\prod_{i \in I} A_i}$
⟨2⟩1. LET: $x \in \prod_{i \in I} \overline{A_i}$
⟨2⟩2. LET: U be a neighbourhood of x
⟨2⟩3. PICK V_i open in X_i such that $x \in \prod_{i \in I} V_i \subseteq U$
⟨2⟩4. For $i \in I$, pick $a_i \in V_i \cap A_i$
PROOF: By Theorem 66 and ⟨2⟩1 using the Axiom of Choice.
⟨2⟩5. U intersects $\prod_{i \in I} A_i$
⟨2⟩6. Q.E.D.
PROOF: $a \in U \cap \prod_{i \in I} A_i$.
□

The following example shows that Theorem 108 fails in the box topology.

Example 137. Define $f : \mathbb{R} \to \mathbb{R}^\omega$ by $f(t) = (t, t, \ldots)$. Then $\pi_n \circ f = \mathrm{id}_\mathbb{R}$ is continuous for all n. But f is not continuous when $\mathbb{R}^\omega$ is given the box topology because the inverse image of

$$(-1, 1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times \cdots$$

is $\{0\}$ which is not open.

The following example shows that Proposition 117 fails in the box topology.

Example 138. Give $\mathbb{R}^\omega$ the box topology. Let $a_n = (1/n, 1/n, \ldots)$ for $n \geq 1$ and $l = (0, 0, \ldots)$. Then $\pi_i(a_n) \to \pi_i(l)$ as $n \to \infty$ for all i, but $a_n \not\to l$ as $n \to \infty$ since the open set

$$(-1, 1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times \cdots$$

contains l but does not contain any a_n.

Example 139. The set $\mathbb{R}^\infty$ is closed in $\mathbb{R}^\omega$ under the box topology. For let (a_n) be any sequence not in $\mathbb{R}^\infty$. Let U_n be an open interval around a_n that does not contain 0 if $a_n \neq 0$, and $U_n = \mathbb{R}$ if $a_n = 0$. Then $\prod_{n \geq 0} U_n$ is a neighbourhood of (a_n) that does not intersect $\mathbb{R}^\infty$.

23 The Metric Topology

Definition 140 (Metric)**.** Let X be a set. A *metric* on X is a function $d : X^2 \to \mathbb{R}$ such that:

1. For all $x, y \in X$, $d(x, y) \geq 0$
2. For all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$
3. For all $x, y \in X$, $d(x, y) = d(y, x)$
4. (*Triangle Inequality*) For all $x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$

We call $d(x, y)$ the *distance* between x and y.

Definition 141 (Open Ball)**.** Let X be a metric space. Let $a \in X$ and $\epsilon > 0$. The *open ball* with *centre* a and *radius* ϵ is

$$B(a, \epsilon) = \{x \in X \mid d(a, x) < \epsilon\} \ .$$

Definition 142 (Metric Topology)**.** Let X be a metric space. The *metric topology* on X is the topology generated by the basis consisting of all the open balls.

We prove this is a basis for a topology.

PROOF:
⟨1⟩1. For every point a, there exists a ball B such that $a \in B$
  PROOF: We have $a \in B(a, 1)$.
⟨1⟩2. For any balls B_1, B_2 and point $a \in B_1 \cap B_2$, there exists a ball B_3 such that $a \in B_3 \subseteq B_1 \cap B_2$
  ⟨2⟩1. LET: $B_1 = B(c_1, \epsilon_1)$ and $B_2 = B(c_2, \epsilon_2)$
  ⟨2⟩2. LET: $\delta = \min(\epsilon_1 - d(c_1, a), \epsilon_2 - d(c_2, a))$
    PROVE: $B(a, \delta) \subseteq B_1 \cap B_2$
  ⟨2⟩3. LET: $x \in B(a, \delta)$
  ⟨2⟩4. $x \in B_1$
    PROOF:

$$\begin{aligned} d(x, c_1) &= d(x, a) + d(a, c_1) \\ &< \delta + d(a, c_1) \\ &\leq \epsilon_1 \end{aligned}$$

  ⟨2⟩5. $x \in B_2$
    PROOF: Similar.

□

Proposition 143. *Let X be a metric space and $U \subseteq X$. Then U is open if and only if, for every $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$.*

PROOF:
$\langle 1 \rangle 1$. If U is open then, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$.
  $\langle 2 \rangle 1$. ASSUME: U is open.
  $\langle 2 \rangle 2$. LET: $x \in U$
  $\langle 2 \rangle 3$. PICK $a \in X$ and $\delta > 0$ such that $x \in B(a, \delta) \subseteq U$
  $\langle 2 \rangle 4$. LET: $\epsilon = \delta - d(a, x)$
    PROVE: $B(x, \epsilon) \subseteq U$
  $\langle 2 \rangle 5$. LET: $y \in B(x, \epsilon)$
  $\langle 2 \rangle 6$. $d(y, a) < \delta$
    PROOF:

$$\begin{aligned} d(y, a) &\leq d(a, x) + d(x, y) \\ &< \delta + d(x, y) \\ &= \epsilon \end{aligned}$$

  $\langle 2 \rangle 7$. $y \in U$
$\langle 1 \rangle 2$. If, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$, then U is open.
  PROOF: Immediate from definitions.
□

Definition 144 (Discrete Metric)**.** Let X be a set. The *discrete metric* on X is defined by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Proposition 145. *The discrete metric induces the discrete topology.*

PROOF: For any (open) set U and point $a \in U$, we have $a \in B(a, 1) \subseteq U$. □

Definition 146 (Standard Metric on $\mathbb{R}$)**.** The *standard metric* on $\mathbb{R}$ is defined by $d(x, y) = |x - y|$.

Proposition 147. *The standard metric on $\mathbb{R}$ induces the standard topology on $\mathbb{R}$.*

PROOF:
$\langle 1 \rangle 1$. Every open ball is open in the standard topology on $\mathbb{R}$.
  PROOF: $B(a, \epsilon) = (a - \epsilon, a + \epsilon)$
$\langle 1 \rangle 2$. For every open set U and point $a \in U$, there exists $\epsilon > 0$ such that $B(a, \epsilon) \subseteq U$
  $\langle 2 \rangle 1$. LET: U be an open set and $a \in U$
  $\langle 2 \rangle 2$. PICK an open interval b, c such that $a \in (b, c) \subseteq U$
  $\langle 2 \rangle 3$. LET: $\epsilon = \min(a - b, c - a)$
  $\langle 2 \rangle 4$. $B(a, \epsilon) \subseteq U$
□

Definition 148 (Metrizable)**.** A topological space X is *metrizable* if and only if there exists a metric on X that induces the topology.

Definition 149 (Bounded)**.** Let X be a metric space and $A \subseteq X$. Then A is *bounded* if and only if there exists M such that, for all $x, y \in A$, we have $d(x, y) \leq M$.

Definition 150 (Diameter)**.** Let X be a metric space and $A \subseteq X$. The *diameter* of A is

$$\operatorname{diam} A = \sup_{x,y \in A} d(x, y) \ .$$

Definition 151 (Standard Bounded Metric)**.** Let d be a metric on X. The *standard bounded metric* corresponding to d is the metric $\overline{d}$ defined by

$$\overline{d}(x, y) = \min(d(x, y), 1) \ .$$

We prove this is a metric.

PROOF:
⟨1⟩1. $\overline{d}(x, y) \geq 0$
  PROOF: Since $d(x, y) \geq 0$
⟨1⟩2. $\overline{d}(x, y) = 0$ if and only if $x = y$
  PROOF: $\overline{d}(x, y) = 0$ if and only if $d(x, y) = 0$ if and only if $x = y$
⟨1⟩3. $\overline{d}(x, y) = \overline{d}(y, x)$
  PROOF: Since $d(x, y) = d(y, x)$
⟨1⟩4. $\overline{d}(x, z) \leq \overline{d}(x, y) + \overline{d}(y, z)$
  PROOF:

$$\begin{aligned}
\overline{d}(x, y) + \overline{d}(y, z) &= \min(d(x, y), 1) + \min(d(y, z), 1) \\
&= \min(d(x, y) + d(y, z), d(x, y) + 1, d(y, z) + 1, 2) \\
&\geq \min(d(x, z), 1) \\
&= \overline{d}(x, z)
\end{aligned}$$

□

Lemma 152. *In any metric space X, the set $\mathcal{B} = \{B(a, \epsilon) \mid a \in X, \epsilon < 1\}$ is a basis for the metric topology.*

PROOF:
⟨1⟩1. Every element of $\mathcal{B}$ is open.
  PROOF: From Lemma 14.
⟨1⟩2. For every open set U and point $a \in U$, there exists $B \in \mathcal{B}$ such that $a \in B \subseteq U$
  ⟨2⟩1. LET: U be an open set and $a \in U$
  ⟨2⟩2. PICK$\epsilon > 0$ such that $B(a, \epsilon) \subseteq U$
  ⟨2⟩3. $B(a, \min(\epsilon, 1/2)) \subseteq U$
⟨1⟩3. Q.E.D.
  PROOF: Lemma 15.
□

Proposition 153. *Let d be a metric on the set X. Then the standard bounded metric $\overline{d}$ induces the same metric as d.*

PROOF: This follows from Lemma 152 since the open balls with radius < 1 are the same under both metrics. □

Lemma 154. *Let d and d' be two metrics on the same set X. Let $\mathcal{T}$ and $\mathcal{T}'$ be the topologies they induce. Then $\mathcal{T} \subseteq \mathcal{T}'$ if and only if, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that*

$$B_{d'}(x, \delta) \subseteq B_d(x, \epsilon) \ .$$

PROOF:
$\langle 1 \rangle 1$. If $\mathcal{T} \subseteq \mathcal{T}'$ then, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$
  PROOF: From Proposition 143 since $x \in B_d(x, \epsilon) \in \mathcal{T}'$.
$\langle 1 \rangle 2$. If, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$, then $\mathcal{T} \subseteq \mathcal{T}'$
  $\langle 2 \rangle 1$. ASSUME: For all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$
  $\langle 2 \rangle 2$. LET: $U \in \mathcal{T}$
  $\langle 2 \rangle 3$. For all $x \in U$ there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq U$.
    $\langle 3 \rangle 1$. LET: $x \in U$
    $\langle 3 \rangle 2$. PICK $\epsilon > 0$ such that $B_d(x, \epsilon) \subseteq U$
      PROOF: Proposition 143
    $\langle 3 \rangle 3$. PICK $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$
      PROOF: By $\langle 2 \rangle 1$
    $\langle 3 \rangle 4$. $B_{d'}(x, \delta) \subseteq U$
  $\langle 2 \rangle 4$. $U \in \mathcal{T}'$
    PROOF: Proposition 143.
□

Proposition 155. $\mathbb{R}^2$ *under the dictionary order topology is metrizable.*

PROOF: Define $d : \mathbb{R}^2 \to \mathbb{R}$ by

$$d((x, y), (x, z)) = \max(|y - z|, 1)$$

$$d((x, y), (x', y')) = 1 \qquad \text{if } x \neq x'$$ □

$\langle 1 \rangle 1$. $x \in \bigcap_{i=1}^{N} {\pi_i}^{-1}() \subseteq B_D(a, \epsilon)$

Proposition 156. *Let $d : X^2 \to \mathbb{R}$ be a metric on X. Then the metric topology on X is the coarsest topology such that d is continuous.*

PROOF:
$\langle 1 \rangle 1$. d is continuous in each variable separately.
  $\langle 2 \rangle 1$. LET: $a \in X$ and $d_a : X \to \mathbb{R}$ be the function $d(a, -)$
  $\langle 2 \rangle 2$. LET: $b \in X$ and $\epsilon > 0$
  $\langle 2 \rangle 3$. For all $x \in X$, if $d(b, x) < \epsilon$ then $|d(a, b) - d(a, x)| < \epsilon$
$\langle 1 \rangle 2$. If $\mathcal{T}$ is any topology under which d is continuous then $\mathcal{T}$ is finer than the metric topology.
  PROOF: Since $B(a, \epsilon) = {d_a}^{-1}((-\infty, \epsilon))$
□

24 Real Linear Algebra

Definition 157 (Square Metric)**.** The *square metric* ρ on $\mathbb{R}^n$ is defined by

$$\rho(\vec{x}, \vec{y}) = \max(|x_1 - y_1|, \ldots, |x_n - y_n|)$$

We prove this is a metric.

PROOF:
$\langle 1\rangle 1.$ $\rho(\vec{x}, \vec{y}) \geq 0$
PROOF: Immediate from definition.
$\langle 1\rangle 2.$ $\rho(\vec{x}, \vec{y}) = 0$ if and only if $\vec{x} = \vec{y}$
PROOF: Immediate from definition.
$\langle 1\rangle 3.$ $\rho(\vec{x}, \vec{y}) = \rho(\vec{y}, \vec{x})$
PROOF: Immediate from definition.
$\langle 1\rangle 4.$ $\rho(\vec{x}, \vec{z}) \leq \rho(\vec{x}, \vec{y}) + \rho(\vec{y}, \vec{z})$
PROOF: Since $|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$.
□

Proposition 158. *The square metric induces the standard topology on* $\mathbb{R}^n$*.*

PROOF:
$\langle 1\rangle 1.$ For every $a \in X$ and $\epsilon > 0$, we have $B_\rho(a, \epsilon)$ is open in the standard product topology.
PROOF:
$$B_\rho(a, \epsilon) = (a_1 - \epsilon, a_1 + \epsilon) \times \cdots \times (a_n - \epsilon, a_n + \epsilon)$$
$\langle 1\rangle 2.$ For any open sets $U_1, \ldots, U_n$ in $\mathbb{R}$, we have $U_1 \times \cdots \times U_n$ is open in the square metric topology.
$\langle 2\rangle 1.$ LET: $\vec{a} \in U_1 \times \cdots \times U_n$
$\langle 2\rangle 2.$ For $i = 1, \ldots, n$, PICK $\epsilon_i > 0$ such that $(a_i - \epsilon_i, a_i + \epsilon_i) \subseteq U_i$
$\langle 2\rangle 3.$ LET: $\epsilon = \min(\epsilon_1, \ldots, \epsilon_n)$
$\langle 2\rangle 4.$ $B_\rho(\vec{a}, \epsilon) \subseteq U$
□

Definition 159. Given $\vec{x}, \vec{y} \in \mathbb{R}^n$, define the *sum* $\vec{x} + \vec{y}$ by

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n) \ .$$

Definition 160. Given $\lambda \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^n$, define the *scalar product* $\lambda\vec{x} \in \mathbb{R}^n$ by

$$\lambda(x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n)$$

Definition 161 (Inner Product)**.** Given $\vec{x}, \vec{y} \in \mathbb{R}^n$, define the *inner product* $\vec{x} \cdot \vec{y} \in \mathbb{R}$ by

$$(x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) = x_1 y_1 + \cdots + x_n y_n \ .$$

We write $\vec{x}^2$ for $\vec{x} \cdot \vec{x}$.

Definition 162 (Norm)**.** Let $n \geq 1$. The *norm* on $\mathbb{R}^n$ is the function $\| \ \| : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\|(x_1, \ldots, x_n)\| = \sqrt{x_1^2 + \cdots + x_n^2}$$

Lemma 163.

$$\|\vec{x}\|^2 = \vec{x}^2$$

PROOF: Immediate from definitions. ∎

Lemma 164.

$$\vec{x} \cdot (\vec{y} + \vec{z}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z}$$

PROOF: Each is equal to $(x_1 y_1 + x_1 z_1, \ldots, x_n y_n + x_n z_n)$. ∎

Lemma 165.

$$|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \|\vec{y}\|$$

PROOF:
⟨1⟩1. ASSUME: $\vec{x} \neq \vec{0} \neq \vec{y}$
  PROOF: Otherwise both sides are 0.
⟨1⟩2. LET: $a = 1/\|\vec{x}\|$
⟨1⟩3. LET: $b = 1/\|\vec{y}\|$
⟨1⟩4. $(a\vec{x} + b\vec{y})^2 \geq 0$ and $(a\vec{x} - b\vec{y})^2 \geq 0$
⟨1⟩5. $a^2\|\vec{x}\|^2 + 2ab\vec{x} \cdot \vec{y} + b^2\|\vec{y}\|^2 \geq 0$ and $a^2\|\vec{x}\|^2 - 2ab\vec{x} \cdot \vec{y} + b^2\|\vec{y}\|^2 \geq 0$
⟨1⟩6. $2ab\vec{x} \cdot \vec{y} + 2 \geq 0$ and $-2ab\vec{x} \cdot \vec{y} + 2 \geq 0$
⟨1⟩7. $\vec{x} \cdot \vec{y} \geq -1/ab$ and $\vec{x} \cdot \vec{y} \leq 1/ab$
⟨1⟩8. $\vec{x} \cdot \vec{y} \geq -\|\vec{x}\|\|\vec{y}\|$ and $\vec{x} \cdot \vec{y} \leq \|\vec{x}\|\|\vec{y}\|$
∎

Lemma 166 (Triangle Inequality)**.**

$$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$$

PROOF:

$$\begin{aligned}
\|\vec{x} + \vec{y}\|^2 &= \|\vec{x}\|^2 + 2\vec{x} \cdot \vec{y} + \|\vec{y}\|^2 \\
&\leq \|\vec{x}\|^2 + 2\|\vec{x}\|\|\vec{y}\| + \|\vec{y}\|^2 && \text{(Lemma 165)} \\
&= (\|\vec{x}\| + \|\vec{y}\|)^2 && \blacksquare
\end{aligned}$$

Definition 167 (Euclidean Metric)**.** Let $n \geq 1$. The *Euclidean metric* on $\mathbb{R}^n$ is defined by

$$d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\| \ .$$

We prove this is a metric.

⟨1⟩1. $d(\vec{x}, \vec{y}) \geq 0$
  PROOF: Immediate from definition.
⟨1⟩2. $d(\vec{x}, \vec{y}) = 0$ if and only if $\vec{x} = \vec{y}$
  PROOF: $d(\vec{x}, \vec{y}) = 0$ if and only if $\vec{x} - \vec{y} = \vec{0}$.

⟨1⟩3. $d(\vec{x}, \vec{y}) = d(\vec{y}, \vec{x})$
PROOF: Immediate from definition.
⟨1⟩4. $d(\vec{x}, \vec{z}) \leq d(\vec{x}, \vec{y}) + d(\vec{y}, \vec{z})$
PROOF:

$$\begin{aligned} \|\vec{x} - \vec{z}\| &= \|(\vec{x} - \vec{y}) + (\vec{y} - \vec{z})\| \\ &\leq \|\vec{x} - \vec{y}\| + \|\vec{y} - \vec{z}\| && \text{(Lemma 166)} \end{aligned}$$

□

Proposition 168. *The Euclidean metric induces the standard topology on* $\mathbb{R}^n$.

PROOF:
⟨1⟩1. LET: ρ be the square metric.
⟨1⟩2. For all $\vec{a} \in \mathbb{R}^n$ and $\epsilon > 0$, we have $B_d(\vec{a}, \epsilon) \subseteq B_\rho(\vec{a}, \epsilon)$
⟨2⟩1. LET: $\vec{x} \in B_d(\vec{a}, \epsilon)$
⟨2⟩2. $\sqrt{(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2} < \epsilon$
⟨2⟩3. $(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2 < \epsilon^2$
⟨2⟩4. For all i we have $(x_i - a_i)^2 < \epsilon^2$
⟨2⟩5. For all i we have $|x_i - a_i| < \epsilon$
⟨2⟩6. $\rho(\vec{x}, \vec{a}) < \epsilon$
⟨1⟩3. For all $\vec{a} \in \mathbb{R}^n$ and $\epsilon > 0$, we have $B_\rho(\vec{a}, \epsilon/\sqrt{n}) \subseteq B_d(\vec{a}, \epsilon)$
⟨2⟩1. LET: $\vec{x} \in B_\rho(\vec{a}, \epsilon/\sqrt{n})$
⟨2⟩2. $\rho(\vec{x}, \vec{a}) < \epsilon/\sqrt{n}$
⟨2⟩3. For all i we have $|x_i - x_a| < \epsilon/\sqrt{n}$
⟨2⟩4. For all i we have $(x_i - x_a)^2 < \epsilon^2/n$
⟨2⟩5. $d(\vec{x}, \vec{a}) < \epsilon$
⟨1⟩4. Q.E.D.
PROOF: By Lemma 154.

□

Proposition 169. *The space* $\mathbb{R}^\omega$ *is metrizable.*

PROOF:
⟨1⟩1. LET: D be the metric on $\mathbb{R}^\omega$ defined by $D(x, y) = \sup_i(\overline{d}(x_i, y_i)/i)$ where $\overline{d}$ is the standard bounded metric.
⟨2⟩1. $D(x, y) \geq 0$
⟨2⟩2. $D(x, y) = 0$ if and only if $x = y$
⟨2⟩3. $D(x, y) = D(y, x)$
⟨2⟩4. $D(x, z) \leq D(x, y) + D(y, z)$
PROOF:

$$\begin{aligned} D(x, z) &= \sup_i \frac{\overline{d}(x_i, z_i)}{i} \\ &\leq \sup_i \frac{\overline{d}(x_i, y_i) + \overline{d}(y_i, z_i)}{i} \\ &\leq \sup_i \frac{\overline{d}(x_i, y_i)}{i} + \sup_i \frac{\overline{d}(y_i, z_i)}{i} \\ &= D(x, y) + D(y, z) \end{aligned}$$

⟨1⟩2. Every open ball $B_D(a, \epsilon)$ is open in the product topology.
  ⟨2⟩1. Pick N such that $1/\epsilon < N$
  ⟨2⟩2. $B_D(a, \epsilon) = \prod_{i=1}^{\infty} U_i$ where $U_i = (a_i - i\epsilon, a_i + i\epsilon)$ if $i \leq N$, and $U_i = \mathbb{R}$ if $i > N$
⟨1⟩3. For any open set U and $a \in U$, there exists $\epsilon > 0$ such that $B(a, \epsilon) \subseteq U$.
  ⟨2⟩1. Let: $n \geq 1$, V be an open set in $\mathbb{R}$ and $a \in {\pi_n}^{-1}(V)$
  ⟨2⟩2. Pick $\epsilon > 0$ such that $(a - \epsilon, a + \epsilon) \subseteq V$
  ⟨2⟩3. $B(a, \epsilon/n) \subseteq {\pi_n}^{-1}(V)$
□

25 The Uniform Topology

Definition 170 (Uniform Metric)**.** Let J be a set. The *uniform metric* $\overline{\rho}$ on $\mathbb{R}^J$ is defined by

$$\overline{\rho}(a, b) = \sup_{j \in J} \overline{d}(a_j, b_j)$$

where $\overline{d}$ is the standard bounded metric on $\mathbb{R}$.

The *uniform topology* on $\mathbb{R}^J$ is the topology induced by the uniform metric.

We prove this is a metric.

Proof:
⟨1⟩1. $\overline{\rho}(a, b) \geq 0$
  Proof: Immediate from definitions.
⟨1⟩2. $\overline{\rho}(a, b) = 0$ if and only if $a = b$
  Proof: Immediate from definitions.
⟨1⟩3. $\overline{\rho}(a, b) = \overline{\rho}(b, a)$
  Proof: Immediate from definitions.
⟨1⟩4. $\overline{\rho}(a, c) \leq \overline{\rho}(a, b) + \overline{\rho}(b, c)$
  Proof:

$$\begin{aligned}
\overline{\rho}(a, c) &= \sup_{j \in J} \overline{d}(a_j, c_j) \\
&\leq \sup_{j \in J} (\overline{d}(a_j, b_j) + \overline{d}(b_j, c_j)) \\
&\leq \sup_{j \in J} \overline{d}(a_j, b_j) + \sup_{j \in J} \overline{d}(b_j, c_j) \\
&= \overline{\rho}(a, b) + \overline{\rho}(b, c)
\end{aligned}$$

□

Proposition 171. *The uniform topology on $\mathbb{R}^J$ is finer than the product topology.*

Proof:
⟨1⟩1. Let: $j \in J$ and U be open in $\mathbb{R}$
  Prove: ${\pi_j}^{-1}(U)$ is open in the uniform topology.
⟨1⟩2. Let: $a \in {\pi_j}^{-1}(U)$
⟨1⟩3. Pick $\epsilon > 0$ such that $(a_j - \epsilon, a_j + \epsilon) \subseteq U$

⟨1⟩4. $B_{\overline{\rho}}(a, \epsilon) \subseteq {\pi_j}^{-1}(U)$
□

Proposition 172. *The uniform topology on $\mathbb{R}^J$ is coarser than the box topology.*

PROOF:
⟨1⟩1. LET: $a \in \mathbb{R}^J$ and $\epsilon > 0$
PROVE: $B(a, \epsilon)$ is open in the box topology.
⟨1⟩2. LET: $b \in B(a, \epsilon)$
⟨1⟩3. For $j \in J$ we have $|a_j - b_j| < \epsilon$
⟨1⟩4. For $j \in J$,
LET: $\delta_j = (\epsilon - |a_j - b_j|)/2$
⟨1⟩5. $\prod_{j \in J}(b_j - \delta_j, b_j + \delta_j) \subseteq B(a, \epsilon)$
□

Proposition 173. *The uniform topology on $\mathbb{R}^J$ is strictly finer than the product topology if and only if J is infinite.*

PROOF:
⟨1⟩1. If J is finite then the uniform and product topologies coincide.
PROOF: The uniform, box and product topologies are all the same.
⟨1⟩2. If J is infinite then the uniform and product topologies are different.
PROOF: The set $B(\vec{0}, 1)$ is open in the uniform topology but not the product topology.
□

Proposition 174 (DC)**.** *The uniform topology on $\mathbb{R}^J$ is strictly coarser than the box topology if and only if J is infinite.*

PROOF:
⟨1⟩1. If J is finite then the uniform and box topologies coincide.
PROOF: The uniform, box and product topologies are all the same.
⟨1⟩2. If J is infinite then the uniform and box topologies are different.
PROOF: Pick an ω-sequence $(j_1, j_2, \ldots)$ in J. Let $U = \prod_{j \in J} U_j$ where $U_{j_i} = (-1/i, 1/i)$ and $U_j = (-1, 1)$ for all other j. Then $\vec{0} \in U$ but there is no $\epsilon > 0$ such that $B(\vec{0}, \epsilon) \subseteq U$.
□