ABSTRACT. The Jones polynomial and the Kauffman bracket are constructed, and their relation with knot and link theory is described. The quantum groups and tangle functor formalisms for understanding these invariants and their descendents are given. The quantum group $U_q(sl_2)$, which gives rise to the Jones polynomial, is constructed explicitly. The 3-manifold invariants and the axiomatic topological quantum field theories which arise from these link invariants at certain values of the parameter are constructed.

LINKS, QUANTUM GROUPS AND TQFT'S

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Introduction

In studying a class of mathematical objects, such as knots, one usually begins by developing ideas and machinery to understand specific features of them. When invariants are discovered, they arise out of such understanding and generally give information about those features. Thus they are often immediately useful for proving theorems, even if they are difficult to compute.

This is not the situation we find ourselves in with the invariants of knots and 3-manifolds which have appeared since the Jones polynomial. We have a wealth of invariants, all readily computable, but standing decidedly outside the traditions of knot and 3-manifold theory. They lack a geometric interpretation, and consequently have been of almost no use in answering questions one might have asked before their creation. In effect, we are left looking for the branch of mathematics from which these should have come organically.

In fact, we have an answer of sorts. In [Wit89b], Witten gave a heuristic definition of the Jones polynomial in terms of a topological quantum field theory, following the outline of a program proposed by Atiyah [Ati88]. Specifically, he considered a knot in a 3-manifold and a connection A on some principal G-bundle, with G a simple Lie group. The Chern-Simons functional associates a number CS(A) to A, but it is well-defined only up to an integer, so the quantity $\exp(2\pi ikCS(A))$ is well-defined. Also, the holonomy of the connection around the knot is an element of G well-defined up to conjugation, so the trace of it with respect to a given representation is well-defined. Multiplying these two gives a number depending on the knot, the manifold, the representation and the connection. The magic comes when we average over all connections and all principal bundles: Of course, this makes no sense, since there is

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no apparent measure on the infinite-dimensional space of connections. But proceeding heuristically, such an average should depend only on the manifold, representation and the isotopy type of the knot. Witten argued using a close correspondence with conformal field theory, that when the manifold is S^3 and the representation the fundamental one, this invariant had combinatorial properties that forced it to be the analogue of the Jones polynomial for the given group (Specifically, the Jones polynomial at certain values of the parameter). Needless to say, a long physics tradition of very successful heuristic reasoning along these lines suggested to Witten that this ill-defined average should make sense in this case.

The fundamental problem of the field, then, is to develop this nonrigorous but beautiful geometric interpretation of the new invariants into a rigorous one that reasonably captures its flavor. This is important for at least two reasons.

First, such an interpretation of the invariants is probably necessary for their application to topological questions. While the invariants have yet to prove themselves in this regard, it is to be hoped that as part of a well-developed theory they could go a long way towards unlocking the mysteries of low-dimensional topology. This theory seems most likely to arise out of something like Witten's construction, which relates the invariants explicitly to gauge theory and other geometry.

Second, Chern-Simons field theory is a field theory which is clearly nontrivial but which admits an exact combinatorial solution. In this sense, interpreting Witten's work rigorously is just part of a large endeavor within mathematical physics to understand mathematically the whole of quantum field theory. Such an understanding seems far more tractable for the Chern-Simons theory, with its combinatorial, finite-dimensional expression, than for genuinely physical theories. From this point of view topology may be seen as a laboratory for a particularly simple kind of physics, the understanding of which may point us towards a rigorous foundation for the more complex and physically interesting theories.

Work towards this goal proceeds along two fronts. On the one hand are efforts to understand the geometry and physics of the Chern-Simons field theory, most notably recent efforts to do perturbation theory in this context [AS92, AS94, Kon]. On the other are combinatorial, algebraic, and topological efforts to understand the invariants themselves, especially with an eye towards structures suggested by the physics. This is the subject of the present article.

We begin in Section 1 with a discussion of the Jones polynomial, or more precisely the Kauffman bracket form of it. We mention other invariants, but throughout our focus will be on this example as the easiest. Section 2 first gives a functorial framework for viewing these invariants, constructing the functor explicitly for the Kauffman bracket, and then shows such a functorial setup arises naturally from a certain algebraic structure: a ribbon Hopf algebra. Section 3 discusses quantum groups, the family of ribbon Hopf algebras associated to each Lie algebra. We construct the quantum group associated to sl_2 explicitly, and show that with the fundamental rep-

resentation it gives the Kauffman bracket as its invariant.

The last four sections give the construction of the topological quantum field theories. These bear further comment. Witten's construction obviously gives a 3-manifold invariant as well, by considering an arbitrary manifold with the empty link. This formed some of the inspiration for Reshetikhin and Turaev's construction of the 3-manifold invariants [RT91] using the algebraic machinery of quantum groups. But Atiyah observed that this averaging process (called path integration by physicists), if taken at its word, implied some strong statements about how the invariants behave under cutting and pasting. These statements were formulated by Atiyah into his axioms for a topological quantum field theory, or TQFT [Ati89, Ati90a]. It was shown in [Wal] and [Tur94] that the 3-manifold invariants satisfy these axioms, although the key ideas of this proof already appeared in [RT91], and in a less rigorous form in [Wit89a, Wit89b]. One of the principal aims of this paper is to give a simple proof of this fact, which is the best rigorous connection we have between these invariants and actual quantum field theory.

Section 4 discusses the properties of the quantum groups at roots of unity, summarized as their being *modular Hopf algebras*, which allow one to construct TQFT's from them. Section 5 gives a categorical formulation of TQFT's. Section 6 sketches a purely combinatorial description of the relevant category, that of biframed 3-dimensional cobordisms, in terms of surgery on links. This allows us in Section 7 to construct the TQFT out of the link invariants we have already constructed.

This paper is intended to introduce and invite a large mathematical audience to this field. The focus is on giving the flavor and illustrating some of the power of these simple ideas. This is attempted by proving an important result with a minimum of machinery, rather than by surveying the whole of the field, or giving a full account of the machinery. Consequently many interesting areas go unmentioned or barely mentioned. By the same token, the references emphasize pointing the interested reader to good introductory accounts, rather than a thorough assignment of credit. I apologize for any omissions, and point the reader to those same introductory works, many of which have excellent bibliographies, for the complete story.

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1. Links and Their Invariants

The study of knots and links begins with some physics which, while a little eccentric sounding to modern ears, and decidedly wrong, bears a striking resemblance to the physics where this story ends. In 1867 Lord Kelvin proposed that atoms were knotted vortices of ether, and molecules were linked atoms [Kel67]. Efforts began to solve the basic problem of knot theory: When are two knots the same? Early efforts to approach

this by careful tabulating and naive searching for invariants failed completely. Modern topology has proved more successful. The approach of focusing exclusively on the knot complement and almost exclusively on its fundamental group has, by dint of great effort, produced a complete but thoroughly impractical algorithm for determining if two prime knots (knots which cannot be cut into two smaller knots) are the same up to orientation of the knot and space [Hem92]. The story for links is more complicated, and not as well understood.

The Jones polynomial, with its descendants, grew from an entirely different branch of mathematics. Its origins bear telling as a study in serendipity, although we will only sketch Jones' construction, which is buried in the construction in Section 2. Jones, in the course of proving an important result about the ways in which certain algebras of operators sit inside each other [Jon83], constructed an algebra with a trace on it. He then noticed that this algebra gave a representation of the braid group (the mapping-class group of the plane with finitely many points removed). Now by Markov's theorem [BZ85], links can be represented by elements of the braid group, with two braids giving the same link exactly when they can be connected by conjugation and another move, called stabilization. Any trace is invariant under conjugation, and this particular trace could easily be normalized so as to be invariant under stabilization. Thus presenting the link as a braid, representing the braid in the algebra, and taking the trace gives an invariant of the link. Since the algebra and the trace depend on a parameter t, so does the invariant, and it turns out the invariant is essentially a Laurent polynomial in t: the celebrated Jones polynomial.

This invariant almost certainly does not come from classical knot theory: It distinguishes links with diffeomorphic complement and detects mirror images for example. It is eminently computable: Although the algorithm is roughly exponential in the number of crossings, and the problem is known to be #P-hard [JVW90], a clever high school student can compute it easily. The collection of these invariants is a fairly good distinguisher of knots, though it does not distinguish all knots, and it is not even known whether it can distinguish a knot from the unknot [Gar]. Maddeningly, these invariants have no geometric interpretation (at least, no rigorous one), and consequently they have provided few purely topological results (chiefly the Tait conjectures, about so-called alternating knots). With such mysteries before us, it is time to do some mathematics.

A link is a smooth embedding of several copies of S^1 into oriented S^3 . A knot is a link with one component. Links are equivalent if there is an orientation-preserving diffeomorphism of S^3 taking one to the other. This notion is equivalent to the corresponding PL notion, and to the corresponding topological notion if we restrict to 'tame' knots [BZ85]. An oriented link has an orientation on each component, and a framed link comes equipped with a nonzero section of the normal bundle, also up to positive diffeomorphism. We will draw links as in Figure 1, with only transverse-double-point crossings. Oriented links will be drawn with arrows indicating orienta-

tion, and framed links will be drawn so that the framing lies in the plane (thus in Figure 1, B represents a different framed knot from A, because the twist cannot be undone). The unknot is the knot which bounds a disk, and has an obvious preferred framing (shown in A). This is all the knot theory we will need to define the Jones polynomial, using a construction due to Kauffman [Kau87].



Figure 1. Some knots and links

Let \mathcal{L} be a projection of a framed, unoriented link L. The Kauffman bracket of \mathcal{L} , $\langle \mathcal{L} \rangle$, is an element of $\mathbb{Z}[A, A^{-1}]$, with A an indeterminate, computed by the following skein relations:

$$\langle \phi \rangle = 1$$

$$\langle \diamondsuit \rangle = A \langle \diamondsuit \rangle + A^{-1} \langle \diamondsuit \rangle$$

(3)
$$\langle \bigcirc \rangle = (-A^2 - A^{-2}) \langle \bigcirc \rangle$$

where ϕ is the link with no components. Equation (2), for example, says that any time you can find three different link projections which look exactly the same except in a small disk, where they look as shown in the equation, then their brackets satisfy this equation. Of course, this means if you happen to know the brackets of the two projections on the right side, this tells you the bracket of the left side. Equation (3) is interpreted similarly, and gives the effect of removing an unlinked unknot from a link.

Theorem 1. [Kau87] The bracket of every projection is uniquely determined by these three conditions, and one can compute it explicitly from them.

Proof. By induction on the number of crossings. \Box

For example,

$$\langle \bigcirc \rangle = A \langle \bigcirc \bigcirc \rangle + A^{-1} \langle \bigcirc \bigcirc \rangle$$

= $A(-A^2 - A^{-2})^2 + A^{-1}(-A^2 - A^{-2}) = -A^3(-A^2 - A^{-2}).$

More generally,

$$\langle \langle \bigcirc \rangle \rangle = A \langle \langle \bigcirc \rangle \rangle + A^{-1} \langle \langle \bigcirc \rangle \rangle = -A^3 \langle \langle \bigcirc \rangle \rangle,$$

i.e., adding a twist multiplies any bracket by $-A^3$. This is actually a positive twist, corresponding to a clockwise rotation of the framing. A negative twist would multiply the bracket by $-A^{-3}$. The reader can compute that the Kauffman bracket of the right-handed trefoil (C in Figure 1) is $A^7 + A^3 + A^{-1} - A^{-9}$.

The point is that the Kauffman bracket does not depend on the projection, but only on the unoriented framed link. This remarkable fact follows from simple calculations, and the framed Reidemeister's theorem, which says that two projections correspond to the same framed unoriented links if and only if they can be connected by a sequence of a certain finite set of local moves which are roughly Moves I, II, III of Theorem 3 in Section 2 (see [BZ85] for Reidemeister's theorem, the framed version follows from [Tra83]).

Theorem 2. [Kau87] The Kauffman bracket takes the same value on two projections of the same framed link L. Henceforth refer to this value as $\langle L \rangle$. \square

We would like a link invariant, rather than a framed link invariant, if for no other reason than that that is what has been studied since Kelvin (although a case can be made that framed links are a more 'natural' object to study: see e.g., Kirby's theorem in Section 5). We have already seen that adding a positive or negative twist to a link changes the bracket by $-A^{\pm 3}$. It also changes the framing by one full turn. The idea is to correct the bracket so that this move leaves it unchanged, without losing the framed link invariance. To do this, consider an *oriented* link projection, and label each crossing as positive or negative according to whether it looks like the first or second crossing appearing in Equation (6): I.e., according to whether one strand rotates clockwise or counterclockwise around the other. Then the *writhe* of an oriented link projection is the number of positive crossings minus the number of negative crossings. For example, the writhe of E in Figure 1 is +3, that of F, its mirror image, is -3. This is an invariant of oriented framed links, which increases by one when a full twist is added. Thus if $\mathcal L$ is a projection of an oriented link L, the quantity

$$(4) (-A)^{-3\operatorname{writhe}(\mathcal{L})}\langle \mathcal{L}\rangle,$$

where $\langle \mathcal{L} \rangle$ is understood to be the bracket of \mathcal{L} with the orientation removed, depends only on the link L.

Even better, notice that in the bracket of the trefoil, exponents increased in steps of four. This is true in general. In fact, a pretty induction on the number of crossing shows that the quantity (4) is an element of $\mathbb{Z}[A^4, A^{-4}]$ if there are an even number of components and is A^2 times such an element if the number of components is odd. This motivates

Definition 1. The quantity (4), with $t^{-1/4}$ substituted for A, is called the Jones polynomial of L, or $V_L(t)$. It sends oriented, unframed links to polynomials in t and t^{-1} (times $t^{1/2}$ if there are an odd number of components) and satisfies the skein relations

$$(5) V_{\emptyset}(t) = 1$$

(6)
$$t^{-1}V_{(i)}(t) - tV_{(i)}(t) = (t^{1/2} - t^{-1/2})V_{(i)}(t)$$

which uniquely determine it.

One can use Equations (5) and (6) to compute the Jones polynomial more efficiently than via the Kauffman bracket, but it is a bit trickier (see, e.g., [LM88]). We should note that the Jones polynomial as it often appears is our Jones polynomial divided by $-t^{1/2} - t^{-1/2}$, so that its value on the unknot is 1.

It is worth doing a small computation to see what we have. From the calculations above, the Jones polynomial of the right-handed trefoil, E in Figure 1, is $t^{1/2}(t^4-t^2-t-1)$. But from the definition, the Jones polynomial of the mirror image of a link (a projection of which is gotten by reversing every crossing of a projection of the link) has the same Jones polynomial, except with t replaced by t^{-1} . Thus the left-handed trefoil, F in Figure 1, has invariant $t^{-1/2}(t^{-4}-t^{-2}-t^{-1}-1)$. Since these are different, the two trefoils are not equivalent! Already we can see there is more afoot here than simply the fundamental group of the complement, the major tool of classical knot theory, which must be the same for both. In fact, it was not until 1914 that Dehn proved the inequivalence of these two.

The Jones polynomial was quickly generalized by a host of people [FYH⁺85, PT87] to a similar, two-variable polynomial, now known as the HOMFLY polynomial (the name, formed from their initials, was unfortunately coined before the work of Przyticki and Traczyk was well known), and by Kauffman to another two-variable polynomial, the Kauffman polynomial, with very similar properties [Kau, Kau87].

Connections with statistical mechanics were quickly noticed [Jon85, Kau88]. Many exactly-solvable statistical mechanical systems had recently been constructed [Bax82] from solutions of the quantum Yang-Baxter Equation (9). Meanwhile, work in inverse scattering theory had found solutions to the classical Yang-Baxter equation associated to Lie algebras [BD82] and efforts to quantize these were just bearing fruit [Dri83, Jim85]. This remarkable confluence resulted in the construction of quantum groups and associated link invariants, which we sketch in the next two sections.

2. Ribbon Hopf Algebras and Tangle Invariants

The modern view of algebraic topology is as being about certain functors from the geometric category of topological spaces and homotopy classes of maps, to the algebraic category of groups and homomorphisms. The tremendous success of this program might suggest considering other geometric categories and hoping to find functors from them into algebraic categories. This approach may be taken as the guiding philosophy behind the invariants discussed in this article. In fact, the principal difference between these modern functorial invariants and the classical ones is that here the topological spaces themselves form the morphisms.

In this section we construct the appropriate geometric category, the category of framed tangles [Yet88, Tur89], and show that the Kauffman bracket arises out of a functor from this category to the category of vector spaces. We then show how to

construct such a functor from the representation theory of a ribbon Hopf algebra, which we will define. In the next section, we construct the underlying ribbon Hopf algebra, and discuss how to construct a large family of similar ribbon Hopf algebras. Despite the first two paragraphs, this whole section should be accessible to someone knowing little or no category theory.

A tangle is the image of a smooth embedding of a union of circles and intervals into the cylinder $D \times I$, where D is the unit disk in \mathbb{C} . The intersection of a tangle with the boundary of the cylinder is required to be transverse, to lie in $X \times (\{0\} \cup \{1\})$, where X is the x-axis in D, and to be exactly the image of the endpoints of the intervals. Tangles are considered up to smooth isotopy of the cylinder leaving $X \times \{0\}$ and $X \times \{1\}$ invariant. Oriented and framed tangles are defined by analogy, but we require the framing to point in the positive x direction at the top and bottom. We define $\mathrm{dom}(T)$ to be the number of intersection points with $X \times \{0\}$, and $\mathrm{codom}(T)$ to be the number of intersection points with $X \times \{1\}$ (for oriented tangles, we must also keep track of the orientations of the points). See Figure 2 for an examples.



FIGURE 2. Composition and tensor product of tangles

Tangles allow two different multiplications. The composition of tangles T_1T_2 is the tangle formed by putting T_1 on top of T_2 , isotoping so that the boundaries match up smoothly. This is only well-defined if $dom(T_1) = codom(T_2)$. The tensor product $T_1 \otimes T_2$ is formed by putting them next to each other, and treating them as a single tangle. Both multiplications are associative. See Figure 2 for examples.

In the language of category theory, let \mathfrak{T} be the category whose objects are nonnegative integers, and whose morphisms from n to m are the tangles T with $\mathrm{dom}(T)=n$ and $\mathrm{codom}(T)=m$. Composition is of course composition of tangles. The tensor product makes this category into what is known as a monoidal category. The analogy to keep in mind is the category \mathfrak{V} , whose objects are finite-dimensional vector spaces and whose morphisms are linear maps. Here the monoidal structure is given by tensor product of linear maps and vector spaces.

In fact every tangle (hence every link, which is a tangle from the object 0 to itself) can be constructed out of a handful of generating tangles. Further, we can say exactly which such combinations give the same tangle. This gives a completely algebraic presentation of this monoidal category. The proof of this theorem is not difficult: It involves projections of tangles, and paths between projections.

Theorem 3. [Tur88, Tur89, FY89] Every unoriented, framed tangle is the composition of tensor products of the five tangles $\langle \langle \rangle, \langle \rangle, \langle \rangle, \langle \rangle, \langle \rangle, \langle \rangle, \langle \rangle$, and two such products correspond to the same tangle if and only if they can be connected by a sequence of the following moves

where T and S are arbitrary tangles. The same is true of oriented, framed tangles, if each generator and each relation above is written with every possible consistent orientation. The same is also true in either case for unframed tangles, if one adds in relation I that both equal the identity tangle. \square

Much of this structure is quite natural algebraically. Moves VI and VII are just restatements of axioms of a monoidal category. More interestingly, the morphism $\langle \cdot \rangle$, which permutes two objects, looks like the canonical vector space map from $V \otimes W$ to $W \otimes V$, except that its square is not one. The vector space map is an example of a symmetry, a concept arising naturally in category theory (see Section 5). Instead, $\langle \cdot \rangle$ and X give the category of tangles a *braiding*, a weakening of symmetry that comes up in studying 2-categories. In fact, the subcategory of \mathfrak{T} generated by X, X and Xis the free braided, monoidal category generated by one object. This subcategory is none other than the braid group mentioned in Section 1 (actually, the braid groupoid: The endomorphisms of the object n form the mapping class group of the plane with n points removed). The morphisms \bigcap and \bigcup are like the canonical vector space map from $V \otimes V^*$ to $\mathbb C$ and its dual, and in particular give the category a rigidity or dual structure, also much-studied in category theory. Leaving off Move I entirely (this corresponds to Kauffman's 'regular isotopy' of links) and orienting the tangles we have the free rigid, braided, monoidal category on one object. There are hints that Move I is also natural in the context of n-categories [FY89, JS93, KV].

We are looking for a monoidal functor from \mathfrak{T} to \mathfrak{V} : That is, an assignment of linear maps to the five generating tangles, such that the seven relations of the previous theorem hold true, with composition interpreted as composition of linear maps and tensor product interpreted as tensor product of linear maps. Call such a map a *tangle functor*. In particular, a tangle functor would send links to linear maps from $\mathbb C$ to itself ($\mathbb C$, like the object 0, is the identity object for tensor multiplication), which are just complex numbers. Thus we get a numerical link invariant.

Hypothesizing that the Kauffman bracket arises from a tangle functor, we see that

Equations (2) and (3) are just two more equations that our five linear maps must satisfy. With patience and the additional hint that all of these equations can be satisfied by operators on a two-dimensional space, you would undoubtedly come up with something like the following [Tur89]: Let V be \mathbb{C}^2 , and define maps

$$\begin{array}{c} \left\langle \ \right\rangle \right\rangle : V \otimes V \to V \otimes V \qquad \left\langle \ \right\rangle \right\rangle : V \otimes V \to V \otimes V \\ \left\langle \ \right\rangle \left\langle \ \right\rangle : V \otimes V \to \mathbb{C} \qquad \left\langle \ \right\rangle \right\rangle : V \to V \qquad \left\langle \ \right\rangle \left\langle \ \right\rangle : \mathbb{C} \to V \otimes V \end{array}$$

by the following, writing the basis for $V \otimes V$ as $(1,0) \otimes (1,0)$, $(1,0) \otimes (0,1)$, $(0,1) \otimes (1,0)$, and $(0,1) \otimes (0,1)$:

$$(8) \qquad \langle \stackrel{\dots}{\cap} \rangle = \begin{bmatrix} 0 & A & -A^{-1} & 0 \end{bmatrix} \qquad \langle \stackrel{\dots}{\mid} \rangle = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \langle \stackrel{\cup}{\dots} \rangle = \begin{bmatrix} 0 & -A & A^{-1} & 0 \end{bmatrix}^T.$$

The reader can check all the relations with nothing but elementary linear algebra (Moves III and V are best checked by applying Equation (2) first). In particular this confirms that the Kauffman bracket is a framed link invariant. Using the above the Jones polynomial can also be written as a tangle functor.

We now give a set of structures and axioms on an algebra \mathcal{A} which guarantee that the category whose objects are finite-dimensional representations and whose morphisms are intertwiners (maps between representations commuting with the action of \mathcal{A}) is naturally the range of such a functor. Let us proceed heuristically.

Since sequences of points are sent to a tensor product of representations, the algebra must act on such tensor products. Likewise, it must act on \mathbb{C} , the image of the object 0. This requires a *bialgebra*. That is, an algebra \mathcal{A} , with homomorphisms $\Delta: \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ (the coproduct) and $\epsilon: \mathcal{A} \to \mathbb{C}$ (the counit) satisfying

$$(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta$$
$$(\epsilon \otimes 1)\Delta = 1 = (1 \otimes \epsilon)\Delta.$$

The reason for the terminology is that the adjoint of Δ together with ϵ give a multiplication and identity on the dual space, making it an algebra. A bialgebra \mathcal{A} acts on \mathbb{C} by $\epsilon: \mathcal{A} \to \mathbb{C} = \operatorname{End}(\mathbb{C})$, and if $\rho: \mathcal{A} \to \operatorname{End}(V)$ and $\tau: \mathcal{A} \to \operatorname{End}(W)$ are representations, then $(\rho \otimes \tau)\Delta: \mathcal{A} \to \operatorname{End}(V) \otimes \operatorname{End}(W)$ is a representation on $V \otimes W$.

Now restricting attention to oriented tangles, recall that Ω and U, with various orientations, were analogous to duality in vector spaces. That is, if a boundary point with one orientation corresponds to a representation V, a point with the other orientation should correspond to V^* , and for example Ω should correspond to the canonical

map from $V^* \otimes V$ to \mathbb{C} . In order for V^* to be a representation, our bialgebra \mathcal{A} must be a *Hopf algebra*: It should have an antihomomorphism $S: \mathcal{A} \to \mathcal{A}$ (antipode) satisfying

$$m(S \otimes 1)\Delta = m(1 \otimes S)\Delta = 1_{\mathcal{A}}\epsilon$$

where m is the multiplication map. If $\rho: \mathcal{A} \to \operatorname{End}(V)$ is a representation, then the dual representation $\rho^*: \mathcal{A} \to \operatorname{End}(V^*)$ is defined by $\rho^*(a)v^* = v^* \circ \rho(S(a))$. The antipode axiom assures that there is a canonical intertwiner from $V \otimes V^*$ to the trivial representation. Hopf algebras have a long and distinguished history [Swe69, LR87] outside the scope of this article.

The last piece of information is the braiding \mathbb{K} , which should be an intertwiner from $V \otimes W$ to $W \otimes V$. It should not be the flip map, $\sigma_{VW} : v \otimes w \mapsto w \otimes v$, because then we would have $\mathbb{K} = \mathbb{K}$ and our tangle functor would be very uninteresting! The flip map will be an intertwiner whenever \mathcal{A} is cocommutative, that is, when all $\Delta(a)$ are symmetric. Thus we will expect interesting link invariants only from noncocommutative Hopf algebras. More precisely, if $\Delta'(a) = \sigma_{\mathcal{A}\mathcal{A}}(\Delta(a))$, then we expect $\Delta \neq \Delta'$. They shouldn't be too different, though, since we want \mathbb{K} to behave somewhat like the flip map. It turns out the right notion is for Δ and Δ' to be connected by an inner automorphism. That is, define a quasitriangular Hopf algebra \mathcal{A} [Dri85] to be a Hopf algebra \mathcal{A} together with an invertible element $R \in \mathcal{A} \otimes \mathcal{A}$, satisfying

$$\Delta'(a) = R\Delta(a)R^{-1}$$
$$(\Delta \otimes 1)R = R_{13}R_{23} \qquad (1 \otimes \Delta)R = R_{13}R_{12}$$

where if $R = \sum_i a_i \otimes b_i$, then $R_{13} = \sum_i a_i \otimes 1 \otimes b_i$, $R_{23} = \sum_i 1 \otimes a_i \otimes b_i$, etc. From these equations one easily gets the Yang-Baxter equation

$$(9) R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

which will ultimately give us Move III.

Notice cocommutative Hopf algebras are always trivially quasitrangular, with $R = 1 \otimes 1$. Drinfeld has shown [CP94, 4.2D][Dri87] that one can combine any Hopf algebra \mathcal{A} with its dual to get a larger Hopf algebra, the 'quantum double,' which is quasitriangular (the ones we are interested in are almost of this form). Thus there are lots of them around. They also have a rich and interesting algebraic structure. For example, if we define $u = \sum_i S(b_i)a_i$, where $R = \sum_i a_i \otimes b_i$, then

$$S^2(a) = uau^{-1}.$$

This implies for example that V is isomorphic as a representation to its double dual V^{**} , though not by the obvious map.

The representation theory of a quasitriangular Hopf algebra will be a rigid, braided, monoidal category. To get Move I requires a *ribbon Hopf algebra* [RT90]. The ribbon structure is not as compelling algebraically as the Hopf and quasitriangular structure,

perhaps because we do not understand Move I categorically as well as the others. A ribbon Hopf algebra is a quasitriangular Hopf algebra with a grouplike element G (i.e., $\Delta(G) = G \otimes G$ and $\epsilon(G) = 1$, hence $S(G) = G^{-1}$) satisfying

$$G^{-1}uG^{-1} = S(u)$$

 $GaG^{-1} = S^{2}(a).$

Given a representation (ρ_V, V) and an element $x \in \text{End}(V)$, define the quantum trace $\text{qtr by } \text{qtr}_V(x) = \text{tr}(\rho_V(G)x)$. The second condition above implies that, identifying End(V) with $V \otimes V^*$, qtr is the unique to scaling intertwiner from this representation to the trivial one. That G is grouplike means that the quantum trace is additive on direct sums and multiplicative on tensor products. Also important is the quantum dimension of V, $\text{qdim}(V) = \text{qtr}_V(1)$.

A simple example is called for. Let G be a finite group, and $\mathbb{C}G$ be the group algebra. Then the maps

$$\Delta(g) = g \otimes g \quad S(g) = g^{-1} \quad \epsilon(g) = 1 \quad (\text{for } g \in G)$$

extend by linearity to make $\mathbb{C}G$ a cocommutative Hopf algebra. This explains the term grouplike. Being cocommutative, it is a ribbon Hopf algebra with

$$R = 1 \otimes 1$$
 $G = 1$.

While these are certainly interesting Hopf algebras, their quasitriangular and ribbon structures are trivial, and we should not expect information about links from them. For that we must wait until the next section.

In putting this together, we see that we have gotten more than we were originally looking for. Specifically, given a ribbon Hopf algebra \mathcal{A} , define a *labeled tangle* to be an oriented tangle with a finite-dimensional representation of \mathcal{A} assigned to each component. Thus the domain and codomain are now sequences of oriented points labeled by representations. Assign to such a sequence a representation of \mathcal{A} by

$$\mathcal{F}((V_1, \varepsilon_1), (V_2, \varepsilon_2), \dots, (V_n, \varepsilon_n)) = \bigotimes_{i=1}^n V_i^{\varepsilon_i},$$

where V_i is a representation, ε_i is + or - indicating the orientation, and $V_i^+ = V_i$, $V_i^- = V_i^*$.

Theorem 4. [RT90] For every ribbon Hopf algebra \mathcal{A} there is a monoidal functor from the category of labeled oriented framed tangles to that of representations and intertwiners of \mathcal{A} . That is, there is a map \mathcal{F} which assigns to each such tangle T an intertwiner $\mathcal{F}(T): \mathcal{F}(\text{dom}(T)) \to \mathcal{F}(\text{codom}(T))$, satisfying $\mathcal{F}(T_1T_2) = \mathcal{F}(T_1)\mathcal{F}(T_2)$ and $\mathcal{F}(T_1 \otimes T_2) = \mathcal{F}(T_1) \otimes \mathcal{F}(T_2)$, \mathcal{F} of the identity tangle is the identity operator, and $\mathcal{F}(\emptyset) = 1$. It is determined by its values on generating tangles, which are shown in the following chart. Here G_i and R_{ij} are the actions of G and G_i on G_i and G_i are the actions of G_i and G_i are the actions of G_i and G_i and G_i and G_i are the action of G_i and G_i and G_i and G_i are the action of G_i and G_i and G_i are the action of G_i and G_i and G_i are the action of G_i and G_i and G_i and G_i are the action of G_i and G_i and G_i are the action of G_i and G_i and G_i are the action of G_i and G_i and G_i are the action of G_i and G_i and G_i are the action of G_i and G_i and G_i are the action of G_i and G_i are the action of G_i and G_i and G_i are the action of G_i and G_i and G_i are the action of G_i and G_i are the action of G_i and G_i are the action

respectively, σ_{ij} is the flip map, and v_{α} is a basis of V_i , v_{α}^* its dual basis of V_i^* . Mirror images of the crossings shown get sent to their inverses.

Proof. We have only to check invariance under Moves I-VII. Moves II, IV, VI and VII are easy to check, Move III is the Yang-Baxter Equation (9). Moves I and V require more involved calculations. \Box

\mathcal{F} of	takes	to	\mathcal{F} of	takes	to
i	$x^* \otimes x$	$x^*(x)$	~∪ _i	c	$c\sum_{\alpha}v_{\alpha}\otimes v_{\alpha}^{*}$
i _O .	$x \otimes x^*$	$x^*(G_i(x))$	i	c	$c \sum_{\alpha} v_{\alpha}^* \otimes G_i^{-1}(v_{\alpha})$
i∑j	$x \otimes y$	$\sigma_{ji}R_{ij}(x\otimes y)$	i	$x^* \otimes y^*$	$\sigma_{j^*i^*} R_{i^*j^*}(x^* \otimes y^*)$
i, j	$x \otimes y^*$	$\sigma_{j^*i}R_{ij^*}(x\otimes y)$	i\delta j	$x^* \otimes y$	$\sigma_{ji^*}R_{i^*j}(x^*\otimes y)$

Remark 1.

- There is a kind of converse of this result, a Tannaka-Krein type theorem: Every functor from \mathfrak{T} to \mathfrak{V} arises in this way from some ribbon Hopf algebra [Maj89, Saw].
- We have focused on oriented tangle functors, although currently our only interesting example is unoriented. It turns out that a self-dual representation of a ribbon Hopf algebra gives (almost) an unoriented tangle functor. As is shown in the next section, the Kauffman bracket functor essentially arises from a Hopf algebra representation corresponding to the fundamental representation of sl_2 , which is self-dual.
- The use of tensor products in discussing Δ and R is subtle in the case of most interest, when \mathcal{A} is infinite-dimensional. All equations make sense if we require only that they hold when represented on an arbitrary finite-dimensional representation. This involves no loss of information, because in the end we are only interested in these representations [Saw].

We need to beef up our invariant in a fairly trivial fashion for future sections. Define a ribbon graph exactly as a tangle, except we allow coupons, squares with a definite top, bottom and orientation, with strands allowed to intersect the coupons at distinct points on the top and bottom, as in Figure 3. Theorem 3 is still true of this larger category if we add the coupon as a generator and add relations VIII (and its mirror image) and IX in Figure 3, understood to apply for coupons with any number of incoming and outgoing strands. Given a ribbon Hopf algebra \mathcal{A} , a labeled ribbon graph is one where the strands are labeled by representations of \mathcal{A} as before, and any coupon with incoming strands labeled V_1, \ldots, V_n and outgoing labeled by W_1, \ldots, W_m is labeled by an intertwiner from $V_1 \otimes \cdots \otimes V_n$ to $W_1 \otimes \cdots \otimes W_m$, with representations replaced by their duals if the corresponding strand is oriented down when it intersects the coupon. It is an easy matter to show that \mathcal{F} extends to a functor from labeled ribbon graphs to the representation theory of \mathcal{A} , with \mathcal{F} of a

coupon labeled by f being simply f. The following summarizes important facts about \mathcal{F} .

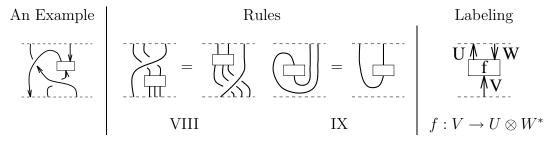


FIGURE 3. Coupons in ribbon graphs

Proposition 1. [RT90]

- (a) \mathcal{F} of a tangle with a component labeled by a direct sum of representations is the direct sum of \mathcal{F} of the tangle labeled by each summand.
- (b) \mathcal{F} of a tangle with a component labeled by a given representation is \mathcal{F} of the same tangle with the component given the reverse orientation and labeled by the dual representation.
- (c) \mathcal{F} of a tangle with one component labeled by the trivial representation is \mathcal{F} of that tangle with that component deleted.
- (d) \mathcal{F} of a tangle with a component labeled by a tensor product of representations is \mathcal{F} of that tangle with the component replaced by two parallel components, following the framing, labeled by the tensor factors.
- (e) \mathcal{F} of a tangle which can be separated by a sphere into a link and a subtangle is \mathcal{F} of the link times \mathcal{F} of the subtangle.
- (f) Let T be a tangle with a component labeled by an irreducible representation V, and let L be a link with a component labeled by V. If T' is the tangle formed by cutting both of these components and gluing them together along the cuts (consistent with the orientations) then $\mathcal{F}(T') = \mathcal{F}(T)\mathcal{F}(L)/\operatorname{qdim}(V)$.
- (g) $\mathcal{F}(\mathbb{G}) = \operatorname{qtr}(f)$. \square

3. Quantum Groups

Up to this point, this article has been close to being self-contained: All that has been left out are computations, and with a few more precise statements and a sprinkling of hints, the industrious reader could probably reproduce them. I hope also that it has been well-motivated: That the idea of a functor from the tangle category to vector spaces is a natural thing to look for, and that it might reasonably lead one to consider ribbon Hopf algebras. The same will not be true of this section.

It is certainly natural to argue, as we do, that Lie algebras give a nice set of ribbon Hopf algebras which offer trivial link information because they are cocommutative.

This suggests deforming them within the set of all ribbon Hopf algebras to get interesting link invariants. If we could argue that there was a unique deformation in the space of ribbon Hopf algebras, or that some sort of quantization described it geometrically, such a construction would seem very natural. Unfortunately, these approaches give only a rough framework in which to proceed, and ultimately one is forced to write down a guess for generators and relations and prove by hand that they give a ribbon Hopf algebra. For this reason we will eschew the deep and significant mathematics in these approaches [CP94, §1-3,6.1,6.2] [FT87, Dri83, Dri87], and present the algebras as if by oracle. Likewise, there is a fair amount of interesting technical machinery to prove the basic facts of these algebras [CP94, §6.4,8.1-8.3,10.1 [Lus93, Lus88, Ros90a, Jim85, KR88, KR90]: We will merely sketch the easy case sl_2 and state the broad results in general. Ultimately, from our point of view these quantum groups are a means for extracting geometric information (link invariants) out of quintessentially geometric objects (Lie groups), and it is certainly to be hoped that in the near future we will have a geometric understanding of how they arise.

Recall from the previous section that the group algebra of a finite group is a ribbon Hopf algebra, although the most information it can give about a link is the number of components. It is a little subtle deciding how to define the analogue of the group algebra for a Lie group, but it is clear that morally the same should be true. The best surrogate for this object turns out to be the universal enveloping algebra of the Lie algebra, $U(\mathfrak{g})$. For the group SL(2), this is the algebra $U(sl_2)$ generated by $\{x, y, h\}$, with relations

$$[h, x] = 2x$$
$$[h, y] = -2y$$
$$[x, y] = h$$

where [a, b] = ab - ba. Its Hopf algebra structure is determined by

$$\Delta(a) = a \otimes 1 + 1 \otimes a$$
 $\epsilon(a) = 0$ $S(a) = -a$

where a is in $\{x, y, h\}$. Its ribbon and quasitriangular structure is trivial, $R = 1 \otimes 1$ and G = 1. The universal enveloping algebras of the other simple Lie algebras have similar, if more complicated presentations [Hum72]. Given such a Lie algebra \mathfrak{g} , there is associated an algebra $U_s(\mathfrak{g})$, the quantum universal enveloping algebra of \mathfrak{g} , depending on a nonzero complex parameter s. It will prove to be a ribbon Hopf algebra, and in an appropriate sense will approach $U(\mathfrak{g})$ as a ribbon Hopf algebra as $s \to 1$. These quantized universal enveloping algebras are collectively called quantum groups, though the term is sometimes used more generally to refer to ribbon Hopf algebras, quasitriangular Hopf algebras, or even all Hopf algebras.

 $U_s(sl_2)$ is the algebra generated by $\{x,y,h\}$ subject to the relations

(10)
$$[h, x] = 2x$$
$$[h, y] = -2y$$
$$[x, y] = (s^{2h} - s^{-2h})/(s^2 - s^{-2}).$$

There are a number of artifices for interpreting the last equation, none entirely satisfactory (the casual reader is encouraged to ignore the problem). Ours is as follows: Let $U(sl_2)'$ be the completion of $U(sl_2)$ in the topology of convergence on every finite-dimensional representation, and let the closure of the subalgebra generated by h be \mathcal{H} . Notice the right side of the last equation is in \mathcal{H} for all $s \neq 0$, because every finite-dimensional representation is spanned by eigenvectors of h with integer eigenvalues. Consider the algebra spanned by free products of x, y and elements of \mathcal{H} , with the topology inherited from \mathcal{H} . The above equations generate an ideal in this algebra, and the quotient by the closure of that ideal is the algebra $U_s(sl_2)$.

This algebra is really just $U(sl_2)$ in disguise. Specifically, if we use \bar{x} , \bar{y} , \bar{h} temporarily to denote the generators of $U(sl_2)$ and recall that the Casimir element is $C = (\bar{h} + 1)^2/4 + \bar{y}\bar{x}$, then the map $\varphi : U_s(sl_2) \to U(sl_2)'$ given by

$$\varphi(h) = \bar{h} \qquad \varphi(y) = \bar{y}$$

$$\varphi(x) = 4(s^{4\sqrt{C}} + s^{-4\sqrt{C}} - s^{2\bar{h}-2} + s^{2-2\bar{h}}) / ((\bar{h} - 1)^2 - 4C)(s^2 - s^{-2})^2 \bar{x}$$

is a homomorphism for all $s \neq 0$. Further, this homomorphism is 1-1 and has dense range if s is not a root of unity (essentially because the coefficient of \bar{x} is invertible) [CP94, §4.6][Jim85]. Thus up to completions $U_s(sl_2)$ and $U(sl_2)$ are isomorphic as algebras. So $U_s(sl_2)$ is semisimple, its finite-dimensional representations are in one to one correspondence with that of $U(sl_2)$, and the actions of h and \bar{h} on these representations are the same. In particular, they have the same formal characters.

The Hopf algebra structure is given by

(11)
$$\Delta(x) = x \otimes s^h + s^{-h} \otimes x \qquad S(x) = -s^2 x \Delta(y) = y \otimes s^h + s^{-h} \otimes y \qquad S(y) = -s^{-2} y$$

and the rest as in the $U(sl_2)$ case. The key observation is that these maps commute with the operator $[h,\cdot]$. From this and the fact that the formal characters are the same as for $U(sl_2)$ it is easy to see that duals of representations and decomposition of tensor products into irreducible representations are the same (away from a root of unity) as in the classical case.

The quasitriangular structure is quite a bit trickier. Define quantum integers

$$[n]_q = (s^{2n} - s^{-2n})/(s^2 - s^{-2}).$$

Then

(12)
$$R = s^{(h \otimes h)} \sum_{n=0}^{\infty} \frac{s^{n(n+1)} (1 - s^{-4})^n}{[n]_a [n-1]_a \cdots [1]_a} x^n \otimes y^n \qquad G = s^{2h}.$$

The infinite sum is well defined in the usual topology, because only finitely many terms are nonzero on any given pair of representations.

If you believe that all the axioms of a ribbon Hopf algebra are satisfied by the algebra above, you should be wondering what the tangle functor looks like. First we must understand the representation theory explicitly. For this recall that sl_2 has an n-dimensional representation for each n > 0, with basis $\{v_i\}_{i=1}^n$ such that

$$hv_i = (n - 2i + 1)v_i$$

 $xv_i = (i - 1)v_{i-1}$
 $yv_i = (n - i)v_{i+1}$

and that the action of h is independent of s (by the isomorphism). It is easy to reconstruct the representation V_n of $U_s(sl_2)$ with basis $\{v_i\}_{i=1}^n$ such that

(13)
$$hv_{i} = (n - 2i + 1)v_{i}$$
$$xv_{i} = [i - 1]_{q}v_{i-1}$$
$$yv_{i} = [n - i]_{q}v_{i+1}.$$

It is now easy to compute the action of R on $V_2 \otimes V_2$, and composing with the flip map gives

Identifying V with V^* by the intertwiner $\alpha: v_1 \mapsto sv_2^*$, $\alpha: v_2 \mapsto -s^{-1}v_1^*$ gives

with the other generating tangles determined by these. Notice this is almost an unoriented tangle functor—in fact the Kauffman bracket functor, Equations (7) and (8), with A = s—except for an annoying minus sign in α . The link invariant will actually be the Kauffman bracket times $(-1)^w$, where w is the total winding number of the link projection $((-1)^w)$ is a framed, unoriented link invariant!). In general any irreducible self-dual representation gives, on choosing such an α , either an unoriented functor or a functor which differs from one by a minus sign in α . Kirillov and Reshetikhin [KR88] nicely fix this problem for sl_2 by constructing $U_q(sl_2)$ as a deformation of $U(sl_2)$ with a nonstandard quasitriangular (really triangular) structure.

We have only described the functor with everything labeled by the two-dimensional representation, but we have really constructed much more than that. There is enough

information above to write down the full theory for all representations, but no neat skein-theory description has been found for the corresponding invariant.

It is now clearer where the skein relation in Equation (2) is coming from. $V_2 \otimes V_2$ breaks up as the sum of two irreducible representations, the trivial one and V_3 , and thus the space of intertwiners on it is two-dimensional. The two fragments pictured on the right side of Equation (2), interpreted as tangles, correspond to the identity intertwiner and a multiple of the projection onto the trivial subrepresentation. They span the space of intertwiners, which includes \mathcal{L} .

As for other Lie algebras, the story is essentially the same. A similar, but more complicated presentation of $U_s(\mathfrak{g})$ can be given, and shown in the same sense to be isomorphic to $U(\mathfrak{g})$ (though in general the isomorphism does not admit explicit formulae as in the sl_2 case). The ribbon Hopf structure can all be written down explicitly, and the representation theory is 'the same' as for the original Lie algebra except at roots of unity [CP94, §8.3,10.1]. The link invariants coming from the fundamental representation of sl_n are particular values of the two-variable HOMFLY polynomial. The polynomials for B_n , C_n , and D_n at the fundamental representation are all special values of the two-variable Kauffman invariant [Tur89, Res88]. Both of these polynomials can be computed by skein-theoretic algorithms similar to that for the Jones polynomial. There is also a skein-theory algorithm for the G_2 invariant at the fundamental representation involving trivalent graphs [Kup94].

Remark 2. A word on notation. It is customary to speak of the quantum groups as $U_q(\mathfrak{g})$, where q is, depending on the author, s^2 [CP94, Kas94, Lus93, Jim85] or s^4 [Res88, Ros90b, Dri87, KM91]. Of course, this requires choosing a second or fourth root of q to write the R matrix. It is also common to call this variable t, or to call the variable in the Jones polynomial q, although these are inverses of each other with the usual conventions. Other minor variations exist on the definition of $U_q(\mathfrak{g})$. We have tried to remain internally consistent and consistent with well-established conventions, such as the skein relations for the Jones polynomial.

4. Modular Hopf Algebras

What happens to the quantum group at a root of unity? The answer is, the algebra ceases to be isomorphic to the unquantized algebra, and in fact ceases to be semisimple. The ribbon Hopf structure and some of the representation theory survive this collapse however, and emerge with subtle properties that allow the link invariant to be extended to a 3-manifold invariant and a topological quantum field theory. This is the subject of the next four sections. We begin with a precise look at the situation for sl_2 .

When s is a root of unity, the representations of $U_s(sl_2)$ given in the previous section are still representations, but may not all be irreducible. It is not hard to see that irreducibility is equivalent to v_1 being the unique h-eigenvector on which x

acts as 0. Thus if we let l be the least natural number such that $s^{4l}=1$ (so that sis a primitive lth or 2lth root of unity for l odd, or a primitive 4lth root of unity), we have $[l]_q = 0$, $[n]_q \neq 0$ for n < l, and thus V_n is still irreducible for $n \leq l$, but is not for larger n, by Equation (13). Recalling that the quantum dimension of a representation is the trace of G in that representation, notice that the quantum dimension of V_n is $[n]_q$, and thus is nonzero for n < l and equal to zero for n = l. It turns out that irreducible representations with quantum dimension zero have an important property: If one takes any representation in the ideal generated by them, i.e., a direct sum of tensor products of these with other representations, then any intertwiner from this representation to itself will have quantum trace zero. But this implies that any link labeled by such a representation has invariant zero. It is easy to check that the tensor product $V_n \otimes V_m$ for n, m < l is a sum of representations V_k for k < l plus a representation in this ideal. Since for purposes of knot theory the representations in this ideal are irrelevant, it makes sense to throw this trivial-trace representation out and define the truncated tensor product to be the rest of the tensor product. With this new tensor product, direct sums of representations V_n for n < lstill form a rigid, braided, tensor category, and give a tangle functor with truncated tensor product. But now there are only finitely many essentially different labels.

The same situation applies to a general quantum group, at least in many cases. If s is a primitive 2lth root of unity for l odd (and also prime to 3 in the G_2 case), then the representation of highest weight λ is still an irreducible representation with the correct character when $\langle \lambda + \rho, \theta \rangle \leq l$, where θ is the highest root (the unique long root in the Weyl chamber), ρ is half the sum of the positive roots, and $\langle \cdot, \cdot \rangle$ is as in [Hum72]. This representation has nonzero quantum dimension exactly when the inequality is strict. Further, if the decomposition of the unquantized tensor product of two representations in the smaller set lies entirely in the larger, the tensor product in this case decomposes the same way. From this it follows that the tensor product of any two representations in the smaller set is a sum of such representations plus a trivial-trace representation. Again, this gives a 'truncated' tangle functor [And92]. Of course, this is only interesting if more than the trivial representation is included, so we usually restrict to l such that $l > \langle \rho, \theta \rangle$. The quantity $l - \langle \rho, \theta \rangle$ is the k occurring in the Chern-Simons action in the introduction.

The other useful property of quantum groups at roots of unity has to do with the value of the invariant on the Hopf link, B in Figure 1. Let $H_{i,j}$ be the value of \mathcal{F} on the Hopf link labeled by representations λ_i and λ_j . For $U_s(sl_2)$, a straightforward calculation shows that $H_{i,j} = [ij]_q$, where of course $\lambda_i = i$. At a root of unity, if we restrict to our preferred representations, these numbers form a matrix, and another calculation shows that this matrix is nonsingular when s is a primitive 4lth root of unity or a primitive 2lth root with l odd [RT91, BHMV92]. Turaev and Wenzl have shown [TW93], by relating the question to work of Kac and Petersen [KP84], that the same is true of this matrix for an arbitrary quantum group when l satisfies the

same restrictions as in the previous paragraph.

This information is encapsulated in the definition of a modular Hopf algebra (see [RT91], in this simplified form [Tur92, TW93]).

Definition 2. A modular Hopf algebra is a ribbon Hopf algebra \mathcal{A} , together with a finite collection of irreducible representations $\lambda_1, \ldots, \lambda_k$, including the trivial representation, closed under duals, and having $\operatorname{qdim}(\lambda_i) \neq 0$, satisfying

- $\lambda_i \otimes \lambda_j = \bigoplus_{m=1}^k (N_{i,j}^m \lambda_m) \oplus \eta_{i,j}$, where $N_{i,j}^m$ is a multiplicity and $\eta_{i,j}$ is a representation on which qtr is zero on all intertwiners.
- The matrix $H_{i,j}$, for $1 \le i, j \le k$ is nonsingular.

Remark 3. That the quantum dimension of each λ_i is nonzero actually follows from the second condition of the definition.

We will now derive the key facts about a modular Hopf algebra that we will need later on. Consider the truncated representation algebra \mathbf{R} of our modular Hopf algebra. That is, the commutative algebra over \mathbb{C} spanned by $\lambda_1, \ldots, \lambda_k$ with $\lambda_i \lambda_j = \sum_m N_{i,j}^m \lambda_m$.

We can extend qdim by linearity to a homomorphism from \mathbf{R} to \mathbb{C} . More generally, if T is a ribbon tangle which is labeled except for one closed component x without coupons, and $a = \sum_i c_i \lambda_i \in \mathbf{R}$ we can define T with x labeled by a to be the formal sum $\sum_i c_i T_{\lambda_i}$, where T_{λ_i} is T with x labeled by λ_i . Then define the value of the invariant

$$\mathcal{F}(T_a) \stackrel{def}{=} \sum_{i=1}^k c_i \mathcal{F}(T_{\lambda_i}).$$

This is consistent with the additivity of the invariant on direct sums.

It is natural to ask if \mathbf{R} is semisimple: That is, is it spanned by minimal idempotents? In fact the dual basis to $\{\lambda_i\}$ with respect to the nondegenerate, symmetric pairing given by $\{H_{i,j}\}$ consists of such minimal idempotents. Minimal idempotents correspond to homomorphisms to \mathbb{C} , and the idempotent dual to the trival representation corresponds to the homomorphism $\operatorname{qdim}(\cdot)$. This minimal idempotent, which is the source of the 3-manifold invariant, is computed explicitly in the following proposition.

Proposition 2. [Saw] Let

$$\omega = \sum_{i=1}^{k} \operatorname{qdim}(\lambda_i) \lambda_i.$$

- (a) $a\omega = \operatorname{qdim}(a)\omega$ for all $a \in \mathbf{R}$.
- (b) the Hopf link labeled by λ_i and ω has nonzero \mathcal{F} if and only if λ_i is trivial (in particular $\operatorname{qdim}(\omega) \neq 0$). \square

5. Axiomatic Topological Quantum Field Theory

The fundamental idea of the first three sections was to get a detailed combinatorial description of an interesting geometric category (\mathfrak{T}) and use it to find functors to categories of vector spaces and linear maps. We do the same thing with the cobordism category, and call it a topological quantum field theory, or TQFT. Now we obtain numerical invariants not of (framed) links, but of (biframed) 3-manifolds. The work is remarkably parallel, the key difference being that this category is not presented as straightforwardly, but indirectly, in terms of links. Thus instead of TQFT's being associated to a specific kind of algebra, we construct them from a specific kind of link invariant: one arising from a modular Hopf algebra.

Let Σ_1 and Σ_2 be smooth oriented d-1-manifolds. A dimension d cobordism \mathfrak{m} with domain Σ_1 and codomain Σ_2 is up to diffeomorphism a triple (M, f_1, f_2) , where M is an oriented smooth d-dimensional manifold with boundary, and f_1 and f_2 are orientation-preserving endomorphisms from Σ_1^* and Σ_2 respectively to ∂M . Here Σ_1^* is Σ_1 with the opposite orientation, and we require of f_1 and f_2 that ∂M be the disjoint union of their ranges. By "up to diffeomorphism" we mean that two triples (M, f_1, f_2) and (M', f'_1, f'_2) are the same morphism if there is a diffeomorphism $F: M_1 \to M_2$ with $f'_1 = Ff_1$ and $f'_2 = Ff_2$. Thus it is a manifold with parameterized boundary, but with some of the boundary considered incoming and some considered outgoing, so that it can be viewed as acting like a function. More precisely, think of the Σ_i as being analogous to vector spaces and cobordisms as being analogous to linear maps.

In this vein, the analogue of composition of linear maps is gluing of cobordisms. That is, if $\mathfrak{m} = (M, f_1, f_2)$ and $\mathfrak{m}' = (M', f_1', f_2')$, define $\mathfrak{m}'\mathfrak{m} = (M' \cup_{f_1'f_2^{-1}} M, f_1, f_2')$, where $M' \cup_{f_1'f_2^{-1}} M$ is the manifold formed by identifying points in $\partial M'$ with ∂M via the orientation reversing map $f_1'f_2^{-1}$. This composition is associative, and its identity is clearly

$$1_{\Sigma} = (\Sigma \times I, \mathrm{id}_0, \mathrm{id}_1),$$

where I is the unit interval and $\mathrm{id}_0: \Sigma^* \to \Sigma \times \{0\}$ and $\mathrm{id}_1: \Sigma \to \Sigma \times \{1\}$ are the identity maps.

Tensor product of vector spaces is analogous to disjoint union $\Sigma_1 \cup \Sigma_2$; \mathbb{C} , the identity for tensor product, corresponds to 0, the empty d-1-manifold. Corresponding to tensor product of linear maps we have the associative operation $\mathfrak{m} \cup \mathfrak{m}' = (M \cup M', f_1 \cup f_1', f_2 \cup f_2')$. The empty d-manifold \emptyset acts like the number 1 as the identity for this product.

Of course $V \otimes W$ and $W \otimes V$ are the same vector space, in the sense that there is a canonical map between them $\sigma_{VW}: v \otimes w \mapsto w \otimes v$. Likewise there is a canonical cobordism from $\Sigma_1 \cup \Sigma_2$ to $\Sigma_2 \cup \Sigma_1$, namely the cobordism $\mathfrak{c}_{\Sigma_1,\Sigma_2} = ((\Sigma_1 \cup \Sigma_2) \times I, f_0, f_1)$, with

$$f_0: \Sigma_1^* \cup \Sigma_2^* \to (\Sigma_1 \cup \Sigma_2) \times \{0\} \quad f_1: \Sigma_2 \cup \Sigma_1 \to (\Sigma_1 \cup \Sigma_2) \times \{1\}$$

being the identity and the order-reversing map respectively.

No mathematician, and certainly no category theorist, would let such a good analogy go unnamed. The definition of cobordisms is designed to give the morphisms of a category, which we'll call \mathfrak{C} , and all the additional structures we discussed are just to say that \mathfrak{C} and \mathfrak{V} are symmetric, monoidal or *tensor* categories (there are some axioms to check, which are trivial in both cases. See [Lan71]).

The point of this is

Definition 3. [Ati89, Ati90a] A d-dimensional axiomatic topological quantum field theory, or TQFT, is a functor \mathcal{Z} of tensor categories from \mathfrak{C} to \mathfrak{V} . That is, a map \mathcal{Z} which sends each oriented d-1-manifold to a finite-dimensional vector space $\mathcal{Z}(\Sigma)$, and each cobordism \mathfrak{m} from Σ_1 to Σ_2 to a linear map $\mathcal{Z}(\mathfrak{m}): \mathcal{Z}(\Sigma_1) \to \mathcal{Z}(\Sigma_2)$, such that $\mathcal{Z}(1_{\Sigma}) = 1_{\mathcal{Z}(\Sigma)}, \mathcal{Z}(\Sigma_1 \cup \Sigma_2) = \mathcal{Z}(\Sigma_1) \otimes \mathcal{Z}(\Sigma_2), \mathcal{Z}(0) = \mathbb{C}, \mathcal{Z}(\mathfrak{m}_1 \cup \mathfrak{m}_2) = \mathcal{Z}(\mathfrak{m}_1) \otimes \mathcal{Z}(\mathfrak{m}_2), \mathcal{Z}(\emptyset) = 1$, and $\mathcal{Z}(\mathfrak{c})$ is the flip map.

Notice that a closed d-manifold is a cobordism from the empty d-1-manifold to itself, and thus gets sent to an operator on \mathbb{C} , or in other words a number. So one thing a TQFT gives is a numerical d-manifold invariant. There is a straightforward algorithm for computing it from a Morse function: If you can compute the operators assigned to cobordisms for each type of singularity, you can multiply them together to get the invariant of the closed manifold. Also, it follows that the mapping class group of any d-1-manifold acts on the vector space associated to that manifold. Thus these contain a lot of topological information [Tur94, §III].

The definition of a TQFT is modeled on what we expect a topological quantum field theory to yield. For the passage from the physics to these axioms, see [Ati90a] or [Axe91].

Let us now fix d=3. We will not actually construct a functor from \mathfrak{C} to \mathfrak{V} , but from the category \mathfrak{FC} of biframed (also called 2-framed) cobordisms to \mathfrak{V} . The reason for this lies buried in the subtleties of the Chern-Simons path integral, from which these examples spring, where a choice of biframing is necessary to regularize certain integrals. The casual reader may safely ignore the technicalities of biframings, and view them as playing a role very similar to that of framings on links for link invariants.

A biframing on a closed 3-manifold is a trivialization of $TM \oplus TM$, up to isotopy. Atiyah argues [Ati90b] that a given trivialization extends to a 4-manifold bounded by that 3-manifold exactly when the 4-manifold has a certain signature, which is a complete invariant of the biframing. A biframing on a manifold with boundary Σ restricts to a map from $T\Sigma \oplus T\Sigma$ to \mathbb{R}^6 . Call a biframed 2-manifold one equipped with such a map. A biframed cobordism is a 3-cobordism together with a trivialization up to isotopy fixing the boundary, and its domain and codomain are thus biframed 2-manifolds. Biframed cobordisms form a compact closed category \mathfrak{FC} exactly as before, with the observation that if Σ is a biframed 2-manifold, Σ^* inherits its biframing, and 1_{Σ} , and $\mathfrak{c}_{\Sigma,\Gamma}$ have canonical biframings.

We can make this category simpler to work with in a number of ways. First, two biframed 2-manifolds of the same genus are easily seen to be connected by an invertible biframed cobordism. Thus their vector spaces are isomorphic, and we only need to pick a vector space for each one (this reduction is what category theorists call 'skeletonization'. The category of tangles is really a skeletonization of a larger category with a geometrically more natural definition). The fact that \mathfrak{FC} has a duality structure similar to that of tangles (view $\Sigma \times I$ as a morphism from $\Sigma^* \cup \Sigma$ to 0) means we really do not have to distinguish between domain and codomain. Finally, the value of the functor on disconnected cobordisms is clearly determined by its value on connected ones.

To state the simplified version, for each g choose a representative biframed genus g surface Σ_g , and a biframed cobordism $\mathfrak{d}_g:\emptyset\to\Sigma_g\cup\Sigma_g$ whose underlying manifold is $\Sigma_g\times I$ and which is symmetric in the sense that $\mathfrak{c}_{\Sigma_g,\Sigma_g}\mathfrak{d}_g=\mathfrak{d}_g$. Also let \mathfrak{e}_g be a biframed cobordism such that $(\mathfrak{e}_g\otimes 1)(1\otimes\mathfrak{d}_g)=1=(1\otimes\mathfrak{e}_g)(\mathfrak{d}_g\otimes 1)$.

Theorem 5. [Saw] For each g, let $\mathcal{Z}(\Sigma_g)$ be a finite-dimensional vector space; for each biframed cobordism $\mathfrak{m}: \bigcup_{i=1}^n \Sigma_{g_i} \to \emptyset$, let $\mathcal{Z}(\mathfrak{m}): \bigotimes_{i=1}^n \mathcal{Z}(\Sigma_{g_i}) \to \mathbb{C}$ be a linear functional; and for each nonempty, closed, biframed 3-manifold M, let $\mathcal{Z}(M)$ be in \mathbb{C} . Then \mathcal{Z} extends uniquely to a biframed TQFT, if and only if the following hold:

- (a) Nondegeneracy: $\mathcal{Z}(\mathfrak{e}_g)$ is a symmetric nondegenerate pairing on $\mathcal{Z}(\Sigma_g)$.
- (b) Symmetry: If \mathfrak{m} has domain $\bigcup_{i=1}^n \Sigma_{g_i}$, and \mathfrak{m}' with domain $\bigcup_{i=1}^n \Sigma_{g_{\sigma(i)}}$ for some permutation σ is the same manifold as \mathfrak{m} with the same parameterization written in a different order, then $\mathcal{Z}(\mathfrak{m}) = \mathcal{Z}(\mathfrak{m}')P_{\sigma}$, where P_{σ} is the map on the appropriate tensor product of vector spaces which permutes the tensor factors.
- (c) Sewing: Suppose \mathfrak{m} has domain $\bigcup_{i=1}^m \Sigma_{g_i}$ and \mathfrak{n} has domain $\bigcup_{i=1}^n \Sigma_{g'_i}$ with $g_m = g'_1$. We can form the biframed manifold $\mathfrak{m} \cup_s \mathfrak{n}$, with domain $\bigcup_{i=1}^{m-1} \Sigma_{g_i} \cup \bigcup_{i=2}^n \Sigma_{g'_i}$, by composing with \mathfrak{d}_g along these boundary components (Figure 4). Then

$$\mathcal{Z}(\mathfrak{m} \cup_m \mathfrak{n}) = \mathcal{Z}(\mathfrak{m}) \otimes \mathcal{Z}(\mathfrak{n}) \circ \alpha$$

where α is the canonical map sending $v_1 \otimes \cdots \otimes v_{m-1} \otimes w_2 \otimes \cdots \otimes w_n$ to $\sum_i v_1 \otimes \cdots v_{m-1} \otimes a_i \otimes b_i \otimes w_2 \otimes \cdots \otimes w_n$, with a_i and b_i dual bases of $\mathcal{Z}(\Sigma_{g_m})$ with respect to the pairing $\mathcal{Z}(\mathfrak{e}_g)$.

(d) Mending: Suppose \mathfrak{m} has domain $\bigcup_{i=1}^n \Sigma_{g_i}$, with $g_1 = g_2$. We can form \mathfrak{m}_m by composing with \mathfrak{d}_g along the first two boundary components (Figure 4). Then

$$\mathcal{Z}(\mathfrak{m}_m) = \mathcal{Z}(\mathfrak{m}) \circ \alpha$$

where α is the canonical map as above. \square

Remark 4.



Figure 4. Examples of sewing and mending

- There is one obvious piece of structure on \mathfrak{C} that we are leaving off: Reversing the orientation on a cobordism from Σ to Σ' gives a cobordism from Σ' to Σ . This is analogous to Hilbert space adjoint. It thus seems natural to replace \mathfrak{V} with the category of finite-dimensional Hilbert spaces and ask that this structure too be preserved. This is called a unitary TQFT, and is the more natural one from the point of view of physics. It turns out that, while we will construct a TQFT for any quantum group and any primitive 2lth root of unity (with the limitations on l given in Section 4) it is only unitary for $s = e^{\pi i/l}$. This is exactly the value of s corresponding to Witten's geometric construction.
- Of course, an ordinary framing on a closed 3-manifold determines a biframing, so in the end we will have an invariant of framed 3-manifolds as well.

6. Surgery and Cobordism

We would like to have an algebraic description of the category \mathfrak{FC} of framed cobordisms in terms of generators and relations. The idea is to define a TQFT by the image of the generating morphisms, and prove that it is a TQFT by confirming that it preserves the relations. A natural choice is Morse theory. Here the generators are the morphisms attaching handles and the relations are those given by Cerf theory [Cer70]. This is the approach taken in [Wal], but it involves a lot of detail checking, because there are a lot of ways to attach a handle.

Surgery on links meets our present need much better. Since surgery describes 3-manifolds in terms of links, and we are constructing TQFT's out of link invariants, we will find many of the details fall into place. Of course, surgery does not apparently handle biframing, or manifolds with boundary, so this section is devoted to extending it to these situations. We begin with a review of (integer) surgery.

Consider a framed unoriented link L in S^3 . Let T be a tubular neighborhood of L. Each component of ∂T has a natural meridian, which bounds a disk in T. It also has a natural longitude, the pushoff of L in the direction of the framing. Remove each component of L and glue it back in by a map sending the meridian to the longitude and the longitude to minus the meridian (you must choose orientations for these curves, but the result does not depend on this choice). The manifold one obtains is called M_L , the result of surgery on L. For example, surgery on the 0-framed unknot is easily seen to be $S^1 \times S^2$, and a little more work shows that surgery on a ± 1 framed

unknot gives S^3 again.

It turns out, by a theorem of Lickorish [Lic62], that every closed, compact, connected, oriented 3-manifold admits such a presentation. Further, by a theorem of Kirby [Kir78], two links present the same 3-manifold if and only if they can be related by a sequence of the moves in Figure 5 and their mirror images, with any number of strands understood to pass through the pictured unknot. Actually, Kirby's original version involved two "nonlocal" moves: Fenn and Rourke [FR79] reduced them to this move.

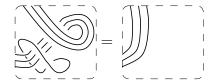


FIGURE 5. The Kirby move

Kirby's proof actually takes place in four dimensions. He views S^3 as bounding B^4 , and interprets L as defining a 4-manifold by attaching 2-handles to tubular neighborhoods of the components. The resulting 4-manifold will have boundary M_L . Thus the more sophisticated view of surgery is that it results in a 3-manifold together with a choice of 4-manifold for it to bound.

This is perfect for us, because this is just what we saw in the previous section determines a biframing. Of course we need a refinement of the Kirby move which preserves the signature of the 4-manifold but is still powerful enough to connect all links giving the same signature. This task at first sounds daunting, but is in fact quite simple. The key observation is that, since each component of L corresponds to adding a handle to the 4-manifold, the second relative cohomology of the 4-manifold M has a basis element for each handle, supported in that handle. Further, the intersection form of two basis elements is the linking number of the corresponding components (the self-intersection is the self-linking number, determined by the framing). Thus the signature of the 4-manifold is exactly the signature of the matrix of linking numbers of L!

I.
$$\left[\underbrace{SS} \right] = \left[\begin{array}{c} \end{array} \right]$$
 II. $\left[\begin{array}{c} \end{array} \right]$

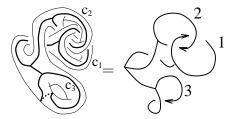
FIGURE 6. The biframed Kirby moves

An easy calculation shows that the Kirby move changes the signature of the linking matrix by 1. Thus the two *biframed Kirby moves* pictured in Figure 6, each a composition of two Kirby moves, do not change the signature. On the other hand, if two

links represent the same 3-manifold and have the same signature, one can convert the sequence of ordinary Kirby moves connecting them to a sequence of biframed Kirby moves.

Theorem 6. [Saw] Closed, oriented, biframed 3-manifolds are in one to one correspondence with equivalence classes of biframed unoriented links in S^3 modulo moves I and II in Figure 6 and their mirror images. \square

The cobordism category is not much harder. For each surface Σ_g choose a biframed handlebody which it bounds, H_g . To represent pictorially an embedding of H_g into S^3 , choose a set of generators for the fundamental group of the handlebody, $\{c_i\}$ for $1 \leq i \leq g$. Represent an embedding of H_g into S^3 as in Figure 7.The embedding is recovered by thickening the framed graph to a handlebody and identifying H_g with the thickening by a map which sends c_i to the boundary above the *i*th circle on the graph. Choose \mathfrak{d}_g so that the identification of Σ_g with Σ_g^* sends each c_i to a meridian intersecting it and vice versa, as illustrated for genus two in Figure 7.







Boundaries identified by \mathfrak{d}_q

FIGURE 7. A pictorial description of the cobordism category

Now if $\mathfrak{m}: \bigcup_{i=1}^n \Sigma_{g_i} \to \emptyset$ is a connected cobordism, consider the closed manifold M formed by gluing $\bigcup_{i=1}^n H_{g_i}$ to \mathfrak{m} along the boundary by the parametrization, and consider the resulting embedding N of $\bigcup_{i=1}^n H_{g_i}$ into M. This pair determines \mathfrak{m} uniquely. M can be presented as M_L for some link L, and the embedding N can be isotoped so as not to intersect the embedded tori coming from surgery on L. so N corresponds to an embedding of $\bigcup_{i=1}^n H_{g_i}$ into S^3 which does not intersect L. Thus \mathfrak{m} is determined by a pair (N, L), where L is an unoriented biframed link and N is an embedding of handlebodies into S^3 which does not intersect L. Specifically, \mathfrak{m} is obtained by doing surgery on L, removing the image of the interiors of the handlebodies, and parametrizing the resulting boundary by N restricted to the boundary. We call (N, L) a presentation of \mathfrak{m} , and in general refer to the cobordism presented by (N, L) as [N, L].

In [Roba] a set of Kirby-type moves is given for manifolds with boundary. From this it follows that two presentations give the same cobordism if and only if they can be connected by isotopy and a sequence of Kirby moves (Figure 5) and their mirror image. The framed unknot in Figure 5 represents a component of L and the strands

passing through it represent pieces of L or N. We thus have a purely combinatorial description of cobordisms.

In fact, (N, L) determines a biframed cobordism, and Figure 6 connects all equivalent presentations of a biframed cobordism. Specifically, having chosen a biframing on H_g , biframings on [N, L] are in one-to-one correspondence with biframings on M_L , which are preserved by the biframed Kirby moves.

Our last goal is to give a presentation of a cobordism obtained by sewing or mending, in terms of the presentation of the manifold(s) to be sewn or mended. The following proposition is not hard to show from the biframed Kirby moves.

Proposition 3. [Saw] Any biframed cobordism can be written as [N, L], where N is in standard position in the sense that it can be projected with no self-crossings, as illustrated in Figure 8.

Thus we may assume the presentations to be sewn or mended are in standard form. It is more clear and convenient to describe the algorithm pictorially, as in Figures 8. The corresponding picture applies for any genus and any number of strands passing through the handles.



FIGURE 8. Sewing and mending

The algorithm for sewing follows straightforwardly from the definition, but the algorithm for mending needs a little justification. We first note that, to form the mending $[N,L]_g$, we might just as well form the mending $[N,\emptyset]_m$ and then do surgery on the image of L in this new manifold. But now because N is in standard position, $[N,\emptyset]_m$ is simply $S^1\times S^2$, and the picture we get is exactly the right side of Figure 8.

7. Constructing The TQFT

We are now ready to construct a TQFT. For this we need to associate vector spaces to surfaces. We first associate excessively large vector spaces to each Σ_g , spanned by labeled ribbon graphs in H_g . Then we give a pairing on this corresponding to \mathfrak{e}_g , and quotient by the null space to force the pairing to be nondegenerate. The value of the invariant on cobordisms is then fairly obvious.

Given a biframed handlebody H_g , a closed labeled ribbon graph in H_g is defined as in Section 2, except that the graph is embedded in H_g , and equivalence is by a positive diffeomorphisms fixing the boundary.

Let V_g be the vector space of all formal linear combinations of closed labeled ribbon graphs in H_g . For each presentation (N, L), where N is an embedding of $\bigcup_{j=1}^n H_{g_i}$ into S^3 , we define a map $f_{(N,L)}: \bigotimes_{j=1}^n V_{g_j} \to \mathbb{C}$, as follows. If h_j is a closed labeled ribbon graph in H_{g_j} for $1 \leq j \leq n$, embed $\bigcup_{j=1}^n h_j$ into S^3 via N, and label each component of L by $\Omega = \operatorname{qdim}^{-1/2}(\omega)\omega$, with ω as in Proposition 2. Let $K = \operatorname{qdim}(\Omega)$, and define $f_{(N,L)}(\bigotimes h_j) = K^{-1}$ times \mathcal{F} of the resulting graph.

Proposition 4.

- (a) If Moves I and II of Figure 6 represent closed labeled ribbon graphs in S^3 with the framed unknots pictured labeled by Ω , the values of \mathcal{F} on both sides of the equal signs agree.
- (b) The function $f_{(N,L)}$ depends only on [N,L], and not on the presentation. Thus we will speak henceforth of $f_{[N,L]}$.
- *Proof.* (a) For Move II, use Proposition 1 and Figure 9. The first step is by parts (a) and (d), the second by part (f), the third by Proposition (2a), and the last by (a) and (d) again.

For Move I, apply Move II to the left side to get a Hopf link with both components labeled by Ω , one having a +1 framing. By Proposition (2b), \mathcal{F} of this is $\operatorname{qdim}(\Omega) \operatorname{qdim}^{-1/2}(\omega) = 1$.

(b) This follows immediately from (a). \Box

$$\mathcal{F}\left(\begin{bmatrix} \overline{\lambda_{i}} \\ \overline{\lambda_{i}} \end{bmatrix} \right) = \sum_{i} N_{\lambda_{i}} \mathcal{F}\left(\begin{bmatrix} \overline{\lambda_{i}} \\ \overline{\lambda_{i}} \end{bmatrix} \right) = \sum_{i} N_{\lambda_{i}} \mathcal{F}\left(\begin{bmatrix} \overline{\lambda_{i}} \\ \overline{\lambda_{i}} \end{bmatrix} \right) / \operatorname{qdim}(\lambda_{i})$$

$$= \sum_{i} N_{\lambda_{i}} \mathcal{F}\left(\begin{bmatrix} \overline{\lambda_{i}} \\ \overline{\lambda_{i}} \end{bmatrix} \right) = \mathcal{F}\left(\begin{bmatrix} \overline{\lambda_{i}} \\ \overline{\lambda_{i}} \end{bmatrix} \right) = \mathcal{F}\left(\begin{bmatrix} \overline{\lambda_{i}} \\ \overline{\lambda_{i}} \end{bmatrix} \right)$$

FIGURE 9. Proof of invariance under Move II

In particular, we have a symmetric bilinear pairing $f_{\mathfrak{e}_g}: V_g \otimes V_g \to \mathbb{C}$. If $N_g = \{v \in V_g: f_{\mathfrak{e}_g}(v, w) = 0 \ \forall w \in V_g\}$, define

(14)
$$\mathcal{Z}(\Sigma_g) = V_g/N_g,$$

and then $f_{\mathfrak{e}_g}$ descends to a nondegenerate pairing on $\mathcal{Z}(\Sigma_g)$.

Proposition 5. $f_{[N,L]}: \bigotimes_j V_{g_j} \to \mathbb{C}$ descends to a map $\mathcal{Z}([N,L]): \bigotimes_j \mathcal{Z}(\Sigma_{g_j}) \to \mathbb{C}$.

Sketch of Proof. This follows easily from the fact that we can treat $f_{[N,L]}(h_1 \otimes \cdots \otimes h_n)$ as $f_{\mathfrak{e}_{g_j}}(h_j \otimes k_j)$, where k_j is the image of the other h_i 's and L under an identification of the complement of the image of H_{g_j} with H_{g_j} . \square

We are now closing in on our prey. The only question remaining is the elusive one of sewing and mending.

Lemma 1. For any sequence of signed, labeled points, there exists a set of intertwiners x_i and y_i , $1 \le i \le k$ such that the equalities hold in Figure 10 for the value of \mathcal{F} on any closed, labeled, ribbon graph containing the pictures.

FIGURE 10. Cutting ribbon graphs

Proof. The sequence of signed labeled points corresponds to some tensor product of representations, say W. Write the identity on W as a sum $\sum_j p_j$ of projections onto irreducible subrepresentations (plus a projection onto a trivial-trace subrepresentation which does not affect the outcome and which we ignore). For the first, we may interpret the fragment T of the graph as a tangle, with $\mathcal{F}(T)$ intertwining W and the trivial representation. Thus $\mathcal{F}(T)p_i$ will be zero unless p_i projects onto a trivial subrepresentation. So we get $\mathcal{F}(T) = \mathcal{F}(T) \sum_i p_i$, with the sum over projections onto trivial subrepresentations. Writing each such p_i as $y_i x_i$, with x_i an intertwiner from W to the trivial representation and y_i an intertwiner the other way, we have the result.

For the second, it follows from Proposition 2 that \mathcal{F} of the tangle shown is a multiple of the projection onto the sum of all trivial subrepresentations. This equates the first and second pictures. The last is just Proposition 1e. \square

The following theorem first appears in [RT91], though not in the language of TQFT's. It was proven for sl_2 , using only the Kauffman bracket formalism without quantum groups, in [BHMV]. Complete proofs of the theorem as stated here first appear in [Wal] and [Tur94, Thm. IV.1.9].

Theorem 7. [RT91] Given a modular Hopf algebra A, the Z defined in this section extends to a TQFT.

Proof. We need to check axioms (a)-(d) from Theorem 5. Nondegeneracy we have already checked, and Symmetry is immediate from the construction.

For Sewing, let $[M, L] = \mathfrak{m}$ and $[N, K] = \mathfrak{n}$ be two presentations in standard form of the cobordisms mentioned in the Sewing axiom. We may as well assume that

each has only one boundary component, since we can check the equality with the maps applied to arbitrary vectors by gluing ribbon graphs into each other boundary component. Let J be the presentation of their sewing described in Section 6, and let j be the closed labeled ribbon graph obtained by labeling every component of J by Ω . Thus

$$\mathcal{Z}(\mathfrak{m}\cup_{s}\mathfrak{n})=K^{-1}\mathcal{F}(j).$$

More precisely, since [M, L] is in standard form, we can glue the complement of the image of M to the underlying manifold of \mathfrak{d}_g and identify the result with H_g . This identification sends L to a ribbon graph $h_{\mathfrak{m}}$ in H_g , such that $\mathcal{Z}(\mathfrak{m})(h) = \langle h_{\mathfrak{m}}, h \rangle$. Likewise we can define $h_{\mathfrak{n}}$ so that $\mathcal{Z}(\mathfrak{n})(h) = \langle h_{\mathfrak{n}}, h \rangle$. Now it is clear from the construction that j is $h_{\mathfrak{m}}$ glued to $h_{\mathfrak{n}}$ via \mathfrak{d}_g , so that

$$\mathcal{F}(j) = K\langle h_{\mathfrak{m}}, h_{\mathfrak{n}} \rangle.$$

On the other hand

$$\begin{split} \langle h_{\mathfrak{m}}, h_{\mathfrak{n}} \rangle &= \sum_{j=1}^{k} \langle h_{\mathfrak{m}}, a_{j} \rangle \langle b_{j}, h_{\mathfrak{n}} \rangle \\ &= \sum_{j=1}^{k} \mathcal{Z}(\mathfrak{m})(a_{j}) \mathcal{Z}(\mathfrak{n})(b_{j}) \end{split}$$

which gives the result.

For Mending, let $[M, L] = \mathfrak{m}$ be a presentation in standard form of the cobordism mentioned in this axiom, and as above assume \mathfrak{m} has only two boundary components, both of genus g. Let a_j and b_j be a basis and dual basis for $\mathcal{Z}(\Sigma_g)$, and choose linear combinations of closed ribbon graphs in H_g to represent them, which we'll also call a_j and b_j . The proof is then contained in Figure 11. The second equality is by Lemma 1a, the third by Proposition 1e and the seventh by Lemma 1b. The combinations a_j and b_j are represented pictorially as one graph for conservation of subscripts: It does not effect the proof. \square

Remark 5.

• These issues of framing are easy to fix for closed manifolds. If $[\emptyset, L]$ is the manifold presented by L, and $\sigma(L)$ is the signature of the linking matrix of L, then a simple computation shows that $C^{\sigma}(L)\mathcal{Z}([\emptyset, L])$ is invariant under framing if $C = 1/u_{+}(\Omega)$. This is the Reshetikhin-Turaev invariant, except they multiply by K to get \mathcal{F} instead of \mathcal{Z} , thus making the invariant multiplicative in connect-sums of manifolds. Their normalization is a bit different, because they in effect label by ω instead of Ω , and thus must correct for the number of components as well as the signature of L.

FIGURE 11. Proof of mending invariance

• The vector spaces $\mathcal{Z}(\Sigma_g)$ have been defined somewhat abstractly, and it is worth noting that they can be constructed quite explicitly. For example, it follows from what we've done that $\mathcal{Z}(\Sigma_2)$ is spanned by graphs as shown in Figure 12, where λ , γ , and δ are any representations and f_1 , f_2 are any appropriate intertwiners. Thus $\mathcal{Z}(\Sigma_2) = \bigoplus_{i,j,m=1}^k W_{i,i^*}^j \otimes W_{m,m^*}^j$, where $W_{i,j}^m$ is the space of intertwiners from $V_{\lambda_i} \otimes V_{\lambda_j}$ to V_{λ_m} , a space of dimension $N_{i,j}^m$.

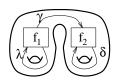


FIGURE 12. Spanning labeled ribbon graphs in H_2

By necessity the material presented here left off much of interest in the general field, and gave only a broad sketch of what it did cover. Below we offer the interested reader some pointers to other important topics and more details.

Good sources for classical knot theory include the excellent but dated [Rol76] and the more up-to-date [BZ85]. A good elementary survey of the knot polynomials can be found in [LM88].

Quite a lot is being done with quantum groups, with an eye towards both topological and algebraic applications, such as Kashiwara and Lusztig's canonical bases and the Kazhdan-Lusztig conjectures. Fortunately, the field has recently benefited from a number of expository books, including [Lus93, CP94, Kas94, SS94], which cover all of this, and all of the background and details skipped in Sections 2 and 3. [Bax82] gives a good account of the integrable models in statistical mechanics,

[Maj90, Fad84] connect them to quantum groups, and [Kau91] presents many ideas of statistical mechanics from a very knot-theoretic viewpoint.

Much has also been written about the 3-manifold invariants constructed in [RT91]. The sl_2 invariant has been constructed in many different forms and from many different perspectives in [KM91, Lic91, Cra91, BHMV92, BHMV, Koh92, Mor92, TW93, Wen93, KL94]. The square norm of the 3-manifold invariant associated to any modular Hopf algebra can be computed as a certain sum over states on a triangulation of the 3-manifold [TV92, Wal, Robb]. There are also a number of 3-manifold invariants which are constructed from the same sorts of data but which appear to be different, including [Kup89, Kup, KR].

The unfortunately unpublished [Wal] gives a thorough and informative account of the TQFT's we construct, extending the formalism to cobordisms, with corners (this is a strong version of the 'duality' which Witten uses to get his solution). It is a good resource for novices and experts. Much general imformation can be found in Quinn's lecture notes on the subject [Qui95]. The other articles in the same volume represent an excellent introduction to many of the physical and geometric aspects of this subject.

There are other ways to get at the link and 3-manifold invariants quite apart from quantum groups. Affine Lie algebras [KP84, KW88], by way of conformal field theory [TK88, Ver88], offer a functor from the tangle category to a sort of intermediate linear category. One can still construct the TQFT in this context. This approach is technically more difficult, because there are infinite-dimensional algebras and vector spaces to contend with, but is directly linked to the physics.

Also closer to the physics are efforts to bring the tools mathematical physics has developed for understanding quantum field theories to bear on Chern-Simons theory. Good overviews of this include [Axe91, Ati90a]. More recent efforts have focused on perturbative approaches to Chern-Simons theory, including [AS92, AS94, Kon]. Perturbation theory when applied to the link invariants is entirely combinatorial, and fits naturally into the framework of Vassiliev invariants, which were developed out of purely topological consideration [Vas90, Vas92]. Good sources for the theory of Vassiliev invariants and their relation with the link invariants in this paper are [BN91, BL91], as well as the excellent expository article which recently appeared in these pages [Bir93] (it also gives a good introduction to the link invariants and their place in the history of knot theory).

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