Topology

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Contents

Chapter 1

Set Theory

1.1 Primitive Notions

Let there be *sets*.

Given sets A and B, let there be *functions* from A to B. We write $f : A \to B$ iff f is a function from A to B.

Given functions $f : A \to B$ and $g : B \to C$, let there be a function $g \circ f : A \to C$, the *composite* of f and g.

1.2 The Axiom of Associativity

Axiom 1.2.1 (Associativity)**.** *Let $f : A \to B$, $g : B \to C$ and $h : C \to D$. Then $h \circ (g \circ f) = (h \circ g) \circ f : A \to D$.*

From now on we write $h \circ g \circ f$ for the composite of f, g and h, and similarly for more than three functions.

1.3 Injective Functions

Definition 1.3.1 (Injective)**.** A function $f : A \to B$ is *injective*, $f : A \rightarrowtail B$, iff, for every set X and functions $g, h : X \to A$, if $f \circ g = f \circ h$ then $g = h$.

Proposition 1.3.2. *Let $f : A \to B$ and $g : B \to C$. If f and g are injective then $g \circ f$ is injective.*

Proof:
$\langle 1 \rangle 1$. Assume: f and g are injective.
$\langle 1 \rangle 2$. Let: X be a set and $x, y : X \to A$
$\langle 1 \rangle 3$. Assume: $g \circ f \circ x = g \circ f \circ y$
$\langle 1 \rangle 4$. $f \circ x = f \circ y$
  Proof: g is injective ($\langle 1 \rangle 1$)
$\langle 1 \rangle 5$. $x = y$

PROOF: f is injective ($\langle 1\rangle 1$)
□

Lemma 1.3.3. *Let $f : A \to B$ and $g : B \to C$. If $g \circ f$ is injective then f is injective.*

PROOF:
$\langle 1\rangle 1$. ASSUME: $g \circ f$ is injective.
$\langle 1\rangle 2$. LET: X be any set and $x, y : X \to A$
$\langle 1\rangle 3$. ASSUME: $f \circ x = f \circ y$
$\langle 1\rangle 4$. $g \circ f \circ x = g \circ f \circ y$
$\langle 1\rangle 5$. $x = y$
PROOF: Using $\langle 1\rangle 1$.
□

1.4 Surjective Functions

Definition 1.4.1 (Surjective)**.** Let $f : A \to B$. Then f is *surjective*, $f : A \twoheadrightarrow B$, iff, for any set X and functions $g, h : B \to X$, if $g \circ f = h \circ f$ then $g = h$.

Lemma 1.4.2. *Let $f : A \to B$ and $g : B \to C$. If f and g are surjective then $g \circ f$ is surjective.*

PROOF: Dual to Lemma 1.3.2. □

Lemma 1.4.3. *Let $f : A \to B$ and $g : B \to C$. If $g \circ f$ is surjective then g is surjective.*

PROOF: Dual to Lemma 1.3.3. □

1.5 Identity Functions

Axiom 1.5.1 (Identity Function)**.** *For any set A, there exists a function $\mathrm{id}_A : A \to A$, the* identity *function on A, such that:*

Left Unit Law *for every set B and function $f : B \to A$ we have $\mathrm{id}_A \circ f = f : B \to A$;*

Right Unit Law *for every set B and function $f : A \to B$ we have $f \circ \mathrm{id}_A = f : A \to B$.*

Proposition 1.5.2. *The identity function on a set is unique.*

PROOF:If $i, j : A \to A$ are both identity functions, then

$$\begin{aligned} i &= i \circ j && \text{(Right Unit Law for } j\text{)} \\ &= j && \text{(Left Unit Law for } i\text{)} \\ &: A \to A && \square \end{aligned}$$

Proposition 1.5.3. *Every identity function is injective.*

PROOF: If $\mathrm{id}_B \circ x = \mathrm{id}_B \circ y$ then $x = y$ by the Left Unit Law. □

Proposition 1.5.4. *Every identity function is surjective.*

PROOF: Dual. □

Definition 1.5.5 (Retraction, Section)**.** Let $r : A \to B$ and $s : B \to A$. Then r is a *retraction* of s, and s is a *section* of r, iff $r \circ s = \mathrm{id}_B$.

Proposition 1.5.6. *If $r_1 : A \to B$ is a retraction of $s_1 : B \to A$ and $r_2 : B \to C$ is a retraction of $s_2 : C \to B$ then $r_2 \circ r_1$ is a retraction of $s_1 \circ s_2$.*

PROOF:

$$\begin{aligned} r_2 \circ r_1 \circ s_1 \circ s_2 &= r_2 \circ \mathrm{id}_B \circ s_2 && (r_1 \text{ is a retraction of } s_1) \\ &= r_2 \circ s_2 && (\text{Unit Laws}) \\ &= \mathrm{id}_C && (r_2 \text{ is a retraction of } s_2) \end{aligned}$$

□

Proposition 1.5.7. *Every retraction is surjective.*

PROOF:
⟨1⟩1. LET: $r : A \to B$ be a retraction of s
⟨1⟩2. LET: X be a set and $x, y : B \to X$ with $x \circ r = y \circ r$
⟨1⟩3. $x \circ r \circ s = y \circ r \circ s$
⟨1⟩4. $x = y$
□

Proposition 1.5.8. *Every section is injective.*

PROOF: Dual. □

Proposition 1.5.9. *Every identity function is a retraction of itself.*

PROOF: Immediate from the Unit Laws. □

Proposition 1.5.10. *If $r : B \to A$ is a retraction of $f : A \to B$ and s is a section of f then $r = s$.*

PROOF:

$$\begin{aligned} r &= r \circ \mathrm{id}_B && (\text{Right Unit Law}) \\ &= r \circ f \circ s && (s \text{ is a section of } f) \\ &= \mathrm{id}_A \circ s && (r \text{ is a retraction of } f) \\ &= s && (\text{Left Unit Law}) \end{aligned}$$

1.5.1 Isomorphisms

Definition 1.5.11 (Isomorphism)**.** Let A and B be sets. A function $i : A \to B$ is an *isomorphism* between A and B, $i : A \cong B$, iff there exists a function $i^{-1} : B \to A$, the *inverse* to i, that is a section and a retraction of i.

Proposition 1.5.12. *The inverse of an isomorphism is unique.*

Proof: Immediate from Proposition 1.5.10. □

Proposition 1.5.13. *Every isomorphism is injective.*

Proof: Immediate from Proposition 1.5.8. □

Proposition 1.5.14. *Every isomorphism is surjective.*

Proof: Immediate from Proposition 1.5.7. □

Proposition 1.5.15. *Every identity function is an isomorphism and is its own inverse.*

Proof: Immediate from Proposition 1.5.9. □

Proposition 1.5.16. *If $i : A \cong B$ is an isomorphism then $i^{-1} : B \cong A$ is an isomorphism and $(i^{-1})^{-1} = i$.*

Proof: Immediate from the definition of isomorphism. □

Proposition 1.5.17. *If $i : A \cong B$ and $j : B \cong C$ then $j \circ i : A \cong C$ and $(j \circ i)^{-1} = i^{-1} \circ j^{-1}$.*

Proof: Immediate from Proposition 1.5.6. □

1.5.2 Parts of a Set

Definition 1.5.18 (Part)**.** A *part* S of a set A consists of:

- a set $\mathrm{dom}\, S$;
- an injective function $i : S \hookrightarrow A$

Definition 1.5.19. Two parts $i : S \hookrightarrow A$, $j : T \hookrightarrow A$ are *equivalent*, $i \equiv_A j$, iff there exists an isomorphism $\phi : S \cong T$ such that $i = j \circ \phi$.

Proposition 1.5.20. *Any part of a set is equivalent to itself.*

Proof:
⟨1⟩1. Let: $i : S \hookrightarrow A$ be a part of A.
Prove: $i \equiv_A i$
⟨1⟩2. $\mathrm{id}_S : S \cong S$
Proof: By Proposition 1.5.15
⟨1⟩3. $i = i \circ \mathrm{id}_S$
Proof: By the Right Unit Law.

□

Proposition 1.5.21. *If $i \equiv_A j$ then $j \equiv_A i$.*

PROOF:
⟨1⟩1. LET: $i : S \hookrightarrow A$ and $j : T \hookrightarrow A$
⟨1⟩2. ASSUME: $i \equiv_A j$
⟨1⟩3. PICK an isomorphism $\phi : S \cong T$ such that $i = j \circ \phi$
⟨1⟩4. $\phi^{-1} : T \cong S$
PROOF: By Proposition 1.5.16.
⟨1⟩5. $j = i \circ \phi^{-1}$
PROOF: Compose both sides of ⟨1⟩3 with ϕ^{-1}.
□

Proposition 1.5.22. *If $i \equiv_A j$ and $j \equiv_A k$ then $i \equiv_A k$.*

PROOF:
⟨1⟩1. LET: $i : R \hookrightarrow A$, $j : S \hookrightarrow A$ and $k : T \to A$
⟨1⟩2. PICK isomorphisms $\phi : R \cong S$ and $\psi : S \cong T$ such that $i = j \circ \phi$ and $j = k \circ \psi$
⟨1⟩3. $\psi \circ \phi : R \cong T$
PROOF: By Proposition 1.5.17.
⟨1⟩4. $i = k \circ \psi \circ \phi$
□

Definition 1.5.23 (Inclusion)**.** Let $i : U \hookrightarrow A$ and $j : V \hookrightarrow A$ be parts of A. Then i is *included* in j, $i \subseteq_A j$, iff there exists a function $\phi : U \to V$ such that $i = j \circ \phi$.

Proposition 1.5.24. *If $i \equiv_A i'$ and $j \equiv_A j'$ and $i \subseteq_A j$ then $i' \subseteq_A j'$.*

PROOF:
⟨1⟩1. LET: $i : S \hookrightarrow A$, $i' : S' \hookrightarrow A$, $j : T \hookrightarrow A$, $j' : T' \hookrightarrow A$
⟨1⟩2. PICK $\phi : S \cong S'$, $\psi : T \cong T'$ and $\chi : S \to T$ such that $i = i' \circ \phi$, $j = j' \circ \psi$ and $i = j \circ \chi$
⟨1⟩3. $\psi \circ \chi \circ \phi^{-1} : S' \to T'$
⟨1⟩4. $i' = j' \circ \psi \circ \chi \circ \phi^{-1}$
□

Proposition 1.5.25. *For any part i of A we have $i \subseteq_A i$.*

PROOF:
⟨1⟩1. LET: $i : S \hookrightarrow A$
⟨1⟩2. $\mathrm{id}_S : S \to S$
⟨1⟩3. $i = i \circ \mathrm{id}_S$
□

Proposition 1.5.26. *If $i \subseteq_A j$ and $j \subseteq_A k$ then $i \subseteq_A k$.*

PROOF:

$\langle 1\rangle 1.$ LET: $i : R \hookrightarrow A$, $j : S \hookrightarrow A$ and $k : T \hookrightarrow A$
$\langle 1\rangle 2.$ PICK $\phi : R \to S$ and $\psi : S \to T$ such that $i = j \circ \phi$ and $j = k \circ \psi$
$\langle 1\rangle 3.$ $\psi \circ \phi : R \to T$
$\langle 1\rangle 4.$ $i = k \circ \psi \circ \phi$
□

Proposition 1.5.27. *If $i \subseteq_A j$ and $j \subseteq_A i$ then $i \equiv_A j$.*

PROOF:
$\langle 1\rangle 1.$ LET: $i : R \hookrightarrow A$, $j : S \hookrightarrow A$
$\langle 1\rangle 2.$ PICK $\phi : R \to S$ and $\phi^{-1} : S \to R$ such that $i = j \circ \phi$ and $j = i \circ \phi^{-1}$
$\langle 1\rangle 3.$ $\phi \circ \phi^{-1} = \mathrm{id}_S$
  $\langle 2\rangle 1.$ $j \circ \phi \circ \phi^{-1} = j$
  $\langle 2\rangle 2.$ Q.E.D.
    PROOF: The result follows because j is injective.
$\langle 1\rangle 4.$ $\phi^{-1} \circ \phi = \mathrm{id}_T$
  PROOF: Similar.
□

1.5.3 The Empty Set

Axiom 1.5.28 (Empty Set)**.** *There exists a set $\emptyset$, the* empty set*, such that, for every set X, there exists a unique function $¡_X : \emptyset \to X$.*

Proposition 1.5.29 (Uniqueness of Empty Set)**.** *Let E be any set. Then E is empty if and only if there exists an isomorphism $E \cong \emptyset$, in which case the isomorphism is unique.*

PROOF:
$\langle 1\rangle 1.$ If E is empty then $E \cong \emptyset$
  $\langle 2\rangle 1.$ ASSUME: E is empty
  $\langle 2\rangle 2.$ LET: ϕ be the unique function $E \to \emptyset$
  $\langle 2\rangle 3.$ $¡_E \circ \phi = \mathrm{id}_E$
    PROOF: There is only one function $E \to E$.
  $\langle 2\rangle 4.$ $\phi \circ ¡_E = \mathrm{id}_\emptyset$
    PROOF: There is only one function $\emptyset \to \emptyset$.
$\langle 1\rangle 2.$ If $E \cong \emptyset$ then E is empty
  $\langle 2\rangle 1.$ LET: $\phi : E \cong \emptyset$
  $\langle 2\rangle 2.$ LET: X be a set
    PROVE: There is a unique function $E \to X$
  $\langle 2\rangle 3.$ $¡_X \circ \phi : E \to X$
  $\langle 2\rangle 4.$ If $f : E \to X$ then $f = ¡_X \circ \phi$
    $\langle 3\rangle 1.$ LET: $f : E \to X$
    $\langle 3\rangle 2.$ $f \circ \phi^{-1} : \emptyset \to X$
    $\langle 3\rangle 3.$ $f \circ \phi^{-1} = ¡_X$
      PROOF: Uniqueness of $¡_X$.
    $\langle 3\rangle 4.$ Q.E.D.

⟨1⟩3. There is at most one isomorphism $E \cong \emptyset$

PROOF: This holds because there is at most one function $E \to \emptyset$.

□

Proposition 1.5.30.

$$¡_\emptyset = \mathrm{id}_\emptyset$$

PROOF: By the uniqueness of $¡_\emptyset$. □

1.5.4 The Terminal Set

Axiom 1.5.31 (Terminal Set)**.** *There exists a set* 1*, the* terminal set*, such that, for every set* X*, there exists a unique function* $!_X : X \to 1$*.*

Proposition 1.5.32 (Uniqueness of Terminal Set)**.** *Let* T *be any set. Then* T *is terminal if and only if there exists an isomorphism* $T \cong 1$*, in which case the isomorphism is unique.*

PROOF: Dual to Proposition 1.5.29.

Proposition 1.5.33.

$$!_1 = \mathrm{id}_1$$

PROOF: From the uniqueness of $!_1$. □

Definition 1.5.34 (Element)**.** An *element* of a set A is a function $1 \to A$. We write $a \in A$ for $a : 1 \to A$. We write $f(a)$ for $f \circ a$ when $f : A \to B$ and $a \in A$.

Axiom 1.5.35 (Extensionality)**.** *Let* A *and* B *be sets and* $f, g : A \to B$ *be functions. If, for all* $a \in A$*, we have* $f(a) = g(a) \in B$*, then* $f = g$*.*

Proposition 1.5.36. *Let* $f : A \to B$*. Then* f *is injective if and only if, for all* $x, y \in A$*, if* $f(x) = f(y) \in B$ *then* $x = y \in A$*.*

PROOF:

⟨1⟩1. If f is injective and $f(x) = f(y) \in B$ then $x = y \in A$

PROOF: Immediate from the definition of injective.

⟨1⟩2. If, for all $x, y \in A$, if $f(x) = f(y) \in B$ then $x = y \in A$

⟨2⟩1. ASSUME: For all $x, y \in A$, if $f(x) = f(y)$, then $x = y$

⟨2⟩2. LET: X be any set and $g, h : X \to A$ with $f \circ g = f \circ h$

PROVE: $g = h$

⟨2⟩3. LET: $x \in X$

PROVE: $g(x) = h(x)$

⟨2⟩4. $f(g(x)) = f(h(x))$

PROOF: From ⟨2⟩2.

⟨2⟩5. $g(x) = h(x)$

PROOF: By ⟨2⟩1

□

Proposition 1.5.37. *Any element* $e \in X$ *is a section of the unique function* $!_X : X \to 1$*.*

PROOF:$!_X \circ e = \mathrm{id}_1$ because there is only one function $1 \to 1$. □

Axiom 1.5.38 (Non-degeneracy)**.** *The empty set $\emptyset$ has no elements.*

Proposition 1.5.39. *For any set X, the function $¡_X : \emptyset \to X$ is injective.*

PROOF: From Proposition 1.5.36. □

Definition 1.5.40 (Empty Part)**.** For any set X, the *empty part* of X is $\emptyset = ¡_X : \emptyset \hookrightarrow X$.

Definition 1.5.41 (Constant Function)**.** A function $f : A \to B$ is *constant* iff there exists $b \in B$ such that $f = b \circ !_A$.

Definition 1.5.42 (Membership)**.** Let $i : U \hookrightarrow A$ be a part of A and $a \in A$. Then a is a *member* of i, $a \in_A i$, iff there exists $\overline{a} \in U$ such that $i(\overline{a}) = a$.

Proposition 1.5.43. *Let A be a set. Let i, j be parts of A and $a \in A$. If $a \in_A i$ and $i \subseteq_A j$ then $a \in_A j$.*

PROOF:
$\langle 1 \rangle 1.$ PICK $\overline{a} \in \mathrm{dom}\, i$ such that $a = i(\overline{a})$.
$\langle 1 \rangle 2.$ PICK $\phi : \mathrm{dom}\, i \to \mathrm{dom}\, j$ such that $i = j \circ \phi$
$\langle 1 \rangle 3.$ $a = j(\phi(\overline{a}))$
□

1.5.5 Products

Axiom 1.5.44 (Products)**.** *For any sets A and B, there exists a set $A \times B$, the* product *of A and B, and functions $\pi_1 : A \times B \to A$, $\pi_2 : A \times B \to B$, the* projections*, such that, for any set C and functions $f : C \to A$, $g : C \to B$, there exists a unique function $\langle f, g \rangle : C \to A \times B$ such that*

$$\pi_1 \circ \langle f, g \rangle = f; \qquad \pi_2 \circ \langle f, g \rangle = g \ .$$

Definition 1.5.45. Given functions $f : A \to B$ and $g : C \to D$, define $f \times g : A \times C \to B \times D$ by

$$f \times g = \langle f \circ \pi_1, g \circ \pi_2 \rangle$$

1.5.6 Coproducts

Axiom 1.5.46 (Coproducts)**.** *For any sets A and B, there exists a set $A \uplus B$, the* coproduct *or* sum *of A and B, and functions $\kappa_1 : A \to A \uplus B$, $\kappa_2 : B \to A \uplus B$, the* injections*, such that, for any set C and functions $f : A \to C$, $g : B \to C$, there exists a unique function $[f, g] : A \uplus B \to C$ such that*

$$[f, g] \circ \kappa_1 = f; \qquad [f, g] \circ \kappa_2 = g \ .$$

Definition 1.5.47 (Complement)**.** Let $i : I \hookrightarrow J$ and $i' : I' \hookrightarrow J$ be parts of J. Then i' is the *complement* of i iff J is the sum of I and I' with injections i and i'.

1.5.7 Equalizers

Axiom 1.5.48 (Equalizers)**.** *For any sets A and B and functions $f, g : A \to B$, there exists a set E and function $e : E \to A$, the* equalizer *of A and B, such that:*

- $f \circ e = g \circ e : E \to B$;
- *For any set C and function $h : C \to A$ such that $f \circ h = g \circ h$, there exists a unique function $\overline{h} : C \to E$ such that $h = e \circ \overline{h}$.*

Proposition 1.5.49. *All equalizers are injective.*

PROOF:
$\langle 1 \rangle 1.$ LET: $e : E \to A$ be the equalizer of $f, g : A \to B$
$\langle 1 \rangle 2.$ LET: $x, y : X \to E$ with $e \circ x = e \circ y$
$\langle 1 \rangle 3.$ $f \circ e \circ x = g \circ e \circ x$
PROOF: $f \circ e = g \circ e$ by $\langle 1 \rangle 11$.
$\langle 1 \rangle 4.$ $x = y$
PROOF: x and y are both the unique $z : X \to E$ such that $e \circ z = e \circ x$.
□

1.5.8 Coequalizers

Axiom 1.5.50 (Coequalizers)**.** *For any sets A and B and functions $f, g : A \to B$, there exists a set C and function $c : B \to C$, the* coequalizer *of f and g, such that:*

- $c \circ f = c \circ g : A \to C$
- *For any set X and function $h : B \to X$ such that $h \circ f = h \circ g$, there exists a unique function $\overline{h} : C \to X$ such that $\overline{h} \circ c = h$.*

1.5.9 Pullbacks

Definition 1.5.51 (Pullback)**.** The diagram below is a *pullback diagram* iff:

- $f \circ p = g \circ q$
- for every set X and functions $x : X \to B$ and $y : X \to C$ such that $f \circ x = g \circ y$, there exists a unique function $\langle x, y \rangle : X \to A$ such that $p \circ \langle x, y \rangle = x$ and $q \circ \langle x, y \rangle = y$.

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[d, "q"'] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

Proposition 1.5.52. *Let $f : A \to C$ and $g : B \to C$. Then f and g have a pullback.*

$$\begin{tikzcd}
E \arrow[dr, "e"] & & \\
& A \times B \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & A \arrow[d, "f"] \\
& B \arrow[r, "g"'] & C
\end{tikzcd}$$

PROOF:
⟨1⟩1. Construct the product $\pi_1 : A \times B \to A$, $\pi_2 : A \times B \to B$.
⟨1⟩2. Construct the equalizer $e : E \to A$ of $f \circ \pi_1$ and $g \circ \pi_2$.
PROVE: $\pi_1 \circ e$ and $\pi_2 \circ e$ form a pullback of f and g
⟨1⟩3. $f \circ \pi_1 \circ e = g \circ \pi_2 \circ e$
⟨1⟩4. LET: X be a set and $x : X \to A$, $y : X \to B$ satisfy $f \circ x = g \circ y$
⟨1⟩5. $f \circ \pi_1 \circ \langle x, y \rangle = g \circ \pi_2 \circ \langle x, y \rangle$
⟨1⟩6. LET: $m : X \to E$ be the function such that $e \circ m = \langle x, y \rangle$
⟨1⟩7. $\pi_1 \circ e \circ m = x$ and $\pi_2 \circ e \circ m = y$
⟨1⟩8. m is unique.
PROOF:
⟨2⟩1. LET: $n : X \to E$ be such that $\pi_1 \circ e \circ n = x$ and $\pi_2 \circ e \circ n = y$
⟨2⟩2. $e \circ n = \langle x, y \rangle$
⟨2⟩3. $n = m$
PROOF: By ⟨1⟩6
□

Proposition 1.5.53. *Pullbbacks are unique up to isomorphism.*

That is, let P be a pullback of $f : A \to C$ and $g : B \to C$ with projections $p : P \to A$ and $q : P \to B$. Let Q be a set and $p' : Q \to A$, $q' : Q \to B$. Then Q is a pullback of f and g with projections p' and q' if and only if there exists a bijection $\phi : Q \cong P$ such that $p \circ \phi = p'$ and $q \circ \phi = q'$, in which case ϕ is unique.

$$\begin{tikzcd}
Q \arrow[dr, "\phi"] \arrow[drr, bend left, "p'"] \arrow[ddr, bend right, "q'"'] & & \\
& P \arrow[r, "p"'] \arrow[d, "q"] & A \arrow[d, "f"] \\
& B \arrow[r, "g"'] & C
\end{tikzcd}$$

PROOF:
⟨1⟩1. If Q is a pullback then there exists a bijection $\phi : Q \cong P$ such that $p \circ \phi = p'$ and $q \circ \phi = q'$
⟨2⟩1. ASSUME: Q is a pullback with projections p' and q'

⟨2⟩2. LET: $\phi : Q \to P$ be the unique function such that $p \circ \phi = p'$ and $q \circ \phi = q'$

PROOF: Such a ϕ exists because $f \circ p' = g \circ q'$.

⟨2⟩3. LET: $\phi^{-1} : P \to Q$ be the unique function such that $p' \circ \phi^{-1} = p$ and $q' \circ \phi^{-1} = q$

PROOF: Such a function exists because $f \circ p = g \circ q$.

⟨2⟩4. $\phi \circ \phi^{-1} = \mathrm{id}_P$

PROOF: Each is the unique function x such that $p \circ x = p$ and $q \circ x = q$.

⟨2⟩5. $\phi^{-1} \circ \phi = \mathrm{id}_Q$

PROOF: Similar.

⟨1⟩2. If $\phi : Q \cong P$ is a bijection then Q is a pullback with projections $p \circ \phi$ and $q \circ \phi$

⟨2⟩1. $f \circ p \circ \phi = g \circ q \circ \phi$

PROOF: This holds because $f \circ p = g \circ q$

⟨2⟩2. For any set X and functions $x : X \to A$, $y : X \to B$ such that $f \circ x = g \circ y$, there exists a unique function $m : X \to Q$ such that $p \circ \phi \circ m = x$ and $q \circ \phi \circ m = y$

PROOF:

$$\begin{aligned} & p \circ \phi \circ m = x \text{ and } q \circ \phi \circ m = y \\ \Leftrightarrow & \phi \circ m = \langle x, y \rangle \\ \Leftrightarrow & m = \phi^{-1} \circ \langle x, y \rangle \end{aligned}$$

⟨1⟩3. If $\phi, \phi' : P \cong Q$ are bijections such that $p \circ \phi = p \circ \phi'$ and $q \circ \phi = q \circ \phi'$

PROOF: This follows from the definition of pullback.

□

Proposition 1.5.54. *The pullback of an injective function is injective.*

That is, if the diagram below is a pullback diagram and f is injective then q is injective.

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[d, tail, "q"'] & B \arrow[d, tail, "f"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

PROOF:

⟨1⟩1. LET: X be a set and $x, y : X \to A$ with $q \circ x = q \circ y$

⟨1⟩2. $f \circ p \circ x = g \circ q \circ x$

⟨1⟩3. LET:

$z : X \to A$ be the function such that $p \circ z = p \circ x$ and $q \circ z = q \circ x$

⟨1⟩4. $z = x$

⟨1⟩5. $z = y$

⟨2⟩1. $q \circ x = q \circ y$

PROOF: By ⟨1⟩1.

⟨2⟩2. $f \circ p \circ x = f \circ p \circ y$

PROOF:

$$
\begin{aligned}
f \circ p \circ x &= g \circ q \circ x && (\langle 1 \rangle 2) \\
&= g \circ q \circ y && (\langle 1 \rangle 1) \\
&= f \circ p \circ y && \text{(the diagram is a pullback)}
\end{aligned}
$$

$\langle 2 \rangle 3.$ $p \circ x = p \circ y$

PROOF: f is injective.

□

1.5.10 Function Sets

Axiom 1.5.55 (Function Sets)**.** *For any sets A and B, there exists a set A^B and a function $\epsilon : A^B \times B \to A$, the* evaluation *function, such that, for any set C and function $f : C \times B \to A$, there exists a unique function $\lambda f : C \to A^B$ such that*

$$\epsilon \circ (\lambda f \times \mathrm{id}_B) = f \quad .$$

1.5.11 The Subset Classifier

Definition 1.5.56. The set 2 is $1 + 1$. We write $\top$ (*truth*) for $\kappa_1 : 1 \to 2$, and $\bot$ (*falsehood*) for $\kappa_2 : 1 \to 2$.

Axiom 1.5.57 (Subset Classifier)**.** *For every injective function $m : A \rightarrowtail B$, there exists a unique function $\chi_m : B \to 2$, the* characteristic function *of m, such that the following diagram is a pullback diagram:*

$$
\begin{tikzcd}
A \arrow[r, "!"] \arrow[d, "m"'] & 1 \arrow[d, "\top"] \\
B \arrow[r, "\chi_m"'] & 2
\end{tikzcd}
$$

Proposition 1.5.58. *Every function $\phi : A \to 2$ is the characteristic function of a part of A.*

PROOF:

$\langle 1 \rangle 1.$ Construct a pullback

$$
\begin{tikzcd}
I \arrow[r] \arrow[d, "q"] & 1 \arrow[d, "\top"] \\
A \arrow[r, "\phi"'] & 2
\end{tikzcd}
$$

PROOF: By Proposition 1.5.52.

$\langle 1 \rangle 2.$ q is injective

PROOF: By Proposition 1.5.54.

□

Axiom 1.5.59 (Boolean)**.** *For any $p \in 2$ we have $p = \top$ or $p = \bot$.*

Proposition 1.5.60. *Let $i : U \hookrightarrow A$ and $j : V \hookrightarrow A$ be parts of A. Then the following are equivalent:*

1. *$i \subseteq_A j$ and $j \subseteq_A i$*
2. *There exist $h : U \to V$ and $k : V \to U$ such that $i = j \circ h$, $j = i \circ k$, $k \circ h = \mathrm{id}_U$ and $h \circ k = \mathrm{id}_V$.*
3. *The characteristic function of i is the characteristic function of j.*

PROOF:
⟨1⟩1. $1 \Rightarrow 2$
  ⟨2⟩1. ASSUME: $i \subseteq_A j$ and $j \subseteq_A i$
  ⟨2⟩2. LET: $h : U \to V$ be such that $i = j \circ h$
  ⟨2⟩3. LET: $k : V \to U$ be such that $j = i \circ k$
  ⟨2⟩4. $k \circ h = \mathrm{id}_U$
    ⟨3⟩1. $i \circ k \circ h = i$
      PROOF: From ⟨2⟩2 and ⟨2⟩3.
    ⟨3⟩2. Q.E.D.
      PROOF: Since i is injective.
  ⟨2⟩5. $h \circ k = \mathrm{id}_V$
    PROOF: Similar.
⟨1⟩2. $2 \Rightarrow 1$
  PROOF: Trivial.
⟨1⟩3. $2 \Rightarrow 3$
  ⟨2⟩1. ASSUME: 2
  ⟨2⟩2. LET: $\phi : A \to 2$ be the characteristic function of i
     PROVE: ϕ is the characteristic function of j

$$\begin{tikzcd}
X \arrow[dr] \arrow[ddrrr] \arrow[dddrr] & & & \\
& V \arrow[dr] \arrow[drr] \arrow[ddr] & & \\
& & U \arrow[r] \arrow[d] & 1 \arrow[d, "\top"] \\
& & A \arrow[r, "\phi"'] & 2
\end{tikzcd}$$

    PROOF: By the Subset Classifier Axiom.
  ⟨2⟩3. LET: X be a set and $x : X \to 1$, $y : X \to A$ satisfy $\phi \circ y = \top \circ x$
  ⟨2⟩4. LET: $\langle x, y \rangle : X \to U$ be the unique function such that $! \circ \langle x, y \rangle = x$ and $i \circ \langle x, y \rangle = y$
    PROOF: By ⟨2⟩2.
  ⟨2⟩5. $h \circ \langle x, y \rangle$ is the unique function $X \to V$ such that $! \circ h \circ \langle x, y \rangle = x$ and $j \circ h \circ \langle x, y \rangle = y$
    ⟨3⟩1. $! \circ h \circ \langle x, y \rangle = x$
      PROOF: Since 1 is terminal.

$\langle 3 \rangle 2.$ $j \circ h \circ \langle x, y \rangle = y$
PROOF: From $\langle 2 \rangle 1$ and $\langle 2 \rangle 4$.
$\langle 3 \rangle 3.$ If $! \circ f = x$ and $j \circ f = y$ then $f = h \circ \langle x, y \rangle$
$\langle 4 \rangle 1.$ LET: $f : X \to V$ satisfy $! \circ f = x$ and $j \circ f = y$
$\langle 4 \rangle 2.$ $! \circ k \circ f = x$
PROOF: As 1 is terminal.
$\langle 4 \rangle 3.$ $i \circ k \circ f = y$
PROOF: From $\langle 2 \rangle 1$ and $\langle 4 \rangle 1$.
$\langle 4 \rangle 4.$ $k \circ f = \langle x, y \rangle$
PROOF: From $\langle 2 \rangle 4$, $\langle 4 \rangle 2$ and $\langle 4 \rangle 3$.
$\langle 4 \rangle 5.$ $f = h \circ \langle x, y \rangle$
PROOF: From $\langle 2 \rangle 1$ and $\langle 4 \rangle 4$.
$\langle 1 \rangle 4.$ $3 \Rightarrow 2$

$$\begin{tikzcd}
U \arrow[dr, shift left] \arrow[drr] \arrow[ddr, "i"'] & & \\
& V \arrow[ul, shift left] \arrow[r] \arrow[d, "j"'] & 1 \arrow[d, "\top"] \\
& A \arrow[r, "\phi"'] & 2
\end{tikzcd}$$

$\langle 2 \rangle 1.$ ASSUME: 3
$\langle 2 \rangle 2.$ LET: ϕ be the characteristic function of i and j
$\langle 2 \rangle 3.$ LET: $h : U \to V$ be the unique function such that $! \circ h =!$ and $j \circ h = i$
$\langle 3 \rangle 1.$ $\top \circ ! = \phi \circ i$
PROOF: This holds because ϕ is the characteristic function of i.
$\langle 3 \rangle 2.$ Q.E.D.
PROOF: Since ϕ is the characteristic function of j.
$\langle 2 \rangle 4.$ LET: $k : V \to U$ be the unique function such that $! \circ k =!$ and $i \circ k = j$
PROOF: Similar.
$\langle 2 \rangle 5.$ $k \circ h = \mathrm{id}_U$
PROOF: Each is the unique function f such that $! \circ f =!$ and $i \circ f = i$
$\langle 2 \rangle 6.$ $h \circ k = \mathrm{id}_V$
PROOF: Each is the unique function f such that $! \circ f =!$ and $j \circ f = j$
□

1.6 The Basics

Lemma 1.6.1. *Let X be a set, $\mathcal{B} \subseteq \mathcal{P}X$ and $U \subseteq X$. Then the following are equivalent:*

1. *For all $x \in U$ there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$.*
2. *There exists $\mathcal{B}_0 \subseteq \mathcal{B}$ such that $U = \bigcup \mathcal{B}_0$.*

PROOF:
$\langle 1 \rangle 1.$ $1 \Rightarrow 2$

PROOF: If 1 is true then $U = \bigcup\{B \in \mathcal{B} : B \subseteq U\}$.
$\langle 1 \rangle 2$. $2 \Rightarrow 1$
PROOF: Trivial.
□

Definition 1.6.2 (Fixed Point)**.** Let X be a set, $f : X \to X$, and $x \in X$. Then x is a *fixed point* of f iff $f(x) = x$.

Definition 1.6.3 (Saturated)**.** Let X, Y be sets and $p : X \twoheadrightarrow Y$ be a surjective function. Let $C \subseteq X$. Then C is *saturated* with respect to p iff, for all $x, x' \in X$, if $x \in C$ and $p(x) = p(x')$ then $x' \in C$.

Definition 1.6.4 (Cover)**.** Let A be a set and $\mathcal{C} \subseteq \mathcal{P}A$. Then $\mathcal{C}$ *covers* A iff $\bigcup \mathcal{C} = A$.

Definition 1.6.5 (Finite Intersection Property)**.** Let X be a set and $\mathcal{C} \subseteq \mathcal{P}X$. Then $\mathcal{C}$ has the *finite intersection property* if and only if every finite nonempty subset of $\mathcal{C}$ has nonempty intersection.

Lemma 1.6.6 (AC)**.** *Let X be a set and $\mathcal{A} \subseteq \mathcal{P}X$ have the finite intersection property. Then there exists a maximal $\mathcal{D} \subseteq \mathcal{P}X$ that has the finite intersection property and includes $\mathcal{A}$.*

PROOF: A straightforward application of Zorn's lemma, since the union of a chain of sets that has the finite intersection property has the finite intersection property. □

Lemma 1.6.7. *Let X be a set and $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the finite intersection property. Then any finite intersection of elements of $\mathcal{D}$ is an element of $\mathcal{D}$.*

PROOF:
$\langle 1 \rangle 1$. LET: A be a finite intersection of elements of $\mathcal{D}$
$\langle 1 \rangle 2$. $\mathcal{D} \cup \{A\}$ has the finite intersection property.
$\langle 1 \rangle 3$. $\mathcal{D} \cup \{A\} = \mathcal{D}$
□

Lemma 1.6.8. *Let X be a set and $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the finite intersection property. If $A \subseteq X$ intersects every element of $\mathcal{D}$ then $A \in \mathcal{D}$.*

PROOF: This holds because $\mathcal{D} \cup \{A\}$ satisfies the finite intersection property. □

Definition 1.6.9 (Graph)**.** Let $f : A \to B$. The *graph* of f is the set $\{(x, f(x)) : x \in A\} \subseteq A \times B$.

Definition 1.6.10 (Point-Finite)**.** Let X be a set and $\{A_\alpha\}_{\alpha \in J}$ be a family of subsets of X. Then $\{A_\alpha\}_{\alpha \in J}$ is *point-finite* iff, for all $x \in X$, there are only finitely many $\alpha \in J$ such that $x \in A_\alpha$.

Definition 1.6.11 (Countable Intersection Property)**.** A family of parts of a set X has the *countable intersection property* iff every countable subfamily has nonempty intersection.

1.7 Refinements

Definition 1.7.1 (Refinement)**.** Let X be a set and $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}X$. Then $\mathcal{B}$ is a *refinement* of $\mathcal{A}$ iff, for all $B \in \mathcal{B}$, there exists $A \in \mathcal{A}$ such that $B \subseteq A$.

1.8 Order Theory

Definition 1.8.1 (Cofinal)**.** Let J be a poset and $K \subseteq J$. Then K is *cofinal* iff, for all $x \in J$, there exists $y \in K$ such that $x \leq y$.

Definition 1.8.2 (Directed Set)**.** A *directed set* is a poset J such that, for all $x, y \in J$, there exists $z \in J$ such that $x \leq z$ and $y \leq z$.

Definition 1.8.3 (Linear Order)**.** Let X be a set. A *linear order* on X is a relation $\leq \subseteq X^2$ such that:

- For all $x \in X$, $x \leq x$
- For all $x, y, z \in X$, if $x \leq y$ and $y \leq z$ then $x \leq z$
- For all $x, y \in X$, if $x \leq y$ and $y \leq x$ then $x = y$
- For all $x, y \in X$, we have $x \leq y$ or $y \leq x$

We write $x < y$ iff $x \leq y$ and $x \neq y$.

A *linearly ordered set* consists of a set and a linear order on the set.

Definition 1.8.4 (Convex)**.** Let L be a linearly ordered set and $A \subseteq L$. Then A is *convex* iff, for all $x, y \in A$ and $z \in L$, if $x < z < y$ then $z \in A$.

Definition 1.8.5 (Least Upper Bound Property)**.** A linearly ordered set L has the *least upper bound property* iff every subset of L bounded above has a least upper bound.

Definition 1.8.6 (Linear Continuum)**.** A *linear continuum* is a linearly ordered set L such that:

- L has the least upper bound property.
- For all $x, y \in L$ with $x < y$, there exists $z \in L$ such that $x < z < y$.

Proposition 1.8.7. *If L is a linear continuum then every convex subset of L is a linear continuum.*

PROOF:
$\langle 1 \rangle 1.$ LET: L be a linear continuum and $C \subseteq L$ be convex
$\langle 1 \rangle 2.$ C satisfies the least upper bound property.
  $\langle 2 \rangle 1.$ LET: $S \subseteq C$ be nonempty and bounded above by u in C.
  $\langle 2 \rangle 2.$ LET: s be the supremum of S in L
  $\langle 2 \rangle 3.$ PICK $x \in S$

$\langle 2\rangle 4.$ $x \leq s \leq u$
$\langle 2\rangle 5.$ $s \in C$
Proof: C is convex.
$\langle 2\rangle 6.$ s is the supremum of S in C
$\langle 1\rangle 3.$ C is dense.
Proof:
$\langle 2\rangle 1.$ Let: $x, y \in C$ satisfy $x < y$
$\langle 2\rangle 2.$ Pick $z \in L$ such that $x < z < y$
$\langle 2\rangle 3.$ $z \in C$
Proof: C is convex.
□

Lemma 1.8.8. *For any real numbers a, b with $a < b$ we have $[a, b) \cong [0, 1)$.*

Proof: The map $\phi : [a, b) \cong [0, 1)$ where $\phi(x) = (x - a)/(b - a)$ is an order isomorphism. □

Proposition 1.8.9. *Let X be a linearly ordered set. Let $a, b, c \in X$ with $a < c < b$. Then $[a, b) \cong [0, 1)$ if and only if $[a, c) \cong [c, b) \cong [0, 1)$.*

Proof:
$\langle 1\rangle 1.$ If $[a, b) \cong [0, 1)$ then $[a, c) \cong [c, b) \cong [0, 1)$.
$\langle 2\rangle 1.$ Assume: $\phi : [a, b) \cong [0, 1)$ is an order isomorphism.
$\langle 2\rangle 2.$ $[a, c) \cong [0, 1)$
Proof:

$$\begin{aligned} [a, c) &\cong [0, \phi(c)) && \text{(under } \phi\text{)} \\ &\cong [0, 1) && \text{(Lemma 1.7.8)} \end{aligned}$$

$\langle 2\rangle 3.$ $[c, b) \cong [0, 1)$
Proof: Similar.
$\langle 1\rangle 2.$ If $[a, c) \cong [c, b) \cong [0, 1)$ then $[a, b) \cong [0, 1)$.
$\langle 2\rangle 1.$ Assume: $[a, c) \cong [c, b) \cong [0, 1)$
$\langle 2\rangle 2.$ Let: $\phi : [a, c) \cong [0, 1/2)$ and $\psi : [c, b) \cong [1/2, 1)$
$\langle 2\rangle 3.$ Let: $\chi : [a, b) \to [0, 1)$ be given by $\chi(x) = \begin{cases} \phi(x) & \text{if } x < c \\ \psi(x) & \text{if } x \geq c \end{cases}$
$\langle 2\rangle 4.$ $\chi : [a, b) \cong [0, 1)$
Proof: Easy to check.
□

Proposition 1.8.10 (CC)**.** *Let X be a linearly ordered set. Let $\{x_n\}_{n \geq 0}$ be an increasing sequence of points of X. Suppose b is the supremum of $\{x_n : n \geq 0\}$. Then $[x_0, b) \cong [0, 1)$ if and only if $[x_i, x_{i+1}) \cong [0, 1)$ for all i.*

Proof:
$\langle 1\rangle 1.$ If $[x_0, b) \cong [0, 1)$ then for all i $[x_i, x_{i+1}) \cong [0, 1)$.
Proof: If $\phi : [x_0, b) \cong [0, 1)$ then $[x_i, x_{i+1}) \cong [\phi(x_i), \phi(x_{i+1})) \cong [0, 1)$ by Lemma 1.7.8.
$\langle 1\rangle 2.$ If for all i $[x_i, x_{i+1}) \cong [0, 1)$ then $[x_0, b) \cong [0, 1)$.

PROOF:
$\langle 2 \rangle 1$. LET: $\phi_i : [x_i, x_{i+1}) \cong [0,1)$ for all i
$\langle 2 \rangle 2$. Define $\phi : [x_0, b) \cong [0,1)$ by: $\phi(y) = \phi_i(y) \qquad (x_0 \leq y < b)$ where i is least such that $y < i_{i+1}$
PROOF: There exists such an i because y is not an upper bound for $\{x_n : n \geq 0\}$.
$\langle 2 \rangle 3$. ϕ is an order isomorphism.
PROOF: Easy to check.
□

Proposition 1.8.11 (CC). *For all $0 < \alpha < \Omega$, the interval $[(0,0),(\alpha,0))$ in $S_\Omega \times [0,1)$ is order isomorphic to $[0,1)$ in $\mathbb{R}$.*

PROOF:
$\langle 1 \rangle 1$. If $[(0,0),(\alpha,0)) \cong [0,1)$ then $[(0,0),(\alpha+1,0)) \cong [0,1)$
PROOF: By Proposition 1.7.9.
$\langle 1 \rangle 2$. Let λ be a limit ordinal, $0 < \lambda < \Omega$. If, for all α with $0 < \alpha < \lambda$, we have $[(0,0),(\alpha,0)) \cong [0,1)$, then $[(0,0),(\lambda,0)) \cong [0,1)$.
PROOF: By Propositoin 1.7.10.
$\langle 1 \rangle 3$. Q.E.D.
PROOF: By transfinite induction.
□

Chapter 2

Real Analysis

Definition 2.0.1 (Cantor Set)**.** Define a sequence of sets $A_n \subseteq [0,1]$ by:

$$\begin{aligned} A_0 &= [0,1] \\ A_{n+1} &= A_n \setminus \bigcup_{k=0}^{3^{n-1}-1} \left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n} \right) \end{aligned}$$

The *Cantor set* is $\bigcap_{n=0}^{\infty} A_n$.

Chapter 3

Topological Spaces

3.1 Topologies

Definition 3.1.1 (Topology)**.** A *topology* on a set X is a set $\mathcal{T} \subseteq \mathcal{P}X$ such that:

1. $X \in \mathcal{T}$;
2. for all $U, V \in \mathcal{T}$, we have $U \cap V \in \mathcal{T}$;
3. For all $\mathcal{A} \subseteq \mathcal{T}$, we have $\bigcup \mathcal{A} \in \mathcal{T}$.

A *topological space* X consists of a set X and a topology on X. The elements of X are called *points* and the elements of $\mathcal{T}$ are called *open sets*.

Proposition 3.1.2. *In any topological space, the empty set is open.*

PROOF: Immediate from axiom 3. □

Definition 3.1.3 (Discrete Topology)**.** The *discrete* topology on a set X is $\mathcal{P}X$.

Definition 3.1.4 (Indiscrete Topology)**.** The *indiscrete* topology on a set X is $\{\emptyset, X\}$.

Definition 3.1.5 (Open Cover)**.** Let X be a topological space. A cover $\mathcal{C} \subseteq \mathcal{P}X$ of X is an *open cover* iff every member of $\mathcal{C}$ is open.

Definition 3.1.6 (Finer, Coarser)**.** Let $\mathcal{T}$, $\mathcal{T}'$ be topologies on a set X. Then $\mathcal{T}$ is *finer* than $\mathcal{T}'$, and $\mathcal{T}'$ is *coarser* than $\mathcal{T}$, iff $\mathcal{T}' \subseteq \mathcal{T}$.

The topology $\mathcal{T}$ is *strictly* finer than $\mathcal{T}'$, and $\mathcal{T}'$ is *strictly* coarser than $\mathcal{T}$, iff $\mathcal{T} \subset \mathcal{T}'$.

The topologies $\mathcal{T}$ and $\mathcal{T}'$ are *comparable* iff $\mathcal{T} \subseteq \mathcal{T}'$ or $\mathcal{T}' \subseteq \mathcal{T}$.

Definition 3.1.7 (Finite Complement Topology)**.** The *finite complement topology* on a set X is $\{U : X \setminus U \text{ is finite}\} \cup \{X\}$.

Definition 3.1.8 (Isolated Point)**.** Let X be a topological space and $a \in X$. Then a is an *isolated point* iff $\{a\}$ is open.

3.2 Neighbourhoods

Definition 3.2.1 (Neighbourhood)**.** Let X be a topological space and $A \subseteq X$. A *neighbourhood* of A is an set that includes an open set that includes A.

A *neighbourhood* of a point a is a neighbourhood of $\{a\}$.

Proposition 3.2.2. *If N is a neighbourhood of A and $B \subseteq A$ then N is a neighbourhood of B.*

PROOF: Immediate from definitions. □

Proposition 3.2.3. *A set U is open if and only if it is a neighbourhood of each of its points.*

PROOF:
⟨1⟩1. LET: X be a topological space and $A \subseteq X$
⟨1⟩2. If U is a neighbourhood of each of its points then A is open.
  ⟨2⟩1. ASSUME: U includes a neighbourhood of each of its points
    PROVE: $U = \bigcup\{V \subseteq U : V \text{ is open}\}$
  ⟨2⟩2. $\bigcup\{V \subseteq U : V \text{ is open}\} \subseteq U$
    PROOF: Set theory.
  ⟨2⟩3. $U \subseteq \bigcup\{V \subseteq U : V \text{ is open}\}$
    PROOF: Immediate from ⟨2⟩1.
⟨1⟩3. If U is open then U is a neighbourhood of each of its points.
  PROOF: Immediate from definitions.
□

Proposition 3.2.4. *If M is a neighbourhood of A and $M \subseteq N$ then N is a neighbourhood of A.*

PROOF: Immediate from definitions. □

Proposition 3.2.5. *If M and N are neighbourhoods of A then $M \cap N$ is a neighbourhood of A.*

PROOF: Pick open sets U and V such that $A \subseteq U \subseteq M$ and $A \subseteq N \subseteq V$. Then $A \subseteq U \cap V \subseteq M \cap N$.

Proposition 3.2.6. *If N is a neighbourhood of x then $x \in N$.*

PROOF: Immediate from definitions. □

Proposition 3.2.7. *If N is a neighbourhood of x then there exists a neighbourhood U of x such that, for all $y \in U$, M is a neighbourhood of y.*

PROOF: Pick an open set U such that $x \in U \subseteq N$. □

Theorem 3.2.8. *Let X be a set and $\rhd \subseteq \mathcal{P}X \times X$ a relation such that:*

1. *If $M \rhd x$ and $M \subseteq N$ then $N \rhd x$*
2. *$X \rhd x$ for all $x \in X$*

3. *If $M \rhd x$ and $N \rhd x$ then $M \cap N \rhd x$*

4. *If $N \rhd x$ then $x \in N$*

5. *If $M \rhd x$ then there exists $N \rhd x$ such that, for all $y \in N$, $M \rhd y$.*

Then there exists a unique topology $\mathcal{T}$ such that $N \rhd x$ iff N is a neighbourhood of x.

PROOF:
⟨1⟩1. LET: $\rhd$ be a relation satisfying 1–3
⟨1⟩2. LET: $\mathcal{T} = \{U \in \mathcal{P}X : \forall x \in U.U \rhd x\}$
⟨1⟩3. $\mathcal{T}$ is a topology.
 ⟨2⟩1. $X \in \mathcal{T}$
  PROOF: By axiom 2
 ⟨2⟩2. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$
  PROOF: By axiom 3
 ⟨2⟩3. For all $\mathcal{A} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{A} \in \mathcal{T}$
  ⟨3⟩1. LET: $x \in \bigcup \mathcal{A}$
  ⟨3⟩2. PICK $U \in \mathcal{A}$ such that $x \in U$
  ⟨3⟩3. $U \rhd x$
  ⟨3⟩4. $\bigcup \mathcal{A} \rhd x$
   PROOF: By axiom 1
⟨1⟩4. In $\mathcal{T}$, $N \rhd x$ iff N is a neighbourhood of x.
 ⟨2⟩1. If $N \rhd x$ then N is a neighbourhood of x
  ⟨3⟩1. ASSUME: $N \rhd x$
  ⟨3⟩2. $x \in N$
   PROOF: By axiom 4
  ⟨3⟩3. LET: $U = \{y \in N : N \rhd y\}$
  ⟨3⟩4. U is open
   ⟨4⟩1. LET: $y \in U$
     PROVE: $U \rhd y$
   ⟨4⟩2. $N \rhd y$
   ⟨4⟩3. PICK $W \rhd y$ such that, for all $z \in W$, $N \rhd z$
    PROOF: By axiom 5
   ⟨4⟩4. $W \subseteq U$
   ⟨4⟩5. $U \rhd y$
    PROOF: By axiom 1
  ⟨3⟩5. $x \in U \subseteq N$
 ⟨2⟩2. If N is a neighbourhood of x then $N \rhd x$
  ⟨3⟩1. LET: N be a neighbourhood of x
  ⟨3⟩2. PICK U open such that $x \in U \subseteq N$
  ⟨3⟩3. $U \rhd x$
   PROOF: By ⟨1⟩2
  ⟨3⟩4. $N \rhd x$
   PROOF: By axiom 1
⟨1⟩5. $\mathcal{T}$ is unique.

PROOF: By Proposition 3.2.3.
∎

Definition 3.2.9 (Sufficiently Close)**.** Let X be a topological space, $a \in X$, and P be a property of points of X. We write "For all x sufficiently close to a, $P(x)$" to mean "There exists a neighbourhood N of a such that, for all $x \in N$, $P(x)$."

3.3 Local Bases

Definition 3.3.1 (Local Basis)**.** Let X be a topological space and $x \in X$. A *local basis* at x is a set $\mathcal{B}$ of open neighbourhoods of x such that every neighbourhood of x includes a member of $\mathcal{B}$. We call the elements of $\mathcal{B}$ *basic open neighbourhoods*.

Proposition 3.3.2. *Let $\mathcal{B}$ be a local basis at x and $M, N \in \mathcal{B}$. Then there exists $P \in \mathcal{B}$ such that $P \subseteq M \cap N$.*

PROOF: This holds because $M \cap N$ is a neighbourhood of x (Proposition 3.2.5).
∎

Proposition 3.3.3. *Let X be a topological space, $x \in X$ and $\mathcal{B} \subseteq \mathcal{P}X$. Then $\mathcal{B}$ is a local basis at x iff $\mathcal{B}$ is a set of open neighbourhoods of x such that every open neighbourhood of x includes a member of $\mathcal{B}$.*

PROOF:
$\langle 1 \rangle 1$. If $\mathcal{B}$ is a local basis at x then $\mathcal{B}$ is a set of open neighbourhoods of x such that every open neighbourhood of x includes a member of $\mathcal{B}$
PROOF: Trivial.
$\langle 1 \rangle 2$. If $\mathcal{B}$ is a set of open neighbourhoods of x such that every open neighbourhood of x includes a member of $\mathcal{B}$ then $\mathcal{B}$ is a local basis at x.
PROOF: Every neighbourhood of x includes an open neighbourhood of x, which therefore includes an element of $\mathcal{B}$.
∎

3.4 Bases

Definition 3.4.1 (Basis for a Topology)**.** Let $(X, \mathcal{T})$ be a topological space. A *basis* for the topology on X is a set of open sets $\mathcal{B}$ such that every open set is a union of members of $\mathcal{B}$. The members of $\mathcal{B}$ are called *basic open sets*, and $\mathcal{T}$ is called the topology *generated* by $\mathcal{B}$.

Proposition 3.4.2. *Let $(X, \mathcal{T})$ be a topological space and $\mathcal{B} \subseteq \mathcal{P}X$. Then the following are equivalent:*

1. *$\mathcal{B}$ is a basis for $\mathcal{T}$.*

2. *A set U is open if and only if, for all $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$.*

3. *$\mathcal{T}$ is the set of all unions of subsets of $\mathcal{B}$.*

4. *Every member of $\mathcal{B}$ is open and, for all $x \in X$ and every open neighbourhood U of x, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$.*

5. *For all $x \in X$, the set $\{B \in \mathcal{B} : x \in B\}$ is a local basis at x.*

PROOF:
$\langle 1 \rangle 1.$ $1 \Rightarrow 2$
  $\langle 2 \rangle 1.$ ASSUME: $\mathcal{B}$ is a basis for the topology $\mathcal{T}$.
  $\langle 2 \rangle 2.$ For all $U \in \mathcal{T}$ and $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B$
    PROOF: Immediate from the definition of basis ($\langle 2 \rangle 1$).
  $\langle 2 \rangle 3.$ For all $U \subseteq X$, if $\forall x \in U . \exists B \in \mathcal{B} . x \in B \subseteq U$ then $U \in \mathcal{T}$
    PROOF: By Proposition 3.2.3.
$\langle 1 \rangle 2.$ $2 \Leftrightarrow 3$
  PROOF: From Lemma 1.6.1.
$\langle 1 \rangle 3.$ $3 \Rightarrow 1$
  PROOF: Trivial.
$\langle 1 \rangle 4.$ $2 \Rightarrow 4$
  PROOF: Trivial.
$\langle 1 \rangle 5.$ $4 \Rightarrow 2$
  PROOF:
  $\langle 2 \rangle 1.$ ASSUME: 4
  $\langle 2 \rangle 2.$ If U is open then, for all $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$
    PROOF: Immediate from $\langle 2 \rangle 1$.
  $\langle 2 \rangle 3.$ If, for all $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$, then U is open.
    PROOF: By Proposition 3.2.3 using the fact that every member of $\mathcal{B}$ is open ($\langle 2 \rangle 1$).
$\langle 1 \rangle 6.$ $4 \Leftrightarrow 5$
  PROOF: From Proposition 3.3.3.
□

Corollary 3.4.2.1. *If $\mathcal{B}$ is a basis for the topology $\mathcal{T}$, then $\mathcal{T}$ is the coarsest topology in which every element of $\mathcal{B}$ is open.*

Lemma 3.4.3. *Let X be a set and $\mathcal{B} \subseteq \mathcal{P}X$. Then $\mathcal{B}$ is a basis for a topology $\mathcal{T}$ on X if and only if:*

1. *$\bigcup \mathcal{B} = X$*

2. *for all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.*

In this case, $\mathcal{T}$ is unique.

PROOF:

$\langle 1\rangle 1.$ If $\mathcal{B}$ is a basis for a topology then $\bigcup \mathcal{B} = X$
$\langle 2\rangle 1.$ ASSUME: $\mathcal{B}$ is a basis for the topology $\mathcal{T}$
$\langle 2\rangle 2.$ LET: $x \in X$
$\langle 2\rangle 3.$ There exists $B \in \mathcal{B}$ such that $x \in B$
PROOF: From the definition of basis, since $X \in \mathcal{T}$. ($\langle 2\rangle 1$, $\langle 2\rangle 2$).
$\langle 1\rangle 2.$ If $\mathcal{B}$ is a basis for a topology then it satisfies condition 2
$\langle 2\rangle 1.$ ASSUME: $\mathcal{B}$ is a basis for the topology $\mathcal{T}$
$\langle 2\rangle 2.$ LET: $B_1, B_2 \in \mathcal{B}$
$\langle 2\rangle 3.$ $B_1, B_2 \in \mathcal{T}$
PROOF: From the definition of basis ($\langle 2\rangle 1$, $\langle 2\rangle 2$).
$\langle 2\rangle 4.$ $B_1 \cap B_2 \in \mathcal{T}$
PROOF: By the definition of topology, the open sets in $\mathcal{T}$ are closed under binary intersection ($\langle 2\rangle 1$, $\langle 2\rangle 3$)
$\langle 2\rangle 5.$ For all $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$
PROOF: From the definition of basis ($\langle 2\rangle 1$, $\langle 2\rangle 4$)
$\langle 1\rangle 3.$ If $\mathcal{B}$ satisfies conditions 1 and 2 then $\mathcal{T} = \{U \subseteq X : \forall x \in U. \exists B \in \mathcal{B}. x \in B \subseteq U\}$ is a topology and $\mathcal{B}$ is a basis for $\mathcal{T}$.
$\langle 2\rangle 1.$ ASSUME: $\mathcal{B}$ satisfies conditions 1 and 2
$\langle 2\rangle 2.$ $X \in \mathcal{T}$
PROOF: For all $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq X$ by condition 1 ($\langle 2\rangle 1$).
$\langle 2\rangle 3.$ For all $\mathcal{A} \subseteq \mathcal{T}$, we have $\bigcup \mathcal{A} \in \mathcal{T}$
$\langle 3\rangle 1.$ LET: $\mathcal{A} \subseteq \mathcal{T}$
$\langle 3\rangle 2.$ LET: $x \in \bigcup \mathcal{A}$
$\langle 3\rangle 3.$ PICK $U \in \mathcal{A}$ such that $x \in U$
PROOF: From $\langle 3\rangle 2$.
$\langle 3\rangle 4.$ PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$
PROOF: Since $U \in \mathcal{T}$, using the definition of $\mathcal{T}$ ($\langle 3\rangle 1$, $\langle 3\rangle 3$)
$\langle 3\rangle 5.$ $x \in B \subseteq \bigcup \mathcal{A}$
PROOF: From $\langle 3\rangle 3$ and $\langle 3\rangle 4$.
$\langle 2\rangle 4.$ For all $U, V \in \mathcal{T}$, we have $U \cap V \in \mathcal{T}$
$\langle 3\rangle 1.$ LET: $U, V \in \mathcal{T}$
$\langle 3\rangle 2.$ LET: $x \in U \cap V$
$\langle 3\rangle 3.$ PICK $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \subseteq U$ and $x \in B_2 \subseteq V$
PROOF: From $\langle 3\rangle 1$, $\langle 3\rangle 2$ and the definition of $\mathcal{T}$.
$\langle 3\rangle 4.$ PICK $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$
PROOF: Using condition 2 ($\langle 2\rangle 1$, $\langle 3\rangle 3$).
$\langle 3\rangle 5.$ $x \in B_3 \subseteq U \cap V$
PROOF: From $\langle 3\rangle 3$ and $\langle 3\rangle 4$.
$\langle 2\rangle 5.$ $\bigcup \mathcal{B} = X$
PROOF: This is condition 1 ($\langle 2\rangle 1$).
$\langle 2\rangle 6.$ For all $U \in \mathcal{T}$ and $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$
PROOF: Immediate from the definition of $\mathcal{T}$.
$\langle 1\rangle 4.$ $\mathcal{T}$ is unique.
PROOF: From Proposition 3.4.2.
$\square$

Corollary 3.4.3.1. *Let X be a set and $\mathcal{B} \subseteq \mathcal{P}X$ be such that $\bigcup \mathcal{B} = X$ and $\mathcal{B}$ is closed under binary intersection. Then $\mathcal{B}$ is a basis for a unique topology on X.*

Lemma 3.4.4. *Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$ on X respectively. Then $\mathcal{T} \subseteq \mathcal{T}'$ if and only if, for all $B \in \mathcal{B}$ and $x \in B$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.*

PROOF:
$\langle 1 \rangle 1$. If $\mathcal{T} \subseteq \mathcal{T}'$ then, for all $B \in \mathcal{B}$ and $x \in B$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.
  $\langle 2 \rangle 1$. ASSUME: $\mathcal{T} \subseteq \mathcal{T}'$
  $\langle 2 \rangle 2$. LET: $B \in \mathcal{B}$ and $x \in B$
  $\langle 2 \rangle 3$. $B \in \mathcal{T}$
    PROOF: This holds because $\mathcal{B} \subseteq \mathcal{T}$ by the definition of basis. ($\langle 2 \rangle 2$)
  $\langle 2 \rangle 4$. $B \in \mathcal{T}'$
    PROOF: From $\langle 2 \rangle 1$ and $\langle 2 \rangle 3$.
  $\langle 2 \rangle 5$. There exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.
$\langle 1 \rangle 2$. If, for all $B \in \mathcal{B}$ and $x \in B$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$, then $\mathcal{T} \subseteq \mathcal{T}'$.
  $\langle 2 \rangle 1$. ASSUME: For all $B \in \mathcal{B}$ and $x \in B$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$
  $\langle 2 \rangle 2$. LET: $U \in \mathcal{T}$
    PROVE: $U \in \mathcal{T}'$
  $\langle 2 \rangle 3$. LET: $x \in U$
  $\langle 2 \rangle 4$. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$
    PROOF: Since $\mathcal{B}$ is a basis for $\mathcal{T}$ ($\langle 2 \rangle 2$, $\langle 2 \rangle 3$).
  $\langle 2 \rangle 5$. PICK $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$
    PROOF: From $\langle 2 \rangle 1$ and $\langle 2 \rangle 4$.
  $\langle 2 \rangle 6$. $x \in B' \subseteq U$
    PROOF: From $\langle 2 \rangle 4$ and $\langle 2 \rangle 5$.
  $\langle 2 \rangle 7$. Q.E.D.
    PROOF: By Proposition 3.4.2.
$\square$

Definition 3.4.5 (Lower Limit Topology)**.** The *lower limit topology* on $\mathbb{R}$ is the one generated by the set of all half-open intervals of the form $[a, b)$. We write $\mathbb{R}_l$ for the topological space consisting of $\mathbb{R}$ under this topology.

We prove this is a topology.

PROOF:
$\langle 1 \rangle 1$. LET: $\mathcal{B}$ be the set of all half-open intervals of the form $[a, b)$.
$\langle 1 \rangle 2$. $\bigcup \mathcal{B} = \mathbb{R}$
  PROOF: For all $x \in \mathbb{R}$, we have $x \in [x, x+1) \in \mathcal{B}$.
$\langle 1 \rangle 3$. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.
  PROOF: If $x \in [a, b) \cap [c, d)$ then $x \in [\max(a, c), \min(b, d)) \subseteq [a, b) \cap [c, d)$.

⟨1⟩4. Q.E.D.
  PROOF: By Lemma 3.4.3.
□

Definition 3.4.6 (K-topology)**.** The *K-topology* on $\mathbb{R}$ is the one generated by the set of all open intervals (a, b) and all sets of the form $(a, b) \setminus K$, where $K = \{1/n : n \in \mathbb{Z}^+\}$. We write $\mathbb{R}_K$ for the topological space consisting of $\mathbb{R}$ under this topology.

We prove this is a topology.

PROOF:
⟨1⟩1. LET: $\mathcal{B} = \{(a, b) : a, b \in \mathbb{R}, a < b\} \cup \{(a, b) \setminus K : a, b \in \mathbb{R}, a < b\}$
⟨1⟩2. $\bigcup \mathcal{B} = \mathbb{R}$
  PROOF: For all $x \in \mathbb{R}$, we have $x \in (x - 1, x + 1) \in \mathcal{B}$.
⟨1⟩3. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.
  ⟨2⟩1. LET: $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$
    PROVE: There exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$
  ⟨2⟩2. CASE: $B_1 = (a, b)$, $B_2 = (c, d)$
    PROOF: Take $B_3 = (\max(a, c), \min(b, d))$
  ⟨2⟩3. CASE: $B_1 = (a, b)$, $B_2 = (c, d) \setminus K$
    PROOF: Take $B_3 = (\max(a, c), \min(b, d)) \setminus K$
  ⟨2⟩4. CASE: $B_1 = (a, b) \setminus K$, $B_2 = (c, d)$
    PROOF: Take $B_3 = (\max(a, c), \min(b, d)) \setminus K$
  ⟨2⟩5. CASE: $B_1 = (a, b) \setminus K$, $B_2 = (c, d) \setminus K$
    PROOF: Take $B_3 = (\max(a, c), \min(b, d)) \setminus K$
⟨1⟩4. Q.E.D.
  PROOF: By Lemma 3.4.3.
□

Lemma 3.4.7. *The lower limit topology and the K-topology are incomparable.*

PROOF: $[0, 1)$ is not open in the K-topology. $(-1, 1) \setminus K$ is not open in the lower limit topology, because there is no half-open interval $[a, b)$ such that $0 \in [a, b) \subseteq (-1, 1) \setminus K$. □

Proposition 3.4.8. *The set of all singletons is a basis for any discrete space.*

PROOF: Easy. □

Definition 3.4.9 (Line with Two Origins)**.** The *line with two origins* is the set $\mathbb{R} \setminus \{0\} \cup \{p, q\}$ under the topology generated by the basis consisting of:

- all open intervals in $\mathbb{R}$ that do not contain 0;
- all sets of the form $(-a, 0) \cup \{p\} \cup (0, a)$ where $a > 0$;
- all sets of the form $(-a, 0) \cup \{q\} \cup (0, a)$ where $a > 0$

3.5 Closed Sets

Definition 3.5.1 (Closed)**.** Let X be a topological space and $A \subseteq X$. Then A is *closed* iff $X \setminus A$ is open.

Proposition 3.5.2. *In any topological space X, the empty set $\emptyset$ is closed.*

PROOF: This holds because $X \setminus \emptyset = X$ is open. ∎

Proposition 3.5.3. *In any topological space X, the set X is closed.*

PROOF: This holds because $X \setminus X = \emptyset$ is open. ∎

Proposition 3.5.4. *The union of two closed sets is closed.*

PROOF: If C and D are closed then $X \setminus (C \cup D) = (X \setminus C) \cup (X \setminus D)$ is open. ∎

Proposition 3.5.5. *In any topological space, the intersection of a nonempty set of closed sets is closed.*

PROOF: Let $\mathcal{C}$ be a nonempty set of closed sets. Then $X \setminus \bigcap \mathcal{C} = \bigcup \{X \setminus C : C \in \mathcal{C}\}$ is open. ∎

Proposition 3.5.6. *Let X be a topological space and $U \subseteq X$. Then U is open if and only if $X \setminus U$ is closed.*

PROOF: Immediate from definitions.

Theorem 3.5.7. *Let X be a set and $\mathcal{C} \subseteq \mathcal{P}X$. Suppose:*

1. *$\emptyset, X \in \mathcal{C}$;*
2. *for all nonempty $\mathcal{A} \subseteq \mathcal{C}$, we have $\bigcap \mathcal{A} \in \mathcal{C}$;*
3. *for all $C, D \in \mathcal{C}$, we have $C \cup D \in \mathcal{C}$.*

Then there exists a unique topology on X under which $\mathcal{C}$ is the set of all closed sets, namely

$$\mathcal{T} = \{U \subseteq X : X \setminus U \in \mathcal{C}\}$$

PROOF:
$\langle 1 \rangle 1$. LET: $\mathcal{C}$ be a set satisfying 1–3
$\langle 1 \rangle 2$. LET: $\mathcal{T} = \{X \setminus C : C \in \mathcal{C}\}$
$\langle 1 \rangle 3$. $\mathcal{T}$ is a topology
  $\langle 2 \rangle 1$. $X \in \mathcal{T}$
    PROOF: $X \setminus X = \emptyset \in \mathcal{C}$ by condition 1.
  $\langle 2 \rangle 2$. For all $\mathcal{A} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{A} \in \mathcal{T}$.
    $\langle 3 \rangle 1$. LET: $\mathcal{A} \subseteq \mathcal{T}$
    $\langle 3 \rangle 2$. CASE: $\mathcal{A} = \emptyset$
      PROOF: In this case, $X \setminus \bigcup \mathcal{A} = X \in \mathcal{C}$ by condition 1.
    $\langle 3 \rangle 3$. CASE: $\mathcal{A}$ is nonempty

PROOF: In this case, we have $X \setminus \bigcup \mathcal{A} = \bigcap\{X \setminus U : U \in \mathcal{A}\} \in \mathcal{C}$ by condition 2.

⟨2⟩3. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$

PROOF: $X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V) \in \mathcal{C}$ by condition 3.

⟨1⟩4. $\mathcal{C}$ is the set of closed sets.

PROOF:

$$\begin{aligned} C \text{ is closed} &\Leftrightarrow X \setminus C \in \mathcal{T} \\ &\Leftrightarrow X \setminus (X \setminus C) \in \mathcal{C} \\ &\Leftrightarrow C \in \mathcal{C} \end{aligned}$$

⟨1⟩5. $\mathcal{T}$ is unique.

PROOF: By Proposition 3.5.6.

□

Definition 3.5.8 (Closed Covering)**.** A *closed covering* of a topological space is a covering in which every member is a closed set.

3.6 Locally Finite Families

Definition 3.6.1 (Locally Finite)**.** Let X be a topological space and $\{A_i\}_{i \in I}$ a family of subsets of X. Then $\{A_i\}_{i \in I}$ is *locally finite* iff, for all $x \in X$, there exists a neighbourhood N of x such that there are only finitely many $i \in I$ such that N intersects A_i.

Proposition 3.6.2. *If $\{A_i\}_{i \in I}$ is locally finite and $B_i \subseteq A_i$ for all i then $\{B_i\}_{i \in I}$ is locally finite.*

PROOF: Immediate from definitions. □

Proposition 3.6.3. *Every finite family of open sets is locally finite.*

PROOF: Trivial. □

3.7 Countably Locally Finite Sets

Definition 3.7.1 (Countably Locally Finite)**.** Let X be a space. A subset of $\mathcal{P}X$ is *countably locally finite* iff it is the union of countably many locally finite sets.

3.8 Closure of a Set

Definition 3.8.1 (Closure)**.** Let X be a topological space and $A \subseteq X$. The *closure* of A, $\mathrm{Cl}\, A$ or $\overline{A}$, is the intersection of all closed sets that include A.

PROOF: This intersection always exists because X is a closed set that includes A. □

Proposition 3.8.2. *Let X be a topological space and $A \subseteq X$. Then $A \subseteq \overline{A}$.*

PROOF: Immediate from definitions. □

Proposition 3.8.3. *Let X be a topological space and $A \subseteq X$. Then $\overline{A}$ is closed.*

PROOF: This follows from Proposition 3.5.5. □

Proposition 3.8.4. *Let X be a topological space and $A, C \subseteq X$. If $A \subseteq C$ and C is closed then $\overline{A} \subseteq C$.*

PROOF: Immediate from definitions. □

Proposition 3.8.5. *Let X be a topological space and $A, B \subseteq X$. If $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$.*

PROOF:
⟨1⟩1. ASSUME: $A \subseteq B$
⟨1⟩2. $A \subseteq \overline{B}$
  PROOF: Proposition 3.8.2.
⟨1⟩3. $\overline{A} \subseteq \overline{B}$
  PROOF: Propositions 3.8.3, 3.8.4.
□

Proposition 3.8.6. *Let X be a set and $A \subseteq X$. Then A is closed if and only if $A = \overline{A}$.*

PROOF:
⟨1⟩1. If A is closed then $A = \overline{A}$
  ⟨2⟩1. ASSUME: A is closed
  ⟨2⟩2. $A \subseteq \overline{A}$
    PROOF: By Proposition 3.8.2.
  ⟨2⟩3. $\overline{A} \subseteq A$
    PROOF: By Proposition 3.8.4 since $A \subseteq A$.
⟨1⟩2. If $A = \overline{A}$ then A is closed.
  PROOF: By Proposition 3.8.3.
□

Corollary 3.8.6.1.

$$\overline{\emptyset} = \emptyset$$

Theorem 3.8.7 (Kuratowski Closure Axioms)**.** *Let X be a set and $\overline{(\_)} : \mathcal{P}X \to \mathcal{P}X$ be a function such that:*

1. $\overline{\emptyset} = \emptyset$
2. *For all $A \subseteq X$, $A \subseteq \overline{A}$*
3. *For all $A \subseteq X$, $\overline{A} = \overline{\overline{A}}$*
4. *For all $A, B \subseteq X$, $\overline{A \cup B} = \overline{A} \cup \overline{B}$*

Then there exists a unique topology $\mathcal{T}$ on X such that $\overline{A}$ is the closure of A for all $A \in \mathcal{P}X$.

PROOF:
⟨1⟩1. For all $C, D \subseteq X$, if $C \subseteq D$ then $\overline{C} \subseteq \overline{D}$
 ⟨2⟩1. ASSUME: $C \subseteq D$
 ⟨2⟩2. $\overline{C} = \overline{D}$
  PROOF:

$$\begin{aligned}\overline{D} &= \overline{C \cup D} && (\langle 2\rangle 1) \\ &= \overline{C} \cup \overline{D} && (\text{axiom } 4)\end{aligned}$$

⟨1⟩2. LET: $\mathcal{T}$ be the topology in which a set C is closed iff $\overline{C} = C$.
 ⟨2⟩1. $\overline{\emptyset} = \emptyset$
  PROOF: This is axiom 1.
 ⟨2⟩2. $\overline{X} = X$
  PROOF: By axiom 2.
 ⟨2⟩3. For any set $\mathcal{A}$ of sets C such that $\overline{C} = C$, we have $\overline{\bigcap \mathcal{A}} = \bigcap \mathcal{A}$
  ⟨3⟩1. $\overline{\bigcap \mathcal{A}} \subseteq \bigcap \mathcal{A}$
   ⟨4⟩1. LET: $C \in \mathcal{A}$
   ⟨4⟩2. $\overline{\bigcap \mathcal{A}} \subseteq C$
    PROOF:

$$\begin{aligned}\overline{\bigcap \mathcal{A}} &\subseteq \overline{C} && (\langle 1\rangle 1) \\ &= C && (\langle 4\rangle 1)\end{aligned}$$

  ⟨3⟩2. Q.E.D.
 ⟨2⟩4. If $\overline{C} = C$ and $\overline{D} = D$ then $\overline{C \cup D} = C \cup D$
  PROOF: By axiom 4.
 ⟨2⟩5. Q.E.D.
  PROOF: By Theorem 3.5.7.
⟨1⟩3. For all $A \subseteq X$, the closure of A in $\mathcal{T}$ is $\overline{A}$
 ⟨2⟩1. $\overline{A}$ is closed
  PROOF: From axiom 3.
 ⟨2⟩2. If $A \subseteq C$ and C is closed then $\overline{A} \subseteq C$
  PROOF:

$$\begin{aligned}C &= \overline{C} && (C \text{ is closed}) \\ &= \overline{A \cup C} && (A \subseteq C) \\ &= \overline{A} \cup \overline{C} && (\text{axiom } 4)\end{aligned}$$

□

Theorem 3.8.8. *Let A be a subset of the topological space X and $\mathcal{B}$ a basis for X. Then $x \in \overline{A}$ if and only if, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A.*

PROOF:
⟨1⟩1. If $x \in \overline{A}$ then, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A.
 PROOF: Immediate from Theorem 3.9.3.
⟨1⟩2. If, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A, then $x \in \overline{A}$.
 ⟨2⟩1. ASSUME: for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A.

⟨2⟩2. LET: U be a neighbourhood of x
⟨2⟩3. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$
PROOF: $\mathcal{B}$ is a basis.
⟨2⟩4. B intersects A.
PROOF: By ⟨2⟩1.
⟨2⟩5. U intersects A.
⟨2⟩6. Q.E.D.
PROOF: By Theorem 3.9.3.
□

Lemma 3.8.9. *If $\{A_i\}_{i \in I}$ is locally finite then so is $\{\overline{A_i}\}_{i \in I}$.*

PROOF:
⟨1⟩1. LET: $\{A_i\}_{i \in I}$ be a locally finite family of subsets of the space X.
⟨1⟩2. LET: $x \in X$
⟨1⟩3. PICK a neighbourhood U of x that intersects only $A_{i_1}, \ldots, A_{i_n}$.
⟨1⟩4. U intersects only $\overline{A_{i_1}}, \ldots, \overline{A_{i_n}}$.
□

Lemma 3.8.10. *Let $\{A_i\}_{i \in I}$ be locally finite. Then $\overline{\bigcup_{i \in I} A_i} = \bigcup_{i \in I} \overline{A_i}$.*

PROOF:
⟨1⟩1. LET: $x \in \overline{\bigcup_{i \in I} A_i}$
⟨1⟩2. PICK a neighbourhood U of x that intersects only $A_{i_1}, \ldots, A_{i_n}$.
⟨1⟩3. $x \in \overline{A_{i_1}} \cup \cdots \cup \overline{A_{i_n}}$
PROOF: If not, then $U - \overline{A_{i_1}} - \cdots - \overline{A_{i_n}}$ would be a neighbourhood of x that does not intersect $\bigcup_{i \in I} A_i$.
□

3.9 Interior of a Set

Definition 3.9.1 (Interior)**.** Let X be a topological space and $A \subseteq X$. The *interior* of A, $\operatorname{Int} A$, is the union of all open sets included in A.

Lemma 3.9.2. *If $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$.*

PROOF: $\overline{B}$ is a closed set that includes B, hence includes A. □

Theorem 3.9.3. *Let A be a subset of the topological space X and $x \in X$. Then $x \in \overline{A}$ if and only if every neighbourhood of x intersects A.*

PROOF:
$$\begin{aligned} x \notin \overline{A} &\Leftrightarrow \exists C \text{ closed } (A \subseteq C \wedge x \notin C) \\ &\Leftrightarrow \exists U \text{ open } (A \subseteq X \setminus U \wedge x \in U) \\ &\Leftrightarrow \exists U \text{ open } (A \text{ intersects } U \wedge x \in U) \end{aligned}$$
□

Lemma 3.9.4.
$$X \setminus \operatorname{Int} A = \overline{X \setminus A}$$

Proof:
⟨1⟩1. $X \setminus \operatorname{Int} A \subseteq \overline{X \setminus A}$
  ⟨2⟩1. $X \setminus A \subseteq \overline{X \setminus A}$
  ⟨2⟩2. $X \setminus \overline{X \setminus A} \subseteq A$
  ⟨2⟩3. $X \setminus \overline{X \setminus A} \subseteq \operatorname{Int} A$
⟨1⟩2. $\overline{X \setminus A} \subseteq X \setminus \operatorname{Int} A$
  ⟨2⟩1. $\operatorname{Int} A \subseteq A$
  ⟨2⟩2. $X \setminus A \subseteq X \setminus \operatorname{Int} A$
  ⟨2⟩3. $\overline{X \setminus A} \subseteq X \setminus \operatorname{Int} A$
□

3.10 Boundary

Definition 3.10.1 (Boundary)**.** Let X be a topological space and $A \subseteq X$. The *boundary* of A, $\operatorname{Bd} A$, is $\overline{A} \cap \overline{X \setminus A}$.

Lemma 3.10.2.

$$\operatorname{Bd} A = \overline{A} \setminus \operatorname{Int} A$$

Proof: From Lemma 3.9.4. □

Lemma 3.10.3. $\overline{A} = \operatorname{Int} A \cup \operatorname{Bd} A$

Proof:

$$\begin{aligned}
\operatorname{Int} A \cup \operatorname{Bd} A &= \operatorname{Int} A \cup (\overline{A} \cap (X \setminus \operatorname{Int} A)) \\
&= \operatorname{Int} A \cup \overline{A} \\
&= \overline{A}
\end{aligned}$$

□

Corollary 3.10.3.1. $\operatorname{Bd} A = \emptyset$ *iff* A *is open and closed.*

Lemma 3.10.4. *For any set* U*, the following are equivalent:*

1. U *is open.*
2. $\operatorname{Bd} U \cap U = \emptyset$
3. $\operatorname{Bd} U = \overline{U} \setminus U$

Proof:
⟨1⟩1. $1 \Rightarrow 3$
  Proof: From Lemma 3.10.2.
⟨1⟩2. $3 \Rightarrow 2$
  Proof: Set theory.
⟨1⟩3. $2 \Rightarrow 1$
  Proof:

$$\begin{aligned}
U &\subseteq \overline{U} \\
&= \operatorname{Int} U \cup \operatorname{Bd} U && \text{(Lemma 3.10.3)} \\
\therefore U &\subseteq \operatorname{Int} U
\end{aligned}$$

□

3.11 Limit Points

Definition 3.11.1 (Limit Point)**.** Let X be a topological space, $A \subseteq X$, and $x \in X$. Then x is a *limit point*, *cluster point* or *point of accumulation* of A iff every neighbourhood of x intersects A in a point other than x.

Lemma 3.11.2. *If $A \subseteq B$ then every limit point of A is a limit point of B.*

PROOF: Immediate from the definition. ∎

Theorem 3.11.3. *Let A be a subset of the topological space X. Let A' be the set of all limit points of A. Then $\overline{A} = A \cup A'$.*

PROOF:
$\langle 1 \rangle 1$. If $x \in \overline{A}$ and $x \notin A$ then $x \in A'$
PROOF: in this case, every neighbourhood of x intersects A in a point other than x.
$\langle 1 \rangle 2$. $A \subseteq \overline{A}$
PROOF: From the definition of $\overline{A}$.
$\langle 1 \rangle 3$. $A' \subseteq \overline{A}$
PROOF: By Theorem 3.9.3.
∎

Corollary 3.11.3.1. *A set is closed if and only if it contains all its limit points.*

3.12 Subbases

Definition 3.12.1 (Subbasis)**.** Let X be a topological space. A *subbasis* for the topology on X is a set $\mathcal{S} \subseteq \mathcal{P}X$ such that, for every open set U and $x \in U$, there exist $S_1, \ldots, S_n \in \mathcal{S}$ such that $x \in S_1 \cap \cdots \cap S_n \subseteq U$. We say the topology is *generated* by $\mathcal{S}$.

Lemma 3.12.2. *Let $\mathcal{T}$ be a topology on X and $\mathcal{S} \subseteq \mathcal{P}X$. Then the following are equivalent:*

1. *$\mathcal{S}$ is a subbasis for $\mathcal{T}$.*
2. *The set of all finite intersections of members of $\mathcal{S}$ is a basis for $\mathcal{T}$*
3. *$\mathcal{T}$ is the set of all unions of finite intersections of members of $\mathcal{S}$.*

PROOF: $1 \Leftrightarrow 2$ holds immediately from the definitions. $2 \Leftrightarrow 3$ holds by Proposition 3.4.2. ∎

Corollary 3.12.2.1. *If $\mathcal{S}$ is a subbasis for the topology $\mathcal{T}$, then $\mathcal{T}$ is the coarsest topology in which every element of $\mathcal{S}$ is open.*

Lemma 3.12.3. *Let X be a set and $\mathcal{S} \subseteq \mathcal{P}X$. Then $\mathcal{S}$ is a subbasis for a topology on X if and only if $\bigcup \mathcal{S} = X$.*

PROOF:
$\langle 1\rangle 1$. If $\mathcal{S}$ is a subbasis for a topology on X then $\bigcup \mathcal{S} = X$
$\langle 2\rangle 1$. ASSUME: $\mathcal{S}$ is a subbasis for a topology $\mathcal{T}$ on X.
$\langle 2\rangle 2$. LET: $x \in X$
$\langle 2\rangle 3$. PICK $S_1, \ldots, S_n \in \mathcal{S}$ such that $x \in S_1 \cap \cdots \cap S_n \subseteq X$
PROOF: From the definition of subbasis ($\langle 2\rangle 1$, $\langle 2\rangle 2$).
$\langle 2\rangle 4$. $x \in \bigcup \mathcal{S}$
PROOF: Immediate from $\langle 2\rangle 3$.
$\langle 1\rangle 2$. If $\bigcup \mathcal{S} = X$ then $\mathcal{S}$ is a subbasis for a topology on X
$\langle 2\rangle 1$. ASSUME: $\bigcup \mathcal{S} = X$
PROVE: The set of all finite intersections of elements of $\mathcal{S}$ is a basis for a topology on X.
$\langle 2\rangle 2$. LET: $\mathcal{B}$ be the set of all finite intersections of elements of $\mathcal{S}$.
$\langle 2\rangle 3$. $\bigcup \mathcal{B} = X$
PROOF: From $\langle 2\rangle 1$ and $\langle 2\rangle 2$.
$\langle 2\rangle 4$. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$
PROOF: Take $B_3 = B_1 \cap B_2$ ($\langle 2\rangle 2$).
$\langle 2\rangle 5$. $\mathcal{B}$ is a basis for a topology on X.
PROOF: By Lemma 3.4.3.
$\langle 2\rangle 6$. Q.E.D.
PROOF: By Lemma 3.12.2.
□

3.13 Convergence

Definition 3.13.1 (Net)**.** Let X be a topological space. A *net* $(x_\alpha)_{\alpha \in J}$ in X consists of a directed set J and a function $x : J \to X$.

Definition 3.13.2 (Convergence)**.** Let $(x_\alpha)_{\alpha \in J}$ be a net in the topological space X, and $l \in X$. Then the net *converges* to l, $x_\alpha \to l$, if and only if, for every neighbourhood U of l, there exists $\alpha \in J$ such that, for all $\beta \geq \alpha$, we have $x_\beta \in U$.

Theorem 3.13.3 (AC)**.** *Let X be a topological space and $A \subseteq X$. Then $x \in \overline{A}$ if and only if there exists a net of points of A converging to x.*

PROOF:
$\langle 1\rangle 1$. If $x \in \overline{A}$ then there exists a net of points of A converging to x.
$\langle 2\rangle 1$. LET: $x \in \overline{A}$
$\langle 2\rangle 2$. LET: J be the poset of neighbourhoods of x under $\supseteq$.
$\langle 2\rangle 3$. For $U \in J$ PICK a point $x_U \in U \cap A$
PROOF: By Theorem 3.9.3
$\langle 2\rangle 4$. $(x_U)_{U \in J}$ is a net
PROOF: Given $U, V \in J$ we have $U \cap V \in J$ and $U \supseteq U \cup V$, $V \supseteq U \cup V$.
$\langle 2\rangle 5$. $x_U \to x$

PROOF: For any neighbourhood U of x we have $U \in J$ and if $U \supseteq V$ then $x_V \in U$.

$\langle 1 \rangle 2$. If there exists a net of points of A converging to x then $x \in \overline{A}$.

$\langle 2 \rangle 1$. LET: $(x_\alpha)_{\alpha \in J}$ be a net of points in A that converges to x.

$\langle 2 \rangle 2$. LET: U be a neighbourhood of x

$\langle 2 \rangle 3$. PICK $\alpha \in J$ such that, for all $\beta \geq \alpha$, we have $x_\beta \in U$

$\langle 2 \rangle 4$. $x_\alpha \in U \cap A$

$\langle 2 \rangle 5$. Q.E.D.

PROOF: By Theorem 3.9.3

□

Theorem 3.13.4. *Let X and Y be topological spaces and $f : X \to Y$. Then f is continuous if and only if, for every net $(x_\alpha)_{\alpha \in J}$ in X, if $x_\alpha \to x$ then $f(x_\alpha) \to f(x)$.*

PROOF:

$\langle 1 \rangle 1$. If f is continuous and $x_\alpha \to x$ then $f(x_\alpha) \to f(x)$

$\langle 2 \rangle 1$. ASSUME: f is continuous.

$\langle 2 \rangle 2$. ASSUME: $x_\alpha \to x$

$\langle 2 \rangle 3$. LET: V be a neighbourhood of $f(x)$

$\langle 2 \rangle 4$. $f^{-1}(V)$ is a neighbourhood of x

$\langle 2 \rangle 5$. PICK α such that, for all $\beta \geq \alpha$, we have $x_\beta \in f^{-1}(V)$

$\langle 2 \rangle 6$. For all $\beta \geq \alpha$ we have $f(x_\beta) \in V$

$\langle 1 \rangle 2$. If, for every net (x_α) in X, if $x_\alpha \to x$ then $f(x_\alpha) \to f(x)$, then f is continous.

$\langle 2 \rangle 1$. ASSUME: for every net (x_α) in X, if $x_\alpha \to x$ then $f(x_\alpha) \to f(x)$

$\langle 2 \rangle 2$. LET: $A \subseteq X$

PROVE: $f(\overline{A}) \subseteq \overline{f(A)}$

$\langle 2 \rangle 3$. LET: $x \in \overline{A}$

$\langle 2 \rangle 4$. PICK a net (x_α) in A such that $x_\alpha \to x$

PROOF: Theorem 3.13.3

$\langle 2 \rangle 5$. $f(x_\alpha) \to f(x)$

PROOF: By $\langle 2 \rangle 1$

$\langle 2 \rangle 6$. $f(x) \in \overline{f(A)}$

PROOF: Theorem 3.13.3

$\langle 2 \rangle 7$. Q.E.D.

PROOF: By Theorem 5.2.2.

□

Definition 3.13.5 (Subnet)**.** Let $(x_\alpha)_{\alpha \in J}$ be a net in X. Let K be a directed set and $g : K \to J$ be a monotone function such that $g(K)$ is cofinal in J. Then the net $(x_{g(\beta)})_{\beta \in K}$ is called a *subnet* of (x_α).

3.14 Accumulation Points

Definition 3.14.1 (Accumulation Point)**.** Let X be a topological space, and $(x_\alpha)_{\alpha \in J}$ a net in X, and $a \in X$. Then a is an *accumulation point* of (x_α) iff, for every neighbourhood U of x, the set $\{\alpha \in J : x_\alpha \in U\}$ is cofinal in J.

Lemma 3.14.2. *Let X be a topological space, $(x_\alpha)_{\alpha \in J}$ be a nonempty net in X and $a \in X$. Then a is an accumulation point of (x_α) if and only if there exists a subnet of (x_α) that converges to a.*

PROOF:
$\langle 1 \rangle 1$. If a is an accumulation point of (x_α) then there exists a subnet of (x_α) that converges to a.
  $\langle 2 \rangle 1$. ASSUME: a is an accumulation point of (x_α).
  $\langle 2 \rangle 2$. LET: K be the poset $\{(\alpha, U) : \alpha \in J, U$ is a neighbourhood of $a, x_\alpha \in U\}$ under: $(\alpha, U) \leq (\beta, V)$ iff $\alpha \leq \beta$ and $U \subseteq V$.
  $\langle 2 \rangle 3$. $(x_\alpha)_{(\alpha,U) \in K}$ is a subnet of $(x_\alpha)_{\alpha \in J}$
    $\langle 3 \rangle 1$. K is directed.
      $\langle 4 \rangle 1$. LET: $(\alpha, U), (\beta, V) \in K$
      $\langle 4 \rangle 2$. PICK $\gamma \in J$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$.
      $\langle 4 \rangle 3$. PICK $\delta \in J$ such that $\gamma \leq \delta$ and $x_\delta \in U \cap V$
        PROOF: By $\langle 2 \rangle 1$.
      $\langle 4 \rangle 4$. $(\delta, U \cap V) \in K$ and $(\alpha, U) \leq (\delta, U \cap V)$, $(\beta, V) \leq (\delta, U \cap V)$
    $\langle 3 \rangle 2$. If $(\alpha, U) \leq (\beta, V)$ then $\alpha \leq \beta$
      PROOF: From $\langle 2 \rangle 2$.
    $\langle 3 \rangle 3$. $\{\alpha : \exists U.(\alpha, U) \in K\}$ is cofinal in J
      PROOF: For $\alpha \in J$ we have $(\alpha, X) \in K$, so in fact $\{\alpha : \exists U.(\alpha, U) \in K\} = J$.
  $\langle 2 \rangle 4$. The subnet converges to a.
    $\langle 3 \rangle 1$. LET: U be a neighbourhood of a.
    $\langle 3 \rangle 2$. PICK $\alpha \in J$
    $\langle 3 \rangle 3$. PICK $\beta \in J$ such that $\alpha \leq \beta$ and $x_\beta \in U$
      PROOF: By $\langle 2 \rangle 1$.
    $\langle 3 \rangle 4$. For all $(\gamma, V) \geq (\beta, U)$ we have $x_\gamma \in U$
      PROOF: $x_\gamma \in V \subseteq U$.
$\langle 1 \rangle 2$. If there exists a subnet of (x_α) that converges to a then a is an accumulation point of (x_α).
  $\langle 2 \rangle 1$. ASSUME: $(x_{g(\beta)})_{\beta \in K}$ converges to a
  $\langle 2 \rangle 2$. LET: U be a neighbourhoof of a
  $\langle 2 \rangle 3$. LET: $\alpha \in J$
    PROVE: There exists $\gamma \geq \alpha$ such that $x_\gamma \in U$
  $\langle 2 \rangle 4$. PICK $\beta \in K$ such that, for all $\beta' \geq \beta$, we have $x_{g(\beta')} \in U$
    PROOF: By $\langle 2 \rangle 1$.
  $\langle 2 \rangle 5$. PICK $\beta' \in K$ such that $g(\beta') \geq \alpha$
    PROOF: Since $g(K)$ is cofinal in J.
  $\langle 2 \rangle 6$. PICK $\beta'' \in K$ such that $\beta \leq \beta''$ and $\beta' \leq \beta''$
    PROOF: K is directed.
  $\langle 2 \rangle 7$. $g(\beta'') \geq \alpha$ and $x_{g(\beta'')} \in U$

$\square$

3.15 Dense Sets

Definition 3.15.1 (Dense)**.** Let X be a topological space and $A \subseteq X$. Then A is *dense* in X iff $\overline{A} = X$.

3.16 G_δ Sets

Definition 3.16.1 (G_δ Set)**.** A G_δ *set* is the intersection of a countable set of open sets.

3.17 Separated Sets

Definition 3.17.1 (Separated Sets)**.** Let X be a topological space and $A, B \subseteq X$. Then A and B are *separated* iff $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$.

3.18 Coherent Topology

Definition 3.18.1 (Coherent Topology)**.** Let $X_1 \subseteq X_2 \subseteq \cdots$ be a sequence of topological spaces such that each X_n is a closed subspace of X_{n+1}. Let $X = \bigcup_{n=1}^{\infty} X_n$. Then the topology on X *coherent* with the subspaces X_n is the topology defined by: $U \subseteq X$ is open iff $U \cap X_n$ is open in X_n for all n.

Chapter 4

Constructions of Topological Spaces

4.1 The Order Topology

Definition 4.1.1 (Order Topology)**.** Let X be a linearly ordered set with more than one element. The *order topology* on X is the topology generated by the basis consisting of:

- all open intervals (a, b)
- all half-open intervals $(a, \top]$ where $\top$ is the greatest element of X, if there is one;
- all half-open intervals $[\bot, a)$ where $\bot$ is the least element of X, if there is one.

We prove this is a basis for a topology.

Proof:
$\langle 1 \rangle 1$. Let: $\mathcal{B}$ be the set of all sets of these three forms.
$\langle 1 \rangle 2$. $\bigcup \mathcal{B} = X$
  $\langle 2 \rangle 1$. Let: $x \in X$
    Prove: There exists $B \in \mathcal{B}$ such that $x \in B$
  $\langle 2 \rangle 2$. Case: x is least in X
    $\langle 3 \rangle 1$. Pick $a \in X$ such that $a > x$
      Proof: X has more than one element.
    $\langle 3 \rangle 2$. $x \in [x, a) \in \mathcal{B}$
  $\langle 2 \rangle 3$. Case: x is greatest in X
    $\langle 3 \rangle 1$. Pick $a \in X$ such that $a < x$
      Proof: X has more than one element.
    $\langle 3 \rangle 2$. $x \in (a, x] \in \mathcal{B}$
  $\langle 2 \rangle 4$. Case: x is neither least nor greatest in X

⟨3⟩1. Pick $a, b \in X$ such that $a < x < b$

⟨3⟩2. $x \in (a, b) \in \mathcal{B}$

⟨1⟩3. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$

⟨2⟩1. Let: $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$

⟨2⟩2. Case: $B_1 = (a, b), B_2 = (c, d)$

Proof: Take $B_3 = (\max(a, c), \min(b, d))$.

⟨2⟩3. Case: $B_1 = (a, b), B_2 = (c, \top]$

Proof: Take $B_3 = (\max(a, c), b)$.

⟨2⟩4. Case: $B_1 = (a, b), B_2 = [\perp, d)$

Proof: Take $B_3 = (a, \min(b, d))$.

⟨2⟩5. Case: $B_1 = (a, \top], B_2 = (c, d)$

Proof: Similar to ⟨2⟩3.

⟨2⟩6. Case: $B_1 = (a, \top], B_2 = (c, \top]$

Proof: Take $B_3 = (\max(a, c), \top]$.

⟨2⟩7. Case: $B_1 = (a, \top], B_2 = [\perp, d)$

Proof: Take $B_3 = (a, d)$.

⟨2⟩8. Case: $B_1 = [\perp, b), B_2 = (c, d)$

Proof: Similar to ⟨2⟩4.

⟨2⟩9. Case: $B_1 = [\perp, b), B_2 = (c, \top]$

Proof: Simlar to ⟨2⟩7.

⟨2⟩10. Case: $B_1 = [\perp, b), B_2 = [\perp, d)$

Proof: Take $B_3 = [\perp, \min(b, d))$.

⟨1⟩4. Q.E.D.

Proof: By Lemma 3.4.3.

□

Lemma 4.1.2. *Let X be a linearly ordered set, $U \subseteq X$ be open, and $a \in U$. Then:*

1. *Either a is greatest in X, or there exists $a' > a$ such that $[a, a') \subseteq U$*
2. *Either a is least in X, or there exists $a' < a$ such that $(a', a] \subseteq U$.*

Proof:

⟨1⟩1. Either a is greatest in X, or there exists $a' > a$ such that $[a, a') \subseteq U$

⟨2⟩1. Assume: a is not greatest in X

⟨2⟩2. Pick a basic open set B such that $a \in B \subseteq U$

⟨2⟩3. Case: $B = (a'', a')$

Proof: $a < a'$ and $[a, a') \subseteq B \subseteq U$

⟨2⟩4. Case: $B = [\perp, a')$

Proof: $a < a'$ and $[a, a') \subseteq B \subseteq U$

⟨2⟩5. Case: $B = (a'', \top]$

Proof: Pick any $a' > a$ (one exists by ⟨2⟩1). Then $[a, a') \subseteq B \subseteq U$.S

⟨1⟩2. Either a is least in X, or there exists $a' < a$ such that $(a', a] \subseteq U$.

Proof: Similar.

□

Lemma 4.1.3. *The open rays form a subbasis for the order topology.*

⟨1⟩1. Let: X be a linearly ordered set with more than one element.
⟨1⟩2. The open rays form a subbasis for a topology.
⟨2⟩1. Let: $x \in X$
Prove: x is an element of an open ray.
⟨2⟩2. Case: x is greatest in X
⟨3⟩1. Pick $a \in X$ such that $a < x$
Proof: X has more than one element (⟨1⟩1).
⟨3⟩2. $x \in (a, +\infty)$
⟨2⟩3. Case: x is not greatest in X
⟨3⟩1. Pick $a \in X$ such that $x < a$
⟨3⟩2. $x \in (-\infty, a)$
⟨2⟩4. Q.E.D.
Proof: By Lemma 3.12.2.
⟨1⟩3. Let: $\mathcal{T}_o$ be the order topology and $\mathcal{T}_S$ be the topology generated by the open rays.
⟨1⟩4. $\mathcal{T}_o \subseteq \mathcal{T}_S$
⟨2⟩1. Every open interval (a, b) is open in $\mathcal{T}_S$
Proof: $(a, b) = (a, +\infty) \cap (-\infty, b)$.
⟨2⟩2. If $\top$ is greatest then $(a, \top]$ is open in $\mathcal{T}_S$
Proof: $(a, \top] = (a, +\infty)$.
⟨2⟩3. If $\bot$ is least then $[\bot, b)$ is open in $\mathcal{T}_S$
Proof: $[\bot, b) = [\bot, +\infty)$.
⟨2⟩4. Q.E.D.
Proof: By Corollary 3.4.2.1.
⟨1⟩5. $\mathcal{T}_S \subseteq \mathcal{T}_o$
⟨2⟩1. For all $a \in X$, we have $(a, +\infty)$ is open in $\mathcal{T}_o$
⟨3⟩1. Let: $x \in (a, +\infty)$
Prove: There exists a basis element B such that $x \in B \subseteq (a, +\infty)$
⟨3⟩2. Case: x is greatest
Proof: Take $B = (a, x]$
⟨3⟩3. Case: x is not greatest
⟨4⟩1. Pick $b > x$
⟨4⟩2. $x \in (a, b) \subseteq (a, +\infty)$
⟨2⟩2. For all $a \in X$, we have $(-\infty, a)$ is open in $\mathcal{T}_o$
Proof: Similar.
⟨2⟩3. Q.E.D.
Proof: By Corollary 3.12.2.1.
□

Lemma 4.1.4. *In a linearly ordered set X under the order topology, the closed intervals and closed rays are closed.*

PROOF:

$$X \setminus [a,b] = (-\infty, a) \cup (b, +\infty)$$
$$X \setminus (-\infty, a] = (a, +\infty)$$
$$X \setminus [a, +\infty) = (-\infty, a)$$

∎

Definition 4.1.5 (Standard Topology on $\mathbb{R}$)**.** The *standard topology* on $\mathbb{R}$ is the order topology.

Lemma 4.1.6. *The standard topology is strictly coarser than the lower limit topology.*

PROOF:
⟨1⟩1. The standard topology is coarser than the lower limit topology.
⟨2⟩1. For every open interval (a,b) and $x \in (a,b)$, there exists a half-open interval $[c,d)$ such that $x \in [c,d) \subseteq (a,b)$
PROOF: Take $[c,d) = [x,b)$.
⟨2⟩2. Q.E.D.
PROOF: By Lemma 3.4.4.
⟨1⟩2. There exists a set U open in the lower limit topology that is not open in the standard topology.
PROOF: Take $U = [0,1)$.
∎

Lemma 4.1.7. *The standard topology is strictly coarser than the K-topology.*

PROOF:
⟨1⟩1. The standard topology is coarser than the K-topology.
PROOF: Every open interval is open in the K-topology.
⟨1⟩2. There exists a set U open in the K-topology that is not open in the standard topology.
PROOF: Take $U = (-1,1) \setminus K$. Then $0 \in U$ but there is no open interval (a,b) such that $0 \in (a,b) \subseteq U$.
∎

Definition 4.1.8 (Ordered Square)**.** The *ordered square* I_o^2 is the topological space $[0,1]^2$ under the order topology induced by the lexicographic order.

Lemma 4.1.9. *Let L be a linear continuum with a greatest element. Then every non-empty closed set in L has a greatest element.*

PROOF:
⟨1⟩1. LET: C be a non-empty closed set in L
⟨1⟩2. LET: u be the supremum of C
⟨1⟩3. $u \in C$
⟨2⟩1. ASSUME: w.l.o.g u is not least in L
PROOF: If u is least then $C = \{u\}$.
⟨2⟩2. LET: U be any open neighbourhood of u
⟨2⟩3. PICK $v < u$ such that $(v,u] \subseteq U$

PROOF: By Lemma 4.1.2.

⟨2⟩4. PICK $x \in C$ such that $v < x$

PROOF: v is not an upper bound for C (⟨1⟩2).

⟨2⟩5. U intersects C in v

⟨2⟩6. Q.E.D.

PROOF: By Theorem 3.9.3.

□

Definition 4.1.10 (Long Line)**.** The *long line* is $(S_\Omega \times [0,1)) \setminus \{(0,0)\}$ under the dictionary order, where S_Ω is the first uncountable ordinal under the order topology.

4.2 The Product Topology

Definition 4.2.1 (Product Topology)**.** Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces. The *product topology* on $\prod_{\alpha \in J} X_\alpha$ is the topology generated by the subbasis consisting of all sets of the form $\pi_\alpha^{-1}(U)$ where $\alpha \in J$ and U is open in X_α. The *product space* of $\{X_\alpha\}_{\alpha \in J}$ is $\prod_{\alpha \in J} X_\alpha$ under the product topology.

Lemma 4.2.2. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces and A_α be closed in X_α for all α. Then $\prod_{\alpha \in J} A_\alpha$ is closed in $\prod_{\alpha \in J} X_\alpha$.*

PROOF: This holds because $\prod_{\alpha \in J} X_\alpha \setminus \prod_{\alpha \in J} A_\alpha = \bigcup_{\alpha \in J} \pi_\alpha^{-1}(X_\alpha \setminus A_\alpha)$. □

Theorem 4.2.3. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces. The set of all sets of the form $\prod_{\alpha \in J} U_\alpha$ where each U_α is open in X_α, and $U_\alpha = X_\alpha$ for all but finitely many α, is a basis for the product topology on $\prod_{\alpha \in J} X_\alpha$.*

PROOF: By Lemma 3.12.2. □

Theorem 4.2.4. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces, and let $\mathcal{B}_\alpha$ be a basis for the topology on X_α for each α. Then*

$$\begin{aligned} \mathcal{B} = \{ \prod_{\alpha \in J} U_\alpha : & \textit{ for finitely many } \alpha \in J, U_\alpha \in \mathcal{B}_\alpha, \\ & \textit{and } U_\alpha = X_\alpha \textit{ for all other values of } \alpha \} \end{aligned}$$

is a basis for the product topology on $\prod_{\alpha \in J} X_\alpha$.

PROOF:

⟨1⟩1. Every member of $\mathcal{B}$ is open in the product topology.

PROOF: Immediate from definitions.

⟨1⟩2. For every open set U and $\{x_\alpha\}_{\alpha \in J} \in U$, there exists $B \in \mathcal{B}$ such that $\{x_\alpha\}_{\alpha \in J} \in B \subseteq U$.

⟨2⟩1. LET: U be open and $\{x_\alpha\}_{\alpha \in J} \in U$

⟨2⟩2. PICK U_α open in X_α for each α such that $\{x_\alpha\}_{\alpha \in J} \in \prod_{\alpha \in J} U_\alpha \subseteq U$ and $U_\alpha = X_\alpha$ for all α except $\alpha_1, \ldots, \alpha_n$.

PROOF: By Theorem 4.2.3.

⟨2⟩3. Pick $B_{\alpha_i} \in \mathcal{B}_{\alpha_i}$ such that $x_\alpha \in B_{\alpha_i} \subseteq U_{\alpha_i}$ for $i = 1, \ldots, n$

⟨2⟩4. $\{x_\alpha\}_{\alpha \in J} \in \prod_{\alpha \in J} V_\alpha \subseteq U$ where $V_{\alpha_i} = B_{\alpha_i}$ for $i = 1, \ldots, n$, and $V_\alpha = X_\alpha$ for all other α.

□

Theorem 4.2.5 (AC)**.** *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces and $A_\alpha \subseteq X_\alpha$ for all α. If $\prod_{\alpha \in J} X_\alpha$ is given the product topology, then*

$$\prod_{\alpha \in J} \overline{A_\alpha} = \overline{\prod_{\alpha \in J} A_\alpha} \ .$$

Proof:

⟨1⟩1. $\prod_{\alpha \in J} \overline{A_\alpha} \subseteq \overline{\prod_{\alpha \in J} A_\alpha}$

⟨2⟩1. Let: $\{x_\alpha\}_{\alpha \in J} \in \prod_{\alpha \in J} \overline{A_\alpha}$

⟨2⟩2. Let: $\prod_{\alpha \in J} U_\alpha$ be a basic neighbourhood of $\{x_\alpha\}_{\alpha \in J}$, where each U_α is open in X_α, and $U_\alpha = X_\alpha$ except for $\alpha = \alpha_1, \ldots, \alpha_n$.

⟨2⟩3. For $\alpha \in J$, Pick $a_\alpha \in A_\alpha \cap U_\alpha$.

Proof: By Theorem 3.9.3, using the Axiom of Choice.

⟨2⟩4. $\{a_\alpha\}_{\alpha \in J} \in \prod_{\alpha \in J} A_\alpha \cap \prod_{\alpha \in J} U_\alpha$

⟨2⟩5. Q.E.D.

Proof: By Theorem 3.9.3.

⟨1⟩2. $\overline{\prod_{\alpha \in J} A_\alpha} \subseteq \prod_{\alpha \in J} \overline{A_\alpha}$

⟨2⟩1. Let: $\{x_\alpha\}_{\alpha \in J} \in \overline{\prod_{\alpha \in J} A_\alpha}$

⟨2⟩2. Let: $\alpha \in J$

Prove: $x_\alpha \in \overline{A_\alpha}$

⟨2⟩3. Let: U be a neighbourhood of x_α in X_α

⟨2⟩4. $\pi_\alpha^{-1}(U)$ is a neighbourhood of $\{x_\alpha\}_{\alpha \in J}$

⟨2⟩5. Pick $\{a_\alpha\}_{\alpha \in J} \in \pi_\alpha^{-1}(U) \cap \prod_{\alpha \in J} A_\alpha$

Proof: By Theorem 3.9.3.

⟨2⟩6. $a_\alpha \in U \cap A_\alpha$

⟨2⟩7. Q.E.D.

Proof: By Theorem 3.9.3.

□

Definition 4.2.6 (Standard Topology on $\mathbb{R}^J$)**.** For J a set, the *standard topology* on $\mathbb{R}^J$ is the product topology where $\mathbb{R}$ is given the standard topology.

Definition 4.2.7 (Closed Unit Ball)**.** The *closed unit ball* B^2 is $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ as a subset of $\mathbb{R}^2$.

Definition 4.2.8 (Sorgenfrey Plane)**.** The *Sorgenfrey plane* is $\mathbb{R}_l^2$.

4.3 The Subspace Topology

Definition 4.3.1 (Subspace Topology)**.** Let X be a topological space and $Y \subseteq X$. The *subspace topology* on Y is $\{Y \cap U : U \text{ open in } X\}$. With this topology, Y is a *subspace* of X.

We prove this is a topology.

PROOF:
$\langle 1\rangle 1$. LET: $\mathcal{T} = \{Y \cap U : U \text{ open in } X\}$
$\langle 1\rangle 2$. $Y \in \mathcal{T}$
PROOF: $Y = Y \cap X$
$\langle 1\rangle 3$. $\mathcal{T}$ is closed under union.
$\langle 2\rangle 1$. LET: $\mathcal{A} \subseteq \mathcal{T}$
PROVE: $\bigcup \mathcal{A} = Y \cap \bigcup\{U \text{ open in } X : Y \cap U \in \mathcal{A}\}$
$\langle 2\rangle 2$. $\bigcup \mathcal{A} \subseteq Y \cap \bigcup\{U \text{ open in } X : Y \cap U \in \mathcal{A}\}$
$\langle 3\rangle 1$. LET: $x \in \bigcup \mathcal{A}$
$\langle 3\rangle 2$. PICK $V \in \mathcal{A}$ such that $x \in V$
$\langle 3\rangle 3$. PICK U open in X such that $V = Y \cap U$
PROOF: By the definition of $\mathcal{T}$ ($\langle 1\rangle 1$, $\langle 2\rangle 1$, $\langle 3\rangle 2$)
$\langle 3\rangle 4$. $x \in Y \cap \bigcup\{U \text{ open in } X : Y \cap U \in \mathcal{A}\}$
$\langle 2\rangle 3$. $Y \cap \bigcup\{U \text{ open in } X : Y \cap U \in \mathcal{A}\} \subseteq \bigcup \mathcal{A}$
PROOF: Set theory.
$\langle 1\rangle 4$. $\mathcal{T}$ is closed under binary intersection.
PROOF: This holds because $(Y \cap U) \cap (Y \cap V) = Y \cap (U \cap V)$.
$\square$

Lemma 4.3.2. *Let X be a topological space, $Y \subseteq X$, and $A \subseteq Y$. Then the topology A inherits as a subspace of X is the same as the topology A inherits as a subspace of Y.*

PROOF:

$$\begin{aligned}
&\text{topology as a subspace of } Y \\
=&\{V \cap A : V \text{ open in } Y\} \\
=&\{V \cap A : \exists U \text{ open in } X. V = U \cap Y\} \\
=&\{U \cap Y \cap A : U \text{ open in } X\} \\
=&\{U \cap A : U \text{ open in } X\} \\
=&\text{topology as a subspace of } X \square
\end{aligned}$$

Lemma 4.3.3. *Let Y be a subspace of X. If U is open in Y and Y is open in X then U is open in X.*

PROOF:
$\langle 1\rangle 1$. PICK V open in X such that $U = Y \cap V$
$\langle 1\rangle 2$. U is open in X
PROOF: The open sets in X are closed under binary intersection.
$\square$

Theorem 4.3.4. *Let Y be a subspace of X. Let $A \subseteq Y$. Let $\overline{A}$ be the closure of A in X. Then the closure of A in Y is $\overline{A} \cap Y$.*

PROOF:
$\langle 1\rangle 1$. $\overline{A} \cap Y$ is a closed set in Y that includes A.
$\langle 2\rangle 1$. $\overline{A} \cap Y$ is closed in Y.
PROOF: By Lemma 4.3.4.1.

⟨2⟩2. $A \subseteq \overline{A} \cap Y$.
⟨1⟩2. If C is any closed set in Y that includes A then $\overline{A} \cap Y \subseteq C$.
⟨2⟩1. Let: C be a closed set in Y that includes A.
⟨2⟩2. Pick D closed in X such that $C = D \cap Y$.
Proof: By Lemma 4.3.4.1.
⟨2⟩3. $\overline{A} \subseteq D$
⟨2⟩4. $\overline{A} \subseteq C$
□

Corollary 4.3.4.1. *Let Y be a subspace of X. Then a set $A \subseteq Y$ is closed in Y if and only if it is the intersection of a closed set in X with Y.*

Corollary 4.3.4.2. *Let Y be a subspace of X. If A is closed in Y and Y is closed in X then A is closed in X.*

Lemma 4.3.5. *Let X be a topological space and $Y \subseteq X$. If $\mathcal{B}$ is a basis for the topology on X then $\{B \cap Y : B \in \mathcal{B}\}$ is a basis for the subspace topology on Y.*

Proof:
⟨1⟩1. For all $B \in \mathcal{B}$, we have $B \cap Y$ is open in Y.
Proof: Immediate from definitions.
⟨1⟩2. For every V open in Y and $y \in V$, there exists $B \in \mathcal{B}$ such that $y \in B \cap Y \subseteq V$.
⟨2⟩1. Let: V be open in Y and $y \in V$
⟨2⟩2. Pick U open in X such that $V = Y \cap U$
⟨2⟩3. Pick $B \in \mathcal{B}$ such that $y \in B \subseteq U$
⟨2⟩4. $y \in B \cap Y \subseteq V$
□

Lemma 4.3.6. *Let X be a topological space and $Y \subseteq X$. If $\mathcal{S}$ is a subbasis for the topology on X then $\{S \cap Y : S \in \mathcal{S}\}$ is a subbasis for the subspace topology on Y.*

Proof:
⟨1⟩1. For all $S \in \mathcal{S}$, we have $S \cap Y$ is open in Y.
Proof: Immediate from definitions.
⟨1⟩2. For every V open in Y and $y \in V$, there exist $S_1, \ldots, S_n \in \mathcal{S}$ such that $y \in (S_1 \cap Y) \cap \cdots \cap (S_n \cap Y) \subseteq V$
⟨2⟩1. Let: V be open in Y and $y \in V$
⟨2⟩2. Pick U open in X such that $V = U \cap Y$
⟨2⟩3. Pick $S_1, \ldots, S_n \in \mathcal{S}$ such that $y \in S_1 \cap \cdots \cap S_n \subseteq U$
⟨2⟩4. $y \in (S_1 \cap Y) \cap \cdots \cap (S_n \cap Y) \subseteq V$
□

Theorem 4.3.7. *Let X be a linearly ordered set in the order topology. Let $Y \subseteq X$ be convex. Then the order topology on Y is the same as the subspace topology.*

Proof:

⟨1⟩1. LET: $\mathcal{T}_o$ be the order topology and $\mathcal{T}_s$ be the subspace topology.
⟨1⟩2. $\mathcal{T}_o \subseteq \mathcal{T}_s$
⟨2⟩1. For all $a \in Y$, we have $\{y \in Y : a < y\} \in \mathcal{T}_s$
PROOF: $\{y \in Y : a < y\} = \{x \in X : a < x\} \cap Y$
⟨2⟩2. For all $a \in Y$, we have $\{y \in Y : y < a\} \in \mathcal{T}_s$
PROOF: Similar.
⟨2⟩3. Q.E.D.
PROOF: Lemma 4.1.3 and Corollary 3.12.2.1.
⟨1⟩3. $\mathcal{T}_s \subseteq \mathcal{T}_o$
⟨2⟩1. The sets $(a, +\infty) \cap Y$ and $(-\infty, a) \cap Y$ for $a \in X$ form a subbasis for $\mathcal{T}_s$
PROOF: Lemma 4.3.6, Lemma 4.1.3.
⟨2⟩2. For all $a \in X$, we have $(a, +\infty) \cap Y \in \mathcal{T}_o$
⟨3⟩1. LET: $a \in X$
⟨3⟩2. CASE: $a \in Y$
PROOF: In this case, $(a, +\infty) \cap Y$ is an open ray in Y.
⟨3⟩3. CASE: For all $y \in Y$ we have $a < y$
PROOF: In this case, $(a, +\infty) \cap Y = Y$.
⟨3⟩4. CASE: For all $y \in Y$ we have $y < a$
PROOF: In this case, $(a, +\infty) \cap Y = \emptyset$.
⟨3⟩5. Q.E.D.
PROOF: These are the only cases because Y is convex.
⟨2⟩3. For all $a \in X$, we have $(-\infty, a) \cap Y \in \mathcal{T}_o$
PROOF: Similar.
⟨2⟩4. Q.E.D.
PROOF: Corollary 3.12.2.1.
□

Theorem 4.3.8. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces, and let A_α be a subspace of X_α for all α. Then the product topology on $\prod_{\alpha \in J} A_\alpha$ is the same as the topology it inherits as a subspace of $\prod_{\alpha \in J} X_\alpha$.*

PROOF: Each is the topology generated by the subbasis consisting of $\pi_\alpha^{-1}(U) \cap \prod_{\alpha \in J} A_\alpha = \pi_\alpha^{-1}(U \cap A_\alpha)$ where $\alpha \in J$ and U is open in X_α, using Lemma 4.3.6.
□

Definition 4.3.9 (Unit Circle)**.** The *unit circle* S^1 is $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ as a subspace of $\mathbb{R}^2$.

Proposition 4.3.10. *Let Y be a subspace of X, $A \subseteq Y$, and $a \in Y$. Then a is a limit point of A in the subspace topology on Y if and only if a is a limit point of A is the topology of X.*

Proof:

a is a limit point of A in Y

$\Leftrightarrow \forall U$ open in $Y(a \in U \Rightarrow U$ intersects A outside $a)$

$\Leftrightarrow \forall V$ open in $X(a \in V \cap Y \Rightarrow V \cap Y$ intersects A outside $a)$

$\Leftrightarrow \forall V$ open in $X(a \in V \Rightarrow V$ intersects A outside $a)$

$(a \in Y, A \subseteq Y)$

$\Leftrightarrow a$ is a limit point of A in X ∎

4.4 The Box Topology

Definition 4.4.1 (Box Topology)**.** Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces. The *box topology* on $\prod_{\alpha \in J} X_\alpha$ is the topology generated by the basis consisting of all sets of the form $\prod_{\alpha \in J} U_\alpha$, where each U_α is open in X_α.

We prove this is a basis.

Proof:

⟨1⟩1. Let: $\mathcal{B}$ be the set of all sets of the form $\prod_{\alpha \in J} U_\alpha$, where each U_α is open in X_α.

⟨1⟩2. $\bigcup \mathcal{B} = \prod_{\alpha \in J} X_\alpha$

Proof: This holds because $\prod_{\alpha \in J} X_\alpha \in \mathcal{B}$.

⟨1⟩3. $\mathcal{B}$ is closed under binary intersection.

Proof: $\prod_{\alpha \in J} U_\alpha \cap \prod_{\alpha \in J} V_\alpha = \prod_{\alpha \in J}(U_\alpha \cap V_\alpha)$.

⟨1⟩4. Q.E.D.

Proof: Corollary 3.4.3.1.

Theorem 4.4.2 (AC)**.** *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces, and let $\mathcal{B}_\alpha$ be a basis for the topology on X_α for each α. Then*

$$\mathcal{B} = \{\prod_{\alpha \in J} B_\alpha : \forall \alpha \in J. B_\alpha \in \mathcal{B}_\alpha\}$$

is a basis for the box topology on $\prod_{\alpha \in J} X_\alpha$.

Proof:

⟨1⟩1. Every member of $\mathcal{B}$ is open in the box topology.

Proof: Immediate from definitions.

⟨1⟩2. For every open set U and $\{x_\alpha\}_{\alpha \in J} \in U$, there exists $B \in \mathcal{B}$ such that $\{x_\alpha\}_{\alpha \in J} \in B \subseteq U$.

⟨2⟩1. Let: U be open and $\{x_\alpha\}_{\alpha \in J} \in U$

⟨2⟩2. Pick U_α open in X_α for each α such that $\{x_\alpha\}_{\alpha \in J} \in \prod_{\alpha \in J} U_\alpha \subseteq U$.

⟨2⟩3. Pick $B_\alpha \in \mathcal{B}_\alpha$ such that $x_\alpha \in B_\alpha \subseteq U_\alpha$ for each α

Proof: Using the Axiom of Choice.

⟨2⟩4. $\{x_\alpha\}_{\alpha \in J} \in \prod_{\alpha \in J} B_\alpha \subseteq U$

∎

Theorem 4.4.3. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces, and let A_α be a subspace of X_α for all α. Let $\prod_{\alpha \in J} X_\alpha$ be given the box topology. Then the box topology on $\prod_{\alpha \in J} A_\alpha$ is the same as the topology it inherits as a subspace of $\prod_{\alpha \in J} X_\alpha$.*

Proof: Each is the topology generated by the basis
$\{\prod_{\alpha \in J}(U_\alpha \cap A_\alpha) : U_\alpha$ is open in $X_\alpha\}$, using Lemma 4.3.5. □

Theorem 4.4.4. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of Hausdorff spaces. Then $\prod_{\alpha \in J} X_\alpha$ is Hausdorff under the box topology.*

Proof:
⟨1⟩1. Let: $\{x_\alpha\}_{\alpha \in J}, \{y_\alpha\}_{\alpha \in J} \in \prod_{\alpha \in J} X_\alpha$ with $\{x_\alpha\}_{\alpha \in J} \neq \{y_\alpha\}_{\alpha \in J}$
⟨1⟩2. Pick $\alpha \in J$ such that $x_\alpha \neq y_\alpha$
⟨1⟩3. Pick disjoint neighbourhoods U of x_α and V of y_α.
⟨1⟩4. $\pi_\alpha^{-1}(U)$ and $\pi_\alpha^{-1}(V)$ are disjoint neighbourhoods of $\{x_\alpha\}_{\alpha \in J}$ and $\{y_\alpha\}_{\alpha \in J}$
□

Corollary 4.4.4.1. *The space $\mathbb{R}^\omega$ under the box topology is Hausdorff.*

Theorem 4.4.5 (AC)**.** *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces and $A_\alpha \subseteq X_\alpha$ for all α. If $\prod_{\alpha \in J} X_\alpha$ is given the box topology, then*

$$\prod_{\alpha \in J} \overline{A_\alpha} = \overline{\prod_{\alpha \in J} A_\alpha} \enspace .$$

Proof:
⟨1⟩1. $\prod_{\alpha \in J} \overline{A_\alpha} \subseteq \overline{\prod_{\alpha \in J} A_\alpha}$
  ⟨2⟩1. Let: $\{x_\alpha\}_{\alpha \in J} \in \prod_{\alpha \in J} \overline{A_\alpha}$
  ⟨2⟩2. Let: $\prod_{\alpha \in J} U_\alpha$ be a basic neighbourhood of $\{x_\alpha\}_{\alpha \in J}$, where each U_α is open in X_α.
  ⟨2⟩3. For $\alpha \in J$, Pick $a_\alpha \in A_\alpha \cap U_\alpha$.
    Proof: By Theorem 3.9.3, using the Axiom of Choice.
  ⟨2⟩4. $\{a_\alpha\}_{\alpha \in J} \in \prod_{\alpha \in J} A_\alpha \cap \prod_{\alpha \in J} U_\alpha$
  ⟨2⟩5. Q.E.D.
    Proof: By Theorem 3.9.3.
⟨1⟩2. $\overline{\prod_{\alpha \in J} A_\alpha} \subseteq \prod_{\alpha \in J} \overline{A_\alpha}$
  ⟨2⟩1. Let: $\{x_\alpha\}_{\alpha \in J} \in \overline{\prod_{\alpha \in J} A_\alpha}$
  ⟨2⟩2. Let: $\alpha \in J$
    Prove: $x_\alpha \in \overline{A_\alpha}$
  ⟨2⟩3. Let: U be a neighbourhood of x_α in X_α
  ⟨2⟩4. $\pi_\alpha^{-1}(U)$ is a neighbourhood of $\{x_\alpha\}_{\alpha \in J}$
  ⟨2⟩5. Pick $\{a_\alpha\}_{\alpha \in J} \in \pi_\alpha^{-1}(U) \cap \prod_{\alpha \in J} A_\alpha$
    Proof: By Theorem 3.9.3.
  ⟨2⟩6. $a_\alpha \in U \cap A_\alpha$
  ⟨2⟩7. Q.E.D.
    Proof: By Theorem 3.9.3.
□

4.5 The Quotient Topology

Definition 4.5.1 (Quotient Map)**.** Let X and Y be topological spaces. Let $p : X \twoheadrightarrow Y$ be a surjective map. Then p is a *quotient map* iff, for all $U \subseteq Y$, we have U is open in Y iff $p^{-1}(U)$ is open in X.

Lemma 4.5.2. *Let X and Y be topological spaces and $p : X \twoheadrightarrow Y$ be surjective and continuous. Then the following are equivalent.*

1. *p is a quotient map.*
2. *p maps saturated open sets to open sets.*
3. *p maps saturated closed sets to closed sets.*

Proof:
⟨1⟩1. $1 \Rightarrow 2$
  ⟨2⟩1. Assume: p is a quotient map.
  ⟨2⟩2. Let: $U \subseteq X$ be a saturated open set.
  ⟨2⟩3. $U = p^{-1}(p(U))$
    ⟨3⟩1. $U \subseteq p^{-1}(p(U))$
      Proof: Set theory.
    ⟨3⟩2. $p^{-1}(p(U)) \subseteq U$
      ⟨4⟩1. Let: $x \in p^{-1}(p(U))$
      ⟨4⟩2. Pick $y \in U$ such that $p(x) = p(y)$
      ⟨4⟩3. $x \in U$
        Proof: ⟨2⟩2, ⟨4⟩2.
  ⟨2⟩4. $p(U)$ is open
    Proof: ⟨2⟩1, ⟨2⟩3.
⟨1⟩2. $2 \Rightarrow 3$
  ⟨2⟩1. Assume: p maps saturated open sets to open sets
  ⟨2⟩2. Let: $C \subseteq X$ be a saturated closed set.
  ⟨2⟩3. $X \setminus C$ is a saturated open set.
    ⟨3⟩1. Let: $x \in X \setminus C$ and $x' \in X$ be such that $p(x) = p(x')$
    ⟨3⟩2. $x' \notin C$
      Proof: If $x' \in C$ then $x \in C$ since C is saturated.
  ⟨2⟩4. $p(X \setminus C)$ is open.
    Proof: By ⟨2⟩1 and ⟨2⟩3.
  ⟨2⟩5. $p(X \setminus C) = Y \setminus p(C)$
    ⟨3⟩1. $p(X \setminus C) \subseteq Y \setminus p(C)$
      ⟨4⟩1. Let: $x \in X \setminus C$
      ⟨4⟩2. Assume: for a contradiction $p(x) \in p(C)$
      ⟨4⟩3. Pick $x' \in C$ such that $p(x) = p(x')$
      ⟨4⟩4. Q.E.D.
        Proof: We have $x \notin C$, $x' \in C$ and $p(x) = p(x')$, contradicting ⟨2⟩2.
    ⟨3⟩2. $Y \setminus p(C) \subseteq p(X \setminus C)$
      ⟨4⟩1. Let: $y \notin p(C)$
      ⟨4⟩2. Pick $x \in X$ such that $p(x) = y$

PROOF: p is surjective.

$\langle 4\rangle 3.$ $x \notin C$

$\langle 1\rangle 3.$ $3 \Rightarrow 1$

$\langle 2\rangle 1.$ ASSUME: p maps saturated closed sets to closed sets

$\langle 2\rangle 2.$ LET: $C \subseteq Y$ be such that $p^{-1}(Y)$ is closed

$\langle 2\rangle 3.$ $p^{-1}(C)$ is saturated

$\langle 3\rangle 1.$ LET: $x \in p^{-1}(C)$, $x' \in X$ and $p(x) = p(x')$

$\langle 3\rangle 2.$ $x' \in p^{-1}(C)$

$\langle 2\rangle 4.$ $p(p^{-1}(C))$ is closed

PROOF: By $\langle 2\rangle 1$ and $\langle 2\rangle 3$.

$\langle 2\rangle 5.$ $C = p(p^{-1}(C))$

PROOF: By set theory, since p is surjective.

$\square$

Corollary 4.5.2.1. *If $p : X \twoheadrightarrow Y$ is a surjective continuous map that is either an open map or a closed map, then p is a quotient map.*

Definition 4.5.3 (Quotient Topology)**.** Let X be a topological space, A a set, and $p : X \twoheadrightarrow A$ a surjective map. Then the *quotient topology* on A induced by p is

$$\{U \subseteq A : p^{-1}(U) \text{ is open in } X\} \enspace .$$

It is easy to check this is a topology.

Lemma 4.5.4. *Let X be a topological space, A a set, and $p : X \twoheadrightarrow A$ a surjective map. Then the quotient topology induced by p is the unique topology on A such that p is a quotient map.*

PROOF: Immediate from definitions. $\square$

Definition 4.5.5 (Quotient Space)**.** Let X be a topological space and X^* a partition of X. Let $p : X \twoheadrightarrow X^*$ be the canonical map. Then X^* under the quotient topology induced by p is called a *quotient space* of X.

Proposition 4.5.6. *Let $p : X \twoheadrightarrow Y$ be a quotient map. Let $A \subseteq X$ be open and saturated. Then $p \restriction_A : A \twoheadrightarrow p(A)$ is a quotient map.*

PROOF:

$\langle 1\rangle 1.$ LET: $q = p \restriction_A : A \twoheadrightarrow p(A)$

$\langle 1\rangle 2.$ For all $V \subseteq p(A)$, we have $q^{-1}(V) = p^{-1}(V)$

$\langle 2\rangle 1.$ $q^{-1}(V) \subseteq p^{-1}(V)$

PROOF: Trivial.

$\langle 2\rangle 2.$ $p^{-1}(V) \subseteq q^{-1}(V)$

$\langle 3\rangle 1.$ LET: $x \in p^{-1}(V)$

$\langle 3\rangle 2.$ PICK $x' \in A$ such that $p(x') = p(x)$

PROOF: One exists because $p(x) \in V \subseteq p(A)$.

$\langle 3\rangle 3.$ $x \in A$

PROOF: This holds because A is saturated.

$\langle 3\rangle 4.$ $x \in q^{-1}(V)$

PROOF: From ⟨3⟩1 and ⟨3⟩3.
⟨1⟩3. For all $U \subseteq X$, we have $p(U \cap A) = p(U) \cap p(A)$
⟨1⟩4. LET: $V \subseteq p(A)$ be such that $q^{-1}(V)$ is open in A.
PROVE: V is open in $p(A)$.
⟨1⟩5. $q^{-1}(V)$ is open in X
⟨1⟩6. $p^{-1}(V)$ is open in X
⟨1⟩7. V is open in Y
⟨1⟩8. V is open in $p(A)$
□

Proposition 4.5.7. *Let $p : X \twoheadrightarrow Y$ be a quotient map. Let $A \subseteq X$ be closed and saturated. Then $p \restriction_A : A \twoheadrightarrow p(A)$ is a quotient map.*

PROOF: Similar. □

Proposition 4.5.8. *Let $p : X \twoheadrightarrow Y$ be an open quotient map. Let $A \subseteq X$ be saturated. Then $p \restriction_A : A \twoheadrightarrow p(A)$ is a quotient map.*

PROOF:
⟨1⟩1. LET: $q = p \restriction_A : A \twoheadrightarrow p(A)$
⟨1⟩2. For all $V \subseteq p(A)$, we have $q^{-1}(V) = p^{-1}(V)$
⟨2⟩1. $q^{-1}(V) \subseteq p^{-1}(V)$
PROOF: Trivial.
⟨2⟩2. $p^{-1}(V) \subseteq q^{-1}(V)$
⟨3⟩1. LET: $x \in p^{-1}(V)$
⟨3⟩2. PICK $x' \in A$ such that $p(x') = p(x)$
PROOF: One exists because $p(x) \in V \subseteq p(A)$.
⟨3⟩3. $x \in A$
PROOF: This holds because A is saturated.
⟨3⟩4. $x \in q^{-1}(V)$
PROOF: From ⟨3⟩1 and ⟨3⟩3.
⟨1⟩3. For all $U \subseteq X$, we have $p(U \cap A) = p(U) \cap p(A)$
⟨2⟩1. $p(U \cap A) \subseteq p(U) \cap p(A)$
PROOF: Set theory.
⟨2⟩2. $p(U) \cap p(A) \subseteq p(U \cap A)$
⟨3⟩1. LET: $x \in U$, $y \in A$, $p(x) = p(y)$
PROVE: $p(x) \in p(U \cap A)$
⟨3⟩2. $x \in A$
PROOF: A is saturated.
⟨3⟩3. $x \in U \cap A$
⟨1⟩4. LET: $V \subseteq p(A)$ be such that $q^{-1}(V)$ is open in A.
PROVE: V is open in $p(A)$.
⟨1⟩5. $p^{-1}(V)$ is open in A
PROOF: By ⟨1⟩2
⟨1⟩6. PICK U open in X such that $p^{-1}(V) = U \cap A$
⟨1⟩7. $V = p(U) \cap p(A)$

PROOF:

$$\begin{aligned} V &= p(p^{-1}(V)) && (p \text{ is surjective}) \\ &= p(U \cap A) && (\langle 1\rangle 6) \\ &= p(U) \cap p(A) && (\langle 1\rangle 3) \end{aligned}$$

$\langle 1\rangle 8$. $p(U)$ is open in Y
 PROOF: $\langle 1\rangle 6$, p is an open map.
$\langle 1\rangle 9$. V is open in $p(A)$
 PROOF: $\langle 1\rangle 7$, $\langle 1\rangle 8$
□

Proposition 4.5.9. *Let $p : X \twoheadrightarrow Y$ be a closed quotient map. Let $A \subseteq X$ be saturated. Then $p \restriction_A : A \twoheadrightarrow p(A)$ is a quotient map.*

PROOF: Similar. □

Proposition 4.5.10. *The composite of two quotient maps is a quotient map.*

PROOF: From Proposition 5.2.22. □

Proposition 4.5.11. *Let X^* be a quotient space of X. If every element of X^* is closed in X, then X^* is T_1.*

PROOF:
$\langle 1\rangle 1$. LET: $C \in X^*$
$\langle 1\rangle 2$. $p^{-1}(\{C\}) = C$
 PROOF: Definition of p.
$\langle 1\rangle 3$. $p^{-1}(\{C\})$ is closed in X
 PROOF: By hypothesis.
$\langle 1\rangle 4$. $\{C\}$ is closed in X^*.
 PROOF: By Proposition 5.2.21.
□

Chapter 5

Functions Between Topological Spaces

5.1 Open Maps

Definition 5.1.1. Let X and Y be topological spaces and $f : X \to Y$. Then f is an *open map* iff, for all U open in X, $f(U)$ is open in Y.

Lemma 5.1.2. *Let X and Y be topological spaces and $f : X \to Y$. Let $\mathcal{B}$ be a basis for the topology on X. Then f is an open map if and only if, for all $B \in \mathcal{B}$, $f(B)$ is open in Y.*

PROOF:
$\langle 1 \rangle 1$. If f is an open map then, for all $B \in \mathcal{B}$, $f(B)$ is open in Y.
  PROOF: Immediate from definitions.
$\langle 1 \rangle 2$. If, for all $B \in \mathcal{B}$, $f(B)$ is open in Y, then f is an open map.
  $\langle 2 \rangle 1$. ASSUME: For all $B \in \mathcal{B}$, $f(B)$ is open in Y.
  $\langle 2 \rangle 2$. LET: U be open in X
    PROVE: $f(U)$ is open in Y
  $\langle 2 \rangle 3$. LET: $\mathcal{B}_0 \subseteq \mathcal{B}$ be such that $U = \bigcup \mathcal{B}_0$
  $\langle 2 \rangle 4$. $f(U) = \bigcup_{B \in \mathcal{B}_0} f(B)$
    PROOF: Set theory.
  $\langle 2 \rangle 5$. $f(U)$ is open in Y.
    PROOF: From $\langle 2 \rangle 1$, $\langle 2 \rangle 4$ and the fact that the open sets are closed under union.
$\square$

Corollary 5.1.2.1. *Let X and Y be topological spaces and $f : X \to Y$. Let $\mathcal{S}$ be a subbasis for the topology on X. Then f is an open map if and only if, for all $S \in \mathcal{S}$, $f(S)$ is open in Y.*

Lemma 5.1.3 (AC)**.** *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces. Then the projection $\pi_\alpha : \prod_{\alpha \in J} X_\alpha \to X_\alpha$ is an open map.*

PROOF:
⟨1⟩1. For U open in X_α, we have $\pi_\alpha(\pi_\alpha^{-1}(U))$ is open in X_α
  PROOF: $\pi_\alpha(\pi_\alpha^{-1}(U)) = U$ if all the other X_α are nonempty, $\emptyset$ otherwise.
⟨1⟩2. For $\beta \neq \alpha$ and U open in X_β, we have $\pi_\alpha(\pi_\beta^{-1}(U))$ is open in X_α
  PROOF: $\pi_\alpha(\pi_\beta^{-1}(U)) = X_\alpha$ if all the X_γ are nonempty for $\gamma \neq \alpha$, $\emptyset$ otherwise.
⟨1⟩3. Q.E.D.
  PROOF: By Corollary 5.1.2.1.

5.2 Continuous Functions

Definition 5.2.1 (Continuous)**.** Let X and Y be topological spaces and $f : X \to Y$ a function. Then f is *continuous* if and only if, for every open set U in Y, the set $f^{-1}(U)$ is open in X.

Theorem 5.2.2. *Let X and Y be topological spaces and $f : X \to Y$. Then the following are equivalent.*

1. *f is continuous.*
2. *For every closed set C in Y, the set $f^{-1}(C)$ is closed in X.*
3. *For every set $A \subseteq X$, we have $f(\overline{A}) \subseteq \overline{f(A)}$.*

PROOF:
⟨1⟩1. $1 \Rightarrow 3$
  ⟨2⟩1. ASSUME: f is continuous.
  ⟨2⟩2. LET: $A \subseteq X$
  ⟨2⟩3. LET: $x \in \overline{A}$
     PROVE: $f(x) \in \overline{f(A)}$
  ⟨2⟩4. LET: V be a neighbourhood of $f(x)$
  ⟨2⟩5. $f^{-1}(V)$ is a neighbourhood of x
    PROOF: ⟨2⟩1, ⟨2⟩3, ⟨2⟩4
  ⟨2⟩6. $f^{-1}(V)$ intersects A in a, say.
    PROOF: ⟨2⟩3, ⟨2⟩5, Theorem 3.9.3.
  ⟨2⟩7. V intersects $f(A)$ in $f(a)$.
  ⟨2⟩8. Q.E.D.
    PROOF: Theorem 3.9.3.
⟨1⟩2. $3 \Rightarrow 2$
  ⟨2⟩1. ASSUME: 3
  ⟨2⟩2. LET: C be a closed set in Y
  ⟨2⟩3. $\overline{f^{-1}(C)} = f^{-1}(C)$
    PROOF:
$$\begin{aligned} f(\overline{f^{-1}(C)}) &\subseteq \overline{f(f^{-1}(C))} && (⟨2⟩1) \\ &\subseteq \overline{C} \end{aligned}$$
⟨1⟩3. $2 \Rightarrow 1$
  ⟨2⟩1. ASSUME: 2

⟨2⟩2. LET: V be open in Y
⟨2⟩3. $f^{-1}(Y \setminus V)$ is closed in X
PROOF: By ⟨2⟩1.
⟨2⟩4. $f^{-1}(V)$ is open in X.
PROOF: $f^{-1}(V) = X \setminus f^{-1}(Y \setminus V)$.
□

Lemma 5.2.3. *If $f : X \to Y$ maps all of X to the single point y_0 of Y, then f is continuous.*

PROOF: For V open in Y, the set $f^{-1}(V)$ is either X (if $y_0 \in V$) or $\emptyset$ (if $y_0 \notin V$).

Definition 5.2.4 (Continuity at a Point)**.** Let X and Y be topological spaces, $f : X \to Y$ a function, and $x \in X$. Then f is *continuous at* x if and only if, for every neighbourhood V of $f(x)$, there exists a neighbourhood U of x such that $f(U) \subseteq V$.

Theorem 5.2.5. *Let X and Y be topological spaces and $f : X \to Y$. Then f is continuous if and only if f is continuous at every point of X.*

PROOF:
⟨1⟩1. If f is continuous then f is continuous at every point of X.
⟨2⟩1. ASSUME: f is continuous
⟨2⟩2. LET: $x \in X$
⟨2⟩3. LET: V be a neighbourhood of $f(x)$
⟨2⟩4. $f^{-1}(V)$ is a neighbourhood of x
⟨2⟩5. $f(f^{-1}(V)) \subseteq V$
⟨1⟩2. If f is continuous at every point of X then f is continuous.
⟨2⟩1. ASSUME: f is continuous at every point of X.
⟨2⟩2. LET: V be open in Y
PROVE: $f^{-1}(V)$ is open in X.
⟨2⟩3. LET: $x \in f^{-1}(V)$
⟨2⟩4. V is a neighbourhood of $f(x)$
⟨2⟩5. PICK a neighbourhood U of x such that $f(U) \subseteq V$
PROOF: By ⟨2⟩1.
⟨2⟩6. $x \in U \subseteq f^{-1}(V)$
⟨2⟩7. Q.E.D.
PROOF: By Proposition 3.2.3.
□

Lemma 5.2.6. *Let X and Y be topological spaces and $f : X \to Y$. Let $\mathcal{B}$ be a basis for the topology on Y. Then f is continuous if and only if, for all $B \in \mathcal{B}$, the set $f^{-1}(B)$ is open in X.*

PROOF:
⟨1⟩1. If f is continuous then, for all $B \in \mathcal{B}$, the set $f^{-1}(B)$ is open in X.
PROOF: Immediate from definitions.
⟨1⟩2. If, for all $B \in \mathcal{B}$, the set $f^{-1}(B)$ is open in X, then f is continuous.

⟨2⟩1. ASSUME: For all $B \in \mathcal{B}$, the set $f^{-1}(B)$ is open in X.
⟨2⟩2. LET: $x \in X$
⟨2⟩3. LET: V be a neighbourhood of $f(x)$
⟨2⟩4. PICK $B \in \mathcal{B}$ such that $f(x) \in B \subseteq V$
⟨2⟩5. $f^{-1}(B)$ is a neighbourhood of x
PROOF: By ⟨2⟩1.
⟨2⟩6. $f(f^{-1}(B)) \subseteq B$
PROOF: Set theory.
⟨2⟩7. Q.E.D.
PROOF: Theorem 5.2.5.
□

Lemma 5.2.7. *The projections $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are continuous.*

PROOF:Immediate from definitions. □

Theorem 5.2.8. *If A is a subspace of X, the inclusion function $j : A \to X$ is continuous.*

PROOF: For V open in X, the set $j^{-1}(V) = V \cap A$ is open in A.

Theorem 5.2.9. *If $f : X \to Y$ and $g : Y \to Z$ are continuous, then the map $g \circ f : X \to Z$ is continuous.*

PROOF:
⟨1⟩1. LET: V be open in Z
⟨1⟩2. $g^{-1}(V)$ is open in Y
⟨1⟩3. $f^{-1}(g^{-1}(V))$ is open in X
□

Theorem 5.2.10. *If $f : X \to Y$ is continuous and if A is a subspace of X, then the restricted function $f \restriction A : A \to Y$ is continuous.*

PROOF: For V open in Y, the set $(f \restriction A)^{-1}(V) = f^{-1}(V) \cap A$ is open in A. □

Theorem 5.2.11. *Let $f : X \to Y$ be continuous. If Z is a subspace of Y that includes the range of f, then the function $g : X \to Z$ obtained by restricting the codomain of f is continuous. If Z is a space having Y as a subspace, then the function $h : X \to Z$ obtained by expanding the codomain of f is continuous.*

PROOF:
⟨1⟩1. If Z is a subspace of Y that includes the range of f, then the function $g : X \to Z$ obtained by restricting the codomain of f is continuous.
⟨2⟩1. LET: V be open in Z
⟨2⟩2. PICK W open in Y such that $V = W \cap Z$
⟨2⟩3. $f^{-1}(W)$ is open in X.
⟨2⟩4. $g^{-1}(V)$ is open in X.
PROOF: $g^{-1}(V) = f^{-1}(W)$.

⟨1⟩2. If Z is a space having Y as a subspace, then the function $h : X \to Z$ obtained by expanding the codomain of f is continuous.

PROOF: For V open in Z, we have $h^{-1}(V) = f^{-1}(V \cap Y)$ is open in X.

□

Theorem 5.2.12. *Let X and Y be topological spaces and $f : X \to Y$. If $x_n \to x$ as $n \to \infty$ in X and f is continuous at x, then $f(x_n) \to f(x)$ as $n \to \infty$ in Y.*

PROOF:

⟨1⟩1. ASSUME: $x_n \to x$ as $n \to \infty$

⟨1⟩2. ASSUME: f is continuous at x

⟨1⟩3. LET: V be a neighbourhood of $f(x)$

⟨1⟩4. PICK a neighbourhood U of x such that $f(U) \subseteq V$

PROOF: By ⟨1⟩2.

⟨1⟩5. PICK N such that, for all $n \geq N$, $x_n \in U$

PROOF: By ⟨1⟩1

⟨1⟩6. For $n \geq N$, $f(x_n) \in V$

PROOF: By ⟨1⟩4.

□

Theorem 5.2.13. *Let X, Y and Z be topological spaces. Let $f : X \to Y$. If there exists a set $\mathcal{A}$ of open sets in X such that:*

- *$\bigcup \mathcal{A} = X$;*
- *for all $U \in \mathcal{A}$, the function $f \restriction U : U \to X$ is continuous;*

then f is continuous.

PROOF:

⟨1⟩1. LET: V be open in Y

⟨1⟩2. For all $U \in \mathcal{A}$, the set $(f \restriction U)^{-1}(V)$ is open in X.

⟨2⟩1. LET: $U \in \mathcal{A}$

⟨2⟩2. $(f \restriction U)^{-1}(V)$ is open in U

PROOF: Since $f \restriction U : U \to X$ is continuous.

⟨2⟩3. Q.E.D.

PROOF: By Lemma 4.3.3.

⟨1⟩3. Q.E.D.

PROOF: Since $f^{-1}(V) = \bigcup_{U \in \mathcal{A}} (f \restriction U)^{-1}(V)$.

Theorem 5.2.14 (The Pasting Lemma)**.** *Let $X = A \cup B$ where A and B are closed in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous. If $f(x) = g(x)$ for every $x \in A \cap B$, then the function $h : X \to Y$ defined by*

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases}$$

is continuous.

PROOF:

⟨1⟩1. LET: C be closed in Y
⟨1⟩2. $f^{-1}(C)$ is closed in A
PROOF: Theorem 5.2.2.
⟨1⟩3. $f^{-1}(C)$ is closed in X
PROOF: Lemma 4.3.4.1.
⟨1⟩4. $g^{-1}(C)$ is closed in B
PROOF: Theorem 5.2.2.
⟨1⟩5. $g^{-1}(C)$ is closed in X
PROOF: Lemma 4.3.4.1.
⟨1⟩6. $h^{-1}(C)$ is closed in X
PROOF: $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$
⟨1⟩7. Q.E.D.
PROOF: Theorem 5.2.2.
□

Theorem 5.2.15. *Let $f : A \to \prod_{\alpha \in J} X_\alpha$ be given by the equation*

$$f(a) = \{f_\alpha(a)\}_{\alpha \in J} ,$$

where $f_\alpha : A \to X_\alpha$ for each α. Let $\prod_{\alpha \in J} X_\alpha$ have the product topology. Then the function f is continuous if and only if each function f_α is continuous.

PROOF:
⟨1⟩1. If f is continuous then each f_α is continuous.
PROOF: This holds because $f_\alpha = \pi_\alpha \circ f$.
⟨1⟩2. If every f_α is continuous then f is continuous.
⟨2⟩1. ASSUME: Every f_α is continuous.
⟨2⟩2. LET: $\alpha \in J$ and U be open in X_α
⟨2⟩3. $f^{-1}(\pi_\alpha^{-1}(U))$ is open in A
PROOF: $f^{-1}(\pi_\alpha^{-1}(U)) = f_\alpha^{-1}(U)$.
□

5.2.1 Homeomorphisms

Definition 5.2.16 (Homeomorphism)**.** Let X and Y be topological spaces and $f : X \to Y$. Then f is a *homeomorphism* between X and Y iff f is a bijection, and f and f^{-1} are both continuous.

Definition 5.2.17 (Topological Property)**.** A property P of topological spaces is a *topological property* iff, for any spaces X and Y, if X is homeomorphic to Y then P holds of X if and only if P holds of Y.

Definition 5.2.18 ((Topological) Imbedding)**.** Let X and Y be topological spaces and $f : X \to Y$. Then f is a *(topological) imbedding* iff f is a homeomorphism between X and $\operatorname{im} f$.

Definition 5.2.19 (Homogeneous)**.** A topological space X is *homogeneous* iff, for all $x, y \in X$, there exists a homeomorphism $f : X \cong X$ such that $f(x) = y$.

5.2.2 Strongly Continuous Functions

Definition 5.2.20 (Strongly Continuous)**.** Let X and Y be topological spaces and $f : X \to Y$. Then f is *strongly continuous* iff, for all $V \subseteq Y$, we have V is open in Y if and only if $f^{-1}(V)$ is open in X.

Proposition 5.2.21. *Let X and Y be topological spaces and $f : X \to Y$. Then f is strongly continuous if and only if, for all $C \subseteq Y$, C is closed in Y if and only if $f^{-1}(C)$ is closed in X.*

PROOF:
⟨1⟩1. If f is strongly continuous then, for all $C \subseteq Y$, we have C is closed in Y if and only if $f^{-1}(C)$ is closed in X.
  PROOF:

$$\begin{aligned} C \text{ is closed in } Y &\Leftrightarrow Y \setminus C \text{ is open in } Y \\ &\Leftrightarrow f^{-1}(Y \setminus C) \text{ is open in } X \\ &\Leftrightarrow X \setminus f^{-1}(C) \text{ is open in } X \\ &\Leftrightarrow f^{-1}(C) \text{ is closed in } X \end{aligned}$$

⟨1⟩2. If, for all $C \subseteq Y$, we have C is closed in Y if and only if $f^{-1}(C)$ is closed in X, then f is strongly continuous.
  PROOF: Similar.
□

Proposition 5.2.22. *The composite of two strongly continuous functions is strongly continuous.*

PROOF:
⟨1⟩1. LET: $f : X \to Y$ and $g : Y \to Z$ be strongly continuous.
⟨1⟩2. LET: $V \subseteq Z$
⟨1⟩3. V is open iff $f^{-1}(g^{-1}(V))$ is open
  PROOF:

$$\begin{aligned} V \text{ is open} &\Leftrightarrow g^{-1}(V) \text{ is open} && (\langle 1\rangle 1) \\ &\Leftrightarrow f^{-1}(g^{-1}(V)) \text{ is open} && (\langle 1\rangle 1) \end{aligned}$$

□

Proposition 5.2.23. *Let X, Y and Z be topological spaces. Let $f : X \to Y$ and $g : Y \to Z$. If f is strongly continuous and $g \circ f$ is continuous, then g is continuous.*

PROOF:
⟨1⟩1. LET: $V \subseteq Z$ be open in Z.
⟨1⟩2. $f^{-1}(g^{-1}(V))$ is open in X.
  PROOF: $g \circ f$ is continuous.
⟨1⟩3. $g^{-1}(V)$ is open in Y.
  PROOF: f is strongly continuous.
□

Proposition 5.2.24. *Let X, Y and Z be topological spaces. Let $f : X \to Y$ and $g : Y \to Z$. If f and $g \circ f$ are strongly continuous, then g is strongly continuous.*

PROOF:
$\langle 1 \rangle 1$. LET: $U \subseteq Z$
$\langle 1 \rangle 2$. U is open in Z iff $g^{-1}(U)$ is open in Y
PROOF:

$$\begin{aligned} U \text{ is open in } Z &\Leftrightarrow f^{-1}(g^{-1}(U)) \text{ is open in } X && (g \circ f \text{ is strongly continuous}) \\ &\Leftrightarrow g^{-1}(U) \text{ is open in } Y && (f \text{ is strongly continuous}) \end{aligned}$$

□

5.3 Closed Maps

Definition 5.3.1 (Closed Map)**.** Let X and Y be topological spaces and $f : X \to Y$. Then f is a *closed map* iff, for every closed set $C \subseteq X$, the set $f(C)$ is closed in Y.

5.4 Local Homeomorphism

Definition 5.4.1 (Locally Homeomorphic)**.** Let X and Y be topological spaces. Then X is *locally homeomorphic* to Y iff every point in X has an open neighborhood that is homeomorphic with an open set in Y.

Proposition 5.4.2. *The long line is locally homeomorphic with* $\mathbb{R}$.

PROOF:
$\langle 1 \rangle 1$. LET: $x \in L$
$\langle 1 \rangle 2$. PICK an ordinal α such that $x < (\alpha, 0)$.
$\langle 1 \rangle 3$. $(-\infty, (\alpha, 0))$ is an open neighbourhood of x that is homeomorphic to $(0, 1)$.
□

5.5 Retracts

Definition 5.5.1 (Retract)**.** Let Z be a topological space. If Y is a subspace of Z, we say that Y is a *retract* of Z iff there exists a continuous function $r : Z \to Y$ such that $r(y) = y$ for all $y \in Y$.

Chapter 6

Separation Axioms

6.1 T_1 Spaces

Definition 6.1.1 (T_1 Space)**.** A topological space X is a T_1 *space* iff every finite set is closed.

Theorem 6.1.2. *Let X be a T_1 space and $A \subseteq X$. Then x is a limit point of A if and only if every neighbourhood of x contains infinitely many points of A.*

Proof:
$\langle 1 \rangle 1$. If some neighbourhood of x contains only finitely many points of A then x is not a limit point of A.
  $\langle 2 \rangle 1$. Assume: Some neighbourhood U of x contains only finite many points $a_1, \ldots, a_n$ of A.
  $\langle 2 \rangle 2$. $X \setminus \{a_1, \ldots, a_n\}$ is open.
   Proof: X is T_1.
  $\langle 2 \rangle 3$. $U \setminus \{a_1, \ldots, a_n\}$ is a neighbourhood of x that does not intersect A.
$\langle 1 \rangle 2$. If every neighbourhood of x contains infinitely many points of A then x is a limit point of A.
  Proof: From the definition of limit point.
□

Proposition 6.1.3. *A subspace of a T_1 space is T_1.*

Proof:
$\langle 1 \rangle 1$. Let: X be a T_1 space and $Y \subseteq X$
$\langle 1 \rangle 2$. Let: $a \in Y$
$\langle 1 \rangle 3$. $\{a\}$ is closed in X
  Proof: By $\langle 1 \rangle 1$.
$\langle 1 \rangle 4$. $\{a\}$ is closed in Y
  Proof: By Corollary 4.3.4.1.
□

Definition 6.1.4 (Separate Points from Closed Sets)**.** Let X be a space and $\{f_\alpha\}_{\alpha \in J}$ be a family of continuous functions $f_\alpha : X \to \mathbb{R}$. Then $\{f_\alpha\}$ *separates points from closed sets* in X iff, for every point $x_0 \in X$ and every neighbourhood U of x_0, there exists $\alpha \in J$ such that f_α is positive at x_0 and vanishes outside U.

Theorem 6.1.5 (Imbedding Theorem)**.** *Let X be a T_1 space and $\{f_\alpha\}_{\alpha \in J}$ be a family of functions $X \to \mathbb{R}$ that separates points from closed sets. Then the function $F : X \to \mathbb{R}^J$ defined by*

$$F(x)_\alpha = f_\alpha(x)$$

is an imbedding. If each f_α maps X into $[0, 1]$ then F is an imbedding $X \to [0, 1]^J$.

Proof:
⟨1⟩1. F is continuous
  Proof: By Theorem 5.2.15.
⟨1⟩2. F is injective
  ⟨2⟩1. Let: $x, y \in X$ with $x \neq y$
  ⟨2⟩2. Pick a neighbourhood U of x such that $y \notin U$
    Proof: X is T_1
  ⟨2⟩3. Pick $\alpha \in J$ such that f_α is positive at x and vanishes outside U
  ⟨2⟩4. $f_\alpha(x) \neq f_\alpha(y)$
  ⟨2⟩5. $F(x) \neq F(y)$
⟨1⟩3. F is open as a map $X \to F(U)$
  ⟨2⟩1. Let: U be open
  ⟨2⟩2. Let: $z \in F(U)$
  ⟨2⟩3. Pick $x \in U$ such that $F(x) = z$
  ⟨2⟩4. Pick $\alpha \in J$ such that f_α is positive at x and vanishes outside U
  ⟨2⟩5. $z \in {\pi_\alpha}^{-1}((0, +\infty)) \cap F(U) \subseteq F(U)$
□

6.2 Hausdorff Spaces

Definition 6.2.1 (Hausdorff Space)**.** A topological space X is a *Hausdorff space* iff, for any points $x, y \in X$ with $x \neq y$, there exist disjoint neighbourhoods U of x and V of y.

Theorem 6.2.2. *Every Hausdorff space is T_1.*

Proof:
⟨1⟩1. Let: X be a Hausdorff space
⟨1⟩2. Let: $a \in X$
    Prove: $\{a\}$ is closed.
⟨1⟩3. Let: $b \in X \setminus \{a\}$
⟨1⟩4. Pick disjoint neighbourhoods U of a and V of b

$\langle 1 \rangle 5.$ $b \in V \subseteq X \setminus \{a\}$
$\langle 1 \rangle 6.$ Q.E.D.
PROOF: By Proposition 3.2.3.
□

Theorem 6.2.3. *In a Hausdorff space, a sequence has at most one limit.*

PROOF:
$\langle 1 \rangle 1.$ ASSUME: for a contradiction $x_n \to l$ and $x_n \to m$ as $n \to \infty$, and $l \neq m$
$\langle 1 \rangle 2.$ PICK disjoint neighbourhoods U of l and V of m
$\langle 1 \rangle 3.$ PICK N such that, for all $n \geq N$, $x_n \in U$ and $x_n \in V$
$\langle 1 \rangle 4.$ $x_N \in U \cap V$
□

Theorem 6.2.4. *Every linearly ordered set is Hausdorff under the order topology.*

PROOF:
$\langle 1 \rangle 1.$ LET: X be a linearly ordered set under the order topology.
$\langle 1 \rangle 2.$ LET: $x, y \in X$ with $x \neq y$
$\langle 1 \rangle 3.$ ASSUME: w.l.o.g. $x < y$
PROVE: There exist disjoint neighbourhoods U of x and V of y.
$\langle 1 \rangle 4.$ CASE: There exists z such that $x < z < y$
PROOF: In this case, take $U = (-\infty, z)$ and $V = (z, +\infty)$.
$\langle 1 \rangle 5.$ CASE: There does not exist z such that $x < z < y$
PROOF: In this case, take $U = (-\infty, y)$ and $V = (x, +\infty)$.
□

Theorem 6.2.5. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of Hausdorff spaces. Then $\prod_{\alpha \in J} X_\alpha$ is Hausdorff under the product topology.*

PROOF:
$\langle 1 \rangle 1.$ LET: $\{x_\alpha\}_{\alpha \in J}, \{y_\alpha\}_{\alpha \in J} \in \prod_{\alpha \in J} X_\alpha$ with $\{x_\alpha\}_{\alpha \in J} \neq \{y_\alpha\}_{\alpha \in J}$
$\langle 1 \rangle 2.$ PICK $\alpha \in J$ such that $x_\alpha \neq y_\alpha$
$\langle 1 \rangle 3.$ PICK disjoint neighbourhoods U of x_α and V of y_α.
$\langle 1 \rangle 4.$ $\pi_\alpha^{-1}(U)$ and $\pi_\alpha^{-1}(V)$ are disjoint neighbourhoods of $\{x_\alpha\}_{\alpha \in J}$ and $\{y_\alpha\}_{\alpha \in J}$
□

Corollary 6.2.5.1. *The Sorgenfrey plane is Hausdorff.*

Corollary 6.2.5.2. *For any set I, the space $\mathbb{R}^I$ is Hausdorff.*

Proposition 6.2.6. *Let X and Y be topological spaces and $f : X \to Y$. If f is continuous and injective and Y is Hausdorff then X is Hausdorff.*

PROOF:
$\langle 1 \rangle 1.$ LET: $x, y \in X$ with $x \neq y$
$\langle 1 \rangle 2.$ $f(x) \neq f(y)$
PROOF: f is injective.

⟨1⟩3. PICK disjoint neighbourhoods U, V of $f(x)$ and $f(y)$
PROOF: Y is Hausdorff.
⟨1⟩4. $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint neighbourhoods of x and y.
□

Corollary 6.2.6.1. *A subspace of a Hausdorff space is Hausdorff.*

Corollary 6.2.6.2. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of nonempty spaces. If $\prod_{\alpha \in J} X_\alpha$ is Hausdorff then so is each X_α.*

Corollary 6.2.6.3. *Let $\mathcal{T}$ and $\mathcal{T}'$ be topologies on the same set X. If $\mathcal{T} \subseteq \mathcal{T}'$ and X is Hausdorff under $\mathcal{T}$ then X is Hausdorff under $\mathcal{T}'$.*

Corollary 6.2.6.4. *The space $\mathbb{R}_K$ is Hausdorff.*

Proposition 6.2.7. *$\mathbb{R}_l$ is Hausdorff.*

PROOF: Let $a, b \in \mathbb{R}_l$ with $a < b$. Then $(-\infty, b)$ and $[b, +\infty)$ are disjoint open sets containing a and b respectively. □

Proposition 6.2.8. *The continuous image of a Hausdorff space is not necessarily Hausdorff.*

PROOF: The identity map from the discrete two-point space to the indiscrete two-point space is continuous. □

Lemma 6.2.9. *Let A be a subspace of X and Z be Hausdorff. Let $f : A \to Z$ be contiuous. Then there is at most one extension of f to a continuous function $\overline{A} \to Z$.*

PROOF:
⟨1⟩1. ASSUME: for a contradiction $g, h : \overline{A} \to Z$ are continuous extensions of f with $g(x) \neq h(x)$
⟨1⟩2. PICK disjoint open neighbourhoods U of $g(x)$ and V of $h(x)$
⟨1⟩3. PICK a point $a \in A \cap g^{-1}(U) \cap h^{-1}(V)$
PROOF: One exists because $g^{-1}(U) \cap h^{-1}(V)$ is a neighbourhood of $x \in \overline{A}$.
⟨1⟩4. $g(a) \in U \cap V$
□

6.3 Regular Spaces

Definition 6.3.1 (Regular)**.** A topological space X is *regular* iff, for every closed set A and point $a \notin A$, there exist disjoint neighbourhoods U of A and V of a.

Proposition 6.3.2. *Let X be a T_1 space. Then X is regular if and only if, for every point x and neighbourhood U of x, there exists a neighbourhood V of x such that $\overline{V} \subseteq U$.*

PROOF:

$\langle 1 \rangle 1$. If X is regular then, for every point x and neighbourhood N of x, there exists a neighbourhood V of x such that $\overline{V} \subseteq N$.
$\langle 2 \rangle 1$. ASSUME: X is regular.
$\langle 2 \rangle 2$. LET: $x \in X$ and N be a neighbourhood of x
$\langle 2 \rangle 3$. PICK an open set U such that $x \in U \subseteq N$
$\langle 2 \rangle 4$. PICK disjoint open sets V, W such that $x \in V$ and $X \setminus U \subseteq W$
$\langle 2 \rangle 5$. $\overline{V} \subseteq N$
PROOF:

$$\begin{aligned} \overline{V} &\subseteq X \setminus W \\ &\subseteq U \\ &\subseteq N \end{aligned}$$

$\langle 1 \rangle 2$. If, for every point x and neighbourhood U of x, there exists a neighbourhood V of x such that $\overline{V} \subseteq U$, then X is regular.
$\langle 2 \rangle 1$. ASSUME: For every point x and neighbourhood U of x, there exists a neighbourhood V of x such that $\overline{V} \subseteq U$.
$\langle 2 \rangle 2$. LET: $x \in X$ and A be a closed set with $x \notin A$
$\langle 2 \rangle 3$. PICK a neighbourhood V of x such that $\overline{V} \subseteq X \setminus A$
$\langle 2 \rangle 4$. $x \in V$ and $A \subseteq X \setminus \overline{V}$
$\square$

Proposition 6.3.3. *Every linearly ordered set under the order topology is regular.*

PROOF:
$\langle 1 \rangle 1$. LET: X be a linearly ordered set under the order topology.
$\langle 1 \rangle 2$. LET: $x \in X$ and U be a neighbourhood of x
PROVE: There exists a neighbourhood V of x with $\overline{V} \subseteq U$
$\langle 1 \rangle 3$. CASE: x is greatest and least in X
PROOF: Take $V = U = X = \{x\}$
$\langle 1 \rangle 4$. CASE: x is greatest in X and there exists $a < x$ such that $(a, x] \subseteq U$
$\langle 2 \rangle 1$. CASE: There exists b such that $a < b < x$
PROOF: Take $V = (b, x]$.
$\langle 2 \rangle 2$. CASE: There is no b such that $a < b < x$
$\langle 3 \rangle 1$. LET: $V = U = \{x\}$
$\langle 3 \rangle 2$. $\overline{V} = V$
PROOF: For any $y \neq x$, we have $(-\infty, x)$ is a neighbourhood of y that does not intersect V.
$\langle 1 \rangle 5$. CASE: x is least in X and there exists $b > x$ such that $[x, b) \subseteq U$
PROOF: Similar.
$\langle 1 \rangle 6$. CASE: There exist $a < x < b$ such that $(a, b) \subseteq U$
$\langle 2 \rangle 1$. PICK a point c such that $a < c < x$ if there is one, otherwise LET: $c = a$
$\langle 2 \rangle 2$. PICK a point d such that $x < d < b$ if there is one, otherwise LET: $d = b$
$\langle 2 \rangle 3$. LET: $V = (c, d)$
$\langle 2 \rangle 4$. $\overline{V} \subseteq U$

PROOF:
$$\begin{aligned}\overline{V} &\subseteq [c, d] \\ &\subseteq (a, b) \\ &\subseteq U\end{aligned}$$

⟨1⟩7. Q.E.D.
PROOF: These cases are exhaustive by Lemma 4.1.2. They prove X is regular by Proposition 6.3.2.

∎

Proposition 6.3.4. *A subspace of a regular space is regular.*

PROOF:
⟨1⟩1. LET: X be a regular space and $Y \subseteq X$
⟨1⟩2. LET: $A \subseteq Y$ be closed in Y and $a \in Y \setminus A$
⟨1⟩3. PICK C closed in X such that $A = C \cap Y$
PROOF: By Corollary 4.3.4.1.
⟨1⟩4. PICK disjoint open sets U, V in X such that $C \subseteq U$ and $a \in V$
⟨1⟩5. $U \cap Y$ and $V \cap Y$ are disjoint open sets in Y such that $A \subseteq U \cap Y$ and $a \in V \cap Y$

∎

Corollary 6.3.4.1. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of nonempty spaces. If $\prod_{\alpha \in J} X_\alpha$ is regular then so is each X_α.*

Proposition 6.3.5 (AC)**.** *The product of a family of regular spaces is regular.*

PROOF:
⟨1⟩1. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of regular spaces.
⟨1⟩2. $\prod_{\alpha \in J} X_\alpha$ is T_1
⟨1⟩3. LET: $\vec{a} \in U$ where U is open in $\prod_{\alpha \in J} X_\alpha$
⟨1⟩4. PICK $\prod_{\alpha \in J} U_\alpha$ such that each U_α is open in X_α, $U_\alpha = X_\alpha$ except at α_1, …, α_n, and $\vec{a} \in \prod_{\alpha \in J} U_\alpha \subseteq U$
⟨1⟩5. For $1 \leq i \leq n$, PICK V_{α_i} open in X_{α_i} such that $a_{\alpha_i} \in V_{\alpha_i}$ and $\overline{V_{\alpha_i}} \subseteq U_{\alpha_i}$
⟨1⟩6. For $\alpha \neq \alpha_1, \ldots, \alpha_n$,
LET: $V_\alpha = X_\alpha$
⟨1⟩7. $\vec{a} \in \prod_{\alpha \in J} V_\alpha$
⟨1⟩8. $\overline{\prod_{\alpha \in J} V_\alpha} \subseteq \prod_{\alpha \in J} U_\alpha$
PROOF: By Theorem 4.2.5.

∎

Corollary 6.3.5.1. *The Sorgenfrey plane is regular.*

Corollary 6.3.5.2. *For any set I, the space $\mathbb{R}^I$ is regular.*

Proposition 6.3.6. *The space $\mathbb{R}_K$ is not regular.*

PROOF:There do not exist disjoint neighbourhoods of 0 and K. ∎

Proposition 6.3.7. *The continuous image of a regular space is not necessarily regular.*

PROOF: The identity map from the discrete two-point space to the indiscrete two-point space is continuous. □

6.4 Completely Regular Spaces

Definition 6.4.1 (Separated by a Continuous Function)**.** Let A and B be subsets of a topological space X. Then A and B can be *separated by a continuous function* iff there exists a continuous $f : X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

Definition 6.4.2 (Completely Regular)**.** A space X is *completely regular* iff X is T_1 and, for every point a and closed set A not containing a, we have that $\{a\}$ and A can be separated by a continuous function.

Theorem 6.4.3. *The product of a family of completely regular spaces is completely regular.*

PROOF:
⟨1⟩1. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of completely regular spaces.
⟨1⟩2. LET: $a \in \prod_{\alpha \in J} X_\alpha$ and A be closed in $\prod_{\alpha \in J} X_\alpha$ such that $a \notin A$
⟨1⟩3. PICK a basic open neighbourhood $\prod_{\alpha \in J} U_\alpha \subseteq \prod_{\alpha \in J} X_\alpha \setminus A$ of a such that $U_\alpha = X_\alpha$ except for $\alpha = \alpha_1, \ldots, \alpha_n$
⟨1⟩4. For $1 \leq i \leq n$, PICK a continuous $f_i : X_{\alpha_i} \to [0,1]$ that is 0 at a_{α_i} and 1 on $X_{\alpha_i} \setminus U_{\alpha_i}$
⟨1⟩5. LET: $f : \prod_{\alpha \in J} X_\alpha \to [0,1]$ be given by $f(x) = \prod_{i=1}^n f_i(x_{\alpha_i})$
⟨1⟩6. $f(a) = 0$
⟨1⟩7. $f(x) = 1$ for $x \in A$
⟨1⟩8. f is continuous
□

Corollary 6.4.3.1. *The Sorgenfrey plane is completely regular.*

Corollary 6.4.3.2. *For any set I, the space $\mathbb{R}^I$ is completely regular.*

Proposition 6.4.4. *For any set J, the space $\mathbb{R}^J$ in the box topology is completely regular.*

PROOF:
⟨1⟩1. LET: $a \in \mathbb{R}^J$ and $A \subseteq \mathbb{R}^J$ be closed with $a \notin A$
PROVE: There exists $f : \mathbb{R}^J_{\text{box}} \to [0,1]$ continuous such that $f(a) = 1$ and $f(A) = \{0\}$
⟨1⟩2. ASSUME: w.l.o.g. $A \cap (-1,1)^J = \emptyset$ and $a = \vec{0}$
⟨2⟩1. PICK a basic open set $\prod_{\alpha \in J} U_\alpha$ such that $a \in \prod_{\alpha \in J} U_\alpha \subseteq \mathbb{R}^J \setminus A$
⟨2⟩2. For $\alpha \in J$, PICK b_α, c_α such that $a_\alpha \in (b_\alpha, c_\alpha) \subseteq U_\alpha$
⟨2⟩3. For $\alpha \in J$, PICK a homeomorphism $f_\alpha : \mathbb{R} \to \mathbb{R}$ that maps b_α to -1, a_α to 0 and c_α to 1
⟨2⟩4. $\prod_{\alpha \in J} f_\alpha$ is an automorphism $\mathbb{R}^J_{\text{box}}$ that maps a to $\vec{0}$ and A to a closed set disjoint from $(-1,1)^J$

⟨1⟩3. PICK a continuous function $f : \mathbb{R}^J_{\text{uniform}} \rightarrow [0,1]$ such that $f(\vec{0}) = 1$ and $f(\mathbb{R}^J \setminus (-1,1)^J) = \{0\}$
⟨1⟩4. f is continuous w.r.t. the box topology
□

Proposition 6.4.5. *Not every regular space is completely regular.*

PROOF:
⟨1⟩1. For $m \in \mathbb{Z}$,
LET: $L_m = \{m\} \times [-1, 0]$
⟨1⟩2. For each odd integer n and each integer $k \geq 2$,
LET: $C_{nk} = (\{n{+}1{-}1/k\} home/robin/fun/RogOMatic/src/actuatortimes [-1, 0]) \cup (\{n - 1 + 1/k\} \times [-1, 0]) \cup \{(x, y) : (x - n)^2 + y^2 = (1 - 1/k)^2, y \geq 0\}$
⟨1⟩3. For each odd integer n and each integer $k \geq 2$,
LET: $p_{nk} = (n, 1 - 1/k)$
⟨1⟩4. PICK two points a, b not in any L_m or C_{nk}
⟨1⟩5. LET: $X = \bigcup_{m \in \mathbb{Z}} L_m \cup \bigcup_{n,k} C_{nk} \cup \{a, b\}$
⟨1⟩6. LET: $\mathcal{B}$ be the set consisting of all subsets of $\mathbb{R}^2$ of the following forms:

1. The intersection of X with a horizontal open line segment that contains none of the points p_{nk}
2. A set formed from one of the sets C_{nk} by deleting finitely many points.
3. For each even integer m, the set $\{a\} \cup \{(x, y) \in X : x < m\}$
4. For each even integer m, the set $\{b\} \cup \{(x, y) \in X : x > m\}$

⟨1⟩7. $\mathcal{B}$ is a basis for a topology on X
⟨2⟩1. For all $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$
⟨2⟩2. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$
⟨3⟩1. CASE: B_1, B_2 are both of type 1
PROOF: Their intersection is of type 1.
⟨3⟩2. CASE: B_1 is of type 1 and B_2 is of type 2
PROOF: Their intersection is of type 2, since a horizontal line segment intersects C_{nk} in at most two points.
⟨3⟩3. CASE: B_1 is of type 1 and B_2 is of type 3
PROOF: Their intersection is of type 1
⟨3⟩4. CASE: B_1 is of type 1 and B_2 is of type 4
PROOF: Their intersection is of type 1
⟨3⟩5. CASE: B_1 is of type 2 and B_2 is of type 2
PROOF: Their intersection is of type 2
⟨3⟩6. CASE: B_1 is of type 2 and B_2 is of type 3
PROOF: Their intersection is B_1
⟨3⟩7. CASE: B_1 is of type 2 and B_2 is of type 4
PROOF: Their intersection is B_1
⟨3⟩8. CASE: B_1 is of type 3 and B_2 is of type 3
PROOF: Their intersection is of type 3
⟨3⟩9. CASE: B_1 is of type 3 and B_2 is of type 4

⟨4⟩1. Let: $B_1 = \{a\} \cup \{(x,y) \in X : x < m\}$ and $B_2 = \{b\} \cup \{(x,y) \in X : x > n\}$

⟨4⟩2. Case: $x = (s, 1 - 1/k)$ for some s and integer $x \geq 2$

Proof: In this case, $x \in C_{nk}$ for some n and $C_{nk} \subseteq B_1 \cap B_2$.

⟨4⟩3. Case: $x = (s,t)$ and $t \neq 1 - 1/k$ for any integer $k \geq 2$

Proof: In this case, $x \in ((n,m) \times \{t\}) \cap X \subseteq B_1 \cap B_2$

⟨3⟩10. Case: B_1 is of type 4 and B_2 is of type 4

Proof: Their intersection is of type 4

⟨1⟩8. For any continuous function $f : X \to \mathbb{R}$, we have $f(a) = f(b)$

⟨2⟩1. Let: $f : X \to \mathbb{R}$ be continuous

⟨2⟩2. For any $c \in \mathbb{R}$, we have $f^{-1}(c)$ is G_δ

Proof: $f^{-1}(c) = \bigcap_{q \in \mathbb{Q}^+} f^{-1}(c - q, c + q)$

⟨2⟩3. Let: $S_{nk} = \{p \in C_{nk} : f(p) \neq f(p_{nk})\}$

⟨2⟩4. For all n, k, we have S_{nk} is countable.

⟨3⟩1. Let: $f^{-1}(p_{nk}) = \bigcap_{m=1}^{\infty} U_m$ where U_m is open in X

⟨3⟩2. For each m, Pick $B_m \in \mathcal{B}$ such that $p_{nk} \in B_m \subseteq U_m$

⟨3⟩3. $S_{nk} \subseteq \bigcup_{m=1}^{\infty} (C_{nk} \setminus B_m)$

⟨3⟩4. Each $C_{nk} \setminus B_m$ is countable

⟨4⟩1. Let: $m \in \mathbb{Z}$

⟨4⟩2. B_m cannot be of type 1

⟨4⟩3. If B_m is of type 2 then $C_{nk} \setminus B_m$ is finite.

⟨4⟩4. If B_m is of type 3 or 4 then $C_{nk} \setminus B_m$ is empty.

⟨2⟩5. Pick $d \in [-1, 0]$ such that $\mathbb{R} \times \{d\}$ intersects none of the sets S_{nk}

⟨2⟩6. For n odd, we have

$$f(n-1, d) = \lim_{k \to \infty} f(p_{nk}) \ .$$

⟨3⟩1. Let: $\epsilon > 0$

⟨3⟩2. Pick $B \in \mathcal{B}$ such that $(n-1, d) \in B \subseteq f^{-1}(f(n-1,d) - \epsilon, f(n-1,d) + \epsilon)$

⟨3⟩3. There exists $\delta > 0$ such that, for $x \in (n - 1 - \delta, n - 1 + \delta)$, we have $(x, d) \in B$

⟨3⟩4. Pick K such that $1/K < \delta$

⟨3⟩5. Let: $k \geq K$

⟨3⟩6. $f(n - 1 + 1/k, d) = f(p_{nk})$

⟨3⟩7. $|f(n-1,d) - f(n-1+1/k, d)| < \epsilon$

⟨3⟩8. $|f(n-1,d) - f(p_{nk})| < \epsilon$

⟨2⟩7. For n odd, we have

$$f(n+1, d) = \lim_{k \to \infty} f(p_{nk}) \ .$$

Proof: Similar.

⟨2⟩8. Q.E.D.

⟨3⟩1. Assume: $f(a) \neq f(b)$

⟨3⟩2. Assume: w.l.o.g. $f(a) < f(b)$

⟨3⟩3. Pick $B \in \mathcal{B}$ such that $a \in B \subseteq f^{-1}(-\infty, (f(a) + f(b))/2)$

⟨3⟩4. Let: m be even such that $B = \{a\} \cup \{(x,y) \in X : x < m\}$

⟨3⟩5. Pick $B \in \mathcal{B}$ such that $b \in B \subseteq f^{-1}((f(a) + f(b))/2, +\infty)$

⟨3⟩6. Let: m' be even such that $B = \{b\} \cup \{(x,y) \in X : x > m'\}$

⟨3⟩7. $f(m,d) = f(m',d)$
⟨3⟩8. Q.E.D.
⟨1⟩9. X is regular.
⟨1⟩10. X is not completely regular.
PROOF: a and b cannot be separated by a continuous function.
□

Theorem 6.4.6 (AC)**.** *A space is completely regular iff it is homeomorphic to a subspace of* $[0,1]^J$ *for some* J*.*

PROOF:
⟨1⟩1. Every completely regular space is homeomorphic to a subspace of $[0,1]^J$ for some J.
⟨2⟩1. LET: X be completely regular
⟨2⟩2. For every point a and open set U that contains a, PICK a continuous function f_{aU} that is positive on a and vanishes outside U
⟨2⟩3. The family $\{f_{aU}\}$ separates points from closed sets
⟨2⟩4. Q.E.D.
PROOF: By the Imbedding Theorem.
⟨1⟩2. Every subspace of $[0,1]^J$ is completely regular.
PROOF: By Theorem 6.4.3 and Proposition 6.3.4.
□

Proposition 6.4.7. *The continuous image of a completely regular space is not necessarily completely regular.*

PROOF: The identity map from the discrete two-point space to the indiscrete two-point space is continuous. □

6.5 Normal Spaces

Definition 6.5.1 (Normal Space)**.** A *normal* space is a T_1 space such that, for any disjoint closed sets A, B, there exist disjoint open sets U, V such that $A \subseteq U$ and $B \subseteq V$.

Theorem 6.5.2. *Every linearly ordered set is normal under the order topology.*

PROOF: See Steen and Steerbach *Counterexamples in Topology* Example 39. □

Proposition 6.5.3. *The product space* $S_\Omega \times \overline{S_\Omega}$ *is not normal.*

PROOF:
⟨1⟩1. LET: $\Delta = \{(x,x) : x \in \overline{S_\Omega}\} \subseteq \overline{S_\Omega} \times \overline{S_\Omega}\}$
⟨1⟩2. Δ is closed in $\overline{S_\Omega} \times \overline{S_\Omega}$
⟨1⟩3. LET: $A = \Delta \cap (S_\Omega \times \overline{S_\Omega})$
⟨1⟩4. A is closed in $S_\Omega \times \overline{S_\Omega}$
⟨1⟩5. LET: $B = S_\Omega \times \{\Omega\}$
⟨1⟩6. B is closed

⟨1⟩7. $A \cap B = \emptyset$
⟨1⟩8. Assume: for a contradiction U and V are disjoint open sets including A and B respectively
⟨1⟩9. For all $x \in S_\Omega$ there exists $\beta \in (x, \Omega)$ such that $(x, \beta) \notin U$
⟨2⟩1. Let: $x \in S_\Omega$
⟨2⟩2. $(x, \Omega) \in V$
Proof: $(x, \Omega) \in B \subseteq V$
⟨2⟩3. Pick $y < \Omega$ such that $\{x\} \times (y, \Omega] \subseteq V$
Proof: By Lemma 4.1.2.
⟨2⟩4. Pick β such that $x, y < \beta < \Omega$
Proof: Such a β exists because Ω is a limit ordinal.
⟨1⟩10. For $x \in S_\Omega$,
Let: $\beta(x)$ be the least element of (x, Ω) such that $(x, \beta(x)) \notin U$
⟨1⟩11. Let: $b = \sup_{n=1}^{\infty} \beta^n(0)$
⟨1⟩12. $\beta^n(0) \to b$ as $n \to \infty$
⟨1⟩13. $(\beta^n(0), \beta^{n+1}(0)) \to (b, b)$ as $n \to \infty$
⟨1⟩14. $(b, b) \in A$
⟨1⟩15. $(b, b) \in U$
⟨1⟩16. For all n we have $(\beta^n(0), \beta^{n+1}(0)) \notin U$
Proof: By ⟨1⟩10.
⟨1⟩17. Q.E.D.
Proof: Steps ⟨1⟩12, ⟨1⟩15 and ⟨1⟩16 form a contradiction.
□

Corollary 6.5.3.1. *Not every completely regular space is normal.*

Corollary 6.5.3.2. *An open subspace of a normal space is not necessarily normal.*

Corollary 6.5.3.3. *The product of two normal spaces is not necessarily normal.*

Proposition 6.5.4. *A closed subspace of a normal space is normal.*

Proof:
⟨1⟩1. Let: X be normal and $C \subseteq X$ be closed.
⟨1⟩2. Let: A and B be closed in C
⟨1⟩3. A and B are closed in X
Proof: By Corollary 4.3.4.2.
⟨1⟩4. Pick disjoint open neighbourhoods U and V of A and B in X
⟨1⟩5. $U \cap C$ and $V \cap C$ are disjoint open neighourhoods of A and B in C
□

Corollary 6.5.4.1. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of nonempty spaces. If $\prod_{\alpha \in J} X_\alpha$ is normal then each X_α is normal.*

Proposition 6.5.5. *If the Continuum Hypothesis then $\mathbb{R}^\omega$ under the box topology is normal.*

Proof: See Rudin. The box product of countably many compact metric spaces. *General Topology and Its Applications*, 2:293–298, 1972. □

Proposition 6.5.6 (Stone (DC)). *If J is uncountable then $\mathbb{R}^J$ is not normal.*

PROOF:
$\langle 1\rangle 1$. LET: $X = (\mathbb{Z}^+)^J$
PROVE: X is not normal.
$\langle 1\rangle 2$. For $x \in X$ and $B \subseteq^{\text{fin}} J$,
LET:

$$U(x, B) = \{y \in X : \forall \alpha \in B . y_\alpha = x_\alpha\} \ .$$

$\langle 1\rangle 3$. $\{U(x, B) : x \in X, B \subseteq^{\text{fin}} J\}$ is a basis for X
$\langle 2\rangle 1$. LET: $x \in X$ and $\prod_{\alpha \in J} U_\alpha$ be a basic open set including x, where $U_\alpha = \mathbb{Z}^+$ for all α except $\alpha_1, \ldots, \alpha_n$
$\langle 2\rangle 2$. $x \in U(x, \{\alpha_1, \ldots, \alpha_n\}) \subseteq \prod_{\alpha \in J} U_\alpha$
$\langle 1\rangle 4$. For $n \in \mathbb{Z}^+$,
LET: $P_n = \{x \in X : x \text{ is injective on } J \setminus x^{-1}(n)\}$
$\langle 1\rangle 5$. P_1 and P_2 are closed and disjoint.
$\langle 2\rangle 1$. P_1 is closed
$\langle 3\rangle 1$. LET: $x \in X \setminus P_1$
$\langle 3\rangle 2$. PICK $\alpha, \beta \in J$ such that $x_\alpha = x_\beta \neq 1$
$\langle 3\rangle 3$. LET: $U_\gamma = \{x_\alpha\}$ if $\gamma = \alpha$ or $\gamma = \beta$, $\mathbb{Z}^+$ for all other $\gamma \in J$
$\langle 3\rangle 4$. $x \in \prod_{\gamma \in J} U_\gamma \subseteq X \setminus P_1$
$\langle 2\rangle 2$. P_2 is closed
PROOF: Similar.
$\langle 2\rangle 3$. $P_1 \cap P_2 = \emptyset$
PROOF: If $x \in P_1 \cap P_2$ then x is injective on J, contradicting the fact that J is uncountable.
$\langle 1\rangle 6$. ASSUME: for a contradiction U and V are disjoint open sets including P_1 and P_2
$\langle 1\rangle 7$. Given a sequence (α_i) of distinct elements of J and a strictly increasing sequence (n_i) of positive integers,
LET:

$$B_i^{\alpha,n} = \{\alpha_1, \ldots, \alpha_{n_i}\}$$
$$x_i^{\alpha,n} \in X$$
$$(x_i^{\alpha,n})_\beta = \begin{cases} j & \text{if } \beta = \alpha_j, 1 \leq j \leq n_{i-1} \\ 1 & \text{for all other values of } \beta \end{cases}$$

for $i \geq 1$
$\langle 1\rangle 8$. PICK sequences (α_i), (n_i) such that, for all $i \geq 1$, we have $U(x_i^{\alpha,n}, B_i^{\alpha,n}) \subseteq U$
$\langle 2\rangle 1$. LET: $x_1 \in X$ be given by $(x_1)_\alpha = 1$ for all $\alpha \in J$
$\langle 2\rangle 2$. $x_1 \in U$
PROOF: $x_1 \in P_1 \subseteq U$
$\langle 2\rangle 3$. PICK $B_1 \subseteq^{\text{fin}} J$ such that $U(x_1, B_1) \subseteq U$
PROOF: By $\langle 1\rangle 3$.
$\langle 2\rangle 4$. LET: $n_1 = |B_1|$ and $B_1 = \{\alpha_1, \ldots, \alpha_{n_1}\}$
$\langle 2\rangle 5$. ASSUME: We have chosen $n_1, \ldots, n_k$ strictly increasing and $\alpha_1, \ldots, \alpha_{n_k}$ such that, for $1 \leq i \leq k$, we have $U(x_i^{\alpha,n}, B_i^{\alpha,n}) \subseteq U$

$\langle 2\rangle 6.$ $x^{\alpha,n}_{i+1} \in U$
PROOF: $x^{\alpha,n}_{i+1} \in P_1 \subseteq U$
$\langle 2\rangle 7.$ PICK $C \subseteq^{\text{fin}} J$ such that $U(x^{\alpha,n}_{i+1}, C) \subseteq U$
$\langle 2\rangle 8.$ LET: n_{i+1} and $\alpha_{n_i+1}, \ldots, \alpha_{n_{i+1}}$ be such that $B^{\alpha,n}_i \cup C = B^{\alpha,n}_{i+1}$
$\langle 2\rangle 9.$ $U(x^{\alpha,n}_{i+1}, B^{\alpha,n}_{i+1}) \subseteq U$
$\langle 1\rangle 9.$ LET: $A = \{\alpha_i : i \geq 1\}$
$\langle 1\rangle 10.$ LET: $y \in X$, $y_\beta = j$ if $\beta = \alpha_j$, $y_\beta = 2$ for $\beta \notin A$
$\langle 1\rangle 11.$ PICK B such that $U(y, B) \subseteq V$
$\langle 1\rangle 12.$ PICK i such that $A \cap B \subseteq B^{\alpha,n}_i$
$\langle 1\rangle 13.$ $U(x^{\alpha,n}_{i+1}, B^{\alpha,n}_{i+1}) \cap U(y, B) \neq \emptyset$
PROOF: $x^{\alpha,n}_{i+1} \in U(x^{\alpha,n}_{i+1}, B^{\alpha,n}_{i+1}) \cap U(y, B)$
$\langle 1\rangle 14.$ Q.E.D.
PROOF: This contradicts the fact that U and V are disjoint ($\langle 1\rangle 6$).
□

Theorem 6.5.7 (Urysohn Lemma)**.** *Let X be a normal space. Let A and B be disjoint closed subsets of X. Then there exists a continuous map $f : X \to [0, 1]$ such that $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$.*

PROOF:
$\langle 1\rangle 1.$ LET: P be the set of all rational numbers in $[0, 1]$
$\langle 1\rangle 2.$ For all $q \in P$, PICK an open set U_q in X such that $A \subseteq U_0$, $U_1 \subseteq X \setminus B$, and whenever $p < q$ then $\overline{U_p} \subseteq U_q$
$\langle 2\rangle 1.$ PICK an enumeration (q_n) of P such that $q_1 = 1$ and $q_2 = 0$
$\langle 2\rangle 2.$ LET: $U_1 = X \setminus B$
$\langle 2\rangle 3.$ PICK an open set U_0 such that $A \subseteq U_0$ and $\overline{U_0} \subseteq U_1$
$\langle 2\rangle 4.$ ASSUME: we have open sets U_1, U_0, …, U_{q_n} such that whenever $p < q$ then $\overline{U_p} \subseteq U_q$
$\langle 2\rangle 5.$ $q_2 < q_{n+1} < q_1$
$\langle 2\rangle 6.$ LET: q_k be greatest among q_1, …, q_n such that $q_k < q_{n+1}$, and q_l be least such that $q_{n+1} < q_l$
$\langle 2\rangle 7.$ PICK an open set $U_{q_{n+1}}$ such that $\overline{U_{q_k}} \subseteq U_{q_{n+1}}$ and $\overline{U_{q_{n+1}}} \subseteq U_{q_l}$
$\langle 2\rangle 8.$ For all $p, q \in \{q_1, \ldots, q_{n+1}\}$, if $p < q$ then $\overline{U_p} \subseteq U_q$
$\langle 1\rangle 3.$ Extend the family (U_q) to $\mathbb{Q}$ by defining: $U_q = \emptyset$ if $q < 0$ and $U_q = X$ if $q > 1$
$\langle 1\rangle 4.$ For all rationals p, q with $p < q$ we have $\overline{U_p} \subseteq U_q$
$\langle 1\rangle 5.$ Define $f : X \to [0, 1]$ by $f(x) = \inf\{q \in \mathbb{Q} : x \in U_q\}$
PROOF: This set is nonempty since $x \in U_1$ and bounded below since if $x \in U_q$ then $q \geq 0$.
$\langle 1\rangle 6.$ For all $x \in A$ we have $f(x) = 0$
$\langle 1\rangle 7.$ For all $x \in B$ we have $f(x) = 1$
$\langle 1\rangle 8.$ If $x \in \overline{U_r}$ then $f(x) \leq r$
$\langle 1\rangle 9.$ If $x \notin U_r$ then $f(x) \geq r$
$\langle 1\rangle 10.$ f is continuous
$\langle 2\rangle 1.$ LET: $x_0 \in X$
$\langle 2\rangle 2.$ LET: (c, d) be an open interval containing $f(x_0)$
PROVE: There exists a neighbourhood U of x_0 such that $f(U) \subseteq (c, d)$

⟨2⟩3. PICK rationals p, q such that $c < p < f(x_0) < q < d$
⟨2⟩4. $x \notin \overline{U_p}$
PROOF: By ⟨1⟩8
⟨2⟩5. $x \in U_q$
PROOF: By ⟨1⟩9
⟨2⟩6. LET: $U = U_q \setminus \overline{U_p}$
∎

Definition 6.5.8 (Vanish Precisely)**.** Let X be a set and $A \subseteq X$. Let $f : X \to [0,1]$. Then f *vanishes precisely* on A iff $f^{-1}(0) = A$.

Theorem 6.5.9 (CC)**.** *Let X be a normal space and $A \subseteq X$. Then there exists a continuous function $f : X \to [0,1]$ such that f vanishes precisely on A if and only if A is a closed G_δ set.*

PROOF:
⟨1⟩1. If there exists f such that f vanishes precisely on A then A is closed.
PROOF: This holds because $A = f^{-1}(0)$.
⟨1⟩2. If there exists f such that f vanishes precisely on A then A is G_δ.
PROOF: This holds because $A = \bigcap_{q \in \mathbb{Q}^+} f^{-1}([0,q))$.
⟨1⟩3. If A is closed and G_δ then there exists f that vanishes precisely on A.
⟨2⟩1. LET: $A = \bigcap_{n=1}^{\infty} U_n$
⟨2⟩2. For $n \geq 1$, PICK $f_n : X \to [0, 1/2^n]$ such that $f(x) = 0$ for $x \in A$ and $f(x) = 1/2^n$ for $x \in X \setminus U_n$
PROOF: By the Urysohn Lemma.
⟨2⟩3. LET: $f : X \to [0,1]$ be given by $f(x) = \sum_{n=1}^{\infty} f_n(x)$
PROOF: The series converges for every x by the Comparison Test.
⟨2⟩4. f is continuous
⟨3⟩1. f_n converges uniformly to f
PROOF: By the Weierstrass M-test.
⟨3⟩2. Q.E.D.
PROOF: By the Uniform Limit Theorem.
⟨2⟩5. $f(x) = 0$ for $x \in A$
PROOF: From ⟨2⟩2.
⟨2⟩6. $f(x) > 0$ for $x \notin A$
⟨3⟩1. LET: $x \notin A$
⟨3⟩2. PICK N such that $x \notin U_N$
⟨3⟩3. Q.E.D.
PROOF:

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} f_n(x) && (\langle 2\rangle 3) \\ &\geq f_N(x) \\ &> 0 && (\langle 2\rangle 2) \end{aligned}$$

∎

Theorem 6.5.10 (Strong Form of Urysohn Lemma)**.** *Let X be a normal space.*

Then there exists a continuous function $f : X \to [0,1]$ such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$ if and only if A and B are disjoint, closed and G_δ.

PROOF:
⟨1⟩1. If there exists a continuous function $f : X \to [0,1]$ such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$ then A and B are disjoint, closed and G_δ
  ⟨2⟩1. ASSUME: there exists a continuous function $f : X \to [0,1]$ such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$
  ⟨2⟩2. A and B are disjoint
  ⟨2⟩3. A is closed and G_δ
    PROOF: By Theorem 6.5.9.
  ⟨2⟩4. B is closed and G_δ
    PROOF: Apply Theorem 6.5.9 to $1 - f$.
⟨1⟩2. If A and B are disjoint, closed and G_δ then there exists a continuous function $f : X \to [0,1]$ such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$
  ⟨2⟩1. ASSUME: A and B are disjoint, closed and G_δ
  ⟨2⟩2. PICK $g : X \to [0,1]$ that vanishes precisely on A and $h : X \to [0,1]$ that vanishes precisely on B
  ⟨2⟩3. LET: $f = g/(g+h)$
□

Definition 6.5.11 (Universal Extension Property)**.** A topological space Y has the *universal extension property* iff, for every normal space X and closed subspace A of X, every continuous function $A \to Y$ can be extended to a continuous function $X \to Y$.

Theorem 6.5.12 (Tietze Extension Theorem (DC))**.** *Let X be a normal space. Let A be closed subspace of X.*

1. *Any continuous function $A \to [a,b]$ can be extended to a continuous function $X \to [a,b]$.*

2. *Any continuous function $A \to \mathbb{R}$ can be extend to a continuous function $X \to \mathbb{R}$.*

PROOF:
⟨1⟩1. Any continuous function $A \to [-1,1]$ can be extended to a continuous function $X \to [-1,1]$
  ⟨2⟩1. For every continuous function $f : A \to [-r,r]$, there exists a continuous $g : X \to \mathbb{R}$ such that

$$|g(x)| \leq \frac{1}{3}r \qquad (x \in X)$$

$$|g(x) - f(x)| \leq \frac{2}{3}r \qquad (x \in A)$$

    ⟨3⟩1. LET: $f : A \to [-r,r]$ be continuous
    ⟨3⟩2. LET: $I_1 = [-r, -\frac{1}{3}r]$
    ⟨3⟩3. LET: $I_2 = [-\frac{1}{3}r, \frac{1}{3}r]$
    ⟨3⟩4. LET: $I_3 = [\frac{1}{3}r, r]$

⟨3⟩5. LET: $B = f^{-1}(I_1)$
⟨3⟩6. LET: $C = f^{-1}(I_3)$
⟨3⟩7. PICK a continuous $g : X \to [-\frac{1}{3}r, \frac{1}{3}r]$ such that $g(x) = -\frac{1}{3}r$ for $x \in B$ and $g(x) = \frac{1}{3}r$ for $x \in C$
PROOF: By the Urysohn Lemma, since B and C are closed disjoint subsets of X.
⟨3⟩8. For all $x \in A$ we have $|g(x) - f(x)| \leq \frac{2}{3}r$
⟨4⟩1. LET: $x \in A$
⟨4⟩2. CASE: $f(x) \in I_1$
PROOF:

$$\begin{aligned} |g(x) - f(x)| &= \left|-\frac{1}{3}r - f(x)\right| && (x \in B) \\ &\leq \frac{2}{3}r && (f(x) \in I_1) \end{aligned}$$

⟨4⟩3. CASE: $f(x) \in I_2$
PROOF: In this case, $|g(x) - f(x)| \leq \frac{2}{3}r$ since $f(x), g(x) \in I_2$.
⟨4⟩4. CASE: $f(x) \in I_3$
PROOF:

$$\begin{aligned} |g(x) - f(x)| &= \left|\frac{1}{3}r - f(x)\right| && (x \in C) \\ &\leq \frac{2}{3}r && (f(x) \in I_3) \end{aligned}$$

⟨2⟩2. LET: $f : A \to [-1, 1]$ be continuous.
⟨2⟩3. PICK a sequence of functions (g_n) such that

$$|g_n(x)| \leq \frac{1}{3}\left(\frac{2}{3}\right)^{n-1} \qquad (x \in X)$$

$$|f(x) - g_1(x) - \cdots - g_n(x)| \leq (2/3)^n \qquad (x \in A)$$

PROOF:Given $g_1, \ldots, g_n$, we apply ⟨2⟩1 with $f = f - g_1 - \cdots - g_n$ and $r = (2/3)^n$.
⟨2⟩4. LET: $g(x) = \sum_{n=1}^{\infty} g_n(x)$ for $x \in X$
PROOF: This series converges by the Comparison Test since $\sum_{n=1}^{\infty}(2/3)^n$ converges.
⟨2⟩5. g is continuous.
⟨3⟩1. $\sum_{n=1}^{N} g_n$ converges to g uniformly
PROOF: By the Weierstrass M-test.
⟨3⟩2. Q.E.D.
PROOF: By the Uniform Limit Theory.
⟨2⟩6. For all $x \in A$ we have $g(x) = f(x)$
PROOF: $|\sum_{n=1}^{N} g_n(x) - f(x)| \leq (2/3)^N \to 0$ as $N \to \infty$.
⟨2⟩7. For all $x \in X$ we have $-1 \leq g(x) \leq 1$

PROOF:

$$\begin{aligned}|\sum_{n=1}^{N} g_n(x)| &\leq \sum_{n=1}^{N} |g_n(x)| \\ &\leq 1/3 \sum_{n=1}^{N} (2/3)^{n-1} \\ &\to 2/3 && \text{as } n \to \infty\end{aligned}$$

⟨1⟩2. Any continuous function $A \to (-1,1)$ can be extend to a continuous function $X \to (-1,1)$

⟨2⟩1. LET: $f : A \to (-1,1)$ be continuous

⟨2⟩2. PICK a continuous $g : X \to [-1,1]$ that extends f

PROOF: By ⟨1⟩1.

⟨2⟩3. LET: $D = g^{-1}(-1) \cup g^{-1}(1)$

⟨2⟩4. D is closed in X

PROOF: Since g is continuous and $\{-1\}$, $\{1\}$ are closed in $[-1,1]$.

⟨2⟩5. $D \cap A = \emptyset$

PROOF: Since $g(A) = f(A) \subseteq (-1,1)$.

⟨2⟩6. PICK a continuous $\phi : X \to [0,1]$ such that $\phi(D) = \{0\}$ and $\phi(A) = \{1\}$

PROOF: By the Urysohn Lemma.

⟨2⟩7. LET: $h = g\phi$

⟨2⟩8. h is continuous

⟨2⟩9. h extends f

⟨2⟩10. $\operatorname{im} h \subseteq (-1,1)$

⟨1⟩3. Q.E.D.

PROOF: The result follows because any closed interval in $\mathbb{R}$ is homeomorphic to $[-1,1]$ and $\mathbb{R} \cong (-1,1)$.

□

Lemma 6.5.13 (Shrinking Lemma (AC)). *Let X be a normal space. Let $\{U_\alpha\}_{\alpha \in J}$ be a point-finite indexed open covering of X. Then there exists an indexed open covering $\{V_\alpha\}_{\alpha \in J}$ such that $\overline{V_\alpha} \subseteq U_\alpha$ for all $\alpha \in J$.*

PROOF:

⟨1⟩1. PICK a well-ordering $\prec$ on J

⟨1⟩2. PICK open sets V_α for $\alpha \in J$ such that $A_\alpha \subseteq V_\alpha$ and $\overline{V_\alpha} \subseteq U_\alpha$, where

$$A_\alpha = X \setminus \bigcup_{\beta \prec \alpha} V_\beta \cup \bigcup_{\alpha \prec \beta} U_\beta$$

PROOF: Apply transfinite induction to Proposition 13.1.16.

⟨1⟩3. $\{V_\alpha\}_{\alpha \in J}$ covers X

⟨2⟩1. LET: $x \in X$

⟨2⟩2. LET: $\alpha_1, \ldots, \alpha_n$ be the elements of J such that $x \in U_{\alpha_i}$, where $\alpha_1 \prec \cdots \prec \alpha_n$

PROVE: $x \in V_{\alpha_i}$ for some i

⟨2⟩3. ASSUME: $x \notin V_{\alpha_1}, \ldots, V_{\alpha_{n-1}}$

⟨2⟩4. $x \in A_{\alpha_n}$

⟨2⟩5. $x \in V_{\alpha_n}$

□

Proposition 6.5.14 (DC)**.** $S_\Omega \times \overline{S_\Omega}$ *is not normal.*

PROOF:
⟨1⟩1. LET: $\Delta = \{(x,x) : x \in \overline{S_\Omega}\}$
⟨1⟩2. Δ is closed in $\overline{S_\Omega}^2$
⟨2⟩1. LET: $(x,y) \in \overline{S_\Omega}^2 \setminus \Delta$
⟨2⟩2. PICK disjoint open sets U, V such that $x \in U$ and $y \in V$
⟨2⟩3. $(x,y) \in U \times V \subseteq \overline{S_\Omega}^2 \setminus \Delta$
⟨1⟩3. LET: $A = \Delta \cap (S_\Omega \times \overline{S_\Omega})$
⟨1⟩4. A is closed in $S_\Omega \times \overline{S_\Omega}$
⟨1⟩5. LET: $B = S_\Omega \times \{\Omega\}$
⟨1⟩6. B is closed in $S_\Omega \times \overline{S_\Omega}$
⟨1⟩7. $A \cap B = \emptyset$
⟨1⟩8. ASSUME: for a contradiction U and V are disjoint open sets including A and B respectively
⟨1⟩9. PICK a sequence x_n in S_Ω such that $x_n < x_{n+1} < \Omega$ and $(x_n, x_{n+1}) \notin U$ for all n
⟨2⟩1. LET: $x_n \in S_\Omega$
⟨2⟩2. $(x_n, \Omega) \in V$
⟨2⟩3. PICK open sets $W \subseteq S_\Omega$, $X \subseteq \overline{S_\Omega}$ such that $x_n \in W$, $\Omega \in X$ and $W \times X \subseteq V$
⟨2⟩4. PICK $y < \Omega$ such that $(x_{n+1}, \Omega] \subseteq X$
⟨2⟩5. LET: $x_{n+1} = y + 1$
⟨1⟩10. LET: b be the supremum of $\{x_n : n \geq 1\}$
⟨1⟩11. $(x_n, x_{n+1}) \to (b,b)$ as $n \to \infty$
⟨1⟩12. $(b,b) \in A$
⟨1⟩13. $(b,b) \in U$
⟨1⟩14. For all n we have $(x_n, x_{n+1}) \notin U$
□

Proposition 6.5.15 (AC)**.** $\mathbb{R}_l$ *is normal.*

PROOF:
⟨1⟩1. LET: A and B be disjoint closed sets in $\mathbb{R}_l$
⟨1⟩2. For $a \in A$, PICK $x_a > a$ such that $[a, x_a)$ not intersecting B
⟨1⟩3. For $b \in B$, PICK $x_b > b$ such that $[b, x_b)$ does not intersect A
⟨1⟩4. LET: $U = \bigcup_{a \in A}[a, x_a)$ and $V = \bigcup_{b \in B}[b, x_b)$
⟨1⟩5. U and V are disjoint open sets including A and B respectively.
□

Lemma 6.5.16. *The set* $L = \{(x, -x); x \in \mathbb{R}\}$ *as a subspace of* $\mathbb{R}_l^2$ *is closed*

⟨1⟩1. LET: $(x,y) \notin L$, so $y \neq -x$
PROVE: There exists a neighbourhood U of (x,y) that does not intersect L
⟨1⟩2. CASE: $y > -x$

PROOF: In this case, take $U = [x, +\infty) \times [y, +\infty)$
$\langle 1 \rangle 3$. CASE: $y < -x$
PROOF: In this case, take $U = [x, (x - y)/2) \times [y, (y - x)/2)$.

Proposition 6.5.17 (AC). *The Sorgenfrey plane is not normal.*

PROOF:
$\langle 1 \rangle 1$. ASSUME: for a contradiction the Sorgenfrey plane is normal.
$\langle 1 \rangle 2$. LET: $L = \{(x, -x); x \in \mathbb{R}\}$ as a subspace of $\mathbb{R}_l^2$
$\langle 1 \rangle 3$. L has the discrete topology.
$\langle 2 \rangle 1$. LET: $(x, -x) \in L$
PROVE: $\{(x, -x)\}$ is open in L
$\langle 2 \rangle 2$. $\{(x, -x)\} = ([x, +\infty) \times [-x, +\infty)) \cap L$
$\langle 1 \rangle 4$. Every subset of L is closed in $\mathbb{R}_l^2$
PROOF: By Corollary 4.3.4.2.
$\langle 1 \rangle 5$. For every nonempty proper subset A of L, PICK disjoint open sets U_A, V_A containing A and $L \setminus A$
PROOF: By $\langle 1 \rangle 1$ and $\langle 1 \rangle 4$.
$\langle 1 \rangle 6$. LET: $D = \mathbb{Q}^2$
$\langle 1 \rangle 7$. D is dense in $\mathbb{R}_l^2$
PROOF: Given any basic open set $[a, b) \times [c, d)$, pick rationals q, r such that $a \leq q < b$ and $c \leq r < d$. Then $(q, r) \in ([a, b) \times [c, d)) \cap D$
$\langle 1 \rangle 8$. LET: $\theta : \mathcal{P}L \to \mathcal{P}D$ be the function

$$\theta(A) = U_A \cap D \qquad (\emptyset \neq A \neq L)$$
$$\theta(\emptyset) = \emptyset$$
$$\theta(L) = D$$

$\langle 1 \rangle 9$. θ is injective
$\langle 2 \rangle 1$. LET: $A, B \subseteq L$ with $\theta(A) = \theta(B)$
PROVE: $A = B$
$\langle 2 \rangle 2$. CASE: $\emptyset \neq A \neq L$ and $\emptyset \neq B \neq L$
$\langle 3 \rangle 1$. $A \subseteq B$
$\langle 4 \rangle 1$. LET: $x \in A$
$\langle 4 \rangle 2$. $x \in U_A$
PROOF: By $\langle 1 \rangle 5$
$\langle 4 \rangle 3$. $x \in U_B$
PROOF: By $\langle 2 \rangle 1$
$\langle 4 \rangle 4$. $x \notin L \setminus B$
PROOF: By $\langle 1 \rangle 5$
$\langle 4 \rangle 5$. $x \in B$
PROOF: Since $x \in L$ by $\langle 4 \rangle 1$
$\langle 3 \rangle 2$. $B \subseteq A$
PROOF: Similar.
$\langle 2 \rangle 3$. CASE: $\emptyset \neq A \neq L$ and $B = \emptyset$
PROOF: This implies $U_A \cap D = \emptyset$ which contradicts the fact that D is dense.
$\langle 2 \rangle 4$. CASE: $\emptyset \neq A \neq L$ and $B = L$
PROOF: This implies $V_A \cap D = \emptyset$ which contradicts the fact that D is dense.

⟨2⟩5. CASE: $A = B = \emptyset$
PROOF: Trivial
⟨2⟩6. CASE: $A = \emptyset$ and $B = L$
PROOF: This implies $D = \emptyset$ which is a contradiction.
⟨2⟩7. CASE: $A = B = L$
PROOF: Trivial
⟨1⟩10. Q.E.D.
PROOF: This is a contradiction since D is countable and L is uncountable.
□

Proposition 6.5.18. *The continuous image of a normal space is not necessarily normal.*

PROOF: The identity map from the discrete two-point space to the indiscrete two-point space is continuous. □

6.6 Completely Normal Spaces

Definition 6.6.1 (Completely Normal)**.** A space X is *completely normal* iff every subspace is normal.

Proposition 6.6.2. *A subspace of a completely normal space is completely normal.*

PROOF: Immediate from definitions. □

Proposition 6.6.3. *Let X be a topological space. Then X is completely normal iff X is T_1 and, for any pair of separated sets A, B in X, there exist disjoint open sets including them.*

PROOF:
⟨1⟩1. If X is completely normal then X is T_1 and, for any pair of separated sets A, B in X, there exist disjoint open sets including them.
⟨2⟩1. ASSUME: X is completely normal.
⟨2⟩2. X is T_1
PROOF: Holds because X is normal.
⟨2⟩3. For any pair of separated sets A, B in X, there exist disjoint open sets including them.
⟨3⟩1. LET: A and B be separated in X
⟨3⟩2. LET: $Y = X \setminus (\overline{A} \cap \overline{B})$
⟨3⟩3. PICK disjoint open sets U, V in Y such that $\overline{A} \cap Y \subseteq U$ and $\overline{B} \cap Y \subseteq V$
PROOF: Y is normal by ⟨2⟩1.
⟨3⟩4. PICK open sets U_0, V_0 in X such that $U = U_0 \cap Y$, $V = V_0 \cap Y$
⟨3⟩5. $A \subseteq U_0 \setminus \overline{B}$ and $B \subseteq V_0 \setminus \overline{A}$
PROOF: Using ⟨3⟩1.
⟨1⟩2. If X is T_1 and, for any pair of separated sets A, B in X, there exist disjoint open sets including them, then X is completely normal.

⟨2⟩1. ASSUME: X is T_1 and, for any pair of separated sets A, B in X, there exist disjoint open sets including them
⟨2⟩2. LET: $Y \subseteq X$
⟨2⟩3. Y is T_1
PROOF: By Proposition 6.1.3.
⟨2⟩4. LET: A and B be disjoint closed sets in Y
⟨2⟩5. A and B are separated in X
⟨3⟩1. $\overline{A} \cap Y = A$ and $\overline{B} \cap Y = B$
PROOF: By Proposition 3.8.6 and Theorem 4.3.4.
⟨3⟩2. $\overline{A} \cap B = \emptyset$

$$\begin{aligned} \overline{A} \cap B &= \overline{A} \cap \overline{B} \cap Y && (\langle 3\rangle 1) \\ &= A \cap B && (\langle 3\rangle 1) \\ &= \emptyset && (\langle 2\rangle 4) \end{aligned}$$

⟨3⟩3. $A \cap \overline{B} = \emptyset$
PROOF: Similar.
⟨2⟩6. PICK disjoint open sets U and V that include A and B respectively.
PROOF: By ⟨2⟩1.
⟨2⟩7. $U \cap Y$ and $V \cap Y$ are disjoint open sets in Y that include A and B respectively.
□

Proposition 6.6.4. *A well-ordered set in the order topology is completely normal.*

PROOF:
⟨1⟩1. LET: X be a well-ordered set.
⟨1⟩2. For all $a, b \in X$ with $a < b$, we have $(a, b]$ is open.
⟨2⟩1. CASE: b is greatest in X
PROOF: This case holds by the definition of the order topology.
⟨2⟩2. CASE: b is not greatest in X
PROOF: In this case, $(a, b] = (a, c)$ where c is the successor of b.
⟨1⟩3. LET: A and B be separated sets in X
PROVE: There exist disjoint open sets U, V including A and B
⟨1⟩4. CASE: The least element of X is not in A or B
⟨2⟩1. LET: $U = \bigcup\{(x, a] : a \in A, x < a, (x, a] \cap B = \emptyset\}$
⟨2⟩2. LET: $V = \bigcup\{(y, b] : b \in B, y < b, (y, b] \cap A = \emptyset\}$
⟨2⟩3. U is open
PROOF: From ⟨1⟩2.
⟨2⟩4. V is open
PROOF: From ⟨1⟩2.
⟨2⟩5. $A \subseteq U$
⟨3⟩1. LET: $a \in A$
⟨3⟩2. PICK W a neighbourhood of a such that $W \cap B = \emptyset$
PROOF: By ⟨1⟩3.
⟨3⟩3. PICK $x < a$ such that $(x, a] \subseteq W$

PROOF: By Lemma 4.1.2
⟨3⟩4. $a \in (x, a] \subseteq U$
⟨2⟩6. $B \subseteq V$
PROOF: Similar.
⟨2⟩7. $U \cap V = \emptyset$
⟨1⟩5. CASE: $\perp \in A$
⟨2⟩1. PICK disjoint open sets U and V that include $A \setminus \{\perp\}$ and B
PROOF: From ⟨1⟩4.
⟨2⟩2. $U \cup \{\perp\}$ and V are disjoint open sets that include A and B
PROOF: $\{\perp\}$ is open because it is $(-\infty, a)$ where a is the successor of $\perp$.
⟨1⟩6. Q.E.D.
PROOF: By Proposition 6.6.3.
□

Proposition 6.6.5. *The product of two completely normal spaces is not necessarily completely normal.*

PROOF:
⟨1⟩1. S_Ω is completely normal.
PROOF: By Proposition 6.6.4
⟨1⟩2. $\overline{S_\Omega}$ is completely normal.
PROOF: By Proposition 6.6.4
⟨1⟩3. $S_\Omega \times \overline{S_\Omega}$ is not completely normal.
PROOF: By Proposition 6.5.3.
□

Proposition 6.6.6. *A compact Hausdorff space is not necessarily completely normal.*

PROOF:
⟨1⟩1. PICK an uncountable set J
⟨1⟩2. $[0, 1]^J$ is compact Hausdorff
PROOF: By Tychonoff's Theorem and Theorem 6.2.5.
⟨1⟩3. $(0, 1)^J$ is not normal.
PROOF: By Proposition 6.5.6, since $(0, 1) \cong \mathbb{R}$.
□

Proposition 6.6.7. *The space $\mathbb{R}_l$ is completely normal.*

PROOF:
⟨1⟩1. LET: $X \subseteq \mathbb{R}_l$
⟨1⟩2. LET: A and B be disjoint closed sets in X.
⟨1⟩3. PICK closed sets C and D such that $A = C \cap X$ and $B = D \cap X$
⟨1⟩4. For $a \in A$, PICK $x_a > a$ such that $[a, x_a) \cap D = \emptyset$
⟨1⟩5. For $b \in B$, PICK $x_b > b$ such that $[b, x_b) \cap C = \emptyset$
⟨1⟩6. $\bigcup_{a \in A} [a, x_a) \cap X$ and $\bigcup_{b \in B} [b, x_b) \cap X$ are disjoint open sets in X that include A and B
□

6.7 Perfectly Normal Spaces

Definition 6.7.1 (Perfectly Normal)**.** A space is *perfectly normal* iff it is normal and every closed set is G_δ.

Proposition 6.7.2. *Every perfectly normal space is completely normal.*

PROOF:
$\langle 1 \rangle 1$. LET: X be perfectly normal.
$\langle 1 \rangle 2$. LET: A and B be separated sets in X
$\langle 1 \rangle 3$. PICK continuous functions $f, g : X \to [0,1]$ that vanish precisely on $\overline{A}$ and $\overline{B}$, respectively.
PROOF: By Theorem 6.5.9.
$\langle 1 \rangle 4$. LET: $h = f - g$
$\langle 1 \rangle 5$. $B \subseteq h^{-1}((0, +\infty))$ and $A \subseteq h^{-1}((-\infty, 0))$
$\langle 1 \rangle 6$. Q.E.D.
PROOF: By Proposition 6.6.3.
□

Proposition 6.7.3. *The space $\overline{S_\Omega}$ is not perfectly normal.*

PROOF: The set $\{\Omega\}$ is not G_δ. □

Chapter 7

Countability Axioms

7.1 The First Countability Axiom

Definition 7.1.1 (First Countability Axiom)**.** A topological space X satisfies the *first countability axiom*, or is *first countable*, iff every point has a countable local basis.

Proposition 7.1.2. *S_Ω is first countable.*

PROOF: For every countable ordinal $\alpha > 0$, the set $\{(\beta, \alpha + 1) : \beta < \alpha\}$ is a local basis at α. The set $\{\{0\}\}$ is a local basis at 0. ∎

Theorem 7.1.3 (The Sequence Lemma (CC))**.** *Let X be a first countable space and $A \subseteq X$. If $x \in \overline{A}$, then there exists a sequence of points of A that converges to x.*

PROOF:
⟨1⟩1. LET: $x \in \overline{A}$
⟨1⟩2. PICK a countable basis $\{B_n\}_{n \in \mathbb{Z}^+}$ at x.
⟨1⟩3. For $n \geq 1$, PICK a point $a_n \in B_1 \cap \cdots \cap B_n \cap A$
PROVE: $a_n \to x$ as $n \to \infty$
PROOF: Using Countable Choice. Such an a_n exists because $B_1 \cap \cdots \cap B_n$ is a neighbourhood of x. Apply Theorem 3.9.3.
⟨1⟩4. LET: U be a neighbourhood of x
⟨1⟩5. PICK N such that $B_N \subseteq U$
PROOF: From ⟨1⟩2.
⟨1⟩6. For $n \geq N$, we have $a_n \in U$
PROOF:

$$\begin{aligned} a_n &\in B_1 \cap \cdots \cap B_n && (\langle 1\rangle 3) \\ &\subseteq B_N && (n \geq N) \\ &\subseteq U && (\langle 1\rangle 5) \end{aligned}$$

∎

Theorem 7.1.4 (CC)**.** *Let X and Y be topological spaces where X is first countable. Let $x \in X$. Suppose that, for every sequence $\{x_n\}_{n\geq 1}$ such that $x_n \to x$ as $n \to \infty$, we have $f(x_n) \to f(x)$ as $n \to \infty$. Then f is continuous at x.*

PROOF:
⟨1⟩1. LET: V be a neighbourhood of $f(x)$
⟨1⟩2. ASSUME: for a contradiction that, for every neighbourhood U of x, $f(U) \nsubseteq V$
⟨1⟩3. PICK a countable local basis $\{B_n\}_{n\geq 1}$
⟨1⟩4. For $n \geq 1$, PICK $a_n \in B_1 \cap \cdots \cap B_n$ such that $f(a_n) \notin V$
⟨1⟩5. $a_n \to x$ as $n \to \infty$
PROOF:
⟨2⟩1. LET: U be a neighbourhood of x
⟨2⟩2. PICK N such that $B_N \subseteq U$
⟨2⟩3. For all $n \geq N$, $a_n \in U$
PROOF:

$$\begin{aligned} a_n &\in B_1 \cap \cdots \cap B_n && (\langle 1\rangle 4) \\ &\subseteq B_N && (n \geq N) \\ &\subseteq U && (\langle 2\rangle 2) \end{aligned}$$

⟨1⟩6. $f(a_n) \to f(x)$ as $n \to \infty$
⟨1⟩7. There exists N such that, for all $n \geq N$, we have $f(a_n) \in V$
⟨1⟩8. Q.E.D.

Lemma 7.1.5 (CC)**.** *$\mathbb{R}^\omega$ under the box topology is not first countable.*

PROOF:
⟨1⟩1. LET: $\{B_n\}_{n\geq 1}$ be any countable set of neighbourhoods of $\vec{0}$
⟨1⟩2. For $n \geq 1$, PICK U_{nm} for $m \geq 1$ such that $\vec{0} \in \prod_{m=1}^\infty U_{nm} \subseteq B_n$
⟨1⟩3. For $n \geq 1$, PICK a_n, b_n such that $0 \in (a_n, b_n) \subseteq U_{nn}$
⟨1⟩4. LET: $U = \prod_{n=1}^\infty (a_n/2, b_n/2)$
⟨1⟩5. $\vec{0} \in U$
⟨1⟩6. For all n, $B_n \nsubseteq U$
□

Lemma 7.1.6 (CC)**.** *If J is uncountable then $\mathbb{R}^J$ is not first countable.*

PROOF:
⟨1⟩1. LET: $\{B_n\}_{n\geq 1}$ be a countable family of neighbourhoods of $\vec{0}$
⟨1⟩2. For $n \geq 1$, PICK $U_{n\alpha}$ such that $\vec{0} \in \prod_{\alpha\in J} U_{n\alpha} \subseteq B_n$ where $U_{n\alpha}$ is open in $\mathbb{R}$ and $U_{n\alpha} = \mathbb{R}$ except for $\alpha = \alpha_{n1}, \ldots, \alpha_{nr_n}$
⟨1⟩3. PICK β such that β is different from α_{ni} for all n, i
⟨1⟩4. LET: $V = \pi_\beta^{-1}((-1, 1))$
⟨1⟩5. $\vec{0} \in V$
⟨1⟩6. $V \nsubseteq B_n$ for all n
□

Lemma 7.1.7. $\mathbb{R}_l$ *is first countable.*

PROOF: For all $x \in \mathbb{R}$, $\{[x, q) : q \in \mathbb{Q}, q > x\}$ is a basis at x. $\square$

Lemma 7.1.8. *The ordered square is first countable.*

PROOF:
⟨1⟩1. LET: $(x, y) \in I_o^2$
PROVE: There exists a countable local basis $\mathcal{B}$ at (x, y)
⟨1⟩2. CASE: $(x, y) = (0, 0)$
PROOF: Take $\mathcal{B} = \{[(0, 0), (0, q)) : q \in \mathbb{Q}, 0 < q < 1\}$.
⟨1⟩3. CASE: $0 < y < 1$
PROOF: Take $\mathcal{B} = \{((x, q), (x, q')) : q, q' \in \mathbb{Q}, q < y < q'\}$.
⟨1⟩4. CASE: $x < 1, y = 1$
PROOF: Take $\mathcal{B} = \{((x, q), (q', 0)) : q, q' \in \mathbb{Q}, 0 < q < 1, x < q' < 1\}$.
⟨1⟩5. CASE: $x > 0, y = 0$
PROOF: Take $\mathcal{B} = \{((q, 1), (x, q')) : q, q' \in \mathbb{Q}, 0 < q < x, 0 < q' < 1\}$
⟨1⟩6. CASE: $(x, y) = (1, 1)$
PROOF: Take $\mathcal{B} = \{((1, q), (1, 1)] : q \in \mathbb{Q}, 0 < q < 1\}$.
$\square$

Proposition 7.1.9. *A subspace of a first countable space is first countable.*

PROOF:
⟨1⟩1. LET: X be a first countable space and $A \subseteq X$
⟨1⟩2. LET: $a \in A$
⟨1⟩3. PICK a countable basis $\mathcal{B}$ at a in X
⟨1⟩4. $\{B \cap A : B \in \mathcal{B}$ is a countable basis at a in A.
$\square$

Proposition 7.1.10 (CC)**.** *A countable product of first countable spaces is first countable.*

PROOF:
⟨1⟩1. LET: $\{X_n\}_{n \in \mathbb{Z}^+}$ be a countable family of first countable spaces.
⟨1⟩2. LET: $\vec{x} \in \prod_{n=1}^{\infty} X_n$
⟨1⟩3. PICK a countable basis $\mathcal{B}_n$ at x_n in X_n for all n
⟨1⟩4. LET: $\mathcal{B}$ be the set of all sets $\prod_{i=1}^{n} U_n$ where $U_n \in \mathcal{B}_n$ for finitely many n and $U_n = X_n$ for all other n.
⟨1⟩5. $\mathcal{B}$ is a countable basis at $\vec{x}$ in $\prod_{n=1}^{\infty} X_n$
$\square$

Corollary 7.1.10.1. *The space* $\mathbb{R}^\omega$ *is first countable.*

Proposition 7.1.11. *The space* S_Ω *is first countable.*

PROOF:
⟨1⟩1. LET: $\alpha \in S_\Omega$
PROVE: α has a countable local basis.

⟨1⟩2. CASE: α is zero or a successor ordinal.
PROOF: In this case, $\{\{\alpha\}\}$ is a local basis.
⟨1⟩3. CASE: α is a limit ordinal.
⟨2⟩1. PICK a countable sequence (β_n) with supremum α
⟨2⟩2. $\{(\beta_n, \alpha + 1) : n \in \mathbb{Z}^+\}$ is a local basis.
□

Proposition 7.1.12. *The space $\overline{S_\Omega}$ is not first countable.*

PROOF:
⟨1⟩1. ASSUME: for a contradiction $\mathcal{B}$ is a countable local basis at Ω
⟨1⟩2. LET: $\alpha = \sup\{\inf B : B \in \mathcal{B}\}$
⟨1⟩3. $\alpha < \Omega$
⟨1⟩4. There is no $B \in \mathcal{B}$ such that $B \subseteq (\alpha, +\infty)$
□

Proposition 7.1.13. *The continuous image of a first countable space is first countable.*

PROOF:
⟨1⟩1. LET: X be a first countable space, Y a space and $f : X \to Y$ continuous.
⟨1⟩2. LET: $y \in f(X)$
⟨1⟩3. PICK $x \in X$ such that $y = f(x)$
⟨1⟩4. PICK a countable local basis $\mathcal{B}$ at x
⟨1⟩5. $\{f(B) : B \in \mathcal{B}\}$ is a countable local basis at y.
□

Proposition 7.1.14. *$S_\Omega \times \overline{S_\Omega}$ is not first countable.*

PROOF: $(0, \Omega)$ has no countable basis. □

Proposition 7.1.15. *The Sorgenfrey plane is first countable.*

PROOF: For any point (a, b), the set $\{[a, a + q) \times [b, b + r) : q, r \in \mathbb{Q}\}$ is a countable local basis at (a, b). □

7.2 Separable Spaces

Definition 7.2.1 (Separable Space)**.** A topological space X is *separable* iff it has a countable dense subset.

Proposition 7.2.2. *The space S_Ω is not separable.*

PROOF:
⟨1⟩1. LET: $D \subseteq S_\Omega$ be countable.
⟨1⟩2. LET: $\alpha = \sup D$
⟨1⟩3. $\overline{D} \subseteq (-\infty, \alpha]$
□

Proposition 7.2.3. *The space $\overline{S_\Omega}$ is not separable.*

PROOF:
⟨1⟩1. LET: $D \subseteq S_\Omega$ be countable.
⟨1⟩2. LET: $\alpha = \sup\{\beta \in D : \beta < \Omega\}$
⟨1⟩3. $\alpha < \Omega$
PROOF: α is the supremum of countably many countable ordinals.
⟨1⟩4. $\overline{D} \subseteq (-\infty, \alpha] \cup \{\Omega\}$
□

Corollary 7.2.3.1. *Not every compact Hausdorff space is separable.*

Proposition 7.2.4. *Every open subspace of a separable space is separable.*

PROOF:
⟨1⟩1. LET: X be a separable space with countable dense subset D.
⟨1⟩2. LET: U be an open subspace of X
PROVE: $D \cap U$ is a countable dense subset of U.
⟨1⟩3. $D \cap U$ is countable.
⟨1⟩4. LET: V be an open set in U.
⟨1⟩5. V is open in X
PROOF: Lemma 4.3.3
⟨1⟩6. V intersects D
⟨1⟩7. V intensects $D \cap U$
□

Proposition 7.2.5 (CC). *The product of a countable family of separable spaces is separable.*

PROOF:
⟨1⟩1. LET: (X_n) be a countable family of separable spaces.
⟨1⟩2. For $n \geq 1$, PICK a dense set D_n in X_n
⟨1⟩3. $\prod_{n=1}^{\infty} D_n$ is dense in $\prod_{n=1}^{\infty} X_n$.
□

Proposition 7.2.6. *The continuous image of a separable space is separable.*

PROOF:
⟨1⟩1. LET: X be a separable space, Y a space and $f : X \to Y$ be continuous.
⟨1⟩2. PICK a countable dense set D in X
⟨1⟩3. $f(D)$ is dense in $f(X)$.
□

Corollary 7.2.6.1. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of nonempty topological spaces. If $\prod_{\alpha \in J} X_\alpha$ is separable then each X_α is separable.*

Corollary 7.2.6.2. *$S_\Omega \times \overline{S_\Omega}$ is not separable.*

Proposition 7.2.7. *The ordered square is not separable.*

PROOF: $\{\{x\} \times (0,1) : x \in [0,1]\}$ is an uncountable set of disjoint open sets. □

Proposition 7.2.8. $\mathbb{R}_l$ *is separable.*

PROOF: $\mathbb{Q}$ is dense. □

Proposition 7.2.9. *The Sorgenfrey plane is separable.*

PROOF: $\mathbb{Q}^2$ is dense. □

Proposition 7.2.10. *Not every closed subspace of a separable space is separable.*

PROOF: $\mathbb{R}_l^2$ is separable but the subspace $\{(x, -x) : x \in \mathbb{R}\}$ is not. □

7.3 The Second Countability Axiom

Definition 7.3.1 (Second Countability Axiom)**.** A topological space satisfies the *second countability axiom*, or is *second countable*, iff it has a countable basis.

Proposition 7.3.2. S_Ω *is not second countable.*

PROOF: $\{\{\alpha\} : \alpha$ is a countable successor ordinal$\}$ is an uncountable set of disjoint open sets. □

Proposition 7.3.3. *A subspace of a second countable space is second countable.*

PROOF:
⟨1⟩1. LET: X be a second countable space and $A \subseteq X$
⟨1⟩2. PICK a countable basis $\mathcal{B}$ for X
⟨1⟩3. $\{B \cap A : B \in \mathcal{B}\}$ is a countable basis for A
□

Proposition 7.3.4 (CC)**.** *The product of countably many second countable spaces is second countable.*

PROOF:
⟨1⟩1. LET: $\{X_n\}_{n \in \mathbb{Z}^+}$ be a countable family of second countable spaces.
⟨1⟩2. For $n \in \mathbb{Z}^+$, PICK a countable basis $\mathcal{B}_n$ for X_n.
⟨1⟩3. LET: $\mathcal{B}$ be the set of all sets of the form $\prod_{n=1}^{\infty} U_n$, where $U_n \in \mathcal{B}_n$ for finitely many n, and $U_n = X_n$ for all other n.
⟨1⟩4. $\mathcal{B}$ is a countable basis for $\prod_{n=1}^{\infty} X_n$
□

Theorem 7.3.5 (CC)**.** *Every second countable space is separable.*

PROOF:
⟨1⟩1. LET: X be a second countable space.
⟨1⟩2. PICK a countable basis $\mathcal{B}$ for X
⟨1⟩3. For $B \in \mathcal{B}$ nonempty, PICK a point $x_B \in B$
⟨1⟩4. $D = \{x_B : B \in \mathcal{B} \setminus \{\emptyset\}\}$ is dense.
⟨2⟩1. LET: $l \in X$

PROVE: $l \in \overline{D}$
⟨2⟩2. LET: $B \in \mathcal{B}$ such that $l \in B$
⟨2⟩3. $x_B \in B \cap D$
⟨2⟩4. Q.E.D.
PROOF:By Theorem 3.8.8

Corollary 7.3.5.1. *$S_\Omega \times \overline{S_\Omega}$ is not second countable.*

Corollary 7.3.5.2. *The space $\mathbb{R}^\omega$ is separable.*

Corollary 7.3.5.3. *If J is uncountable then $\mathbb{R}^J$ is not second countable.*

Proposition 7.3.6. *The ordered square is not second countable.*

PROOF:
⟨1⟩1. LET: $\mathcal{B}$ be any basis
⟨1⟩2. For $x \in [0, 1]$, PICK B_x such that $x \in B_x \subseteq ((x, 0), (x, 1))$
⟨1⟩3. The function $B_{(-)}$ is an injective function $[0, 1] \to \mathcal{B}$
⟨1⟩4. $\mathcal{B}$ is uncountable.
□

Proposition 7.3.7. *The space $\overline{S_\Omega}$ is not second countable.*

PROOF: It is not first countable (Proposition 7.1.12). □

Proposition 7.3.8. *The continuous image of a second countable space is second countable.*

PROOF:
⟨1⟩1. LET: X be a second countable space, Y a space and $f : X \to Y$ be continuous.
⟨1⟩2. PICK a countable basis $\mathcal{B}$ for X.
⟨1⟩3. $\{f(B) : B \in \mathcal{B}$ is a countable basis for $f(X)$
□

Theorem 7.3.9. *Every regular Lindelöf space is normal.*

PROOF:
⟨1⟩1. LET: X be a regular Lindelöf space.
⟨1⟩2. LET: A and B be disjoint closed sets in X.
⟨1⟩3. $\{U$ open in $X : \overline{U} \cap B = \emptyset\}$ covers A
PROOF: Proposition 6.3.2.
⟨1⟩4. PICK a countable open covering $\{U_n : n \in \mathbb{Z}^+\}$ of A such that $\overline{U_n} \cap B = \emptyset$ for all n
⟨1⟩5. PICK a countable open covering $\{V_n : n \in \mathbb{Z}^+\}$ of B such that $\overline{V_n} \cap A = \emptyset$ for all n
PROOF: Similar.
⟨1⟩6. For $n \in \mathbb{Z}^+$,
LET: $U'_n = U_n \setminus \bigcup_{i=1}^n \overline{V_i}$ and $V'_n = V_n \setminus \bigcup_{i=1}^n \overline{U_i}$
⟨1⟩7. LET: $U' = \bigcup_{n=1}^\infty U'_n$ and $V = \bigcup_{n=1}^\infty V'_n$

⟨1⟩8. $A \subseteq U'$ and $B \subseteq V'$
⟨1⟩9. $U' \cap V' = \emptyset$
□

Corollary 7.3.9.1. *If J is uncountable then $\mathbb{R}^J$ is not Lindelöf.*

Proposition 7.3.10. *Every second countable regular space is completely normal.*

Proof:
⟨1⟩1. Let: X be second countable and regular and $Y \subseteq X$
⟨1⟩2. Y is second countable
Proof: Proposition 7.3.3.
⟨1⟩3. Y is regular
Proof: Proposition 6.3.4
⟨1⟩4. Y is normal
Proof: Theorem 7.3.9
□

Proposition 7.3.11. *The space $\mathbb{R}^\omega$ is second countable.*

Proof: The sets $\prod_{n=0}^{\infty} U_n$ form a basis, where U_n is an interval of the form (q, r) for $q, r \in \mathbb{Q}$ for finitely many n, and $U_n = \mathbb{R}$ for all other n. □

Proposition 7.3.12 (CC)**.** *In a second countable space, every discrete subspace is countable.*

Proof:
⟨1⟩1. Let: X be a second countable space
⟨1⟩2. Pick a countable basis $\mathcal{B}$
⟨1⟩3. Let: $D \subseteq X$ be discrete
⟨1⟩4. For $a \in D$, Pick $B_a \in \mathcal{B}$ such that $B_a \cap D = \{a\}$
⟨1⟩5. $a \mapsto B_a$ is injective
□

Proposition 7.3.13. *The space $\mathbb{R}_K$ is second countable.*

Proof: $\{(a, b) : a, b \in \mathbb{R}\} \cup \{(a, b) - K : a, b \in \mathbb{Q}\}$ is a basis. □

Corollary 7.3.13.1. *The space $\mathbb{R}_K$ is first countable.*

Corollary 7.3.13.2. *The space $\mathbb{R}_K$ is separable.*

Proposition 7.3.14. *Let J be a set with $|J| > |\mathbb{R}|$. Then $\mathbb{R}^J$ is not separable.*

Proof:
⟨1⟩1. Assume: D is countable and dense in $\mathbb{R}^J$
Prove: $|J| \leq |\mathbb{R}|$
⟨1⟩2. Define $f : J \to \mathcal{P}D$ by $f(\alpha) = D \cap {\pi_\alpha}^{-1}((0, 1))$
⟨1⟩3. f is injective

⟨2⟩1. LET: $\alpha, \beta \in J$ with $\alpha \neq \beta$
⟨2⟩2. PICK $x \in D \cap {\pi_\alpha}^{-1}((0,1)) \cap {\pi_\beta}^{-1}((2,3))$
⟨2⟩3. $x \in f(\alpha)$ but $x \notin f(\beta)$
□

Corollary 7.3.14.1. *The product of a family of separable spaces is not necessarily separable.*

Chapter 8

Connectedness

8.1 Connected Spaces

Definition 8.1.1 (Separation)**.** Let X be a topological space. A *separation* of X is a pair of disjoint nonempty subsets whose union in X.

Definition 8.1.2 (Connected)**.** A topological space is *connected* iff it has no separation.

Proposition 8.1.3. *S_Ω is not connected.*

Proof: $\{0\}$ and $S_\Omega \setminus \{0\}$ form a separation. □

Proposition 8.1.4. *A space X is connected if and only if the only sets that are both closed and open are $\emptyset$ and X.*

Proof: Immediate from definitions. □

Proposition 8.1.5. *Let Y be a subspace of X. Then a separation of Y is a pair of disjoint nonempty sets A, B such that $A \cup B = Y$ and neither of A, B contains a limit point of the other.*

Proof:
$\langle 1 \rangle 1$. If A and B form a separation of Y then A and B are disjoint and nonempty, $A \cup B = Y$, and neither of A, B contains a limit point of the other.
  $\langle 2 \rangle 1$. Let: A and B be a separation of Y
  $\langle 2 \rangle 2$. A and B are disjoint and nonempty and $A \cup B = Y$
    Proof: Immediate from the definition of separation.
  $\langle 2 \rangle 3$. A does not contain a limit point of B
    Proof: B is closed in Y, hence contains all its limit points (Corollary 3.11.3.1), and so the result follows because A and B are disjoint.
  $\langle 2 \rangle 4$. B does not contain a limit point of A
    Proof: Similar.
$\langle 1 \rangle 2$. If A and B are disjoint and nonempty, $A \cup B = Y$, and neither of A, B contains a limit point of the other, then A and B are a separation of Y.

$\langle 2\rangle 1$. Assume: A and B are disjoint and nonempty, $A \cup B = Y$, and neither of A, B contains a limit point of the other

$\langle 2\rangle 2$. A is closed in Y

Proof: Every limit point of A is not in B, so is in A. Apply Corollary 3.11.3.1.

$\langle 2\rangle 3$. B is open in Y

Proof:$B = Y \setminus A$

$\langle 2\rangle 4$. A is open in Y

Proof: Similar.

□

Proposition 8.1.6. *If the sets C and D form a separation of X, and Y is a connected subspace of X, then $Y \subseteq C$ or $Y \subseteq D$.*

Proof: Otherwise, $Y \cap C$ and $Y \cap D$ would be a separation of Y. □

Proposition 8.1.7. *The union of a set of connected subspaces of X that have a point in common is connected.*

Proof:

$\langle 1\rangle 1$. Let: $\mathcal{S}$ be a set of connected subspaces that have the point a in common.

$\langle 1\rangle 2$. Assume: for a contradiction U and V form a separation of $\bigcup \mathcal{S}$

$\langle 1\rangle 3$. Assume: w.l.o.g. $a \in U$

$\langle 1\rangle 4$. For all $Y \in \mathcal{S}$ we have $Y \subseteq U$

Proof: By Proposition 8.1.6.

$\langle 1\rangle 5$. $V = \emptyset$

$\langle 1\rangle 6$. Q.E.D.

Proof: This contradicts $\langle 1\rangle 2$.

□

Theorem 8.1.8. *Let A be a connected subspace of X. If $A \subseteq B \subseteq \overline{A}$ then B is connected.*

Proof:

$\langle 1\rangle 1$. Assume: for a contradiction U and V are a separation of B

$\langle 1\rangle 2$. $A \subseteq U$ or $A \subseteq V$

Proof: By Proposition 8.1.6.

$\langle 1\rangle 3$. Assume: w.l.o.g. $A \subseteq U$

$\langle 1\rangle 4$. $\overline{A} \subseteq \overline{U}$

Proof: By Proposition 3.8.5.

$\langle 1\rangle 5$. $B \subseteq \overline{U}$

Proof: Since $B \subseteq \overline{A}$.

$\langle 1\rangle 6$. The closure of U in B is B

Proof: By Theorem 4.3.4.

$\langle 1\rangle 7$. $U = B$

Proof: Since U is closed in B.

$\langle 1\rangle 8$. Q.E.D.

Proof: This contradicts $\langle 1\rangle 1$.

∎

Theorem 8.1.9. *The image of a connected space under a continuous map is connected.*

PROOF: Let X be a connected space, Y a topological space, and $f : X \twoheadrightarrow Y$ be surjective. If U and V form a separation of Y, then $f^{-1}(U)$ and $f^{-1}(V)$ form a separation of X. ∎

Corollary 8.1.9.1. *Let $\mathcal{T}$ and $\mathcal{T}'$ be topologies on the same set X. If $\mathcal{T} \subseteq \mathcal{T}'$ and X is connected under $\mathcal{T}'$ then X is connected under $\mathcal{T}$.*

Corollary 8.1.9.2. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces. If $\prod_{\alpha \in J} X_\alpha$ is connected then each X_α is connected.*

Corollary 8.1.9.3. *The Sorgenfrey plane is disconnected.*

Proposition 8.1.10. *The product of a family of connected spaces is connected.*

PROOF:
⟨1⟩1. The product of two connected spaces is connected.
  PROOF:
  ⟨2⟩1. LET: X and Y be connected spaces.
  ⟨2⟩2. ASSUME: w.l.o.g. X and Y are nonempty.
    PROOF: If either is empty then $X \times Y = \emptyset$ is connected.
  ⟨2⟩3. ASSUME: for a contradiction U and V are a separation of $X \times Y$.
  ⟨2⟩4. PICK $b \in Y$
    PROOF: By ⟨2⟩2.
  ⟨2⟩5. For all $x \in X$,
      LET: $T_x = (X \times \{b\}) \cup (\{x\} \times Y)$
  ⟨2⟩6. For all $x \in X$, T_x is connected
    ⟨3⟩1. $X \times \{b\}$ is connected
      PROOF: It is homeomorphic to X.
    ⟨3⟩2. $\{x\} \times Y$ is connected
      PROOF: It is homeomorphic to Y.
    ⟨3⟩3. Q.E.D.
      PROOF: By Proposition 8.1.7.
  ⟨2⟩7. $X \times Y = \bigcup_{x \in X} T_x$
  ⟨2⟩8. Q.E.D.
    ⟨3⟩1. PICK $a \in X$
      PROOF: By ⟨2⟩2.
    ⟨3⟩2. $(a, b) \in T_x$ for all $x \in X$
    ⟨3⟩3. Q.E.D.
      PROOF: By Proposition 8.1.7.
⟨1⟩2. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of connected spaces.
⟨1⟩3. ASSUME: w.l.o.g. $\prod_{\alpha \in J} X_\alpha$ is nonempty
⟨1⟩4. PICK $\vec{a} \in \prod_{\alpha \in J} X_\alpha$
⟨1⟩5. For K a finite subset of J,

LET: $X_K = \{\vec{x} \in \prod_{\alpha \in J} X_\alpha : x_\alpha = a_\alpha \text{ for all } \alpha \in J \setminus K\}$

$\langle 1 \rangle 6.$ For all K, X_K is connected.

PROOF: It is homeomorphic to $\prod_{\alpha \in K} X_\alpha$, so it is connected by $\langle 1 \rangle 1$.

$\langle 1 \rangle 7.$ $\bigcup_{K \subseteq^{\text{fin}} J} X_K$ is connected.

PROOF: By Proposition 8.1.7 since $\vec{a} \in X_K$ for all K.

$\langle 1 \rangle 8.$ $\prod_{\alpha \in J} X_\alpha = \overline{\bigcup_{K \subseteq^{\text{fin}} J} X_K}$

$\langle 2 \rangle 1.$ LET: $\vec{x} \in \prod_{\alpha \in J} X_\alpha$

$\langle 2 \rangle 2.$ LET: U be an open neighbourhood of $\vec{x}$

$\langle 2 \rangle 3.$ PICK a basic open set $\prod_{\alpha \in J} V_\alpha$ such that $\vec{x} \in \prod_{\alpha \in J} V_\alpha \subseteq U$, where each V_α is open in X_α, and $V_\alpha = X_\alpha$ except for $\alpha \in K$ for some finite $K \subseteq J$

PROVE: U intersects X_K

$\langle 2 \rangle 4.$ LET: $\vec{y} \in \prod_{\alpha \in J} X_\alpha$ with $y_\alpha = x_\alpha$ for $\alpha \in K$, $y_\alpha = a_\alpha$ for $\alpha \notin K$

$\langle 2 \rangle 5.$ $\vec{y} \in U \cap X_K$

$\langle 1 \rangle 9.$ Q.E.D.

Corollary 8.1.10.1. *For any set I, the space $\mathbb{R}^I$ under the product topology is connected.*

Proposition 8.1.11. $\mathbb{R}^\omega$ *under the box topology is disconnected.*

PROOF: The set of all bounded sequences and the set of all unbounded sequences form a separation. □

Definition 8.1.12 (Totally Disconnected)**.** A space is *totally disconnected* iff the only connected subspaces are the singletons.

Theorem 8.1.13. *Let L be a linearly ordered set under the order topology. Then L is connected if and only if L is a linear continuum.*

PROOF:

$\langle 1 \rangle 1.$ If L is a linear continuum then L is connected.

$\langle 2 \rangle 1.$ LET: L be a linear continuum.

$\langle 2 \rangle 2.$ ASSUME: for a contradiction U and V are a separation of L.

$\langle 2 \rangle 3.$ PICK $a \in U$ and $b \in V$

$\langle 2 \rangle 4.$ ASSUME: w.l.o.g. $a < b$

$\langle 2 \rangle 5.$ LET: $l = \sup\{x \in A : x < b\}$

$\langle 2 \rangle 6.$ CASE: $l \in A$

$\langle 3 \rangle 1.$ PICK $a' > l$ such that $[l, a') \subseteq A$

PROOF: By Lemma 4.1.2. We know l is not greatest in X because $l < b$.

$\langle 3 \rangle 2.$ PICK a^* such that $l < a^* < a'$

PROOF: L is dense.

$\langle 3 \rangle 3.$ $l < a^*$, $a^* \in A$, $a^* < b$

PROOF: If $b < a^*$ then $b \in A$ by $\langle 3 \rangle 1$.

$\langle 3 \rangle 4.$ Q.E.D.

PROOF: This contradicts $\langle 2 \rangle 5$.

$\langle 2 \rangle 7.$ CASE: $l \in B$

$\langle 3 \rangle 1.$ PICK $b' < l$ such that $(b', l] \subseteq B$

Proof: By Lemma 4.1.2. We know l is not least in X because $a < l$.

$\langle 3 \rangle 2$. Pick b^* such that $b' < b^* < l$

Prove: b^* is an upper bound for $\{x \in A : x < b\}$

$\langle 3 \rangle 3$. Let: $x \in A$ and $x < b$

$\langle 3 \rangle 4$. $x \leq b^*$

Proof: If $b^* < x$ then $b^* < x \leq l$ and so $x \in B$ by $\langle 3 \rangle 1$.

$\langle 3 \rangle 5$. Q.E.D.

Proof: This contradicts $\langle 2 \rangle 5$.

$\langle 1 \rangle 2$. If L is connected then L is a linear continuum.

$\langle 2 \rangle 1$. Assume: L is connected

$\langle 2 \rangle 2$. L has the least upper bound property

$\langle 3 \rangle 1$. Assume: for a contradiction $A \subseteq L$ is bounded above with no least upper bound

$\langle 3 \rangle 2$. Let: U be the set of upper bounds of A

$\langle 3 \rangle 3$. U is open

$\langle 4 \rangle 1$. Let: $u \in U$

$\langle 4 \rangle 2$. Pick an upper bound v for A with $v < u$

Proof: u is not the least upper bound for A ($\langle 3 \rangle 1$)

$\langle 4 \rangle 3$. $u \in (v, +\infty) \subseteq U$

$\langle 3 \rangle 4$. Let: V be the set of lower bounds of U

$\langle 3 \rangle 5$. U and V form a separation of L

$\langle 4 \rangle 1$. V is open

Proof: Similar to $\langle 3 \rangle 3$.

$\langle 4 \rangle 2$. U and V are disjoint

$\langle 5 \rangle 1$. Assume: for a contradiction $x \in U \cap V$

$\langle 5 \rangle 2$. Pick $u \in U$ such that $u < x$

Proof: x is not the lowest upper bound of A

$\langle 5 \rangle 3$. $x \leq u < x$

$\langle 4 \rangle 3$. $U \cup V = L$

$\langle 5 \rangle 1$. Let: $x \in L \setminus U$

$\langle 5 \rangle 2$. Pick $a \in A$ such that $x < a$

$\langle 5 \rangle 3$. $a \in V$

$\langle 5 \rangle 4$. $x \in V$

$\langle 2 \rangle 3$. For all $x, y \in L$, there exists $z \in L$ such that $x < z < y$

Proof: Otherwise $(-\infty, y)$ and $(x, +\infty)$ would form a separation of L.

$\square$

Corollary 8.1.13.1. *The real line $\mathbb{R}$ is connected, and so is every ray and interval in $\mathbb{R}$.*

Corollary 8.1.13.2. *The ordered square is connected.*

Corollary 8.1.13.3. *Not every closed subspace of a connected space is connected.*

Proof: The set $\{0, 1\}$ is disconnected as a subspace of $\mathbb{R}$.

Corollary 8.1.13.4. *Not every open subspace of a connected space is connected.*

PROOF: The space $\mathbb{R} \setminus \{0\}$ is a disconnected open subspace of $\mathbb{R}$. ∎

Theorem 8.1.14 (Intermediate Value Theorem)**.** *Let X be a connected space and Y a linearly ordered set under the order topology. Let $f : X \to Y$ be continuous. Let $a, b \in X$ and $r \in Y$. If $f(a) < r < f(b)$, then there exists $c \in X$ such that $f(c) = r$.*

PROOF: If not, then $f^{-1}((-\infty, r))$ and $f^{-1}((r, +\infty))$ would be a separation of X. ∎

Proposition 8.1.15. *Every connected regular space with more than one point is uncountable.*

PROOF:
⟨1⟩1. Every connected completely regular space with more than one point is uncountable.
  ⟨2⟩1. LET: X be connected and completely regular and $a, b \in X$ with $a \neq b$
  ⟨2⟩2. PICK a continuous $f : X \to [0, 1]$ such that $f(a) = 0$ and $f(b) = 1$
  ⟨2⟩3. f is surjective.
    PROOF: By the Intermediate Value Theorem.
⟨1⟩2. Every connected regular space with more than one point is uncountable.
  ⟨2⟩1. ASSUME: for a contradiction X is connected, regular and countable with more than one point.
  ⟨2⟩2. X is Lindelöf
  ⟨2⟩3. X is normal
    PROOF: By Theorem 7.3.9
  ⟨2⟩4. Q.E.D.
    PROOF: Contraditcting ⟨1⟩1.
∎

Proposition 8.1.16. *$\overline{S_\Omega}$ is not conneced.*

PROOF: $\{0\}$ is clopen. ∎

Proposition 8.1.17. *$\mathbb{R}_l$ is not connected.*

PROOF: The set $[0, +\infty)$ is clopen. ∎

Proposition 8.1.18. *The space $\mathbb{R}^\omega$ under the uniform topology is not connected.*

PROOF: The set of all bounded sequences and the set of all unbounded sequences form a separation. ∎

Proposition 8.1.19. *The space $\mathbb{R}_K$ is connected.*

PROOF: Easy. ∎

8.2 Components and Local Connectedness

Definition 8.2.1 ((Connected) Component)**.** Let X be a topological space. Define an equivalence relation $\sim$ on X by: $x \sim y$ iff there exists a connected subspace $U \subseteq X$ such that $x \in U$ and $y \in U$. The *(connected) components* of X are the equivalence classes under $\sim$.

We prove this is an equivalence relation.

PROOF:
$\langle 1 \rangle 1$. For all $x \in X$ we have $x \sim x$.
PROOF: The subspace $\{x\} \subseteq X$ is connected.
$\langle 1 \rangle 2$. For all $x, y \in X$, if $x \sim y$ then $y \sim x$.
PROOF: Immediate from definitions.
$\langle 1 \rangle 3$. For all $x, y, z \in X$, if $x \sim y$ and $y \sim z$ then $x \sim z$.
PROOF: By Proposition 8.1.7.
$\square$

Proposition 8.2.2. *Let X be a topological space. If $C \subseteq X$ is connected and nonempty, then there exists a unique component D of X such that $C \subseteq D$.*

PROOF:
$\langle 1 \rangle 1$. PICK $a \in C$
$\langle 1 \rangle 2$. LET: D be the $\sim$-equivalence class of A
$\langle 1 \rangle 3$. $C \subseteq D$
PROOF: For all $x \in C$ we have $a \sim x$ by definition.
$\langle 1 \rangle 4$. D is unique
PROOF: This holds because the components are disjoint.
$\square$

Proposition 8.2.3 (AC)**.** *Every component is connected.*

PROOF:
$\langle 1 \rangle 1$. LET: C be a component of the topological space X
$\langle 1 \rangle 2$. PICK $a \in C$
$\langle 1 \rangle 3$. For all $x \in C$, PICK a connected subspace C_x of X containing both a and x.
PROOF: Such a C_x exists since $a \sim x$.
$\langle 1 \rangle 4$. $C = \bigcup_{x \in C} C_x$
PROOF: This holds because $C_x \subseteq C$ by Proposition 8.2.2.
$\langle 1 \rangle 5$. Q.E.D.
PROOF: It follows that C is connected by Proposition 8.1.7.
$\square$

Proposition 8.2.4. *Every component is closed.*

PROOF: From Theorem 8.1.8. $\square$

Proposition 8.2.5. *The component of $\vec{a}$ in $\mathbb{R}^\omega$ under the uniform topology is $\{\vec{b} : \vec{b} - \vec{a}$ is bounded$\}$.*

PROOF:
⟨1⟩1. $C = \{\vec{b} : \vec{b} - \vec{a} \text{ is bounded}\}$ is connected.
⟨2⟩1. ASSUME: $C = U \cup V$ is a separation of C with $\vec{a} \in U$
⟨2⟩2. PICK $\vec{b} \in V$
⟨2⟩3. $\{\epsilon : \epsilon\vec{b} + (1-\epsilon)\vec{a} \in U\}$ and $\{\epsilon : \epsilon\vec{b} + (1-\epsilon)\vec{a} \in V\}$ form a separation of $[0, 1]$
⟨1⟩2. If $\vec{a}, \vec{b} \in C$ and $\vec{b} - \vec{a}$ is unbounded then C is disconnected.
PROOF: $\{\vec{c} : \vec{c} - \vec{a} \text{ is bounded}\}$ and $\{\vec{c} : \vec{c} - \vec{a} \text{ is unbounded}\}$
□

Proposition 8.2.6. *Let $x, y \in \mathbb{R}^\omega$ under the box topology. Then x and y are in the same component iff $x - y$ is eventually zero.*

PROOF:
⟨1⟩1. For all $x \in \mathbb{R}^\omega$ the set $\{y : x - y \text{ is eventually zero}\}$ is connected
PROOF: It is the union of the sets $C_N = \{y : \forall n \geq N. y_n = 0\}$, each of which is connected because it is homeomorphic to $\mathbb{R}^{N-1}$.
⟨1⟩2. If $x - y$ is not eventually zero then x and y are in different components
⟨2⟩1. ASSUME: $x - y$ is not eventually zero
⟨2⟩2. Define $h : \mathbb{R}^\omega \to \mathbb{R}^\omega$ by: $h(z)_n = \begin{cases} z_n - x_n & \text{if } x_n = y_n \\ n(z_n - x_n)/(y_n - x_n) & \text{if } x_n \neq y_n \end{cases}$
⟨2⟩3. h is an automorphism of $\mathbb{R}^\omega$ under the box topology
⟨2⟩4. $h(x) = 0$
⟨2⟩5. $h(y)$ is unbounded
⟨2⟩6. Q.E.D.
PROOF: The inverse image under h of the set of bounded sequences and the set of unbounded sequences form a separation of $\mathbb{R}^\omega$ with x and y in different sets.
□
□

8.3 Path Connectedness

Definition 8.3.1 (Path)**.** Let X be a topological space and $a, b \in X$. A *path* from a to b is a continuous function $p : [0, 1] \to X$ such that $p(0) = a$ and $p(1) = b$.

Definition 8.3.2 (Path Connected)**.** A topological space is *path connected* iff there exists a path between any two points.

Proposition 8.3.3. *Every path connected space is connected.*

PROOF:
⟨1⟩1. LET: X be a path connected space
⟨1⟩2. ASSUME: for a contradiction U and V are a separation of X.
⟨1⟩3. PICK $a \in U$ and $b \in V$

⟨1⟩4. PICK a path $p : [0,1] \to X$ from a to b
⟨1⟩5. $p^{-1}(U)$ and $p^{-1}(V)$ form a separation of $[0,1]$.
⟨1⟩6. Q.E.D.
PROOF: This contradicts Corollary 8.1.13.1.

Corollary 8.3.3.1. *S_Ω is not path connected.*

Corollary 8.3.3.2. *$\overline{S_\Omega}$ is not path connected.*

Corollary 8.3.3.3. *$\mathbb{R}_l$ is not path connected.*

Corollary 8.3.3.4. *The Sorgenfrey plane is not path connected.*

Corollary 8.3.3.5. *The space $\mathbb{R}^\omega$ under the uniform topology is not path connected. connected.*

Corollary 8.3.3.6. *The space $\mathbb{R}^\omega$ under the box topology is not path connected.*

Proposition 8.3.4. *The long line is path connected.*

PROOF:
⟨1⟩1. LET: $a, b \in L$
⟨1⟩2. PICK an ordinal α such that $a, b < (\alpha, 0)$
⟨1⟩3. There exists a path from a to b
PROOF: This holds because $[(0,0),(\alpha,0))$ is homeomorphic to $[0,1)$ by Proposition 1.7.11.
□

Corollary 8.3.4.1. *Not every closed subspace of a path connected space is path connected.*

PROOF: Take any two-element subspace of the long line.

Corollary 8.3.4.2. *Not every open subspace of a path connected space is path connected.*

PROOF: The space $\mathbb{R} \setminus \{0\}$ is not path connected as a subspace of $\mathbb{R}$. □

Definition 8.3.5 (Path Component)**.** Let X be a topological space. Define an equivalence relation $\sim$ on X by: $x \sim y$ iff there exists a path from x to y. The equivalence classes are called the *path components* of X.

We prove this is an equivalence relation.

PROOF:
⟨1⟩1. For all $x \in X$ we have $x \sim x$
PROOF: The constant path $p : [0,1] \to X$ where $p(t) = x$ is a path from x to x.
⟨1⟩2. If $x \sim y$ then $y \sim x$
PROOF: If $p : [0,1] \to X$ is a path from x to y then $\lambda t.p(1-t)$ is a path from y to x.
⟨1⟩3. If $x \sim y$ and $y \sim z$ then $x \sim z$

$\langle 2\rangle 1$. Let: p be a path from x to y and q be a path from y to z.
$\langle 2\rangle 2$. Let: $r : [0,1] \to X$ where
$$r(t) = \begin{cases} p(2t) & \text{if } 0 \leq t \leq 1/2 \\ q(2t-1) & \text{if } 1/2 \leq t \leq 1 \end{cases}$$
$\langle 2\rangle 3$. r is a path from x to z.
Proof: r is continuous by the Pasting Lemma.
$\square$

Proposition 8.3.6. *Every path component is path connected.*

Proof: By definition, if x and y are in the same path component then there is a path from x to y. $\square$

Proposition 8.3.7. *If A is a nonempty path connected subspace of the space X, then A is included in a unique path component.*

Proof:
$\langle 1\rangle 1$. Pick $a \in A$
$\langle 1\rangle 2$. Let: C be the equivalence class of a under $\sim$
$\langle 1\rangle 3$. $A \subseteq C$
Proof: For all $x \in A$, there exists a path from a to x.
$\langle 1\rangle 4$. C is unique
Proof: C is the unique path component such that $a \in C$.
$\square$

Proposition 8.3.8. *Every path component is included in a component.*

Proof: From Propositions 8.3.3 and 8.2.2. $\square$

Proposition 8.3.9. *The ordered square is not path connected.*

Proof:
$\langle 1\rangle 1$. Assume: for a contradiction $p : [0,1] \to I_o^2$ is a path from $(0,0)$ to $(1,1)$.
$\langle 1\rangle 2$. For all $x \in [0,1]$, $p^{-1}(\{x\} \times (0,1))$ is open in $[0,1]$
$\langle 1\rangle 3$. For all $x \in [0,1]$, Pick a rational $q_x \in p^{-1}(\{x\} \times (0,1))$
$\langle 1\rangle 4$. $\{q_x : x \in [0,1]\}$ is an uncountable set of rationals.
$\square$

Proposition 8.3.10 (AC)**.** *The product of a family of path connected spaces is path connected.*

Proof:
$\langle 1\rangle 1$. Let: $\{X_\alpha\}_{\alpha \in J}$ be a family of path connected spaces and $a, b \in \prod_{\alpha \in J} X_\alpha$
$\langle 1\rangle 2$. For $\alpha \in J$, Pick a path $p_\alpha : [0,1] \to X_\alpha$ from a_α to b_α
$\langle 1\rangle 3$. Define $p : [0,1] \to \prod_{\alpha \in J} X_\alpha$ by $p(t)_\alpha = p_\alpha(t)$
$\langle 1\rangle 4$. p is a path from a to b
Proof: By Theorem 5.2.15.
$\square$

Corollary 8.3.10.1. *For any set I, the space $\mathbb{R}^I$ in the product topology is path connected.*

Proposition 8.3.11. *The space $\mathbb{R}_K$ is not path connected.*

PROOF:
$\langle 1 \rangle 1$. ASSUME: for a contradiction $p : [0,1] \to \mathbb{R}_K$ is a path from 0 to 1
$\langle 1 \rangle 2$. LET: $p : [0,1] \to \mathbb{R}_K$ be a path from 0 to 1
$\langle 1 \rangle 3$. $p([0,1])$ is compact and connected in $\mathbb{R}_K$.
PROOF: Theorem 8.1.9 and Proposition 9.4.10.
$\langle 1 \rangle 4$. $p([0,1])$ is connected in $\mathbb{R}$.
PROOF: Corollary 8.1.9.1
$\langle 1 \rangle 5$. $[0,1] \subseteq p([0,1])$
PROOF: For any $x \in [0,1]$, if $x \notin p([0,1])$ then $p([0,1]) \cap (-\infty, x)$ and $p([0,1]) \cap (x, +\infty)$ form a separation of $p([0,1])$.
$\langle 1 \rangle 6$. $[0,1]$ is compact in $\mathbb{R}_K$
PROOF: Proposition 9.4.6.
$\langle 1 \rangle 7$. Q.E.D.
PROOF: This contradicts Corollary 9.4.11.2.
□

Proposition 8.3.12. *Let $f : X \to Y$ be continuous and surjective. If X is path connected then Y is path connected.*

PROOF:
$\langle 1 \rangle 1$. LET: $a, b \in Y$
$\langle 1 \rangle 2$. PICK $x, y \in X$ such that $f(x) = a$ and $f(y) = b$
$\langle 1 \rangle 3$. PICK a path $p : [0,1] \to X$ such that $p(0) = x$ and $p(1) = y$
$\langle 1 \rangle 4$. $f \circ p$ is a path from a to b
□

Corollary 8.3.12.1. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of non-empty topological spaces. If $\prod_{\alpha \in J} X_\alpha$ is path connected then each X_α is path connected.*

8.4 Connected Subspaces of Euclidean Space

Definition 8.4.1 (Unit 2-Sphere)**.** The *unit 2-sphere* is $S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ as a subspace of $\mathbb{R}^3$.

Definition 8.4.2 (Unit Ball)**.** For any $n \geq 1$, the *closed unit ball* in $\mathbb{R}^n$ is

$$B^n = \{\vec{x} \in \mathbb{R}^n : \|\vec{x}\| \leq 1\} \ .$$

Proposition 8.4.3. *Every open unit ball and closed unit ball in $\mathbb{R}^n$ is path connected.*

PROOF: The straight line between any two points is a path in the ball. □

Definition 8.4.4 (Punctured Euclidean Space)**.** For $n \geq 1$, *punctured Euclidean space* is $\mathbb{R}^n \setminus \{\vec{0}\}$.

Proposition 8.4.5. *Punctured Euclidean space in $\mathbb{R}^n$ is path connected iff $n > 1$.*

PROOF: Easy. □

Definition 8.4.6 (Unit Sphere)**.** For $n \geq 1$, the *unit sphere* S^n is $\{\vec{x} \in \mathbb{R}^{n+1} : \|\vec{x}\| = 1\}$.

Proposition 8.4.7. *In any number of dimensions, the unit sphere is path connected.*

PROOF: Easy. □

Definition 8.4.8 (Topologist's Sine Curve)**.** The *topologist's sine curve* is the closure of

$$S = \{(x, \sin 1/x) : x \in \mathbb{R}\}$$

in $\mathbb{R}^2$.

Proposition 8.4.9. *The topologist's sine curve is connected.*

PROOF:
$\langle 1 \rangle 1$. $S = \{(x, \sin 1/x) : x \in \mathbb{R}\}$ is connected.
  $\langle 2 \rangle 1$. The function $f : \mathbb{R} \to \mathbb{R}^2$ given by $f(x) = (x, \sin 1/x)$ is continuous.
    PROOF: By Theorem 5.2.15.
  $\langle 2 \rangle 2$. Q.E.D.
    PROOF: By Theorem 8.1.9.
$\langle 1 \rangle 2$. Q.E.D.
  PROOF: By Theorem 8.1.8.
□

Proposition 8.4.10 (CC)**.** *The topologist's sine curve is not path connected.*

PROOF:
$\langle 1 \rangle 1$. LET: $S = \{(x, \sin 1/x) : x \in \mathbb{R}\}$
$\langle 1 \rangle 2$. ASSUME: for a contradiction $p : [0, 1] \to \overline{S}$ is a path from $(0, 0)$ to $(1, \sin 1)$.
$\langle 1 \rangle 3$. $p^{-1}(\{0\} \times [-1, 1])$ is closed.
$\langle 1 \rangle 4$. $p^{-1}(\{0\} \times [-1, 1])$ has a greatest element.
  PROOF: By Lemma 4.1.9.
$\langle 1 \rangle 5$. LET: $q : [0, 1] \to \overline{S}$ be a path such that:
- $q(0) \in \{0\} \times [-1, 1]$
- $q(x) \in S$ for $x > 0$

  PROOF: Let b be greatest in $p^{-1}(\{0\} \times [-1, 1])$. Then q is obtained by rescaling p restricted to $[b, 1]$.
$\langle 1 \rangle 6$. LET: $q(t) = (x(t), y(t))$ for $0 \leq t \leq 1$
$\langle 1 \rangle 7$. $x(0) = 0$

⟨1⟩8. $x(t) > 0$ for $t > 0$
⟨1⟩9. $y(t) = \sin 1/x(t)$ for $t > 0$
⟨1⟩10. There exists a sequence $t_n \in [0, 1]$ such that $t_n \to 0$ as $n \to \infty$ and $y(t_n) = (-1)^n$ for all n.
  ⟨2⟩1. For each n, PICK u_n such that $0 < u_n < x(1/n)$ and $\sin 1/u_n = (-1)^n$.
    PROOF: Such a u_n exists because $\sin 1/x$ takes values 1 and -1 infinitely often in $(0, x(1/n))$.
  ⟨2⟩2. For each n, PICK t_n such that $0 < t_n < 1/n$ and $x(t_n) = u$
    PROOF: By the Intermediate Value Theorem.
⟨1⟩11. Q.E.D.
  PROOF: This is a contradiction as $y(t_n) \to y(0)$ as $n \to \infty$ because y is continuous.

□

8.5 Local Connectedness

Definition 8.5.1 (Locally Connected)**.** Let X be a topological space and $x \in X$. Then X is *locally connected* at x iff every neighbourhood of x includes a connected neighbourhood of x.

The space X is *locally connected* iff it is locally connected at every point.

Proposition 8.5.2. *S_Ω is not locally connected.*

PROOF: There is no connected neighbourhood of ω. □

Proposition 8.5.3. *$\overline{S_\Omega}$ is not locally connected.*

PROOF: There is no connected neighbourhood of ω. □

Proposition 8.5.4. *For any set I, the space $\mathbb{R}^I$ is locally connected.*

PROOF: Every basic open set is the product of connected spaces, hence connected. □

Proposition 8.5.5. *Let X be a topological space. Then X is locally connected if and only if, for every open set U in X, every component of U is open in X.*

PROOF:
⟨1⟩1. If X is locally connected then, for every open set U in X, every component of U is open in X.
  ⟨2⟩1. ASSUME: X is locally connected.
  ⟨2⟩2. LET: U be open in X.
  ⟨2⟩3. LET: C be a component of U.
  ⟨2⟩4. LET: $x \in C$
    PROVE: C is a neighbourhood of x
  ⟨2⟩5. U is a neighbourhood of x in X.
    PROOF: From ⟨2⟩2, ⟨2⟩3 and ⟨2⟩4.
  ⟨2⟩6. PICK a connected neighbourhood V of x such that $V \subseteq U$.

PROOF: Using $\langle 2\rangle 1$.
$\langle 2\rangle 7$. $V \subseteq C$
PROOF: By Proposition 8.2.2.
$\langle 2\rangle 8$. C is a neighbourhood of x
PROOF: By Proposition 3.2.4.
$\langle 2\rangle 9$. Q.E.D.
PROOF: By Proposition 3.2.3.
$\langle 1\rangle 2$. If, for every open set U in X, every component of U is open in X, then X is locally connected.
$\langle 2\rangle 1$. ASSUME: For every open set U in X, every component of U is open in X.
$\langle 2\rangle 2$. LET: $x \in X$ and N be a neighbourhood of x
$\langle 2\rangle 3$. PICK U open such that $x \in U \subseteq N$
$\langle 2\rangle 4$. LET: C be the component of U that contains x
$\langle 2\rangle 5$. C is open in X
PROOF: By $\langle 2\rangle 1$.
$\langle 2\rangle 6$. C is a connected neighbourhood of x that is included in N
□

Corollary 8.5.5.1. *In a locally connected space, every component is open.*

Corollary 8.5.5.2. *The space $\mathbb{R}^\omega$ under the box topology is not locally connected.*

Corollary 8.5.5.3. *Not every closed subspace of a locally connected space is locally connected.*

PROOF: The topologist's sine curve is not locally connected. □

Proposition 8.5.6. *$S_\Omega \times \overline{S_\Omega}$ is not locally connected.*

(ω, ω) has no connected neighbourhood. □

Proposition 8.5.7. *$\mathbb{R}_l$ is not locally connected.*

PROOF: 0 has no connected neighbourhood. □

Proposition 8.5.8. *The Sorgenfrey plane is not locally connected.*

PROOF: Any basic open set $[a, b) \times [c, d)$ can be separated into $[a, b) \times [c, e)$ and $[a, b) \times [e, d)$ for some $c < e < d$. □

Proposition 8.5.9. *The space $\mathbb{R}^\omega$ under the uniform topology is locally connected.*

PROOF: For any neighbourhood U of a point x, the neighbourhood $U \cap \{y : y - x \text{ is bounded}\}$ is connected. □

Proposition 8.5.10. *The space $\mathbb{R}_K$ is not locally connected.*

PROOF: The open set $(-1, 1) - K$ does not include a connected neighbourhood of 0. □

Proposition 8.5.11. *Every open subspace of a locally connected space is locally connected.*

PROOF: Follows easily from definition. ∎

Proposition 8.5.12 (AC)**.** *The product of a family of locally connected spaces is locally connected.*

PROOF:
⟨1⟩1. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of locally connected spaces and $\vec{x} \in \prod_{\alpha \in J} X_\alpha$
⟨1⟩2. LET: $\prod_{\alpha \in J} U_\alpha$ be any basic neighbourhood of $\vec{x}$, where each U_α is open in X_α, and $U_\alpha = X_\alpha$ except for $\alpha = \alpha_1, \dots, \alpha_n$
⟨1⟩3. For $\alpha \in J$, PICK a connected neighbourhood C_α of x_α with $C_\alpha \subseteq U_\alpha$
⟨1⟩4. $\prod_{\alpha \in J} C_\alpha$ is connected
PROOF: Proposition 8.1.10
∎

Proposition 8.5.13. *Every discrete space is locally connected.*

PROOF: For any point x, the set $\{x\}$ is a connected neighbourhood of x. ∎

Corollary 8.5.13.1. *The continuous image of a locally connected space is not necessarily locally connected.*

8.6 Local Path Connectedness

Definition 8.6.1 (Locally Path Connected)**.** Let X be a topological space and $x \in X$. Then X is *locally path connected at x* iff every neighbourhood of x includes a path connected neighbourhood of x.

The space X is *locally path connected* iff it is locally path connected at every point.

Proposition 8.6.2. S_Ω *is not locally path connected.*

PROOF: There is no path connected neighbourhood of ω. ∎

Proposition 8.6.3. $\overline{S_\Omega}$ *is not locally path connected.*

PROOF: There is no path connected neighbourhood of ω. ∎

Proposition 8.6.4. *Not every closed subspace of a locally path connected space is locally path connected.*

PROOF: The topologist's sine curve is not loally path connected. ∎

Proposition 8.6.5. *Every open subspace of a locally path connected space is locally path connected.*

PROOF: Follows easily from definition. ∎

Proposition 8.6.6. *Every locally path connected space is locally connected.*

PROOF: From Proposition 8.3.3. ∎

Corollary 8.6.6.1. *$\mathbb{R}_l$ is not locally path connected.*

Corollary 8.6.6.2. *The Sorgenfrey plane is not locally path connected.*

Corollary 8.6.6.3. *The space $\mathbb{R}^\omega$ under the box topology is not locally path connected.*

Corollary 8.6.6.4. *The space $\mathbb{R}_K$ is not locally path connected.*

Corollary 8.6.6.5. *The topologist's sine curve is not locally path connected.*

Proposition 8.6.7 (AC)**.** *The product of a family of locally path connected spaces is locally path connected.*

PROOF:
⟨1⟩1. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of locally connected spaces and $\vec{x} \in \prod_{\alpha \in J} X_\alpha$
⟨1⟩2. LET: $\prod_{\alpha \in J} U_\alpha$ be any basic neighbourhood of $\vec{x}$, where each U_α is open in X_α, and $U_\alpha = X_\alpha$ except for $\alpha = \alpha_1, \ldots, \alpha_n$
⟨1⟩3. For $\alpha \in J$, PICK a path connected neighbourhood C_α of x_α with $C_\alpha \subseteq U_\alpha$
⟨1⟩4. $\prod_{\alpha \in J} C_\alpha$ is path connected
PROOF: Proposition **??**
∎

Proposition 8.6.8. *Let X be a topological space. Then X is locally path connected if and only if, for every open set U in X, every path component of U is open in X.*

PROOF:
⟨1⟩1. If X is locally path connected then, for every open set U in X, every path component of U is open in X.
⟨2⟩1. ASSUME: X is locally path connected.
⟨2⟩2. LET: U be open in X.
⟨2⟩3. LET: C be a path component of U.
⟨2⟩4. LET: $x \in C$
PROVE: C is a neighbourhood of x
⟨2⟩5. U is a neighbourhood of x in X.
PROOF: From ⟨2⟩2, ⟨2⟩3 and ⟨2⟩4.
⟨2⟩6. PICK a path connected neighbourhood V of x such that $V \subseteq U$.
PROOF: Using ⟨2⟩1.
⟨2⟩7. $V \subseteq C$
PROOF: By Proposition 8.3.7.
⟨2⟩8. C is a neighbourhood of x
PROOF: By Proposition 3.2.4.
⟨2⟩9. Q.E.D.
PROOF: By Proposition 3.2.3.
⟨1⟩2. If, for every open set U in X, every path component of U is open in X, then X is locally path connected.

⟨2⟩1. Assume: For every open set U in X, every path component of U is open in X.
⟨2⟩2. Let: $x \in X$ and N be a neighbourhood of x
⟨2⟩3. Pick U open such that $x \in U \subseteq N$
⟨2⟩4. Let: C be the path component of U that contains x
⟨2⟩5. C is open in X
Proof: By ⟨2⟩1.
⟨2⟩6. C is a path connected neighbourhood of x that is included in N
□

Theorem 8.6.9 (AC)**.** *Let X be a topological space. If X is locally path connected, then its components and its path components are the same.*

Proof:
⟨1⟩1. Let: P be a path component of X
⟨1⟩2. Let: C be the component such that $P \subseteq C$
Prove: $P = C$
⟨1⟩3. Let: $Q = C \setminus P$
⟨1⟩4. P is open in X
Proof: By Proposition 8.6.8.
⟨1⟩5. Q is open in X
Proof: By Proposition 8.6.8 since Q is the union of the path components included in C other than P.
⟨1⟩6. $Q = \emptyset$
Proof: Otherwise P and Q would form a separation of C, contradicting 8.2.3.
□

Proposition 8.6.10. *$S_\Omega \times \overline{S_\Omega}$ is not locally path connected.*

Proof: (ω, ω) has no path connected neighbourhood. □

Proposition 8.6.11. *The ordered square is not locally path connected.*

Proof:
⟨1⟩1. Assume: for a contradiction $(1/2, 0)$ has a path connected neighbourhod U
⟨1⟩2. Pick $a < 1/2$ such that $((a,1),(1/2,0)) \subseteq U$
⟨1⟩3. Let: $p : [0,1] \to I_o^2$ be a path from $(a,1)$ to $(1/2,0)$
⟨1⟩4. For every x such that $a < x < 1/2$, Pick a rational q_x such that $p(q_x) \in ((x,0),(x,1))$
⟨1⟩5. $\{q_x : a < x < 1/2\}$ is an uncountable set of rationals.
□

Proposition 8.6.12. *For any set I, the space $\mathbb{R}^I$ is locally path connected.*

Proof: Every basic open set is the product of path connected spaces, hence path connected. □

Proposition 8.6.13. *The space $\mathbb{R}^\omega$ under the uniform topology is locally path connected.*

PROOF: Its components and path components are the same. ∎

Proposition 8.6.14. *Every discrete space is locally path connected.*

PROOF: For any point x, the set $\{x\}$ is a path connected neighbourhood of x. ∎

Corollary 8.6.14.1. *The continuous image of a locally path connected space is not necessarily locally path connected.*

8.7 Weak Local Connectedness

Definition 8.7.1 (Weakly Locally Connected)**.** Let X be a topological space and $x \in X$. Then X is *weakly locally connected at* x iff every neighbourhood of x contains a connected subspace that contains a neighbourhood of x.

Chapter 9

Compact Spaces

9.1 Countable Compactness

Definition 9.1.1 (Countably Compact)**.** A topological space is *countably compact* iff every countable open covering has a finite subcovering.

9.2 Limit Point Compactness

Definition 9.2.1 (Limit Point Compact)**.** A space is *limit point compact* iff every infinite set has a limit point.

Proposition 9.2.2 (CC)**.** *$S_\Omega \times \overline{S_\Omega}$ is limit point compact.*

Proof:
⟨1⟩1. Let: $A \subseteq S_\Omega \times \overline{S_\Omega}$ be infinite
⟨1⟩2. Case: $\pi_1(A)$ is finite.
⟨2⟩1. Pick x such that there are infinitely many y such that $(x, y) \in A$
⟨2⟩2. Pick a limit point l of $\{y : (x, y) \in A\}$
⟨2⟩3. (x, l) is a limit point of A
⟨1⟩3. Case: $\pi_1(A)$ is infinite.
⟨2⟩1. Pick a limit point l of $\pi_1(A)$.
⟨2⟩2. l is a limit ordinal
⟨2⟩3. Pick a countable sequence x_n with limit l
⟨2⟩4. For $n \geq 1$, Pick $a_n > x_n$ and y_n such that $(a_n, y_n) \in A$
⟨2⟩5. Case: $\{y_n : n \geq 1\}$ is finite
⟨3⟩1. Pick y such that $y = y_n$ for infinitely many n
⟨3⟩2. (l, y) is a limit point for A
¡
⟨2⟩6. Case: $\{y_n : n \geq 1\}$ is infinite
⟨3⟩1. Pick a limit point m for $\{y_n : n \geq 1\}$
⟨3⟩2. (l, m) is a limit point for A
□

Proposition 9.2.3. *The Sorgenfrey plane is not limit point compact.*

PROOF: $\mathbb{Z}^2$ has no limit point. □

Proposition 9.2.4. *The space $\mathbb{R}^\omega$ under the box topology is not limit point compact.*

PROOF:The set of all constant sequences of integers is an infinite set with no limit point. □

Proposition 9.2.5. *Not every open subspace of a limit point compact space is limit point compact.*

PROOF: The space $[0, 1]$ is limit point compact but $(0, 1)$ is not. □

Proposition 9.2.6. *The product of two limit point compact spaces is not necessarily limit point compact.*

PROOF: See Steen and Seebach *Counterexamples in Topology* Example 112. □

Proposition 9.2.7. *The continuous image of a limit point comapct space is not necessarily limit point comapct.*

PROOF: Let Y be a two-point set under the indiscrete topology. Then $\mathbb{N} \times Y$ is limit point compact, but $\mathbb{N}$ is not. □

9.3 Lindelöf Spaces

Definition 9.3.1 (Lindelöf Space)**.** A topological space X is *Lindelöf* iff every open covering has a countable subcovering.

Theorem 9.3.2 (CC)**.** *Every second countable space is Lindelöf.*

PROOF:
⟨1⟩1. LET: X be a second countable space
⟨1⟩2. PICK a countable basis $\mathcal{B}$ for X.
⟨1⟩3. LET: $\mathcal{A}$ be an open cover of X
⟨1⟩4. For every $B \in \mathcal{B}$ such that there exists $U \in \mathcal{A}$ such that $B \subseteq U$, PICK $U_B \in \mathcal{A}$ such that $B \subseteq U_B$
⟨1⟩5. $\{U_B : B \in \mathcal{B}, \exists U \in \mathcal{A}.B \subseteq U\}$ covers X.
  ⟨2⟩1. LET: $x \in X$
  ⟨2⟩2. PICK $U \in \mathcal{A}$ such that $x \in U$
  ⟨2⟩3. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$
  ⟨2⟩4. $x \in U_B$
□

Corollary 9.3.2.1. *The space $\mathbb{R}^\omega$ is Lindelöf.*

Corollary 9.3.2.2. *The space $\mathbb{R}_K$ is Lindelöf.*

Proposition 9.3.3. *The space S_Ω is not Lindelöf.*

PROOF:$\{(-\infty, \alpha) : \alpha \in S_\Omega\}$ is an open cover that has no countable subcover. □

Proposition 9.3.4 (CC)**.** *The space $\overline{S_\Omega}$ is Lindelöf.*

PROOF:
⟨1⟩1. LET: $\mathcal{A}$ be an open cover of $\overline{S_\Omega}$
⟨1⟩2. PICK $U \in \mathcal{A}$ such that $\Omega \in U$
⟨1⟩3. PICK $\alpha < \Omega$ such that $(\alpha, \Omega] \subseteq U$
⟨1⟩4. For $\beta \leq \alpha$, PICK $U_\beta \in \mathcal{A}$ such that $\beta \in U_\beta$
⟨1⟩5. $\{U\} \cup \{U_\beta : \beta \leq \alpha\}$ is a countable subcover of $\mathcal{A}$.
□

Proposition 9.3.5 (CC)**.** *The continuous image of a Lindelöf space is Lindelöf.*

PROOF:
⟨1⟩1. LET: X be a Lindelöf space, Y a space and $f : X \to Y$ continuous.
⟨1⟩2. LET: $\mathcal{A}$ be an open covering of Y
⟨1⟩3. $\{f^{-1}(V) : V \in \mathcal{A}\}$ is an open covering of X
⟨1⟩4. PICK a countable subcovering $\{f^{-1}(V_1), f^{-1}(V_2), \ldots\}$ of $\{f^{-1}(V) : V \in \mathcal{A}\}$
⟨1⟩5. $\{V_1, V_2, \ldots\}$ is a countable subcovering of $\mathcal{A}$
□

Proposition 9.3.6. *The Sorgenfrey plane is not Lindelöf.*

PROOF:
⟨1⟩1. LET: $L = \{(x, -x) : x \in \mathbb{R}\}$
⟨1⟩2. L is closed in $\mathbb{R}_l^2$
⟨2⟩1. LET: $(x, y) \notin L$, so $y \neq -x$
PROVE: There exists a neighbourhood U of (x, y) that does not intersect L
⟨2⟩2. CASE: $y > -x$
PROOF: In this case, take $U = [x, +\infty) \times [y, +\infty)$
⟨2⟩3. CASE: $y < -x$
PROOF: In this case, take $U = [x, (x - y)/2) \times [y, (y - x)/2)$.
⟨1⟩3. LET: $\mathcal{U} = \{\mathbb{R}_l^2 \setminus L\} \cup \{[a, b) \times [-a, d) : a, b, d \in \mathbb{R}\}$
⟨1⟩4. $\mathcal{U}$ is an open covering of $\mathbb{R}_l^2$
⟨1⟩5. No countable subset of $\mathcal{U}$ covers $\mathbb{R}_l^2$
PROOF: Every set $[a, b) \times [-a, d)$ intersects L in exactly one point, namely $(a, -a)$.
□

Corollary 9.3.6.1. *The Sorgenfrey plane is not second countable.*

Corollary 9.3.6.2. *The product of two Lindelöf spaces is not necessarily Lindelöf.*

Proposition 9.3.7. *The space $\mathbb{R}^\omega$ under the box topology is not Lindelöf.*

PROOF: The set $\{\prod_{n=0}^{\infty}(a_n, a_n + 1) : \forall n.a_n \in \mathbb{Z}\}$ covers the space but has no countable subcover. $\square$

Proposition 9.3.8. *Not every open subspace of a Lindelöf space is Lindelöf.*

PROOF: The ordered square is Lindelöf but the subspace $[0, 1]times(0, 1)$ is not. $\square$

9.4 Compactness

Definition 9.4.1 (Compact)**.** A topological space is *compact* iff every open cover has a finite subcover.

Proposition 9.4.2. *S_Ω is not compact.*

PROOF: The open covering $\{(-\infty, \alpha) : \alpha \in S_\Omega\}$ has no finite subcovering. $\square$

Proposition 9.4.3. *$\mathbb{R}_l$ is not compact.*

PROOF: $\{[n, n + 1) : n \in \mathbb{Z}\}$ has no finite subcover. $\square$

Proposition 9.4.4. *The space $\mathbb{R}^\omega$ under the box topology is not compact.*

PROOF: The set $\{\prod_{n=0}^{\infty}(a_n, a_n+1) : n \in \mathbb{Z}\}$ is a cover that has no finite subcover. $\square$

Proposition 9.4.5. *Let Y be a subspace of X. Then Y is compact if and only if every covering of Y by sets open in X contains a finite subcollection covering Y.*

PROOF:
⟨1⟩1. If Y is compact then every covering of Y by sets open in X contains a finite subcollection covering Y.
  ⟨2⟩1. ASSUME: Y is compact.
  ⟨2⟩2. LET: $\mathcal{A}$ be a covering of Y by sets open in X.
  ⟨2⟩3. $\{U \cap Y : U \in \mathcal{A}\}$ is an open covering of Y.
  ⟨2⟩4. PICK a finite subcovering $V_1, \ldots, V_n$ of $\{U \cap Y : U \in \mathcal{A}\}$
  ⟨2⟩5. For $1 \leq i \leq n$, PICK $U_i \in \mathcal{A}$ such that $V_i = U_i \cap Y$.
  ⟨2⟩6. $\{U_1, \ldots, U_n\}$ is a finite subset of $\mathcal{A}$ that covers Y.
⟨1⟩2. If every covering of Y by sets open in X contains a finite subcollection covering Y then Y is compact.
  ⟨2⟩1. ASSUME: Every covering of Y by sets open in X contains a finite subcollection covering Y.
  ⟨2⟩2. LET: $\mathcal{A}$ be an open covering of Y
  ⟨2⟩3. LET: $\mathcal{B} = \{U$ open in $X : U \cap Y \in \mathcal{A}\}$
  ⟨2⟩4. $\mathcal{B}$ covers Y
  ⟨2⟩5. PICK a finite subcollection $\{U_1, \ldots, U_n\} \subseteq \mathcal{B}$ that covers Y
  ⟨2⟩6. $\{U_1 \cap Y, \ldots, U_n \cap Y\}$ is a finite subcover of $\mathcal{A}$.
$\square$

Proposition 9.4.6. *Every closed subspace of a compact space is compact.*

PROOF:
$\langle 1\rangle 1.$ LET: X be a compact space and $Y \subseteq X$ be closed.
$\langle 1\rangle 2.$ LET: $\mathcal{A}$ be a covering of Y by spaces open in X
$\langle 1\rangle 3.$ $\mathcal{A} \cup \{X \setminus Y\}$ is an open covering of X.
$\langle 1\rangle 4.$ PICK a finite subcovering $\{U_1, \ldots, U_n\}$ or $\{U_1, \ldots, U_n, X \setminus Y\}$
$\langle 1\rangle 5.$ $\{U_1, \ldots, U_n\}$ is a finite subset of $\mathcal{A}$ that covers Y.
$\langle 1\rangle 6.$ Q.E.D.
PROOF: Proposition 9.4.5.
□

Corollary 9.4.6.1. *Not every compact Hausdorff space is connected.*

PROOF: The space $[0, 1] \cup [2, 3]$ is compact Hausdorff and disconnected. □

Corollary 9.4.6.2. *Not every compact Hausdorff space is path connected.*

Corollary 9.4.6.3. *Not every compact Hausdorff space is locally connected.*

The space $[0, 1] \cap \mathbb{Q}$ is not locally connected.

Corollary 9.4.6.4. *Not every compact Hausdorff space is locally path connected.*

Proposition 9.4.7. *Not every open subspace of a compact space is compact.*

PROOF: The space $[0, 1]$ is compact but $(0, 1)$ is not. □

Lemma 9.4.8. *If Y is a compact subspace of the Hausdorff space X and $a \notin Y$, then there exist disjoint open sets U and V of X containing a and Y, respectively.*

PROOF:
$\langle 1\rangle 1.$ For $y \in Y$, there exist disjoint open sets U and V such that $a \in U$ and $y \in V$.
$\langle 1\rangle 2.$ $\{V \text{ open in } X : \exists U \text{ open and disjoint from } V.a \in U\}$ is a covering of Y by open sets in X.
$\langle 1\rangle 3.$ PICK a finite subset $\{V_1, \ldots, V_n\}$ that covers Y.
$\langle 1\rangle 4.$ For $1 \leq i \leq n$, PICKU_i disjoint from V_i such that $a \in U_i$
$\langle 1\rangle 5.$ LET: $U = U_1 \cap \cdots \cap U_n$ and $V = V_1 \cup \cdots \cup V_n$
□

Proposition 9.4.9. *Every compact subspace of a Hausdorff space is closed.*

PROOF:
$\langle 1\rangle 1.$ LET: X be a Hausdorff space and $Y \subseteq X$ be compact.
$\langle 1\rangle 2.$ Every point $a \notin Y$ has an open neighbourhood disjoint from Y.
PROOF: By Lemma 9.4.8.
$\langle 1\rangle 3.$ Q.E.D.
PROOF: By Proposition 3.2.3.

Proposition 9.4.10. *The image of a compact space under a continuous map is compact.*

PROOF:
$\langle 1 \rangle 1$. LET: $f : X \to Y$ be continuous where X is compact.
$\langle 1 \rangle 2$. LET: $\mathcal{A}$ be a covering of $f(X)$ by open sets in Y.
$\langle 1 \rangle 3$. $\{f^{-1}(U) : U \in \mathcal{A}\}$ is an open covering of X.
$\langle 1 \rangle 4$. PICK a finite subcovering $\{f^{-1}(U_1), \ldots, f^{-1}(U_n)\}$
$\langle 1 \rangle 5$. $\{U_1, \ldots, U_n\}$ is a finite subset of $\mathcal{A}$ that covers $f(X)$.
$\langle 1 \rangle 6$. Q.E.D.
PROOF: By Proposition 9.4.5.
□

Corollary 9.4.10.1. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces. If $\prod_{\alpha \in J} X_\alpha$ is compact then each X_α is compact.*

Corollary 9.4.10.2. *$S_\Omega \times \overline{S_\Omega}$ is compact.*

Corollary 9.4.10.3. *The Sorgenfrey plane is not compact.*

Corollary 9.4.10.4. *For any nonempty set I, the sapce $\mathbb{R}^I$ is not compact.*

Corollary 9.4.10.5. *Let $\mathcal{T}$ and $\mathcal{T}'$ be topologies on the same set X. If $\mathcal{T} \subseteq \mathcal{T}'$ and $\mathcal{T}'$ is compact then $\mathcal{T}$ is compact.*

Corollary 9.4.10.6. *The space $\mathbb{R}_K$ is not compact.*

Theorem 9.4.11. *Let $f : X \to Y$ be a bijective continuous function. If X is compact and Y is Hausdorff then f is a homeomorphism.*

PROOF:
$\langle 1 \rangle 1$. LET: C be closed in X
$\langle 1 \rangle 2$. C is compact
PROOF: Proposition 9.4.6.
$\langle 1 \rangle 3$. $f(C)$ is compact
PROOF: Proposition 9.4.10
$\langle 1 \rangle 4$. $f(C)$ is closed
PROOF: Proposition 9.4.9.
$\langle 1 \rangle 5$. Q.E.D.
PROOF: By Theorem 5.2.2 we have that f^{-1} is continuous.
□

Corollary 9.4.11.1. *Let $\mathcal{T}$ and $\mathcal{T}'$ be topologies on the same set X. If $\mathcal{T} \subseteq \mathcal{T}'$, $\mathcal{T}$ is Hausdorff and $\mathcal{T}'$ is compact then $\mathcal{T} = \mathcal{T}'$.*

Corollary 9.4.11.2. *The space $[0, 1]$ is not compact as a subspace of $\mathbb{R}_K$.*

Theorem 9.4.12 (Tube Lemma)**.** *Let A and B be subspaces of X and Y, respectively; let N be an open set in $X \times Y$ including $A \times B$. If A and B are compact, then there exist open sets U and V in X and Y, respectively, such that*

$$A \times B \subseteq U \times V \subseteq N \ .$$

PROOF:
⟨1⟩1. For all $a \in A$, there exist open sets U and V in X and Y, respectively, such that
$$\{a\} \times B \subseteq U \times V \subseteq N \ .$$
⟨2⟩1. LET: $a \in A$
⟨2⟩2. For all $b \in B$, there exist open sets U and V in X and Y, respectively, such that $(a, b) \in U \times V \subseteq N$.
⟨2⟩3. $\{V \text{ open in } Y : \exists U \text{ open in } X.a \in U, U \times V \subseteq N\}$ covers B
⟨2⟩4. PICK a finite subset $\{V_1, \ldots, V_n\}$ that covers B.
⟨2⟩5. For $1 \leq i \leq n$, PICK U_i open in X such that $a \in U_i$ and $U_i \times V_i \subseteq N$
⟨2⟩6. LET: $U = U_1 \cap \cdots \cap U_n$ and $V = V_1 \cup \cdots \cup V_n$
⟨1⟩2. $\{U \text{ open in } X : \exists V \text{ open in } Y.B \subseteq V \text{ and } U \times V \subseteq N\}$ covers A.
⟨1⟩3. PICK a finite subset $\{U_1, \ldots, U_n\}$ that covers A.
⟨1⟩4. For $1 \leq i \leq n$, PICK V_i open in B such that $B \subseteq V_i$ and $U_i \times V_i \subseteq N$.
⟨1⟩5. LET: $U = U_1 \cup \cdots \cup U_n$ and $V = V_1 \cap \cdots \cap V_n$
⟨1⟩6. $A \times B \subseteq U \times V \subseteq N$
□

Lemma 9.4.13. *Let $\mathcal{A}$ be a set of basis elements for $X \times Y$ such that no finite subset of $\mathcal{A}$ covers $X \times Y$. If X is compact, then there exists a point $x \in X$ such that no finite subset of $\mathcal{A}$ covers $\{x\} \times Y$.*

PROOF:
⟨1⟩1. ASSUME: X is compact.
⟨1⟩2. ASSUME: For all $x \in X$, there is a finite subset of $\mathcal{A}$ that covers $\{x\} \times Y$
PROVE: A finite subset of $\mathcal{A}$ covers $X \times Y$
⟨1⟩3. $\{U \text{ open in } X : \exists U_1 \times V_1, \ldots, U_r \times V_r \in \mathcal{A}.U = U_1 \cap \cdots \cap U_r, Y = V_1 \cup \cdots \cup V_r\}$ covers X.
⟨1⟩4. PICK a finite subcover $\{U_1, \ldots, U_n\}$
⟨1⟩5. For $1 \leq i \leq n$, PICK$U_{i1} \times V_{i1}, \ldots, U_{ir_i} \times V_{ir_i} \in \mathcal{A}$ such that $U_i = U_{i1} \cap \cdots \cap U_{ir_i}$ and $Y = V_{i1} \cup \cdots \cup V_{ir_i}$
⟨1⟩6. $\{U_{ij} : 1 \leq i \leq n, 1 \leq j \leq r_i\}$ covers $X \times Y$
□

Proposition 9.4.14. *The product of two compact spaces is compact.*

PROOF:
⟨1⟩1. LET: X and Y be compact spaces.
⟨1⟩2. LET: $\mathcal{A}$ be an open covering of $X \times Y$
⟨1⟩3. For all $x \in X$, there exists a neighbourhood W of x such that $W \times Y$ is covered by finitely many elements of $\mathcal{A}$.
⟨2⟩1. LET: $x \in X$
⟨2⟩2. $\{x\} \times Y$ is compact.
PROOF: It is homeomorphic to Y.
⟨2⟩3. PICK a finite subset $\{U_1, \ldots, U_m\}$ of $\mathcal{A}$ that covers $\{x\} \times Y$
PROOF: By Proposition 9.4.5.
⟨2⟩4. There exists a neighbourhood W of x such that $W \times Y \subseteq U_1 \cup \cdots \cup U_m$

PROOF: By the Tube Lemma.

⟨1⟩4. $\{W$ open in $X : W \times Y$ is covered by finitely many elements of $\mathcal{A}\}$ is an open covering of X.

⟨1⟩5. PICK a finite subcovering $\{W_1, \ldots, W_n\}$

⟨1⟩6. For $1 \leq i \leq n$, PICK a finite subset $\{U_{i1}, \ldots, U_{ir_i}\}$ of $\mathcal{A}$ that covers $W_i \times Y$

⟨1⟩7. $\{U_{11}, \ldots, U_{nr_n}\}$ is a finite subcovering of $\mathcal{A}$.

□

Proposition 9.4.15. *A topological space is compact if and only if every nonempty set of closed sets that has the finite intersection property has nonempty intersection.*

PROOF: Immediate from definitions. □

Lemma 9.4.16. *If Y is compact then $\pi_1 : X \times Y \to X$ is a closed map.*

PROOF:

⟨1⟩1. LET: $C \subseteq X \times Y$ be closed

⟨1⟩2. LET: $x \in X \setminus \pi_1(C)$

⟨1⟩3. For all $y \in Y$, we have $(x, y) \notin C$

⟨1⟩4. For all $y \in Y$, there exist open neighbourhoods U of x and V of y such that $U \times V \subseteq (X \times Y) \setminus C$

⟨1⟩5. $\{V$ open in $Y : \exists U$ an open neighbourhood of x such that $U \times V \subseteq (X \times Y) \setminus C\}$ is an open covering of Y.

⟨1⟩6. PICK a finite subcovering $\{V_1, \ldots, V_n\}$

⟨1⟩7. For $1 \leq i \leq n$, PICK an open neighbourhood U_i of x such that $U_i \times V_i \subseteq (X \times Y) \setminus C$

⟨1⟩8. $x \in U_1 \cap \cdots \cap U_n \subseteq X \setminus \pi_1(C)$

□

Theorem 9.4.17. *Let X be a compact space. Let $f_n : X \to \mathbb{R}$ be a sequence of continuous functions such that, for all $x \in X$, $f_n(x) \to f(x)$ as $n \to \infty$. If f is continuous, and if the sequence $(f_n)_n$ is monotone increasing, and if X is compact, then the convergence is uniform.*

PROOF:

⟨1⟩1. LET: $\epsilon > 0$

PROVE: There exists N such that, for all $n \geq N$, we have $|f_n(x) - f(x)| < \epsilon$

⟨1⟩2. For $n \in \mathbb{Z}^+$,

LET: $U_n = \{x \in X : f(x) - f_n(x) < \epsilon\}$

⟨1⟩3. Each U_n is open

PROOF: Let $g(x) = f(x) - f_n(x)$. Then g is continuous and $U_n = g^{-1}((-\infty, \epsilon))$.

⟨1⟩4. $\{U_n : n \geq 1\}$ is an open covering of X

⟨2⟩1. LET: $x \in X$

⟨2⟩2. PICK N such that, for all $n \geq N$, $|f(x) - f_n(x)| < \epsilon$

PROOF: $f_n(x) \to f(x)$ as $n \to \infty$

⟨2⟩3. $f(x) - f_N(x) < \epsilon$

PROOF: This holds since the sequece $(f_n)_n$ is monotone.

$\langle 1\rangle 5.$ PICK a finite subcovering $\{U_{n_1}, \ldots, U_{n_k}\}$

$\langle 1\rangle 6.$ LET: $N = \max(n_1, \ldots, n_k)$

$\langle 1\rangle 7.$ For all $n \geq N$ we have $|f_n(x) - f(x)| < \epsilon$

□

Lemma 9.4.18. *Every compact Hausdorff space is normal.*

PROOF:

$\langle 1\rangle 1.$ LET: A and B be disjoint closed sets in the compact Hausdorff space X.

$\langle 1\rangle 2.$ For all $a \in A$, there exist disjoint open sets U and V such that $a \in U$ and $B \subseteq V$.

PROOF: By Lemma 9.4.8.

$\langle 1\rangle 3.$ $\{U \text{ open in } X : \exists V \text{ open in } Y.U \cap V = \emptyset, B \subseteq V\}$ is an open covering of A

$\langle 1\rangle 4.$ PICK a finite subcovering $\{U_1, \ldots, U_n\}$

$\langle 1\rangle 5.$ For $1 \leq i \leq n$, PICK V_i open in Y such that $U_i \cap V_i = \emptyset$ and $B \subseteq V_i$

$\langle 1\rangle 6.$ LET: $U = U_1 \cup \cdots \cup U_n$ and $V = V_1 \cap \cdots \cap V_n$

□

Theorem 9.4.19. *Let X be a complete linearly ordered set under the order topology. Then every closed interval in X is compact.*

PROOF:

$\langle 1\rangle 1.$ LET: X be a complete linearly ordered set in the order topology

$\langle 1\rangle 2.$ LET: $a, b \in X$, $a < b$

PROVE: $[a, b]$ is compact

$\langle 1\rangle 3.$ LET: $\mathcal{A}$ be a set of open sets that covers $[a, b]$

$\langle 1\rangle 4.$ For all $x \in [a, b)$, there exists $y \in (x, b]$ such that $[x, y]$ is covered by at most two points of $\mathcal{A}$

$\langle 2\rangle 1.$ LET: $x \in [a, b]$

$\langle 2\rangle 2.$ PICK $U \in \mathcal{A}$ such that $x \in U$

PROOF: By $\langle 1\rangle 3$ and $\langle 2\rangle 1$

$\langle 2\rangle 3.$ PICK $y \in (x, b]$ such that $[x, y) \subseteq U$

PROOF: By Lemma 4.1.2.

$\langle 2\rangle 4.$ PICK $V \in \mathcal{A}$ such that $y \in V$

PROOF: By $\langle 1\rangle 3$ and $\langle 2\rangle 3$.

$\langle 2\rangle 5.$ $[x, y]$ is covered by $\{U, V\}$

PROOF: By $\langle 2\rangle 3$ and $\langle 2\rangle 4$.

$\langle 1\rangle 5.$ LET: $C = \{y \in (a, b] : [a, y] \text{ is covered by a finite subset of } \mathcal{A}\}$

$\langle 1\rangle 6.$ C is nonempty

PROOF: By $\langle 1\rangle 4$.

$\langle 1\rangle 7.$ LET: $c = \sup C$

PROOF: By $\langle 1\rangle 1$.

$\langle 1\rangle 8.$ $c \in C$

$\langle 2\rangle 1.$ PICK $U \in \mathcal{A}$ such that $c \in U$

$\langle 2\rangle 2.$ PICK $y \in [a, c)$ such that $(y, c] \subseteq U$

PROOF: By Lemma 4.1.2
⟨2⟩3. PICK z such that $y < z$ and $z \in C$
PROOF: This exists because y is not an upper bound for C.
⟨2⟩4. PICK a finite $\mathcal{A}_0 \subseteq \mathcal{A}$ such that $[a, z]$ is covered by $\mathcal{A}_0$
⟨2⟩5. $[a, c]$ is covered by $\mathcal{A}_0 \cup \{U\}$
⟨1⟩9. $c = b$
⟨2⟩1. ASSUME: for a contradiction $c < b$
⟨2⟩2. PICK $y \in (c, b]$ such that $[c, y]$ is covered by at most two elements of $\mathcal{A}$.
PROOF: By ⟨1⟩4
⟨2⟩3. $y > c$ and $y \in C$
⟨2⟩4. Q.E.D.
PROOF: This contradicts ⟨1⟩7.
⟨1⟩10. Q.E.D.

Corollary 9.4.19.1. *Every closed interval in $\mathbb{R}$ is compact.*

Corollary 9.4.19.2 (CC)**.** *S_Ω is limit point compact.*

PROOF:
⟨1⟩1. LET: A be an infinite subset of S_Ω
⟨1⟩2. PICK a countably infinite subset $B \subseteq A$
⟨1⟩3. LET: $b = \sup B$
⟨1⟩4. $B \subseteq [0, b]$
⟨1⟩5. $[0, b]$ is compact
PROOF: By the theorem.
⟨1⟩6. B has a limit point in $[0, b]$
⟨1⟩7. A has a limit point in $[0, b]$
□

Corollary 9.4.19.3. *The ordered square is compact.*

Corollary 9.4.19.4. *The ordered square is limit point compact.*

Corollary 9.4.19.5. *Not every subspace of a compact space is compact.*

PROOF: $[0, 1]$ is compact but $(0, 1)$ is not. □

Theorem 9.4.20 (Extreme Value Theorem)**.** *Let $f : X \to Y$ be continuous where Y is a linearly ordered set in the order topology. If X is compact, then there exist $c, d \in X$ such that, for all $x \in X$, we have $f(c) \leq f(x) \leq f(d)$.*

PROOF:
⟨1⟩1. $f(X)$ is compact.
PROOF: By Proposition 9.4.10.
⟨1⟩2. $f(X)$ has a greatest element.
⟨2⟩1. ASSUME: for a contradiction $f(X)$ has no greatest element.
⟨2⟩2. $\{(-\infty, f(x)) : x \in X\}$ is a set of open sets that covers $f(X)$.
⟨2⟩3. PICK a finite subset $\{(-\infty, f(x_1)), \ldots, (-\infty, f(x_n))\}$ that covers $f(X)$.
PROOF: By Proposition 9.4.5

⟨2⟩4. Let: $f(x_N)$ be largest out of $f(x_1)$, …, $f(x_n)$
⟨2⟩5. $f(x_N) < f(x_N)$
⟨2⟩6. Q.E.D.
Proof: This is a contradiction.
⟨1⟩3. $f(X)$ has a least element.
Proof: Similar.
□

Theorem 9.4.21 (DC). *A nonempty compact Hausdorff space with no isolated points is uncountable.*

Proof:
⟨1⟩1. Let: X be a nonempty compact Hausdorff space with no isolated points.
⟨1⟩2. For every nonempty open $U \subseteq X$ and point $x \in X$, there exists a nonempty open $V \subseteq U$ such that $x \notin \overline{V}$
⟨2⟩1. Let: $U \subseteq X$ be nonempty and open and $x \in X$
⟨2⟩2. Pick $y \in U$ such that $y \neq x$
Proof: This is possible because $U \neq \{x\}$ since x is not an isolated point.
⟨2⟩3. Pick disjoint open neighbourhoods W_1 and W_2 of x and y
Proof: Since X is Hausdorff
⟨2⟩4. Let: $V = U \cap W_2$
⟨2⟩5. $x \notin \overline{V}$
Proof: We have $\overline{V} \subseteq \overline{W_2} \subseteq X \setminus W_1$.
⟨1⟩3. Let: $f : \mathbb{Z}^+ \to X$
Prove: f is not surjective
⟨1⟩4. Pick a sequence of open sets $V_1 \supseteq V_2 \supseteq \cdots$ such that $f(n) \notin \overline{V_n}$
Proof: By ⟨1⟩2 and Dependent Choice.
⟨1⟩5. Pick a point $b \in \bigcap_{i=1}^{\infty} \overline{V_i}$
Proof: By Proposition 9.4.15.
⟨1⟩6. $b \neq f(n)$ for all n
Proof: For each n we have $b \in \overline{V_n}$ (⟨1⟩5) and $f(n) \notin \overline{V_n}$ (⟨1⟩4).
□

Corollary 9.4.21.1. *Every closed interval in $\mathbb{R}$ is uncountable.*

Theorem 9.4.22. *Every compact space is limit point compact.*

Proof:
⟨1⟩1. Let: X be a compact space.
⟨1⟩2. Let: $A \subseteq X$ be a set with no limit points.
Prove: A is finite.
⟨1⟩3. A is closed.
Proof: By Corollary 3.11.3.1.
⟨1⟩4. A is compact.
Proof: By Proposition 9.4.6.
⟨1⟩5. $\{U$ open in $X : U \cap A$ is a singleton$\}$ covers A
⟨2⟩1. Let: $a \in A$

⟨2⟩2. PICK an open neighbourhood U of a such that U does not intersect A at a point other than a

PROOF: One must exist because a is not a limit point of A (⟨1⟩2).

⟨2⟩3. $U \cap A = \{a\}$

⟨1⟩6. PICK a finite subcover $\{U_1, \ldots, U_n\}$

PROOF: By ⟨1⟩4 using Proposition 9.4.5.

⟨1⟩7. For $1 \leq i \leq n$,

LET: $U_i \cap A = \{a_i\}$

⟨1⟩8. $A = \{a_1, \ldots, a_n\}$

□

Proposition 9.4.23. *Let X be a space and $C, D \subseteq X$ be compact. Then $C \cup D$ is compact.*

PROOF:

⟨1⟩1. LET: $\mathcal{A}$ be a set of open sets that covers $C \cup D$

⟨1⟩2. PICK a finite subset $\mathcal{A}_1$ that covers C and a finite subset $\mathcal{A}_2$ that covers D.

⟨1⟩3. $\mathcal{A}_1 \cup \mathcal{A}_2$ is a finite subset of $\mathcal{A}$ that covers $C \cup D$.

⟨1⟩4. Q.E.D.

Theorem 9.4.24. *Every compact Hausdorff space is normal.*

PROOF:

⟨1⟩1. LET: X be a compact Hausdorff space.

⟨1⟩2. LET: A and B be disjoint closed sets in X.

⟨1⟩3. $\{U \text{ open in } X : \exists V \text{ open in } X. B \subseteq V \wedge U \cap V = \emptyset\}$ covers A

⟨2⟩1. B is compact

PROOF: By Proposition 9.4.6.

⟨2⟩2. Q.E.D.

PROOF: By Lemma 9.4.8.

⟨1⟩4. PICK a finite subcover $\{U_1, \ldots, U_n\}$

PROOF: A is compact by Proposition 9.4.6.

⟨1⟩5. For $1 \leq i \leq n$, PICK V_i open in X such that $B \subseteq V_i$ and $U_i \cap V_i = \emptyset$

⟨1⟩6. LET: $U = U_1 \cup \cdots \cup U_n$ and $V = V_1 \cap \cdots \cap V_n$

⟨1⟩7. U and V are disjoint open sets, $A \subseteq U$ and $B \subseteq V$

□

Corollary 9.4.24.1. *The ordered square is normal.*

Proposition 9.4.25. *Not every compact Hausdorff space is first countable.*

PROOF: The space $\overline{S_\Omega}$ is compact Hausdorff but not first countable. □

Corollary 9.4.25.1. *Not every compact Hausdorff space is second countable.*

Theorem 9.4.26 (Tychonoff (AC))**.** *The product of a family of compact spaces is compact.*

PROOF:

⟨1⟩1. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of compact spaces.
LET: $X = \prod_{\alpha \in J} X_\alpha$
⟨1⟩2. LET: $\mathcal{A} \subseteq \mathcal{P}X$ satisfy the finite intersection property.
PROVE: $\bigcap_{A \in \mathcal{A}} \overline{A}$ is nonempty.
⟨1⟩3. PICK a set $\mathcal{D} \subseteq \mathcal{P}X$ that includes $\mathcal{A}$ and is maximal with respect to the finite intersection property.
PROOF: By Lemma 1.6.6.
⟨1⟩4. For $\alpha \in J$, PICK $x_\alpha \in \bigcap_{D \in \mathcal{D}} \overline{\pi_\alpha(D)}$
⟨2⟩1. LET: $\alpha \in J$
⟨2⟩2. $\{\overline{\pi_\alpha(D)} : D \in \mathcal{D}\}$ satisfies the finite intersection property.
⟨2⟩3. Q.E.D.
PROOF: By Proposition 9.4.15
⟨1⟩5. LET: $x = (x_\alpha)_{\alpha \in J}$
⟨1⟩6. For all $D \in \mathcal{D}$ we have $(x_\alpha)_{\alpha \in J} \in \overline{D}$
PROOF:
⟨2⟩1. Every subbasis element containing x intersects every member of $\mathcal{D}$
⟨3⟩1. LET: $\pi_\alpha(U)^{-1}$ be a subbasis element containing x where U is open in X_α
⟨3⟩2. LET: $D \in \mathcal{D}$
⟨3⟩3. U intersects $\pi_\alpha(D)$
⟨2⟩2. Every subbasis element containing x is a member of $\mathcal{D}$
PROOF: By Lemma 1.6.8
⟨2⟩3. Every basis element containing x is a member of $\mathcal{D}$
PROOF: By Lemma 1.6.7
⟨2⟩4. Every basis element containing x intersects every member of $\mathcal{D}$
PROOF: This follows because $\mathcal{D}$ satisfies the finite intersection property.
⟨1⟩7. Q.E.D.
PROOF: By Proposition 9.4.15
□

PROOF:
⟨1⟩1. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of compact spaces and $X = \prod_{\alpha \in J} X_\alpha$.
⟨1⟩2. PICK a well-ordering $<$ of J such that J has a greatest elemeent $\top$
⟨1⟩3. For all $\alpha \in J$ and every family of points $p = \{p_i \in X_i\}_{i \leq \alpha}$,
LET: $Y_\alpha(p) = \{x \in X : \forall i \leq \alpha . x_i = p_i\}$
⟨1⟩4. For all $\beta \in J$ and every family of points $p = \{p_i \in X_i\}_{i < \beta}$,
LET: $Z_\beta(p) = \bigcap_{\alpha < \beta} Y_\alpha = \{x \in X : \forall i < \beta . x_i = p_i\}$
⟨1⟩5. Given $\beta \in J$, a family of points $\{p_i \in X_i\}_{i < \beta}$, and a finite set $\mathcal{A}$ of basis elements that covers $Z_\beta(p)$, there exists $\alpha < \beta$ such that $\mathcal{A}$ covers $Y_\alpha(p)$
⟨2⟩1. ASSUME: (
w.l.o.g. β has no immediate predecessor)
⟨2⟩2. For $A \in \mathcal{A}$,
LET: $J_A = \{i < \beta : \pi_i(A) \neq X_i\}$
⟨2⟩3. LET: α be the largest element of $\bigcup_{A \in \mathcal{A}} J_A$
PROOF: The set has a greatest element because each J_A is finite and $\mathcal{A}$ is finite.

⟨2⟩4. $\mathcal{A}$ covers $Y_\alpha(p)$
⟨3⟩1. LET: $x \in Y_\alpha(p)$
⟨3⟩2. LET: $y \in Z_\beta(p)$ be the point with

$$y_i = \begin{cases} p_i & \text{if } i < \beta \\ x_i & \text{if } i \geq \beta \end{cases}$$

⟨3⟩3. PICK $A \in \mathcal{A}$ such that $y \in A$
⟨3⟩4. $x \in A$
⟨4⟩1. For $i \leq \alpha$ we have $x_i \in \pi_i(A)$
⟨5⟩1. $x_i = p_i$
PROOF: From ⟨3⟩1 and ⟨1⟩3.
⟨5⟩2. $y_i = p_i$
PROOF: From ⟨3⟩2
⟨5⟩3. $y_i \in \pi_i(A)$
PROOF: From ⟨3⟩3.
⟨4⟩2. For $\alpha < i < \beta$ we have $x_i \in \pi_i(A)$
⟨5⟩1. $i \notin J_A$
PROOF: From ⟨2⟩3
⟨5⟩2. $\pi_i(A) = X_i$
PROOF: From ⟨2⟩2
⟨4⟩3. For $i \geq \beta$ we have $x_i \in \pi_i(A)$
⟨5⟩1. $x_i = y_i$
PROOF: By ⟨3⟩2
⟨5⟩2. $y_i \in \pi_i(A)$
PROOF: By ⟨3⟩3

⟨1⟩6. ASSUME: for a contradiction $\mathcal{A}$ is a set of basis elements such that no finite subset covers X
⟨1⟩7. For all $\alpha \in J$ there exists a family of points $\{p_i \in X_i\}_{i \leq \alpha}$ such that no finite subset of $\mathcal{A}$ covers $Y_\alpha(p)$
⟨2⟩1. ASSUME: as induction hypothesis $\beta \in J$ and p_i has been chosen for all $i < \beta$ such that, for all $\alpha < \beta$, no finite subset of $\mathcal{A}$ covers $Y_\alpha(p)$
⟨2⟩2. No finite subset of $\mathcal{A}$ covers $Z_\beta(p)$
PROOF: By ⟨1⟩5
⟨2⟩3. PICK $p_\beta \in X_\beta$ such that no finite subset of $\mathcal{A}$ covers $Z_\beta(p) \times \{p_\beta\} = Y_\beta(p)$
PROOF: By Lemma 9.4.13.
⟨1⟩8. Q.E.D.
PROOF: This is a contradiction since $Y_\top(p) = \{p\}$ and so must be covered by a single element of $\mathcal{A}$.
□

Theorem 9.4.27. *In a compact Hausdorff space, the components and the quasicomponents coincide.*

PROOF:
⟨1⟩1. LET: X be a compact Hausdorff space and $x, y \in X$ lie in the same

quasicomponent.

PROVE: x and y are in the same component.

⟨1⟩2. LET: $\mathcal{A}$ be the set of all closed subspaces A of X such that x and y lie in the same quasicomponent of A.

⟨1⟩3. Every chain in $\mathcal{A}$ has a lower bound.

⟨2⟩1. LET: $\mathcal{B} \subseteq \mathcal{A}$ be a chain

PROVE: $Y = \bigcap \mathcal{B} \in \mathcal{A}$

⟨2⟩2. ASSUME: for a contradiction $Y = C \cup D$ were C and D are disjoint and open in Y, $x \in C$ and $y \in D$

⟨2⟩3. PICK disjoint open sets U and V in X such that $C \subseteq U$ and $D \subseteq V$

PROOF: By Lemma 9.4.18.

⟨2⟩4. $\{B \setminus (U \cup V) : B \in \mathcal{B}\}$ satisfies the finite intersection property.

⟨3⟩1. LET: $B_1, \dots, B_n \in \mathcal{B}$

⟨3⟩2. $B_1 \cap \cdots \cap B_n \in \mathcal{B}$

PROOF: By ⟨2⟩1.

⟨3⟩3. $B_1 \cap \cdots \cap B_n \setminus (U \cap V)$ is nonempty

PROOF: $B_1 \cap \cdots \cap B_n \cap U$ and $B_1 \cap \cdots \cap B_n \cap V$ cannot be disjoint, because x and y are in the same quasicomponent of $B_1 \cap \cdots \cap B_n$.

⟨2⟩5. $Y \setminus (U \cup V)$ is nonempty.

PROOF: By Proposition 9.4.15.

⟨2⟩6. Q.E.D.

PROOF: This is a contradiction since $Y \setminus (U \cup V) = Y \setminus (C \cup D)$.

⟨1⟩4. PICK a minimal element $D \in \mathcal{A}$

PROOF: One exists by Zorn's Lemma.

⟨1⟩5. D is connected.

⟨2⟩1. ASSUME: [
for a contradiction $D = U \uplus V$ is a separation of D]

⟨2⟩2. CASE: $x, y \in U$

PROOF: In this case we have $U \in \mathcal{A}$ contradicting the minimality of D.

⟨2⟩3. CASE: $x \in U, y \in V$

PROOF: This is a contradiction because x and y are in the same quasicomponent of D.

⟨2⟩4. CASE: $x \in V, y \in U$

PROOF: Similar to ⟨2⟩3.

⟨2⟩5. CASE: $x, y \in V$

PROOF: Similar to ⟨2⟩2.

□

9.5 Perfect Maps

Proposition 9.5.1. *Let $p : X \twoheadrightarrow Y$ be a closed continuous surjective map. For all $y \in Y$ and U an open neighbourhood of $p^{-1}(y)$, there exists an open neighbourhood W of y such that $p^{-1}(W) \subseteq U$.*

PROOF: Take $W = Y \setminus p(X \setminus U)$. □

Proposition 9.5.2 (AC)**.** *Let $p : X \twoheadrightarrow Y$ be a closed continuous surjective map. If X is normal then Y is normal.*

Proof:
$\langle 1\rangle 1.$ Let: $A, B \subseteq Y$ be closed
$\langle 1\rangle 2.$ $p^{-1}(A)$, $p^{-1}(B)$ are closed in X.
$\langle 1\rangle 3.$ Pick disjoint open sets U, V of $p^{-1}(A)$, $p^{-1}(B)$ respectively.
$\langle 1\rangle 4.$ For all $a \in A$, Pick an open neighbourhood W_a of a such that $p^{-1}(W_a) \subseteq U$
Proof: By Proposition 9.5.1.
$\langle 1\rangle 5.$ For all $b \in B$, Pick an open neighbourhood W'_b of b such that $p^{-1}(W'_b) \subseteq V$
Proof: By Proposition 9.5.1.
$\langle 1\rangle 6.$ Let: $W = \bigcup_{a \in A} W_a$ and $W' = \bigcup_{b \in B} W'_b$
$\langle 1\rangle 7.$ $W \cap W' = \emptyset$
Proof: This holds because $p^{-1}(W) \subseteq U$, $p^{-1}(W') \subseteq V$, and p is surjective.
□

Definition 9.5.3 (Perfect Map)**.** Let X and Y be topological spaces and $p : X \to Y$. Then p is *perfect* iff p is closed, continuous, surjective, and $p^{-1}(y)$ is compact for all $y \in Y$.

Proposition 9.5.4. *Let $p : X \to Y$ be a perfect map. If X is Hausdorff then so is Y.*

Proof:
$\langle 1\rangle 1.$ Let: $a, b \in Y$ with $a \neq b$
$\langle 1\rangle 2.$ Pick disjoint open neighbourhoods U and V of $\pi^{-1}(a)$ and $\pi^{-1}(b)$, respectively.
Proof: By Lemma 9.4.18.
$\langle 1\rangle 3.$ Pick open neighbourhoods W and W' of a and b such that $\pi^{-1}(W) \subseteq U$ and $\pi^{-1}(W') \subseteq V$
Proof: By Proposition 9.5.1.
$\langle 1\rangle 4.$ W and W' are disjoint.
□

Proposition 9.5.5. *Let $p : X \twoheadrightarrow Y$ be perfect. If X is regular then so is Y.*

Proof:
$\langle 1\rangle 1.$ Y is T_1
Proof: By Proposition 9.5.4.
$\langle 1\rangle 2.$ Let: $C \subseteq Y$ be closed and $a \in Y \setminus C$
$\langle 1\rangle 3.$ $p^{-1}(C)$ is closed and $p^{-1}(a)$ is disjoint from $p^{-1}(C)$.
$\langle 1\rangle 4.$ Pick disjoint open neighbourhoods U, V of $p^{-1}(C)$, $p^{-1}(a)$ respectively.
Proof: By Lemma 9.4.8.
$\langle 1\rangle 5.$ Pick an open neighbourhood W' of a such that $p^{-1}(W') \subseteq V$
Proof: By Proposition 9.5.1.
$\langle 1\rangle 6.$ For $c \in C$, Pick an open neighbourhood W_c such that $p^{-1}(W_c) \subseteq U$

PROOF: By Proposition 9.5.1.
⟨1⟩7. $W = \bigcup_{c \in C} W_c$ is an open neighbourhood of C disjoint from W'
∎

Proposition 9.5.6 (AC)**.** *Let $p : X \twoheadrightarrow Y$ be perfect. If X is locally compact then so is Y.*

PROOF:
⟨1⟩1. LET: $b \in Y$
⟨1⟩2. $\{U \text{ open in } X : \exists C \subseteq X \text{ compact}.U \subseteq C\}$ covers $p^{-1}(b)$
⟨1⟩3. PICK a finite subcover $\{U_1, \ldots, U_n\}$
⟨1⟩4. For $1 \leq i \leq n$, PICK a compact $C_i \subseteq X$ such that $U_i \subseteq C_i$
⟨1⟩5. For $1 \leq i \leq n$, PICK a neighbourhood W_i of b such that $p^{-1}(W_i) \subseteq U_i$
PROOF: By Proposition 9.5.1
⟨1⟩6. $b \in W_1 \cup \cdots \cup W_n \subseteq p(C_1) \cup \cdots \cup p(C_n)$
⟨1⟩7. $p(C_1) \cup \cdots \cup p(C_n)$ is compact.
⟨2⟩1. Each $p(C_i)$ is compact.
PROOF: By Proposition 9.4.10.
⟨2⟩2. Q.E.D.
PROOF: By Proposition 9.4.23.
∎

9.6 Sequential Compactness

Definition 9.6.1 (Sequentially Compact)**.** A space is *sequentially compact* iff every sequence has a convergent subsequence.

Proposition 9.6.2. *$\overline{S_\Omega}$ is not sequentially compact.*

PROOF: Ω is a limit point of S_Ω but is not the limit of any sequence of points in S_Ω. ∎

9.7 Local Compactness

Definition 9.7.1 (Local Compactness)**.** Let X be a topological space.

For $x \in X$, the space X is *locally compact* at x iff there exists a compact subspace $C \subseteq X$ that includes a neighbourhood of x.

The space X is *locally compact* iff it is locally compact at every point.

Proposition 9.7.2. *Every complete linearly ordered set is locally compact under the order topology.*

PROOF:
⟨1⟩1. LET: L be a complete linearly ordered set and $x \in L$
PROVE: There exists a compact subspace $C \subseteq L$ that includes a neighbourhood U of x
⟨1⟩2. CASE: x is least and greatest in L

PROOF: In this case, $L = \{x\}$ is compact.
⟨1⟩3. CASE: x is least in L but not greatest
⟨2⟩1. PICK $a < x$
⟨2⟩2. Take $C = [a, x]$ and $U = (a, x]$
⟨1⟩4. CASE: x is greatest in L but not least
PROOF: Similar.
⟨1⟩5. CASE: x is neither least nor greatest
⟨2⟩1. PICK $a < x$ and $b > x$
⟨2⟩2. Take $C = [a, b]$ and $U = (a, b)$
□

Corollary 9.7.2.1. *For every ordinal α, the space S_α is locally compact.*

Theorem 9.7.3. *Every closed subspace of a locally compact Hausdorff space is locally compact.*

PROOF:
⟨1⟩1. LET: X be locally compact Hausdorff and $C \subseteq X$ be closed.
⟨1⟩2. LET: $x \in C$
⟨1⟩3. PICK $D \subseteq X$ compact and $U \subseteq D$ open such that $x \in U$
⟨1⟩4. D is closed.
PROOF: Proposition 9.4.9.
⟨1⟩5. $C \cap D$ is closed
PROOF: Propositon 3.5.5.
⟨1⟩6. $C \cap D$ is compact
PROOF: Proposition 9.4.6.
⟨1⟩7. Q.E.D.
PROOF: $C \cap D \subseteq C$ is compact and includes the open neighbourhood $U \cap C$ of x.
□

Proposition 9.7.4. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of nonempty topological spaces. If $\prod_{\alpha \in J} X_\alpha$ is locally compact, then each X_α is locally compact.*

PROOF:
⟨1⟩1. LET: $\alpha \in J$ and $x_\alpha \in X_\alpha$
⟨1⟩2. PICK $x_\beta \in X_\beta$ for all $\beta \in J \setminus \{\alpha\}$
⟨1⟩3. PICK a compact subspace $C \subseteq \prod_{\alpha \in J} X_\alpha$ that a neighbourhood U of x included in C
⟨1⟩4. PICK a basic open set $\prod_{\alpha \in J} U_\alpha$ such that $x \in \prod_{\alpha \in J} U_\alpha \subseteq U$
⟨1⟩5. $x_\alpha \in U_\alpha \subseteq \pi_\alpha(C)$
⟨1⟩6. $\pi_\alpha(C)$ is compact.
PROOF: By Proposition 9.4.10.
□

Corollary 9.7.4.1. *The Sorgenfrey plane is not locally compact.*

Proposition 9.7.5. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of locally compact spaces such that X_α is compact for all but finitely many values of α. Then $\prod_{\alpha \in J} X_\alpha$ is locally compact.*

PROOF:
⟨1⟩1. ASSUME: X_α is compact if $\alpha \neq \alpha_1, \ldots, \alpha_n$
⟨1⟩2. LET: $\vec{x} \in \prod_{\alpha \in J} X_\alpha$
⟨1⟩3. For $1 \leq i \leq n$, PICK $C_{\alpha_i} \subseteq X_{\alpha_i}$ compact and U_{α_i} open such that $x_{\alpha_i} \in U_{\alpha_i} \subseteq C_{\alpha_i}$
⟨1⟩4. For $\alpha \neq \alpha_1, \ldots, \alpha_n$,
LET: $C_\alpha = U_\alpha = X_\alpha$
⟨1⟩5. $\vec{x} \in \prod_{\alpha \in J} U_\alpha \subseteq \prod_{\alpha \in J} C_\alpha$
⟨1⟩6. $\prod_{\alpha \in J} C_\alpha$ is compact
PROOF: By Tychonoff's Theorem.
□

Proposition 9.7.6. $\mathbb{R}_l$ *is not locally compact.*

PROOF: $[0, +\infty)$ can be partitioned into infinitely many disjoint open sets, which therefore do not have a finite subcover. □

Proposition 9.7.7. *Let* $\{X_\alpha\}_{\alpha \in J}$ *be a family of nonempty topological spaces. If* $\prod_{\alpha \in J} X_\alpha$ *is locally compact, then all but finitely many of the* X_α *are compact.*

PROOF:
⟨1⟩1. PICK a point $a = (a_\alpha)_{\alpha \in J} \in \prod_{\alpha \in J} X_\alpha$
⟨1⟩2. PICK a compact $C \subseteq \prod_{\alpha \in J} X_\alpha$ that includes the basic neighbourhood $\prod_{\alpha \in J} U_\alpha$ of a, where $U_\alpha = X_\alpha$ for all α except $\alpha = \alpha_1, \ldots, \alpha_n$
⟨1⟩3. For $\alpha \neq \alpha_1, \ldots, \alpha_n$, we have X_α is compact.
PROOF: X_α is homeomorphic to a closed subspace of C.
□

Corollary 9.7.7.1. *For any infinite set* I*, the space* $\mathbb{R}^I$ *is not locally compact.*

Proposition 9.7.8. $[0, 1]^\omega$ *is not compact under the uniform topology.*

PROOF:$\{a_i : i \geq 0\}$ is an infinite set with no limit point, where a_i is the point with ith component 1 and all other components 0. □

Corollary 9.7.8.1. $\mathbb{R}^\omega$ *under the uniform topology is not locally compact.*

PROOF:
⟨1⟩1. ASSUME: $\mathbb{R}^\omega$ is locally compact
⟨1⟩2. LET: C be a compact subspace such that $B(\vec{0}, \epsilon) \subseteq C$
⟨1⟩3. $\overline{B(\vec{0}, \epsilon)}$ is compact.
⟨1⟩4. Q.E.D.
PROOF: This contradicts the proposition.
□

Proposition 9.7.9. *Not every subspace of a locally compact Hausdorff space is locally compact.*

PROOF: $\mathbb{R}$ is locally compact Hausdoff, $\mathbb{Q}$ is not locally compact. □

Proposition 9.7.10. *The continuous image of a locally compact Hausdorff space is not necessarily locally compact.*

PROOF:
⟨1⟩1. LET: $\{q_0, q_1, \ldots\}$ be an enumeration of $[0,1] \cap \mathbb{Q}$.
⟨1⟩2. Define $f : (0, +\infty) \setminus \mathbb{Z} \to [0,1] \cap \mathbb{Q}$ by: $f(x) = q_n$ for $x \in (n, n+1)$
⟨1⟩3. f is continuous.
PROOF: The inverse image of any set is a union of open intervals.
□

9.8 Compactifications

Definition 9.8.1 (Compactification)**.** Let X and Y be spaces. Then Y is a *compactification* of X iff Y is a compact Hausdorff space and X is a subspace of Y with $\overline{X} = Y$.

Two compcactificatons Y_1, Y_2 of X are *equivalent* iff there exists a homeomorphism between Y_1 and Y_2 that is the identity on X.

Lemma 9.8.2. *Let $h : X \to Z$ be an imbedding. Then there exists a compactification $c : X \to Y$ of X, unique up to equivalence, and an imbedding $i : Y \to Z$ such that $h = i \circ c$.*

PROOF: Simply take Y to be the closure of X in Z. □

Definition 9.8.3 (One-Point Compactification)**.** A *one-point compactification* of X is a compactification Y of X such that $Y \setminus X$ is a singleton.

Theorem 9.8.4. *Let X be a topological space. Then X is locally compact Hausdorff if and only if there exists a space Y such that:*

1. *X is a subspace of Y*
2. *The set $Y \setminus X$ is a singleton.*
3. *Y is a compact Hausdorff space.*

If Y and Y' are two spaces satisfying these conditions, then there exists a unique homeomorphism between Y and Y' that is the identity on X.

PROOF:
⟨1⟩1. If X is locally compact Hausdorff then there exists a space Y satisfying 1–3.
⟨2⟩1. LET: $Y = X \cup \{\infty\}$ under the topology $\mathcal{T} = \{U \subseteq X : U \text{ is open in } X\} \cup \{Y \setminus C : C \subseteq X \text{ is compact}\}$.
⟨3⟩1. $Y \in \mathcal{T}$
PROOF: This holds because $Y = Y \setminus \emptyset$.
⟨3⟩2. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$.
⟨4⟩1. LET: $U, V \in \mathcal{T}$
⟨4⟩2. CASE: U, V are open in X

PROOF: In this case, $U \cap V$ is open in X.
⟨4⟩3. CASE: U is open in X, $V = Y \setminus C$ where $C \subseteq X$ is compact.
⟨5⟩1. $U \cap V = U \setminus C$
⟨5⟩2. C is closed in X
PROOF: Proposition 9.4.9.
⟨5⟩3. $U \cap V$ is open in X
⟨4⟩4. CASE: $U = Y \setminus C$ where $C \subseteq X$ is compact, V is open in X.
PROOF: Similar.
⟨4⟩5. CASE: $U = Y \setminus C$, $V = Y \setminus D$ where $C, D \subseteq X$ are compact.
⟨5⟩1. $U \cap V = Y \setminus (C \cup D)$
⟨5⟩2. C and D are closed in X
PROOF: Proposition 9.4.9.
⟨5⟩3. $C \cup D$ is closed in X
PROOF: Proposition 3.5.4.
⟨5⟩4. $C \cup D$ is compact.
PROOF: By Proposition 9.4.23. □
⟨3⟩3. For all $\mathcal{A} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{A} \in \mathcal{T}$.
⟨4⟩1. LET: $\mathcal{A} \subseteq \mathcal{T}$
⟨4⟩2. CASE: Every element of $\mathcal{A}$ is an open set in X.
PROOF: In this case, $\bigcup \mathcal{A}$ is open in X.
⟨4⟩3. CASE: There exists C compact in X such that $Y \setminus C \in \mathcal{A}$
⟨5⟩1. $\bigcup \mathcal{A} = Y \setminus (\bigcap \{D \subseteq X : D \text{ compact}, Y \setminus D \in \mathcal{A}\} \setminus \bigcup \{U \text{ open in } X : U \in \mathcal{A}\})$
PROOF: Set theory.
⟨5⟩2. $\bigcap \{D \subseteq X : D \text{ compact}, Y \setminus D \in \mathcal{A}\} \setminus \bigcup \{U \text{ open in } X : U \in \mathcal{A}\}$ is compact.
PROOF: It is a closed subset of the compact set C.
⟨2⟩2. X is a subspace of Y
⟨3⟩1. For every open set U of X, there exists V open in Y such that $U = V \cap X$
PROOF: Take $V = U$.
⟨3⟩2. For every open set V in Y, we have $V \cap X$ is open in X.
⟨4⟩1. LET: V be open in Y
⟨4⟩2. CASE: V is open in X
PROOF: In this case, $V \cap X = V$.
⟨4⟩3. CASE: $V = Y \setminus C$ where $C \subseteq X$ is compact.
⟨5⟩1. C is closed in X.
PROOF: By Proposition 9.4.9.
⟨5⟩2. $V \cap X = X \setminus C$
⟨2⟩3. $Y \setminus X = \{\infty\}$
⟨2⟩4. Y is compact.
⟨3⟩1. LET: $\mathcal{A}$ be an open covering of Y
⟨3⟩2. PICK $U \in \mathcal{A}$ such that $\infty \in U$
⟨3⟩3. PICK $C \subseteq X$ compact such that $U = Y \setminus C$.
⟨3⟩4. $\{V \cap X : V \in \mathcal{A}\}$ is set of open sets that covers C
⟨3⟩5. PICK a finite subset $\{V_1, \ldots, V_n\}$ such that $\{V_1 \cap X, \ldots, V_n \cap X\}$

covers C.

$\langle 3 \rangle 6.$ $\{U, V_1, \ldots, V_n\}$ is a finite subcover of Y.

$\langle 2 \rangle 5.$ Y is Hausdorff.

$\langle 3 \rangle 1.$ LET: $x, y \in Y$ with $x \neq y$

PROVE: There exist disjoint open neighbourhoods U, V of x and y.

$\langle 3 \rangle 2.$ CASE: $x, y \in X$

PROOF: In this case, we just use the fact that X is Hausdorff.

$\langle 3 \rangle 3.$ CASE: $x = \infty$, $y \in X$

$\langle 4 \rangle 1.$ PICK $C \subseteq X$ compact such that C includes an open neighbourhood V of y

$\langle 4 \rangle 2.$ LET: $U = Y \setminus C$

$\langle 3 \rangle 4.$ CASE: $x \in X$, $y = \infty$

PROOF: Simlar.

$\langle 1 \rangle 2.$ If there exists a space Y satisfying 1–3 then X is locally compact Hausdorff.

$\langle 2 \rangle 1.$ LET: Y be a space satisfying 1–3

$\langle 2 \rangle 2.$ LET: ∞ be the point in $Y \setminus X$

$\langle 2 \rangle 3.$ X is locally compact

$\langle 3 \rangle 1.$ LET: $x \in X$

$\langle 3 \rangle 2.$ PICK disjoint open neighbourhoods U of x and V of ∞

$\langle 3 \rangle 3.$ $X \setminus V$ is compact and includes U

PROOF: $X \setminus V = Y \setminus V$ is compact because it is a closed subset of Y (Proposition 9.4.6).

$\langle 2 \rangle 4.$ X is Hausdorff.

PROOF: By Corollary 6.2.6.1.

$\langle 1 \rangle 3.$ If Y and Y' are two spaces satisfying 1–3 then there exists a unique homemorphism between Y and Y' that is the identity on X.

$\langle 2 \rangle 1.$ LET: Y and Y' be two spaces that satisfy 1–3.

$\langle 2 \rangle 2.$ LET: $Y \setminus X = \{p\}$ and $Y' \setminus X = \{q\}$

$\langle 2 \rangle 3.$ LET: $h : Y \to Y'$ be given by

$$h(x) = x \qquad\qquad (x \in X)$$
$$h(p) = q$$

$\langle 2 \rangle 4.$ h is a homeomorphism

$\langle 3 \rangle 1.$ h is bijective.

$\langle 3 \rangle 2.$ h is continuous.

$\langle 4 \rangle 1.$ LET: $V \subseteq Y'$ be open.

PROVE: $h^{-1}(V)$ is open.

$\langle 4 \rangle 2.$ CASE: $V \subseteq X$

$\langle 5 \rangle 1.$ $h^{-1}(V) = V$

$\langle 5 \rangle 2.$ V is open in X

PROOF: Condition 1 for Y'.

$\langle 5 \rangle 3.$ V is open in Y

PROOF: Condition 1 for Y.

$\langle 4 \rangle 3.$ CASE: $q \in V$

$\langle 5 \rangle 1.$ $Y' \setminus V$ is compact.

PROOF: Proposition 9.4.6.

⟨5⟩2. $Y' \setminus V$ is closed in Y.

PROOF: Proposition 9.4.9.

⟨5⟩3. $h^{-1}(V) = Y \setminus (Y' \setminus V)$

⟨3⟩3. h^{-1} is continuous.

PROOF: Similar.

⟨2⟩5. If $h' : Y \to Y'$ is a homeomorphism such that $h' \restriction_X = \mathrm{id}_X$ then $h' = h$

□

Theorem 9.8.5. *Let X be a Hausdorff space. Then X is locally compact if and only if, for all $x \in X$ and any neighbourhood U of x, there exists an open neighbourhood V of x such that $\overline{V}$ is compact and $\overline{V} \subseteq U$.*

PROOF:

⟨1⟩1. If X is locally compact then, for all $x \in X$ and any neighbourhood U of x, there exists an open neighbourhood V of x such that $\overline{V}$ is compact and $\overline{V} \subseteq U$.

⟨2⟩1. ASSUME: X is locally compact.

⟨2⟩2. LET: $x \in X$ and U be a neighbourhood of x.

⟨2⟩3. LET: Y be the one-point compactification of X.

PROOF: By Theorem 9.8.4.

⟨2⟩4. LET: $C = Y \setminus U$

⟨2⟩5. C is compact

PROOF: By Proposition 9.4.6.

⟨2⟩6. PICK disjoint open sets V, W containing x and C

PROOF: Lemma 9.4.8

⟨2⟩7. V is open in X

PROOF: $V \subseteq X$ since $\infty \in W$.

⟨2⟩8. The closure of V in X is compact

⟨3⟩1. The closure of V is X is the same as the closure of V in Y.

PROOF: The point ∞ cannot be a limit point of V since W is a neighbourhood disjoint from V.

⟨3⟩2. The closure of V in Y is compact.

PROOF: By Proposition 9.4.6.

⟨2⟩9. $\overline{V} \subseteq U$

PROOF:

$$\begin{aligned} \overline{V} &\subseteq Y \setminus W \\ &\subseteq Y \setminus C \\ &= U \end{aligned}$$

⟨1⟩2. If, for all $x \in X$ and any neighbourhood U of x, there exists an open neighbourhood V of x such that $\overline{V}$ is compact and $\overline{V} \subseteq U$, then X is locally compact.

⟨2⟩1. ASSUME: for all $x \in X$ and any neighbourhood U of x, there exists an open neighbourhood V of x such that $\overline{V}$ is compact and $\overline{V} \subseteq U$

⟨2⟩2. LET: $x \in X$

PROVE: There exists $C \subseteq X$ compact such that C includes a neighbourhood U of x

⟨2⟩3. PICK an open neighbourhood V of x such that $\overline{V}$ is compact and $\overline{V} \subseteq X$

⟨2⟩4. Take $C = \overline{V}$ and $U = V$

□

Corollary 9.8.5.1. *Every open subspace of a locally compact Hausdorff space is locally compact.*

Corollary 9.8.5.2. *A space is locally compact Hausdorff if and only if it is an open subspace of a compact Hausdorff space.*

Corollary 9.8.5.3. *Every locally compact Hausdorff space is completely regular.*

Corollary 9.8.5.4. *The space $\mathbb{R}_K$ is not locally compact.*

Lemma 9.8.6 (AC)**.** *If $p : X \to Y$ is a quotient map and Z is a locally compact Hausdorff space, then the map*

$$\pi = p \times \mathrm{id}_Z : X \times Z \to Y \times Z$$

is a quotient map.

PROOF:

⟨1⟩1. π is surjective.

PROOF: This holds because p is surjective.

⟨1⟩2. π is continuous.

PROOF: By Theorem 5.2.15.

⟨1⟩3. For $A \subseteq Y \times Z$, if $\pi^{-1}(A)$ is open in $X \times Z$ then A is open in $Y \times Z$.

⟨2⟩1. LET: $A \subseteq Y \times Z$

⟨2⟩2. ASSUME: $\pi^{-1}(A)$ is open in $X \times Z$

⟨2⟩3. LET: $(y, z) \in A$

⟨2⟩4. PICK $x \in X$ such that $p(x) = y$

PROOF: Since p is surjective.

⟨2⟩5. PICK open sets U_1, V with $\overline{V}$ compact such that $(x, y) \in U_1 \times V$ and $U_1 \times \overline{V} \subseteq \pi^{-1}(A)$

PROOF: Using Theorem 9.8.5

⟨2⟩6. PICK a sequence of open sets $U_1, U_2, \ldots$ in X such that $p^{-1}(p(U_n)) \subseteq U_{n+1}$ and $U_n \times \overline{V} \subseteq \pi^{-1}(A)$ for all n

⟨3⟩1. LET: U be open with $U \times \overline{V} \subseteq \pi^{-1}(A)$

PROVE: There exists W open with $p^{-1}(p(U)) \subseteq W$ and $W \times \overline{V} \subseteq \pi^{-1}(A)$

⟨3⟩2. For all $x \in p^{-1}(p(U))$, PICK open sets U_x, V_x such that $x \in U_x$, $\overline{V} \subseteq V_x$ and $U_x \times V_x \subseteq \pi^{-1}(A)$

PROOF: By the Tube Lemma.

⟨3⟩3. LET: $W = \bigcup_{x \in p^{-1}(p(U))} U_x$

⟨2⟩7. LET: $U = \bigcup_{n=1}^{\infty} U_n$

⟨2⟩8. U is saturated with respect to p

⟨3⟩1. LET: $a \in U$, $b \in X$, $p(a) = p(b)$

⟨3⟩2. PICK n such that $a \in U_n$

⟨3⟩3. $b \in p^{-1}(p(U_n))$

⟨3⟩4. $b \in U_{n+1}$

⟨3⟩5. $b \in U$

⟨2⟩9. $p(U)$ is open in Y

PROOF: By Lemma 4.5.2.

⟨2⟩10. $(y, z) \in p(U) \times V \subseteq A$

⟨2⟩11. Q.E.D.

PROOF: By Proposition 3.2.3.

□

Theorem 9.8.7. *Let $p : A \to B$ and $q : C \to D$ be quotient maps. If B and C are locally compact Hausdorff spaces, then $p \times q : A \times C \to B \times D$ is a quotient map.*

PROOF: This holds by Lemma 9.8.6 and Proposition 4.5.10 because $p \times q = (\mathrm{id}_B \times q) \circ (p \times \mathrm{id}_C)$. □

Theorem 9.8.8. *Let X be a completely regular space. Let Y be a compactification of X such that every bounded continuous map $X \to \mathbb{R}$ extends uniquely to a continuous map $Y \to \mathbb{R}$. Then, for every compact Hausdorff space C, every continuous map $X \to C$ extends uniquely to a continuous map $Y \to C$.*

PROOF:

⟨1⟩1. LET: C be a compact Hausdorff space and $f : X \to C$ a continuous function

⟨1⟩2. PICK a set J and an imedding $C \subseteq [0, 1]^J$

⟨2⟩1. C is normal

PROOF: By Lemma 9.4.18

⟨2⟩2. Q.E.D.

PROOF: By Theorem 6.4.6.

⟨1⟩3. For $\alpha \in J$,

LET: $g_\alpha : Y \to \mathbb{R}$ be the unique continuous extension of $\pi_\alpha \circ f$

⟨1⟩4. Define $g : Y \to \mathbb{R}^J$ by $g(y)_\alpha = g_\alpha(y)$

⟨1⟩5. g is continuous

PROOF: By Theorem 5.2.15.

⟨1⟩6. g extends f

⟨1⟩7. We have $g : Y \to C$

PROOF:

$$\begin{aligned} g(Y) &= g(\overline{X}) \\ &\subseteq \overline{g(X)} && \text{(Theorem 5.2.2)} \\ &= \overline{f(X)} && (\langle 1 \rangle 6) \\ &\subseteq \overline{C} \\ &= C && \text{(Proposition 9.4.9)} \end{aligned}$$

⟨1⟩8. g is unique

⟨2⟩1. LET: $h : Y \to C$ be a continuous extension of f
⟨2⟩2. For all $\alpha \in J$, $\pi_\alpha \circ h$ extends $\pi_\alpha \circ f$
⟨2⟩3. For all $\alpha \in J$, $\pi_\alpha \circ h = g_\alpha$
PROOF: By ⟨1⟩3
⟨2⟩4. $h = g$
PROOF: By ⟨1⟩4
□

Corollary 9.8.8.1. *Let X be a completely regular space. Let Y_1 and Y_2 be compactifications of X such that every bounded continuous map $X \to \mathbb{R}$ extends uniquely to a continuous map $Y_i \to \mathbb{R}$. Then Y_1 and Y_2 are equivalent.*

Definition 9.8.9 (Stone-Čech Compactification)**.** Let X be a completely regular space. The *Stone-Čech compactification* of X, $\beta(X)$, is the compactification of X such that, for every compact Hausdorff space C, every continuous function $X \to C$ extends uniquely to a continuous function $\beta(X) \to C$.

Chapter 10

Metric Spaces

10.1 The Metric Topology

Definition 10.1.1 (Metric)**.** A *metric* on a set X is a function $d : X \times X \to \mathbb{R}$ such that, for all $x, y, z \in X$:

1. $d(x, y) \geq 0$;
2. $d(x, y) = 0$ if and only if $x = y$;
3. $d(x, y) = d(y, x)$;
4. **Triangle Inequality**
   $d(x, z) \leq d(x, y) + d(y, z)$

A *metric space* X consists of a set X and a metric on X. We call $d(x, y)$ the *distance* between x and y.

Definition 10.1.2 (Open Ball)**.** Let X be a metric space with metric d, $x \in X$ and $\epsilon > 0$. The *open ball* with *centre* x and *radius* ϵ is

$$B_d(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\} \ .$$

Lemma 10.1.3. *Let X be a metric space, $x, y \in X$ and $\epsilon > 0$. If $y \in B(x, \epsilon)$, then there exists δ such that $0 < \delta < \epsilon$ and*

$$B(y, \delta) \subseteq B(x, \epsilon) \ .$$

Proof:
$\langle 1 \rangle 1.$ Let: $\delta = \epsilon - d(x, y)$
$\langle 1 \rangle 2.$ Let: $z \in B(y, \delta)$
$\langle 1 \rangle 3.$ $d(x, z) < \epsilon$

PROOF:

$$\begin{aligned}
d(x,z) &\leq d(x,y) + d(y,z) && \text{(Triangle Inequality)} \\
&< d(x,y) + \delta && (\langle 1\rangle 2) \\
&= \epsilon && (\langle 1\rangle 1)
\end{aligned}$$

□

Definition 10.1.4 (Metric Topology)**.** Let d be a metric on X. The *metric topology* on X induced by d is the topology generated by the basis consisting of the open balls.

We prove this is a topology.

PROOF:
⟨1⟩1. Every point is in an open ball.
PROOF: $x \in B(x,1)$
⟨1⟩2. If B_1, B_2 are open balls and $x \in B_1 \cap B_2$, then there exists an open ball B_3 such that $x \in B_3 \subseteq B_1 \cap B_2$.
⟨2⟩1. LET: $x \in B(y,\epsilon_1) \cap B(z,\epsilon_2)$
⟨2⟩2. PICK δ_1, δ_2 such that $0 < \delta_1 < \epsilon_1$, $0 < \delta_2 < \epsilon_2$, $B(x,\delta_1) \subseteq B(y,\epsilon_1)$ and $B(x,\delta_2) \subseteq B(z,\epsilon_2)$.
PROOF: Lemma 10.1.3.
⟨2⟩3. LET: $\delta = \min(\delta_1,\delta_2)$
⟨2⟩4. $x \in B(x,\delta) \subseteq B(y,\epsilon_1) \cap B(y,\epsilon_2)$
⟨1⟩3. Q.E.D.
PROOF: Lemma 3.4.3.

Lemma 10.1.5. *A set U is open in the metric topology induced by d if and only if, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x,\epsilon) \subseteq U$.*

PROOF:
⟨1⟩1. If U is open then, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x,\epsilon) \subseteq U$.
⟨2⟩1. ASSUME: U is open.
⟨2⟩2. LET: $x \in U$
⟨2⟩3. PICK $B(y,\delta)$ such that $x \in B(y,\delta) \subseteq U$
⟨2⟩4. PICK ϵ such that $0 < \epsilon < \delta$ and $B(x,\epsilon) \subseteq B(y,\delta)$
PROOF: Lemma 10.1.3.
⟨2⟩5. $B(x,\epsilon) \subseteq U$
PROOF: From ⟨2⟩3 and ⟨2⟩4.
⟨1⟩2. If, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x,\epsilon) \subseteq U$, then U is open.
PROOF: Immediate from definition of metric topology.

□

Lemma 10.1.6. *Let d and d' be two metrics on the set X. Let $\mathcal{T}$ and $\mathcal{T}'$ be the topologies the induce, respectively. Then $\mathcal{T} \subseteq \mathcal{T}'$ if and only if, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x,\delta) \subseteq B_d(x,\epsilon)$.*

PROOF:
⟨1⟩1. If $\mathcal{T} \subseteq \mathcal{T}'$ then, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x,\delta) \subseteq B_d(x,\epsilon)$.

⟨2⟩1. ASSUME: $\mathcal{T} \subseteq \mathcal{T}'$
⟨2⟩2. LET: $x \in X$ and $\epsilon > 0$
⟨2⟩3. $B_d(x, \epsilon) \in \mathcal{T}'$
PROOF: From ⟨2⟩1.
⟨2⟩4. There exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$
PROOF: By Lemma 10.1.5.

⟨1⟩2. If, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$, then $\mathcal{T} \subseteq \mathcal{T}'$
⟨2⟩1. ASSUME: For all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$.
⟨2⟩2. LET: $U \in \mathcal{T}$
PROVE: $U \in \mathcal{T}'$
⟨2⟩3. LET: $x \in U$
⟨2⟩4. PICK $\epsilon > 0$ be such that $B_d(x, \epsilon) \subseteq U$
PROOF: By Lemma 10.1.5.
⟨2⟩5. PICK $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$
PROOF: By ⟨2⟩1.
⟨2⟩6. $B_{d'}(x, \delta) \subseteq U$
PROOF: By ⟨2⟩4 and ⟨2⟩5.
⟨2⟩7. Q.E.D.
PROOF: By Lemma 10.1.5.

□

Definition 10.1.7 (Metrizable)**.** A topological space is *metrizable* if and only if there exists a metric that induces its topology.

Lemma 10.1.8. *Every discrete space is metrizable.*

PROOF: The discrete topology is induced by the metric $d(x, y) = 1$ if $x \neq y$, 0 if $x = y$. □

Proposition 10.1.9. *The continuous image of a metrizable space is not necessarily metrizable.*

PROOF: The identity map from the discrete two-point space to the indiscrete two-point space is continuous. □

Lemma 10.1.10. $\mathbb{R}$ *is metrizable.*

PROOF: The standard topology is induced by the metric $d(x, y) = |x - y|$. □

Definition 10.1.11 (Bounded)**.** Let X be a metric space and $A \subseteq X$. Then A is *bounded* iff $\{d(x, y) : x, y \in A\}$ is bounded above, in which case its *diameter* is

$$\operatorname{diam} A = \sup_{x,y \in A} d(x, y) \quad .$$

Lemma 10.1.12. *Let* (X, d) *be a metric space and* $A \subseteq X$*. Then* $d \restriction_{A \times A}$ *is a metric on* A *that induces the subspace topology.*

PROOF:
⟨1⟩1. $d\restriction_{A\times A}$ is a metric on A.
  PROOF: Each of the axioms for a metric follows immediately from the same axiom for d.
⟨1⟩2. The topology induced by $d\restriction_{A\times A}$ is the product topology.
  PROOF: Both are the topology generated by the basis consisting of all the open balls $B_{d\restriction_{A\times A}}(a,\epsilon) = B_d(a,\epsilon)\cap A$.
□

Lemma 10.1.13. *Every metric space is Hausdorff.*

PROOF:
⟨1⟩1. LET: X be a metric space and $x, y \in X$ with $x \neq y$.
⟨1⟩2. LET: $\epsilon = d(x, y)$
⟨1⟩3. $B(x, \epsilon/2)$ and $B(y, \epsilon/2)$ are disjoint neighbourhoods of x and y.
□

Theorem 10.1.14. *Every metric space is first countable.*

PROOF: $\{B(x,q) : q \in \mathbb{Q}^+\}$ is a local basis at x. □

Corollary 10.1.14.1. *If J is infinite then the space $\mathbb{R}^J$ is not metrizable.*

Definition 10.1.15 (Standard Bounded Metric)**.** Let d be a metric on X. The *standard bounded metric* corresponding to d is

$$\overline{d}(x,y) = \min(d(x,y),1) \ .$$

We prove this is a metric.

PROOF:
⟨1⟩1. $\overline{d}(x,y) \geq 0$
  PROOF: This holds because $d(x,y) \geq 0$ (d is a metric) and $1 > 0$.
⟨1⟩2. $\overline{d}(x,y) = 0$ iff $x = y$
  PROOF: Immediate from definition.
⟨1⟩3. $\overline{d}(x,y) = \overline{d}(y,x)$
  PROOF: Immediate from definition.
⟨1⟩4. $\overline{d}(x,z) \leq \overline{d}(x,y) + \overline{d}(y,z)$
  ⟨2⟩1. CASE: $d(x,y) \leq 1$, $d(y,z) \leq 1$
    PROOF:

$$\begin{aligned}\overline{d}(x,z) &\leq d(x,z) \\ &\leq d(x,y) + d(y,z) \\ &= \overline{d}(x,y) + \overline{d}(y,z)\end{aligned}$$

  ⟨2⟩2. CASE: $d(y,z) > 1$
    PROOF:

$$\begin{aligned}\overline{d}(x,z) &\leq 1 \\ &\leq \overline{d}(x,y) + 1 \\ &= \overline{d}(x,y) + \overline{d}(y,z)\end{aligned}$$

$\langle 2\rangle 3$. CASE: $d(x, y) > 1$
PROOF: Similar.
□

Theorem 10.1.16. *Let d be a metric on X. Then the standard bounded metric $\overline{d}$ corresponding to d induces the same topology as d.*

PROOF:
$\langle 1\rangle 1$. LET: $\mathcal{T}$ be the topology induced by d and $\mathcal{T}'$ be the topology induced by $\overline{d}$.
$\langle 1\rangle 2$. $\mathcal{T} \subseteq \mathcal{T}'$
$\langle 2\rangle 1$. LET: $x \in X$ and $\epsilon > 0$
$\langle 2\rangle 2$. LET: $\delta = \min(\epsilon, 1/2)$
$\langle 2\rangle 3$. $B_{\overline{d}}(x, \delta) \subseteq B_d(x, \epsilon)$
$\langle 3\rangle 1$. LET: $y \in B_{\overline{d}}(x, \delta)$
$\langle 3\rangle 2$. $\overline{d}(x, y) < \delta$
$\langle 3\rangle 3$. $\overline{d}(x, y) < 1$
PROOF: From $\langle 2\rangle 2$ and $\langle 3\rangle 2$.
$\langle 3\rangle 4$. $\overline{d}(x, y) = d(x, y)$
PROOF: From $\langle 3\rangle 3$ and the definition of $\overline{d}$.
$\langle 3\rangle 5$. $d(x, y) < \epsilon$
PROOF: By $\langle 2\rangle 2$ and $\langle 3\rangle 2$ and $\langle 3\rangle 4$.
$\langle 1\rangle 3$. $\mathcal{T}' \subseteq \mathcal{T}$
$\langle 2\rangle 1$. LET: $x \in X$ and $\epsilon > 0$
$\langle 2\rangle 2$. $B_d(x, \epsilon) \subseteq B_{\overline{d}}(x, \epsilon)$
PROOF: This holds because $\overline{d}(x, y) \leq d(x, y)$.
□

Definition 10.1.17 (Square Metric)**.** The *square metric* on $\mathbb{R}^n$ is defined by

$$\rho(\vec{x}, \vec{y}) = \max(|x_1 - y_1|, \ldots, |x_n - y_n|) \ .$$

We prove this is a metric.

PROOF:
$\langle 1\rangle 1$. $\rho(\vec{x}, \vec{y}) \geq 0$
PROOF: Immediate from definitions.
$\langle 1\rangle 2$. $\rho(\vec{x}, \vec{y}) = 0$ iff $\vec{x} = \vec{y}$
PROOF: Immediate from definitions.
$\langle 1\rangle 3$. $\rho(\vec{x}, \vec{y}) = \rho(\vec{y}, \vec{x})$
PROOF: Immediate from definitions.
$\langle 1\rangle 4$. $\rho(\vec{x}, \vec{z}) \leq \rho(\vec{x}, \vec{y}) + \rho(\vec{y}, \vec{z})$
$\langle 2\rangle 1$. For all i, we have $|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$
$\langle 2\rangle 2$. For all i, $|x_i - z_i| \leq \rho(\vec{x}, \vec{y}) + \rho(\vec{y}, \vec{z})$
$\langle 2\rangle 3$. $\rho(\vec{x}, \vec{z}) \leq \rho(\vec{x}, \vec{y}) + \rho(\vec{y}, \vec{z})$
□

Theorem 10.1.18. *The square metric induces the standard topology on $\mathbb{R}^n$.*

PROOF:
⟨1⟩1. LET: $\mathcal{T}_\rho$ be the topology induced by the square metric and $\mathcal{T}_s$ the standard topology.
⟨1⟩2. $\mathcal{T}_\rho \subseteq \mathcal{T}_s$
PROOF: This holds because $B_\rho(\vec{x}, \epsilon) = (x_1 - \epsilon, x_1 + \epsilon) \times \cdots \times (x_n - \epsilon, x_n + \epsilon)$.
⟨1⟩3. $\mathcal{T}_s \subseteq \mathcal{T}_\rho$
⟨2⟩1. LET: $B = U_1 \times \cdots \times U_n$ be a basic open set in $\mathcal{T}_s$, where each U_i is open in $\mathbb{R}$.
⟨2⟩2. LET: $\vec{x} \in B$
⟨2⟩3. For $1 \leq i \leq n$, PICK $\epsilon_i > 0$ such that $(x_i - \epsilon_i, x_i + \epsilon_i) \subseteq U_i$
⟨2⟩4. LET: $\epsilon = \min(\epsilon_1, \ldots, \epsilon_n)$
⟨2⟩5. $B_\rho(\vec{x}, \epsilon) \subseteq B$
□

Lemma 10.1.19. *The product of a countable family of metrizable spaces is metrizable.*

PROOF:
⟨1⟩1. LET: $\{X_n\}_{n \in \mathbb{Z}^+}$ be a family of metric spaces with metrics bounded by 1, $X = \prod_{n=1}^\infty X_n$.
⟨1⟩2. LET: $D : X \times X \rightarrow \mathbb{R}$ be given by
$$D(\vec{x}, \vec{y}) = \sup_{n \geq 1} \frac{d(x_n, y_n)}{n} \quad .$$
⟨1⟩3. D is a metric on X.
⟨2⟩1. $D(\vec{x}, \vec{y}) \geq 0$
PROOF: Immediate from definitions.
⟨2⟩2. $D(\vec{x}, \vec{y}) = 0$ iff $\vec{x} = \vec{y}$
PROOF: Immediate from definitions.
⟨2⟩3. $D(\vec{x}, \vec{y}) = D(\vec{y}, \vec{x})$
PROOF: Immediate from definitions.
⟨2⟩4. $D(\vec{x}, \vec{z}) \leq D(\vec{x}, \vec{y}) + D(\vec{y}, \vec{z})$
⟨3⟩1. For all n, we have $\frac{d(x_n, z_n)}{n} \leq \frac{d(x_n, y_n)}{n} + \frac{d(y_n, z_n)}{n}$
⟨3⟩2. For all n, we have $\frac{d(x_n, z_n)}{n} \leq D(\vec{x}, \vec{y}) + D(\vec{y}, \vec{z})$
⟨3⟩3. $D(\vec{x}, \vec{z}) \leq D(\vec{x}, \vec{y}) + D(\vec{y}, \vec{z})$
⟨1⟩4. LET: $\mathcal{T}_D$ be the topology induced by D and $\mathcal{T}_p$ the product topology.
⟨1⟩5. $\mathcal{T}_D \subseteq \mathcal{T}_p$
⟨2⟩1. LET: $U \in \mathcal{T}_D$
PROVE: $U \in \mathcal{T}_p$
⟨2⟩2. LET: $\vec{x} \in U$
⟨2⟩3. PICK $\epsilon > 0$ such that $B_D(\vec{x}, \epsilon) \subseteq U$
⟨2⟩4. PICK N such that $1/N < \epsilon$
⟨2⟩5. LET: $V = B(x_1, \epsilon) \times \cdots \times B(x_N, \epsilon) \times X_{N+1} \times X_{N+2} \times \cdots$
⟨2⟩6. $\vec{x} \in V \subseteq B_D(\vec{x}, \epsilon)$
⟨1⟩6. $\mathcal{T}_p \subseteq \mathcal{T}_D$
⟨2⟩1. LET: $U = \prod_{n=1}^\infty U_n$ be a basic open set in $\mathcal{T}_p$, where each U_n is open in X_n, and $U_n = X_n$ for $n > N$.

⟨2⟩2. LET: $\vec{x} \in U$
PROVE: There exists $\epsilon > 0$ such that $B_D(\vec{x}, \epsilon) \subseteq U$.
⟨2⟩3. For $n \leq N$, PICK $\epsilon_n > 0$ such that $B(x_n, \epsilon_n) \subseteq U_n$
⟨2⟩4. LET: $\epsilon = \min(\epsilon_1, \epsilon_2/2, \ldots, \epsilon_n/n)$
⟨2⟩5. LET: $\vec{y} \in B_D(\vec{x}, \epsilon)$
⟨2⟩6. For $n \leq N$, $y_n \in U_n$
⟨3⟩1. $D(\vec{x}, \vec{y}) < \epsilon$
⟨3⟩2. $d(x_n, y_n)/n < \epsilon$
⟨3⟩3. $d(x_n, y_n)/n < \epsilon_n/n$
⟨3⟩4. Q.E.D.
PROOF: By ⟨2⟩3.
□

Corollary 10.1.19.1. *The space $\mathbb{R}^\omega$ is metrizable.*

Definition 10.1.20 (Uniform Metric)**.** Let J be a set. The *uniform metric* $\overline{\rho}$ on $\mathbb{R}^J$ is defined by

$$\overline{\rho}(\vec{x}, \vec{y}) = \sup_{\alpha \in J} \overline{d}(x_\alpha, y_\alpha) \ .$$

where $\overline{d}$ is the standard bounded metric on $\mathbb{R}$. The *uniform topology* is the topology induced by the uniform metric.

We prove this is a metric.

PROOF:
⟨1⟩1. $\overline{\rho}(\vec{x}, \vec{y}) \geq 0$
PROOF: Immediate from definitions.
⟨1⟩2. $\overline{\rho}(\vec{x}, \vec{y}) = 0$ iff $\vec{x} = \vec{y}$
PROOF: Immediate from definitions.
⟨1⟩3. $\overline{\rho}(\vec{x}, \vec{y}) = \overline{\rho}(\vec{y}, \vec{x})$
PROOF: Immediate from definitions.
⟨1⟩4. $\overline{\rho}(\vec{x}, \vec{z}) \leq \overline{\rho}(\vec{x}, \vec{y}) + \overline{\rho}(\vec{y}, \vec{z})$
PROOF:
⟨2⟩1. For all $\alpha \in J$, $\overline{d}(x_\alpha, z_\alpha) \leq \overline{d}(x_\alpha, y_\alpha) + \overline{d}(y_\alpha, z_\alpha)$
⟨2⟩2. For all $\alpha \in J$, $\overline{d}(x_\alpha, z_\alpha) \leq \overline{\rho}(\vec{x}, \vec{y}) + \overline{\rho}(\vec{y}, \vec{z})$
⟨2⟩3. $\overline{\rho}(\vec{x}, \vec{z}) \leq \overline{\rho}(\vec{x}, \vec{y}) + \overline{\rho}(\vec{y}, \vec{z})$
□

Theorem 10.1.21 (DC)**.** *The uniform topology on $\mathbb{R}^J$ is finer than the product topology and coarser than the box topology. These three topologies are different iff J is infinite.*

PROOF:
⟨1⟩1. The uniform topology is finer than the product topology.
⟨2⟩1. LET: $B = \prod_{\alpha \in J} U_\alpha$ be a basic open set in the product topology, where each U_α is open in $\mathbb{R}$, and $U_\alpha = \mathbb{R}$ except for $\alpha = \alpha_1, \ldots, \alpha_n$.
⟨2⟩2. LET: $\vec{x} \in U$
⟨2⟩3. For $1 \leq i \leq n$, PICK $0 < \epsilon_i < 1$ such that $(x_{\alpha_i} - \epsilon_i, x_{\alpha_i} + \epsilon_i) \subseteq U_{\alpha_i}$.

$\langle 2\rangle 4$. LET: $\epsilon = \min(\epsilon_1, \ldots, \epsilon_n)$
$\langle 2\rangle 5$. $B_{\overline{\rho}}(\vec{x}, \epsilon) \subseteq B$
$\langle 3\rangle 1$. LET: $\vec{y} \in B_{\overline{\rho}}(\vec{x}, \epsilon)$
$\langle 3\rangle 2$. For $1 \leq i \leq n$, we have $y_i \in U_{\alpha_i}$
$\langle 4\rangle 1$. LET: $1 \leq i \leq n$
$\langle 4\rangle 2$. $\overline{d}(x_{\alpha_i}, y_{\alpha_i}) < \epsilon_i$
PROOF: From $\langle 2\rangle 4$ and $\langle 3\rangle 1$.
$\langle 4\rangle 3$. $d(x_{\alpha_i}, y_{\alpha_i}) < \epsilon_i$
PROOF: From $\langle 4\rangle 2$ since $\epsilon_i < 1$ ($\langle 2\rangle 3$).
$\langle 4\rangle 4$. Q.E.D.
PROOF: By $\langle 2\rangle 3$.
$\langle 1\rangle 2$. The uniform topology is coarser than the box topology.
$\langle 2\rangle 1$. LET: $\vec{x} \in \mathbb{R}^J$ and $\epsilon > 0$
PROVE: $B_{\overline{\rho}}(\vec{x}, \epsilon)$ is open in the box topology.
$\langle 2\rangle 2$. CASE: $\epsilon < 1$
PROOF: In this case, $B(\vec{x}, \epsilon) = \prod_{\alpha \in J}(x_\alpha - \epsilon, x_\alpha + \epsilon)$.
$\langle 2\rangle 3$. CASE: $\epsilon \geq 1$
PROOF: In this case, $B(\vec{x}, \epsilon) = \mathbb{R}^J$.
$\langle 1\rangle 3$. If J is finite then the product topology is the same as the box topology.
PROOF: Immediate from definitions.
$\langle 1\rangle 4$. If J is infinite then the uniform topology is distinct from the product topology.
$\langle 2\rangle 1$. $B(\vec{0}, 1/2)$ is not open in the product topology.
$\langle 3\rangle 1$. $\vec{0} \in B(\vec{0}, 1/2)$
$\langle 3\rangle 2$. LET: $\prod_{\alpha \in J} U_\alpha$ be any basic open set containing $\vec{0}$, where U_α is open in $\mathbb{R}$ for all α, and $U_\alpha = \mathbb{R}$ except for $\alpha = \alpha_1, \ldots, \alpha_n$
$\langle 3\rangle 3$. PICK $\alpha_0 \in J$ such that $\alpha_0 \neq \alpha_1, \ldots, \alpha_n$
$\langle 3\rangle 4$. LET: $\vec{x}$ be such that $x_{\alpha_0} = 1$, and $x_\alpha = 0$ for $\alpha \neq \alpha_0$.
$\langle 3\rangle 5$. $\vec{x} \in \prod_{\alpha \in J} U_\alpha$
$\langle 3\rangle 6$. $\vec{x} \notin B(\vec{0}, 1/2)$
$\langle 1\rangle 5$. If J is infinite then the uniform topology is distinct from the box topology.
$\langle 2\rangle 1$. PICK a countable sequence α_1, α_2, … in J
$\langle 2\rangle 2$. LET: $U = \prod_{\alpha \in J} U_\alpha$, where $U_{\alpha_n} = (-1/n, 1/n)$ for all n, and $U_\alpha = \mathbb{R}$ for all other α.
PROVE: U is not open in the uniform topology.
$\langle 2\rangle 3$. $\vec{0} \in U$
$\langle 2\rangle 4$. LET: $\epsilon > 0$
PROVE: $B(\vec{0}, \epsilon) \nsubseteq U$
$\langle 2\rangle 5$. PICK N such that $1/N < \epsilon$
$\langle 2\rangle 6$. LET: $\vec{x}$ be such that $x_{\alpha_N} = 1/N$ and $x_\alpha = 0$ for all other α
$\langle 2\rangle 7$. $\vec{x} \in B(\vec{0}, \epsilon)$
$\langle 2\rangle 8$. $\vec{x} \notin U$
$\square$

Proposition 10.1.22. *The space $\mathbb{R}^\omega$ under the uniform topology is not second countable.*

PROOF: The set of all sequences of 0s and 1s is discrete but uncountable. ∎

Corollary 10.1.22.1. *Not every metric space is second countable.*

Theorem 10.1.23. *Let X and Y be metric spaces. Let $f : X \to Y$ and $x \in X$. Then f is continuous at x if and only if, for every $\epsilon > 0$, there exists $\delta > 0$ such that, for all $x' \in X$, if $d(x, x') < \delta$ then $d(f(x), f(x')) < \epsilon$.*

PROOF:
⟨1⟩1. If f is continuous at x then, for every $\epsilon > 0$, there exists $\delta > 0$ such that, for all $x' \in X$, if $d(x, x') < \delta$ then $d(f(x), f(x')) < \epsilon$.
⟨2⟩1. ASSUME: f is continuous at x.
⟨2⟩2. LET: $\epsilon > 0$
⟨2⟩3. PICK a neighbourhood U of x such that $f(U) \subseteq B(f(x), \epsilon)$
PROOF: One exists by ⟨2⟩1, since $B(f(x), \epsilon)$ is a neighbourhood of $f(x)$.
⟨2⟩4. PICK $\delta > 0$ such that $B(x, \delta) \subseteq U$
PROOF: By ⟨2⟩3 and Lemma 10.1.5.
⟨2⟩5. LET: $x' \in X$ with $d(x, x') < \delta$
⟨2⟩6. $x' \in U$
PROOF: From ⟨2⟩4 and ⟨2⟩5.
⟨2⟩7. $f(x') \in B(f(x), \epsilon)$
PROOF: From ⟨2⟩3 and ⟨2⟩6.
⟨1⟩2. If, for all $\epsilon > 0$, there exists $\delta > 0$ such that, for all $x' \in X$, if $d(x, x') < \delta$ then $d(f(x), f(x')) < \epsilon$, then f is continuous at x.
⟨2⟩1. ASSUME: For all $\epsilon > 0$ there exists $\delta > 0$ such that, for all $x' \in X$, if $d(x, x') < \delta$ then $d(f(x), f(x')) < \epsilon$.
⟨2⟩2. LET: V be a neighbourhood of $f(x)$
⟨2⟩3. PICK $\epsilon > 0$ such that $B(f(x), \epsilon) \subseteq V$
PROOF: By Lemma 10.1.5.
⟨2⟩4. PICK $\delta > 0$ such that, for all $x' \in X$, if $d(x, x') < \delta$ then $d(f(x), f(x')) < \epsilon$.
PROOF: By ⟨2⟩1 and ⟨2⟩3.
⟨2⟩5. $B(x, \delta)$ is a neighbourhood of x
PROOF: By the definition of the metric topology.
⟨2⟩6. $f(B(x, \delta)) \subseteq V$
⟨3⟩1. LET: $x' \in B(x, \delta)$
⟨3⟩2. $d(f(x), f(x')) < \epsilon$
PROOF: From ⟨2⟩4.
⟨3⟩3. $x' \in V$
PROOF: From ⟨2⟩3.
∎

Lemma 10.1.24. *Addition is a continuous function $\mathbb{R}^2 \to \mathbb{R}$.*

PROOF:
⟨1⟩1. LET: $(x, y) \in \mathbb{R}^2$ and $\epsilon > 0$
⟨1⟩2. LET: $\delta = \epsilon/2$

$\langle 1 \rangle 3$. Let: $(x', y') \in \mathbb{R}^2$ be such that $\rho((x, y), (x', y')) < \delta$, where ρ is the square metric

$\langle 1 \rangle 4$. $|x - x'| < \delta$ and $|y - y'| < \delta$

$\langle 1 \rangle 5$. $|(x + y) - (x' + y')| < \epsilon$

Proof:

$$\begin{aligned} |(x + y) - (x' + y')| &\leq |x - x'| + |y - y'| \\ &< 2\delta && (\langle 1 \rangle 4) \\ &= \epsilon && (\langle 1 \rangle 2) \end{aligned}$$

$\langle 1 \rangle 6$. Q.E.D.

Proof: By Theorem 10.1.23.

□

Lemma 10.1.25. *Additive inverse is a continuous function* $- : \mathbb{R} \to \mathbb{R}$.

Proof: If $|x - y| < \epsilon$ then $|(-x) - (-y)| < \epsilon$. □

Lemma 10.1.26. *Multiplication is a continuous function* $\cdot : \mathbb{R}^2 \to \mathbb{R}$.

Proof:

$\langle 1 \rangle 1$. Let: $(x, y) \in \mathbb{R}^2$ and $\epsilon > 0$

$\langle 1 \rangle 2$. Let: $\delta = \min(1, \epsilon/(|x| + |y| + 1)$

$\langle 1 \rangle 3$. Let: $(x', y') \in \mathbb{R}^2$ and $\rho((x, y), (x', y')) < \delta$

$\langle 1 \rangle 4$. $|xy - x'y'| < \epsilon$

Proof:

$$\begin{aligned} |xy - x'y'| &= |x(y' - y) + y(x' - x) + (x - x')(y - y')| \\ &\leq |x||y' - y| + |y||x' - x| + |x - x'||y - y'| \\ &< |x|\delta + |y|\delta + \delta^2 && (\langle 1 \rangle 3) \\ &= \delta(|x| + |y| + \delta) \\ &\leq \delta(|x| + |y| + 1) && (\langle 1 \rangle 2) \\ &\leq \epsilon && (\langle 1 \rangle 2) \end{aligned}$$

□

Lemma 10.1.27. *Multiplicative inverse is a continuous function* $(\ )^{-1} : \mathbb{R} \setminus \{0\} \to \mathbb{R}$.

Proof:

$\langle 1 \rangle 1$. Let: $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be defined by $f(x) = x^{-1}$.

$\langle 1 \rangle 2$. Let: $a, b \in \mathbb{R}$ with $a < b$

Prove: $f^{-1}((a, b))$ is open

$\langle 1 \rangle 3$. Case: $0 < a < b$

Proof: $f^{-1}((a, b)) = (b^{-1}, a^{-1})$

$\langle 1 \rangle 4$. Case: $a < 0 < b$

Proof: $f^{-1}((a, b)) = (-\infty, a^{-1}) \cup (b^{-1}, +\infty)$

$\langle 1 \rangle 5$. Case: $a < b < 0$

Proof: $f^{-1}((a, b)) = (b^{-1}, a^{-1})$

□

Definition 10.1.28 (Uniform Convergence)**.** Let X be a set and Y a metric space. Let $f_n : X \to Y$ for $n \geq 1$, and $f : X \to Y$. Then f_n *converges uniformly* to f as $n \to \infty$ iff, for all $\epsilon > 0$, there exists N such that, for all $x \in X$ and $n \geq N$, $d(f_n(x), f(x)) < \epsilon$.

Theorem 10.1.29 (Uniform Limit Theorem)**.** *Let X be a topological space and Y a metric space. Let $f_n : X \to Y$ for $n \geq 1$ and $f : X \to Y$. If f_n converges uniformly to f and each f_n is continuous, then f is continuous.*

Proof:
$\langle 1 \rangle$1. Let: $x \in X$ and $\epsilon > 0$
$\langle 1 \rangle$2. Pick N such that, for all $x' \in X$ and $\delta > 0$, $d(f_n(x'), f(x')) < \epsilon/3$
$\langle 1 \rangle$3. Pick $\delta > 0$ such that, for all $x' \in X$, if $d(x, x') < \delta$ then $d(f_N(x), f_N(x')) < \epsilon/3$
$\langle 1 \rangle$4. For all $x' \in X$, if $d(x, x') < \delta$ then $d(f(x), f(x')) < \epsilon$
Proof:
$$\begin{aligned} d(f(x), f(x')) &\leq d(f(x), f_N(x)) + d(f_N(x), f_N(x')) + d(f_N(x'), f(x')) \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 \\ &= \epsilon \end{aligned}$$
□

Lemma 10.1.30. *Let X be a set. Let $f_n : X \to \mathbb{R}$ for $n \geq 1$ and $f : X \to \mathbb{R}$. Then f_n converges uniformly to f if and only if f_n converges to f in $\mathbb{R}^X$ under the uniform topology.*

Proof:
$\langle 1 \rangle$1. If f_n converges uniformly to f then f_n converges to f under the uniform topology.
$\langle 2 \rangle$1. Assume: f_n converges uniformly to f
$\langle 2 \rangle$2. Let: $\epsilon > 0$
$\langle 2 \rangle$3. Pick N such that, for all $x \in X$ and $n \geq N$, $d(f_n(x), f(x)) < \epsilon/2$
$\langle 2 \rangle$4. $\overline{\rho}(f_n, f) \leq \epsilon/2$
$\langle 2 \rangle$5. $\overline{\rho}(f_n, f) < \epsilon$
$\langle 1 \rangle$2. If f_n converges to f under the uniform topology then f_n converges uniformly to f.
$\langle 2 \rangle$1. Assume: f_n converges to f under the uniform topology.
$\langle 2 \rangle$2. Let: $\epsilon > 0$
$\langle 2 \rangle$3. Pick N such that, for all $n \geq N$, $\overline{\rho}(f_n, f) < \epsilon$
$\langle 2 \rangle$4. For all $n \geq N$ and $x \in X$, $d(f_n(x), f(x)) < \epsilon$
□

Theorem 10.1.31. *Every monotone increasing sequence of real numbers that is bounded above converges to its supremum.*

Proof:
$\langle 1 \rangle$1. Let: $\{s_n\}_{n \geq 1}$ be a monotone increasing sequence of real numbers bounded above with supremum l.

$\langle 1 \rangle 2.$ LET: $\epsilon > 0$
$\langle 1 \rangle 3.$ $l - \epsilon$ is not an upper bound for $\{s_n : n \geq 1\}$.
$\langle 1 \rangle 4.$ PICK N such that $x_N > l - \epsilon$
$\langle 1 \rangle 5.$ For all $n \geq N$, we hawe $l - \epsilon < x_n \leq l$
$\langle 1 \rangle 6.$ For all $n \geq N$, we have $|x_n - l| < \epsilon$
□

Definition 10.1.32 (Infinite Series)**.** Let $\{a_n\}_{n \geq 1}$ be a sequence of real numbers. The *infinite series* $\sum_{n=1}^{\infty} a_n$ *converges* to s iff $\sum_{n=1}^{N} a_n \to s$ as $N \to \infty$.

Proposition 10.1.33. *If* $\sum_{n=1}^{\infty} a_n = s$ *and* $\sum_{n=1}^{\infty} b_n = t$ *then* $\sum_{n=1}^{\infty} (ca - n + b_n) = cs + t$.

PROOF:This holds because $\sum_{n=1}^{N} (ca_n + b_n) = c \sum_{n=1}^{N} a_n + \sum_{n=1}^{N} b_n \to cs + t$ as $N \to \infty$. □

Theorem 10.1.34 (Comparison Test)**.** *If* $|a_i| \leq b_i$ *for all* i *and* $\sum_{i=1}^{\infty} b_i$ *converges, then* $\sum_{i=1}^{\infty} a_i$ *converges.*

PROOF:
$\langle 1 \rangle 1.$ $\sum_{i=1}^{\infty} |a_i|$ converges
  PROOF: $\sum_{i=1}^{N} |a_i|$ is a monotone increasing sequence bounded above by $\sum_{i=1}^{\infty} b_i$.
$\langle 1 \rangle 2.$ LET: $c_i = |a_i| + a_i$
$\langle 1 \rangle 3.$ $\sum_{i=1}^{\infty} c_i$ converges
  PROOF: $\sum_{i=1}^{N} c_i$ is a monotone increasing sequence bounded above by $2 \sum_{i=1}^{\infty} |a_i|$.
$\langle 1 \rangle 4.$ Q.E.D.
  PROOF: Since $a_i = c_i - |a_i|$.

Lemma 10.1.35. *If* $\sum_{n=1}^{\infty} a_n$ *converges then* $\sum_{n=N}^{\infty} a_n \to 0$ *as* $N \to \infty$.

PROOF:

$$\begin{aligned} \sum_{n=N}^{\infty} a_n &= \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N-1} a_n \\ &\to \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} a_n \\ &= 0 \end{aligned}$$

as $N \to \infty$. □

Theorem 10.1.36 (Weierstrass M-Test)**.** *Let* X *be a set and* $f_n : X \to \mathbb{R}$ *for* $n \geq 1$. *If* $|f_n(x)| \leq M_n$ *for all* $n \geq 1$ *and all* $x \in X$, *and if* $\sum_{n=1}^{\infty} M_n$ *converges, then*

$$\sum_{n=1}^{N} f_n(x) \to \sum_{n=1}^{\infty} f_n(x)$$

uniformly in x *as* $N \to \infty$.

PROOF:

⟨1⟩1. For $N \geq 1$,
LET: $s_N : X \to \mathbb{R}$, $s_N(x) = \sum_{n=1}^N f_n(x)$
⟨1⟩2. For all $x \in X$, $\sum_{n=1}^\infty f_n(x)$ converges.
PROOF: By the Comparison Test.
⟨1⟩3. LET: $s : X \to \mathbb{R}$, $s(x) = \sum_{n=1}^\infty f_n(x)$.
⟨1⟩4. For $N \geq 1$,
LET: $r_N = \sum_{n=N+1}^\infty M_n$
⟨1⟩5. For $1 \leq N < K$, we have $|s_K(x) - s_N(x)| \leq r_N$ for all $x \in X$
PROOF:

$$\begin{aligned}|s_K(x) - s_N(x)| &= \left|\sum_{n=N+1}^{K} f_n(x)\right| \\ &\leq \sum_{n=N+1}^{K} |f_n(x)| \\ &\leq \sum_{n=N+1}^{K} M_n \\ &\leq \sum_{n=N+1}^{\infty} M_n \\ &= r_N\end{aligned}$$

⟨1⟩6. For $N \geq 1$ and $x \in X$ we have $|s(x) - s_N(x)| \leq r_N$
PROOF: Let $K \to \infty$ in ⟨1⟩5.
⟨1⟩7. LET: $\epsilon > 0$
⟨1⟩8. PICK N such that, for all $N' \geq N$, we have $r_{N'} < \epsilon$
PROOF: Such an N exists by Lemma 10.1.35.
⟨1⟩9. For all $N' \geq N$ and $x \in X$ we have $|s_{N'}(x) - s(x)| < \epsilon$
□

Definition 10.1.37. Let X be a metric space. Let $x \in X$ and $A \subseteq X$ be nonempty. The *distance* from x to A is

$$d(x, A) = \inf_{a \in A} d(x, a) \ .$$

Lemma 10.1.38. *Let X be a metric space and $A \subseteq X$ be nonempty. Then the function $d(-, A) : X \to \mathbb{R}$ is continuous.*

PROOF:
⟨1⟩1. LET: $x \in X$ and $\epsilon > 0$
⟨1⟩2. LET: $y \in X$ with $d(x, y) < \epsilon$
⟨1⟩3. $|d(x, A) - d(y, A)| < \epsilon$
PROOF:
⟨2⟩1. $d(x, A) - d(y, A) < \epsilon$

PROOF:

$$\begin{aligned} d(x,A) &= \inf_{a \in A} d(x,a) \\ &\leq \inf_{a \in A} (d(x,y) + d(y,a)) \\ &= d(x,y) + \inf_{a \in A} d(y,a) \\ &= d(x,y) + d(y,A) \\ &< \epsilon + d(y,A) \end{aligned}$$

⟨2⟩2. $d(y,A) - d(x,A) < \epsilon$

PROOF: Similar.

⟨1⟩4. Q.E.D.

PROOF: By Theorem 10.1.23.

□

Definition 10.1.39 (Shrinking Map)**.** Let X be a metric space and $f : X \to X$. Then f is a *shrinking map* iff, for all $x, y \in X$ with $x \neq y$, we have $d(f(x), f(y)) < d(x, y)$.

Definition 10.1.40 (Contraction)**.** Let X be a metric space and $f : X \to X$. Then f is a *contraction* iff there exists $\alpha < 1$ such that, for all $x, y \in X$,

$$d(f(x), f(y)) \leq \alpha d(x, y) \ .$$

Proposition 10.1.41. *Every separable metric space is second countable.*

PROOF:

⟨1⟩1. LET: X be a separable metric space.

⟨1⟩2. PICK a countable dense set D

⟨1⟩3. LET: $\mathcal{B} = \{B(d,q) : d \in D, q \in \mathbb{Q}^+\}$

⟨1⟩4. $\mathcal{B}$ is a countable basis for X

□

Corollary 10.1.41.1. *The space $\mathbb{R}^\omega$ under the uniform topology is not separable.*

Corollary 10.1.41.2. *Not every metric space is separable.*

Corollary 10.1.41.3. *The space $\mathbb{R}^\omega$ under the box topology is not separable.*

Proposition 10.1.42 (CC)**.** *Every Lindelöf metric space is second countable.*

PROOF:

⟨1⟩1. LET: X be a Lindelöf metric space.

⟨1⟩2. For all $n \in \mathbb{Z}^+$, PICK a countable covering $\mathcal{A}_n$ of X by $1/n$-balls

PROOF: One exists by the Lindelöf condition, since the set of all $1/n$-balls covers X.

⟨1⟩3. $\bigcup_{n=1}^\infty \mathcal{A}_n$ is a countable basis.

□

Corollary 10.1.42.1. *The space $\mathbb{R}^\omega$ under the uniform topology is not Lindelöf.*

Corollary 10.1.42.2. *Not every metric space is Lindelöf.*

Proposition 10.1.43. *The space $\mathbb{R}_l$ is not metrizable.*

Proof: It is Lindelöf but not second countable. ∎

Proposition 10.1.44. *The ordered square is not metrizable.*

Proof: It is compact but not second countable. ∎

Proposition 10.1.45. *The space $\mathbb{R}^\omega$ under the uniform topology is not second countable.*

Proof: It contains a subspace homeomorphic to $\mathbb{R}$. ∎

Theorem 10.1.46 (AC)**.** *Every metrizable space is normal.*

Proof:
⟨1⟩1. Let: X be a metric space.
⟨1⟩2. Let: A and B be disjoint closed subspaces of X.
⟨1⟩3. For $a \in A$, Pick $\epsilon_a > 0$ such that $B(a, \epsilon_a)$ does not intersect B.
⟨1⟩4. For $b \in B$, Pick $\epsilon_b > 0$ such that $B(b, \epsilon_b)$ does not intersect A.
⟨1⟩5. Let: $U = \bigcup_{a \in A} B(a, \epsilon_a/2)$
⟨1⟩6. Let: $V = \bigcup_{b \in B} B(b, \epsilon_b/2)$
⟨1⟩7. $U \cap V = \emptyset$
⟨2⟩1. Let: $z \in U \cap V$
⟨2⟩2. Pick $a \in A$ and $b \in B$ such that $z \in B(a, \epsilon_a/2)$ and $z \in B(b, \epsilon_b/2)$
⟨2⟩3. Assume: w.l.o.g. $\epsilon_a \leq \epsilon_b$
⟨2⟩4. $a \in B(b, \epsilon_b)$
Proof:

$$\begin{aligned} d(a,b) &\leq d(a,z) + d(b,z) && \text{(Triangle Inequality)} \\ &< \epsilon_a/2 + \epsilon_b/2 && (\langle 2\rangle 2) \\ &\leq \epsilon_b && (\langle 2\rangle 3) \end{aligned}$$

⟨2⟩5. Q.E.D.
Proof: This contradicts ⟨1⟩4.
∎

Corollary 10.1.46.1. *The space $\mathbb{R}^\omega$ is normal.*

Corollary 10.1.46.2. *The space $\mathbb{R}_K$ is not metlizable.*

Proposition 10.1.47. *Every metrizable space is completely normal.*

Proof: Every subspace is metrizable (Lemma 10.1.12) hence normal (Theorem 10.1.46). ∎

Proposition 10.1.48. *Every metrizable space is perfectly normal.*

Proof:

$\langle 1\rangle 1$. LET: X be a metric space.
$\langle 1\rangle 2$. X is normal.
PROOF: Theorem 10.1.46
$\langle 1\rangle 3$. Every closed set is G_δ.
PROOF: If A is closed then $A = \bigcap_{q \in \mathbb{Q}^+} \{x \in X : d(A,x) < q\}$.
□

Theorem 10.1.49 (Urysohn Metrization Theorem (CC))**.** *Every second countable regular space is metrizable.*

PROOF:
$\langle 1\rangle 1$. LET: X be a second countable regular space.
$\langle 1\rangle 2$. X is normal.
$\langle 1\rangle 3$. PICK a countable basis $\mathcal{B} = \{B_n : n \in \mathbb{Z}^+\}$
$\langle 1\rangle 4$. For every pair of integers m, n with $\overline{B_m} \subseteq B_n$, PICK a continuous function $g_{mn} : X \to [0,1]$ such that $g_{mn}(\overline{B_m}) = \{1\}$ and $g_{mn}(X \setminus B_n) = \{0\}$
PROOF: By the Urysohn Lemma.
$\langle 1\rangle 5$. The set $\{g_{mn} : \overline{U_m} \subseteq U_n\}$ separates points from closed sets in X
$\langle 2\rangle 1$. LET: $x \in X$ and U be a neighbourhood of x
$\langle 2\rangle 2$. PICK $B_n \in \mathcal{B}$ such that $x \in B_n \subseteq U$
$\langle 2\rangle 3$. PICK V open such that $x \in V$ and $\overline{V} \subseteq B_n$
$\langle 2\rangle 4$. PICK $B_m \in \mathcal{B}$ such that $x \in B_m \subseteq V$
$\langle 2\rangle 5$. $g_{mn}(x) = 1$ and g_{mn} vanishes outside U
$\langle 1\rangle 6$. X is imbeddable in $[0,1]^\omega$
PROOF: By the Imbedding Theorem.
$\langle 1\rangle 7$. Q.E.D.

Corollary 10.1.49.1. *The space $\mathbb{R}^\omega$ under the box topology is not second countable.*

Proposition 10.1.50. *Not every second countable Hausdorff space is metrizable.*

PROOF: $\mathbb{R}_K$ is second countable and Hausdorff but not metrizable (because it is not regular). □

Proposition 10.1.51. *There exists a space that is completely normal, first countable, Lindelöf and separable but not metrizable.*

PROOF: The space $\mathbb{R}_l$ is all of these. □

Proposition 10.1.52. *$\overline{S_\Omega}$ is not metrizable.*

PROOF: It is compact but not sequentially compact. □

Proposition 10.1.53. *Every compact metric space is second countable.*

PROOF:
$\langle 1\rangle 1$. LET: X be a compact etric space

⟨1⟩2. For every $n \geq 1$, Pick a finite covering $\mathcal{A}_n$ of X by open balls of radius $1/n$

Proof: Such a covering exists because $\{B_{1/n}(x) : x \in X\}$ covers X.

⟨1⟩3. $\bigcup_{n=1}^{\infty} \mathcal{A}_n$ is a countable basis for X

□

Corollary 10.1.53.1. *The space $\mathbb{R}^\omega$ under the uniform topology is not compact.*

Corollary 10.1.53.2. *The space $\mathbb{R}^\omega$ under the uniform topology is not limit point compact.*

Proposition 10.1.54. *The space $\mathbb{R}^\omega$ under the box topology is not locally compact.*

Proof:

⟨1⟩1. Assume: $\mathbb{R}^\omega$ under the box topology is locally compact.

⟨1⟩2. For every point x, there exists a basic open set $B = \prod_{i=0}^{\infty} U_i$ such that $x \in B$ and $\overline{B}$ is compact.

⟨1⟩3. The box topology on $\overline{B}$ is the same as the product topology on $\overline{B}$

Proof: By Corollary 9.4.11.1.

⟨1⟩4. The box topology on $\overline{B}$ is strictly finer than the product topology.

Proof:By Theorem 10.1.21.

□

Proposition 10.1.55. *Not every metrizable space is connected.*

Proof: The discrete space with two points is metrizable but not connected. □

Corollary 10.1.55.1. *Not every metrizable space is path connected.*

Proposition 10.1.56. *Not every metric space is limit point compact.*

Proof: The space $\mathbb{R}$ is not limit point compact. □

Proposition 10.1.57. *Not every metric space is locally compact.*

The space $\mathbb{R}^\omega$ in the uniform topology is not locally compact.

10.2 Isometries

Definition 10.2.1 (Isometry)**.** Let X be a metric space. An *isometry* of X is a function $f : X \to X$ such that, for all $x, y \in X$,

$$d(f(x), f(y)) = d(x, y) \ .$$

10.3 Lebesgue Numbers

Definition 10.3.1 (Lebesgue Number)**.** Let X be a metric space and $\mathcal{A}$ an open covering of X. A *Lebesgue number* for $\mathcal{A}$ is a real $\delta > 0$ such that, for every nonempty set $A \subseteq X$ of diameter $< \delta$, there exists $U \in \mathcal{A}$ such that $A \subseteq U$.

Lemma 10.3.2 (Lebesgue Number Lemma)**.** *In a compact metric space, every open covering has a Lebesgue number.*

PROOF:
⟨1⟩1. LET: X be a compact metric space and $\mathcal{A}$ an open covering of X
PROVE: There exists a Lebesgue number δ for $\mathcal{A}$.
⟨1⟩2. ASSUME: w.l.o.g. $X \notin \mathcal{A}$
PROOF: If $X \in \mathcal{A}$ then we can take $\delta = 1$.
⟨1⟩3. PICK a finite subcovering $\{U_1, \ldots, U_n\} \subseteq \mathcal{A}$ that covers X
⟨1⟩4. For $1 \leq i \leq n$,
LET: $C_i = X \setminus U_i$
⟨1⟩5. LET: $f : X \to \mathbb{R}$ be defined by

$$f(x) = 1/n \sum_{i=1}^{n} d(x, C_i) \ .$$

PROOF: Each C_i is nonempty by ⟨1⟩2.
⟨1⟩6. For all $x \in X$ we have $f(x) > 0$
⟨2⟩1. LET: $x \in X$
⟨2⟩2. PICK i such that $x \in U_i$
PROOF: By ⟨1⟩3.
⟨2⟩3. PICK $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U_i$
PROOF: By Lemma 10.1.5.
⟨2⟩4. $d(x, C_i) \geq \epsilon$
⟨1⟩7. f is continuous
PROOF: From Lemma 10.1.38.
⟨1⟩8. LET: $\delta = \min f(X)$
PROVE: For every nonempty set $A \subseteq X$ with diameter $< \delta$, there exists $U \in \mathcal{A}$ such that $A \subseteq U$
PROOF: $f(X)$ has a minimum by the Extreme Value Theorem.
⟨1⟩9. LET: $A \subseteq X$ be nonempty with $\operatorname{diam} A < \delta$
⟨1⟩10. PICK $x_0 \in A$
⟨1⟩11. LET: i be such that $d(x_0, C_i)$ is greatest among $d(x_0, C_1)$, …, $d(x_0, C_n)$
⟨1⟩12. $\delta \leq d(x_0, C_i)$
PROOF:

$$\begin{aligned}
\delta &\leq f(x_0) && (\langle 1\rangle 8) \\
&= 1/n \sum_{j=1}^{n} d(x_0, C_j) && (\langle 1\rangle 5) \\
&\leq 1/n \sum_{j=1}^{n} d(x_0, C_i) && (\langle 1\rangle 11) \\
&= d(x_0, C_i)
\end{aligned}$$

⟨1⟩13. $x_0 \in U_i$
PROOF: $x_0 \notin C_i$ because $d(x_0, C_i) > 0$.
□

Theorem 10.3.3 (DC)**.** *Let X be a metrizable space. Then the following are* *equivalent:*

1. *X is compact.*
2. *X is limit point compact.*
3. *X is sequentially compact.*

PROOF:
$\langle 1\rangle 1.$ $1 \Rightarrow 2$
PROOF: Theorem 9.4.22.
$\langle 1\rangle 2.$ $2 \Rightarrow 3$
$\langle 2\rangle 1.$ ASSUME: X is limit point compact.
$\langle 2\rangle 2.$ LET: (x_n) be a sequence in X
PROVE: (x_n) has a convergent subsequence.
$\langle 2\rangle 3.$ CASE: $\{x_n : n \in \mathbb{Z}^+\}$ is finite.
PROOF: In this case, (x_n) has a constant subsequence.
$\langle 2\rangle 4.$ CASE: $\{x_n : n \in \mathbb{Z}^+\}$ is infinite.
$\langle 3\rangle 1.$ PICK a limit point l of $\{x_n : n \in \mathbb{Z}^+\}$
$\langle 3\rangle 2.$ For every poisitive integer r, PICK n_r such that $n_r > n_{r-1}$ and $d(x_{n_r}, l) < 1/r$
PROOF: There always exists such an n_r since $B(l, 1/r)$ intersects $\{x_n : n \in \mathbb{Z}^+\}$ in infinitely many points by Theorem 6.1.2.
$\langle 3\rangle 3.$ $x_{n_r} \to l$ as $r \to \infty$
$\langle 1\rangle 3.$ $3 \Rightarrow 1$
$\langle 2\rangle 1.$ ASSUME: X is sequentially compact.
$\langle 2\rangle 2.$ Every open covering of X has a Lebesgue number.
$\langle 3\rangle 1.$ LET: $\mathcal{A}$ be an open covering of X.
$\langle 3\rangle 2.$ ASSUME: for a contradiction that, for all $\delta > 0$, there exists a set $C \subseteq X$ with $\operatorname{diam} C < \delta$ such that there is no $U \in \mathcal{A}$ such that $C \subseteq U$
$\langle 3\rangle 3.$ For $n \geq 1$, PICK $C_n \subseteq X$ with $\operatorname{diam} C_n < 1/n$ such that there is no $U \in \mathcal{A}$ such that $C_n \subseteq U$
$\langle 3\rangle 4.$ For $n \geq 1$, PICK $x_n \in C_n$
$\langle 3\rangle 5.$ PICK a convergent subsequence (x_{n_r}) of (x_n)
PROOF: By $\langle 2\rangle 1$.
$\langle 3\rangle 6.$ LET: $x_{n_r} \to l$ as $r \to \infty$
$\langle 3\rangle 7.$ PICK $U \in \mathcal{A}$ with $l \in U$
PROOF: By $\langle 3\rangle 1$
$\langle 3\rangle 8.$ PICK $\epsilon > 0$ such that $B(l, \epsilon) \subseteq U$
PROOF: By Lemma 10.1.5.
$\langle 3\rangle 9.$ PICK R such that $1/n_R < \epsilon/2$ and $d(x_{n_R}, l) < \epsilon/2$
PROOF: By $\langle 3\rangle 6$
$\langle 3\rangle 10.$ $C_{n_R} \subseteq U$

PROOF:

$$\begin{aligned} C_{n_R} &\subseteq B(x_{n_R}, 1/n_R) && (\langle 3\rangle 3,\ \langle 3\rangle 4) \\ &\subseteq B(x_{n_R}, \epsilon/2) && (\langle 3\rangle 9) \\ &\subseteq B(l, \epsilon) && (\langle 3\rangle 9) \\ &\subseteq U && (\langle 3\rangle 8) \end{aligned}$$

$\langle 3\rangle$11. Q.E.D.
PROOF: This contradicts $\langle 3\rangle$3.

$\langle 2\rangle$3. For all $\epsilon > 0$, there exists a finite covering of X by ϵ-balls.
$\langle 3\rangle$1. LET: $\epsilon > 0$
$\langle 3\rangle$2. ASSUME: for a contradiction there is no finite covering of X by ϵ-balls.
$\langle 3\rangle$3. PICK a sequence (x_n) in X such that, for all n,

$$x_n \notin B(x_1, \epsilon) \cup \cdots \cup B(x_{n-1}, \epsilon) \ .$$

$\langle 3\rangle$4. For all m, n with $m > n$ we have $d(x_m, x_n) \geq \epsilon$
$\langle 3\rangle$5. Any $\epsilon/2$-ball contains at most one element of (x_n).
$\langle 3\rangle$6. (x_n) has no convergent subsequence.
$\langle 3\rangle$7. Q.E.D.
PROOF: This contradicts $\langle 2\rangle$1.

$\langle 2\rangle$4. LET: $\mathcal{A}$ be an open covering of X
$\langle 2\rangle$5. PICK a Lebesgue number δ for $\mathcal{A}$
PROOF: By $\langle 2\rangle$2.
$\langle 2\rangle$6. PICK a finite covering $\{B_1, \ldots, B_n\}$ of X by $\delta/3$-balls.
PROOF: By $\langle 2\rangle$3.
$\langle 2\rangle$7. For $1 \leq i \leq n$, PICK $U_i \in \mathcal{A}$ such that $B_i \subseteq U_i$
$\langle 2\rangle$8. $\{U_1, \ldots, U_n\}$ covers X.

□

Corollary 10.3.3.1. *S_Ω is not metrizable.*

PROOF: It is limit point compact (Corollary 9.4.19.2) but not compact (Proposition 9.4.2). □

Corollary 10.3.3.2. *The space $\mathbb{R}^\omega$ is not limit point compact.*

10.4 Uniform Continuity

Definition 10.4.1 (Uniform Continuity)**.** Let X and Y be metric spaces and $f : X \to Y$. Then f is *uniformly continuous* iff, for all $\epsilon > 0$, there exists $\delta > 0$ such that, for all $x, y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$.

Theorem 10.4.2 (Uniform Continuity Theorem)**.** *Let X be a compact metric space, Y a metric space, and $f : X \to Y$ be continuous. Then f is uniformly continuous.*

PROOF:
$\langle 1\rangle$1. LET: $\epsilon > 0$
PROVE: There exists $\delta > 0$ such that, for all $x, y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$.

$\langle 1 \rangle 2$. LET: $\mathcal{A} = \{f^{-1}(B(y, \epsilon/2)) : y \in Y\}$
$\langle 1 \rangle 3$. $\mathcal{A}$ is an open covering of X
$\langle 1 \rangle 4$. PICK a Lebesgue number δ for $\mathcal{A}$.
PROVE: For all $x, y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$
PROOF: By the Lebesgue Number Lemma
$\langle 1 \rangle 5$. LET: $x, y \in X$ with $d(x, y) < \delta$
$\langle 1 \rangle 6$. $\mathrm{diam}\{x, y\} < \delta$
$\langle 1 \rangle 7$. PICK $z \in Y$ such that $\{x, y\} \subseteq f^{-1}(B(z, \epsilon/2))$
$\langle 1 \rangle 8$. $d(f(x), f(y)) < \epsilon$
□

10.5 Locally Metrizable Spaces

Definition 10.5.1 (Locally Metrizable)**.** A space is *locally metrizable* iff every point has a metrizable neighbourhood.

Proposition 10.5.2. *Every metrizable space is locally metrizable.*

PROOF: Trivial. □

Corollary 10.5.2.1. *The space $\mathbb{R}^\omega$ is locally metrizable.*

Proposition 10.5.3. *A compact Hausdorff space is metrizable if and only if it is locally metrizable.*

PROOF:
$\langle 1 \rangle 1$. LET: X be a locally metrizable compact Hausdorff space
$\langle 1 \rangle 2$. X is regular
PROOF: Lemma 9.4.18
$\langle 1 \rangle 3$. X is second countable
$\langle 2 \rangle 1$. $\{U : U$ open in X and metrizable$\}$ covers X
$\langle 2 \rangle 2$. PICK a finite subcover $U_1, \ldots, U_n$
$\langle 2 \rangle 3$. For $1 \leq i \leq n$, PICK a countable basis $\mathcal{B}_i$ of U_i
$\langle 2 \rangle 4$. $\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_n$ is a basis for X
$\langle 1 \rangle 4$. Q.E.D.
PROOF: By the Urysohn Metrization Theorem.
□

Corollary 10.5.3.1. *$\overline{S_\Omega}$ is not locally metrizable.*

Corollary 10.5.3.2. *The ordered square is not locally metrizable.*

Proposition 10.5.4. *Every subspace of a locally metrizable space is locally metrizable.*

PROOF:
$\langle 1 \rangle 1$. LET: X be locally metrizable and $Y \subseteq X$
$\langle 1 \rangle 2$. LET: $y \in Y$

$\langle 1 \rangle 3$. PICK a metrizable neighbourhood U of y in X
$\langle 1 \rangle 4$. $U \cap Y$ is a metrizable neighbourhood of y in Y
□

Corollary 10.5.4.1. *$S_\Omega \times \overline{S_\Omega}$ is not locally metrizable.*

PROOF: It has a subspace homeomorphic to $\overline{S_\Omega}$. □

Proposition 10.5.5 (CC)**.** *Every locally metrizable regular Lindelöf space is metrizable.*

PROOF:
$\langle 1 \rangle 1$. LET: X be a locally metrizable regular Lindelöf space.
$\langle 1 \rangle 2$. Every point in X has an open second countable neighbourhood.
  $\langle 2 \rangle 1$. LET: $x \in X$
  $\langle 2 \rangle 2$. PICK an open metrizable U containing x
    PROOF: X is locally metrizable ($\langle 1 \rangle 1$)
  $\langle 2 \rangle 3$. PICK an open V such that $x \in V \subseteq \overline{V} \subseteq U$
    PROOF: Proposition 6.3.2
  $\langle 2 \rangle 4$. $\overline{V}$ is Lindelöf
    PROOF: Proposition 13.1.32
  $\langle 2 \rangle 5$. $\overline{V}$ is second countable
    PROOF: Proposition 10.1.42
$\langle 1 \rangle 3$. PICK a countable covering of secound countable open sets $\mathcal{U}$
  PROOF: X is Lindelöf ($\langle 1 \rangle 1$)
$\langle 1 \rangle 4$. For $U \in \mathcal{U}$, PICK a countable basis $\mathcal{B}_U$
$\langle 1 \rangle 5$. $\bigcup_{U \in \mathcal{U}} \mathcal{B}_U$ is a countable basis for X
  $\langle 2 \rangle 1$. LET: $x \in U$ where U is open in X
  $\langle 2 \rangle 2$. PICK $V \in \mathcal{U}$ such that $x \in V$
  $\langle 2 \rangle 3$. There exists $B \in \mathcal{B}_V$ such that $x \in B \subseteq U \cap V$
$\langle 1 \rangle 6$. Q.E.D.
  PROOF: By the Urysohn Metrization Theorem.
□

Corollary 10.5.5.1. *$\mathbb{R}_l$ is not locally metrizable.*

Proposition 10.5.6. *The Sorgenfrey plane is not locally metrizable.*

PROOF:
$\langle 1 \rangle 1$. LET: U be any neighbourhood of $(0,0)$
    PROVE: U is not Lindelöf
$\langle 1 \rangle 2$. PICK $a > 0$ such that $[0,a)^2 \subseteq U$
$\langle 1 \rangle 3$. LET: $L = \{(x, a-x) : 0 < x < a\}$
$\langle 1 \rangle 4$. L is closed in U
  PROOF: By Lemma 6.5.16 since $(x,y) \mapsto (x, a+y)$ is a homeomorphism of $\mathbb{R}_l^2$ with itself.
$\langle 1 \rangle 5$. LET: $\mathcal{U} = \{U \setminus L\} \cup \{([x,b) \times [a-x,c)) \cap U : b > a, c > a-x\}$
$\langle 1 \rangle 6$. $\mathcal{U}$ covers U

⟨1⟩7. No countable subset of $\mathcal{U}$ covers U

PROOF: Every set of the for $[x, b) \times [a - x, c)$ intersects L in exactly one point.

□

Corollary 10.5.6.1. *The Sorgenfrey plane is not metrizable.*

Proposition 10.5.7. *The space $\mathbb{R}_K$ is locally metrizable.*

PROOF: The set $(-1, 1) - K$ is a metrizable neighbourhood of 0. For any other point p, pick an open interval around p that does not contain 0. □

Proposition 10.5.8. *The product of two locally metrizable spaces is locally metrizable.*

PROOF:

⟨1⟩1. LET: X and Y be locally metrizable

⟨1⟩2. LET: $(a, b) \in X \times Y$

⟨1⟩3. PICK metrizable neighbourhoods U of a and V of b

⟨1⟩4. $U \times V$ is a metrizable neighbourhood of (a, b).

PROOF: By Lemma 10.1.19.

□

Proposition 10.5.9. *The product of two locally metrizable spaces is locally metrizable.*

PROOF:

⟨1⟩1. LET: X and Y be locally metrizable

⟨1⟩2. LET: $(a, b) \in X \times Y$

⟨1⟩3. PICK metrizable neighbourhoods U of a and V of b

⟨1⟩4. $U \times V$ is a metrizable neighbourhood of (a, b).

PROOF: By Lemma 10.1.19.

□

Proposition 10.5.10. *The space $\mathbb{R}_K^\omega$ is not locally metrizable.*

PROOF: If it were, then there would be a basic open set $\prod_n U_n$ that is metrizable, but then $\mathbb{R}_K$ would be metrizable as it is homeomorphic to a subspace of $\prod_n U_n$.

□

Corollary 10.5.10.1. *The product of a countable family of locally metrizable spaces is not necessarily locally metrizable.*

Proposition 10.5.11. *The continuous image of a locally metrizable space is not necessarily locally metrizable.*

PROOF: The identity map from the discrete two-point space to the indiscrete two-point space is continuous. □

Chapter 11

Manifolds

11.1 Manifolds

Definition 11.1.1 (Manifold)**.** Let $m \geq 1$. An *m-manifold* is a second countable Hausdorff space such that each point has a neighbourhood that is homeomorphic to an open subset of $\mathbb{R}^m$.

A *curve* is a 1-manifold and a *surface* is a 2-manifold.

Definition 11.1.2 (Support)**.** Let X be a topological space and $\phi : X \to \mathbb{R}$ be a function. Then the *support* of ϕ is the closure of $\phi^{-1}(\mathbb{R} \setminus \{0\})$.

Definition 11.1.3 (Partition of Unity)**.** Let X be a topological space. Let $\{U_1, \dots, U_n\}$ be a finite indexed open covering of X. An indexed family of continuous functions $\phi_1, \dots, \phi_n : X \to [0, 1]$ is a *partition of unity* dominated by $\{U_1, \dots, U_n\}$ iff:

1. $\operatorname{supp} \phi_i \subseteq U_i$ for all i;
2. $\sum_{i=1}^{n} \phi_i(x) = 1$ for all $x \in X$.

Theorem 11.1.4 (Existence of Finite Partitions of Unity)**.** *Let X be a normal space. Let $\{U_1, \dots, U_n\}$ be a finite indexed open covering of X. Then there exists a partition of unity dominated by $\{U_1, \dots, U_n\}$.*

PROOF:

⟨1⟩1. For every finite indexed open covering $\{U_1, \dots, U_n\}$ of X, there exists a finite indexed open covering $\{V_1, \dots, V_n\}$ such that $\overline{V_i} \subseteq U_i$

⟨2⟩1. For $1 \leq k \leq n$, there exist open sets $V_1, \dots, V_k$ such that $\overline{V_i} \subseteq U_i$ for all i and $\{V_1, \dots, V_k, U_{k+1}, \dots, U_n\}$ covers X

⟨3⟩1. ASSUME: as an induction hypothesis that $0leqk < k$ and there exist open sets $V_1, \dots, V_k$ such that $\overline{V_i} \subseteq U_i$ for all i and $\{V_1, \dots, V_k, U_{k+1}, \dots, U_n\}$ covers X

⟨3⟩2. LET: $A = X \setminus (V_1 \cup \cdots \cup V_k) \setminus (U_{k+2} \cup \cdots \cup U_n)$

⟨3⟩3. A is closed

$\langle 3\rangle 4.$ $A \subseteq U_{k+1}$

Proof: Since $\{V_1, \ldots, V_k, U_{k+1}, \ldots, U_n\}$ covers X

$\langle 3\rangle 5.$ Pick an open set V_{k+1} such that $A \subseteq V_{k+1}$ and $\overline{V_{k+1}} \subseteq U_{k+1}$

Proof: By Proposition 6.3.2

$\langle 3\rangle 6.$ $\{V_1, \ldots, V_k, V_{k+1}, U_{k+2}, \ldots, U_n\}$ covers X

$\langle 1\rangle 2.$ Pick an open covering $\{V_1, \ldots, V_n\}$ with $\overline{V_i} \subseteq U_i$ for all i

Proof: By $\langle 1\rangle 1$.

$\langle 1\rangle 3.$ Pick an open covering $\{W_1, \ldots, W_n\}$ with $\overline{W_i} \subseteq V_i$ for all i

Proof: By $\langle 1\rangle 1$.

$\langle 1\rangle 4.$ For $1 \leq i \leq n$, Pick a continuous function $\psi_i : X \to [0, 1]$ such that $\psi_i(\overline{W_i}) = \{1\}$ and $\psi_i(X \setminus V_i) = \{0\}$

Proof: By the Urysohn Lemma.

$\langle 1\rangle 5.$ Let: $\Psi : X \to \mathbb{R}$ where $\Psi(x) = \sum_{i=1}^{n} \psi_i(x)$

$\langle 1\rangle 6.$ $\Psi(x) > 0$ for all $x \in X$

$\langle 2\rangle 1.$ Let: $x \in X$

$\langle 2\rangle 2.$ Pick i such that $x \in W_i$

$\langle 2\rangle 3.$ $\psi_i(x) = 1$

$\langle 1\rangle 7.$ For $1 \leq j \leq n$,

Let: $\phi_j(x) = \frac{\psi_j(x)}{\Psi(x)}$

$\langle 1\rangle 8.$ $\psi_1, \ldots, \psi_n$ are a partition of unity dominated by $\{U_1, \ldots, U_n\}$

$\langle 2\rangle 1.$ $\operatorname{supp} \psi_i \subseteq U_i$

$\langle 3\rangle 1.$ ${\psi_i}^{-1}(\mathbb{R} \setminus \{0\}) \subseteq V_i$

Proof: By $\langle 1\rangle 4$

$\langle 3\rangle 2.$ $\operatorname{supp} \psi_i \subseteq \overline{V_i}$

Proof: Proposition 3.8.5

$\langle 2\rangle 2.$ $\sum_{i=1}^{n} \psi_i(x) = 1$ for all $x \in X$

$\square$

Theorem 11.1.5. *Let X be a compact Hausdorff space. Suppose that, for every $x \in X$, there exists a neighbourhood U of x and a positive integer k such that U can be imbedded in $\mathbb{R}^k$. Then there exists a positive integer N such that X can be imbedded in $\mathbb{R}^N$.*

Proof:

$\langle 1\rangle 1.$ Pick a finite open covering $\{U_1, \ldots, U_n\}$ of X such that each U_i can be imbedded in $\mathbb{R}^k$ for some k

Proof: Since $\{U$ open in $X : U$ can be imbedded in $\mathbb{R}^k$ for some $k\}$ covers X.

$\langle 1\rangle 2.$ For $1 \leq i \leq n$, Pick a positive integer k_i and an imbedding $g_i : U_i \to \mathbb{R}^{k_i}$

$\langle 1\rangle 3.$ Pick a partition of unity $\phi_1, \ldots, \phi_n$ dominated by $\{U_1, \ldots, U_n\}$

$\langle 2\rangle 1.$ X is normal

Proof: By Lemma 9.4.18.

$\langle 2\rangle 2.$ Q.E.D.

Proof: Theorem 11.1.4

$\langle 1\rangle 4.$ For $1 \leq i \leq n$,

Let: $A_i = \operatorname{supp} \phi_i$

⟨1⟩5. For $1 \leq i \leq n$,
LET: $h_i : X \to \mathbb{R}^{k_i}$ be defined by

$$h_i(x) = \begin{cases} \phi_i(x)g_i(x) & \text{for } x \in U_i \\ \vec{0} & \text{for } x \in X \setminus A_i \end{cases}$$

PROOF: If $x \in U_i$ and $x \in X \setminus A_i$ then $x \notin \operatorname{supp} \phi_i$ so $\phi_i(x) = 0$

⟨1⟩6. LET: $N = n + k_1 + \cdots + k_n$

⟨1⟩7. LET: $F : X \to \mathbb{R}^N$ be the function

$$F(x) = (\phi_1(x), \ldots, \phi_n(x), h_1(x), \ldots, h_n(x))$$

⟨1⟩8. F is an imbedding
⟨2⟩1. F is continuous
PROOF: Each h_i is continuous by Theorem 5.2.13.
⟨2⟩2. F is injective
⟨3⟩1. ASSUME: $F(x) = F(y)$
⟨3⟩2. PICK i such that $\phi_i(x) > 0$
PROOF: Since $\sum_i \phi_i(x) = 1$ (⟨1⟩3)
⟨3⟩3. $\phi_i(y) = 0$
PROOF: By ⟨3⟩1
⟨3⟩4. $x, y \in U_i$
PROOF: Since $\operatorname{supp} \phi_i \subseteq U_i$
⟨3⟩5. $h_i(x) = h_i(y)$
PROOF: By ⟨3⟩1
⟨3⟩6. $g_i(x) = g_i(y)$
PROOF: By ⟨1⟩5
⟨3⟩7. $x = y$
PROOF: By ⟨1⟩2
⟨2⟩3. Q.E.D.
PROOF: By Theorem 9.4.11

□

Corollary 11.1.5.1. *Every compact manifold can be imbedded in $\mathbb{R}^N$ for some N.*

Proposition 11.1.6. *The line with two origins is a second countable T_1 space where every point has a neighbourhood that is homeomorphic to an open subset of $\mathbb{R}$, but it is not a 1-manifold.*

Chapter 12

Normed Spaces

12.1 The Norm on $\mathbb{R}^n$

Definition 12.1.1 (Norm)**.** Given $\vec{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, the *norm* $\|\vec{x}\|$ is defined by

$$\|\vec{x}\| = \sqrt{x_1^2 + \cdots + x_n^2} \ .$$

Definition 12.1.2 (Vector Sum)**.** Define the *sum* of $\vec{x}, \vec{y} \in \mathbb{R}^n$ by

$$\vec{x} + \vec{y} = (x_1 + y_1, \ldots, x_n + y_n) \ .$$

Definition 12.1.3 (Scalar Product)**.** Given $c \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^n$, define the *scalar product* $c\vec{x}$ to be

$$c\vec{x} = (cx_1, \ldots, cx_n) \ .$$

Definition 12.1.4 (Inner Product)**.** The *inner product* of $\vec{x}, \vec{y} \in \mathbb{R}^n$ is

$$\vec{x} \cdot \vec{y} = x_1 y_1 + \cdots + x_n y_n \ .$$

Lemma 12.1.5.

$$\vec{x} \cdot (\vec{y} + \vec{z}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z}$$

Proof: Both are equal to $\sum_{i=1}^n (x_i y_i + x_i z_i)$. ☐

Lemma 12.1.6.

$$|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \|\vec{y}\|$$

Proof:
$\langle 1 \rangle 1$. Case: $\vec{x} = \vec{0}$ or $\vec{y} = \vec{0}$
Proof: In this case, both sides are 0.
$\langle 1 \rangle 2$. Case: $\vec{x} \neq \vec{0} \neq \vec{y}$
$\langle 2 \rangle 1$. Let: $a = 1/\|\vec{x}\|$, $b = 1/\|\vec{y}\|$
$\langle 2 \rangle 2$. $2 + 2ab\vec{x} \cdot \vec{y} \geq 0$
$\langle 3 \rangle 1$. $\|a\vec{x} + b\vec{y}\|^2 \geq 0$

$\langle 3\rangle 2.$ $\sum_{i=1}^{n}(ax_i + by_i)^2 \geq 0$
$\langle 3\rangle 3.$ $a^2 \sum_{i=1}^{n} x_i^2 + b^2 \sum_{i=1}^{n} y_i^2 + 2ab \sum_{i=1}^{n} x_i y_i \geq 0$
$\langle 3\rangle 4.$ $a^2 \|\vec{x}\|^2 + b^2 \|\vec{y}\|^2 + 2ab\vec{x} \cdot \vec{y} \geq 0$
$\langle 2\rangle 3.$ $2 - 2ab\vec{x} \cdot \vec{y} \geq 0$
PROOF: Similar.
$\langle 2\rangle 4.$ $2 - 2ab|\vec{x} \cdot \vec{y}| \geq 0$
PROOF: From $\langle 2\rangle 2$ and $\langle 2\rangle 3$.
$\langle 2\rangle 5.$ $|\vec{x} \cdot \vec{y}| \leq 1/ab$
∎

Lemma 12.1.7.

$$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$$

PROOF:

$$\begin{aligned}
\|\vec{x} + \vec{y}\|^2 &= (\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y}) \\
&= \|\vec{x}\|^2 + 2\vec{x} \cdot \vec{y} + \|\vec{y}\|^2 && \text{(Lemma 12.1.5)} \\
&\leq \|\vec{x}\|^2 + 2\|\vec{x}\|\|\vec{y}\| + \|\vec{y}\|^2 && \text{(Lemma 12.1.6)} \\
&= (\|\vec{x}\| + \|\vec{y}\|)^2
\end{aligned}$$

∎

Definition 12.1.8 (Euclidean Metric)**.** The *euclidean metric* on $\mathbb{R}^n$ is given by

$$d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|$$

We prove this is a metric.

PROOF:
$\langle 1\rangle 1.$ $d(\vec{x}, \vec{y}) \geq 0$
PROOF: Immediate from definitions.
$\langle 1\rangle 2.$ $d(\vec{x}, \vec{y}) = 0$ iff $\vec{x} = \vec{y}$
PROOF: Immediate from definitions.
$\langle 1\rangle 3.$ $d(\vec{x}, \vec{y}) = d(\vec{y}, \vec{x})$
PROOF: Immediate from definitions.
$\langle 1\rangle 4.$ $d(\vec{x}, \vec{z}) \leq d(\vec{x}, \vec{y}) + d(\vec{y}, \vec{z})$
PROOF: From Lemma 12.1.7.
∎

Lemma 12.1.9. *Let d be the euclidean topology on $\mathbb{R}^n$ and ρ the square topology. Then, for all $x, y \in \mathbb{R}^n$, we have*

$$\rho(x, y) \leq d(x, y) \leq \sqrt{n}\rho(x, y)$$

PROOF:
$\langle 1\rangle 1.$ $\rho(x, y) \leq d(x, y)$
$\langle 2\rangle 1.$ For $1 \leq i \leq n$ we have $|x_i - y_i| \leq d(x, y)$
PROOF: By the definition of the euclidean metric.
$\langle 2\rangle 2.$ Q.E.D.
PROOF: By the definition of the square metric.

$\langle 1 \rangle 2.$ $d(x,y) \leq \sqrt{n}\rho(x,y)$

PROOF:

$$\begin{aligned} d(x,y) &= \sqrt{(x_1-y_1)^2+\cdots+(x_n-y_n)^2} \\ &\leq \sqrt{\rho(x,y)^2+\cdots+\rho(x,y)^2} \\ &= \sqrt{n\rho(x,y)^2} \\ &= \sqrt{n}\rho(x,y) \end{aligned}$$

$\square$

Corollary 12.1.9.1. *The euclidean metric induces the standard topology on* $\mathbb{R}^n$.

Definition 12.1.10. Let l_2 be the set of sequences $\vec{a} \in \mathbb{R}^\omega$ such that $\sum_{n=1}^\infty a_n^2 < \infty$.

Lemma 12.1.11. *If* $\vec{a}, \vec{b} \in l_2$ *then* $\sum_{n=1}^\infty |a_n b_n| < \infty$.

PROOF:

$$\begin{aligned} \sum_{n=1}^N |a_n b_n| &\leq \sqrt{(\sum_{n=1}^N a_n^2)(\sum_{n=1}^N b_n^2)} && \text{(Lemma 12.1.6)} \\ &\to \sqrt{\sum_{n=1}^\infty a_n^2)(\sum_{n=1}^\infty b_n^2)} \text{ as } n \to \infty \end{aligned}$$

$\square$

Lemma 12.1.12. *If* $\vec{a}, \vec{b} \in l_2$ *then* $\vec{a} + \vec{b} \in l_2$.

PROOF:

$$\begin{aligned} \sum_{n=1}^\infty (a_n+b_n)^2 &= \sum_{n=1}^\infty a_n^2 + 2\sum_{n=1}^\infty a_n b_n + \sum_{n=1}^\infty b_n^2 \\ &\leq \sum_{n=1}^\infty a_n^2 + 2\sum_{n=1}^\infty |a_n b_n| + \sum_{n=1}^\infty b_n^2 \\ &< \infty && \text{(Lemma 12.1.11)} \end{aligned}$$

$\square$

Lemma 12.1.13. *If* $c \in \mathbb{R}$ *and* $\vec{a} \in l_2$ *then* $c\vec{a} \in l_2$.

PROOF:$\sum_{n=1}^\infty (ca_n)^2 = c^2 \sum_{n=1}^\infty a_n^2$. $\square$

Definition 12.1.14 (The l^2-metric)**.** The l^2*-metric* is defined on l_2 by

$$d(\vec{a}, \vec{b}) = \left[\sum_{n=1}^\infty (a_n - b_n)^2\right]^{\frac{1}{2}} .$$

The topology induced by this metric is the l^2*-topology*. We write l_2 for this set under the l^2-topology.

We prove this is a metric.

PROOF:
⟨1⟩1. $d(\vec{a},\vec{b}) \geq 0$

PROOF: Immediate from definitions.

⟨1⟩2. $d(\vec{a},\vec{b}) = 0$ iff $\vec{a} = \vec{b}$

PROOF: Immediate from definitions.

⟨1⟩3. $d(\vec{a},\vec{b}) = d(\vec{b},\vec{a})$

PROOF: Immediate from definitions.

⟨1⟩4. $d(\vec{a},\vec{c}) \leq d(\vec{a},\vec{b}) + d(\vec{b},\vec{c})$

PROOF: $\sqrt{\sum_{i=1}^{N}(a_n - c_n)^2} \leq \sqrt{\sum_{i=1}^{N}(a_n - b_n)^2} + \sqrt{\sum_{i=1}^{N}(b_n - c_n)^2}$ since the euclidean metric on $\mathbb{R}^N$ is a metric.

□

Definition 12.1.15 (Hilbert Cube)**.** The *Hilbert cube* is $\prod_{n=1}^{\infty}[0, 1/n]$ as a subspace of the l_2.

Definition 12.1.16 (Isometric Imbedding)**.** Let X, Y be metric spaces and $f : X \to Y$. Then f is an *isometric imbedding* iff, for all $x, y \in X$, $d(f(x), f(y)) = d(x, y)$.

Lemma 12.1.17. *Every isometric imbedding is an imbedding.*

PROOF:
⟨1⟩1. LET: $f : X \to Y$ be an isometric imbedding.

⟨1⟩2. f is continuous.

PROOF: If $d(x, y) < \epsilon$ then $d(f(x), f(y)) < \epsilon$.

⟨1⟩3. f is injective.

PROOF: If $f(x) = f(y)$ then $d(f(x), f(y)) = 0$ so $d(x, y) = 0$ hence $x = y$.

⟨1⟩4. $f^{-1} : f(X) \to X$ is continuous.

PROOF: If $d(f^{-1}(x), f^{-1}(y)) < \epsilon$ then $d(x, y) < \epsilon$.

□

Chapter 13

Topological Groups

13.1 Topological Groups

Definition 13.1.1 (Topological Group)**.** A *topological group* G consists of a group G that is also a T_1 space such that $\cdot : G^2 \to G$ and $(\ )^{-1} : G \to G$ are continuous.

Proposition 13.1.2. *Every topological group is homogeneous.*

PROOF:
⟨1⟩1. LET: G be a topological group.
⟨1⟩2. LET: $x, y \in G$
⟨1⟩3. LET: $f : G \to G$ be given by $f(g) = yx^{-1}z$
⟨1⟩4. f is a homeomorphism
⟨1⟩5. $f(x) = y$
□

Definition 13.1.3 (Symmetric)**.** Let G be a topological group. A neighbourhood U of e is *symmetric* iff $U = U^{-1}$.

Proposition 13.1.4. *For every neighbourhood U of e, there exists a symmetric neighbourhood V of e such that $VV \subseteq U$.*

PROOF:
⟨1⟩1. LET: $m : G^2 \to G$ be the multiplication function
⟨1⟩2. $ee \in U$
⟨1⟩3. $(e, e) \in m^{-1}(U)$
⟨1⟩4. PICK neighbourhoods U_1, U_2 of e such that $(e, e) \in U_1 \times U_2 \subseteq m^{-1}(U)$
⟨1⟩5. LET: $V' = U_1 \cap U_2$
⟨1⟩6. $V'V' \subseteq U$
⟨1⟩7. LET: $f : G^2 \to G$ be the function $f(x, y) = xy^{-1}$
⟨1⟩8. $(e, e) \in f^{-1}(V')$
⟨1⟩9. PICK a neighbourhood W of e such that $WW^{-1} \subseteq V'$
⟨1⟩10. LET: $V = WW^{-1}$

⟨1⟩11. V is a neighbourhood of e

PROOF: V is open because $V = \bigcup_{a \in W^{-1}} Wa$.

⟨1⟩12. V is symmetric

⟨1⟩13. $VV \subseteq U$

□

Proposition 13.1.5. *Every topological group is regular.*

PROOF:

⟨1⟩1. LET: G be a topological group

⟨1⟩2. LET: $A \subseteq G$ be closed and $a \notin A$

⟨1⟩3. $G \setminus Aa^{-1}$ is a neighbourhood of e

⟨1⟩4. PICK a symmetric neighbourhood V of e such that $VV \subseteq G \setminus Aa^{-1}$

PROOF: Proposition 13.1.4.

⟨1⟩5. VA and Va are disjoint neighbourhoods of A and a

□

Proposition 13.1.6. *The long line is not second countable.*

PROOF:Let $\mathcal{B}$ be a basis for L. Then, for every countable ordinal α, $\mathcal{B}$ mst contain a basic open set that contains $(\alpha, 1/2)$ but not $(\beta, 1/2)$ for any other β. Therefore, $\mathcal{B}$ is uncountable. □

Corollary 13.1.6.1. *The long line cannot be imbedded in* $\mathbb{R}$.

Theorem 13.1.7. *Let $f : X \to Y$. Let Y be compact Hausdorff. Then f is continuous if and only if the graph of f is closed in $X \times Y$.*

PROOF:

⟨1⟩1. LET: G_f be the graph of f.

⟨1⟩2. If f is continuous then the graph of f is closed.

⟨2⟩1. ASSUME: f is continuous.

⟨2⟩2. LET: $(x, y) \in (X \times Y) \setminus G_f$

⟨2⟩3. $y \neq f(x)$

⟨2⟩4. PICK disjoint open neighbourhoods U of $f(x)$ and V of y

PROOF: Y is Hausdorff.

⟨2⟩5. $(x, y) \in f^{-1}(U) \times V \subseteq (X \times Y) \setminus G_f$

⟨2⟩6. Q.E.D.

⟨1⟩3. If the graph of f is closed then f is continuous.

⟨2⟩1. ASSUME: G_f is closed.

⟨2⟩2. LET: $x_0 \in X$ and V be an open neighbourhood of $f(x_0)$

⟨2⟩3. $G_f \cap (X \times (Y \setminus V))$ is closed

⟨2⟩4. $\pi_1(G_f \cap (X \times (Y \setminus V)))$ is closed

PROOF: Lemma 9.4.16

⟨2⟩5. $x_0 \in X \setminus \pi_1(G_f \cap (X \times (Y \setminus V))) \subseteq f^{-1}(V)$

⟨2⟩6. Q.E.D.

□

Theorem 13.1.8. *Let X be a compact Hausdorff space. Let $\mathcal{A}$ be a set of closed connected subspaces of X that is linearly ordered by proper inclusion. Then*

$$Y = \bigcap \mathcal{A}$$

is connected.

PROOF:
⟨1⟩1. ASSUME: for a contradiction C and D form a separation of Y
⟨1⟩2. PICK disjoint U and V open in X such that $C = U \cap Y$ and $D = V \cap Y$
  ⟨2⟩1. C and D are compact
    ⟨3⟩1. Y is compact
      PROOF: Y is a closed subset of X, hence compact by Proposition 9.4.6.
    ⟨3⟩2. Q.E.D.
      PROOF: C and D are closed subsets of Y hence compact by Proposition 9.4.6.
  ⟨2⟩2. Q.E.D.
    PROOF: By Lemma 9.4.18.
⟨1⟩3. For all $A \in \mathcal{A}$, we have $A \setminus (U \cup V)$ is nonempty
  PROOF: Since A is connected.
⟨1⟩4. $\{A \setminus (U \cup V) : A \in \mathcal{A}\}$ has the finite intersection property
  PROOF: This holds because $\mathcal{A}$ is linearly ordered under proper inclusion.
⟨1⟩5. $\bigcap_{A \in \mathcal{A}} (A \setminus (U \cup V))$ is nonempty
  PROOF: By Proposition 9.4.15.
□

Theorem 13.1.9. *Let $A \subseteq \mathbb{R}^n$. Then the following are equivalent:*

1. *A is compact.*
2. *A is closed and bounded under the euclidean metric.*
3. *A is closed and bounded under the square metric.*

PROOF:
⟨1⟩1. $1 \Rightarrow 2$
  ⟨2⟩1. ASSUME: A is compact.
  ⟨2⟩2. A is closed.
    PROOF: By Proposition 9.4.9.
  ⟨2⟩3. $\{B(\vec{0}, n) : n \in \mathbb{Z}^+\}$ covers A
  ⟨2⟩4. PICK a finite subcover $\{B(\vec{0}, n_1), \ldots, B(\vec{0}, n_k)\}$
  ⟨2⟩5. LET: $N = \max(n_1, \ldots, n_k)$
  ⟨2⟩6. For all $x, y \in A$ we have $d(x, y) < 2N$
    PROOF: We have $d(x, y) \leq d(\vec{0}, x) + d(\vec{0}, y) < N + N$.
⟨1⟩2. $2 \Rightarrow 3$
  PROOF: If $d(x, y) < \epsilon$ for all $x, y \in A$ then $\rho(x, y) < \epsilon\sqrt{n}$ by Lemma 12.1.9.
⟨1⟩3. $3 \Rightarrow 1$
  ⟨2⟩1. ASSUME: A is closed and $\rho(x, y) < \epsilon$ for all $x, y \in A$

⟨2⟩2. PICK $x_0 \in A$
⟨2⟩3. LET: $b = \rho(\vec{0}, x_0)$
⟨2⟩4. LET: $P = \epsilon + b$
⟨2⟩5. $A \subseteq [-P, P]^n$
PROOF:For any $y \in A$ we have

$$\begin{aligned} \rho(\vec{0}, y) &\leq \rho(\vec{0}, x_0) + \rho(x_0, y) && \text{(Triangle Inequality)} \\ &< b + \epsilon && (\langle 2\rangle 3,\ \langle 2\rangle 1) \\ &= P && (\langle 2\rangle 4) \end{aligned}$$

⟨2⟩6. $[-P, P]^n$ is compact.
PROOF: By Corollary 9.4.19.1 and Proposition 9.4.14.
⟨2⟩7. Q.E.D.
PROOF: By Proposition 9.4.6.
□

Theorem 13.1.10 (AC)**.** *Let X be a topological space. Then X is compact if and only if every nonempty net in X has a convergent subnet.*

PROOF:
⟨1⟩1. If X is compact then every nonempty net in X has a convergent subnet.
⟨2⟩1. ASSUME: X is compact.
⟨2⟩2. LET: $(x_\alpha)_{\alpha \in J}$ be a nonempty net in X
⟨2⟩3. For $\alpha \in J$,
LET: $B_\alpha = \{\beta \in J : \alpha \leq \beta\}$.
⟨2⟩4. $\{B_\alpha : \alpha \in J\}$ has the finite intersection property.
⟨3⟩1. LET: $\alpha_1, \ldots, \alpha_n \in J$
⟨3⟩2. PICK $\beta \in J$ such that $\alpha_1 \leq \beta, \ldots, \alpha_n \leq \beta$
⟨3⟩3. $x_\beta \in B_{\alpha_1} \cap \cdots \cap B_{\alpha_n}$
⟨2⟩5. PICK $l \in \bigcap_{\alpha \in J} B_\alpha$
PROOF: Proposition 9.4.15.
⟨2⟩6. LET: $K = \{\alpha \in J : x_\alpha = l\}$
⟨2⟩7. K is cofinal in J
⟨3⟩1. LET: $\alpha \in J$
⟨3⟩2. $l \in B_\alpha$
PROOF: By ⟨2⟩5.
⟨3⟩3. There exists $\beta \geq \alpha$ such that $x_\beta = l$.
⟨2⟩8. $(x_\alpha)_{\alpha \in K}$ is a subnet of $(x_\alpha)_{\alpha \in J}$ that converges to l.
⟨1⟩2. If every nonempty net in X has a convergent subnet then X is compact.
⟨2⟩1. ASSUME: Every nonempty net in X has a convergent subnet
⟨2⟩2. LET: $\mathcal{A}$ be a nonempty set of closed sets with the finite intersection property.
⟨2⟩3. LET: J be the poset of all finite intersections of elements of $\mathcal{A}$ under $\supseteq$
⟨2⟩4. PICK $x_C \in C$ for all $C \in J$
PROOF: These are all nonempty by ⟨2⟩2.
⟨2⟩5. PICK an accumulation point l of (x_C)
PROVE: $l \in \bigcap \mathcal{A}$
PROOF: One exists by Lemma 3.14.2.

$\langle 2 \rangle 6$. LET: $C \in \mathcal{A}$
PROVE: $l \in C$
$\langle 2 \rangle 7$. LET: U be a neighbourhood of l
PROVE: U intersects C
$\langle 2 \rangle 8$. PICK $D \subseteq C$ such that $x_D \in U$
PROOF: By $\langle 2 \rangle 5$.
$\langle 2 \rangle 9$. U intersects C
$\langle 2 \rangle 10$. $l \in C$
PROOF: By Theorem 3.9.3 since C is closed ($\langle 2 \rangle 2$).
$\langle 2 \rangle 11$. Q.E.D.
PROOF: Proposition 9.4.15.
□

Corollary 13.1.10.1 (AC). *Let G be a topological group. Let A and B be subsets of G. If A is closed in G and B is compact then AB is closed in G.*

PROOF:
$\langle 1 \rangle 1$. LET: $c \in \overline{AB}$
PROVE: $c \in AB$
$\langle 1 \rangle 2$. PICK a net $(x_\alpha)_{\alpha \in J}$ that converges to c
PROOF: By Theorem 3.13.3.
$\langle 1 \rangle 3$. For $\alpha \in J$, PICK $a_\alpha \in A$ and $b_\alpha \in B$ such that $x_\alpha = a_\alpha b_\alpha$
$\langle 1 \rangle 4$. PICK a convergent subnet $(b_{g(\beta)})_{\beta \in K}$ of $(b_\alpha)_{\alpha \in J}$
PROOF: By Theorem 13.1.10.
$\langle 1 \rangle 5$. LET: $b_{g(\beta)} \to b$
$\langle 1 \rangle 6$. $b \in B$
$\langle 2 \rangle 1$. B is closed
PROOF: By Proposition 9.4.9.
$\langle 2 \rangle 2$. Q.E.D.
PROOF: By Theorem 3.13.3
$\langle 1 \rangle 7$. $a_{g(\beta)} \to cb^{-1}$
PROOF: By Theorem 3.13.4
$\langle 1 \rangle 8$. $cb^{-1} \in A$
PROOF: By Theorem 3.13.3
$\langle 1 \rangle 9$. $c \in AB$
$\langle 1 \rangle 10$. Q.E.D.
PROOF: By Proposition 3.8.6.

Proposition 13.1.11. *Let $A_0 + A_1$ be the sum of A_0 and A_1 with injections $i_0 : A_0 \to A_0 + A_1$ and $i_1 : A_1 \to A_0 + A_1$.*

Let $g : B \to A_0 + A_1$ be a function.

Let B_0 be the pullback of i_0 and g with projections $j_0 : B_0 \to B$ and $k_0 : B_0 \to A_0$.

Let B_1 be the pullback of i_1 and g with projection s$j_1 : B_1 \to B$ and $k_1 : B_1 \to A_1$.

Then B is the sum of B_0 and B_1 with injections j_0 and j_1.

$$\begin{tikzcd} B_0 \arrow[r, "j_0"] \arrow[d] & B \arrow[d, "g"] & B_1 \arrow[l, "j_1"'] \arrow[d] \\ A_0 \arrow[r, "i_0"'] & A_0 + A_1 & A_1 \arrow[l, "i_1"'] \end{tikzcd}$$

PROOF:
$\langle 1\rangle 1$. LET: X be any set and $x : B_0 \to X$, $y : B_1 \to X$

Proposition 13.1.12 (CC)**.** *Let X be a space and $\mathcal{B}$ be a basis for X. Suppose that every subset of $\mathcal{B}$ that covers X has a countable subcover. Then X is Lindelöf.*

PROOF:
$\langle 1\rangle 1$. LET: $\mathcal{A}$ be an open cover of X.
$\langle 1\rangle 2$. $\{B \in \mathcal{B} : \exists U \in \mathcal{A}.B \subseteq U\}$ covers X.
$\langle 1\rangle 3$. PICK a countable subcover $\mathcal{B}_0$
$\langle 1\rangle 4$. For $B \in \mathcal{B}_0$, PICK $U_B \in \mathcal{A}$ such that $B \subseteq U_B$
$\langle 1\rangle 5$. $\{U_B : B \in \mathcal{B}_0\}$ is a countable subcover of $\mathcal{A}$.
□

Proposition 13.1.13 (CC)**.** *The space $\mathbb{R}_l$ is Lindelöf.*

PROOF:
$\langle 1\rangle 1$. LET: $\mathcal{A}$ be a set of basis elements $[a, b)$ that covers X
PROVE: $\mathcal{A}$ has a countable subcover.
$\langle 1\rangle 2$. LET: $C = \bigcup\{(a, b) : [a, b) \in \mathcal{A}\}$
$\langle 1\rangle 3$. $\mathbb{R} \setminus C$ is countable.
$\langle 2\rangle 1$. For all $x \in \mathbb{R} \setminus C$, PICK a rational q_x such that there exists b such that $q_x \in [x, b) \in \mathcal{A}$
$\langle 3\rangle 1$. PICK $[a, b) \in \mathcal{A}$ such that $x \in [a, b)$
$\langle 3\rangle 2$. $x = a$
PROOF: If not we would have $x \in C$
$\langle 3\rangle 3$. There exists a rational in (a, b)
$\langle 2\rangle 2$. For $x, y \in \mathbb{R} \setminus C$, if $x < y$ then $q_x < q_y$
$\langle 3\rangle 1$. PICK b, c such that $q_x \in [x, b) \in \mathcal{A}$ and $q_y \in [y, c) \in \mathcal{A}$
PROOF: By $\langle 2\rangle 1$.
$\langle 3\rangle 2$. $b \leq y$
PROOF: Otherwise we would have $y \in (x, b) \subseteq C$.
$\langle 3\rangle 3$. $q_x < q_y$
PROOF: $q_x < b \leq y \leq q_y$
$\langle 2\rangle 3$. The map $q_- : \mathbb{R} \setminus C \to \mathbb{Q}$ is injective.
$\langle 1\rangle 4$. For $x \in \mathbb{R} \setminus C$, PICK $[a_x, b_x) \in \mathcal{A}$ such that $a_x \leq x < b_x$
$\langle 1\rangle 5$. PICK a countable subset $((a_n, b_n))_{n \in \mathbb{Z}^+}$ of $\{(a, b) : [a, b) \in \mathcal{A}\}$ that covers C
$\langle 2\rangle 1$. The set C as a subspace of $\mathbb{R}$ with the standard topology is second countable.

⟨2⟩2. The set C as a subspace of $\mathbb{R}$ with the standard topology is Lindelöf.
PROOF: By Theorem 9.3.2.
⟨1⟩6. $\{[a_x, b_x) : x \in \mathbb{R} \setminus C\} \cup \{[a_n, b_n) : n \in \mathbb{Z}^+\}$ is a countable subcover of $\mathcal{A}$.
⟨1⟩7. Q.E.D.
PROOF: By Proposition 13.1.12.
□

Proposition 13.1.14 (AC). *The space $\mathbb{R}_l$ is not second countable.*

PROOF:
⟨1⟩1. LET: $\mathcal{B}$ be any basis for $\mathbb{R}_l$
⟨1⟩2. For $x \in \mathbb{R}$, PICK $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq [x, x+1)$
⟨1⟩3. The mapping $B_{(-)}$ is an injective function $\mathbb{R} \to \mathcal{B}$
PROOF: For any x we have $x = \min B_x$.
⟨1⟩4. $\mathcal{B}$ is uncountable.
□

Proposition 13.1.15. *The product of a Lindelöf space and a compact space is Lindelöf.*

PROOF:
⟨1⟩1. LET: X be a Lindelöf space and Y a compact space.
⟨1⟩2. LET: $\mathcal{A}$ be an open covering of $X \times Y$
⟨1⟩3. For all $x \in X$, there exists a neighbourhood W of x such that $W \times Y$ is covered by finitely many elements of $\mathcal{A}$.
⟨2⟩1. LET: $x \in X$
⟨2⟩2. $\{x\} \times Y$ is compact.
PROOF: It is homeomorphic to Y.
⟨2⟩3. PICK a finite subset $\{U_1, \ldots, U_m\}$ of $\mathcal{A}$ that covers $\{x\} \times Y$
PROOF: By Proposition 9.4.5.
⟨2⟩4. There exists a neighbourhood W of x such that $W \times Y \subseteq U_1 \cup \cdots \cup U_m$
PROOF: By the Tube Lemma.
⟨1⟩4. $\{W \text{ open in } X : W \times Y \text{ is covered by finitely many elements of } \mathcal{A}\}$ is an open covering of X.
⟨1⟩5. PICK a countable subcovering $\{W_1, W_2, \ldots\}$
⟨1⟩6. For $i \geq 1$, PICK a finite subset $\{U_{i1}, \ldots, U_{ir_i}\}$ of $\mathcal{A}$ that covers $W_i \times Y$
⟨1⟩7. $\{U_{1j} : i \geq 1, 1 \leq j \leq r_i\}$ is a countable subcovering of $\mathcal{A}$.
□

Proposition 13.1.16. *Let X be a T_1 space. Then X is normal if and only if, for every closed set A and open set $U \supseteq A$, there exists an open set $V \supseteq A$ such that $\overline{V} \subseteq U$.*

PROOF:
⟨1⟩1. If X is normal then, for every closed set A and open set $U \supseteq A$, there exists an open set $V \supseteq A$ such that $\overline{V} \subseteq U$.
⟨2⟩1. ASSUME: X is normal.
⟨2⟩2. LET: A be a closed set and U an open set with $A \subseteq U$

⟨2⟩3. PICK disjoint open sets V, W such that $A \subseteq V$ andn $X \setminus U \subseteq W$
⟨2⟩4. $\overline{V} \subseteq U$
PROOF:

$$\begin{aligned} \overline{V} &\subseteq X \setminus W \\ &\subseteq U \end{aligned}$$

⟨1⟩2. If, for every closed set A and open set $U \supseteq A$, there exists an open set $V \supseteq A$ such that $\overline{V} \subseteq U$, then X is normal.
⟨2⟩1. ASSUME: for every closed set A and open set $U \supseteq A$, there exists an open set $V \supseteq A$ such that $\overline{V} \subseteq U$.
⟨2⟩2. LET: A, B be disjoint closed sets
⟨2⟩3. PICK an open set V such that $A \subseteq V$ and $\overline{V} \subseteq X \setminus B$
⟨2⟩4. $A \subseteq V$ and $B \subseteq X \setminus \overline{V}$
□

Definition 13.1.17 (Action)**.** Let G be a topological group and X a topological space. An *action* of G on X is a continuous function $\cdot : G \times X \to X$ such that, for all $g, h \in G$ and $x \in X$:

1. $e \cdot x = x$
2. $g \cdot (h \cdot x) = gh \cdot x$

Definition 13.1.18 (Orbit Space)**.** Let G be a topological group, X a topological space, and $\cdot : G \times X \to X$ an action of G on X. Then the *orbit space* X/G is the quotient space of X by the equivalence relation $\sim$ generated by $x \sim g \cdot x$ for all $x \in X$, $g \in G$.

Theorem 13.1.19. *Let G be a topological group. Let X be a topological space. Let $\cdot : G \times X \to X$ be an action of G on X. Then the canonical map $\pi : X \twoheadrightarrow X/G$ is perfect.*

⟨1⟩1. π is closed.
⟨2⟩1. LET: $A \subseteq X$ be closed.
⟨2⟩2. $GA = \{g \cdot a : g \in G, a \in A\}$ is closed
⟨3⟩1. LET: $z \notin GA$
⟨3⟩2. For all $g \in G$ we have $g \cdot z \notin A$
⟨3⟩3. For $g \in G$, there exist U an open neighbourhood of g and V an open neighbourhood of z such that UV does not intersect A
⟨3⟩4. $\{U \text{ open in } G : \exists V \text{ an open neighbourhood of } z. UV \cap A = \emptyset\}$ covers G
⟨3⟩5. PICK a finite subcover $\{U_1, \ldots, U_n\}$
⟨3⟩6. For $1 \leq i \leq n$, PICK V_i an open neighbourhood of z such that $U_i V_i \cap A = \emptyset$
⟨3⟩7. $z \in V_1 \cap \cdots \cap V_n \subseteq X \setminus GA$
⟨2⟩3. $\pi(A)$ is closed
$\pi^{-1}(\pi(A)) = GA$
⟨1⟩2. π is continuous.
PROOF: By definition of the quotient topology.

⟨1⟩3. π is surjective.
  PROOF: By definition.
⟨1⟩4. For all $a \in X/G$ we have $\pi^{-1}(a)$ is compact.
  ⟨2⟩1. LET: $a \in X/G$
  ⟨2⟩2. PICK $x \in X$ such that $a = \pi(x)$
  ⟨2⟩3. $\pi^{-1}(a) = \{gx : g \in G\}$
  ⟨2⟩4. $\pi^{-1}(a)$ is homeomorphic to G
□

Corollary 13.1.19.1. *If X is Hausdorff then so is X/G.*

Corollary 13.1.19.2. *If X is regular then so is X/G.*

Corollary 13.1.19.3. *If X is normal then so is X/G.*

Corollary 13.1.19.4. *If X is locally compact then so is X/G.*

Corollary 13.1.19.5. *If X is second countable then so is X/G.*

Proposition 13.1.20. *Let $p : X \twoheadrightarrow Y$ be perfect. If X is second countable then so is Y.*

PROOF:
⟨1⟩1. PICK a countable basis $\mathcal{B}$ for X
⟨1⟩2. LET: $\mathcal{J} = \{J \subseteq^{\text{fin}} \mathcal{B} : \exists W \text{ open in } Y . p^{-1}(W) \subseteq \bigcup J\}$
⟨1⟩3. For every $J \in \mathcal{J}$,
    LET: $W_J = \bigcup\{W \text{ open in } Y : p^{-1}(W) \subseteq \bigcup J\}$.
    PROVE: $\{W_J : J \in \mathcal{J}\}$ is a basis for Y.
⟨1⟩4. $y \in V$ where V is open in Y
⟨1⟩5. $\{B \in \mathcal{B} : x \in B \subseteq p^{-1}(V)\}$ covers $p^{-1}(y)$
⟨1⟩6. PICK a countable subcover $J \subseteq^{\text{fin}} \mathcal{B}$
⟨1⟩7. $y \in W_J \subseteq V$
  ⟨2⟩1. $p^{-1}(y) \subseteq \bigcup J$
  ⟨2⟩2. PICK an open neighbourhood W of y such that $p^{-1}(W) \subseteq \bigcup J$
    PROOF: By Proposition 9.5.1.
  ⟨2⟩3. $W \subseteq W_J$
□

Proposition 13.1.21. *A subspace of a T_1 space is T_1.*

PROOF:
⟨1⟩1. LET: X be T_1 and $Y \subseteq X$
⟨1⟩2. LET: $a \in Y$
⟨1⟩3. $\{a\}$ is closed in X
⟨1⟩4. $\{a\}$ is closed in Y
  PROOF: By Corollary 4.3.4.1.
□

Proposition 13.1.22 (DC)**.** *Not every topological group is normal.*

PROOF: From Proposition 6.5.6. ∎

Theorem 13.1.23. *A subspace of a completely regular space is completely regular.*

PROOF:
⟨1⟩1. LET: X be completely regular and $Y \subseteq X$
⟨1⟩2. LET: $a \in Y$ and A be closed in Y such that $a \notin A$
⟨1⟩3. PICK C closed in X such that $A = X \cap C$
⟨1⟩4. PICK a continuous function $f : X \to [0, 1]$ such that $f(a) = 0$ and $f(C) = \{1\}$
⟨1⟩5. $f \restriction Y : Y \to [0, 1]$ is a continuous function such that $(f \restriction Y)(a) = 0$ and $(f \restriction Y)(A) = \{1\}$
∎

Proposition 13.1.24 (DC)**.** *Every topological group is completely regular.*

PROOF:
⟨1⟩1. LET: G be a topological group
⟨1⟩2. LET: $x \in G$ and $A \subseteq G$ be closed such that $x \notin A$
PROVE: There exists a continuous $f : G \to [0, 1]$ such that $f(x) = 0$ and $f(A) = \{1\}$
⟨1⟩3. ASSUME: w.l.o.g. $x = e$
PROOF: $\lambda y.x^{-1}y$ is an automorphism of G that maps x to e.
⟨1⟩4. PICK a sequence V_n $(n \geq 0)$ of symmetric neighbourhoods of e disjoint from A such that $V_n V_n \subseteq V_{n-1}$ for all n
⟨2⟩1. LET: $V_0 = X \setminus A$
⟨2⟩2. Given V_n, PICK a symmetric neighbourhood V_{n+1} of e such that $V_{n+1}V_{n+1} \subseteq V_n$
PROOF: By Proposition 13.1.4.
⟨1⟩5. For every dyadic rational p, define an open set $U(p)$ as follows:

$$\begin{aligned} U(1/2^n) &= V_n && (n \geq 0) \\ U((2k+1)/2^{n+1}) &= V_{n+1}U(k/2^n) && (0 < k < 2^n) \\ U(p) &= \emptyset && (p \leq 0) \\ U(p) &= G && (p > 1) \end{aligned}$$

⟨1⟩6. For all k and n, we have

$$V_n U(k/2^n) \subseteq U((k+1)/2^n)$$

⟨2⟩1. $k \leq 0$
PROOF: In this case, $V_n U(k/2^n) = \emptyset$
⟨2⟩2. $k = 1$ and $n > 0$
PROOF:

$$\begin{aligned} V_n U(1/2^n) &= V_n V_n \\ &\subseteq V_{n-1} \\ &= U(1/2^{n-1}) \end{aligned}$$

⟨2⟩3. $k = 2a$ for some $0 < a < 2^{n-1}$

PROOF:

$$\begin{aligned} V_n U(2a/2^n) &= V_n U(a/2^{n-1}) \\ &= U(2a + 1/2^n) \end{aligned}$$

⟨2⟩4. $k = 2a + 1$ for some $0 < a < 2^{n-1}$

PROOF:

$$\begin{aligned} V_n U((2a+1)/2^n) &= V_n V_n U(a/2^{n-1}) \\ &\subseteq V_{n-1} U(a/2^{n-1}) \\ &\subseteq U((a+1)/2^{n-1}) \end{aligned}$$

⟨2⟩5. $k \geq 2^n$

PROOF: In this case, $U((k+1)/2^n) = G$.

⟨1⟩7. Define $f : G \to [0,1]$ by

$$f(x) = \inf\{p : x \in U(p)\}$$

PROOF: This set is nonempty because $x \in U(1)$ and bounded below because if $x \in U(p)$ then $p > 0$.

⟨1⟩8. For $n > 0$ we have $\overline{U(k/2^n)} \subseteq V_n U(k/2^n)$

⟨2⟩1. LET: $x \in \overline{U(k/2^n)}$

⟨2⟩2. $V_n x$ is a neighbourhood of x

⟨2⟩3. PICK $y \in V_n x \cap U(k/2^n)$

⟨2⟩4. PICK $z \in V_n$ such that $y = zx$

⟨2⟩5. $x = z^{-1} y$

⟨1⟩9. For p and q dyadic rationals, if $p < q$ then $\overline{U(p)} \subseteq U(q)$

⟨1⟩10. If $x \in \overline{U(p)}$ then $f(x) \leq p$

⟨2⟩1. For all $q > p$ we have $x \in U(q)$

⟨2⟩2. For all $q > p$ we have $f(x) \leq q$

⟨1⟩11. If $x \notin U(p)$ then $f(x) \geq p$

PROOF: If $x \notin U(p)$ and $x \in U(q)$ then $q > p$.

⟨1⟩12. f is continuous

⟨2⟩1. LET: $x_0 \in X$

⟨2⟩2. LET: $c < f(x_0) < d$

PROVE: There exist a neighbourhood U of x_0 such that $f(U) \subseteq (c,d)$

⟨2⟩3. PICK rational numbers p, q such that $c < p < f(x_0) < q < d$

⟨2⟩4. $x \notin \overline{U(p)}$

⟨2⟩5. $x \in U(q)$

⟨2⟩6. Take $U = U(q) \setminus \overline{U(p)}$

⟨1⟩13. $f(e) = 0$

PROOF: We have $e \in U(1/2^n)$ for all n.

⟨1⟩14. $f(A) = \{1\}$

PROOF: If $x \in A$ and $x \in U(p)$ then $p > 1$.

□

Definition 13.1.25 (Bijection)**.** A function $f : A \to B$ is a *bijection*, $f : A \cong B$, iff there exists a function $f^{-1} : B \to A$, the *inverse* of f, such that $f^{-1} \circ f = \mathrm{id}_A$ and $f \circ f^{-1} = \mathrm{id}_B$.

Theorem 13.1.26. *Let Y be a normal space. Then Y is an absolute retract if and only if Y has the universal extension property.*

PROOF:

⟨1⟩1. If Y is an absolute retract then Y has the universal extension property.

⟨2⟩1. ASSUME: Y is an absolute retract.

⟨2⟩2. LET: X be a normal space, A a closed subspace of X and $f : A \to Y$ a continuous function.

⟨2⟩3. LET: Z_f be the quotient space of $X \cup Y$ under: $a \sim f(a)$ for all $a \in A$

⟨2⟩4. LET: $p : X \cup Y \twoheadrightarrow Z_f$ be the quotient map

⟨2⟩5. For all $x_1, x_2 \in X$ we have $p(x_1) = p(x_2)$ iff $x_1 = x_2$ or $x_1, x_2 inA$ and $f(x_1) = f(x_2)$; and for $x \in X$ and $y \in Y$ we have $p(x) = p(y)$ iff $f(x) = y$; and for $y_1, y_2 \in Y$ we have $p(y_1) = p(y_2)$ iff $y_1 = y_2$

⟨2⟩6. p imbeds Y into a closed subspace of Z_f

⟨3⟩1. p is injective on Y

⟨3⟩2. $p^{-1} : p(Y) \to Y$ is continuous

⟨4⟩1. LET: $U \subseteq Y$ be open

PROVE: $p(U)$ is open

⟨4⟩2. $p^{-1}(p(U)) = f^{-1}(U) \cup U$

⟨3⟩3. $p(Y)$ is closed

PROOF: $p^{-1}(p(Y)) = A \cup Y$

⟨2⟩7. Z_f is normal

⟨3⟩1. Z_f is T_1

PROOF: For $y \in Y$ we have $p^{-1}(y) = f^{-1}(y) \cup \{y\}$ which is closed.

⟨3⟩2. Any two disjoint closed sets in Z_f can be separated by a continuous function.

⟨4⟩1. LET: C and D be disjoint closed sets in Z_f

⟨4⟩2. PICK $g : Y \to [0, 1]$ such that $g(Y \cap p^{-1}(C)) = \{0\}$ and $g(Y \cap p^{-1}(D)) = \{1\}$

PROOF: By the Urysohn Lemma.

⟨4⟩3. PICK $h : X \to [0, 1]$ such that $h(X \cap p^{-1}(C)) = \{0\}$ and $h(X \cap p^{-1}(D)) = \{1\}$ and h agrees with $g \circ f$ on A

PROOF: By the Tietze Extension Theorem applied to $A \cup (X \cap p^{-1}(C)) \cup (X \cap p^{-1}(D))$.

⟨4⟩4. LET: $k : Z_f \to [0, 1]$ be the continuous function such that $k(p(x)) = h(x)$ for $x \in X$ and $k(p(y)) = g(y)$ for $y \in Y$

PROOF: By the Pasting Lemma

⟨4⟩5. $k(C) = \{0\}$

⟨4⟩6. $k(D) = \{1\}$

⟨3⟩3. Q.E.D.

PROOF: If g is such a continuous function then $g^{-1}([0, 1/2))$ and $g^{-1}((1/2, 1])$ are disjoint open sets that include A and B respectively.

⟨2⟩8. PICK a retraction $r : Z_f \to p(Y)$

⟨2⟩9. $p^{-1} \circ r \circ p : X \to Y$ extends f

⟨1⟩2. If Y has the universal extension property then Y is an absolute retract.

⟨2⟩1. ASSUME: Y has the universal extension property

⟨2⟩2. LET: Z be a normal space, Y_0 a closed subspace of Z, and $\phi : Y \cong Y_0$ a homeomorphism

⟨2⟩3. PICK a continuous extension $f : Z \to Y$ of ϕ^{-1}

⟨2⟩4. $\phi \circ f$ is a retraction
□

Theorem 13.1.27. *Every manifold is metrizable.*

Proof:
⟨1⟩1. Let: X be an m-manifold.
⟨1⟩2. X is regular.
⟨2⟩1. X is T_1
⟨2⟩2. Let: $x \in X$ and U be a neighbourhood of x
⟨2⟩3. Pick a neighbourhood V of x that is imbeddable in $\mathbb{R}^m$
⟨2⟩4. Pick a neighbourhood W of x such that $\overline{W} \subseteq U \cap V$
Proof: One exists since V is regular (Proposition 6.3.4)
⟨2⟩5. $x \in W$ and $\overline{W} \subseteq U$
⟨2⟩6. Q.E.D.
Proof: Proposition 6.3.2
⟨1⟩3. Q.E.D.
Proof: By the Urysohn Metrization Theorem.
□

Theorem 13.1.28. *Let X be a compact Hausdorff space in which every point has a neighbourhood that is imbeddable in $\mathbb{R}^m$. Then X is an m-manifold.*

Proof:
⟨1⟩1. There exists N such that X is imbeddable in $\mathbb{R}^N$
Proof: Theorem 11.1.5
⟨1⟩2. X is second countable.
Proof: Proposition 7.3.3
□

Proposition 13.1.29. *S_Ω is locally metrizable.*

Proof: For any $\alpha \in S_\Omega$, the neighbourhood $[0, \alpha] = (-\infty, \alpha + 1)$ is imbeddable in $\mathbb{R}$. □

Proposition 13.1.30 (DC)**.** *$\overline{S_\Omega}$ is compact.*

Proof:Proof:
⟨1⟩1. Let: $\mathcal{A}$ be an open cover of $\overline{S_\Omega}$
⟨1⟩2. Assume: for a contradiction there is no finite subcover of $\mathcal{A}$
⟨1⟩3. There exists a sequence of sets $U_n \in \mathcal{A}$ and ordinals α_n such that $\alpha_{n+1} < \alpha_n$ for all n and $\alpha_n \in U_n$ for all n
⟨2⟩1. Let: $\alpha_1 = \Omega$
⟨2⟩2. Given $\alpha_1, \ldots, \alpha_n$ and $U_1, \ldots, U_{n-1}$ with $0 \neq \alpha_n < \alpha_{n-1} < \cdots < \alpha_1$ and $\alpha_i \in U_i$ for $i < n$, Pick $U_n \in \mathcal{A}$ with $\alpha_n \in U_n$
Proof: By ⟨1⟩1.
⟨2⟩3. Pick $\alpha_{n+1} < \alpha_n$ such that $(\alpha_{n+1}, \alpha_n] \subseteq U_n$
Proof: By Lemma 4.1.2.
⟨2⟩4. $\alpha_{n+1} \neq 0$

PROOF: If $\alpha_{n+1} = 0$ then $U_1, \ldots, U_n$ cover $\overline{S_\Omega}$, contradicting $\langle 1 \rangle 2$.

$\langle 1 \rangle 4$. Q.E.D.

PROOF: This is a contradiction because the ordinals are well-ordered.

□

Proposition 13.1.31. *$\mathbb{R}_l$ is not limit point compact.*

PROOF: $\mathbb{Z}$ has no limit point. □

Proposition 13.1.32. *Every closed subspace of a Lindelöf space is Lindelöf.*

PROOF:

$\langle 1 \rangle 1$. LET: X be Lindelöf and $A \subseteq X$ be closed

$\langle 1 \rangle 2$. LET: $\mathcal{U}$ be an open covering of A

$\langle 1 \rangle 3$. $\{U$ open in $X : U \cap A \in \mathcal{U}\} \cup \{X \setminus A\}$ covers X

$\langle 1 \rangle 4$. PICK a countable subcovering $\mathcal{V}$

$\langle 1 \rangle 5$. $\{U \cap A : U \in \mathcal{V}, U \neq X \setminus A\}$ is a countable subcover of $\mathcal{U}$

□

Proposition 13.1.33. *$\mathbb{R}^\omega$ is locally connected.*

PROOF:This holds because every basic open set is connected, being the product of a family of connected spaces. □

Proposition 13.1.34. *The space $\mathbb{R}^\omega$ under the box topology is not first countable.*

PROOF:

$\langle 1 \rangle 1$. ASSUME: for a contradiction $\{U_n\}_{n \geq 0}$ is a countable basis at 0.

$\langle 1 \rangle 2$. For $n \geq 1$, PICK a basic open set $B_n = \prod_{j=0}^{\infty} (a_{nj}, b_{nj})$ such that $0 \in B_n \subseteq U_n$

$\langle 1 \rangle 3$. $\prod_{n=0}^{\infty} (a_{nn}/2, b_{nn}/2)$ is a neighbourhood of 0 that does not include any U_n

□

Proposition 13.1.35. *The space $\mathbb{R}^\omega$ under the box topology is not locally metrizable.*

PROOF:

$\langle 1 \rangle 1$. LET: U be any neighbourhood of 0

$\langle 1 \rangle 2$. LET: A be the set of all sequences in U with all coordinates positive

$\langle 1 \rangle 3$. $0 \in \overline{A}$

$\langle 1 \rangle 4$. There is no sequence of points of A converging to 0.

$\langle 1 \rangle 5$. U is not metrizable.

PROOF: By the Sequence Lemma.

□

Proposition 13.1.36. *For any nonempty set I, the space $\mathbb{R}^I$ is not limit point compact.*

PROOF: $\mathbb{Z}^I$ is an infinite set with no limit point. □

Proposition 13.1.37. *The space* $\mathbb{R}^{[0,1]}$ *is separable.*

PROOF: The set D is dense where D is the set of all functions $f : [0,1] \to \mathbb{Q}$ such that there exists a sequence of rationals $0 = q_0 < q_1 < \cdots < q_N = 1$ such that f is constant on $[q_i, q_{i+1})$ for $0 \leq i < N$. $\square$

Proposition 13.1.38. *If* J *is uncountable then* $\mathbb{R}^J$ *is not locally metrizable.*

PROOF: Every point has a neighbourhood homeomorphic to $\mathbb{R}^J$. $\square$

Proposition 13.1.39. *The space* $\mathbb{R}_K$ *is not limit point compact.*

PROOF: The set $\mathbb{Z}$ has no limit point. $\square$

Proposition 13.1.40. *The topologist's sine curve is not locally connected.*

PROOF: There is no connected neighbourhood of $(0,0)$. $\square$

Corollary 13.1.40.1. *Not every metric space is locally connected.*

Corollary 13.1.40.2. *Not every metric space is locally path connected.*

Proposition 13.1.41. *Not every metric space is compact.*

PROOF: The space $\mathbb{R}$ is not compact. $\square$

Proposition 13.1.42. *Every closed subspace of a limit point compact space is limit point compact.*

PROOF:
$\langle 1 \rangle 1$. LET: X be a limit point compact space and $C \subseteq X$ be closed.
$\langle 1 \rangle 2$. LET: $A \subseteq C$ be infinite.
$\langle 1 \rangle 3$. PICK a limit point l of A in X
$\langle 1 \rangle 4$. $l \in C$
  $\langle 2 \rangle 1$. l is a limt point of C
    PROOF: By Lemma 3.11.2.
  $\langle 2 \rangle 2$. Q.E.D.
    PROOF: By Corollary 3.11.3.1.
$\langle 1 \rangle 5$. l is a limit point of A in C.
  PROOF: By Proposition 4.3.10.
$\square$

Proposition 13.1.43. *For any part* $i : S \hookrightarrow X$ *of a set* X*, we have* $\emptyset \subseteq_X i$.

PROOF: We have $i \circ ¡_S = ¡_X$ by the uniqueness of $¡_X$. $\square$

Theorem 13.1.44. *Let* X *be a completely regular space. Then there exists a compactification* Y *of* X *such that every bounded continuous map* $X \to \mathbb{R}$ *extends uniquely to a continuous map* $Y \to \mathbb{R}$.

PROOF:
$\langle 1 \rangle 1$. LET: J be the set of all bounded continuous functions $X \to \mathbb{R}$

$\langle 1\rangle 2.$ For $\alpha \in J$,
LET: $I_\alpha = [\inf \alpha, \sup \alpha]$
$\langle 1\rangle 3.$ LET: $Z = \prod_{\alpha \in J} I_\alpha$
$\langle 1\rangle 4.$ LET: $h : X \to Z$ be defined by

$$h(x)_\alpha = \alpha(x)$$

$\langle 1\rangle 5.$ Z is compact Hausdorff
$\langle 2\rangle 1.$ Z is compact
PROOF: By Tychonoff's Theorem.
$\langle 2\rangle 2.$ Z is Hausdorff
PROOF: By Theorem 6.2.5
$\langle 1\rangle 6.$ h is an imbedding
$\langle 2\rangle 1.$ The set J separates points from closed sets
PROOF: This holds because X is completely regular.
$\langle 2\rangle 2.$ Q.E.D.
PROOF: By the Imbedding Theorem.
$\langle 1\rangle 7.$ LET: Y be the compactification of X such that $X \subseteq Y \to Z$ factors h
PROOF: By Lemma 9.8.2
$\langle 1\rangle 8.$ Every bounded continuous map $X \to \mathbb{R}$ extends uniquely to a continuous map $Y \to \mathbb{R}$
$\langle 2\rangle 1.$ LET: $\alpha : X \to \mathbb{R}$ be a bounded continuous function
$\langle 2\rangle 2.$ LET: $k : Y \to Z$ be the imbedding from $\langle 1\rangle 7$
$\langle 2\rangle 3.$ LET: $\overline{\alpha} = \pi_\alpha \circ k : Y \to \mathbb{R}$
$\langle 2\rangle 4.$ $\overline{\alpha}$ extends α
PROOF:For $x \in X$, we have

$$\begin{aligned} \overline{\alpha}(x) &= k(x)_\alpha \\ &= h(x)_\alpha \\ &= \alpha(x) \end{aligned}$$

$\langle 2\rangle 5.$ If $f : Y \to Z$ is continuous and extends α then $f = \overline{\alpha}$
PROOF: By Lemma 6.2.9.
□