Topology

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Contents

Chapter 1

Set Theory

1.1 Unions

Proposition 1.1.1. *Let X be a set and $U \subseteq X$ and $\mathcal{B} \subseteq \mathcal{P}X$. Then the following are equivalent:*

1. *there exists $\mathcal{B}_0 \subseteq \mathcal{B}$ such that $U = \bigcup \mathcal{B}_0$*
2. *for all x, we have $x \in U$ iff there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$*

PROOF:
⟨1⟩1. $1 \Rightarrow 2$
  ⟨2⟩1. ASSUME: 1
  ⟨2⟩2. PICK $\mathcal{B}_0 \subseteq \mathcal{B}$ such that $U = \bigcup \mathcal{B}_0$
  ⟨2⟩3. If $x \in U$ then there exists $B \in \mathcal{B}_0$ such that $x \in B \subseteq U$
  ⟨2⟩4. If there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$ then $x \in U$
⟨1⟩2. $2 \Rightarrow 1$
  ⟨2⟩1. ASSUME: 2
  ⟨2⟩2. LET: $\mathcal{B}_0 = \{B \in \mathcal{B} : B \subseteq U\}$
  ⟨2⟩3. For all $x \in U$, there exists $B \in \mathcal{B}_0$ such that $x \in B$
  ⟨2⟩4. For all $B \in \mathcal{B}_0$ and $x \in B$ we have $x \in U$
□

1.2 Saturated Sets

Definition 1.2.1 (Saturated Set)**.** Let $f : X \to Y$ and $A \subseteq X$. Then A is *saturated* w.r.t. f iff, for all $x \in A$ and $x' \in X$, if $f(x) = f(x')$ then $x' \in A$.

Proposition 1.2.2. *Let $f : X \to Y$ and $A \subseteq X$. The following are equivalent:*

1. *A is saturated*
2. *$A = f^{-1}(B)$ for some $B \subseteq Y$*

3. $A = f^{-1}(f(A))$.

PROOF: Easy. □

Chapter 2

Order Theory

Definition 2.0.1 (Linear Continuum)**.** A *linear continuum* is a complete dense linearly ordered set with more than one element.

Chapter 3

Real Analysis

Definition 3.0.1. Let $\mathbb{R}^\infty$ be the set of all sequences in $\mathbb{R}^\omega$ that are eventually zero.

Definition 3.0.2 (Hilbert Cube)**.** The *Hilbert cube* is the set $\prod_{n=1}^{\infty}[0, 1/n]$, a subset of $\mathbb{R}^\omega$.

Proposition 3.0.3. *A bounded increasing sequence of real numbers converges to its supremum.*

Proof:
⟨1⟩1. Let: (s_n) be a bounded increasing sequence.
⟨1⟩2. Let: $l = \sup_{n \in \mathbb{Z}^+} s_n$
⟨1⟩3. Let: $\epsilon > 0$
⟨1⟩4. Pick N such that $s_N > l - \epsilon$
  Proof: Such an N exists because $l - \epsilon$ is not an upper bound for (s_n).
⟨1⟩5. For $n \geq N$ we have $l - \epsilon < s_n \leq l$
□

Proposition 3.0.4 (Comparison Test)**.** *Suppose* $|a_n| \leq b_n$ *for all* n*. If* $\sum_{n=1}^{\infty} b_n < \infty$ *then* $\sum_{n=1}^{\infty} a_n < \infty$*.*

Proof:
⟨1⟩1. $\sum_{n=1}^{\infty} |a_n| < \infty$
  Proof: $\sum_{n=1}^{N} |a_n|$ is an increasing sequence bounded above by $\sum_{n=1}^{\infty} b_n$.
⟨1⟩2. Let: $c_n = |a_n| + a_n$ for all n
⟨1⟩3. $\sum_{n=1}^{\infty} c_n < \infty$
  Proof: $\sum_{n=1}^{N} c_n$ is an increasing sequence bounded above by $2\sum_{n=1}^{\infty} |a_n|$.
⟨1⟩4. Q.E.D.
  Proof: $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} c_n - \sum_{n=1}^{\infty} |a_n|$.

Proposition 3.0.5 (Weierstrass M-test)**.** *Let* X *be a set. Let* $f_n : X \to \mathbb{R}$ *be a sequence of functions. Let*

$$s_n(x) = \sum_{i=1}^{n} f_i(x)$$

for $n \geq 1$ *and* $x \in X$*. Let* $M_n > 0$ *for all* n*. Suppose* $|f_n(x)| \leq M_n$ *for all* n *and all* $x \in X$*. If* $\sum_{n=1}^{\infty} M_n < \infty$*, then* s_n *converges uniformly to some function.*

PROOF:
⟨1⟩1. LET: $s(x) = \sum_{i=1}^{\infty} f_i(x)$ for all $x \in X$
  PROOF: This sequence converges by the Comparison Test.
⟨1⟩2. LET: $r_n = \sum_{i=n+1}^{\infty} M_i$ for all n.
⟨1⟩3. For $k > n$ we have $|s_k(x) - s_n(x)| < r_n$ for all $x \in X$
  PROOF:

$$\begin{aligned} |s_k(x) - s_n(x)| &= \sum_{i=n+1}^{k} f_i(x) \\ &\leq \sum_{i=n+1}^{k} M_i \\ &< r_n \end{aligned}$$

⟨1⟩4. For all n and $x \in X$ we have $|s(x) - s_n(x)| \leq r_n$
  PROOF: Take the limit as $k \to \infty$.
⟨1⟩5. s_n converges uniformly to s as $n \to \infty$.
  ⟨2⟩1. LET: $\epsilon > 0$
  ⟨2⟩2. PICK N such that for all $n \geq N$ we have $r_n < \epsilon$
  ⟨2⟩3. For $n \geq N$ and $x \in X$ we have $|s(x) - s_n(x)| < \epsilon$
□

Chapter 4

Topological Spaces

4.1 Topologies

Definition 4.1.1 (Topology)**.** Let X be a set. A *topology* on X is a set $\mathcal{T} \subseteq \mathcal{P}X$ such that:

- $\emptyset \in \mathcal{T}$
- $X \in \mathcal{T}$
- for all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$
- for all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$

A *topological space* X consists of a set X and a topology $\mathcal{T}$ on X. We call the elements of X *points* and the elements of $\mathcal{T}$ *open sets.*

Definition 4.1.2 (Discrete Topology)**.** For any set X, the *discrete topology* on X is $\mathcal{P}X$.

Definition 4.1.3 (Indiscrete Topology)**.** For any set X, the *indiscrete topology* on X is $\{\emptyset, X\}$.

Definition 4.1.4 (Finite Complement Topology)**.** For any set X, the *finite complement topology* on X is $\{U \in \mathcal{P}X : X \setminus U \text{ is finite}\}$.

Definition 4.1.5 (Countable Complement Topology)**.** For any set X, the *countable complement topology* on X is $\{U \in \mathcal{P}X : X \setminus U \text{ is countable}\}$.

Definition 4.1.6 (Finer, Coarser)**.** Let $\mathcal{T}$ and $\mathcal{T}'$ be topologies on the same set X. Then $\mathcal{T}$ is *finer* than $\mathcal{T}'$, and $\mathcal{T}'$ is *coarser* than $\mathcal{T}$, iff $\mathcal{T}' \subseteq \mathcal{T}$. We say $\mathcal{T}$ is *strictly* finer than $\mathcal{T}'$, and $\mathcal{T}'$ is *strictly* coarser than $\mathcal{T}$, iff $\mathcal{T}' \subset \mathcal{T}$.

We say $\mathcal{T}$ and $\mathcal{T}'$ are *comparable* iff one is finer than the other.

Note. On any set X, the discrete topology is the finest topology and the indiscrete topology is the coarsest topology. The finite complement topology is finer than the countable complement topology, and strictly finer iff X is infinite.

Proposition 4.1.7. *Let X be a set. The intersection of a nonempty set of topologies on X is a topology on X.*

PROOF:
$\langle 1\rangle 1.$ LET: $\mathbb{T}$ be a nonempty set of topologies on X.
$\langle 1\rangle 2.$ $\emptyset \in \bigcap \mathbb{T}$
PROOF: For all $\mathcal{T} \in \mathbb{T}$ we have $\emptyset \in \mathcal{T}$.
$\langle 1\rangle 3.$ $X \in \bigcap \mathbb{T}$
PROOF: For all $\mathcal{T} \in \mathbb{T}$ we have $X \in \mathcal{T}$.
$\langle 1\rangle 4.$ For all $\mathcal{U} \subseteq \bigcap \mathbb{T}$ we have $\bigcup \mathcal{U} \in \bigcap \mathbb{T}$
$\langle 2\rangle 1.$ LET: $\mathcal{U} \subseteq \bigcap \mathbb{T}$
$\langle 2\rangle 2.$ LET: $\mathcal{T} \in \mathbb{T}$
$\langle 2\rangle 3.$ $\mathcal{U} \subseteq \mathcal{T}$
$\langle 2\rangle 4.$ $\bigcup \mathcal{U} \in \mathcal{T}$
$\langle 1\rangle 5.$ For all $U, V \in \bigcap \mathbb{T}$ we have $U \cap V \in \bigcap \mathbb{T}$
$\langle 2\rangle 1.$ LET: $U, V \in \bigcap \mathbb{T}$
$\langle 2\rangle 2.$ LET: $\mathcal{T} \in \mathbb{T}$
$\langle 2\rangle 3.$ $U, V \in \mathcal{T}$
$\langle 2\rangle 4.$ $U \cap V \in \mathcal{T}$
□

Corollary 4.1.7.1. *For any set $\mathbb{T}$ of topologies on a set X, there exists a unique coarsest topology $\mathcal{T}_0$ that is finer than every element of $\mathbb{T}$.*

PROOF: Take $\mathcal{T}_0$ to be the intersection of all the topologies $\mathcal{T}$ such that $\bigcup \mathbb{T} \subseteq \mathcal{T}$. The intersection is nonempty because the discrete topology is one such $\mathcal{T}$. □

4.2 Closed Sets

Definition 4.2.1 (Closed Set)**.** Let X be a topological space and $C \subseteq X$. Then C is *closed* iff $X \setminus C$ is open.

Proposition 4.2.2. *Let X be a set and $\mathcal{C} \subseteq \mathcal{P}X$. Then there exists a topology $\mathcal{T}$ with respect to which $\mathcal{C}$ is the set of closed sets if and only if*

1. $\emptyset \in \mathcal{C}$
2. $X \in \mathcal{C}$
3. *for all nonempty* $\mathcal{A} \subseteq \mathcal{C}$ *we have* $\bigcap \mathcal{A} \in \mathcal{C}$
4. *for all* $C, D \in \mathcal{C}$ *we have* $C \cup D \in \mathcal{C}$

In this case, $\mathcal{T}$ is unique and is given by $\mathcal{T} = \{U \in \mathcal{P}X : X \setminus U \in \mathcal{C}\}$.

PROOF:
$\langle 1\rangle 1.$ If $\mathcal{C}$ is the set of closed sets in a topology then conditions 1–4 hold.
$\langle 2\rangle 1.$ ASSUME: $\mathcal{C}$ is the set of closed sets with respect to some topology $\mathcal{T}$.
$\langle 2\rangle 2.$ $\emptyset \in \mathcal{C}$

PROOF: This holds because $X \setminus \emptyset = X$ is open.
⟨2⟩3. $X \in \mathcal{C}$
PROOF: This holds because $X \setminus X = \emptyset$ is open.
⟨2⟩4. For any nonempty $\mathcal{A} \subseteq \mathcal{C}$ we have $\bigcap \mathcal{A} \in \mathcal{C}$.
PROOF: This holds because $X \setminus \bigcap \mathcal{A} = \bigcup \{X \setminus U : U \in \mathcal{A}\}$ is open.
⟨2⟩5. For all $C, D \in \mathcal{C}$ we have $C \cup D \in \mathcal{C}$
PROOF: This holds because $X \setminus (C \cup D) = (X \setminus C) \cap (X \setminus D)$ is open.
⟨1⟩2. If conditions 1–4 hold then $\mathcal{T} = \{U \in \mathcal{P}X : X \setminus U \in \mathcal{C}\}$ is a topology with respect to which $\mathcal{C}$ is the set of closed sets.
⟨2⟩1. ASSUME: conditions 1–4 hold
⟨2⟩2. LET: $\mathcal{T} = \{U \in \mathcal{P}X : X \setminus U \in \mathcal{C}\}$
⟨2⟩3. $\mathcal{T}$ is a topology.
⟨3⟩1. $\emptyset \in \mathcal{T}$
PROOF: This holds because $X \setminus \emptyset = X \in \mathcal{C}$ by condition 2.
⟨3⟩2. $X \in \mathcal{T}$
PROOF: This holds because $X \setminus X = \emptyset \in \mathcal{C}$ by condition 1.
⟨3⟩3. For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$
⟨4⟩1. ASSUME: w.l.o.g. $\mathcal{U} \neq \emptyset$
PROOF: $\bigcup \emptyset = \emptyset \in \mathcal{T}$ by ⟨3⟩1
⟨4⟩2. $X \setminus \bigcup \mathcal{U} \in \mathcal{C}$
⟨5⟩1. $X \setminus \bigcup \mathcal{U} = \bigcap \{X \setminus U : U \in \mathcal{U}\}$
PROOF: De Morgan's law.
⟨5⟩2. $\bigcap \{X \setminus U : U \in \mathcal{U}\} \in \mathcal{C}$
PROOF: Condition 3.
⟨3⟩4. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$
PROOF: $X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V) \in \mathcal{C}$ by condition 4.
⟨2⟩4. A set C is closed in $\mathcal{T}$ iff $C \in \mathcal{C}$.
PROOF:

$$\begin{aligned} C \text{ is closed} &\Leftrightarrow X \setminus C \in \mathcal{T} && \text{(definition of closed set)} \\ &\Leftrightarrow X \setminus (X \setminus C) \in \mathcal{C} && (\langle 2 \rangle 2) \\ &\Leftrightarrow C \in \mathcal{C} \end{aligned}$$

⟨1⟩3. In any topology, U is open iff $X \setminus U$ is closed.
PROOF: From the definition of closed set.
$\square$

Example 4.2.3.

1. Every subset of a discrete space is closed.
2. The closed sets in an indiscrete space X are $\emptyset$ and X.
3. In the finite complement topology on a set X, the closed sets are X and the finite subsets of X.
4. In the countable complement topology on a set X, the closed sets are X and the countable subsets of X.

Proposition 4.2.4. *Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on the set X. Then $\mathcal{T}$ is finer than $\mathcal{T}'$ iff every set that is closed under $\mathcal{T}'$ is closed under $\mathcal{T}$.*

PROOF: From definitions. ∎

4.3 Interior

Definition 4.3.1 (Interior)**.** The *interior* of a set A, $\operatorname{Int} A$, is the union of the open sets included in A.

Proposition 4.3.2. *Let X be a set and $\operatorname{Int} : \mathcal{P}X \to \mathcal{P}X$. There exists a topology $\mathcal{T}$ such that $\operatorname{Int} A$ is the interior of A with respect to $\mathcal{T}$ for all A if and only if the following hold for all $A, B \subseteq X$:*

1. $\operatorname{Int} X = X$
2. $\operatorname{Int} A \subseteq A$
3. $\operatorname{Int}(\operatorname{Int} A) = \operatorname{Int} A$
4. $\operatorname{Int}(A \cap B) = \operatorname{Int} A \cap \operatorname{Int} B$

In this case, $\mathcal{T}$ is unique and is given by $\mathcal{T} = \{U \in \mathcal{P}X : \operatorname{Int} U = U\}$.

PROOF:
⟨1⟩1. If $\operatorname{Int} A$ is the interior of A for all A with respect to $\mathcal{T}$ then conditions 1 – 4 hold.
⟨2⟩1. $\operatorname{Int} X = X$
PROOF: This holds because X is open.
⟨2⟩2. $\operatorname{Int} A \subseteq A$
PROOF: From definition.
⟨2⟩3. $\operatorname{Int}(\operatorname{Int} A) = \operatorname{Int} A$
PROOF: This holds because $\operatorname{Int} A$ is open.
⟨2⟩4. $\operatorname{Int}(A \cap B) = \operatorname{Int} A \cap \operatorname{Int} B$
⟨3⟩1. $\operatorname{Int}(A \cap B) \subseteq \operatorname{Int} A$
PROOF: This holds because $\operatorname{Int}(A \cap B)$ is an open subset of A.
⟨3⟩2. $\operatorname{Int}(A \cap B) \subseteq \operatorname{Int} B$
PROOF: Similar.
⟨3⟩3. $\operatorname{Int} A \cap \operatorname{Int} B \subseteq \operatorname{Int}(A \cap B)$
PROOF: This holds because $\operatorname{Int} A \cap \operatorname{Int} B$ is an open subset of $A \cap B$.
⟨1⟩2. If conditions 1–4 hold then $\mathcal{T} = \{U \in \mathcal{P}X : \operatorname{Int} U = U\}$ is a topology with respect to which $\operatorname{Int} A$ is the interior of A for all A.
⟨2⟩1. ASSUME: conditions 1 – 4 hold
⟨2⟩2. For all $A, B \subseteq X$, if $A \subseteq B$ then $\operatorname{Int} A \subseteq \operatorname{Int} B$
PROOF:

$$\begin{aligned}
\operatorname{Int} A &= \operatorname{Int}(A \cap B) \\
&= \operatorname{Int} A \cap \operatorname{Int} B && \text{(condition 4)} \\
&\subseteq \operatorname{Int} B
\end{aligned}$$

$\langle 2\rangle 3.$ $\mathcal{T}$ is a topology.
$\langle 3\rangle 1.$ $\emptyset \in \mathcal{T}$
Proof: $\text{Int}\,\emptyset = \emptyset$ by condition 2.
$\langle 3\rangle 2.$ $X \in \mathcal{T}$
Proof: Condition 1.
$\langle 3\rangle 3.$ For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$
$\langle 4\rangle 1.$ Let: $\mathcal{U} \subseteq \mathcal{T}$
$\langle 4\rangle 2.$ $\text{Int} \bigcup \mathcal{U} \subseteq \bigcup \mathcal{U}$
Proof: Condition 2.
$\langle 4\rangle 3.$ $\bigcup \mathcal{U} \subseteq \text{Int} \bigcup \mathcal{U}$
$\langle 5\rangle 1.$ Let: $U \in \mathcal{U}$
$\langle 5\rangle 2.$ $U \subseteq \text{Int} \bigcup \mathcal{U}$
Proof:

$$\begin{aligned} U &= \text{Int}\, U \\ &\subseteq \text{Int} \bigcup \mathcal{U} \end{aligned}$$

$\langle 3\rangle 4.$ For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$
Proof:

$$\begin{aligned} \text{Int}(U \cap V) &= \text{Int}\, U \cap \text{Int}\, V && \text{(condition 4)} \\ &= U \cap V \end{aligned}$$

$\langle 2\rangle 4.$ For all $A \subseteq X$ we have $\text{Int}\, A$ is the interior of A with respect to $\mathcal{T}$.
$\langle 3\rangle 1.$ If U is open and $U \subseteq A$ then $U \subseteq \text{Int}\, A$
Proof:

$$\begin{aligned} U &= \text{Int}\, U && (U \text{ is open}) \\ &\subseteq \text{Int}\, A && (\langle 2\rangle 2) \end{aligned}$$

$\langle 3\rangle 2.$ $\text{Int}\, A$ is an open set that is a subset of A
Proof: From conditions 2 and 3.
$\langle 1\rangle 3.$ In any topology, U is open if and only if $\text{Int}\, U = U$
$\langle 2\rangle 1.$ If U is open then $\text{Int}\, U = U$.
$\langle 3\rangle 1.$ $\text{Int}\, U \subseteq U$
Proof: By definition.
$\langle 3\rangle 2.$ $U \subseteq \text{Int}\, U$
Proof: This holds because U is an open subset of U.
$\langle 2\rangle 2.$ If $\text{Int}\, U = U$ then U is open.
Proof: This holds because $\text{Int}\, U$ is open.
□

Example 4.3.3.

1. In the discrete topology, $\text{Int}\, A = A$ for all A.
2. In the indiscrete topology, $\text{Int}\, X = X$ and $\text{Int}\, A = \emptyset$ for all other A.
3. In the finite complement topology, $\text{Int}\, A = A$ if A is cofinite and $\text{Int}\, A = \emptyset$ for all other A.
4. In the countable complement topology on X, $\text{Int}\, A = A$ if $X \setminus A$ is countable, and $\text{Int}\, A = \emptyset$ for all other A.

Proposition 4.3.4. *Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on the set X. Then $\mathcal{T}$ is finer than $\mathcal{T}'$ iff, for every set A, the interior of A under $\mathcal{T}'$ is a subset of the interior of A under $\mathcal{T}$.*

PROOF:
$\langle 1\rangle 1$. LET: $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on the set X.
$\langle 1\rangle 2$. If $\mathcal{T}$ is finer than $\mathcal{T}'$ then, for every set A, the interior of A under $\mathcal{T}'$ is a subset of the interior of A under $\mathcal{T}$.
  $\langle 2\rangle 1$. ASSUME: $\mathcal{T}' \subseteq \mathcal{T}$
  $\langle 2\rangle 2$. LET: $A \subseteq X$
  $\langle 2\rangle 3$. LET: $\mathrm{Int}_{\mathcal{T}'} A$ be the interior of A under $\mathcal{T}'$ and $\mathrm{Int}_{\mathcal{T}} A$ the interior of A under $\mathcal{T}$
  $\langle 2\rangle 4$. $\mathrm{Int}_{\mathcal{T}'} A \subseteq \mathrm{Int}_{\mathcal{T}} A$
   PROOF: $\mathrm{Int}_{\mathcal{T}'} A$ is a subset of A that is open in $\mathcal{T}$.
$\langle 1\rangle 3$. If, for every set A, the interior of A under $\mathcal{T}'$ is a subset of the interior of A under $\mathcal{T}$, then $\mathcal{T}$ is finer than $\mathcal{T}'$.
  $\langle 2\rangle 1$. ASSUME: for every set A, the interior of A under $\mathcal{T}'$ is a subset of the interior of A under $\mathcal{T}$
  $\langle 2\rangle 2$. LET: $U \in \mathcal{T}'$
  $\langle 2\rangle 3$. The interior of U under $\mathcal{T}$ is U
  $\langle 2\rangle 4$. $U \in \mathcal{T}$
$\square$

4.4 Closure

Definition 4.4.1 (Closure)**.** Let X be a topological space and $A \subseteq X$. The *closure* of A, $\overline{A}$, is the intersection of the closed sets that include A.

This intersection is nonempty because X is a closed set that includes A.

Proposition 4.4.2. *Let X be a set and $\overline{()} : \mathcal{P}X \to \mathcal{P}X$. There exists a topology $\mathcal{T}$ such that $\overline{A}$ is the closure of A with respect to $\mathcal{T}$ for all A if and only if the following hold for all $A, B \subseteq X$:*

1. $\overline{\emptyset} = \emptyset$
2. $A \subseteq \overline{A}$
3. $\overline{\overline{A}} = \overline{A}$
4. $\overline{A \cup B} = \overline{A} \cup \overline{B}$

In this case, $\mathcal{T}$ is unique and is the topology such that a set C is closed if and only if $C = \overline{C}$.

PROOF: Dual to Proposition 4.3.2.

Example 4.4.3.

1. In the discrete topology, $\overline{A} = A$ for all A.

2. In the indiscrete topology on X, we have $\overline{\emptyset} = \emptyset$ and $\overline{A} = X$ for all other A.

3. In the finite complement topology on X, we have $\overline{A} = A$ for A finite and $\overline{A} = X$ for all other A.

4. In the countable complement topology on X, we have $\overline{A} = A$ for A countable and $\overline{A} = X$ for all other X.

Proposition 4.4.4. *Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on the set X. Then $\mathcal{T}$ is finer than $\mathcal{T}'$ iff, for every set A, the closure of A under $\mathcal{T}$ is a subset of the closure of A under $\mathcal{T}'$.*

PROOF: Dual to Proposition 4.3.4.

Proposition 4.4.5.

$$\operatorname{Int} A = X \setminus \overline{X \setminus A}$$

PROOF:
⟨1⟩1. $X \setminus \overline{X \setminus A} \subseteq \operatorname{Int} A$
⟨2⟩1. $X \setminus \overline{X \setminus A}$ is open
⟨2⟩2. $X \setminus \overline{X \setminus A} \subseteq A$
PROOF: This holds because $X \setminus A \subseteq \overline{X \setminus A}$.
⟨1⟩2. $\operatorname{Int} A \subseteq X \setminus \overline{X \setminus A}$
⟨2⟩1. $\overline{X \setminus A} \subseteq X \setminus \operatorname{Int} A$
⟨3⟩1. $X \setminus \operatorname{Int} A$ is closed
⟨3⟩2. $X \setminus A \subseteq X \setminus \operatorname{Int} A$
PROOF: This holds because $\operatorname{Int} A \subseteq A$
□

Corollary 4.4.5.1.

$$\overline{A} = X \setminus \operatorname{Int}(X \setminus A)$$

4.5 Boundary

Definition 4.5.1 (Boundary)**.** Let X be a topological space and $A \subseteq X$. Then the *boundary* of A is $\partial A = \overline{A} \cap \overline{X \setminus A}$.

Proposition 4.5.2.

$$\operatorname{Int} A \cap \partial A = \emptyset$$

PROOF:

$$\begin{aligned} \operatorname{Int} A \cap \partial A &\subseteq \operatorname{Int} A \cap \overline{X \setminus A} \\ &= \emptyset && \text{(Proposition 4.4.5)} \end{aligned}$$

□

Proposition 4.5.3.

$$\overline{A} = \operatorname{Int} A \cup \partial A$$

PROOF:

$$
\begin{aligned}
\operatorname{Int} A \cup \partial A &= \operatorname{Int} A \cup (\overline{A} \cap \overline{X \setminus A}) \\
&= (\operatorname{Int} A \cup \overline{A}) \cap (\operatorname{Int} A \cup \overline{X \setminus A}) \\
&= \overline{A} \cap X && \text{(Proposition 4.4.5)} \\
&= \overline{A}
\end{aligned}
$$

∎

Corollary 4.5.3.1.

$$\operatorname{Int} A = \overline{A} \setminus \partial A$$

PROOF: Using Proposition 4.5.2

Proposition 4.5.4. *A set U is open iff $\partial U = \overline{U} \setminus U$*

PROOF:
$\langle 1 \rangle 1$. If U is open then $\partial U = \overline{U} \setminus U$
PROOF:

$$
\begin{aligned}
\partial U &= \overline{U} \cap \overline{X \setminus U} \\
&= \overline{U} \cap (X \setminus U) && \text{(Proposition 4.4.2)} \\
&= \overline{U} \setminus U
\end{aligned}
$$

$\langle 1 \rangle 2$. If $\partial U = \overline{U} \setminus U$ then U is open
PROOF:

$$
\begin{aligned}
\operatorname{Int} U &= \overline{U} \setminus \partial U && \text{(Corollary 4.5.3.1)} \\
&= \overline{U} \setminus (\overline{U} \setminus U) \\
&= U && (U \subseteq \overline{U})
\end{aligned}
$$

∎

4.6 Neighbourhoods

Definition 4.6.1 (Neighbourhood)**.** Let X be a topological space. A *neighbourhood* of x is a set N such that there exists an open set U such that $x \in U \subseteq N$.

Proposition 4.6.2. *Let X be a set and $\mathcal{N} : X \to \mathcal{P}X$. Then there exists a topology $\mathcal{T}$ such that $\mathcal{N}(x)$ is the set of neighbourhoods of x for all x if and only if:*

1. *For all $x \in X$, $\mathcal{N}(x)$ is nonempty.*
2. *For all $x \in X$ and $N \in \mathcal{N}(x)$ we have $x \in N$.*
3. *For all $x \in X$ and $M, N \subseteq X$, if $M \in \mathcal{N}(x)$ and $M \subseteq N$ then $N \in \mathcal{N}(x)$.*
4. *For all $x \in X$ and $M, N \in \mathcal{N}(x)$ we have $M \cap N \in \mathcal{N}(x)$*
5. *For all $x \in X$ and $N \in \mathcal{N}(x)$, there exists $M \in \mathcal{N}(x)$ such that $M \subseteq N$ and $\forall y \in M.M \in \mathcal{N}(y)$.*

In this case, $\mathcal{T}$ is unique and is given by $\mathcal{T} = \{U \subseteq X : \forall x \in U.U \in \mathcal{N}(x)\}$.

PROOF:

$\langle 1 \rangle 1$. If $\mathcal{N}(x)$ is the set of neighbourhoods of x with respect to some topology for all $x \in X$ then conditions 1–5 hold.

$\langle 2 \rangle 1$. Condition 1 holds.

PROOF: This holds because X is a neighbourhood of x for all x.

$\langle 2 \rangle 2$. Condition 2 holds.

PROOF: If there exists an open U such that $x \in U \subseteq N$ then $x \in N$.

$\langle 2 \rangle 3$. Condition 3 holds.

$\langle 3 \rangle 1$. LET: $x \in X$ and $M, N \subseteq X$

$\langle 3 \rangle 2$. ASSUME: $M \in \mathcal{N}(x)$ and $M \subseteq N$

$\langle 3 \rangle 3$. PICK U open such that $x \in U \subseteq M$

$\langle 3 \rangle 4$. $x \in U \subseteq N$

$\langle 2 \rangle 4$. Condition 4 holds.

$\langle 3 \rangle 1$. LET: $x \in X$ and $M, N \in \mathcal{N}(x)$

$\langle 3 \rangle 2$. PICK U, V open such that $x \in U \subseteq M$ and $x \in V \subseteq N$

$\langle 3 \rangle 3$. $x \in U \cap V \subseteq M \cap N$

$\langle 2 \rangle 5$. Condition 5 holds.

$\langle 3 \rangle 1$. LET: $x \in X$ and $N \in \mathcal{N}(x)$

$\langle 3 \rangle 2$. PICK U open such that $x \in U \subseteq N$

$\langle 3 \rangle 3$. $U \in \mathcal{N}(x)$

PROOF: We have $x \in U \subseteq U$

$\langle 3 \rangle 4$. For all $y \in U$ we have $U \in \mathcal{N}(y)$

PROOF: For all $y \in U$ we have U is an open set such that $y \in U \subseteq U$, so U is a neighbourhood of y.

$\langle 1 \rangle 2$. If conditions 1–5 hold then $\mathcal{T} = \{U \subseteq X : \forall x \in U.U \in \mathcal{N}(x)\}$ is a topology with respect to which $\mathcal{N}(x)$ is the set of neighbourhoods of x for all x.

$\langle 2 \rangle 1$. $\mathcal{T}$ is a topology.

$\langle 3 \rangle 1$. $\emptyset \in \mathcal{T}$

PROOF: Vacuously, $\forall x \in \emptyset.\emptyset \in \mathcal{N}(x)$

$\langle 3 \rangle 2$. $X \in \mathcal{T}$

$\langle 4 \rangle 1$. LET: $x \in X$

$\langle 4 \rangle 2$. PICK $N \in \mathcal{N}(x)$

PROOF: By condition 1

$\langle 4 \rangle 3$. $X \in \mathcal{N}(x)$

PROOF: By condition 3

$\langle 3 \rangle 3$. For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$

$\langle 4 \rangle 1$. LET: $\mathcal{U} \subseteq \mathcal{T}$

$\langle 4 \rangle 2$. LET: $x \in \bigcup \mathcal{U}$

$\langle 4 \rangle 3$. PICK $U \in \mathcal{U}$ such that $x \in U$

$\langle 4 \rangle 4$. $U \in \mathcal{N}(x)$

PROOF: $\langle 4 \rangle 1$, $\langle 4 \rangle 3$

$\langle 4 \rangle 5$. $\bigcup \mathcal{U} \in \mathcal{N}(x)$

PROOF: By condition 3

$\langle 3 \rangle 4$. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$

$\langle 4\rangle 1$. LET: $U, V \in \mathcal{T}$
$\langle 4\rangle 2$. LET: $x \in U \cap V$
$\langle 4\rangle 3$. $U, V \in \mathcal{N}(x)$
PROOF: $\langle 4\rangle 1$, $\langle 4\rangle 2$
$\langle 4\rangle 4$. $U \cap V \in \mathcal{N}(x)$
PROOF: By condition 4.
$\langle 2\rangle 2$. For all $x \in X$, we have $\mathcal{N}(x)$ is the set of neighbourhoods of x.
$\langle 3\rangle 1$. For all $N \in \mathcal{N}(x)$ we have N is a neighbourhood of x.
$\langle 4\rangle 1$. LET: $N \in \mathcal{N}(x)$
$\langle 4\rangle 2$. PICK $U \in \mathcal{N}(x)$ such that $U \subseteq N$ and U is open.
PROOF: By condition 5
$\langle 4\rangle 3$. $x \in U \subseteq N$
PROOF: By condition 2
$\langle 3\rangle 2$. If N is a neighbourhood of x then $N \in \mathcal{N}(x)$
$\langle 4\rangle 1$. LET: U be open such that $x \in U \subseteq N$
$\langle 4\rangle 2$. $U \in \mathcal{N}(x)$
PROOF: Definition of $\mathcal{T}$
$\langle 4\rangle 3$. $N \in \mathcal{N}(x)$
PROOF: By condition 3
$\langle 1\rangle 3$. In any topology, a set is open if and only if it is a neighbourhood of all of its points.
$\langle 2\rangle 1$. If U is open and $x \in U$ then U is a neighbourhood of x.
PROOF: Immediate from the definition of neighbourhood.
$\langle 2\rangle 2$. If U is a neighbourhood of all of its points then U is open.
PROOF: In this case $U = \bigcup\{V \text{ open } : V \subseteq U\}$
□

Example 4.6.3.

1. In the discrete topology, A is a neighbourhood of x iff $x \in A$.

2. In the indiscrete topology on X, the only neighbourhood of a point x is X.

3. In the finite complement topology on X, we have A is a neighbourhood of x iff $x \in A$ and $X \setminus A$ is finite.

4. In the countable complement topology on X, we have A is a neighbourhood of x iff $x \in A$ and $X \setminus A$ is countable.

Proposition 4.6.4. *Let $\mathcal{T}$ and $\mathcal{T}'$ be topologies on a set X. Then $\mathcal{T}$ is finer than $\mathcal{T}'$ iff, for all $x \in X$, every neighbourhood of x under $\mathcal{T}'$ is a neighbourhood of x under $\mathcal{T}$.*

PROOF:
$\langle 1\rangle 1$. LET: $\mathcal{T}$ and $\mathcal{T}'$ be topologies on a set X.
$\langle 1\rangle 2$. If $\mathcal{T}$ is finer than $\mathcal{T}'$ then, for all $x \in X$, every neighbourhood of x under $\mathcal{T}'$ is a neighbourhood of x under $\mathcal{T}$

PROOF: Immediate from definitions.

⟨1⟩3. If, for all $x \in X$, every neighbourhood of x under $\mathcal{T}'$ is a neighbourhood of x under $\mathcal{T}$, then $\mathcal{T}$ is finer than $\mathcal{T}'$

PROOF: From the fact that a set is open iff it is a neighbourhood of all its points (Proposition 4.6.2).

□

Proposition 4.6.5. *Let X be a topological space, $A \subseteq X$ and $x \in X$. Then $x \in \overline{A}$ if and only if every neighbourhood of x intersects A.*

PROOF:

$$\begin{aligned} x \in \overline{A} &\Leftrightarrow \forall C \text{ closed}. A \subseteq C \Rightarrow x \in C \\ &\Leftrightarrow \forall U \text{ open}. A \subseteq X \setminus U \Rightarrow x \notin U \\ &\Leftrightarrow \forall U \text{ open}. x \in U \Rightarrow A \cap U \neq \emptyset \end{aligned}$$

□

Proposition 4.6.6. *In a topological space X, if N is a neighbourhood of x, C is closed and $x \notin C$ then $N \setminus C$ is a neighbourhood of x.*

PROOF: This holds because N and $X \setminus C$ are neighbourhoods of x. □

4.7 Limit Points

Definition 4.7.1 (Limit Point)**.** Let X be a topological space, $x \in X$ and $A \subseteq X$. Then x is a *limit point*, *cluster point* or *point of accumulation* for A iff every neighbourhood of x intersects A in a point other than x.

Proposition 4.7.2. *Let X be a topological space. Let $A \subseteq X$. Let A' be the set of limit points of A. Then*

$$\overline{A} = A \cup A'$$

PROOF:

⟨1⟩1. LET: X be a topological space.

⟨1⟩2. LET: $A \subseteq X$

⟨1⟩3. $\overline{A} \subseteq A \cup A'$

PROOF:

⟨2⟩1. LET: $x \in \overline{A} \setminus A$

⟨2⟩2. Every neighbourhood of x intersects A

PROOF: Proposition 4.6.5, ⟨1⟩1, ⟨1⟩2, ⟨2⟩1.

⟨2⟩3. Every neighbourhood of x intersects A in a point other than x.

PROOF: ⟨2⟩1, ⟨2⟩2

⟨1⟩4. $A \subseteq \overline{A}$

PROOF: Proposition 4.4.2, ⟨1⟩1, ⟨1⟩2.

⟨1⟩5. $A' \subseteq \overline{A}$

PROOF: Proposition 4.6.5, ⟨1⟩1, ⟨1⟩2.

□

Corollary 4.7.2.1. *Let X be a topological space and $A \subseteq X$. Then A is closed if and only if it contains all its limit points.*

PROOF:
$\langle 1 \rangle 1$. LET: X be a topological space
$\langle 1 \rangle 2$. LET: $A \subseteq X$
$\langle 1 \rangle 3$. LET: A' be the set of limit points of A
$\langle 1 \rangle 4$. A is closed if and only if $A' \subseteq A$
PROOF:

$$\begin{aligned} A \text{ is closed} &\Leftrightarrow A = \overline{A} && \text{(Proposition 4.4.2)} \\ &\Leftrightarrow A = A' \cup A && \text{(Proposition 4.7.2)} \end{aligned}$$

4.8 Convergence

Definition 4.8.1 (Convergence)**.** Let X be a topological space. Let (x_n) be a sequence in X and $l \in X$. Then (x_n) *converges* to l, $x_n \to l$ as $n \to \infty$, iff for every neighbourhood M of l there exists an integer N such that, for all $n \geq N$, we have $x_n \in M$.

Proposition 4.8.2. *Let X be a topological space, $A \subseteq X$, and (x_n) be a sequence in A. If $x_n \to l$ as $n \to \infty$ then $l \in \overline{A}$.*

PROOF: From Proposition 4.6.5. □

4.9 Bases for topologies

Definition 4.9.1 (Basis)**.** Let X be a topological space. A *basis* for the topology on X is a set $\mathcal{B} \subseteq \mathcal{P}X$ such that:

- every element of $\mathcal{B}$ is open
- for every open set U and every point $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

We say the topology on X is the topology *generated* by $\mathcal{B}$.

Proposition 4.9.2. *Let X be a set and $\mathcal{B} \subseteq \mathcal{P}X$. Then $\mathcal{B}$ is a basis for a topology $\mathcal{T}$ on X if and only if:*

1. $\bigcup \mathcal{B} = X$
2. *for all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.*

In this case, $\mathcal{T}$ is unique and is the set of all unions of subsets of $\mathcal{B}$.

PROOF:
$\langle 1 \rangle 1$. If $\mathcal{B}$ is a basis for $\mathcal{T}$ then conditions 1 and 2 hold.

$\langle 2\rangle 1$. ASSUME: $\mathcal{B}$ is a basis for $\mathcal{T}$.

$\langle 2\rangle 2$. Condition 1 holds

$\langle 3\rangle 1$. LET: $x \in X$

$\langle 3\rangle 2$. X is an open set and $x \in X$

$\langle 3\rangle 3$. There exists $B \in \mathcal{B}$ such that $x \in B \subseteq X$.

PROOF: Since $\mathcal{B}$ is a basis ($\langle 2\rangle 1$)

$\langle 2\rangle 3$. Condition 2 holds

$\langle 3\rangle 1$. LET: $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$

$\langle 3\rangle 2$. $B_1 \cap B_2$ is open

$\langle 3\rangle 3$. There exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$

$\langle 1\rangle 2$. If conditions 1 and 2 hold then $\mathcal{B}$ is a basis for the set of all unions of subsets of $\mathcal{B}$.

$\langle 2\rangle 1$. ASSUME: Conditions 1 and 2 hold.

$\langle 2\rangle 2$. LET: $\mathcal{T}$ be the set of all unions of subsets of $\mathcal{B}$.

$\langle 2\rangle 3$. $\mathcal{T}$ is a topology.

$\langle 3\rangle 1$. $\emptyset \in \mathcal{T}$

PROOF: This holds because $\emptyset = \bigcup \emptyset$.

$\langle 3\rangle 2$. $X \in \mathcal{T}$

PROOF: This holds by Condition 1.

$\langle 3\rangle 3$. For all $\mathcal{U} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{U} \in \mathcal{T}$.

$\langle 4\rangle 1$. LET: $\mathcal{U} \subseteq \mathcal{T}$

$\langle 4\rangle 2$. LET: $x \in \bigcup \mathcal{U}$

$\langle 4\rangle 3$. PICK $U \in \mathcal{U}$ such that $x \in U$

$\langle 4\rangle 4$. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$

PROOF: By Proposition 1.1.1

$\langle 4\rangle 5$. $x \in B \subseteq \bigcup \mathcal{U}$

$\langle 4\rangle 6$. Q.E.D.

PROOF: By Proposition 1.1.1

$\langle 3\rangle 4$. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$.

$\langle 4\rangle 1$. LET: $U, V \in \mathcal{T}$

$\langle 4\rangle 2$. LET: $x \in U \cap V$

$\langle 4\rangle 3$. PICK $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \subseteq U$ and $x \in B_2 \subseteq V$

PROOF: By Proposition 1.1.1

$\langle 4\rangle 4$. PICK $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$

PROOF: By Condition 2.

$\langle 4\rangle 5$. $x \in B_3 \subseteq U \cap V$

$\langle 4\rangle 6$. Q.E.D.

PROOF: By Proposition 1.1.1

$\langle 2\rangle 4$. $\mathcal{B}$ is a basis for $\mathcal{T}$.

$\langle 3\rangle 1$. Every element of $\mathcal{B}$ is in $\mathcal{T}$.

PROOF: For $B \in \mathcal{B}$ we have $B = \bigcup \{B\}$.

$\langle 3\rangle 2$. For all $U \in \mathcal{T}$ and $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

PROOF: By Proposition 1.1.1

$\langle 1\rangle 3$. If $\mathcal{B}$ is a basis for $\mathcal{T}$ then $\mathcal{T}$ is the set of all unions of subsets of $\mathcal{B}$.

$\langle 2\rangle 1$. ASSUME: $\mathcal{B}$ is a basis for $\mathcal{T}$

$\langle 2\rangle 2$. Every union of a subset of $\mathcal{B}$ is in $\mathcal{T}$

  PROOF: This holds because every element of $\mathcal{B}$ is open.
$\langle 2 \rangle 3$. Every set in $\mathcal{T}$ is the union of a subset of $\mathcal{B}$
  PROOF: By Proposition 1.1.1
$\square$

Example 4.9.3. The set of all one-point sets is a basis for the discrete topology on any set.

Proposition 4.9.4. *The topology generated by a basis $\mathcal{B}$ is the coarsest topology in which every element of $\mathcal{B}$ is open.*

PROOF: If every element of $\mathcal{B}$ is open in a topology $\mathcal{T}$ then every union of elements of $\mathcal{B}$ is open in $\mathcal{T}$. $\square$

Proposition 4.9.5. *Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$ on X respectively. Then the following are equivalent.*

1. $\mathcal{T} \subseteq \mathcal{T}'$
2. *For every $B \in \mathcal{B}$ and $x \in B$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.*

PROOF:
$\langle 1 \rangle 1$. $1 \Rightarrow 2$
  PROOF: If $\mathcal{T} \subseteq \mathcal{T}'$ then every element of $\mathcal{B}$ is open in $\mathcal{T}'$.
$\langle 1 \rangle 2$. $2 \Rightarrow 1$
  $\langle 2 \rangle 1$. ASSUME: 2
  $\langle 2 \rangle 2$. LET: $U \in \mathcal{T}$
  $\langle 2 \rangle 3$. LET: $x \in U$
  $\langle 2 \rangle 4$. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$
  $\langle 2 \rangle 5$. PICK $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$
  $\langle 2 \rangle 6$. $x \in B' \subseteq U$
$\square$

Proposition 4.9.6. *Let $\mathcal{B}$ be a basis for the topology on X and $\mathcal{B}' \subseteq \mathcal{P}X$. Then $\mathcal{B}'$ is a basis for the topology on X if and only if:*

1. *Every element of $\mathcal{B}'$ is open.*
2. *For every $B \in \mathcal{B}$ and $x \in B$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.*

PROOF:
$\langle 1 \rangle 1$. If $\mathcal{B}'$ is a basis then conditions 1 and 2 hold.
  PROOF: Immediate from the definition of basis.
$\langle 1 \rangle 2$. If conditions 1 and 2 hold then $\mathcal{B}'$ is a basis.
  $\langle 2 \rangle 1$. ASSUME: conditions 1 and 2 hold
  $\langle 2 \rangle 2$. LET: U be an open set and $x \in U$
    PROVE: There exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq U$
  $\langle 2 \rangle 3$. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$
  $\langle 2 \rangle 4$. PICK $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$
  $\langle 2 \rangle 5$. $x \in B' \subseteq U$

□

Proposition 4.9.7. *Let $\mathcal{B}$ be a basis for the topology on X, $A \subseteq X$ and $x \in X$. Then $x \in \overline{A}$ iff every element of $\mathcal{B}$ that contains x intersects A.*

Proof:
$\langle 1 \rangle 1$. If $x \in \overline{A}$ then every element of $\mathcal{B}$ that contains x intersects A.
Proof: By Proposition 4.6.5.
$\langle 1 \rangle 2$. If every element of $\mathcal{B}$ that contains x intersects A then $x \in \overline{A}$.
$\langle 2 \rangle 1$. Assume: every element of $\mathcal{B}$ that contains x intersects A.
$\langle 2 \rangle 2$. Let: U be an open set that contains x
Prove: U intersects A.
$\langle 2 \rangle 3$. Pick $B \in \mathcal{B}$ such that $x \in B \subseteq U$
Proof: Since $\mathcal{B}$ is a basis.
$\langle 2 \rangle 4$. B intersects A
Proof: $\langle 2 \rangle 1$, $\langle 2 \rangle 3$
$\langle 2 \rangle 5$. U intersects A
Proof: $\langle 2 \rangle 3$, $\langle 2 \rangle 4$
$\langle 2 \rangle 6$. Q.E.D.
Proof: Proposition 4.6.5

□

Definition 4.9.8 (Lower Limit Topology)**.** The *lower limit topology* on the real line is the topology generated by the basis consisting of all half-open intervals $[a, b)$ with $a < b$.

We write $\mathbb{R}_l$ for the set of real numbers under this topology.

We prove this is a basis for a topology.

Proof:
$\langle 1 \rangle 1$. Every real number is in a half-open interval.
Proof: We have $x \in [x, x + 1)$.
$\langle 1 \rangle 2$. Given open intervals I and J with $x \in I \cap J$, there exists an open interval K such that $x \in K \subseteq I \cap J$.
Proof: If $I = [a, b)$ and $J = [c, d)$ then take $K = [\max(a, c), \min(b, d))$.
$\langle 1 \rangle 3$. Q.E.D.
Proof: By Proposition 4.9.2.

□

Definition 4.9.9 (K-topology)**.** Let $K = \{1/n : n \in \mathbb{Z}^+\}$. The *$K$-topology* on the real line is the topology generated by the basis

$$\mathcal{B} = \{(a, b) : a < b\} \cup \{(a, b) \setminus K : a < b\} .$$

We write $\mathbb{R}_K$ for the set of real numbers under this topology.

We prove this is a basis for a topology.

Proof:
$\langle 1 \rangle 1$. For all $x \in \mathbb{R}$ there exists $B \in \mathcal{B}$ with $x \in B$.

PROOF: We have $x \in (x-1, x+1)$.
⟨1⟩2. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ with $x \in B_3 \subseteq B_1 \cap B_2$.
⟨2⟩1. CASE: $B_1 = (a,b), B_2 = (c,d)$
PROOF: Take $B_3 = (\max(a,c), \min(b,d))$.
⟨2⟩2. CASE: $B_1 = (a,b), B_2 = (c,d) \setminus K$
PROOF: Take $B_3 = (\max(a,c), \min(b,d)) \setminus K$.
⟨2⟩3. CASE: $B_1 = (a,b) \setminus K, B_2 = (c,d)$
PROOF: Take $B_3 = (\max(a,c), \min(b,d)) \setminus K$.
⟨2⟩4. CASE: $B_1 = (a,b) \setminus K, B_2 = (c,d) \setminus K$
PROOF: Take $B_3 = (\max(a,c), \min(b,d)) \setminus K$.
⟨1⟩3. Q.E.D.
PROOF: By Proposition 4.9.2.
□

Proposition 4.9.10. *The lower limit topology and the K-topology are incomparable.*

PROOF:
The set $[0,1)$ is open in the lower limit topology but not in the K-topology.
The set $(-1,1) \setminus K$ is open in the K-topology but not in the lower limit topology.
□

4.10 Subbases

Definition 4.10.1 (Subbasis)**.** Let X be a topological space. A *subbasis* for the topology on X is a set $\mathcal{S} \subseteq \mathcal{P}X$ such that the open sets are exactly the unions of finite intersections of elements of $\mathcal{S}$.

Proposition 4.10.2. *Let X be a set and $\mathcal{S} \subseteq \mathcal{P}X$. Then $\mathcal{S}$ is a subbasis for a topology $\mathcal{T}$ on X if and only if $\bigcup \mathcal{S} = X$, in which case $\mathcal{T}$ is unique and the set of all finite intersections of elements of $\mathcal{S}$ is a basis for $\mathcal{T}$*

PROOF:
⟨1⟩1. If $\mathcal{S}$ is a subbasis for a topology in X then $\bigcup \mathcal{S} = X$
⟨2⟩1. LET: $X = \bigcup \mathcal{A}$ where every member of $\mathcal{A}$ is a finite intersection of elements of $\mathcal{S}$
⟨2⟩2. LET: $x \in X$
⟨2⟩3. PICK $A \in \mathcal{A}$ such that $x \in A$
⟨2⟩4. PICK $U_1, \ldots, U_n \in \mathcal{S}$ such that $A = U_1 \cap \cdots \cap U_n$
⟨2⟩5. $x \in U_1 \in \mathcal{S}$
⟨1⟩2. If $\bigcup \mathcal{S} = X$ then $\mathcal{S}$ is a subbasis for the topology on X generated by the set of all finite intersections of elements of $\mathcal{S}$
⟨2⟩1. ASSUME: $\bigcup \mathcal{S} = X$
⟨2⟩2. LET: $\mathcal{B}$ be the set of all finite intersections of elements of $\mathcal{S}$
⟨2⟩3. LET: $\mathcal{T}$ be the topology generated by the basis $\mathcal{B}$
⟨3⟩1. For all $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$

⟨4⟩1. LET: $x \in X$
⟨4⟩2. PICK $S \in \mathcal{S}$ such that $x \in S$
PROOF: ⟨2⟩1, ⟨4⟩1
⟨4⟩3. $x \in S \in \mathcal{B}$
PROOF: ⟨2⟩2, ⟨4⟩2
⟨3⟩2. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$
PROOF: Take $B_3 = B_1 \cap B_2$
⟨3⟩3. Q.E.D.
PROOF: Proposition 4.9.2.
⟨2⟩4. $\mathcal{T}$ is the set of all unions of finite intersections of elements of $\mathcal{S}$
⟨1⟩3. If $\mathcal{S}$ is a subbasis for $\mathcal{T}$ and $\mathcal{T}'$ then $\mathcal{T} = \mathcal{T}'$
PROOF: $\mathcal{T}$ and $\mathcal{T}'$ are both the set of all unions of finite intersections of elements of $\mathcal{S}$.
□

Proposition 4.10.3. *The topology generated by a subbasis $\mathcal{S}$ is the coarsest topology in which every element of $\mathcal{S}$ is open.*

PROOF: If every element of $\mathcal{S}$ is open in a topology $\mathcal{T}$ then every union of finite intersections of elements of $\mathcal{S}$ is open in $\mathcal{T}$. □

Proposition 4.10.4. *Let $\mathcal{B}$ be a basis for the topology on X and $\mathcal{S} \subseteq \mathcal{P}X$. Then $\mathcal{S}$ is a subbasis for the topology on X if and only if:*

1. *Every element of $\mathcal{S}$ is open.*
2. *Every element of $\mathcal{B}$ is a union of finite intersections of elements of $\mathcal{S}$.*

PROOF:
⟨1⟩1. If $\mathcal{S}$ is a subbasis then conditions 1 and 2 hold.
PROOF: Immediate from definitions.
⟨1⟩2. If conditions 1 and 2 hold then $\mathcal{S}$ is a subbasis.
⟨2⟩1. ASSUME: conditions 1 and 2 hold.
⟨2⟩2. Every open set is a union of finite intersections of elements of $\mathcal{S}$.
PROOF: From condition 2 since every open set is a union of elements of $\mathcal{B}$.
⟨2⟩3. Every union of finite intersections of elements of $\mathcal{S}$ is open.
PROOF: From condition 1.
□

4.11 Local Bases

Definition 4.11.1 (Local Basis)**.** Let X be a topological space and $a \in X$. A *(local) basis at x* is a set $\mathcal{B}$ of neighbourhoods of x such that every neighbourhood of x includes at least one element of $\mathcal{B}$.

Chapter 5

Constructions of Topological Spaces

5.1 The Order Topology

Definition 5.1.1 (Order Topology)**.** Let X be a linearly ordered set with more than one element. The *order topology* on X is the topology generated by the subbasis consisting of the open rays.

Proposition 5.1.2. *Let X be a linearly ordered set with more than one element. The following sets together form a basis $\mathcal{B}$ for the order topology:*

- *All open intervals (a, b) for $a, b \in X$, $a < b$.*
- *All intervals of the form $[\bot, a)$ for $a \in X$ where $\bot$ is the least element in X, if it exists.*
- *All intervals of the form $(a, \top]$ for $a \in X$ where $\top$ is the greatest element in X, if it exists.*

PROOF:
$\langle 1 \rangle 1$. LET: X be a linearly ordered set with more than one element.
$\langle 1 \rangle 2$. $\mathcal{B}$ is a basis for a topology on X.
  $\langle 2 \rangle 1$. $\bigcup \mathcal{B} = X$.
    $\langle 3 \rangle 1$. LET: $x \in X$
      PROVE: There exists $B \in \mathcal{B}$ with $x \in B$
    $\langle 3 \rangle 2$. CASE: x is least in X
      $\langle 4 \rangle 1$. PICK $a \in X$ with $a \neq x$
        PROOF: $\langle 1 \rangle 1$
      $\langle 4 \rangle 2$. $x \in [x, a) \in \mathcal{B}$
    $\langle 3 \rangle 3$. CASE: x is greatest in X
      $\langle 4 \rangle 1$. PICK $a \in X$ with $a \neq x$
        PROOF: $\langle 1 \rangle 1$

⟨4⟩2. $x \in (a, x] \in \mathcal{B}$

⟨3⟩4. CASE: x is neither least nor greatest in X

⟨4⟩1. PICK $a, b \in X$ with $a < x < b$

⟨4⟩2. $x \in (a, b) \in \mathcal{B}$

⟨2⟩2. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

PROOF: Take $B_3 = B_1 \cap B_2$.

⟨2⟩3. Q.E.D.

PROOF: Proposition 4.9.2

⟨1⟩3. Every element of $\mathcal{B}$ is open in the order topology.

PROOF: We have $(a, b) = (a, +\infty) \cap (-\infty, b)$, and $[\bot, b) = (-\infty, b)$, and $(a, \top] = (a, +\infty)$.

⟨1⟩4. Every open ray is open in the topology generated by $\mathcal{B}$.

⟨2⟩1. If $x \in (a, +\infty)$ then there exists $B \in \mathcal{B}$ such that $x \in B \subseteq (a, +\infty)$.

⟨3⟩1. CASE: x is greatest in X

PROOF: Take $B = (a, x]$

⟨3⟩2. CASE: x is not greatest in X

PROOF: Pick b such that $x < b$ and take $B = (a, b)$.

⟨2⟩2. If $x \in (-\infty, b)$ then there exists $B \in \mathcal{B}$ such that $x \in B \subseteq (-\infty, b)$.

PROOF: Similar.

⟨1⟩5. Q.E.D.

PROOF: Using Proposition 4.9.4 and 4.10.3.

□

Proposition 5.1.3. *Let X be a linearly ordered set under the order topology and $U \subseteq X$ be open. If $a \in U$ then either a is greatest in X or there exists $b > a$ such that $[a, b) \subseteq U$.*

PROOF: From definitions.

Proposition 5.1.4. *Let X be a linearly ordered set under the order topology and $U \subseteq X$ be open. If $a \in U$ then either a is least in X or there exists $b < a$ such that $(b, a] \subseteq U$.*

PROOF: From definitions.

Proposition 5.1.5. *Let X be a linearly ordered set in the order topology and $A \subseteq X$ be nonempty. If A has a supremum s then $s \in \overline{A}$.*

PROOF:

⟨1⟩1. LET: X be a linearly ordered set in the order topology.

⟨1⟩2. LET: $A \subseteq X$ be nonempty

⟨1⟩3. LET: s be the supremum of A.

⟨1⟩4. LET: U be an open set containing s

PROVE: U intersects A

⟨1⟩5. CASE: s is least in X

⟨2⟩1. PICK $a \in A$

PROOF: A is nonempty (⟨1⟩2).

⟨2⟩2. $a \leq s$
PROOF: ⟨1⟩3, ⟨2⟩1.
⟨2⟩3. $s = a$
PROOF: ⟨1⟩5, ⟨2⟩2.
⟨2⟩4. $s \in A \cap U$
PROOF: ⟨1⟩4, ⟨2⟩1, ⟨2⟩3.
⟨1⟩6. CASE: s is not least in X
⟨2⟩1. PICK $l < s$ such that $(l, s] \subseteq U$
PROOF: Proposition 5.1.4, ⟨1⟩1, ⟨1⟩4,⟨1⟩6.
⟨2⟩2. PICK $a \in A$ such that $l < a$
PROOF: ⟨1⟩3, ⟨2⟩1.
⟨2⟩3. $a \in A \cap U$
⟨3⟩1. $l < a \leq s$
PROOF: ⟨2⟩1, ⟨2⟩2.
⟨3⟩2. $l \in U$
PROOF: ⟨2⟩1, ⟨3⟩1.
⟨3⟩3. Q.E.D.
PROOF: With ⟨2⟩2

Proposition 5.1.6. *Let X be a complete linearly ordered set in the order topology and $A \subseteq X$. If A is nonempty, closed and bounded above then A has a greatest element.*

PROOF:
⟨1⟩1. LET: X be a complete linearly ordered set in the order topology.
⟨1⟩2. LET: $A \subseteq X$
⟨1⟩3. ASSUME: A is nonempty, closed and bounded above.
⟨1⟩4. LET: $s = \sup A$
PROOF: X is complete (⟨1⟩1) and A is nonempty and bounded above (⟨1⟩3).
⟨1⟩5. $s \in \overline{A}$
PROOF: Proposition 5.1.5, ⟨1⟩1, ⟨1⟩2, ⟨1⟩3, ⟨1⟩4.
⟨1⟩6. $s \in A$
PROOF: Proposition 4.4.2, ⟨1⟩3, ⟨1⟩5.
□

5.1.1 The Standard Topology on the Real Line

Definition 5.1.7 (Standard Topology on the Real Line)**.** The *standard topology* on the real line $\mathbb{R}$ is the order topology.

We write $\mathbb{R}$ for the topological space consisting of the real numbers under the standard topology.

Proposition 5.1.8. *The lower limit topology is strictly finer than the standard topology.*

PROOF:
⟨1⟩1. The lower limit topology is finer than the standard topology.

⟨2⟩1. LET: $x \in (a, b)]$
⟨2⟩2. $x \in [x, b) \subseteq (a, b)]$
⟨2⟩3. Q.E.D.
PROOF: By Proposition 4.9.5.
⟨1⟩2. There exists a set that is open in the lower limit topology but not in the standard topology.
PROOF: $[0, 1)$ is not open in the standard topology because there is no open interval (a, b) such that $0 \in (a, b) \subseteq [0, 1)$.
□

Proposition 5.1.9. *The K-topology is strictly finer than the standard topology.*

PROOF:
⟨1⟩1. The K-topology is finer than the standard topology.
PROOF: Every basic open set for the standard topology is a basic open set in the K-topology.
⟨1⟩2. There is a set open in the K-topology that is not open in the standard topology.
PROOF: The set $(-1, 1) \setminus K$ is not open in the standard topology because there is no open interval (a, b) such that $0 \in (a, b) \subseteq (-1, 1) \setminus K$.
□

5.1.2 The Ordered Square

Definition 5.1.10 (Ordered Square)**.** The *ordered square* I_o^2 is $[0, 1]^2$ under the dictionary order topology.

5.2 The Product Topology

Definition 5.2.1 (Product Topology)**.** Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces. The *product topology* on $\prod_{\alpha \in J} X_\alpha$ is the topology generated by the subbasis consisting of all sets ${\pi_\alpha}^{-1}(U)$ where $\alpha \in J$ and U is open in X_α.

Proposition 5.2.2. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces and $(x_\alpha) \in \prod_{\alpha \in J} X_\alpha$. Let $\alpha \in J$. If M is a neighbourhood of x_α in X_α then ${\pi_\alpha}^{-1}(M)$ is a neighbourhood of (x_α).*

PROOF:
⟨1⟩1. PICK U open in X_α such that $x_\alpha \in U \subseteq M$
⟨1⟩2. $(x_\alpha) \in {\pi_\alpha}^{-1}(U) \subseteq {\pi_\alpha}^{-1}(M)$
□

Proposition 5.2.3. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces. Then the product topology is the topology generated by the basis consisting of all sets of the form $\prod_{\alpha \in J} U_\alpha$, where each U_α is open in X_α, and $U_\alpha = X_\alpha$ for all but finitely many α.*

PROOF: From Proposition 4.10.2. □

Proposition 5.2.4. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces and $\mathcal{B}_\alpha$ be a basis for X_α for all $\alpha \in J$. Let $\mathcal{B}$ consisting of all sets of the form $\prod_{\alpha \in J} U_\alpha$ where $U_\alpha \in \mathcal{B}_\alpha$ for finitely many α, and $U_\alpha = X_\alpha$ for all other α. Then $\mathcal{B}$ is a basis for the product topology on $\prod_{\alpha \in J} X_\alpha$.*

PROOF:
⟨1⟩1. Every element of $\mathcal{B}$ is open.
  PROOF: From definitions.
⟨1⟩2. $\bigcup \mathcal{B} = \prod_{\alpha \in J} X_\alpha$
  PROOF: This holds because $\prod_{\alpha \in J} X_\alpha \in \mathcal{B}$.
⟨1⟩3. For every open U and point $(x_\alpha) \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$.
  ⟨2⟩1. PICK U_α for each α such that each U_α is open in X_α, $U_\alpha = X_\alpha$ except for $\alpha = \alpha_1, \ldots, \alpha_n$, and $(x_\alpha) \in \prod_{\alpha \in J} U_\alpha \subseteq U$
  ⟨2⟩2. For $1 \leq i \leq n$ PICK $B_i \in \mathcal{B}_{\alpha_i}$ such that $x_{\alpha_i} \in B_i \subseteq U_{\alpha_i}$
  ⟨2⟩3. $(x_\alpha) \in \prod_{\alpha \in J} V_\alpha \subseteq U$, where $V_{\alpha_i} = B_i$ for $1 \leq i \leq n$, and $V_\alpha = X_\alpha$ for all other α.

Proposition 5.2.5. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces and C_α be closed in X_α for all $\alpha \in J$. Then $\prod_{\alpha \in J} C_\alpha$ is closed in $\prod_{\alpha \in J} X_\alpha$.*

PROOF: This holds because $\prod_{\alpha \in J} X_\alpha \setminus \prod_{\alpha \in J} C_\alpha = \bigcup_{\alpha \in J} {\pi_\alpha}^{-1}(X_\alpha \setminus C_\alpha)$. □

Proposition 5.2.6 (AC)**.** *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces and $A_\alpha \subseteq X_\alpha$ for all $\alpha \in J$. Then*

$$\prod_{\alpha \in J} \overline{A_\alpha} = \overline{\prod_{\alpha \in J} A_\alpha} \ .$$

(This requires the Axiom of Countable Choice if J is countably infinite, and the Axiom of Choice if J is uncountable.)

PROOF:
⟨1⟩1. $\prod_{\alpha \in J} \overline{A_\alpha} \subseteq \overline{\prod_{\alpha \in J} A_\alpha}$
  ⟨2⟩1. LET: $(x_\alpha) \in \prod_{\alpha \in J} \overline{A_\alpha}$
  ⟨2⟩2. LET: U_α be open in X_α for all α with $U_\alpha = X_\alpha$ for all α except α_1, …, α_n and $(x_\alpha) \in \prod_{\alpha \in J} U_\alpha$
  ⟨2⟩3. For all α we have $x_\alpha \in \overline{A_\alpha}$
    PROOF: From ⟨2⟩1
  ⟨2⟩4. For all α we have $x_\alpha \in U_\alpha$
    PROOF: From ⟨2⟩2
  ⟨2⟩5. For all α we have U_α intersects A_α
    PROOF: Proposition 4.6.5, ⟨2⟩3, ⟨2⟩4
  ⟨2⟩6. $\prod_{\alpha \in J} U_\alpha$ intersects $\prod_{\alpha \in J} A_\alpha$
    PROOF: From ⟨2⟩5 using the Axiom of Choice.
  ⟨2⟩7. Q.E.D.
    PROOF: Proposition 4.9.7.
⟨1⟩2. $\overline{\prod_{\alpha \in J} A_\alpha} \subseteq \prod_{\alpha \in J} \overline{A_\alpha}$

$\langle 2 \rangle 1$. LET: $(x_\alpha) \in \overline{\prod_{\alpha \in J} A_\alpha}$
$\langle 2 \rangle 2$. LET: $\alpha \in J$
PROVE: $x_\alpha \in \overline{A_\alpha}$
$\langle 2 \rangle 3$. LET: N be a neighbourhood of x_α in X_α
PROVE: N intersects A_α
$\langle 2 \rangle 4$. LET: $M = {\pi_\alpha}^{-1}(N)$
$\langle 2 \rangle 5$. M is a neighbourhood of (x_α)
PROOF: Proposition 5.2.2, $\langle 2 \rangle 3$, $\langle 2 \rangle 4$
$\langle 2 \rangle 6$. M intersects $\prod_{\alpha \in J} A_\alpha$
PROOF: Proposition 4.6.5, $\langle 2 \rangle 1$, $\langle 2 \rangle 5$.
$\langle 2 \rangle 7$. PICK $(y_\alpha) \in M \cap \prod_{\alpha \in J} A_\alpha$
PROOF: $\langle 2 \rangle 6$
$\langle 2 \rangle 8$. $y_\alpha \in N \cap A_\alpha$
PROOF: $\langle 2 \rangle 4$, $\langle 2 \rangle 7$
$\langle 2 \rangle 9$. Q.E.D.
PROOF: Proposition 4.6.5
□

5.3 The Subspace Topology

Definition 5.3.1 (Subspace Topology)**.** Let $(X, \mathcal{T})$ be a topological space and $Y \subseteq X$. The *subspace topology* on Y is $\mathcal{T}' = \{U \cap Y : U \in \mathcal{T}\}$.

A *subspace* of $(X, \mathcal{T})$ is a topological space consisting of a subset of X under the subspace topology.

We prove this is a topology.

PROOF:
$\langle 1 \rangle 1$. $\emptyset \in \mathcal{T}'$
PROOF: This holds because $\emptyset = \emptyset \cap Y$.
$\langle 1 \rangle 2$. $Y \in \mathcal{T}'$
PROOF: This holds because $Y = X \cap Y$.
$\langle 1 \rangle 3$. For all $\mathcal{U} \subseteq \mathcal{T}'$ we have $\bigcup \mathcal{U} \in \mathcal{T}'$
PROOF: This holds because $\bigcup \mathcal{U} = \bigcup \{U \in \mathcal{T} : U \cap Y \in \mathcal{U}\} \cap Y$.
$\langle 1 \rangle 4$. For all $U, V \in \mathcal{T}'$ we have $U \cap V \in \mathcal{T}'$
$\langle 2 \rangle 1$. PICK $U' \in \mathcal{T}$ and $V' \in \mathcal{T}$ such that $U = U' \cap Y$ and $V = V' \cap Y$
$\langle 2 \rangle 2$. $U \cap V = U' \cap V' \cap Y$
□

Proposition 5.3.2. *Let X be a topological space, $Y \subseteq X$ and $Z \subseteq Y$. The subspace topology on Z as a subspace of Y is the same as the subspace topology on Z as a subspace of X.*

PROOF:The subspace topology on Z as a subspace of Y is

$$\begin{aligned} \{V \cap Z : V \text{ open in } Y\} &= \{(U \cap Y) \cap Z : U \text{ open in } X\} \\ &= \{U \cap Z : U \text{ open in } X\} \ . \end{aligned}$$

□

Proposition 5.3.3. *Let Y be a subspace of X. If U is open in Y and Y is open in X then U is open in X.*

PROOF: There exists V open in X such that $U = V \cap Y$. □

Proposition 5.3.4. *Let X be a topological space, Y a subspace of X, and $A \subseteq Y$. Then A is closed in Y if and only if there exists a closed C in X such that $A = C \cap Y$.*

PROOF:
⟨1⟩1. If A is closed in Y then there exists C closed in X such that $A = C \cap Y$
  ⟨2⟩1. ASSUME: A is closed in Y
  ⟨2⟩2. PICK U open in X such that $Y \setminus A = U \cap Y$
  ⟨2⟩3. LET: $C = X \setminus U$
  ⟨2⟩4. $A = C \cap Y$
⟨1⟩2. For all C closed in X, we have $C \cap Y$ is closed in Y.
  ⟨2⟩1. LET: C be closed in X
  ⟨2⟩2. $Y \setminus (C \cap Y)$ is open in Y
    PROOF: $Y \setminus (C \cap Y) = (X \setminus C) \cap Y$
□

Corollary 5.3.4.1. *Let X be a topological space, Y a subspace of X and $C \subseteq Y$. If C is closed in Y and Y is closed in X then C is closed in X.*

PROOF: There exists D closed in X such that $C = D \cap Y$. □

Proposition 5.3.5. *Let X be a topological space, Y a subspace of X and $A \subseteq Y$. Let $\overline{A}$ be the closure of A in X. Then the closure of A in Y is $\overline{A} \cap Y$.*

PROOF:
⟨1⟩1. $\overline{A} \cap Y$ is closed in Y
  PROOF: Proposition 5.3.4.
⟨1⟩2. $\overline{A} \cap Y \subseteq A$
⟨1⟩3. If C is a closed set in Y and $C \subseteq A$ then $C \subseteq \overline{A} \cap Y$.
  ⟨2⟩1. PICK D closed in X such that $C = D \cap Y$
    PROOF: Proposition 5.3.4.
  ⟨2⟩2. $D \subseteq \overline{A}$
  ⟨2⟩3. $C \subseteq \overline{A} \cap Y$
□

Proposition 5.3.6. *Let X be a topological space, Y a subspace of X, $N \subseteq Y$ and $y \in Y$. Then N is a neighbourhood of y in Y iff there exists a neighbourhood M of y in X such that $N = M \cap Y$.*

PROOF:
⟨1⟩1. If N is a neighbourhood of y in Y then there exists a neighbourhood M of y in X such that $N = M \cap Y$
  ⟨2⟩1. ASSUME: N is a neighbourhood of y in Y
  ⟨2⟩2. PICK a set U open in Y such that $y \in U \subseteq N$

⟨2⟩3. PICK a set V open in X such that $U = V \cap Y$
⟨2⟩4. Take $M = V \cup N$
⟨1⟩2. For any neighbourhood M of y in X, we have $M \cap Y$ is a neighbourhood of y in Y.
⟨2⟩1. PICK U open in X such that $y \in U \subseteq M$
⟨2⟩2. $y \in U \cap Y \subseteq M \cap Y$
□

Proposition 5.3.7. *Let X be a topological space and Y a subspace of X. If $\mathcal{B}$ is a basis for the topology on X then $\mathcal{B}' = \{B \cap Y : B \in \mathcal{B}\}$ is a basis for the subspace topology on Y.*

PROOF:
⟨1⟩1. Every element of $\mathcal{B}'$ is open.
PROOF: Immediate from definitions.
⟨1⟩2. For every open U in Y and $y \in U$ there exists $B' \in \mathcal{B}'$ such that $y \in B' \subseteq U$.
⟨2⟩1. PICK V open in X such that $U = V \cap Y$
⟨2⟩2. PICK $B \in \mathcal{B}$ such that $y \in B \subseteq V$
⟨2⟩3. Take $B' = B \cap Y$
□

Proposition 5.3.8. *Let X be a topological space and Y a subspace of X. If $\mathcal{S}$ is a subbasis for the topology on X then $\mathcal{S}' = \{S \cap Y : S \in \mathcal{S}\}$ is a subbasis for the subspace topology on Y.*

PROOF:
⟨1⟩1. $\{S_1 \cap \cdots \cap S_n \cap Y : S_1, \ldots, S_n \in \mathcal{S}\}$ is a basis for the subspace topology.
PROOF: Proposition 5.3.7.
⟨1⟩2. Every element of $\mathcal{S}'$ is open in the subspace topology.
PROOF: Immediate from definitions.
⟨1⟩3. For all $S_1, \ldots, S_n \in \mathcal{S}$ we have $S_1 \cap \cdots \cap S_n \cap Y$ is a union of finite intersections of elements of $\mathcal{S}'$
PROOF: $S_1 \cap \cdots \cap S_n \cap Y = (S_1 \cap Y) \cap \cdots \cap (S_n \cap Y)$.
⟨1⟩4. Q.E.D.
PROOF: Proposition 4.10.4.
□

Proposition 5.3.9. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces and $A_\alpha \subseteq X_\alpha$ for all $\alpha \in J$. Then the product topology on $\prod_{\alpha \in J} A_\alpha$ is the same as the topology that $\prod_{\alpha \in J} A_\alpha$ inherits as a subspace of $\prod_{\alpha \in J} X_\alpha$.*

PROOF:
⟨1⟩1. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces and $A_\alpha \subseteq X_\alpha$ for all $\alpha \in J$
⟨1⟩2. LET: $\pi_\alpha : \prod_{\alpha \in J} X_\alpha \to X_\alpha$ be the αth projection on $\prod_{\alpha \in J} X_\alpha$
⟨1⟩3. LET: $p_\alpha : \prod_{\alpha \in J} A_\alpha \to A_\alpha$ be the αth projection on $\prod_{\alpha \in J} A_\alpha$
⟨1⟩4. the product topology on $\prod_{\alpha \in J} A_\alpha$ is the same as the topology that $\prod_{\alpha \in J} A_\alpha$ inherits as a subspace of $\prod_{\alpha \in J} X_\alpha$.

PROOF: Each is the topology generated by the subbasis
$\{p_\alpha{}^{-1}(U) : \alpha \in J, U \text{ open in } A_\alpha\} = \{p_\alpha{}^{-1}(U \cap A_\alpha) : \alpha \in J, U \text{ open in } X_\alpha\} \quad = \{\pi_\alpha{}^{-1}(U) \cap \prod_{\alpha \in J} A_\alpha : \alpha \in$

using Proposition 5.3.8.
□

Proposition 5.3.10. *Let X be a linearly ordered set under the order topology. Let $Y \subseteq X$ be convex. Then the order topology on Y is the same as the subspace topology.*

PROOF:
⟨1⟩1. LET: $\mathcal{T}_o$ be the order topology on Y and $\mathcal{T}_s$ the subspace topology.
⟨1⟩2. $\mathcal{T}_o \subseteq \mathcal{T}_s$
⟨2⟩1. For all $a \in Y$ we have $\{x \in Y : x < a\} \in \mathcal{T}_s$
PROOF: The set is $(-\infty, a) \cap Y$.
⟨2⟩2. For all $a \in Y$ we have $\{x \in Y : x > a\} \in \mathcal{T}_s$
PROOF: The set is $(a, +\infty) \cap Y$.
⟨2⟩3. Q.E.D.
PROOF: Proposition 4.10.3.
⟨1⟩3. $\mathcal{T}_s \subseteq \mathcal{T}_o$
⟨2⟩1. For all $a \in X$ we have $(-\infty, a) \cap Y \in \mathcal{T}_o$
⟨3⟩1. CASE: $a \in Y$
PROOF: In this case, $(-\infty, a) \cap Y = \{x \in Y : x < a\}$.
⟨3⟩2. CASE: a is less than every element of Y
PROOF: In this case, $(-\infty, a) \cap Y = \emptyset$.
⟨3⟩3. CASE: a is greater than every element of Y
PROOF: In this case, $(-\infty, a) \cap Y = Y$.
⟨3⟩4. Q.E.D.
PROOF: These are the only three possibilities because Y is convex.
⟨2⟩2. For all $a \in X$ we have $(a, +\infty) \cap Y \in \mathcal{T}_o$
PROOF: Similar.
⟨2⟩3. Q.E.D.
PROOF: Proposition 4.10.3.
□

5.3.1 Unit Sphere

Definition 5.3.11 (Unit n-sphere)**.** For $n \geq 1$, the *unit n-sphere* S^n is the space $\{(x_1, \dots, x_{n+1}) : x_1^2 + \cdots + x_{n+1}^2 = 1\}$ as a subspace of $\mathbb{R}^{n+1}$.

5.3.2 Unit Ball

Definition 5.3.12 (Unit n-ball)**.** For $n \geq 1$, the *unit n-ball* B^n is the space $\{(x_1, \dots, x_{n+1}) : x_1^2 + \cdots + x_{n+1}^2 \leq 1\}$ as a subspace of $\mathbb{R}^{n+1}$.

5.3.3 Punctured Euclidean Space

Definition 5.3.13 (Punctured Euclidean Space)**.** The space *n-dimensional punctured Euclidean space* is $\mathbb{R}^n \setminus \{0\}$.

5.3.4 Topologist's Sine Curve

Definition 5.3.14 (Topologist's Sine Curve)**.** The *topologist's sine curve* is the closure of $\{(x, \sin 1/x) : 0 < x \leq 1\}$ in $\mathbb{R}^2$.

5.4 The Box Topology

Definition 5.4.1 (Box Topology)**.** Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces. The *box topology* on $\prod_{\alpha \in J} X_\alpha$ is the topology generated by the basis $\mathcal{B}$ consisting of all sets of the form $\prod_{\alpha \in J} U_\alpha$ where each U_α is open in X_α.

We prove $\mathcal{B}$ is a basis for a topology.

PROOF:
⟨1⟩1. $\bigcup \mathcal{B} = \prod_{\alpha \in J} X_\alpha$
PROOF: This holds because $\prod_{\alpha \in J} X_\alpha \in \mathcal{B}$.
⟨1⟩2. For all $B_1, B_2 \in \mathcal{B}$ and $(x_\alpha) \in B_1 \cap B_2$ there exists $B_3 \in \mathcal{B}$ such that $(x_\alpha) \in B_3 \subseteq B_1 \cap B_2$
PROOF: If $B_1 = \prod_{\alpha \in J} U_\alpha$ and $B_2 = \prod_{\alpha \in J} V_\alpha$ take $B_3 = \prod_{\alpha \in J} (U_\alpha \cap V_\alpha)$.
⟨1⟩3. Q.E.D.
PROOF: Proposition 4.9.2
□

Proposition 5.4.2. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces and $(x_\alpha) \in \prod_{\alpha \in J} X_\alpha$ under the box topology. Let $\alpha \in J$. If M is a neighbourhood of x_α in X_α then ${\pi_\alpha}^{-1}(M)$ is a neighbourhood of (x_α).*

PROOF:
⟨1⟩1. PICK U open in X_α such that $x_\alpha \in U \subseteq M$
⟨1⟩2. $(x_\alpha) \in {\pi_\alpha}^{-1}(U) \subseteq {\pi_\alpha}^{-1}(M)$
□

Proposition 5.4.3 (AC)**.** *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces. Suppose $\mathcal{B}_\alpha$ is a basis for the topology on X_α for all $\alpha \in J$. Then $\{\prod_{\alpha \in J} B_\alpha : \forall \alpha \in J. B_\alpha \in \mathcal{B}_\alpha\}$ is a basis for the box topology on $\prod_{\alpha \in J} X_\alpha$.*

PROOF:
⟨1⟩1. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces.
⟨1⟩2. LET: $\mathcal{B}_\alpha$ be a basis for the topology on X_α for all $\alpha \in J$.
⟨1⟩3. LET: $\mathcal{B} = \{\prod_{\alpha \in J} B_\alpha : \forall \alpha \in J. B_\alpha \in \mathcal{B}_\alpha\}$
⟨1⟩4. Every element of $\mathcal{B}$ is open.
PROOF: From ⟨1⟩2, ⟨1⟩3
⟨1⟩5. For every open set U and point $(x_\alpha) \in U$, there exists $B \in \mathcal{B}$ such that $(x_\alpha) \in B \subseteq U$.

⟨2⟩1. LET: U be open and $(x_\alpha) \in U$
⟨2⟩2. PICK U_α open in X_α for all α such that $(x_\alpha) \in \prod_{\alpha \in J} U_\alpha \subseteq U$
PROOF: ⟨2⟩2
⟨2⟩3. For $\alpha \in J$, PICK $B_\alpha \in \mathcal{B}_\alpha$ such that $x_\alpha \in B_\alpha \subseteq U_\alpha$
PROOF: ⟨1⟩2, ⟨2⟩2
⟨2⟩4. $(x_\alpha) \in \prod_{\alpha \in J} B_\alpha \subseteq U$
PROOF: ⟨2⟩2, ⟨2⟩3
□

Proposition 5.4.4. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces and $A_\alpha \subseteq X_\alpha$ for all $\alpha \in J$. Then the box topology on $\prod_{\alpha \in J} A_\alpha$ is the same as the topology that $\prod_{\alpha \in J} A_\alpha$ inherits as a subspace of $\prod_{\alpha \in J} X_\alpha$ under the box topology.*

PROOF:Each is the topology generated by the basis

$$\begin{aligned} &\{\prod_{\alpha \in J} U_\alpha : \forall \alpha \in J. U_\alpha \text{ open in } A_\alpha\} \\ =&\{\prod_{\alpha \in J} (V_\alpha \cap A_\alpha) : \forall \alpha \in J. V_\alpha \text{ open in } X_\alpha\} \\ =&\{\prod_{\alpha \in J} V_\alpha \cap \prod_{\alpha \in J} A_\alpha : \forall \alpha \in J. V_\alpha \text{ open in } X_\alpha\} \end{aligned}$$

using Proposition 5.3.7. □

Proposition 5.4.5 (AC)**.** *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces and $A_\alpha \subseteq X_\alpha$ for all $\alpha \in J$. Then*

$$\prod_{\alpha \in J} \overline{A_\alpha} = \overline{\prod_{\alpha \in J} A_\alpha}$$

under the box topology.

PROOF:
⟨1⟩1. $\prod_{\alpha \in J} \overline{A_\alpha} \subseteq \overline{\prod_{\alpha \in J} A_\alpha}$
⟨2⟩1. LET: $(x_\alpha) \in \prod_{\alpha \in J} \overline{A_\alpha}$
⟨2⟩2. LET: U_α be open in X_α for all α with $(x_\alpha) \in \prod_{\alpha \in J} U_\alpha$
⟨2⟩3. For all α we have $x_\alpha \in \overline{A_\alpha}$
PROOF: From ⟨2⟩1
⟨2⟩4. For all α we have $x_\alpha \in U_\alpha$
PROOF: From ⟨2⟩2
⟨2⟩5. For all α we have U_α intersects A_α
PROOF: Proposition 4.6.5, ⟨2⟩3, ⟨2⟩4
⟨2⟩6. $\prod_{\alpha \in J} U_\alpha$ intersects $\prod_{\alpha \in J} A_\alpha$
PROOF: From ⟨2⟩5 using the Axiom of Choice.
⟨2⟩7. Q.E.D.
PROOF: Proposition 4.9.7.
⟨1⟩2. $\overline{\prod_{\alpha \in J} A_\alpha} \subseteq \prod_{\alpha \in J} \overline{A_\alpha}$
⟨2⟩1. LET: $(x_\alpha) \in \overline{\prod_{\alpha \in J} A_\alpha}$

⟨2⟩2. LET: $\alpha \in J$
PROVE: $x_\alpha \in \overline{A_\alpha}$
⟨2⟩3. LET: N be a neighbourhood of x_α in X_α
PROVE: N intersects A_α
⟨2⟩4. LET: $M = {\pi_\alpha}^{-1}(N)$
⟨2⟩5. M is a neighbourhood of (x_α)
PROOF: Proposition 5.4.2, ⟨2⟩3, ⟨2⟩4
⟨2⟩6. M intersects $\prod_{\alpha \in J} A_\alpha$
PROOF: Proposition 4.6.5, ⟨2⟩1, ⟨2⟩5.
⟨2⟩7. PICK $(y_\alpha) \in M \cap \prod_{\alpha \in J} A_\alpha$
PROOF: ⟨2⟩6
⟨2⟩8. $y_\alpha \in N \cap A_\alpha$
PROOF: ⟨2⟩4, ⟨2⟩7
⟨2⟩9. Q.E.D.
PROOF: Proposition 4.6.5

□

Chapter 6

Functions Between Topological Spaces

6.1 Continuous Functions

Definition 6.1.1 (Continuous)**.** Let X and Y be topological spaces. A function $f : X \to Y$ is *continuous* iff, for every open set V in Y, we have $f^{-1}(V)$ is open in X.

Proposition 6.1.2. *Let X and Y be topological spaces and $f : X \to Y$. Then f is continuous if and only if, for every closed set C in Y, we have $f^{-1}(C)$ is closed in X.*

PROOF:

$$\begin{aligned} f \text{ is continuous} &\Leftrightarrow (\forall V \subseteq Y. V \text{ open} \Rightarrow f^{-1}(V) \text{ open}) \\ &\Leftrightarrow (\forall V \subseteq Y. Y \setminus V \text{ closed} \Rightarrow X \setminus f^{-1}(V) \text{ closed}) \quad \text{(Proposition 4.2.2)} \\ &\Leftrightarrow (\forall V \subseteq Y. Y \setminus V \text{ closed} \Rightarrow f^{-1}(Y \setminus V) \text{ closed}) \\ &\Leftrightarrow (\forall C \subseteq Y. C \text{ closed} \Rightarrow f^{-1}(C) \text{ closed}) \end{aligned}$$

□

Proposition 6.1.3. *Let X and Y be topological spaces and $f : X \to Y$. Then f is continuous if and only if, for all $A \subseteq X$, we have $f(\overline{A}) \subseteq \overline{f(A)}$.*

PROOF:
⟨1⟩1. LET: X and Y be topological spaces and $f : X \to Y$
⟨1⟩2. If f is continuous then, for all $A \subseteq X$, we have $f(\overline{A}) \subseteq \overline{f(A)}$.
  PROOF:
  ⟨2⟩1. ASSUME: f is continuous.
  ⟨2⟩2. LET: $A \subseteq X$
  ⟨2⟩3. $f^{-1}(\overline{f(A)})$ is closed
    PROOF: Proposition 6.1.2
  ⟨2⟩4. $A \subseteq f^{-1}(\overline{f(A)})$
    PROOF: This holds because $f(A) \subseteq \overline{f(A)}$

⟨2⟩5. $\overline{A} \subseteq f^{-1}(\overline{f(A)})$
PROOF: From ⟨2⟩3 and ⟨2⟩4
⟨2⟩6. $f(\overline{A}) \subseteq \overline{f(A)}$
PROOF: From ⟨2⟩5
⟨1⟩3. If, for all $A \subseteq X$, we have $f(\overline{A}) \subseteq \overline{f(A)}$, then f is continuous.
⟨2⟩1. ASSUME: for all $A \subseteq X$, we have $f(\overline{A}) \subseteq \overline{f(A)}$
⟨2⟩2. LET: $C \subseteq Y$ be closed
⟨2⟩3. $f(\overline{f^{-1}(C)}) \subseteq \overline{C}$
PROOF: ⟨2⟩1
⟨2⟩4. $f(\overline{f^{-1}(C)}) \subseteq C$
PROOF: Proposition 4.4.2, ⟨2⟩2, ⟨2⟩3
⟨2⟩5. $\overline{f^{-1}(C)} \subseteq f^{-1}(C)$
PROOF: ⟨2⟩4
⟨2⟩6. $\inf f(C)$ is closed
PROOF: Proposition 4.4.2, ⟨2⟩5
⟨2⟩7. Q.E.D.
PROOF: Proposition 6.1.2

□

Proposition 6.1.4. *Let X and Y be topological spaces and $f : X \to Y$. Let $\mathcal{B}$ be a basis for the topology on Y. Then f is continuous if and only if, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X.*

PROOF:
⟨1⟩1. LET: X and Y be topological spaces and $f : X \to Y$
⟨1⟩2. LET: $\mathcal{B}$ be a basis for the topology on Y
⟨1⟩3. If f is continuous then, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X.
PROOF: Immediate from definitions and the fact that every element of $\mathcal{B}$ is open in Y (⟨1⟩2).
⟨1⟩4. If, for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X, then f is continuous.
⟨2⟩1. ASSUME: for all $B \in \mathcal{B}$, we have $f^{-1}(B)$ is open in X
⟨2⟩2. LET: V be open in Y
⟨2⟩3. LET: $x \in f^{-1}(V)$
⟨2⟩4. PICK $B \in \mathcal{B}$ such that $f(x) \in B \subseteq V$
PROOF: ⟨1⟩2, ⟨2⟩2, ⟨2⟩3
⟨2⟩5. $x \in f^{-1}(B) \subseteq \inf f(V)$
PROOF: ⟨2⟩4
⟨2⟩6. $f^{-1}(V)$ is a neighbourhood of x
PROOF: ⟨2⟩1, ⟨2⟩4, ⟨2⟩5
⟨2⟩7. Q.E.D.
PROOF: Proposition 4.6.2

□

Proposition 6.1.5. *Let X and Y be topological spaces and $f : X \to Y$. Let $\mathcal{S}$ be a basis for the topology on Y. Then f is continuous if and only if, for all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in X.*

PROOF:

$\langle 1\rangle 1$. LET: X and Y be topological spaces and $f : X \to Y$
$\langle 1\rangle 2$. LET: $\mathcal{S}$ be a subbasis for the topology on Y
$\langle 1\rangle 3$. If f is continuous then, for all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in X
PROOF: Immediate from definitions and the fact that every member of $\mathcal{S}$ is open in Y ($\langle 1\rangle 2$).
$\langle 1\rangle 4$. If, for all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in X, then f is continuous.
$\langle 2\rangle 1$. ASSUME: for all $S \in \mathcal{S}$, we have $f^{-1}(S)$ is open in X
$\langle 2\rangle 2$. The set of all finite intersections of elements of $\mathcal{S}$ form a basis $\mathcal{B}$ for Y
PROOF: Proposition 4.10.2.
$\langle 2\rangle 3$. LET: $B \in \mathcal{B}$
PROVE: $f^{-1}(B)$ is open in X
$\langle 2\rangle 4$. PICK $S_1, \dots, S_n \in \mathcal{S}$ such that $B = S_1 \cap \cdots \cap S_n$
PROOF: $\langle 2\rangle 2$, $\langle 2\rangle 3$
$\langle 2\rangle 5$. $f^{-1}(B) = f^{-1}(S_1) \cap \cdots \cap f^{-1}(S_n)$
PROOF: From $\langle 2\rangle 4$
$\langle 2\rangle 6$. $f^{-1}(B)$ is open.
PROOF: From $\langle 2\rangle 1$, $\langle 2\rangle 4$, $\langle 2\rangle 5$.
$\langle 2\rangle 7$. Q.E.D.
PROOF: Proposition 6.1.4.
□

Proposition 6.1.6. *Let X and Y be topological functions. Every constant function $f : X \to Y$ is continuous.*

PROOF: For $V \subseteq Y$ open, we have $f^{-1}(V)$ is either $\emptyset$ or X. □

Proposition 6.1.7. *Let X be a topological space and Y a subspace of X. Then the inclusion $i : Y \hookrightarrow X$ is continuous.*

PROOF: For $U \subseteq X$ open, we have $i^{-1}(U) = U \cap Y$ is open in Y. □

Corollary 6.1.7.1. *The identity function on a topological space X is continuous.*

Proposition 6.1.8. *If $f : X \to Y$ and $g : Y \to Z$ are continuous then $g \circ f : X \to Z$ is continuous.*

PROOF:
$\langle 1\rangle 1$. LET: $U \subseteq Z$ be open.
$\langle 1\rangle 2$. $g^{-1}(U)$ is open in Y
$\langle 1\rangle 3$. $\inf f(\inf g(U))$ is open in X.
□

Proposition 6.1.9. *Let $f : X \to Y$ be continuous and $A \subseteq X$. Then the restriction $f \restriction A : A \to Y$ is continuous.*

PROOF: From Propositions 6.1.7 and 6.1.8 since $f \restriction A = f \circ i$ where $i : A \hookrightarrow X$ is the inclusion. □

Proposition 6.1.10. *Let $f : X \to Y$ be continuous and $f(X) \subseteq Z \subseteq Y$. Then f is continuous considered as a function $X \to Z$.*

Proof:
$\langle 1\rangle 1$. Let: $f : X \to Y$ be continuous
$\langle 1\rangle 2$. Let: $f(X) \subseteq Z \subseteq Y$
$\langle 1\rangle 3$. Let: $V \subseteq Z$ be open
$\langle 1\rangle 4$. Pick U open in Y such that $V = U \cap Z$
  Proof: $\langle 1\rangle 3$
$\langle 1\rangle 5$. $f^{-1}(U)$ is open
  Proof:$\langle 1\rangle 1$, $\langle 1\rangle 4$
$\langle 1\rangle 6$. $f^{-1}(U) = f^{-1}(V)$
  Proof: $\langle 1\rangle 2$, $\langle 1\rangle 4$
$\langle 1\rangle 7$. $f^{-1}(V)$ is open
  Proof: $\langle 1\rangle 5$, $\langle 1\rangle 6$
□

Proposition 6.1.11. *Let X and Y be topological spaces and $Z \subseteq Y$. Let $f : X \to Z$. If f is continuous as a function $X \to Z$, then f is continuous as a function $X \to Y$.*

Proof: From Propositions 6.1.8 and 6.1.7 since $f = i \circ f$ where $i : Z \to Y$ is the inclusion. □

Proposition 6.1.12. *Let $f : X \to Y$ be continuous. Let $\mathcal{U} \subseteq \mathcal{P}X$. If $X = \bigcup \mathcal{U}$ and $f \restriction U : U \to Y$ is continuous for all $U \in \mathcal{U}$, then f is continuous.*

Proof:
$\langle 1\rangle 1$. Let: $f : X \to Y$ be continuous
$\langle 1\rangle 2$. Let: $\mathcal{U} \subseteq \mathcal{P}X$
$\langle 1\rangle 3$. Assume: $X = \bigcup \mathcal{U}$
$\langle 1\rangle 4$. Assume: $f \restriction U : U \to Y$ is continuous for all $U \in \mathcal{U}$
$\langle 1\rangle 5$. Let: $V \subseteq Y$ be open
$\langle 1\rangle 6$. $f^{-1}(V)$ is open.
  $\langle 2\rangle 1$. Let: $x \in f^{-1}(V)$
  $\langle 2\rangle 2$. Pick $U \in \mathcal{U}$ such that $x \in U$
    Proof: $\langle 1\rangle 3$
  $\langle 2\rangle 3$. $f^{-1}(V) \cap U$ is open
    Proof: By $\langle 1\rangle 4$ since $f^{-1}(V) \cap U = (f \restriction U)^{-1}(V)$
  $\langle 2\rangle 4$. $x \in f^{-1}(V) \cap U \subseteq f^{-1}(V)$
    Proof: $\langle 2\rangle 1$, $\langle 2\rangle 2$
  $\langle 2\rangle 5$. Q.E.D.
    Proof: Proposition 4.6.2.
□

Proposition 6.1.13 (Pasting Lemma)**.** *Let X and Y be topological spaces. Let $X = A \cup B$ where A and B are closed in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous, and suppose $f(x) = g(x)$ for all $x \in A \cap B$. Then the function $h : X \to Y$ defined by*

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases}$$

is continuous.

PROOF:
$\langle 1\rangle 1$. LET: X and Y be topological spaces.
$\langle 1\rangle 2$. LET: $X = A \cup B$ where A and B are closed in X.
$\langle 1\rangle 3$. LET: $f : A \to Y$ and $g : B \to Y$ be continuous.
$\langle 1\rangle 4$. ASSUME: $f(x) = g(x)$ for all $x \in A \cap B$
$\langle 1\rangle 5$. LET: $h : X \to Y$ be defined by

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases}$$

PROOF: This is well-defined by $\langle 1\rangle 2$ and $\langle 1\rangle 4$.
$\langle 1\rangle 6$. LET: $C \subseteq Y$ be closed
$\langle 1\rangle 7$. $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$
PROOF: $\langle 1\rangle 5$
$\langle 1\rangle 8$. $f^{-1}(C)$ is closed in X
$\langle 2\rangle 1$. $f^{-1}(C)$ is closed in A
PROOF: Proposition 6.1.2, $\langle 1\rangle 3$, $\langle 1\rangle 6$.
$\langle 2\rangle 2$. Q.E.D.
PROOF: Proposition 5.3.4, $\langle 1\rangle 2$, $\langle 1\rangle 6$.
$\langle 1\rangle 9$. $g^{-1}(C)$ is closed in X
PROOF: Similar.
$\langle 1\rangle 10$. $h^{-1}(C)$ is closed in X
PROOF: Proposition 4.2.2, $\langle 1\rangle 7$, $\langle 1\rangle 8$, $\langle 1\rangle 9$.
$\langle 1\rangle 11$. Q.E.D.
PROOF: Proposition 6.1.2.
□

Proposition 6.1.14. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces. For all $\alpha \in J$, the projection $\pi_\alpha : \prod_{\alpha \in J} X_\alpha \to X_\alpha$ is continuous.*

PROOF: Immediate from definitions. □

Proposition 6.1.15. *Let X be a space and $\{Y_\alpha\}_{\alpha \in J}$ a family of spaces. Let $f_\alpha : X \to Y_\alpha$ for all $\alpha \in J$. Then $\langle f_\alpha \mid \alpha \in J\rangle : X \to \prod_{\alpha \in J} Y_\alpha$ is continuous iff each f_α is continuous.*

PROOF:
$\langle 1\rangle 1$. LET: X be a space and $\{Y_\alpha\}_{\alpha \in J}$ a family of spaces.
$\langle 1\rangle 2$. LET: $f_\alpha : X \to Y_\alpha$ for all $\alpha \in J$.
$\langle 1\rangle 3$. If $\langle f_\alpha \mid \alpha \in J\rangle$ is continuous then each f_α is continuous.
PROOF: From Propositions 6.1.8 and 6.1.14 since $f_\alpha = \pi_\alpha \circ \langle f_\alpha \mid \alpha \in J\rangle$.
$\langle 1\rangle 4$. If each f_α is continuous then $\langle f_\alpha \mid \alpha \in J\rangle$ is continuous.
$\langle 2\rangle 1$. ASSUME: each f_α is continuous.
$\langle 2\rangle 2$. LET: $\alpha \in J$ and U be open in α
$\langle 2\rangle 3$. $\langle f_\alpha \mid \alpha \in J\rangle^{-1}({\pi_\alpha}^{-1}(U)) = {f_\alpha}^{-1}(U)$
$\langle 2\rangle 4$. $\langle f_\alpha \mid \alpha \in J\rangle^{-1}({\pi_\alpha}^{-1}(U))$ is open in X
PROOF: $\langle 2\rangle 1$, $\langle 2\rangle 2$, $\langle 2\rangle 3$.

$\langle 2\rangle 5.$ Q.E.D.
PROOF: Proposition 6.1.5
□

Definition 6.1.16 (Continuity at a Point)**.** Let X and Y be topological spaces, $f : X \to Y$ and $x \in X$. Then f is *continuous at* x iff, for every neighbourhood N of $f(x)$, there exists a neighbourhood M of x such that $f(M) \subseteq N$.

Proposition 6.1.17. *Let X and Y be topological spaces and $f : X \to Y$. Then f is continuous if and only if it is continuous at every point.*

PROOF:
$\langle 1\rangle 1.$ LET: X and Y be topological spaces and $f : X \to Y$
$\langle 1\rangle 2.$ If f is continuous then f is continuous at every point.
$\langle 2\rangle 1.$ ASSUME: f is continuous.
$\langle 2\rangle 2.$ LET: $x \in X$
$\langle 2\rangle 3.$ LET: N be a neighbourhood of $f(x)$
$\langle 2\rangle 4.$ PICK V open in Y such that $f(x) \in V \subseteq Y$
PROOF: $\langle 2\rangle 3$
$\langle 2\rangle 5.$ LET: $M = f^{-1}(V)$
$\langle 2\rangle 6.$ M is a neighbourhood of x
$\langle 3\rangle 1.$ M is open
PROOF: $\langle 2\rangle 1$, $\langle 2\rangle 4$, $\langle 2\rangle 5$
$\langle 3\rangle 2.$ $x \in M$
PROOF: $\langle 2\rangle 4$, $\langle 2\rangle 5$
$\langle 3\rangle 3.$ Q.E.D.
PROOF: Proposition 4.6.2
$\langle 2\rangle 7.$ $f(M) \subseteq N$
PROOF: $\langle 2\rangle 4$, $\langle 2\rangle 5$
$\langle 1\rangle 3.$ If f is continuous at every point then f is continuous.
$\langle 2\rangle 1.$ ASSUME: f is continuous at every point
$\langle 2\rangle 2.$ LET: V be open in Y
PROVE: $f^{-1}(V)$ is open in X
$\langle 2\rangle 3.$ LET: $x \in f^{-1}(V)$
$\langle 2\rangle 4.$ PICK a neighbourhood M of x such that $f(M) \subseteq V$
$\langle 3\rangle 1.$ V is a neighbourhood of $f(x)$
PROOF: $\langle 2\rangle 2$, $\langle 2\rangle 3$
$\langle 3\rangle 2.$ Q.E.D.
PROOF: $\langle 2\rangle 1$
$\langle 2\rangle 5.$ $M \subseteq f^{-1}(V)$
PROOF: $\langle 2\rangle 4$
$\langle 2\rangle 6.$ $f^{-1}(V)$ is a neighbourhood of x
PROOF: Proposition 4.6.2, $\langle 2\rangle 4$, $\langle 2\rangle 5$
$\langle 2\rangle 7.$ Q.E.D.
PROOF: Proposition 4.6.2
□

Proposition 6.1.18. *Let X and Y be topological spaces and $f : X \to Y$. If $x_n \to l$ as $n \to \infty$ in X and f is continuous at l then $f(x_n) \to f(l)$ as $n \to \infty$.*

PROOF:
$\langle 1 \rangle 1$. LET: X and Y be topological spaces and $f : X \to Y$ be continuous.
$\langle 1 \rangle 2$. ASSUME: $x_n \to l$ as $n \to \infty$
$\langle 1 \rangle 3$. ASSUME: f is continuous at l.
$\langle 1 \rangle 4$. LET: N be a neighbourhood of $f(l)$
$\langle 1 \rangle 5$. PICK a neighbourhood M of l such that $f(M) \subseteq N$
PROOF: $\langle 1 \rangle 3$, $\langle 1 \rangle 1$, $\langle 1 \rangle 4$.
$\langle 1 \rangle 6$. PICK N such that, for all $n \geq N$, we have $x_n \in M$
PROOF: $\langle 1 \rangle 2$, $\langle 1 \rangle 5$.
$\langle 1 \rangle 7$. For all $n \geq N$ we have $f(x_n) \in M$
PROOF: $\langle 1 \rangle 5$, $\langle 1 \rangle 6$.
□

6.2 Homeomorphisms

Definition 6.2.1 (Homeomorphism)**.** Let X and Y be topological spaces. A *homeomorphism* $f : X \cong Y$ is a bijective function $f : X \to Y$ such that f and $f^{-1} : Y \to X$ are continuous.

Spaces X and Y are *homeomorphic*, $X \cong Y$, iff there exists a homeomorphism between them.

Proposition 6.2.2. *Let X and Y be topological spaces and $f : X \to Y$. Then f is a homeomorphism iff f is a bijection and, for all $U \subseteq X$, we have U is open iff $f(U)$ is open.*

PROOF: Immediate from definitions. □

Definition 6.2.3 (Topological Property)**.** A property P of topological spaces is a *topological property* iff, given homeomorphic spaces $X \cong Y$, we have that X has the property P if and only if Y has the property P.

Definition 6.2.4 (Homogeneous)**.** A topological space X is *homogeneous* iff, for all $x, y \in X$, there exists a homeomorphism $\phi : X \cong X$ such that $\phi(x) = y$.

6.3 Topological Imbeddings

Definition 6.3.1 (Topological Imbedding)**.** Let X and Y be topological spaces and $f : X \to Y$. Then f is a *(topological) imbedding* iff f is a homeomorphism between X and $f(X)$.

Proposition 6.3.2. *Let X and Y be topological spaces and $f : X \to Y$. Then f is an imbedding iff f is continuous, injective, and $f(U)$ is open in $f(X)$ for all U open in X.*

PROOF:
⟨1⟩1. LET: X and Y be topological spaces and $f : X \to Y$.
⟨1⟩2. If f is an imbedding then f is continuous.
PROOF: From Proposition 6.1.11.
⟨1⟩3. If f is an imbedding then f is injective.
PROOF: Immediate from definition.
⟨1⟩4. If f is an imbedding then $f(U)$ is open in $f(X)$ for all U open in X
PROOF: From the fact that $f^{-1} : f(X) \to X$ is continuous.
⟨1⟩5. If f is continuous, injective, and $f(U)$ is open in $f(X)$ for all U open in X, then f is an imbedding.
⟨2⟩1. ASSUME: f is continuous
⟨2⟩2. ASSUME: f is injective
⟨2⟩3. ASSUME: $f(U)$ is open in $f(X)$ for all U open in X
⟨2⟩4. f is a bijection between X and $f(X)$
PROOF: From ⟨2⟩2
⟨2⟩5. $f : X \to f(X)$ is continuous.
PROOF: Proposition 6.1.10, ⟨2⟩1.
⟨2⟩6. $f^{-1} : f(X) \to X$ is continuous.
PROOF: From ⟨2⟩3
□

Proposition 6.3.3. *Let $\{X_\alpha\}_{\alpha\in J}$ be a family of topological spaces. Let $\alpha \in J$. Pick $a_\beta \in X_\beta$ for all $\beta \neq \alpha$. Then the function $f : X_\alpha \to \prod_{\alpha\in J} X_\alpha$ given by*

$$\begin{aligned} f(x)_\alpha &= x \\ f(x)_\beta &= a_beta && (\beta \neq \alpha) \end{aligned}$$

is an imbedding.

PROOF:
⟨1⟩1. LET: $\{X_\alpha\}_{\alpha\in J}$ be a family of topological spaces.
⟨1⟩2. LET: $\alpha \in J$
⟨1⟩3. LET: $a_\beta \in X_\beta$ for all $\beta \neq \alpha$
⟨1⟩4. LET: $f : X_\alpha \to \prod_{\alpha\in J} X_\alpha$ given by

$$\begin{aligned} f(x)_\alpha &= x \\ f(x)_\beta &= a_beta && (\beta \neq \alpha) \end{aligned}$$

⟨1⟩5. f is continuous.
PROOF: From Proposition 6.1.15, Corollary 6.1.7.1, Proposition 6.1.6.
⟨1⟩6. f is injective
PROOF: From ⟨1⟩4.
⟨1⟩7. For all U open in X_α we have $f(U)$ is open in $f(X_\alpha)$
PROOF: For $U \subseteq X_\alpha$ open we have $f(U) = {\pi_\alpha}^{-1}(U) \cap f(X_\alpha)$ is open in $f(X_\alpha)$.
⟨1⟩8. Q.E.D.
PROOF: Proposition 6.3.2.
□

6.4 Open Maps

Definition 6.4.1 (Open Map)**.** Let X and Y be topological spaces and $f : X \to Y$. Then f is an *open map* iff, for every U open in X, we have $f(U)$ is open in Y.

Proposition 6.4.2. *Let X and Y be topological spaces and $f : X \to Y$. Let $\mathcal{B}$ be a basis for X. Then f is an open map iff, for all $B \in \mathcal{B}$, we have $f(B)$ is open in Y.*

PROOF:
⟨1⟩1. If f is open then, for all $B \in \mathcal{B}$, we have $f(B)$ is open.
  PROOF: This holds because every element of $\mathcal{B}$ is open.
⟨1⟩2. If, for all $B \in \mathcal{B}$, $f(B)$ is open, then f is open.
  ⟨2⟩1. ASSUME: for all $B \in \mathcal{B}$, $f(B)$ is open.
  ⟨2⟩2. LET: $U \subseteq X$ be open
  ⟨2⟩3. $f(U)$ is open.
    PROOF:

$$\begin{aligned} f(U) &= f\left(\bigcup\{B \in \mathcal{B} : B \subseteq U\}\right) && (\mathcal{B} \text{ is a basis}) \\ &= \bigcup\{f(B) : B \in \mathcal{B}, B \subseteq U\} \end{aligned}$$

□

Corollary 6.4.2.1. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces. Each projection $\pi_\alpha : \prod_{\alpha \in J} X_\alpha \to X_\alpha$ is an open map.*

6.5 Closed Maps

Definition 6.5.1 (Closed Map)**.** Let X and Y be topological spaces and $f : X \to Y$. Then f is a *closed map* iff, for every closed $C \subseteq X$, we have $f(C)$ is closed.

6.6 Strong Continuity

Definition 6.6.1 (Strong Continuity)**.** Let X and Y be topological spaces and $f : X \to Y$. Then f is *strongly continuous* iff, for all $U \subseteq Y$, we have U is open in Y if and only if $f^{-1}(U)$ is open in X.

Proposition 6.6.2. *Let X and Y be topological spaces and $f : X \to Y$. Then f is strongly continuous iff, for all $C \subseteq Y$, we have C is closed in Y iff $f^{-1}(C)$ is closed in X.*

PROOF: Easy. □

Proposition 6.6.3. *Let X and Y be topological spaces and $p : X \to Y$. The following are equivalent.*

1. *p is strongly continuous.*

2. *p is continuous and maps saturated open sets to open sets.*

3. *p is continuous and maps saturated closed sets to closed sets.*

PROOF: From definitions. □

Corollary 6.6.3.1. 1. *Every open continuous map is strongly continuous.*

2. *Every closed continuous map is strongly continuous.*

Proposition 6.6.4. *The composite of two strongly continuous functions is strongly continuous.*

PROOF: Easy. □

Proposition 6.6.5. *Let $p : X \to Y$ be strongly continuous and $f : Y \to Z$. If $f \circ p$ is continuous then f is continuous.*

PROOF: For $V \subseteq Z$ open we have $p^{-1}(f^{-1}(V))$ is open, hence $f^{-1}(V)$ open.

Proposition 6.6.6. *Let $f : X \to Y$ be strongly continuous and $g : Y \to Z$. If $g \circ f$ is strongly continuous then g is strongly continuous.*

PROOF: For $V \subseteq Z$ we have V is open iff $f^{-1}(g^{-1}(V))$ is open iff $g^{-1}(V)$ is open. □

Proposition 6.6.7. *Let $p : A \to B$ and $q : C \to D$ be open strongly continuous maps. Then $p \times q : A \times C \to B \times D$ is open and strongly continuous.*

PROOF:
$\langle 1 \rangle 1.$ $p \times q$ is an open map.
  $\langle 2 \rangle 1.$ LET: U be open in A and V open in C
  $\langle 2 \rangle 2.$ $(p \times q)(U \times V)$ is open in $B \times D$
  $\langle 2 \rangle 3.$ Q.E.D.
    PROOF: Proposition 6.4.2.
$\langle 1 \rangle 2.$ $p \times q$ is strongly continuous.
  PROOF: Corollary 6.6.3.1.
□

6.7 Quotient Maps

Definition 6.7.1 (Quotient Map)**.** Let X and Y be topological spaces and $p : X \to Y$. Then p is a *quotient map* iff p is surjective and strongly continuous.

Example 6.7.2. For $\{X_\alpha\}_{\alpha \in J}$ a family of topological spaces, the projections $\pi_\alpha : \prod_{\alpha \in J} X_\alpha \to X_\alpha$ are quotient maps, because they are open maps, continuous and surjective.

Proposition 6.7.3. *Let $p : X \twoheadrightarrow Y$ be a quotient map. Let $A \subseteq X$ be saturated w.r.t. p. Let $q : A \to p(A)$ be the restriction of p to A.*

1. *If A is either open or closed in X then q is a quotient map.*

2. *If p is either an open map or a closed map then q is a quotient map.*

PROOF:
⟨1⟩1. For all $V \subseteq p(A)$ we have $q^{-1}(V) = p^{-1}(V)$
PROOF: From the fact that A is saturated.
⟨1⟩2. For all $U \subseteq X$ we have $p(U \cap A) = p(U) \cap p(A)$
PROOF: From the fact that A is saturated.
⟨1⟩3. If either A is open or p is an open map then q is a quotient map.
⟨2⟩1. ASSUME: Either A is open or p is an open map.
PROVE: For all $V \subseteq p(A)$, if $q^{-1}(V)$ is open in A then V is open in $p(A)$.
⟨2⟩2. LET: $V \subseteq p(A)$
⟨2⟩3. ASSUME: $q^{-1}(V)$ is open in A
⟨2⟩4. CASE: A is open
⟨3⟩1. $q^{-1}(V)$ is open in X.
PROOF: Proposition 5.3.3
⟨3⟩2. $p^{-1}(V)$ is open in X.
PROOF: By ⟨1⟩1.
⟨3⟩3. V is open in Y
PROOF: Since p is a quotient map.
⟨3⟩4. V is open in $p(A)$
⟨2⟩5. CASE: p is an open map
⟨3⟩1. PICK U open in X such that $p^{-1}(V) = q^{-1}(V) = U \cap A$
⟨3⟩2. $V = p(U) \cap p(A)$
PROOF:

$$\begin{aligned} V &= p(p^{-1}(V)) && (p \text{ is surjective}) \\ &= p(U \cap A) && (\langle 3\rangle 1) \\ &= p(U) \cap p(A) && (\langle 1\rangle 2) \end{aligned}$$

⟨1⟩4. If either A is closed or p is a closed map then q is a quotient map.
PROOF: Similar.
□

Proposition 6.7.4. *Let X be a topological space and Y a set. Let $p : X \to Y$ be surjective. Then there exists a unique topology on Y with respect to which p is a quotient map.*

PROOF: The topology is given by: U is open if and only if $p^{-1}(U)$ is open. □

Definition 6.7.5 (Quotient Topology)**.** Let X be a topological space and Y a set. Let $p : X \to Y$ be surjective. The *quotient topology* on Y induced by p is the unique topology with respect to which p is a quotient map.

Definition 6.7.6 (Quotient Space)**.** Let X be a topological space and $\sim$ an equivalence relation on X. The *quotient space*, *identification space* or *decomposition space* $X/\sim$ is the quotient set $X/\sim$ under the quotient topology induced by the canonical map $X \to X/\sim$.

Chapter 7

Separation Axioms

7.1 T_1 Spaces

Definition 7.1.1 (T_1 Space)**.** A T_1 *space* is a topological space in which every one-point set is closed.

Example 7.1.2.

1. The discrete topology is T_1.
2. The indiscrete topology on X is T_1 iff X is a singleton.
3. The finite complement topology is T_1.
4. The countable complement topology is T_1.
5. The lower limit topology on $\mathbb{R}$ is T_1.
6. The K-topology on $\mathbb{R}$ is T_1.
7. If $\mathcal{T}$ is finer than $\mathcal{T}'$ and $\mathcal{T}'$ is T_1 then $\mathcal{T}$ is T_1.
8. The order topology on a linearly ordered set X is T_1, because $X \setminus \{a\} = (-\infty, a) \cup (a, +\infty)$ is open.
9. A subspace of a T_1 space is T_1.
10. The ordered square is T_1.

Proposition 7.1.3. *A topological space is a T_1 space if and only if every finite set is closed.*

Proof: By Proposition 4.2.2, a finite union of closed sets is closed.

Proposition 7.1.4. *Let X be a T_1 space, $A \subseteq X$ and $a \in X$. Then a is a limit point of A if and only if every neighbourhood of a contains infinitely many points of A.*

PROOF:
$\langle 1 \rangle 1.$ LET: X be a T_1 space.
$\langle 1 \rangle 2.$ LET: $A \subseteq X$
$\langle 1 \rangle 3.$ LET: $a \in X$
$\langle 1 \rangle 4.$ If a is a limit point of A then every neighbourhood of a contains infinitely many points of A.
$\langle 2 \rangle 1.$ ASSUME: a is a limit point of A
$\langle 2 \rangle 2.$ ASSUME: for a contradiction N is a neighbourhood of a that contains only finitely many points of A, say a_1, ..., a_n and also possibly a.
$\langle 2 \rangle 3.$ $N \setminus \{a_1, \ldots, a_n\}$ is a neighbourhood of a that does not intersect A except possibly at a.
PROOF: Proposition 4.6.6.
$\langle 2 \rangle 4.$ Q.E.D.
PROOF: $\langle 2 \rangle 1$ and $\langle 2 \rangle 3$ are a contradiction.
$\langle 1 \rangle 5.$ If every neighbourhood of a contains infinitely many points of A then a is a limit point of A.
PROOF: Immediate from definitions.
□

Proposition 7.1.5. *The product of a family of T_1 spaces is T_1.*

PROOF: From Proposition 5.2.5. □

Proposition 7.1.6. *A topological space X is T_1 if and only if, for any points $a, b \in X$ with $a \neq b$, there exists a neighbourhood N of a that does not contain b.*

PROOF: Immediate from definitions. □

7.2 Hausdorff Spaces

Definition 7.2.1 (Hausdorff Space)**.** A *Hausdorff space* is a topological space such that any two distinct points have two disjoint neighbourhoods.

Example 7.2.2.

1. The discrete topology is Hausdorff because $\{a\}$ and $\{b\}$ are two disjoint neighbourhoods of a and b if $a \neq b$.
2. The indiscrete topology on X is Hausdorff iff X is a singleton.
3. The finite complement topology on X is Hausdorff iff X is finite.
4. The countable complement topology on X is Hausdorff iff X is countable.
5. The lower limit topology on $\mathbb{R}$ is Hausdorff.
6. The K-topology on $\mathbb{R}$ is Hausdorff.

Proposition 7.2.3. *Every Hausdroff space is T_1.*

PROOF:
$\langle 1 \rangle 1$. LET: X be a Hausdorff space.
$\langle 1 \rangle 2$. LET: $a \in X$
PROVE: $X \setminus \{a\}$ is open.
$\langle 1 \rangle 3$. LET: $b \in X \setminus \{a\}$
$\langle 1 \rangle 4$. PICK disjoint neighbourhoods M of a and N of b.
PROOF: $\langle 1 \rangle 3$, $\langle 1 \rangle 3$
$\langle 1 \rangle 5$. $N \subseteq X \setminus \{a\}$
PROOF: From $\langle 1 \rangle 4$
$\langle 1 \rangle 6$. $X \setminus \{a\}$ is a neighbourhood of b
PROOF: Proposition 4.6.2, $\langle 1 \rangle 4$, $\langle 1 \rangle 5$
$\langle 1 \rangle 7$. Q.E.D.
PROOF: Proposition 4.6.2.

Proposition 7.2.4. *In a Hausdorff space, a sequence has at most one limit.*

PROOF:
$\langle 1 \rangle 1$. LET: X be a Hausdorff space.
$\langle 1 \rangle 2$. ASSUME: for a contradiction $x_n \to l$ as $n \to \infty$ and $x_n \to m$ as $n \to \infty$ and $l \neq m$
$\langle 1 \rangle 3$. PICK disjoint neighbourhoods L and M of l and m respectively.
PROOF: $\langle 1 \rangle 3$, $\langle 1 \rangle 2$
$\langle 1 \rangle 4$. PICK N such that, for all $n \geq N$, we have $x_n \in L$ and $x_n \in M$
PROOF: $\langle 1 \rangle 2$, $\langle 1 \rangle 3$
$\langle 1 \rangle 5$. Q.E.D.
PROOF: $\langle 1 \rangle 4$ contradicts the fact that L and M are disjoint ($\langle 1 \rangle 3$).
$\square$

Proposition 7.2.5. *Every linearly ordered set is Hausdorff under the order topology.*

PROOF:
$\langle 1 \rangle 1$. LET: X be a linearly ordered set under the order topology.
$\langle 1 \rangle 2$. LET: $a, b \in X$ with $a \neq b$
PROVE: There exist disjoint neighbourhoods L and M of a and b respectively.
$\langle 1 \rangle 3$. ASSUME: w.l.o.g. $a < b$
$\langle 1 \rangle 4$. CASE: There exists c such that $a < c < b$
PROOF: Take $L = (-\infty, c)$ and $M = (c, +\infty)$.
$\langle 1 \rangle 5$. CASE: There is no c such that $a < c < b$
PROOF: Take $L = (-\infty, b)$ and $M = (a, +\infty)$.
$\square$

Proposition 7.2.6. *The product of a family of Hausdorff spaces is Hausdorff.*

PROOF:

⟨1⟩1. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of Hausdorff spaces.
⟨1⟩2. LET: $(a_\alpha), (b_\alpha) \in \prod_{\alpha \in J} X_\alpha$ be distinct.
⟨1⟩3. PICK $\alpha \in J$ such that $a_\alpha \neq b_\alpha$
⟨1⟩4. PICK disjoint neighbourhoods L and M of a_α and b_α in X_α
⟨1⟩5. ${\pi_\alpha}^{-1}(L)$ and ${\pi_\alpha}^{-1}(M)$ are disjoint neighbourhoods of (a_α) and (b_α) respectively.
□

Corollary 7.2.6.1. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of Hausdorff spaces. Then $\prod_{\alpha \in J} X_\alpha$ under the box topology is Hausdorff.*

PROOF: The box topology is finer than the product topology. □

Proposition 7.2.7. *A subspace of a Hausdorff space is Hausdorff.*

PROOF:
⟨1⟩1. LET: X be Hausdorff and $Y \subseteq X$
⟨1⟩2. LET: $a, b \in Y$ with $a \neq b$
⟨1⟩3. PICK disjoint neighbourhoods M and N of a and b in X
⟨1⟩4. $M \cap Y$ and $N \cap Y$ are disjoint neighbourhoods of a and b in Y
PROOF: Proposition 5.3.6
□

Corollary 7.2.7.1. *The ordered square is Hausdorff.*

Proposition 7.2.8. *A topological space X is Hausdorff iff the diagonal $\Delta = \{(x, x) : x \in X\}$ is closed in $X \times X$.*

PROOF: Both 'X is Hausdorff' and '$(X \times X) \setminus \Delta$ is open' are equivalent to the statement: for all $(a, b) \in X \times X$, if $(a, b) \notin \Delta$ then there exist open U, V such that $a \in U$, $b \in V$ and $U \times V \subseteq (X \times X) \setminus \Delta$. □

7.3 Regular Spaces

Definition 7.3.1 (Regular)**.** A topological space X is *regular* iff it is T_1 and, for every closed $A \subseteq X$ and point $a \in X \setminus A$, there exist disjoint neighbourhoods M and N of A and a respectively.

Chapter 8

Countability Axioms

8.1 The First Countability Axiom

Definition 8.1.1 (First Countable)**.** A topological space satisfies the *first countability axiom*, or is *first countable*, iff every point has a countable local basis.

Proposition 8.1.2. *A space is first countable iff every point has a countable basis* $\{B_1, B_2, \ldots\}$ *such that* $B_1 \supseteq B_2 \supseteq \cdots$.

Proof: If a point has a countable basis $\{C_1, C_2, \ldots\}$, then take $B_n = C_1 \cap \cdots \cap C_n$. ∎

Proposition 8.1.3 (Sequence Lemma (CC))**.** *Let* X *be a first countable space,* $A \subseteq X$ *and* $a \in \overline{A}$*. Then there exists a sequence in* A *that converges to* a*.*

Proof:
⟨1⟩1. Pick a basis $\{B_1, B_2, \ldots\}$ at a with $B_1 \supseteq B_2 \supseteq \cdots$
⟨1⟩2. For $n = 1, 2, \ldots$, Pick $a_n \in B_n \cap A$
Prove: For every neighbourhood U of a, there exists N such that, for all $n \geq N$, we have $a_n \in U$.
Proof: Using Proposition 4.6.5.
⟨1⟩3. Let: U be a neighbourhood of a.
⟨1⟩4. Pick N such that $B_N \subseteq U$
⟨1⟩5. For all $n \geq N$ we have $a_n \in U$
Proof: Since $a_n \in B_n \subseteq B_N \subseteq U$.
∎

Example 8.1.4.
1. The space $\mathbb{R}_l$ is first countable. For any $a \in \mathbb{R}$ the set of all intervals $[a, q)$ for q rational is a local basis at a.
2. The ordered square is first countable. For:
   - A basis at $(0, 0)$ is the set of all intervals of the form $((0, 0), (0, q))$ with $q > 0$ rational.

- For $0 < y < 1$, a basis at (x, y) is the set of all intervals of the form $((x, q), (x, r))$ with q, r rational, $q < y < r$.
- For $x < 1$, a basis at $(x, 1)$ is the set of all intervals of the form $((x, q), (r, 0))$ with q, r rational, $q < 1$ and $r > x$.
- For $x > 0$, a basis at $(x, 0)$ is the set of all intervals of the form $((q, 0), (x, r))$ with q, r rational, $q < x$ and $r > 0$.
- A basis at $(1, 1)$ is the set of all intervals of the form $((1, q), (1, 1))$ with $q < 1$ rational.

3. The space $\mathbb{R}^\omega$ under the box topology is not first countable. Let A be the set of all sequences whose members are all positive. Then $\vec{0} \in \overline{A}$ but there is no sequence in A that converges to $\vec{0}$. For, given any sequence $((a_{mn})_n)_m$ in A, the open set

$$\prod_{n=1}^{\infty}(-a_{nn}, a_{nn})$$

contains $\vec{0}$ but does not contain any member of the sequence.

Example 8.1.5. For J uncountable, the space $\mathbb{R}^J$ under the product topology is not first countable.

PROOF:
⟨1⟩1. LET: $A = \{(x_\alpha) \in \mathbb{R}^J : x_\alpha = 1$ for all but finitely many $\alpha\}$
⟨1⟩2. $\vec{0} \in \overline{A}$
 ⟨2⟩1. LET: $\vec{0} \in \prod_{\alpha \in J} U_\alpha$ where each U_α is open in $\mathbb{R}$ and $U_\alpha = \mathbb{R}$ for all but finitely many α, say $\alpha_1, \ldots, \alpha_n$
 ⟨2⟩2. LET: $x_\alpha = 0$ for $\alpha = \alpha_1, \ldots, \alpha_n$ and $x_\alpha = 1$ for all other α
 ⟨2⟩3. $(x_\alpha) \in \prod_{\alpha \in J} U_\alpha \cap A$
⟨1⟩3. There is no sequence of points in A that converges to $\vec{0}$.
 ⟨2⟩1. ASSUME: for a contradiction $((a_{n\alpha})_\alpha)_n$ is a sequence in A that converges to $\vec{0}$
 ⟨2⟩2. For $n = 1, 2, \ldots$,
 LET: $J_n = \{\alpha \in J : a_{n\alpha} \neq 1\}$
 ⟨2⟩3. PICK $\beta \in J \setminus \bigcup_{n=1}^{\infty} J_n$
 PROOF: This is possible because J is uncountable.
 ⟨2⟩4. $\vec{0} \in {\pi_\beta}^{-1}((-1, 1))$
 ⟨2⟩5. There is no n such that $(a_{n\alpha})_\alpha \in {\pi_\beta}^{-1}((-1, 1))$
 PROOF: $a_{n\beta} = 1$ for all n.
□

Chapter 9

Connectedness

9.1 Connectedness

Definition 9.1.1 (Connected)**.** Let X be a topological space. A *separation* of X is a pair of nonempty disjoint open sets U, V such that $U \cup V = X$. The space X is *connected* iff it does not have a separation.

Proposition 9.1.2. *A space X is connected iff the only sets that are both open and closed are $\emptyset$ and X.*

PROOF: Immediate from definitions. □

Proposition 9.1.3. *Let X be a topological space and Y a subspace of X. Then a separation of Y is a pair of disjoint nonempty sets A and B such that $A \cup B = Y$ and neither of A, B contains a limit point of the other.*

PROOF: If A and B are disjoint, nonempty, and $A \cup B = Y$, then we have

A and B form a separation of Y

$\Leftrightarrow$ A and B are open in Y

$\Leftrightarrow$ A and B are closed in Y

$\Leftrightarrow$ A and B each contain all their limit points in Y

$\Leftrightarrow$ neither of A, B contains a point that is a limit point of the other in Y

$\Leftrightarrow$ neither of A, B contains a point that is a limit point of the other in X

□

Proposition 9.1.4. *If C and D form a separation of X and Y is a connected subspace of X then $Y \subseteq C$ or $Y \subseteq D$.*

PROOF: Otherwise $C \cap Y$ and $D \cap Y$ would form a separation of Y. □

Proposition 9.1.5. *Let $\mathcal{A}$ be a set of connected subspaces of a space X and B a connected subspace of X such that every element of $\mathcal{A}$ intersects B. Then $\bigcup \mathcal{A} \cup B$ is connected.*

PROOF:
$\langle 1 \rangle 1$. LET: $\mathcal{A}$ be a set of connected subspaces of a space X and B a connected subspace of X such that every element of $\mathcal{A}$ intersects B.
$\langle 1 \rangle 2$. ASSUME: for a contradiction C and D form a separation of $\bigcup \mathcal{A} \cup B$
$\langle 1 \rangle 3$. ASSUME: w.l.o.g. $B \subseteq C$
PROOF: Proposition 9.1.4, $\langle 1 \rangle 1$, $\langle 1 \rangle 2$.
$\langle 1 \rangle 4$. For all $A \in \mathcal{A}$ we have $A \subseteq C$
PROOF: Proposition 9.1.4, $\langle 1 \rangle 1$, $\langle 1 \rangle 2$, $\langle 1 \rangle 3$.
$\langle 1 \rangle 5$. $\bigcup \mathcal{A} \cup B \subseteq C$
PROOF: $\langle 1 \rangle 3$, $\langle 1 \rangle 4$.
$\langle 1 \rangle 6$. Q.E.D.
PROOF: This contradicts the fact that D is nonempty.
□

Corollary 9.1.5.1. *The union of a set of connected subspaces of a space X that have a point in common is connected.*

Proposition 9.1.6. *Let (A_n) be a sequence of connected subspaces of X such that A_n intersects A_{n+1} for all n. Then $\bigcup_n A_n$ is connected.*

PROOF:
$\langle 1 \rangle 1$. LET: (A_n) be a sequence of connected subspaces of X such that A_n intersects A_{n+1} for all n
$\langle 1 \rangle 2$. ASSUME: for a contradiction C and D form a separation of $\bigcup_n A_n$
$\langle 1 \rangle 3$. ASSUME: w.l.o.g. $A_1 \subseteq C$
PROOF: Proposition 9.1.4, $\langle 1 \rangle 1$, $\langle 1 \rangle 2$.
$\langle 1 \rangle 4$. For all n, $A_n \subseteq C$
$\langle 2 \rangle 1$. ASSUME: $A_n \subseteq C$
$\langle 2 \rangle 2$. PICK $a \in A_n \cap A_{n+1}$
PROOF: $\langle 1 \rangle 1$.
$\langle 2 \rangle 3$. $a \in C$
PROOF: $\langle 2 \rangle 1$, $\langle 2 \rangle 2$
$\langle 2 \rangle 4$. $A_{n+1} \subseteq C$
PROOF: Proposition 9.1.4, $\langle 1 \rangle 1$, $\langle 1 \rangle 2$, $\langle 2 \rangle 2$, $\langle 2 \rangle 3$.
$\langle 2 \rangle 5$. Q.E.D.
PROOF: By induction using $\langle 1 \rangle 3$.
$\langle 1 \rangle 5$. Q.E.D.
PROOF: This contradicts the fact that D is nonempty.

Proposition 9.1.7. *Let A be a connected subspace of X. If $A \subseteq B \subseteq \overline{A}$ then B is connected.*

PROOF:
$\langle 1 \rangle 1$. ASSUME: for a contradiction C and D form a separation of B
$\langle 1 \rangle 2$. ASSUME: w.l.o.g. $A \subseteq C$
PROOF: Using Proposition 9.1.4.
$\langle 1 \rangle 3$. PICK $x \in D$
PROOF: Since D is nonempty.

$\langle 1\rangle 4.$ x is a limit point of A

PROOF: Since $x \in \overline{A}$ and $x \notin A$.

$\langle 1\rangle 5.$ x is a limit point of C

PROOF: By $\langle 1\rangle 2$

$\langle 1\rangle 6.$ Q.E.D.

PROOF: This contradicts Proposition 9.1.3.

□

Corollary 9.1.7.1. *Let X be a topological space. The closure of a connected subspace of X is connected.*

Proposition 9.1.8. *The continuous image of a connected space is connected.*

PROOF: Let $f : X \to Y$ be continuous and X be connected. If C and D formed a separation of $f(X)$ then $f^{-1}(C)$ and $f^{-1}(D)$ would form a separation of X.

□

Proposition 9.1.9. *The product of two connected spaces is connected.*

PROOF:

$\langle 1\rangle 1.$ LET: X and Y be connected.

$\langle 1\rangle 2.$ PICK $(a, b) \in X \times Y$

PROOF: If either X or Y is empty then $X \times Y$ is empty and hence connected.

$\langle 1\rangle 3.$ $X \times \{b\}$ is connected.

PROOF: It is homeomorphic to X.

$\langle 1\rangle 4.$ For all $x \in X$ we have $\{x\} \times Y$ is connected.

PROOF: It is homeomorphic to Y.

$\langle 1\rangle 5.$ For $x \in X$

LET: $T_x = (X \times \{b\}) \cup (\{x\} \times Y)$

$\langle 1\rangle 6.$ For all $x \in X$ we have T_x is connected.

PROOF: Corollary 9.1.5.1 since the two sets have (x, b) is common.

$\langle 1\rangle 7.$ $X \times Y = \bigcup_{x \in X} T_x$

$\langle 1\rangle 8.$ $X \times Y$ is connected.

PROOF: Corollary 9.1.5.1 since the sets all have (a, b) in common.

□

Proposition 9.1.10. *The product of a family of connected spaces is connected.*

PROOF:

$\langle 1\rangle 1.$ LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of connected spaces.

$\langle 1\rangle 2.$ LET: $X = \prod_{\alpha \in J} X_\alpha$

$\langle 1\rangle 3.$ PICK $(a_\alpha) \in X$

PROOF: We may assume X is nonempty as the empty space is trivially connected.

$\langle 1\rangle 4.$ For $K \subseteq^{\text{fin}} J$,

LET: $X_K = \{(x_\alpha) \in X : \forall \alpha \notin K. x_\alpha = a_\alpha\}$

$\langle 1\rangle 5.$ For $K \subseteq^{\text{fin}} J$, we have X_K is connected.

PROOF: This holds using Proposition 9.1.9 because $X_K \cong \prod_{\alpha \in K} X_\alpha$.

$\langle 1\rangle 6$. LET: $Y = \bigcup_{K \subseteq^{\text{fin}} J} X_K$
$\langle 1\rangle 7$. Y is connected.
PROOF: Corollary 9.1.5.1
$\langle 1\rangle 8$. $\overline{Y} = X$
$\langle 2\rangle 1$. LET: $(x_\alpha) \in X$
$\langle 2\rangle 2$. LET: $(x_\alpha) \in \prod_{\alpha \in J} U_\alpha$ where each U_α is open in X_α and $U_\alpha = X_\alpha$ for all but finitely many α
$\langle 2\rangle 3$. LET: $K = \{\alpha \in J : U_\alpha \neq X_\alpha\}$
$\langle 2\rangle 4$. LET: $(y_a lpha)$ be the point with $y_\alpha = x_\alpha$ for $\alpha \in K$ and $y_\alpha = a_\alpha$ for all other α
$\langle 2\rangle 5$. $(y_\alpha) \in \prod_{\alpha \in J} U_\alpha$
$\langle 2\rangle 6$. $(y_\alpha) \in X_K$
$\langle 2\rangle 7$. $\prod_{\alpha \in J} U_\alpha$ intersects X_K
$\langle 2\rangle 8$. Q.E.D.
PROOF: Proposition 4.6.5.
□
$\langle 1\rangle 9$. X is connected.
PROOF: Proposition 9.1.7.
□

Proposition 9.1.11. *Any linear continuum is connected under the order topology.*

PROOF:
$\langle 1\rangle 1$. LET: L be a linear continuum under the order topology.
$\langle 1\rangle 2$. ASSUME: for a contradiction U and V form a separation of L.
$\langle 1\rangle 3$. PICK $a \in U$ and $b \in V$
$\langle 1\rangle 4$. ASSUME: w.l.o.g. $a < b$
$\langle 1\rangle 5$. LET: $s = \sup\{x \in U : a \leq x < b\}$
PROOF: L is complete.
$\langle 1\rangle 6$. CASE: $s \in U$
$\langle 2\rangle 1$. $s < b$
$\langle 2\rangle 2$. PICK $u > s$ such that $[s, u) \subseteq U$
PROOF: Proposition 5.1.3
$\langle 2\rangle 3$. PICK t such that $s < t < u$
PROOF: L is dense.
$\langle 2\rangle 4$. $t \in U$ and $a \leq t < b$
$\langle 2\rangle 5$. Q.E.D.
PROOF: This contradicts $\langle 1\rangle 5$.
$\langle 1\rangle 7$. CASE: $s \in V$
$\langle 2\rangle 1$. $a < s$
$\langle 2\rangle 2$. PICK $u < s$ such that $(u, s] \subseteq V$
PROOF: Proposition 5.1.4
$\langle 2\rangle 3$. u is an upper bound for $\{x \in U : a \leq x < b\}$
$\langle 2\rangle 4$. Q.E.D.
PROOF: This contradicts $\langle 1\rangle 5$.
□

Corollary 9.1.11.1. *If L is a linear continuum under the order topology, then every interval and ray in L is connected.*

Example 9.1.12.
1. The real line $\mathbb{R}$ is connected, and so is every interval and ray in $\mathbb{R}$.
2. Any set with more than one point is disconnected under the discrete topology.
3. Any set is connected under the indiscrete topology.
4. Any infinite set is connected under the finite complement topology.
5. Any uncountable set is connected under the countable complement topology.
6. The space $\mathbb{R}_l$ is not connected because $(-\infty, 0)$ and $[0, +\infty)$ form a separation.
7. The space $\mathbb{R}^\omega$ in the uniform topology is not connected. The set of bounded sequences and the set of unbounded sequences form a separation.
8. The space $\mathbb{R}^\omega$ in the box topology is not connected because it is finer than the uniform topology.
9. The ordered square is connected, because it is a linear continuum.
10. The topologist's sine curve is connected. The set $\{(x, \sin 1/x) : 0 < x \leq 1\}$ is connected by Proposition 9.1.8, and so the topologist's sine curve is connected by Corollary 9.1.7.1.

Proposition 9.1.13. *Let X be a topological space, C a connected subspace of X, and $A \subseteq X$. If C intersects A and $X \setminus A$ then C intersects ∂A.*

PROOF: Otherwise $C \cap \operatorname{Int} A$ and $C \cap \operatorname{Int}(X \setminus A)$ would form a separation of C. □

Theorem 9.1.14 (Intermediate Value Theorem)**.** *Let X be a connected space and Y a linearly ordered set under the order topology. Let $f : X \to Y$ be continuous. Let $a, b \in X$ and $c \in Y$. If $f(a) < c < f(b)$, then there exists $x \in X$ such that $f(x) = c$.*

PROOF: If not, then $(-\infty, c) \cap f(X)$ and $(c, +\infty) \cap f(X)$ would form a separation of $f(X)$, contradicting Proposition 9.1.8. □

9.2 Totally Disconnected Spaces

Definition 9.2.1 (Totally Disconnected)**.** A topological space X is *totally disconnected* iff the only nonempty connected subspaces are the one-point sets.

Example 9.2.2.
1. Any set under the discrete topology is totally disconnected.
2. The space $\mathbb{Q}$ is totally disconnected.

9.3 Path Connected Spaces

Definition 9.3.1 (Path)**.** Let X be a topological space and $a, b \in X$. A *path* from a to b is a continuous function $f : [0, 1] \to X$ such that $f(0) = a$ and $f(1) = b$.

Definition 9.3.2 (Path Connected)**.** A topological space X is *path connected* iff, for any points a, b, there exists a path from a to b.

Proposition 9.3.3. *Every path connected space is connected.*

PROOF:
$\langle 1 \rangle 1$. LET: X be a path connected space.
$\langle 1 \rangle 2$. ASSUME: for a contradiction U and V form a separation of X.
$\langle 1 \rangle 3$. PICK $a \in U$ and $b \in V$
$\langle 1 \rangle 4$. PICK a path $p : [0, 1] \to X$ from a to b
$\langle 1 \rangle 5$. $p([0, 1]) \cap U$ and $p([0, 1]) \cap V$ form a separation of $p([0, 1])$
$\langle 1 \rangle 6$. Q.E.D.

Proposition 9.3.4. *The continuous image of a path connected space is path connected.*

PROOF:
$\langle 1 \rangle 1$. LET: X be path connected and $f : X \twoheadrightarrow Y$ surjective continuous.
$\langle 1 \rangle 2$. LET: $a, b \in Y$.
$\langle 1 \rangle 3$. PICK $x, y \in X$ such that $f(x) = a$ and $f(y) = b$.
$\langle 1 \rangle 4$. PICK a path $p : [0, 1] \to X$ from x to y.
$\langle 1 \rangle 5$. $f \circ p$ is a path from a to b.
$\square$

Example 9.3.5.
1. For $n \geq 1$, the unit n-ball is path connected, hence connected. Given $a, b \in B^n$, the function $p : [0, 1] \to B^n$ given by $p(t) = (1 - t)a + tb$ is a path from a to b.
2. For $n > 1$, n-dimensional punctured Euclidean space is path connected, hence connected. Given $a, b \in \mathbb{R}^n \setminus \{0\}$, we can find a path from a to b that consists of at most two line segments.
3. For $n \geq 1$, the unit n-sphere is path connected, hence connected. For the map $g : \mathbb{R}^{n+1} \setminus \{0\} \twoheadrightarrow S^n$ given by $g(x_1, \ldots, x_n) = (x_1, \ldots, x_n)/\sqrt{x_1^2 + \cdots + x_n^2}$ is continuous.
4. The ordered square is not path connected. If $p : [0, 1] \to I_o^2$ is a path from $(0, 0)$ to $(1, 1)$ then $\{p^{-1}(\{x\} \times (0, 1)) : x \in [0, 1]\}$ is a set of uncountably many disjoint open sets in $[0, 1]$, which is impossible because each must contain a different rational.

Proposition 9.3.6 (DC)**.** *The topologist's sine curve is not path connected.*

PROOF:
$\langle 1 \rangle 1$. LET: $S = \{(x, \sin 1/x) : 0 < x \leq 1\}$.

$\langle 1 \rangle 2$. ASSUME: for a contradiction $p : [0,1] \to \overline{S}$ is a path from $(0,0)$ to $(1, \sin 1)$.
$\langle 1 \rangle 3$. LET: $p(t) = (x(t), y(t))$ for $t \in [0,1]$
$\langle 1 \rangle 4$. x and y are continuous.
PROOF: Proposition 6.1.15, $\langle 1 \rangle 2$, $\langle 1 \rangle 3$.
$\langle 1 \rangle 5$. $\{t \in [0,1] : x(t) = 0\}$ is closed.
PROOF: From $\langle 1 \rangle 4$ since the set is $x^{-1}(\{0\})$.
$\langle 1 \rangle 6$. LET: b be the greatest element of $[0,1]$ such that $x(b) = 0$ for some y.
PROOF: Proposition 5.1.6
$\langle 1 \rangle 7$. There exists a sequence (t_n) in $(b,1]$ such that $t_n \to b$ as $n \to \infty$ and $y(t_n) = (-1)^n$
$\langle 2 \rangle 1$. LET: $U = 1 - b$
$\langle 2 \rangle 2$. For all n, PICK u_n with $0 < u_n < x(b+U/n)$ such that $\sin 1/u_n = (-1)^n$
PROOF: This is possible since $\sin 1/x$ takes values 1 and -1 infinitely often in $(0, \epsilon)$ for any ϵ.
$\langle 2 \rangle 3$. For all n, PICK t_n with $b < t_n < b + U/n$ such that $x(t_n) = u_n$
PROOF: One such value must exist by the Intermediate Value Theorem.
$\langle 2 \rangle 4$. $t_n \to b$ as $n \to \infty$
$\langle 2 \rangle 5$. $y(t_n) = (-1)^n$ for all n
$\langle 1 \rangle 8$. Q.E.D.
PROOF: This contradicts Proposition 9.3.6.

Chapter 10

Metric Spaces

10.1 The Metric Topology

Definition 10.1.1 (Metric)**.** A *metric* on a set X is a function $d : X^2 \to \mathbb{R}$ such that, for all $x, y, z \in X$:

1. $d(x, y) \geq 0$
2. $d(x, y) = 0$ if and only if $x = y$
3. $d(x, y) = d(y, x)$
4. **Triangle Inequality**
   $d(x, z) \leq d(x, y) + d(y, z)$

A *metric space* consists of a set X and a metric on X. We call the elements of X *points* and $d(x, y)$ the *distance* between x and y.

Definition 10.1.2 (Open Ball)**.** Let $a \in X$ and $\epsilon > 0$. The *open ball* with *center* a and *radius* ϵ is

$$B(a, \epsilon) = \{x \in X : d(x, a) < \epsilon\} \ .$$

Definition 10.1.3 (Metric Topology)**.** The *metric topology* induced by d is the topology generated by the basis consisting of the open balls.

A topological space is *metrizable* iff there exists a metric that induces its topology.

We prove this is a basis.

Proof:
⟨1⟩1. Let: $\mathcal{B}$ be the set of all open balls.
⟨1⟩2. $\bigcup \mathcal{B} = X$
  Proof: This holds because $a \in B(a, 1)$ for all $a \in X$.
⟨1⟩3. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.
  ⟨2⟩1. Let: $B_1 = B(a, \epsilon_1)$ and $B_2 = B(b, \epsilon_2)$

$\langle 2\rangle 2$. LET: $\delta = \min(\epsilon_1 - d(a,x), \epsilon_2 - d(b,x))$
$\langle 2\rangle 3$. LET: $B_3 = B(x,\delta)$
$\langle 2\rangle 4$. $x \in B_3 \subseteq B_1 \cap B_2$
$\langle 3\rangle 1$. $B_3 \subseteq B_1$
$\langle 4\rangle 1$. LET: $y \in B_3$
$\langle 4\rangle 2$. $d(y,a) < \epsilon_1$
PROOF:

$$\begin{aligned} d(y,a) &\leq d(y,x) + d(x,a) && \text{(Triangle inequality)} \\ &< \delta + d(x,a) && (\langle 2\rangle 3,\ \langle 4\rangle 1) \\ &\leq \epsilon_1 && (\langle 2\rangle 2) \end{aligned}$$

$\langle 3\rangle 2$. $B_3 \subseteq B_2$
PROOF: Similar.
$\langle 1\rangle 4$. Q.E.D.
PROOF: Proposition 4.9.2
□

Proposition 10.1.4. *A set U is open in the metric topology if and only if, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x,\epsilon) \subseteq U$.*

PROOF:
$\langle 1\rangle 1$. If U is open then, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x,\epsilon) \subseteq U$
$\langle 2\rangle 1$. ASSUME: U is open
$\langle 2\rangle 2$. LET: $x \in U$
$\langle 2\rangle 3$. PICK an open ball $B(a,\delta)$ such that $x \in B(a,\delta) \subseteq U$
$\langle 2\rangle 4$. LET: $\epsilon = \delta - d(a,x)$
$\langle 2\rangle 5$. $B(x,\epsilon) \subseteq U$
$\langle 3\rangle 1$. LET: $y \in B(x,\epsilon)$
$\langle 3\rangle 2$. $y \in B(a,\delta)$
PROOF:

$$\begin{aligned} d(a,y) &\leq d(a,x) + d(x,y) && \text{(Triangle Inequality)} \\ &< d(a,x) + \epsilon && (\langle 3\rangle 1) \\ &= \delta && (\langle 2\rangle 4) \end{aligned}$$

$\langle 3\rangle 3$. $y \in U$
PROOF: $\langle 2\rangle 3$, $\langle 3\rangle 2$
$\langle 1\rangle 2$. If, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x,\epsilon) \subseteq U$, then U is open.
PROOF: Immediate from definitions.
□

Example 10.1.5.

1. The discrete topology on X is induced by the *discrete metric*

$$d(x,y) = \begin{cases} 0 & \text{if } x \neq y \\ 1 & \text{if } x = y \end{cases}$$

2. The standard topology on $\mathbb{R}$ is induced by the *Euclidean metric* $d(x,y) = |x-y|$.

Proposition 10.1.6. *Let d and d' be two metrics on the set X. Then the topology induced by d is finer than the topology induced by d' if and only if, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that*

$$B_d(x, \delta) \subseteq B_{d'}(x, \epsilon)$$

PROOF:
⟨1⟩1. If the topology induced by d is finer than the topology induced by d' if and only if, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_d(x, \delta) \subseteq B_{d'}(x, \epsilon)$.
  PROOF: From Proposition 10.1.4.
⟨1⟩2. If, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_d(x, \delta) \subseteq B_{d'}(x, \epsilon)$, then the topology induced by d is finer than the topology induced by d'
  ⟨2⟩1. ASSUME: for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_d(x, \delta) \subseteq B_{d'}(x, \epsilon)$
  ⟨2⟩2. LET: $x \in X$ and $\epsilon > 0$
    PROVE: $B_{d'}(x, \epsilon)$ is open under d
  ⟨2⟩3. LET: $y \in B_{d'}(x, \epsilon)$
  ⟨2⟩4. PICK $\epsilon' > 0$ such that $B_{d'}(y, \epsilon') \subseteq B_{d'}(x, \epsilon)$
    PROOF: Proposition 10.1.4, ⟨2⟩3
  ⟨2⟩5. PICK $\delta > 0$ such that $B_d(y, \delta) \subseteq B_{d'}(y, \epsilon')$
    PROOF: ⟨2⟩1, ⟨2⟩4
  ⟨2⟩6. $B_d(y, \delta) \subseteq B_{d'}(x, \epsilon)$
    PROOF: ⟨2⟩4, ⟨2⟩5
  ⟨2⟩7. Q.E.D.
    PROOF: Proposition 4.9.4, Proposition 10.1.4.
□

Proposition 10.1.7. *The metric topology is the coarsest topology such that $d : X^2 \to \mathbb{R}$ is continuous.*

PROOF:
⟨1⟩1. d is continuous with respect to the metric topology.
  ⟨2⟩1. LET: $(x, y) \in X^2$ and N be a neighbourhood of $d(x, y)$
  ⟨2⟩2. PICK $\epsilon > 0$ such that $(d(x, y) - \epsilon, d(x, y) + \epsilon) \subseteq N$
  ⟨2⟩3. LET: $M = B(x, \epsilon/2) \times B(y, \epsilon/2)$
    PROVE: $d(M) \subseteq N$
  ⟨2⟩4. LET: $(a, b) \in M$
  ⟨2⟩5. $d(a, b) - d(x, y) < \epsilon$
    PROOF:

$$\begin{aligned} d(a, b) &\leq d(a, x) + d(x, y) + d(y, b) \\ &< \epsilon/2 + d(x, y) + \epsilon/2 \end{aligned}$$

  ⟨2⟩6. $d(x, y) - d(a, b) < \epsilon$
    PROOF: Similar.
⟨1⟩2. If $\mathcal{T}$ is a topology with respect to which d is continuous then $\mathcal{T}$ is finer than the metric topology.

⟨2⟩1. Let: $a \in X$ and $\epsilon > 0$
Prove: $B(a, \epsilon)$ is open in $\mathcal{T}$
⟨2⟩2. The function $d_a = \lambda x.d(a, x)$ is continuous.
⟨2⟩3. $B(a, \epsilon) = {d_a}^{-1}((-\epsilon, \epsilon))$
□

Proposition 10.1.8. *Every metric space is Hausdorff.*

Proof: Let X be a metric space and $a, b \in X$ with $a \neq b$. Let $\epsilon = d(a, b)$. Then $B(a, \epsilon/2)$ and $B(b, \epsilon/2)$ are disjoint neighbourhoods of a and b. □

Proposition 10.1.9. *Every metric space is first countable.*

Proof: For any point a, the open balls $B(a, q)$ for q a positive rational form a basis at a. □

Example 10.1.10. The space $\mathbb{R}^\omega$ under the box topology is not metrizable, because it is not first countable.

Definition 10.1.11 (Bounded)**.** Let X be a metric space and $A \subseteq X$. Then A is *bounded* iff $\{d(x, y) : x, y \in A\}$ is bounded above, in which case its *diameter* is

$$\operatorname{diam} A = \sup\{d(x, y) : x, y \in A\}$$

10.2 Subspaces

Proposition 10.2.1. *Let (X, d) be a metric space and $A \subseteq X$. Then the restriction of d to A^2 is a metric on A that induces the subspace topology on A.*

Proof:
⟨1⟩1. Let: d' be the restriction of d to A^2.
⟨1⟩2. d' is a metric on A.
Proof: Easy.
⟨1⟩3. d' induces the subspace topology on A.
Proof: The topology induced by d' and the subspace topology are each the topology with basis $B_{d'}(x, \epsilon) = B_d(x, \epsilon) \cap A$ for $x \in A$ and $\epsilon > 0$.
□

10.3 Continuous Functions

Proposition 10.3.1. *Let X and Y be metric spaces, $f : X \to Y$ and $x \in X$. Then f is continuous at x if and only if, for all $\epsilon > 0$, there exists $\delta > 0$ such that, for all $y \in X$, if $d_X(x, y) < \delta$ then $d_Y(f(x), f(y)) < \epsilon$.*

Proof:
⟨1⟩1. If f is continuous at x then the right-hand side holds.
⟨2⟩1. Assume: f is continuous at x

⟨2⟩2. LET: $\epsilon > 0$
⟨2⟩3. PICK a neighbourhood M of x such that $f(M) \subseteq B_{d_Y}(f(x), \epsilon)$
⟨2⟩4. PICK $\delta > 0$ such that $B_{d_X}(x, \delta) \subseteq M$
⟨2⟩5. LET: $y \in X$ with $d_X(x, y) < \delta$
⟨2⟩6. $y \in M$
⟨2⟩7. $f(y) \in B_{d_Y}(f(x), \epsilon)$
⟨1⟩2. If the right-hand side holds then f is continuous at x.
⟨2⟩1. ASSUME: the right-hand side holds
⟨2⟩2. LET: N be a neighbourhood of $f(x)$
⟨2⟩3. PICK $\epsilon > 0$ such that $B_{d_Y}(f(x), \epsilon) \subseteq N$
⟨2⟩4. PICK $\delta > 0$ such that, for all $y \in X$, if $d_X(x, y) < \delta$ then $d_Y(f(x), f(y)) < \epsilon$
⟨2⟩5. LET: $M = B_{d_X}(x, \delta)$
⟨2⟩6. $f(M) \subseteq N$
□

10.4 Uniform Convergence

Definition 10.4.1 (Uniform Convergence)**.** Let X be a set and Y a metric space. Let (f_n) be a sequence of functions $X \to Y$ and $f : X \to Y$. Then f_n *converges uniformly* to f as $n \to \infty$ iff, for all $\epsilon > 0$, there exists N such that, for all $n \geq N$ and $x \in X$,

$$d(f_n(x), f(x)) < \epsilon \ .$$

Theorem 10.4.2 (Uniform Limit Theorem)**.** *Let X be a topological space and Y a metric space. Let (f_n) be a sequence of continuous functions $X \to Y$ and $f : X \to Y$. If f_n converges uniformly to f as $n \to \infty$ then f is continuous.*

PROOF:
⟨1⟩1. LET: $x \in X$ and V be a neighbourhood of $f(x)$
⟨1⟩2. PICK $\epsilon > 0$ such that $B(f(x), \epsilon) \subseteq V$
⟨1⟩3. PICK N such that, for all $n \geq N$ and $y \in X$, we have $d(f_n(y), f(y)) < \epsilon/3$
⟨1⟩4. PICK a neighbourhood M of x such that $f_N(M) \subseteq B(f_N(x), \epsilon/3)$
⟨1⟩5. $f(M) \subseteq V$
PROOF: For $y \in M$ we have

$$\begin{aligned} d(f(x), f(y)) &\leq d(f(x), f_N(x)) + d(f_N(x), f_N(y)) + d(f_N(y), f(y)) \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 \\ &= \epsilon \end{aligned}$$

□

10.5 Isometric Imbeddings

Definition 10.5.1 (Isometric Imbedding)**.** Let X and Y be metric spaces and $f : X \to Y$. Then f is an *isometric imbedding* iff, for all $x, y \in X$, we have $d_Y(f(x), f(y)) = d_X(x, y)$.

Proposition 10.5.2. *Every isometric imbedding is a topological imbedding.*

PROOF:
$\langle 1 \rangle 1$. LET: $f : X \to Y$ be an isometric imbedding.
$\langle 1 \rangle 2$. f is continuous
PROOF: If $d_X(x, y) < \epsilon$ then $d_Y(f(x), f(y)) < \epsilon$.
$\langle 1 \rangle 3$. f is injective
PROOF: $f(x) = f(y) \Leftrightarrow d_Y(f(x), f(y)) = 0 \Leftrightarrow d_X(x, y) = 0 \Leftrightarrow x = y$.
$\langle 1 \rangle 4$. For all U open in X we have $f(U)$ is open in $f(X)$.
$\langle 2 \rangle 1$. LET: U be open in X
$\langle 2 \rangle 2$. LET: $y \in f(U)$
$\langle 2 \rangle 3$. PICK $x \in U$ such that $f(x) = y$
$\langle 2 \rangle 4$. PICK $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$
$\langle 2 \rangle 5$. $B(y, \epsilon) \cap f(X) \subseteq f(U)$
$\langle 3 \rangle 1$. LET: $y' \in B(y, \epsilon) \cap f(X)$
$\langle 3 \rangle 2$. PICK $x' \in X$ such that $f(x') = y'$
$\langle 3 \rangle 3$. $d(y, y') < \epsilon$
$\langle 3 \rangle 4$. $d(x, x') < \epsilon$
$\langle 3 \rangle 5$. $x' \in U$
□

10.6 The Square Metric

Definition 10.6.1 (Square Metric)**.** The *square metric* ρ on $\mathbb{R}^n$ is defined by

$$\rho((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \max(|x_1 - y_1|, \ldots, |x_n - y_n|)$$

We prove this is a metric.

PROOF:
$\langle 1 \rangle 1$. $\rho(\vec{x}, \vec{y}) \geq 0$
PROOF: Immediate from definitions.
$\langle 1 \rangle 2$. $\rho(\vec{x}, \vec{y}) = 0$ iff $\vec{x} = \vec{y}$
PROOF: Immediate from definitions.
$\langle 1 \rangle 3$. $\rho(\vec{x}, \vec{y}) = \rho(\vec{y}, \vec{x})$
PROOF: Immediate from definitions.
$\langle 1 \rangle 4$. $\rho(\vec{x}, \vec{z}) \leq \rho(\vec{x}, \vec{y}) + \rho(\vec{y}, \vec{z})$
$\langle 2 \rangle 1$. For $1 \leq i \leq n$ we have $|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$
$\langle 2 \rangle 2$. For $1 \leq i \leq n$ we have $|x_1 - z_i| \leq \rho(\vec{x}, \vec{y}) + \rho(\vec{y}, \vec{z})$
$\langle 2 \rangle 3$. $\rho(\vec{x}, \vec{z}) \leq \rho(\vec{x}, \vec{y}) + \rho(\vec{y}, \vec{z})$
□

Proposition 10.6.2. *The square metric induces the product topology on* $\mathbb{R}^n$.

PROOF:
$\langle 1 \rangle 1$. For all $\vec{a} \in \mathbb{R}^n$ and $\epsilon > 0$ we have that $B_\rho(\vec{a}, \epsilon)$ is open in the product topology.

PROOF: This holds because $B_\rho(\vec{a}, \epsilon) = (a_1 - \epsilon, a_1 + \epsilon) \times \cdots \times (a_n - \epsilon, a_n + \epsilon)$.

$\langle 1 \rangle 2$. The set of all sets of the form $(a_1, b_1) \times \cdots \times (a_n, b_n)$ is a basis for the product topology on $\mathbb{R}^n$

PROOF: Propositions 5.1.2 and 5.2.4.

$\langle 1 \rangle 3$. Every set of the form $(a_1, b_1) \times \cdots \times (a_n, b_n)$ is open under ρ.

$\langle 2 \rangle 1$. LET: $\vec{x} \in (a_1, b_1) \times \cdots \times (a_n, b_n)$

$\langle 2 \rangle 2$. LET: $\epsilon = \min(x_1 - a_1, b_1 - x_1, \ldots, x_n - a_n, b_n - x_n)$

$\langle 2 \rangle 3$. $B_\rho(\vec{x}, \epsilon) \subseteq (a_1, b_1) \times \cdots \times (a_n, b_n)$

$\langle 2 \rangle 4$. Q.E.D.

PROOF: From Proposition 10.1.4.

□

Proposition 10.6.3. *Addition is continuous* $+ : \mathbb{R}^2 \to \mathbb{R}$.

PROOF:

$\langle 1 \rangle 1$. LET: $(a, b) \in \mathbb{R}^2$ and $\epsilon > 0$

PROVE: There exists $\delta > 0$, such that, for all $(x, y) \in \mathbb{R}^2$, if $\rho((a, b), (x, y)) < \delta$ then $|(a + b) - (x + y)| < \epsilon$

$\langle 1 \rangle 2$. LET: $\delta = \epsilon/2$

$\langle 1 \rangle 3$. LET: $(x, y) \in \mathbb{R}^2$ with $\rho((a, b), (x, y)) < \delta$

$\langle 1 \rangle 4$. $|(a + b) - (x + y)| < \epsilon$

PROOF:

$$\begin{aligned} |(a + b) - (x + y)| &= |a - x| + |b - y| \\ &< \delta + \delta \\ &= \epsilon \end{aligned}$$

□

Proposition 10.6.4. *Multiplication is continuous* $\cdot : \mathbb{R}^2 \to \mathbb{R}$.

PROOF:

$\langle 1 \rangle 1$. LET: $(a, b) \in \mathbb{R}^2$ and $\epsilon > 0$

PROVE: There exists $\delta > 0$, such that, for all $(x, y) \in \mathbb{R}^2$, if $\rho((a, b), (x, y)) < \delta$ then $|ab - xy| < \epsilon$

$\langle 1 \rangle 2$. LET: $\delta = \min(\epsilon/(|a| + |b| + 1), 1/2)$

$\langle 1 \rangle 3$. LET: $(x, y) \in \mathbb{R}^2$ with $\rho((a, b), (x, y)) < \delta$

$\langle 1 \rangle 4$. $|ab - xy| < \epsilon$

PROOF:

$$\begin{aligned} |xy - ab| &\leq |a||y - b| + |b||x - a| + |x - a||y - b| \\ &< |a|\delta + |b|\delta + \delta^2 \\ &< |a|\delta + |b|\delta + \delta \\ &= (|a| + |b| + 1)\delta \\ &\leq \epsilon \end{aligned}$$

□

Corollary 10.6.4.1. *Additive inverse is continuous* $- : \mathbb{R} \to \mathbb{R}$.

Proposition 10.6.5. *Multiplicative inverse is continuous* $m = (\ )^{-1} : \mathbb{R} \setminus \{0\} \to \mathbb{R}$.

Proof:
⟨1⟩1. Let: $a, b \in \mathbb{R}$ with $a < b$
Prove: $m^{-1}((a, b) \setminus \{0\})$ is open.
⟨1⟩2. Case: $b < 0$
Proof: In this case $m^{-1}((a, b)) = (1/b, 1/a)$.
⟨1⟩3. Case: $b = 0$
Proof: In this case $m^{-1}((a, b)) = (-\infty, 1/a)$.
⟨1⟩4. Case: $a < 0 < b$
Proof: In this case $m^{-1}((a, b)) = (-\infty, 1/a) \cup (1/b, +\infty)$.
⟨1⟩5. Case: $a = 0$
Proof: In this case $m^{-1}((a, b)) = (1/b, +\infty)$.
⟨1⟩6. Case: $a > 0$
Proof: In this case $m^{-1}((a, b)) = (1/b, 1/a)$.
□

10.7 The Standard Bounded Metric

Definition 10.7.1 (Standard Bounded Metric)**.** Let d be a metric on X. The *standard bounded metric* corresponding to d is the function $\overline{d} : X^2 \to \mathbb{R}$ given by

$$\overline{d}(x, y) = \min(d(x, y), 1)$$

We prove this is a metric.

Proof:
⟨1⟩1. $\overline{d}(x, y) \geq 0$
Proof: Immediate from definition.
⟨1⟩2. $\overline{d}(x, y) = 0$ iff $x = y$
Proof: This holds because $\overline{d}(x, y) = 0$ iff $d(x, y) = 0$.
⟨1⟩3. $\overline{d}(x, y) = \overline{d}(y, x)$
Proof: Immediate from definition.
⟨1⟩4. $\overline{d}(x, z) \leq \overline{d}(x, y) + \overline{d}(y, z)$
⟨2⟩1. Case: $d(x, y) \geq 1$
Proof:

$$\begin{aligned}\overline{d}(x, z) &\leq 1 \\ &\leq 1 + \overline{d}(y, z) \\ &= \overline{d}(x, y) + \overline{d}(y, z)\end{aligned}$$

⟨2⟩2. Case: $d(y, z) \geq 1$
Proof: Similar.
⟨2⟩3. Case: $d(x, y) < 1$ and $d(y, z) < 1$

PROOF:

$$\begin{aligned}\overline{d}(x,z) &\leq d(x,z) \\ &\leq d(x,y)+d(y,z) \\ &= \overline{d}(x,y)+\overline{d}(y,z)\end{aligned}$$

□

Proposition 10.7.2. *The standard bounded metric corresponding to d induces the same topology as d.*

PROOF:
⟨1⟩1. Every d-ball is open under $\overline{d}$
⟨2⟩1. LET: $x \in B_d(a,\epsilon)$
PROVE: There exists $\delta > 0$ such that $B_{\overline{d}}(x,\delta) \subseteq B_d(a,\epsilon)$
⟨2⟩2. PICK $\gamma > 0$ such that $B_d(x,\gamma) \subseteq B_d(a,\epsilon)$
⟨2⟩3. LET: $\delta = \min(\gamma, 1/2)$
⟨2⟩4. $B_{\overline{d}}(x,\delta) \subseteq B_d(a,\epsilon)$
PROOF:

$$\begin{aligned}B_{\overline{d}}(x,\delta) &= B_d(x,\delta) && (\delta < 1) \\ &\subseteq B_d(x,\gamma) && (\delta \leq \gamma) \\ &\subseteq B_d(a,\epsilon) && (\langle 2\rangle 2)\end{aligned}$$

⟨1⟩2. Every $\overline{d}$-ball is open under d
PROOF: $B_{\overline{d}}(a,\epsilon) = B_d(a,\epsilon)$ if $\epsilon < 1$, X if $\epsilon \geq 1$.
⟨1⟩3. Q.E.D.
PROOF: Using Proposition 4.9.4

□

Corollary 10.7.2.1. *For every metrizable space X, there exists a bounded metric that induces the topology on X.*

Proposition 10.7.3. *The product of a countable family of metrizable spaces is metrizable.*

PROOF:
⟨1⟩1. LET: $\{(X_n, d_n)\}_{n \geq 1}$ be a countable family of metric spaces.
⟨1⟩2. ASSUME: w.l.o.g. every d_n is bounded by 1
⟨1⟩3. LET: $D : (\prod_{n=1}^{\infty} X_n)^2 \to \mathbb{R}$ be defined by

$$D(\vec{x},\vec{y}) = \sup_{i \geq 1}(d_i(x_i,y_i)/i)$$

⟨1⟩4. D is a metric.
⟨2⟩1. $D(\vec{x},\vec{y}) \geq 0$
PROOF: Immediate from definition.
⟨2⟩2. $D(\vec{x},\vec{y}) = 0$ iff $\vec{x} = \vec{y}$
PROOF: Immediate from definition.
⟨2⟩3. $D(\vec{x},\vec{y}) = D(\vec{y},\vec{x})$
PROOF: Immediate from definition.
⟨2⟩4. $D(\vec{x},\vec{z}) \leq D(\vec{x},\vec{y}) + D(\vec{y},\vec{z})$
⟨3⟩1. For all i we have $d_i(x_i,z_i)/i \leq d_i(x_i,y_i)/i + d_i(y_i,z_i)/i$

$\langle 3 \rangle 2$. For all i we have $d_i(x_i, z_i)/i \leq D(\vec{x}, \vec{y}) + D(\vec{y}, \vec{z})$
$\langle 3 \rangle 3$. $D(\vec{x}, \vec{z}) \leq D(\vec{x}, \vec{y}) + D(\vec{y}, \vec{z})$
$\langle 1 \rangle 5$. D induces the product topology on $\mathbb{R}^\omega$.
$\langle 2 \rangle 1$. The product topology is finer than the topology induced by D.
$\langle 3 \rangle 1$. LET: U be open in the topology induced by D.
$\langle 3 \rangle 2$. LET: $\vec{x} \in U$
$\langle 3 \rangle 3$. PICK $\epsilon > 0$ such that $B_D(\vec{x}, \epsilon) \subseteq U$
$\langle 3 \rangle 4$. PICK N such that $1/N < \epsilon$
$\langle 3 \rangle 5$. LET: $V = B_{d_1}(x_1, \epsilon) \times \cdots \times B_{d_N}(x_N, \epsilon) \times X_{N+1} \times X_{N+2} \times \cdots$
$\langle 3 \rangle 6$. $\vec{x} \in V \subseteq U$
$\langle 2 \rangle 2$. The topology induced by D is finer than the product topology.
$\langle 3 \rangle 1$. LET: $n \geq 1$ and U be open in X_n
PROVE: ${\pi_n}^{-1}(U)$ is open in the topology induced by D.
$\langle 3 \rangle 2$. LET: $\vec{x} \in {\pi_n}^{-1}(U)$
$\langle 3 \rangle 3$. PICK $\epsilon > 0$ such that $B_{d_n}(x_n, \epsilon) \subseteq U$
$\langle 3 \rangle 4$. $B_D(\vec{x}, \epsilon/n) \subseteq {\pi_n}^{-1}(U)$
□

Example 10.7.4. For J uncountable, the space $\mathbb{R}^J$ is not metrizable, because it is not first countable.

10.8 The Uniform Metric

Definition 10.8.1 (Uniform Metric)**.** Let J be a nonempty set and (X, d) a metric space. The *uniform metric* $\overline{\rho}$ on X^J is defined by

$$\overline{\rho}((x_\alpha), (y_\alpha)) = \sup_{\alpha \in J} \overline{d}(x_\alpha, y_\alpha)$$

where $\overline{d}$ is the standard bounded metric corresponding to X.

The *uniform topology* on X^J is the topology induced by the uniform metric.

It is easy to check that this is a metric.

Proposition 10.8.2. *Let J be a set, (f_n) a sequence of functions $J \to X$ and $f : J \to X$. Then f_n converges uniformly to f as $n \to \infty$ if and only if $f_n \to f$ as $n \to \infty$ in X^J under the uniform topology.*

PROOF: Easy. □

Proposition 10.8.3 (DC)**.** *The uniform topology is finer than the product topology and coarser than the box topology.*

PROOF:
$\langle 1 \rangle 1$. The uniform topology is finer than the product topology.
$\langle 2 \rangle 1$. LET: $\alpha \in J$ and U be open in X
PROVE: ${\pi_\alpha}^{-1}(U)$ is open in the uniform topology.
$\langle 2 \rangle 2$. LET: $(x_\alpha) \in {\pi_\alpha}^{-1}(U)$

⟨2⟩3. PICK $\epsilon > 0$ such that $B_d(x_\alpha, \epsilon) \subseteq U$
PROOF: Proposition 10.1.4, ⟨2⟩1, ⟨2⟩2
⟨2⟩4. ASSUME: w.l.o.g. $\epsilon < 1$
⟨2⟩5. $B_{\overline{\rho}}((x_\alpha), \epsilon) \subseteq {\pi_\alpha}^{-1}(U)$
⟨1⟩2. The uniform topology is coarser than the box topology.
⟨2⟩1. LET: $(x_\alpha) \in X^J$ and $\epsilon > 0$
PROVE: $U = B_{\overline{\rho}}((x_\alpha), \epsilon)$ is open in the box topology.
⟨2⟩2. CASE: $\epsilon > 1$
PROOF: In this case $U = X^J$
⟨2⟩3. CASE: $\epsilon \leq 1$
PROOF: For $(y_\alpha) \in U$ we have $(y_\alpha) \in \prod_{\alpha \in J} B_d(y_\alpha, \epsilon/2) \subseteq U$.
□

Proposition 10.8.4. *Let X be a topological space and Y a metric space. Let $\mathcal{C}(X, Y)$ be the set of all continuous functions $X \to Y$ under the uniform topology. Then the evaluation map $\mathcal{C}(X, Y) \times X \to Y$ is continuous.*

PROOF:
⟨1⟩1. LET: (
$(f, x) \in \mathcal{C}(X, Y))$
⟨1⟩2. LET: V be a neighbourhood of $f(x)$
⟨1⟩3. PICK $\epsilon > 0$ such that $B(f(x), \epsilon) \subseteq V$
⟨1⟩4. PICK a neighbourhood U of x such that $f(U) \subseteq B(f(x), \epsilon/2)$
PROVE: For all $(g, y) \in B_{\overline{\rho}}(f, \epsilon/2) \times U$ we have $g(y) \in V$
⟨1⟩5. LET: $(g, y) \in B_{\overline{\rho}}(f, \epsilon/2) \times U$
⟨1⟩6. $d(g(y), f(x)) < \epsilon$
PROOF:

$$\begin{aligned} d(g(y), f(x)) &\leq d(g(y), f(y)) + d(f(y), f(x)) \\ &< \overline{\rho}(g, f) + d(f(y), f(x)) \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon \end{aligned}$$

□

Chapter 11

Normed Spaces

Definition 11.0.1 (Norm)**.** Let K be either $\mathbb{R}$ or $\mathbb{C}$. Let X be a K-vector space. A *norm* on X is a function $\| \; \| : X \to K$ such that, for all $\alpha \in K$ and $x, y \in X$:

1. $\|x\| \geq 0$
2. $\|x\| = 0$ if and only if $x = 0$
3. $\|\alpha x\| = |\alpha| \|x\|$
4. **Triangle Inequality**
   $\|x + y\| \leq \|x\| + \|y\|$

Definition 11.0.2. Given a norm $\| \; \|$ on X, the metric d *induced* by the norm is

$$d(x, y) = \|x - y\| \quad .$$

We prove this is a metric.

Proof:
⟨1⟩1. $d(x, y) \geq 0$
  Proof: Immediate from definitions.
⟨1⟩2. $d(x, y) = 0$ if and only if $x = y$
  Proof: Immediate from definitions.
⟨1⟩3. $d(x, y) = d(y, x)$
  Proof: Immediate from definitions.
⟨1⟩4. $d(x, z) \leq d(x, y) + d(y, z)$
  Proof:

$$\begin{aligned} d(x, z) &= \|x - z\| \\ &= \|(x - y) + (y - z)\| \\ &\leq \|x - y\| + \|y - z\| \end{aligned}$$

□

Proposition 11.0.3. *Let X be a normed space. Vector addition is a continuous function $X^2 \to X$.*

PROOF:
$\langle 1\rangle 1$. Give X^2 the square metric ρ.
$\langle 1\rangle 2$. LET: $(a, b) \in X^2$ and $\epsilon > 0$
$\langle 1\rangle 3$. LET: $\delta = \epsilon/2$
$\langle 1\rangle 4$. LET: $(x, y) \in X^2$ with $\rho((a, b), (x, y)) < \delta$
$\langle 1\rangle 5$. $\|(a + b) - (x + y)\| < \epsilon$
PROOF:

$$\begin{aligned} \|(a+b)-(x+y)\| &\leq \|a-x\| + \|b-y\| && \text{(Triangle inequality)} \\ &< \delta + \delta && (\langle 1\rangle 4) \\ &= \epsilon && (\langle 1\rangle 3) \end{aligned}$$

□

Proposition 11.0.4. *Let X be a normed space. Scalar multiplication is a continuous function $K \times X \to X$.*

PROOF:
$\langle 1\rangle 1$. Give $K \times X$ the square metric.
$\langle 1\rangle 2$. LET: $(\lambda, a) \in K \times X$ and $\epsilon > 0$
$\langle 1\rangle 3$. LET: $\delta = \min(\epsilon/(|\lambda| + \|a\| + 1), 1/2)$
$\langle 1\rangle 4$. LET: $(\mu, x) \in K \times X$ with $\rho((\lambda, a), (\mu, x)) < \delta$
$\langle 1\rangle 5$. $\|\lambda a - \mu x\| < \epsilon$
PROOF:

$$\begin{aligned} \|\mu x - \lambda a\| &\leq |\lambda|\|x - a\| + |\mu - \lambda|\|a\| + \|\mu - \lambda\|\|x - a\| \\ &< |\lambda|\delta + \|a\|\delta + \delta^2 \\ &< |\lambda|\delta + \|a\|\delta + \delta \\ &\leq \epsilon \end{aligned}$$

□

Proposition 11.0.5. *Every open ball and closed ball in a normed space is path connected.*

PROOF:
$\langle 1\rangle 1$. LET: B be either the open ball or closed ball with center c and radius r in the normed space X.
$\langle 1\rangle 2$. LET: $a, b \in B$
$\langle 1\rangle 3$. LET: $p : [0, 1] \to B$

$$p(t) = (1 - t)a + tb$$

PROOF: For $t \in [0, 1]$ we have

$$\begin{aligned} d(p(t), c) &= \|p(t) - c\| \\ &= \|(1 - t)a + tb - c\| \\ &= \|(1 - t)(a - c) + t(b - c)\| \\ &\leq (1 - t)\|a - c\| + t\|b - c\| \end{aligned}$$

and this is $< r$ if B is an open ball, $\leq r$ if B is a closed ball.
$\langle 1\rangle 4$. p is a path from a to b
□

Chapter 12

Inner Product Spaces

Definition 12.0.1 (Inner Product)**.** Let K be either $\mathbb{R}$ or $\mathbb{C}$. Let X be a K-vector space. An *inner product* on X is a function $\cdot : X^2 \to K$ such that, for all $\alpha \in K$ and $x, y, z \in X$:

1. $\alpha x \cdot y = \alpha(x \cdot y)$
2. $(x + y) \cdot z = x \cdot z + y \cdot z$
3. $x \cdot y = \overline{y \cdot x}$
4. If $x \neq 0$ then $x \cdot x > 0$.

Example 12.0.2. The *standard inner product* on K^n is given by

$$(x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) = x_1\overline{y_1} + \cdots + x_n\overline{y_n}$$

It is easy to check this is an inner product.

Proposition 12.0.3.

$$x \cdot x \geq 0$$

PROOF:
$\langle 1 \rangle$1. $0 \cdot 0 = 0$
PROOF:

$$\begin{aligned} 0 \cdot 0 &= (0 + 0) \cdot 0 \\ &= 0 \cdot 0 + 0 \cdot 0 \end{aligned}$$

$\langle 1 \rangle$2. For $x \neq 0$ we have $x \cdot x > 0$
□

Proposition 12.0.4. *lCauchy-Schwarz Inequality]*

$$|x \cdot y|^2 \leq (x \cdot x)(y \cdot y)$$

PROOF:
$\langle 1 \rangle$1. CASE: $y = 0$
PROOF: In this case, both sides are 0.

$\langle 1 \rangle 2$. CASE: $y \neq 0$

$\langle 2 \rangle 1$. LET: $\lambda = x \cdot y / y \cdot y$

$\langle 2 \rangle 2$. $x \cdot x - |x \cdot y|^2 / y \cdot y \geq 0$

PROOF:

$$\begin{aligned} 0 &\leq (x - \lambda y) \cdot (x - \lambda y) \\ &= x \cdot x - \overline{\lambda}(x \cdot y) - \lambda(y \cdot x) + |\lambda|^2 y \cdot y \\ &= x \cdot x - |x \cdot y|^2 / y \cdot y - |x \cdot y|^2 / y \cdot y + |x \cdot y|^2 / y \cdot y \\ &= x \cdot x - |x \cdot y|^2 / y \cdot y \end{aligned}$$

□

Definition 12.0.5. Given an inner product on X, the norm *induced* by the inner product is defined by

$$\|x\| = \sqrt{x \cdot x}$$

We prove this is a norm.

PROOF:

$\langle 1 \rangle 1$. $\|x\| \geq 0$

PROOF: Proposition 12.0.3

$\langle 1 \rangle 2$. $\|x\| = 0$ iff $x = 0$

PROOF: We have $\|x\| = 0$ iff $x \cdot x = 0$ iff $x = 0$.

$\langle 1 \rangle 3$. $\|\alpha x\| = |\alpha| \|x\|$

PROOF:

$$\begin{aligned} \|\alpha x\|^2 &= \alpha x \cdot \alpha x \\ &= \alpha \overline{\alpha}(x \cdot x) \\ &= |\alpha|^2 \|x\|^2 \end{aligned}$$

$\langle 1 \rangle 4$. $\|x + y\| \leq \|x\| + \|y\|$

PROOF:

$$\begin{aligned} \|x + y\|^2 &= (x + y) \cdot (x + y) \\ &= \|x\|^2 + x \cdot y + y \cdot x + \|y\|^2 \\ &= \|x\|^2 + x \cdot y + \overline{x \cdot y} + \|y\|^2 \\ &\leq \|x\|^2 + 2|x \cdot y| + \|y\|^2 \\ &\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 && \text{(Cauchy-Schwarz)} \\ &= (\|x\| + \|y\|)^2 \end{aligned}$$

□

Proposition 12.0.6. *The topology induced by the standard inner product on $\mathbb{R}^n$ is the standard topology.*

PROOF:

$\langle 1 \rangle 1$. LET: d be the topology induced by the standard inner product and ρ the square topology.

$\langle 1 \rangle 2$. For all $\vec{x}, \vec{y} \in \mathbb{R}^n$ we have $\rho(\vec{x}, \vec{y}) \leq d(\vec{x}, \vec{y})$

PROOF:

$$\begin{aligned}\rho(\vec{x},\vec{y})^2 &= \max(|x_1-y_1|,\ldots,|x_n-y_n|)^2 \\ &\leq |x_1-y_1|^2+\cdots+|x_n-y_n|^2 \\ &= d(\vec{x},\vec{y})^2\end{aligned}$$

⟨1⟩3. For all $\vec{x},\vec{y}\in\mathbb{R}^n$ we have $d(\vec{x},\vec{y})\leq\sqrt{n}\rho(\vec{x},\vec{y})$

PROOF:

$$\begin{aligned}d(\vec{x},\vec{y})^2 &= |x_1-y_1|^2+\cdots+|x_n-y_n|^2 \\ &\leq n\rho(\vec{x},\vec{y})^2\end{aligned}$$

⟨1⟩4. d and ρ induce the same topology.

PROOF: Proposition 10.1.6.

⟨1⟩5. Q.E.D.

PROOF: Proposition 10.6.2.

□

Definition 12.0.7 (l^2-inner product)**.** Let

$$l^2=\left\{\vec{x}\in\mathbb{R}^\omega:\sum_{n=1}^\infty x_n^2<\infty\right\}$$

The l^2-*inner product* on l^2 is defined by

$$\vec{x}\cdot\vec{y}=\sum_{n=1}^\infty x_ny_n$$

The norm (metric, topology) induced by this inner product is the l^2-*norm (metric, topology).*

We prove this is an inner product.

PROOF:

⟨1⟩1. For all $\vec{x},\vec{y}\in l^2$ we have $\sum_{n=1}^\infty x_ny_n<\infty$

PROOF:We have

$$\begin{aligned}\left|\sum_{n=1}^N x_ny_n\right|^2 &\leq \left(\sum_{n=1}^N x_n^2\right)\left(\sum_{n=1}^N y_n^2\right) && \text{(Cauchy-Schwarz)} \\ &\leq \left(\sum_{n=1}^\infty x_n^2\right)\left(\sum_{n=1}^\infty y_n^2\right)\end{aligned}$$

Hence $\sum_{n=1}^\infty x_ny_n$ converges by the Comparison Test.

⟨1⟩2. $\alpha\vec{x}\cdot\vec{y}=\alpha(\vec{x}\cdot\vec{y})$

PROOF: Immediate from definitions.

⟨1⟩3. $(\vec{x}+\vec{y})\cdot\vec{z}=\vec{x}\cdot\vec{z}+\vec{y}\cdot\vec{z}$

PROOF: Immediate from definitions.

⟨1⟩4. $\vec{x}\cdot\vec{y}=\vec{y}\cdot\vec{x}$

PROOF: Immediate from definitions.

⟨1⟩5. If $\vec{x}\neq 0$ then $\vec{x}\cdot\vec{x}>0$

PROOF: Immediate from definitions.

□

Proposition 12.0.8. *The l^2-topology is strictly coarser than the box topology. In fact, the l^2-topology is strictly coarser than the box topology on $\mathbb{R}^\infty$.*

PROOF:
⟨1⟩1. The l^2-topology is coarser than the box topology.
⟨2⟩1. LET: U be open in the l^2-topology and $\vec{x} \in U$
⟨2⟩2. PICK $\epsilon > 0$ such that $B_{l^2}(\vec{x}, \epsilon) \subseteq U$
⟨2⟩3. PICK a sequence of positive real numbers $a_1, a_2, \ldots$ such that $\sum_{n=1}^\infty a_n^2 < \epsilon^2$
⟨2⟩4. $\vec{x} \in \prod_{n=1}^\infty (x_n - a_n, x_n + a_n) \subseteq U$
PROOF: For $\vec{y} \in \prod_{n=1}^\infty (x_n - a_n, x_n + a_n)$ we have

$$\begin{aligned} d_{l^2}(\vec{x}, \vec{y})^2 &= \sum_{n=1}^\infty (x_n - y_n)^2 \\ &< \sum_{n=1}^\infty a_n^2 \\ &< \epsilon^2 \end{aligned}$$

⟨1⟩2. There exists a subset of $\mathbb{R}^\infty$ that is open in the box topology but not in the l^2-topology.
PROOF: The set $\prod_{n=1}^\infty (-1/n, 1/n) \cap \mathbb{R}^\infty$ is open in the box topology but not in the l^2-topology.
□

Proposition 12.0.9. *The l^2-topology is strictly finer than the uniform topology. In fact, the l^2-topology is strictly finer than the uniform topology on $\mathbb{R}^\infty$.*

PROOF:
⟨1⟩1. The l^2-topology is finer than the uniform topology.
⟨2⟩1. LET: $\vec{x} \in l^2$ and $\epsilon > 0$
PROVE: For all $\vec{y}$, if $d_{l^2}(\vec{x}, \vec{y}) < 2\epsilon$ then $\overline{\rho}(\vec{x}, \vec{y}) < \epsilon$
⟨2⟩2. LET: $\vec{y} \in l^2$ with $d_{l^2}(\vec{x}, \vec{y}) < 2\epsilon$
⟨2⟩3. $\sum (x_n - y_n)^2 < 4\epsilon^2$
⟨2⟩4. For all n we have $|x_n - y_n| < 2\epsilon$
⟨2⟩5. $\overline{\rho}(\vec{x}, \vec{y}) \leq 2\epsilon$
⟨1⟩2. There exists a subset of $\mathbb{R}^\infty$ that is open in the l^2-topology but not in the uniform topology.
PROOF: $B_{l^2}(0, 1)$ is open in the l^2-topology but not in the uniform topology.
□

Chapter 13

Topological Groups

Definition 13.0.1 (Topological Group)**.** A *topological group* G is a group with a T_1-topology such that the multiplication map $\cdot : G^2 \to G$ and inverse map $()^{-1} : G \to G$ are continuous.

Proposition 13.0.2. *Every subgroup of a topological group is a topological group under the subspace topology.*

PROOF: Easy.

Proposition 13.0.3. *Every topological group is homogeneous.*

PROOF: Let G be a topological group and $x, y \in G$. Then $\lambda z.yx^{-1}z$ is a homeomorphism that maps x to y. □

Proposition 13.0.4. *Let G be a topological group and H be a subgroup of G. Then $\overline{H}$ is a topological group under the subspace topology.*

PROOF:
$\langle 1 \rangle 1$. $\overline{H}$ is a subgroup of G.
  $\langle 2 \rangle 1$. LET: $f : G^2 \to G$, $f(x, y) = xy^{-1}$
  $\langle 2 \rangle 2$. $f(\overline{H} \times \overline{H}) \subseteq \overline{H}$
    PROOF:

$$\begin{aligned} f(\overline{H} \times \overline{H}) &= f(\overline{H \times H}) && \text{(Proposition 5.2.6)} \\ &\subseteq \overline{f(H \times H)} && \text{(Proposition 6.1.3)} \\ &\subseteq \overline{H} && (H \text{ is a subgroup of } G) \end{aligned}$$

□

Proposition 13.0.5. *Let G be a topological group and $H \leq G$. Give G/H the quotient topology induced by the canonical map $G \to G/H$. Then G/H is homogeneous.*

PROOF:
$\langle 1 \rangle 1$. LET: $aH, bH \in G/H$

⟨1⟩2. LET: $\phi : G/H \to G/H$ be the function $\phi(xH) = (ba^{-1}xH)$

⟨2⟩1. LET: $xH = yH$

PROVE: $ba^{-1}xH = ba^{-1}yH$

⟨2⟩2. $x^{-1}y \in H$

⟨2⟩3. $(ba^{-1}x)^{-1}ba^{-1}y \in H$

PROOF:

$$\begin{aligned}(ba^{-1}x)^{-1}ba^{-1}y &= x^{-1}ab^{-1}ba^{-1}y \\ &= x^{-1}y\end{aligned}$$

⟨1⟩3. ϕ is a homeomorphism.

□

Proposition 13.0.6. *Let G be a topological group and H a closed subgroup of G. Give G/H the quotient topology induced by the canonical map $\pi : G \to G/H$. Then G/H is T_1.*

PROOF: Let $a \in G$. Then $\pi^{-1}(\{aH\}) = aH = f_a(H)$ where $f_a(x) = ax$ for $x \in G$. This set is closed, because f_a is a homeomorphism. Hence $\{aH\}$ is closed in G/H.

Proposition 13.0.7. *Let G be a topological group and H a subgroup of G. Give G/H the quotient topology induced by the canonical map $\pi : G \to G/H$. Then π is an open map.*

PROOF: This holds because, for U open in G,

$$\begin{aligned}\pi^{-1}(\pi(U)) &= \{xh : x \in U, h \in H\} \\ &= \bigcup_{h \in H} \phi_h(U)\end{aligned}$$

where ϕ_h is the automorphism that maps x to xh. □

Proposition 13.0.8. *Let G be a topological group and H a closed normal subgroup of G. Give G/H the quotient topology induced by the canonical map $\pi : G \to G/H$. Then G/H is a topological group.*

PROOF:

⟨1⟩1. G/H is T_1.

PROOF: Proposition 13.0.6.

⟨1⟩2. Multiplication in G/H is continuous.

⟨2⟩1. LET: $m : G^2 \to G$ be multiplication in G

⟨2⟩2. LET: $n : (G/H)^2 \to G/H$ be multiplication in G/H

⟨2⟩3. $n \circ (\pi \times \pi) = \pi \circ m : G^2 \to G/H$

⟨2⟩4. $n \circ (\pi \times \pi)$ is continuous.

PROOF: $n \circ (\pi \times \pi) = \pi \circ m$ and π and m are both continuous.

⟨2⟩5. π is an open map

PROOF: Proposition 13.0.7.

⟨2⟩6. $\pi \times \pi$ is strongly continuous.

PROOF: Proposition 6.6.7.

⟨2⟩7. n is continuous.

PROOF: Proposition 6.6.5.
⟨1⟩3. The inverse map in G/H is continuous.
PROOF: Similar.
□

Definition 13.0.9 (Symmetric Neighbourhood of e)**.** A neighbourhood U of e is *symmetric* iff $U = U^{-1}$.

Proposition 13.0.10. *For every neighbourhood U of e, there exists a symmetric neighbourhood V of e such that $V \cdot V \subseteq U$.*

PROOF:
⟨1⟩1. PICK a neighbourhood V' of e such that $V' \cdot V' \subseteq U$
⟨2⟩1. PICK a neighbourhood N of (e, e) in $G \times G$ such that, for all $x, y \in N$ we have $xy \in U$
PROOF: Since multiplication is continuous.
⟨2⟩2. PICK neighbourhoods V_1, V_2 of e such that $V_1 \times V_2 \subseteq N$
⟨2⟩3. LET: $V' = V_1 \cap V_2$
⟨1⟩2. PICK a neighbourhood W of e such that $W \cdot W^{-1} \subseteq V'$
PROOF: Similar, since $\lambda x, y.xy^{-1}$ is continuous.
⟨1⟩3. LET: $V = W \cdot W^{-1}$
⟨1⟩4. V is a neighbourhood of e
⟨2⟩1. $e \in V$
PROOF: This holds because $e \in W$ and $e = ee^{-1}$.
⟨2⟩2. PICK an open set U_1 such that $e \in U_1 \subseteq W$
⟨2⟩3. $U_1 \cdot {U_1}^{-1}$ is open
PROOF: This holds because $U_1 \cdot {U_1}^{-1} = \bigcup_{x \in U_1} U_1 x^{-1}$, and each $U_1 x^{-1}$ is open since it is an automorphic image of U_1.
⟨2⟩4. $e \in U_1 \cdot {U_1}^{-1} \subseteq V$
⟨1⟩5. V is symmetric
⟨2⟩1. LET: $x \in V$
⟨2⟩2. PICK $y, z \in W$ such that $x = yz^{-1}$
⟨2⟩3. $x^{-1} \in V$
PROOF:$x^{-1} = zy^{-1}$
⟨1⟩6. $V \cdot V \subseteq U$
PROOF: $V \cdot V \subseteq V' \cdot V' \subseteq U$ by ⟨1⟩1, ⟨1⟩2, ⟨1⟩3.
□

Proposition 13.0.11. *Every topological group is regular.*

PROOF:
⟨1⟩1. LET: G be a topological group.
⟨1⟩2. LET: $A \subseteq G$ be closed and $a \in G \setminus A$
⟨1⟩3. Aa^{-1} is closed
PROOF: Since the map $\lambda x.xa^{-1}$ is an automorphism.
⟨1⟩4. $G \setminus Aa^{-1}$ is a neighbourhood of e
PROOF: If $e \in Aa^{-1}$ then $a \in A$.
⟨1⟩5. PICK a symmetric neighbourhood V of e such that $V \cdot V \subseteq G \setminus Aa^{-1}$

PROOF: Proposition 13.0.10.
⟨1⟩6. $V \cdot A$ and Va are disjoint neighbourhoods of A and a respectively.
⟨2⟩1. $V \cdot A \cap Va = \emptyset$
⟨3⟩1. ASSUME: for a contradiction $xy = za$ where $x, z \in V$ and $y \in A$
⟨3⟩2. $ya^{-1} = x^{-1}z$
⟨3⟩3. $x^{-1} \in V$
⟨3⟩4. $x^{-1}z \in G \setminus Aa^{-1}$
⟨2⟩2. $V \cdot A$ is a neighbourhood of A
⟨3⟩1. PICK an open U such that $e \in U \subseteq V$
⟨3⟩2. $e \in U \cdot A \subseteq V \cdot A$
⟨3⟩3. $U \cdot A$ is open
PROOF: $U \cdot A = \bigcup_{x \in A} Ux$
⟨2⟩3. Va is a neighbourhood of a
PROOF: Similar.
□

Proposition 13.0.12. *Let G be a topological group and H a closed subgroup of G. Give G/H the quotient topology induced by the canonical map $\pi : G \to G/H$. Then G/H is regular.*

PROOF:
⟨1⟩1. G/H is T_1
PROOF: Proposition 13.0.6.
⟨1⟩2. The closed subsets of G/H are the sets $\pi(A)$ where A is a saturated closed set in G.
⟨2⟩1. For every closed $C \subseteq G/H$ we have $C = \pi(\pi^{-1}(C))$
⟨2⟩2. If A is a saturated closed set in G then $\pi(A)$ is closed in G/H
PROOF: Proposition 6.6.3.
⟨1⟩3. LET: $\pi(A)$ be a closed set in G/H and $aH \in G/H \setminus \pi(A)$, where A is a saturated closed set in G.
⟨1⟩4. Aa^{-1} is closed
PROOF: Since $\lambda t.ta^{-1}$ is an automorphism of G.
⟨1⟩5. LET: $U = G \setminus Aa^{-1}$
⟨1⟩6. U is a neighbourhood of e
PROOF: If $e \in Aa^{-1}$ then $a \in A$.
⟨1⟩7. PICK a symmetric neighbourhood V of e such that $V \cdot V \subseteq U$
PROOF: Proposition 13.0.10.
⟨1⟩8. $\pi(V \cdot A)$ and $\pi(Va)$ are disjoint neighbourhoods of $\pi(A)$ and aH
⟨2⟩1. $\pi(V \cdot A) \cap \pi(Va) = \emptyset$
⟨3⟩1. ASSUME: for a contradiction $xyH = zaH$ where $x, z \in V$ and $y \in A$
⟨3⟩2. $yH = x^{-1}zaH$
⟨3⟩3. $x^{-1}za \in A$
PROOF: Since A is saturated.
⟨3⟩4. $x^{-1} \in V$
PROOF: Since V is symmetric.
⟨3⟩5. $x^{-1}z \in U$
PROOF: From ⟨1⟩7

$\langle 3 \rangle 6.$ $x^{-1}z \notin Aa^{-1}$

PROOF: From $\langle 1 \rangle 5$

$\langle 3 \rangle 7.$ Q.E.D.

PROOF: $\langle 3 \rangle 3$ and $\langle 3 \rangle 6$ form a contradiction.

$\langle 2 \rangle 2.$ $\pi(V \cdot A)$ is a neighbourhood of $\pi(A)$

PROOF: Using Proposition 13.0.7

$\langle 2 \rangle 3.$ $\pi(Va)$ is a neighbourhood of aH

PROOF: Using Proposition 13.0.7

□