

Topology

Robin Adams

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Contents

1	Set Theory	4
1.1	Primitive Notions	4
1.2	The Axiom of Associativity	4
1.3	Injective Functions	4
1.4	Surjective Functions	5
1.5	Retractions and Sections	5
1.6	Identity Functions	6
1.6.1	Isomorphisms	7
1.6.2	Parts of a Set	7
1.6.3	The Empty Set	9
1.6.4	The Terminal Set	10
1.6.5	Products	11
1.6.6	Coproducts	12
1.6.7	Equalizers	12
1.6.8	Coequalizers	12
1.6.9	Pullbacks	13
1.6.10	Function Sets	15
1.6.11	The Subset Classifier	15
1.7	The Basics	18
1.8	Refinements	19
1.9	Order Theory	19
2	Real Analysis	22
3	Topological Spaces	23
3.1	Topologies	23
3.2	Neighbourhoods	24
3.3	Open Refinements	26
3.4	Local Bases	26
3.5	Bases	26
3.6	Closed Sets	31
3.7	Closed Refinements	32
3.8	Locally Finite Families	32
3.9	Countably Locally Finite Sets	32

3.10	Closure of a Set	33
3.11	Interior of a Set	35
3.12	Boundary	36
3.13	Limit Points	37
3.14	Subbases	37
3.15	Convergence	38
3.16	Accumulation Points	40
3.17	Dense Sets	41
3.18	G_δ Sets	41
3.19	Separated Sets	41
3.20	Coherent Topology	41
4	Constructions of Topological Spaces	42
4.1	The Order Topology	42
4.2	The Product Topology	46
4.3	The Subspace Topology	47
4.4	The Box Topology	51
4.5	The Quotient Topology	53
5	Functions Between Topological Spaces	57
5.1	Open Maps	57
5.2	Continuous Functions	58
5.2.1	Homeomorphisms	62
5.2.2	Strongly Continuous Functions	63
5.3	Closed Maps	64
5.4	Local Homeomorphism	64
5.5	Retracts	64
6	Separation Axioms	65
6.1	T_1 Spaces	65
6.2	Hausdorff Spaces	66
6.3	Regular Spaces	68
6.4	Completely Regular Spaces	71
6.5	Normal Spaces	74
6.6	Completely Normal Spaces	86
6.7	Perfectly Normal Spaces	89
7	Countability Axioms	90
7.1	The First Countability Axiom	90
7.2	Separable Spaces	93
7.3	The Second Countability Axiom	95
8	Connectedness	99
8.1	Connected Spaces	99
8.2	Components and Local Connectedness	105
8.3	Path Connectedness	106

8.4	Connected Subspaces of Euclidean Space	109
8.5	Local Connectedness	111
8.6	Local Path Connectedness	113
8.7	Weak Local Connectedness	116
9	Compact Spaces	117
9.1	Countable Compactness	117
9.2	Limit Point Compactness	117
9.3	Lindelöf Spaces	118
9.4	Compactness	120
9.5	Perfect Maps	131
9.6	Sequential Compactness	133
9.7	Local Compactness	133
9.8	Compactifications	136
10	Metric Spaces	143
10.1	The Metric Topology	143
10.2	Isometries	161
10.3	Lebesgue Numbers	161
10.4	Uniform Continuity	164
10.5	Locally Metrizable Spaces	164
11	Manifolds	168
11.1	Manifolds	168
12	Normed Spaces	171
12.1	The Norm on \mathbb{R}^n	171
13	Topological Groups	175
13.1	Topological Groups	175

Chapter 1

Set Theory

1.1 Primitive Notions

Let there be *sets*.

Given sets A and B , let there be *functions* from A to B . We write $f : A \rightarrow B$ iff f is a function from A to B .

Given functions $f : A \rightarrow B$ and $g : B \rightarrow C$, let there be a function $g \circ f : A \rightarrow C$, the *composite* of f and g .

1.2 The Axiom of Associativity

Axiom 1.2.1 (Associativity). Let $f : A \rightarrow B$, $g : B \rightarrow C$ and $h : C \rightarrow D$. Then $h \circ (g \circ f) = (h \circ g) \circ f : A \rightarrow D$.

From now on we write $h \circ g \circ f$ for the composite of f , g and h , and similarly for more than three functions.

1.3 Injective Functions

Definition 1.3.1 (Injective). A function $f : A \rightarrow B$ is *injective*, $f : A \rightarrowtail B$, iff, for every set X and functions $g, h : X \rightarrow A$, if $f \circ g = f \circ h$ then $g = h$.

Proposition 1.3.2. Let $f : A \rightarrow B$ and $g : B \rightarrow C$. If f and g are injective then $g \circ f$ is injective.

PROOF:

$\langle 1 \rangle 1$. ASSUME: f and g are injective.

$\langle 1 \rangle 2$. LET: X be a set and $x, y : X \rightarrow A$

$\langle 1 \rangle 3$. ASSUME: $g \circ f \circ x = g \circ f \circ y$

$\langle 1 \rangle 4$. $f \circ x = f \circ y$

PROOF: g is injective ($\langle 1 \rangle 1$)

$\langle 1 \rangle 5$. $x = y$

PROOF: f is injective ($\langle 1 \rangle 1$)
 \square

Lemma 1.3.3. *Let $f : A \rightarrow B$ and $g : B \rightarrow C$. If $g \circ f$ is injective then f is injective.*

PROOF:
 $\langle 1 \rangle 1$. ASSUME: $g \circ f$ is injective.
 $\langle 1 \rangle 2$. LET: X be any set and $x, y : X \rightarrow A$
 $\langle 1 \rangle 3$. ASSUME: $f \circ x = f \circ y$
 $\langle 1 \rangle 4$. $g \circ f \circ x = g \circ f \circ y$
 $\langle 1 \rangle 5$. $x = y$
 PROOF: Using $\langle 1 \rangle 1$.
 \square

1.4 Surjective Functions

Definition 1.4.1 (Surjective). Let $f : A \rightarrow B$. Then f is *surjective*, $f : A \rightarrow B$, iff, for any set X and functions $g, h : B \rightarrow X$, if $g \circ f = h \circ f$ then $g = h$.

Lemma 1.4.2. *Let $f : A \rightarrow B$ and $g : B \rightarrow C$. If f and g are surjective then $g \circ f$ is surjective.*

PROOF: Dual to Lemma 1.3.2. \square

Lemma 1.4.3. *Let $f : A \rightarrow B$ and $g : B \rightarrow C$. If $g \circ f$ is surjective then g is surjective.*

PROOF: Dual to Lemma 1.3.3. \square

1.5 Retractions and Sections

Definition 1.5.1 (Retraction, Section). Let $r : A \rightarrow B$ and $s : B \rightarrow A$. Then r is a *retraction* of s , and s is a *section* of r , iff $r \circ s = \text{id}_B$.

Proposition 1.5.2. *If $r_1 : A \rightarrow B$ is a retraction of $s_1 : B \rightarrow A$ and $r_2 : B \rightarrow C$ is a retraction of $s_2 : C \rightarrow B$ then $r_2 \circ r_1$ is a retraction of $s_1 \circ s_2$.*

PROOF:

$$\begin{aligned} r_2 \circ r_1 \circ s_1 \circ s_2 &= r_2 \circ \text{id}_B \circ s_2 && (r_1 \text{ is a retraction of } s_1) \\ &= r_2 \circ s_2 && (\text{Unit Laws}) \\ &= \text{id}_C && (r_2 \text{ is a retraction of } s_2) \end{aligned}$$

\square

Proposition 1.5.3. *Every section is injective.*

PROOF:
 $\langle 1 \rangle 1$. LET: $s : A \rightarrow B$ be a section of $r : B \rightarrow A$

$\langle 1 \rangle 2$. LET: $x, y : X \rightarrow A$ satisfy $s \circ x = s \circ y$

$\langle 1 \rangle 3$. $x = y$

PROOF:

$$\begin{aligned}
 x &= \text{id}_A \circ x && \text{(Left Unit Law)} \\
 &= r \circ s \circ x && (\langle 1 \rangle 1) \\
 &= r \circ s \circ y && (\langle 1 \rangle 2) \\
 &= \text{id}_A \circ y && (\langle 1 \rangle 1) \\
 &= y && \text{(Left Unit Law)}
 \end{aligned}$$

□

Proposition 1.5.4. *Every retraction is surjective.*

PROOF: Dual. □

1.6 Identity Functions

Axiom 1.6.1 (Identity Function). *For any set A , there exists a function $\text{id}_A : A \rightarrow A$, the identity function on A , such that:*

Left Unit Law *for every set B and function $f : B \rightarrow A$ we have $\text{id}_A \circ f = f : B \rightarrow A$;*

Right Unit Law *for every set B and function $f : A \rightarrow B$ we have $f \circ \text{id}_A = f : A \rightarrow B$.*

Proposition 1.6.2. *The identity function on a set is unique.*

PROOF: If $i, j : A \rightarrow A$ are both identity functions, then

$$\begin{aligned}
 i &= i \circ j && \text{(Right Unit Law for } j) \\
 &= j && \text{(Left Unit Law for } i) \\
 &: A \rightarrow A && \square
 \end{aligned}$$

Proposition 1.6.3. *Every identity function is injective.*

PROOF: If $\text{id}_B \circ x = \text{id}_B \circ y$ then $x = y$ by the Left Unit Law. □

Proposition 1.6.4. *Every identity function is surjective.*

PROOF: Dual. □

Proposition 1.6.5. *Every identity function is a retraction of itself.*

PROOF: Immediate from the Unit Laws. □

Proposition 1.6.6. *If $r : B \rightarrow A$ is a retraction of $f : A \rightarrow B$ and s is a section of f then $r = s$.*

PROOF:

$$\begin{aligned}
r &= r \circ \text{id}_B && \text{(Right Unit Law)} \\
&= r \circ f \circ s && (s \text{ is a section of } f) \\
&= \text{id}_A \circ s && (r \text{ is a retraction of } f) \\
&= s && \text{(Left Unit Law)}
\end{aligned}$$

1.6.1 Isomorphisms

Definition 1.6.7 (Isomorphism). Let A and B be sets. A function $i : A \rightarrow B$ is an *isomorphism* between A and B , $i : A \cong B$, iff there exists a function $i^{-1} : B \rightarrow A$, the *inverse* to i , that is a section and a retraction of i .

Proposition 1.6.8. *The inverse of an isomorphism is unique.*

PROOF: Immediate from Proposition 1.6.6. \square

Proposition 1.6.9. *Every isomorphism is injective.*

PROOF: Immediate from Proposition 1.5.3. \square

Proposition 1.6.10. *Every isomorphism is surjective.*

PROOF: Immediate from Proposition 1.5.4. \square

Proposition 1.6.11. *Every identity function is an isomorphism and is its own inverse.*

PROOF: Immediate from Proposition 1.6.5. \square

Proposition 1.6.12. *If $i : A \cong B$ is an isomorphism then $i^{-1} : B \cong A$ is an isomorphism and $(i^{-1})^{-1} = i$.*

PROOF: Immediate from the definition of isomorphism. \square

Proposition 1.6.13. *If $i : A \cong B$ and $j : B \cong C$ then $j \circ i : A \cong C$ and $(j \circ i)^{-1} = i^{-1} \circ j^{-1}$.*

PROOF: Immediate from Proposition 1.5.2. \square

1.6.2 Parts of a Set

Definition 1.6.14 (Part). A *part* S of a set A consists of:

- a set $\text{dom } S$;
- an injective function $i : S \hookrightarrow A$

Definition 1.6.15. Two parts $i : S \hookrightarrow A$, $j : T \hookrightarrow A$ are *equivalent*, $i \equiv_A j$, iff there exists an isomorphism $\phi : S \cong T$ such that $i = j \circ \phi$.

Proposition 1.6.16. *Any part of a set is equivalent to itself.*

PROOF:

$\langle 1 \rangle 1$. LET: $i : S \hookrightarrow A$ be a part of A .

PROVE: $i \equiv_A i$

$\langle 1 \rangle 2$. $\text{id}_S : S \cong S$

PROOF: By Proposition 1.6.11

$\langle 1 \rangle 3$. $i = i \circ \text{id}_S$

PROOF: By the Right Unit Law.

□

Proposition 1.6.17. *If $i \equiv_A j$ then $j \equiv_A i$.*

PROOF:

$\langle 1 \rangle 1$. LET: $i : S \hookrightarrow A$ and $j : T \hookrightarrow A$

$\langle 1 \rangle 2$. ASSUME: $i \equiv_A j$

$\langle 1 \rangle 3$. PICK an isomorphism $\phi : S \cong T$ such that $i = j \circ \phi$

$\langle 1 \rangle 4$. $\phi^{-1} : T \cong S$

PROOF: By Proposition 1.6.12.

$\langle 1 \rangle 5$. $j = i \circ \phi^{-1}$

PROOF: Compose both sides of $\langle 1 \rangle 3$ with ϕ^{-1} .

□

Proposition 1.6.18. *If $i \equiv_A j$ and $j \equiv_A k$ then $i \equiv_A k$.*

PROOF:

$\langle 1 \rangle 1$. LET: $i : R \hookrightarrow A$, $j : S \hookrightarrow A$ and $k : T \rightarrow A$

$\langle 1 \rangle 2$. PICK isomorphisms $\phi : R \cong S$ and $\psi : S \cong T$ such that $i = j \circ \phi$ and $j = k \circ \psi$

$\langle 1 \rangle 3$. $\psi \circ \phi : R \cong T$

PROOF: By Proposition 1.6.13.

$\langle 1 \rangle 4$. $i = k \circ \psi \circ \phi$

□

Definition 1.6.19 (Inclusion). Let $i : U \hookrightarrow A$ and $j : V \hookrightarrow A$ be parts of A . Then i is *included* in j , $i \subseteq_A j$, iff there exists a function $\phi : U \rightarrow V$ such that $i = j \circ \phi$.

Proposition 1.6.20. *If $i \equiv_A i'$ and $j \equiv_A j'$ and $i \subseteq_A j$ then $i' \subseteq_A j'$.*

PROOF:

$\langle 1 \rangle 1$. LET: $i : S \hookrightarrow A$, $i' : S' \hookrightarrow A$, $j : T \hookrightarrow A$, $j' : T' \hookrightarrow A$

$\langle 1 \rangle 2$. PICK $\phi : S \cong S'$, $\psi : T \cong T'$ and $\chi : S \rightarrow T$ such that $i = i' \circ \phi$, $j = j' \circ \psi$ and $i = j \circ \chi$

$\langle 1 \rangle 3$. $\psi \circ \chi \circ \phi^{-1} : S' \rightarrow T'$

$\langle 1 \rangle 4$. $i' = j' \circ \psi \circ \chi \circ \phi^{-1}$

□

Proposition 1.6.21. *For any part i of A we have $i \subseteq_A i$.*

PROOF:

- $\langle 1 \rangle 1.$ LET: $i : S \hookrightarrow A$
 - $\langle 1 \rangle 2.$ $\text{id}_S : S \rightarrow S$
 - $\langle 1 \rangle 3.$ $i = i \circ \text{id}_S$
-

Proposition 1.6.22. *If $i \subseteq_A j$ and $j \subseteq_A k$ then $i \subseteq_A k$.*

PROOF:

- $\langle 1 \rangle 1.$ LET: $i : R \hookrightarrow A$, $j : S \hookrightarrow A$ and $k : T \hookrightarrow A$
 - $\langle 1 \rangle 2.$ PICK $\phi : R \rightarrow S$ and $\psi : S \rightarrow T$ such that $i = j \circ \phi$ and $j = k \circ \psi$
 - $\langle 1 \rangle 3.$ $\psi \circ \phi : R \rightarrow T$
 - $\langle 1 \rangle 4.$ $i = k \circ \psi \circ \phi$
-

Proposition 1.6.23. *If $i \subseteq_A j$ and $j \subseteq_A i$ then $i \equiv_A j$.*

PROOF:

- $\langle 1 \rangle 1.$ LET: $i : R \hookrightarrow A$, $j : S \hookrightarrow A$
- $\langle 1 \rangle 2.$ PICK $\phi : R \rightarrow S$ and $\phi^{-1} : S \rightarrow R$ such that $i = j \circ \phi$ and $j = i \circ \phi^{-1}$
- $\langle 1 \rangle 3.$ $\phi \circ \phi^{-1} = \text{id}_S$
 - $\langle 2 \rangle 1.$ $j \circ \phi \circ \phi^{-1} = j$
 - $\langle 2 \rangle 2.$ Q.E.D.

PROOF: The result follows because j is injective.

- $\langle 1 \rangle 4.$ $\phi^{-1} \circ \phi = \text{id}_R$

PROOF: Similar.

□

1.6.3 The Empty Set

Axiom 1.6.24 (Empty Set). *There exists a set \emptyset , the empty set, such that, for every set X , there exists a unique function $\text{id}_X : \emptyset \rightarrow X$.*

Proposition 1.6.25 (Uniqueness of Empty Set). *Let E be any set. Then E is empty if and only if there exists an isomorphism $E \cong \emptyset$, in which case the isomorphism is unique.*

PROOF:

- $\langle 1 \rangle 1.$ If E is empty then $E \cong \emptyset$
 - $\langle 2 \rangle 1.$ ASSUME: E is empty
 - $\langle 2 \rangle 2.$ LET: ϕ be the unique function $E \rightarrow \emptyset$
 - $\langle 2 \rangle 3.$ $\text{id}_E \circ \phi = \text{id}_E$
 - PROOF: There is only one function $E \rightarrow E$.
 - $\langle 2 \rangle 4.$ $\phi \circ \text{id}_E = \text{id}_\emptyset$
 - PROOF: There is only one function $\emptyset \rightarrow \emptyset$.
- $\langle 1 \rangle 2.$ If $E \cong \emptyset$ then E is empty
 - $\langle 2 \rangle 1.$ LET: $\phi : E \cong \emptyset$
 - $\langle 2 \rangle 2.$ LET: X be a set
 - PROVE: There is a unique function $E \rightarrow X$

(2)3. $i_X \circ \phi : E \rightarrow X$
 (2)4. If $f : E \rightarrow X$ then $f = i_X \circ \phi$
 (3)1. LET: $f : E \rightarrow X$
 (3)2. $f \circ \phi^{-1} : \emptyset \rightarrow X$
 (3)3. $f \circ \phi^{-1} = i_X$
 PROOF: Uniqueness of i_X .
 (3)4. Q.E.D.
 (1)3. There is at most one isomorphism $E \cong \emptyset$
 PROOF: This holds because there is at most one function $E \rightarrow \emptyset$.
 \square

Proposition 1.6.26.

$$i_\emptyset = \text{id}_\emptyset$$

PROOF: By the uniqueness of i_\emptyset . \square

1.6.4 The Terminal Set

Axiom 1.6.27 (Terminal Set). *There exists a set 1 , the terminal set, such that, for every set X , there exists a unique function $!_X : X \rightarrow 1$.*

Proposition 1.6.28 (Uniqueness of Terminal Set). *Let T be any set. Then T is terminal if and only if there exists an isomorphism $T \cong 1$, in which case the isomorphism is unique.*

PROOF: Dual to Proposition 1.6.25.

Proposition 1.6.29.

$$!_1 = \text{id}_1$$

PROOF: From the uniqueness of $!_1$. \square

Definition 1.6.30 (Element). An *element* of a set A is a function $1 \rightarrow A$. We write $a \in A$ for $a : 1 \rightarrow A$. We write $f(a)$ for $f \circ a$ when $f : A \rightarrow B$ and $a \in A$.

Axiom 1.6.31 (Extensionality). *Let A and B be sets and $f, g : A \rightarrow B$ be functions. If, for all $a \in A$, we have $f(a) = g(a) \in B$, then $f = g$.*

Proposition 1.6.32. *Let $f : A \rightarrow B$. Then f is injective if and only if, for all $x, y \in A$, if $f(x) = f(y) \in B$ then $x = y \in A$.*

PROOF:

(1)1. If f is injective and $f(x) = f(y) \in B$ then $x = y \in A$
 PROOF: Immediate from the definition of injective.
 (1)2. If, for all $x, y \in A$, if $f(x) = f(y) \in B$ then $x = y \in A$
 (2)1. ASSUME: For all $x, y \in A$, if $f(x) = f(y)$, then $x = y$
 (2)2. LET: X be any set and $g, h : X \rightarrow A$ with $f \circ g = f \circ h$
 PROVE: $g = h$
 (2)3. LET: $x \in X$
 PROVE: $g(x) = h(x)$

$\langle 2 \rangle 4. f(g(x)) = f(h(x))$

PROOF: From $\langle 2 \rangle 2$.

$\langle 2 \rangle 5. g(x) = h(x)$

PROOF: By $\langle 2 \rangle 1$

□

Proposition 1.6.33. *Any element $e \in X$ is a section of the unique function $!_X : X \rightarrow 1$.*

PROOF: $!_X \circ e = \text{id}_1$ because there is only one function $1 \rightarrow 1$. □

Axiom 1.6.34 (Non-degeneracy). *The empty set \emptyset has no elements.*

Proposition 1.6.35. *For any set X , the function $!_X : \emptyset \rightarrow X$ is injective.*

PROOF: From Proposition 1.6.32. □

Definition 1.6.36 (Empty Part). For any set X , the *empty part* of X is $\emptyset = !_X : \emptyset \hookrightarrow X$.

Definition 1.6.37 (Constant Function). A function $f : A \rightarrow B$ is *constant* iff there exists $b \in B$ such that $f = b \circ !_A$.

Definition 1.6.38 (Membership). Let $i : U \hookrightarrow A$ be a part of A and $a \in A$. Then a is a *member* of i , $a \in_A i$, iff there exists $\bar{a} \in U$ such that $i(\bar{a}) = a$.

Proposition 1.6.39. *Let A be a set. Let i, j be parts of A and $a \in A$. If $a \in_A i$ and $i \subseteq_A j$ then $a \in_A j$.*

PROOF:

$\langle 1 \rangle 1$. PICK $\bar{a} \in \text{dom } i$ such that $a = i(\bar{a})$.

$\langle 1 \rangle 2$. PICK $\phi : \text{dom } i \rightarrow \text{dom } j$ such that $i = j \circ \phi$

$\langle 1 \rangle 3$. $a = j(\phi(\bar{a}))$

□

1.6.5 Products

Axiom 1.6.40 (Products). *For any sets A and B , there exists a set $A \times B$, the product of A and B , and functions $\pi_1 : A \times B \rightarrow A$, $\pi_2 : A \times B \rightarrow B$, the projections, such that, for any set C and functions $f : C \rightarrow A$, $g : C \rightarrow B$, there exists a unique function $\langle f, g \rangle : C \rightarrow A \times B$ such that*

$$\pi_1 \circ \langle f, g \rangle = f; \quad \pi_2 \circ \langle f, g \rangle = g .$$

Definition 1.6.41. Given functions $f : A \rightarrow B$ and $g : C \rightarrow D$, define $f \times g : A \times C \rightarrow B \times D$ by

$$f \times g = \langle f \circ \pi_1, g \circ \pi_2 \rangle$$

1.6.6 Coproducts

Axiom 1.6.42 (Coproducts). *For any sets A and B , there exists a set $A \uplus B$, the coproduct or sum of A and B , and functions $\kappa_1 : A \rightarrow A \uplus B$, $\kappa_2 : B \rightarrow A \uplus B$, the injections, such that, for any set C and functions $f : A \rightarrow C$, $g : B \rightarrow C$, there exists a unique function $[f, g] : A \uplus B \rightarrow C$ such that*

$$[f, g] \circ \kappa_1 = f; \quad [f, g] \circ \kappa_2 = g \quad .$$

Definition 1.6.43 (Complement). Let $i : I \hookrightarrow J$ and $i' : I' \hookrightarrow J$ be parts of J . Then i' is the *complement* of i iff J is the sum of I and I' with injections i and i' .

1.6.7 Equalizers

Axiom 1.6.44 (Equalizers). *For any sets A and B and functions $f, g : A \rightarrow B$, there exists a set E and function $e : E \rightarrow A$, the equalizer of A and B , such that:*

- $f \circ e = g \circ e : E \rightarrow B$;
- For any set C and function $h : C \rightarrow A$ such that $f \circ h = g \circ h$, there exists a unique function $\bar{h} : C \rightarrow E$ such that $h = e \circ \bar{h}$.

Proposition 1.6.45. *All equalizers are injective.*

PROOF:

$\langle 1 \rangle 1$. LET: $e : E \rightarrow A$ be the equalizer of $f, g : A \rightarrow B$

$\langle 1 \rangle 2$. LET: $x, y : X \rightarrow E$ with $e \circ x = e \circ y$

$\langle 1 \rangle 3$. $f \circ e \circ x = g \circ e \circ x$

PROOF: $f \circ e = g \circ e$ by $\langle 1 \rangle 1$.

$\langle 1 \rangle 4$. $x = y$

PROOF: x and y are both the unique $z : X \rightarrow E$ such that $e \circ z = e \circ x$.

□

1.6.8 Coequalizers

Axiom 1.6.46 (Coequalizers). *For any sets A and B and functions $f, g : A \rightarrow B$, there exists a set C and function $c : B \rightarrow C$, the coequalizer of f and g , such that:*

- $c \circ f = c \circ g : A \rightarrow C$
- For any set X and function $h : B \rightarrow X$ such that $h \circ f = h \circ g$, there exists a unique function $\bar{h} : C \rightarrow X$ such that $\bar{h} \circ c = h$.

1.6.9 Pullbacks

Definition 1.6.47 (Pullback). The diagram below is a *pullback diagram* iff:

- $f \circ p = g \circ q$
- for every set X and functions $x : X \rightarrow B$ and $y : X \rightarrow C$ such that $f \circ x = g \circ y$, there exists a unique function $\langle x, y \rangle : X \rightarrow A$ such that $p \circ \langle x, y \rangle = x$ and $q \circ \langle x, y \rangle = y$.

$$\begin{array}{ccc} A & \xrightarrow{p} & B \\ q \downarrow & & \downarrow f \\ C & \xrightarrow{g} & D \end{array}$$

Proposition 1.6.48. Let $f : A \rightarrow C$ and $g : B \rightarrow C$. Then f and g have a pullback.

$$\begin{array}{ccccc} E & & & & \\ & \searrow e & & & \\ & A \times B & \xrightarrow{\pi_1} & A & \\ & \pi_2 \downarrow & & \downarrow f & \\ & B & \xrightarrow{g} & C & \end{array}$$

PROOF:

- (1)1. Construct the product $\pi_1 : A \times B \rightarrow A$, $\pi_2 : A \times B \rightarrow B$.
- (1)2. Construct the equalizer $e : E \rightarrow A \times B$ of $f \circ \pi_1$ and $g \circ \pi_2$.
PROVE: $\pi_1 \circ e$ and $\pi_2 \circ e$ form a pullback of f and g
- (1)3. $f \circ \pi_1 \circ e = g \circ \pi_2 \circ e$
- (1)4. LET: X be a set and $x : X \rightarrow A$, $y : X \rightarrow B$ satisfy $f \circ x = g \circ y$
- (1)5. $f \circ \pi_1 \circ \langle x, y \rangle = g \circ \pi_2 \circ \langle x, y \rangle$
- (1)6. LET: $m : X \rightarrow E$ be the function such that $e \circ m = \langle x, y \rangle$
- (1)7. $\pi_1 \circ e \circ m = x$ and $\pi_2 \circ e \circ m = y$
- (1)8. m is unique.

PROOF:

- (2)1. LET: $n : X \rightarrow E$ be such that $\pi_1 \circ e \circ n = x$ and $\pi_2 \circ e \circ n = y$
- (2)2. $e \circ n = \langle x, y \rangle$
- (2)3. $n = m$

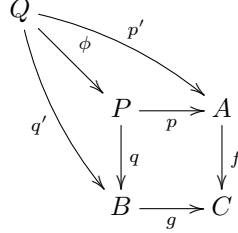
PROOF: By (1)6

□

Proposition 1.6.49. Pullbacks are unique up to isomorphism.

That is, let P be a pullback of $f : A \rightarrow C$ and $g : B \rightarrow C$ with projections $p : P \rightarrow A$ and $q : P \rightarrow B$. Let Q be a set and $p' : Q \rightarrow A$, $q' : Q \rightarrow B$. Then Q is a pullback of f and g with projections p' and q' if and only if there exists

a bijection $\phi : Q \cong P$ such that $p \circ \phi = p'$ and $q \circ \phi = q'$, in which case ϕ is unique.



PROOF:

(1)1. If Q is a pullback then there exists a bijection $\phi : Q \cong P$ such that $p \circ \phi = p'$ and $q \circ \phi = q'$

(2)1. ASSUME: Q is a pullback with projections p' and q'

(2)2. LET: $\phi : Q \rightarrow P$ be the unique function such that $p \circ \phi = p'$ and $q \circ \phi = q'$

PROOF: Such a ϕ exists because $f \circ p' = g \circ q'$.

(2)3. LET: $\phi^{-1} : P \rightarrow Q$ be the unique function such that $p' \circ \phi^{-1} = p$ and $q' \circ \phi^{-1} = q$

PROOF: Such a function exists because $f \circ p = g \circ q$.

(2)4. $\phi \circ \phi^{-1} = \text{id}_P$

PROOF: Each is the unique function x such that $p \circ x = p$ and $q \circ x = q$.

(2)5. $\phi^{-1} \circ \phi = \text{id}_Q$

PROOF: Similar.

(1)2. If $\phi : Q \cong P$ is a bijection then Q is a pullback with projections $p \circ \phi$ and $q \circ \phi$

(2)1. $f \circ p \circ \phi = g \circ q \circ \phi$

PROOF: This holds because $f \circ p = g \circ q$

(2)2. For any set X and functions $x : X \rightarrow A$, $y : X \rightarrow B$ such that $f \circ x = g \circ y$, there exists a unique function $m : X \rightarrow Q$ such that $p \circ \phi \circ m = x$ and $q \circ \phi \circ m = y$

PROOF:

$$p \circ \phi \circ m = x \text{ and } q \circ \phi \circ m = y$$

$$\Leftrightarrow \phi \circ m = \langle x, y \rangle$$

$$\Leftrightarrow m = \phi^{-1} \circ \langle x, y \rangle$$

(1)3. If $\phi, \phi' : P \cong Q$ are bijections such that $p \circ \phi = p \circ \phi'$ and $q \circ \phi = q \circ \phi'$

PROOF: This follows from the definition of pullback.

□

Proposition 1.6.50. *The pullback of an injective function is injective.*

That is, if the diagram below is a pullback diagram and f is injective then q

is injective.

$$\begin{array}{ccc} A & \xrightarrow{p} & B \\ q \downarrow & & \downarrow f \\ C & \xrightarrow{g} & D \end{array}$$

PROOF:

$\langle 1 \rangle 1$. LET: X be a set and $x, y : X \rightarrow A$ with $q \circ x = q \circ y$

$\langle 1 \rangle 2$. $f \circ p \circ x = g \circ q \circ x$

$\langle 1 \rangle 3$. LET:

$z : X \rightarrow A$ be the function such that $p \circ z = p \circ x$ and $q \circ z = q \circ x$

$\langle 1 \rangle 4$. $z = x$

$\langle 1 \rangle 5$. $z = y$

$\langle 2 \rangle 1$. $q \circ x = q \circ y$

PROOF: By $\langle 1 \rangle 1$.

$\langle 2 \rangle 2$. $f \circ p \circ x = f \circ p \circ y$

PROOF:

$$f \circ p \circ x = g \circ q \circ x \quad (\langle 1 \rangle 2)$$

$$= g \circ q \circ y \quad (\langle 1 \rangle 1)$$

$$= f \circ p \circ y \quad (\text{the diagram is a pullback})$$

$\langle 2 \rangle 3$. $p \circ x = p \circ y$

PROOF: f is injective.

□

1.6.10 Function Sets

Axiom 1.6.51 (Function Sets). *For any sets A and B , there exists a set A^B and a function $\epsilon : A^B \times B \rightarrow A$, the evaluation function, such that, for any set C and function $f : C \times B \rightarrow A$, there exists a unique function $\lambda f : C \rightarrow A^B$ such that*

$$\epsilon \circ (\lambda f \times \text{id}_B) = f \quad .$$

1.6.11 The Subset Classifier

Definition 1.6.52. The set 2 is $1 + 1$. We write \top (*truth*) for $\kappa_1 : 1 \rightarrow 2$, and \perp (*falsehood*) for $\kappa_2 : 1 \rightarrow 2$.

Axiom 1.6.53 (Subset Classifier). *For every injective function $m : A \rightarrow B$, there exists a unique function $\chi_m : B \rightarrow 2$, the characteristic function of m , such that the following diagram is a pullback diagram:*

$$\begin{array}{ccc} A & \xrightarrow{!} & 1 \\ m \downarrow & & \downarrow \top \\ B & \xrightarrow{\chi_m} & 2 \end{array}$$

Proposition 1.6.54. *Every function $\phi : A \rightarrow 2$ is the characteristic function of a part of A .*

PROOF:

$\langle 1 \rangle 1$. Construct a pullback

$$\begin{array}{ccc} I & \longrightarrow & 1 \\ \downarrow q & & \downarrow \top \\ A & \xrightarrow{\phi} & 2 \end{array}$$

PROOF: By Proposition 1.6.48.

$\langle 1 \rangle 2$. q is injective

PROOF: By Proposition 1.6.50.

□

Axiom 1.6.55 (Boolean). *For any $p \in 2$ we have $p = \top$ or $p = \perp$.*

Proposition 1.6.56. *Let $i : U \hookrightarrow A$ and $j : V \hookrightarrow A$ be parts of A . Then the following are equivalent:*

1. $i \subseteq_A j$ and $j \subseteq_A i$
2. There exist $h : U \rightarrow V$ and $k : V \rightarrow U$ such that $i = j \circ h$, $j = i \circ k$, $k \circ h = \text{id}_U$ and $h \circ k = \text{id}_V$.
3. The characteristic function of i is the characteristic function of j .

PROOF:

$\langle 1 \rangle 1$. $1 \Rightarrow 2$

$\langle 2 \rangle 1$. ASSUME: $i \subseteq_A j$ and $j \subseteq_A i$

$\langle 2 \rangle 2$. LET: $h : U \rightarrow V$ be such that $i = j \circ h$

$\langle 2 \rangle 3$. LET: $k : V \rightarrow U$ be such that $j = i \circ k$

$\langle 2 \rangle 4$. $k \circ h = \text{id}_U$

$\langle 3 \rangle 1$. $i \circ k \circ h = i$

PROOF: From $\langle 2 \rangle 2$ and $\langle 2 \rangle 3$.

$\langle 3 \rangle 2$. Q.E.D.

PROOF: Since i is injective.

$\langle 2 \rangle 5$. $h \circ k = \text{id}_V$

PROOF: Similar.

$\langle 1 \rangle 2$. $2 \Rightarrow 1$

PROOF: Trivial.

$\langle 1 \rangle 3$. $2 \Rightarrow 3$

$\langle 2 \rangle 1$. ASSUME: 2

$\langle 2 \rangle 2$. LET: $\phi : A \rightarrow 2$ be the characteristic function of i

PROVE: ϕ is the characteristic function of j

⟨3⟩2. Q.E.D.

PROOF: Since ϕ is the characteristic function of j .

⟨2⟩4. LET: $k : V \rightarrow U$ be the unique function such that $! \circ k = !$ and $i \circ k = j$

PROOF: Similar.

⟨2⟩5. $k \circ h = \text{id}_U$

PROOF: Each is the unique function f such that $! \circ f = !$ and $i \circ f = i$

⟨2⟩6. $h \circ k = \text{id}_V$

PROOF: Each is the unique function f such that $! \circ f = !$ and $j \circ f = j$

□

1.7 The Basics

Lemma 1.7.1. *Let X be a set, $\mathcal{B} \subseteq \mathcal{P}X$ and $U \subseteq X$. Then the following are equivalent:*

1. *For all $x \in U$ there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$.*
2. *There exists $\mathcal{B}_0 \subseteq \mathcal{B}$ such that $U = \bigcup \mathcal{B}_0$.*

PROOF:

⟨1⟩1. $1 \Rightarrow 2$

PROOF: If 1 is true then $U = \bigcup \{B \in \mathcal{B} : B \subseteq U\}$.

⟨1⟩2. $2 \Rightarrow 1$

PROOF: Trivial.

□

Definition 1.7.2 (Fixed Point). Let X be a set, $f : X \rightarrow X$, and $x \in X$. Then x is a *fixed point* of f iff $f(x) = x$.

Definition 1.7.3 (Saturated). Let X, Y be sets and $p : X \rightarrow Y$ be a surjective function. Let $C \subseteq X$. Then C is *saturated* with respect to p iff, for all $x, x' \in X$, if $x \in C$ and $p(x) = p(x')$ then $x' \in C$.

Definition 1.7.4 (Cover). Let A be a set and $\mathcal{C} \subseteq \mathcal{P}A$. Then \mathcal{C} *covers* A iff $\bigcup \mathcal{C} = A$.

Definition 1.7.5 (Finite Intersection Property). Let X be a set and $\mathcal{C} \subseteq \mathcal{P}X$. Then \mathcal{C} has the *finite intersection property* if and only if every finite nonempty subset of \mathcal{C} has nonempty intersection.

Lemma 1.7.6 (AC). *Let X be a set and $\mathcal{A} \subseteq \mathcal{P}X$ have the finite intersection property. Then there exists a maximal $\mathcal{D} \subseteq \mathcal{P}X$ that has the finite intersection property and includes \mathcal{A} .*

PROOF: A straightforward application of Zorn's lemma, since the union of a chain of sets that has the finite intersection property has the finite intersection property. □

Lemma 1.7.7. *Let X be a set and $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the finite intersection property. Then any finite intersection of elements of \mathcal{D} is an element of \mathcal{D} .*

PROOF:

- $\langle 1 \rangle 1.$ LET: A be a finite intersection of elements of \mathcal{D}
- $\langle 1 \rangle 2.$ $\mathcal{D} \cup \{A\}$ has the finite intersection property.
- $\langle 1 \rangle 3.$ $\mathcal{D} \cup \{A\} = \mathcal{D}$

□

Lemma 1.7.8. *Let X be a set and $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the finite intersection property. If $A \subseteq X$ intersects every element of \mathcal{D} then $A \in \mathcal{D}$.*

PROOF: This holds because $\mathcal{D} \cup \{A\}$ satisfies the finite intersection property. □

Definition 1.7.9 (Graph). Let $f : A \rightarrow B$. The *graph* of f is the set $\{(x, f(x)) : x \in A\} \subseteq A \times B$.

Definition 1.7.10 (Point-Finite). Let X be a set and $\{A_\alpha\}_{\alpha \in J}$ be a family of subsets of X . Then $\{A_\alpha\}_{\alpha \in J}$ is *point-finite* iff, for all $x \in X$, there are only finitely many $\alpha \in J$ such that $x \in A_\alpha$.

Definition 1.7.11 (Countable Intersection Property). A family of parts of a set X has the *countable intersection property* iff every countable subfamily has nonempty intersection.

1.8 Refinements

Definition 1.8.1 (Refinement). Let X be a set and $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}X$. Then \mathcal{B} is a *refinement* of \mathcal{A} iff, for all $B \in \mathcal{B}$, there exists $A \in \mathcal{A}$ such that $B \subseteq A$.

1.9 Order Theory

Definition 1.9.1 (Cofinal). Let J be a poset and $K \subseteq J$. Then K is *cofinal* iff, for all $x \in J$, there exists $y \in K$ such that $x \leq y$.

Definition 1.9.2 (Directed Set). A *directed set* is a poset J such that, for all $x, y \in J$, there exists $z \in J$ such that $x \leq z$ and $y \leq z$.

Definition 1.9.3 (Linear Order). Let X be a set. A *linear order* on X is a relation $\leq \subseteq X^2$ such that:

- For all $x \in X$, $x \leq x$
- For all $x, y, z \in X$, if $x \leq y$ and $y \leq z$ then $x \leq z$
- For all $x, y \in X$, if $x \leq y$ and $y \leq x$ then $x = y$
- For all $x, y \in X$, we have $x \leq y$ or $y \leq x$

We write $x < y$ iff $x \leq y$ and $x \neq y$.

A *linearly ordered set* consists of a set and a linear order on the set.

Definition 1.9.4 (Convex). Let L be a linearly ordered set and $A \subseteq L$. Then A is *convex* iff, for all $x, y \in A$ and $z \in L$, if $x < z < y$ then $z \in A$.

Definition 1.9.5 (Least Upper Bound Property). A linearly ordered set L has the *least upper bound property* iff every subset of L bounded above has a least upper bound.

Definition 1.9.6 (Linear Continuum). A *linear continuum* is a linearly ordered set L such that:

- L has the least upper bound property.
- For all $x, y \in L$ with $x < y$, there exists $z \in L$ such that $x < z < y$.

Proposition 1.9.7. *If L is a linear continuum then every convex subset of L is a linear continuum.*

PROOF:

<1>1. LET: L be a linear continuum and $C \subseteq L$ be convex

<1>2. C satisfies the least upper bound property.

<2>1. LET: $S \subseteq C$ be nonempty and bounded above by u in C .

<2>2. LET: s be the supremum of S in L

<2>3. PICK $x \in S$

<2>4. $x \leq s \leq u$

<2>5. $s \in C$

PROOF: C is convex.

<2>6. s is the supremum of S in C

<1>3. C is dense.

PROOF:

<2>1. LET: $x, y \in C$ satisfy $x < y$

<2>2. PICK $z \in L$ such that $x < z < y$

<2>3. $z \in C$

PROOF: C is convex.

□

Lemma 1.9.8. *For any real numbers a, b with $a < b$ we have $[a, b] \cong [0, 1]$.*

PROOF: The map $\phi : [a, b] \cong [0, 1]$ where $\phi(x) = (x - a)/(b - a)$ is an order isomorphism. □

Proposition 1.9.9. *Let X be a linearly ordered set. Let $a, b, c \in X$ with $a < c < b$. Then $[a, b] \cong [0, 1]$ if and only if $[a, c] \cong [c, b] \cong [0, 1]$.*

PROOF:

<1>1. If $[a, b] \cong [0, 1]$ then $[a, c] \cong [c, b] \cong [0, 1]$.

<2>1. ASSUME: $\phi : [a, b] \cong [0, 1]$ is an order isomorphism.

<2>2. $[a, c] \cong [0, 1]$

PROOF:

$$\begin{aligned} [a, c] &\cong [0, \phi(c)] && \text{(under } \phi) \\ &\cong [0, 1) && \text{(Lemma 1.9.8)} \end{aligned}$$

(2)3. $[c, b] \cong [0, 1)$

PROOF: Similar.

(1)2. If $[a, c] \cong [c, b] \cong [0, 1)$ then $[a, b] \cong [0, 1)$.

(2)1. ASSUME: $[a, c] \cong [c, b] \cong [0, 1)$

(2)2. LET: $\phi : [a, c] \cong [0, 1/2)$ and $\psi : [c, b] \cong [1/2, 1)$

(2)3. LET: $\chi : [a, b] \rightarrow [0, 1)$ be given by $\chi(x) = \begin{cases} \phi(x) & \text{if } x < c \\ \psi(x) & \text{if } x \geq c \end{cases}$

(2)4. $\chi : [a, b] \cong [0, 1)$

PROOF: Easy to check.

□

Proposition 1.9.10 (CC). *Let X be a linearly ordered set. Let $\{x_n\}_{n \geq 0}$ be an increasing sequence of points of X . Suppose b is the supremum of $\{x_n : n \geq 0\}$. Then $[x_0, b] \cong [0, 1)$ if and only if $[x_i, x_{i+1}] \cong [0, 1)$ for all i .*

PROOF:

(1)1. If $[x_0, b] \cong [0, 1)$ then for all i $[x_i, x_{i+1}] \cong [0, 1)$.

PROOF: If $[x_0, b] \cong [0, 1)$ then $[x_i, x_{i+1}] \cong [\phi(x_i), \phi(x_{i+1})) \cong [0, 1)$ by Lemma 1.9.8.

(1)2. If for all i $[x_i, x_{i+1}] \cong [0, 1)$ then $[x_0, b] \cong [0, 1)$.

PROOF:

(2)1. LET: $\phi_i : [x_i, x_{i+1}] \cong [0, 1)$ for all i

(2)2. Define $\phi : [x_0, b] \cong [0, 1)$ by: $\phi(y) = \phi_i(y)$ ($x_0 \leq y < b$) where i is least such that $y < x_{i+1}$

PROOF: There exists such an i because y is not an upper bound for $\{x_n : n \geq 0\}$.

(2)3. ϕ is an order isomorphism.

PROOF: Easy to check.

□

Proposition 1.9.11 (CC). *For all $0 < \alpha < \Omega$, the interval $[(0, 0), (\alpha, 0))$ in $S_\Omega \times [0, 1)$ is order isomorphic to $[0, 1)$ in \mathbb{R} .*

PROOF:

(1)1. If $[(0, 0), (\alpha, 0)) \cong [0, 1)$ then $[(0, 0), (\alpha + 1, 0)) \cong [0, 1)$

PROOF: By Proposition 1.9.9.

(1)2. Let λ be a limit ordinal, $0 < \lambda < \Omega$. If, for all α with $0 < \alpha < \lambda$, we have $[(0, 0), (\alpha, 0)) \cong [0, 1)$, then $[(0, 0), (\lambda, 0)) \cong [0, 1)$.

PROOF: By Proposition 1.9.10.

(1)3. Q.E.D.

PROOF: By transfinite induction.

□

Chapter 2

Real Analysis

Definition 2.0.1 (Cantor Set). Define a sequence of sets $A_n \subseteq [0, 1]$ by:

$$A_0 = [0, 1]$$
$$A_{n+1} = A_n \setminus \bigcup_{k=0}^{3^{n-1}-1} \left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n} \right)$$

The *Cantor set* is $\bigcap_{n=0}^{\infty} A_n$.

Chapter 3

Topological Spaces

3.1 Topologies

Definition 3.1.1 (Topology). A *topology* on a set X is a set $\mathcal{T} \subseteq \mathcal{P}X$ such that:

1. $X \in \mathcal{T}$;
2. for all $U, V \in \mathcal{T}$, we have $U \cap V \in \mathcal{T}$;
3. For all $\mathcal{A} \subseteq \mathcal{T}$, we have $\bigcup \mathcal{A} \in \mathcal{T}$.

A *topological space* X consists of a set X and a topology on X . The elements of X are called *points* and the elements of \mathcal{T} are called *open sets*.

Proposition 3.1.2. *In any topological space, the empty set is open.*

PROOF: Immediate from axiom 3. \square

Definition 3.1.3 (Discrete Topology). The *discrete* topology on a set X is $\mathcal{P}X$.

Definition 3.1.4 (Indiscrete Topology). The *indiscrete* topology on a set X is $\{\emptyset, X\}$.

Definition 3.1.5 (Open Cover). Let X be a topological space. A cover $\mathcal{C} \subseteq \mathcal{P}X$ of X is an *open cover* iff every member of \mathcal{C} is open.

Definition 3.1.6 (Finer, Coarser). Let $\mathcal{T}, \mathcal{T}'$ be topologies on a set X . Then \mathcal{T} is *finer* than \mathcal{T}' , and \mathcal{T}' is *coarser* than \mathcal{T} , iff $\mathcal{T}' \subseteq \mathcal{T}$.

The topology \mathcal{T} is *strictly finer* than \mathcal{T}' , and \mathcal{T}' is *strictly coarser* than \mathcal{T} , iff $\mathcal{T} \subset \mathcal{T}'$.

The topologies \mathcal{T} and \mathcal{T}' are *comparable* iff $\mathcal{T} \subseteq \mathcal{T}'$ or $\mathcal{T}' \subseteq \mathcal{T}$.

Definition 3.1.7 (Finite Complement Topology). The *finite complement topology* on a set X is $\{U : X \setminus U \text{ is finite}\} \cup \{X\}$.

Definition 3.1.8 (Isolated Point). Let X be a topological space and $a \in X$. Then a is an *isolated point* iff $\{a\}$ is open.

3.2 Neighbourhoods

Definition 3.2.1 (Neighbourhood). Let X be a topological space and $A \subseteq X$. A *neighbourhood* of A is an set that includes an open set that includes A .

A *neighbourhood* of a point a is a neighbourhood of $\{a\}$.

Proposition 3.2.2. *If N is a neighbourhood of A and $B \subseteq A$ then N is a neighbourhood of B .*

PROOF: Immediate from definitions. \square

Proposition 3.2.3. *A set U is open if and only if it is a neighbourhood of each of its points.*

PROOF:

$\langle 1 \rangle 1$. LET: X be a topological space and $A \subseteq X$

$\langle 1 \rangle 2$. If U is a neighbourhood of each of its points then A is open.

$\langle 2 \rangle 1$. ASSUME: U includes a neighbourhood of each of its points

PROVE: $U = \bigcup \{V \subseteq U : V \text{ is open}\}$

$\langle 2 \rangle 2$. $\bigcup \{V \subseteq U : V \text{ is open}\} \subseteq U$

PROOF: Set theory.

$\langle 2 \rangle 3$. $U \subseteq \bigcup \{V \subseteq U : V \text{ is open}\}$

PROOF: Immediate from $\langle 2 \rangle 1$.

$\langle 1 \rangle 3$. If U is open then U is a neighbourhood of each of its points.

PROOF: Immediate from definitions.

\square

Proposition 3.2.4. *If M is a neighbourhood of A and $M \subseteq N$ then N is a neighbourhood of A .*

PROOF: Immediate from definitions. \square

Proposition 3.2.5. *If M and N are neighbourhoods of A then $M \cap N$ is a neighbourhood of A .*

PROOF: Pick open sets U and V such that $A \subseteq U \subseteq M$ and $A \subseteq N \subseteq V$. Then $A \subseteq U \cap V \subseteq M \cap N$.

Proposition 3.2.6. *If N is a neighbourhood of x then $x \in N$.*

PROOF: Immediate from definitions. \square

Proposition 3.2.7. *If N is a neighbourhood of x then there exists a neighbourhood U of x such that, for all $y \in U$, M is a neighbourhood of y .*

PROOF: Pick an open set U such that $x \in U \subseteq N$. \square

Theorem 3.2.8. *Let X be a set and $\triangleright \subseteq \mathcal{P}X \times X$ a relation such that:*

1. *If $M \triangleright x$ and $M \subseteq N$ then $N \triangleright x$*
2. *$X \triangleright x$ for all $x \in X$*

3. If $M \triangleright x$ and $N \triangleright x$ then $M \cap N \triangleright x$
4. If $N \triangleright x$ then $x \in N$
5. If $M \triangleright x$ then there exists $N \triangleright x$ such that, for all $y \in N$, $M \triangleright y$.

Then there exists a unique topology \mathcal{T} such that $N \triangleright x$ iff N is a neighbourhood of x .

PROOF:

- $\langle 1 \rangle 1$. LET: \triangleright be a relation satisfying 1–3
- $\langle 1 \rangle 2$. LET: $\mathcal{T} = \{U \in \mathcal{P}X : \forall x \in U. U \triangleright x\}$
- $\langle 1 \rangle 3$. \mathcal{T} is a topology.
 - $\langle 2 \rangle 1$. $X \in \mathcal{T}$
PROOF: By axiom 2
 - $\langle 2 \rangle 2$. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$
PROOF: By axiom 3
 - $\langle 2 \rangle 3$. For all $\mathcal{A} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{A} \in \mathcal{T}$
 - $\langle 3 \rangle 1$. LET: $x \in \bigcup \mathcal{A}$
 - $\langle 3 \rangle 2$. PICK $U \in \mathcal{A}$ such that $x \in U$
 - $\langle 3 \rangle 3$. $U \triangleright x$
 - $\langle 3 \rangle 4$. $\bigcup \mathcal{A} \triangleright x$
PROOF: By axiom 1
- $\langle 1 \rangle 4$. In \mathcal{T} , $N \triangleright x$ iff N is a neighbourhood of x .
 - $\langle 2 \rangle 1$. If $N \triangleright x$ then N is a neighbourhood of x
 - $\langle 3 \rangle 1$. ASSUME: $N \triangleright x$
 - $\langle 3 \rangle 2$. $x \in N$
PROOF: By axiom 4
 - $\langle 3 \rangle 3$. LET: $U = \{y \in N : N \triangleright y\}$
 - $\langle 3 \rangle 4$. U is open
 - $\langle 4 \rangle 1$. LET: $y \in U$
PROVE: $U \triangleright y$
 - $\langle 4 \rangle 2$. $N \triangleright y$
 - $\langle 4 \rangle 3$. PICK $W \triangleright y$ such that, for all $z \in W$, $N \triangleright z$
PROOF: By axiom 5
 - $\langle 4 \rangle 4$. $W \subseteq U$
 - $\langle 4 \rangle 5$. $U \triangleright y$
PROOF: By axiom 1
 - $\langle 3 \rangle 5$. $x \in U \subseteq N$
 - $\langle 2 \rangle 2$. If N is a neighbourhood of x then $N \triangleright x$
 - $\langle 3 \rangle 1$. LET: N be a neighbourhood of x
 - $\langle 3 \rangle 2$. PICK U open such that $x \in U \subseteq N$
 - $\langle 3 \rangle 3$. $U \triangleright x$
PROOF: By $\langle 1 \rangle 2$
 - $\langle 3 \rangle 4$. $N \triangleright x$
PROOF: By axiom 1
- $\langle 1 \rangle 5$. \mathcal{T} is unique.

PROOF: By Proposition 3.2.3.

□

Definition 3.2.9 (Sufficiently Close). Let X be a topological space, $a \in X$, and P be a property of points of X . We write “For all x sufficiently close to a , $P(x)$ ” to mean “There exists a neighbourhood N of a such that, for all $x \in N$, $P(x)$.”

3.3 Open Refinements

Definition 3.3.1 (Open Refinement). Let X be a space and $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}X$. Then \mathcal{B} is an *open refinement* of \mathcal{A} iff \mathcal{B} is a refinement of \mathcal{A} and every member of \mathcal{B} is open.

3.4 Local Bases

Definition 3.4.1 (Local Basis). Let X be a topological space and $x \in X$. A *local basis* at x is a set \mathcal{B} of open neighbourhoods of x such that every neighbourhood of x includes a member of \mathcal{B} . We call the elements of \mathcal{B} *basic open neighbourhoods*.

Proposition 3.4.2. Let \mathcal{B} be a local basis at x and $M, N \in \mathcal{B}$. Then there exists $P \in \mathcal{B}$ such that $P \subseteq M \cap N$.

PROOF: This holds because $M \cap N$ is a neighbourhood of x (Proposition 3.2.5).

□

Proposition 3.4.3. Let X be a topological space, $x \in X$ and $\mathcal{B} \subseteq \mathcal{P}X$. Then \mathcal{B} is a local basis at x iff \mathcal{B} is a set of open neighbourhoods of x such that every open neighbourhood of x includes a member of \mathcal{B} .

PROOF:

⟨1⟩1. If \mathcal{B} is a local basis at x then \mathcal{B} is a set of open neighbourhoods of x such that every open neighbourhood of x includes a member of \mathcal{B}

PROOF: Trivial.

⟨1⟩2. If \mathcal{B} is a set of open neighbourhoods of x such that every open neighbourhood of x includes a member of \mathcal{B} then \mathcal{B} is a local basis at x .

PROOF: Every neighbourhood of x includes an open neighbourhood of x , which therefore includes an element of \mathcal{B} .

□

3.5 Bases

Definition 3.5.1 (Basis for a Topology). Let (X, \mathcal{T}) be a topological space. A *basis* for the topology on X is a set of open sets \mathcal{B} such that every open set is a union of members of \mathcal{B} . The members of \mathcal{B} are called *basic open sets*, and \mathcal{T} is called the topology *generated* by \mathcal{B} .

Proposition 3.5.2. *Let (X, \mathcal{T}) be a topological space and $\mathcal{B} \subseteq \mathcal{P}X$. Then the following are equivalent:*

1. \mathcal{B} is a basis for \mathcal{T} .
2. A set U is open if and only if, for all $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$.
3. \mathcal{T} is the set of all unions of subsets of \mathcal{B} .
4. Every member of \mathcal{B} is open and, for all $x \in X$ and every open neighbourhood U of x , there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$.
5. For all $x \in X$, the set $\{B \in \mathcal{B} : x \in B\}$ is a local basis at x .

PROOF:

$\langle 1 \rangle 1. 1 \Rightarrow 2$

$\langle 2 \rangle 1.$ ASSUME: \mathcal{B} is a basis for the topology \mathcal{T} .

$\langle 2 \rangle 2.$ For all $U \in \mathcal{T}$ and $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B$

PROOF: Immediate from the definition of basis ($\langle 2 \rangle 1$).

$\langle 2 \rangle 3.$ For all $U \subseteq X$, if $\forall x \in U. \exists B \in \mathcal{B}. x \in B \subseteq U$ then $U \in \mathcal{T}$

PROOF: By Proposition 3.2.3.

$\langle 1 \rangle 2. 2 \Leftrightarrow 3$

PROOF: From Lemma 1.7.1.

$\langle 1 \rangle 3. 3 \Rightarrow 1$

PROOF: Trivial.

$\langle 1 \rangle 4. 2 \Rightarrow 4$

PROOF: Trivial.

$\langle 1 \rangle 5. 4 \Rightarrow 2$

PROOF:

$\langle 2 \rangle 1.$ ASSUME: 4

$\langle 2 \rangle 2.$ If U is open then, for all $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$

PROOF: Immediate from $\langle 2 \rangle 1$.

$\langle 2 \rangle 3.$ If, for all $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$, then U is open.

PROOF: By Proposition 3.2.3 using the fact that every member of \mathcal{B} is open ($\langle 2 \rangle 1$).

$\langle 1 \rangle 6. 4 \Leftrightarrow 5$

PROOF: From Proposition 3.4.3.

□

Corollary 3.5.2.1. *If \mathcal{B} is a basis for the topology \mathcal{T} , then \mathcal{T} is the coarsest topology in which every element of \mathcal{B} is open.*

Lemma 3.5.3. *Let X be a set and $\mathcal{B} \subseteq \mathcal{P}X$. Then \mathcal{B} is a basis for a topology \mathcal{T} on X if and only if:*

1. $\bigcup \mathcal{B} = X$

2. for all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

In this case, \mathcal{T} is unique.

PROOF:

- (1)1. If \mathcal{B} is a basis for a topology then $\bigcup \mathcal{B} = X$
- (2)1. ASSUME: \mathcal{B} is a basis for the topology \mathcal{T}
- (2)2. LET: $x \in X$
- (2)3. There exists $B \in \mathcal{B}$ such that $x \in B$
 PROOF: From the definition of basis, since $X \in \mathcal{T}$. ((2)1, (2)2).
- (1)2. If \mathcal{B} is a basis for a topology then it satisfies condition 2
- (2)1. ASSUME: \mathcal{B} is a basis for the topology \mathcal{T}
- (2)2. LET: $B_1, B_2 \in \mathcal{B}$
- (2)3. $B_1, B_2 \in \mathcal{T}$
 PROOF: From the definition of basis ((2)1, (2)2).
- (2)4. $B_1 \cap B_2 \in \mathcal{T}$
 PROOF: By the definition of topology, the open sets in \mathcal{T} are closed under binary intersection ((2)1, (2)3)
- (2)5. For all $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$
 PROOF: From the definition of basis ((2)1, (2)4)
- (1)3. If \mathcal{B} satisfies conditions 1 and 2 then $\mathcal{T} = \{U \subseteq X : \forall x \in U. \exists B \in \mathcal{B}. x \in B \subseteq U\}$ is a topology and \mathcal{B} is a basis for \mathcal{T} .
- (2)1. ASSUME: \mathcal{B} satisfies conditions 1 and 2
- (2)2. $X \in \mathcal{T}$
 PROOF: For all $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq X$ by condition 1 ((2)1).
- (2)3. For all $\mathcal{A} \subseteq \mathcal{T}$, we have $\bigcup \mathcal{A} \in \mathcal{T}$
 - (3)1. LET: $\mathcal{A} \subseteq \mathcal{T}$
 - (3)2. LET: $x \in \bigcup \mathcal{A}$
 - (3)3. PICK $U \in \mathcal{A}$ such that $x \in U$
 PROOF: From (3)2.
 - (3)4. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$
 PROOF: Since $U \in \mathcal{T}$, using the definition of \mathcal{T} ((3)1, (3)3)
 - (3)5. $x \in B \subseteq \bigcup \mathcal{A}$
 PROOF: From (3)3 and (3)4.
- (2)4. For all $U, V \in \mathcal{T}$, we have $U \cap V \in \mathcal{T}$
 - (3)1. LET: $U, V \in \mathcal{T}$
 - (3)2. LET: $x \in U \cap V$
 - (3)3. PICK $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \subseteq U$ and $x \in B_2 \subseteq V$
 PROOF: From (3)1, (3)2 and the definition of \mathcal{T} .
 - (3)4. PICK $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$
 PROOF: Using condition 2 ((2)1, (3)3).
 - (3)5. $x \in B_3 \subseteq U \cap V$
 PROOF: From (3)3 and (3)4.
- (2)5. $\bigcup \mathcal{B} = X$
 PROOF: This is condition 1 ((2)1).

⟨2⟩6. For all $U \in \mathcal{T}$ and $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$

PROOF: Immediate from the definition of \mathcal{T} .

⟨1⟩4. \mathcal{T} is unique.

PROOF: From Proposition 3.5.2.

□

Corollary 3.5.3.1. *Let X be a set and $\mathcal{B} \subseteq \mathcal{P}X$ be such that $\bigcup \mathcal{B} = X$ and \mathcal{B} is closed under binary intersection. Then \mathcal{B} is a basis for a unique topology on X .*

Lemma 3.5.4. *Let \mathcal{B} and \mathcal{B}' be bases for the topologies \mathcal{T} and \mathcal{T}' on X respectively. Then $\mathcal{T} \subseteq \mathcal{T}'$ if and only if, for all $B \in \mathcal{B}$ and $x \in B$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.*

PROOF:

⟨1⟩1. If $\mathcal{T} \subseteq \mathcal{T}'$ then, for all $B \in \mathcal{B}$ and $x \in B$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.

⟨2⟩1. ASSUME: $\mathcal{T} \subseteq \mathcal{T}'$

⟨2⟩2. LET: $B \in \mathcal{B}$ and $x \in B$

⟨2⟩3. $B \in \mathcal{T}$

PROOF: This holds because $\mathcal{B} \subseteq \mathcal{T}$ by the definition of basis. (⟨2⟩2)

⟨2⟩4. $B \in \mathcal{T}'$

PROOF: From ⟨2⟩1 and ⟨2⟩3.

⟨2⟩5. There exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.

⟨1⟩2. If, for all $B \in \mathcal{B}$ and $x \in B$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$, then $\mathcal{T} \subseteq \mathcal{T}'$.

⟨2⟩1. ASSUME: For all $B \in \mathcal{B}$ and $x \in B$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$

⟨2⟩2. LET: $U \in \mathcal{T}$

PROVE: $U \in \mathcal{T}'$

⟨2⟩3. LET: $x \in U$

⟨2⟩4. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$

PROOF: Since \mathcal{B} is a basis for \mathcal{T} (⟨2⟩2, ⟨2⟩3).

⟨2⟩5. PICK $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$

PROOF: From ⟨2⟩1 and ⟨2⟩4.

⟨2⟩6. $x \in B' \subseteq U$

PROOF: From ⟨2⟩4 and ⟨2⟩5.

⟨2⟩7. Q.E.D.

PROOF: By Proposition 3.5.2.

□

Definition 3.5.5 (Lower Limit Topology). The *lower limit topology* on \mathbb{R} is the one generated by the set of all half-open intervals of the form $[a, b)$. We write \mathbb{R}_l for the topological space consisting of \mathbb{R} under this topology.

We prove this is a topology.

PROOF:

⟨1⟩1. LET: \mathcal{B} be the set of all half-open intervals of the form $[a, b)$.

⟨1⟩2. $\bigcup \mathcal{B} = \mathbb{R}$

PROOF: For all $x \in \mathbb{R}$, we have $x \in [x, x+1) \in \mathcal{B}$.

⟨1⟩3. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

PROOF: If $x \in [a, b) \cap [c, d)$ then $x \in [\max(a, c), \min(b, d)) \subseteq [a, b) \cap [c, d)$.

⟨1⟩4. Q.E.D.

PROOF: By Lemma 3.5.3.

□

Definition 3.5.6 (*K-topology*). The *K-topology* on \mathbb{R} is the one generated by the set of all open intervals (a, b) and all sets of the form $(a, b) \setminus K$, where $K = \{1/n : n \in \mathbb{Z}^+\}$. We write \mathbb{R}_K for the topological space consisting of \mathbb{R} under this topology.

We prove this is a topology.

PROOF:

⟨1⟩1. LET: $\mathcal{B} = \{(a, b) : a, b \in \mathbb{R}, a < b\} \cup \{(a, b) \setminus K : a, b \in \mathbb{R}, a < b\}$

⟨1⟩2. $\bigcup \mathcal{B} = \mathbb{R}$

PROOF: For all $x \in \mathbb{R}$, we have $x \in (x-1, x+1) \in \mathcal{B}$.

⟨1⟩3. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

⟨2⟩1. LET: $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$

PROVE: There exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$

⟨2⟩2. CASE: $B_1 = (a, b)$, $B_2 = (c, d)$

PROOF: Take $B_3 = (\max(a, c), \min(b, d))$

⟨2⟩3. CASE: $B_1 = (a, b)$, $B_2 = (c, d) \setminus K$

PROOF: Take $B_3 = (\max(a, c), \min(b, d)) \setminus K$

⟨2⟩4. CASE: $B_1 = (a, b) \setminus K$, $B_2 = (c, d)$

PROOF: Take $B_3 = (\max(a, c), \min(b, d)) \setminus K$

⟨2⟩5. CASE: $B_1 = (a, b) \setminus K$, $B_2 = (c, d) \setminus K$

PROOF: Take $B_3 = (\max(a, c), \min(b, d)) \setminus K$

⟨1⟩4. Q.E.D.

PROOF: By Lemma 3.5.3.

□

Lemma 3.5.7. *The lower limit topology and the K-topology are incomparable.*

PROOF: $[0, 1)$ is not open in the *K-topology*. $(-1, 1) \setminus K$ is not open in the lower limit topology, because there is no half-open interval $[a, b)$ such that $0 \in [a, b) \subseteq (-1, 1) \setminus K$. □

Proposition 3.5.8. *The set of all singletons is a basis for any discrete space.*

PROOF: Easy. □

Definition 3.5.9 (*Line with Two Origins*). The *line with two origins* is the set $\mathbb{R} \setminus \{0\} \cup \{p, q\}$ under the topology generated by the basis consisting of:

- all open intervals in \mathbb{R} that do not contain 0;

- all sets of the form $(-a, 0) \cup \{p\} \cup (0, a)$ where $a > 0$;
- all sets of the form $(-a, 0) \cup \{q\} \cup (0, a)$ where $a > 0$

3.6 Closed Sets

Definition 3.6.1 (Closed). Let X be a topological space and $A \subseteq X$. Then A is *closed* iff $X \setminus A$ is open.

Proposition 3.6.2. *In any topological space X , the empty set \emptyset is closed.*

PROOF: This holds because $X \setminus \emptyset = X$ is open. \square

Proposition 3.6.3. *In any topological space X , the set X is closed.*

PROOF: This holds because $X \setminus X = \emptyset$ is open. \square

Proposition 3.6.4. *The union of two closed sets is closed.*

PROOF: If C and D are closed then $X \setminus (C \cup D) = (X \setminus C) \cup (X \setminus D)$ is open. \square

Proposition 3.6.5. *In any topological space, the intersection of a nonempty set of closed sets is closed.*

PROOF: Let \mathcal{C} be a nonempty set of closed sets. Then $X \setminus \bigcap \mathcal{C} = \bigcup \{X \setminus C : C \in \mathcal{C}\}$ is open. \square

Proposition 3.6.6. *Let X be a topological space and $U \subseteq X$. Then U is open if and only if $X \setminus U$ is closed.*

PROOF: Immediate from definitions.

Theorem 3.6.7. *Let X be a set and $\mathcal{C} \subseteq \mathcal{P}X$. Suppose:*

1. $\emptyset, X \in \mathcal{C}$;
2. for all nonempty $\mathcal{A} \subseteq \mathcal{C}$, we have $\bigcap \mathcal{A} \in \mathcal{C}$;
3. for all $C, D \in \mathcal{C}$, we have $C \cup D \in \mathcal{C}$.

Then there exists a unique topology on X under which \mathcal{C} is the set of all closed sets, namely

$$\mathcal{T} = \{U \subseteq X : X \setminus U \in \mathcal{C}\}$$

PROOF:

$\langle 1 \rangle$ 1. LET: \mathcal{C} be a set satisfying 1–3

$\langle 1 \rangle$ 2. LET: $\mathcal{T} = \{X \setminus C : C \in \mathcal{C}\}$

$\langle 1 \rangle$ 3. \mathcal{T} is a topology

$\langle 2 \rangle$ 1. $X \in \mathcal{T}$

PROOF: $X \setminus X = \emptyset \in \mathcal{C}$ by condition 1.

$\langle 2 \rangle$ 2. For all $\mathcal{A} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{A} \in \mathcal{T}$.

- ⟨3⟩1. LET: $\mathcal{A} \subseteq \mathcal{T}$
 ⟨3⟩2. CASE: $\mathcal{A} = \emptyset$
 PROOF: In this case, $X \setminus \bigcup \mathcal{A} = X \in \mathcal{C}$ by condition 1.
 ⟨3⟩3. CASE: \mathcal{A} is nonempty
 PROOF: In this case, we have $X \setminus \bigcup \mathcal{A} = \bigcap \{X \setminus U : U \in \mathcal{A}\} \in \mathcal{C}$ by condition 2.
 ⟨2⟩3. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$
 PROOF: $X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V) \in \mathcal{C}$ by condition 3.
 ⟨1⟩4. \mathcal{C} is the set of closed sets.
 PROOF:

$$\begin{aligned}
 C \text{ is closed} &\Leftrightarrow X \setminus C \in \mathcal{T} \\
 &\Leftrightarrow X \setminus (X \setminus C) \in \mathcal{C} \\
 &\Leftrightarrow C \in \mathcal{C}
 \end{aligned}$$

- ⟨1⟩5. \mathcal{T} is unique.
 PROOF: By Proposition 3.6.6.

□

Definition 3.6.8 (Closed Covering). A *closed covering* of a topological space is a covering in which every member is a closed set.

3.7 Closed Refinements

Definition 3.7.1 (Closed Refinement). Let X be a space and $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}X$. Then \mathcal{B} is an *closed refinement* of \mathcal{A} iff \mathcal{B} is a refinement of \mathcal{A} and every member of \mathcal{B} is closed.

3.8 Locally Finite Families

Definition 3.8.1 (Locally Finite). Let X be a topological space and $\{A_i\}_{i \in I}$ a family of subsets of X . Then $\{A_i\}_{i \in I}$ is *locally finite* iff, for all $x \in X$, there exists a neighbourhood N of x such that there are only finitely many $i \in I$ such that N intersects A_i .

Proposition 3.8.2. *If $\{A_i\}_{i \in I}$ is locally finite and $B_i \subseteq A_i$ for all i then $\{B_i\}_{i \in I}$ is locally finite.*

PROOF: Immediate from definitions. □

Proposition 3.8.3. *Every finite family of open sets is locally finite.*

PROOF: Trivial. □

3.9 Countably Locally Finite Sets

Definition 3.9.1 (Countably Locally Finite). Let X be a space. A subset of $\mathcal{P}X$ is *countably locally finite* iff it is the union of countably many locally finite sets.

3.10 Closure of a Set

Definition 3.10.1 (Closure). Let X be a topological space and $A \subseteq X$. The *closure* of A , $\text{Cl } A$ or \overline{A} , is the intersection of all closed sets that include A .

PROOF: This intersection always exists because X is a closed set that includes A . \square

Proposition 3.10.2. *Let X be a topological space and $A \subseteq X$. Then $A \subseteq \overline{A}$.*

PROOF: Immediate from definitions. \square

Proposition 3.10.3. *Let X be a topological space and $A \subseteq X$. Then \overline{A} is closed.*

PROOF: This follows from Proposition 3.6.5. \square

Proposition 3.10.4. *Let X be a topological space and $A, C \subseteq X$. If $A \subseteq C$ and C is closed then $\overline{A} \subseteq C$.*

PROOF: Immediate from definitions. \square

Proposition 3.10.5. *Let X be a topological space and $A, B \subseteq X$. If $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$.*

PROOF:

$\langle 1 \rangle 1$. ASSUME: $A \subseteq B$

$\langle 1 \rangle 2$. $A \subseteq \overline{B}$

PROOF: Proposition 3.10.2.

$\langle 1 \rangle 3$. $\overline{A} \subseteq \overline{B}$

PROOF: Propositions 3.10.3, 3.10.4.

\square

Proposition 3.10.6. *Let X be a set and $A \subseteq X$. Then A is closed if and only if $A = \overline{A}$.*

PROOF:

$\langle 1 \rangle 1$. If A is closed then $A = \overline{A}$

$\langle 2 \rangle 1$. ASSUME: A is closed

$\langle 2 \rangle 2$. $A \subseteq \overline{A}$

PROOF: By Proposition 3.10.2.

$\langle 2 \rangle 3$. $\overline{A} \subseteq A$

PROOF: By Proposition 3.10.4 since $A \subseteq A$.

$\langle 1 \rangle 2$. If $A = \overline{A}$ then A is closed.

PROOF: By Proposition 3.10.3.

\square

Corollary 3.10.6.1.

$$\overline{\emptyset} = \emptyset$$

Theorem 3.10.7 (Kuratowski Closure Axioms). *Let X be a set and $(-) : \mathcal{P}X \rightarrow \mathcal{P}X$ be a function such that:*

1. $\bar{\emptyset} = \emptyset$
2. For all $A \subseteq X$, $A \subseteq \bar{A}$
3. For all $A \subseteq X$, $\bar{A} = \overline{\bar{A}}$
4. For all $A, B \subseteq X$, $\overline{A \cup B} = \bar{A} \cup \bar{B}$

Then there exists a unique topology \mathcal{T} on X such that \bar{A} is the closure of A for all $A \in \mathcal{P}X$.

PROOF:

$\langle 1 \rangle 1$. For all $C, D \subseteq X$, if $C \subseteq D$ then $\bar{C} \subseteq \bar{D}$

$\langle 2 \rangle 1$. ASSUME: $C \subseteq D$

$\langle 2 \rangle 2$. $\bar{C} = \bar{D}$

PROOF:

$$\bar{D} = \overline{\bar{C} \cup D} \quad (\langle 2 \rangle 1)$$

$$= \bar{C} \cup \bar{D} \quad (\text{axiom 4})$$

$\langle 1 \rangle 2$. LET: \mathcal{T} be the topology in which a set C is closed iff $\bar{C} = C$.

$\langle 2 \rangle 1$. $\bar{\emptyset} = \emptyset$

PROOF: This is axiom 1.

$\langle 2 \rangle 2$. $\bar{X} = X$

PROOF: By axiom 2.

$\langle 2 \rangle 3$. For any set \mathcal{A} of sets C such that $\bar{C} = C$, we have $\overline{\bigcap \mathcal{A}} = \bigcap \mathcal{A}$

$\langle 3 \rangle 1$. $\bigcap \mathcal{A} \subseteq \bigcap \mathcal{A}$

$\langle 4 \rangle 1$. LET: $C \in \mathcal{A}$

$\langle 4 \rangle 2$. $\bigcap \mathcal{A} \subseteq C$

PROOF:

$$\overline{\bigcap \mathcal{A}} \subseteq \bar{C} \quad (\langle 1 \rangle 1)$$

$$= C \quad (\langle 4 \rangle 1)$$

$\langle 3 \rangle 2$. Q.E.D.

$\langle 2 \rangle 4$. If $\bar{C} = C$ and $\bar{D} = D$ then $\overline{C \cup D} = C \cup D$

PROOF: By axiom 4.

$\langle 2 \rangle 5$. Q.E.D.

PROOF: By Theorem 3.6.7.

$\langle 1 \rangle 3$. For all $A \subseteq X$, the closure of A in \mathcal{T} is \bar{A}

$\langle 2 \rangle 1$. \bar{A} is closed

PROOF: From axiom 3.

$\langle 2 \rangle 2$. If $A \subseteq C$ and C is closed then $\bar{A} \subseteq C$

PROOF:

$$C = \bar{C} \quad (C \text{ is closed})$$

$$= \overline{A \cup C} \quad (A \subseteq C)$$

$$= \bar{A} \cup \bar{C} \quad (\text{axiom 4})$$

□

Theorem 3.10.8. *Let A be a subset of the topological space X and \mathcal{B} a basis for X . Then $x \in \overline{A}$ if and only if, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A .*

PROOF:

$\langle 1 \rangle 1$. If $x \in \overline{A}$ then, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A .

PROOF: Immediate from Theorem 3.11.3.

$\langle 1 \rangle 2$. If, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A , then $x \in \overline{A}$.

$\langle 2 \rangle 1$. ASSUME: for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A .

$\langle 2 \rangle 2$. LET: U be a neighbourhood of x

$\langle 2 \rangle 3$. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$

PROOF: \mathcal{B} is a basis.

$\langle 2 \rangle 4$. B intersects A .

PROOF: By $\langle 2 \rangle 1$.

$\langle 2 \rangle 5$. U intersects A .

$\langle 2 \rangle 6$. Q.E.D.

PROOF: By Theorem 3.11.3.

□

Lemma 3.10.9. *If $\{A_i\}_{i \in I}$ is locally finite then so is $\{\overline{A_i}\}_{i \in I}$.*

PROOF:

$\langle 1 \rangle 1$. LET: $\{A_i\}_{i \in I}$ be a locally finite family of subsets of the space X .

$\langle 1 \rangle 2$. LET: $x \in X$

$\langle 1 \rangle 3$. PICK a neighbourhood U of x that intersects only A_{i_1}, \dots, A_{i_n} .

$\langle 1 \rangle 4$. U intersects only $\overline{A_{i_1}}, \dots, \overline{A_{i_n}}$.

□

Lemma 3.10.10. *Let $\{A_i\}_{i \in I}$ be locally finite. Then $\overline{\bigcup_{i \in I} A_i} = \bigcup_{i \in I} \overline{A_i}$.*

PROOF:

$\langle 1 \rangle 1$. LET: $x \in \overline{\bigcup_{i \in I} A_i}$

$\langle 1 \rangle 2$. PICK a neighbourhood U of x that intersects only A_{i_1}, \dots, A_{i_n} .

$\langle 1 \rangle 3$. $x \in \overline{A_{i_1}} \cup \dots \cup \overline{A_{i_n}}$

PROOF: If not, then $U - \overline{A_{i_1}} - \dots - \overline{A_{i_n}}$ would be a neighbourhood of x that does not intersect $\bigcup_{i \in I} A_i$.

□

3.11 Interior of a Set

Definition 3.11.1 (Interior). Let X be a topological space and $A \subseteq X$. The *interior* of A , $\text{Int } A$, is the union of all open sets included in A .

Lemma 3.11.2. *If $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$.*

PROOF: \overline{B} is a closed set that includes B , hence includes A . □

Theorem 3.11.3. *Let A be a subset of the topological space X and $x \in X$. Then $x \in \overline{A}$ if and only if every neighbourhood of x intersects A .*

PROOF:

$$\begin{aligned}
x \notin \overline{A} &\Leftrightarrow \exists C \text{ closed } (A \subseteq C \wedge x \notin C) \\
&\Leftrightarrow \exists U \text{ open } (A \subseteq X \setminus U \wedge x \in U) \\
&\Leftrightarrow \exists U \text{ open } (A \text{ intersects } U \wedge x \in U)
\end{aligned}$$

□

Lemma 3.11.4.

$$X \setminus \text{Int } A = \overline{X \setminus A}$$

PROOF:

$$\begin{aligned}
\langle 1 \rangle 1. & X \setminus \text{Int } A \subseteq \overline{X \setminus A} \\
\langle 2 \rangle 1. & X \setminus A \subseteq \overline{X \setminus A} \\
\langle 2 \rangle 2. & X \setminus \overline{X \setminus A} \subseteq A \\
\langle 2 \rangle 3. & \overline{X \setminus A} \subseteq X \setminus \text{Int } A \\
\langle 1 \rangle 2. & \overline{X \setminus A} \subseteq X \setminus \text{Int } A \\
\langle 2 \rangle 1. & \text{Int } A \subseteq A \\
\langle 2 \rangle 2. & X \setminus A \subseteq X \setminus \text{Int } A \\
\langle 2 \rangle 3. & \overline{X \setminus A} \subseteq X \setminus \text{Int } A
\end{aligned}$$

□

3.12 Boundary

Definition 3.12.1 (Boundary). Let X be a topological space and $A \subseteq X$. The *boundary* of A , $\text{Bd } A$, is $\overline{A} \cap X \setminus A$.

Lemma 3.12.2.

$$\text{Bd } A = \overline{A} \setminus \text{Int } A$$

PROOF: From Lemma 3.11.4. □

Lemma 3.12.3. $\overline{A} = \text{Int } A \cup \text{Bd } A$

PROOF:

$$\begin{aligned}
\text{Int } A \cup \text{Bd } A &= \text{Int } A \cup (\overline{A} \cap (X \setminus \text{Int } A)) \\
&= \text{Int } A \cup \overline{A} \\
&= \overline{A}
\end{aligned}$$

□

Corollary 3.12.3.1. $\text{Bd } A = \emptyset$ iff A is open and closed.

Lemma 3.12.4. For any set U , the following are equivalent:

1. U is open.
2. $\text{Bd } U \cap U = \emptyset$
3. $\text{Bd } U = \overline{U} \setminus U$

PROOF:

$$\langle 1 \rangle 1. 1 \Rightarrow 3$$

PROOF: From Lemma 3.12.2.

⟨1⟩2. $3 \Rightarrow 2$

PROOF: Set theory.

⟨1⟩3. $2 \Rightarrow 1$

PROOF:

$$\begin{aligned} U &\subseteq \overline{U} \\ &= \text{Int } U \cup \text{Bd } U && (\text{Lemma 3.12.3}) \\ \therefore U &\subseteq \text{Int } U \end{aligned}$$

□

3.13 Limit Points

Definition 3.13.1 (Limit Point). Let X be a topological space, $A \subseteq X$, and $x \in X$. Then x is a *limit point*, *cluster point* or *point of accumulation* of A iff every neighbourhood of x intersects A in a point other than x .

Lemma 3.13.2. *If $A \subseteq B$ then every limit point of A is a limit point of B .*

PROOF: Immediate from the definition. □

Theorem 3.13.3. *Let A be a subset of the topological space X . Let A' be the set of all limit points of A . Then $\overline{A} = A \cup A'$.*

PROOF:

⟨1⟩1. If $x \in \overline{A}$ and $x \notin A$ then $x \in A'$

PROOF: in this case, every neighbourhood of x intersects A in a point other than x .

⟨1⟩2. $A \subseteq \overline{A}$

PROOF: From the definition of \overline{A} .

⟨1⟩3. $A' \subseteq \overline{A}$

PROOF: By Theorem 3.11.3.

□

Corollary 3.13.3.1. *A set is closed if and only if it contains all its limit points.*

3.14 Subbases

Definition 3.14.1 (Subbasis). Let X be a topological space. A *subbasis* for the topology on X is a set $\mathcal{S} \subseteq \mathcal{P}X$ such that, for every open set U and $x \in U$, there exist $S_1, \dots, S_n \in \mathcal{S}$ such that $x \in S_1 \cap \dots \cap S_n \subseteq U$. We say the topology is *generated* by \mathcal{S} .

Lemma 3.14.2. *Let \mathcal{T} be a topology on X and $\mathcal{S} \subseteq \mathcal{P}X$. Then the following are equivalent:*

1. \mathcal{S} is a subbasis for \mathcal{T} .

2. The set of all finite intersections of members of \mathcal{S} is a basis for \mathcal{T}

3. \mathcal{T} is the set of all unions of finite intersections of members of \mathcal{S} .

PROOF: $1 \Leftrightarrow 2$ holds immediately from the definitions. $2 \Leftrightarrow 3$ holds by Proposition 3.5.2. \square

Corollary 3.14.2.1. *If \mathcal{S} is a subbasis for the topology \mathcal{T} , then \mathcal{T} is the coarsest topology in which every element of \mathcal{S} is open.*

Lemma 3.14.3. *Let X be a set and $\mathcal{S} \subseteq \mathcal{P}X$. Then \mathcal{S} is a subbasis for a topology on X if and only if $\bigcup \mathcal{S} = X$.*

PROOF:

$\langle 1 \rangle 1$. If \mathcal{S} is a subbasis for a topology on X then $\bigcup \mathcal{S} = X$

$\langle 2 \rangle 1$. ASSUME: \mathcal{S} is a subbasis for a topology \mathcal{T} on X .

$\langle 2 \rangle 2$. LET: $x \in X$

$\langle 2 \rangle 3$. PICK $S_1, \dots, S_n \in \mathcal{S}$ such that $x \in S_1 \cap \dots \cap S_n \subseteq X$

PROOF: From the definition of subbasis ($\langle 2 \rangle 1$, $\langle 2 \rangle 2$).

$\langle 2 \rangle 4$. $x \in \bigcup \mathcal{S}$

PROOF: Immediate from $\langle 2 \rangle 3$.

$\langle 1 \rangle 2$. If $\bigcup \mathcal{S} = X$ then \mathcal{S} is a subbasis for a topology on X

$\langle 2 \rangle 1$. ASSUME: $\bigcup \mathcal{S} = X$

PROVE: The set of all finite intersections of elements of \mathcal{S} is a basis for a topology on X .

$\langle 2 \rangle 2$. LET: \mathcal{B} be the set of all finite intersections of elements of \mathcal{S} .

$\langle 2 \rangle 3$. $\bigcup \mathcal{B} = X$

PROOF: From $\langle 2 \rangle 1$ and $\langle 2 \rangle 2$.

$\langle 2 \rangle 4$. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$

PROOF: Take $B_3 = B_1 \cap B_2$ ($\langle 2 \rangle 2$).

$\langle 2 \rangle 5$. \mathcal{B} is a basis for a topology on X .

PROOF: By Lemma 3.5.3.

$\langle 2 \rangle 6$. Q.E.D.

PROOF: By Lemma 3.14.2.

\square

3.15 Convergence

Definition 3.15.1 (Net). Let X be a topological space. A *net* $(x_\alpha)_{\alpha \in J}$ in X consists of a directed set J and a function $x : J \rightarrow X$.

Definition 3.15.2 (Convergence). Let $(x_\alpha)_{\alpha \in J}$ be a net in the topological space X , and $l \in X$. Then the net *converges* to l , $x_\alpha \rightarrow l$, if and only if, for every neighbourhood U of l , there exists $\alpha \in J$ such that, for all $\beta \geq \alpha$, we have $x_\beta \in U$.

Theorem 3.15.3 (AC). *Let X be a topological space and $A \subseteq X$. Then $x \in \overline{A}$ if and only if there exists a net of points of A converging to x .*

PROOF:

- (1)1. If $x \in \bar{A}$ then there exists a net of points of A converging to x .
- (2)1. LET: $x \in \bar{A}$
- (2)2. LET: J be the poset of neighbourhoods of x under \supseteq .
- (2)3. For $U \in J$ PICK a point $x_U \in U \cap A$
PROOF: By Theorem 3.11.3
- (2)4. $(x_U)_{U \in J}$ is a net
PROOF: Given $U, V \in J$ we have $U \cap V \in J$ and $U \supseteq U \cup V, V \supseteq U \cup V$.
- (2)5. $x_U \rightarrow x$
PROOF: For any neighbourhood U of x we have $U \in J$ and if $U \supseteq V$ then $x_V \in U$.
- (1)2. If there exists a net of points of A converging to x then $x \in \bar{A}$.
- (2)1. LET: $(x_\alpha)_{\alpha \in J}$ be a net of points in A that converges to x .
- (2)2. LET: U be a neighbourhood of x
- (2)3. PICK $\alpha \in J$ such that, for all $\beta \geq \alpha$, we have $x_\beta \in U$
- (2)4. $x_\alpha \in U \cap A$
- (2)5. Q.E.D.
PROOF: By Theorem 3.11.3

□

Theorem 3.15.4. *Let X and Y be topological spaces and $f : X \rightarrow Y$. Then f is continuous if and only if, for every net $(x_\alpha)_{\alpha \in J}$ in X , if $x_\alpha \rightarrow x$ then $f(x_\alpha) \rightarrow f(x)$.*

PROOF:

- (1)1. If f is continuous and $x_\alpha \rightarrow x$ then $f(x_\alpha) \rightarrow f(x)$
- (2)1. ASSUME: f is continuous.
- (2)2. ASSUME: $x_\alpha \rightarrow x$
- (2)3. LET: V be a neighbourhood of $f(x)$
- (2)4. $f^{-1}(V)$ is a neighbourhood of x
- (2)5. PICK α such that, for all $\beta \geq \alpha$, we have $x_\beta \in f^{-1}(V)$
- (2)6. For all $\beta \geq \alpha$ we have $f(x_\beta) \in V$
- (1)2. If, for every net (x_α) in X , if $x_\alpha \rightarrow x$ then $f(x_\alpha) \rightarrow f(x)$, then f is continuous.
- (2)1. ASSUME: for every net (x_α) in X , if $x_\alpha \rightarrow x$ then $f(x_\alpha) \rightarrow f(x)$
- (2)2. LET: $A \subseteq X$
PROVE: $f(\bar{A}) \subseteq \overline{f(A)}$
- (2)3. LET: $x \in \bar{A}$
- (2)4. PICK a net (x_α) in A such that $x_\alpha \rightarrow x$
PROOF: Theorem 3.15.3
- (2)5. $f(x_\alpha) \rightarrow f(x)$
PROOF: By (2)1
- (2)6. $f(x) \in \overline{f(A)}$
PROOF: Theorem 3.15.3
- (2)7. Q.E.D.
PROOF: By Theorem 5.2.2.

□

Definition 3.15.5 (Subnet). Let $(x_\alpha)_{\alpha \in J}$ be a net in X . Let K be a directed set and $g : K \rightarrow J$ be a monotone function such that $g(K)$ is cofinal in J . Then the net $(x_{g(\beta)})_{\beta \in K}$ is called a *subnet* of (x_α) .

3.16 Accumulation Points

Definition 3.16.1 (Accumulation Point). Let X be a topological space, and $(x_\alpha)_{\alpha \in J}$ a net in X , and $a \in X$. Then a is an *accumulation point* of (x_α) iff, for every neighbourhood U of x , the set $\{\alpha \in J : x_\alpha \in U\}$ is cofinal in J .

Lemma 3.16.2. *Let X be a topological space, $(x_\alpha)_{\alpha \in J}$ be a nonempty net in X and $a \in X$. Then a is an accumulation point of (x_α) if and only if there exists a subnet of (x_α) that converges to a .*

PROOF:

- (1)1. If a is an accumulation point of (x_α) then there exists a subnet of (x_α) that converges to a .
- (2)1. ASSUME: a is an accumulation point of (x_α) .
- (2)2. LET: K be the poset $\{(\alpha, U) : \alpha \in J, U \text{ is a neighbourhood of } a, x_\alpha \in U\}$ under: $(\alpha, U) \leq (\beta, V)$ iff $\alpha \leq \beta$ and $U \subseteq V$.
- (2)3. $(x_\alpha)_{(\alpha, U) \in K}$ is a subnet of $(x_\alpha)_{\alpha \in J}$
 - (3)1. K is directed.
 - (4)1. LET: $(\alpha, U), (\beta, V) \in K$
 - (4)2. PICK $\gamma \in J$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$.
 - (4)3. PICK $\delta \in J$ such that $\gamma \leq \delta$ and $x_\delta \in U \cap V$
 - PROOF: By (2)1.
 - (4)4. $(\delta, U \cap V) \in K$ and $(\alpha, U) \leq (\delta, U \cap V)$, $(\beta, V) \leq (\delta, U \cap V)$
- (3)2. If $(\alpha, U) \leq (\beta, V)$ then $\alpha \leq \beta$
 - PROOF: From (2)2.
- (3)3. $\{\alpha : \exists U. (\alpha, U) \in K\}$ is cofinal in J
 - PROOF: For $\alpha \in J$ we have $(\alpha, X) \in K$, so in fact $\{\alpha : \exists U. (\alpha, U) \in K\} = J$.
- (2)4. The subnet converges to a .
 - (3)1. LET: U be a neighbourhood of a .
 - (3)2. PICK $\alpha \in J$
 - (3)3. PICK $\beta \in J$ such that $\alpha \leq \beta$ and $x_\beta \in U$
 - PROOF: By (2)1.
 - (3)4. For all $(\gamma, V) \geq (\beta, U)$ we have $x_\gamma \in U$
 - PROOF: $x_\gamma \in V \subseteq U$.
- (1)2. If there exists a subnet of (x_α) that converges to a then a is an accumulation point of (x_α) .
 - (2)1. ASSUME: $(x_{g(\beta)})_{\beta \in K}$ converges to a
 - (2)2. LET: U be a neighbourhood of a
 - (2)3. LET: $\alpha \in J$
 - PROVE: There exists $\gamma \geq \alpha$ such that $x_\gamma \in U$
 - (2)4. PICK $\beta \in K$ such that, for all $\beta' \geq \beta$, we have $x_{g(\beta')} \in U$

PROOF: By $\langle 2 \rangle 1$.

$\langle 2 \rangle 5$. PICK $\beta' \in K$ such that $g(\beta') \geq \alpha$

PROOF: Since $g(K)$ is cofinal in J .

$\langle 2 \rangle 6$. PICK $\beta'' \in K$ such that $\beta \leq \beta''$ and $\beta' \leq \beta''$

PROOF: K is directed.

$\langle 2 \rangle 7$. $g(\beta'') \geq \alpha$ and $x_{g(\beta'')} \in U$

□

3.17 Dense Sets

Definition 3.17.1 (Dense). Let X be a topological space and $A \subseteq X$. Then A is *dense* in X iff $\overline{A} = X$.

3.18 G_δ Sets

Definition 3.18.1 (G_δ Set). A G_δ set is the intersection of a countable set of open sets.

3.19 Separated Sets

Definition 3.19.1 (Separated Sets). Let X be a topological space and $A, B \subseteq X$. Then A and B are *separated* iff $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$.

3.20 Coherent Topology

Definition 3.20.1 (Coherent Topology). Let $X_1 \subseteq X_2 \subseteq \cdots$ be a sequence of topological spaces such that each X_n is a closed subspace of X_{n+1} . Let $X = \bigcup_{n=1}^{\infty} X_n$. Then the topology on X *coherent* with the subspaces X_n is the topology defined by: $U \subseteq X$ is open iff $U \cap X_n$ is open in X_n for all n .

Chapter 4

Constructions of Topological Spaces

4.1 The Order Topology

Definition 4.1.1 (Order Topology). Let X be a linearly ordered set with more than one element. The *order topology* on X is the topology generated by the basis consisting of:

- all open intervals (a, b)
- all half-open intervals $(a, \top]$ where \top is the greatest element of X , if there is one;
- all half-open intervals $[\perp, a)$ where \perp is the least element of X , if there is one.

We prove this is a basis for a topology.

PROOF:

$\langle 1 \rangle 1$. LET: \mathcal{B} be the set of all sets of these three forms.

$\langle 1 \rangle 2$. $\bigcup \mathcal{B} = X$

$\langle 2 \rangle 1$. LET: $x \in X$

PROVE: There exists $B \in \mathcal{B}$ such that $x \in B$

$\langle 2 \rangle 2$. CASE: x is least in X

$\langle 3 \rangle 1$. PICK $a \in X$ such that $a > x$

PROOF: X has more than one element.

$\langle 3 \rangle 2$. $x \in [x, a) \in \mathcal{B}$

$\langle 2 \rangle 3$. CASE: x is greatest in X

$\langle 3 \rangle 1$. PICK $a \in X$ such that $a < x$

PROOF: X has more than one element.

$\langle 3 \rangle 2$. $x \in (a, x] \in \mathcal{B}$

$\langle 2 \rangle 4$. CASE: x is neither least nor greatest in X

Lemma 4.1.3. *The open rays form a subbasis for the order topology.*

- ⟨1⟩1. LET: X be a linearly ordered set with more than one element.
- ⟨1⟩2. The open rays form a subbasis for a topology.
 - ⟨2⟩1. LET: $x \in X$
 - PROVE: x is an element of an open ray.
 - ⟨2⟩2. CASE: x is greatest in X
 - ⟨3⟩1. PICK $a \in X$ such that $a < x$
 - PROOF: X has more than one element (⟨1⟩1).
 - ⟨3⟩2. $x \in (a, +\infty)$
 - ⟨2⟩3. CASE: x is not greatest in X
 - ⟨3⟩1. PICK $a \in X$ such that $x < a$
 - ⟨3⟩2. $x \in (-\infty, a)$
 - ⟨2⟩4. Q.E.D.
 - PROOF: By Lemma 3.14.2.
- ⟨1⟩3. LET: \mathcal{T}_o be the order topology and \mathcal{T}_S be the topology generated by the open rays.
 - ⟨1⟩4. $\mathcal{T}_o \subseteq \mathcal{T}_S$
 - ⟨2⟩1. Every open interval (a, b) is open in \mathcal{T}_S
 - PROOF: $(a, b) = (a, +\infty) \cap (-\infty, b)$.
 - ⟨2⟩2. If \top is greatest then $(a, \top]$ is open in \mathcal{T}_S
 - PROOF: $(a, \top] = (a, +\infty)$.
 - ⟨2⟩3. If \perp is least then $[\perp, b)$ is open in \mathcal{T}_S
 - PROOF: $[\perp, b) = [\perp, +\infty)$.
 - ⟨2⟩4. Q.E.D.
 - PROOF: By Corollary 3.5.2.1.
 - ⟨1⟩5. $\mathcal{T}_S \subseteq \mathcal{T}_o$
 - ⟨2⟩1. For all $a \in X$, we have $(a, +\infty)$ is open in \mathcal{T}_o
 - ⟨3⟩1. LET: $x \in (a, +\infty)$
 - PROVE: There exists a basis element B such that $x \in B \subseteq (a, +\infty)$
 - ⟨3⟩2. CASE: x is greatest
 - PROOF: Take $B = (a, x]$
 - ⟨3⟩3. CASE: x is not greatest
 - ⟨4⟩1. PICK $b > x$
 - ⟨4⟩2. $x \in (a, b) \subseteq (a, +\infty)$
 - ⟨2⟩2. For all $a \in X$, we have $(-\infty, a)$ is open in \mathcal{T}_o
 - PROOF: Similar.
 - ⟨2⟩3. Q.E.D.
 - PROOF: By Corollary 3.14.2.1.

□

Lemma 4.1.4. *In a linearly ordered set X under the order topology, the closed intervals and closed rays are closed.*

PROOF:

$$\begin{aligned} X \setminus [a, b] &= (-\infty, a) \cup (b, +\infty) \\ X \setminus (-\infty, a] &= (a, +\infty) \\ X \setminus [a, +\infty) &= (-\infty, a) \end{aligned} \quad \square$$

Definition 4.1.5 (Standard Topology on \mathbb{R}). The *standard topology* on \mathbb{R} is the order topology.

Lemma 4.1.6. *The standard topology is strictly coarser than the lower limit topology.*

PROOF:

- $\langle 1 \rangle 1$. The standard topology is coarser than the lower limit topology.
 - $\langle 2 \rangle 1$. For every open interval (a, b) and $x \in (a, b)$, there exists a half-open interval $[c, d)$ such that $x \in [c, d) \subseteq (a, b)$.

PROOF: Take $[c, d) = [x, b)$.
 - $\langle 2 \rangle 2$. Q.E.D.

PROOF: By Lemma 3.5.4.
- $\langle 1 \rangle 2$. There exists a set U open in the lower limit topology that is not open in the standard topology.

PROOF: Take $U = [0, 1)$.

\square

Lemma 4.1.7. *The standard topology is strictly coarser than the K -topology.*

PROOF:

- $\langle 1 \rangle 1$. The standard topology is coarser than the K -topology.

PROOF: Every open interval is open in the K -topology.
- $\langle 1 \rangle 2$. There exists a set U open in the K -topology that is not open in the standard topology.

PROOF: Take $U = (-1, 1) \setminus K$. Then $0 \in U$ but there is no open interval (a, b) such that $0 \in (a, b) \subseteq U$.

\square

Definition 4.1.8 (Ordered Square). The *ordered square* I_o^2 is the topological space $[0, 1]^2$ under the order topology induced by the lexicographic order.

Lemma 4.1.9. *Let L be a linear continuum with a greatest element. Then every non-empty closed set in L has a greatest element.*

PROOF:

- $\langle 1 \rangle 1$. LET: C be a non-empty closed set in L
- $\langle 1 \rangle 2$. LET: u be the supremum of C
- $\langle 1 \rangle 3$. $u \in C$
 - $\langle 2 \rangle 1$. ASSUME: w.l.o.g u is not least in L

PROOF: If u is least then $C = \{u\}$.
 - $\langle 2 \rangle 2$. LET: U be any open neighbourhood of u
 - $\langle 2 \rangle 3$. PICK $v < u$ such that $(v, u] \subseteq U$

PROOF: By Lemma 4.1.2.

(2)4. PICK $x \in C$ such that $v < x$

PROOF: v is not an upper bound for C ($\langle 1 \rangle 2$).

(2)5. U intersects C in v

(2)6. Q.E.D.

PROOF: By Theorem 3.11.3.

□

Definition 4.1.10 (Long Line). The *long line* is $(S_\Omega \times [0, 1)) \setminus \{(0, 0)\}$ under the dictionary order, where S_Ω is the first uncountable ordinal under the order topology.

4.2 The Product Topology

Definition 4.2.1 (Product Topology). Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces. The *product topology* on $\prod_{\alpha \in J} X_\alpha$ is the topology generated by the subbasis consisting of all sets of the form $\pi_\alpha^{-1}(U)$ where $\alpha \in J$ and U is open in X_α . The *product space* of $\{X_\alpha\}_{\alpha \in J}$ is $\prod_{\alpha \in J} X_\alpha$ under the product topology.

Lemma 4.2.2. Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces and A_α be closed in X_α for all α . Then $\prod_{\alpha \in J} A_\alpha$ is closed in $\prod_{\alpha \in J} X_\alpha$.

PROOF: This holds because $\prod_{\alpha \in J} X_\alpha \setminus \prod_{\alpha \in J} A_\alpha = \bigcup_{\alpha \in J} \pi_\alpha^{-1}(X_\alpha \setminus A_\alpha)$. □

Theorem 4.2.3. Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces. The set of all sets of the form $\prod_{\alpha \in J} U_\alpha$ where each U_α is open in X_α , and $U_\alpha = X_\alpha$ for all but finitely many α , is a basis for the product topology on $\prod_{\alpha \in J} X_\alpha$.

PROOF: By Lemma 3.14.2. □

Theorem 4.2.4. Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces, and let \mathcal{B}_α be a basis for the topology on X_α for each α . Then

$$\mathcal{B} = \left\{ \prod_{\alpha \in J} U_\alpha : \text{for finitely many } \alpha \in J, U_\alpha \in \mathcal{B}_\alpha, \right. \\ \left. \text{and } U_\alpha = X_\alpha \text{ for all other values of } \alpha \right\}$$

is a basis for the product topology on $\prod_{\alpha \in J} X_\alpha$.

PROOF:

(1)1. Every member of \mathcal{B} is open in the product topology.

PROOF: Immediate from definitions.

(1)2. For every open set U and $\{x_\alpha\}_{\alpha \in J} \in U$, there exists $B \in \mathcal{B}$ such that $\{x_\alpha\}_{\alpha \in J} \in B \subseteq U$.

(2)1. LET: U be open and $\{x_\alpha\}_{\alpha \in J} \in U$

(2)2. PICK U_α open in X_α for each α such that $\{x_\alpha\}_{\alpha \in J} \in \prod_{\alpha \in J} U_\alpha \subseteq U$ and $U_\alpha = X_\alpha$ for all α except $\alpha_1, \dots, \alpha_n$.

PROOF: By Theorem 4.2.3.

- (2)3. PICK $B_{\alpha_i} \in \mathcal{B}_{\alpha_i}$ such that $x_\alpha \in B_{\alpha_i} \subseteq U_{\alpha_i}$ for $i = 1, \dots, n$
 (2)4. $\{x_\alpha\}_{\alpha \in J} \in \prod_{\alpha \in J} V_\alpha \subseteq U$ where $V_{\alpha_i} = B_{\alpha_i}$ for $i = 1, \dots, n$, and $V_\alpha = X_\alpha$ for all other α .

□

Theorem 4.2.5 (AC). *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces and $A_\alpha \subseteq X_\alpha$ for all α . If $\prod_{\alpha \in J} X_\alpha$ is given the product topology, then*

$$\prod_{\alpha \in J} \overline{A_\alpha} = \overline{\prod_{\alpha \in J} A_\alpha}.$$

PROOF:

- (1)1. $\prod_{\alpha \in J} \overline{A_\alpha} \subseteq \overline{\prod_{\alpha \in J} A_\alpha}$
 (2)1. LET: $\{x_\alpha\}_{\alpha \in J} \in \prod_{\alpha \in J} \overline{A_\alpha}$
 (2)2. LET: $\prod_{\alpha \in J} U_\alpha$ be a basic neighbourhood of $\{x_\alpha\}_{\alpha \in J}$, where each U_α is open in X_α , and $U_\alpha = X_\alpha$ except for $\alpha = \alpha_1, \dots, \alpha_n$.
 (2)3. For $\alpha \in J$, PICK $a_\alpha \in A_\alpha \cap U_\alpha$.
 PROOF: By Theorem 3.11.3, using the Axiom of Choice.
 (2)4. $\{a_\alpha\}_{\alpha \in J} \in \prod_{\alpha \in J} A_\alpha \cap \prod_{\alpha \in J} U_\alpha$
 (2)5. Q.E.D.

PROOF: By Theorem 3.11.3.

- (1)2. $\overline{\prod_{\alpha \in J} A_\alpha} \subseteq \prod_{\alpha \in J} \overline{A_\alpha}$
 (2)1. LET: $\{x_\alpha\}_{\alpha \in J} \in \overline{\prod_{\alpha \in J} A_\alpha}$
 (2)2. LET: $\alpha \in J$
 PROVE: $x_\alpha \in \overline{A_\alpha}$
 (2)3. LET: U be a neighbourhood of x_α in X_α
 (2)4. $\pi_\alpha^{-1}(U)$ is a neighbourhood of $\{x_\alpha\}_{\alpha \in J}$
 (2)5. PICK $\{a_\alpha\}_{\alpha \in J} \in \pi_\alpha^{-1}(U) \cap \prod_{\alpha \in J} A_\alpha$
 PROOF: By Theorem 3.11.3.
 (2)6. $a_\alpha \in U \cap A_\alpha$
 (2)7. Q.E.D.

PROOF: By Theorem 3.11.3.

□

Definition 4.2.6 (Standard Topology on \mathbb{R}^J). For J a set, the *standard topology* on \mathbb{R}^J is the product topology where \mathbb{R} is given the standard topology.

Definition 4.2.7 (Closed Unit Ball). The *closed unit ball* B^2 is $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ as a subset of \mathbb{R}^2 .

Definition 4.2.8 (Sorgenfrey Plane). The *Sorgenfrey plane* is \mathbb{R}_l^2 .

4.3 The Subspace Topology

Definition 4.3.1 (Subspace Topology). Let X be a topological space and $Y \subseteq X$. The *subspace topology* on Y is $\{Y \cap U : U \text{ open in } X\}$. With this topology, Y is a *subspace* of X .

We prove this is a topology.

PROOF:

$\langle 1 \rangle 1$. LET: $\mathcal{T} = \{Y \cap U : U \text{ open in } X\}$

$\langle 1 \rangle 2$. $Y \in \mathcal{T}$

PROOF: $Y = Y \cap X$

$\langle 1 \rangle 3$. \mathcal{T} is closed under union.

$\langle 2 \rangle 1$. LET: $\mathcal{A} \subseteq \mathcal{T}$

PROVE: $\bigcup \mathcal{A} = Y \cap \bigcup \{U \text{ open in } X : Y \cap U \in \mathcal{A}\}$

$\langle 2 \rangle 2$. $\bigcup \mathcal{A} \subseteq Y \cap \bigcup \{U \text{ open in } X : Y \cap U \in \mathcal{A}\}$

$\langle 3 \rangle 1$. LET: $x \in \bigcup \mathcal{A}$

$\langle 3 \rangle 2$. PICK $V \in \mathcal{A}$ such that $x \in V$

$\langle 3 \rangle 3$. PICK U open in X such that $V = Y \cap U$

PROOF: By the definition of \mathcal{T} ($\langle 1 \rangle 1$, $\langle 2 \rangle 1$, $\langle 3 \rangle 2$)

$\langle 3 \rangle 4$. $x \in Y \cap \bigcup \{U \text{ open in } X : Y \cap U \in \mathcal{A}\}$

$\langle 2 \rangle 3$. $Y \cap \bigcup \{U \text{ open in } X : Y \cap U \in \mathcal{A}\} \subseteq \bigcup \mathcal{A}$

PROOF: Set theory.

$\langle 1 \rangle 4$. \mathcal{T} is closed under binary intersection.

PROOF: This holds because $(Y \cap U) \cap (Y \cap V) = Y \cap (U \cap V)$.

□

Lemma 4.3.2. *Let X be a topological space, $Y \subseteq X$, and $A \subseteq Y$. Then the topology A inherits as a subspace of X is the same as the topology A inherits as a subspace of Y .*

PROOF:

topology as a subspace of Y

$= \{V \cap A : V \text{ open in } Y\}$

$= \{V \cap A : \exists U \text{ open in } X. V = U \cap Y\}$

$= \{U \cap Y \cap A : U \text{ open in } X\}$

$= \{U \cap A : U \text{ open in } X\}$

$= \text{topology as a subspace of } X$ □

Lemma 4.3.3. *Let Y be a subspace of X . If U is open in Y and Y is open in X then U is open in X .*

PROOF:

$\langle 1 \rangle 1$. PICK V open in X such that $U = Y \cap V$

$\langle 1 \rangle 2$. U is open in X

PROOF: The open sets in X are closed under binary intersection.

□

Theorem 4.3.4. *Let Y be a subspace of X . Let $A \subseteq Y$. Let \bar{A} be the closure of A in X . Then the closure of A in Y is $\bar{A} \cap Y$.*

PROOF:

$\langle 1 \rangle 1$. $\bar{A} \cap Y$ is a closed set in Y that includes A .

$\langle 2 \rangle 1$. $\bar{A} \cap Y$ is closed in Y .

PROOF: By Lemma 4.3.4.1.

- (2)2. $A \subseteq \overline{A} \cap Y$.
 (1)2. If C is any closed set in Y that includes A then $\overline{A} \cap Y \subseteq C$.
 (2)1. LET: C be a closed set in Y that includes A .
 (2)2. PICK D closed in X such that $C = D \cap Y$.
 PROOF: By Lemma 4.3.4.1.
 (2)3. $\overline{A} \subseteq D$
 (2)4. $\overline{A} \subseteq C$

□

Corollary 4.3.4.1. *Let Y be a subspace of X . Then a set $A \subseteq Y$ is closed in Y if and only if it is the intersection of a closed set in X with Y .*

Corollary 4.3.4.2. *Let Y be a subspace of X . If A is closed in Y and Y is closed in X then A is closed in X .*

Lemma 4.3.5. *Let X be a topological space and $Y \subseteq X$. If \mathcal{B} is a basis for the topology on X then $\{B \cap Y : B \in \mathcal{B}\}$ is a basis for the subspace topology on Y .*

PROOF:

- (1)1. For all $B \in \mathcal{B}$, we have $B \cap Y$ is open in Y .
 PROOF: Immediate from definitions.
 (1)2. For every V open in Y and $y \in V$, there exists $B \in \mathcal{B}$ such that $y \in B \cap Y \subseteq V$.
 (2)1. LET: V be open in Y and $y \in V$
 (2)2. PICK U open in X such that $V = Y \cap U$
 (2)3. PICK $B \in \mathcal{B}$ such that $y \in B \subseteq U$
 (2)4. $y \in B \cap Y \subseteq V$

□

Lemma 4.3.6. *Let X be a topological space and $Y \subseteq X$. If \mathcal{S} is a subbasis for the topology on X then $\{S \cap Y : S \in \mathcal{S}\}$ is a subbasis for the subspace topology on Y .*

PROOF:

- (1)1. For all $S \in \mathcal{S}$, we have $S \cap Y$ is open in Y .
 PROOF: Immediate from definitions.
 (1)2. For every V open in Y and $y \in V$, there exist $S_1, \dots, S_n \in \mathcal{S}$ such that $y \in (S_1 \cap Y) \cap \dots \cap (S_n \cap Y) \subseteq V$
 (2)1. LET: V be open in Y and $y \in V$
 (2)2. PICK U open in X such that $V = U \cap Y$
 (2)3. PICK $S_1, \dots, S_n \in \mathcal{S}$ such that $y \in S_1 \cap \dots \cap S_n \subseteq U$
 (2)4. $y \in (S_1 \cap Y) \cap \dots \cap (S_n \cap Y) \subseteq V$

□

Theorem 4.3.7. *Let X be a linearly ordered set in the order topology. Let $Y \subseteq X$ be convex. Then the order topology on Y is the same as the subspace topology.*

PROOF:

- (1)1. LET: \mathcal{T}_o be the order topology and \mathcal{T}_s be the subspace topology.
 (1)2. $\mathcal{T}_o \subseteq \mathcal{T}_s$
 (2)1. For all $a \in Y$, we have $\{y \in Y : a < y\} \in \mathcal{T}_s$
 PROOF: $\{y \in Y : a < y\} = \{x \in X : a < x\} \cap Y$
 (2)2. For all $a \in Y$, we have $\{y \in Y : y < a\} \in \mathcal{T}_s$
 PROOF: Similar.
 (2)3. Q.E.D.
 PROOF: Lemma 4.1.3 and Corollary 3.14.2.1.
 (1)3. $\mathcal{T}_s \subseteq \mathcal{T}_o$
 (2)1. The sets $(a, +\infty) \cap Y$ and $(-\infty, a) \cap Y$ for $a \in X$ form a subbasis for \mathcal{T}_s
 PROOF: Lemma 4.3.6, Lemma 4.1.3.
 (2)2. For all $a \in X$, we have $(a, +\infty) \cap Y \in \mathcal{T}_o$
 (3)1. LET: $a \in X$
 (3)2. CASE: $a \in Y$
 PROOF: In this case, $(a, +\infty) \cap Y$ is an open ray in Y .
 (3)3. CASE: For all $y \in Y$ we have $a < y$
 PROOF: In this case, $(a, +\infty) \cap Y = Y$.
 (3)4. CASE: For all $y \in Y$ we have $y < a$
 PROOF: In this case, $(a, +\infty) \cap Y = \emptyset$.
 (3)5. Q.E.D.
 PROOF: These are the only cases because Y is convex.
 (2)3. For all $a \in X$, we have $(-\infty, a) \cap Y \in \mathcal{T}_o$
 PROOF: Similar.
 (2)4. Q.E.D.
 PROOF: Corollary 3.14.2.1.

□

Theorem 4.3.8. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces, and let A_α be a subspace of X_α for all α . Then the product topology on $\prod_{\alpha \in J} A_\alpha$ is the same as the topology it inherits as a subspace of $\prod_{\alpha \in J} X_\alpha$.*

PROOF: Each is the topology generated by the subbasis consisting of $\pi_\alpha^{-1}(U) \cap \prod_{\alpha \in J} A_\alpha = \pi_\alpha^{-1}(U \cap A_\alpha)$ where $\alpha \in J$ and U is open in X_α , using Lemma 4.3.6.

□

Definition 4.3.9 (Unit Circle). The *unit circle* S^1 is $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ as a subspace of \mathbb{R}^2 .

Proposition 4.3.10. *Let Y be a subspace of X , $A \subseteq Y$, and $a \in Y$. Then a is a limit point of A in the subspace topology on Y if and only if a is a limit point of A in the topology of X .*

PROOF:

$$\begin{aligned}
& a \text{ is a limit point of } A \text{ in } Y \\
& \Leftrightarrow \forall U \text{ open in } Y (a \in U \Rightarrow U \text{ intersects } A \text{ outside } a) \\
& \Leftrightarrow \forall V \text{ open in } X (a \in V \cap Y \Rightarrow V \cap Y \text{ intersects } A \text{ outside } a) \\
& \Leftrightarrow \forall V \text{ open in } X (a \in V \Rightarrow V \text{ intersects } A \text{ outside } a) \\
& \quad (a \in Y, A \subseteq Y) \\
& \Leftrightarrow a \text{ is a limit point of } A \text{ in } X
\end{aligned}$$

□

4.4 The Box Topology

Definition 4.4.1 (Box Topology). Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces. The *box topology* on $\prod_{\alpha \in J} X_\alpha$ is the topology generated by the basis consisting of all sets of the form $\prod_{\alpha \in J} U_\alpha$, where each U_α is open in X_α .

We prove this is a basis.

PROOF:

- ⟨1⟩1. LET: \mathcal{B} be the set of all sets of the form $\prod_{\alpha \in J} U_\alpha$, where each U_α is open in X_α .
- ⟨1⟩2. $\bigcup \mathcal{B} = \prod_{\alpha \in J} X_\alpha$
PROOF: This holds because $\prod_{\alpha \in J} X_\alpha \in \mathcal{B}$.
- ⟨1⟩3. \mathcal{B} is closed under binary intersection.
PROOF: $\prod_{\alpha \in J} U_\alpha \cap \prod_{\alpha \in J} V_\alpha = \prod_{\alpha \in J} (U_\alpha \cap V_\alpha)$.
- ⟨1⟩4. Q.E.D.
PROOF: Corollary 3.5.3.1.

Theorem 4.4.2 (AC). Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces, and let \mathcal{B}_α be a basis for the topology on X_α for each α . Then

$$\mathcal{B} = \left\{ \prod_{\alpha \in J} B_\alpha : \forall \alpha \in J, B_\alpha \in \mathcal{B}_\alpha \right\}$$

is a basis for the box topology on $\prod_{\alpha \in J} X_\alpha$.

PROOF:

- ⟨1⟩1. Every member of \mathcal{B} is open in the box topology.
PROOF: Immediate from definitions.
- ⟨1⟩2. For every open set U and $\{x_\alpha\}_{\alpha \in J} \in U$, there exists $B \in \mathcal{B}$ such that $\{x_\alpha\}_{\alpha \in J} \in B \subseteq U$.
 - ⟨2⟩1. LET: U be open and $\{x_\alpha\}_{\alpha \in J} \in U$
 - ⟨2⟩2. PICK U_α open in X_α for each α such that $\{x_\alpha\}_{\alpha \in J} \in \prod_{\alpha \in J} U_\alpha \subseteq U$.
 - ⟨2⟩3. PICK $B_\alpha \in \mathcal{B}_\alpha$ such that $x_\alpha \in B_\alpha \subseteq U_\alpha$ for each α
PROOF: Using the Axiom of Choice.
 - ⟨2⟩4. $\{x_\alpha\}_{\alpha \in J} \in \prod_{\alpha \in J} B_\alpha \subseteq U$

□

Theorem 4.4.3. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces, and let A_α be a subspace of X_α for all α . Let $\prod_{\alpha \in J} X_\alpha$ be given the box topology. Then the box topology on $\prod_{\alpha \in J} A_\alpha$ is the same as the topology it inherits as a subspace of $\prod_{\alpha \in J} X_\alpha$.*

PROOF: Each is the topology generated by the basis $\{\prod_{\alpha \in J} (U_\alpha \cap A_\alpha) : U_\alpha \text{ is open in } X_\alpha\}$, using Lemma 4.3.5. \square

Theorem 4.4.4. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of Hausdorff spaces. Then $\prod_{\alpha \in J} X_\alpha$ is Hausdorff under the box topology.*

PROOF:

- $\langle 1 \rangle 1$. LET: $\{x_\alpha\}_{\alpha \in J}, \{y_\alpha\}_{\alpha \in J} \in \prod_{\alpha \in J} X_\alpha$ with $\{x_\alpha\}_{\alpha \in J} \neq \{y_\alpha\}_{\alpha \in J}$
 - $\langle 1 \rangle 2$. PICK $\alpha \in J$ such that $x_\alpha \neq y_\alpha$
 - $\langle 1 \rangle 3$. PICK disjoint neighbourhoods U of x_α and V of y_α .
 - $\langle 1 \rangle 4$. $\pi_\alpha^{-1}(U)$ and $\pi_\alpha^{-1}(V)$ are disjoint neighbourhoods of $\{x_\alpha\}_{\alpha \in J}$ and $\{y_\alpha\}_{\alpha \in J}$
- \square

Corollary 4.4.4.1. *The space \mathbb{R}^ω under the box topology is Hausdorff.*

Theorem 4.4.5 (AC). *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces and $A_\alpha \subseteq X_\alpha$ for all α . If $\prod_{\alpha \in J} X_\alpha$ is given the box topology, then*

$$\prod_{\alpha \in J} \overline{A_\alpha} = \overline{\prod_{\alpha \in J} A_\alpha}.$$

PROOF:

- $\langle 1 \rangle 1$. $\prod_{\alpha \in J} \overline{A_\alpha} \subseteq \overline{\prod_{\alpha \in J} A_\alpha}$
 - $\langle 2 \rangle 1$. LET: $\{x_\alpha\}_{\alpha \in J} \in \prod_{\alpha \in J} \overline{A_\alpha}$
 - $\langle 2 \rangle 2$. LET: $\prod_{\alpha \in J} U_\alpha$ be a basic neighbourhood of $\{x_\alpha\}_{\alpha \in J}$, where each U_α is open in X_α .
 - $\langle 2 \rangle 3$. For $\alpha \in J$, PICK $a_\alpha \in A_\alpha \cap U_\alpha$.
- PROOF: By Theorem 3.11.3, using the Axiom of Choice.
- $\langle 2 \rangle 4$. $\{a_\alpha\}_{\alpha \in J} \in \prod_{\alpha \in J} A_\alpha \cap \prod_{\alpha \in J} U_\alpha$
 - $\langle 2 \rangle 5$. Q.E.D.

PROOF: By Theorem 3.11.3.

- $\langle 1 \rangle 2$. $\overline{\prod_{\alpha \in J} A_\alpha} \subseteq \prod_{\alpha \in J} \overline{A_\alpha}$
 - $\langle 2 \rangle 1$. LET: $\{x_\alpha\}_{\alpha \in J} \in \overline{\prod_{\alpha \in J} A_\alpha}$
 - $\langle 2 \rangle 2$. LET: $\alpha \in J$
- PROVE: $x_\alpha \in \overline{A_\alpha}$
- $\langle 2 \rangle 3$. LET: U be a neighbourhood of x_α in X_α
 - $\langle 2 \rangle 4$. $\pi_\alpha^{-1}(U)$ is a neighbourhood of $\{x_\alpha\}_{\alpha \in J}$
 - $\langle 2 \rangle 5$. PICK $\{a_\alpha\}_{\alpha \in J} \in \pi_\alpha^{-1}(U) \cap \prod_{\alpha \in J} A_\alpha$
- PROOF: By Theorem 3.11.3.
- $\langle 2 \rangle 6$. $a_\alpha \in U \cap A_\alpha$
 - $\langle 2 \rangle 7$. Q.E.D.

PROOF: By Theorem 3.11.3.

\square

4.5 The Quotient Topology

Definition 4.5.1 (Quotient Map). Let X and Y be topological spaces. Let $p : X \twoheadrightarrow Y$ be a surjective map. Then p is a *quotient map* iff, for all $U \subseteq Y$, we have U is open in Y iff $p^{-1}(U)$ is open in X .

Lemma 4.5.2. Let X and Y be topological spaces and $p : X \twoheadrightarrow Y$ be surjective and continuous. Then the following are equivalent.

1. p is a quotient map.
2. p maps saturated open sets to open sets.
3. p maps saturated closed sets to closed sets.

PROOF:

$\langle 1 \rangle 1. 1 \Rightarrow 2$

$\langle 2 \rangle 1.$ ASSUME: p is a quotient map.

$\langle 2 \rangle 2.$ LET: $U \subseteq Y$ be a saturated open set.

$\langle 2 \rangle 3.$ $U = p^{-1}(p(U))$

$\langle 3 \rangle 1.$ $U \subseteq p^{-1}(p(U))$

PROOF: Set theory.

$\langle 3 \rangle 2.$ $p^{-1}(p(U)) \subseteq U$

$\langle 4 \rangle 1.$ LET: $x \in p^{-1}(p(U))$

$\langle 4 \rangle 2.$ PICK $y \in U$ such that $p(x) = p(y)$

$\langle 4 \rangle 3.$ $x \in U$

PROOF: $\langle 2 \rangle 2, \langle 4 \rangle 2.$

$\langle 2 \rangle 4.$ $p(U)$ is open

PROOF: $\langle 2 \rangle 1, \langle 2 \rangle 3.$

$\langle 1 \rangle 2. 2 \Rightarrow 3$

$\langle 2 \rangle 1.$ ASSUME: p maps saturated open sets to open sets

$\langle 2 \rangle 2.$ LET: $C \subseteq X$ be a saturated closed set.

$\langle 2 \rangle 3.$ $X \setminus C$ is a saturated open set.

$\langle 3 \rangle 1.$ LET: $x \in X \setminus C$ and $x' \in X$ be such that $p(x) = p(x')$

$\langle 3 \rangle 2.$ $x' \notin C$

PROOF: If $x' \in C$ then $x \in C$ since C is saturated.

$\langle 2 \rangle 4.$ $p(X \setminus C)$ is open.

PROOF: By $\langle 2 \rangle 1$ and $\langle 2 \rangle 3.$

$\langle 2 \rangle 5.$ $p(X \setminus C) = Y \setminus p(C)$

$\langle 3 \rangle 1.$ $p(X \setminus C) \subseteq Y \setminus p(C)$

$\langle 4 \rangle 1.$ LET: $x \in X \setminus C$

$\langle 4 \rangle 2.$ ASSUME: for a contradiction $p(x) \in p(C)$

$\langle 4 \rangle 3.$ PICK $x' \in C$ such that $p(x) = p(x')$

$\langle 4 \rangle 4.$ Q.E.D.

PROOF: We have $x \notin C, x' \in C$ and $p(x) = p(x')$, contradicting $\langle 2 \rangle 2.$

$\langle 3 \rangle 2.$ $Y \setminus p(C) \subseteq p(X \setminus C)$

$\langle 4 \rangle 1.$ LET: $y \notin p(C)$

$\langle 4 \rangle 2.$ PICK $x \in X$ such that $p(x) = y$

PROOF: p is surjective.

$\langle 4 \rangle 3$. $x \notin C$

$\langle 1 \rangle 3$. $3 \Rightarrow 1$

$\langle 2 \rangle 1$. ASSUME: p maps saturated closed sets to closed sets

$\langle 2 \rangle 2$. LET: $C \subseteq Y$ be such that $p^{-1}(Y)$ is closed

$\langle 2 \rangle 3$. $p^{-1}(C)$ is saturated

$\langle 3 \rangle 1$. LET: $x \in p^{-1}(C)$, $x' \in X$ and $p(x) = p(x')$

$\langle 3 \rangle 2$. $x' \in p^{-1}(C)$

$\langle 2 \rangle 4$. $p(p^{-1}(C))$ is closed

PROOF: By $\langle 2 \rangle 1$ and $\langle 2 \rangle 3$.

$\langle 2 \rangle 5$. $C = p(p^{-1}(C))$

PROOF: By set theory, since p is surjective.

□

Corollary 4.5.2.1. *If $p : X \rightarrow Y$ is a surjective continuous map that is either an open map or a closed map, then p is a quotient map.*

Definition 4.5.3 (Quotient Topology). Let X be a topological space, A a set, and $p : X \rightarrow A$ a surjective map. Then the *quotient topology* on A induced by p is

$$\{U \subseteq A : p^{-1}(U) \text{ is open in } X\} .$$

It is easy to check this is a topology.

Lemma 4.5.4. *Let X be a topological space, A a set, and $p : X \rightarrow A$ a surjective map. Then the quotient topology induced by p is the unique topology on A such that p is a quotient map.*

PROOF: Immediate from definitions. □

Definition 4.5.5 (Quotient Space). Let X be a topological space and X^* a partition of X . Let $p : X \rightarrow X^*$ be the canonical map. Then X^* under the quotient topology induced by p is called a *quotient space* of X .

Proposition 4.5.6. *Let $p : X \rightarrow Y$ be a quotient map. Let $A \subseteq X$ be open and saturated. Then $p \upharpoonright_A : A \rightarrow p(A)$ is a quotient map.*

PROOF:

$\langle 1 \rangle 1$. LET: $q = p \upharpoonright_A : A \rightarrow p(A)$

$\langle 1 \rangle 2$. For all $V \subseteq p(A)$, we have $q^{-1}(V) = p^{-1}(V)$

$\langle 2 \rangle 1$. $q^{-1}(V) \subseteq p^{-1}(V)$

PROOF: Trivial.

$\langle 2 \rangle 2$. $p^{-1}(V) \subseteq q^{-1}(V)$

$\langle 3 \rangle 1$. LET: $x \in p^{-1}(V)$

$\langle 3 \rangle 2$. PICK $x' \in A$ such that $p(x') = p(x)$

PROOF: One exists because $p(x) \in V \subseteq p(A)$.

$\langle 3 \rangle 3$. $x \in A$

PROOF: This holds because A is saturated.

$\langle 3 \rangle 4$. $x \in q^{-1}(V)$

PROOF: From $\langle 3 \rangle 1$ and $\langle 3 \rangle 3$.

- $\langle 1 \rangle 3$. For all $U \subseteq X$, we have $p(U \cap A) = p(U) \cap p(A)$
- $\langle 1 \rangle 4$. LET: $V \subseteq p(A)$ be such that $q^{-1}(V)$ is open in A .
PROVE: V is open in $p(A)$.
- $\langle 1 \rangle 5$. $q^{-1}(V)$ is open in X
- $\langle 1 \rangle 6$. $p^{-1}(V)$ is open in X
- $\langle 1 \rangle 7$. V is open in Y
- $\langle 1 \rangle 8$. V is open in $p(A)$

□

Proposition 4.5.7. *Let $p : X \rightarrow Y$ be a quotient map. Let $A \subseteq X$ be closed and saturated. Then $p \upharpoonright_A : A \rightarrow p(A)$ is a quotient map.*

PROOF: Similar. □

Proposition 4.5.8. *Let $p : X \rightarrow Y$ be an open quotient map. Let $A \subseteq X$ be saturated. Then $p \upharpoonright_A : A \rightarrow p(A)$ is a quotient map.*

PROOF:

- $\langle 1 \rangle 1$. LET: $q = p \upharpoonright_A : A \rightarrow p(A)$
- $\langle 1 \rangle 2$. For all $V \subseteq p(A)$, we have $q^{-1}(V) = p^{-1}(V)$
 - $\langle 2 \rangle 1$. $q^{-1}(V) \subseteq p^{-1}(V)$
PROOF: Trivial.
 - $\langle 2 \rangle 2$. $p^{-1}(V) \subseteq q^{-1}(V)$
 - $\langle 3 \rangle 1$. LET: $x \in p^{-1}(V)$
 - $\langle 3 \rangle 2$. PICK $x' \in A$ such that $p(x') = p(x)$
PROOF: One exists because $p(x) \in V \subseteq p(A)$.
 - $\langle 3 \rangle 3$. $x \in A$
PROOF: This holds because A is saturated.
 - $\langle 3 \rangle 4$. $x \in q^{-1}(V)$
PROOF: From $\langle 3 \rangle 1$ and $\langle 3 \rangle 3$.
- $\langle 1 \rangle 3$. For all $U \subseteq X$, we have $p(U \cap A) = p(U) \cap p(A)$
 - $\langle 2 \rangle 1$. $p(U \cap A) \subseteq p(U) \cap p(A)$
PROOF: Set theory.
 - $\langle 2 \rangle 2$. $p(U) \cap p(A) \subseteq p(U \cap A)$
 - $\langle 3 \rangle 1$. LET: $x \in U, y \in A, p(x) = p(y)$
PROVE: $p(x) \in p(U \cap A)$
 - $\langle 3 \rangle 2$. $x \in A$
PROOF: A is saturated.
 - $\langle 3 \rangle 3$. $x \in U \cap A$
- $\langle 1 \rangle 4$. LET: $V \subseteq p(A)$ be such that $q^{-1}(V)$ is open in A .
PROVE: V is open in $p(A)$.
- $\langle 1 \rangle 5$. $p^{-1}(V)$ is open in A
PROOF: By $\langle 1 \rangle 2$
- $\langle 1 \rangle 6$. PICK U open in X such that $p^{-1}(V) = U \cap A$
- $\langle 1 \rangle 7$. $V = p(U) \cap p(A)$

PROOF:

$$\begin{aligned}
 V &= p(p^{-1}(V)) && (p \text{ is surjective}) \\
 &= p(U \cap A) && (\langle 1 \rangle 6) \\
 &= p(U) \cap p(A) && (\langle 1 \rangle 3)
 \end{aligned}$$

$\langle 1 \rangle 8$. $p(U)$ is open in Y

PROOF: $\langle 1 \rangle 6$, p is an open map.

$\langle 1 \rangle 9$. V is open in $p(A)$

PROOF: $\langle 1 \rangle 7$, $\langle 1 \rangle 8$

□

Proposition 4.5.9. *Let $p : X \twoheadrightarrow Y$ be a closed quotient map. Let $A \subseteq X$ be saturated. Then $p \restriction_A : A \twoheadrightarrow p(A)$ is a quotient map.*

PROOF: Similar. □

Proposition 4.5.10. *The composite of two quotient maps is a quotient map.*

PROOF: From Proposition 5.2.22. □

Proposition 4.5.11. *Let X^* be a quotient space of X . If every element of X^* is closed in X , then X^* is T_1 .*

PROOF:

$\langle 1 \rangle 1$. LET: $C \in X^*$

$\langle 1 \rangle 2$. $p^{-1}(\{C\}) = C$

PROOF: Definition of p .

$\langle 1 \rangle 3$. $p^{-1}(\{C\})$ is closed in X

PROOF: By hypothesis.

$\langle 1 \rangle 4$. $\{C\}$ is closed in X^* .

PROOF: By Proposition 5.2.21.

□

Chapter 5

Functions Between Topological Spaces

5.1 Open Maps

Definition 5.1.1. Let X and Y be topological spaces and $f : X \rightarrow Y$. Then f is an *open map* iff, for all U open in X , $f(U)$ is open in Y .

Lemma 5.1.2. Let X and Y be topological spaces and $f : X \rightarrow Y$. Let \mathcal{B} be a basis for the topology on X . Then f is an open map if and only if, for all $B \in \mathcal{B}$, $f(B)$ is open in Y .

PROOF:

$\langle 1 \rangle 1$. If f is an open map then, for all $B \in \mathcal{B}$, $f(B)$ is open in Y .

PROOF: Immediate from definitions.

$\langle 1 \rangle 2$. If, for all $B \in \mathcal{B}$, $f(B)$ is open in Y , then f is an open map.

$\langle 2 \rangle 1$. ASSUME: For all $B \in \mathcal{B}$, $f(B)$ is open in Y .

$\langle 2 \rangle 2$. LET: U be open in X

PROVE: $f(U)$ is open in Y

$\langle 2 \rangle 3$. LET: $\mathcal{B}_0 \subseteq \mathcal{B}$ be such that $U = \bigcup \mathcal{B}_0$

$\langle 2 \rangle 4$. $f(U) = \bigcup_{B \in \mathcal{B}_0} f(B)$

PROOF: Set theory.

$\langle 2 \rangle 5$. $f(U)$ is open in Y .

PROOF: From $\langle 2 \rangle 1$, $\langle 2 \rangle 4$ and the fact that the open sets are closed under union.

□

Corollary 5.1.2.1. Let X and Y be topological spaces and $f : X \rightarrow Y$. Let \mathcal{S} be a subbasis for the topology on X . Then f is an open map if and only if, for all $S \in \mathcal{S}$, $f(S)$ is open in Y .

Lemma 5.1.3 (AC). Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces. Then the projection $\pi_\alpha : \prod_{\alpha \in J} X_\alpha \rightarrow X_\alpha$ is an open map.

PROOF:

$\langle 1 \rangle 1$. For U open in X_α , we have $\pi_\alpha(\pi_\alpha^{-1}(U))$ is open in X_α

PROOF: $\pi_\alpha(\pi_\alpha^{-1}(U)) = U$ if all the other X_α are nonempty, \emptyset otherwise.

$\langle 1 \rangle 2$. For $\beta \neq \alpha$ and U open in X_β , we have $\pi_\alpha(\pi_\beta^{-1}(U))$ is open in X_α

PROOF: $\pi_\alpha(\pi_\beta^{-1}(U)) = X_\alpha$ if all the X_γ are nonempty for $\gamma \neq \alpha$, \emptyset otherwise.

$\langle 1 \rangle 3$. Q.E.D.

PROOF: By Corollary 5.1.2.1.

5.2 Continuous Functions

Definition 5.2.1 (Continuous). Let X and Y be topological spaces and $f : X \rightarrow Y$ a function. Then f is *continuous* if and only if, for every open set U in Y , the set $f^{-1}(U)$ is open in X .

Theorem 5.2.2. Let X and Y be topological spaces and $f : X \rightarrow Y$. Then the following are equivalent.

1. f is continuous.
2. For every closed set C in Y , the set $f^{-1}(C)$ is closed in X .
3. For every set $A \subseteq X$, we have $f(\overline{A}) \subseteq \overline{f(A)}$.

PROOF:

$\langle 1 \rangle 1$. $1 \Rightarrow 3$

$\langle 2 \rangle 1$. ASSUME: f is continuous.

$\langle 2 \rangle 2$. LET: $A \subseteq X$

$\langle 2 \rangle 3$. LET: $x \in \overline{A}$

PROVE: $f(x) \in \overline{f(A)}$

$\langle 2 \rangle 4$. LET: V be a neighbourhood of $f(x)$

$\langle 2 \rangle 5$. $f^{-1}(V)$ is a neighbourhood of x

PROOF: $\langle 2 \rangle 1$, $\langle 2 \rangle 3$, $\langle 2 \rangle 4$

$\langle 2 \rangle 6$. $f^{-1}(V)$ intersects A in a , say.

PROOF: $\langle 2 \rangle 3$, $\langle 2 \rangle 5$, Theorem 3.11.3.

$\langle 2 \rangle 7$. V intersects $f(A)$ in $f(a)$.

$\langle 2 \rangle 8$. Q.E.D.

PROOF: Theorem 3.11.3.

$\langle 1 \rangle 2$. $3 \Rightarrow 2$

$\langle 2 \rangle 1$. ASSUME: 3

$\langle 2 \rangle 2$. LET: C be a closed set in Y

$\langle 2 \rangle 3$. $f^{-1}(C) = \overline{f^{-1}(C)}$

PROOF:

$$\begin{aligned} f(\overline{f^{-1}(C)}) &\subseteq \overline{f(f^{-1}(C))} & (\langle 2 \rangle 1) \\ &\subseteq \overline{C} \end{aligned}$$

$\langle 1 \rangle 3$. $2 \Rightarrow 1$

$\langle 2 \rangle 1$. ASSUME: 2

- ⟨2⟩2. LET: V be open in Y
- ⟨2⟩3. $f^{-1}(Y \setminus V)$ is closed in X
- PROOF: By ⟨2⟩1.
- ⟨2⟩4. $f^{-1}(V)$ is open in X .
- PROOF: $f^{-1}(V) = X \setminus f^{-1}(Y \setminus V)$.

□

Lemma 5.2.3. *If $f : X \rightarrow Y$ maps all of X to the single point y_0 of Y , then f is continuous.*

PROOF: For V open in Y , the set $f^{-1}(V)$ is either X (if $y_0 \in V$) or \emptyset (if $y_0 \notin V$).

Definition 5.2.4 (Continuity at a Point). Let X and Y be topological spaces, $f : X \rightarrow Y$ a function, and $x \in X$. Then f is *continuous at x* if and only if, for every neighbourhood V of $f(x)$, there exists a neighbourhood U of x such that $f(U) \subseteq V$.

Theorem 5.2.5. *Let X and Y be topological spaces and $f : X \rightarrow Y$. Then f is continuous if and only if f is continuous at every point of X .*

PROOF:

- ⟨1⟩1. If f is continuous then f is continuous at every point of X .
- ⟨2⟩1. ASSUME: f is continuous
- ⟨2⟩2. LET: $x \in X$
- ⟨2⟩3. LET: V be a neighbourhood of $f(x)$
- ⟨2⟩4. $f^{-1}(V)$ is a neighbourhood of x
- ⟨2⟩5. $f(f^{-1}(V)) \subseteq V$
- ⟨1⟩2. If f is continuous at every point of X then f is continuous.
- ⟨2⟩1. ASSUME: f is continuous at every point of X .
- ⟨2⟩2. LET: V be open in Y
- PROVE: $f^{-1}(V)$ is open in X .
- ⟨2⟩3. LET: $x \in f^{-1}(V)$
- ⟨2⟩4. V is a neighbourhood of $f(x)$
- ⟨2⟩5. PICK a neighbourhood U of x such that $f(U) \subseteq V$
- PROOF: By ⟨2⟩1.
- ⟨2⟩6. $x \in U \subseteq f^{-1}(V)$
- ⟨2⟩7. Q.E.D.
- PROOF: By Proposition 3.2.3.

□

Lemma 5.2.6. *Let X and Y be topological spaces and $f : X \rightarrow Y$. Let \mathcal{B} be a basis for the topology on Y . Then f is continuous if and only if, for all $B \in \mathcal{B}$, the set $f^{-1}(B)$ is open in X .*

PROOF:

- ⟨1⟩1. If f is continuous then, for all $B \in \mathcal{B}$, the set $f^{-1}(B)$ is open in X .
- PROOF: Immediate from definitions.
- ⟨1⟩2. If, for all $B \in \mathcal{B}$, the set $f^{-1}(B)$ is open in X , then f is continuous.

⟨2⟩1. ASSUME: For all $B \in \mathcal{B}$, the set $f^{-1}(B)$ is open in X .

⟨2⟩2. LET: $x \in X$

⟨2⟩3. LET: V be a neighbourhood of $f(x)$

⟨2⟩4. PICK $B \in \mathcal{B}$ such that $f(x) \in B \subseteq V$

⟨2⟩5. $f^{-1}(B)$ is a neighbourhood of x

PROOF: By ⟨2⟩1.

⟨2⟩6. $f(f^{-1}(B)) \subseteq B$

PROOF: Set theory.

⟨2⟩7. Q.E.D.

PROOF: Theorem 5.2.5.

□

Lemma 5.2.7. *The projections $\pi_1 : X \times Y \rightarrow X$ and $\pi_2 : X \times Y \rightarrow Y$ are continuous.*

PROOF: Immediate from definitions. □

Theorem 5.2.8. *If A is a subspace of X , the inclusion function $j : A \rightarrow X$ is continuous.*

PROOF: For V open in X , the set $j^{-1}(V) = V \cap A$ is open in A .

Theorem 5.2.9. *If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are continuous, then the map $g \circ f : X \rightarrow Z$ is continuous.*

PROOF:

⟨1⟩1. LET: V be open in Z

⟨1⟩2. $g^{-1}(V)$ is open in Y

⟨1⟩3. $f^{-1}(g^{-1}(V))$ is open in X

□

Theorem 5.2.10. *If $f : X \rightarrow Y$ is continuous and if A is a subspace of X , then the restricted function $f \upharpoonright A : A \rightarrow Y$ is continuous.*

PROOF: For V open in Y , the set $(f \upharpoonright A)^{-1}(V) = f^{-1}(V) \cap A$ is open in A . □

Theorem 5.2.11. *Let $f : X \rightarrow Y$ be continuous. If Z is a subspace of Y that includes the range of f , then the function $g : X \rightarrow Z$ obtained by restricting the codomain of f is continuous. If Z is a space having Y as a subspace, then the function $h : X \rightarrow Z$ obtained by expanding the codomain of f is continuous.*

PROOF:

⟨1⟩1. If Z is a subspace of Y that includes the range of f , then the function $g : X \rightarrow Z$ obtained by restricting the codomain of f is continuous.

⟨2⟩1. LET: V be open in Z

⟨2⟩2. PICK W open in Y such that $V = W \cap Z$

⟨2⟩3. $f^{-1}(W)$ is open in X .

⟨2⟩4. $g^{-1}(V)$ is open in X .

PROOF: $g^{-1}(V) = f^{-1}(W)$.

⟨1⟩2. If Z is a space having Y as a subspace, then the function $h : X \rightarrow Z$ obtained by expanding the codomain of f is continuous.

PROOF: For V open in Z , we have $h^{-1}(V) = f^{-1}(V \cap Y)$ is open in X .
□

Theorem 5.2.12. *Let X and Y be topological spaces and $f : X \rightarrow Y$. If $x_n \rightarrow x$ as $n \rightarrow \infty$ in X and f is continuous at x , then $f(x_n) \rightarrow f(x)$ as $n \rightarrow \infty$ in Y .*

PROOF:

⟨1⟩1. ASSUME: $x_n \rightarrow x$ as $n \rightarrow \infty$

⟨1⟩2. ASSUME: f is continuous at x

⟨1⟩3. LET: V be a neighbourhood of $f(x)$

⟨1⟩4. PICK a neighbourhood U of x such that $f(U) \subseteq V$

PROOF: By ⟨1⟩2.

⟨1⟩5. PICK N such that, for all $n \geq N$, $x_n \in U$

PROOF: By ⟨1⟩1

⟨1⟩6. For $n \geq N$, $f(x_n) \in V$

PROOF: By ⟨1⟩4.

□

Theorem 5.2.13. *Let X , Y and Z be topological spaces. Let $f : X \rightarrow Y$. If there exists a set \mathcal{A} of open sets in X such that:*

- $\bigcup \mathcal{A} = X$;
- for all $U \in \mathcal{A}$, the function $f \upharpoonright U : U \rightarrow Y$ is continuous;

then f is continuous.

PROOF:

⟨1⟩1. LET: V be open in Y

⟨1⟩2. For all $U \in \mathcal{A}$, the set $(f \upharpoonright U)^{-1}(V)$ is open in X .

⟨2⟩1. LET: $U \in \mathcal{A}$

⟨2⟩2. $(f \upharpoonright U)^{-1}(V)$ is open in U

PROOF: Since $f \upharpoonright U : U \rightarrow Y$ is continuous.

⟨2⟩3. Q.E.D.

PROOF: By Lemma 4.3.3.

⟨1⟩3. Q.E.D.

PROOF: Since $f^{-1}(V) = \bigcup_{U \in \mathcal{A}} (f \upharpoonright U)^{-1}(V)$.

Theorem 5.2.14 (The Pasting Lemma). *Let $X = A \cup B$ where A and B are closed in X . Let $f : A \rightarrow Y$ and $g : B \rightarrow Y$ be continuous. If $f(x) = g(x)$ for every $x \in A \cap B$, then the function $h : X \rightarrow Y$ defined by*

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases}$$

is continuous.

PROOF:

- ⟨1⟩1. LET: C be closed in Y
- ⟨1⟩2. $f^{-1}(C)$ is closed in A
PROOF: Theorem 5.2.2.
- ⟨1⟩3. $f^{-1}(C)$ is closed in X
PROOF: Lemma 4.3.4.1.
- ⟨1⟩4. $g^{-1}(C)$ is closed in B
PROOF: Theorem 5.2.2.
- ⟨1⟩5. $g^{-1}(C)$ is closed in X
PROOF: Lemma 4.3.4.1.
- ⟨1⟩6. $h^{-1}(C)$ is closed in X
PROOF: $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$
- ⟨1⟩7. Q.E.D.
PROOF: Theorem 5.2.2.

□

Theorem 5.2.15. Let $f : A \rightarrow \prod_{\alpha \in J} X_\alpha$ be given by the equation

$$f(a) = \{f_\alpha(a)\}_{\alpha \in J} ,$$

where $f_\alpha : A \rightarrow X_\alpha$ for each α . Let $\prod_{\alpha \in J} X_\alpha$ have the product topology. Then the function f is continuous if and only if each function f_α is continuous.

PROOF:

- ⟨1⟩1. If f is continuous then each f_α is continuous.
PROOF: This holds because $f_\alpha = \pi_\alpha \circ f$.
- ⟨1⟩2. If every f_α is continuous then f is continuous.
- ⟨2⟩1. ASSUME: Every f_α is continuous.
- ⟨2⟩2. LET: $\alpha \in J$ and U be open in X_α
- ⟨2⟩3. $f^{-1}(\pi_\alpha^{-1}(U))$ is open in A
PROOF: $f^{-1}(\pi_\alpha^{-1}(U)) = f_\alpha^{-1}(U)$.

□

5.2.1 Homeomorphisms

Definition 5.2.16 (Homeomorphism). Let X and Y be topological spaces and $f : X \rightarrow Y$. Then f is a *homeomorphism* between X and Y iff f is a bijection, and f and f^{-1} are both continuous.

Definition 5.2.17 (Topological Property). A property P of topological spaces is a *topological property* iff, for any spaces X and Y , if X is homeomorphic to Y then P holds of X if and only if P holds of Y .

Definition 5.2.18 ((Topological) Imbedding). Let X and Y be topological spaces and $f : X \rightarrow Y$. Then f is a *(topological) imbedding* iff f is a homeomorphism between X and $\text{im } f$.

Definition 5.2.19 (Homogeneous). A topological space X is *homogeneous* iff, for all $x, y \in X$, there exists a homeomorphism $f : X \cong X$ such that $f(x) = y$.

5.2.2 Strongly Continuous Functions

Definition 5.2.20 (Strongly Continuous). Let X and Y be topological spaces and $f : X \rightarrow Y$. Then f is *strongly continuous* iff, for all $V \subseteq Y$, we have V is open in Y if and only if $f^{-1}(V)$ is open in X .

Proposition 5.2.21. *Let X and Y be topological spaces and $f : X \rightarrow Y$. Then f is strongly continuous if and only if, for all $C \subseteq Y$, C is closed in Y if and only if $f^{-1}(C)$ is closed in X .*

PROOF:

(1)1. If f is strongly continuous then, for all $C \subseteq Y$, we have C is closed in Y if and only if $f^{-1}(C)$ is closed in X .

PROOF:

$$\begin{aligned} C \text{ is closed in } Y &\Leftrightarrow Y \setminus C \text{ is open in } Y \\ &\Leftrightarrow f^{-1}(Y \setminus C) \text{ is open in } X \\ &\Leftrightarrow X \setminus f^{-1}(C) \text{ is open in } X \\ &\Leftrightarrow f^{-1}(C) \text{ is closed in } X \end{aligned}$$

(1)2. If, for all $C \subseteq Y$, we have C is closed in Y if and only if $f^{-1}(C)$ is closed in X , then f is strongly continuous.

PROOF: Similar.

□

Proposition 5.2.22. *The composite of two strongly continuous functions is strongly continuous.*

PROOF:

(1)1. LET: $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be strongly continuous.

(1)2. LET: $V \subseteq Z$

(1)3. V is open iff $f^{-1}(g^{-1}(V))$ is open

PROOF:

$$\begin{aligned} V \text{ is open} &\Leftrightarrow g^{-1}(V) \text{ is open} && ((1)1) \\ &\Leftrightarrow f^{-1}(g^{-1}(V)) \text{ is open} && ((1)1) \end{aligned}$$

□

Proposition 5.2.23. *Let X , Y and Z be topological spaces. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$. If f is strongly continuous and $g \circ f$ is continuous, then g is continuous.*

PROOF:

(1)1. LET: $V \subseteq Z$ be open in Z .

(1)2. $f^{-1}(g^{-1}(V))$ is open in X .

PROOF: $g \circ f$ is continuous.

(1)3. $g^{-1}(V)$ is open in Y .

PROOF: f is strongly continuous.

□

Proposition 5.2.24. *Let X, Y and Z be topological spaces. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$. If f and $g \circ f$ are strongly continuous, then g is strongly continuous.*

PROOF:

$\langle 1 \rangle 1$. LET: $U \subseteq Z$

$\langle 1 \rangle 2$. U is open in Z iff $g^{-1}(U)$ is open in Y

PROOF:

U is open in $Z \Leftrightarrow f^{-1}(g^{-1}(U))$ is open in X ($g \circ f$ is strongly continuous)

$\Leftrightarrow g^{-1}(U)$ is open in Y (f is strongly continuous)

□

5.3 Closed Maps

Definition 5.3.1 (Closed Map). Let X and Y be topological spaces and $f : X \rightarrow Y$. Then f is a *closed map* iff, for every closed set $C \subseteq X$, the set $f(C)$ is closed in Y .

5.4 Local Homeomorphism

Definition 5.4.1 (Locally Homeomorphic). Let X and Y be topological spaces. Then X is *locally homeomorphic* to Y iff every point in X has an open neighborhood that is homeomorphic with an open set in Y .

Proposition 5.4.2. *The long line is locally homeomorphic with \mathbb{R} .*

PROOF:

$\langle 1 \rangle 1$. LET: $x \in L$

$\langle 1 \rangle 2$. PICK an ordinal α such that $x < (\alpha, 0)$.

$\langle 1 \rangle 3$. $(-\infty, (\alpha, 0))$ is an open neighbourhood of x that is homeomorphic to $(0, 1)$.

□

5.5 Retracts

Definition 5.5.1 (Retract). Let Z be a topological space. If Y is a subspace of Z , we say that Y is a *retract* of Z iff there exists a continuous function $r : Z \rightarrow Y$ such that $r(y) = y$ for all $y \in Y$.

Chapter 6

Separation Axioms

6.1 T_1 Spaces

Definition 6.1.1 (T_1 Space). A topological space X is a T_1 space iff every finite set is closed.

Theorem 6.1.2. Let X be a T_1 space and $A \subseteq X$. Then x is a limit point of A if and only if every neighbourhood of x contains infinitely many points of A .

PROOF:

$\langle 1 \rangle 1$. If some neighbourhood of x contains only finitely many points of A then x is not a limit point of A .

$\langle 2 \rangle 1$. ASSUME: Some neighbourhood U of x contains only finite many points a_1, \dots, a_n of A .

$\langle 2 \rangle 2$. $X \setminus \{a_1, \dots, a_n\}$ is open.

PROOF: X is T_1 .

$\langle 2 \rangle 3$. $U \setminus \{a_1, \dots, a_n\}$ is a neighbourhood of x that does not intersect A .

$\langle 1 \rangle 2$. If every neighbourhood of x contains infinitely many points of A then x is a limit point of A .

PROOF: From the definition of limit point.

□

Proposition 6.1.3. A subspace of a T_1 space is T_1 .

PROOF:

$\langle 1 \rangle 1$. LET: X be a T_1 space and $Y \subseteq X$

$\langle 1 \rangle 2$. LET: $a \in Y$

$\langle 1 \rangle 3$. $\{a\}$ is closed in X

PROOF: By $\langle 1 \rangle 1$.

$\langle 1 \rangle 4$. $\{a\}$ is closed in Y

PROOF: By Corollary 4.3.4.1.

□

Definition 6.1.4 (Separate Points from Closed Sets). Let X be a space and $\{f_\alpha\}_{\alpha \in J}$ be a family of continuous functions $f_\alpha : X \rightarrow \mathbb{R}$. Then $\{f_\alpha\}$ *separates points from closed sets* in X iff, for every point $x_0 \in X$ and every neighbourhood U of x_0 , there exists $\alpha \in J$ such that f_α is positive at x_0 and vanishes outside U .

Theorem 6.1.5 (Imbedding Theorem). Let X be a T_1 space and $\{f_\alpha\}_{\alpha \in J}$ be a family of functions $X \rightarrow \mathbb{R}$ that separates points from closed sets. Then the function $F : X \rightarrow \mathbb{R}^J$ defined by

$$F(x)_\alpha = f_\alpha(x)$$

is an imbedding. If each f_α maps X into $[0, 1]$ then F is an imbedding $X \rightarrow [0, 1]^J$.

PROOF:

(1)1. F is continuous

PROOF: By Theorem 5.2.15.

(1)2. F is injective

(2)1. LET: $x, y \in X$ with $x \neq y$

(2)2. PICK a neighbourhood U of x such that $y \notin U$

PROOF: X is T_1

(2)3. PICK $\alpha \in J$ such that f_α is positive at x and vanishes outside U

(2)4. $f_\alpha(x) \neq f_\alpha(y)$

(2)5. $F(x) \neq F(y)$

(1)3. F is open as a map $X \rightarrow F(U)$

(2)1. LET: U be open

(2)2. LET: $z \in F(U)$

(2)3. PICK $x \in U$ such that $F(x) = z$

(2)4. PICK $\alpha \in J$ such that f_α is positive at x and vanishes outside U

(2)5. $z \in \pi_\alpha^{-1}((0, +\infty)) \cap F(U) \subseteq F(U)$

□

6.2 Hausdorff Spaces

Definition 6.2.1 (Hausdorff Space). A topological space X is a *Hausdorff space* iff, for any points $x, y \in X$ with $x \neq y$, there exist disjoint neighbourhoods U of x and V of y .

Theorem 6.2.2. Every Hausdorff space is T_1 .

PROOF:

(1)1. LET: X be a Hausdorff space

(1)2. LET: $a \in X$

PROVE: $\{a\}$ is closed.

(1)3. LET: $b \in X \setminus \{a\}$

(1)4. PICK disjoint neighbourhoods U of a and V of b

⟨1⟩5. $b \in V \subseteq X \setminus \{a\}$

⟨1⟩6. Q.E.D.

PROOF: By Proposition 3.2.3.

□

Theorem 6.2.3. *In a Hausdorff space, a sequence has at most one limit.*

PROOF:

⟨1⟩1. ASSUME: for a contradiction $x_n \rightarrow l$ and $x_n \rightarrow m$ as $n \rightarrow \infty$, and $l \neq m$

⟨1⟩2. PICK disjoint neighbourhoods U of l and V of m

⟨1⟩3. PICK N such that, for all $n \geq N$, $x_n \in U$ and $x_n \in V$

⟨1⟩4. $x_N \in U \cap V$

□

Theorem 6.2.4. *Every linearly ordered set is Hausdorff under the order topology.*

PROOF:

⟨1⟩1. LET: X be a linearly ordered set under the order topology.

⟨1⟩2. LET: $x, y \in X$ with $x \neq y$

⟨1⟩3. ASSUME: w.l.o.g. $x < y$

PROVE: There exist disjoint neighbourhoods U of x and V of y .

⟨1⟩4. CASE: There exists z such that $x < z < y$

PROOF: In this case, take $U = (-\infty, z)$ and $V = (z, +\infty)$.

⟨1⟩5. CASE: There does not exist z such that $x < z < y$

PROOF: In this case, take $U = (-\infty, y)$ and $V = (x, +\infty)$.

□

Theorem 6.2.5. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of Hausdorff spaces. Then $\prod_{\alpha \in J} X_\alpha$ is Hausdorff under the product topology.*

PROOF:

⟨1⟩1. LET: $\{x_\alpha\}_{\alpha \in J}, \{y_\alpha\}_{\alpha \in J} \in \prod_{\alpha \in J} X_\alpha$ with $\{x_\alpha\}_{\alpha \in J} \neq \{y_\alpha\}_{\alpha \in J}$

⟨1⟩2. PICK $\alpha \in J$ such that $x_\alpha \neq y_\alpha$

⟨1⟩3. PICK disjoint neighbourhoods U of x_α and V of y_α .

⟨1⟩4. $\pi_\alpha^{-1}(U)$ and $\pi_\alpha^{-1}(V)$ are disjoint neighbourhoods of $\{x_\alpha\}_{\alpha \in J}$ and $\{y_\alpha\}_{\alpha \in J}$

□

Corollary 6.2.5.1. *The Sorgenfrey plane is Hausdorff.*

Corollary 6.2.5.2. *For any set I , the space \mathbb{R}^I is Hausdorff.*

Proposition 6.2.6. *Let X and Y be topological spaces and $f : X \rightarrow Y$. If f is continuous and injective and Y is Hausdorff then X is Hausdorff.*

PROOF:

⟨1⟩1. LET: $x, y \in X$ with $x \neq y$

⟨1⟩2. $f(x) \neq f(y)$

PROOF: f is injective.

⟨1⟩3. PICK disjoint neighbourhoods U, V of $f(x)$ and $f(y)$

PROOF: Y is Hausdorff.

⟨1⟩4. $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint neighbourhoods of x and y .

□

Corollary 6.2.6.1. *A subspace of a Hausdorff space is Hausdorff.*

Corollary 6.2.6.2. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of nonempty spaces. If $\prod_{\alpha \in J} X_\alpha$ is Hausdorff then so is each X_α .*

Corollary 6.2.6.3. *Let \mathcal{T} and \mathcal{T}' be topologies on the same set X . If $\mathcal{T} \subseteq \mathcal{T}'$ and X is Hausdorff under \mathcal{T} then X is Hausdorff under \mathcal{T}' .*

Corollary 6.2.6.4. *The space \mathbb{R}_K is Hausdorff.*

Proposition 6.2.7. *\mathbb{R}_l is Hausdorff.*

PROOF: Let $a, b \in \mathbb{R}_l$ with $a < b$. Then $(-\infty, b)$ and $[b, +\infty)$ are disjoint open sets containing a and b respectively. □

Proposition 6.2.8. *The continuous image of a Hausdorff space is not necessarily Hausdorff.*

PROOF: The identity map from the discrete two-point space to the indiscrete two-point space is continuous. □

Lemma 6.2.9. *Let A be a subspace of X and Z be Hausdorff. Let $f : A \rightarrow Z$ be continuous. Then there is at most one extension of f to a continuous function $\bar{A} \rightarrow Z$.*

PROOF:

⟨1⟩1. ASSUME: for a contradiction $g, h : \bar{A} \rightarrow Z$ are continuous extensions of f with $g(x) \neq h(x)$

⟨1⟩2. PICK disjoint open neighbourhoods U of $g(x)$ and V of $h(x)$

⟨1⟩3. PICK a point $a \in A \cap g^{-1}(U) \cap h^{-1}(V)$

PROOF: One exists because $g^{-1}(U) \cap h^{-1}(V)$ is a neighbourhood of $x \in \bar{A}$.

⟨1⟩4. $g(a) \in U \cap V$

□

6.3 Regular Spaces

Definition 6.3.1 (Regular). A topological space X is *regular* iff, for every closed set A and point $a \notin A$, there exist disjoint neighbourhoods U of A and V of a .

Proposition 6.3.2. *Let X be a T_1 space. Then X is regular if and only if, for every point x and neighbourhood U of x , there exists a neighbourhood V of x such that $\bar{V} \subseteq U$.*

PROOF:

- ⟨1⟩1. If X is regular then, for every point x and neighbourhood N of x , there exists a neighbourhood V of x such that $\overline{V} \subseteq N$.
- ⟨2⟩1. ASSUME: X is regular.
- ⟨2⟩2. LET: $x \in X$ and N be a neighbourhood of x
- ⟨2⟩3. PICK an open set U such that $x \in U \subseteq N$
- ⟨2⟩4. PICK disjoint open sets V, W such that $x \in V$ and $X \setminus U \subseteq W$
- ⟨2⟩5. $\overline{V} \subseteq N$

PROOF:

$$\begin{aligned}\overline{V} &\subseteq X \setminus W \\ &\subseteq U \\ &\subseteq N\end{aligned}$$

- ⟨1⟩2. If, for every point x and neighbourhood U of x , there exists a neighbourhood V of x such that $\overline{V} \subseteq U$, then X is regular.
- ⟨2⟩1. ASSUME: For every point x and neighbourhood U of x , there exists a neighbourhood V of x such that $\overline{V} \subseteq U$.
- ⟨2⟩2. LET: $x \in X$ and A be a closed set with $x \notin A$
- ⟨2⟩3. PICK a neighbourhood V of x such that $\overline{V} \subseteq X \setminus A$
- ⟨2⟩4. $x \in V$ and $A \subseteq X \setminus \overline{V}$

□

Proposition 6.3.3. *Every linearly ordered set under the order topology is regular.*

PROOF:

- ⟨1⟩1. LET: X be a linearly ordered set under the order topology.
- ⟨1⟩2. LET: $x \in X$ and U be a neighbourhood of x
PROVE: There exists a neighbourhood V of x with $\overline{V} \subseteq U$
- ⟨1⟩3. CASE: x is greatest and least in X
PROOF: Take $V = U = X = \{x\}$
- ⟨1⟩4. CASE: x is greatest in X and there exists $a < x$ such that $(a, x] \subseteq U$
 - ⟨2⟩1. CASE: There exists b such that $a < b < x$
PROOF: Take $V = (b, x]$.
 - ⟨2⟩2. CASE: There is no b such that $a < b < x$
 - ⟨3⟩1. LET: $V = U = \{x\}$
 - ⟨3⟩2. $\overline{V} = V$
PROOF: For any $y \neq x$, we have $(-\infty, x)$ is a neighbourhood of y that does not intersect V .
- ⟨1⟩5. CASE: x is least in X and there exists $b > x$ such that $[x, b) \subseteq U$
PROOF: Similar.
- ⟨1⟩6. CASE: There exist $a < x < b$ such that $(a, b) \subseteq U$
 - ⟨2⟩1. PICK a point c such that $a < c < x$ if there is one, otherwise
LET: $c = a$
 - ⟨2⟩2. PICK a point d such that $x < d < b$ if there is one, otherwise
LET: $d = b$
 - ⟨2⟩3. LET: $V = (c, d)$
 - ⟨2⟩4. $\overline{V} \subseteq U$

PROOF:

$$\begin{aligned}\bar{V} &\subseteq [c, d] \\ &\subseteq (a, b) \\ &\subseteq U\end{aligned}$$

⟨1⟩7. Q.E.D.

PROOF: These cases are exhaustive by Lemma 4.1.2. They prove X is regular by Proposition 6.3.2.

□

Proposition 6.3.4. *A subspace of a regular space is regular.*

PROOF:

- ⟨1⟩1. LET: X be a regular space and $Y \subseteq X$
- ⟨1⟩2. LET: $A \subseteq Y$ be closed in Y and $a \in Y \setminus A$
- ⟨1⟩3. PICK C closed in X such that $A = C \cap Y$

PROOF: By Corollary 4.3.4.1.

- ⟨1⟩4. PICK disjoint open sets U, V in X such that $C \subseteq U$ and $a \in V$
- ⟨1⟩5. $U \cap Y$ and $V \cap Y$ are disjoint open sets in Y such that $A \subseteq U \cap Y$ and $a \in V \cap Y$

□

Corollary 6.3.4.1. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of nonempty spaces. If $\prod_{\alpha \in J} X_\alpha$ is regular then so is each X_α .*

Proposition 6.3.5 (AC). *The product of a family of regular spaces is regular.*

PROOF:

- ⟨1⟩1. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of regular spaces.
- ⟨1⟩2. $\prod_{\alpha \in J} X_\alpha$ is T_1
- ⟨1⟩3. LET: $\vec{a} \in U$ where U is open in $\prod_{\alpha \in J} X_\alpha$
- ⟨1⟩4. PICK $\prod_{\alpha \in J} U_\alpha$ such that each U_α is open in X_α , $U_\alpha = X_\alpha$ except at $\alpha_1, \dots, \alpha_n$, and $\vec{a} \in \prod_{\alpha \in J} U_\alpha \subseteq U$
- ⟨1⟩5. For $1 \leq i \leq n$, PICK V_{α_i} open in X_{α_i} such that $a_{\alpha_i} \in V_{\alpha_i}$ and $\overline{V_{\alpha_i}} \subseteq U_{\alpha_i}$
- ⟨1⟩6. For $\alpha \neq \alpha_1, \dots, \alpha_n$,
LET: $V_\alpha = X_\alpha$
- ⟨1⟩7. $\vec{a} \in \prod_{\alpha \in J} V_\alpha$
- ⟨1⟩8. $\overline{\prod_{\alpha \in J} V_\alpha} \subseteq \prod_{\alpha \in J} U_\alpha$

PROOF: By Theorem 4.2.5.

□

Corollary 6.3.5.1. *The Sorgenfrey plane is regular.*

Corollary 6.3.5.2. *For any set I , the space \mathbb{R}^I is regular.*

Proposition 6.3.6. *The space \mathbb{R}_K is not regular.*

PROOF: There do not exist disjoint neighbourhoods of 0 and K . □

Proposition 6.3.7. *The continuous image of a regular space is not necessarily regular.*

PROOF: The identity map from the discrete two-point space to the indiscrete two-point space is continuous. \square

6.4 Completely Regular Spaces

Definition 6.4.1 (Separated by a Continuous Function). Let A and B be subsets of a topological space X . Then A and B can be *separated by a continuous function* iff there exists a continuous $f : X \rightarrow [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

Definition 6.4.2 (Completely Regular). A space X is *completely regular* iff X is T_1 and, for every point a and closed set A not containing a , we have that $\{a\}$ and A can be separated by a continuous function.

Theorem 6.4.3. *The product of a family of completely regular spaces is completely regular.*

PROOF:

- $\langle 1 \rangle 1$. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of completely regular spaces.
- $\langle 1 \rangle 2$. LET: $a \in \prod_{\alpha \in J} X_\alpha$ and A be closed in $\prod_{\alpha \in J} X_\alpha$ such that $a \notin A$
- $\langle 1 \rangle 3$. PICK a basic open neighbourhood $\prod_{\alpha \in J} U_\alpha \subseteq \prod_{\alpha \in J} X_\alpha \setminus A$ of a such that $U_\alpha = X_\alpha$ except for $\alpha = \alpha_1, \dots, \alpha_n$
- $\langle 1 \rangle 4$. For $1 \leq i \leq n$, PICK a continuous $f_i : X_{\alpha_i} \rightarrow [0, 1]$ that is 0 at a_{α_i} and 1 on $X_{\alpha_i} \setminus U_{\alpha_i}$
- $\langle 1 \rangle 5$. LET: $f : \prod_{\alpha \in J} X_\alpha \rightarrow [0, 1]$ be given by $f(x) = \prod_{i=1}^n f_i(x_{\alpha_i})$
- $\langle 1 \rangle 6$. $f(a) = 0$
- $\langle 1 \rangle 7$. $f(x) = 1$ for $x \in A$
- $\langle 1 \rangle 8$. f is continuous

\square

Corollary 6.4.3.1. *The Sorgenfrey plane is completely regular.*

Corollary 6.4.3.2. *For any set I , the space \mathbb{R}^I is completely regular.*

Proposition 6.4.4. *For any set J , the space \mathbb{R}^J in the box topology is completely regular.*

PROOF:

- $\langle 1 \rangle 1$. LET: $a \in \mathbb{R}^J$ and $A \subseteq \mathbb{R}^J$ be closed with $a \notin A$
 PROVE: There exists $f : \mathbb{R}_{\text{box}}^J \rightarrow [0, 1]$ continuous such that $f(a) = 1$ and $f(A) = \{0\}$
- $\langle 1 \rangle 2$. ASSUME: w.l.o.g. $A \cap (-1, 1)^J = \emptyset$ and $a = \vec{0}$
 - $\langle 2 \rangle 1$. PICK a basic open set $\prod_{\alpha \in J} U_\alpha$ such that $a \in \prod_{\alpha \in J} U_\alpha \subseteq \mathbb{R}^J \setminus A$
 - $\langle 2 \rangle 2$. For $\alpha \in J$, PICK b_α, c_α such that $a_\alpha \in (b_\alpha, c_\alpha) \subseteq U_\alpha$
 - $\langle 2 \rangle 3$. For $\alpha \in J$, PICK a homeomorphism $f_\alpha : \mathbb{R} \rightarrow \mathbb{R}$ that maps b_α to -1 , a_α to 0 and c_α to 1
 - $\langle 2 \rangle 4$. $\prod_{\alpha \in J} f_\alpha$ is an automorphism $\mathbb{R}_{\text{box}}^J$ that maps a to $\vec{0}$ and A to a closed set disjoint from $(-1, 1)^J$

- (1)3. PICK a continuous function $f : \mathbb{R}_{\text{uniform}}^J \rightarrow [0, 1]$ such that $f(\vec{0}) = 1$ and $f(\mathbb{R}^J \setminus (-1, 1)^J) = \{0\}$
- (1)4. f is continuous w.r.t. the box topology
-

Proposition 6.4.5. *Not every regular space is completely regular.*

PROOF:

- (1)1. For $m \in \mathbb{Z}$,
LET: $L_m = \{m\} \times [-1, 0]$
- (1)2. For each odd integer n and each integer $k \geq 2$,
LET: $C_{nk} = (\{n+1-1/k\} \text{home/robin/fun/RogOMatic/src/actuatortimes}[-1, 0]) \cup (\{n-1+1/k\} \times [-1, 0]) \cup \{(x, y) : (x-n)^2 + y^2 = (1-1/k)^2, y \geq 0\}$
- (1)3. For each odd integer n and each integer $k \geq 2$,
LET: $p_{nk} = (n, 1-1/k)$
- (1)4. PICK two points a, b not in any L_m or C_{nk}
- (1)5. LET: $X = \bigcup_{m \in \mathbb{Z}} L_m \cup \bigcup_{n,k} C_{nk} \cup \{a, b\}$
- (1)6. LET: \mathcal{B} be the set consisting of all subsets of \mathbb{R}^2 of the following forms:
1. The intersection of X with a horizontal open line segment that contains none of the points p_{nk}
 2. A set formed from one of the sets C_{nk} by deleting finitely many points.
 3. For each even integer m , the set $\{a\} \cup \{(x, y) \in X : x < m\}$
 4. For each even integer m , the set $\{b\} \cup \{(x, y) \in X : x > m\}$
- (1)7. \mathcal{B} is a basis for a topology on X
- (2)1. For all $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$
- (2)2. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$
- (3)1. CASE: B_1, B_2 are both of type 1
PROOF: Their intersection is of type 1.
- (3)2. CASE: B_1 is of type 1 and B_2 is of type 2
PROOF: Their intersection is of type 2, since a horizontal line segment intersects C_{nk} in at most two points.
- (3)3. CASE: B_1 is of type 1 and B_2 is of type 3
PROOF: Their intersection is of type 1
- (3)4. CASE: B_1 is of type 1 and B_2 is of type 4
PROOF: Their intersection is of type 1
- (3)5. CASE: B_1 is of type 2 and B_2 is of type 2
PROOF: Their intersection is of type 2
- (3)6. CASE: B_1 is of type 2 and B_2 is of type 3
PROOF: Their intersection is B_1
- (3)7. CASE: B_1 is of type 2 and B_2 is of type 4
PROOF: Their intersection is B_1
- (3)8. CASE: B_1 is of type 3 and B_2 is of type 3
PROOF: Their intersection is of type 3
- (3)9. CASE: B_1 is of type 3 and B_2 is of type 4

- (4)1. LET: $B_1 = \{a\} \cup \{(x, y) \in X : x < m\}$ and $B_2 = \{b\} \cup \{(x, y) \in X : x > n\}$
 (4)2. CASE: $x = (s, 1 - 1/k)$ for some s and integer $x \geq 2$
 PROOF: In this case, $x \in C_{nk}$ for some n and $C_{nk} \subseteq B_1 \cap B_2$.
 (4)3. CASE: $x = (s, t)$ and $t \neq 1 - 1/k$ for any integer $k \geq 2$
 PROOF: In this case, $x \in ((n, m) \times \{t\}) \cap X \subseteq B_1 \cap B_2$
 (3)10. CASE: B_1 is of type 4 and B_2 is of type 4
 PROOF: Their intersection is of type 4
 (2)8. For any continuous function $f : X \rightarrow \mathbb{R}$, we have $f(a) = f(b)$
 (2)1. LET: $f : X \rightarrow \mathbb{R}$ be continuous
 (2)2. For any $c \in \mathbb{R}$, we have $f^{-1}(c)$ is G_δ
 PROOF: $f^{-1}(c) = \bigcap_{q \in \mathbb{Q}^+} f^{-1}(c - q, c + q)$
 (2)3. LET: $S_{nk} = \{p \in C_{nk} : f(p) \neq f(p_{nk})\}$
 (2)4. For all n, k , we have S_{nk} is countable.
 (3)1. LET: $f^{-1}(p_{nk}) = \bigcap_{m=1}^{\infty} U_m$ where U_m is open in X
 (3)2. For each m , PICK $B_m \in \mathcal{B}$ such that $p_{nk} \in B_m \subseteq U_m$
 (3)3. $S_{nk} \subseteq \bigcup_{m=1}^{\infty} (C_{nk} \setminus B_m)$
 (3)4. Each $C_{nk} \setminus B_m$ is countable
 (4)1. LET: $m \in \mathbb{Z}$
 (4)2. B_m cannot be of type 1
 (4)3. If B_m is of type 2 then $C_{nk} \setminus B_m$ is finite.
 (4)4. If B_m is of type 3 or 4 then $C_{nk} \setminus B_m$ is empty.
 (2)5. PICK $d \in [-1, 0]$ such that $\mathbb{R} \times \{d\}$ intersects none of the sets S_{nk}
 (2)6. For n odd, we have

$$f(n-1, d) = \lim_{k \rightarrow \infty} f(p_{nk}) .$$

 (3)1. LET: $\epsilon > 0$
 (3)2. PICK $B \in \mathcal{B}$ such that $(n-1, d) \in B \subseteq f^{-1}(f(n-1, d) - \epsilon, f(n-1, d) + \epsilon)$
 (3)3. There exists $\delta > 0$ such that, for $x \in (n-1-\delta, n-1+\delta)$, we have $(x, d) \in B$
 (3)4. PICK K such that $1/K < \delta$
 (3)5. LET: $k \geq K$
 (3)6. $f(n-1+1/k, d) = f(p_{nk})$
 (3)7. $|f(n-1, d) - f(n-1+1/k, d)| < \epsilon$
 (3)8. $|f(n-1, d) - f(p_{nk})| < \epsilon$
 (2)7. For n odd, we have

$$f(n+1, d) = \lim_{k \rightarrow \infty} f(p_{nk}) .$$

 PROOF: Similar.
 (2)8. Q.E.D.
 (3)1. ASSUME: $f(a) \neq f(b)$
 (3)2. ASSUME: w.l.o.g. $f(a) < f(b)$
 (3)3. PICK $B \in \mathcal{B}$ such that $a \in B \subseteq f^{-1}(-\infty, (f(a) + f(b))/2)$
 (3)4. LET: m be even such that $B = \{a\} \cup \{(x, y) \in X : x < m\}$
 (3)5. PICK $B \in \mathcal{B}$ such that $b \in B \subseteq f^{-1}((f(a) + f(b))/2, +\infty)$
 (3)6. LET: m' be even such that $B = \{b\} \cup \{(x, y) \in X : x > m'\}$

- ⟨3⟩7. $f(m, d) = f(m', d)$
 ⟨3⟩8. Q.E.D.
 ⟨1⟩9. X is regular.
 ⟨1⟩10. X is not completely regular.
 PROOF: a and b cannot be separated by a continuous function.
 □

Theorem 6.4.6 (AC). *A space is completely regular iff it is homeomorphic to a subspace of $[0, 1]^J$ for some J .*

PROOF:

- ⟨1⟩1. Every completely regular space is homeomorphic to a subspace of $[0, 1]^J$ for some J .
 ⟨2⟩1. LET: X be completely regular
 ⟨2⟩2. For every point a and open set U that contains a , PICK a continuous function f_{aU} that is positive on a and vanishes outside U
 ⟨2⟩3. The family $\{f_{aU}\}$ separates points from closed sets
 ⟨2⟩4. Q.E.D.

PROOF: By the Imbedding Theorem.

- ⟨1⟩2. Every subspace of $[0, 1]^J$ is completely regular.

PROOF: By Theorem 6.4.3 and Proposition 6.3.4.

□

Proposition 6.4.7. *The continuous image of a completely regular space is not necessarily completely regular.*

PROOF: The identity map from the discrete two-point space to the indiscrete two-point space is continuous. □

6.5 Normal Spaces

Definition 6.5.1 (Normal Space). A *normal* space is a T_1 space such that, for any disjoint closed sets A, B , there exist disjoint open sets U, V such that $A \subseteq U$ and $B \subseteq V$.

Theorem 6.5.2. *Every linearly ordered set is normal under the order topology.*

PROOF: See Steen and Steerbach *Counterexamples in Topology* Example 39. □

Proposition 6.5.3. *The product space $S_\Omega \times \overline{S_\Omega}$ is not normal.*

PROOF:

- ⟨1⟩1. LET: $\Delta = \{(x, x) : x \in \overline{S_\Omega}\} \subseteq \overline{S_\Omega} \times \overline{S_\Omega}$
 ⟨1⟩2. Δ is closed in $\overline{S_\Omega} \times \overline{S_\Omega}$
 ⟨1⟩3. LET: $A = \Delta \cap (S_\Omega \times \overline{S_\Omega})$
 ⟨1⟩4. A is closed in $S_\Omega \times \overline{S_\Omega}$
 ⟨1⟩5. LET: $B = S_\Omega \times \{\Omega\}$
 ⟨1⟩6. B is closed

- (1)7. $A \cap B = \emptyset$
 (1)8. ASSUME: for a contradiction U and V are disjoint open sets including A and B respectively
 (1)9. For all $x \in S_\Omega$ there exists $\beta \in (x, \Omega)$ such that $(x, \beta) \notin U$
 (2)1. LET: $x \in S_\Omega$
 (2)2. $(x, \Omega) \in V$
 PROOF: $(x, \Omega) \in B \subseteq V$
 (2)3. PICK $y < \Omega$ such that $\{x\} \times (y, \Omega] \subseteq V$
 PROOF: By Lemma 4.1.2.
 (2)4. PICK β such that $x, y < \beta < \Omega$
 PROOF: Such a β exists because Ω is a limit ordinal.
 (1)10. For $x \in S_\Omega$,
 LET: $\beta(x)$ be the least element of (x, Ω) such that $(x, \beta(x)) \notin U$
 (1)11. LET: $b = \sup_{n=1}^\infty \beta^n(0)$
 (1)12. $\beta^n(0) \rightarrow b$ as $n \rightarrow \infty$
 (1)13. $(\beta^n(0), \beta^{n+1}(0)) \rightarrow (b, b)$ as $n \rightarrow \infty$
 (1)14. $(b, b) \in A$
 (1)15. $(b, b) \in U$
 (1)16. For all n we have $(\beta^n(0), \beta^{n+1}(0)) \notin U$
 PROOF: By (1)10.
 (1)17. Q.E.D.
 PROOF: Steps (1)12, (1)15 and (1)16 form a contradiction.

□

Corollary 6.5.3.1. *Not every completely regular space is normal.*

Corollary 6.5.3.2. *An open subspace of a normal space is not necessarily normal.*

Corollary 6.5.3.3. *The product of two normal spaces is not necessarily normal.*

Proposition 6.5.4. *A closed subspace of a normal space is normal.*

PROOF:

- (1)1. LET: X be normal and $C \subseteq X$ be closed.
 (1)2. LET: A and B be closed in C
 (1)3. A and B are closed in X
 PROOF: By Corollary 4.3.4.2.
 (1)4. PICK disjoint open neighbourhoods U and V of A and B in X
 (1)5. $U \cap C$ and $V \cap C$ are disjoint open neighbourhoods of A and B in C

□

Corollary 6.5.4.1. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of nonempty spaces. If $\prod_{\alpha \in J} X_\alpha$ is normal then each X_α is normal.*

Proposition 6.5.5. *If the Continuum Hypothesis then \mathbb{R}^ω under the box topology is normal.*

PROOF: See Rudin. The box product of countably many compact metric spaces. *General Topology and Its Applications*, 2:293–298, 1972. □

Proposition 6.5.6 (Stone (DC)). *If J is uncountable then \mathbb{R}^J is not normal.*

PROOF:

$\langle 1 \rangle 1$. LET: $X = (\mathbb{Z}^+)^J$

PROVE: X is not normal.

$\langle 1 \rangle 2$. For $x \in X$ and $B \subseteq^{\text{fin}} J$,

LET:

$$U(x, B) = \{y \in X : \forall \alpha \in B. y_\alpha = x_\alpha\} .$$

$\langle 1 \rangle 3$. $\{U(x, B) : x \in X, B \subseteq^{\text{fin}} J\}$ is a basis for X

$\langle 2 \rangle 1$. LET: $x \in X$ and $\prod_{\alpha \in J} U_\alpha$ be a basic open set including x , where $U_\alpha = \mathbb{Z}^+$ for all α except $\alpha_1, \dots, \alpha_n$

$\langle 2 \rangle 2$. $x \in U(x, \{\alpha_1, \dots, \alpha_n\}) \subseteq \prod_{\alpha \in J} U_\alpha$

$\langle 1 \rangle 4$. For $n \in \mathbb{Z}^+$,

LET: $P_n = \{x \in X : x \text{ is injective on } J \setminus x^{-1}(n)\}$

$\langle 1 \rangle 5$. P_1 and P_2 are closed and disjoint.

$\langle 2 \rangle 1$. P_1 is closed

$\langle 3 \rangle 1$. LET: $x \in X \setminus P_1$

$\langle 3 \rangle 2$. PICK $\alpha, \beta \in J$ such that $x_\alpha = x_\beta \neq 1$

$\langle 3 \rangle 3$. LET: $U_\gamma = \{x_\alpha\}$ if $\gamma = \alpha$ or $\gamma = \beta$, \mathbb{Z}^+ for all other $\gamma \in J$

$\langle 3 \rangle 4$. $x \in \prod_{\gamma \in J} U_\gamma \subseteq X \setminus P_1$

$\langle 2 \rangle 2$. P_2 is closed

PROOF: Similar.

$\langle 2 \rangle 3$. $P_1 \cap P_2 = \emptyset$

PROOF: If $x \in P_1 \cap P_2$ then x is injective on J , contradicting the fact that J is uncountable.

$\langle 1 \rangle 6$. ASSUME: for a contradiction U and V are disjoint open sets including P_1 and P_2

$\langle 1 \rangle 7$. Given a sequence (α_i) of distinct elements of J and a strictly increasing sequence (n_i) of positive integers,

LET:

$$B_i^{\alpha, n} = \{\alpha_1, \dots, \alpha_{n_i}\}$$

$$x_i^{\alpha, n} \in X$$

$$(x_i^{\alpha, n})_\beta = \begin{cases} j & \text{if } \beta = \alpha_j, 1 \leq j \leq n_{i-1} \\ 1 & \text{for all other values of } \beta \end{cases}$$

for $i \geq 1$

$\langle 1 \rangle 8$. PICK sequences (α_i) , (n_i) such that, for all $i \geq 1$, we have $U(x_i^{\alpha, n}, B_i^{\alpha, n}) \subseteq U$

$\langle 2 \rangle 1$. LET: $x_1 \in X$ be given by $(x_1)_\alpha = 1$ for all $\alpha \in J$

$\langle 2 \rangle 2$. $x_1 \in U$

PROOF: $x_1 \in P_1 \subseteq U$

$\langle 2 \rangle 3$. PICK $B_1 \subseteq^{\text{fin}} J$ such that $U(x_1, B_1) \subseteq U$

PROOF: By $\langle 1 \rangle 3$.

$\langle 2 \rangle 4$. LET: $n_1 = |B_1|$ and $B_1 = \{\alpha_1, \dots, \alpha_{n_1}\}$

$\langle 2 \rangle 5$. ASSUME: We have chosen n_1, \dots, n_k strictly increasing and $\alpha_1, \dots, \alpha_{n_k}$ such that, for $1 \leq i \leq k$, we have $U(x_i^{\alpha, n}, B_i^{\alpha, n}) \subseteq U$

- (2)6. $x_{i+1}^{\alpha,n} \in U$
 PROOF: $x_{i+1}^{\alpha,n} \in P_1 \subseteq U$
 (2)7. PICK $C \subseteq^{\text{fin}} J$ such that $U(x_{i+1}^{\alpha,n}, C) \subseteq U$
 (2)8. LET: n_{i+1} and $\alpha_{n_{i+1}+1}, \dots, \alpha_{n_{i+1}}$ be such that $B_i^{\alpha,n} \cup C = B_{i+1}^{\alpha,n}$
 (2)9. $U(x_{i+1}^{\alpha,n}, B_{i+1}^{\alpha,n}) \subseteq U$
 (1)9. LET: $A = \{\alpha_i : i \geq 1\}$
 (1)10. LET: $y \in X$, $y_\beta = j$ if $\beta = \alpha_j$, $y_\beta = 2$ for $\beta \notin A$
 (1)11. PICK B such that $U(y, B) \subseteq V$
 (1)12. PICK i such that $A \cap B \subseteq B_i^{\alpha,n}$
 (1)13. $U(x_{i+1}^{\alpha,n}, B_{i+1}^{\alpha,n}) \cap U(y, B) \neq \emptyset$
 PROOF: $x_{i+1}^{\alpha,n} \in U(x_{i+1}^{\alpha,n}, B_{i+1}^{\alpha,n}) \cap U(y, B)$
 (1)14. Q.E.D.
 PROOF: This contradicts the fact that U and V are disjoint ((1)6).
 □

Theorem 6.5.7 (Urysohn Lemma). *Let X be a normal space. Let A and B be disjoint closed subsets of X . Then there exists a continuous map $f : X \rightarrow [0, 1]$ such that $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$.*

PROOF:

- (1)1. LET: P be the set of all rational numbers in $[0, 1]$
 (1)2. For all $q \in P$, PICK an open set U_q in X such that $A \subseteq U_0$, $U_1 \subseteq X \setminus B$,
 and whenever $p < q$ then $\overline{U_p} \subseteq U_q$
 (2)1. PICK an enumeration (q_n) of P such that $q_1 = 1$ and $q_2 = 0$
 (2)2. LET: $U_1 = X \setminus B$
 (2)3. PICK an open set U_0 such that $A \subseteq U_0$ and $\overline{U_0} \subseteq U_1$
 (2)4. ASSUME: we have open sets U_1, U_0, \dots, U_{q_n} such that whenever $p < q$
 then $\overline{U_p} \subseteq U_q$
 (2)5. $q_2 < q_{n+1} < q_1$
 (2)6. LET: q_k be greatest among q_1, \dots, q_n such that $q_k < q_{n+1}$, and q_l be
 least such that $q_{n+1} < q_l$
 (2)7. PICK an open set $U_{q_{n+1}}$ such that $\overline{U_{q_k}} \subseteq U_{q_{n+1}}$ and $\overline{U_{q_{n+1}}} \subseteq U_{q_l}$
 (2)8. For all $p, q \in \{q_1, \dots, q_{n+1}\}$, if $p < q$ then $\overline{U_p} \subseteq U_q$
 (1)3. Extend the family (U_q) to \mathbb{Q} by defining: $U_q = \emptyset$ if $q < 0$ and $U_q = X$ if
 $q > 1$
 (1)4. For all rationals p, q with $p < q$ we have $\overline{U_p} \subseteq U_q$
 (1)5. Define $f : X \rightarrow [0, 1]$ by $f(x) = \inf\{q \in \mathbb{Q} : x \in U_q\}$
 PROOF: This set is nonempty since $x \in U_1$ and bounded below since if $x \in U_q$
 then $q \geq 0$.
 (1)6. For all $x \in A$ we have $f(x) = 0$
 (1)7. For all $x \in B$ we have $f(x) = 1$
 (1)8. If $x \in \overline{U_r}$ then $f(x) \leq r$
 (1)9. If $x \notin U_r$ then $f(x) \geq r$
 (1)10. f is continuous
 (2)1. LET: $x_0 \in X$
 (2)2. LET: (c, d) be an open interval containing $f(x_0)$
 PROVE: There exists a neighbourhood U of x_0 such that $f(U) \subseteq (c, d)$

- (2)3. PICK rationals p, q such that $c < p < f(x_0) < q < d$
 (2)4. $x \notin \overline{U_p}$
 PROOF: By (1)8
 (2)5. $x \in U_q$
 PROOF: By (1)9
 (2)6. LET: $U = U_q \setminus \overline{U_p}$

□

Definition 6.5.8 (Vanish Precisely). Let X be a set and $A \subseteq X$. Let $f : X \rightarrow [0, 1]$. Then f *vanishes precisely* on A iff $f^{-1}(0) = A$.

Theorem 6.5.9 (CC). Let X be a normal space and $A \subseteq X$. Then there exists a continuous function $f : X \rightarrow [0, 1]$ such that f vanishes precisely on A if and only if A is a closed G_δ set.

PROOF:

- (1)1. If there exists f such that f vanishes precisely on A then A is closed.
 PROOF: This holds because $A = f^{-1}(0)$.
 (1)2. If there exists f such that f vanishes precisely on A then A is G_δ .
 PROOF: This holds because $A = \bigcap_{q \in \mathbb{Q}^+} f^{-1}([0, q))$.
 (1)3. If A is closed and G_δ then there exists f that vanishes precisely on A .
 (2)1. LET: $A = \bigcap_{n=1}^{\infty} U_n$
 (2)2. For $n \geq 1$, PICK $f_n : X \rightarrow [0, 1/2^n]$ such that $f_n(x) = 0$ for $x \in A$ and $f_n(x) = 1/2^n$ for $x \in X \setminus U_n$
 PROOF: By the Urysohn Lemma.
 (2)3. LET: $f : X \rightarrow [0, 1]$ be given by $f(x) = \sum_{n=1}^{\infty} f_n(x)$
 PROOF: The series converges for every x by the Comparison Test.
 (2)4. f is continuous
 (3)1. f_n converges uniformly to f
 PROOF: By the Weierstrass M-test.
 (3)2. Q.E.D.
 PROOF: By the Uniform Limit Theorem.
 (2)5. $f(x) = 0$ for $x \in A$
 PROOF: From (2)2.
 (2)6. $f(x) > 0$ for $x \notin A$
 (3)1. LET: $x \notin A$
 (3)2. PICK N such that $x \notin U_N$
 (3)3. Q.E.D.
 PROOF:

$$\begin{aligned}
 f(x) &= \sum_{n=1}^{\infty} f_n(x) & ((2)3) \\
 &\geq f_N(x) \\
 &> 0 & ((2)2)
 \end{aligned}$$

□

Theorem 6.5.10 (Strong Form of Urysohn Lemma). Let X be a normal space.

Then there exists a continuous function $f : X \rightarrow [0, 1]$ such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$ if and only if A and B are disjoint, closed and G_δ .

PROOF:

- ⟨1⟩1. If there exists a continuous function $f : X \rightarrow [0, 1]$ such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$ then A and B are disjoint, closed and G_δ
- ⟨2⟩1. ASSUME: there exists a continuous function $f : X \rightarrow [0, 1]$ such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$
- ⟨2⟩2. A and B are disjoint
- ⟨2⟩3. A is closed and G_δ
PROOF: By Theorem 6.5.9.
- ⟨2⟩4. B is closed and G_δ
PROOF: Apply Theorem 6.5.9 to $1 - f$.
- ⟨1⟩2. If A and B are disjoint, closed and G_δ then there exists a continuous function $f : X \rightarrow [0, 1]$ such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$
- ⟨2⟩1. ASSUME: A and B are disjoint, closed and G_δ
- ⟨2⟩2. PICK $g : X \rightarrow [0, 1]$ that vanishes precisely on A and $h : X \rightarrow [0, 1]$ that vanishes precisely on B
- ⟨2⟩3. LET: $f = g/(g + h)$

□

Definition 6.5.11 (Universal Extension Property). A topological space Y has the *universal extension property* iff, for every normal space X and closed subspace A of X , every continuous function $A \rightarrow Y$ can be extended to a continuous function $X \rightarrow Y$.

Theorem 6.5.12 (Tietze Extension Theorem (DC)). Let X be a normal space. Let A be closed subspace of X .

- 1. Any continuous function $A \rightarrow [a, b]$ can be extended to a continuous function $X \rightarrow [a, b]$.
- 2. Any continuous function $A \rightarrow \mathbb{R}$ can be extended to a continuous function $X \rightarrow \mathbb{R}$.

PROOF:

- ⟨1⟩1. Any continuous function $A \rightarrow [-1, 1]$ can be extended to a continuous function $X \rightarrow [-1, 1]$
- ⟨2⟩1. For every continuous function $f : A \rightarrow [-r, r]$, there exists a continuous $g : X \rightarrow \mathbb{R}$ such that

$$|g(x)| \leq \frac{1}{3}r \quad (x \in X)$$

$$|g(x) - f(x)| \leq \frac{2}{3}r \quad (x \in A)$$

- ⟨3⟩1. LET: $f : A \rightarrow [-r, r]$ be continuous
- ⟨3⟩2. LET: $I_1 = [-r, -\frac{1}{3}r]$
- ⟨3⟩3. LET: $I_2 = [-\frac{1}{3}r, \frac{1}{3}r]$
- ⟨3⟩4. LET: $I_3 = [\frac{1}{3}r, r]$

⟨3⟩5. LET: $B = f^{-1}(I_1)$

⟨3⟩6. LET: $C = f^{-1}(I_3)$

⟨3⟩7. PICK a continuous $g : X \rightarrow [-\frac{1}{3}r, \frac{1}{3}r]$ such that $g(x) = -\frac{1}{3}r$ for $x \in B$ and $g(x) = \frac{1}{3}r$ for $x \in C$

PROOF: By the Urysohn Lemma, since B and C are closed disjoint subsets of X .

⟨3⟩8. For all $x \in A$ we have $|g(x) - f(x)| \leq \frac{2}{3}r$

⟨4⟩1. LET: $x \in A$

⟨4⟩2. CASE: $f(x) \in I_1$

PROOF:

$$\begin{aligned} |g(x) - f(x)| &= \left| -\frac{1}{3}r - f(x) \right| & (x \in B) \\ &\leq \frac{2}{3}r & (f(x) \in I_1) \end{aligned}$$

⟨4⟩3. CASE: $f(x) \in I_2$

PROOF: In this case, $|g(x) - f(x)| \leq \frac{2}{3}r$ since $f(x), g(x) \in I_2$.

⟨4⟩4. CASE: $f(x) \in I_3$

PROOF:

$$\begin{aligned} |g(x) - f(x)| &= \left| \frac{1}{3}r - f(x) \right| & (x \in C) \\ &\leq \frac{2}{3}r & (f(x) \in I_3) \end{aligned}$$

⟨2⟩2. LET: $f : A \rightarrow [-1, 1]$ be continuous.

⟨2⟩3. PICK a sequence of functions (g_n) such that

$$|g_n(x)| \leq \frac{1}{3} \left(\frac{2}{3} \right)^{n-1} \quad (x \in X)$$

$$|f(x) - g_1(x) - \cdots - g_n(x)| \leq (2/3)^n \quad (x \in A)$$

PROOF: Given g_1, \dots, g_n , we apply ⟨2⟩1 with $f = f - g_1 - \cdots - g_n$ and $r = (2/3)^n$.

⟨2⟩4. LET: $g(x) = \sum_{n=1}^{\infty} g_n(x)$ for $x \in X$

PROOF: This series converges by the Comparison Test since $\sum_{n=1}^{\infty} (2/3)^n$ converges.

⟨2⟩5. g is continuous.

⟨3⟩1. $\sum_{n=1}^N g_n$ converges to g uniformly

PROOF: By the Weierstrass M -test.

⟨3⟩2. Q.E.D.

PROOF: By the Uniform Limit Theory.

⟨2⟩6. For all $x \in A$ we have $g(x) = f(x)$

PROOF: $|\sum_{n=1}^N g_n(x) - f(x)| \leq (2/3)^N \rightarrow 0$ as $N \rightarrow \infty$.

⟨2⟩7. For all $x \in X$ we have $-1 \leq g(x) \leq 1$

PROOF:

$$\begin{aligned} \left| \sum_{n=1}^N g_n(x) \right| &\leq \sum_{n=1}^N |g_n(x)| \\ &\leq 1/3 \sum_{n=1}^N (2/3)^{n-1} \\ &\rightarrow 2/3 \quad \text{as } n \rightarrow \infty \end{aligned}$$

$\langle 1 \rangle 2$. Any continuous function $A \rightarrow (-1, 1)$ can be extended to a continuous function $X \rightarrow (-1, 1)$

$\langle 2 \rangle 1$. LET: $f : A \rightarrow (-1, 1)$ be continuous

$\langle 2 \rangle 2$. PICK a continuous $g : X \rightarrow [-1, 1]$ that extends f

PROOF: By $\langle 1 \rangle 1$.

$\langle 2 \rangle 3$. LET: $D = g^{-1}(-1) \cup g^{-1}(1)$

$\langle 2 \rangle 4$. D is closed in X

PROOF: Since g is continuous and $\{-1\}, \{1\}$ are closed in $[-1, 1]$.

$\langle 2 \rangle 5$. $D \cap A = \emptyset$

PROOF: Since $g(A) = f(A) \subseteq (-1, 1)$.

$\langle 2 \rangle 6$. PICK a continuous $\phi : X \rightarrow [0, 1]$ such that $\phi(D) = \{0\}$ and $\phi(A) = \{1\}$

PROOF: By the Urysohn Lemma.

$\langle 2 \rangle 7$. LET: $h = g\phi$

$\langle 2 \rangle 8$. h is continuous

$\langle 2 \rangle 9$. h extends f

$\langle 2 \rangle 10$. $\text{im } h \subseteq (-1, 1)$

$\langle 1 \rangle 3$. Q.E.D.

PROOF: The result follows because any closed interval in \mathbb{R} is homeomorphic to $[-1, 1]$ and $\mathbb{R} \cong (-1, 1)$.

□

Lemma 6.5.13 (Shrinking Lemma (AC)). *Let X be a normal space. Let $\{U_\alpha\}_{\alpha \in J}$ be a point-finite indexed open covering of X . Then there exists an indexed open covering $\{V_\alpha\}_{\alpha \in J}$ such that $\overline{V_\alpha} \subseteq U_\alpha$ for all $\alpha \in J$.*

PROOF:

$\langle 1 \rangle 1$. PICK a well-ordering \prec on J

$\langle 1 \rangle 2$. PICK open sets V_α for $\alpha \in J$ such that $A_\alpha \subseteq V_\alpha$ and $\overline{V_\alpha} \subseteq U_\alpha$, where

$$A_\alpha = X \setminus \bigcup_{\beta \prec \alpha} V_\beta \cup \bigcup_{\alpha \prec \beta} U_\beta$$

PROOF: Apply transfinite induction to Proposition 13.1.16.

$\langle 1 \rangle 3$. $\{V_\alpha\}_{\alpha \in J}$ covers X

$\langle 2 \rangle 1$. LET: $x \in X$

$\langle 2 \rangle 2$. LET: $\alpha_1, \dots, \alpha_n$ be the elements of J such that $x \in U_{\alpha_i}$, where $\alpha_1 \prec \dots \prec \alpha_n$

PROVE: $x \in V_{\alpha_i}$ for some i

$\langle 2 \rangle 3$. ASSUME: $x \notin V_{\alpha_1}, \dots, V_{\alpha_{n-1}}$

$\langle 2 \rangle 4$. $x \in A_{\alpha_n}$

$\langle 2 \rangle 5$. $x \in V_{\alpha_n}$

□

Proposition 6.5.14 (DC). $S_\Omega \times \overline{S_\Omega}$ is not normal.

PROOF:

- ⟨1⟩1. LET: $\Delta = \{(x, x) : x \in \overline{S_\Omega}\}$
- ⟨1⟩2. Δ is closed in $\overline{S_\Omega}^2$
 - ⟨2⟩1. LET: $(x, y) \in \overline{S_\Omega}^2 \setminus \Delta$
 - ⟨2⟩2. PICK disjoint open sets U, V such that $x \in U$ and $y \in V$
 - ⟨2⟩3. $(x, y) \in U \times V \subseteq \overline{S_\Omega}^2 \setminus \Delta$
- ⟨1⟩3. LET: $A = \Delta \cap (S_\Omega \times \overline{S_\Omega})$
- ⟨1⟩4. A is closed in $S_\Omega \times \overline{S_\Omega}$
- ⟨1⟩5. LET: $B = S_\Omega \times \{\Omega\}$
- ⟨1⟩6. B is closed in $S_\Omega \times \overline{S_\Omega}$
- ⟨1⟩7. $A \cap B = \emptyset$
- ⟨1⟩8. ASSUME: for a contradiction U and V are disjoint open sets including A and B respectively
- ⟨1⟩9. PICK a sequence x_n in S_Ω such that $x_n < x_{n+1} < \Omega$ and $(x_n, x_{n+1}) \notin U$ for all n
 - ⟨2⟩1. LET: $x_n \in S_\Omega$
 - ⟨2⟩2. $(x_n, \Omega) \in V$
 - ⟨2⟩3. PICK open sets $W \subseteq S_\Omega, X \subseteq \overline{S_\Omega}$ such that $x_n \in W, \Omega \in X$ and $W \times X \subseteq V$
 - ⟨2⟩4. PICK $y < \Omega$ such that $(x_{n+1}, \Omega] \subseteq X$
 - ⟨2⟩5. LET: $x_{n+1} = y + 1$
- ⟨1⟩10. LET: b be the supremum of $\{x_n : n \geq 1\}$
- ⟨1⟩11. $(x_n, x_{n+1}) \rightarrow (b, b)$ as $n \rightarrow \infty$
- ⟨1⟩12. $(b, b) \in A$
- ⟨1⟩13. $(b, b) \in U$
- ⟨1⟩14. For all n we have $(x_n, x_{n+1}) \notin U$

□

Proposition 6.5.15 (AC). \mathbb{R}_I is normal.

PROOF:

- ⟨1⟩1. LET: A and B be disjoint closed sets in \mathbb{R}_I
- ⟨1⟩2. For $a \in A$, PICK $x_a > a$ such that $[a, x_a)$ not intersecting B
- ⟨1⟩3. For $b \in B$, PICK $x_b > b$ such that $[b, x_b)$ does not intersect A
- ⟨1⟩4. LET: $U = \bigcup_{a \in A} [a, x_a)$ and $V = \bigcup_{b \in B} [b, x_b)$
- ⟨1⟩5. U and V are disjoint open sets including A and B respectively.

□

Lemma 6.5.16. The set $L = \{(x, -x) : x \in \mathbb{R}\}$ as a subspace of \mathbb{R}_I^2 is closed

- ⟨1⟩1. LET: $(x, y) \notin L$, so $y \neq -x$
 PROVE: There exists a neighbourhood U of (x, y) that does not intersect L
- ⟨1⟩2. CASE: $y > -x$

PROOF: In this case, take $U = [x, +\infty) \times [y, +\infty)$
 (1)3. CASE: $y < -x$
 PROOF: In this case, take $U = [x, (x - y)/2) \times [y, (y - x)/2)$.

Proposition 6.5.17 (AC). *The Sorgenfrey plane is not normal.*

PROOF:

- (1)1. ASSUME: for a contradiction the Sorgenfrey plane is normal.
- (1)2. LET: $L = \{(x, -x); x \in \mathbb{R}\}$ as a subspace of \mathbb{R}_l^2
- (1)3. L has the discrete topology.
 - (2)1. LET: $(x, -x) \in L$
 - PROVE: $\{(x, -x)\}$ is open in L
 - (2)2. $\{(x, -x)\} = ([x, +\infty) \times [-x, +\infty)) \cap L$
- (1)4. Every subset of L is closed in \mathbb{R}_l^2
 PROOF: By Corollary 4.3.4.2.
- (1)5. For every nonempty proper subset A of L , PICK disjoint open sets U_A , V_A containing A and $L \setminus A$
 PROOF: By (1)1 and (1)4.
- (1)6. LET: $D = \mathbb{Q}^2$
- (1)7. D is dense in \mathbb{R}_l^2
 PROOF: Given any basic open set $[a, b) \times [c, d)$, pick rationals q, r such that $a \leq q < b$ and $c \leq r < d$. Then $(q, r) \in ([a, b) \times [c, d)) \cap D$
- (1)8. LET: $\theta : \mathcal{P}L \rightarrow \mathcal{P}D$ be the function

$$\begin{aligned} \theta(A) &= U_A \cap D & (\emptyset \neq A \neq L) \\ \theta(\emptyset) &= \emptyset \\ \theta(L) &= D \end{aligned}$$
- (1)9. θ is injective
 - (2)1. LET: $A, B \subseteq L$ with $\theta(A) = \theta(B)$
 - PROVE: $A = B$
 - (2)2. CASE: $\emptyset \neq A \neq L$ and $\emptyset \neq B \neq L$
 - (3)1. $A \subseteq B$
 - (4)1. LET: $x \in A$
 - (4)2. $x \in U_A$
 - PROOF: By (1)5
 - (4)3. $x \in U_B$
 - PROOF: By (2)1
 - (4)4. $x \notin L \setminus B$
 - PROOF: By (1)5
 - (4)5. $x \in B$
 - PROOF: Since $x \in L$ by (4)1
 - (3)2. $B \subseteq A$
 - PROOF: Similar.
 - (2)3. CASE: $\emptyset \neq A \neq L$ and $B = \emptyset$
 - PROOF: This implies $U_A \cap D = \emptyset$ which contradicts the fact that D is dense.
 - (2)4. CASE: $\emptyset \neq A \neq L$ and $B = L$
 - PROOF: This implies $V_A \cap D = \emptyset$ which contradicts the fact that D is dense.

- (2)5. CASE: $A = B = \emptyset$
 PROOF: Trivial
 (2)6. CASE: $A = \emptyset$ and $B = L$
 PROOF: This implies $D = \emptyset$ which is a contradiction.
 (2)7. CASE: $A = B = L$
 PROOF: Trivial
 (1)10. Q.E.D.
 PROOF: This is a contradiction since D is countable and L is uncountable.

□

Proposition 6.5.18. *The continuous image of a normal space is not necessarily normal.*

PROOF: The identity map from the discrete two-point space to the indiscrete two-point space is continuous. □

Lemma 6.5.19. *Let X be a regular space with a countably locally finite basis. Then X is normal and every closed set is G_δ .*

PROOF:

- (1)1. LET: X be regular with a countably locally finite basis.
 (1)2. For every open set W , there exists a countable set \mathcal{U} of open sets such that $W = \bigcup \mathcal{U} = \bigcup_{U \in \mathcal{U}} \overline{U}$
 (2)1. PICK a locally finite set \mathcal{B}_n for $n \in \mathbb{N}$ such that $\bigcup_{n=0}^{\infty} \mathcal{B}_n$ is a basis.
 PROOF: By (1)1.
 (2)2. For $n \in \mathbb{N}$,
 LET: $\mathcal{C}_n = \{B \in \mathcal{B}_n : \overline{B} \subseteq W\}$
 (2)3. For $n \in \mathbb{N}$, \mathcal{C}_n is locally finite.
 PROOF: This holds because $\mathcal{C}_n \subseteq \mathcal{B}_n$ ((2)1, (2)2).
 (2)4. For $n \in \mathbb{N}$,
 LET: $U_n = \bigcup \mathcal{C}_n$
 (2)5. For $n \in \mathbb{N}$, U_n is open.
 PROOF: This holds because every element of \mathcal{C}_n is open ((2)1, (2)2, (2)4).
 (2)6. For $n \in \mathbb{N}$, $\overline{U_n} = \bigcup_{B \in \mathcal{C}_n} \overline{B}$
 PROOF: By Lemma 3.10.10.
 (2)7. For $n \in \mathbb{N}$, $\overline{U_n} \subseteq W$
 PROOF: From (2)2 and (2)6.
 (2)8. $W \subseteq \bigcup_{n=0}^{\infty} U_n$
 (3)1. LET: $x \in W$
 (3)2. PICK a neighbourhood U of x such that $\overline{U} \subseteq W$
 PROOF: By Proposition 6.3.2 and (3)1 since X is regular ((1)1).
 (3)3. PICK $n \in \mathbb{N}$ and $B \in \mathcal{B}_n$ such that $x \in B \subseteq U$
 PROOF: By (2)1 and (3)2.
 (3)4. $B \in \mathcal{C}_n$
 (4)1. $\overline{B} \subseteq W$
 PROOF:

$$\begin{aligned}
 \overline{B} &\subseteq \overline{U} && \text{(Proposition 3.10.5, (3)3)} \\
 &\subseteq W && \text{((3)2)}
 \end{aligned}$$

$\langle 4 \rangle 2$. Q.E.D.
 PROOF: $\langle 2 \rangle 2$, $\langle 3 \rangle 3$, $\langle 4 \rangle 1$
 $\langle 3 \rangle 5$. $x \in U_n$
 PROOF: $\langle 2 \rangle 4$, $\langle 3 \rangle 3$, $\langle 3 \rangle 4$.
 $\langle 1 \rangle 3$. Every closed set is G_δ
 PROOF:
 $\langle 2 \rangle 1$. LET: C be closed
 $\langle 2 \rangle 2$. PICK a countable set \mathcal{U} of open sets such that $X \setminus C = \bigcup \mathcal{U} = \bigcup_{U \in \mathcal{U}} \overline{U}$
 PROOF: By $\langle 1 \rangle 2$
 $\langle 2 \rangle 3$. $C = \bigcap_{U \in \mathcal{U}} X \setminus \overline{U}$
 PROOF: From $\langle 2 \rangle 2$ and De Morgan's laws.
 $\langle 1 \rangle 4$. X is normal
 $\langle 2 \rangle 1$. LET: C and D be disjoint closed sets.
 $\langle 2 \rangle 2$. PICK a countable sequence of open sets U_n such that $X \setminus D = \bigcup_{n=0}^{\infty} U_n = \bigcup_{n=0}^{\infty} \overline{U_n}$
 PROOF: By $\langle 1 \rangle 2$ and $\langle 2 \rangle 1$.
 $\langle 2 \rangle 3$. PICK a countable sequence of open sets V_n such that $X \setminus C = \bigcup_{n=0}^{\infty} V_n = \bigcup_{n=0}^{\infty} \overline{V_n}$
 PROOF: By $\langle 1 \rangle 2$ and $\langle 2 \rangle 1$.
 $\langle 2 \rangle 4$. For $n \in \mathbb{N}$,
 LET: $U'_n = U_n \setminus \bigcup_{i=0}^n \overline{V_i}$
 $\langle 2 \rangle 5$. For $n \in \mathbb{N}$,
 LET: $V'_n = V_n \setminus \bigcup_{i=0}^n \overline{U_i}$
 $\langle 2 \rangle 6$. LET: $U = \bigcup_{n=0}^{\infty} U'_n$
 $\langle 2 \rangle 7$. LET: $V = \bigcup_{n=0}^{\infty} V'_n$
 $\langle 2 \rangle 8$. U is open
 $\langle 3 \rangle 1$. For each n , U'_n is open
 $\langle 4 \rangle 1$. LET: $n \in \mathbb{N}$
 $\langle 4 \rangle 2$. U_n is open
 PROOF: By $\langle 2 \rangle 2$.
 $\langle 4 \rangle 3$. $\bigcup_{i=0}^n \overline{V_i}$ is closed
 PROOF: By Proposition 3.6.4 and Proposition 3.10.3.
 $\langle 4 \rangle 4$. Q.E.D.
 PROOF: Since $U'_n = U_n \cap (X \setminus \bigcup_{i=0}^n \overline{V_i})$
 $\langle 3 \rangle 2$. Q.E.D.
 PROOF: By $\langle 2 \rangle 6$
 $\langle 2 \rangle 9$. V is open
 PROOF: Similar.
 $\langle 2 \rangle 10$. $U \cap V = \emptyset$
 $\langle 3 \rangle 1$. ASSUME: for a contradiction $x \in U \cap V$
 $\langle 3 \rangle 2$. PICK m, n such that $x \in U'_m$ and $x \in V'_n$
 PROOF: $\langle 2 \rangle 6$, $\langle 2 \rangle 7$, $\langle 3 \rangle 1$
 $\langle 3 \rangle 3$. ASSUME: w.l.o.g. $m \leq n$
 $\langle 3 \rangle 4$. $x \in V'_n$ and $x \in U_m$
 PROOF: From $\langle 2 \rangle 4$ and $\langle 3 \rangle 2$.
 $\langle 3 \rangle 5$. Q.E.D.

PROOF: This contradicts $\langle 2 \rangle 5$.
 $\langle 2 \rangle 11$. $C \subseteq U$
 $\langle 3 \rangle 1$. LET: $x \in C$
 $\langle 3 \rangle 2$. $x \in X \setminus D$
PROOF: By $\langle 2 \rangle 1$ and $\langle 3 \rangle 1$.
 $\langle 3 \rangle 3$. PICK n such that $x \in U_n$
PROOF: By $\langle 2 \rangle 2$ and $\langle 3 \rangle 2$.
 $\langle 3 \rangle 4$. $x \in U'_n$
 $\langle 4 \rangle 1$. For all i , $x \notin V_i$
PROOF: From $\langle 2 \rangle 3$ and $\langle 3 \rangle 4$.
 $\langle 4 \rangle 2$. Q.E.D.
PROOF: From $\langle 2 \rangle 4$ and $\langle 3 \rangle 3$ and $\langle 4 \rangle 1$.
 $\langle 3 \rangle 5$. Q.E.D.
PROOF: By $\langle 2 \rangle 6$.
 $\langle 2 \rangle 12$. $D \subseteq V$
PROOF: Similar.

□

Lemma 6.5.20. *Let X be a normal space. Let A be a closed G_δ set in X . Then there exists a continuous $f : X \rightarrow [0, 1]$ such that $f(x) = 0$ for $x \in A$ and $f(x) > 0$ for $x \notin A$.*

PROOF:
 $\langle 1 \rangle 1$. LET: X be a normal space.
 $\langle 1 \rangle 2$. LET: A be a closed G_δ set in X .
 $\langle 1 \rangle 3$. PICK open sets U_n such that $A = \bigcup_{n=0}^{\infty} U_n$
PROOF: From $\langle 1 \rangle 2$
 $\langle 1 \rangle 4$. For $n \in \mathbb{N}$, PICK $f_n : X \rightarrow [0, 1]$ continuous such that $f(x) = 0$ for $x \in A$ and $f(x) = 1$ for $x \notin U_n$
PROOF: By the Urysohn lemma, $\langle 1 \rangle 1$, $\langle 1 \rangle 2$ and $\langle 1 \rangle 3$.
 $\langle 1 \rangle 5$. LET: $f : X \rightarrow [0, 1]$ with $f(x) = \sum_{n=0}^{\infty} f_n(x)/2^{n+1}$
PROOF: The sequence converges by the Comparison Test with $\sum_{n=0}^{\infty} 1/2^{n+1}$.
 $\langle 1 \rangle 6$. f is continuous
PROOF: By the Weierstrass M-test and the Uniform Limit Theorem.
 $\langle 1 \rangle 7$. f vanishes on A
 $\langle 1 \rangle 8$. f is positive on $X \setminus A$

□

6.6 Completely Normal Spaces

Definition 6.6.1 (Completely Normal). A space X is *completely normal* iff every subspace is normal.

Proposition 6.6.2. *A subspace of a completely normal space is completely normal.*

PROOF: Immediate from definitions. □

Proposition 6.6.3. *Let X be a topological space. Then X is completely normal iff X is T_1 and, for any pair of separated sets A, B in X , there exist disjoint open sets including them.*

PROOF:

⟨1⟩1. If X is completely normal then X is T_1 and, for any pair of separated sets A, B in X , there exist disjoint open sets including them.

⟨2⟩1. ASSUME: X is completely normal.

⟨2⟩2. X is T_1

PROOF: Holds because X is normal.

⟨2⟩3. For any pair of separated sets A, B in X , there exist disjoint open sets including them.

⟨3⟩1. LET: A and B be separated in X

⟨3⟩2. LET: $Y = X \setminus (\bar{A} \cap \bar{B})$

⟨3⟩3. PICK disjoint open sets U, V in Y such that $\bar{A} \cap Y \subseteq U$ and $\bar{B} \cap Y \subseteq V$
PROOF: Y is normal by ⟨2⟩1.

⟨3⟩4. PICK open sets U_0, V_0 in X such that $U = U_0 \cap Y, V = V_0 \cap Y$

⟨3⟩5. $A \subseteq U_0 \setminus \bar{B}$ and $B \subseteq V_0 \setminus \bar{A}$

PROOF: Using ⟨3⟩1.

⟨1⟩2. If X is T_1 and, for any pair of separated sets A, B in X , there exist disjoint open sets including them, then X is completely normal.

⟨2⟩1. ASSUME: X is T_1 and, for any pair of separated sets A, B in X , there exist disjoint open sets including them

⟨2⟩2. LET: $Y \subseteq X$

⟨2⟩3. Y is T_1

PROOF: By Proposition 6.1.3.

⟨2⟩4. LET: A and B be disjoint closed sets in Y

⟨2⟩5. A and B are separated in X

⟨3⟩1. $\bar{A} \cap Y = A$ and $\bar{B} \cap Y = B$

PROOF: By Proposition 3.10.6 and Theorem 4.3.4.

⟨3⟩2. $\bar{A} \cap B = \emptyset$

$$\bar{A} \cap B = \bar{A} \cap \bar{B} \cap Y \quad (\langle 3 \rangle 1)$$

$$= A \cap B \quad (\langle 3 \rangle 1)$$

$$= \emptyset \quad (\langle 2 \rangle 4)$$

⟨3⟩3. $A \cap \bar{B} = \emptyset$

PROOF: Similar.

⟨2⟩6. PICK disjoint open sets U and V that include A and B respectively.

PROOF: By ⟨2⟩1.

⟨2⟩7. $U \cap Y$ and $V \cap Y$ are disjoint open sets in Y that include A and B respectively.

□

Proposition 6.6.4. *A well-ordered set in the order topology is completely normal.*

PROOF:

- ⟨1⟩1. LET: X be a well-ordered set.
- ⟨1⟩2. For all $a, b \in X$ with $a < b$, we have $(a, b]$ is open.
 - ⟨2⟩1. CASE: b is greatest in X
 - PROOF: This case holds by the definition of the order topology.
 - ⟨2⟩2. CASE: b is not greatest in X
 - PROOF: In this case, $(a, b] = (a, c)$ where c is the successor of b .
- ⟨1⟩3. LET: A and B be separated sets in X
 - PROVE: There exist disjoint open sets U, V including A and B
- ⟨1⟩4. CASE: The least element of X is not in A or B
 - ⟨2⟩1. LET: $U = \bigcup \{(x, a] : a \in A, x < a, (x, a] \cap B = \emptyset\}$
 - ⟨2⟩2. LET: $V = \bigcup \{(y, b] : b \in B, y < b, (y, b] \cap A = \emptyset\}$
 - ⟨2⟩3. U is open
 - PROOF: From ⟨1⟩2.
 - ⟨2⟩4. V is open
 - PROOF: From ⟨1⟩2.
 - ⟨2⟩5. $A \subseteq U$
 - ⟨3⟩1. LET: $a \in A$
 - ⟨3⟩2. PICK W a neighbourhood of a such that $W \cap B = \emptyset$
 - PROOF: By ⟨1⟩3.
 - ⟨3⟩3. PICK $x < a$ such that $(x, a] \subseteq W$
 - PROOF: By Lemma 4.1.2
 - ⟨3⟩4. $a \in (x, a] \subseteq U$
 - ⟨2⟩6. $B \subseteq V$
 - PROOF: Similar.
 - ⟨2⟩7. $U \cap V = \emptyset$
- ⟨1⟩5. CASE: $\perp \in A$
 - ⟨2⟩1. PICK disjoint open sets U and V that include $A \setminus \{\perp\}$ and B
 - PROOF: From ⟨1⟩4.
 - ⟨2⟩2. $U \cup \{\perp\}$ and V are disjoint open sets that include A and B
 - PROOF: $\{\perp\}$ is open because it is $(-\infty, a)$ where a is the successor of \perp .
- ⟨1⟩6. Q.E.D.
 - PROOF: By Proposition 6.6.3.

□

Proposition 6.6.5. *The product of two completely normal spaces is not necessarily completely normal.*

PROOF:

- ⟨1⟩1. S_Ω is completely normal.
 - PROOF: By Proposition 6.6.4
- ⟨1⟩2. $\overline{S_\Omega}$ is completely normal.
 - PROOF: By Proposition 6.6.4
- ⟨1⟩3. $S_\Omega \times \overline{S_\Omega}$ is not completely normal.
 - PROOF: By Proposition 6.5.3.

□

Proposition 6.6.6. *A compact Hausdorff space is not necessarily completely normal.*

PROOF:

⟨1⟩1. PICK an uncountable set J

⟨1⟩2. $[0, 1]^J$ is compact Hausdorff

PROOF: By Tychonoff's Theorem and Theorem 6.2.5.

⟨1⟩3. $(0, 1)^J$ is not normal.

PROOF: By Proposition 6.5.6, since $(0, 1) \cong \mathbb{R}$.

□

Proposition 6.6.7. *The space \mathbb{R}_l is completely normal.*

PROOF:

⟨1⟩1. LET: $X \subseteq \mathbb{R}_l$

⟨1⟩2. LET: A and B be disjoint closed sets in X .

⟨1⟩3. PICK closed sets C and D such that $A = C \cap X$ and $B = D \cap X$

⟨1⟩4. For $a \in A$, PICK $x_a > a$ such that $[a, x_a) \cap D = \emptyset$

⟨1⟩5. For $b \in B$, PICK $x_b > b$ such that $[b, x_b) \cap C = \emptyset$

⟨1⟩6. $\bigcup_{a \in A} [a, x_a) \cap X$ and $\bigcup_{b \in B} [b, x_b) \cap X$ are disjoint open sets in X that include A and B

□

6.7 Perfectly Normal Spaces

Definition 6.7.1 (Perfectly Normal). A space is *perfectly normal* iff it is normal and every closed set is G_δ .

Proposition 6.7.2. *Every perfectly normal space is completely normal.*

PROOF:

⟨1⟩1. LET: X be perfectly normal.

⟨1⟩2. LET: A and B be separated sets in X

⟨1⟩3. PICK continuous functions $f, g : X \rightarrow [0, 1]$ that vanish precisely on \overline{A} and \overline{B} , respectively.

PROOF: By Theorem 6.5.9.

⟨1⟩4. LET: $h = f - g$

⟨1⟩5. $B \subseteq h^{-1}((0, +\infty))$ and $A \subseteq h^{-1}((-\infty, 0))$

⟨1⟩6. Q.E.D.

PROOF: By Proposition 6.6.3.

□

Proposition 6.7.3. *The space $\overline{S_\Omega}$ is not perfectly normal.*

PROOF: The set $\{\Omega\}$ is not G_δ . □

Chapter 7

Countability Axioms

7.1 The First Countability Axiom

Definition 7.1.1 (First Countability Axiom). A topological space X satisfies the *first countability axiom*, or is *first countable*, iff every point has a countable local basis.

Proposition 7.1.2. S_Ω is first countable.

PROOF: For every countable ordinal $\alpha > 0$, the set $\{(\beta, \alpha + 1) : \beta < \alpha\}$ is a local basis at α . The set $\{\{0\}\}$ is a local basis at 0. \square

Theorem 7.1.3 (The Sequence Lemma (CC)). *Let X be a first countable space and $A \subseteq X$. If $x \in \overline{A}$, then there exists a sequence of points of A that converges to x .*

PROOF:

$\langle 1 \rangle 1$. LET: $x \in \overline{A}$

$\langle 1 \rangle 2$. PICK a countable basis $\{B_n\}_{n \in \mathbb{Z}^+}$ at x .

$\langle 1 \rangle 3$. For $n \geq 1$, PICK a point $a_n \in B_1 \cap \cdots \cap B_n \cap A$

PROVE: $a_n \rightarrow x$ as $n \rightarrow \infty$

PROOF: Using Countable Choice. Such an a_n exists because $B_1 \cap \cdots \cap B_n$ is a neighbourhood of x . Apply Theorem 3.11.3.

$\langle 1 \rangle 4$. LET: U be a neighbourhood of x

$\langle 1 \rangle 5$. PICK N such that $B_N \subseteq U$

PROOF: From $\langle 1 \rangle 2$.

$\langle 1 \rangle 6$. For $n \geq N$, we have $a_n \in U$

PROOF:

$$\begin{aligned} a_n &\in B_1 \cap \cdots \cap B_n && (\langle 1 \rangle 3) \\ &\subseteq B_N && (n \geq N) \\ &\subseteq U && (\langle 1 \rangle 5) \end{aligned}$$

\square

Theorem 7.1.4 (CC). *Let X and Y be topological spaces where X is first countable. Let $x \in X$. Suppose that, for every sequence $\{x_n\}_{n \geq 1}$ such that $x_n \rightarrow x$ as $n \rightarrow \infty$, we have $f(x_n) \rightarrow f(x)$ as $n \rightarrow \infty$. Then f is continuous at x .*

PROOF:

- $\langle 1 \rangle 1$. LET: V be a neighbourhood of $f(x)$
- $\langle 1 \rangle 2$. ASSUME: for a contradiction that, for every neighbourhood U of x , $f(U) \not\subseteq V$
- $\langle 1 \rangle 3$. PICK a countable local basis $\{B_n\}_{n \geq 1}$
- $\langle 1 \rangle 4$. For $n \geq 1$, PICK $a_n \in B_1 \cap \dots \cap B_n$ such that $f(a_n) \notin V$
- $\langle 1 \rangle 5$. $a_n \rightarrow x$ as $n \rightarrow \infty$

PROOF:

- $\langle 2 \rangle 1$. LET: U be a neighbourhood of x
- $\langle 2 \rangle 2$. PICK N such that $B_N \subseteq U$
- $\langle 2 \rangle 3$. For all $n \geq N$, $a_n \in U$

PROOF:

$$\begin{aligned} a_n &\in B_1 \cap \dots \cap B_n && (\langle 1 \rangle 4) \\ &\subseteq B_N && (n \geq N) \\ &\subseteq U && (\langle 2 \rangle 2) \end{aligned}$$

- $\langle 1 \rangle 6$. $f(a_n) \rightarrow f(x)$ as $n \rightarrow \infty$
- $\langle 1 \rangle 7$. There exists N such that, for all $n \geq N$, we have $f(a_n) \in V$
- $\langle 1 \rangle 8$. Q.E.D.

Lemma 7.1.5 (CC). \mathbb{R}^ω under the box topology is not first countable.

PROOF:

- $\langle 1 \rangle 1$. LET: $\{B_n\}_{n \geq 1}$ be any countable set of neighbourhoods of $\vec{0}$
- $\langle 1 \rangle 2$. For $n \geq 1$, PICK U_{nm} for $m \geq 1$ such that $\vec{0} \in \prod_{m=1}^\infty U_{nm} \subseteq B_n$
- $\langle 1 \rangle 3$. For $n \geq 1$, PICK a_n, b_n such that $0 \in (a_n, b_n) \subseteq U_{nn}$
- $\langle 1 \rangle 4$. LET: $U = \prod_{n=1}^\infty (a_n/2, b_n/2)$
- $\langle 1 \rangle 5$. $\vec{0} \in U$
- $\langle 1 \rangle 6$. For all n , $B_n \not\subseteq U$

□

Lemma 7.1.6 (CC). *If J is uncountable then \mathbb{R}^J is not first countable.*

PROOF:

- $\langle 1 \rangle 1$. LET: $\{B_n\}_{n \geq 1}$ be a countable family of neighbourhoods of $\vec{0}$
- $\langle 1 \rangle 2$. For $n \geq 1$, PICK $U_{n\alpha}$ such that $\vec{0} \in \prod_{\alpha \in J} U_{n\alpha} \subseteq B_n$ where $U_{n\alpha}$ is open in \mathbb{R} and $U_{n\alpha} = \mathbb{R}$ except for $\alpha = \alpha_{n1}, \dots, \alpha_{nr_n}$
- $\langle 1 \rangle 3$. PICK β such that β is different from α_{ni} for all n, i
- $\langle 1 \rangle 4$. LET: $V = \pi_\beta^{-1}((-1, 1))$
- $\langle 1 \rangle 5$. $\vec{0} \in V$
- $\langle 1 \rangle 6$. $V \not\subseteq B_n$ for all n

□

Lemma 7.1.7. \mathbb{R}_l is first countable.

PROOF: For all $x \in \mathbb{R}$, $\{[x, q) : q \in \mathbb{Q}, q > x\}$ is a basis at x . \square

Lemma 7.1.8. The ordered square is first countable.

PROOF:

(1)1. LET: $(x, y) \in I_o^2$

PROVE: There exists a countable local basis \mathcal{B} at (x, y)

(1)2. CASE: $(x, y) = (0, 0)$

PROOF: Take $\mathcal{B} = \{[(0, 0), (0, q)) : q \in \mathbb{Q}, 0 < q < 1\}$.

(1)3. CASE: $0 < y < 1$

PROOF: Take $\mathcal{B} = \{((x, q), (x, q')) : q, q' \in \mathbb{Q}, q < y < q'\}$.

(1)4. CASE: $x < 1, y = 1$

PROOF: Take $\mathcal{B} = \{((x, q), (q', 0)) : q, q' \in \mathbb{Q}, 0 < q < 1, x < q' < 1\}$.

(1)5. CASE: $x > 0, y = 0$

PROOF: Take $\mathcal{B} = \{((q, 1), (x, q')) : q, q' \in \mathbb{Q}, 0 < q < x, 0 < q' < 1\}$

(1)6. CASE: $(x, y) = (1, 1)$

PROOF: Take $\mathcal{B} = \{((1, q), (1, 1]) : q \in \mathbb{Q}, 0 < q < 1\}$.

\square

Proposition 7.1.9. A subspace of a first countable space is first countable.

PROOF:

(1)1. LET: X be a first countable space and $A \subseteq X$

(1)2. LET: $a \in A$

(1)3. PICK a countable basis \mathcal{B} at a in X

(1)4. $\{B \cap A : B \in \mathcal{B} \text{ is a countable basis at } a \text{ in } X\}$

\square

Proposition 7.1.10 (CC). A countable product of first countable spaces is first countable.

PROOF:

(1)1. LET: $\{X_n\}_{n \in \mathbb{Z}^+}$ be a countable family of first countable spaces.

(1)2. LET: $\vec{x} \in \prod_{n=1}^{\infty} X_n$

(1)3. PICK a countable basis \mathcal{B}_n at x_n in X_n for all n

(1)4. LET: \mathcal{B} be the set of all sets $\prod_{i=1}^n U_i$ where $U_i \in \mathcal{B}_i$ for finitely many i and $U_i = X_i$ for all other i .

(1)5. \mathcal{B} is a countable basis at \vec{x} in $\prod_{n=1}^{\infty} X_n$

\square

Corollary 7.1.10.1. The space \mathbb{R}^{ω} is first countable.

Proposition 7.1.11. The space S_{Ω} is first countable.

PROOF:

(1)1. LET: $\alpha \in S_{\Omega}$

PROVE: α has a countable local basis.

- (1)2. CASE: α is zero or a successor ordinal.
 PROOF: In this case, $\{\{\alpha\}\}$ is a local basis.
 (1)3. CASE: α is a limit ordinal.
 (2)1. PICK a countable sequence (β_n) with supremum α
 (2)2. $\{(\beta_n, \alpha + 1) : n \in \mathbb{Z}^+\}$ is a local basis.

□

Proposition 7.1.12. *The space $\overline{S_\Omega}$ is not first countable.*

PROOF:

- (1)1. ASSUME: for a contradiction \mathcal{B} is a countable local basis at Ω
 (1)2. LET: $\alpha = \sup\{\inf B : B \in \mathcal{B}\}$
 (1)3. $\alpha < \Omega$
 (1)4. There is no $B \in \mathcal{B}$ such that $B \subseteq (\alpha, +\infty)$

□

Proposition 7.1.13. *The continuous image of a first countable space is first countable.*

PROOF:

- (1)1. LET: X be a first countable space, Y a space and $f : X \rightarrow Y$ continuous.
 (1)2. LET: $y \in f(X)$
 (1)3. PICK $x \in X$ such that $y = f(x)$
 (1)4. PICK a countable local basis \mathcal{B} at x
 (1)5. $\{f(B) : B \in \mathcal{B}\}$ is a countable local basis at y .

□

Proposition 7.1.14. *$S_\Omega \times \overline{S_\Omega}$ is not first countable.*

PROOF: $(0, \Omega)$ has no countable basis. □

Proposition 7.1.15. *The Sorgenfrey plane is first countable.*

PROOF: For any point (a, b) , the set $\{[a, a + q) \times [b, b + r) : q, r \in \mathbb{Q}\}$ is a countable local basis at (a, b) . □

7.2 Separable Spaces

Definition 7.2.1 (Separable Space). A topological space X is *separable* iff it has a countable dense subset.

Proposition 7.2.2. *The space S_Ω is not separable.*

PROOF:

- (1)1. LET: $D \subseteq S_\Omega$ be countable.
 (1)2. LET: $\alpha = \sup D$
 (1)3. $\overline{D} \subseteq (-\infty, \alpha]$

□

Proposition 7.2.3. *The space $\overline{S_\Omega}$ is not separable.*

PROOF:

- ⟨1⟩1. LET: $D \subseteq S_\Omega$ be countable.
- ⟨1⟩2. LET: $\alpha = \sup\{\beta \in D : \beta < \Omega\}$
- ⟨1⟩3. $\alpha < \Omega$

PROOF: α is the supremum of countably many countable ordinals.

- ⟨1⟩4. $\overline{D} \subseteq (-\infty, \alpha] \cup \{\Omega\}$

□

Corollary 7.2.3.1. *Not every compact Hausdorff space is separable.*

Proposition 7.2.4. *Every open subspace of a separable space is separable.*

PROOF:

- ⟨1⟩1. LET: X be a separable space with countable dense subset D .
- ⟨1⟩2. LET: U be an open subspace of X
PROVE: $D \cap U$ is a countable dense subset of U .
- ⟨1⟩3. $D \cap U$ is countable.
- ⟨1⟩4. LET: V be an open set in U .
- ⟨1⟩5. V is open in X

PROOF: Lemma 4.3.3

- ⟨1⟩6. V intersects D
- ⟨1⟩7. V intersects $D \cap U$

□

Proposition 7.2.5 (CC). *The product of a countable family of separable spaces is separable.*

PROOF:

- ⟨1⟩1. LET: (X_n) be a countable family of separable spaces.
- ⟨1⟩2. For $n \geq 1$, PICK a dense set D_n in X_n
- ⟨1⟩3. $\prod_{n=1}^{\infty} D_n$ is dense in $\prod_{n=1}^{\infty} X_n$.

□

Proposition 7.2.6. *The continuous image of a separable space is separable.*

PROOF:

- ⟨1⟩1. LET: X be a separable space, Y a space and $f : X \rightarrow Y$ be continuous.
- ⟨1⟩2. PICK a countable dense set D in X
- ⟨1⟩3. $f(D)$ is dense in $f(X)$.

□

Corollary 7.2.6.1. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of nonempty topological spaces. If $\prod_{\alpha \in J} X_\alpha$ is separable then each X_α is separable.*

Corollary 7.2.6.2. $S_\Omega \times \overline{S_\Omega}$ is not separable.

Proposition 7.2.7. *The ordered square is not separable.*

PROOF: $\{\{x\} \times (0, 1) : x \in [0, 1]\}$ is an uncountable set of disjoint open sets. □

Proposition 7.2.8. \mathbb{R}_l is separable.

PROOF: \mathbb{Q} is dense. \square

Proposition 7.2.9. The Sorgenfrey plane is separable.

PROOF: \mathbb{Q}^2 is dense. \square

Proposition 7.2.10. Not every closed subspace of a separable space is separable.

PROOF: \mathbb{R}_l^2 is separable but the subspace $\{(x, -x) : x \in \mathbb{R}\}$ is not. \square

7.3 The Second Countability Axiom

Definition 7.3.1 (Second Countability Axiom). A topological space satisfies the *second countability axiom*, or is *second countable*, iff it has a countable basis.

Proposition 7.3.2. S_Ω is not second countable.

PROOF: $\{\{\alpha\} : \alpha \text{ is a countable successor ordinal}\}$ is an uncountable set of disjoint open sets. \square

Proposition 7.3.3. A subspace of a second countable space is second countable.

PROOF:

$\langle 1 \rangle 1.$ LET: X be a second countable space and $A \subseteq X$

$\langle 1 \rangle 2.$ PICK a countable basis \mathcal{B} for X

$\langle 1 \rangle 3.$ $\{B \cap A : B \in \mathcal{B}\}$ is a countable basis for A

\square

Proposition 7.3.4 (CC). The product of countably many second countable spaces is second countable.

PROOF:

$\langle 1 \rangle 1.$ LET: $\{X_n\}_{n \in \mathbb{Z}^+}$ be a countable family of second countable spaces.

$\langle 1 \rangle 2.$ For $n \in \mathbb{Z}^+$, PICK a countable basis \mathcal{B}_n for X_n .

$\langle 1 \rangle 3.$ LET: \mathcal{B} be the set of all sets of the form $\prod_{n=1}^{\infty} U_n$, where $U_n \in \mathcal{B}_n$ for finitely many n , and $U_n = X_n$ for all other n .

$\langle 1 \rangle 4.$ \mathcal{B} is a countable basis for $\prod_{n=1}^{\infty} X_n$

\square

Theorem 7.3.5 (CC). Every second countable space is separable.

PROOF:

$\langle 1 \rangle 1.$ LET: X be a second countable space.

$\langle 1 \rangle 2.$ PICK a countable basis \mathcal{B} for X

$\langle 1 \rangle 3.$ For $B \in \mathcal{B}$ nonempty, PICK a point $x_B \in B$

$\langle 1 \rangle 4.$ $D = \{x_B : B \in \mathcal{B} \setminus \{\emptyset\}\}$ is dense.

$\langle 2 \rangle 1.$ LET: $l \in X$

PROVE: $l \in \overline{D}$

(2)2. LET: $B \in \mathcal{B}$ such that $l \in B$

(2)3. $x_B \in B \cap D$

(2)4. Q.E.D.

PROOF: By Theorem 3.10.8

Corollary 7.3.5.1. $S_\Omega \times \overline{S_\Omega}$ is not second countable.

Corollary 7.3.5.2. The space \mathbb{R}^ω is separable.

Corollary 7.3.5.3. If J is uncountable then \mathbb{R}^J is not second countable.

Proposition 7.3.6. The ordered square is not second countable.

PROOF:

(1)1. LET: \mathcal{B} be any basis

(1)2. For $x \in [0, 1]$, PICK B_x such that $x \in B_x \subseteq ((x, 0), (x, 1))$

(1)3. The function $B_{(-)}$ is an injective function $[0, 1] \rightarrow \mathcal{B}$

(1)4. \mathcal{B} is uncountable.

□

Proposition 7.3.7. The space $\overline{S_\Omega}$ is not second countable.

PROOF: It is not first countable (Proposition 7.1.12). □

Proposition 7.3.8. The continuous image of a second countable space is second countable.

PROOF:

(1)1. LET: X be a second countable space, Y a space and $f : X \rightarrow Y$ be continuous.

(1)2. PICK a countable basis \mathcal{B} for X .

(1)3. $\{f(B) : B \in \mathcal{B}\}$ is a countable basis for $f(X)$

□

Theorem 7.3.9. Every regular Lindelöf space is normal.

PROOF:

(1)1. LET: X be a regular Lindelöf space.

(1)2. LET: A and B be disjoint closed sets in X .

(1)3. $\{U \text{ open in } X : \overline{U} \cap B = \emptyset\}$ covers A

PROOF: Proposition 6.3.2.

(1)4. PICK a countable open covering $\{U_n : n \in \mathbb{Z}^+\}$ of A such that $\overline{U_n} \cap B = \emptyset$ for all n

(1)5. PICK a countable open covering $\{V_n : n \in \mathbb{Z}^+\}$ of B such that $\overline{V_n} \cap A = \emptyset$ for all n

PROOF: Similar.

(1)6. For $n \in \mathbb{Z}^+$,

LET: $U'_n = U_n \setminus \bigcup_{i=1}^n \overline{V_i}$ and $V'_n = V_n \setminus \bigcup_{i=1}^n \overline{U_i}$

(1)7. LET: $U' = \bigcup_{n=1}^\infty U'_n$ and $V = \bigcup_{n=1}^\infty V'_n$

⟨1⟩8. $A \subseteq U'$ and $B \subseteq V'$

⟨1⟩9. $U' \cap V' = \emptyset$

□

Corollary 7.3.9.1. *If J is uncountable then \mathbb{R}^J is not Lindelöf.*

Proposition 7.3.10. *Every second countable regular space is completely normal.*

PROOF:

⟨1⟩1. LET: X be second countable and regular and $Y \subseteq X$

⟨1⟩2. Y is second countable

PROOF: Proposition 7.3.3.

⟨1⟩3. Y is regular

PROOF: Proposition 6.3.4

⟨1⟩4. Y is normal

PROOF: Theorem 7.3.9

□

Proposition 7.3.11. *The space \mathbb{R}^ω is second countable.*

PROOF: The sets $\prod_{n=0}^\infty U_n$ form a basis, where U_n is an interval of the form (q, r) for $q, r \in \mathbb{Q}$ for finitely many n , and $U_n = \mathbb{R}$ for all other n . □

Proposition 7.3.12 (CC). *In a second countable space, every discrete subspace is countable.*

PROOF:

⟨1⟩1. LET: X be a second countable space

⟨1⟩2. PICK a countable basis \mathcal{B}

⟨1⟩3. LET: $D \subseteq X$ be discrete

⟨1⟩4. For $a \in D$, PICK $B_a \in \mathcal{B}$ such that $B_a \cap D = \{a\}$

⟨1⟩5. $a \mapsto B_a$ is injective

□

Proposition 7.3.13. *The space \mathbb{R}_K is second countable.*

PROOF: $\{(a, b) : a, b \in \mathbb{R}\} \cup \{(a, b) - K : a, b \in \mathbb{Q}\}$ is a basis. □

Corollary 7.3.13.1. *The space \mathbb{R}_K is first countable.*

Corollary 7.3.13.2. *The space \mathbb{R}_K is separable.*

Proposition 7.3.14. *Let J be a set with $|J| > |\mathbb{R}|$. Then \mathbb{R}^J is not separable.*

PROOF:

⟨1⟩1. ASSUME: D is countable and dense in \mathbb{R}^J

PROVE: $|J| \leq |\mathbb{R}|$

⟨1⟩2. Define $f : J \rightarrow \mathcal{P}D$ by $f(\alpha) = D \cap \pi_\alpha^{-1}((0, 1))$

⟨1⟩3. f is injective

- $\langle 2 \rangle 1$. LET: $\alpha, \beta \in J$ with $\alpha \neq \beta$
 - $\langle 2 \rangle 2$. PICK $x \in D \cap \pi_\alpha^{-1}((0, 1)) \cap \pi_\beta^{-1}((2, 3))$
 - $\langle 2 \rangle 3$. $x \in f(\alpha)$ but $x \notin f(\beta)$
-

Corollary 7.3.14.1. *The product of a family of separable spaces is not necessarily separable.*

Chapter 8

Connectedness

8.1 Connected Spaces

Definition 8.1.1 (Separation). Let X be a topological space. A *separation* of X is a pair of disjoint nonempty subsets whose union is X .

Definition 8.1.2 (Connected). A topological space is *connected* iff it has no separation.

Proposition 8.1.3. S_Ω is not connected.

PROOF: $\{0\}$ and $S_\Omega \setminus \{0\}$ form a separation. \square

Proposition 8.1.4. A space X is connected if and only if the only sets that are both closed and open are \emptyset and X .

PROOF: Immediate from definitions. \square

Proposition 8.1.5. Let Y be a subspace of X . Then a separation of Y is a pair of disjoint nonempty sets A, B such that $A \cup B = Y$ and neither of A, B contains a limit point of the other.

PROOF:

- $\langle 1 \rangle 1$. If A and B form a separation of Y then A and B are disjoint and nonempty, $A \cup B = Y$, and neither of A, B contains a limit point of the other.
- $\langle 2 \rangle 1$. LET: A and B be a separation of Y
- $\langle 2 \rangle 2$. A and B are disjoint and nonempty and $A \cup B = Y$
PROOF: Immediate from the definition of separation.
- $\langle 2 \rangle 3$. A does not contain a limit point of B
PROOF: B is closed in Y , hence contains all its limit points (Corollary 3.13.3.1), and so the result follows because A and B are disjoint.
- $\langle 2 \rangle 4$. B does not contain a limit point of A
PROOF: Similar.
- $\langle 1 \rangle 2$. If A and B are disjoint and nonempty, $A \cup B = Y$, and neither of A, B contains a limit point of the other, then A and B are a separation of Y .

- (2)1. ASSUME: A and B are disjoint and nonempty, $A \cup B = Y$, and neither of A, B contains a limit point of the other
 (2)2. A is closed in Y
 PROOF: Every limit point of A is not in B , so is in A . Apply Corollary 3.13.3.1.
 (2)3. B is open in Y
 PROOF: $B = Y \setminus A$
 (2)4. A is open in Y
 PROOF: Similar.

□

Proposition 8.1.6. *If the sets C and D form a separation of X , and Y is a connected subspace of X , then $Y \subseteq C$ or $Y \subseteq D$.*

PROOF: Otherwise, $Y \cap C$ and $Y \cap D$ would be a separation of Y . □

Proposition 8.1.7. *The union of a set of connected subspaces of X that have a point in common is connected.*

PROOF:

- (1)1. LET: \mathcal{S} be a set of connected subspaces that have the point a in common.
 (1)2. ASSUME: for a contradiction U and V form a separation of $\bigcup \mathcal{S}$
 (1)3. ASSUME: w.l.o.g. $a \in U$
 (1)4. For all $Y \in \mathcal{S}$ we have $Y \subseteq U$
 PROOF: By Proposition 8.1.6.
 (1)5. $V = \emptyset$
 (1)6. Q.E.D.
 PROOF: This contradicts (1)2.

□

Theorem 8.1.8. *Let A be a connected subspace of X . If $A \subseteq B \subseteq \overline{A}$ then B is connected.*

PROOF:

- (1)1. ASSUME: for a contradiction U and V are a separation of B
 (1)2. $A \subseteq U$ or $A \subseteq V$
 PROOF: By Proposition 8.1.6.
 (1)3. ASSUME: w.l.o.g. $A \subseteq U$
 (1)4. $\overline{A} \subseteq \overline{U}$
 PROOF: By Proposition 3.10.5.
 (1)5. $B \subseteq \overline{U}$
 PROOF: Since $B \subseteq \overline{A}$.
 (1)6. The closure of U in B is B
 PROOF: By Theorem 4.3.4.
 (1)7. $U = B$
 PROOF: Since U is closed in B .
 (1)8. Q.E.D.
 PROOF: This contradicts (1)1.

□

Theorem 8.1.9. *The image of a connected space under a continuous map is connected.*

PROOF: Let X be a connected space, Y a topological space, and $f : X \rightarrow Y$ be surjective. If U and V form a separation of Y , then $f^{-1}(U)$ and $f^{-1}(V)$ form a separation of X . □

Corollary 8.1.9.1. *Let \mathcal{T} and \mathcal{T}' be topologies on the same set X . If $\mathcal{T} \subseteq \mathcal{T}'$ and X is connected under \mathcal{T}' then X is connected under \mathcal{T} .*

Corollary 8.1.9.2. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces. If $\prod_{\alpha \in J} X_\alpha$ is connected then each X_α is connected.*

Corollary 8.1.9.3. *The Sorgenfrey plane is disconnected.*

Proposition 8.1.10. *The product of a family of connected spaces is connected.*

PROOF:

⟨1⟩1. The product of two connected spaces is connected.

PROOF:

⟨2⟩1. LET: X and Y be connected spaces.

⟨2⟩2. ASSUME: w.l.o.g. X and Y are nonempty.

PROOF: If either is empty then $X \times Y = \emptyset$ is connected.

⟨2⟩3. ASSUME: for a contradiction U and V are a separation of $X \times Y$.

⟨2⟩4. PICK $b \in Y$

PROOF: By ⟨2⟩2.

⟨2⟩5. For all $x \in X$,

LET: $T_x = (X \times \{b\}) \cup (\{x\} \times Y)$

⟨2⟩6. For all $x \in X$, T_x is connected

⟨3⟩1. $X \times \{b\}$ is connected

PROOF: It is homeomorphic to X .

⟨3⟩2. $\{x\} \times Y$ is connected

PROOF: It is homeomorphic to Y .

⟨3⟩3. Q.E.D.

PROOF: By Proposition 8.1.7.

⟨2⟩7. $X \times Y = \bigcup_{x \in X} T_x$

⟨2⟩8. Q.E.D.

⟨3⟩1. PICK $a \in X$

PROOF: By ⟨2⟩2.

⟨3⟩2. $(a, b) \in T_x$ for all $x \in X$

⟨3⟩3. Q.E.D.

PROOF: By Proposition 8.1.7.

⟨1⟩2. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of connected spaces.

⟨1⟩3. ASSUME: w.l.o.g. $\prod_{\alpha \in J} X_\alpha$ is nonempty

⟨1⟩4. PICK $\vec{a} \in \prod_{\alpha \in J} X_\alpha$

⟨1⟩5. For K a finite subset of J ,

- LET: $X_K = \{\vec{x} \in \prod_{\alpha \in J} X_\alpha : x_\alpha = a_\alpha \text{ for all } \alpha \in J \setminus K\}$
- $\langle 1 \rangle 6$. For all K , X_K is connected.
- PROOF: It is homeomorphic to $\prod_{\alpha \in K} X_\alpha$, so it is connected by $\langle 1 \rangle 1$.
- $\langle 1 \rangle 7$. $\bigcup_{K \subseteq \text{fin } J} X_K$ is connected.
- PROOF: By Proposition 8.1.7 since $\vec{a} \in X_K$ for all K .
- $\langle 1 \rangle 8$. $\prod_{\alpha \in J} X_\alpha = \bigcup_{K \subseteq \text{fin } J} X_K$
- $\langle 2 \rangle 1$. LET: $\vec{x} \in \prod_{\alpha \in J} X_\alpha$
- $\langle 2 \rangle 2$. LET: U be an open neighbourhood of \vec{x}
- $\langle 2 \rangle 3$. PICK a basic open set $\prod_{\alpha \in J} V_\alpha$ such that $\vec{x} \in \prod_{\alpha \in J} V_\alpha \subseteq U$, where each V_α is open in X_α , and $V_\alpha = X_\alpha$ except for $\alpha \in K$ for some finite $K \subseteq J$
- PROVE: U intersects X_K
- $\langle 2 \rangle 4$. LET: $\vec{y} \in \prod_{\alpha \in J} X_\alpha$ with $y_\alpha = x_\alpha$ for $\alpha \in K$, $y_\alpha = a_\alpha$ for $\alpha \notin K$
- $\langle 2 \rangle 5$. $\vec{y} \in U \cap X_K$
- $\langle 1 \rangle 9$. Q.E.D.

Corollary 8.1.10.1. *For any set I , the space \mathbb{R}^I under the product topology is connected.*

Proposition 8.1.11. \mathbb{R}^ω under the box topology is disconnected.

PROOF: The set of all bounded sequences and the set of all unbounded sequences form a separation. \square

Definition 8.1.12 (Totally Disconnected). A space is *totally disconnected* iff the only connected subspaces are the singletons.

Theorem 8.1.13. *Let L be a linearly ordered set under the order topology. Then L is connected if and only if L is a linear continuum.*

PROOF:

- $\langle 1 \rangle 1$. If L is a linear continuum then L is connected.
- $\langle 2 \rangle 1$. LET: L be a linear continuum.
- $\langle 2 \rangle 2$. ASSUME: for a contradiction U and V are a separation of L .
- $\langle 2 \rangle 3$. PICK $a \in U$ and $b \in V$
- $\langle 2 \rangle 4$. ASSUME: w.l.o.g. $a < b$
- $\langle 2 \rangle 5$. LET: $l = \sup\{x \in A : x < b\}$
- $\langle 2 \rangle 6$. CASE: $l \in A$
- $\langle 3 \rangle 1$. PICK $a' > l$ such that $[l, a') \subseteq A$
- PROOF: By Lemma 4.1.2. We know l is not greatest in X because $l < b$.
- $\langle 3 \rangle 2$. PICK a^* such that $l < a^* < a'$
- PROOF: L is dense.
- $\langle 3 \rangle 3$. $l < a^*$, $a^* \in A$, $a^* < b$
- PROOF: If $b < a^*$ then $b \in A$ by $\langle 3 \rangle 1$.
- $\langle 3 \rangle 4$. Q.E.D.
- PROOF: This contradicts $\langle 2 \rangle 5$.
- $\langle 2 \rangle 7$. CASE: $l \in B$
- $\langle 3 \rangle 1$. PICK $b' < l$ such that $(b', l] \subseteq B$

PROOF: By Lemma 4.1.2. We know l is not least in X because $a < l$.

⟨3⟩2. PICK b^* such that $b' < b^* < l$

PROVE: b^* is an upper bound for $\{x \in A : x < b\}$

⟨3⟩3. LET: $x \in A$ and $x < b$

⟨3⟩4. $x \leq b^*$

PROOF: If $b^* < x$ then $b^* < x \leq l$ and so $x \in B$ by ⟨3⟩1.

⟨3⟩5. Q.E.D.

PROOF: This contradicts ⟨2⟩5.

⟨1⟩2. If L is connected then L is a linear continuum.

⟨2⟩1. ASSUME: L is connected

⟨2⟩2. L has the least upper bound property

⟨3⟩1. ASSUME: for a contradiction $A \subseteq L$ is bounded above with no least upper bound

⟨3⟩2. LET: U be the set of upper bounds of A

⟨3⟩3. U is open

⟨4⟩1. LET: $u \in U$

⟨4⟩2. PICK an upper bound v for A with $v < u$

PROOF: u is not the least upper bound for A (⟨3⟩1)

⟨4⟩3. $u \in (v, +\infty) \subseteq U$

⟨3⟩4. LET: V be the set of lower bounds of U

⟨3⟩5. U and V form a separation of L

⟨4⟩1. V is open

PROOF: Similar to ⟨3⟩3.

⟨4⟩2. U and V are disjoint

⟨5⟩1. ASSUME: for a contradiction $x \in U \cap V$

⟨5⟩2. PICK $u \in U$ such that $u < x$

PROOF: x is not the lowest upper bound of A

⟨5⟩3. $x \leq u < x$

⟨4⟩3. $U \cup V = L$

⟨5⟩1. LET: $x \in L \setminus U$

⟨5⟩2. PICK $a \in A$ such that $x < a$

⟨5⟩3. $a \in V$

⟨5⟩4. $x \in V$

⟨2⟩3. For all $x, y \in L$, there exists $z \in L$ such that $x < z < y$

PROOF: Otherwise $(-\infty, y)$ and $(x, +\infty)$ would form a separation of L .

□

Corollary 8.1.13.1. *The real line \mathbb{R} is connected, and so is every ray and interval in \mathbb{R} .*

Corollary 8.1.13.2. *The ordered square is connected.*

Corollary 8.1.13.3. *Not every closed subspace of a connected space is connected.*

PROOF: The set $\{0, 1\}$ is disconnected as a subspace of \mathbb{R} .

Corollary 8.1.13.4. *Not every open subspace of a connected space is connected.*

PROOF: The space $\mathbb{R} \setminus \{0\}$ is a disconnected open subspace of \mathbb{R} . \square

Theorem 8.1.14 (Intermediate Value Theorem). *Let X be a connected space and Y a linearly ordered set under the order topology. Let $f : X \rightarrow Y$ be continuous. Let $a, b \in X$ and $r \in Y$. If $f(a) < r < f(b)$, then there exists $c \in X$ such that $f(c) = r$.*

PROOF: If not, then $f^{-1}((-\infty, r))$ and $f^{-1}((r, +\infty))$ would be a separation of X . \square

Proposition 8.1.15. *Every connected regular space with more than one point is uncountable.*

PROOF:

$\langle 1 \rangle 1$. Every connected completely regular space with more than one point is uncountable.

$\langle 2 \rangle 1$. LET: X be connected and completely regular and $a, b \in X$ with $a \neq b$

$\langle 2 \rangle 2$. PICK a continuous $f : X \rightarrow [0, 1]$ such that $f(a) = 0$ and $f(b) = 1$

$\langle 2 \rangle 3$. f is surjective.

PROOF: By the Intermediate Value Theorem.

$\langle 1 \rangle 2$. Every connected regular space with more than one point is uncountable.

$\langle 2 \rangle 1$. ASSUME: for a contradiction X is connected, regular and countable with more than one point.

$\langle 2 \rangle 2$. X is Lindelöf

$\langle 2 \rangle 3$. X is normal

PROOF: By Theorem 7.3.9

$\langle 2 \rangle 4$. Q.E.D.

PROOF: Contradicting $\langle 1 \rangle 1$.

\square

Proposition 8.1.16. $\overline{S_\Omega}$ is not connected.

PROOF: $\{0\}$ is clopen. \square

Proposition 8.1.17. \mathbb{R}_l is not connected.

PROOF: The set $[0, +\infty)$ is clopen. \square

Proposition 8.1.18. *The space \mathbb{R}^ω under the uniform topology is not connected.*

PROOF: The set of all bounded sequences and the set of all unbounded sequences form a separation. \square

Proposition 8.1.19. *The space \mathbb{R}_K is connected.*

PROOF: Easy. \square

8.2 Components and Local Connectedness

Definition 8.2.1 ((Connected) Component). Let X be a topological space. Define an equivalence relation \sim on X by: $x \sim y$ iff there exists a connected subspace $U \subseteq X$ such that $x \in U$ and $y \in U$. The *(connected) components* of X are the equivalence classes under \sim .

We prove this is an equivalence relation.

PROOF:

$\langle 1 \rangle 1$. For all $x \in X$ we have $x \sim x$.

PROOF: The subspace $\{x\} \subseteq X$ is connected.

$\langle 1 \rangle 2$. For all $x, y \in X$, if $x \sim y$ then $y \sim x$.

PROOF: Immediate from definitions.

$\langle 1 \rangle 3$. For all $x, y, z \in X$, if $x \sim y$ and $y \sim z$ then $x \sim z$.

PROOF: By Proposition 8.1.7.

□

Proposition 8.2.2. *Let X be a topological space. If $C \subseteq X$ is connected and nonempty, then there exists a unique component D of X such that $C \subseteq D$.*

PROOF:

$\langle 1 \rangle 1$. PICK $a \in C$

$\langle 1 \rangle 2$. LET: D be the \sim -equivalence class of A

$\langle 1 \rangle 3$. $C \subseteq D$

PROOF: For all $x \in C$ we have $a \sim x$ by definition.

$\langle 1 \rangle 4$. D is unique

PROOF: This holds because the components are disjoint.

□

Proposition 8.2.3 (AC). *Every component is connected.*

PROOF:

$\langle 1 \rangle 1$. LET: C be a component of the topological space X

$\langle 1 \rangle 2$. PICK $a \in C$

$\langle 1 \rangle 3$. For all $x \in C$, PICK a connected subspace C_x of X containing both a and x .

PROOF: Such a C_x exists since $a \sim x$.

$\langle 1 \rangle 4$. $C = \bigcup_{x \in C} C_x$

PROOF: This holds because $C_x \subseteq C$ by Proposition 8.2.2.

$\langle 1 \rangle 5$. Q.E.D.

PROOF: It follows that C is connected by Proposition 8.1.7.

□

Proposition 8.2.4. *Every component is closed.*

PROOF: From Theorem 8.1.8. □

Proposition 8.2.5. *The component of \vec{a} in \mathbb{R}^ω under the uniform topology is $\{\vec{b} : \vec{b} - \vec{a} \text{ is bounded}\}$.*

PROOF:

- ⟨1⟩1. $C = \{\vec{b} : \vec{b} - \vec{a} \text{ is bounded}\}$ is connected.
 - ⟨2⟩1. ASSUME: $C = U \cup V$ is a separation of C with $\vec{a} \in U$
 - ⟨2⟩2. PICK $\vec{b} \in V$
 - ⟨2⟩3. $\{\epsilon : \epsilon\vec{b} + (1 - \epsilon)\vec{a} \in U\}$ and $\{\epsilon : \epsilon\vec{b} + (1 - \epsilon)\vec{a} \in V\}$ form a separation of $[0, 1]$
 - ⟨1⟩2. If $\vec{a}, \vec{b} \in C$ and $\vec{b} - \vec{a}$ is unbounded then C is disconnected.
- PROOF: $\{\vec{c} : \vec{c} - \vec{a} \text{ is bounded}\}$ and $\{\vec{c} : \vec{c} - \vec{a} \text{ is unbounded}\}$
-

Proposition 8.2.6. *Let $x, y \in \mathbb{R}^\omega$ under the box topology. Then x and y are in the same component iff $x - y$ is eventually zero.*

PROOF:

- ⟨1⟩1. For all $x \in \mathbb{R}^\omega$ the set $\{y : x - y \text{ is eventually zero}\}$ is connected
- PROOF: It is the union of the sets $C_N = \{y : \forall n \geq N. y_n = 0\}$, each of which is connected because it is homeomorphic to \mathbb{R}^{N-1} .
- ⟨1⟩2. If $x - y$ is not eventually zero then x and y are in different components
 - ⟨2⟩1. ASSUME: $x - y$ is not eventually zero
 - ⟨2⟩2. Define $h : \mathbb{R}^\omega \rightarrow \mathbb{R}^\omega$ by: $h(z)_n = \begin{cases} z_n - x_n & \text{if } x_n = y_n \\ n(z_n - x_n)/(y_n - x_n) & \text{if } x_n \neq y_n \end{cases}$
 - ⟨2⟩3. h is an automorphism of \mathbb{R}^ω under the box topology
 - ⟨2⟩4. $h(x) = 0$
 - ⟨2⟩5. $h(y)$ is unbounded
 - ⟨2⟩6. Q.E.D.
- PROOF: The inverse image under h of the set of bounded sequences and the set of unbounded sequences form a separation of \mathbb{R}^ω with x and y in different sets.

□

□

8.3 Path Connectedness

Definition 8.3.1 (Path). Let X be a topological space and $a, b \in X$. A *path* from a to b is a continuous function $p : [0, 1] \rightarrow X$ such that $p(0) = a$ and $p(1) = b$.

Definition 8.3.2 (Path Connected). A topological space is *path connected* iff there exists a path between any two points.

Proposition 8.3.3. *Every path connected space is connected.*

PROOF:

- ⟨1⟩1. LET: X be a path connected space
- ⟨1⟩2. ASSUME: for a contradiction U and V are a separation of X .
- ⟨1⟩3. PICK $a \in U$ and $b \in V$

- ⟨1⟩4. PICK a path $p : [0, 1] \rightarrow X$ from a to b
- ⟨1⟩5. $p^{-1}(U)$ and $p^{-1}(V)$ form a separation of $[0, 1]$.
- ⟨1⟩6. Q.E.D.

PROOF: This contradicts Corollary 8.1.13.1.

Corollary 8.3.3.1. S_Ω is not path connected.

Corollary 8.3.3.2. $\overline{S_\Omega}$ is not path connected.

Corollary 8.3.3.3. \mathbb{R}_l is not path connected.

Corollary 8.3.3.4. The Sorgenfrey plane is not path connected.

Corollary 8.3.3.5. The space \mathbb{R}^ω under the uniform topology is not path connected. connected.

Corollary 8.3.3.6. The space \mathbb{R}^ω under the box topology is not path connected.

Proposition 8.3.4. The long line is path connected.

PROOF:

- ⟨1⟩1. LET: $a, b \in L$
- ⟨1⟩2. PICK an ordinal α such that $a, b < (\alpha, 0)$
- ⟨1⟩3. There exists a path from a to b

PROOF: This holds because $[(0, 0), (\alpha, 0))$ is homeomorphic to $[0, 1)$ by Proposition 1.9.11.

□

Corollary 8.3.4.1. Not every closed subspace of a path connected space is path connected.

PROOF: Take any two-element subspace of the long line.

Corollary 8.3.4.2. Not every open subspace of a path connected space is path connected.

PROOF: The space $\mathbb{R} \setminus \{0\}$ is not path connected as a subspace of \mathbb{R} . □

Definition 8.3.5 (Path Component). Let X be a topological space. Define an equivalence relation \sim on X by: $x \sim y$ iff there exists a path from x to y . The equivalence classes are called the *path components* of X .

We prove this is an equivalence relation.

PROOF:

- ⟨1⟩1. For all $x \in X$ we have $x \sim x$

PROOF: The constant path $p : [0, 1] \rightarrow X$ where $p(t) = x$ is a path from x to x .

- ⟨1⟩2. If $x \sim y$ then $y \sim x$

PROOF: If $p : [0, 1] \rightarrow X$ is a path from x to y then $\lambda t.p(1 - t)$ is a path from y to x .

- ⟨1⟩3. If $x \sim y$ and $y \sim z$ then $x \sim z$

(2)1. LET: p be a path from x to y and q be a path from y to z .

(2)2. LET: $r : [0, 1] \rightarrow X$ where

$$r(t) = \begin{cases} p(2t) & \text{if } 0 \leq t \leq 1/2 \\ q(2t - 1) & \text{if } 1/2 \leq t \leq 1 \end{cases}$$

(2)3. r is a path from x to z .

PROOF: r is continuous by the Pasting Lemma.

□

Proposition 8.3.6. *Every path component is path connected.*

PROOF: By definition, if x and y are in the same path component then there is a path from x to y . □

Proposition 8.3.7. *If A is a nonempty path connected subspace of the space X , then A is included in a unique path component.*

PROOF:

(1)1. PICK $a \in A$

(1)2. LET: C be the equivalence class of a under \sim

(1)3. $A \subseteq C$

PROOF: For all $x \in A$, there exists a path from a to x .

(1)4. C is unique

PROOF: C is the unique path component such that $a \in C$.

□

Proposition 8.3.8. *Every path component is included in a component.*

PROOF: From Propositions 8.3.3 and 8.2.2. □

Proposition 8.3.9. *The ordered square is not path connected.*

PROOF:

(1)1. ASSUME: for a contradiction $p : [0, 1] \rightarrow I_o^2$ is a path from $(0, 0)$ to $(1, 1)$.

(1)2. For all $x \in [0, 1]$, $p^{-1}(\{x\} \times (0, 1))$ is open in $[0, 1]$

(1)3. For all $x \in [0, 1]$, PICK a rational $q_x \in p^{-1}(\{x\} \times (0, 1))$

(1)4. $\{q_x : x \in [0, 1]\}$ is an uncountable set of rationals.

□

Proposition 8.3.10 (AC). *The product of a family of path connected spaces is path connected.*

PROOF:

(1)1. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of path connected spaces and $a, b \in \prod_{\alpha \in J} X_\alpha$

(1)2. For $\alpha \in J$, PICK a path $p_\alpha : [0, 1] \rightarrow X_\alpha$ from a_α to b_α

(1)3. Define $p : [0, 1] \rightarrow \prod_{\alpha \in J} X_\alpha$ by $p(t)_\alpha = p_\alpha(t)$

(1)4. p is a path from a to b

PROOF: By Theorem 5.2.15.

□

Corollary 8.3.10.1. *For any set I , the space \mathbb{R}^I in the product topology is path connected.*

Proposition 8.3.11. *The space \mathbb{R}_K is not path connected.*

PROOF:

<1>1. ASSUME: for a contradiction $p : [0, 1] \rightarrow \mathbb{R}_K$ is a path from 0 to 1

<1>2. LET: $p : [0, 1] \rightarrow \mathbb{R}_K$ be a path from 0 to 1

<1>3. $p([0, 1])$ is compact and connected in \mathbb{R}_K .

PROOF: Theorem 8.1.9 and Proposition 9.4.10.

<1>4. $p([0, 1])$ is connected in \mathbb{R} .

PROOF: Corollary 8.1.9.1

<1>5. $[0, 1] \subseteq p([0, 1])$

PROOF: For any $x \in [0, 1]$, if $x \notin p([0, 1])$ then $p([0, 1]) \cap (-\infty, x)$ and $p([0, 1]) \cap (x, +\infty)$ form a separation of $p([0, 1])$.

<1>6. $[0, 1]$ is compact in \mathbb{R}_K

PROOF: Proposition 9.4.6.

<1>7. Q.E.D.

PROOF: This contradicts Corollary 9.4.11.2.

□

Proposition 8.3.12. *Let $f : X \rightarrow Y$ be continuous and surjective. If X is path connected then Y is path connected.*

PROOF:

<1>1. LET: $a, b \in Y$

<1>2. PICK $x, y \in X$ such that $f(x) = a$ and $f(y) = b$

<1>3. PICK a path $p : [0, 1] \rightarrow X$ such that $p(0) = x$ and $p(1) = y$

<1>4. $f \circ p$ is a path from a to b

□

Corollary 8.3.12.1. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of non-empty topological spaces. If $\prod_{\alpha \in J} X_\alpha$ is path connected then each X_α is path connected.*

8.4 Connected Subspaces of Euclidean Space

Definition 8.4.1 (Unit 2-Sphere). The *unit 2-sphere* is $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ as a subspace of \mathbb{R}^3 .

Definition 8.4.2 (Unit Ball). For any $n \geq 1$, the *closed unit ball* in \mathbb{R}^n is

$$B^n = \{\vec{x} \in \mathbb{R}^n : \|\vec{x}\| \leq 1\} .$$

Proposition 8.4.3. *Every open unit ball and closed unit ball in \mathbb{R}^n is path connected.*

PROOF: The straight line between any two points is a path in the ball. □

Definition 8.4.4 (Punctured Euclidean Space). For $n \geq 1$, *punctured Euclidean space* is $\mathbb{R}^n \setminus \{\vec{0}\}$.

Proposition 8.4.5. *Punctured Euclidean space in \mathbb{R}^n is path connected iff $n > 1$.*

PROOF: Easy. \square

Definition 8.4.6 (Unit Sphere). For $n \geq 1$, the *unit sphere* S^n is $\{\vec{x} \in \mathbb{R}^{n+1} : \|\vec{x}\| = 1\}$.

Proposition 8.4.7. *In any number of dimensions, the unit sphere is path connected.*

PROOF: Easy. \square

Definition 8.4.8 (Topologist's Sine Curve). The *topologist's sine curve* is the closure of

$$S = \{(x, \sin 1/x) : x \in \mathbb{R}\}$$

in \mathbb{R}^2 .

Proposition 8.4.9. *The topologist's sine curve is connected.*

PROOF:

$\langle 1 \rangle 1$. $S = \{(x, \sin 1/x) : x \in \mathbb{R}\}$ is connected.

$\langle 2 \rangle 1$. The function $f : \mathbb{R} \rightarrow \mathbb{R}^2$ given by $f(x) = (x, \sin 1/x)$ is continuous.

PROOF: By Theorem 5.2.15.

$\langle 2 \rangle 2$. Q.E.D.

PROOF: By Theorem 8.1.9.

$\langle 1 \rangle 2$. Q.E.D.

PROOF: By Theorem 8.1.8.

\square

Proposition 8.4.10 (CC). *The topologist's sine curve is not path connected.*

PROOF:

$\langle 1 \rangle 1$. LET: $S = \{(x, \sin 1/x) : x \in \mathbb{R}\}$

$\langle 1 \rangle 2$. ASSUME: for a contradiction $p : [0, 1] \rightarrow \bar{S}$ is a path from $(0, 0)$ to $(1, \sin 1)$.

$\langle 1 \rangle 3$. $p^{-1}(\{0\} \times [-1, 1])$ is closed.

$\langle 1 \rangle 4$. $p^{-1}(\{0\} \times [-1, 1])$ has a greatest element.

PROOF: By Lemma 4.1.9.

$\langle 1 \rangle 5$. LET: $q : [0, 1] \rightarrow \bar{S}$ be a path such that:

- $q(0) \in \{0\} \times [-1, 1]$
- $q(x) \in S$ for $x > 0$

PROOF: Let b be greatest in $p^{-1}(\{0\} \times [-1, 1])$. Then q is obtained by rescaling p restricted to $[b, 1]$.

$\langle 1 \rangle 6$. LET: $q(t) = (x(t), y(t))$ for $0 \leq t \leq 1$

$\langle 1 \rangle 7$. $x(0) = 0$

- ⟨1⟩8. $x(t) > 0$ for $t > 0$
 ⟨1⟩9. $y(t) = \sin 1/x(t)$ for $t > 0$
 ⟨1⟩10. There exists a sequence $t_n \in [0, 1]$ such that $t_n \rightarrow 0$ as $n \rightarrow \infty$ and $y(t_n) = (-1)^n$ for all n .
 ⟨2⟩1. For each n , PICK u_n such that $0 < u_n < x(1/n)$ and $\sin 1/u_n = (-1)^n$.
 PROOF: Such a u_n exists because $\sin 1/x$ takes values 1 and -1 infinitely often in $(0, x(1/n))$.
 ⟨2⟩2. For each n , PICK t_n such that $0 < t_n < 1/n$ and $x(t_n) = u$
 PROOF: By the Intermediate Value Theorem.
 ⟨1⟩11. Q.E.D.
 PROOF: This is a contradiction as $y(t_n) \rightarrow y(0)$ as $n \rightarrow \infty$ because y is continuous.

□

8.5 Local Connectedness

Definition 8.5.1 (Locally Connected). Let X be a topological space and $x \in X$. Then X is *locally connected* at x iff every neighbourhood of x includes a connected neighbourhood of x .

The space X is *locally connected* iff it is locally connected at every point.

Proposition 8.5.2. S_Ω is not locally connected.

PROOF: There is no connected neighbourhood of ω . □

Proposition 8.5.3. $\overline{S_\Omega}$ is not locally connected.

PROOF: There is no connected neighbourhood of ω . □

Proposition 8.5.4. For any set I , the space \mathbb{R}^I is locally connected.

PROOF: Every basic open set is the product of connected spaces, hence connected. □

Proposition 8.5.5. Let X be a topological space. Then X is locally connected if and only if, for every open set U in X , every component of U is open in X .

PROOF:

- ⟨1⟩1. If X is locally connected then, for every open set U in X , every component of U is open in X .
 ⟨2⟩1. ASSUME: X is locally connected.
 ⟨2⟩2. LET: U be open in X .
 ⟨2⟩3. LET: C be a component of U .
 ⟨2⟩4. LET: $x \in C$
 PROVE: C is a neighbourhood of x
 ⟨2⟩5. U is a neighbourhood of x in X .
 PROOF: From ⟨2⟩2, ⟨2⟩3 and ⟨2⟩4.
 ⟨2⟩6. PICK a connected neighbourhood V of x such that $V \subseteq U$.

PROOF: Using $\langle 2 \rangle 1$.

$\langle 2 \rangle 7$. $V \subseteq C$

PROOF: By Proposition 8.2.2.

$\langle 2 \rangle 8$. C is a neighbourhood of x

PROOF: By Proposition 3.2.4.

$\langle 2 \rangle 9$. Q.E.D.

PROOF: By Proposition 3.2.3.

$\langle 1 \rangle 2$. If, for every open set U in X , every component of U is open in X , then X is locally connected.

$\langle 2 \rangle 1$. ASSUME: For every open set U in X , every component of U is open in X .

$\langle 2 \rangle 2$. LET: $x \in X$ and N be a neighbourhood of x

$\langle 2 \rangle 3$. PICK U open such that $x \in U \subseteq N$

$\langle 2 \rangle 4$. LET: C be the component of U that contains x

$\langle 2 \rangle 5$. C is open in X

PROOF: By $\langle 2 \rangle 1$.

$\langle 2 \rangle 6$. C is a connected neighbourhood of x that is included in N

□

Corollary 8.5.5.1. *In a locally connected space, every component is open.*

Corollary 8.5.5.2. *The space \mathbb{R}^ω under the box topology is not locally connected.*

Corollary 8.5.5.3. *Not every closed subspace of a locally connected space is locally connected.*

PROOF: The topologist's sine curve is not locally connected. □

Proposition 8.5.6. *$S_\Omega \times \overline{S_\Omega}$ is not locally connected.*

(ω, ω) has no connected neighbourhood. □

Proposition 8.5.7. *\mathbb{R}_l is not locally connected.*

PROOF: 0 has no connected neighbourhood. □

Proposition 8.5.8. *The Sorgenfrey plane is not locally connected.*

PROOF: Any basic open set $[a, b) \times [c, d)$ can be separated into $[a, b) \times [c, e)$ and $[a, b) \times [e, d)$ for some $c < e < d$. □

Proposition 8.5.9. *The space \mathbb{R}^ω under the uniform topology is locally connected.*

PROOF: For any neighbourhood U of a point x , the neighbourhood $U \cap \{y : y - x \text{ is bounded}\}$ is connected. □

Proposition 8.5.10. *The space \mathbb{R}_K is not locally connected.*

PROOF: The open set $(-1, 1) - K$ does not include a connected neighbourhood of 0. □

Proposition 8.5.11. *Every open subspace of a locally connected space is locally connected.*

PROOF: Follows easily from definition. \square

Proposition 8.5.12 (AC). *The product of a family of locally connected spaces is locally connected.*

PROOF:

- (1)1. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of locally connected spaces and $\vec{x} \in \prod_{\alpha \in J} X_\alpha$
- (1)2. LET: $\prod_{\alpha \in J} U_\alpha$ be any basic neighbourhood of \vec{x} , where each U_α is open in X_α , and $U_\alpha = X_\alpha$ except for $\alpha = \alpha_1, \dots, \alpha_n$
- (1)3. For $\alpha \in J$, PICK a connected neighbourhood C_α of x_α with $C_\alpha \subseteq U_\alpha$
- (1)4. $\prod_{\alpha \in J} C_\alpha$ is connected

PROOF: Proposition 8.1.10

\square

Proposition 8.5.13. *Every discrete space is locally connected.*

PROOF: For any point x , the set $\{x\}$ is a connected neighbourhood of x . \square

Corollary 8.5.13.1. *The continuous image of a locally connected space is not necessarily locally connected.*

8.6 Local Path Connectedness

Definition 8.6.1 (Locally Path Connected). Let X be a topological space and $x \in X$. Then X is *locally path connected at x* iff every neighbourhood of x includes a path connected neighbourhood of x .

The space X is *locally path connected* iff it is locally path connected at every point.

Proposition 8.6.2. S_Ω is not locally path connected.

PROOF: There is no path connected neighbourhood of ω . \square

Proposition 8.6.3. $\overline{S_\Omega}$ is not locally path connected.

PROOF: There is no path connected neighbourhood of ω . \square

Proposition 8.6.4. *Not every closed subspace of a locally path connected space is locally path connected.*

PROOF: The topologist's sine curve is not locally path connected. \square

Proposition 8.6.5. *Every open subspace of a locally path connected space is locally path connected.*

PROOF: Follows easily from definition. \square

Proposition 8.6.6. *Every locally path connected space is locally connected.*

PROOF: From Proposition 8.3.3. \square

Corollary 8.6.6.1. \mathbb{R}_l is not locally path connected.

Corollary 8.6.6.2. The Sorgenfrey plane is not locally path connected.

Corollary 8.6.6.3. The space \mathbb{R}^ω under the box topology is not locally path connected.

Corollary 8.6.6.4. The space \mathbb{R}_K is not locally path connected.

Corollary 8.6.6.5. The topologist's sine curve is not locally path connected.

Proposition 8.6.7 (AC). The product of a family of locally path connected spaces is locally path connected.

PROOF:

- $\langle 1 \rangle 1$. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of locally connected spaces and $\vec{x} \in \prod_{\alpha \in J} X_\alpha$
- $\langle 1 \rangle 2$. LET: $\prod_{\alpha \in J} U_\alpha$ be any basic neighbourhood of \vec{x} , where each U_α is open in X_α , and $U_\alpha = X_\alpha$ except for $\alpha = \alpha_1, \dots, \alpha_n$
- $\langle 1 \rangle 3$. For $\alpha \in J$, PICK a path connected neighbourhood C_α of x_α with $C_\alpha \subseteq U_\alpha$
- $\langle 1 \rangle 4$. $\prod_{\alpha \in J} C_\alpha$ is path connected

PROOF: Proposition ??

\square

Proposition 8.6.8. Let X be a topological space. Then X is locally path connected if and only if, for every open set U in X , every path component of U is open in X .

PROOF:

- $\langle 1 \rangle 1$. If X is locally path connected then, for every open set U in X , every path component of U is open in X .
- $\langle 2 \rangle 1$. ASSUME: X is locally path connected.
- $\langle 2 \rangle 2$. LET: U be open in X .
- $\langle 2 \rangle 3$. LET: C be a path component of U .
- $\langle 2 \rangle 4$. LET: $x \in C$
PROVE: C is a neighbourhood of x
- $\langle 2 \rangle 5$. U is a neighbourhood of x in X .
PROOF: From $\langle 2 \rangle 2$, $\langle 2 \rangle 3$ and $\langle 2 \rangle 4$.
- $\langle 2 \rangle 6$. PICK a path connected neighbourhood V of x such that $V \subseteq U$.
PROOF: Using $\langle 2 \rangle 1$.
- $\langle 2 \rangle 7$. $V \subseteq C$
PROOF: By Proposition 8.3.7.
- $\langle 2 \rangle 8$. C is a neighbourhood of x
PROOF: By Proposition 3.2.4.
- $\langle 2 \rangle 9$. Q.E.D.
PROOF: By Proposition 3.2.3.
- $\langle 1 \rangle 2$. If, for every open set U in X , every path component of U is open in X , then X is locally path connected.

- (2)1. ASSUME: For every open set U in X , every path component of U is open in X .
 (2)2. LET: $x \in X$ and N be a neighbourhood of x
 (2)3. PICK U open such that $x \in U \subseteq N$
 (2)4. LET: C be the path component of U that contains x
 (2)5. C is open in X
 PROOF: By (2)1.
 (2)6. C is a path connected neighbourhood of x that is included in N

□

Theorem 8.6.9 (AC). *Let X be a topological space. If X is locally path connected, then its components and its path components are the same.*

PROOF:

- (1)1. LET: P be a path component of X
 (1)2. LET: C be the component such that $P \subseteq C$
 PROVE: $P = C$
 (1)3. LET: $Q = C \setminus P$
 (1)4. P is open in X
 PROOF: By Proposition 8.6.8.
 (1)5. Q is open in X
 PROOF: By Proposition 8.6.8 since Q is the union of the path components included in C other than P .
 (1)6. $Q = \emptyset$
 PROOF: Otherwise P and Q would form a separation of C , contradicting 8.2.3.

□

Proposition 8.6.10. $S_\Omega \times \overline{S_\Omega}$ is not locally path connected.

PROOF: (ω, ω) has no path connected neighbourhood. □

Proposition 8.6.11. *The ordered square is not locally path connected.*

PROOF:

- (1)1. ASSUME: for a contradiction $(1/2, 0)$ has a path connected neighbourhood U
 (1)2. PICK $a < 1/2$ such that $((a, 1), (1/2, 0)) \subseteq U$
 (1)3. LET: $p : [0, 1] \rightarrow I_o^2$ be a path from $(a, 1)$ to $(1/2, 0)$
 (1)4. For every x such that $a < x < 1/2$, PICK a rational q_x such that $p(q_x) \in ((x, 0), (x, 1))$
 (1)5. $\{q_x : a < x < 1/2\}$ is an uncountable set of rationals.

□

Proposition 8.6.12. *For any set I , the space \mathbb{R}^I is locally path connected.*

PROOF: Every basic open set is the product of path connected spaces, hence path connected. □

Proposition 8.6.13. *The space \mathbb{R}^ω under the uniform topology is locally path connected.*

PROOF: Its components and path components are the same. \square

Proposition 8.6.14. *Every discrete space is locally path connected.*

PROOF: For any point x , the set $\{x\}$ is a path connected neighbourhood of x . \square

Corollary 8.6.14.1. *The continuous image of a locally path connected space is not necessarily locally path connected.*

8.7 Weak Local Connectedness

Definition 8.7.1 (Weakly Locally Connected). Let X be a topological space and $x \in X$. Then X is *weakly locally connected at x* iff every neighbourhood of x contains a connected subspace that contains a neighbourhood of x .

Chapter 9

Compact Spaces

9.1 Countable Compactness

Definition 9.1.1 (Countably Compact). A topological space is *countably compact* iff every countable open covering has a finite subcovering.

9.2 Limit Point Compactness

Definition 9.2.1 (Limit Point Compact). A space is *limit point compact* iff every infinite set has a limit point.

Proposition 9.2.2 (CC). $S_\Omega \times \overline{S_\Omega}$ is limit point compact.

PROOF:

- $\langle 1 \rangle 1$. LET: $A \subseteq S_\Omega \times \overline{S_\Omega}$ be infinite
- $\langle 1 \rangle 2$. CASE: $\pi_1(A)$ is finite.
 - $\langle 2 \rangle 1$. PICK x such that there are infinitely many y such that $(x, y) \in A$
 - $\langle 2 \rangle 2$. PICK a limit point l of $\{y : (x, y) \in A\}$
 - $\langle 2 \rangle 3$. (x, l) is a limit point of A
- $\langle 1 \rangle 3$. CASE: $\pi_1(A)$ is infinite.
 - $\langle 2 \rangle 1$. PICK a limit point l of $\pi_1(A)$.
 - $\langle 2 \rangle 2$. l is a limit ordinal
 - $\langle 2 \rangle 3$. PICK a countable sequence x_n with limit l
 - $\langle 2 \rangle 4$. For $n \geq 1$, PICK $a_n > x_n$ and y_n such that $(a_n, y_n) \in A$
 - $\langle 2 \rangle 5$. CASE: $\{y_n : n \geq 1\}$ is finite
 - $\langle 3 \rangle 1$. PICK y such that $y = y_n$ for infinitely many n
 - $\langle 3 \rangle 2$. (l, y) is a limit point for A
 - $\langle 2 \rangle 6$. CASE: $\{y_n : n \geq 1\}$ is infinite
 - $\langle 3 \rangle 1$. PICK a limit point m for $\{y_n : n \geq 1\}$
 - $\langle 3 \rangle 2$. (l, m) is a limit point for A

□

Proposition 9.2.3. *The Sorgenfrey plane is not limit point compact.*

PROOF: \mathbb{Z}^2 has no limit point. \square

Proposition 9.2.4. *The space \mathbb{R}^ω under the box topology is not limit point compact.*

PROOF: The set of all constant sequences of integers is an infinite set with no limit point. \square

Proposition 9.2.5. *Not every open subspace of a limit point compact space is limit point compact.*

PROOF: The space $[0, 1]$ is limit point compact but $(0, 1)$ is not. \square

Proposition 9.2.6. *The product of two limit point compact spaces is not necessarily limit point compact.*

PROOF: See Steen and Seebach *Counterexamples in Topology* Example 112. \square

Proposition 9.2.7. *The continuous image of a limit point compact space is not necessarily limit point compact.*

PROOF: Let Y be a two-point set under the indiscrete topology. Then $\mathbb{N} \times Y$ is limit point compact, but \mathbb{N} is not. \square

9.3 Lindelöf Spaces

Definition 9.3.1 (Lindelöf Space). A topological space X is *Lindelöf* iff every open covering has a countable subcovering.

Theorem 9.3.2 (CC). *Every second countable space is Lindelöf.*

PROOF:

- $\langle 1 \rangle 1$. LET: X be a second countable space
- $\langle 1 \rangle 2$. PICK a countable basis \mathcal{B} for X .
- $\langle 1 \rangle 3$. LET: \mathcal{A} be an open cover of X
- $\langle 1 \rangle 4$. For every $B \in \mathcal{B}$ such that there exists $U \in \mathcal{A}$ such that $B \subseteq U$, PICK $U_B \in \mathcal{A}$ such that $B \subseteq U_B$
- $\langle 1 \rangle 5$. $\{U_B : B \in \mathcal{B}, \exists U \in \mathcal{A}. B \subseteq U\}$ covers X .
- $\langle 2 \rangle 1$. LET: $x \in X$
- $\langle 2 \rangle 2$. PICK $U \in \mathcal{A}$ such that $x \in U$
- $\langle 2 \rangle 3$. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$
- $\langle 2 \rangle 4$. $x \in U_B$

\square

Corollary 9.3.2.1. *The space \mathbb{R}^ω is Lindelöf.*

Corollary 9.3.2.2. *The space \mathbb{R}_K is Lindelöf.*

Proposition 9.3.3. *The space S_Ω is not Lindelöf.*

PROOF: $\{(-\infty, \alpha) : \alpha \in S_\Omega\}$ is an open cover that has no countable subcover. \square

Proposition 9.3.4 (CC). *The space $\overline{S_\Omega}$ is Lindelöf.*

PROOF:

- $\langle 1 \rangle 1$. LET: \mathcal{A} be an open cover of $\overline{S_\Omega}$
 - $\langle 1 \rangle 2$. PICK $U \in \mathcal{A}$ such that $\Omega \in U$
 - $\langle 1 \rangle 3$. PICK $\alpha < \Omega$ such that $(\alpha, \Omega] \subseteq U$
 - $\langle 1 \rangle 4$. For $\beta \leq \alpha$, PICK $U_\beta \in \mathcal{A}$ such that $\beta \in U_\beta$
 - $\langle 1 \rangle 5$. $\{U\} \cup \{U_\beta : \beta \leq \alpha\}$ is a countable subcover of \mathcal{A} .
- \square

Proposition 9.3.5 (CC). *The continuous image of a Lindelöf space is Lindelöf.*

PROOF:

- $\langle 1 \rangle 1$. LET: X be a Lindelöf space, Y a space and $f : X \rightarrow Y$ continuous.
 - $\langle 1 \rangle 2$. LET: \mathcal{A} be an open covering of Y
 - $\langle 1 \rangle 3$. $\{f^{-1}(V) : V \in \mathcal{A}\}$ is an open covering of X
 - $\langle 1 \rangle 4$. PICK a countable subcovering $\{f^{-1}(V_1), f^{-1}(V_2), \dots\}$ of $\{f^{-1}(V) : V \in \mathcal{A}\}$
 - $\langle 1 \rangle 5$. $\{V_1, V_2, \dots\}$ is a countable subcovering of \mathcal{A}
- \square

Proposition 9.3.6. *The Sorgenfrey plane is not Lindelöf.*

PROOF:

- $\langle 1 \rangle 1$. LET: $L = \{(x, -x) : x \in \mathbb{R}\}$
 - $\langle 1 \rangle 2$. L is closed in \mathbb{R}_l^2
 - $\langle 2 \rangle 1$. LET: $(x, y) \notin L$, so $y \neq -x$

PROVE: There exists a neighbourhood U of (x, y) that does not intersect L
 - $\langle 2 \rangle 2$. CASE: $y > -x$

PROOF: In this case, take $U = [x, +\infty) \times [y, +\infty)$
 - $\langle 2 \rangle 3$. CASE: $y < -x$

PROOF: In this case, take $U = [x, (x - y)/2) \times [y, (y - x)/2)$.
 - $\langle 1 \rangle 3$. LET: $\mathcal{U} = \{\mathbb{R}_l^2 \setminus L\} \cup \{[a, b) \times [-a, d) : a, b, d \in \mathbb{R}\}$
 - $\langle 1 \rangle 4$. \mathcal{U} is an open covering of \mathbb{R}_l^2
 - $\langle 1 \rangle 5$. No countable subset of \mathcal{U} covers \mathbb{R}_l^2

PROOF: Every set $[a, b) \times [-a, d)$ intersects L in exactly one point, namely $(a, -a)$.
- \square

Corollary 9.3.6.1. *The Sorgenfrey plane is not second countable.*

Corollary 9.3.6.2. *The product of two Lindelöf spaces is not necessarily Lindelöf.*

Proposition 9.3.7. *The space \mathbb{R}^ω under the box topology is not Lindelöf.*

PROOF: The set $\{\prod_{n=0}^{\infty}(a_n, a_n + 1) : \forall n. a_n \in \mathbb{Z}\}$ covers the space but has no countable subcover. \square

Proposition 9.3.8. *Not every open subspace of a Lindelöf space is Lindelöf.*

PROOF: The ordered square is Lindelöf but the subspace $[0, 1] \text{ times } (0, 1)$ is not. \square

9.4 Compactness

Definition 9.4.1 (Compact). A topological space is *compact* iff every open cover has a finite subcover.

Proposition 9.4.2. S_{Ω} is not compact.

PROOF: The open covering $\{(-\infty, \alpha) : \alpha \in S_{\Omega}\}$ has no finite subcovering. \square

Proposition 9.4.3. \mathbb{R}_l is not compact.

PROOF: $\{[n, n + 1) : n \in \mathbb{Z}\}$ has no finite subcover. \square

Proposition 9.4.4. The space \mathbb{R}^{ω} under the box topology is not compact.

PROOF: The set $\{\prod_{n=0}^{\infty}(a_n, a_n + 1) : n \in \mathbb{Z}\}$ is a cover that has no finite subcover. \square

Proposition 9.4.5. *Let Y be a subspace of X . Then Y is compact if and only if every covering of Y by sets open in X contains a finite subcollection covering Y .*

PROOF:

- (1)1. If Y is compact then every covering of Y by sets open in X contains a finite subcollection covering Y .
- (2)1. ASSUME: Y is compact.
- (2)2. LET: \mathcal{A} be a covering of Y by sets open in X .
- (2)3. $\{U \cap Y : U \in \mathcal{A}\}$ is an open covering of Y .
- (2)4. PICK a finite subcovering V_1, \dots, V_n of $\{U \cap Y : U \in \mathcal{A}\}$
- (2)5. For $1 \leq i \leq n$, PICK $U_i \in \mathcal{A}$ such that $V_i = U_i \cap Y$.
- (2)6. $\{U_1, \dots, U_n\}$ is a finite subset of \mathcal{A} that covers Y .
- (1)2. If every covering of Y by sets open in X contains a finite subcollection covering Y then Y is compact.
- (2)1. ASSUME: Every covering of Y by sets open in X contains a finite subcollection covering Y .
- (2)2. LET: \mathcal{A} be an open covering of Y
- (2)3. LET: $\mathcal{B} = \{U \text{ open in } X : U \cap Y \in \mathcal{A}\}$
- (2)4. \mathcal{B} covers Y
- (2)5. PICK a finite subcollection $\{U_1, \dots, U_n\} \subseteq \mathcal{B}$ that covers Y
- (2)6. $\{U_1 \cap Y, \dots, U_n \cap Y\}$ is a finite subcover of \mathcal{A} .

\square

Proposition 9.4.6. *Every closed subspace of a compact space is compact.*

PROOF:

- ⟨1⟩1. LET: X be a compact space and $Y \subseteq X$ be closed.
- ⟨1⟩2. LET: \mathcal{A} be a covering of Y by spaces open in X
- ⟨1⟩3. $\mathcal{A} \cup \{X \setminus Y\}$ is an open covering of X .
- ⟨1⟩4. PICK a finite subcovering $\{U_1, \dots, U_n\}$ or $\{U_1, \dots, U_n, X \setminus Y\}$
- ⟨1⟩5. $\{U_1, \dots, U_n\}$ is a finite subset of \mathcal{A} that covers Y .
- ⟨1⟩6. Q.E.D.

PROOF: Proposition 9.4.5.

□

Corollary 9.4.6.1. *Not every compact Hausdorff space is connected.*

PROOF: The space $[0, 1] \cup [2, 3]$ is compact Hausdorff and disconnected. □

Corollary 9.4.6.2. *Not every compact Hausdorff space is path connected.*

Corollary 9.4.6.3. *Not every compact Hausdorff space is locally connected.*

The space $[0, 1] \cap \mathbb{Q}$ is not locally connected.

Corollary 9.4.6.4. *Not every compact Hausdorff space is locally path connected.*

Proposition 9.4.7. *Not every open subspace of a compact space is compact.*

PROOF: The space $[0, 1]$ is compact but $(0, 1)$ is not. □

Lemma 9.4.8. *If Y is a compact subspace of the Hausdorff space X and $a \notin Y$, then there exist disjoint open sets U and V of X containing a and Y , respectively.*

PROOF:

- ⟨1⟩1. For $y \in Y$, there exist disjoint open sets U and V such that $a \in U$ and $y \in V$.
- ⟨1⟩2. $\{V \text{ open in } X : \exists U \text{ open and disjoint from } V, a \in U\}$ is a covering of Y by open sets in X .
- ⟨1⟩3. PICK a finite subset $\{V_1, \dots, V_n\}$ that covers Y .
- ⟨1⟩4. For $1 \leq i \leq n$, PICK U_i disjoint from V_i such that $a \in U_i$
- ⟨1⟩5. LET: $U = U_1 \cap \dots \cap U_n$ and $V = V_1 \cup \dots \cup V_n$

□

Proposition 9.4.9. *Every compact subspace of a Hausdorff space is closed.*

PROOF:

- ⟨1⟩1. LET: X be a Hausdorff space and $Y \subseteq X$ be compact.
- ⟨1⟩2. Every point $a \notin Y$ has an open neighbourhood disjoint from Y .

PROOF: By Lemma 9.4.8.

- ⟨1⟩3. Q.E.D.

PROOF: By Proposition 3.2.3.

Proposition 9.4.10. *The image of a compact space under a continuous map is compact.*

PROOF:

- ⟨1⟩1. LET: $f : X \rightarrow Y$ be continuous where X is compact.
- ⟨1⟩2. LET: \mathcal{A} be a covering of $f(X)$ by open sets in Y .
- ⟨1⟩3. $\{f^{-1}(U) : U \in \mathcal{A}\}$ is an open covering of X .
- ⟨1⟩4. PICK a finite subcovering $\{f^{-1}(U_1), \dots, f^{-1}(U_n)\}$
- ⟨1⟩5. $\{U_1, \dots, U_n\}$ is a finite subset of \mathcal{A} that covers $f(X)$.
- ⟨1⟩6. Q.E.D.

PROOF: By Proposition 9.4.5.

□

Corollary 9.4.10.1. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces. If $\prod_{\alpha \in J} X_\alpha$ is compact then each X_α is compact.*

Corollary 9.4.10.2. *$S_\Omega \times \overline{S_\Omega}$ is compact.*

Corollary 9.4.10.3. *The Sorgenfrey plane is not compact.*

Corollary 9.4.10.4. *For any nonempty set I , the space \mathbb{R}^I is not compact.*

Corollary 9.4.10.5. *Let \mathcal{T} and \mathcal{T}' be topologies on the same set X . If $\mathcal{T} \subseteq \mathcal{T}'$ and \mathcal{T}' is compact then \mathcal{T} is compact.*

Corollary 9.4.10.6. *The space \mathbb{R}_K is not compact.*

Theorem 9.4.11. *Let $f : X \rightarrow Y$ be a bijective continuous function. If X is compact and Y is Hausdorff then f is a homeomorphism.*

PROOF:

- ⟨1⟩1. LET: C be closed in X
- ⟨1⟩2. C is compact
PROOF: Proposition 9.4.6.
- ⟨1⟩3. $f(C)$ is compact
PROOF: Proposition 9.4.10
- ⟨1⟩4. $f(C)$ is closed
PROOF: Proposition 9.4.9.
- ⟨1⟩5. Q.E.D.

PROOF: By Theorem 5.2.2 we have that f^{-1} is continuous.

□

Corollary 9.4.11.1. *Let \mathcal{T} and \mathcal{T}' be topologies on the same set X . If $\mathcal{T} \subseteq \mathcal{T}'$, \mathcal{T} is Hausdorff and \mathcal{T}' is compact then $\mathcal{T} = \mathcal{T}'$.*

Corollary 9.4.11.2. *The space $[0, 1]$ is not compact as a subspace of \mathbb{R}_K .*

Theorem 9.4.12 (Tube Lemma). *Let A and B be subspaces of X and Y , respectively; let N be an open set in $X \times Y$ including $A \times B$. If A and B are compact, then there exist open sets U and V in X and Y , respectively, such that*

$$A \times B \subseteq U \times V \subseteq N .$$

PROOF:

- (1)1. For all $a \in A$, there exist open sets U and V in X and Y , respectively, such that

$$\{a\} \times B \subseteq U \times V \subseteq N.$$

- (2)1. LET: $a \in A$
 (2)2. For all $b \in B$, there exist open sets U and V in X and Y , respectively, such that $(a, b) \in U \times V \subseteq N$.
 (2)3. $\{V \text{ open in } Y : \exists U \text{ open in } X. a \in U, U \times V \subseteq N\}$ covers B
 (2)4. PICK a finite subset $\{V_1, \dots, V_n\}$ that covers B .
 (2)5. For $1 \leq i \leq n$, PICK U_i open in X such that $a \in U_i$ and $U_i \times V_i \subseteq N$
 (2)6. LET: $U = U_1 \cap \dots \cap U_n$ and $V = V_1 \cup \dots \cup V_n$
 (1)2. $\{U \text{ open in } X : \exists V \text{ open in } Y. B \subseteq V \text{ and } U \times V \subseteq N\}$ covers A .
 (1)3. PICK a finite subset $\{U_1, \dots, U_n\}$ that covers A .
 (1)4. For $1 \leq i \leq n$, PICK V_i open in B such that $B \subseteq V_i$ and $U_i \times V_i \subseteq N$.
 (1)5. LET: $U = U_1 \cup \dots \cup U_n$ and $V = V_1 \cap \dots \cap V_n$
 (1)6. $A \times B \subseteq U \times V \subseteq N$

□

Lemma 9.4.13. *Let \mathcal{A} be a set of basis elements for $X \times Y$ such that no finite subset of \mathcal{A} covers $X \times Y$. If X is compact, then there exists a point $x \in X$ such that no finite subset of \mathcal{A} covers $\{x\} \times Y$.*

PROOF:

- (1)1. ASSUME: X is compact.
 (1)2. ASSUME: For all $x \in X$, there is a finite subset of \mathcal{A} that covers $\{x\} \times Y$
 PROVE: A finite subset of \mathcal{A} covers $X \times Y$
 (1)3. $\{U \text{ open in } X : \exists U_1 \times V_1, \dots, U_r \times V_r \in \mathcal{A}. U = U_1 \cap \dots \cap U_r, Y = V_1 \cup \dots \cup V_r\}$ covers X .
 (1)4. PICK a finite subcover $\{U_1, \dots, U_n\}$
 (1)5. For $1 \leq i \leq n$, PICK $U_{i1} \times V_{i1}, \dots, U_{ir_i} \times V_{ir_i} \in \mathcal{A}$ such that $U_i = U_{i1} \cap \dots \cap U_{ir_i}$ and $Y = V_{i1} \cup \dots \cup V_{ir_i}$
 (1)6. $\{U_{ij} : 1 \leq i \leq n, 1 \leq j \leq r_i\}$ covers $X \times Y$

□

Proposition 9.4.14. *The product of two compact spaces is compact.*

PROOF:

- (1)1. LET: X and Y be compact spaces.
 (1)2. LET: \mathcal{A} be an open covering of $X \times Y$
 (1)3. For all $x \in X$, there exists a neighbourhood W of x such that $W \times Y$ is covered by finitely many elements of \mathcal{A} .
 (2)1. LET: $x \in X$
 (2)2. $\{x\} \times Y$ is compact.
 PROOF: It is homeomorphic to Y .
 (2)3. PICK a finite subset $\{U_1, \dots, U_m\}$ of \mathcal{A} that covers $\{x\} \times Y$
 PROOF: By Proposition 9.4.5.
 (2)4. There exists a neighbourhood W of x such that $W \times Y \subseteq U_1 \cup \dots \cup U_m$

PROOF: By the Tube Lemma.

- ⟨1⟩4. $\{W \text{ open in } X : W \times Y \text{ is covered by finitely many elements of } \mathcal{A}\}$ is an open covering of X .
- ⟨1⟩5. PICK a finite subcovering $\{W_1, \dots, W_n\}$
- ⟨1⟩6. For $1 \leq i \leq n$, PICK a finite subset $\{U_{i1}, \dots, U_{ir_i}\}$ of \mathcal{A} that covers $W_i \times Y$
- ⟨1⟩7. $\{U_{11}, \dots, U_{nr_n}\}$ is a finite subcovering of \mathcal{A} .

□

Proposition 9.4.15. *A topological space is compact if and only if every nonempty set of closed sets that has the finite intersection property has nonempty intersection.*

PROOF: Immediate from definitions. □

Lemma 9.4.16. *If Y is compact then $\pi_1 : X \times Y \rightarrow X$ is a closed map.*

PROOF:

- ⟨1⟩1. LET: $C \subseteq X \times Y$ be closed
- ⟨1⟩2. LET: $x \in X \setminus \pi_1(C)$
- ⟨1⟩3. For all $y \in Y$, we have $(x, y) \notin C$
- ⟨1⟩4. For all $y \in Y$, there exist open neighbourhoods U of x and V of y such that $U \times V \subseteq (X \times Y) \setminus C$
- ⟨1⟩5. $\{V \text{ open in } Y : \exists U \text{ an open neighbourhood of } x \text{ such that } U \times V \subseteq (X \times Y) \setminus C\}$ is an open covering of Y .
- ⟨1⟩6. PICK a finite subcovering $\{V_1, \dots, V_n\}$
- ⟨1⟩7. For $1 \leq i \leq n$, PICK an open neighbourhood U_i of x such that $U_i \times V_i \subseteq (X \times Y) \setminus C$
- ⟨1⟩8. $x \in U_1 \cap \dots \cap U_n \subseteq X \setminus \pi_1(C)$

□

Theorem 9.4.17. *Let X be a compact space. Let $f_n : X \rightarrow \mathbb{R}$ be a sequence of continuous functions such that, for all $x \in X$, $f_n(x) \rightarrow f(x)$ as $n \rightarrow \infty$. If f is continuous, and if the sequence $(f_n)_n$ is monotone increasing, and if X is compact, then the convergence is uniform.*

PROOF:

- ⟨1⟩1. LET: $\epsilon > 0$
 PROVE: There exists N such that, for all $n \geq N$, we have $|f_n(x) - f(x)| < \epsilon$
- ⟨1⟩2. For $n \in \mathbb{Z}^+$,
 LET: $U_n = \{x \in X : f(x) - f_n(x) < \epsilon\}$
- ⟨1⟩3. Each U_n is open
 PROOF: Let $g(x) = f(x) - f_n(x)$. Then g is continuous and $U_n = g^{-1}((-\infty, \epsilon))$.
- ⟨1⟩4. $\{U_n : n \geq 1\}$ is an open covering of X
- ⟨2⟩1. LET: $x \in X$
- ⟨2⟩2. PICK N such that, for all $n \geq N$, $|f(x) - f_n(x)| < \epsilon$
 PROOF: $f_n(x) \rightarrow f(x)$ as $n \rightarrow \infty$
- ⟨2⟩3. $f(x) - f_N(x) < \epsilon$

PROOF: This holds since the sequence $(f_n)_n$ is monotone.

- ⟨1⟩5. PICK a finite subcovering $\{U_{n_1}, \dots, U_{n_k}\}$
 - ⟨1⟩6. LET: $N = \max(n_1, \dots, n_k)$
 - ⟨1⟩7. For all $n \geq N$ we have $|f_n(x) - f(x)| < \epsilon$
-

Lemma 9.4.18. *Every compact Hausdorff space is normal.*

PROOF:

- ⟨1⟩1. LET: A and B be disjoint closed sets in the compact Hausdorff space X .
- ⟨1⟩2. For all $a \in A$, there exist disjoint open sets U and V such that $a \in U$ and $B \subseteq V$.

PROOF: By Lemma 9.4.8.

- ⟨1⟩3. $\{U \text{ open in } X : \exists V \text{ open in } Y, U \cap V = \emptyset, B \subseteq V\}$ is an open covering of A
 - ⟨1⟩4. PICK a finite subcovering $\{U_1, \dots, U_n\}$
 - ⟨1⟩5. For $1 \leq i \leq n$, PICK V_i open in Y such that $U_i \cap V_i = \emptyset$ and $B \subseteq V_i$
 - ⟨1⟩6. LET: $U = U_1 \cup \dots \cup U_n$ and $V = V_1 \cap \dots \cap V_n$
-

Theorem 9.4.19. *Let X be a complete linearly ordered set under the order topology. Then every closed interval in X is compact.*

PROOF:

- ⟨1⟩1. LET: X be a complete linearly ordered set in the order topology
- ⟨1⟩2. LET: $a, b \in X, a < b$
PROVE: $[a, b]$ is compact
- ⟨1⟩3. LET: \mathcal{A} be a set of open sets that covers $[a, b]$
- ⟨1⟩4. For all $x \in [a, b)$, there exists $y \in (x, b]$ such that $[x, y]$ is covered by at most two points of \mathcal{A}
 - ⟨2⟩1. LET: $x \in [a, b]$
 - ⟨2⟩2. PICK $U \in \mathcal{A}$ such that $x \in U$
PROOF: By ⟨1⟩3 and ⟨2⟩1
 - ⟨2⟩3. PICK $y \in (x, b]$ such that $[x, y] \subseteq U$
PROOF: By Lemma 4.1.2.
 - ⟨2⟩4. PICK $V \in \mathcal{A}$ such that $y \in V$
PROOF: By ⟨1⟩3 and ⟨2⟩3.
 - ⟨2⟩5. $[x, y]$ is covered by $\{U, V\}$
PROOF: By ⟨2⟩3 and ⟨2⟩4.
- ⟨1⟩5. LET: $C = \{y \in (a, b] : [a, y] \text{ is covered by a finite subset of } \mathcal{A}\}$
- ⟨1⟩6. C is nonempty
PROOF: By ⟨1⟩4.
- ⟨1⟩7. LET: $c = \sup C$
PROOF: By ⟨1⟩1.
- ⟨1⟩8. $c \in C$
 - ⟨2⟩1. PICK $U \in \mathcal{A}$ such that $c \in U$
 - ⟨2⟩2. PICK $y \in [a, c)$ such that $(y, c] \subseteq U$

PROOF: By Lemma 4.1.2

$\langle 2 \rangle 3$. PICK z such that $y < z$ and $z \in C$

PROOF: This exists because y is not an upper bound for C .

$\langle 2 \rangle 4$. PICK a finite $\mathcal{A}_0 \subseteq \mathcal{A}$ such that $[a, z]$ is covered by \mathcal{A}_0

$\langle 2 \rangle 5$. $[a, c]$ is covered by $\mathcal{A}_0 \cup \{U\}$

$\langle 1 \rangle 9$. $c = b$

$\langle 2 \rangle 1$. ASSUME: for a contradiction $c < b$

$\langle 2 \rangle 2$. PICK $y \in (c, b]$ such that $[c, y]$ is covered by at most two elements of \mathcal{A} .

PROOF: By $\langle 1 \rangle 4$

$\langle 2 \rangle 3$. $y > c$ and $y \in C$

$\langle 2 \rangle 4$. Q.E.D.

PROOF: This contradicts $\langle 1 \rangle 7$.

$\langle 1 \rangle 10$. Q.E.D.

Corollary 9.4.19.1. *Every closed interval in \mathbb{R} is compact.*

Corollary 9.4.19.2 (CC). *S_Ω is limit point compact.*

PROOF:

$\langle 1 \rangle 1$. LET: A be an infinite subset of S_Ω

$\langle 1 \rangle 2$. PICK a countably infinite subset $B \subseteq A$

$\langle 1 \rangle 3$. LET: $b = \sup B$

$\langle 1 \rangle 4$. $B \subseteq [0, b]$

$\langle 1 \rangle 5$. $[0, b]$ is compact

PROOF: By the theorem.

$\langle 1 \rangle 6$. B has a limit point in $[0, b]$

$\langle 1 \rangle 7$. A has a limit point in $[0, b]$

□

Corollary 9.4.19.3. *The ordered square is compact.*

Corollary 9.4.19.4. *The ordered square is limit point compact.*

Corollary 9.4.19.5. *Not every subspace of a compact space is compact.*

PROOF: $[0, 1]$ is compact but $(0, 1)$ is not. □

Theorem 9.4.20 (Extreme Value Theorem). *Let $f : X \rightarrow Y$ be continuous where Y is a linearly ordered set in the order topology. If X is compact, then there exist $c, d \in X$ such that, for all $x \in X$, we have $f(c) \leq f(x) \leq f(d)$.*

PROOF:

$\langle 1 \rangle 1$. $f(X)$ is compact.

PROOF: By Proposition 9.4.10.

$\langle 1 \rangle 2$. $f(X)$ has a greatest element.

$\langle 2 \rangle 1$. ASSUME: for a contradiction $f(X)$ has no greatest element.

$\langle 2 \rangle 2$. $\{(-\infty, f(x)) : x \in X\}$ is a set of open sets that covers $f(X)$.

$\langle 2 \rangle 3$. PICK a finite subset $\{(-\infty, f(x_1)), \dots, (-\infty, f(x_n))\}$ that covers $f(X)$.

PROOF: By Proposition 9.4.5

⟨2⟩4. LET: $f(x_N)$ be largest out of $f(x_1), \dots, f(x_n)$

⟨2⟩5. $f(x_N) < f(x_N)$

⟨2⟩6. Q.E.D.

PROOF: This is a contradiction.

⟨1⟩3. $f(X)$ has a least element.

PROOF: Similar.

□

Theorem 9.4.21 (DC). *A nonempty compact Hausdorff space with no isolated points is uncountable.*

PROOF:

⟨1⟩1. LET: X be a nonempty compact Hausdorff space with no isolated points.

⟨1⟩2. For every nonempty open $U \subseteq X$ and point $x \in X$, there exists a nonempty open $V \subseteq U$ such that $x \notin \overline{V}$

⟨2⟩1. LET: $U \subseteq X$ be nonempty and open and $x \in X$

⟨2⟩2. PICK $y \in U$ such that $y \neq x$

PROOF: This is possible because $U \neq \{x\}$ since x is not an isolated point.

⟨2⟩3. PICK disjoint open neighbourhoods W_1 and W_2 of x and y

PROOF: Since X is Hausdorff

⟨2⟩4. LET: $V = U \cap W_2$

⟨2⟩5. $x \notin \overline{V}$

PROOF: We have $\overline{V} \subseteq \overline{W_2} \subseteq X \setminus W_1$.

⟨1⟩3. LET: $f : \mathbb{Z}^+ \rightarrow X$

PROVE: f is not surjective

⟨1⟩4. PICK a sequence of open sets $V_1 \supseteq V_2 \supseteq \dots$ such that $f(n) \notin \overline{V_n}$

PROOF: By ⟨1⟩2 and Dependent Choice.

⟨1⟩5. PICK a point $b \in \bigcap_{i=1}^{\infty} \overline{V_i}$

PROOF: By Proposition 9.4.15.

⟨1⟩6. $b \neq f(n)$ for all n

PROOF: For each n we have $b \in \overline{V_n}$ (⟨1⟩5) and $f(n) \notin \overline{V_n}$ (⟨1⟩4).

□

Corollary 9.4.21.1. *Every closed interval in \mathbb{R} is uncountable.*

Theorem 9.4.22. *Every compact space is limit point compact.*

PROOF:

⟨1⟩1. LET: X be a compact space.

⟨1⟩2. LET: $A \subseteq X$ be a set with no limit points.

PROVE: A is finite.

⟨1⟩3. A is closed.

PROOF: By Corollary 3.13.3.1.

⟨1⟩4. A is compact.

PROOF: By Proposition 9.4.6.

⟨1⟩5. $\{U \text{ open in } X : U \cap A \text{ is a singleton}\}$ covers A

⟨2⟩1. LET: $a \in A$

- (2)2. PICK an open neighbourhood U of a such that U does not intersect A at a point other than a
 PROOF: One must exist because a is not a limit point of A ((1)2).
 (2)3. $U \cap A = \{a\}$
 (1)6. PICK a finite subcover $\{U_1, \dots, U_n\}$
 PROOF: By (1)4 using Proposition 9.4.5.
 (1)7. For $1 \leq i \leq n$,
 LET: $U_i \cap A = \{a_i\}$
 (1)8. $A = \{a_1, \dots, a_n\}$
 \square

Proposition 9.4.23. *Let X be a space and $C, D \subseteq X$ be compact. Then $C \cup D$ is compact.*

- PROOF:
 (1)1. LET: \mathcal{A} be a set of open sets that covers $C \cup D$
 (1)2. PICK a finite subset \mathcal{A}_1 that covers C and a finite subset \mathcal{A}_2 that covers D .
 (1)3. $\mathcal{A}_1 \cup \mathcal{A}_2$ is a finite subset of \mathcal{A} that covers $C \cup D$.
 (1)4. Q.E.D.

Theorem 9.4.24. *Every compact Hausdorff space is normal.*

- PROOF:
 (1)1. LET: X be a compact Hausdorff space.
 (1)2. LET: A and B be disjoint closed sets in X .
 (1)3. $\{U \text{ open in } X : \exists V \text{ open in } X. B \subseteq V \wedge U \cap V = \emptyset\}$ covers A
 (2)1. B is compact
 PROOF: By Proposition 9.4.6.
 (2)2. Q.E.D.
 PROOF: By Lemma 9.4.8.
 (1)4. PICK a finite subcover $\{U_1, \dots, U_n\}$
 PROOF: A is compact by Proposition 9.4.6.
 (1)5. For $1 \leq i \leq n$, PICK V_i open in X such that $B \subseteq V_i$ and $U_i \cap V_i = \emptyset$
 (1)6. LET: $U = U_1 \cup \dots \cup U_n$ and $V = V_1 \cap \dots \cap V_n$
 (1)7. U and V are disjoint open sets, $A \subseteq U$ and $B \subseteq V$
 \square

Corollary 9.4.24.1. *The ordered square is normal.*

Proposition 9.4.25. *Not every compact Hausdorff space is first countable.*

PROOF: The space $\overline{S_\Omega}$ is compact Hausdorff but not first countable. \square

Corollary 9.4.25.1. *Not every compact Hausdorff space is second countable.*

Theorem 9.4.26 (Tychonoff (AC)). *The product of a family of compact spaces is compact.*

PROOF:

- (1)1. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of compact spaces.
 LET: $X = \prod_{\alpha \in J} X_\alpha$
- (1)2. LET: $\mathcal{A} \subseteq \mathcal{P}X$ satisfy the finite intersection property.
 PROVE: $\bigcap_{A \in \mathcal{A}} \overline{A}$ is nonempty.
- (1)3. PICK a set $\mathcal{D} \subseteq \mathcal{P}X$ that includes \mathcal{A} and is maximal with respect to the finite intersection property.
 PROOF: By Lemma 1.7.6.
- (1)4. For $\alpha \in J$, PICK $x_\alpha \in \bigcap_{D \in \mathcal{D}} \overline{\pi_\alpha(D)}$
- (2)1. LET: $\alpha \in J$
- (2)2. $\{\pi_\alpha(D) : D \in \mathcal{D}\}$ satisfies the finite intersection property.
- (2)3. Q.E.D.
 PROOF: By Proposition 9.4.15
- (1)5. LET: $x = (x_\alpha)_{\alpha \in J}$
- (1)6. For all $D \in \mathcal{D}$ we have $(x_\alpha)_{\alpha \in J} \in \overline{D}$
 PROOF:
- (2)1. Every subbasis element containing x intersects every member of \mathcal{D}
- (3)1. LET: $\pi_\alpha(U)^{-1}$ be a subbasis element containing x where U is open in X_α
- (3)2. LET: $D \in \mathcal{D}$
- (3)3. U intersects $\pi_\alpha(D)$
- (2)2. Every subbasis element containing x is a member of \mathcal{D}
 PROOF: By Lemma 1.7.8
- (2)3. Every basis element containing x is a member of \mathcal{D}
 PROOF: By Lemma 1.7.7
- (2)4. Every basis element containing x intersects every member of \mathcal{D}
 PROOF: This follows because \mathcal{D} satisfies the finite intersection property.
- (1)7. Q.E.D.
 PROOF: By Proposition 9.4.15
-
- PROOF:
- (1)1. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of compact spaces and $X = \prod_{\alpha \in J} X_\alpha$.
- (1)2. PICK a well-ordering $<$ of J such that J has a greatest element \top
- (1)3. For all $\alpha \in J$ and every family of points $p = \{p_i \in X_i\}_{i \leq \alpha}$,
 LET: $Y_\alpha(p) = \{x \in X : \forall i \leq \alpha. x_i = p_i\}$
- (1)4. For all $\beta \in J$ and every family of points $p = \{p_i \in X_i\}_{i < \beta}$,
 LET: $Z_\beta(p) = \bigcap_{\alpha < \beta} Y_\alpha = \{x \in X : \forall i < \beta. x_i = p_i\}$
- (1)5. Given $\beta \in J$, a family of points $\{p_i \in X_i\}_{i < \beta}$, and a finite set \mathcal{A} of basis elements that covers $Z_\beta(p)$, there exists $\alpha < \beta$ such that \mathcal{A} covers $Y_\alpha(p)$
- (2)1. ASSUME: (
 w.l.o.g. β has no immediate predecessor)
- (2)2. For $A \in \mathcal{A}$,
 LET: $J_A = \{i < \beta : \pi_i(A) \neq X_i\}$
- (2)3. LET: α be the largest element of $\bigcup_{A \in \mathcal{A}} J_A$
 PROOF: The set has a greatest element because each J_A is finite and \mathcal{A} is finite.

quasicomponent.

PROVE: x and y are in the same component.

$\langle 1 \rangle 2$. LET: \mathcal{A} be the set of all closed subspaces A of X such that x and y lie in the same quasicomponent of A .

$\langle 1 \rangle 3$. Every chain in \mathcal{A} has a lower bound.

$\langle 2 \rangle 1$. LET: $\mathcal{B} \subseteq \mathcal{A}$ be a chain

PROVE: $Y = \bigcap \mathcal{B} \in \mathcal{A}$

$\langle 2 \rangle 2$. ASSUME: for a contradiction $Y = C \cup D$ where C and D are disjoint and open in Y , $x \in C$ and $y \in D$

$\langle 2 \rangle 3$. PICK disjoint open sets U and V in X such that $C \subseteq U$ and $D \subseteq V$

PROOF: By Lemma 9.4.18.

$\langle 2 \rangle 4$. $\{B \setminus (U \cup V) : B \in \mathcal{B}\}$ satisfies the finite intersection property.

$\langle 3 \rangle 1$. LET: $B_1, \dots, B_n \in \mathcal{B}$

$\langle 3 \rangle 2$. $B_1 \cap \dots \cap B_n \in \mathcal{B}$

PROOF: By $\langle 2 \rangle 1$.

$\langle 3 \rangle 3$. $B_1 \cap \dots \cap B_n \setminus (U \cup V)$ is nonempty

PROOF: $B_1 \cap \dots \cap B_n \cap U$ and $B_1 \cap \dots \cap B_n \cap V$ cannot be disjoint, because x and y are in the same quasicomponent of $B_1 \cap \dots \cap B_n$.

$\langle 2 \rangle 5$. $Y \setminus (U \cup V)$ is nonempty.

PROOF: By Proposition 9.4.15.

$\langle 2 \rangle 6$. Q.E.D.

PROOF: This is a contradiction since $Y \setminus (U \cup V) = Y \setminus (C \cup D)$.

$\langle 1 \rangle 4$. PICK a minimal element $D \in \mathcal{A}$

PROOF: One exists by Zorn's Lemma.

$\langle 1 \rangle 5$. D is connected.

$\langle 2 \rangle 1$. ASSUME: [

for a contradiction $D = U \uplus V$ is a separation of D]

$\langle 2 \rangle 2$. CASE: $x, y \in U$

PROOF: In this case we have $U \in \mathcal{A}$ contradicting the minimality of D .

$\langle 2 \rangle 3$. CASE: $x \in U, y \in V$

PROOF: This is a contradiction because x and y are in the same quasicomponent of D .

$\langle 2 \rangle 4$. CASE: $x \in V, y \in U$

PROOF: Similar to $\langle 2 \rangle 3$.

$\langle 2 \rangle 5$. CASE: $x, y \in V$

PROOF: Similar to $\langle 2 \rangle 2$.

□

9.5 Perfect Maps

Proposition 9.5.1. *Let $p : X \twoheadrightarrow Y$ be a closed continuous surjective map. For all $y \in Y$ and U an open neighbourhood of $p^{-1}(y)$, there exists an open neighbourhood W of y such that $p^{-1}(W) \subseteq U$.*

PROOF: Take $W = Y \setminus p(X \setminus U)$. □

Proposition 9.5.2 (AC). *Let $p : X \rightarrow Y$ be a closed continuous surjective map. If X is normal then Y is normal.*

PROOF:

- $\langle 1 \rangle 1$. LET: $A, B \subseteq Y$ be closed
- $\langle 1 \rangle 2$. $p^{-1}(A), p^{-1}(B)$ are closed in X .
- $\langle 1 \rangle 3$. PICK disjoint open sets U, V of $p^{-1}(A), p^{-1}(B)$ respectively.
- $\langle 1 \rangle 4$. For all $a \in A$, PICK an open neighbourhood W_a of a such that $p^{-1}(W_a) \subseteq U$

PROOF: By Proposition 9.5.1.

- $\langle 1 \rangle 5$. For all $b \in B$, PICK an open neighbourhood W'_b of b such that $p^{-1}(W'_b) \subseteq V$

PROOF: By Proposition 9.5.1.

- $\langle 1 \rangle 6$. LET: $W = \bigcup_{a \in A} W_a$ and $W' = \bigcup_{b \in B} W'_b$
- $\langle 1 \rangle 7$. $W \cap W' = \emptyset$

PROOF: This holds because $p^{-1}(W) \subseteq U, p^{-1}(W') \subseteq V$, and p is surjective.

□

Definition 9.5.3 (Perfect Map). Let X and Y be topological spaces and $p : X \rightarrow Y$. Then p is *perfect* iff p is closed, continuous, surjective, and $p^{-1}(y)$ is compact for all $y \in Y$.

Proposition 9.5.4. *Let $p : X \rightarrow Y$ be a perfect map. If X is Hausdorff then so is Y .*

PROOF:

- $\langle 1 \rangle 1$. LET: $a, b \in Y$ with $a \neq b$
- $\langle 1 \rangle 2$. PICK disjoint open neighbourhoods U and V of $\pi^{-1}(a)$ and $\pi^{-1}(b)$, respectively.

PROOF: By Lemma 9.4.18.

- $\langle 1 \rangle 3$. PICK open neighbourhoods W and W' of a and b such that $\pi^{-1}(W) \subseteq U$ and $\pi^{-1}(W') \subseteq V$

PROOF: By Proposition 9.5.1.

- $\langle 1 \rangle 4$. W and W' are disjoint.

□

Proposition 9.5.5. *Let $p : X \rightarrow Y$ be perfect. If X is regular then so is Y .*

PROOF:

- $\langle 1 \rangle 1$. Y is T_1

PROOF: By Proposition 9.5.4.

- $\langle 1 \rangle 2$. LET: $C \subseteq Y$ be closed and $a \in Y \setminus C$
- $\langle 1 \rangle 3$. $p^{-1}(C)$ is closed and $p^{-1}(a)$ is disjoint from $p^{-1}(C)$.
- $\langle 1 \rangle 4$. PICK disjoint open neighbourhoods U, V of $p^{-1}(C), p^{-1}(a)$ respectively.

PROOF: By Lemma 9.4.8.

- $\langle 1 \rangle 5$. PICK an open neighbourhood W' of a such that $p^{-1}(W') \subseteq V$

PROOF: By Proposition 9.5.1.

- $\langle 1 \rangle 6$. For $c \in C$, PICK an open neighbourhood W_c such that $p^{-1}(W_c) \subseteq U$

PROOF: By Proposition 9.5.1.

$\langle 1 \rangle 7$. $W = \bigcup_{c \in C} W_c$ is an open neighbourhood of C disjoint from W'

□

Proposition 9.5.6 (AC). *Let $p : X \rightarrow Y$ be perfect. If X is locally compact then so is Y .*

PROOF:

$\langle 1 \rangle 1$. LET: $b \in Y$

$\langle 1 \rangle 2$. $\{U \text{ open in } X : \exists C \subseteq X \text{ compact. } U \subseteq C\}$ covers $p^{-1}(b)$

$\langle 1 \rangle 3$. PICK a finite subcover $\{U_1, \dots, U_n\}$

$\langle 1 \rangle 4$. For $1 \leq i \leq n$, PICK a compact $C_i \subseteq X$ such that $U_i \subseteq C_i$

$\langle 1 \rangle 5$. For $1 \leq i \leq n$, PICK a neighbourhood W_i of b such that $p^{-1}(W_i) \subseteq U_i$

PROOF: By Proposition 9.5.1

$\langle 1 \rangle 6$. $b \in W_1 \cup \dots \cup W_n \subseteq p(C_1) \cup \dots \cup p(C_n)$

$\langle 1 \rangle 7$. $p(C_1) \cup \dots \cup p(C_n)$ is compact.

$\langle 2 \rangle 1$. Each $p(C_i)$ is compact.

PROOF: By Proposition 9.4.10.

$\langle 2 \rangle 2$. Q.E.D.

PROOF: By Proposition 9.4.23.

□

9.6 Sequential Compactness

Definition 9.6.1 (Sequentially Compact). A space is *sequentially compact* iff every sequence has a convergent subsequence.

Proposition 9.6.2. $\overline{S_\Omega}$ is not sequentially compact.

PROOF: Ω is a limit point of S_Ω but is not the limit of any sequence of points in S_Ω . □

9.7 Local Compactness

Definition 9.7.1 (Local Compactness). Let X be a topological space.

For $x \in X$, the space X is *locally compact* at x iff there exists a compact subspace $C \subseteq X$ that includes a neighbourhood of x .

The space X is *locally compact* iff it is locally compact at every point.

Proposition 9.7.2. *Every complete linearly ordered set is locally compact under the order topology.*

PROOF:

$\langle 1 \rangle 1$. LET: L be a complete linearly ordered set and $x \in L$

PROVE: There exists a compact subspace $C \subseteq L$ that includes a neighbourhood U of x

$\langle 1 \rangle 2$. CASE: x is least and greatest in L

- PROOF: In this case, $L = \{x\}$ is compact.
- ⟨1⟩3. CASE: x is least in L but not greatest
 - ⟨2⟩1. PICK $a < x$
 - ⟨2⟩2. Take $C = [a, x]$ and $U = (a, x]$
 - ⟨1⟩4. CASE: x is greatest in L but not least

PROOF: Similar.
 - ⟨1⟩5. CASE: x is neither least nor greatest
 - ⟨2⟩1. PICK $a < x$ and $b > x$
 - ⟨2⟩2. Take $C = [a, b]$ and $U = (a, b)$

□

Corollary 9.7.2.1. *For every ordinal α , the space S_α is locally compact.*

Theorem 9.7.3. *Every closed subspace of a locally compact Hausdorff space is locally compact.*

PROOF:

- ⟨1⟩1. LET: X be locally compact Hausdorff and $C \subseteq X$ be closed.
- ⟨1⟩2. LET: $x \in C$
- ⟨1⟩3. PICK $D \subseteq X$ compact and $U \subseteq D$ open such that $x \in U$
- ⟨1⟩4. D is closed.

PROOF: Proposition 9.4.9.
- ⟨1⟩5. $C \cap D$ is closed

PROOF: Proposition 3.6.5.
- ⟨1⟩6. $C \cap D$ is compact

PROOF: Proposition 9.4.6.
- ⟨1⟩7. Q.E.D.

PROOF: $C \cap D \subseteq C$ is compact and includes the open neighbourhood $U \cap C$ of x .

□

Proposition 9.7.4. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of nonempty topological spaces. If $\prod_{\alpha \in J} X_\alpha$ is locally compact, then each X_α is locally compact.*

PROOF:

- ⟨1⟩1. LET: $\alpha \in J$ and $x_\alpha \in X_\alpha$
- ⟨1⟩2. PICK $x_\beta \in X_\beta$ for all $\beta \in J \setminus \{\alpha\}$
- ⟨1⟩3. PICK a compact subspace $C \subseteq \prod_{\alpha \in J} X_\alpha$ that a neighbourhood U of x included in C
- ⟨1⟩4. PICK a basic open set $\prod_{\alpha \in J} U_\alpha$ such that $x \in \prod_{\alpha \in J} U_\alpha \subseteq U$
- ⟨1⟩5. $x_\alpha \in U_\alpha \subseteq \pi_\alpha(C)$
- ⟨1⟩6. $\pi_\alpha(C)$ is compact.

PROOF: By Proposition 9.4.10.

□

Corollary 9.7.4.1. *The Sorgenfrey plane is not locally compact.*

Proposition 9.7.5. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of locally compact spaces such that X_α is compact for all but finitely many values of α . Then $\prod_{\alpha \in J} X_\alpha$ is locally compact.*

PROOF:

- ⟨1⟩1. ASSUME: X_α is compact if $\alpha \neq \alpha_1, \dots, \alpha_n$
 - ⟨1⟩2. LET: $\vec{x} \in \prod_{\alpha \in J} X_\alpha$
 - ⟨1⟩3. For $1 \leq i \leq n$, PICK $C_{\alpha_i} \subseteq X_{\alpha_i}$ compact and U_{α_i} open such that $x_{\alpha_i} \in U_{\alpha_i} \subseteq C_{\alpha_i}$
 - ⟨1⟩4. For $\alpha \neq \alpha_1, \dots, \alpha_n$,
LET: $C_\alpha = U_\alpha = X_\alpha$
 - ⟨1⟩5. $\vec{x} \in \prod_{\alpha \in J} U_\alpha \subseteq \prod_{\alpha \in J} C_\alpha$
 - ⟨1⟩6. $\prod_{\alpha \in J} C_\alpha$ is compact
- PROOF: By Tychonoff's Theorem.

□

Proposition 9.7.6. \mathbb{R}_l is not locally compact.

PROOF: $[0, +\infty)$ can be partitioned into infinitely many disjoint open sets, which therefore do not have a finite subcover. □

Proposition 9.7.7. Let $\{X_\alpha\}_{\alpha \in J}$ be a family of nonempty topological spaces. If $\prod_{\alpha \in J} X_\alpha$ is locally compact, then all but finitely many of the X_α are compact.

PROOF:

- ⟨1⟩1. PICK a point $a = (a_\alpha)_{\alpha \in J} \in \prod_{\alpha \in J} X_\alpha$
- ⟨1⟩2. PICK a compact $C \subseteq \prod_{\alpha \in J} X_\alpha$ that includes the basic neighbourhood $\prod_{\alpha \in J} U_\alpha$ of a , where $U_\alpha = X_\alpha$ for all α except $\alpha = \alpha_1, \dots, \alpha_n$
- ⟨1⟩3. For $\alpha \neq \alpha_1, \dots, \alpha_n$, we have X_α is compact.

PROOF: X_α is homeomorphic to a closed subspace of C .

□

Corollary 9.7.7.1. For any infinite set I , the space \mathbb{R}^I is not locally compact.

Proposition 9.7.8. $[0, 1]^\omega$ is not compact under the uniform topology.

PROOF: $\{a_i : i \geq 0\}$ is an infinite set with no limit point, where a_i is the point with i th component 1 and all other components 0. □

Corollary 9.7.8.1. \mathbb{R}^ω under the uniform topology is not locally compact.

PROOF:

- ⟨1⟩1. ASSUME: \mathbb{R}^ω is locally compact
- ⟨1⟩2. LET: C be a compact subspace such that $B(\vec{0}, \epsilon) \subseteq C$
- ⟨1⟩3. $\overline{B(\vec{0}, \epsilon)}$ is compact.
- ⟨1⟩4. Q.E.D.

PROOF: This contradicts the proposition.

□

Proposition 9.7.9. Not every subspace of a locally compact Hausdorff space is locally compact.

PROOF: \mathbb{R} is locally compact Hausdorff, \mathbb{Q} is not locally compact. □

Proposition 9.7.10. *The continuous image of a locally compact Hausdorff space is not necessarily locally compact.*

PROOF:

- ⟨1⟩1. LET: $\{q_0, q_1, \dots\}$ be an enumeration of $[0, 1] \cap \mathbb{Q}$.
- ⟨1⟩2. Define $f : (0, +\infty) \setminus \mathbb{Z} \rightarrow [0, 1] \cap \mathbb{Q}$ by: $f(x) = q_n$ for $x \in (n, n+1)$
- ⟨1⟩3. f is continuous.

PROOF: The inverse image of any set is a union of open intervals.

□

9.8 Compactifications

Definition 9.8.1 (Compactification). Let X and Y be spaces. Then Y is a *compactification* of X iff Y is a compact Hausdorff space and X is a subspace of Y with $\overline{X} = Y$.

Two compactifications Y_1, Y_2 of X are *equivalent* iff there exists a homeomorphism between Y_1 and Y_2 that is the identity on X .

Lemma 9.8.2. *Let $h : X \rightarrow Z$ be an imbedding. Then there exists a compactification $c : X \rightarrow Y$ of X , unique up to equivalence, and an imbedding $i : Y \rightarrow Z$ such that $h = i \circ c$.*

PROOF: Simply take Y to be the closure of X in Z . □

Definition 9.8.3 (One-Point Compactification). A *one-point compactification* of X is a compactification Y of X such that $Y \setminus X$ is a singleton.

Theorem 9.8.4. *Let X be a topological space. Then X is locally compact Hausdorff if and only if there exists a space Y such that:*

1. X is a subspace of Y
2. The set $Y \setminus X$ is a singleton.
3. Y is a compact Hausdorff space.

If Y and Y' are two spaces satisfying these conditions, then there exists a unique homeomorphism between Y and Y' that is the identity on X .

PROOF:

- ⟨1⟩1. If X is locally compact Hausdorff then there exists a space Y satisfying 1–3.
- ⟨2⟩1. LET: $Y = X \cup \{\infty\}$ under the topology $\mathcal{T} = \{U \subseteq X : U \text{ is open in } X\} \cup \{Y \setminus C : C \subseteq X \text{ is compact}\}$.
- ⟨3⟩1. $Y \in \mathcal{T}$
PROOF: This holds because $Y = Y \setminus \emptyset$.
- ⟨3⟩2. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$.
- ⟨4⟩1. LET: $U, V \in \mathcal{T}$
- ⟨4⟩2. CASE: U, V are open in X

- PROOF: In this case, $U \cap V$ is open in X .
- ⟨4⟩3. CASE: U is open in X , $V = Y \setminus C$ where $C \subseteq X$ is compact.
- ⟨5⟩1. $U \cap V = U \setminus C$
- ⟨5⟩2. C is closed in X
- PROOF: Proposition 9.4.9.
- ⟨5⟩3. $U \cap V$ is open in X
- ⟨4⟩4. CASE: $U = Y \setminus C$ where $C \subseteq X$ is compact, V is open in X .
- PROOF: Similar.
- ⟨4⟩5. CASE: $U = Y \setminus C$, $V = Y \setminus D$ where $C, D \subseteq X$ are compact.
- ⟨5⟩1. $U \cap V = Y \setminus (C \cup D)$
- ⟨5⟩2. C and D are closed in X
- PROOF: Proposition 9.4.9.
- ⟨5⟩3. $C \cup D$ is closed in X
- PROOF: Proposition 3.6.4.
- ⟨5⟩4. $C \cup D$ is compact.
- PROOF: By Proposition 9.4.23. \square
- ⟨3⟩3. For all $\mathcal{A} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{A} \in \mathcal{T}$.
- ⟨4⟩1. LET: $\mathcal{A} \subseteq \mathcal{T}$
- ⟨4⟩2. CASE: Every element of \mathcal{A} is an open set in X .
- PROOF: In this case, $\bigcup \mathcal{A}$ is open in X .
- ⟨4⟩3. CASE: There exists C compact in X such that $Y \setminus C \in \mathcal{A}$
- ⟨5⟩1. $\bigcup \mathcal{A} = Y \setminus (\bigcap \{D \subseteq X : D \text{ compact}, Y \setminus D \in \mathcal{A}\} \setminus \bigcup \{U \text{ open in } X : U \in \mathcal{A}\})$
- PROOF: Set theory.
- ⟨5⟩2. $\bigcap \{D \subseteq X : D \text{ compact}, Y \setminus D \in \mathcal{A}\} \setminus \bigcup \{U \text{ open in } X : U \in \mathcal{A}\}$ is compact.
- PROOF: It is a closed subset of the compact set C .
- ⟨2⟩2. X is a subspace of Y
- ⟨3⟩1. For every open set U of X , there exists V open in Y such that $U = V \cap X$
- PROOF: Take $V = U$.
- ⟨3⟩2. For every open set V in Y , we have $V \cap X$ is open in X .
- ⟨4⟩1. LET: V be open in Y
- ⟨4⟩2. CASE: V is open in X
- PROOF: In this case, $V \cap X = V$.
- ⟨4⟩3. CASE: $V = Y \setminus C$ where $C \subseteq X$ is compact.
- ⟨5⟩1. C is closed in X .
- PROOF: By Proposition 9.4.9.
- ⟨5⟩2. $V \cap X = X \setminus C$
- ⟨2⟩3. $Y \setminus X = \{\infty\}$
- ⟨2⟩4. Y is compact.
- ⟨3⟩1. LET: \mathcal{A} be an open covering of Y
- ⟨3⟩2. PICK $U \in \mathcal{A}$ such that $\infty \in U$
- ⟨3⟩3. PICK $C \subseteq X$ compact such that $U = Y \setminus C$.
- ⟨3⟩4. $\{V \cap X : V \in \mathcal{A}\}$ is set of open sets that covers C
- ⟨3⟩5. PICK a finite subset $\{V_1, \dots, V_n\}$ such that $\{V_1 \cap X, \dots, V_n \cap X\}$

- covers C .
- ⟨3⟩6. $\{U, V_1, \dots, V_n\}$ is a finite subcover of Y .
- ⟨2⟩5. Y is Hausdorff.
- ⟨3⟩1. LET: $x, y \in Y$ with $x \neq y$
PROVE: There exist disjoint open neighbourhoods U, V of x and y .
- ⟨3⟩2. CASE: $x, y \in X$
PROOF: In this case, we just use the fact that X is Hausdorff.
- ⟨3⟩3. CASE: $x = \infty, y \in X$
- ⟨4⟩1. PICK $C \subseteq X$ compact such that C includes an open neighbourhood V of y
- ⟨4⟩2. LET: $U = Y \setminus C$
- ⟨3⟩4. CASE: $x \in X, y = \infty$
PROOF: Similar.
- ⟨1⟩2. If there exists a space Y satisfying 1–3 then X is locally compact Hausdorff.
- ⟨2⟩1. LET: Y be a space satisfying 1–3
- ⟨2⟩2. LET: ∞ be the point in $Y \setminus X$
- ⟨2⟩3. X is locally compact
- ⟨3⟩1. LET: $x \in X$
- ⟨3⟩2. PICK disjoint open neighbourhoods U of x and V of ∞
- ⟨3⟩3. $X \setminus V$ is compact and includes U
PROOF: $X \setminus V = Y \setminus V$ is compact because it is a closed subset of Y (Proposition 9.4.6).
- ⟨2⟩4. X is Hausdorff.
PROOF: By Corollary 6.2.6.1.
- ⟨1⟩3. If Y and Y' are two spaces satisfying 1–3 then there exists a unique homeomorphism between Y and Y' that is the identity on X .
- ⟨2⟩1. LET: Y and Y' be two spaces that satisfy 1–3.
- ⟨2⟩2. LET: $Y \setminus X = \{p\}$ and $Y' \setminus X = \{q\}$
- ⟨2⟩3. LET: $h : Y \rightarrow Y'$ be given by
- $$\begin{aligned} h(x) &= x & (x \in X) \\ h(p) &= q \end{aligned}$$
- ⟨2⟩4. h is a homeomorphism
- ⟨3⟩1. h is bijective.
- ⟨3⟩2. h is continuous.
- ⟨4⟩1. LET: $V \subseteq Y'$ be open.
PROVE: $h^{-1}(V)$ is open.
- ⟨4⟩2. CASE: $V \subseteq X$
- ⟨5⟩1. $h^{-1}(V) = V$
- ⟨5⟩2. V is open in X
PROOF: Condition 1 for Y' .
- ⟨5⟩3. V is open in Y
PROOF: Condition 1 for Y .
- ⟨4⟩3. CASE: $q \in V$
- ⟨5⟩1. $Y' \setminus V$ is compact.

PROOF: Proposition 9.4.6.

(5)2. $Y' \setminus V$ is closed in Y .

PROOF: Proposition 9.4.9.

(5)3. $h^{-1}(V) = Y \setminus (Y' \setminus V)$

(3)3. h^{-1} is continuous.

PROOF: Similar.

(2)5. If $h' : Y \rightarrow Y'$ is a homeomorphism such that $h' \upharpoonright_X = \text{id}_X$ then $h' = h$

□

Theorem 9.8.5. *Let X be a Hausdorff space. Then X is locally compact if and only if, for all $x \in X$ and any neighbourhood U of x , there exists an open neighbourhood V of x such that \overline{V} is compact and $\overline{V} \subseteq U$.*

PROOF:

(1)1. If X is locally compact then, for all $x \in X$ and any neighbourhood U of x , there exists an open neighbourhood V of x such that \overline{V} is compact and $\overline{V} \subseteq U$.

(2)1. ASSUME: X is locally compact.

(2)2. LET: $x \in X$ and U be a neighbourhood of x .

(2)3. LET: Y be the one-point compactification of X .

PROOF: By Theorem 9.8.4.

(2)4. LET: $C = Y \setminus U$

(2)5. C is compact

PROOF: By Proposition 9.4.6.

(2)6. PICK disjoint open sets V, W containing x and C

PROOF: Lemma 9.4.8

(2)7. V is open in X

PROOF: $V \subseteq X$ since $\infty \in W$.

(2)8. The closure of V in X is compact

(3)1. The closure of V in X is the same as the closure of V in Y .

PROOF: The point ∞ cannot be a limit point of V since W is a neighbourhood disjoint from V .

(3)2. The closure of V in Y is compact.

PROOF: By Proposition 9.4.6.

(2)9. $\overline{V} \subseteq U$

PROOF:

$$\begin{aligned}\overline{V} &\subseteq Y \setminus W \\ &\subseteq Y \setminus C \\ &= U\end{aligned}$$

(1)2. If, for all $x \in X$ and any neighbourhood U of x , there exists an open neighbourhood V of x such that \overline{V} is compact and $\overline{V} \subseteq U$, then X is locally compact.

(2)1. ASSUME: for all $x \in X$ and any neighbourhood U of x , there exists an open neighbourhood V of x such that \overline{V} is compact and $\overline{V} \subseteq U$

(2)2. LET: $x \in X$

PROVE: There exists $C \subseteq X$ compact such that C includes a neighbourhood U of x

(2)3. PICK an open neighbourhood V of x such that \bar{V} is compact and $\bar{V} \subseteq X$

(2)4. Take $C = \bar{V}$ and $U = V$

□

Corollary 9.8.5.1. *Every open subspace of a locally compact Hausdorff space is locally compact.*

Corollary 9.8.5.2. *A space is locally compact Hausdorff if and only if it is an open subspace of a compact Hausdorff space.*

Corollary 9.8.5.3. *Every locally compact Hausdorff space is completely regular.*

Corollary 9.8.5.4. *The space \mathbb{R}_K is not locally compact.*

Lemma 9.8.6 (AC). *If $p : X \rightarrow Y$ is a quotient map and Z is a locally compact Hausdorff space, then the map*

$$\pi = p \times \text{id}_Z : X \times Z \rightarrow Y \times Z$$

is a quotient map.

PROOF:

(1)1. π is surjective.

PROOF: This holds because p is surjective.

(1)2. π is continuous.

PROOF: By Theorem 5.2.15.

(1)3. For $A \subseteq Y \times Z$, if $\pi^{-1}(A)$ is open in $X \times Z$ then A is open in $Y \times Z$.

(2)1. LET: $A \subseteq Y \times Z$

(2)2. ASSUME: $\pi^{-1}(A)$ is open in $X \times Z$

(2)3. LET: $(y, z) \in A$

(2)4. PICK $x \in X$ such that $p(x) = y$

PROOF: Since p is surjective.

(2)5. PICK open sets U_1, V with \bar{V} compact such that $(x, y) \in U_1 \times V$ and $U_1 \times \bar{V} \subseteq \pi^{-1}(A)$

PROOF: Using Theorem 9.8.5

(2)6. PICK a sequence of open sets U_1, U_2, \dots in X such that $p^{-1}(p(U_n)) \subseteq U_{n+1}$ and $U_n \times \bar{V} \subseteq \pi^{-1}(A)$ for all n

(3)1. LET: U be open with $U \times \bar{V} \subseteq \pi^{-1}(A)$

PROVE: There exists W open with $p^{-1}(p(U)) \subseteq W$ and $W \times \bar{V} \subseteq \pi^{-1}(A)$

(3)2. For all $x \in p^{-1}(p(U))$, PICK open sets U_x, V_x such that $x \in U_x$, $\bar{V} \subseteq V_x$ and $U_x \times V_x \subseteq \pi^{-1}(A)$

PROOF: By the Tube Lemma.

(3)3. LET: $W = \bigcup_{x \in p^{-1}(p(U))} U_x$

(2)7. LET: $U = \bigcup_{n=1}^{\infty} U_n$

(2)8. U is saturated with respect to p

- ⟨3⟩1. LET: $a \in U, b \in X, p(a) = p(b)$
- ⟨3⟩2. PICK n such that $a \in U_n$
- ⟨3⟩3. $b \in p^{-1}(p(U_n))$
- ⟨3⟩4. $b \in U_{n+1}$
- ⟨3⟩5. $b \in U$
- ⟨2⟩9. $p(U)$ is open in Y
- PROOF: By Lemma 4.5.2.
- ⟨2⟩10. $(y, z) \in p(U) \times V \subseteq A$
- ⟨2⟩11. Q.E.D.
- PROOF: By Proposition 3.2.3.

□

Theorem 9.8.7. *Let $p : A \rightarrow B$ and $q : C \rightarrow D$ be quotient maps. If B and C are locally compact Hausdorff spaces, then $p \times q : A \times C \rightarrow B \times D$ is a quotient map.*

PROOF: This holds by Lemma 9.8.6 and Proposition 4.5.10 because $p \times q = (\text{id}_B \times q) \circ (p \times \text{id}_C)$. □

Theorem 9.8.8. *Let X be a completely regular space. Let Y be a compactification of X such that every bounded continuous map $X \rightarrow \mathbb{R}$ extends uniquely to a continuous map $Y \rightarrow \mathbb{R}$. Then, for every compact Hausdorff space C , every continuous map $X \rightarrow C$ extends uniquely to a continuous map $Y \rightarrow C$.*

PROOF:

- ⟨1⟩1. LET: C be a compact Hausdorff space and $f : X \rightarrow C$ a continuous function
- ⟨1⟩2. PICK a set J and an imedding $C \subseteq [0, 1]^J$
- ⟨2⟩1. C is normal
- PROOF: By Lemma 9.4.18
- ⟨2⟩2. Q.E.D.
- PROOF: By Theorem 6.4.6.
- ⟨1⟩3. For $\alpha \in J$,
- LET: $g_\alpha : Y \rightarrow \mathbb{R}$ be the unique continuous extension of $\pi_\alpha \circ f$
- ⟨1⟩4. Define $g : Y \rightarrow \mathbb{R}^J$ by $g(y)_\alpha = g_\alpha(y)$
- ⟨1⟩5. g is continuous
- PROOF: By Theorem 5.2.15.
- ⟨1⟩6. g extends f
- ⟨1⟩7. We have $g : Y \rightarrow C$

PROOF:

$$\begin{aligned}
 g(Y) &= g(\overline{X}) \\
 &\subseteq \overline{g(X)} && \text{(Theorem 5.2.2)} \\
 &= \overline{f(X)} && \text{(\langle 1 \rangle 6)} \\
 &\subseteq \overline{C} \\
 &= C && \text{(Proposition 9.4.9)}
 \end{aligned}$$

- ⟨1⟩8. g is unique

$\langle 2 \rangle 1$. LET: $h : Y \rightarrow C$ be a continuous extension of f

$\langle 2 \rangle 2$. For all $\alpha \in J$, $\pi_\alpha \circ h$ extends $\pi_\alpha \circ f$

$\langle 2 \rangle 3$. For all $\alpha \in J$, $\pi_\alpha \circ h = g_\alpha$

PROOF: By $\langle 1 \rangle 3$

$\langle 2 \rangle 4$. $h = g$

PROOF: By $\langle 1 \rangle 4$

□

Corollary 9.8.8.1. *Let X be a completely regular space. Let Y_1 and Y_2 be compactifications of X such that every bounded continuous map $X \rightarrow \mathbb{R}$ extends uniquely to a continuous map $Y_i \rightarrow \mathbb{R}$. Then Y_1 and Y_2 are equivalent.*

Definition 9.8.9 (Stone-Čech Compactification). Let X be a completely regular space. The *Stone-Čech compactification* of X , $\beta(X)$, is the compactification of X such that, for every compact Hausdorff space C , every continuous function $X \rightarrow C$ extends uniquely to a continuous function $\beta(X) \rightarrow C$.

Chapter 10

Metric Spaces

10.1 The Metric Topology

Definition 10.1.1 (Metric). A *metric* on a set X is a function $d : X \times X \rightarrow \mathbb{R}$ such that, for all $x, y, z \in X$:

1. $d(x, y) \geq 0$;
2. $d(x, y) = 0$ if and only if $x = y$;
3. $d(x, y) = d(y, x)$;
4. **Triangle Inequality**

$$d(x, z) \leq d(x, y) + d(y, z)$$

A *metric space* X consists of a set X and a metric on X . We call $d(x, y)$ the *distance* between x and y .

Definition 10.1.2 (Open Ball). Let X be a metric space with metric d , $x \in X$ and $\epsilon > 0$. The *open ball* with *centre* x and *radius* ϵ is

$$B_d(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\} .$$

Lemma 10.1.3. Let X be a metric space, $x, y \in X$ and $\epsilon > 0$. If $y \in B(x, \epsilon)$, then there exists δ such that $0 < \delta < \epsilon$ and

$$B(y, \delta) \subseteq B(x, \epsilon) .$$

PROOF:

- ⟨1⟩1. LET: $\delta = \epsilon - d(x, y)$
⟨1⟩2. LET: $z \in B(y, \delta)$
⟨1⟩3. $d(x, z) < \epsilon$

PROOF:

$$\begin{aligned}
 d(x, z) &\leq d(x, y) + d(y, z) && \text{(Triangle Inequality)} \\
 &< d(x, y) + \delta && ((1)2) \\
 &= \epsilon && ((1)1)
 \end{aligned}$$

□

Definition 10.1.4 (Metric Topology). Let d be a metric on X . The *metric topology* on X induced by d is the topology generated by the basis consisting of the open balls.

We prove this is a topology.

PROOF:

⟨1⟩1. Every point is in an open ball.

PROOF: $x \in B(x, 1)$

⟨1⟩2. If B_1, B_2 are open balls and $x \in B_1 \cap B_2$, then there exists an open ball B_3 such that $x \in B_3 \subseteq B_1 \cap B_2$.

⟨2⟩1. LET: $x \in B(y, \epsilon_1) \cap B(z, \epsilon_2)$

⟨2⟩2. PICK δ_1, δ_2 such that $0 < \delta_1 < \epsilon_1, 0 < \delta_2 < \epsilon_2, B(x, \delta_1) \subseteq B(y, \epsilon_1)$ and $B(x, \delta_2) \subseteq B(z, \epsilon_2)$.

PROOF: Lemma 10.1.3.

⟨2⟩3. LET: $\delta = \min(\delta_1, \delta_2)$

⟨2⟩4. $x \in B(x, \delta) \subseteq B(y, \epsilon_1) \cap B(z, \epsilon_2)$

⟨1⟩3. Q.E.D.

PROOF: Lemma 3.5.3.

Lemma 10.1.5. A set U is open in the metric topology induced by d if and only if, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$.

PROOF:

⟨1⟩1. If U is open then, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$.

⟨2⟩1. ASSUME: U is open.

⟨2⟩2. LET: $x \in U$

⟨2⟩3. PICK $B(y, \delta)$ such that $x \in B(y, \delta) \subseteq U$

⟨2⟩4. PICK ϵ such that $0 < \epsilon < \delta$ and $B(x, \epsilon) \subseteq B(y, \delta)$

PROOF: Lemma 10.1.3.

⟨2⟩5. $B(x, \epsilon) \subseteq U$

PROOF: From ⟨2⟩3 and ⟨2⟩4.

⟨1⟩2. If, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$, then U is open.

PROOF: Immediate from definition of metric topology.

□

Lemma 10.1.6. Let d and d' be two metrics on the set X . Let \mathcal{T} and \mathcal{T}' be the topologies they induce, respectively. Then $\mathcal{T} \subseteq \mathcal{T}'$ if and only if, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$.

PROOF:

⟨1⟩1. If $\mathcal{T} \subseteq \mathcal{T}'$ then, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$.

(2)1. ASSUME: $\mathcal{T} \subseteq \mathcal{T}'$
 (2)2. LET: $x \in X$ and $\epsilon > 0$
 (2)3. $B_d(x, \epsilon) \in \mathcal{T}'$
 PROOF: From (2)1.
 (2)4. There exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$
 PROOF: By Lemma 10.1.5.
 (1)2. If, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$, then $\mathcal{T} \subseteq \mathcal{T}'$
 (2)1. ASSUME: For all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$.
 (2)2. LET: $U \in \mathcal{T}$
 PROVE: $U \in \mathcal{T}'$
 (2)3. LET: $x \in U$
 (2)4. PICK $\epsilon > 0$ be such that $B_d(x, \epsilon) \subseteq U$
 PROOF: By Lemma 10.1.5.
 (2)5. PICK $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$
 PROOF: By (2)1.
 (2)6. $B_{d'}(x, \delta) \subseteq U$
 PROOF: By (2)4 and (2)5.
 (2)7. Q.E.D.
 PROOF: By Lemma 10.1.5.

□

Definition 10.1.7 (Metrisable). A topological space is *metrisable* if and only if there exists a metric that induces its topology.

Lemma 10.1.8. *Every discrete space is metrisable.*

PROOF: The discrete topology is induced by the metric $d(x, y) = 1$ if $x \neq y$, 0 if $x = y$. □

Proposition 10.1.9. *The continuous image of a metrisable space is not necessarily metrisable.*

PROOF: The identity map from the discrete two-point space to the indiscrete two-point space is continuous. □

Lemma 10.1.10. \mathbb{R} is metrisable.

PROOF: The standard topology is induced by the metric $d(x, y) = |x - y|$. □

Definition 10.1.11 (Bounded). Let X be a metric space and $A \subseteq X$. Then A is *bounded* iff $\{d(x, y) : x, y \in A\}$ is bounded above, in which case its *diameter* is

$$\text{diam } A = \sup_{x, y \in A} d(x, y) .$$

Lemma 10.1.12. *Let (X, d) be a metric space and $A \subseteq X$. Then $d \upharpoonright_{A \times A}$ is a metric on A that induces the subspace topology.*

PROOF:

⟨1⟩1. $d \upharpoonright_{A \times A}$ is a metric on A .

PROOF: Each of the axioms for a metric follows immediately from the same axiom for d .

⟨1⟩2. The topology induced by $d \upharpoonright_{A \times A}$ is the product topology.

PROOF: Both are the topology generated by the basis consisting of all the open balls $B_{d \upharpoonright_{A \times A}}(a, \epsilon) = B_d(a, \epsilon) \cap A$.

□

Lemma 10.1.13. *Every metric space is Hausdorff.*

PROOF:

⟨1⟩1. LET: X be a metric space and $x, y \in X$ with $x \neq y$.

⟨1⟩2. LET: $\epsilon = d(x, y)$

⟨1⟩3. $B(x, \epsilon/2)$ and $B(y, \epsilon/2)$ are disjoint neighbourhoods of x and y .

□

Theorem 10.1.14. *Every metric space is first countable.*

PROOF: $\{B(x, q) : q \in \mathbb{Q}^+\}$ is a local basis at x . □

Corollary 10.1.14.1. *If J is infinite then the space \mathbb{R}^J is not metrizable.*

Definition 10.1.15 (Standard Bounded Metric). Let d be a metric on X . The *standard bounded metric* corresponding to d is

$$\bar{d}(x, y) = \min(d(x, y), 1) .$$

We prove this is a metric.

PROOF:

⟨1⟩1. $\bar{d}(x, y) \geq 0$

PROOF: This holds because $d(x, y) \geq 0$ (d is a metric) and $1 > 0$.

⟨1⟩2. $\bar{d}(x, y) = 0$ iff $x = y$

PROOF: Immediate from definition.

⟨1⟩3. $\bar{d}(x, y) = \bar{d}(y, x)$

PROOF: Immediate from definition.

⟨1⟩4. $\bar{d}(x, z) \leq \bar{d}(x, y) + \bar{d}(y, z)$

⟨2⟩1. CASE: $d(x, y) \leq 1, d(y, z) \leq 1$

PROOF:

$$\begin{aligned} \bar{d}(x, z) &\leq d(x, z) \\ &\leq d(x, y) + d(y, z) \\ &= \bar{d}(x, y) + \bar{d}(y, z) \end{aligned}$$

⟨2⟩2. CASE: $d(y, z) > 1$

PROOF:

$$\begin{aligned} \bar{d}(x, z) &\leq 1 \\ &\leq \bar{d}(x, y) + 1 \\ &= \bar{d}(x, y) + \bar{d}(y, z) \end{aligned}$$

⟨2⟩3. CASE: $d(x, y) > 1$

PROOF: Similar.

□

Theorem 10.1.16. *Let d be a metric on X . Then the standard bounded metric \bar{d} corresponding to d induces the same topology as d .*

PROOF:

⟨1⟩1. LET: \mathcal{T} be the topology induced by d and \mathcal{T}' be the topology induced by \bar{d} .

⟨1⟩2. $\mathcal{T} \subseteq \mathcal{T}'$

⟨2⟩1. LET: $x \in X$ and $\epsilon > 0$

⟨2⟩2. LET: $\delta = \min(\epsilon, 1/2)$

⟨2⟩3. $B_{\bar{d}}(x, \delta) \subseteq B_d(x, \epsilon)$

⟨3⟩1. LET: $y \in B_{\bar{d}}(x, \delta)$

⟨3⟩2. $\bar{d}(x, y) < \delta$

⟨3⟩3. $\bar{d}(x, y) < 1$

PROOF: From ⟨2⟩2 and ⟨3⟩2.

⟨3⟩4. $\bar{d}(x, y) = d(x, y)$

PROOF: From ⟨3⟩3 and the definition of \bar{d} .

⟨3⟩5. $d(x, y) < \epsilon$

PROOF: By ⟨2⟩2 and ⟨3⟩2 and ⟨3⟩4.

⟨1⟩3. $\mathcal{T}' \subseteq \mathcal{T}$

⟨2⟩1. LET: $x \in X$ and $\epsilon > 0$

⟨2⟩2. $B_d(x, \epsilon) \subseteq B_{\bar{d}}(x, \epsilon)$

PROOF: This holds because $\bar{d}(x, y) \leq d(x, y)$.

□

Definition 10.1.17 (Square Metric). The *square metric* on \mathbb{R}^n is defined by

$$\rho(\vec{x}, \vec{y}) = \max(|x_1 - y_1|, \dots, |x_n - y_n|) \ .$$

We prove this is a metric.

PROOF:

⟨1⟩1. $\rho(\vec{x}, \vec{y}) \geq 0$

PROOF: Immediate from definitions.

⟨1⟩2. $\rho(\vec{x}, \vec{y}) = 0$ iff $\vec{x} = \vec{y}$

PROOF: Immediate from definitions.

⟨1⟩3. $\rho(\vec{x}, \vec{y}) = \rho(\vec{y}, \vec{x})$

PROOF: Immediate from definitions.

⟨1⟩4. $\rho(\vec{x}, \vec{z}) \leq \rho(\vec{x}, \vec{y}) + \rho(\vec{y}, \vec{z})$

⟨2⟩1. For all i , we have $|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$

⟨2⟩2. For all i , $|x_i - z_i| \leq \rho(\vec{x}, \vec{y}) + \rho(\vec{y}, \vec{z})$

⟨2⟩3. $\rho(\vec{x}, \vec{z}) \leq \rho(\vec{x}, \vec{y}) + \rho(\vec{y}, \vec{z})$

□

Theorem 10.1.18. *The square metric induces the standard topology on \mathbb{R}^n .*

PROOF:

⟨1⟩1. LET: \mathcal{T}_ρ be the topology induced by the square metric and \mathcal{T}_s the standard topology.

⟨1⟩2. $\mathcal{T}_\rho \subseteq \mathcal{T}_s$

PROOF: This holds because $B_\rho(\vec{x}, \epsilon) = (x_1 - \epsilon, x_1 + \epsilon) \times \cdots \times (x_n - \epsilon, x_n + \epsilon)$.

⟨1⟩3. $\mathcal{T}_s \subseteq \mathcal{T}_\rho$

⟨2⟩1. LET: $B = U_1 \times \cdots \times U_n$ be a basic open set in \mathcal{T}_s , where each U_i is open in \mathbb{R} .

⟨2⟩2. LET: $\vec{x} \in B$

⟨2⟩3. For $1 \leq i \leq n$, PICK $\epsilon_i > 0$ such that $(x_i - \epsilon_i, x_i + \epsilon_i) \subseteq U_i$

⟨2⟩4. LET: $\epsilon = \min(\epsilon_1, \dots, \epsilon_n)$

⟨2⟩5. $B_\rho(\vec{x}, \epsilon) \subseteq B$

□

Lemma 10.1.19. *The product of a countable family of metrizable spaces is metrizable.*

PROOF:

⟨1⟩1. LET: $\{X_n\}_{n \in \mathbb{Z}^+}$ be a family of metric spaces with metrics bounded by 1,
 $X = \prod_{n=1}^{\infty} X_n$.

⟨1⟩2. LET: $D : X \times X \rightarrow \mathbb{R}$ be given by

$$D(\vec{x}, \vec{y}) = \sup_{n \geq 1} \frac{d(x_n, y_n)}{n}.$$

⟨1⟩3. D is a metric on X .

⟨2⟩1. $D(\vec{x}, \vec{y}) \geq 0$

PROOF: Immediate from definitions.

⟨2⟩2. $D(\vec{x}, \vec{y}) = 0$ iff $\vec{x} = \vec{y}$

PROOF: Immediate from definitions.

⟨2⟩3. $D(\vec{x}, \vec{y}) = D(\vec{y}, \vec{x})$

PROOF: Immediate from definitions.

⟨2⟩4. $D(\vec{x}, \vec{z}) \leq D(\vec{x}, \vec{y}) + D(\vec{y}, \vec{z})$

⟨3⟩1. For all n , we have $\frac{d(x_n, z_n)}{n} \leq \frac{d(x_n, y_n)}{n} + \frac{d(y_n, z_n)}{n}$

⟨3⟩2. For all n , we have $\frac{d(x_n, z_n)}{n} \leq D(\vec{x}, \vec{y}) + D(\vec{y}, \vec{z})$

⟨3⟩3. $D(\vec{x}, \vec{z}) \leq D(\vec{x}, \vec{y}) + D(\vec{y}, \vec{z})$

⟨1⟩4. LET: \mathcal{T}_D be the topology induced by D and \mathcal{T}_p the product topology.

⟨1⟩5. $\mathcal{T}_D \subseteq \mathcal{T}_p$

⟨2⟩1. LET: $U \in \mathcal{T}_D$

PROVE: $U \in \mathcal{T}_p$

⟨2⟩2. LET: $\vec{x} \in U$

⟨2⟩3. PICK $\epsilon > 0$ such that $B_D(\vec{x}, \epsilon) \subseteq U$

⟨2⟩4. PICK N such that $1/N < \epsilon$

⟨2⟩5. LET: $V = B(x_1, \epsilon) \times \cdots \times B(x_N, \epsilon) \times X_{N+1} \times X_{N+2} \times \cdots$

⟨2⟩6. $\vec{x} \in V \subseteq B_D(\vec{x}, \epsilon)$

⟨1⟩6. $\mathcal{T}_p \subseteq \mathcal{T}_D$

⟨2⟩1. LET: $U = \prod_{n=1}^{\infty} U_n$ be a basic open set in \mathcal{T}_p , where each U_n is open in X_n , and $U_n = X_n$ for $n > N$.

- (2)2. LET: $\vec{x} \in U$
 PROVE: There exists $\epsilon > 0$ such that $B_D(\vec{x}, \epsilon) \subseteq U$.
 (2)3. For $n \leq N$, PICK $\epsilon_n > 0$ such that $B(x_n, \epsilon_n) \subseteq U_n$
 (2)4. LET: $\epsilon = \min(\epsilon_1, \epsilon_2/2, \dots, \epsilon_n/n)$
 (2)5. LET: $\vec{y} \in B_D(\vec{x}, \epsilon)$
 (2)6. For $n \leq N$, $y_n \in U_n$
 (3)1. $D(\vec{x}, \vec{y}) < \epsilon$
 (3)2. $d(x_n, y_n)/n < \epsilon$
 (3)3. $d(x_n, y_n)/n < \epsilon_n/n$
 (3)4. Q.E.D.
 PROOF: By (2)3.

□

Corollary 10.1.19.1. *The space \mathbb{R}^ω is metrizable.*

Definition 10.1.20 (Uniform Metric). Let J be a set. The *uniform metric* $\bar{\rho}$ on \mathbb{R}^J is defined by

$$\bar{\rho}(\vec{x}, \vec{y}) = \sup_{\alpha \in J} \bar{d}(x_\alpha, y_\alpha) .$$

where \bar{d} is the standard bounded metric on \mathbb{R} . The *uniform topology* is the topology induced by the uniform metric.

We prove this is a metric.

PROOF:

- (1)1. $\bar{\rho}(\vec{x}, \vec{y}) \geq 0$
 PROOF: Immediate from definitions.
 (1)2. $\bar{\rho}(\vec{x}, \vec{y}) = 0$ iff $\vec{x} = \vec{y}$
 PROOF: Immediate from definitions.
 (1)3. $\bar{\rho}(\vec{x}, \vec{y}) = \bar{\rho}(\vec{y}, \vec{x})$
 PROOF: Immediate from definitions.
 (1)4. $\bar{\rho}(\vec{x}, \vec{z}) \leq \bar{\rho}(\vec{x}, \vec{y}) + \bar{\rho}(\vec{y}, \vec{z})$
 PROOF:
 (2)1. For all $\alpha \in J$, $\bar{d}(x_\alpha, z_\alpha) \leq \bar{d}(x_\alpha, y_\alpha) + \bar{d}(y_\alpha, z_\alpha)$
 (2)2. For all $\alpha \in J$, $\bar{d}(x_\alpha, z_\alpha) \leq \bar{\rho}(\vec{x}, \vec{y}) + \bar{\rho}(\vec{y}, \vec{z})$
 (2)3. $\bar{\rho}(\vec{x}, \vec{z}) \leq \bar{\rho}(\vec{x}, \vec{y}) + \bar{\rho}(\vec{y}, \vec{z})$

□

Theorem 10.1.21 (DC). *The uniform topology on \mathbb{R}^J is finer than the product topology and coarser than the box topology. These three topologies are different iff J is infinite.*

PROOF:

- (1)1. The uniform topology is finer than the product topology.
 (2)1. LET: $B = \prod_{\alpha \in J} U_\alpha$ be a basic open set in the product topology, where each U_α is open in \mathbb{R} , and $U_\alpha = \mathbb{R}$ except for $\alpha = \alpha_1, \dots, \alpha_n$.
 (2)2. LET: $\vec{x} \in U$
 (2)3. For $1 \leq i \leq n$, PICK $0 < \epsilon_i < 1$ such that $(x_{\alpha_i} - \epsilon_i, x_{\alpha_i} + \epsilon_i) \subseteq U_{\alpha_i}$.

PROOF: The set of all sequences of 0s and 1s is discrete but uncountable. \square

Corollary 10.1.22.1. *Not every metric space is second countable.*

Theorem 10.1.23. *Let X and Y be metric spaces. Let $f : X \rightarrow Y$ and $x \in X$. Then f is continuous at x if and only if, for every $\epsilon > 0$, there exists $\delta > 0$ such that, for all $x' \in X$, if $d(x, x') < \delta$ then $d(f(x), f(x')) < \epsilon$.*

PROOF:

- $\langle 1 \rangle 1$. If f is continuous at x then, for every $\epsilon > 0$, there exists $\delta > 0$ such that, for all $x' \in X$, if $d(x, x') < \delta$ then $d(f(x), f(x')) < \epsilon$.
- $\langle 2 \rangle 1$. ASSUME: f is continuous at x .
- $\langle 2 \rangle 2$. LET: $\epsilon > 0$
- $\langle 2 \rangle 3$. PICK a neighbourhood U of x such that $f(U) \subseteq B(f(x), \epsilon)$
PROOF: One exists by $\langle 2 \rangle 1$, since $B(f(x), \epsilon)$ is a neighbourhood of $f(x)$.
- $\langle 2 \rangle 4$. PICK $\delta > 0$ such that $B(x, \delta) \subseteq U$
PROOF: By $\langle 2 \rangle 3$ and Lemma 10.1.5.
- $\langle 2 \rangle 5$. LET: $x' \in X$ with $d(x, x') < \delta$
- $\langle 2 \rangle 6$. $x' \in U$
PROOF: From $\langle 2 \rangle 4$ and $\langle 2 \rangle 5$.
- $\langle 2 \rangle 7$. $f(x') \in B(f(x), \epsilon)$
PROOF: From $\langle 2 \rangle 3$ and $\langle 2 \rangle 6$.
- $\langle 1 \rangle 2$. If, for all $\epsilon > 0$, there exists $\delta > 0$ such that, for all $x' \in X$, if $d(x, x') < \delta$ then $d(f(x), f(x')) < \epsilon$, then f is continuous at x .
- $\langle 2 \rangle 1$. ASSUME: For all $\epsilon > 0$ there exists $\delta > 0$ such that, for all $x' \in X$, if $d(x, x') < \delta$ then $d(f(x), f(x')) < \epsilon$.
- $\langle 2 \rangle 2$. LET: V be a neighbourhood of $f(x)$
- $\langle 2 \rangle 3$. PICK $\epsilon > 0$ such that $B(f(x), \epsilon) \subseteq V$
PROOF: By Lemma 10.1.5.
- $\langle 2 \rangle 4$. PICK $\delta > 0$ such that, for all $x' \in X$, if $d(x, x') < \delta$ then $d(f(x), f(x')) < \epsilon$.
PROOF: By $\langle 2 \rangle 1$ and $\langle 2 \rangle 3$.
- $\langle 2 \rangle 5$. $B(x, \delta)$ is a neighbourhood of x
PROOF: By the definition of the metric topology.
- $\langle 2 \rangle 6$. $f(B(x, \delta)) \subseteq V$
 - $\langle 3 \rangle 1$. LET: $x' \in B(x, \delta)$
 - $\langle 3 \rangle 2$. $d(f(x), f(x')) < \epsilon$
PROOF: From $\langle 2 \rangle 4$.
 - $\langle 3 \rangle 3$. $x' \in V$
PROOF: From $\langle 2 \rangle 3$.

\square

Lemma 10.1.24. *Addition is a continuous function $\mathbb{R}^2 \rightarrow \mathbb{R}$.*

PROOF:

- $\langle 1 \rangle 1$. LET: $(x, y) \in \mathbb{R}^2$ and $\epsilon > 0$
- $\langle 1 \rangle 2$. LET: $\delta = \epsilon/2$

⟨1⟩3. LET: $(x', y') \in \mathbb{R}^2$ be such that $\rho((x, y), (x', y')) < \delta$, where ρ is the square metric

⟨1⟩4. $|x - x'| < \delta$ and $|y - y'| < \delta$

⟨1⟩5. $|(x + y) - (x' + y')| < \epsilon$

PROOF:

$$\begin{aligned} |(x + y) - (x' + y')| &\leq |x - x'| + |y - y'| \\ &< 2\delta && \langle 1 \rangle 4 \\ &= \epsilon && \langle 1 \rangle 2 \end{aligned}$$

⟨1⟩6. Q.E.D.

PROOF: By Theorem 10.1.23.

□

Lemma 10.1.25. *Additive inverse is a continuous function $- : \mathbb{R} \rightarrow \mathbb{R}$.*

PROOF: If $|x - y| < \epsilon$ then $|(-x) - (-y)| < \epsilon$. □

Lemma 10.1.26. *Multiplication is a continuous function $\cdot : \mathbb{R}^2 \rightarrow \mathbb{R}$.*

PROOF:

⟨1⟩1. LET: $(x, y) \in \mathbb{R}^2$ and $\epsilon > 0$

⟨1⟩2. LET: $\delta = \min(1, \epsilon/(|x| + |y| + 1))$

⟨1⟩3. LET: $(x', y') \in \mathbb{R}^2$ and $\rho((x, y), (x', y')) < \delta$

⟨1⟩4. $|xy - x'y'| < \epsilon$

PROOF:

$$\begin{aligned} |xy - x'y'| &= |x(y' - y) + y(x' - x) + (x - x')(y - y')| \\ &\leq |x||y' - y| + |y||x' - x| + |x - x'||y - y'| \\ &< |x|\delta + |y|\delta + \delta^2 && \langle 1 \rangle 3 \\ &= \delta(|x| + |y| + \delta) \\ &\leq \delta(|x| + |y| + 1) && \langle 1 \rangle 2 \\ &\leq \epsilon && \langle 1 \rangle 2 \end{aligned}$$

□

Lemma 10.1.27. *Multiplicative inverse is a continuous function $(\)^{-1} : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$.*

PROOF:

⟨1⟩1. LET: $f : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ be defined by $f(x) = x^{-1}$.

⟨1⟩2. LET: $a, b \in \mathbb{R}$ with $a < b$

PROVE: $f^{-1}((a, b))$ is open

⟨1⟩3. CASE: $0 < a < b$

PROOF: $f^{-1}((a, b)) = (b^{-1}, a^{-1})$

⟨1⟩4. CASE: $a < 0 < b$

PROOF: $f^{-1}((a, b)) = (-\infty, a^{-1}) \cup (b^{-1}, +\infty)$

⟨1⟩5. CASE: $a < b < 0$

PROOF: $f^{-1}((a, b)) = (b^{-1}, a^{-1})$

□

Definition 10.1.28 (Uniform Convergence). Let X be a set and Y a metric space. Let $f_n : X \rightarrow Y$ for $n \geq 1$, and $f : X \rightarrow Y$. Then f_n converges uniformly to f as $n \rightarrow \infty$ iff, for all $\epsilon > 0$, there exists N such that, for all $x \in X$ and $n \geq N$, $d(f_n(x), f(x)) < \epsilon$.

Theorem 10.1.29 (Uniform Limit Theorem). Let X be a topological space and Y a metric space. Let $f_n : X \rightarrow Y$ for $n \geq 1$ and $f : X \rightarrow Y$. If f_n converges uniformly to f and each f_n is continuous, then f is continuous.

PROOF:

- (1)1. LET: $x \in X$ and $\epsilon > 0$
- (1)2. PICK N such that, for all $x' \in X$ and $\delta > 0$, $d(f_n(x'), f(x')) < \epsilon/3$
- (1)3. PICK $\delta > 0$ such that, for all $x' \in X$, if $d(x, x') < \delta$ then $d(f_N(x), f_N(x')) < \epsilon/3$
- (1)4. For all $x' \in X$, if $d(x, x') < \delta$ then $d(f(x), f(x')) < \epsilon$

PROOF:

$$\begin{aligned} d(f(x), f(x')) &\leq d(f(x), f_N(x)) + d(f_N(x), f_N(x')) + d(f_N(x'), f(x')) \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 \\ &= \epsilon \end{aligned}$$

□

Lemma 10.1.30. Let X be a set. Let $f_n : X \rightarrow \mathbb{R}$ for $n \geq 1$ and $f : X \rightarrow \mathbb{R}$. Then f_n converges uniformly to f if and only if f_n converges to f in \mathbb{R}^X under the uniform topology.

PROOF:

- (1)1. If f_n converges uniformly to f then f_n converges to f under the uniform topology.
 - (2)1. ASSUME: f_n converges uniformly to f
 - (2)2. LET: $\epsilon > 0$
 - (2)3. PICK N such that, for all $x \in X$ and $n \geq N$, $d(f_n(x), f(x)) < \epsilon/2$
 - (2)4. $\bar{\rho}(f_n, f) \leq \epsilon/2$
 - (2)5. $\bar{\rho}(f_n, f) < \epsilon$
- (1)2. If f_n converges to f under the uniform topology then f_n converges uniformly to f .
 - (2)1. ASSUME: f_n converges to f under the uniform topology.
 - (2)2. LET: $\epsilon > 0$
 - (2)3. PICK N such that, for all $n \geq N$, $\bar{\rho}(f_n, f) < \epsilon$
 - (2)4. For all $n \geq N$ and $x \in X$, $d(f_n(x), f(x)) < \epsilon$

□

Theorem 10.1.31. Every monotone increasing sequence of real numbers that is bounded above converges to its supremum.

PROOF:

- (1)1. LET: $\{s_n\}_{n \geq 1}$ be a monotone increasing sequence of real numbers bounded above with supremum l .

- (1)2. LET: $\epsilon > 0$
 (1)3. $l - \epsilon$ is not an upper bound for $\{s_n : n \geq 1\}$.
 (1)4. PICK N such that $x_N > l - \epsilon$
 (1)5. For all $n \geq N$, we have $l - \epsilon < x_n \leq l$
 (1)6. For all $n \geq N$, we have $|x_n - l| < \epsilon$
 \square

Definition 10.1.32 (Infinite Series). Let $\{a_n\}_{n \geq 1}$ be a sequence of real numbers. The *infinite series* $\sum_{n=1}^{\infty} a_n$ *converges* to s iff $\sum_{n=1}^N a_n \rightarrow s$ as $N \rightarrow \infty$.

Proposition 10.1.33. If $\sum_{n=1}^{\infty} a_n = s$ and $\sum_{n=1}^{\infty} b_n = t$ then $\sum_{n=1}^{\infty} (ca_n + b_n) = cs + t$.

PROOF: This holds because $\sum_{n=1}^N (ca_n + b_n) = c \sum_{n=1}^N a_n + \sum_{n=1}^N b_n \rightarrow cs + t$ as $N \rightarrow \infty$. \square

Theorem 10.1.34 (Comparison Test). If $|a_i| \leq b_i$ for all i and $\sum_{i=1}^{\infty} b_i$ converges, then $\sum_{i=1}^{\infty} a_i$ converges.

PROOF:

- (1)1. $\sum_{i=1}^{\infty} |a_i|$ converges

PROOF: $\sum_{i=1}^N |a_i|$ is a monotone increasing sequence bounded above by $\sum_{i=1}^{\infty} b_i$.

- (1)2. LET: $c_i = |a_i| + a_i$

- (1)3. $\sum_{i=1}^{\infty} c_i$ converges

PROOF: $\sum_{i=1}^N c_i$ is a monotone increasing sequence bounded above by $2 \sum_{i=1}^{\infty} |a_i|$.

- (1)4. Q.E.D.

PROOF: Since $a_i = c_i - |a_i|$.

Lemma 10.1.35. If $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=N}^{\infty} a_n \rightarrow 0$ as $N \rightarrow \infty$.

PROOF:

$$\begin{aligned}
 \sum_{n=N}^{\infty} a_n &= \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N-1} a_n \\
 &\rightarrow \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} a_n \\
 &= 0
 \end{aligned}$$

as $N \rightarrow \infty$. \square

Theorem 10.1.36 (Weierstrass M-Test). Let X be a set and $f_n : X \rightarrow \mathbb{R}$ for $n \geq 1$. If $|f_n(x)| \leq M_n$ for all $n \geq 1$ and all $x \in X$, and if $\sum_{n=1}^{\infty} M_n$ converges, then

$$\sum_{n=1}^N f_n(x) \rightarrow \sum_{n=1}^{\infty} f_n(x)$$

uniformly in x as $N \rightarrow \infty$.

PROOF:

⟨1⟩1. For $N \geq 1$,

LET: $s_N : X \rightarrow \mathbb{R}$, $s_N(x) = \sum_{n=1}^N f_n(x)$

⟨1⟩2. For all $x \in X$, $\sum_{n=1}^{\infty} f_n(x)$ converges.

PROOF: By the Comparison Test.

⟨1⟩3. LET: $s : X \rightarrow \mathbb{R}$, $s(x) = \sum_{n=1}^{\infty} f_n(x)$.

⟨1⟩4. For $N \geq 1$,

LET: $r_N = \sum_{n=N+1}^{\infty} M_n$

⟨1⟩5. For $1 \leq N < K$, we have $|s_K(x) - s_N(x)| \leq r_N$ for all $x \in X$

PROOF:

$$\begin{aligned} |s_K(x) - s_N(x)| &= \left| \sum_{n=N+1}^K f_n(x) \right| \\ &\leq \sum_{n=N+1}^K |f_n(x)| \\ &\leq \sum_{n=N+1}^K M_n \\ &\leq \sum_{n=N+1}^{\infty} M_n \\ &= r_N \end{aligned}$$

⟨1⟩6. For $N \geq 1$ and $x \in X$ we have $|s(x) - s_N(x)| \leq r_N$

PROOF: Let $K \rightarrow \infty$ in ⟨1⟩5.

⟨1⟩7. LET: $\epsilon > 0$

⟨1⟩8. PICK N such that, for all $N' \geq N$, we have $r_{N'} < \epsilon$

PROOF: Such an N exists by Lemma 10.1.35.

⟨1⟩9. For all $N' \geq N$ and $x \in X$ we have $|s_{N'}(x) - s(x)| < \epsilon$

□

Definition 10.1.37. Let X be a metric space. Let $x \in X$ and $A \subseteq X$ be nonempty. The *distance* from x to A is

$$d(x, A) = \inf_{a \in A} d(x, a) .$$

Lemma 10.1.38. Let X be a metric space and $A \subseteq X$ be nonempty. Then the function $d(-, A) : X \rightarrow \mathbb{R}$ is continuous.

PROOF:

⟨1⟩1. LET: $x \in X$ and $\epsilon > 0$

⟨1⟩2. LET: $y \in X$ with $d(x, y) < \epsilon$

⟨1⟩3. $|d(x, A) - d(y, A)| < \epsilon$

PROOF:

⟨2⟩1. $d(x, A) - d(y, A) < \epsilon$

PROOF:

$$\begin{aligned}
d(x, A) &= \inf_{a \in A} d(x, a) \\
&\leq \inf_{a \in A} (d(x, y) + d(y, a)) \\
&= d(x, y) + \inf_{a \in A} d(y, a) \\
&= d(x, y) + d(y, A) \\
&< \epsilon + d(y, A)
\end{aligned}$$

(2)2. $d(y, A) - d(x, A) < \epsilon$

PROOF: Similar.

(1)4. Q.E.D.

PROOF: By Theorem 10.1.23.

□

Definition 10.1.39 (Shrinking Map). Let X be a metric space and $f : X \rightarrow X$. Then f is a *shrinking map* iff, for all $x, y \in X$ with $x \neq y$, we have $d(f(x), f(y)) < d(x, y)$.

Definition 10.1.40 (Contraction). Let X be a metric space and $f : X \rightarrow X$. Then f is a *contraction* iff there exists $\alpha < 1$ such that, for all $x, y \in X$,

$$d(f(x), f(y)) \leq \alpha d(x, y) .$$

Proposition 10.1.41. *Every separable metric space is second countable.*

PROOF:

(1)1. LET: X be a separable metric space.

(1)2. PICK a countable dense set D

(1)3. LET: $\mathcal{B} = \{B(d, q) : d \in D, q \in \mathbb{Q}^+\}$

(1)4. \mathcal{B} is a countable basis for X

□

Corollary 10.1.41.1. *The space \mathbb{R}^ω under the uniform topology is not separable.*

Corollary 10.1.41.2. *Not every metric space is separable.*

Corollary 10.1.41.3. *The space \mathbb{R}^ω under the box topology is not separable.*

Proposition 10.1.42 (CC). *Every Lindelöf metric space is second countable.*

PROOF:

(1)1. LET: X be a Lindelöf metric space.

(1)2. For all $n \in \mathbb{Z}^+$, PICK a countable covering \mathcal{A}_n of X by $1/n$ -balls

PROOF: One exists by the Lindelöf condition, since the set of all $1/n$ -balls covers X .

(1)3. $\bigcup_{n=1}^{\infty} \mathcal{A}_n$ is a countable basis.

□

Corollary 10.1.42.1. *The space \mathbb{R}^ω under the uniform topology is not Lindelöf.*

Corollary 10.1.42.2. *Not every metric space is Lindelöf.*

Proposition 10.1.43. *The space \mathbb{R}_l is not metrizable.*

PROOF: It is Lindelöf but not second countable. \square

Proposition 10.1.44. *The ordered square is not metrizable.*

PROOF: It is compact but not second countable. \square

Proposition 10.1.45. *The space \mathbb{R}^ω under the uniform topology is not second countable.*

PROOF: It contains a subspace homeomorphic to \mathbb{R} . \square

Theorem 10.1.46 (AC). *Every metrizable space is normal.*

PROOF:

$\langle 1 \rangle 1$. LET: X be a metric space.

$\langle 1 \rangle 2$. LET: A and B be disjoint closed subspaces of X .

$\langle 1 \rangle 3$. For $a \in A$, PICK $\epsilon_a > 0$ such that $B(a, \epsilon_a)$ does not intersect B .

$\langle 1 \rangle 4$. For $b \in B$, PICK $\epsilon_b > 0$ such that $B(b, \epsilon_b)$ does not intersect A .

$\langle 1 \rangle 5$. LET: $U = \bigcup_{a \in A} B(a, \epsilon_a/2)$

$\langle 1 \rangle 6$. LET: $V = \bigcup_{b \in B} B(b, \epsilon_b/2)$

$\langle 1 \rangle 7$. $U \cap V = \emptyset$

$\langle 2 \rangle 1$. LET: $z \in U \cap V$

$\langle 2 \rangle 2$. PICK $a \in A$ and $b \in B$ such that $z \in B(a, \epsilon_a/2)$ and $z \in B(b, \epsilon_b/2)$

$\langle 2 \rangle 3$. ASSUME: w.l.o.g. $\epsilon_a \leq \epsilon_b$

$\langle 2 \rangle 4$. $a \in B(b, \epsilon_b)$

PROOF:

$$d(a, b) \leq d(a, z) + d(b, z) \quad (\text{Triangle Inequality})$$

$$< \epsilon_a/2 + \epsilon_b/2 \quad (\langle 2 \rangle 2)$$

$$\leq \epsilon_b \quad (\langle 2 \rangle 3)$$

$\langle 2 \rangle 5$. Q.E.D.

PROOF: This contradicts $\langle 1 \rangle 4$.

\square

Corollary 10.1.46.1. *The space \mathbb{R}^ω is normal.*

Corollary 10.1.46.2. *The space \mathbb{R}_K is not metrizable.*

Proposition 10.1.47. *Every metrizable space is completely normal.*

PROOF: Every subspace is metrizable (Lemma 10.1.12) hence normal (Theorem 10.1.46). \square

Proposition 10.1.48. *Every metrizable space is perfectly normal.*

PROOF:

⟨1⟩1. LET: X be a metric space.

⟨1⟩2. X is normal.

PROOF: Theorem 10.1.46

⟨1⟩3. Every closed set is G_δ .

PROOF: If A is closed then $A = \bigcap_{q \in \mathbb{Q}^+} \{x \in X : d(A, x) < q\}$.

□

Theorem 10.1.49 (Urysohn Metrization Theorem (CC)). *Every second countable regular space is metrizable.*

PROOF:

⟨1⟩1. LET: X be a second countable regular space.

⟨1⟩2. X is normal.

⟨1⟩3. PICK a countable basis $\mathcal{B} = \{B_n : n \in \mathbb{Z}^+\}$

⟨1⟩4. For every pair of integers m, n with $\overline{B_m} \subseteq B_n$, PICK a continuous function $g_{mn} : X \rightarrow [0, 1]$ such that $g_{mn}(\overline{B_m}) = \{1\}$ and $g_{mn}(X \setminus B_n) = \{0\}$

PROOF: By the Urysohn Lemma.

⟨1⟩5. The set $\{g_{mn} : \overline{B_m} \subseteq B_n\}$ separates points from closed sets in X

⟨2⟩1. LET: $x \in X$ and U be a neighbourhood of x

⟨2⟩2. PICK $B_n \in \mathcal{B}$ such that $x \in B_n \subseteq U$

⟨2⟩3. PICK V open such that $x \in V$ and $\overline{V} \subseteq B_n$

⟨2⟩4. PICK $B_m \in \mathcal{B}$ such that $x \in B_m \subseteq V$

⟨2⟩5. $g_{mn}(x) = 1$ and g_{mn} vanishes outside U

⟨1⟩6. X is imbeddable in $[0, 1]^\omega$

PROOF: By the Imbedding Theorem.

⟨1⟩7. Q.E.D.

Corollary 10.1.49.1. *The space \mathbb{R}^ω under the box topology is not second countable.*

Proposition 10.1.50. *Not every second countable Hausdorff space is metrizable.*

PROOF: \mathbb{R}_K is second countable and Hausdorff but not metrizable (because it is not regular). □

Proposition 10.1.51. *There exists a space that is completely normal, first countable, Lindelöf and separable but not metrizable.*

PROOF: The space \mathbb{R}_l is all of these. □

Proposition 10.1.52. *$\overline{S_\Omega}$ is not metrizable.*

PROOF: It is compact but not sequentially compact. □

Proposition 10.1.53. *Every compact metric space is second countable.*

PROOF:

⟨1⟩1. LET: X be a compact metric space

⟨1⟩2. For every $n \geq 1$, PICK a finite covering \mathcal{A}_n of X by open balls of radius $1/n$

PROOF: Such a covering exists because $\{B_{1/n}(x) : x \in X\}$ covers X .

⟨1⟩3. $\bigcup_{n=1}^{\infty} \mathcal{A}_n$ is a countable basis for X

□

Corollary 10.1.53.1. *The space \mathbb{R}^{ω} under the uniform topology is not compact.*

Corollary 10.1.53.2. *The space \mathbb{R}^{ω} under the uniform topology is not limit point compact.*

Proposition 10.1.54. *The space \mathbb{R}^{ω} under the box topology is not locally compact.*

PROOF:

⟨1⟩1. ASSUME: \mathbb{R}^{ω} under the box topology is locally compact.

⟨1⟩2. For every point x , there exists a basic open set $B = \prod_{i=0}^{\infty} U_i$ such that $x \in B$ and \overline{B} is compact.

⟨1⟩3. The box topology on \overline{B} is the same as the product topology on \overline{B}

PROOF: By Corollary 9.4.11.1.

⟨1⟩4. The box topology on \overline{B} is strictly finer than the product topology.

PROOF: By Theorem 10.1.21.

□

Proposition 10.1.55. *Not every metrizable space is connected.*

PROOF: The discrete space with two points is metrizable but not connected. □

Corollary 10.1.55.1. *Not every metrizable space is path connected.*

Proposition 10.1.56. *Not every metric space is limit point compact.*

PROOF: The space \mathbb{R} is not limit point compact. □

Proposition 10.1.57. *Not every metric space is locally compact.*

The space \mathbb{R}^{ω} in the uniform topology is not locally compact.

Lemma 10.1.58 (AC). *Let X be a metrizable space. Then every open covering \mathcal{A} of X has a countably locally finite open refinement \mathcal{E} that covers X .*

PROOF:

⟨1⟩1. LET: X be a metric space.

⟨1⟩2. PICK a well-ordering $<$ for \mathcal{A} .

⟨1⟩3. For $n \in \mathbb{Z}^+$ and $U \in \mathcal{A}$,

LET:

$$S_n(U) = \{x \in X : B(x, 1/n) \subseteq U\}$$

⟨1⟩4. For $n \in \mathbb{Z}^+$ and $U \in \mathcal{A}$,

LET:

$$T_n(U) = S_n(U) - \bigcup_{V < U} V$$

⟨1⟩5. For $n \in \mathbb{Z}^+$ and $U \in \mathcal{A}$,
 LET:

$$E_n(U) = \bigcup_{x \in T_n(U)} B(x, 1/3n)$$

⟨1⟩6. For $n \in \mathbb{Z}^+$,
 LET:

$$\mathcal{E}_n = \{E_n(U) : U \in \mathcal{A}\}$$

⟨1⟩7. LET:

$$\mathcal{E} = \bigcup_{n \in \mathbb{Z}^+} \mathcal{E}_n$$

⟨1⟩8. \mathcal{E} is countably locally finite

⟨2⟩1. For all n , \mathcal{E}_n is locally finite.

⟨3⟩1. For all $x \in X$, we have $B(x, 1/6n)$ intersects at most one element of \mathcal{E}_n

⟨4⟩1. ASSUME: for a contradiction $a \in B(x, 1/6n) \cap E_n(U)$ and $b \in B(x, 1/6n) \cap E_n(V)$

⟨4⟩2. PICK $c \in T_n(U)$ such that $d(a, c) < 1/3n$ and $d \in T_n(V)$ such that $d(b, d) < 1/3n$

⟨4⟩3. ASSUME: w.l.o.g. $V < U$

⟨4⟩4. $c \in V$

⟨5⟩1. $d(c, d) < 1/n$

PROOF:

$$\begin{aligned} d(c, d) &\leq d(c, a) + d(a, x) + d(x, b) + d(b, d) \quad (\text{Triangle Inequality}) \\ &< 1/3n + 1/6n + 1/6n + 1/3n \quad (\langle 4 \rangle 1, \langle 4 \rangle 2) \\ &= 1/n \end{aligned}$$

⟨5⟩2. $B(d, 1/n) \subseteq V$

⟨6⟩1. $d \in S_n(V)$

PROOF: From ⟨1⟩4 and ⟨4⟩2.

⟨6⟩2. Q.E.D.

PROOF: From ⟨1⟩3

⟨4⟩5. Q.E.D.

PROOF: This is a contradiction because $c \in T_n(U)$ (⟨4⟩2) so $c \notin V$ (⟨1⟩4, ⟨4⟩3).

⟨1⟩9. \mathcal{E} is an open refinement of \mathcal{A}

⟨2⟩1. \mathcal{E} is a refinement of \mathcal{A}

⟨3⟩1. For every n , we have \mathcal{E}_n is a refinement of \mathcal{A} .

⟨4⟩1. LET: n be a positive integer

⟨4⟩2. For every $U \in \mathcal{A}$ we have $E_n(U) \subseteq U$

⟨5⟩1. LET: $U \in \mathcal{A}$ and $x \in E_n(U)$

⟨5⟩2. PICK $y \in T_n(U)$ such that $x \in B(y, 1/3n)$

PROOF: ⟨1⟩5, ⟨5⟩1.

⟨5⟩3. $y \in S_n(U)$

PROOF: ⟨1⟩4, ⟨5⟩2

⟨5⟩4. $x \in U$

PROOF: ⟨1⟩3, ⟨5⟩2, ⟨5⟩3

(2)2. Every member of \mathcal{E} is open.
 (3)1. For all n , every member of \mathcal{E}_n is open.
 (4)1. LET: n be a positive integer
 (4)2. For all $U \in \mathcal{A}$, $E_n(U)$ is open.
 PROOF: By (1)5, $E_n(U)$ is a union of open balls.
 (4)3. Q.E.D.
 PROOF: By (1)6
 (3)2. Q.E.D.
 PROOF: By (1)7.
 (1)10. \mathcal{E} covers X
 (2)1. LET: $x \in X$
 (2)2. LET: U be the least member of \mathcal{A} such that $x \in U$
 (2)3. PICK n such that $B(x, 1/n) \subseteq U$
 (2)4. $x \in E_n(U) \in \mathcal{E}$

□

10.2 Isometries

Definition 10.2.1 (Isometry). Let X be a metric space. An *isometry* of X is a function $f : X \rightarrow X$ such that, for all $x, y \in X$,

$$d(f(x), f(y)) = d(x, y) .$$

10.3 Lebesgue Numbers

Definition 10.3.1 (Lebesgue Number). Let X be a metric space and \mathcal{A} an open covering of X . A *Lebesgue number* for \mathcal{A} is a real $\delta > 0$ such that, for every nonempty set $A \subseteq X$ of diameter $< \delta$, there exists $U \in \mathcal{A}$ such that $A \subseteq U$.

Lemma 10.3.2 (Lebesgue Number Lemma). *In a compact metric space, every open covering has a Lebesgue number.*

PROOF:

(1)1. LET: X be a compact metric space and \mathcal{A} an open covering of X
 PROVE: There exists a Lebesgue number δ for \mathcal{A} .
 (1)2. ASSUME: w.l.o.g. $X \notin \mathcal{A}$
 PROOF: If $X \in \mathcal{A}$ then we can take $\delta = 1$.
 (1)3. PICK a finite subcovering $\{U_1, \dots, U_n\} \subseteq \mathcal{A}$ that covers X
 (1)4. For $1 \leq i \leq n$,
 LET: $C_i = X \setminus U_i$
 (1)5. LET: $f : X \rightarrow \mathbb{R}$ be defined by

$$f(x) = 1/n \sum_{i=1}^n d(x, C_i) .$$

PROOF: Each C_i is nonempty by (1)2.

⟨1⟩6. For all $x \in X$ we have $f(x) > 0$
 ⟨2⟩1. LET: $x \in X$
 ⟨2⟩2. PICK i such that $x \in U_i$
 PROOF: By ⟨1⟩3.
 ⟨2⟩3. PICK $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U_i$
 PROOF: By Lemma 10.1.5.
 ⟨2⟩4. $d(x, C_i) \geq \epsilon$
 ⟨1⟩7. f is continuous
 PROOF: From Lemma 10.1.38.
 ⟨1⟩8. LET: $\delta = \min f(X)$
 PROVE: For every nonempty set $A \subseteq X$ with diameter $< \delta$, there exists
 $U \in \mathcal{A}$ such that $A \subseteq U$
 PROOF: $f(X)$ has a minimum by the Extreme Value Theorem.
 ⟨1⟩9. LET: $A \subseteq X$ be nonempty with $\text{diam } A < \delta$
 ⟨1⟩10. PICK $x_0 \in A$
 ⟨1⟩11. LET: i be such that $d(x_0, C_i)$ is greatest among $d(x_0, C_1), \dots, d(x_0, C_n)$
 ⟨1⟩12. $\delta \leq d(x_0, C_i)$
 PROOF:

$$\delta \leq f(x_0) \tag{⟨1⟩8}$$

$$= 1/n \sum_{j=1}^n d(x_0, C_j) \tag{⟨1⟩5}$$

$$\leq 1/n \sum_{j=1}^n d(x_0, C_i) \tag{⟨1⟩11}$$

$$= d(x_0, C_i)$$
 ⟨1⟩13. $x_0 \in U_i$
 PROOF: $x_0 \notin C_i$ because $d(x_0, C_i) > 0$.
 □

Theorem 10.3.3 (DC). *Let X be a metrizable space. Then the following are equivalent:*

1. X is compact.
2. X is limit point compact.
3. X is sequentially compact.

PROOF:

⟨1⟩1. $1 \Rightarrow 2$

PROOF: Theorem 9.4.22.

⟨1⟩2. $2 \Rightarrow 3$

⟨2⟩1. ASSUME: X is limit point compact.

⟨2⟩2. LET: (x_n) be a sequence in X

PROVE: (x_n) has a convergent subsequence.

⟨2⟩3. CASE: $\{x_n : n \in \mathbb{Z}^+\}$ is finite.

PROOF: In this case, (x_n) has a constant subsequence.

⟨2⟩4. CASE: $\{x_n : n \in \mathbb{Z}^+\}$ is infinite.

⟨3⟩1. PICK a limit point l of $\{x_n : n \in \mathbb{Z}^+\}$

⟨3⟩2. For every positive integer r , PICK n_r such that $n_r > n_{r-1}$ and $d(x_{n_r}, l) < 1/r$

PROOF: There always exists such an n_r since $B(l, 1/r)$ intersects $\{x_n : n \in \mathbb{Z}^+\}$ in infinitely many points by Theorem 6.1.2.

⟨3⟩3. $x_{n_r} \rightarrow l$ as $r \rightarrow \infty$

⟨1⟩3. $3 \Rightarrow 1$

⟨2⟩1. ASSUME: X is sequentially compact.

⟨2⟩2. Every open covering of X has a Lebesgue number.

⟨3⟩1. LET: \mathcal{A} be an open covering of X .

⟨3⟩2. ASSUME: for a contradiction that, for all $\delta > 0$, there exists a set $C \subseteq X$ with $\text{diam } C < \delta$ such that there is no $U \in \mathcal{A}$ such that $C \subseteq U$

⟨3⟩3. For $n \geq 1$, PICK $C_n \subseteq X$ with $\text{diam } C_n < 1/n$ such that there is no $U \in \mathcal{A}$ such that $C_n \subseteq U$

⟨3⟩4. For $n \geq 1$, PICK $x_n \in C_n$

⟨3⟩5. PICK a convergent subsequence (x_{n_r}) of (x_n)

PROOF: By ⟨2⟩1.

⟨3⟩6. LET: $x_{n_r} \rightarrow l$ as $r \rightarrow \infty$

⟨3⟩7. PICK $U \in \mathcal{A}$ with $l \in U$

PROOF: By ⟨3⟩1

⟨3⟩8. PICK $\epsilon > 0$ such that $B(l, \epsilon) \subseteq U$

PROOF: By Lemma 10.1.5.

⟨3⟩9. PICK R such that $1/n_R < \epsilon/2$ and $d(x_{n_R}, l) < \epsilon/2$

PROOF: By ⟨3⟩6

⟨3⟩10. $C_{n_R} \subseteq U$

PROOF:

$$\begin{aligned} C_{n_R} &\subseteq B(x_{n_R}, 1/n_R) && (\langle 3 \rangle 3, \langle 3 \rangle 4) \\ &\subseteq B(x_{n_R}, \epsilon/2) && (\langle 3 \rangle 9) \\ &\subseteq B(l, \epsilon) && (\langle 3 \rangle 9) \\ &\subseteq U && (\langle 3 \rangle 8) \end{aligned}$$

⟨3⟩11. Q.E.D.

PROOF: This contradicts ⟨3⟩3.

⟨2⟩3. For all $\epsilon > 0$, there exists a finite covering of X by ϵ -balls.

⟨3⟩1. LET: $\epsilon > 0$

⟨3⟩2. ASSUME: for a contradiction there is no finite covering of X by ϵ -balls.

⟨3⟩3. PICK a sequence (x_n) in X such that, for all n ,

$$x_n \notin B(x_1, \epsilon) \cup \cdots \cup B(x_{n-1}, \epsilon) .$$

⟨3⟩4. For all m, n with $m > n$ we have $d(x_m, x_n) \geq \epsilon$

⟨3⟩5. Any $\epsilon/2$ -ball contains at most one element of (x_n) .

⟨3⟩6. (x_n) has no convergent subsequence.

⟨3⟩7. Q.E.D.

PROOF: This contradicts ⟨2⟩1.

- (2)4. LET: \mathcal{A} be an open covering of X
 (2)5. PICK a Lebesgue number δ for \mathcal{A}
 PROOF: By (2)2.
 (2)6. PICK a finite covering $\{B_1, \dots, B_n\}$ of X by $\delta/3$ -balls.
 PROOF: By (2)3.
 (2)7. For $1 \leq i \leq n$, PICK $U_i \in \mathcal{A}$ such that $B_i \subseteq U_i$
 (2)8. $\{U_1, \dots, U_n\}$ covers X .

□

Corollary 10.3.3.1. S_Ω is not metrizable.

PROOF: It is limit point compact (Corollary 9.4.19.2) but not compact (Proposition 9.4.2). □

Corollary 10.3.3.2. The space \mathbb{R}^ω is not limit point compact.

10.4 Uniform Continuity

Definition 10.4.1 (Uniform Continuity). Let X and Y be metric spaces and $f : X \rightarrow Y$. Then f is *uniformly continuous* iff, for all $\epsilon > 0$, there exists $\delta > 0$ such that, for all $x, y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$.

Theorem 10.4.2 (Uniform Continuity Theorem). Let X be a compact metric space, Y a metric space, and $f : X \rightarrow Y$ be continuous. Then f is uniformly continuous.

PROOF:

- (1)1. LET: $\epsilon > 0$
 PROVE: There exists $\delta > 0$ such that, for all $x, y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$.
 (1)2. LET: $\mathcal{A} = \{f^{-1}(B(y, \epsilon/2)) : y \in Y\}$
 (1)3. \mathcal{A} is an open covering of X
 (1)4. PICK a Lebesgue number δ for \mathcal{A} .
 PROVE: For all $x, y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$
 PROOF: By the Lebesgue Number Lemma
 (1)5. LET: $x, y \in X$ with $d(x, y) < \delta$
 (1)6. $\text{diam}\{x, y\} < \delta$
 (1)7. PICK $z \in Y$ such that $\{x, y\} \subseteq f^{-1}(B(z, \epsilon/2))$
 (1)8. $d(f(x), f(y)) < \epsilon$

□

10.5 Locally Metrizable Spaces

Definition 10.5.1 (Locally Metrizable). A space is *locally metrizable* iff every point has a metrizable neighbourhood.

Proposition 10.5.2. Every metrizable space is locally metrizable.

PROOF: Trivial. \square

Corollary 10.5.2.1. *The space \mathbb{R}^ω is locally metrizable.*

Proposition 10.5.3. *A compact Hausdorff space is metrizable if and only if it is locally metrizable.*

PROOF:

$\langle 1 \rangle 1$. LET: X be a locally metrizable compact Hausdorff space

$\langle 1 \rangle 2$. X is regular

PROOF: Lemma 9.4.18

$\langle 1 \rangle 3$. X is second countable

$\langle 2 \rangle 1$. $\{U : U \text{ open in } X \text{ and metrizable}\}$ covers X

$\langle 2 \rangle 2$. PICK a finite subcover U_1, \dots, U_n

$\langle 2 \rangle 3$. For $1 \leq i \leq n$, PICK a countable basis \mathcal{B}_i of U_i

$\langle 2 \rangle 4$. $\mathcal{B}_1 \cup \dots \cup \mathcal{B}_n$ is a basis for X

$\langle 1 \rangle 4$. Q.E.D.

PROOF: By the Urysohn Metrization Theorem.

\square

Corollary 10.5.3.1. *$\overline{S_\Omega}$ is not locally metrizable.*

Corollary 10.5.3.2. *The ordered square is not locally metrizable.*

Proposition 10.5.4. *Every subspace of a locally metrizable space is locally metrizable.*

PROOF:

$\langle 1 \rangle 1$. LET: X be locally metrizable and $Y \subseteq X$

$\langle 1 \rangle 2$. LET: $y \in Y$

$\langle 1 \rangle 3$. PICK a metrizable neighbourhood U of y in X

$\langle 1 \rangle 4$. $U \cap Y$ is a metrizable neighbourhood of y in Y

\square

Corollary 10.5.4.1. *$S_\Omega \times \overline{S_\Omega}$ is not locally metrizable.*

PROOF: It has a subspace homeomorphic to $\overline{S_\Omega}$. \square

Proposition 10.5.5 (CC). *Every locally metrizable regular Lindelöf space is metrizable.*

PROOF:

$\langle 1 \rangle 1$. LET: X be a locally metrizable regular Lindelöf space.

$\langle 1 \rangle 2$. Every point in X has an open second countable neighbourhood.

$\langle 2 \rangle 1$. LET: $x \in X$

$\langle 2 \rangle 2$. PICK an open metrizable U containing x

PROOF: X is locally metrizable ($\langle 1 \rangle 1$)

$\langle 2 \rangle 3$. PICK an open V such that $x \in V \subseteq \overline{V} \subseteq U$

PROOF: Proposition 6.3.2

$\langle 2 \rangle 4$. \overline{V} is Lindelöf

PROOF: Proposition 13.1.32

⟨2⟩5. \bar{V} is second countable

PROOF: Proposition 10.1.42

⟨1⟩3. PICK a countable covering of second countable open sets \mathcal{U}

PROOF: X is Lindelöf (⟨1⟩1)

⟨1⟩4. For $U \in \mathcal{U}$, PICK a countable basis \mathcal{B}_U

⟨1⟩5. $\bigcup_{U \in \mathcal{U}} \mathcal{B}_U$ is a countable basis for X

⟨2⟩1. LET: $x \in U$ where U is open in X

⟨2⟩2. PICK $V \in \mathcal{U}$ such that $x \in V$

⟨2⟩3. There exists $B \in \mathcal{B}_V$ such that $x \in B \subseteq U \cap V$

⟨1⟩6. Q.E.D.

PROOF: By the Urysohn Metrization Theorem.

□

Corollary 10.5.5.1. \mathbb{R}_l is not locally metrizable.

Proposition 10.5.6. The Sorgenfrey plane is not locally metrizable.

PROOF:

⟨1⟩1. LET: U be any neighbourhood of $(0, 0)$

PROVE: U is not Lindelöf

⟨1⟩2. PICK $a > 0$ such that $[0, a]^2 \subseteq U$

⟨1⟩3. LET: $L = \{(x, a - x) : 0 < x < a\}$

⟨1⟩4. L is closed in U

PROOF: By Lemma 6.5.16 since $(x, y) \mapsto (x, a + y)$ is a homeomorphism of \mathbb{R}_l^2 with itself.

⟨1⟩5. LET: $\mathcal{U} = \{U \setminus L\} \cup \{([x, b] \times [a - x, c]) \cap U : b > a, c > a - x\}$

⟨1⟩6. \mathcal{U} covers U

⟨1⟩7. No countable subset of \mathcal{U} covers U

PROOF: Every set of the form $[x, b] \times [a - x, c]$ intersects L in exactly one point.

□

Corollary 10.5.6.1. The Sorgenfrey plane is not metrizable.

Proposition 10.5.7. The space \mathbb{R}_K is locally metrizable.

PROOF: The set $(-1, 1) - K$ is a metrizable neighbourhood of 0. For any other point p , pick an open interval around p that does not contain 0. □

Proposition 10.5.8. The product of two locally metrizable spaces is locally metrizable.

PROOF:

⟨1⟩1. LET: X and Y be locally metrizable

⟨1⟩2. LET: $(a, b) \in X \times Y$

⟨1⟩3. PICK metrizable neighbourhoods U of a and V of b

⟨1⟩4. $U \times V$ is a metrizable neighbourhood of (a, b) .

PROOF: By Lemma 10.1.19.

□

Proposition 10.5.9. *The product of two locally metrizable spaces is locally metrizable.*

PROOF:

- ⟨1⟩1. LET: X and Y be locally metrizable
- ⟨1⟩2. LET: $(a, b) \in X \times Y$
- ⟨1⟩3. PICK metrizable neighbourhoods U of a and V of b
- ⟨1⟩4. $U \times V$ is a metrizable neighbourhood of (a, b) .

PROOF: By Lemma 10.1.19.

□

Proposition 10.5.10. *The space \mathbb{R}_K^ω is not locally metrizable.*

PROOF: If it were, then there would be a basic open set $\prod_n U_n$ that is metrizable, but then \mathbb{R}_K would be metrizable as it is homeomorphic to a subspace of $\prod_n U_n$.

□

Corollary 10.5.10.1. *The product of a countable family of locally metrizable spaces is not necessarily locally metrizable.*

Proposition 10.5.11. *The continuous image of a locally metrizable space is not necessarily locally metrizable.*

PROOF: The identity map from the discrete two-point space to the indiscrete two-point space is continuous. □

Chapter 11

Manifolds

11.1 Manifolds

Definition 11.1.1 (Manifold). Let $m \geq 1$. An m -manifold is a second countable Hausdorff space such that each point has a neighbourhood that is homeomorphic to an open subset of \mathbb{R}^m .

A *curve* is a 1-manifold and a *surface* is a 2-manifold.

Definition 11.1.2 (Support). Let X be a topological space and $\phi : X \rightarrow \mathbb{R}$ be a function. Then the *support* of ϕ is the closure of $\phi^{-1}(\mathbb{R} \setminus \{0\})$.

Definition 11.1.3 (Partition of Unity). Let X be a topological space. Let $\{U_1, \dots, U_n\}$ be a finite indexed open covering of X . An indexed family of continuous functions $\phi_1, \dots, \phi_n : X \rightarrow [0, 1]$ is a *partition of unity* dominated by $\{U_1, \dots, U_n\}$ iff:

1. $\text{supp } \phi_i \subseteq U_i$ for all i ;
2. $\sum_{i=1}^n \phi_i(x) = 1$ for all $x \in X$.

Theorem 11.1.4 (Existence of Finite Partitions of Unity). *Let X be a normal space. Let $\{U_1, \dots, U_n\}$ be a finite indexed open covering of X . Then there exists a partition of unity dominated by $\{U_1, \dots, U_n\}$.*

PROOF:

- (1)1. For every finite indexed open covering $\{U_1, \dots, U_n\}$ of X , there exists a finite indexed open covering $\{V_1, \dots, V_n\}$ such that $\overline{V_i} \subseteq U_i$
- (2)1. For $1 \leq k \leq n$, there exist open sets V_1, \dots, V_k such that $\overline{V_i} \subseteq U_i$ for all i and $\{V_1, \dots, V_k, U_{k+1}, \dots, U_n\}$ covers X
- (3)1. ASSUME: as an induction hypothesis that $0 \leq k < n$ and there exist open sets V_1, \dots, V_k such that $\overline{V_i} \subseteq U_i$ for all i and $\{V_1, \dots, V_k, U_{k+1}, \dots, U_n\}$ covers X
- (3)2. LET: $A = X \setminus (V_1 \cup \dots \cup V_k) \setminus (U_{k+2} \cup \dots \cup U_n)$
- (3)3. A is closed

PROOF: Since $\{V_1, \dots, V_k, U_{k+1}, \dots, U_n\}$ covers X

PROOF: By Proposition 6.3.2

⟨1⟩2. PICK an open covering $\{V_1, \dots, V_n\}$ with $\overline{V_i} \subseteq U_i$ for all i

$\langle 1 \rangle 3$. PICK an open covering $\{W_1, \dots, W_n\}$ with $\overline{W_i} \subseteq V_i$ for all i

⟨1⟩4. For $1 \leq i \leq n$, PICK a continuous function $\psi_i : X \rightarrow [0, 1]$ such that $\psi_i(\overline{W_i}) = \{1\}$ and $\psi_i(X \setminus V_i) = \{0\}$

(1)5. LET: $\Psi : X \rightarrow \mathbb{R}$ where $\Psi(x) = \sum_{i=1}^n \psi_i(x)$

⟨2⟩1. LET: $x \in X$

$\langle 2 \rangle 3.$ $\psi_i(x) = 1$

LET: $\phi_j(x) = \frac{\psi_j(x)}{\Psi(x)}$

$\langle 2 \rangle 1.$ $\text{supp } \psi_i \subseteq U_i$

PROOF: By $\langle 1 \rangle_4$

PROOF: Proposition 3.10.5

$$\langle 2 \rangle 2. \sum_{i=1}^n \psi_i(x) = 1 \text{ for all } x \in X$$

PROOF:

PROOF: Since $\{U \text{ open in } X : U \text{ can be imbedded in } \mathbb{R}^k \text{ for some } k\}$ covers X .

(1)3. PICK a partition of unity ϕ_1, \dots, ϕ_n dominated by $\{U_1, \dots, U_n\}$

PROOF: By Lemma 9.4.18.

PROOF: Theorem 11.1.4

LET: $A_i = \text{supp } \phi_i$

⟨1⟩5. For $1 \leq i \leq n$,

LET: $h_i : X \rightarrow \mathbb{R}^{k_i}$ be defined by

$$h_i(x) = \begin{cases} \phi_i(x)g_i(x) & \text{for } x \in U_i \\ \vec{0} & \text{for } x \in X \setminus A_i \end{cases}$$

PROOF: If $x \in U_i$ and $x \in X \setminus A_i$ then $x \notin \text{supp } \phi_i$ so $\phi_i(x) = 0$

⟨1⟩6. LET: $N = n + k_1 + \cdots + k_n$

⟨1⟩7. LET: $F : X \rightarrow \mathbb{R}^N$ be the function

$$F(x) = (\phi_1(x), \dots, \phi_n(x), h_1(x), \dots, h_n(x))$$

⟨1⟩8. F is an imbedding

⟨2⟩1. F is continuous

PROOF: Each h_i is continuous by Theorem 5.2.13.

⟨2⟩2. F is injective

⟨3⟩1. ASSUME: $F(x) = F(y)$

⟨3⟩2. PICK i such that $\phi_i(x) > 0$

PROOF: Since $\sum_i \phi_i(x) = 1$ (⟨1⟩3)

⟨3⟩3. $\phi_i(y) = 0$

PROOF: By ⟨3⟩1

⟨3⟩4. $x, y \in U_i$

PROOF: Since $\text{supp } \phi_i \subseteq U_i$

⟨3⟩5. $h_i(x) = h_i(y)$

PROOF: By ⟨3⟩1

⟨3⟩6. $g_i(x) = g_i(y)$

PROOF: By ⟨1⟩5

⟨3⟩7. $x = y$

PROOF: By ⟨1⟩2

⟨2⟩3. Q.E.D.

PROOF: By Theorem 9.4.11

□

Corollary 11.1.5.1. *Every compact manifold can be imbedded in \mathbb{R}^N for some N .*

Proposition 11.1.6. *The line with two origins is a second countable T_1 space where every point has a neighbourhood that is homeomorphic to an open subset of \mathbb{R} , but it is not a 1-manifold.*

Chapter 12

Normed Spaces

12.1 The Norm on \mathbb{R}^n

Definition 12.1.1 (Norm). Given $\vec{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, the *norm* $\|\vec{x}\|$ is defined by

$$\|\vec{x}\| = \sqrt{x_1^2 + \dots + x_n^2} .$$

Definition 12.1.2 (Vector Sum). Define the *sum* of $\vec{x}, \vec{y} \in \mathbb{R}^n$ by

$$\vec{x} + \vec{y} = (x_1 + y_1, \dots, x_n + y_n) .$$

Definition 12.1.3 (Scalar Product). Given $c \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^n$, define the *scalar product* $c\vec{x}$ to be

$$c\vec{x} = (cx_1, \dots, cx_n) .$$

Definition 12.1.4 (Inner Product). The *inner product* of $\vec{x}, \vec{y} \in \mathbb{R}^n$ is

$$\vec{x} \cdot \vec{y} = x_1 y_1 + \dots + x_n y_n .$$

Lemma 12.1.5.

$$\vec{x} \cdot (\vec{y} + \vec{z}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z}$$

PROOF: Both are equal to $\sum_{i=1}^n (x_i y_i + x_i z_i)$. \square

Lemma 12.1.6.

$$|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \|\vec{y}\|$$

PROOF:

$\langle 1 \rangle 1$. CASE: $\vec{x} = \vec{0}$ or $\vec{y} = \vec{0}$

PROOF: In this case, both sides are 0.

$\langle 1 \rangle 2$. CASE: $\vec{x} \neq \vec{0} \neq \vec{y}$

$\langle 2 \rangle 1$. LET: $a = 1/\|\vec{x}\|$, $b = 1/\|\vec{y}\|$

$\langle 2 \rangle 2$. $2 + 2ab\vec{x} \cdot \vec{y} \geq 0$

$\langle 3 \rangle 1$. $\|a\vec{x} + b\vec{y}\|^2 \geq 0$

- $\langle 3 \rangle 2. \sum_{i=1}^n (ax_i + by_i)^2 \geq 0$
 $\langle 3 \rangle 3. a^2 \sum_{i=1}^n x_i^2 + b^2 \sum_{i=1}^n y_i^2 + 2ab \sum_{i=1}^n x_i y_i \geq 0$
 $\langle 3 \rangle 4. a^2 \|\vec{x}\|^2 + b^2 \|\vec{y}\|^2 + 2ab \vec{x} \cdot \vec{y} \geq 0$
 $\langle 2 \rangle 3. 2 - 2ab \vec{x} \cdot \vec{y} \geq 0$
 PROOF: Similar.
 $\langle 2 \rangle 4. 2 - 2ab |\vec{x} \cdot \vec{y}| \geq 0$
 PROOF: From $\langle 2 \rangle 2$ and $\langle 2 \rangle 3$.
 $\langle 2 \rangle 5. |\vec{x} \cdot \vec{y}| \leq 1/ab$
 \square

Lemma 12.1.7.

$$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$$

PROOF:

$$\begin{aligned}
 \|\vec{x} + \vec{y}\|^2 &= (\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y}) \\
 &= \|\vec{x}\|^2 + 2\vec{x} \cdot \vec{y} + \|\vec{y}\|^2 && \text{(Lemma 12.1.5)} \\
 &\leq \|\vec{x}\|^2 + 2\|\vec{x}\|\|\vec{y}\| + \|\vec{y}\|^2 && \text{(Lemma 12.1.6)} \\
 &= (\|\vec{x}\| + \|\vec{y}\|)^2 && \square
 \end{aligned}$$

Definition 12.1.8 (Euclidean Metric). The *euclidean metric* on \mathbb{R}^n is given by

$$d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|$$

We prove this is a metric.

PROOF:

- $\langle 1 \rangle 1. d(\vec{x}, \vec{y}) \geq 0$
 PROOF: Immediate from definitions.
 $\langle 1 \rangle 2. d(\vec{x}, \vec{y}) = 0$ iff $\vec{x} = \vec{y}$
 PROOF: Immediate from definitions.
 $\langle 1 \rangle 3. d(\vec{x}, \vec{y}) = d(\vec{y}, \vec{x})$
 PROOF: Immediate from definitions.
 $\langle 1 \rangle 4. d(\vec{x}, \vec{z}) \leq d(\vec{x}, \vec{y}) + d(\vec{y}, \vec{z})$
 PROOF: From Lemma 12.1.7.
 \square

Lemma 12.1.9. Let d be the euclidean topology on \mathbb{R}^n and ρ the square topology. Then, for all $x, y \in \mathbb{R}^n$, we have

$$\rho(x, y) \leq d(x, y) \leq \sqrt{n}\rho(x, y)$$

PROOF:

- $\langle 1 \rangle 1. \rho(x, y) \leq d(x, y)$
 $\langle 2 \rangle 1.$ For $1 \leq i \leq n$ we have $|x_i - y_i| \leq d(x, y)$
 PROOF: By the definition of the euclidean metric.
 $\langle 2 \rangle 2.$ Q.E.D.
 PROOF: By the definition of the square metric.

(1)2. $d(x, y) \leq \sqrt{n}\rho(x, y)$

PROOF:

$$\begin{aligned} d(x, y) &= \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2} \\ &\leq \sqrt{\rho(x, y)^2 + \cdots + \rho(x, y)^2} \\ &= \sqrt{n\rho(x, y)^2} \\ &= \sqrt{n}\rho(x, y) \end{aligned}$$

□

Corollary 12.1.9.1. *The euclidean metric induces the standard topology on \mathbb{R}^n .*

Definition 12.1.10. Let l_2 be the set of sequences $\vec{a} \in \mathbb{R}^\omega$ such that $\sum_{n=1}^\infty a_n^2 < \infty$.

Lemma 12.1.11. *If $\vec{a}, \vec{b} \in l_2$ then $\sum_{n=1}^\infty |a_n b_n| < \infty$.*

PROOF:

$$\begin{aligned} \sum_{n=1}^N |a_n b_n| &\leq \sqrt{\left(\sum_{n=1}^N a_n^2\right)\left(\sum_{n=1}^N b_n^2\right)} && \text{(Lemma 12.1.6)} \\ &\rightarrow \sqrt{\sum_{n=1}^\infty a_n^2 \sum_{n=1}^\infty b_n^2} \text{ as } n \rightarrow \infty \end{aligned}$$

□

Lemma 12.1.12. *If $\vec{a}, \vec{b} \in l_2$ then $\vec{a} + \vec{b} \in l_2$.*

PROOF:

$$\begin{aligned} \sum_{n=1}^\infty (a_n + b_n)^2 &= \sum_{n=1}^\infty a_n^2 + 2 \sum_{n=1}^\infty a_n b_n + \sum_{n=1}^\infty b_n^2 \\ &\leq \sum_{n=1}^\infty a_n^2 + 2 \sum_{n=1}^\infty |a_n b_n| + \sum_{n=1}^\infty b_n^2 \\ &< \infty && \text{(Lemma 12.1.11)} \end{aligned}$$

□

Lemma 12.1.13. *If $c \in \mathbb{R}$ and $\vec{a} \in l_2$ then $c\vec{a} \in l_2$.*

PROOF: $\sum_{n=1}^\infty (ca_n)^2 = c^2 \sum_{n=1}^\infty a_n^2$. □

Definition 12.1.14 (The l^2 -metric). The l^2 -metric is defined on l_2 by

$$d(\vec{a}, \vec{b}) = \left[\sum_{n=1}^\infty (a_n - b_n)^2 \right]^{\frac{1}{2}}.$$

The topology induced by this metric is the l^2 -topology. We write l_2 for this set under the l^2 -topology.

We prove this is a metric.

PROOF:

⟨1⟩1. $d(\vec{a}, \vec{b}) \geq 0$

PROOF: Immediate from definitions.

⟨1⟩2. $d(\vec{a}, \vec{b}) = 0$ iff $\vec{a} = \vec{b}$

PROOF: Immediate from definitions.

⟨1⟩3. $d(\vec{a}, \vec{b}) = d(\vec{b}, \vec{a})$

PROOF: Immediate from definitions.

⟨1⟩4. $d(\vec{a}, \vec{c}) \leq d(\vec{a}, \vec{b}) + d(\vec{b}, \vec{c})$

PROOF: $\sqrt{\sum_{i=1}^N (a_i - c_i)^2} \leq \sqrt{\sum_{i=1}^N (a_i - b_i)^2} + \sqrt{\sum_{i=1}^N (b_i - c_i)^2}$ since the euclidean metric on \mathbb{R}^N is a metric.

□

Definition 12.1.15 (Hilbert Cube). The *Hilbert cube* is $\prod_{n=1}^{\infty} [0, 1/n]$ as a subspace of the l_2 .

Definition 12.1.16 (Isometric Imbedding). Let X, Y be metric spaces and $f : X \rightarrow Y$. Then f is an *isometric imbedding* iff, for all $x, y \in X$, $d(f(x), f(y)) = d(x, y)$.

Lemma 12.1.17. *Every isometric imbedding is an imbedding.*

PROOF:

⟨1⟩1. LET: $f : X \rightarrow Y$ be an isometric imbedding.

⟨1⟩2. f is continuous.

PROOF: If $d(x, y) < \epsilon$ then $d(f(x), f(y)) < \epsilon$.

⟨1⟩3. f is injective.

PROOF: If $f(x) = f(y)$ then $d(f(x), f(y)) = 0$ so $d(x, y) = 0$ hence $x = y$.

⟨1⟩4. $f^{-1} : f(X) \rightarrow X$ is continuous.

PROOF: If $d(f^{-1}(x), f^{-1}(y)) < \epsilon$ then $d(x, y) < \epsilon$.

□

Chapter 13

Topological Groups

13.1 Topological Groups

Definition 13.1.1 (Topological Group). A *topological group* G consists of a group G that is also a T_1 space such that $\cdot : G^2 \rightarrow G$ and $(\)^{-1} : G \rightarrow G$ are continuous.

Proposition 13.1.2. *Every topological group is homogeneous.*

PROOF:

- $\langle 1 \rangle 1$. LET: G be a topological group.
- $\langle 1 \rangle 2$. LET: $x, y \in G$
- $\langle 1 \rangle 3$. LET: $f : G \rightarrow G$ be given by $f(g) = yx^{-1}g$
- $\langle 1 \rangle 4$. f is a homeomorphism
- $\langle 1 \rangle 5$. $f(x) = y$

□

Definition 13.1.3 (Symmetric). Let G be a topological group. A neighbourhood U of e is *symmetric* iff $U = U^{-1}$.

Proposition 13.1.4. *For every neighbourhood U of e , there exists a symmetric neighbourhood V of e such that $VV \subseteq U$.*

PROOF:

- $\langle 1 \rangle 1$. LET: $m : G^2 \rightarrow G$ be the multiplication function
- $\langle 1 \rangle 2$. $ee \in U$
- $\langle 1 \rangle 3$. $(e, e) \in m^{-1}(U)$
- $\langle 1 \rangle 4$. PICK neighbourhoods U_1, U_2 of e such that $(e, e) \in U_1 \times U_2 \subseteq m^{-1}(U)$
- $\langle 1 \rangle 5$. LET: $V' = U_1 \cap U_2$
- $\langle 1 \rangle 6$. $V'V' \subseteq U$
- $\langle 1 \rangle 7$. LET: $f : G^2 \rightarrow G$ be the function $f(x, y) = xy^{-1}$
- $\langle 1 \rangle 8$. $(e, e) \in f^{-1}(V')$
- $\langle 1 \rangle 9$. PICK a neighbourhood W of e such that $WW^{-1} \subseteq V'$
- $\langle 1 \rangle 10$. LET: $V = WW^{-1}$

- ⟨1⟩11. V is a neighbourhood of e
 PROOF: V is open because $V = \bigcup_{a \in W^{-1}} Wa$.
 ⟨1⟩12. V is symmetric
 ⟨1⟩13. $VV \subseteq U$
 □

Proposition 13.1.5. *Every topological group is regular.*

PROOF:

- ⟨1⟩1. LET: G be a topological group
 ⟨1⟩2. LET: $A \subseteq G$ be closed and $a \notin A$
 ⟨1⟩3. $G \setminus Aa^{-1}$ is a neighbourhood of e
 ⟨1⟩4. PICK a symmetric neighbourhood V of e such that $VV \subseteq G \setminus Aa^{-1}$
 PROOF: Proposition 13.1.4.
 ⟨1⟩5. VA and Va are disjoint neighbourhoods of A and a
 □

Proposition 13.1.6. *The long line is not second countable.*

PROOF: Let \mathcal{B} be a basis for L . Then, for every countable ordinal α , \mathcal{B} must contain a basic open set that contains $(\alpha, 1/2)$ but not $(\beta, 1/2)$ for any other β . Therefore, \mathcal{B} is uncountable. □

Corollary 13.1.6.1. *The long line cannot be imbedded in \mathbb{R} .*

Theorem 13.1.7. *Let $f : X \rightarrow Y$. Let Y be compact Hausdorff. Then f is continuous if and only if the graph of f is closed in $X \times Y$.*

PROOF:

- ⟨1⟩1. LET: G_f be the graph of f .
 ⟨1⟩2. If f is continuous then the graph of f is closed.
 ⟨2⟩1. ASSUME: f is continuous.
 ⟨2⟩2. LET: $(x, y) \in (X \times Y) \setminus G_f$
 ⟨2⟩3. $y \neq f(x)$
 ⟨2⟩4. PICK disjoint open neighbourhoods U of $f(x)$ and V of y
 PROOF: Y is Hausdorff.
 ⟨2⟩5. $(x, y) \in f^{-1}(U) \times V \subseteq (X \times Y) \setminus G_f$
 ⟨2⟩6. Q.E.D.
 ⟨1⟩3. If the graph of f is closed then f is continuous.
 ⟨2⟩1. ASSUME: G_f is closed.
 ⟨2⟩2. LET: $x_0 \in X$ and V be an open neighbourhood of $f(x_0)$
 ⟨2⟩3. $G_f \cap (X \times (Y \setminus V))$ is closed
 ⟨2⟩4. $\pi_1(G_f \cap (X \times (Y \setminus V)))$ is closed
 PROOF: Lemma 9.4.16
 ⟨2⟩5. $x_0 \in X \setminus \pi_1(G_f \cap (X \times (Y \setminus V))) \subseteq f^{-1}(V)$
 ⟨2⟩6. Q.E.D.
 □

Theorem 13.1.8. *Let X be a compact Hausdorff space. Let \mathcal{A} be a set of closed connected subspaces of X that is linearly ordered by proper inclusion. Then*

$$Y = \bigcap \mathcal{A}$$

is connected.

PROOF:

⟨1⟩1. ASSUME: for a contradiction C and D form a separation of Y

⟨1⟩2. PICK disjoint U and V open in X such that $C = U \cap Y$ and $D = V \cap Y$

⟨2⟩1. C and D are compact

⟨3⟩1. Y is compact

PROOF: Y is a closed subset of X , hence compact by Proposition 9.4.6.

⟨3⟩2. Q.E.D.

PROOF: C and D are closed subsets of Y hence compact by Proposition 9.4.6.

⟨2⟩2. Q.E.D.

PROOF: By Lemma 9.4.18.

⟨1⟩3. For all $A \in \mathcal{A}$, we have $A \setminus (U \cup V)$ is nonempty

PROOF: Since A is connected.

⟨1⟩4. $\{A \setminus (U \cup V) : A \in \mathcal{A}\}$ has the finite intersection property

PROOF: This holds because \mathcal{A} is linearly ordered under proper inclusion.

⟨1⟩5. $\bigcap_{A \in \mathcal{A}} (A \setminus (U \cup V))$ is nonempty

PROOF: By Proposition 9.4.15.

□

Theorem 13.1.9. *Let $A \subseteq \mathbb{R}^n$. Then the following are equivalent:*

1. A is compact.
2. A is closed and bounded under the euclidean metric.
3. A is closed and bounded under the square metric.

PROOF:

⟨1⟩1. $1 \Rightarrow 2$

⟨2⟩1. ASSUME: A is compact.

⟨2⟩2. A is closed.

PROOF: By Proposition 9.4.9.

⟨2⟩3. $\{B(\vec{0}, n) : n \in \mathbb{Z}^+\}$ covers A

⟨2⟩4. PICK a finite subcover $\{B(\vec{0}, n_1), \dots, B(\vec{0}, n_k)\}$

⟨2⟩5. LET: $N = \max(n_1, \dots, n_k)$

⟨2⟩6. For all $x, y \in A$ we have $d(x, y) < 2N$

PROOF: We have $d(x, y) \leq d(\vec{0}, x) + d(\vec{0}, y) < N + N$.

⟨1⟩2. $2 \Rightarrow 3$

PROOF: If $d(x, y) < \epsilon$ for all $x, y \in A$ then $\rho(x, y) < \epsilon\sqrt{n}$ by Lemma 12.1.9.

⟨1⟩3. $3 \Rightarrow 1$

⟨2⟩1. ASSUME: A is closed and $\rho(x, y) < \epsilon$ for all $x, y \in A$

- (2)2. PICK $x_0 \in A$
 (2)3. LET: $b = \rho(\vec{0}, x_0)$
 (2)4. LET: $P = \epsilon + b$
 (2)5. $A \subseteq [-P, P]^n$
 PROOF: For any $y \in A$ we have

$$\begin{aligned} \rho(\vec{0}, y) &\leq \rho(\vec{0}, x_0) + \rho(x_0, y) && \text{(Triangle Inequality)} \\ &< b + \epsilon && ((2)3, (2)1) \\ &= P && ((2)4) \end{aligned}$$

 (2)6. $[-P, P]^n$ is compact.
 PROOF: By Corollary 9.4.19.1 and Proposition 9.4.14.
 (2)7. Q.E.D.
 PROOF: By Proposition 9.4.6.

□

Theorem 13.1.10 (AC). *Let X be a topological space. Then X is compact if and only if every nonempty net in X has a convergent subnet.*

PROOF:

- (1)1. If X is compact then every nonempty net in X has a convergent subnet.
 (2)1. ASSUME: X is compact.
 (2)2. LET: $(x_\alpha)_{\alpha \in J}$ be a nonempty net in X
 (2)3. For $\alpha \in J$,
 LET: $B_\alpha = \{\beta \in J : \alpha \leq \beta\}$.
 (2)4. $\{B_\alpha : \alpha \in J\}$ has the finite intersection property.
 (3)1. LET: $\alpha_1, \dots, \alpha_n \in J$
 (3)2. PICK $\beta \in J$ such that $\alpha_1 \leq \beta, \dots, \alpha_n \leq \beta$
 (3)3. $x_\beta \in B_{\alpha_1} \cap \dots \cap B_{\alpha_n}$
 (2)5. PICK $l \in \bigcap_{\alpha \in J} B_\alpha$
 PROOF: Proposition 9.4.15.
 (2)6. LET: $K = \{\alpha \in J : x_\alpha = l\}$
 (2)7. K is cofinal in J
 (3)1. LET: $\alpha \in J$
 (3)2. $l \in B_\alpha$
 PROOF: By (2)5.
 (3)3. There exists $\beta \geq \alpha$ such that $x_\beta = l$.
 (2)8. $(x_\alpha)_{\alpha \in K}$ is a subnet of $(x_\alpha)_{\alpha \in J}$ that converges to l .
 (1)2. If every nonempty net in X has a convergent subnet then X is compact.
 (2)1. ASSUME: Every nonempty net in X has a convergent subnet
 (2)2. LET: \mathcal{A} be a nonempty set of closed sets with the finite intersection property.
 (2)3. LET: J be the poset of all finite intersections of elements of \mathcal{A} under \supseteq
 (2)4. PICK $x_C \in C$ for all $C \in J$
 PROOF: These are all nonempty by (2)2.
 (2)5. PICK an accumulation point l of (x_C)
 PROVE: $l \in \bigcap \mathcal{A}$
 PROOF: One exists by Lemma 3.16.2.

- (2)6. LET: $C \in \mathcal{A}$
 PROVE: $l \in C$
 (2)7. LET: U be a neighbourhood of l
 PROVE: U intersects C
 (2)8. PICK $D \subseteq C$ such that $x_D \in U$
 PROOF: By (2)5.
 (2)9. U intersects C
 (2)10. $l \in C$
 PROOF: By Theorem 3.11.3 since C is closed ((2)2).
 (2)11. Q.E.D.
 PROOF: Proposition 9.4.15.

□

Corollary 13.1.10.1 (AC). *Let G be a topological group. Let A and B be subsets of G . If A is closed in G and B is compact then AB is closed in G .*

PROOF:

- (1)1. LET: $c \in \overline{AB}$
 PROVE: $c \in AB$
 (1)2. PICK a net $(x_\alpha)_{\alpha \in J}$ that converges to c
 PROOF: By Theorem 3.15.3.
 (1)3. For $\alpha \in J$, PICK $a_\alpha \in A$ and $b_\alpha \in B$ such that $x_\alpha = a_\alpha b_\alpha$
 (1)4. PICK a convergent subnet $(b_{g(\beta)})_{\beta \in K}$ of $(b_\alpha)_{\alpha \in J}$
 PROOF: By Theorem 13.1.10.
 (1)5. LET: $b_{g(\beta)} \rightarrow b$
 (1)6. $b \in B$
 (2)1. B is closed
 PROOF: By Proposition 9.4.9.
 (2)2. Q.E.D.
 PROOF: By Theorem 3.15.3
 (1)7. $a_{g(\beta)} \rightarrow cb^{-1}$
 PROOF: By Theorem 3.15.4
 (1)8. $cb^{-1} \in A$
 PROOF: By Theorem 3.15.3
 (1)9. $c \in AB$
 (1)10. Q.E.D.
 PROOF: By Proposition 3.10.6.

Proposition 13.1.11. *Let $A_0 + A_1$ be the sum of A_0 and A_1 with injections $i_0 : A_0 \rightarrow A_0 + A_1$ and $i_1 : A_1 \rightarrow A_0 + A_1$.*

Let $g : B \rightarrow A_0 + A_1$ be a function.

Let B_0 be the pullback of i_0 and g with projections $j_0 : B_0 \rightarrow B$ and $k_0 : B_0 \rightarrow A_0$.

Let B_1 be the pullback of i_1 and g with projection $sj_1 : B_1 \rightarrow B$ and $k_1 : B_1 \rightarrow A_1$.

Then B is the sum of B_0 and B_1 with injections j_0 and j_1 .

$$\begin{array}{ccccc}
B_0 & \xrightarrow{j_0} & B & \xleftarrow{j_1} & B_1 \\
\downarrow & & \downarrow g & & \downarrow \\
A_0 & \xrightarrow{i_0} & A_0 + A_1 & \xleftarrow{i_1} & A_1
\end{array}$$

PROOF:

(1)1. LET: X be any set and $x : B_0 \rightarrow X, y : B_1 \rightarrow X$

Proposition 13.1.12 (CC). *Let X be a space and \mathcal{B} be a basis for X . Suppose that every subset of \mathcal{B} that covers X has a countable subcover. Then X is Lindelöf.*

PROOF:

- (1)1. LET: \mathcal{A} be an open cover of X .
- (1)2. $\{B \in \mathcal{B} : \exists U \in \mathcal{A}. B \subseteq U\}$ covers X .
- (1)3. PICK a countable subcover \mathcal{B}_0
- (1)4. For $B \in \mathcal{B}_0$, PICK $U_B \in \mathcal{A}$ such that $B \subseteq U_B$
- (1)5. $\{U_B : B \in \mathcal{B}_0\}$ is a countable subcover of \mathcal{A} .

□

Proposition 13.1.13 (CC). *The space \mathbb{R}_l is Lindelöf.*

PROOF:

- (1)1. LET: \mathcal{A} be a set of basis elements $[a, b)$ that covers X
 PROVE: \mathcal{A} has a countable subcover.
- (1)2. LET: $C = \bigcup \{(a, b) : [a, b) \in \mathcal{A}\}$
- (1)3. $\mathbb{R} \setminus C$ is countable.
 - (2)1. For all $x \in \mathbb{R} \setminus C$, PICK a rational q_x such that there exists b such that $q_x \in [x, b) \in \mathcal{A}$
 - (3)1. PICK $[a, b) \in \mathcal{A}$ such that $x \in [a, b)$
 - (3)2. $x = a$
 PROOF: If not we would have $x \in C$
 - (3)3. There exists a rational in (a, b)
 - (2)2. For $x, y \in \mathbb{R} \setminus C$, if $x < y$ then $q_x < q_y$
 - (3)1. PICK b, c such that $q_x \in [x, b) \in \mathcal{A}$ and $q_y \in [y, c) \in \mathcal{A}$
 PROOF: By (2)1.
 - (3)2. $b \leq y$
 PROOF: Otherwise we would have $y \in (x, b) \subseteq C$.
 - (3)3. $q_x < q_y$
 PROOF: $q_x < b \leq y \leq q_y$
 - (2)3. The map $q_- : \mathbb{R} \setminus C \rightarrow \mathbb{Q}$ is injective.
- (1)4. For $x \in \mathbb{R} \setminus C$, PICK $[a_x, b_x) \in \mathcal{A}$ such that $a_x \leq x < b_x$
- (1)5. PICK a countable subset $((a_n, b_n))_{n \in \mathbb{Z}^+}$ of $\{(a, b) : [a, b) \in \mathcal{A}\}$ that covers C
 - (2)1. The set C as a subspace of \mathbb{R} with the standard topology is second countable.

⟨2⟩2. The set C as a subspace of \mathbb{R} with the standard topology is Lindelöf.

PROOF: By Theorem 9.3.2.

⟨1⟩6. $\{[a_x, b_x) : x \in \mathbb{R} \setminus C\} \cup \{[a_n, b_n) : n \in \mathbb{Z}^+\}$ is a countable subcover of \mathcal{A} .

⟨1⟩7. Q.E.D.

PROOF: By Proposition 13.1.12.

□

Proposition 13.1.14 (AC). *The space \mathbb{R}_l is not second countable.*

PROOF:

⟨1⟩1. LET: \mathcal{B} be any basis for \mathbb{R}_l

⟨1⟩2. For $x \in \mathbb{R}$, PICK $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq [x, x+1)$

⟨1⟩3. The mapping $B_{(-)}$ is an injective function $\mathbb{R} \rightarrow \mathcal{B}$

PROOF: For any x we have $x = \min B_x$.

⟨1⟩4. \mathcal{B} is uncountable.

□

Proposition 13.1.15. *The product of a Lindelöf space and a compact space is Lindelöf.*

PROOF:

⟨1⟩1. LET: X be a Lindelöf space and Y a compact space.

⟨1⟩2. LET: \mathcal{A} be an open covering of $X \times Y$

⟨1⟩3. For all $x \in X$, there exists a neighbourhood W of x such that $W \times Y$ is covered by finitely many elements of \mathcal{A} .

⟨2⟩1. LET: $x \in X$

⟨2⟩2. $\{x\} \times Y$ is compact.

PROOF: It is homeomorphic to Y .

⟨2⟩3. PICK a finite subset $\{U_1, \dots, U_m\}$ of \mathcal{A} that covers $\{x\} \times Y$

PROOF: By Proposition 9.4.5.

⟨2⟩4. There exists a neighbourhood W of x such that $W \times Y \subseteq U_1 \cup \dots \cup U_m$

PROOF: By the Tube Lemma.

⟨1⟩4. $\{W \text{ open in } X : W \times Y \text{ is covered by finitely many elements of } \mathcal{A}\}$ is an open covering of X .

⟨1⟩5. PICK a countable subcovering $\{W_1, W_2, \dots\}$

⟨1⟩6. For $i \geq 1$, PICK a finite subset $\{U_{i1}, \dots, U_{ir_i}\}$ of \mathcal{A} that covers $W_i \times Y$

⟨1⟩7. $\{U_{1j} : i \geq 1, 1 \leq j \leq r_i\}$ is a countable subcovering of \mathcal{A} .

□

Proposition 13.1.16. *Let X be a T_1 space. Then X is normal if and only if, for every closed set A and open set $U \supseteq A$, there exists an open set $V \supseteq A$ such that $\bar{V} \subseteq U$.*

PROOF:

⟨1⟩1. If X is normal then, for every closed set A and open set $U \supseteq A$, there exists an open set $V \supseteq A$ such that $\bar{V} \subseteq U$.

⟨2⟩1. ASSUME: X is normal.

⟨2⟩2. LET: A be a closed set and U an open set with $A \subseteq U$

- (2)3. PICK disjoint open sets V, W such that $A \subseteq V$ and $X \setminus U \subseteq W$
 (2)4. $\overline{V} \subseteq U$

PROOF:

$$\begin{aligned}\overline{V} &\subseteq X \setminus W \\ &\subseteq U\end{aligned}$$

- (1)2. If, for every closed set A and open set $U \supseteq A$, there exists an open set $V \supseteq A$ such that $\overline{V} \subseteq U$, then X is normal.
 (2)1. ASSUME: for every closed set A and open set $U \supseteq A$, there exists an open set $V \supseteq A$ such that $\overline{V} \subseteq U$.
 (2)2. LET: A, B be disjoint closed sets
 (2)3. PICK an open set V such that $A \subseteq V$ and $\overline{V} \subseteq X \setminus B$
 (2)4. $A \subseteq V$ and $B \subseteq X \setminus \overline{V}$

□

Definition 13.1.17 (Action). Let G be a topological group and X a topological space. An *action* of G on X is a continuous function $\cdot : G \times X \rightarrow X$ such that, for all $g, h \in G$ and $x \in X$:

1. $e \cdot x = x$
2. $g \cdot (h \cdot x) = gh \cdot x$

Definition 13.1.18 (Orbit Space). Let G be a topological group, X a topological space, and $\cdot : G \times X \rightarrow X$ an action of G on X . Then the *orbit space* X/G is the quotient space of X by the equivalence relation \sim generated by $x \sim g \cdot x$ for all $x \in X, g \in G$.

Theorem 13.1.19. Let G be a topological group. Let X be a topological space. Let $\cdot : G \times X \rightarrow X$ be an action of G on X . Then the canonical map $\pi : X \rightarrow X/G$ is perfect.

- (1)1. π is closed.
 (2)1. LET: $A \subseteq X$ be closed.
 (2)2. $GA = \{g \cdot a : g \in G, a \in A\}$ is closed
 (3)1. LET: $z \notin GA$
 (3)2. For all $g \in G$ we have $g \cdot z \notin A$
 (3)3. For $g \in G$, there exist U an open neighbourhood of g and V an open neighbourhood of z such that UV does not intersect A
 (3)4. $\{U \text{ open in } G : \exists V \text{ an open neighbourhood of } z. UV \cap A = \emptyset\}$ covers G
 (3)5. PICK a finite subcover $\{U_1, \dots, U_n\}$
 (3)6. For $1 \leq i \leq n$, PICK V_i an open neighbourhood of z such that $U_i V_i \cap A = \emptyset$
 (3)7. $z \in V_1 \cap \dots \cap V_n \subseteq X \setminus GA$
 (2)3. $\pi(A)$ is closed
 $\pi^{-1}(\pi(A)) = GA$
 (1)2. π is continuous.
 PROOF: By definition of the quotient topology.

⟨1⟩3. π is surjective.

PROOF: By definition.

⟨1⟩4. For all $a \in X/G$ we have $\pi^{-1}(a)$ is compact.

⟨2⟩1. LET: $a \in X/G$

⟨2⟩2. PICK $x \in X$ such that $a = \pi(x)$

⟨2⟩3. $\pi^{-1}(a) = \{gx : g \in G\}$

⟨2⟩4. $\pi^{-1}(a)$ is homeomorphic to G

□

Corollary 13.1.19.1. *If X is Hausdorff then so is X/G .*

Corollary 13.1.19.2. *If X is regular then so is X/G .*

Corollary 13.1.19.3. *If X is normal then so is X/G .*

Corollary 13.1.19.4. *If X is locally compact then so is X/G .*

Corollary 13.1.19.5. *If X is second countable then so is X/G .*

Proposition 13.1.20. *Let $p : X \twoheadrightarrow Y$ be perfect. If X is second countable then so is Y .*

PROOF:

⟨1⟩1. PICK a countable basis \mathcal{B} for X

⟨1⟩2. LET: $\mathcal{J} = \{J \subseteq^{\text{fin}} \mathcal{B} : \exists W \text{ open in } Y. p^{-1}(W) \subseteq \bigcup J\}$

⟨1⟩3. For every $J \in \mathcal{J}$,

LET: $W_J = \bigcup \{W \text{ open in } Y : p^{-1}(W) \subseteq \bigcup J\}$.

PROVE: $\{W_J : J \in \mathcal{J}\}$ is a basis for Y .

⟨1⟩4. $y \in V$ where V is open in Y

⟨1⟩5. $\{B \in \mathcal{B} : x \in B \subseteq p^{-1}(V)\}$ covers $p^{-1}(y)$

⟨1⟩6. PICK a countable subcover $J \subseteq^{\text{fin}} \mathcal{B}$

⟨1⟩7. $y \in W_J \subseteq V$

⟨2⟩1. $p^{-1}(y) \subseteq \bigcup J$

⟨2⟩2. PICK an open neighbourhood W of y such that $p^{-1}(W) \subseteq \bigcup J$

PROOF: By Proposition 9.5.1.

⟨2⟩3. $W \subseteq W_J$

□

Proposition 13.1.21. *A subspace of a T_1 space is T_1 .*

PROOF:

⟨1⟩1. LET: X be T_1 and $Y \subseteq X$

⟨1⟩2. LET: $a \in Y$

⟨1⟩3. $\{a\}$ is closed in X

⟨1⟩4. $\{a\}$ is closed in Y

PROOF: By Corollary 4.3.4.1.

□

Proposition 13.1.22 (DC). *Not every topological group is normal.*

PROOF: From Proposition 6.5.6. \square

Theorem 13.1.23. *A subspace of a completely regular space is completely regular.*

PROOF:

- $\langle 1 \rangle 1$. LET: X be completely regular and $Y \subseteq X$
- $\langle 1 \rangle 2$. LET: $a \in Y$ and A be closed in Y such that $a \notin A$
- $\langle 1 \rangle 3$. PICK C closed in X such that $A = X \cap C$
- $\langle 1 \rangle 4$. PICK a continuous function $f : X \rightarrow [0, 1]$ such that $f(a) = 0$ and $f(C) = \{1\}$
- $\langle 1 \rangle 5$. $f \upharpoonright Y : Y \rightarrow [0, 1]$ is a continuous function such that $(f \upharpoonright Y)(a) = 0$ and $(f \upharpoonright Y)(A) = \{1\}$

\square

Proposition 13.1.24 (DC). *Every topological group is completely regular.*

PROOF:

- $\langle 1 \rangle 1$. LET: G be a topological group
- $\langle 1 \rangle 2$. LET: $x \in G$ and $A \subseteq G$ be closed such that $x \notin A$
 PROVE: There exists a continuous $f : G \rightarrow [0, 1]$ such that $f(x) = 0$ and $f(A) = \{1\}$
- $\langle 1 \rangle 3$. ASSUME: w.l.o.g. $x = e$
 PROOF: $\lambda y.x^{-1}y$ is an automorphism of G that maps x to e .
- $\langle 1 \rangle 4$. PICK a sequence V_n ($n \geq 0$) of symmetric neighbourhoods of e disjoint from A such that $V_n V_n \subseteq V_{n-1}$ for all n
- $\langle 2 \rangle 1$. LET: $V_0 = X \setminus A$
- $\langle 2 \rangle 2$. Given V_n , PICK a symmetric neighbourhood V_{n+1} of e such that $V_{n+1} V_{n+1} \subseteq V_n$

PROOF: By Proposition 13.1.4.

- $\langle 1 \rangle 5$. For every dyadic rational p , define an open set $U(p)$ as follows:

$$\begin{aligned} U(1/2^n) &= V_n & (n \geq 0) \\ U((2k+1)/2^{n+1}) &= V_{n+1}U(k/2^n) & (0 < k < 2^n) \\ U(p) &= \emptyset & (p \leq 0) \\ U(p) &= G & (p > 1) \end{aligned}$$

- $\langle 1 \rangle 6$. For all k and n , we have

$$V_n U(k/2^n) \subseteq U((k+1)/2^n)$$

- $\langle 2 \rangle 1$. $k \leq 0$

PROOF: In this case, $V_n U(k/2^n) = \emptyset$

- $\langle 2 \rangle 2$. $k = 1$ and $n > 0$

PROOF:

$$\begin{aligned} V_n U(1/2^n) &= V_n V_n \\ &\subseteq V_{n-1} \\ &= U(1/2^{n-1}) \end{aligned}$$

- $\langle 2 \rangle 3$. $k = 2a$ for some $0 < a < 2^{n-1}$

PROOF:

$$\begin{aligned} V_n U(2a/2^n) &= V_n U(a/2^{n-1}) \\ &= U(2a+1/2^n) \end{aligned}$$

(2)4. $k = 2a + 1$ for some $0 < a < 2^{n-1}$

PROOF:

$$\begin{aligned} V_n U((2a+1)/2^n) &= V_n V_n U(a/2^{n-1}) \\ &\subseteq V_{n-1} U(a/2^{n-1}) \\ &\subseteq U((a+1)/2^{n-1}) \end{aligned}$$

(2)5. $k \geq 2^n$

PROOF: In this case, $U((k+1)/2^n) = G$.

(1)7. Define $f : G \rightarrow [0, 1]$ by

$$f(x) = \inf\{p : x \in U(p)\}$$

PROOF: This set is nonempty because $x \in U(1)$ and bounded below because if $x \in U(p)$ then $p > 0$.

(1)8. For $n > 0$ we have $\overline{U(k/2^n)} \subseteq V_n U(k/2^n)$

(2)1. LET: $x \in \overline{U(k/2^n)}$

(2)2. $V_n x$ is a neighbourhood of x

(2)3. PICK $y \in V_n x \cap U(k/2^n)$

(2)4. PICK $z \in V_n$ such that $y = zx$

(2)5. $x = z^{-1}y$

(1)9. For p and q dyadic rationals, if $p < q$ then $\overline{U(p)} \subseteq U(q)$

(1)10. If $x \in \overline{U(p)}$ then $f(x) \leq p$

(2)1. For all $q > p$ we have $x \in U(q)$

(2)2. For all $q > p$ we have $f(x) \leq q$

(1)11. If $x \notin U(p)$ then $f(x) \geq p$

PROOF: If $x \notin U(p)$ and $x \in U(q)$ then $q > p$.

(1)12. f is continuous

(2)1. LET: $x_0 \in X$

(2)2. LET: $c < f(x_0) < d$

PROVE: There exist a neighbourhood U of x_0 such that $f(U) \subseteq (c, d)$

(2)3. PICK rational numbers p, q such that $c < p < f(x_0) < q < d$

(2)4. $x \notin \overline{U(p)}$

(2)5. $x \in U(q)$

(2)6. Take $U = U(q) \setminus \overline{U(p)}$

(1)13. $f(e) = 0$

PROOF: We have $e \in U(1/2^n)$ for all n .

(1)14. $f(A) = \{1\}$

PROOF: If $x \in A$ and $x \in U(p)$ then $p > 1$.

□

Definition 13.1.25 (Bijection). A function $f : A \rightarrow B$ is a *bijection*, $f : A \cong B$, iff there exists a function $f^{-1} : B \rightarrow A$, the *inverse* of f , such that $f^{-1} \circ f = \text{id}_A$ and $f \circ f^{-1} = \text{id}_B$.

Theorem 13.1.26. Let Y be a normal space. Then Y is an absolute retract if and only if Y has the universal extension property.

PROOF:

- ⟨1⟩1. If Y is an absolute retract then Y has the universal extension property.
- ⟨2⟩1. ASSUME: Y is an absolute retract.
- ⟨2⟩2. LET: X be a normal space, A a closed subspace of X and $f : A \rightarrow Y$ a continuous function.
- ⟨2⟩3. LET: Z_f be the quotient space of $X \cup Y$ under: $a \sim f(a)$ for all $a \in A$
- ⟨2⟩4. LET: $p : X \cup Y \rightarrow Z_f$ be the quotient map
- ⟨2⟩5. For all $x_1, x_2 \in X$ we have $p(x_1) = p(x_2)$ iff $x_1 = x_2$ or $x_1, x_2 \in A$ and $f(x_1) = f(x_2)$; and for $x \in X$ and $y \in Y$ we have $p(x) = p(y)$ iff $f(x) = y$; and for $y_1, y_2 \in Y$ we have $p(y_1) = p(y_2)$ iff $y_1 = y_2$
- ⟨2⟩6. p imbeds Y into a closed subspace of Z_f
 - ⟨3⟩1. p is injective on Y
 - ⟨3⟩2. $p^{-1} : p(Y) \rightarrow Y$ is continuous
 - ⟨4⟩1. LET: $U \subseteq Y$ be open
PROVE: $p(U)$ is open
 - ⟨4⟩2. $p^{-1}(p(U)) = f^{-1}(U) \cup U$
 - ⟨3⟩3. $p(Y)$ is closed
PROOF: $p^{-1}(p(Y)) = A \cup Y$
- ⟨2⟩7. Z_f is normal
 - ⟨3⟩1. Z_f is T_1
PROOF: For $y \in Y$ we have $p^{-1}(y) = f^{-1}(y) \cup \{y\}$ which is closed.
 - ⟨3⟩2. Any two disjoint closed sets in Z_f can be separated by a continuous function.
 - ⟨4⟩1. LET: C and D be disjoint closed sets in Z_f
 - ⟨4⟩2. PICK $g : Y \rightarrow [0, 1]$ such that $g(Y \cap p^{-1}(C)) = \{0\}$ and $g(Y \cap p^{-1}(D)) = \{1\}$
PROOF: By the Urysohn Lemma.
 - ⟨4⟩3. PICK $h : X \rightarrow [0, 1]$ such that $h(X \cap p^{-1}(C)) = \{0\}$ and $h(X \cap p^{-1}(D)) = \{1\}$ and h agrees with $g \circ f$ on A
PROOF: By the Tietze Extension Theorem applied to $A \cup (X \cap p^{-1}(C)) \cup (X \cap p^{-1}(D))$.
 - ⟨4⟩4. LET: $k : Z_f \rightarrow [0, 1]$ be the continuous function such that $k(p(x)) = h(x)$ for $x \in X$ and $k(p(y)) = g(y)$ for $y \in Y$
PROOF: By the Pasting Lemma
 - ⟨4⟩5. $k(C) = \{0\}$
 - ⟨4⟩6. $k(D) = \{1\}$
 - ⟨3⟩3. Q.E.D.
PROOF: If g is such a continuous function then $g^{-1}([0, 1/2])$ and $g^{-1}((1/2, 1])$ are disjoint open sets that include A and B respectively.
- ⟨2⟩8. PICK a retraction $r : Z_f \rightarrow p(Y)$
- ⟨2⟩9. $p^{-1} \circ r \circ p : X \rightarrow Y$ extends f
- ⟨1⟩2. If Y has the universal extension property then Y is an absolute retract.
 - ⟨2⟩1. ASSUME: Y has the universal extension property
 - ⟨2⟩2. LET: Z be a normal space, Y_0 a closed subspace of Z , and $\phi : Y \cong Y_0$ a homeomorphism
 - ⟨2⟩3. PICK a continuous extension $f : Z \rightarrow Y$ of ϕ^{-1}

□ $\langle 2 \rangle 4$. $\phi \circ f$ is a retraction

Theorem 13.1.27. *Every manifold is metrizable.*

PROOF:

$\langle 1 \rangle 1$. LET: X be an m -manifold.

$\langle 1 \rangle 2$. X is regular.

$\langle 2 \rangle 1$. X is T_1

$\langle 2 \rangle 2$. LET: $x \in X$ and U be a neighbourhood of x

$\langle 2 \rangle 3$. PICK a neighbourhood V of x that is imbeddable in \mathbb{R}^m

$\langle 2 \rangle 4$. PICK a neighbourhood W of x such that $\overline{W} \subseteq U \cap V$

PROOF: One exists since V is regular (Proposition 6.3.4)

$\langle 2 \rangle 5$. $x \in W$ and $\overline{W} \subseteq U$

$\langle 2 \rangle 6$. Q.E.D.

PROOF: Proposition 6.3.2

$\langle 1 \rangle 3$. Q.E.D.

PROOF: By the Urysohn Metrization Theorem.

□

Theorem 13.1.28. *Let X be a compact Hausdorff space in which every point has a neighbourhood that is imbeddable in \mathbb{R}^m . Then X is an m -manifold.*

PROOF:

$\langle 1 \rangle 1$. There exists N such that X is imbeddable in \mathbb{R}^N

PROOF: Theorem 11.1.5

$\langle 1 \rangle 2$. X is second countable.

PROOF: Proposition 7.3.3

□

Proposition 13.1.29. *S_Ω is locally metrizable.*

PROOF: For any $\alpha \in S_\Omega$, the neighbourhood $[0, \alpha] = (-\infty, \alpha + 1)$ is imbeddable in \mathbb{R} . □

Proposition 13.1.30 (DC). *$\overline{S_\Omega}$ is compact.*

PROOF:PROOF:

$\langle 1 \rangle 1$. LET: \mathcal{A} be an open cover of $\overline{S_\Omega}$

$\langle 1 \rangle 2$. ASSUME: for a contradiction there is no finite subcover of \mathcal{A}

$\langle 1 \rangle 3$. There exists a sequence of sets $U_n \in \mathcal{A}$ and ordinals α_n such that $\alpha_{n+1} < \alpha_n$ for all n and $\alpha_n \in U_n$ for all n

$\langle 2 \rangle 1$. LET: $\alpha_1 = \Omega$

$\langle 2 \rangle 2$. Given $\alpha_1, \dots, \alpha_n$ and U_1, \dots, U_{n-1} with $0 \neq \alpha_n < \alpha_{n-1} < \dots < \alpha_1$ and $\alpha_i \in U_i$ for $i < n$, PICK $U_n \in \mathcal{A}$ with $\alpha_n \in U_n$

PROOF: By $\langle 1 \rangle 1$.

$\langle 2 \rangle 3$. PICK $\alpha_{n+1} < \alpha_n$ such that $(\alpha_{n+1}, \alpha_n] \subseteq U_n$

PROOF: By Lemma 4.1.2.

$\langle 2 \rangle 4$. $\alpha_{n+1} \neq 0$

PROOF: If $\alpha_{n+1} = 0$ then U_1, \dots, U_n cover $\overline{S_\Omega}$, contradicting $\langle 1 \rangle 2$.
 $\langle 1 \rangle 4$. Q.E.D.

PROOF: This is a contradiction because the ordinals are well-ordered.
 \square

Proposition 13.1.31. \mathbb{R}_l is not limit point compact.

PROOF: \mathbb{Z} has no limit point. \square

Proposition 13.1.32. Every closed subspace of a Lindelöf space is Lindelöf.

PROOF:

- $\langle 1 \rangle 1$. LET: X be Lindelöf and $A \subseteq X$ be closed
 - $\langle 1 \rangle 2$. LET: \mathcal{U} be an open covering of A
 - $\langle 1 \rangle 3$. $\{U \text{ open in } X : U \cap A \in \mathcal{U}\} \cup \{X \setminus A\}$ covers X
 - $\langle 1 \rangle 4$. PICK a countable subcovering \mathcal{V}
 - $\langle 1 \rangle 5$. $\{U \cap A : U \in \mathcal{V}, U \neq X \setminus A\}$ is a countable subcover of \mathcal{U}
- \square

Proposition 13.1.33. \mathbb{R}^ω is locally connected.

PROOF: This holds because every basic open set is connected, being the product of a family of connected spaces. \square

Proposition 13.1.34. The space \mathbb{R}^ω under the box topology is not first countable.

PROOF:

- $\langle 1 \rangle 1$. ASSUME: for a contradiction $\{U_n\}_{n \geq 0}$ is a countable basis at 0.
 - $\langle 1 \rangle 2$. For $n \geq 1$, PICK a basic open set $B_n = \prod_{j=0}^\infty (a_{nj}, b_{nj})$ such that $0 \in B_n \subseteq U_n$
 - $\langle 1 \rangle 3$. $\prod_{n=0}^\infty (a_{nn}/2, b_{nn}/2)$ is a neighbourhood of 0 that does not include any U_n
- \square

Proposition 13.1.35. The space \mathbb{R}^ω under the box topology is not locally metrizable.

PROOF:

- $\langle 1 \rangle 1$. LET: U be any neighbourhood of 0
- $\langle 1 \rangle 2$. LET: A be the set of all sequences in U with all coordinates positive
- $\langle 1 \rangle 3$. $0 \in \overline{A}$
- $\langle 1 \rangle 4$. There is no sequence of points of A converging to 0.
- $\langle 1 \rangle 5$. U is not metrizable.

PROOF: By the Sequence Lemma.
 \square

Proposition 13.1.36. For any nonempty set I , the space \mathbb{R}^I is not limit point compact.

PROOF: \mathbb{Z}^I is an infinite set with no limit point. \square

Proposition 13.1.37. *The space $\mathbb{R}^{[0,1]}$ is separable.*

PROOF: The set D is dense where D is the set of all functions $f : [0, 1] \rightarrow \mathbb{Q}$ such that there exists a sequence of rationals $0 = q_0 < q_1 < \cdots < q_N = 1$ such that f is constant on $[q_i, q_{i+1})$ for $0 \leq i < N$. \square

Proposition 13.1.38. *If J is uncountable then \mathbb{R}^J is not locally metrizable.*

PROOF: Every point has a neighbourhood homeomorphic to \mathbb{R}^J . \square

Proposition 13.1.39. *The space \mathbb{R}_K is not limit point compact.*

PROOF: The set \mathbb{Z} has no limit point. \square

Proposition 13.1.40. *The topologist's sine curve is not locally connected.*

PROOF: There is no connected neighbourhood of $(0, 0)$. \square

Corollary 13.1.40.1. *Not every metric space is locally connected.*

Corollary 13.1.40.2. *Not every metric space is locally path connected.*

Proposition 13.1.41. *Not every metric space is compact.*

PROOF: The space \mathbb{R} is not compact. \square

Proposition 13.1.42. *Every closed subspace of a limit point compact space is limit point compact.*

PROOF:

$\langle 1 \rangle 1$. LET: X be a limit point compact space and $C \subseteq X$ be closed.

$\langle 1 \rangle 2$. LET: $A \subseteq C$ be infinite.

$\langle 1 \rangle 3$. PICK a limit point l of A in X

$\langle 1 \rangle 4$. $l \in C$

$\langle 2 \rangle 1$. l is a limit point of C

PROOF: By Lemma 3.13.2.

$\langle 2 \rangle 2$. Q.E.D.

PROOF: By Corollary 3.13.3.1.

$\langle 1 \rangle 5$. l is a limit point of A in C .

PROOF: By Proposition 4.3.10.

\square

Proposition 13.1.43. *For any part $i : S \hookrightarrow X$ of a set X , we have $\emptyset \subseteq_X i$.*

PROOF: We have $i \circ j_S = j_X$ by the uniqueness of j_X . \square

Theorem 13.1.44. *Let X be a completely regular space. Then there exists a compactification Y of X such that every bounded continuous map $X \rightarrow \mathbb{R}$ extends uniquely to a continuous map $Y \rightarrow \mathbb{R}$.*

PROOF:

$\langle 1 \rangle 1$. LET: J be the set of all bounded continuous functions $X \rightarrow \mathbb{R}$

- (1)2. For $\alpha \in J$,
 LET: $I_\alpha = [\inf \alpha, \sup \alpha]$
 (1)3. LET: $Z = \prod_{\alpha \in J} I_\alpha$
 (1)4. LET: $h : X \rightarrow Z$ be defined by $h(x)_\alpha = \alpha(x)$
 (1)5. Z is compact Hausdorff
 (2)1. Z is compact
 PROOF: By Tychonoff's Theorem.
 (2)2. Z is Hausdorff
 PROOF: By Theorem 6.2.5
 (1)6. h is an imbedding
 (2)1. The set J separates points from closed sets
 PROOF: This holds because X is completely regular.
 (2)2. Q.E.D.
 PROOF: By the Imbedding Theorem.
 (1)7. LET: Y be the compactification of X such that $X \subseteq Y \rightarrow Z$ factors h
 PROOF: By Lemma 9.8.2
 (1)8. Every bounded continuous map $X \rightarrow \mathbb{R}$ extends uniquely to a continuous map $Y \rightarrow \mathbb{R}$
 (2)1. LET: $\alpha : X \rightarrow \mathbb{R}$ be a bounded continuous function
 (2)2. LET: $k : Y \rightarrow Z$ be the imbedding from (1)7
 (2)3. LET: $\bar{\alpha} = \pi_\alpha \circ k : Y \rightarrow \mathbb{R}$
 (2)4. $\bar{\alpha}$ extends α
 PROOF: For $x \in X$, we have

$$\begin{aligned}
 \bar{\alpha}(x) &= k(x)_\alpha \\
 &= h(x)_\alpha \\
 &= \alpha(x)
 \end{aligned}$$

- (2)5. If $f : Y \rightarrow Z$ is continuous and extends α then $f = \bar{\alpha}$
 PROOF: By Lemma 6.2.9.

□