Topology

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## Chapter 1

# Set Theory

## 1.1 Primitive Notions

Let there be sets.

Given sets A and B, let there be functions from A to B. We write  $f:A\to B$  iff f is a function from A to B.

Given functions  $f:A\to B$  and  $g:B\to C$ , let there be a function  $g\circ f:A\to C$ , the *composite* of f and g.

## 1.2 The Axiom of Associativity

**Axiom 1.2.1** (Associativity). Let  $f: A \to B$ ,  $g: B \to C$  and  $h: C \to D$ . Then  $h \circ (g \circ f) = (h \circ g) \circ f: A \to D$ .

From now on we write  $h \circ g \circ f$  for the composite of f, g and h, and similarly for more than three functions.

## 1.3 Injective Functions

**Definition 1.3.1** (Injective). A function  $f: A \to B$  is *injective*,  $f: A \rightarrowtail B$ , iff, for every set X and functions  $g, h: X \to A$ , if  $f \circ g = f \circ h$  then g = h.

**Proposition 1.3.2.** Let  $f: A \to B$  and  $g: B \to C$ . If f and g are injective then  $g \circ f$  is injective.

#### Proof:

- $\langle 1 \rangle 1$ . Assume: f and g are injective.
- $\langle 1 \rangle 2$ . Let: X be a set and  $x, y : X \to A$
- $\langle 1 \rangle 3$ . Assume:  $g \circ f \circ x = g \circ f \circ y$
- $\langle 1 \rangle 4$ .  $f \circ x = f \circ y$

PROOF: g is injective  $(\langle 1 \rangle 1)$ 

 $\langle 1 \rangle 5. \ x = y$ 

```
PROOF: f is injective (\langle 1 \rangle 1)
```

**Lemma 1.3.3.** Let  $f: A \to B$  and  $g: B \to C$ . If  $g \circ f$  is injective then f is injective.

#### Proof:

- $\langle 1 \rangle 1$ . Assume:  $g \circ f$  is injective.
- $\langle 1 \rangle 2$ . Let: X be any set and  $x, y: X \to A$
- $\langle 1 \rangle 3$ . Assume:  $f \circ x = f \circ y$
- $\langle 1 \rangle 4. \ g \circ f \circ x = g \circ f \circ y$
- $\langle 1 \rangle 5. \ x = y$

Proof: Using  $\langle 1 \rangle 1$ .

## 1.4 Surjective Functions

**Definition 1.4.1** (Surjective). Let  $f: A \to B$ . Then f is *surjective*,  $f: A \to B$ , iff, for any set X and functions  $g, h: B \to X$ , if  $g \circ f = h \circ f$  then g = h.

**Lemma 1.4.2.** Let  $f: A \to B$  and  $g: B \to C$ . If f and g are surjective then  $g \circ f$  is surjective.

Proof: Dual to Lemma 1.3.2.  $\Box$ 

**Lemma 1.4.3.** Let  $f: A \to B$  and  $g: B \to C$ . If  $g \circ f$  is surjective then g is surjective.

PROOF: Dual to Lemma 1.3.3.  $\square$ 

## 1.5 Identity Functions

**Axiom 1.5.1** (Identity Function). For any set A, there exists a function  $id_A : A \to A$ , the identity function on A, such that:

**Left Unit Law** for every set B and function  $f: B \to A$  we have  $id_A \circ f = f: B \to A$ ;

**Right Unit Law** for every set B and function  $f: A \to B$  we have  $f \circ id_A = f: A \to B$ .

**Proposition 1.5.2.** The identity function on a set is unique.

PROOF:If  $i, j: A \to A$  are both identity functions, then  $i = i \circ j$  (Right Unit Law for j)  $= j \qquad \qquad \text{(Left Unit Law for } i\text{)}$   $: A \to A \qquad \qquad \square$ 

**Proposition 1.5.3.** Every identity function is injective.

PROOF: If  $id_B \circ x = id_B \circ y$  then x = y by the Left Unit Law.  $\square$ 

**Proposition 1.5.4.** Every identity function is surjective.

Proof: Dual.

**Definition 1.5.5** (Retraction, Section). Let  $r: A \to B$  and  $s: B \to A$ . Then r is a retraction of s, and s is a section of r, iff  $r \circ s = \mathrm{id}_B$ .

**Proposition 1.5.6.** If  $r_1: A \to B$  is a retraction of  $s_1: B \to A$  and  $r_2: B \to C$  is a retraction of  $s_2: C \to B$  then  $r_2 \circ r_1$  is a retraction of  $s_1 \circ s_2$ .

Proof:

$$r_2 \circ r_1 \circ s_1 \circ s_2 = r_2 \circ \mathrm{id}_B \circ s_2$$
  $(r_1 \text{ is a retraction of } s_1)$   
=  $r_2 \circ s_2$  (Unit Laws)  
=  $\mathrm{id}_C$   $(r_2 \text{ is a retraction of } s_2)$ 

Proposition 1.5.7. Every retraction is surjective.

Proof:

$$\langle 1 \rangle 1.$$
 Let:  $r:A \to B$  be a retraction of  $s$   $\langle 1 \rangle 2.$  Let:  $X$  be a set and  $x,y:B \to X$  with  $x \circ r = y \circ r$   $\langle 1 \rangle 3.$   $x \circ r \circ s = y \circ r \circ s$   $\langle 1 \rangle 4.$   $x = y$ 

Proposition 1.5.8. Every section is injective.

Proof: Dual.

**Proposition 1.5.9.** Every identity function is a retraction of itself.

PROOF: Immediate from the Unit Laws.

**Proposition 1.5.10.** If  $r: B \to A$  is a retraction of  $f: A \to B$  and s is a section of f then r = s.

Proof:

$$r = r \circ id_B$$
 (Right Unit Law)  
 $= r \circ f \circ s$  (s is a section of f)  
 $= id_A \circ s$  (r is a retraction of f)  
 $= s$  (Left Unit Law)

## 1.5.1 Isomorphisms

**Definition 1.5.11** (Isomorphism). Let A and B be sets. A function  $i: A \to B$  is an *isomorphism* between A and B,  $i: A \cong B$ , iff there exists a function  $i^{-1}: B \to A$ , the *inverse* to i, that is a section and a retraction of i.

**Proposition 1.5.12.** The inverse of an isomorphism is unique.

PROOF: Immediate from Proposition 1.5.10.  $\Box$ 

**Proposition 1.5.13.** Every isomorphism is injective.

PROOF: Immediate from Proposition 1.5.8.

Proposition 1.5.14. Every isomorphism is surjective.

PROOF: Immediate from Proposition 1.5.7.

**Proposition 1.5.15.** Every identity function is an isomorphism and is its own inverse.

PROOF: Immediate from Proposition 1.5.9.

**Proposition 1.5.16.** If  $i: A \cong B$  is an isomorphism then  $i^{-1}: B \cong A$  is an isomorphism and  $(i^{-1})^{-1} = i$ .

Proof: Immediate from the definition of isomorphism.  $\Box$ 

**Proposition 1.5.17.** *If*  $i : A \cong B$  *and*  $j : B \cong C$  *then*  $j \circ i : A \cong C$  *and*  $(j \circ i)^{-1} = i^{-1} \circ j^{-1}$ .

PROOF: Immediate from Proposition 1.5.6.

#### 1.5.2 Parts of a Set

**Definition 1.5.18** (Part). A part S of a set A consists of:

- a set dom S;
- an injective function  $i: S \hookrightarrow A$

**Definition 1.5.19.** Two parts  $i: S \hookrightarrow A$ ,  $j: T \hookrightarrow A$  are equivalent,  $i \equiv_A j$ , iff there exists an isomorphism  $\phi: S \cong T$  such that  $i = j \circ \phi$ .

**Proposition 1.5.20.** Any part of a set is equivalent to itself.

Proof:

 $\langle 1 \rangle 1$ . Let:  $i: S \hookrightarrow A$  be a part of A. Prove:  $i \equiv_A i$ 

 $\langle 1 \rangle 2$ .  $\mathrm{id}_S : S \cong S$ 

Proof: By Proposition 1.5.15

 $\langle 1 \rangle 3. \ i = i \circ id_S$ 

PROOF: By the Right Unit Law.

**Proposition 1.5.21.** *If*  $i \equiv_A j$  *then*  $j \equiv_A i$ .

#### Proof:

- $\langle 1 \rangle 1$ . Let:  $i: S \hookrightarrow A$  and  $j: T \hookrightarrow A$
- $\langle 1 \rangle 2$ . Assume:  $i \equiv_A j$
- $\langle 1 \rangle 3$ . PICK an isomorphism  $\phi : S \cong T$  such that  $i = j \circ \phi$
- $\langle 1 \rangle 4. \ \phi^{-1} : T \cong S$

Proof: By Proposition 1.5.16.

 $\langle 1 \rangle 5. \ j = i \circ \phi^{-1}$ 

PROOF: Compose both sides of  $\langle 1 \rangle 3$  with  $\phi^{-1}$ .

**Proposition 1.5.22.** If  $i \equiv_A j$  and  $j \equiv_A k$  then  $i \equiv_A k$ .

#### Proof:

- $\langle 1 \rangle 1$ . Let:  $i: R \hookrightarrow A, j: S \hookrightarrow A \text{ and } k: T \to A$
- $\langle 1 \rangle 2.$  Pick isomorphisms  $\phi:R\cong S$  and  $\psi:S\cong T$  such that  $i=j\circ \phi$  and  $j=k\circ \psi$
- $\langle 1 \rangle 3. \ \psi \circ \phi : R \cong T$

Proof: By Proposition 1.5.17.

 $\langle 1 \rangle 4. \ i = k \circ \psi \circ \phi$ 

**Definition 1.5.23** (Inclusion). Let  $i: U \hookrightarrow A$  and  $j: V \hookrightarrow A$  be parts of A. Then i is *included* in j,  $i \subseteq_A j$ , iff there exists a function  $\phi: U \to V$  such that  $i = j \circ \phi$ .

**Proposition 1.5.24.** If  $i \equiv_A i'$  and  $j \equiv_A j'$  and  $i \subseteq_A j$  then  $i' \subseteq_A j'$ .

#### Proof:

- $\langle 1 \rangle 1$ . Let:  $i: S \hookrightarrow A, i': S' \hookrightarrow A, j: T \hookrightarrow A, j': T' \hookrightarrow A$
- $\langle 1 \rangle 2$ . PICK  $\phi: S \cong S', \psi: T \cong T'$  and  $\chi: S \to T$  such that  $i = i' \circ \phi, j = j' \circ \psi$  and  $i = j \circ \chi$
- $\langle 1 \rangle 3. \ \psi \circ \chi \circ \phi^{-1} : S' \to T'$
- $\langle 1 \rangle 4. \ i' = j' \circ \psi \circ \chi \circ \phi^{-1}$

**Proposition 1.5.25.** For any part i of A we have  $i \subseteq_A i$ .

#### Proof:

- $\langle 1 \rangle 1$ . Let:  $i: S \hookrightarrow A$
- $\langle 1 \rangle 2$ . id<sub>S</sub> :  $S \to S$
- $\langle 1 \rangle 3. \ i = i \circ \mathrm{id}_S$

**Proposition 1.5.26.** If  $i \subseteq_A j$  and  $j \subseteq_A k$  then  $i \subseteq_A k$ .

Proof:

## 1.5.3 The Empty Set

PROOF: Similar.

**Axiom 1.5.28** (Empty Set). There exists a set  $\emptyset$ , the empty set, such that, for every set X, there exists a unique function  $X : \emptyset \to X$ .

**Proposition 1.5.29** (Uniqueness of Empty Set). Let E be any set. Then E is empty if and only if there exists an isomorphism  $E \cong \emptyset$ , in which case the isomorphism is unique.

```
Proof:
```

```
\langle 1 \rangle 1. If E is empty then E \cong \emptyset
   \langle 2 \rangle 1. Assume: E is empty
   \langle 2 \rangle 2. Let: \phi be the unique function E \to \emptyset
   \langle 2 \rangle 3. i_E \circ \phi = id_E
       PROOF: There is only one function E \to E.
   \langle 2 \rangle 4. \ \phi \circ i_E = id_\emptyset
       PROOF: There is only one function \emptyset \to \emptyset.
\langle 1 \rangle 2. If E \cong \emptyset then E is empty
   \langle 2 \rangle 1. Let: \phi : E \cong \emptyset
   \langle 2 \rangle 2. Let: X be a set
             PROVE: There is a unique function E \to X
   \langle 2 \rangle 3. \mid_X \circ \phi : E \to X
   \langle 2 \rangle 4. If f: E \to X then f = \mathcal{I}_X \circ \phi
       \langle 3 \rangle 1. Let: f: E \to X
       \langle 3 \rangle 2. \ f \circ \phi^{-1} : \emptyset \to X
       \langle 3 \rangle 3. \ f \circ \phi^{-1} = i_X
           PROOF: Uniqueness of X.
       \langle 3 \rangle 4. Q.E.D.
```

 $\langle 1 \rangle 3$ . There is at most one isomorphism  $E \cong \emptyset$ PROOF: This holds because there is at most one function  $E \to \emptyset$ .

Proposition 1.5.30.

$$i\emptyset=\mathrm{id}_\emptyset$$

PROOF: By the uniqueness of  $i_{\emptyset}$ .

#### 1.5.4 The Terminal Set

**Axiom 1.5.31** (Terminal Set). There exists a set 1, the terminal set, such that, for every set X, there exists a unique function  $!_X : X \to 1$ .

**Proposition 1.5.32** (Uniqueness of Terminal Set). Let T be any set. Then T is terminal if and only if there exists an isomorphism  $T \cong 1$ , in which case the isomorphism is unique.

PROOF: Dual to Proposition 1.5.29.

Proposition 1.5.33.

$$!_1 = id_1$$

PROOF: From the uniqueness of  $!_1$ .  $\square$ 

**Definition 1.5.34** (Element). An *element* of a set A is a function  $1 \to A$ . We write  $a \in A$  for  $a: 1 \to A$ . We write f(a) for  $f \circ a$  when  $f: A \to B$  and  $a \in A$ .

**Axiom 1.5.35** (Extensionality). Let A and B be sets and  $f, g : A \to B$  be functions. If, for all  $a \in A$ , we have  $f(a) = g(a) \in B$ , then f = g.

**Proposition 1.5.36.** *Let*  $f : A \to B$ . *Then* f *is injective if and only if, for all*  $x, y \in A$ , *if*  $f(x) = f(y) \in B$  *then*  $x = y \in A$ .

Proof:

 $\langle 1 \rangle 1$ . If f is injective and  $f(x) = f(y) \in B$  then  $x = y \in A$ 

PROOF: Immediate from the definition of injective.

- $\langle 1 \rangle 2$ . If, for all  $x, y \in A$ , if  $f(x) = f(y) \in B$  then  $x = y \in A$ 
  - $\langle 2 \rangle 1$ . Assume: For all  $x, y \in A$ , if f(x) = f(y), then x = y
  - $\langle 2 \rangle$ 2. Let: X be any set and  $g,h:X\to A$  with  $f\circ g=f\circ h$  Prove: g=h

 $\langle 2 \rangle 3$ . Let:  $x \in X$ 

PROVE: g(x) = h(x)

 $\langle 2 \rangle 4$ . f(g(x)) = f(h(x))

Proof: From  $\langle 2 \rangle 2$ .

 $\langle 2 \rangle 5.$  g(x) = h(x)

Proof: By  $\langle 2 \rangle 1$ 

**Proposition 1.5.37.** Any element  $e \in X$  is a section of the unique function  $!_X : X \to 1$ .

PROOF:  $!_X \circ e = \mathrm{id}_1$  because there is only one function  $1 \to 1$ .

**Axiom 1.5.38** (Non-degeneracy). The empty set  $\emptyset$  has no elements.

**Proposition 1.5.39.** For any set X, the function  $j_X : \emptyset \to X$  is injective.

PROOF: From Proposition 1.5.36.  $\square$ 

**Definition 1.5.40** (Empty Part). For any set X, the *empty part* of X is  $\emptyset = i_X : \emptyset \hookrightarrow X$ .

**Definition 1.5.41** (Constant Function). A function  $f: A \to B$  is constant iff there exists  $b \in B$  such that  $f = b \circ !_A$ .

**Definition 1.5.42** (Membership). Let  $i: U \hookrightarrow A$  be a part of A and  $a \in A$ . Then a is a *member* of i,  $a \in_A i$ , iff there exists  $\overline{a} \in U$  such that  $i(\overline{a}) = a$ .

**Proposition 1.5.43.** *Let* A *be a set. Let* i, j *be parts of* A *and*  $a \in A$ . *If*  $a \in_A i$  *and*  $i \subseteq_A j$  *then*  $a \in_A j$ .

#### Proof:

- $\langle 1 \rangle 1$ . Pick  $\overline{a} \in \text{dom } i \text{ such that } a = i(\overline{a})$ .
- $\langle 1 \rangle 2$ . Pick  $\phi : \text{dom } i \to \text{dom } j \text{ such that } i = j \circ \phi$
- $\langle 1 \rangle 3. \ a = j(\phi(\overline{a}))$

#### 1.5.5 Products

**Axiom 1.5.44** (Products). For any sets A and B, there exists a set  $A \times B$ , the product of A and B, and functions  $\pi_1 : A \times B \to A$ ,  $\pi_2 : A \times B \to B$ , the projections, such that, for any set C and functions  $f : C \to A$ ,  $g : C \to B$ , there exists a unique function  $\langle f, g \rangle : C \to A \times B$  such that

$$\pi_1 \circ \langle f, g \rangle = f; \qquad \pi_2 \circ \langle f, g \rangle = g.$$

**Definition 1.5.45.** Given functions  $f:A\to B$  and  $g:C\to D$ , define  $f\times g:A\times C\to B\times D$  by

$$f \times g = \langle f \circ \pi_1, g \circ \pi_2 \rangle$$

#### 1.5.6 Coproducts

**Axiom 1.5.46** (Coproducts). For any sets A and B, there exists a set  $A \uplus B$ , the coproduct or sum of A and B, and functions  $\kappa_1 : A \to A \uplus B$ ,  $\kappa_2 : B \to A \uplus B$ , the injections, such that, for any set C and functions  $f : A \to C$ ,  $g : B \to C$ , there exists a unique function  $[f,g] : A \uplus B \to C$  such that

$$[f,g] \circ \kappa_1 = f;$$
  $[f,g] \circ \kappa_2 = g$ .

**Definition 1.5.47** (Complement). Let  $i: I \hookrightarrow J$  and  $i': I' \hookrightarrow J$  be parts of J. Then i' is the *complement* of i iff J is the sum of I and I' with injections i and i'.

## 1.5.7 Equalizers

**Axiom 1.5.48** (Equalizers). For any sets A and B and functions  $f, g: A \to B$ , there exists a set E and function  $e: E \to A$ , the equalizer of A and B, such that:

- $f \circ e = g \circ e : E \to B$ ;
- For any set C and function  $h: C \to A$  such that  $f \circ h = g \circ h$ , there exists a unique function  $\overline{h}: C \to E$  such that  $h = e \circ \overline{h}$ .

Proposition 1.5.49. All equalizers are injective.

Proof:

```
\langle 1 \rangle 1. Let: e: E \to A be the equalizer of f, g: A \to B
```

 $\langle 1 \rangle 2$ . Let:  $x, y : X \to E$  with  $e \circ x = e \circ y$ 

 $\langle 1 \rangle 3. \ f \circ e \circ x = g \circ e \circ x$ 

PROOF:  $f \circ e = g \circ e$  by  $\langle 1 \rangle 11$ .

 $\langle 1 \rangle 4. \ x = y$ 

PROOF: x and y are both the unique  $z: X \to E$  such that  $e \circ z = e \circ x$ .

## 1.5.8 Coequalizers

**Axiom 1.5.50** (Coequalizers). For any sets A and B and functions  $f, g : A \to B$ , there exists a set C and function  $c : B \to C$ , the coequalizer of f and g, such that:

- $c \circ f = c \circ g : A \to C$
- For any set X and function  $h: B \to X$  such that  $h \circ f = h \circ g$ , there exists a unique function  $\overline{h}: C \to X$  such that  $\overline{h} \circ c = h$ .

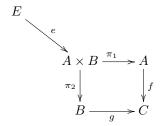
#### 1.5.9 Pullbacks

**Definition 1.5.51** (Pullback). The diagram below is a *pullback diagram* iff:

- $f \circ p = g \circ q$
- for every set X and functions  $x: X \to B$  and  $y: X \to C$  such that  $f \circ x = g \circ y$ , there exists a unique function  $\langle x, y \rangle : X \to A$  such that  $p \circ \langle x, y \rangle = x$  and  $q \circ \langle x, y \rangle = y$ .

$$\begin{array}{ccc}
A & \xrightarrow{p} & B \\
\downarrow q & & \downarrow f \\
C & \xrightarrow{q} & D
\end{array}$$

**Proposition 1.5.52.** Let  $f: A \to C$  and  $g: B \to C$ . Then f and g have a pullback.



Proof:

- $\langle 1 \rangle 1$ . Construct the product  $\pi_1 : A \times B \to A$ ,  $\pi_2 : A \times B \to B$ .
- $\langle 1 \rangle$ 2. Construct the equalizer  $e: E \to A$  of  $f \circ \pi_1$  and  $g \circ \pi_2$ . PROVE:  $\pi_1 \circ e$  and  $\pi_2 \circ e$  form a pullback of f and g
- $\langle 1 \rangle 3. \ f \circ \pi_1 \circ e = g \circ \pi_2 \circ e$
- $\langle 1 \rangle 4$ . Let: X be a set and  $x: X \to A, y: X \to B$  satisfy  $f \circ x = g \circ y$
- $\langle 1 \rangle 5. \ f \circ \pi_1 \circ \langle x, y \rangle = g \circ \pi_2 \circ \langle x, y \rangle$
- $\langle 1 \rangle$ 6. Let:  $m: X \to E$  be the function such that  $e \circ m = \langle x, y \rangle$
- $\langle 1 \rangle 7$ .  $\pi_1 \circ e \circ m = x$  and  $\pi_2 \circ e \circ m = y$
- $\langle 1 \rangle 8$ . m is unique.

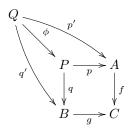
Proof:

- $\langle 2 \rangle 1$ . Let:  $n: X \to E$  be such that  $\pi_1 \circ e \circ n = x$  and  $\pi_2 \circ e \circ n = y$
- $\langle 2 \rangle 2$ .  $e \circ n = \langle x, y \rangle$
- $\langle 2 \rangle 3$ . n=m

Proof: By  $\langle 1 \rangle 6$ 

**Proposition 1.5.53.** Pullbbacks are unique up to isomorphism.

That is, let P be a pullback of  $f:A\to C$  and  $g:B\to C$  with projections  $p:P\to A$  and  $q:P\to B$ . Let Q be a set and  $p':Q\to A$ ,  $q':Q\to B$ . Then Q is a pullback of f and g with projections p' and q' if and only if there exists a bijection  $\phi:Q\cong P$  such that  $p\circ\phi=p'$  and  $q\circ\phi=q'$ , in which case  $\phi$  is unique.



Proof:

- $\langle 1 \rangle 1.$  If Q is a pullback then there exists a bijection  $\phi:Q\cong P$  such that  $p\circ \phi=p'$  and  $q\circ \phi=q'$ 
  - $\langle 2 \rangle 1$ . Assume: Q is a pullback with projections p' and q'

 $\langle 2 \rangle 2.$  Let:  $\phi:Q\to P$  be the unique function such that  $p\circ\phi=p'$  and  $q\circ\phi=q'$ 

PROOF: Such a  $\phi$  exists because  $f \circ p' = g \circ q'$ .

 $\langle 2 \rangle$ 3. Let:  $\phi^{-1}: P \to Q$  be the unique function such that  $p' \circ \phi^{-1} = p$  and  $q' \circ \phi^{-1} = q$ 

PROOF: Such a function exists because  $f \circ p = g \circ q$ .

 $\langle 2 \rangle 4. \ \phi \circ \phi^{-1} = \mathrm{id}_P$ 

PROOF: Each is the unique function x such that  $p \circ x = p$  and  $q \circ x = q$ .

 $\langle 2 \rangle 5. \ \phi^{-1} \circ \phi = \mathrm{id}_Q$ 

PROOF: Similar.

- $\langle 1 \rangle 2$ . If  $\phi: Q \cong P$  is a bijection then Q is a pullback with projections  $p \circ \phi$  and  $q \circ \phi$ 
  - $\langle 2 \rangle 1$ .  $f \circ p \circ \phi = g \circ q \circ \phi$

PROOF: This holds because  $f \circ p = g \circ q$ 

 $\langle 2 \rangle 2$ . For any set X and functions  $x: X \to A$ ,  $y: X \to B$  such that  $f \circ x = g \circ y$ , there exists a unique function  $m: X \to Q$  such that  $p \circ \phi \circ m = x$  and  $q \circ \phi \circ m = y$ 

Proof:

$$p \circ \phi \circ m = x \text{ and } q \circ \phi \circ m = y$$
  
$$\Leftrightarrow \phi \circ m = \langle x, y \rangle$$
  
$$\Leftrightarrow m = \phi^{-1} \circ \langle x, y \rangle$$

 $\langle 1 \rangle 3$ . If  $\phi, \phi' : P \cong Q$  are bijections such that  $p \circ \phi = p \circ \phi'$  and  $q \circ \phi = q \circ \phi'$  PROOF: This follows from the definition of pullback.

Proposition 1.5.54. The pullback of an injective function is injective.

That is, if the diagram below is a pullback diagram and f is injective then q is injective.



PROOF:

- $\langle 1 \rangle 1$ . Let: X be a set and  $x, y : X \to A$  with  $q \circ x = q \circ y$
- $\langle 1 \rangle 2$ .  $f \circ p \circ x = g \circ q \circ x$
- $\langle 1 \rangle 3$ . Let:

 $z: X \to A$  be the function such that  $p \circ z = p \circ x$  and  $q \circ z = q \circ x$ 

- $\langle 1 \rangle 4. \ z = x$
- $\langle 1 \rangle 5. \ z = y$ 
  - $\langle 2 \rangle 1. \ q \circ x = q \circ y$

PROOF: By  $\langle 1 \rangle 1$ .

 $\langle 2 \rangle 2$ .  $f \circ p \circ x = f \circ p \circ y$ 

Proof:

$$\begin{split} f \circ p \circ x &= g \circ q \circ x \\ &= g \circ q \circ y \\ &= f \circ p \circ y \end{split} \qquad \begin{aligned} &(\langle 1 \rangle 2) \\ &(\langle 1 \rangle 1) \end{aligned}$$

 $\langle 2 \rangle 3. \ p \circ x = p \circ y$ 

PROOF: f is injective.

#### 1.5.10 Function Sets

**Axiom 1.5.55** (Function Sets). For any sets A and B, there exists a set  $A^B$  and a function  $\epsilon: A^B \times B \to A$ , the evaluation function, such that, for any set C and function  $f: C \times B \to A$ , there exists a unique function  $\lambda f: C \to A^B$  such that

$$\epsilon \circ (\lambda f \times \mathrm{id}_B) = f$$
.

#### 1.5.11 The Subset Classifier

**Definition 1.5.56.** The set 2 is 1+1. We write  $\top$  (*truth*) for  $\kappa_1: 1 \to 2$ , and  $\bot$  (*falsehood*) for  $\kappa_2: 1 \to 2$ .

**Axiom 1.5.57** (Subset Classifier). For every injective function  $m: A \rightarrow B$ , there exists a unique function  $\chi_m: B \rightarrow 2$ , the characteristic function of m, such that the following diagram is a pullback diagram:



**Proposition 1.5.58.** Every function  $\phi: A \to 2$  is the characteristic function of a part of A.

#### Proof:

 $\langle 1 \rangle 1$ . Construct a pullback



Proof: By Proposition 1.5.52.

 $\langle 1 \rangle 2$ . q is injective

Proof: By Proposition 1.5.54.

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**Axiom 1.5.59** (Boolean). For any  $p \in 2$  we have  $p = \top$  or  $p = \bot$ .

**Proposition 1.5.60.** Let  $i: U \hookrightarrow A$  and  $j: V \hookrightarrow A$  be parts of A. Then the following are equivalent:

- 1.  $i \subseteq_A j$  and  $j \subseteq_A i$
- 2. There exist  $h:U\to V$  and  $k:V\to U$  such that  $i=j\circ h,\ j=i\circ k,$   $k\circ h=\mathrm{id}_U$  and  $h\circ k=\mathrm{id}_V.$
- 3. The characteristic function of i is the characteristic function of j.

#### Proof:

- $\langle 1 \rangle 1. \ 1 \Rightarrow 2$ 
  - $\langle 2 \rangle 1$ . Assume:  $i \subseteq_A j$  and  $j \subseteq_A i$
  - $\langle 2 \rangle 2$ . Let:  $h: U \to V$  be such that  $i = j \circ h$
  - $\langle 2 \rangle 3$ . Let:  $k: V \to U$  be such that  $j = i \circ k$
  - $\langle 2 \rangle 4$ .  $k \circ h = \mathrm{id}_U$ 
    - $\langle 3 \rangle 1. \ i \circ k \circ h = i$

PROOF: From  $\langle 2 \rangle 2$  and  $\langle 2 \rangle 3$ .

 $\langle 3 \rangle 2$ . Q.E.D.

Proof: Since i is injective.

 $\langle 2 \rangle 5. \ h \circ k = \mathrm{id}_V$ 

Proof: Similar.

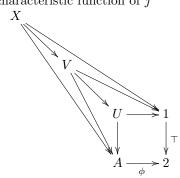
 $\langle 1 \rangle 2. \ 2 \Rightarrow 1$ 

PROOF: Trivial.

 $\langle 1 \rangle 3. \ 2 \Rightarrow 3$ 

 $\langle 2 \rangle 1$ . Assume: 2

 $\langle 2 \rangle 2$ . Let:  $\phi: A \to 2$  be the characteristic function of i Prove:  $\phi$  is the characteristic function of j



PROOF: By the Subset Classifier Axiom.

- $\langle 2 \rangle 3$ . Let: X be a set and  $x: X \to 1, \ y: X \to A$  satisfy  $\phi \circ y = \top \circ x$
- $\langle 2 \rangle$ 4. Let:  $\langle x,y \rangle: X \to U$  be the unique function such that  $! \circ \langle x,y \rangle = x$  and  $i \circ \langle x,y \rangle = y$

PROOF: By  $\langle 2 \rangle 2$ .

- $\langle 2 \rangle$ 5.  $h \circ \langle x, y \rangle$  is the unique function  $X \to V$  such that  $! \circ h \circ \langle x, y \rangle = x$  and  $j \circ h \circ \langle x, y \rangle = y$ 
  - $\langle 3 \rangle 1. ! \circ h \circ \langle x, y \rangle = x$

PROOF: Since 1 is terminal.

 $\langle 3 \rangle 2. \ j \circ h \circ \langle x, y \rangle = y$ 

PROOF: From  $\langle 2 \rangle 1$  and  $\langle 2 \rangle 4$ .

 $\langle 3 \rangle 3$ . If !  $\circ f = x$  and  $j \circ f = y$  then  $f = h \circ \langle x, y \rangle$ 

 $\langle 4 \rangle 1$ . Let:  $f: X \to V$  satisfy  $! \circ f = x$  and  $j \circ f = y$ 

 $\langle 4 \rangle 2$ . !  $\circ k \circ f = x$ 

PROOF: As 1 is terminal.

 $\langle 4 \rangle 3. \ i \circ k \circ f = y$ 

PROOF: From  $\langle 2 \rangle 1$  and  $\langle 4 \rangle 1$ .

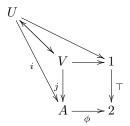
 $\langle 4 \rangle 4$ .  $k \circ f = \langle x, y \rangle$ 

PROOF: From  $\langle 2 \rangle 4$ ,  $\langle 4 \rangle 2$  and  $\langle 4 \rangle 3$ .

 $\langle 4 \rangle 5.$   $f = h \circ \langle x, y \rangle$ 

PROOF: From  $\langle 2 \rangle 1$  and  $\langle 4 \rangle 4$ .

 $\langle 1 \rangle 4. \ 3 \Rightarrow 2$ 



- $\langle 2 \rangle 1$ . Assume: 3
- $\langle 2 \rangle 2$ . Let:  $\phi$  be the characteristic function of i and j
- $\langle 2 \rangle 3$ . Let:  $h: U \to V$  be the unique function such that  $! \circ h = !$  and  $j \circ h = i$

 $\langle 3 \rangle 1$ .  $\forall \circ ! = \phi \circ i$ 

PROOF: This holds because  $\phi$  is the characteristic function of i.

 $\langle 3 \rangle 2$ . Q.E.D.

PROOF: Since  $\phi$  is the characteristic function of j.

- $\langle 2 \rangle 4.$  Let:  $k:V \to U$  be the unique function such that  $! \circ k = !$  and  $i \circ k = j$  Proof: Similar.
- $\langle 2 \rangle 5.$   $k \circ h = \mathrm{id}_U$

PROOF: Each is the unique function f such that  $! \circ f = !$  and  $i \circ f = i$ 

 $\langle 2 \rangle 6. \ h \circ k = \mathrm{id}_V$ 

PROOF: Each is the unique function f such that  $! \circ f = !$  and  $j \circ f = j$ 

### 1.6 The Basics

**Lemma 1.6.1.** Let X be a set,  $\mathcal{B} \subseteq \mathcal{P}X$  and  $U \subseteq X$ . Then the following are equivalent:

- 1. For all  $x \in U$  there exists  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$ .
- 2. There exists  $\mathcal{B}_0 \subseteq \mathcal{B}$  such that  $U = \bigcup \mathcal{B}_0$ .

Proof:

 $\langle 1 \rangle 1. \ 1 \Rightarrow 2$ 

```
PROOF: If 1 is true then U=\bigcup\{B\in\mathcal{B}:B\subseteq U\}. \langle 1\rangle 2.\ 2\Rightarrow 1 PROOF: Trivial.
```

**Definition 1.6.2** (Fixed Point). Let X be a set,  $f: X \to X$ , and  $x \in X$ . Then x is a fixed point of f iff f(x) = x.

**Definition 1.6.3** (Saturated). Let X, Y be sets and  $p: X \to Y$  be a surjective function. Let  $C \subseteq X$ . Then C is *saturated* with respect to p iff, for all  $x, x' \in X$ , if  $x \in C$  and p(x) = p(x') then  $x' \in C$ .

**Definition 1.6.4** (Cover). Let A be a set and  $C \subseteq PA$ . Then C covers A iff  $\bigcup C = A$ .

**Definition 1.6.5** (Finite Intersection Property). Let X be a set and  $C \subseteq \mathcal{P}X$ . Then C has the *finite intersection property* if and only if every finite nonempty subset of C has nonempty intersection.

**Lemma 1.6.6** (AC). Let X be a set and  $A \subseteq \mathcal{P}X$  have the finite intersection property. Then there exists a maximal  $\mathcal{D} \subseteq \mathcal{P}X$  that has the finite intersection property and includes A.

PROOF: A straightforward application of Zorn's lemma, since the union of a chain of sets that has the finite intersection property has the finite intersection property.  $\Box$ 

**Lemma 1.6.7.** Let X be a set and  $\mathcal{D} \subseteq \mathcal{P}X$  be maximal with respect to the finite intersection property. Then any finite intersection of elements of  $\mathcal{D}$  is an element of  $\mathcal{D}$ .

#### Proof:

 $\langle 1 \rangle 1$ . Let: A be a finite intersection of elements of  $\mathcal{D}$ 

 $\langle 1 \rangle 2$ .  $\mathcal{D} \cup \{A\}$  has the finite intersection property.

$$\langle 1 \rangle 3. \ \mathcal{D} \cup \{A\} = \mathcal{D}$$

**Lemma 1.6.8.** Let X be a set and  $\mathcal{D} \subseteq \mathcal{P}X$  be maximal with respect to the finite intersection property. If  $A \subseteq X$  intersects every element of  $\mathcal{D}$  then  $A \in \mathcal{D}$ .

PROOF: This holds because  $\mathcal{D} \cup \{A\}$  satisfies the finite intersection property.  $\square$ 

**Definition 1.6.9** (Graph). Let  $f: A \to B$ . The *graph* of f is the set  $\{(x, f(x)) : x \in A\} \subseteq A \times B$ .

**Definition 1.6.10** (Point-Finite). Let X be a set and  $\{A_{\alpha}\}_{{\alpha}\in J}$  be a family of subsets of X. Then  $\{A_{\alpha}\}_{{\alpha}\in J}$  is *point-finite* iff, for all  $x\in X$ , there are only finitely many  ${\alpha}\in J$  such that  $x\in A_{\alpha}$ .

**Definition 1.6.11** (Countable Intersection Property). A family of parts of a set X has the *countable intersection property* iff every countable subfamily has nonempty intersection.

## 1.7 Order Theory

**Definition 1.7.1** (Cofinal). Let J be a poset and  $K \subseteq J$ . Then K is *cofinal* iff, for all  $x \in J$ , there exists  $y \in K$  such that  $x \leq y$ .

**Definition 1.7.2** (Directed Set). A *directed set* is a poset J such that, for all  $x, y \in J$ , there exists  $z \in J$  such that  $x \leq z$  and  $y \leq z$ .

**Definition 1.7.3** (Linear Order). Let X be a set. A *linear order* on X is a relation  $\leq \subseteq X^2$  such that:

- For all  $x \in X$ ,  $x \le x$
- For all  $x, y, z \in X$ , if  $x \leq y$  and  $y \leq z$  then  $x \leq z$
- For all  $x, y \in X$ , if  $x \leq y$  and  $y \leq x$  then x = y
- For all  $x, y \in X$ , we have  $x \leq y$  or  $y \leq x$

We write x < y iff  $x \le y$  and  $x \ne y$ .

A linearly ordered set consists of a set and a linear order on the set.

**Definition 1.7.4** (Convex). Let L be a linearly ordered set and  $A \subseteq L$ . Then A is *convex* iff, for all  $x, y \in A$  and  $z \in L$ , if x < z < y then  $z \in A$ .

**Definition 1.7.5** (Least Upper Bound Property). A linearly ordered set L has the *least upper bound property* iff every subset of L bounded above has a least upper bound.

**Definition 1.7.6** (Linear Continuum). A *linear continuum* is a linearly ordered set L such that:

- L has the least upper bound property.
- For all  $x, y \in L$  with x < y, there exists  $z \in L$  such that x < z < y.

**Proposition 1.7.7.** If L is a linear continuum then every convex subset of L is a linear continuum.

#### Proof:

- $\langle 1 \rangle 1$ . Let: L be a linear continuum and  $C \subseteq L$  be convex
- $\langle 1 \rangle 2$ . C satisfies the least upper bound property.
  - $\langle 2 \rangle 1$ . Let:  $S \subseteq C$  be nonempty and bounded above by u in C.
  - $\langle 2 \rangle 2$ . Let: s be the supremum of S in L
  - $\langle 2 \rangle 3$ . Pick  $x \in S$
  - $\langle 2 \rangle 4. \ x \leq s \leq u$
  - $\langle 2 \rangle 5. \ s \in C$

Proof: C is convex.

- $\langle 2 \rangle 6$ . s is the supremum of S in C
- $\langle 1 \rangle 3$ . C is dense.

Proof:

```
\langle 2 \rangle 1. Let: x, y \in C satisfy x < y
```

- $\langle 2 \rangle 2$ . Pick  $z \in L$  such that x < z < y
- $\langle 2 \rangle 3. \ z \in C$

Proof: C is convex.

**Lemma 1.7.8.** For any real numbers a, b with a < b we have  $[a, b) \cong [0, 1)$ .

PROOF: The map  $\phi:[a,b)\cong [0,1)$  where  $\phi(x)=(x-a)/(b-a)$  is an order isomorphism.  $\square$ 

**Proposition 1.7.9.** Let X be a linearly ordered set. Let  $a, b, c \in X$  with a < c < b. Then  $[a, b) \cong [0, 1)$  if and only if  $[a, c) \cong [c, b) \cong [0, 1)$ .

#### PROOF:

- (1)1. If  $[a, b) \cong [0, 1)$  then  $[a, c) \cong [c, b) \cong [0, 1)$ .
  - $\langle 2 \rangle 1$ . Assume:  $\phi : [a,b) \cong [0,1)$  is an order isomorphism.
  - $\langle 2 \rangle 2$ .  $[a,c) \cong [0,1)$

Proof:

$$[a,c) \cong [0,\phi(c))$$
 (under  $\phi$ )  
 $\cong [0,1)$  (Lemma 1.7.8)

 $\langle 2 \rangle 3. \ [c,b) \cong [0,1)$ 

Proof: Similar.

- $\langle 1 \rangle 2$ . If  $[a,c) \cong [c,b) \cong [0,1)$  then  $[a,b) \cong [0,1)$ .
  - $\langle 2 \rangle 1$ . Assume:  $[a,c) \cong [c,b) \cong [0,1)$
  - $\langle 2 \rangle 2$ . Let:  $\phi : [a,c) \cong [0,1/2)$  and  $\psi : [c,b) \cong [1/2,1)$

$$\langle 2 \rangle$$
3. Let:  $\chi : [a,b) \to [0,1)$  be given by  $\chi(x) = \begin{cases} \phi(x) & \text{if } x < c \\ \psi(x) & \text{if } x \ge c \end{cases}$ 

 $\langle 2 \rangle 4. \ \chi : [a,b) \cong [0,1)$ 

PROOF: Easy to check.

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**Proposition 1.7.10** (CC). Let X be a linearly ordered set. Let  $\{x_n\}_{n\geq 0}$  be an increasing sequence of points of X. Suppose b is the supremum of  $\{x_n : n \geq 0\}$ . Then  $[x_0, b) \cong [0, 1)$  if and only if  $[x_i, x_{i+1}) \cong [0, 1)$  for all i.

#### PROOF

 $\langle 1 \rangle 1$ . If  $[x_0, b) \cong [0, 1)$  then for all  $i [x_i, x_{i+1}) \cong [0, 1)$ .

PROOF: If  $\phi : [x_0, b) \cong [0, 1)$  then  $[x_i, x_{i+1}) \cong [\phi(x_i), \phi(x_{i+1})) \cong [0, 1)$  by Lemma 1.7.8.

- $\langle 1 \rangle 2$ . If for all i  $[x_i, x_{i+1}) \cong [0, 1)$  then  $[x_0, b) \cong [0, 1)$ . PROOF:
  - $\langle 2 \rangle 1$ . Let:  $\phi_i : [x_i, x_{i+1}) \cong [0, 1)$  for all i
  - $\langle 2 \rangle 2$ . Define  $\phi: [x_0, b) \cong [0, 1)$  by:  $\phi(y) = \phi_i(y)$   $(x_0 \leq y < b)$  where i is least such that  $y < i_{i+1}$

PROOF: There exists such an i because y is not an upper bound for  $\{x_n : n \ge 0\}$ .

```
\langle 2 \rangle 3. \phi is an order isomorphism. PROOF: Easy to check. \Box

Proposition 1.7.11 (CC). For all 0 < \alpha < \Omega, the interval [(0,0),(\alpha,0)) in S_{\Omega} \times [0,1) is order isomorphic to [0,1) in \mathbb{R}.

PROOF: \langle 1 \rangle 1. If [(0,0),(\alpha,0)) \cong [0,1) then [(0,0),(\alpha+1,0)) \cong [0,1) PROOF: By Proposition 1.7.9. \langle 1 \rangle 2. Let \lambda be a limit ordinal, 0 < \lambda < \Omega. If, for all \alpha with 0 < \alpha < \lambda, we have [(0,0),(\alpha,0)) \cong [0,1), then [(0,0),(\lambda,0)) \cong [0,1). PROOF: By Proposition 1.7.10. \langle 1 \rangle 3. Q.E.D. PROOF: By transfinite induction.
```

# Chapter 2

# Real Analysis

**Definition 2.0.1** (Cantor Set). Define a sequence of sets  $A_n \subseteq [0,1]$  by:

$$A_0 = [0, 1]$$

$$A_{n+1} = A_n \setminus \bigcup_{k=0}^{3^{n-1}-1} \left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n}\right)$$

The Cantor set is  $\bigcap_{n=0}^{\infty} A_n$ .

## Chapter 3

# **Topological Spaces**

## 3.1 Topologies

**Definition 3.1.1** (Topology). A topology on a set X is a set  $\mathcal{T} \subseteq \mathcal{P}X$  such that:

- 1.  $X \in \mathcal{T}$ ;
- 2. for all  $U, V \in \mathcal{T}$ , we have  $U \cap V \in \mathcal{T}$ ;
- 3. For all  $A \subseteq \mathcal{T}$ , we have  $\bigcup A \in \mathcal{T}$ .

A topological space X consists of a set X and a topology on X. The elements of X are called *points* and the elements of  $\mathcal{T}$  are called *open sets*.

**Proposition 3.1.2.** In any topological space, the empty set is open.

PROOF: Immediate from axiom 3.

**Definition 3.1.3** (Discrete Topology). The *discrete* topology on a set X is  $\mathcal{P}X$ .

**Definition 3.1.4** (Indiscrete Topology). The *indiscrete* topology on a set X is  $\{\emptyset, X\}$ .

**Definition 3.1.5** (Open Cover). Let X be a topological space. A cover  $\mathcal{C} \subseteq \mathcal{P}X$  of X is an *open cover* iff every member of  $\mathcal{C}$  is open.

**Definition 3.1.6** (Finer, Coarser). Let  $\mathcal{T}$ ,  $\mathcal{T}'$  be topologies on a set X. Then  $\mathcal{T}$  is *finer* than  $\mathcal{T}'$ , and  $\mathcal{T}'$  is *coarser* than  $\mathcal{T}$ , iff  $\mathcal{T}' \subseteq \mathcal{T}$ .

The topology  $\mathcal{T}$  is *strictly* finer than  $\mathcal{T}'$ , and  $\mathcal{T}'$  is *strictly* coarser than  $\mathcal{T}$ , iff  $\mathcal{T} \subset \mathcal{T}'$ .

The topologies  $\mathcal{T}$  and  $\mathcal{T}'$  are *comparable* iff  $\mathcal{T} \subseteq \mathcal{T}'$  or  $\mathcal{T}' \subseteq \mathcal{T}$ .

**Definition 3.1.7** (Finite Complement Topology). The *finite complement topology* on a set X is  $\{U: X \setminus U \text{ is finite}\} \cup \{X\}$ .

**Definition 3.1.8** (Isolated Point). Let X be a topological space and  $a \in X$ . Then a is an *isolated point* iff  $\{a\}$  is open.

## 3.2 Neighbourhoods

**Definition 3.2.1** (Neighbourhood). Let X be a topological space and  $A \subseteq X$ . A *neighbourhood* of A is an set that includes an open set that includes A. A *neighbourhood* of a point a is a neighbourhood of  $\{a\}$ .

**Proposition 3.2.2.** If N is a neighbourhood of A and  $B \subseteq A$  then N is a neighbourhood of B.

PROOF: Immediate from definitions.

**Proposition 3.2.3.** A set U is open if and only if it is a neighbourhood of each of its points.

#### Proof:

- $\langle 1 \rangle 1$ . Let: X be a topological space and  $A \subseteq X$
- $\langle 1 \rangle 2$ . If U is a neighbourhood of each of its points then A is open.
  - $\langle 2 \rangle$ 1. Assume: U includes a neighbourhood of each of its points Prove:  $U = \bigcup \{V \subset U : V \text{ is open}\}$
  - $\langle 2 \rangle 2$ .  $\bigcup \{ V \subseteq U : V \text{ is open} \} \subseteq U$

PROOF: Set theory.

 $\langle 2 \rangle 3. \ U \subseteq \bigcup \{V \subseteq U : V \text{ is open}\}\$ 

PROOF: Immediate from  $\langle 2 \rangle 1$ .

 $\langle 1 \rangle 3$ . If U is open then U is a neighbourhood of each of its points.

PROOF: Immediate from definitions.

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**Proposition 3.2.4.** If M is a neighbourhood of A and  $M \subseteq N$  then N is a neighbourhood of A.

PROOF: Immediate from definitions.  $\square$ 

**Proposition 3.2.5.** If M and N are neighbourhoods of A then  $M \cap N$  is a neighbourhood of A.

PROOF: Pick open sets U and V such that  $A \subseteq U \subseteq M$  and  $A \subseteq N \subseteq V$ . Then  $A \subseteq U \cap V \subseteq M \cap N$ .

**Proposition 3.2.6.** If N is a neighbourhood of x then  $x \in N$ .

PROOF: Immediate from definitions.  $\square$ 

**Proposition 3.2.7.** If N is a neighbourhood of x then there exists a neighbourhood U of x such that, for all  $y \in U$ , M is a neighbourhood of y.

PROOF: Pick an open set U such that  $x \in U \subseteq N$ .  $\square$ 

**Theorem 3.2.8.** Let X be a set and  $\triangleright \subseteq \mathcal{P}X \times X$  a relation such that:

- 1. If  $M \triangleright x$  and  $M \subseteq N$  then  $N \triangleright x$
- 2.  $X \triangleright x$  for all  $x \in X$

```
3. If M \triangleright x and N \triangleright x then M \cap N \triangleright x
```

- 4. If  $N \triangleright x$  then  $x \in N$
- 5. If  $M \triangleright x$  then there exists  $N \triangleright x$  such that, for all  $y \in N$ ,  $M \triangleright y$ .

Then there exists a unique topology  $\mathcal{T}$  such that  $N \triangleright x$  iff N is a neighbourhood of x.

```
PROOF:
```

```
\langle 1 \rangle 1. Let: \triangleright be a relation satisfying 1–3
```

$$\langle 1 \rangle 2$$
. Let:  $\mathcal{T} = \{ U \in \mathcal{P}X : \forall x \in U.U \rhd x \}$ 

- $\langle 1 \rangle 3$ .  $\mathcal{T}$  is a topology.
  - $\langle 2 \rangle 1. \ X \in \mathcal{T}$

PROOF: By axiom 2

 $\langle 2 \rangle 2$ . For all  $U, V \in \mathcal{T}$  we have  $U \cap V \in \mathcal{T}$ 

PROOF: By axiom 3

 $\langle 2 \rangle 3$ . For all  $\mathcal{A} \subseteq \mathcal{T}$  we have  $\bigcup \mathcal{A} \in \mathcal{T}$ 

- $\langle 3 \rangle 1$ . Let:  $x \in \bigcup A$
- $\langle 3 \rangle 2$ . PICK  $U \in \mathcal{A}$  such that  $x \in U$
- $\langle 3 \rangle 3$ .  $U \rhd x$
- $\langle 3 \rangle 4. \bigcup \mathcal{A} \rhd x$

PROOF: By axiom 1

- $\langle 1 \rangle 4$ . In  $\mathcal{T}$ ,  $N \triangleright x$  iff N is a neighbourhood of x.
  - $\langle 2 \rangle 1$ . If  $N \rhd x$  then N is a neighbourhood of x
    - $\langle 3 \rangle 1$ . Assume:  $N \rhd x$
    - $\langle 3 \rangle 2. \ x \in N$

Proof: By axiom 4

- $\langle 3 \rangle 3$ . Let:  $U = \{ y \in N : N \rhd y \}$
- $\langle 3 \rangle 4$ . *U* is open
  - $\langle 4 \rangle 1$ . Let:  $y \in U$

Prove:  $U \triangleright y$ 

- $\langle 4 \rangle 2$ .  $N \rhd y$
- $\langle 4 \rangle 3$ . PICK  $W \triangleright y$  such that, for all  $z \in W$ ,  $N \triangleright z$

PROOF: By axiom 5

- $\langle 4 \rangle 4. \ W \subseteq U$
- $\langle 4 \rangle 5$ .  $U \rhd y$

PROOF: By axiom 1

- $\langle 3 \rangle 5. \ x \in U \subseteq N$
- $\langle 2 \rangle 2$ . If N is a neighbourhood of x then  $N \triangleright x$ 
  - $\langle 3 \rangle 1$ . Let: N be a neighbourhood of x
  - $\langle 3 \rangle 2$ . PICK U open such that  $x \in U \subseteq N$
  - $\langle 3 \rangle 3. \ U \rhd x$

Proof: By  $\langle 1 \rangle 2$ 

 $\langle 3 \rangle 4. \ N \rhd x$ 

PROOF: By axiom 1

 $\langle 1 \rangle 5$ .  $\mathcal{T}$  is unique.

PROOF: By Proposition 3.2.3.

**Definition 3.2.9** (Sufficiently Close). Let X be a topological space,  $a \in X$ , and P be a property of points of X. We write "For all x sufficiently close to a, P(x)" to mean "There exists a neighbourhood N of a such that, for all  $x \in N$ , P(x)."

## 3.3 Local Bases

**Definition 3.3.1** (Local Basis). Let X be a topological space and  $x \in X$ . A *local basis* at x is a set  $\mathcal{B}$  of open neighbourhoods of x such that every neighbourhood of x includes a member of  $\mathcal{B}$ . We call the elements of  $\mathcal{B}$  basic open neighbourhoods.

**Proposition 3.3.2.** Let  $\mathcal{B}$  be a local basis at x and  $M, N \in \mathcal{B}$ . Then there exists  $P \in \mathcal{B}$  such that  $P \subseteq M \cap N$ .

PROOF: This holds because  $M \cap N$  is a neighbourhood of x (Proposition 3.2.5).  $\sqcap$ 

**Proposition 3.3.3.** Let X be a topological space,  $x \in X$  and  $\mathcal{B} \subseteq \mathcal{P}X$ . Then  $\mathcal{B}$  is a local basis at x iff  $\mathcal{B}$  is a set of open neighbourhoods of x such that every open neighbourhood of x includes a member of  $\mathcal{B}$ .

#### Proof:

- $\langle 1 \rangle 1$ . If  $\mathcal{B}$  is a local basis at x then  $\mathcal{B}$  is a set of open neighbourhoods of x such that every open neighbourhood of x includes a member of  $\mathcal{B}$  Proof: Trivial.
- $\langle 1 \rangle 2$ . If  $\mathcal{B}$  is a set of open neighbourhoods of x such that every open neighbourhood of x includes a member of  $\mathcal{B}$  then  $\mathcal{B}$  is a local basis at x.

PROOF: Every neighbourhood of x includes an open neighbourhood of x, which therefore includes an element of  $\mathcal{B}$ .

### 3.4 Bases

**Definition 3.4.1** (Basis for a Topology). Let  $(X, \mathcal{T})$  be a topological space. A basis for the topology on X is a set of open sets  $\mathcal{B}$  such that every open set is a union of members of  $\mathcal{B}$ . The members of  $\mathcal{B}$  are called basic open sets, and  $\mathcal{T}$  is called the topology generated by  $\mathcal{B}$ .

**Proposition 3.4.2.** *Let*  $(X, \mathcal{T})$  *be a topological space and*  $\mathcal{B} \subseteq \mathcal{P}X$ *. Then the following are equivalent:* 

1.  $\mathcal{B}$  is a basis for  $\mathcal{T}$ .

- 2. A set U is open if and only if, for all  $x \in U$ , there exists  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$ .
- 3.  $\mathcal{T}$  is the set of all unions of subsets of  $\mathcal{B}$ .
- 4. Every member of  $\mathcal{B}$  is open and, for all  $x \in X$  and every open neighbourhood U of x, there exists  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$ .
- 5. For all  $x \in X$ , the set  $\{B \in \mathcal{B} : x \in B\}$  is a local basis at x.

#### Proof:

- $\langle 1 \rangle 1. \ 1 \Rightarrow 2$ 
  - $\langle 2 \rangle 1$ . Assume:  $\mathcal{B}$  is a basis for the topology  $\mathcal{T}$ .
  - $\langle 2 \rangle 2$ . For all  $U \in \mathcal{T}$  and  $x \in U$ , there exists  $B \in \mathcal{B}$  such that  $x \in B$  PROOF: Immediate from the definition of basis  $(\langle 2 \rangle 1)$ .
  - $\langle 2 \rangle$ 3. For all  $U \subseteq X$ , if  $\forall x \in U. \exists B \in \mathcal{B}. x \in B \subseteq U$  then  $U \in \mathcal{T}$  PROOF: By Proposition 3.2.3.
- $\langle 1 \rangle 2$ .  $2 \Leftrightarrow 3$

PROOF: From Lemma 1.6.1.

 $\langle 1 \rangle 3. \ 3 \Rightarrow 1$ 

PROOF: Trivial.

 $\langle 1 \rangle 4. \ 2 \Rightarrow 4$ 

Proof: Trivial.

 $\langle 1 \rangle 5. \ 4 \Rightarrow 2$ 

Proof:

- $\langle 2 \rangle 1$ . Assume: 4
- $\langle 2 \rangle 2$ . If U is open then, for all  $x \in U$ , there exists  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$  PROOF: Immediate from  $\langle 2 \rangle 1$ .
- $\langle 2 \rangle 3$ . If, for all  $x \in U$ , there exists  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$ , then U is open.

PROOF: By Proposition 3.2.3 using the fact that every member of  $\mathcal{B}$  is open  $(\langle 2 \rangle 1)$ .

 $\langle 1 \rangle 6. \ 4 \Leftrightarrow 5$ 

PROOF: From Proposition 3.3.3.

П

**Corollary 3.4.2.1.** If  $\mathcal{B}$  is a basis for the topology  $\mathcal{T}$ , then  $\mathcal{T}$  is the coarsest topology in which every element of  $\mathcal{B}$  is open.

**Lemma 3.4.3.** Let X be a set and  $\mathcal{B} \subseteq \mathcal{P}X$ . Then  $\mathcal{B}$  is a basis for a topology  $\mathcal{T}$  on X if and only if:

1. 
$$\bigcup \mathcal{B} = X$$

2. for all  $B_1, B_2 \in \mathcal{B}$  and  $x \in B_1 \cap B_2$ , there exists  $B_3 \in \mathcal{B}$  such that  $x \in B_3 \subseteq B_1 \cap B_2$ .

In this case,  $\mathcal{T}$  is unique.

Proof:

```
\langle 2 \rangle 1. Assume: \mathcal{B} is a basis for the topology \mathcal{T}
    \langle 2 \rangle 2. Let: x \in X
   \langle 2 \rangle 3. There exists B \in \mathcal{B} such that x \in B
       PROOF: From the definition of basis, since X \in \mathcal{T}. (\langle 2 \rangle 1, \langle 2 \rangle 2).
\langle 1 \rangle 2. If \mathcal{B} is a basis for a topology then it satisfies condition 2
    \langle 2 \rangle 1. Assume: \mathcal{B} is a basis for the topology \mathcal{T}
    \langle 2 \rangle 2. Let: B_1, B_2 \in \mathcal{B}
    \langle 2 \rangle 3. \ B_1, B_2 \in \mathcal{T}
       PROOF: From the definition of basis (\langle 2 \rangle 1, \langle 2 \rangle 2).
    \langle 2 \rangle 4. B_1 \cap B_2 \in \mathcal{T}
       Proof: By the definition of topology, the open sets in \mathcal{T} are closed under
       binary intersection (\langle 2 \rangle 1, \langle 2 \rangle 3)
    \langle 2 \rangle 5. For all x \in B_1 \cap B_2, there exists B_3 \in \mathcal{B} such that x \in B_3 \subseteq B_1 \cap B_2
       PROOF: From the definition of basis (\langle 2 \rangle 1, \langle 2 \rangle 4)
\langle 1 \rangle 3. If \mathcal{B} satisfies conditions 1 and 2 then \mathcal{T} = \{ U \subseteq X : \forall x \in U : \exists B \in \mathcal{B} : x \in \mathcal{A} \}
          B \subseteq U} is a topology and \mathcal{B} is a basis for \mathcal{T}.
    \langle 2 \rangle 1. Assume: \mathcal{B} satisfies conditions 1 and 2
   \langle 2 \rangle 2. \ X \in \mathcal{T}
       PROOF: For all x \in X, there exists B \in \mathcal{B} such that x \in B \subseteq X by
       condition 1 (\langle 2 \rangle 1).
    \langle 2 \rangle 3. For all \mathcal{A} \subseteq \mathcal{T}, we have \bigcup \mathcal{A} \in \mathcal{T}
        \langle 3 \rangle 1. Let: \mathcal{A} \subseteq \mathcal{T}
        \langle 3 \rangle 2. Let: x \in \bigcup \mathcal{A}
       \langle 3 \rangle 3. PICK U \in \mathcal{A} such that x \in U
           PROOF: From \langle 3 \rangle 2.
        \langle 3 \rangle 4. Pick B \in \mathcal{B} such that x \in B \subseteq U
           PROOF: Since U \in \mathcal{T}, using the definition of \mathcal{T} (\langle 3 \rangle 1, \langle 3 \rangle 3)
        \langle 3 \rangle 5. \ x \in B \subseteq \bigcup A
           PROOF: From \langle 3 \rangle 3 and \langle 3 \rangle 4.
    \langle 2 \rangle 4. For all U, V \in \mathcal{T}, we have U \cap V \in \mathcal{T}
        \langle 3 \rangle 1. Let: U, V \in \mathcal{T}
        \langle 3 \rangle 2. Let: x \in U \cap V
        \langle 3 \rangle 3. Pick B_1, B_2 \in \mathcal{B} such that x \in B_1 \subseteq U and x \in B_2 \subseteq V
           PROOF: From \langle 3 \rangle 1, \langle 3 \rangle 2 and the definition of \mathcal{T}.
       \langle 3 \rangle 4. PICK B_3 \in \mathcal{B} such that x \in B_3 \subseteq B_1 \cap B_2
           PROOF: Using condition 2 (\langle 2 \rangle 1, \langle 3 \rangle 3).
        \langle 3 \rangle 5. \ x \in B_3 \subseteq U \cap V
           PROOF: From \langle 3 \rangle 3 and \langle 3 \rangle 4.
    \langle 2 \rangle 5. \bigcup \mathcal{B} = X
       PROOF: This is condition 1 (\langle 2 \rangle 1).
    \langle 2 \rangle 6. For all U \in \mathcal{T} and x \in U, there exists B \in \mathcal{B} such that x \in B \subseteq U
       PROOF: Immediate from the definition of \mathcal{T}.
\langle 1 \rangle 4. \mathcal{T} is unique.
   Proof: From Proposition 3.4.2.
```

 $\langle 1 \rangle 1$ . If  $\mathcal{B}$  is a basis for a topology then  $\bigcup \mathcal{B} = X$ 

**Corollary 3.4.3.1.** Let X be a set and  $\mathcal{B} \subseteq \mathcal{P}X$  be such that  $\bigcup \mathcal{B} = X$  and  $\mathcal{B}$  is closed under binary intersection. Then  $\mathcal{B}$  is a basis for a unique topology on X.

**Lemma 3.4.4.** Let  $\mathcal{B}$  and  $\mathcal{B}'$  be bases for the topologies  $\mathcal{T}$  and  $\mathcal{T}'$  on X respectively. Then  $\mathcal{T} \subseteq \mathcal{T}'$  if and only if, for all  $B \in \mathcal{B}$  and  $x \in B$ , there exists  $B' \in \mathcal{B}'$  such that  $x \in B' \subseteq B$ .

#### PROOF:

- $\langle 1 \rangle 1$ . If  $\mathcal{T} \subseteq \mathcal{T}'$  then, for all  $B \in \mathcal{B}$  and  $x \in B$ , there exists  $B' \in \mathcal{B}'$  such that  $x \in B' \subseteq B$ .
  - $\langle 2 \rangle 1$ . Assume:  $\mathcal{T} \subseteq \mathcal{T}'$
  - $\langle 2 \rangle 2$ . Let:  $B \in \mathcal{B}$  and  $x \in B$
  - $\langle 2 \rangle 3. \ B \in \mathcal{T}$

PROOF: This holds because  $\mathcal{B} \subseteq \mathcal{T}$  by the definition of basis.  $(\langle 2 \rangle 2)$ 

 $\langle 2 \rangle 4. \ B \in \mathcal{T}'$ 

PROOF: From  $\langle 2 \rangle 1$  and  $\langle 2 \rangle 3$ .

- $\langle 2 \rangle$ 5. There exists  $B' \in \mathcal{B}'$  such that  $x \in B' \subseteq B$ .
- $\langle 1 \rangle 2$ . If, for all  $B \in \mathcal{B}$  and  $x \in B$ , there exists  $B' \in \mathcal{B}'$  such that  $x \in B' \subseteq B$ , then  $\mathcal{T} \subseteq \mathcal{T}'$ .
  - $\langle 2 \rangle 1$ . Assume: For all  $B \in \mathcal{B}$  and  $x \in B$ , there exists  $B' \in \mathcal{B}'$  such that  $x \in B' \subseteq B$
  - $\langle 2 \rangle 2$ . Let:  $U \in \mathcal{T}$ Prove:  $U \in \mathcal{T}'$
  - $\langle 2 \rangle 3$ . Let:  $x \in U$
  - $\langle 2 \rangle 4$ . Pick  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$

PROOF: Since  $\mathcal{B}$  is a basis for  $\mathcal{T}$  ( $\langle 2 \rangle 2, \langle 2 \rangle 3$ ).

 $\langle 2 \rangle$ 5. Pick  $B' \in \mathcal{B}'$  such that  $x \in B' \subseteq B$ 

PROOF: From  $\langle 2 \rangle 1$  and  $\langle 2 \rangle 4$ .

 $\langle 2 \rangle 6. \ x \in B' \subseteq U$ 

PROOF: From  $\langle 2 \rangle 4$  and  $\langle 2 \rangle 5$ .

 $\langle 2 \rangle$ 7. Q.E.D.

Proof: By Proposition 3.4.2.

**Definition 3.4.5** (Lower Limit Topology). The *lower limit topology* on  $\mathbb{R}$  is the one generated by the set of all half-open intervals of the form [a,b). We write  $\mathbb{R}_l$  for the topological space consisting of  $\mathbb{R}$  under this topology.

We prove this is a topology.

#### Proof:

- $\langle 1 \rangle 1$ . Let:  $\mathcal{B}$  be the set of all half-open intervals of the form [a,b).
- $\langle 1 \rangle 2. \bigcup \mathcal{B} = \mathbb{R}$

PROOF: For all  $x \in \mathbb{R}$ , we have  $x \in [x, x + 1) \in \mathcal{B}$ .

 $\langle 1 \rangle 3$ . For all  $B_1, B_2 \in \mathcal{B}$  and  $x \in B_1 \cap B_2$ , there exists  $B_3 \in \mathcal{B}$  such that  $x \in B_3 \subseteq B_1 \cap B_2$ .

PROOF: If  $x \in [a, b) \cap [c, d)$  then  $x \in [\max(a, c), \min(b, d)) \subseteq [a, b) \cap [c, d)$ .

```
\langle 1 \rangle4. Q.E.D. PROOF: By Lemma 3.4.3.
```

**Definition 3.4.6** (K-topology). The K-topology on  $\mathbb{R}$  is the one generated by the set of all open intervals (a,b) and all sets of the form  $(a,b) \setminus K$ , where  $K = \{1/n : n \in \mathbb{Z}^+\}$ . We write  $\mathbb{R}_K$  for the topological space consisting of  $\mathbb{R}$  under this topology.

We prove this is a topology.

```
PROOF:
\langle 1 \rangle 1. Let: \mathcal{B} = \{(a,b) : a,b \in \mathbb{R}, a < b\} \cup \{(a,b) \setminus K : a,b \in \mathbb{R}, a < b\}
\langle 1 \rangle 2. \bigcup \mathcal{B} = \mathbb{R}
   PROOF: For all x \in \mathbb{R}, we have x \in (x - 1, x + 1) \in \mathcal{B}.
\langle 1 \rangle 3. For all B_1, B_2 \in \mathcal{B} and x \in B_1 \cap B_2, there exists B_3 \in \mathcal{B} such that
        x \in B_3 \subseteq B_1 \cap B_2.
   \langle 2 \rangle 1. Let: B_1, B_2 \in \mathcal{B} and x \in B_1 \cap B_2
            PROVE: There exists B_3 \in \mathcal{B} such that x \in B_3 \subseteq B_1 \cap B_2
   \langle 2 \rangle 2. Case: B_1 = (a, b), B_2 = (c, d)
       PROOF: Take B_3 = (\max(a, c), \min(b, d))
   \langle 2 \rangle 3. Case: B_1 = (a, b), B_2 = (c, d) \setminus K
       PROOF: Take B_3 = (\max(a, c), \min(b, d)) \setminus K
   \langle 2 \rangle 4. Case: B_1 = (a, b) \setminus K, B_2 = (c, d)
       PROOF: Take B_3 = (\max(a, c), \min(b, d)) \setminus K
   \langle 2 \rangle5. Case: B_1 = (a,b) \setminus K, B_2 = (c,d) \setminus K
       PROOF: Take B_3 = (\max(a, c), \min(b, d)) \setminus K
\langle 1 \rangle 4. Q.E.D.
   Proof: By Lemma 3.4.3.
```

**Lemma 3.4.7.** The lower limit topology and the K-topology are incomparable.

PROOF: [0,1) is not open in the K-topology.  $(-1,1)\setminus K$  is not open in the lower limit topology, because there is no half-open interval [a,b) such that  $0 \in [a,b) \subseteq (-1,1)\setminus K$ .  $\square$ 

**Proposition 3.4.8.** The set of all singletons is a basis for any discrete space.

Proof: Easy.

**Definition 3.4.9** (Line with Two Origins). The *line with two origins* is the set  $\mathbb{R} \setminus \{0\} \cup \{p,q\}$  under the topology generated by the basis consisting of:

- all open intervals in  $\mathbb{R}$  that do not contain 0;
- all sets of the form  $(-a,0) \cup \{p\} \cup (0,a)$  where a > 0;
- all sets of the form  $(-a,0) \cup \{q\} \cup (0,a)$  where a>0

## 3.5 Closed Sets

**Definition 3.5.1** (Closed). Let X be a topological space and  $A \subseteq X$ . Then A is *closed* iff  $X \setminus A$  is open.

**Proposition 3.5.2.** In any topological space X, the empty set  $\emptyset$  is closed.

PROOF: This holds because  $X \setminus \emptyset = X$  is open.  $\square$ 

**Proposition 3.5.3.** In any topological space X, the set X is closed.

PROOF: This holds because  $X \setminus X = \emptyset$  is open.  $\square$ 

Proposition 3.5.4. The union of two closed sets is closed.

PROOF: If C and D are closed then  $X \setminus (C \cup D) = (X \setminus C) \cup (X \setminus D)$  is open.  $\square$ 

**Proposition 3.5.5.** In any topological space, the intersection of a nonempty set of closed sets is closed.

PROOF: Let  $\mathcal{C}$  be a nonempty set of closed sets. Then  $X \setminus \bigcap \mathcal{C} = \bigcup \{X \setminus C : C \in \mathcal{C}\}$  is open.  $\square$ 

**Proposition 3.5.6.** Let X be a topological space and  $U \subseteq X$ . Then U is open if and only if  $X \setminus U$  is closed.

PROOF: Immediate from definitions.

**Theorem 3.5.7.** Let X be a set and  $C \subseteq PX$ . Suppose:

- 1.  $\emptyset, X \in \mathcal{C}$ ;
- 2. for all nonempty  $A \subseteq C$ , we have  $\bigcap A \in C$ ;
- 3. for all  $C, D \in \mathcal{C}$ , we have  $C \cup D \in \mathcal{C}$ .

Then there exists a unique topology on X under which  $\mathcal C$  is the set of all closed sets, namely

$$\mathcal{T} = \{U \subseteq X : X \setminus U \in \mathcal{C}\}$$

Proof:

- $\langle 1 \rangle 1$ . Let:  $\mathcal{C}$  be a set satisfying 1–3
- $\langle 1 \rangle 2$ . Let:  $\mathcal{T} = \{ X \setminus C : C \in \mathcal{C} \}$
- $\langle 1 \rangle 3$ .  $\mathcal{T}$  is a topology
  - $\langle 2 \rangle 1. \ X \in \mathcal{T}$

PROOF:  $X \setminus X = \emptyset \in \mathcal{C}$  by condition 1.

- $\langle 2 \rangle 2$ . For all  $\mathcal{A} \subseteq \mathcal{T}$  we have  $\bigcup \mathcal{A} \in \mathcal{T}$ .
  - $\langle 3 \rangle 1$ . Let:  $\mathcal{A} \subseteq \mathcal{T}$
  - $\langle 3 \rangle 2$ . Case:  $\mathcal{A} = \emptyset$

PROOF: In this case,  $X \setminus \bigcup A = X \in C$  by condition 1.

 $\langle 3 \rangle 3$ . Case:  $\mathcal{A}$  is nonempty

PROOF: In this case, we have  $X \setminus \bigcup \mathcal{A} = \bigcap \{X \setminus U : U \in \mathcal{A}\} \in \mathcal{C}$  by condition 2.

 $\langle 2 \rangle 3$ . For all  $U, V \in \mathcal{T}$  we have  $U \cap V \in \mathcal{T}$ 

PROOF:  $X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V) \in \mathcal{C}$  by condition 3.

 $\langle 1 \rangle 4$ . C is the set of closed sets.

Proof:

$$C \text{ is closed} \Leftrightarrow X \setminus C \in \mathcal{T}$$
 
$$\Leftrightarrow X \setminus (X \setminus C) \in \mathcal{C}$$
 
$$\Leftrightarrow C \in \mathcal{C}$$

 $\langle 1 \rangle 5$ .  $\mathcal{T}$  is unique.

Proof: By Proposition 3.5.6.

**Definition 3.5.8** (Closed Covering). A *closed covering* of a topological space is a covering in which every member is a closed set.

## 3.6 Locally Finite Families

**Definition 3.6.1** (Locally Finite). Let X be a topological space and  $\{A_i\}_{i\in I}$  a family of subsets of X. Then  $\{A_i\}_{i\in I}$  is *locally finite* iff, for all  $x\in X$ , there exists a neighbourhood N of x such that there are only finitely many  $i\in I$  such that N intersects  $A_i$ .

**Proposition 3.6.2.** If  $\{A_i\}_{i\in I}$  is locally finite and  $B_i\subseteq A_i$  for all i then  $\{B_i\}_{i\in I}$  is locally finite.

PROOF: Immediate from definitions.

Proposition 3.6.3. Every finite family of open sets is locally finite.

Proof: Trivial.

### 3.7 Closure of a Set

**Definition 3.7.1** (Closure). Let X be a topological space and  $A \subseteq X$ . The *closure* of A, Cl A or  $\overline{A}$ , is the intersection of all closed sets that include A.

PROOF: This intersection always exists because X is a closed set that includes A.  $\square$ 

**Proposition 3.7.2.** Let X be a topological space and  $A \subseteq X$ . Then  $A \subseteq \overline{A}$ .

PROOF: Immediate from definitions.

**Proposition 3.7.3.** Let X be a topological space and  $A \subseteq X$ . Then  $\overline{A}$  is closed.

PROOF: This follows from Proposition 3.5.5.

**Proposition 3.7.4.** Let X be a topological space and  $A, C \subseteq X$ . If  $A \subseteq C$  and C is closed then  $\overline{A} \subseteq C$ .

Proof: Immediate from definitions.  $\Box$ 

**Proposition 3.7.5.** Let X be a topological space and  $A, B \subseteq X$ . If  $A \subseteq B$  then  $\overline{A} \subseteq \overline{B}$ .

Proof:

 $\langle 1 \rangle 1$ . Assume:  $A \subseteq B$ 

 $\langle 1 \rangle 2$ .  $A \subseteq \overline{B}$ 

Proof: Proposition 3.7.2.

 $\langle 1 \rangle 3. \ \overline{A} \subseteq \overline{B}$ 

Proof: Propositions 3.7.3, 3.7.4.

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**Proposition 3.7.6.** Let X be a set and  $A \subseteq X$ . Then A is closed if and only if  $A = \overline{A}$ .

Proof:

- $\langle 1 \rangle 1$ . If A is closed then  $A = \overline{A}$ 
  - $\langle 2 \rangle 1$ . Assume: A is closed
  - $\langle 2 \rangle 2$ .  $A \subseteq \overline{A}$

PROOF: By Proposition 3.7.2.

 $\langle 2 \rangle 3. \ \overline{A} \subseteq A$ 

PROOF: By Proposition 3.7.4 since  $A \subseteq A$ .

 $\langle 1 \rangle 2$ . If  $A = \overline{A}$  then A is closed.

Proof: By Proposition 3.7.3.

П

Corollary 3.7.6.1.

$$\overline{\emptyset} = \emptyset$$

**Theorem 3.7.7** (Kuratowski Closure Axioms). Let X be a set and  $(-): \mathcal{P}X \to \mathcal{P}X$  be a function such that:

- 1.  $\overline{\emptyset} = \emptyset$
- 2. For all  $A \subseteq X$ ,  $A \subseteq \overline{A}$
- 3. For all  $A \subseteq X$ ,  $\overline{A} = \overline{\overline{A}}$
- 4. For all  $A, B \subseteq X$ ,  $\overline{A \cup B} = \overline{A} \cup \overline{B}$

Then there exists a unique topology  $\mathcal{T}$  on X such that  $\overline{A}$  is the closure of A for all  $A \in \mathcal{P}X$ .

PROOF:

- $\langle 1 \rangle 1$ . For all  $C, D \subseteq X$ , if  $C \subseteq D$  then  $\overline{C} \subseteq \overline{D}$ 
  - $\langle 2 \rangle 1$ . Assume:  $C \subseteq D$

$$\langle 2 \rangle 2. \ \overline{C} = \overline{D}$$

Proof:

$$\overline{D} = \overline{C \cup D}$$
 (\langle 2\rangle 1)  
=  $\overline{C} \cup \overline{D}$  (axiom 4)

 $\langle 1 \rangle 2$ . Let:  $\mathcal{T}$  be the topology in which a set C is closed iff  $\overline{C} = C$ .

 $\langle 2 \rangle 1. \ \overline{\emptyset} = \emptyset$ 

PROOF: This is axiom 1.

 $\langle 2 \rangle 2. \ \overline{X} = X$ 

PROOF: By axiom 2.

 $\langle 2 \rangle 3$ . For any set  $\mathcal{A}$  of sets C such that  $\overline{C} = C$ , we have  $\overline{\bigcap \mathcal{A}} = \bigcap \mathcal{A}$ 

$$\langle 3 \rangle 1. \ \overline{\bigcap \mathcal{A}} \subseteq \bigcap \mathcal{A}$$

 $\langle 4 \rangle 1$ . Let:  $C \in \mathcal{A}$ 

$$\langle 4 \rangle 2. \ \overline{\bigcap \mathcal{A}} \subseteq C$$

Proof:

 $\langle 3 \rangle 2$ . Q.E.D.

 $\langle 2 \rangle 4$ . If  $\overline{C} = C$  and  $\overline{D} = D$  then  $\overline{C \cup D} = C \cup D$ 

PROOF: By axiom 4.

 $\langle 2 \rangle$ 5. Q.E.D.

PROOF: By Theorem 3.5.7.

 $\langle 1 \rangle 3$ . For all  $A \subseteq X$ , the closure of A in  $\mathcal{T}$  is  $\overline{A}$ 

 $\langle 2 \rangle 1$ .  $\overline{A}$  is closed

Proof: From axiom 3.

 $\langle 2 \rangle 2$ . If  $A \subseteq C$  and C is closed then  $\overline{A} \subseteq C$ 

Proof:

$$C = \overline{C}$$
 (C is closed)  
 $= \overline{A \cup C}$  ( $A \subseteq C$ )  
 $= \overline{A} \cup \overline{C}$  (axiom 4)

**Theorem 3.7.8.** Let A be a subset of the topological space X and  $\mathcal{B}$  a basis for X. Then  $x \in \overline{A}$  if and only if, for all  $B \in \mathcal{B}$ , if  $x \in B$  then B intersects A.

#### PROOF:

 $\langle 1 \rangle 1$ . If  $x \in \overline{A}$  then, for all  $B \in \mathcal{B}$ , if  $x \in B$  then B intersects A.

PROOF: Immediate from Theorem 3.8.3.

- $\langle 1 \rangle 2$ . If, for all  $B \in \mathcal{B}$ , if  $x \in B$  then B intersects A, then  $x \in \overline{A}$ .
  - $\langle 2 \rangle 1$ . Assume: for all  $B \in \mathcal{B}$ , if  $x \in B$  then B intersects A.
  - $\langle 2 \rangle 2$ . Let: U be a neighbourhood of x
  - $\langle 2 \rangle 3$ . Pick  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$

PROOF:  $\mathcal{B}$  is a basis.

 $\langle 2 \rangle 4$ . B intersects A.

PROOF: By  $\langle 2 \rangle 1$ .

 $\langle 2 \rangle 5$ . U intersects A.

 $\langle 2 \rangle 6$ . Q.E.D.

PROOF: By Theorem 3.8.3.

**Lemma 3.7.9.** If  $\{A_i\}_{i\in I}$  is locally finite then so is  $\{\overline{A_i}\}_{i\in I}$ .

### Proof:

- $\langle 1 \rangle 1$ . Let:  $\{A_i\}_{i \in I}$  be a locally finite family of subsets of the space X.
- $\langle 1 \rangle 2$ . Let:  $x \in X$
- $\langle 1 \rangle 3$ . PICK a neighbourhood U of x that intersects only  $A_{i_1}, \ldots, A_{i_n}$ .
- $\langle 1 \rangle 4$ . *U* intersects only  $\overline{A_{i_1}}, \ldots, \overline{A_{i_n}}$ .

**Lemma 3.7.10.** Let  $\{A_i\}_{i\in I}$  be locally finite. Then  $\overline{\bigcup_{i\in I} A_i} = \bigcup_{i\in I} \overline{A_i}$ .

### Proof:

- $\langle 1 \rangle 1$ . Let:  $x \in \bigcup_{i \in I} A_i$
- $\langle 1 \rangle 2$ . PICK a neighbourhood U of x that intersects only  $A_{i_1}, \ldots, A_{i_n}$ .

 $\langle 1 \rangle 3.$   $x \in \overline{A_{i_1}} \cup \cdots \cup \overline{A_{i_n}}$ PROOF: If not, then  $U - \overline{A_{i_1}} - \cdots - \overline{A_{i_n}}$  would be a neighbourhood of x that does not intersect  $\bigcup_{i \in I} A_i$ .

#### Interior of a Set 3.8

**Definition 3.8.1** (Interior). Let X be a topological space and  $A \subseteq X$ . The interior of A, Int A, is the union of all open sets included in A.

**Lemma 3.8.2.** If  $A \subseteq B$  then  $\overline{A} \subseteq \overline{B}$ .

PROOF:  $\overline{B}$  is a closed set that includes B, hence includes A.  $\square$ 

**Theorem 3.8.3.** Let A be a subset of the topological space X and  $x \in X$ . Then  $x \in \overline{A}$  if and only if every neighbourhood of x intersects A.

PROOF:

$$x \notin \overline{A} \Leftrightarrow \exists C \text{ closed } (A \subseteq C \land x \notin C)$$
  
 $\Leftrightarrow \exists U \text{ open } (A \subseteq X \setminus U \land x \in U)$   
 $\Leftrightarrow \exists U \text{ open } (A \text{ intersects } U \land x \in U)$ 

Lemma 3.8.4.

$$X \setminus \operatorname{Int} A = \overline{X \setminus A}$$

Proof:

- $\langle 1 \rangle 1. \ X \setminus \operatorname{Int} A \subseteq \overline{X \setminus A}$ 
  - $\langle 2 \rangle 1. \ X \setminus A \subseteq \overline{X \setminus A}$
  - $\langle 2 \rangle 2. \ X \setminus \overline{X \setminus A} \subseteq A$

## 3.9 Boundary

**Definition 3.9.1** (Boundary). Let X be a topological space and  $A \subseteq X$ . The boundary of A, Bd A, is  $\overline{A} \cap \overline{X} \setminus \overline{A}$ .

Lemma 3.9.2.

$$\operatorname{Bd} A = \overline{A} \setminus \operatorname{Int} A$$

PROOF: From Lemma 3.8.4.  $\square$ 

**Lemma 3.9.3.**  $\overline{A} = \operatorname{Int} A \cup \operatorname{Bd} A$ 

PROOF:

$$\operatorname{Int} A \cup \operatorname{Bd} A = \operatorname{Int} A \cup (\overline{A} \cap (X \setminus \operatorname{Int} A))$$

$$= \operatorname{Int} A \cup \overline{A}$$

$$= \overline{A}$$

Corollary 3.9.3.1. Bd  $A = \emptyset$  iff A is open and closed.

**Lemma 3.9.4.** For any set U, the following are equivalent:

- 1. U is open.
- 2. Bd  $U \cap U = \emptyset$
- 3. Bd  $U = \overline{U} \setminus U$

Proof:

 $\langle 1 \rangle 1. \ 1 \Rightarrow 3$ 

PROOF: From Lemma 3.9.2.

 $\langle 1 \rangle 2. \ 3 \Rightarrow 2$ 

PROOF: Set theory.

 $\langle 1 \rangle 3. \ 2 \Rightarrow 1$ 

Proof:

$$\begin{split} U \subseteq \overline{U} \\ &= \operatorname{Int} U \cup \operatorname{Bd} U \\ & :: U \subseteq \operatorname{Int} U \end{split}$$
 (Lemma 3.9.3)

### 3.10 Limit Points

**Definition 3.10.1** (Limit Point). Let X be a topological space,  $A \subseteq X$ , and  $x \in X$ . Then x is a *limit point*, *cluster point* or *point of accumulation* of A iff every neighbourhood of x intersects A in a point other than x.

**Lemma 3.10.2.** If  $A \subseteq B$  then every limit point of A is a limit point of B.

PROOF: Immediate from the definition.  $\square$ 

**Theorem 3.10.3.** Let A be a subset of the topological space X. Let A' be the set of all limit points of A. Then  $\overline{A} = A \cup A'$ .

#### Proof:

 $\langle 1 \rangle 1$ . If  $x \in \overline{A}$  and  $x \notin A$  then  $x \in A'$ 

PROOF: in this case, every neighbourhood of x intersects A in a point other than x.

 $\langle 1 \rangle 2$ .  $A \subseteq \overline{A}$ 

PROOF: From the definition of  $\overline{A}$ .

 $\langle 1 \rangle 3. \ A' \subseteq \overline{A}$ 

PROOF: By Theorem 3.8.3.

Corollary 3.10.3.1. A set is closed if and only if it contains all its limit points.

#### 3.11 Subbases

**Definition 3.11.1** (Subbasis). Let X be a topological space. A *subbasis* for the topology on X is a set  $S \subseteq \mathcal{P}X$  such that, for every open set U and  $x \in U$ , there exist  $S_1, \ldots, S_n \in \mathcal{S}$  such that  $x \in S_1 \cap \cdots \cap S_n \subseteq U$ . We say the topology is *generated* by S.

**Lemma 3.11.2.** Let  $\mathcal{T}$  be a topology on X and  $S \subseteq \mathcal{P}X$ . Then the following are equivalent:

- 1. S is a subbasis for T.
- 2. The set of all finite intersections of members of S is a basis for T
- 3.  $\mathcal{T}$  is the set of all unions of finite intersections of members of  $\mathcal{S}$ .

PROOF: 1  $\Leftrightarrow$  2 holds immediately from the definitions. 2  $\Leftrightarrow$  3 holds by Proposition 3.4.2.  $\Box$ 

**Corollary 3.11.2.1.** If S is a subbasis for the topology T, then T is the coarsest topology in which every element of S is open.

**Lemma 3.11.3.** Let X be a set and  $S \subseteq PX$ . Then S is a subbasis for a topology on X if and only if  $\bigcup S = X$ .

```
Proof:
```

```
\langle 1 \rangle 1. If S is a subbasis for a topology on X then \bigcup S = X
   \langle 2 \rangle 1. Assume: S is a subbasis for a topology T on X.
   \langle 2 \rangle 2. Let: x \in X
   \langle 2 \rangle 3. Pick S_1, \ldots, S_n \in \mathcal{S} such that x \in S_1 \cap \cdots \cap S_n \subseteq X
      PROOF: From the definition of subbasis (\langle 2 \rangle 1, \langle 2 \rangle 2).
   \langle 2 \rangle 4. \ x \in \bigcup S
      PROOF: Immediate from \langle 2 \rangle 3.
\langle 1 \rangle 2. If \bigcup S = X then S is a subbasis for a topology on X
   \langle 2 \rangle 1. Assume: \bigcup S = X
            Prove: The set of all finite intersections of elements of S is a basis
                           for a topology on X.
   \langle 2 \rangle 2. Let: \mathcal{B} be the set of all finite intersections of elements of \mathcal{S}.
   \langle 2 \rangle 3. | \mathcal{B} = X
      PROOF: From \langle 2 \rangle 1 and \langle 2 \rangle 2.
   \langle 2 \rangle 4. For all B_1, B_2 \in \mathcal{B} and x \in B_1 \cap B_2, there exists B_3 \in \mathcal{B} such that
            x \in B_3 \subseteq B_1 \cap B_2
      PROOF: Take B_3 = B_1 \cap B_2 (\langle 2 \rangle 2).
   \langle 2 \rangle 5. \mathcal{B} is a basis for a topology on X.
      Proof: By Lemma 3.4.3.
   \langle 2 \rangle 6. Q.E.D.
      Proof: By Lemma 3.11.2.
```

## 3.12 Convergence

**Definition 3.12.1** (Net). Let X be a topological space. A net  $(x_{\alpha})_{\alpha \in J}$  in X consists of a directed set J and a function  $x: J \to X$ .

**Definition 3.12.2** (Convergence). Let  $(x_{\alpha})_{\alpha \in J}$  be a net in the topological space X, and  $l \in X$ . Then the net *converges* to l,  $x_{\alpha} \to l$ , if and only if, for every neighbourhood U of l, there exists  $\alpha \in J$  such that, for all  $\beta \geq \alpha$ , we have  $x_{\beta} \in U$ .

**Theorem 3.12.3** (AC). Let X be a topological space and  $A \subseteq X$ . Then  $x \in \overline{A}$  if and only if there exists a net of points of A converging to x.

#### Proof:

- $\langle 1 \rangle 1$ . If  $x \in \overline{A}$  then there exists a net of points of A converging to x.
  - $\langle 2 \rangle 1$ . Let:  $x \in \overline{A}$
  - $\langle 2 \rangle 2$ . Let: J be the poset of neighbourhoods of x under  $\supseteq$ .
  - $\langle 2 \rangle 3$ . For  $U \in J$  Pick a point  $x_U \in U \cap A$

PROOF: By Theorem 3.8.3

 $\langle 2 \rangle 4$ .  $(x_U)_{U \in J}$  is a net

PROOF: Given  $U, V \in J$  we have  $U \cap V \in J$  and  $U \supseteq U \cup V$ ,  $V \supseteq U \cup V$ .

 $\langle 2 \rangle 5. \ x_U \to x$ 

```
PROOF: For any neighbourhood U of x we have U \in J and if U \supseteq V then x_U \in U
```

- $\langle 1 \rangle 2$ . If there exists a net of points of A converging to x then  $x \in \overline{A}$ .
  - $\langle 2 \rangle 1$ . Let:  $(x_{\alpha})_{\alpha \in J}$  be a net of points in A that converges to x.
  - $\langle 2 \rangle 2$ . Let: U be a neighbourhood of x
  - $\langle 2 \rangle 3$ . PICK  $\alpha \in J$  such that, for all  $\beta \geq \alpha$ , we have  $x_{\beta} \in U$
  - $\langle 2 \rangle 4. \ x_{\alpha} \in U \cap A$
  - $\langle 2 \rangle$ 5. Q.E.D.

PROOF: By Theorem 3.8.3

**Theorem 3.12.4.** Let X and Y be topological spaces and  $f: X \to Y$ . Then f is continuous if and only if, for every net  $(x_{\alpha})_{\alpha \in J}$  in X, if  $x_{\alpha} \to x$  then  $f(x_{\alpha}) \to f(x)$ .

#### PROOF:

- $\langle 1 \rangle 1$ . If f is continuous and  $x_{\alpha} \to x$  then  $f(x_{\alpha}) \to f(x)$ 
  - $\langle 2 \rangle 1$ . Assume: f is continuous.
  - $\langle 2 \rangle 2$ . Assume:  $x_{\alpha} \to x$
  - $\langle 2 \rangle 3$ . Let: V be a neighbourhood of f(x)
  - $\langle 2 \rangle 4$ .  $f^{-1}(V)$  is a neighbourhood of x
  - $\langle 2 \rangle 5$ . PICK  $\alpha$  such that, for all  $\beta \geq \alpha$ , we have  $x_{\beta} \in f^{-1}(V)$
  - $\langle 2 \rangle 6$ . For all  $\beta \geq \alpha$  we have  $f(x_{\beta}) \in V$
- $\langle 1 \rangle 2$ . If, for every net  $(x_{\alpha})$  in X, if  $x_{\alpha} \to x$  then  $f(x_{\alpha}) \to f(x)$ , then f is continuous.
  - $\langle 2 \rangle 1$ . Assume: for every net  $(x_{\alpha})$  in X, if  $x_{\alpha} \to x$  then  $f(x_{\alpha}) \to f(x)$
  - $\langle 2 \rangle 2$ . Let:  $A \subseteq X$ Prove:  $f(\overline{A}) \subseteq \overline{f(A)}$
  - $\langle 2 \rangle 3$ . Let:  $x \in \overline{A}$
  - $\langle 2 \rangle 4$ . PICK a net  $(x_{\alpha})$  in A such that  $x_{\alpha} \to x$

PROOF: Theorem 3.12.3

 $\langle 2 \rangle 5. \ f(x_{\alpha}) \to f(x)$ 

Proof: By  $\langle 2 \rangle 1$ 

 $\langle 2 \rangle 6. \ f(x) \in f(A)$ 

Proof: Theorem 3.12.3

 $\langle 2 \rangle 7$ . Q.E.D.

PROOF: By Theorem 5.2.2.

**Definition 3.12.5** (Subnet). Let  $(x_{\alpha})_{\alpha \in J}$  be a net in X. Let K be a directed set and  $g: K \to J$  be a monotone function such that g(K) is cofinal in J. Then the net  $(x_{g(\beta)})_{\beta \in K}$  is called a *subnet* of  $(x_{\alpha})$ .

#### 3.13 Accumulation Points

**Definition 3.13.1** (Accumulation Point). Let X be a topological space, and  $(x_{\alpha})_{\alpha \in J}$  a net in X, and  $a \in X$ . Then a is an accumulation point of  $(x_{\alpha})$  iff,

for every neighbourhood U of x, the set  $\{\alpha \in J : x_{\alpha} \in U\}$  is cofinal in J.

**Lemma 3.13.2.** Let X be a topological space,  $(x_{\alpha})_{\alpha \in J}$  be a nonempty net in X and  $a \in X$ . Then a is an accumulation point of  $(x_{\alpha})$  if and only if there exists a subnet of  $(x_{\alpha})$  that converges to a.

#### Proof:

- $\langle 1 \rangle 1$ . If a is an accumulation point of  $(x_{\alpha})$  then there exists a subnet of  $(x_{\alpha})$  that converges to a.
  - $\langle 2 \rangle 1$ . Assume: a is an accumulation point of  $(x_{\alpha})$ .
  - $\langle 2 \rangle 2$ . Let: K be the poset  $\{(\alpha, U) : \alpha \in J, U \text{ is a neighbourhood of } a, x_{\alpha} \in U\}$  under:  $(\alpha, U) \leq (\beta, V)$  iff  $\alpha \leq \beta$  and  $U \subseteq V$ .
  - $\langle 2 \rangle 3. \ (x_{\alpha})_{(\alpha,U) \in K}$  is a subnet of  $(x_{\alpha})_{\alpha \in J}$ 
    - $\langle 3 \rangle 1$ . K is directed.
      - $\langle 4 \rangle 1$ . Let:  $(\alpha, U), (\beta, V) \in K$
      - $\langle 4 \rangle 2$ . PICK  $\gamma \in J$  such that  $\alpha \leq \gamma$  and  $\beta \leq \gamma$ .
      - $\langle 4 \rangle 3$ . PICK  $\delta \in J$  such that  $\gamma \leq \delta$  and  $x_{\delta} \in U \cap V$ PROOF: By  $\langle 2 \rangle 1$ .
      - $\langle 4 \rangle 4$ .  $(\delta, U \cap V) \in K$  and  $(\alpha, U) \leq (\delta, U \cap V)$ ,  $(\beta, V) \leq (\delta, U \cap V)$
    - $\langle 3 \rangle 2$ . If  $(\alpha, U) \leq (\beta, V)$  then  $\alpha \leq \beta$

PROOF: From  $\langle 2 \rangle 2$ .

 $\langle 3 \rangle 3$ .  $\{\alpha : \exists U.(\alpha, U) \in K\}$  is cofinal in J

PROOF: For  $\alpha \in J$  we have  $(\alpha, X) \in K$ , so in fact  $\{\alpha : \exists U . (\alpha, U) \in K\} = J$ .

- $\langle 2 \rangle 4$ . The subnet converges to a.
  - $\langle 3 \rangle 1$ . Let: *U* be a neighbourhood of *a*.
  - $\langle 3 \rangle 2$ . Pick  $\alpha \in J$
  - $\langle 3 \rangle 3$ . PICK  $\beta \in J$  such that  $\alpha \leq \beta$  and  $x_{\beta} \in U$

Proof: By  $\langle 2 \rangle 1$ .

- $\langle 3 \rangle 4$ . For all  $(\gamma, V) \geq (\beta, U)$  we have  $x_{\gamma} \in U$
- PROOF:  $x_{\gamma} \in V \subseteq U$ .  $\langle 1 \rangle 2$ . If there exists a subnet of  $(x_{\alpha})$  that converges to a then a is an accumu-
- lation point of  $(x_{\alpha})$ .  $\langle 2 \rangle 1$ . Assume:  $(x_{q(\beta)})_{\beta \in K}$  converges to a
  - $\langle 2 \rangle$ 1. ASSUME.  $(a_{g(\beta)})_{\beta \in K}$  converges to  $\langle 2 \rangle$ 2. Let: U be a neighbourhoof of a
  - $\langle 2 \rangle 3$ . Let:  $\alpha \in J$

PROVE: There exists  $\gamma \geq \alpha$  such that  $x_{\gamma} \in U$ 

- $\langle 2 \rangle 4$ . PICK  $\beta \in K$  such that, for all  $\beta' \geq \beta$ , we have  $x_{g(\beta')} \in U$  PROOF: By  $\langle 2 \rangle 1$ .
- $\langle 2 \rangle$ 5. PICK  $\beta' \in K$  such that  $g(\beta') \geq \alpha$

PROOF: Since g(K) is cofinal in J.

 $\langle 2 \rangle 6$ . Pick  $\beta'' \in K$  such that  $\beta \leq \beta''$  and  $\beta' \leq \beta''$ 

PROOF: K is directed.

 $\langle 2 \rangle$ 7.  $g(\beta'') \geq \alpha$  and  $x_{g(\beta'')} \in U$ 

## 3.14 Dense Sets

**Definition 3.14.1** (Dense). Let X be a topological space and  $A \subseteq X$ . Then A is *dense* in X iff  $\overline{A} = X$ .

## 3.15 $G_{\delta}$ Sets

**Definition 3.15.1** ( $G_{\delta}$  Set). A  $G_{\delta}$  set is the intersection of a countable set of open sets.

## 3.16 Separated Sets

**Definition 3.16.1** (Separated Sets). Let X be a topological space and  $A, B \subseteq X$ . Then A and B are *separated* iff  $\overline{A} \cap B = \emptyset$  and  $A \cap \overline{B} = \emptyset$ .

## 3.17 Coherent Topology

**Definition 3.17.1** (Coherent Topology). Let  $X_1 \subseteq X_2 \subseteq \cdots$  be a sequence of topological spaces such that each  $X_n$  is a closed subspace of  $X_{n+1}$ . Let  $X = \bigcup_{n=1}^{\infty} X_n$ . Then the topology on X coherent with the subspaces  $X_n$  is the topology defined by:  $U \subseteq X$  is open iff  $U \cap X_n$  is open in  $X_n$  for all n.

## Chapter 4

# Constructions of Topological Spaces

## 4.1 The Order Topology

**Definition 4.1.1** (Order Topology). Let X be a linearly ordered set with more than one element. The *order topology* on X is the topology generated by the basis consisting of:

- all open intervals (a, b)
- all half-open intervals  $(a, \top]$  where  $\top$  is the greatest element of X, if there is one:
- all half-open intervals  $[\bot, a)$  where  $\bot$  is the least element of X, if there is one

We prove this is a basis for a topology.

#### Proof:

```
\langle 1 \rangle 1. Let: \mathcal{B} be the set of all sets of these three forms.
```

- $\langle 1 \rangle 2. \ \bigcup \mathcal{B} = X$ 
  - $\langle 2 \rangle 1$ . Let:  $x \in X$

PROVE: There exists  $B \in \mathcal{B}$  such that  $x \in B$ 

- $\langle 2 \rangle 2$ . Case: x is least in X
  - $\langle 3 \rangle 1$ . PICK  $a \in X$  such that a > x

PROOF: X has more than one element.

- $\langle 3 \rangle 2. \ x \in [x, a) \in \mathcal{B}$
- $\langle 2 \rangle 3$ . Case: x is greatest in X
  - $\langle 3 \rangle 1$ . PICK  $a \in X$  such that a < x

PROOF: X has more than one element.

- $\langle 3 \rangle 2. \ x \in (a, x] \in \mathcal{B}$
- $\langle 2 \rangle 4$ . Case: x is neither least nor greatest in X

```
\langle 3 \rangle 1. PICK a, b \in X such that a < x < b
      \langle 3 \rangle 2. \ x \in (a,b) \in \mathcal{B}
\langle 1 \rangle 3. For all B_1, B_2 \in \mathcal{B} and x \in B_1 \cap B_2, there exists B_3 \in \mathcal{B} such that
       x \in B_3 \subseteq B_1 \cap B_2
   \langle 2 \rangle 1. Let: B_1, B_2 \in \mathcal{B} and x \in B_1 \cap B_2
  \langle 2 \rangle 2. Case: B_1 = (a, b), B_2 = (c, d)
      PROOF: Take B_3 = (\max(a, c), \min(b, d)).
   \langle 2 \rangle 3. Case: B_1 = (a, b), B_2 = (c, \top]
      PROOF: Take B_3 = (\max(a, c), b).
   \langle 2 \rangle 4. Case: B_1 = (a, b), B_2 = [\bot, d)
      PROOF: Take B_3 = (a, \min(b, d)).
   \langle 2 \rangle 5. Case: B_1 = (a, \top], B_2 = (c, d)
      Proof: Similar to \langle 2 \rangle 3.
   \langle 2 \rangle 6. Case: B_1 = (a, \top], B_2 = (c, \top]
      PROOF: Take B_3 = (\max(a, c), \top].
   \langle 2 \rangle 7. Case: B_1 = (a, \top], B_2 = [\bot, d)
      PROOF: Take B_3 = (a, d).
   \langle 2 \rangle 8. Case: B_1 = [\bot, b), B_2 = (c, d)
      Proof: Similar to \langle 2 \rangle 4.
   \langle 2 \rangle 9. Case: B_1 = [\bot, b), B_2 = (c, \top]
      PROOF: Simlar to \langle 2 \rangle 7.
   \langle 2 \rangle 10. Case: B_1 = [\bot, b), B_2 = [\bot, d)
      PROOF: Take B_3 = [\bot, \min(b, d)).
\langle 1 \rangle 4. Q.E.D.
  Proof: By Lemma 3.4.3.
Then:
   1. Either a is greatest in X, or there exists a' > a such that [a, a') \subseteq U
```

**Lemma 4.1.2.** Let X be a linearly ordered set,  $U \subseteq X$  be open, and  $a \in U$ .

2. Either a is least in X, or there exists a' < a such that  $(a', a] \subseteq U$ .

```
\langle 1 \rangle 1. Either a is greatest in X, or there exists a' > a such that [a, a') \subseteq U
```

- $\langle 2 \rangle 1$ . Assume: a is not greatest in X
- $\langle 2 \rangle 2$ . PICK a basic open set B such that  $a \in B \subseteq U$
- $\langle 2 \rangle 3$ . Case: B = (a'', a')

Proof: a < a' and  $[a, a') \subseteq B \subseteq U$ 

 $\langle 2 \rangle 4$ . Case:  $B = [\bot, a')$ 

Proof: a < a' and  $[a, a') \subseteq B \subseteq U$ 

 $\langle 2 \rangle 5$ . Case:  $B = (a'', \top]$ 

PROOF: Pick any a' > a (one exists by  $\langle 2 \rangle 1$ ). Then  $[a, a') \subseteq B \subseteq U.S$ 

 $\langle 1 \rangle 2$ . Either a is least in X, or there exists a' < a such that  $(a', a] \subseteq U$ . Proof: Similar.

```
Lemma 4.1.3. The open rays form a subbasis for the order topology.
```

```
\langle 1 \rangle 1. Let: X be a linearly ordered set with more than one element.
\langle 1 \rangle 2. The open rays form a subbasis for a topology.
    \langle 2 \rangle 1. Let: x \in X
            PROVE: x is an element of an open ray.
   \langle 2 \rangle 2. Case: x is greatest in X
       \langle 3 \rangle 1. PICK a \in X such that a < x
          PROOF: X has more than one element (\langle 1 \rangle 1).
       \langle 3 \rangle 2. \ x \in (a, +\infty)
    \langle 2 \rangle 3. Case: x is not greatest in X
       \langle 3 \rangle 1. PICK a \in X such that x < a
       \langle 3 \rangle 2. \ x \in (-\infty, a)
    \langle 2 \rangle 4. Q.E.D.
       Proof: By Lemma 3.11.2.
\langle 1 \rangle 3. Let: \mathcal{T}_o be the order topology and \mathcal{T}_S be the topology generated by the
                  open rays.
\langle 1 \rangle 4. \mathcal{T}_o \subseteq \mathcal{T}_S
    \langle 2 \rangle 1. Every open interval (a,b) is open in \mathcal{T}_S
       PROOF: (a, b) = (a, +\infty) \cap (-\infty, b).
    \langle 2 \rangle 2. If \top is greatest then (a, \top] is open in \mathcal{T}_S
       PROOF: (a, \top] = (a, +\infty).
    \langle 2 \rangle 3. If \perp is least then [\perp, b) is open in \mathcal{T}_S
       PROOF: [\bot, b) = [\bot, +\infty).
    \langle 2 \rangle 4. Q.E.D.
       Proof: By Corollary 3.4.2.1.
\langle 1 \rangle 5. \mathcal{T}_S \subseteq \mathcal{T}_o
    \langle 2 \rangle 1. For all a \in X, we have (a, +\infty) is open in \mathcal{T}_o
       \langle 3 \rangle 1. Let: x \in (a, +\infty)
                PROVE: There exists a basis element B such that x \in B \subseteq (a, +\infty)
       \langle 3 \rangle 2. Case: x is greatest
          PROOF: Take B = (a, x]
       \langle 3 \rangle 3. Case: x is not greatest
          \langle 4 \rangle 1. Pick b > x
          \langle 4 \rangle 2. \ x \in (a,b) \subseteq (a,+\infty)
    \langle 2 \rangle 2. For all a \in X, we have (-\infty, a) is open in \mathcal{T}_a
       Proof: Similar.
   \langle 2 \rangle 3. Q.E.D.
       Proof: By Corollary 3.11.2.1.
```

**Lemma 4.1.4.** In a linearly ordered set X under the order topology, the closed intervals and closed rays are closed.

Proof:

$$X \setminus [a, b] = (-\infty, a) \cup (b, +\infty)$$

$$X \setminus (-\infty, a] = (a, +\infty)$$

$$X \setminus [a, +\infty) = (-\infty, a)$$

**Definition 4.1.5** (Standard Topology on  $\mathbb{R}$ ). The *standard topology* on  $\mathbb{R}$  is the order topology.

**Lemma 4.1.6.** The standard topology is strictly coarser than the lower limit topology.

#### PROOF:

- $\langle 1 \rangle 1$ . The standard topology is coarser than the lower limit topology.
  - $\langle 2 \rangle 1$ . For every open interval (a,b) and  $x \in (a,b)$ , there exists a half-open interval [c,d) such that  $x \in [c,d) \subseteq (a,b)$

PROOF: Take [c,d) = [x,b).

 $\langle 2 \rangle 2$ . Q.E.D.

Proof: By Lemma 3.4.4.

 $\langle 1 \rangle$ 2. There exists a set U open in the lower limit topology that is not open in the standard topology.

PROOF: Take U = [0, 1).

**Lemma 4.1.7.** The standard topology is strictly coarser than the K-topology.

#### PROOF

 $\langle 1 \rangle 1$ . The standard topology is coarser than the K-topology.

PROOF: Every open interval is open in the K-topology.

 $\langle 1 \rangle 2$ . There exists a set U open in the K-topology that is not open in the standard topology.

PROOF: Take  $U = (-1,1) \setminus K$ . Then  $0 \in U$  but there is no open interval (a,b) such that  $0 \in (a,b) \subseteq U$ .

**Definition 4.1.8** (Ordered Square). The ordered square  $I_o^2$  is the topological space  $[0,1]^2$  under the order topology induced by the lexicographic order.

**Lemma 4.1.9.** Let L be a linear continuum with a greatest element. Then every non-empty closed set in L has a greatest element.

#### Proof:

- $\langle 1 \rangle 1$ . Let: C be a non-empty closed set in L
- $\langle 1 \rangle 2$ . Let: u be the supremum of C
- $\langle 1 \rangle 3. \ u \in C$ 
  - $\langle 2 \rangle 1$ . Assume: w.l.o.g u is not least in L

PROOF: If u is least then  $C = \{u\}$ .

- $\langle 2 \rangle 2$ . Let: U be any open neighbourhood of u
- $\langle 2 \rangle 3$ . Pick v < u such that  $(v, u] \subseteq U$

PROOF: By Lemma 4.1.2.  $\langle 2 \rangle$ 4. PICK  $x \in C$  such that v < x PROOF: v is not an upper bound for C ( $\langle 1 \rangle$ 2).  $\langle 2 \rangle$ 5. U intersects C in v  $\langle 2 \rangle$ 6. Q.E.D. PROOF: By Theorem 3.8.3.

**Definition 4.1.10** (Long Line). The *long line* is  $(S_{\Omega} \times [0,1)) \setminus \{(0,0)\}$  under the dictionary order, where  $S_{\Omega}$  is the first uncountable ordinal under the order topology.

## 4.2 The Product Topology

**Definition 4.2.1** (Product Topology). Let  $\{X_{\alpha}\}_{{\alpha}\in J}$  be a family of topological spaces. The *product topology* on  $\prod_{{\alpha}\in J} X_{\alpha}$  is the topology generated by the subbasis consisting of all sets of the form  $\pi_{\alpha}^{-1}(U)$  where  ${\alpha}\in J$  and U is open in  $X_{\alpha}$ . The *product space* of  $\{X_{\alpha}\}_{{\alpha}\in J}$  is  $\prod_{{\alpha}\in J} X_{\alpha}$  under the product topology.

**Lemma 4.2.2.** Let  $\{X_{\alpha}\}_{{\alpha}\in J}$  be a family of topological spaces and  $A_{\alpha}$  be closed in  $X_{\alpha}$  for all  $\alpha$ . Then  $\prod_{{\alpha}\in J}A_{\alpha}$  is closed in  $\prod_{{\alpha}\in J}X_{\alpha}$ .

PROOF: This holds because  $\prod_{\alpha \in J} X_{\alpha} \setminus \prod_{\alpha \in J} A_{\alpha} = \bigcup_{\alpha \in J} \pi_{\alpha}^{-1}(X_{\alpha} \setminus A_{\alpha})$ .  $\square$ 

**Theorem 4.2.3.** Let  $\{X_{\alpha}\}_{{\alpha}\in J}$  be a family of topological spaces. The set of all sets of the form  $\prod_{{\alpha}\in J}U_{\alpha}$  where each  $U_{\alpha}$  is open in  $X_{\alpha}$ , and  $U_{\alpha}=X_{\alpha}$  for all but finitely many  $\alpha$ , is a basis for the product topology on  $\prod_{{\alpha}\in J}X_{\alpha}$ .

PROOF: By Lemma 3.11.2.

**Theorem 4.2.4.** Let  $\{X_{\alpha}\}_{{\alpha}\in J}$  be a family of topological spaces, and let  $\mathcal{B}_{\alpha}$  be a basis for the topology on  $X_{\alpha}$  for each  $\alpha$ . Then

$$\mathcal{B} = \{ \prod_{\alpha \in J} U_{\alpha} : \text{for finitely many } \alpha \in J, U_{\alpha} \in \mathcal{B}_{\alpha},$$

$$\text{and } U_{\alpha} = X_{\alpha} \text{ for all other values of } \alpha \}$$

is a basis for the product topology on  $\prod_{\alpha \in J} X_{\alpha}$ .

#### PROOF:

- $\langle 1 \rangle 1$ . Every member of  $\mathcal{B}$  is open in the product topology. PROOF: Immediate from definitions.
- $\langle 1 \rangle 2$ . For every open set U and  $\{x_{\alpha}\}_{{\alpha} \in J} \in U$ , there exists  $B \in \mathcal{B}$  such that  $\{x_{\alpha}\}_{{\alpha} \in J} \in B \subseteq U$ .
  - $\langle 2 \rangle 1$ . Let: U be open and  $\{x_{\alpha}\}_{{\alpha} \in J} \in U$
  - $\langle 2 \rangle 2$ . PICK  $U_{\alpha}$  open in  $X_{\alpha}$  for each  $\alpha$  such that  $\{x_{\alpha}\}_{{\alpha} \in J} \in \prod_{{\alpha} \in J} U_{\alpha} \subseteq U$  and  $U_{\alpha} = X_{\alpha}$  for all  $\alpha$  except  $\alpha_1, \ldots, \alpha_n$ .

PROOF: By Theorem 4.2.3.

 $\langle 2 \rangle 3$ . Pick  $B_{\alpha_i} \in \mathcal{B}_{\alpha_i}$  such that  $x_{\alpha} \in B_{\alpha_i} \subseteq U_{\alpha_i}$  for  $i = 1, \ldots, n$ 

 $\langle 2 \rangle 4$ .  $\{x_{\alpha}\}_{{\alpha} \in J} \in \prod_{{\alpha} \in J} V_{\alpha} \subseteq U \text{ where } V_{\alpha_i} = B_{\alpha_i} \text{ for } i = 1, \ldots, n, \text{ and } V_{\alpha} = X_{\alpha} \text{ for all other } \alpha.$ 

**Theorem 4.2.5** (AC). Let  $\{X_{\alpha}\}_{{\alpha}\in J}$  be a family of topological spaces and  $A_{\alpha}\subseteq X_{\alpha}$  for all  $\alpha$ . If  $\prod_{{\alpha}\in J} X_{\alpha}$  is given the product topology, then

$$\prod_{\alpha \in J} \overline{A_{\alpha}} = \overline{\prod_{\alpha \in J} A_{\alpha}} .$$

Proof:

 $\langle 1 \rangle 1. \prod_{\alpha \in J} \overline{A_{\alpha}} \subseteq \overline{\prod_{\alpha \in J} A_{\alpha}}$ 

 $\langle 2 \rangle 1$ . Let:  $\{x_{\alpha}\}_{{\alpha} \in J} \in \prod_{{\alpha} \in J} \overline{A_{\alpha}}$ 

 $\langle 2 \rangle 2$ . Let:  $\prod_{\alpha \in J} U_{\alpha}$  be a basic neighbourhood of  $\{x_{\alpha}\}_{\alpha \in J}$ , where each  $U_{\alpha}$  is open in  $X_{\alpha}$ , and  $U_{\alpha} = X_{\alpha}$  except for  $\alpha = \alpha_1, \ldots, \alpha_n$ .

 $\langle 2 \rangle 3$ . For  $\alpha \in J$ , Pick  $a_{\alpha} \in A_{\alpha} \cap U_{\alpha}$ .

PROOF: By Theorem 3.8.3, using the Axiom of Choice.

 $\langle 2 \rangle 4. \ \{a_{\alpha}\}_{{\alpha} \in J} \in \prod_{{\alpha} \in J} A_{\alpha} \cap \prod_{{\alpha} \in J} U_{\alpha}$ 

 $\langle 2 \rangle$ 5. Q.E.D.

PROOF: By Theorem 3.8.3.

 $\langle 1 \rangle 2$ .  $\overline{\prod_{\alpha \in J} A_{\alpha}} \subseteq \prod_{\alpha \in J} \overline{A_{\alpha}}$ 

 $\langle 2 \rangle 1$ . Let:  $\{x_{\alpha}\}_{{\alpha} \in J} \in \overline{\prod_{{\alpha} \in J} A_{\alpha}}$ 

 $\langle 2 \rangle 2$ . Let:  $\alpha \in J$ Prove:  $x_{\alpha} \in \overline{A_{\alpha}}$ 

 $\langle 2 \rangle 3$ . Let: U be a neighbourhood of  $x_{\alpha}$  in  $X_{\alpha}$ 

 $\langle 2 \rangle 4$ .  $\pi_{\alpha}^{-1}(U)$  is a neighbourhood of  $\{x_{\alpha}\}_{{\alpha} \in J}$ 

 $\langle 2 \rangle$ 5. PICK  $\{a_{\alpha}\}_{{\alpha} \in J} \in \pi_{\alpha}^{-1}(U) \cap \prod_{{\alpha} \in J} A_{\alpha}$ 

PROOF: By Theorem 3.8.3.

 $\langle 2 \rangle 6. \ a_{\alpha} \in U \cap A_{\alpha}$ 

 $\langle 2 \rangle 7$ . Q.E.D.

PROOF: By Theorem 3.8.3.

Ш

**Definition 4.2.6** (Standard Topology on  $\mathbb{R}^J$ ). For J a set, the *standard topology* on  $\mathbb{R}^J$  is the product topology where  $\mathbb{R}$  is given the standard topology.

**Definition 4.2.7** (Closed Unit Ball). The closed unit ball  $B^2$  is  $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$  as a subset of  $\mathbb{R}^2$ .

**Definition 4.2.8** (Sorgenfrey Plane). The Sorgenfrey plane is  $\mathbb{R}^2_l$ .

## 4.3 The Subspace Topology

**Definition 4.3.1** (Subspace Topology). Let X be a topological space and  $Y \subseteq X$ . The *subspace topology* on Y is  $\{Y \cap U : U \text{ open in } X\}$ . With this topology, Y is a *subspace* of X.

We prove this is a topology.

```
Proof:
\langle 1 \rangle 1. Let: \mathcal{T} = \{ Y \cap U : U \text{ open in } X \}
\langle 1 \rangle 2. \ Y \in \mathcal{T}
    Proof: Y = Y \cap X
\langle 1 \rangle 3. \mathcal{T} is closed under union.
    \langle 2 \rangle 1. Let: \mathcal{A} \subseteq \mathcal{T}
               PROVE: \bigcup \mathcal{A} = Y \cap \bigcup \{U \text{ open in } X : Y \cap U \in \mathcal{A}\}
    \langle 2 \rangle 2. \bigcup \mathcal{A} \subseteq Y \cap \bigcup \{U \text{ open in } X : Y \cap U \in \mathcal{A}\}
         \langle 3 \rangle 1. Let: x \in \bigcup A
         \langle 3 \rangle 2. PICK V \in \mathcal{A} such that x \in V
        \langle 3 \rangle 3. Pick U open in X such that V = Y \cap U
            PROOF: By the definition of \mathcal{T} (\langle 1 \rangle 1, \langle 2 \rangle 1, \langle 3 \rangle 2)
         \langle 3 \rangle 4. \ x \in Y \cap \bigcup \{U \text{ open in } X : Y \cap U \in \mathcal{A} \}
    \langle 2 \rangle 3. \ Y \cap \bigcup \{U \text{ open in } X : Y \cap U \in \mathcal{A}\} \subseteq \bigcup \mathcal{A}
        Proof: Set theory.
\langle 1 \rangle 4. \mathcal{T} is closed under binary intersection.
    PROOF: This holds because (Y \cap U) \cap (Y \cap V) = Y \cap (U \cap V).
```

**Lemma 4.3.2.** Let X be a topological space,  $Y \subseteq X$ , and  $A \subseteq Y$ . Then the topology A inherits as a subspace of X is the same as the topology A inherits as a subspace of Y.

Proof:

topology as a subspace of 
$$Y$$
  
= $\{V \cap A : V \text{ open in } Y\}$   
= $\{V \cap A : \exists U \text{ open in } X.V = U \cap Y\}$   
= $\{U \cap Y \cap A : U \text{ open in } X\}$   
= $\{U \cap A : U \text{ open in } X\}$   
=topology as a subspace of  $X\square$ 

**Lemma 4.3.3.** Let Y be a subspace of X. If U is open in Y and Y is open in X then U is open in X.

```
Proof:
```

- $\langle 1 \rangle 1$ . Pick V open in X such that  $U = Y \cap V$
- $\langle 1 \rangle 2$ . U is open in X

PROOF: The open sets in X are closed under binary intersection.

**Theorem 4.3.4.** Let Y be a subspace of X. Let  $A \subseteq Y$ . Let  $\overline{A}$  be the closure of A in X. Then the closure of A in Y is  $\overline{A} \cap Y$ .

#### Proof:

- $\langle 1 \rangle 1$ .  $\overline{A} \cap Y$  is a closed set in Y that includes A.
  - $\langle 2 \rangle 1$ .  $\overline{A} \cap Y$  is closed in Y.

Proof: By Lemma 4.3.4.1.

- $\langle 2 \rangle 2$ .  $A \subseteq \overline{A} \cap Y$ .
- $\langle 1 \rangle 2$ . If C is any closed set in Y that includes A then  $\overline{A} \cap Y \subseteq C$ .
  - $\langle 2 \rangle 1$ . Let: C be a closed set in Y that includes A.
  - $\langle 2 \rangle 2$ . Pick D closed in X such that  $C = D \cap Y$ .

PROOF: By Lemma 4.3.4.1.

- $\langle 2 \rangle 3. \ \overline{A} \subseteq D$
- $\langle 2 \rangle 4. \ \overline{A} \subseteq C$

**Corollary 4.3.4.1.** Let Y be a subspace of X. Then a set  $A \subseteq Y$  is closed in Y if and only if it is the intersection of a closed set in X with Y.

Corollary 4.3.4.2. Let Y be a subspace of X. If A is closed in Y and Y is closed in X then A is closed in X.

**Lemma 4.3.5.** Let X be a topological space and  $Y \subseteq X$ . If  $\mathcal{B}$  is a basis for the topology on X then  $\{B \cap Y : B \in \mathcal{B}\}$  is a basis for the subspace topology on Y.

#### PROOF:

 $\langle 1 \rangle 1$ . For all  $B \in \mathcal{B}$ , we have  $B \cap Y$  is open in Y.

Proof: Immediate from definitions.

- $\langle 1 \rangle 2$ . For every V open in Y and  $y \in V$ , there exists  $B \in \mathcal{B}$  such that  $y \in B \cap Y \subseteq V$ .
  - $\langle 2 \rangle 1$ . Let: V be open in Y and  $y \in V$
  - $\langle 2 \rangle 2$ . PICK U open in X such that  $V = Y \cap U$
  - $\langle 2 \rangle 3$ . Pick  $B \in \mathcal{B}$  such that  $y \in B \subseteq U$
- $\langle 2 \rangle 4. \ y \in B \cap Y \subseteq V$

**Lemma 4.3.6.** Let X be a topological space and  $Y \subseteq X$ . If S is a subbasis for the topology on X then  $\{S \cap Y : S \in S\}$  is a subbasis for the subspace topology on Y.

#### Proof:

 $\langle 1 \rangle 1$ . For all  $S \in \mathcal{S}$ , we have  $S \cap Y$  is open in Y.

Proof: Immediate from definitions.

- $\langle 1 \rangle 2$ . For every V open in Y and  $y \in V$ , there exist  $S_1, \ldots, S_n \in \mathcal{S}$  such that  $y \in (S_1 \cap Y) \cap \cdots \cap (S_n \cap Y) \subseteq V$ 
  - $\langle 2 \rangle 1$ . Let: V be open in Y and  $y \in V$
  - $\langle 2 \rangle 2$ . PICK U open in X such that  $V = U \cap Y$
  - $\langle 2 \rangle 3$ . Pick  $S_1, \ldots, S_n \in \mathcal{S}$  such that  $y \in S_1 \cap \cdots \cap S_n \subseteq U$
- $\langle 2 \rangle 4. \ y \in (S_1 \cap Y) \cap \cdots \cap (S_n \cap Y) \subseteq V$

**Theorem 4.3.7.** Let X be a linearly ordered set in the order topology. Let  $Y \subseteq X$  be convex. Then the order topology on Y is the same as the subspace topology.

Proof:

```
\langle 1 \rangle 1. Let: \mathcal{T}_o be the order topology and \mathcal{T}_s be the subspace topology.
\langle 1 \rangle 2. \mathcal{T}_o \subseteq \mathcal{T}_s
   \langle 2 \rangle 1. For all a \in Y, we have \{ y \in Y : a < y \} \in \mathcal{T}_s
      PROOF: \{y \in Y : a < y\} = \{x \in X : a < x\} \cap Y
   \langle 2 \rangle 2. For all a \in Y, we have \{y \in Y : y < a\} \in \mathcal{T}_s
      Proof: Similar.
   \langle 2 \rangle 3. Q.E.D.
      PROOF: Lemma 4.1.3 and Corollary 3.11.2.1.
\langle 1 \rangle 3. \ \mathcal{T}_s \subseteq \mathcal{T}_o
   \langle 2 \rangle 1. The sets (a, +\infty) \cap Y and (-\infty, a) \cap Y for a \in X form a subbasis for \mathcal{T}_s
      Proof: Lemma 4.3.6, Lemma 4.1.3.
   \langle 2 \rangle 2. For all a \in X, we have (a, +\infty) \cap Y \in \mathcal{T}_o
      \langle 3 \rangle 1. Let: a \in X
      \langle 3 \rangle 2. Case: a \in Y
         PROOF: In this case, (a, +\infty) \cap Y is an open ray in Y.
      \langle 3 \rangle 3. Case: For all y \in Y we have a < y
         PROOF: In this case, (a, +\infty) \cap Y = Y.
      \langle 3 \rangle 4. Case: For all y \in Y we have y < a
         PROOF: In this case, (a, +\infty) \cap Y = \emptyset.
      \langle 3 \rangle 5. Q.E.D.
         PROOF: These are the only cases because Y is convex.
   \langle 2 \rangle 3. For all a \in X, we have (-\infty, a) \cap Y \in \mathcal{T}_o
      PROOF: Similar.
   \langle 2 \rangle 4. Q.E.D.
      Proof: Corollary 3.11.2.1.
```

**Theorem 4.3.8.** Let  $\{X_{\alpha}\}_{{\alpha}\in J}$  be a family of topological spaces, and let  $A_{\alpha}$  be a subspace of  $X_{\alpha}$  for all  $\alpha$ . Then the product topology on  $\prod_{{\alpha}\in J}A_{\alpha}$  is the same as the topology it inherits as a subspace of  $\prod_{{\alpha}\in J}X_{\alpha}$ .

PROOF: Each is the topology generated by the subbasis consisting of  $\pi_{\alpha}^{-1}(U) \cap \prod_{\alpha \in J} A_{\alpha} = \pi_{\alpha}^{-1}(U \cap A_{\alpha})$  where  $\alpha \in J$  and U is open in  $X_{\alpha}$ , using Lemma 4.3.6.

**Definition 4.3.9** (Unit Circle). The unit circle  $S^1$  is  $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$  as a subspace of  $\mathbb{R}^2$ .

**Proposition 4.3.10.** Let Y be a subspace of X,  $A \subseteq Y$ , and  $a \in Y$ . Then a is a limit point of A in the subspace topology on Y if and only if a is a limit point of A is the topology of X.

Proof:

a is a limit point of A in Y  $\Leftrightarrow \forall U$  open in  $Y(a \in U \Rightarrow U \text{ intersects } A \text{ outside } a)$   $\Leftrightarrow \forall V \text{ open in } X(a \in V \cap Y \Rightarrow V \cap Y \text{ intersects } A \text{ outside } a)$   $\Leftrightarrow \forall V \text{ open in } X(a \in V \Rightarrow V \text{ intersects } A \text{ outside } a)$   $(a \in Y, A \subseteq Y)$  $\Leftrightarrow a$  is a limit point of A in X

## 4.4 The Box Topology

**Definition 4.4.1** (Box Topology). Let  $\{X_{\alpha}\}_{{\alpha}\in J}$  be a family of topological spaces. The box topology on  $\prod_{{\alpha}\in J} X_{\alpha}$  is the topology generated by the basis consisting of all sets of the form  $\prod_{{\alpha}\in J} U_{\alpha}$ , where each  $U_{\alpha}$  is open in  $X_{\alpha}$ .

We prove this is a basis.

Proof:

 $\langle 1 \rangle 1$ . Let:  $\mathcal{B}$  be the set of all sets of the form  $\prod_{\alpha \in J} U_{\alpha}$ , where each  $U_{\alpha}$  is open in  $X_{\alpha}$ .

 $\langle 1 \rangle 2. \ \bigcup \mathcal{B} = \prod_{\alpha \in J} X_{\alpha}$ 

PROOF: This holds because  $\prod_{\alpha \in J} X_{\alpha} \in \mathcal{B}$ .

 $\langle 1 \rangle 3$ .  $\mathcal{B}$  is closed under binary intersection.

PROOF:  $\prod_{\alpha \in J} U_{\alpha} \cap \prod_{\alpha \in J} V_{\alpha} = \prod_{\alpha \in J} (U_{\alpha} \cap V_{\alpha}).$ 

 $\langle 1 \rangle 4$ . Q.E.D.

Proof: Corollary 3.4.3.1.

**Theorem 4.4.2** (AC). Let  $\{X_{\alpha}\}_{{\alpha}\in J}$  be a family of topological spaces, and let  $\mathcal{B}_{\alpha}$  be a basis for the topology on  $X_{\alpha}$  for each  $\alpha$ . Then

$$\mathcal{B} = \{ \prod_{\alpha \in J} B_{\alpha} : \forall \alpha \in J.B_{\alpha} \in \mathcal{B}_{\alpha} \}$$

is a basis for the box topology on  $\prod_{\alpha \in J} X_{\alpha}$ .

Proof:

 $\langle 1 \rangle 1$ . Every member of  $\mathcal{B}$  is open in the box topology.

PROOF: Immediate from definitions.

 $\langle 1 \rangle 2$ . For every open set U and  $\{x_{\alpha}\}_{{\alpha} \in J} \in U$ , there exists  $B \in \mathcal{B}$  such that  $\{x_{\alpha}\}_{{\alpha} \in J} \in B \subseteq U$ .

 $\langle 2 \rangle 1$ . Let: U be open and  $\{x_{\alpha}\}_{{\alpha} \in J} \in U$ 

 $\langle 2 \rangle 2$ . PICK  $U_{\alpha}$  open in  $X_{\alpha}$  for each  $\alpha$  such that  $\{x_{\alpha}\}_{{\alpha} \in J} \in \prod_{{\alpha} \in J} U_{\alpha} \subseteq U$ .

 $\langle 2 \rangle 3$ . PICK  $B_{\alpha} \in \mathcal{B}_{\alpha}$  such that  $x_{\alpha} \in B_{\alpha} \subseteq U_{\alpha}$  for each  $\alpha$ 

PROOF: Using the Axiom of Choice.

 $\langle 2 \rangle 4. \ \{x_{\alpha}\}_{{\alpha} \in J} \in \prod_{{\alpha} \in J} B_{\alpha} \subseteq U$ 

**Theorem 4.4.3.** Let  $\{X_{\alpha}\}_{{\alpha}\in J}$  be a family of topological spaces, and let  $A_{\alpha}$  be a subspace of  $X_{\alpha}$  for all  $\alpha$ . Let  $\prod_{{\alpha}\in J} X_{\alpha}$  be given the box topology. Then the box topology on  $\prod_{{\alpha}\in J} A_{\alpha}$  is the same as the topology it inherits as a subspace of  $\prod_{{\alpha}\in J} X_{\alpha}$ .

PROOF: Each is the topology generated by the basis  $\{\prod_{\alpha\in J}(U_{\alpha}\cap A_{\alpha}): U_{\alpha} \text{ is open in } X_{\alpha}\}$ , using Lemma 4.3.5.  $\square$ 

**Theorem 4.4.4.** Let  $\{X_{\alpha}\}_{{\alpha}\in J}$  be a family of Hausdorff spaces. Then  $\prod_{{\alpha}\in J} X_{\alpha}$  is Hausdorff under the box topology.

#### Proof:

 $\langle 1 \rangle 1$ . Let:  $\{x_{\alpha}\}_{{\alpha} \in J}, \{y_{\alpha}\}_{{\alpha} \in J} \in \prod_{{\alpha} \in J} X_{\alpha} \text{ with } \{x_{\alpha}\}_{{\alpha} \in J} \neq \{y_{\alpha}\}_{{\alpha} \in J}$ 

 $\langle 1 \rangle 2$ . Pick  $\alpha \in J$  such that  $x_{\alpha} \neq y_{\alpha}$ 

 $\langle 1 \rangle 3$ . PICK disjoint neighbourhoods U of  $x_{\alpha}$  and V of  $y_{\alpha}$ .

 $\langle 1 \rangle 4$ .  $\pi_{\alpha}^{-1}(U)$  and  $\pi_{\alpha}^{-1}(V)$  are disjoint neighbourhoods of  $\{x_{\alpha}\}_{{\alpha} \in J}$  and  $\{y_{\alpha}\}_{{\alpha} \in J}$ 

Corollary 4.4.4.1. The space  $\mathbb{R}^{\omega}$  under the box topology is Hausdorff.

**Theorem 4.4.5** (AC). Let  $\{X_{\alpha}\}_{{\alpha}\in J}$  be a family of topological spaces and  $A_{\alpha}\subseteq X_{\alpha}$  for all  $\alpha$ . If  $\prod_{{\alpha}\in J} X_{\alpha}$  is given the box topology, then

$$\prod_{\alpha \in J} \overline{A_{\alpha}} = \overline{\prod_{\alpha \in J} A_{\alpha}} .$$

### Proof:

 $\langle 1 \rangle 1. \prod_{\alpha \in J} \overline{A_{\alpha}} \subseteq \overline{\prod_{\alpha \in J} A_{\alpha}}$ 

 $\langle 2 \rangle 1$ . Let:  $\{x_{\alpha}\}_{{\alpha} \in J} \in \prod_{{\alpha} \in J} A_{\alpha}$ 

 $\langle 2 \rangle 2$ . Let:  $\prod_{\alpha \in J} U_{\alpha}$  be a basic neighbourhood of  $\{x_{\alpha}\}_{\alpha \in J}$ , where each  $U_{\alpha}$  is open in  $X_{\alpha}$ .

 $\langle 2 \rangle 3$ . For  $\alpha \in J$ , PICK  $a_{\alpha} \in A_{\alpha} \cap U_{\alpha}$ .

PROOF: By Theorem 3.8.3, using the Axiom of Choice.

- $\langle 2 \rangle 4. \ \{a_{\alpha}\}_{{\alpha} \in J} \in \prod_{{\alpha} \in J} A_{\alpha} \cap \prod_{{\alpha} \in J} U_{\alpha}$
- $\langle 2 \rangle$ 5. Q.E.D.

PROOF: By Theorem 3.8.3.

- $\langle 1 \rangle 2$ .  $\overline{\prod_{\alpha \in J} A_{\alpha}} \subseteq \prod_{\alpha \in J} \overline{A_{\alpha}}$ 
  - $\langle 2 \rangle 1$ . Let:  $\{x_{\alpha}\}_{{\alpha} \in J} \in \overline{\prod_{{\alpha} \in J} A_{\alpha}}$
  - $\langle 2 \rangle 2$ . Let:  $\alpha \in J$

PROVE:  $x_{\alpha} \in \overline{A_{\alpha}}$ 

- $\langle 2 \rangle 3$ . Let: U be a neighbourhood of  $x_{\alpha}$  in  $X_{\alpha}$
- $\langle 2 \rangle 4$ .  $\pi_{\alpha}^{-1}(U)$  is a neighbourhood of  $\{x_{\alpha}\}_{{\alpha} \in J}$
- $\langle 2 \rangle$ 5. PICK  $\{a_{\alpha}\}_{{\alpha} \in J} \in \pi_{\alpha}^{-1}(U) \cap \prod_{{\alpha} \in J} A_{\alpha}$ PROOF: By Theorem 3.8.3.
- $\langle 2 \rangle 6. \ a_{\alpha} \in U \cap A_{\alpha}$
- $\langle 2 \rangle$ 7. Q.E.D.

PROOF: By Theorem 3.8.3.

## 4.5 The Quotient Topology

2. p maps saturated open sets to open sets.

1. p is a quotient map.

**Definition 4.5.1** (Quotient Map). Let X and Y be topological spaces. Let  $p: X \to Y$  be a surjective map. Then p is a *quotient map* iff, for all  $U \subseteq Y$ , we have U is open in Y iff  $p^{-1}(U)$  is open in X.

**Lemma 4.5.2.** Let X and Y be topological spaces and  $p: X \to Y$  be surjective and continuous. Then the following are equivalent.

```
3. p maps saturated closed sets to closed sets.
PROOF:
\langle 1 \rangle 1. \ 1 \Rightarrow 2
    \langle 2 \rangle 1. Assume: p is a quotient map.
    \langle 2 \rangle 2. Let: U \subseteq X be a saturated open set.
    \langle 2 \rangle 3. \ U = p^{-1}(p(U))
       \langle 3 \rangle 1. \ U \subseteq p^{-1}(p(U))
           PROOF: Set theory.
       \langle 3 \rangle 2. \ p^{-1}(p(U)) \subseteq U
            \langle 4 \rangle 1. Let: x \in p^{-1}(p(U))
           \langle 4 \rangle 2. PICK y \in U such that p(x) = p(y)
           \langle 4 \rangle 3. \ x \in U
               Proof: \langle 2 \rangle 2, \langle 4 \rangle 2.
    \langle 2 \rangle 4. p(U) is open
       Proof: \langle 2 \rangle 1, \langle 2 \rangle 3.
\langle 1 \rangle 2. \ 2 \Rightarrow 3
    \langle 2 \rangle 1. Assume: p maps saturated open sets to open sets
    \langle 2 \rangle 2. Let: C \subseteq X be a saturated closed set.
    \langle 2 \rangle 3. X \setminus C is a saturated open set.
        \langle 3 \rangle 1. Let: x \in X \setminus C and x' \in X be such that p(x) = p(x')
        \langle 3 \rangle 2. \ x' \notin C
           PROOF: If x' \in C then x \in C since C is saturated.
    \langle 2 \rangle 4. p(X \setminus C) is open.
       PROOF: By \langle 2 \rangle 1 and \langle 2 \rangle 3.
    \langle 2 \rangle 5. \ p(X \setminus C) = Y \setminus p(C)
       \langle 3 \rangle 1. \ p(X \setminus C) \subseteq Y \setminus p(C)
            \langle 4 \rangle 1. Let: x \in X \setminus C
           \langle 4 \rangle 2. Assume: for a contradiction p(x) \in p(C)
           \langle 4 \rangle 3. Pick x' \in C such that p(x) = p(x')
           \langle 4 \rangle 4. Q.E.D.
               PROOF: We have x \notin C, x' \in C and p(x) = p(x'), contradicting \langle 2 \rangle 2.
        \langle 3 \rangle 2. \ Y \setminus p(C) \subseteq p(X \setminus C)
           \langle 4 \rangle 1. Let: y \notin p(C)
```

 $\langle 4 \rangle 2$ . PICK  $x \in X$  such that p(x) = y

```
PROOF: p is surjective. \langle 4 \rangle 3. \ x \notin C \langle 1 \rangle 3. \ 3 \Rightarrow 1 \langle 2 \rangle 1. Assume: p maps saturated closed sets to closed sets \langle 2 \rangle 2. Let: C \subseteq Y be such that p^{-1}(Y) is closed \langle 2 \rangle 3. \ p^{-1}(C) is saturated \langle 3 \rangle 1. Let: x \in p^{-1}(C), \ x' \in X and p(x) = p(x') \langle 3 \rangle 2. \ x' \in p^{-1}(C) is closed Proof: By \langle 2 \rangle 1 and \langle 2 \rangle 3. \langle 2 \rangle 5. \ C = p(p^{-1}(C)) Proof: By set theory, since p is surjective.
```

**Corollary 4.5.2.1.** If  $p: X \rightarrow Y$  is a surjective continuous map that is either an open map or a closed map, then p is a quotient map.

**Definition 4.5.3** (Quotient Topology). Let X be a topological space, A a set, and p: X woheadrightarrow A a surjective map. Then the *quotient topology* on A induced by p is

$$\{U \subseteq A : p^{-1}(U) \text{ is open in } X\}$$
.

It is easy to check this is a topology.

**Lemma 4.5.4.** Let X be a topological space, A a set, and  $p: X \rightarrow A$  a surjective map. Then the quotient topology induced by p is the unique topology on A such that p is a quotient map.

PROOF: Immediate from definitions.

**Definition 4.5.5** (Quotient Space). Let X be a topological space and  $X^*$  a partition of X. Let  $p: X woheadrightarrow X^*$  be the canonical map. Then  $X^*$  under the quotient topology induced by p is called a *quotient space* of X.

**Proposition 4.5.6.** Let  $p: X \to Y$  be a quotient map. Let  $A \subseteq X$  be open and saturated. Then  $p \upharpoonright_A: A \to p(A)$  is a quotient map.

Proof:

```
\begin{split} \langle 1 \rangle 1. & \text{ Let: } q = p \upharpoonright_A: A \twoheadrightarrow p(A) \\ \langle 1 \rangle 2. & \text{ For all } V \subseteq p(A), \text{ we have } q^{-1}(V) = p^{-1}(V) \\ \langle 2 \rangle 1. & q^{-1}(V) \subseteq p^{-1}(V) \\ & \text{ Proof: Trivial.} \\ \langle 2 \rangle 2. & p^{-1}(V) \subseteq q^{-1}(V) \\ & \langle 3 \rangle 1. & \text{ Let: } x \in p^{-1}(V) \\ & \langle 3 \rangle 2. & \text{ Pick } x' \in A \text{ such that } p(x') = p(x) \\ & \text{ Proof: One exists because } p(x) \in V \subseteq p(A). \\ & \langle 3 \rangle 3. & x \in A \\ & \text{ Proof: This holds because } A \text{ is saturated.} \\ & \langle 3 \rangle 4. & x \in q^{-1}(V) \end{split}
```

```
PROOF: From \langle 3 \rangle 1 and \langle 3 \rangle 3.
\langle 1 \rangle 3. For all U \subseteq X, we have p(U \cap A) = p(U) \cap p(A)
\langle 1 \rangle 4. Let: V \subseteq p(A) be such that q^{-1}(V) is open in A.
        PROVE: V is open in p(A).
\langle 1 \rangle 5. q^{-1}(V) is open in X
\langle 1 \rangle 6. \ p^{-1}(V) is open in X
\langle 1 \rangle 7. V is open in Y
\langle 1 \rangle 8. V is open in p(A)
Proposition 4.5.7. Let p: X \to Y be a quotient map. Let A \subseteq X be closed
and saturated. Then p \upharpoonright_A : A \rightarrow p(A) is a quotient map.
Proof: Similar.
Proposition 4.5.8. Let p: X \to Y be an open quotient map. Let A \subseteq X be
saturated. Then p \upharpoonright_A : A \to p(A) is a quotient map.
Proof:
\langle 1 \rangle 1. Let: q = p \upharpoonright_A : A \rightarrow p(A)
\langle 1 \rangle 2. For all V \subseteq p(A), we have q^{-1}(V) = p^{-1}(V)
   \langle 2 \rangle 1. \ q^{-1}(V) \subseteq p^{-1}(V)
      PROOF: Trivial.
   \langle 2 \rangle 2. \ p^{-1}(V) \subseteq q^{-1}(V)
       \langle 3 \rangle 1. Let: x \in p^{-1}(V)
      \langle 3 \rangle 2. PICK x' \in A such that p(x') = p(x)
          PROOF: One exists because p(x) \in V \subseteq p(A).
       \langle 3 \rangle 3. \ x \in A
          Proof: This holds because A is saturated.
      \langle 3 \rangle 4. \ x \in q^{-1}(V)
          PROOF: From \langle 3 \rangle 1 and \langle 3 \rangle 3.
\langle 1 \rangle 3. For all U \subseteq X, we have p(U \cap A) = p(U) \cap p(A)
   \langle 2 \rangle 1. \ p(U \cap A) \subseteq p(U) \cap p(A)
      PROOF: Set theory.
   \langle 2 \rangle 2. p(U) \cap p(A) \subseteq p(U \cap A)
       \langle 3 \rangle 1. Let: x \in U, y \in A, p(x) = p(y)
               PROVE: p(x) \in p(U \cap A)
       \langle 3 \rangle 2. \ x \in A
          PROOF: A is saturated.
       \langle 3 \rangle 3. \ x \in U \cap A
\langle 1 \rangle 4. Let: V \subseteq p(A) be such that q^{-1}(V) is open in A.
        PROVE: V is open in p(A).
\langle 1 \rangle 5. p^{-1}(V) is open in A
   Proof: By \langle 1 \rangle 2
```

 $\langle 1 \rangle 6$ . Pick U open in X such that  $p^{-1}(V) = U \cap A$ 

 $\langle 1 \rangle 7. \ V = p(U) \cap p(A)$ 

Proof:

$$V = p(p^{-1}(V)) (p \text{ is surjective})$$

$$= p(U \cap A) (\langle 1 \rangle 6)$$

$$= p(U) \cap p(A) (\langle 1 \rangle 3)$$

 $\langle 1 \rangle 8. \ p(U)$  is open in Y

PROOF:  $\langle 1 \rangle 6$ , p is an open map.

 $\langle 1 \rangle 9$ . V is open in p(A)PROOF:  $\langle 1 \rangle 7$ ,  $\langle 1 \rangle 8$ 

**Proposition 4.5.9.** Let  $p: X \to Y$  be a closed quotient map. Let  $A \subseteq X$  be saturated. Then  $p \upharpoonright_A: A \to p(A)$  is a quotient map.

Proof: Similar.  $\square$ 

**Proposition 4.5.10.** The composite of two quotient maps is a quotient map.

Proof: From Proposition 5.2.22.  $\Box$ 

**Proposition 4.5.11.** Let  $X^*$  be a quotient space of X. If every element of  $X^*$  is closed in X, then  $X^*$  is  $T_1$ .

Proof:

 $\langle 1 \rangle 1$ . Let:  $C \in X^*$ 

 $\langle 1 \rangle 2. \ p^{-1}(\{C\}) = C$ 

PROOF: Definition of p.

 $\langle 1 \rangle 3. \ p^{-1}(\{C\})$  is closed in X PROOF: By hypothesis.

 $\langle 1 \rangle 4$ .  $\{C\}$  is closed in  $X^*$ .

Proof: By Proposition 5.2.21.

## Chapter 5

## Functions Between Topological Spaces

### 5.1 Open Maps

**Definition 5.1.1.** Let X and Y be topological spaces and  $f: X \to Y$ . Then f is an *open map* iff, for all U open in X, f(U) is open in Y.

**Lemma 5.1.2.** Let X and Y be topological spaces and  $f: X \to Y$ . Let  $\mathcal{B}$  be a basis for the topology on X. Then f is an open map if and only if, for all  $B \in \mathcal{B}$ , f(B) is open in Y.

#### Proof:

- $\langle 1 \rangle 1$ . If f is an open map then, for all  $B \in \mathcal{B}$ , f(B) is open in Y.
  - Proof: Immediate from definitions.
- $\langle 1 \rangle 2$ . If, for all  $B \in \mathcal{B}$ , f(B) is open in Y, then f is an open map.
  - $\langle 2 \rangle 1$ . Assume: For all  $B \in \mathcal{B}$ , f(B) is open in Y.
  - $\langle 2 \rangle 2$ . Let: U be open in X
    - PROVE: f(U) is open in Y
  - $\langle 2 \rangle 3$ . Let:  $\mathcal{B}_0 \subseteq \mathcal{B}$  be such that  $U = \bigcup \mathcal{B}_0$
  - $\langle 2 \rangle 4. \ f(U) = \bigcup_{B \in \mathcal{B}_0} f(B)$ 
    - Proof: Set theory.
  - $\langle 2 \rangle 5$ . f(U) is open in Y.

PROOF: From  $\langle 2 \rangle 1, \ \langle 2 \rangle 4$  and the fact that the open sets are closed under union.

**Corollary 5.1.2.1.** Let X and Y be topological spaces and  $f: X \to Y$ . Let S be a subbasis for the topology on X. Then f is an open map if and only if, for all  $S \in S$ , f(S) is open in Y.

**Lemma 5.1.3** (AC). Let  $\{X_{\alpha}\}_{{\alpha}\in J}$  be a family of topological spaces. Then the projection  $\pi_{\alpha}: \prod_{{\alpha}\in J} X_{\alpha} \to X_{\alpha}$  is an open map.

Proof:

 $\langle 1 \rangle 1$ . For U open in  $X_{\alpha}$ , we have  $\pi_{\alpha}(\pi_{\alpha}^{-1}(U))$  is open in  $X_{\alpha}$  PROOF:  $\pi_{\alpha}(\pi_{\alpha}^{-1}(U)) = U$  if all the other  $X_{\alpha}$  are nonempty,  $\emptyset$  otherwise.

 $\langle 1 \rangle 2$ . For  $\beta \neq \alpha$  and U open in  $X_{\beta}$ , we have  $\pi_{\alpha}(\pi_{\beta}^{-1}(U))$  is open in  $X_{\alpha}$ 

PROOF:  $\pi_{\alpha}(\pi_{\beta}^{-1}(U)) = X_{\alpha}$  if all the  $X_{\gamma}$  are nonempty for  $\gamma \neq \alpha, \emptyset$  otherwise.  $\langle 1 \rangle 3$ . Q.E.D.

Proof: By Corollary 5.1.2.1.

#### 5.2 Continuous Functions

**Definition 5.2.1** (Continuous). Let X and Y be topological spaces and f:  $X \to Y$  a function. Then f is continuous if and only if, for every open set U in Y, the set  $f^{-1}(U)$  is open in X.

**Theorem 5.2.2.** Let X and Y be topological spaces and  $f: X \to Y$ . Then the following are equivalent.

- 1. f is continuous.
- 2. For every closed set C in Y, the set  $f^{-1}(C)$  is closed in X.
- 3. For every set  $A \subseteq X$ , we have  $f(\overline{A}) \subseteq \overline{f(A)}$ .

PROOF:

- $\langle 1 \rangle 1. \ 1 \Rightarrow 3$ 
  - $\langle 2 \rangle 1$ . Assume: f is continuous.
  - $\langle 2 \rangle 2$ . Let:  $A \subseteq X$
  - $\langle 2 \rangle 3$ . Let:  $x \in A$

PROVE:  $f(x) \in f(A)$ 

- $\langle 2 \rangle 4$ . Let: V be a neighbourhood of f(x)
- $\langle 2 \rangle 5$ .  $f^{-1}(V)$  is a neighbourhood of x

Proof:  $\langle 2 \rangle 1$ ,  $\langle 2 \rangle 3$ ,  $\langle 2 \rangle 4$ 

 $\langle 2 \rangle 6.$   $f^{-1}(V)$  intersects A in a, say.

PROOF:  $\langle 2 \rangle 3$ ,  $\langle 2 \rangle 5$ , Theorem 3.8.3.

- $\langle 2 \rangle 7$ . V intersects f(A) in f(a).
- $\langle 2 \rangle 8$ . Q.E.D.

PROOF: Theorem 3.8.3.

- $\langle 1 \rangle 2. \ 3 \Rightarrow 2$ 
  - $\langle 2 \rangle 1$ . Assume: 3
  - $\langle 2 \rangle 2$ . Let: C be a closed set in Y
  - $\langle 2 \rangle 3. \ \overline{f^{-1}(C)} = f^{-1}(C)$

Proof:

$$f(\overline{f^{-1}(C)}) \subseteq \overline{f(f^{-1}(C))}$$

$$\subseteq \overline{C}$$

$$(\langle 2 \rangle 1)$$

- $\langle 1 \rangle 3. \ 2 \Rightarrow 1$ 
  - $\langle 2 \rangle 1$ . Assume: 2

```
\langle 2 \rangle 2. Let: V be open in Y
    \langle 2 \rangle 3. f^{-1}(Y \setminus V) is closed in X
       Proof: By \langle 2 \rangle 1.
   \langle 2 \rangle 4. f^{-1}(V) is open in X.
       PROOF: f^{-1}(V) = X \setminus f^{-1}(Y \setminus V).
```

**Lemma 5.2.3.** If  $f: X \to Y$  maps all of X to the single point  $y_0$  of Y, then f is continuous.

PROOF: For V open in Y, the set  $f^{-1}(V)$  is either X (if  $y_0 \in V$ ) or  $\emptyset$  (if  $y_0 \notin V$ ).

**Definition 5.2.4** (Continuity at a Point). Let X and Y be topological spaces,  $f: X \to Y$  a function, and  $x \in X$ . Then f is continuous at x if and only if, for every neighbourhood V of f(x), there exists a neighbourhood U of x such that  $f(U) \subseteq V$ .

**Theorem 5.2.5.** Let X and Y be topological spaces and  $f: X \to Y$ . Then f is continuous if and only if f is continuous at every point of X.

#### Proof:

- $\langle 1 \rangle 1$ . If f is continuous then f is continuous at every point of X.
  - $\langle 2 \rangle 1$ . Assume: f is continuous
  - $\langle 2 \rangle 2$ . Let:  $x \in X$
  - $\langle 2 \rangle 3$ . Let: V be a neighbourhood of f(x)
  - $\langle 2 \rangle 4$ .  $f^{-1}(V)$  is a neighbourhood of x
  - $\langle 2 \rangle 5.$   $f(f^{-1}(V)) \subseteq V$
- $\langle 1 \rangle 2$ . If f is continuous at every point of X then f is continuous.
  - $\langle 2 \rangle 1$ . Assume: f is continuous at every point of X.
  - $\langle 2 \rangle 2$ . Let: V be open in Y PROVE:  $f^{-1}(V)$  is open in X.
  - $\langle 2 \rangle 3$ . Let:  $x \in f^{-1}(V)$
  - $\langle 2 \rangle 4$ . V is a neighbourhood of f(x)
  - $\langle 2 \rangle$ 5. Pick a neighbourhood U of x such that  $f(U) \subseteq V$ Proof: By  $\langle 2 \rangle 1$ .

- $\langle 2 \rangle 6. \ x \in U \subseteq f^{-1}(V)$
- $\langle 2 \rangle$ 7. Q.E.D.

Proof: By Proposition 3.2.3.

**Lemma 5.2.6.** Let X and Y be topological spaces and  $f: X \to Y$ . Let  $\mathcal{B}$  be a basis for the topology on Y. Then f is continuous if and only if, for all  $B \in \mathcal{B}$ , the set  $f^{-1}(B)$  is open in X.

#### Proof:

- $\langle 1 \rangle 1$ . If f is continuous then, for all  $B \in \mathcal{B}$ , the set  $f^{-1}(B)$  is open in X. PROOF: Immediate from definitions.
- $\langle 1 \rangle 2$ . If, for all  $B \in \mathcal{B}$ , the set  $f^{-1}(B)$  is open in X, then f is continuous.

```
\langle 2 \rangle 1. Assume: For all B \in \mathcal{B}, the set f^{-1}(B) is open in X. \langle 2 \rangle 2. Let: x \in X \langle 2 \rangle 3. Let: V be a neighbourhood of f(x) \langle 2 \rangle 4. Pick B \in \mathcal{B} such that f(x) \in B \subseteq V \langle 2 \rangle 5. f^{-1}(B) is a neighbourhood of x Proof: By \langle 2 \rangle 1. \langle 2 \rangle 6. f(f^{-1}(B)) \subseteq B Proof: Set theory. \langle 2 \rangle 7. Q.E.D. Proof: Theorem 5.2.5.
```

**Lemma 5.2.7.** The projections  $\pi_1: X \times Y \to X$  and  $\pi_2: X \times Y \to Y$  are continuous.

Proof:Immediate from definitions.

**Theorem 5.2.8.** If A is a subspace of X, the inclusion function  $j : A \to X$  is continuous.

PROOF: For V open in X, the set  $j^{-1}(V) = V \cap A$  is open in A.

**Theorem 5.2.9.** If  $f: X \to Y$  and  $g: Y \to Z$  are continuous, then the map  $g \circ f: X \to Z$  is continuous.

#### Proof:

 $\langle 1 \rangle 1$ . Let: V be open in Z  $\langle 1 \rangle 2$ .  $g^{-1}(V)$  is open in Y $\langle 1 \rangle 3$ .  $f^{-1}(g^{-1}(V))$  is open in X

**Theorem 5.2.10.** If  $f: X \to Y$  is continuous and if A is a subspace of X, then the restricted function  $f \upharpoonright A: A \to Y$  is continuous.

PROOF: For V open in Y, the set  $(f \upharpoonright A)^{-1}(V) = f^{-1}(V) \cap A$  is open in A.  $\square$ 

**Theorem 5.2.11.** Let  $f: X \to Y$  be continuous. If Z is a subspace of Y that includes the range of f, then the function  $g: X \to Z$  obtained by restricting the codomain of f is continuous. If Z is a space having Y as a subspace, then the function  $h: X \to Z$  obtained by expanding the codomain of f is continuous.

#### PROOF:

- $\langle 1 \rangle 1$ . If Z is a subspace of Y that includes the range of f, then the function  $g: X \to Z$  obtained by restricting the codomain of f is continuous.
  - $\langle 2 \rangle 1$ . Let: V be open in Z
  - $\langle 2 \rangle 2$ . PICK W open in Y such that  $V = W \cap Z$
  - $\langle 2 \rangle 3$ .  $f^{-1}(W)$  is open in X.
  - $\langle 2 \rangle 4$ .  $g^{-1}(V)$  is open in X.

PROOF:  $g^{-1}(V) = f^{-1}(W)$ .

 $\langle 1 \rangle 2$ . If Z is a space having Y as a subspace, then the function  $h: X \to Z$  obtained by expanding the codomain of f is continuous.

PROOF: For V open in Z, we have  $h^{-1}(V) = f^{-1}(V \cap Y)$  is open in X.

**Theorem 5.2.12.** Let X and Y be topological spaces and  $f: X \to Y$ . If  $x_n \to x$  as  $n \to \infty$  in X and f is continuous at x, then  $f(x_n) \to f(x)$  as  $n \to \infty$  in Y.

PROOF:

- $\langle 1 \rangle 1$ . Assume:  $x_n \to x$  as  $n \to \infty$
- $\langle 1 \rangle 2$ . Assume: f is continuous at x
- $\langle 1 \rangle 3$ . Let: V be a neighbourhood of f(x)
- $\langle 1 \rangle 4$ . PICK a neighbourhood U of x such that  $f(U) \subseteq V$  PROOF: By  $\langle 1 \rangle 2$ .
- $\langle 1 \rangle 5$ . PICK N such that, for all  $n \geq N$ ,  $x_n \in U$

Proof: By  $\langle 1 \rangle 1$ 

 $\langle 1 \rangle 6$ . For  $n \geq N$ ,  $f(x_n) \in V$ 

PROOF: By  $\langle 1 \rangle 4$ .

**Theorem 5.2.13.** Let X, Y and Z be topological spaces. Let  $f: X \to Y$ . If there exists a set A of open sets in X such that:

- $\bigcup A = X$ ;
- for all  $U \in \mathcal{A}$ , the function  $f \upharpoonright U : U \to X$  is continuous;

then f is continuous.

Proof:

- $\langle 1 \rangle 1$ . Let: V be open in Y
- $\langle 1 \rangle 2$ . For all  $U \in \mathcal{A}$ , the set  $(f \upharpoonright U)^{-1}(V)$  is open in X.
  - $\langle 2 \rangle 1$ . Let:  $U \in \mathcal{A}$
  - $\langle 2 \rangle 2$ .  $(f \upharpoonright U)^{-1}(V)$  is open in U

PROOF: Since  $f \upharpoonright U : U \to X$  is continuous.

 $\langle 2 \rangle 3$ . Q.E.D.

PROOF: By Lemma 4.3.3.

 $\langle 1 \rangle 3$ . Q.E.D.

PROOF: Since  $f^{-1}(V) = \bigcup_{U \in \Delta} (f \upharpoonright U)^{-1}(V)$ .

**Theorem 5.2.14** (The Pasting Lemma). Let  $X = A \cup B$  where A and B are closed in X. Let  $f: A \to Y$  and  $g: B \to Y$  be continuous. If f(x) = g(x) for every  $x \in A \cap B$ , then the function  $h: X \to Y$  defined by

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases}$$

is continuous.

Proof:

```
⟨1⟩1. Let: C be closed in Y ⟨1⟩2. f^{-1}(C) is closed in A PROOF: Theorem 5.2.2. ⟨1⟩3. f^{-1}(C) is closed in X PROOF: Lemma 4.3.4.1. ⟨1⟩4. g^{-1}(C) is closed in B PROOF: Theorem 5.2.2. ⟨1⟩5. g^{-1}(C) is closed in X PROOF: Lemma 4.3.4.1. ⟨1⟩6. h^{-1}(C) is closed in X PROOF: h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C) ⟨1⟩7. Q.E.D. PROOF: Theorem 5.2.2.
```

**Theorem 5.2.15.** Let  $f: A \to \prod_{\alpha \in J} X_{\alpha}$  be given by the equation

$$f(a) = \{ f_{\alpha}(a) \}_{\alpha \in J} ,$$

where  $f_{\alpha}: A \to X_{\alpha}$  for each  $\alpha$ . Let  $\prod_{\alpha \in J} X_{\alpha}$  have the product topology. Then the function f is continuous if and only if each function  $f_{\alpha}$  is continuous.

#### PROOF:

 $\begin{array}{l} \langle 1 \rangle 1. \ \ \text{If} \ f \ \ \text{is continuous then each} \ f_{\alpha} \ \ \text{is continuous.} \\ \text{Proof: This holds because} \ f_{\alpha} = \pi_{\alpha} \circ f. \\ \langle 1 \rangle 2. \ \ \text{If every} \ f_{\alpha} \ \ \text{is continuous then} \ f \ \ \text{is continuous.} \\ \langle 2 \rangle 1. \ \ \text{Assume: Every} \ f_{\alpha} \ \ \text{is continuous.} \\ \langle 2 \rangle 2. \ \ \text{Let:} \ \alpha \in J \ \ \text{and} \ U \ \ \text{be open in} \ \ X_{\alpha} \\ \langle 2 \rangle 3. \ \ f^{-1}(\pi_{\alpha}^{-1}(U)) \ \ \text{is open in} \ A \\ \text{Proof:} \ \ f^{-1}(\pi_{\alpha}^{-1}(U)) = f_{\alpha}^{-1}(U). \\ \end{array}$ 

#### 5.2.1 Homeomorphisms

**Definition 5.2.16** (Homeomorphism). Let X and Y be topological spaces and  $f: X \to Y$ . Then f is a homeomorphism between X and Y iff f is a bijection, and f and  $f^{-1}$  are both continuous.

**Definition 5.2.17** (Topological Property). A property P of topological spaces is a *topological property* iff, for any spaces X and Y, if X is homeomorphic to Y then P holds of X if and only if P holds of Y.

**Definition 5.2.18** ((Topological) Imbedding). Let X and Y be topological spaces and  $f: X \to Y$ . Then f is a (topological) imbedding iff f is a homeomorphism between X and im f.

**Definition 5.2.19** (Homogeneous). A topological space X is homogeneous iff, for all  $x, y \in X$ , there exists a homeomorphism  $f: X \cong X$  such that f(x) = y.

### 5.2.2 Strongly Continuous Functions

**Definition 5.2.20** (Strongly Continuous). Let X and Y be topological spaces and  $f: X \to Y$ . Then f is *strongly continuous* iff, for all  $V \subseteq Y$ , we have V is open in Y if and only if  $f^{-1}(V)$  is open in X.

**Proposition 5.2.21.** Let X and Y be topological spaces and  $f: X \to Y$ . Then f is strongly continuous if and only if, for all  $C \subseteq Y$ , C is closed in Y if and only if  $f^{-1}(C)$  is closed in X.

#### PROOF:

 $\langle 1 \rangle 1$ . If f is strongly continuous then, for all  $C \subseteq Y$ , we have C is closed in Y if and only if  $f^{-1}(C)$  is closed in X.

Proof:

$$C$$
 is closed in  $Y \Leftrightarrow Y \setminus C$  is open in  $Y$   
 $\Leftrightarrow f^{-1}(Y \setminus C)$  is open in  $X$   
 $\Leftrightarrow X \setminus f^{-1}(C)$  is open in  $X$   
 $\Leftrightarrow f^{-1}(C)$  is closed in  $X$ 

 $\langle 1 \rangle 2$ . If, for all  $C \subseteq Y$ , we have C is closed in Y if and only if  $f^{-1}(C)$  is closed in X, then f is strongly continuous.

PROOF: Similar.

П

**Proposition 5.2.22.** The composite of two strongly continuous functions is strongly continuous.

#### Proof:

- $\langle 1 \rangle 1$ . Let:  $f: X \to Y$  and  $g: Y \to Z$  be strongly continuous.
- $\langle 1 \rangle 2$ . Let:  $V \subseteq Z$
- $\langle 1 \rangle 3$ . V is open iff  $f^{-1}(g^{-1}(V))$  is open

Proof:

$$V$$
 is open  $\Leftrightarrow g^{-1}(V)$  is open  $(\langle 1 \rangle 1)$   
 $\Leftrightarrow f^{-1}(g^{-1}(V))$  is open  $(\langle 1 \rangle 1)$ 

**Proposition 5.2.23.** Let X, Y and Z be topological spaces. Let  $f: X \to Y$  and  $g: Y \to Z$ . If f is strongly continuous and  $g \circ f$  is continuous, then g is continuous.

#### Proof:

- $\langle 1 \rangle 1$ . Let:  $V \subseteq Z$  be open in Z.
- $\langle 1 \rangle 2$ .  $f^{-1}(g^{-1}(V))$  is open in X.

Proof:  $g \circ f$  is continuous.

 $\langle 1 \rangle 3. \ g^{-1}(V)$  is open in Y.

Proof: f is strongly continuous.

**Proposition 5.2.24.** Let X, Y and Z be topological spaces. Let  $f: X \to Y$  and  $g: Y \to Z$ . If f and  $g \circ f$  are strongly continuous, then g is strongly continuous.

## 5.3 Closed Maps

**Definition 5.3.1** (Closed Map). Let X and Y be topological spaces and  $f: X \to Y$ . Then f is a *closed map* iff, for every closed set  $C \subseteq X$ , the set f(C) is closed in Y.

### 5.4 Local Homeomorphism

**Definition 5.4.1** (Locally Homeomorphic). Let X and Y be topological spaces. Then X is *locally homeomorphic* to Y iff every point in X has an open neighborhood that is homeomorphic with an open set in Y.

**Proposition 5.4.2.** The long line is locally homeomorphic with  $\mathbb{R}$ .

```
PROOF: \begin{array}{l} \langle 1 \rangle 1. \ \ \text{Let:} \ \ x \in L \\ \langle 1 \rangle 2. \ \ \text{Pick an ordinal} \ \ \alpha \ \text{such that} \ \ x < (\alpha,0). \\ \langle 1 \rangle 3. \ \ (-\infty,(\alpha,0)) \ \text{is an open neighbourhood of} \ x \ \text{that is homeomorphic to} \ (0,1). \\ \hline \\ \end{array}
```

#### 5.5 Retracts

**Definition 5.5.1** (Retract). Let Z be a topological space. If Y is a subspace of Z, we say that Y is a *retract* of Z iff there exists a continuous function  $r:Z\to Y$  such that r(y)=y for all  $y\in Y$ .

## Chapter 6

## Separation Axioms

## 6.1 $T_1$ Spaces

**Definition 6.1.1** ( $T_1$  Space). A topological space X is a  $T_1$  space iff every finite set is closed.

**Theorem 6.1.2.** Let X be a  $T_1$  space and  $A \subseteq X$ . Then x is a limit point of A if and only if every neighbourhood of x contains infinitely many points of A.

#### Proof:

- $\langle 1 \rangle 1$ . If some neighbourhood of x contains only finitely many points of A then x is not a limit point of A.
  - $\langle 2 \rangle 1$ . Assume: Some neighbourhood U of x contains only finite many points  $a_1, \ldots, a_n$  of A.
  - $\langle 2 \rangle 2$ .  $X \setminus \{a_1, \dots, a_n\}$  is open. PROOF: X is  $T_1$ .
  - $\langle 2 \rangle 3$ .  $U \setminus \{a_1, \ldots, a_n\}$  is a neighbourhood of x that does not intersect A.
- $\langle 1 \rangle 2$ . If every neighbourhood of x contains infinitely many points of A then x is a limit point of A.

PROOF: From the definition of limit point.

**Proposition 6.1.3.** A subspace of a  $T_1$  space is  $T_1$ .

#### Proof:

- $\langle 1 \rangle 1$ . Let: X be a  $T_1$  space and  $Y \subseteq X$
- $\langle 1 \rangle 2$ . Let:  $a \in Y$
- $\langle 1 \rangle 3$ .  $\{a\}$  is closed in X

PROOF: By  $\langle 1 \rangle 1$ .

 $\langle 1 \rangle 4$ .  $\{a\}$  is closed in Y

PROOF: By Corollary 4.3.4.1.

**Definition 6.1.4** (Separate Points from Closed Sets). Let X be a space and  $\{f_{\alpha}\}_{{\alpha}\in J}$  be a family of continuous functions  $f_{\alpha}:X\to\mathbb{R}$ . Then  $\{f_{\alpha}\}$  separates points from closed sets in X iff, for every point  $x_0\in X$  and every neighbourhood U of  $x_0$ , there exists  $\alpha\in J$  such that  $f_{\alpha}$  is positive at  $x_0$  and vanishes outside U

**Theorem 6.1.5** (Imbedding Theorem). Let X be a  $T_1$  space and  $\{f_{\alpha}\}_{{\alpha}\in J}$  be a family of functions  $X\to\mathbb{R}$  that separates points from closed sets. Then the function  $F:X\to\mathbb{R}^J$  defined by

$$F(x)_{\alpha} = f_{\alpha}(x)$$

is an imbedding. If each  $f_{\alpha}$  maps X into [0,1] then F is an imbedding  $X \to [0,1]^J$ .

#### Proof:

 $\langle 1 \rangle 1$ . F is continuous

PROOF: By Theorem 5.2.15.

 $\langle 1 \rangle 2$ . F is injective

 $\langle 2 \rangle 1$ . Let:  $x, y \in X$  with  $x \neq y$ 

 $\langle 2 \rangle 2$ . PICK a neighbourhood U of x such that  $y \notin U$ 

Proof: X is  $T_1$ 

 $\langle 2 \rangle 3$ . PICK  $\alpha \in J$  such that  $f_{\alpha}$  is positive at x and vanishes outside U

 $\langle 2 \rangle 4. \ f_{\alpha}(x) \neq f_{\alpha}(y)$ 

 $\langle 2 \rangle 5. \ F(x) \neq F(y)$ 

 $\langle 1 \rangle 3$ . F is open as a map  $X \to F(U)$ 

 $\langle 2 \rangle 1$ . Let: U be open

 $\langle 2 \rangle 2$ . Let:  $z \in F(U)$ 

 $\langle 2 \rangle 3$ . Pick  $x \in U$  such that F(x) = z

 $\langle 2 \rangle 4$ . PICK  $\alpha \in J$  such that  $f_{\alpha}$  is positive at x and vanishes outside U

 $\langle 2 \rangle 5. \ z \in \pi_{\alpha}^{-1}((0, +\infty)) \cap F(U) \subseteq F(U)$ 

## 6.2 Hausdorff Spaces

**Definition 6.2.1** (Hausdorff Space). A topological space X is a *Hausdorff space* iff, for any points  $x, y \in X$  with  $x \neq y$ , there exist disjoint neighbourhoods U of x and Y of y.

**Theorem 6.2.2.** Every Hausdorff space is  $T_1$ .

#### Proof:

 $\langle 1 \rangle 1$ . Let: X be a Hausdorff space

 $\langle 1 \rangle 2$ . Let:  $a \in X$ 

PROVE:  $\{a\}$  is closed.

 $\langle 1 \rangle 3$ . Let:  $b \in X \setminus \{a\}$ 

 $\langle 1 \rangle 4$ . PICK disjoint neighbourhoods U of a and V of b

```
 \begin{array}{l} \langle 1 \rangle 5. \;\; b \in V \subseteq X \setminus \{a\} \\ \langle 1 \rangle 6. \;\; \text{Q.E.D.} \\ \text{PROOF: By Proposition 3.2.3.} \end{array}
```

**Theorem 6.2.3.** In a Hausdorff space, a sequence has at most one limit.

#### Proof:

- $\langle 1 \rangle 1$ . Assume: for a contradiction  $x_n \to l$  and  $x_n \to m$  as  $n \to \infty$ , and  $l \neq m$
- $\langle 1 \rangle 2$ . PICK disjoint neighbourhoods U of l and V of m
- $\langle 1 \rangle 3$ . PICK N such that, for all  $n \geq N$ ,  $x_n \in U$  and  $x_n \in V$
- $\langle 1 \rangle 4. \ x_N \in U \cap V$

**Theorem 6.2.4.** Every linearly ordered set is Hausdorff under the order topology.

#### Proof:

- $\langle 1 \rangle 1$ . Let: X be a linearly ordered set under the order topology.
- $\langle 1 \rangle 2$ . Let:  $x, y \in X$  with  $x \neq y$
- $\langle 1 \rangle 3$ . Assume: w.l.o.g. x < y

PROVE: There exist disjoint neighbourhoods U of x and V of y.

 $\langle 1 \rangle 4$ . Case: There exists z such that x < z < y

PROOF: In this case, take  $U = (-\infty, z)$  and  $V = (z, +\infty)$ .

 $\langle 1 \rangle$ 5. Case: There does not exist z such that x < z < y

PROOF: In this case, take  $U = (-\infty, y)$  and  $V = (x, +\infty)$ .

**Theorem 6.2.5.** Let  $\{X_{\alpha}\}_{{\alpha}\in J}$  be a family of Hausdorff spaces. Then  $\prod_{{\alpha}\in J} X_{\alpha}$  is Hausdorff under the product topology.

#### Proof:

- $\langle 1 \rangle 1$ . Let:  $\{x_{\alpha}\}_{\alpha \in J}, \{y_{\alpha}\}_{\alpha \in J} \in \prod_{\alpha \in J} X_{\alpha} \text{ with } \{x_{\alpha}\}_{\alpha \in J} \neq \{y_{\alpha}\}_{\alpha \in J}$
- $\langle 1 \rangle 2$ . PICK  $\alpha \in J$  such that  $x_{\alpha} \neq y_{\alpha}$
- $\langle 1 \rangle 3$ . PICK disjoint neighbourhoods U of  $x_{\alpha}$  and V of  $y_{\alpha}$ .
- $\langle 1 \rangle$ 4.  $\pi_{\alpha}^{-1}(U)$  and  $\pi_{\alpha}^{-1}(V)$  are disjoint neighbourhoods of  $\{x_{\alpha}\}_{{\alpha}\in J}$  and  $\{y_{\alpha}\}_{{\alpha}\in J}$

Corollary 6.2.5.1. The Sorgenfrey plane is Hausdorff.

Corollary 6.2.5.2. For any set I, the space  $\mathbb{R}^I$  is Hausdorff.

**Proposition 6.2.6.** Let X and Y be topological spaces and  $f: X \to Y$ . If f is continuous and injective and Y is Hausdorff then X is Hausdorff.

#### Proof:

- $\langle 1 \rangle 1$ . Let:  $x, y \in X$  with  $x \neq y$
- $\langle 1 \rangle 2. \ f(x) \neq f(y)$

PROOF: f is injective.

```
\langle 1 \rangle3. PICK disjoint neighbourhoods U, V of f(x) and f(y) PROOF: Y is Hausdorff. \langle 1 \rangle4. f^{-1}(U) and f^{-1}(V) are disjoint neighbourhoods of x and y.
```

Corollary 6.2.6.1. A subspace of a Hausdorff space is Hausdorff.

Corollary 6.2.6.2. Let  $\{X_{\alpha}\}_{{\alpha}\in J}$  be a family of nonempty spaces. If  $\prod_{{\alpha}\in J} X_{\alpha}$  is Hausdorff then so is each  $X_{\alpha}$ .

**Corollary 6.2.6.3.** Let  $\mathcal{T}$  and  $\mathcal{T}'$  be topologies on the same set X. If  $\mathcal{T} \subseteq \mathcal{T}'$  and X is Hausdorff under  $\mathcal{T}$  then X is Hausdorff under  $\mathcal{T}'$ .

Corollary 6.2.6.4. The space  $\mathbb{R}_K$  is Hausdorff.

**Proposition 6.2.7.**  $\mathbb{R}_l$  is Hausdorff.

PROOF: Let  $a, b \in \mathbb{R}_l$  with a < b. Then  $(-\infty, b)$  and  $[b, +\infty)$  are disjoint open sets containing a and b respectively.  $\square$ 

**Proposition 6.2.8.** The continuous image of a Hausdorff space is not necessarily Hausdorff.

PROOF: The identity map from the discrete two-point space to the indiscrete two-point space is continuous.  $\Box$ 

**Lemma 6.2.9.** Let A be a subspace of X and Z be Hausdorff. Let  $f: A \to Z$  be continuous. Then there is at most one extension of f to a continuous function  $\overline{A} \to Z$ .

#### PROOF:

- $\langle 1 \rangle 1$ . Assume: for a contradiction  $g, h : \overline{A} \to Z$  are continuous extensions of f with  $g(x) \neq h(x)$
- $\langle 1 \rangle 2$ . PICK disjoint open neighbourhoods U of g(x) and V of h(x)
- $\langle 1 \rangle$ 3. PICK a point  $a \in A \cap g^{-1}(U) \cap h^{-1}(V)$ PROOF: One exists because  $g^{-1}(U) \cap h^{-1}(V)$  is a neighbourhood of  $x \in \overline{A}$ .  $\langle 1 \rangle$ 4.  $g(a) \in U \cap V$

## 6.3 Regular Spaces

**Definition 6.3.1** (Regular). A topological space X is regular iff, for every closed set A and point  $a \notin A$ , there exist disjoint neighbourhoods U of A and V of a.

**Proposition 6.3.2.** Let X be a  $T_1$  space. Then X is regular if and only if, for every point x and neighbourhood U of x, there exists a neighbourhood V of x such that  $\overline{V} \subseteq U$ .

Proof:

- $\langle 1 \rangle 1$ . If X is regular then, for every point x and neighbourhood N of x, there exists a neighbourhood V of x such that  $\overline{V} \subseteq N$ .
  - $\langle 2 \rangle 1$ . Assume: X is regular.
  - $\langle 2 \rangle 2$ . Let:  $x \in X$  and N be a neighbourhood of x
  - $\langle 2 \rangle$ 3. PICK an open set U such that  $x \in U \subseteq N$
  - $\langle 2 \rangle 4$ . Pick disjoint open sets V, W such that  $x \in V$  and  $X \setminus U \subseteq W$
  - $\langle 2 \rangle 5. \ \overline{V} \subseteq N$

Proof:

$$\overline{V} \subseteq X \setminus W$$
 
$$\subseteq U$$
 
$$\subseteq N$$

- $\langle 1 \rangle 2$ . If, for every point x and neighbourhood U of x, there exists a neighbourhood V of x such that  $\overline{V} \subseteq U$ , then X is regular.
  - $\langle 2 \rangle$ 1. Assume: For every point x and neighbourhood U of x, there exists a neighbourhood V of x such that  $\overline{V} \subseteq U$ .
  - $\langle 2 \rangle 2$ . Let:  $x \in X$  and A be a closed set with  $x \notin A$
  - $\langle 2 \rangle 3$ . PICK a neighbourhood V of x such that  $\overline{V} \subseteq X \setminus A$
- $\langle 2 \rangle 4. \ x \in V \text{ and } A \subseteq X \setminus \overline{V}$

**Proposition 6.3.3.** Every linearly ordered set under the order topology is regular.

#### PROOF:

- $\langle 1 \rangle 1$ . Let: X be a linearly ordered set under the order topology.
- $\langle 1 \rangle$ 2. Let:  $x \in X$  and U be a neighbourhood of x Prove: There exists a neighbourhood V of x with  $\overline{V} \subseteq U$
- $\langle 1 \rangle 3$ . Case: x is greatest and least in X

PROOF: Take  $V = U = X = \{x\}$ 

- (1)4. Case: x is greatest in X and there exists a < x such that  $(a, x] \subseteq U$ 
  - $\langle 2 \rangle 1$ . Case: There exists b such that a < b < x

PROOF: Take V = (b, x].

- $\langle 2 \rangle 2$ . Case: There is no b such that a < b < x
  - $\langle 3 \rangle 1$ . Let:  $V = U = \{x\}$
  - $\langle 3 \rangle 2$ .  $\overline{V} = V$

PROOF: For any  $y \neq x$ , we have  $(-\infty, x)$  is a neighbourhood of y that does not intersect V.

- $\langle 1 \rangle$ 5. Case: x is least in X and there exists b > x such that  $[x,b) \subseteq U$  Proof: Similar.
- $\langle 1 \rangle 6$ . Case: There exist a < x < b such that  $(a, b) \subseteq U$ 
  - $\langle 2 \rangle 1.$  Pick a point c such that a < c < x if there is one, otherwise Let: c = a
  - $\langle 2 \rangle 2$ . PICK a point d such that x < d < b if there is one, otherwise Let: d = b
  - $\langle 2 \rangle 3$ . Let: V = (c, d)
  - $\langle 2 \rangle 4. \ \overline{V} \subseteq U$

Proof:

$$\overline{V} \subseteq [c,d]$$

$$\subseteq (a,b)$$

$$\subseteq U$$

 $\langle 1 \rangle 7$ . Q.E.D.

PROOF: These cases are exhaustive by Lemma 4.1.2. They prove X is regular by Proposition 6.3.2.

Proposition 6.3.4. A subspace of a regular space is regular.

#### PROOF:

- $\langle 1 \rangle 1$ . Let: X be a regular space and  $Y \subseteq X$
- $\langle 1 \rangle 2$ . Let:  $A \subseteq Y$  be closed in Y and  $a \in Y \setminus A$
- $\langle 1 \rangle$ 3. PICK C closed in X such that  $A = C \cap Y$  PROOF: By Corollary 4.3.4.1.
- $\langle 1 \rangle 4$ . PICK disjoint open sets U, V in X such that  $C \subseteq U$  and  $a \in V$
- $\langle 1 \rangle$ 5.  $U \cap Y$  and  $V \cap Y$  are disjoint open sets in Y such that  $A \subseteq U \cap Y$  and  $a \in V \cap Y$

П

**Corollary 6.3.4.1.** Let  $\{X_{\alpha}\}_{{\alpha}\in J}$  be a family of nonempty spaces. If  $\prod_{{\alpha}\in J} X_{\alpha}$  is regular then so is each  $X_{\alpha}$ .

**Proposition 6.3.5** (AC). The product of a family of regular spaces is regular.

#### PROOF

- $\langle 1 \rangle 1$ . Let:  $\{X_{\alpha}\}_{{\alpha} \in J}$  be a family of regular spaces.
- $\langle 1 \rangle 2$ .  $\prod_{\alpha \in J} X_{\alpha}$  is  $T_1$
- $\langle 1 \rangle 3$ . Let:  $\vec{a} \in U$  where U is open in  $\prod_{\alpha \in J} X_{\alpha}$
- (1)4. PICK  $\prod_{\alpha \in J} U_{\alpha}$  such that each  $U_{\alpha}$  is open in  $X_{\alpha}$ ,  $U_{\alpha} = X_{\alpha}$  except at  $\alpha_1$ , ...,  $\alpha_n$ , and  $\vec{a} \in \prod_{\alpha \in J} U_{\alpha} \subseteq U$
- $\langle 1 \rangle$ 5. For  $1 \leq i \leq n$ , PICK  $V_{\alpha_i}$  open in  $X_{\alpha_i}$  such that  $a_{\alpha_i} \in V_{\alpha_i}$  and  $\overline{V_{\alpha_i}} \subseteq U_{\alpha_i}$
- $\langle 1 \rangle 6$ . For  $\alpha \neq \alpha_1, \dots, \alpha_n$ , LET:  $V_{\alpha} = X_{\alpha}$
- $\langle 1 \rangle 7. \ \vec{a} \in \prod_{\alpha \in J} V_{\alpha}$
- $\langle 1 \rangle 8. \prod_{\alpha \in J} V_{\alpha} \subseteq \prod_{\alpha \in J} U_{\alpha}$ PROOF: By Theorem 4.2.5.

П

Corollary 6.3.5.1. The Sorgenfrey plane is regular.

Corollary 6.3.5.2. For any set I, the space  $\mathbb{R}^I$  is regular.

**Proposition 6.3.6.** The space  $\mathbb{R}_K$  is not regular.

Proof: There do not exist disjoint neighbourhoods of 0 and K.  $\square$ 

**Proposition 6.3.7.** The continuous image of a regular space is not necessarily regular.

PROOF: The identity map from the discrete two-point space to the indiscrete two-point space is continuous.  $\square$ 

#### 6.4Completely Regular Spaces

**Definition 6.4.1** (Separated by a Continuous Function). Let A and B be subsets of a topological space X. Then A and B can be separated by a continuous function iff there exists a continuous  $f: X \to [0,1]$  such that  $f(A) = \{0\}$  and  $f(B) = \{1\}.$ 

**Definition 6.4.2** (Completely Regular). A space X is completely regular iff X is  $T_1$  and, for every point a and closed set A not containing a, we have that  $\{a\}$ and A can be separated by a continuous function.

**Theorem 6.4.3.** The product of a family of completely regular spaces is completely regular.

#### Proof:

- $\langle 1 \rangle 1$ . Let:  $\{X_{\alpha}\}_{{\alpha} \in J}$  be a family of completely regular spaces.
- $\langle 1 \rangle 2$ . Let:  $a \in \prod_{\alpha \in J} X_{\alpha}$  and A be closed in  $\prod_{\alpha \in J} X_{\alpha}$  such that  $a \notin A$   $\langle 1 \rangle 3$ . Pick a basic open neighbourhood  $\prod_{\alpha \in J} U_{\alpha} \subseteq \prod_{\alpha \in J} X_{\alpha} \setminus A$  of a such that  $U_{\alpha} = X_{\alpha}$  except for  $\alpha = \alpha_1, \dots, \alpha_n$
- $\langle 1 \rangle 4$ . For  $1 \leq i \leq n$ , PICK a continuous  $f_i: X_{\alpha_i} \to [0,1]$  that is 0 at  $a_{\alpha_i}$  and 1 on  $X_{\alpha_i} \setminus U_{\alpha_i}$
- $\langle 1 \rangle$ 5. Let:  $f: \prod_{\alpha \in I} X_{\alpha} \to [0,1]$  be given by  $f(x) = \prod_{i=1}^{n} f_i(x_{\alpha_i})$
- $\langle 1 \rangle 6.$  f(a) = 0
- $\langle 1 \rangle 7$ . f(x) = 1 for  $x \in A$
- $\langle 1 \rangle 8$ . f is continuous

Corollary 6.4.3.1. The Sorgenfrey plane is completely regular.

Corollary 6.4.3.2. For any set I, the space  $\mathbb{R}^I$  is completely regular.

**Proposition 6.4.4.** For any set J, the space  $\mathbb{R}^J$  in the box topology is completely regular.

#### PROOF:

- $\langle 1 \rangle 1$ . Let:  $a \in \mathbb{R}^J$  and  $A \subseteq \mathbb{R}^J$  be closed with  $a \notin A$ Prove: There exists  $f: \mathbb{R}^J_{\text{box}} \to [0,1]$  continuous such that f(a) = 1and  $f(A) = \{0\}$
- $\langle 1 \rangle 2$ . Assume: w.l.o.g.  $A \cap (-1,1)^J = \emptyset$  and  $a = \vec{0}$ 
  - $\langle 2 \rangle 1$ . Pick a basic open set  $\prod_{\alpha \in J} U_{\alpha}$  such that  $a \in \prod_{\alpha \in J} U_{\alpha} \subseteq \mathbb{R}^{J} \setminus A$
  - $\langle 2 \rangle 2$ . For  $\alpha \in J$ , PICK  $b_{\alpha}, c_{\alpha}$  such that  $a_{\alpha} \in (b_{\alpha}, c_{\alpha}) \subseteq U_{\alpha}$
  - $\langle 2 \rangle 3$ . For  $\alpha \in J$ , Pick a homeomorphism  $f_{\alpha} : \mathbb{R} \to \mathbb{R}$  that maps  $b_{\alpha}$  to -1,  $a_{\alpha}$  to 0 and  $c_{\alpha}$  to 1
  - $\langle 2 \rangle 4$ .  $\prod_{\alpha \in J} f_{\alpha}$  is an automorphism  $\mathbb{R}^{J}_{\text{box}}$  that maps a to  $\vec{0}$  and A to a closed set disjoint from  $(-1,1)^J$

```
\langle 1 \rangle 3. Pick a continuous function f: \mathbb{R}^J_{\mathrm{uniform}} \to [0,1] such that f(\vec{0}) = 1 and f(\mathbb{R}^J \setminus (-1,1)^J) = \{0\}
```

 $\langle 1 \rangle 4$ . f is continuous w.r.t. the box topology

### Proposition 6.4.5. Not every regular space is completely regular.

#### Proof:

 $\langle 1 \rangle 1$ . For  $m \in \mathbb{Z}$ , Let:  $L_m = \{m\} \times [-1, 0]$ 

 $\langle 1 \rangle 2$ . For each odd integer n and each integer  $k \geq 2$ , Let:  $C_{nk} = (\{n+1-1/k\} home/robin/fun/RogOMatic/src/actuatortimes[-1,0]) \cup (\{n-1+1/k\} \times [-1,0]) \cup \{(x,y): (x-n)^2 + y^2 = (1-1/k)^2, y \geq 0\}$ 

 $\langle 1 \rangle 3$ . For each odd integer n and each integer  $k \geq 2$ , Let:  $p_{nk} = (n, 1 - 1/k)$ 

- $\langle 1 \rangle 4$ . PICK two points a, b not in any  $L_m$  or  $C_{nk}$
- $\langle 1 \rangle$ 5. Let:  $X = \bigcup_{m \in \mathbb{Z}} L_m \cup \bigcup_{n,k} C_{nk} \cup \{a,b\}$
- $\langle 1 \rangle 6$ . Let:  $\mathcal{B}$  be the set consisting of all subsets of  $\mathbb{R}^2$  of the following forms:
  - 1. The intersection of X with a horizontal open line segment that contains none of the points  $p_{nk}$
  - 2. A set formed from one of the sets  $C_{nk}$  by deleting finitely many points.
  - 3. For each even integer m, the set  $\{a\} \cup \{(x,y) \in X : x < m\}$
  - 4. For each even integer m, the set  $\{b\} \cup \{(x,y) \in X : x > m\}$
- $\langle 1 \rangle 7$ .  $\mathcal{B}$  is a basis for a topology on X
  - $\langle 2 \rangle 1$ . For all  $x \in X$ , there exists  $B \in \mathcal{B}$  such that  $x \in B$
  - $\langle 2 \rangle 2$ . For all  $B_1, B_2 \in \mathcal{B}$  and  $x \in B_1 \cap B_2$ , there exists  $B_3 \in \mathcal{B}$  such that  $x \in B_3 \subseteq B_1 \cap B_2$ 
    - $\langle 3 \rangle 1$ . Case:  $B_1$ ,  $B_2$  are both of type 1 Proof: Their intersection is of type 1.
    - $\langle 3 \rangle$ 2. Case:  $B_1$  is of type 1 and  $B_2$  is of type 2 PROOF: Their intersection is of type 2, since a horizontal line segment intersects  $C_{nk}$  in at most two points.
    - $\langle 3 \rangle 3$ . CASE:  $B_1$  is of type 1 and  $B_2$  is of type 3 PROOF: Their intersection is of type 1
    - $\langle 3 \rangle 4$ . Case:  $B_1$  is of type 1 and  $B_2$  is of type 4 PROOF: Their intersection is of type 1
    - $\langle 3 \rangle$ 5. CASE:  $B_1$  is of type 2 and  $B_2$  is of type 2 PROOF: Their intersection is of type 2
    - $\langle 3 \rangle$ 6. Case:  $B_1$  is of type 2 and  $B_2$  is of type 3 Proof: Their intersection is  $B_1$
    - $\langle 3 \rangle$ 7. Case:  $B_1$  is of type 2 and  $B_2$  is of type 4 Proof: Their intersection is  $B_1$
    - $\langle 3 \rangle 8$ . Case:  $B_1$  is of type 3 and  $B_2$  is of type 3 PROOF: Their intersection is of type 3
    - $\langle 3 \rangle 9$ . Case:  $B_1$  is of type 3 and  $B_2$  is of type 4

- $\langle 4 \rangle 1$ . Let:  $B_1 = \{a\} \cup \{(x,y) \in X : x < m\}$  and  $B_2 = \{b\} \cup \{(x,y) \in X : x < m\}$ X: x > n
- $\langle 4 \rangle 2$ . Case: x = (s, 1 1/k) for some s and integer  $x \geq 2$

PROOF: In this case,  $x \in C_{nk}$  for some n and  $C_{nk} \subseteq B_1 \cap B_2$ .

- $\langle 4 \rangle 3$ . Case: x = (s, t) and  $t \neq 1 1/k$  for any integer  $k \geq 2$ PROOF: In this case,  $x \in ((n, m) \times \{t\}) \cap X \subseteq B_1 \cap B_2$
- $\langle 3 \rangle 10$ . Case:  $B_1$  is of type 4 and  $B_2$  is of type 4

Proof: Their intersection is of type 4

- $\langle 1 \rangle 8$ . For any continuous function  $f: X \to \mathbb{R}$ , we have f(a) = f(b)
  - $\langle 2 \rangle 1$ . Let:  $f: X \to \mathbb{R}$  be continuous
  - $\langle 2 \rangle$ 2. For any  $c \in \mathbb{R}$ , we have  $f^{-1}(c)$  is  $G_{\delta}$  PROOF:  $f^{-1}(c) = \bigcap_{q \in \mathbb{Q}^+} f^{-1}(c-q, c+q)$   $\langle 2 \rangle$ 3. Let:  $S_{nk} = \{ p \in C_{nk} : f(p) \neq f(p_{nk}) \}$

- $\langle 2 \rangle 4$ . For all n, k, we have  $S_{nk}$  is countable.
  - $\langle 3 \rangle 1$ . Let:  $f^{-1}(p_{nk}) = \bigcap_{m=1}^{\infty} U_m$  where  $U_m$  is open in X
  - $\langle 3 \rangle 2$ . For each m, Pick  $B_m \in \mathcal{B}$  such that  $p_{nk} \in B_m \subseteq U_m$
  - $\langle 3 \rangle 3. \ S_{nk} \subseteq \bigcup_{m=1}^{\infty} (C_{nk} \setminus B_m)$
  - $\langle 3 \rangle 4$ . Each  $C_{nk} \setminus B_m$  is countable
    - $\langle 4 \rangle 1$ . Let:  $m \in \mathbb{Z}$
    - $\langle 4 \rangle 2$ .  $B_m$  cannot be of type 1
    - $\langle 4 \rangle 3$ . If  $B_m$  is of type 2 then  $C_{nk} \setminus B_m$  is finite.
    - $\langle 4 \rangle 4$ . If  $B_m$  is of type 3 or 4 then  $C_{nk} \setminus B_m$  is empty.
- $\langle 2 \rangle$ 5. PICK  $d \in [-1,0]$  such that  $\mathbb{R} \times \{d\}$  intersects none of the sets  $S_{nk}$
- $\langle 2 \rangle 6$ . For n odd, we have

$$f(n-1,d) = \lim_{k \to \infty} f(p_{nk}) .$$

- $\langle 3 \rangle 1$ . Let:  $\epsilon > 0$
- $\langle 3 \rangle 2$ . Pick  $B \in \mathcal{B}$  such that  $(n-1,d) \in B \subseteq f^{-1}(f(n-1,d)-\epsilon,f(n-1,d))$
- $\langle 3 \rangle 3$ . There exists  $\delta > 0$  such that, for  $x \in (n-1-\delta, n-1+\delta)$ , we have  $(x,d) \in B$
- $\langle 3 \rangle 4$ . Pick K such that  $1/K < \delta$
- $\langle 3 \rangle 5$ . Let:  $k \geq K$
- $\langle 3 \rangle 6. \ f(n-1+1/k,d) = f(p_{nk})$
- $\langle 3 \rangle 7. |f(n-1,d) f(n-1+1/k,d)| < \epsilon$
- $\langle 3 \rangle 8. |f(n-1,d) f(p_{nk})| < \epsilon$
- $\langle 2 \rangle$ 7. For *n* odd, we have

$$f(n+1,d) = \lim_{k \to \infty} f(p_{nk}) .$$

Proof: Similar.

- $\langle 2 \rangle 8$ . Q.E.D.
  - $\langle 3 \rangle 1$ . Assume:  $f(a) \neq f(b)$
  - $\langle 3 \rangle 2$ . Assume: w.l.o.g. f(a) < f(b)
  - $\langle 3 \rangle 3$ . Pick  $B \in \mathcal{B}$  such that  $a \in B \subseteq f^{-1}(-\infty, (f(a) + f(b))/2)$
  - $\langle 3 \rangle 4$ . Let: m be even such that  $B = \{a\} \cup \{(x,y) \in X : x < m\}$
  - $\langle 3 \rangle 5$ . Pick  $B \in \mathcal{B}$  such that  $b \in B \subseteq f^{-1}((f(a) + f(b))/2, +\infty)$
  - $\langle 3 \rangle 6$ . Let: m' be even such that  $B = \{b\} \cup \{(x,y) \in X : x > m'\}$

```
\langle 3 \rangle 7. f(m,d) = f(m',d)
```

 $\langle 3 \rangle 8$ . Q.E.D.

 $\langle 1 \rangle 9$ . X is regular.

 $\langle 1 \rangle 10$ . X is not completely regular.

PROOF: a and b cannot be separated by a continuous function.

**Theorem 6.4.6** (AC). A space is completely regular iff it is homeomorphic to a subspace of  $[0,1]^J$  for some J.

#### PROOF:

- $\langle 1 \rangle 1$ . Every completely regular space is homeomorphic to a subspace of  $[0,1]^J$  for some J.
  - $\langle 2 \rangle 1$ . Let: X be completely regular
  - $\langle 2 \rangle 2$ . For every point a and open set U that contains a, PICK a continuous function  $f_{aU}$  that is positive on a and vanishes outside U
  - $\langle 2 \rangle 3$ . The family  $\{f_{aU}\}$  separates points from closed sets
  - $\langle 2 \rangle 4$ . Q.E.D.

PROOF: By the Imbedding Theorem.

 $\langle 1 \rangle 2$ . Every subspace of  $[0,1]^J$  is completely regular.

PROOF: By Theorem 6.4.3 and Proposition 6.3.4.

**Proposition 6.4.7.** The continuous image of a completely regular space is not necessarily completely regular.

Proof: The identity map from the discrete two-point space to the indiscrete two-point space is continuous.  $\Box$ 

# 6.5 Normal Spaces

**Definition 6.5.1** (Normal Space). A *normal* space is a  $T_1$  space such that, for any disjoint closed sets A, B, there exist disjoint open sets U, V such that  $A \subseteq U$  and  $B \subseteq V$ .

**Theorem 6.5.2.** Every linearly ordered set is normal under the order topology.

PROOF: See Steen and Steerbach Counterexamples in Topology Example 39.

**Proposition 6.5.3.** The product space  $S_{\Omega} \times \overline{S_{\Omega}}$  is not normal.

#### Proof:

- $\langle 1 \rangle 1$ . Let:  $\Delta = \{(x, x) : x \in \overline{S_{\Omega}}\} \subseteq \overline{S_{\Omega}} \times \overline{S_{\Omega}}\}$
- $\langle 1 \rangle 2$ .  $\Delta$  is closed in  $\overline{S_{\Omega}} \times \overline{S_{\Omega}}$
- $\langle 1 \rangle 3$ . Let:  $A = \Delta \cap (S_{\Omega} \times \overline{S_{\Omega}})$
- $\langle 1 \rangle 4$ . A is closed in  $S_{\Omega} \times \overline{S_{\Omega}}$
- $\langle 1 \rangle 5$ . Let:  $B = S_{\Omega} \times \{\Omega\}$
- $\langle 1 \rangle 6$ . B is closed

```
\langle 1 \rangle 7. A \cap B = \emptyset
\langle 1 \rangle 8. Assume: for a contradiction U and V are disjoint open sets including A
                          and B respectively
\langle 1 \rangle 9. For all x \in S_{\Omega} there exists \beta \in (x, \Omega) such that (x, \beta) \notin U
    \langle 2 \rangle 1. Let: x \in S_{\Omega}
   \langle 2 \rangle 2. \ (x, \Omega) \in V
       Proof: (x, \Omega) \in B \subseteq V
   \langle 2 \rangle 3. PICK y < \Omega such that \{x\} \times (y, \Omega] \subseteq V
       PROOF: By Lemma 4.1.2.
   \langle 2 \rangle 4. Pick \beta such that x, y < \beta < \Omega
       PROOF: Such a \beta exists because \Omega is a limit ordinal.
\langle 1 \rangle 10. For x \in S_{\Omega},
          Let: \beta(x) be the least element of (x,\Omega) such that (x,\beta(x)) \notin U
\langle 1 \rangle 11. Let: b = \sup_{n=1}^{\infty} \beta^n(0)
\langle 1 \rangle 12. \beta^n(0) \to b as n \to \infty
\langle 1 \rangle 13. \ (\beta^n(0), \beta^{n+1}(0)) \to (b, b) \text{ as } n \to \infty
\langle 1 \rangle 14. \ (b,b) \in A
\langle 1 \rangle 15. \ (b,b) \in U
\langle 1 \rangle 16. For all n we have (\beta^n(0), \beta^{n+1}(0)) \notin U
   PROOF: By \langle 1 \rangle 10.
\langle 1 \rangle 17. Q.E.D.
   PROOF: Steps \langle 1 \rangle 12, \langle 1 \rangle 15 and \langle 1 \rangle 16 form a contradiction.
```

Corollary 6.5.3.1. Not every completely regular space is normal.

Corollary 6.5.3.2. An open subspace of a normal space is not necessarily normal.

**Corollary 6.5.3.3.** *The product of two normal spaces is not necessarily normal.* 

**Proposition 6.5.4.** A closed subspace of a normal space is normal.

#### Proof:

- $\langle 1 \rangle 1$ . Let: X be normal and  $C \subseteq X$  be closed.
- $\langle 1 \rangle 2$ . Let: A and B be closed in C
- $\langle 1 \rangle 3$ . A and B are closed in X

PROOF: By Corollary 4.3.4.2.

- $\langle 1 \rangle 4$ . PICK disjoint open neighbourhoods U and V of A and B in X
- $\langle 1 \rangle 5.\ U \cap C$  and  $V \cap C$  are disjoint open neighburhoods of A and B in C  $\sqcap$

**Corollary 6.5.4.1.** Let  $\{X_{\alpha}\}_{{\alpha}\in J}$  be a family of nonempty spaces. If  $\prod_{{\alpha}\in J} X_{\alpha}$  is normal then each  $X_{\alpha}$  is normal.

**Proposition 6.5.5.** If the Continuum Hypothesis then  $\mathbb{R}^{\omega}$  under the box topology is normal.

PROOF: See Rudin. The box product of countably many compact metric spaces. General Topology and Its Applications, 2:293–298, 1972.  $\Box$ 

### **Proposition 6.5.6** (Stone (DC)). If J is uncountable then $\mathbb{R}^J$ is not normal.

Proof:

 $\langle 1 \rangle 1$ . Let:  $X = (\mathbb{Z}^+)^J$ 

Prove: X is not normal.

 $\langle 1 \rangle 2$ . For  $x \in X$  and  $B \subseteq^{\text{fin}} J$ , Let:

$$U(x,B) = \{ y \in X : \forall \alpha \in B. y_{\alpha} = x_{\alpha} \}$$

- $U(x,B)=\{y\in X: \forall \alpha\in B. y_\alpha=x_\alpha\}\ .$   $\langle 1\rangle 3.\ \{U(x,B): x\in X, B\subseteq^{\mathrm{fin}}J\}$  is a basis for X
  - $\langle 2 \rangle 1$ . Let:  $x \in X$  and  $\prod_{\alpha \in J} U_{\alpha}$  be a basic open set including x, where  $U_{\alpha} =$  $\mathbb{Z}^+$  for all  $\alpha$  except  $\alpha_1, \ldots, \alpha_n$
  - $\langle 2 \rangle 2$ .  $x \in U(x, \{\alpha_1, \dots, \alpha_n\}) \subseteq \prod_{\alpha \in I} U_\alpha$
- $\langle 1 \rangle 4$ . For  $n \in \mathbb{Z}^+$ ,

Let:  $P_n = \{x \in X : x \text{ is injective on } J \setminus x^{-1}(n)\}$ 

- $\langle 1 \rangle 5$ .  $P_1$  and  $P_2$  are closed and disjoint.
  - $\langle 2 \rangle 1$ .  $P_1$  is closed
    - $\langle 3 \rangle 1$ . Let:  $x \in X \setminus P_1$
    - $\langle 3 \rangle 2$ . PICK  $\alpha, \beta \in J$  such that  $x_{\alpha} = x_{\beta} \neq 1$
    - $\langle 3 \rangle 3$ . Let:  $U_{\gamma} = \{x_{\alpha}\}$  if  $\gamma = \alpha$  or  $\gamma = \beta$ ,  $\mathbb{Z}^+$  for all other  $\gamma \in J$
    - $\langle 3 \rangle 4. \ x \in \prod_{\gamma \in J} U_{\gamma} \subseteq X \setminus P_1$
  - $\langle 2 \rangle 2$ .  $P_2$  is closed

PROOF: Similar.

 $\langle 2 \rangle 3. P_1 \cap P_2 = \emptyset$ 

PROOF: If  $x \in P_1 \cap P_2$  then x is injective on J, contradicting the fact that J is uncountable.

- (1)6. Assume: for a contradiction U and V are disjoint open sets including  $P_1$ and  $P_2$
- $\langle 1 \rangle 7$ . Given a sequence  $(\alpha_i)$  of distinct elements of J and a strictly increasing sequence  $(n_i)$  of positive integers, Let:

$$B_i^{\alpha,n} = \{\alpha_1, \dots, \alpha_{n_i}\}$$

$$x_i^{\alpha,n} \in X$$

$$(x_i^{\alpha,n})_{\beta} = \begin{cases} j & \text{if } \beta = \alpha_j, 1 \le j \le n_{i-1} \\ 1 & \text{for all other values of } \beta \end{cases}$$

for i > 1

- (1)8. PICK sequences  $(\alpha_i)$ ,  $(n_i)$  such that, for all  $i \geq 1$ , we have  $U(x_i^{\alpha,n}, B_i^{\alpha,n}) \subseteq$ 
  - $\langle 2 \rangle 1$ . Let:  $x_1 \in X$  be given by  $(x_1)_{\alpha} = 1$  for all  $\alpha \in J$
  - $\langle 2 \rangle 2. \ x_1 \in U$

PROOF:  $x_1 \in P_1 \subseteq U$  $\langle 2 \rangle 3$ . PICK  $B_1 \subseteq^{\text{fin}} J$  such that  $U(x_1, B_1) \subseteq U$ 

Proof: By  $\langle 1 \rangle 3$ .

- $\langle 2 \rangle 4$ . Let:  $n_1 = |B_1|$  and  $B_1 = \{\alpha_1, \dots, \alpha_{n_1}\}$
- $\langle 2 \rangle$ 5. Assume: We have chosen  $n_1, \ldots, n_k$  strictly increasing and  $\alpha_1, \ldots, \alpha_k$  $\alpha_{n_k}$  such that, for  $1 \leq i \leq k$ , we have  $U(x_i^{\alpha,n}, B_i^{\alpha,n}) \subseteq U$

**Theorem 6.5.7** (Urysohn Lemma). Let X be a normal space. Let A and B be disjoint closed subsets of X. Then there exists a continuous map  $f: X \to [0,1]$  such that f(x) = 0 for all  $x \in A$  and f(x) = 1 for all  $x \in B$ .

#### PROOF:

- $\langle 1 \rangle 1$ . Let: P be the set of all rational numbers in [0,1]
- $\langle 1 \rangle$ 2. For all  $q \in P$ , PICK an open set  $U_q$  in X such that  $A \subseteq U_0$ ,  $U_1 \subseteq X \setminus B$ , and whenever p < q then  $\overline{U_p} \subseteq U_q$ 
  - $\langle 2 \rangle 1$ . Pick an enumeration  $(q_n)$  of P such that  $q_1 = 1$  and  $q_2 = 0$
  - $\langle 2 \rangle 2$ . Let:  $U_1 = X \setminus B$
  - $\langle 2 \rangle 3$ . PICK an open set  $U_0$  such that  $A \subseteq U_0$  and  $\overline{U_0} \subseteq U_1$
  - $\langle 2 \rangle 4$ . Assume: we have open sets  $U_1, U_0, \ldots, U_{q_n}$  such that whenever p < q then  $\overline{U_p} \subseteq U_q$
  - $\langle 2 \rangle 5. \ q_2 < q_{n+1} < q_1$
  - $\langle 2 \rangle$ 6. Let:  $q_k$  be greatest among  $q_1, \ldots, q_n$  such that  $q_k < q_{n+1}$ , and  $q_l$  be least such that  $q_{n+1} < q_l$
  - $\langle 2 \rangle$ 7. PICK an open set  $U_{q_{n+1}}$  such that  $\overline{U_{q_k}} \subseteq U_{q_{n+1}}$  and  $\overline{U_{q_{n+1}}} \subseteq U_{q_l}$
  - $\langle 2 \rangle 8$ . For all  $p, q \in \{q_1, \dots, q_{n+1}\}$ , if p < q then  $\overline{U_p} \subseteq U_q$
- $\langle 1 \rangle 3$ . Extend the family  $(U_q)$  to  $\mathbb Q$  by defining:  $U_q = \emptyset$  if q < 0 and  $U_q = X$  if q > 1
- $\langle 1 \rangle 4$ . For all rationals p, q with p < q we have  $\overline{U}_p \subseteq U_q$
- $\langle 1 \rangle$ 5. Define  $f: X \to [0,1]$  by  $f(x) = \inf\{q \in \mathbb{Q} : x \in U_q\}$

PROOF: This set is nonempty since  $x \in U_1$  and bounded below since if  $x \in U_q$  then  $q \ge 0$ .

- $\langle 1 \rangle 6$ . For all  $x \in A$  we have f(x) = 0
- $\langle 1 \rangle 7$ . For all  $x \in B$  we have f(x) = 1
- $\langle 1 \rangle 8$ . If  $x \in \overline{U_r}$  then  $f(x) \leq r$
- $\langle 1 \rangle 9$ . If  $x \notin U_r$  then  $f(x) \geq r$
- $\langle 1 \rangle 10$ . f is continuous
  - $\langle 2 \rangle 1$ . Let:  $x_0 \in X$
  - $\langle 2 \rangle$ 2. Let: (c,d) be an open interval containing  $f(x_0)$ PROVE: There exists a neighbourhood U of  $x_0$  such that  $f(U) \subseteq (c,d)$

```
\langle 2 \rangle 3. PICK rationals p, q such that c  <math>\langle 2 \rangle 4. x \notin \overline{U_p} PROOF: By \langle 1 \rangle 8 \langle 2 \rangle 5. x \in U_q PROOF: By \langle 1 \rangle 9 \langle 2 \rangle 6. Let: U = U_q \setminus \overline{U_p} Definition 6.5.8 (Vanish Precisely). Let X be a set and A \subseteq X. Let f: X \to [0,1]. Then f vanishes precisely on A iff f^{-1}(0) = A.
```

Proof:

 $\langle 1 \rangle 1$ . If there exists f such that f vanishes precisely on A then A is closed. PROOF: This holds because  $A = f^{-1}(0)$ .

a continuous function  $f: X \to [0,1]$  such that f vanishes precisely on A if and

 $\langle 1 \rangle 2$ . If there exists f such that f vanishes precisely on A then A is  $G_{\delta}$ . PROOF: This holds because  $A = \bigcap_{q \in \mathbb{Q}^+} f^{-1}([0,q))$ .

 $\langle 1 \rangle 3$ . If A is closed and  $G_{\delta}$  then there exists f that vanishes precisely on A.

 $\langle 2 \rangle 1$ . Let:  $A = \bigcap_{n=1}^{\infty} U_n$ 

only if A is a closed  $G_{\delta}$  set.

 $\langle 2 \rangle 2$ . For  $n \geq 1$ , PICK  $f_n : X \to [0, 1/2^n]$  such that f(x) = 0 for  $x \in A$  and  $f(x) = 1/2^n$  for  $x \in X \setminus U_n$ 

PROOF: By the Urysohn Lemma.

 $\langle 2 \rangle 3$ . Let:  $f: X \to [0,1]$  be given by  $f(x) = \sum_{n=1}^{\infty} f_n(x)$ 

PROOF: The series converges for every x by the Comparison Test.

 $\langle 2 \rangle 4$ . f is continuous

 $\langle 3 \rangle 1$ .  $f_n$  converges uniformly to f

PROOF: By the Weierstrass M-test.

 $\langle 3 \rangle 2$ . Q.E.D.

PROOF: By the Uniform Limit Theorem.

 $\langle 2 \rangle 5$ . f(x) = 0 for  $x \in A$ 

PROOF: From  $\langle 2 \rangle 2$ .

 $\langle 2 \rangle 6$ . f(x) > 0 for  $x \notin A$ 

 $\langle 3 \rangle 1$ . Let:  $x \notin A$ 

 $\langle 3 \rangle 2$ . PICK N such that  $x \notin U_N$ 

 $\langle 3 \rangle 3$ . Q.E.D.

Proof:

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \qquad (\langle 2 \rangle 3)$$
  

$$\geq f_N(x)$$
  

$$> 0 \qquad (\langle 2 \rangle 2)$$

**Theorem 6.5.10** (Strong Form of Urysohn Lemma). Let X be a normal space.

Then there exists a continuous function  $f: X \to [0,1]$  such that  $f^{-1}(0) = A$ and  $f^{-1}(1) = B$  if and only if A and B are disjoint, closed and  $G_{\delta}$ .

#### Proof:

- $\langle 1 \rangle 1$ . If there exists a continuous function  $f: X \to [0,1]$  such that  $f^{-1}(0) = A$ and  $f^{-1}(1) = B$  then A and B are disjoint, closed and  $G_{\delta}$ 
  - $\langle 2 \rangle 1$ . Assume: there exists a continuous function  $f: X \to [0,1]$  such that  $f^{-1}(0) = A$  and  $f^{-1}(1) = B$
  - $\langle 2 \rangle 2$ . A and B are disjoint
  - $\langle 2 \rangle 3$ . A is closed and  $G_{\delta}$

PROOF: By Theorem 6.5.9.

 $\langle 2 \rangle 4$ . B is closed and  $G_{\delta}$ 

PROOF: Apply Theorem 6.5.9 to 1 - f.

- $\langle 1 \rangle 2$ . If A and B are disjoint, closed and  $G_{\delta}$  then there exists a continuous function  $f: X \to [0,1]$  such that  $f^{-1}(0) = A$  and  $f^{-1}(1) = B$ 
  - $\langle 2 \rangle 1$ . Assume: A and B are disjoint, closed and  $G_{\delta}$
  - $\langle 2 \rangle 2$ . PICK  $g: X \to [0,1]$  that vanishes precisely on A and  $h: X \to [0,1]$  that vanishes precisely on B
- $\langle 2 \rangle 3$ . Let: f = g/(g+h)

**Definition 6.5.11** (Universal Extension Property). A topological space Y has the universal extension property iff, for every normal space X and closed subspace A of X, every continuous function  $A \to Y$  can be extended to a continuous function  $X \to Y$ .

**Theorem 6.5.12** (Tietze Extension Theorem (DC)). Let X be a normal space. Let A be closed subspace of X.

- 1. Any continuous function  $A \rightarrow [a,b]$  can be extended to a continuous func $tion X \rightarrow [a,b].$
- 2. Any continuous function  $A \to \mathbb{R}$  can be extend to a continuous function  $X \to \mathbb{R}$ .

#### Proof:

- $\langle 1 \rangle 1$ . Any continuous function  $A \to [-1,1]$  can be extended to a continuous function  $X \to [-1, 1]$ 
  - $\langle 2 \rangle 1$ . For every continuous function  $f: A \to [-r, r]$ , there exists a continuous  $g: X \to \mathbb{R}$  such that

$$|g(x)| \le \frac{1}{3}r \qquad (x \in X)$$

$$|g(x)-f(x)| \leq \frac{2}{3}r \qquad (x \in A)$$
   
 (3)1. Let:  $f:A \to [-r,r]$  be continuous

- $\langle 3 \rangle 2$ . Let:  $I_1 = [-r, -\frac{1}{3}r]$   $\langle 3 \rangle 3$ . Let:  $I_2 = [-\frac{1}{3}r, \frac{1}{3}r]$   $\langle 3 \rangle 4$ . Let:  $I_3 = [\frac{1}{3}r, r]$

- $\langle 3 \rangle 5$ . Let:  $B = f^{-1}(I_1)$
- $\langle 3 \rangle 6$ . Let:  $C = f^{-1}(I_3)$
- $\langle 3 \rangle 7$ . PICK a continuous  $g: X \to [-\frac{1}{3}r, \frac{1}{3}r]$  such that  $g(x) = -\frac{1}{3}r$  for  $x \in B$ and  $g(x) = \frac{1}{3}r$  for  $x \in C$

PROOF: By the Urysohn Lemma, since B and C are closed disjoint subsets of X.

- $\langle 3 \rangle 8$ . For all  $x \in A$  we have  $|g(x) f(x)| \leq \frac{2}{3}r$ 
  - $\langle 4 \rangle 1$ . Let:  $x \in A$
  - $\langle 4 \rangle 2$ . Case:  $f(x) \in I_1$

Proof:

$$|g(x) - f(x)| = \left| -\frac{1}{3}r - f(x) \right| \qquad (x \in B)$$

$$\leq \frac{2}{3}r \qquad (f(x) \in I_1)$$

 $\langle 4 \rangle 3$ . Case:  $f(x) \in I_2$ 

PROOF: In this case,  $|g(x) - f(x)| \le \frac{2}{3}r$  since  $f(x), g(x) \in I_2$ .

 $\langle 4 \rangle 4$ . Case:  $f(x) \in I_3$ 

Proof:

$$|g(x) - f(x)| = \left| \frac{1}{3}r - f(x) \right| \qquad (x \in C)$$

$$\leq \frac{2}{3}r \qquad (f(x) \in I_3)$$

- $\langle 2 \rangle 2$ . Let:  $f: A \to [-1,1]$  be continuous.
- $\langle 2 \rangle$ 3. PICK a sequence of functions  $(g_n)$  such that

$$|g_n(x)| \le \frac{1}{3} \left(\frac{2}{3}\right)^{n-1} \qquad (x \in X)$$

$$|f(x) - g_1(x) - \dots - g_n(x)| \le (2/3)^n$$
  $(x \in A)$ 

 $|f(x)-g_1(x)-\cdots-g_n(x)|\leq (2/3)^n$   $(x\in A)$ PROOF:Given  $g_1,\ldots,g_n$ , we apply  $\langle 2\rangle 1$  with  $f=f-g_1-\cdots-g_n$  and  $r = (2/3)^n$ .

 $\langle 2 \rangle 4$ . Let:  $g(x) = \sum_{n=1}^{\infty} g_n(x)$  for  $x \in X$ 

PROOF: This series converges by the Comparison Test since  $\sum_{n=1}^{\infty} (2/3)^n$ converges.

- $\langle 2 \rangle$ 5. g is continuous.
  - $\langle 3 \rangle 1$ .  $\sum_{n=1}^{N} g_n$  converges to g uniformly

Proof: By the Weierstrass M-test.

 $\langle 3 \rangle 2$ . Q.E.D.

PROOF: By the Uniform Limit Theory.

 $\langle 2 \rangle 6$ . For all  $x \in A$  we have g(x) = f(x)

PROOF:  $\left|\sum_{n=1}^{N} g_n(x) - f(x)\right| \le (2/3)^N \to 0 \text{ as } N \to \infty.$   $\langle 2 \rangle 7$ . For all  $x \in X$  we have  $-1 \le g(x) \le 1$ 

Proof:

$$\left| \sum_{n=1}^{N} g_n(x) \right| \le \sum_{n=1}^{N} |g_n(x)|$$

$$\le 1/3 \sum_{n=1}^{N} (2/3)^{n-1}$$

$$\to 2/3$$

- $\langle 1 \rangle 2$ . Any continuous function  $A \to (-1,1)$  can be extend to a continuous function  $X \to (-1,1)$ 
  - $\langle 2 \rangle 1$ . Let:  $f: A \to (-1,1)$  be continuous
  - $\langle 2 \rangle 2$ . PICK a continuous  $g: X \to [-1, 1]$  that extends f Proof: By  $\langle 1 \rangle 1$ .
  - $\langle 2 \rangle 3$ . Let:  $D = g^{-1}(-1) \cup g^{-1}(1)$
  - $\langle 2 \rangle 4$ . D is closed in X

PROOF: Since g is continuous and  $\{-1\}$ ,  $\{1\}$  are closed in [-1,1].

 $\langle 2 \rangle 5$ .  $D \cap A = \emptyset$ 

PROOF: Since  $g(A) = f(A) \subseteq (-1, 1)$ .

- $\langle 2 \rangle 6$ . PICK a continuous  $\phi: X \to [0,1]$  such that  $\phi(D) = \{0\}$  and  $\phi(A) = \{1\}$ PROOF: By the Urysohn Lemma.
- $\langle 2 \rangle 7$ . Let:  $h = g\phi$
- $\langle 2 \rangle 8$ . h is continuous
- $\langle 2 \rangle 9$ . h extends f
- $\langle 2 \rangle 10$ . im  $h \subseteq (-1,1)$
- $\langle 1 \rangle 3$ . Q.E.D.

PROOF: The result follows because any closed interval in  $\mathbb{R}$  is homeomorphic to [-1, 1] and  $\mathbb{R} \cong (-1, 1)$ .

Lemma 6.5.13 (Shrinking Lemma (AC)). Let X be a normal space. Let  $\{U_{\alpha}\}_{{\alpha}\in J}$  be a point-finite indexed open covering of X. Then there exists an indexed open covering  $\{V_{\alpha}\}_{{\alpha}\in J}$  such that  $\overline{V_{\alpha}}\subseteq U_{\alpha}$  for all  ${\alpha}\in J$ .

### PROOF:

- $\langle 1 \rangle 1$ . Pick a well-ordering  $\prec$  on J
- $\langle 1 \rangle$ 2. PICK open sets  $V_{\alpha}$  for  $\alpha \in J$  such that  $A_{\alpha} \subseteq V_{\alpha}$  and  $\overline{V_{\alpha}} \subseteq U_{\alpha}$ , where  $A_{\alpha} = X \setminus \bigcup_{\beta \prec \alpha} V_{\beta} \cup \bigcup_{\alpha \prec \beta} U_{\beta}$

$$A_{\alpha} = X \setminus \bigcup_{\beta \prec \alpha} V_{\beta} \cup \bigcup_{\alpha \prec \beta} U_{\beta}$$

PROOF: Apply transfinite induction to Proposition 13.1.16.

- $\langle 1 \rangle 3. \ \{V_{\alpha}\}_{{\alpha} \in J} \text{ covers } X$ 
  - $\langle 2 \rangle 1$ . Let:  $x \in X$
  - $\langle 2 \rangle 2$ . Let:  $\alpha_1, \ldots, \alpha_n$  be the elements of J such that  $x \in U_{\alpha_i}$ , where  $\alpha_1 \prec \alpha_1 = \alpha_1 + \alpha_2 = \alpha_1 + \alpha_2 = \alpha_1 + \alpha_2 = \alpha_1 + \alpha_2 = \alpha_2 = \alpha_2 = \alpha_1 + \alpha_2 = \alpha_2 =$  $\cdots \prec \alpha_n$

PROVE:  $x \in V_{\alpha_i}$  for some i

- $\langle 2 \rangle 3$ . Assume:  $x \notin V_{\alpha_1}, \dots, V_{\alpha_{n-1}}$
- $\langle 2 \rangle 4. \ x \in A_{\alpha_n}$
- $\langle 2 \rangle 5. \ x \in V_{\alpha_n}$

**Proposition 6.5.14** (DC).  $S_{\Omega} \times \overline{S_{\Omega}}$  is not normal.

#### Proof:

- $\langle 1 \rangle 1$ . Let:  $\Delta = \{(x, x) : x \in \overline{S_{\Omega}}\}$
- $\langle 1 \rangle 2$ .  $\Delta$  is closed in  $\overline{S_{\Omega}}^2$ 
  - $\langle 2 \rangle 1$ . Let:  $(x,y) \in \overline{S_{\Omega}}^2 \setminus \Delta$
  - $\langle 2 \rangle 2$ . PICK disjoint open sets U, V such that  $x \in U$  and  $y \in V$
- $\langle 2 \rangle 3. \ (x,y) \in U \times V \subseteq \overline{S_{\Omega}}^2 \setminus \Delta$   $\langle 1 \rangle 3. \ \text{Let:} \ A = \Delta \cap (S_{\Omega} \times \overline{S_{\Omega}})$
- $\langle 1 \rangle 4$ . A is closed in  $S_{\Omega} \times \overline{S_{\Omega}}$
- $\langle 1 \rangle$ 5. Let:  $B = S_{\Omega} \times \{\Omega\}$
- $\langle 1 \rangle 6$ . B is closed in  $S_{\Omega} \times \overline{S_{\Omega}}$
- $\langle 1 \rangle 7$ .  $A \cap B = \emptyset$
- $\langle 1 \rangle 8$ . Assume: for a contradiction U and V are disjoint open sets including A and B respectively
- $\langle 1 \rangle 9$ . Pick a sequence  $x_n$  in  $S_{\Omega}$  such that  $x_n < x_{n+1} < \Omega$  and  $(x_n, x_{n+1}) \notin U$ for all n
  - $\langle 2 \rangle 1$ . Let:  $x_n \in S_{\Omega}$
  - $\langle 2 \rangle 2. \ (x_n, \Omega) \in V$
  - $\langle 2 \rangle 3$ . Pick open sets  $W \subseteq S_{\Omega}, X \subseteq \overline{S_{\Omega}}$  such that  $x_n \in W, \Omega \in X$  and  $W\times X\subseteq V$
  - $\langle 2 \rangle 4$ . Pick  $y < \Omega$  such that  $(x_{n+1}, \Omega] \subseteq X$
  - $\langle 2 \rangle 5$ . Let:  $x_{n+1} = y + 1$
- $\langle 1 \rangle 10$ . Let: b be the supremum of  $\{x_n : n \geq 1\}$
- $\langle 1 \rangle 11. \ (x_n, x_{n+1}) \to (b, b) \text{ as } n \to \infty$
- $\langle 1 \rangle 12. \ (b,b) \in A$
- $\langle 1 \rangle 13. \ (b,b) \in U$
- $\langle 1 \rangle 14$ . For all n we have  $(x_n, x_{n+1}) \notin U$

#### **Proposition 6.5.15** (AC). $\mathbb{R}_l$ is normal.

#### Proof:

- $\langle 1 \rangle 1$ . Let: A and B be disjoint closed sets in  $\mathbb{R}_l$
- $\langle 1 \rangle 2$ . For  $a \in A$ , PICK  $x_a > a$  such that  $[a, x_a)$  not intersecting B
- $\langle 1 \rangle 3$ . For  $b \in B$ , PICK  $x_b > b$  such that  $[b, x_b)$  does not intersect A
- $\langle 1 \rangle 4$ . Let:  $U = \bigcup_{a \in A} [a, x_a)$  and  $V = \bigcup_{b \in B} [b, x_b)$
- $\langle 1 \rangle 5$ . U and V are disjoint open sets including A and B respectively.

**Lemma 6.5.16.** The set  $L = \{(x, -x); x \in \mathbb{R}\}$  as a subspace of  $\mathbb{R}^2_l$  is closed

- $\langle 1 \rangle 1$ . Let:  $(x,y) \notin L$ , so  $y \neq -x$ PROVE: There exists a neighbourhood U of (x, y) that does not intersect
- $\langle 1 \rangle 2$ . Case: y > -x

```
PROOF: In this case, take U = [x, +\infty) \times [y, +\infty)
\langle 1 \rangle 3. Case: y < -x
   PROOF: In this case, take U = [x, (x - y)/2) \times [y, (y - x)/2).
Proposition 6.5.17 (AC). The Sorgenfrey plane is not normal.
Proof:
\langle 1 \rangle 1. Assume: for a contradiction the Sorgenfrey plane is normal.
\langle 1 \rangle 2. Let: L = \{(x, -x); x \in \mathbb{R}\} as a subspace of \mathbb{R}^2
\langle 1 \rangle 3. L has the discrete topology.
   \langle 2 \rangle 1. Let: (x, -x) \in L
            PROVE: \{(x, -x)\} is open in L
   \langle 2 \rangle 2. \{(x, -x)\} = ([x, +\infty) \times [-x, +\infty)) \cap L
\langle 1 \rangle 4. Every subset of L is closed in \mathbb{R}^2
   Proof: By Corollary 4.3.4.2.
\langle 1 \rangle5. For every nonempty proper subset A of L, Pick disjoint open sets U_A,
         V_A containing A and L \setminus A
   PROOF: By \langle 1 \rangle 1 and \langle 1 \rangle 4.
\langle 1 \rangle 6. Let: D = \mathbb{Q}^2
\langle 1 \rangle 7. D is dense in \mathbb{R}^2
   PROOF: Given any basic open set [a,b) \times [c,d), pick rationals q, r such that
   a \leq q < b \text{ and } c \leq r < d. Then (q, r) \in ([a, b) \times [c, d)) \cap D
\langle 1 \rangle 8. Let: \theta : \mathcal{P}L \to \mathcal{P}D be the function
                                                                                (\emptyset \neq A \neq L)
                                     \theta(A) = U_A \cap D
                                      \theta(\emptyset) = \emptyset
                                      \theta(L) = D
\langle 1 \rangle 9. \theta is injective
   \langle 2 \rangle 1. Let: A, B \subseteq L with \theta(A) = \theta(B)
            Prove: A = B
   \langle 2 \rangle 2. Case: \emptyset \neq A \neq L and \emptyset \neq B \neq L
       \langle 3 \rangle 1. \ A \subseteq B
          \langle 4 \rangle 1. Let: x \in A
          \langle 4 \rangle 2. \ x \in U_A
             Proof: By \langle 1 \rangle 5
          \langle 4 \rangle 3. \ x \in U_B
              Proof: By \langle 2 \rangle 1
          \langle 4 \rangle 4. \ x \notin L \setminus B
              Proof: By \langle 1 \rangle 5
          \langle 4 \rangle 5. \ x \in B
              PROOF: Since x \in L by \langle 4 \rangle 1
       \langle 3 \rangle 2. B \subseteq A
          Proof: Similar.
   \langle 2 \rangle 3. Case: \emptyset \neq A \neq L and B = \emptyset
       PROOF: This implies U_A \cap D = \emptyset which contradicts the fact that D is dense.
   \langle 2 \rangle 4. Case: \emptyset \neq A \neq L and B = L
       PROOF: This implies V_A \cap D = \emptyset which contradicts the fact that D is dense.
```

 $\langle 2 \rangle$ 5. Case:  $A = B = \emptyset$ 

PROOF: Trivial

 $\langle 2 \rangle 6$ . Case:  $A = \emptyset$  and B = L

PROOF: This implies  $D = \emptyset$  which is a contradiction.

 $\langle 2 \rangle$ 7. Case: A = B = L

Proof: Trivial

 $\langle 1 \rangle 10$ . Q.E.D.

PROOF: This is a contradiction since D is countable and L is uncountable.

**Proposition 6.5.18.** The continuous image of a normal space is not necessarily normal.

PROOF: The identity map from the discrete two-point space to the indiscrete two-point space is continuous.  $\Box$ 

# 6.6 Completely Normal Spaces

**Definition 6.6.1** (Completely Normal). A space X is *completely normal* iff every subspace is normal.

**Proposition 6.6.2.** A subspace of a completely normal space is completely normal.

PROOF: Immediate from definitions.

**Proposition 6.6.3.** Let X be a topological space. Then X is completely normal iff X is  $T_1$  and, for any pair of separated sets A, B in X, there exist disjoint open sets including them.

#### Proof:

- $\langle 1 \rangle 1$ . If X is completely normal then X is  $T_1$  and, for any pair of separated sets A, B in X, there exist disjoint open sets including them.
  - $\langle 2 \rangle$ 1. Assume: X is completely normal.
  - $\langle 2 \rangle 2$ . X is  $T_1$

PROOF: Holds because X is normal.

- $\langle 2 \rangle 3$ . For any pair of separated sets A, B in X, there exist disjoint open sets including them.
  - $\langle 3 \rangle 1$ . Let: A and B be separated in X
  - $\langle 3 \rangle 2$ . Let:  $Y = X \setminus (\overline{A} \cap \overline{B})$
  - $\langle 3 \rangle 3$ . PICK disjoint open sets U,V in Y such that  $\overline{A} \cap Y \subseteq U$  and  $\overline{B} \cap Y \subseteq V$  PROOF: Y is normal by  $\langle 2 \rangle 1$ .
  - $\langle 3 \rangle 4$ . PICK open sets  $U_0, V_0$  in X such that  $U = U_0 \cap Y, V = V_0 \cap Y$
  - $\langle 3 \rangle 5$ .  $A \subseteq U_0 \setminus \overline{B}$  and  $B \subseteq V_0 \setminus \overline{A}$ PROOF: Using  $\langle 3 \rangle 1$ .
- $\langle 1 \rangle$ 2. If X is  $T_1$  and, for any pair of separated sets A, B in X, there exist disjoint open sets including them, then X is completely normal.

- $\langle 2 \rangle 1$ . Assume: X is  $T_1$  and, for any pair of separated sets A, B in X, there exist disjoint open sets including them
- $\langle 2 \rangle 2$ . Let:  $Y \subseteq X$
- $\langle 2 \rangle 3$ . Y is  $T_1$

PROOF: By Proposition 6.1.3.

- $\langle 2 \rangle 4$ . Let: A and B be disjoint closed sets in Y
- $\langle 2 \rangle 5$ . A and B are separated in X
  - $\langle 3 \rangle 1$ .  $\overline{A} \cap Y = A$  and  $\overline{B} \cap Y = B$

PROOF: By Proposition 3.7.6 and Theorem 4.3.4.

 $\langle 3 \rangle 2$ .  $\overline{A} \cap B = \emptyset$ 

$$\overline{A} \cap B = \overline{A} \cap \overline{B} \cap Y \tag{(3)1)}$$

$$= A \cap B \tag{(3)1}$$

$$=\emptyset \qquad \qquad (\langle 2 \rangle 4)$$

 $\langle 3 \rangle 3. \ A \cap \overline{B} = \emptyset$ 

PROOF: Similar.

- $\langle 2 \rangle$ 6. Pick disjoint open sets U and V that include A and B respectively. Proof: By  $\langle 2 \rangle$ 1.
- $\langle 2 \rangle$ 7.  $U \cap Y$  and  $V \cap Y$  are disjoint open sets in Y that include A and B respectively.

П

**Proposition 6.6.4.** A well-ordered set in the order topology is completely normal.

#### Proof:

- $\langle 1 \rangle 1$ . Let: X be a well-ordered set.
- $\langle 1 \rangle 2$ . For all  $a, b \in X$  with a < b, we have (a, b] is open.
  - $\langle 2 \rangle 1$ . Case: b is greatest in X

PROOF: This case holds by the definition of the order topology.

 $\langle 2 \rangle 2$ . Case: b is not greatest in X

PROOF: In this case, (a, b] = (a, c) where c is the successor of b.

 $\langle 1 \rangle 3$ . Let: A and B be separated sets in X

Prove: There exist disjoint open sets U, V including A and B

- $\langle 1 \rangle 4$ . Case: The least element of X is not in A or B
  - $\langle 2 \rangle 1$ . Let:  $U = \bigcup \{ (x, a] : a \in A, x < a, (x, a] \cap B = \emptyset \}$
  - $\langle 2 \rangle 2$ . Let:  $V = \bigcup \{ (y, b] : b \in B, y < b, (y, b] \cap A = \emptyset \}$
  - $\langle 2 \rangle 3$ . U is open

PROOF: From  $\langle 1 \rangle 2$ .

 $\langle 2 \rangle 4$ . V is open

PROOF: From  $\langle 1 \rangle 2$ .

- $\langle 2 \rangle 5$ .  $A \subseteq U$ 
  - $\langle 3 \rangle 1$ . Let:  $a \in A$
  - $\langle 3 \rangle$ 2. PICK W a neighbourhood of a such that  $W \cap B = \emptyset$ PROOF: By  $\langle 1 \rangle$ 3.
  - $\langle 3 \rangle 3$ . Pick x < a such that  $(x, a] \subseteq W$

```
PROOF: By Lemma 4.1.2
      \langle 3 \rangle 4. \ a \in (x, a] \subseteq U
   \langle 2 \rangle 6. \ B \subseteq V
      PROOF: Similar.
   \langle 2 \rangle 7. U \cap V = \emptyset
\langle 1 \rangle 5. Case: \bot \in A
   \langle 2 \rangle 1. PICK disjoint open sets U and V that include A \setminus \{\bot\} and B
      PROOF: From \langle 1 \rangle 4.
   \langle 2 \rangle 2. U \cup \{\bot\} and V are disjoint open sets that include A and B
      PROOF: \{\bot\} is open because it is (-\infty, a) where a is the successor of \bot.
\langle 1 \rangle 6. Q.E.D.
   Proof: By Proposition 6.6.3.
Proposition 6.6.5. The product of two completely normal spaces is not neces-
sarily completely normal.
PROOF:
\langle 1 \rangle 1. S_{\Omega} is completely normal.
   Proof: By Proposition 6.6.4
\langle 1 \rangle 2. \overline{S_{\Omega}} is completely normal.
   Proof: By Proposition 6.6.4
\langle 1 \rangle 3. S_{\Omega} \times \overline{S_{\Omega}} is not completely normal.
   Proof: By Proposition 6.5.3.
Proposition 6.6.6. A compact Hausdorff space is not necessarily completely
normal.
PROOF:
\langle 1 \rangle 1. PICK an uncountable set J
\langle 1 \rangle 2. [0,1]^J is compact Hausdorff
   PROOF: By Tychonoff's Theorem and Theorem 6.2.5.
\langle 1 \rangle 3. (0,1)^J is not normal.
   PROOF: By Proposition 6.5.6, since (0,1) \cong \mathbb{R}.
Proposition 6.6.7. The space \mathbb{R}_l is completely normal.
PROOF:
\langle 1 \rangle 1. Let: X \subseteq \mathbb{R}_l
\langle 1 \rangle 2. Let: A and B be disjoint closed sets in X.
\langle 1 \rangle 3. PICK closed sets C and D such that A = C \cap X and B = D \cap X
\langle 1 \rangle 4. For a \in A, Pick x_a > a such that [a, x_a) \cap D = \emptyset
\langle 1 \rangle 5. For b \in B, PICK x_b > b such that [b, x_b) \cap C = \emptyset
\langle 1 \rangle 6. \bigcup_{a \in A} [a, x_a) \cap X and \bigcup_{b \in B} [b, x_b) \cap X are disjoint open sets in X that
        include A and B
```

# 6.7 Perfectly Normal Spaces

**Definition 6.7.1** (Perfectly Normal). A space is *perfectly normal* iff it is normal and every closed set is  $G_{\delta}$ .

**Proposition 6.7.2.** Every perfectly normal space is completely normal.

```
PROOF: \langle 1 \rangle 1. LET: X be perfectly normal. \langle 1 \rangle 2. LET: A and B be separated sets in X \langle 1 \rangle 3. PICK continuous functions f,g:X \to [0,1] that vanish precisely on \overline{A} and \overline{B}, respectively.

PROOF: By Theorem 6.5.9. \langle 1 \rangle 4. LET: h = f - g \langle 1 \rangle 5. B \subseteq h^{-1}((0, +\infty)) and A \subseteq h^{-1}((-\infty, 0)) \langle 1 \rangle 6. Q.E.D.

PROOF: By Proposition 6.6.3.
```

**Proposition 6.7.3.** The space  $\overline{S_{\Omega}}$  is not perfectly normal.

PROOF: The set  $\{\Omega\}$  is not  $G_{\delta}$ .  $\square$ 

# Chapter 7

# Countability Axioms

# 7.1 The First Countability Axiom

**Definition 7.1.1** (First Countability Axiom). A topological space X satisfies the *first countability axiom*, or is *first countable*, iff every point has a countable local basis.

**Proposition 7.1.2.**  $S_{\Omega}$  is first countable.

PROOF: For every countable ordinal  $\alpha > 0$ , the set  $\{(\beta, \alpha + 1) : \beta < \alpha\}$  is a local basis at  $\alpha$ . The set  $\{\{0\}\}$  is a local basis at 0.  $\square$ 

**Theorem 7.1.3** (The Sequence Lemma (CC)). Let X be a first countable space and  $A \subseteq X$ . If  $x \in \overline{A}$ , then there exists a sequence of points of A that converges to x.

#### Proof:

- $\langle 1 \rangle 1$ . Let:  $x \in \overline{A}$
- $\langle 1 \rangle 2$ . PICK a countable basis  $\{B_n\}_{n \in \mathbb{Z}^+}$  at x.
- (1)3. For  $n \ge 1$ , PICK a point  $a_n \in B_1 \cap \cdots \cap B_n \cap A$ PROVE:  $a_n \to x$  as  $n \to \infty$

PROOF: Using Countable Choice. Such an  $a_n$  exists because  $B_1 \cap \cdots \cap B_n$  is a neighbourhood of x. Apply Theorem 3.8.3.

- $\langle 1 \rangle 4$ . Let: U be a neighbourhood of x
- $\langle 1 \rangle$ 5. PICK N such that  $B_N \subseteq U$

PROOF: From  $\langle 1 \rangle 2$ .

 $\langle 1 \rangle 6$ . For  $n \geq N$ , we have  $a_n \in U$ 

Proof:

$$a_n \in B_1 \cap \dots \cap B_n$$
  $(\langle 1 \rangle 3)$   
 $\subseteq B_N$   $(n \ge N)$   
 $\subseteq U$   $(\langle 1 \rangle 5)$ 

**Theorem 7.1.4** (CC). Let X and Y be topological spaces where X is first countable. Let  $x \in X$ . Suppose that, for every sequence  $\{x_n\}_{n\geq 1}$  such that  $x_n \to x$  as  $n \to \infty$ , we have  $f(x_n) \to f(x)$  as  $n \to \infty$ . Then f is continuous at x.

#### Proof:

- $\langle 1 \rangle 1$ . Let: V be a neighbourhood of f(x)
- $\langle 1 \rangle 2.$  Assume: for a contradiction that, for every neighbourhood U of  $x,\,f(U) \not\subseteq V$
- $\langle 1 \rangle 3$ . PICK a countable local basis  $\{B_n\}_{n\geq 1}$
- $\langle 1 \rangle 4$ . For  $n \geq 1$ , PICK  $a_n \in B_1 \cap \cdots \cap B_n$  such that  $f(a_n) \notin V$
- $\langle 1 \rangle 5$ .  $a_n \to x$  as  $n \to \infty$

#### Proof:

- $\langle 2 \rangle 1$ . Let: U be a neighbourhood of x
- $\langle 2 \rangle 2$ . PICK N such that  $B_N \subseteq U$
- $\langle 2 \rangle 3$ . For all  $n \geq N$ ,  $a_n \in U$

PROOF:

$$a_n \in B_1 \cap \dots \cap B_n$$
  $(\langle 1 \rangle 4)$   
 $\subseteq B_N$   $(n \ge N)$   
 $\subseteq U$   $(\langle 2 \rangle 2)$ 

- $\langle 1 \rangle 6. \ f(a_n) \to f(x) \text{ as } n \to \infty$
- $\langle 1 \rangle 7$ . There exists N such that, for all  $n \geq N$ , we have  $f(a_n) \in V$
- $\langle 1 \rangle 8$ . Q.E.D.

**Lemma 7.1.5** (CC).  $\mathbb{R}^{\omega}$  under the box topology is not first countable.

#### Proof:

- $\langle 1 \rangle 1$ . Let:  $\{B_n\}_{n\geq 1}$  be any countable set of neighbourhoods of  $\vec{0}$
- $\langle 1 \rangle 2$ . For  $n \geq 1$ , Pick  $U_{nm}$  for  $m \geq 1$  such that  $\vec{0} \in \prod_{m=1}^{\infty} U_{nm} \subseteq B_n$
- $\langle 1 \rangle 3$ . For  $n \geq 1$ , PICK  $a_n$ ,  $b_n$  such that  $0 \in (a_n, b_n) \subseteq U_{nn}$
- (1)4. Let:  $U = \prod_{n=1}^{\infty} (a_n/2, b_n/2)$
- $\langle 1 \rangle 5. \ \vec{0} \in U$
- $\langle 1 \rangle 6$ . For all  $n, B_n \nsubseteq U$

**Lemma 7.1.6** (CC). If J is uncountable then  $\mathbb{R}^J$  is not first countable.

#### Proof

- $\langle 1 \rangle 1$ . Let:  $\{B_n\}_{n \geq 1}$  be a countable family of neighbourhoods of 0
- $\langle 1 \rangle 2$ . For  $n \geq 1$ , PICK  $U_{n\alpha}$  such that  $\vec{0} \in \prod_{\alpha \in J} U_{n\alpha} \subseteq B_n$  where  $U_{n\alpha}$  is open in  $\mathbb{R}$  and  $U_{n\alpha} = \mathbb{R}$  except for  $\alpha = \alpha_{n1}, \ldots, \alpha_{nr_n}$
- $\langle 1 \rangle 3$ . PICK  $\beta$  such that  $\beta$  is different from  $\alpha_{ni}$  for all n, i
- $\langle 1 \rangle 4$ . Let:  $V = \pi_{\beta}^{-1}((-1,1))$
- $\langle 1 \rangle 5. \ \vec{0} \in V$
- $\langle 1 \rangle 6$ .  $V \not\subseteq B_n$  for all n

#### **Lemma 7.1.7.** $\mathbb{R}_l$ is first countable.

PROOF: For all  $x \in \mathbb{R}$ ,  $\{[x,q) : q \in \mathbb{Q}, q > x\}$  is a basis at x.  $\sqcup$ 

**Lemma 7.1.8.** The ordered square is first countable.

#### Proof:

 $\langle 1 \rangle 1$ . Let:  $(x,y) \in I_o^2$ 

PROVE: There exists a countable local basis  $\mathcal{B}$  at (x,y)

 $\langle 1 \rangle 2$ . Case: (x, y) = (0, 0)

PROOF: Take  $\mathcal{B} = \{[(0,0),(0,q)) : q \in \mathbb{Q}, 0 < q < 1\}.$ 

 $\langle 1 \rangle 3$ . Case: 0 < y < 1

PROOF: Take  $\mathcal{B} = \{((x, q), (x, q')) : q, q' \in \mathbb{Q}, q < y < q'\}.$ 

 $\langle 1 \rangle 4$ . Case: x < 1, y = 1

Proof: Take  $\mathcal{B} = \{((x,q),(q',0)): q,q' \in \mathbb{Q}, 0 < q < 1, x < q' < 1\}.$ 

 $\langle 1 \rangle 5$ . Case: x > 0, y = 0

PROOF: Take  $\mathcal{B} = \{((q, 1), (x, q')) : q, q' \in \mathbb{Q}, 0 < q < x, 0 < q' < 1\}$ 

 $\langle 1 \rangle 6$ . Case: (x,y) = (1,1)

PROOF: Take  $\mathcal{B} = \{((1, q), (1, 1)] : q \in \mathbb{Q}, 0 < q < 1\}.$ 

**Proposition 7.1.9.** A subspace of a first countable space is first countable.

#### PROOF:

 $\langle 1 \rangle 1$ . Let: X be a first countable space and  $A \subseteq X$ 

 $\langle 1 \rangle 2$ . Let:  $a \in A$ 

 $\langle 1 \rangle 3$ . PICK a countable basis  $\mathcal{B}$  at a in X

 $\langle 1 \rangle 4$ .  $\{B \cap A : B \in \mathcal{B} \text{ is a countable basis at } a \text{ in } A$ .

**Proposition 7.1.10** (CC). A countable product of first countable spaces is first countable.

#### Proof:

 $\langle 1 \rangle 1$ . Let:  $\{X_n\}_{n \in \mathbb{Z}^+}$  be a countable family of first countable spaces.  $\langle 1 \rangle 2$ . Let:  $\vec{x} \in \prod_{n=1}^{\infty} X_n$ 

 $\langle 1 \rangle$ 3. PICK a countable basis  $\mathcal{B}_n$  at  $x_n$  in  $X_n$  for all n  $\langle 1 \rangle$ 4. Let:  $\mathcal{B}$  be the set of all sets  $\prod_{i=1}^n U_n$  where  $U_n \in \mathcal{B}_n$  for finitely many nand  $U_n = X_n$  for all other n.

 $\langle 1 \rangle 5$ .  $\mathcal{B}$  is a countable basis at  $\vec{x}$  in  $\prod_{n=1}^{\infty} X_n$ 

Corollary 7.1.10.1. The space  $\mathbb{R}^{\omega}$  is first countable.

**Proposition 7.1.11.** The space  $S_{\Omega}$  is first countable.

#### PROOF:

 $\langle 1 \rangle 1$ . Let:  $\alpha \in S_{\Omega}$ 

Prove:  $\alpha$  has a countable local basis.

 $\langle 1 \rangle 2$ . Case:  $\alpha$  is zero or a successor ordinal.

PROOF: In this case,  $\{\{\alpha\}\}\$  is a local basis.

- $\langle 1 \rangle 3$ . Case:  $\alpha$  is a limit ordinal.
  - $\langle 2 \rangle 1$ . PICK a countable sequence  $(\beta_n)$  with supremum  $\alpha$
- $\langle 2 \rangle 2$ .  $\{(\beta_n, \alpha + 1) : n \in \mathbb{Z}^+\}$  is a local basis.

**Proposition 7.1.12.** The space  $\overline{S_{\Omega}}$  is not first countable.

#### PROOF

- $\langle 1 \rangle 1$ . Assume: for a contradiction  $\mathcal{B}$  is a countable local basis at  $\Omega$
- $\langle 1 \rangle 2$ . Let:  $\alpha = \sup\{\inf B : B \in \mathcal{B}\}$
- $\langle 1 \rangle 3. \ \alpha < \Omega$
- $\langle 1 \rangle$ 4. There is no  $B \in \mathcal{B}$  such that  $B \subseteq (\alpha, +\infty)$

**Proposition 7.1.13.** The continuous image of a first countable space is first countable.

#### Proof:

- $\langle 1 \rangle 1$ . Let: X be a first countable space, Y a space and  $f: X \to Y$  continuous.
- $\langle 1 \rangle 2$ . Let:  $y \in f(X)$
- $\langle 1 \rangle 3$ . PICK  $x \in X$  such that y = f(x)
- $\langle 1 \rangle 4$ . PICK a countable local basis  $\mathcal{B}$  at x
- $\langle 1 \rangle$ 5.  $\{f(B) : B \in \mathcal{B}\}$  is a countable local basis at y.

**Proposition 7.1.14.**  $S_{\Omega} \times \overline{S_{\Omega}}$  is not first countable.

PROOF:  $(0,\Omega)$  has no countable basis.  $\sqcup$ 

**Proposition 7.1.15.** The Sorgenfrey plane is first countable.

PROOF: For any point (a,b), the set  $\{[a,a+q)\times[b,b+r):q,r\in\mathbb{Q}\}$  is a countable local basis at (a,b).  $\square$ 

# 7.2 Separable Spaces

**Definition 7.2.1** (Separable Space). A topological space X is separable iff it has a countable dense subset.

**Proposition 7.2.2.** The space  $S_{\Omega}$  is not separable.

#### Proof:

- $\langle 1 \rangle 1$ . Let:  $D \subseteq S_{\Omega}$  be countable.
- $\langle 1 \rangle 2$ . Let:  $\alpha = \sup D$
- $\langle 1 \rangle 3. \ \overline{D} \subseteq (-\infty, \alpha]$

**Proposition 7.2.3.** The space  $\overline{S_{\Omega}}$  is not separable.

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Proof:
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- $\langle 1 \rangle 1$ . Let:  $D \subseteq S_{\Omega}$  be countable.
- $\langle 1 \rangle 2$ . Let:  $\alpha = \sup \{ \beta \in D : \beta < \Omega \}$
- $\langle 1 \rangle 3. \ \alpha < \Omega$

PROOF:  $\alpha$  is the supremum of countably many countable ordinals.

$$\langle 1 \rangle 4. \ \overline{D} \subseteq (-\infty, \alpha] \cup \{\Omega\}$$

Corollary 7.2.3.1. Not every compact Hausdorff space is separable.

**Proposition 7.2.4.** Every open subspace of a separable space is separable.

#### Proof:

- $\langle 1 \rangle 1$ . Let: X be a separable space with countable dense subset D.
- $\langle 1 \rangle 2$ . Let: U be an open subspace of X PROVE:  $D \cap U$  is a countable dense subset of U.
- $\langle 1 \rangle 3$ .  $D \cap U$  is countable.
- $\langle 1 \rangle 4$ . Let: V be an open set in U.
- $\langle 1 \rangle 5$ . V is open in X

Proof: Lemma 4.3.3

- $\langle 1 \rangle 6$ . V intersects D
- $\langle 1 \rangle 7$ . V intensects  $D \cap U$

**Proposition 7.2.5** (CC). The product of a countable family of separable spaces is separable.

#### Proof:

- $\langle 1 \rangle 1$ . Let:  $(X_n)$  be a countable family of separable spaces.
- $\langle 1 \rangle 2$ . For  $n \geq 1$ , PICK a dense set  $D_n$  in  $X_n$   $\langle 1 \rangle 3$ .  $\prod_{n=1}^{\infty} D_n$  is dense in  $\prod_{n=1}^{\infty} X_n$ .

**Proposition 7.2.6.** The continuous image of a separable space is separable.

- $\langle 1 \rangle 1$ . Let: X be a separable space, Y a space and  $f: X \to Y$  be continuous.
- $\langle 1 \rangle 2$ . Pick a countable dense set D in X
- $\langle 1 \rangle 3$ . f(D) is dense in f(X).

Corollary 7.2.6.1. Let  $\{X_{\alpha}\}_{{\alpha}\in J}$  be a family of nonempty topological spaces. If  $\prod_{\alpha \in I} X_{\alpha}$  is separable then each  $X_{\alpha}$  is separable.

Corollary 7.2.6.2.  $S_{\Omega} \times \overline{S_{\Omega}}$  is not separable.

Proposition 7.2.7. The ordered square is not separable.

PROOF:  $\{x\} \times (0,1) : x \in [0,1]\}$  is an uncountable set of disjoint open sets.  $\square$ 

Proposition 7.2.8. $\mathbb{R}_l$ is separable.
Proof: $\mathbb{Q}$ is dense. $\square$
Proposition 7.2.9. The Sorgenfrey plane is separable.
PROOF: $\mathbb{Q}^2$ is dense. $\square$
<b>Proposition 7.2.10.</b> Not every closed subspace of a separable space is separable.
PROOF: $\mathbb{R}^2_l$ is separable but the subspace $\{(x,-x):x\in\mathbb{R}\}$ is not. $\square$
7.3 The Second Countability Axiom
<b>Definition 7.3.1</b> (Second Countability Axiom). A topological space satisfies
the second countability axiom, or is second countable, iff it has a countable basis
the second countability axiom, or is second countable, iff it has a countable basis <b>Proposition 7.3.2.</b> $S_{\Omega}$ is not second countable.
Proposition 7.3.2. $S_{\Omega}$ is not second countable. PROOF: $\{\{\alpha\}: \alpha \text{ is a countable successor ordinal}\}$ is an uncountable set of dis

**Proposition 7.3.4** (CC). The product of countably many second countable spaces is second countable.

#### Proof:

- $\langle 1 \rangle 1$ . Let:  $\{X_n\}_{n \in \mathbb{Z}^+}$  be a countable family of second countable spaces.
- $\langle 1 \rangle 2$ . For  $n \in \mathbb{Z}^+$ , PICK a countable basis  $\mathcal{B}_n$  for  $X_n$ .  $\langle 1 \rangle 3$ . Let:  $\mathcal{B}$  be the set of all sets of the form  $\prod_{n=1}^{\infty} U_n$ , where  $U_n \in \mathcal{B}_n$  for finitely many n, and  $U_n = X_n$  for all other n.  $\langle 1 \rangle 4$ .  $\mathcal{B}$  is a countable basis for  $\prod_{n=1}^{\infty} X_n$

 $\langle 1 \rangle 2$ . PICK a countable basis  $\mathcal{B}$  for X

 $\langle 1 \rangle 3$ .  $\{B \cap A : B \in \mathcal{B}\}$  is a countable basis for A

**Theorem 7.3.5** (CC). Every second countable space is separable.

#### PROOF:

- $\langle 1 \rangle 1$ . Let: X be a second countable space.
- $\langle 1 \rangle 2$ . Pick a countable basis  $\mathcal{B}$  for X
- $\langle 1 \rangle 3$ . For  $B \in \mathcal{B}$  nonempty, PICK a point  $x_B \in B$
- $\langle 1 \rangle 4$ .  $D = \{x_B : B \in \mathcal{B} \setminus \{\emptyset\}\}$  is dense.  $\langle 2 \rangle 1$ . Let:  $l \in X$

Prove:  $l \in \overline{D}$ 

- $\langle 2 \rangle 2$ . Let:  $B \in \mathcal{B}$  such that  $l \in B$
- $\langle 2 \rangle 3. \ x_B \in B \cap D$
- $\langle 2 \rangle 4$ . Q.E.D.

PROOF:By Theorem 3.7.8

Corollary 7.3.5.1.  $S_{\Omega} \times \overline{S_{\Omega}}$  is not second countable.

Corollary 7.3.5.2. The space  $\mathbb{R}^{\omega}$  is separable.

Corollary 7.3.5.3. If J is uncountable then  $\mathbb{R}^J$  is not second countable.

**Proposition 7.3.6.** The ordered square is not second countable.

#### PROOF:

- $\langle 1 \rangle 1$ . Let:  $\mathcal{B}$  be any basis
- $\langle 1 \rangle 2$ . For  $x \in [0,1]$ , PICK  $B_x$  such that  $x \in B_x \subseteq ((x,0),(x,1))$
- $\langle 1 \rangle 3$ . The function  $B_{(-)}$  is an injective function  $[0,1] \to \mathcal{B}$
- $\langle 1 \rangle 4$ .  $\mathcal{B}$  is uncountable.

**Proposition 7.3.7.** The space  $\overline{S_{\Omega}}$  is not second countable.

PROOF: It is not first countable (Proposition 7.1.12).  $\square$ 

**Proposition 7.3.8.** The continuous image of a second countable space is second countable.

#### Proof:

- $\langle 1 \rangle 1$ . Let: X be a second countable space, Y a space and  $f: X \to Y$  be continuous.
- $\langle 1 \rangle 2$ . PICK a countable basis  $\mathcal{B}$  for X.
- $\langle 1 \rangle 3. \{ f(B) : B \in \mathcal{B} \text{ is a countable basis for } f(X) \}$

**Theorem 7.3.9.** Every regular Lindelöf space is normal.

- $\langle 1 \rangle 1$ . Let: X be a regular Lindelöf space.
- $\langle 1 \rangle 2$ . Let: A and B be disjoint closed sets in X.
- $\langle 1 \rangle 3$ .  $\{ U \text{ open in } X : \overline{U} \cap B = \emptyset \} \text{ covers } A$ Proof: Proposition 6.3.2.
- $\langle 1 \rangle 4$ . Pick a countable open covering  $\{U_n : n \in \mathbb{Z}^+\}$  of A such that  $\overline{U_n} \cap B = \emptyset$
- (1)5. PICK a countable open covering  $\{V_n : n \in \mathbb{Z}^+\}$  of B such that  $\overline{V_n} \cap A = \emptyset$ for all n

PROOF: Similar.

 $\langle 1 \rangle 6$ . For  $n \in \mathbb{Z}^+$ ,

Let: 
$$U'_n = U_n \setminus \bigcup_{i=1}^n \overline{V_i}$$
 and  $V'_n = V_n \setminus \bigcup_{i=1}^n \overline{U_i}$   $\langle 1 \rangle 7$ . Let:  $U' = \bigcup_{n=1}^{\infty} U'_n$  and  $V = \bigcup_{n=1}^{\infty} V'_n$ 

$$\langle 1 \rangle 8. \ A \subseteq U' \text{ and } B \subseteq V'$$
  
 $\langle 1 \rangle 9. \ U' \cap V' = \emptyset$ 

Corollary 7.3.9.1. If J is uncountable then  $\mathbb{R}^J$  is not Lindelöf.

**Proposition 7.3.10.** Every second countable regular space is completely normal.

#### PROOF:

- $\langle 1 \rangle 1$ . Let: X be second countable and regular and  $Y \subseteq X$
- $\langle 1 \rangle 2$ . Y is second countable

Proof: Proposition 7.3.3.

 $\langle 1 \rangle 3$ . Y is regular

Proof: Proposition 6.3.4

 $\langle 1 \rangle 4$ . Y is normal

Proof: Theorem 7.3.9

**Proposition 7.3.11.** The space  $\mathbb{R}^{\omega}$  is second countable.

PROOF: The sets  $\prod_{n=0}^{\infty} U_n$  form a basis, where  $U_n$  is an interval of the form (q,r) for  $q,r \in \mathbb{Q}$  for finitely many n, and  $U_n = \mathbb{R}$  for all other n.  $\square$ 

**Proposition 7.3.12** (CC). In a second countable space, every discrete subspace is countable.

#### Proof:

- $\langle 1 \rangle 1$ . Let: X be a second countable space
- $\langle 1 \rangle 2$ . PICK a countable basis  $\mathcal{B}$
- $\langle 1 \rangle 3$ . Let:  $D \subseteq X$  be discrete
- $\langle 1 \rangle 4$ . For  $a \in D$ , PICK  $B_a \in \mathcal{B}$  such that  $B_a \cap D = \{a\}$
- $\langle 1 \rangle 5. \ a \mapsto B_a \text{ is injective}$

**Proposition 7.3.13.** The space  $\mathbb{R}_K$  is second countable.

PROOF:  $\{(a,b): a,b \in \mathbb{R}\} \cup \{(a,b)-K: a,b \in \mathbb{Q}\}$  is a basis.  $\square$ 

Corollary 7.3.13.1. The space  $\mathbb{R}_K$  is first countable.

Corollary 7.3.13.2. The space  $\mathbb{R}_K$  is separable.

**Proposition 7.3.14.** Let J be a set with  $|J| > |\mathbb{R}|$ . Then  $\mathbb{R}^J$  is not separable.

#### Proof:

- $\langle 1 \rangle 1$ . Assume: D is countable and dense in  $\mathbb{R}^J$  Prove:  $|J| \leq |\mathbb{R}|$
- $\langle 1 \rangle 2$ . Define  $f: J \to \mathcal{P}D$  by  $f(\alpha) = D \cap \pi_{\alpha}^{-1}((0,1))$
- $\langle 1 \rangle 3$ . f is injective

```
\begin{array}{l} \langle 2 \rangle 1. \ \text{Let:} \ \alpha,\beta \in J \ \text{with} \ \alpha \neq \beta \\ \langle 2 \rangle 2. \ \text{Pick} \ x \in D \cap \pi_{\alpha}^{-1}((0,1)) \cap \pi_{\beta}^{-1}((2,3)) \\ \langle 2 \rangle 3. \ x \in f(\alpha) \ \text{but} \ x \notin f(\beta) \end{array}
```

Corollary 7.3.14.1. The product of a family of separable spaces is not necessarily separable.

# Chapter 8

# Connectedness

## 8.1 Connected Spaces

**Definition 8.1.1** (Separation). Let X be a topological space. A *separation* of X is a pair of disjoint nonempty subsets whose union in X.

**Definition 8.1.2** (Connected). A topological space is *connected* iff it has no separation.

**Proposition 8.1.3.**  $S_{\Omega}$  is not connected.

PROOF:  $\{0\}$  and  $S_{\Omega} \setminus \{0\}$  form a separation.  $\square$ 

**Proposition 8.1.4.** A space X is connected if and only if the only sets that are both closed and open are  $\emptyset$  and X.

Proof: Immediate from definitions.

**Proposition 8.1.5.** Let Y be a subspace of X. Then a separation of Y is a pair of disjoint nonempty sets A, B such that  $A \cup B = Y$  and neither of A, B contains a limit point of the other.

#### Proof:

- $\langle 1 \rangle 1$ . If A and B form a separation of Y then A and B are disjoint and nonempty,  $A \cup B = Y$ , and neither of A, B contains a limit point of the other.
  - $\langle 2 \rangle 1$ . Let: A and B be a separation of Y
  - $\langle 2 \rangle 2$ . A and B are disjoint and nonempty and  $A \cup B = Y$  PROOF: Immediate from the definition of separation.
  - $\langle 2 \rangle$ 3. A does not contain a limit point of B PROOF: B is closed in Y, hence contains all its limit points (Corollary 3.10.3.1), and so the result follows because A and B are disjoint.
  - $\langle 2 \rangle$ 4. B does not contain a limit point of A PROOF: Similar.
- $\langle 1 \rangle 2$ . If A and B are disjoint and nonempty,  $A \cup B = Y$ , and neither of A, B contains a limit point of the other, then A and B are a separation of Y.

- $\langle 2 \rangle 1$ . Assume: A and B are disjoint and nonempty,  $A \cup B = Y$ , and neither of A, B contains a limit point of the other
- $\langle 2 \rangle 2$ . A is closed in Y

PROOF: Every limit point of A is not in B, so is in A. Apply Corollary 3.10.3.1.

 $\langle 2 \rangle 3$ . B is open in Y

Proof: $B = Y \setminus A$ 

 $\langle 2 \rangle 4$ . A is open in Y

PROOF: Similar.

**Proposition 8.1.6.** If the sets C and D form a separation of X, and Y is a connected subspace of X, then  $Y \subseteq C$  or  $Y \subseteq D$ .

PROOF: Otherwise,  $Y \cap C$  and  $Y \cap D$  would be a separation of Y.  $\square$ 

**Proposition 8.1.7.** The union of a set of connected subspaces of X that have a point in common is connected.

#### PROOF:

- $\langle 1 \rangle 1$ . Let: S be a set of connected subspaces that have the point a in common.
- $\langle 1 \rangle 2$ . Assume: for a contradiction U and V form a separation of  $\bigcup S$
- $\langle 1 \rangle 3$ . Assume: w.l.o.g.  $a \in U$
- $\langle 1 \rangle 4$ . For all  $Y \in \mathcal{S}$  we have  $Y \subseteq U$

Proof: By Proposition 8.1.6.

- $\langle 1 \rangle 5. V = \emptyset$
- $\langle 1 \rangle 6$ . Q.E.D.

PROOF: This contradicts  $\langle 1 \rangle 2$ .

**Theorem 8.1.8.** Let A be a connected subspace of X. If  $A \subseteq B \subseteq \overline{A}$  then B is connected.

#### Proof:

- $\langle 1 \rangle 1$ . Assume: for a contradiction U and V are a separation of B
- $\langle 1 \rangle 2$ .  $A \subseteq U$  or  $A \subseteq V$

Proof: By Proposition 8.1.6.

- $\langle 1 \rangle 3$ . Assume: w.l.o.g.  $A \subseteq U$
- $\langle 1 \rangle 4. \ \overline{A} \subseteq \overline{U}$

Proof: By Proposition 3.7.5.

 $\langle 1 \rangle 5. \ B \subseteq \overline{U}$ 

PROOF: Since  $B \subseteq \overline{A}$ .

 $\langle 1 \rangle 6$ . The closure of *U* in *B* is *B* 

PROOF: By Theorem 4.3.4.

 $\langle 1 \rangle 7. \ U = B$ 

PROOF: Since U is closed in B.

 $\langle 1 \rangle 8$ . Q.E.D.

PROOF: This contradicts  $\langle 1 \rangle 1$ .

**Theorem 8.1.9.** The image of a connected space under a continuous map is connected.

PROOF: Let X be a connected space, Y a topological space, and  $f: X \to Y$  be surjective. If U and V form a separation of Y, then  $f^{-1}(U)$  and  $f^{-1}(V)$  form a separation of X.  $\square$ 

**Corollary 8.1.9.1.** Let  $\mathcal{T}$  and  $\mathcal{T}'$  be topologies on the same set X. If  $\mathcal{T} \subseteq \mathcal{T}'$  and X is connected under  $\mathcal{T}'$  then X is connected under  $\mathcal{T}$ .

**Corollary 8.1.9.2.** Let  $\{X_{\alpha}\}_{{\alpha}\in J}$  be a family of topological spaces. If  $\prod_{{\alpha}\in J} X_{\alpha}$  is connected then each  $X_{\alpha}$  is connected.

Corollary 8.1.9.3. The Sorgenfrey plane is disconnected.

**Proposition 8.1.10.** The product of a family of connected spaces is connected.

#### PROOF:

 $\langle 1 \rangle 1$ . The product of two connected spaces is connected.

Proof:

- $\langle 2 \rangle 1$ . Let: X and Y be connected spaces.
- $\langle 2 \rangle 2$ . Assume: w.l.o.g. X and Y are nonempty. Proof: If either is empty then  $X \times Y = \emptyset$  is connected.

 $\langle 2 \rangle 3$ . Assume: for a contradiction U and V are a separation of  $X \times Y$ .

 $\langle 2 \rangle 4$ . Pick  $b \in Y$ 

PROOF: By  $\langle 2 \rangle 2$ .

 $\langle 2 \rangle 5$ . For all  $x \in X$ ,

Let: 
$$T_x = (X \times \{b\}) \cup (\{x\} \times Y)$$

- $\langle 2 \rangle 6$ . For all  $x \in X$ ,  $T_x$  is connected
  - $\langle 3 \rangle 1. \ X \times \{b\}$  is connected

PROOF: It is homeomorphic to X.

 $\langle 3 \rangle 2$ .  $\{x\} \times Y$  is connected

PROOF: It is homeomorphic to Y.

 $\langle 3 \rangle 3$ . Q.E.D.

Proof: By Proposition 8.1.7.

- $\langle 2 \rangle 7. \ X \times Y = \bigcup_{x \in X} T_x$
- $\langle 2 \rangle 8$ . Q.E.D.
  - $\langle 3 \rangle 1$ . Pick  $a \in X$

Proof: By  $\langle 2 \rangle 2$ .

- $\langle 3 \rangle 2$ .  $(a,b) \in T_x$  for all  $x \in X$
- $\langle 3 \rangle 3$ . Q.E.D.

Proof: By Proposition 8.1.7.

- $\langle 1 \rangle 2$ . Let:  $\{X_{\alpha}\}_{{\alpha} \in J}$  be a family of connected spaces.
- $\langle 1 \rangle 3$ . Assume: w.l.o.g.  $\prod_{\alpha \in J} X_{\alpha}$  is nonempty
- $\langle 1 \rangle 4$ . Pick  $\vec{a} \in \prod_{\alpha \in J} X_{\alpha}$
- $\langle 1 \rangle 5$ . For K a finite subset of J,

Let:  $X_K = \{\vec{x} \in \prod_{\alpha \in J} X_\alpha : x_\alpha = a_\alpha \text{ for all } \alpha \in J \setminus K\}$ 

 $\langle 1 \rangle 6$ . For all  $K, X_K$  is connected.

PROOF: It is homeomorphic to  $\prod_{\alpha \in K} X_{\alpha}$ , so it is connected by  $\langle 1 \rangle 1$ .

 $\langle 1 \rangle 7$ .  $\bigcup_{K \subset \text{fin } I} X_K$  is connected.

Proof: By Proposition 8.1.7 since  $\vec{a} \in X_K$  for all K.

- $\begin{array}{l} \langle 1 \rangle 8. \ \prod_{\alpha \in J} X_{\alpha} = \overline{\bigcup_{K \subseteq ^{\text{fin}} J} X_K} \\ \langle 2 \rangle 1. \ \text{Let:} \ \vec{x} \in \prod_{\alpha \in J} X_{\alpha} \end{array}$ 

  - $\langle 2 \rangle 2$ . Let: U be an open neighbourhood of  $\vec{x}$
  - $\langle 2 \rangle 3$ . PICK a basic open set  $\prod_{\alpha \in J} V_{\alpha}$  such that  $\vec{x} \in \prod_{\alpha \in J} V_{\alpha} \subseteq U$ , where each  $V_{\alpha}$  is open in  $X_{\alpha}$ , and  $V_{\alpha} = X_{\alpha}$  except for  $\alpha \in K$  for some finite  $K \subseteq J$

Prove: U intersects  $X_K$ 

- $\langle 2 \rangle 4$ . Let:  $\vec{y} \in \prod_{\alpha \in I} X_{\alpha}$  with  $y_{\alpha} = x_{\alpha}$  for  $\alpha \in K$ ,  $y_{\alpha} = a_{\alpha}$  for  $\alpha \notin K$
- $\langle 2 \rangle 5. \ \vec{y} \in U \cap X_K$
- $\langle 1 \rangle 9$ . Q.E.D.

**Corollary 8.1.10.1.** For any set I, the space  $\mathbb{R}^I$  under the product topology is connected.

**Proposition 8.1.11.**  $\mathbb{R}^{\omega}$  under the box topology is disconnected.

PROOF: The set of all bounded sequences and the set of all unbounded sequences form a separation.  $\square$ 

**Definition 8.1.12** (Totally Disconnected). A space is totally disconnected iff the only connected subspaces are the singletons.

**Theorem 8.1.13.** Let L be a linearly ordered set under the order topology. Then L is connected if and only if L is a linear continuum.

#### Proof:

- $\langle 1 \rangle 1$ . If L is a linear continuum then L is connected.
  - $\langle 2 \rangle 1$ . Let: L be a linear continuum.
  - $\langle 2 \rangle 2$ . Assume: for a contradiction U and V are a separation of L.
  - $\langle 2 \rangle 3$ . Pick  $a \in U$  and  $b \in V$
  - $\langle 2 \rangle 4$ . Assume: w.l.o.g. a < b
  - $\langle 2 \rangle 5$ . Let:  $l = \sup\{x \in A : x < b\}$
  - $\langle 2 \rangle 6$ . Case:  $l \in A$ 
    - $\langle 3 \rangle 1$ . Pick a' > l such that  $[l, a') \subseteq A$

PROOF: By Lemma 4.1.2. We know l is not greatest in X because l < b.

 $\langle 3 \rangle 2$ . Pick  $a^*$  such that  $l < a^* < a'$ 

Proof: L is dense.

 $\langle 3 \rangle 3. \ l < a^*, a^* \in A, a^* < b$ 

PROOF: If  $b < a^*$  then  $b \in A$  by  $\langle 3 \rangle 1$ .

 $\langle 3 \rangle 4$ . Q.E.D.

PROOF: This contradicts  $\langle 2 \rangle 5$ .

- $\langle 2 \rangle 7$ . Case:  $l \in B$ 
  - $\langle 3 \rangle 1$ . Pick b' < l such that  $(b', l] \subseteq B$

```
PROOF: By Lemma 4.1.2. We know l is not least in X because a < l.
      \langle 3 \rangle 2. Pick b^* such that b' < b^* < l
              PROVE: b^* is an upper bound for \{x \in A : x < b\}
      \langle 3 \rangle 3. Let: x \in A and x < b
      \langle 3 \rangle 4. \ x \leq b^*
         PROOF: If b^* < x then b^* < x \le l and so x \in B by \langle 3 \rangle 1.
      \langle 3 \rangle5. Q.E.D.
         Proof: This contradicts \langle 2 \rangle 5.
\langle 1 \rangle 2. If L is connected then L is a linear continuum.
   \langle 2 \rangle 1. Assume: L is connected
   \langle 2 \rangle 2. L has the least upper bound property
      \langle 3 \rangle 1. Assume: for a contradiction A \subseteq L is bounded above with no least
                            upper bound
      \langle 3 \rangle 2. Let: U be the set of upper bounds of A
      \langle 3 \rangle 3. U is open
          \langle 4 \rangle 1. Let: u \in U
         \langle 4 \rangle 2. PICK an upper bound v for A with v < u
             PROOF: u is not the least upper bound for A(\langle 3 \rangle 1)
         \langle 4 \rangle 3. \ u \in (v, +\infty) \subseteq U
      \langle 3 \rangle 4. Let: V be the set of lower bounds of U
      \langle 3 \rangle 5. U and V form a separation of L
         \langle 4 \rangle 1. V is open
             Proof: Similar to \langle 3 \rangle 3.
         \langle 4 \rangle 2. U and V are disjoint
             \langle 5 \rangle 1. Assume: for a contradiction x \in U \cap V
             \langle 5 \rangle 2. Pick u \in U such that u < x
                PROOF: x is not the lowest upper bound of A
             \langle 5 \rangle 3. \ x \leq u < x
         \langle 4 \rangle 3. \ U \cup V = L
             \langle 5 \rangle 1. Let: x \in L \setminus U
             \langle 5 \rangle 2. Pick a \in A such that x < a
             \langle 5 \rangle 3. \ a \in V
             \langle 5 \rangle 4. \ x \in V
   \langle 2 \rangle 3. For all x, y \in L, there exists z \in L such that x < z < y
      PROOF: Otherwise (-\infty, y) and (x, +\infty) would form a separation of L.
```

**Corollary 8.1.13.1.** The real line  $\mathbb{R}$  is connected, and so is every ray and interval in  $\mathbb{R}$ .

Corollary 8.1.13.2. The ordered square is connected.

Corollary 8.1.13.3. Not every closed subspace of a connected space is connected.

PROOF: The set  $\{0,1\}$  is disconnected as a subspace of  $\mathbb{R}$ .

Corollary 8.1.13.4. Not every open subspace of a connected space is connected.

PROOF: The space $\mathbb{R} \setminus \{0\}$ is a disconnected open subspace of $\mathbb{R}$ . $\square$
<b>Theorem 8.1.14</b> (Intermediate Value Theorem). Let $X$ be a connected space and $Y$ a linearly ordered set under the order topology. Let $f: X \to Y$ be continuous. Let $a, b \in X$ and $r \in Y$ . If $f(a) < r < f(b)$ , then there exists $c \in X$ such that $f(c) = r$ .
PROOF: If not, then $f^{-1}((-\infty,r))$ and $f^{-1}((r,+\infty))$ would be a separation of $X$ . $\square$
<b>Proposition 8.1.15.</b> Every connected regular space with more than one point is uncountable.
Proof: $ \langle 1 \rangle 1. \text{ Every connected completely regular space with more than one point is uncountable.} $ $ \langle 2 \rangle 1. \text{ Let: } X \text{ be connected and completely regular and } a,b \in X \text{ with } a \neq b $ $ \langle 2 \rangle 2. \text{ Pick a continuous } f: X \to [0,1] \text{ such that } f(a) = 0 \text{ and } f(b) = 1 $ $ \langle 2 \rangle 3.  f \text{ is surjective.} $ Proof: By the Intermediate Value Theorem.} $ \langle 1 \rangle 2. \text{ Every connected regular space with more than one point is uncountable.} $ $ \langle 2 \rangle 1. \text{ Assume: for a contradiction } X \text{ is connected, regular and countable with more than one point.} $ $ \langle 2 \rangle 2.  X \text{ is Lindel\"{o}f} $ $ \langle 2 \rangle 3.  X \text{ is normal } $ Proof: By Theorem 7.3.9 $ \langle 2 \rangle 4. \text{ Q.E.D.} $ Proof: Contradicting $\langle 1 \rangle 1.$
<b>Proposition 8.1.16.</b> $\overline{S_{\Omega}}$ is not conneced.
Proof: $\{0\}$ is clopen. $\square$
<b>Proposition 8.1.17.</b> $\mathbb{R}_l$ is not connected.
PROOF: The set $[0, +\infty)$ is clopen. $\square$
<b>Proposition 8.1.18.</b> The space $\mathbb{R}^{\omega}$ under the uniform topology is not connected.
Proof: The set of all bounded sequences and the set of all unbounded sequences form a separation. $\Box$
<b>Proposition 8.1.19.</b> The space $\mathbb{R}_K$ is connected.
Proof: Easy. $\square$

## 8.2 Components and Local Connectedness

**Definition 8.2.1** ((Connected) Component). Let X be a topological space. Define an equivalence relation  $\sim$  on X by:  $x \sim y$  iff there exists a connected subspace  $U \subseteq X$  such that  $x \in U$  and  $y \in U$ . The (connected) components of X are the equivalence classes under  $\sim$ .

We prove this is an equivalence relation.

```
PROOF:  \langle 1 \rangle 1. \text{ For all } x \in X \text{ we have } x \sim x. \\ \text{PROOF: The subspace } \{x\} \subseteq X \text{ is connected.} \\ \langle 1 \rangle 2. \text{ For all } x,y \in X, \text{ if } x \sim y \text{ then } y \sim x. \\ \text{PROOF: Immediate from definitions.} \\ \langle 1 \rangle 3. \text{ For all } x,y,z \in X, \text{ if } x \sim y \text{ and } y \sim z \text{ then } x \sim z. \\ \text{PROOF: By Proposition 8.1.7.}
```

**Proposition 8.2.2.** Let X be a topological space. If  $C \subseteq X$  is connected and nonempty, then there exists a unique component D of X such that  $C \subseteq D$ .

```
PROOF: \begin{split} &\langle 1 \rangle 1. \text{ PICK } a \in C \\ &\langle 1 \rangle 2. \text{ Let: } D \text{ be the $\sim$-equivalence class of } A \\ &\langle 1 \rangle 3. \ C \subseteq D \\ &\text{ PROOF: For all } x \in C \text{ we have } a \sim x \text{ by definition.} \\ &\langle 1 \rangle 4. \ D \text{ is unique} \\ &\text{ PROOF: This holds because the components are disjoint.} \\ &\sqcap \end{split}
```

Proposition 8.2.3 (AC). Every component is connected.

#### PROOF

- $\langle 1 \rangle 1$ . Let: C be a component of the topological space X
- $\langle 1 \rangle 2$ . Pick  $a \in C$
- $\langle 1 \rangle 3$ . For all  $x \in C$ , PICK a connected subspace  $C_x$  of X containing both a and x.

PROOF: Such a  $C_x$  exists since  $a \sim x$ .

 $\langle 1 \rangle 4$ .  $C = \bigcup_{x \in C} C_x$ 

PROOF: This holds because  $C_x \subseteq C$  by Proposition 8.2.2.

 $\langle 1 \rangle 5$ . Q.E.D.

PROOF: It follows that C is connected by Proposition 8.1.7.

**Proposition 8.2.4.** Every component is closed.

PROOF: From Theorem 8.1.8.  $\square$ 

**Proposition 8.2.5.** The component of  $\vec{a}$  in  $\mathbb{R}^{\omega}$  under the uniform topology is  $\{\vec{b}:\vec{b}-\vec{a} \text{ is bounded}\}.$ 

Proof:

- $\langle 1 \rangle 1$ .  $C = \{ \vec{b} : \vec{b} \vec{a} \text{ is bounded} \}$  is connected.
  - $\langle 2 \rangle 1$ . Assume:  $C = U \cup V$  is a separation of C with  $\vec{a} \in U$
  - $\langle 2 \rangle 2$ . Pick  $\vec{b} \in V$
  - $\langle 2 \rangle 3. \ \{\epsilon : \epsilon \vec{b} + (1 \epsilon) \vec{a} \in U\} \ \text{and} \ \{\epsilon : \epsilon \vec{b} + (1 \epsilon) \vec{a} \in V\} \ \text{form a separation of} \ [0, 1]$
- $\langle 1 \rangle 2$ . If  $\vec{a}, \vec{b} \in C$  and  $\vec{b} \vec{a}$  is unbounded then C is disconnected.

PROOF:  $\{\vec{c}: \vec{c} - \vec{a} \text{ is bounded}\}\$  and  $\{\vec{c}: \vec{c} - \vec{a} \text{ is unbounded}\}\$ 

**Proposition 8.2.6.** Let  $x, y \in \mathbb{R}^{\omega}$  under the box topology. Then x and y are in the same component iff x - y is eventually zero.

PROOF:

- $\langle 1 \rangle 1$ . For all  $x \in \mathbb{R}^{\omega}$  the set  $\{y : x-y \text{ is eventulally zero}\}$  is connected PROOF: It is the union of the sets  $C_N = \{y : \forall n \geq N. y_n = 0\}$ , each of which is connected because it is homeomorphic to  $\mathbb{R}^{N-1}$ .
- $\langle 1 \rangle 2$ . If x y is not eventually zero then x and y are in different components
  - $\langle 2 \rangle 1$ . Assume: x y is not eventually zero
  - $\langle 2 \rangle 2$ . Define  $h: \mathbb{R}^{\omega} \to \mathbb{R}^{\omega}$  by:  $h(z)_n = \begin{cases} z_n x_n & \text{if } x_n = y_n \\ n(z_n x_n)/(y_n x_n) & \text{if } x_n \neq y_n \end{cases}$
  - $\langle 2 \rangle 3$ . h is an automorphism of  $\mathbb{R}^{\omega}$  under the box topology
  - $\langle 2 \rangle 4$ . h(x) = 0
  - $\langle 2 \rangle 5$ . h(y) is unbounded
  - $\langle 2 \rangle 6$ . Q.E.D.

PROOF: The inverse image under h of the set of bounded sequences and the set of unbounded sequences form a separation of  $\mathbb{R}^{\omega}$  with x and y in different sets.

## 8.3 Path Connectedness

**Definition 8.3.1** (Path). Let X be a topological space and  $a, b \in X$ . A path from a to b is a continuous function  $p : [0,1] \to X$  such that p(0) = a and p(1) = b.

**Definition 8.3.2** (Path Connected). A topological space is *path connected* iff there exists a path between any two points.

**Proposition 8.3.3.** Every path connected space is connected.

Proof:

- $\langle 1 \rangle 1$ . Let: X be a path connected space
- $\langle 1 \rangle 2$ . Assume: for a contradiction U and V are a separation of X.
- $\langle 1 \rangle 3$ . Pick  $a \in U$  and  $b \in V$

- $\langle 1 \rangle 4$ . PICK a path  $p:[0,1] \to X$  from a to b
- $\langle 1 \rangle 5$ .  $p^{-1}(U)$  and  $p^{-1}(V)$  form a separation of [0,1].
- $\langle 1 \rangle 6$ . Q.E.D.

Proof: This contradicts Corollary 8.1.13.1.

Corollary 8.3.3.1.  $S_{\Omega}$  is not path connected.

Corollary 8.3.3.2.  $\overline{S_{\Omega}}$  is not path connected.

Corollary 8.3.3.3.  $\mathbb{R}_l$  is not path connected.

Corollary 8.3.3.4. The Sorgenfrey plane is not path connected.

Corollary 8.3.3.5. The space  $\mathbb{R}^{\omega}$  under the uniform topology is not path connected. connected.

Corollary 8.3.3.6. The space  $\mathbb{R}^{\omega}$  under the box topology is not path connected.

**Proposition 8.3.4.** The long line is path connected.

#### Proof:

- $\langle 1 \rangle 1$ . Let:  $a, b \in L$
- $\langle 1 \rangle 2$ . PICK an ordinal  $\alpha$  such that  $a, b < (\alpha, 0)$
- $\langle 1 \rangle 3$ . There exists a path from a to b

PROOF: This holds because  $[(0,0),(\alpha,0))$  is homeomorphic to [0,1) by Proposition 1.7.11.

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Corollary 8.3.4.1. Not every closed subspace of a path connected space is path connected.

PROOF: Take any two-element subspace of the long line.

Corollary 8.3.4.2. Not every open subspace of a path connected space is path connected.

PROOF: The space  $\mathbb{R} \setminus \{0\}$  is not path connected as a subspace of  $\mathbb{R}$ .  $\square$ 

**Definition 8.3.5** (Path Component). Let X be a topological space. Define an equivalence relation  $\sim$  on X by:  $x \sim y$  iff there exists a path from x to y. The equivalence classes are called the *path components* of X.

We prove this is an equivalence relation.

#### Proof:

- $\langle 1 \rangle 1$ . For all  $x \in X$  we have  $x \sim x$ 
  - PROOF: The constant path  $p:[0,1]\to X$  where p(t)=x is a path from x to x.
- $\langle 1 \rangle 2$ . If  $x \sim y$  then  $y \sim x$

PROOF: If  $p:[0,1] \to X$  is a path from x to y then  $\lambda t.p(1-t)$  is a path from y to x.

 $\langle 1 \rangle 3$ . If  $x \sim y$  and  $y \sim z$  then  $x \sim z$ 

- $\langle 2 \rangle 1$ . Let: p be a path from x to y and q be a path from y to z.
- $\langle 2 \rangle 2$ . Let:  $r: [0,1] \to X$  where

$$r(t) = \begin{cases} p(2t) & \text{if } 0 \le t \le 1/2\\ q(2t-1) & \text{if } 1/2 \le t \le 1 \end{cases}$$

 $\langle 2 \rangle 3$ . r is a path from x to z.

PROOF: r is continuous by the Pasting Lemma.

Proposition 8.3.6. Every path component is path connected.

PROOF: By definition, if x and y are in the same path component then there is a path from x to y.  $\square$ 

**Proposition 8.3.7.** If A is a nonempty path connected subspace of the space X, then A is included in a unique path component.

#### Proof:

- $\langle 1 \rangle 1$ . Pick  $a \in A$
- $\langle 1 \rangle 2$ . Let: C be the equivalence class of a under  $\sim$
- $\langle 1 \rangle 3. \ A \subseteq C$

PROOF: For all  $x \in A$ , there exists a path from a to x.

 $\langle 1 \rangle 4$ . C is unique

PROOF: C is the unique path component such that  $a \in C$ .

Proposition 8.3.8. Every path component is included in a component.

PROOF: From Propositions 8.3.3 and 8.2.2.  $\square$ 

**Proposition 8.3.9.** The ordered square is not path connected.

#### Proof:

- $\langle 1 \rangle 1$ . Assume: for a contradiction  $p:[0,1] \to I_o^2$  is a path from (0,0) to (1,1).
- $\langle 1 \rangle 2$ . For all  $x \in [0,1]$ ,  $p^{-1}(\{x\} \times (0,1))$  is open in [0,1]
- $\langle 1 \rangle 3$ . For all  $x \in [0,1]$ , PICK a rational  $q_x \in p^{-1}(\{x\} \times (0,1))$
- $\langle 1 \rangle 4$ .  $\{q_x : x \in [0,1]\}$  is an uncountable set of rationals.

**Proposition 8.3.10** (AC). The product of a family of path connected spaces is path connected.

### Proof:

- $\langle 1 \rangle 1$ . Let:  $\{X_{\alpha}\}_{{\alpha} \in J}$  be a family of path connected spaces and  $a, b \in \prod_{{\alpha} \in J} X_{\alpha}$
- $\langle 1 \rangle 2$ . For  $\alpha \in J$ , PICK a path  $p_{\alpha} : [0,1] \to X_{\alpha}$  from  $a_{\alpha}$  to  $b_{\alpha}$
- $\langle 1 \rangle 3$ . Define  $p: [0,1] \to \prod_{\alpha \in J} X_{\alpha}$  by  $p(t)_{\alpha} = p_{\alpha}(t)$
- $\langle 1 \rangle 4$ . p is a path from a to b

PROOF: By Theorem 5.2.15.

**Corollary 8.3.10.1.** For any set I, the space  $\mathbb{R}^I$  in the product topology is path connected.

**Proposition 8.3.11.** The space  $\mathbb{R}_K$  is not path connected.

#### PROOF

- $\langle 1 \rangle 1$ . Assume: for a contradiction  $p:[0,1] \to \mathbb{R}_K$  is a path from 0 to 1
- $\langle 1 \rangle 2$ . Let:  $p:[0,1] \to \mathbb{R}_K$  be a path from 0 to 1
- $\langle 1 \rangle 3$ . p([0,1]) is compact and connected in  $\mathbb{R}_K$ .

PROOF: Theorem 8.1.9 and Proposition 9.4.10.

 $\langle 1 \rangle 4$ . p([0,1]) is connected in  $\mathbb{R}$ .

Proof: Corollary 8.1.9.1

 $\langle 1 \rangle 5. \ [0,1] \subseteq p([0,1])$ 

PROOF: For any  $x \in [0, 1]$ , if  $x \notin p([0, 1])$  then  $p([0, 1]) \cap (-\infty, x)$  and  $p([0, 1]) \cap (x, +\infty)$  form a separation of p([0, 1]).

 $\langle 1 \rangle 6$ . [0, 1] is compact in  $\mathbb{R}_K$ 

Proof: Proposition 9.4.6.

 $\langle 1 \rangle 7$ . Q.E.D.

Proof: This contradicts Corollary 9.4.11.2.

**Proposition 8.3.12.** Let  $f: X \to Y$  be continuous and surjective. If X is path connected then Y is path connected.

#### Proof:

- $\langle 1 \rangle 1$ . Let:  $a, b \in Y$
- $\langle 1 \rangle 2$ . PICK  $x, y \in X$  such that f(x) = a and f(y) = b
- $\langle 1 \rangle 3$ . PICK a path  $p:[0,1] \to X$  such that p(0)=x and p(1)=y
- $\langle 1 \rangle 4$ .  $f \circ p$  is a path from a to b

**Corollary 8.3.12.1.** Let  $\{X_{\alpha}\}_{{\alpha}\in J}$  be a family of non-empty topological spaces. If  $\prod_{{\alpha}\in J} X_{\alpha}$  is path connected then each  $X_{\alpha}$  is path connected.

# 8.4 Connected Subspaces of Euclidean Space

**Definition 8.4.1** (Unit 2-Sphere). The unit 2-sphere is  $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$  as a subspace of  $\mathbb{R}^3$ .

**Definition 8.4.2** (Unit Ball). For any  $n \geq 1$ , the closed unit ball in  $\mathbb{R}^n$  is

$$B^n = \{ \vec{x} \in \mathbb{R}^n : ||\vec{x}|| \le 1 \}$$
.

**Proposition 8.4.3.** Every open unit ball and closed unit ball in  $\mathbb{R}^n$  is path connected.

PROOF: The straight line between any two points is a path in the ball.  $\square$ 

**Definition 8.4.4** (Punctured Euclidean Space). For  $n \geq 1$ , punctured Euclidean space is  $\mathbb{R}^n \setminus \{\vec{0}\}$ .

**Proposition 8.4.5.** Punctured Euclidean space in  $\mathbb{R}^n$  is path connected iff n > 1.

Proof: Easy.

**Definition 8.4.6** (Unit Sphere). For  $n \ge 1$ , the unit sphere  $S^n$  is  $\{\vec{x} \in \mathbb{R}^{n+1} : \|\vec{x}\| = 1\}$ .

**Proposition 8.4.7.** In any number of dimensions, the unit sphere is path connected.

Proof: Easy.  $\square$ 

**Definition 8.4.8** (Topologist's Sine Curve). The *topologist's sine curve* is the closure of

$$S = \{(x, \sin 1/x) : x \in \mathbb{R}\}$$

in  $\mathbb{R}^2$ .

**Proposition 8.4.9.** The topologist's sine curve is connected.

PROOF:

 $\langle 1 \rangle 1$ .  $S = \{(x, \sin 1/x) : x \in \mathbb{R}\}$  is connected.

 $\langle 2 \rangle 1$ . The function  $f: \mathbb{R} \to \mathbb{R}^2$  given by  $f(x) = (x, \sin 1/x)$  is continuous.

PROOF: By Theorem 5.2.15.

 $\langle 2 \rangle 2$ . Q.E.D.

PROOF: By Theorem 8.1.9.

 $\langle 1 \rangle 2$ . Q.E.D.

Proof: By Theorem 8.1.8.

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**Proposition 8.4.10** (CC). The topologist's sine curve is not path connected.

Proof:

```
\langle 1 \rangle 1. Let: S = \{(x, \sin 1/x) : x \in \mathbb{R}\}
```

 $\langle 1 \rangle 2$ . Assume: for a contradiction  $p:[0,1] \to \overline{S}$  is a path from (0,0) to  $(1,\sin 1)$ .

 $\langle 1 \rangle 3. \ p^{-1}(\{0\} \times [-1,1])$  is closed.

 $\langle 1 \rangle 4$ .  $p^{-1}(\{0\} \times [-1,1])$  has a greatest element.

PROOF: By Lemma 4.1.9.

 $\langle 1 \rangle 5$ . Let:  $q:[0,1] \to \overline{S}$  be a path such that:

• 
$$q(0) \in \{0\} \times [-1, 1]$$

• 
$$q(x) \in S \text{ for } x > 0$$

PROOF: Let b be greatest in  $p^{-1}(\{0\}\times[-1,1])$ . Then q is obtained by rescaling p restricted to [b,1].

$$\langle 1 \rangle 6$$
. Let:  $q(t) = (x(t), y(t))$  for  $0 \le t \le 1$ 

$$\langle 1 \rangle 7. \ \ x(0) = 0$$

- $\langle 1 \rangle 8. \ x(t) > 0 \text{ for } t > 0$
- $\langle 1 \rangle 9. \ y(t) = \sin 1/x(t) \text{ for } t > 0$
- $\langle 1 \rangle 10$ . There exists a sequence  $t_n \in [0,1]$  such that  $t_n \to 0$  as  $n \to \infty$  and  $y(t_n) = (-1)^n$  for all n.
  - $\langle 2 \rangle 1$ . For each n, PICK  $u_n$  such that  $0 < u_n < x(1/n)$  and  $\sin 1/u_n = (-1)^n$ . PROOF: Such a  $u_n$  exists because  $\sin 1/x$  takes values 1 and -1 infinitely often in (0, x(1/n)).
  - $\langle 2 \rangle 2$ . For each n, PICK  $t_n$  such that  $0 < t_n < 1/n$  and  $x(t_n) = u$  PROOF: By the Intermediate Value Theorem.
- $\langle 1 \rangle 11$ . Q.E.D.

PROOF: This is a contradiction as  $y(t_n) \to y(0)$  as  $n \to \infty$  because y is continuous.

#### 8.5 Local Connectedness

**Definition 8.5.1** (Locally Connected). Let X be a topological space and  $x \in X$ . Then X is *locally connected* at x iff every neighbourhood of x includes a connected neighbourhood of x.

The space X is *locally connected* iff it is locally connected at every point.

**Proposition 8.5.2.**  $S_{\Omega}$  is not locally connected.

PROOF: There is no connected neighbourhood of  $\omega$ .  $\sqcup$ 

**Proposition 8.5.3.**  $\overline{S_{\Omega}}$  is not locally connected.

PROOF: There is no connected neighbourhood of  $\omega$ .  $\square$ 

**Proposition 8.5.4.** For any set I, the space  $\mathbb{R}^I$  is locally connected.

PROOF: Every basic open set is the product of connected spaces, hence connected.  $\Box$ 

**Proposition 8.5.5.** Let X be a topological space. Then X is locally connected if and only if, for every open set U in X, every component of U is open in X.

#### Proof:

- $\langle 1 \rangle 1$ . If X is locally connected then, for every open set U in X, every component of U is open in X.
  - $\langle 2 \rangle 1$ . Assume: X is locally connected.
  - $\langle 2 \rangle 2$ . Let: *U* be open in *X*.
  - $\langle 2 \rangle 3$ . Let: C be a component of U.
  - $\langle 2 \rangle 4$ . Let:  $x \in C$

Prove: C is a neighbourhood of x

 $\langle 2 \rangle 5$ . U is a neighbourhood of x in X.

PROOF: From  $\langle 2 \rangle 2$ ,  $\langle 2 \rangle 3$  and  $\langle 2 \rangle 4$ .

 $\langle 2 \rangle$ 6. PICK a connected neighbourhood V of x such that  $V \subseteq U$ .

PROOF: Using $\langle 2 \rangle 1$ . $\langle 2 \rangle 7$ . $V \subseteq C$ PROOF: By Proposition 8.2.2. $\langle 2 \rangle 8$ . $C$ is a neighbourhood of $x$ PROOF: By Proposition 3.2.4. $\langle 2 \rangle 9$ . Q.E.D. PROOF: By Proposition 3.2.3. $\langle 1 \rangle 2$ . If, for every open set $U$ in $X$ , every component of $U$ is open in $X$ is locally connected. $\langle 2 \rangle 1$ . Assume: For every open set $U$ in $X$ , every component of $U$ is open in $X$ . $\langle 2 \rangle 2$ . Let: $x \in X$ and $X$ be a neighbourhood of $X$ $\langle 2 \rangle 3$ . Pick $U$ open such that $X \in U \subseteq X$ $\langle 2 \rangle 4$ . Let: $X \in X$ be the component of $X \in X$ that is included in $X \in X$ Proof: By $X \in X$ by $X \in X$ is a connected neighbourhood of $X \in X$ that is included in $X \in X$ is a connected neighbourhood of $X \in X$ that is included in $X \in X$ is a connected neighbourhood of $X \in X$ that is included in $X \in X$ is a connected neighbourhood of $X \in X$ that is included in $X \in X$ is a connected neighbourhood of $X \in X$ that is included in $X \in X$ is a connected neighbourhood of $X \in X$ that is included in $X \in X$ is a connected neighbourhood of $X \in X$ that is included in $X \in X$ is a connected neighbourhood of $X \in X$ that is included in $X \in X$ is a connected neighbourhood of $X \in X$ that is included in $X \in X$ is a connected neighbourhood of $X \in X$ that is included in $X \in X$ is a connected neighbourhood of $X \in X$ that is included in $X \in X$ that inclu
Corollary 8.5.5.1. In a locally connected space, every component is open.
Corollary 8.5.5.2. The space $\mathbb{R}^{\omega}$ under the box topology is not locally connected.
Corollary 8.5.5.3. Not every closed subspace of a locally connected space is locally connected.
Proof: The topologist's sine curve is not locally connected.
<b>Proposition 8.5.6.</b> $S_{\Omega} \times \overline{S_{\Omega}}$ is not locally connected.
$(\omega,\omega)$ has no connected neighbourhood. $\square$
<b>Proposition 8.5.7.</b> $\mathbb{R}_l$ is not locally connected.
Proof: 0 has no connected neighbourhood. $\square$
Proposition 8.5.8. The Sorgenfrey plane is not locally connected.
PROOF: Any basic open set $[a,b)\times[c,d)$ can be separated into $[a,b)\times[c,e)$ and $[a,b)\times[e,d)$ for some $c< e< d$ . $\square$
<b>Proposition 8.5.9.</b> The space $\mathbb{R}^{\omega}$ under the uniform topology is locally connected.
PROOF: For any neighbourhood $U$ of a point $x$ , the neighbourhood $U \cap \{y: y-x \text{ is bounded}\}$ is connected. $\square$
<b>Proposition 8.5.10.</b> The space $\mathbb{R}_K$ is not locally connected.
PROOF: The open set $(-1,1)-K$ does not include a connected neighbourhood of 0. $\square$

<b>Proposition 8.5.11.</b> Every open subspace of a locally connected space is locally connected.
Proof: Follows easily from definition. $\Box$
<b>Proposition 8.5.12</b> (AC). The product of a family of locally connected spaces is locally connected.
PROOF: $\langle 1 \rangle 1$ . Let: $\{X_{\alpha}\}_{\alpha \in J}$ be a family of locally connected spaces and $\vec{x} \in \prod_{\alpha \in J} X_{\alpha} \langle 1 \rangle 2$ . Let: $\prod_{\alpha \in J} U_{\alpha}$ be any basic neighbourhood of $\vec{x}$ , where each $U_{\alpha}$ is open in $X_{\alpha}$ , and $U_{\alpha} = X_{\alpha}$ except for $\alpha = \alpha_1, \ldots, \alpha_n$ $\langle 1 \rangle 3$ . For $\alpha \in J$ , Pick a connected neighbourhood $C_{\alpha}$ of $x_{\alpha}$ with $C_{\alpha} \subseteq U_{\alpha}$ $\langle 1 \rangle 4$ . $\prod_{\alpha \in J} C_{\alpha}$ is connected Proof: Proposition 8.1.10
Proposition 8.5.13. Every discrete space is locally connected.
PROOF: For any point $x$ , the set $\{x\}$ is a connected neighbourhood of $x$ . $\square$
Corollary 8.5.13.1. The continuous image of a locally connected space is not necessarily locally connected.
8.6 Local Path Connectedness
<b>Definition 8.6.1</b> (Locally Path Connected). Let $X$ be a topological space and $x \in X$ . Then $X$ is locally path connected at $x$ iff every neighbourhood of $x$ includes a path connected neighbourhood of $x$ .  The space $X$ is locally path connected iff it is locally path connected at every
point.
point.
point. <b>Proposition 8.6.2.</b> $S_{\Omega}$ is not locally path connected.
point. <b>Proposition 8.6.2.</b> $S_{\Omega}$ is not locally path connected. PROOF: There is no path connected neighbourhood of $\omega$ .
point. <b>Proposition 8.6.2.</b> $S_{\Omega}$ is not locally path connected. PROOF: There is no path connected neighbourhood of $\omega$ . <b>Proposition 8.6.3.</b> $\overline{S_{\Omega}}$ is not locally path connected.
point. <b>Proposition 8.6.2.</b> $S_{\Omega}$ is not locally path connected. PROOF: There is no path connected neighbourhood of $\omega$ . <b>Proposition 8.6.3.</b> $\overline{S_{\Omega}}$ is not locally path connected. PROOF: There is no path connected neighbourhood of $\omega$ . <b>Proposition 8.6.4.</b> Not every closed subspace of a locally path connected space
point.  Proposition 8.6.2. $S_{\Omega}$ is not locally path connected.  Proposition 8.6.3. $\overline{S_{\Omega}}$ is not locally path connected.  Proposition 8.6.3. $\overline{S_{\Omega}}$ is not locally path connected.  Proposition 8.6.4. Not every closed subspace of a locally path connected space is locally path connected.
Proposition 8.6.2. $S_{\Omega}$ is not locally path connected.  Proposition 8.6.3. $\overline{S_{\Omega}}$ is not locally path connected.  Proposition 8.6.3. $\overline{S_{\Omega}}$ is not locally path connected.  Proposition 8.6.4. Not every closed subspace of a locally path connected space is locally path connected.  Proposition 8.6.4. Not every closed subspace of a locally path connected space is locally path connected.  Proposition 8.6.5. Every open subspace of a locally path connected space is

Proof: From Proposition 8.3.3.  $\square$ 

Corollary 8.6.6.1.  $\mathbb{R}_l$  is not locally path connected.

Corollary 8.6.6.2. The Sorgenfrey plane is not locally path connected.

**Corollary 8.6.6.3.** The space  $\mathbb{R}^{\omega}$  under the box topology is not locally path connected.

Corollary 8.6.6.4. The space  $\mathbb{R}_K$  is not locally path connected.

Corollary 8.6.6.5. The topologist's sine curve is not locally path connected.

**Proposition 8.6.7** (AC). The product of a family of locally path connected spaces is locally path connected.

#### Proof:

- $\langle 1 \rangle 1$ . Let:  $\{X_{\alpha}\}_{{\alpha} \in J}$  be a family of locally connected spaces and  $\vec{x} \in \prod_{{\alpha} \in J} X_{\alpha}$
- $\langle 1 \rangle 2$ . Let:  $\prod_{\alpha \in J} U_{\alpha}$  be any basic neighbourhood of  $\vec{x}$ , where each  $U_{\alpha}$  is open in  $X_{\alpha}$ , and  $U_{\alpha} = X_{\alpha}$  except for  $\alpha = \alpha_1, \ldots, \alpha_n$
- $\langle 1 \rangle 3$ . For  $\alpha \in J$ , Pick a path connected neighbourhood  $C_{\alpha}$  of  $x_{\alpha}$  with  $C_{\alpha} \subseteq U_{\alpha}$
- $\langle 1 \rangle 4$ .  $\prod_{\alpha \in J} C_{\alpha}$  is path connected

PROOF: Proposition ??

**Proposition 8.6.8.** Let X be a topological space. Then X is locally path connected if and only if, for every open set U in X, every path component of U is open in X.

#### Proof:

- $\langle 1 \rangle 1$ . If X is locally path connected then, for every open set U in X, every path component of U is open in X.
  - $\langle 2 \rangle 1$ . Assume: X is locally path connected.
  - $\langle 2 \rangle 2$ . Let: U be open in X.
  - $\langle 2 \rangle 3$ . Let: C be a path component of U.
  - $\langle 2 \rangle 4$ . Let:  $x \in C$

Prove: C is a neighbourhood of x

 $\langle 2 \rangle 5$ . U is a neighbourhood of x in X.

PROOF: From  $\langle 2 \rangle 2$ ,  $\langle 2 \rangle 3$  and  $\langle 2 \rangle 4$ .

- $\langle 2 \rangle$ 6. PICK a path connected neighbourhood V of x such that  $V \subseteq U$ . PROOF: Using  $\langle 2 \rangle$ 1.
- $\langle 2 \rangle 7. \ V \subseteq C$

Proof: By Proposition 8.3.7.

 $\langle 2 \rangle 8$ . C is a neighbourhood of x

PROOF: By Proposition 3.2.4.

 $\langle 2 \rangle 9$ . Q.E.D.

Proof: By Proposition 3.2.3.

 $\langle 1 \rangle 2$ . If, for every open set U in X, every path component of U is open in X, then X is locally path connected.

 $\langle 2 \rangle 1$ . Assume: For every open set U in X, every path component of U is open in X.  $\langle 2 \rangle 2$ . Let:  $x \in X$  and N be a neighbourhood of x  $\langle 2 \rangle 3$ . Pick U open such that  $x \in U \subseteq N$  $\langle 2 \rangle 4$ . Let: C be the path component of U that contains x  $\langle 2 \rangle 5$ . C is open in X Proof: By  $\langle 2 \rangle 1$ .  $\langle 2 \rangle$ 6. C is a path connected neighbourhood of x that is included in N **Theorem 8.6.9** (AC). Let X be a topological space. If X is locally path connected, then its components and its path components are the same. Proof:  $\langle 1 \rangle 1$ . Let: P be a path component of X  $\langle 1 \rangle 2$ . Let: C be the component such that  $P \subseteq C$ Prove: P = C $\langle 1 \rangle 3$ . Let:  $Q = C \setminus P$  $\langle 1 \rangle 4$ . P is open in X Proof: By Proposition 8.6.8.  $\langle 1 \rangle 5$ . Q is open in X PROOF: By Proposition 8.6.8 since Q is the union of the path components included in C other than P.  $\langle 1 \rangle 6. \ Q = \emptyset$ PROOF: Otherwise P and Q would form a separation of C, contradicting 8.2.3. **Proposition 8.6.10.**  $S_{\Omega} \times \overline{S_{\Omega}}$  is not locally path connected. PROOF:  $(\omega, \omega)$  has no path connected neighbourhood. **Proposition 8.6.11.** The ordered square is not locally path connected. PROOF:

 $\langle 1 \rangle 1$ . Assume: for a contradiction (1/2,0) has a path connected neighbourhod

 $\langle 1 \rangle 2$ . PICK a < 1/2 such that  $((a,1),(1/2,0)) \subseteq U$ 

 $\langle 1 \rangle 3$ . Let:  $p:[0,1] \to I_o^2$  be a path from (a,1) to (1/2,0)

(1)4. For every x such that a < x < 1/2, PICK a rational  $q_x$  such that  $p(q_x) \in$ ((x,0),(x,1))

 $\langle 1 \rangle 5$ .  $\{q_x : a < x < 1/2\}$  is an uncountable set of rationals.

**Proposition 8.6.12.** For any set I, the space  $\mathbb{R}^I$  is locally path connected.

PROOF: Every basic open set is the product of path connected spaces, hence path connected.  $\square$ 

**Proposition 8.6.13.** The space  $\mathbb{R}^{\omega}$  under the uniform topology is locally path connected.

Proof: Its components and path components are the same.  $\Box$ 

Proposition 8.6.14. Every discrete space is locally path connected.

PROOF: For any point x, the set  $\{x\}$  is a path connected neighbourhood of x.

Corollary 8.6.14.1. The continuous image of a locally path connected space is not necessarily locally path connected.

### 8.7 Weak Local Connectedness

**Definition 8.7.1** (Weakly Locally Connected). Let X be a topological space and  $x \in X$ . Then X is weakly locally connected at x iff every neighbourhood of x contains a connected subspace that contains a neighbourhood of x.

## Chapter 9

# Compact Spaces

### 9.1 Countable Compactness

**Definition 9.1.1** (Countably Compact). A topological space is *countably compact* iff every countable open covering has a finite subcovering.

### 9.2 Limit Point Compactness

**Definition 9.2.1** (Limit Point Compact). A space is *limit point compact* iff every infinite set has a limit point.

**Proposition 9.2.2** (CC).  $S_{\Omega} \times \overline{S_{\Omega}}$  is limit point compact.

```
Proof:
\langle 1 \rangle 1. Let: A \subseteq S_{\Omega} \times \overline{S_{\Omega}} be infinite
\langle 1 \rangle 2. Case: \pi_1(A) is finite.
    \langle 2 \rangle 1. PICK x such that there are infinitely many y such that (x,y) \in A
    \langle 2 \rangle 2. PICK a limit point l of \{y : (x,y) \in A\}
    \langle 2 \rangle 3. (x, l) is a limit point of A
\langle 1 \rangle 3. Case: \pi_1(A) is infinite.
    \langle 2 \rangle 1. Pick a limit point l of \pi_1(A).
    \langle 2 \rangle 2. l is a limit ordinal
    \langle 2 \rangle 3. Pick a countable sequence x_n with limit l
    \langle 2 \rangle 4. For n \geq 1, PICK a_n > x_n and y_n such that (a_n, y_n) \in A
   \langle 2 \rangle5. Case: \{y_n : n \geq 1\} is finite
       \langle 3 \rangle 1. PICK y such that y = y_n for infinitely many n
       \langle 3 \rangle 2. (l, y) is a limit point for A
   \langle 2 \rangle 6. Case: \{y_n : n \geq 1\} is infinite
       \langle 3 \rangle 1. PICK a limit point m for \{y_n : n \geq 1\}
       \langle 3 \rangle 2. (l, m) is a limit point for A
```

Proposition 9.2.3. The Sorgenfrey plane is not limit point compact.

PROOF:  $\mathbb{Z}^2$  has no limit point.  $\square$ Proposition 9.2.4. The space  $\mathbb{R}^\omega$  under the box topology is not limit point compact.

PROOF: The set of all constant sequences of integers is an infinite set with no limit point.  $\square$ Proposition 9.2.5. Not every open subspace of a limit point compact space is limit point compact.

PROOF: The space [0,1] is limit point compact but (0,1) is not.  $\square$ Proposition 9.2.6. The product of two limit point compact spaces is not necessarily limit point compact.

PROOF: See Steen and Seebach Countexamples in Topology Example 112.  $\square$ Proposition 9.2.7. The continuous image of a limit point comapct space is not necessarily limit point comapct.

### 9.3 Lindelöf Spaces

limit point compact, but  $\mathbb{N}$  is not.  $\square$ 

**Definition 9.3.1** (Lindelöf Space). A topological space X is  $Lindel\"{o}f$  iff every open covering has a countable subcovering.

PROOF: Let Y be a two-point set under the indiscrete topology. Then  $\mathbb{N} \times Y$  is

**Theorem 9.3.2** (CC). Every second countable space is Lindelöf.

#### Proof:

- $\langle 1 \rangle 1$ . Let: X be a second countable space
- $\langle 1 \rangle 2$ . PICK a countable basis  $\mathcal{B}$  for X.
- $\langle 1 \rangle 3$ . Let:  $\mathcal{A}$  be an open cover of X
- $\langle 1 \rangle$ 4. For every  $B \in \mathcal{B}$  such that there exists  $U \in \mathcal{A}$  such that  $B \subseteq U$ , PICK  $U_B \in \mathcal{A}$  such that  $B \subseteq U_B$
- $\langle 1 \rangle 5. \{ U_B : B \in \mathcal{B}, \exists U \in \mathcal{A}.B \subseteq U \} \text{ covers } X.$ 
  - $\langle 2 \rangle 1$ . Let:  $x \in X$
  - $\langle 2 \rangle 2$ . PICK  $U \in \mathcal{A}$  such that  $x \in U$
  - $\langle 2 \rangle 3$ . Pick  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$
- $\langle 2 \rangle 4. \ x \in U_B$

Corollary 9.3.2.1. The space  $\mathbb{R}^{\omega}$  is Lindelöf.

Corollary 9.3.2.2. The space  $\mathbb{R}_K$  is Lindelöf.

**Proposition 9.3.3.** The space  $S_{\Omega}$  is not Lindelöf.

PROOF: $\{(-\infty, \alpha) : \alpha \in S_{\Omega}\}$  is an open cover that has no countable subcover.  $\square$ 

**Proposition 9.3.4** (CC). The space  $\overline{S_{\Omega}}$  is Lindelöf.

#### Proof:

- $\langle 1 \rangle 1$ . Let:  $\mathcal{A}$  be an open cover of  $\overline{S_{\Omega}}$
- $\langle 1 \rangle 2$ . PICK  $U \in \mathcal{A}$  such that  $\Omega \in U$
- $\langle 1 \rangle 3$ . PICK  $\alpha < \Omega$  such that  $(\alpha, \Omega] \subseteq U$
- $\langle 1 \rangle 4$ . For  $\beta \leq \alpha$ , PICK  $U_{\beta} \in \mathcal{A}$  such that  $\beta \in U_{\beta}$
- $\langle 1 \rangle$ 5.  $\{U\} \cup \{U_{\beta} : \beta \leq \alpha\}$  is a countable subcover of  $\mathcal{A}$ .

**Proposition 9.3.5** (CC). The continuous image of a Lindelöf space is Lindelöf.

#### PROOF:

- $\langle 1 \rangle 1$ . Let: X be a Lindelöf space, Y a space and  $f: X \to Y$  continuous.
- $\langle 1 \rangle 2$ . Let:  $\mathcal{A}$  be an open covering of Y
- $\langle 1 \rangle 3. \{ f^{-1}(V) : V \in \mathcal{A} \}$  is an open covering of X
- $\langle 1 \rangle$ 4. PICK a countable subcovering  $\{f^{-1}(V_1), f^{-1}(V_2), \ldots\}$  of  $\{f^{-1}(V) : V \in \mathcal{A}\}$
- $\langle 1 \rangle$ 5.  $\{ \acute{V}_1, V_2, \ldots \}$  is a countable subcovering of  ${\mathcal A}$

**Proposition 9.3.6.** The Sorgenfrey plane is not Lindelöf.

#### Proof:

- $\langle 1 \rangle 1$ . Let:  $L = \{(x, -x) : x \in \mathbb{R}\}$
- $\langle 1 \rangle 2$ . L is closed in  $\mathbb{R}^2$ 
  - $\langle 2 \rangle 1$ . Let:  $(x,y) \notin L$ , so  $y \neq -x$

Prove: There exists a neighbourhood U of (x,y) that does not intersect L

 $\langle 2 \rangle 2$ . Case: y > -x

PROOF: In this case, take  $U = [x, +\infty) \times [y, +\infty)$ 

 $\langle 2 \rangle 3$ . Case: y < -x

PROOF: In this case, take  $U = [x, (x - y)/2) \times [y, (y - x)/2)$ .

- $\langle 1 \rangle 3$ . Let:  $\mathcal{U} = \{ \mathbb{R}^2_l \setminus L \} \cup \{ [a,b) \times [-a,d) : a,b,d \in \mathbb{R} \}$
- $\langle 1 \rangle 4$ .  $\mathcal{U}$  is an open covering of  $\mathbb{R}^2$
- $\langle 1 \rangle 5$ . No countable subset of  $\mathcal{U}$  covers  $\mathbb{R}^2_I$

PROOF: Every set  $[a,b) \times [-a,d)$  intersects L in exactly one point, namely (a,-a).

Corollary 9.3.6.1. The Sorgenfrey plane is not second countable.

**Corollary 9.3.6.2.** The product of two Lindelöf spaces is not necessarily Lindelöf.

**Proposition 9.3.7.** The space  $\mathbb{R}^{\omega}$  under the box topology is not Lindelöf.

PROOF: The set  $\{\prod_{n=0}^{\infty}(a_n, a_n + 1) : \forall n.a_n \in \mathbb{Z}\}$  covers the space but has no countable subcover.  $\square$ 

Proposition 9.3.8. Not every open subspace of a Lindelöf space is Lindelöf.

PROOF: The ordered square is Lindelöf but the subspace [0,1]times(0,1) is not.  $\Box$ 

### 9.4 Compactness

**Definition 9.4.1** (Compact). A topological space is *compact* iff every open cover has a finite subcover.

**Proposition 9.4.2.**  $S_{\Omega}$  is not compact.

PROOF: The open covering  $\{(-\infty, \alpha) : \alpha \in S_{\Omega}\}$  has no finite subcovering.  $\square$ 

**Proposition 9.4.3.**  $\mathbb{R}_l$  is not compact.

PROOF:  $\{[n, n+1) : n \in \mathbb{Z}\}$  has no finite subcover.  $\square$ 

**Proposition 9.4.4.** The space  $\mathbb{R}^{\omega}$  under the box topology is not compact.

PROOF: The set  $\{\prod_{n=0}^{\infty}(a_n, a_n+1) : n \in \mathbb{Z}\}$  is a cover that has no finite subcover.

**Proposition 9.4.5.** Let Y be a subspace of X. Then Y is compact if and only if every covering of Y by sets open in X contains a finite subcollection covering Y.

#### Proof:

- $\langle 1 \rangle 1$ . If Y is compact then every covering of Y by sets open in X contains a finite subcollection covering Y.
  - $\langle 2 \rangle 1$ . Assume: Y is compact.
  - $\langle 2 \rangle 2$ . Let:  $\mathcal{A}$  be a covering of Y by sets open in X.
  - $\langle 2 \rangle 3$ .  $\{ U \cap Y : U \in \mathcal{A} \}$  is an open covering of Y.
  - $\langle 2 \rangle 4$ . PICK a finite subcovering  $V_1, \ldots, V_n$  of  $\{U \cap Y : U \in A\}$
  - $\langle 2 \rangle 5$ . For  $1 \leq i \leq n$ , PICK  $U_i \in \mathcal{A}$  such that  $V_i = U_i \cap Y$ .
  - $\langle 2 \rangle 6$ .  $\{U_1, \ldots, U_n\}$  is a finite subset of  $\mathcal{A}$  that covers Y.
- $\langle 1 \rangle 2$ . If every covering of Y by sets open in X contains a finite subcollection covering Y then Y is compact.
  - $\langle 2 \rangle 1$ . Assume: Every covering of Y by sets open in X contains a finite sub-collection covering Y.
  - $\langle 2 \rangle 2$ . Let:  $\mathcal{A}$  be an open covering of Y
  - $\langle 2 \rangle 3$ . Let:  $\mathcal{B} = \{ U \text{ open in } X : U \cap Y \in \mathcal{A} \}$
  - $\langle 2 \rangle 4$ .  $\mathcal{B}$  covers Y
  - $\langle 2 \rangle 5$ . Pick a finite subcollection  $\{U_1, \ldots, U_n\} \subseteq \mathcal{B}$  that covers Y
- $\langle 2 \rangle 6$ .  $\{U_1 \cap Y, \dots, U_n \cap Y\}$  is a finite subcover of  $\mathcal{A}$ .

**Proposition 9.4.6.** Every closed subspace of a compact space is compact.

#### PROOF:

- $\langle 1 \rangle 1$ . Let: X be a compact space and  $Y \subseteq X$  be closed.
- $\langle 1 \rangle 2$ . Let:  $\mathcal{A}$  be a covering of Y by spaces open in X
- $\langle 1 \rangle 3$ .  $\mathcal{A} \cup \{X \setminus Y\}$  is an open covering of X.
- $\langle 1 \rangle 4$ . PICK a finite subcovering  $\{U_1, \ldots, U_n\}$  or  $\{U_1, \ldots, U_n, X \setminus Y\}$
- $\langle 1 \rangle 5$ .  $\{U_1, \ldots, U_n\}$  is a finite subset of  $\mathcal{A}$  that covers Y.
- $\langle 1 \rangle 6$ . Q.E.D.

Proof: Proposition 9.4.5.

П

Corollary 9.4.6.1. Not every compact Hausdorff space is connected.

PROOF: The space  $[0,1] \cup [2,3]$  is compact Hausdorff and disconnected.  $\square$ 

Corollary 9.4.6.2. Not every compact Hausdorff space is path connected.

Corollary 9.4.6.3. Not every compact Hausdorff space is locally connected.

The space  $[0,1] \cap \mathbb{Q}$  is not locally connected.

Corollary 9.4.6.4. Not every compact Hausdorff space is locally path connected.

Proposition 9.4.7. Not every open subspace of a compact space is compact.

PROOF: The space [0,1] is compact but (0,1) is not.  $\square$ 

**Lemma 9.4.8.** If Y is a compact subspace of the Hausdorff space X and  $a \notin Y$ , then there exist disjoint open sets U and V of X containing a and Y, respectively.

#### PROOF:

- $\langle 1 \rangle 1$ . For  $y \in Y$ , there exist disjoint open sets U and V such that  $a \in U$  and  $u \in V$ .
- $\langle 1 \rangle 2$ . {V open in  $X : \exists U$  open and disjoint from  $V.a \in U$ } is a covering of Y by open sets in X.
- $\langle 1 \rangle 3$ . PICK a finite subset  $\{V_1, \ldots, V_n\}$  that covers Y.
- $\langle 1 \rangle 4$ . For  $1 \leq i \leq n$ , PICK $U_i$  disjoint from  $V_i$  such that  $a \in U_i$
- $\langle 1 \rangle$ 5. Let:  $U = U_1 \cap \cdots \cap U_n$  and  $V = V_1 \cup \cdots \cup V_n$

**Proposition 9.4.9.** Every compact subspace of a Hausdorff space is closed.

#### Proof

- $\langle 1 \rangle 1$ . Let: X be a Hausdorff space and  $Y \subseteq X$  be compact.
- $\langle 1 \rangle$ 2. Every point  $a \notin Y$  has an open neighbourhood disjoint from Y. PROOF: By Lemma 9.4.8.
- $\langle 1 \rangle 3$ . Q.E.D.

Proof: By Proposition 3.2.3.

**Proposition 9.4.10.** The image of a compact space under a continuous map is compact.

#### Proof:

- $\langle 1 \rangle 1$ . Let:  $f: X \to Y$  be continuous where X is compact.
- $\langle 1 \rangle 2$ . Let:  $\mathcal{A}$  be a covering of f(X) by open sets in Y.
- $\langle 1 \rangle 3$ .  $\{ f^{-1}(U) : U \in \mathcal{A} \}$  is an open covering of X.
- $\langle 1 \rangle 4$ . PICK a finite subcovering  $\{f^{-1}(U_1), \dots, f^{-1}(U_n)\}$
- $\langle 1 \rangle 5$ .  $\{U_1, \ldots, U_n\}$  is a finite subset of  $\mathcal{A}$  that covers f(X).
- $\langle 1 \rangle 6$ . Q.E.D.

Proof: By Proposition 9.4.5.

**Corollary 9.4.10.1.** Let  $\{X_{\alpha}\}_{{\alpha}\in J}$  be a family of topological spaces. If  $\prod_{{\alpha}\in J} X_{\alpha}$  is compact then each  $X_{\alpha}$  is compact.

Corollary 9.4.10.2.  $S_{\Omega} \times \overline{S_{\Omega}}$  is compact.

Corollary 9.4.10.3. The Sorgenfrey plane is not compact.

Corollary 9.4.10.4. For any nonempty set I, the sapce  $\mathbb{R}^I$  is not compact.

**Corollary 9.4.10.5.** Let  $\mathcal{T}$  and  $\mathcal{T}'$  be topologies on the same set X. If  $\mathcal{T} \subseteq \mathcal{T}'$  and  $\mathcal{T}'$  is compact then  $\mathcal{T}$  is compact.

Corollary 9.4.10.6. The space  $\mathbb{R}_K$  is not compact.

**Theorem 9.4.11.** Let  $f: X \to Y$  be a bijective continuous function. If X is compact and Y is Hausdorff then f is a homeomorphism.

#### Proof:

- $\langle 1 \rangle 1$ . Let: C be closed in X
- $\langle 1 \rangle 2$ . C is compact

Proof: Proposition 9.4.6.

 $\langle 1 \rangle 3$ . f(C) is compact

Proof: Proposition 9.4.10

 $\langle 1 \rangle 4$ . f(C) is closed

Proof: Proposition 9.4.9.

 $\langle 1 \rangle 5$ . Q.E.D.

PROOF: By Theorem 5.2.2 we have that  $f^{-1}$  is continuous.

П

**Corollary 9.4.11.1.** Let  $\mathcal{T}$  and  $\mathcal{T}'$  be topologies on the same set X. If  $\mathcal{T} \subseteq \mathcal{T}'$ ,  $\mathcal{T}$  is Hausdorff and  $\mathcal{T}'$  is compact then  $\mathcal{T} = \mathcal{T}'$ .

Corollary 9.4.11.2. The space [0,1] is not compact as a subspace of  $\mathbb{R}_K$ .

**Theorem 9.4.12** (Tube Lemma). Let A and B be subspaces of X and Y, respectively; let N be an open set in  $X \times Y$  including  $A \times B$ . If A and B are compact, then there exist open sets U and V in X and Y, respectively, such that

$$A \times B \subseteq U \times V \subseteq N$$
.

#### PROOF:

 $\langle 1 \rangle 1$ . For all  $a \in A$ , there exist open sets U and V in X and Y, respectively, such that

$$\{a\} \times B \subseteq U \times V \subseteq N$$
.

- $\langle 2 \rangle 1$ . Let:  $a \in A$
- $\langle 2 \rangle 2$ . For all  $b \in B$ , there exist open sets U and V in X and Y, respectively, such that  $(a,b) \in U \times V \subseteq N$ .
- $\langle 2 \rangle 3$ .  $\{ V \text{ open in } Y : \exists U \text{ open in } X.a \in U, U \times V \subseteq N \}$  covers B
- $\langle 2 \rangle 4$ . PICK a finite subset  $\{V_1, \ldots, V_n\}$  that covers B.
- $\langle 2 \rangle$ 5. For  $1 \leq i \leq n$ , PICK  $U_i$  open in X such that  $a \in U_i$  and  $U_i \times V_i \subseteq N$
- $\langle 2 \rangle 6$ . Let:  $U = U_1 \cap \cdots \cap U_n$  and  $V = V_1 \cup \cdots \cup V_n$
- $\langle 1 \rangle 2$ . {U open in  $X : \exists V$  open in  $Y.B \subseteq V$  and  $U \times V \subseteq N$ } covers A.
- $\langle 1 \rangle 3$ . PICK a finite subset  $\{U_1, \ldots, U_n\}$  that covers A.
- (1)4. For  $1 \leq i \leq n$ , PICK  $V_i$  open in B such that  $B \subseteq V_i$  and  $U_i \times V_i \subseteq N$ .
- $\langle 1 \rangle 5$ . Let:  $U = U_1 \cup \cdots \cup U_n$  and  $V = V_1 \cap \cdots \cap V_n$
- $\langle 1 \rangle 6. \ A \times B \subseteq U \times V \subseteq N$

**Lemma 9.4.13.** Let A be a set of basis elements for  $X \times Y$  such that no finite subset of A covers  $X \times Y$ . If X is compact, then there exists a point  $x \in X$  such that no finite subset of A covers  $\{x\} \times Y$ .

#### Proof:

- $\langle 1 \rangle 1$ . Assume: X is compact.
- $\langle 1 \rangle$ 2. Assume: For all  $x \in X$ , there is a finite subset of  $\mathcal{A}$  that covers  $\{x\} \times Y$  Prove: A finite subset of  $\mathcal{A}$  covers  $X \times Y$
- $\langle 1 \rangle 3$ .  $\{ U \text{ open in } X : \exists U_1 \times V_1, \dots, U_r \times V_r \in \mathcal{A}. U = U_1 \cap \dots \cap U_r, Y = V_1 \cup \dots \cup V_r \} \text{ covers } X.$
- $\langle 1 \rangle 4$ . PICK a finite subcover  $\{U_1, \ldots, U_n\}$
- $\langle 1 \rangle$ 5. For  $1 \leq i \leq n$ , PICK $U_{i1} \times V_{i1}, \ldots, U_{ir_i} \times V_{ir_i} \in \mathcal{A}$  such that  $U_i = U_{i1} \cap \cdots \cap U_{ir_i}$  and  $Y = V_{i1} \cup \cdots \cup V_{ir_i}$
- $\langle 1 \rangle 6. \ \{ U_{ij} : 1 \le i \le n, 1 \le j \le r_i \} \text{ covers } X \times Y$

**Proposition 9.4.14.** The product of two compact spaces is compact.

#### PROOF

- $\langle 1 \rangle 1$ . Let: X and Y be compact spaces.
- $\langle 1 \rangle 2$ . Let: A be an open covering of  $X \times Y$
- $\langle 1 \rangle 3$ . For all  $x \in X$ , there exists a neighbourhood W of x such that  $W \times Y$  is covered by finitely many elements of A.
  - $\langle 2 \rangle 1$ . Let:  $x \in X$
  - $\langle 2 \rangle 2$ .  $\{x\} \times Y$  is compact.

PROOF: It is homeomorphic to Y.

- $\langle 2 \rangle 3$ . PICK a finite subset  $\{U_1, \ldots, U_m\}$  of  $\mathcal{A}$  that covers  $\{x\} \times Y$  PROOF: By Proposition 9.4.5.
- $\langle 2 \rangle 4$ . There exists a neighbourhood W of x such that  $W \times Y \subseteq U_1 \cup \cdots \cup U_m$

PROOF: By the Tube Lemma.

- $\langle 1 \rangle 4$ .  $\{ W \text{ open in } X : W \times Y \text{ is covered by finitely many elements of } \mathcal{A} \}$  is an open covering of X.
- $\langle 1 \rangle$ 5. PICK a finite subcovering  $\{W_1, \dots, W_n\}$
- $\langle 1 \rangle 6$ . For  $1 \leq i \leq n$ , PICK a finite subset  $\{U_{i1}, \ldots, U_{ir_i}\}$  of  $\mathcal{A}$  that covers  $W_i \times Y$
- $\langle 1 \rangle 7$ .  $\{U_{11}, \dots, \overline{U_{nr_n}}\}$  is a finite subcovering of  $\mathcal{A}$ .

**Proposition 9.4.15.** A topological space is compact if and only if every nonempty set of closed sets that has the finite intersection property has nonempty intersection.

Proof: Immediate from definitions.  $\Box$ 

**Lemma 9.4.16.** If Y is compact then  $\pi_1: X \times Y \to X$  is a closed map.

#### Proof:

- $\langle 1 \rangle 1$ . Let:  $C \subseteq X \times Y$  be closed
- $\langle 1 \rangle 2$ . Let:  $x \in X \setminus \pi_1(C)$
- $\langle 1 \rangle 3$ . For all  $y \in Y$ , we have  $(x, y) \notin C$
- $\langle 1 \rangle 4$ . For all  $y \in Y$ , there exist open neighbourhoods U of x and V of y such that  $U \times V \subseteq (X \times Y) \setminus C$
- $\langle 1 \rangle$ 5.  $\{ V \text{ open in } Y : \exists U \text{ an open neighbourhood of } x \text{ such that } U \times V \subseteq (X \times Y) \setminus C \}$  is an open covering of Y.
- $\langle 1 \rangle 6$ . PICK a finite subcovering  $\{V_1, \ldots, V_n\}$
- $\langle 1 \rangle$ 7. For  $1 \leq i \leq n$ , PICK an open neighbourhood  $U_i$  of x such that  $U_i \times V_i \subseteq (X \times Y) \setminus C$
- $\langle 1 \rangle 8. \ x \in U_1 \cap \cdots \cap U_n \subseteq X \setminus \pi_1(C)$

**Theorem 9.4.17.** Let X be a compact space. Let  $f_n: X \to \mathbb{R}$  be a sequence of continuous functions such that, for all  $x \in X$ ,  $f_n(x) \to f(x)$  as  $n \to \infty$ . If f is continuous, and if the sequence  $(f_n)_n$  is monotone increasing, and if X is compact, then the convergence is uniform.

#### Proof:

 $\langle 1 \rangle 1$ . Let:  $\epsilon > 0$ 

PROVE: There exists N such that, for all  $n \ge N$ , we have  $|f_n(x) - f(x)| < \epsilon$ 

 $\langle 1 \rangle 2$ . For  $n \in \mathbb{Z}^+$ ,

Let: 
$$U_n = \{x \in X : f(x) - f_n(x) < \epsilon\}$$

 $\langle 1 \rangle 3$ . Each  $U_n$  is open

PROOF: Let  $g(x) = f(x) - f_n(x)$ . Then g is continuous and  $U_n = g^{-1}((-\infty, \epsilon))$ .

- $\langle 1 \rangle 4$ .  $\{U_n : n \geq 1\}$  is an open covering of X
  - $\langle 2 \rangle 1$ . Let:  $x \in X$
  - $\langle 2 \rangle 2$ . PICK N such that, for all  $n \geq N$ ,  $|f(x) f_n(x)| < \epsilon$

PROOF:  $f_n(x) \to f(x)$  as  $n \to \infty$ 

 $\langle 2 \rangle 3. \ f(x) - f_N(x) < \epsilon$ 

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PROOF: This holds since the sequece (f_n)_n is monotone.
\langle 1 \rangle5. PICK a finite subcovering \{U_{n_1}, \ldots, U_{n_k}\}
\langle 1 \rangle 6. Let: N = \max(n_1, \ldots, n_k)
\langle 1 \rangle 7. For all n \geq N we have |f_n(x) - f(x)| < \epsilon
Lemma 9.4.18. Every compact Hausdorff space is normal.
Proof:
\langle 1 \rangle 1. Let: A and B be disjoint closed sets in the compact Hausdorff space X.
\langle 1 \rangle 2. For all a \in A, there exist disjoint open sets U and V such that a \in U and
        B \subseteq V.
   PROOF: By Lemma 9.4.8.
\langle 1 \rangle 3. {U open in X : \exists V open in Y.U \cap V = \emptyset, B \subseteq V} is an open covering of
\langle 1 \rangle 4. PICK a finite subcovering \{U_1, \ldots, U_n\}
\langle 1 \rangle5. For 1 \leq i \leq n, PICK V_i open in Y such that U_i \cap V_i = \emptyset and B \subseteq V_i
\langle 1 \rangle 6. Let: U = U_1 \cup \cdots \cup U_n and V = V_1 \cap \cdots \cap V_n
Theorem 9.4.19. Let X be a complete linearly ordered set under the order
topology. Then every closed interval in X is compact.
PROOF:
\langle 1 \rangle 1. Let: X be a complete linearly ordered set in the order topology
\langle 1 \rangle 2. Let: a, b \in X, a < b
        Prove: [a, b] is compact
\langle 1 \rangle 3. Let: A be a set of open sets that covers [a, b]
\langle 1 \rangle 4. For all x \in [a,b), there exists y \in (x,b] such that [x,y] is covered by at
        most two points of A
   \langle 2 \rangle 1. Let: x \in [a, b]
   \langle 2 \rangle 2. Pick U \in \mathcal{A} such that x \in U
      PROOF: By \langle 1 \rangle 3 and \langle 2 \rangle 1
   \langle 2 \rangle 3. Pick y \in (x, b] such that [x, y) \subseteq U
      Proof: By Lemma 4.1.2.
   \langle 2 \rangle 4. PICK V \in \mathcal{A} such that y \in V
      PROOF: By \langle 1 \rangle 3 and \langle 2 \rangle 3.
   \langle 2 \rangle 5. [x,y] is covered by \{U,V\}
      PROOF: By \langle 2 \rangle 3 and \langle 2 \rangle 4.
\langle 1 \rangle5. Let: C = \{ y \in (a, b] : [a, y] \text{ is covered by a finite subset of } \mathcal{A} \}
\langle 1 \rangle 6. C is nonempty
   Proof: By \langle 1 \rangle 4.
\langle 1 \rangle 7. Let: c = \sup C
   PROOF: By \langle 1 \rangle 1.
\langle 1 \rangle 8. \ c \in C
   \langle 2 \rangle 1. PICK U \in \mathcal{A} such that c \in U
   \langle 2 \rangle 2. Pick y \in [a, c) such that (y, c] \subseteq U
```

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Proof: By Lemma 4.1.2
   \langle 2 \rangle 3. Pick z such that y < z and z \in C
      PROOF: This exists because y is not an upper bound for C.
   \langle 2 \rangle 4. PICK a finite \mathcal{A}_0 \subseteq \mathcal{A} such that [a, z] is covered by \mathcal{A}_0
   \langle 2 \rangle 5. [a, c] is covered by \mathcal{A}_0 \cup \{U\}
\langle 1 \rangle 9. \ c = b
   \langle 2 \rangle 1. Assume: for a contradiction c < b
   \langle 2 \rangle 2. Pick y \in (c, b] such that [c, y] is covered by at most two elements of A.
      Proof: By \langle 1 \rangle 4
   \langle 2 \rangle 3. \ y > c \text{ and } y \in C
   \langle 2 \rangle 4. Q.E.D.
      PROOF: This contradicts \langle 1 \rangle 7.
\langle 1 \rangle 10. Q.E.D.
Corollary 9.4.19.1. Every closed interval in \mathbb{R} is compact.
Corollary 9.4.19.2 (CC). S_{\Omega} is limit point compact.
Proof:
\langle 1 \rangle 1. Let: A be an infinite subset of S_{\Omega}
\langle 1 \rangle 2. Pick a countably infinite subset B \subseteq A
\langle 1 \rangle 3. Let: b = \sup B
\langle 1 \rangle 4. B \subseteq [0, b]
\langle 1 \rangle 5. [0, b] is compact
  PROOF: By the theorem.
\langle 1 \rangle 6. B has a limit point in [0, b]
\langle 1 \rangle 7. A has a limit point in [0, b]
Corollary 9.4.19.3. The ordered square is compact.
Corollary 9.4.19.4. The ordered square is limit point compact.
Corollary 9.4.19.5. Not every subspace of a compact space is compact.
PROOF: [0,1] is compact but (0,1) is not. \square
Theorem 9.4.20 (Extreme Value Theorem). Let f: X \to Y be continuous
where Y is a linearly ordered set in the order topology. If X is compact, then
there exist c, d \in X such that, for all x \in X, we have f(c) \leq f(x) \leq f(d).
PROOF:
\langle 1 \rangle 1. f(X) is compact.
  Proof: By Proposition 9.4.10.
\langle 1 \rangle 2. f(X) has a greatest element.
   \langle 2 \rangle 1. Assume: for a contradiction f(X) has no greatest element.
   \langle 2 \rangle 2. \{(-\infty, f(x)) : x \in X\} is a set of open sets that covers f(X).
   \langle 2 \rangle 3. PICK a finite subset \{(-\infty, f(x_1)), \dots, (-\infty, f(x_n))\} that covers f(X).
```

Proof: By Proposition 9.4.5

```
\langle 2 \rangle 4. Let: f(x_N) be largest out of f(x_1), \ldots, f(x_n)
   \langle 2 \rangle 5. f(x_N) < f(x_N)
   \langle 2 \rangle 6. Q.E.D.
      PROOF: This is a contradiction.
\langle 1 \rangle 3. f(X) has a least element.
   PROOF: Similar.
Theorem 9.4.21 (DC). A nonempty compact Hausdorff space with no isolated
points is uncountable.
Proof:
\langle 1 \rangle 1. Let: X be a nonempty compact Hausdorff space with no isolated points.
\langle 1 \rangle 2. For every nonempty open U \subseteq X and point x \in X, there exists a
        nonempty open V \subseteq U such that x \notin \overline{V}
   \langle 2 \rangle 1. Let: U \subseteq X be nonempty and open and x \in X
   \langle 2 \rangle 2. PICK y \in U such that y \neq x
      PROOF: This is possible because U \neq \{x\} since x is not an isolated point.
   \langle 2 \rangle 3. PICK disjoint open neighbourhoods W_1 and W_2 of x and y
      Proof: Since X is Hausdorff
   \langle 2 \rangle 4. Let: V = U \cap W_2
   \langle 2 \rangle 5. \ x \notin \overline{V}
      PROOF: We have \overline{V} \subseteq \overline{W_2} \subseteq X \setminus W_1.
\langle 1 \rangle 3. Let: f: \mathbb{Z}^+ \to X
        Prove: f is not surjective
\langle 1 \rangle 4. PICK a sequence of open sets V_1 \supseteq V_2 \supseteq \cdots such that f(n) \notin \overline{V_n}
   PROOF: By \langle 1 \rangle 2 and Dependent Choice.
\langle 1 \rangle 5. Pick a point b \in \bigcap_{i=1}^{\infty} \overline{V_i}
   PROOF: By Proposition 9.4.15.
\langle 1 \rangle 6. b \neq f(n) for all n
   PROOF: For each n we have b \in \overline{V_n} (\langle 1 \rangle 5) and f(n) \notin \overline{V_n} (\langle 1 \rangle 4).
```

#### Corollary 9.4.21.1. Every closed interval in $\mathbb{R}$ is uncountable.

**Theorem 9.4.22.** Every compact space is limit point compact.

#### PROOF:

- $\langle 1 \rangle 1$ . Let: X be a compact space.
- $\langle 1 \rangle$ 2. Let:  $A \subseteq X$  be a set with no limit points. Prove: A is finite.

 $\langle 1 \rangle 3$ . A is closed.

PROOF: By Corollary 3.10.3.1.

 $\langle 1 \rangle 4$ . A is compact.

Proof: By Proposition 9.4.6.

- $\langle 1 \rangle$ 5. {U open in  $X: U \cap A$  is a singleton} covers A
  - $\langle 2 \rangle 1$ . Let:  $a \in A$

 $\langle 2 \rangle 2$ . PICK an open neighbourhood U of a such that U does not intersect A at a point other than a PROOF: One must exist because a is not a limit point of A ( $\langle 1 \rangle 2$ ).

 $\langle 2 \rangle 3$ .  $U \cap A = \{a\}$ 

 $\langle 1 \rangle 6$ . PICK a finite subcover  $\{U_1, \ldots, U_n\}$ 

PROOF: By  $\langle 1 \rangle 4$  using Proposition 9.4.5.  $\langle 1 \rangle 7$ . For  $1 \leq i \leq n$ ,

LET:  $U_i \cap A = \{a_i\}$  $\langle 1 \rangle 8. \ A = \{a_1, \dots, a_n\}$ 

**Proposition 9.4.23.** Let X be a space and  $C, D \subseteq X$  be compact. Then  $C \cup D$  is compact.

#### Proof:

- $\langle 1 \rangle 1$ . Let:  $\mathcal{A}$  be a set of open sets that covers  $C \cup D$
- $\langle 1 \rangle$ 2. PICK a finite subset  $A_1$  that covers C and a finite subset  $A_2$  that covers D.
- $\langle 1 \rangle 3$ .  $A_1 \cup A_2$  is a finite subset of A that covers  $C \cup D$ .
- $\langle 1 \rangle 4$ . Q.E.D.

**Theorem 9.4.24.** Every compact Hausdorff space is normal.

#### Proof:

- $\langle 1 \rangle 1$ . Let: X be a compact Hausdorff space.
- $\langle 1 \rangle 2$ . Let: A and B be disjoint closed sets in X.
- $\langle 1 \rangle 3$ .  $\{ U \text{ open in } X : \exists V \text{ open in } X.B \subseteq V \land U \cap V = \emptyset \} \text{ covers } A$ 
  - $\langle 2 \rangle 1$ . B is compact

PROOF: By Proposition 9.4.6.

 $\langle 2 \rangle 2$ . Q.E.D.

Proof: By Lemma 9.4.8.

 $\langle 1 \rangle 4$ . PICK a finite subcover  $\{U_1, \ldots, U_n\}$ 

Proof: A is compact by Proposition 9.4.6.

- $\langle 1 \rangle 5$ . For  $1 \leq i \leq n$ , PICK  $V_i$  open in X such that  $B \subseteq V_i$  and  $U_i \cap V_i = \emptyset$
- $\langle 1 \rangle 6$ . Let:  $U = U_1 \cup \cdots \cup U_n$  and  $V = V_1 \cap \cdots \cap V_n$
- $\langle 1 \rangle 7$ . U and V are disjoint open sets,  $A \subseteq U$  and  $B \subseteq V$

Corollary 9.4.24.1. The ordered square is normal.

**Proposition 9.4.25.** Not every compact Hausdorff space is first countable.

PROOF: The space  $\overline{S_{\Omega}}$  is compact Hausdorff but not first countable.  $\square$ 

Corollary 9.4.25.1. Not every compact Hausdorff space is second countable.

**Theorem 9.4.26** (Tychonoff (AC)). The product of a family of compact spaces is compact.

Proof:

```
\begin{array}{ll} \langle 1 \rangle 1. \ \ \mathrm{Let:} \ \ \{X_{\alpha}\}_{\alpha \in J} \ \mathrm{be} \ \mathrm{a} \ \mathrm{family} \ \mathrm{of} \ \mathrm{compact} \ \mathrm{spaces}. \\ \mathrm{Let:} \ \ X = \prod_{\alpha \in J} X_{\alpha} \\ \langle 1 \rangle 2. \ \ \mathrm{Let:} \ \ \mathcal{A} \subseteq \mathcal{P}X \ \mathrm{satisfy} \ \mathrm{the} \ \mathrm{finite} \ \mathrm{intersection} \ \mathrm{property}. \\ \mathrm{Prove:} \ \ \bigcap_{A \in \mathcal{A}} \overline{A} \ \mathrm{is} \ \mathrm{nonempty}. \end{array}
```

 $\langle 1 \rangle$ 3. Pick a set  $\mathcal{D} \subseteq \mathcal{P}X$  that includes  $\mathcal{A}$  and is maximal with respect to the finite intersection property.

Proof: By Lemma 1.6.6.

```
\langle 1 \rangle 4. For \alpha \in J, PICK x_{\alpha} \in \bigcap_{D \in \mathcal{D}} \overline{\pi_{\alpha}(D)}
```

 $\langle 2 \rangle 1$ . Let:  $\alpha \in J$ 

 $\langle 2 \rangle 2$ .  $\{ \overline{\pi_{\alpha}(D)} : D \in \mathcal{D} \}$  satisfies the finite intersection property.

 $\langle 2 \rangle 3$ . Q.E.D.

Proof: By Proposition 9.4.15

 $\langle 1 \rangle 5$ . Let:  $x = (x_{\alpha})_{\alpha \in J}$ 

 $\langle 1 \rangle 6$ . For all  $D \in \mathcal{D}$  we have  $(x_{\alpha})_{\alpha \in J} \in \overline{D}$ 

PROOF:

 $\langle 2 \rangle 1$ . Every subbasis element containing x intersects every member of  $\mathcal{D}$ 

- $\langle 3 \rangle 1$ . Let:  $\pi_{\alpha}(U)^{-1}$  be a subbasis element containing x where U is open in  $X_{\alpha}$
- $\langle 3 \rangle 2$ . Let:  $D \in \mathcal{D}$
- $\langle 3 \rangle 3$ . U intersects  $\pi_{\alpha}(D)$
- $\langle 2 \rangle$ 2. Every subbasis element containing x is a member of  $\mathcal{D}$  Proof: By Lemma 1.6.8
- $\langle 2 \rangle$ 3. Every basis element containing x is a member of  $\mathcal{D}$  PROOF: By Lemma 1.6.7
- $\langle 2 \rangle$ 4. Every basis element containing x intersects every member of  $\mathcal{D}$  PROOF: This follows because  $\mathcal{D}$  satisfies the finite intersection property.  $\langle 1 \rangle$ 7. Q.E.D.

PROOF: By Proposition 9.4.15

#### Proof:

- $\langle 1 \rangle 1$ . Let:  $\{X_{\alpha}\}_{{\alpha} \in J}$  be a family of compact spaces and  $X = \prod_{{\alpha} \in J} X_{\alpha}$ .
- $\langle 1 \rangle 2$ . Pick a well-ordering  $\langle$  of J such that J has a greatest element  $\top$
- $\langle 1 \rangle$ 3. For all  $\alpha \in J$  and every family of points  $p = \{p_i \in X_i\}_{i \leq \alpha}$ , Let:  $Y_{\alpha}(p) = \{x \in X : \forall i \leq \alpha. x_i = p_i\}$
- $\langle 1 \rangle 4$ . For all  $\beta \in J$  and every family of points  $p = \{p_i \in X_i\}_{i < \beta}$ , Let:  $Z_{\beta}(p) = \bigcap_{\alpha < \beta} Y_{\alpha} = \{x \in X : \forall i < \beta. x_i = p_i\}$
- $\langle 1 \rangle$ 5. Given  $\beta \in J$ , a family of points  $\{p_i \in X_i\}_{i < \beta}$ , and a finite set  $\mathcal{A}$  of basis elements that covers  $Z_{\beta}(p)$ , there exists  $\alpha < \beta$  such that  $\mathcal{A}$  covers  $Y_{\alpha}(p)$ 
  - $\langle 2 \rangle 1.$  Assume: ( w.l.o.g.  $\beta$  has no immediate predecessor)
  - $\langle 2 \rangle 2$ . For  $A \in \mathcal{A}$ , LET:  $J_A = \{i < \beta : \pi_i(A) \neq X_i\}$
  - $\langle 2 \rangle$ 3. Let:  $\alpha$  be the largest element of  $\bigcup_{A \in \mathcal{A}} J_A$ Proof: The set has a greatest element because each  $J_A$  is finite and  $\mathcal{A}$  is finite.

```
\langle 2 \rangle 4. \mathcal{A} covers Y_{\alpha}(p)

\langle 3 \rangle 1. Let: x \in Y_{\alpha}(p)

\langle 3 \rangle 2. Let: y \in Z_{\beta}(p) be the point with
```

$$y_i = \begin{cases} p_i & \text{if } i < \beta \\ x_i & \text{if } i \ge \beta \end{cases}$$

 $\langle 3 \rangle 3$ . PICK  $A \in \mathcal{A}$  such that  $y \in A$ 

 $\langle 3 \rangle 4. \ x \in A$ 

 $\langle 4 \rangle 1$ . For  $i \leq \alpha$  we have  $x_i \in \pi_i(A)$ 

 $\langle 5 \rangle 1. \ x_i = p_i$ 

PROOF: From  $\langle 3 \rangle 1$  and  $\langle 1 \rangle 3$ .

 $\langle 5 \rangle 2. \ y_i = p_i$ 

Proof: From  $\langle 3 \rangle 2$ 

 $\langle 5 \rangle 3. \ y_i \in \pi_i(A)$ 

PROOF: From  $\langle 3 \rangle 3$ .

 $\langle 4 \rangle 2$ . For  $\alpha < i < \beta$  we have  $x_i \in \pi_i(A)$ 

 $\langle 5 \rangle 1. \ i \notin J_A$ 

Proof: From  $\langle 2 \rangle 3$ 

 $\langle 5 \rangle 2$ .  $\pi_i(A) = X_i$ 

Proof: From  $\langle 2 \rangle 2$ 

 $\langle 4 \rangle 3$ . For  $i \geq \beta$  we have  $x_i \in \pi_i(A)$ 

 $\langle 5 \rangle 1. \ x_i = y_i$ 

Proof: By  $\langle 3 \rangle 2$ 

 $\langle 5 \rangle 2. \ y_i \in \pi_i(A)$ 

Proof: By  $\langle 3 \rangle 3$ 

- (1)6. Assume: for a contradiction  $\mathcal A$  is a set of basis elements such that no finite subset covers X
- $\langle 1 \rangle$ 7. For all  $\alpha \in J$  there exists a family of points  $\{p_i \in X_i\}_{i \leq \alpha}$  such that no finite subset of  $\mathcal{A}$  covers  $Y_{\alpha}(p)$ 
  - $\langle 2 \rangle 1$ . Assume: as induction hypothesis  $\beta \in J$  and  $p_i$  has been chosen for all  $i < \beta$  such that, for all  $\alpha < \beta$ , no finite subset of  $\mathcal{A}$  covers  $Y_{\alpha}(p)$

 $\langle 2 \rangle 2$ . No finite subset of  $\mathcal{A}$  covers  $Z_{\beta}(p)$ 

Proof: By  $\langle 1 \rangle 5$ 

(2)3. PICK  $p_{\beta} \in X_{\beta}$  such that no finite subset of  $\mathcal{A}$  covers  $Z_{\beta}(p) \times \{p_{\beta}\} = Y_{\beta}(p)$ 

Proof: By Lemma 9.4.13.

 $\langle 1 \rangle 8$ . Q.E.D.

PROOF: This is a contradiction since  $Y_{\top}(p) = \{p\}$  and so must be covered by a single element of A.

**Theorem 9.4.27.** In a compact Hausdorff space, the components and the quasicomponents coincide.

#### Proof:

 $\langle 1 \rangle 1$ . Let: X be a compact Hausdorff space and  $x, y \in X$  lie in the same

quasicomponent.

PROVE: x and y are in the same component.

- $\langle 1 \rangle$ 2. Let:  $\mathcal{A}$  be the set of all closed subspaces A of X such that x and y lie in the same quasicomponent of A.
- $\langle 1 \rangle 3$ . Every chain in  $\mathcal{A}$  has a lower bound.
  - $\langle 2 \rangle 1$ . Let:  $\mathcal{B} \subseteq \mathcal{A}$  be a chain

Prove:  $Y = \bigcap \mathcal{B} \in \mathcal{A}$ 

- $\langle 2 \rangle 2.$  Assume: for a contradiction  $Y=C \cup D$  were C and D are disjoint and open in  $Y, \ x \in C$  and  $y \in D$
- $\langle 2 \rangle$ 3. PICK disjoint open sets U and V in X such that  $C \subseteq U$  and  $D \subseteq V$  PROOF: By Lemma 9.4.18.
- $\langle 2 \rangle 4$ .  $\{B \setminus (U \cup V) : B \in \mathcal{B}\}$  satisfies the finite intersection property.
  - $\langle 3 \rangle 1$ . Let:  $B_1, \ldots, B_n \in \mathcal{B}$
  - $\langle 3 \rangle 2. \ B_1 \cap \cdots \cap B_n \in \mathcal{B}$

Proof: By  $\langle 2 \rangle 1$ .

 $\langle 3 \rangle 3$ .  $B_1 \cap \cdots \cap B_n \setminus (U \cap V)$  is nonempty

PROOF:  $B_1 \cap \cdots \cap B_n \cap U$  and  $B_1 \cap \cdots \cap B_n \cap V$  cannot be disjoint, because x and y are in the same quasicomponent of  $B_1 \cap \cdots \cap B_n$ .

 $\langle 2 \rangle 5$ .  $Y \setminus (U \cup V)$  is nonempty.

Proof: By Proposition 9.4.15.

 $\langle 2 \rangle 6$ . Q.E.D.

PROOF: This is a contradiction since  $Y \setminus (U \cup V) = Y \setminus (C \cup D)$ .

 $\langle 1 \rangle 4$ . Pick a minimal element  $D \in \mathcal{A}$ 

PROOF: One exists by Zorn's Lemma.

- $\langle 1 \rangle 5$ . D is connected.
  - $\langle 2 \rangle 1$ . Assume: [

for a contradiction  $D = U \uplus V$  is a separation of D

 $\langle 2 \rangle 2$ . Case:  $x, y \in U$ 

PROOF: In this case we have  $U \in \mathcal{A}$  contradicting the minimality of D.

 $\langle 2 \rangle 3$ . Case:  $x \in U, y \in V$ 

PROOF: This is a contradiction because x and y are in the same quasicomponent of D.

 $\langle 2 \rangle 4$ . Case:  $x \in V, y \in U$ 

Proof: Similar to  $\langle 2 \rangle 3$ .

 $\langle 2 \rangle 5$ . Case:  $x, y \in V$ 

Proof: Similar to  $\langle 2 \rangle 2$ .

### 9.5 Perfect Maps

**Proposition 9.5.1.** Let  $p: X \to Y$  be a closed continuous surjective map. For all  $y \in Y$  and U an open neighbourhood of  $p^{-1}(y)$ , there exists an open neighbourhood W of y such that  $p^{-1}(W) \subseteq U$ .

PROOF: Take  $W = Y \setminus p(X \setminus U)$ .  $\square$ 

**Proposition 9.5.2** (AC). Let  $p: X \rightarrow Y$  be a closed continuous surjective map. If X is normal then Y is normal.

#### Proof:

- $\langle 1 \rangle 1$ . Let:  $A, B \subseteq Y$  be closed
- $\langle 1 \rangle 2$ .  $p^{-1}(A)$ ,  $p^{-1}(B)$  are closed in X.
- $\langle 1 \rangle 3$ . PICK disjoint open sets U, V of  $p^{-1}(A), p^{-1}(B)$  respectively.
- (1)4. For all  $a \in A$ , PICK an open neighbourhood  $W_a$  of a such that  $p^{-1}(W_a) \subseteq U$

Proof: By Proposition 9.5.1.

 $\langle 1 \rangle$ 5. For all  $b \in B$ , PICK an open neighbourhood  $W_b'$  of b such that  $p^{-1}(W_b') \subseteq V$ 

Proof: By Proposition 9.5.1.

- $\langle 1 \rangle 6$ . Let:  $W = \bigcup_{a \in A} W_a$  and  $W' = \bigcup_{b \in B} W'_b$
- $\langle 1 \rangle 7. \ W \cap W' = \emptyset$

PROOF: This holds because  $p^{-1}(W) \subseteq U$ ,  $p^{-1}(W') \subseteq V$ , and p is surjective.

**Definition 9.5.3** (Perfect Map). Let X and Y be topological spaces and  $p: X \to Y$ . Then p is *perfect* iff p is closed, continuous, surjective, and  $p^{-1}(y)$  is compact for all  $y \in Y$ .

**Proposition 9.5.4.** Let  $p: X \to Y$  be a perfect map. If X is Hausdorff then so is Y.

#### Proof:

- $\langle 1 \rangle 1$ . Let:  $a, b \in Y$  with  $a \neq b$
- $\langle 1 \rangle 2$ . PICK disjoint open neighbourhoods U and V of  $\pi^{-1}(a)$  and  $\pi^{-1}(b)$ , respectively.

PROOF: By Lemma 9.4.18.

 $\langle 1 \rangle 3$ . Pick open neighbourhoods W and W' of a and b such that  $\pi^{-1}(W) \subseteq U$  and  $\pi^{-1}(W') \subseteq V$ 

Proof: By Proposition 9.5.1.

 $\langle 1 \rangle 4$ . W and W' are disjoint.

**Proposition 9.5.5.** Let p: X woheadrightarrow Y be perfect. If X is regular then so is Y.

#### Proof:

 $\langle 1 \rangle 1$ . Y is  $T_1$ 

Proof: By Proposition 9.5.4.

- $\langle 1 \rangle 2$ . Let:  $C \subseteq Y$  be closed and  $a \in Y \setminus C$
- $\langle 1 \rangle 3. \ p^{-1}(C)$  is closed and  $p^{-1}(a)$  is disjoint from  $p^{-1}(C)$ .
- $\langle 1 \rangle 4$ . PICK disjoint open neighbourhoods U, V of  $p^{-1}(C), p^{-1}(a)$  respectively. PROOF: By Lemma 9.4.8.
- $\langle 1 \rangle$ 5. PICK an open neighbourhood W' of a such that  $p^{-1}(W') \subseteq V$  PROOF: By Proposition 9.5.1.
- $\langle 1 \rangle 6$ . For  $c \in C$ , PICK an open neighbourhood  $W_c$  such that  $p^{-1}(W_c) \subseteq U$

```
Proof: By Proposition 9.5.1.
\langle 1 \rangle 7. W = \bigcup_{c \in C} W_c is an open neighbourhood of C disjoint from W'
Proposition 9.5.6 (AC). Let p: X \rightarrow Y be perfect. If X is locally compact
then so is Y.
Proof:
\langle 1 \rangle 1. Let: b \in Y
\langle 1 \rangle 2. {U open in X : \exists C \subseteq X \text{ compact.} U \subseteq C} covers p^{-1}(b)
\langle 1 \rangle 3. PICK a finite subcover \{U_1, \ldots, U_n\}
\langle 1 \rangle 4. For 1 \leq i \leq n, PICK a compact C_i \subseteq X such that U_i \subseteq C_i
\langle 1 \rangle5. For 1 \leq i \leq n, PICK a neighbourhood W_i of b such that p^{-1}(W_i) \subseteq U_i
   Proof: By Proposition 9.5.1
\langle 1 \rangle 6. \ b \in W_1 \cup \cdots \cup W_n \subseteq p(C_1) \cup \cdots \cup p(C_n)
\langle 1 \rangle 7. p(C_1) \cup \cdots \cup p(C_n) is compact.
   \langle 2 \rangle 1. Each p(C_i) is compact.
      Proof: By Proposition 9.4.10.
   \langle 2 \rangle 2. Q.E.D.
```

### 9.6 Sequential Compactness

PROOF: By Proposition 9.4.23.

**Definition 9.6.1** (Sequentially Compact). A space is *sequentially compact* iff every sequence has a convergent subsequence.

**Proposition 9.6.2.**  $\overline{S_{\Omega}}$  is not sequentially compact.

PROOF:  $\Omega$  is a limit point of  $S_{\Omega}$  but is not the limit of any sequence of points in  $S_{\Omega}$ .  $\square$ 

### 9.7 Local Compactness

**Definition 9.7.1** (Local Compactness). Let X be a topological space.

For  $x \in X$ , the space X is *locally compact* at x iff there exists a compact subspace  $C \subseteq X$  that includes a neighbourhood of x.

The space X is *locally compact* iff it is locally compact at every point.

**Proposition 9.7.2.** Every complete linearly ordered set is locally compact under the order topology.

#### PROOF:

 $\langle 1 \rangle 1$ . Let: L be a complete linearly ordered set and  $x \in L$ 

PROVE: There exists a compact subspace  $C \subseteq L$  that includes a neighbourhood U of x

 $\langle 1 \rangle 2$ . Case: x is least and greatest in L

```
PROOF: In this case, L = \{x\} is compact.
\langle 1 \rangle 3. Case: x is least in L but not greatest
   \langle 2 \rangle 1. Pick a < x
   \langle 2 \rangle 2. Take C = [a, x] and U = (a, x]
\langle 1 \rangle 4. Case: x is greatest in L but not least
   PROOF: Similar.
\langle 1 \rangle 5. Case: x is neither least nor greatest
   \langle 2 \rangle 1. Pick a < x and b > x
   \langle 2 \rangle 2. Take C = [a, b] and U = (a, b)
П
Corollary 9.7.2.1. For every ordinal \alpha, the space S_{\alpha} is locally compact.
Theorem 9.7.3. Every closed subspace of a locally compact Hausdorff space is
locally\ compact.
Proof:
\langle 1 \rangle 1. Let: X be locally compact Hausdorff and C \subseteq X be closed.
\langle 1 \rangle 2. Let: x \in C
\langle 1 \rangle 3. PICK D \subseteq X compact and U \subseteq D open such that x \in U
\langle 1 \rangle 4. D is closed.
   Proof: Proposition 9.4.9.
\langle 1 \rangle 5. C \cap D is closed
   Proof: Proposition 3.5.5.
\langle 1 \rangle 6. C \cap D is compact
   Proof: Proposition 9.4.6.
\langle 1 \rangle 7. Q.E.D.
   PROOF: C \cap D \subseteq C is compact and includes the open neighbourhood U \cap C
   of x.
Proposition 9.7.4. Let \{X_{\alpha}\}_{{\alpha}\in J} be a family of nonempty topological spaces.
If \prod_{\alpha \in I} X_{\alpha} is locally compact, then each X_{\alpha} is locally compact.
Proof:
\langle 1 \rangle 1. Let: \alpha \in J and x_{\alpha} \in X_{\alpha}
\langle 1 \rangle 2. Pick x_{\beta} \in X_{\beta} for all \beta \in J \setminus \{\alpha\}
\langle 1 \rangle 3. PICK a compact subspace C \subseteq \prod_{\alpha \in I} X_{\alpha} that a neighbourhood U of x
        included in C
\langle 1 \rangle 4. PICK a basic open set \prod_{\alpha \in J} U_{\alpha} such that x \in \prod_{\alpha \in J} U_{\alpha} \subseteq U
\langle 1 \rangle 5. \ x_{\alpha} \in U_{\alpha} \subseteq \pi_{\alpha}(C)
\langle 1 \rangle 6. \pi_{\alpha}(C) is compact.
   Proof: By Proposition 9.4.10.
```

Corollary 9.7.4.1. The Sorgenfrey plane is not locally compact.

**Proposition 9.7.5.** Let  $\{X_{\alpha}\}_{{\alpha}\in J}$  be a family of locally compact spaces such that  $X_{\alpha}$  is compact for all but finitely many values of  $\alpha$ . Then  $\prod_{{\alpha}\in J} X_{\alpha}$  is locally compact.

Proof:
$\langle 1 \rangle 1$ . Assume: $X_{\alpha}$ is compact if $\alpha \neq \alpha_1, \ldots, \alpha_n$
$\langle 1 \rangle 2$ . Let: $\vec{x} \in \prod_{\alpha \in J} X_{\alpha}$
$\langle 1 \rangle 3$ . For $1 \leq i \leq n$ , Pick $C_{\alpha_i} \subseteq X_{\alpha_i}$ compact and $U_{\alpha_i}$ open such that $x_{\alpha_i} \in \mathcal{C}$
$U_{\alpha_i} \subseteq C_{\alpha_i}$
$\langle 1 \rangle 4$ . For $\alpha \neq \alpha_1, \ldots, \alpha_n$ ,
LET: $C_{\alpha} = U_{\alpha} = X_{\alpha}$
$\langle 1 \rangle 5. \ \vec{x} \in \prod_{\alpha \in J} U_{\alpha} \subseteq \prod_{\alpha \in J} C_{\alpha}$
/1/6. $\Pi = C$ is compact
$\langle 1 \rangle 6$ . $\prod_{\alpha \in J} C_{\alpha}$ is compact
PROOF: By Tychonoff's Theorem.
<b>Proposition 9.7.6.</b> $\mathbb{R}_l$ is not locally compact.
PROOF: $[0, +\infty)$ can be partitioned into infinitely many disjoint open sets, which
therefore do not have a finite subcover. $\Box$
thorotoro do not have a mino papeover.
<b>Proposition 9.7.7.</b> Let $\{X_{\alpha}\}_{{\alpha}\in J}$ be a family of nonempty topological spaces.
If $\prod_{\alpha \in J} X_{\alpha}$ is locally compact, then all but finitely many of the $X_{\alpha}$ are compact.
Proof:
$\langle 1 \rangle 1$ . PICK a point $a = (a_{\alpha})_{\alpha \in J} \in \prod_{\alpha \in J} X_{\alpha}$
$\langle 1 \rangle 2$ . PICK a compact $C \subseteq \prod_{\alpha \in J} X_{\alpha}$ that includes the basic neighbourhood
$\prod_{\alpha \in J} U_{\alpha}$ of a, where $U_{\alpha} = X_{\alpha}$ for all $\alpha$ except $\alpha = \alpha_1, \ldots, \alpha_n$
$\langle 1 \rangle 3$ . For $\alpha \neq \alpha_1, \ldots, \alpha_n$ , we have $X_{\alpha}$ is compact.
PROOF: $X_{\alpha}$ is homeomorphic to a closed subspace of $C$ .
Corollary 9.7.7.1. For any infinite set $I$ , the space $\mathbb{R}^{I}$ is not locally compact.
<b>Proposition 9.7.8.</b> $[0,1]^{\omega}$ is not compact under the uniform topology.
PROOF: $\{a_i : i \geq 0\}$ is an infinite set with no limit point, where $a_i$ is the point
with ith component 1 and all other components 0. $\square$
Corollary 9.7.8.1. $\mathbb{R}^{\omega}$ under the uniform topology is not locally compact.
Proof:
$\langle 1 \rangle 1$ . Assume: $\mathbb{R}^{\omega}$ is locally compact
$\langle 1 \rangle$ 2. Let: $C$ be a compact subspace such that $B(\vec{0}, \epsilon) \subseteq C$
$\langle 1 \rangle 3. \ B(\vec{0}, \epsilon)$ is compact.
$\langle 1 \rangle 4$ . Q.E.D.
PROOF: This contradicts the proposition.
Proposition 9.7.9. Not every subspace of a locally compact Hausdorff space is
locally compact.
cocarry correpaids.
PROOF: $\mathbb{R}$ is locally compact Hausdoff, $\mathbb{Q}$ is not locally compact. $\square$

**Proposition 9.7.10.** The continuous image of a locally compact Hausdorff space is not necessarily locally compact.

#### Proof:

```
\langle 1 \rangle 1. Let: \{q_0, q_1, \ldots\} be an enumeration of [0, 1] \cap \mathbb{Q}.
```

- $\langle 1 \rangle 2$ . Define  $f: (0, +\infty) \setminus \mathbb{Z} \to [0, 1] \cap \mathbb{Q}$  by:  $f(x) = q_n$  for  $x \in (n, n+1)$
- $\langle 1 \rangle 3$ . f is continuous.

Proof: The inverse image of any set is a union of open intervals.  $\Box$ 

### 9.8 Compactifications

**Definition 9.8.1** (Compactification). Let X and Y be spaces. Then Y is a compactification of X iff Y is a compact Hausdorff space and X is a subspace of Y with  $\overline{X} = Y$ .

Two compcactifications  $Y_1$ ,  $Y_2$  of X are equivalent iff there exists a homeomorphism between  $Y_1$  and  $Y_2$  that is the identity on X.

**Lemma 9.8.2.** Let  $h: X \to Z$  be an imbedding. Then there exists a compactification  $c: X \to Y$  of X, unique up to equivalence, and an imbedding  $i: Y \to Z$  such that  $h = i \circ c$ .

PROOF: Simply take Y to be the closure of X in Z.  $\square$ 

**Definition 9.8.3** (One-Point Compactification). A *one-point compactification* of X is a compactification Y of X such that  $Y \setminus X$  is a singleton.

**Theorem 9.8.4.** Let X be a topological space. Then X is locally compact Hausdorff if and only if there exists a space Y such that:

- 1. X is a subspace of Y
- 2. The set  $Y \setminus X$  is a singleton.
- 3. Y is a compact Hausdorff space.

If Y and Y' are two spaces satisfying these conditions, then there exists a unique homeomorphism between Y and Y' that is the identity on X.

#### Proof:

- $\langle 1 \rangle 1$ . If X is locally compact Hausdorff then there exists a space Y satisfying 1–3.
  - $\langle 2 \rangle 1$ . Let:  $Y = X \cup \{\infty\}$  under the topology  $\mathcal{T} = \{U \subseteq X : U \text{ is open in } X\} \cup \{Y \setminus C : C \subseteq X \text{ is compact}\}.$ 
    - $\langle 3 \rangle 1. \ Y \in \mathcal{T}$

PROOF: This holds because  $Y = Y \setminus \emptyset$ .

- $\langle 3 \rangle 2$ . For all  $U, V \in \mathcal{T}$  we have  $U \cap V \in \mathcal{T}$ .
  - $\langle 4 \rangle 1$ . Let:  $U, V \in \mathcal{T}$
  - $\langle 4 \rangle 2$ . Case: U, V are open in X

PROOF: In this case,  $U \cap V$  is open in X.

- $\langle 4 \rangle 3$ . Case: U is open in  $X, V = Y \setminus C$  where  $C \subseteq X$  is compact.
  - $\langle 5 \rangle 1. \ U \cap V = U \setminus C$
  - $\langle 5 \rangle 2$ . C is closed in X

Proof: Proposition 9.4.9.

- $\langle 5 \rangle 3$ .  $U \cap V$  is open in X
- $\langle 4 \rangle 4.$  Case:  $U = Y \setminus C$  where  $C \subseteq X$  is compact, V is open in X. Proof: Similar.
- $\langle 4 \rangle$ 5. Case:  $U = Y \setminus C$ ,  $V = Y \setminus D$  where  $C, D \subseteq X$  are compact.
  - $\langle 5 \rangle 1. \ U \cap V = Y \setminus (C \cup D)$
  - $\langle 5 \rangle 2$ . C and D are closed in X

Proof: Proposition 9.4.9.

 $\langle 5 \rangle 3$ .  $C \cup D$  is closed in X

Proof: Proposition 3.5.4.

 $\langle 5 \rangle 4$ .  $C \cup D$  is compact.

Proof: By Proposition 9.4.23.  $\square$ 

- $\langle 3 \rangle 3$ . For all  $\mathcal{A} \subseteq \mathcal{T}$  we have  $\bigcup \mathcal{A} \in \mathcal{T}$ .
  - $\langle 4 \rangle 1$ . Let:  $\mathcal{A} \subseteq \mathcal{T}$
  - $\langle 4 \rangle 2$ . Case: Every element of  $\mathcal{A}$  is an open set in X.

PROOF: In this case,  $\bigcup A$  is open in X.

- $\langle 4 \rangle$ 3. Case: There exists C compact in X such that  $Y \setminus C \in \mathcal{A}$ 
  - $\langle 5 \rangle 1. \ \bigcup \mathcal{A} = Y \backslash (\bigcap \{D \subseteq X : D \text{ compact}, Y \backslash D \in \mathcal{A} \} \backslash \bigcup \{U \text{ open in } X : U \in \mathcal{A} \})$

PROOF: Set theory.

 $\langle 5 \rangle 2$ .  $\bigcap \{D \subseteq X : D \text{ compact}, Y \setminus D \in \mathcal{A}\} \setminus \bigcup \{U \text{ open in } X : U \in \mathcal{A}\}$  is compact.

PROOF: It is a closed subset of the compact set C.

- $\langle 2 \rangle 2$ . X is a subspace of Y
  - $\langle 3 \rangle 1.$  For every open set U of X, there exists V open in Y such that  $U=V\cap X$

PROOF: Take V = U.

- $\langle 3 \rangle 2$ . For every open set V in Y, we have  $V \cap X$  is open in X.
  - $\langle 4 \rangle 1$ . Let: V be open in Y
  - $\langle 4 \rangle 2$ . Case: V is open in X

PROOF: In this case,  $V \cap X = V$ .

- $\langle 4 \rangle 3$ . Case:  $V = Y \setminus C$  where  $C \subseteq X$  is compact.
  - $\langle 5 \rangle 1$ . C is closed in X.

PROOF: By Proposition 9.4.9.

 $\langle 5 \rangle 2. \ V \cap X = X \setminus C$ 

- $\langle 2 \rangle 3. \ Y \setminus X = \{\infty\}$
- $\langle 2 \rangle 4$ . Y is compact.
  - $\langle 3 \rangle 1$ . Let:  $\mathcal{A}$  be an open covering of Y
  - $\langle 3 \rangle 2$ . PICK  $U \in \mathcal{A}$  such that  $\infty \in U$
  - $\langle 3 \rangle 3$ . Pick  $C \subseteq X$  compact such that  $U = Y \setminus C$ .
  - $\langle 3 \rangle 4. \{ V \cap X : V \in \mathcal{A} \}$  is set of open sets that covers C
  - $\langle 3 \rangle$ 5. PICK a finite subset  $\{V_1, \ldots, V_n\}$  such that  $\{V_1 \cap X, \ldots, V_n \cap X\}$

covers C.

- $\langle 3 \rangle 6. \{U, V_1, \dots, V_n\}$  is a finite subcover of Y.
- $\langle 2 \rangle 5$ . Y is Hausdorff.
  - $\langle 3 \rangle 1$ . Let:  $x, y \in Y$  with  $x \neq y$

Prove: There exist disjoint open neighbourhoods U, V of x and y.

 $\langle 3 \rangle 2$ . Case:  $x, y \in X$ 

PROOF: In this case, we just use the fact that X is Hausdorff.

- $\langle 3 \rangle 3$ . Case:  $x = \infty, y \in X$ 
  - $\langle 4 \rangle 1.$  Pick  $C \subseteq X$  compact such that C includes an open neighbourhood V of y
  - $\langle 4 \rangle 2$ . Let:  $U = Y \setminus C$
- $\langle 3 \rangle 4$ . Case:  $x \in X$ ,  $y = \infty$

PROOF: Simlar.

- $\langle 1 \rangle 2.$  If there exists a space Y satisfying 1–3 then X is locally compact Hausdorff.
  - $\langle 2 \rangle 1$ . Let: Y be a space satisfying 1–3
  - $\langle 2 \rangle 2$ . Let:  $\infty$  be the point in  $Y \setminus X$
  - $\langle 2 \rangle 3$ . X is locally compact
    - $\langle 3 \rangle 1$ . Let:  $x \in X$
    - $\langle 3 \rangle 2$ . Pick disjoint open neighbourhoods U of x and V of  $\infty$
    - $\langle 3 \rangle$ 3.  $X \setminus V$  is compact and includes UPROOF:  $X \setminus V = Y \setminus V$  is compact because it is a closed subset of Y (Proposition 9.4.6).
  - $\langle 2 \rangle 4$ . X is Hausdorff.

PROOF: By Corollary 6.2.6.1.

- $\langle 1 \rangle$ 3. If Y and Y' are two spaces satisfying 1–3 then there exists a unique homemorphism between Y and Y' that is the identity on X.
  - $\langle 2 \rangle 1$ . Let: Y and Y' be two spaces that satisfy 1–3.
  - $\langle 2 \rangle 2$ . Let:  $Y \setminus X = \{p\}$  and  $Y' \setminus X = \{q\}$
  - $\langle 2 \rangle 3$ . Let:  $h: Y \to Y'$  be given by

$$h(x) = x (x \in X)$$
  
$$h(p) = q$$

- $\langle 2 \rangle 4$ . h is a homeomorphism
  - $\langle 3 \rangle 1$ . h is bijective.
  - $\langle 3 \rangle 2$ . h is continuous.
    - $\langle 4 \rangle$ 1. Let:  $V \subseteq Y'$  be open. PROVE:  $h^{-1}(V)$  is open.
    - $\langle 4 \rangle 2$ . Case:  $V \subseteq X$ 
      - $\langle 5 \rangle 1. \ h^{-1}(V) = V$
      - $\langle 5 \rangle 2$ . V is open in X

PROOF: Condition 1 for Y'.

 $\langle 5 \rangle 3$ . V is open in Y

PROOF: Condition 1 for Y.

- $\langle 4 \rangle 3$ . Case:  $q \in V$ 
  - $\langle 5 \rangle 1. \ Y' \setminus V$  is compact.

Proof: Proposition 9.4.6.

 $\langle 5 \rangle 2$ .  $Y' \setminus V$  is closed in Y.

Proof: Proposition 9.4.9.

$$\langle 5 \rangle 3. \ h^{-1}(V) = Y \setminus (Y' \setminus V)$$

 $\langle 3 \rangle 3$ .  $h^{-1}$  is continuous.

PROOF: Similar.

 $\langle 2 \rangle$ 5. If  $h': Y \to Y'$  is a homeomorphism such that  $h' \upharpoonright_X = \mathrm{id}_X$  then h' = h

**Theorem 9.8.5.** Let X be a Hausdorff space. Then X is locally compact if and only if, for all  $x \in X$  and any neighbourhood U of x, there exists an open neighbourhood V of x such that  $\overline{V}$  is compact and  $\overline{V} \subseteq U$ .

#### Proof:

- $\langle 1 \rangle 1$ . If X is locally compact then, for all  $x \in X$  and any neighbourhood U of x, there exists an open neighbourhood V of x such that  $\overline{V}$  is compact and  $\overline{V} \subset U$ .
  - $\langle 2 \rangle 1$ . Assume: X is locally compact.
  - $\langle 2 \rangle 2$ . Let:  $x \in X$  and U be a neighbourhood of x.
  - $\langle 2 \rangle$ 3. Let: Y be the one-point compactification of X.

Proof: By Theorem 9.8.4.

- $\langle 2 \rangle 4$ . Let:  $C = Y \setminus U$
- $\langle 2 \rangle$ 5. C is compact

Proof: By Proposition 9.4.6.

 $\langle 2 \rangle$ 6. PICK disjoint open sets V, W containing x and C

Proof: Lemma 9.4.8

 $\langle 2 \rangle 7$ . V is open in X

PROOF:  $V \subseteq X$  since  $\infty \in W$ .

- $\langle 2 \rangle 8$ . The closure of V in X is compact
  - $\langle 3 \rangle 1$ . The closure of V is X is the same as the closure of V in Y.

PROOF: The point  $\infty$  cannot be a limit point of V since W is a neighbourhood disjoint from V.

 $\langle 3 \rangle 2$ . The closure of V in Y is compact.

PROOF: By Proposition 9.4.6.

 $\langle 2 \rangle 9. \ \overline{V} \subset U$ 

Proof:

$$\overline{V} \subseteq Y \setminus W$$
$$\subseteq Y \setminus C$$
$$= U$$

- $\langle 1 \rangle 2$ . If, for all  $x \in X$  and any neighbourhood U of x, there exists an open neighbourhood V of x such that  $\overline{V}$  is compact and  $\overline{V} \subseteq U$ , then X is locally compact.
  - $\langle 2\rangle 1.$  Assume: for all  $x\in X$  and any neighbourhood U of x, there exists an open neighbourhood V of x such that  $\overline{V}$  is compact and  $\overline{V}\subseteq U$
  - $\langle 2 \rangle 2$ . Let:  $x \in X$

PROVE: There exists  $C \subseteq X$  compact such that C includes a neighbourhood U of x

 $\langle 2 \rangle 3$ . PICK an open neighbourhood V of x such that  $\overline{V}$  is compact and  $\overline{V} \subseteq X$ 

 $\langle 2 \rangle 4$ . Take  $C = \overline{V}$  and U = V

**Corollary 9.8.5.1.** Every open subspace of a locally compact Hausdorff space is locally compact.

**Corollary 9.8.5.2.** A space is locally compact Hausdorff if and only if it is an open subspace of a compact Hausdorff space.

Corollary 9.8.5.3. Every locally compact Hausdorff space is completely regular.

Corollary 9.8.5.4. The space  $\mathbb{R}_K$  is not locally compact.

**Lemma 9.8.6** (AC). If  $p: X \to Y$  is a quotient map and Z is a locally compact Hausdorff space, then the map

$$\pi = p \times \mathrm{id}_Z : X \times Z \to Y \times Z$$

is a quotient map.

#### Proof:

 $\langle 1 \rangle 1$ .  $\pi$  is surjective.

PROOF: This holds because p is surjective.

 $\langle 1 \rangle 2$ .  $\pi$  is continuous.

PROOF: By Theorem 5.2.15.

- $\langle 1 \rangle 3$ . For  $A \subseteq Y \times Z$ , if  $\pi^{-1}(A)$  is open in  $X \times Z$  then A is open in  $Y \times Z$ .
  - $\langle 2 \rangle 1$ . Let:  $A \subseteq Y \times Z$
  - $\langle 2 \rangle 2$ . Assume:  $\pi^{-1}(A)$  is open in  $X \times Z$
  - $\langle 2 \rangle 3$ . Let:  $(y,z) \in A$
  - $\langle 2 \rangle 4$ . PICK  $x \in X$  such that p(x) = y

Proof: Since p is surjective.

 $\langle 2 \rangle$ 5. PICK open sets  $U_1$ , V with  $\overline{V}$  compact such that  $(x,y) \in U_1 \times V$  and  $U_1 \times \overline{V} \subseteq \pi^{-1}(A)$ 

PROOF: Using Theorem 9.8.5

- (2)6. PICK a sequence of open sets  $U_1, U_2, \ldots$  in X such that  $p^{-1}(p(U_n)) \subseteq U_{n+1}$  and  $U_n \times \overline{V} \subseteq \pi^{-1}(A)$  for all n
  - $\langle 3 \rangle$ 1. Let: U be open with  $U \times \overline{V} \subseteq \pi^{-1}(A)$ PROVE: There exists W open with  $p^{-1}(p(U)) \subseteq W$  and  $W \times \overline{V} \subseteq \pi^{-1}(A)$
  - $\langle 3 \rangle 2$ . For all  $x \in p^{-1}(p(U))$ , PICK open sets  $U_x$ ,  $V_x$  such that  $x \in U_x$ ,  $\overline{V} \subseteq V_x$  and  $U_x \times V_x \subseteq \pi^{-1}(A)$

PROOF: By the Tube Lemma.

- $\langle 3 \rangle 3$ . Let:  $W = \bigcup_{x \in p^{-1}(p(U))} U_x$
- $\langle 2 \rangle 7$ . Let:  $U = \bigcup_{n=1}^{\infty} U_n$
- $\langle 2 \rangle 8$ . U is saturated with respect to p

```
⟨3⟩1. Let: a \in U, b \in X, p(a) = p(b) ⟨3⟩2. Pick n such that a \in U_n ⟨3⟩3. b \in p^{-1}(p(U_n)) ⟨3⟩4. b \in U_{n+1} ⟨3⟩5. b \in U ⟨2⟩9. p(U) is open in Y Proof: By Lemma 4.5.2. ⟨2⟩10. (y,z) \in p(U) \times V \subseteq A ⟨2⟩11. Q.E.D. Proof: By Proposition 3.2.3.
```

**Theorem 9.8.7.** Let  $p:A \to B$  and  $q:C \to D$  be quotient maps. If B and C are locally compact Hausdorff spaces, then  $p \times q:A \times C \to B \times D$  is a quotient map.

PROOF: This holds by Lemma 9.8.6 and Proposition 4.5.10 because  $p \times q = (\mathrm{id}_B \times q) \circ (p \times \mathrm{id}_C)$ .  $\square$ 

**Theorem 9.8.8.** Let X be a completely regular space. Let Y be a compactification of X such that every bounded continuous map  $X \to \mathbb{R}$  extends uniquely to a continuous map  $Y \to \mathbb{R}$ . Then, for every compact Hausdorff space C, every continuous map  $X \to C$  extends uniquely to a continuous map  $Y \to C$ .

#### Proof:

П

- $\langle 1 \rangle 1$ . Let: C be a compact Hausdorff space and  $f: X \to C$  a continuous function
- $\langle 1 \rangle 2$ . Pick a set J and an imedding  $C \subseteq [0,1]^J$ 
  - $\langle 2 \rangle 1$ . C is normal

PROOF: By Lemma 9.4.18

 $\langle 2 \rangle 2$ . Q.E.D.

PROOF: By Theorem 6.4.6.

 $\langle 1 \rangle 3$ . For  $\alpha \in J$ ,

Let:  $g_{\alpha}: Y \to \mathbb{R}$  be the unique continuous extension of  $\pi_{\alpha} \circ f$ 

- $\langle 1 \rangle 4$ . Define  $g: Y \to \mathbb{R}^J$  by  $g(y)_\alpha = g_\alpha(y)$
- $\langle 1 \rangle 5$ . g is continuous

PROOF: By Theorem 5.2.15.

- $\langle 1 \rangle 6$ . g extends f
- $\langle 1 \rangle 7$ . We have  $g: Y \to C$

Proof:

$$g(Y) = g(\overline{X})$$

$$\subseteq \overline{g(X)} \qquad (Theorem 5.2.2)$$

$$= \overline{f(X)} \qquad (\langle 1 \rangle 6)$$

$$\subseteq \overline{C}$$

$$= C \qquad (Proposition 9.4.9)$$

 $\langle 1 \rangle 8$ . g is unique

```
\langle 2 \rangle 1. Let: h: Y \to C be a continuous extension of f \langle 2 \rangle 2. For all \alpha \in J, \pi_{\alpha} \circ h extends \pi_{\alpha} \circ f \langle 2 \rangle 3. For all \alpha \in J, \pi_{\alpha} \circ h = g_{\alpha} Proof: By \langle 1 \rangle 3 \langle 2 \rangle 4. h = g Proof: By \langle 1 \rangle 4
```

**Corollary 9.8.8.1.** Let X be a completely regular space. Let  $Y_1$  and  $Y_2$  be compactifications of X such that every bounded continuous map  $X \to \mathbb{R}$  extends uniquely to a continuous map  $Y_i \to \mathbb{R}$ . Then  $Y_1$  and  $Y_2$  are equivalent.

**Definition 9.8.9** (Stone-Čech Compactification). Let X be a completely regular space. The *Stone-Čech compactification* of X,  $\beta(X)$ , is the compactification of X such that, for every compact Hausdorff space C, every continuous function  $X \to C$  extends uniquely to a continuous function  $\beta(X) \to C$ .

## Chapter 10

# Metric Spaces

### 10.1 The Metric Topology

**Definition 10.1.1** (Metric). A *metric* on a set X is a function  $d: X \times X \to \mathbb{R}$  such that, for all  $x, y, z \in X$ :

- 1.  $d(x,y) \ge 0$ ;
- 2. d(x,y) = 0 if and only if x = y;
- 3. d(x,y) = d(y,x);
- 4. Triangle Inequality

$$d(x,z) \le d(x,y) + d(y,z)$$

A metric space X consists of a set X and a metric on X. We call d(x,y) the distance between x and y.

**Definition 10.1.2** (Open Ball). Let X be a metric space with metric  $d, x \in X$  and  $\epsilon > 0$ . The *open ball* with *centre* x and *radius*  $\epsilon$  is

$$B_d(x,\epsilon) = \{ y \in X : d(x,y) < \epsilon \} .$$

**Lemma 10.1.3.** Let X be a metric space,  $x, y \in X$  and  $\epsilon > 0$ . If  $y \in B(x, \epsilon)$ , then there exists  $\delta$  such that  $0 < \delta < \epsilon$  and

$$B(y, \delta) \subseteq B(x, \epsilon)$$
.

PROOF:

- $\langle 1 \rangle 1$ . Let:  $\delta = \epsilon d(x, y)$
- $\langle 1 \rangle 2$ . Let:  $z \in B(y, \delta)$
- $\langle 1 \rangle 3. \ d(x,z) < \epsilon$

Proof:

$$\begin{aligned} d(x,z) & \leq d(x,y) + d(y,z) & \text{(Triangle Inequality)} \\ & < d(x,y) + \delta & \text{($\langle 1 \rangle 2$)} \\ & = \epsilon & \text{($\langle 1 \rangle 1$)} \end{aligned}$$

**Definition 10.1.4** (Metric Topology). Let d be a metric on X. The *metric topology* on X induced by d is the topology generated by the basis consisting of the open balls.

We prove this is a topology.

#### Proof:

 $\langle 1 \rangle 1$ . Every point is in an open ball.

PROOF:  $x \in B(x, 1)$ 

- $\langle 1 \rangle 2$ . If  $B_1$ ,  $B_2$  are open balls and  $x \in B_1 \cap B_2$ , then there exists an open ball  $B_3$  such that  $x \in B_3 \subseteq B_1 \cap B_2$ .
  - $\langle 2 \rangle 1$ . Let:  $x \in B(y, \epsilon_1) \cap B(z, \epsilon_2)$
  - $\langle 2 \rangle 2$ . PICK  $\delta_1$ ,  $\delta_2$  such that  $0 < \delta_1 < \epsilon_1$ ,  $0 < \delta_2 < \epsilon_2$ ,  $B(x, \delta_1) \subseteq B(y, \epsilon_1)$  and  $B(x, \delta_2) \subseteq B(z, \epsilon_2)$ .

Proof: Lemma 10.1.3.

- $\langle 2 \rangle 3$ . Let:  $\delta = \min(\delta_1, \delta_2)$
- $\langle 2 \rangle 4. \ x \in B(x, \delta) \subseteq B(y, \epsilon_1) \cap B(y, \epsilon_2)$
- $\langle 1 \rangle 3$ . Q.E.D.

Proof: Lemma 3.4.3.

**Lemma 10.1.5.** A set U is open in the metric topology induced by d if and only if, for all  $x \in U$ , there exists  $\epsilon > 0$  such that  $B(x, \epsilon) \subseteq U$ .

#### Proof:

- $\langle 1 \rangle 1$ . If U is open then, for all  $x \in U$ , there exists  $\epsilon > 0$  such that  $B(x, \epsilon) \subseteq U$ .
  - $\langle 2 \rangle 1$ . Assume: U is open.
  - $\langle 2 \rangle 2$ . Let:  $x \in U$
  - $\langle 2 \rangle 3$ . Pick  $B(y, \delta)$  such that  $x \in B(y, \delta) \subseteq U$
  - $\langle 2 \rangle 4$ . PICK  $\epsilon$  such that  $0 < \epsilon < \delta$  and  $B(x, \epsilon) \subseteq B(y, \delta)$

PROOF: Lemma 10.1.3.

 $\langle 2 \rangle 5. \ B(x, \epsilon) \subseteq U$ 

PROOF: From  $\langle 2 \rangle 3$  and  $\langle 2 \rangle 4$ .

 $\langle 1 \rangle 2$ . If, for all  $x \in U$ , there exists  $\epsilon > 0$  such that  $B(x, \epsilon) \subseteq U$ , then U is open. PROOF: Immediate from definition of metric topology.

**Lemma 10.1.6.** Let d and d' be two metrics on the set X. Let  $\mathcal{T}$  and  $\mathcal{T}'$  be the topologies the induce, respectively. Then  $\mathcal{T} \subseteq \mathcal{T}'$  if and only if, for all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$ .

#### Proof:

 $\langle 1 \rangle 1$ . If  $\mathcal{T} \subseteq \mathcal{T}'$  then, for all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $B_{d'}(x,\delta) \subseteq B_d(x,\epsilon)$ .

 $\langle 2 \rangle 1$ . Assume:  $\mathcal{T} \subseteq \mathcal{T}'$  $\langle 2 \rangle 2$ . Let:  $x \in X$  and  $\epsilon > 0$  $\langle 2 \rangle 3. \ B_d(x, \epsilon) \in \mathcal{T}'$ PROOF: From  $\langle 2 \rangle 1$ .  $\langle 2 \rangle 4$ . There exists  $\delta > 0$  such that  $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$ Proof: By Lemma 10.1.5.  $\langle 1 \rangle 2$ . If, for all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $B_{d'}(x,\delta) \subseteq$  $B_d(x,\epsilon)$ , then  $\mathcal{T} \subseteq \mathcal{T}'$  $\langle 2 \rangle$ 1. Assume: For all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $B_{d'}(x,\delta) \subseteq B_d(x,\epsilon).$  $\langle 2 \rangle 2$ . Let:  $U \in \mathcal{T}$ Prove:  $U \in \mathcal{T}'$  $\langle 2 \rangle 3$ . Let:  $x \in U$  $\langle 2 \rangle 4$ . Pick  $\epsilon > 0$  be such that  $B_d(x, \epsilon) \subseteq U$ Proof: By Lemma 10.1.5.  $\langle 2 \rangle$ 5. Pick  $\delta > 0$  such that  $B_{d'}(x,\delta) \subseteq B_d(x,\epsilon)$ Proof: By  $\langle 2 \rangle 1$ .  $\langle 2 \rangle 6. \ B_{d'}(x,\delta) \subseteq U$ PROOF: By  $\langle 2 \rangle 4$  and  $\langle 2 \rangle 5$ .  $\langle 2 \rangle 7$ . Q.E.D.

**Definition 10.1.7** (Metrizable). A topological space is *metrizable* if and only if there exists a metric that induces its topology.

Lemma 10.1.8. Every discrete space is metrizable.

PROOF: The discrete topology is induced by the metric d(x,y) = 1 if  $x \neq y$ , 0 if x = y.  $\square$ 

**Proposition 10.1.9.** The continuous image of a metrizable space is not necessarily metrizable.

PROOF: The identity map from the discrete two-point space to the indiscrete two-point space is continuous.  $\Box$ 

**Lemma 10.1.10.**  $\mathbb{R}$  *is metrizable.* 

Proof: By Lemma 10.1.5.

PROOF: The standard topology is induced by the metric d(x,y) = |x-y|.  $\square$ 

**Definition 10.1.11** (Bounded). Let X be a metric space and  $A \subseteq X$ . Then A is bounded iff  $\{d(x,y): x,y \in A\}$  is bounded above, in which case its diameter is

$$\operatorname{diam} A = \sup_{x,y \in A} d(x,y) .$$

**Lemma 10.1.12.** Let (X,d) be a metric space and  $A \subseteq X$ . Then  $d \upharpoonright_{A \times A}$  is a metric on A that induces the subspace topology.

 $\langle 1 \rangle 1$ .  $d \upharpoonright_{A \times A}$  is a metric on A.

PROOF: Each of the axioms for a metric follows immediately from the same axiom for d.

 $\langle 1 \rangle 2$ . The topology induced by  $d \upharpoonright_{A \times A}$  is the product topology.

PROOF: Both are the topology generated by the basis consisting of all the open balls  $B_{d\restriction_{A\times A}}(a,\epsilon)=B_d(a,\epsilon)\cap A$ .

П

Lemma 10.1.13. Every metric space is Hausdorff.

Proof:

 $\langle 1 \rangle 1$ . Let: X be a metric space and  $x, y \in X$  with  $x \neq y$ .

 $\langle 1 \rangle 2$ . Let:  $\epsilon = d(x, y)$ 

 $\langle 1 \rangle$ 3.  $B(x, \epsilon/2)$  and  $B(y, \epsilon/2)$  are disjoint neighbourhoods of x and y.

Theorem 10.1.14. Every metric space is first countable.

PROOF:  $\{B(x,q): q \in \mathbb{Q}^+\}$  is a local basis at x.  $\square$ 

Corollary 10.1.14.1. If J is infinite then the space  $\mathbb{R}^J$  is not metrizable.

**Definition 10.1.15** (Standard Bounded Metric). Let d be a metric on X. The standard bounded metric corresponding to d is

$$\overline{d}(x,y) = \min(d(x,y),1) .$$

We prove this is a metric.

PROOF:

 $\langle 1 \rangle 1$ .  $\overline{d}(x,y) > 0$ 

PROOF: This holds because  $d(x,y) \ge 0$  (d is a metric) and 1 > 0.

 $\langle 1 \rangle 2$ .  $\overline{d}(x,y) = 0$  iff x = y

PROOF: Immediate from definition.

 $\langle 1 \rangle 3. \ \overline{d}(x,y) = \overline{d}(y,x)$ 

PROOF: Immediate from definition.

 $\langle 1 \rangle 4. \ \overline{d}(x,z) \leq \overline{d}(x,y) + \overline{d}(y,z)$ 

 $\langle 2 \rangle 1$ . Case:  $d(x,y) \leq 1$ ,  $d(y,z) \leq 1$ 

Proof:

$$\overline{d}(x,z) \le d(x,z)$$

$$\le d(x,y) + d(y,z)$$

$$= \overline{d}(x,y) + \overline{d}(y,z)$$

 $\langle 2 \rangle 2$ . Case: d(y, z) > 1Proof:

$$\begin{split} \overline{d}(x,z) &\leq 1 \\ &\leq \overline{d}(x,y) + 1 \\ &= \overline{d}(x,y) + \overline{d}(y,z) \end{split}$$

```
\langle 2 \rangle 3. Case: d(x,y) > 1
        PROOF: Similar.
Theorem 10.1.16. Let d be a metric on X. Then the standard bounded metric
d corresponding to d induces the same topology as d.
\langle 1 \rangle 1. Let: \mathcal{T} be the topology induced by d and \mathcal{T}' be the topology induced by
\langle 1 \rangle 2. \mathcal{T} \subseteq \mathcal{T}'
    \langle 2 \rangle 1. Let: x \in X and \epsilon > 0
    \langle 2 \rangle 2. Let: \delta = \min(\epsilon, 1/2)
    \langle 2 \rangle 3. \ B_{\overline{d}}(x,\delta) \subseteq B_d(x,\epsilon)
        \langle 3 \rangle 1. Let: y \in B_{\overline{d}}(x, \delta)
        \langle 3 \rangle 2. \overline{d}(x,y) < \delta
        \langle 3 \rangle 3. \ \overline{d}(x,y) < 1
            PROOF: From \langle 2 \rangle 2 and \langle 3 \rangle 2.
        \langle 3 \rangle 4. \ d(x,y) = d(x,y)
            PROOF: From \langle 3 \rangle 3 and the definition of \overline{d}.
        \langle 3 \rangle 5. \ d(x,y) < \epsilon
            PROOF: By \langle 2 \rangle 2 and \langle 3 \rangle 2 and \langle 3 \rangle 4.
\langle 1 \rangle 3. \ \mathcal{T}' \subseteq \mathcal{T}
    \langle 2 \rangle 1. Let: x \in X and \epsilon > 0
    \langle 2 \rangle 2. B_d(x, \epsilon) \subseteq B_{\overline{d}}(x, \epsilon)
```

**Definition 10.1.17** (Square Metric). The square metric on  $\mathbb{R}^n$  is defined by

$$\rho(\vec{x}, \vec{y}) = \max(|x_1 - y_1|, \dots, |x_n - y_n|)$$
.

We prove this is a metric.

```
Proof:
```

```
\langle 1 \rangle 1. \ \rho(\vec{x}, \vec{y}) \ge 0
```

PROOF: Immediate from definitions.

$$\langle 1 \rangle 2$$
.  $\rho(\vec{x}, \vec{y}) = 0$  iff  $\vec{x} = \vec{y}$ 

PROOF: Immediate from definitions.

$$\langle 1 \rangle 3. \ \rho(\vec{x}, \vec{y}) = \rho(\vec{y}, \vec{x})$$

PROOF: Immediate from definitions.

$$\langle 1 \rangle 4. \ \rho(\vec{x}, \vec{z}) \le \rho(\vec{x}, \vec{y}) + \rho(\vec{y}, \vec{z})$$

 $\langle 2 \rangle 1$ . For all i, we have  $|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$ 

PROOF: This holds because  $\overline{d}(x,y) \leq d(x,y)$ .

- $\langle 2 \rangle 2$ . For all i,  $|x_i z_i| \le \rho(\vec{x}, \vec{y}) + \rho(\vec{y}, \vec{z})$
- $\langle 2 \rangle 3. \ \rho(\vec{x}, \vec{z}) \le \rho(\vec{x}, \vec{y}) + \rho(\vec{y}, \vec{z})$

**Theorem 10.1.18.** The square metric induces the standard topology on  $\mathbb{R}^n$ .

- $\langle 1 \rangle 1$ . Let:  $\mathcal{T}_{\rho}$  be the topology induced by the square metric and  $\mathcal{T}_s$  the standard topology.
- $\langle 1 \rangle 2$ .  $\mathcal{T}_{\rho} \subseteq \mathcal{T}_{s}$

PROOF: This holds because  $B_{\rho}(\vec{x}, \epsilon) = (x_1 - \epsilon, x_1 + \epsilon) \times \cdots \times (x_n - \epsilon, x_n + \epsilon)$ .

- $\langle 1 \rangle 3. \ \mathcal{T}_s \subseteq \mathcal{T}_\rho$ 
  - $\langle 2 \rangle 1$ . Let:  $B = U_1 \times \cdots \times U_n$  be a basic open set in  $\mathcal{T}_s$ , where each  $U_i$  is open in  $\mathbb{R}$ .
  - $\langle 2 \rangle 2$ . Let:  $\vec{x} \in B$
  - $\langle 2 \rangle 3$ . For  $1 \leq i \leq n$ , PICK  $\epsilon_i > 0$  such that  $(x_i \epsilon_i, x_i + \epsilon_i) \subseteq U_i$
  - $\langle 2 \rangle 4$ . Let:  $\epsilon = \min(\epsilon_1, \dots, \epsilon_n)$
- $\langle 2 \rangle 5. \ B_{\rho}(\vec{x}, \epsilon) \subseteq B$

**Lemma 10.1.19.** The product of a countable family of metrizable spaces is metrizable.

PROOF:

- $\langle 1 \rangle 1$ . Let:  $\{X_n\}_{n \in \mathbb{Z}^+}$  be a family of metric spaces with metrics bounded by 1,  $X = \prod_{n=1}^{\infty} X_n$ .
- $\langle 1 \rangle 2$ . Let:  $D: X \times X \to \mathbb{R}$  be given by

$$D(\vec{x}, \vec{y}) = \sup_{n>1} \frac{d(x_n, y_n)}{n} .$$

- $\langle 1 \rangle 3$ . D is a metric on X.
  - $\langle 2 \rangle 1$ .  $D(\vec{x}, \vec{y}) \geq 0$

PROOF: Immediate from definitions.

 $\langle 2 \rangle 2$ .  $D(\vec{x}, \vec{y}) = 0$  iff  $\vec{x} = \vec{y}$ 

PROOF: Immediate from definitions.

 $\langle 2 \rangle 3. \ D(\vec{x}, \vec{y}) = D(\vec{y}, \vec{x})$ 

Proof: Immediate from definitions.

- $\begin{array}{l} \langle 2 \rangle 4. \ D(\vec{x},\vec{z}) \leq D(\vec{x},\vec{y}) + D(\vec{y},\vec{z}) \\ \langle 3 \rangle 1. \ \text{For all } n, \text{ we have } \frac{d(x_n,z_n)}{n} \leq \frac{d(x_n,y_n)}{n} + \frac{d(y_n,z_n)}{n} \\ \langle 3 \rangle 2. \ \text{For all } n, \text{ we have } \frac{d(x_n,z_n)}{n} \leq D(\vec{x},\vec{y}) + D(\vec{y},\vec{z}) \end{array}$ 

  - $\langle 3 \rangle 3. \ D(\vec{x}, \vec{z}) \leq D(\vec{x}, \vec{y}) + D(\vec{y}, \vec{z})$
- $\langle 1 \rangle 4$ . Let:  $\mathcal{T}_D$  be the topology induced by D and  $\mathcal{T}_p$  the product topology.
- $\langle 1 \rangle 5$ .  $\mathcal{T}_D \subseteq \mathcal{T}_p$ 
  - $\langle 2 \rangle 1$ . Let:  $U \in \mathcal{T}_D$

PROVE:  $U \in \mathcal{T}_p$ 

- $\langle 2 \rangle 2$ . Let:  $\vec{x} \in U$
- $\langle 2 \rangle 3$ . Pick  $\epsilon > 0$  such that  $B_D(\vec{x}, \epsilon) \subseteq U$
- $\langle 2 \rangle 4$ . Pick N such that  $1/N < \epsilon$
- $\langle 2 \rangle 5$ . Let:  $V = B(x_1, \epsilon) \times \cdots \times B(x_N, \epsilon) \times X_{N+1} \times X_{N+2} \times \cdots$
- $\langle 2 \rangle 6. \ \vec{x} \in V \subseteq B_D(\vec{x}, \epsilon)$
- $\langle 1 \rangle 6. \ \mathcal{T}_p \subseteq \mathcal{T}_D$ 
  - $\langle 2 \rangle 1$ . Let:  $U = \prod_{n=1}^{\infty} U_n$  be a basic open set in  $\mathcal{T}_p$ , where each  $U_n$  is open in  $X_n$ , and  $U_n = X_n$  for n > N.

```
 \langle 2 \rangle 2. \text{ Let: } \vec{x} \in U  Prove: There exists \epsilon > 0 such that B_D(\vec{x}, \epsilon) \subseteq U.  \langle 2 \rangle 3. \text{ For } n \leq N, \text{ Pick } \epsilon_n > 0 \text{ such that } B(x_n, \epsilon_n) \subseteq U_n   \langle 2 \rangle 4. \text{ Let: } \epsilon = \min(\epsilon_1, \epsilon_2/2, \dots, \epsilon_n/n)   \langle 2 \rangle 5. \text{ Let: } \vec{y} \in B_D(\vec{x}, \epsilon)   \langle 2 \rangle 6. \text{ For } n \leq N, y_n \in U_n   \langle 3 \rangle 1. D(\vec{x}, \vec{y}) < \epsilon   \langle 3 \rangle 2. d(x_n, y_n)/n < \epsilon   \langle 3 \rangle 3. d(x_n, y_n)/n < \epsilon_n/n   \langle 3 \rangle 4. \text{ Q.E.D.}  Proof: By \langle 2 \rangle 3.
```

Corollary 10.1.19.1. The space  $\mathbb{R}^{\omega}$  is metrizable.

**Definition 10.1.20** (Uniform Metric). Let J be a set. The *uniform metric*  $\overline{\rho}$  on  $\mathbb{R}^J$  is defined by

$$\overline{\rho}(\vec{x}, \vec{y}) = \sup_{\alpha \in J} \overline{d}(x_{\alpha}, y_{\alpha}) .$$

where  $\overline{d}$  is the standard bounded metric on  $\mathbb{R}$ . The uniform topology is the topology induced by the uniform metric.

We prove this is a metric.

### Proof:

 $\langle 1 \rangle 1. \ \overline{\rho}(\vec{x}, \vec{y}) > 0$ 

PROOF: Immediate from definitions.

 $\langle 1 \rangle 2$ .  $\overline{\rho}(\vec{x}, \vec{y}) = 0$  iff  $\vec{x} = \vec{y}$ 

PROOF: Immediate from definitions.

 $\langle 1 \rangle 3. \ \overline{\rho}(\vec{x}, \vec{y}) = \overline{\rho}(\vec{y}, \vec{x})$ 

Proof: Immediate from definitions.

 $\langle 1 \rangle 4. \ \overline{\rho}(\vec{x}, \vec{z}) \leq \overline{\rho}(\vec{x}, \vec{y}) + \overline{\rho}(\vec{y}, \vec{z})$ 

Proof:

- $\langle 2 \rangle 1$ . For all  $\alpha \in J$ ,  $\overline{d}(x_{\alpha}, z_{\alpha}) \leq \overline{d}(x_{\alpha}, y_{\alpha}) + \overline{d}(y_{\alpha}, z_{\alpha})$
- $\langle 2 \rangle 2$ . For all  $\alpha \in J$ ,  $\overline{d}(x_{\alpha}, z_{\alpha}) \leq \overline{\rho}(\vec{x}, \vec{y}) + \overline{\rho}(\vec{y}, \vec{z})$
- $\langle 2 \rangle 3. \ \overline{\rho}(\vec{x}, \vec{z}) \le \overline{\rho}(\vec{x}, \vec{y}) + \overline{\rho}(\vec{y}, \vec{z})$

**Theorem 10.1.21** (DC). The uniform topology on  $\mathbb{R}^J$  is finer than the product topology and coarser than the box topology. These three topologies are different iff J is infinite.

### Proof:

- $\langle 1 \rangle 1$ . The uniform topology is finer than the product topology.
  - $\langle 2 \rangle$ 1. Let:  $B = \prod_{\alpha \in J} U_{\alpha}$  be a basic open set in the product topology, where each  $U_{\alpha}$  is open in  $\mathbb{R}$ , and  $U_{\alpha} = \mathbb{R}$  except for  $\alpha = \alpha_1, \ldots, \alpha_n$ .
  - $\langle 2 \rangle 2$  Let  $\vec{r} \in U$
  - $\langle 2 \rangle 3$ . For  $1 \leq i \leq n$ , Pick  $0 < \epsilon_i < 1$  such that  $(x_{\alpha_i} \epsilon_i, x_{\alpha_i} + \epsilon_i) \subseteq U_{\alpha_i}$ .

```
\langle 2 \rangle 5. \ B_{\overline{\rho}}(\vec{x}, \epsilon) \subseteq B
        \langle 3 \rangle 1. Let: \vec{y} \in B_{\overline{\rho}}(\vec{x}, \epsilon)
        \langle 3 \rangle 2. For 1 \leq i \leq n, we have y_i \in U_{\alpha_i}
             \langle 4 \rangle 1. Let: 1 \le i \le n
            \langle 4 \rangle 2. \ \overline{d}(x_{\alpha_i}, y_{\alpha_i}) < \epsilon_i
                PROOF: From \langle 2 \rangle 4 and \langle 3 \rangle 1.
            \langle 4 \rangle 3. \ d(x_{\alpha_i}, y_{\alpha_i}) < \epsilon_i
                PROOF: From \langle 4 \rangle 2 since \epsilon_i < 1 (\langle 2 \rangle 3).
            \langle 4 \rangle 4. Q.E.D.
                Proof: By \langle 2 \rangle 3.
\langle 1 \rangle 2. The uniform topology is coarser than the box topology.
    \langle 2 \rangle 1. Let: \vec{x} \in \mathbb{R}^J and \epsilon > 0
               PROVE: B_{\overline{o}}(\vec{x}, \epsilon) is open in the box topology.
    \langle 2 \rangle 2. Case: \epsilon < 1
        PROOF: In this case, B(\vec{x}, \epsilon) = \prod_{\alpha \in I} (x_{\alpha} - \epsilon, x_{\alpha} + \epsilon).
    \langle 2 \rangle 3. Case: \epsilon \geq 1
        PROOF: In this case, B(\vec{x}, \epsilon) = \mathbb{R}^J.
\langle 1 \rangle 3. If J is finite then the product topology is the same as the box topology.
    Proof: Immediate from definitions.
\langle 1 \rangle 4. If J is infinite then the uniform topology is distinct from the product
          topology.
    \langle 2 \rangle 1. B(\vec{0}, 1/2) is not open in the product topology.
        \langle 3 \rangle 1. \ \vec{0} \in B(\vec{0}, 1/2)
        \langle 3 \rangle 2. Let: \prod_{\alpha \in J} U_{\alpha} be any basic open set containing \vec{0}, where U_{\alpha} is open
                             in \mathbb{R} for all \alpha, and U_{\alpha} = \mathbb{R} except for \alpha = \alpha_1, \ldots, \alpha_n
        \langle 3 \rangle 3. Pick \alpha_0 \in J such that \alpha_0 \neq \alpha_1, \ldots, \alpha_n
        \langle 3 \rangle 4. Let: \vec{x} be such that x_{\alpha_0} = 1, and x_{\alpha} = 0 for \alpha \neq \alpha_0.
        \langle 3 \rangle 5. \ \vec{x} \in \prod_{\alpha \in J} U_{\alpha}
        \langle 3 \rangle 6. \ \vec{x} \notin B(\vec{0}, 1/2)
\langle 1 \rangle 5. If J is infinite then the uniform topology is distinct from the box topology.
    \langle 2 \rangle 1. Pick a countable sequence \alpha_1, \alpha_2, \ldots in J
    \langle 2 \rangle 2. Let: U = \prod_{\alpha \in J} U_{\alpha}, where U_{\alpha_n} = (-1/n, 1/n) for all n, and U_{\alpha} = \mathbb{R}
                         for all other \alpha.
               Prove: U is not open in the uniform topology.
    \langle 2 \rangle 3. \ \vec{0} \in U
    \langle 2 \rangle 4. Let: \epsilon > 0
               Prove: B(\vec{0}, \epsilon) \nsubseteq U
    \langle 2 \rangle5. PICK N such that 1/N < \epsilon
    \langle 2 \rangle6. Let: \vec{x} be such that x_{\alpha_N} = 1/N and x_{\alpha} = 0 for all other \alpha
    \langle 2 \rangle 7. \ \vec{x} \in B(\vec{0}, \epsilon)
    \langle 2 \rangle 8. \ \vec{x} \notin U
```

 $\langle 2 \rangle 4$ . Let:  $\epsilon = \min(\epsilon_1, \dots, \epsilon_n)$ 

**Proposition 10.1.22.** The space  $\mathbb{R}^{\omega}$  under the uniform topology is not second countable.

```
Corollary 10.1.22.1. Not every metric space is second countable.
Theorem 10.1.23. Let X and Y be metric spaces. Let f: X \to Y and x \in X.
Then f is continuous at x if and only if, for every \epsilon > 0, there exists \delta > 0 such
that, for all x' \in X, if d(x, x') < \delta then d(f(x), f(x')) < \epsilon.
Proof:
\langle 1 \rangle 1. If f is continuous at x then, for every \epsilon > 0, there exists \delta > 0 such that,
        for all x' \in X, if d(x, x') < \delta then d(f(x), f(x')) < \epsilon.
    \langle 2 \rangle 1. Assume: f is continuous at x.
    \langle 2 \rangle 2. Let: \epsilon > 0
    \langle 2 \rangle 3. PICK a neighbourhood U of x such that f(U) \subseteq B(f(x), \epsilon)
       PROOF: One exists by \langle 2 \rangle 1, since B(f(x), \epsilon) is a neighbourhood of f(x).
    \langle 2 \rangle 4. Pick \delta > 0 such that B(x, \delta) \subseteq U
       PROOF: By \langle 2 \rangle 3 and Lemma 10.1.5.
    \langle 2 \rangle 5. Let: x' \in X with d(x, x') < \delta
    \langle 2 \rangle 6. \ x' \in U
       PROOF: From \langle 2 \rangle 4 and \langle 2 \rangle 5.
    \langle 2 \rangle 7. \ f(x') \in B(f(x), \epsilon)
       PROOF: From \langle 2 \rangle 3 and \langle 2 \rangle 6.
\langle 1 \rangle 2. If, for all \epsilon > 0, there exists \delta > 0 such that, for all x' \in X, if d(x, x') < \delta
         then d(f(x), f(x')) < \epsilon, then f is continuous at x.
    \langle 2 \rangle 1. Assume: For all \epsilon > 0 there exists \delta > 0 such that, for all x' \in X, if
                          d(x, x') < \delta then d(f(x), f(x')) < \epsilon.
    \langle 2 \rangle 2. Let: V be a neighbourhood of f(x)
    \langle 2 \rangle 3. Pick \epsilon > 0 such that B(f(x), \epsilon) \subseteq V
       PROOF: By Lemma 10.1.5.
    \langle 2 \rangle 4. PICK \delta > 0 such that, for all x' \in X, if d(x, x') < \delta then d(f(x), f(x')) < \delta
       PROOF: By \langle 2 \rangle 1 and \langle 2 \rangle 3.
    \langle 2 \rangle 5. B(x, \delta) is a neighbourhood of x
       PROOF: By the definition of the metric topology.
    \langle 2 \rangle 6. \ f(B(x,\delta)) \subseteq V
       \langle 3 \rangle 1. Let: x' \in B(x, \delta)
       \langle 3 \rangle 2. \ d(f(x), f(x')) < \epsilon
          PROOF: From \langle 2 \rangle 4.
       \langle 3 \rangle 3. \ x' \in V
          PROOF: From \langle 2 \rangle 3.
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Lemma 10.1.24. Addition is a continuous function \mathbb{R}^2 \to \mathbb{R}.
PROOF:
\langle 1 \rangle 1. Let: (x,y) \in \mathbb{R}^2 and \epsilon > 0
\langle 1 \rangle 2. Let: \delta = \epsilon/2
```

PROOF: The set of all sequences of 0s and 1s is discrete but uncountable.  $\square$ 

```
\langle 1 \rangle 3. Let: (x',y') \in \mathbb{R}^2 be such that \rho((x,y),(x',y')) < \delta, where \rho is the square metric
```

$$\langle 1 \rangle 4$$
.  $|x - x'| < \delta$  and  $|y - y'| < \delta$ 

$$\langle 1 \rangle 5. |(x+y) - (x'+y')| < \epsilon$$

$$|(x+y) - (x'+y')| \le |x-x'| + |y-y'|$$

$$< 2\delta \qquad (\langle 1 \rangle 4)$$

$$= \epsilon \qquad (\langle 1 \rangle 2)$$

 $\langle 1 \rangle 6$ . Q.E.D.

PROOF: By Theorem 10.1.23.

**Lemma 10.1.25.** Additive inverse is a continuous function  $-: \mathbb{R} \to \mathbb{R}$ .

PROOF: If 
$$|x-y| < \epsilon$$
 then  $|(-x) - (-y)| < \epsilon$ .  $\square$ 

**Lemma 10.1.26.** *Multiplication is a continuous function*  $\cdot : \mathbb{R}^2 \to \mathbb{R}$ .

Proof:

$$\langle 1 \rangle 1$$
. Let:  $(x,y) \in \mathbb{R}^2$  and  $\epsilon > 0$ 

$$\langle 1 \rangle 2$$
. Let:  $\delta = \min(1, \epsilon/(|x| + |y| + 1))$ 

$$\langle 1 \rangle 3$$
. Let:  $(x', y') \in \mathbb{R}^2$  and  $\rho((x, y), (x', y')) < \delta$ 

$$\langle 1 \rangle 4$$
.  $|xy - x'y'| < \epsilon$ 

Proof:

$$|xy - x'y'| = |x(y' - y) + y(x' - x) + (x - x')(y - y')|$$

$$\leq |x||y' - y| + |y||x' - x| + |x - x'||y - y'|$$

$$< |x|\delta + |y|\delta + \delta^{2}$$

$$= \delta(|x| + |y| + \delta)$$

$$\leq \delta(|x| + |y| + 1)$$

$$\leq \epsilon$$

$$(\langle 1 \rangle 2)$$

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**Lemma 10.1.27.** Multiplicative inverse is a continuous function ( )<sup>-1</sup> :  $\mathbb{R} \setminus \{0\} \to \mathbb{R}$ .

Proof:

$$\langle 1 \rangle 1$$
. Let:  $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$  be defined by  $f(x) = x^{-1}$ .

$$\langle 1 \rangle 2$$
. Let:  $a, b \in \mathbb{R}$  with  $a < b$ 

PROVE:  $f^{-1}((a,b))$  is open

 $\langle 1 \rangle 3$ . Case: 0 < a < b

PROOF:  $f^{-1}((a,b)) = (b^{-1}, a^{-1})$ 

 $\langle 1 \rangle 4$ . Case: a < 0 < b

PROOF: 
$$f^{-1}((a,b)) = (-\infty, a^{-1}) \cup (b^{-1}, +\infty)$$

 $\langle 1 \rangle 5$ . Case: a < b < 0

PROOF: 
$$f^{-1}((a,b)) = (b^{-1}, a^{-1})$$

**Definition 10.1.28** (Uniform Convergence). Let X be a set and Y a metric space. Let  $f_n: X \to Y$  for  $n \ge 1$ , and  $f: X \to Y$ . Then  $f_n$  converges uniformly to f as  $n \to \infty$  iff, for all  $\epsilon > 0$ , there exists N such that, for all  $x \in X$  and  $n \ge N$ ,  $d(f_n(x), f(x)) < \epsilon$ .

**Theorem 10.1.29** (Uniform Limit Theorem). Let X be a topological space and Y a metric space. Let  $f_n: X \to Y$  for  $n \ge 1$  and  $f: X \to Y$ . If  $f_n$  converges uniformly to f and each  $f_n$  is continuous, then f is continuous.

### Proof:

- $\langle 1 \rangle 1$ . Let:  $x \in X$  and  $\epsilon > 0$
- $\langle 1 \rangle 2$ . PICK N such that, for all  $x' \in X$  and  $\delta > 0$ ,  $d(f_n(x'), f(x')) < \epsilon/3$
- $\langle 1 \rangle 3$ . PICK  $\delta > 0$  such that, for all  $x' \in X$ , if  $d(x, x') < \delta$  then  $d(f_N(x), f_N(x')) < \epsilon/3$
- $\langle 1 \rangle 4$ . For all  $x' \in X$ , if  $d(x, x') < \delta$  then  $d(f(x), f(x')) < \epsilon$ PROOF:

$$d(f(x), f(x')) \le d(f(x), f_N(x)) + d(f_N(x), f_N(x')) + d(f_N(x'), f(x'))$$
$$< \epsilon/3 + \epsilon/3 + \epsilon/3$$
$$= \epsilon$$

**Lemma 10.1.30.** Let X be a set. Let  $f_n : X \to \mathbb{R}$  for  $n \ge 1$  and  $f : X \to \mathbb{R}$ . Then  $f_n$  converges uniformly to f if and only if  $f_n$  converges to f in  $\mathbb{R}^X$  under the uniform topology.

### Proof:

- $\langle 1 \rangle 1$ . If  $f_n$  converges uniformly to f then  $f_n$  converges to f under the uniform topology.
  - $\langle 2 \rangle 1$ . Assume:  $f_n$  converges uniformly to f
  - $\langle 2 \rangle 2$ . Let:  $\epsilon > 0$
  - $\langle 2 \rangle 3$ . Pick N such that, for all  $x \in X$  and  $n \geq N$ ,  $d(f_n(x), f(x)) < \epsilon/2$
  - $\langle 2 \rangle 4. \ \overline{\rho}(f_n, f) \le \epsilon/2$
  - $\langle 2 \rangle 5. \ \overline{\rho}(f_n, f) < \epsilon$
- $\langle 1 \rangle 2$ . If  $f_n$  converges to f under the uniform topology then  $f_n$  converges uniformly to f.
  - $\langle 2 \rangle 1$ . Assume:  $f_n$  converges to f under the uniform topology.
  - $\langle 2 \rangle 2$ . Let:  $\epsilon > 0$
  - $\langle 2 \rangle 3$ . Pick N such that, for all  $n \geq N$ ,  $\overline{\rho}(f_n, f) < \epsilon$
  - $\langle 2 \rangle 4$ . For all  $n \geq N$  and  $x \in X$ ,  $d(f_n(x), f(x)) < \epsilon$

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**Theorem 10.1.31.** Every monotone increasing sequence of real numbers that is bounded above converges to its supremum.

### PROOF:

 $\langle 1 \rangle 1$ . Let:  $\{s_n\}_{n\geq 1}$  be a monotone increasing sequence of real numbers bounded above with supremum l.

 $\langle 1 \rangle 2$ . Let:  $\epsilon > 0$ 

 $\langle 1 \rangle 3$ .  $l - \epsilon$  is not an upper bound for  $\{s_n : n \geq 1\}$ .

 $\langle 1 \rangle 4$ . PICK N such that  $x_N > l - \epsilon$ 

 $\langle 1 \rangle 5$ . For all  $n \geq N$ , we have  $l - \epsilon < x_n \leq l$ 

 $\langle 1 \rangle 6$ . For all  $n \geq N$ , we have  $|x_n - l| < \epsilon$ 

**Definition 10.1.32** (Infinite Series). Let  $\{a_n\}_{n\geq 1}$  be a sequence of real numbers. The *infinite series*  $\sum_{n=1}^{\infty} a_n$  converges to s iff  $\sum_{n=1}^{N} a_n \to s$  as  $N \to \infty$ .

**Proposition 10.1.33.** If  $\sum_{n=1}^{\infty} a_n = s \text{ and } \sum_{n=1}^{\infty} b_n = t \text{ then } \sum_{n=1}^{\infty} (ca - n + a) = t \text{ then } \sum_{n=1}^{\infty} (ca - n) = t \text{ th$  $b_n) = cs + t.$ 

PROOF: This holds because  $\sum_{n=1}^{N} (ca_n + b_n) = c \sum_{n=1}^{N} a_n + \sum_{n=1}^{N} b_n \to cs + t$ as  $N \to \infty$ .

**Theorem 10.1.34** (Comparison Test). If  $|a_i| \leq b_i$  for all i and  $\sum_{i=1}^{\infty} b_i$  converges, then  $\sum_{i=1}^{\infty} a_i$  converges.

Proof:

PROOF:  $\sum_{i=1}^{\infty} |a_i|$  converges
PROOF:  $\sum_{i=1}^{N} |a_i|$  is a monotone increasing sequence bounded above by  $\sum_{i=1}^{\infty} b_i$ .

 $\langle 1 \rangle 2$ . Let:  $c_i = |a_i| + a_i$   $\langle 1 \rangle 3$ .  $\sum_{i=1}^{\infty} c_i$  converges PROOF:  $\sum_{i=1}^{N} c_i$  is a monotone increasing sequence bounded above by  $2 \sum_{i=1}^{\infty} |a_i|$ .  $\langle 1 \rangle 4$ . Q.E.D.

PROOF: Since  $a_i = c_i - |a_i|$ 

**Lemma 10.1.35.** If  $\sum_{n=1}^{\infty} a_n$  converges then  $\sum_{n=N}^{\infty} a_n \to 0$  as  $N \to \infty$ .

PROOF:

$$\sum_{n=N}^{\infty} a_n = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N-1} a_n$$

$$\rightarrow \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} a_n$$

$$= 0$$

as  $N \to \infty$ .

**Theorem 10.1.36** (Weierstrass M-Test). Let X be a set and  $f_n: X \to \mathbb{R}$  for  $n \ge 1$ . If  $|f_n(x)| \le M_n$  for all  $n \ge 1$  and all  $x \in X$ , and if  $\sum_{n=1}^{\infty} M_n$  converges, then

$$\sum_{n=1}^{N} f_n(x) \to \sum_{n=1}^{\infty} f_n(x)$$

uniformly in x as  $N \to \infty$ .

Proof:

 $\langle 1 \rangle 1$ . For  $N \geq 1$ ,

LET:  $s_N: X \to \mathbb{R}, s_N(x) = \sum_{n=1}^N f_n(x)$  $\langle 1 \rangle 2$ . For all  $x \in X$ ,  $\sum_{n=1}^\infty f_n(x)$  converges.

PROOF: By the Comparison Test.

- $\langle 1 \rangle 3$ . Let:  $s: X \to \mathbb{R}, s(x) = \sum_{n=1}^{\infty} f_n(x)$ .
- $\langle 1 \rangle 4$ . For  $N \geq 1$ ,

Let:  $r_N = \sum_{n=N+1}^{\infty} M_n$   $\langle 1 \rangle$ 5. For  $1 \leq N < K$ , we have  $|s_K(x) - s_N(x)| \leq r_N$  for all  $x \in X$ Proof:

$$|s_K(x) - s_N(x)| = \left| \sum_{n=N+1}^K f_n(x) \right|$$

$$\leq \sum_{n=N+1}^K |f_n(x)|$$

$$\leq \sum_{n=N+1}^K M_n$$

$$\leq \sum_{n=N+1}^\infty M_n$$

- $\langle 1 \rangle 6$ . For  $N \geq 1$  and  $x \in X$  we have  $|s(x) s_N(x)| \leq r_N$ PROOF: Let  $K \to \infty$  in  $\langle 1 \rangle 5$ .
- $\langle 1 \rangle 7$ . Let:  $\epsilon > 0$
- $\langle 1 \rangle 8$ . PICK N such that, for all  $N' \geq N$ , we have  $r_{N'} < \epsilon$ PROOF: Such an N exists by Lemma 10.1.35.
- $\langle 1 \rangle 9$ . For all  $N' \geq N$  and  $x \in X$  we have  $|s_{N'}(x) s(x)| < \epsilon$

**Definition 10.1.37.** Let X be a metric space. Let  $x \in X$  and  $A \subseteq X$  be nonempty. The distance from x to A is

$$d(x,A) = \inf_{a \in A} d(x,a) .$$

**Lemma 10.1.38.** Let X be a metric space and  $A \subseteq X$  be nonempty. Then the function  $d(-,A): X \to \mathbb{R}$  is continuous.

PROOF:

- $\langle 1 \rangle 1$ . Let:  $x \in X$  and  $\epsilon > 0$
- $\langle 1 \rangle 2$ . Let:  $y \in X$  with  $d(x,y) < \epsilon$
- $\langle 1 \rangle 3. |d(x,A) d(y,A)| < \epsilon$

Proof:

$$\langle 2 \rangle 1. \ d(x,A) - d(y,A) < \epsilon$$

$$d(x,A) = \inf_{a \in A} d(x,a)$$

$$\leq \inf_{a \in A} (d(x,y) + d(y,a))$$

$$= d(x,y) + \inf_{a \in A} d(y,a)$$

$$= d(x,y) + d(y,A)$$

$$< \epsilon + d(y,A)$$

 $\langle 2 \rangle 2$ .  $d(y, A) - d(x, A) < \epsilon$ PROOF: Similar.

 $\langle 1 \rangle 4$ . Q.E.D.

PROOF: By Theorem 10.1.23.

**Definition 10.1.39** (Shrinking Map). Let X be a metric space and  $f: X \to X$ . Then f is a *shrinking map* iff, for all  $x, y \in X$  with  $x \neq y$ , we have d(f(x), f(y)) < d(x, y).

**Definition 10.1.40** (Contraction). Let X be a metric space and  $f: X \to X$ . Then f is a *contraction* iff there exists  $\alpha < 1$  such that, for all  $x, y \in X$ ,

$$d(f(x), f(y)) \le \alpha d(x, y)$$
.

**Proposition 10.1.41.** Every separable metric space is second countable.

### Proof:

- $\langle 1 \rangle 1$ . Let: X be a separable metric space.
- $\langle 1 \rangle 2$ . Pick a countable dense set D
- $\langle 1 \rangle 3$ . Let:  $\mathcal{B} = \{ B(d,q) : d \in D, q \in \mathbb{Q}^+ \}$
- $\langle 1 \rangle 4$ .  $\mathcal{B}$  is a countable basis for X

 $\stackrel{\cap}{\vdash}$ 

Corollary 10.1.41.1. The space  $\mathbb{R}^{\omega}$  under the uniform topology is not separable.

Corollary 10.1.41.2. Not every metric space is separable.

Corollary 10.1.41.3. The space  $\mathbb{R}^{\omega}$  under the box topology is not separable.

**Proposition 10.1.42** (CC). Every Lindelöf metric space is second countable.

### PROOF:

- $\langle 1 \rangle 1$ . Let: X be a Lindelöf metric space.
- $\langle 1 \rangle$ 2. For all  $n \in \mathbb{Z}^+$ , PICK a countable covering  $\mathcal{A}_n$  of X by 1/n-balls PROOF: One exists by the Lindelöf condition, since the set of all 1/n-balls covers X.
- $\langle 1 \rangle 3$ .  $\bigcup_{n=1}^{\infty} A_n$  is a countable basis.

Corollary 10.1.42.1. The space  $\mathbb{R}^{\omega}$  under the uniform topology is not Lindelöf. Corollary 10.1.42.2. Not every metric space is Lindelöf. **Proposition 10.1.43.** The space  $\mathbb{R}_l$  is not metrizable. Proof: It is Lindelöf but not second countable. Proposition 10.1.44. The ordered square is not metrizable. Proof: It is compact but not second countable.  $\Box$ **Proposition 10.1.45.** The space  $\mathbb{R}^{\omega}$  under the uniform topology is not second countable.PROOF: It contains a subspace homeomorphic to  $\mathbb{R}$ .  $\square$ Theorem 10.1.46 (AC). Every metrizable space is normal. PROOF:  $\langle 1 \rangle 1$ . Let: X be a metric space.  $\langle 1 \rangle 2$ . Let: A and B be disjoint closed subspaces of X.  $\langle 1 \rangle 3$ . For  $a \in A$ , PICK  $\epsilon_a > 0$  such that  $B(a, \epsilon_a)$  does not intersect B.  $\langle 1 \rangle 4$ . For  $b \in B$ , PICK  $\epsilon_b > 0$  such that  $B(b, \epsilon_b)$  does not intersect A.  $\langle 1 \rangle$ 5. Let:  $U = \bigcup_{a \in A} B(a, \epsilon_a/2)$  $\langle 1 \rangle$ 6. Let:  $V = \bigcup_{b \in B} B(b, \epsilon_b/2)$  $\langle 1 \rangle 7. \ U \cap V = \emptyset$  $\langle 2 \rangle 1$ . Let:  $z \in U \cap V$  $\langle 2 \rangle 2$ . Pick  $a \in A$  and  $b \in B$  such that  $z \in B(a, \epsilon_a/2)$  and  $z \in B(b, \epsilon_b/2)$  $\langle 2 \rangle 3$ . Assume: w.l.o.g.  $\epsilon_a \leq \epsilon_b$ 

> $d(a,b) \le d(a,z) + d(b,z)$  (Triangle Inequality)  $< \epsilon_a/2 + \epsilon_b/2$  ( $\langle 2 \rangle 2$ )

((2) 2)

 $\leq \epsilon_b$  ( $\langle 2 \rangle 3$ )

 $\langle 2 \rangle$ 5. Q.E.D.

 $\langle 2 \rangle 4. \ a \in B(b, \epsilon_b)$ PROOF:

PROOF: This contradicts  $\langle 1 \rangle 4$ .

Corollary 10.1.46.1. The space  $\mathbb{R}^{\omega}$  is normal.

Corollary 10.1.46.2. The space  $\mathbb{R}_K$  is not methizable.

**Proposition 10.1.47.** Every metrizable space is completely normal.

PROOF: Every subspace is metrizable (Lemma 10.1.12) hence normal (Theorem 10.1.46).  $\Box$ 

Proposition 10.1.48. Every metrizable space is perfectly normal.

Proof:

 $\langle 1 \rangle 1$ . Let: X be a metric space.  $\langle 1 \rangle 2$ . X is normal. Proof: Theorem 10.1.46  $\langle 1 \rangle 3$ . Every closed set is  $G_{\delta}$ . Proof: If A is closed then  $A = \bigcap_{q \in \mathbb{Q}^+} \{x \in X : d(A, x) < q\}.$ **Theorem 10.1.49** (Urysohn Metrization Theorem (CC)). Every second countable regular space is metrizable. Proof:  $\langle 1 \rangle 1$ . Let: X be a second countable regular space.  $\langle 1 \rangle 2$ . X is normal.  $\langle 1 \rangle 3$ . PICK a countable basis  $\mathcal{B} = \{B_n : n \in \mathbb{Z}^+\}$  $\langle 1 \rangle 4$ . For every pair of integers m, n with  $\overline{B_m} \subseteq B_n$ , PICK a continuous function  $g_{mn}: X \to [0,1]$  such that  $g_{mn}(\overline{B_m}) = \{1\}$  and  $g_{mn}(X \setminus B_n) = \{0\}$ Proof: By the Urysohn Lemma.  $\langle 1 \rangle 5$ . The set  $\{g_{mn} : \overline{U_m} \subseteq U_n\}$  separates points from closed sets in X  $\langle 2 \rangle 1$ . Let:  $x \in X$  and U be a neighbourhood of x  $\langle 2 \rangle 2$ . Pick  $B_n \in \mathcal{B}$  such that  $x \in B_n \subseteq U$  $\langle 2 \rangle 3$ . Pick V open such that  $x \in V$  and  $\overline{V} \subseteq B_n$  $\langle 2 \rangle 4$ . PICK  $B_m \in \mathcal{B}$  such that  $x \in B_m \subseteq V$  $\langle 2 \rangle 5$ .  $g_{mn}(x) = 1$  and  $g_{mn}$  vanishes outside U $\langle 1 \rangle 6$ . X is imbeddable in  $[0,1]^{\omega}$ PROOF: By the Imbedding Theorem.  $\langle 1 \rangle 7$ . Q.E.D. Corollary 10.1.49.1. The space  $\mathbb{R}^{\omega}$  under the box topology is not second count-Proposition 10.1.50. Not every second countable Hausdorff space is metrizable.PROOF:  $\mathbb{R}_K$  is second countable and Hausdorff but not metrizable (because it is not regular).  $\square$ **Proposition 10.1.51.** There exists a space that is completely normal, first countable, Lindelöf and separable but not metrizable. PROOF: The space  $\mathbb{R}_l$  is all of these.  $\square$ **Proposition 10.1.52.**  $\overline{S_{\Omega}}$  is not metrizable. Proof: It is compact but not sequentially compact. **Proposition 10.1.53.** Every compact metric space is second countable.

 $\langle 1 \rangle 1$ . Let: X be a compact etric space

 $\langle 1 \rangle 2$ . For every  $n \geq 1$ , PICK a finite covering  $\mathcal{A}_n$  of X by open balls of radius 1/n

PROOF: Such a covering exists because  $\{B_{1/n}(x): x \in X\}$  covers X.

 $\langle 1 \rangle 3. \bigcup_{n=1}^{\infty} A_n$  is a countable basis for X

Corollary 10.1.53.1. The space  $\mathbb{R}^{\omega}$  under the uniform topology is not compact.

Corollary 10.1.53.2. The space  $\mathbb{R}^{\omega}$  under the uniform topology is not limit point compact.

**Proposition 10.1.54.** The space  $\mathbb{R}^{\omega}$  under the box topology is not locally compact.

#### Proof:

- $\langle 1 \rangle 1$ . Assume:  $\mathbb{R}^{\omega}$  under the box topology is locally compact.
- $\langle 1 \rangle 2$ . For every point x, there exists a basic open set  $B = \prod_{i=0}^{\infty} U_i$  such that  $x \in B$  and  $\overline{B}$  is compact.
- $\langle 1 \rangle 3$ . The box topology on  $\overline{B}$  is the same as the product topology on  $\overline{B}$  PROOF: By Corollary 9.4.11.1.
- $\langle 1 \rangle 4.$  The box topology on  $\overline{B}$  is strictly finer than the product topology. Proof:By Theorem 10.1.21.

Proposition 10.1.55. Not every metrizable space is connected.

PROOF: The discrete space with two points is metrizable but not connected.  $\Box$ 

Corollary 10.1.55.1. Not every metrizable space is path connected.

Proposition 10.1.56. Not every metric space is limit point compact.

PROOF: The space  $\mathbb{R}$  is not limit point compact.  $\square$ 

**Proposition 10.1.57.** Not every metric space is locally compact.

The space  $\mathbb{R}^{\omega}$  in the uniform topology is not locally compact.

### 10.2 Isometries

**Definition 10.2.1** (Isometry). Let X be a metric space. An *isometry* of X is a function  $f: X \to X$  such that, for all  $x, y \in X$ ,

$$d(f(x), f(y)) = d(x, y) .$$

# 10.3 Lebesgue Numbers

**Definition 10.3.1** (Lebesgue Number). Let X be a metric space and  $\mathcal{A}$  an open covering of X. A Lebesgue number for  $\mathcal{A}$  is a real  $\delta > 0$  such that, for every nonempty set  $A \subseteq X$  of diameter  $< \delta$ , there exists  $U \in \mathcal{A}$  such that  $A \subseteq U$ .

**Lemma 10.3.2** (Lebesgue Number Lemma). In a compact metric space, every open covering has a Lebesgue number.

### PROOF:

 $\langle 1 \rangle$ 1. Let: X be a compact metric space and  $\mathcal{A}$  an open covering of X Prove: There exists a Lebesgue number  $\delta$  for  $\mathcal{A}$ .

 $\langle 1 \rangle 2$ . Assume: w.l.o.g.  $X \notin \mathcal{A}$ 

PROOF: If  $X \in \mathcal{A}$  then we can take  $\delta = 1$ .

 $\langle 1 \rangle 3$ . Pick a finite subcovering  $\{U_1, \ldots, U_n\} \subseteq \mathcal{A}$  that covers X

 $\langle 1 \rangle 4$ . For  $1 \leq i \leq n$ ,

Let:  $C_i = X \setminus U_i$ 

 $\langle 1 \rangle$ 5. Let:  $f: X \to \mathbb{R}$  be defined by

$$f(x) = 1/n \sum_{i=1}^{n} d(x, C_i)$$
.

PROOF: Each  $C_i$  is nonempty by  $\langle 1 \rangle 2$ .

 $\langle 1 \rangle 6$ . For all  $x \in X$  we have f(x) > 0

 $\langle 2 \rangle 1$ . Let:  $x \in X$ 

 $\langle 2 \rangle 2$ . PICK i such that  $x \in U_i$ 

Proof: By  $\langle 1 \rangle 3$ .

 $\langle 2 \rangle 3$ . Pick  $\epsilon > 0$  such that  $B(x, \epsilon) \subseteq U_i$ 

PROOF: By Lemma 10.1.5.

 $\langle 2 \rangle 4. \ d(x, C_i) \ge \epsilon$ 

 $\langle 1 \rangle 7$ . f is continuous

PROOF: From Lemma 10.1.38.

 $\langle 1 \rangle 8$ . Let:  $\delta = \min f(X)$ 

PROVE: For every nonempty set  $A\subseteq X$  with diameter  $<\delta,$  there exists  $U\in\mathcal{A}$  such that  $A\subseteq U$ 

PROOF: f(X) has a minimum by the Extreme Value Theorem.

 $\langle 1 \rangle 9$ . Let:  $A \subseteq X$  be nonempty with diam  $A < \delta$ 

 $\langle 1 \rangle 10$ . Pick  $x_0 \in A$ 

 $\langle 1 \rangle 11$ . Let: i be such that  $d(x_0, C_i)$  is greatest among  $d(x_0, C_1), \ldots, d(x_0, C_n)$ 

 $\langle 1 \rangle 12. \ \delta \leq d(x_0, C_i)$ 

Proof:

$$\delta \le f(x_0) \tag{\langle 1 \rangle 8}$$

$$= 1/n \sum_{j=1}^{n} d(x_0, C_j)$$
 (\langle 1\rangle 5)

$$\leq 1/n \sum_{j=1}^{n} d(x_0, C_i) \tag{\langle 1 \rangle 11}$$

$$=d(x_0,C_i)$$

 $\langle 1 \rangle 13. \ x_0 \in U_i$ 

PROOF:  $x_0 \notin C_i$  because  $d(x_0, C_i) > 0$ .

**Theorem 10.3.3** (DC). Let X be a metrizable space. Then the following are

### equivalent:

- 1. X is compact.
- 2. X is limit point compact.
- 3. X is sequentially compact.

#### Proof:

 $\langle 1 \rangle 1. \ 1 \Rightarrow 2$ 

PROOF: Theorem 9.4.22.

- $\langle 1 \rangle 2. \ 2 \Rightarrow 3$ 
  - $\langle 2 \rangle 1$ . Assume: X is limit point compact.
  - $\langle 2 \rangle 2$ . Let:  $(x_n)$  be a sequence in X Prove:  $(x_n)$  has a convergent subsequence.
  - $\langle 2 \rangle 3$ . Case:  $\{x_n : n \in \mathbb{Z}^+\}$  is finite.

PROOF: In this case,  $(x_n)$  has a constant subsequence.

- $\langle 2 \rangle 4$ . Case:  $\{x_n : n \in \mathbb{Z}^+\}$  is infinite.
  - $\langle 3 \rangle 1$ . PICK a limit point l of  $\{x_n : n \in \mathbb{Z}^+\}$
  - $\langle 3 \rangle 2.$  For every poisitive integer r, PICK  $n_r$  such that  $n_r > n_{r-1}$  and  $d(x_{n_r},l) < 1/r$

PROOF: There always exists such an  $n_r$  since B(l, 1/r) intersects  $\{x_n : n \in \mathbb{Z}^+\}$  in infinitely many points by Theorem 6.1.2.

- $\langle 3 \rangle 3. \ x_{n_r} \to l \text{ as } r \to \infty$
- $\langle 1 \rangle 3. \ 3 \Rightarrow 1$ 
  - $\langle 2 \rangle 1$ . Assume: X is sequentially compact.
  - $\langle 2 \rangle 2$ . Every open covering of X has a Lebesgue number.
    - $\langle 3 \rangle 1$ . Let:  $\mathcal{A}$  be an open covering of X.
    - $\langle 3 \rangle 2$ . Assume: for a contradiction that, for all  $\delta > 0$ , there exists a set  $C \subseteq X$  with diam  $C < \delta$  such that there is no  $U \in \mathcal{A}$  such that  $C \subset U$
    - $\langle 3 \rangle 3$ . For  $n \geq 1$ , PICK  $C_n \subseteq X$  with diam  $C_n < 1/n$  such that there is no  $U \in \mathcal{A}$  such that  $C_n \subseteq U$
    - $\langle 3 \rangle 4$ . For  $n \geq 1$ , PICK  $x_n \in C_n$
    - $\langle 3 \rangle$ 5. PICK a convergent subsequence  $(x_{n_r})$  of  $(x_n)$ 
      - Proof: By  $\langle 2 \rangle 1$ .
    - $\langle 3 \rangle 6$ . Let:  $x_{n_r} \to l$  as  $r \to \infty$  $\langle 3 \rangle 7$ . Pick  $U \in \mathcal{A}$  with  $l \in U$ 
      - PROOF: By  $\langle 3 \rangle 1$
    - $\langle 3 \rangle 8$ . Pick  $\epsilon > 0$  such that  $B(l, \epsilon) \subseteq U$

PROOF: By Lemma 10.1.5.

 $\langle 3 \rangle 9$ . PICK R such that  $1/n_R < \epsilon/2$  and  $d(x_{n_R}, l) < \epsilon/2$ 

Proof: By  $\langle 3 \rangle 6$ 

 $\langle 3 \rangle 10. \ C_{n_R} \subseteq U$ 

$$C_{n_R} \subseteq B(x_{n_R}, 1/n_R) \qquad (\langle 3 \rangle 3, \langle 3 \rangle 4)$$

$$\subseteq B(x_{n_R}, \epsilon/2) \qquad (\langle 3 \rangle 9)$$

$$\subseteq B(l, \epsilon) \qquad (\langle 3 \rangle 9)$$

$$\subseteq U \qquad (\langle 3 \rangle 8)$$

 $\langle 3 \rangle 11$ . Q.E.D.

Proof: This contradicts  $\langle 3 \rangle 3$ .

- $\langle 2 \rangle 3$ . For all  $\epsilon > 0$ , there exists a finite covering of X by  $\epsilon$ -balls.
  - $\langle 3 \rangle 1$ . Let:  $\epsilon > 0$
  - $\langle 3 \rangle 2$ . Assume: for a contradiction there is no finite covering of X by  $\epsilon$ -balls.
  - $\langle 3 \rangle 3$ . PICK a sequence  $(x_n)$  in X such that, for all n,

$$x_n \notin B(x_1, \epsilon) \cup \cdots \cup B(x_{n-1}, \epsilon)$$
.

- $\langle 3 \rangle 4$ . For all m, n with m > n we have  $d(x_m, x_n) \geq \epsilon$
- $\langle 3 \rangle 5$ . Any  $\epsilon/2$ -ball contains at most one element of  $(x_n)$ .
- $\langle 3 \rangle 6$ .  $(x_n)$  has no convergent subsequence.
- $\langle 3 \rangle$ 7. Q.E.D.

PROOF: This contradicts  $\langle 2 \rangle 1$ .

- $\langle 2 \rangle 4$ . Let:  $\mathcal{A}$  be an open covering of X
- $\langle 2 \rangle$ 5. Pick a Lebesgue number  $\delta$  for  $\mathcal{A}$

Proof: By  $\langle 2 \rangle 2$ .

 $\langle 2 \rangle$ 6. PICK a finite covering  $\{B_1, \ldots, B_n\}$  of X by  $\delta/3$ -balls.

Proof: By  $\langle 2 \rangle 3$ .

- $\langle 2 \rangle 7$ . For  $1 \leq i \leq n$ , PICK  $U_i \in \mathcal{A}$  such that  $B_i \subseteq U_i$
- $\langle 2 \rangle 8. \{U_1, \ldots, U_n\} \text{ covers } X.$

Corollary 10.3.3.1.  $S_{\Omega}$  is not metrizable.

PROOF: It is limit point compact (Corollary 9.4.19.2) but not compact (Proposition 9.4.2).  $\Box$ 

Corollary 10.3.3.2. The space  $\mathbb{R}^{\omega}$  is not limit point compact.

# 10.4 Uniform Continuity

**Definition 10.4.1** (Uniform Continuity). Let X and Y be metric spaces and  $f: X \to Y$ . Then f is uniformly continuous iff, for all  $\epsilon > 0$ , there exists  $\delta > 0$  such that, for all  $x, y \in X$ , if  $d(x, y) < \delta$  then  $d(f(x), f(y)) < \epsilon$ .

**Theorem 10.4.2** (Uniform Continuity Theorem). Let X be a compact metric space, Y a metric space, and  $f: X \to Y$  be continuous. Then f is uniformly continuous.

### Proof:

 $\langle 1 \rangle 1$ . Let:  $\epsilon > 0$ 

PROVE: There exists  $\delta > 0$  such that, for all  $x, y \in X$ , if  $d(x, y) < \delta$  then  $d(f(x), f(y)) < \epsilon$ .

### 10.5 Locally Metrizable Spaces

**Definition 10.5.1** (Locally Metrizable). A space is *locally metrizable* iff every point has a metrizable neighbourhood.

**Proposition 10.5.2.** Every metrizable space is locally metrizable.

Proof: Trivial.

Corollary 10.5.2.1. The space  $\mathbb{R}^{\omega}$  is locally metrizable.

**Proposition 10.5.3.** A compact Hausdorff space is metrizable if and only if it is locally metrizable.

### Proof:

- $\langle 1 \rangle 1$ . Let: X be a locally metrizable compact Hausdorff space
- $\langle 1 \rangle 2$ . X is regular

PROOF: Lemma 9.4.18

- $\langle 1 \rangle 3$ . X is second countable
  - $\langle 2 \rangle 1$ .  $\{U: U \text{ open in } X \text{ and metrizable} \}$  covers X
  - $\langle 2 \rangle 2$ . Pick a finite subcover  $U_1, \ldots, U_n$
  - $\langle 2 \rangle 3$ . For  $1 \leq i \leq n$ , PICK a countable basis  $\mathcal{B}_i$  of  $U_i$
  - $\langle 2 \rangle 4$ .  $\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_n$  is a basis for X
- $\langle 1 \rangle 4$ . Q.E.D.

Proof: By the Urysohn Metrization Theorem.

Corollary 10.5.3.1.  $\overline{S_{\Omega}}$  is not locally metrizable.

Corollary 10.5.3.2. The ordered square is not locally metrizable.

**Proposition 10.5.4.** Every subspace of a locally metrizable space is locally metrizable.

### Proof:

- $\langle 1 \rangle 1$ . Let: X be locally metrizable and  $Y \subseteq X$
- $\langle 1 \rangle 2$ . Let:  $y \in Y$

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\langle 1 \rangle 3. Pick a metrizable neighbourhood U of y in X
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 $\langle 1 \rangle 4$ .  $U \cap Y$  is a metrizable neighbourhood of y in Y

Corollary 10.5.4.1.  $S_{\Omega} \times \overline{S_{\Omega}}$  is not locally metrizable.

PROOF: It has a subspace homeomorphic to  $\overline{S_{\Omega}}$ .  $\square$ 

**Proposition 10.5.5** (CC). Every locally metrizable regular Lindelöf space is metrizable.

### Proof:

- $\langle 1 \rangle 1$ . Let: X be a locally metrizable regular Lindelöf space.
- $\langle 1 \rangle 2$ . Every point in X has an open second countable neighbourhood.
  - $\langle 2 \rangle 1$ . Let:  $x \in X$
  - $\langle 2 \rangle 2$ . Pick an open metrizable U containing x

PROOF: X is locally metrizable  $(\langle 1 \rangle 1)$ 

 $\langle 2 \rangle$ 3. PICK an open V such that  $x \in V \subseteq \overline{V} \subseteq U$ 

PROOF: Proposition 6.3.2

 $\langle 2 \rangle 4$ .  $\overline{V}$  is Lindelöf

Proof: Proposition 13.1.32

 $\langle 2 \rangle 5$ .  $\overline{V}$  is second countable

Proof: Proposition 10.1.42

 $\langle 1 \rangle 3$ . Pick a countable covering of secound countable open sets  $\mathcal{U}$ 

PROOF: X is Lindelöf ( $\langle 1 \rangle 1$ )

- $\langle 1 \rangle 4$ . For  $U \in \mathcal{U}$ , PICK a countable basis  $\mathcal{B}_U$
- $\langle 1 \rangle 5$ .  $\bigcup_{U \in \mathcal{U}} \mathcal{B}_U$  is a countable basis for X
  - $\langle 2 \rangle 1$ . Let:  $x \in U$  where U is open in X
  - $\langle 2 \rangle 2$ . Pick  $V \in \mathcal{U}$  such that  $x \in V$
  - $\langle 2 \rangle 3$ . There exists  $B \in \mathcal{B}_V$  such that  $x \in B \subseteq U \cap V$
- $\langle 1 \rangle 6$ . Q.E.D.

PROOF: By the Urysohn Metrization Theorem.

Corollary 10.5.5.1.  $\mathbb{R}_l$  is not locally metrizable.

**Proposition 10.5.6.** The Sorgenfrey plane is not locally metrizable.

#### Proof:

 $\langle 1 \rangle 1$ . Let: U be any neighbourhood of (0,0)

Prove: U is not Lindelöf

- $\langle 1 \rangle 2$ . Pick a > 0 such that  $[0, a)^2 \subseteq U$
- $\langle 1 \rangle 3$ . Let:  $L = \{(x, a x) : 0 < x < a\}$
- $\langle 1 \rangle 4$ . L is closed in U

PROOF: By Lemma 6.5.16 since  $(x,y) \mapsto (x,a+y)$  is a homeomorphism of  $\mathbb{R}^2_t$  with itself.

- (1)5. Let:  $\mathcal{U} = \{U \setminus L\} \cup \{([x,b) \times [a-x,c)) \cap U : b > a,c > a-x\}$
- $\langle 1 \rangle 6$ .  $\mathcal{U}$  covers U

$\langle 1 \rangle$ 7. No countable subset of $\mathcal U$ covers $U$ PROOF: Every set of the for $[x,b) \times [a-x,c)$ intersects $L$ in exactly one point $\square$
Corollary 10.5.6.1. The Sorgenfrey plane is not metrizable.
<b>Proposition 10.5.7.</b> The space $\mathbb{R}_K$ is locally metrizable.
PROOF: The set $(-1,1)-K$ is a metrizable neighbourhood of 0. For any other point $p$ , pick an open interval around $p$ that does not contain 0. $\square$
<b>Proposition 10.5.8.</b> The product of two locally metrizable spaces is locally metrizable.
PROOF: $ \langle 1 \rangle 1. \text{ Let: } X \text{ and } Y \text{ be locally metrizable} \\ \langle 1 \rangle 2. \text{ Let: } (a,b) \in X \times Y \\ \langle 1 \rangle 3. \text{ Pick metrizable neighbourhoods } U \text{ of } a \text{ and } V \text{ of } b \\ \langle 1 \rangle 4. \ U \times V \text{ is a metrizable neighbourhood of } (a,b). \\ \text{Proof: By Lemma 10.1.19.} \\ \square$
<b>Proposition 10.5.9.</b> The product of two locally metrizable spaces is locally metrizable.
PROOF: $ \langle 1 \rangle 1. \text{ Let: } X \text{ and } Y \text{ be locally metrizable}                                    $
<b>Proposition 10.5.10.</b> The space $\mathbb{R}_K^{\omega}$ is not locally metrizable.
PROOF: If it were, then there would be a basic open set $\prod_n U_n$ that is metrizable but then $\mathbb{R}_K$ would be metrizable as it is homeomorphic to a subspace of $\prod_n U_n$
Corollary 10.5.10.1. The product of a countable family of locally metrizable spaces is not necessarily locally metrizable.
Proposition 10.5.11. The continuous image of a locally metrizable space is not necessarily locally metrizable.

PROOF: The identity map from the discrete two-point space to the indiscrete

two-point space is continuous.  $\square$ 

# Chapter 11

# Manifolds

### 11.1 Manifolds

**Definition 11.1.1** (Manifold). Let  $m \geq 1$ . An m-manifold is a second countable Hausdorff space such that each point has a neighbourhood that is homeomorphic to an open subset of  $\mathbb{R}^m$ .

A curve is a 1-manifold and a surface is a 2-manifold.

**Definition 11.1.2** (Support). Let X be a topological space and  $\phi: X \to \mathbb{R}$  be a function. Then the *support* of  $\phi$  is the closure of  $\phi^{-1}(\mathbb{R} \setminus \{0\})$ .

**Definition 11.1.3** (Partition of Unity). Let X be a topological space. Let  $\{U_1, \ldots, U_n\}$  be a finite indexed open covering of X. An indexed family of continuous functions  $\phi_1, \ldots, \phi_n : X \to [0,1]$  is a partition of unity dominated by  $\{U_1, \ldots, U_n\}$  iff:

- 1. supp  $\phi_i \subseteq U_i$  for all i;
- 2.  $\sum_{i=1}^{n} \phi_i(x) = 1$  for all  $x \in X$ .

**Theorem 11.1.4** (Existence of Finite Partitions of Unity). Let X be a normal space. Let  $\{U_1, \ldots, U_n\}$  be a finite indexed open covering of X. Then there exists a partition of unity dominated by  $\{U_1, \ldots, U_n\}$ .

### Proof:

- $\langle 1 \rangle 1$ . For every finite indexed open covering  $\{U_1, \ldots, U_n\}$  of X, there exists a finite indexed open covering  $\{V_1, \ldots, V_n\}$  such that  $\overline{V_i} \subseteq U_i$ 
  - $\langle 2 \rangle 1$ . For  $1 \leq k \leq n$ , there exist open sets  $V_1, \ldots, V_k$  such that  $\overline{V_i} \subseteq U_i$  for all i and  $\{V_1, \ldots, V_k, U_{k+1}, \ldots, U_n\}$  covers X
    - $\langle 3 \rangle$ 1. Assume: as an induction hypothesis that 0 leq k < k and there exist open sets  $V_1, \ldots, V_k$  such that  $\overline{V_i} \subseteq U_i$  for all i and  $\{V_1, \ldots, V_k, U_{k+1}, \ldots, U_n\}$  covers X
    - $\langle 3 \rangle 2$ . Let:  $A = X \setminus (V_1 \cup \cdots \cup V_k) \setminus (U_{k+2} \cup \cdots \cup U_n)$
    - $\langle 3 \rangle 3$ . A is closed

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\langle 3 \rangle 4. \ A \subseteq U_{k+1}
           PROOF: Since \{V_1, \ldots, V_k, U_{k+1}, \ldots, U_n\} covers X
        \langle 3 \rangle 5. Pick an open set V_{k+1} such that A \subseteq V_{k+1} and \overline{V_{k+1}} \subseteq U_{k+1}
           Proof: By Proposition 6.3.2
        \langle 3 \rangle 6. \{V_1, \dots, V_k, V_{k+1}, U_{k+2}, \dots, U_n\} \text{ covers } X
\langle 1 \rangle 2. PICK an open covering \{V_1, \ldots, V_n\} with \overline{V_i} \subseteq U_i for all i
    Proof: By \langle 1 \rangle 1.
\langle 1 \rangle 3. Pick an open covering \{W_1, \ldots, W_n\} with \overline{W_i} \subseteq V_i for all i
    Proof: By \langle 1 \rangle 1.
(1)4. For 1 \leq i \leq n, PICK a continuous function \psi_i : X \to [0,1] such that
          \psi_i(\overline{W_i}) = \{1\} \text{ and } \psi_i(X \setminus V_i) = \{0\}
   PROOF: By the Urysohn Lemma.
\langle 1 \rangle 5. Let: \Psi: X \to \mathbb{R} where \Psi(x) = \sum_{i=1}^n \psi_i(x)
\langle 1 \rangle 6. \ \Psi(x) > 0 \text{ for all } x \in X
    \langle 2 \rangle 1. Let: x \in X
    \langle 2 \rangle 2. PICK i such that x \in W_i
    \langle 2 \rangle 3. \ \psi_i(x) = 1
\langle 1 \rangle 7. For 1 \leq j \leq n,
Let: \phi_j(x) = \frac{\psi_j(x)}{\Psi(x)}
\langle 1 \rangle 8. \ \psi_1, \ldots, \psi_n are a partition of unity dominated by \{U_1, \ldots, U_n\}
    \langle 2 \rangle 1. supp \psi_i \subseteq U_i
        \langle 3 \rangle 1. \ \psi_i^{-1}(\mathbb{R} \setminus \{0\}) \subseteq V_i
           Proof: By \langle 1 \rangle 4
        \langle 3 \rangle 2. supp \psi_i \subseteq \overline{V_i}
           Proof: Proposition 3.7.5
    \langle 2 \rangle 2. \sum_{i=1}^{n} \psi_i(x) = 1 for all x \in X
П
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**Theorem 11.1.5.** Let X be a compact Hausdorff space. Suppose that, for every  $x \in X$ , there exists a neighbourhood U of x and a positive integer k such that U can be imbedded in  $\mathbb{R}^k$ . Then there exists a positive integer N such that X can be imbedded in  $\mathbb{R}^N$ .

#### Proof:

 $\langle 1 \rangle 1$ . PICK a finite open covering  $\{U_1, \ldots, U_n\}$  of X such that each  $U_i$  can be imbedded in  $\mathbb{R}^k$  for some k

PROOF: Since  $\{U \text{ open in } X : U \text{ can be imbedded in } \mathbb{R}^k \text{ for some } k\}$  covers X.

- $\langle 1 \rangle 2$ . For  $1 \leq i \leq n$ , PICK a positive integer  $k_i$  and an imbedding  $g_i : U_i \to \mathbb{R}^{k_i}$
- $\langle 1 \rangle 3$ . Pick a partition of unity  $\phi_1, \ldots, \phi_n$  dominated by  $\{U_1, \ldots, U_n\}$ 
  - $\langle 2 \rangle 1$ . X is normal

Proof: By Lemma 9.4.18.

 $\langle 2 \rangle 2$ . Q.E.D.

Proof: Theorem 11.1.4

 $\langle 1 \rangle 4$ . For  $1 \leq i \leq n$ , LET:  $A_i = \operatorname{supp} \phi_i$ 

```
\langle 1 \rangle 5. For 1 \leq i \leq n,
   LET: h_i: X \to \mathbb{R}^{k_i} be defined by h_i(x) = \begin{cases} \phi_i(x)g_i(x) & \text{for } x \in U_i \\ \vec{0} & \text{for } x \in X \setminus A_i \end{cases}
PROOF: If x \in U_i and x \in X \setminus A_i then x \notin \text{supp } \phi_i so \phi_i(x) = 0
\langle 1 \rangle 6. Let: N = n + k_1 + \dots + k_n
\langle 1 \rangle7. Let: F: X \to \mathbb{R}^N be the function
                                            F(x) = (\phi_1(x), \dots, \phi_n(x), h_1(x), \dots, h_n(x))
\langle 1 \rangle 8. F is an imbedding
    \langle 2 \rangle 1. F is continuous
        PROOF: Each h_i is continuous by Theorem 5.2.13.
    \langle 2 \rangle 2. F is injective
         \langle 3 \rangle 1. Assume: F(x) = F(y)
        \langle 3 \rangle 2. PICK i such that \phi_i(x) > 0
            PROOF: Since \sum_{i} \phi_i(x) = 1 \ (\langle 1 \rangle 3)
         \langle 3 \rangle 3. \ \phi_i(y) = 0
            Proof: By \langle 3 \rangle 1
        \langle 3 \rangle 4. \ x, y \in U_i
            PROOF: Since supp \phi_i \subseteq U_i
         \langle 3 \rangle 5. h_i(x) = h_i(y)
            Proof: By \langle 3 \rangle 1
         \langle 3 \rangle 6. g_i(x) = g_i(y)
            Proof: By \langle 1 \rangle 5
         \langle 3 \rangle 7. \ x = y
            Proof: By \langle 1 \rangle 2
    \langle 2 \rangle 3. Q.E.D.
        PROOF: By Theorem 9.4.11
```

Corollary 11.1.5.1. Every compact manifold can be imbedded in  $\mathbb{R}^N$  for some N.

**Proposition 11.1.6.** The line with two origins is a second countable  $T_1$  space where every point has a neighbourhood that is homeomorphic to an open subset of  $\mathbb{R}$ , but it is not a 1-manifold.

# Chapter 12

# Normed Spaces

### 12.1 The Norm on $\mathbb{R}^n$

**Definition 12.1.1** (Norm). Given  $\vec{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ , the norm  $\|\vec{x}\|$  is defined by

 $\|\vec{x}\| = \sqrt{x_1^2 + \dots + x_n^2}$ .

**Definition 12.1.2** (Vector Sum). Define the sum of  $\vec{x}, \vec{y} \in \mathbb{R}^n$  by

$$\vec{x} + \vec{y} = (x_1 + y_1, \dots, x_n + y_n)$$
.

**Definition 12.1.3** (Scalar Product). Given  $c \in \mathbb{R}$  and  $\vec{x} \in \mathbb{R}^n$ , define the scalar product  $c\vec{x}$  to be

$$c\vec{x} = (cx_1, \dots, cx_n)$$
.

**Definition 12.1.4** (Inner Product). The *inner product* of  $\vec{x}, \vec{y} \in \mathbb{R}^n$  is

$$\vec{x} \cdot \vec{y} = x_1 y_1 + \dots + x_n y_n .$$

Lemma 12.1.5.

$$\vec{x} \cdot (\vec{y} + \vec{z}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z}$$

PROOF: Both are equal to  $\sum_{i=1}^{n} (x_i y_i + x_i z_i)$ .

Lemma 12.1.6.

$$|\vec{x} \cdot \vec{y}| \le ||\vec{x}|| ||\vec{y}||$$

Proof:

 $\langle 1 \rangle 1$ . Case:  $\vec{x} = \vec{0}$  or  $\vec{y} = \vec{0}$ 

PROOF: In this case, both sides are 0.

 $\langle 1 \rangle 2$ . Case:  $\vec{x} \neq \vec{0} \neq \vec{y}$ 

 $\langle 2 \rangle 1$ . Let:  $a = 1/\|\vec{x}\|, b = 1/\|\vec{y}\|$ 

 $\langle 2 \rangle 2$ .  $2 + 2ab\vec{x} \cdot \vec{y} \ge 0$ 

 $\langle 3 \rangle 1. \ \|a\vec{x} + b\vec{y}\|^2 \ge 0$ 

$$\begin{array}{l} \langle 3 \rangle 2. \ \sum_{i=1}^{n} (ax_{i} + by_{i})^{2} \geq 0 \\ \langle 3 \rangle 3. \ a^{2} \sum_{i=1}^{n} x_{i}^{2} + b^{2} \sum_{i=1}^{n} y_{i}^{2} + 2ab \sum_{i=1}^{n} x_{i} y_{i} \geq 0 \\ \langle 3 \rangle 4. \ a^{2} \|\vec{x}\|^{2} + b^{2} \|\vec{y}\|^{2} + 2ab\vec{x} \cdot \vec{y} \geq 0 \\ \langle 2 \rangle 3. \ 2 - 2ab\vec{x} \cdot \vec{y} \geq 0 \\ \text{PROOF: Similar.} \\ \langle 2 \rangle 4. \ 2 - 2ab |\vec{x} \cdot \vec{y}| \geq 0 \\ \text{PROOF: From } \langle 2 \rangle 2 \text{ and } \langle 2 \rangle 3. \\ \langle 2 \rangle 5. \ |\vec{x} \cdot \vec{y}| \leq 1/ab \\ \end{array}$$

### Lemma 12.1.7.

$$\|\vec{x} + \vec{y}\| \le \|\vec{x}\| + \|\vec{y}\|$$

Proof:

$$\begin{split} \|\vec{x} + \vec{y}\|^2 &= (\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y}) \\ &= \|\vec{x}\|^2 + 2\vec{x} \cdot \vec{y} + \|\vec{y}\|^2 \\ &\leq \|\vec{x}\|^2 + 2\|\vec{x}\| \|\vec{y}\| + \|\vec{y}\|^2 \\ &= (\|\vec{x}\| + \|\vec{y}\|)^2 \end{split}$$
 (Lemma 12.1.6)

**Definition 12.1.8** (Euclidean Metric). The *euclidean metric* on  $\mathbb{R}^n$  is given by

$$d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|$$

We prove this is a metric.

Proof:

 $\langle 1 \rangle 1. \ d(\vec{x}, \vec{y}) \geq 0$ 

PROOF: Immediate from definitions.

 $\langle 1 \rangle 2$ .  $d(\vec{x}, \vec{y}) = 0$  iff  $\vec{x} = \vec{y}$ 

PROOF: Immediate from definitions.

 $\langle 1 \rangle 3. \ d(\vec{x}, \vec{y}) = d(\vec{y}, \vec{x})$ 

PROOF: Immediate from definitions.

 $\langle 1 \rangle 4. \ d(\vec{x}, \vec{z}) \leq d(\vec{x}, \vec{y}) + d(\vec{y}, \vec{z})$ 

PROOF: From Lemma 12.1.7.

**Lemma 12.1.9.** Let d be the euclidean topology on  $\mathbb{R}^n$  and  $\rho$  the square topology. Then, for all  $x, y \in \mathbb{R}^n$ , we have

$$\rho(x,y) \le d(x,y) \le \sqrt{n}\rho(x,y)$$

Proof:

 $\langle 1 \rangle 1. \ \rho(x,y) \leq d(x,y)$ 

 $\langle 2 \rangle 1$ . For  $1 \leq i \leq n$  we have  $|x_i - y_i| \leq d(x, y)$ 

PROOF: By the definition of the euclidean metric.

 $\langle 2 \rangle 2$ . Q.E.D.

PROOF: By the definition of the square metric.

$$\langle 1 \rangle 2. \ d(x,y) \le \sqrt{n} \rho(x,y)$$

$$d(x,y) = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2}$$

$$\leq \sqrt{\rho(x,y)^2 + \dots + \rho(x,y)^2}$$

$$= \sqrt{n\rho(x,y)^2}$$

$$= \sqrt{n}\rho(x,y)$$

Corollary 12.1.9.1. The euclidean metric induces the standard topology on  $\mathbb{R}^n$ .

**Definition 12.1.10.** Let  $l_2$  be the set of sequences  $\vec{a} \in \mathbb{R}^{\omega}$  such that  $\sum_{n=1}^{\infty} a_n^2 < \infty$ .

**Lemma 12.1.11.** If  $\vec{a}, \vec{b} \in l_2 \text{ then } \sum_{n=1}^{\infty} |a_n b_n| < \infty$ .

PROOF:

$$\sum_{n=1}^{N} |a_n b_n| \le \sqrt{(\sum_{n=1}^{N} a_n^2)(\sum_{n=1}^{N} b_n^2)}$$
 (Lemma 12.1.6)  
 
$$\to \sqrt{\sum_{n=1}^{\infty} a_n^2)(\sum_{n=1}^{\infty} b_n^2)} \text{ as } n \to \infty$$

**Lemma 12.1.12.** If  $\vec{a}, \vec{b} \in l_2$  then  $\vec{a} + \vec{b} \in l_2$ .

Proof:

$$\sum_{n=1}^{\infty} (a_n + b_n)^2 = \sum_{n=1}^{\infty} a_n^2 + 2 \sum_{n=1}^{\infty} a_n b_n + \sum_{n=1}^{\infty} b_n^2$$

$$\leq \sum_{n=1}^{\infty} a_n^2 + 2 \sum_{n=1}^{\infty} |a_n b_n| + \sum_{n=1}^{\infty} b_n^2$$

$$< \infty$$
(Lemma 12.1.11)

**Lemma 12.1.13.** If  $c \in \mathbb{R}$  and  $\vec{a} \in l_2$  then  $c\vec{a} \in l_2$ .

Proof: 
$$\sum_{n=1}^{\infty} (ca_n)^2 = c^2 \sum_{n=1}^{\infty} a_n^2$$
.

**Definition 12.1.14** (The  $l^2$ -metric). The  $l^2$ -metric is defined on  $l_2$  by

$$d(\vec{a}, \vec{b}) = \left[\sum_{n=1}^{\infty} (a_n - b_n)^2\right]^{\frac{1}{2}}$$
.

The topology induced by this metric is the  $l^2$ -topology. We write  $l_2$  for this set under the  $l^2$ -topology.

We prove this is a metric.

```
Proof:
```

 $\langle 1 \rangle 1. \ d(\vec{a}, \vec{b}) \geq 0$ 

PROOF: Immediate from definitions.

 $\langle 1 \rangle 2$ .  $d(\vec{a}, \vec{b}) = 0$  iff  $\vec{a} = \vec{b}$ 

PROOF: Immediate from definitions.

 $\langle 1 \rangle 3. \ d(\vec{a}, \vec{b}) = d(\vec{b}, \vec{a})$ 

PROOF: Immediate from definitions.

 $\langle 1 \rangle 4. \ d(\vec{a}, \vec{c}) \leq d(\vec{a}, \vec{b}) + d(\vec{b}, \vec{c})$ 

1)4. 
$$d(\vec{a}, \vec{c}) \leq d(\vec{a}, b) + d(b, \vec{c})$$
  
PROOF:  $\sqrt{\sum_{i=1}^{N} (a_n - c_n)^2} \leq \sqrt{\sum_{i=1}^{N} (a_n - b_n)^2} + \sqrt{\sum_{i=1}^{N} (b_n - c_n)^2}$  since the euclidean metric on  $\mathbb{R}^N$  is a metric.

**Definition 12.1.15** (Hilbert Cube). The *Hilbert cube* is  $\prod_{n=1}^{\infty} [0, 1/n]$  as a subspace of the  $l_2$ .

**Definition 12.1.16** (Isometric Imbedding). Let X, Y be metric spaces and f:  $X \to Y$ . Then f is an isometric imbedding iff, for all  $x, y \in X$ , d(f(x), f(y)) =d(x,y).

Lemma 12.1.17. Every isometric imbedding is an imbedding.

- $\langle 1 \rangle 1$ . Let:  $f: X \to Y$  be an isometric imbedding.
- $\langle 1 \rangle 2$ . f is continuous.

PROOF: If  $d(x,y) < \epsilon$  then  $d(f(x),f(y)) < \epsilon$ .

 $\langle 1 \rangle 3$ . f is injective.

PROOF: If f(x) = f(y) then d(f(x), f(y)) = 0 so d(x, y) = 0 hence x = y.

 $\langle 1 \rangle 4. \ f^{-1}: f(X) \to X$  is continuous.

PROOF: If  $d(f^{-1}(x), f^{-1}(y)) < \epsilon$  then  $d(x, y) < \epsilon$ .

# Chapter 13

# Topological Groups

### 13.1 Topological Groups

**Definition 13.1.1** (Topological Group). A topological group G consists of a group G that is also a  $T_1$  space such that  $\cdot: G^2 \to G$  and  $()^{-1}: G \to G$  are continuous.

**Proposition 13.1.2.** Every topological group is homogeneous.

```
Proof:
```

```
\begin{array}{ll} \langle 1 \rangle 1. \ \ \text{Let:} \ \ G \ \ \text{be a topological group.} \\ \langle 1 \rangle 2. \ \ \text{Let:} \ \ x,y \in G \\ \langle 1 \rangle 3. \ \ \text{Let:} \ \ f:G \to G \ \ \text{be given by} \ \ f(g) = yx^{-1}z \\ \langle 1 \rangle 4. \ \ f \ \ \text{is a homeomorphism} \\ \langle 1 \rangle 5. \ \ f(x) = y \end{array}
```

**Definition 13.1.3** (Symmetric). Let G be a topological group. A neighbourhood U of e is symmetric iff  $U = U^{-1}$ .

**Proposition 13.1.4.** For every neighbourhood U of e, there exists a symmetric neighbourhood V of e such that  $VV \subseteq U$ .

### PROOF:

```
\langle 1 \rangle 1. Let: m:G^2 \to G be the multiplication function \langle 1 \rangle 2. ee \in U
```

 $\langle 1 \rangle 3. \ (e,e) \in m^{-1}(U)$ 

 $\langle 1 \rangle 4$ . PICK neighbourhoods  $U_1, U_2$  of e such that  $(e, e) \in U_1 \times U_2 \subseteq m^{-1}(U)$ 

 $\langle 1 \rangle 5$ . Let:  $V' = U_1 \cap U_2$ 

 $\langle 1 \rangle 6. \ V'V' \subseteq U$ 

 $\langle 1 \rangle$ 7. Let:  $f: G^2 \to G$  be the function  $f(x,y) = xy^{-1}$ 

 $\langle 1 \rangle 8. \ (e,e) \in f^{-1}(V')$ 

 $\langle 1 \rangle 9$ . PICK a neighbourhood W of e such that  $WW^{-1} \subseteq V'$ 

 $\langle 1 \rangle 10$ . Let:  $V = WW^{-1}$ 

```
\langle 1 \rangle11. V is a neighbourhood of e PROOF: V is open because V = \bigcup_{a \in W^{-1}} Wa. \langle 1 \rangle12. V is symmetric \langle 1 \rangle13. VV \subseteq U
```

**Proposition 13.1.5.** Every topological group is regular.

### PROOF:

- $\langle 1 \rangle 1$ . Let: G be a topological group
- $\langle 1 \rangle 2$ . Let:  $A \subseteq G$  be closed and  $a \notin A$
- $\langle 1 \rangle 3$ .  $G \setminus Aa^{-1}$  is a neighbourhood of e
- $\langle 1 \rangle$ 4. PICK a symmetric neighbourhood V of e such that  $VV \subseteq G \setminus Aa^{-1}$  PROOF: Proposition 13.1.4.
- $\langle 1 \rangle 5.~VA$  and Va are disjoint neighbourhoods of A and a  $\Box$

**Proposition 13.1.6.** The long line is not second countable.

PROOF:Let  $\mathcal{B}$  be a basis for L. Then, for every countable ordinal  $\alpha$ ,  $\mathcal{B}$  mst contain a basic open set that contains  $(\alpha, 1/2)$  but not  $(\beta, 1/2)$  for any other  $\beta$ . Therefore,  $\mathcal{B}$  is uncountable.  $\square$ 

Corollary 13.1.6.1. The long line cannot be imbedded in  $\mathbb{R}$ .

**Theorem 13.1.7.** Let  $f: X \to Y$ . Let Y be compact Hausdorff. Then f is continuous if and only if the graph of f is closed in  $X \times Y$ .

### Proof:

- $\langle 1 \rangle 1$ . Let:  $G_f$  be the graph of f.
- $\langle 1 \rangle 2$ . If f is continuous then the graph of f is closed.
  - $\langle 2 \rangle 1$ . Assume: f is continuous.
  - $\langle 2 \rangle 2$ . Let:  $(x,y) \in (X \times Y) \setminus G_f$
  - $\langle 2 \rangle 3. \ y \neq f(x)$
  - $\langle 2 \rangle$ 4. PICK disjoint open neighbourhoods U of f(x) and V of y PROOF: Y is Hausdorff.
  - $\langle 2 \rangle 5. \ (x,y) \in f^{-1}(U) \times V \subseteq (X \times Y) \setminus G_f$
  - $\langle 2 \rangle 6$ . Q.E.D.
- $\langle 1 \rangle 3$ . If the graph of f is closed then f is continuous.
  - $\langle 2 \rangle 1$ . Assume:  $G_f$  is closed.
  - $\langle 2 \rangle 2$ . Let:  $x_0 \in X$  and V be an open neighbourhood of  $f(x_0)$
  - $\langle 2 \rangle 3.$   $G_f \cap (X \times (Y \setminus V))$  is closed
  - $\langle 2 \rangle 4$ .  $\pi_1(G_f \cap (X \times (Y \setminus V)))$  is closed

Proof: Lemma 9.4.16

 $\langle 2 \rangle 5. \ x_0 \in X \setminus \pi_1(G_f \cap (X \times (Y \setminus V))) \subseteq f^{-1}(V)$ 

 $\langle 2 \rangle 6$ . Q.E.D.

**Theorem 13.1.8.** Let X be a compact Hausdorff space. Let A be a set of closed connected subspaces of X that is linearly ordered by proper inclusion. Then

$$Y = \bigcap \mathcal{A}$$

is connected.

### Proof:

- $\langle 1 \rangle 1$ . Assume: for a contradiction C and D form a separation of Y
- $\langle 1 \rangle 2$ . Pick disjoint U and V open in X such that  $C = U \cap Y$  and  $D = V \cap Y$ 
  - $\langle 2 \rangle 1$ . C and D are compact
    - $\langle 3 \rangle 1$ . Y is compact

PROOF: Y is a closed subset of X, hence compact by Proposition 9.4.6.

 $\langle 3 \rangle 2$ . Q.E.D.

PROOF: C and D are closed subsets of Y hence compact by Proposition 9.4.6.

 $\langle 2 \rangle 2$ . Q.E.D.

PROOF: By Lemma 9.4.18.

 $\langle 1 \rangle 3$ . For all  $A \in \mathcal{A}$ , we have  $A \setminus (U \cup V)$  is nonempty

Proof: Since A is connected.

 $\langle 1 \rangle 4$ .  $\{ A \setminus (U \cup V) : A \in \mathcal{A} \}$  has the finite intersection property

PROOF: This holds because A is linearly ordered under proper inclusion.

 $\langle 1 \rangle 5. \bigcap_{A \in \mathcal{A}} (A \setminus (U \cup V))$  is nonempty

PROOF: By Proposition 9.4.15.

П

**Theorem 13.1.9.** Let  $A \subseteq \mathbb{R}^n$ . Then the following are equivalent:

- 1. A is compact.
- 2. A is closed and bounded under the euclidean metric.
- 3. A is closed and bounded under the square metric.

### Proof:

- $\langle 1 \rangle 1. \ 1 \Rightarrow 2$ 
  - $\langle 2 \rangle 1$ . Assume: A is compact.
  - $\langle 2 \rangle 2$ . A is closed.

Proof: By Proposition 9.4.9.

- $\langle 2 \rangle 3. \{B(\vec{0}, n) : n \in \mathbb{Z}^+\} \text{ covers } A$
- $\langle 2 \rangle 4$ . PICK a finite subcover  $\{B(\vec{0}, n_1), \dots, B(\vec{0}, n_k)\}$
- $\langle 2 \rangle 5$ . Let:  $N = \max(n_1, \ldots, n_k)$
- $\langle 2 \rangle 6$ . For all  $x, y \in A$  we have d(x, y) < 2N

PROOF: We have  $d(x,y) \le d(\vec{0},x) + d(\vec{0},y) < N + N$ .

 $\langle 1 \rangle 2. \ 2 \Rightarrow 3$ 

PROOF: If  $d(x,y) < \epsilon$  for all  $x,y \in A$  then  $\rho(x,y) < \epsilon \sqrt{n}$  by Lemma 12.1.9.  $\langle 1 \rangle 3$ .  $3 \Rightarrow 1$ 

 $\langle 2 \rangle 1$ . Assume: A is closed and  $\rho(x,y) < \epsilon$  for all  $x,y \in A$ 

```
\begin{array}{ll} \langle 2 \rangle 2. & \text{Pick } x_0 \in A \\ \langle 2 \rangle 3. & \text{Let: } b = \rho(\vec{0}, x_0) \\ \langle 2 \rangle 4. & \text{Let: } P = \epsilon + b \\ \langle 2 \rangle 5. & A \subseteq [-P, P]^n \\ & \text{Proof:For any } y \in A \text{ we have} \\ & \rho(\vec{0}, y) \leq \rho(\vec{0}, x_0) + \rho(x_0, y) \\ & < b + \epsilon \\ & = P \\ & (\langle 2 \rangle 3, \, \langle 2 \rangle 1) \\ & = P \\ & (\langle 2 \rangle 4) \\ \langle 2 \rangle 6. & [-P, P]^n \text{ is compact.} \\ & \text{Proof: By Corollary } 9.4.19.1 \text{ and Proposition } 9.4.14. \\ \langle 2 \rangle 7. & \text{Q.E.D.} \\ & \text{Proof: By Proposition } 9.4.6. \end{array}
```

**Theorem 13.1.10** (AC). Let X be a topological space. Then X is compact if and only if every nonempty net in X has a convergent subnet.

#### PROOF

- $\langle 1 \rangle 1$ . If X is compact then every nonempty net in X has a convergent subnet.
  - $\langle 2 \rangle 1$ . Assume: X is compact.
  - $\langle 2 \rangle 2$ . Let:  $(x_{\alpha})_{\alpha \in J}$  be a nonempty net in X
  - $\langle 2 \rangle 3$ . For  $\alpha \in J$ , LET:  $B_{\alpha} = \{ \beta \in J : \alpha \leq \beta \}$ .
  - $\langle 2 \rangle 4$ .  $\{B_{\alpha} : \alpha \in J\}$  has the finite intersection property.
    - $\langle 3 \rangle 1$ . Let:  $\alpha_1, \ldots, \alpha_n \in J$
    - $\langle 3 \rangle 2$ . Pick  $\beta \in J$  such that  $\alpha_1 \leq \beta, \ldots, \alpha_n \leq \beta$
    - $\langle 3 \rangle 3. \ x_{\beta} \in B_{\alpha_1} \cap \cdots \cap B_{\alpha_n}$
  - $\langle 2 \rangle$ 5. Pick  $l \in \bigcap_{\alpha \in J} B_{\alpha}$

Proof: Proposition 9.4.15.

- $\langle 2 \rangle 6$ . Let:  $K = \{ \alpha \in J : x_{\alpha} = l \}$
- $\langle 2 \rangle 7$ . K is cofinal in J
  - $\langle 3 \rangle 1$ . Let:  $\alpha \in J$
  - $\langle 3 \rangle 2. \ l \in B_{\alpha}$

Proof: By  $\langle 2 \rangle 5$ .

- $\langle 3 \rangle 3$ . There exists  $\beta \geq \alpha$  such that  $x_{\beta} = l$ .
- $\langle 2 \rangle 8$ .  $(x_{\alpha})_{\alpha \in K}$  is a subnet of  $(x_{\alpha})_{\alpha \in J}$  that converges to l.
- $\langle 1 \rangle 2$ . If every nonempty net in X has a convergent subnet then X is compact.
  - $\langle 2 \rangle 1$ . Assume: Every nonempty net in X has a convergent subnet
  - $\langle 2 \rangle$ 2. Let:  $\mathcal{A}$  be a nonempty set of closed sets with the finite intersection property.
  - $\langle 2 \rangle$ 3. Let: J be the poset of all finite intersections of elements of A under  $\supseteq$
  - $\langle 2 \rangle 4$ . Pick  $x_C \in C$  for all  $C \in J$

PROOF: These are all nonempty by  $\langle 2 \rangle 2$ .

 $\langle 2 \rangle$ 5. Pick an accumulation point l of  $(x_C)$ 

Prove:  $l \in \bigcap \mathcal{A}$ 

Proof: One exists by Lemma 3.13.2.

```
\langle 2 \rangle 6. Let: C \in \mathcal{A}
           Prove: l \in C
   \langle 2 \rangle 7. Let: U be a neighbourhood of l
          Prove: U intersects C
   \langle 2 \rangle 8. Pick D \subseteq C such that x_D \in U
      Proof: By \langle 2 \rangle 5.
   \langle 2 \rangle 9. U intersects C
   \langle 2 \rangle 10. \ l \in C
      PROOF: By Theorem 3.8.3 since C is closed (\langle 2 \rangle 2).
   \langle 2 \rangle 11. Q.E.D.
      Proof: Proposition 9.4.15.
Corollary 13.1.10.1 (AC). Let G be a topological group. Let A and B be
subsets of G. If A is closed in G and B is compact then AB is closed in G.
Proof:
\langle 1 \rangle 1. Let: c \in \overline{AB}
        Prove: c \in AB
\langle 1 \rangle 2. PICK a net (x_{\alpha})_{{\alpha} \in J} that converges to c
   PROOF: By Theorem 3.12.3.
\langle 1 \rangle 3. For \alpha \in J, PICK a_{\alpha} \in A and b_{\alpha} \in B such that x_{\alpha} = a_{\alpha} b_{\alpha}
\langle 1 \rangle 4. PICK a convergent subnet (b_{q(\beta)})_{\beta \in K} of (b_{\alpha})_{\alpha \in J}
   PROOF: By Theorem 13.1.10.
\langle 1 \rangle 5. Let: b_{g(\beta)} \to b
\langle 1 \rangle 6. \ b \in B
   \langle 2 \rangle 1. B is closed
      Proof: By Proposition 9.4.9.
   \langle 2 \rangle 2. Q.E.D.
      PROOF: By Theorem 3.12.3
\langle 1 \rangle 7. \ a_{g(\beta)} \to cb^{-1}
   PROOF: By Theorem 3.12.4
\langle 1 \rangle 8. \ cb^{-1} \in A
   PROOF: By Theorem 3.12.3
\langle 1 \rangle 9. \ c \in AB
\langle 1 \rangle 10. Q.E.D.
   Proof: By Proposition 3.7.6.
Proposition 13.1.11. Let A_0 + A_1 be the sum of A_0 and A_1 with injections
i_0: A_0 \to A_0 + A_1 \text{ and } i_1: A_1 \to A_0 + A_1.
    Let g: B \to A_0 + A_1 be a function.
    Let B_0 be the pullback of i_0 and g with projections j_0: B_0 \to B and k_0:
B_0 \to A_0.
    Let B_1 be the pullback of i_1 and g with projection sj_1: B_1 \to B and k_1:
```

Then B is the sum of  $B_0$  and  $B_1$  with injections  $j_0$  and  $j_1$ .

 $B_1 \to A_1$ .

$$B_0 \xrightarrow{j_0} B \xleftarrow{j_1} B_1$$

$$\downarrow \qquad \qquad \downarrow g \qquad \qquad \downarrow \downarrow$$

$$A_0 \xrightarrow{i_0} A_0 + A_1 \xleftarrow{i_1} A_1$$

 $\langle 1 \rangle 1$ . Let: X be any set and  $x: B_0 \to X, y: B_1 \to X$ 

**Proposition 13.1.12** (CC). Let X be a space and  $\mathcal{B}$  be a basis for X. Suppose that every subset of  $\mathcal{B}$  that covers X has a countable subcover. Then X is Lindelöf.

### Proof:

- $\langle 1 \rangle 1$ . Let:  $\mathcal{A}$  be an open cover of X.
- $\langle 1 \rangle 2$ .  $\{ B \in \mathcal{B} : \exists U \in \mathcal{A}.B \subseteq U \}$  covers X.
- $\langle 1 \rangle 3$ . Pick a countable subcover  $\mathcal{B}_0$
- $\langle 1 \rangle 4$ . For  $B \in \mathcal{B}_0$ , PICK  $U_B \in \mathcal{A}$  such that  $B \subseteq U_B$
- $\langle 1 \rangle$ 5.  $\{U_B : B \in \mathcal{B}_0\}$  is a countable subcover of  $\mathcal{A}$ .

### **Proposition 13.1.13** (CC). The space $\mathbb{R}_l$ is Lindelöf.

### Proof:

- $\langle 1 \rangle$ 1. Let:  $\mathcal{A}$  be a set of basis elements [a,b) that covers X Prove:  $\mathcal{A}$  has a countable subcover.
- $\langle 1 \rangle 2$ . Let:  $C = \bigcup \{(a,b) : [a,b) \in \mathcal{A}\}$
- $\langle 1 \rangle 3$ .  $\mathbb{R} \setminus C$  is countable.
  - $\langle 2 \rangle$ 1. For all  $x \in \mathbb{R} \setminus C$ , PICK a rational  $q_x$  such that there exists b such that  $q_x \in [x,b) \in \mathcal{A}$ 
    - $\langle 3 \rangle 1$ . PICK  $[a,b) \in \mathcal{A}$  such that  $x \in [a,b)$
    - $\langle 3 \rangle 2. \ x = a$

PROOF: If not we would have  $x \in C$ 

- $\langle 3 \rangle 3$ . There exists a rational in (a, b)
- $\langle 2 \rangle 2$ . For  $x, y \in \mathbb{R} \setminus C$ , if x < y then  $q_x < q_y$ 
  - $\langle 3 \rangle 1$ . PICK b, c such that  $q_x \in [x, b) \in \mathcal{A}$  and  $q_y \in [y, c) \in \mathcal{A}$  PROOF: By  $\langle 2 \rangle 1$ .
  - $\langle 3 \rangle 2. \ b \leq y$

PROOF: Otherwise we would have  $y \in (x, b) \subseteq C$ .

 $\langle 3 \rangle 3. \ q_x < q_y$ 

Proof:  $q_x < b \le y \le q_y$ 

- $\langle 2 \rangle 3$ . The map  $q_- : \mathbb{R} \setminus C \to \mathbb{Q}$  is injective.
- $\langle 1 \rangle 4$ . For  $x \in \mathbb{R} \setminus C$ , PICK  $[a_x, b_x) \in \mathcal{A}$  such that  $a_x \leq x < b_x$
- $\langle 1 \rangle$ 5. PICK a countable subset  $((a_n, b_n))_{n \in \mathbb{Z}^+}$  of  $\{(a, b) : [a, b) \in \mathcal{A}\}$  that covers C
  - $\langle 2 \rangle 1.$  The set C as a subspace of  $\mathbb R$  with the standard topology is second countable.

- $\langle 2 \rangle 2$ . The set C as a subspace of  $\mathbb{R}$  with the standard topology is Lindelöf. PROOF: By Theorem 9.3.2.
- $\langle 1 \rangle 6. \{ [a_x, b_x) : x \in \mathbb{R} \setminus C \} \cup \{ [a_n, b_n) : n \in \mathbb{Z}^+ \}$  is a countable subcover of A.  $\langle 1 \rangle 7$ . Q.E.D.

Proof: By Proposition 13.1.12.

**Proposition 13.1.14** (AC). The space  $\mathbb{R}_l$  is not second countable.

### Proof:

- $\langle 1 \rangle 1$ . Let:  $\mathcal{B}$  be any basis for  $\mathbb{R}_l$
- $\langle 1 \rangle 2$ . For  $x \in \mathbb{R}$ , Pick  $B_x \in \mathcal{B}$  such that  $x \in B_x \subseteq [x, x+1)$
- $\langle 1 \rangle 3$ . The mapping  $B_{(-)}$  is an injective function  $\mathbb{R} \to \mathcal{B}$

PROOF: For any x we have  $x = \min B_x$ .

 $\langle 1 \rangle 4$ .  $\mathcal{B}$  is uncountable.

**Proposition 13.1.15.** The product of a Lindelöf space and a compact space is Lindelöf.

### PROOF:

- $\langle 1 \rangle 1$ . Let: X be a Lindelöf space and Y a compact space.
- $\langle 1 \rangle 2$ . Let: A be an open covering of  $X \times Y$
- $\langle 1 \rangle 3$ . For all  $x \in X$ , there exists a neighbourhood W of x such that  $W \times Y$  is covered by finitely many elements of A.
  - $\langle 2 \rangle 1$ . Let:  $x \in X$
  - $\langle 2 \rangle 2$ .  $\{x\} \times Y$  is compact.

PROOF: It is homeomorphic to Y.

- $\langle 2 \rangle 3$ . PICK a finite subset  $\{U_1, \ldots, U_m\}$  of  $\mathcal{A}$  that covers  $\{x\} \times Y$ Proof: By Proposition 9.4.5.
- $\langle 2 \rangle 4$ . There exists a neighbourhood W of x such that  $W \times Y \subseteq U_1 \cup \cdots \cup U_m$ PROOF: By the Tube Lemma.
- $\langle 1 \rangle 4$ . {W open in  $X: W \times Y$  is covered by finitely many elements of  $\mathcal{A}$ } is an open covering of X.
- $\langle 1 \rangle$ 5. PICK a countable subcovering  $\{W_1, W_2, \ldots\}$
- $\langle 1 \rangle$ 6. For  $i \geq 1$ , PICK a finite subset  $\{U_{i1}, \ldots, U_{ir_i}\}$  of  $\mathcal{A}$  that covers  $W_i \times Y$
- $\langle 1 \rangle 7$ .  $\{U_{1j} : i \geq 1, 1 \leq j \leq r_i\}$  is a countable subcovering of  $\mathcal{A}$ .

**Proposition 13.1.16.** Let X be a  $T_1$  space. Then X is normal if and only if, for every closed set A and open set  $U \supseteq A$ , there exists an open set  $V \supseteq A$  such that  $\overline{V} \subseteq U$ .

### PROOF:

- $\langle 1 \rangle 1$ . If X is normal then, for every closed set A and open set  $U \supseteq A$ , there exists an open set  $V \supseteq A$  such that  $\overline{V} \subseteq U$ .
  - $\langle 2 \rangle 1$ . Assume: X is normal.
  - $\langle 2 \rangle 2$ . Let: A be a closed set and U an open set with  $A \subseteq U$

- $\langle 2 \rangle 3$ . PICK disjoint open sets V, W such that  $A \subseteq V$  and  $X \setminus U \subseteq W$
- $\langle 2 \rangle 4. \ \overline{V} \subseteq U$

$$\overline{V} \subseteq X \setminus W$$
 
$$\subseteq U$$

- $\langle 1 \rangle 2$ . If, for every closed set A and open set  $U \supseteq A$ , there exists an open set  $V \supseteq A$  such that  $\overline{V} \subseteq U$ , then X is normal.
  - $\langle 2 \rangle$ 1. Assume: for every closed set A and open set  $U \supseteq A$ , there exists an open set  $V \supseteq A$  such that  $\overline{V} \subseteq U$ .
  - $\langle 2 \rangle 2$ . Let: A, B be disjoint closed sets
  - $\langle 2 \rangle 3$ . PICK an open set V such that  $A \subseteq V$  and  $\overline{V} \subseteq X \setminus B$
- $\langle 2 \rangle 4$ .  $A \subseteq V$  and  $B \subseteq X \setminus \overline{V}$

**Definition 13.1.17** (Action). Let G be a topological group and X a topological space. An *action* of G on X is a continuous function  $\cdot : G \times X \to X$  such that, for all  $g, h \in G$  and  $x \in X$ :

- 1.  $e \cdot x = x$
- 2.  $g \cdot (h \cdot x) = gh \cdot x$

**Definition 13.1.18** (Orbit Space). Let G be a topological group, X a topological space, and  $\cdot: G \times X \to X$  an action of G on X. Then the *orbit space* X/G is the quotient space of X by the equivalence relation  $\sim$  generated by  $x \sim g \cdot x$  for all  $x \in X$ ,  $g \in G$ .

**Theorem 13.1.19.** Let G be a topological group. Let X be a topological space. Let  $\cdot : G \times X \to X$  be an action of G on X. Then the canonical map  $\pi : X \twoheadrightarrow X/G$  is perfect.

- $\langle 1 \rangle 1$ .  $\pi$  is closed.
  - $\langle 2 \rangle 1$ . Let:  $A \subseteq X$  be closed.
  - $\langle 2 \rangle 2$ .  $GA = \{g \cdot a : g \in G, a \in A\}$  is closed
    - $\langle 3 \rangle 1$ . Let:  $z \notin GA$
    - $\langle 3 \rangle 2$ . For all  $g \in G$  we have  $g \cdot z \notin A$
    - $\langle 3 \rangle 3$ . For  $g \in G$ , there exist U an open neighbourhood of g and V an open neighbourhood of z such that UV does not intersect A
    - $\langle 3 \rangle 4. \ \{U \ \text{open in} \ G: \exists V \ \text{an open neighbourhood of} \ z.UV \cap A = \emptyset \}$  covers G
    - $\langle 3 \rangle 5$ . PICK a finite subcover  $\{U_1, \ldots, U_n\}$
    - (3)6. For  $1 \leq i \leq n$ , PICK  $V_i$  an open neighbourhood of z such that  $U_i V_i \cap A = \emptyset$
    - $\langle 3 \rangle 7. \ z \in V_1 \cap \cdots \cap V_n \subseteq X \setminus GA$
  - $\langle 2 \rangle 3$ .  $\pi(A)$  is closed
    - $\pi^{-1}(\pi(A)) = GA$
- $\langle 1 \rangle 2$ .  $\pi$  is continuous.

PROOF: By definition of the quotient topology.

```
\langle 1 \rangle 3. \pi is surjective.
   PROOF: By definition.
\langle 1 \rangle 4. For all a \in X/G we have \pi^{-1}(a) is compact.
   \langle 2 \rangle 1. Let: a \in X/G
   \langle 2 \rangle 2. PICK x \in X such that a = \pi(x)
   \langle 2 \rangle 3. \ \pi^{-1}(a) = \{ gx : g \in G \}
   \langle 2 \rangle 4. \pi^{-1}(a) is homeomorphic to G
Corollary 13.1.19.1. If X is Hausdorff then so is X/G.
Corollary 13.1.19.2. If X is regular then so is X/G.
Corollary 13.1.19.3. If X is normal then so is X/G.
Corollary 13.1.19.4. If X is locally compact then so is X/G.
Corollary 13.1.19.5. If X is second countable then so is X/G.
Proposition 13.1.20. Let p: X \to Y be perfect. If X is second countable then
so is Y.
Proof:
\langle 1 \rangle 1. PICK a countable basis \mathcal{B} for X
\langle 1 \rangle 2. Let: \mathcal{J} = \{ J \subseteq^{\text{fin}} \mathcal{B} : \exists W \text{ open in } Y.p^{-1}(W) \subseteq \bigcup J \}
\langle 1 \rangle 3. For every J \in \mathcal{J},
        Let: W_J = \bigcup \{ W \text{ open in } Y : p^{-1}(W) \subseteq \bigcup J \}.
        PROVE: \{W_J : J \in \mathcal{J}\} is a basis for Y.
\langle 1 \rangle 4. \ y \in V \text{ where } V \text{ is open in } Y
\langle 1 \rangle 5. \{ B \in \mathcal{B} : x \in B \subseteq p^{-1}(V) \} covers p^{-1}(y)
\langle 1 \rangle6. Pick a countable subcover J \subseteq^{\text{fin}} \mathcal{B}
\langle 1 \rangle 7. \ y \in W_J \subseteq V
   \langle 2 \rangle 1. \ p^{-1}(y) \subseteq \bigcup J
   \langle 2 \rangle 2. Pick an open neighbourhood W of y such that p^{-1}(W) \subseteq \bigcup J
      Proof: By Proposition 9.5.1.
   \langle 2 \rangle 3. \ W \subseteq W_J
П
Proposition 13.1.21. A subspace of a T_1 space is T_1.
PROOF:
\langle 1 \rangle 1. Let: X be T_1 and Y \subseteq X
\langle 1 \rangle 2. Let: a \in Y
\langle 1 \rangle 3. \{a\} is closed in X
\langle 1 \rangle 4. \{a\} is closed in Y
   PROOF: By Corollary 4.3.4.1.
Proposition 13.1.22 (DC). Not every topological group is normal.
```

Proof: From Proposition 6.5.6.  $\square$ 

**Theorem 13.1.23.** A subspace of a completely regular space is completely regular.

### PROOF:

- $\langle 1 \rangle 1$ . Let: X be completely regular and  $Y \subseteq X$
- $\langle 1 \rangle 2$ . Let:  $a \in Y$  and A be closed in Y such that  $a \notin A$
- $\langle 1 \rangle 3$ . PICK C closed in X such that  $A = X \cap C$
- $\langle 1 \rangle 4.$  Pick a continuous function  $f: X \to [0,1]$  such that f(a) = 0 and  $f(C) = \{1\}$
- $\langle 1 \rangle 5.$   $f \upharpoonright Y:Y \to [0,1]$  is a continuous function such that  $(f \upharpoonright Y)(a)=0$  and  $(f \upharpoonright Y)(A)=\{1\}$

**Proposition 13.1.24** (DC). Every topological group is completely regular.

### PROOF:

- $\langle 1 \rangle 1$ . Let: G be a topological group
- $\langle 1 \rangle 2$ . Let:  $x \in G$  and  $A \subseteq G$  be closed such that  $x \notin A$  Prove: There exists a continuous  $f: G \to [0,1]$  such that f(x) = 0 and  $f(A) = \{1\}$
- $\langle 1 \rangle 3$ . Assume: w.l.o.g. x = e

PROOF:  $\lambda y.x^{-1}y$  is an automorphism of G that maps x to e.

- $\langle 1 \rangle$ 4. PICK a sequence  $V_n$   $(n \geq 0)$  of symmetric neighbourhoods of e disjoint from A such that  $V_n V_n \subseteq V_{n-1}$  for all n
  - $\langle 2 \rangle 1$ . Let:  $V_0 = X \setminus A$
  - $\langle 2 \rangle$ 2. Given  $V_n$ , PICK a symmetric neighbourhood  $V_{n+1}$  of e such that  $V_{n+1}V_{n+1} \subseteq V_n$

PROOF: By Proposition 13.1.4.

 $\langle 1 \rangle 5$ . For every dyadic rational p, define an open set U(p) as follows:

$$U(1/2^{n}) = V_{n} (n \ge 0)$$

$$U((2k+1)/2^{n+1}) = V_{n+1}U(k/2^{n}) (0 < k < 2^{n})$$

$$U(p) = \emptyset (p \le 0)$$

$$U(p) = G (p > 1)$$

 $\langle 1 \rangle 6$ . For all k and n, we have

$$V_n U(k/2^n) \subseteq U((k+1)/2^n)$$

 $\langle 2 \rangle 1. \ k \leq 0$ 

PROOF: In this case,  $V_nU(k/2^n) = \emptyset$ 

 $\langle 2 \rangle 2$ . k = 1 and n > 0

Proof:

$$V_n U(1/2^n) = V_n V_n$$

$$\subseteq V_{n-1}$$

$$= U(1/2^{n-1})$$

 $\langle 2 \rangle 3$ . k = 2a for some  $0 < a < 2^{n-1}$ 

$$V_n U(2a/2^n) = V_n U(a/2^{n-1})$$
 
$$= U(2a+1/2^n)$$
 
$$\langle 2 \rangle 4. \ k=2a+1 \text{ for some } 0 < a < 2^{n-1}$$
 Proof:

Proof:

$$V_n U((2a+1)/2^n) = V_n V_n U(a/2^{n-1})$$

$$\subseteq V_{n-1} U(a/2^{n-1})$$

$$\subseteq U((a+1)/2^{n-1})$$

 $\langle 2 \rangle 5. \ k \geq 2^n$ 

PROOF: In this case,  $U((k+1)/2^n) = G$ .

 $\langle 1 \rangle 7$ . Define  $f: G \to [0,1]$  by

$$f(x) = \inf\{p : x \in U(p)\}\$$

PROOF: This set is nonempty because  $x \in U(1)$  and bounded below because if  $x \in U(p)$  then p > 0.

- $\langle 1 \rangle 8$ . For n > 0 we have  $\overline{U(k/2^n)} \subseteq V_n U(k/2^n)$ 
  - $\langle 2 \rangle 1$ . Let:  $x \in U(k/2^n)$
  - $\langle 2 \rangle 2$ .  $V_n x$  is a neighbourhood of x
  - $\langle 2 \rangle 3$ . Pick  $y \in V_n x \cap U(k/2^n)$
  - $\langle 2 \rangle 4$ . Pick  $z \in V_n$  such that y = zx
  - $\langle 2 \rangle 5. \ \ x = z^{-1}y$
- $\langle 1 \rangle 9$ . For p and q dyadic rationals, if p < q then  $\overline{U(p)} \subseteq U(q)$
- $\langle 1 \rangle 10$ . If  $x \in \overline{U(p)}$  then  $f(x) \leq p$ 
  - $\langle 2 \rangle 1$ . For all q > p we have  $x \in U(q)$
  - $\langle 2 \rangle 2$ . For all q > p we have  $f(x) \leq q$
- $\langle 1 \rangle 11$ . If  $x \notin U(p)$  then  $f(x) \geq p$

PROOF: If  $x \notin U(p)$  and  $x \in U(q)$  then q > p.

- $\langle 1 \rangle 12$ . f is continuous
  - $\langle 2 \rangle 1$ . Let:  $x_0 \in X$
  - $\langle 2 \rangle 2$ . Let:  $c < f(x_0) < d$

PROVE: There exist a neighbourhood U of  $x_0$  such that  $f(U) \subseteq (c,d)$ 

- $\langle 2 \rangle 3$ . Pick rational numbers p, q such that c
- $\langle 2 \rangle 4. \ x \notin \overline{U(p)}$
- $\langle 2 \rangle 5. \ x \in U(q)$
- $\langle 2 \rangle 6$ . Take  $U = U(q) \setminus \overline{U(p)}$
- $\langle 1 \rangle 13. \ f(e) = 0$

PROOF: We have  $e \in U(1/2^n)$  for all n.

 $\langle 1 \rangle 14. \ f(A) = \{1\}$ 

PROOF: If  $x \in A$  and  $x \in U(p)$  then p > 1.

**Definition 13.1.25** (Bijection). A function  $f: A \to B$  is a bijection,  $f: A \cong B$ , iff there exists a function  $f^{-1}: B \to A$ , the inverse of f, such that  $f^{-1} \circ f = \mathrm{id}_A$ and  $f \circ f^{-1} = id_B$ .

**Theorem 13.1.26.** Let Y be a normal space. Then Y is an absolute retract if and only if Y has the universal extension property.

### PROOF:

- $\langle 1 \rangle 1$ . If Y is an absolute retract then Y has the universal extension property.
  - $\langle 2 \rangle 1$ . Assume: Y is an absolute retract.
  - $\langle 2 \rangle 2$ . Let: X be a normal space, A a closed subspace of X and  $f: A \to Y$  a continuous function.
  - $\langle 2 \rangle 3$ . Let:  $Z_f$  be the quotient space of  $X \cup Y$  under:  $a \sim f(a)$  for all  $a \in A$
  - $\langle 2 \rangle 4$ . Let:  $p: X \cup Y \rightarrow Z_f$  be the quotient map
  - $\langle 2 \rangle$ 5. For all  $x_1, x_2 \in X$  we have  $p(x_1) = p(x_2)$  iff  $x_1 = x_2$  or  $x_1, x_2 inA$  and  $f(x_1) = f(x_2)$ ; and for  $x \in X$  and  $y \in Y$  we have p(x) = p(y) iff f(x) = y; and for  $y_1, y_2 \in Y$  we have  $p(y_1) = p(y_2)$  iff  $y_1 = y_2$
  - $\langle 2 \rangle 6$ . p imbeds Y into a closed subspace of  $Z_f$ 
    - $\langle 3 \rangle 1$ . p is injective on Y
    - $\langle 3 \rangle 2. \ p^{-1} : p(Y) \to Y \text{ is continuous}$ 
      - $\langle 4 \rangle$ 1. Let:  $U \subseteq Y$  be open
      - PROVE: p(U) is open  $\langle 4 \rangle 2$ .  $p^{-1}(p(U)) = f^{-1}(U) \cup U$
    - $\langle 3 \rangle 3$ . p(Y) is closed
      - PROOF:  $p^{-1}(p(Y)) = A \cup Y$
  - $\langle 2 \rangle 7$ .  $Z_f$  is normal
    - $\langle 3 \rangle 1$ .  $Z_f$  is  $T_1$

PROOF: For  $y \in Y$  we have  $p^{-1}(y) = f^{-1}(y) \cup \{y\}$  which is closed.

- $\langle 3 \rangle 2$ . Any two disjoint closed sets in  $Z_f$  can be separated by a continuous function.
  - $\langle 4 \rangle 1$ . Let: C and D be disjoint closed sets in  $Z_f$
  - $\langle 4 \rangle 2$ . PICK  $g: Y \to [0,1]$  such that  $g(Y \cap p^{-1}(C)) = \{0\}$  and  $g(Y \cap p^{-1}(D)) = \{1\}$

PROOF: By the Urysohn Lemma.

 $\langle 4 \rangle 3$ . Pick  $h: X \to [0,1]$  such that  $h(X \cap p^{-1}(C)) = \{0\}$  and  $h(X \cap p^{-1}(D)) = \{1\}$  and h agrees with  $g \circ f$  on A

PROOF: By the Tietze Extension Theorem applied to  $A \cup (X \cap p^{-1}(C)) \cup (X \cap p^{-1}(D))$ .

(4)4. Let:  $k: Z_f \to [0,1]$  be the continuous function such that k(p(x)) = h(x) for  $x \in X$  and k(p(y)) = g(y) for  $y \in Y$ 

Proof: By the Pasting Lemma

- $\langle 4 \rangle 5. \ k(C) = \{0\}$
- $\langle 4 \rangle 6. \ k(D) = \{1\}$
- $\langle 3 \rangle 3$ . Q.E.D.

PROOF: If g is such a continuous function then  $g^{-1}([0,1/2))$  and  $g^{-1}((1/2,1])$  are disjoint open sets that include A and B respectively.

- $\langle 2 \rangle 8$ . PICK a retraction  $r: Z_f \to p(Y)$
- $\langle 2 \rangle 9. \ p^{-1} \circ r \circ p : X \to Y \text{ extends } f$
- $\langle 1 \rangle 2$ . If Y has the universal extension property then Y is an absolute retract.
  - $\langle 2 \rangle 1$ . Assume: Y has the universal extension property
  - $\langle 2 \rangle$ 2. Let: Z be a normal space,  $Y_0$  a closed subspace of Z, and  $\phi: Y \cong Y_0$  a homeomorphism
  - $\langle 2 \rangle 3$ . PICK a continuous extension  $f: Z \to Y$  of  $\phi^{-1}$

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Theorem 13.1.27. Every manifold is metrizable.
\langle 1 \rangle 1. Let: X be an m-manifold.
\langle 1 \rangle 2. X is regular.
   \langle 2 \rangle 1. X is T_1
   \langle 2 \rangle 2. Let: x \in X and U be a neighbourhood of x
   \langle 2 \rangle 3. PICK a neighbourhood V of x that is imbeddable in \mathbb{R}^m
   \langle 2 \rangle 4. PICK a neighbourhood W of x such that \overline{W} \subseteq U \cap V
      PROOF: One exists since V is regular (Proposition 6.3.4)
   \langle 2 \rangle 5. \ x \in W \text{ and } \overline{W} \subseteq U
   \langle 2 \rangle 6. Q.E.D.
      Proof: Proposition 6.3.2
\langle 1 \rangle 3. Q.E.D.
   PROOF: By the Urysohn Metrization Theorem.
Theorem 13.1.28. Let X be a compact Hausdorff space in which every point
has a neighbourhood that is imbeddable in \mathbb{R}^m. Then X is an m-manifold.
Proof:
\langle 1 \rangle 1. There exists N such that X is imbeddable in \mathbb{R}^N
   PROOF: Theorem 11.1.5
\langle 1 \rangle 2. X is second countable.
   Proof: Proposition 7.3.3
Proposition 13.1.29. S_{\Omega} is locally metrizable.
PROOF: For any \alpha \in S_{\Omega}, the neighbourhood [0, \alpha] = (-\infty, \alpha + 1) is imbeddable
Proposition 13.1.30 (DC). \overline{S_{\Omega}} is compact.
PROOF: PROOF:
\langle 1 \rangle 1. Let: \mathcal{A} be an open cover of \overline{S_{\Omega}}
\langle 1 \rangle 2. Assume: for a contradiction there is no finite subcover of \mathcal{A}
\langle 1 \rangle 3. There exists a sequence of sets U_n \in \mathcal{A} and ordinals \alpha_n such that \alpha_{n+1} < 1
        \alpha_n for all n and \alpha_n \in U_n for all n
   \langle 2 \rangle 1. Let: \alpha_1 = \Omega
   \langle 2 \rangle 2. Given \alpha_1, \ldots, \alpha_n and U_1, \ldots, U_{n-1} with 0 \neq \alpha_n < \alpha_{n-1} < \cdots < \alpha_1
           and \alpha_i \in U_i for i < n, PICK U_n \in \mathcal{A} with \alpha_n \in U_n
      Proof: By \langle 1 \rangle 1.
   \langle 2 \rangle 3. PICK \alpha_{n+1} < \alpha_n such that (\alpha_{n+1}, \alpha_n] \subseteq U_n
      PROOF: By Lemma 4.1.2.
   \langle 2 \rangle 4. \ \alpha_{n+1} \neq 0
```

 $\langle 2 \rangle 4$ .  $\phi \circ f$  is a retraction

PROOF: If $\alpha_{n+1} = 0$ then $U_1, \ldots, U_n$ cover $S_{\Omega}$ , contradicting $\langle 1 \rangle 2$ . $\langle 1 \rangle 4$ . Q.E.D.  PROOF: This is a contradiction because the ordinals are well-ordered.
<b>Proposition 13.1.31.</b> $\mathbb{R}_l$ is not limit point compact.
Proof: $\mathbb{Z}$ has no limit point. $\square$
<b>Proposition 13.1.32.</b> Every closed subspace of a Lindelöf space is Lindelöf.
PROOF: $ \langle 1 \rangle 1. \text{ Let: } X \text{ be Lindel\"of and } A \subseteq X \text{ be closed} \\ \langle 1 \rangle 2. \text{ Let: } \mathcal{U} \text{ be an open covering of } A \\ \langle 1 \rangle 3. \{ U \text{ open in } X : U \cap A \in \mathcal{U} \} \cup \{ X \setminus A \} \text{ covers } X \\ \langle 1 \rangle 4. \text{ Pick a countable subcovering } \mathcal{V} \\ \langle 1 \rangle 5. \{ U \cap A : U \in \mathcal{V}, U \neq X \setminus A \} \text{ is a countable subcover of } \mathcal{U} \\ \square$
<b>Proposition 13.1.33.</b> $\mathbb{R}^{\omega}$ is locally connected.
Proof:This holds because every basic open set is connected, being the product of a family of connected spaces. $\Box$
<b>Proposition 13.1.34.</b> The space $\mathbb{R}^{\omega}$ under the box topology is not first countable.
PROOF: $ \langle 1 \rangle 1. \text{ Assume: for a contradiction } \{U_n\}_{n \geq 0} \text{ is a countable basis at } 0. \\ \langle 1 \rangle 2. \text{ For } n \geq 1, \text{ Pick a basic open set } B_n = \prod_{j=0}^{\infty} (a_{nj}, b_{nj}) \text{ such that } 0 \in B_n \subseteq U_n \\ \langle 1 \rangle 3. \prod_{n=0}^{\infty} (a_{nn}/2, b_{nn}/2) \text{ is a neighbourhood of } 0 \text{ that does not include any } U_n $
<b>Proposition 13.1.35.</b> The space $\mathbb{R}^{\omega}$ under the box topology is not locally metrizable.
PROOF: $ \langle 1 \rangle 1. \text{ Let: } U \text{ be any neighbourhood of } 0 \\ \langle 1 \rangle 2. \text{ Let: } A \text{ be the set of all sequences in } U \text{ with all coordinates positive } \\ \langle 1 \rangle 3. \ 0 \in \overline{A} \\ \langle 1 \rangle 4. \text{ There is no sequence of points of } A \text{ converging to } 0. \\ \langle 1 \rangle 5. \ U \text{ is not metrizable.} \\ \text{PROOF: By the Sequence Lemma.} $
<b>Proposition 13.1.36.</b> For any nonempty set $I$ , the space $\mathbb{R}^I$ is not limit point compact.
PROOF: $\mathbb{Z}^I$ is an infinite set with no limit point. $\square$

**Proposition 13.1.37.** The space  $\mathbb{R}^{[0,1]}$  is separable. PROOF: The set D is dense where D is the set of all functions  $f:[0,1]\to\mathbb{Q}$ such that there exists a sequence of rationals  $0 = q_0 < q_1 < \cdots < q_N = 1$  such that f is constant on  $[q_i, q_{i+1})$  for  $0 \le i < N$ .  $\square$ **Proposition 13.1.38.** If J is uncountable then  $\mathbb{R}^J$  is not locally metrizable. PROOF: Every point has a neighbourhood homeomorphic to  $\mathbb{R}^J$ .  $\square$ **Proposition 13.1.39.** The space  $\mathbb{R}_K$  is not limit point compact. PROOF: The set  $\mathbb{Z}$  has no limit point.  $\square$ **Proposition 13.1.40.** The topologist's sine curve is not locally connected. PROOF: There is no connected neighbourhood of (0,0).  $\square$ Corollary 13.1.40.1. Not every metric space is locally connected. Corollary 13.1.40.2. Not every metric space is locally path connected. **Proposition 13.1.41.** Not every metric space is compact. PROOF: The space  $\mathbb{R}$  is not compact.  $\square$ **Proposition 13.1.42.** Every closed subspace of a limit point compact space is limit point compact. Proof:  $\langle 1 \rangle 1$ . Let: X be a limit point compact space and  $C \subseteq X$  be closed.  $\langle 1 \rangle 2$ . Let:  $A \subseteq C$  be infinite.  $\langle 1 \rangle 3$ . Pick a limit point l of A in X $\langle 1 \rangle 4. \ l \in C$  $\langle 2 \rangle 1$ . l is a limt point of C PROOF: By Lemma 3.10.2.  $\langle 2 \rangle 2$ . Q.E.D. Proof: By Corollary 3.10.3.1.  $\langle 1 \rangle 5$ . *l* is a limit point of *A* in *C*. Proof: By Proposition 4.3.10. **Proposition 13.1.43.** For any part  $i: S \hookrightarrow X$  of a set X, we have  $\emptyset \subseteq_X i$ . PROOF: We have  $i \circ i_S = i_X$  by the uniqueness of  $i_X$ .  $\square$ 

 $\langle 1 \rangle 1$ . Let: J be the set of all bounded continuous functions  $X \to \mathbb{R}$ 

**Theorem 13.1.44.** Let X be a completely regular space. Then there exists a compactification Y of X such that every bounded continuous map  $X \to \mathbb{R}$ 

extends uniquely to a continuous map  $Y \to \mathbb{R}$ .

Proof:

 $\langle 1 \rangle 2$ . For  $\alpha \in J$ ,

Let:  $I_{\alpha} = [\inf \alpha, \sup \alpha]$ 

- $\langle 1 \rangle 3$ . Let:  $Z = \prod_{\alpha \in J} I_{\alpha}$   $\langle 1 \rangle 4$ . Let:  $h: X \to Z$  be defined by

$$h(x)_{\alpha} = \alpha(x)$$

- $\langle 1 \rangle 5$ . Z is compact Hausdorff
  - $\langle 2 \rangle 1$ . Z is compact

PROOF: By Tychonoff's Theorem.

 $\langle 2 \rangle 2$ . Z is Hausdorff

PROOF: By Theorem 6.2.5

- $\langle 1 \rangle 6$ . h is an imbedding
  - $\langle 2 \rangle 1$ . The set J separates points from closed sets

PROOF: This holds because X is completely regular.

 $\langle 2 \rangle 2$ . Q.E.D.

PROOF: By the Imbedding Theorem.

- $\langle 1 \rangle 7$ . Let: Y be the compactification of X such that  $X \subseteq Y \to Z$  factors h Proof: By Lemma 9.8.2
- $\langle 1 \rangle 8$ . Every bounded continuous map  $X \to \mathbb{R}$  extends uniquely to a continuous map  $Y \to \mathbb{R}$ 
  - $\langle 2 \rangle 1$ . Let:  $\alpha: X \to \mathbb{R}$  be a bounded continuous function
  - $\langle 2 \rangle 2$ . Let:  $k: Y \to Z$  be the imbedding from  $\langle 1 \rangle 7$
  - $\langle 2 \rangle 3$ . Let:  $\overline{\alpha} = \pi_{\alpha} \circ k : Y \to \mathbb{R}$
  - $\langle 2 \rangle 4$ .  $\overline{\alpha}$  extends  $\alpha$

PROOF:For  $x \in X$ , we have

$$\overline{\alpha}(x) = k(x)_{\alpha}$$

$$= h(x)_{\alpha}$$

$$= \alpha(x)$$

 $\langle 2 \rangle 5$ . If  $f: Y \to Z$  is continuous and extends  $\alpha$  then  $f = \overline{\alpha}$ PROOF: By Lemma 6.2.9.