

Topology

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Chapter 1

Set Theory

1.1 Primitive Notions

Let there be *sets*.

Given sets A and B , let there be *functions* from A to B . We write $f : A \rightarrow B$ iff f is a function from A to B .

Given functions $f : A \rightarrow B$ and $g : B \rightarrow C$, let there be a function $g \circ f : A \rightarrow C$, the *composite* of f and g .

Definition 1.1.1 (Injective). A function $f : A \rightarrow B$ is *injective*, $f : A \rightarrowtail B$, iff, for every set X and functions $g, h : X \rightarrow A$, if $f \circ g = f \circ h$ then $g = h$.

Definition 1.1.2 (Surjective). Let $f : A \rightarrow B$. Then f is *surjective*, $f : A \twoheadrightarrow B$, iff, for any set X and functions $g, h : B \rightarrow X$, if $g \circ f = h \circ f$ then $g = h$.

1.2 Axioms

1.2.1 Axioms for a Category

Axiom 1.2.1 (Associativity). Let $f : A \rightarrow B$, $g : B \rightarrow C$ and $h : C \rightarrow D$. Then $h \circ (g \circ f) = (h \circ g) \circ f : A \rightarrow D$.

From now on we write $h \circ g \circ f$ for the composite of f , g and h , and similarly for more than three functions.

Lemma 1.2.2. Let $f : A \rightarrow B$ and $g : B \rightarrow C$. If f and g are injective then $g \circ f$ is injective.

PROOF:

- $\langle 1 \rangle 1$. ASSUME: f and g are injective.
- $\langle 1 \rangle 2$. LET: X be a set and $x, y : X \rightarrow A$
- $\langle 1 \rangle 3$. ASSUME: $g \circ f \circ x = g \circ f \circ y$
- $\langle 1 \rangle 4$. $f \circ x = f \circ y$

PROOF: g is injective ($\langle 1 \rangle 1$)
 $\langle 1 \rangle 5. x = y$
 PROOF: f is injective ($\langle 1 \rangle 1$)
 \square

Lemma 1.2.3. *Let $f : A \rightarrow B$ and $g : B \rightarrow C$. If f and g are surjective then $g \circ f$ is surjective.*

PROOF: Dual. \square

Lemma 1.2.4. *Let $f : A \rightarrow B$ and $g : B \rightarrow C$. If $g \circ f$ is injective then f is injective.*

PROOF:
 $\langle 1 \rangle 1.$ ASSUME: $g \circ f$ is injective.
 $\langle 1 \rangle 2.$ LET: X be any set and $x, y : X \rightarrow A$
 $\langle 1 \rangle 3.$ ASSUME: $f \circ x = f \circ y$
 $\langle 1 \rangle 4.$ $g \circ f \circ x = g \circ f \circ y$
 $\langle 1 \rangle 5. x = y$
 PROOF: Using $\langle 1 \rangle 1.$
 \square

Lemma 1.2.5. *Let $f : A \rightarrow B$ and $g : B \rightarrow C$. If $g \circ f$ is surjective then g is surjective.*

PROOF: Dual. \square

Axiom 1.2.6 (Identity Function). *For any set A , there exists a function $\text{id}_A : A \rightarrow A$, the identity function on A , such that:*

Left Unit Law *for every set B and function $f : B \rightarrow A$ we have $\text{id}_A \circ f = f : B \rightarrow A$;*

Right Unit Law *for every set B and function $f : A \rightarrow B$ we have $f \circ \text{id}_A = f : A \rightarrow B$.*

Proposition 1.2.7. *The identity function on a set is unique.*

PROOF: If $i, j : A \rightarrow A$ are both identity functions, then
 $i = i \circ j$ (Right Unit Law for j)
 $= j$ (Left Unit Law for i)
 $: A \rightarrow A$ \square

Proposition 1.2.8. *Every identity function is injective.*

PROOF: If $\text{id}_B \circ x = \text{id}_B \circ y$ then $x = y$ by the Left Unit Law. \square

Proposition 1.2.9. *Every identity function is surjective.*

PROOF: Dual. \square

Definition 1.2.10 (Retraction, Section). Let $r : A \rightarrow B$ and $s : B \rightarrow A$. Then r is a *retraction* of s , and s is a *section* of r , iff $r \circ s = \text{id}_B$.

Proposition 1.2.11. If $r_1 : A \rightarrow B$ is a retraction of $s_1 : B \rightarrow A$ and $r_2 : B \rightarrow C$ is a retraction of $s_2 : C \rightarrow B$ then $r_2 \circ r_1$ is a retraction of $s_1 \circ s_2$.

PROOF:

$$\begin{aligned} r_2 \circ r_1 \circ s_1 \circ s_2 &= r_2 \circ \text{id}_B \circ s_2 && (r_1 \text{ is a retraction of } s_1) \\ &= r_2 \circ s_2 && (\text{Unit Laws}) \\ &= \text{id}_C && (r_2 \text{ is a retraction of } s_2) \end{aligned}$$

□

Proposition 1.2.12. Every retraction is surjective.

PROOF:

- <1>1. LET: $r : A \rightarrow B$ be a retraction of s
- <1>2. LET: X be a set and $x, y : B \rightarrow X$ with $x \circ r = y \circ r$
- <1>3. $x \circ r \circ s = y \circ r \circ s$
- <1>4. $x = y$

□

Proposition 1.2.13. Every section is injective.

PROOF: Dual. □

Proposition 1.2.14. Every identity function is a retraction of itself.

PROOF: Immediate from the Unit Laws. □

Proposition 1.2.15. If $r : B \rightarrow A$ is a retraction of $f : A \rightarrow B$ and s is a section of f then $r = s$.

PROOF:

$$\begin{aligned} r &= r \circ \text{id}_B && (\text{Right Unit Law}) \\ &= r \circ f \circ s && (s \text{ is a section of } f) \\ &= \text{id}_A \circ s && (r \text{ is a retraction of } f) \\ &= s && (\text{Left Unit Law}) \end{aligned}$$

1.2.2 Isomorphisms

Definition 1.2.16 (Isomorphism). Let A and B be sets. A function $i : A \rightarrow B$ is an *isomorphism* between A and B , $i : A \cong B$, iff there exists a function $i^{-1} : B \rightarrow A$, the *inverse* to i , that is a section and a retraction of i .

Proposition 1.2.17. The inverse of an isomorphism is unique.

PROOF: Immediate from Proposition 1.2.15. □

Proposition 1.2.18. Every isomorphism is injective.

PROOF: Immediate from Proposition 1.2.13. \square

Proposition 1.2.19. *Every isomorphism is surjective.*

PROOF: Immediate from Proposition 1.2.12. \square

Proposition 1.2.20. *Every identity function is an isomorphism and is its own inverse.*

PROOF: Immediate from Proposition 1.2.14. \square

Proposition 1.2.21. *If $i : A \cong B$ is an isomorphism then $i^{-1} : B \cong A$ is an isomorphism and $(i^{-1})^{-1} = i$.*

PROOF: Immediate from the definition of isomorphism. \square

Proposition 1.2.22. *If $i : A \cong B$ and $j : B \cong C$ then $j \circ i : A \cong C$ and $(j \circ i)^{-1} = i^{-1} \circ j^{-1}$.*

PROOF: Immediate from Proposition 1.2.11. \square

1.2.3 Parts of a Set

Definition 1.2.23 (Part). A *part* S of a set A consists of:

- a set $\text{dom } S$;
- an injective function $i : S \hookrightarrow A$

Definition 1.2.24. Two parts $i : S \hookrightarrow A$, $j : T \hookrightarrow A$ are *equivalent*, $i \equiv_A j$, iff there exists an isomorphism $\phi : S \cong T$ such that $i = j \circ \phi$.

Proposition 1.2.25. *Any part of a set is equivalent to itself.*

PROOF:

$\langle 1 \rangle 1$. LET: $i : S \hookrightarrow A$ be a part of A .

PROVE: $i \equiv_A i$

$\langle 1 \rangle 2$. $\text{id}_S : S \cong S$

PROOF: By Proposition 1.2.20

$\langle 1 \rangle 3$. $i = i \circ \text{id}_S$

PROOF: By the Right Unit Law.

\square

Proposition 1.2.26. *If $i \equiv_A j$ then $j \equiv_A i$.*

PROOF:

$\langle 1 \rangle 1$. LET: $i : S \hookrightarrow A$ and $j : T \hookrightarrow A$

$\langle 1 \rangle 2$. ASSUME: $i \equiv_A j$

$\langle 1 \rangle 3$. PICK an isomorphism $\phi : S \cong T$ such that $i = j \circ \phi$

$\langle 1 \rangle 4$. $\phi^{-1} : T \cong S$

PROOF: By Proposition 1.2.21.

$\langle 1 \rangle 5. j = i \circ \phi^{-1}$

PROOF: Compose both sides of $\langle 1 \rangle 3$ with ϕ^{-1} .

□

Proposition 1.2.27. *If $i \equiv_A j$ and $j \equiv_A k$ then $i \equiv_A k$.*

PROOF:

$\langle 1 \rangle 1.$ LET: $i : R \hookrightarrow A, j : S \hookrightarrow A$ and $k : T \rightarrow A$

$\langle 1 \rangle 2.$ PICK isomorphisms $\phi : R \cong S$ and $\psi : S \cong T$ such that $i = j \circ \phi$ and $j = k \circ \psi$

$\langle 1 \rangle 3.$ $\psi \circ \phi : R \cong T$

PROOF: By Proposition 1.2.22.

$\langle 1 \rangle 4.$ $i = k \circ \psi \circ \phi$

□

Definition 1.2.28 (Inclusion). Let $i : U \hookrightarrow A$ and $j : V \hookrightarrow A$ be parts of A . Then i is *included* in j , $i \subseteq_A j$, iff there exists a function $\phi : U \rightarrow V$ such that $i = j \circ \phi$.

Proposition 1.2.29. *If $i \equiv_A i'$ and $j \equiv_A j'$ and $i \subseteq_A j$ then $i' \subseteq_A j'$.*

PROOF:

$\langle 1 \rangle 1.$ LET: $i : S \hookrightarrow A, i' : S' \hookrightarrow A, j : T \hookrightarrow A, j' : T' \hookrightarrow A$

$\langle 1 \rangle 2.$ PICK $\phi : S \cong S', \psi : T \cong T'$ and $\chi : S \rightarrow T$ such that $i = i' \circ \phi, j = j' \circ \psi$ and $i = j \circ \chi$

$\langle 1 \rangle 3.$ $\psi \circ \chi \circ \phi^{-1} : S' \rightarrow T'$

$\langle 1 \rangle 4.$ $i' = j' \circ \psi \circ \chi \circ \phi^{-1}$

□

Proposition 1.2.30. *For any part i of A we have $i \subseteq_A i$.*

PROOF:

$\langle 1 \rangle 1.$ LET: $i : S \hookrightarrow A$

$\langle 1 \rangle 2.$ $\text{id}_S : S \rightarrow S$

$\langle 1 \rangle 3.$ $i = i \circ \text{id}_S$

□

Proposition 1.2.31. *If $i \subseteq_A j$ and $j \subseteq_A k$ then $i \subseteq_A k$.*

PROOF:

$\langle 1 \rangle 1.$ LET: $i : R \hookrightarrow A, j : S \hookrightarrow A$ and $k : T \hookrightarrow A$

$\langle 1 \rangle 2.$ PICK $\phi : R \rightarrow S$ and $\psi : S \rightarrow T$ such that $i = j \circ \phi$ and $j = k \circ \psi$

$\langle 1 \rangle 3.$ $\psi \circ \phi : R \rightarrow T$

$\langle 1 \rangle 4.$ $i = k \circ \psi \circ \phi$

□

Proposition 1.2.32. *If $i \subseteq_A j$ and $j \subseteq_A i$ then $i \equiv_A j$.*

PROOF:

$\langle 1 \rangle 1.$ LET: $i : R \hookrightarrow A, j : S \hookrightarrow A$

- ⟨1⟩2. PICK $\phi : R \rightarrow S$ and $\phi^{-1} : S \rightarrow R$ such that $i = j \circ \phi$ and $j = i \circ \phi^{-1}$
- ⟨1⟩3. $\phi \circ \phi^{-1} = \text{id}_S$
- ⟨2⟩1. $j \circ \phi \circ \phi^{-1} = j$
- ⟨2⟩2. Q.E.D.

PROOF: The result follows because j is injective.

- ⟨1⟩4. $\phi^{-1} \circ \phi = \text{id}_T$

PROOF: Similar.

□

1.2.4 The Empty Set

Axiom 1.2.33 (Empty Set). *There exists a set \emptyset , the empty set, such that, for every set X , there exists a unique function $i_X : \emptyset \rightarrow X$.*

Proposition 1.2.34 (Uniqueness of Empty Set). *Let E be any set. Then E is empty if and only if there exists an isomorphism $E \cong \emptyset$, in which case the isomorphism is unique.*

PROOF:

- ⟨1⟩1. If E is empty then $E \cong \emptyset$
- ⟨2⟩1. ASSUME: E is empty
- ⟨2⟩2. LET: ϕ be the unique function $E \rightarrow \emptyset$
- ⟨2⟩3. $i_E \circ \phi = \text{id}_E$
- PROOF: There is only one function $E \rightarrow E$.
- ⟨2⟩4. $\phi \circ i_E = \text{id}_{\emptyset}$
- PROOF: There is only one function $\emptyset \rightarrow \emptyset$.
- ⟨1⟩2. If $E \cong \emptyset$ then E is empty
- ⟨2⟩1. LET: $\phi : E \cong \emptyset$
- ⟨2⟩2. LET: X be a set
- PROVE: There is a unique function $E \rightarrow X$
- ⟨2⟩3. $i_X \circ \phi : E \rightarrow X$
- ⟨2⟩4. If $f : E \rightarrow X$ then $f = i_X \circ \phi$
- ⟨3⟩1. LET: $f : E \rightarrow X$
- ⟨3⟩2. $f \circ \phi^{-1} : \emptyset \rightarrow X$
- ⟨3⟩3. $f \circ \phi^{-1} = i_X$
- PROOF: Uniqueness of i_X .
- ⟨3⟩4. Q.E.D.
- ⟨1⟩3. There is at most one isomorphism $E \cong \emptyset$
- PROOF: This holds because there is at most one function $E \rightarrow \emptyset$.

□

Proposition 1.2.35.

$$i_{\emptyset} = \text{id}_{\emptyset}$$

PROOF: By the uniqueness of i_{\emptyset} . □

1.2.5 The Terminal Set

Axiom 1.2.36 (Terminal Set). *There exists a set 1 , the terminal set, such that, for every set X , there exists a unique function $!_X : X \rightarrow 1$.*

Proposition 1.2.37 (The Terminal Set is Unique up to Unique Canonical Isomorphism). *If T and T' are terminal sets with unique functions $b_X : X \rightarrow T$ and $b'_X : X \rightarrow T'$ for all X , then there exists a unique isomorphism $\phi : T \cong T'$ such that, for all X , we have $\phi \circ b_X = b'_X$.*

PROOF: Dual to Proposition ??.

Proposition 1.2.38.

$$!_1 = \text{id}_1$$

PROOF: From the uniqueness of $!_1$. \square

Definition 1.2.39 (Element). An *element* of a set A is a function $1 \rightarrow A$. We write $a \in A$ for $a : 1 \rightarrow A$. We write $f(a)$ for $f \circ a$ when $f : A \rightarrow B$ and $a \in A$.

Axiom 1.2.40 (Extensionality). *Let A and B be sets and $f, g : A \rightarrow B$ be functions. If, for all $a \in A$, we have $f(a) = g(a) \in B$, then $f = g$.*

Proposition 1.2.41. *Let $f : A \rightarrow B$. Then f is injective if and only if, for all $x, y \in A$, if $f(x) = f(y) \in B$ then $x = y \in A$.*

PROOF:

$\langle 1 \rangle 1$. If f is injective and $f(x) = f(y) \in B$ then $x = y \in A$

PROOF: Immediate from the definition of injective.

$\langle 2 \rangle 1$. If, for all $x, y \in A$, if $f(x) = f(y) \in B$ then $x = y \in A$

$\langle 2 \rangle 1$. ASSUME: For all $x, y \in A$, if $f(x) = f(y)$, then $x = y$

$\langle 2 \rangle 2$. LET: X be any set and $g, h : X \rightarrow A$ with $f \circ g = f \circ h$

PROVE: $g = h$

$\langle 2 \rangle 3$. LET: $x \in X$

PROVE: $g(x) = h(x)$

$\langle 2 \rangle 4$. $f(g(x)) = f(h(x))$

PROOF: From $\langle 2 \rangle 2$.

$\langle 2 \rangle 5$. $g(x) = h(x)$

PROOF: By $\langle 2 \rangle 1$

\square

Proposition 1.2.42. *Any element $e \in X$ is a section of the unique function $!_X : X \rightarrow 1$.*

PROOF: $!_X \circ e = \text{id}_1$ because there is only one function $1 \rightarrow 1$. \square

Axiom 1.2.43 (Non-degeneracy). *The empty set \emptyset has no elements.*

Proposition 1.2.44. *For any set X , the function $!_X : \emptyset \rightarrow X$ is injective.*

PROOF: From Proposition 1.2.41. \square

Definition 1.2.45 (Empty Part). For any set X , the *empty part* of X is $\emptyset = \text{id}_X : \emptyset \hookrightarrow X$.

Definition 1.2.46 (Constant Function). A function $f : A \rightarrow B$ is *constant* iff there exists $b \in B$ such that $f = b \circ !_A$.

Definition 1.2.47 (Membership). Let $i : U \hookrightarrow A$ be a part of A and $a \in A$. Then a is a *member* of i , $a \in_A i$, iff there exists $\bar{a} \in U$ such that $i(\bar{a}) = a$.

Proposition 1.2.48. Let A be a set. Let i, j be parts of A and $a \in A$. If $a \in_A i$ and $i \subseteq_A j$ then $a \in_A j$.

PROOF:

$\langle 1 \rangle 1$. PICK $\bar{a} \in \text{dom } i$ such that $a = i(\bar{a})$.

$\langle 1 \rangle 2$. PICK $\phi : \text{dom } i \rightarrow \text{dom } j$ such that $i = j \circ \phi$

$\langle 1 \rangle 3$. $a = j(\phi(\bar{a}))$

□

1.2.6 Products

Axiom 1.2.49 (Products). For any sets A and B , there exists a set $A \times B$, the product of A and B , and functions $\pi_1 : A \times B \rightarrow A$, $\pi_2 : A \times B \rightarrow B$, the projections, such that, for any set C and functions $f : C \rightarrow A$, $g : C \rightarrow B$, there exists a unique function $\langle f, g \rangle : C \rightarrow A \times B$ such that

$$\pi_1 \circ \langle f, g \rangle = f; \quad \pi_2 \circ \langle f, g \rangle = g .$$

Definition 1.2.50. Given functions $f : A \rightarrow B$ and $g : C \rightarrow D$, define $f \times g : A \times C \rightarrow B \times D$ by

$$f \times g = \langle f \circ \pi_1, g \circ \pi_2 \rangle$$

1.2.7 Coproducts

Axiom 1.2.51 (Coproducts). For any sets A and B , there exists a set $A \uplus B$, the coproduct or sum of A and B , and functions $\kappa_1 : A \rightarrow A \uplus B$, $\kappa_2 : B \rightarrow A \uplus B$, the injections, such that, for any set C and functions $f : A \rightarrow C$, $g : B \rightarrow C$, there exists a unique function $[f, g] : A \uplus B \rightarrow C$ such that

$$[f, g] \circ \kappa_1 = f; \quad [f, g] \circ \kappa_2 = g .$$

Definition 1.2.52 (Complement). Let $i : I \hookrightarrow J$ and $i' : I' \hookrightarrow J$ be parts of J . Then i' is the *complement* of i iff J is the sum of I and I' with injections i and i' .

1.2.8 Equalizers

Axiom 1.2.53 (Equalizers). For any sets A and B and functions $f, g : A \rightarrow B$, there exists a set E and function $e : E \rightarrow A$, the equalizer of A and B , such that:

- $f \circ e = g \circ e : E \rightarrow B$;
- For any set C and function $h : C \rightarrow A$ such that $f \circ h = g \circ h$, there exists a unique function $\bar{h} : C \rightarrow E$ such that $h = e \circ \bar{h}$.

Proposition 1.2.54. *All equalizers are injective.*

PROOF:

$\langle 1 \rangle 1$. LET: $e : E \rightarrow A$ be the equalizer of $f, g : A \rightarrow B$

$\langle 1 \rangle 2$. LET: $x, y : X \rightarrow E$ with $e \circ x = e \circ y$

$\langle 1 \rangle 3$. $f \circ e \circ x = g \circ e \circ x$

PROOF: $f \circ e = g \circ e$ by $\langle 1 \rangle 1$.

$\langle 1 \rangle 4$. $x = y$

PROOF: x and y are both the unique $z : X \rightarrow E$ such that $e \circ z = e \circ x$.

□

1.2.9 Coequalizers

Axiom 1.2.55 (Coequalizers). *For any sets A and B and functions $f, g : A \rightarrow B$, there exists a set C and function $c : B \rightarrow C$, the coequalizer of f and g , such that:*

- $c \circ f = c \circ g : A \rightarrow C$
- For any set X and function $h : B \rightarrow X$ such that $h \circ f = h \circ g$, there exists a unique function $\bar{h} : C \rightarrow X$ such that $\bar{h} \circ c = h$.

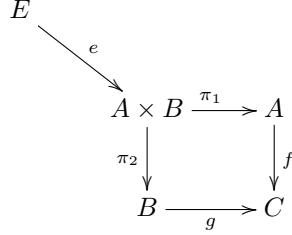
1.2.10 Pullbacks

Definition 1.2.56 (Pullback). The diagram below is a *pullback diagram* iff:

- $f \circ p = g \circ q$
- for every set X and functions $x : X \rightarrow B$ and $y : X \rightarrow C$ such that $f \circ x = g \circ y$, there exists a unique function $\langle x, y \rangle : X \rightarrow A$ such that $p \circ \langle x, y \rangle = x$ and $q \circ \langle x, y \rangle = y$.

$$\begin{array}{ccc} A & \xrightarrow{p} & B \\ q \downarrow & & \downarrow f \\ C & \xrightarrow{g} & D \end{array}$$

Proposition 1.2.57. *Let $f : A \rightarrow C$ and $g : B \rightarrow C$. Then f and g have a pullback.*



PROOF:

- <1>1. Construct the product $\pi_1 : A \times B \rightarrow A$, $\pi_2 : A \times B \rightarrow B$.
- <1>2. Construct the equalizer $e : E \rightarrow A \times B$ of $f \circ \pi_1$ and $g \circ \pi_2$.
 PROVE: $\pi_1 \circ e$ and $\pi_2 \circ e$ form a pullback of f and g
- <1>3. $f \circ \pi_1 \circ e = g \circ \pi_2 \circ e$
- <1>4. LET: X be a set and $x : X \rightarrow A$, $y : X \rightarrow B$ satisfy $f \circ x = g \circ y$
- <1>5. $f \circ \pi_1 \circ \langle x, y \rangle = g \circ \pi_2 \circ \langle x, y \rangle$
- <1>6. LET: $m : X \rightarrow E$ be the function such that $e \circ m = \langle x, y \rangle$
- <1>7. $\pi_1 \circ e \circ m = x$ and $\pi_2 \circ e \circ m = y$
- <1>8. m is unique.

PROOF:

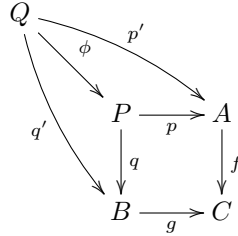
- <2>1. LET: $n : X \rightarrow E$ be such that $\pi_1 \circ e \circ n = x$ and $\pi_2 \circ e \circ n = y$
- <2>2. $e \circ n = \langle x, y \rangle$
- <2>3. $n = m$

PROOF: By <1>6

□

Proposition 1.2.58. *Pullbacks are unique up to isomorphism.*

That is, let P be a pullback of $f : A \rightarrow C$ and $g : B \rightarrow C$ with projections $p : P \rightarrow A$ and $q : P \rightarrow B$. Let Q be a set and $p' : Q \rightarrow A$, $q' : Q \rightarrow B$. Then Q is a pullback of f and g with projections p' and q' if and only if there exists a bijection $\phi : Q \cong P$ such that $p \circ \phi = p'$ and $q \circ \phi = q'$, in which case ϕ is unique.



PROOF:

- <1>1. If Q is a pullback then there exists a bijection $\phi : Q \cong P$ such that $p \circ \phi = p'$ and $q \circ \phi = q'$
- <2>1. ASSUME: Q is a pullback with projections p' and q'

- (2)2. LET: $\phi : Q \rightarrow P$ be the unique function such that $p \circ \phi = p'$ and $q \circ \phi = q'$
PROOF: Such a ϕ exists because $f \circ p' = g \circ q'$.
- (2)3. LET: $\phi^{-1} : P \rightarrow Q$ be the unique function such that $p' \circ \phi^{-1} = p$ and $q' \circ \phi^{-1} = q$
PROOF: Such a function exists because $f \circ p = g \circ q$.
- (2)4. $\phi \circ \phi^{-1} = \text{id}_P$
PROOF: Each is the unique function x such that $p \circ x = p$ and $q \circ x = q$.
- (2)5. $\phi^{-1} \circ \phi = \text{id}_Q$
PROOF: Similar.
- (1)2. If $\phi : Q \cong P$ is a bijection then Q is a pullback with projections $p \circ \phi$ and $q \circ \phi$
(2)1. $f \circ p \circ \phi = g \circ q \circ \phi$
PROOF: This holds because $f \circ p = g \circ q$
- (2)2. For any set X and functions $x : X \rightarrow A$, $y : X \rightarrow B$ such that $f \circ x = g \circ y$, there exists a unique function $m : X \rightarrow Q$ such that $p \circ \phi \circ m = x$ and $q \circ \phi \circ m = y$
PROOF:

$$\begin{aligned} p \circ \phi \circ m &= x \text{ and } q \circ \phi \circ m = y \\ \Leftrightarrow \phi \circ m &= \langle x, y \rangle \\ \Leftrightarrow m &= \phi^{-1} \circ \langle x, y \rangle \end{aligned}$$

- (1)3. If $\phi, \phi' : P \cong Q$ are bijections such that $p \circ \phi = p \circ \phi'$ and $q \circ \phi = q \circ \phi'$
PROOF: This follows from the definition of pullback.

□

Proposition 1.2.59. *The pullback of an injective function is injective.*

That is, if the diagram below is a pullback diagram and f is injective then q is injective.

$$\begin{array}{ccc} A & \xrightarrow{p} & B \\ q \downarrow & & \downarrow f \\ C & \xrightarrow{g} & D \end{array}$$

PROOF:

- (1)1. LET: X be a set and $x, y : X \rightarrow A$ with $q \circ x = q \circ y$
(1)2. $f \circ p \circ x = g \circ q \circ x$
(1)3. LET:
 $z : X \rightarrow A$ be the function such that $p \circ z = p \circ x$ and $q \circ z = q \circ x$
(1)4. $z = x$
(1)5. $z = y$
(2)1. $q \circ x = q \circ y$
PROOF: By (1)1.
(2)2. $f \circ p \circ x = f \circ p \circ y$

PROOF:

$$\begin{aligned}
 f \circ p \circ x &= g \circ q \circ x && (\langle 1 \rangle 2) \\
 &= g \circ q \circ y && (\langle 1 \rangle 1) \\
 &= f \circ p \circ y && \text{(the diagram is a pullback)}
 \end{aligned}$$

$\langle 2 \rangle 3$. $p \circ x = p \circ y$

PROOF: f is injective.

□

1.2.11 Function Sets

Axiom 1.2.60 (Function Sets). *For any sets A and B , there exists a set A^B and a function $\epsilon : A^B \times B \rightarrow A$, the evaluation function, such that, for any set C and function $f : C \times B \rightarrow A$, there exists a unique function $\lambda f : C \rightarrow A^B$ such that*

$$\epsilon \circ (\lambda f \times \text{id}_B) = f \text{ .}$$

1.2.12 The Subset Classifier

Definition 1.2.61. The set 2 is $1 + 1$. We write \top (*truth*) for $\kappa_1 : 1 \rightarrow 2$, and \perp (*falsehood*) for $\kappa_2 : 1 \rightarrow 2$.

Axiom 1.2.62 (Subset Classifier). *For every injective function $m : A \rightarrowtail B$, there exists a unique function $\chi_m : B \rightarrow 2$, the characteristic function of m , such that the following diagram is a pullback diagram:*

$$\begin{array}{ccc}
 A & \xrightarrow{!} & 1 \\
 m \downarrow & & \downarrow \top \\
 B & \xrightarrow{\chi_m} & 2
 \end{array}$$

Proposition 1.2.63. *Every function $\phi : A \rightarrow 2$ is the characteristic function of a part of A .*

PROOF:

$\langle 1 \rangle 1$. Construct a pullback

$$\begin{array}{ccc}
 I & \xrightarrow{\quad} & 1 \\
 q \downarrow & & \downarrow \top \\
 A & \xrightarrow[\phi]{} & 2
 \end{array}$$

PROOF: By Proposition 1.2.57.

$\langle 1 \rangle 2$. q is injective

PROOF: By Proposition 1.2.59.

□

Axiom 1.2.64 (Boolean). *For any $p \in 2$ we have $p = \top$ or $p = \perp$.*

Proposition 1.2.65. *Let $i : U \hookrightarrow A$ and $j : V \hookrightarrow A$ be parts of A . Then the following are equivalent:*

1. $i \subseteq_A j$ and $j \subseteq_A i$
2. There exist $h : U \rightarrow V$ and $k : V \rightarrow U$ such that $i = j \circ h$, $j = i \circ k$, $k \circ h = \text{id}_U$ and $h \circ k = \text{id}_V$.
3. The characteristic function of i is the characteristic function of j .

PROOF:

$$\langle 1 \rangle 1. \quad 1 \Rightarrow 2$$

$\langle 2 \rangle 1$. ASSUME: $i \subseteq_A j$ and $j \subseteq_A i$

$\langle 2 \rangle 2$. LET: $h : U \rightarrow V$ be such that $i = j \circ h$

⟨2⟩3. LET: $k : V \rightarrow U$ be such that $j = i \circ k$

$\langle 2 \rangle 4.$ $k \circ h = \text{id}_U$

$\langle 3 \rangle 1. \ i \circ k \circ h = i$

PROOF: From $\langle 2 \rangle 2$ and $\langle 2 \rangle 3$.

⟨3⟩2. Q.E.D.

PROOF: Since i is injective.

2.5. $h \circ k = \text{id}_V$

PROOF: Similar.

$$\langle 1 \rangle 2. \quad 2 \Rightarrow 1$$

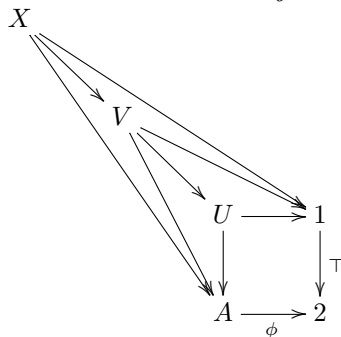
PROOF: Trivial.

$$\langle 1 \rangle 3. \quad 2 \Rightarrow 3$$

$\langle 2 \rangle 1$. ASSUME: 2

$\langle 2 \rangle$ 2. LET: $\phi : A \rightarrow 2$ be the characteristic function of i

PROVE: ϕ is the characteristic function of j



PROOF: By the Subset Classifier Axiom.

⟨2⟩3. LET: X be a set and $x : X \rightarrow 1$, $y : X \rightarrow A$ satisfy $\phi \circ y = \top \circ x$

(2)4. LET: $\langle x, y \rangle : X \rightarrow U$ be the unique function such that $! \circ \langle x, y \rangle = x$ and

$$i \circ \langle x, y \rangle = y$$

PROOF: By $\langle 2 \rangle 2$.

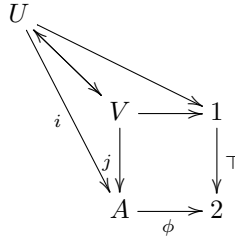
(2)5. $h \circ \langle x, y \rangle$ is the unique function $X \rightarrow V$ such that $! \circ h \circ \langle x, y \rangle = x$ and

$$j \circ h \circ \langle x, y \rangle = y$$

$$\langle 3 \rangle 1. \quad ! \circ h \circ \langle x, y \rangle = x$$

PROOF: Since 1 is terminal.

$\langle 3 \rangle 2. j \circ h \circ \langle x, y \rangle = y$
 PROOF: From $\langle 2 \rangle 1$ and $\langle 2 \rangle 4$.
 $\langle 3 \rangle 3.$ If $! \circ f = x$ and $j \circ f = y$ then $f = h \circ \langle x, y \rangle$
 $\langle 4 \rangle 1.$ LET: $f : X \rightarrow V$ satisfy $! \circ f = x$ and $j \circ f = y$
 $\langle 4 \rangle 2. ! \circ k \circ f = x$
 PROOF: As 1 is terminal.
 $\langle 4 \rangle 3. i \circ k \circ f = y$
 PROOF: From $\langle 2 \rangle 1$ and $\langle 4 \rangle 1$.
 $\langle 4 \rangle 4. k \circ f = \langle x, y \rangle$
 PROOF: From $\langle 2 \rangle 4$, $\langle 4 \rangle 2$ and $\langle 4 \rangle 3$.
 $\langle 4 \rangle 5. f = h \circ \langle x, y \rangle$
 PROOF: From $\langle 2 \rangle 1$ and $\langle 4 \rangle 4$.
 $\langle 1 \rangle 4. 3 \Rightarrow 2$



$\langle 2 \rangle 1.$ ASSUME: 3
 $\langle 2 \rangle 2.$ LET: ϕ be the characteristic function of i and j
 $\langle 2 \rangle 3.$ LET: $h : U \rightarrow V$ be the unique function such that $! \circ h = !$ and $j \circ h = i$
 $\langle 3 \rangle 1. \top \circ ! = \phi \circ i$
 PROOF: This holds because ϕ is the characteristic function of i .
 $\langle 3 \rangle 2.$ Q.E.D.
 PROOF: Since ϕ is the characteristic function of j .
 $\langle 2 \rangle 4.$ LET: $k : V \rightarrow U$ be the unique function such that $! \circ k = !$ and $i \circ k = j$
 PROOF: Similar.
 $\langle 2 \rangle 5. k \circ h = \text{id}_U$
 PROOF: Each is the unique function f such that $! \circ f = !$ and $i \circ f = i$
 $\langle 2 \rangle 6. h \circ k = \text{id}_V$
 PROOF: Each is the unique function f such that $! \circ f = !$ and $j \circ f = j$
 \square

1.3 The Basics

Lemma 1.3.1. *Let X be a set, $\mathcal{B} \subseteq \mathcal{P}X$ and $U \subseteq X$. Then the following are equivalent:*

1. *For all $x \in U$ there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$.*
2. *There exists $\mathcal{B}_0 \subseteq \mathcal{B}$ such that $U = \bigcup \mathcal{B}_0$.*

PROOF:

$\langle 1 \rangle 1. 1 \Rightarrow 2$

PROOF: If 1 is true then $U = \bigcup \{B \in \mathcal{B} : B \subseteq U\}$.

$\langle 1 \rangle 2. 2 \Rightarrow 1$

PROOF: Trivial.

□

Definition 1.3.2 (Fixed Point). Let X be a set, $f : X \rightarrow X$, and $x \in X$. Then x is a *fixed point* of f iff $f(x) = x$.

Definition 1.3.3 (Saturated). Let X, Y be sets and $p : X \rightarrow Y$ be a surjective function. Let $C \subseteq X$. Then C is *saturated* with respect to p iff, for all $x, x' \in X$, if $x \in C$ and $p(x) = p(x')$ then $x' \in C$.

Definition 1.3.4 (Cover). Let A be a set and $\mathcal{C} \subseteq \mathcal{P}A$. Then \mathcal{C} *covers* A iff $\bigcup \mathcal{C} = A$.

Definition 1.3.5 (Finite Intersection Property). Let X be a set and $\mathcal{C} \subseteq \mathcal{P}X$. Then \mathcal{C} has the *finite intersection property* if and only if every finite nonempty subset of \mathcal{C} has nonempty intersection.

Lemma 1.3.6 (AC). *Let X be a set and $\mathcal{A} \subseteq \mathcal{P}X$ have the finite intersection property. Then there exists a maximal $\mathcal{D} \subseteq \mathcal{P}X$ that has the finite intersection property and includes \mathcal{A} .*

PROOF: A straightforward application of Zorn's lemma, since the union of a chain of sets that has the finite intersection property has the finite intersection property. □

Lemma 1.3.7. *Let X be a set and $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the finite intersection property. Then any finite intersection of elements of \mathcal{D} is an element of \mathcal{D} .*

PROOF:

$\langle 1 \rangle 1.$ LET: A be a finite intersection of elements of \mathcal{D}

$\langle 1 \rangle 2.$ $\mathcal{D} \cup \{A\}$ has the finite intersection property.

$\langle 1 \rangle 3.$ $\mathcal{D} \cup \{A\} = \mathcal{D}$

□

Lemma 1.3.8. *Let X be a set and $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the finite intersection property. If $A \subseteq X$ intersects every element of \mathcal{D} then $A \in \mathcal{D}$.*

PROOF: This holds because $\mathcal{D} \cup \{A\}$ satisfies the finite intersection property. □

Definition 1.3.9 (Graph). Let $f : A \rightarrow B$. The *graph* of f is the set $\{(x, f(x)) : x \in A\} \subseteq A \times B$.

Definition 1.3.10 (Point-Finite). Let X be a set and $\{A_\alpha\}_{\alpha \in J}$ be a family of subsets of X . Then $\{A_\alpha\}_{\alpha \in J}$ is *point-finite* iff, for all $x \in X$, there are only finitely many $\alpha \in J$ such that $x \in A_\alpha$.

Definition 1.3.11 (Countable Intersection Property). A family of parts of a set X has the *countable intersection property* iff every countable subfamily has nonempty intersection.

1.4 Order Theory

Definition 1.4.1 (Cofinal). Let J be a poset and $K \subseteq J$. Then K is *cofinal* iff, for all $x \in J$, there exists $y \in K$ such that $x \leq y$.

Definition 1.4.2 (Directed Set). A *directed set* is a poset J such that, for all $x, y \in J$, there exists $z \in J$ such that $x \leq z$ and $y \leq z$.

Definition 1.4.3 (Linear Order). Let X be a set. A *linear order* on X is a relation $\leq \subseteq X^2$ such that:

- For all $x \in X$, $x \leq x$
- For all $x, y, z \in X$, if $x \leq y$ and $y \leq z$ then $x \leq z$
- For all $x, y \in X$, if $x \leq y$ and $y \leq x$ then $x = y$
- For all $x, y \in X$, we have $x \leq y$ or $y \leq x$

We write $x < y$ iff $x \leq y$ and $x \neq y$.

A *linearly ordered set* consists of a set and a linear order on the set.

Definition 1.4.4 (Convex). Let L be a linearly ordered set and $A \subseteq L$. Then A is *convex* iff, for all $x, y \in A$ and $z \in L$, if $x < z < y$ then $z \in A$.

Definition 1.4.5 (Least Upper Bound Property). A linearly ordered set L has the *least upper bound property* iff every subset of L bounded above has a least upper bound.

Definition 1.4.6 (Linear Continuum). A *linear continuum* is a linearly ordered set L such that:

- L has the least upper bound property.
- For all $x, y \in L$ with $x < y$, there exists $z \in L$ such that $x < z < y$.

Proposition 1.4.7. *If L is a linear continuum then every convex subset of L is a linear continuum.*

PROOF:

$\langle 1 \rangle 1$. LET: L be a linear continuum and $C \subseteq L$ be convex

$\langle 1 \rangle 2$. C satisfies the least upper bound property.

$\langle 2 \rangle 1$. LET: $S \subseteq C$ be nonempty and bounded above by u in C .

$\langle 2 \rangle 2$. LET: s be the supremum of S in L

$\langle 2 \rangle 3$. PICK $x \in S$

$\langle 2 \rangle 4$. $x \leq s \leq u$

$\langle 2 \rangle 5$. $s \in C$

PROOF: C is convex.

$\langle 2 \rangle 6$. s is the supremum of S in C

$\langle 1 \rangle 3$. C is dense.

PROOF:

- (2)1. LET: $x, y \in C$ satisfy $x < y$
- (2)2. PICK $z \in L$ such that $x < z < y$
- (2)3. $z \in C$

PROOF: C is convex.

□

Lemma 1.4.8. *For any real numbers a, b with $a < b$ we have $[a, b] \cong [0, 1]$.*

PROOF: The map $\phi : [a, b] \cong [0, 1]$ where $\phi(x) = (x - a)/(b - a)$ is an order isomorphism. □

Proposition 1.4.9. *Let X be a linearly ordered set. Let $a, b, c \in X$ with $a < c < b$. Then $[a, b] \cong [0, 1]$ if and only if $[a, c] \cong [c, b] \cong [0, 1]$.*

PROOF:

- (1)1. If $[a, b] \cong [0, 1]$ then $[a, c] \cong [c, b] \cong [0, 1]$.
- (2)1. ASSUME: $\phi : [a, b] \cong [0, 1]$ is an order isomorphism.
- (2)2. $[a, c] \cong [0, 1]$

PROOF:

$$\begin{aligned} [a, c] &\cong [0, \phi(c)] && \text{(under } \phi) \\ &\cong [0, 1] && \text{(Lemma 1.4.8)} \end{aligned}$$

- (2)3. $[c, b] \cong [0, 1]$

PROOF: Similar.

- (1)2. If $[a, c] \cong [c, b] \cong [0, 1]$ then $[a, b] \cong [0, 1]$.
- (2)1. ASSUME: $[a, c] \cong [c, b] \cong [0, 1]$
- (2)2. LET: $\phi : [a, c] \cong [0, 1/2]$ and $\psi : [c, b] \cong [1/2, 1]$

- (2)3. LET: $\chi : [a, b] \rightarrow [0, 1]$ be given by $\chi(x) = \begin{cases} \phi(x) & \text{if } x < c \\ \psi(x) & \text{if } x \geq c \end{cases}$

- (2)4. $\chi : [a, b] \cong [0, 1]$

PROOF: Easy to check.

□

Proposition 1.4.10 (CC). *Let X be a linearly ordered set. Let $\{x_n\}_{n \geq 0}$ be an increasing sequence of points of X . Suppose b is the supremum of $\{x_n : n \geq 0\}$. Then $[x_0, b] \cong [0, 1]$ if and only if $[x_i, x_{i+1}] \cong [0, 1]$ for all i .*

PROOF:

- (1)1. If $[x_0, b] \cong [0, 1]$ then for all i $[x_i, x_{i+1}] \cong [0, 1]$.

PROOF: If $\phi : [x_0, b] \cong [0, 1]$ then $[x_i, x_{i+1}] \cong [\phi(x_i), \phi(x_{i+1})) \cong [0, 1]$ by Lemma 1.4.8.

- (1)2. If for all i $[x_i, x_{i+1}] \cong [0, 1]$ then $[x_0, b] \cong [0, 1]$.

PROOF:

- (2)1. LET: $\phi_i : [x_i, x_{i+1}] \cong [0, 1]$ for all i

- (2)2. Define $\phi : [x_0, b] \cong [0, 1]$ by: $\phi(y) = \phi_i(y)$ ($x_0 \leq y < b$) where i is least such that $y < x_{i+1}$

PROOF: There exists such an i because y is not an upper bound for $\{x_n : n \geq 0\}$.

$\langle 2 \rangle 3$. ϕ is an order isomorphism.

PROOF: Easy to check.

□

Proposition 1.4.11 (CC). *For all $0 < \alpha < \Omega$, the interval $[(0, 0), (\alpha, 0))$ in $S_\Omega \times [0, 1)$ is order isomorphic to $[0, 1)$ in \mathbb{R} .*

PROOF:

$\langle 1 \rangle 1$. If $[(0, 0), (\alpha, 0)) \cong [0, 1)$ then $[(0, 0), (\alpha + 1, 0)) \cong [0, 1)$

PROOF: By Proposition 1.4.9.

$\langle 1 \rangle 2$. Let λ be a limit ordinal, $0 < \lambda < \Omega$. If, for all α with $0 < \alpha < \lambda$, we have $[(0, 0), (\alpha, 0)) \cong [0, 1)$, then $[(0, 0), (\lambda, 0)) \cong [0, 1)$.

PROOF: By Proposition 1.4.10.

$\langle 1 \rangle 3$. Q.E.D.

PROOF: By transfinite induction.

□

Chapter 2

Real Analysis

Definition 2.0.1 (Cantor Set). Define a sequence of sets $A_n \subseteq [0, 1]$ by:

$$A_0 = [0, 1]$$
$$A_{n+1} = A_n \setminus \bigcup_{k=0}^{3^{n-1}-1} \left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n} \right)$$

The *Cantor set* is $\bigcap_{n=0}^{\infty} A_n$.

Chapter 3

Topological Spaces

3.1 Topologies

Definition 3.1.1 (Topology). A *topology* on a set X is a set $\mathcal{T} \subseteq \mathcal{P}X$ such that:

1. $X \in \mathcal{T}$;
2. for all $U, V \in \mathcal{T}$, we have $U \cap V \in \mathcal{T}$;
3. For all $\mathcal{A} \subseteq \mathcal{T}$, we have $\bigcup \mathcal{A} \in \mathcal{T}$.

A *topological space* X consists of a set X and a topology on X . The elements of X are called *points* and the elements of \mathcal{T} are called *open sets*.

Proposition 3.1.2. *In any topological space, the empty set is open.*

PROOF: Immediate from axiom 3. \square

Definition 3.1.3 (Discrete Topology). The *discrete* topology on a set X is $\mathcal{P}X$.

Definition 3.1.4 (Indiscrete Topology). The *indiscrete* topology on a set X is $\{\emptyset, X\}$.

Definition 3.1.5 (Open Cover). Let X be a topological space. A cover $\mathcal{C} \subseteq \mathcal{P}X$ of X is an *open cover* iff every member of \mathcal{C} is open.

Definition 3.1.6 (Finer, Coarser). Let $\mathcal{T}, \mathcal{T}'$ be topologies on a set X . Then \mathcal{T} is *finer* than \mathcal{T}' , and \mathcal{T}' is *coarser* than \mathcal{T} , iff $\mathcal{T}' \subseteq \mathcal{T}$.

The topology \mathcal{T} is *strictly finer* than \mathcal{T}' , and \mathcal{T}' is *strictly coarser* than \mathcal{T} , iff $\mathcal{T} \subset \mathcal{T}'$.

The topologies \mathcal{T} and \mathcal{T}' are *comparable* iff $\mathcal{T} \subseteq \mathcal{T}'$ or $\mathcal{T}' \subseteq \mathcal{T}$.

Definition 3.1.7 (Finite Complement Topology). The *finite complement topology* on a set X is $\{U : X \setminus U \text{ is finite}\} \cup \{X\}$.

Definition 3.1.8 (Isolated Point). Let X be a topological space and $a \in X$. Then a is an *isolated point* iff $\{a\}$ is open.

3.2 Neighbourhoods

Definition 3.2.1 (Neighbourhood). Let X be a topological space and $A \subseteq X$. A *neighbourhood* of A is an set that includes an open set that includes A .

A *neighbourhood* of a point a is a neighbourhood of $\{a\}$.

Proposition 3.2.2. *If N is a neighbourhood of A and $B \subseteq A$ then N is a neighbourhood of B .*

PROOF: Immediate from definitions. \square

Proposition 3.2.3. *A set U is open if and only if it is a neighbourhood of each of its points.*

PROOF:

$\langle 1 \rangle 1$. LET: X be a topological space and $A \subseteq X$

$\langle 1 \rangle 2$. If U is a neighbourhood of each of its points then A is open.

$\langle 2 \rangle 1$. ASSUME: U includes a neighbourhood of each of its points

PROVE: $U = \bigcup \{V \subseteq U : V \text{ is open}\}$

$\langle 2 \rangle 2$. $\bigcup \{V \subseteq U : V \text{ is open}\} \subseteq U$

PROOF: Set theory.

$\langle 2 \rangle 3$. $U \subseteq \bigcup \{V \subseteq U : V \text{ is open}\}$

PROOF: Immediate from $\langle 2 \rangle 1$.

$\langle 1 \rangle 3$. If U is open then U is a neighbourhood of each of its points.

PROOF: Immediate from definitions.

\square

Proposition 3.2.4. *If M is a neighbourhood of A and $M \subseteq N$ then N is a neighbourhood of A .*

PROOF: Immediate from definitions. \square

Proposition 3.2.5. *If M and N are neighbourhoods of A then $M \cap N$ is a neighbourhood of A .*

PROOF: Pick open sets U and V such that $A \subseteq U \subseteq M$ and $A \subseteq V \subseteq N$. Then $A \subseteq U \cap V \subseteq M \cap N$.

Proposition 3.2.6. *If N is a neighbourhood of x then $x \in N$.*

PROOF: Immediate from definitions. \square

Proposition 3.2.7. *If N is a neighbourhood of x then there exists a neighbourhood U of x such that, for all $y \in U$, M is a neighbourhood of y .*

PROOF: Pick an open set U such that $x \in U \subseteq N$. \square

Theorem 3.2.8. *Let X be a set and $\triangleright \subseteq \mathcal{P}X \times X$ a relation such that:*

1. *If $M \triangleright x$ and $M \subseteq N$ then $N \triangleright x$*
2. *$X \triangleright x$ for all $x \in X$*

3. If $M \triangleright x$ and $N \triangleright x$ then $M \cap N \triangleright x$
4. If $N \triangleright x$ then $x \in N$
5. If $M \triangleright x$ then there exists $N \triangleright x$ such that, for all $y \in N$, $M \triangleright y$.

Then there exists a unique topology \mathcal{T} such that $N \triangleright x$ iff N is a neighbourhood of x .

PROOF:

- $\langle 1 \rangle 1$. LET: \triangleright be a relation satisfying 1–3
- $\langle 1 \rangle 2$. LET: $\mathcal{T} = \{U \in \mathcal{P}X : \forall x \in U. U \triangleright x\}$
- $\langle 1 \rangle 3$. \mathcal{T} is a topology.
 - $\langle 2 \rangle 1$. $X \in \mathcal{T}$
PROOF: By axiom 2
 - $\langle 2 \rangle 2$. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$
PROOF: By axiom 3
 - $\langle 2 \rangle 3$. For all $\mathcal{A} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{A} \in \mathcal{T}$
 - $\langle 3 \rangle 1$. LET: $x \in \bigcup \mathcal{A}$
 - $\langle 3 \rangle 2$. PICK $U \in \mathcal{A}$ such that $x \in U$
 - $\langle 3 \rangle 3$. $U \triangleright x$
 - $\langle 3 \rangle 4$. $\bigcup \mathcal{A} \triangleright x$
PROOF: By axiom 1
- $\langle 1 \rangle 4$. In \mathcal{T} , $N \triangleright x$ iff N is a neighbourhood of x .
 - $\langle 2 \rangle 1$. If $N \triangleright x$ then N is a neighbourhood of x
 - $\langle 3 \rangle 1$. ASSUME: $N \triangleright x$
 - $\langle 3 \rangle 2$. $x \in N$
PROOF: By axiom 4
 - $\langle 3 \rangle 3$. LET: $U = \{y \in N : N \triangleright y\}$
 - $\langle 3 \rangle 4$. U is open
 - $\langle 4 \rangle 1$. LET: $y \in U$
PROVE: $U \triangleright y$
 - $\langle 4 \rangle 2$. $N \triangleright y$
 - $\langle 4 \rangle 3$. PICK $W \triangleright y$ such that, for all $z \in W$, $N \triangleright z$
PROOF: By axiom 5
 - $\langle 4 \rangle 4$. $W \subseteq U$
 - $\langle 4 \rangle 5$. $U \triangleright y$
PROOF: By axiom 1
 - $\langle 3 \rangle 5$. $x \in U \subseteq N$
 - $\langle 2 \rangle 2$. If N is a neighbourhood of x then $N \triangleright x$
 - $\langle 3 \rangle 1$. LET: N be a neighbourhood of x
 - $\langle 3 \rangle 2$. PICK U open such that $x \in U \subseteq N$
 - $\langle 3 \rangle 3$. $U \triangleright x$
PROOF: By $\langle 1 \rangle 2$
 - $\langle 3 \rangle 4$. $N \triangleright x$
PROOF: By axiom 1
- $\langle 1 \rangle 5$. \mathcal{T} is unique.

PROOF: By Proposition 3.2.3.

□

Definition 3.2.9 (Sufficiently Close). Let X be a topological space, $a \in X$, and P be a property of points of X . We write “For all x sufficiently close to a , $P(x)$ ” to mean “There exists a neighbourhood N of a such that, for all $x \in N$, $P(x)$.”

3.3 Local Bases

Definition 3.3.1 (Local Basis). Let X be a topological space and $x \in X$. A *local basis* at x is a set \mathcal{B} of open neighbourhoods of x such that every neighbourhood of x includes a member of \mathcal{B} . We call the elements of \mathcal{B} *basic open neighbourhoods*.

Proposition 3.3.2. Let \mathcal{B} be a local basis at x and $M, N \in \mathcal{B}$. Then there exists $P \in \mathcal{B}$ such that $P \subseteq M \cap N$.

PROOF: This holds because $M \cap N$ is a neighbourhood of x (Proposition 3.2.5).

□

Proposition 3.3.3. Let X be a topological space, $x \in X$ and $\mathcal{B} \subseteq \mathcal{P}X$. Then \mathcal{B} is a local basis at x iff \mathcal{B} is a set of open neighbourhoods of x such that every open neighbourhood of x includes a member of \mathcal{B} .

PROOF:

⟨1⟩1. If \mathcal{B} is a local basis at x then \mathcal{B} is a set of open neighbourhoods of x such that every open neighbourhood of x includes a member of \mathcal{B}

PROOF: Trivial.

⟨1⟩2. If \mathcal{B} is a set of open neighbourhoods of x such that every open neighbourhood of x includes a member of \mathcal{B} then \mathcal{B} is a local basis at x .

PROOF: Every neighbourhood of x includes an open neighbourhood of x , which therefore includes an element of \mathcal{B} .

□

3.4 Bases

Definition 3.4.1 (Basis for a Topology). Let (X, \mathcal{T}) be a topological space. A *basis* for the topology on X is a set of open sets \mathcal{B} such that every open set is a union of members of \mathcal{B} . The members of \mathcal{B} are called *basic open sets*, and \mathcal{T} is called the topology *generated* by \mathcal{B} .

Proposition 3.4.2. Let (X, \mathcal{T}) be a topological space and $\mathcal{B} \subseteq \mathcal{P}X$. Then the following are equivalent:

1. \mathcal{B} is a basis for \mathcal{T} .

2. A set U is open if and only if, for all $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$.
3. \mathcal{T} is the set of all unions of subsets of \mathcal{B} .
4. Every member of \mathcal{B} is open and, for all $x \in X$ and every open neighbourhood U of x , there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$.
5. For all $x \in X$, the set $\{B \in \mathcal{B} : x \in B\}$ is a local basis at x .

PROOF:

$\langle 1 \rangle 1. 1 \Rightarrow 2$

$\langle 2 \rangle 1.$ ASSUME: \mathcal{B} is a basis for the topology \mathcal{T} .

$\langle 2 \rangle 2.$ For all $U \in \mathcal{T}$ and $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B$

PROOF: Immediate from the definition of basis ($\langle 2 \rangle 1$).

$\langle 2 \rangle 3.$ For all $U \subseteq X$, if $\forall x \in U. \exists B \in \mathcal{B}. x \in B \subseteq U$ then $U \in \mathcal{T}$

PROOF: By Proposition 3.2.3.

$\langle 1 \rangle 2. 2 \Leftrightarrow 3$

PROOF: From Lemma 1.3.1.

$\langle 1 \rangle 3. 3 \Rightarrow 1$

PROOF: Trivial.

$\langle 1 \rangle 4. 2 \Rightarrow 4$

PROOF: Trivial.

$\langle 1 \rangle 5. 4 \Rightarrow 2$

PROOF:

$\langle 2 \rangle 1.$ ASSUME: 4

$\langle 2 \rangle 2.$ If U is open then, for all $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$

PROOF: Immediate from $\langle 2 \rangle 1$.

$\langle 2 \rangle 3.$ If, for all $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$, then U is open.

PROOF: By Proposition 3.2.3 using the fact that every member of \mathcal{B} is open ($\langle 2 \rangle 1$).

$\langle 1 \rangle 6. 4 \Leftrightarrow 5$

PROOF: From Proposition 3.3.3.

□

Corollary 3.4.2.1. If \mathcal{B} is a basis for the topology \mathcal{T} , then \mathcal{T} is the coarsest topology in which every element of \mathcal{B} is open.

Lemma 3.4.3. Let X be a set and $\mathcal{B} \subseteq \mathcal{P}X$. Then \mathcal{B} is a basis for a topology \mathcal{T} on X if and only if:

1. $\bigcup \mathcal{B} = X$
2. for all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

In this case, \mathcal{T} is unique.

PROOF:

- ⟨1⟩1. If \mathcal{B} is a basis for a topology then $\bigcup \mathcal{B} = X$
- ⟨2⟩1. ASSUME: \mathcal{B} is a basis for the topology \mathcal{T}
- ⟨2⟩2. LET: $x \in X$
- ⟨2⟩3. There exists $B \in \mathcal{B}$ such that $x \in B$
 PROOF: From the definition of basis, since $X \in \mathcal{T}$. (⟨2⟩1, ⟨2⟩2).
- ⟨1⟩2. If \mathcal{B} is a basis for a topology then it satisfies condition 2
- ⟨2⟩1. ASSUME: \mathcal{B} is a basis for the topology \mathcal{T}
- ⟨2⟩2. LET: $B_1, B_2 \in \mathcal{B}$
- ⟨2⟩3. $B_1, B_2 \in \mathcal{T}$
 PROOF: From the definition of basis (⟨2⟩1, ⟨2⟩2).
- ⟨2⟩4. $B_1 \cap B_2 \in \mathcal{T}$
 PROOF: By the definition of topology, the open sets in \mathcal{T} are closed under binary intersection (⟨2⟩1, ⟨2⟩3)
- ⟨2⟩5. For all $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$
 PROOF: From the definition of basis (⟨2⟩1, ⟨2⟩4)
- ⟨1⟩3. If \mathcal{B} satisfies conditions 1 and 2 then $\mathcal{T} = \{U \subseteq X : \forall x \in U. \exists B \in \mathcal{B}. x \in B \subseteq U\}$ is a topology and \mathcal{B} is a basis for \mathcal{T} .
- ⟨2⟩1. ASSUME: \mathcal{B} satisfies conditions 1 and 2
- ⟨2⟩2. $X \in \mathcal{T}$
 PROOF: For all $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq X$ by condition 1 (⟨2⟩1).
- ⟨2⟩3. For all $\mathcal{A} \subseteq \mathcal{T}$, we have $\bigcup \mathcal{A} \in \mathcal{T}$
- ⟨3⟩1. LET: $\mathcal{A} \subseteq \mathcal{T}$
- ⟨3⟩2. LET: $x \in \bigcup \mathcal{A}$
- ⟨3⟩3. PICK $U \in \mathcal{A}$ such that $x \in U$
 PROOF: From ⟨3⟩2.
- ⟨3⟩4. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$
 PROOF: Since $U \in \mathcal{T}$, using the definition of \mathcal{T} (⟨3⟩1, ⟨3⟩3)
- ⟨3⟩5. $x \in B \subseteq \bigcup \mathcal{A}$
 PROOF: From ⟨3⟩3 and ⟨3⟩4.
- ⟨2⟩4. For all $U, V \in \mathcal{T}$, we have $U \cap V \in \mathcal{T}$
- ⟨3⟩1. LET: $U, V \in \mathcal{T}$
- ⟨3⟩2. LET: $x \in U \cap V$
- ⟨3⟩3. PICK $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \subseteq U$ and $x \in B_2 \subseteq V$
 PROOF: From ⟨3⟩1, ⟨3⟩2 and the definition of \mathcal{T} .
- ⟨3⟩4. PICK $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$
 PROOF: Using condition 2 (⟨2⟩1, ⟨3⟩3).
- ⟨3⟩5. $x \in B_3 \subseteq U \cap V$
 PROOF: From ⟨3⟩3 and ⟨3⟩4.
- ⟨2⟩5. $\bigcup \mathcal{B} = X$
 PROOF: This is condition 1 (⟨2⟩1).
- ⟨2⟩6. For all $U \in \mathcal{T}$ and $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$
 PROOF: Immediate from the definition of \mathcal{T} .
- ⟨1⟩4. \mathcal{T} is unique.
 PROOF: From Proposition 3.4.2.

□

Corollary 3.4.3.1. *Let X be a set and $\mathcal{B} \subseteq \mathcal{P}X$ be such that $\bigcup \mathcal{B} = X$ and \mathcal{B} is closed under binary intersection. Then \mathcal{B} is a basis for a unique topology on X .*

Lemma 3.4.4. *Let \mathcal{B} and \mathcal{B}' be bases for the topologies \mathcal{T} and \mathcal{T}' on X respectively. Then $\mathcal{T} \subseteq \mathcal{T}'$ if and only if, for all $B \in \mathcal{B}$ and $x \in B$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.*

PROOF:

$\langle 1 \rangle 1$. If $\mathcal{T} \subseteq \mathcal{T}'$ then, for all $B \in \mathcal{B}$ and $x \in B$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.

$\langle 2 \rangle 1$. ASSUME: $\mathcal{T} \subseteq \mathcal{T}'$

$\langle 2 \rangle 2$. LET: $B \in \mathcal{B}$ and $x \in B$

$\langle 2 \rangle 3$. $B \in \mathcal{T}$

PROOF: This holds because $\mathcal{B} \subseteq \mathcal{T}$ by the definition of basis. ($\langle 2 \rangle 2$)

$\langle 2 \rangle 4$. $B \in \mathcal{T}'$

PROOF: From $\langle 2 \rangle 1$ and $\langle 2 \rangle 3$.

$\langle 2 \rangle 5$. There exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.

$\langle 1 \rangle 2$. If, for all $B \in \mathcal{B}$ and $x \in B$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$, then $\mathcal{T} \subseteq \mathcal{T}'$.

$\langle 2 \rangle 1$. ASSUME: For all $B \in \mathcal{B}$ and $x \in B$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$

$\langle 2 \rangle 2$. LET: $U \in \mathcal{T}$

PROVE: $U \in \mathcal{T}'$

$\langle 2 \rangle 3$. LET: $x \in U$

$\langle 2 \rangle 4$. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$

PROOF: Since \mathcal{B} is a basis for \mathcal{T} ($\langle 2 \rangle 2$, $\langle 2 \rangle 3$).

$\langle 2 \rangle 5$. PICK $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$

PROOF: From $\langle 2 \rangle 1$ and $\langle 2 \rangle 4$.

$\langle 2 \rangle 6$. $x \in B' \subseteq U$

PROOF: From $\langle 2 \rangle 4$ and $\langle 2 \rangle 5$.

$\langle 2 \rangle 7$. Q.E.D.

PROOF: By Proposition 3.4.2.

□

Definition 3.4.5 (Lower Limit Topology). The *lower limit topology* on \mathbb{R} is the one generated by the set of all half-open intervals of the form $[a, b)$. We write \mathbb{R}_l for the topological space consisting of \mathbb{R} under this topology.

We prove this is a topology.

PROOF:

$\langle 1 \rangle 1$. LET: \mathcal{B} be the set of all half-open intervals of the form $[a, b)$.

$\langle 1 \rangle 2$. $\bigcup \mathcal{B} = \mathbb{R}$

PROOF: For all $x \in \mathbb{R}$, we have $x \in [x, x+1) \in \mathcal{B}$.

$\langle 1 \rangle 3$. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

PROOF: If $x \in [a, b) \cap [c, d)$ then $x \in [\max(a, c), \min(b, d)) \subseteq [a, b) \cap [c, d)$.

⟨1⟩4. Q.E.D.

PROOF: By Lemma 3.4.3.

□

Definition 3.4.6 (K -topology). The K -topology on \mathbb{R} is the one generated by the set of all open intervals (a, b) and all sets of the form $(a, b) \setminus K$, where $K = \{1/n : n \in \mathbb{Z}^+\}$. We write \mathbb{R}_K for the topological space consisting of \mathbb{R} under this topology.

We prove this is a topology.

PROOF:

⟨1⟩1. LET: $\mathcal{B} = \{(a, b) : a, b \in \mathbb{R}, a < b\} \cup \{(a, b) \setminus K : a, b \in \mathbb{R}, a < b\}$

⟨1⟩2. $\bigcup \mathcal{B} = \mathbb{R}$

PROOF: For all $x \in \mathbb{R}$, we have $x \in (x-1, x+1) \in \mathcal{B}$.

⟨1⟩3. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

⟨2⟩1. LET: $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$

PROVE: There exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$

⟨2⟩2. CASE: $B_1 = (a, b)$, $B_2 = (c, d)$

PROOF: Take $B_3 = (\max(a, c), \min(b, d))$

⟨2⟩3. CASE: $B_1 = (a, b)$, $B_2 = (c, d) \setminus K$

PROOF: Take $B_3 = (\max(a, c), \min(b, d)) \setminus K$

⟨2⟩4. CASE: $B_1 = (a, b) \setminus K$, $B_2 = (c, d)$

PROOF: Take $B_3 = (\max(a, c), \min(b, d)) \setminus K$

⟨2⟩5. CASE: $B_1 = (a, b) \setminus K$, $B_2 = (c, d) \setminus K$

PROOF: Take $B_3 = (\max(a, c), \min(b, d)) \setminus K$

⟨1⟩4. Q.E.D.

PROOF: By Lemma 3.4.3.

□

Lemma 3.4.7. The lower limit topology and the K -topology are incomparable.

PROOF: $[0, 1)$ is not open in the K -topology. $(-1, 1) \setminus K$ is not open in the lower limit topology, because there is no half-open interval $[a, b)$ such that $0 \in [a, b) \subseteq (-1, 1) \setminus K$. □

Proposition 3.4.8. The set of all singletons is a basis for any discrete space.

PROOF: Easy. □

Definition 3.4.9 (Line with Two Origins). The *line with two origins* is the set $\mathbb{R} \setminus \{0\} \cup \{p, q\}$ under the topology generated by the basis consisting of:

- all open intervals in \mathbb{R} that do not contain 0;
- all sets of the form $(-a, 0) \cup \{p\} \cup (0, a)$ where $a > 0$;
- all sets of the form $(-a, 0) \cup \{q\} \cup (0, a)$ where $a > 0$

3.5 Closed Sets

Definition 3.5.1 (Closed). Let X be a topological space and $A \subseteq X$. Then A is *closed* iff $X \setminus A$ is open.

Proposition 3.5.2. *In any topological space X , the empty set \emptyset is closed.*

PROOF: This holds because $X \setminus \emptyset = X$ is open. \square

Proposition 3.5.3. *In any topological space X , the set X is closed.*

PROOF: This holds because $X \setminus X = \emptyset$ is open. \square

Proposition 3.5.4. *The union of two closed sets is closed.*

PROOF: If C and D are closed then $X \setminus (C \cup D) = (X \setminus C) \cup (X \setminus D)$ is open. \square

Proposition 3.5.5. *In any topological space, the intersection of a nonempty set of closed sets is closed.*

PROOF: Let \mathcal{C} be a nonempty set of closed sets. Then $X \setminus \bigcap \mathcal{C} = \bigcup \{X \setminus C : C \in \mathcal{C}\}$ is open. \square

Proposition 3.5.6. *Let X be a topological space and $U \subseteq X$. Then U is open if and only if $X \setminus U$ is closed.*

PROOF: Immediate from definitions.

Theorem 3.5.7. *Let X be a set and $\mathcal{C} \subseteq \mathcal{P}X$. Suppose:*

1. $\emptyset, X \in \mathcal{C}$;
2. for all nonempty $\mathcal{A} \subseteq \mathcal{C}$, we have $\bigcap \mathcal{A} \in \mathcal{C}$;
3. for all $C, D \in \mathcal{C}$, we have $C \cup D \in \mathcal{C}$.

Then there exists a unique topology on X under which \mathcal{C} is the set of all closed sets, namely

$$\mathcal{T} = \{U \subseteq X : X \setminus U \in \mathcal{C}\}$$

PROOF:

$\langle 1 \rangle 1$. LET: \mathcal{C} be a set satisfying 1–3

$\langle 1 \rangle 2$. LET: $\mathcal{T} = \{X \setminus C : C \in \mathcal{C}\}$

$\langle 1 \rangle 3$. \mathcal{T} is a topology

$\langle 2 \rangle 1$. $X \in \mathcal{T}$

PROOF: $X \setminus X = \emptyset \in \mathcal{C}$ by condition 1.

$\langle 2 \rangle 2$. For all $\mathcal{A} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{A} \in \mathcal{T}$.

$\langle 3 \rangle 1$. LET: $\mathcal{A} \subseteq \mathcal{T}$

$\langle 3 \rangle 2$. CASE: $\mathcal{A} = \emptyset$

PROOF: In this case, $X \setminus \bigcup \mathcal{A} = X \in \mathcal{C}$ by condition 1.

$\langle 3 \rangle 3$. CASE: \mathcal{A} is nonempty

PROOF: In this case, we have $X \setminus \bigcup \mathcal{A} = \bigcap \{X \setminus U : U \in \mathcal{A}\} \in \mathcal{C}$ by condition 2.

(2)3. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$

PROOF: $X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V) \in \mathcal{C}$ by condition 3.

(1)4. \mathcal{C} is the set of closed sets.

PROOF:

$$\begin{aligned} C \text{ is closed} &\Leftrightarrow X \setminus C \in \mathcal{T} \\ &\Leftrightarrow X \setminus (X \setminus C) \in \mathcal{C} \\ &\Leftrightarrow C \in \mathcal{C} \end{aligned}$$

(1)5. \mathcal{T} is unique.

PROOF: By Proposition 3.5.6.

□

Definition 3.5.8 (Closed Covering). A *closed covering* of a topological space is a covering in which every member is a closed set.

3.6 Locally Finite Families

Definition 3.6.1 (Locally Finite). Let X be a topological space and $\{A_i\}_{i \in I}$ a family of subsets of X . Then $\{A_i\}_{i \in I}$ is *locally finite* iff, for all $x \in X$, there exists a neighbourhood N of x such that there are only finitely many $i \in I$ such that N intersects A_i .

Proposition 3.6.2. If $\{A_i\}_{i \in I}$ is locally finite and $B_i \subseteq A_i$ for all i then $\{B_i\}_{i \in I}$ is locally finite.

PROOF: Immediate from definitions. □

Proposition 3.6.3. Every finite family of open sets is locally finite.

PROOF: Trivial. □

3.7 Closure of a Set

Definition 3.7.1 (Closure). Let X be a topological space and $A \subseteq X$. The *closure* of A , $\text{Cl } A$ or \overline{A} , is the intersection of all closed sets that include A .

PROOF: This intersection always exists because X is a closed set that includes A . □

Proposition 3.7.2. Let X be a topological space and $A \subseteq X$. Then $A \subseteq \overline{A}$.

PROOF: Immediate from definitions. □

Proposition 3.7.3. Let X be a topological space and $A \subseteq X$. Then \overline{A} is closed.

PROOF: This follows from Proposition 3.5.5. □

Proposition 3.7.4. *Let X be a topological space and $A, C \subseteq X$. If $A \subseteq C$ and C is closed then $\overline{A} \subseteq C$.*

PROOF: Immediate from definitions. \square

Proposition 3.7.5. *Let X be a topological space and $A, B \subseteq X$. If $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$.*

PROOF:

$\langle 1 \rangle 1.$ ASSUME: $A \subseteq B$

$\langle 1 \rangle 2.$ $A \subseteq \overline{B}$

PROOF: Proposition 3.7.2.

$\langle 1 \rangle 3.$ $\overline{A} \subseteq \overline{B}$

PROOF: Propositions 3.7.3, 3.7.4.

\square

Proposition 3.7.6. *Let X be a set and $A \subseteq X$. Then A is closed if and only if $A = \overline{A}$.*

PROOF:

$\langle 1 \rangle 1.$ If A is closed then $A = \overline{A}$

$\langle 2 \rangle 1.$ ASSUME: A is closed

$\langle 2 \rangle 2.$ $A \subseteq \overline{A}$

PROOF: By Proposition 3.7.2.

$\langle 2 \rangle 3.$ $\overline{A} \subseteq A$

PROOF: By Proposition 3.7.4 since $A \subseteq A$.

$\langle 1 \rangle 2.$ If $A = \overline{A}$ then A is closed.

PROOF: By Proposition 3.7.3.

\square

Corollary 3.7.6.1.

$$\overline{\emptyset} = \emptyset$$

Theorem 3.7.7 (Kuratowski Closure Axioms). *Let X be a set and $(-) : \mathcal{P}X \rightarrow \mathcal{P}X$ be a function such that:*

1. $\overline{\emptyset} = \emptyset$
2. For all $A \subseteq X$, $A \subseteq \overline{A}$
3. For all $A \subseteq X$, $\overline{A} = \overline{\overline{A}}$
4. For all $A, B \subseteq X$, $\overline{A \cup B} = \overline{A} \cup \overline{B}$

Then there exists a unique topology \mathcal{T} on X such that \overline{A} is the closure of A for all $A \in \mathcal{P}X$.

PROOF:

$\langle 1 \rangle 1.$ For all $C, D \subseteq X$, if $C \subseteq D$ then $\overline{C} \subseteq \overline{D}$

$\langle 2 \rangle 1.$ ASSUME: $C \subseteq D$

⟨2⟩2. $\overline{C} = \overline{D}$

PROOF:

$$\begin{aligned}\overline{D} &= \overline{C \cup D} && (\langle 2 \rangle 1) \\ &= \overline{C} \cup \overline{D} && (\text{axiom 4})\end{aligned}$$

⟨1⟩2. LET: \mathcal{T} be the topology in which a set C is closed iff $\overline{C} = C$.

⟨2⟩1. $\overline{\emptyset} = \emptyset$

PROOF: This is axiom 1.

⟨2⟩2. $\overline{X} = X$

PROOF: By axiom 2.

⟨2⟩3. For any set \mathcal{A} of sets C such that $\overline{C} = C$, we have $\overline{\bigcap \mathcal{A}} = \bigcap \mathcal{A}$

⟨3⟩1. $\bigcap \mathcal{A} \subseteq \bigcap \mathcal{A}$

⟨4⟩1. LET: $C \in \mathcal{A}$

⟨4⟩2. $\bigcap \mathcal{A} \subseteq C$

PROOF:

$$\begin{aligned}\overline{\bigcap \mathcal{A}} &\subseteq \overline{C} && (\langle 1 \rangle 1) \\ &= C && (\langle 4 \rangle 1)\end{aligned}$$

⟨3⟩2. Q.E.D.

⟨2⟩4. If $\overline{C} = C$ and $\overline{D} = D$ then $\overline{C \cup D} = C \cup D$

PROOF: By axiom 4.

⟨2⟩5. Q.E.D.

PROOF: By Theorem 3.5.7.

⟨1⟩3. For all $A \subseteq X$, the closure of A in \mathcal{T} is \overline{A}

⟨2⟩1. \overline{A} is closed

PROOF: From axiom 3.

⟨2⟩2. If $A \subseteq C$ and C is closed then $\overline{A} \subseteq C$

PROOF:

$$\begin{aligned}C &= \overline{C} && (C \text{ is closed}) \\ &= \overline{A \cup C} && (A \subseteq C) \\ &= \overline{A} \cup \overline{C} && (\text{axiom 4})\end{aligned}$$

□

Theorem 3.7.8. *Let A be a subset of the topological space X and \mathcal{B} a basis for X . Then $x \in \overline{A}$ if and only if, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A .*

PROOF:

⟨1⟩1. If $x \in \overline{A}$ then, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A .

PROOF: Immediate from Theorem 3.8.3.

⟨1⟩2. If, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A , then $x \in \overline{A}$.

⟨2⟩1. ASSUME: for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A .

⟨2⟩2. LET: U be a neighbourhood of x

⟨2⟩3. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$

PROOF: \mathcal{B} is a basis.

⟨2⟩4. B intersects A .

PROOF: By ⟨2⟩1.

$\langle 2 \rangle 5$. U intersects A .

$\langle 2 \rangle 6$. Q.E.D.

PROOF: By Theorem 3.8.3.

□

3.8 Interior of a Set

Definition 3.8.1 (Interior). Let X be a topological space and $A \subseteq X$. The *interior* of A , $\text{Int } A$, is the union of all open sets included in A .

Lemma 3.8.2. If $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$.

PROOF: \overline{B} is a closed set that includes B , hence includes A . □

Theorem 3.8.3. Let A be a subset of the topological space X and $x \in X$. Then $x \in \overline{A}$ if and only if every neighbourhood of x intersects A .

PROOF:

$$\begin{aligned} x \notin \overline{A} &\Leftrightarrow \exists C \text{ closed } (A \subseteq C \wedge x \notin C) \\ &\Leftrightarrow \exists U \text{ open } (A \subseteq X \setminus U \wedge x \in U) \\ &\Leftrightarrow \exists U \text{ open } (A \text{ intersects } U \wedge x \in U) \end{aligned} \quad \square$$

Lemma 3.8.4.

$$X \setminus \text{Int } A = \overline{X \setminus A}$$

PROOF:

$$\begin{aligned} \langle 1 \rangle 1. & X \setminus \text{Int } A \subseteq \overline{X \setminus A} \\ \langle 2 \rangle 1. & X \setminus A \subseteq \overline{X \setminus A} \\ \langle 2 \rangle 2. & X \setminus \overline{X \setminus A} \subseteq A \\ \langle 2 \rangle 3. & X \setminus \overline{X \setminus A} \subseteq \text{Int } A \\ \langle 1 \rangle 2. & \overline{X \setminus A} \subseteq X \setminus \text{Int } A \\ \langle 2 \rangle 1. & \text{Int } A \subseteq A \\ \langle 2 \rangle 2. & X \setminus A \subseteq X \setminus \text{Int } A \\ \langle 2 \rangle 3. & \overline{X \setminus A} \subseteq X \setminus \text{Int } A \end{aligned}$$

□

3.9 Boundary

Definition 3.9.1 (Boundary). Let X be a topological space and $A \subseteq X$. The *boundary* of A , $\text{Bd } A$, is $\overline{A} \cap \overline{X \setminus A}$.

Lemma 3.9.2.

$$\text{Bd } A = \overline{A} \setminus \text{Int } A$$

PROOF: From Lemma 3.8.4. □

Lemma 3.9.3. $\overline{A} = \text{Int } A \cup \text{Bd } A$

PROOF:

$$\begin{aligned}
 \text{Int } A \cup \text{Bd } A &= \text{Int } A \cup (\overline{A} \cap (X \setminus \text{Int } A)) \\
 &= \text{Int } A \cup \overline{A} \\
 &= \overline{A}
 \end{aligned}
 \quad \square$$

Corollary 3.9.3.1. $\text{Bd } A = \emptyset$ iff A is open and closed.

Lemma 3.9.4. For any set U , the following are equivalent:

1. U is open.
2. $\text{Bd } U \cap U = \emptyset$
3. $\text{Bd } U = \overline{U} \setminus U$

PROOF:

$\langle 1 \rangle 1. 1 \Rightarrow 3$

PROOF: From Lemma 3.9.2.

$\langle 1 \rangle 2. 3 \Rightarrow 2$

PROOF: Set theory.

$\langle 1 \rangle 3. 2 \Rightarrow 1$

PROOF:

$$\begin{aligned}
 U &\subseteq \overline{U} \\
 &= \text{Int } U \cup \text{Bd } U \quad (\text{Lemma 3.9.3}) \\
 \therefore U &\subseteq \text{Int } U
 \end{aligned}$$

\square

3.10 Limit Points

Definition 3.10.1 (Limit Point). Let X be a topological space, $A \subseteq X$, and $x \in X$. Then x is a *limit point*, *cluster point* or *point of accumulation* of A iff every neighbourhood of x intersects A in a point other than x .

Lemma 3.10.2. If $A \subseteq B$ then every limit point of A is a limit point of B .

PROOF: Immediate from the definition. \square

Theorem 3.10.3. Let A be a subset of the topological space X . Let A' be the set of all limit points of A . Then $\overline{A} = A \cup A'$.

PROOF:

$\langle 1 \rangle 1.$ If $x \in \overline{A}$ and $x \notin A$ then $x \in A'$

PROOF: in this case, every neighbourhood of x intersects A in a point other than x .

$\langle 1 \rangle 2. A \subseteq \overline{A}$

PROOF: From the definition of \overline{A} .

$\langle 1 \rangle 3. A' \subseteq \overline{A}$

PROOF: By Theorem 3.8.3.

\square

Corollary 3.10.3.1. A set is closed if and only if it contains all its limit points.

3.11 Subbases

Definition 3.11.1 (Subbasis). Let X be a topological space. A *subbasis* for the topology on X is a set $\mathcal{S} \subseteq \mathcal{P}X$ such that, for every open set U and $x \in U$, there exist $S_1, \dots, S_n \in \mathcal{S}$ such that $x \in S_1 \cap \dots \cap S_n \subseteq U$. We say the topology is *generated* by \mathcal{S} .

Lemma 3.11.2. Let \mathcal{T} be a topology on X and $\mathcal{S} \subseteq \mathcal{P}X$. Then the following are equivalent:

1. \mathcal{S} is a subbasis for \mathcal{T} .
2. The set of all finite intersections of members of \mathcal{S} is a basis for \mathcal{T}
3. \mathcal{T} is the set of all unions of finite intersections of members of \mathcal{S} .

PROOF: $1 \Leftrightarrow 2$ holds immediately from the definitions. $2 \Leftrightarrow 3$ holds by Proposition 3.4.2. \square

Corollary 3.11.2.1. If \mathcal{S} is a subbasis for the topology \mathcal{T} , then \mathcal{T} is the coarsest topology in which every element of \mathcal{S} is open.

Lemma 3.11.3. Let X be a set and $\mathcal{S} \subseteq \mathcal{P}X$. Then \mathcal{S} is a subbasis for a topology on X if and only if $\bigcup \mathcal{S} = X$.

PROOF:

$\langle 1 \rangle 1$. If \mathcal{S} is a subbasis for a topology on X then $\bigcup \mathcal{S} = X$

$\langle 2 \rangle 1$. ASSUME: \mathcal{S} is a subbasis for a topology \mathcal{T} on X .

$\langle 2 \rangle 2$. LET: $x \in X$

$\langle 2 \rangle 3$. PICK $S_1, \dots, S_n \in \mathcal{S}$ such that $x \in S_1 \cap \dots \cap S_n \subseteq X$

PROOF: From the definition of subbasis ($\langle 2 \rangle 1$, $\langle 2 \rangle 2$).

$\langle 2 \rangle 4$. $x \in \bigcup \mathcal{S}$

PROOF: Immediate from $\langle 2 \rangle 3$.

$\langle 1 \rangle 2$. If $\bigcup \mathcal{S} = X$ then \mathcal{S} is a subbasis for a topology on X

$\langle 2 \rangle 1$. ASSUME: $\bigcup \mathcal{S} = X$

PROVE: The set of all finite intersections of elements of \mathcal{S} is a basis for a topology on X .

$\langle 2 \rangle 2$. LET: \mathcal{B} be the set of all finite intersections of elements of \mathcal{S} .

$\langle 2 \rangle 3$. $\bigcup \mathcal{B} = X$

PROOF: From $\langle 2 \rangle 1$ and $\langle 2 \rangle 2$.

$\langle 2 \rangle 4$. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that

$x \in B_3 \subseteq B_1 \cap B_2$

PROOF: Take $B_3 = B_1 \cap B_2$ ($\langle 2 \rangle 2$).

$\langle 2 \rangle 5$. \mathcal{B} is a basis for a topology on X .

PROOF: By Lemma 3.4.3.

$\langle 2 \rangle 6$. Q.E.D.

PROOF: By Lemma 3.11.2.

\square

3.12 Convergence

Definition 3.12.1 (Net). Let X be a topological space. A *net* $(x_\alpha)_{\alpha \in J}$ in X consists of a directed set J and a function $x : J \rightarrow X$.

Definition 3.12.2 (Convergence). Let $(x_\alpha)_{\alpha \in J}$ be a net in the topological space X , and $l \in X$. Then the net *converges* to l , $x_\alpha \rightarrow l$, if and only if, for every neighbourhood U of l , there exists $\alpha \in J$ such that, for all $\beta \geq \alpha$, we have $x_\beta \in U$.

Theorem 3.12.3 (AC). Let X be a topological space and $A \subseteq X$. Then $x \in \overline{A}$ if and only if there exists a net of points of A converging to x .

PROOF:

$\langle 1 \rangle 1$. If $x \in \overline{A}$ then there exists a net of points of A converging to x .

$\langle 2 \rangle 1$. LET: $x \in \overline{A}$

$\langle 2 \rangle 2$. LET: J be the poset of neighbourhoods of x under \supseteq .

$\langle 2 \rangle 3$. For $U \in J$ PICK a point $x_U \in U \cap A$

PROOF: By Theorem 3.8.3

$\langle 2 \rangle 4$. $(x_U)_{U \in J}$ is a net

PROOF: Given $U, V \in J$ we have $U \cap V \in J$ and $U \supseteq U \cup V, V \supseteq U \cup V$.

$\langle 2 \rangle 5$. $x_U \rightarrow x$

PROOF: For any neighbourhood U of x we have $U \in J$ and if $U \supseteq V$ then $x_V \in U$.

$\langle 1 \rangle 2$. If there exists a net of points of A converging to x then $x \in \overline{A}$.

$\langle 2 \rangle 1$. LET: $(x_\alpha)_{\alpha \in J}$ be a net of points in A that converges to x .

$\langle 2 \rangle 2$. LET: U be a neighbourhood of x

$\langle 2 \rangle 3$. PICK $\alpha \in J$ such that, for all $\beta \geq \alpha$, we have $x_\beta \in U$

$\langle 2 \rangle 4$. $x_\alpha \in U \cap A$

$\langle 2 \rangle 5$. Q.E.D.

PROOF: By Theorem 3.8.3

□

Theorem 3.12.4. Let X and Y be topological spaces and $f : X \rightarrow Y$. Then f is continuous if and only if, for every net $(x_\alpha)_{\alpha \in J}$ in X , if $x_\alpha \rightarrow x$ then $f(x_\alpha) \rightarrow f(x)$.

PROOF:

$\langle 1 \rangle 1$. If f is continuous and $x_\alpha \rightarrow x$ then $f(x_\alpha) \rightarrow f(x)$

$\langle 2 \rangle 1$. ASSUME: f is continuous.

$\langle 2 \rangle 2$. ASSUME: $x_\alpha \rightarrow x$

$\langle 2 \rangle 3$. LET: V be a neighbourhood of $f(x)$

$\langle 2 \rangle 4$. $f^{-1}(V)$ is a neighbourhood of x

$\langle 2 \rangle 5$. PICK α such that, for all $\beta \geq \alpha$, we have $x_\beta \in f^{-1}(V)$

$\langle 2 \rangle 6$. For all $\beta \geq \alpha$ we have $f(x_\beta) \in V$

$\langle 1 \rangle 2$. If, for every net (x_α) in X , if $x_\alpha \rightarrow x$ then $f(x_\alpha) \rightarrow f(x)$, then f is continuous.

$\langle 2 \rangle 1$. ASSUME: for every net (x_α) in X , if $x_\alpha \rightarrow x$ then $f(x_\alpha) \rightarrow f(x)$

- ⟨2⟩2. LET: $A \subseteq X$
PROVE: $\overline{f(A)} \subseteq f(\overline{A})$
- ⟨2⟩3. LET: $x \in \overline{A}$
- ⟨2⟩4. PICK a net (x_α) in A such that $x_\alpha \rightarrow x$
PROOF: Theorem 3.12.3
- ⟨2⟩5. $f(x_\alpha) \rightarrow f(x)$
PROOF: By ⟨2⟩1
- ⟨2⟩6. $f(x) \in \overline{f(A)}$
PROOF: Theorem 3.12.3
- ⟨2⟩7. Q.E.D.
PROOF: By Theorem 5.2.2.

□

Definition 3.12.5 (Subnet). Let $(x_\alpha)_{\alpha \in J}$ be a net in X . Let K be a directed set and $g : K \rightarrow J$ be a monotone function such that $g(K)$ is cofinal in J . Then the net $(x_{g(\beta)})_{\beta \in K}$ is called a *subnet* of (x_α) .

3.13 Accumulation Points

Definition 3.13.1 (Accumulation Point). Let X be a topological space, and $(x_\alpha)_{\alpha \in J}$ a net in X , and $a \in X$. Then a is an *accumulation point* of (x_α) iff, for every neighbourhood U of x , the set $\{\alpha \in J : x_\alpha \in U\}$ is cofinal in J .

Lemma 3.13.2. Let X be a topological space, $(x_\alpha)_{\alpha \in J}$ be a nonempty net in X and $a \in X$. Then a is an accumulation point of (x_α) if and only if there exists a subnet of (x_α) that converges to a .

PROOF:

- ⟨1⟩1. If a is an accumulation point of (x_α) then there exists a subnet of (x_α) that converges to a .
- ⟨2⟩1. ASSUME: a is an accumulation point of (x_α) .
- ⟨2⟩2. LET: K be the poset $\{(\alpha, U) : \alpha \in J, U \text{ is a neighbourhood of } a, x_\alpha \in U\}$ under: $(\alpha, U) \leq (\beta, V)$ iff $\alpha \leq \beta$ and $U \subseteq V$.
- ⟨2⟩3. $(x_\alpha)_{(\alpha, U) \in K}$ is a subnet of $(x_\alpha)_{\alpha \in J}$
 - ⟨3⟩1. K is directed.
 - ⟨4⟩1. LET: $(\alpha, U), (\beta, V) \in K$
 - ⟨4⟩2. PICK $\gamma \in J$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$.
 - ⟨4⟩3. PICK $\delta \in J$ such that $\gamma \leq \delta$ and $x_\delta \in U \cap V$
PROOF: By ⟨2⟩1.
 - ⟨4⟩4. $(\delta, U \cap V) \in K$ and $(\alpha, U) \leq (\delta, U \cap V), (\beta, V) \leq (\delta, U \cap V)$
 - ⟨3⟩2. If $(\alpha, U) \leq (\beta, V)$ then $\alpha \leq \beta$
PROOF: From ⟨2⟩2.
 - ⟨3⟩3. $\{\alpha : \exists U. (\alpha, U) \in K\}$ is cofinal in J
PROOF: For $\alpha \in J$ we have $(\alpha, X) \in K$, so in fact $\{\alpha : \exists U. (\alpha, U) \in K\} = J$.
- ⟨2⟩4. The subnet converges to a .

Chapter 4

Constructions of Topological Spaces

4.1 The Order Topology

Definition 4.1.1 (Order Topology). Let X be a linearly ordered set with more than one element. The *order topology* on X is the topology generated by the basis consisting of:

- all open intervals (a, b)
- all half-open intervals $(a, \top]$ where \top is the greatest element of X , if there is one;
- all half-open intervals $[\perp, a)$ where \perp is the least element of X , if there is one.

We prove this is a basis for a topology.

PROOF:

$\langle 1 \rangle 1$. LET: \mathcal{B} be the set of all sets of these three forms.

$\langle 1 \rangle 2$. $\bigcup \mathcal{B} = X$

$\langle 2 \rangle 1$. LET: $x \in X$

PROVE: There exists $B \in \mathcal{B}$ such that $x \in B$

$\langle 2 \rangle 2$. CASE: x is least in X

$\langle 3 \rangle 1$. PICK $a \in X$ such that $a > x$

PROOF: X has more than one element.

$\langle 3 \rangle 2$. $x \in [x, a) \in \mathcal{B}$

$\langle 2 \rangle 3$. CASE: x is greatest in X

$\langle 3 \rangle 1$. PICK $a \in X$ such that $a < x$

PROOF: X has more than one element.

$\langle 3 \rangle 2$. $x \in (a, x] \in \mathcal{B}$

$\langle 2 \rangle 4$. CASE: x is neither least nor greatest in X

Lemma 4.1.3. *The open rays form a subbasis for the order topology.*

- ⟨1⟩1. LET: X be a linearly ordered set with more than one element.
- ⟨1⟩2. The open rays form a subbasis for a topology.
 - ⟨2⟩1. LET: $x \in X$
 - PROVE: x is an element of an open ray.
 - ⟨2⟩2. CASE: x is greatest in X
 - ⟨3⟩1. PICK $a \in X$ such that $a < x$
 - PROOF: X has more than one element (⟨1⟩1).
 - ⟨3⟩2. $x \in (a, +\infty)$
 - ⟨2⟩3. CASE: x is not greatest in X
 - ⟨3⟩1. PICK $a \in X$ such that $x < a$
 - ⟨3⟩2. $x \in (-\infty, a)$
 - ⟨2⟩4. Q.E.D.
 - PROOF: By Lemma 3.11.2.
- ⟨1⟩3. LET: \mathcal{T}_o be the order topology and \mathcal{T}_S be the topology generated by the open rays.
 - ⟨1⟩4. $\mathcal{T}_o \subseteq \mathcal{T}_S$
 - ⟨2⟩1. Every open interval (a, b) is open in \mathcal{T}_S
 - PROOF: $(a, b) = (a, +\infty) \cap (-\infty, b)$.
 - ⟨2⟩2. If \top is greatest then $(a, \top]$ is open in \mathcal{T}_S
 - PROOF: $(a, \top] = (a, +\infty)$.
 - ⟨2⟩3. If \perp is least then $[\perp, b)$ is open in \mathcal{T}_S
 - PROOF: $[\perp, b) = [\perp, +\infty)$.
 - ⟨2⟩4. Q.E.D.
 - PROOF: By Corollary 3.4.2.1.
 - ⟨1⟩5. $\mathcal{T}_S \subseteq \mathcal{T}_o$
 - ⟨2⟩1. For all $a \in X$, we have $(a, +\infty)$ is open in \mathcal{T}_o
 - ⟨3⟩1. LET: $x \in (a, +\infty)$
 - PROVE: There exists a basis element B such that $x \in B \subseteq (a, +\infty)$
 - ⟨3⟩2. CASE: x is greatest
 - PROOF: Take $B = (a, x]$
 - ⟨3⟩3. CASE: x is not greatest
 - ⟨4⟩1. PICK $b > x$
 - ⟨4⟩2. $x \in (a, b) \subseteq (a, +\infty)$
 - ⟨2⟩2. For all $a \in X$, we have $(-\infty, a)$ is open in \mathcal{T}_o
 - PROOF: Similar.
 - ⟨2⟩3. Q.E.D.
 - PROOF: By Corollary 3.11.2.1.

□

Lemma 4.1.4. *In a linearly ordered set X under the order topology, the closed intervals and closed rays are closed.*

PROOF:

$$\begin{aligned} X \setminus [a, b] &= (-\infty, a) \cup (b, +\infty) \\ X \setminus (-\infty, a] &= (a, +\infty) \\ X \setminus [a, +\infty) &= (-\infty, a) \end{aligned} \quad \square$$

Definition 4.1.5 (Standard Topology on \mathbb{R}). The *standard topology* on \mathbb{R} is the order topology.

Lemma 4.1.6. *The standard topology is strictly coarser than the lower limit topology.*

PROOF:

- $\langle 1 \rangle 1$. The standard topology is coarser than the lower limit topology.
 - $\langle 2 \rangle 1$. For every open interval (a, b) and $x \in (a, b)$, there exists a half-open interval $[c, d)$ such that $x \in [c, d) \subseteq (a, b)$.

PROOF: Take $[c, d) = [x, b)$.
 - $\langle 2 \rangle 2$. Q.E.D.

PROOF: By Lemma 3.4.4.
- $\langle 1 \rangle 2$. There exists a set U open in the lower limit topology that is not open in the standard topology.

PROOF: Take $U = [0, 1)$.

\square

Lemma 4.1.7. *The standard topology is strictly coarser than the K -topology.*

PROOF:

- $\langle 1 \rangle 1$. The standard topology is coarser than the K -topology.

PROOF: Every open interval is open in the K -topology.
- $\langle 1 \rangle 2$. There exists a set U open in the K -topology that is not open in the standard topology.

PROOF: Take $U = (-1, 1) \setminus K$. Then $0 \in U$ but there is no open interval (a, b) such that $0 \in (a, b) \subseteq U$.

\square

Definition 4.1.8 (Ordered Square). The *ordered square* I_o^2 is the topological space $[0, 1]^2$ under the order topology induced by the lexicographic order.

Lemma 4.1.9. *Let L be a linear continuum with a greatest element. Then every non-empty closed set in L has a greatest element.*

PROOF:

- $\langle 1 \rangle 1$. LET: C be a non-empty closed set in L
- $\langle 1 \rangle 2$. LET: u be the supremum of C
- $\langle 1 \rangle 3$. $u \in C$
 - $\langle 2 \rangle 1$. ASSUME: w.l.o.g u is not least in L

PROOF: If u is least then $C = \{u\}$.
 - $\langle 2 \rangle 2$. LET: U be any open neighbourhood of u
 - $\langle 2 \rangle 3$. PICK $v < u$ such that $(v, u] \subseteq U$

PROOF: By Lemma 4.1.2.

(2)4. PICK $x \in C$ such that $v < x$

PROOF: v is not an upper bound for C ($\langle 1 \rangle 2$).

(2)5. U intersects C in v

(2)6. Q.E.D.

PROOF: By Theorem 3.8.3.

□

Definition 4.1.10 (Long Line). The *long line* is $(S_\Omega \times [0, 1)) \setminus \{(0, 0)\}$ under the dictionary order, where S_Ω is the first uncountable ordinal under the order topology.

4.2 The Product Topology

Definition 4.2.1 (Product Topology). Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces. The *product topology* on $\prod_{\alpha \in J} X_\alpha$ is the topology generated by the subbasis consisting of all sets of the form $\pi_\alpha^{-1}(U)$ where $\alpha \in J$ and U is open in X_α . The *product space* of $\{X_\alpha\}_{\alpha \in J}$ is $\prod_{\alpha \in J} X_\alpha$ under the product topology.

Lemma 4.2.2. Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces and A_α be closed in X_α for all α . Then $\prod_{\alpha \in J} A_\alpha$ is closed in $\prod_{\alpha \in J} X_\alpha$.

PROOF: This holds because $\prod_{\alpha \in J} X_\alpha \setminus \prod_{\alpha \in J} A_\alpha = \bigcup_{\alpha \in J} \pi_\alpha^{-1}(X_\alpha \setminus A_\alpha)$. □

Theorem 4.2.3. Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces. The set of all sets of the form $\prod_{\alpha \in J} U_\alpha$ where each U_α is open in X_α , and $U_\alpha = X_\alpha$ for all but finitely many α , is a basis for the product topology on $\prod_{\alpha \in J} X_\alpha$.

PROOF: By Lemma 3.11.2. □

Theorem 4.2.4. Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces, and let \mathcal{B}_α be a basis for the topology on X_α for each α . Then

$$\mathcal{B} = \left\{ \prod_{\alpha \in J} U_\alpha : \text{for finitely many } \alpha \in J, U_\alpha \in \mathcal{B}_\alpha, \right. \\ \left. \text{and } U_\alpha = X_\alpha \text{ for all other values of } \alpha \right\}$$

is a basis for the product topology on $\prod_{\alpha \in J} X_\alpha$.

PROOF:

(1)1. Every member of \mathcal{B} is open in the product topology.

PROOF: Immediate from definitions.

(1)2. For every open set U and $\{x_\alpha\}_{\alpha \in J} \in U$, there exists $B \in \mathcal{B}$ such that $\{x_\alpha\}_{\alpha \in J} \in B \subseteq U$.

(2)1. LET: U be open and $\{x_\alpha\}_{\alpha \in J} \in U$

(2)2. PICK U_α open in X_α for each α such that $\{x_\alpha\}_{\alpha \in J} \in \prod_{\alpha \in J} U_\alpha \subseteq U$ and $U_\alpha = X_\alpha$ for all α except $\alpha_1, \dots, \alpha_n$.

PROOF: By Theorem 4.2.3.

- (2)3. PICK $B_{\alpha_i} \in \mathcal{B}_{\alpha_i}$ such that $x_\alpha \in B_{\alpha_i} \subseteq U_{\alpha_i}$ for $i = 1, \dots, n$
 (2)4. $\{x_\alpha\}_{\alpha \in J} \in \prod_{\alpha \in J} V_\alpha \subseteq U$ where $V_{\alpha_i} = B_{\alpha_i}$ for $i = 1, \dots, n$, and $V_\alpha = X_\alpha$ for all other α .

□

Theorem 4.2.5 (AC). *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces and $A_\alpha \subseteq X_\alpha$ for all α . If $\prod_{\alpha \in J} X_\alpha$ is given the product topology, then*

$$\prod_{\alpha \in J} \overline{A_\alpha} = \overline{\prod_{\alpha \in J} A_\alpha}.$$

PROOF:

- (1)1. $\prod_{\alpha \in J} \overline{A_\alpha} \subseteq \overline{\prod_{\alpha \in J} A_\alpha}$
 (2)1. LET: $\{x_\alpha\}_{\alpha \in J} \in \prod_{\alpha \in J} \overline{A_\alpha}$
 (2)2. LET: $\prod_{\alpha \in J} U_\alpha$ be a basic neighbourhood of $\{x_\alpha\}_{\alpha \in J}$, where each U_α is open in X_α , and $U_\alpha = X_\alpha$ except for $\alpha = \alpha_1, \dots, \alpha_n$.
 (2)3. For $\alpha \in J$, PICK $a_\alpha \in A_\alpha \cap U_\alpha$.
 PROOF: By Theorem 3.8.3, using the Axiom of Choice.
 (2)4. $\{a_\alpha\}_{\alpha \in J} \in \prod_{\alpha \in J} A_\alpha \cap \prod_{\alpha \in J} U_\alpha$
 (2)5. Q.E.D.

PROOF: By Theorem 3.8.3.

- (1)2. $\overline{\prod_{\alpha \in J} A_\alpha} \subseteq \prod_{\alpha \in J} \overline{A_\alpha}$
 (2)1. LET: $\{x_\alpha\}_{\alpha \in J} \in \overline{\prod_{\alpha \in J} A_\alpha}$
 (2)2. LET: $\alpha \in J$
 PROVE: $x_\alpha \in \overline{A_\alpha}$
 (2)3. LET: U be a neighbourhood of x_α in X_α
 (2)4. $\pi_\alpha^{-1}(U)$ is a neighbourhood of $\{x_\alpha\}_{\alpha \in J}$
 (2)5. PICK $\{a_\alpha\}_{\alpha \in J} \in \pi_\alpha^{-1}(U) \cap \prod_{\alpha \in J} A_\alpha$
 PROOF: By Theorem 3.8.3.
 (2)6. $a_\alpha \in U \cap A_\alpha$
 (2)7. Q.E.D.

PROOF: By Theorem 3.8.3.

□

Definition 4.2.6 (Standard Topology on \mathbb{R}^J). For J a set, the *standard topology* on \mathbb{R}^J is the product topology where \mathbb{R} is given the standard topology.

Definition 4.2.7 (Closed Unit Ball). The *closed unit ball* B^2 is $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ as a subset of \mathbb{R}^2 .

Definition 4.2.8 (Sorgenfrey Plane). The *Sorgenfrey plane* is \mathbb{R}_l^2 .

4.3 The Subspace Topology

Definition 4.3.1 (Subspace Topology). Let X be a topological space and $Y \subseteq X$. The *subspace topology* on Y is $\{Y \cap U : U \text{ open in } X\}$. With this topology, Y is a *subspace* of X .

We prove this is a topology.

PROOF:

$\langle 1 \rangle 1$. LET: $\mathcal{T} = \{Y \cap U : U \text{ open in } X\}$

$\langle 1 \rangle 2$. $Y \in \mathcal{T}$

PROOF: $Y = Y \cap X$

$\langle 1 \rangle 3$. \mathcal{T} is closed under union.

$\langle 2 \rangle 1$. LET: $\mathcal{A} \subseteq \mathcal{T}$

PROVE: $\bigcup \mathcal{A} = Y \cap \bigcup \{U \text{ open in } X : Y \cap U \in \mathcal{A}\}$

$\langle 2 \rangle 2$. $\bigcup \mathcal{A} \subseteq Y \cap \bigcup \{U \text{ open in } X : Y \cap U \in \mathcal{A}\}$

$\langle 3 \rangle 1$. LET: $x \in \bigcup \mathcal{A}$

$\langle 3 \rangle 2$. PICK $V \in \mathcal{A}$ such that $x \in V$

$\langle 3 \rangle 3$. PICK U open in X such that $V = Y \cap U$

PROOF: By the definition of \mathcal{T} ($\langle 1 \rangle 1$, $\langle 2 \rangle 1$, $\langle 3 \rangle 2$)

$\langle 3 \rangle 4$. $x \in Y \cap \bigcup \{U \text{ open in } X : Y \cap U \in \mathcal{A}\}$

$\langle 2 \rangle 3$. $Y \cap \bigcup \{U \text{ open in } X : Y \cap U \in \mathcal{A}\} \subseteq \bigcup \mathcal{A}$

PROOF: Set theory.

$\langle 1 \rangle 4$. \mathcal{T} is closed under binary intersection.

PROOF: This holds because $(Y \cap U) \cap (Y \cap V) = Y \cap (U \cap V)$.

□

Lemma 4.3.2. *Let X be a topological space, $Y \subseteq X$, and $A \subseteq Y$. Then the topology A inherits as a subspace of X is the same as the topology A inherits as a subspace of Y .*

PROOF:

$$\begin{aligned} & \text{topology as a subspace of } Y \\ &= \{V \cap A : V \text{ open in } Y\} \\ &= \{V \cap A : \exists U \text{ open in } X. V = U \cap Y\} \\ &= \{U \cap Y \cap A : U \text{ open in } X\} \\ &= \{U \cap A : U \text{ open in } X\} \\ &= \text{topology as a subspace of } X \square \end{aligned}$$

Lemma 4.3.3. *Let Y be a subspace of X . If U is open in Y and Y is open in X then U is open in X .*

PROOF:

$\langle 1 \rangle 1$. PICK V open in X such that $U = Y \cap V$

$\langle 1 \rangle 2$. U is open in X

PROOF: The open sets in X are closed under binary intersection.

□

Theorem 4.3.4. *Let Y be a subspace of X . Let $A \subseteq Y$. Let \bar{A} be the closure of A in X . Then the closure of A in Y is $\bar{A} \cap Y$.*

PROOF:

$\langle 1 \rangle 1$. $\bar{A} \cap Y$ is a closed set in Y that includes A .

$\langle 2 \rangle 1$. $\bar{A} \cap Y$ is closed in Y .

PROOF: By Lemma 4.3.4.1.

- (2)2. $A \subseteq \overline{A} \cap Y$.
- (1)2. If C is any closed set in Y that includes A then $\overline{A} \cap Y \subseteq C$.
- (2)1. LET: C be a closed set in Y that includes A .
- (2)2. PICK D closed in X such that $C = D \cap Y$.
- PROOF: By Lemma 4.3.4.1.
- (2)3. $\overline{A} \subseteq D$
- (2)4. $\overline{A} \subseteq C$

□

Corollary 4.3.4.1. *Let Y be a subspace of X . Then a set $A \subseteq Y$ is closed in Y if and only if it is the intersection of a closed set in X with Y .*

Corollary 4.3.4.2. *Let Y be a subspace of X . If A is closed in Y and Y is closed in X then A is closed in X .*

Lemma 4.3.5. *Let X be a topological space and $Y \subseteq X$. If \mathcal{B} is a basis for the topology on X then $\{B \cap Y : B \in \mathcal{B}\}$ is a basis for the subspace topology on Y .*

PROOF:

- (1)1. For all $B \in \mathcal{B}$, we have $B \cap Y$ is open in Y .
- PROOF: Immediate from definitions.
- (1)2. For every V open in Y and $y \in V$, there exists $B \in \mathcal{B}$ such that $y \in B \cap Y \subseteq V$.
- (2)1. LET: V be open in Y and $y \in V$
- (2)2. PICK U open in X such that $V = Y \cap U$
- (2)3. PICK $B \in \mathcal{B}$ such that $y \in B \subseteq U$
- (2)4. $y \in B \cap Y \subseteq V$

□

Lemma 4.3.6. *Let X be a topological space and $Y \subseteq X$. If \mathcal{S} is a subbasis for the topology on X then $\{S \cap Y : S \in \mathcal{S}\}$ is a subbasis for the subspace topology on Y .*

PROOF:

- (1)1. For all $S \in \mathcal{S}$, we have $S \cap Y$ is open in Y .
- PROOF: Immediate from definitions.
- (1)2. For every V open in Y and $y \in V$, there exist $S_1, \dots, S_n \in \mathcal{S}$ such that $y \in (S_1 \cap Y) \cap \dots \cap (S_n \cap Y) \subseteq V$
- (2)1. LET: V be open in Y and $y \in V$
- (2)2. PICK U open in X such that $V = U \cap Y$
- (2)3. PICK $S_1, \dots, S_n \in \mathcal{S}$ such that $y \in S_1 \cap \dots \cap S_n \subseteq U$
- (2)4. $y \in (S_1 \cap Y) \cap \dots \cap (S_n \cap Y) \subseteq V$

□

Theorem 4.3.7. *Let X be a linearly ordered set in the order topology. Let $Y \subseteq X$ be convex. Then the order topology on Y is the same as the subspace topology.*

PROOF:

- (1)1. LET: \mathcal{T}_o be the order topology and \mathcal{T}_s be the subspace topology.
 (1)2. $\mathcal{T}_o \subseteq \mathcal{T}_s$
 (2)1. For all $a \in Y$, we have $\{y \in Y : a < y\} \in \mathcal{T}_s$
 PROOF: $\{y \in Y : a < y\} = \{x \in X : a < x\} \cap Y$
 (2)2. For all $a \in Y$, we have $\{y \in Y : y < a\} \in \mathcal{T}_s$
 PROOF: Similar.
 (2)3. Q.E.D.
 PROOF: Lemma 4.1.3 and Corollary 3.11.2.1.
 (1)3. $\mathcal{T}_s \subseteq \mathcal{T}_o$
 (2)1. The sets $(a, +\infty) \cap Y$ and $(-\infty, a) \cap Y$ for $a \in X$ form a subbasis for \mathcal{T}_s
 PROOF: Lemma 4.3.6, Lemma 4.1.3.
 (2)2. For all $a \in X$, we have $(a, +\infty) \cap Y \in \mathcal{T}_o$
 (3)1. LET: $a \in X$
 (3)2. CASE: $a \in Y$
 PROOF: In this case, $(a, +\infty) \cap Y$ is an open ray in Y .
 (3)3. CASE: For all $y \in Y$ we have $a < y$
 PROOF: In this case, $(a, +\infty) \cap Y = Y$.
 (3)4. CASE: For all $y \in Y$ we have $y < a$
 PROOF: In this case, $(a, +\infty) \cap Y = \emptyset$.
 (3)5. Q.E.D.
 PROOF: These are the only cases because Y is convex.
 (2)3. For all $a \in X$, we have $(-\infty, a) \cap Y \in \mathcal{T}_o$
 PROOF: Similar.
 (2)4. Q.E.D.
 PROOF: Corollary 3.11.2.1.

□

Theorem 4.3.8. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces, and let A_α be a subspace of X_α for all α . Then the product topology on $\prod_{\alpha \in J} A_\alpha$ is the same as the topology it inherits as a subspace of $\prod_{\alpha \in J} X_\alpha$.*

PROOF: Each is the topology generated by the subbasis consisting of $\pi_\alpha^{-1}(U) \cap \prod_{\alpha \in J} A_\alpha = \pi_\alpha^{-1}(U \cap A_\alpha)$ where $\alpha \in J$ and U is open in X_α , using Lemma 4.3.6.

□

Definition 4.3.9 (Unit Circle). The *unit circle* S^1 is $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ as a subspace of \mathbb{R}^2 .

Proposition 4.3.10. *Let Y be a subspace of X , $A \subseteq Y$, and $a \in Y$. Then a is a limit point of A in the subspace topology on Y if and only if a is a limit point of A in the topology of X .*

PROOF:

$$\begin{aligned}
& a \text{ is a limit point of } A \text{ in } Y \\
& \Leftrightarrow \forall U \text{ open in } Y (a \in U \Rightarrow U \text{ intersects } A \text{ outside } a) \\
& \Leftrightarrow \forall V \text{ open in } X (a \in V \cap Y \Rightarrow V \cap Y \text{ intersects } A \text{ outside } a) \\
& \Leftrightarrow \forall V \text{ open in } X (a \in V \Rightarrow V \text{ intersects } A \text{ outside } a) \\
& \quad (a \in Y, A \subseteq Y) \\
& \Leftrightarrow a \text{ is a limit point of } A \text{ in } X
\end{aligned}$$

□

4.4 The Box Topology

Definition 4.4.1 (Box Topology). Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces. The *box topology* on $\prod_{\alpha \in J} X_\alpha$ is the topology generated by the basis consisting of all sets of the form $\prod_{\alpha \in J} U_\alpha$, where each U_α is open in X_α .

We prove this is a basis.

PROOF:

- ⟨1⟩1. LET: \mathcal{B} be the set of all sets of the form $\prod_{\alpha \in J} U_\alpha$, where each U_α is open in X_α .
- ⟨1⟩2. $\bigcup \mathcal{B} = \prod_{\alpha \in J} X_\alpha$
PROOF: This holds because $\prod_{\alpha \in J} X_\alpha \in \mathcal{B}$.
- ⟨1⟩3. \mathcal{B} is closed under binary intersection.
PROOF: $\prod_{\alpha \in J} U_\alpha \cap \prod_{\alpha \in J} V_\alpha = \prod_{\alpha \in J} (U_\alpha \cap V_\alpha)$.
- ⟨1⟩4. Q.E.D.
PROOF: Corollary 3.4.3.1.

Theorem 4.4.2 (AC). Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces, and let \mathcal{B}_α be a basis for the topology on X_α for each α . Then

$$\mathcal{B} = \left\{ \prod_{\alpha \in J} B_\alpha : \forall \alpha \in J, B_\alpha \in \mathcal{B}_\alpha \right\}$$

is a basis for the box topology on $\prod_{\alpha \in J} X_\alpha$.

PROOF:

- ⟨1⟩1. Every member of \mathcal{B} is open in the box topology.
PROOF: Immediate from definitions.
- ⟨1⟩2. For every open set U and $\{x_\alpha\}_{\alpha \in J} \in U$, there exists $B \in \mathcal{B}$ such that $\{x_\alpha\}_{\alpha \in J} \in B \subseteq U$.
 - ⟨2⟩1. LET: U be open and $\{x_\alpha\}_{\alpha \in J} \in U$
 - ⟨2⟩2. PICK U_α open in X_α for each α such that $\{x_\alpha\}_{\alpha \in J} \in \prod_{\alpha \in J} U_\alpha \subseteq U$.
 - ⟨2⟩3. PICK $B_\alpha \in \mathcal{B}_\alpha$ such that $x_\alpha \in B_\alpha \subseteq U_\alpha$ for each α
PROOF: Using the Axiom of Choice.
 - ⟨2⟩4. $\{x_\alpha\}_{\alpha \in J} \in \prod_{\alpha \in J} B_\alpha \subseteq U$

□

Theorem 4.4.3. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces, and let A_α be a subspace of X_α for all α . Let $\prod_{\alpha \in J} X_\alpha$ be given the box topology. Then the box topology on $\prod_{\alpha \in J} A_\alpha$ is the same as the topology it inherits as a subspace of $\prod_{\alpha \in J} X_\alpha$.*

PROOF: Each is the topology generated by the basis $\{\prod_{\alpha \in J} (U_\alpha \cap A_\alpha) : U_\alpha \text{ is open in } X_\alpha\}$, using Lemma 4.3.5. \square

Theorem 4.4.4. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of Hausdorff spaces. Then $\prod_{\alpha \in J} X_\alpha$ is Hausdorff under the box topology.*

PROOF:

- $\langle 1 \rangle 1$. LET: $\{x_\alpha\}_{\alpha \in J}, \{y_\alpha\}_{\alpha \in J} \in \prod_{\alpha \in J} X_\alpha$ with $\{x_\alpha\}_{\alpha \in J} \neq \{y_\alpha\}_{\alpha \in J}$
 - $\langle 1 \rangle 2$. PICK $\alpha \in J$ such that $x_\alpha \neq y_\alpha$
 - $\langle 1 \rangle 3$. PICK disjoint neighbourhoods U of x_α and V of y_α .
 - $\langle 1 \rangle 4$. $\pi_\alpha^{-1}(U)$ and $\pi_\alpha^{-1}(V)$ are disjoint neighbourhoods of $\{x_\alpha\}_{\alpha \in J}$ and $\{y_\alpha\}_{\alpha \in J}$
- \square

Corollary 4.4.4.1. *The space \mathbb{R}^ω under the box topology is Hausdorff.*

Theorem 4.4.5 (AC). *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces and $A_\alpha \subseteq X_\alpha$ for all α . If $\prod_{\alpha \in J} X_\alpha$ is given the box topology, then*

$$\prod_{\alpha \in J} \overline{A_\alpha} = \overline{\prod_{\alpha \in J} A_\alpha}.$$

PROOF:

- $\langle 1 \rangle 1$. $\prod_{\alpha \in J} \overline{A_\alpha} \subseteq \overline{\prod_{\alpha \in J} A_\alpha}$
 - $\langle 2 \rangle 1$. LET: $\{x_\alpha\}_{\alpha \in J} \in \prod_{\alpha \in J} \overline{A_\alpha}$
 - $\langle 2 \rangle 2$. LET: $\prod_{\alpha \in J} U_\alpha$ be a basic neighbourhood of $\{x_\alpha\}_{\alpha \in J}$, where each U_α is open in X_α .
 - $\langle 2 \rangle 3$. For $\alpha \in J$, PICK $a_\alpha \in A_\alpha \cap U_\alpha$.
- PROOF: By Theorem 3.8.3, using the Axiom of Choice.
- $\langle 2 \rangle 4$. $\{a_\alpha\}_{\alpha \in J} \in \prod_{\alpha \in J} A_\alpha \cap \prod_{\alpha \in J} U_\alpha$
 - $\langle 2 \rangle 5$. Q.E.D.

PROOF: By Theorem 3.8.3.

- $\langle 1 \rangle 2$. $\overline{\prod_{\alpha \in J} A_\alpha} \subseteq \prod_{\alpha \in J} \overline{A_\alpha}$
 - $\langle 2 \rangle 1$. LET: $\{x_\alpha\}_{\alpha \in J} \in \overline{\prod_{\alpha \in J} A_\alpha}$
 - $\langle 2 \rangle 2$. LET: $\alpha \in J$
- PROVE: $x_\alpha \in \overline{A_\alpha}$
- $\langle 2 \rangle 3$. LET: U be a neighbourhood of x_α in X_α
 - $\langle 2 \rangle 4$. $\pi_\alpha^{-1}(U)$ is a neighbourhood of $\{x_\alpha\}_{\alpha \in J}$
 - $\langle 2 \rangle 5$. PICK $\{a_\alpha\}_{\alpha \in J} \in \pi_\alpha^{-1}(U) \cap \prod_{\alpha \in J} A_\alpha$
- PROOF: By Theorem 3.8.3.
- $\langle 2 \rangle 6$. $a_\alpha \in U \cap A_\alpha$
 - $\langle 2 \rangle 7$. Q.E.D.

PROOF: By Theorem 3.8.3.

\square

4.5 The Quotient Topology

Definition 4.5.1 (Quotient Map). Let X and Y be topological spaces. Let $p : X \twoheadrightarrow Y$ be a surjective map. Then p is a *quotient map* iff, for all $U \subseteq Y$, we have U is open in Y iff $p^{-1}(U)$ is open in X .

Lemma 4.5.2. Let X and Y be topological spaces and $p : X \twoheadrightarrow Y$ be surjective and continuous. Then the following are equivalent.

1. p is a quotient map.
2. p maps saturated open sets to open sets.
3. p maps saturated closed sets to closed sets.

PROOF:

$\langle 1 \rangle 1. 1 \Rightarrow 2$

$\langle 2 \rangle 1.$ ASSUME: p is a quotient map.

$\langle 2 \rangle 2.$ LET: $U \subseteq Y$ be a saturated open set.

$\langle 2 \rangle 3.$ $U = p^{-1}(p(U))$

$\langle 3 \rangle 1.$ $U \subseteq p^{-1}(p(U))$

PROOF: Set theory.

$\langle 3 \rangle 2.$ $p^{-1}(p(U)) \subseteq U$

$\langle 4 \rangle 1.$ LET: $x \in p^{-1}(p(U))$

$\langle 4 \rangle 2.$ PICK $y \in U$ such that $p(x) = p(y)$

$\langle 4 \rangle 3.$ $x \in U$

PROOF: $\langle 2 \rangle 2, \langle 4 \rangle 2.$

$\langle 2 \rangle 4.$ $p(U)$ is open

PROOF: $\langle 2 \rangle 1, \langle 2 \rangle 3.$

$\langle 1 \rangle 2. 2 \Rightarrow 3$

$\langle 2 \rangle 1.$ ASSUME: p maps saturated open sets to open sets

$\langle 2 \rangle 2.$ LET: $C \subseteq X$ be a saturated closed set.

$\langle 2 \rangle 3.$ $X \setminus C$ is a saturated open set.

$\langle 3 \rangle 1.$ LET: $x \in X \setminus C$ and $x' \in X$ be such that $p(x) = p(x')$

$\langle 3 \rangle 2.$ $x' \notin C$

PROOF: If $x' \in C$ then $x \in C$ since C is saturated.

$\langle 2 \rangle 4.$ $p(X \setminus C)$ is open.

PROOF: By $\langle 2 \rangle 1$ and $\langle 2 \rangle 3.$

$\langle 2 \rangle 5.$ $p(X \setminus C) = Y \setminus p(C)$

$\langle 3 \rangle 1.$ $p(X \setminus C) \subseteq Y \setminus p(C)$

$\langle 4 \rangle 1.$ LET: $x \in X \setminus C$

$\langle 4 \rangle 2.$ ASSUME: for a contradiction $p(x) \in p(C)$

$\langle 4 \rangle 3.$ PICK $x' \in C$ such that $p(x) = p(x')$

$\langle 4 \rangle 4.$ Q.E.D.

PROOF: We have $x \notin C, x' \in C$ and $p(x) = p(x')$, contradicting $\langle 2 \rangle 2.$

$\langle 3 \rangle 2.$ $Y \setminus p(C) \subseteq p(X \setminus C)$

$\langle 4 \rangle 1.$ LET: $y \notin p(C)$

$\langle 4 \rangle 2.$ PICK $x \in X$ such that $p(x) = y$

PROOF: p is surjective.

$\langle 4 \rangle 3. x \notin C$

$\langle 1 \rangle 3. 3 \Rightarrow 1$

$\langle 2 \rangle 1.$ ASSUME: p maps saturated closed sets to closed sets

$\langle 2 \rangle 2.$ LET: $C \subseteq Y$ be such that $p^{-1}(Y)$ is closed

$\langle 2 \rangle 3. p^{-1}(C)$ is saturated

$\langle 3 \rangle 1.$ LET: $x \in p^{-1}(C)$, $x' \in X$ and $p(x) = p(x')$

$\langle 3 \rangle 2. x' \in p^{-1}(C)$

$\langle 2 \rangle 4. p(p^{-1}(C))$ is closed

PROOF: By $\langle 2 \rangle 1$ and $\langle 2 \rangle 3$.

$\langle 2 \rangle 5. C = p(p^{-1}(C))$

PROOF: By set theory, since p is surjective.

□

Corollary 4.5.2.1. *If $p : X \rightarrow Y$ is a surjective continuous map that is either an open map or a closed map, then p is a quotient map.*

Definition 4.5.3 (Quotient Topology). Let X be a topological space, A a set, and $p : X \rightarrow A$ a surjective map. Then the *quotient topology* on A induced by p is

$$\{U \subseteq A : p^{-1}(U) \text{ is open in } X\} .$$

It is easy to check this is a topology.

Lemma 4.5.4. *Let X be a topological space, A a set, and $p : X \rightarrow A$ a surjective map. Then the quotient topology induced by p is the unique topology on A such that p is a quotient map.*

PROOF: Immediate from definitions. □

Definition 4.5.5 (Quotient Space). Let X be a topological space and X^* a partition of X . Let $p : X \rightarrow X^*$ be the canonical map. Then X^* under the quotient topology induced by p is called a *quotient space* of X .

Proposition 4.5.6. *Let $p : X \rightarrow Y$ be a quotient map. Let $A \subseteq X$ be open and saturated. Then $p \upharpoonright_A : A \rightarrow p(A)$ is a quotient map.*

PROOF:

$\langle 1 \rangle 1.$ LET: $q = p \upharpoonright_A : A \rightarrow p(A)$

$\langle 1 \rangle 2.$ For all $V \subseteq p(A)$, we have $q^{-1}(V) = p^{-1}(V)$

$\langle 2 \rangle 1. q^{-1}(V) \subseteq p^{-1}(V)$

PROOF: Trivial.

$\langle 2 \rangle 2. p^{-1}(V) \subseteq q^{-1}(V)$

$\langle 3 \rangle 1.$ LET: $x \in p^{-1}(V)$

$\langle 3 \rangle 2.$ PICK $x' \in A$ such that $p(x') = p(x)$

PROOF: One exists because $p(x) \in V \subseteq p(A)$.

$\langle 3 \rangle 3. x \in A$

PROOF: This holds because A is saturated.

$\langle 3 \rangle 4. x \in q^{-1}(V)$

PROOF: From $\langle 3 \rangle 1$ and $\langle 3 \rangle 3$.

- $\langle 1 \rangle 3$. For all $U \subseteq X$, we have $p(U \cap A) = p(U) \cap p(A)$
- $\langle 1 \rangle 4$. LET: $V \subseteq p(A)$ be such that $q^{-1}(V)$ is open in A .
PROVE: V is open in $p(A)$.
- $\langle 1 \rangle 5$. $q^{-1}(V)$ is open in X
- $\langle 1 \rangle 6$. $p^{-1}(V)$ is open in X
- $\langle 1 \rangle 7$. V is open in Y
- $\langle 1 \rangle 8$. V is open in $p(A)$

□

Proposition 4.5.7. *Let $p : X \twoheadrightarrow Y$ be a quotient map. Let $A \subseteq X$ be closed and saturated. Then $p \upharpoonright_A : A \twoheadrightarrow p(A)$ is a quotient map.*

PROOF: Similar. □

Proposition 4.5.8. *Let $p : X \twoheadrightarrow Y$ be an open quotient map. Let $A \subseteq X$ be saturated. Then $p \upharpoonright_A : A \twoheadrightarrow p(A)$ is a quotient map.*

PROOF:

- $\langle 1 \rangle 1$. LET: $q = p \upharpoonright_A : A \twoheadrightarrow p(A)$
- $\langle 1 \rangle 2$. For all $V \subseteq p(A)$, we have $q^{-1}(V) = p^{-1}(V)$
 - $\langle 2 \rangle 1$. $q^{-1}(V) \subseteq p^{-1}(V)$
PROOF: Trivial.
 - $\langle 2 \rangle 2$. $p^{-1}(V) \subseteq q^{-1}(V)$
 - $\langle 3 \rangle 1$. LET: $x \in p^{-1}(V)$
 - $\langle 3 \rangle 2$. PICK $x' \in A$ such that $p(x') = p(x)$
PROOF: One exists because $p(x) \in V \subseteq p(A)$.
 - $\langle 3 \rangle 3$. $x \in A$
PROOF: This holds because A is saturated.
 - $\langle 3 \rangle 4$. $x \in q^{-1}(V)$
PROOF: From $\langle 3 \rangle 1$ and $\langle 3 \rangle 3$.
- $\langle 1 \rangle 3$. For all $U \subseteq X$, we have $p(U \cap A) = p(U) \cap p(A)$
 - $\langle 2 \rangle 1$. $p(U \cap A) \subseteq p(U) \cap p(A)$
PROOF: Set theory.
 - $\langle 2 \rangle 2$. $p(U) \cap p(A) \subseteq p(U \cap A)$
 - $\langle 3 \rangle 1$. LET: $x \in U$, $y \in A$, $p(x) = p(y)$
PROVE: $p(x) \in p(U \cap A)$
 - $\langle 3 \rangle 2$. $x \in A$
PROOF: A is saturated.
 - $\langle 3 \rangle 3$. $x \in U \cap A$
- $\langle 1 \rangle 4$. LET: $V \subseteq p(A)$ be such that $q^{-1}(V)$ is open in A .
PROVE: V is open in $p(A)$.
- $\langle 1 \rangle 5$. $p^{-1}(V)$ is open in A
PROOF: By $\langle 1 \rangle 2$
- $\langle 1 \rangle 6$. PICK U open in X such that $p^{-1}(V) = U \cap A$
- $\langle 1 \rangle 7$. $V = p(U) \cap p(A)$

PROOF:

$$\begin{aligned}
 V &= p(p^{-1}(V)) && (p \text{ is surjective}) \\
 &= p(U \cap A) && (\langle 1 \rangle 6) \\
 &= p(U) \cap p(A) && (\langle 1 \rangle 3)
 \end{aligned}$$

$\langle 1 \rangle 8$. $p(U)$ is open in Y

PROOF: $\langle 1 \rangle 6$, p is an open map.

$\langle 1 \rangle 9$. V is open in $p(A)$

PROOF: $\langle 1 \rangle 7$, $\langle 1 \rangle 8$

□

Proposition 4.5.9. *Let $p : X \twoheadrightarrow Y$ be a closed quotient map. Let $A \subseteq X$ be saturated. Then $p \restriction_A : A \twoheadrightarrow p(A)$ is a quotient map.*

PROOF: Similar. □

Proposition 4.5.10. *The composite of two quotient maps is a quotient map.*

PROOF: From Proposition 5.2.22. □

Proposition 4.5.11. *Let X^* be a quotient space of X . If every element of X^* is closed in X , then X^* is T_1 .*

PROOF:

$\langle 1 \rangle 1$. LET: $C \in X^*$

$\langle 1 \rangle 2$. $p^{-1}(\{C\}) = C$

PROOF: Definition of p .

$\langle 1 \rangle 3$. $p^{-1}(\{C\})$ is closed in X

PROOF: By hypothesis.

$\langle 1 \rangle 4$. $\{C\}$ is closed in X^* .

PROOF: By Proposition 5.2.21.

□

Chapter 5

Functions Between Topological Spaces

5.1 Open Maps

Definition 5.1.1. Let X and Y be topological spaces and $f : X \rightarrow Y$. Then f is an *open map* iff, for all U open in X , $f(U)$ is open in Y .

Lemma 5.1.2. Let X and Y be topological spaces and $f : X \rightarrow Y$. Let \mathcal{B} be a basis for the topology on X . Then f is an open map if and only if, for all $B \in \mathcal{B}$, $f(B)$ is open in Y .

PROOF:

$\langle 1 \rangle 1$. If f is an open map then, for all $B \in \mathcal{B}$, $f(B)$ is open in Y .

PROOF: Immediate from definitions.

$\langle 1 \rangle 2$. If, for all $B \in \mathcal{B}$, $f(B)$ is open in Y , then f is an open map.

$\langle 2 \rangle 1$. ASSUME: For all $B \in \mathcal{B}$, $f(B)$ is open in Y .

$\langle 2 \rangle 2$. LET: U be open in X

PROVE: $f(U)$ is open in Y

$\langle 2 \rangle 3$. LET: $\mathcal{B}_0 \subseteq \mathcal{B}$ be such that $U = \bigcup \mathcal{B}_0$

$\langle 2 \rangle 4$. $f(U) = \bigcup_{B \in \mathcal{B}_0} f(B)$

PROOF: Set theory.

$\langle 2 \rangle 5$. $f(U)$ is open in Y .

PROOF: From $\langle 2 \rangle 1$, $\langle 2 \rangle 4$ and the fact that the open sets are closed under union.

□

Corollary 5.1.2.1. Let X and Y be topological spaces and $f : X \rightarrow Y$. Let \mathcal{S} be a subbasis for the topology on X . Then f is an open map if and only if, for all $S \in \mathcal{S}$, $f(S)$ is open in Y .

Lemma 5.1.3 (AC). Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces. Then the projection $\pi_\alpha : \prod_{\alpha \in J} X_\alpha \rightarrow X_\alpha$ is an open map.

PROOF:

$\langle 1 \rangle 1$. For U open in X_α , we have $\pi_\alpha(\pi_\alpha^{-1}(U))$ is open in X_α

PROOF: $\pi_\alpha(\pi_\alpha^{-1}(U)) = U$ if all the other X_α are nonempty, \emptyset otherwise.

$\langle 1 \rangle 2$. For $\beta \neq \alpha$ and U open in X_β , we have $\pi_\alpha(\pi_\beta^{-1}(U))$ is open in X_α

PROOF: $\pi_\alpha(\pi_\beta^{-1}(U)) = X_\alpha$ if all the X_γ are nonempty for $\gamma \neq \alpha$, \emptyset otherwise.

$\langle 1 \rangle 3$. Q.E.D.

PROOF: By Corollary 5.1.2.1.

5.2 Continuous Functions

Definition 5.2.1 (Continuous). Let X and Y be topological spaces and $f : X \rightarrow Y$ a function. Then f is *continuous* if and only if, for every open set U in Y , the set $f^{-1}(U)$ is open in X .

Theorem 5.2.2. Let X and Y be topological spaces and $f : X \rightarrow Y$. Then the following are equivalent.

1. f is continuous.
2. For every closed set C in Y , the set $f^{-1}(C)$ is closed in X .
3. For every set $A \subseteq X$, we have $f(\overline{A}) \subseteq \overline{f(A)}$.

PROOF:

$\langle 1 \rangle 1$. $1 \Rightarrow 3$

$\langle 2 \rangle 1$. ASSUME: f is continuous.

$\langle 2 \rangle 2$. LET: $A \subseteq X$

$\langle 2 \rangle 3$. LET: $x \in \overline{A}$

PROVE: $f(x) \in \overline{f(A)}$

$\langle 2 \rangle 4$. LET: V be a neighbourhood of $f(x)$

$\langle 2 \rangle 5$. $f^{-1}(V)$ is a neighbourhood of x

PROOF: $\langle 2 \rangle 1$, $\langle 2 \rangle 3$, $\langle 2 \rangle 4$

$\langle 2 \rangle 6$. $f^{-1}(V)$ intersects A in a , say.

PROOF: $\langle 2 \rangle 3$, $\langle 2 \rangle 5$, Theorem 3.8.3.

$\langle 2 \rangle 7$. V intersects $f(A)$ in $f(a)$.

$\langle 2 \rangle 8$. Q.E.D.

PROOF: Theorem 3.8.3.

$\langle 1 \rangle 2$. $3 \Rightarrow 2$

$\langle 2 \rangle 1$. ASSUME: 3

$\langle 2 \rangle 2$. LET: C be a closed set in Y

$\langle 2 \rangle 3$. $f^{-1}(C) = \overline{f^{-1}(C)}$

PROOF:

$$\begin{aligned} f(\overline{f^{-1}(C)}) &\subseteq \overline{f(f^{-1}(C))} & (\langle 2 \rangle 1) \\ &\subseteq \overline{C} \end{aligned}$$

$\langle 1 \rangle 3$. $2 \Rightarrow 1$

$\langle 2 \rangle 1$. ASSUME: 2

- ⟨2⟩2. LET: V be open in Y
- ⟨2⟩3. $f^{-1}(Y \setminus V)$ is closed in X
PROOF: By ⟨2⟩1.
- ⟨2⟩4. $f^{-1}(V)$ is open in X .
PROOF: $f^{-1}(V) = X \setminus f^{-1}(Y \setminus V)$.

□

Lemma 5.2.3. *If $f : X \rightarrow Y$ maps all of X to the single point y_0 of Y , then f is continuous.*

PROOF: For V open in Y , the set $f^{-1}(V)$ is either X (if $y_0 \in V$) or \emptyset (if $y_0 \notin V$).

Definition 5.2.4 (Continuity at a Point). Let X and Y be topological spaces, $f : X \rightarrow Y$ a function, and $x \in X$. Then f is *continuous at x* if and only if, for every neighbourhood V of $f(x)$, there exists a neighbourhood U of x such that $f(U) \subseteq V$.

Theorem 5.2.5. *Let X and Y be topological spaces and $f : X \rightarrow Y$. Then f is continuous if and only if f is continuous at every point of X .*

PROOF:

- ⟨1⟩1. If f is continuous then f is continuous at every point of X .
 - ⟨2⟩1. ASSUME: f is continuous
 - ⟨2⟩2. LET: $x \in X$
 - ⟨2⟩3. LET: V be a neighbourhood of $f(x)$
 - ⟨2⟩4. $f^{-1}(V)$ is a neighbourhood of x
 - ⟨2⟩5. $f(f^{-1}(V)) \subseteq V$
- ⟨1⟩2. If f is continuous at every point of X then f is continuous.
 - ⟨2⟩1. ASSUME: f is continuous at every point of X .
 - ⟨2⟩2. LET: V be open in Y
PROVE: $f^{-1}(V)$ is open in X .
 - ⟨2⟩3. LET: $x \in f^{-1}(V)$
 - ⟨2⟩4. V is a neighbourhood of $f(x)$
 - ⟨2⟩5. PICK a neighbourhood U of x such that $f(U) \subseteq V$
PROOF: By ⟨2⟩1.
 - ⟨2⟩6. $x \in U \subseteq f^{-1}(V)$
 - ⟨2⟩7. Q.E.D.

□

Lemma 5.2.6. *Let X and Y be topological spaces and $f : X \rightarrow Y$. Let \mathcal{B} be a basis for the topology on Y . Then f is continuous if and only if, for all $B \in \mathcal{B}$, the set $f^{-1}(B)$ is open in X .*

PROOF:

- ⟨1⟩1. If f is continuous then, for all $B \in \mathcal{B}$, the set $f^{-1}(B)$ is open in X .
PROOF: Immediate from definitions.
- ⟨1⟩2. If, for all $B \in \mathcal{B}$, the set $f^{-1}(B)$ is open in X , then f is continuous.

⟨2⟩1. ASSUME: For all $B \in \mathcal{B}$, the set $f^{-1}(B)$ is open in X .

⟨2⟩2. LET: $x \in X$

⟨2⟩3. LET: V be a neighbourhood of $f(x)$

⟨2⟩4. PICK $B \in \mathcal{B}$ such that $f(x) \in B \subseteq V$

⟨2⟩5. $f^{-1}(B)$ is a neighbourhood of x

PROOF: By ⟨2⟩1.

⟨2⟩6. $f(f^{-1}(B)) \subseteq B$

PROOF: Set theory.

⟨2⟩7. Q.E.D.

PROOF: Theorem 5.2.5.

□

Lemma 5.2.7. *The projections $\pi_1 : X \times Y \rightarrow X$ and $\pi_2 : X \times Y \rightarrow Y$ are continuous.*

PROOF: Immediate from definitions. □

Theorem 5.2.8. *If A is a subspace of X , the inclusion function $j : A \rightarrow X$ is continuous.*

PROOF: For V open in X , the set $j^{-1}(V) = V \cap A$ is open in A .

Theorem 5.2.9. *If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are continuous, then the map $g \circ f : X \rightarrow Z$ is continuous.*

PROOF:

⟨1⟩1. LET: V be open in Z

⟨1⟩2. $g^{-1}(V)$ is open in Y

⟨1⟩3. $f^{-1}(g^{-1}(V))$ is open in X

□

Theorem 5.2.10. *If $f : X \rightarrow Y$ is continuous and if A is a subspace of X , then the restricted function $f \upharpoonright A : A \rightarrow Y$ is continuous.*

PROOF: For V open in Y , the set $(f \upharpoonright A)^{-1}(V) = f^{-1}(V) \cap A$ is open in A . □

Theorem 5.2.11. *Let $f : X \rightarrow Y$ be continuous. If Z is a subspace of Y that includes the range of f , then the function $g : X \rightarrow Z$ obtained by restricting the codomain of f is continuous. If Z is a space having Y as a subspace, then the function $h : X \rightarrow Z$ obtained by expanding the codomain of f is continuous.*

PROOF:

⟨1⟩1. If Z is a subspace of Y that includes the range of f , then the function $g : X \rightarrow Z$ obtained by restricting the codomain of f is continuous.

⟨2⟩1. LET: V be open in Z

⟨2⟩2. PICK W open in Y such that $V = W \cap Z$

⟨2⟩3. $f^{-1}(W)$ is open in X .

⟨2⟩4. $g^{-1}(V)$ is open in X .

PROOF: $g^{-1}(V) = f^{-1}(W)$.

⟨1⟩2. If Z is a space having Y as a subspace, then the function $h : X \rightarrow Z$ obtained by expanding the codomain of f is continuous.

PROOF: For V open in Z , we have $h^{-1}(V) = f^{-1}(V \cap Y)$ is open in X .
□

Theorem 5.2.12. *Let X and Y be topological spaces and $f : X \rightarrow Y$. If $x_n \rightarrow x$ as $n \rightarrow \infty$ in X and f is continuous at x , then $f(x_n) \rightarrow f(x)$ as $n \rightarrow \infty$ in Y .*

PROOF:

⟨1⟩1. ASSUME: $x_n \rightarrow x$ as $n \rightarrow \infty$

⟨1⟩2. ASSUME: f is continuous at x

⟨1⟩3. LET: V be a neighbourhood of $f(x)$

⟨1⟩4. PICK a neighbourhood U of x such that $f(U) \subseteq V$

PROOF: By ⟨1⟩2.

⟨1⟩5. PICK N such that, for all $n \geq N$, $x_n \in U$

PROOF: By ⟨1⟩1

⟨1⟩6. For $n \geq N$, $f(x_n) \in V$

PROOF: By ⟨1⟩4.

□

Theorem 5.2.13. *Let X , Y and Z be topological spaces. Let $f : X \rightarrow Y$. If there exists a set \mathcal{A} of open sets in X such that:*

- $\bigcup \mathcal{A} = X$;
- for all $U \in \mathcal{A}$, the function $f \upharpoonright U : U \rightarrow Y$ is continuous;

then f is continuous.

PROOF:

⟨1⟩1. LET: V be open in Y

⟨1⟩2. For all $U \in \mathcal{A}$, the set $(f \upharpoonright U)^{-1}(V)$ is open in X .

⟨2⟩1. LET: $U \in \mathcal{A}$

⟨2⟩2. $(f \upharpoonright U)^{-1}(V)$ is open in U

PROOF: Since $f \upharpoonright U : U \rightarrow Y$ is continuous.

⟨2⟩3. Q.E.D.

PROOF: By Lemma 4.3.3.

⟨1⟩3. Q.E.D.

PROOF: Since $f^{-1}(V) = \bigcup_{U \in \mathcal{A}} (f \upharpoonright U)^{-1}(V)$.

Theorem 5.2.14 (The Pasting Lemma). *Let $X = A \cup B$ where A and B are closed in X . Let $f : A \rightarrow Y$ and $g : B \rightarrow Y$ be continuous. If $f(x) = g(x)$ for every $x \in A \cap B$, then the function $h : X \rightarrow Y$ defined by*

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases}$$

is continuous.

PROOF:

- ⟨1⟩1. LET: C be closed in Y
- ⟨1⟩2. $f^{-1}(C)$ is closed in A
PROOF: Theorem 5.2.2.
- ⟨1⟩3. $f^{-1}(C)$ is closed in X
PROOF: Lemma 4.3.4.1.
- ⟨1⟩4. $g^{-1}(C)$ is closed in B
PROOF: Theorem 5.2.2.
- ⟨1⟩5. $g^{-1}(C)$ is closed in X
PROOF: Lemma 4.3.4.1.
- ⟨1⟩6. $h^{-1}(C)$ is closed in X
PROOF: $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$
- ⟨1⟩7. Q.E.D.
PROOF: Theorem 5.2.2.

□

Theorem 5.2.15. Let $f : A \rightarrow \prod_{\alpha \in J} X_\alpha$ be given by the equation

$$f(a) = \{f_\alpha(a)\}_{\alpha \in J} ,$$

where $f_\alpha : A \rightarrow X_\alpha$ for each α . Let $\prod_{\alpha \in J} X_\alpha$ have the product topology. Then the function f is continuous if and only if each function f_α is continuous.

PROOF:

- ⟨1⟩1. If f is continuous then each f_α is continuous.
PROOF: This holds because $f_\alpha = \pi_\alpha \circ f$.
- ⟨1⟩2. If every f_α is continuous then f is continuous.
⟨2⟩1. ASSUME: Every f_α is continuous.
⟨2⟩2. LET: $\alpha \in J$ and U be open in X_α
⟨2⟩3. $f^{-1}(\pi_\alpha^{-1}(U))$ is open in A
PROOF: $f^{-1}(\pi_\alpha^{-1}(U)) = f_\alpha^{-1}(U)$.

□

5.2.1 Homeomorphisms

Definition 5.2.16 (Homeomorphism). Let X and Y be topological spaces and $f : X \rightarrow Y$. Then f is a *homeomorphism* between X and Y iff f is a bijection, and f and f^{-1} are both continuous.

Definition 5.2.17 (Topological Property). A property P of topological spaces is a *topological property* iff, for any spaces X and Y , if X is homeomorphic to Y then P holds of X if and only if P holds of Y .

Definition 5.2.18 ((Topological) Imbedding). Let X and Y be topological spaces and $f : X \rightarrow Y$. Then f is a *(topological) imbedding* iff f is a homeomorphism between X and $\text{im } f$.

Definition 5.2.19 (Homogeneous). A topological space X is *homogeneous* iff, for all $x, y \in X$, there exists a homeomorphism $f : X \cong X$ such that $f(x) = y$.

5.2.2 Strongly Continuous Functions

Definition 5.2.20 (Strongly Continuous). Let X and Y be topological spaces and $f : X \rightarrow Y$. Then f is *strongly continuous* iff, for all $V \subseteq Y$, we have V is open in Y if and only if $f^{-1}(V)$ is open in X .

Proposition 5.2.21. *Let X and Y be topological spaces and $f : X \rightarrow Y$. Then f is strongly continuous if and only if, for all $C \subseteq Y$, C is closed in Y if and only if $f^{-1}(C)$ is closed in X .*

PROOF:

(1)1. If f is strongly continuous then, for all $C \subseteq Y$, we have C is closed in Y if and only if $f^{-1}(C)$ is closed in X .

PROOF:

$$\begin{aligned} C \text{ is closed in } Y &\Leftrightarrow Y \setminus C \text{ is open in } Y \\ &\Leftrightarrow f^{-1}(Y \setminus C) \text{ is open in } X \\ &\Leftrightarrow X \setminus f^{-1}(C) \text{ is open in } X \\ &\Leftrightarrow f^{-1}(C) \text{ is closed in } X \end{aligned}$$

(1)2. If, for all $C \subseteq Y$, we have C is closed in Y if and only if $f^{-1}(C)$ is closed in X , then f is strongly continuous.

PROOF: Similar.

□

Proposition 5.2.22. *The composite of two strongly continuous functions is strongly continuous.*

PROOF:

(1)1. LET: $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be strongly continuous.

(1)2. LET: $V \subseteq Z$

(1)3. V is open iff $f^{-1}(g^{-1}(V))$ is open

PROOF:

$$\begin{aligned} V \text{ is open} &\Leftrightarrow g^{-1}(V) \text{ is open} && ((1)1) \\ &\Leftrightarrow f^{-1}(g^{-1}(V)) \text{ is open} && ((1)1) \end{aligned}$$

□

Proposition 5.2.23. *Let X , Y and Z be topological spaces. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$. If f is strongly continuous and $g \circ f$ is continuous, then g is continuous.*

PROOF:

(1)1. LET: $V \subseteq Z$ be open in Z .

(1)2. $f^{-1}(g^{-1}(V))$ is open in X .

PROOF: $g \circ f$ is continuous.

(1)3. $g^{-1}(V)$ is open in Y .

PROOF: f is strongly continuous.

□

Proposition 5.2.24. *Let X, Y and Z be topological spaces. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$. If f and $g \circ f$ are strongly continuous, then g is strongly continuous.*

PROOF:

$\langle 1 \rangle 1$. LET: $U \subseteq Z$

$\langle 1 \rangle 2$. U is open in Z iff $g^{-1}(U)$ is open in Y

PROOF:

U is open in $Z \Leftrightarrow f^{-1}(g^{-1}(U))$ is open in X ($g \circ f$ is strongly continuous)

$\Leftrightarrow g^{-1}(U)$ is open in Y (f is strongly continuous)

□

5.3 Closed Maps

Definition 5.3.1 (Closed Map). Let X and Y be topological spaces and $f : X \rightarrow Y$. Then f is a *closed map* iff, for every closed set $C \subseteq X$, the set $f(C)$ is closed in Y .

5.4 Local Homeomorphism

Definition 5.4.1 (Locally Homeomorphic). Let X and Y be topological spaces. Then X is *locally homeomorphic* to Y iff every point in X has an open neighborhood that is homeomorphic with an open set in Y .

Proposition 5.4.2. *The long line is locally homeomorphic with \mathbb{R} .*

PROOF:

$\langle 1 \rangle 1$. LET: $x \in L$

$\langle 1 \rangle 2$. PICK an ordinal α such that $x < (\alpha, 0)$.

$\langle 1 \rangle 3$. $(-\infty, (\alpha, 0))$ is an open neighbourhood of x that is homeomorphic to $(0, 1)$.

□

5.5 Retracts

Definition 5.5.1 (Retract). Let Z be a topological space. If Y is a subspace of Z , we say that Y is a *retract* of Z iff there exists a continuous function $r : Z \rightarrow Y$ such that $r(y) = y$ for all $y \in Y$.

Chapter 6

Separation Axioms

6.1 T_1 Spaces

Definition 6.1.1 (T_1 Space). A topological space X is a T_1 space iff every finite set is closed.

Theorem 6.1.2. Let X be a T_1 space and $A \subseteq X$. Then x is a limit point of A if and only if every neighbourhood of x contains infinitely many points of A .

PROOF:

$\langle 1 \rangle 1$. If some neighbourhood of x contains only finitely many points of A then x is not a limit point of A .

$\langle 2 \rangle 1$. ASSUME: Some neighbourhood U of x contains only finite many points a_1, \dots, a_n of A .

$\langle 2 \rangle 2$. $X \setminus \{a_1, \dots, a_n\}$ is open.

PROOF: X is T_1 .

$\langle 2 \rangle 3$. $U \setminus \{a_1, \dots, a_n\}$ is a neighbourhood of x that does not intersect A .

$\langle 1 \rangle 2$. If every neighbourhood of x contains infinitely many points of A then x is a limit point of A .

PROOF: From the definition of limit point.

□

Proposition 6.1.3. A subspace of a T_1 space is T_1 .

PROOF:

$\langle 1 \rangle 1$. LET: X be a T_1 space and $Y \subseteq X$

$\langle 1 \rangle 2$. LET: $a \in Y$

$\langle 1 \rangle 3$. $\{a\}$ is closed in X

PROOF: By $\langle 1 \rangle 1$.

$\langle 1 \rangle 4$. $\{a\}$ is closed in Y

PROOF: By Corollary 4.3.4.1.

□

Definition 6.1.4 (Separate Points from Closed Sets). Let X be a space and $\{f_\alpha\}_{\alpha \in J}$ be a family of continuous functions $f_\alpha : X \rightarrow \mathbb{R}$. Then $\{f_\alpha\}$ *separates points from closed sets* in X iff, for every point $x_0 \in X$ and every neighbourhood U of x_0 , there exists $\alpha \in J$ such that f_α is positive at x_0 and vanishes outside U .

Theorem 6.1.5 (Imbedding Theorem). Let X be a T_1 space and $\{f_\alpha\}_{\alpha \in J}$ be a family of functions $X \rightarrow \mathbb{R}$ that separates points from closed sets. Then the function $F : X \rightarrow \mathbb{R}^J$ defined by

$$F(x)_\alpha = f_\alpha(x)$$

is an imbedding. If each f_α maps X into $[0, 1]$ then F is an imbedding $X \rightarrow [0, 1]^J$.

PROOF:

(1)1. F is continuous

PROOF: By Theorem 5.2.15.

(1)2. F is injective

(2)1. LET: $x, y \in X$ with $x \neq y$

(2)2. PICK a neighbourhood U of x such that $y \notin U$

PROOF: X is T_1

(2)3. PICK $\alpha \in J$ such that f_α is positive at x and vanishes outside U

(2)4. $f_\alpha(x) \neq f_\alpha(y)$

(2)5. $F(x) \neq F(y)$

(1)3. F is open as a map $X \rightarrow F(U)$

(2)1. LET: U be open

(2)2. LET: $z \in F(U)$

(2)3. PICK $x \in U$ such that $F(x) = z$

(2)4. PICK $\alpha \in J$ such that f_α is positive at x and vanishes outside U

(2)5. $z \in \pi_\alpha^{-1}((0, +\infty)) \cap F(U) \subseteq F(U)$

□

6.2 Hausdorff Spaces

Definition 6.2.1 (Hausdorff Space). A topological space X is a *Hausdorff space* iff, for any points $x, y \in X$ with $x \neq y$, there exist disjoint neighbourhoods U of x and V of y .

Theorem 6.2.2. Every Hausdorff space is T_1 .

PROOF:

(1)1. LET: X be a Hausdorff space

(1)2. LET: $a \in X$

PROVE: $\{a\}$ is closed.

(1)3. LET: $b \in X \setminus \{a\}$

(1)4. PICK disjoint neighbourhoods U of a and V of b

⟨1⟩5. $b \in V \subseteq X \setminus \{a\}$

⟨1⟩6. Q.E.D.

PROOF: By Proposition 3.2.3.

□

Theorem 6.2.3. *In a Hausdorff space, a sequence has at most one limit.*

PROOF:

⟨1⟩1. ASSUME: for a contradiction $x_n \rightarrow l$ and $x_n \rightarrow m$ as $n \rightarrow \infty$, and $l \neq m$

⟨1⟩2. PICK disjoint neighbourhoods U of l and V of m

⟨1⟩3. PICK N such that, for all $n \geq N$, $x_n \in U$ and $x_n \in V$

⟨1⟩4. $x_N \in U \cap V$

□

Theorem 6.2.4. *Every linearly ordered set is Hausdorff under the order topology.*

PROOF:

⟨1⟩1. LET: X be a linearly ordered set under the order topology.

⟨1⟩2. LET: $x, y \in X$ with $x \neq y$

⟨1⟩3. ASSUME: w.l.o.g. $x < y$

PROVE: There exist disjoint neighbourhoods U of x and V of y .

⟨1⟩4. CASE: There exists z such that $x < z < y$

PROOF: In this case, take $U = (-\infty, z)$ and $V = (z, +\infty)$.

⟨1⟩5. CASE: There does not exist z such that $x < z < y$

PROOF: In this case, take $U = (-\infty, y)$ and $V = (x, +\infty)$.

□

Theorem 6.2.5. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of Hausdorff spaces. Then $\prod_{\alpha \in J} X_\alpha$ is Hausdorff under the product topology.*

PROOF:

⟨1⟩1. LET: $\{x_\alpha\}_{\alpha \in J}, \{y_\alpha\}_{\alpha \in J} \in \prod_{\alpha \in J} X_\alpha$ with $\{x_\alpha\}_{\alpha \in J} \neq \{y_\alpha\}_{\alpha \in J}$

⟨1⟩2. PICK $\alpha \in J$ such that $x_\alpha \neq y_\alpha$

⟨1⟩3. PICK disjoint neighbourhoods U of x_α and V of y_α .

⟨1⟩4. $\pi_\alpha^{-1}(U)$ and $\pi_\alpha^{-1}(V)$ are disjoint neighbourhoods of $\{x_\alpha\}_{\alpha \in J}$ and $\{y_\alpha\}_{\alpha \in J}$

□

Corollary 6.2.5.1. *The Sorgenfrey plane is Hausdorff.*

Corollary 6.2.5.2. *For any set I , the space \mathbb{R}^I is Hausdorff.*

Proposition 6.2.6. *Let X and Y be topological spaces and $f : X \rightarrow Y$. If f is continuous and injective and Y is Hausdorff then X is Hausdorff.*

PROOF:

⟨1⟩1. LET: $x, y \in X$ with $x \neq y$

⟨1⟩2. $f(x) \neq f(y)$

PROOF: f is injective.

⟨1⟩3. PICK disjoint neighbourhoods U, V of $f(x)$ and $f(y)$

PROOF: Y is Hausdorff.

⟨1⟩4. $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint neighbourhoods of x and y .

□

Corollary 6.2.6.1. *A subspace of a Hausdorff space is Hausdorff.*

Corollary 6.2.6.2. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of nonempty spaces. If $\prod_{\alpha \in J} X_\alpha$ is Hausdorff then so is each X_α .*

Corollary 6.2.6.3. *Let \mathcal{T} and \mathcal{T}' be topologies on the same set X . If $\mathcal{T} \subseteq \mathcal{T}'$ and X is Hausdorff under \mathcal{T} then X is Hausdorff under \mathcal{T}' .*

Corollary 6.2.6.4. *The space \mathbb{R}_K is Hausdorff.*

Proposition 6.2.7. *\mathbb{R}_l is Hausdorff.*

PROOF: Let $a, b \in \mathbb{R}_l$ with $a < b$. Then $(-\infty, b)$ and $[b, +\infty)$ are disjoint open sets containing a and b respectively. □

Proposition 6.2.8. *The continuous image of a Hausdorff space is not necessarily Hausdorff.*

PROOF: The identity map from the discrete two-point space to the indiscrete two-point space is continuous. □

6.3 Regular Spaces

Definition 6.3.1 (Regular). A topological space X is *regular* iff, for every closed set A and point $a \notin A$, there exist disjoint neighbourhoods U of A and V of a .

Proposition 6.3.2. *Let X be a T_1 space. Then X is regular if and only if, for every point x and neighbourhood U of x , there exists a neighbourhood V of x such that $\bar{V} \subseteq U$.*

PROOF:

⟨1⟩1. If X is regular then, for every point x and neighbourhood N of x , there exists a neighbourhood V of x such that $\bar{V} \subseteq N$.

⟨2⟩1. ASSUME: X is regular.

⟨2⟩2. LET: $x \in X$ and N be a neighbourhood of x

⟨2⟩3. PICK an open set U such that $x \in U \subseteq N$

⟨2⟩4. PICK disjoint open sets V, W such that $x \in V$ and $X \setminus U \subseteq W$

⟨2⟩5. $\bar{V} \subseteq N$

PROOF:

$$\begin{aligned}\bar{V} &\subseteq X \setminus W \\ &\subseteq U \\ &\subseteq N\end{aligned}$$

- ⟨1⟩2. If, for every point x and neighbourhood U of x , there exists a neighbourhood V of x such that $\overline{V} \subseteq U$, then X is regular.
- ⟨2⟩1. ASSUME: For every point x and neighbourhood U of x , there exists a neighbourhood V of x such that $\overline{V} \subseteq U$.
- ⟨2⟩2. LET: $x \in X$ and A be a closed set with $x \notin A$
- ⟨2⟩3. PICK a neighbourhood V of x such that $\overline{V} \subseteq X \setminus A$
- ⟨2⟩4. $x \in V$ and $A \subseteq X \setminus \overline{V}$

□

Proposition 6.3.3. *Every linearly ordered set under the order topology is regular.*

PROOF:

- ⟨1⟩1. LET: X be a linearly ordered set under the order topology.
- ⟨1⟩2. LET: $x \in X$ and U be a neighbourhood of x
PROVE: There exists a neighbourhood V of x with $\overline{V} \subseteq U$
- ⟨1⟩3. CASE: x is greatest and least in X
PROOF: Take $V = U = X = \{x\}$
- ⟨1⟩4. CASE: x is greatest in X and there exists $a < x$ such that $(a, x] \subseteq U$
 - ⟨2⟩1. CASE: There exists b such that $a < b < x$
PROOF: Take $V = (b, x]$.
 - ⟨2⟩2. CASE: There is no b such that $a < b < x$
 - ⟨3⟩1. LET: $V = U = \{x\}$
 - ⟨3⟩2. $\overline{V} = V$
PROOF: For any $y \neq x$, we have $(-\infty, x)$ is a neighbourhood of y that does not intersect V .
- ⟨1⟩5. CASE: x is least in X and there exists $b > x$ such that $[x, b) \subseteq U$
PROOF: Similar.
- ⟨1⟩6. CASE: There exist $a < x < b$ such that $(a, b) \subseteq U$
 - ⟨2⟩1. PICK a point c such that $a < c < x$ if there is one, otherwise
LET: $c = a$
 - ⟨2⟩2. PICK a point d such that $x < d < b$ if there is one, otherwise
LET: $d = b$
 - ⟨2⟩3. LET: $V = (c, d)$
 - ⟨2⟩4. $\overline{V} \subseteq U$
PROOF:
$$\begin{aligned}\overline{V} &\subseteq [c, d] \\ &\subseteq (a, b) \\ &\subseteq U\end{aligned}$$
- ⟨1⟩7. Q.E.D.

PROOF: These cases are exhaustive by Lemma 4.1.2. They prove X is regular by Proposition 6.3.2.

□

Proposition 6.3.4. *A subspace of a regular space is regular.*

PROOF:

- (1)1. LET: X be a regular space and $Y \subseteq X$
 (1)2. LET: $A \subseteq Y$ be closed in Y and $a \in Y \setminus A$
 (1)3. PICK C closed in X such that $A = C \cap Y$
 PROOF: By Corollary 4.3.4.1.
 (1)4. PICK disjoint open sets U, V in X such that $C \subseteq U$ and $a \in V$
 (1)5. $U \cap Y$ and $V \cap Y$ are disjoint open sets in Y such that $A \subseteq U \cap Y$ and $a \in V \cap Y$

□

Corollary 6.3.4.1. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of nonempty spaces. If $\prod_{\alpha \in J} X_\alpha$ is regular then so is each X_α .*

Proposition 6.3.5 (AC). *The product of a family of regular spaces is regular.*

PROOF:

- (1)1. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of regular spaces.
 (1)2. $\prod_{\alpha \in J} X_\alpha$ is T_1
 (1)3. LET: $\vec{a} \in U$ where U is open in $\prod_{\alpha \in J} X_\alpha$
 (1)4. PICK $\prod_{\alpha \in J} U_\alpha$ such that each U_α is open in X_α , $U_\alpha = X_\alpha$ except at $\alpha_1, \dots, \alpha_n$, and $\vec{a} \in \prod_{\alpha \in J} U_\alpha \subseteq U$
 (1)5. For $1 \leq i \leq n$, PICK V_{α_i} open in X_{α_i} such that $a_{\alpha_i} \in V_{\alpha_i}$ and $\overline{V_{\alpha_i}} \subseteq U_{\alpha_i}$
 (1)6. For $\alpha \neq \alpha_1, \dots, \alpha_n$,
 LET: $V_\alpha = X_\alpha$
 (1)7. $\vec{a} \in \prod_{\alpha \in J} V_\alpha$
 (1)8. $\overline{\prod_{\alpha \in J} V_\alpha} \subseteq \prod_{\alpha \in J} U_\alpha$
 PROOF: By Theorem 4.2.5.

□

Corollary 6.3.5.1. *The Sorgenfrey plane is regular.*

Corollary 6.3.5.2. *For any set I , the space \mathbb{R}^I is regular.*

Proposition 6.3.6. *The space \mathbb{R}_K is not regular.*

PROOF: There do not exist disjoint neighbourhoods of 0 and K . □

Proposition 6.3.7. *The continuous image of a regular space is not necessarily regular.*

PROOF: The identity map from the discrete two-point space to the indiscrete two-point space is continuous. □

6.4 Completely Regular Spaces

Definition 6.4.1 (Separated by a Continuous Function). Let A and B be subsets of a topological space X . Then A and B can be *separated by a continuous function* iff there exists a continuous $f : X \rightarrow [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

Definition 6.4.2 (Completely Regular). A space X is *completely regular* iff X is T_1 and, for every point a and closed set A not containing a , we have that $\{a\}$ and A can be separated by a continuous function.

Theorem 6.4.3. *The product of a family of completely regular spaces is completely regular.*

PROOF:

- ⟨1⟩1. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of completely regular spaces.
 - ⟨1⟩2. LET: $a \in \prod_{\alpha \in J} X_\alpha$ and A be closed in $\prod_{\alpha \in J} X_\alpha$ such that $a \notin A$
 - ⟨1⟩3. PICK a basic open neighbourhood $\prod_{\alpha \in J} U_\alpha \subseteq \prod_{\alpha \in J} X_\alpha \setminus A$ of a such that $U_\alpha = X_\alpha$ except for $\alpha = \alpha_1, \dots, \alpha_n$
 - ⟨1⟩4. For $1 \leq i \leq n$, PICK a continuous $f_i : X_{\alpha_i} \rightarrow [0, 1]$ that is 0 at a_{α_i} and 1 on $X_{\alpha_i} \setminus U_{\alpha_i}$
 - ⟨1⟩5. LET: $f : \prod_{\alpha \in J} X_\alpha \rightarrow [0, 1]$ be given by $f(x) = \prod_{i=1}^n f_i(x_{\alpha_i})$
 - ⟨1⟩6. $f(a) = 0$
 - ⟨1⟩7. $f(x) = 1$ for $x \in A$
 - ⟨1⟩8. f is continuous
-

Corollary 6.4.3.1. *The Sorgenfrey plane is completely regular.*

Corollary 6.4.3.2. *For any set I , the space \mathbb{R}^I is completely regular.*

Proposition 6.4.4. *For any set J , the space \mathbb{R}^J in the box topology is completely regular.*

PROOF:

- ⟨1⟩1. LET: $a \in \mathbb{R}^J$ and $A \subseteq \mathbb{R}^J$ be closed with $a \notin A$
 PROVE: There exists $f : \mathbb{R}_{\text{box}}^J \rightarrow [0, 1]$ continuous such that $f(a) = 1$ and $f(A) = \{0\}$
 - ⟨1⟩2. ASSUME: w.l.o.g. $A \cap (-1, 1)^J = \emptyset$ and $a = \vec{0}$
 - ⟨2⟩1. PICK a basic open set $\prod_{\alpha \in J} U_\alpha$ such that $a \in \prod_{\alpha \in J} U_\alpha \subseteq \mathbb{R}^J \setminus A$
 - ⟨2⟩2. For $\alpha \in J$, PICK b_α, c_α such that $a_\alpha \in (b_\alpha, c_\alpha) \subseteq U_\alpha$
 - ⟨2⟩3. For $\alpha \in J$, PICK a homeomorphism $f_\alpha : \mathbb{R} \rightarrow \mathbb{R}$ that maps b_α to -1 , a_α to 0 and c_α to 1
 - ⟨2⟩4. $\prod_{\alpha \in J} f_\alpha$ is an automorphism $\mathbb{R}_{\text{box}}^J$ that maps a to $\vec{0}$ and A to a closed set disjoint from $(-1, 1)^J$
 - ⟨1⟩3. PICK a continuous function $f : \mathbb{R}_{\text{uniform}}^J \rightarrow [0, 1]$ such that $f(\vec{0}) = 1$ and $f(\mathbb{R}^J \setminus (-1, 1)^J) = \{0\}$
 - ⟨1⟩4. f is continuous w.r.t. the box topology
-

Proposition 6.4.5. *Not every regular space is completely regular.*

PROOF:

- ⟨1⟩1. For $m \in \mathbb{Z}$,
 LET: $L_m = \{m\} \times [-1, 0]$
- ⟨1⟩2. For each odd integer n and each integer $k \geq 2$,

- LET: $C_{nk} = (\{n+1-1/k\}home/robin/fun/RogOMatic/src/actuator\times[-1, 0]) \cup (\{n-1+1/k\} \times [-1, 0]) \cup \{(x, y) : (x-n)^2 + y^2 = (1-1/k)^2, y \geq 0\}$
- (1)3. For each odd integer n and each integer $k \geq 2$,
 LET: $p_{nk} = (n, 1-1/k)$
- (1)4. PICK two points a, b not in any L_m or C_{nk}
- (1)5. LET: $X = \bigcup_{m \in \mathbb{Z}} L_m \cup \bigcup_{n,k} C_{nk} \cup \{a, b\}$
- (1)6. LET: \mathcal{B} be the set consisting of all subsets of \mathbb{R}^2 of the following forms:
1. The intersection of X with a horizontal open line segment that contains none of the points p_{nk}
 2. A set formed from one of the sets C_{nk} by deleting finitely many points.
 3. For each even integer m , the set $\{a\} \cup \{(x, y) \in X : x < m\}$
 4. For each even integer m , the set $\{b\} \cup \{(x, y) \in X : x > m\}$
- (1)7. \mathcal{B} is a basis for a topology on X
- (2)1. For all $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$
- (2)2. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$
- (3)1. CASE: B_1, B_2 are both of type 1
 PROOF: Their intersection is of type 1.
- (3)2. CASE: B_1 is of type 1 and B_2 is of type 2
 PROOF: Their intersection is of type 2, since a horizontal line segment intersects C_{nk} in at most two points.
- (3)3. CASE: B_1 is of type 1 and B_2 is of type 3
 PROOF: Their intersection is of type 1
- (3)4. CASE: B_1 is of type 1 and B_2 is of type 4
 PROOF: Their intersection is of type 1
- (3)5. CASE: B_1 is of type 2 and B_2 is of type 2
 PROOF: Their intersection is of type 2
- (3)6. CASE: B_1 is of type 2 and B_2 is of type 3
 PROOF: Their intersection is B_1
- (3)7. CASE: B_1 is of type 2 and B_2 is of type 4
 PROOF: Their intersection is B_1
- (3)8. CASE: B_1 is of type 3 and B_2 is of type 3
 PROOF: Their intersection is of type 3
- (3)9. CASE: B_1 is of type 3 and B_2 is of type 4
- (4)1. LET: $B_1 = \{a\} \cup \{(x, y) \in X : x < m\}$ and $B_2 = \{b\} \cup \{(x, y) \in X : x > n\}$
- (4)2. CASE: $x = (s, 1-1/k)$ for some s and integer $x \geq 2$
 PROOF: In this case, $x \in C_{nk}$ for some n and $C_{nk} \subseteq B_1 \cap B_2$.
- (4)3. CASE: $x = (s, t)$ and $t \neq 1-1/k$ for any integer $k \geq 2$
 PROOF: In this case, $x \in ((n, m) \times \{t\}) \cap X \subseteq B_1 \cap B_2$
- (3)10. CASE: B_1 is of type 4 and B_2 is of type 4
 PROOF: Their intersection is of type 4
- (1)8. For any continuous function $f : X \rightarrow \mathbb{R}$, we have $f(a) = f(b)$
- (2)1. LET: $f : X \rightarrow \mathbb{R}$ be continuous

- (2)2. For any $c \in \mathbb{R}$, we have $f^{-1}(c)$ is G_δ
PROOF: $f^{-1}(c) = \bigcap_{q \in \mathbb{Q}^+} f^{-1}(c - q, c + q)$
(2)3. LET: $S_{nk} = \{p \in C_{nk} : f(p) \neq f(p_{nk})\}$
(2)4. For all n, k , we have S_{nk} is countable.
(3)1. LET: $f^{-1}(p_{nk}) = \bigcap_{m=1}^{\infty} U_m$ where U_m is open in X
(3)2. For each m , PICK $B_m \in \mathcal{B}$ such that $p_{nk} \in B_m \subseteq U_m$
(3)3. $S_{nk} \subseteq \bigcup_{m=1}^{\infty} (C_{nk} \setminus B_m)$
(3)4. Each $C_{nk} \setminus B_m$ is countable
(4)1. LET: $m \in \mathbb{Z}$
(4)2. B_m cannot be of type 1
(4)3. If B_m is of type 2 then $C_{nk} \setminus B_m$ is finite.
(4)4. If B_m is of type 3 or 4 then $C_{nk} \setminus B_m$ is empty.
(2)5. PICK $d \in [-1, 0]$ such that $\mathbb{R} \times \{d\}$ intersects none of the sets S_{nk}
(2)6. For n odd, we have

$$f(n-1, d) = \lim_{k \rightarrow \infty} f(p_{nk}) .$$

- (3)1. LET: $\epsilon > 0$
(3)2. PICK $B \in \mathcal{B}$ such that $(n-1, d) \in B \subseteq f^{-1}(f(n-1, d) - \epsilon, f(n-1, d) + \epsilon)$
(3)3. There exists $\delta > 0$ such that, for $x \in (n-1-\delta, n-1+\delta)$, we have $(x, d) \in B$
(3)4. PICK K such that $1/K < \delta$
(3)5. LET: $k \geq K$
(3)6. $f(n-1+1/k, d) = f(p_{nk})$
(3)7. $|f(n-1, d) - f(n-1+1/k, d)| < \epsilon$
(3)8. $|f(n-1, d) - f(p_{nk})| < \epsilon$
(2)7. For n odd, we have

$$f(n+1, d) = \lim_{k \rightarrow \infty} f(p_{nk}) .$$

PROOF: Similar.

- (2)8. Q.E.D.
(3)1. ASSUME: $f(a) \neq f(b)$
(3)2. ASSUME: w.l.o.g. $f(a) < f(b)$
(3)3. PICK $B \in \mathcal{B}$ such that $a \in B \subseteq f^{-1}(-\infty, (f(a) + f(b))/2)$
(3)4. LET: m be even such that $B = \{a\} \cup \{(x, y) \in X : x < m\}$
(3)5. PICK $B \in \mathcal{B}$ such that $b \in B \subseteq f^{-1}((f(a) + f(b))/2, +\infty)$
(3)6. LET: m' be even such that $B = \{b\} \cup \{(x, y) \in X : x > m'\}$
(3)7. $f(m, d) = f(m', d)$
(3)8. Q.E.D.

(1)9. X is regular.

(1)10. X is not completely regular.

PROOF: a and b cannot be separated by a continuous function.

□

Theorem 6.4.6 (AC). *A space is completely regular iff it is homeomorphic to a subspace of $[0, 1]^J$ for some J .*

PROOF:

- ⟨1⟩1. Every completely regular space is homeomorphic to a subspace of $[0, 1]^J$ for some J .
- ⟨2⟩1. LET: X be completely regular
- ⟨2⟩2. For every point a and open set U that contains a , PICK a continuous function f_{aU} that is positive on a and vanishes outside U
- ⟨2⟩3. The family $\{f_{aU}\}$ separates points from closed sets
- ⟨2⟩4. Q.E.D.
- PROOF: By the Imbedding Theorem.
- ⟨1⟩2. Every subspace of $[0, 1]^J$ is completely regular.
- PROOF: By Theorem 6.4.3 and Proposition 6.3.4.
-

Proposition 6.4.7. *The continuous image of a completely regular space is not necessarily completely regular.*

PROOF: The identity map from the discrete two-point space to the indiscrete two-point space is continuous. □

6.5 Normal Spaces

Definition 6.5.1 (Normal Space). A *normal* space is a T_1 space such that, for any disjoint closed sets A, B , there exist disjoint open sets U, V such that $A \subseteq U$ and $B \subseteq V$.

Theorem 6.5.2. *Every linearly ordered set is normal under the order topology.*

PROOF: See Steen and Steerbach *Counterexamples in Topology* Example 39. □

Proposition 6.5.3. *The product space $S_\Omega \times \overline{S_\Omega}$ is not normal.*

PROOF:

- ⟨1⟩1. LET: $\Delta = \{(x, x) : x \in \overline{S_\Omega}\} \subseteq \overline{S_\Omega} \times \overline{S_\Omega}$
- ⟨1⟩2. Δ is closed in $\overline{S_\Omega} \times \overline{S_\Omega}$
- ⟨1⟩3. LET: $A = \Delta \cap (S_\Omega \times \overline{S_\Omega})$
- ⟨1⟩4. A is closed in $S_\Omega \times \overline{S_\Omega}$
- ⟨1⟩5. LET: $B = S_\Omega \times \{\Omega\}$
- ⟨1⟩6. B is closed
- ⟨1⟩7. $A \cap B = \emptyset$
- ⟨1⟩8. ASSUME: for a contradiction U and V are disjoint open sets including A and B respectively
- ⟨1⟩9. For all $x \in S_\Omega$ there exists $\beta \in (x, \Omega)$ such that $(x, \beta) \notin U$
 - ⟨2⟩1. LET: $x \in S_\Omega$
 - ⟨2⟩2. $(x, \Omega) \in V$
 - PROOF: $(x, \Omega) \in B \subseteq V$
 - ⟨2⟩3. PICK $y < \Omega$ such that $\{x\} \times (y, \Omega] \subseteq V$
 - PROOF: By Lemma 4.1.2.
 - ⟨2⟩4. PICK β such that $x, y < \beta < \Omega$

PROOF: Such a β exists because Ω is a limit ordinal.

$\langle 1 \rangle 10$. For $x \in S_\Omega$,

LET: $\beta(x)$ be the least element of (x, Ω) such that $(x, \beta(x)) \notin U$

$\langle 1 \rangle 11$. LET: $b = \sup_{n=1}^\infty \beta^n(0)$

$\langle 1 \rangle 12$. $\beta^n(0) \rightarrow b$ as $n \rightarrow \infty$

$\langle 1 \rangle 13$. $(\beta^n(0), \beta^{n+1}(0)) \rightarrow (b, b)$ as $n \rightarrow \infty$

$\langle 1 \rangle 14$. $(b, b) \in A$

$\langle 1 \rangle 15$. $(b, b) \in U$

$\langle 1 \rangle 16$. For all n we have $(\beta^n(0), \beta^{n+1}(0)) \notin U$

PROOF: By $\langle 1 \rangle 10$.

$\langle 1 \rangle 17$. Q.E.D.

PROOF: Steps $\langle 1 \rangle 12$, $\langle 1 \rangle 15$ and $\langle 1 \rangle 16$ form a contradiction.

□

Corollary 6.5.3.1. *Not every completely regular space is normal.*

Corollary 6.5.3.2. *An open subspace of a normal space is not necessarily normal.*

Corollary 6.5.3.3. *The product of two normal spaces is not necessarily normal.*

Proposition 6.5.4. *A closed subspace of a normal space is normal.*

PROOF:

$\langle 1 \rangle 1$. LET: X be normal and $C \subseteq X$ be closed.

$\langle 1 \rangle 2$. LET: A and B be closed in C

$\langle 1 \rangle 3$. A and B are closed in X

PROOF: By Corollary 4.3.4.2.

$\langle 1 \rangle 4$. PICK disjoint open neighbourhoods U and V of A and B in X

$\langle 1 \rangle 5$. $U \cap C$ and $V \cap C$ are disjoint open neighbourhoods of A and B in C

□

Corollary 6.5.4.1. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of nonempty spaces. If $\prod_{\alpha \in J} X_\alpha$ is normal then each X_α is normal.*

Proposition 6.5.5. *If the Continuum Hypothesis then \mathbb{R}^ω under the box topology is normal.*

PROOF: See Rudin. The box product of countably many compact metric spaces. *General Topology and Its Applications*, 2:293–298, 1972. □

Proposition 6.5.6 (Stone (DC)). *If J is uncountable then \mathbb{R}^J is not normal.*

PROOF:

$\langle 1 \rangle 1$. LET: $X = (\mathbb{Z}^+)^J$

PROVE: X is not normal.

$\langle 1 \rangle 2$. For $x \in X$ and $B \subseteq^{\text{fin}} J$,

LET:

$$U(x, B) = \{y \in X : \forall \alpha \in B. y_\alpha = x_\alpha\} .$$

- (1)3. $\{U(x, B) : x \in X, B \subseteq^{\text{fin}} J\}$ is a basis for X
 (2)1. LET: $x \in X$ and $\prod_{\alpha \in J} U_\alpha$ be a basic open set including x , where $U_\alpha = \mathbb{Z}^+$ for all α except $\alpha_1, \dots, \alpha_n$
 (2)2. $x \in U(x, \{\alpha_1, \dots, \alpha_n\}) \subseteq \prod_{\alpha \in J} U_\alpha$
 (1)4. For $n \in \mathbb{Z}^+$,
 LET: $P_n = \{x \in X : x \text{ is injective on } J \setminus x^{-1}(n)\}$
 (1)5. P_1 and P_2 are closed and disjoint.
 (2)1. P_1 is closed
 (3)1. LET: $x \in X \setminus P_1$
 (3)2. PICK $\alpha, \beta \in J$ such that $x_\alpha = x_\beta \neq 1$
 (3)3. LET: $U_\gamma = \{x_\alpha\}$ if $\gamma = \alpha$ or $\gamma = \beta$, \mathbb{Z}^+ for all other $\gamma \in J$
 (3)4. $x \in \prod_{\gamma \in J} U_\gamma \subseteq X \setminus P_1$
 (2)2. P_2 is closed
 PROOF: Similar.
 (2)3. $P_1 \cap P_2 = \emptyset$
 PROOF: If $x \in P_1 \cap P_2$ then x is injective on J , contradicting the fact that J is uncountable.
 (1)6. ASSUME: for a contradiction U and V are disjoint open sets including P_1 and P_2
 (1)7. Given a sequence (α_i) of distinct elements of J and a strictly increasing sequence (n_i) of positive integers,
 LET:

$$B_i^{\alpha, n} = \{\alpha_1, \dots, \alpha_{n_i}\}$$

$$x_i^{\alpha, n} \in X$$

$$(x_i^{\alpha, n})_\beta = \begin{cases} j & \text{if } \beta = \alpha_j, 1 \leq j \leq n_{i-1} \\ 1 & \text{for all other values of } \beta \end{cases}$$
 for $i \geq 1$
 (1)8. PICK sequences (α_i) , (n_i) such that, for all $i \geq 1$, we have $U(x_i^{\alpha, n}, B_i^{\alpha, n}) \subseteq U$
 (2)1. LET: $x_1 \in X$ be given by $(x_1)_\alpha = 1$ for all $\alpha \in J$
 (2)2. $x_1 \in U$
 PROOF: $x_1 \in P_1 \subseteq U$
 (2)3. PICK $B_1 \subseteq^{\text{fin}} J$ such that $U(x_1, B_1) \subseteq U$
 PROOF: By (1)3.
 (2)4. LET: $n_1 = |B_1|$ and $B_1 = \{\alpha_1, \dots, \alpha_{n_1}\}$
 (2)5. ASSUME: We have chosen n_1, \dots, n_k strictly increasing and $\alpha_1, \dots, \alpha_{n_k}$ such that, for $1 \leq i \leq k$, we have $U(x_i^{\alpha, n}, B_i^{\alpha, n}) \subseteq U$
 (2)6. $x_{i+1}^{\alpha, n} \in U$
 PROOF: $x_{i+1}^{\alpha, n} \in P_1 \subseteq U$
 (2)7. PICK $C \subseteq^{\text{fin}} J$ such that $U(x_{i+1}^{\alpha, n}, C) \subseteq U$
 (2)8. LET: n_{i+1} and $\alpha_{n_{i+1}+1}, \dots, \alpha_{n_{i+1}}$ be such that $B_i^{\alpha, n} \cup C = B_{i+1}^{\alpha, n}$
 (2)9. $U(x_{i+1}^{\alpha, n}, B_{i+1}^{\alpha, n}) \subseteq U$
 (1)9. LET: $A = \{\alpha_i : i \geq 1\}$
 (1)10. LET: $y \in X$, $y_\beta = j$ if $\beta = \alpha_j$, $y_\beta = 2$ for $\beta \notin A$

- (1)11. PICK B such that $U(y, B) \subseteq V$
 (1)12. PICK i such that $A \cap B \subseteq B_i^{\alpha, n}$
 (1)13. $U(x_{i+1}^{\alpha, n}, B_{i+1}^{\alpha, n}) \cap U(y, B) \neq \emptyset$
 PROOF: $x_{i+1}^{\alpha, n} \in U(x_{i+1}^{\alpha, n}, B_{i+1}^{\alpha, n}) \cap U(y, B)$
 (1)14. Q.E.D.
 PROOF: This contradicts the fact that U and V are disjoint ((1)6).
 \square

Theorem 6.5.7 (Urysohn Lemma). *Let X be a normal space. Let A and B be disjoint closed subsets of X . Then there exists a continuous map $f : X \rightarrow [0, 1]$ such that $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$.*

PROOF:

- (1)1. LET: P be the set of all rational numbers in $[0, 1]$
 (1)2. For all $q \in P$, PICK an open set U_q in X such that $A \subseteq U_0$, $U_1 \subseteq X \setminus B$,
 and whenever $p < q$ then $\overline{U_p} \subseteq U_q$
 (2)1. PICK an enumeration (q_n) of P such that $q_1 = 1$ and $q_2 = 0$
 (2)2. LET: $U_1 = X \setminus B$
 (2)3. PICK an open set U_0 such that $A \subseteq U_0$ and $\overline{U_0} \subseteq U_1$
 (2)4. ASSUME: we have open sets U_1, U_0, \dots, U_{q_n} such that whenever $p < q$
 then $\overline{U_p} \subseteq U_q$
 (2)5. $q_2 < q_{n+1} < q_1$
 (2)6. LET: q_k be greatest among q_1, \dots, q_n such that $q_k < q_{n+1}$, and q_l be
 least such that $q_{n+1} < q_l$
 (2)7. PICK an open set $U_{q_{n+1}}$ such that $\overline{U_{q_k}} \subseteq U_{q_{n+1}}$ and $\overline{U_{q_{n+1}}} \subseteq U_{q_l}$
 (2)8. For all $p, q \in \{q_1, \dots, q_{n+1}\}$, if $p < q$ then $\overline{U_p} \subseteq U_q$
 (1)3. Extend the family (U_q) to \mathbb{Q} by defining: $U_q = \emptyset$ if $q < 0$ and $U_q = X$ if
 $q > 1$
 (1)4. For all rationals p, q with $p < q$ we have $\overline{U_p} \subseteq U_q$
 (1)5. Define $f : X \rightarrow [0, 1]$ by $f(x) = \inf\{q \in \mathbb{Q} : x \in U_q\}$
 PROOF: This set is nonempty since $x \in U_1$ and bounded below since if $x \in U_q$
 then $q \geq 0$.
 (1)6. For all $x \in A$ we have $f(x) = 0$
 (1)7. For all $x \in B$ we have $f(x) = 1$
 (1)8. If $x \in \overline{U_r}$ then $f(x) \leq r$
 (1)9. If $x \notin U_r$ then $f(x) \geq r$
 (1)10. f is continuous
 (2)1. LET: $x_0 \in X$
 (2)2. LET: (c, d) be an open interval containing $f(x_0)$
 PROVE: There exists a neighbourhood U of x_0 such that $f(U) \subseteq (c, d)$
 (2)3. PICK rationals p, q such that $c < p < f(x_0) < q < d$
 (2)4. $x \notin \overline{U_p}$
 PROOF: By (1)8
 (2)5. $x \in U_q$
 PROOF: By (1)9
 (2)6. LET: $U = U_q \setminus \overline{U_p}$
 \square

Definition 6.5.8 (Vanish Precisely). Let X be a set and $A \subseteq X$. Let $f : X \rightarrow [0, 1]$. Then f *vanishes precisely* on A iff $f^{-1}(0) = A$.

Theorem 6.5.9 (CC). Let X be a normal space and $A \subseteq X$. Then there exists a continuous function $f : X \rightarrow [0, 1]$ such that f vanishes precisely on A if and only if A is a closed G_δ set.

PROOF:

⟨1⟩1. If there exists f such that f vanishes precisely on A then A is closed.

PROOF: This holds because $A = f^{-1}(0)$.

⟨1⟩2. If there exists f such that f vanishes precisely on A then A is G_δ .

PROOF: This holds because $A = \bigcap_{q \in \mathbb{Q}^+} f^{-1}([0, q])$.

⟨1⟩3. If A is closed and G_δ then there exists f that vanishes precisely on A .

⟨2⟩1. LET: $A = \bigcap_{n=1}^{\infty} U_n$

⟨2⟩2. For $n \geq 1$, PICK $f_n : X \rightarrow [0, 1/2^n]$ such that $f_n(x) = 0$ for $x \in A$ and $f_n(x) = 1/2^n$ for $x \in X \setminus U_n$

PROOF: By the Urysohn Lemma.

⟨2⟩3. LET: $f : X \rightarrow [0, 1]$ be given by $f(x) = \sum_{n=1}^{\infty} f_n(x)$

PROOF: The series converges for every x by the Comparison Test.

⟨2⟩4. f is continuous

⟨3⟩1. f_n converges uniformly to f

PROOF: By the Weierstrass M-test.

⟨3⟩2. Q.E.D.

PROOF: By the Uniform Limit Theorem.

⟨2⟩5. $f(x) = 0$ for $x \in A$

PROOF: From ⟨2⟩2.

⟨2⟩6. $f(x) > 0$ for $x \notin A$

⟨3⟩1. LET: $x \notin A$

⟨3⟩2. PICK N such that $x \notin U_N$

⟨3⟩3. Q.E.D.

PROOF:

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \quad (\langle 2 \rangle 3)$$

$$\geq f_N(x)$$

$$> 0 \quad (\langle 2 \rangle 2)$$

□

Theorem 6.5.10 (Strong Form of Urysohn Lemma). Let X be a normal space. Then there exists a continuous function $f : X \rightarrow [0, 1]$ such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$ if and only if A and B are disjoint, closed and G_δ .

PROOF:

⟨1⟩1. If there exists a continuous function $f : X \rightarrow [0, 1]$ such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$ then A and B are disjoint, closed and G_δ

⟨2⟩1. ASSUME: there exists a continuous function $f : X \rightarrow [0, 1]$ such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$

- ⟨2⟩2. A and B are disjoint
- ⟨2⟩3. A is closed and G_δ
PROOF: By Theorem 6.5.9.
- ⟨2⟩4. B is closed and G_δ
PROOF: Apply Theorem 6.5.9 to $1 - f$.
- ⟨1⟩2. If A and B are disjoint, closed and G_δ then there exists a continuous function $f : X \rightarrow [0, 1]$ such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$
- ⟨2⟩1. ASSUME: A and B are disjoint, closed and G_δ
- ⟨2⟩2. PICK $g : X \rightarrow [0, 1]$ that vanishes precisely on A and $h : X \rightarrow [0, 1]$ that vanishes precisely on B
- ⟨2⟩3. LET: $f = g/(g + h)$

□

Definition 6.5.11 (Universal Extension Property). A topological space Y has the *universal extension property* iff, for every normal space X and closed subspace A of X , every continuous function $A \rightarrow Y$ can be extended to a continuous function $X \rightarrow Y$.

Theorem 6.5.12 (Tietze Extension Theorem (DC)). *Let X be a normal space. Let A be closed subspace of X .*

- 1. *Any continuous function $A \rightarrow [a, b]$ can be extended to a continuous function $X \rightarrow [a, b]$.*
- 2. *Any continuous function $A \rightarrow \mathbb{R}$ can be extended to a continuous function $X \rightarrow \mathbb{R}$.*

PROOF:

- ⟨1⟩1. Any continuous function $A \rightarrow [-1, 1]$ can be extended to a continuous function $X \rightarrow [-1, 1]$
- ⟨2⟩1. For every continuous function $f : A \rightarrow [-r, r]$, there exists a continuous $g : X \rightarrow \mathbb{R}$ such that

$$|g(x)| \leq \frac{1}{3}r \quad (x \in X)$$

$$|g(x) - f(x)| \leq \frac{2}{3}r \quad (x \in A)$$

- ⟨3⟩1. LET: $f : A \rightarrow [-r, r]$ be continuous

- ⟨3⟩2. LET: $I_1 = [-r, -\frac{1}{3}r]$

- ⟨3⟩3. LET: $I_2 = [-\frac{1}{3}r, \frac{1}{3}r]$

- ⟨3⟩4. LET: $I_3 = [\frac{1}{3}r, r]$

- ⟨3⟩5. LET: $B = f^{-1}(I_1)$

- ⟨3⟩6. LET: $C = f^{-1}(I_3)$

- ⟨3⟩7. PICK a continuous $g : X \rightarrow [-\frac{1}{3}r, \frac{1}{3}r]$ such that $g(x) = -\frac{1}{3}r$ for $x \in B$ and $g(x) = \frac{1}{3}r$ for $x \in C$

PROOF: By the Urysohn Lemma, since B and C are closed disjoint subsets of X .

- ⟨3⟩8. For all $x \in A$ we have $|g(x) - f(x)| \leq \frac{2}{3}r$

- ⟨4⟩1. LET: $x \in A$

⟨4⟩2. CASE: $f(x) \in I_1$

PROOF:

$$\begin{aligned} |g(x) - f(x)| &= \left| -\frac{1}{3}r - f(x) \right| & (x \in B) \\ &\leq \frac{2}{3}r & (f(x) \in I_1) \end{aligned}$$

⟨4⟩3. CASE: $f(x) \in I_2$

PROOF: In this case, $|g(x) - f(x)| \leq \frac{2}{3}r$ since $f(x), g(x) \in I_2$.

⟨4⟩4. CASE: $f(x) \in I_3$

PROOF:

$$\begin{aligned} |g(x) - f(x)| &= \left| \frac{1}{3}r - f(x) \right| & (x \in C) \\ &\leq \frac{2}{3}r & (f(x) \in I_3) \end{aligned}$$

⟨2⟩2. LET: $f : A \rightarrow [-1, 1]$ be continuous.

⟨2⟩3. PICK a sequence of functions (g_n) such that

$$|g_n(x)| \leq \frac{1}{3} \left(\frac{2}{3} \right)^{n-1} \quad (x \in X)$$

$$|f(x) - g_1(x) - \cdots - g_n(x)| \leq (2/3)^n \quad (x \in A)$$

PROOF: Given g_1, \dots, g_n , we apply ⟨2⟩1 with $f = f - g_1 - \cdots - g_n$ and $r = (2/3)^n$.

⟨2⟩4. LET: $g(x) = \sum_{n=1}^{\infty} g_n(x)$ for $x \in X$

PROOF: This series converges by the Comparison Test since $\sum_{n=1}^{\infty} (2/3)^n$ converges.

⟨2⟩5. g is continuous.

⟨3⟩1. $\sum_{n=1}^N g_n$ converges to g uniformly

PROOF: By the Weierstrass M -test.

⟨3⟩2. Q.E.D.

PROOF: By the Uniform Limit Theory.

⟨2⟩6. For all $x \in A$ we have $g(x) = f(x)$

PROOF: $|\sum_{n=1}^N g_n(x) - f(x)| \leq (2/3)^N \rightarrow 0$ as $N \rightarrow \infty$.

⟨2⟩7. For all $x \in X$ we have $-1 \leq g(x) \leq 1$

PROOF:

$$\begin{aligned} \left| \sum_{n=1}^N g_n(x) \right| &\leq \sum_{n=1}^N |g_n(x)| \\ &\leq \frac{1}{3} \sum_{n=1}^N (2/3)^{n-1} \\ &\rightarrow 2/3 & \text{as } n \rightarrow \infty \end{aligned}$$

⟨1⟩2. Any continuous function $A \rightarrow (-1, 1)$ can be extended to a continuous function $X \rightarrow (-1, 1)$

⟨2⟩1. LET: $f : A \rightarrow (-1, 1)$ be continuous

⟨2⟩2. PICK a continuous $g : X \rightarrow [-1, 1]$ that extends f

PROOF: By ⟨1⟩1.

⟨2⟩3. LET: $D = g^{-1}(-1) \cup g^{-1}(1)$

- (2)4. D is closed in X
 PROOF: Since g is continuous and $\{-1\}, \{1\}$ are closed in $[-1, 1]$.
 (2)5. $D \cap A = \emptyset$
 PROOF: Since $g(A) = f(A) \subseteq (-1, 1)$.
 (2)6. PICK a continuous $\phi : X \rightarrow [0, 1]$ such that $\phi(D) = \{0\}$ and $\phi(A) = \{1\}$
 PROOF: By the Urysohn Lemma.
 (2)7. LET: $h = g\phi$
 (2)8. h is continuous
 (2)9. h extends f
 (2)10. $\text{im } h \subseteq (-1, 1)$
 (1)3. Q.E.D.
 PROOF: The result follows because any closed interval in \mathbb{R} is homeomorphic to $[-1, 1]$ and $\mathbb{R} \cong (-1, 1)$.
 □

Lemma 6.5.13 (Shrinking Lemma (AC)). *Let X be a normal space. Let $\{U_\alpha\}_{\alpha \in J}$ be a point-finite indexed open covering of X . Then there exists an indexed open covering $\{V_\alpha\}_{\alpha \in J}$ such that $\overline{V_\alpha} \subseteq U_\alpha$ for all $\alpha \in J$.*

PROOF:

- (1)1. PICK a well-ordering \prec on J
 (1)2. PICK open sets V_α for $\alpha \in J$ such that $A_\alpha \subseteq V_\alpha$ and $\overline{V_\alpha} \subseteq U_\alpha$, where

$$A_\alpha = X \setminus \bigcup_{\beta \prec \alpha} V_\beta \cup \bigcup_{\alpha \prec \beta} U_\beta$$

PROOF: Apply transfinite induction to Proposition 13.1.16.

- (1)3. $\{V_\alpha\}_{\alpha \in J}$ covers X
 (2)1. LET: $x \in X$
 (2)2. LET: $\alpha_1, \dots, \alpha_n$ be the elements of J such that $x \in U_{\alpha_i}$, where $\alpha_1 \prec \dots \prec \alpha_n$
 PROVE: $x \in V_{\alpha_i}$ for some i
 (2)3. ASSUME: $x \notin V_{\alpha_1}, \dots, V_{\alpha_{n-1}}$
 (2)4. $x \in A_{\alpha_n}$
 (2)5. $x \in V_{\alpha_n}$
 □

Proposition 6.5.14 (DC). $S_\Omega \times \overline{S_\Omega}$ is not normal.

PROOF:

- (1)1. LET: $\Delta = \{(x, x) : x \in \overline{S_\Omega}\}$
 (1)2. Δ is closed in $\overline{S_\Omega}^2$
 (2)1. LET: $(x, y) \in \overline{S_\Omega}^2 \setminus \Delta$
 (2)2. PICK disjoint open sets U, V such that $x \in U$ and $y \in V$
 (2)3. $(x, y) \in U \times V \subseteq \overline{S_\Omega}^2 \setminus \Delta$
 (1)3. LET: $A = \Delta \cap (S_\Omega \times \overline{S_\Omega})$
 (1)4. A is closed in $S_\Omega \times \overline{S_\Omega}$
 (1)5. LET: $B = S_\Omega \times \{\Omega\}$
 (1)6. B is closed in $S_\Omega \times \overline{S_\Omega}$

- ⟨1⟩7. $A \cap B = \emptyset$
- ⟨1⟩8. ASSUME: for a contradiction U and V are disjoint open sets including A and B respectively
- ⟨1⟩9. PICK a sequence x_n in S_Ω such that $x_n < x_{n+1} < \Omega$ and $(x_n, x_{n+1}) \notin U$ for all n
 - ⟨2⟩1. LET: $x_n \in S_\Omega$
 - ⟨2⟩2. $(x_n, \Omega) \in V$
 - ⟨2⟩3. PICK open sets $W \subseteq S_\Omega$, $X \subseteq \overline{S_\Omega}$ such that $x_n \in W$, $\Omega \in X$ and $W \times X \subseteq V$
 - ⟨2⟩4. PICK $y < \Omega$ such that $(x_{n+1}, \Omega] \subseteq X$
 - ⟨2⟩5. LET: $x_{n+1} = y + 1$
- ⟨1⟩10. LET: b be the supremum of $\{x_n : n \geq 1\}$
- ⟨1⟩11. $(x_n, x_{n+1}) \rightarrow (b, b)$ as $n \rightarrow \infty$
- ⟨1⟩12. $(b, b) \in A$
- ⟨1⟩13. $(b, b) \in U$
- ⟨1⟩14. For all n we have $(x_n, x_{n+1}) \notin U$

□

Proposition 6.5.15 (AC). \mathbb{R}_l is normal.

PROOF:

- ⟨1⟩1. LET: A and B be disjoint closed sets in \mathbb{R}_l
- ⟨1⟩2. For $a \in A$, PICK $x_a > a$ such that $[a, x_a)$ not intersecting B
- ⟨1⟩3. For $b \in B$, PICK $x_b > b$ such that $[b, x_b)$ does not intersect A
- ⟨1⟩4. LET: $U = \bigcup_{a \in A} [a, x_a)$ and $V = \bigcup_{b \in B} [b, x_b)$
- ⟨1⟩5. U and V are disjoint open sets including A and B respectively.

□

Lemma 6.5.16. The set $L = \{(x, -x); x \in \mathbb{R}\}$ as a subspace of \mathbb{R}_l^2 is closed

- ⟨1⟩1. LET: $(x, y) \notin L$, so $y \neq -x$
 - PROVE: There exists a neighbourhood U of (x, y) that does not intersect L
- ⟨1⟩2. CASE: $y > -x$
 - PROOF: In this case, take $U = [x, +\infty) \times [y, +\infty)$
- ⟨1⟩3. CASE: $y < -x$
 - PROOF: In this case, take $U = [x, (x - y)/2) \times [y, (y - x)/2)$.

Proposition 6.5.17 (AC). The Sorgenfrey plane is not normal.

PROOF:

- ⟨1⟩1. ASSUME: for a contradiction the Sorgenfrey plane is normal.
- ⟨1⟩2. LET: $L = \{(x, -x); x \in \mathbb{R}\}$ as a subspace of \mathbb{R}_l^2
- ⟨1⟩3. L has the discrete topology.
 - ⟨2⟩1. LET: $(x, -x) \in L$
 - PROVE: $\{(x, -x)\}$ is open in L
 - ⟨2⟩2. $\{(x, -x)\} = ([x, +\infty) \times [-x, +\infty)) \cap L$
- ⟨1⟩4. Every subset of L is closed in \mathbb{R}_l^2

PROOF: By Corollary 4.3.4.2.

⟨1⟩5. For every nonempty proper subset A of L , PICK disjoint open sets U_A , V_A containing A and $L \setminus A$

PROOF: By ⟨1⟩1 and ⟨1⟩4.

⟨1⟩6. LET: $D = \mathbb{Q}^2$

⟨1⟩7. D is dense in \mathbb{R}_l^2

PROOF: Given any basic open set $[a, b) \times [c, d)$, pick rationals q, r such that $a \leq q < b$ and $c \leq r < d$. Then $(q, r) \in ([a, b) \times [c, d)) \cap D$

⟨1⟩8. LET: $\theta : \mathcal{P}L \rightarrow \mathcal{P}D$ be the function

$$\begin{aligned}\theta(A) &= U_A \cap D & (\emptyset \neq A \neq L) \\ \theta(\emptyset) &= \emptyset \\ \theta(L) &= D\end{aligned}$$

⟨1⟩9. θ is injective

⟨2⟩1. LET: $A, B \subseteq L$ with $\theta(A) = \theta(B)$

PROVE: $A = B$

⟨2⟩2. CASE: $\emptyset \neq A \neq L$ and $\emptyset \neq B \neq L$

⟨3⟩1. $A \subseteq B$

⟨4⟩1. LET: $x \in A$

⟨4⟩2. $x \in U_A$

PROOF: By ⟨1⟩5

⟨4⟩3. $x \in U_B$

PROOF: By ⟨2⟩1

⟨4⟩4. $x \notin L \setminus B$

PROOF: By ⟨1⟩5

⟨4⟩5. $x \in B$

PROOF: Since $x \in L$ by ⟨4⟩1

⟨3⟩2. $B \subseteq A$

PROOF: Similar.

⟨2⟩3. CASE: $\emptyset \neq A \neq L$ and $B = \emptyset$

PROOF: This implies $U_A \cap D = \emptyset$ which contradicts the fact that D is dense.

⟨2⟩4. CASE: $\emptyset \neq A \neq L$ and $B = L$

PROOF: This implies $V_A \cap D = \emptyset$ which contradicts the fact that D is dense.

⟨2⟩5. CASE: $A = B = \emptyset$

PROOF: Trivial

⟨2⟩6. CASE: $A = \emptyset$ and $B = L$

PROOF: This implies $D = \emptyset$ which is a contradiction.

⟨2⟩7. CASE: $A = B = L$

PROOF: Trivial

⟨1⟩10. Q.E.D.

PROOF: This is a contradiction since D is countable and L is uncountable.

□

Proposition 6.5.18. *The continuous image of a normal space is not necessarily normal.*

PROOF: The identity map from the discrete two-point space to the indiscrete

two-point space is continuous. \square

6.6 Completely Normal Spaces

Definition 6.6.1 (Completely Normal). A space X is *completely normal* iff every subspace is normal.

Proposition 6.6.2. *A subspace of a completely normal space is completely normal.*

PROOF: Immediate from definitions. \square

Proposition 6.6.3. *Let X be a topological space. Then X is completely normal iff X is T_1 and, for any pair of separated sets A, B in X , there exist disjoint open sets including them.*

PROOF:

$\langle 1 \rangle 1$. If X is completely normal then X is T_1 and, for any pair of separated sets A, B in X , there exist disjoint open sets including them.

$\langle 2 \rangle 1$. ASSUME: X is completely normal.

$\langle 2 \rangle 2$. X is T_1

PROOF: Holds because X is normal.

$\langle 2 \rangle 3$. For any pair of separated sets A, B in X , there exist disjoint open sets including them.

$\langle 3 \rangle 1$. LET: A and B be separated in X

$\langle 3 \rangle 2$. LET: $Y = X \setminus (\overline{A} \cap \overline{B})$

$\langle 3 \rangle 3$. PICK disjoint open sets U, V in Y such that $\overline{A} \cap Y \subseteq U$ and $\overline{B} \cap Y \subseteq V$

PROOF: Y is normal by $\langle 2 \rangle 1$.

$\langle 3 \rangle 4$. PICK open sets U_0, V_0 in X such that $U = U_0 \cap Y, V = V_0 \cap Y$

$\langle 3 \rangle 5$. $A \subseteq U_0 \setminus \overline{B}$ and $B \subseteq V_0 \setminus \overline{A}$

PROOF: Using $\langle 3 \rangle 1$.

$\langle 1 \rangle 2$. If X is T_1 and, for any pair of separated sets A, B in X , there exist disjoint open sets including them, then X is completely normal.

$\langle 2 \rangle 1$. ASSUME: X is T_1 and, for any pair of separated sets A, B in X , there exist disjoint open sets including them

$\langle 2 \rangle 2$. LET: $Y \subseteq X$

$\langle 2 \rangle 3$. Y is T_1

PROOF: By Proposition 6.1.3.

$\langle 2 \rangle 4$. LET: A and B be disjoint closed sets in Y

$\langle 2 \rangle 5$. A and B are separated in X

$\langle 3 \rangle 1$. $\overline{A} \cap Y = A$ and $\overline{B} \cap Y = B$

PROOF: By Proposition 3.7.6 and Theorem 4.3.4.

$\langle 3 \rangle 2$. $\overline{A} \cap B = \emptyset$

$$\begin{aligned} \overline{A} \cap B &= \overline{A} \cap \overline{B} \cap Y & (\langle 3 \rangle 1) \\ &= A \cap B & (\langle 3 \rangle 1) \\ &= \emptyset & (\langle 2 \rangle 4) \end{aligned}$$

⟨3⟩3. $A \cap \overline{B} = \emptyset$

PROOF: Similar.

⟨2⟩6. PICK disjoint open sets U and V that include A and B respectively.

PROOF: By ⟨2⟩1.

⟨2⟩7. $U \cap Y$ and $V \cap Y$ are disjoint open sets in Y that include A and B respectively.

□

Proposition 6.6.4. *A well-ordered set in the order topology is completely normal.*

PROOF:

⟨1⟩1. LET: X be a well-ordered set.

⟨1⟩2. For all $a, b \in X$ with $a < b$, we have $(a, b]$ is open.

⟨2⟩1. CASE: b is greatest in X

PROOF: This case holds by the definition of the order topology.

⟨2⟩2. CASE: b is not greatest in X

PROOF: In this case, $(a, b] = (a, c)$ where c is the successor of b .

⟨1⟩3. LET: A and B be separated sets in X

PROVE: There exist disjoint open sets U, V including A and B

⟨1⟩4. CASE: The least element of X is not in A or B

⟨2⟩1. LET: $U = \bigcup \{(x, a] : a \in A, x < a, (x, a] \cap B = \emptyset\}$

⟨2⟩2. LET: $V = \bigcup \{(y, b] : b \in B, y < b, (y, b] \cap A = \emptyset\}$

⟨2⟩3. U is open

PROOF: From ⟨1⟩2.

⟨2⟩4. V is open

PROOF: From ⟨1⟩2.

⟨2⟩5. $A \subseteq U$

⟨3⟩1. LET: $a \in A$

⟨3⟩2. PICK W a neighbourhood of a such that $W \cap B = \emptyset$

PROOF: By ⟨1⟩3.

⟨3⟩3. PICK $x < a$ such that $(x, a] \subseteq W$

PROOF: By Lemma 4.1.2

⟨3⟩4. $a \in (x, a] \subseteq U$

⟨2⟩6. $B \subseteq V$

PROOF: Similar.

⟨2⟩7. $U \cap V = \emptyset$

⟨1⟩5. CASE: $\perp \in A$

⟨2⟩1. PICK disjoint open sets U and V that include $A \setminus \{\perp\}$ and B

PROOF: From ⟨1⟩4.

⟨2⟩2. $U \cup \{\perp\}$ and V are disjoint open sets that include A and B

PROOF: $\{\perp\}$ is open because it is $(-\infty, a)$ where a is the successor of \perp .

⟨1⟩6. Q.E.D.

PROOF: By Proposition 6.6.3.

□

Proposition 6.6.5. *The product of two completely normal spaces is not necessarily completely normal.*

PROOF:

⟨1⟩1. S_Ω is completely normal.

PROOF: By Proposition 6.6.4

⟨1⟩2. $\overline{S_\Omega}$ is completely normal.

PROOF: By Proposition 6.6.4

⟨1⟩3. $S_\Omega \times \overline{S_\Omega}$ is not completely normal.

PROOF: By Proposition 6.5.3.

□

Proposition 6.6.6. *A compact Hausdorff space is not necessarily completely normal.*

PROOF:

⟨1⟩1. PICK an uncountable set J

⟨1⟩2. $[0, 1]^J$ is compact Hausdorff

PROOF: By Tychonoff's Theorem and Theorem 6.2.5.

⟨1⟩3. $(0, 1)^J$ is not normal.

PROOF: By Proposition 6.5.6, since $(0, 1) \cong \mathbb{R}$.

□

Proposition 6.6.7. *The space \mathbb{R}_l is completely normal.*

PROOF:

⟨1⟩1. LET: $X \subseteq \mathbb{R}_l$

⟨1⟩2. LET: A and B be disjoint closed sets in X .

⟨1⟩3. PICK closed sets C and D such that $A = C \cap X$ and $B = D \cap X$

⟨1⟩4. For $a \in A$, PICK $x_a > a$ such that $[a, x_a) \cap D = \emptyset$

⟨1⟩5. For $b \in B$, PICK $x_b > b$ such that $[b, x_b) \cap C = \emptyset$

⟨1⟩6. $\bigcup_{a \in A} [a, x_a) \cap X$ and $\bigcup_{b \in B} [b, x_b) \cap X$ are disjoint open sets in X that include A and B

□

6.7 Perfectly Normal Spaces

Definition 6.7.1 (Perfectly Normal). A space is *perfectly normal* iff it is normal and every closed set is G_δ .

Proposition 6.7.2. *Every perfectly normal space is completely normal.*

PROOF:

⟨1⟩1. LET: X be perfectly normal.

⟨1⟩2. LET: A and B be separated sets in X

⟨1⟩3. PICK continuous functions $f, g : X \rightarrow [0, 1]$ that vanish precisely on \overline{A} and \overline{B} , respectively.

PROOF: By Theorem 6.5.9.

⟨1⟩4. LET: $h = f - g$

⟨1⟩5. $B \subseteq h^{-1}((0, +\infty))$ and $A \subseteq h^{-1}((-\infty, 0))$

⟨1⟩6. Q.E.D.

PROOF: By Proposition 6.6.3.

□

Proposition 6.7.3. *The space $\overline{S_\Omega}$ is not perfectly normal.*

PROOF: The set $\{\Omega\}$ is not G_δ . □

Chapter 7

Countability Axioms

7.1 The First Countability Axiom

Definition 7.1.1 (First Countability Axiom). A topological space X satisfies the *first countability axiom*, or is *first countable*, iff every point has a countable local basis.

Proposition 7.1.2. S_Ω is first countable.

PROOF: For every countable ordinal $\alpha > 0$, the set $\{(\beta, \alpha + 1) : \beta < \alpha\}$ is a local basis at α . The set $\{\{0\}\}$ is a local basis at 0. \square

Theorem 7.1.3 (The Sequence Lemma (CC)). *Let X be a first countable space and $A \subseteq X$. If $x \in \overline{A}$, then there exists a sequence of points of A that converges to x .*

PROOF:

$\langle 1 \rangle 1$. LET: $x \in \overline{A}$

$\langle 1 \rangle 2$. PICK a countable basis $\{B_n\}_{n \in \mathbb{Z}^+}$ at x .

$\langle 1 \rangle 3$. For $n \geq 1$, PICK a point $a_n \in B_1 \cap \cdots \cap B_n \cap A$

PROVE: $a_n \rightarrow x$ as $n \rightarrow \infty$

PROOF: Using Countable Choice. Such an a_n exists because $B_1 \cap \cdots \cap B_n$ is a neighbourhood of x . Apply Theorem 3.8.3.

$\langle 1 \rangle 4$. LET: U be a neighbourhood of x

$\langle 1 \rangle 5$. PICK N such that $B_N \subseteq U$

PROOF: From $\langle 1 \rangle 2$.

$\langle 1 \rangle 6$. For $n \geq N$, we have $a_n \in U$

PROOF:

$$\begin{aligned} a_n &\in B_1 \cap \cdots \cap B_n && (\langle 1 \rangle 3) \\ &\subseteq B_N && (n \geq N) \\ &\subseteq U && (\langle 1 \rangle 5) \end{aligned}$$

\square

Theorem 7.1.4 (CC). *Let X and Y be topological spaces where X is first countable. Let $x \in X$. Suppose that, for every sequence $\{x_n\}_{n \geq 1}$ such that $x_n \rightarrow x$ as $n \rightarrow \infty$, we have $f(x_n) \rightarrow f(x)$ as $n \rightarrow \infty$. Then f is continuous at x .*

PROOF:

- $\langle 1 \rangle 1$. LET: V be a neighbourhood of $f(x)$
- $\langle 1 \rangle 2$. ASSUME: for a contradiction that, for every neighbourhood U of x , $f(U) \not\subseteq V$
- $\langle 1 \rangle 3$. PICK a countable local basis $\{B_n\}_{n \geq 1}$
- $\langle 1 \rangle 4$. For $n \geq 1$, PICK $a_n \in B_1 \cap \dots \cap B_n$ such that $f(a_n) \notin V$
- $\langle 1 \rangle 5$. $a_n \rightarrow x$ as $n \rightarrow \infty$

PROOF:

- $\langle 2 \rangle 1$. LET: U be a neighbourhood of x
- $\langle 2 \rangle 2$. PICK N such that $B_N \subseteq U$
- $\langle 2 \rangle 3$. For all $n \geq N$, $a_n \in U$

PROOF:

$$\begin{aligned} a_n &\in B_1 \cap \dots \cap B_n && (\langle 1 \rangle 4) \\ &\subseteq B_N && (n \geq N) \\ &\subseteq U && (\langle 2 \rangle 2) \end{aligned}$$

- $\langle 1 \rangle 6$. $f(a_n) \rightarrow f(x)$ as $n \rightarrow \infty$
- $\langle 1 \rangle 7$. There exists N such that, for all $n \geq N$, we have $f(a_n) \in V$
- $\langle 1 \rangle 8$. Q.E.D.

Lemma 7.1.5 (CC). \mathbb{R}^ω under the box topology is not first countable.

PROOF:

- $\langle 1 \rangle 1$. LET: $\{B_n\}_{n \geq 1}$ be any countable set of neighbourhoods of $\vec{0}$
 - $\langle 1 \rangle 2$. For $n \geq 1$, PICK U_{nm} for $m \geq 1$ such that $\vec{0} \in \prod_{m=1}^\infty U_{nm} \subseteq B_n$
 - $\langle 1 \rangle 3$. For $n \geq 1$, PICK a_n, b_n such that $0 \in (a_n, b_n) \subseteq U_{nn}$
 - $\langle 1 \rangle 4$. LET: $U = \prod_{n=1}^\infty (a_n/2, b_n/2)$
 - $\langle 1 \rangle 5$. $\vec{0} \in U$
 - $\langle 1 \rangle 6$. For all n , $B_n \not\subseteq U$
-

Lemma 7.1.6 (CC). *If J is uncountable then \mathbb{R}^J is not first countable.*

PROOF:

- $\langle 1 \rangle 1$. LET: $\{B_n\}_{n \geq 1}$ be a countable family of neighbourhoods of $\vec{0}$
 - $\langle 1 \rangle 2$. For $n \geq 1$, PICK $U_{n\alpha}$ such that $\vec{0} \in \prod_{\alpha \in J} U_{n\alpha} \subseteq B_n$ where $U_{n\alpha}$ is open in \mathbb{R} and $U_{n\alpha} = \mathbb{R}$ except for $\alpha = \alpha_{n1}, \dots, \alpha_{nr_n}$
 - $\langle 1 \rangle 3$. PICK β such that β is different from α_{ni} for all n, i
 - $\langle 1 \rangle 4$. LET: $V = \pi_\beta^{-1}((-1, 1))$
 - $\langle 1 \rangle 5$. $\vec{0} \in V$
 - $\langle 1 \rangle 6$. $V \not\subseteq B_n$ for all n
-

Lemma 7.1.7. \mathbb{R}_l is first countable.

PROOF: For all $x \in \mathbb{R}$, $\{[x, q) : q \in \mathbb{Q}, q > x\}$ is a basis at x . \square

Lemma 7.1.8. The ordered square is first countable.

PROOF:

$\langle 1 \rangle 1$. LET: $(x, y) \in I_o^2$

PROVE: There exists a countable local basis \mathcal{B} at (x, y)

$\langle 1 \rangle 2$. CASE: $(x, y) = (0, 0)$

PROOF: Take $\mathcal{B} = \{[(0, 0), (0, q)) : q \in \mathbb{Q}, 0 < q < 1\}$.

$\langle 1 \rangle 3$. CASE: $0 < y < 1$

PROOF: Take $\mathcal{B} = \{((x, q), (x, q')) : q, q' \in \mathbb{Q}, q < y < q'\}$.

$\langle 1 \rangle 4$. CASE: $x < 1, y = 1$

PROOF: Take $\mathcal{B} = \{((x, q), (q', 0)) : q, q' \in \mathbb{Q}, 0 < q < 1, x < q' < 1\}$.

$\langle 1 \rangle 5$. CASE: $x > 0, y = 0$

PROOF: Take $\mathcal{B} = \{((q, 1), (x, q')) : q, q' \in \mathbb{Q}, 0 < q < x, 0 < q' < 1\}$

$\langle 1 \rangle 6$. CASE: $(x, y) = (1, 1)$

PROOF: Take $\mathcal{B} = \{((1, q), (1, 1)) : q \in \mathbb{Q}, 0 < q < 1\}$.

\square

Proposition 7.1.9. A subspace of a first countable space is first countable.

PROOF:

$\langle 1 \rangle 1$. LET: X be a first countable space and $A \subseteq X$

$\langle 1 \rangle 2$. LET: $a \in A$

$\langle 1 \rangle 3$. PICK a countable basis \mathcal{B} at a in X

$\langle 1 \rangle 4$. $\{B \cap A : B \in \mathcal{B} \text{ is a countable basis at } a \text{ in } X\}$

\square

Proposition 7.1.10 (CC). A countable product of first countable spaces is first countable.

PROOF:

$\langle 1 \rangle 1$. LET: $\{X_n\}_{n \in \mathbb{Z}^+}$ be a countable family of first countable spaces.

$\langle 1 \rangle 2$. LET: $\vec{x} \in \prod_{n=1}^{\infty} X_n$

$\langle 1 \rangle 3$. PICK a countable basis \mathcal{B}_n at x_n in X_n for all n

$\langle 1 \rangle 4$. LET: \mathcal{B} be the set of all sets $\prod_{i=1}^n U_i$ where $U_i \in \mathcal{B}_i$ for finitely many i and $U_i = X_i$ for all other i .

$\langle 1 \rangle 5$. \mathcal{B} is a countable basis at \vec{x} in $\prod_{n=1}^{\infty} X_n$

\square

Corollary 7.1.10.1. The space \mathbb{R}^{ω} is first countable.

Proposition 7.1.11. The space S_{Ω} is first countable.

PROOF:

$\langle 1 \rangle 1$. LET: $\alpha \in S_{\Omega}$

PROVE: α has a countable local basis.

- (1)2. CASE: α is zero or a successor ordinal.
 PROOF: In this case, $\{\{\alpha\}\}$ is a local basis.
 (1)3. CASE: α is a limit ordinal.
 (2)1. PICK a countable sequence (β_n) with supremum α
 (2)2. $\{(\beta_n, \alpha + 1) : n \in \mathbb{Z}^+\}$ is a local basis.

□

Proposition 7.1.12. *The space $\overline{S_\Omega}$ is not first countable.*

PROOF:

- (1)1. ASSUME: for a contradiction \mathcal{B} is a countable local basis at Ω
 (1)2. LET: $\alpha = \sup\{\inf B : B \in \mathcal{B}\}$
 (1)3. $\alpha < \Omega$
 (1)4. There is no $B \in \mathcal{B}$ such that $B \subseteq (\alpha, +\infty)$

□

Proposition 7.1.13. *The continuous image of a first countable space is first countable.*

PROOF:

- (1)1. LET: X be a first countable space, Y a space and $f : X \rightarrow Y$ continuous.
 (1)2. LET: $y \in f(X)$
 (1)3. PICK $x \in X$ such that $y = f(x)$
 (1)4. PICK a countable local basis \mathcal{B} at x
 (1)5. $\{f(B) : B \in \mathcal{B}\}$ is a countable local basis at y .

□

Proposition 7.1.14. *$S_\Omega \times \overline{S_\Omega}$ is not first countable.*

PROOF: $(0, \Omega)$ has no countable basis. □

Proposition 7.1.15. *The Sorgenfrey plane is first countable.*

PROOF: For any point (a, b) , the set $\{[a, a + q) \times [b, b + r) : q, r \in \mathbb{Q}\}$ is a countable local basis at (a, b) . □

7.2 Separable Spaces

Definition 7.2.1 (Separable Space). A topological space X is *separable* iff it has a countable dense subset.

Proposition 7.2.2. *The space S_Ω is not separable.*

PROOF:

- (1)1. LET: $D \subseteq S_\Omega$ be countable.
 (1)2. LET: $\alpha = \sup D$
 (1)3. $\overline{D} \subseteq (-\infty, \alpha]$

□

Proposition 7.2.3. *The space $\overline{S_\Omega}$ is not separable.*

PROOF:

- ⟨1⟩1. LET: $D \subseteq S_\Omega$ be countable.
- ⟨1⟩2. LET: $\alpha = \sup\{\beta \in D : \beta < \Omega\}$
- ⟨1⟩3. $\alpha < \Omega$

PROOF: α is the supremum of countably many countable ordinals.

- ⟨1⟩4. $\overline{D} \subseteq (-\infty, \alpha] \cup \{\Omega\}$

□

Corollary 7.2.3.1. *Not every compact Hausdorff space is separable.*

Proposition 7.2.4. *Every open subspace of a separable space is separable.*

PROOF:

- ⟨1⟩1. LET: X be a separable space with countable dense subset D .
- ⟨1⟩2. LET: U be an open subspace of X
PROVE: $D \cap U$ is a countable dense subset of U .
- ⟨1⟩3. $D \cap U$ is countable.
- ⟨1⟩4. LET: V be an open set in U .
- ⟨1⟩5. V is open in X

PROOF: Lemma 4.3.3

- ⟨1⟩6. V intersects D
- ⟨1⟩7. V intersects $D \cap U$

□

Proposition 7.2.5 (CC). *The product of a countable family of separable spaces is separable.*

PROOF:

- ⟨1⟩1. LET: (X_n) be a countable family of separable spaces.
- ⟨1⟩2. For $n \geq 1$, PICK a dense set D_n in X_n
- ⟨1⟩3. $\prod_{n=1}^{\infty} D_n$ is dense in $\prod_{n=1}^{\infty} X_n$.

□

Proposition 7.2.6. *The continuous image of a separable space is separable.*

PROOF:

- ⟨1⟩1. LET: X be a separable space, Y a space and $f : X \rightarrow Y$ be continuous.
- ⟨1⟩2. PICK a countable dense set D in X
- ⟨1⟩3. $f(D)$ is dense in $f(X)$.

□

Corollary 7.2.6.1. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of nonempty topological spaces. If $\prod_{\alpha \in J} X_\alpha$ is separable then each X_α is separable.*

Corollary 7.2.6.2. $S_\Omega \times \overline{S_\Omega}$ is not separable.

Proposition 7.2.7. *The ordered square is not separable.*

PROOF: $\{\{x\} \times (0, 1) : x \in [0, 1]\}$ is an uncountable set of disjoint open sets. □

Proposition 7.2.8. \mathbb{R}_l is separable.

PROOF: \mathbb{Q} is dense. \square

Proposition 7.2.9. The Sorgenfrey plane is separable.

PROOF: \mathbb{Q}^2 is dense. \square

Proposition 7.2.10. Not every closed subspace of a separable space is separable.

PROOF: \mathbb{R}_l^2 is separable but the subspace $\{(x, -x) : x \in \mathbb{R}\}$ is not. \square

7.3 The Second Countability Axiom

Definition 7.3.1 (Second Countability Axiom). A topological space satisfies the *second countability axiom*, or is *second countable*, iff it has a countable basis.

Proposition 7.3.2. S_Ω is not second countable.

PROOF: $\{\{\alpha\} : \alpha \text{ is a countable successor ordinal}\}$ is an uncountable set of disjoint open sets. \square

Proposition 7.3.3. A subspace of a second countable space is second countable.

PROOF:

$\langle 1 \rangle 1.$ LET: X be a second countable space and $A \subseteq X$

$\langle 1 \rangle 2.$ PICK a countable basis \mathcal{B} for X

$\langle 1 \rangle 3.$ $\{B \cap A : B \in \mathcal{B}\}$ is a countable basis for A

\square

Proposition 7.3.4 (CC). The product of countably many second countable spaces is second countable.

PROOF:

$\langle 1 \rangle 1.$ LET: $\{X_n\}_{n \in \mathbb{Z}^+}$ be a countable family of second countable spaces.

$\langle 1 \rangle 2.$ For $n \in \mathbb{Z}^+$, PICK a countable basis \mathcal{B}_n for X_n .

$\langle 1 \rangle 3.$ LET: \mathcal{B} be the set of all sets of the form $\prod_{n=1}^{\infty} U_n$, where $U_n \in \mathcal{B}_n$ for finitely many n , and $U_n = X_n$ for all other n .

$\langle 1 \rangle 4.$ \mathcal{B} is a countable basis for $\prod_{n=1}^{\infty} X_n$

\square

Theorem 7.3.5 (CC). Every second countable space is separable.

PROOF:

$\langle 1 \rangle 1.$ LET: X be a second countable space.

$\langle 1 \rangle 2.$ PICK a countable basis \mathcal{B} for X

$\langle 1 \rangle 3.$ For $B \in \mathcal{B}$ nonempty, PICK a point $x_B \in B$

$\langle 1 \rangle 4.$ $D = \{x_B : B \in \mathcal{B} \setminus \{\emptyset\}\}$ is dense.

$\langle 2 \rangle 1.$ LET: $l \in X$

PROVE: $l \in \overline{D}$

(2)2. LET: $B \in \mathcal{B}$ such that $l \in B$

(2)3. $x_B \in B \cap D$

(2)4. Q.E.D.

PROOF: By Theorem 3.7.8

Corollary 7.3.5.1. $S_\Omega \times \overline{S_\Omega}$ is not second countable.

Corollary 7.3.5.2. The space \mathbb{R}^ω is separable.

Corollary 7.3.5.3. If J is uncountable then \mathbb{R}^J is not second countable.

Proposition 7.3.6. The ordered square is not second countable.

PROOF:

(1)1. LET: \mathcal{B} be any basis

(1)2. For $x \in [0, 1]$, PICK B_x such that $x \in B_x \subseteq ((x, 0), (x, 1))$

(1)3. The function $B_{(-)}$ is an injective function $[0, 1] \rightarrow \mathcal{B}$

(1)4. \mathcal{B} is uncountable.

□

Proposition 7.3.7. The space $\overline{S_\Omega}$ is not second countable.

PROOF: It is not first countable (Proposition 7.1.12). □

Proposition 7.3.8. The continuous image of a second countable space is second countable.

PROOF:

(1)1. LET: X be a second countable space, Y a space and $f : X \rightarrow Y$ be continuous.

(1)2. PICK a countable basis \mathcal{B} for X .

(1)3. $\{f(B) : B \in \mathcal{B}\}$ is a countable basis for $f(X)$

□

Theorem 7.3.9. Every regular Lindelöf space is normal.

PROOF:

(1)1. LET: X be a regular Lindelöf space.

(1)2. LET: A and B be disjoint closed sets in X .

(1)3. $\{U \text{ open in } X : \overline{U} \cap B = \emptyset\}$ covers A

PROOF: Proposition 6.3.2.

(1)4. PICK a countable open covering $\{U_n : n \in \mathbb{Z}^+\}$ of A such that $\overline{U_n} \cap B = \emptyset$ for all n

(1)5. PICK a countable open covering $\{V_n : n \in \mathbb{Z}^+\}$ of B such that $\overline{V_n} \cap A = \emptyset$ for all n

PROOF: Similar.

(1)6. For $n \in \mathbb{Z}^+$,

LET: $U'_n = U_n \setminus \bigcup_{i=1}^n \overline{V_i}$ and $V'_n = V_n \setminus \bigcup_{i=1}^n \overline{U_i}$

(1)7. LET: $U' = \bigcup_{n=1}^\infty U'_n$ and $V = \bigcup_{n=1}^\infty V'_n$

⟨1⟩8. $A \subseteq U'$ and $B \subseteq V'$

⟨1⟩9. $U' \cap V' = \emptyset$

□

Corollary 7.3.9.1. *If J is uncountable then \mathbb{R}^J is not Lindelöf.*

Proposition 7.3.10. *Every second countable regular space is completely normal.*

PROOF:

⟨1⟩1. LET: X be second countable and regular and $Y \subseteq X$

⟨1⟩2. Y is second countable

PROOF: Proposition 7.3.3.

⟨1⟩3. Y is regular

PROOF: Proposition 6.3.4

⟨1⟩4. Y is normal

PROOF: Theorem 7.3.9

□

Proposition 7.3.11. *The space \mathbb{R}^ω is second countable.*

PROOF: The sets $\prod_{n=0}^\infty U_n$ form a basis, where U_n is an interval of the form (q, r) for $q, r \in \mathbb{Q}$ for finitely many n , and $U_n = \mathbb{R}$ for all other n . □

Proposition 7.3.12 (CC). *In a second countable space, every discrete subspace is countable.*

PROOF:

⟨1⟩1. LET: X be a second countable space

⟨1⟩2. PICK a countable basis \mathcal{B}

⟨1⟩3. LET: $D \subseteq X$ be discrete

⟨1⟩4. For $a \in D$, PICK $B_a \in \mathcal{B}$ such that $B_a \cap D = \{a\}$

⟨1⟩5. $a \mapsto B_a$ is injective

□

Proposition 7.3.13. *The space \mathbb{R}_K is second countable.*

PROOF: $\{(a, b) : a, b \in \mathbb{R}\} \cup \{(a, b) - K : a, b \in \mathbb{Q}\}$ is a basis. □

Corollary 7.3.13.1. *The space \mathbb{R}_K is first countable.*

Corollary 7.3.13.2. *The space \mathbb{R}_K is separable.*

Proposition 7.3.14. *Let J be a set with $|J| > |\mathbb{R}|$. Then \mathbb{R}^J is not separable.*

PROOF:

⟨1⟩1. ASSUME: D is countable and dense in \mathbb{R}^J

PROVE: $|J| \leq |\mathbb{R}|$

⟨1⟩2. Define $f : J \rightarrow \mathcal{P}D$ by $f(\alpha) = D \cap \pi_\alpha^{-1}((0, 1))$

⟨1⟩3. f is injective

- $\langle 2 \rangle 1$. LET: $\alpha, \beta \in J$ with $\alpha \neq \beta$
 - $\langle 2 \rangle 2$. PICK $x \in D \cap \pi_\alpha^{-1}((0, 1)) \cap \pi_\beta^{-1}((2, 3))$
 - $\langle 2 \rangle 3$. $x \in f(\alpha)$ but $x \notin f(\beta)$
-

Corollary 7.3.14.1. *The product of a family of separable spaces is not necessarily separable.*

Chapter 8

Connectedness

8.1 Connected Spaces

Definition 8.1.1 (Separation). Let X be a topological space. A *separation* of X is a pair of disjoint nonempty subsets whose union is X .

Definition 8.1.2 (Connected). A topological space is *connected* iff it has no separation.

Proposition 8.1.3. S_Ω is not connected.

PROOF: $\{0\}$ and $S_\Omega \setminus \{0\}$ form a separation. \square

Proposition 8.1.4. A space X is connected if and only if the only sets that are both closed and open are \emptyset and X .

PROOF: Immediate from definitions. \square

Proposition 8.1.5. Let Y be a subspace of X . Then a separation of Y is a pair of disjoint nonempty sets A, B such that $A \cup B = Y$ and neither of A, B contains a limit point of the other.

PROOF:

$\langle 1 \rangle 1$. If A and B form a separation of Y then A and B are disjoint and nonempty, $A \cup B = Y$, and neither of A, B contains a limit point of the other.

$\langle 2 \rangle 1$. LET: A and B be a separation of Y

$\langle 2 \rangle 2$. A and B are disjoint and nonempty and $A \cup B = Y$

PROOF: Immediate from the definition of separation.

$\langle 2 \rangle 3$. A does not contain a limit point of B

PROOF: B is closed in Y , hence contains all its limit points (Corollary 3.10.3.1), and so the result follows because A and B are disjoint.

$\langle 2 \rangle 4$. B does not contain a limit point of A

PROOF: Similar.

$\langle 1 \rangle 2$. If A and B are disjoint and nonempty, $A \cup B = Y$, and neither of A, B contains a limit point of the other, then A and B are a separation of Y .

- (2)1. ASSUME: A and B are disjoint and nonempty, $A \cup B = Y$, and neither of A, B contains a limit point of the other
 (2)2. A is closed in Y
 PROOF: Every limit point of A is not in B , so is in A . Apply Corollary 3.10.3.1.
 (2)3. B is open in Y
 PROOF: $B = Y \setminus A$
 (2)4. A is open in Y
 PROOF: Similar.

□

Proposition 8.1.6. *If the sets C and D form a separation of X , and Y is a connected subspace of X , then $Y \subseteq C$ or $Y \subseteq D$.*

PROOF: Otherwise, $Y \cap C$ and $Y \cap D$ would be a separation of Y . □

Proposition 8.1.7. *The union of a set of connected subspaces of X that have a point in common is connected.*

PROOF:

- (1)1. LET: \mathcal{S} be a set of connected subspaces that have the point a in common.
 (1)2. ASSUME: for a contradiction U and V form a separation of $\bigcup \mathcal{S}$
 (1)3. ASSUME: w.l.o.g. $a \in U$
 (1)4. For all $Y \in \mathcal{S}$ we have $Y \subseteq U$
 PROOF: By Proposition 8.1.6.
 (1)5. $V = \emptyset$
 (1)6. Q.E.D.
 PROOF: This contradicts (1)2.

□

Theorem 8.1.8. *Let A be a connected subspace of X . If $A \subseteq B \subseteq \overline{A}$ then B is connected.*

PROOF:

- (1)1. ASSUME: for a contradiction U and V are a separation of B
 (1)2. $A \subseteq U$ or $A \subseteq V$
 PROOF: By Proposition 8.1.6.
 (1)3. ASSUME: w.l.o.g. $A \subseteq U$
 (1)4. $\overline{A} \subseteq \overline{U}$
 PROOF: By Proposition 3.7.5.
 (1)5. $B \subseteq \overline{U}$
 PROOF: Since $B \subseteq \overline{A}$.
 (1)6. The closure of U in B is B
 PROOF: By Theorem 4.3.4.
 (1)7. $U = B$
 PROOF: Since U is closed in B .
 (1)8. Q.E.D.
 PROOF: This contradicts (1)1.

□

Theorem 8.1.9. *The image of a connected space under a continuous map is connected.*

PROOF: Let X be a connected space, Y a topological space, and $f : X \rightarrow Y$ be surjective. If U and V form a separation of Y , then $f^{-1}(U)$ and $f^{-1}(V)$ form a separation of X . □

Corollary 8.1.9.1. *Let \mathcal{T} and \mathcal{T}' be topologies on the same set X . If $\mathcal{T} \subseteq \mathcal{T}'$ and X is connected under \mathcal{T}' then X is connected under \mathcal{T} .*

Corollary 8.1.9.2. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces. If $\prod_{\alpha \in J} X_\alpha$ is connected then each X_α is connected.*

Corollary 8.1.9.3. *The Sorgenfrey plane is disconnected.*

Proposition 8.1.10. *The product of a family of connected spaces is connected.*

PROOF:

⟨1⟩1. The product of two connected spaces is connected.

PROOF:

⟨2⟩1. LET: X and Y be connected spaces.

⟨2⟩2. ASSUME: w.l.o.g. X and Y are nonempty.

PROOF: If either is empty then $X \times Y = \emptyset$ is connected.

⟨2⟩3. ASSUME: for a contradiction U and V are a separation of $X \times Y$.

⟨2⟩4. PICK $b \in Y$

PROOF: By ⟨2⟩2.

⟨2⟩5. For all $x \in X$,

LET: $T_x = (X \times \{b\}) \cup (\{x\} \times Y)$

⟨2⟩6. For all $x \in X$, T_x is connected

⟨3⟩1. $X \times \{b\}$ is connected

PROOF: It is homeomorphic to X .

⟨3⟩2. $\{x\} \times Y$ is connected

PROOF: It is homeomorphic to Y .

⟨3⟩3. Q.E.D.

PROOF: By Proposition 8.1.7.

⟨2⟩7. $X \times Y = \bigcup_{x \in X} T_x$

⟨2⟩8. Q.E.D.

⟨3⟩1. PICK $a \in X$

PROOF: By ⟨2⟩2.

⟨3⟩2. $(a, b) \in T_x$ for all $x \in X$

⟨3⟩3. Q.E.D.

PROOF: By Proposition 8.1.7.

⟨1⟩2. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of connected spaces.

⟨1⟩3. ASSUME: w.l.o.g. $\prod_{\alpha \in J} X_\alpha$ is nonempty

⟨1⟩4. PICK $\vec{a} \in \prod_{\alpha \in J} X_\alpha$

⟨1⟩5. For K a finite subset of J ,

- LET: $X_K = \{\vec{x} \in \prod_{\alpha \in J} X_\alpha : x_\alpha = a_\alpha \text{ for all } \alpha \in J \setminus K\}$
- $\langle 1 \rangle 6$. For all K , X_K is connected.
- PROOF: It is homeomorphic to $\prod_{\alpha \in K} X_\alpha$, so it is connected by $\langle 1 \rangle 1$.
- $\langle 1 \rangle 7$. $\bigcup_{K \subseteq \text{fin } J} X_K$ is connected.
- PROOF: By Proposition 8.1.7 since $\vec{a} \in X_K$ for all K .
- $\langle 1 \rangle 8$. $\prod_{\alpha \in J} X_\alpha = \bigcup_{K \subseteq \text{fin } J} X_K$
- $\langle 2 \rangle 1$. LET: $\vec{x} \in \prod_{\alpha \in J} X_\alpha$
- $\langle 2 \rangle 2$. LET: U be an open neighbourhood of \vec{x}
- $\langle 2 \rangle 3$. PICK a basic open set $\prod_{\alpha \in J} V_\alpha$ such that $\vec{x} \in \prod_{\alpha \in J} V_\alpha \subseteq U$, where each V_α is open in X_α , and $V_\alpha = X_\alpha$ except for $\alpha \in K$ for some finite $K \subseteq J$
- PROVE: U intersects X_K
- $\langle 2 \rangle 4$. LET: $\vec{y} \in \prod_{\alpha \in J} X_\alpha$ with $y_\alpha = x_\alpha$ for $\alpha \in K$, $y_\alpha = a_\alpha$ for $\alpha \notin K$
- $\langle 2 \rangle 5$. $\vec{y} \in U \cap X_K$
- $\langle 1 \rangle 9$. Q.E.D.

Corollary 8.1.10.1. *For any set I , the space \mathbb{R}^I under the product topology is connected.*

Proposition 8.1.11. \mathbb{R}^ω under the box topology is disconnected.

PROOF: The set of all bounded sequences and the set of all unbounded sequences form a separation. \square

Definition 8.1.12 (Totally Disconnected). A space is *totally disconnected* iff the only connected subspaces are the singletons.

Theorem 8.1.13. *Let L be a linearly ordered set under the order topology. Then L is connected if and only if L is a linear continuum.*

PROOF:

- $\langle 1 \rangle 1$. If L is a linear continuum then L is connected.
- $\langle 2 \rangle 1$. LET: L be a linear continuum.
- $\langle 2 \rangle 2$. ASSUME: for a contradiction U and V are a separation of L .
- $\langle 2 \rangle 3$. PICK $a \in U$ and $b \in V$
- $\langle 2 \rangle 4$. ASSUME: w.l.o.g. $a < b$
- $\langle 2 \rangle 5$. LET: $l = \sup\{x \in A : x < b\}$
- $\langle 2 \rangle 6$. CASE: $l \in A$
- $\langle 3 \rangle 1$. PICK $a' > l$ such that $[l, a') \subseteq A$
- PROOF: By Lemma 4.1.2. We know l is not greatest in X because $l < b$.
- $\langle 3 \rangle 2$. PICK a^* such that $l < a^* < a'$
- PROOF: L is dense.
- $\langle 3 \rangle 3$. $l < a^*$, $a^* \in A$, $a^* < b$
- PROOF: If $b < a^*$ then $b \in A$ by $\langle 3 \rangle 1$.
- $\langle 3 \rangle 4$. Q.E.D.
- PROOF: This contradicts $\langle 2 \rangle 5$.
- $\langle 2 \rangle 7$. CASE: $l \in B$
- $\langle 3 \rangle 1$. PICK $b' < l$ such that $(b', l] \subseteq B$

PROOF: By Lemma 4.1.2. We know l is not least in X because $a < l$.

⟨3⟩2. PICK b^* such that $b' < b^* < l$

PROVE: b^* is an upper bound for $\{x \in A : x < b\}$

⟨3⟩3. LET: $x \in A$ and $x < b$

⟨3⟩4. $x \leq b^*$

PROOF: If $b^* < x$ then $b^* < x \leq l$ and so $x \in B$ by ⟨3⟩1.

⟨3⟩5. Q.E.D.

PROOF: This contradicts ⟨2⟩5.

⟨1⟩2. If L is connected then L is a linear continuum.

⟨2⟩1. ASSUME: L is connected

⟨2⟩2. L has the least upper bound property

⟨3⟩1. ASSUME: for a contradiction $A \subseteq L$ is bounded above with no least upper bound

⟨3⟩2. LET: U be the set of upper bounds of A

⟨3⟩3. U is open

⟨4⟩1. LET: $u \in U$

⟨4⟩2. PICK an upper bound v for A with $v < u$

PROOF: u is not the least upper bound for A (⟨3⟩1)

⟨4⟩3. $u \in (v, +\infty) \subseteq U$

⟨3⟩4. LET: V be the set of lower bounds of U

⟨3⟩5. U and V form a separation of L

⟨4⟩1. V is open

PROOF: Similar to ⟨3⟩3.

⟨4⟩2. U and V are disjoint

⟨5⟩1. ASSUME: for a contradiction $x \in U \cap V$

⟨5⟩2. PICK $u \in U$ such that $u < x$

PROOF: x is not the lowest upper bound of A

⟨5⟩3. $x \leq u < x$

⟨4⟩3. $U \cup V = L$

⟨5⟩1. LET: $x \in L \setminus U$

⟨5⟩2. PICK $a \in A$ such that $x < a$

⟨5⟩3. $a \in V$

⟨5⟩4. $x \in V$

⟨2⟩3. For all $x, y \in L$, there exists $z \in L$ such that $x < z < y$

PROOF: Otherwise $(-\infty, y)$ and $(x, +\infty)$ would form a separation of L .

□

Corollary 8.1.13.1. *The real line \mathbb{R} is connected, and so is every ray and interval in \mathbb{R} .*

Corollary 8.1.13.2. *The ordered square is connected.*

Corollary 8.1.13.3. *Not every closed subspace of a connected space is connected.*

PROOF: The set $\{0, 1\}$ is disconnected as a subspace of \mathbb{R} .

Corollary 8.1.13.4. *Not every open subspace of a connected space is connected.*

PROOF: The space $\mathbb{R} \setminus \{0\}$ is a disconnected open subspace of \mathbb{R} . \square

Theorem 8.1.14 (Intermediate Value Theorem). *Let X be a connected space and Y a linearly ordered set under the order topology. Let $f : X \rightarrow Y$ be continuous. Let $a, b \in X$ and $r \in Y$. If $f(a) < r < f(b)$, then there exists $c \in X$ such that $f(c) = r$.*

PROOF: If not, then $f^{-1}((-\infty, r))$ and $f^{-1}((r, +\infty))$ would be a separation of X . \square

Proposition 8.1.15. *Every connected regular space with more than one point is uncountable.*

PROOF:

$\langle 1 \rangle 1$. Every connected completely regular space with more than one point is uncountable.

$\langle 2 \rangle 1$. LET: X be connected and completely regular and $a, b \in X$ with $a \neq b$

$\langle 2 \rangle 2$. PICK a continuous $f : X \rightarrow [0, 1]$ such that $f(a) = 0$ and $f(b) = 1$

$\langle 2 \rangle 3$. f is surjective.

PROOF: By the Intermediate Value Theorem.

$\langle 1 \rangle 2$. Every connected regular space with more than one point is uncountable.

$\langle 2 \rangle 1$. ASSUME: for a contradiction X is connected, regular and countable with more than one point.

$\langle 2 \rangle 2$. X is Lindelöf

$\langle 2 \rangle 3$. X is normal

PROOF: By Theorem 7.3.9

$\langle 2 \rangle 4$. Q.E.D.

PROOF: Contradicting $\langle 1 \rangle 1$.

\square

Proposition 8.1.16. $\overline{S_\Omega}$ is not connected.

PROOF: $\{0\}$ is clopen. \square

Proposition 8.1.17. \mathbb{R}_l is not connected.

PROOF: The set $[0, +\infty)$ is clopen. \square

Proposition 8.1.18. *The space \mathbb{R}^ω under the uniform topology is not connected.*

PROOF: The set of all bounded sequences and the set of all unbounded sequences form a separation. \square

Proposition 8.1.19. *The space \mathbb{R}_K is connected.*

PROOF: Easy. \square

8.2 Components and Local Connectedness

Definition 8.2.1 ((Connected) Component). Let X be a topological space. Define an equivalence relation \sim on X by: $x \sim y$ iff there exists a connected subspace $U \subseteq X$ such that $x \in U$ and $y \in U$. The *(connected) components* of X are the equivalence classes under \sim .

We prove this is an equivalence relation.

PROOF:

$\langle 1 \rangle 1$. For all $x \in X$ we have $x \sim x$.

PROOF: The subspace $\{x\} \subseteq X$ is connected.

$\langle 1 \rangle 2$. For all $x, y \in X$, if $x \sim y$ then $y \sim x$.

PROOF: Immediate from definitions.

$\langle 1 \rangle 3$. For all $x, y, z \in X$, if $x \sim y$ and $y \sim z$ then $x \sim z$.

PROOF: By Proposition 8.1.7.

□

Proposition 8.2.2. *Let X be a topological space. If $C \subseteq X$ is connected and nonempty, then there exists a unique component D of X such that $C \subseteq D$.*

PROOF:

$\langle 1 \rangle 1$. PICK $a \in C$

$\langle 1 \rangle 2$. LET: D be the \sim -equivalence class of A

$\langle 1 \rangle 3$. $C \subseteq D$

PROOF: For all $x \in C$ we have $a \sim x$ by definition.

$\langle 1 \rangle 4$. D is unique

PROOF: This holds because the components are disjoint.

□

Proposition 8.2.3 (AC). *Every component is connected.*

PROOF:

$\langle 1 \rangle 1$. LET: C be a component of the topological space X

$\langle 1 \rangle 2$. PICK $a \in C$

$\langle 1 \rangle 3$. For all $x \in C$, PICK a connected subspace C_x of X containing both a and x .

PROOF: Such a C_x exists since $a \sim x$.

$\langle 1 \rangle 4$. $C = \bigcup_{x \in C} C_x$

PROOF: This holds because $C_x \subseteq C$ by Proposition 8.2.2.

$\langle 1 \rangle 5$. Q.E.D.

PROOF: It follows that C is connected by Proposition 8.1.7.

□

Proposition 8.2.4. *Every component is closed.*

PROOF: From Theorem 8.1.8. □

Proposition 8.2.5. *The component of \vec{a} in \mathbb{R}^ω under the uniform topology is $\{\vec{b} : \vec{b} - \vec{a} \text{ is bounded}\}$.*

PROOF:

- ⟨1⟩1. $C = \{\vec{b} : \vec{b} - \vec{a} \text{ is bounded}\}$ is connected.
- ⟨2⟩1. ASSUME: $C = U \cup V$ is a separation of C with $\vec{a} \in U$
- ⟨2⟩2. PICK $\vec{b} \in V$
- ⟨2⟩3. $\{\epsilon : \epsilon\vec{b} + (1 - \epsilon)\vec{a} \in U\}$ and $\{\epsilon : \epsilon\vec{b} + (1 - \epsilon)\vec{a} \in V\}$ form a separation of $[0, 1]$
- ⟨1⟩2. If $\vec{a}, \vec{b} \in C$ and $\vec{b} - \vec{a}$ is unbounded then C is disconnected.

PROOF: $\{\vec{c} : \vec{c} - \vec{a} \text{ is bounded}\}$ and $\{\vec{c} : \vec{c} - \vec{a} \text{ is unbounded}\}$

□

Proposition 8.2.6. *Let $x, y \in \mathbb{R}^\omega$ under the box topology. Then x and y are in the same component iff $x - y$ is eventually zero.*

PROOF:

- ⟨1⟩1. For all $x \in \mathbb{R}^\omega$ the set $\{y : x - y \text{ is eventually zero}\}$ is connected
- PROOF: It is the union of the sets $C_N = \{y : \forall n \geq N. y_n = 0\}$, each of which is connected because it is homeomorphic to \mathbb{R}^{N-1} .
- ⟨1⟩2. If $x - y$ is not eventually zero then x and y are in different components
- ⟨2⟩1. ASSUME: $x - y$ is not eventually zero
- ⟨2⟩2. Define $h : \mathbb{R}^\omega \rightarrow \mathbb{R}^\omega$ by: $h(z)_n = \begin{cases} z_n - x_n & \text{if } x_n = y_n \\ n(z_n - x_n)/(y_n - x_n) & \text{if } x_n \neq y_n \end{cases}$
- ⟨2⟩3. h is an automorphism of \mathbb{R}^ω under the box topology
- ⟨2⟩4. $h(x) = 0$
- ⟨2⟩5. $h(y)$ is unbounded
- ⟨2⟩6. Q.E.D.

PROOF: The inverse image under h of the set of bounded sequences and the set of unbounded sequences form a separation of \mathbb{R}^ω with x and y in different sets.

□

□

8.3 Path Connectedness

Definition 8.3.1 (Path). Let X be a topological space and $a, b \in X$. A *path* from a to b is a continuous function $p : [0, 1] \rightarrow X$ such that $p(0) = a$ and $p(1) = b$.

Definition 8.3.2 (Path Connected). A topological space is *path connected* iff there exists a path between any two points.

Proposition 8.3.3. *Every path connected space is connected.*

PROOF:

- ⟨1⟩1. LET: X be a path connected space
- ⟨1⟩2. ASSUME: for a contradiction U and V are a separation of X .
- ⟨1⟩3. PICK $a \in U$ and $b \in V$

- ⟨1⟩4. PICK a path $p : [0, 1] \rightarrow X$ from a to b
- ⟨1⟩5. $p^{-1}(U)$ and $p^{-1}(V)$ form a separation of $[0, 1]$.
- ⟨1⟩6. Q.E.D.

PROOF: This contradicts Corollary 8.1.13.1.

Corollary 8.3.3.1. S_Ω is not path connected.

Corollary 8.3.3.2. $\overline{S_\Omega}$ is not path connected.

Corollary 8.3.3.3. \mathbb{R}_l is not path connected.

Corollary 8.3.3.4. The Sorgenfrey plane is not path connected.

Corollary 8.3.3.5. The space \mathbb{R}^ω under the uniform topology is not path connected. connected.

Corollary 8.3.3.6. The space \mathbb{R}^ω under the box topology is not path connected.

Proposition 8.3.4. The long line is path connected.

PROOF:

- ⟨1⟩1. LET: $a, b \in L$
- ⟨1⟩2. PICK an ordinal α such that $a, b < (\alpha, 0)$
- ⟨1⟩3. There exists a path from a to b

PROOF: This holds because $[(0, 0), (\alpha, 0))$ is homeomorphic to $[0, 1)$ by Proposition 1.4.11.

□

Corollary 8.3.4.1. Not every closed subspace of a path connected space is path connected.

PROOF: Take any two-element subspace of the long line.

Corollary 8.3.4.2. Not every open subspace of a path connected space is path connected.

PROOF: The space $\mathbb{R} \setminus \{0\}$ is not path connected as a subspace of \mathbb{R} . □

Definition 8.3.5 (Path Component). Let X be a topological space. Define an equivalence relation \sim on X by: $x \sim y$ iff there exists a path from x to y . The equivalence classes are called the *path components* of X .

We prove this is an equivalence relation.

PROOF:

- ⟨1⟩1. For all $x \in X$ we have $x \sim x$

PROOF: The constant path $p : [0, 1] \rightarrow X$ where $p(t) = x$ is a path from x to x .

- ⟨1⟩2. If $x \sim y$ then $y \sim x$

PROOF: If $p : [0, 1] \rightarrow X$ is a path from x to y then $\lambda t.p(1 - t)$ is a path from y to x .

- ⟨1⟩3. If $x \sim y$ and $y \sim z$ then $x \sim z$

(2)1. LET: p be a path from x to y and q be a path from y to z .

(2)2. LET: $r : [0, 1] \rightarrow X$ where

$$r(t) = \begin{cases} p(2t) & \text{if } 0 \leq t \leq 1/2 \\ q(2t - 1) & \text{if } 1/2 \leq t \leq 1 \end{cases}$$

(2)3. r is a path from x to z .

PROOF: r is continuous by the Pasting Lemma.

□

Proposition 8.3.6. *Every path component is path connected.*

PROOF: By definition, if x and y are in the same path component then there is a path from x to y . □

Proposition 8.3.7. *If A is a nonempty path connected subspace of the space X , then A is included in a unique path component.*

PROOF:

(1)1. PICK $a \in A$

(1)2. LET: C be the equivalence class of a under \sim

(1)3. $A \subseteq C$

PROOF: For all $x \in A$, there exists a path from a to x .

(1)4. C is unique

PROOF: C is the unique path component such that $a \in C$.

□

Proposition 8.3.8. *Every path component is included in a component.*

PROOF: From Propositions 8.3.3 and 8.2.2. □

Proposition 8.3.9. *The ordered square is not path connected.*

PROOF:

(1)1. ASSUME: for a contradiction $p : [0, 1] \rightarrow I_o^2$ is a path from $(0, 0)$ to $(1, 1)$.

(1)2. For all $x \in [0, 1]$, $p^{-1}(\{x\} \times (0, 1))$ is open in $[0, 1]$

(1)3. For all $x \in [0, 1]$, PICK a rational $q_x \in p^{-1}(\{x\} \times (0, 1))$

(1)4. $\{q_x : x \in [0, 1]\}$ is an uncountable set of rationals.

□

Proposition 8.3.10 (AC). *The product of a family of path connected spaces is path connected.*

PROOF:

(1)1. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of path connected spaces and $a, b \in \prod_{\alpha \in J} X_\alpha$

(1)2. For $\alpha \in J$, PICK a path $p_\alpha : [0, 1] \rightarrow X_\alpha$ from a_α to b_α

(1)3. Define $p : [0, 1] \rightarrow \prod_{\alpha \in J} X_\alpha$ by $p(t)_\alpha = p_\alpha(t)$

(1)4. p is a path from a to b

PROOF: By Theorem 5.2.15.

□

Corollary 8.3.10.1. *For any set I , the space \mathbb{R}^I in the product topology is path connected.*

Proposition 8.3.11. *The space \mathbb{R}_K is not path connected.*

PROOF:

<1>1. ASSUME: for a contradiction $p : [0, 1] \rightarrow \mathbb{R}_K$ is a path from 0 to 1

<1>2. LET: $p : [0, 1] \rightarrow \mathbb{R}_K$ be a path from 0 to 1

<1>3. $p([0, 1])$ is compact and connected in \mathbb{R}_K .

PROOF: Theorem 8.1.9 and Proposition 9.4.10.

<1>4. $p([0, 1])$ is connected in \mathbb{R} .

PROOF: Corollary 8.1.9.1

<1>5. $[0, 1] \subseteq p([0, 1])$

PROOF: For any $x \in [0, 1]$, if $x \notin p([0, 1])$ then $p([0, 1]) \cap (-\infty, x)$ and $p([0, 1]) \cap (x, +\infty)$ form a separation of $p([0, 1])$.

<1>6. $[0, 1]$ is compact in \mathbb{R}_K

PROOF: Proposition 9.4.6.

<1>7. Q.E.D.

PROOF: This contradicts Corollary 9.4.11.2.

□

Proposition 8.3.12. *Let $f : X \rightarrow Y$ be continuous and surjective. If X is path connected then Y is path connected.*

PROOF:

<1>1. LET: $a, b \in Y$

<1>2. PICK $x, y \in X$ such that $f(x) = a$ and $f(y) = b$

<1>3. PICK a path $p : [0, 1] \rightarrow X$ such that $p(0) = x$ and $p(1) = y$

<1>4. $f \circ p$ is a path from a to b

□

Corollary 8.3.12.1. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of non-empty topological spaces. If $\prod_{\alpha \in J} X_\alpha$ is path connected then each X_α is path connected.*

8.4 Connected Subspaces of Euclidean Space

Definition 8.4.1 (Unit 2-Sphere). The *unit 2-sphere* is $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ as a subspace of \mathbb{R}^3 .

Definition 8.4.2 (Unit Ball). For any $n \geq 1$, the *closed unit ball* in \mathbb{R}^n is

$$B^n = \{\vec{x} \in \mathbb{R}^n : \|\vec{x}\| \leq 1\} .$$

Proposition 8.4.3. *Every open unit ball and closed unit ball in \mathbb{R}^n is path connected.*

PROOF: The straight line between any two points is a path in the ball. □

Definition 8.4.4 (Punctured Euclidean Space). For $n \geq 1$, *punctured Euclidean space* is $\mathbb{R}^n \setminus \{\vec{0}\}$.

Proposition 8.4.5. *Punctured Euclidean space in \mathbb{R}^n is path connected iff $n > 1$.*

PROOF: Easy. \square

Definition 8.4.6 (Unit Sphere). For $n \geq 1$, the *unit sphere* S^n is $\{\vec{x} \in \mathbb{R}^{n+1} : \|\vec{x}\| = 1\}$.

Proposition 8.4.7. *In any number of dimensions, the unit sphere is path connected.*

PROOF: Easy. \square

Definition 8.4.8 (Topologist's Sine Curve). The *topologist's sine curve* is the closure of

$$S = \{(x, \sin 1/x) : x \in \mathbb{R}\}$$

in \mathbb{R}^2 .

Proposition 8.4.9. *The topologist's sine curve is connected.*

PROOF:

$\langle 1 \rangle 1$. $S = \{(x, \sin 1/x) : x \in \mathbb{R}\}$ is connected.

$\langle 2 \rangle 1$. The function $f : \mathbb{R} \rightarrow \mathbb{R}^2$ given by $f(x) = (x, \sin 1/x)$ is continuous.

PROOF: By Theorem 5.2.15.

$\langle 2 \rangle 2$. Q.E.D.

PROOF: By Theorem 8.1.9.

$\langle 1 \rangle 2$. Q.E.D.

PROOF: By Theorem 8.1.8.

\square

Proposition 8.4.10 (CC). *The topologist's sine curve is not path connected.*

PROOF:

$\langle 1 \rangle 1$. LET: $S = \{(x, \sin 1/x) : x \in \mathbb{R}\}$

$\langle 1 \rangle 2$. ASSUME: for a contradiction $p : [0, 1] \rightarrow \bar{S}$ is a path from $(0, 0)$ to $(1, \sin 1)$.

$\langle 1 \rangle 3$. $p^{-1}(\{0\} \times [-1, 1])$ is closed.

$\langle 1 \rangle 4$. $p^{-1}(\{0\} \times [-1, 1])$ has a greatest element.

PROOF: By Lemma 4.1.9.

$\langle 1 \rangle 5$. LET: $q : [0, 1] \rightarrow \bar{S}$ be a path such that:

- $q(0) \in \{0\} \times [-1, 1]$
- $q(x) \in S$ for $x > 0$

PROOF: Let b be greatest in $p^{-1}(\{0\} \times [-1, 1])$. Then q is obtained by rescaling p restricted to $[b, 1]$.

$\langle 1 \rangle 6$. LET: $q(t) = (x(t), y(t))$ for $0 \leq t \leq 1$

$\langle 1 \rangle 7$. $x(0) = 0$

- ⟨1⟩8. $x(t) > 0$ for $t > 0$
 ⟨1⟩9. $y(t) = \sin 1/x(t)$ for $t > 0$
 ⟨1⟩10. There exists a sequence $t_n \in [0, 1]$ such that $t_n \rightarrow 0$ as $n \rightarrow \infty$ and $y(t_n) = (-1)^n$ for all n .
 ⟨2⟩1. For each n , PICK u_n such that $0 < u_n < x(1/n)$ and $\sin 1/u_n = (-1)^n$.
 PROOF: Such a u_n exists because $\sin 1/x$ takes values 1 and -1 infinitely often in $(0, x(1/n))$.
 ⟨2⟩2. For each n , PICK t_n such that $0 < t_n < 1/n$ and $x(t_n) = u$
 PROOF: By the Intermediate Value Theorem.
 ⟨1⟩11. Q.E.D.
 PROOF: This is a contradiction as $y(t_n) \rightarrow y(0)$ as $n \rightarrow \infty$ because y is continuous.

□

8.5 Local Connectedness

Definition 8.5.1 (Locally Connected). Let X be a topological space and $x \in X$. Then X is *locally connected* at x iff every neighbourhood of x includes a connected neighbourhood of x .

The space X is *locally connected* iff it is locally connected at every point.

Proposition 8.5.2. S_Ω is not locally connected.

PROOF: There is no connected neighbourhood of ω . □

Proposition 8.5.3. $\overline{S_\Omega}$ is not locally connected.

PROOF: There is no connected neighbourhood of ω . □

Proposition 8.5.4. For any set I , the space \mathbb{R}^I is locally connected.

PROOF: Every basic open set is the product of connected spaces, hence connected. □

Proposition 8.5.5. Let X be a topological space. Then X is locally connected if and only if, for every open set U in X , every component of U is open in X .

PROOF:

- ⟨1⟩1. If X is locally connected then, for every open set U in X , every component of U is open in X .
 ⟨2⟩1. ASSUME: X is locally connected.
 ⟨2⟩2. LET: U be open in X .
 ⟨2⟩3. LET: C be a component of U .
 ⟨2⟩4. LET: $x \in C$
 PROVE: C is a neighbourhood of x
 ⟨2⟩5. U is a neighbourhood of x in X .
 PROOF: From ⟨2⟩2, ⟨2⟩3 and ⟨2⟩4.
 ⟨2⟩6. PICK a connected neighbourhood V of x such that $V \subseteq U$.

PROOF: Using $\langle 2 \rangle 1$.

$\langle 2 \rangle 7$. $V \subseteq C$

PROOF: By Proposition 8.2.2.

$\langle 2 \rangle 8$. C is a neighbourhood of x

PROOF: By Proposition 3.2.4.

$\langle 2 \rangle 9$. Q.E.D.

PROOF: By Proposition 3.2.3.

$\langle 1 \rangle 2$. If, for every open set U in X , every component of U is open in X , then X is locally connected.

$\langle 2 \rangle 1$. ASSUME: For every open set U in X , every component of U is open in X .

$\langle 2 \rangle 2$. LET: $x \in X$ and N be a neighbourhood of x

$\langle 2 \rangle 3$. PICK U open such that $x \in U \subseteq N$

$\langle 2 \rangle 4$. LET: C be the component of U that contains x

$\langle 2 \rangle 5$. C is open in X

PROOF: By $\langle 2 \rangle 1$.

$\langle 2 \rangle 6$. C is a connected neighbourhood of x that is included in N

□

Corollary 8.5.5.1. *In a locally connected space, every component is open.*

Corollary 8.5.5.2. *The space \mathbb{R}^ω under the box topology is not locally connected.*

Corollary 8.5.5.3. *Not every closed subspace of a locally connected space is locally connected.*

PROOF: The topologist's sine curve is not locally connected. □

Proposition 8.5.6. *$S_\Omega \times \overline{S_\Omega}$ is not locally connected.*

(ω, ω) has no connected neighbourhood. □

Proposition 8.5.7. *\mathbb{R}_l is not locally connected.*

PROOF: 0 has no connected neighbourhood. □

Proposition 8.5.8. *The Sorgenfrey plane is not locally connected.*

PROOF: Any basic open set $[a, b) \times [c, d)$ can be separated into $[a, b) \times [c, e)$ and $[a, b) \times [e, d)$ for some $c < e < d$. □

Proposition 8.5.9. *The space \mathbb{R}^ω under the uniform topology is locally connected.*

PROOF: For any neighbourhood U of a point x , the neighbourhood $U \cap \{y : y - x \text{ is bounded}\}$ is connected. □

Proposition 8.5.10. *The space \mathbb{R}_K is not locally connected.*

PROOF: The open set $(-1, 1) - K$ does not include a connected neighbourhood of 0. □

Proposition 8.5.11. *Every open subspace of a locally connected space is locally connected.*

PROOF: Follows easily from definition. \square

Proposition 8.5.12 (AC). *The product of a family of locally connected spaces is locally connected.*

PROOF:

$\langle 1 \rangle 1$. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of locally connected spaces and $\vec{x} \in \prod_{\alpha \in J} X_\alpha$

$\langle 1 \rangle 2$. LET: $\prod_{\alpha \in J} U_\alpha$ be any basic neighbourhood of \vec{x} , where each U_α is open in X_α , and $U_\alpha = X_\alpha$ except for $\alpha = \alpha_1, \dots, \alpha_n$

$\langle 1 \rangle 3$. For $\alpha \in J$, PICK a connected neighbourhood C_α of x_α with $C_\alpha \subseteq U_\alpha$

$\langle 1 \rangle 4$. $\prod_{\alpha \in J} C_\alpha$ is connected

PROOF: Proposition 8.1.10

\square

Proposition 8.5.13. *Every discrete space is locally connected.*

PROOF: For any point x , the set $\{x\}$ is a connected neighbourhood of x . \square

Corollary 8.5.13.1. *The continuous image of a locally connected space is not necessarily locally connected.*

8.6 Local Path Connectedness

Definition 8.6.1 (Locally Path Connected). Let X be a topological space and $x \in X$. Then X is *locally path connected at x* iff every neighbourhood of x includes a path connected neighbourhood of x .

The space X is *locally path connected* iff it is locally path connected at every point.

Proposition 8.6.2. S_Ω is not locally path connected.

PROOF: There is no path connected neighbourhood of ω . \square

Proposition 8.6.3. $\overline{S_\Omega}$ is not locally path connected.

PROOF: There is no path connected neighbourhood of ω . \square

Proposition 8.6.4. *Not every closed subspace of a locally path connected space is locally path connected.*

PROOF: The topologist's sine curve is not locally path connected. \square

Proposition 8.6.5. *Every open subspace of a locally path connected space is locally path connected.*

PROOF: Follows easily from definition. \square

Proposition 8.6.6. *Every locally path connected space is locally connected.*

PROOF: From Proposition 8.3.3. \square

Corollary 8.6.6.1. \mathbb{R}_l is not locally path connected.

Corollary 8.6.6.2. The Sorgenfrey plane is not locally path connected.

Corollary 8.6.6.3. The space \mathbb{R}^ω under the box topology is not locally path connected.

Corollary 8.6.6.4. The space \mathbb{R}_K is not locally path connected.

Corollary 8.6.6.5. The topologist's sine curve is not locally path connected.

Proposition 8.6.7 (AC). The product of a family of locally path connected spaces is locally path connected.

PROOF:

- $\langle 1 \rangle 1$. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of locally connected spaces and $\vec{x} \in \prod_{\alpha \in J} X_\alpha$
- $\langle 1 \rangle 2$. LET: $\prod_{\alpha \in J} U_\alpha$ be any basic neighbourhood of \vec{x} , where each U_α is open in X_α , and $U_\alpha = X_\alpha$ except for $\alpha = \alpha_1, \dots, \alpha_n$
- $\langle 1 \rangle 3$. For $\alpha \in J$, PICK a path connected neighbourhood C_α of x_α with $C_\alpha \subseteq U_\alpha$
- $\langle 1 \rangle 4$. $\prod_{\alpha \in J} C_\alpha$ is path connected

PROOF: Proposition ??

\square

Proposition 8.6.8. Let X be a topological space. Then X is locally path connected if and only if, for every open set U in X , every path component of U is open in X .

PROOF:

- $\langle 1 \rangle 1$. If X is locally path connected then, for every open set U in X , every path component of U is open in X .
- $\langle 2 \rangle 1$. ASSUME: X is locally path connected.
- $\langle 2 \rangle 2$. LET: U be open in X .
- $\langle 2 \rangle 3$. LET: C be a path component of U .
- $\langle 2 \rangle 4$. LET: $x \in C$
PROVE: C is a neighbourhood of x
- $\langle 2 \rangle 5$. U is a neighbourhood of x in X .
PROOF: From $\langle 2 \rangle 2$, $\langle 2 \rangle 3$ and $\langle 2 \rangle 4$.
- $\langle 2 \rangle 6$. PICK a path connected neighbourhood V of x such that $V \subseteq U$.
PROOF: Using $\langle 2 \rangle 1$.
- $\langle 2 \rangle 7$. $V \subseteq C$
PROOF: By Proposition 8.3.7.
- $\langle 2 \rangle 8$. C is a neighbourhood of x
PROOF: By Proposition 3.2.4.
- $\langle 2 \rangle 9$. Q.E.D.
PROOF: By Proposition 3.2.3.
- $\langle 1 \rangle 2$. If, for every open set U in X , every path component of U is open in X , then X is locally path connected.

- (2)1. ASSUME: For every open set U in X , every path component of U is open in X .
 (2)2. LET: $x \in X$ and N be a neighbourhood of x
 (2)3. PICK U open such that $x \in U \subseteq N$
 (2)4. LET: C be the path component of U that contains x
 (2)5. C is open in X
 PROOF: By (2)1.
 (2)6. C is a path connected neighbourhood of x that is included in N

□

Theorem 8.6.9 (AC). *Let X be a topological space. If X is locally path connected, then its components and its path components are the same.*

PROOF:

- (1)1. LET: P be a path component of X
 (1)2. LET: C be the component such that $P \subseteq C$
 PROVE: $P = C$
 (1)3. LET: $Q = C \setminus P$
 (1)4. P is open in X
 PROOF: By Proposition 8.6.8.
 (1)5. Q is open in X
 PROOF: By Proposition 8.6.8 since Q is the union of the path components included in C other than P .
 (1)6. $Q = \emptyset$
 PROOF: Otherwise P and Q would form a separation of C , contradicting 8.2.3.

□

Proposition 8.6.10. $S_\Omega \times \overline{S_\Omega}$ is not locally path connected.

PROOF: (ω, ω) has no path connected neighbourhood. □

Proposition 8.6.11. *The ordered square is not locally path connected.*

PROOF:

- (1)1. ASSUME: for a contradiction $(1/2, 0)$ has a path connected neighbourhood U
 (1)2. PICK $a < 1/2$ such that $((a, 1), (1/2, 0)) \subseteq U$
 (1)3. LET: $p : [0, 1] \rightarrow I_o^2$ be a path from $(a, 1)$ to $(1/2, 0)$
 (1)4. For every x such that $a < x < 1/2$, PICK a rational q_x such that $p(q_x) \in ((x, 0), (x, 1))$
 (1)5. $\{q_x : a < x < 1/2\}$ is an uncountable set of rationals.

□

Proposition 8.6.12. *For any set I , the space \mathbb{R}^I is locally path connected.*

PROOF: Every basic open set is the product of path connected spaces, hence path connected. □

Proposition 8.6.13. *The space \mathbb{R}^ω under the uniform topology is locally path connected.*

PROOF: Its components and path components are the same. \square

Proposition 8.6.14. *Every discrete space is locally path connected.*

PROOF: For any point x , the set $\{x\}$ is a path connected neighbourhood of x . \square

Corollary 8.6.14.1. *The continuous image of a locally path connected space is not necessarily locally path connected.*

8.7 Weak Local Connectedness

Definition 8.7.1 (Weakly Locally Connected). Let X be a topological space and $x \in X$. Then X is *weakly locally connected at x* iff every neighbourhood of x contains a connected subspace that contains a neighbourhood of x .

Chapter 9

Compact Spaces

9.1 Countable Compactness

Definition 9.1.1 (Countably Compact). A topological space is *countably compact* iff every countable open covering has a finite subcovering.

9.2 Limit Point Compactness

Definition 9.2.1 (Limit Point Compact). A space is *limit point compact* iff every infinite set has a limit point.

Proposition 9.2.2 (CC). $S_\Omega \times \overline{S_\Omega}$ is limit point compact.

PROOF:

- $\langle 1 \rangle 1$. LET: $A \subseteq S_\Omega \times \overline{S_\Omega}$ be infinite
- $\langle 1 \rangle 2$. CASE: $\pi_1(A)$ is finite.
 - $\langle 2 \rangle 1$. PICK x such that there are infinitely many y such that $(x, y) \in A$
 - $\langle 2 \rangle 2$. PICK a limit point l of $\{y : (x, y) \in A\}$
 - $\langle 2 \rangle 3$. (x, l) is a limit point of A
- $\langle 1 \rangle 3$. CASE: $\pi_1(A)$ is infinite.
 - $\langle 2 \rangle 1$. PICK a limit point l of $\pi_1(A)$.
 - $\langle 2 \rangle 2$. l is a limit ordinal
 - $\langle 2 \rangle 3$. PICK a countable sequence x_n with limit l
 - $\langle 2 \rangle 4$. For $n \geq 1$, PICK $a_n > x_n$ and y_n such that $(a_n, y_n) \in A$
 - $\langle 2 \rangle 5$. CASE: $\{y_n : n \geq 1\}$ is finite
 - $\langle 3 \rangle 1$. PICK y such that $y = y_n$ for infinitely many n
 - $\langle 3 \rangle 2$. (l, y) is a limit point for A
 - $\langle 2 \rangle 6$. CASE: $\{y_n : n \geq 1\}$ is infinite
 - $\langle 3 \rangle 1$. PICK a limit point m for $\{y_n : n \geq 1\}$
 - $\langle 3 \rangle 2$. (l, m) is a limit point for A

□

Proposition 9.2.3. *The Sorgenfrey plane is not limit point compact.*

PROOF: \mathbb{Z}^2 has no limit point. \square

Proposition 9.2.4. *The space \mathbb{R}^ω under the box topology is not limit point compact.*

PROOF: The set of all constant sequences of integers is an infinite set with no limit point. \square

Proposition 9.2.5. *Not every open subspace of a limit point compact space is limit point compact.*

PROOF: The space $[0, 1]$ is limit point compact but $(0, 1)$ is not. \square

Proposition 9.2.6. *The product of two limit point compact spaces is not necessarily limit point compact.*

PROOF: See Steen and Seebach *Counterexamples in Topology* Example 112. \square

Proposition 9.2.7. *The continuous image of a limit point compact space is not necessarily limit point compact.*

PROOF: Let Y be a two-point set under the indiscrete topology. Then $\mathbb{N} \times Y$ is limit point compact, but \mathbb{N} is not. \square

9.3 Lindelöf Spaces

Definition 9.3.1 (Lindelöf Space). A topological space X is *Lindelöf* iff every open covering has a countable subcovering.

Theorem 9.3.2 (CC). *Every second countable space is Lindelöf.*

PROOF:

- $\langle 1 \rangle 1$. LET: X be a second countable space
- $\langle 1 \rangle 2$. PICK a countable basis \mathcal{B} for X .
- $\langle 1 \rangle 3$. LET: \mathcal{A} be an open cover of X
- $\langle 1 \rangle 4$. For every $B \in \mathcal{B}$ such that there exists $U \in \mathcal{A}$ such that $B \subseteq U$, PICK $U_B \in \mathcal{A}$ such that $B \subseteq U_B$
- $\langle 1 \rangle 5$. $\{U_B : B \in \mathcal{B}, \exists U \in \mathcal{A}. B \subseteq U\}$ covers X .
- $\langle 2 \rangle 1$. LET: $x \in X$
- $\langle 2 \rangle 2$. PICK $U \in \mathcal{A}$ such that $x \in U$
- $\langle 2 \rangle 3$. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$
- $\langle 2 \rangle 4$. $x \in U_B$

\square

Corollary 9.3.2.1. *The space \mathbb{R}^ω is Lindelöf.*

Corollary 9.3.2.2. *The space \mathbb{R}_K is Lindelöf.*

Proposition 9.3.3. *The space S_Ω is not Lindelöf.*

PROOF: $\{(-\infty, \alpha) : \alpha \in S_\Omega\}$ is an open cover that has no countable subcover. \square

Proposition 9.3.4 (CC). *The space $\overline{S_\Omega}$ is Lindelöf.*

PROOF:

- $\langle 1 \rangle 1$. LET: \mathcal{A} be an open cover of $\overline{S_\Omega}$
 - $\langle 1 \rangle 2$. PICK $U \in \mathcal{A}$ such that $\Omega \in U$
 - $\langle 1 \rangle 3$. PICK $\alpha < \Omega$ such that $(\alpha, \Omega] \subseteq U$
 - $\langle 1 \rangle 4$. For $\beta \leq \alpha$, PICK $U_\beta \in \mathcal{A}$ such that $\beta \in U_\beta$
 - $\langle 1 \rangle 5$. $\{U\} \cup \{U_\beta : \beta \leq \alpha\}$ is a countable subcover of \mathcal{A} .
- \square

Proposition 9.3.5 (CC). *The continuous image of a Lindelöf space is Lindelöf.*

PROOF:

- $\langle 1 \rangle 1$. LET: X be a Lindelöf space, Y a space and $f : X \rightarrow Y$ continuous.
 - $\langle 1 \rangle 2$. LET: \mathcal{A} be an open covering of Y
 - $\langle 1 \rangle 3$. $\{f^{-1}(V) : V \in \mathcal{A}\}$ is an open covering of X
 - $\langle 1 \rangle 4$. PICK a countable subcovering $\{f^{-1}(V_1), f^{-1}(V_2), \dots\}$ of $\{f^{-1}(V) : V \in \mathcal{A}\}$
 - $\langle 1 \rangle 5$. $\{V_1, V_2, \dots\}$ is a countable subcovering of \mathcal{A}
- \square

Proposition 9.3.6. *The Sorgenfrey plane is not Lindelöf.*

PROOF:

- $\langle 1 \rangle 1$. LET: $L = \{(x, -x) : x \in \mathbb{R}\}$
 - $\langle 1 \rangle 2$. L is closed in \mathbb{R}_l^2
 - $\langle 2 \rangle 1$. LET: $(x, y) \notin L$, so $y \neq -x$

PROVE: There exists a neighbourhood U of (x, y) that does not intersect L
 - $\langle 2 \rangle 2$. CASE: $y > -x$

PROOF: In this case, take $U = [x, +\infty) \times [y, +\infty)$
 - $\langle 2 \rangle 3$. CASE: $y < -x$

PROOF: In this case, take $U = [x, (x - y)/2) \times [y, (y - x)/2)$.
 - $\langle 1 \rangle 3$. LET: $\mathcal{U} = \{\mathbb{R}_l^2 \setminus L\} \cup \{[a, b) \times [-a, d) : a, b, d \in \mathbb{R}\}$
 - $\langle 1 \rangle 4$. \mathcal{U} is an open covering of \mathbb{R}_l^2
 - $\langle 1 \rangle 5$. No countable subset of \mathcal{U} covers \mathbb{R}_l^2

PROOF: Every set $[a, b) \times [-a, d)$ intersects L in exactly one point, namely $(a, -a)$.
- \square

Corollary 9.3.6.1. *The Sorgenfrey plane is not second countable.*

Corollary 9.3.6.2. *The product of two Lindelöf spaces is not necessarily Lindelöf.*

Proposition 9.3.7. *The space \mathbb{R}^ω under the box topology is not Lindelöf.*

PROOF: The set $\{\prod_{n=0}^{\infty}(a_n, a_n + 1) : \forall n. a_n \in \mathbb{Z}\}$ covers the space but has no countable subcover. \square

Proposition 9.3.8. *Not every open subspace of a Lindelöf space is Lindelöf.*

PROOF: The ordered square is Lindelöf but the subspace $[0, 1] \text{ times } (0, 1)$ is not. \square

9.4 Compactness

Definition 9.4.1 (Compact). A topological space is *compact* iff every open cover has a finite subcover.

Proposition 9.4.2. S_{Ω} is not compact.

PROOF: The open covering $\{(-\infty, \alpha) : \alpha \in S_{\Omega}\}$ has no finite subcovering. \square

Proposition 9.4.3. \mathbb{R}_l is not compact.

PROOF: $\{[n, n + 1) : n \in \mathbb{Z}\}$ has no finite subcover. \square

Proposition 9.4.4. The space \mathbb{R}^{ω} under the box topology is not compact.

PROOF: The set $\{\prod_{n=0}^{\infty}(a_n, a_n + 1) : n \in \mathbb{Z}\}$ is a cover that has no finite subcover. \square

Proposition 9.4.5. *Let Y be a subspace of X . Then Y is compact if and only if every covering of Y by sets open in X contains a finite subcollection covering Y .*

PROOF:

- (1)1. If Y is compact then every covering of Y by sets open in X contains a finite subcollection covering Y .
- (2)1. ASSUME: Y is compact.
- (2)2. LET: \mathcal{A} be a covering of Y by sets open in X .
- (2)3. $\{U \cap Y : U \in \mathcal{A}\}$ is an open covering of Y .
- (2)4. PICK a finite subcovering V_1, \dots, V_n of $\{U \cap Y : U \in \mathcal{A}\}$
- (2)5. For $1 \leq i \leq n$, PICK $U_i \in \mathcal{A}$ such that $V_i = U_i \cap Y$.
- (2)6. $\{U_1, \dots, U_n\}$ is a finite subset of \mathcal{A} that covers Y .
- (1)2. If every covering of Y by sets open in X contains a finite subcollection covering Y then Y is compact.
- (2)1. ASSUME: Every covering of Y by sets open in X contains a finite subcollection covering Y .
- (2)2. LET: \mathcal{A} be an open covering of Y
- (2)3. LET: $\mathcal{B} = \{U \text{ open in } X : U \cap Y \in \mathcal{A}\}$
- (2)4. \mathcal{B} covers Y
- (2)5. PICK a finite subcollection $\{U_1, \dots, U_n\} \subseteq \mathcal{B}$ that covers Y
- (2)6. $\{U_1 \cap Y, \dots, U_n \cap Y\}$ is a finite subcover of \mathcal{A} .

\square

Proposition 9.4.6. *Every closed subspace of a compact space is compact.*

PROOF:

- ⟨1⟩1. LET: X be a compact space and $Y \subseteq X$ be closed.
- ⟨1⟩2. LET: \mathcal{A} be a covering of Y by spaces open in X
- ⟨1⟩3. $\mathcal{A} \cup \{X \setminus Y\}$ is an open covering of X .
- ⟨1⟩4. PICK a finite subcovering $\{U_1, \dots, U_n\}$ or $\{U_1, \dots, U_n, X \setminus Y\}$
- ⟨1⟩5. $\{U_1, \dots, U_n\}$ is a finite subset of \mathcal{A} that covers Y .
- ⟨1⟩6. Q.E.D.

PROOF: Proposition 9.4.5.

□

Corollary 9.4.6.1. *Not every compact Hausdorff space is connected.*

PROOF: The space $[0, 1] \cup [2, 3]$ is compact Hausdorff and disconnected. □

Corollary 9.4.6.2. *Not every compact Hausdorff space is path connected.*

Corollary 9.4.6.3. *Not every compact Hausdorff space is locally connected.*

The space $[0, 1] \cap \mathbb{Q}$ is not locally connected.

Corollary 9.4.6.4. *Not every compact Hausdorff space is locally path connected.*

Proposition 9.4.7. *Not every open subspace of a compact space is compact.*

PROOF: The space $[0, 1]$ is compact but $(0, 1)$ is not. □

Lemma 9.4.8. *If Y is a compact subspace of the Hausdorff space X and $a \notin Y$, then there exist disjoint open sets U and V of X containing a and Y , respectively.*

PROOF:

- ⟨1⟩1. For $y \in Y$, there exist disjoint open sets U and V such that $a \in U$ and $y \in V$.
- ⟨1⟩2. $\{V \text{ open in } X : \exists U \text{ open and disjoint from } V, a \in U\}$ is a covering of Y by open sets in X .
- ⟨1⟩3. PICK a finite subset $\{V_1, \dots, V_n\}$ that covers Y .
- ⟨1⟩4. For $1 \leq i \leq n$, PICK U_i disjoint from V_i such that $a \in U_i$
- ⟨1⟩5. LET: $U = U_1 \cap \dots \cap U_n$ and $V = V_1 \cup \dots \cup V_n$

□

Proposition 9.4.9. *Every compact subspace of a Hausdorff space is closed.*

PROOF:

- ⟨1⟩1. LET: X be a Hausdorff space and $Y \subseteq X$ be compact.
- ⟨1⟩2. Every point $a \notin Y$ has an open neighbourhood disjoint from Y .

PROOF: By Lemma 9.4.8.

- ⟨1⟩3. Q.E.D.

PROOF: By Proposition 3.2.3.

Proposition 9.4.10. *The image of a compact space under a continuous map is compact.*

PROOF:

- ⟨1⟩1. LET: $f : X \rightarrow Y$ be continuous where X is compact.
- ⟨1⟩2. LET: \mathcal{A} be a covering of $f(X)$ by open sets in Y .
- ⟨1⟩3. $\{f^{-1}(U) : U \in \mathcal{A}\}$ is an open covering of X .
- ⟨1⟩4. PICK a finite subcovering $\{f^{-1}(U_1), \dots, f^{-1}(U_n)\}$
- ⟨1⟩5. $\{U_1, \dots, U_n\}$ is a finite subset of \mathcal{A} that covers $f(X)$.
- ⟨1⟩6. Q.E.D.

PROOF: By Proposition 9.4.5.

□

Corollary 9.4.10.1. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces. If $\prod_{\alpha \in J} X_\alpha$ is compact then each X_α is compact.*

Corollary 9.4.10.2. $S_\Omega \times \overline{S_\Omega}$ is compact.

Corollary 9.4.10.3. *The Sorgenfrey plane is not compact.*

Corollary 9.4.10.4. *For any nonempty set I , the space \mathbb{R}^I is not compact.*

Corollary 9.4.10.5. *Let \mathcal{T} and \mathcal{T}' be topologies on the same set X . If $\mathcal{T} \subseteq \mathcal{T}'$ and \mathcal{T}' is compact then \mathcal{T} is compact.*

Corollary 9.4.10.6. *The space \mathbb{R}_K is not compact.*

Theorem 9.4.11. *Let $f : X \rightarrow Y$ be a bijective continuous function. If X is compact and Y is Hausdorff then f is a homeomorphism.*

PROOF:

- ⟨1⟩1. LET: C be closed in X
- ⟨1⟩2. C is compact
PROOF: Proposition 9.4.6.
- ⟨1⟩3. $f(C)$ is compact
PROOF: Proposition 9.4.10
- ⟨1⟩4. $f(C)$ is closed
PROOF: Proposition 9.4.9.
- ⟨1⟩5. Q.E.D.

PROOF: By Theorem 5.2.2 we have that f^{-1} is continuous.

□

Corollary 9.4.11.1. *Let \mathcal{T} and \mathcal{T}' be topologies on the same set X . If $\mathcal{T} \subseteq \mathcal{T}'$, \mathcal{T} is Hausdorff and \mathcal{T}' is compact then $\mathcal{T} = \mathcal{T}'$.*

Corollary 9.4.11.2. *The space $[0, 1]$ is not compact as a subspace of \mathbb{R}_K .*

Theorem 9.4.12 (Tube Lemma). *Let A and B be subspaces of X and Y , respectively; let N be an open set in $X \times Y$ including $A \times B$. If A and B are compact, then there exist open sets U and V in X and Y , respectively, such that*

$$A \times B \subseteq U \times V \subseteq N .$$

PROOF:

- (1)1. For all $a \in A$, there exist open sets U and V in X and Y , respectively, such that

$$\{a\} \times B \subseteq U \times V \subseteq N.$$

- (2)1. LET: $a \in A$
 (2)2. For all $b \in B$, there exist open sets U and V in X and Y , respectively, such that $(a, b) \in U \times V \subseteq N$.
 (2)3. $\{V \text{ open in } Y : \exists U \text{ open in } X. a \in U, U \times V \subseteq N\}$ covers B
 (2)4. PICK a finite subset $\{V_1, \dots, V_n\}$ that covers B .
 (2)5. For $1 \leq i \leq n$, PICK U_i open in X such that $a \in U_i$ and $U_i \times V_i \subseteq N$
 (2)6. LET: $U = U_1 \cap \dots \cap U_n$ and $V = V_1 \cup \dots \cup V_n$
 (1)2. $\{U \text{ open in } X : \exists V \text{ open in } Y. B \subseteq V \text{ and } U \times V \subseteq N\}$ covers A .
 (1)3. PICK a finite subset $\{U_1, \dots, U_n\}$ that covers A .
 (1)4. For $1 \leq i \leq n$, PICK V_i open in B such that $B \subseteq V_i$ and $U_i \times V_i \subseteq N$.
 (1)5. LET: $U = U_1 \cup \dots \cup U_n$ and $V = V_1 \cap \dots \cap V_n$
 (1)6. $A \times B \subseteq U \times V \subseteq N$

□

Lemma 9.4.13. *Let \mathcal{A} be a set of basis elements for $X \times Y$ such that no finite subset of \mathcal{A} covers $X \times Y$. If X is compact, then there exists a point $x \in X$ such that no finite subset of \mathcal{A} covers $\{x\} \times Y$.*

PROOF:

- (1)1. ASSUME: X is compact.
 (1)2. ASSUME: For all $x \in X$, there is a finite subset of \mathcal{A} that covers $\{x\} \times Y$
 PROVE: A finite subset of \mathcal{A} covers $X \times Y$
 (1)3. $\{U \text{ open in } X : \exists U_1 \times V_1, \dots, U_r \times V_r \in \mathcal{A}. U = U_1 \cap \dots \cap U_r, Y = V_1 \cup \dots \cup V_r\}$ covers X .
 (1)4. PICK a finite subcover $\{U_1, \dots, U_n\}$
 (1)5. For $1 \leq i \leq n$, PICK $U_{i1} \times V_{i1}, \dots, U_{ir_i} \times V_{ir_i} \in \mathcal{A}$ such that $U_i = U_{i1} \cap \dots \cap U_{ir_i}$ and $Y = V_{i1} \cup \dots \cup V_{ir_i}$
 (1)6. $\{U_{ij} : 1 \leq i \leq n, 1 \leq j \leq r_i\}$ covers $X \times Y$

□

Proposition 9.4.14. *The product of two compact spaces is compact.*

PROOF:

- (1)1. LET: X and Y be compact spaces.
 (1)2. LET: \mathcal{A} be an open covering of $X \times Y$
 (1)3. For all $x \in X$, there exists a neighbourhood W of x such that $W \times Y$ is covered by finitely many elements of \mathcal{A} .
 (2)1. LET: $x \in X$
 (2)2. $\{x\} \times Y$ is compact.
 PROOF: It is homeomorphic to Y .
 (2)3. PICK a finite subset $\{U_1, \dots, U_m\}$ of \mathcal{A} that covers $\{x\} \times Y$
 PROOF: By Proposition 9.4.5.
 (2)4. There exists a neighbourhood W of x such that $W \times Y \subseteq U_1 \cup \dots \cup U_m$

PROOF: By the Tube Lemma.

- ⟨1⟩4. $\{W \text{ open in } X : W \times Y \text{ is covered by finitely many elements of } \mathcal{A}\}$ is an open covering of X .
- ⟨1⟩5. PICK a finite subcovering $\{W_1, \dots, W_n\}$
- ⟨1⟩6. For $1 \leq i \leq n$, PICK a finite subset $\{U_{i1}, \dots, U_{ir_i}\}$ of \mathcal{A} that covers $W_i \times Y$
- ⟨1⟩7. $\{U_{11}, \dots, U_{nr_n}\}$ is a finite subcovering of \mathcal{A} .

□

Proposition 9.4.15. *A topological space is compact if and only if every nonempty set of closed sets that has the finite intersection property has nonempty intersection.*

PROOF: Immediate from definitions. □

Lemma 9.4.16. *If Y is compact then $\pi_1 : X \times Y \rightarrow X$ is a closed map.*

PROOF:

- ⟨1⟩1. LET: $C \subseteq X \times Y$ be closed
- ⟨1⟩2. LET: $x \in X \setminus \pi_1(C)$
- ⟨1⟩3. For all $y \in Y$, we have $(x, y) \notin C$
- ⟨1⟩4. For all $y \in Y$, there exist open neighbourhoods U of x and V of y such that $U \times V \subseteq (X \times Y) \setminus C$
- ⟨1⟩5. $\{V \text{ open in } Y : \exists U \text{ an open neighbourhood of } x \text{ such that } U \times V \subseteq (X \times Y) \setminus C\}$ is an open covering of Y .
- ⟨1⟩6. PICK a finite subcovering $\{V_1, \dots, V_n\}$
- ⟨1⟩7. For $1 \leq i \leq n$, PICK an open neighbourhood U_i of x such that $U_i \times V_i \subseteq (X \times Y) \setminus C$
- ⟨1⟩8. $x \in U_1 \cap \dots \cap U_n \subseteq X \setminus \pi_1(C)$

□

Theorem 9.4.17. *Let X be a compact space. Let $f_n : X \rightarrow \mathbb{R}$ be a sequence of continuous functions such that, for all $x \in X$, $f_n(x) \rightarrow f(x)$ as $n \rightarrow \infty$. If f is continuous, and if the sequence $(f_n)_n$ is monotone increasing, and if X is compact, then the convergence is uniform.*

PROOF:

- ⟨1⟩1. LET: $\epsilon > 0$
 PROVE: There exists N such that, for all $n \geq N$, we have $|f_n(x) - f(x)| < \epsilon$
- ⟨1⟩2. For $n \in \mathbb{Z}^+$,
 LET: $U_n = \{x \in X : f(x) - f_n(x) < \epsilon\}$
- ⟨1⟩3. Each U_n is open
 PROOF: Let $g(x) = f(x) - f_n(x)$. Then g is continuous and $U_n = g^{-1}((-\infty, \epsilon))$.
- ⟨1⟩4. $\{U_n : n \geq 1\}$ is an open covering of X
- ⟨2⟩1. LET: $x \in X$
- ⟨2⟩2. PICK N such that, for all $n \geq N$, $|f(x) - f_n(x)| < \epsilon$
 PROOF: $f_n(x) \rightarrow f(x)$ as $n \rightarrow \infty$
- ⟨2⟩3. $f(x) - f_N(x) < \epsilon$

PROOF: This holds since the sequence $(f_n)_n$ is monotone.

- ⟨1⟩5. PICK a finite subcovering $\{U_{n_1}, \dots, U_{n_k}\}$
 - ⟨1⟩6. LET: $N = \max(n_1, \dots, n_k)$
 - ⟨1⟩7. For all $n \geq N$ we have $|f_n(x) - f(x)| < \epsilon$
-

Lemma 9.4.18. *Every compact Hausdorff space is normal.*

PROOF:

- ⟨1⟩1. LET: A and B be disjoint closed sets in the compact Hausdorff space X .
- ⟨1⟩2. For all $a \in A$, there exist disjoint open sets U and V such that $a \in U$ and $B \subseteq V$.

PROOF: By Lemma 9.4.8.

- ⟨1⟩3. $\{U \text{ open in } X : \exists V \text{ open in } Y, U \cap V = \emptyset, B \subseteq V\}$ is an open covering of A
 - ⟨1⟩4. PICK a finite subcovering $\{U_1, \dots, U_n\}$
 - ⟨1⟩5. For $1 \leq i \leq n$, PICK V_i open in Y such that $U_i \cap V_i = \emptyset$ and $B \subseteq V_i$
 - ⟨1⟩6. LET: $U = U_1 \cup \dots \cup U_n$ and $V = V_1 \cap \dots \cap V_n$
-

Theorem 9.4.19. *Let X be a complete linearly ordered set under the order topology. Then every closed interval in X is compact.*

PROOF:

- ⟨1⟩1. LET: X be a complete linearly ordered set in the order topology
- ⟨1⟩2. LET: $a, b \in X, a < b$
PROVE: $[a, b]$ is compact
- ⟨1⟩3. LET: \mathcal{A} be a set of open sets that covers $[a, b]$
- ⟨1⟩4. For all $x \in [a, b)$, there exists $y \in (x, b]$ such that $[x, y]$ is covered by at most two points of \mathcal{A}
 - ⟨2⟩1. LET: $x \in [a, b]$
 - ⟨2⟩2. PICK $U \in \mathcal{A}$ such that $x \in U$
PROOF: By ⟨1⟩3 and ⟨2⟩1
 - ⟨2⟩3. PICK $y \in (x, b]$ such that $[x, y] \subseteq U$
PROOF: By Lemma 4.1.2.
 - ⟨2⟩4. PICK $V \in \mathcal{A}$ such that $y \in V$
PROOF: By ⟨1⟩3 and ⟨2⟩3.
 - ⟨2⟩5. $[x, y]$ is covered by $\{U, V\}$
PROOF: By ⟨2⟩3 and ⟨2⟩4.
- ⟨1⟩5. LET: $C = \{y \in (a, b] : [a, y] \text{ is covered by a finite subset of } \mathcal{A}\}$
- ⟨1⟩6. C is nonempty
PROOF: By ⟨1⟩4.
- ⟨1⟩7. LET: $c = \sup C$
PROOF: By ⟨1⟩1.
- ⟨1⟩8. $c \in C$
 - ⟨2⟩1. PICK $U \in \mathcal{A}$ such that $c \in U$
 - ⟨2⟩2. PICK $y \in [a, c)$ such that $(y, c] \subseteq U$

PROOF: By Lemma 4.1.2

⟨2⟩3. PICK z such that $y < z$ and $z \in C$

PROOF: This exists because y is not an upper bound for C .

⟨2⟩4. PICK a finite $\mathcal{A}_0 \subseteq \mathcal{A}$ such that $[a, z]$ is covered by \mathcal{A}_0

⟨2⟩5. $[a, c]$ is covered by $\mathcal{A}_0 \cup \{U\}$

⟨1⟩9. $c = b$

⟨2⟩1. ASSUME: for a contradiction $c < b$

⟨2⟩2. PICK $y \in (c, b]$ such that $[c, y]$ is covered by at most two elements of \mathcal{A} .

PROOF: By ⟨1⟩4

⟨2⟩3. $y > c$ and $y \in C$

⟨2⟩4. Q.E.D.

PROOF: This contradicts ⟨1⟩7.

⟨1⟩10. Q.E.D.

Corollary 9.4.19.1. *Every closed interval in \mathbb{R} is compact.*

Corollary 9.4.19.2 (CC). *S_Ω is limit point compact.*

PROOF:

⟨1⟩1. LET: A be an infinite subset of S_Ω

⟨1⟩2. PICK a countably infinite subset $B \subseteq A$

⟨1⟩3. LET: $b = \sup B$

⟨1⟩4. $B \subseteq [0, b]$

⟨1⟩5. $[0, b]$ is compact

PROOF: By the theorem.

⟨1⟩6. B has a limit point in $[0, b]$

⟨1⟩7. A has a limit point in $[0, b]$

□

Corollary 9.4.19.3. *The ordered square is compact.*

Corollary 9.4.19.4. *The ordered square is limit point compact.*

Corollary 9.4.19.5. *Not every subspace of a compact space is compact.*

PROOF: $[0, 1]$ is compact but $(0, 1)$ is not. □

Theorem 9.4.20 (Extreme Value Theorem). *Let $f : X \rightarrow Y$ be continuous where Y is a linearly ordered set in the order topology. If X is compact, then there exist $c, d \in X$ such that, for all $x \in X$, we have $f(c) \leq f(x) \leq f(d)$.*

PROOF:

⟨1⟩1. $f(X)$ is compact.

PROOF: By Proposition 9.4.10.

⟨1⟩2. $f(X)$ has a greatest element.

⟨2⟩1. ASSUME: for a contradiction $f(X)$ has no greatest element.

⟨2⟩2. $\{(-\infty, f(x)) : x \in X\}$ is a set of open sets that covers $f(X)$.

⟨2⟩3. PICK a finite subset $\{(-\infty, f(x_1)), \dots, (-\infty, f(x_n))\}$ that covers $f(X)$.

PROOF: By Proposition 9.4.5

⟨2⟩4. LET: $f(x_N)$ be largest out of $f(x_1), \dots, f(x_n)$

⟨2⟩5. $f(x_N) < f(x_N)$

⟨2⟩6. Q.E.D.

PROOF: This is a contradiction.

⟨1⟩3. $f(X)$ has a least element.

PROOF: Similar.

□

Theorem 9.4.21 (DC). *A nonempty compact Hausdorff space with no isolated points is uncountable.*

PROOF:

⟨1⟩1. LET: X be a nonempty compact Hausdorff space with no isolated points.

⟨1⟩2. For every nonempty open $U \subseteq X$ and point $x \in X$, there exists a nonempty open $V \subseteq U$ such that $x \notin \overline{V}$

⟨2⟩1. LET: $U \subseteq X$ be nonempty and open and $x \in X$

⟨2⟩2. PICK $y \in U$ such that $y \neq x$

PROOF: This is possible because $U \neq \{x\}$ since x is not an isolated point.

⟨2⟩3. PICK disjoint open neighbourhoods W_1 and W_2 of x and y

PROOF: Since X is Hausdorff

⟨2⟩4. LET: $V = U \cap W_2$

⟨2⟩5. $x \notin \overline{V}$

PROOF: We have $\overline{V} \subseteq \overline{W_2} \subseteq X \setminus W_1$.

⟨1⟩3. LET: $f : \mathbb{Z}^+ \rightarrow X$

PROVE: f is not surjective

⟨1⟩4. PICK a sequence of open sets $V_1 \supseteq V_2 \supseteq \dots$ such that $f(n) \notin \overline{V_n}$

PROOF: By ⟨1⟩2 and Dependent Choice.

⟨1⟩5. PICK a point $b \in \bigcap_{i=1}^{\infty} \overline{V_i}$

PROOF: By Proposition 9.4.15.

⟨1⟩6. $b \neq f(n)$ for all n

PROOF: For each n we have $b \in \overline{V_n}$ (⟨1⟩5) and $f(n) \notin \overline{V_n}$ (⟨1⟩4).

□

Corollary 9.4.21.1. *Every closed interval in \mathbb{R} is uncountable.*

Theorem 9.4.22. *Every compact space is limit point compact.*

PROOF:

⟨1⟩1. LET: X be a compact space.

⟨1⟩2. LET: $A \subseteq X$ be a set with no limit points.

PROVE: A is finite.

⟨1⟩3. A is closed.

PROOF: By Corollary 3.10.3.1.

⟨1⟩4. A is compact.

PROOF: By Proposition 9.4.6.

⟨1⟩5. $\{U \text{ open in } X : U \cap A \text{ is a singleton}\}$ covers A

⟨2⟩1. LET: $a \in A$

- (2)2. PICK an open neighbourhood U of a such that U does not intersect A at a point other than a
 PROOF: One must exist because a is not a limit point of A ((1)2).
 (2)3. $U \cap A = \{a\}$
 (1)6. PICK a finite subcover $\{U_1, \dots, U_n\}$
 PROOF: By (1)4 using Proposition 9.4.5.
 (1)7. For $1 \leq i \leq n$,
 LET: $U_i \cap A = \{a_i\}$
 (1)8. $A = \{a_1, \dots, a_n\}$
 \square

Proposition 9.4.23. *Let X be a space and $C, D \subseteq X$ be compact. Then $C \cup D$ is compact.*

- PROOF:
 (1)1. LET: \mathcal{A} be a set of open sets that covers $C \cup D$
 (1)2. PICK a finite subset \mathcal{A}_1 that covers C and a finite subset \mathcal{A}_2 that covers D .
 (1)3. $\mathcal{A}_1 \cup \mathcal{A}_2$ is a finite subset of \mathcal{A} that covers $C \cup D$.
 (1)4. Q.E.D.

Theorem 9.4.24. *Every compact Hausdorff space is normal.*

- PROOF:
 (1)1. LET: X be a compact Hausdorff space.
 (1)2. LET: A and B be disjoint closed sets in X .
 (1)3. $\{U \text{ open in } X : \exists V \text{ open in } X. B \subseteq V \wedge U \cap V = \emptyset\}$ covers A
 (2)1. B is compact
 PROOF: By Proposition 9.4.6.
 (2)2. Q.E.D.
 PROOF: By Lemma 9.4.8.
 (1)4. PICK a finite subcover $\{U_1, \dots, U_n\}$
 PROOF: A is compact by Proposition 9.4.6.
 (1)5. For $1 \leq i \leq n$, PICK V_i open in X such that $B \subseteq V_i$ and $U_i \cap V_i = \emptyset$
 (1)6. LET: $U = U_1 \cup \dots \cup U_n$ and $V = V_1 \cap \dots \cap V_n$
 (1)7. U and V are disjoint open sets, $A \subseteq U$ and $B \subseteq V$
 \square

Corollary 9.4.24.1. *The ordered square is normal.*

Proposition 9.4.25. *Not every compact Hausdorff space is first countable.*

PROOF: The space $\overline{S_\Omega}$ is compact Hausdorff but not first countable. \square

Corollary 9.4.25.1. *Not every compact Hausdorff space is second countable.*

Theorem 9.4.26 (Tychonoff (AC)). *The product of a family of compact spaces is compact.*

PROOF:

- (1)1. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of compact spaces.
 LET: $X = \prod_{\alpha \in J} X_\alpha$
- (1)2. LET: $\mathcal{A} \subseteq \mathcal{P}X$ satisfy the finite intersection property.
 PROVE: $\bigcap_{A \in \mathcal{A}} \overline{A}$ is nonempty.
- (1)3. PICK a set $\mathcal{D} \subseteq \mathcal{P}X$ that includes \mathcal{A} and is maximal with respect to the finite intersection property.
 PROOF: By Lemma 1.3.6.
- (1)4. For $\alpha \in J$, PICK $x_\alpha \in \bigcap_{D \in \mathcal{D}} \overline{\pi_\alpha(D)}$
- (2)1. LET: $\alpha \in J$
- (2)2. $\{\pi_\alpha(D) : D \in \mathcal{D}\}$ satisfies the finite intersection property.
- (2)3. Q.E.D.
 PROOF: By Proposition 9.4.15
- (1)5. LET: $x = (x_\alpha)_{\alpha \in J}$
- (1)6. For all $D \in \mathcal{D}$ we have $(x_\alpha)_{\alpha \in J} \in \overline{D}$
 PROOF:
- (2)1. Every subbasis element containing x intersects every member of \mathcal{D}
- (3)1. LET: $\pi_\alpha(U)^{-1}$ be a subbasis element containing x where U is open in X_α
- (3)2. LET: $D \in \mathcal{D}$
- (3)3. U intersects $\pi_\alpha(D)$
- (2)2. Every subbasis element containing x is a member of \mathcal{D}
 PROOF: By Lemma 1.3.8
- (2)3. Every basis element containing x is a member of \mathcal{D}
 PROOF: By Lemma 1.3.7
- (2)4. Every basis element containing x intersects every member of \mathcal{D}
 PROOF: This follows because \mathcal{D} satisfies the finite intersection property.
- (1)7. Q.E.D.
 PROOF: By Proposition 9.4.15
-
- PROOF:
- (1)1. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of compact spaces and $X = \prod_{\alpha \in J} X_\alpha$.
- (1)2. PICK a well-ordering $<$ of J such that J has a greatest element \top
- (1)3. For all $\alpha \in J$ and every family of points $p = \{p_i \in X_i\}_{i \leq \alpha}$,
 LET: $Y_\alpha(p) = \{x \in X : \forall i \leq \alpha. x_i = p_i\}$
- (1)4. For all $\beta \in J$ and every family of points $p = \{p_i \in X_i\}_{i < \beta}$,
 LET: $Z_\beta(p) = \bigcap_{\alpha < \beta} Y_\alpha = \{x \in X : \forall i < \beta. x_i = p_i\}$
- (1)5. Given $\beta \in J$, a family of points $\{p_i \in X_i\}_{i < \beta}$, and a finite set \mathcal{A} of basis elements that covers $Z_\beta(p)$, there exists $\alpha < \beta$ such that \mathcal{A} covers $Y_\alpha(p)$
- (2)1. ASSUME: (
 w.l.o.g. β has no immediate predecessor)
- (2)2. For $A \in \mathcal{A}$,
 LET: $J_A = \{i < \beta : \pi_i(A) \neq X_i\}$
- (2)3. LET: α be the largest element of $\bigcup_{A \in \mathcal{A}} J_A$
 PROOF: The set has a greatest element because each J_A is finite and \mathcal{A} is finite.

quasicomponent.

PROVE: x and y are in the same component.

$\langle 1 \rangle 2$. LET: \mathcal{A} be the set of all closed subspaces A of X such that x and y lie in the same quasicomponent of A .

$\langle 1 \rangle 3$. Every chain in \mathcal{A} has a lower bound.

$\langle 2 \rangle 1$. LET: $\mathcal{B} \subseteq \mathcal{A}$ be a chain

PROVE: $Y = \bigcap \mathcal{B} \in \mathcal{A}$

$\langle 2 \rangle 2$. ASSUME: for a contradiction $Y = C \cup D$ where C and D are disjoint and open in Y , $x \in C$ and $y \in D$

$\langle 2 \rangle 3$. PICK disjoint open sets U and V in X such that $C \subseteq U$ and $D \subseteq V$

PROOF: By Lemma 9.4.18.

$\langle 2 \rangle 4$. $\{B \setminus (U \cup V) : B \in \mathcal{B}\}$ satisfies the finite intersection property.

$\langle 3 \rangle 1$. LET: $B_1, \dots, B_n \in \mathcal{B}$

$\langle 3 \rangle 2$. $B_1 \cap \dots \cap B_n \in \mathcal{B}$

PROOF: By $\langle 2 \rangle 1$.

$\langle 3 \rangle 3$. $B_1 \cap \dots \cap B_n \setminus (U \cup V)$ is nonempty

PROOF: $B_1 \cap \dots \cap B_n \cap U$ and $B_1 \cap \dots \cap B_n \cap V$ cannot be disjoint, because x and y are in the same quasicomponent of $B_1 \cap \dots \cap B_n$.

$\langle 2 \rangle 5$. $Y \setminus (U \cup V)$ is nonempty.

PROOF: By Proposition 9.4.15.

$\langle 2 \rangle 6$. Q.E.D.

PROOF: This is a contradiction since $Y \setminus (U \cup V) = Y \setminus (C \cup D)$.

$\langle 1 \rangle 4$. PICK a minimal element $D \in \mathcal{A}$

PROOF: One exists by Zorn's Lemma.

$\langle 1 \rangle 5$. D is connected.

$\langle 2 \rangle 1$. ASSUME: [

for a contradiction $D = U \uplus V$ is a separation of D]

$\langle 2 \rangle 2$. CASE: $x, y \in U$

PROOF: In this case we have $U \in \mathcal{A}$ contradicting the minimality of D .

$\langle 2 \rangle 3$. CASE: $x \in U, y \in V$

PROOF: This is a contradiction because x and y are in the same quasicomponent of D .

$\langle 2 \rangle 4$. CASE: $x \in V, y \in U$

PROOF: Similar to $\langle 2 \rangle 3$.

$\langle 2 \rangle 5$. CASE: $x, y \in V$

PROOF: Similar to $\langle 2 \rangle 2$.

□

9.5 Perfect Maps

Proposition 9.5.1. *Let $p : X \twoheadrightarrow Y$ be a closed continuous surjective map. For all $y \in Y$ and U an open neighbourhood of $p^{-1}(y)$, there exists an open neighbourhood W of y such that $p^{-1}(W) \subseteq U$.*

PROOF: Take $W = Y \setminus p(X \setminus U)$. □

Proposition 9.5.2 (AC). *Let $p : X \rightarrow Y$ be a closed continuous surjective map. If X is normal then Y is normal.*

PROOF:

- $\langle 1 \rangle 1$. LET: $A, B \subseteq Y$ be closed
- $\langle 1 \rangle 2$. $p^{-1}(A), p^{-1}(B)$ are closed in X .
- $\langle 1 \rangle 3$. PICK disjoint open sets U, V of $p^{-1}(A), p^{-1}(B)$ respectively.
- $\langle 1 \rangle 4$. For all $a \in A$, PICK an open neighbourhood W_a of a such that $p^{-1}(W_a) \subseteq U$

PROOF: By Proposition 9.5.1.

- $\langle 1 \rangle 5$. For all $b \in B$, PICK an open neighbourhood W'_b of b such that $p^{-1}(W'_b) \subseteq V$

PROOF: By Proposition 9.5.1.

- $\langle 1 \rangle 6$. LET: $W = \bigcup_{a \in A} W_a$ and $W' = \bigcup_{b \in B} W'_b$
- $\langle 1 \rangle 7$. $W \cap W' = \emptyset$

PROOF: This holds because $p^{-1}(W) \subseteq U, p^{-1}(W') \subseteq V$, and p is surjective.

□

Definition 9.5.3 (Perfect Map). Let X and Y be topological spaces and $p : X \rightarrow Y$. Then p is *perfect* iff p is closed, continuous, surjective, and $p^{-1}(y)$ is compact for all $y \in Y$.

Proposition 9.5.4. *Let $p : X \rightarrow Y$ be a perfect map. If X is Hausdorff then so is Y .*

PROOF:

- $\langle 1 \rangle 1$. LET: $a, b \in Y$ with $a \neq b$
- $\langle 1 \rangle 2$. PICK disjoint open neighbourhoods U and V of $\pi^{-1}(a)$ and $\pi^{-1}(b)$, respectively.

PROOF: By Lemma 9.4.18.

- $\langle 1 \rangle 3$. PICK open neighbourhoods W and W' of a and b such that $\pi^{-1}(W) \subseteq U$ and $\pi^{-1}(W') \subseteq V$

PROOF: By Proposition 9.5.1.

- $\langle 1 \rangle 4$. W and W' are disjoint.

□

Proposition 9.5.5. *Let $p : X \rightarrow Y$ be perfect. If X is regular then so is Y .*

PROOF:

- $\langle 1 \rangle 1$. Y is T_1

PROOF: By Proposition 9.5.4.

- $\langle 1 \rangle 2$. LET: $C \subseteq Y$ be closed and $a \in Y \setminus C$
- $\langle 1 \rangle 3$. $p^{-1}(C)$ is closed and $p^{-1}(a)$ is disjoint from $p^{-1}(C)$.
- $\langle 1 \rangle 4$. PICK disjoint open neighbourhoods U, V of $p^{-1}(C), p^{-1}(a)$ respectively.

PROOF: By Lemma 9.4.8.

- $\langle 1 \rangle 5$. PICK an open neighbourhood W' of a such that $p^{-1}(W') \subseteq V$

PROOF: By Proposition 9.5.1.

- $\langle 1 \rangle 6$. For $c \in C$, PICK an open neighbourhood W_c such that $p^{-1}(W_c) \subseteq U$

PROOF: By Proposition 9.5.1.

(1)7. $W = \bigcup_{c \in C} W_c$ is an open neighbourhood of C disjoint from W'

□

Proposition 9.5.6 (AC). *Let $p : X \rightarrow Y$ be perfect. If X is locally compact then so is Y .*

PROOF:

(1)1. LET: $b \in Y$

(1)2. $\{U \text{ open in } X : \exists C \subseteq X \text{ compact. } U \subseteq C\}$ covers $p^{-1}(b)$

(1)3. PICK a finite subcover $\{U_1, \dots, U_n\}$

(1)4. For $1 \leq i \leq n$, PICK a compact $C_i \subseteq X$ such that $U_i \subseteq C_i$

(1)5. For $1 \leq i \leq n$, PICK a neighbourhood W_i of b such that $p^{-1}(W_i) \subseteq U_i$

PROOF: By Proposition 9.5.1

(1)6. $b \in W_1 \cup \dots \cup W_n \subseteq p(C_1) \cup \dots \cup p(C_n)$

(1)7. $p(C_1) \cup \dots \cup p(C_n)$ is compact.

(2)1. Each $p(C_i)$ is compact.

PROOF: By Proposition 9.4.10.

(2)2. Q.E.D.

PROOF: By Proposition 9.4.23.

□

9.6 Sequential Compactness

Definition 9.6.1 (Sequentially Compact). A space is *sequentially compact* iff every sequence has a convergent subsequence.

Proposition 9.6.2. $\overline{S_\Omega}$ is not sequentially compact.

PROOF: Ω is a limit point of S_Ω but is not the limit of any sequence of points in S_Ω . □

9.7 Local Compactness

Definition 9.7.1 (Local Compactness). Let X be a topological space.

For $x \in X$, the space X is *locally compact* at x iff there exists a compact subspace $C \subseteq X$ that includes a neighbourhood of x .

The space X is *locally compact* iff it is locally compact at every point.

Proposition 9.7.2. *Every complete linearly ordered set is locally compact under the order topology.*

PROOF:

(1)1. LET: L be a complete linearly ordered set and $x \in L$

PROVE: There exists a compact subspace $C \subseteq L$ that includes a neighbourhood U of x

(1)2. CASE: x is least and greatest in L

- PROOF: In this case, $L = \{x\}$ is compact.
- ⟨1⟩3. CASE: x is least in L but not greatest
 - ⟨2⟩1. PICK $a < x$
 - ⟨2⟩2. Take $C = [a, x]$ and $U = (a, x]$
 - ⟨1⟩4. CASE: x is greatest in L but not least

PROOF: Similar.
 - ⟨1⟩5. CASE: x is neither least nor greatest
 - ⟨2⟩1. PICK $a < x$ and $b > x$
 - ⟨2⟩2. Take $C = [a, b]$ and $U = (a, b)$

□

Corollary 9.7.2.1. *For every ordinal α , the space S_α is locally compact.*

Theorem 9.7.3. *Every closed subspace of a locally compact Hausdorff space is locally compact.*

PROOF:

- ⟨1⟩1. LET: X be locally compact Hausdorff and $C \subseteq X$ be closed.
- ⟨1⟩2. LET: $x \in C$
- ⟨1⟩3. PICK $D \subseteq X$ compact and $U \subseteq D$ open such that $x \in U$
- ⟨1⟩4. D is closed.

PROOF: Proposition 9.4.9.
- ⟨1⟩5. $C \cap D$ is closed

PROOF: Proposition 3.5.5.
- ⟨1⟩6. $C \cap D$ is compact

PROOF: Proposition 9.4.6.
- ⟨1⟩7. Q.E.D.

PROOF: $C \cap D \subseteq C$ is compact and includes the open neighbourhood $U \cap C$ of x .

□

Proposition 9.7.4. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of nonempty topological spaces. If $\prod_{\alpha \in J} X_\alpha$ is locally compact, then each X_α is locally compact.*

PROOF:

- ⟨1⟩1. LET: $\alpha \in J$ and $x_\alpha \in X_\alpha$
- ⟨1⟩2. PICK $x_\beta \in X_\beta$ for all $\beta \in J \setminus \{\alpha\}$
- ⟨1⟩3. PICK a compact subspace $C \subseteq \prod_{\alpha \in J} X_\alpha$ that a neighbourhood U of x included in C
- ⟨1⟩4. PICK a basic open set $\prod_{\alpha \in J} U_\alpha$ such that $x \in \prod_{\alpha \in J} U_\alpha \subseteq U$
- ⟨1⟩5. $x_\alpha \in U_\alpha \subseteq \pi_\alpha(C)$
- ⟨1⟩6. $\pi_\alpha(C)$ is compact.

PROOF: By Proposition 9.4.10.

□

Corollary 9.7.4.1. *The Sorgenfrey plane is not locally compact.*

Proposition 9.7.5. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of locally compact spaces such that X_α is compact for all but finitely many values of α . Then $\prod_{\alpha \in J} X_\alpha$ is locally compact.*

PROOF:

- ⟨1⟩1. ASSUME: X_α is compact if $\alpha \neq \alpha_1, \dots, \alpha_n$
 - ⟨1⟩2. LET: $\vec{x} \in \prod_{\alpha \in J} X_\alpha$
 - ⟨1⟩3. For $1 \leq i \leq n$, PICK $C_{\alpha_i} \subseteq X_{\alpha_i}$ compact and U_{α_i} open such that $x_{\alpha_i} \in U_{\alpha_i} \subseteq C_{\alpha_i}$
 - ⟨1⟩4. For $\alpha \neq \alpha_1, \dots, \alpha_n$,
LET: $C_\alpha = U_\alpha = X_\alpha$
 - ⟨1⟩5. $\vec{x} \in \prod_{\alpha \in J} U_\alpha \subseteq \prod_{\alpha \in J} C_\alpha$
 - ⟨1⟩6. $\prod_{\alpha \in J} C_\alpha$ is compact
- PROOF: By Tychonoff's Theorem.

□

Proposition 9.7.6. \mathbb{R}_l is not locally compact.

PROOF: $[0, +\infty)$ can be partitioned into infinitely many disjoint open sets, which therefore do not have a finite subcover. □

Proposition 9.7.7. Let $\{X_\alpha\}_{\alpha \in J}$ be a family of nonempty topological spaces. If $\prod_{\alpha \in J} X_\alpha$ is locally compact, then all but finitely many of the X_α are compact.

PROOF:

- ⟨1⟩1. PICK a point $a = (a_\alpha)_{\alpha \in J} \in \prod_{\alpha \in J} X_\alpha$
- ⟨1⟩2. PICK a compact $C \subseteq \prod_{\alpha \in J} X_\alpha$ that includes the basic neighbourhood $\prod_{\alpha \in J} U_\alpha$ of a , where $U_\alpha = X_\alpha$ for all α except $\alpha = \alpha_1, \dots, \alpha_n$
- ⟨1⟩3. For $\alpha \neq \alpha_1, \dots, \alpha_n$, we have X_α is compact.

PROOF: X_α is homeomorphic to a closed subspace of C .

□

Corollary 9.7.7.1. For any infinite set I , the space \mathbb{R}^I is not locally compact.

Proposition 9.7.8. $[0, 1]^\omega$ is not compact under the uniform topology.

PROOF: $\{a_i : i \geq 0\}$ is an infinite set with no limit point, where a_i is the point with i th component 1 and all other components 0. □

Corollary 9.7.8.1. \mathbb{R}^ω under the uniform topology is not locally compact.

PROOF:

- ⟨1⟩1. ASSUME: \mathbb{R}^ω is locally compact
- ⟨1⟩2. LET: C be a compact subspace such that $B(\vec{0}, \epsilon) \subseteq C$
- ⟨1⟩3. $\overline{B(\vec{0}, \epsilon)}$ is compact.
- ⟨1⟩4. Q.E.D.

PROOF: This contradicts the proposition.

□

Proposition 9.7.9. Not every subspace of a locally compact Hausdorff space is locally compact.

PROOF: \mathbb{R} is locally compact Hausdorff, \mathbb{Q} is not locally compact. □

Proposition 9.7.10. *The continuous image of a locally compact Hausdorff space is not necessarily locally compact.*

PROOF:

- ⟨1⟩1. LET: $\{q_0, q_1, \dots\}$ be an enumeration of $[0, 1] \cap \mathbb{Q}$.
- ⟨1⟩2. Define $f : (0, +\infty) \setminus \mathbb{Z} \rightarrow [0, 1] \cap \mathbb{Q}$ by: $f(x) = q_n$ for $x \in (n, n+1)$
- ⟨1⟩3. f is continuous.

PROOF: The inverse image of any set is a union of open intervals.

□

9.8 Compactifications

Definition 9.8.1 (Compactification). Let X and Y be spaces. Then Y is a *compactification* of X iff Y is a compact Hausdorff space and X is a subspace of Y with $\overline{X} = Y$.

Two compactifications Y_1, Y_2 of X are *equivalent* iff there exists a homeomorphism between Y_1 and Y_2 that is the identity on X .

Definition 9.8.2 (One-Point Compactification). A *one-point compactification* of X is a compactification Y of X such that $Y \setminus X$ is a singleton.

Theorem 9.8.3. *Let X be a topological space. Then X is locally compact Hausdorff if and only if there exists a space Y such that:*

1. X is a subspace of Y
2. The set $Y \setminus X$ is a singleton.
3. Y is a compact Hausdorff space.

If Y and Y' are two spaces satisfying these conditions, then there exists a unique homeomorphism between Y and Y' that is the identity on X .

PROOF:

- ⟨1⟩1. If X is locally compact Hausdorff then there exists a space Y satisfying 1–3.
- ⟨2⟩1. LET: $Y = X \cup \{\infty\}$ under the topology $\mathcal{T} = \{U \subseteq X : U \text{ is open in } X\} \cup \{Y \setminus C : C \subseteq X \text{ is compact}\}$.
- ⟨3⟩1. $Y \in \mathcal{T}$
PROOF: This holds because $Y = Y \setminus \emptyset$.
- ⟨3⟩2. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$.
 - ⟨4⟩1. LET: $U, V \in \mathcal{T}$
 - ⟨4⟩2. CASE: U, V are open in X
PROOF: In this case, $U \cap V$ is open in X .
 - ⟨4⟩3. CASE: U is open in X , $V = Y \setminus C$ where $C \subseteq X$ is compact.
 - ⟨5⟩1. $U \cap V = U \setminus C$
 - ⟨5⟩2. C is closed in X
PROOF: Proposition 9.4.9.

PROOF: In this case, we just use the fact that X is Hausdorff.

⟨4⟩1. PICK $C \subseteq X$ compact such that C includes an open neighbourhood V of y

⟨3⟩4. CASE: $x \in X, y = \infty$

(1)2. If there exists a space Y satisfying 1–3 then X is locally compact Hausdorff.

(2)3. X is locally compact

⟨3⟩3. $X \setminus V$ is compact and includes U

PROOF: By Corollary 6.2.6.1.

2. LET: $Y \setminus X = \{p\}$ and $Y' \setminus X = \{q\}$

$$h(p) = q$$

3. h is continuous.

PROVE: $h^{-1}(V)$ is open.

$\langle 5 \rangle 1. \quad h^{-1}(V) = V$

PROOF: Condition 1 for Y' .

⟨5⟩3. V is open in Y

PROOF: Condition 1 for Y .

4.3. CASE: $q \in V$

$\langle 5 \rangle 1$. $Y' \setminus V$ is compact.

PROOF: Proposition 9.4.6.

⟨5⟩2. $Y' \setminus V$ is closed in Y .

PROOF: Proposition 9.4.9.

⟨5⟩3. $h^{-1}(V) = Y \setminus (Y' \setminus V)$

$\langle 3 \rangle 3$. h^{-1} is continuous.

PROOF: Similar.

(2)5. If $h' : Y \rightarrow Y'$ is a homeomorphism such that $h' \upharpoonright_X = \text{id}_X$ then $h' = h$
 \square

Theorem 9.8.4. *Let X be a Hausdorff space. Then X is locally compact if and only if, for all $x \in X$ and any neighbourhood U of x , there exists an open neighbourhood V of x such that \overline{V} is compact and $\overline{V} \subseteq U$.*

PROOF:

(1)1. If X is locally compact then, for all $x \in X$ and any neighbourhood U of x , there exists an open neighbourhood V of x such that \overline{V} is compact and $\overline{V} \subseteq U$.

(2)1. ASSUME: X is locally compact.

(2)2. LET: $x \in X$ and U be a neighbourhood of x .

(2)3. LET: Y be the one-point compactification of X .

PROOF: By Theorem 9.8.3.

(2)4. LET: $C = Y \setminus U$

(2)5. C is compact

PROOF: By Proposition 9.4.6.

(2)6. PICK disjoint open sets V, W containing x and C

PROOF: Lemma 9.4.8

(2)7. V is open in X

PROOF: $V \subseteq X$ since $\infty \in W$.

(2)8. The closure of V in X is compact

(3)1. The closure of V in X is the same as the closure of V in Y .

PROOF: The point ∞ cannot be a limit point of V since W is a neighbourhood disjoint from V .

(3)2. The closure of V in Y is compact.

PROOF: By Proposition 9.4.6.

(2)9. $\overline{V} \subseteq U$

PROOF:

$$\begin{aligned}\overline{V} &\subseteq Y \setminus W \\ &\subseteq Y \setminus C \\ &= U\end{aligned}$$

(1)2. If, for all $x \in X$ and any neighbourhood U of x , there exists an open neighbourhood V of x such that \overline{V} is compact and $\overline{V} \subseteq U$, then X is locally compact.

(2)1. ASSUME: for all $x \in X$ and any neighbourhood U of x , there exists an open neighbourhood V of x such that \overline{V} is compact and $\overline{V} \subseteq U$

(2)2. LET: $x \in X$

PROVE: There exists $C \subseteq X$ compact such that C includes a neighbourhood U of x

(2)3. PICK an open neighbourhood V of x such that \overline{V} is compact and $\overline{V} \subseteq X$

(2)4. Take $C = \overline{V}$ and $U = V$

\square

Corollary 9.8.4.1. *Every open subspace of a locally compact Hausdorff space is locally compact.*

Corollary 9.8.4.2. *A space is locally compact Hausdorff if and only if it is an open subspace of a compact Hausdorff space.*

Corollary 9.8.4.3. *Every locally compact Hausdorff space is completely regular.*

Corollary 9.8.4.4. *The space \mathbb{R}_K is not locally compact.*

Lemma 9.8.5 (AC). *If $p : X \rightarrow Y$ is a quotient map and Z is a locally compact Hausdorff space, then the map*

$$\pi = p \times \text{id}_Z : X \times Z \rightarrow Y \times Z$$

is a quotient map.

PROOF:

⟨1⟩1. π is surjective.

PROOF: This holds because p is surjective.

⟨1⟩2. π is continuous.

PROOF: By Theorem 5.2.15.

⟨1⟩3. For $A \subseteq Y \times Z$, if $\pi^{-1}(A)$ is open in $X \times Z$ then A is open in $Y \times Z$.

⟨2⟩1. LET: $A \subseteq Y \times Z$

⟨2⟩2. ASSUME: $\pi^{-1}(A)$ is open in $X \times Z$

⟨2⟩3. LET: $(y, z) \in A$

⟨2⟩4. PICK $x \in X$ such that $p(x) = y$

PROOF: Since p is surjective.

⟨2⟩5. PICK open sets U_1, V with \bar{V} compact such that $(x, y) \in U_1 \times V$ and $U_1 \times \bar{V} \subseteq \pi^{-1}(A)$

PROOF: Using Theorem 9.8.4

⟨2⟩6. PICK a sequence of open sets U_1, U_2, \dots in X such that $p^{-1}(p(U_n)) \subseteq U_{n+1}$ and $U_n \times \bar{V} \subseteq \pi^{-1}(A)$ for all n

⟨3⟩1. LET: U be open with $U \times \bar{V} \subseteq \pi^{-1}(A)$

PROVE: There exists W open with $p^{-1}(p(U)) \subseteq W$ and $W \times \bar{V} \subseteq \pi^{-1}(A)$

⟨3⟩2. For all $x \in p^{-1}(p(U))$, PICK open sets U_x, V_x such that $x \in U_x$, $\bar{V} \subseteq V_x$ and $U_x \times V_x \subseteq \pi^{-1}(A)$

PROOF: By the Tube Lemma.

⟨3⟩3. LET: $W = \bigcup_{x \in p^{-1}(p(U))} U_x$

⟨2⟩7. LET: $U = \bigcup_{n=1}^{\infty} U_n$

⟨2⟩8. U is saturated with respect to p

⟨3⟩1. LET: $a \in U, b \in X, p(a) = p(b)$

⟨3⟩2. PICK n such that $a \in U_n$

⟨3⟩3. $b \in p^{-1}(p(U_n))$

⟨3⟩4. $b \in U_{n+1}$

⟨3⟩5. $b \in U$

⟨2⟩9. $p(U)$ is open in Y

PROOF: By Lemma 4.5.2.

⟨2⟩10. $(y, z) \in p(U) \times V \subseteq A$

⟨2⟩11. Q.E.D.

PROOF: By Proposition 3.2.3.

□

Theorem 9.8.6. *Let $p : A \rightarrow B$ and $q : C \rightarrow D$ be quotient maps. If B and C are locally compact Hausdorff spaces, then $p \times q : A \times C \rightarrow B \times D$ is a quotient map.*

PROOF: This holds by Lemma 9.8.5 and Proposition 4.5.10 because $p \times q = (\text{id}_B \times q) \circ (p \times \text{id}_C)$. □

Chapter 10

Metric Spaces

10.1 The Metric Topology

Definition 10.1.1 (Metric). A *metric* on a set X is a function $d : X \times X \rightarrow \mathbb{R}$ such that, for all $x, y, z \in X$:

1. $d(x, y) \geq 0$;
2. $d(x, y) = 0$ if and only if $x = y$;
3. $d(x, y) = d(y, x)$;
4. **Triangle Inequality**

$$d(x, z) \leq d(x, y) + d(y, z)$$

A *metric space* X consists of a set X and a metric on X . We call $d(x, y)$ the *distance* between x and y .

Definition 10.1.2 (Open Ball). Let X be a metric space with metric d , $x \in X$ and $\epsilon > 0$. The *open ball* with *centre* x and *radius* ϵ is

$$B_d(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\} .$$

Lemma 10.1.3. Let X be a metric space, $x, y \in X$ and $\epsilon > 0$. If $y \in B(x, \epsilon)$, then there exists δ such that $0 < \delta < \epsilon$ and

$$B(y, \delta) \subseteq B(x, \epsilon) .$$

PROOF:

- ⟨1⟩1. LET: $\delta = \epsilon - d(x, y)$
⟨1⟩2. LET: $z \in B(y, \delta)$
⟨1⟩3. $d(x, z) < \epsilon$

PROOF:

$$\begin{aligned}
 d(x, z) &\leq d(x, y) + d(y, z) && \text{(Triangle Inequality)} \\
 &< d(x, y) + \delta && (\langle 1 \rangle 2) \\
 &= \epsilon && (\langle 1 \rangle 1)
 \end{aligned}$$

□

Definition 10.1.4 (Metric Topology). Let d be a metric on X . The *metric topology* on X induced by d is the topology generated by the basis consisting of the open balls.

We prove this is a topology.

PROOF:

$\langle 1 \rangle 1$. Every point is in an open ball.

PROOF: $x \in B(x, 1)$

$\langle 1 \rangle 2$. If B_1, B_2 are open balls and $x \in B_1 \cap B_2$, then there exists an open ball B_3 such that $x \in B_3 \subseteq B_1 \cap B_2$.

$\langle 2 \rangle 1$. LET: $x \in B(y, \epsilon_1) \cap B(z, \epsilon_2)$

$\langle 2 \rangle 2$. PICK δ_1, δ_2 such that $0 < \delta_1 < \epsilon_1, 0 < \delta_2 < \epsilon_2, B(x, \delta_1) \subseteq B(y, \epsilon_1)$ and $B(x, \delta_2) \subseteq B(z, \epsilon_2)$.

PROOF: Lemma 10.1.3.

$\langle 2 \rangle 3$. LET: $\delta = \min(\delta_1, \delta_2)$

$\langle 2 \rangle 4$. $x \in B(x, \delta) \subseteq B(y, \epsilon_1) \cap B(z, \epsilon_2)$

$\langle 1 \rangle 3$. Q.E.D.

PROOF: Lemma 3.4.3.

Lemma 10.1.5. A set U is open in the metric topology induced by d if and only if, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$.

PROOF:

$\langle 1 \rangle 1$. If U is open then, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$.

$\langle 2 \rangle 1$. ASSUME: U is open.

$\langle 2 \rangle 2$. LET: $x \in U$

$\langle 2 \rangle 3$. PICK $B(y, \delta)$ such that $x \in B(y, \delta) \subseteq U$

$\langle 2 \rangle 4$. PICK ϵ such that $0 < \epsilon < \delta$ and $B(x, \epsilon) \subseteq B(y, \delta)$

PROOF: Lemma 10.1.3.

$\langle 2 \rangle 5$. $B(x, \epsilon) \subseteq U$

PROOF: From $\langle 2 \rangle 3$ and $\langle 2 \rangle 4$.

$\langle 1 \rangle 2$. If, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$, then U is open.

PROOF: Immediate from definition of metric topology.

□

Lemma 10.1.6. Let d and d' be two metrics on the set X . Let \mathcal{T} and \mathcal{T}' be the topologies they induce, respectively. Then $\mathcal{T} \subseteq \mathcal{T}'$ if and only if, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$.

PROOF:

$\langle 1 \rangle 1$. If $\mathcal{T} \subseteq \mathcal{T}'$ then, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$.

(2)1. ASSUME: $\mathcal{T} \subseteq \mathcal{T}'$
 (2)2. LET: $x \in X$ and $\epsilon > 0$
 (2)3. $B_d(x, \epsilon) \in \mathcal{T}'$
 PROOF: From (2)1.
 (2)4. There exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$
 PROOF: By Lemma 10.1.5.
 (1)2. If, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$, then $\mathcal{T} \subseteq \mathcal{T}'$
 (2)1. ASSUME: For all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$.
 (2)2. LET: $U \in \mathcal{T}$
 PROVE: $U \in \mathcal{T}'$
 (2)3. LET: $x \in U$
 (2)4. PICK $\epsilon > 0$ be such that $B_d(x, \epsilon) \subseteq U$
 PROOF: By Lemma 10.1.5.
 (2)5. PICK $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$
 PROOF: By (2)1.
 (2)6. $B_{d'}(x, \delta) \subseteq U$
 PROOF: By (2)4 and (2)5.
 (2)7. Q.E.D.
 PROOF: By Lemma 10.1.5.

□

Definition 10.1.7 (Metrisable). A topological space is *metrisable* if and only if there exists a metric that induces its topology.

Lemma 10.1.8. *Every discrete space is metrisable.*

PROOF: The discrete topology is induced by the metric $d(x, y) = 1$ if $x \neq y$, 0 if $x = y$. □

Proposition 10.1.9. *The continuous image of a metrisable space is not necessarily metrisable.*

PROOF: The identity map from the discrete two-point space to the indiscrete two-point space is continuous. □

Lemma 10.1.10. \mathbb{R} is metrisable.

PROOF: The standard topology is induced by the metric $d(x, y) = |x - y|$. □

Definition 10.1.11 (Bounded). Let X be a metric space and $A \subseteq X$. Then A is *bounded* iff $\{d(x, y) : x, y \in A\}$ is bounded above, in which case its *diameter* is

$$\text{diam } A = \sup_{x, y \in A} d(x, y) .$$

Lemma 10.1.12. *Let (X, d) be a metric space and $A \subseteq X$. Then $d \upharpoonright_{A \times A}$ is a metric on A that induces the subspace topology.*

PROOF:

⟨1⟩1. $d \upharpoonright_{A \times A}$ is a metric on A .

PROOF: Each of the axioms for a metric follows immediately from the same axiom for d .

⟨1⟩2. The topology induced by $d \upharpoonright_{A \times A}$ is the product topology.

PROOF: Both are the topology generated by the basis consisting of all the open balls $B_{d \upharpoonright_{A \times A}}(a, \epsilon) = B_d(a, \epsilon) \cap A$.

□

Lemma 10.1.13. *Every metric space is Hausdorff.*

PROOF:

⟨1⟩1. LET: X be a metric space and $x, y \in X$ with $x \neq y$.

⟨1⟩2. LET: $\epsilon = d(x, y)$

⟨1⟩3. $B(x, \epsilon/2)$ and $B(y, \epsilon/2)$ are disjoint neighbourhoods of x and y .

□

Theorem 10.1.14. *Every metric space is first countable.*

PROOF: $\{B(x, q) : q \in \mathbb{Q}^+\}$ is a local basis at x . □

Corollary 10.1.14.1. *If J is infinite then the space \mathbb{R}^J is not metrizable.*

Definition 10.1.15 (Standard Bounded Metric). Let d be a metric on X . The *standard bounded metric* corresponding to d is

$$\bar{d}(x, y) = \min(d(x, y), 1) .$$

We prove this is a metric.

PROOF:

⟨1⟩1. $\bar{d}(x, y) \geq 0$

PROOF: This holds because $d(x, y) \geq 0$ (d is a metric) and $1 > 0$.

⟨1⟩2. $\bar{d}(x, y) = 0$ iff $x = y$

PROOF: Immediate from definition.

⟨1⟩3. $\bar{d}(x, y) = \bar{d}(y, x)$

PROOF: Immediate from definition.

⟨1⟩4. $\bar{d}(x, z) \leq \bar{d}(x, y) + \bar{d}(y, z)$

⟨2⟩1. CASE: $d(x, y) \leq 1, d(y, z) \leq 1$

PROOF:

$$\begin{aligned} \bar{d}(x, z) &\leq d(x, z) \\ &\leq d(x, y) + d(y, z) \\ &= \bar{d}(x, y) + \bar{d}(y, z) \end{aligned}$$

⟨2⟩2. CASE: $d(y, z) > 1$

PROOF:

$$\begin{aligned} \bar{d}(x, z) &\leq 1 \\ &\leq \bar{d}(x, y) + 1 \\ &= \bar{d}(x, y) + \bar{d}(y, z) \end{aligned}$$

⟨2⟩3. CASE: $d(x, y) > 1$

PROOF: Similar.

□

Theorem 10.1.16. *Let d be a metric on X . Then the standard bounded metric \bar{d} corresponding to d induces the same topology as d .*

PROOF:

⟨1⟩1. LET: \mathcal{T} be the topology induced by d and \mathcal{T}' be the topology induced by \bar{d} .

⟨1⟩2. $\mathcal{T} \subseteq \mathcal{T}'$

⟨2⟩1. LET: $x \in X$ and $\epsilon > 0$

⟨2⟩2. LET: $\delta = \min(\epsilon, 1/2)$

⟨2⟩3. $B_{\bar{d}}(x, \delta) \subseteq B_d(x, \epsilon)$

⟨3⟩1. LET: $y \in B_{\bar{d}}(x, \delta)$

⟨3⟩2. $\bar{d}(x, y) < \delta$

⟨3⟩3. $\bar{d}(x, y) < 1$

PROOF: From ⟨2⟩2 and ⟨3⟩2.

⟨3⟩4. $\bar{d}(x, y) = d(x, y)$

PROOF: From ⟨3⟩3 and the definition of \bar{d} .

⟨3⟩5. $d(x, y) < \epsilon$

PROOF: By ⟨2⟩2 and ⟨3⟩2 and ⟨3⟩4.

⟨1⟩3. $\mathcal{T}' \subseteq \mathcal{T}$

⟨2⟩1. LET: $x \in X$ and $\epsilon > 0$

⟨2⟩2. $B_d(x, \epsilon) \subseteq B_{\bar{d}}(x, \epsilon)$

PROOF: This holds because $\bar{d}(x, y) \leq d(x, y)$.

□

Definition 10.1.17 (Square Metric). The *square metric* on \mathbb{R}^n is defined by

$$\rho(\vec{x}, \vec{y}) = \max(|x_1 - y_1|, \dots, |x_n - y_n|) \ .$$

We prove this is a metric.

PROOF:

⟨1⟩1. $\rho(\vec{x}, \vec{y}) \geq 0$

PROOF: Immediate from definitions.

⟨1⟩2. $\rho(\vec{x}, \vec{y}) = 0$ iff $\vec{x} = \vec{y}$

PROOF: Immediate from definitions.

⟨1⟩3. $\rho(\vec{x}, \vec{y}) = \rho(\vec{y}, \vec{x})$

PROOF: Immediate from definitions.

⟨1⟩4. $\rho(\vec{x}, \vec{z}) \leq \rho(\vec{x}, \vec{y}) + \rho(\vec{y}, \vec{z})$

⟨2⟩1. For all i , we have $|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$

⟨2⟩2. For all i , $|x_i - z_i| \leq \rho(\vec{x}, \vec{y}) + \rho(\vec{y}, \vec{z})$

⟨2⟩3. $\rho(\vec{x}, \vec{z}) \leq \rho(\vec{x}, \vec{y}) + \rho(\vec{y}, \vec{z})$

□

Theorem 10.1.18. *The square metric induces the standard topology on \mathbb{R}^n .*

PROOF:

⟨1⟩1. LET: \mathcal{T}_ρ be the topology induced by the square metric and \mathcal{T}_s the standard topology.

⟨1⟩2. $\mathcal{T}_\rho \subseteq \mathcal{T}_s$

PROOF: This holds because $B_\rho(\vec{x}, \epsilon) = (x_1 - \epsilon, x_1 + \epsilon) \times \cdots \times (x_n - \epsilon, x_n + \epsilon)$.

⟨1⟩3. $\mathcal{T}_s \subseteq \mathcal{T}_\rho$

⟨2⟩1. LET: $B = U_1 \times \cdots \times U_n$ be a basic open set in \mathcal{T}_s , where each U_i is open in \mathbb{R} .

⟨2⟩2. LET: $\vec{x} \in B$

⟨2⟩3. For $1 \leq i \leq n$, PICK $\epsilon_i > 0$ such that $(x_i - \epsilon_i, x_i + \epsilon_i) \subseteq U_i$

⟨2⟩4. LET: $\epsilon = \min(\epsilon_1, \dots, \epsilon_n)$

⟨2⟩5. $B_\rho(\vec{x}, \epsilon) \subseteq B$

□

Lemma 10.1.19. *The product of a countable family of metrizable spaces is metrizable.*

PROOF:

⟨1⟩1. LET: $\{X_n\}_{n \in \mathbb{Z}^+}$ be a family of metric spaces with metrics bounded by 1,
 $X = \prod_{n=1}^{\infty} X_n$.

⟨1⟩2. LET: $D : X \times X \rightarrow \mathbb{R}$ be given by

$$D(\vec{x}, \vec{y}) = \sup_{n \geq 1} \frac{d(x_n, y_n)}{n}.$$

⟨1⟩3. D is a metric on X .

⟨2⟩1. $D(\vec{x}, \vec{y}) \geq 0$

PROOF: Immediate from definitions.

⟨2⟩2. $D(\vec{x}, \vec{y}) = 0$ iff $\vec{x} = \vec{y}$

PROOF: Immediate from definitions.

⟨2⟩3. $D(\vec{x}, \vec{y}) = D(\vec{y}, \vec{x})$

PROOF: Immediate from definitions.

⟨2⟩4. $D(\vec{x}, \vec{z}) \leq D(\vec{x}, \vec{y}) + D(\vec{y}, \vec{z})$

⟨3⟩1. For all n , we have $\frac{d(x_n, z_n)}{n} \leq \frac{d(x_n, y_n)}{n} + \frac{d(y_n, z_n)}{n}$

⟨3⟩2. For all n , we have $\frac{d(x_n, z_n)}{n} \leq D(\vec{x}, \vec{y}) + D(\vec{y}, \vec{z})$

⟨3⟩3. $D(\vec{x}, \vec{z}) \leq D(\vec{x}, \vec{y}) + D(\vec{y}, \vec{z})$

⟨1⟩4. LET: \mathcal{T}_D be the topology induced by D and \mathcal{T}_p the product topology.

⟨1⟩5. $\mathcal{T}_D \subseteq \mathcal{T}_p$

⟨2⟩1. LET: $U \in \mathcal{T}_D$

PROVE: $U \in \mathcal{T}_p$

⟨2⟩2. LET: $\vec{x} \in U$

⟨2⟩3. PICK $\epsilon > 0$ such that $B_D(\vec{x}, \epsilon) \subseteq U$

⟨2⟩4. PICK N such that $1/N < \epsilon$

⟨2⟩5. LET: $V = B(x_1, \epsilon) \times \cdots \times B(x_N, \epsilon) \times X_{N+1} \times X_{N+2} \times \cdots$

⟨2⟩6. $\vec{x} \in V \subseteq B_D(\vec{x}, \epsilon)$

⟨1⟩6. $\mathcal{T}_p \subseteq \mathcal{T}_D$

⟨2⟩1. LET: $U = \prod_{n=1}^{\infty} U_n$ be a basic open set in \mathcal{T}_p , where each U_n is open in X_n , and $U_n = X_n$ for $n > N$.

- (2)2. LET: $\vec{x} \in U$
 PROVE: There exists $\epsilon > 0$ such that $B_D(\vec{x}, \epsilon) \subseteq U$.
 (2)3. For $n \leq N$, PICK $\epsilon_n > 0$ such that $B(x_n, \epsilon_n) \subseteq U_n$
 (2)4. LET: $\epsilon = \min(\epsilon_1, \epsilon_2/2, \dots, \epsilon_n/n)$
 (2)5. LET: $\vec{y} \in B_D(\vec{x}, \epsilon)$
 (2)6. For $n \leq N$, $y_n \in U_n$
 (3)1. $D(\vec{x}, \vec{y}) < \epsilon$
 (3)2. $d(x_n, y_n)/n < \epsilon$
 (3)3. $d(x_n, y_n)/n < \epsilon_n/n$
 (3)4. Q.E.D.
 PROOF: By (2)3.

□

Corollary 10.1.19.1. *The space \mathbb{R}^ω is metrizable.*

Definition 10.1.20 (Uniform Metric). Let J be a set. The *uniform metric* $\bar{\rho}$ on \mathbb{R}^J is defined by

$$\bar{\rho}(\vec{x}, \vec{y}) = \sup_{\alpha \in J} \bar{d}(x_\alpha, y_\alpha) .$$

where \bar{d} is the standard bounded metric on \mathbb{R} . The *uniform topology* is the topology induced by the uniform metric.

We prove this is a metric.

PROOF:

- (1)1. $\bar{\rho}(\vec{x}, \vec{y}) \geq 0$
 PROOF: Immediate from definitions.
 (1)2. $\bar{\rho}(\vec{x}, \vec{y}) = 0$ iff $\vec{x} = \vec{y}$
 PROOF: Immediate from definitions.
 (1)3. $\bar{\rho}(\vec{x}, \vec{y}) = \bar{\rho}(\vec{y}, \vec{x})$
 PROOF: Immediate from definitions.
 (1)4. $\bar{\rho}(\vec{x}, \vec{z}) \leq \bar{\rho}(\vec{x}, \vec{y}) + \bar{\rho}(\vec{y}, \vec{z})$
 PROOF:
 (2)1. For all $\alpha \in J$, $\bar{d}(x_\alpha, z_\alpha) \leq \bar{d}(x_\alpha, y_\alpha) + \bar{d}(y_\alpha, z_\alpha)$
 (2)2. For all $\alpha \in J$, $\bar{d}(x_\alpha, z_\alpha) \leq \bar{\rho}(\vec{x}, \vec{y}) + \bar{\rho}(\vec{y}, \vec{z})$
 (2)3. $\bar{\rho}(\vec{x}, \vec{z}) \leq \bar{\rho}(\vec{x}, \vec{y}) + \bar{\rho}(\vec{y}, \vec{z})$

□

Theorem 10.1.21 (DC). *The uniform topology on \mathbb{R}^J is finer than the product topology and coarser than the box topology. These three topologies are different iff J is infinite.*

PROOF:

- (1)1. The uniform topology is finer than the product topology.
 (2)1. LET: $B = \prod_{\alpha \in J} U_\alpha$ be a basic open set in the product topology, where each U_α is open in \mathbb{R} , and $U_\alpha = \mathbb{R}$ except for $\alpha = \alpha_1, \dots, \alpha_n$.
 (2)2. LET: $\vec{x} \in U$
 (2)3. For $1 \leq i \leq n$, PICK $0 < \epsilon_i < 1$ such that $(x_{\alpha_i} - \epsilon_i, x_{\alpha_i} + \epsilon_i) \subseteq U_{\alpha_i}$.

- (2)4. LET: $\epsilon = \min(\epsilon_1, \dots, \epsilon_n)$
- (2)5. $B_{\vec{\rho}}(\vec{x}, \epsilon) \subseteq B$
 - (3)1. LET: $\vec{y} \in B_{\vec{\rho}}(\vec{x}, \epsilon)$
 - (3)2. For $1 \leq i \leq n$, we have $y_i \in U_{\alpha_i}$
 - (4)1. LET: $1 \leq i \leq n$
 - (4)2. $\vec{d}(x_{\alpha_i}, y_{\alpha_i}) < \epsilon_i$

PROOF: From (2)4 and (3)1.
 - (4)3. $d(x_{\alpha_i}, y_{\alpha_i}) < \epsilon_i$

PROOF: From (4)2 since $\epsilon_i < 1$ ((2)3).
 - (4)4. Q.E.D.

PROOF: By (2)3.
- (1)2. The uniform topology is coarser than the box topology.
 - (2)1. LET: $\vec{x} \in \mathbb{R}^J$ and $\epsilon > 0$

PROVE: $B_{\vec{\rho}}(\vec{x}, \epsilon)$ is open in the box topology.
 - (2)2. CASE: $\epsilon < 1$

PROOF: In this case, $B(\vec{x}, \epsilon) = \prod_{\alpha \in J} (x_\alpha - \epsilon, x_\alpha + \epsilon)$.
 - (2)3. CASE: $\epsilon \geq 1$

PROOF: In this case, $B(\vec{x}, \epsilon) = \mathbb{R}^J$.
- (1)3. If J is finite then the product topology is the same as the box topology.

PROOF: Immediate from definitions.
- (1)4. If J is infinite then the uniform topology is distinct from the product topology.
 - (2)1. $B(\vec{0}, 1/2)$ is not open in the product topology.
 - (3)1. $\vec{0} \in B(\vec{0}, 1/2)$
 - (3)2. LET: $\prod_{\alpha \in J} U_\alpha$ be any basic open set containing $\vec{0}$, where U_α is open in \mathbb{R} for all α , and $U_\alpha = \mathbb{R}$ except for $\alpha = \alpha_1, \dots, \alpha_n$
 - (3)3. PICK $\alpha_0 \in J$ such that $\alpha_0 \neq \alpha_1, \dots, \alpha_n$
 - (3)4. LET: \vec{x} be such that $x_{\alpha_0} = 1$, and $x_\alpha = 0$ for $\alpha \neq \alpha_0$.
 - (3)5. $\vec{x} \in \prod_{\alpha \in J} U_\alpha$
 - (3)6. $\vec{x} \notin B(\vec{0}, 1/2)$
- (1)5. If J is infinite then the uniform topology is distinct from the box topology.
 - (2)1. PICK a countable sequence $\alpha_1, \alpha_2, \dots$ in J
 - (2)2. LET: $U = \prod_{\alpha \in J} U_\alpha$, where $U_{\alpha_n} = (-1/n, 1/n)$ for all n , and $U_\alpha = \mathbb{R}$ for all other α .

PROVE: U is not open in the uniform topology.
 - (2)3. $\vec{0} \in U$
 - (2)4. LET: $\epsilon > 0$

PROVE: $B(\vec{0}, \epsilon) \not\subseteq U$
 - (2)5. PICK N such that $1/N < \epsilon$
 - (2)6. LET: \vec{x} be such that $x_{\alpha_N} = 1/N$ and $x_\alpha = 0$ for all other α
 - (2)7. $\vec{x} \in B(\vec{0}, \epsilon)$
 - (2)8. $\vec{x} \notin U$

Proposition 10.1.22. *The space \mathbb{R}^ω under the uniform topology is not second countable.*

PROOF: The set of all sequences of 0s and 1s is discrete but uncountable. \square

Corollary 10.1.22.1. *Not every metric space is second countable.*

Theorem 10.1.23. *Let X and Y be metric spaces. Let $f : X \rightarrow Y$ and $x \in X$. Then f is continuous at x if and only if, for every $\epsilon > 0$, there exists $\delta > 0$ such that, for all $x' \in X$, if $d(x, x') < \delta$ then $d(f(x), f(x')) < \epsilon$.*

PROOF:

$\langle 1 \rangle 1$. If f is continuous at x then, for every $\epsilon > 0$, there exists $\delta > 0$ such that, for all $x' \in X$, if $d(x, x') < \delta$ then $d(f(x), f(x')) < \epsilon$.

$\langle 2 \rangle 1$. ASSUME: f is continuous at x .

$\langle 2 \rangle 2$. LET: $\epsilon > 0$

$\langle 2 \rangle 3$. PICK a neighbourhood U of x such that $f(U) \subseteq B(f(x), \epsilon)$

PROOF: One exists by $\langle 2 \rangle 1$, since $B(f(x), \epsilon)$ is a neighbourhood of $f(x)$.

$\langle 2 \rangle 4$. PICK $\delta > 0$ such that $B(x, \delta) \subseteq U$

PROOF: By $\langle 2 \rangle 3$ and Lemma 10.1.5.

$\langle 2 \rangle 5$. LET: $x' \in X$ with $d(x, x') < \delta$

$\langle 2 \rangle 6$. $x' \in U$

PROOF: From $\langle 2 \rangle 4$ and $\langle 2 \rangle 5$.

$\langle 2 \rangle 7$. $f(x') \in B(f(x), \epsilon)$

PROOF: From $\langle 2 \rangle 3$ and $\langle 2 \rangle 6$.

$\langle 1 \rangle 2$. If, for all $\epsilon > 0$, there exists $\delta > 0$ such that, for all $x' \in X$, if $d(x, x') < \delta$ then $d(f(x), f(x')) < \epsilon$, then f is continuous at x .

$\langle 2 \rangle 1$. ASSUME: For all $\epsilon > 0$ there exists $\delta > 0$ such that, for all $x' \in X$, if $d(x, x') < \delta$ then $d(f(x), f(x')) < \epsilon$.

$\langle 2 \rangle 2$. LET: V be a neighbourhood of $f(x)$

$\langle 2 \rangle 3$. PICK $\epsilon > 0$ such that $B(f(x), \epsilon) \subseteq V$

PROOF: By Lemma 10.1.5.

$\langle 2 \rangle 4$. PICK $\delta > 0$ such that, for all $x' \in X$, if $d(x, x') < \delta$ then $d(f(x), f(x')) < \epsilon$.

PROOF: By $\langle 2 \rangle 1$ and $\langle 2 \rangle 3$.

$\langle 2 \rangle 5$. $B(x, \delta)$ is a neighbourhood of x

PROOF: By the definition of the metric topology.

$\langle 2 \rangle 6$. $f(B(x, \delta)) \subseteq V$

$\langle 3 \rangle 1$. LET: $x' \in B(x, \delta)$

$\langle 3 \rangle 2$. $d(f(x), f(x')) < \epsilon$

PROOF: From $\langle 2 \rangle 4$.

$\langle 3 \rangle 3$. $x' \in V$

PROOF: From $\langle 2 \rangle 3$.

\square

Lemma 10.1.24. *Addition is a continuous function $\mathbb{R}^2 \rightarrow \mathbb{R}$.*

PROOF:

$\langle 1 \rangle 1$. LET: $(x, y) \in \mathbb{R}^2$ and $\epsilon > 0$

$\langle 1 \rangle 2$. LET: $\delta = \epsilon/2$

⟨1⟩3. LET: $(x', y') \in \mathbb{R}^2$ be such that $\rho((x, y), (x', y')) < \delta$, where ρ is the square metric

⟨1⟩4. $|x - x'| < \delta$ and $|y - y'| < \delta$

⟨1⟩5. $|(x + y) - (x' + y')| < \epsilon$

PROOF:

$$\begin{aligned} |(x + y) - (x' + y')| &\leq |x - x'| + |y - y'| \\ &< 2\delta && (\langle 1 \rangle 4) \\ &= \epsilon && (\langle 1 \rangle 2) \end{aligned}$$

⟨1⟩6. Q.E.D.

PROOF: By Theorem 10.1.23.

□

Lemma 10.1.25. *Additive inverse is a continuous function $- : \mathbb{R} \rightarrow \mathbb{R}$.*

PROOF: If $|x - y| < \epsilon$ then $|(-x) - (-y)| < \epsilon$. □

Lemma 10.1.26. *Multiplication is a continuous function $\cdot : \mathbb{R}^2 \rightarrow \mathbb{R}$.*

PROOF:

⟨1⟩1. LET: $(x, y) \in \mathbb{R}^2$ and $\epsilon > 0$

⟨1⟩2. LET: $\delta = \min(1, \epsilon/(|x| + |y| + 1))$

⟨1⟩3. LET: $(x', y') \in \mathbb{R}^2$ and $\rho((x, y), (x', y')) < \delta$

⟨1⟩4. $|xy - x'y'| < \epsilon$

PROOF:

$$\begin{aligned} |xy - x'y'| &= |x(y' - y) + y(x' - x) + (x - x')(y - y')| \\ &\leq |x||y' - y| + |y||x' - x| + |x - x'||y - y'| \\ &< |x|\delta + |y|\delta + \delta^2 && (\langle 1 \rangle 3) \\ &= \delta(|x| + |y| + \delta) \\ &\leq \delta(|x| + |y| + 1) && (\langle 1 \rangle 2) \\ &\leq \epsilon && (\langle 1 \rangle 2) \end{aligned}$$

□

Lemma 10.1.27. *Multiplicative inverse is a continuous function $(\)^{-1} : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$.*

PROOF:

⟨1⟩1. LET: $f : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ be defined by $f(x) = x^{-1}$.

⟨1⟩2. LET: $a, b \in \mathbb{R}$ with $a < b$

PROVE: $f^{-1}((a, b))$ is open

⟨1⟩3. CASE: $0 < a < b$

PROOF: $f^{-1}((a, b)) = (b^{-1}, a^{-1})$

⟨1⟩4. CASE: $a < 0 < b$

PROOF: $f^{-1}((a, b)) = (-\infty, a^{-1}) \cup (b^{-1}, +\infty)$

⟨1⟩5. CASE: $a < b < 0$

PROOF: $f^{-1}((a, b)) = (b^{-1}, a^{-1})$

□

Definition 10.1.28 (Uniform Convergence). Let X be a set and Y a metric space. Let $f_n : X \rightarrow Y$ for $n \geq 1$, and $f : X \rightarrow Y$. Then f_n converges uniformly to f as $n \rightarrow \infty$ iff, for all $\epsilon > 0$, there exists N such that, for all $x \in X$ and $n \geq N$, $d(f_n(x), f(x)) < \epsilon$.

Theorem 10.1.29 (Uniform Limit Theorem). Let X be a topological space and Y a metric space. Let $f_n : X \rightarrow Y$ for $n \geq 1$ and $f : X \rightarrow Y$. If f_n converges uniformly to f and each f_n is continuous, then f is continuous.

PROOF:

- (1)1. LET: $x \in X$ and $\epsilon > 0$
- (1)2. PICK N such that, for all $x' \in X$ and $\delta > 0$, $d(f_n(x'), f(x')) < \epsilon/3$
- (1)3. PICK $\delta > 0$ such that, for all $x' \in X$, if $d(x, x') < \delta$ then $d(f_N(x), f_N(x')) < \epsilon/3$
- (1)4. For all $x' \in X$, if $d(x, x') < \delta$ then $d(f(x), f(x')) < \epsilon$

PROOF:

$$\begin{aligned} d(f(x), f(x')) &\leq d(f(x), f_N(x)) + d(f_N(x), f_N(x')) + d(f_N(x'), f(x')) \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 \\ &= \epsilon \end{aligned}$$

□

Lemma 10.1.30. Let X be a set. Let $f_n : X \rightarrow \mathbb{R}$ for $n \geq 1$ and $f : X \rightarrow \mathbb{R}$. Then f_n converges uniformly to f if and only if f_n converges to f in \mathbb{R}^X under the uniform topology.

PROOF:

- (1)1. If f_n converges uniformly to f then f_n converges to f under the uniform topology.
 - (2)1. ASSUME: f_n converges uniformly to f
 - (2)2. LET: $\epsilon > 0$
 - (2)3. PICK N such that, for all $x \in X$ and $n \geq N$, $d(f_n(x), f(x)) < \epsilon/2$
 - (2)4. $\bar{\rho}(f_n, f) \leq \epsilon/2$
 - (2)5. $\bar{\rho}(f_n, f) < \epsilon$
- (1)2. If f_n converges to f under the uniform topology then f_n converges uniformly to f .
 - (2)1. ASSUME: f_n converges to f under the uniform topology.
 - (2)2. LET: $\epsilon > 0$
 - (2)3. PICK N such that, for all $n \geq N$, $\bar{\rho}(f_n, f) < \epsilon$
 - (2)4. For all $n \geq N$ and $x \in X$, $d(f_n(x), f(x)) < \epsilon$

□

Theorem 10.1.31. Every monotone increasing sequence of real numbers that is bounded above converges to its supremum.

PROOF:

- (1)1. LET: $\{s_n\}_{n \geq 1}$ be a monotone increasing sequence of real numbers bounded above with supremum l .

- (1)2. LET: $\epsilon > 0$
 (1)3. $l - \epsilon$ is not an upper bound for $\{s_n : n \geq 1\}$.
 (1)4. PICK N such that $x_N > l - \epsilon$
 (1)5. For all $n \geq N$, we have $l - \epsilon < x_n \leq l$
 (1)6. For all $n \geq N$, we have $|x_n - l| < \epsilon$
 \square

Definition 10.1.32 (Infinite Series). Let $\{a_n\}_{n \geq 1}$ be a sequence of real numbers. The *infinite series* $\sum_{n=1}^{\infty} a_n$ *converges* to s iff $\sum_{n=1}^N a_n \rightarrow s$ as $N \rightarrow \infty$.

Proposition 10.1.33. If $\sum_{n=1}^{\infty} a_n = s$ and $\sum_{n=1}^{\infty} b_n = t$ then $\sum_{n=1}^{\infty} (ca_n + b_n) = cs + t$.

PROOF: This holds because $\sum_{n=1}^N (ca_n + b_n) = c \sum_{n=1}^N a_n + \sum_{n=1}^N b_n \rightarrow cs + t$ as $N \rightarrow \infty$. \square

Theorem 10.1.34 (Comparison Test). If $|a_i| \leq b_i$ for all i and $\sum_{i=1}^{\infty} b_i$ converges, then $\sum_{i=1}^{\infty} a_i$ converges.

PROOF:

- (1)1. $\sum_{i=1}^{\infty} |a_i|$ converges

PROOF: $\sum_{i=1}^N |a_i|$ is a monotone increasing sequence bounded above by $\sum_{i=1}^{\infty} b_i$.

- (1)2. LET: $c_i = |a_i| + a_i$

- (1)3. $\sum_{i=1}^{\infty} c_i$ converges

PROOF: $\sum_{i=1}^N c_i$ is a monotone increasing sequence bounded above by $2 \sum_{i=1}^{\infty} |a_i|$.

- (1)4. Q.E.D.

PROOF: Since $a_i = c_i - |a_i|$.

Lemma 10.1.35. If $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=N}^{\infty} a_n \rightarrow 0$ as $N \rightarrow \infty$.

PROOF:

$$\begin{aligned}
 \sum_{n=N}^{\infty} a_n &= \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N-1} a_n \\
 &\rightarrow \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} a_n \\
 &= 0
 \end{aligned}$$

as $N \rightarrow \infty$. \square

Theorem 10.1.36 (Weierstrass M-Test). Let X be a set and $f_n : X \rightarrow \mathbb{R}$ for $n \geq 1$. If $|f_n(x)| \leq M_n$ for all $n \geq 1$ and all $x \in X$, and if $\sum_{n=1}^{\infty} M_n$ converges, then

$$\sum_{n=1}^N f_n(x) \rightarrow \sum_{n=1}^{\infty} f_n(x)$$

uniformly in x as $N \rightarrow \infty$.

PROOF:

⟨1⟩1. For $N \geq 1$,

LET: $s_N : X \rightarrow \mathbb{R}$, $s_N(x) = \sum_{n=1}^N f_n(x)$

⟨1⟩2. For all $x \in X$, $\sum_{n=1}^{\infty} f_n(x)$ converges.

PROOF: By the Comparison Test.

⟨1⟩3. LET: $s : X \rightarrow \mathbb{R}$, $s(x) = \sum_{n=1}^{\infty} f_n(x)$.

⟨1⟩4. For $N \geq 1$,

LET: $r_N = \sum_{n=N+1}^{\infty} M_n$

⟨1⟩5. For $1 \leq N < K$, we have $|s_K(x) - s_N(x)| \leq r_N$ for all $x \in X$

PROOF:

$$\begin{aligned} |s_K(x) - s_N(x)| &= \left| \sum_{n=N+1}^K f_n(x) \right| \\ &\leq \sum_{n=N+1}^K |f_n(x)| \\ &\leq \sum_{n=N+1}^K M_n \\ &\leq \sum_{n=N+1}^{\infty} M_n \\ &= r_N \end{aligned}$$

⟨1⟩6. For $N \geq 1$ and $x \in X$ we have $|s(x) - s_N(x)| \leq r_N$

PROOF: Let $K \rightarrow \infty$ in ⟨1⟩5.

⟨1⟩7. LET: $\epsilon > 0$

⟨1⟩8. PICK N such that, for all $N' \geq N$, we have $r_{N'} < \epsilon$

PROOF: Such an N exists by Lemma 10.1.35.

⟨1⟩9. For all $N' \geq N$ and $x \in X$ we have $|s_{N'}(x) - s(x)| < \epsilon$

□

Definition 10.1.37. Let X be a metric space. Let $x \in X$ and $A \subseteq X$ be nonempty. The *distance* from x to A is

$$d(x, A) = \inf_{a \in A} d(x, a) .$$

Lemma 10.1.38. Let X be a metric space and $A \subseteq X$ be nonempty. Then the function $d(-, A) : X \rightarrow \mathbb{R}$ is continuous.

PROOF:

⟨1⟩1. LET: $x \in X$ and $\epsilon > 0$

⟨1⟩2. LET: $y \in X$ with $d(x, y) < \epsilon$

⟨1⟩3. $|d(x, A) - d(y, A)| < \epsilon$

PROOF:

⟨2⟩1. $d(x, A) - d(y, A) < \epsilon$

PROOF:

$$\begin{aligned}
d(x, A) &= \inf_{a \in A} d(x, a) \\
&\leq \inf_{a \in A} (d(x, y) + d(y, a)) \\
&= d(x, y) + \inf_{a \in A} d(y, a) \\
&= d(x, y) + d(y, A) \\
&< \epsilon + d(y, A)
\end{aligned}$$

(2)2. $d(y, A) - d(x, A) < \epsilon$

PROOF: Similar.

(1)4. Q.E.D.

PROOF: By Theorem 10.1.23.

□

Definition 10.1.39 (Shrinking Map). Let X be a metric space and $f : X \rightarrow X$. Then f is a *shrinking map* iff, for all $x, y \in X$ with $x \neq y$, we have $d(f(x), f(y)) < d(x, y)$.

Definition 10.1.40 (Contraction). Let X be a metric space and $f : X \rightarrow X$. Then f is a *contraction* iff there exists $\alpha < 1$ such that, for all $x, y \in X$,

$$d(f(x), f(y)) \leq \alpha d(x, y) .$$

Proposition 10.1.41. *Every separable metric space is second countable.*

PROOF:

(1)1. LET: X be a separable metric space.

(1)2. PICK a countable dense set D

(1)3. LET: $\mathcal{B} = \{B(d, q) : d \in D, q \in \mathbb{Q}^+\}$

(1)4. \mathcal{B} is a countable basis for X

□

Corollary 10.1.41.1. *The space \mathbb{R}^ω under the uniform topology is not separable.*

Corollary 10.1.41.2. *Not every metric space is separable.*

Corollary 10.1.41.3. *The space \mathbb{R}^ω under the box topology is not separable.*

Proposition 10.1.42 (CC). *Every Lindelöf metric space is second countable.*

PROOF:

(1)1. LET: X be a Lindelöf metric space.

(1)2. For all $n \in \mathbb{Z}^+$, PICK a countable covering \mathcal{A}_n of X by $1/n$ -balls

PROOF: One exists by the Lindelöf condition, since the set of all $1/n$ -balls covers X .

(1)3. $\bigcup_{n=1}^{\infty} \mathcal{A}_n$ is a countable basis.

□

Corollary 10.1.42.1. *The space \mathbb{R}^ω under the uniform topology is not Lindelöf.*

Corollary 10.1.42.2. *Not every metric space is Lindelöf.*

Proposition 10.1.43. *The space \mathbb{R}_l is not metrizable.*

PROOF: It is Lindelöf but not second countable. \square

Proposition 10.1.44. *The ordered square is not metrizable.*

PROOF: It is compact but not second countable. \square

Proposition 10.1.45. *The space \mathbb{R}^ω under the uniform topology is not second countable.*

PROOF: It contains a subspace homeomorphic to \mathbb{R} . \square

Theorem 10.1.46 (AC). *Every metrizable space is normal.*

PROOF:

$\langle 1 \rangle 1$. LET: X be a metric space.

$\langle 1 \rangle 2$. LET: A and B be disjoint closed subspaces of X .

$\langle 1 \rangle 3$. For $a \in A$, PICK $\epsilon_a > 0$ such that $B(a, \epsilon_a)$ does not intersect B .

$\langle 1 \rangle 4$. For $b \in B$, PICK $\epsilon_b > 0$ such that $B(b, \epsilon_b)$ does not intersect A .

$\langle 1 \rangle 5$. LET: $U = \bigcup_{a \in A} B(a, \epsilon_a/2)$

$\langle 1 \rangle 6$. LET: $V = \bigcup_{b \in B} B(b, \epsilon_b/2)$

$\langle 1 \rangle 7$. $U \cap V = \emptyset$

$\langle 2 \rangle 1$. LET: $z \in U \cap V$

$\langle 2 \rangle 2$. PICK $a \in A$ and $b \in B$ such that $z \in B(a, \epsilon_a/2)$ and $z \in B(b, \epsilon_b/2)$

$\langle 2 \rangle 3$. ASSUME: w.l.o.g. $\epsilon_a \leq \epsilon_b$

$\langle 2 \rangle 4$. $a \in B(b, \epsilon_b)$

PROOF:

$$d(a, b) \leq d(a, z) + d(b, z) \quad (\text{Triangle Inequality})$$

$$< \epsilon_a/2 + \epsilon_b/2 \quad (\langle 2 \rangle 2)$$

$$\leq \epsilon_b \quad (\langle 2 \rangle 3)$$

$\langle 2 \rangle 5$. Q.E.D.

PROOF: This contradicts $\langle 1 \rangle 4$.

\square

Corollary 10.1.46.1. *The space \mathbb{R}^ω is normal.*

Corollary 10.1.46.2. *The space \mathbb{R}_K is not metrizable.*

Proposition 10.1.47. *Every metrizable space is completely normal.*

PROOF: Every subspace is metrizable (Lemma 10.1.12) hence normal (Theorem 10.1.46). \square

Proposition 10.1.48. *Every metrizable space is perfectly normal.*

PROOF:

⟨1⟩1. LET: X be a metric space.

⟨1⟩2. X is normal.

PROOF: Theorem 10.1.46

⟨1⟩3. Every closed set is G_δ .

PROOF: If A is closed then $A = \bigcap_{q \in \mathbb{Q}^+} \{x \in X : d(A, x) < q\}$.

□

Theorem 10.1.49 (Urysohn Metrization Theorem (CC)). *Every second countable regular space is metrizable.*

PROOF:

⟨1⟩1. LET: X be a second countable regular space.

⟨1⟩2. X is normal.

⟨1⟩3. PICK a countable basis $\mathcal{B} = \{B_n : n \in \mathbb{Z}^+\}$

⟨1⟩4. For every pair of integers m, n with $\overline{B_m} \subseteq B_n$, PICK a continuous function $g_{mn} : X \rightarrow [0, 1]$ such that $g_{mn}(\overline{B_m}) = \{1\}$ and $g_{mn}(X \setminus B_n) = \{0\}$

PROOF: By the Urysohn Lemma.

⟨1⟩5. The set $\{g_{mn} : \overline{B_m} \subseteq B_n\}$ separates points from closed sets in X

⟨2⟩1. LET: $x \in X$ and U be a neighbourhood of x

⟨2⟩2. PICK $B_n \in \mathcal{B}$ such that $x \in B_n \subseteq U$

⟨2⟩3. PICK V open such that $x \in V$ and $\overline{V} \subseteq B_n$

⟨2⟩4. PICK $B_m \in \mathcal{B}$ such that $x \in B_m \subseteq V$

⟨2⟩5. $g_{mn}(x) = 1$ and g_{mn} vanishes outside U

⟨1⟩6. X is imbeddable in $[0, 1]^\omega$

PROOF: By the Imbedding Theorem.

⟨1⟩7. Q.E.D.

Corollary 10.1.49.1. *The space \mathbb{R}^ω under the box topology is not second countable.*

Proposition 10.1.50. *Not every second countable Hausdorff space is metrizable.*

PROOF: \mathbb{R}_K is second countable and Hausdorff but not metrizable (because it is not regular). □

Proposition 10.1.51. *There exists a space that is completely normal, first countable, Lindelöf and separable but not metrizable.*

PROOF: The space \mathbb{R}_l is all of these. □

Proposition 10.1.52. *$\overline{S_\Omega}$ is not metrizable.*

PROOF: It is compact but not sequentially compact. □

Proposition 10.1.53. *Every compact metric space is second countable.*

PROOF:

⟨1⟩1. LET: X be a compact metric space

(1)2. For every $n \geq 1$, PICK a finite covering \mathcal{A}_n of X by open balls of radius $1/n$

PROOF: Such a covering exists because $\{B_{1/n}(x) : x \in X\}$ covers X .

(1)3. $\bigcup_{n=1}^{\infty} \mathcal{A}_n$ is a countable basis for X

□

Corollary 10.1.53.1. *The space \mathbb{R}^{ω} under the uniform topology is not compact.*

Corollary 10.1.53.2. *The space \mathbb{R}^{ω} under the uniform topology is not limit point compact.*

Proposition 10.1.54. *The space \mathbb{R}^{ω} under the box topology is not locally compact.*

PROOF:

(1)1. ASSUME: \mathbb{R}^{ω} under the box topology is locally compact.

(1)2. For every point x , there exists a basic open set $B = \prod_{i=0}^{\infty} U_i$ such that $x \in B$ and \overline{B} is compact.

(1)3. The box topology on \overline{B} is the same as the product topology on \overline{B}

PROOF: By Corollary 9.4.11.1.

(1)4. The box topology on \overline{B} is strictly finer than the product topology.

PROOF: By Theorem 10.1.21.

□

Proposition 10.1.55. *Not every metrizable space is connected.*

PROOF: The discrete space with two points is metrizable but not connected. □

Corollary 10.1.55.1. *Not every metrizable space is path connected.*

Proposition 10.1.56. *Not every metric space is limit point compact.*

PROOF: The space \mathbb{R} is not limit point compact. □

Proposition 10.1.57. *Not every metric space is locally compact.*

The space \mathbb{R}^{ω} in the uniform topology is not locally compact.

10.2 Isometries

Definition 10.2.1 (Isometry). Let X be a metric space. An *isometry* of X is a function $f : X \rightarrow X$ such that, for all $x, y \in X$,

$$d(f(x), f(y)) = d(x, y) .$$

10.3 Lebesgue Numbers

Definition 10.3.1 (Lebesgue Number). Let X be a metric space and \mathcal{A} an open covering of X . A *Lebesgue number* for \mathcal{A} is a real $\delta > 0$ such that, for every nonempty set $A \subseteq X$ of diameter $< \delta$, there exists $U \in \mathcal{A}$ such that $A \subseteq U$.

Lemma 10.3.2 (Lebesgue Number Lemma). *In a compact metric space, every open covering has a Lebesgue number.*

PROOF:

⟨1⟩1. LET: X be a compact metric space and \mathcal{A} an open covering of X

PROVE: There exists a Lebesgue number δ for \mathcal{A} .

⟨1⟩2. ASSUME: w.l.o.g. $X \notin \mathcal{A}$

PROOF: If $X \in \mathcal{A}$ then we can take $\delta = 1$.

⟨1⟩3. PICK a finite subcovering $\{U_1, \dots, U_n\} \subseteq \mathcal{A}$ that covers X

⟨1⟩4. For $1 \leq i \leq n$,

LET: $C_i = X \setminus U_i$

⟨1⟩5. LET: $f : X \rightarrow \mathbb{R}$ be defined by

$$f(x) = 1/n \sum_{i=1}^n d(x, C_i) .$$

PROOF: Each C_i is nonempty by ⟨1⟩2.

⟨1⟩6. For all $x \in X$ we have $f(x) > 0$

⟨2⟩1. LET: $x \in X$

⟨2⟩2. PICK i such that $x \in U_i$

PROOF: By ⟨1⟩3.

⟨2⟩3. PICK $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U_i$

PROOF: By Lemma 10.1.5.

⟨2⟩4. $d(x, C_i) \geq \epsilon$

⟨1⟩7. f is continuous

PROOF: From Lemma 10.1.38.

⟨1⟩8. LET: $\delta = \min f(X)$

PROVE: For every nonempty set $A \subseteq X$ with diameter $< \delta$, there exists $U \in \mathcal{A}$ such that $A \subseteq U$

PROOF: $f(X)$ has a minimum by the Extreme Value Theorem.

⟨1⟩9. LET: $A \subseteq X$ be nonempty with $\text{diam } A < \delta$

⟨1⟩10. PICK $x_0 \in A$

⟨1⟩11. LET: i be such that $d(x_0, C_i)$ is greatest among $d(x_0, C_1), \dots, d(x_0, C_n)$

⟨1⟩12. $\delta \leq d(x_0, C_i)$

PROOF:

$$\delta \leq f(x_0) \tag{⟨1⟩8}$$

$$= 1/n \sum_{j=1}^n d(x_0, C_j) \tag{⟨1⟩5}$$

$$\leq 1/n \sum_{j=1}^n d(x_0, C_i) \tag{⟨1⟩11}$$

$$= d(x_0, C_i)$$

⟨1⟩13. $x_0 \in U_i$

PROOF: $x_0 \notin C_i$ because $d(x_0, C_i) > 0$.

□

Theorem 10.3.3 (DC). *Let X be a metrizable space. Then the following are*

equivalent:

1. X is compact.
2. X is limit point compact.
3. X is sequentially compact.

PROOF:

$\langle 1 \rangle 1. 1 \Rightarrow 2$

PROOF: Theorem 9.4.22.

$\langle 1 \rangle 2. 2 \Rightarrow 3$

$\langle 2 \rangle 1.$ ASSUME: X is limit point compact.

$\langle 2 \rangle 2.$ LET: (x_n) be a sequence in X

PROVE: (x_n) has a convergent subsequence.

$\langle 2 \rangle 3.$ CASE: $\{x_n : n \in \mathbb{Z}^+\}$ is finite.

PROOF: In this case, (x_n) has a constant subsequence.

$\langle 2 \rangle 4.$ CASE: $\{x_n : n \in \mathbb{Z}^+\}$ is infinite.

$\langle 3 \rangle 1.$ PICK a limit point l of $\{x_n : n \in \mathbb{Z}^+\}$

$\langle 3 \rangle 2.$ For every positive integer r , PICK n_r such that $n_r > n_{r-1}$ and $d(x_{n_r}, l) < 1/r$

PROOF: There always exists such an n_r since $B(l, 1/r)$ intersects $\{x_n : n \in \mathbb{Z}^+\}$ in infinitely many points by Theorem 6.1.2.

$\langle 3 \rangle 3.$ $x_{n_r} \rightarrow l$ as $r \rightarrow \infty$

$\langle 1 \rangle 3. 3 \Rightarrow 1$

$\langle 2 \rangle 1.$ ASSUME: X is sequentially compact.

$\langle 2 \rangle 2.$ Every open covering of X has a Lebesgue number.

$\langle 3 \rangle 1.$ LET: \mathcal{A} be an open covering of X .

$\langle 3 \rangle 2.$ ASSUME: for a contradiction that, for all $\delta > 0$, there exists a set $C \subseteq X$ with $\text{diam } C < \delta$ such that there is no $U \in \mathcal{A}$ such that $C \subseteq U$

$\langle 3 \rangle 3.$ For $n \geq 1$, PICK $C_n \subseteq X$ with $\text{diam } C_n < 1/n$ such that there is no $U \in \mathcal{A}$ such that $C_n \subseteq U$

$\langle 3 \rangle 4.$ For $n \geq 1$, PICK $x_n \in C_n$

$\langle 3 \rangle 5.$ PICK a convergent subsequence (x_{n_r}) of (x_n)

PROOF: By $\langle 2 \rangle 1.$

$\langle 3 \rangle 6.$ LET: $x_{n_r} \rightarrow l$ as $r \rightarrow \infty$

$\langle 3 \rangle 7.$ PICK $U \in \mathcal{A}$ with $l \in U$

PROOF: By $\langle 3 \rangle 1$

$\langle 3 \rangle 8.$ PICK $\epsilon > 0$ such that $B(l, \epsilon) \subseteq U$

PROOF: By Lemma 10.1.5.

$\langle 3 \rangle 9.$ PICK R such that $1/n_R < \epsilon/2$ and $d(x_{n_R}, l) < \epsilon/2$

PROOF: By $\langle 3 \rangle 6$

$\langle 3 \rangle 10.$ $C_{n_R} \subseteq U$

PROOF:

$$\begin{aligned}
C_{n_R} &\subseteq B(x_{n_R}, 1/n_R) & (\langle 3 \rangle 3, \langle 3 \rangle 4) \\
&\subseteq B(x_{n_R}, \epsilon/2) & (\langle 3 \rangle 9) \\
&\subseteq B(l, \epsilon) & (\langle 3 \rangle 9) \\
&\subseteq U & (\langle 3 \rangle 8)
\end{aligned}$$

$\langle 3 \rangle 11$. Q.E.D.

PROOF: This contradicts $\langle 3 \rangle 3$.

$\langle 2 \rangle 3$. For all $\epsilon > 0$, there exists a finite covering of X by ϵ -balls.

$\langle 3 \rangle 1$. LET: $\epsilon > 0$

$\langle 3 \rangle 2$. ASSUME: for a contradiction there is no finite covering of X by ϵ -balls.

$\langle 3 \rangle 3$. PICK a sequence (x_n) in X such that, for all n ,

$$x_n \notin B(x_1, \epsilon) \cup \dots \cup B(x_{n-1}, \epsilon) .$$

$\langle 3 \rangle 4$. For all m, n with $m > n$ we have $d(x_m, x_n) \geq \epsilon$

$\langle 3 \rangle 5$. Any $\epsilon/2$ -ball contains at most one element of (x_n) .

$\langle 3 \rangle 6$. (x_n) has no convergent subsequence.

$\langle 3 \rangle 7$. Q.E.D.

PROOF: This contradicts $\langle 2 \rangle 1$.

$\langle 2 \rangle 4$. LET: \mathcal{A} be an open covering of X

$\langle 2 \rangle 5$. PICK a Lebesgue number δ for \mathcal{A}

PROOF: By $\langle 2 \rangle 2$.

$\langle 2 \rangle 6$. PICK a finite covering $\{B_1, \dots, B_n\}$ of X by $\delta/3$ -balls.

PROOF: By $\langle 2 \rangle 3$.

$\langle 2 \rangle 7$. For $1 \leq i \leq n$, PICK $U_i \in \mathcal{A}$ such that $B_i \subseteq U_i$

$\langle 2 \rangle 8$. $\{U_1, \dots, U_n\}$ covers X .

□

Corollary 10.3.3.1. S_Ω is not metrizable.

PROOF: It is limit point compact (Corollary 9.4.19.2) but not compact (Proposition 9.4.2). □

Corollary 10.3.3.2. The space \mathbb{R}^ω is not limit point compact.

10.4 Uniform Continuity

Definition 10.4.1 (Uniform Continuity). Let X and Y be metric spaces and $f : X \rightarrow Y$. Then f is *uniformly continuous* iff, for all $\epsilon > 0$, there exists $\delta > 0$ such that, for all $x, y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$.

Theorem 10.4.2 (Uniform Continuity Theorem). *Let X be a compact metric space, Y a metric space, and $f : X \rightarrow Y$ be continuous. Then f is uniformly continuous.*

PROOF:

$\langle 1 \rangle 1$. LET: $\epsilon > 0$

PROVE: There exists $\delta > 0$ such that, for all $x, y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$.

- ⟨1⟩2. LET: $\mathcal{A} = \{f^{-1}(B(y, \epsilon/2)) : y \in Y\}$
 - ⟨1⟩3. \mathcal{A} is an open covering of X
 - ⟨1⟩4. PICK a Lebesgue number δ for \mathcal{A} .
PROVE: For all $x, y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$
PROOF: By the Lebesgue Number Lemma
 - ⟨1⟩5. LET: $x, y \in X$ with $d(x, y) < \delta$
 - ⟨1⟩6. $\text{diam}\{x, y\} < \delta$
 - ⟨1⟩7. PICK $z \in Y$ such that $\{x, y\} \subseteq f^{-1}(B(z, \epsilon/2))$
 - ⟨1⟩8. $d(f(x), f(y)) < \epsilon$
-

10.5 Locally Metrizable Spaces

Definition 10.5.1 (Locally Metrizable). A space is *locally metrizable* iff every point has a metrizable neighbourhood.

Proposition 10.5.2. *Every metrizable space is locally metrizable.*

PROOF: Trivial. □

Corollary 10.5.2.1. *The space \mathbb{R}^ω is locally metrizable.*

Proposition 10.5.3. *A compact Hausdorff space is metrizable if and only if it is locally metrizable.*

PROOF:

- ⟨1⟩1. LET: X be a locally metrizable compact Hausdorff space
- ⟨1⟩2. X is regular
PROOF: Lemma 9.4.18
- ⟨1⟩3. X is second countable
 - ⟨2⟩1. $\{U : U \text{ open in } X \text{ and metrizable}\}$ covers X
 - ⟨2⟩2. PICK a finite subcover U_1, \dots, U_n
 - ⟨2⟩3. For $1 \leq i \leq n$, PICK a countable basis \mathcal{B}_i of U_i
 - ⟨2⟩4. $\mathcal{B}_1 \cup \dots \cup \mathcal{B}_n$ is a basis for X
- ⟨1⟩4. Q.E.D.

PROOF: By the Urysohn Metrization Theorem.
□

Corollary 10.5.3.1. $\overline{S_\Omega}$ is not locally metrizable.

Corollary 10.5.3.2. *The ordered square is not locally metrizable.*

Proposition 10.5.4. *Every subspace of a locally metrizable space is locally metrizable.*

PROOF:

- ⟨1⟩1. LET: X be locally metrizable and $Y \subseteq X$
- ⟨1⟩2. LET: $y \in Y$

- ⟨1⟩3. PICK a metrizable neighbourhood U of y in X
- ⟨1⟩4. $U \cap Y$ is a metrizable neighbourhood of y in Y

□

Corollary 10.5.4.1. $S_\Omega \times \overline{S_\Omega}$ is not locally metrizable.

PROOF: It has a subspace homeomorphic to $\overline{S_\Omega}$. □

Proposition 10.5.5 (CC). *Every locally metrizable regular Lindelöf space is metrizable.*

PROOF:

- ⟨1⟩1. LET: X be a locally metrizable regular Lindelöf space.
- ⟨1⟩2. Every point in X has an open second countable neighbourhood.
 - ⟨2⟩1. LET: $x \in X$
 - ⟨2⟩2. PICK an open metrizable U containing x
 - PROOF: X is locally metrizable (⟨1⟩1)
 - ⟨2⟩3. PICK an open V such that $x \in V \subseteq \overline{V} \subseteq U$
 - PROOF: Proposition 6.3.2
 - ⟨2⟩4. \overline{V} is Lindelöf
 - PROOF: Proposition 13.1.32
 - ⟨2⟩5. \overline{V} is second countable
 - PROOF: Proposition 10.1.42
- ⟨1⟩3. PICK a countable covering of second countable open sets \mathcal{U}
 - PROOF: X is Lindelöf (⟨1⟩1)
- ⟨1⟩4. For $U \in \mathcal{U}$, PICK a countable basis \mathcal{B}_U
- ⟨1⟩5. $\bigcup_{U \in \mathcal{U}} \mathcal{B}_U$ is a countable basis for X
 - ⟨2⟩1. LET: $x \in U$ where U is open in X
 - ⟨2⟩2. PICK $V \in \mathcal{U}$ such that $x \in V$
 - ⟨2⟩3. There exists $B \in \mathcal{B}_V$ such that $x \in B \subseteq U \cap V$
- ⟨1⟩6. Q.E.D.
 - PROOF: By the Urysohn Metrization Theorem.

□

Corollary 10.5.5.1. \mathbb{R}_l is not locally metrizable.

Proposition 10.5.6. *The Sorgenfrey plane is not locally metrizable.*

PROOF:

- ⟨1⟩1. LET: U be any neighbourhood of $(0, 0)$
 - PROVE: U is not Lindelöf
- ⟨1⟩2. PICK $a > 0$ such that $[0, a]^2 \subseteq U$
- ⟨1⟩3. LET: $L = \{(x, a - x) : 0 < x < a\}$
- ⟨1⟩4. L is closed in U
 - PROOF: By Lemma 6.5.16 since $(x, y) \mapsto (x, a + y)$ is a homeomorphism of \mathbb{R}_l^2 with itself.
- ⟨1⟩5. LET: $\mathcal{U} = \{U \setminus L\} \cup \{([x, b] \times [a - x, c]) \cap U : b > a, c > a - x\}$
- ⟨1⟩6. \mathcal{U} covers U

⟨1⟩7. No countable subset of \mathcal{U} covers U

PROOF: Every set of the form $[x, b) \times [a - x, c)$ intersects L in exactly one point.
□

Corollary 10.5.6.1. *The Sorgenfrey plane is not metrizable.*

Proposition 10.5.7. *The space \mathbb{R}_K is locally metrizable.*

PROOF: The set $(-1, 1) - K$ is a metrizable neighbourhood of 0. For any other point p , pick an open interval around p that does not contain 0. □

Proposition 10.5.8. *The product of two locally metrizable spaces is locally metrizable.*

PROOF:

⟨1⟩1. LET: X and Y be locally metrizable

⟨1⟩2. LET: $(a, b) \in X \times Y$

⟨1⟩3. PICK metrizable neighbourhoods U of a and V of b

⟨1⟩4. $U \times V$ is a metrizable neighbourhood of (a, b) .

PROOF: By Lemma 10.1.19.
□

Proposition 10.5.9. *The product of two locally metrizable spaces is locally metrizable.*

PROOF:

⟨1⟩1. LET: X and Y be locally metrizable

⟨1⟩2. LET: $(a, b) \in X \times Y$

⟨1⟩3. PICK metrizable neighbourhoods U of a and V of b

⟨1⟩4. $U \times V$ is a metrizable neighbourhood of (a, b) .

PROOF: By Lemma 10.1.19.
□

Proposition 10.5.10. *The space \mathbb{R}_K^ω is not locally metrizable.*

PROOF: If it were, then there would be a basic open set $\prod_n U_n$ that is metrizable, but then \mathbb{R}_K would be metrizable as it is homeomorphic to a subspace of $\prod_n U_n$.
□

Corollary 10.5.10.1. *The product of a countable family of locally metrizable spaces is not necessarily locally metrizable.*

Proposition 10.5.11. *The continuous image of a locally metrizable space is not necessarily locally metrizable.*

PROOF: The identity map from the discrete two-point space to the indiscrete two-point space is continuous. □

Chapter 11

Manifolds

11.1 Manifolds

Definition 11.1.1 (Manifold). Let $m \geq 1$. An m -manifold is a second countable Hausdorff space such that each point has a neighbourhood that is homeomorphic to an open subset of \mathbb{R}^m .

A *curve* is a 1-manifold and a *surface* is a 2-manifold.

Definition 11.1.2 (Support). Let X be a topological space and $\phi : X \rightarrow \mathbb{R}$ be a function. Then the *support* of ϕ is the closure of $\phi^{-1}(\mathbb{R} \setminus \{0\})$.

Definition 11.1.3 (Partition of Unity). Let X be a topological space. Let $\{U_1, \dots, U_n\}$ be a finite indexed open covering of X . An indexed family of continuous functions $\phi_1, \dots, \phi_n : X \rightarrow [0, 1]$ is a *partition of unity* dominated by $\{U_1, \dots, U_n\}$ iff:

1. $\text{supp } \phi_i \subseteq U_i$ for all i ;
2. $\sum_{i=1}^n \phi_i(x) = 1$ for all $x \in X$.

Theorem 11.1.4 (Existence of Finite Partitions of Unity). *Let X be a normal space. Let $\{U_1, \dots, U_n\}$ be a finite indexed open covering of X . Then there exists a partition of unity dominated by $\{U_1, \dots, U_n\}$.*

PROOF:

- (1)1. For every finite indexed open covering $\{U_1, \dots, U_n\}$ of X , there exists a finite indexed open covering $\{V_1, \dots, V_n\}$ such that $\overline{V_i} \subseteq U_i$
- (2)1. For $1 \leq k \leq n$, there exist open sets V_1, \dots, V_k such that $\overline{V_i} \subseteq U_i$ for all i and $\{V_1, \dots, V_k, U_{k+1}, \dots, U_n\}$ covers X
- (3)1. ASSUME: as an induction hypothesis that $0 \leq k < n$ and there exist open sets V_1, \dots, V_k such that $\overline{V_i} \subseteq U_i$ for all i and $\{V_1, \dots, V_k, U_{k+1}, \dots, U_n\}$ covers X
- (3)2. LET: $A = X \setminus (V_1 \cup \dots \cup V_k) \setminus (U_{k+2} \cup \dots \cup U_n)$
- (3)3. A is closed

- ⟨3⟩4. $A \subseteq U_{k+1}$
 PROOF: Since $\{V_1, \dots, V_k, U_{k+1}, \dots, U_n\}$ covers X
 ⟨3⟩5. PICK an open set V_{k+1} such that $A \subseteq V_{k+1}$ and $\overline{V_{k+1}} \subseteq U_{k+1}$
 PROOF: By Proposition 6.3.2
 ⟨3⟩6. $\{V_1, \dots, V_k, V_{k+1}, U_{k+2}, \dots, U_n\}$ covers X
 ⟨1⟩2. PICK an open covering $\{V_1, \dots, V_n\}$ with $\overline{V_i} \subseteq U_i$ for all i
 PROOF: By ⟨1⟩1.
 ⟨1⟩3. PICK an open covering $\{W_1, \dots, W_n\}$ with $\overline{W_i} \subseteq V_i$ for all i
 PROOF: By ⟨1⟩1.
 ⟨1⟩4. For $1 \leq i \leq n$, PICK a continuous function $\psi_i : X \rightarrow [0, 1]$ such that
 $\psi_i(\overline{W_i}) = \{1\}$ and $\psi_i(X \setminus V_i) = \{0\}$
 PROOF: By the Urysohn Lemma.
 ⟨1⟩5. LET: $\Psi : X \rightarrow \mathbb{R}$ where $\Psi(x) = \sum_{i=1}^n \psi_i(x)$
 ⟨1⟩6. $\Psi(x) > 0$ for all $x \in X$
 ⟨2⟩1. LET: $x \in X$
 ⟨2⟩2. PICK i such that $x \in W_i$
 ⟨2⟩3. $\psi_i(x) = 1$
 ⟨1⟩7. For $1 \leq j \leq n$,
 LET: $\phi_j(x) = \frac{\psi_j(x)}{\Psi(x)}$
 ⟨1⟩8. ψ_1, \dots, ψ_n are a partition of unity dominated by $\{U_1, \dots, U_n\}$
 ⟨2⟩1. $\text{supp } \psi_i \subseteq U_i$
 ⟨3⟩1. $\psi_i^{-1}(\mathbb{R} \setminus \{0\}) \subseteq V_i$
 PROOF: By ⟨1⟩4
 ⟨3⟩2. $\text{supp } \psi_i \subseteq \overline{V_i}$
 PROOF: Proposition 3.7.5
 ⟨2⟩2. $\sum_{i=1}^n \psi_i(x) = 1$ for all $x \in X$
 □

Theorem 11.1.5. *Let X be a compact Hausdorff space. Suppose that, for every $x \in X$, there exists a neighbourhood U of x and a positive integer k such that U can be imbedded in \mathbb{R}^k . Then there exists a positive integer N such that X can be imbedded in \mathbb{R}^N .*

PROOF:

- ⟨1⟩1. PICK a finite open covering $\{U_1, \dots, U_n\}$ of X such that each U_i can be imbedded in \mathbb{R}^k for some k
 PROOF: Since $\{U \text{ open in } X : U \text{ can be imbedded in } \mathbb{R}^k \text{ for some } k\}$ covers X .
 ⟨1⟩2. For $1 \leq i \leq n$, PICK a positive integer k_i and an imbedding $g_i : U_i \rightarrow \mathbb{R}^{k_i}$
 ⟨1⟩3. PICK a partition of unity ϕ_1, \dots, ϕ_n dominated by $\{U_1, \dots, U_n\}$
 ⟨2⟩1. X is normal
 PROOF: By Lemma 9.4.18.
 ⟨2⟩2. Q.E.D.
 PROOF: Theorem 11.1.4
 ⟨1⟩4. For $1 \leq i \leq n$,
 LET: $A_i = \text{supp } \phi_i$

⟨1⟩5. For $1 \leq i \leq n$,

LET: $h_i : X \rightarrow \mathbb{R}^{k_i}$ be defined by

$$h_i(x) = \begin{cases} \phi_i(x)g_i(x) & \text{for } x \in U_i \\ \vec{0} & \text{for } x \in X \setminus A_i \end{cases}$$

PROOF: If $x \in U_i$ and $x \in X \setminus A_i$ then $x \notin \text{supp } \phi_i$ so $\phi_i(x) = 0$

⟨1⟩6. LET: $N = n + k_1 + \cdots + k_n$

⟨1⟩7. LET: $F : X \rightarrow \mathbb{R}^N$ be the function

$$F(x) = (\phi_1(x), \dots, \phi_n(x), h_1(x), \dots, h_n(x))$$

⟨1⟩8. F is an imbedding

⟨2⟩1. F is continuous

PROOF: Each h_i is continuous by Theorem 5.2.13.

⟨2⟩2. F is injective

⟨3⟩1. ASSUME: $F(x) = F(y)$

⟨3⟩2. PICK i such that $\phi_i(x) > 0$

PROOF: Since $\sum_i \phi_i(x) = 1$ (⟨1⟩3)

⟨3⟩3. $\phi_i(y) = 0$

PROOF: By ⟨3⟩1

⟨3⟩4. $x, y \in U_i$

PROOF: Since $\text{supp } \phi_i \subseteq U_i$

⟨3⟩5. $h_i(x) = h_i(y)$

PROOF: By ⟨3⟩1

⟨3⟩6. $g_i(x) = g_i(y)$

PROOF: By ⟨1⟩5

⟨3⟩7. $x = y$

PROOF: By ⟨1⟩2

⟨2⟩3. Q.E.D.

PROOF: By Theorem 9.4.11

□

Corollary 11.1.5.1. *Every compact manifold can be imbedded in \mathbb{R}^N for some N .*

Proposition 11.1.6. *The line with two origins is a second countable T_1 space where every point has a neighbourhood that is homeomorphic to an open subset of \mathbb{R} , but it is not a 1-manifold.*

Chapter 12

Normed Spaces

12.1 The Norm on \mathbb{R}^n

Definition 12.1.1 (Norm). Given $\vec{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, the *norm* $\|\vec{x}\|$ is defined by

$$\|\vec{x}\| = \sqrt{x_1^2 + \dots + x_n^2} .$$

Definition 12.1.2 (Vector Sum). Define the *sum* of $\vec{x}, \vec{y} \in \mathbb{R}^n$ by

$$\vec{x} + \vec{y} = (x_1 + y_1, \dots, x_n + y_n) .$$

Definition 12.1.3 (Scalar Product). Given $c \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^n$, define the *scalar product* $c\vec{x}$ to be

$$c\vec{x} = (cx_1, \dots, cx_n) .$$

Definition 12.1.4 (Inner Product). The *inner product* of $\vec{x}, \vec{y} \in \mathbb{R}^n$ is

$$\vec{x} \cdot \vec{y} = x_1 y_1 + \dots + x_n y_n .$$

Lemma 12.1.5.

$$\vec{x} \cdot (\vec{y} + \vec{z}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z}$$

PROOF: Both are equal to $\sum_{i=1}^n (x_i y_i + x_i z_i)$. \square

Lemma 12.1.6.

$$|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \|\vec{y}\|$$

PROOF:

$\langle 1 \rangle 1$. CASE: $\vec{x} = \vec{0}$ or $\vec{y} = \vec{0}$

PROOF: In this case, both sides are 0.

$\langle 1 \rangle 2$. CASE: $\vec{x} \neq \vec{0} \neq \vec{y}$

$\langle 2 \rangle 1$. LET: $a = 1/\|\vec{x}\|$, $b = 1/\|\vec{y}\|$

$\langle 2 \rangle 2$. $2 + 2ab\vec{x} \cdot \vec{y} \geq 0$

$\langle 3 \rangle 1$. $\|a\vec{x} + b\vec{y}\|^2 \geq 0$

- $\langle 3 \rangle 2. \sum_{i=1}^n (ax_i + by_i)^2 \geq 0$
 $\langle 3 \rangle 3. a^2 \sum_{i=1}^n x_i^2 + b^2 \sum_{i=1}^n y_i^2 + 2ab \sum_{i=1}^n x_i y_i \geq 0$
 $\langle 3 \rangle 4. a^2 \|\vec{x}\|^2 + b^2 \|\vec{y}\|^2 + 2ab \vec{x} \cdot \vec{y} \geq 0$
 $\langle 2 \rangle 3. 2 - 2ab \vec{x} \cdot \vec{y} \geq 0$
 PROOF: Similar.
 $\langle 2 \rangle 4. 2 - 2ab |\vec{x} \cdot \vec{y}| \geq 0$
 PROOF: From $\langle 2 \rangle 2$ and $\langle 2 \rangle 3$.
 $\langle 2 \rangle 5. |\vec{x} \cdot \vec{y}| \leq 1/ab$
 \square

Lemma 12.1.7.

$$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$$

PROOF:

$$\begin{aligned}
 \|\vec{x} + \vec{y}\|^2 &= (\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y}) \\
 &= \|\vec{x}\|^2 + 2\vec{x} \cdot \vec{y} + \|\vec{y}\|^2 && \text{(Lemma 12.1.5)} \\
 &\leq \|\vec{x}\|^2 + 2\|\vec{x}\|\|\vec{y}\| + \|\vec{y}\|^2 && \text{(Lemma 12.1.6)} \\
 &= (\|\vec{x}\| + \|\vec{y}\|)^2 && \square
 \end{aligned}$$

Definition 12.1.8 (Euclidean Metric). The *euclidean metric* on \mathbb{R}^n is given by

$$d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|$$

We prove this is a metric.

PROOF:

- $\langle 1 \rangle 1. d(\vec{x}, \vec{y}) \geq 0$
 PROOF: Immediate from definitions.
 $\langle 1 \rangle 2. d(\vec{x}, \vec{y}) = 0$ iff $\vec{x} = \vec{y}$
 PROOF: Immediate from definitions.
 $\langle 1 \rangle 3. d(\vec{x}, \vec{y}) = d(\vec{y}, \vec{x})$
 PROOF: Immediate from definitions.
 $\langle 1 \rangle 4. d(\vec{x}, \vec{z}) \leq d(\vec{x}, \vec{y}) + d(\vec{y}, \vec{z})$
 PROOF: From Lemma 12.1.7.
 \square

Lemma 12.1.9. Let d be the euclidean topology on \mathbb{R}^n and ρ the square topology. Then, for all $x, y \in \mathbb{R}^n$, we have

$$\rho(x, y) \leq d(x, y) \leq \sqrt{n}\rho(x, y)$$

PROOF:

- $\langle 1 \rangle 1. \rho(x, y) \leq d(x, y)$
 $\langle 2 \rangle 1.$ For $1 \leq i \leq n$ we have $|x_i - y_i| \leq d(x, y)$
 PROOF: By the definition of the euclidean metric.
 $\langle 2 \rangle 2.$ Q.E.D.
 PROOF: By the definition of the square metric.

(1)2. $d(x, y) \leq \sqrt{n}\rho(x, y)$

PROOF:

$$\begin{aligned} d(x, y) &= \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2} \\ &\leq \sqrt{\rho(x, y)^2 + \cdots + \rho(x, y)^2} \\ &= \sqrt{n\rho(x, y)^2} \\ &= \sqrt{n}\rho(x, y) \end{aligned}$$

□

Corollary 12.1.9.1. *The euclidean metric induces the standard topology on \mathbb{R}^n .*

Definition 12.1.10. Let l_2 be the set of sequences $\vec{a} \in \mathbb{R}^\omega$ such that $\sum_{n=1}^\infty a_n^2 < \infty$.

Lemma 12.1.11. *If $\vec{a}, \vec{b} \in l_2$ then $\sum_{n=1}^\infty |a_n b_n| < \infty$.*

PROOF:

$$\begin{aligned} \sum_{n=1}^N |a_n b_n| &\leq \sqrt{\left(\sum_{n=1}^N a_n^2\right)\left(\sum_{n=1}^N b_n^2\right)} && \text{(Lemma 12.1.6)} \\ &\rightarrow \sqrt{\sum_{n=1}^\infty a_n^2 \sum_{n=1}^\infty b_n^2} \text{ as } n \rightarrow \infty \end{aligned}$$

□

Lemma 12.1.12. *If $\vec{a}, \vec{b} \in l_2$ then $\vec{a} + \vec{b} \in l_2$.*

PROOF:

$$\begin{aligned} \sum_{n=1}^\infty (a_n + b_n)^2 &= \sum_{n=1}^\infty a_n^2 + 2 \sum_{n=1}^\infty a_n b_n + \sum_{n=1}^\infty b_n^2 \\ &\leq \sum_{n=1}^\infty a_n^2 + 2 \sum_{n=1}^\infty |a_n b_n| + \sum_{n=1}^\infty b_n^2 \\ &< \infty && \text{(Lemma 12.1.11)} \end{aligned}$$

□

Lemma 12.1.13. *If $c \in \mathbb{R}$ and $\vec{a} \in l_2$ then $c\vec{a} \in l_2$.*

PROOF: $\sum_{n=1}^\infty (ca_n)^2 = c^2 \sum_{n=1}^\infty a_n^2$. □

Definition 12.1.14 (The l^2 -metric). The l^2 -metric is defined on l_2 by

$$d(\vec{a}, \vec{b}) = \left[\sum_{n=1}^\infty (a_n - b_n)^2 \right]^{\frac{1}{2}}.$$

The topology induced by this metric is the l^2 -topology. We write l_2 for this set under the l^2 -topology.

We prove this is a metric.

PROOF:

⟨1⟩1. $d(\vec{a}, \vec{b}) \geq 0$

PROOF: Immediate from definitions.

⟨1⟩2. $d(\vec{a}, \vec{b}) = 0$ iff $\vec{a} = \vec{b}$

PROOF: Immediate from definitions.

⟨1⟩3. $d(\vec{a}, \vec{b}) = d(\vec{b}, \vec{a})$

PROOF: Immediate from definitions.

⟨1⟩4. $d(\vec{a}, \vec{c}) \leq d(\vec{a}, \vec{b}) + d(\vec{b}, \vec{c})$

PROOF: $\sqrt{\sum_{i=1}^N (a_i - c_i)^2} \leq \sqrt{\sum_{i=1}^N (a_i - b_i)^2} + \sqrt{\sum_{i=1}^N (b_i - c_i)^2}$ since the euclidean metric on \mathbb{R}^N is a metric.

□

Definition 12.1.15 (Hilbert Cube). The *Hilbert cube* is $\prod_{n=1}^{\infty} [0, 1/n]$ as a subspace of the l_2 .

Definition 12.1.16 (Isometric Imbedding). Let X, Y be metric spaces and $f : X \rightarrow Y$. Then f is an *isometric imbedding* iff, for all $x, y \in X$, $d(f(x), f(y)) = d(x, y)$.

Lemma 12.1.17. *Every isometric imbedding is an imbedding.*

PROOF:

⟨1⟩1. LET: $f : X \rightarrow Y$ be an isometric imbedding.

⟨1⟩2. f is continuous.

PROOF: If $d(x, y) < \epsilon$ then $d(f(x), f(y)) < \epsilon$.

⟨1⟩3. f is injective.

PROOF: If $f(x) = f(y)$ then $d(f(x), f(y)) = 0$ so $d(x, y) = 0$ hence $x = y$.

⟨1⟩4. $f^{-1} : f(X) \rightarrow X$ is continuous.

PROOF: If $d(f^{-1}(x), f^{-1}(y)) < \epsilon$ then $d(x, y) < \epsilon$.

□

Chapter 13

Topological Groups

13.1 Topological Groups

Definition 13.1.1 (Topological Group). A *topological group* G consists of a group G that is also a T_1 space such that $\cdot : G^2 \rightarrow G$ and $(\)^{-1} : G \rightarrow G$ are continuous.

Proposition 13.1.2. *Every topological group is homogeneous.*

PROOF:

- $\langle 1 \rangle 1$. LET: G be a topological group.
- $\langle 1 \rangle 2$. LET: $x, y \in G$
- $\langle 1 \rangle 3$. LET: $f : G \rightarrow G$ be given by $f(g) = yx^{-1}g$
- $\langle 1 \rangle 4$. f is a homeomorphism
- $\langle 1 \rangle 5$. $f(x) = y$

□

Definition 13.1.3 (Symmetric). Let G be a topological group. A neighbourhood U of e is *symmetric* iff $U = U^{-1}$.

Proposition 13.1.4. *For every neighbourhood U of e , there exists a symmetric neighbourhood V of e such that $VV \subseteq U$.*

PROOF:

- $\langle 1 \rangle 1$. LET: $m : G^2 \rightarrow G$ be the multiplication function
- $\langle 1 \rangle 2$. $ee \in U$
- $\langle 1 \rangle 3$. $(e, e) \in m^{-1}(U)$
- $\langle 1 \rangle 4$. PICK neighbourhoods U_1, U_2 of e such that $(e, e) \in U_1 \times U_2 \subseteq m^{-1}(U)$
- $\langle 1 \rangle 5$. LET: $V' = U_1 \cap U_2$
- $\langle 1 \rangle 6$. $V'V' \subseteq U$
- $\langle 1 \rangle 7$. LET: $f : G^2 \rightarrow G$ be the function $f(x, y) = xy^{-1}$
- $\langle 1 \rangle 8$. $(e, e) \in f^{-1}(V')$
- $\langle 1 \rangle 9$. PICK a neighbourhood W of e such that $WW^{-1} \subseteq V'$
- $\langle 1 \rangle 10$. LET: $V = WW^{-1}$

- (1)11. V is a neighbourhood of e
 PROOF: V is open because $V = \bigcup_{a \in W^{-1}} Wa$.
 (1)12. V is symmetric
 (1)13. $VV \subseteq U$
 \square

Proposition 13.1.5. *Every topological group is regular.*

PROOF:

- (1)1. LET: G be a topological group
 (1)2. LET: $A \subseteq G$ be closed and $a \notin A$
 (1)3. $G \setminus Aa^{-1}$ is a neighbourhood of e
 (1)4. PICK a symmetric neighbourhood V of e such that $VV \subseteq G \setminus Aa^{-1}$
 PROOF: Proposition 13.1.4.
 (1)5. VA and Va are disjoint neighbourhoods of A and a
 \square

Proposition 13.1.6. *The long line is not second countable.*

PROOF: Let \mathcal{B} be a basis for L . Then, for every countable ordinal α , \mathcal{B} must contain a basic open set that contains $(\alpha, 1/2)$ but not $(\beta, 1/2)$ for any other β . Therefore, \mathcal{B} is uncountable. \square

Corollary 13.1.6.1. *The long line cannot be imbedded in \mathbb{R} .*

Theorem 13.1.7. *Let $f : X \rightarrow Y$. Let Y be compact Hausdorff. Then f is continuous if and only if the graph of f is closed in $X \times Y$.*

PROOF:

- (1)1. LET: G_f be the graph of f .
 (1)2. If f is continuous then the graph of f is closed.
 (2)1. ASSUME: f is continuous.
 (2)2. LET: $(x, y) \in (X \times Y) \setminus G_f$
 (2)3. $y \neq f(x)$
 (2)4. PICK disjoint open neighbourhoods U of $f(x)$ and V of y
 PROOF: Y is Hausdorff.
 (2)5. $(x, y) \in f^{-1}(U) \times V \subseteq (X \times Y) \setminus G_f$
 (2)6. Q.E.D.
 (1)3. If the graph of f is closed then f is continuous.
 (2)1. ASSUME: G_f is closed.
 (2)2. LET: $x_0 \in X$ and V be an open neighbourhood of $f(x_0)$
 (2)3. $G_f \cap (X \times (Y \setminus V))$ is closed
 (2)4. $\pi_1(G_f \cap (X \times (Y \setminus V)))$ is closed
 PROOF: Lemma 9.4.16
 (2)5. $x_0 \in X \setminus \pi_1(G_f \cap (X \times (Y \setminus V))) \subseteq f^{-1}(V)$
 (2)6. Q.E.D.
 \square

Theorem 13.1.8. *Let X be a compact Hausdorff space. Let \mathcal{A} be a set of closed connected subspaces of X that is linearly ordered by proper inclusion. Then*

$$Y = \bigcap \mathcal{A}$$

is connected.

PROOF:

⟨1⟩1. ASSUME: for a contradiction C and D form a separation of Y

⟨1⟩2. PICK disjoint U and V open in X such that $C = U \cap Y$ and $D = V \cap Y$

⟨2⟩1. C and D are compact

⟨3⟩1. Y is compact

PROOF: Y is a closed subset of X , hence compact by Proposition 9.4.6.

⟨3⟩2. Q.E.D.

PROOF: C and D are closed subsets of Y hence compact by Proposition 9.4.6.

⟨2⟩2. Q.E.D.

PROOF: By Lemma 9.4.18.

⟨1⟩3. For all $A \in \mathcal{A}$, we have $A \setminus (U \cup V)$ is nonempty

PROOF: Since A is connected.

⟨1⟩4. $\{A \setminus (U \cup V) : A \in \mathcal{A}\}$ has the finite intersection property

PROOF: This holds because \mathcal{A} is linearly ordered under proper inclusion.

⟨1⟩5. $\bigcap_{A \in \mathcal{A}} (A \setminus (U \cup V))$ is nonempty

PROOF: By Proposition 9.4.15.

□

Theorem 13.1.9. *Let $A \subseteq \mathbb{R}^n$. Then the following are equivalent:*

1. A is compact.
2. A is closed and bounded under the euclidean metric.
3. A is closed and bounded under the square metric.

PROOF:

⟨1⟩1. $1 \Rightarrow 2$

⟨2⟩1. ASSUME: A is compact.

⟨2⟩2. A is closed.

PROOF: By Proposition 9.4.9.

⟨2⟩3. $\{B(\vec{0}, n) : n \in \mathbb{Z}^+\}$ covers A

⟨2⟩4. PICK a finite subcover $\{B(\vec{0}, n_1), \dots, B(\vec{0}, n_k)\}$

⟨2⟩5. LET: $N = \max(n_1, \dots, n_k)$

⟨2⟩6. For all $x, y \in A$ we have $d(x, y) < 2N$

PROOF: We have $d(x, y) \leq d(\vec{0}, x) + d(\vec{0}, y) < N + N$.

⟨1⟩2. $2 \Rightarrow 3$

PROOF: If $d(x, y) < \epsilon$ for all $x, y \in A$ then $\rho(x, y) < \epsilon\sqrt{n}$ by Lemma 12.1.9.

⟨1⟩3. $3 \Rightarrow 1$

⟨2⟩1. ASSUME: A is closed and $\rho(x, y) < \epsilon$ for all $x, y \in A$

- (2)2. PICK $x_0 \in A$
 (2)3. LET: $b = \rho(\vec{0}, x_0)$
 (2)4. LET: $P = \epsilon + b$
 (2)5. $A \subseteq [-P, P]^n$
 PROOF: For any $y \in A$ we have

$$\begin{aligned} \rho(\vec{0}, y) &\leq \rho(\vec{0}, x_0) + \rho(x_0, y) && \text{(Triangle Inequality)} \\ &< b + \epsilon && ((2)3, (2)1) \\ &= P && ((2)4) \end{aligned}$$

 (2)6. $[-P, P]^n$ is compact.
 PROOF: By Corollary 9.4.19.1 and Proposition 9.4.14.
 (2)7. Q.E.D.
 PROOF: By Proposition 9.4.6.

□

Theorem 13.1.10 (AC). *Let X be a topological space. Then X is compact if and only if every nonempty net in X has a convergent subnet.*

PROOF:

- (1)1. If X is compact then every nonempty net in X has a convergent subnet.
 (2)1. ASSUME: X is compact.
 (2)2. LET: $(x_\alpha)_{\alpha \in J}$ be a nonempty net in X
 (2)3. For $\alpha \in J$,
 LET: $B_\alpha = \{\beta \in J : \alpha \leq \beta\}$.
 (2)4. $\{B_\alpha : \alpha \in J\}$ has the finite intersection property.
 (3)1. LET: $\alpha_1, \dots, \alpha_n \in J$
 (3)2. PICK $\beta \in J$ such that $\alpha_1 \leq \beta, \dots, \alpha_n \leq \beta$
 (3)3. $x_\beta \in B_{\alpha_1} \cap \dots \cap B_{\alpha_n}$
 (2)5. PICK $l \in \bigcap_{\alpha \in J} B_\alpha$
 PROOF: Proposition 9.4.15.
 (2)6. LET: $K = \{\alpha \in J : x_\alpha = l\}$
 (2)7. K is cofinal in J
 (3)1. LET: $\alpha \in J$
 (3)2. $l \in B_\alpha$
 PROOF: By (2)5.
 (3)3. There exists $\beta \geq \alpha$ such that $x_\beta = l$.
 (2)8. $(x_\alpha)_{\alpha \in K}$ is a subnet of $(x_\alpha)_{\alpha \in J}$ that converges to l .
 (1)2. If every nonempty net in X has a convergent subnet then X is compact.
 (2)1. ASSUME: Every nonempty net in X has a convergent subnet
 (2)2. LET: \mathcal{A} be a nonempty set of closed sets with the finite intersection property.
 (2)3. LET: J be the poset of all finite intersections of elements of \mathcal{A} under \supseteq
 (2)4. PICK $x_C \in C$ for all $C \in J$
 PROOF: These are all nonempty by (2)2.
 (2)5. PICK an accumulation point l of (x_C)
 PROVE: $l \in \bigcap \mathcal{A}$
 PROOF: One exists by Lemma 3.13.2.

- (2)6. LET: $C \in \mathcal{A}$
 PROVE: $l \in C$
 (2)7. LET: U be a neighbourhood of l
 PROVE: U intersects C
 (2)8. PICK $D \subseteq C$ such that $x_D \in U$
 PROOF: By (2)5.
 (2)9. U intersects C
 (2)10. $l \in C$
 PROOF: By Theorem 3.8.3 since C is closed ((2)2).
 (2)11. Q.E.D.
 PROOF: Proposition 9.4.15.

□

Corollary 13.1.10.1 (AC). *Let G be a topological group. Let A and B be subsets of G . If A is closed in G and B is compact then AB is closed in G .*

PROOF:

- (1)1. LET: $c \in \overline{AB}$
 PROVE: $c \in AB$
 (1)2. PICK a net $(x_\alpha)_{\alpha \in J}$ that converges to c
 PROOF: By Theorem 3.12.3.
 (1)3. For $\alpha \in J$, PICK $a_\alpha \in A$ and $b_\alpha \in B$ such that $x_\alpha = a_\alpha b_\alpha$
 (1)4. PICK a convergent subnet $(b_{g(\beta)})_{\beta \in K}$ of $(b_\alpha)_{\alpha \in J}$
 PROOF: By Theorem 13.1.10.
 (1)5. LET: $b_{g(\beta)} \rightarrow b$
 (1)6. $b \in B$
 (2)1. B is closed
 PROOF: By Proposition 9.4.9.
 (2)2. Q.E.D.
 PROOF: By Theorem 3.12.3
 (1)7. $a_{g(\beta)} \rightarrow cb^{-1}$
 PROOF: By Theorem 3.12.4
 (1)8. $cb^{-1} \in A$
 PROOF: By Theorem 3.12.3
 (1)9. $c \in AB$
 (1)10. Q.E.D.
 PROOF: By Proposition 3.7.6.

Proposition 13.1.11. *Let $A_0 + A_1$ be the sum of A_0 and A_1 with injections $i_0 : A_0 \rightarrow A_0 + A_1$ and $i_1 : A_1 \rightarrow A_0 + A_1$.*

Let $g : B \rightarrow A_0 + A_1$ be a function.

Let B_0 be the pullback of i_0 and g with projections $j_0 : B_0 \rightarrow B$ and $k_0 : B_0 \rightarrow A_0$.

Let B_1 be the pullback of i_1 and g with projection $sj_1 : B_1 \rightarrow B$ and $k_1 : B_1 \rightarrow A_1$.

Then B is the sum of B_0 and B_1 with injections j_0 and j_1 .

$$\begin{array}{ccccc}
B_0 & \xrightarrow{j_0} & B & \xleftarrow{j_1} & B_1 \\
\downarrow & & \downarrow g & & \downarrow \\
A_0 & \xrightarrow{i_0} & A_0 + A_1 & \xleftarrow{i_1} & A_1
\end{array}$$

PROOF:

(1)1. LET: X be any set and $x : B_0 \rightarrow X, y : B_1 \rightarrow X$

Proposition 13.1.12 (CC). *Let X be a space and \mathcal{B} be a basis for X . Suppose that every subset of \mathcal{B} that covers X has a countable subcover. Then X is Lindelöf.*

PROOF:

- (1)1. LET: \mathcal{A} be an open cover of X .
- (1)2. $\{B \in \mathcal{B} : \exists U \in \mathcal{A}. B \subseteq U\}$ covers X .
- (1)3. PICK a countable subcover \mathcal{B}_0
- (1)4. For $B \in \mathcal{B}_0$, PICK $U_B \in \mathcal{A}$ such that $B \subseteq U_B$
- (1)5. $\{U_B : B \in \mathcal{B}_0\}$ is a countable subcover of \mathcal{A} .

□

Proposition 13.1.13 (CC). *The space \mathbb{R}_l is Lindelöf.*

PROOF:

- (1)1. LET: \mathcal{A} be a set of basis elements $[a, b)$ that covers X
 PROVE: \mathcal{A} has a countable subcover.
- (1)2. LET: $C = \bigcup \{(a, b) : [a, b) \in \mathcal{A}\}$
- (1)3. $\mathbb{R} \setminus C$ is countable.
 - (2)1. For all $x \in \mathbb{R} \setminus C$, PICK a rational q_x such that there exists b such that $q_x \in [x, b) \in \mathcal{A}$
 - (3)1. PICK $[a, b) \in \mathcal{A}$ such that $x \in [a, b)$
 - (3)2. $x = a$
 PROOF: If not we would have $x \in C$
 - (3)3. There exists a rational in (a, b)
 - (2)2. For $x, y \in \mathbb{R} \setminus C$, if $x < y$ then $q_x < q_y$
 - (3)1. PICK b, c such that $q_x \in [x, b) \in \mathcal{A}$ and $q_y \in [y, c) \in \mathcal{A}$
 PROOF: By (2)1.
 - (3)2. $b \leq y$
 PROOF: Otherwise we would have $y \in (x, b) \subseteq C$.
 - (3)3. $q_x < q_y$
 PROOF: $q_x < b \leq y \leq q_y$
 - (2)3. The map $q_- : \mathbb{R} \setminus C \rightarrow \mathbb{Q}$ is injective.
- (1)4. For $x \in \mathbb{R} \setminus C$, PICK $[a_x, b_x) \in \mathcal{A}$ such that $a_x \leq x < b_x$
- (1)5. PICK a countable subset $((a_n, b_n))_{n \in \mathbb{Z}^+}$ of $\{(a, b) : [a, b) \in \mathcal{A}\}$ that covers C
 - (2)1. The set C as a subspace of \mathbb{R} with the standard topology is second countable.

⟨2⟩2. The set C as a subspace of \mathbb{R} with the standard topology is Lindelöf.

PROOF: By Theorem 9.3.2.

⟨1⟩6. $\{[a_x, b_x) : x \in \mathbb{R} \setminus C\} \cup \{[a_n, b_n) : n \in \mathbb{Z}^+\}$ is a countable subcover of \mathcal{A} .

⟨1⟩7. Q.E.D.

PROOF: By Proposition 13.1.12.

□

Proposition 13.1.14 (AC). *The space \mathbb{R}_l is not second countable.*

PROOF:

⟨1⟩1. LET: \mathcal{B} be any basis for \mathbb{R}_l

⟨1⟩2. For $x \in \mathbb{R}$, PICK $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq [x, x+1)$

⟨1⟩3. The mapping $B_{(-)}$ is an injective function $\mathbb{R} \rightarrow \mathcal{B}$

PROOF: For any x we have $x = \min B_x$.

⟨1⟩4. \mathcal{B} is uncountable.

□

Proposition 13.1.15. *The product of a Lindelöf space and a compact space is Lindelöf.*

PROOF:

⟨1⟩1. LET: X be a Lindelöf space and Y a compact space.

⟨1⟩2. LET: \mathcal{A} be an open covering of $X \times Y$

⟨1⟩3. For all $x \in X$, there exists a neighbourhood W of x such that $W \times Y$ is covered by finitely many elements of \mathcal{A} .

⟨2⟩1. LET: $x \in X$

⟨2⟩2. $\{x\} \times Y$ is compact.

PROOF: It is homeomorphic to Y .

⟨2⟩3. PICK a finite subset $\{U_1, \dots, U_m\}$ of \mathcal{A} that covers $\{x\} \times Y$

PROOF: By Proposition 9.4.5.

⟨2⟩4. There exists a neighbourhood W of x such that $W \times Y \subseteq U_1 \cup \dots \cup U_m$

PROOF: By the Tube Lemma.

⟨1⟩4. $\{W \text{ open in } X : W \times Y \text{ is covered by finitely many elements of } \mathcal{A}\}$ is an open covering of X .

⟨1⟩5. PICK a countable subcovering $\{W_1, W_2, \dots\}$

⟨1⟩6. For $i \geq 1$, PICK a finite subset $\{U_{i1}, \dots, U_{ir_i}\}$ of \mathcal{A} that covers $W_i \times Y$

⟨1⟩7. $\{U_{1j} : i \geq 1, 1 \leq j \leq r_i\}$ is a countable subcovering of \mathcal{A} .

□

Proposition 13.1.16. *Let X be a T_1 space. Then X is normal if and only if, for every closed set A and open set $U \supseteq A$, there exists an open set $V \supseteq A$ such that $\bar{V} \subseteq U$.*

PROOF:

⟨1⟩1. If X is normal then, for every closed set A and open set $U \supseteq A$, there exists an open set $V \supseteq A$ such that $\bar{V} \subseteq U$.

⟨2⟩1. ASSUME: X is normal.

⟨2⟩2. LET: A be a closed set and U an open set with $A \subseteq U$

- (2)3. PICK disjoint open sets V, W such that $A \subseteq V$ and $X \setminus U \subseteq W$
 (2)4. $\overline{V} \subseteq U$

PROOF:

$$\begin{aligned}\overline{V} &\subseteq X \setminus W \\ &\subseteq U\end{aligned}$$

- (1)2. If, for every closed set A and open set $U \supseteq A$, there exists an open set $V \supseteq A$ such that $\overline{V} \subseteq U$, then X is normal.
 (2)1. ASSUME: for every closed set A and open set $U \supseteq A$, there exists an open set $V \supseteq A$ such that $\overline{V} \subseteq U$.
 (2)2. LET: A, B be disjoint closed sets
 (2)3. PICK an open set V such that $A \subseteq V$ and $\overline{V} \subseteq X \setminus B$
 (2)4. $A \subseteq V$ and $B \subseteq X \setminus \overline{V}$

□

Definition 13.1.17 (Action). Let G be a topological group and X a topological space. An *action* of G on X is a continuous function $\cdot : G \times X \rightarrow X$ such that, for all $g, h \in G$ and $x \in X$:

1. $e \cdot x = x$
2. $g \cdot (h \cdot x) = gh \cdot x$

Definition 13.1.18 (Orbit Space). Let G be a topological group, X a topological space, and $\cdot : G \times X \rightarrow X$ an action of G on X . Then the *orbit space* X/G is the quotient space of X by the equivalence relation \sim generated by $x \sim g \cdot x$ for all $x \in X, g \in G$.

Theorem 13.1.19. Let G be a topological group. Let X be a topological space. Let $\cdot : G \times X \rightarrow X$ be an action of G on X . Then the canonical map $\pi : X \rightarrow X/G$ is perfect.

- (1)1. π is closed.
 (2)1. LET: $A \subseteq X$ be closed.
 (2)2. $GA = \{g \cdot a : g \in G, a \in A\}$ is closed
 (3)1. LET: $z \notin GA$
 (3)2. For all $g \in G$ we have $g \cdot z \notin A$
 (3)3. For $g \in G$, there exist U an open neighbourhood of g and V an open neighbourhood of z such that UV does not intersect A
 (3)4. $\{U \text{ open in } G : \exists V \text{ an open neighbourhood of } z. UV \cap A = \emptyset\}$ covers G
 (3)5. PICK a finite subcover $\{U_1, \dots, U_n\}$
 (3)6. For $1 \leq i \leq n$, PICK V_i an open neighbourhood of z such that $U_i V_i \cap A = \emptyset$
 (3)7. $z \in V_1 \cap \dots \cap V_n \subseteq X \setminus GA$
 (2)3. $\pi(A)$ is closed
 $\pi^{-1}(\pi(A)) = GA$
 (1)2. π is continuous.
 PROOF: By definition of the quotient topology.

⟨1⟩3. π is surjective.

PROOF: By definition.

⟨1⟩4. For all $a \in X/G$ we have $\pi^{-1}(a)$ is compact.

⟨2⟩1. LET: $a \in X/G$

⟨2⟩2. PICK $x \in X$ such that $a = \pi(x)$

⟨2⟩3. $\pi^{-1}(a) = \{gx : g \in G\}$

⟨2⟩4. $\pi^{-1}(a)$ is homeomorphic to G

□

Corollary 13.1.19.1. *If X is Hausdorff then so is X/G .*

Corollary 13.1.19.2. *If X is regular then so is X/G .*

Corollary 13.1.19.3. *If X is normal then so is X/G .*

Corollary 13.1.19.4. *If X is locally compact then so is X/G .*

Corollary 13.1.19.5. *If X is second countable then so is X/G .*

Proposition 13.1.20. *Let $p : X \twoheadrightarrow Y$ be perfect. If X is second countable then so is Y .*

PROOF:

⟨1⟩1. PICK a countable basis \mathcal{B} for X

⟨1⟩2. LET: $\mathcal{J} = \{J \subseteq^{\text{fin}} \mathcal{B} : \exists W \text{ open in } Y. p^{-1}(W) \subseteq \bigcup J\}$

⟨1⟩3. For every $J \in \mathcal{J}$,

LET: $W_J = \bigcup \{W \text{ open in } Y : p^{-1}(W) \subseteq \bigcup J\}$.

PROVE: $\{W_J : J \in \mathcal{J}\}$ is a basis for Y .

⟨1⟩4. $y \in V$ where V is open in Y

⟨1⟩5. $\{B \in \mathcal{B} : x \in B \subseteq p^{-1}(V)\}$ covers $p^{-1}(y)$

⟨1⟩6. PICK a countable subcover $J \subseteq^{\text{fin}} \mathcal{B}$

⟨1⟩7. $y \in W_J \subseteq V$

⟨2⟩1. $p^{-1}(y) \subseteq \bigcup J$

⟨2⟩2. PICK an open neighbourhood W of y such that $p^{-1}(W) \subseteq \bigcup J$

PROOF: By Proposition 9.5.1.

⟨2⟩3. $W \subseteq W_J$

□

Proposition 13.1.21. *A subspace of a T_1 space is T_1 .*

PROOF:

⟨1⟩1. LET: X be T_1 and $Y \subseteq X$

⟨1⟩2. LET: $a \in Y$

⟨1⟩3. $\{a\}$ is closed in X

⟨1⟩4. $\{a\}$ is closed in Y

PROOF: By Corollary 4.3.4.1.

□

Proposition 13.1.22 (DC). *Not every topological group is normal.*

PROOF: From Proposition 6.5.6. \square

Theorem 13.1.23. *A subspace of a completely regular space is completely regular.*

PROOF:

- $\langle 1 \rangle 1$. LET: X be completely regular and $Y \subseteq X$
- $\langle 1 \rangle 2$. LET: $a \in Y$ and A be closed in Y such that $a \notin A$
- $\langle 1 \rangle 3$. PICK C closed in X such that $A = X \cap C$
- $\langle 1 \rangle 4$. PICK a continuous function $f : X \rightarrow [0, 1]$ such that $f(a) = 0$ and $f(C) = \{1\}$
- $\langle 1 \rangle 5$. $f \upharpoonright Y : Y \rightarrow [0, 1]$ is a continuous function such that $(f \upharpoonright Y)(a) = 0$ and $(f \upharpoonright Y)(A) = \{1\}$

\square

Proposition 13.1.24 (DC). *Every topological group is completely regular.*

PROOF:

- $\langle 1 \rangle 1$. LET: G be a topological group
- $\langle 1 \rangle 2$. LET: $x \in G$ and $A \subseteq G$ be closed such that $x \notin A$
 PROVE: There exists a continuous $f : G \rightarrow [0, 1]$ such that $f(x) = 0$ and $f(A) = \{1\}$
- $\langle 1 \rangle 3$. ASSUME: w.l.o.g. $x = e$
 PROOF: $\lambda y.x^{-1}y$ is an automorphism of G that maps x to e .
- $\langle 1 \rangle 4$. PICK a sequence V_n ($n \geq 0$) of symmetric neighbourhoods of e disjoint from A such that $V_n V_n \subseteq V_{n-1}$ for all n
- $\langle 2 \rangle 1$. LET: $V_0 = X \setminus A$
- $\langle 2 \rangle 2$. Given V_n , PICK a symmetric neighbourhood V_{n+1} of e such that $V_{n+1} V_{n+1} \subseteq V_n$

PROOF: By Proposition 13.1.4.

- $\langle 1 \rangle 5$. For every dyadic rational p , define an open set $U(p)$ as follows:

$$\begin{aligned} U(1/2^n) &= V_n & (n \geq 0) \\ U((2k+1)/2^{n+1}) &= V_{n+1}U(k/2^n) & (0 < k < 2^n) \\ U(p) &= \emptyset & (p \leq 0) \\ U(p) &= G & (p > 1) \end{aligned}$$

- $\langle 1 \rangle 6$. For all k and n , we have

$$V_n U(k/2^n) \subseteq U((k+1)/2^n)$$

- $\langle 2 \rangle 1$. $k \leq 0$

PROOF: In this case, $V_n U(k/2^n) = \emptyset$

- $\langle 2 \rangle 2$. $k = 1$ and $n > 0$

PROOF:

$$\begin{aligned} V_n U(1/2^n) &= V_n V_n \\ &\subseteq V_{n-1} \\ &= U(1/2^{n-1}) \end{aligned}$$

- $\langle 2 \rangle 3$. $k = 2a$ for some $0 < a < 2^{n-1}$

PROOF:

$$\begin{aligned} V_n U(2a/2^n) &= V_n U(a/2^{n-1}) \\ &= U(2a+1/2^n) \end{aligned}$$

(2)4. $k = 2a + 1$ for some $0 < a < 2^{n-1}$

PROOF:

$$\begin{aligned} V_n U((2a+1)/2^n) &= V_n V_n U(a/2^{n-1}) \\ &\subseteq V_{n-1} U(a/2^{n-1}) \\ &\subseteq U((a+1)/2^{n-1}) \end{aligned}$$

(2)5. $k \geq 2^n$

PROOF: In this case, $U((k+1)/2^n) = G$.

(1)7. Define $f : G \rightarrow [0, 1]$ by

$$f(x) = \inf\{p : x \in U(p)\}$$

PROOF: This set is nonempty because $x \in U(1)$ and bounded below because if $x \in U(p)$ then $p > 0$.

(1)8. For $n > 0$ we have $\overline{U(k/2^n)} \subseteq V_n U(k/2^n)$

(2)1. LET: $x \in \overline{U(k/2^n)}$

(2)2. $V_n x$ is a neighbourhood of x

(2)3. PICK $y \in V_n x \cap U(k/2^n)$

(2)4. PICK $z \in V_n$ such that $y = zx$

(2)5. $x = z^{-1}y$

(1)9. For p and q dyadic rationals, if $p < q$ then $\overline{U(p)} \subseteq U(q)$

(1)10. If $x \in \overline{U(p)}$ then $f(x) \leq p$

(2)1. For all $q > p$ we have $x \in U(q)$

(2)2. For all $q > p$ we have $f(x) \leq q$

(1)11. If $x \notin U(p)$ then $f(x) \geq p$

PROOF: If $x \notin U(p)$ and $x \in U(q)$ then $q > p$.

(1)12. f is continuous

(2)1. LET: $x_0 \in X$

(2)2. LET: $c < f(x_0) < d$

PROVE: There exist a neighbourhood U of x_0 such that $f(U) \subseteq (c, d)$

(2)3. PICK rational numbers p, q such that $c < p < f(x_0) < q < d$

(2)4. $x \notin \overline{U(p)}$

(2)5. $x \in U(q)$

(2)6. Take $U = U(q) \setminus \overline{U(p)}$

(1)13. $f(e) = 0$

PROOF: We have $e \in U(1/2^n)$ for all n .

(1)14. $f(A) = \{1\}$

PROOF: If $x \in A$ and $x \in U(p)$ then $p > 1$.

□

Definition 13.1.25 (Bijection). A function $f : A \rightarrow B$ is a *bijection*, $f : A \cong B$, iff there exists a function $f^{-1} : B \rightarrow A$, the *inverse* of f , such that $f^{-1} \circ f = \text{id}_A$ and $f \circ f^{-1} = \text{id}_B$.

Theorem 13.1.26. Let Y be a normal space. Then Y is an absolute retract if and only if Y has the universal extension property.

PROOF:

- ⟨1⟩1. If Y is an absolute retract then Y has the universal extension property.
- ⟨2⟩1. ASSUME: Y is an absolute retract.
- ⟨2⟩2. LET: X be a normal space, A a closed subspace of X and $f : A \rightarrow Y$ a continuous function.
- ⟨2⟩3. LET: Z_f be the quotient space of $X \cup Y$ under: $a \sim f(a)$ for all $a \in A$
- ⟨2⟩4. LET: $p : X \cup Y \rightarrow Z_f$ be the quotient map
- ⟨2⟩5. For all $x_1, x_2 \in X$ we have $p(x_1) = p(x_2)$ iff $x_1 = x_2$ or $x_1, x_2 \in A$ and $f(x_1) = f(x_2)$; and for $x \in X$ and $y \in Y$ we have $p(x) = p(y)$ iff $f(x) = y$; and for $y_1, y_2 \in Y$ we have $p(y_1) = p(y_2)$ iff $y_1 = y_2$
- ⟨2⟩6. p imbeds Y into a closed subspace of Z_f
 - ⟨3⟩1. p is injective on Y
 - ⟨3⟩2. $p^{-1} : p(Y) \rightarrow Y$ is continuous
 - ⟨4⟩1. LET: $U \subseteq Y$ be open
PROVE: $p(U)$ is open
 - ⟨4⟩2. $p^{-1}(p(U)) = f^{-1}(U) \cup U$
 - ⟨3⟩3. $p(Y)$ is closed
PROOF: $p^{-1}(p(Y)) = A \cup Y$
- ⟨2⟩7. Z_f is normal
 - ⟨3⟩1. Z_f is T_1
PROOF: For $y \in Y$ we have $p^{-1}(y) = f^{-1}(y) \cup \{y\}$ which is closed.
 - ⟨3⟩2. Any two disjoint closed sets in Z_f can be separated by a continuous function.
 - ⟨4⟩1. LET: C and D be disjoint closed sets in Z_f
 - ⟨4⟩2. PICK $g : Y \rightarrow [0, 1]$ such that $g(Y \cap p^{-1}(C)) = \{0\}$ and $g(Y \cap p^{-1}(D)) = \{1\}$
PROOF: By the Urysohn Lemma.
 - ⟨4⟩3. PICK $h : X \rightarrow [0, 1]$ such that $h(X \cap p^{-1}(C)) = \{0\}$ and $h(X \cap p^{-1}(D)) = \{1\}$ and h agrees with $g \circ f$ on A
PROOF: By the Tietze Extension Theorem applied to $A \cup (X \cap p^{-1}(C)) \cup (X \cap p^{-1}(D))$.
 - ⟨4⟩4. LET: $k : Z_f \rightarrow [0, 1]$ be the continuous function such that $k(p(x)) = h(x)$ for $x \in X$ and $k(p(y)) = g(y)$ for $y \in Y$
PROOF: By the Pasting Lemma
 - ⟨4⟩5. $k(C) = \{0\}$
 - ⟨4⟩6. $k(D) = \{1\}$
 - ⟨3⟩3. Q.E.D.
PROOF: If g is such a continuous function then $g^{-1}([0, 1/2])$ and $g^{-1}((1/2, 1])$ are disjoint open sets that include A and B respectively.
- ⟨2⟩8. PICK a retraction $r : Z_f \rightarrow p(Y)$
- ⟨2⟩9. $p^{-1} \circ r \circ p : X \rightarrow Y$ extends f
- ⟨1⟩2. If Y has the universal extension property then Y is an absolute retract.
 - ⟨2⟩1. ASSUME: Y has the universal extension property
 - ⟨2⟩2. LET: Z be a normal space, Y_0 a closed subspace of Z , and $\phi : Y \cong Y_0$ a homeomorphism
 - ⟨2⟩3. PICK a continuous extension $f : Z \rightarrow Y$ of ϕ^{-1}

□ $\langle 2 \rangle 4$. $\phi \circ f$ is a retraction

Theorem 13.1.27. *Every manifold is metrizable.*

PROOF:

$\langle 1 \rangle 1$. LET: X be an m -manifold.

$\langle 1 \rangle 2$. X is regular.

$\langle 2 \rangle 1$. X is T_1

$\langle 2 \rangle 2$. LET: $x \in X$ and U be a neighbourhood of x

$\langle 2 \rangle 3$. PICK a neighbourhood V of x that is imbeddable in \mathbb{R}^m

$\langle 2 \rangle 4$. PICK a neighbourhood W of x such that $\overline{W} \subseteq U \cap V$

PROOF: One exists since V is regular (Proposition 6.3.4)

$\langle 2 \rangle 5$. $x \in W$ and $\overline{W} \subseteq U$

$\langle 2 \rangle 6$. Q.E.D.

PROOF: Proposition 6.3.2

$\langle 1 \rangle 3$. Q.E.D.

PROOF: By the Urysohn Metrization Theorem.

□

Theorem 13.1.28. *Let X be a compact Hausdorff space in which every point has a neighbourhood that is imbeddable in \mathbb{R}^m . Then X is an m -manifold.*

PROOF:

$\langle 1 \rangle 1$. There exists N such that X is imbeddable in \mathbb{R}^N

PROOF: Theorem 11.1.5

$\langle 1 \rangle 2$. X is second countable.

PROOF: Proposition 7.3.3

□

Proposition 13.1.29. *S_Ω is locally metrizable.*

PROOF: For any $\alpha \in S_\Omega$, the neighbourhood $[0, \alpha] = (-\infty, \alpha + 1)$ is imbeddable in \mathbb{R} . □

Proposition 13.1.30 (DC). *$\overline{S_\Omega}$ is compact.*

PROOF:PROOF:

$\langle 1 \rangle 1$. LET: \mathcal{A} be an open cover of $\overline{S_\Omega}$

$\langle 1 \rangle 2$. ASSUME: for a contradiction there is no finite subcover of \mathcal{A}

$\langle 1 \rangle 3$. There exists a sequence of sets $U_n \in \mathcal{A}$ and ordinals α_n such that $\alpha_{n+1} < \alpha_n$ for all n and $\alpha_n \in U_n$ for all n

$\langle 2 \rangle 1$. LET: $\alpha_1 = \Omega$

$\langle 2 \rangle 2$. Given $\alpha_1, \dots, \alpha_n$ and U_1, \dots, U_{n-1} with $0 \neq \alpha_n < \alpha_{n-1} < \dots < \alpha_1$ and $\alpha_i \in U_i$ for $i < n$, PICK $U_n \in \mathcal{A}$ with $\alpha_n \in U_n$

PROOF: By $\langle 1 \rangle 1$.

$\langle 2 \rangle 3$. PICK $\alpha_{n+1} < \alpha_n$ such that $(\alpha_{n+1}, \alpha_n] \subseteq U_n$

PROOF: By Lemma 4.1.2.

$\langle 2 \rangle 4$. $\alpha_{n+1} \neq 0$

PROOF: If $\alpha_{n+1} = 0$ then U_1, \dots, U_n cover $\overline{S_\Omega}$, contradicting $\langle 1 \rangle 2$.
 $\langle 1 \rangle 4$. Q.E.D.

PROOF: This is a contradiction because the ordinals are well-ordered.
 \square

Proposition 13.1.31. \mathbb{R}_l is not limit point compact.

PROOF: \mathbb{Z} has no limit point. \square

Proposition 13.1.32. Every closed subspace of a Lindelöf space is Lindelöf.

PROOF:

- $\langle 1 \rangle 1$. LET: X be Lindelöf and $A \subseteq X$ be closed
 - $\langle 1 \rangle 2$. LET: \mathcal{U} be an open covering of A
 - $\langle 1 \rangle 3$. $\{U \text{ open in } X : U \cap A \in \mathcal{U}\} \cup \{X \setminus A\}$ covers X
 - $\langle 1 \rangle 4$. PICK a countable subcovering \mathcal{V}
 - $\langle 1 \rangle 5$. $\{U \cap A : U \in \mathcal{V}, U \neq X \setminus A\}$ is a countable subcover of \mathcal{U}
- \square

Proposition 13.1.33. \mathbb{R}^ω is locally connected.

PROOF: This holds because every basic open set is connected, being the product of a family of connected spaces. \square

Proposition 13.1.34. The space \mathbb{R}^ω under the box topology is not first countable.

PROOF:

- $\langle 1 \rangle 1$. ASSUME: for a contradiction $\{U_n\}_{n \geq 0}$ is a countable basis at 0.
 - $\langle 1 \rangle 2$. For $n \geq 1$, PICK a basic open set $B_n = \prod_{j=0}^\infty (a_{nj}, b_{nj})$ such that $0 \in B_n \subseteq U_n$
 - $\langle 1 \rangle 3$. $\prod_{n=0}^\infty (a_{nn}/2, b_{nn}/2)$ is a neighbourhood of 0 that does not include any U_n
- \square

Proposition 13.1.35. The space \mathbb{R}^ω under the box topology is not locally metrizable.

PROOF:

- $\langle 1 \rangle 1$. LET: U be any neighbourhood of 0
- $\langle 1 \rangle 2$. LET: A be the set of all sequences in U with all coordinates positive
- $\langle 1 \rangle 3$. $0 \in \overline{A}$
- $\langle 1 \rangle 4$. There is no sequence of points of A converging to 0.
- $\langle 1 \rangle 5$. U is not metrizable.

PROOF: By the Sequence Lemma.
 \square

Proposition 13.1.36. For any nonempty set I , the space \mathbb{R}^I is not limit point compact.

PROOF: \mathbb{Z}^I is an infinite set with no limit point. \square

Proposition 13.1.37. *The space $\mathbb{R}^{[0,1]}$ is separable.*

PROOF: The set D is dense where D is the set of all functions $f : [0, 1] \rightarrow \mathbb{Q}$ such that there exists a sequence of rationals $0 = q_0 < q_1 < \cdots < q_N = 1$ such that f is constant on $[q_i, q_{i+1})$ for $0 \leq i < N$. \square

Proposition 13.1.38. *If J is uncountable then \mathbb{R}^J is not locally metrizable.*

PROOF: Every point has a neighbourhood homeomorphic to \mathbb{R}^J . \square

Proposition 13.1.39. *The space \mathbb{R}_K is not limit point compact.*

PROOF: The set \mathbb{Z} has no limit point. \square

Proposition 13.1.40. *The topologist's sine curve is not locally connected.*

PROOF: There is no connected neighbourhood of $(0, 0)$. \square

Corollary 13.1.40.1. *Not every metric space is locally connected.*

Corollary 13.1.40.2. *Not every metric space is locally path connected.*

Proposition 13.1.41. *Not every metric space is compact.*

PROOF: The space \mathbb{R} is not compact. \square

Proposition 13.1.42. *Every closed subspace of a limit point compact space is limit point compact.*

PROOF:

$\langle 1 \rangle 1$. LET: X be a limit point compact space and $C \subseteq X$ be closed.

$\langle 1 \rangle 2$. LET: $A \subseteq C$ be infinite.

$\langle 1 \rangle 3$. PICK a limit point l of A in X

$\langle 1 \rangle 4$. $l \in C$

$\langle 2 \rangle 1$. l is a limit point of C

PROOF: By Lemma 3.10.2.

$\langle 2 \rangle 2$. Q.E.D.

PROOF: By Corollary 3.10.3.1.

$\langle 1 \rangle 5$. l is a limit point of A in C .

PROOF: By Proposition 4.3.10.

\square

Proposition 13.1.43. *For any part $i : S \hookrightarrow X$ of a set X , we have $\emptyset \subseteq_X i$.*

PROOF: We have $i \circ i_S = i_X$ by the uniqueness of i_X . \square