Topology

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Contents

Chapter 1

Set Theory

1.1 Sets and Functions

1.1.1 Primitive Notions

Let there be *sets*.

Given sets A and B, let there be *functions* from A to B. We write $f : A \to B$ iff f is a function from A to B, and we call A the *domain* of f and B the *codomain* of f.

Given sets A, B, C and functions $f : A \to B$ and $g : B \to C$, let there be a function $g \circ f : A \to C$, the *composite* of f and g.

1.1.2 The Axiom of Associativity

Axiom 1.1.1 (Axiom of Associativity)**.** *Let A, B and C be sets. Let $f : A \to B$, $g : B \to C$ and $h : C \to D$. Then $h \circ (g \circ f) = (h \circ g) \circ f : A \to D$.*

From now on we write $h \circ g \circ f$ for the composite of f, g and h, and similarly for more than three functions.

1.1.3 Identity Functions

Definition 1.1.2 (Identity Function)**.** Let A be a set. An *identity function* on A is a function $i : A \to A$ such that:

Left Unit Law For every set X and function $f : X \to A$, we have $i \circ f = f : X \to A$.

Right Unit Law For every set X and function $f : A \to X$, we have $f \circ i = f : A \to X$.

Proposition 1.1.3. *Any two identity functions on a set are equal.*

PROOF:

⟨1⟩1. Let: A be a set.
⟨1⟩2. Let: $i, j : A \to A$ be identity functions on A.
⟨1⟩3. $i = j : A \to A$
Proof:

$$\begin{aligned} i &= i \circ j && \text{(Right Unit Law for } j\text{, ⟨1⟩2)} \\ &= j && \text{(Left Unit Law for } i\text{, ⟨1⟩2)} \end{aligned}$$

□

Axiom 1.1.4 (Identity Functions)**.** *Every set has an identity function.*

Given a set A, we write id_A for the identity function on A.

1.1.4 Isomorphisms

Definition 1.1.5 (Isomorphism)**.** A function $i : A \to B$ is an *isomorphism*, $i : A \cong B$, iff there exists a function $i^{-1} : B \to A$, the *inverse* of i, such that $i^{-1} \circ i = \mathrm{id}_A : A \to A$ and $i \circ i^{-1} = \mathrm{id}_B : B \to B$.

Two sets A and B are *isomorphic*, $A \cong B$, iff there exists an isomorphism between A and B.

1.2 The Empty Set

Definition 1.2.1 (Empty Set)**.** A set E is *empty* iff, for every set X, there exists exactly one function $E \to X$.

Proposition 1.2.2 (Uniqueness of the Empty Set)**.** *Let E be an empty set. Then a set E' is empty if and only if $E \cong E'$, in which case the isomorphism between E and E' is unique.*

Proof:
⟨1⟩1. If $E \cong E'$ then E' is empty.
⟨2⟩1. Let: $\phi : E \cong E'$ be an isomorphism.
Prove: E' is empty.
⟨2⟩2. Let: X be a set.
Prove: There is exactly one function $E' \to X$.
⟨2⟩3. Let: $f : E \to X$ be the unique function $E \to X$.
⟨2⟩4. $f \circ \phi^{-1} : E' \to X$
⟨2⟩5. For any $g : E' \to X$ we have $g = f \circ \phi^{-1} : E' \to X$
⟨3⟩1. Let: $g : E' \to X$
⟨3⟩2. $g \circ \phi = f : E \to X$
Proof: By the uniqueness of f
⟨3⟩3. $g = f \circ \phi^{-1} : E' \to X$
Proof:

$$\begin{aligned} g &= g \circ \mathrm{id}_E && \text{(Right Unit Law)} \\ &= g \circ \phi \circ \phi^{-1} && (\phi \text{ is an isomorphism}) \\ &= f \circ \phi^{-1} && \text{(⟨3⟩2)} \end{aligned}$$

⟨1⟩2. If E' is empty then there exists a unique isomorphism $E \cong E'$.
⟨2⟩1. Let: E' be empty
⟨2⟩2. Let: $\phi : E \to E'$ be the unique function $E \to E'$
⟨2⟩3. Let: $\phi^{-1} : E' \to E$ be the unique function $E' \to E$
⟨2⟩4. $\phi^{-1} \circ \phi = \mathrm{id}_E$
Proof: Each is the unique function $E \to E$.
⟨2⟩5. $\phi \circ \phi^{-1} = \mathrm{id}_{E'}$
Proof: Each is the unique function $E' \to E'$.
□

Axiom 1.2.3 (Empty Set)**.** *There exists an empty set.*

We write $\emptyset$ for the empty set. For any set A, we write $¡_A$ for the unique function $\emptyset \to A$.

1.3 The Terminal Set

Definition 1.3.1 (Terminal Set)**.** A set T is *terminal* iff, for every set X, there exists exactly one function $X \to T$.

Proposition 1.3.2 (Uniqueness of the Terminal Set)**.** *Let T be a terminal set. Then a set T' is terminal if and only if $T \cong T'$, in which case the isomorphism between T and T' is unique.*

Proof:
⟨1⟩1. If $T \cong T'$ then T' is terminal.
⟨2⟩1. Let: $\phi : T \cong T'$ be an isomorphism.
Prove: T' is empty.
⟨2⟩2. Let: X be a set.
Prove: There is exactly one function $X \to T'$.
⟨2⟩3. Let: $f : X \to T$ be the unique function $X \to T$.
⟨2⟩4. $\phi \circ f : X \to T'$
⟨2⟩5. For any $g : X \to T'$ we have $g = \phi \circ f : X \to T'$
⟨3⟩1. Let: $g : X \to T'$
⟨3⟩2. $\phi^{-1} \circ g = f : E \to X$
Proof: By the uniqueness of f
⟨3⟩3. $g = \phi \circ f : E' \to X$
Proof:

$$\begin{aligned} g &= \mathrm{id}_{T'} \circ g && \text{(Left Unit Law)} \\ &= \phi \circ \phi^{-1} \circ g && \text{(ϕ is an isomorphism)} \\ &= \phi \circ f && \text{(⟨3⟩2)} \end{aligned}$$

⟨1⟩2. If T' is terminal then there exists a unique isomorphism $T \cong T'$.
⟨2⟩1. Let: T' be terminal
⟨2⟩2. Let: $\phi : T \to T'$ be the unique function $T \to T'$
⟨2⟩3. Let: $\phi^{-1} : T' \to T$ be the unique function $T' \to T$
⟨2⟩4. $\phi^{-1} \circ \phi = \mathrm{id}_T$

PROOF: Each is the unique function $T \to T$.
$\langle 2 \rangle 5.$ $\phi \circ \phi^{-1} = \mathrm{id}_{T'}$
PROOF: Each is the unique function $T' \to T'$.
□

Axiom 1.3.3 (Terminal Set)**.** *There exists a terminal set.*

We write 1 for the terminal set. For any set A, we write $!_A$ for the unique function $A \to 1$.

1.4 Product Sets

Definition 1.4.1 (Product)**.** Let A and B be sets. A *product* of A and B consists of:

- a set P, also called the *product*;
- a function $\pi_1 : P \to A$, the *first projection*;
- a function $\pi_2 : P \to B$, the *second projection*

such that, for any set X and functions $f : X \to A$ and $g : X \to B$, there exists a unique function $\langle f, g \rangle : X \to P$, the *pairing* of f and g, such that $\pi_1 \circ \langle f, g \rangle = f : X \to A$ and $\pi_2 \circ \langle f, g \rangle = g : X \to B$.

Proposition 1.4.2 (Uniqueness of Product)**.** *Let A and B be sets and P a product of A and B with projections $\pi_1 : P \to A$ and $\pi_2 : P \to B$. Let Q be a set and $p : Q \to A$ and $q : Q \to B$. Then Q is a product of A and B with projections p and q iff there exists an isomorphism $\phi : P \cong Q$ such that $p \circ \phi = \pi_1 : P \to A$ and $q \circ \phi = \pi_2 : P \to B$, in which case ϕ is unique.*

PROOF:
$\langle 1 \rangle 1.$ If Q is a product of A and B with projections p and q then there exists a unique isomorphism $\phi : P \cong Q$ such that $p \circ \phi = \pi_1 : P \to A$ and $q \circ \phi = \pi_2$
$\langle 2 \rangle 1.$ ASSUME: Q is a product of A and B with projections p and q.
$\langle 2 \rangle 2.$ LET: $\phi : P \to Q$ be the unique function with $p \circ \phi = \pi_1$ and $q \circ \phi = \pi_2$
$\langle 2 \rangle 3.$ LET: $\phi^{-1} : Q \to P$ be the unique function with $\pi_1 \circ \phi^{-1} = p$ and $\pi_2 \circ \phi^{-1} = q$
$\langle 2 \rangle 4.$ $\phi^{-1} \circ \phi = \mathrm{id}_P$
$\langle 3 \rangle 1.$ $\pi_1 \circ \phi^{-1} \circ \phi = \pi_1$
PROOF:

$$\begin{aligned} \pi_1 \circ \phi^{-1} \circ \phi &= p \circ \phi && (\langle 2 \rangle 3) \\ &= \pi_1 && (\langle 2 \rangle 2) \end{aligned}$$

$\langle 3 \rangle 2.$ $\pi_2 \circ \phi^{-1} \circ \phi = \pi_2$
PROOF:

$$\begin{aligned} \pi_2 \circ \phi^{-1} \circ \phi &= q \circ \phi && (\langle 2 \rangle 3) \\ &= \pi_2 && (\langle 2 \rangle 2) \end{aligned}$$

$\langle 3\rangle 3.$ $\pi_1 \circ \mathrm{id}_P = \pi_1$

PROOF: Right Unit Law.

$\langle 3\rangle 4.$ $\pi_2 \circ \mathrm{id}_P = \pi_2$

PROOF: Right Unit Law.

$\langle 3\rangle 5.$ Q.E.D.

PROOF: By the uniqueness of the function $x : P \to P$ such that $\pi_1 \circ x = \pi_1$ and $\pi_2 \circ x = \pi_2$.

$\langle 2\rangle 5.$ $\phi \circ \phi^{-1} = \mathrm{id}_Q$

$\langle 3\rangle 1.$ $p \circ \phi \circ \phi^{-1} = p$

PROOF:

$$\begin{aligned} p \circ \phi \circ \phi^{-1} &= \pi_1 \circ \phi^{-1} && (\langle 2\rangle 2) \\ &= p && (\langle 2\rangle 3) \end{aligned}$$

$\langle 3\rangle 2.$ $q \circ \phi \circ \phi^{-1} = q$

PROOF:

$$\begin{aligned} q \circ \phi \circ \phi^{-1} &= \pi_2 \circ \phi^{-1} && (\langle 2\rangle 2) \\ &= q && (\langle 2\rangle 3) \end{aligned}$$

$\langle 3\rangle 3.$ $p \circ \mathrm{id}_Q = p$

PROOF: Right Unit Law

$\langle 3\rangle 4.$ $q \circ \mathrm{id}_Q = q$

PROOF: Right Unit Law

$\langle 3\rangle 5.$ Q.E.D.

PROOF: By the uniqueness of the function $x : Q \to Q$ such that $p \circ x = p$ and $q \circ x = q$.

$\langle 1\rangle 2.$ If $\phi : P \cong Q$ is an isomorphism and $p \circ \phi = \pi_1$ and $q \circ \phi = \pi_2$ then Q is a product of A and B with projections p and q.

$\langle 2\rangle 1.$ LET: $\phi : P \cong Q$ be an isomorphism with $p \circ \phi = \pi_1$ and $q \circ \phi = \pi_2$

$\langle 2\rangle 2.$ LET: X be any set and $f : X \to A$ and $g : X \to B$

$\langle 2\rangle 3.$ LET: $\langle f, g\rangle : X \to P$ be the unique function such that $\pi_1 \circ \langle f, g\rangle = f$ and $\pi_2 \circ \langle f, g\rangle = g$

$\langle 2\rangle 4.$ $p \circ \phi \circ \langle f, g\rangle = f$

PROOF: From $\langle 2\rangle 1$ and $\langle 2\rangle 3$

$\langle 2\rangle 5.$ $q \circ \phi \circ \langle f, g\rangle = g$

PROOF: From $\langle 2\rangle 1$ and $\langle 2\rangle 3$

$\langle 2\rangle 6.$ If $x : X \to Q$ satisfies $p \circ x = f$ and $q \circ x = g$ then $x = \phi \circ \langle f, g\rangle$

$\langle 3\rangle 1.$ ASSUME: $p \circ x = f$ and $q \circ x = g$

$\langle 3\rangle 2.$ $\phi^{-1} \circ x = \langle f, g\rangle$

$\langle 4\rangle 1.$ $\pi_1 \circ \phi^{-1} \circ x = f$

PROOF:

$$\begin{aligned} \pi_1 \circ \phi^{-1} \circ x &= p \circ \phi \circ \phi^{-1} \circ x && (\langle 2\rangle 1) \\ &= p \circ \mathrm{id}_Q \circ x && (\phi \text{ is iso}) \\ &= p \circ x && (\text{Left Unit Law}) \\ &= f && (\langle 3\rangle 1) \end{aligned}$$

$\langle 4\rangle 2.$ $\pi_2 \circ \phi^{-1} \circ x = g$

PROOF:

$$\begin{aligned}
\pi_2 \circ \phi^{-1} \circ x &= q \circ \phi \circ \phi^{-1} \circ x && (\langle 2\rangle 1) \\
&= q \circ \mathrm{id}_Q \circ x && (\phi \text{ is iso}) \\
&= q \circ x && (\text{Left Unit Law}) \\
&= g && (\langle 3\rangle 1)
\end{aligned}$$

□

Axiom 1.4.3 (Product)**.** *Any two sets have a product.*

Given sets A and B, we write $A \times B$ for the product of A and B, $\pi_1 : A \times B \to A$ for the first projection, and $\pi_2 : A \times B \to B$ for the second projection.

1.5 Coproduct Sets

Definition 1.5.1 (Coproduct)**.** Let A and B be sets. A *coproduct* of A and B consists of:

- a set C, also called the *coproduct*;
- a function $\kappa_1 : A \to C$, the *first injection*;
- a function $\kappa_2 : P \to B$, the *second injection*

such that, for any set X and functions $f : A \to X$ and $g : B \to X$, there exists a unique function $[f, g] : C \to X$, the *copairing* of f and g, such that $[f, g] \circ \kappa_1 = f : A \to X$ and $[f, g] \circ \kappa_2 = g : B \to X$.

Proposition 1.5.2 (Uniqueness of coproduct)**.** *Let A and B be sets and C a coproduct of A and B with injections $\kappa_1 : A \to C$ and $\kappa_2 : B \to C$. Let D be a set and $p : A \to D$ and $q : B \to D$. Then D is a coproduct of A and B with injections p and q iff there exists an isomorphism $\phi : C \cong D$ such that $\phi \circ \kappa_1 = p : A \to D$ and $\phi \circ \kappa_2 = q : B \to D$, in which case ϕ is unique.*

PROOF:

$\langle 1\rangle 1.$ If D is a coproduct of A and B with injections p and q then there exists a unique isomorphism $\phi : C \cong D$ such that $\phi \circ \kappa_1 = p : A \to D$ and $\phi \circ \kappa_2 = q$

$\langle 2\rangle 1.$ ASSUME: D is a coproduct of A and B with injections p and q.

$\langle 2\rangle 2.$ LET: $\phi : C \to D$ be the unique function with $\phi \circ \kappa_1 = p$ and $\phi \circ \kappa_2 = q$

$\langle 2\rangle 3.$ LET: $\phi^{-1} : D \to C$ be the unique function with $\phi^{-1} \circ p = \kappa_1$ and $\phi^{-1} \circ q = \kappa_2$

$\langle 2\rangle 4.$ $\phi^{-1} \circ \phi = \mathrm{id}_C$

$\langle 3\rangle 1.$ $\phi^{-1} \circ \phi \circ \kappa_1 = \kappa_1$

PROOF:

$$\begin{aligned}
\phi^{-1} \circ \phi \circ \kappa_1 &= \phi^{-1} \circ p && (\langle 2\rangle 2) \\
&= \kappa_1 && (\langle 2\rangle 3)
\end{aligned}$$

$\langle 3\rangle 2.$ $\kappa_2 \circ \phi^{-1} \circ \phi = \kappa_2$

PROOF:

$$\begin{aligned} \phi^{-1} \circ \phi \circ \kappa_2 &= \phi^{-1} \circ q && (\langle 2\rangle 2) \\ &= \kappa_2 && (\langle 2\rangle 3) \end{aligned}$$

$\langle 3\rangle 3$. $\mathrm{id}_P \circ \kappa_1 = \kappa_1$

PROOF: Left Unit Law.

$\langle 3\rangle 4$. $\mathrm{id}_P \circ \kappa_2 = \kappa_2$

PROOF: Left Unit Law.

$\langle 3\rangle 5$. Q.E.D.

PROOF: By the uniqueness of the function $x : P \to P$ such that $x \circ \kappa_1 = \kappa_1$ and $x \circ \kappa_2 = \kappa_2$.

$\langle 2\rangle 5$. $\phi \circ \phi^{-1} = \mathrm{id}_Q$

$\langle 3\rangle 1$. $\phi \circ \phi^{-1} \circ p = p$

PROOF:

$$\begin{aligned} \phi \circ \phi^{-1} \circ p &= \phi \circ \kappa_1 && (\langle 2\rangle 3) \\ &= p && (\langle 2\rangle 2) \end{aligned}$$

$\langle 3\rangle 2$. $q \circ \phi \circ \phi^{-1} = q$

PROOF:

$$\begin{aligned} \phi \circ \phi^{-1} \circ q &= \phi \circ \kappa_2 && (\langle 2\rangle 3) \\ &= q && (\langle 2\rangle 2) \end{aligned}$$

$\langle 3\rangle 3$. $\mathrm{id}_Q \circ p = p$

PROOF: Left Unit Law

$\langle 3\rangle 4$. $\mathrm{id}_Q \circ q = q$

PROOF: Left Unit Law

$\langle 3\rangle 5$. Q.E.D.

PROOF: By the uniqueness of the function $x : Q \to Q$ such that $x \circ p = p$ and $x \circ q = q$.

$\langle 1\rangle 2$. If $\phi : P \cong Q$ is an isomorphism and $\phi \circ \kappa_1 = p$ and $\phi \circ \kappa_2 = q$ then Q is a coproduct of A and B with injections p and q.

$\langle 2\rangle 1$. LET: $\phi : P \cong Q$ be an isomorphism with $\phi \circ \kappa_1 = p$ and $\phi \circ \kappa_2 = q$

$\langle 2\rangle 2$. $\phi^{-1} \circ p = \kappa_1$

PROOF:

$$\begin{aligned} \phi^{-1} \circ p &= \phi^{-1} \circ \phi \circ \kappa_1 && (\langle 2\rangle 1) \\ &= \mathrm{id}_P \circ \kappa_1 && (\phi \text{ is iso}) \\ &= \kappa_1 && (\text{Left Unit Law}) \end{aligned}$$

$\langle 2\rangle 3$. $\phi^{-1} \circ q = \kappa_2$

PROOF:

$$\begin{aligned} \phi^{-1} \circ q &= \phi^{-1} \circ \phi \circ \kappa_2 && (\langle 2\rangle 1) \\ &= \mathrm{id}_P \circ \kappa_2 && (\phi \text{ is iso}) \\ &= \kappa_2 && (\text{Left Unit Law}) \end{aligned}$$

$\langle 2\rangle 4$. LET: X be any set and $f : A \to X$ and $g : B \to X$

$\langle 2\rangle 5$. LET: $[f, g] : C \to X$ be the unique function such that $[f, g] \circ \kappa_1 = f$ and $[f, g] \circ \kappa_2 = g$

$\langle 2\rangle 6$. $[f, g] \circ \phi^{-1} \circ p = f$

PROOF: From $\langle 2\rangle 2$ and $\langle 2\rangle 5$

$\langle 2\rangle 7$. $[f,g] \circ \phi^{-1} \circ q = g$
PROOF: From $\langle 2\rangle 3$ and $\langle 2\rangle 5$
$\langle 2\rangle 8$. If $x : X \to Q$ satisfies $x \circ p = f$ and $x \circ q = g$ then $x = [f,g] \circ \phi^{-1}$
$\langle 3\rangle 1$. ASSUME: $x \circ p = f$ and $x \circ q = g$
$\langle 3\rangle 2$. $x \circ \phi = [f,g]$
$\langle 4\rangle 1$. $x \circ \phi \circ \kappa_1 = f$
PROOF: From $\langle 2\rangle 1$ and $\langle 3\rangle 1$.
$\langle 4\rangle 2$. $x \circ \phi \circ \kappa_2 = g$
PROOF: From $\langle 2\rangle 1$ and $\langle 3\rangle 1$.
□

Axiom 1.5.3 (Coproduct)**.** *Any two sets have a coproduct.*

Given sets A and B, we write $A+B$ for the coproduct of A and B, $\kappa_1 : A \to A+B$ for the first injection, and $\kappa_2 : B \to A+B$ for the second injection.

1.6 Equalizers

Definition 1.6.1 (Equalizer)**.** Let A and B be sets and $f, g : A \to B$. An *equalizer* of f and g consists of:

- a set E
- a function $e : E \to A$

such that:

- $f \circ e = g \circ e$
- For any set X and function $x : X \to A$ such that $f \circ x = g \circ x$, there exists a unique function $\overline{x} : X \to E$ such that $x = e \circ \overline{x}$

Proposition 1.6.2 (Uniqueness of Equalizers)**.** *Let $e : E \to A$ be an equalizer of $f, g : A \to B$. Let $e' : E' \to A$. Then e' is an equalizer of f and g if and only if there exists an isomorphism $\phi : E \cong E'$ such that $e' \circ \phi = e$, in which case ϕ is unique.*

PROOF:
$\langle 1\rangle 1$. If e' is an equalizer of f and g then there exists a unique isomorphism $\phi : E \cong E'$ such that $e' \circ \phi = e$
$\langle 2\rangle 1$. ASSUME: e' is an equalizer of f and g.
$\langle 2\rangle 2$. LET: $\phi : E \to E'$ be the unique function such that $e' \circ \phi = e$
$\langle 2\rangle 3$. LET: $\phi^{-1} : E' \to E$ be the unique function such that $e \circ \phi^{-1} = e'$
$\langle 2\rangle 4$. $\phi^{-1} \circ \phi = \mathrm{id}_E$
$\langle 2\rangle 5$. $\phi \circ \phi^{-1} = \mathrm{id}_{E'}$
$\langle 1\rangle 2$. If there exists an isomorphism $\phi : E \cong E'$ with $e' \circ \phi = e$ then e' is an equalizer of f and g.
□

1.7 Preliminary Definitions

Definition 1.7.1 (Identity Function)**.** Let A be a set. A function $i : A \to A$ is an *identity function* on A iff:

Left Unit Law For every set X and function $f : X \to A$, we have $i \circ f = f : X \to A$;

Right Unit Law For every set X and function $f : A \to X$, we have $f \circ i = f : A \to X$.

Definition 1.7.2 (Empty Set)**.** A set $\emptyset$ is *empty* iff, for every set X, there exists exactly one function $\emptyset \to X$.

Definition 1.7.3 (Terminal Set)**.** A set T is *terminal* iff, for every set X, there exists exactly one function $X \to T$.

Definition 1.7.4 (Pullback)**.** Let $f : A \to C$ and $g : B \to C$. A *pullback* of f and g consists of:

- a set P, also called the *pullback*;
- functions $p : P \to A$ and $q : P \to B$, the *projections*

such that:

- $f \circ p = g \circ q : P \to C$
- For any set X and functions $x : X \to A$ and $y : X \to B$ such that $f \circ x = g \circ y : X \to C$, there exists a unique function $\langle x, y\rangle : X \to P$ such that $p \circ \langle x, y\rangle = x : X \to A$ and $q \circ \langle x, y\rangle = y : X \to B$.

Definition 1.7.5 (Pushout)**.** Let $f : A \to B$ and $g : A \to C$. A *pushout* of f and g consists of:

- a set P, also called the *pushout*;
- functions $i : B \to P$ and $j : C \to P$, the *injections*

such that:

- $i \circ f = j \circ g : A \to P$;
- For any set X and functions $x : B \to X$ and $y : C \to X$ such that $x \circ f = y \circ g : A \to X$, there exists a unique function $[x, y] : P \to X$ such that $[x, y] \circ i = x : B \to X$ and $[x, y] \circ j = y : C \to X$.

1.8 The Axioms

Axiom 1.8.1 (Axiom of Identity Functions)**.** *Every set has an identity function.*

Axiom 1.8.2 (Empty Set Axiom)**.** *There exists an empty set.*

Axiom 1.8.3 (Terminal Set Axiom)**.** *There exists a terminal set.*

Axiom 1.8.4 (Pullback Axiom)**.** *Any two functions with common codomain have a pullback.*

Axiom 1.8.5 (Pushout Axiom)**.** *Any two functions with common domain have a pushout.*

1.9 Injective Functions

Definition 1.9.1 (Injective)**.** A function $f : A \to B$ is *injective*, $f : A \rightarrowtail B$, iff, for every set X and functions $g, h : X \to A$, if $f \circ g = f \circ h$ then $g = h$.

Proposition 1.9.2. *Let $f : A \to B$ and $g : B \to C$. If f and g are injective then $g \circ f$ is injective.*

PROOF:
$\langle 1 \rangle$1. ASSUME: f is injective.
$\langle 1 \rangle$2. ASSUME: g is injective.
$\langle 1 \rangle$3. LET: X be a set and $x, y : X \to A$.
$\langle 1 \rangle$4. ASSUME: $g \circ f \circ x = g \circ f \circ y$
$\langle 1 \rangle$5. $f \circ x = f \circ y$
  PROOF: $\langle 1 \rangle$2, $\langle 1 \rangle$4.
$\langle 1 \rangle$6. $x = y$
  PROOF: $\langle 1 \rangle$1, $\langle 1 \rangle$5.
□

Proposition 1.9.3. *Let $f : A \to B$ and $g : B \to C$. If $g \circ f$ is injective then f is injective.*

PROOF:
$\langle 1 \rangle$1. ASSUME: $g \circ f$ is injective.
$\langle 1 \rangle$2. LET: X be any set and $x, y : X \to A$.
$\langle 1 \rangle$3. ASSUME: $f \circ x = f \circ y$
$\langle 1 \rangle$4. $g \circ f \circ x = g \circ f \circ y$
  PROOF: $\langle 1 \rangle$3
$\langle 1 \rangle$5. $x = y$
  PROOF: $\langle 1 \rangle$1, $\langle 1 \rangle$4.
□

1.10 Surjective Functions

Definition 1.10.1 (Surjective)**.** Let $f : A \to B$. Then f is *surjective*, $f : A \twoheadrightarrow B$, iff, for any set X and functions $g, h : B \to X$, if $g \circ f = h \circ f$ then $g = h$.

Lemma 1.10.2. *Let $f : A \to B$ and $g : B \to C$. If f and g are surjective then $g \circ f$ is surjective.*

PROOF: Dual to Proposition 1.9.2. □

Lemma 1.10.3. *Let $f : A \to B$ and $g : B \to C$. If $g \circ f$ is surjective then g is surjective.*

PROOF: Dual to Proposition 1.9.3. □

1.11 Retractions and Sections

Definition 1.11.1 (Retraction, Section)**.** Let $r : A \to B$ and $s : B \to A$. Then r is a *retraction* of s, and s is a *section* of r, iff $r \circ s = \mathrm{id}_B$.

Proposition 1.11.2. *If $r_1 : A \to B$ is a retraction of $s_1 : B \to A$ and $r_2 : B \to C$ is a retraction of $s_2 : C \to B$ then $r_2 \circ r_1$ is a retraction of $s_1 \circ s_2$.*

PROOF:

$$
\begin{aligned}
r_2 \circ r_1 \circ s_1 \circ s_2 &= r_2 \circ \mathrm{id}_B \circ s_2 && (r_1 \text{ is a retraction of } s_1) \\
&= r_2 \circ s_2 && \text{(Unit Laws)} \\
&= \mathrm{id}_C && (r_2 \text{ is a retraction of } s_2)
\end{aligned}
$$

□

Proposition 1.11.3. *Every section is injective.*

PROOF:
⟨1⟩1. LET: $s : A \to B$ be a section of $r : B \to A$
⟨1⟩2. LET: $x, y : X \to A$ satisfy $s \circ x = s \circ y$
⟨1⟩3. $x = y$
PROOF:

$$
\begin{aligned}
x &= \mathrm{id}_A \circ x && \text{(Left Unit Law)} \\
&= r \circ s \circ x && (\langle 1\rangle 1) \\
&= r \circ s \circ y && (\langle 1\rangle 2) \\
&= \mathrm{id}_A \circ y && (\langle 1\rangle 1) \\
&= y && \text{(Left Unit Law)}
\end{aligned}
$$

□

Proposition 1.11.4. *Every retraction is surjective.*

PROOF: Dual. □

1.12 Identity Functions

Axiom 1.12.1 (Identity Function)**.** *For any set A, there exists a function $\mathrm{id}_A : A \to A$, the* identity *function on A, such that:*

Left Unit Law *for every set B and function $f : B \to A$ we have $\mathrm{id}_A \circ f = f : B \to A$;*

Right Unit Law *for every set B and function $f : A \to B$ we have $f \circ \mathrm{id}_A = f : A \to B$.*

Proposition 1.12.2. *The identity function on a set is unique.*

Proof:If $i, j : A \to A$ are both identity functions, then

$$\begin{aligned} i &= i \circ j && \text{(Right Unit Law for } j\text{)} \\ &= j && \text{(Left Unit Law for } i\text{)} \\ &: A \to A \end{aligned}$$

□

Proposition 1.12.3. *Every identity function is a retraction of itself.*

Proof: Immediate from the Unit Laws. □

Proposition 1.12.4. *Every identity function is injective.*

Proof: From Proposition 1.11.3 and 1.12.3. □

Proposition 1.12.5. *Every identity function is surjective.*

Proof: From Proposition 1.11.4 and 1.12.3. □

Proposition 1.12.6. *If $r : B \to A$ is a retraction of $f : A \to B$ and s is a section of f then $r = s$.*

Proof:

$$\begin{aligned} r &= r \circ \mathrm{id}_B && \text{(Right Unit Law)} \\ &= r \circ f \circ s && (s \text{ is a section of } f) \\ &= \mathrm{id}_A \circ s && (r \text{ is a retraction of } f) \\ &= s && \text{(Left Unit Law)} \end{aligned}$$

1.13 Isomorphisms

Definition 1.13.1 (Isomorphism)**.** Let A and B be sets. A function $i : A \to B$ is an *isomorphism* between A and B, $i : A \cong B$, iff there exists a function $i^{-1} : B \to A$, the *inverse* to i, that is a section and a retraction of i.

Proposition 1.13.2. *The inverse of an isomorphism is unique.*

Proof: Immediate from Proposition 1.12.6. □

Proposition 1.13.3. *Every isomorphism is injective.*

PROOF: Immediate from Proposition 1.11.3. ∎

Proposition 1.13.4. *Every isomorphism is surjective.*

PROOF: Immediate from Proposition 1.11.4. ∎

Proposition 1.13.5. *Every identity function is an isomorphism and is its own inverse.*

PROOF: Immediate from Proposition 1.12.3. ∎

Proposition 1.13.6. *If $i : A \cong B$ is an isomorphism then $i^{-1} : B \cong A$ is an isomorphism and $(i^{-1})^{-1} = i$.*

PROOF: Immediate from the definition of isomorphism. ∎

Proposition 1.13.7. *If $i : A \cong B$ and $j : B \cong C$ then $j \circ i : A \cong C$ and $(j \circ i)^{-1} = i^{-1} \circ j^{-1}$.*

PROOF: Immediate from Proposition 1.11.2. ∎

1.14 Parts of a Set

Definition 1.14.1 (Part)**.** A *part* S of a set A consists of:

- a set $\mathrm{dom}\, S$;
- an injective function $i : S \hookrightarrow A$

Definition 1.14.2. Two parts $i : S \hookrightarrow A$, $j : T \hookrightarrow A$ are *equivalent*, $i \equiv_A j$, iff there exists an isomorphism $\phi : S \cong T$ such that $i = j \circ \phi$.

Proposition 1.14.3. *Any part of a set is equivalent to itself.*

PROOF:For any part $i : X \hookrightarrow A$ of A we have $i = i \circ \mathrm{id}_X$ by the Right Unit Law. ∎

Proposition 1.14.4. *If $i \equiv_A j$ then $j \equiv_A i$.*

PROOF:
⟨1⟩1. LET: $i : S \hookrightarrow A$ and $j : T \hookrightarrow A$
⟨1⟩2. ASSUME: $i \equiv_A j$
⟨1⟩3. PICK an isomorphism $\phi : S \cong T$ such that $i = j \circ \phi$
  PROOF: From ⟨1⟩2
⟨1⟩4. $\phi^{-1} : T \cong S$
  PROOF: By Proposition 1.13.6.
⟨1⟩5. $j = i \circ \phi^{-1}$
  PROOF:

$$\begin{aligned} j &= j \circ \mathrm{id}_T && \text{(Right Unit Law)} \\ &= j \circ \phi \circ \phi^{-1} && (\langle 1\rangle 3) \\ &= i \circ \phi^{-1} && (\langle 1\rangle 3) \end{aligned}$$

□

Proposition 1.14.5. *If $i \equiv_A j$ and $j \equiv_A k$ then $i \equiv_A k$.*

PROOF:
⟨1⟩1. LET: $i : R \hookrightarrow A$, $j : S \hookrightarrow A$ and $k : T \to A$
⟨1⟩2. PICK isomorphisms $\phi : R \cong S$ and $\psi : S \cong T$ such that $i = j \circ \phi$ and $j = k \circ \psi$
⟨1⟩3. $\psi \circ \phi : R \cong T$
PROOF: By Proposition 1.13.7.
⟨1⟩4. $i = k \circ \psi \circ \phi$
□

Definition 1.14.6. Given a set A, we write A for the part $\mathrm{id}_A : A \hookrightarrow A$.

(This is a part by Proposition 1.12.4.)

Definition 1.14.7 (Inclusion)**.** Let $i : U \hookrightarrow A$ and $j : V \hookrightarrow A$ be parts of A. Then i is *included* in j, $i \subseteq_A j$, iff there exists a function $\phi : U \to V$ such that $i = j \circ \phi$.

Proposition 1.14.8. *If $i \equiv_A i'$ and $j \equiv_A j'$ and $i \subseteq_A j$ then $i' \subseteq_A j'$.*

PROOF:
⟨1⟩1. LET: $i : S \hookrightarrow A$, $i' : S' \hookrightarrow A$, $j : T \hookrightarrow A$, $j' : T' \hookrightarrow A$
⟨1⟩2. PICK $\phi : S \cong S'$, $\psi : T \cong T'$ and $\chi : S \to T$ such that $i = i' \circ \phi$, $j = j' \circ \psi$ and $i = j \circ \chi$
⟨1⟩3. $\psi \circ \chi \circ \phi^{-1} : S' \to T'$
⟨1⟩4. $i' = j' \circ \psi \circ \chi \circ \phi^{-1}$
□

Proposition 1.14.9. *For any part i of A we have $i \subseteq_A i$.*

PROOF:
⟨1⟩1. LET: $i : S \hookrightarrow A$
⟨1⟩2. $\mathrm{id}_S : S \to S$
⟨1⟩3. $i = i \circ \mathrm{id}_S$
□

Proposition 1.14.10. *If $i \subseteq_A j$ and $j \subseteq_A k$ then $i \subseteq_A k$.*

PROOF:
⟨1⟩1. LET: $i : R \hookrightarrow A$, $j : S \hookrightarrow A$ and $k : T \hookrightarrow A$
⟨1⟩2. PICK $\phi : R \to S$ and $\psi : S \to T$ such that $i = j \circ \phi$ and $j = k \circ \psi$
⟨1⟩3. $\psi \circ \phi : R \to T$
⟨1⟩4. $i = k \circ \psi \circ \phi$
□

Proposition 1.14.11. *If $i \subseteq_A j$ and $j \subseteq_A i$ then $i \equiv_A j$.*

PROOF:

⟨1⟩1. LET: $i : R \hookrightarrow A$, $j : S \hookrightarrow A$
⟨1⟩2. PICK $\phi : R \to S$ and $\phi^{-1} : S \to R$ such that $i = j \circ \phi$ and $j = i \circ \phi^{-1}$
⟨1⟩3. $\phi \circ \phi^{-1} = \mathrm{id}_S$
⟨2⟩1. $j \circ \phi \circ \phi^{-1} = j$
⟨2⟩2. Q.E.D.
PROOF: The result follows because j is injective.
⟨1⟩4. $\phi^{-1} \circ \phi = \mathrm{id}_T$
PROOF: Similar.
□

Proposition 1.14.12. *For any part i of A we have $i \subseteq_A A$.*

PROOF: For any part i of A, we have $i = \mathrm{id}_A \circ i$ by the Left Unit Law. □

Definition 1.14.13 (Restriction)**.** Let $f : A \to B$ and $i : S \hookrightarrow A$ be a part of A. Then the *restriction* of f to i, $f \restriction i$, is the function $f \circ i : S \to B$.

1.15 The Empty Set

Axiom 1.15.1 (Empty Set)**.** *There exists a set $\emptyset$, the* empty set*, such that, for every set X, there exists a unique function $¡_X : \emptyset \to X$.*

Proposition 1.15.2 (Uniqueness of Empty Set)**.** *Let E be any set. Then E is empty if and only if there exists an isomorphism $E \cong \emptyset$, in which case the isomorphism is unique.*

PROOF:
⟨1⟩1. If E is empty then $E \cong \emptyset$
⟨2⟩1. ASSUME: E is empty
⟨2⟩2. LET: ϕ be the unique function $E \to \emptyset$
⟨2⟩3. $¡_E \circ \phi = \mathrm{id}_E$
PROOF: There is only one function $E \to E$.
⟨2⟩4. $\phi \circ ¡_E = \mathrm{id}_\emptyset$
PROOF: There is only one function $\emptyset \to \emptyset$.
⟨1⟩2. If $E \cong \emptyset$ then E is empty
⟨2⟩1. LET: $\phi : E \cong \emptyset$
⟨2⟩2. LET: X be a set
PROVE: There is a unique function $E \to X$
⟨2⟩3. $¡_X \circ \phi : E \to X$
⟨2⟩4. If $f : E \to X$ then $f = ¡_X \circ \phi$
⟨3⟩1. LET: $f : E \to X$
⟨3⟩2. $f \circ \phi^{-1} : \emptyset \to X$
⟨3⟩3. $f \circ \phi^{-1} = ¡_X$
PROOF: Uniqueness of $¡_X$.
⟨3⟩4. Q.E.D.
⟨1⟩3. There is at most one isomorphism $E \cong \emptyset$
PROOF: This holds because there is at most one function $E \to \emptyset$.

□

Proposition 1.15.3.

$$¡_\emptyset = \mathrm{id}_\emptyset$$

Proof: By the uniqueness of $¡_\emptyset$. □

1.16 The Terminal Set

Axiom 1.16.1 (Terminal Set)**.** *There exists a set* 1*, the* terminal set*, such that, for every set X, there exists a unique function $!_X : X \to 1$.*

Proposition 1.16.2 (Uniqueness of Terminal Set)**.** *Let T be any set. Then T is terminal if and only if there exists an isomorphism $T \cong 1$, in which case the isomorphism is unique.*

Proof: Dual to Proposition 1.15.2.

Proposition 1.16.3.

$$!_1 = \mathrm{id}_1$$

Proof: From the uniqueness of $!_1$. □

1.17 Elements

Definition 1.17.1 (Element)**.** An *element* of a set A is a function $1 \to A$. We write $a \in A$ for $a : 1 \to A$. We write $f(a)$ for $f \circ a$ when $f : A \to B$ and $a \in A$.

1.17.1 The Axiom of Extensionality

Axiom 1.17.2 (Extensionality)**.** *Let A and B be sets and $f, g : A \to B$ be functions. If, for all $a \in A$, we have $f(a) = g(a) \in B$, then $f = g$.*

Proposition 1.17.3. *Let $f : A \to B$. Then f is injective if and only if, for all $x, y \in A$, if $f(x) = f(y) \in B$ then $x = y \in A$.*

Proof:
$\langle 1 \rangle 1$. If f is injective and $f(x) = f(y) \in B$ then $x = y \in A$
  Proof: Immediate from the definition of injective.
$\langle 1 \rangle 2$. If, for all $x, y \in A$, if $f(x) = f(y) \in B$ then $x = y \in A$
  $\langle 2 \rangle 1$. Assume: For all $x, y \in A$, if $f(x) = f(y)$, then $x = y$
  $\langle 2 \rangle 2$. Let: X be any set and $g, h : X \to A$ with $f \circ g = f \circ h$
    Prove: $g = h$
  $\langle 2 \rangle 3$. Let: $x \in X$
    Prove: $g(x) = h(x)$
  $\langle 2 \rangle 4$. $f(g(x)) = f(h(x))$
    Proof: From $\langle 2 \rangle 2$.
  $\langle 2 \rangle 5$. $g(x) = h(x)$

PROOF: By ⟨2⟩1
$\square$

Proposition 1.17.4. *Any element $e \in X$ is a section of the unique function $!_X : X \to 1$.*

PROOF:$!_X \circ e = \mathrm{id}_1$ because there is only one function $1 \to 1$. $\square$

Axiom 1.17.5 (Non-degeneracy)**.** *The empty set $\emptyset$ has no elements.*

Proposition 1.17.6. *For any set X, the function $¡_X : \emptyset \to X$ is injective.*

PROOF: From Proposition 1.17.3. $\square$

Definition 1.17.7 (Empty Part)**.** For any set X, the *empty part* of X is $\emptyset = ¡_X : \emptyset \hookrightarrow X$.

Definition 1.17.8 (Constant Function)**.** A function $f : A \to B$ is *constant* iff there exists $b \in B$ such that $f = b \circ !_A$.

Definition 1.17.9 (Membership)**.** Let $i : U \hookrightarrow A$ be a part of A and $a \in A$. Then a is a *member* of i, $a \in_A i$, iff there exists $\overline{a} \in U$ such that $i(\overline{a}) = a$.

Proposition 1.17.10. *Let A be a set. Let i, j be parts of A and $a \in A$. If $a \in_A i$ and $i \subseteq_A j$ then $a \in_A j$.*

PROOF:
⟨1⟩1. PICK $\overline{a} \in \mathrm{dom}\, i$ such that $a = i(\overline{a})$.
⟨1⟩2. PICK $\phi : \mathrm{dom}\, i \to \mathrm{dom}\, j$ such that $i = j \circ \phi$
⟨1⟩3. $a = j(\phi(\overline{a}))$
$\square$

1.17.2 Products

Axiom 1.17.11 (Products)**.** *For any sets A and B, there exists a set $A \times B$, the* product *of A and B, and functions $\pi_1 : A \times B \to A$, $\pi_2 : A \times B \to B$, the* projections*, such that, for any set C and functions $f : C \to A$, $g : C \to B$, there exists a unique function $\langle f, g\rangle : C \to A \times B$ such that*

$$\pi_1 \circ \langle f, g\rangle = f; \qquad \pi_2 \circ \langle f, g\rangle = g \ .$$

Definition 1.17.12. Given functions $f : A \to B$ and $g : C \to D$, define $f \times g : A \times C \to B \times D$ by

$$f \times g = \langle f \circ \pi_1, g \circ \pi_2\rangle$$

1.17.3 Coproducts

Axiom 1.17.13 (Coproducts)**.** *For any sets A and B, there exists a set $A \uplus B$, the* coproduct *or* sum *of A and B, and functions $\kappa_1 : A \to A \uplus B$, $\kappa_2 : B \to A \uplus B$, the* injections*, such that, for any set C and functions $f : A \to C$, $g : B \to C$, there exists a unique function $[f, g] : A \uplus B \to C$ such that*

$$[f, g] \circ \kappa_1 = f; \qquad [f, g] \circ \kappa_2 = g \ .$$

Definition 1.17.14 (Complement)**.** Let $i : I \hookrightarrow J$ and $i' : I' \hookrightarrow J$ be parts of J. Then i' is the *complement* of i iff J is the sum of I and I' with injections i and i'.

1.17.4 Equalizers

Axiom 1.17.15 (Equalizers)**.** *For any sets A and B and functions $f, g : A \to B$, there exists a set E and function $e : E \to A$, the* equalizer *of A and B, such that:*

- $f \circ e = g \circ e : E \to B$;
- *For any set C and function $h : C \to A$ such that $f \circ h = g \circ h$, there exists a unique function $\overline{h} : C \to E$ such that $h = e \circ \overline{h}$.*

Proposition 1.17.16. *All equalizers are injective.*

Proof:
⟨1⟩1. Let: $e : E \to A$ be the equalizer of $f, g : A \to B$
⟨1⟩2. Let: $x, y : X \to E$ with $e \circ x = e \circ y$
⟨1⟩3. $f \circ e \circ x = g \circ e \circ x$
Proof: $f \circ e = g \circ e$ by ⟨1⟩11.
⟨1⟩4. $x = y$
Proof: x and y are both the unique $z : X \to E$ such that $e \circ z = e \circ x$.
□

1.17.5 Coequalizers

Axiom 1.17.17 (Coequalizers)**.** *For any sets A and B and functions $f, g : A \to B$, there exists a set C and function $c : B \to C$, the* coequalizer *of f and g, such that:*

- $c \circ f = c \circ g : A \to C$
- *For any set X and function $h : B \to X$ such that $h \circ f = h \circ g$, there exists a unique function $\overline{h} : C \to X$ such that $\overline{h} \circ c = h$.*

1.17.6 Pullbacks

Definition 1.17.18 (Pullback)**.** The diagram below is a *pullback diagram* iff:

- $f \circ p = g \circ q$
- for every set X and functions $x : X \to B$ and $y : X \to C$ such that $f \circ x = g \circ y$, there exists a unique function $\langle x, y \rangle : X \to A$ such that $p \circ \langle x, y \rangle = x$ and $q \circ \langle x, y \rangle = y$.

$$\begin{tikzcd}
A \arrow[r, "p"] \arrow[d, "q"'] & B \arrow[d, "f"] \\
C \arrow[r, "g"'] & D
\end{tikzcd}$$

Proposition 1.17.19. *Let $f : A \to C$ and $g : B \to C$. Then f and g have a pullback.*

$$\begin{tikzcd}
E \arrow[dr, "e"] & & \\
& A \times B \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & A \arrow[d, "f"] \\
& B \arrow[r, "g"'] & C
\end{tikzcd}$$

PROOF:
$\langle 1 \rangle 1$. Construct the product $\pi_1 : A \times B \to A$, $\pi_2 : A \times B \to B$.
$\langle 1 \rangle 2$. Construct the equalizer $e : E \to A$ of $f \circ \pi_1$ and $g \circ \pi_2$.
PROVE: $\pi_1 \circ e$ and $\pi_2 \circ e$ form a pullback of f and g
$\langle 1 \rangle 3$. $f \circ \pi_1 \circ e = g \circ \pi_2 \circ e$
$\langle 1 \rangle 4$. LET: X be a set and $x : X \to A$, $y : X \to B$ satisfy $f \circ x = g \circ y$
$\langle 1 \rangle 5$. $f \circ \pi_1 \circ \langle x, y \rangle = g \circ \pi_2 \circ \langle x, y \rangle$
$\langle 1 \rangle 6$. LET: $m : X \to E$ be the function such that $e \circ m = \langle x, y \rangle$
$\langle 1 \rangle 7$. $\pi_1 \circ e \circ m = x$ and $\pi_2 \circ e \circ m = y$
$\langle 1 \rangle 8$. m is unique.
PROOF:
$\langle 2 \rangle 1$. LET: $n : X \to E$ be such that $\pi_1 \circ e \circ n = x$ and $\pi_2 \circ e \circ n = y$
$\langle 2 \rangle 2$. $e \circ n = \langle x, y \rangle$
$\langle 2 \rangle 3$. $n = m$
PROOF: By $\langle 1 \rangle 6$
$\square$

Proposition 1.17.20. *Pullbbacks are unique up to isomorphism.*

That is, let P be a pullback of $f : A \to C$ and $g : B \to C$ with projections $p : P \to A$ and $q : P \to B$. Let Q be a set and $p' : Q \to A$, $q' : Q \to B$. Then Q is a pullback of f and g with projections p' and q' if and only if there exists a bijection $\phi : Q \cong P$ such that $p \circ \phi = p'$ and $q \circ \phi = q'$, in which case ϕ is unique.

$$\begin{tikzcd}
Q \arrow[dr, "\phi"] \arrow[drr, bend left, "p'"] \arrow[ddr, bend right, "q'"'] & & \\
& P \arrow[r, "p"'] \arrow[d, "q"] & A \arrow[d, "f"] \\
& B \arrow[r, "g"'] & C
\end{tikzcd}$$

Proof:

$\langle 1 \rangle 1$. If Q is a pullback then there exists a bijection $\phi : Q \cong P$ such that $p \circ \phi = p'$ and $q \circ \phi = q'$

$\langle 2 \rangle 1$. Assume: Q is a pullback with projections p' and q'

$\langle 2 \rangle 2$. Let: $\phi : Q \to P$ be the unique function such that $p \circ \phi = p'$ and $q \circ \phi = q'$

Proof: Such a ϕ exists because $f \circ p' = g \circ q'$.

$\langle 2 \rangle 3$. Let: $\phi^{-1} : P \to Q$ be the unique function such that $p' \circ \phi^{-1} = p$ and $q' \circ \phi^{-1} = q$

Proof: Such a function exists because $f \circ p = g \circ q$.

$\langle 2 \rangle 4$. $\phi \circ \phi^{-1} = \mathrm{id}_P$

Proof: Each is the unique function x such that $p \circ x = p$ and $q \circ x = q$.

$\langle 2 \rangle 5$. $\phi^{-1} \circ \phi = \mathrm{id}_Q$

Proof: Similar.

$\langle 1 \rangle 2$. If $\phi : Q \cong P$ is a bijection then Q is a pullback with projections $p \circ \phi$ and $q \circ \phi$

$\langle 2 \rangle 1$. $f \circ p \circ \phi = g \circ q \circ \phi$

Proof: This holds because $f \circ p = g \circ q$

$\langle 2 \rangle 2$. For any set X and functions $x : X \to A$, $y : X \to B$ such that $f \circ x = g \circ y$, there exists a unique function $m : X \to Q$ such that $p \circ \phi \circ m = x$ and $q \circ \phi \circ m = y$

Proof:

$$\begin{aligned}
& p \circ \phi \circ m = x \text{ and } q \circ \phi \circ m = y \\
\Leftrightarrow & \phi \circ m = \langle x, y \rangle \\
\Leftrightarrow & m = \phi^{-1} \circ \langle x, y \rangle
\end{aligned}$$

$\langle 1 \rangle 3$. If $\phi, \phi' : P \cong Q$ are bijections such that $p \circ \phi = p \circ \phi'$ and $q \circ \phi = q \circ \phi'$

Proof: This follows from the definition of pullback.

□

Proposition 1.17.21 (Pullback Lemma)**.** *In the diagram below, if both squares are pullbacks, then the outer rectangle is a pullback.*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[r, "h"] \arrow[d, "k"] & C \arrow[d, "l"] \\
D \arrow[r, "m"'] & E \arrow[r, "n"'] & F
\end{tikzcd}$$

Proof:
⟨1⟩1. Let: $x : X \to C$ and $y : X \to D$ be such that $l \circ x = n \circ m \circ y$
⟨1⟩2. Let: $z : X \to B$ be the unique function such that $h \circ z = x$ and $k \circ z = m \circ y$
  Proof: z exists because $l \circ x = n \circ m \circ y$
⟨1⟩3. Let: $a : X \to A$ be the unique function such that $f \circ a = z$ and $g \circ a = y$
  Proof: a exists because $k \circ z = m \circ y$
⟨1⟩4. $h \circ f \circ a = x$ and $g \circ a = y$
⟨1⟩5. a is unique such that $h \circ f \circ a = x$ and $g \circ a = y$
  ⟨2⟩1. Let: $a' : X \to A$
  ⟨2⟩2. Assume: $h \circ f \circ a' = x$ and $g \circ a' = y$
  ⟨2⟩3. $f \circ a' = z$
    ⟨3⟩1. $h \circ f \circ a' = x$
      Proof: ⟨2⟩2
    ⟨3⟩2. $k \circ f \circ a' = m \circ y$
      Proof:

$$\begin{aligned} k \circ f \circ a' &= m \circ g \circ a' \\ &= m \circ y \end{aligned}$$

    ⟨3⟩3. Q.E.D.
      Proof: By ⟨1⟩2.
  ⟨2⟩4. $a' = a$
    ⟨1⟩3, ⟨2⟩2, ⟨2⟩3

□

Proposition 1.17.22. *The pullback of an injective function is injective.*

That is, if the diagram below is a pullback diagram and f is injective then q is injective.

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[d, "q"'] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

Proof:
⟨1⟩1. Let: X be a set and $x, y : X \to A$ with $q \circ x = q \circ y$
⟨1⟩2. $f \circ p \circ x = g \circ q \circ x$
⟨1⟩3. Let:
    $z : X \to A$ be the function such that $p \circ z = p \circ x$ and $q \circ z = q \circ x$
⟨1⟩4. $z = x$
⟨1⟩5. $z = y$
  ⟨2⟩1. $q \circ x = q \circ y$
    Proof: By ⟨1⟩1.
  ⟨2⟩2. $f \circ p \circ x = f \circ p \circ y$
    Proof:

$$\begin{aligned} f \circ p \circ x &= g \circ q \circ x && (\langle 1 \rangle 2) \\ &= g \circ q \circ y && (\langle 1 \rangle 1) \\ &= f \circ p \circ y && \text{(the diagram is a pullback)} \end{aligned}$$

$\langle 2\rangle 3.$ $p \circ x = p \circ y$
PROOF: f is injective.
□

Proposition 1.17.23. *In the diagram below, let f and g be a pullback of h and k. Let $\phi : B \cong B'$ be an isomorphism. Then $\phi \circ f$ and g form a pullback of $h \circ \phi^{-1}$ and k.*

$$\begin{tikzcd} & & B' \\ A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[ur, "\phi"] \arrow[d, "h"] & \\ C \arrow[r, "k"'] & D & \end{tikzcd}$$

PROOF:
$\langle 1\rangle 1.$ LET: X be a set and $x : X \to B'$ and $y : X \to C$ satisfiy $h \circ \phi^{-1} \circ x = k \circ y$
$\langle 1\rangle 2.$ There exists a unique $m : X \to A$ such that $f \circ m = \phi^{-1} \circ x$ and $g \circ m = y$
$\langle 1\rangle 3.$ There exists a unique $m : X \to A$ such that $\phi \circ f \circ m = x$ and $g \circ m = y$
□

1.17.7 Inverse Image

Definition 1.17.24 (Inverse Image)**.** Let $f : A \to B$ and $S = (i : \mathrm{dom}\, i \hookrightarrow B)$ be a part of B. The *inverse image* of S underf f, $f^{-1}(S) = (j : \mathrm{dom}\, j \hookrightarrow A)$, is the part of A such that the diagram below is a pullback.

$$\begin{tikzcd} \mathrm{dom}\, j \arrow[r] \arrow[d, tail, "j"] & \mathrm{dom}\, i \arrow[d, tail, "i"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

This is well-defined by Proposition 1.17.23.

1.17.8 Function Sets

Axiom 1.17.25 (Function Sets)**.** *For any sets A and B, there exists a set A^B and a function $\epsilon : A^B \times B \to A$, the* evaluation *function, such that, for any set C and function $f : C \times B \to A$, there exists a unique function $\lambda f : C \to A^B$ such that*

$$\epsilon \circ (\lambda f \times \mathrm{id}_B) = f \quad .$$

1.17.9 The Subset Classifier

Definition 1.17.26. The set 2 is $1 + 1$. We write $\top$ (*truth*) for $\kappa_1 : 1 \to 2$, and $\perp$ (*falsehood*) for $\kappa_2 : 1 \to 2$.

Axiom 1.17.27 (Subset Classifier)**.** *For every injective function $m : A \rightarrowtail B$, there exists a unique function $\chi_m : B \to 2$, the* characteristic function *of m, such that the following diagram is a pullback diagram:*

$$\begin{tikzcd} A \arrow[r, "!"] \arrow[d, "m"'] & 1 \arrow[d, "\top"] \\ B \arrow[r, "\chi_m"'] & 2 \end{tikzcd}$$

Proposition 1.17.28. *Every function $\phi : A \to 2$ is the characteristic function of a part of A.*

Proof:
$\langle 1 \rangle 1$. Construct a pullback

$$\begin{tikzcd} I \arrow[r] \arrow[d, "q"] & 1 \arrow[d, "\top"] \\ A \arrow[r, "\phi"'] & 2 \end{tikzcd}$$

Proof: By Proposition 1.17.19.
$\langle 1 \rangle 2$. q is injective
Proof: By Proposition 1.17.22.
□

Proposition 1.17.29. *Let S be a part of A and $\phi : A \to 2$ be its characteristic function. Then, for all $x \in A$, we have $\phi(x) = \top$ if and only if $x \in_A S$.*

Proof:
$\langle 1 \rangle 1$. Let: $x \in A$
$\langle 1 \rangle 2$. If $\phi(x) = \top$ then $x \in_A S$.
Proof: If $\phi(x) = \top$ then there exists $y \in \operatorname{dom} S$ such that $S(y) = x$.
$\langle 1 \rangle 3$. If $x \in_A S$ then $\phi(x) = \top$.
Proof: If $y \in \operatorname{dom} S$ and $S(y) = x$ then

$$\begin{aligned} \phi(x) &= \phi(S(y)) \\ &= \top \circ ! \circ y \\ &= y \end{aligned}$$

□

Corollary 1.17.29.1. *Two parts of a set are equivalent if and only if they have the same characteristic function.*

Proposition 1.17.30. *Let $f : X \to Y$ and S be a part of Y. If $\psi : Y \to 2$ is the characteristic function of S then $\psi \circ f$ is the characteristic function of $f^{-1}(S)$.*

Proof: From the Pullback Lemma. □

Axiom 1.17.31 (Boolean)**.** *For any $p \in 2$ we have $p = \top$ or $p = \bot$.*

1.18 The Basics

Lemma 1.18.1. *Let X be a set, $\mathcal{B} \subseteq \mathcal{P}X$ and $U \subseteq X$. Then the following are equivalent:*

1. *For all $x \in U$ there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$.*
2. *There exists $\mathcal{B}_0 \subseteq \mathcal{B}$ such that $U = \bigcup \mathcal{B}_0$.*

Proof:
$\langle 1 \rangle 1$. $1 \Rightarrow 2$
Proof: If 1 is true then $U = \bigcup \{B \in \mathcal{B} : B \subseteq U\}$.
$\langle 1 \rangle 2$. $2 \Rightarrow 1$
Proof: Trivial.
□

Definition 1.18.2 (Fixed Point)**.** Let X be a set, $f : X \to X$, and $x \in X$. Then x is a *fixed point* of f iff $f(x) = x$.

Definition 1.18.3 (Saturated)**.** Let X, Y be sets and $p : X \twoheadrightarrow Y$ be a surjective function. Let $C \subseteq X$. Then C is *saturated* with respect to p iff, for all $x, x' \in X$, if $x \in C$ and $p(x) = p(x')$ then $x' \in C$.

Definition 1.18.4 (Cover)**.** Let A be a set and $\mathcal{C} \subseteq \mathcal{P}A$. Then $\mathcal{C}$ *covers* A iff $\bigcup \mathcal{C} = A$.

Definition 1.18.5 (Finite Intersection Property)**.** Let X be a set and $\mathcal{C} \subseteq \mathcal{P}X$. Then $\mathcal{C}$ has the *finite intersection property* if and only if every finite nonempty subset of $\mathcal{C}$ has nonempty intersection.

Lemma 1.18.6 (AC)**.** *Let X be a set and $\mathcal{A} \subseteq \mathcal{P}X$ have the finite intersection property. Then there exists a maximal $\mathcal{D} \subseteq \mathcal{P}X$ that has the finite intersection property and includes $\mathcal{A}$.*

Proof: A straightforward application of Zorn's lemma, since the union of a chain of sets that has the finite intersection property has the finite intersection property. □

Lemma 1.18.7. *Let X be a set and $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the finite intersection property. Then any finite intersection of elements of $\mathcal{D}$ is an element of $\mathcal{D}$.*

Proof:
$\langle 1 \rangle 1$. Let: A be a finite intersection of elements of $\mathcal{D}$
$\langle 1 \rangle 2$. $\mathcal{D} \cup \{A\}$ has the finite intersection property.
$\langle 1 \rangle 3$. $\mathcal{D} \cup \{A\} = \mathcal{D}$
□

Lemma 1.18.8. *Let X be a set and $\mathcal{D} \subseteq \mathcal{P}X$ be maximal with respect to the finite intersection property. If $A \subseteq X$ intersects every element of $\mathcal{D}$ then $A \in \mathcal{D}$.*

PROOF: This holds because $\mathcal{D} \cup \{A\}$ satisfies the finite intersection property. $\square$

Definition 1.18.9 (Graph)**.** Let $f : A \to B$. The *graph* of f is the set $\{(x, f(x)) : x \in A\} \subseteq A \times B$.

Definition 1.18.10 (Point-Finite)**.** Let X be a set and $\{A_\alpha\}_{\alpha \in J}$ be a family of subsets of X. Then $\{A_\alpha\}_{\alpha \in J}$ is *point-finite* iff, for all $x \in X$, there are only finitely many $\alpha \in J$ such that $x \in A_\alpha$.

Definition 1.18.11 (Countable Intersection Property)**.** A family of parts of a set X has the *countable intersection property* iff every countable subfamily has nonempty intersection.

1.19 Refinements

Definition 1.19.1 (Refinement)**.** Let X be a set and $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}X$. Then $\mathcal{B}$ is a *refinement* of $\mathcal{A}$ iff, for all $B \in \mathcal{B}$, there exists $A \in \mathcal{A}$ such that $B \subseteq A$.

1.20 Order Theory

Definition 1.20.1 (Cofinal)**.** Let J be a poset and $K \subseteq J$. Then K is *cofinal* iff, for all $x \in J$, there exists $y \in K$ such that $x \leq y$.

Definition 1.20.2 (Directed Set)**.** A *directed set* is a poset J such that, for all $x, y \in J$, there exists $z \in J$ such that $x \leq z$ and $y \leq z$.

Definition 1.20.3 (Linear Order)**.** Let X be a set. A *linear order* on X is a relation $\leq \subseteq X^2$ such that:

- For all $x \in X$, $x \leq x$
- For all $x, y, z \in X$, if $x \leq y$ and $y \leq z$ then $x \leq z$
- For all $x, y \in X$, if $x \leq y$ and $y \leq x$ then $x = y$
- For all $x, y \in X$, we have $x \leq y$ or $y \leq x$

We write $x < y$ iff $x \leq y$ and $x \neq y$.

A *linearly ordered set* consists of a set and a linear order on the set.

Definition 1.20.4 (Convex)**.** Let L be a linearly ordered set and $A \subseteq L$. Then A is *convex* iff, for all $x, y \in A$ and $z \in L$, if $x < z < y$ then $z \in A$.

Definition 1.20.5 (Least Upper Bound Property)**.** A linearly ordered set L has the *least upper bound property* iff every subset of L bounded above has a least upper bound.

Definition 1.20.6 (Linear Continuum)**.** A *linear continuum* is a linearly ordered set L such that:

- L has the least upper bound property.
- For all $x, y \in L$ with $x < y$, there exists $z \in L$ such that $x < z < y$.

Proposition 1.20.7. *If L is a linear continuum then every convex subset of L is a linear continuum.*

PROOF:
$\langle 1 \rangle 1$. LET: L be a linear continuum and $C \subseteq L$ be convex
$\langle 1 \rangle 2$. C satisfies the least upper bound property.
  $\langle 2 \rangle 1$. LET: $S \subseteq C$ be nonempty and bounded above by u in C.
  $\langle 2 \rangle 2$. LET: s be the supremum of S in L
  $\langle 2 \rangle 3$. PICK $x \in S$
  $\langle 2 \rangle 4$. $x \leq s \leq u$
  $\langle 2 \rangle 5$. $s \in C$
    PROOF: C is convex.
  $\langle 2 \rangle 6$. s is the supremum of S in C
$\langle 1 \rangle 3$. C is dense.
  PROOF:
  $\langle 2 \rangle 1$. LET: $x, y \in C$ satisfy $x < y$
  $\langle 2 \rangle 2$. PICK $z \in L$ such that $x < z < y$
  $\langle 2 \rangle 3$. $z \in C$
    PROOF: C is convex.
□

Lemma 1.20.8. *For any real numbers a, b with $a < b$ we have $[a, b) \cong [0, 1)$.*

PROOF: The map $\phi : [a, b) \cong [0, 1)$ where $\phi(x) = (x - a)/(b - a)$ is an order isomorphism. □

Proposition 1.20.9. *Let X be a linearly ordered set. Let $a, b, c \in X$ with $a < c < b$. Then $[a, b) \cong [0, 1)$ if and only if $[a, c) \cong [c, b) \cong [0, 1)$.*

PROOF:
$\langle 1 \rangle 1$. If $[a, b) \cong [0, 1)$ then $[a, c) \cong [c, b) \cong [0, 1)$.
  $\langle 2 \rangle 1$. ASSUME: $\phi : [a, b) \cong [0, 1)$ is an order isomorphism.
  $\langle 2 \rangle 2$. $[a, c) \cong [0, 1)$
    PROOF:

$$\begin{aligned} [a, c) &\cong [0, \phi(c)) && \text{(under } \phi\text{)} \\ &\cong [0, 1) && \text{(Lemma 1.20.8)} \end{aligned}$$

  $\langle 2 \rangle 3$. $[c, b) \cong [0, 1)$
    PROOF: Similar.
$\langle 1 \rangle 2$. If $[a, c) \cong [c, b) \cong [0, 1)$ then $[a, b) \cong [0, 1)$.
  $\langle 2 \rangle 1$. ASSUME: $[a, c) \cong [c, b) \cong [0, 1)$
  $\langle 2 \rangle 2$. LET: $\phi : [a, c) \cong [0, 1/2)$ and $\psi : [c, b) \cong [1/2, 1)$
  $\langle 2 \rangle 3$. LET: $\chi : [a, b) \to [0, 1)$ be given by $\chi(x) = \begin{cases} \phi(x) & \text{if } x < c \\ \psi(x) & \text{if } x \geq c \end{cases}$

$\langle 2\rangle 4.$ $\chi : [a, b) \cong [0, 1)$
PROOF: Easy to check.
□

Proposition 1.20.10 (CC)**.** *Let X be a linearly ordered set. Let $\{x_n\}_{n \geq 0}$ be an increasing sequence of points of X. Suppose b is the supremum of $\{x_n : n \geq 0\}$. Then $[x_0, b) \cong [0, 1)$ if and only if $[x_i, x_{i+1}) \cong [0, 1)$ for all i.*

PROOF:
$\langle 1\rangle 1.$ If $[x_0, b) \cong [0, 1)$ then for all i $[x_i, x_{i+1}) \cong [0, 1)$.
PROOF: If $\phi : [x_0, b) \cong [0, 1)$ then $[x_i, x_{i+1}) \cong [\phi(x_i), \phi(x_{i+1})) \cong [0, 1)$ by Lemma 1.20.8.
$\langle 1\rangle 2.$ If for all i $[x_i, x_{i+1}) \cong [0, 1)$ then $[x_0, b) \cong [0, 1)$.
PROOF:
$\langle 2\rangle 1.$ LET: $\phi_i : [x_i, x_{i+1}) \cong [0, 1)$ for all i
$\langle 2\rangle 2.$ Define $\phi : [x_0, b) \cong [0, 1)$ by: $\phi(y) = \phi_i(y) \qquad (x_0 \leq y < b)$ where i is least such that $y < i_{i+1}$
PROOF: There exists such an i because y is not an upper bound for $\{x_n : n \geq 0\}$.
$\langle 2\rangle 3.$ ϕ is an order isomorphism.
PROOF: Easy to check.
□

Proposition 1.20.11 (CC)**.** *For all $0 < \alpha < \Omega$, the interval $[(0, 0), (\alpha, 0))$ in $S_\Omega \times [0, 1)$ is order isomorphic to $[0, 1)$ in $\mathbb{R}$.*

PROOF:
$\langle 1\rangle 1.$ If $[(0, 0), (\alpha, 0)) \cong [0, 1)$ then $[(0, 0), (\alpha + 1, 0)) \cong [0, 1)$
PROOF: By Proposition 1.20.9.
$\langle 1\rangle 2.$ Let λ be a limit ordinal, $0 < \lambda < \Omega$. If, for all α with $0 < \alpha < \lambda$, we have $[(0, 0), (\alpha, 0)) \cong [0, 1)$, then $[(0, 0), (\lambda, 0)) \cong [0, 1)$.
PROOF: By Propositoin 1.20.10.
$\langle 1\rangle 3.$ Q.E.D.
PROOF: By transfinite induction.
□

Chapter 2

Real Analysis

Lemma 2.0.1. *Let $f, g : X \to \mathbb{R}$. If $f(X)$ and $g(X)$ are bounded above then $\{f(x) + g(x) : x \in X\}$ is bounded above and*

$$\sup_{x \in X}(f(x) + g(x)) \leq \sup_{x \in X} f(x) + \sup_{x \in X} g(x)$$

PROOF: For $x \in X$ we have $f(x) + g(x) \leq \sup_{x \in X} f(x) + \sup_{x \in X} g(x)$. □

Definition 2.0.2 (Cantor Set)**.** Define a sequence of sets $A_n \subseteq [0, 1]$ by:

$$\begin{aligned} A_0 &= [0, 1] \\ A_{n+1} &= A_n \setminus \bigcup_{k=0}^{3^{n-1}-1} \left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n} \right) \end{aligned}$$

The *Cantor set* is $\bigcap_{n=0}^{\infty} A_n$.

Chapter 3

Topological Spaces

3.1 Topologies

Definition 3.1.1 (Topology)**.** A *topology* on a set X is a set $\mathcal{T} \subseteq \mathcal{P}X$ such that:

1. $X \in \mathcal{T}$;
2. for all $U, V \in \mathcal{T}$, we have $U \cap V \in \mathcal{T}$;
3. For all $\mathcal{A} \subseteq \mathcal{T}$, we have $\bigcup \mathcal{A} \in \mathcal{T}$.

A *topological space* X consists of a set X and a topology on X. The elements of X are called *points* and the elements of $\mathcal{T}$ are called *open sets*.

Proposition 3.1.2. *In any topological space, the empty set is open.*

Proof: Immediate from axiom 3. □

Definition 3.1.3 (Discrete Topology)**.** The *discrete* topology on a set X is $\mathcal{P}X$.

Definition 3.1.4 (Indiscrete Topology)**.** The *indiscrete* topology on a set X is $\{\emptyset, X\}$.

Definition 3.1.5 (Open Cover)**.** Let X be a topological space. A cover $\mathcal{C} \subseteq \mathcal{P}X$ of X is an *open cover* iff every member of $\mathcal{C}$ is open.

Definition 3.1.6 (Finer, Coarser)**.** Let $\mathcal{T}$, $\mathcal{T}'$ be topologies on a set X. Then $\mathcal{T}$ is *finer* than $\mathcal{T}'$, and $\mathcal{T}'$ is *coarser* than $\mathcal{T}$, iff $\mathcal{T}' \subseteq \mathcal{T}$.

The topology $\mathcal{T}$ is *strictly* finer than $\mathcal{T}'$, and $\mathcal{T}'$ is *strictly* coarser than $\mathcal{T}$, iff $\mathcal{T} \subset \mathcal{T}'$.

The topologies $\mathcal{T}$ and $\mathcal{T}'$ are *comparable* iff $\mathcal{T} \subseteq \mathcal{T}'$ or $\mathcal{T}' \subseteq \mathcal{T}$.

Definition 3.1.7 (Finite Complement Topology)**.** The *finite complement topology* on a set X is $\{U : X \setminus U \text{ is finite}\} \cup \{X\}$.

Definition 3.1.8 (Isolated Point)**.** Let X be a topological space and $a \in X$. Then a is an *isolated point* iff $\{a\}$ is open.

3.2 Neighbourhoods

Definition 3.2.1 (Neighbourhood)**.** Let X be a topological space and $A \subseteq X$. A *neighbourhood* of A is an set that includes an open set that includes A.

A *neighbourhood* of a point a is a neighbourhood of $\{a\}$.

Proposition 3.2.2. *If N is a neighbourhood of A and $B \subseteq A$ then N is a neighbourhood of B.*

PROOF: Immediate from definitions. □

Proposition 3.2.3. *A set U is open if and only if it is a neighbourhood of each of its points.*

PROOF:
⟨1⟩1. LET: X be a topological space and $A \subseteq X$
⟨1⟩2. If U is a neighbourhood of each of its points then A is open.
  ⟨2⟩1. ASSUME: U includes a neighbourhood of each of its points
    PROVE: $U = \bigcup\{V \subseteq U : V \text{ is open}\}$
  ⟨2⟩2. $\bigcup\{V \subseteq U : V \text{ is open}\} \subseteq U$
    PROOF: Set theory.
  ⟨2⟩3. $U \subseteq \bigcup\{V \subseteq U : V \text{ is open}\}$
    PROOF: Immediate from ⟨2⟩1.
⟨1⟩3. If U is open then U is a neighbourhood of each of its points.
  PROOF: Immediate from definitions.
□

Proposition 3.2.4. *If M is a neighbourhood of A and $M \subseteq N$ then N is a neighbourhood of A.*

PROOF: Immediate from definitions. □

Proposition 3.2.5. *If M and N are neighbourhoods of A then $M \cap N$ is a neighbourhood of A.*

PROOF: Pick open sets U and V such that $A \subseteq U \subseteq M$ and $A \subseteq N \subseteq V$. Then $A \subseteq U \cap V \subseteq M \cap N$.

Proposition 3.2.6. *If N is a neighbourhood of x then $x \in N$.*

PROOF: Immediate from definitions. □

Proposition 3.2.7. *If N is a neighbourhood of x then there exists a neighbourhood U of x such that, for all $y \in U$, M is a neighbourhood of y.*

PROOF: Pick an open set U such that $x \in U \subseteq N$. □

Theorem 3.2.8. *Let X be a set and $\rhd \subseteq \mathcal{P}X \times X$ a relation such that:*

1. *If $M \rhd x$ and $M \subseteq N$ then $N \rhd x$*
2. *$X \rhd x$ for all $x \in X$*

3. *If $M \rhd x$ and $N \rhd x$ then $M \cap N \rhd x$*

4. *If $N \rhd x$ then $x \in N$*

5. *If $M \rhd x$ then there exists $N \rhd x$ such that, for all $y \in N$, $M \rhd y$.*

Then there exists a unique topology $\mathcal{T}$ such that $N \rhd x$ iff N is a neighbourhood of x.

PROOF:
⟨1⟩1. LET: $\rhd$ be a relation satisfying 1–3
⟨1⟩2. LET: $\mathcal{T} = \{U \in \mathcal{P}X : \forall x \in U.U \rhd x\}$
⟨1⟩3. $\mathcal{T}$ is a topology.
  ⟨2⟩1. $X \in \mathcal{T}$
    PROOF: By axiom 2
  ⟨2⟩2. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$
    PROOF: By axiom 3
  ⟨2⟩3. For all $\mathcal{A} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{A} \in \mathcal{T}$
    ⟨3⟩1. LET: $x \in \bigcup \mathcal{A}$
    ⟨3⟩2. PICK $U \in \mathcal{A}$ such that $x \in U$
    ⟨3⟩3. $U \rhd x$
    ⟨3⟩4. $\bigcup \mathcal{A} \rhd x$
      PROOF: By axiom 1
⟨1⟩4. In $\mathcal{T}$, $N \rhd x$ iff N is a neighbourhood of x.
  ⟨2⟩1. If $N \rhd x$ then N is a neighbourhood of x
    ⟨3⟩1. ASSUME: $N \rhd x$
    ⟨3⟩2. $x \in N$
      PROOF: By axiom 4
    ⟨3⟩3. LET: $U = \{y \in N : N \rhd y\}$
    ⟨3⟩4. U is open
      ⟨4⟩1. LET: $y \in U$
          PROVE: $U \rhd y$
      ⟨4⟩2. $N \rhd y$
      ⟨4⟩3. PICK $W \rhd y$ such that, for all $z \in W$, $N \rhd z$
        PROOF: By axiom 5
      ⟨4⟩4. $W \subseteq U$
      ⟨4⟩5. $U \rhd y$
        PROOF: By axiom 1
    ⟨3⟩5. $x \in U \subseteq N$
  ⟨2⟩2. If N is a neighbourhood of x then $N \rhd x$
    ⟨3⟩1. LET: N be a neighbourhood of x
    ⟨3⟩2. PICK U open such that $x \in U \subseteq N$
    ⟨3⟩3. $U \rhd x$
      PROOF: By ⟨1⟩2
    ⟨3⟩4. $N \rhd x$
      PROOF: By axiom 1
⟨1⟩5. $\mathcal{T}$ is unique.

PROOF: By Proposition 3.2.3.
∎

Definition 3.2.9 (Sufficiently Close)**.** Let X be a topological space, $a \in X$, and P be a property of points of X. We write "For all x sufficiently close to a, $P(x)$" to mean "There exists a neighbourhood N of a such that, for all $x \in N$, $P(x)$."

3.3 Open Refinements

Definition 3.3.1 (Open Refinement)**.** Let X be a space and $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}X$. Then $\mathcal{B}$ is an *open refinement* of $\mathcal{A}$ iff $\mathcal{B}$ is a refinement of $\mathcal{A}$ and every member of $\mathcal{B}$ is open.

3.4 Local Bases

Definition 3.4.1 (Local Basis)**.** Let X be a topological space and $x \in X$. A *local basis* at x is a set $\mathcal{B}$ of open neighbourhoods of x such that every neighbourhood of x includes a member of $\mathcal{B}$. We call the elements of $\mathcal{B}$ *basic open neighbourhoods.*

Proposition 3.4.2. *Let $\mathcal{B}$ be a local basis at x and $M, N \in \mathcal{B}$. Then there exists $P \in \mathcal{B}$ such that $P \subseteq M \cap N$.*

PROOF: This holds because $M \cap N$ is a neighbourhood of x (Proposition 3.2.5).
∎

Proposition 3.4.3. *Let X be a topological space, $x \in X$ and $\mathcal{B} \subseteq \mathcal{P}X$. Then $\mathcal{B}$ is a local basis at x iff $\mathcal{B}$ is a set of open neighbourhoods of x such that every open neighbourhood of x includes a member of $\mathcal{B}$.*

PROOF:
$\langle 1 \rangle 1$. If $\mathcal{B}$ is a local basis at x then $\mathcal{B}$ is a set of open neighbourhoods of x such that every open neighbourhood of x includes a member of $\mathcal{B}$
PROOF: Trivial.
$\langle 1 \rangle 2$. If $\mathcal{B}$ is a set of open neighbourhoods of x such that every open neighbourhood of x includes a member of $\mathcal{B}$ then $\mathcal{B}$ is a local basis at x.
PROOF: Every neighbourhood of x includes an open neighbourhood of x, which therefore includes an element of $\mathcal{B}$.
∎

3.5 Bases

Definition 3.5.1 (Basis for a Topology)**.** Let $(X, \mathcal{T})$ be a topological space. A *basis* for the topology on X is a set of open sets $\mathcal{B}$ such that every open set is a union of members of $\mathcal{B}$. The members of $\mathcal{B}$ are called *basic open sets*, and $\mathcal{T}$ is called the topology *generated* by $\mathcal{B}$.

Proposition 3.5.2. *Let $(X, \mathcal{T})$ be a topological space and $\mathcal{B} \subseteq \mathcal{P}X$. Then the following are equivalent:*

1. *$\mathcal{B}$ is a basis for $\mathcal{T}$.*
2. *A set U is open if and only if, for all $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$.*
3. *$\mathcal{T}$ is the set of all unions of subsets of $\mathcal{B}$.*
4. *Every member of $\mathcal{B}$ is open and, for all $x \in X$ and every open neighbourhood U of x, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$.*
5. *For all $x \in X$, the set $\{B \in \mathcal{B} : x \in B\}$ is a local basis at x.*

PROOF:
$\langle 1 \rangle 1.$ $1 \Rightarrow 2$
  $\langle 2 \rangle 1.$ ASSUME: $\mathcal{B}$ is a basis for the topology $\mathcal{T}$.
  $\langle 2 \rangle 2.$ For all $U \in \mathcal{T}$ and $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B$
    PROOF: Immediate from the definition of basis ($\langle 2 \rangle 1$).
  $\langle 2 \rangle 3.$ For all $U \subseteq X$, if $\forall x \in U. \exists B \in \mathcal{B}. x \in B \subseteq U$ then $U \in \mathcal{T}$
    PROOF: By Proposition 3.2.3.
$\langle 1 \rangle 2.$ $2 \Leftrightarrow 3$
  PROOF: From Lemma 1.18.1.
$\langle 1 \rangle 3.$ $3 \Rightarrow 1$
  PROOF: Trivial.
$\langle 1 \rangle 4.$ $2 \Rightarrow 4$
  PROOF: Trivial.
$\langle 1 \rangle 5.$ $4 \Rightarrow 2$
  PROOF:
  $\langle 2 \rangle 1.$ ASSUME: 4
  $\langle 2 \rangle 2.$ If U is open then, for all $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$
    PROOF: Immediate from $\langle 2 \rangle 1$.
  $\langle 2 \rangle 3.$ If, for all $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$, then U is open.
    PROOF: By Proposition 3.2.3 using the fact that every member of $\mathcal{B}$ is open ($\langle 2 \rangle 1$).
$\langle 1 \rangle 6.$ $4 \Leftrightarrow 5$
  PROOF: From Proposition 3.4.3.
□

Corollary 3.5.2.1. *If $\mathcal{B}$ is a basis for the topology $\mathcal{T}$, then $\mathcal{T}$ is the coarsest topology in which every element of $\mathcal{B}$ is open.*

Lemma 3.5.3. *Let X be a set and $\mathcal{B} \subseteq \mathcal{P}X$. Then $\mathcal{B}$ is a basis for a topology $\mathcal{T}$ on X if and only if:*

1. $\bigcup \mathcal{B} = X$

2. *for all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.*

In this case, $\mathcal{T}$ is unique.

PROOF:
$\langle 1 \rangle 1$. If $\mathcal{B}$ is a basis for a topology then $\bigcup \mathcal{B} = X$
$\langle 2 \rangle 1$. ASSUME: $\mathcal{B}$ is a basis for the topology $\mathcal{T}$
$\langle 2 \rangle 2$. LET: $x \in X$
$\langle 2 \rangle 3$. There exists $B \in \mathcal{B}$ such that $x \in B$
PROOF: From the definition of basis, since $X \in \mathcal{T}$. ($\langle 2 \rangle 1$, $\langle 2 \rangle 2$).
$\langle 1 \rangle 2$. If $\mathcal{B}$ is a basis for a topology then it satisfies condition 2
$\langle 2 \rangle 1$. ASSUME: $\mathcal{B}$ is a basis for the topology $\mathcal{T}$
$\langle 2 \rangle 2$. LET: $B_1, B_2 \in \mathcal{B}$
$\langle 2 \rangle 3$. $B_1, B_2 \in \mathcal{T}$
PROOF: From the definition of basis ($\langle 2 \rangle 1$, $\langle 2 \rangle 2$).
$\langle 2 \rangle 4$. $B_1 \cap B_2 \in \mathcal{T}$
PROOF: By the definition of topology, the open sets in $\mathcal{T}$ are closed under binary intersection ($\langle 2 \rangle 1$, $\langle 2 \rangle 3$)
$\langle 2 \rangle 5$. For all $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$
PROOF: From the definition of basis ($\langle 2 \rangle 1$, $\langle 2 \rangle 4$)
$\langle 1 \rangle 3$. If $\mathcal{B}$ satisfies conditions 1 and 2 then $\mathcal{T} = \{U \subseteq X : \forall x \in U. \exists B \in \mathcal{B}. x \in B \subseteq U\}$ is a topology and $\mathcal{B}$ is a basis for $\mathcal{T}$.
$\langle 2 \rangle 1$. ASSUME: $\mathcal{B}$ satisfies conditions 1 and 2
$\langle 2 \rangle 2$. $X \in \mathcal{T}$
PROOF: For all $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq X$ by condition 1 ($\langle 2 \rangle 1$).
$\langle 2 \rangle 3$. For all $\mathcal{A} \subseteq \mathcal{T}$, we have $\bigcup \mathcal{A} \in \mathcal{T}$
$\langle 3 \rangle 1$. LET: $\mathcal{A} \subseteq \mathcal{T}$
$\langle 3 \rangle 2$. LET: $x \in \bigcup \mathcal{A}$
$\langle 3 \rangle 3$. PICK $U \in \mathcal{A}$ such that $x \in U$
PROOF: From $\langle 3 \rangle 2$.
$\langle 3 \rangle 4$. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$
PROOF: Since $U \in \mathcal{T}$, using the definition of $\mathcal{T}$ ($\langle 3 \rangle 1$, $\langle 3 \rangle 3$)
$\langle 3 \rangle 5$. $x \in B \subseteq \bigcup \mathcal{A}$
PROOF: From $\langle 3 \rangle 3$ and $\langle 3 \rangle 4$.
$\langle 2 \rangle 4$. For all $U, V \in \mathcal{T}$, we have $U \cap V \in \mathcal{T}$
$\langle 3 \rangle 1$. LET: $U, V \in \mathcal{T}$
$\langle 3 \rangle 2$. LET: $x \in U \cap V$
$\langle 3 \rangle 3$. PICK $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \subseteq U$ and $x \in B_2 \subseteq V$
PROOF: From $\langle 3 \rangle 1$, $\langle 3 \rangle 2$ and the definition of $\mathcal{T}$.
$\langle 3 \rangle 4$. PICK $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$
PROOF: Using condition 2 ($\langle 2 \rangle 1$, $\langle 3 \rangle 3$).
$\langle 3 \rangle 5$. $x \in B_3 \subseteq U \cap V$
PROOF: From $\langle 3 \rangle 3$ and $\langle 3 \rangle 4$.
$\langle 2 \rangle 5$. $\bigcup \mathcal{B} = X$
PROOF: This is condition 1 ($\langle 2 \rangle 1$).

⟨2⟩6. For all $U \in \mathcal{T}$ and $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$

PROOF: Immediate from the definition of $\mathcal{T}$.

⟨1⟩4. $\mathcal{T}$ is unique.

PROOF: From Proposition 3.5.2.

□

Corollary 3.5.3.1. *Let X be a set and $\mathcal{B} \subseteq \mathcal{P}X$ be such that $\bigcup \mathcal{B} = X$ and $\mathcal{B}$ is closed under binary intersection. Then $\mathcal{B}$ is a basis for a unique topology on X.*

Lemma 3.5.4. *Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$ on X respectively. Then $\mathcal{T} \subseteq \mathcal{T}'$ if and only if, for all $B \in \mathcal{B}$ and $x \in B$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.*

PROOF:

⟨1⟩1. If $\mathcal{T} \subseteq \mathcal{T}'$ then, for all $B \in \mathcal{B}$ and $x \in B$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.

⟨2⟩1. ASSUME: $\mathcal{T} \subseteq \mathcal{T}'$

⟨2⟩2. LET: $B \in \mathcal{B}$ and $x \in B$

⟨2⟩3. $B \in \mathcal{T}$

PROOF: This holds because $\mathcal{B} \subseteq \mathcal{T}$ by the definition of basis. (⟨2⟩2)

⟨2⟩4. $B \in \mathcal{T}'$

PROOF: From ⟨2⟩1 and ⟨2⟩3.

⟨2⟩5. There exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.

⟨1⟩2. If, for all $B \in \mathcal{B}$ and $x \in B$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$, then $\mathcal{T} \subseteq \mathcal{T}'$.

⟨2⟩1. ASSUME: For all $B \in \mathcal{B}$ and $x \in B$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$

⟨2⟩2. LET: $U \in \mathcal{T}$

PROVE: $U \in \mathcal{T}'$

⟨2⟩3. LET: $x \in U$

⟨2⟩4. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$

PROOF: Since $\mathcal{B}$ is a basis for $\mathcal{T}$ (⟨2⟩2, ⟨2⟩3).

⟨2⟩5. PICK $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$

PROOF: From ⟨2⟩1 and ⟨2⟩4.

⟨2⟩6. $x \in B' \subseteq U$

PROOF: From ⟨2⟩4 and ⟨2⟩5.

⟨2⟩7. Q.E.D.

PROOF: By Proposition 3.5.2.

□

Definition 3.5.5 (Lower Limit Topology)**.** The *lower limit topology* on $\mathbb{R}$ is the one generated by the set of all half-open intervals of the form $[a, b)$. We write $\mathbb{R}_l$ for the topological space consisting of $\mathbb{R}$ under this topology.

We prove this is a topology.

PROOF:

⟨1⟩1. LET: $\mathcal{B}$ be the set of all half-open intervals of the form $[a, b)$.

⟨1⟩2. $\bigcup \mathcal{B} = \mathbb{R}$

Proof: For all $x \in \mathbb{R}$, we have $x \in [x, x+1) \in \mathcal{B}$.

⟨1⟩3. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Proof: If $x \in [a, b) \cap [c, d)$ then $x \in [\max(a, c), \min(b, d)) \subseteq [a, b) \cap [c, d)$.

⟨1⟩4. Q.E.D.

Proof: By Lemma 3.5.3.

□

Definition 3.5.6 (K-topology)**.** The *K-topology* on $\mathbb{R}$ is the one generated by the set of all open intervals (a, b) and all sets of the form $(a, b) \setminus K$, where $K = \{1/n : n \in \mathbb{Z}^+\}$. We write $\mathbb{R}_K$ for the topological space consisting of $\mathbb{R}$ under this topology.

We prove this is a topology.

Proof:

⟨1⟩1. Let: $\mathcal{B} = \{(a, b) : a, b \in \mathbb{R}, a < b\} \cup \{(a, b) \setminus K : a, b \in \mathbb{R}, a < b\}$

⟨1⟩2. $\bigcup \mathcal{B} = \mathbb{R}$

Proof: For all $x \in \mathbb{R}$, we have $x \in (x-1, x+1) \in \mathcal{B}$.

⟨1⟩3. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

⟨2⟩1. Let: $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$

Prove: There exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$

⟨2⟩2. Case: $B_1 = (a, b)$, $B_2 = (c, d)$

Proof: Take $B_3 = (\max(a, c), \min(b, d))$

⟨2⟩3. Case: $B_1 = (a, b)$, $B_2 = (c, d) \setminus K$

Proof: Take $B_3 = (\max(a, c), \min(b, d)) \setminus K$

⟨2⟩4. Case: $B_1 = (a, b) \setminus K$, $B_2 = (c, d)$

Proof: Take $B_3 = (\max(a, c), \min(b, d)) \setminus K$

⟨2⟩5. Case: $B_1 = (a, b) \setminus K$, $B_2 = (c, d) \setminus K$

Proof: Take $B_3 = (\max(a, c), \min(b, d)) \setminus K$

⟨1⟩4. Q.E.D.

Proof: By Lemma 3.5.3.

□

Lemma 3.5.7. *The lower limit topology and the K-topology are incomparable.*

Proof: $[0, 1)$ is not open in the K-topology. $(-1, 1) \setminus K$ is not open in the lower limit topology, because there is no half-open interval $[a, b)$ such that $0 \in [a, b) \subseteq (-1, 1) \setminus K$. □

Proposition 3.5.8. *The set of all singletons is a basis for any discrete space.*

Proof: Easy. □

Definition 3.5.9 (Line with Two Origins)**.** The *line with two origins* is the set $\mathbb{R} \setminus \{0\} \cup \{p, q\}$ under the topology generated by the basis consisting of:

- all open intervals in $\mathbb{R}$ that do not contain 0;

- all sets of the form $(-a, 0) \cup \{p\} \cup (0, a)$ where $a > 0$;
- all sets of the form $(-a, 0) \cup \{q\} \cup (0, a)$ where $a > 0$

3.6 Closed Sets

Definition 3.6.1 (Closed)**.** Let X be a topological space and $A \subseteq X$. Then A is *closed* iff $X \setminus A$ is open.

Proposition 3.6.2. *In any topological space X, the empty set $\emptyset$ is closed.*

PROOF: This holds because $X \setminus \emptyset = X$ is open. □

Proposition 3.6.3. *In any topological space X, the set X is closed.*

PROOF: This holds because $X \setminus X = \emptyset$ is open. □

Proposition 3.6.4. *The union of two closed sets is closed.*

PROOF: If C and D are closed then $X \setminus (C \cup D) = (X \setminus C) \cup (X \setminus D)$ is open. □

Proposition 3.6.5. *In any topological space, the intersection of a nonempty set of closed sets is closed.*

PROOF: Let $\mathcal{C}$ be a nonempty set of closed sets. Then $X \setminus \bigcap \mathcal{C} = \bigcup \{X \setminus C : C \in \mathcal{C}\}$ is open. □

Proposition 3.6.6. *Let X be a topological space and $U \subseteq X$. Then U is open if and only if $X \setminus U$ is closed.*

PROOF: Immediate from definitions.

Theorem 3.6.7. *Let X be a set and $\mathcal{C} \subseteq \mathcal{P}X$. Suppose:*

1. *$\emptyset, X \in \mathcal{C}$;*
2. *for all nonempty $\mathcal{A} \subseteq \mathcal{C}$, we have $\bigcap \mathcal{A} \in \mathcal{C}$;*
3. *for all $C, D \in \mathcal{C}$, we have $C \cup D \in \mathcal{C}$.*

Then there exists a unique topology on X under which $\mathcal{C}$ is the set of all closed sets, namely

$$\mathcal{T} = \{U \subseteq X : X \setminus U \in \mathcal{C}\}$$

PROOF:
$\langle 1 \rangle 1$. LET: $\mathcal{C}$ be a set satisfying 1–3
$\langle 1 \rangle 2$. LET: $\mathcal{T} = \{X \setminus C : C \in \mathcal{C}\}$
$\langle 1 \rangle 3$. $\mathcal{T}$ is a topology
  $\langle 2 \rangle 1$. $X \in \mathcal{T}$
    PROOF: $X \setminus X = \emptyset \in \mathcal{C}$ by condition 1.
  $\langle 2 \rangle 2$. For all $\mathcal{A} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{A} \in \mathcal{T}$.

⟨3⟩1. LET: $\mathcal{A} \subseteq \mathcal{T}$

⟨3⟩2. CASE: $\mathcal{A} = \emptyset$

PROOF: In this case, $X \setminus \bigcup \mathcal{A} = X \in \mathcal{C}$ by condition 1.

⟨3⟩3. CASE: $\mathcal{A}$ is nonempty

PROOF: In this case, we have $X \setminus \bigcup \mathcal{A} = \bigcap \{X \setminus U : U \in \mathcal{A}\} \in \mathcal{C}$ by condition 2.

⟨2⟩3. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$

PROOF: $X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V) \in \mathcal{C}$ by condition 3.

⟨1⟩4. $\mathcal{C}$ is the set of closed sets.

PROOF:

$$\begin{aligned} C \text{ is closed} &\Leftrightarrow X \setminus C \in \mathcal{T} \\ &\Leftrightarrow X \setminus (X \setminus C) \in \mathcal{C} \\ &\Leftrightarrow C \in \mathcal{C} \end{aligned}$$

⟨1⟩5. $\mathcal{T}$ is unique.

PROOF: By Proposition 3.6.6.

□

Definition 3.6.8 (Closed Covering). A *closed covering* of a topological space is a covering in which every member is a closed set.

3.7 Closed Refinements

Definition 3.7.1 (Closed Refinement). Let X be a space and $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}X$. Then $\mathcal{B}$ is an *closed refinement* of $\mathcal{A}$ iff $\mathcal{B}$ is a refinement of $\mathcal{A}$ and every member of $\mathcal{B}$ is closed.

3.8 Locally Finite Families

Definition 3.8.1 (Locally Finite). Let X be a topological space and $\{A_i\}_{i \in I}$ a family of subsets of X. Then $\{A_i\}_{i \in I}$ is *locally finite* iff, for all $x \in X$, there exists a neighbourhood N of x such that there are only finitely many $i \in I$ such that N intersects A_i.

Proposition 3.8.2. *If $\{A_i\}_{i \in I}$ is locally finite and $B_i \subseteq A_i$ for all i then $\{B_i\}_{i \in I}$ is locally finite.*

PROOF: Immediate from definitions. □

Proposition 3.8.3. *Every finite family of open sets is locally finite.*

PROOF: Trivial. □

3.9 Countably Locally Finite Sets

Definition 3.9.1 (Countably Locally Finite). Let X be a space. A subset of $\mathcal{P}X$ is *countably locally finite* iff it is the union of countably many locally finite sets.

3.10 Locally Discrete Sets

Definition 3.10.1 (Locally Discrete)**.** Let X be a topological space and $\{A_i\}_{i \in I}$ a family of subsets of X. Then $\{A_i\}_{i \in I}$ is *locally discrete* iff, for all $x \in X$, there exists a neighbourhood U of x such that there is at most one $i \in I$ such that U intersects A_i.

3.11 Countably Locally Discrete

Definition 3.11.1 (Countably Locally Discrete)**.** Let X be a topological space and $\mathcal{A} \subseteq \mathcal{P}X$. Then the set $\mathcal{A}$ is *countably locally discrete* iff it is the union of countably many locally discrete sets.

3.12 Closure of a Set

Definition 3.12.1 (Closure)**.** Let X be a topological space and $A \subseteq X$. The *closure* of A, $\mathrm{Cl}\, A$ or $\overline{A}$, is the intersection of all closed sets that include A.

Proof: This intersection always exists because X is a closed set that includes A. □

Proposition 3.12.2. *Let X be a topological space and $A \subseteq X$. Then $A \subseteq \overline{A}$.*

Proof: Immediate from definitions. □

Proposition 3.12.3. *Let X be a topological space and $A \subseteq X$. Then $\overline{A}$ is closed.*

Proof: This follows from Proposition 3.6.5. □

Proposition 3.12.4. *Let X be a topological space and $A, C \subseteq X$. If $A \subseteq C$ and C is closed then $\overline{A} \subseteq C$.*

Proof: Immediate from definitions. □

Proposition 3.12.5. *Let X be a topological space and $A, B \subseteq X$. If $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$.*

Proof:
$\langle 1 \rangle 1$. Assume: $A \subseteq B$
$\langle 1 \rangle 2$. $A \subseteq \overline{B}$
  Proof: Proposition 3.12.2.
$\langle 1 \rangle 3$. $\overline{A} \subseteq \overline{B}$
  Proof: Propositions 3.12.3, 3.12.4.
□

Proposition 3.12.6. *Let X be a set and $A \subseteq X$. Then A is closed if and only if $A = \overline{A}$.*

PROOF:
⟨1⟩1. If A is closed then $A = \overline{A}$
⟨2⟩1. ASSUME: A is closed
⟨2⟩2. $A \subseteq \overline{A}$
PROOF: By Proposition 3.12.2.
⟨2⟩3. $\overline{A} \subseteq A$
PROOF: By Proposition 3.12.4 since $A \subseteq A$.
⟨1⟩2. If $A = \overline{A}$ then A is closed.
PROOF: By Proposition 3.12.3.
□

Corollary 3.12.6.1.

$$\overline{\emptyset} = \emptyset$$

Theorem 3.12.7 (Kuratowski Closure Axioms)**.** *Let X be a set and $\overline{(\text{-})} : \mathcal{P}X \to \mathcal{P}X$ be a function such that:*

1. $\overline{\emptyset} = \emptyset$
2. *For all* $A \subseteq X$, $A \subseteq \overline{A}$
3. *For all* $A \subseteq X$, $\overline{A} = \overline{\overline{A}}$
4. *For all* $A, B \subseteq X$, $\overline{A \cup B} = \overline{A} \cup \overline{B}$

Then there exists a unique topology $\mathcal{T}$ on X such that $\overline{A}$ is the closure of A for all $A \in \mathcal{P}X$.

PROOF:
⟨1⟩1. For all $C, D \subseteq X$, if $C \subseteq D$ then $\overline{C} \subseteq \overline{D}$
⟨2⟩1. ASSUME: $C \subseteq D$
⟨2⟩2. $\overline{C} = \overline{D}$
PROOF:

$$\begin{aligned} \overline{D} &= \overline{C \cup D} && (⟨2⟩1) \\ &= \overline{C} \cup \overline{D} && (\text{axiom } 4) \end{aligned}$$

⟨1⟩2. LET: $\mathcal{T}$ be the topology in which a set C is closed iff $\overline{C} = C$.
⟨2⟩1. $\overline{\emptyset} = \emptyset$
PROOF: This is axiom 1.
⟨2⟩2. $\overline{X} = X$
PROOF: By axiom 2.
⟨2⟩3. For any set $\mathcal{A}$ of sets C such that $\overline{C} = C$, we have $\overline{\bigcap \mathcal{A}} = \bigcap \mathcal{A}$
⟨3⟩1. $\overline{\bigcap \mathcal{A}} \subseteq \bigcap \mathcal{A}$
⟨4⟩1. LET: $C \in \mathcal{A}$
⟨4⟩2. $\overline{\bigcap \mathcal{A}} \subseteq C$
PROOF:

$$\begin{aligned} \overline{\bigcap \mathcal{A}} &\subseteq \overline{C} && (⟨1⟩1) \\ &= C && (⟨4⟩1) \end{aligned}$$

$\langle 3\rangle 2$. Q.E.D.
$\langle 2\rangle 4$. If $\overline{C} = C$ and $\overline{D} = D$ then $\overline{C \cup D} = C \cup D$
PROOF: By axiom 4.
$\langle 2\rangle 5$. Q.E.D.
PROOF: By Theorem 3.6.7.
$\langle 1\rangle 3$. For all $A \subseteq X$, the closure of A in $\mathcal{T}$ is $\overline{A}$
$\langle 2\rangle 1$. $\overline{A}$ is closed
PROOF: From axiom 3.
$\langle 2\rangle 2$. If $A \subseteq C$ and C is closed then $\overline{A} \subseteq C$
PROOF:

$$\begin{aligned} C &= \overline{C} && (C \text{ is closed}) \\ &= \overline{A \cup C} && (A \subseteq C) \\ &= \overline{A} \cup \overline{C} && (\text{axiom 4}) \end{aligned}$$

□

Theorem 3.12.8. *Let A be a subset of the topological space X and $\mathcal{B}$ a basis for X. Then $x \in \overline{A}$ if and only if, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A.*

PROOF:
$\langle 1\rangle 1$. If $x \in \overline{A}$ then, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A.
PROOF: Immediate from Theorem 3.13.3.
$\langle 1\rangle 2$. If, for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A, then $x \in \overline{A}$.
$\langle 2\rangle 1$. ASSUME: for all $B \in \mathcal{B}$, if $x \in B$ then B intersects A.
$\langle 2\rangle 2$. LET: U be a neighbourhood of x
$\langle 2\rangle 3$. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$
PROOF: $\mathcal{B}$ is a basis.
$\langle 2\rangle 4$. B intersects A.
PROOF: By $\langle 2\rangle 1$.
$\langle 2\rangle 5$. U intersects A.
$\langle 2\rangle 6$. Q.E.D.
PROOF: By Theorem 3.13.3.

□

Lemma 3.12.9. *If $\{A_i\}_{i \in I}$ is locally finite then so is $\{\overline{A_i}\}_{i \in I}$.*

PROOF:
$\langle 1\rangle 1$. LET: $\{A_i\}_{i \in I}$ be a locally finite family of subsets of the space X.
$\langle 1\rangle 2$. LET: $x \in X$
$\langle 1\rangle 3$. PICK a neighbourhood U of x that intersects only $A_{i_1}, \ldots, A_{i_n}$.
$\langle 1\rangle 4$. U intersects only $\overline{A_{i_1}}, \ldots, \overline{A_{i_n}}$.

□

Lemma 3.12.10. *Let $\{A_i\}_{i \in I}$ be locally finite. Then $\overline{\bigcup_{i \in I} A_i} = \bigcup_{i \in I} \overline{A_i}$.*

PROOF:
$\langle 1\rangle 1$. LET: $x \in \overline{\bigcup_{i \in I} A_i}$
$\langle 1\rangle 2$. PICK a neighbourhood U of x that intersects only $A_{i_1}, \ldots, A_{i_n}$.

$\langle 1\rangle 3.$ $x \in \overline{A_{i_1}} \cup \cdots \cup \overline{A_{i_n}}$

PROOF: If not, then $U - \overline{A_{i_1}} - \cdots - \overline{A_{i_n}}$ would be a neighbourhood of x that does not intersect $\bigcup_{i \in I} A_i$.

□

Definition 3.12.11 (Precise Refinement)**.** Let X be a topological space and $\{U_\alpha\}_{\alpha \in J}$ be a family of subsets of X. Then a *precise refinement* of $\{U_\alpha\}_{\alpha \in J}$ is a family $\{V_\alpha\}_{\alpha \in J}$ such that, for all $\alpha \in J$, we have $\overline{V_\alpha} \subseteq U_\alpha$.

Definition 3.12.12 (Support)**.** Let X be a topological space and $\phi : X \to \mathbb{R}$ be a function. Then the *support* of ϕ is the closure of $\phi^{-1}(\mathbb{R} \setminus \{0\})$.

Lemma 3.12.13. *Let X be a topological space and $\{f_\alpha : X \to \mathbb{R}\}_{\alpha \in J}$ be a family of continuous functions. If $\{\operatorname{supp} f_\alpha\}_{\alpha \in J}$ is locally finite then, for all $x \in X$, we have $f_\alpha(x) = 0$ for all but finitely many $\alpha \in J$.*

PROOF:

$\langle 1\rangle 1.$ ASSUME: $\{\operatorname{supp} f_\alpha\}_{\alpha \in J}$ is locally finite.

$\langle 1\rangle 2.$ LET: $x \in X$

$\langle 1\rangle 3.$ PICK an open neighbourhood U of x that intersects only $\operatorname{supp} f_\alpha$ for only finitely many α, say $\alpha_1, \ldots, \alpha_n$

PROOF: $\langle 1\rangle 1$, $\langle 1\rangle 2$

$\langle 1\rangle 4.$ For all $\alpha \in J$, if $f_\alpha(x) = 0$ then α is one of $\alpha_1, \ldots, \alpha_n$.

PROOF: $\langle 1\rangle 3$, Proposition 3.12.2.

□

Definition 3.12.14 (Partition of Unity)**.** Let X be a topological space. Let $\{U_\alpha\}_{\alpha \in J}$ be an open covering of X. A *partition of unity dominated by* $\{U_\alpha\}_{\alpha \in J}$ is a family of continuous functions $\{\phi_\alpha : X \to [0, 1]\}_{\alpha \in J}$ such that:

1. for all $\alpha \in J$, $\operatorname{supp} \phi_\alpha \subseteq U_\alpha$;
2. the family $\{\operatorname{supp} \phi_\alpha\}_{\alpha \in J}$ is locally finite;
3. $\sum_{\alpha \in J} \phi_\alpha(x) = 1$

3.13 Interior of a Set

Definition 3.13.1 (Interior)**.** Let X be a topological space and $A \subseteq X$. The *interior* of A, $\operatorname{Int} A$, is the union of all open sets included in A.

Lemma 3.13.2. *If $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$.*

PROOF: $\overline{B}$ is a closed set that includes B, hence includes A. □

Theorem 3.13.3. *Let A be a subset of the topological space X and $x \in X$. Then $x \in \overline{A}$ if and only if every neighbourhood of x intersects A.*

PROOF:

$$\begin{aligned} x \notin \overline{A} &\Leftrightarrow \exists C \text{ closed } (A \subseteq C \wedge x \notin C) \\ &\Leftrightarrow \exists U \text{ open } (A \subseteq X \setminus U \wedge x \in U) \\ &\Leftrightarrow \exists U \text{ open } (A \text{ intersects } U \wedge x \in U) \end{aligned}$$

□

Lemma 3.13.4.

$$X \setminus \operatorname{Int} A = \overline{X \setminus A}$$

PROOF:
⟨1⟩1. $X \setminus \operatorname{Int} A \subseteq \overline{X \setminus A}$
  ⟨2⟩1. $X \setminus A \subseteq \overline{X \setminus A}$
  ⟨2⟩2. $X \setminus \overline{X \setminus A} \subseteq A$
  ⟨2⟩3. $X \setminus \overline{X \setminus A} \subseteq \operatorname{Int} A$
⟨1⟩2. $\overline{X \setminus A} \subseteq X \setminus \operatorname{Int} A$
  ⟨2⟩1. $\operatorname{Int} A \subseteq A$
  ⟨2⟩2. $X \setminus A \subseteq X \setminus \operatorname{Int} A$
  ⟨2⟩3. $\overline{X \setminus A} \subseteq X \setminus \operatorname{Int} A$
□

3.14 Boundary

Definition 3.14.1 (Boundary)**.** Let X be a topological space and $A \subseteq X$. The *boundary* of A, $\operatorname{Bd} A$, is $\overline{A} \cap \overline{X \setminus A}$.

Lemma 3.14.2.

$$\operatorname{Bd} A = \overline{A} \setminus \operatorname{Int} A$$

PROOF: From Lemma 3.13.4. □

Lemma 3.14.3. $\overline{A} = \operatorname{Int} A \cup \operatorname{Bd} A$

PROOF:

$$\begin{aligned} \operatorname{Int} A \cup \operatorname{Bd} A &= \operatorname{Int} A \cup (\overline{A} \cap (X \setminus \operatorname{Int} A)) \\ &= \operatorname{Int} A \cup \overline{A} \\ &= \overline{A} \end{aligned}$$

□

Corollary 3.14.3.1. $\operatorname{Bd} A = \emptyset$ *iff A is open and closed.*

Lemma 3.14.4. *For any set U, the following are equivalent:*

1. *U is open.*
2. $\operatorname{Bd} U \cap U = \emptyset$
3. $\operatorname{Bd} U = \overline{U} \setminus U$

PROOF:
⟨1⟩1. $1 \Rightarrow 3$

Proof: From Lemma 3.14.2.

⟨1⟩2. $3 \Rightarrow 2$

Proof: Set theory.

⟨1⟩3. $2 \Rightarrow 1$

Proof:

$$\begin{aligned} U &\subseteq \overline{U} \\ &= \operatorname{Int} U \cup \operatorname{Bd} U && \text{(Lemma 3.14.3)} \\ \therefore U &\subseteq \operatorname{Int} U \end{aligned}$$

□

3.15 Limit Points

Definition 3.15.1 (Limit Point)**.** Let X be a topological space, $A \subseteq X$, and $x \in X$. Then x is a *limit point*, *cluster point* or *point of accumulation* of A iff every neighbourhood of x intersects A in a point other than x.

Lemma 3.15.2. *If $A \subseteq B$ then every limit point of A is a limit point of B.*

Proof: Immediate from the definition. □

Theorem 3.15.3. *Let A be a subset of the topological space X. Let A' be the set of all limit points of A. Then $\overline{A} = A \cup A'$.*

Proof:

⟨1⟩1. If $x \in \overline{A}$ and $x \notin A$ then $x \in A'$

Proof: in this case, every neighbourhood of x intersects A in a point other than x.

⟨1⟩2. $A \subseteq \overline{A}$

Proof: From the definition of $\overline{A}$.

⟨1⟩3. $A' \subseteq \overline{A}$

Proof: By Theorem 3.13.3.

□

Corollary 3.15.3.1. *A set is closed if and only if it contains all its limit points.*

3.16 Subbases

Definition 3.16.1 (Subbasis)**.** Let X be a topological space. A *subbasis* for the topology on X is a set $\mathcal{S} \subseteq \mathcal{P}X$ such that, for every open set U and $x \in U$, there exist $S_1, \ldots, S_n \in \mathcal{S}$ such that $x \in S_1 \cap \cdots \cap S_n \subseteq U$. We say the topology is *generated* by $\mathcal{S}$.

Lemma 3.16.2. *Let $\mathcal{T}$ be a topology on X and $\mathcal{S} \subseteq \mathcal{P}X$. Then the following are equivalent:*

1. *$\mathcal{S}$ is a subbasis for $\mathcal{T}$.*

2. *The set of all finite intersections of members of $\mathcal{S}$ is a basis for $\mathcal{T}$*

3. *$\mathcal{T}$ is the set of all unions of finite intersections of members of $\mathcal{S}$.*

PROOF: $1 \Leftrightarrow 2$ holds immediately from the definitions. $2 \Leftrightarrow 3$ holds by Proposition 3.5.2. □

Corollary 3.16.2.1. *If $\mathcal{S}$ is a subbasis for the topology $\mathcal{T}$, then $\mathcal{T}$ is the coarsest topology in which every element of $\mathcal{S}$ is open.*

Lemma 3.16.3. *Let X be a set and $\mathcal{S} \subseteq \mathcal{P}X$. Then $\mathcal{S}$ is a subbasis for a topology on X if and only if $\bigcup \mathcal{S} = X$.*

PROOF:
⟨1⟩1. If $\mathcal{S}$ is a subbasis for a topology on X then $\bigcup \mathcal{S} = X$
  ⟨2⟩1. ASSUME: $\mathcal{S}$ is a subbasis for a topology $\mathcal{T}$ on X.
  ⟨2⟩2. LET: $x \in X$
  ⟨2⟩3. PICK $S_1, \ldots, S_n \in \mathcal{S}$ such that $x \in S_1 \cap \cdots \cap S_n \subseteq X$
    PROOF: From the definition of subbasis (⟨2⟩1, ⟨2⟩2).
  ⟨2⟩4. $x \in \bigcup \mathcal{S}$
    PROOF: Immediate from ⟨2⟩3.
⟨1⟩2. If $\bigcup \mathcal{S} = X$ then $\mathcal{S}$ is a subbasis for a topology on X
  ⟨2⟩1. ASSUME: $\bigcup \mathcal{S} = X$
    PROVE: The set of all finite intersections of elements of $\mathcal{S}$ is a basis for a topology on X.
  ⟨2⟩2. LET: $\mathcal{B}$ be the set of all finite intersections of elements of $\mathcal{S}$.
  ⟨2⟩3. $\bigcup \mathcal{B} = X$
    PROOF: From ⟨2⟩1 and ⟨2⟩2.
  ⟨2⟩4. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$
    PROOF: Take $B_3 = B_1 \cap B_2$ (⟨2⟩2).
  ⟨2⟩5. $\mathcal{B}$ is a basis for a topology on X.
    PROOF: By Lemma 3.5.3.
  ⟨2⟩6. Q.E.D.
    PROOF: By Lemma 3.16.2.

□

3.17 Convergence

Definition 3.17.1 (Net)**.** Let X be a topological space. A *net* $(x_\alpha)_{\alpha \in J}$ in X consists of a directed set J and a function $x : J \to X$.

Definition 3.17.2 (Convergence)**.** Let $(x_\alpha)_{\alpha \in J}$ be a net in the topological space X, and $l \in X$. Then the net *converges* to l, $x_\alpha \to l$, if and only if, for every neighbourhood U of l, there exists $\alpha \in J$ such that, for all $\beta \geq \alpha$, we have $x_\beta \in U$.

Theorem 3.17.3 (AC)**.** *Let X be a topological space and $A \subseteq X$. Then $x \in \overline{A}$ if and only if there exists a net of points of A converging to x.*

PROOF:
$\langle 1\rangle 1$. If $x \in \overline{A}$ then there exists a net of points of A converging to x.
  $\langle 2\rangle 1$. LET: $x \in \overline{A}$
  $\langle 2\rangle 2$. LET: J be the poset of neighbourhoods of x under $\supseteq$.
  $\langle 2\rangle 3$. For $U \in J$ PICK a point $x_U \in U \cap A$
    PROOF: By Theorem 3.13.3
  $\langle 2\rangle 4$. $(x_U)_{U \in J}$ is a net
    PROOF: Given $U, V \in J$ we have $U \cap V \in J$ and $U \supseteq U \cup V$, $V \supseteq U \cup V$.
  $\langle 2\rangle 5$. $x_U \to x$
    PROOF: For any neighbourhood U of x we have $U \in J$ and if $U \supseteq V$ then $x_V \in U$.
$\langle 1\rangle 2$. If there exists a net of points of A converging to x then $x \in \overline{A}$.
  $\langle 2\rangle 1$. LET: $(x_\alpha)_{\alpha \in J}$ be a net of points in A that converges to x.
  $\langle 2\rangle 2$. LET: U be a neighbourhood of x
  $\langle 2\rangle 3$. PICK $\alpha \in J$ such that, for all $\beta \geq \alpha$, we have $x_\beta \in U$
  $\langle 2\rangle 4$. $x_\alpha \in U \cap A$
  $\langle 2\rangle 5$. Q.E.D.
    PROOF: By Theorem 3.13.3

□

Theorem 3.17.4. *Let X and Y be topological spaces and $f : X \to Y$. Then f is continuous if and only if, for every net $(x_\alpha)_{\alpha \in J}$ in X, if $x_\alpha \to x$ then $f(x_\alpha) \to f(x)$.*

PROOF:
$\langle 1\rangle 1$. If f is continuous and $x_\alpha \to x$ then $f(x_\alpha) \to f(x)$
  $\langle 2\rangle 1$. ASSUME: f is continuous.
  $\langle 2\rangle 2$. ASSUME: $x_\alpha \to x$
  $\langle 2\rangle 3$. LET: V be a neighbourhood of $f(x)$
  $\langle 2\rangle 4$. $f^{-1}(V)$ is a neighbourhood of x
  $\langle 2\rangle 5$. PICK α such that, for all $\beta \geq \alpha$, we have $x_\beta \in f^{-1}(V)$
  $\langle 2\rangle 6$. For all $\beta \geq \alpha$ we have $f(x_\beta) \in V$
$\langle 1\rangle 2$. If, for every net (x_α) in X, if $x_\alpha \to x$ then $f(x_\alpha) \to f(x)$, then f is continous.
  $\langle 2\rangle 1$. ASSUME: for every net (x_α) in X, if $x_\alpha \to x$ then $f(x_\alpha) \to f(x)$
  $\langle 2\rangle 2$. LET: $A \subseteq X$
    PROVE: $f(\overline{A}) \subseteq \overline{f(A)}$
  $\langle 2\rangle 3$. LET: $x \in \overline{A}$
  $\langle 2\rangle 4$. PICK a net (x_α) in A such that $x_\alpha \to x$
    PROOF: Theorem 3.17.3
  $\langle 2\rangle 5$. $f(x_\alpha) \to f(x)$
    PROOF: By $\langle 2\rangle 1$
  $\langle 2\rangle 6$. $f(x) \in \overline{f(A)}$
    PROOF: Theorem 3.17.3
  $\langle 2\rangle 7$. Q.E.D.
    PROOF: By Theorem 5.2.2.

□

Definition 3.17.5 (Subnet)**.** Let $(x_\alpha)_{\alpha\in J}$ be a net in X. Let K be a directed set and $g : K \to J$ be a monotone function such that $g(K)$ is cofinal in J. Then the net $(x_{g(\beta)})_{\beta\in K}$ is called a *subnet* of (x_α).

3.18 Accumulation Points

Definition 3.18.1 (Accumulation Point)**.** Let X be a topological space, and $(x_\alpha)_{\alpha\in J}$ a net in X, and $a \in X$. Then a is an *accumulation point* of (x_α) iff, for every neighbourhood U of x, the set $\{\alpha \in J : x_\alpha \in U\}$ is cofinal in J.

Lemma 3.18.2. *Let X be a topological space, $(x_\alpha)_{\alpha\in J}$ be a nonempty net in X and $a \in X$. Then a is an accumulation point of (x_α) if and only if there exists a subnet of (x_α) that converges to a.*

PROOF:
$\langle 1\rangle 1$. If a is an accumulation point of (x_α) then there exists a subnet of (x_α) that converges to a.
  $\langle 2\rangle 1$. ASSUME: a is an accumulation point of (x_α).
  $\langle 2\rangle 2$. LET: K be the poset $\{(\alpha, U) : \alpha \in J, U$ is a neighbourhood of $a, x_\alpha \in U\}$ under: $(\alpha, U) \leq (\beta, V)$ iff $\alpha \leq \beta$ and $U \subseteq V$.
  $\langle 2\rangle 3$. $(x_\alpha)_{(\alpha,U)\in K}$ is a subnet of $(x_\alpha)_{\alpha\in J}$
    $\langle 3\rangle 1$. K is directed.
      $\langle 4\rangle 1$. LET: $(\alpha, U), (\beta, V) \in K$
      $\langle 4\rangle 2$. PICK $\gamma \in J$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$.
      $\langle 4\rangle 3$. PICK $\delta \in J$ such that $\gamma \leq \delta$ and $x_\delta \in U \cap V$
        PROOF: By $\langle 2\rangle 1$.
      $\langle 4\rangle 4$. $(\delta, U \cap V) \in K$ and $(\alpha, U) \leq (\delta, U \cap V)$, $(\beta, V) \leq (\delta, U \cap V)$
    $\langle 3\rangle 2$. If $(\alpha, U) \leq (\beta, V)$ then $\alpha \leq \beta$
      PROOF: From $\langle 2\rangle 2$.
    $\langle 3\rangle 3$. $\{\alpha : \exists U.(\alpha, U) \in K\}$ is cofinal in J
      PROOF: For $\alpha \in J$ we have $(\alpha, X) \in K$, so in fact $\{\alpha : \exists U.(\alpha, U) \in K\} = J$.
  $\langle 2\rangle 4$. The subnet converges to a.
    $\langle 3\rangle 1$. LET: U be a neighbourhood of a.
    $\langle 3\rangle 2$. PICK $\alpha \in J$
    $\langle 3\rangle 3$. PICK $\beta \in J$ such that $\alpha \leq \beta$ and $x_\beta \in U$
      PROOF: By $\langle 2\rangle 1$.
    $\langle 3\rangle 4$. For all $(\gamma, V) \geq (\beta, U)$ we have $x_\gamma \in U$
      PROOF: $x_\gamma \in V \subseteq U$.
$\langle 1\rangle 2$. If there exists a subnet of (x_α) that converges to a then a is an accumulation point of (x_α).
  $\langle 2\rangle 1$. ASSUME: $(x_{g(\beta)})_{\beta\in K}$ converges to a
  $\langle 2\rangle 2$. LET: U be a neighbourhoof of a
  $\langle 2\rangle 3$. LET: $\alpha \in J$
    PROVE: There exists $\gamma \geq \alpha$ such that $x_\gamma \in U$
  $\langle 2\rangle 4$. PICK $\beta \in K$ such that, for all $\beta' \geq \beta$, we have $x_{g(\beta')} \in U$

PROOF: By ⟨2⟩1.

⟨2⟩5. PICK $\beta' \in K$ such that $g(\beta') \geq \alpha$

PROOF: Since $g(K)$ is cofinal in J.

⟨2⟩6. PICK $\beta'' \in K$ such that $\beta \leq \beta''$ and $\beta' \leq \beta''$

PROOF: K is directed.

⟨2⟩7. $g(\beta'') \geq \alpha$ and $x_{g(\beta'')} \in U$

$\square$

3.19 Dense Sets

Definition 3.19.1 (Dense)**.** Let X be a topological space and $A \subseteq X$. Then A is *dense* in X iff $\overline{A} = X$.

3.20 G_δ Sets

Definition 3.20.1 (G_δ Set)**.** A G_δ *set* is the intersection of a countable set of open sets.

Definition 3.20.2 (F_σ Set)**.** Let X be a topological space and $A \subseteq X$. Then A is an F_σ*-set* iff it is a countable union of closed sets.

3.21 Separated Sets

Definition 3.21.1 (Separated Sets)**.** Let X be a topological space and $A, B \subseteq X$. Then A and B are *separated* iff $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$.

3.22 Coherent Topology

Definition 3.22.1 (Coherent Topology)**.** Let $X_1 \subseteq X_2 \subseteq \cdots$ be a sequence of topological spaces such that each X_n is a closed subspace of X_{n+1}. Let $X = \bigcup_{n=1}^{\infty} X_n$. Then the topology on X *coherent* with the subspaces X_n is the topology defined by: $U \subseteq X$ is open iff $U \cap X_n$ is open in X_n for all n.

Chapter 4

Constructions of Topological Spaces

4.1 The Order Topology

Definition 4.1.1 (Order Topology)**.** Let X be a linearly ordered set with more than one element. The *order topology* on X is the topology generated by the basis consisting of:

- all open intervals (a, b)
- all half-open intervals $(a, \top]$ where $\top$ is the greatest element of X, if there is one;
- all half-open intervals $[\bot, a)$ where $\bot$ is the least element of X, if there is one.

We prove this is a basis for a topology.

PROOF:
$\langle 1 \rangle$1. LET: $\mathcal{B}$ be the set of all sets of these three forms.
$\langle 1 \rangle$2. $\bigcup \mathcal{B} = X$
  $\langle 2 \rangle$1. LET: $x \in X$
    PROVE: There exists $B \in \mathcal{B}$ such that $x \in B$
  $\langle 2 \rangle$2. CASE: x is least in X
    $\langle 3 \rangle$1. PICK $a \in X$ such that $a > x$
      PROOF: X has more than one element.
    $\langle 3 \rangle$2. $x \in [x, a) \in \mathcal{B}$
  $\langle 2 \rangle$3. CASE: x is greatest in X
    $\langle 3 \rangle$1. PICK $a \in X$ such that $a < x$
      PROOF: X has more than one element.
    $\langle 3 \rangle$2. $x \in (a, x] \in \mathcal{B}$
  $\langle 2 \rangle$4. CASE: x is neither least nor greatest in X

⟨3⟩1. Pick $a, b \in X$ such that $a < x < b$
⟨3⟩2. $x \in (a, b) \in \mathcal{B}$
⟨1⟩3. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$
⟨2⟩1. Let: $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$
⟨2⟩2. Case: $B_1 = (a, b), B_2 = (c, d)$
Proof: Take $B_3 = (\max(a, c), \min(b, d))$.
⟨2⟩3. Case: $B_1 = (a, b), B_2 = (c, \top]$
Proof: Take $B_3 = (\max(a, c), b)$.
⟨2⟩4. Case: $B_1 = (a, b), B_2 = [\bot, d)$
Proof: Take $B_3 = (a, \min(b, d))$.
⟨2⟩5. Case: $B_1 = (a, \top], B_2 = (c, d)$
Proof: Similar to ⟨2⟩3.
⟨2⟩6. Case: $B_1 = (a, \top], B_2 = (c, \top]$
Proof: Take $B_3 = (\max(a, c), \top]$.
⟨2⟩7. Case: $B_1 = (a, \top], B_2 = [\bot, d)$
Proof: Take $B_3 = (a, d)$.
⟨2⟩8. Case: $B_1 = [\bot, b), B_2 = (c, d)$
Proof: Similar to ⟨2⟩4.
⟨2⟩9. Case: $B_1 = [\bot, b), B_2 = (c, \top]$
Proof: Simlar to ⟨2⟩7.
⟨2⟩10. Case: $B_1 = [\bot, b), B_2 = [\bot, d)$
Proof: Take $B_3 = [\bot, \min(b, d))$.
⟨1⟩4. Q.E.D.
Proof: By Lemma 3.5.3.
□

Lemma 4.1.2. *Let X be a linearly ordered set, $U \subseteq X$ be open, and $a \in U$. Then:*

1. *Either a is greatest in X, or there exists $a' > a$ such that $[a, a') \subseteq U$*
2. *Either a is least in X, or there exists $a' < a$ such that $(a', a] \subseteq U$.*

Proof:
⟨1⟩1. Either a is greatest in X, or there exists $a' > a$ such that $[a, a') \subseteq U$
⟨2⟩1. Assume: a is not greatest in X
⟨2⟩2. Pick a basic open set B such that $a \in B \subseteq U$
⟨2⟩3. Case: $B = (a'', a')$
Proof: $a < a'$ and $[a, a') \subseteq B \subseteq U$
⟨2⟩4. Case: $B = [\bot, a')$
Proof: $a < a'$ and $[a, a') \subseteq B \subseteq U$
⟨2⟩5. Case: $B = (a'', \top]$
Proof: Pick any $a' > a$ (one exists by ⟨2⟩1). Then $[a, a') \subseteq B \subseteq U$.S
⟨1⟩2. Either a is least in X, or there exists $a' < a$ such that $(a', a] \subseteq U$.
Proof: Similar.
□

Lemma 4.1.3. *The open rays form a subbasis for the order topology.*

⟨1⟩1. LET: X be a linearly ordered set with more than one element.
⟨1⟩2. The open rays form a subbasis for a topology.
⟨2⟩1. LET: $x \in X$
PROVE: x is an element of an open ray.
⟨2⟩2. CASE: x is greatest in X
⟨3⟩1. PICK $a \in X$ such that $a < x$
PROOF: X has more than one element (⟨1⟩1).
⟨3⟩2. $x \in (a, +\infty)$
⟨2⟩3. CASE: x is not greatest in X
⟨3⟩1. PICK $a \in X$ such that $x < a$
⟨3⟩2. $x \in (-\infty, a)$
⟨2⟩4. Q.E.D.
PROOF: By Lemma 3.16.2.
⟨1⟩3. LET: $\mathcal{T}_o$ be the order topology and $\mathcal{T}_S$ be the topology generated by the open rays.
⟨1⟩4. $\mathcal{T}_o \subseteq \mathcal{T}_S$
⟨2⟩1. Every open interval (a, b) is open in $\mathcal{T}_S$
PROOF: $(a, b) = (a, +\infty) \cap (-\infty, b)$.
⟨2⟩2. If $\top$ is greatest then $(a, \top]$ is open in $\mathcal{T}_S$
PROOF: $(a, \top] = (a, +\infty)$.
⟨2⟩3. If $\perp$ is least then $[\perp, b)$ is open in $\mathcal{T}_S$
PROOF: $[\perp, b) = [\perp, +\infty)$.
⟨2⟩4. Q.E.D.
PROOF: By Corollary 3.5.2.1.
⟨1⟩5. $\mathcal{T}_S \subseteq \mathcal{T}_o$
⟨2⟩1. For all $a \in X$, we have $(a, +\infty)$ is open in $\mathcal{T}_o$
⟨3⟩1. LET: $x \in (a, +\infty)$
PROVE: There exists a basis element B such that $x \in B \subseteq (a, +\infty)$
⟨3⟩2. CASE: x is greatest
PROOF: Take $B = (a, x]$
⟨3⟩3. CASE: x is not greatest
⟨4⟩1. PICK $b > x$
⟨4⟩2. $x \in (a, b) \subseteq (a, +\infty)$
⟨2⟩2. For all $a \in X$, we have $(-\infty, a)$ is open in $\mathcal{T}_o$
PROOF: Similar.
⟨2⟩3. Q.E.D.
PROOF: By Corollary 3.16.2.1.

□

Lemma 4.1.4. *In a linearly ordered set X under the order topology, the closed intervals and closed rays are closed.*

PROOF:

$$\begin{aligned} X \setminus [a,b] &= (-\infty, a) \cup (b, +\infty) \\ X \setminus (-\infty, a] &= (a, +\infty) \\ X \setminus [a, +\infty) &= (-\infty, a) \end{aligned}$$

∎

Definition 4.1.5 (Standard Topology on $\mathbb{R}$)**.** The *standard topology* on $\mathbb{R}$ is the order topology.

Lemma 4.1.6. *The standard topology is strictly coarser than the lower limit topology.*

PROOF:
⟨1⟩1. The standard topology is coarser than the lower limit topology.
⟨2⟩1. For every open interval (a,b) and $x \in (a,b)$, there exists a half-open interval $[c,d)$ such that $x \in [c,d) \subseteq (a,b)$
PROOF: Take $[c,d) = [x,b)$.
⟨2⟩2. Q.E.D.
PROOF: By Lemma 3.5.4.
⟨1⟩2. There exists a set U open in the lower limit topology that is not open in the standard topology.
PROOF: Take $U = [0,1)$.
∎

Lemma 4.1.7. *The standard topology is strictly coarser than the K-topology.*

PROOF:
⟨1⟩1. The standard topology is coarser than the K-topology.
PROOF: Every open interval is open in the K-topology.
⟨1⟩2. There exists a set U open in the K-topology that is not open in the standard topology.
PROOF: Take $U = (-1,1) \setminus K$. Then $0 \in U$ but there is no open interval (a,b) such that $0 \in (a,b) \subseteq U$.
∎

Definition 4.1.8 (Ordered Square)**.** The *ordered square* I_o^2 is the topological space $[0,1]^2$ under the order topology induced by the lexicographic order.

Lemma 4.1.9. *Let L be a linear continuum with a greatest element. Then every non-empty closed set in L has a greatest element.*

PROOF:
⟨1⟩1. LET: C be a non-empty closed set in L
⟨1⟩2. LET: u be the supremum of C
⟨1⟩3. $u \in C$
⟨2⟩1. ASSUME: w.l.o.g u is not least in L
PROOF: If u is least then $C = \{u\}$.
⟨2⟩2. LET: U be any open neighbourhood of u
⟨2⟩3. PICK $v < u$ such that $(v,u] \subseteq U$

PROOF: By Lemma 4.1.2.

⟨2⟩4. PICK $x \in C$ such that $v < x$

PROOF: v is not an upper bound for C (⟨1⟩2).

⟨2⟩5. U intersects C in v

⟨2⟩6. Q.E.D.

PROOF: By Theorem 3.13.3.

□

Definition 4.1.10 (Long Line)**.** The *long line* is $(S_\Omega \times [0, 1)) \setminus \{(0, 0)\}$ under the dictionary order, where S_Ω is the first uncountable ordinal under the order topology.

4.2 The Product Topology

Definition 4.2.1 (Product Topology)**.** Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces. The *product topology* on $\prod_{\alpha \in J} X_\alpha$ is the topology generated by the subbasis consisting of all sets of the form $\pi_\alpha^{-1}(U)$ where $\alpha \in J$ and U is open in X_α. The *product space* of $\{X_\alpha\}_{\alpha \in J}$ is $\prod_{\alpha \in J} X_\alpha$ under the product topology.

Lemma 4.2.2. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces and A_α be closed in X_α for all α. Then $\prod_{\alpha \in J} A_\alpha$ is closed in $\prod_{\alpha \in J} X_\alpha$.*

PROOF: This holds because $\prod_{\alpha \in J} X_\alpha \setminus \prod_{\alpha \in J} A_\alpha = \bigcup_{\alpha \in J} \pi_\alpha^{-1}(X_\alpha \setminus A_\alpha)$. □

Theorem 4.2.3. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces. The set of all sets of the form $\prod_{\alpha \in J} U_\alpha$ where each U_α is open in X_α, and $U_\alpha = X_\alpha$ for all but finitely many α, is a basis for the product topology on $\prod_{\alpha \in J} X_\alpha$.*

PROOF: By Lemma 3.16.2. □

Theorem 4.2.4. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces, and let $\mathcal{B}_\alpha$ be a basis for the topology on X_α for each α. Then*

$$\mathcal{B} = \{\prod_{\alpha \in J} U_\alpha : \textit{for finitely many } \alpha \in J, U_\alpha \in \mathcal{B}_\alpha, \\ \textit{and } U_\alpha = X_\alpha \textit{ for all other values of } \alpha\}$$

is a basis for the product topology on $\prod_{\alpha \in J} X_\alpha$.

PROOF:

⟨1⟩1. Every member of $\mathcal{B}$ is open in the product topology.

PROOF: Immediate from definitions.

⟨1⟩2. For every open set U and $\{x_\alpha\}_{\alpha \in J} \in U$, there exists $B \in \mathcal{B}$ such that $\{x_\alpha\}_{\alpha \in J} \in B \subseteq U$.

⟨2⟩1. LET: U be open and $\{x_\alpha\}_{\alpha \in J} \in U$

⟨2⟩2. PICK U_α open in X_α for each α such that $\{x_\alpha\}_{\alpha \in J} \in \prod_{\alpha \in J} U_\alpha \subseteq U$ and $U_\alpha = X_\alpha$ for all α except $\alpha_1, \ldots, \alpha_n$.

PROOF: By Theorem 4.2.3.

⟨2⟩3. PICK $B_{\alpha_i} \in \mathcal{B}_{\alpha_i}$ such that $x_\alpha \in B_{\alpha_i} \subseteq U_{\alpha_i}$ for $i = 1, \ldots, n$

⟨2⟩4. $\{x_\alpha\}_{\alpha \in J} \in \prod_{\alpha \in J} V_\alpha \subseteq U$ where $V_{\alpha_i} = B_{\alpha_i}$ for $i = 1, \ldots, n$, and $V_\alpha = X_\alpha$ for all other α.

□

Theorem 4.2.5 (AC)**.** *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces and $A_\alpha \subseteq X_\alpha$ for all α. If $\prod_{\alpha \in J} X_\alpha$ is given the product topology, then*

$$\prod_{\alpha \in J} \overline{A_\alpha} = \overline{\prod_{\alpha \in J} A_\alpha} \ .$$

PROOF:

⟨1⟩1. $\prod_{\alpha \in J} \overline{A_\alpha} \subseteq \overline{\prod_{\alpha \in J} A_\alpha}$

⟨2⟩1. LET: $\{x_\alpha\}_{\alpha \in J} \in \prod_{\alpha \in J} \overline{A_\alpha}$

⟨2⟩2. LET: $\prod_{\alpha \in J} U_\alpha$ be a basic neighbourhood of $\{x_\alpha\}_{\alpha \in J}$, where each U_α is open in X_α, and $U_\alpha = X_\alpha$ except for $\alpha = \alpha_1, \ldots, \alpha_n$.

⟨2⟩3. For $\alpha \in J$, PICK $a_\alpha \in A_\alpha \cap U_\alpha$.

PROOF: By Theorem 3.13.3, using the Axiom of Choice.

⟨2⟩4. $\{a_\alpha\}_{\alpha \in J} \in \prod_{\alpha \in J} A_\alpha \cap \prod_{\alpha \in J} U_\alpha$

⟨2⟩5. Q.E.D.

PROOF: By Theorem 3.13.3.

⟨1⟩2. $\overline{\prod_{\alpha \in J} A_\alpha} \subseteq \prod_{\alpha \in J} \overline{A_\alpha}$

⟨2⟩1. LET: $\{x_\alpha\}_{\alpha \in J} \in \overline{\prod_{\alpha \in J} A_\alpha}$

⟨2⟩2. LET: $\alpha \in J$

PROVE: $x_\alpha \in \overline{A_\alpha}$

⟨2⟩3. LET: U be a neighbourhood of x_α in X_α

⟨2⟩4. $\pi_\alpha^{-1}(U)$ is a neighbourhood of $\{x_\alpha\}_{\alpha \in J}$

⟨2⟩5. PICK $\{a_\alpha\}_{\alpha \in J} \in \pi_\alpha^{-1}(U) \cap \prod_{\alpha \in J} A_\alpha$

PROOF: By Theorem 3.13.3.

⟨2⟩6. $a_\alpha \in U \cap A_\alpha$

⟨2⟩7. Q.E.D.

PROOF: By Theorem 3.13.3.

□

Definition 4.2.6 (Standard Topology on $\mathbb{R}^J$)**.** For J a set, the *standard topology* on $\mathbb{R}^J$ is the product topology where $\mathbb{R}$ is given the standard topology.

Definition 4.2.7 (Closed Unit Ball)**.** The *closed unit ball* B^2 is $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ as a subset of $\mathbb{R}^2$.

Definition 4.2.8 (Sorgenfrey Plane)**.** The *Sorgenfrey plane* is $\mathbb{R}_l^2$.

4.3 The Subspace Topology

Definition 4.3.1 (Subspace Topology)**.** Let X be a topological space and $Y \subseteq X$. The *subspace topology* on Y is $\{Y \cap U : U \text{ open in } X\}$. With this topology, Y is a *subspace* of X.

We prove this is a topology.

PROOF:
$\langle 1\rangle 1$. LET: $\mathcal{T} = \{Y \cap U : U \text{ open in } X\}$
$\langle 1\rangle 2$. $Y \in \mathcal{T}$
PROOF: $Y = Y \cap X$
$\langle 1\rangle 3$. $\mathcal{T}$ is closed under union.
$\langle 2\rangle 1$. LET: $\mathcal{A} \subseteq \mathcal{T}$
PROVE: $\bigcup \mathcal{A} = Y \cap \bigcup\{U \text{ open in } X : Y \cap U \in \mathcal{A}\}$
$\langle 2\rangle 2$. $\bigcup \mathcal{A} \subseteq Y \cap \bigcup\{U \text{ open in } X : Y \cap U \in \mathcal{A}\}$
$\langle 3\rangle 1$. LET: $x \in \bigcup \mathcal{A}$
$\langle 3\rangle 2$. PICK $V \in \mathcal{A}$ such that $x \in V$
$\langle 3\rangle 3$. PICK U open in X such that $V = Y \cap U$
PROOF: By the definition of $\mathcal{T}$ ($\langle 1\rangle 1$, $\langle 2\rangle 1$, $\langle 3\rangle 2$)
$\langle 3\rangle 4$. $x \in Y \cap \bigcup\{U \text{ open in } X : Y \cap U \in \mathcal{A}\}$
$\langle 2\rangle 3$. $Y \cap \bigcup\{U \text{ open in } X : Y \cap U \in \mathcal{A}\} \subseteq \bigcup \mathcal{A}$
PROOF: Set theory.
$\langle 1\rangle 4$. $\mathcal{T}$ is closed under binary intersection.
PROOF: This holds because $(Y \cap U) \cap (Y \cap V) = Y \cap (U \cap V)$.
□

Lemma 4.3.2. *Let X be a topological space, $Y \subseteq X$, and $A \subseteq Y$. Then the topology A inherits as a subspace of X is the same as the topology A inherits as a subspace of Y.*

PROOF:

$$\begin{aligned} & \text{topology as a subspace of } Y \\ =& \{V \cap A : V \text{ open in } Y\} \\ =& \{V \cap A : \exists U \text{ open in } X . V = U \cap Y\} \\ =& \{U \cap Y \cap A : U \text{ open in } X\} \\ =& \{U \cap A : U \text{ open in } X\} \\ =& \text{topology as a subspace of } X \square \end{aligned}$$

Lemma 4.3.3. *Let Y be a subspace of X. If U is open in Y and Y is open in X then U is open in X.*

PROOF:
$\langle 1\rangle 1$. PICK V open in X such that $U = Y \cap V$
$\langle 1\rangle 2$. U is open in X
PROOF: The open sets in X are closed under binary intersection.
□

Theorem 4.3.4. *Let Y be a subspace of X. Let $A \subseteq Y$. Let $\overline{A}$ be the closure of A in X. Then the closure of A in Y is $\overline{A} \cap Y$.*

PROOF:
$\langle 1\rangle 1$. $\overline{A} \cap Y$ is a closed set in Y that includes A.
$\langle 2\rangle 1$. $\overline{A} \cap Y$ is closed in Y.
PROOF: By Lemma 4.3.4.1.

⟨2⟩2. $A \subseteq \overline{A} \cap Y$.
⟨1⟩2. If C is any closed set in Y that includes A then $\overline{A} \cap Y \subseteq C$.
⟨2⟩1. LET: C be a closed set in Y that includes A.
⟨2⟩2. PICK D closed in X such that $C = D \cap Y$.
PROOF: By Lemma 4.3.4.1.
⟨2⟩3. $\overline{A} \subseteq D$
⟨2⟩4. $\overline{A} \subseteq C$
□

Corollary 4.3.4.1. *Let Y be a subspace of X. Then a set $A \subseteq Y$ is closed in Y if and only if it is the intersection of a closed set in X with Y.*

Corollary 4.3.4.2. *Let Y be a subspace of X. If A is closed in Y and Y is closed in X then A is closed in X.*

Lemma 4.3.5. *Let X be a topological space and $Y \subseteq X$. If $\mathcal{B}$ is a basis for the topology on X then $\{B \cap Y : B \in \mathcal{B}\}$ is a basis for the subspace topology on Y.*

PROOF:
⟨1⟩1. For all $B \in \mathcal{B}$, we have $B \cap Y$ is open in Y.
PROOF: Immediate from definitions.
⟨1⟩2. For every V open in Y and $y \in V$, there exists $B \in \mathcal{B}$ such that $y \in B \cap Y \subseteq V$.
⟨2⟩1. LET: V be open in Y and $y \in V$
⟨2⟩2. PICK U open in X such that $V = Y \cap U$
⟨2⟩3. PICK $B \in \mathcal{B}$ such that $y \in B \subseteq U$
⟨2⟩4. $y \in B \cap Y \subseteq V$
□

Lemma 4.3.6. *Let X be a topological space and $Y \subseteq X$. If $\mathcal{S}$ is a subbasis for the topology on X then $\{S \cap Y : S \in \mathcal{S}\}$ is a subbasis for the subspace topology on Y.*

PROOF:
⟨1⟩1. For all $S \in \mathcal{S}$, we have $S \cap Y$ is open in Y.
PROOF: Immediate from definitions.
⟨1⟩2. For every V open in Y and $y \in V$, there exist $S_1, \ldots, S_n \in \mathcal{S}$ such that $y \in (S_1 \cap Y) \cap \cdots \cap (S_n \cap Y) \subseteq V$
⟨2⟩1. LET: V be open in Y and $y \in V$
⟨2⟩2. PICK U open in X such that $V = U \cap Y$
⟨2⟩3. PICK $S_1, \ldots, S_n \in \mathcal{S}$ such that $y \in S_1 \cap \cdots \cap S_n \subseteq U$
⟨2⟩4. $y \in (S_1 \cap Y) \cap \cdots \cap (S_n \cap Y) \subseteq V$
□

Theorem 4.3.7. *Let X be a linearly ordered set in the order topology. Let $Y \subseteq X$ be convex. Then the order topology on Y is the same as the subspace topology.*

PROOF:

⟨1⟩1. LET: $\mathcal{T}_o$ be the order topology and $\mathcal{T}_s$ be the subspace topology.
⟨1⟩2. $\mathcal{T}_o \subseteq \mathcal{T}_s$
  ⟨2⟩1. For all $a \in Y$, we have $\{y \in Y : a < y\} \in \mathcal{T}_s$
    PROOF: $\{y \in Y : a < y\} = \{x \in X : a < x\} \cap Y$
  ⟨2⟩2. For all $a \in Y$, we have $\{y \in Y : y < a\} \in \mathcal{T}_s$
    PROOF: Similar.
  ⟨2⟩3. Q.E.D.
    PROOF: Lemma 4.1.3 and Corollary 3.16.2.1.
⟨1⟩3. $\mathcal{T}_s \subseteq \mathcal{T}_o$
  ⟨2⟩1. The sets $(a, +\infty) \cap Y$ and $(-\infty, a) \cap Y$ for $a \in X$ form a subbasis for $\mathcal{T}_s$
    PROOF: Lemma 4.3.6, Lemma 4.1.3.
  ⟨2⟩2. For all $a \in X$, we have $(a, +\infty) \cap Y \in \mathcal{T}_o$
    ⟨3⟩1. LET: $a \in X$
    ⟨3⟩2. CASE: $a \in Y$
      PROOF: In this case, $(a, +\infty) \cap Y$ is an open ray in Y.
    ⟨3⟩3. CASE: For all $y \in Y$ we have $a < y$
      PROOF: In this case, $(a, +\infty) \cap Y = Y$.
    ⟨3⟩4. CASE: For all $y \in Y$ we have $y < a$
      PROOF: In this case, $(a, +\infty) \cap Y = \emptyset$.
    ⟨3⟩5. Q.E.D.
      PROOF: These are the only cases because Y is convex.
  ⟨2⟩3. For all $a \in X$, we have $(-\infty, a) \cap Y \in \mathcal{T}_o$
    PROOF: Similar.
  ⟨2⟩4. Q.E.D.
    PROOF: Corollary 3.16.2.1.

□

Theorem 4.3.8. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces, and let A_α be a subspace of X_α for all α. Then the product topology on $\prod_{\alpha \in J} A_\alpha$ is the same as the topology it inherits as a subspace of $\prod_{\alpha \in J} X_\alpha$.*

PROOF: Each is the topology generated by the subbasis consisting of $\pi_\alpha^{-1}(U) \cap \prod_{\alpha \in J} A_\alpha = \pi_\alpha^{-1}(U \cap A_\alpha)$ where $\alpha \in J$ and U is open in X_α, using Lemma 4.3.6.

□

Definition 4.3.9 (Unit Circle)**.** The *unit circle* S^1 is $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ as a subspace of $\mathbb{R}^2$.

Proposition 4.3.10. *Let Y be a subspace of X, $A \subseteq Y$, and $a \in Y$. Then a is a limit point of A in the subspace topology on Y if and only if a is a limit point of A is the topology of X.*

PROOF:

a is a limit point of A in Y

$\Leftrightarrow \forall U$ open in $Y(a \in U \Rightarrow U$ intersects A outside $a)$

$\Leftrightarrow \forall V$ open in $X(a \in V \cap Y \Rightarrow V \cap Y$ intersects A outside $a)$

$\Leftrightarrow \forall V$ open in $X(a \in V \Rightarrow V$ intersects A outside $a)$

$(a \in Y, A \subseteq Y)$

$\Leftrightarrow a$ is a limit point of A in X ∎

4.4 The Box Topology

Definition 4.4.1 (Box Topology)**.** Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces. The *box topology* on $\prod_{\alpha \in J} X_\alpha$ is the topology generated by the basis consisting of all sets of the form $\prod_{\alpha \in J} U_\alpha$, where each U_α is open in X_α.

We prove this is a basis.

PROOF:

$\langle 1 \rangle 1$. LET: $\mathcal{B}$ be the set of all sets of the form $\prod_{\alpha \in J} U_\alpha$, where each U_α is open in X_α.

$\langle 1 \rangle 2$. $\bigcup \mathcal{B} = \prod_{\alpha \in J} X_\alpha$

PROOF: This holds because $\prod_{\alpha \in J} X_\alpha \in \mathcal{B}$.

$\langle 1 \rangle 3$. $\mathcal{B}$ is closed under binary intersection.

PROOF: $\prod_{\alpha \in J} U_\alpha \cap \prod_{\alpha \in J} V_\alpha = \prod_{\alpha \in J} (U_\alpha \cap V_\alpha)$.

$\langle 1 \rangle 4$. Q.E.D.

PROOF: Corollary 3.5.3.1.

Theorem 4.4.2 (AC)**.** *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces, and let $\mathcal{B}_\alpha$ be a basis for the topology on X_α for each α. Then*

$$\mathcal{B} = \{\prod_{\alpha \in J} B_\alpha : \forall \alpha \in J. B_\alpha \in \mathcal{B}_\alpha\}$$

is a basis for the box topology on $\prod_{\alpha \in J} X_\alpha$.

PROOF:

$\langle 1 \rangle 1$. Every member of $\mathcal{B}$ is open in the box topology.

PROOF: Immediate from definitions.

$\langle 1 \rangle 2$. For every open set U and $\{x_\alpha\}_{\alpha \in J} \in U$, there exists $B \in \mathcal{B}$ such that $\{x_\alpha\}_{\alpha \in J} \in B \subseteq U$.

$\langle 2 \rangle 1$. LET: U be open and $\{x_\alpha\}_{\alpha \in J} \in U$

$\langle 2 \rangle 2$. PICK U_α open in X_α for each α such that $\{x_\alpha\}_{\alpha \in J} \in \prod_{\alpha \in J} U_\alpha \subseteq U$.

$\langle 2 \rangle 3$. PICK $B_\alpha \in \mathcal{B}_\alpha$ such that $x_\alpha \in B_\alpha \subseteq U_\alpha$ for each α

PROOF: Using the Axiom of Choice.

$\langle 2 \rangle 4$. $\{x_\alpha\}_{\alpha \in J} \in \prod_{\alpha \in J} B_\alpha \subseteq U$

∎

Theorem 4.4.3. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces, and let A_α be a subspace of X_α for all α. Let $\prod_{\alpha \in J} X_\alpha$ be given the box topology. Then the box topology on $\prod_{\alpha \in J} A_\alpha$ is the same as the topology it inherits as a subspace of $\prod_{\alpha \in J} X_\alpha$.*

PROOF: Each is the topology generated by the basis
$\{\prod_{\alpha \in J}(U_\alpha \cap A_\alpha) : U_\alpha$ is open in $X_\alpha\}$, using Lemma 4.3.5. ∎

Theorem 4.4.4. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of Hausdorff spaces. Then $\prod_{\alpha \in J} X_\alpha$ is Hausdorff under the box topology.*

PROOF:
⟨1⟩1. LET: $\{x_\alpha\}_{\alpha \in J}, \{y_\alpha\}_{\alpha \in J} \in \prod_{\alpha \in J} X_\alpha$ with $\{x_\alpha\}_{\alpha \in J} \neq \{y_\alpha\}_{\alpha \in J}$
⟨1⟩2. PICK $\alpha \in J$ such that $x_\alpha \neq y_\alpha$
⟨1⟩3. PICK disjoint neighbourhoods U of x_α and V of y_α.
⟨1⟩4. $\pi_\alpha^{-1}(U)$ and $\pi_\alpha^{-1}(V)$ are disjoint neighbourhoods of $\{x_\alpha\}_{\alpha \in J}$ and $\{y_\alpha\}_{\alpha \in J}$
∎

Corollary 4.4.4.1. *The space $\mathbb{R}^\omega$ under the box topology is Hausdorff.*

Theorem 4.4.5 (AC)**.** *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces and $A_\alpha \subseteq X_\alpha$ for all α. If $\prod_{\alpha \in J} X_\alpha$ is given the box topology, then*

$$\prod_{\alpha \in J} \overline{A_\alpha} = \overline{\prod_{\alpha \in J} A_\alpha} \ .$$

PROOF:
⟨1⟩1. $\prod_{\alpha \in J} \overline{A_\alpha} \subseteq \overline{\prod_{\alpha \in J} A_\alpha}$
  ⟨2⟩1. LET: $\{x_\alpha\}_{\alpha \in J} \in \prod_{\alpha \in J} \overline{A_\alpha}$
  ⟨2⟩2. LET: $\prod_{\alpha \in J} U_\alpha$ be a basic neighbourhood of $\{x_\alpha\}_{\alpha \in J}$, where each U_α is open in X_α.
  ⟨2⟩3. For $\alpha \in J$, PICK $a_\alpha \in A_\alpha \cap U_\alpha$.
    PROOF: By Theorem 3.13.3, using the Axiom of Choice.
  ⟨2⟩4. $\{a_\alpha\}_{\alpha \in J} \in \prod_{\alpha \in J} A_\alpha \cap \prod_{\alpha \in J} U_\alpha$
  ⟨2⟩5. Q.E.D.
    PROOF: By Theorem 3.13.3.
⟨1⟩2. $\overline{\prod_{\alpha \in J} A_\alpha} \subseteq \prod_{\alpha \in J} \overline{A_\alpha}$
  ⟨2⟩1. LET: $\{x_\alpha\}_{\alpha \in J} \in \overline{\prod_{\alpha \in J} A_\alpha}$
  ⟨2⟩2. LET: $\alpha \in J$
    PROVE: $x_\alpha \in \overline{A_\alpha}$
  ⟨2⟩3. LET: U be a neighbourhood of x_α in X_α
  ⟨2⟩4. $\pi_\alpha^{-1}(U)$ is a neighbourhood of $\{x_\alpha\}_{\alpha \in J}$
  ⟨2⟩5. PICK $\{a_\alpha\}_{\alpha \in J} \in \pi_\alpha^{-1}(U) \cap \prod_{\alpha \in J} A_\alpha$
    PROOF: By Theorem 3.13.3.
  ⟨2⟩6. $a_\alpha \in U \cap A_\alpha$
  ⟨2⟩7. Q.E.D.
    PROOF: By Theorem 3.13.3.
∎

4.5 The Quotient Topology

Definition 4.5.1 (Quotient Map)**.** Let X and Y be topological spaces. Let $p : X \twoheadrightarrow Y$ be a surjective map. Then p is a *quotient map* iff, for all $U \subseteq Y$, we have U is open in Y iff $p^{-1}(U)$ is open in X.

Lemma 4.5.2. *Let X and Y be topological spaces and $p : X \twoheadrightarrow Y$ be surjective and continuous. Then the following are equivalent.*

1. *p is a quotient map.*
2. *p maps saturated open sets to open sets.*
3. *p maps saturated closed sets to closed sets.*

PROOF:
$\langle 1\rangle 1$. $1 \Rightarrow 2$
  $\langle 2\rangle 1$. ASSUME: p is a quotient map.
  $\langle 2\rangle 2$. LET: $U \subseteq X$ be a saturated open set.
  $\langle 2\rangle 3$. $U = p^{-1}(p(U))$
    $\langle 3\rangle 1$. $U \subseteq p^{-1}(p(U))$
      PROOF: Set theory.
    $\langle 3\rangle 2$. $p^{-1}(p(U)) \subseteq U$
      $\langle 4\rangle 1$. LET: $x \in p^{-1}(p(U))$
      $\langle 4\rangle 2$. PICK $y \in U$ such that $p(x) = p(y)$
      $\langle 4\rangle 3$. $x \in U$
        PROOF: $\langle 2\rangle 2$, $\langle 4\rangle 2$.
  $\langle 2\rangle 4$. $p(U)$ is open
    PROOF: $\langle 2\rangle 1$, $\langle 2\rangle 3$.
$\langle 1\rangle 2$. $2 \Rightarrow 3$
  $\langle 2\rangle 1$. ASSUME: p maps saturated open sets to open sets
  $\langle 2\rangle 2$. LET: $C \subseteq X$ be a saturated closed set.
  $\langle 2\rangle 3$. $X \setminus C$ is a saturated open set.
    $\langle 3\rangle 1$. LET: $x \in X \setminus C$ and $x' \in X$ be such that $p(x) = p(x')$
    $\langle 3\rangle 2$. $x' \notin C$
      PROOF: If $x' \in C$ then $x \in C$ since C is saturated.
  $\langle 2\rangle 4$. $p(X \setminus C)$ is open.
    PROOF: By $\langle 2\rangle 1$ and $\langle 2\rangle 3$.
  $\langle 2\rangle 5$. $p(X \setminus C) = Y \setminus p(C)$
    $\langle 3\rangle 1$. $p(X \setminus C) \subseteq Y \setminus p(C)$
      $\langle 4\rangle 1$. LET: $x \in X \setminus C$
      $\langle 4\rangle 2$. ASSUME: for a contradiction $p(x) \in p(C)$
      $\langle 4\rangle 3$. PICK $x' \in C$ such that $p(x) = p(x')$
      $\langle 4\rangle 4$. Q.E.D.
        PROOF: We have $x \notin C$, $x' \in C$ and $p(x) = p(x')$, contradicting $\langle 2\rangle 2$.
    $\langle 3\rangle 2$. $Y \setminus p(C) \subseteq p(X \setminus C)$
      $\langle 4\rangle 1$. LET: $y \notin p(C)$
      $\langle 4\rangle 2$. PICK $x \in X$ such that $p(x) = y$

PROOF: p is surjective.
$\langle 4 \rangle 3.$ $x \notin C$
$\langle 1 \rangle 3.$ $3 \Rightarrow 1$
$\langle 2 \rangle 1.$ ASSUME: p maps saturated closed sets to closed sets
$\langle 2 \rangle 2.$ LET: $C \subseteq Y$ be such that $p^{-1}(Y)$ is closed
$\langle 2 \rangle 3.$ $p^{-1}(C)$ is saturated
$\langle 3 \rangle 1.$ LET: $x \in p^{-1}(C)$, $x' \in X$ and $p(x) = p(x')$
$\langle 3 \rangle 2.$ $x' \in p^{-1}(C)$
$\langle 2 \rangle 4.$ $p(p^{-1}(C))$ is closed
PROOF: By $\langle 2 \rangle 1$ and $\langle 2 \rangle 3$.
$\langle 2 \rangle 5.$ $C = p(p^{-1}(C))$
PROOF: By set theory, since p is surjective.
□

Corollary 4.5.2.1. *If $p : X \twoheadrightarrow Y$ is a surjective continuous map that is either an open map or a closed map, then p is a quotient map.*

Definition 4.5.3 (Quotient Topology)**.** Let X be a topological space, A a set, and $p : X \twoheadrightarrow A$ a surjective map. Then the *quotient topology* on A induced by p is

$$\{U \subseteq A : p^{-1}(U) \text{ is open in } X\} \ .$$

It is easy to check this is a topology.

Lemma 4.5.4. *Let X be a topological space, A a set, and $p : X \twoheadrightarrow A$ a surjective map. Then the quotient topology induced by p is the unique topology on A such that p is a quotient map.*

PROOF: Immediate from definitions. □

Definition 4.5.5 (Quotient Space)**.** Let X be a topological space and X^* a partition of X. Let $p : X \twoheadrightarrow X^*$ be the canonical map. Then X^* under the quotient topology induced by p is called a *quotient space* of X.

Proposition 4.5.6. *Let $p : X \twoheadrightarrow Y$ be a quotient map. Let $A \subseteq X$ be open and saturated. Then $p \restriction_A : A \twoheadrightarrow p(A)$ is a quotient map.*

PROOF:
$\langle 1 \rangle 1.$ LET: $q = p \restriction_A : A \twoheadrightarrow p(A)$
$\langle 1 \rangle 2.$ For all $V \subseteq p(A)$, we have $q^{-1}(V) = p^{-1}(V)$
$\langle 2 \rangle 1.$ $q^{-1}(V) \subseteq p^{-1}(V)$
PROOF: Trivial.
$\langle 2 \rangle 2.$ $p^{-1}(V) \subseteq q^{-1}(V)$
$\langle 3 \rangle 1.$ LET: $x \in p^{-1}(V)$
$\langle 3 \rangle 2.$ PICK $x' \in A$ such that $p(x') = p(x)$
PROOF: One exists because $p(x) \in V \subseteq p(A)$.
$\langle 3 \rangle 3.$ $x \in A$
PROOF: This holds because A is saturated.
$\langle 3 \rangle 4.$ $x \in q^{-1}(V)$

PROOF: From ⟨3⟩1 and ⟨3⟩3.
⟨1⟩3. For all $U \subseteq X$, we have $p(U \cap A) = p(U) \cap p(A)$
⟨1⟩4. LET: $V \subseteq p(A)$ be such that $q^{-1}(V)$ is open in A.
PROVE: V is open in $p(A)$.
⟨1⟩5. $q^{-1}(V)$ is open in X
⟨1⟩6. $p^{-1}(V)$ is open in X
⟨1⟩7. V is open in Y
⟨1⟩8. V is open in $p(A)$
□

Proposition 4.5.7. *Let $p : X \twoheadrightarrow Y$ be a quotient map. Let $A \subseteq X$ be closed and saturated. Then $p \restriction_A : A \twoheadrightarrow p(A)$ is a quotient map.*

PROOF: Similar. □

Proposition 4.5.8. *Let $p : X \twoheadrightarrow Y$ be an open quotient map. Let $A \subseteq X$ be saturated. Then $p \restriction_A : A \twoheadrightarrow p(A)$ is a quotient map.*

PROOF:
⟨1⟩1. LET: $q = p \restriction_A : A \twoheadrightarrow p(A)$
⟨1⟩2. For all $V \subseteq p(A)$, we have $q^{-1}(V) = p^{-1}(V)$
⟨2⟩1. $q^{-1}(V) \subseteq p^{-1}(V)$
PROOF: Trivial.
⟨2⟩2. $p^{-1}(V) \subseteq q^{-1}(V)$
⟨3⟩1. LET: $x \in p^{-1}(V)$
⟨3⟩2. PICK $x' \in A$ such that $p(x') = p(x)$
PROOF: One exists because $p(x) \in V \subseteq p(A)$.
⟨3⟩3. $x \in A$
PROOF: This holds because A is saturated.
⟨3⟩4. $x \in q^{-1}(V)$
PROOF: From ⟨3⟩1 and ⟨3⟩3.
⟨1⟩3. For all $U \subseteq X$, we have $p(U \cap A) = p(U) \cap p(A)$
⟨2⟩1. $p(U \cap A) \subseteq p(U) \cap p(A)$
PROOF: Set theory.
⟨2⟩2. $p(U) \cap p(A) \subseteq p(U \cap A)$
⟨3⟩1. LET: $x \in U$, $y \in A$, $p(x) = p(y)$
PROVE: $p(x) \in p(U \cap A)$
⟨3⟩2. $x \in A$
PROOF: A is saturated.
⟨3⟩3. $x \in U \cap A$
⟨1⟩4. LET: $V \subseteq p(A)$ be such that $q^{-1}(V)$ is open in A.
PROVE: V is open in $p(A)$.
⟨1⟩5. $p^{-1}(V)$ is open in A
PROOF: By ⟨1⟩2
⟨1⟩6. PICK U open in X such that $p^{-1}(V) = U \cap A$
⟨1⟩7. $V = p(U) \cap p(A)$

PROOF:

$$\begin{aligned} V &= p(p^{-1}(V)) && (p \text{ is surjective}) \\ &= p(U \cap A) && (\langle 1\rangle 6) \\ &= p(U) \cap p(A) && (\langle 1\rangle 3) \end{aligned}$$

$\langle 1\rangle 8$. $p(U)$ is open in Y

PROOF: $\langle 1\rangle 6$, p is an open map.

$\langle 1\rangle 9$. V is open in $p(A)$

PROOF: $\langle 1\rangle 7$, $\langle 1\rangle 8$

$\square$

Proposition 4.5.9. *Let $p : X \twoheadrightarrow Y$ be a closed quotient map. Let $A \subseteq X$ be saturated. Then $p \restriction_A : A \twoheadrightarrow p(A)$ is a quotient map.*

PROOF: Similar. $\square$

Proposition 4.5.10. *The composite of two quotient maps is a quotient map.*

PROOF: From Proposition 5.2.22. $\square$

Proposition 4.5.11. *Let X^* be a quotient space of X. If every element of X^* is closed in X, then X^* is T_1.*

PROOF:

$\langle 1\rangle 1$. LET: $C \in X^*$

$\langle 1\rangle 2$. $p^{-1}(\{C\}) = C$

PROOF: Definition of p.

$\langle 1\rangle 3$. $p^{-1}(\{C\})$ is closed in X

PROOF: By hypothesis.

$\langle 1\rangle 4$. $\{C\}$ is closed in X^*.

PROOF: By Proposition 5.2.21.

$\square$

Chapter 5

Functions Between Topological Spaces

5.1 Open Maps

Definition 5.1.1. Let X and Y be topological spaces and $f : X \to Y$. Then f is an *open map* iff, for all U open in X, $f(U)$ is open in Y.

Lemma 5.1.2. *Let X and Y be topological spaces and $f : X \to Y$. Let $\mathcal{B}$ be a basis for the topology on X. Then f is an open map if and only if, for all $B \in \mathcal{B}$, $f(B)$ is open in Y.*

PROOF:
$\langle 1\rangle 1$. If f is an open map then, for all $B \in \mathcal{B}$, $f(B)$ is open in Y.
  PROOF: Immediate from definitions.
$\langle 1\rangle 2$. If, for all $B \in \mathcal{B}$, $f(B)$ is open in Y, then f is an open map.
  $\langle 2\rangle 1$. ASSUME: For all $B \in \mathcal{B}$, $f(B)$ is open in Y.
  $\langle 2\rangle 2$. LET: U be open in X
    PROVE: $f(U)$ is open in Y
  $\langle 2\rangle 3$. LET: $\mathcal{B}_0 \subseteq \mathcal{B}$ be such that $U = \bigcup \mathcal{B}_0$
  $\langle 2\rangle 4$. $f(U) = \bigcup_{B \in \mathcal{B}_0} f(B)$
    PROOF: Set theory.
  $\langle 2\rangle 5$. $f(U)$ is open in Y.
    PROOF: From $\langle 2\rangle 1$, $\langle 2\rangle 4$ and the fact that the open sets are closed under union.
□

Corollary 5.1.2.1. *Let X and Y be topological spaces and $f : X \to Y$. Let $\mathcal{S}$ be a subbasis for the topology on X. Then f is an open map if and only if, for all $S \in \mathcal{S}$, $f(S)$ is open in Y.*

Lemma 5.1.3 (AC)**.** *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces. Then the projection $\pi_\alpha : \prod_{\alpha \in J} X_\alpha \to X_\alpha$ is an open map.*

PROOF:
$\langle 1\rangle 1.$ For U open in X_α, we have $\pi_\alpha(\pi_\alpha^{-1}(U))$ is open in X_α
PROOF: $\pi_\alpha(\pi_\alpha^{-1}(U)) = U$ if all the other X_α are nonempty, $\emptyset$ otherwise.
$\langle 1\rangle 2.$ For $\beta \neq \alpha$ and U open in X_β, we have $\pi_\alpha(\pi_\beta^{-1}(U))$ is open in X_α
PROOF: $\pi_\alpha(\pi_\beta^{-1}(U)) = X_\alpha$ if all the X_γ are nonempty for $\gamma \neq \alpha$, $\emptyset$ otherwise.
$\langle 1\rangle 3.$ Q.E.D.
PROOF: By Corollary 5.1.2.1.

5.2 Continuous Functions

Definition 5.2.1 (Continuous)**.** Let X and Y be topological spaces and $f : X \to Y$ a function. Then f is *continuous* if and only if, for every open set U in Y, the set $f^{-1}(U)$ is open in X.

Theorem 5.2.2. *Let X and Y be topological spaces and $f : X \to Y$. Then the following are equivalent.*

1. *f is continuous.*
2. *For every closed set C in Y, the set $f^{-1}(C)$ is closed in X.*
3. *For every set $A \subseteq X$, we have $f(\overline{A}) \subseteq \overline{f(A)}$.*

PROOF:
$\langle 1\rangle 1.$ $1 \Rightarrow 3$
$\langle 2\rangle 1.$ ASSUME: f is continuous.
$\langle 2\rangle 2.$ LET: $A \subseteq X$
$\langle 2\rangle 3.$ LET: $x \in \overline{A}$
PROVE: $f(x) \in \overline{f(A)}$
$\langle 2\rangle 4.$ LET: V be a neighbourhood of $f(x)$
$\langle 2\rangle 5.$ $f^{-1}(V)$ is a neighbourhood of x
PROOF: $\langle 2\rangle 1$, $\langle 2\rangle 3$, $\langle 2\rangle 4$
$\langle 2\rangle 6.$ $f^{-1}(V)$ intersects A in a, say.
PROOF: $\langle 2\rangle 3$, $\langle 2\rangle 5$, Theorem 3.13.3.
$\langle 2\rangle 7.$ V intersects $f(A)$ in $f(a)$.
$\langle 2\rangle 8.$ Q.E.D.
PROOF: Theorem 3.13.3.
$\langle 1\rangle 2.$ $3 \Rightarrow 2$
$\langle 2\rangle 1.$ ASSUME: 3
$\langle 2\rangle 2.$ LET: C be a closed set in Y
$\langle 2\rangle 3.$ $\overline{f^{-1}(C)} = f^{-1}(C)$
PROOF:
$$\begin{aligned} f(\overline{f^{-1}(C)}) &\subseteq \overline{f(f^{-1}(C))} && (\langle 2\rangle 1) \\ &\subseteq \overline{C} \end{aligned}$$
$\langle 1\rangle 3.$ $2 \Rightarrow 1$
$\langle 2\rangle 1.$ ASSUME: 2

$\langle 2 \rangle 2$. LET: V be open in Y
$\langle 2 \rangle 3$. $f^{-1}(Y \setminus V)$ is closed in X
PROOF: By $\langle 2 \rangle 1$.
$\langle 2 \rangle 4$. $f^{-1}(V)$ is open in X.
PROOF: $f^{-1}(V) = X \setminus f^{-1}(Y \setminus V)$.
□

Lemma 5.2.3. *If $f : X \to Y$ maps all of X to the single point y_0 of Y, then f is continuous.*

PROOF: For V open in Y, the set $f^{-1}(V)$ is either X (if $y_0 \in V$) or $\emptyset$ (if $y_0 \notin V$).

Definition 5.2.4 (Continuity at a Point)**.** Let X and Y be topological spaces, $f : X \to Y$ a function, and $x \in X$. Then f is *continuous at x* if and only if, for every neighbourhood V of $f(x)$, there exists a neighbourhood U of x such that $f(U) \subseteq V$.

Theorem 5.2.5. *Let X and Y be topological spaces and $f : X \to Y$. Then f is continuous if and only if f is continuous at every point of X.*

PROOF:
$\langle 1 \rangle 1$. If f is continuous then f is continuous at every point of X.
$\langle 2 \rangle 1$. ASSUME: f is continuous
$\langle 2 \rangle 2$. LET: $x \in X$
$\langle 2 \rangle 3$. LET: V be a neighbourhood of $f(x)$
$\langle 2 \rangle 4$. $f^{-1}(V)$ is a neighbourhood of x
$\langle 2 \rangle 5$. $f(f^{-1}(V)) \subseteq V$
$\langle 1 \rangle 2$. If f is continuous at every point of X then f is continuous.
$\langle 2 \rangle 1$. ASSUME: f is continuous at every point of X.
$\langle 2 \rangle 2$. LET: V be open in Y
PROVE: $f^{-1}(V)$ is open in X.
$\langle 2 \rangle 3$. LET: $x \in f^{-1}(V)$
$\langle 2 \rangle 4$. V is a neighbourhood of $f(x)$
$\langle 2 \rangle 5$. PICK a neighbourhood U of x such that $f(U) \subseteq V$
PROOF: By $\langle 2 \rangle 1$.
$\langle 2 \rangle 6$. $x \in U \subseteq f^{-1}(V)$
$\langle 2 \rangle 7$. Q.E.D.
PROOF: By Proposition 3.2.3.
□

Lemma 5.2.6. *Let X and Y be topological spaces and $f : X \to Y$. Let $\mathcal{B}$ be a basis for the topology on Y. Then f is continuous if and only if, for all $B \in \mathcal{B}$, the set $f^{-1}(B)$ is open in X.*

PROOF:
$\langle 1 \rangle 1$. If f is continuous then, for all $B \in \mathcal{B}$, the set $f^{-1}(B)$ is open in X.
PROOF: Immediate from definitions.
$\langle 1 \rangle 2$. If, for all $B \in \mathcal{B}$, the set $f^{-1}(B)$ is open in X, then f is continuous.

⟨2⟩1. Assume: For all $B \in \mathcal{B}$, the set $f^{-1}(B)$ is open in X.
⟨2⟩2. Let: $x \in X$
⟨2⟩3. Let: V be a neighbourhood of $f(x)$
⟨2⟩4. Pick $B \in \mathcal{B}$ such that $f(x) \in B \subseteq V$
⟨2⟩5. $f^{-1}(B)$ is a neighbourhood of x
Proof: By ⟨2⟩1.
⟨2⟩6. $f(f^{-1}(B)) \subseteq B$
Proof: Set theory.
⟨2⟩7. Q.E.D.
Proof: Theorem 5.2.5.
□

Lemma 5.2.7. *The projections $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are continuous.*

Proof:Immediate from definitions. □

Theorem 5.2.8. *If A is a subspace of X, the inclusion function $j : A \to X$ is continuous.*

Proof: For V open in X, the set $j^{-1}(V) = V \cap A$ is open in A.

Theorem 5.2.9. *If $f : X \to Y$ and $g : Y \to Z$ are continuous, then the map $g \circ f : X \to Z$ is continuous.*

Proof:
⟨1⟩1. Let: V be open in Z
⟨1⟩2. $g^{-1}(V)$ is open in Y
⟨1⟩3. $f^{-1}(g^{-1}(V))$ is open in X
□

Theorem 5.2.10. *If $f : X \to Y$ is continuous and if A is a subspace of X, then the restricted function $f \restriction A : A \to Y$ is continuous.*

Proof: For V open in Y, the set $(f \restriction A)^{-1}(V) = f^{-1}(V) \cap A$ is open in A. □

Theorem 5.2.11. *Let $f : X \to Y$ be continuous. If Z is a subspace of Y that includes the range of f, then the function $g : X \to Z$ obtained by restricting the codomain of f is continuous. If Z is a space having Y as a subspace, then the function $h : X \to Z$ obtained by expanding the codomain of f is continuous.*

Proof:
⟨1⟩1. If Z is a subspace of Y that includes the range of f, then the function $g : X \to Z$ obtained by restricting the codomain of f is continuous.
⟨2⟩1. Let: V be open in Z
⟨2⟩2. Pick W open in Y such that $V = W \cap Z$
⟨2⟩3. $f^{-1}(W)$ is open in X.
⟨2⟩4. $g^{-1}(V)$ is open in X.
Proof: $g^{-1}(V) = f^{-1}(W)$.

⟨1⟩2. If Z is a space having Y as a subspace, then the function $h : X \to Z$ obtained by expanding the codomain of f is continuous.

PROOF: For V open in Z, we have $h^{-1}(V) = f^{-1}(V \cap Y)$ is open in X.

□

Theorem 5.2.12. *Let X and Y be topological spaces and $f : X \to Y$. If $x_n \to x$ as $n \to \infty$ in X and f is continuous at x, then $f(x_n) \to f(x)$ as $n \to \infty$ in Y.*

PROOF:

⟨1⟩1. ASSUME: $x_n \to x$ as $n \to \infty$

⟨1⟩2. ASSUME: f is continuous at x

⟨1⟩3. LET: V be a neighbourhood of $f(x)$

⟨1⟩4. PICK a neighbourhood U of x such that $f(U) \subseteq V$

PROOF: By ⟨1⟩2.

⟨1⟩5. PICK N such that, for all $n \geq N$, $x_n \in U$

PROOF: By ⟨1⟩1

⟨1⟩6. For $n \geq N$, $f(x_n) \in V$

PROOF: By ⟨1⟩4.

□

Corollary 5.2.12.1. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces and (x_n) a family of points in $\prod_{\alpha \in J} X_\alpha$. We have $x_n \to l$ as $n \to \infty$ if and only if, for all $\alpha \in J$, $\pi_\alpha(x_n) \to \pi_\alpha(l)$ as $n \to \infty$.*

PROOF:

⟨1⟩1. If $x_n \to l$ as $n \to \infty$ then, for all $\alpha \in J$, $\pi_\alpha(x_n) \to \pi_\alpha(l)$ as $n \to \infty$

PROOF: Theorem 5.2.12 and Proposition 5.2.7.

⟨1⟩2. If, for all $\alpha \in J$,we have $\pi_\alpha(x_n) \to \pi_\alpha(l)$ as $n \to \infty$, then $x_n \to l$ as $n \to \infty$

⟨2⟩1. ASSUME: For all $\alpha \in J$,we have $\pi_\alpha(x_n) \to \pi_\alpha(l)$ as $n \to \infty$

⟨2⟩2. LET: $B = \prod_{\alpha \in J} U_\alpha$ be a basic open neighbourhood of l, where $U_\alpha = X_\alpha$ except for $\alpha = \alpha_1, \ldots, \alpha_k$

⟨2⟩3. PICK N such that, for all $n \geq N$ and $1 \leq i \leq k$, we have $\pi_i(x_n) \in U_{\alpha_i}$

⟨2⟩4. For $n \geq N$ we have $x_n \in B$

□

Theorem 5.2.13. *Let X and Y be topological spaces. Let $f : X \to Y$. If there exists a set $\mathcal{A}$ of open sets in X such that:*

- *$\bigcup \mathcal{A} = X$;*
- *for all $U \in \mathcal{A}$, the function $f \restriction U : U \to X$ is continuous;*

then f is continuous.

PROOF:

⟨1⟩1. LET: V be open in Y

⟨1⟩2. For all $U \in \mathcal{A}$, the set $(f \restriction U)^{-1}(V)$ is open in X.

⟨2⟩1. LET: $U \in \mathcal{A}$

$\langle 2\rangle 2.$ $(f \upharpoonright U)^{-1}(V)$ is open in U

PROOF: Since $f \upharpoonright U : U \to X$ is continuous.

$\langle 2\rangle 3.$ Q.E.D.

PROOF: By Lemma 4.3.3.

$\langle 1\rangle 3.$ Q.E.D.

PROOF: Since $f^{-1}(V) = \bigcup_{U \in \mathcal{A}} (f \upharpoonright U)^{-1}(V)$.

Theorem 5.2.14 (The Pasting Lemma)**.** *Let $X = A \cup B$ where A and B are closed in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous. If $f(x) = g(x)$ for every $x \in A \cap B$, then the function $h : X \to Y$ defined by*

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases}$$

is continuous.

PROOF:

$\langle 1\rangle 1.$ LET: C be closed in Y

$\langle 1\rangle 2.$ $f^{-1}(C)$ is closed in A

PROOF: Theorem 5.2.2.

$\langle 1\rangle 3.$ $f^{-1}(C)$ is closed in X

PROOF: Lemma 4.3.4.1.

$\langle 1\rangle 4.$ $g^{-1}(C)$ is closed in B

PROOF: Theorem 5.2.2.

$\langle 1\rangle 5.$ $g^{-1}(C)$ is closed in X

PROOF: Lemma 4.3.4.1.

$\langle 1\rangle 6.$ $h^{-1}(C)$ is closed in X

PROOF: $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$

$\langle 1\rangle 7.$ Q.E.D.

PROOF: Theorem 5.2.2.

□

Theorem 5.2.15. *Let $f : A \to \prod_{\alpha \in J} X_\alpha$ be given by the equation*

$$f(a) = \{f_\alpha(a)\}_{\alpha \in J} \ ,$$

where $f_\alpha : A \to X_\alpha$ for each α. Let $\prod_{\alpha \in J} X_\alpha$ have the product topology. Then the function f is continuous if and only if each function f_α is continuous.

PROOF:

$\langle 1\rangle 1.$ If f is continuous then each f_α is continuous.

PROOF: This holds because $f_\alpha = \pi_\alpha \circ f$.

$\langle 1\rangle 2.$ If every f_α is continuous then f is continuous.

$\langle 2\rangle 1.$ ASSUME: Every f_α is continuous.

$\langle 2\rangle 2.$ LET: $\alpha \in J$ and U be open in X_α

$\langle 2\rangle 3.$ $f^{-1}(\pi_\alpha^{-1}(U))$ is open in A

PROOF: $f^{-1}(\pi_\alpha^{-1}(U)) = f_\alpha^{-1}(U)$.

□

5.2.1 Homeomorphisms

Definition 5.2.16 (Homeomorphism)**.** Let X and Y be topological spaces and $f : X \to Y$. Then f is a *homeomorphism* between X and Y iff f is a bijection, and f and f^{-1} are both continuous.

Definition 5.2.17 (Topological Property)**.** A property P of topological spaces is a *topological property* iff, for any spaces X and Y, if X is homeomorphic to Y then P holds of X if and only if P holds of Y.

Definition 5.2.18 ((Topological) Imbedding)**.** Let X and Y be topological spaces and $f : X \to Y$. Then f is a *(topological) imbedding* iff f is a homeomorphism between X and $\operatorname{im} f$.

Definition 5.2.19 (Homogeneous)**.** A topological space X is *homogeneous* iff, for all $x, y \in X$, there exists a homeomorphism $f : X \cong X$ such that $f(x) = y$.

5.2.2 Strongly Continuous Functions

Definition 5.2.20 (Strongly Continuous)**.** Let X and Y be topological spaces and $f : X \to Y$. Then f is *strongly continuous* iff, for all $V \subseteq Y$, we have V is open in Y if and only if $f^{-1}(V)$ is open in X.

Proposition 5.2.21. *Let X and Y be topological spaces and $f : X \to Y$. Then f is strongly continuous if and only if, for all $C \subseteq Y$, C is closed in Y if and only if $f^{-1}(C)$ is closed in X.*

PROOF:
$\langle 1 \rangle 1$. If f is strongly continuous then, for all $C \subseteq Y$, we have C is closed in Y if and only if $f^{-1}(C)$ is closed in X.
PROOF:

$$\begin{aligned} C \text{ is closed in } Y &\Leftrightarrow Y \setminus C \text{ is open in } Y \\ &\Leftrightarrow f^{-1}(Y \setminus C) \text{ is open in } X \\ &\Leftrightarrow X \setminus f^{-1}(C) \text{ is open in } X \\ &\Leftrightarrow f^{-1}(C) \text{ is closed in } X \end{aligned}$$

$\langle 1 \rangle 2$. If, for all $C \subseteq Y$, we have C is closed in Y if and only if $f^{-1}(C)$ is closed in X, then f is strongly continuous.
PROOF: Similar.
□

Proposition 5.2.22. *The composite of two strongly continuous functions is strongly continuous.*

PROOF:
$\langle 1 \rangle 1$. LET: $f : X \to Y$ and $g : Y \to Z$ be strongly continuous.
$\langle 1 \rangle 2$. LET: $V \subseteq Z$
$\langle 1 \rangle 3$. V is open iff $f^{-1}(g^{-1}(V))$ is open

PROOF:

$$\begin{aligned} V \text{ is open} &\Leftrightarrow g^{-1}(V) \text{ is open} && (\langle 1\rangle 1) \\ &\Leftrightarrow f^{-1}(g^{-1}(V)) \text{ is open} && (\langle 1\rangle 1) \end{aligned}$$

∎

Proposition 5.2.23. *Let X, Y and Z be topological spaces. Let $f : X \to Y$ and $g : Y \to Z$. If f is strongly continuous and $g \circ f$ is continuous, then g is continuous.*

PROOF:
$\langle 1\rangle 1$. LET: $V \subseteq Z$ be open in Z.
$\langle 1\rangle 2$. $f^{-1}(g^{-1}(V))$ is open in X.
PROOF: $g \circ f$ is continuous.
$\langle 1\rangle 3$. $g^{-1}(V)$ is open in Y.
PROOF: f is strongly continuous.
∎

Proposition 5.2.24. *Let X, Y and Z be topological spaces. Let $f : X \to Y$ and $g : Y \to Z$. If f and $g \circ f$ are strongly continuous, then g is strongly continuous.*

PROOF:
$\langle 1\rangle 1$. LET: $U \subseteq Z$
$\langle 1\rangle 2$. U is open in Z iff $g^{-1}(U)$ is open in Y
PROOF:

$$\begin{aligned} U \text{ is open in } Z &\Leftrightarrow f^{-1}(g^{-1}(U)) \text{ is open in } X && (g \circ f \text{ is strongly continuous}) \\ &\Leftrightarrow g^{-1}(U) \text{ is open in } Y && (f \text{ is strongly continuous}) \end{aligned}$$

∎

5.3 Closed Maps

Definition 5.3.1 (Closed Map)**.** Let X and Y be topological spaces and $f : X \to Y$. Then f is a *closed map* iff, for every closed set $C \subseteq X$, the set $f(C)$ is closed in Y.

Lemma 5.3.2. *Let $p : X \to Y$ be a closed map. Let $B \subseteq Y$. Let U be an open neighbourhood of $p^{-1}(B)$. Then there exists an open neighbourhood V of B such that $p^{-1}(V) \subseteq U$.*

PROOF:
$\langle 1\rangle 1$. LET: $V = Y \setminus p(X \setminus U)$
$\langle 1\rangle 2$. V is open
$\langle 1\rangle 3$. $p^{-1}(V) \subseteq U$
∎

5.4 Local Homeomorphism

Definition 5.4.1 (Locally Homeomorphic)**.** Let X and Y be topological spaces. Then X is *locally homeomorphic* to Y iff every point in X has an open neighborhood that is homeomorphic with an open set in Y.

Proposition 5.4.2. *The long line is locally homeomorphic with* $\mathbb{R}$*.*

PROOF:
$\langle 1 \rangle 1$. LET: $x \in L$
$\langle 1 \rangle 2$. PICK an ordinal α such that $x < (\alpha, 0)$.
$\langle 1 \rangle 3$. $(-\infty, (\alpha, 0))$ is an open neighbourhood of x that is homeomorphic to $(0, 1)$.
$\square$

5.5 Retracts

Definition 5.5.1 (Retract)**.** Let Z be a topological space. If Y is a subspace of Z, we say that Y is a *retract* of Z iff there exists a continuous function $r : Z \to Y$ such that $r(y) = y$ for all $y \in Y$.

Chapter 6

Separation Axioms

6.1 T_1 Spaces

Definition 6.1.1 (T_1 Space)**.** A topological space X is a T_1 *space* iff every finite set is closed.

Theorem 6.1.2. *Let X be a T_1 space and $A \subseteq X$. Then x is a limit point of A if and only if every neighbourhood of x contains infinitely many points of A.*

PROOF:
$\langle 1 \rangle 1$. If some neighbourhood of x contains only finitely many points of A then x is not a limit point of A.
  $\langle 2 \rangle 1$. ASSUME: Some neighbourhood U of x contains only finite many points $a_1, \ldots, a_n$ of A.
  $\langle 2 \rangle 2$. $X \setminus \{a_1, \ldots, a_n\}$ is open.
    PROOF: X is T_1.
  $\langle 2 \rangle 3$. $U \setminus \{a_1, \ldots, a_n\}$ is a neighbourhood of x that does not intersect A.
$\langle 1 \rangle 2$. If every neighbourhood of x contains infinitely many points of A then x is a limit point of A.
  PROOF: From the definition of limit point.
□

Proposition 6.1.3. *A subspace of a T_1 space is T_1.*

PROOF:
$\langle 1 \rangle 1$. LET: X be a T_1 space and $Y \subseteq X$
$\langle 1 \rangle 2$. LET: $a \in Y$
$\langle 1 \rangle 3$. $\{a\}$ is closed in X
  PROOF: By $\langle 1 \rangle 1$.
$\langle 1 \rangle 4$. $\{a\}$ is closed in Y
  PROOF: By Corollary 4.3.4.1.
□

Definition 6.1.4 (Separate Points from Closed Sets)**.** Let X be a space and $\{f_\alpha\}_{\alpha \in J}$ be a family of continuous functions $f_\alpha : X \to \mathbb{R}$. Then $\{f_\alpha\}$ *separates points from closed sets* in X iff, for every point $x_0 \in X$ and every neighbourhood U of x_0, there exists $\alpha \in J$ such that f_α is positive at x_0 and vanishes outside U.

Theorem 6.1.5 (Imbedding Theorem)**.** *Let X be a T_1 space and $\{f_\alpha\}_{\alpha \in J}$ be a family of functions $X \to \mathbb{R}$ that separates points from closed sets. Then the function $F : X \to \mathbb{R}^J$ defined by*

$$F(x)_\alpha = f_\alpha(x)$$

is an imbedding. If each f_α maps X into $[0,1]$ then F is an imbedding $X \to [0,1]^J$.

Proof:
$\langle 1 \rangle 1.$ F is continuous
  Proof: By Theorem 5.2.15.
$\langle 1 \rangle 2.$ F is injective
  $\langle 2 \rangle 1.$ Let: $x, y \in X$ with $x \neq y$
  $\langle 2 \rangle 2.$ Pick a neighbourhood U of x such that $y \notin U$
    Proof: X is T_1
  $\langle 2 \rangle 3.$ Pick $\alpha \in J$ such that f_α is positive at x and vanishes outside U
  $\langle 2 \rangle 4.$ $f_\alpha(x) \neq f_\alpha(y)$
  $\langle 2 \rangle 5.$ $F(x) \neq F(y)$
$\langle 1 \rangle 3.$ F is open as a map $X \to F(U)$
  $\langle 2 \rangle 1.$ Let: U be open
  $\langle 2 \rangle 2.$ Let: $z \in F(U)$
  $\langle 2 \rangle 3.$ Pick $x \in U$ such that $F(x) = z$
  $\langle 2 \rangle 4.$ Pick $\alpha \in J$ such that f_α is positive at x and vanishes outside U
  $\langle 2 \rangle 5.$ $z \in {\pi_\alpha}^{-1}((0, +\infty)) \cap F(U) \subseteq F(U)$
$\square$

6.2 Hausdorff Spaces

Definition 6.2.1 (Hausdorff Space)**.** A topological space X is a *Hausdorff space* iff, for any points $x, y \in X$ with $x \neq y$, there exist disjoint neighbourhoods U of x and V of y.

Theorem 6.2.2. *Every Hausdorff space is T_1.*

Proof:
$\langle 1 \rangle 1.$ Let: X be a Hausdorff space
$\langle 1 \rangle 2.$ Let: $a \in X$
    Prove: $\{a\}$ is closed.
$\langle 1 \rangle 3.$ Let: $b \in X \setminus \{a\}$
$\langle 1 \rangle 4.$ Pick disjoint neighbourhoods U of a and V of b

⟨1⟩5. $b \in V \subseteq X \setminus \{a\}$
⟨1⟩6. Q.E.D.
PROOF: By Proposition 3.2.3.
□

Theorem 6.2.3. *In a Hausdorff space, a sequence has at most one limit.*

PROOF:
⟨1⟩1. ASSUME: for a contradiction $x_n \to l$ and $x_n \to m$ as $n \to \infty$, and $l \neq m$
⟨1⟩2. PICK disjoint neighbourhoods U of l and V of m
⟨1⟩3. PICK N such that, for all $n \geq N$, $x_n \in U$ and $x_n \in V$
⟨1⟩4. $x_N \in U \cap V$
□

Theorem 6.2.4. *Every linearly ordered set is Hausdorff under the order topology.*

PROOF:
⟨1⟩1. LET: X be a linearly ordered set under the order topology.
⟨1⟩2. LET: $x, y \in X$ with $x \neq y$
⟨1⟩3. ASSUME: w.l.o.g. $x < y$
PROVE: There exist disjoint neighbourhoods U of x and V of y.
⟨1⟩4. CASE: There exists z such that $x < z < y$
PROOF: In this case, take $U = (-\infty, z)$ and $V = (z, +\infty)$.
⟨1⟩5. CASE: There does not exist z such that $x < z < y$
PROOF: In this case, take $U = (-\infty, y)$ and $V = (x, +\infty)$.
□

Theorem 6.2.5. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of Hausdorff spaces. Then $\prod_{\alpha \in J} X_\alpha$ is Hausdorff under the product topology.*

PROOF:
⟨1⟩1. LET: $\{x_\alpha\}_{\alpha \in J}, \{y_\alpha\}_{\alpha \in J} \in \prod_{\alpha \in J} X_\alpha$ with $\{x_\alpha\}_{\alpha \in J} \neq \{y_\alpha\}_{\alpha \in J}$
⟨1⟩2. PICK $\alpha \in J$ such that $x_\alpha \neq y_\alpha$
⟨1⟩3. PICK disjoint neighbourhoods U of x_α and V of y_α.
⟨1⟩4. $\pi_\alpha^{-1}(U)$ and $\pi_\alpha^{-1}(V)$ are disjoint neighbourhoods of $\{x_\alpha\}_{\alpha \in J}$ and $\{y_\alpha\}_{\alpha \in J}$
□

Corollary 6.2.5.1. *The Sorgenfrey plane is Hausdorff.*

Corollary 6.2.5.2. *For any set I, the space $\mathbb{R}^I$ is Hausdorff.*

Proposition 6.2.6. *Let X and Y be topological spaces and $f : X \to Y$. If f is continuous and injective and Y is Hausdorff then X is Hausdorff.*

PROOF:
⟨1⟩1. LET: $x, y \in X$ with $x \neq y$
⟨1⟩2. $f(x) \neq f(y)$
PROOF: f is injective.

⟨1⟩3. Pick disjoint neighbourhoods U, V of $f(x)$ and $f(y)$
  Proof: Y is Hausdorff.
⟨1⟩4. $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint neighbourhoods of x and y.
□

Corollary 6.2.6.1. *A subspace of a Hausdorff space is Hausdorff.*

Corollary 6.2.6.2. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of nonempty spaces. If $\prod_{\alpha \in J} X_\alpha$ is Hausdorff then so is each X_α.*

Corollary 6.2.6.3. *Let $\mathcal{T}$ and $\mathcal{T}'$ be topologies on the same set X. If $\mathcal{T} \subseteq \mathcal{T}'$ and X is Hausdorff under $\mathcal{T}$ then X is Hausdorff under $\mathcal{T}'$.*

Corollary 6.2.6.4. *The space $\mathbb{R}_K$ is Hausdorff.*

Proposition 6.2.7. *$\mathbb{R}_l$ is Hausdorff.*

Proof: Let $a, b \in \mathbb{R}_l$ with $a < b$. Then $(-\infty, b)$ and $[b, +\infty)$ are disjoint open sets containing a and b respectively. □

Proposition 6.2.8. *The continuous image of a Hausdorff space is not necessarily Hausdorff.*

Proof: The identity map from the discrete two-point space to the indiscrete two-point space is continuous. □

Lemma 6.2.9. *Let A be a subspace of X and Z be Hausdorff. Let $f : A \to Z$ be contiuous. Then there is at most one extension of f to a continuous function $\overline{A} \to Z$.*

Proof:
⟨1⟩1. Assume: for a contradiction $g, h : \overline{A} \to Z$ are continuous extensions of f with $g(x) \neq h(x)$
⟨1⟩2. Pick disjoint open neighbourhoods U of $g(x)$ and V of $h(x)$
⟨1⟩3. Pick a point $a \in A \cap g^{-1}(U) \cap h^{-1}(V)$
  Proof: One exists because $g^{-1}(U) \cap h^{-1}(V)$ is a neighbourhood of $x \in \overline{A}$.
⟨1⟩4. $g(a) \in U \cap V$
□

6.3 Regular Spaces

Definition 6.3.1 (Regular)**.** A topological space X is *regular* iff, for every closed set A and point $a \notin A$, there exist disjoint neighbourhoods U of A and V of a.

Proposition 6.3.2. *Let X be a T_1 space. Then X is regular if and only if, for every point x and neighbourhood U of x, there exists a neighbourhood V of x such that $\overline{V} \subseteq U$.*

Proof:

⟨1⟩1. If X is regular then, for every point x and neighbourhood N of x, there exists a neighbourhood V of x such that $\overline{V} \subseteq N$.

⟨2⟩1. Assume: X is regular.

⟨2⟩2. Let: $x \in X$ and N be a neighbourhood of x

⟨2⟩3. Pick an open set U such that $x \in U \subseteq N$

⟨2⟩4. Pick disjoint open sets V, W such that $x \in V$ and $X \setminus U \subseteq W$

⟨2⟩5. $\overline{V} \subseteq N$

Proof:

$$\begin{aligned} \overline{V} &\subseteq X \setminus W \\ &\subseteq U \\ &\subseteq N \end{aligned}$$

⟨1⟩2. If, for every point x and neighbourhood U of x, there exists a neighbourhood V of x such that $\overline{V} \subseteq U$, then X is regular.

⟨2⟩1. Assume: For every point x and neighbourhood U of x, there exists a neighbourhood V of x such that $\overline{V} \subseteq U$.

⟨2⟩2. Let: $x \in X$ and A be a closed set with $x \notin A$

⟨2⟩3. Pick a neighbourhood V of x such that $\overline{V} \subseteq X \setminus A$

⟨2⟩4. $x \in V$ and $A \subseteq X \setminus \overline{V}$

□

Proposition 6.3.3. *Every linearly ordered set under the order topology is regular.*

Proof:

⟨1⟩1. Let: X be a linearly ordered set under the order topology.

⟨1⟩2. Let: $x \in X$ and U be a neighbourhood of x

Prove: There exists a neighbourhood V of x with $\overline{V} \subseteq U$

⟨1⟩3. Case: x is greatest and least in X

Proof: Take $V = U = X = \{x\}$

⟨1⟩4. Case: x is greatest in X and there exists $a < x$ such that $(a, x] \subseteq U$

⟨2⟩1. Case: There exists b such that $a < b < x$

Proof: Take $V = (b, x]$.

⟨2⟩2. Case: There is no b such that $a < b < x$

⟨3⟩1. Let: $V = U = \{x\}$

⟨3⟩2. $\overline{V} = V$

Proof: For any $y \neq x$, we have $(-\infty, x)$ is a neighbourhood of y that does not intersect V.

⟨1⟩5. Case: x is least in X and there exists $b > x$ such that $[x, b) \subseteq U$

Proof: Similar.

⟨1⟩6. Case: There exist $a < x < b$ such that $(a, b) \subseteq U$

⟨2⟩1. Pick a point c such that $a < c < x$ if there is one, otherwise Let: $c = a$

⟨2⟩2. Pick a point d such that $x < d < b$ if there is one, otherwise Let: $d = b$

⟨2⟩3. Let: $V = (c, d)$

⟨2⟩4. $\overline{V} \subseteq U$

PROOF:

$$\begin{aligned}\overline{V} &\subseteq [c,d] \\ &\subseteq (a,b) \\ &\subseteq U\end{aligned}$$

⟨1⟩7. Q.E.D.

PROOF: These cases are exhaustive by Lemma 4.1.2. They prove X is regular by Proposition 6.3.2.

□

Proposition 6.3.4. *A subspace of a regular space is regular.*

PROOF:

⟨1⟩1. LET: X be a regular space and $Y \subseteq X$

⟨1⟩2. LET: $A \subseteq Y$ be closed in Y and $a \in Y \setminus A$

⟨1⟩3. PICK C closed in X such that $A = C \cap Y$

PROOF: By Corollary 4.3.4.1.

⟨1⟩4. PICK disjoint open sets U, V in X such that $C \subseteq U$ and $a \in V$

⟨1⟩5. $U \cap Y$ and $V \cap Y$ are disjoint open sets in Y such that $A \subseteq U \cap Y$ and $a \in V \cap Y$

□

Corollary 6.3.4.1. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of nonempty spaces. If $\prod_{\alpha \in J} X_\alpha$ is regular then so is each X_α.*

Proposition 6.3.5 (AC)**.** *The product of a family of regular spaces is regular.*

PROOF:

⟨1⟩1. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of regular spaces.

⟨1⟩2. $\prod_{\alpha \in J} X_\alpha$ is T_1

⟨1⟩3. LET: $\vec{a} \in U$ where U is open in $\prod_{\alpha \in J} X_\alpha$

⟨1⟩4. PICK $\prod_{\alpha \in J} U_\alpha$ such that each U_α is open in X_α, $U_\alpha = X_\alpha$ except at α_1, …, α_n, and $\vec{a} \in \prod_{\alpha \in J} U_\alpha \subseteq U$

⟨1⟩5. For $1 \leq i \leq n$, PICK V_{α_i} open in X_{α_i} such that $a_{\alpha_i} \in V_{\alpha_i}$ and $\overline{V_{\alpha_i}} \subseteq U_{\alpha_i}$

⟨1⟩6. For $\alpha \neq \alpha_1, \ldots, \alpha_n$,

LET: $V_\alpha = X_\alpha$

⟨1⟩7. $\vec{a} \in \prod_{\alpha \in J} V_\alpha$

⟨1⟩8. $\overline{\prod_{\alpha \in J} V_\alpha} \subseteq \prod_{\alpha \in J} U_\alpha$

PROOF: By Theorem 4.2.5.

□

Corollary 6.3.5.1. *The Sorgenfrey plane is regular.*

Corollary 6.3.5.2. *For any set I, the space $\mathbb{R}^I$ is regular.*

Proposition 6.3.6. *The space $\mathbb{R}_K$ is not regular.*

PROOF:There do not exist disjoint neighbourhoods of 0 and K. □

Proposition 6.3.7. *The continuous image of a regular space is not necessarily regular.*

PROOF: The identity map from the discrete two-point space to the indiscrete two-point space is continuous. ∎

6.4 Completely Regular Spaces

Definition 6.4.1 (Separated by a Continuous Function)**.** Let A and B be subsets of a topological space X. Then A and B can be *separated by a continuous function* iff there exists a continuous $f : X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

Definition 6.4.2 (Completely Regular)**.** A space X is *completely regular* iff X is T_1 and, for every point a and closed set A not containing a, we have that $\{a\}$ and A can be separated by a continuous function.

Theorem 6.4.3. *The product of a family of completely regular spaces is completely regular.*

PROOF:
$\langle 1\rangle 1.$ LET: $\{X_\alpha\}_{\alpha\in J}$ be a family of completely regular spaces.
$\langle 1\rangle 2.$ LET: $a \in \prod_{\alpha\in J} X_\alpha$ and A be closed in $\prod_{\alpha\in J} X_\alpha$ such that $a \notin A$
$\langle 1\rangle 3.$ PICK a basic open neighbourhood $\prod_{\alpha\in J} U_\alpha \subseteq \prod_{\alpha\in J} X_\alpha \setminus A$ of a such that $U_\alpha = X_\alpha$ except for $\alpha = \alpha_1, \ldots, \alpha_n$
$\langle 1\rangle 4.$ For $1 \leq i \leq n$, PICK a continuous $f_i : X_{\alpha_i} \to [0,1]$ that is 0 at a_{α_i} and 1 on $X_{\alpha_i} \setminus U_{\alpha_i}$
$\langle 1\rangle 5.$ LET: $f : \prod_{\alpha\in J} X_\alpha \to [0,1]$ be given by $f(x) = \prod_{i=1}^n f_i(x_{\alpha_i})$
$\langle 1\rangle 6.$ $f(a) = 0$
$\langle 1\rangle 7.$ $f(x) = 1$ for $x \in A$
$\langle 1\rangle 8.$ f is continuous
∎

Corollary 6.4.3.1. *The Sorgenfrey plane is completely regular.*

Corollary 6.4.3.2. *For any set I, the space $\mathbb{R}^I$ is completely regular.*

Proposition 6.4.4. *For any set J, the space $\mathbb{R}^J$ in the box topology is completely regular.*

PROOF:
$\langle 1\rangle 1.$ LET: $a \in \mathbb{R}^J$ and $A \subseteq \mathbb{R}^J$ be closed with $a \notin A$
PROVE: There exists $f : \mathbb{R}^J_{\text{box}} \to [0,1]$ continuous such that $f(a) = 1$ and $f(A) = \{0\}$
$\langle 1\rangle 2.$ ASSUME: w.l.o.g. $A \cap (-1,1)^J = \emptyset$ and $a = \vec{0}$
$\langle 2\rangle 1.$ PICK a basic open set $\prod_{\alpha\in J} U_\alpha$ such that $a \in \prod_{\alpha\in J} U_\alpha \subseteq \mathbb{R}^J \setminus A$
$\langle 2\rangle 2.$ For $\alpha \in J$, PICK b_α, c_α such that $a_\alpha \in (b_\alpha, c_\alpha) \subseteq U_\alpha$
$\langle 2\rangle 3.$ For $\alpha \in J$, PICK a homeomorphism $f_\alpha : \mathbb{R} \to \mathbb{R}$ that maps b_α to -1, a_α to 0 and c_α to 1
$\langle 2\rangle 4.$ $\prod_{\alpha\in J} f_\alpha$ is an automorphism $\mathbb{R}^J_{\text{box}}$ that maps a to $\vec{0}$ and A to a closed set disjoint from $(-1,1)^J$

⟨1⟩3. PICK a continuous function $f : \mathbb{R}^J_{\text{uniform}} \to [0,1]$ such that $f(\vec{0}) = 1$ and $f(\mathbb{R}^J \setminus (-1,1)^J) = \{0\}$
⟨1⟩4. f is continuous w.r.t. the box topology
□

Proposition 6.4.5. *Not every regular space is completely regular.*

PROOF:
⟨1⟩1. For $m \in \mathbb{Z}$,
LET: $L_m = \{m\} \times [-1,0]$
⟨1⟩2. For each odd integer n and each integer $k \geq 2$,
LET: $C_{nk} = (\{n+1-1/k\} home/robin/fun/RogOMatic/src/actuatortimes [-1,0]) \cup (\{n-1+1/k\} \times [-1,0]) \cup \{(x,y) : (x-n)^2 + y^2 = (1-1/k)^2, y \geq 0\}$
⟨1⟩3. For each odd integer n and each integer $k \geq 2$,
LET: $p_{nk} = (n, 1-1/k)$
⟨1⟩4. PICK two points a, b not in any L_m or C_{nk}
⟨1⟩5. LET: $X = \bigcup_{m \in \mathbb{Z}} L_m \cup \bigcup_{n,k} C_{nk} \cup \{a,b\}$
⟨1⟩6. LET: $\mathcal{B}$ be the set consisting of all subsets of $\mathbb{R}^2$ of the following forms:

1. The intersection of X with a horizontal open line segment that contains none of the points p_{nk}
2. A set formed from one of the sets C_{nk} by deleting finitely many points.
3. For each even integer m, the set $\{a\} \cup \{(x,y) \in X : x < m\}$
4. For each even integer m, the set $\{b\} \cup \{(x,y) \in X : x > m\}$

⟨1⟩7. $\mathcal{B}$ is a basis for a topology on X
⟨2⟩1. For all $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$
⟨2⟩2. For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$
⟨3⟩1. CASE: B_1, B_2 are both of type 1
PROOF: Their intersection is of type 1.
⟨3⟩2. CASE: B_1 is of type 1 and B_2 is of type 2
PROOF: Their intersection is of type 2, since a horizontal line segment intersects C_{nk} in at most two points.
⟨3⟩3. CASE: B_1 is of type 1 and B_2 is of type 3
PROOF: Their intersection is of type 1
⟨3⟩4. CASE: B_1 is of type 1 and B_2 is of type 4
PROOF: Their intersection is of type 1
⟨3⟩5. CASE: B_1 is of type 2 and B_2 is of type 2
PROOF: Their intersection is of type 2
⟨3⟩6. CASE: B_1 is of type 2 and B_2 is of type 3
PROOF: Their intersection is B_1
⟨3⟩7. CASE: B_1 is of type 2 and B_2 is of type 4
PROOF: Their intersection is B_1
⟨3⟩8. CASE: B_1 is of type 3 and B_2 is of type 3
PROOF: Their intersection is of type 3
⟨3⟩9. CASE: B_1 is of type 3 and B_2 is of type 4

⟨4⟩1. Let: $B_1 = \{a\} \cup \{(x,y) \in X : x < m\}$ and $B_2 = \{b\} \cup \{(x,y) \in X : x > n\}$

⟨4⟩2. Case: $x = (s, 1 - 1/k)$ for some s and integer $x \geq 2$

Proof: In this case, $x \in C_{nk}$ for some n and $C_{nk} \subseteq B_1 \cap B_2$.

⟨4⟩3. Case: $x = (s,t)$ and $t \neq 1 - 1/k$ for any integer $k \geq 2$

Proof: In this case, $x \in ((n,m) \times \{t\}) \cap X \subseteq B_1 \cap B_2$

⟨3⟩10. Case: B_1 is of type 4 and B_2 is of type 4

Proof: Their intersection is of type 4

⟨1⟩8. For any continuous function $f : X \to \mathbb{R}$, we have $f(a) = f(b)$

⟨2⟩1. Let: $f : X \to \mathbb{R}$ be continuous

⟨2⟩2. For any $c \in \mathbb{R}$, we have $f^{-1}(c)$ is G_δ

Proof: $f^{-1}(c) = \bigcap_{q \in \mathbb{Q}^+} f^{-1}(c - q, c + q)$

⟨2⟩3. Let: $S_{nk} = \{p \in C_{nk} : f(p) \neq f(p_{nk})\}$

⟨2⟩4. For all n, k, we have S_{nk} is countable.

⟨3⟩1. Let: $f^{-1}(p_{nk}) = \bigcap_{m=1}^{\infty} U_m$ where U_m is open in X

⟨3⟩2. For each m, Pick $B_m \in \mathcal{B}$ such that $p_{nk} \in B_m \subseteq U_m$

⟨3⟩3. $S_{nk} \subseteq \bigcup_{m=1}^{\infty} (C_{nk} \setminus B_m)$

⟨3⟩4. Each $C_{nk} \setminus B_m$ is countable

⟨4⟩1. Let: $m \in \mathbb{Z}$

⟨4⟩2. B_m cannot be of type 1

⟨4⟩3. If B_m is of type 2 then $C_{nk} \setminus B_m$ is finite.

⟨4⟩4. If B_m is of type 3 or 4 then $C_{nk} \setminus B_m$ is empty.

⟨2⟩5. Pick $d \in [-1, 0]$ such that $\mathbb{R} \times \{d\}$ intersects none of the sets S_{nk}

⟨2⟩6. For n odd, we have

$$f(n-1, d) = \lim_{k \to \infty} f(p_{nk}) \ .$$

⟨3⟩1. Let: $\epsilon > 0$

⟨3⟩2. Pick $B \in \mathcal{B}$ such that $(n-1, d) \in B \subseteq f^{-1}(f(n-1, d) - \epsilon, f(n-1, d) + \epsilon)$

⟨3⟩3. There exists $\delta > 0$ such that, for $x \in (n - 1 - \delta, n - 1 + \delta)$, we have $(x, d) \in B$

⟨3⟩4. Pick K such that $1/K < \delta$

⟨3⟩5. Let: $k \geq K$

⟨3⟩6. $f(n - 1 + 1/k, d) = f(p_{nk})$

⟨3⟩7. $|f(n-1, d) - f(n - 1 + 1/k, d)| < \epsilon$

⟨3⟩8. $|f(n-1, d) - f(p_{nk})| < \epsilon$

⟨2⟩7. For n odd, we have

$$f(n+1, d) = \lim_{k \to \infty} f(p_{nk}) \ .$$

Proof: Similar.

⟨2⟩8. Q.E.D.

⟨3⟩1. Assume: $f(a) \neq f(b)$

⟨3⟩2. Assume: w.l.o.g. $f(a) < f(b)$

⟨3⟩3. Pick $B \in \mathcal{B}$ such that $a \in B \subseteq f^{-1}(-\infty, (f(a) + f(b))/2)$

⟨3⟩4. Let: m be even such that $B = \{a\} \cup \{(x,y) \in X : x < m\}$

⟨3⟩5. Pick $B \in \mathcal{B}$ such that $b \in B \subseteq f^{-1}((f(a) + f(b))/2, +\infty)$

⟨3⟩6. Let: m' be even such that $B = \{b\} \cup \{(x,y) \in X : x > m'\}$

⟨3⟩7. $f(m,d) = f(m',d)$
⟨3⟩8. Q.E.D.
⟨1⟩9. X is regular.
⟨1⟩10. X is not completely regular.
PROOF: a and b cannot be separated by a continuous function.
□

Theorem 6.4.6 (AC)**.** *A space is completely regular iff it is homeomorphic to a subspace of* $[0,1]^J$ *for some* J*.*

PROOF:
⟨1⟩1. Every completely regular space is homeomorphic to a subspace of $[0,1]^J$ for some J.
⟨2⟩1. LET: X be completely regular
⟨2⟩2. For every point a and open set U that contains a, PICK a continuous function f_{aU} that is positive on a and vanishes outside U
⟨2⟩3. The family $\{f_{aU}\}$ separates points from closed sets
⟨2⟩4. Q.E.D.
PROOF: By the Imbedding Theorem.
⟨1⟩2. Every subspace of $[0,1]^J$ is completely regular.
PROOF: By Theorem 6.4.3 and Proposition 6.3.4.
□

Proposition 6.4.7. *The continuous image of a completely regular space is not necessarily completely regular.*

PROOF: The identity map from the discrete two-point space to the indiscrete two-point space is continuous. □

6.5 Normal Spaces

Definition 6.5.1 (Normal Space)**.** A *normal* space is a T_1 space such that, for any disjoint closed sets A, B, there exist disjoint open sets U, V such that $A \subseteq U$ and $B \subseteq V$.

Theorem 6.5.2. *Every linearly ordered set is normal under the order topology.*

PROOF: See Steen and Steerbach *Counterexamples in Topology* Example 39. □

Proposition 6.5.3. *The product space* $S_\Omega \times \overline{S_\Omega}$ *is not normal.*

PROOF:
⟨1⟩1. LET: $\Delta = \{(x,x) : x \in \overline{S_\Omega}\} \subseteq \overline{S_\Omega} \times \overline{S_\Omega}\}$
⟨1⟩2. Δ is closed in $\overline{S_\Omega} \times \overline{S_\Omega}$
⟨1⟩3. LET: $A = \Delta \cap (S_\Omega \times \overline{S_\Omega})$
⟨1⟩4. A is closed in $S_\Omega \times \overline{S_\Omega}$
⟨1⟩5. LET: $B = S_\Omega \times \{\Omega\}$
⟨1⟩6. B is closed

$\langle 1\rangle 7.$ $A \cap B = \emptyset$
$\langle 1\rangle 8.$ ASSUME: for a contradiction U and V are disjoint open sets including A and B respectively
$\langle 1\rangle 9.$ For all $x \in S_\Omega$ there exists $\beta \in (x, \Omega)$ such that $(x, \beta) \notin U$
$\langle 2\rangle 1.$ LET: $x \in S_\Omega$
$\langle 2\rangle 2.$ $(x, \Omega) \in V$
PROOF: $(x, \Omega) \in B \subseteq V$
$\langle 2\rangle 3.$ PICK $y < \Omega$ such that $\{x\} \times (y, \Omega] \subseteq V$
PROOF: By Lemma 4.1.2.
$\langle 2\rangle 4.$ PICK β such that $x, y < \beta < \Omega$
PROOF: Such a β exists because Ω is a limit ordinal.
$\langle 1\rangle 10.$ For $x \in S_\Omega$,
LET: $\beta(x)$ be the least element of (x, Ω) such that $(x, \beta(x)) \notin U$
$\langle 1\rangle 11.$ LET: $b = \sup_{n=1}^{\infty} \beta^n(0)$
$\langle 1\rangle 12.$ $\beta^n(0) \to b$ as $n \to \infty$
$\langle 1\rangle 13.$ $(\beta^n(0), \beta^{n+1}(0)) \to (b, b)$ as $n \to \infty$
$\langle 1\rangle 14.$ $(b, b) \in A$
$\langle 1\rangle 15.$ $(b, b) \in U$
$\langle 1\rangle 16.$ For all n we have $(\beta^n(0), \beta^{n+1}(0)) \notin U$
PROOF: By $\langle 1\rangle 10$.
$\langle 1\rangle 17.$ Q.E.D.
PROOF: Steps $\langle 1\rangle 12$, $\langle 1\rangle 15$ and $\langle 1\rangle 16$ form a contradiction.
□

Corollary 6.5.3.1. *Not every completely regular space is normal.*

Corollary 6.5.3.2. *An open subspace of a normal space is not necessarily normal.*

Corollary 6.5.3.3. *The product of two normal spaces is not necessarily normal.*

Proposition 6.5.4. *A closed subspace of a normal space is normal.*

PROOF:
$\langle 1\rangle 1.$ LET: X be normal and $C \subseteq X$ be closed.
$\langle 1\rangle 2.$ LET: A and B be closed in C
$\langle 1\rangle 3.$ A and B are closed in X
PROOF: By Corollary 4.3.4.2.
$\langle 1\rangle 4.$ PICK disjoint open neighbourhoods U and V of A and B in X
$\langle 1\rangle 5.$ $U \cap C$ and $V \cap C$ are disjoint open neighourhoods of A and B in C
□

Corollary 6.5.4.1. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of nonempty spaces. If $\prod_{\alpha \in J} X_\alpha$ is normal then each X_α is normal.*

Proposition 6.5.5. *If the Continuum Hypothesis then $\mathbb{R}^\omega$ under the box topology is normal.*

PROOF: See Rudin. The box product of countably many compact metric spaces. *General Topology and Its Applications*, 2:293–298, 1972. □

Proposition 6.5.6 (Stone (DC))**.** *If J is uncountable then $\mathbb{R}^J$ is not normal.*

PROOF:
⟨1⟩1. LET: $X = (\mathbb{Z}^+)^J$
PROVE: X is not normal.
⟨1⟩2. For $x \in X$ and $B \subseteq^{\text{fin}} J$,
LET:
$$U(x, B) = \{y \in X : \forall \alpha \in B . y_\alpha = x_\alpha\} \enspace .$$
⟨1⟩3. $\{U(x, B) : x \in X, B \subseteq^{\text{fin}} J\}$ is a basis for X
⟨2⟩1. LET: $x \in X$ and $\prod_{\alpha \in J} U_\alpha$ be a basic open set including x, where $U_\alpha = \mathbb{Z}^+$ for all α except $\alpha_1, \ldots, \alpha_n$
⟨2⟩2. $x \in U(x, \{\alpha_1, \ldots, \alpha_n\}) \subseteq \prod_{\alpha \in J} U_\alpha$
⟨1⟩4. For $n \in \mathbb{Z}^+$,
LET: $P_n = \{x \in X : x \text{ is injective on } J \setminus x^{-1}(n)\}$
⟨1⟩5. P_1 and P_2 are closed and disjoint.
⟨2⟩1. P_1 is closed
⟨3⟩1. LET: $x \in X \setminus P_1$
⟨3⟩2. PICK $\alpha, \beta \in J$ such that $x_\alpha = x_\beta \neq 1$
⟨3⟩3. LET: $U_\gamma = \{x_\alpha\}$ if $\gamma = \alpha$ or $\gamma = \beta$, $\mathbb{Z}^+$ for all other $\gamma \in J$
⟨3⟩4. $x \in \prod_{\gamma \in J} U_\gamma \subseteq X \setminus P_1$
⟨2⟩2. P_2 is closed
PROOF: Similar.
⟨2⟩3. $P_1 \cap P_2 = \emptyset$
PROOF: If $x \in P_1 \cap P_2$ then x is injective on J, contradicting the fact that J is uncountable.
⟨1⟩6. ASSUME: for a contradiction U and V are disjoint open sets including P_1 and P_2
⟨1⟩7. Given a sequence (α_i) of distinct elements of J and a strictly increasing sequence (n_i) of positive integers,
LET:
$$B_i^{\alpha,n} = \{\alpha_1, \ldots, \alpha_{n_i}\}$$
$$x_i^{\alpha,n} \in X$$
$$(x_i^{\alpha,n})_\beta = \begin{cases} j & \text{if } \beta = \alpha_j, 1 \leq j \leq n_{i-1} \\ 1 & \text{for all other values of } \beta \end{cases}$$
for $i \geq 1$
⟨1⟩8. PICK sequences (α_i), (n_i) such that, for all $i \geq 1$, we have $U(x_i^{\alpha,n}, B_i^{\alpha,n}) \subseteq U$
⟨2⟩1. LET: $x_1 \in X$ be given by $(x_1)_\alpha = 1$ for all $\alpha \in J$
⟨2⟩2. $x_1 \in U$
PROOF: $x_1 \in P_1 \subseteq U$
⟨2⟩3. PICK $B_1 \subseteq^{\text{fin}} J$ such that $U(x_1, B_1) \subseteq U$
PROOF: By ⟨1⟩3.
⟨2⟩4. LET: $n_1 = |B_1|$ and $B_1 = \{\alpha_1, \ldots, \alpha_{n_1}\}$
⟨2⟩5. ASSUME: We have chosen $n_1, \ldots, n_k$ strictly increasing and $\alpha_1, \ldots, \alpha_{n_k}$ such that, for $1 \leq i \leq k$, we have $U(x_i^{\alpha,n}, B_i^{\alpha,n}) \subseteq U$

⟨2⟩6. $x^{\alpha,n}_{i+1} \in U$
PROOF: $x^{\alpha,n}_{i+1} \in P_1 \subseteq U$
⟨2⟩7. PICK $C \subseteq^{\mathrm{fin}} J$ such that $U(x^{\alpha,n}_{i+1}, C) \subseteq U$
⟨2⟩8. LET: n_{i+1} and $\alpha_{n_i+1}, \ldots, \alpha_{n_{i+1}}$ be such that $B^{\alpha,n}_i \cup C = B^{\alpha,n}_{i+1}$
⟨2⟩9. $U(x^{\alpha,n}_{i+1}, B^{\alpha,n}_{i+1}) \subseteq U$
⟨1⟩9. LET: $A = \{\alpha_i : i \geq 1\}$
⟨1⟩10. LET: $y \in X$, $y_\beta = j$ if $\beta = \alpha_j$, $y_\beta = 2$ for $\beta \notin A$
⟨1⟩11. PICK B such that $U(y, B) \subseteq V$
⟨1⟩12. PICK i such that $A \cap B \subseteq B^{\alpha,n}_i$
⟨1⟩13. $U(x^{\alpha,n}_{i+1}, B^{\alpha,n}_{i+1}) \cap U(y, B) \neq \emptyset$
PROOF: $x^{\alpha,n}_{i+1} \in U(x^{\alpha,n}_{i+1}, B^{\alpha,n}_{i+1}) \cap U(y, B)$
⟨1⟩14. Q.E.D.
PROOF: This contradicts the fact that U and V are disjoint (⟨1⟩6).
□

Theorem 6.5.7 (Urysohn Lemma)**.** *Let X be a normal space. Let A and B be disjoint closed subsets of X. Then there exists a continuous map $f : X \to [0, 1]$ such that $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$.*

PROOF:
⟨1⟩1. LET: P be the set of all rational numbers in $[0, 1]$
⟨1⟩2. For all $q \in P$, PICK an open set U_q in X such that $A \subseteq U_0$, $U_1 \subseteq X \setminus B$, and whenever $p < q$ then $\overline{U_p} \subseteq U_q$
⟨2⟩1. PICK an enumeration (q_n) of P such that $q_1 = 1$ and $q_2 = 0$
⟨2⟩2. LET: $U_1 = X \setminus B$
⟨2⟩3. PICK an open set U_0 such that $A \subseteq U_0$ and $\overline{U_0} \subseteq U_1$
⟨2⟩4. ASSUME: we have open sets U_1, U_0, …, U_{q_n} such that whenever $p < q$ then $\overline{U_p} \subseteq U_q$
⟨2⟩5. $q_2 < q_{n+1} < q_1$
⟨2⟩6. LET: q_k be greatest among q_1, …, q_n such that $q_k < q_{n+1}$, and q_l be least such that $q_{n+1} < q_l$
⟨2⟩7. PICK an open set $U_{q_{n+1}}$ such that $\overline{U_{q_k}} \subseteq U_{q_{n+1}}$ and $\overline{U_{q_{n+1}}} \subseteq U_{q_l}$
⟨2⟩8. For all $p, q \in \{q_1, \ldots, q_{n+1}\}$, if $p < q$ then $\overline{U_p} \subseteq U_q$
⟨1⟩3. Extend the family (U_q) to $\mathbb{Q}$ by defining: $U_q = \emptyset$ if $q < 0$ and $U_q = X$ if $q > 1$
⟨1⟩4. For all rationals p, q with $p < q$ we have $\overline{U_p} \subseteq U_q$
⟨1⟩5. Define $f : X \to [0, 1]$ by $f(x) = \inf\{q \in \mathbb{Q} : x \in U_q\}$
PROOF: This set is nonempty since $x \in U_1$ and bounded below since if $x \in U_q$ then $q \geq 0$.
⟨1⟩6. For all $x \in A$ we have $f(x) = 0$
⟨1⟩7. For all $x \in B$ we have $f(x) = 1$
⟨1⟩8. If $x \in \overline{U_r}$ then $f(x) \leq r$
⟨1⟩9. If $x \notin U_r$ then $f(x) \geq r$
⟨1⟩10. f is continuous
⟨2⟩1. LET: $x_0 \in X$
⟨2⟩2. LET: (c, d) be an open interval containing $f(x_0)$
PROVE: There exists a neighbourhood U of x_0 such that $f(U) \subseteq (c, d)$

⟨2⟩3. PICK rationals p, q such that $c < p < f(x_0) < q < d$

⟨2⟩4. $x \notin \overline{U_p}$

PROOF: By ⟨1⟩8

⟨2⟩5. $x \in U_q$

PROOF: By ⟨1⟩9

⟨2⟩6. LET: $U = U_q \setminus \overline{U_p}$

□

Definition 6.5.8 (Vanish Precisely). Let X be a set and $A \subseteq X$. Let $f : X \to [0,1]$. Then f *vanishes precisely* on A iff $f^{-1}(0) = A$.

Theorem 6.5.9 (CC). *Let X be a normal space and $A \subseteq X$. Then there exists a continuous function $f : X \to [0,1]$ such that f vanishes precisely on A if and only if A is a closed G_δ set.*

PROOF:

⟨1⟩1. If there exists f such that f vanishes precisely on A then A is closed.

PROOF: This holds because $A = f^{-1}(0)$.

⟨1⟩2. If there exists f such that f vanishes precisely on A then A is G_δ.

PROOF: This holds because $A = \bigcap_{q \in \mathbb{Q}^+} f^{-1}([0,q))$.

⟨1⟩3. If A is closed and G_δ then there exists f that vanishes precisely on A.

⟨2⟩1. LET: $A = \bigcap_{n=1}^{\infty} U_n$

⟨2⟩2. For $n \geq 1$, PICK $f_n : X \to [0, 1/2^n]$ such that $f(x) = 0$ for $x \in A$ and $f(x) = 1/2^n$ for $x \in X \setminus U_n$

PROOF: By the Urysohn Lemma.

⟨2⟩3. LET: $f : X \to [0,1]$ be given by $f(x) = \sum_{n=1}^{\infty} f_n(x)$

PROOF: The series converges for every x by the Comparison Test.

⟨2⟩4. f is continuous

⟨3⟩1. f_n converges uniformly to f

PROOF: By the Weierstrass M-test.

⟨3⟩2. Q.E.D.

PROOF: By the Uniform Limit Theorem.

⟨2⟩5. $f(x) = 0$ for $x \in A$

PROOF: From ⟨2⟩2.

⟨2⟩6. $f(x) > 0$ for $x \notin A$

⟨3⟩1. LET: $x \notin A$

⟨3⟩2. PICK N such that $x \notin U_N$

⟨3⟩3. Q.E.D.

PROOF:

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} f_n(x) && (\langle 2 \rangle 3) \\ &\geq f_N(x) \\ &> 0 && (\langle 2 \rangle 2) \end{aligned}$$

□

Theorem 6.5.10 (Strong Form of Urysohn Lemma). *Let X be a normal space.*

Then there exists a continuous function $f : X \to [0,1]$ such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$ if and only if A and B are disjoint, closed and G_δ.

PROOF:
⟨1⟩1. If there exists a continuous function $f : X \to [0,1]$ such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$ then A and B are disjoint, closed and G_δ
⟨2⟩1. ASSUME: there exists a continuous function $f : X \to [0,1]$ such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$
⟨2⟩2. A and B are disjoint
⟨2⟩3. A is closed and G_δ
PROOF: By Theorem 6.5.9.
⟨2⟩4. B is closed and G_δ
PROOF: Apply Theorem 6.5.9 to $1 - f$.
⟨1⟩2. If A and B are disjoint, closed and G_δ then there exists a continuous function $f : X \to [0,1]$ such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$
⟨2⟩1. ASSUME: A and B are disjoint, closed and G_δ
⟨2⟩2. PICK $g : X \to [0,1]$ that vanishes precisely on A and $h : X \to [0,1]$ that vanishes precisely on B
⟨2⟩3. LET: $f = g/(g+h)$
□

Definition 6.5.11 (Universal Extension Property)**.** A topological space Y has the *universal extension property* iff, for every normal space X and closed subspace A of X, every continuous function $A \to Y$ can be extended to a continuous function $X \to Y$.

Theorem 6.5.12 (Tietze Extension Theorem (DC))**.** *Let X be a normal space. Let A be closed subspace of X.*

1. *Any continuous function $A \to [a,b]$ can be extended to a continuous function $X \to [a,b]$.*
2. *Any continuous function $A \to \mathbb{R}$ can be extend to a continuous function $X \to \mathbb{R}$.*

PROOF:
⟨1⟩1. Any continuous function $A \to [-1,1]$ can be extended to a continuous function $X \to [-1,1]$
⟨2⟩1. For every continuous function $f : A \to [-r,r]$, there exists a continuous $g : X \to \mathbb{R}$ such that

$$|g(x)| \leq \frac{1}{3}r \qquad (x \in X)$$
$$|g(x) - f(x)| \leq \frac{2}{3}r \qquad (x \in A)$$

⟨3⟩1. LET: $f : A \to [-r,r]$ be continuous
⟨3⟩2. LET: $I_1 = [-r, -\frac{1}{3}r]$
⟨3⟩3. LET: $I_2 = [-\frac{1}{3}r, \frac{1}{3}r]$
⟨3⟩4. LET: $I_3 = [\frac{1}{3}r, r]$

⟨3⟩5. LET: $B = f^{-1}(I_1)$
⟨3⟩6. LET: $C = f^{-1}(I_3)$
⟨3⟩7. PICK a continuous $g : X \to [-\frac{1}{3}r, \frac{1}{3}r]$ such that $g(x) = -\frac{1}{3}r$ for $x \in B$ and $g(x) = \frac{1}{3}r$ for $x \in C$
PROOF: By the Urysohn Lemma, since B and C are closed disjoint subsets of X.
⟨3⟩8. For all $x \in A$ we have $|g(x) - f(x)| \leq \frac{2}{3}r$
⟨4⟩1. LET: $x \in A$
⟨4⟩2. CASE: $f(x) \in I_1$
PROOF:

$$\begin{aligned}
|g(x) - f(x)| &= \left| -\frac{1}{3}r - f(x) \right| && (x \in B) \\
&\leq \frac{2}{3}r && (f(x) \in I_1)
\end{aligned}$$

⟨4⟩3. CASE: $f(x) \in I_2$
PROOF: In this case, $|g(x) - f(x)| \leq \frac{2}{3}r$ since $f(x), g(x) \in I_2$.
⟨4⟩4. CASE: $f(x) \in I_3$
PROOF:

$$\begin{aligned}
|g(x) - f(x)| &= \left| \frac{1}{3}r - f(x) \right| && (x \in C) \\
&\leq \frac{2}{3}r && (f(x) \in I_3)
\end{aligned}$$

⟨2⟩2. LET: $f : A \to [-1, 1]$ be continuous.
⟨2⟩3. PICK a sequence of functions (g_n) such that

$$\begin{aligned}
|g_n(x)| &\leq \frac{1}{3}\left(\frac{2}{3}\right)^{n-1} && (x \in X) \\
|f(x) - g_1(x) - \cdots - g_n(x)| &\leq (2/3)^n && (x \in A)
\end{aligned}$$

PROOF:Given $g_1, \ldots, g_n$, we apply ⟨2⟩1 with $f = f - g_1 - \cdots - g_n$ and $r = (2/3)^n$.
⟨2⟩4. LET: $g(x) = \sum_{n=1}^{\infty} g_n(x)$ for $x \in X$
PROOF: This series converges by the Comparison Test since $\sum_{n=1}^{\infty}(2/3)^n$ converges.
⟨2⟩5. g is continuous.
⟨3⟩1. $\sum_{n=1}^{N} g_n$ converges to g uniformly
PROOF: By the Weierstrass M-test.
⟨3⟩2. Q.E.D.
PROOF: By the Uniform Limit Theory.
⟨2⟩6. For all $x \in A$ we have $g(x) = f(x)$
PROOF: $|\sum_{n=1}^{N} g_n(x) - f(x)| \leq (2/3)^N \to 0$ as $N \to \infty$.
⟨2⟩7. For all $x \in X$ we have $-1 \leq g(x) \leq 1$

PROOF:

$$\begin{aligned}|\sum_{n=1}^{N} g_n(x)| &\leq \sum_{n=1}^{N} |g_n(x)| \\ &\leq 1/3 \sum_{n=1}^{N} (2/3)^{n-1} \\ &\to 2/3 && \text{as } n \to \infty\end{aligned}$$

⟨1⟩2. Any continuous function $A \to (-1,1)$ can be extend to a continuous function $X \to (-1,1)$

⟨2⟩1. LET: $f : A \to (-1,1)$ be continuous

⟨2⟩2. PICK a continuous $g : X \to [-1,1]$ that extends f

PROOF: By ⟨1⟩1.

⟨2⟩3. LET: $D = g^{-1}(-1) \cup g^{-1}(1)$

⟨2⟩4. D is closed in X

PROOF: Since g is continuous and $\{-1\}$, $\{1\}$ are closed in $[-1,1]$.

⟨2⟩5. $D \cap A = \emptyset$

PROOF: Since $g(A) = f(A) \subseteq (-1,1)$.

⟨2⟩6. PICK a continuous $\phi : X \to [0,1]$ such that $\phi(D) = \{0\}$ and $\phi(A) = \{1\}$

PROOF: By the Urysohn Lemma.

⟨2⟩7. LET: $h = g\phi$

⟨2⟩8. h is continuous

⟨2⟩9. h extends f

⟨2⟩10. $\operatorname{im} h \subseteq (-1,1)$

⟨1⟩3. Q.E.D.

PROOF: The result follows because any closed interval in $\mathbb{R}$ is homeomorphic to $[-1,1]$ and $\mathbb{R} \cong (-1,1)$.

□

Lemma 6.5.13 (Shrinking Lemma (AC)). *Let X be a normal space. Let $\{U_\alpha\}_{\alpha \in J}$ be a point-finite indexed open covering of X. Then there exists an indexed open covering $\{V_\alpha\}_{\alpha \in J}$ such that $\overline{V_\alpha} \subseteq U_\alpha$ for all $\alpha \in J$.*

PROOF:

⟨1⟩1. PICK a well-ordering $\prec$ on J

⟨1⟩2. PICK open sets V_α for $\alpha \in J$ such that $A_\alpha \subseteq V_\alpha$ and $\overline{V_\alpha} \subseteq U_\alpha$, where

$$A_\alpha = X \setminus \bigcup_{\beta \prec \alpha} V_\beta \cup \bigcup_{\alpha \prec \beta} U_\beta$$

PROOF: Apply transfinite induction to Proposition 13.1.16.

⟨1⟩3. $\{V_\alpha\}_{\alpha \in J}$ covers X

⟨2⟩1. LET: $x \in X$

⟨2⟩2. LET: $\alpha_1, \ldots, \alpha_n$ be the elements of J such that $x \in U_{\alpha_i}$, where $\alpha_1 \prec \cdots \prec \alpha_n$

PROVE: $x \in V_{\alpha_i}$ for some i

⟨2⟩3. ASSUME: $x \notin V_{\alpha_1}, \ldots, V_{\alpha_{n-1}}$

⟨2⟩4. $x \in A_{\alpha_n}$

⟨2⟩5. $x \in V_{\alpha_n}$

□

Proposition 6.5.14 (DC). *$S_\Omega \times \overline{S_\Omega}$ is not normal.*

Proof:

$\langle 1\rangle 1$. Let: $\Delta = \{(x,x) : x \in \overline{S_\Omega}\}$

$\langle 1\rangle 2$. Δ is closed in $\overline{S_\Omega}^2$

$\langle 2\rangle 1$. Let: $(x,y) \in \overline{S_\Omega}^2 \setminus \Delta$

$\langle 2\rangle 2$. Pick disjoint open sets U, V such that $x \in U$ and $y \in V$

$\langle 2\rangle 3$. $(x,y) \in U \times V \subseteq \overline{S_\Omega}^2 \setminus \Delta$

$\langle 1\rangle 3$. Let: $A = \Delta \cap (S_\Omega \times \overline{S_\Omega})$

$\langle 1\rangle 4$. A is closed in $S_\Omega \times \overline{S_\Omega}$

$\langle 1\rangle 5$. Let: $B = S_\Omega \times \{\Omega\}$

$\langle 1\rangle 6$. B is closed in $S_\Omega \times \overline{S_\Omega}$

$\langle 1\rangle 7$. $A \cap B = \emptyset$

$\langle 1\rangle 8$. Assume: for a contradiction U and V are disjoint open sets including A and B respectively

$\langle 1\rangle 9$. Pick a sequence x_n in S_Ω such that $x_n < x_{n+1} < \Omega$ and $(x_n, x_{n+1}) \notin U$ for all n

$\langle 2\rangle 1$. Let: $x_n \in S_\Omega$

$\langle 2\rangle 2$. $(x_n, \Omega) \in V$

$\langle 2\rangle 3$. Pick open sets $W \subseteq S_\Omega$, $X \subseteq \overline{S_\Omega}$ such that $x_n \in W$, $\Omega \in X$ and $W \times X \subseteq V$

$\langle 2\rangle 4$. Pick $y < \Omega$ such that $(x_{n+1}, \Omega] \subseteq X$

$\langle 2\rangle 5$. Let: $x_{n+1} = y + 1$

$\langle 1\rangle 10$. Let: b be the supremum of $\{x_n : n \geq 1\}$

$\langle 1\rangle 11$. $(x_n, x_{n+1}) \to (b,b)$ as $n \to \infty$

$\langle 1\rangle 12$. $(b,b) \in A$

$\langle 1\rangle 13$. $(b,b) \in U$

$\langle 1\rangle 14$. For all n we have $(x_n, x_{n+1}) \notin U$

□

Proposition 6.5.15 (AC). *$\mathbb{R}_l$ is normal.*

Proof:

$\langle 1\rangle 1$. Let: A and B be disjoint closed sets in $\mathbb{R}_l$

$\langle 1\rangle 2$. For $a \in A$, Pick $x_a > a$ such that $[a, x_a)$ not intersecting B

$\langle 1\rangle 3$. For $b \in B$, Pick $x_b > b$ such that $[b, x_b)$ does not intersect A

$\langle 1\rangle 4$. Let: $U = \bigcup_{a \in A} [a, x_a)$ and $V = \bigcup_{b \in B} [b, x_b)$

$\langle 1\rangle 5$. U and V are disjoint open sets including A and B respectively.

□

Lemma 6.5.16. *The set $L = \{(x, -x); x \in \mathbb{R}\}$ as a subspace of $\mathbb{R}_l^2$ is closed*

$\langle 1\rangle 1$. Let: $(x,y) \notin L$, so $y \neq -x$

Prove: There exists a neighbourhood U of (x,y) that does not intersect L

$\langle 1\rangle 2$. Case: $y > -x$

PROOF: In this case, take $U = [x, +\infty) \times [y, +\infty)$
$\langle 1 \rangle 3$. CASE: $y < -x$
PROOF: In this case, take $U = [x, (x - y)/2) \times [y, (y - x)/2)$.

Proposition 6.5.17 (AC)**.** *The Sorgenfrey plane is not normal.*

PROOF:
$\langle 1 \rangle 1$. ASSUME: for a contradiction the Sorgenfrey plane is normal.
$\langle 1 \rangle 2$. LET: $L = \{(x, -x); x \in \mathbb{R}\}$ as a subspace of $\mathbb{R}_l^2$
$\langle 1 \rangle 3$. L has the discrete topology.
$\langle 2 \rangle 1$. LET: $(x, -x) \in L$
PROVE: $\{(x, -x)\}$ is open in L
$\langle 2 \rangle 2$. $\{(x, -x)\} = ([x, +\infty) \times [-x, +\infty)) \cap L$
$\langle 1 \rangle 4$. Every subset of L is closed in $\mathbb{R}_l^2$
PROOF: By Corollary 4.3.4.2.
$\langle 1 \rangle 5$. For every nonempty proper subset A of L, PICK disjoint open sets U_A, V_A containing A and $L \setminus A$
PROOF: By $\langle 1 \rangle 1$ and $\langle 1 \rangle 4$.
$\langle 1 \rangle 6$. LET: $D = \mathbb{Q}^2$
$\langle 1 \rangle 7$. D is dense in $\mathbb{R}_l^2$
PROOF: Given any basic open set $[a, b) \times [c, d)$, pick rationals q, r such that $a \leq q < b$ and $c \leq r < d$. Then $(q, r) \in ([a, b) \times [c, d)) \cap D$
$\langle 1 \rangle 8$. LET: $\theta : \mathcal{P}L \to \mathcal{P}D$ be the function

$$\theta(A) = U_A \cap D \qquad\qquad (\emptyset \neq A \neq L)$$
$$\theta(\emptyset) = \emptyset$$
$$\theta(L) = D$$

$\langle 1 \rangle 9$. θ is injective
$\langle 2 \rangle 1$. LET: $A, B \subseteq L$ with $\theta(A) = \theta(B)$
PROVE: $A = B$
$\langle 2 \rangle 2$. CASE: $\emptyset \neq A \neq L$ and $\emptyset \neq B \neq L$
$\langle 3 \rangle 1$. $A \subseteq B$
$\langle 4 \rangle 1$. LET: $x \in A$
$\langle 4 \rangle 2$. $x \in U_A$
PROOF: By $\langle 1 \rangle 5$
$\langle 4 \rangle 3$. $x \in U_B$
PROOF: By $\langle 2 \rangle 1$
$\langle 4 \rangle 4$. $x \notin L \setminus B$
PROOF: By $\langle 1 \rangle 5$
$\langle 4 \rangle 5$. $x \in B$
PROOF: Since $x \in L$ by $\langle 4 \rangle 1$
$\langle 3 \rangle 2$. $B \subseteq A$
PROOF: Similar.
$\langle 2 \rangle 3$. CASE: $\emptyset \neq A \neq L$ and $B = \emptyset$
PROOF: This implies $U_A \cap D = \emptyset$ which contradicts the fact that D is dense.
$\langle 2 \rangle 4$. CASE: $\emptyset \neq A \neq L$ and $B = L$
PROOF: This implies $V_A \cap D = \emptyset$ which contradicts the fact that D is dense.

⟨2⟩5. CASE: $A = B = \emptyset$

PROOF: Trivial

⟨2⟩6. CASE: $A = \emptyset$ and $B = L$

PROOF: This implies $D = \emptyset$ which is a contradiction.

⟨2⟩7. CASE: $A = B = L$

PROOF: Trivial

⟨1⟩10. Q.E.D.

PROOF: This is a contradiction since D is countable and L is uncountable.

□

Proposition 6.5.18. *The continuous image of a normal space is not necessarily normal.*

PROOF: The identity map from the discrete two-point space to the indiscrete two-point space is continuous. □

Lemma 6.5.19. *Let X be a regular space with a countably locally finite basis. Then X is normal and every closed set is G_δ.*

PROOF:

⟨1⟩1. LET: X be regular with a countably locally finite basis.

⟨1⟩2. For every open set W, there exists a countable set $\mathcal{U}$ of open sets such that $W = \bigcup \mathcal{U} = \bigcup_{U \in \mathcal{U}} \overline{U}$

⟨2⟩1. PICK a locally finite set $\mathcal{B}_n$ for $n \in \mathbb{N}$ such that $\bigcup_{n=0}^{\infty} \mathcal{B}_n$ is a basis.

PROOF: By ⟨1⟩1.

⟨2⟩2. For $n \in \mathbb{N}$,

LET: $\mathcal{C}_n = \{B \in \mathcal{B}_n : \overline{B} \subseteq W\}$

⟨2⟩3. For $n \in \mathbb{N}$, $\mathcal{C}_n$ is locally finite.

PROOF: This holds because $\mathcal{C}_n \subseteq \mathcal{B}_n$ (⟨2⟩1, ⟨2⟩2).

⟨2⟩4. For $n \in \mathbb{N}$,

LET: $U_n = \bigcup \mathcal{C}_n$

⟨2⟩5. For $n \in \mathbb{N}$, U_n is open.

PROOF: This holds because every element of $\mathcal{C}_n$ is open (⟨2⟩1, ⟨2⟩2, ⟨2⟩4).

⟨2⟩6. For $n \in \mathbb{N}$, $\overline{U_n} = \bigcup_{B \in \mathcal{C}_n} \overline{B}$

PROOF: By Lemma 3.12.10.

⟨2⟩7. For $n \in \mathbb{N}$, $\overline{U_n} \subseteq W$

PROOF: From ⟨2⟩2 and ⟨2⟩6.

⟨2⟩8. $W \subseteq \bigcup_{n=0}^{\infty} U_n$

⟨3⟩1. LET: $x \in W$

⟨3⟩2. PICK a neighbourhood U of x such that $\overline{U} \subseteq W$

PROOF: By Proposition 6.3.2 and ⟨3⟩1 since X is regular (⟨1⟩1).

⟨3⟩3. PICK $n \in \mathbb{N}$ and $B \in \mathcal{B}_n$ such that $x \in B \subseteq U$

PROOF: By ⟨2⟩1 and ⟨3⟩2.

⟨3⟩4. $B \in \mathcal{C}_n$

⟨4⟩1. $\overline{B} \subseteq W$

PROOF:

$$\begin{aligned} \overline{B} &\subseteq \overline{U} && \text{(Proposition 3.12.5, ⟨3⟩3)} \\ &\subseteq W && \text{(⟨3⟩2)} \end{aligned}$$

⟨4⟩2. Q.E.D.

PROOF: ⟨2⟩2, ⟨3⟩3, ⟨4⟩1

⟨3⟩5. $x \in U_n$

PROOF: ⟨2⟩4, ⟨3⟩3, ⟨3⟩4.

⟨1⟩3. Every closed set is G_δ

PROOF:

⟨2⟩1. LET: C be closed

⟨2⟩2. PICK a countable set $\mathcal{U}$ of open sets such that $X \setminus C = \bigcup \mathcal{U} = \bigcup_{U \in \mathcal{U}} \overline{U}$

PROOF: By ⟨1⟩2

⟨2⟩3. $C = \bigcap_{U \in \mathcal{U}} X \setminus \overline{U}$

PROOF: From ⟨2⟩2 and De Morgan's laws.

⟨1⟩4. X is normal

⟨2⟩1. LET: C and D be disjoint closed sets.

⟨2⟩2. PICK a countable sequence of open sets U_n such that $X \setminus D = \bigcup_{n=0}^{\infty} U_n = \bigcup_{n=0}^{\infty} \overline{U_n}$

PROOF: By ⟨1⟩2 and ⟨2⟩1.

⟨2⟩3. PICK a countable sequence of open sets V_n such that $X \setminus C = \bigcup_{n=0}^{\infty} V_n = \bigcup_{n=0}^{\infty} \overline{V_n}$

PROOF: By ⟨1⟩2 and ⟨2⟩1.

⟨2⟩4. For $n \in \mathbb{N}$,

LET: $U'_n = U_n \setminus \bigcup_{i=0}^{n} \overline{V_i}$

⟨2⟩5. For $n \in \mathbb{N}$,

LET: $V'_n = V_n \setminus \bigcup_{i=0}^{n} \overline{U_i}$

⟨2⟩6. LET: $U = \bigcup_{n=0}^{\infty} U'_n$

⟨2⟩7. LET: $V = \bigcup_{n=0}^{\infty} V'_n$

⟨2⟩8. U is open

⟨3⟩1. For each n, U'_n is open

⟨4⟩1. LET: $n \in \mathbb{N}$

⟨4⟩2. U_n is open

PROOF: By ⟨2⟩2.

⟨4⟩3. $\bigcup_{i=0}^{n} \overline{V_i}$ is closed

PROOF: By Proposition 3.6.4 and Proposition 3.12.3.

⟨4⟩4. Q.E.D.

PROOF: Since $U'_n = U_n \cap (X \setminus \bigcup_{i=0}^{n} \overline{V_i})$

⟨3⟩2. Q.E.D.

PROOF: By ⟨2⟩6

⟨2⟩9. V is open

PROOF: Similar.

⟨2⟩10. $U \cap V = \emptyset$

⟨3⟩1. ASSUME: for a contradiction $x \in U \cap V$

⟨3⟩2. PICK m, n such that $x \in U'_m$ and $x \in V'_n$

PROOF: ⟨2⟩6, ⟨2⟩7, ⟨3⟩1

⟨3⟩3. ASSUME: w.l.o.g. $m \leq n$

⟨3⟩4. $x \in V'_n$ and $x \in U_m$

PROOF: From ⟨2⟩4 and ⟨3⟩2.

⟨3⟩5. Q.E.D.

PROOF: This contradicts $\langle 2\rangle 5$.

$\langle 2\rangle 11.$ $C \subseteq U$

$\langle 3\rangle 1.$ LET: $x \in C$

$\langle 3\rangle 2.$ $x \in X \setminus D$

PROOF: By $\langle 2\rangle 1$ and $\langle 3\rangle 1$.

$\langle 3\rangle 3.$ PICK n such that $x \in U_n$

PROOF: By $\langle 2\rangle 2$ and $\langle 3\rangle 2$.

$\langle 3\rangle 4.$ $x \in U'_n$

$\langle 4\rangle 1.$ For all i, $x \notin V_i$

PROOF: From $\langle 2\rangle 3$ and $\langle 3\rangle 4$.

$\langle 4\rangle 2.$ Q.E.D.

PROOF: From $\langle 2\rangle 4$ and $\langle 3\rangle 3$ and $\langle 4\rangle 1$.

$\langle 3\rangle 5.$ Q.E.D.

PROOF: By $\langle 2\rangle 6$.

$\langle 2\rangle 12.$ $D \subseteq V$

PROOF: Similar.

□

Lemma 6.5.20. *Let X be a normal space. Let A be a closed G_δ set in X. Then there exists a continuous $f : X \to [0,1]$ such that $f(x) = 0$ for $x \in A$ and $f(x) > 0$ for $x \notin A$.*

PROOF:

$\langle 1\rangle 1.$ LET: X be a normal space.

$\langle 1\rangle 2.$ LET: A be a closed G_δ set in X.

$\langle 1\rangle 3.$ PICK open sets U_n such that $A = \bigcup_{n=0}^{\infty} U_n$

PROOF: From $\langle 1\rangle 2$

$\langle 1\rangle 4.$ For $n \in \mathbb{N}$, PICK $f_n : X \to [0,1]$ continuous such that $f(x) = 0$ for $x \in A$ and $f(x) = 1$ for $x \notin U_n$

PROOF: By the Urysohn lemma, $\langle 1\rangle 1$, $\langle 1\rangle 2$ and $\langle 1\rangle 3$.

$\langle 1\rangle 5.$ LET: $f : X \to [0,1]$ with $f(x) = \sum_{n=0}^{\infty} f_n(x)/2^{n+1}$

PROOF: The sequence converges by the Comparison Test with $\sum_{n=0}^{\infty} 1/2^{n+1}$.

$\langle 1\rangle 6.$ f is continuous

PROOF: By the Weierstrass M-test and the Uniform Limit Theorem.

$\langle 1\rangle 7.$ f vanishes on A

$\langle 1\rangle 8.$ f is positive on $X \setminus A$

□

6.6 Completely Normal Spaces

Definition 6.6.1 (Completely Normal)**.** A space X is *completely normal* iff every subspace is normal.

Proposition 6.6.2. *A subspace of a completely normal space is completely normal.*

PROOF: Immediate from definitions. □

Proposition 6.6.3. *Let X be a topological space. Then X is completely normal iff X is T_1 and, for any pair of separated sets A, B in X, there exist disjoint open sets including them.*

PROOF:
⟨1⟩1. If X is completely normal then X is T_1 and, for any pair of separated sets A, B in X, there exist disjoint open sets including them.
⟨2⟩1. ASSUME: X is completely normal.
⟨2⟩2. X is T_1
PROOF: Holds because X is normal.
⟨2⟩3. For any pair of separated sets A, B in X, there exist disjoint open sets including them.
⟨3⟩1. LET: A and B be separated in X
⟨3⟩2. LET: $Y = X \setminus (\overline{A} \cap \overline{B})$
⟨3⟩3. PICK disjoint open sets U, V in Y such that $\overline{A} \cap Y \subseteq U$ and $\overline{B} \cap Y \subseteq V$
PROOF: Y is normal by ⟨2⟩1.
⟨3⟩4. PICK open sets U_0, V_0 in X such that $U = U_0 \cap Y$, $V = V_0 \cap Y$
⟨3⟩5. $A \subseteq U_0 \setminus \overline{B}$ and $B \subseteq V_0 \setminus \overline{A}$
PROOF: Using ⟨3⟩1.
⟨1⟩2. If X is T_1 and, for any pair of separated sets A, B in X, there exist disjoint open sets including them, then X is completely normal.
⟨2⟩1. ASSUME: X is T_1 and, for any pair of separated sets A, B in X, there exist disjoint open sets including them
⟨2⟩2. LET: $Y \subseteq X$
⟨2⟩3. Y is T_1
PROOF: By Proposition 6.1.3.
⟨2⟩4. LET: A and B be disjoint closed sets in Y
⟨2⟩5. A and B are separated in X
⟨3⟩1. $\overline{A} \cap Y = A$ and $\overline{B} \cap Y = B$
PROOF: By Proposition 3.12.6 and Theorem 4.3.4.
⟨3⟩2. $\overline{A} \cap B = \emptyset$

$$\begin{aligned} \overline{A} \cap B &= \overline{A} \cap \overline{B} \cap Y && (\langle 3 \rangle 1) \\ &= A \cap B && (\langle 3 \rangle 1) \\ &= \emptyset && (\langle 2 \rangle 4) \end{aligned}$$

⟨3⟩3. $A \cap \overline{B} = \emptyset$
PROOF: Similar.
⟨2⟩6. PICK disjoint open sets U and V that include A and B respectively.
PROOF: By ⟨2⟩1.
⟨2⟩7. $U \cap Y$ and $V \cap Y$ are disjoint open sets in Y that include A and B respectively.
□

Proposition 6.6.4. *A well-ordered set in the order topology is completely normal.*

PROOF:

⟨1⟩1. LET: X be a well-ordered set.
⟨1⟩2. For all $a, b \in X$ with $a < b$, we have $(a, b]$ is open.
⟨2⟩1. CASE: b is greatest in X
PROOF: This case holds by the definition of the order topology.
⟨2⟩2. CASE: b is not greatest in X
PROOF: In this case, $(a, b] = (a, c)$ where c is the successor of b.
⟨1⟩3. LET: A and B be separated sets in X
PROVE: There exist disjoint open sets U, V including A and B
⟨1⟩4. CASE: The least element of X is not in A or B
⟨2⟩1. LET: $U = \bigcup\{(x, a] : a \in A, x < a, (x, a] \cap B = \emptyset\}$
⟨2⟩2. LET: $V = \bigcup\{(y, b] : b \in B, y < b, (y, b] \cap A = \emptyset\}$
⟨2⟩3. U is open
PROOF: From ⟨1⟩2.
⟨2⟩4. V is open
PROOF: From ⟨1⟩2.
⟨2⟩5. $A \subseteq U$
⟨3⟩1. LET: $a \in A$
⟨3⟩2. PICK W a neighbourhood of a such that $W \cap B = \emptyset$
PROOF: By ⟨1⟩3.
⟨3⟩3. PICK $x < a$ such that $(x, a] \subseteq W$
PROOF: By Lemma 4.1.2
⟨3⟩4. $a \in (x, a] \subseteq U$
⟨2⟩6. $B \subseteq V$
PROOF: Similar.
⟨2⟩7. $U \cap V = \emptyset$
⟨1⟩5. CASE: $\perp \in A$
⟨2⟩1. PICK disjoint open sets U and V that include $A \setminus \{\perp\}$ and B
PROOF: From ⟨1⟩4.
⟨2⟩2. $U \cup \{\perp\}$ and V are disjoint open sets that include A and B
PROOF: $\{\perp\}$ is open because it is $(-\infty, a)$ where a is the successor of $\perp$.
⟨1⟩6. Q.E.D.
PROOF: By Proposition 6.6.3.
□

Proposition 6.6.5. *The product of two completely normal spaces is not necessarily completely normal.*

PROOF:
⟨1⟩1. S_Ω is completely normal.
PROOF: By Proposition 6.6.4
⟨1⟩2. $\overline{S_\Omega}$ is completely normal.
PROOF: By Proposition 6.6.4
⟨1⟩3. $S_\Omega \times \overline{S_\Omega}$ is not completely normal.
PROOF: By Proposition 6.5.3.
□

Proposition 6.6.6. *A compact Hausdorff space is not necessarily completely normal.*

PROOF:
$\langle 1 \rangle 1$. PICK an uncountable set J
$\langle 1 \rangle 2$. $[0,1]^J$ is compact Hausdorff
PROOF: By Tychonoff's Theorem and Theorem 6.2.5.
$\langle 1 \rangle 3$. $(0,1)^J$ is not normal.
PROOF: By Proposition 6.5.6, since $(0,1) \cong \mathbb{R}$.
□

Proposition 6.6.7. *The space $\mathbb{R}_l$ is completely normal.*

PROOF:
$\langle 1 \rangle 1$. LET: $X \subseteq \mathbb{R}_l$
$\langle 1 \rangle 2$. LET: A and B be disjoint closed sets in X.
$\langle 1 \rangle 3$. PICK closed sets C and D such that $A = C \cap X$ and $B = D \cap X$
$\langle 1 \rangle 4$. For $a \in A$, PICK $x_a > a$ such that $[a, x_a) \cap D = \emptyset$
$\langle 1 \rangle 5$. For $b \in B$, PICK $x_b > b$ such that $[b, x_b) \cap C = \emptyset$
$\langle 1 \rangle 6$. $\bigcup_{a \in A}[a, x_a) \cap X$ and $\bigcup_{b \in B}[b, x_b) \cap X$ are disjoint open sets in X that include A and B
□

6.7 Perfectly Normal Spaces

Definition 6.7.1 (Perfectly Normal)**.** A space is *perfectly normal* iff it is normal and every closed set is G_δ.

Proposition 6.7.2. *Every perfectly normal space is completely normal.*

PROOF:
$\langle 1 \rangle 1$. LET: X be perfectly normal.
$\langle 1 \rangle 2$. LET: A and B be separated sets in X
$\langle 1 \rangle 3$. PICK continuous functions $f, g : X \to [0,1]$ that vanish precisely on $\overline{A}$ and $\overline{B}$, respectively.
PROOF: By Theorem 6.5.9.
$\langle 1 \rangle 4$. LET: $h = f - g$
$\langle 1 \rangle 5$. $B \subseteq h^{-1}((0, +\infty))$ and $A \subseteq h^{-1}((-\infty, 0))$
$\langle 1 \rangle 6$. Q.E.D.
PROOF: By Proposition 6.6.3.
□

Proposition 6.7.3. *The space $\overline{S_\Omega}$ is not perfectly normal.*

PROOF: The set $\{\Omega\}$ is not G_δ. □

Chapter 7

Countability Axioms

7.1 The First Countability Axiom

Definition 7.1.1 (First Countability Axiom)**.** A topological space X satisfies the *first countability axiom*, or is *first countable*, iff every point has a countable local basis.

Proposition 7.1.2. *S_Ω is first countable.*

PROOF: For every countable ordinal $\alpha > 0$, the set $\{(\beta, \alpha+1) : \beta < \alpha\}$ is a local basis at α. The set $\{\{0\}\}$ is a local basis at 0. ∎

Theorem 7.1.3 (The Sequence Lemma (CC))**.** *Let X be a first countable space and $A \subseteq X$. If $x \in \overline{A}$, then there exists a sequence of points of A that converges to x.*

PROOF:
$\langle 1 \rangle 1$. LET: $x \in \overline{A}$
$\langle 1 \rangle 2$. PICK a countable basis $\{B_n\}_{n \in \mathbb{Z}^+}$ at x.
$\langle 1 \rangle 3$. For $n \geq 1$, PICK a point $a_n \in B_1 \cap \cdots \cap B_n \cap A$
PROVE: $a_n \to x$ as $n \to \infty$
PROOF: Using Countable Choice. Such an a_n exists because $B_1 \cap \cdots \cap B_n$ is a neighbourhood of x. Apply Theorem 3.13.3.
$\langle 1 \rangle 4$. LET: U be a neighbourhood of x
$\langle 1 \rangle 5$. PICK N such that $B_N \subseteq U$
PROOF: From $\langle 1 \rangle 2$.
$\langle 1 \rangle 6$. For $n \geq N$, we have $a_n \in U$
PROOF:

$$\begin{aligned} a_n &\in B_1 \cap \cdots \cap B_n && (\langle 1 \rangle 3) \\ &\subseteq B_N && (n \geq N) \\ &\subseteq U && (\langle 1 \rangle 5) \end{aligned}$$

∎

Theorem 7.1.4 (CC)**.** *Let X and Y be topological spaces where X is first countable. Let $x \in X$. Suppose that, for every sequence $\{x_n\}_{n\geq 1}$ such that $x_n \to x$ as $n \to \infty$, we have $f(x_n) \to f(x)$ as $n \to \infty$. Then f is continuous at x.*

PROOF:
⟨1⟩1. LET: V be a neighbourhood of $f(x)$
⟨1⟩2. ASSUME: for a contradiction that, for every neighbourhood U of x, $f(U) \nsubseteq V$
⟨1⟩3. PICK a countable local basis $\{B_n\}_{n\geq 1}$
⟨1⟩4. For $n \geq 1$, PICK $a_n \in B_1 \cap \cdots \cap B_n$ such that $f(a_n) \notin V$
⟨1⟩5. $a_n \to x$ as $n \to \infty$
PROOF:
⟨2⟩1. LET: U be a neighbourhood of x
⟨2⟩2. PICK N such that $B_N \subseteq U$
⟨2⟩3. For all $n \geq N$, $a_n \in U$
PROOF:

$$\begin{aligned} a_n &\in B_1 \cap \cdots \cap B_n && (\langle 1\rangle 4) \\ &\subseteq B_N && (n \geq N) \\ &\subseteq U && (\langle 2\rangle 2) \end{aligned}$$

⟨1⟩6. $f(a_n) \to f(x)$ as $n \to \infty$
⟨1⟩7. There exists N such that, for all $n \geq N$, we have $f(a_n) \in V$
⟨1⟩8. Q.E.D.

Lemma 7.1.5 (CC)**.** *$\mathbb{R}^\omega$ under the box topology is not first countable.*

PROOF:
⟨1⟩1. LET: $\{B_n\}_{n\geq 1}$ be any countable set of neighbourhoods of $\vec{0}$
⟨1⟩2. For $n \geq 1$, PICK U_{nm} for $m \geq 1$ such that $\vec{0} \in \prod_{m=1}^{\infty} U_{nm} \subseteq B_n$
⟨1⟩3. For $n \geq 1$, PICK a_n, b_n such that $0 \in (a_n, b_n) \subseteq U_{nn}$
⟨1⟩4. LET: $U = \prod_{n=1}^{\infty}(a_n/2, b_n/2)$
⟨1⟩5. $\vec{0} \in U$
⟨1⟩6. For all n, $B_n \nsubseteq U$
□

Lemma 7.1.6 (CC)**.** *If J is uncountable then $\mathbb{R}^J$ is not first countable.*

PROOF:
⟨1⟩1. LET: $\{B_n\}_{n\geq 1}$ be a countable family of neighbourhoods of $\vec{0}$
⟨1⟩2. For $n \geq 1$, PICK $U_{n\alpha}$ such that $\vec{0} \in \prod_{\alpha \in J} U_{n\alpha} \subseteq B_n$ where $U_{n\alpha}$ is open in $\mathbb{R}$ and $U_{n\alpha} = \mathbb{R}$ except for $\alpha = \alpha_{n1}, \ldots, \alpha_{nr_n}$
⟨1⟩3. PICK β such that β is different from α_{ni} for all n, i
⟨1⟩4. LET: $V = \pi_\beta^{-1}((-1, 1))$
⟨1⟩5. $\vec{0} \in V$
⟨1⟩6. $V \nsubseteq B_n$ for all n
□

Lemma 7.1.7. $\mathbb{R}_l$ *is first countable.*

PROOF: For all $x \in \mathbb{R}$, $\{[x, q) : q \in \mathbb{Q}, q > x\}$ is a basis at x. ∎

Lemma 7.1.8. *The ordered square is first countable.*

PROOF:
⟨1⟩1. LET: $(x, y) \in I_o^2$
PROVE: There exists a countable local basis $\mathcal{B}$ at (x, y)
⟨1⟩2. CASE: $(x, y) = (0, 0)$
PROOF: Take $\mathcal{B} = \{[(0, 0), (0, q)) : q \in \mathbb{Q}, 0 < q < 1\}$.
⟨1⟩3. CASE: $0 < y < 1$
PROOF: Take $\mathcal{B} = \{((x, q), (x, q')) : q, q' \in \mathbb{Q}, q < y < q'\}$.
⟨1⟩4. CASE: $x < 1, y = 1$
PROOF: Take $\mathcal{B} = \{((x, q), (q', 0)) : q, q' \in \mathbb{Q}, 0 < q < 1, x < q' < 1\}$.
⟨1⟩5. CASE: $x > 0, y = 0$
PROOF: Take $\mathcal{B} = \{((q, 1), (x, q')) : q, q' \in \mathbb{Q}, 0 < q < x, 0 < q' < 1\}$
⟨1⟩6. CASE: $(x, y) = (1, 1)$
PROOF: Take $\mathcal{B} = \{((1, q), (1, 1)] : q \in \mathbb{Q}, 0 < q < 1\}$.
∎

Proposition 7.1.9. *A subspace of a first countable space is first countable.*

PROOF:
⟨1⟩1. LET: X be a first countable space and $A \subseteq X$
⟨1⟩2. LET: $a \in A$
⟨1⟩3. PICK a countable basis $\mathcal{B}$ at a in X
⟨1⟩4. $\{B \cap A : B \in \mathcal{B}$ is a countable basis at a in A.
∎

Proposition 7.1.10 (CC)**.** *A countable product of first countable spaces is first countable.*

PROOF:
⟨1⟩1. LET: $\{X_n\}_{n \in \mathbb{Z}^+}$ be a countable family of first countable spaces.
⟨1⟩2. LET: $\vec{x} \in \prod_{n=1}^{\infty} X_n$
⟨1⟩3. PICK a countable basis $\mathcal{B}_n$ at x_n in X_n for all n
⟨1⟩4. LET: $\mathcal{B}$ be the set of all sets $\prod_{i=1}^{n} U_n$ where $U_n \in \mathcal{B}_n$ for finitely many n and $U_n = X_n$ for all other n.
⟨1⟩5. $\mathcal{B}$ is a countable basis at $\vec{x}$ in $\prod_{n=1}^{\infty} X_n$
∎

Corollary 7.1.10.1. *The space* $\mathbb{R}^\omega$ *is first countable.*

Proposition 7.1.11. *The space* S_Ω *is first countable.*

PROOF:
⟨1⟩1. LET: $\alpha \in S_\Omega$
PROVE: α has a countable local basis.

$\langle 1 \rangle 2$. CASE: α is zero or a successor ordinal.
PROOF: In this case, $\{\{\alpha\}\}$ is a local basis.
$\langle 1 \rangle 3$. CASE: α is a limit ordinal.
$\langle 2 \rangle 1$. PICK a countable sequence (β_n) with supremum α
$\langle 2 \rangle 2$. $\{(\beta_n, \alpha + 1) : n \in \mathbb{Z}^+\}$ is a local basis.
□

Proposition 7.1.12. *The space $\overline{S_\Omega}$ is not first countable.*

PROOF:
$\langle 1 \rangle 1$. ASSUME: for a contradiction $\mathcal{B}$ is a countable local basis at Ω
$\langle 1 \rangle 2$. LET: $\alpha = \sup\{\inf B : B \in \mathcal{B}\}$
$\langle 1 \rangle 3$. $\alpha < \Omega$
$\langle 1 \rangle 4$. There is no $B \in \mathcal{B}$ such that $B \subseteq (\alpha, +\infty)$
□

Proposition 7.1.13. *The continuous image of a first countable space is first countable.*

PROOF:
$\langle 1 \rangle 1$. LET: X be a first countable space, Y a space and $f : X \to Y$ continuous.
$\langle 1 \rangle 2$. LET: $y \in f(X)$
$\langle 1 \rangle 3$. PICK $x \in X$ such that $y = f(x)$
$\langle 1 \rangle 4$. PICK a countable local basis $\mathcal{B}$ at x
$\langle 1 \rangle 5$. $\{f(B) : B \in \mathcal{B}\}$ is a countable local basis at y.
□

Proposition 7.1.14. *$S_\Omega \times \overline{S_\Omega}$ is not first countable.*

PROOF: $(0, \Omega)$ has no countable basis. □

Proposition 7.1.15. *The Sorgenfrey plane is first countable.*

PROOF: For any point (a, b), the set $\{[a, a + q) \times [b, b + r) : q, r \in \mathbb{Q}\}$ is a countable local basis at (a, b). □

7.2 Separable Spaces

Definition 7.2.1 (Separable Space)**.** A topological space X is *separable* iff it has a countable dense subset.

Proposition 7.2.2. *The space S_Ω is not separable.*

PROOF:
$\langle 1 \rangle 1$. LET: $D \subseteq S_\Omega$ be countable.
$\langle 1 \rangle 2$. LET: $\alpha = \sup D$
$\langle 1 \rangle 3$. $\overline{D} \subseteq (-\infty, \alpha]$
□

Proposition 7.2.3. *The space $\overline{S_\Omega}$ is not separable.*

PROOF:
⟨1⟩1. LET: $D \subseteq S_\Omega$ be countable.
⟨1⟩2. LET: $\alpha = \sup\{\beta \in D : \beta < \Omega\}$
⟨1⟩3. $\alpha < \Omega$
PROOF: α is the supremum of countably many countable ordinals.
⟨1⟩4. $\overline{D} \subseteq (-\infty, \alpha] \cup \{\Omega\}$
□

Corollary 7.2.3.1. *Not every compact Hausdorff space is separable.*

Proposition 7.2.4. *Every open subspace of a separable space is separable.*

PROOF:
⟨1⟩1. LET: X be a separable space with countable dense subset D.
⟨1⟩2. LET: U be an open subspace of X
PROVE: $D \cap U$ is a countable dense subset of U.
⟨1⟩3. $D \cap U$ is countable.
⟨1⟩4. LET: V be an open set in U.
⟨1⟩5. V is open in X
PROOF: Lemma 4.3.3
⟨1⟩6. V intersects D
⟨1⟩7. V intensects $D \cap U$
□

Proposition 7.2.5 (CC). *The product of a countable family of separable spaces is separable.*

PROOF:
⟨1⟩1. LET: (X_n) be a countable family of separable spaces.
⟨1⟩2. For $n \geq 1$, PICK a dense set D_n in X_n
⟨1⟩3. $\prod_{n=1}^{\infty} D_n$ is dense in $\prod_{n=1}^{\infty} X_n$.
□

Proposition 7.2.6. *The continuous image of a separable space is separable.*

PROOF:
⟨1⟩1. LET: X be a separable space, Y a space and $f : X \to Y$ be continuous.
⟨1⟩2. PICK a countable dense set D in X
⟨1⟩3. $f(D)$ is dense in $f(X)$.
□

Corollary 7.2.6.1. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of nonempty topological spaces. If $\prod_{\alpha \in J} X_\alpha$ is separable then each X_α is separable.*

Corollary 7.2.6.2. $S_\Omega \times \overline{S_\Omega}$ *is not separable.*

Proposition 7.2.7. *The ordered square is not separable.*

PROOF: $\{\{x\} \times (0,1) : x \in [0,1]\}$ is an uncountable set of disjoint open sets. □

Proposition 7.2.8. $\mathbb{R}_l$ *is separable.*

PROOF: $\mathbb{Q}$ is dense. $\square$

Proposition 7.2.9. *The Sorgenfrey plane is separable.*

PROOF: $\mathbb{Q}^2$ is dense. $\square$

Proposition 7.2.10. *Not every closed subspace of a separable space is separable.*

PROOF: $\mathbb{R}_l^2$ is separable but the subspace $\{(x, -x) : x \in \mathbb{R}\}$ is not. $\square$

7.3 The Second Countability Axiom

Definition 7.3.1 (Second Countability Axiom)**.** A topological space satisfies the *second countability axiom*, or is *second countable*, iff it has a countable basis.

Proposition 7.3.2. S_Ω *is not second countable.*

PROOF: $\{\{\alpha\} : \alpha$ is a countable successor ordinal$\}$ is an uncountable set of disjoint open sets. $\square$

Proposition 7.3.3. *A subspace of a second countable space is second countable.*

PROOF:
$\langle 1 \rangle 1$. LET: X be a second countable space and $A \subseteq X$
$\langle 1 \rangle 2$. PICK a countable basis $\mathcal{B}$ for X
$\langle 1 \rangle 3$. $\{B \cap A : B \in \mathcal{B}\}$ is a countable basis for A
$\square$

Proposition 7.3.4 (CC)**.** *The product of countably many second countable spaces is second countable.*

PROOF:
$\langle 1 \rangle 1$. LET: $\{X_n\}_{n \in \mathbb{Z}^+}$ be a countable family of second countable spaces.
$\langle 1 \rangle 2$. For $n \in \mathbb{Z}^+$, PICK a countable basis $\mathcal{B}_n$ for X_n.
$\langle 1 \rangle 3$. LET: $\mathcal{B}$ be the set of all sets of the form $\prod_{n=1}^{\infty} U_n$, where $U_n \in \mathcal{B}_n$ for finitely many n, and $U_n = X_n$ for all other n.
$\langle 1 \rangle 4$. $\mathcal{B}$ is a countable basis for $\prod_{n=1}^{\infty} X_n$
$\square$

Theorem 7.3.5 (CC)**.** *Every second countable space is separable.*

PROOF:
$\langle 1 \rangle 1$. LET: X be a second countable space.
$\langle 1 \rangle 2$. PICK a countable basis $\mathcal{B}$ for X
$\langle 1 \rangle 3$. For $B \in \mathcal{B}$ nonempty, PICK a point $x_B \in B$
$\langle 1 \rangle 4$. $D = \{x_B : B \in \mathcal{B} \setminus \{\emptyset\}\}$ is dense.
  $\langle 2 \rangle 1$. LET: $l \in X$

PROVE: $l \in \overline{D}$

$\langle 2\rangle 2$. LET: $B \in \mathcal{B}$ such that $l \in B$

$\langle 2\rangle 3$. $x_B \in B \cap D$

$\langle 2\rangle 4$. Q.E.D.

PROOF:By Theorem 3.12.8

Corollary 7.3.5.1. *$S_\Omega \times \overline{S_\Omega}$ is not second countable.*

Corollary 7.3.5.2. *The space $\mathbb{R}^\omega$ is separable.*

Corollary 7.3.5.3. *If J is uncountable then $\mathbb{R}^J$ is not second countable.*

Proposition 7.3.6. *The ordered square is not second countable.*

PROOF:

$\langle 1\rangle 1$. LET: $\mathcal{B}$ be any basis

$\langle 1\rangle 2$. For $x \in [0,1]$, PICK B_x such that $x \in B_x \subseteq ((x,0),(x,1))$

$\langle 1\rangle 3$. The function $B_{(-)}$ is an injective function $[0,1] \to \mathcal{B}$

$\langle 1\rangle 4$. $\mathcal{B}$ is uncountable.

□

Proposition 7.3.7. *The space $\overline{S_\Omega}$ is not second countable.*

PROOF: It is not first countable (Proposition 7.1.12). □

Proposition 7.3.8. *The continuous image of a second countable space is second countable.*

PROOF:

$\langle 1\rangle 1$. LET: X be a second countable space, Y a space and $f : X \to Y$ be continuous.

$\langle 1\rangle 2$. PICK a countable basis $\mathcal{B}$ for X.

$\langle 1\rangle 3$. $\{f(B) : B \in \mathcal{B}$ is a countable basis for $f(X)$

□

Theorem 7.3.9. *Every regular Lindelöf space is normal.*

PROOF:

$\langle 1\rangle 1$. LET: X be a regular Lindelöf space.

$\langle 1\rangle 2$. LET: A and B be disjoint closed sets in X.

$\langle 1\rangle 3$. $\{U \text{ open in } X : \overline{U} \cap B = \emptyset\}$ covers A

PROOF: Proposition 6.3.2.

$\langle 1\rangle 4$. PICK a countable open covering $\{U_n : n \in \mathbb{Z}^+\}$ of A such that $\overline{U_n} \cap B = \emptyset$ for all n

$\langle 1\rangle 5$. PICK a countable open covering $\{V_n : n \in \mathbb{Z}^+\}$ of B such that $\overline{V_n} \cap A = \emptyset$ for all n

PROOF: Similar.

$\langle 1\rangle 6$. For $n \in \mathbb{Z}^+$,

LET: $U'_n = U_n \setminus \bigcup_{i=1}^n \overline{V_i}$ and $V'_n = V_n \setminus \bigcup_{i=1}^n \overline{U_i}$

$\langle 1\rangle 7$. LET: $U' = \bigcup_{n=1}^\infty U'_n$ and $V = \bigcup_{n=1}^\infty V'_n$

⟨1⟩8. $A \subseteq U'$ and $B \subseteq V'$
⟨1⟩9. $U' \cap V' = \emptyset$
□

Corollary 7.3.9.1. *If J is uncountable then $\mathbb{R}^J$ is not Lindelöf.*

Proposition 7.3.10. *Every second countable regular space is completely normal.*

PROOF:
⟨1⟩1. LET: X be second countable and regular and $Y \subseteq X$
⟨1⟩2. Y is second countable
  PROOF: Proposition 7.3.3.
⟨1⟩3. Y is regular
  PROOF: Proposition 6.3.4
⟨1⟩4. Y is normal
  PROOF: Theorem 7.3.9
□

Proposition 7.3.11. *The space $\mathbb{R}^\omega$ is second countable.*

PROOF: The sets $\prod_{n=0}^{\infty} U_n$ form a basis, where U_n is an interval of the form (q, r) for $q, r \in \mathbb{Q}$ for finitely many n, and $U_n = \mathbb{R}$ for all other n. □

Proposition 7.3.12 (CC)**.** *In a second countable space, every discrete subspace is countable.*

PROOF:
⟨1⟩1. LET: X be a second countable space
⟨1⟩2. PICK a countable basis $\mathcal{B}$
⟨1⟩3. LET: $D \subseteq X$ be discrete
⟨1⟩4. For $a \in D$, PICK $B_a \in \mathcal{B}$ such that $B_a \cap D = \{a\}$
⟨1⟩5. $a \mapsto B_a$ is injective
□

Proposition 7.3.13. *The space $\mathbb{R}_K$ is second countable.*

PROOF: $\{(a, b) : a, b \in \mathbb{R}\} \cup \{(a, b) - K : a, b \in \mathbb{Q}\}$ is a basis. □

Corollary 7.3.13.1. *The space $\mathbb{R}_K$ is first countable.*

Corollary 7.3.13.2. *The space $\mathbb{R}_K$ is separable.*

Proposition 7.3.14. *Let J be a set with $|J| > |\mathbb{R}|$. Then $\mathbb{R}^J$ is not separable.*

PROOF:
⟨1⟩1. ASSUME: D is countable and dense in $\mathbb{R}^J$
  PROVE: $|J| \leq |\mathbb{R}|$
⟨1⟩2. Define $f : J \to \mathcal{P}D$ by $f(\alpha) = D \cap {\pi_\alpha}^{-1}((0, 1))$
⟨1⟩3. f is injective

⟨2⟩1. Let: $\alpha, \beta \in J$ with $\alpha \neq \beta$
⟨2⟩2. Pick $x \in D \cap {\pi_\alpha}^{-1}((0,1)) \cap {\pi_\beta}^{-1}((2,3))$
⟨2⟩3. $x \in f(\alpha)$ but $x \notin f(\beta)$
□

Corollary 7.3.14.1. *The product of a family of separable spaces is not necessarily separable.*

Chapter 8

Connectedness

8.1 Connected Spaces

Definition 8.1.1 (Separation)**.** Let X be a topological space. A *separation* of X is a pair of disjoint nonempty subsets whose union in X.

Definition 8.1.2 (Connected)**.** A topological space is *connected* iff it has no separation.

Proposition 8.1.3. *S_Ω is not connected.*

PROOF: $\{0\}$ and $S_\Omega \setminus \{0\}$ form a separation. □

Proposition 8.1.4. *A space X is connected if and only if the only sets that are both closed and open are $\emptyset$ and X.*

PROOF: Immediate from definitions. □

Proposition 8.1.5. *Let Y be a subspace of X. Then a separation of Y is a pair of disjoint nonempty sets A, B such that $A \cup B = Y$ and neither of A, B contains a limit point of the other.*

PROOF:
$\langle 1 \rangle 1$. If A and B form a separation of Y then A and B are disjoint and nonempty, $A \cup B = Y$, and neither of A, B contains a limit point of the other.
  $\langle 2 \rangle 1$. LET: A and B be a separation of Y
  $\langle 2 \rangle 2$. A and B are disjoint and nonempty and $A \cup B = Y$
  PROOF: Immediate from the definition of separation.
  $\langle 2 \rangle 3$. A does not contain a limit point of B
  PROOF: B is closed in Y, hence contains all its limit points (Corollary 3.15.3.1), and so the result follows because A and B are disjoint.
  $\langle 2 \rangle 4$. B does not contain a limit point of A
  PROOF: Similar.
$\langle 1 \rangle 2$. If A and B are disjoint and nonempty, $A \cup B = Y$, and neither of A, B contains a limit point of the other, then A and B are a separation of Y.

⟨2⟩1. ASSUME: A and B are disjoint and nonempty, $A \cup B = Y$, and neither of A, B contains a limit point of the other

⟨2⟩2. A is closed in Y

PROOF: Every limit point of A is not in B, so is in A. Apply Corollary 3.15.3.1.

⟨2⟩3. B is open in Y

PROOF: $B = Y \setminus A$

⟨2⟩4. A is open in Y

PROOF: Similar.

□

Proposition 8.1.6. *If the sets C and D form a separation of X, and Y is a connected subspace of X, then $Y \subseteq C$ or $Y \subseteq D$.*

PROOF: Otherwise, $Y \cap C$ and $Y \cap D$ would be a separation of Y. □

Proposition 8.1.7. *The union of a set of connected subspaces of X that have a point in common is connected.*

PROOF:

⟨1⟩1. LET: $\mathcal{S}$ be a set of connected subspaces that have the point a in common.

⟨1⟩2. ASSUME: for a contradiction U and V form a separation of $\bigcup \mathcal{S}$

⟨1⟩3. ASSUME: w.l.o.g. $a \in U$

⟨1⟩4. For all $Y \in \mathcal{S}$ we have $Y \subseteq U$

PROOF: By Proposition 8.1.6.

⟨1⟩5. $V = \emptyset$

⟨1⟩6. Q.E.D.

PROOF: This contradicts ⟨1⟩2.

□

Theorem 8.1.8. *Let A be a connected subspace of X. If $A \subseteq B \subseteq \overline{A}$ then B is connected.*

PROOF:

⟨1⟩1. ASSUME: for a contradiction U and V are a separation of B

⟨1⟩2. $A \subseteq U$ or $A \subseteq V$

PROOF: By Proposition 8.1.6.

⟨1⟩3. ASSUME: w.l.o.g. $A \subseteq U$

⟨1⟩4. $\overline{A} \subseteq \overline{U}$

PROOF: By Proposition 3.12.5.

⟨1⟩5. $B \subseteq \overline{U}$

PROOF: Since $B \subseteq \overline{A}$.

⟨1⟩6. The closure of U in B is B

PROOF: By Theorem 4.3.4.

⟨1⟩7. $U = B$

PROOF: Since U is closed in B.

⟨1⟩8. Q.E.D.

PROOF: This contradicts ⟨1⟩1.

□

Theorem 8.1.9. *The image of a connected space under a continuous map is connected.*

PROOF: Let X be a connected space, Y a topological space, and $f : X \twoheadrightarrow Y$ be surjective. If U and V form a separation of Y, then $f^{-1}(U)$ and $f^{-1}(V)$ form a separation of X. □

Corollary 8.1.9.1. *Let $\mathcal{T}$ and $\mathcal{T}'$ be topologies on the same set X. If $\mathcal{T} \subseteq \mathcal{T}'$ and X is connected under $\mathcal{T}'$ then X is connected under $\mathcal{T}$.*

Corollary 8.1.9.2. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces. If $\prod_{\alpha \in J} X_\alpha$ is connected then each X_α is connected.*

Corollary 8.1.9.3. *The Sorgenfrey plane is disconnected.*

Proposition 8.1.10. *The product of a family of connected spaces is connected.*

PROOF:
$\langle 1 \rangle 1$. The product of two connected spaces is connected.
  PROOF:
  $\langle 2 \rangle 1$. LET: X and Y be connected spaces.
  $\langle 2 \rangle 2$. ASSUME: w.l.o.g. X and Y are nonempty.
    PROOF: If either is empty then $X \times Y = \emptyset$ is connected.
  $\langle 2 \rangle 3$. ASSUME: for a contradiction U and V are a separation of $X \times Y$.
  $\langle 2 \rangle 4$. PICK $b \in Y$
    PROOF: By $\langle 2 \rangle 2$.
  $\langle 2 \rangle 5$. For all $x \in X$,
      LET: $T_x = (X \times \{b\}) \cup (\{x\} \times Y)$
  $\langle 2 \rangle 6$. For all $x \in X$, T_x is connected
    $\langle 3 \rangle 1$. $X \times \{b\}$ is connected
      PROOF: It is homeomorphic to X.
    $\langle 3 \rangle 2$. $\{x\} \times Y$ is connected
      PROOF: It is homeomorphic to Y.
    $\langle 3 \rangle 3$. Q.E.D.
      PROOF: By Proposition 8.1.7.
  $\langle 2 \rangle 7$. $X \times Y = \bigcup_{x \in X} T_x$
  $\langle 2 \rangle 8$. Q.E.D.
    $\langle 3 \rangle 1$. PICK $a \in X$
      PROOF: By $\langle 2 \rangle 2$.
    $\langle 3 \rangle 2$. $(a, b) \in T_x$ for all $x \in X$
    $\langle 3 \rangle 3$. Q.E.D.
      PROOF: By Proposition 8.1.7.
$\langle 1 \rangle 2$. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of connected spaces.
$\langle 1 \rangle 3$. ASSUME: w.l.o.g. $\prod_{\alpha \in J} X_\alpha$ is nonempty
$\langle 1 \rangle 4$. PICK $\vec{a} \in \prod_{\alpha \in J} X_\alpha$
$\langle 1 \rangle 5$. For K a finite subset of J,

LET: $X_K = \{\vec{x} \in \prod_{\alpha \in J} X_\alpha : x_\alpha = a_\alpha \text{ for all } \alpha \in J \setminus K\}$

$\langle 1 \rangle 6$. For all K, X_K is connected.

PROOF: It is homeomorphic to $\prod_{\alpha \in K} X_\alpha$, so it is connected by $\langle 1 \rangle 1$.

$\langle 1 \rangle 7$. $\bigcup_{K \subseteq^{\text{fin}} J} X_K$ is connected.

PROOF: By Proposition 8.1.7 since $\vec{a} \in X_K$ for all K.

$\langle 1 \rangle 8$. $\prod_{\alpha \in J} X_\alpha = \overline{\bigcup_{K \subseteq^{\text{fin}} J} X_K}$

$\langle 2 \rangle 1$. LET: $\vec{x} \in \prod_{\alpha \in J} X_\alpha$

$\langle 2 \rangle 2$. LET: U be an open neighbourhood of $\vec{x}$

$\langle 2 \rangle 3$. PICK a basic open set $\prod_{\alpha \in J} V_\alpha$ such that $\vec{x} \in \prod_{\alpha \in J} V_\alpha \subseteq U$, where each V_α is open in X_α, and $V_\alpha = X_\alpha$ except for $\alpha \in K$ for some finite $K \subseteq J$

PROVE: U intersects X_K

$\langle 2 \rangle 4$. LET: $\vec{y} \in \prod_{\alpha \in J} X_\alpha$ with $y_\alpha = x_\alpha$ for $\alpha \in K$, $y_\alpha = a_\alpha$ for $\alpha \notin K$

$\langle 2 \rangle 5$. $\vec{y} \in U \cap X_K$

$\langle 1 \rangle 9$. Q.E.D.

Corollary 8.1.10.1. *For any set I, the space $\mathbb{R}^I$ under the product topology is connected.*

Proposition 8.1.11. *$\mathbb{R}^\omega$ under the box topology is disconnected.*

PROOF: The set of all bounded sequences and the set of all unbounded sequences form a separation. ☐

Definition 8.1.12 (Totally Disconnected)**.** A space is *totally disconnected* iff the only connected subspaces are the singletons.

Theorem 8.1.13. *Let L be a linearly ordered set under the order topology. Then L is connected if and only if L is a linear continuum.*

PROOF:

$\langle 1 \rangle 1$. If L is a linear continuum then L is connected.

$\langle 2 \rangle 1$. LET: L be a linear continuum.

$\langle 2 \rangle 2$. ASSUME: for a contradiction U and V are a separation of L.

$\langle 2 \rangle 3$. PICK $a \in U$ and $b \in V$

$\langle 2 \rangle 4$. ASSUME: w.l.o.g. $a < b$

$\langle 2 \rangle 5$. LET: $l = \sup\{x \in A : x < b\}$

$\langle 2 \rangle 6$. CASE: $l \in A$

$\langle 3 \rangle 1$. PICK $a' > l$ such that $[l, a') \subseteq A$

PROOF: By Lemma 4.1.2. We know l is not greatest in X because $l < b$.

$\langle 3 \rangle 2$. PICK a^* such that $l < a^* < a'$

PROOF: L is dense.

$\langle 3 \rangle 3$. $l < a^*$, $a^* \in A$, $a^* < b$

PROOF: If $b < a^*$ then $b \in A$ by $\langle 3 \rangle 1$.

$\langle 3 \rangle 4$. Q.E.D.

PROOF: This contradicts $\langle 2 \rangle 5$.

$\langle 2 \rangle 7$. CASE: $l \in B$

$\langle 3 \rangle 1$. PICK $b' < l$ such that $(b', l] \subseteq B$

PROOF: By Lemma 4.1.2. We know l is not least in X because $a < l$.

⟨3⟩2. PICK b^* such that $b' < b^* < l$

PROVE: b^* is an upper bound for $\{x \in A : x < b\}$

⟨3⟩3. LET: $x \in A$ and $x < b$

⟨3⟩4. $x \leq b^*$

PROOF: If $b^* < x$ then $b^* < x \leq l$ and so $x \in B$ by ⟨3⟩1.

⟨3⟩5. Q.E.D.

PROOF: This contradicts ⟨2⟩5.

⟨1⟩2. If L is connected then L is a linear continuum.

⟨2⟩1. ASSUME: L is connected

⟨2⟩2. L has the least upper bound property

⟨3⟩1. ASSUME: for a contradiction $A \subseteq L$ is bounded above with no least upper bound

⟨3⟩2. LET: U be the set of upper bounds of A

⟨3⟩3. U is open

⟨4⟩1. LET: $u \in U$

⟨4⟩2. PICK an upper bound v for A with $v < u$

PROOF: u is not the least upper bound for A (⟨3⟩1)

⟨4⟩3. $u \in (v, +\infty) \subseteq U$

⟨3⟩4. LET: V be the set of lower bounds of U

⟨3⟩5. U and V form a separation of L

⟨4⟩1. V is open

PROOF: Similar to ⟨3⟩3.

⟨4⟩2. U and V are disjoint

⟨5⟩1. ASSUME: for a contradiction $x \in U \cap V$

⟨5⟩2. PICK $u \in U$ such that $u < x$

PROOF: x is not the lowest upper bound of A

⟨5⟩3. $x \leq u < x$

⟨4⟩3. $U \cup V = L$

⟨5⟩1. LET: $x \in L \setminus U$

⟨5⟩2. PICK $a \in A$ such that $x < a$

⟨5⟩3. $a \in V$

⟨5⟩4. $x \in V$

⟨2⟩3. For all $x, y \in L$, there exists $z \in L$ such that $x < z < y$

PROOF: Otherwise $(-\infty, y)$ and $(x, +\infty)$ would form a separation of L.

□

Corollary 8.1.13.1. *The real line $\mathbb{R}$ is connected, and so is every ray and interval in $\mathbb{R}$.*

Corollary 8.1.13.2. *The ordered square is connected.*

Corollary 8.1.13.3. *Not every closed subspace of a connected space is connected.*

PROOF: The set $\{0, 1\}$ is disconnected as a subspace of $\mathbb{R}$.

Corollary 8.1.13.4. *Not every open subspace of a connected space is connected.*

PROOF: The space $\mathbb{R} \setminus \{0\}$ is a disconnected open subspace of $\mathbb{R}$. □

Theorem 8.1.14 (Intermediate Value Theorem)**.** *Let X be a connected space and Y a linearly ordered set under the order topology. Let $f : X \to Y$ be continuous. Let $a, b \in X$ and $r \in Y$. If $f(a) < r < f(b)$, then there exists $c \in X$ such that $f(c) = r$.*

PROOF: If not, then $f^{-1}((-\infty, r))$ and $f^{-1}((r, +\infty))$ would be a separation of X. □

Proposition 8.1.15. *Every connected regular space with more than one point is uncountable.*

PROOF:
⟨1⟩1. Every connected completely regular space with more than one point is uncountable.
  ⟨2⟩1. LET: X be connected and completely regular and $a, b \in X$ with $a \neq b$
  ⟨2⟩2. PICK a continuous $f : X \to [0, 1]$ such that $f(a) = 0$ and $f(b) = 1$
  ⟨2⟩3. f is surjective.
    PROOF: By the Intermediate Value Theorem.
⟨1⟩2. Every connected regular space with more than one point is uncountable.
  ⟨2⟩1. ASSUME: for a contradiction X is connected, regular and countable with more than one point.
  ⟨2⟩2. X is Lindelöf
  ⟨2⟩3. X is normal
    PROOF: By Theorem 7.3.9
  ⟨2⟩4. Q.E.D.
    PROOF: Contraditcting ⟨1⟩1.
□

Proposition 8.1.16. *$\overline{S_\Omega}$ is not conneced.*

PROOF: $\{0\}$ is clopen. □

Proposition 8.1.17. *$\mathbb{R}_l$ is not connected.*

PROOF: The set $[0, +\infty)$ is clopen. □

Proposition 8.1.18. *The space $\mathbb{R}^\omega$ under the uniform topology is not connected.*

PROOF: The set of all bounded sequences and the set of all unbounded sequences form a separation. □

Proposition 8.1.19. *The space $\mathbb{R}_K$ is connected.*

PROOF: Easy. □

8.2 Components and Local Connectedness

Definition 8.2.1 ((Connected) Component)**.** Let X be a topological space. Define an equivalence relation $\sim$ on X by: $x \sim y$ iff there exists a connected subspace $U \subseteq X$ such that $x \in U$ and $y \in U$. The *(connected) components* of X are the equivalence classes under $\sim$.

We prove this is an equivalence relation.

PROOF:
$\langle 1 \rangle 1$. For all $x \in X$ we have $x \sim x$.
PROOF: The subspace $\{x\} \subseteq X$ is connected.
$\langle 1 \rangle 2$. For all $x, y \in X$, if $x \sim y$ then $y \sim x$.
PROOF: Immediate from definitions.
$\langle 1 \rangle 3$. For all $x, y, z \in X$, if $x \sim y$ and $y \sim z$ then $x \sim z$.
PROOF: By Proposition 8.1.7.
∎

Proposition 8.2.2. *Let X be a topological space. If $C \subseteq X$ is connected and nonempty, then there exists a unique component D of X such that $C \subseteq D$.*

PROOF:
$\langle 1 \rangle 1$. PICK $a \in C$
$\langle 1 \rangle 2$. LET: D be the $\sim$-equivalence class of A
$\langle 1 \rangle 3$. $C \subseteq D$
PROOF: For all $x \in C$ we have $a \sim x$ by definition.
$\langle 1 \rangle 4$. D is unique
PROOF: This holds because the components are disjoint.
∎

Proposition 8.2.3 (AC)**.** *Every component is connected.*

PROOF:
$\langle 1 \rangle 1$. LET: C be a component of the topological space X
$\langle 1 \rangle 2$. PICK $a \in C$
$\langle 1 \rangle 3$. For all $x \in C$, PICK a connected subspace C_x of X containing both a and x.
PROOF: Such a C_x exists since $a \sim x$.
$\langle 1 \rangle 4$. $C = \bigcup_{x \in C} C_x$
PROOF: This holds because $C_x \subseteq C$ by Proposition 8.2.2.
$\langle 1 \rangle 5$. Q.E.D.
PROOF: It follows that C is connected by Proposition 8.1.7.
∎

Proposition 8.2.4. *Every component is closed.*

PROOF: From Theorem 8.1.8. ∎

Proposition 8.2.5. *The component of $\vec{a}$ in $\mathbb{R}^\omega$ under the uniform topology is $\{\vec{b} : \vec{b} - \vec{a}$ is bounded$\}$.*

PROOF:
⟨1⟩1. $C = \{\vec{b} : \vec{b} - \vec{a} \text{ is bounded}\}$ is connected.
⟨2⟩1. ASSUME: $C = U \cup V$ is a separation of C with $\vec{a} \in U$
⟨2⟩2. PICK $\vec{b} \in V$
⟨2⟩3. $\{\epsilon : \epsilon\vec{b} + (1-\epsilon)\vec{a} \in U\}$ and $\{\epsilon : \epsilon\vec{b} + (1-\epsilon)\vec{a} \in V\}$ form a separation of $[0, 1]$
⟨1⟩2. If $\vec{a}, \vec{b} \in C$ and $\vec{b} - \vec{a}$ is unbounded then C is disconnected.
PROOF: $\{\vec{c} : \vec{c} - \vec{a} \text{ is bounded}\}$ and $\{\vec{c} : \vec{c} - \vec{a} \text{ is unbounded}\}$
□

Proposition 8.2.6. *Let $x, y \in \mathbb{R}^\omega$ under the box topology. Then x and y are in the same component iff $x - y$ is eventually zero.*

PROOF:
⟨1⟩1. For all $x \in \mathbb{R}^\omega$ the set $\{y : x - y \text{ is eventulally zero}\}$ is connected
PROOF: It is the union of the sets $C_N = \{y : \forall n \geq N. y_n = 0\}$, each of which is connected because it is homeomorphic to $\mathbb{R}^{N-1}$.
⟨1⟩2. If $x - y$ is not eventually zero then x and y are in different components
⟨2⟩1. ASSUME: $x - y$ is not eventually zero
⟨2⟩2. Define $h : \mathbb{R}^\omega \to \mathbb{R}^\omega$ by: $h(z)_n = \begin{cases} z_n - x_n & \text{if } x_n = y_n \\ n(z_n - x_n)/(y_n - x_n) & \text{if } x_n \neq y_n \end{cases}$
⟨2⟩3. h is an automorphism of $\mathbb{R}^\omega$ under the box topology
⟨2⟩4. $h(x) = 0$
⟨2⟩5. $h(y)$ is unbounded
⟨2⟩6. Q.E.D.
PROOF: The inverse image under h of the set of bounded sequences and the set of unbounded sequences form a separation of $\mathbb{R}^\omega$ with x and y in different sets.
□
□

8.3 Path Connectedness

Definition 8.3.1 (Path)**.** Let X be a topological space and $a, b \in X$. A *path* from a to b is a continuous function $p : [0, 1] \to X$ such that $p(0) = a$ and $p(1) = b$.

Definition 8.3.2 (Path Connected)**.** A topological space is *path connected* iff there exists a path between any two points.

Proposition 8.3.3. *Every path connected space is connected.*

PROOF:
⟨1⟩1. LET: X be a path connected space
⟨1⟩2. ASSUME: for a contradiction U and V are a separation of X.
⟨1⟩3. PICK $a \in U$ and $b \in V$

$\langle 1 \rangle 4$. PICK a path $p : [0,1] \to X$ from a to b
$\langle 1 \rangle 5$. $p^{-1}(U)$ and $p^{-1}(V)$ form a separation of $[0,1]$.
$\langle 1 \rangle 6$. Q.E.D.
PROOF: This contradicts Corollary 8.1.13.1.

Corollary 8.3.3.1. *S_Ω is not path connected.*

Corollary 8.3.3.2. *$\overline{S_\Omega}$ is not path connected.*

Corollary 8.3.3.3. *$\mathbb{R}_l$ is not path connected.*

Corollary 8.3.3.4. *The Sorgenfrey plane is not path connected.*

Corollary 8.3.3.5. *The space $\mathbb{R}^\omega$ under the uniform topology is not path connected. connected.*

Corollary 8.3.3.6. *The space $\mathbb{R}^\omega$ under the box topology is not path connected.*

Proposition 8.3.4. *The long line is path connected.*

PROOF:
$\langle 1 \rangle 1$. LET: $a, b \in L$
$\langle 1 \rangle 2$. PICK an ordinal α such that $a, b < (\alpha, 0)$
$\langle 1 \rangle 3$. There exists a path from a to b
PROOF: This holds because $[(0,0), (\alpha, 0))$ is homeomorphic to $[0,1)$ by Proposition 1.20.11.
□

Corollary 8.3.4.1. *Not every closed subspace of a path connected space is path connected.*

PROOF: Take any two-element subspace of the long line.

Corollary 8.3.4.2. *Not every open subspace of a path connected space is path connected.*

PROOF: The space $\mathbb{R} \setminus \{0\}$ is not path connected as a subspace of $\mathbb{R}$. □

Definition 8.3.5 (Path Component)**.** Let X be a topological space. Define an equivalence relation $\sim$ on X by: $x \sim y$ iff there exists a path from x to y. The equivalence classes are called the *path components* of X.

We prove this is an equivalence relation.

PROOF:
$\langle 1 \rangle 1$. For all $x \in X$ we have $x \sim x$
PROOF: The constant path $p : [0,1] \to X$ where $p(t) = x$ is a path from x to x.
$\langle 1 \rangle 2$. If $x \sim y$ then $y \sim x$
PROOF: If $p : [0,1] \to X$ is a path from x to y then $\lambda t.p(1-t)$ is a path from y to x.
$\langle 1 \rangle 3$. If $x \sim y$ and $y \sim z$ then $x \sim z$

⟨2⟩1. LET: p be a path from x to y and q be a path from y to z.
⟨2⟩2. LET: $r : [0,1] \to X$ where

$$r(t) = \begin{cases} p(2t) & \text{if } 0 \leq t \leq 1/2 \\ q(2t-1) & \text{if } 1/2 \leq t \leq 1 \end{cases}$$

⟨2⟩3. r is a path from x to z.
PROOF: r is continuous by the Pasting Lemma.
□

Proposition 8.3.6. *Every path component is path connected.*

PROOF: By definition, if x and y are in the same path component then there is a path from x to y. □

Proposition 8.3.7. *If A is a nonempty path connected subspace of the space X, then A is included in a unique path component.*

PROOF:
⟨1⟩1. PICK $a \in A$
⟨1⟩2. LET: C be the equivalence class of a under $\sim$
⟨1⟩3. $A \subseteq C$
PROOF: For all $x \in A$, there exists a path from a to x.
⟨1⟩4. C is unique
PROOF: C is the unique path component such that $a \in C$.
□

Proposition 8.3.8. *Every path component is included in a component.*

PROOF: From Propositions 8.3.3 and 8.2.2. □

Proposition 8.3.9. *The ordered square is not path connected.*

PROOF:
⟨1⟩1. ASSUME: for a contradiction $p : [0,1] \to I_o^2$ is a path from $(0,0)$ to $(1,1)$.
⟨1⟩2. For all $x \in [0,1]$, $p^{-1}(\{x\} \times (0,1))$ is open in $[0,1]$
⟨1⟩3. For all $x \in [0,1]$, PICK a rational $q_x \in p^{-1}(\{x\} \times (0,1))$
⟨1⟩4. $\{q_x : x \in [0,1]\}$ is an uncountable set of rationals.
□

Proposition 8.3.10 (AC)**.** *The product of a family of path connected spaces is path connected.*

PROOF:
⟨1⟩1. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of path connected spaces and $a, b \in \prod_{\alpha \in J} X_\alpha$
⟨1⟩2. For $\alpha \in J$, PICK a path $p_\alpha : [0,1] \to X_\alpha$ from a_α to b_α
⟨1⟩3. Define $p : [0,1] \to \prod_{\alpha \in J} X_\alpha$ by $p(t)_\alpha = p_\alpha(t)$
⟨1⟩4. p is a path from a to b
PROOF: By Theorem 5.2.15.
□

Corollary 8.3.10.1. *For any set I, the space $\mathbb{R}^I$ in the product topology is path connected.*

Proposition 8.3.11. *The space $\mathbb{R}_K$ is not path connected.*

PROOF:
$\langle 1 \rangle 1$. ASSUME: for a contradiction $p : [0,1] \to \mathbb{R}_K$ is a path from 0 to 1
$\langle 1 \rangle 2$. LET: $p : [0,1] \to \mathbb{R}_K$ be a path from 0 to 1
$\langle 1 \rangle 3$. $p([0,1])$ is compact and connected in $\mathbb{R}_K$.
  PROOF: Theorem 8.1.9 and Proposition 9.5.10.
$\langle 1 \rangle 4$. $p([0,1])$ is connected in $\mathbb{R}$.
  PROOF: Corollary 8.1.9.1
$\langle 1 \rangle 5$. $[0,1] \subseteq p([0,1])$
  PROOF: For any $x \in [0,1]$, if $x \notin p([0,1])$ then $p([0,1]) \cap (-\infty, x)$ and $p([0,1]) \cap (x, +\infty)$ form a separation of $p([0,1])$.
$\langle 1 \rangle 6$. $[0,1]$ is compact in $\mathbb{R}_K$
  PROOF: Proposition 9.5.6.
$\langle 1 \rangle 7$. Q.E.D.
  PROOF: This contradicts Corollary 9.5.11.2.
□

Proposition 8.3.12. *Let $f : X \to Y$ be continuous and surjective. If X is path connected then Y is path connected.*

PROOF:
$\langle 1 \rangle 1$. LET: $a, b \in Y$
$\langle 1 \rangle 2$. PICK $x, y \in X$ such that $f(x) = a$ and $f(y) = b$
$\langle 1 \rangle 3$. PICK a path $p : [0,1] \to X$ such that $p(0) = x$ and $p(1) = y$
$\langle 1 \rangle 4$. $f \circ p$ is a path from a to b
□

Corollary 8.3.12.1. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of non-empty topological spaces. If $\prod_{\alpha \in J} X_\alpha$ is path connected then each X_α is path connected.*

8.4 Connected Subspaces of Euclidean Space

Definition 8.4.1 (Unit 2-Sphere)**.** The *unit 2-sphere* is $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ as a subspace of $\mathbb{R}^3$.

Definition 8.4.2 (Unit Ball)**.** For any $n \geq 1$, the *closed unit ball* in $\mathbb{R}^n$ is

$$B^n = \{\vec{x} \in \mathbb{R}^n : \|\vec{x}\| \leq 1\} \ .$$

Proposition 8.4.3. *Every open unit ball and closed unit ball in $\mathbb{R}^n$ is path connected.*

PROOF: The straight line between any two points is a path in the ball. □

Definition 8.4.4 (Punctured Euclidean Space)**.** For $n \geq 1$, *punctured Euclidean space* is $\mathbb{R}^n \setminus \{\vec{0}\}$.

Proposition 8.4.5. *Punctured Euclidean space in $\mathbb{R}^n$ is path connected iff $n > 1$.*

PROOF: Easy. ∎

Definition 8.4.6 (Unit Sphere)**.** For $n \geq 1$, the *unit sphere* S^n is $\{\vec{x} \in \mathbb{R}^{n+1} : \|\vec{x}\| = 1\}$.

Proposition 8.4.7. *In any number of dimensions, the unit sphere is path connected.*

PROOF: Easy. ∎

Definition 8.4.8 (Topologist's Sine Curve)**.** The *topologist's sine curve* is the closure of

$$S = \{(x, \sin 1/x) : x \in \mathbb{R}\}$$

in $\mathbb{R}^2$.

Proposition 8.4.9. *The topologist's sine curve is connected.*

PROOF:
$\langle 1 \rangle 1$. $S = \{(x, \sin 1/x) : x \in \mathbb{R}\}$ is connected.
  $\langle 2 \rangle 1$. The function $f : \mathbb{R} \to \mathbb{R}^2$ given by $f(x) = (x, \sin 1/x)$ is continuous.
    PROOF: By Theorem 5.2.15.
  $\langle 2 \rangle 2$. Q.E.D.
    PROOF: By Theorem 8.1.9.
$\langle 1 \rangle 2$. Q.E.D.
  PROOF: By Theorem 8.1.8.
∎

Proposition 8.4.10 (CC)**.** *The topologist's sine curve is not path connected.*

PROOF:
$\langle 1 \rangle 1$. LET: $S = \{(x, \sin 1/x) : x \in \mathbb{R}\}$
$\langle 1 \rangle 2$. ASSUME: for a contradiction $p : [0,1] \to \overline{S}$ is a path from $(0,0)$ to $(1, \sin 1)$.
$\langle 1 \rangle 3$. $p^{-1}(\{0\} \times [-1,1])$ is closed.
$\langle 1 \rangle 4$. $p^{-1}(\{0\} \times [-1,1])$ has a greatest element.
  PROOF: By Lemma 4.1.9.
$\langle 1 \rangle 5$. LET: $q : [0,1] \to \overline{S}$ be a path such that:
- $q(0) \in \{0\} \times [-1,1]$
- $q(x) \in S$ for $x > 0$

  PROOF: Let b be greatest in $p^{-1}(\{0\} \times [-1,1])$. Then q is obtained by rescaling p restricted to $[b,1]$.
$\langle 1 \rangle 6$. LET: $q(t) = (x(t), y(t))$ for $0 \leq t \leq 1$
$\langle 1 \rangle 7$. $x(0) = 0$

⟨1⟩8. $x(t) > 0$ for $t > 0$
⟨1⟩9. $y(t) = \sin 1/x(t)$ for $t > 0$
⟨1⟩10. There exists a sequence $t_n \in [0,1]$ such that $t_n \to 0$ as $n \to \infty$ and $y(t_n) = (-1)^n$ for all n.
⟨2⟩1. For each n, PICK u_n such that $0 < u_n < x(1/n)$ and $\sin 1/u_n = (-1)^n$.
PROOF: Such a u_n exists because $\sin 1/x$ takes values 1 and -1 infinitely often in $(0, x(1/n))$.
⟨2⟩2. For each n, PICK t_n such that $0 < t_n < 1/n$ and $x(t_n) = u$
PROOF: By the Intermediate Value Theorem.
⟨1⟩11. Q.E.D.
PROOF: This is a contradiction as $y(t_n) \to y(0)$ as $n \to \infty$ because y is continuous.
□

8.5 Local Connectedness

Definition 8.5.1 (Locally Connected)**.** Let X be a topological space and $x \in X$. Then X is *locally connected* at x iff every neighbourhood of x includes a connected neighbourhood of x.

The space X is *locally connected* iff it is locally connected at every point.

Proposition 8.5.2. *S_Ω is not locally connected.*

PROOF: There is no connected neighbourhood of ω. □

Proposition 8.5.3. *$\overline{S_\Omega}$ is not locally connected.*

PROOF: There is no connected neighbourhood of ω. □

Proposition 8.5.4. *For any set I, the space $\mathbb{R}^I$ is locally connected.*

PROOF: Every basic open set is the product of connected spaces, hence connected. □

Proposition 8.5.5. *Let X be a topological space. Then X is locally connected if and only if, for every open set U in X, every component of U is open in X.*

PROOF:
⟨1⟩1. If X is locally connected then, for every open set U in X, every component of U is open in X.
⟨2⟩1. ASSUME: X is locally connected.
⟨2⟩2. LET: U be open in X.
⟨2⟩3. LET: C be a component of U.
⟨2⟩4. LET: $x \in C$
PROVE: C is a neighbourhood of x
⟨2⟩5. U is a neighbourhood of x in X.
PROOF: From ⟨2⟩2, ⟨2⟩3 and ⟨2⟩4.
⟨2⟩6. PICK a connected neighbourhood V of x such that $V \subseteq U$.

PROOF: Using $\langle 2 \rangle 1$.

$\langle 2 \rangle 7$. $V \subseteq C$

PROOF: By Proposition 8.2.2.

$\langle 2 \rangle 8$. C is a neighbourhood of x

PROOF: By Proposition 3.2.4.

$\langle 2 \rangle 9$. Q.E.D.

PROOF: By Proposition 3.2.3.

$\langle 1 \rangle 2$. If, for every open set U in X, every component of U is open in X, then X is locally connected.

$\langle 2 \rangle 1$. ASSUME: For every open set U in X, every component of U is open in X.

$\langle 2 \rangle 2$. LET: $x \in X$ and N be a neighbourhood of x

$\langle 2 \rangle 3$. PICK U open such that $x \in U \subseteq N$

$\langle 2 \rangle 4$. LET: C be the component of U that contains x

$\langle 2 \rangle 5$. C is open in X

PROOF: By $\langle 2 \rangle 1$.

$\langle 2 \rangle 6$. C is a connected neighbourhood of x that is included in N

∎

Corollary 8.5.5.1. *In a locally connected space, every component is open.*

Corollary 8.5.5.2. *The space $\mathbb{R}^\omega$ under the box topology is not locally connected.*

Corollary 8.5.5.3. *Not every closed subspace of a locally connected space is locally connected.*

PROOF: The topologist's sine curve is not locally connected. ∎

Proposition 8.5.6. *$S_\Omega \times \overline{S_\Omega}$ is not locally connected.*

(ω, ω) has no connected neighbourhood. ∎

Proposition 8.5.7. *$\mathbb{R}_l$ is not locally connected.*

PROOF: 0 has no connected neighbourhood. ∎

Proposition 8.5.8. *The Sorgenfrey plane is not locally connected.*

PROOF: Any basic open set $[a, b) \times [c, d)$ can be separated into $[a, b) \times [c, e)$ and $[a, b) \times [e, d)$ for some $c < e < d$. ∎

Proposition 8.5.9. *The space $\mathbb{R}^\omega$ under the uniform topology is locally connected.*

PROOF: For any neighbourhood U of a point x, the neighbourhood $U \cap \{y : y - x \text{ is bounded}\}$ is connected. ∎

Proposition 8.5.10. *The space $\mathbb{R}_K$ is not locally connected.*

PROOF: The open set $(-1, 1) - K$ does not include a connected neighbourhood of 0. ∎

Proposition 8.5.11. *Every open subspace of a locally connected space is locally connected.*

PROOF: Follows easily from definition. □

Proposition 8.5.12 (AC)**.** *The product of a family of locally connected spaces is locally connected.*

PROOF:
⟨1⟩1. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of locally connected spaces and $\vec{x} \in \prod_{\alpha \in J} X_\alpha$
⟨1⟩2. LET: $\prod_{\alpha \in J} U_\alpha$ be any basic neighbourhood of $\vec{x}$, where each U_α is open in X_α, and $U_\alpha = X_\alpha$ except for $\alpha = \alpha_1, \ldots, \alpha_n$
⟨1⟩3. For $\alpha \in J$, PICK a connected neighbourhood C_α of x_α with $C_\alpha \subseteq U_\alpha$
⟨1⟩4. $\prod_{\alpha \in J} C_\alpha$ is connected
  PROOF: Proposition 8.1.10
□

Proposition 8.5.13. *Every discrete space is locally connected.*

PROOF: For any point x, the set $\{x\}$ is a connected neighbourhood of x. □

Corollary 8.5.13.1. *The continuous image of a locally connected space is not necessarily locally connected.*

8.6 Local Path Connectedness

Definition 8.6.1 (Locally Path Connected)**.** Let X be a topological space and $x \in X$. Then X is *locally path connected at x* iff every neighbourhood of x includes a path connected neighbourhood of x.

The space X is *locally path connected* iff it is locally path connected at every point.

Proposition 8.6.2. S_Ω *is not locally path connected.*

PROOF: There is no path connected neighbourhood of ω. □

Proposition 8.6.3. $\overline{S_\Omega}$ *is not locally path connected.*

PROOF: There is no path connected neighbourhood of ω. □

Proposition 8.6.4. *Not every closed subspace of a locally path connected space is locally path connected.*

PROOF: The topologist's sine curve is not loally path connected. □

Proposition 8.6.5. *Every open subspace of a locally path connected space is locally path connected.*

PROOF: Follows easily from definition. □

Proposition 8.6.6. *Every locally path connected space is locally connected.*

PROOF: From Proposition 8.3.3. ∎

Corollary 8.6.6.1. *$\mathbb{R}_l$ is not locally path connected.*

Corollary 8.6.6.2. *The Sorgenfrey plane is not locally path connected.*

Corollary 8.6.6.3. *The space $\mathbb{R}^\omega$ under the box topology is not locally path connected.*

Corollary 8.6.6.4. *The space $\mathbb{R}_K$ is not locally path connected.*

Corollary 8.6.6.5. *The topologist's sine curve is not locally path connected.*

Proposition 8.6.7 (AC)**.** *The product of a family of locally path connected spaces is locally path connected.*

PROOF:
⟨1⟩1. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of locally connected spaces and $\vec{x} \in \prod_{\alpha \in J} X_\alpha$
⟨1⟩2. LET: $\prod_{\alpha \in J} U_\alpha$ be any basic neighbourhood of $\vec{x}$, where each U_α is open in X_α, and $U_\alpha = X_\alpha$ except for $\alpha = \alpha_1, \ldots, \alpha_n$
⟨1⟩3. For $\alpha \in J$, PICK a path connected neighbourhood C_α of x_α with $C_\alpha \subseteq U_\alpha$
⟨1⟩4. $\prod_{\alpha \in J} C_\alpha$ is path connected
PROOF: Proposition **??**
∎

Proposition 8.6.8. *Let X be a topological space. Then X is locally path connected if and only if, for every open set U in X, every path component of U is open in X.*

PROOF:
⟨1⟩1. If X is locally path connected then, for every open set U in X, every path component of U is open in X.
⟨2⟩1. ASSUME: X is locally path connected.
⟨2⟩2. LET: U be open in X.
⟨2⟩3. LET: C be a path component of U.
⟨2⟩4. LET: $x \in C$
PROVE: C is a neighbourhood of x
⟨2⟩5. U is a neighbourhood of x in X.
PROOF: From ⟨2⟩2, ⟨2⟩3 and ⟨2⟩4.
⟨2⟩6. PICK a path connected neighbourhood V of x such that $V \subseteq U$.
PROOF: Using ⟨2⟩1.
⟨2⟩7. $V \subseteq C$
PROOF: By Proposition 8.3.7.
⟨2⟩8. C is a neighbourhood of x
PROOF: By Proposition 3.2.4.
⟨2⟩9. Q.E.D.
PROOF: By Proposition 3.2.3.
⟨1⟩2. If, for every open set U in X, every path component of U is open in X, then X is locally path connected.

⟨2⟩1. ASSUME: For every open set U in X, every path component of U is open in X.
⟨2⟩2. LET: $x \in X$ and N be a neighbourhood of x
⟨2⟩3. PICK U open such that $x \in U \subseteq N$
⟨2⟩4. LET: C be the path component of U that contains x
⟨2⟩5. C is open in X
PROOF: By ⟨2⟩1.
⟨2⟩6. C is a path connected neighbourhood of x that is included in N
□

Theorem 8.6.9 (AC)**.** *Let X be a topological space. If X is locally path connected, then its components and its path components are the same.*

PROOF:
⟨1⟩1. LET: P be a path component of X
⟨1⟩2. LET: C be the component such that $P \subseteq C$
PROVE: $P = C$
⟨1⟩3. LET: $Q = C \setminus P$
⟨1⟩4. P is open in X
PROOF: By Proposition 8.6.8.
⟨1⟩5. Q is open in X
PROOF: By Proposition 8.6.8 since Q is the union of the path components included in C other than P.
⟨1⟩6. $Q = \emptyset$
PROOF: Otherwise P and Q would form a separation of C, contradicting 8.2.3.
□

Proposition 8.6.10. *$S_\Omega \times \overline{S_\Omega}$ is not locally path connected.*

PROOF: (ω, ω) has no path connected neighbourhood. □

Proposition 8.6.11. *The ordered square is not locally path connected.*

PROOF:
⟨1⟩1. ASSUME: for a contradiction $(1/2, 0)$ has a path connected neighbourhod U
⟨1⟩2. PICK $a < 1/2$ such that $((a, 1), (1/2, 0)) \subseteq U$
⟨1⟩3. LET: $p : [0, 1] \to I_o^2$ be a path from $(a, 1)$ to $(1/2, 0)$
⟨1⟩4. For every x such that $a < x < 1/2$, PICK a rational q_x such that $p(q_x) \in ((x, 0), (x, 1))$
⟨1⟩5. $\{q_x : a < x < 1/2\}$ is an uncountable set of rationals.
□

Proposition 8.6.12. *For any set I, the space $\mathbb{R}^I$ is locally path connected.*

PROOF: Every basic open set is the product of path connected spaces, hence path connected. □

Proposition 8.6.13. *The space $\mathbb{R}^\omega$ under the uniform topology is locally path connected.*

PROOF: Its components and path components are the same. ∎

Proposition 8.6.14. *Every discrete space is locally path connected.*

PROOF: For any point x, the set $\{x\}$ is a path connected neighbourhood of x. ∎

Corollary 8.6.14.1. *The continuous image of a locally path connected space is not necessarily locally path connected.*

Proposition 8.6.15. *A quotient of a locally connected space is locally connected.*

PROOF:
$\langle 1 \rangle 1$. LET: $p : X \twoheadrightarrow Y$ be a quotient map where X is locally connected.
$\langle 1 \rangle 2$. LET: U be open in Y
$\langle 1 \rangle 3$. LET: C be a component of U
PROVE: C is open
$\langle 1 \rangle 4$. $p^{-1}(C)$ is a union of components of $p^{-1}(U)$
$\langle 2 \rangle 1$. LET: $x \in p^{-1}(C)$ and D be the component of $p^{-1}(U)$ that contains x
$\langle 2 \rangle 2$. $p(D)$ is connected.
PROOF: Theorem 8.1.9.
$\langle 2 \rangle 3$. $p(D) \subseteq U$
PROOF: Because $D \subseteq p^{-1}(U)$
$\langle 2 \rangle 4$. $p(D)$ intersects C
PROOF: Both contain $p(x)$
$\langle 2 \rangle 5$. $p(D) \subseteq C$
PROOF: From $\langle 1 \rangle 3$ and $\langle 2 \rangle 2$ and $\langle 2 \rangle 4$
$\langle 2 \rangle 6$. $D \subseteq p^{-1}(C)$
PROOF: From $\langle 2 \rangle 5$
$\langle 1 \rangle 5$. Every component of $p^{-1}(U)$ is open in X
$\langle 2 \rangle 1$. $p^{-1}(U)$ is open.
$\langle 2 \rangle 2$. $p^{-1}(U)$ is locally connected.
$\langle 2 \rangle 3$. Every component of $p^{-1}(U)$ is open in $p^{-1}(U)$
$\langle 2 \rangle 4$. Every component of $p^{-1}(U)$ is open in X.
$\langle 1 \rangle 6$. $p^{-1}(C)$ is a saturated open set.
$\langle 2 \rangle 1$. $p^{-1}(C)$ is saturated.
PROOF: If $x \in p^{-1}(C)$ and $p(x) = p(y)$ then $p(y) \in C$ so $y \in p^{-1}(C)$.
$\langle 2 \rangle 2$. $p^{-1}(C)$ is open.
PROOF: By $\langle 1 \rangle 4$ and $\langle 1 \rangle 5$.
$\langle 1 \rangle 7$. C is open.
PROOF: Lemma 4.5.2.
$\langle 1 \rangle 8$. Q.E.D.
PROOF: Proposition 8.5.5
∎

8.7 Weak Local Connectedness

Definition 8.7.1 (Weakly Locally Connected)**.** Let X be a topological space and $x \in X$. Then X is *weakly locally connected at x* iff every neighbourhood of x contains a connected subspace that contains a neighbourhood of x.

Chapter 9

Compact Spaces

9.1 Countable Compactness

Definition 9.1.1 (Countably Compact)**.** A topological space is *countably compact* iff every countable open covering has a finite subcovering.

9.2 Limit Point Compactness

Definition 9.2.1 (Limit Point Compact)**.** A space is *limit point compact* iff every infinite set has a limit point.

Proposition 9.2.2 (CC)**.** *$S_\Omega \times \overline{S_\Omega}$ is limit point compact.*

Proof:
$\langle 1 \rangle 1$. Let: $A \subseteq S_\Omega \times \overline{S_\Omega}$ be infinite
$\langle 1 \rangle 2$. Case: $\pi_1(A)$ is finite.
  $\langle 2 \rangle 1$. Pick x such that there are infinitely many y such that $(x, y) \in A$
  $\langle 2 \rangle 2$. Pick a limit point l of $\{y : (x, y) \in A\}$
  $\langle 2 \rangle 3$. (x, l) is a limit point of A
$\langle 1 \rangle 3$. Case: $\pi_1(A)$ is infinite.
  $\langle 2 \rangle 1$. Pick a limit point l of $\pi_1(A)$.
  $\langle 2 \rangle 2$. l is a limit ordinal
  $\langle 2 \rangle 3$. Pick a countable sequence x_n with limit l
  $\langle 2 \rangle 4$. For $n \geq 1$, Pick $a_n > x_n$ and y_n such that $(a_n, y_n) \in A$
  $\langle 2 \rangle 5$. Case: $\{y_n : n \geq 1\}$ is finite
    $\langle 3 \rangle 1$. Pick y such that $y = y_n$ for infinitely many n
    $\langle 3 \rangle 2$. (l, y) is a limit point for A
  ¡
  $\langle 2 \rangle 6$. Case: $\{y_n : n \geq 1\}$ is infinite
    $\langle 3 \rangle 1$. Pick a limit point m for $\{y_n : n \geq 1\}$
    $\langle 3 \rangle 2$. (l, m) is a limit point for A
□

Proposition 9.2.3. *The Sorgenfrey plane is not limit point compact.*

PROOF: $\mathbb{Z}^2$ has no limit point. ∎

Proposition 9.2.4. *The space $\mathbb{R}^\omega$ under the box topology is not limit point compact.*

PROOF:The set of all constant sequences of integers is an infinite set with no limit point. ∎

Proposition 9.2.5. *Not every open subspace of a limit point compact space is limit point compact.*

PROOF: The space $[0, 1]$ is limit point compact but $(0, 1)$ is not. ∎

Proposition 9.2.6. *The product of two limit point compact spaces is not necessarily limit point compact.*

PROOF: See Steen and Seebach *Countexamples in Topology* Example 112. ∎

Proposition 9.2.7. *The continuous image of a limit point comapct space is not necessarily limit point comapct.*

PROOF: Let Y be a two-point set under the indiscrete topology. Then $\mathbb{N} \times Y$ is limit point compact, but $\mathbb{N}$ is not. ∎

9.3 Lindelöf Spaces

Definition 9.3.1 (Lindelöf Space)**.** A topological space X is *Lindelöf* iff every open covering has a countable subcovering.

Theorem 9.3.2 (CC)**.** *Every second countable space is Lindelöf.*

PROOF:
⟨1⟩1. LET: X be a second countable space
⟨1⟩2. PICK a countable basis $\mathcal{B}$ for X.
⟨1⟩3. LET: $\mathcal{A}$ be an open cover of X
⟨1⟩4. For every $B \in \mathcal{B}$ such that there exists $U \in \mathcal{A}$ such that $B \subseteq U$, PICK $U_B \in \mathcal{A}$ such that $B \subseteq U_B$
⟨1⟩5. $\{U_B : B \in \mathcal{B}, \exists U \in \mathcal{A}. B \subseteq U\}$ covers X.
  ⟨2⟩1. LET: $x \in X$
  ⟨2⟩2. PICK $U \in \mathcal{A}$ such that $x \in U$
  ⟨2⟩3. PICK $B \in \mathcal{B}$ such that $x \in B \subseteq U$
  ⟨2⟩4. $x \in U_B$
∎

Corollary 9.3.2.1. *The space $\mathbb{R}^\omega$ is Lindelöf.*

Corollary 9.3.2.2. *The space $\mathbb{R}_K$ is Lindelöf.*

Proposition 9.3.3. *The space S_Ω is not Lindelöf.*

PROOF:$\{(-\infty, \alpha) : \alpha \in S_\Omega\}$ is an open cover that has no countable subcover. ∎

Proposition 9.3.4 (CC)**.** *The space $\overline{S_\Omega}$ is Lindelöf.*

PROOF:
⟨1⟩1. LET: $\mathcal{A}$ be an open cover of $\overline{S_\Omega}$
⟨1⟩2. PICK $U \in \mathcal{A}$ such that $\Omega \in U$
⟨1⟩3. PICK $\alpha < \Omega$ such that $(\alpha, \Omega] \subseteq U$
⟨1⟩4. For $\beta \leq \alpha$, PICK $U_\beta \in \mathcal{A}$ such that $\beta \in U_\beta$
⟨1⟩5. $\{U\} \cup \{U_\beta : \beta \leq \alpha\}$ is a countable subcover of $\mathcal{A}$.
∎

Proposition 9.3.5 (CC)**.** *The continuous image of a Lindelöf space is Lindelöf.*

PROOF:
⟨1⟩1. LET: X be a Lindelöf space, Y a space and $f : X \to Y$ continuous.
⟨1⟩2. LET: $\mathcal{A}$ be an open covering of Y
⟨1⟩3. $\{f^{-1}(V) : V \in \mathcal{A}\}$ is an open covering of X
⟨1⟩4. PICK a countable subcovering $\{f^{-1}(V_1), f^{-1}(V_2), \ldots\}$ of $\{f^{-1}(V) : V \in \mathcal{A}\}$
⟨1⟩5. $\{V_1, V_2, \ldots\}$ is a countable subcovering of $\mathcal{A}$
∎

Proposition 9.3.6. *The Sorgenfrey plane is not Lindelöf.*

PROOF:
⟨1⟩1. LET: $L = \{(x, -x) : x \in \mathbb{R}\}$
⟨1⟩2. L is closed in $\mathbb{R}_l^2$
⟨2⟩1. LET: $(x, y) \notin L$, so $y \neq -x$
PROVE: There exists a neighbourhood U of (x, y) that does not intersect L
⟨2⟩2. CASE: $y > -x$
PROOF: In this case, take $U = [x, +\infty) \times [y, +\infty)$
⟨2⟩3. CASE: $y < -x$
PROOF: In this case, take $U = [x, (x-y)/2) \times [y, (y-x)/2)$.
⟨1⟩3. LET: $\mathcal{U} = \{\mathbb{R}_l^2 \setminus L\} \cup \{[a, b) \times [-a, d) : a, b, d \in \mathbb{R}\}$
⟨1⟩4. $\mathcal{U}$ is an open covering of $\mathbb{R}_l^2$
⟨1⟩5. No countable subset of $\mathcal{U}$ covers $\mathbb{R}_l^2$
PROOF: Every set $[a, b) \times [-a, d)$ intersects L in exactly one point, namely $(a, -a)$.
∎

Corollary 9.3.6.1. *The Sorgenfrey plane is not second countable.*

Corollary 9.3.6.2. *The product of two Lindelöf spaces is not necessarily Lindelöf.*

Proposition 9.3.7. *The space $\mathbb{R}^\omega$ under the box topology is not Lindelöf.*

PROOF: The set $\{\prod_{n=0}^{\infty}(a_n, a_n + 1) : \forall n.a_n \in \mathbb{Z}\}$ covers the space but has no countable subcover. $\square$

Proposition 9.3.8. *Not every open subspace of a Lindelöf space is Lindelöf.*

PROOF: The ordered square is Lindelöf but the subspace $[0, 1] times (0, 1)$ is not. $\square$

9.4 Paracompactness

Definition 9.4.1 (Paracompact)**.** A topological space X is *paracompact* iff every open covering of X has a locally finite open refinement that covers X.

Theorem 9.4.2. *Every paracompact Hausdorff space is normal.*

PROOF:
⟨1⟩1. LET: X be a paracompact Hausdorff space.
⟨1⟩2. X is regular.
⟨2⟩1. LET: A be a closed set.
⟨2⟩2. LET: $a \notin A$
⟨2⟩3. For all $x \in A$, there exists an open set U such that $x \in U$ and $a \notin \overline{U}$
⟨3⟩1. LET: $x \in A$
⟨3⟩2. $x \neq a$
PROOF: ⟨2⟩2, ⟨3⟩1
⟨3⟩3. PICK disjoint open neighbourhoods U of x and V of a
PROOF: ⟨1⟩1, ⟨3⟩2
⟨3⟩4. $a \notin \overline{U}$
PROOF: Theorem 3.13.3, ⟨3⟩3.
⟨2⟩4. PICK a locally finite open refinement $\mathcal{C}$ of $\{U \text{ open in } X : a \notin \overline{U}\} \cup \{X \setminus A\}$ that covers X
PROOF: By ⟨2⟩3, $\{U \text{ open in } X : a \notin \overline{U}\} \cup \{X \setminus A\}$ is an open covering of X.
⟨2⟩5. LET: $\mathcal{D} = \{U \in \mathcal{C} : U \cap A \neq \emptyset\}$
⟨2⟩6. $\mathcal{D}$ covers A
PROOF: From ⟨2⟩4 and ⟨2⟩5.
⟨2⟩7. For all $U \in \mathcal{D}$ we have $a \notin \overline{U}$
PROOF: From ⟨2⟩4 and ⟨2⟩5.
⟨2⟩8. LET: $V = \bigcup \mathcal{D}$
⟨2⟩9. V is open
⟨3⟩1. Every member of $\mathcal{D}$ is open.
PROOF: From ⟨2⟩4 and ⟨2⟩5.
⟨3⟩2. Q.E.D.
PROOF: By ⟨2⟩8.
⟨2⟩10. $A \subseteq V$
PROOF: From ⟨2⟩6 and ⟨2⟩7.
⟨2⟩11. $a \notin \overline{V}$

⟨3⟩1. $\mathcal{D}$ is locally finite.

PROOF: Lemma 13.1.45, ⟨2⟩4, ⟨2⟩5.

⟨3⟩2. $\overline{V} = \bigcup_{U \in \mathcal{D}} \overline{U}$

PROOF: By Lemma 3.12.10, ⟨2⟩8 and ⟨3⟩1.

⟨3⟩3. Q.E.D.

PROOF: By ⟨2⟩7.

⟨2⟩12. Q.E.D.

PROOF: Proposition 6.3.2.

⟨1⟩3. X is normal.

⟨2⟩1. LET: A, B be disjoint closed sets.

⟨2⟩2. For all $x \in A$, there exists an open set U such that $x \in U$ and B is disjoint from $\overline{U}$

⟨3⟩1. LET: $x \in A$

⟨3⟩2. $x \notin B$

PROOF: ⟨2⟩2, ⟨3⟩1

⟨3⟩3. PICK disjoint open neighbourhoods U of x and V of B

PROOF: ⟨1⟩2, ⟨3⟩2

⟨3⟩4. B is disjoint from $\overline{U}$

PROOF: $B \subseteq V \subseteq X \setminus \overline{U}$

⟨2⟩3. PICK a locally finite open refinement $\mathcal{C}$ of $\{U \text{ open in } X : B \cap \overline{U} = \emptyset\} \cup \{X \setminus A\}$ that covers X

PROOF: By ⟨2⟩2, $\{U \text{ open in } X : B \cap \overline{U} = \emptyset\} \cup \{X \setminus A\}$ is an open covering of X.

⟨2⟩4. LET: $\mathcal{D} = \{U \in \mathcal{C} : U \cap A \neq \emptyset\}$

⟨2⟩5. $\mathcal{D}$ covers A

PROOF: From ⟨2⟩3 and ⟨2⟩4.

⟨2⟩6. For all $U \in \mathcal{D}$ we have $B \cap \overline{U} = \emptyset$

PROOF: From ⟨2⟩3 and ⟨2⟩4.

⟨2⟩7. LET: $V = \bigcup \mathcal{D}$

⟨2⟩8. V is open

⟨3⟩1. Every member of $\mathcal{D}$ is open.

PROOF: From ⟨2⟩3 and ⟨2⟩4.

⟨3⟩2. Q.E.D.

PROOF: By ⟨2⟩7.

⟨2⟩9. $A \subseteq V$

PROOF: From ⟨2⟩5 and ⟨2⟩6.

⟨2⟩10. $B \cap \overline{V} = \emptyset$

⟨3⟩1. $\mathcal{D}$ is locally finite.

PROOF: Lemma 13.1.45, ⟨2⟩3, ⟨2⟩4.

⟨3⟩2. $\overline{V} = \bigcup_{U \in \mathcal{D}} \overline{U}$

PROOF: By Lemma 3.12.10, ⟨2⟩7 and ⟨3⟩1.

⟨3⟩3. Q.E.D.

PROOF: By ⟨2⟩6.

⟨2⟩11. Q.E.D.

PROOF: V and $X \setminus \overline{V}$ are disjoint open neighbourhoods of A and B respectively.

□

Theorem 9.4.3. *Every closed subspace of a paracompact space is paracompact.*

PROOF:
⟨1⟩1. LET: X be a paracompact space.
⟨1⟩2. LET: Y be closed in X.
⟨1⟩3. LET: $\mathcal{A}$ be an open covering of Y.
⟨1⟩4. $\{U \text{ open in } X : U \cap Y \in \mathcal{A}\} \cup \{X \setminus Y\}$ is an open covering of X.
⟨1⟩5. PICK a locally finite open refinement $\mathcal{B}$ that covers X.
⟨1⟩6. $\{U \cap Y : U \in \mathcal{B}\}$ is a locally finite open refinement of $\mathcal{A}$ that covers Y.
⟨2⟩1. LET: $\mathcal{C} = \{U \cap Y : U \in \mathcal{B}\}$
⟨2⟩2. $\mathcal{C}$ is locally finite.
PROOF: Proposition 3.8.2,⟨1⟩5, ⟨2⟩1.
⟨2⟩3. $\mathcal{C}$ refines $\mathcal{A}$

□

Lemma 9.4.4 (E. Michael (AC))**.** *Let X be a regular space. Then the following are equivalent.*

1. *Every open covering of X has a countably locally finite open refinement that covers X.*
2. *Every open covering of X has a locally finite refinement that covers X.*
3. *Every open covering of X has a locally finite closed refinement that covers X.*
4. *X is paracompact.*

PROOF:
⟨1⟩1. LET: X be a regular space.
⟨1⟩2. $1 \Rightarrow 2$
⟨2⟩1. ASSUME: 1
⟨2⟩2. LET: $\mathcal{A}$ be an open covering of X.
⟨2⟩3. PICK a countably locally finite open refinement $\mathcal{B}$ of $\mathcal{A}$ that covers X.
PROOF: ⟨2⟩1, ⟨2⟩2
⟨2⟩4. PICK locally finite sets $\mathcal{B}_n$ for $n \in \mathbb{N}$ such that $\mathcal{B} = \bigcup_{n=0}^{\infty} \mathcal{B}_n$
PROOF: From ⟨2⟩3
⟨2⟩5. For $n \in \mathbb{N}$,
LET: $V_n = \bigcup \mathcal{B}_n$
⟨2⟩6. For $n \in \mathbb{N}$ and $U \in \mathcal{B}_n$,
LET: $S_n(U) = U \setminus \bigcup_{i<n} V_i$
⟨2⟩7. For $n \in \mathbb{N}$,
LET: $\mathcal{C}_n = \{S_n(U) : U \in \mathcal{B}_n\}$
⟨2⟩8. For $n \in \mathbb{N}$, we have $\mathcal{C}_n$ refines $\mathcal{B}_n$
PROOF: This holds because $S_n(U) \subseteq U$.
⟨2⟩9. LET: $\mathcal{C} = \bigcup_n \mathcal{C}_n$
⟨2⟩10. $\mathcal{C}$ is locally finite

$\langle 3\rangle 1$. LET: $x \in X$

$\langle 3\rangle 2$. LET: N be least such that there exists $U \in \mathcal{B}_N$ such that $x \in U$

PROOF: By $\langle 2\rangle 3$ and $\langle 2\rangle 4$

$\langle 3\rangle 3$. PICK $U \in \mathcal{B}_N$ such that $x \in U$

$\langle 3\rangle 4$. For $1 \leq i \leq N$, PICK a neighbourhood W_i of x that intersects only finitely many elements of $\mathcal{B}_i$

PROOF: By $\langle 2\rangle 4$

$\langle 3\rangle 5$. For $1 \leq i \leq N$, W_i intersects only finitely many elements of $\mathcal{C}_i$

PROOF: If W_i intersects $S_i(U)$ then W_i intersects U.

$\langle 3\rangle 6$. LET: $W = U \cap W_1 \cap \cdots \cap W_N$

$\langle 3\rangle 7$. W intersects only finitely many elements of $\mathcal{C}$

$\langle 4\rangle 1$. For $i \leq N$, W intersects only finitely many elements of $\mathcal{C}_i$

PROOF: From $\langle 3\rangle 5$ and $\langle 3\rangle 6$.

$\langle 4\rangle 2$. For $i > N$, W intersects no elements of $\mathcal{C}_i$.

PROOF: This holds because $W \subseteq U \subseteq V_N$.

$\langle 2\rangle 11$. $\mathcal{C}$ refines $\mathcal{A}$

PROOF: From $\langle 2\rangle 3$ and $\langle 2\rangle 8$

$\langle 2\rangle 12$. $\mathcal{C}$ covers X

$\langle 3\rangle 1$. LET: $x \in X$

$\langle 3\rangle 2$. LET: N be least such that there exists $U \in \mathcal{B}_N$ such that $x \in U$

$\langle 3\rangle 3$. PICK $U \in \mathcal{B}_N$ such that $x \in U$

$\langle 3\rangle 4$. $x \in S_N(U)$

$\langle 1\rangle 3$. $2 \Rightarrow 3$

$\langle 2\rangle 1$. ASSUME: 2

$\langle 2\rangle 2$. LET: $\mathcal{A}$ be an open covering of X.

$\langle 2\rangle 3$. LET: $\mathcal{B} = \{U \text{ open in } X : \exists V \in \mathcal{A}. \overline{U} \subseteq V\}$

$\langle 2\rangle 4$. $\mathcal{B}$ covers X

$\langle 3\rangle 1$. LET: $x \in X$

$\langle 3\rangle 2$. PICK $V \in \mathcal{A}$ such that $x \in V$

PROOF: From $\langle 2\rangle 2$

$\langle 3\rangle 3$. PICK U an open neighbourhood of x such that $\overline{U} \subseteq V$

PROOF: From Proposition 6.3.2, $\langle 1\rangle 1$, $\langle 3\rangle 1$, $\langle 3\rangle 3$.

$\langle 3\rangle 4$. $U \in \mathcal{B}$

PROOF: $\langle 2\rangle 3$, $\langle 3\rangle 2$, $\langle 3\rangle 3$.

$\langle 2\rangle 5$. PICK a locally finite refinement $\mathcal{C}$ of $\mathcal{B}$ that covers X.

PROOF: $\langle 2\rangle 1$, $\langle 2\rangle 3$, $\langle 2\rangle 4$.

$\langle 2\rangle 6$. LET: $\mathcal{D} = \{\overline{C} : C \in \mathcal{C}\}$

$\langle 2\rangle 7$. $\mathcal{D}$ is a locally finite closed refinement of $\mathcal{A}$ that covers X.

$\langle 3\rangle 1$. $\mathcal{D}$ is locally finite.

PROOF: Lemma 3.12.9, $\langle 2\rangle 5$, $\langle 2\rangle 6$.

$\langle 3\rangle 2$. Every member of $\mathcal{D}$ is closed.

PROOF: Proposition 3.12.3, $\langle 2\rangle 6$.

$\langle 3\rangle 3$. $\mathcal{D}$ refines $\mathcal{A}$.

$\langle 4\rangle 1$. LET: $D \in \mathcal{D}$

$\langle 4\rangle 2$. PICK $C \in \mathcal{C}$ such that $D = \overline{C}$

PROOF: $\langle 2\rangle 6$, $\langle 4\rangle 1$

$\langle 4\rangle 3$. PICK $U \in \mathcal{B}$ such that $C \subseteq U$

PROOF: $\langle 2\rangle 5$, $\langle 4\rangle 2$

$\langle 4\rangle 4$. PICK $V \in \mathcal{A}$ such that $\overline{U} \subseteq V$

PROOF: $\langle 2\rangle 3$, $\langle 4\rangle 3$

$\langle 4\rangle 5$. $D \subseteq V$

PROOF:

$$\begin{aligned} D &= \overline{C} && (\langle 4\rangle 2) \\ &\subseteq \overline{U} && (\langle 4\rangle 3, \text{Proposition } 3.12.5) \\ &\subseteq V && (\langle 4\rangle 4) \end{aligned}$$

$\langle 3\rangle 4$. $\mathcal{D}$ covers X.

$\langle 4\rangle 1$. LET: $x \in X$

$\langle 4\rangle 2$. PICK $C \in \mathcal{C}$ such that $x \in C$

PROOF: $\langle 2\rangle 5$, $\langle 4\rangle 1$

$\langle 4\rangle 3$. $x \in \overline{C} \in \mathcal{D}$

$\langle 5\rangle 1$. $x \in \overline{C}$

PROOF: Proposition 3.12.2, $\langle 4\rangle 2$.

$\langle 5\rangle 2$. $\overline{C} \in \mathcal{D}$

PROOF: $\langle 2\rangle 6$, $\langle 4\rangle 2$.

$\langle 1\rangle 4$. $3 \Rightarrow 4$

$\langle 2\rangle 1$. ASSUME: 3

$\langle 2\rangle 2$. LET: $\mathcal{A}$ be an open covering of X

$\langle 2\rangle 3$. PICK a locally finite refinement $\mathcal{B}$ of $\mathcal{A}$ that covers X.

PROOF: $\langle 2\rangle 1$, $\langle 2\rangle 2$

$\langle 2\rangle 4$. $\{U \text{ open in } X : U \text{ intersects only finitely many elements of } \mathcal{B}\}$ is an open covering of X.

PROOF: From $\langle 2\rangle 3$

$\langle 2\rangle 5$. PICK a locally finite closed refinement $\mathcal{C}$ of $\{U \text{ open in } X : U \text{ intersects only finitely many elements}$ that covers X.

PROOF: $\langle 2\rangle 1$, $\langle 2\rangle 4$.

$\langle 2\rangle 6$. Every element of $\mathcal{C}$ intersects only finitely many elements of $\mathcal{B}$

$\langle 3\rangle 1$. LET: $C \in \mathcal{C}$

$\langle 3\rangle 2$. There exists U open in X such that U intersects only finitely many elements of $\mathcal{B}$ and $C \subseteq U$

PROOF: $\langle 2\rangle 5$, $\langle 3\rangle 1$

$\langle 3\rangle 3$. C intersects only finitely many elements of $\mathcal{B}$

PROOF: From $\langle 3\rangle 2$

$\langle 2\rangle 7$. For $B \in \mathcal{B}$,

LET: $C(B) = \{C \in \mathcal{C} : C \subseteq X \setminus B\}$

$\langle 2\rangle 8$. For $B \in \mathcal{B}$,

LET: $E(B) = X \setminus \bigcup C(B)$

$\langle 2\rangle 9$. The union of any subset of $\mathcal{C}$ is closed.

PROOF: Lemma 3.12.10, $\langle 2\rangle 5$.

$\langle 2\rangle 10$. For all $B \in \mathcal{B}$, we have $E(B)$ is open.

PROOF: $\langle 2\rangle 7$, $\langle 2\rangle 8$, $\langle 2\rangle 9$.

$\langle 2\rangle 11$. For all $B \in \mathcal{B}$, we have $B \subseteq E(B)$.

PROOF: $\langle 2\rangle 7$, $\langle 2\rangle 8$.

$\langle 2\rangle 12$. For $B \in \mathcal{B}$, PICK $F(B) \in \mathcal{A}$ such that $B \subseteq F(B)$.

PROOF: $\langle 2\rangle 3$

$\langle 2\rangle 13$. LET: $\mathcal{D} = \{E(B) \cap F(B) : B \in \mathcal{B}\}$

$\langle 2\rangle 14$. $\mathcal{D}$ refines $\mathcal{A}$.

PROOF: $\langle 2\rangle 12$, $\langle 2\rangle 13$

$\langle 2\rangle 15$. $\mathcal{D}$ covers X.

$\langle 3\rangle 1$. LET: $x \in X$

$\langle 3\rangle 2$. PICK $B \in \mathcal{B}$ such that $x \in B$

PROOF: $\langle 2\rangle 3$, $\langle 3\rangle 1$.

$\langle 3\rangle 3$. $x \in E(B) \cap F(B) \in \mathcal{D}$

PROOF: $\langle 2\rangle 11$, $\langle 2\rangle 12$, $\langle 2\rangle 13$, $\langle 3\rangle 2$.

$\langle 2\rangle 16$. $\mathcal{D}$ is locally finite.

$\langle 3\rangle 1$. LET: $x \in X$

$\langle 3\rangle 2$. PICK an open neighbourhood W of x that intersects only finitely many elements of $\mathcal{C}$, say $C_1, \ldots, C_k$.

PROVE: W intersects only finitely many elements of $\mathcal{D}$.

PROOF: $\langle 2\rangle 5$, $\langle 3\rangle 1$

$\langle 3\rangle 3$. W is covered by $C_1, \ldots, C_k$.

PROOF: $\langle 2\rangle 5$, $\langle 3\rangle 2$.

$\langle 3\rangle 4$. Every element of $\mathcal{C}$ intersects only finitely many elements of $\mathcal{D}$.

$\langle 4\rangle 1$. LET: $C \in \mathcal{C}$

$\langle 4\rangle 2$. If C intersects $E(B) \cap F(B)$ for $B \in \mathcal{B}$ then C intersects B

$\langle 5\rangle 1$. LET: $x \in C \cap E(B) \cap F(B)$

$\langle 5\rangle 2$. $C \notin C(B)$

PROOF: $\langle 2\rangle 8$, $\langle 5\rangle 1$

$\langle 5\rangle 3$. C intersects B

PROOF: $\langle 2\rangle 7$, $\langle 5\rangle 2$

$\langle 4\rangle 3$. C intersects only finitely many elements of $\mathcal{B}$

PROOF: $\langle 2\rangle 6$, $\langle 4\rangle 1$

$\langle 4\rangle 4$. Q.E.D.

PROOF: Using $\langle 2\rangle 13$.

$\langle 2\rangle 17$. Every element of $\mathcal{D}$ is open.

$\langle 3\rangle 1$. LET: $B \in \mathcal{B}$.

$\langle 3\rangle 2$. $E(B)$ is open.

PROOF: $\langle 2\rangle 10$, $\langle 3\rangle 1$.

$\langle 3\rangle 3$. $F(B)$ is open.

PROOF: $\langle 2\rangle 2$, $\langle 2\rangle 12$

$\langle 3\rangle 4$. Q.E.D.

PROOF: Using $\langle 2\rangle 13$.

$\langle 1\rangle 5$. $4 \Rightarrow 1$

PROOF: Trivial.

$\square$

Corollary 9.4.4.1. *Every regular Lindelöf space is paracompact.*

Lemma 9.4.5 (Shrinking Lemma (AC))**.** *Let X be a paracompact Hausdorff* *space. Let $\{U_\alpha\}_{\alpha in J}$ be a family of open sets that covers X. Then there exists a locally finite family $\{V_\alpha\}_{\alpha\in J}$ of open sets that covers X such that, for all $\alpha \in J$, we have $\overline{V_\alpha} \subseteq U_\alpha$.*

PROOF:
$\langle 1\rangle 1$. LET: X be a paracompact Hausdorff space.
$\langle 1\rangle 2$. LET: $\{U_\alpha\}_{\alpha\in J}$ be a family of open sets that covers X.
$\langle 1\rangle 3$. LET: $\mathcal{A} = \{V \text{ open in } X : \exists \alpha \in J.\overline{V} \subseteq U_\alpha\}$.
$\langle 1\rangle 4$. $\mathcal{A}$ covers X.
$\langle 2\rangle 1$. LET: $x \in X$.
$\langle 2\rangle 2$. PICK $\alpha \in J$ such that $x \in U_\alpha$.
PROOF: $\langle 1\rangle 2$
$\langle 2\rangle 3$. PICK V open such that $x \in V$ and $\overline{V} \subseteq U_\alpha$
PROOF: Theorem 9.4.2, $\langle 2\rangle 2$.
$\langle 2\rangle 4$. $x \in V \in \mathcal{A}$
PROOF: $\langle 1\rangle 3$, $\langle 2\rangle 3$
$\langle 1\rangle 5$. PICK a locally finite open refinment $\mathcal{B}$ of $\mathcal{A}$ that covers X.
PROOF: $\langle 1\rangle 1$, $\langle 1\rangle 3$, $\langle 1\rangle 4$
$\langle 1\rangle 6$. For $B \in \mathcal{B}$ PICK $f(B) \in J$ such that $\overline{B} \subseteq U_{f(B)}$
$\langle 2\rangle 1$. LET: $B \in \mathcal{B}$
$\langle 2\rangle 2$. PICK $V \in \mathcal{A}$ such that $B \subseteq V$
PROOF: $\langle 1\rangle 5$, $\langle 2\rangle 1$
$\langle 2\rangle 3$. PICK $\alpha \in J$ such that $\overline{V} \subseteq U_\alpha$.
PROOF: $\langle 1\rangle 3$, $\langle 2\rangle 2$
$\langle 2\rangle 4$. $\overline{B} \subseteq U_\alpha$
PROOF:
$$\begin{aligned} \overline{B} &\subseteq \overline{V} && \text{(Proposition 3.12.5, } \langle 2\rangle 2) \\ &\subseteq U_\alpha && (\langle 2\rangle 3) \end{aligned}$$
$\langle 1\rangle 7$. For $\alpha \in J$
LET: $V_\alpha = \bigcup_{f(B)=\alpha} B$
$\langle 1\rangle 8$. For all $\alpha \in J$ we have $\overline{V_\alpha} \subseteq U_\alpha$
$\langle 2\rangle 1$. LET: $\alpha \in J$
$\langle 2\rangle 2$. $\overline{V_\alpha} \subseteq U_\alpha$
PROOF:
$$\begin{aligned} \overline{V_\alpha} &= \overline{\bigcup_{f(B)=\alpha} B} && (\langle 1\rangle 7) \\ &= \bigcup_{f(B)=\alpha} \overline{B} && \text{(Lemma 3.12.10, Lemma 13.1.45, } \langle 1\rangle 5) \\ &\subseteq \bigcup_{f(B)=\alpha} U_{f(B)} && (\langle 1\rangle 6) \\ &= U_\alpha \end{aligned}$$
$\langle 1\rangle 9$. $\{V_\alpha\}_{\alpha\in J}$ is locally finite.
$\langle 2\rangle 1$. LET: $x \in X$
$\langle 2\rangle 2$. PICK an open neighbourhood W of x that intersects only finitely many

elements of $\mathcal{B}$, say $B_1, \ldots, B_n$

PROOF: $\langle 1\rangle 5$, $\langle 2\rangle 1$

$\langle 2\rangle 3$. For all $\alpha \in J$, if W intersects V_α then α is one of $f(B_1), \ldots, f(B_n)$.

$\langle 3\rangle 1$. LET: $\alpha \in J$

$\langle 3\rangle 2$. ASSUME: W intersects V_α

$\langle 3\rangle 3$. PICK $y \in W \cap V_\alpha$

PROOF: $\langle 3\rangle 2$

$\langle 3\rangle 4$. PICK B such that $f(B) = \alpha$ and $y \in B$

PROOF: $\langle 1\rangle 7$, $\langle 3\rangle 3$

$\langle 3\rangle 5$. B is one of $B_1, \ldots, B_n$

PROOF: $\langle 2\rangle 2$, $\langle 3\rangle 3$, $\langle 3\rangle 4$

$\langle 2\rangle 4$. W intersects only finitely many V_α

PROOF: $\langle 2\rangle 3$

$\square$

Theorem 9.4.6. *Let X be a paracompact Hausdorff space. Let $\mathcal{C} \subseteq \mathcal{P}X$ be locally finite. For $C \in \mathcal{C}$ let $\epsilon_C > 0$. Then there exists a continuous function $f : X \to \mathbb{R}$ such that $f(x) > 0$ for all $x \in X$, and $f(x) \leq \epsilon_C$ for all $C \in \mathcal{C}$ and $x \in C$.*

PROOF:

$\langle 1\rangle 1$. LET: $\mathcal{A} = \{U \text{ open in } X : U \text{ intersects at most finitely many elements of } \mathcal{C}\}$

$\langle 1\rangle 2$. $\mathcal{A}$ covers X.

PROOF: Holds since $\mathcal{C}$ is locally finite.

$\langle 1\rangle 3$. PICK a partition of unity $\{\phi_U\}_{U\in\mathcal{A}}$ dominated by $\{U\}_{U\in\mathcal{A}}$.

PROOF: Theorem 10.2.58, $\langle 1\rangle 1$, $\langle 1\rangle 2$.

$\langle 1\rangle 4$. For $U \in \mathcal{A}$,

LET:

$$\delta_U = \begin{cases} \min\{\epsilon_C : C \in \mathcal{C}, C \cap \operatorname{supp} \ \phi_U \neq \emptyset\} & \text{if there exists at least one such } C \\ 1 & \text{if not} \end{cases}$$

$\langle 1\rangle 5$. LET: $f(x) = \sum_{U\in\mathcal{A}} \delta_U \phi_U(x)$

$\langle 2\rangle 1$. For $x \in X$ we have $\phi_U(x) = 0$ for all but finitely many U

$\langle 3\rangle 1$. LET: $x \in X$

$\langle 3\rangle 2$. PICK an open neighbourhood W of x that intersects $\operatorname{supp} \phi_U$ for only finitely many U, say $U_1, \ldots, U_n$

PROOF: $\langle 1\rangle 3$, $\langle 3\rangle 1$

$\langle 3\rangle 3$. For all $U \in \mathcal{A}$, if $\phi_U(x) \neq 0$ then U is one of $U_1, \ldots, U_n$

$\langle 4\rangle 1$. LET: $U \in \mathcal{A}$

$\langle 4\rangle 2$. ASSUME: $\phi_U(x) \neq 0$

$\langle 4\rangle 3$. $x \in \operatorname{supp} \phi_U$

PROOF: Proposition 3.12.2, $\langle 4\rangle 2$.

$\langle 4\rangle 4$. U is one of $U_1, \ldots, U_n$

PROOF: $\langle 3\rangle 2$, $\langle 4\rangle 3$

$\langle 1\rangle 6$. $f(x) > 0$ for all $x \in X$.

$\langle 2\rangle 1$. LET: $x \in X$

$\langle 2\rangle 2$. PICK $U \in \mathcal{A}$ such that $\phi_U(x) > 0$

PROOF: Such a U exists since $\sum_{U \in \mathcal{A}} \phi_U(x) = 1$ by $\langle 1 \rangle 3$.

$\langle 2 \rangle 3$. $\delta_U > 0$

PROOF: $\langle 1 \rangle 4$

$\langle 2 \rangle 4$. Q.E.D.

PROOF: $\langle 1 \rangle 5$

$\langle 1 \rangle 7$. For $C \in \mathcal{C}$ and $x \in C$ we have $f(x) \leq \epsilon_C$.

$\langle 2 \rangle 1$. LET: $C \in \mathcal{C}$

$\langle 2 \rangle 2$. LET: $x \in C$

$\langle 2 \rangle 3$. For all $U \in \mathcal{A}$ we have $\delta_U \phi_U(x) \leq \epsilon_C \phi_U(x)$

$\langle 3 \rangle 1$. LET: $U \in \mathcal{A}$

PROVE: $\delta_U \phi_U(x) \leq \epsilon_C \phi_U(x)$

$\langle 3 \rangle 2$. CASE: $x \in \operatorname{supp} \phi_U$

PROOF: In this case, $\delta_U \leq \epsilon_C$ by $\langle 1 \rangle 4$, $\langle 2 \rangle 2$.

$\langle 3 \rangle 3$. CASE: $x \notin \operatorname{supp} \phi_U$

PROOF: In this case we have $\phi_U(x) = 0$ by Proposition 3.12.2.

$\langle 2 \rangle 4$. $f(x) \leq \epsilon_C$

PROOF:

$$\begin{aligned}
f(x) &= \sum_{U \in \mathcal{A}} \delta_U \phi_U(x) && (\langle 1 \rangle 5) \\
&\leq \sum_{U \in \mathcal{A}} \epsilon_C \phi_U(x) && (\langle 2 \rangle 3) \\
&= \epsilon_C \sum_{U \in \mathcal{A}} \phi_U(x) \\
&= \epsilon_C && (\langle 1 \rangle 3)
\end{aligned}$$

□

Lemma 9.4.7 (Expansion Lemma)**.** *Let $\{B_\alpha\}_{\alpha \in J}$ be a locally finite family of subsets of the paracompact Hausdorff space X. Then there exists a locally finite family $\{U_\alpha\}_{\alpha \in J}$ of open sets such that $B_\alpha \subseteq U_\alpha$ for all $\alpha \in J$.*

PROOF:

$\langle 1 \rangle 1$. LET: X be a paracompact Hausdorff space.

$\langle 1 \rangle 2$. LET: $\{B_\alpha\}_{\alpha \in J}$ be locally finite

$\langle 1 \rangle 3$. LET: $\mathcal{A} = \{U \text{ open in } X : U \text{ intersects } B_\alpha \text{ for only finitely many } \alpha\}$

$\langle 1 \rangle 4$. PICK a locally finite open refinement $\mathcal{B}$ of $\mathcal{A}$ that covers X.

$\langle 2 \rangle 1$. Every element of $\mathcal{A}$ is open.

PROOF: From $\langle 1 \rangle 3$.

$\langle 2 \rangle 2$. $\mathcal{A}$ covers X

PROOF: From $\langle 1 \rangle 2$, $\langle 1 \rangle 3$.

$\langle 2 \rangle 3$. Q.E.D.

PROOF: From $\langle 1 \rangle 1$.

$\langle 1 \rangle 5$. For $\alpha \in J$,

LET: $U_\alpha = \bigcup \{V \in \mathcal{B} : V \cap B_\alpha \neq \emptyset\}$

$\langle 1 \rangle 6$. $\{U_\alpha\}_{\alpha \in J}$ is locally finite.

$\langle 2 \rangle 1$. Every element of $\mathcal{B}$ intersects B_α for only finitely many α.

$\langle 3 \rangle 1$. LET: $V \in \mathcal{B}$

$\langle 3 \rangle 2$. PICK $U \in \mathcal{A}$ such that $U \subseteq V$
PROOF: $\langle 1 \rangle 4$, $\langle 3 \rangle 1$
$\langle 3 \rangle 3$. U intersects B_α for only finitely many α
PROOF: $\langle 1 \rangle 3$, $\langle 3 \rangle 2$
$\langle 3 \rangle 4$. V intersects B_α for only finitely many α
PROOF: $\langle 3 \rangle 2$, $\langle 3 \rangle 3$
$\langle 2 \rangle 2$. LET: $x \in X$
$\langle 2 \rangle 3$. PICK an open neighbourhood W of x that intersects only finitely many elements of $\mathcal{B}$, say $V_1, \ldots, V_n$.
PROOF: $\langle 1 \rangle 4$, $\langle 2 \rangle 2$
$\langle 2 \rangle 4$. For $1 \leq i \leq n$,
LET: $\alpha_{i1}, \ldots, \alpha_{ir_i}$ be the finitely many values of α such that V_i intersects B_α
PROVE: If W intersects B_α then $\alpha = \alpha_{ij}$ for some i, j
PROOF: $\langle 2 \rangle 1$, $\langle 2 \rangle 3$.
$\langle 2 \rangle 5$. LET: $y \in W \cap B_\alpha$
$\langle 2 \rangle 6$. PICK $V \in \mathcal{B}$ such that $y \in V$
PROOF: $\langle 1 \rangle 4$
$\langle 2 \rangle 7$. LET: $V = V_i$
PROOF: $\langle 2 \rangle 3$, $\langle 2 \rangle 5$, $\langle 2 \rangle 6$
$\langle 2 \rangle 8$. V_i intersects B_α
PROOF: $\langle 2 \rangle 5$, $\langle 2 \rangle 6$, $\langle 2 \rangle 7$
$\langle 2 \rangle 9$. $\alpha = \alpha_{ij}$ for some j.
PROOF: $\langle 2 \rangle 4$, $\langle 2 \rangle 8$
$\langle 1 \rangle 7$. For all $\alpha \in J$, we have U_α is open.
PROOF: $\langle 1 \rangle 5$
$\langle 1 \rangle 8$. For all $\alpha \in J$, we have $B_\alpha \subseteq U_\alpha$.
$\langle 2 \rangle 1$. LET: $\alpha \in J$
$\langle 2 \rangle 2$. LET: $x \in B_\alpha$
$\langle 2 \rangle 3$. PICK $V \in \mathcal{B}$ such that $x \in V$
PROOF: $\langle 1 \rangle 4$
$\langle 2 \rangle 4$. $V \cap B_\alpha \neq \emptyset$
PROOF: $\langle 2 \rangle 2$, $\langle 2 \rangle 3$
$\langle 2 \rangle 5$. $x \in U_\alpha$
PROOF: $\langle 1 \rangle 5$, $\langle 2 \rangle 3$, $\langle 2 \rangle 4$
$\square$

9.5 Compactness

Definition 9.5.1 (Compact)**.** A topological space is *compact* iff every open cover has a finite subcover.

Proposition 9.5.2. *S_Ω is not compact.*

PROOF: The open covering $\{(-\infty, \alpha) : \alpha \in S_\Omega\}$ has no finite subcovering. $\square$

Proposition 9.5.3. *$\mathbb{R}_l$ is not compact.*

PROOF: $\{[n, n+1) : n \in \mathbb{Z}\}$ has no finite subcover. ∎

Proposition 9.5.4. *The space $\mathbb{R}^\omega$ under the box topology is not compact.*

PROOF: The set $\{\prod_{n=0}^{\infty}(a_n, a_n+1) : n \in \mathbb{Z}\}$ is a cover that has no finite subcover.
∎

Proposition 9.5.5. *Let Y be a subspace of X. Then Y is compact if and only if every covering of Y by sets open in X contains a finite subcollection covering Y.*

PROOF:
$\langle 1 \rangle 1$. If Y is compact then every covering of Y by sets open in X contains a finite subcollection covering Y.
  $\langle 2 \rangle 1$. ASSUME: Y is compact.
  $\langle 2 \rangle 2$. LET: $\mathcal{A}$ be a covering of Y by sets open in X.
  $\langle 2 \rangle 3$. $\{U \cap Y : U \in \mathcal{A}\}$ is an open covering of Y.
  $\langle 2 \rangle 4$. PICK a finite subcovering $V_1, \ldots, V_n$ of $\{U \cap Y : U \in \mathcal{A}\}$
  $\langle 2 \rangle 5$. For $1 \leq i \leq n$, PICK $U_i \in \mathcal{A}$ such that $V_i = U_i \cap Y$.
  $\langle 2 \rangle 6$. $\{U_1, \ldots, U_n\}$ is a finite subset of $\mathcal{A}$ that covers Y.
$\langle 1 \rangle 2$. If every covering of Y by sets open in X contains a finite subcollection covering Y then Y is compact.
  $\langle 2 \rangle 1$. ASSUME: Every covering of Y by sets open in X contains a finite subcollection covering Y.
  $\langle 2 \rangle 2$. LET: $\mathcal{A}$ be an open covering of Y
  $\langle 2 \rangle 3$. LET: $\mathcal{B} = \{U \text{ open in } X : U \cap Y \in \mathcal{A}\}$
  $\langle 2 \rangle 4$. $\mathcal{B}$ covers Y
  $\langle 2 \rangle 5$. PICK a finite subcollection $\{U_1, \ldots, U_n\} \subseteq \mathcal{B}$ that covers Y
  $\langle 2 \rangle 6$. $\{U_1 \cap Y, \ldots, U_n \cap Y\}$ is a finite subcover of $\mathcal{A}$.
∎

Proposition 9.5.6. *Every closed subspace of a compact space is compact.*

PROOF:
$\langle 1 \rangle 1$. LET: X be a compact space and $Y \subseteq X$ be closed.
$\langle 1 \rangle 2$. LET: $\mathcal{A}$ be a covering of Y by spaces open in X
$\langle 1 \rangle 3$. $\mathcal{A} \cup \{X \setminus Y\}$ is an open covering of X.
$\langle 1 \rangle 4$. PICK a finite subcovering $\{U_1, \ldots, U_n\}$ or $\{U_1, \ldots, U_n, X \setminus Y\}$
$\langle 1 \rangle 5$. $\{U_1, \ldots, U_n\}$ is a finite subset of $\mathcal{A}$ that covers Y.
$\langle 1 \rangle 6$. Q.E.D.
  PROOF: Proposition 9.5.5.
∎

Corollary 9.5.6.1. *Not every compact Hausdorff space is connected.*

PROOF: The space $[0, 1] \cup [2, 3]$ is compact Hausdorff and disconnected. ∎

Corollary 9.5.6.2. *Not every compact Hausdorff space is path connected.*

Corollary 9.5.6.3. *Not every compact Hausdorff space is locally connected.*

The space $[0,1] \cap \mathbb{Q}$ is not locally connected.

Corollary 9.5.6.4. *Not every compact Hausdorff space is locally path connected.*

Proposition 9.5.7. *Not every open subspace of a compact space is compact.*

PROOF: The space $[0,1]$ is compact but $(0,1)$ is not. ∎

Lemma 9.5.8. *If Y is a compact subspace of the Hausdorff space X and $a \notin Y$, then there exist disjoint open sets U and V of X containing a and Y, respectively.*

PROOF:
⟨1⟩1. For $y \in Y$, there exist disjoint open sets U and V such that $a \in U$ and $y \in V$.
⟨1⟩2. $\{V \text{ open in } X : \exists U \text{ open and disjoint from } V. a \in U\}$ is a covering of Y by open sets in X.
⟨1⟩3. PICK a finite subset $\{V_1, \ldots, V_n\}$ that covers Y.
⟨1⟩4. For $1 \leq i \leq n$, PICKU_i disjoint from V_i such that $a \in U_i$
⟨1⟩5. LET: $U = U_1 \cap \cdots \cap U_n$ and $V = V_1 \cup \cdots \cup V_n$
∎

Proposition 9.5.9. *Every compact subspace of a Hausdorff space is closed.*

PROOF:
⟨1⟩1. LET: X be a Hausdorff space and $Y \subseteq X$ be compact.
⟨1⟩2. Every point $a \notin Y$ has an open neighbourhood disjoint from Y.
PROOF: By Lemma 9.5.8.
⟨1⟩3. Q.E.D.
PROOF: By Proposition 3.2.3.

Proposition 9.5.10. *The image of a compact space under a continuous map is compact.*

PROOF:
⟨1⟩1. LET: $f : X \to Y$ be continuous where X is compact.
⟨1⟩2. LET: $\mathcal{A}$ be a covering of $f(X)$ by open sets in Y.
⟨1⟩3. $\{f^{-1}(U) : U \in \mathcal{A}\}$ is an open covering of X.
⟨1⟩4. PICK a finite subcovering $\{f^{-1}(U_1), \ldots, f^{-1}(U_n)\}$
⟨1⟩5. $\{U_1, \ldots, U_n\}$ is a finite subset of $\mathcal{A}$ that covers $f(X)$.
⟨1⟩6. Q.E.D.
PROOF: By Proposition 9.5.5.
∎

Corollary 9.5.10.1. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces. If $\prod_{\alpha \in J} X_\alpha$ is compact then each X_α is compact.*

Corollary 9.5.10.2. $S_\Omega \times \overline{S_\Omega}$ *is compact.*

Corollary 9.5.10.3. *The Sorgenfrey plane is not compact.*

Corollary 9.5.10.4. *For any nonempty set I, the sapce $\mathbb{R}^I$ is not compact.*

Corollary 9.5.10.5. *Let $\mathcal{T}$ and $\mathcal{T}'$ be topologies on the same set X. If $\mathcal{T} \subseteq \mathcal{T}'$ and $\mathcal{T}'$ is compact then $\mathcal{T}$ is compact.*

Corollary 9.5.10.6. *The space $\mathbb{R}_K$ is not compact.*

Theorem 9.5.11. *Let $f : X \to Y$ be a bijective continuous function. If X is compact and Y is Hausdorff then f is a homeomorphism.*

PROOF:
$\langle 1 \rangle 1.$ LET: C be closed in X
$\langle 1 \rangle 2.$ C is compact
  PROOF: Proposition 9.5.6.
$\langle 1 \rangle 3.$ $f(C)$ is compact
  PROOF: Proposition 9.5.10
$\langle 1 \rangle 4.$ $f(C)$ is closed
  PROOF: Proposition 9.5.9.
$\langle 1 \rangle 5.$ Q.E.D.
  PROOF: By Theorem 5.2.2 we have that f^{-1} is continuous.
□

Corollary 9.5.11.1. *Let $\mathcal{T}$ and $\mathcal{T}'$ be topologies on the same set X. If $\mathcal{T} \subseteq \mathcal{T}'$, $\mathcal{T}$ is Hausdorff and $\mathcal{T}'$ is compact then $\mathcal{T} = \mathcal{T}'$.*

Corollary 9.5.11.2. *The space $[0, 1]$ is not compact as a subspace of $\mathbb{R}_K$.*

Theorem 9.5.12 (Tube Lemma)**.** *Let A and B be subspaces of X and Y, respectively; let N be an open set in $X \times Y$ including $A \times B$. If A and B are compact, then there exist open sets U and V in X and Y, respectively, such that*

$$A \times B \subseteq U \times V \subseteq N \ .$$

PROOF:
$\langle 1 \rangle 1.$ For all $a \in A$, there exist open sets U and V in X and Y, respectively, such that

$$\{a\} \times B \subseteq U \times V \subseteq N \ .$$

  $\langle 2 \rangle 1.$ LET: $a \in A$
  $\langle 2 \rangle 2.$ For all $b \in B$, there exist open sets U and V in X and Y, respectively, such that $(a, b) \in U \times V \subseteq N$.
  $\langle 2 \rangle 3.$ $\{V \text{ open in } Y : \exists U \text{ open in } X. a \in U, U \times V \subseteq N\}$ covers B
  $\langle 2 \rangle 4.$ PICK a finite subset $\{V_1, \ldots, V_n\}$ that covers B.
  $\langle 2 \rangle 5.$ For $1 \leq i \leq n$, PICK U_i open in X such that $a \in U_i$ and $U_i \times V_i \subseteq N$
  $\langle 2 \rangle 6.$ LET: $U = U_1 \cap \cdots \cap U_n$ and $V = V_1 \cup \cdots \cup V_n$
$\langle 1 \rangle 2.$ $\{U \text{ open in } X : \exists V \text{ open in } Y. B \subseteq V \text{ and } U \times V \subseteq N\}$ covers A.
$\langle 1 \rangle 3.$ PICK a finite subset $\{U_1, \ldots, U_n\}$ that covers A.
$\langle 1 \rangle 4.$ For $1 \leq i \leq n$, PICK V_i open in B such that $B \subseteq V_i$ and $U_i \times V_i \subseteq N$.
$\langle 1 \rangle 5.$ LET: $U = U_1 \cup \cdots \cup U_n$ and $V = V_1 \cap \cdots \cap V_n$
$\langle 1 \rangle 6.$ $A \times B \subseteq U \times V \subseteq N$

∎

Lemma 9.5.13. *Let $\mathcal{A}$ be a set of basis elements for $X \times Y$ such that no finite subset of $\mathcal{A}$ covers $X \times Y$. If X is compact, then there exists a point $x \in X$ such that no finite subset of $\mathcal{A}$ covers $\{x\} \times Y$.*

PROOF:
⟨1⟩1. ASSUME: X is compact.
⟨1⟩2. ASSUME: For all $x \in X$, there is a finite subset of $\mathcal{A}$ that covers $\{x\} \times Y$
PROVE: A finite subset of $\mathcal{A}$ covers $X \times Y$
⟨1⟩3. $\{U \text{ open in } X : \exists U_1 \times V_1, \ldots, U_r \times V_r \in \mathcal{A}. U = U_1 \cap \cdots \cap U_r, Y = V_1 \cup \cdots \cup V_r\}$ covers X.
⟨1⟩4. PICK a finite subcover $\{U_1, \ldots, U_n\}$
⟨1⟩5. For $1 \leq i \leq n$, PICK$U_{i1} \times V_{i1}, \ldots, U_{ir_i} \times V_{ir_i} \in \mathcal{A}$ such that $U_i = U_{i1} \cap \cdots \cap U_{ir_i}$ and $Y = V_{i1} \cup \cdots \cup V_{ir_i}$
⟨1⟩6. $\{U_{ij} : 1 \leq i \leq n, 1 \leq j \leq r_i\}$ covers $X \times Y$
∎

Proposition 9.5.14. *The product of two compact spaces is compact.*

PROOF:
⟨1⟩1. LET: X and Y be compact spaces.
⟨1⟩2. LET: $\mathcal{A}$ be an open covering of $X \times Y$
⟨1⟩3. For all $x \in X$, there exists a neighbourhood W of x such that $W \times Y$ is covered by finitely many elements of $\mathcal{A}$.
⟨2⟩1. LET: $x \in X$
⟨2⟩2. $\{x\} \times Y$ is compact.
PROOF: It is homeomorphic to Y.
⟨2⟩3. PICK a finite subset $\{U_1, \ldots, U_m\}$ of $\mathcal{A}$ that covers $\{x\} \times Y$
PROOF: By Proposition 9.5.5.
⟨2⟩4. There exists a neighbourhood W of x such that $W \times Y \subseteq U_1 \cup \cdots \cup U_m$
PROOF: By the Tube Lemma.
⟨1⟩4. $\{W \text{ open in } X : W \times Y \text{ is covered by finitely many elements of } \mathcal{A}\}$ is an open covering of X.
⟨1⟩5. PICK a finite subcovering $\{W_1, \ldots, W_n\}$
⟨1⟩6. For $1 \leq i \leq n$, PICK a finite subset $\{U_{i1}, \ldots, U_{ir_i}\}$ of $\mathcal{A}$ that covers $W_i \times Y$
⟨1⟩7. $\{U_{11}, \ldots, U_{nr_n}\}$ is a finite subcovering of $\mathcal{A}$.
∎

Proposition 9.5.15. *A topological space is compact if and only if every nonempty set of closed sets that has the finite intersection property has nonempty intersection.*

PROOF: Immediate from definitions. ∎

Lemma 9.5.16. *If Y is compact then $\pi_1 : X \times Y \to X$ is a closed map.*

PROOF:
⟨1⟩1. LET: $C \subseteq X \times Y$ be closed

⟨1⟩2. LET: $x \in X \setminus \pi_1(C)$
⟨1⟩3. For all $y \in Y$, we have $(x, y) \notin C$
⟨1⟩4. For all $y \in Y$, there exist open neighbourhoods U of x and V of y such that $U \times V \subseteq (X \times Y) \setminus C$
⟨1⟩5. $\{V$ open in $Y : \exists U$ an open neighbourhood of x such that $U \times V \subseteq (X \times Y) \setminus C\}$ is an open covering of Y.
⟨1⟩6. PICK a finite subcovering $\{V_1, \ldots, V_n\}$
⟨1⟩7. For $1 \leq i \leq n$, PICK an open neighbourhood U_i of x such that $U_i \times V_i \subseteq (X \times Y) \setminus C$
⟨1⟩8. $x \in U_1 \cap \cdots \cap U_n \subseteq X \setminus \pi_1(C)$
□

Theorem 9.5.17. *Let X be a compact space. Let $f_n : X \to \mathbb{R}$ be a sequence of continuous functions such that, for all $x \in X$, $f_n(x) \to f(x)$ as $n \to \infty$. If f is continuous, and if the sequence $(f_n)_n$ is monotone increasing, and if X is compact, then the convergence is uniform.*

PROOF:
⟨1⟩1. LET: $\epsilon > 0$
PROVE: There exists N such that, for all $n \geq N$, we have $|f_n(x) - f(x)| < \epsilon$
⟨1⟩2. For $n \in \mathbb{Z}^+$,
LET: $U_n = \{x \in X : f(x) - f_n(x) < \epsilon\}$
⟨1⟩3. Each U_n is open
PROOF: Let $g(x) = f(x) - f_n(x)$. Then g is continuous and $U_n = g^{-1}((-\infty, \epsilon))$.
⟨1⟩4. $\{U_n : n \geq 1\}$ is an open covering of X
⟨2⟩1. LET: $x \in X$
⟨2⟩2. PICK N such that, for all $n \geq N$, $|f(x) - f_n(x)| < \epsilon$
PROOF: $f_n(x) \to f(x)$ as $n \to \infty$
⟨2⟩3. $f(x) - f_N(x) < \epsilon$
PROOF: This holds since the sequece $(f_n)_n$ is monotone.
⟨1⟩5. PICK a finite subcovering $\{U_{n_1}, \ldots, U_{n_k}\}$
⟨1⟩6. LET: $N = \max(n_1, \ldots, n_k)$
⟨1⟩7. For all $n \geq N$ we have $|f_n(x) - f(x)| < \epsilon$
□

Lemma 9.5.18. *Every compact Hausdorff space is normal.*

PROOF:From Thearem 9.4.2

Corollary 9.5.18.1. *The ordered square is normal.*

Theorem 9.5.19. *Let X be a complete linearly ordered set under the order topology. Then every closed interval in X is compact.*

PROOF:
⟨1⟩1. LET: X be a complete linearly ordered set in the order topology
⟨1⟩2. LET: $a, b \in X$, $a < b$

PROVE: $[a, b]$ is compact

$\langle 1\rangle 3$. LET: $\mathcal{A}$ be a set of open sets that covers $[a, b]$

$\langle 1\rangle 4$. For all $x \in [a, b)$, there exists $y \in (x, b]$ such that $[x, y]$ is covered by at most two points of $\mathcal{A}$

$\langle 2\rangle 1$. LET: $x \in [a, b]$

$\langle 2\rangle 2$. PICK $U \in \mathcal{A}$ such that $x \in U$

PROOF: By $\langle 1\rangle 3$ and $\langle 2\rangle 1$

$\langle 2\rangle 3$. PICK $y \in (x, b]$ such that $[x, y) \subseteq U$

PROOF: By Lemma 4.1.2.

$\langle 2\rangle 4$. PICK $V \in \mathcal{A}$ such that $y \in V$

PROOF: By $\langle 1\rangle 3$ and $\langle 2\rangle 3$.

$\langle 2\rangle 5$. $[x, y]$ is covered by $\{U, V\}$

PROOF: By $\langle 2\rangle 3$ and $\langle 2\rangle 4$.

$\langle 1\rangle 5$. LET: $C = \{y \in (a, b] : [a, y]$ is covered by a finite subset of $\mathcal{A}\}$

$\langle 1\rangle 6$. C is nonempty

PROOF: By $\langle 1\rangle 4$.

$\langle 1\rangle 7$. LET: $c = \sup C$

PROOF: By $\langle 1\rangle 1$.

$\langle 1\rangle 8$. $c \in C$

$\langle 2\rangle 1$. PICK $U \in \mathcal{A}$ such that $c \in U$

$\langle 2\rangle 2$. PICK $y \in [a, c)$ such that $(y, c] \subseteq U$

PROOF: By Lemma 4.1.2

$\langle 2\rangle 3$. PICK z such that $y < z$ and $z \in C$

PROOF: This exists because y is not an upper bound for C.

$\langle 2\rangle 4$. PICK a finite $\mathcal{A}_0 \subseteq \mathcal{A}$ such that $[a, z]$ is covered by $\mathcal{A}_0$

$\langle 2\rangle 5$. $[a, c]$ is covered by $\mathcal{A}_0 \cup \{U\}$

$\langle 1\rangle 9$. $c = b$

$\langle 2\rangle 1$. ASSUME: for a contradiction $c < b$

$\langle 2\rangle 2$. PICK $y \in (c, b]$ such that $[c, y]$ is covered by at most two elements of $\mathcal{A}$.

PROOF: By $\langle 1\rangle 4$

$\langle 2\rangle 3$. $y > c$ and $y \in C$

$\langle 2\rangle 4$. Q.E.D.

PROOF: This contradicts $\langle 1\rangle 7$.

$\langle 1\rangle 10$. Q.E.D.

Corollary 9.5.19.1. *Every closed interval in $\mathbb{R}$ is compact.*

Corollary 9.5.19.2 (CC)**.** *S_Ω is limit point compact.*

PROOF:

$\langle 1\rangle 1$. LET: A be an infinite subset of S_Ω

$\langle 1\rangle 2$. PICK a countably infinite subset $B \subseteq A$

$\langle 1\rangle 3$. LET: $b = \sup B$

$\langle 1\rangle 4$. $B \subseteq [0, b]$

$\langle 1\rangle 5$. $[0, b]$ is compact

PROOF: By the theorem.

$\langle 1\rangle 6$. B has a limit point in $[0, b]$

$\langle 1\rangle 7$. A has a limit point in $[0, b]$
□

Corollary 9.5.19.3. *The ordered square is compact.*

Corollary 9.5.19.4. *The ordered square is limit point compact.*

Corollary 9.5.19.5. *Not every subspace of a compact space is compact.*

PROOF: $[0,1]$ is compact but $(0,1)$ is not. □

Theorem 9.5.20 (Extreme Value Theorem)**.** *Let $f : X \to Y$ be continuous where Y is a linearly ordered set in the order topology. If X is compact, then there exist $c, d \in X$ such that, for all $x \in X$, we have $f(c) \leq f(x) \leq f(d)$.*

PROOF:
$\langle 1\rangle 1$. $f(X)$ is compact.
  PROOF: By Proposition 9.5.10.
$\langle 1\rangle 2$. $f(X)$ has a greatest element.
  $\langle 2\rangle 1$. ASSUME: for a contradiction $f(X)$ has no greatest element.
  $\langle 2\rangle 2$. $\{(-\infty, f(x)) : x \in X\}$ is a set of open sets that covers $f(X)$.
  $\langle 2\rangle 3$. PICK a finite subset $\{(-\infty, f(x_1)), \ldots, (-\infty, f(x_n))\}$ that covers $f(X)$.
    PROOF: By Proposition 9.5.5
  $\langle 2\rangle 4$. LET: $f(x_N)$ be largest out of $f(x_1)$, …, $f(x_n)$
  $\langle 2\rangle 5$. $f(x_N) < f(x_N)$
  $\langle 2\rangle 6$. Q.E.D.
    PROOF: This is a contradiction.
$\langle 1\rangle 3$. $f(X)$ has a least element.
  PROOF: Similar.
□

Theorem 9.5.21 (DC)**.** *A nonempty compact Hausdorff space with no isolated points is uncountable.*

PROOF:
$\langle 1\rangle 1$. LET: X be a nonempty compact Hausdorff space with no isolated points.
$\langle 1\rangle 2$. For every nonempty open $U \subseteq X$ and point $x \in X$, there exists a nonempty open $V \subseteq U$ such that $x \notin \overline{V}$
  $\langle 2\rangle 1$. LET: $U \subseteq X$ be nonempty and open and $x \in X$
  $\langle 2\rangle 2$. PICK $y \in U$ such that $y \neq x$
    PROOF: This is possible because $U \neq \{x\}$ since x is not an isolated point.
  $\langle 2\rangle 3$. PICK disjoint open neighbourhoods W_1 and W_2 of x and y
    PROOF: Since X is Hausdorff
  $\langle 2\rangle 4$. LET: $V = U \cap W_2$
  $\langle 2\rangle 5$. $x \notin \overline{V}$
    PROOF: We have $\overline{V} \subseteq \overline{W_2} \subseteq X \setminus W_1$.
$\langle 1\rangle 3$. LET: $f : \mathbb{Z}^+ \to X$
    PROVE: f is not surjective
$\langle 1\rangle 4$. PICK a sequence of open sets $V_1 \supseteq V_2 \supseteq \cdots$ such that $f(n) \notin \overline{V_n}$

PROOF: By $\langle 1\rangle 2$ and Dependent Choice.
$\langle 1\rangle 5$. PICK a point $b \in \bigcap_{i=1}^{\infty} \overline{V_i}$
PROOF: By Proposition 9.5.15.
$\langle 1\rangle 6$. $b \neq f(n)$ for all n
PROOF: For each n we have $b \in \overline{V_n}$ ($\langle 1\rangle 5$) and $f(n) \notin \overline{V_n}$ ($\langle 1\rangle 4$).
□

Corollary 9.5.21.1. *Every closed interval in $\mathbb{R}$ is uncountable.*

Theorem 9.5.22. *Every compact space is limit point compact.*

PROOF:
$\langle 1\rangle 1$. LET: X be a compact space.
$\langle 1\rangle 2$. LET: $A \subseteq X$ be a set with no limit points.
PROVE: A is finite.
$\langle 1\rangle 3$. A is closed.
PROOF: By Corollary 3.15.3.1.
$\langle 1\rangle 4$. A is compact.
PROOF: By Proposition 9.5.6.
$\langle 1\rangle 5$. $\{U \text{ open in } X : U \cap A \text{ is a singleton}\}$ covers A
$\langle 2\rangle 1$. LET: $a \in A$
$\langle 2\rangle 2$. PICK an open neighbourhood U of a such that U does not intersect A at a point other than a
PROOF: One must exist because a is not a limit point of A ($\langle 1\rangle 2$).
$\langle 2\rangle 3$. $U \cap A = \{a\}$
$\langle 1\rangle 6$. PICK a finite subcover $\{U_1, \ldots, U_n\}$
PROOF: By $\langle 1\rangle 4$ using Proposition 9.5.5.
$\langle 1\rangle 7$. For $1 \leq i \leq n$,
LET: $U_i \cap A = \{a_i\}$
$\langle 1\rangle 8$. $A = \{a_1, \ldots, a_n\}$
□

Proposition 9.5.23. *Let X be a space and $C, D \subseteq X$ be compact. Then $C \cup D$ is compact.*

PROOF:
$\langle 1\rangle 1$. LET: $\mathcal{A}$ be a set of open sets that covers $C \cup D$
$\langle 1\rangle 2$. PICK a finite subset $\mathcal{A}_1$ that covers C and a finite subset $\mathcal{A}_2$ that covers D.
$\langle 1\rangle 3$. $\mathcal{A}_1 \cup \mathcal{A}_2$ is a finite subset of $\mathcal{A}$ that covers $C \cup D$.
$\langle 1\rangle 4$. Q.E.D.

Proposition 9.5.24. *Not every compact Hausdorff space is first countable.*

PROOF: The space $\overline{S_\Omega}$ is compact Hausdorff but not first countable. □

Corollary 9.5.24.1. *Not every compact Hausdorff space is second countable.*

Theorem 9.5.25 (Tychonoff (AC))**.** *The product of a family of compact spaces is compact.*

PROOF:
⟨1⟩1. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of compact spaces.
LET: $X = \prod_{\alpha \in J} X_\alpha$
⟨1⟩2. LET: $\mathcal{A} \subseteq \mathcal{P}X$ satisfy the finite intersection property.
PROVE: $\bigcap_{A \in \mathcal{A}} \overline{A}$ is nonempty.
⟨1⟩3. PICK a set $\mathcal{D} \subseteq \mathcal{P}X$ that includes $\mathcal{A}$ and is maximal with respect to the finite intersection property.
PROOF: By Lemma 1.18.6.
⟨1⟩4. For $\alpha \in J$, PICK $x_\alpha \in \bigcap_{D \in \mathcal{D}} \overline{\pi_\alpha(D)}$
⟨2⟩1. LET: $\alpha \in J$
⟨2⟩2. $\{\overline{\pi_\alpha(D)} : D \in \mathcal{D}\}$ satisfies the finite intersection property.
⟨2⟩3. Q.E.D.
PROOF: By Proposition 9.5.15
⟨1⟩5. LET: $x = (x_\alpha)_{\alpha \in J}$
⟨1⟩6. For all $D \in \mathcal{D}$ we have $(x_\alpha)_{\alpha \in J} \in \overline{D}$
PROOF:
⟨2⟩1. Every subbasis element containing x intersects every member of $\mathcal{D}$
⟨3⟩1. LET: $\pi_\alpha(U)^{-1}$ be a subbasis element containing x where U is open in X_α
⟨3⟩2. LET: $D \in \mathcal{D}$
⟨3⟩3. U intersects $\pi_\alpha(D)$
⟨2⟩2. Every subbasis element containing x is a member of $\mathcal{D}$
PROOF: By Lemma 1.18.8
⟨2⟩3. Every basis element containing x is a member of $\mathcal{D}$
PROOF: By Lemma 1.18.7
⟨2⟩4. Every basis element containing x intersects every member of $\mathcal{D}$
PROOF: This follows because $\mathcal{D}$ satisfies the finite intersection property.
⟨1⟩7. Q.E.D.
PROOF: By Proposition 9.5.15
□

PROOF:
⟨1⟩1. LET: $\{X_\alpha\}_{\alpha \in J}$ be a family of compact spaces and $X = \prod_{\alpha \in J} X_\alpha$.
⟨1⟩2. PICK a well-ordering $<$ of J such that J has a greatest elemeent $\top$
⟨1⟩3. For all $\alpha \in J$ and every family of points $p = \{p_i \in X_i\}_{i \leq \alpha}$,
LET: $Y_\alpha(p) = \{x \in X : \forall i \leq \alpha . x_i = p_i\}$
⟨1⟩4. For all $\beta \in J$ and every family of points $p = \{p_i \in X_i\}_{i < \beta}$,
LET: $Z_\beta(p) = \bigcap_{\alpha < \beta} Y_\alpha = \{x \in X : \forall i < \beta . x_i = p_i\}$
⟨1⟩5. Given $\beta \in J$, a family of points $\{p_i \in X_i\}_{i < \beta}$, and a finite set $\mathcal{A}$ of basis elements that covers $Z_\beta(p)$, there exists $\alpha < \beta$ such that $\mathcal{A}$ covers $Y_\alpha(p)$
⟨2⟩1. ASSUME: (
w.l.o.g. β has no immediate predecessor)
⟨2⟩2. For $A \in \mathcal{A}$,
LET: $J_A = \{i < \beta : \pi_i(A) \neq X_i\}$
⟨2⟩3. LET: α be the largest element of $\bigcup_{A \in \mathcal{A}} J_A$
PROOF: The set has a greatest element because each J_A is finite and $\mathcal{A}$ is

finite.

⟨2⟩4. $\mathcal{A}$ covers $Y_\alpha(p)$

⟨3⟩1. LET: $x \in Y_\alpha(p)$

⟨3⟩2. LET: $y \in Z_\beta(p)$ be the point with

$$y_i = \begin{cases} p_i & \text{if } i < \beta \\ x_i & \text{if } i \geq \beta \end{cases}$$

⟨3⟩3. PICK $A \in \mathcal{A}$ such that $y \in A$

⟨3⟩4. $x \in A$

⟨4⟩1. For $i \leq \alpha$ we have $x_i \in \pi_i(A)$

⟨5⟩1. $x_i = p_i$

PROOF: From ⟨3⟩1 and ⟨1⟩3.

⟨5⟩2. $y_i = p_i$

PROOF: From ⟨3⟩2

⟨5⟩3. $y_i \in \pi_i(A)$

PROOF: From ⟨3⟩3.

⟨4⟩2. For $\alpha < i < \beta$ we have $x_i \in \pi_i(A)$

⟨5⟩1. $i \notin J_A$

PROOF: From ⟨2⟩3

⟨5⟩2. $\pi_i(A) = X_i$

PROOF: From ⟨2⟩2

⟨4⟩3. For $i \geq \beta$ we have $x_i \in \pi_i(A)$

⟨5⟩1. $x_i = y_i$

PROOF: By ⟨3⟩2

⟨5⟩2. $y_i \in \pi_i(A)$

PROOF: By ⟨3⟩3

⟨1⟩6. ASSUME: for a contradiction $\mathcal{A}$ is a set of basis elements such that no finite subset covers X

⟨1⟩7. For all $\alpha \in J$ there exists a family of points $\{p_i \in X_i\}_{i \leq \alpha}$ such that no finite subset of $\mathcal{A}$ covers $Y_\alpha(p)$

⟨2⟩1. ASSUME: as induction hypothesis $\beta \in J$ and p_i has been chosen for all $i < \beta$ such that, for all $\alpha < \beta$, no finite subset of $\mathcal{A}$ covers $Y_\alpha(p)$

⟨2⟩2. No finite subset of $\mathcal{A}$ covers $Z_\beta(p)$

PROOF: By ⟨1⟩5

⟨2⟩3. PICK $p_\beta \in X_\beta$ such that no finite subset of $\mathcal{A}$ covers $Z_\beta(p) \times \{p_\beta\} = Y_\beta(p)$

PROOF: By Lemma 9.5.13.

⟨1⟩8. Q.E.D.

PROOF: This is a contradiction since $Y_\top(p) = \{p\}$ and so must be covered by a single element of $\mathcal{A}$.

□

Theorem 9.5.26. *In a compact Hausdorff space, the components and the quasicomponents coincide.*

PROOF:

⟨1⟩1. Let: X be a compact Hausdorff space and $x, y \in X$ lie in the same quasicomponent.
Prove: x and y are in the same component.
⟨1⟩2. Let: $\mathcal{A}$ be the set of all closed subspaces A of X such that x and y lie in the same quasicomponent of A.
⟨1⟩3. Every chain in $\mathcal{A}$ has a lower bound.
⟨2⟩1. Let: $\mathcal{B} \subseteq \mathcal{A}$ be a chain
Prove: $Y = \bigcap \mathcal{B} \in \mathcal{A}$
⟨2⟩2. Assume: for a contradiction $Y = C \cup D$ were C and D are disjoint and open in Y, $x \in C$ and $y \in D$
⟨2⟩3. Pick disjoint open sets U and V in X such that $C \subseteq U$ and $D \subseteq V$
Proof: By Lemma 9.5.18.
⟨2⟩4. $\{B \setminus (U \cup V) : B \in \mathcal{B}\}$ satisfies the finite intersection property.
⟨3⟩1. Let: $B_1, \ldots, B_n \in \mathcal{B}$
⟨3⟩2. $B_1 \cap \cdots \cap B_n \in \mathcal{B}$
Proof: By ⟨2⟩1.
⟨3⟩3. $B_1 \cap \cdots \cap B_n \setminus (U \cap V)$ is nonempty
Proof: $B_1 \cap \cdots \cap B_n \cap U$ and $B_1 \cap \cdots \cap B_n \cap V$ cannot be disjoint, because x and y are in the same quasicomponent of $B_1 \cap \cdots \cap B_n$.
⟨2⟩5. $Y \setminus (U \cup V)$ is nonempty.
Proof: By Proposition 9.5.15.
⟨2⟩6. Q.E.D.
Proof: This is a contradiction since $Y \setminus (U \cup V) = Y \setminus (C \cup D)$.
⟨1⟩4. Pick a minimal element $D \in \mathcal{A}$
Proof: One exists by Zorn's Lemma.
⟨1⟩5. D is connected.
⟨2⟩1. Assume: [
for a contradiction $D = U \uplus V$ is a separation of D]
⟨2⟩2. Case: $x, y \in U$
Proof: In this case we have $U \in \mathcal{A}$ contradicting the minimality of D.
⟨2⟩3. Case: $x \in U, y \in V$
Proof: This is a contradiction because x and y are in the same quasicomponent of D.
⟨2⟩4. Case: $x \in V, y \in U$
Proof: Similar to ⟨2⟩3.
⟨2⟩5. Case: $x, y \in V$
Proof: Similar to ⟨2⟩2.
∎

9.6 Perfect Maps

Proposition 9.6.1. *Let $p : X \twoheadrightarrow Y$ be a closed continuous surjective map. For all $y \in Y$ and U an open neighbourhood of $p^{-1}(y)$, there exists an open neighbourhood W of y such that $p^{-1}(W) \subseteq U$.*

Proof: Take $W = Y \setminus p(X \setminus U)$. ∎

Proposition 9.6.2 (AC)**.** *Let $p : X \twoheadrightarrow Y$ be a closed continuous surjective map. If X is normal then Y is normal.*

PROOF:
$\langle 1 \rangle 1$. LET: $A, B \subseteq Y$ be closed
$\langle 1 \rangle 2$. $p^{-1}(A)$, $p^{-1}(B)$ are closed in X.
$\langle 1 \rangle 3$. PICK disjoint open sets U, V of $p^{-1}(A)$, $p^{-1}(B)$ respectively.
$\langle 1 \rangle 4$. For all $a \in A$, PICK an open neighbourhood W_a of a such that $p^{-1}(W_a) \subseteq U$
PROOF: By Proposition 9.6.1.
$\langle 1 \rangle 5$. For all $b \in B$, PICK an open neighbourhood W'_b of b such that $p^{-1}(W'_b) \subseteq V$
PROOF: By Proposition 9.6.1.
$\langle 1 \rangle 6$. LET: $W = \bigcup_{a \in A} W_a$ and $W' = \bigcup_{b \in B} W'_b$
$\langle 1 \rangle 7$. $W \cap W' = \emptyset$
PROOF: This holds because $p^{-1}(W) \subseteq U$, $p^{-1}(W') \subseteq V$, and p is surjective.
□

Definition 9.6.3 (Perfect Map)**.** Let X and Y be topological spaces and $p : X \to Y$. Then p is *perfect* iff p is closed, continuous, surjective, and $p^{-1}(y)$ is compact for all $y \in Y$.

Proposition 9.6.4. *Let $p : X \to Y$ be a perfect map. If X is Hausdorff then so is Y.*

PROOF:
$\langle 1 \rangle 1$. LET: $a, b \in Y$ with $a \neq b$
$\langle 1 \rangle 2$. PICK disjoint open neighbourhoods U and V of $\pi^{-1}(a)$ and $\pi^{-1}(b)$, respectively.
PROOF: By Lemma 9.5.18.
$\langle 1 \rangle 3$. PICK open neighbourhoods W and W' of a and b such that $\pi^{-1}(W) \subseteq U$ and $\pi^{-1}(W') \subseteq V$
PROOF: By Proposition 9.6.1.
$\langle 1 \rangle 4$. W and W' are disjoint.
□

Proposition 9.6.5. *Let $p : X \twoheadrightarrow Y$ be perfect. If X is regular then so is Y.*

PROOF:
$\langle 1 \rangle 1$. Y is T_1
PROOF: By Proposition 9.6.4.
$\langle 1 \rangle 2$. LET: $C \subseteq Y$ be closed and $a \in Y \setminus C$
$\langle 1 \rangle 3$. $p^{-1}(C)$ is closed and $p^{-1}(a)$ is disjoint from $p^{-1}(C)$.
$\langle 1 \rangle 4$. PICK disjoint open neighbourhoods U, V of $p^{-1}(C)$, $p^{-1}(a)$ respectively.
PROOF: By Lemma 9.5.8.
$\langle 1 \rangle 5$. PICK an open neighbourhood W' of a such that $p^{-1}(W') \subseteq V$
PROOF: By Proposition 9.6.1.
$\langle 1 \rangle 6$. For $c \in C$, PICK an open neighbourhood W_c such that $p^{-1}(W_c) \subseteq U$

PROOF: By Proposition 9.6.1.
⟨1⟩7. $W = \bigcup_{c \in C} W_c$ is an open neighbourhood of C disjoint from W'
□

Proposition 9.6.6 (AC)**.** *Let $p : X \twoheadrightarrow Y$ be perfect. If X is locally compact then so is Y.*

PROOF:
⟨1⟩1. LET: $b \in Y$
⟨1⟩2. $\{U \text{ open in } X : \exists C \subseteq X \text{ compact}.U \subseteq C\}$ covers $p^{-1}(b)$
⟨1⟩3. PICK a finite subcover $\{U_1, \ldots, U_n\}$
⟨1⟩4. For $1 \leq i \leq n$, PICK a compact $C_i \subseteq X$ such that $U_i \subseteq C_i$
⟨1⟩5. For $1 \leq i \leq n$, PICK a neighbourhood W_i of b such that $p^{-1}(W_i) \subseteq U_i$
PROOF: By Proposition 9.6.1
⟨1⟩6. $b \in W_1 \cup \cdots \cup W_n \subseteq p(C_1) \cup \cdots \cup p(C_n)$
⟨1⟩7. $p(C_1) \cup \cdots \cup p(C_n)$ is compact.
⟨2⟩1. Each $p(C_i)$ is compact.
PROOF: By Proposition 9.5.10.
⟨2⟩2. Q.E.D.
PROOF: By Proposition 9.5.23.
□

Proposition 9.6.7. *The image of a regular space under a perfect map is regular.*

PROOF:
⟨1⟩1. LET: $p : X \twoheadrightarrow Y$ be a perfect map where X is regular.
⟨1⟩2. LET: $A \subseteq Y$ be closed and $a \notin A$.
⟨1⟩3. PICK disjoint open neighbourhoods U and V of $p^{-1}(A)$ and $p^{-1}(a)$ respectively.
PROOF: Lemma 9.5.8
⟨1⟩4. PICK neighbourhoods U' of A and V' of a such that $p^{-1}(U') \subseteq U$ and $p^{-1}(V') \subseteq V$.
PROOF: Lemma 5.3.2.
⟨1⟩5. U' and V' are disjoint.
□

9.7 Sequential Compactness

Definition 9.7.1 (Sequentially Compact)**.** A space is *sequentially compact* iff every sequence has a convergent subsequence.

Proposition 9.7.2. *$\overline{S_\Omega}$ is not sequentially compact.*

PROOF: Ω is a limit point of S_Ω but is not the limit of any sequence of points in S_Ω. □

9.8 Local Compactness

Definition 9.8.1 (Local Compactness)**.** Let X be a topological space.

For $x \in X$, the space X is *locally compact* at x iff there exists a compact subspace $C \subseteq X$ that includes a neighbourhood of x.

The space X is *locally compact* iff it is locally compact at every point.

Proposition 9.8.2. *Every complete linearly ordered set is locally compact under the order topology.*

PROOF:
$\langle 1 \rangle$1. LET: L be a complete linearly ordered set and $x \in L$
  PROVE: There exists a compact subspace $C \subseteq L$ that includes a neighbourhood U of x
$\langle 1 \rangle$2. CASE: x is least and greatest in L
  PROOF: In this case, $L = \{x\}$ is compact.
$\langle 1 \rangle$3. CASE: x is least in L but not greatest
  $\langle 2 \rangle$1. PICK $a < x$
  $\langle 2 \rangle$2. Take $C = [a, x]$ and $U = (a, x]$
$\langle 1 \rangle$4. CASE: x is greatest in L but not least
  PROOF: Similar.
$\langle 1 \rangle$5. CASE: x is neither least nor greatest
  $\langle 2 \rangle$1. PICK $a < x$ and $b > x$
  $\langle 2 \rangle$2. Take $C = [a, b]$ and $U = (a, b)$
□

Corollary 9.8.2.1. *For every ordinal α, the space S_α is locally compact.*

Theorem 9.8.3. *Every closed subspace of a locally compact Hausdorff space is locally compact.*

PROOF:
$\langle 1 \rangle$1. LET: X be locally compact Hausdorff and $C \subseteq X$ be closed.
$\langle 1 \rangle$2. LET: $x \in C$
$\langle 1 \rangle$3. PICK $D \subseteq X$ compact and $U \subseteq D$ open such that $x \in U$
$\langle 1 \rangle$4. D is closed.
  PROOF: Proposition 9.5.9.
$\langle 1 \rangle$5. $C \cap D$ is closed
  PROOF: Propositon 3.6.5.
$\langle 1 \rangle$6. $C \cap D$ is compact
  PROOF: Proposition 9.5.6.
$\langle 1 \rangle$7. Q.E.D.
  PROOF: $C \cap D \subseteq C$ is compact and includes the open neighbourhood $U \cap C$ of x.
□

Proposition 9.8.4. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of nonempty topological spaces. If $\prod_{\alpha \in J} X_\alpha$ is locally compact, then each X_α is locally compact.*

PROOF:
$\langle 1\rangle 1.$ LET: $\alpha \in J$ and $x_\alpha \in X_\alpha$
$\langle 1\rangle 2.$ PICK $x_\beta \in X_\beta$ for all $\beta \in J \setminus \{\alpha\}$
$\langle 1\rangle 3.$ PICK a compact subspace $C \subseteq \prod_{\alpha \in J} X_\alpha$ that a neighbourhood U of x included in C
$\langle 1\rangle 4.$ PICK a basic open set $\prod_{\alpha \in J} U_\alpha$ such that $x \in \prod_{\alpha \in J} U_\alpha \subseteq U$
$\langle 1\rangle 5.$ $x_\alpha \in U_\alpha \subseteq \pi_\alpha(C)$
$\langle 1\rangle 6.$ $\pi_\alpha(C)$ is compact.
PROOF: By Proposition 9.5.10.
□

Proposition 9.8.5. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of locally compact spaces such that X_α is compact for all but finitely many values of α. Then $\prod_{\alpha \in J} X_\alpha$ is locally compact.*

PROOF:
$\langle 1\rangle 1.$ ASSUME: X_α is compact if $\alpha \neq \alpha_1, \ldots, \alpha_n$
$\langle 1\rangle 2.$ LET: $\vec{x} \in \prod_{\alpha \in J} X_\alpha$
$\langle 1\rangle 3.$ For $1 \leq i \leq n$, PICK $C_{\alpha_i} \subseteq X_{\alpha_i}$ compact and U_{α_i} open such that $x_{\alpha_i} \in U_{\alpha_i} \subseteq C_{\alpha_i}$
$\langle 1\rangle 4.$ For $\alpha \neq \alpha_1, \ldots, \alpha_n$,
LET: $C_\alpha = U_\alpha = X_\alpha$
$\langle 1\rangle 5.$ $\vec{x} \in \prod_{\alpha \in J} U_\alpha \subseteq \prod_{\alpha \in J} C_\alpha$
$\langle 1\rangle 6.$ $\prod_{\alpha \in J} C_\alpha$ is compact
PROOF: By Tychonoff's Theorem.
□

Proposition 9.8.6. *$\mathbb{R}_l$ is not locally compact.*

PROOF: $[0, +\infty)$ can be partitioned into infinitely many disjoint open sets, which therefore do not have a finite subcover. □

Corollary 9.8.6.1. *The Sorgenfrey plane is not locally compact.*

Proposition 9.8.7. *Let $\{X_\alpha\}_{\alpha \in J}$ be a family of nonempty topological spaces. If $\prod_{\alpha \in J} X_\alpha$ is locally compact, then all but finitely many of the X_α are compact.*

PROOF:
$\langle 1\rangle 1.$ PICK a point $a = (a_\alpha)_{\alpha \in J} \in \prod_{\alpha \in J} X_\alpha$
$\langle 1\rangle 2.$ PICK a compact $C \subseteq \prod_{\alpha \in J} X_\alpha$ that includes the basic neighbourhood $\prod_{\alpha \in J} U_\alpha$ of a, where $U_\alpha = X_\alpha$ for all α except $\alpha = \alpha_1, \ldots, \alpha_n$
$\langle 1\rangle 3.$ For $\alpha \neq \alpha_1, \ldots, \alpha_n$, we have X_α is compact.
PROOF: X_α is homeomorphic to a closed subspace of C.
□

Corollary 9.8.7.1. *For any infinite set I, the space $\mathbb{R}^I$ is not locally compact.*

Proposition 9.8.8. *$[0,1]^\omega$ is not compact under the uniform topology.*

PROOF:$\{a_i : i \geq 0\}$ is an infinite set with no limit point, where a_i is the point with ith component 1 and all other components 0. □

Corollary 9.8.8.1. $\mathbb{R}^\omega$ *under the uniform topology is not locally compact.*

PROOF:
$\langle 1\rangle 1$. ASSUME: $\mathbb{R}^\omega$ is locally compact
$\langle 1\rangle 2$. LET: C be a compact subspace such that $B(\vec{0}, \epsilon) \subseteq C$
$\langle 1\rangle 3$. $\overline{B(\vec{0}, \epsilon)}$ is compact.
$\langle 1\rangle 4$. Q.E.D.
PROOF: This contradicts the proposition.
□

Proposition 9.8.9. *Not every subspace of a locally compact Hausdorff space is locally compact.*

PROOF: $\mathbb{R}$ is locally compact Hausdoff, $\mathbb{Q}$ is not locally compact. □

Proposition 9.8.10. *The continuous image of a locally compact Hausdorff space is not necessarily locally compact.*

PROOF:
$\langle 1\rangle 1$. LET: $\{q_0, q_1, \ldots\}$ be an enumeration of $[0,1] \cap \mathbb{Q}$.
$\langle 1\rangle 2$. Define $f : (0, +\infty) \setminus \mathbb{Z} \to [0,1] \cap \mathbb{Q}$ by: $f(x) = q_n$ for $x \in (n, n+1)$
$\langle 1\rangle 3$. f is continuous.
PROOF: The inverse image of any set is a union of open intervals.
□

9.9 Compactifications

Definition 9.9.1 (Compactification)**.** Let X and Y be spaces. Then Y is a *compactification* of X iff Y is a compact Hausdorff space and X is a subspace of Y with $\overline{X} = Y$.

Two compcactificatons Y_1, Y_2 of X are *equivalent* iff there exists a homeomorphism between Y_1 and Y_2 that is the identity on X.

Lemma 9.9.2. *Let $h : X \to Z$ be an imbedding. Then there exists a compactification $c : X \to Y$ of X, unique up to equivalence, and an imbedding $i : Y \to Z$ such that $h = i \circ c$.*

PROOF: Simply take Y to be the closure of X in Z. □

Definition 9.9.3 (One-Point Compactification)**.** A *one-point compactification* of X is a compactification Y of X such that $Y \setminus X$ is a singleton.

Theorem 9.9.4. *Let X be a topological space. Then X is locally compact Hausdorff if and only if there exists a space Y such that:*

1. *X is a subspace of Y*
2. *The set $Y \setminus X$ is a singleton.*
3. *Y is a compact Hausdorff space.*

If Y and Y' are two spaces satisfying these conditions, then there exists a unique homeomorphism between Y and Y' that is the identity on X.

PROOF:

$\langle 1 \rangle 1$. If X is locally compact Hausdorff then there exists a space Y satisfying 1–3.

$\langle 2 \rangle 1$. LET: $Y = X \cup \{\infty\}$ under the topology $\mathcal{T} = \{U \subseteq X : U$ is open in $X\} \cup \{Y \setminus C : C \subseteq X$ is compact$\}$.

$\langle 3 \rangle 1$. $Y \in \mathcal{T}$

PROOF: This holds because $Y = Y \setminus \emptyset$.

$\langle 3 \rangle 2$. For all $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$.

$\langle 4 \rangle 1$. LET: $U, V \in \mathcal{T}$

$\langle 4 \rangle 2$. CASE: U, V are open in X

PROOF: In this case, $U \cap V$ is open in X.

$\langle 4 \rangle 3$. CASE: U is open in X, $V = Y \setminus C$ where $C \subseteq X$ is compact.

$\langle 5 \rangle 1$. $U \cap V = U \setminus C$

$\langle 5 \rangle 2$. C is closed in X

PROOF: Proposition 9.5.9.

$\langle 5 \rangle 3$. $U \cap V$ is open in X

$\langle 4 \rangle 4$. CASE: $U = Y \setminus C$ where $C \subseteq X$ is compact, V is open in X.

PROOF: Similar.

$\langle 4 \rangle 5$. CASE: $U = Y \setminus C$, $V = Y \setminus D$ where $C, D \subseteq X$ are compact.

$\langle 5 \rangle 1$. $U \cap V = Y \setminus (C \cup D)$

$\langle 5 \rangle 2$. C and D are closed in X

PROOF: Proposition 9.5.9.

$\langle 5 \rangle 3$. $C \cup D$ is closed in X

PROOF: Proposition 3.6.4.

$\langle 5 \rangle 4$. $C \cup D$ is compact.

PROOF: By Proposition 9.5.23. □

$\langle 3 \rangle 3$. For all $\mathcal{A} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{A} \in \mathcal{T}$.

$\langle 4 \rangle 1$. LET: $\mathcal{A} \subseteq \mathcal{T}$

$\langle 4 \rangle 2$. CASE: Every element of $\mathcal{A}$ is an open set in X.

PROOF: In this case, $\bigcup \mathcal{A}$ is open in X.

$\langle 4 \rangle 3$. CASE: There exists C compact in X such that $Y \setminus C \in \mathcal{A}$

$\langle 5 \rangle 1$. $\bigcup \mathcal{A} = Y \setminus (\bigcap \{D \subseteq X : D \text{ compact}, Y \setminus D \in \mathcal{A}\} \setminus \bigcup \{U \text{ open in } X : U \in \mathcal{A}\})$

PROOF: Set theory.

$\langle 5 \rangle 2$. $\bigcap \{D \subseteq X : D \text{ compact}, Y \setminus D \in \mathcal{A}\} \setminus \bigcup \{U \text{ open in } X : U \in \mathcal{A}\}$ is compact.

PROOF: It is a closed subset of the compact set C.

$\langle 2 \rangle 2$. X is a subspace of Y

$\langle 3 \rangle 1$. For every open set U of X, there exists V open in Y such that $U = V \cap X$

PROOF: Take $V = U$.

$\langle 3 \rangle 2$. For every open set V in Y, we have $V \cap X$ is open in X.

$\langle 4 \rangle 1$. LET: V be open in Y

$\langle 4 \rangle 2$. CASE: V is open in X

PROOF: In this case, $V \cap X = V$.

$\langle 4 \rangle 3$. CASE: $V = Y \setminus C$ where $C \subseteq X$ is compact.

$\langle 5 \rangle 1$. C is closed in X.

PROOF: By Proposition 9.5.9.

$\langle 5 \rangle 2$. $V \cap X = X \setminus C$

$\langle 2 \rangle 3$. $Y \setminus X = \{\infty\}$

$\langle 2 \rangle 4$. Y is compact.

$\langle 3 \rangle 1$. LET: $\mathcal{A}$ be an open covering of Y

$\langle 3 \rangle 2$. PICK $U \in \mathcal{A}$ such that $\infty \in U$

$\langle 3 \rangle 3$. PICK $C \subseteq X$ compact such that $U = Y \setminus C$.

$\langle 3 \rangle 4$. $\{V \cap X : V \in \mathcal{A}\}$ is set of open sets that covers C

$\langle 3 \rangle 5$. PICK a finite subset $\{V_1, \ldots, V_n\}$ such that $\{V_1 \cap X, \ldots, V_n \cap X\}$ covers C.

$\langle 3 \rangle 6$. $\{U, V_1, \ldots, V_n\}$ is a finite subcover of Y.

$\langle 2 \rangle 5$. Y is Hausdorff.

$\langle 3 \rangle 1$. LET: $x, y \in Y$ with $x \neq y$

PROVE: There exist disjoint open neighbourhoods U, V of x and y.

$\langle 3 \rangle 2$. CASE: $x, y \in X$

PROOF: In this case, we just use the fact that X is Hausdorff.

$\langle 3 \rangle 3$. CASE: $x = \infty$, $y \in X$

$\langle 4 \rangle 1$. PICK $C \subseteq X$ compact such that C includes an open neighbourhood V of y

$\langle 4 \rangle 2$. LET: $U = Y \setminus C$

$\langle 3 \rangle 4$. CASE: $x \in X$, $y = \infty$

PROOF: Simlar.

$\langle 1 \rangle 2$. If there exists a space Y satisfying 1–3 then X is locally compact Hausdorff.

$\langle 2 \rangle 1$. LET: Y be a space satisfying 1–3

$\langle 2 \rangle 2$. LET: ∞ be the point in $Y \setminus X$

$\langle 2 \rangle 3$. X is locally compact

$\langle 3 \rangle 1$. LET: $x \in X$

$\langle 3 \rangle 2$. PICK disjoint open neighbourhoods U of x and V of ∞

$\langle 3 \rangle 3$. $X \setminus V$ is compact and includes U

PROOF: $X \setminus V = Y \setminus V$ is compact because it is a closed subset of Y (Proposition 9.5.6).

$\langle 2 \rangle 4$. X is Hausdorff.

PROOF: By Corollary 6.2.6.1.

$\langle 1 \rangle 3$. If Y and Y' are two spaces satisfying 1–3 then there exists a unique homemorphism between Y and Y' that is the identity on X.

$\langle 2 \rangle 1$. LET: Y and Y' be two spaces that satisfy 1–3.

$\langle 2 \rangle 2$. LET: $Y \setminus X = \{p\}$ and $Y' \setminus X = \{q\}$

$\langle 2 \rangle 3$. LET: $h : Y \to Y'$ be given by

$$h(x) = x \qquad (x \in X)$$
$$h(p) = q$$

⟨2⟩4. h is a homeomorphism
⟨3⟩1. h is bijective.
⟨3⟩2. h is continuous.
⟨4⟩1. Let: $V \subseteq Y'$ be open.
Prove: $h^{-1}(V)$ is open.
⟨4⟩2. Case: $V \subseteq X$
⟨5⟩1. $h^{-1}(V) = V$
⟨5⟩2. V is open in X
Proof: Condition 1 for Y'.
⟨5⟩3. V is open in Y
Proof: Condition 1 for Y.
⟨4⟩3. Case: $q \in V$
⟨5⟩1. $Y' \setminus V$ is compact.
Proof: Proposition 9.5.6.
⟨5⟩2. $Y' \setminus V$ is closed in Y.
Proof: Proposition 9.5.9.
⟨5⟩3. $h^{-1}(V) = Y \setminus (Y' \setminus V)$
⟨3⟩3. h^{-1} is continuous.
Proof: Similar.
⟨2⟩5. If $h' : Y \to Y'$ is a homeomorphism such that $h' \restriction_X = \mathrm{id}_X$ then $h' = h$
□

Theorem 9.9.5. *Let X be a Hausdorff space. Then X is locally compact if and only if, for all $x \in X$ and any neighbourhood U of x, there exists an open neighbourhood V of x such that $\overline{V}$ is compact and $\overline{V} \subseteq U$.*

Proof:
⟨1⟩1. If X is locally compact then, for all $x \in X$ and any neighbourhood U of x, there exists an open neighbourhood V of x such that $\overline{V}$ is compact and $\overline{V} \subseteq U$.
⟨2⟩1. Assume: X is locally compact.
⟨2⟩2. Let: $x \in X$ and U be a neighbourhood of x.
⟨2⟩3. Let: Y be the one-point compactification of X.
Proof: By Theorem 9.9.4.
⟨2⟩4. Let: $C = Y \setminus U$
⟨2⟩5. C is compact
Proof: By Proposition 9.5.6.
⟨2⟩6. Pick disjoint open sets V, W containing x and C
Proof: Lemma 9.5.8
⟨2⟩7. V is open in X
Proof: $V \subseteq X$ since $\infty \in W$.
⟨2⟩8. The closure of V in X is compact
⟨3⟩1. The closure of V is X is the same as the closure of V in Y.
Proof: The point ∞ cannot be a limit point of V since W is a neighbourhood disjoint from V.
⟨3⟩2. The closure of V in Y is compact.
Proof: By Proposition 9.5.6.

⟨2⟩9. $\overline{V} \subseteq U$

Proof:

$$\begin{aligned}\overline{V} &\subseteq Y \setminus W \\ &\subseteq Y \setminus C \\ &= U\end{aligned}$$

⟨1⟩2. If, for all $x \in X$ and any neighbourhood U of x, there exists an open neighbourhood V of x such that $\overline{V}$ is compact and $\overline{V} \subseteq U$, then X is locally compact.

⟨2⟩1. Assume: for all $x \in X$ and any neighbourhood U of x, there exists an open neighbourhood V of x such that $\overline{V}$ is compact and $\overline{V} \subseteq U$

⟨2⟩2. Let: $x \in X$

Prove: There exists $C \subseteq X$ compact such that C includes a neighbourhood U of x

⟨2⟩3. Pick an open neighbourhood V of x such that $\overline{V}$ is compact and $\overline{V} \subseteq X$

⟨2⟩4. Take $C = \overline{V}$ and $U = V$

□

Corollary 9.9.5.1. *Every open subspace of a locally compact Hausdorff space is locally compact.*

Corollary 9.9.5.2. *A space is locally compact Hausdorff if and only if it is an open subspace of a compact Hausdorff space.*

Corollary 9.9.5.3. *Every locally compact Hausdorff space is completely regular.*

Corollary 9.9.5.4. *The space $\mathbb{R}_K$ is not locally compact.*

Lemma 9.9.6 (AC)**.** *If $p : X \to Y$ is a quotient map and Z is a locally compact Hausdorff space, then the map*

$$\pi = p \times \mathrm{id}_Z : X \times Z \to Y \times Z$$

is a quotient map.

Proof:

⟨1⟩1. π is surjective.

Proof: This holds because p is surjective.

⟨1⟩2. π is continuous.

Proof: By Theorem 5.2.15.

⟨1⟩3. For $A \subseteq Y \times Z$, if $\pi^{-1}(A)$ is open in $X \times Z$ then A is open in $Y \times Z$.

⟨2⟩1. Let: $A \subseteq Y \times Z$

⟨2⟩2. Assume: $\pi^{-1}(A)$ is open in $X \times Z$

⟨2⟩3. Let: $(y, z) \in A$

⟨2⟩4. Pick $x \in X$ such that $p(x) = y$

Proof: Since p is surjective.

⟨2⟩5. Pick open sets U_1, V with $\overline{V}$ compact such that $(x, y) \in U_1 \times V$ and $U_1 \times \overline{V} \subseteq \pi^{-1}(A)$

PROOF: Using Theorem 9.9.5

⟨2⟩6. PICK a sequence of open sets U_1, U_2, … in X such that $p^{-1}(p(U_n)) \subseteq U_{n+1}$ and $U_n \times \overline{V} \subseteq \pi^{-1}(A)$ for all n

⟨3⟩1. LET: U be open with $U \times \overline{V} \subseteq \pi^{-1}(A)$

PROVE: There exists W open with $p^{-1}(p(U)) \subseteq W$ and $W \times \overline{V} \subseteq \pi^{-1}(A)$

⟨3⟩2. For all $x \in p^{-1}(p(U))$, PICK open sets U_x, V_x such that $x \in U_x$, $\overline{V} \subseteq V_x$ and $U_x \times V_x \subseteq \pi^{-1}(A)$

PROOF: By the Tube Lemma.

⟨3⟩3. LET: $W = \bigcup_{x \in p^{-1}(p(U))} U_x$

⟨2⟩7. LET: $U = \bigcup_{n=1}^{\infty} U_n$

⟨2⟩8. U is saturated with respect to p

⟨3⟩1. LET: $a \in U$, $b \in X$, $p(a) = p(b)$

⟨3⟩2. PICK n such that $a \in U_n$

⟨3⟩3. $b \in p^{-1}(p(U_n))$

⟨3⟩4. $b \in U_{n+1}$

⟨3⟩5. $b \in U$

⟨2⟩9. $p(U)$ is open in Y

PROOF: By Lemma 4.5.2.

⟨2⟩10. $(y, z) \in p(U) \times V \subseteq A$

⟨2⟩11. Q.E.D.

PROOF: By Proposition 3.2.3.

□

Theorem 9.9.7. *Let $p : A \to B$ and $q : C \to D$ be quotient maps. If B and C are locally compact Hausdorff spaces, then $p \times q : A \times C \to B \times D$ is a quotient map.*

PROOF: This holds by Lemma 9.9.6 and Proposition 4.5.10 because $p \times q = (\mathrm{id}_B \times q) \circ (p \times \mathrm{id}_C)$. □

Theorem 9.9.8. *Let X be a completely regular space. Let Y be a compactification of X such that every bounded continuous map $X \to \mathbb{R}$ extends uniquely to a continuous map $Y \to \mathbb{R}$. Then, for every compact Hausdorff space C, every continuous map $X \to C$ extends uniquely to a continuous map $Y \to C$.*

PROOF:

⟨1⟩1. LET: C be a compact Hausdorff space and $f : X \to C$ a continuous function

⟨1⟩2. PICK a set J and an imedding $C \subseteq [0, 1]^J$

⟨2⟩1. C is normal

PROOF: By Lemma 9.5.18

⟨2⟩2. Q.E.D.

PROOF: By Theorem 6.4.6.

⟨1⟩3. For $\alpha \in J$,

LET: $g_\alpha : Y \to \mathbb{R}$ be the unique continuous extension of $\pi_\alpha \circ f$

⟨1⟩4. Define $g : Y \to \mathbb{R}^J$ by $g(y)_\alpha = g_\alpha(y)$

$\langle 1 \rangle 5$. g is continuous

PROOF: By Theorem 5.2.15.

$\langle 1 \rangle 6$. g extends f

$\langle 1 \rangle 7$. We have $g : Y \to C$

PROOF:

$$\begin{aligned} g(Y) &= g(\overline{X}) \\ &\subseteq \overline{g(X)} && \text{(Theorem 5.2.2)} \\ &= \overline{f(X)} && (\langle 1 \rangle 6) \\ &\subseteq \overline{C} \\ &= C && \text{(Proposition 9.5.9)} \end{aligned}$$

$\langle 1 \rangle 8$. g is unique

$\langle 2 \rangle 1$. LET: $h : Y \to C$ be a continuous extension of f

$\langle 2 \rangle 2$. For all $\alpha \in J$, $\pi_\alpha \circ h$ extends $\pi_\alpha \circ f$

$\langle 2 \rangle 3$. For all $\alpha \in J$, $\pi_\alpha \circ h = g_\alpha$

PROOF: By $\langle 1 \rangle 3$

$\langle 2 \rangle 4$. $h = g$

PROOF: By $\langle 1 \rangle 4$

□

Corollary 9.9.8.1. *Let X be a completely regular space. Let Y_1 and Y_2 be compactifications of X such that every bounded continuous map $X \to \mathbb{R}$ extends uniquely to a continuous map $Y_i \to \mathbb{R}$. Then Y_1 and Y_2 are equivalent.*

Definition 9.9.9 (Stone-Čech Compactification)**.** Let X be a completely regular space. The *Stone-Čech compactification* of X, $\beta(X)$, is the compactification of X such that, for every compact Hausdorff space C, every continuous function $X \to C$ extends uniquely to a continuous function $\beta(X) \to C$.

Chapter 10

Metric Spaces

10.1 Metrics

Definition 10.1.1 (Metric)**.** A *metric* on a set X is a function $d : X \times X \to \mathbb{R}$ such that, for all $x, y, z \in X$:

1. $d(x, y) \geq 0$;
2. $d(x, y) = 0$ if and only if $x = y$;
3. $d(x, y) = d(y, x)$;
4. **Triangle Inequality**
   $d(x, z) \leq d(x, y) + d(y, z)$

A *metric space* X consists of a set X and a metric on X. We call $d(x, y)$ the *distance* between x and y.

10.1.1 Open Balls

Definition 10.1.2 (Open Ball)**.** Let X be a metric space with metric d, $x \in X$ and $\epsilon > 0$. The *open ball* with *centre* x and *radius* ϵ is

$$B_d(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\} \ .$$

Lemma 10.1.3. *Let X be a metric space, $x, y \in X$ and $\epsilon > 0$. If $y \in B(x, \epsilon)$, then there exists δ such that $0 < \delta < \epsilon$ and*

$$B(y, \delta) \subseteq B(x, \epsilon) \ .$$

Proof:
$\langle 1 \rangle 1.$ Let: $\delta = \epsilon - d(x, y)$
$\langle 1 \rangle 2.$ Let: $z \in B(y, \delta)$
$\langle 1 \rangle 3.$ $d(x, z) < \epsilon$

PROOF:

$$\begin{aligned} d(x,z) &\leq d(x,y) + d(y,z) && \text{(Triangle Inequality)} \\ &< d(x,y) + \delta && (\langle 1\rangle 2) \\ &= \epsilon && (\langle 1\rangle 1) \end{aligned}$$

□

10.1.2 Bounded Sets

Definition 10.1.4 (Bounded)**.** Let X be a metric space and $A \subseteq X$. Then A is *bounded* iff $\{d(x,y) : x, y \in A\}$ is bounded above, in which case its *diameter* is

$$\operatorname{diam} A = \sup_{x,y \in A} d(x,y) \enspace .$$

10.1.3 Bounded Functions

Definition 10.1.5 (Bounded Function)**.** Let X be a set and Y a metric space. A function $f : X \to Y$ is *bounded* iff $\operatorname{ran} f$ is bounded.

We write $\mathcal{B}(X,Y)$ for the set of all bounded functions $X \to Y$.

The Sup Metric

Definition 10.1.6 (Sup Metric)**.** Let X be a nonempty set and Y a metric space. The *sup-metric* ρ on $\mathcal{B}(X,Y)$ is defined by

$$\rho(f,g) = \sup_{x \in X} d(f(x), g(x)) \enspace .$$

We write $\mathcal{B}(X,Y)$ for the metric space of all bounded functions $X \to Y$ under the sup-metric.

We prove this is well-defined and is a metric.

PROOF:
$\langle 1\rangle 1$. LET: X be a nonempty set.
$\langle 1\rangle 2$. LET: Y be a metric space.
$\langle 1\rangle 3$. For all $f, g \in \mathcal{B}(X,Y)$, the set $\{d(f(x), g(x)) : x \in X\}$ is bounded above.
  $\langle 2\rangle 1$. LET: $f, g \in \mathcal{B}(X,Y)$
  $\langle 2\rangle 2$. LET: $M = \operatorname{diam} f(X)$ and $N = \operatorname{diam} g(X)$
  $\langle 2\rangle 3$. PICK $x_0 \in X$
    PROOF: $\langle 1\rangle 1$
  $\langle 2\rangle 4$. LET: $D = d(f(x_0), g(x_0))$
  $\langle 2\rangle 5$. LET: $x \in X$
  $\langle 2\rangle 6$. $d(f(x), g(x)) \leq M + N + D$
    PROOF:

$$\begin{aligned} d(f(x), g(x)) &\leq d(f(x), f(x_0)) + d(f(x_0), g(x_0)) + d(g(x_0), g(x)) && \text{(Triangle inequality)} \\ &\leq M + D + N && (\langle 2\rangle 2,\ \langle 2\rangle 4) \end{aligned}$$

$\langle 1\rangle 4$. For all $f, g \in \mathcal{B}(X,Y)$ we have $\rho(f,g) \geq 0$

⟨2⟩1. Let: $f, g \in \mathcal{B}(X, Y)$
⟨2⟩2. Pick $x_0 \in X$
Proof: ⟨1⟩1
⟨2⟩3. $\rho(f, g) \geq 0$
Proof:

$$\begin{aligned} \rho(f,g) &\geq d(f(x_0), g(x_0)) && \text{(Definition of } \rho\text{)} \\ &\geq 0 && (\langle 1\rangle 2) \end{aligned}$$

⟨1⟩5. For all $f \in \mathcal{B}(X, Y)$ we have $\rho(f, f) = 0$
Proof: This holds because $d(f(x), f(x)) = 0$ for all $x \in X$.
⟨1⟩6. For all $f, g \in \mathcal{B}(X, Y)$ we have $\rho(f, g) = \rho(g, f)$
Proof:

$$\begin{aligned} \rho(f,g) &= \sup_{x \in X} d(f(x), g(x)) && \text{(definition of } \rho\text{)} \\ &= \sup_{x \in X} d(g(x), f(x)) && (\langle 1\rangle 2) \\ &= \rho(g, f) && \text{(definition of } \rho\text{)} \end{aligned}$$

⟨1⟩7. For all $f, g, h \in \mathcal{B}(X, Y)$ we have $\rho(f, h) \leq \rho(f, g) + \rho(g, h)$
Proof:

$$\begin{aligned} \rho(f,h) &= \sup_{x \in X} d(f(x), h(x)) && \text{(definition of } \rho\text{)} \\ &\leq \sup_{x \in X} (d(f(x), g(x)) + d(g(x), h(x))) && \text{(Triangle inequality)} \\ &\leq \sup_{x \in X} d(f(x), g(x)) + \sup_{x \in X} d(g(x), h(x)) && \text{(Lemma 2.0.1)} \\ &= \rho(f, g) + \rho(g, h) && \text{(definition of } \rho\text{)} \end{aligned}$$

□

10.1.4 Totally Bounded Metric Spaces

Definition 10.1.7 (Totally Bounded)**.** A metric space X is *totally bounded* iff, for every $\epsilon > 0$, there exists a finite covering of X by ϵ-balls.

10.2 The Metric Topology

Definition 10.2.1 (Metric Topology)**.** Let d be a metric on X. The *metric topology* on X induced by d is the topology generated by the basis consisting of the open balls.

We prove this is a topology.

Proof:
⟨1⟩1. Every point is in an open ball.
Proof: $x \in B(x, 1)$
⟨1⟩2. If B_1, B_2 are open balls and $x \in B_1 \cap B_2$, then there exists an open ball B_3 such that $x \in B_3 \subseteq B_1 \cap B_2$.
⟨2⟩1. Let: $x \in B(y, \epsilon_1) \cap B(z, \epsilon_2)$

⟨2⟩2. PICK δ_1, δ_2 such that $0 < \delta_1 < \epsilon_1$, $0 < \delta_2 < \epsilon_2$, $B(x, \delta_1) \subseteq B(y, \epsilon_1)$ and $B(x, \delta_2) \subseteq B(z, \epsilon_2)$.
PROOF: Lemma 10.1.3.
⟨2⟩3. LET: $\delta = \min(\delta_1, \delta_2)$
⟨2⟩4. $x \in B(x, \delta) \subseteq B(y, \epsilon_1) \cap B(y, \epsilon_2)$
⟨1⟩3. Q.E.D.
PROOF: Lemma 3.5.3.

Lemma 10.2.2. *A set U is open in the metric topology induced by d if and only if, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$.*

PROOF:
⟨1⟩1. If U is open then, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$.
⟨2⟩1. ASSUME: U is open.
⟨2⟩2. LET: $x \in U$
⟨2⟩3. PICK $B(y, \delta)$ such that $x \in B(y, \delta) \subseteq U$
⟨2⟩4. PICK ϵ such that $0 < \epsilon < \delta$ and $B(x, \epsilon) \subseteq B(y, \delta)$
PROOF: Lemma 10.1.3.
⟨2⟩5. $B(x, \epsilon) \subseteq U$
PROOF: From ⟨2⟩3 and ⟨2⟩4.
⟨1⟩2. If, for all $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$, then U is open.
PROOF: Immediate from definition of metric topology.
□

Lemma 10.2.3. *Let d and d' be two metrics on the set X. Let $\mathcal{T}$ and $\mathcal{T}'$ be the topologies the induce, respectively. Then $\mathcal{T} \subseteq \mathcal{T}'$ if and only if, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$.*

PROOF:
⟨1⟩1. If $\mathcal{T} \subseteq \mathcal{T}'$ then, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$.
⟨2⟩1. ASSUME: $\mathcal{T} \subseteq \mathcal{T}'$
⟨2⟩2. LET: $x \in X$ and $\epsilon > 0$
⟨2⟩3. $B_d(x, \epsilon) \in \mathcal{T}'$
PROOF: From ⟨2⟩1.
⟨2⟩4. There exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$
PROOF: By Lemma 10.2.2.
⟨1⟩2. If, for all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$, then $\mathcal{T} \subseteq \mathcal{T}'$
⟨2⟩1. ASSUME: For all $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$.
⟨2⟩2. LET: $U \in \mathcal{T}$
PROVE: $U \in \mathcal{T}'$
⟨2⟩3. LET: $x \in U$
⟨2⟩4. PICK $\epsilon > 0$ be such that $B_d(x, \epsilon) \subseteq U$
PROOF: By Lemma 10.2.2.
⟨2⟩5. PICK $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$

PROOF: By ⟨2⟩1.
⟨2⟩6. $B_{d'}(x, \delta) \subseteq U$
PROOF: By ⟨2⟩4 and ⟨2⟩5.
⟨2⟩7. Q.E.D.
PROOF: By Lemma 10.2.2.
□

Definition 10.2.4 (Metrizable)**.** A topological space is *metrizable* if and only if there exists a metric that induces its topology.

Lemma 10.2.5. *Every discrete space is metrizable.*

PROOF: The discrete topology is induced by the metric $d(x, y) = 1$ if $x \neq y$, 0 if $x = y$. □

Proposition 10.2.6. *The continuous image of a metrizable space is not necessarily metrizable.*

PROOF: The identity map from the discrete two-point space to the indiscrete two-point space is continuous. □

Lemma 10.2.7. $\mathbb{R}$ *is metrizable.*

PROOF: The standard topology is induced by the metric $d(x, y) = |x - y|$. □

Lemma 10.2.8. *Let (X, d) be a metric space and $A \subseteq X$. Then $d\restriction_{A \times A}$ is a metric on A that induces the subspace topology.*

PROOF:
⟨1⟩1. $d\restriction_{A \times A}$ is a metric on A.
PROOF: Each of the axioms for a metric follows immediately from the same axiom for d.
⟨1⟩2. The topology induced by $d\restriction_{A \times A}$ is the product topology.
PROOF: Both are the topology generated by the basis consisting of all the open balls $B_{d\restriction_{A \times A}}(a, \epsilon) = B_d(a, \epsilon) \cap A$.
□

Lemma 10.2.9. *Every metric space is Hausdorff.*

PROOF:
⟨1⟩1. LET: X be a metric space and $x, y \in X$ with $x \neq y$.
⟨1⟩2. LET: $\epsilon = d(x, y)$
⟨1⟩3. $B(x, \epsilon/2)$ and $B(y, \epsilon/2)$ are disjoint neighbourhoods of x and y.
□

Theorem 10.2.10. *Every metric space is first countable.*

PROOF: $\{B(x, q) : q \in \mathbb{Q}^+\}$ is a local basis at x. □

Corollary 10.2.10.1. *If J is infinite then the space $\mathbb{R}^J$ is not metrizable.*

Definition 10.2.11 (Standard Bounded Metric)**.** Let d be a metric on X. The *standard bounded metric* corresponding to d is

$$\overline{d}(x,y) = \min(d(x,y),1) \enspace .$$

We prove this is a metric.

PROOF:
$\langle 1 \rangle 1$. $\overline{d}(x,y) \geq 0$
PROOF: This holds because $d(x,y) \geq 0$ (d is a metric) and $1 > 0$.
$\langle 1 \rangle 2$. $\overline{d}(x,y) = 0$ iff $x = y$
PROOF: Immediate from definition.
$\langle 1 \rangle 3$. $\overline{d}(x,y) = \overline{d}(y,x)$
PROOF: Immediate from definition.
$\langle 1 \rangle 4$. $\overline{d}(x,z) \leq \overline{d}(x,y) + \overline{d}(y,z)$
$\langle 2 \rangle 1$. CASE: $d(x,y) \leq 1$, $d(y,z) \leq 1$
PROOF:

$$\begin{aligned} \overline{d}(x,z) &\leq d(x,z) \\ &\leq d(x,y) + d(y,z) \\ &= \overline{d}(x,y) + \overline{d}(y,z) \end{aligned}$$

$\langle 2 \rangle 2$. CASE: $d(y,z) > 1$
PROOF:

$$\begin{aligned} \overline{d}(x,z) &\leq 1 \\ &\leq \overline{d}(x,y) + 1 \\ &= \overline{d}(x,y) + \overline{d}(y,z) \end{aligned}$$

$\langle 2 \rangle 3$. CASE: $d(x,y) > 1$
PROOF: Similar.
$\square$

Theorem 10.2.12. *Let d be a metric on X. Then the standard bounded metric $\overline{d}$ corresponding to d induces the same topology as d.*

PROOF:
$\langle 1 \rangle 1$. LET: $\mathcal{T}$ be the topology induced by d and $\mathcal{T}'$ be the topology induced by $\overline{d}$.
$\langle 1 \rangle 2$. $\mathcal{T} \subseteq \mathcal{T}'$
$\langle 2 \rangle 1$. LET: $x \in X$ and $\epsilon > 0$
$\langle 2 \rangle 2$. LET: $\delta = \min(\epsilon, 1/2)$
$\langle 2 \rangle 3$. $B_{\overline{d}}(x,\delta) \subseteq B_d(x,\epsilon)$
$\langle 3 \rangle 1$. LET: $y \in B_{\overline{d}}(x,\delta)$
$\langle 3 \rangle 2$. $\overline{d}(x,y) < \delta$
$\langle 3 \rangle 3$. $\overline{d}(x,y) < 1$
PROOF: From $\langle 2 \rangle 2$ and $\langle 3 \rangle 2$.
$\langle 3 \rangle 4$. $\overline{d}(x,y) = d(x,y)$
PROOF: From $\langle 3 \rangle 3$ and the definition of $\overline{d}$.
$\langle 3 \rangle 5$. $d(x,y) < \epsilon$

PROOF: By ⟨2⟩2 and ⟨3⟩2 and ⟨3⟩4.
⟨1⟩3. $\mathcal{T}' \subseteq \mathcal{T}$
⟨2⟩1. LET: $x \in X$ and $\epsilon > 0$
⟨2⟩2. $B_d(x, \epsilon) \subseteq B_{\overline{d}}(x, \epsilon)$
PROOF: This holds because $\overline{d}(x, y) \leq d(x, y)$.
∎

Definition 10.2.13 (Square Metric)**.** The *square metric* on $\mathbb{R}^n$ is defined by

$$\rho(\vec{x}, \vec{y}) = \max(|x_1 - y_1|, \ldots, |x_n - y_n|) \ .$$

We prove this is a metric.

PROOF:
⟨1⟩1. $\rho(\vec{x}, \vec{y}) \geq 0$
PROOF: Immediate from definitions.
⟨1⟩2. $\rho(\vec{x}, \vec{y}) = 0$ iff $\vec{x} = \vec{y}$
PROOF: Immediate from definitions.
⟨1⟩3. $\rho(\vec{x}, \vec{y}) = \rho(\vec{y}, \vec{x})$
PROOF: Immediate from definitions.
⟨1⟩4. $\rho(\vec{x}, \vec{z}) \leq \rho(\vec{x}, \vec{y}) + \rho(\vec{y}, \vec{z})$
⟨2⟩1. For all i, we have $|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$
⟨2⟩2. For all i, $|x_i - z_i| \leq \rho(\vec{x}, \vec{y}) + \rho(\vec{y}, \vec{z})$
⟨2⟩3. $\rho(\vec{x}, \vec{z}) \leq \rho(\vec{x}, \vec{y}) + \rho(\vec{y}, \vec{z})$
∎

Theorem 10.2.14. *The square metric induces the standard topology on* $\mathbb{R}^n$*.*

PROOF:
⟨1⟩1. LET: $\mathcal{T}_\rho$ be the topology induced by the square metric and $\mathcal{T}_s$ the standard topology.
⟨1⟩2. $\mathcal{T}_\rho \subseteq \mathcal{T}_s$
PROOF: This holds because $B_\rho(\vec{x}, \epsilon) = (x_1 - \epsilon, x_1 + \epsilon) \times \cdots \times (x_n - \epsilon, x_n + \epsilon)$.
⟨1⟩3. $\mathcal{T}_s \subseteq \mathcal{T}_\rho$
⟨2⟩1. LET: $B = U_1 \times \cdots \times U_n$ be a basic open set in $\mathcal{T}_s$, where each U_i is open in $\mathbb{R}$.
⟨2⟩2. LET: $\vec{x} \in B$
⟨2⟩3. For $1 \leq i \leq n$, PICK $\epsilon_i > 0$ such that $(x_i - \epsilon_i, x_i + \epsilon_i) \subseteq U_i$
⟨2⟩4. LET: $\epsilon = \min(\epsilon_1, \ldots, \epsilon_n)$
⟨2⟩5. $B_\rho(\vec{x}, \epsilon) \subseteq B$
∎

Lemma 10.2.15. *The product of a countable family of metrizable spaces is metrizable.*

PROOF:
⟨1⟩1. LET: $\{X_n\}_{n \in \mathbb{Z}^+}$ be a family of metric spaces with metrics bounded by 1, $X = \prod_{n=1}^{\infty} X_n$.

⟨1⟩2. LET: $D : X \times X \to \mathbb{R}$ be given by

$$D(\vec{x}, \vec{y}) = \sup_{n \geq 1} \frac{d(x_n, y_n)}{n} \enspace .$$

⟨1⟩3. D is a metric on X.

⟨2⟩1. $D(\vec{x}, \vec{y}) \geq 0$

PROOF: Immediate from definitions.

⟨2⟩2. $D(\vec{x}, \vec{y}) = 0$ iff $\vec{x} = \vec{y}$

PROOF: Immediate from definitions.

⟨2⟩3. $D(\vec{x}, \vec{y}) = D(\vec{y}, \vec{x})$

PROOF: Immediate from definitions.

⟨2⟩4. $D(\vec{x}, \vec{z}) \leq D(\vec{x}, \vec{y}) + D(\vec{y}, \vec{z})$

⟨3⟩1. For all n, we have $\frac{d(x_n,z_n)}{n} \leq \frac{d(x_n,y_n)}{n} + \frac{d(y_n,z_n)}{n}$

⟨3⟩2. For all n, we have $\frac{d(x_n,z_n)}{n} \leq D(\vec{x}, \vec{y}) + D(\vec{y}, \vec{z})$

⟨3⟩3. $D(\vec{x}, \vec{z}) \leq D(\vec{x}, \vec{y}) + D(\vec{y}, \vec{z})$

⟨1⟩4. LET: $\mathcal{T}_D$ be the topology induced by D and $\mathcal{T}_p$ the product topology.

⟨1⟩5. $\mathcal{T}_D \subseteq \mathcal{T}_p$

⟨2⟩1. LET: $U \in \mathcal{T}_D$

PROVE: $U \in \mathcal{T}_p$

⟨2⟩2. LET: $\vec{x} \in U$

⟨2⟩3. PICK $\epsilon > 0$ such that $B_D(\vec{x}, \epsilon) \subseteq U$

⟨2⟩4. PICK N such that $1/N < \epsilon$

⟨2⟩5. LET: $V = B(x_1, \epsilon) \times \cdots \times B(x_N, \epsilon) \times X_{N+1} \times X_{N+2} \times \cdots$

⟨2⟩6. $\vec{x} \in V \subseteq B_D(\vec{x}, \epsilon)$

⟨1⟩6. $\mathcal{T}_p \subseteq \mathcal{T}_D$

⟨2⟩1. LET: $U = \prod_{n=1}^{\infty} U_n$ be a basic open set in $\mathcal{T}_p$, where each U_n is open in X_n, and $U_n = X_n$ for $n > N$.

⟨2⟩2. LET: $\vec{x} \in U$

PROVE: There exists $\epsilon > 0$ such that $B_D(\vec{x}, \epsilon) \subseteq U$.

⟨2⟩3. For $n \leq N$, PICK $\epsilon_n > 0$ such that $B(x_n, \epsilon_n) \subseteq U_n$

⟨2⟩4. LET: $\epsilon = \min(\epsilon_1, \epsilon_2/2, \ldots, \epsilon_n/n)$

⟨2⟩5. LET: $\vec{y} \in B_D(\vec{x}, \epsilon)$

⟨2⟩6. For $n \leq N$, $y_n \in U_n$

⟨3⟩1. $D(\vec{x}, \vec{y}) < \epsilon$

⟨3⟩2. $d(x_n, y_n)/n < \epsilon$

⟨3⟩3. $d(x_n, y_n)/n < \epsilon_n/n$

⟨3⟩4. Q.E.D.

PROOF: By ⟨2⟩3.

□

Corollary 10.2.15.1. *The space $\mathbb{R}^\omega$ is metrizable.*

Definition 10.2.16 (Uniform Metric)**.** Let (X, d) be a metric space and J be a set. The *uniform metric* $\overline{\rho}$ on X^J is defined by

$$\overline{\rho}(\vec{x}, \vec{y}) = \sup_{\alpha \in J} \overline{d}(x_\alpha, y_\alpha) \enspace .$$

where $\overline{d}$ is the standard bounded metric

$$\overline{d}(x,y) = \min(d(x,y),1) \enspace .$$

The *uniform topology* is the topology induced by the uniform metric.

We prove this is a metric.

PROOF:
⟨1⟩1. $\overline{\rho}(\vec{x},\vec{y}) \geq 0$
PROOF: Immediate from definitions.
⟨1⟩2. $\overline{\rho}(\vec{x},\vec{y}) = 0$ iff $\vec{x} = \vec{y}$
PROOF: Immediate from definitions.
⟨1⟩3. $\overline{\rho}(\vec{x},\vec{y}) = \overline{\rho}(\vec{y},\vec{x})$
PROOF: Immediate from definitions.
⟨1⟩4. $\overline{\rho}(\vec{x},\vec{z}) \leq \overline{\rho}(\vec{x},\vec{y}) + \overline{\rho}(\vec{y},\vec{z})$
PROOF:
⟨2⟩1. For all $\alpha \in J$, $\overline{d}(x_\alpha, z_\alpha) \leq \overline{d}(x_\alpha, y_\alpha) + \overline{d}(y_\alpha, z_\alpha)$
⟨2⟩2. For all $\alpha \in J$, $\overline{d}(x_\alpha, z_\alpha) \leq \overline{\rho}(\vec{x},\vec{y}) + \overline{\rho}(\vec{y},\vec{z})$
⟨2⟩3. $\overline{\rho}(\vec{x},\vec{z}) \leq \overline{\rho}(\vec{x},\vec{y}) + \overline{\rho}(\vec{y},\vec{z})$
□

Theorem 10.2.17 (DC)**.** *The uniform topology on $\mathbb{R}^J$ is finer than the product topology and coarser than the box topology. These three topologies are different iff J is infinite.*

PROOF:
⟨1⟩1. The uniform topology is finer than the product topology.
⟨2⟩1. LET: $B = \prod_{\alpha \in J} U_\alpha$ be a basic open set in the product topology, where each U_α is open in $\mathbb{R}$, and $U_\alpha = \mathbb{R}$ except for $\alpha = \alpha_1, \ldots, \alpha_n$.
⟨2⟩2. LET: $\vec{x} \in U$
⟨2⟩3. For $1 \leq i \leq n$, PICK $0 < \epsilon_i < 1$ such that $(x_{\alpha_i} - \epsilon_i, x_{\alpha_i} + \epsilon_i) \subseteq U_{\alpha_i}$.
⟨2⟩4. LET: $\epsilon = \min(\epsilon_1, \ldots, \epsilon_n)$
⟨2⟩5. $B_{\overline{\rho}}(\vec{x},\epsilon) \subseteq B$
⟨3⟩1. LET: $\vec{y} \in B_{\overline{\rho}}(\vec{x},\epsilon)$
⟨3⟩2. For $1 \leq i \leq n$, we have $y_i \in U_{\alpha_i}$
⟨4⟩1. LET: $1 \leq i \leq n$
⟨4⟩2. $\overline{d}(x_{\alpha_i}, y_{\alpha_i}) < \epsilon_i$
PROOF: From ⟨2⟩4 and ⟨3⟩1.
⟨4⟩3. $d(x_{\alpha_i}, y_{\alpha_i}) < \epsilon_i$
PROOF: From ⟨4⟩2 since $\epsilon_i < 1$ (⟨2⟩3).
⟨4⟩4. Q.E.D.
PROOF: By ⟨2⟩3.
⟨1⟩2. The uniform topology is coarser than the box topology.
⟨2⟩1. LET: $\vec{x} \in \mathbb{R}^J$ and $\epsilon > 0$
PROVE: $B_{\overline{\rho}}(\vec{x},\epsilon)$ is open in the box topology.
⟨2⟩2. CASE: $\epsilon < 1$
PROOF: In this case, $B(\vec{x},\epsilon) = \prod_{\alpha \in J}(x_\alpha - \epsilon, x_\alpha + \epsilon)$.

⟨2⟩3. CASE: $\epsilon \geq 1$
PROOF: In this case, $B(\vec{x}, \epsilon) = \mathbb{R}^J$.
⟨1⟩3. If J is finite then the product topology is the same as the box topology.
PROOF: Immediate from definitions.
⟨1⟩4. If J is infinite then the uniform topology is distinct from the product topology.
⟨2⟩1. $B(\vec{0}, 1/2)$ is not open in the product topology.
⟨3⟩1. $\vec{0} \in B(\vec{0}, 1/2)$
⟨3⟩2. LET: $\prod_{\alpha \in J} U_\alpha$ be any basic open set containing $\vec{0}$, where U_α is open in $\mathbb{R}$ for all α, and $U_\alpha = \mathbb{R}$ except for $\alpha = \alpha_1, \ldots, \alpha_n$
⟨3⟩3. PICK $\alpha_0 \in J$ such that $\alpha_0 \neq \alpha_1, \ldots, \alpha_n$
⟨3⟩4. LET: $\vec{x}$ be such that $x_{\alpha_0} = 1$, and $x_\alpha = 0$ for $\alpha \neq \alpha_0$.
⟨3⟩5. $\vec{x} \in \prod_{\alpha \in J} U_\alpha$
⟨3⟩6. $\vec{x} \notin B(\vec{0}, 1/2)$
⟨1⟩5. If J is infinite then the uniform topology is distinct from the box topology.
⟨2⟩1. PICK a countable sequence α_1, α_2, ... in J
⟨2⟩2. LET: $U = \prod_{\alpha \in J} U_\alpha$, where $U_{\alpha_n} = (-1/n, 1/n)$ for all n, and $U_\alpha = \mathbb{R}$ for all other α.
PROVE: U is not open in the uniform topology.
⟨2⟩3. $\vec{0} \in U$
⟨2⟩4. LET: $\epsilon > 0$
PROVE: $B(\vec{0}, \epsilon) \nsubseteq U$
⟨2⟩5. PICK N such that $1/N < \epsilon$
⟨2⟩6. LET: $\vec{x}$ be such that $x_{\alpha_N} = 1/N$ and $x_\alpha = 0$ for all other α
⟨2⟩7. $\vec{x} \in B(\vec{0}, \epsilon)$
⟨2⟩8. $\vec{x} \notin U$
□

Proposition 10.2.18. *The space $\mathbb{R}^\omega$ under the uniform topology is not second countable.*

PROOF: The set of all sequences of 0s and 1s is discrete but uncountable. □

Corollary 10.2.18.1. *Not every metric space is second countable.*

Theorem 10.2.19. *Let X and Y be metric spaces. Let $f : X \to Y$ and $x \in X$. Then f is continuous at x if and only if, for every $\epsilon > 0$, there exists $\delta > 0$ such that, for all $x' \in X$, if $d(x, x') < \delta$ then $d(f(x), f(x')) < \epsilon$.*

PROOF:
⟨1⟩1. If f is continuous at x then, for every $\epsilon > 0$, there exists $\delta > 0$ such that, for all $x' \in X$, if $d(x, x') < \delta$ then $d(f(x), f(x')) < \epsilon$.
⟨2⟩1. ASSUME: f is continuous at x.
⟨2⟩2. LET: $\epsilon > 0$
⟨2⟩3. PICK a neighbourhood U of x such that $f(U) \subseteq B(f(x), \epsilon)$
PROOF: One exists by ⟨2⟩1, since $B(f(x), \epsilon)$ is a neighbourhood of $f(x)$.
⟨2⟩4. PICK $\delta > 0$ such that $B(x, \delta) \subseteq U$
PROOF: By ⟨2⟩3 and Lemma 10.2.2.

⟨2⟩5. LET: $x' \in X$ with $d(x, x') < \delta$

⟨2⟩6. $x' \in U$

PROOF: From ⟨2⟩4 and ⟨2⟩5.

⟨2⟩7. $f(x') \in B(f(x), \epsilon)$

PROOF: From ⟨2⟩3 and ⟨2⟩6.

⟨1⟩2. If, for all $\epsilon > 0$, there exists $\delta > 0$ such that, for all $x' \in X$, if $d(x, x') < \delta$ then $d(f(x), f(x')) < \epsilon$, then f is continuous at x.

⟨2⟩1. ASSUME: For all $\epsilon > 0$ there exists $\delta > 0$ such that, for all $x' \in X$, if $d(x, x') < \delta$ then $d(f(x), f(x')) < \epsilon$.

⟨2⟩2. LET: V be a neighbourhood of $f(x)$

⟨2⟩3. PICK $\epsilon > 0$ such that $B(f(x), \epsilon) \subseteq V$

PROOF: By Lemma 10.2.2.

⟨2⟩4. PICK $\delta > 0$ such that, for all $x' \in X$, if $d(x, x') < \delta$ then $d(f(x), f(x')) < \epsilon$.

PROOF: By ⟨2⟩1 and ⟨2⟩3.

⟨2⟩5. $B(x, \delta)$ is a neighbourhood of x

PROOF: By the definition of the metric topology.

⟨2⟩6. $f(B(x, \delta)) \subseteq V$

⟨3⟩1. LET: $x' \in B(x, \delta)$

⟨3⟩2. $d(f(x), f(x')) < \epsilon$

PROOF: From ⟨2⟩4.

⟨3⟩3. $x' \in V$

PROOF: From ⟨2⟩3.

□

Lemma 10.2.20. *Let X be a metric space. Then the metric $d : X^2 \to \mathbb{R}$ is continuous.*

PROOF:

⟨1⟩1. Give X^2 the square metric.

⟨1⟩2. LET: $x, y \in X$ and $\epsilon > 0$

⟨1⟩3. LET: $\delta = \epsilon/2$

⟨1⟩4. LET: $x', y' \in X$ with $d((x, y), (x', y')) < \delta$

⟨1⟩5. $|d(x, y) - d(x', y')| < \epsilon$

⟨2⟩1. $d(x, y) < d(x', y') + \epsilon$

PROOF:

$$\begin{aligned} d(x,y) &\leq d(x,x') + d(x',y') + d(y,y') && \text{(Triangle inequality)} \\ &< d(x',y') + 2\delta && (\langle 1\rangle 4) \\ &= d(x',y') + \epsilon && (\langle 1\rangle 3) \end{aligned}$$

⟨2⟩2. $d(x', y') < d(x, y) + \epsilon$

PROOF: Similar.

Lemma 10.2.21. *Addition is a continuous function $\mathbb{R}^2 \to \mathbb{R}$.*

PROOF:

⟨1⟩1. LET: $(x, y) \in \mathbb{R}^2$ and $\epsilon > 0$

⟨1⟩2. LET: $\delta = \epsilon/2$

⟨1⟩3. LET: $(x', y') \in \mathbb{R}^2$ be such that $\rho((x,y),(x',y')) < \delta$, where ρ is the square metric

⟨1⟩4. $|x - x'| < \delta$ and $|y - y'| < \delta$

⟨1⟩5. $|(x+y) - (x'+y')| < \epsilon$

PROOF:

$$\begin{aligned} |(x+y)-(x'+y')| &\leq |x-x'| + |y-y'| \\ &< 2\delta && (\langle 1\rangle 4) \\ &= \epsilon && (\langle 1\rangle 2) \end{aligned}$$

⟨1⟩6. Q.E.D.

PROOF: By Theorem 10.2.19.

□

Lemma 10.2.22. *Additive inverse is a continuous function* $- : \mathbb{R} \to \mathbb{R}$.

PROOF: If $|x - y| < \epsilon$ then $|(-x) - (-y)| < \epsilon$. □

Lemma 10.2.23. *Multiplication is a continuous function* $\cdot : \mathbb{R}^2 \to \mathbb{R}$.

PROOF:

⟨1⟩1. LET: $(x, y) \in \mathbb{R}^2$ and $\epsilon > 0$

⟨1⟩2. LET: $\delta = \min(1, \epsilon/(|x| + |y| + 1)$

⟨1⟩3. LET: $(x', y') \in \mathbb{R}^2$ and $\rho((x,y),(x',y')) < \delta$

⟨1⟩4. $|xy - x'y'| < \epsilon$

PROOF:

$$\begin{aligned} |xy - x'y'| &= |x(y'-y) + y(x'-x) + (x-x')(y-y')| \\ &\leq |x||y'-y| + |y||x'-x| + |x-x'||y-y'| \\ &< |x|\delta + |y|\delta + \delta^2 && (\langle 1\rangle 3) \\ &= \delta(|x| + |y| + \delta) \\ &\leq \delta(|x| + |y| + 1) && (\langle 1\rangle 2) \\ &\leq \epsilon && (\langle 1\rangle 2) \end{aligned}$$

□

Lemma 10.2.24. *Multiplicative inverse is a continuous function* $(\ )^{-1} : \mathbb{R} \setminus \{0\} \to \mathbb{R}$.

PROOF:

⟨1⟩1. LET: $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be defined by $f(x) = x^{-1}$.

⟨1⟩2. LET: $a, b \in \mathbb{R}$ with $a < b$

PROVE: $f^{-1}((a,b))$ is open

⟨1⟩3. CASE: $0 < a < b$

PROOF: $f^{-1}((a,b)) = (b^{-1}, a^{-1})$

⟨1⟩4. CASE: $a < 0 < b$

PROOF: $f^{-1}((a,b)) = (-\infty, a^{-1}) \cup (b^{-1}, +\infty)$

⟨1⟩5. CASE: $a < b < 0$

PROOF: $f^{-1}((a,b)) = (b^{-1}, a^{-1})$

□

Definition 10.2.25 (Uniform Convergence)**.** Let X be a set and Y a metric space. Let $f_n : X \to Y$ for $n \geq 1$, and $f : X \to Y$. Then f_n *converges uniformly* to f as $n \to \infty$ iff, for all $\epsilon > 0$, there exists N such that, for all $x \in X$ and $n \geq N$, $d(f_n(x), f(x)) < \epsilon$.

Theorem 10.2.26 (Uniform Limit Theorem)**.** *Let X be a topological space and Y a metric space. Let $f_n : X \to Y$ for $n \geq 1$ and $f : X \to Y$. If f_n converges uniformly to f and each f_n is continuous, then f is continuous.*

PROOF:
⟨1⟩1. LET: $x \in X$ and $\epsilon > 0$
⟨1⟩2. PICK N such that, for all $x' \in X$ and $\delta > 0$, $d(f_n(x'), f(x')) < \epsilon/3$
⟨1⟩3. PICK $\delta > 0$ such that, for all $x' \in X$, if $d(x, x') < \delta$ then $d(f_N(x), f_N(x')) < \epsilon/3$
⟨1⟩4. For all $x' \in X$, if $d(x, x') < \delta$ then $d(f(x), f(x')) < \epsilon$
PROOF:
$$\begin{aligned} d(f(x), f(x')) &\leq d(f(x), f_N(x)) + d(f_N(x), f_N(x')) + d(f_N(x'), f(x')) \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 \\ &= \epsilon \end{aligned}$$
□

Lemma 10.2.27. *Let X be a set and Y a metric space. Let $f_n : X \to Y$ for $n \geq 1$ and $f : X \to Y$. Then f_n converges uniformly to f if and only if f_n converges to f in Y^X under the uniform topology.*

PROOF:
⟨1⟩1. If f_n converges uniformly to f then f_n converges to f under the uniform topology.
⟨2⟩1. ASSUME: f_n converges uniformly to f
⟨2⟩2. LET: $\epsilon > 0$
⟨2⟩3. PICK N such that, for all $x \in X$ and $n \geq N$, $d(f_n(x), f(x)) < \epsilon/2$
⟨2⟩4. $\overline{\rho}(f_n, f) \leq \epsilon/2$
⟨2⟩5. $\overline{\rho}(f_n, f) < \epsilon$
⟨1⟩2. If f_n converges to f under the uniform topology then f_n converges uniformly to f.
⟨2⟩1. ASSUME: f_n converges to f under the uniform topology.
⟨2⟩2. LET: $\epsilon > 0$
⟨2⟩3. PICK N such that, for all $n \geq N$, $\overline{\rho}(f_n, f) < \epsilon$
⟨2⟩4. For all $n \geq N$ and $x \in X$, $d(f_n(x), f(x)) < \epsilon$
□

Theorem 10.2.28. *Every monotone increasing sequence of real numbers that is bounded above converges to its supremum.*

PROOF:
⟨1⟩1. LET: $\{s_n\}_{n \geq 1}$ be a monotone increasing sequence of real numbers bounded above with supremum l.

⟨1⟩2. LET: $\epsilon > 0$
⟨1⟩3. $l - \epsilon$ is not an upper bound for $\{s_n : n \geq 1\}$.
⟨1⟩4. PICK N such that $x_N > l - \epsilon$
⟨1⟩5. For all $n \geq N$, we hawe $l - \epsilon < x_n \leq l$
⟨1⟩6. For all $n \geq N$, we have $|x_n - l| < \epsilon$
□

Definition 10.2.29 (Infinite Series)**.** Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers. The *infinite series* $\sum_{n=1}^{\infty} a_n$ *converges* to s iff $\sum_{n=1}^{N} a_n \to s$ as $N \to \infty$.

Proposition 10.2.30. *If* $\sum_{n=1}^{\infty} a_n = s$ *and* $\sum_{n=1}^{\infty} b_n = t$ *then* $\sum_{n=1}^{\infty}(ca - n + b_n) = cs + t$.

PROOF:This holds because $\sum_{n=1}^{N}(ca_n + b_n) = c\sum_{n=1}^{N} a_n + \sum_{n=1}^{N} b_n \to cs + t$ as $N \to \infty$. □

Theorem 10.2.31 (Comparison Test)**.** *If* $|a_i| \leq b_i$ *for all* i *and* $\sum_{i=1}^{\infty} b_i$ *converges, then* $\sum_{i=1}^{\infty} a_i$ *converges.*

PROOF:
⟨1⟩1. $\sum_{i=1}^{\infty} |a_i|$ converges
  PROOF: $\sum_{i=1}^{N} |a_i|$ is a monotone increasing sequence bounded above by $\sum_{i=1}^{\infty} b_i$.
⟨1⟩2. LET: $c_i = |a_i| + a_i$
⟨1⟩3. $\sum_{i=1}^{\infty} c_i$ converges
  PROOF: $\sum_{i=1}^{N} c_i$ is a monotone increasing sequence bounded above by $2\sum_{i=1}^{\infty} |a_i|$.
⟨1⟩4. Q.E.D.
  PROOF: Since $a_i = c_i - |a_i|$.

Lemma 10.2.32. *If* $\sum_{n=1}^{\infty} a_n$ *converges then* $\sum_{n=N}^{\infty} a_n \to 0$ *as* $N \to \infty$.

PROOF:

$$\begin{aligned}\sum_{n=N}^{\infty} a_n &= \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N-1} a_n \\ &\to \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} a_n \\ &= 0\end{aligned}$$

as $N \to \infty$. □

Theorem 10.2.33 (Weierstrass M-Test)**.** *Let* X *be a set and* $f_n : X \to \mathbb{R}$ *for* $n \geq 1$. *If* $|f_n(x)| \leq M_n$ *for all* $n \geq 1$ *and all* $x \in X$, *and if* $\sum_{n=1}^{\infty} M_n$ *converges, then*

$$\sum_{n=1}^{N} f_n(x) \to \sum_{n=1}^{\infty} f_n(x)$$

uniformly in x *as* $N \to \infty$.

PROOF:

$\langle 1\rangle 1$. For $N \geq 1$,
    LET: $s_N : X \to \mathbb{R}$, $s_N(x) = \sum_{n=1}^{N} f_n(x)$
$\langle 1\rangle 2$. For all $x \in X$, $\sum_{n=1}^{\infty} f_n(x)$ converges.
  PROOF: By the Comparison Test.
$\langle 1\rangle 3$. LET: $s : X \to \mathbb{R}$, $s(x) = \sum_{n=1}^{\infty} f_n(x)$.
$\langle 1\rangle 4$. For $N \geq 1$,
    LET: $r_N = \sum_{n=N+1}^{\infty} M_n$
$\langle 1\rangle 5$. For $1 \leq N < K$, we have $|s_K(x) - s_N(x)| \leq r_N$ for all $x \in X$
  PROOF:

$$\begin{aligned} |s_K(x) - s_N(x)| &= \left| \sum_{n=N+1}^{K} f_n(x) \right| \\ &\leq \sum_{n=N+1}^{K} |f_n(x)| \\ &\leq \sum_{n=N+1}^{K} M_n \\ &\leq \sum_{n=N+1}^{\infty} M_n \\ &= r_N \end{aligned}$$

$\langle 1\rangle 6$. For $N \geq 1$ and $x \in X$ we have $|s(x) - s_N(x)| \leq r_N$
  PROOF: Let $K \to \infty$ in $\langle 1\rangle 5$.
$\langle 1\rangle 7$. LET: $\epsilon > 0$
$\langle 1\rangle 8$. PICK N such that, for all $N' \geq N$, we have $r_{N'} < \epsilon$
  PROOF: Such an N exists by Lemma 10.2.32.
$\langle 1\rangle 9$. For all $N' \geq N$ and $x \in X$ we have $|s_{N'}(x) - s(x)| < \epsilon$
$\square$

Definition 10.2.34. Let X be a metric space. Let $x \in X$ and $A \subseteq X$ be nonempty. The *distance* from x to A is

$$d(x, A) = \inf_{a \in A} d(x, a) \quad .$$

Lemma 10.2.35. *Let X be a metric space and $A \subseteq X$ be nonempty. Then the function $d(-, A) : X \to \mathbb{R}$ is continuous.*

PROOF:
$\langle 1\rangle 1$. LET: $x \in X$ and $\epsilon > 0$
$\langle 1\rangle 2$. LET: $y \in X$ with $d(x, y) < \epsilon$
$\langle 1\rangle 3$. $|d(x, A) - d(y, A)| < \epsilon$
  PROOF:
  $\langle 2\rangle 1$. $d(x, A) - d(y, A) < \epsilon$

PROOF:

$$\begin{aligned} d(x,A) &= \inf_{a \in A} d(x,a) \\ &\leq \inf_{a \in A} (d(x,y) + d(y,a)) \\ &= d(x,y) + \inf_{a \in A} d(y,a) \\ &= d(x,y) + d(y,A) \\ &< \epsilon + d(y,A) \end{aligned}$$

⟨2⟩2. $d(y,A) - d(x,A) < \epsilon$
PROOF: Similar.
⟨1⟩4. Q.E.D.
PROOF: By Theorem 10.2.19.
□

Definition 10.2.36 (Shrinking Map)**.** Let X be a metric space and $f : X \to X$. Then f is a *shrinking map* iff, for all $x, y \in X$ with $x \neq y$, we have $d(f(x), f(y)) < d(x, y)$.

Definition 10.2.37 (Contraction)**.** Let X be a metric space and $f : X \to X$. Then f is a *contraction* iff there exists $\alpha < 1$ such that, for all $x, y \in X$,

$$d(f(x), f(y)) \leq \alpha d(x, y) \enspace .$$

Proposition 10.2.38. *Every separable metric space is second countable.*

PROOF:
⟨1⟩1. LET: X be a separable metric space.
⟨1⟩2. PICK a countable dense set D
⟨1⟩3. LET: $\mathcal{B} = \{B(d,q) : d \in D, q \in \mathbb{Q}^+\}$
⟨1⟩4. $\mathcal{B}$ is a countable basis for X
□

Corollary 10.2.38.1. *The space $\mathbb{R}^\omega$ under the uniform topology is not separable.*

Corollary 10.2.38.2. *Not every metric space is separable.*

Corollary 10.2.38.3. *The space $\mathbb{R}^\omega$ under the box topology is not separable.*

Proposition 10.2.39 (CC)**.** *Every Lindelöf metric space is second countable.*

PROOF:
⟨1⟩1. LET: X be a Lindelöf metric space.
⟨1⟩2. For all $n \in \mathbb{Z}^+$, PICK a countable covering $\mathcal{A}_n$ of X by $1/n$-balls
PROOF: One exists by the Lindelöf condition, since the set of all $1/n$-balls covers X.
⟨1⟩3. $\bigcup_{n=1}^\infty \mathcal{A}_n$ is a countable basis.
□

Corollary 10.2.39.1. *The space $\mathbb{R}^\omega$ under the uniform topology is not Lindelöf.*

Corollary 10.2.39.2. *Not every metric space is Lindelöf.*

Proposition 10.2.40. *The space $\mathbb{R}_l$ is not metrizable.*

PROOF: It is Lindelöf but not second countable. $\square$

Proposition 10.2.41. *The ordered square is not metrizable.*

PROOF: It is compact but not second countable. $\square$

Proposition 10.2.42. *The space $\mathbb{R}^\omega$ under the uniform topology is not second countable.*

PROOF: It contains a subspace homeomorphic to $\mathbb{R}$. $\square$

Theorem 10.2.43 (AC)**.** *Every metrizable space is normal.*

PROOF:
$\langle 1 \rangle 1$. LET: X be a metric space.
$\langle 1 \rangle 2$. LET: A and B be disjoint closed subspaces of X.
$\langle 1 \rangle 3$. For $a \in A$, PICK $\epsilon_a > 0$ such that $B(a, \epsilon_a)$ does not intersect B.
$\langle 1 \rangle 4$. For $b \in B$, PICK $\epsilon_b > 0$ such that $B(b, \epsilon_b)$ does not intersect A.
$\langle 1 \rangle 5$. LET: $U = \bigcup_{a \in A} B(a, \epsilon_a/2)$
$\langle 1 \rangle 6$. LET: $V = \bigcup_{b \in B} B(b, \epsilon_b/2)$
$\langle 1 \rangle 7$. $U \cap V = \emptyset$
  $\langle 2 \rangle 1$. LET: $z \in U \cap V$
  $\langle 2 \rangle 2$. PICK $a \in A$ and $b \in B$ such that $z \in B(a, \epsilon_a/2)$ and $z \in B(b, \epsilon_b/2)$
  $\langle 2 \rangle 3$. ASSUME: w.l.o.g. $\epsilon_a \leq \epsilon_b$
  $\langle 2 \rangle 4$. $a \in B(b, \epsilon_b)$
    PROOF:

$$\begin{aligned} d(a,b) &\leq d(a,z) + d(b,z) && \text{(Triangle Inequality)} \\ &< \epsilon_a/2 + \epsilon_b/2 && (\langle 2 \rangle 2) \\ &\leq \epsilon_b && (\langle 2 \rangle 3) \end{aligned}$$

  $\langle 2 \rangle 5$. Q.E.D.
    PROOF: This contradicts $\langle 1 \rangle 4$.

$\square$

Corollary 10.2.43.1. *The space $\mathbb{R}^\omega$ is normal.*

Corollary 10.2.43.2. *The space $\mathbb{R}_K$ is not metlizable.*

Proposition 10.2.44. *Every metrizable space is completely normal.*

PROOF: Every subspace is metrizable (Lemma 10.2.8) hence normal (Theorem 10.2.43). $\square$

Proposition 10.2.45. *Every metrizable space is perfectly normal.*

PROOF:

⟨1⟩1. LET: X be a metric space.
⟨1⟩2. X is normal.
PROOF: Theorem 10.2.43
⟨1⟩3. Every closed set is G_δ.
PROOF: If A is closed then $A = \bigcap_{q \in \mathbb{Q}^+} \{x \in X : d(A, x) < q\}$.
□

Theorem 10.2.46 (Urysohn Metrization Theorem (CC))**.** *Every second countable regular space is metrizable.*

PROOF:
⟨1⟩1. LET: X be a second countable regular space.
⟨1⟩2. X is normal.
⟨1⟩3. PICK a countable basis $\mathcal{B} = \{B_n : n \in \mathbb{Z}^+\}$
⟨1⟩4. For every pair of integers m, n with $\overline{B_m} \subseteq B_n$, PICK a continuous function $g_{mn} : X \to [0, 1]$ such that $g_{mn}(\overline{B_m}) = \{1\}$ and $g_{mn}(X \setminus B_n) = \{0\}$
PROOF: By the Urysohn Lemma.
⟨1⟩5. The set $\{g_{mn} : \overline{U_m} \subseteq U_n\}$ separates points from closed sets in X
⟨2⟩1. LET: $x \in X$ and U be a neighbourhood of x
⟨2⟩2. PICK $B_n \in \mathcal{B}$ such that $x \in B_n \subseteq U$
⟨2⟩3. PICK V open such that $x \in V$ and $\overline{V} \subseteq B_n$
⟨2⟩4. PICK $B_m \in \mathcal{B}$ such that $x \in B_m \subseteq V$
⟨2⟩5. $g_{mn}(x) = 1$ and g_{mn} vanishes outside U
⟨1⟩6. X is imbeddable in $[0, 1]^\omega$
PROOF: By the Imbedding Theorem.
⟨1⟩7. Q.E.D.

Corollary 10.2.46.1. *The space $\mathbb{R}^\omega$ under the box topology is not second countable.*

Proposition 10.2.47. *Not every second countable Hausdorff space is metrizable.*

PROOF: $\mathbb{R}_K$ is second countable and Hausdorff but not metrizable (because it is not regular). □

Proposition 10.2.48. *There exists a space that is completely normal, first countable, Lindelöf and separable but not metrizable.*

PROOF: The space $\mathbb{R}_l$ is all of these. □

Proposition 10.2.49. *$\overline{S_\Omega}$ is not metrizable.*

PROOF: It is compact but not sequentially compact. □

Proposition 10.2.50. *Every compact metric space is second countable.*

PROOF:
⟨1⟩1. LET: X be a compact etric space

⟨1⟩2. For every $n \geq 1$, PICK a finite covering $\mathcal{A}_n$ of X by open balls of radius $1/n$

PROOF: Such a covering exists because $\{B_{1/n}(x) : x \in X\}$ covers X.

⟨1⟩3. $\bigcup_{n=1}^{\infty} \mathcal{A}_n$ is a countable basis for X

□

Corollary 10.2.50.1. *The space $\mathbb{R}^\omega$ under the uniform topology is not compact.*

Corollary 10.2.50.2. *The space $\mathbb{R}^\omega$ under the uniform topology is not limit point compact.*

Proposition 10.2.51. *The space $\mathbb{R}^\omega$ under the box topology is not locally compact.*

PROOF:

⟨1⟩1. ASSUME: $\mathbb{R}^\omega$ under the box topology is locally compact.

⟨1⟩2. For every point x, there exists a basic open set $B = \prod_{i=0}^{\infty} U_i$ such that $x \in B$ and $\overline{B}$ is compact.

⟨1⟩3. The box topology on $\overline{B}$ is the same as the product topology on $\overline{B}$

PROOF: By Corollary 9.5.11.1.

⟨1⟩4. The box topology on $\overline{B}$ is strictly finer than the product topology.

PROOF:By Theorem 10.2.17.

□

Proposition 10.2.52. *Not every metrizable space is connected.*

PROOF: The discrete space with two points is metrizable but not connected. □

Corollary 10.2.52.1. *Not every metrizable space is path connected.*

Proposition 10.2.53. *Not every metric space is limit point compact.*

PROOF: The space $\mathbb{R}$ is not limit point compact. □

Proposition 10.2.54. *Not every metric space is locally compact.*

The space $\mathbb{R}^\omega$ in the uniform topology is not locally compact.

Lemma 10.2.55 (AC)**.** *Let X be a metrizable space. Then every open covering $\mathcal{A}$ of X has a countably locally discrete open refinement $\mathcal{E}$ that covers X.*

PROOF:

⟨1⟩1. LET: X be a metric space.

⟨1⟩2. PICK a well-ordering $<$ for $\mathcal{A}$.

⟨1⟩3. For $n \in \mathbb{Z}^+$ and $U \in \mathcal{A}$,

LET:

$$S_n(U) = \{x \in X : B(x, 1/n) \subseteq U\}$$

⟨1⟩4. For $n \in \mathbb{Z}^+$ and $U \in \mathcal{A}$,

LET:

$$T_n(U) = S_n(U) - \bigcup_{V<U} V$$

⟨1⟩5. For $n \in \mathbb{Z}^+$ and $U \in \mathcal{A}$,
Let:
$$E_n(U) = \bigcup_{x \in T_n(U)} B(x, 1/3n)$$
⟨1⟩6. For $n \in \mathbb{Z}^+$,
Let:
$$\mathcal{E}_n = \{E_n(U) : U \in \mathcal{A}\}$$
⟨1⟩7. Let:
$$\mathcal{E} = \bigcup_{n \in \mathbb{Z}^+} \mathcal{E}_n$$
⟨1⟩8. $\mathcal{E}$ is countably locally discrete
⟨2⟩1. For all n, $\mathcal{E}_n$ is locally discrete.
⟨3⟩1. For all $x \in X$, we have $B(x, 1/6n)$ intersects at most one element of $\mathcal{E}_n$
⟨4⟩1. Assume: for a contradiction $a \in B(x, 1/6n) \cap E_n(U)$ and $b \in B(x, 1/6n) \cap E_n(V)$
⟨4⟩2. Pick $c \in T_n(U)$ such that $d(a, c) < 1/3n$ and $d \in T_n(V)$ such that $d(b, d) < 1/3n$
⟨4⟩3. Assume: w.l.o.g. $V < U$
⟨4⟩4. $c \in V$
⟨5⟩1. $d(c, d) < 1/n$
Proof:
$$\begin{aligned} d(c,d) &\leq d(c,a) + d(a,x) + d(x,b) + d(b,d) && \text{(Triangle Inequality)} \\ &< 1/3n + 1/6n + 1/6n + 1/3n && (\langle 4\rangle 1,\ \langle 4\rangle 2) \\ &= 1/n \end{aligned}$$
⟨5⟩2. $B(d, 1/n) \subseteq V$
⟨6⟩1. $d \in S_n(V)$
Proof: From ⟨1⟩4 and ⟨4⟩2.
⟨6⟩2. Q.E.D.
Proof: From ⟨1⟩3
⟨4⟩5. Q.E.D.
Proof: This is a contradiction because $c \in T_n(U)$ (⟨4⟩2) so $c \notin V$ (⟨1⟩4, ⟨4⟩3).
⟨1⟩9. $\mathcal{E}$ is an open refinement of $\mathcal{A}$
⟨2⟩1. $\mathcal{E}$ is a refinement of $\mathcal{A}$
⟨3⟩1. For every n, we have $\mathcal{E}_n$ is a refinement of $\mathcal{A}$.
⟨4⟩1. Let: n be a positive integer
⟨4⟩2. For every $U \in \mathcal{A}$ we have $E_n(U) \subseteq U$
⟨5⟩1. Let: $U \in \mathcal{A}$ and $x \in E_n(U)$
⟨5⟩2. Pick $y \in T_n(U)$ such that $x \in B(y, 1/3n)$
Proof: ⟨1⟩5, ⟨5⟩1.
⟨5⟩3. $y \in S_n(U)$
Proof: ⟨1⟩4, ⟨5⟩2
⟨5⟩4. $x \in U$
Proof: ⟨1⟩3, ⟨5⟩2, ⟨5⟩3

$\langle 2\rangle 2$. Every member of $\mathcal{E}$ is open.
$\langle 3\rangle 1$. For all n, every member of $\mathcal{E}_n$ is open.
$\langle 4\rangle 1$. Let: n be a positive integer
$\langle 4\rangle 2$. For all $U \in \mathcal{A}$, $E_n(U)$ is open.
Proof: By $\langle 1\rangle 5$, $E_n(U)$ is a union of open balls.
$\langle 4\rangle 3$. Q.E.D.
Proof: By $\langle 1\rangle 6$
$\langle 3\rangle 2$. Q.E.D.
Proof: By $\langle 1\rangle 7$.
$\langle 1\rangle 10$. $\mathcal{E}$ covers X
$\langle 2\rangle 1$. Let: $x \in X$
$\langle 2\rangle 2$. Let: U be the least member of $\mathcal{A}$ such that $x \in U$
$\langle 2\rangle 3$. Pick n such that $B(x, 1/n) \subseteq U$
$\langle 2\rangle 4$. $x \in E_n(U) \in \mathcal{E}$
$\square$

Theorem 10.2.56. *Every metrizable space is paracompact.*

Proof: From Michael's Lemma and Lemma 10.2.55.

Theorem 10.2.57 (Bing-Nagata-Smirnov Metrization Theorem (AC))**.** *Let X be a topological space. Then the following are equivalent.*

1. *X is metrizable.*
2. *X is regular and has a countably locally finite basis.*
3. *X is regular and has a countably locally discrete basis.*

Proof:
$\langle 1\rangle 1$. Every regular space with a countably locally finite basis is metrizable.
$\langle 2\rangle 1$. Let: X be a regular space with a countably locally finite basis $\mathcal{B}$.
$\langle 2\rangle 2$. X is normal.
Proof: Lemma 6.5.19, $\langle 2\rangle 1$.
$\langle 2\rangle 3$. Every closed set in X is G_δ.
Proof: Lemma 6.5.19, $\langle 2\rangle 1$.
$\langle 2\rangle 4$. Pick locally finite sets $\mathcal{B}_n$ such that $\mathcal{B} = \bigcup_{n=0}^{\infty} \mathcal{B}_n$.
Proof: From $\langle 2\rangle 1$.
$\langle 2\rangle 5$. For $n \in \mathbb{N}$ and $B \in \mathcal{B}_n$, Pick a continuous function $f_{nB} : X \to [0, 1/n]$ such that $f_{nB}(x) > 0$ for $x \in B$ and $f_{nB}(x) = 0$ for $x \notin B$
$\langle 3\rangle 1$. Let: $n \in \mathbb{N}$ and $B \in \mathcal{B}_n$
$\langle 3\rangle 2$. B is open.
$\langle 4\rangle 1$. $B \in \mathcal{B}$.
Proof: $\langle 2\rangle 4$, $\langle 3\rangle 1$
$\langle 4\rangle 2$. Q.E.D.
Proof: $\langle 2\rangle 1$, $\langle 4\rangle 1$
$\langle 3\rangle 3$. $X \setminus B$ is closed and G_δ.
$\langle 4\rangle 1$. $X \setminus B$ is closed.

PROOF: Proposition 3.6.6, ⟨3⟩2.

⟨4⟩2. $X \setminus B$ is G_δ.

PROOF: ⟨2⟩3, ⟨4⟩1.

⟨3⟩4. PICK $g : X \to [0, 1]$ that vanishes precisely on $X \setminus B$.

PROOF: Theorem 6.5.9, ⟨2⟩2,⟨3⟩3.

⟨3⟩5. Q.E.D.

PROOF: Let $f(x) = g(x)/n$.

⟨2⟩6. $\{f_{nB}\}_{n\in\mathbb{N},B\in\mathcal{B}_n}$ separates points from closed sets in X.

⟨3⟩1. LET: $x_0 \in X$ and U be a neighbourhood of x_0

⟨3⟩2. PICK $n \in \mathbb{N}$ and $B \in \mathcal{B}_n$ such that $x_0 \in B \subseteq U$

⟨4⟩1. PICK $B \in \mathcal{B}$ such that $x_0 \in B \subseteq U$

PROOF: ⟨2⟩1, ⟨3⟩1.

⟨4⟩2. PICK $n \in \mathbb{N}$ such that $B \in \mathcal{B}_n$

PROOF: ⟨2⟩4, ⟨4⟩1.

⟨3⟩3. $f_{nB}(x_0) > 0$

PROOF: ⟨2⟩5, ⟨3⟩2.

⟨3⟩4. f_{nB} vanishes outside U.

PROOF: ⟨2⟩5, ⟨3⟩2.

⟨2⟩7. LET: $J = \sum_{n\in\mathbb{N}} \mathcal{B}_n$

⟨2⟩8. LET: $F : X \to [0, 1]^J$ be the function $F(x)(n, B) = f_{nB}(x)$

⟨2⟩9. F is an imbedding relative to the product topology on $[0, 1]^J$

PROOF: By the Imbedding Theorem and ⟨2⟩6.

⟨2⟩10. F is an imbedding relative to the uniform topology on $[0, 1]^J$

⟨3⟩1. F is injective.

PROOF: From ⟨2⟩9

⟨3⟩2. F is an open map relative to the uniform topology.

PROOF: From ⟨2⟩9 and Theorem 10.2.17.

⟨3⟩3. F is continuous relative to the uniform topology.

⟨4⟩1. LET: $x_0 \in X$

⟨4⟩2. LET: $\epsilon > 0$

⟨4⟩3. For all $n \in \mathbb{N}$, PICK a neighbourhood V_n of x_0 such that, for all $B \in \mathcal{B}_n$, f_{nB} varies by at most $\epsilon/2$ on V_n.

⟨5⟩1. LET:
$n \in \mathbb{N}$

⟨5⟩2. PICK a neighbourhood U of x_0 that intersects only finitely many elements of $\mathcal{B}_n$, say $B_1, \ldots, B_k$

PROOF: By ⟨2⟩4 and ⟨4⟩1.

⟨5⟩3. For $j = 1, \ldots, k$, PICK a neighbourhood W_j of x_0 such that f_{nB_j} varies by at most $\epsilon/2$ on W_j

PROOF: By ⟨2⟩5.

⟨5⟩4. LET: $V_n = U \cap W_1 \cap \cdots \cap W_k$

⟨5⟩5. Q.E.D.

⟨6⟩1. LET: $B \in \mathcal{B}_n$

PROVE: f_{nB} varies by at most $\epsilon/2$ on V_n

⟨6⟩2. CASE: B is one of $B_1, \ldots, B_j$

PROOF: From ⟨5⟩3 and ⟨5⟩4

⟨6⟩3. CASE: B is not one of $B_1, \ldots, B_j$
⟨7⟩1. f_{nB} is zero on U
PROOF: ⟨2⟩5, ⟨5⟩2
⟨7⟩2. f_{nB} is zero on V_n
PROOF: ⟨5⟩4, ⟨7⟩1
⟨4⟩4. PICK N such that $1/N \leq \epsilon/2$
PROOF: Using ⟨4⟩2
⟨4⟩5. LET: $W = V_0 \cap V_1 \cap \cdots \cap V_N$
⟨4⟩6. For all $x \in W$, we have $\rho(F(x), F(x_0)) < \epsilon$
⟨5⟩1. LET: $x \in W$
⟨5⟩2. For $n \leq N$ and $B \in \mathcal{B}_n$ we have $|f_{nB}(x) - f_{nB}(x_0)| \leq \epsilon/2$
PROOF: ⟨4⟩3, ⟨4⟩5
⟨5⟩3. For $n > N$ and $B \in \mathcal{B}_n$ we have $|f_{nB}(x) - f_{nB}(x_0)| \leq \epsilon/2$
PROOF: ⟨2⟩5, ⟨4⟩4
⟨5⟩4. $\rho(F(x), F(x_0)) \leq \epsilon/2$
PROOF: ⟨2⟩8, ⟨5⟩2, ⟨5⟩3
⟨3⟩4. Q.E.D.
⟨1⟩2. Every metrizable space is regular.
PROOF: Theorem 10.2.43.
⟨1⟩3. Every metrizable space has a countably locally discrete basis.
⟨2⟩1. LET: X be a metric space.
⟨2⟩2. For $n \in \mathbb{Z}^+$,
LET: $\mathcal{A}_n$ be the set of all open balls of radius $1/n$.
⟨2⟩3. For $n \in \mathbb{Z}^+$, PICK a locally finite open refinement $\mathcal{B}_n$ of $\mathcal{A}_n$ that covers X.
PROOF: Lemma **??**.
⟨2⟩4. LET: $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$
⟨2⟩5. $\mathcal{B}$ is countably locally finite.
PROOF: ⟨2⟩3, ⟨2⟩4
⟨2⟩6. $\mathcal{B}$ is a basis for X.
⟨3⟩1. Every element of $\mathcal{B}$ is open.
PROOF: ⟨2⟩3, ⟨2⟩4
⟨3⟩2. For every open set U and $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$
⟨4⟩1. LET: U be an open set and $x \in U$.
⟨4⟩2. PICK n such that $B(x, 1/n) \subseteq U$
PROOF: ⟨4⟩1
⟨4⟩3. PICK $B \in \mathcal{B}_n$ such that $x \in B \subseteq B(x, 1/n)$
⟨5⟩1. $B(x, 1/n) \in \mathcal{A}_n$
PROOF: ⟨2⟩2, ⟨4⟩1
⟨5⟩2. Q.E.D.
PROOF: ⟨2⟩3, ⟨5⟩1
⟨4⟩4. $B \in \mathcal{B}$
PROOF: ⟨2⟩4, ⟨4⟩3
⟨3⟩3. Q.E.D.
PROOF: Proposition 3.5.2

$\square$

Theorem 10.2.58 (AC)**.** *Let X be a paracompact Hausdorff space. Let $\{U_\alpha\}_{\alpha \in J}$ be an open covering of X. Then there exists a partition of unity on X dominated by $\{U_\alpha\}_{\alpha \in J}$.*

PROOF:
⟨1⟩1. PICK a locally finite open cover $\{V_\alpha\}_{\alpha \in J}$ of X such that $\overline{V_\alpha} \subseteq U_\alpha$ for all α.
PROOF: By the Shrinking Lemma.
⟨1⟩2. PICK a locally finite open cover $\{W_\alpha\}_{\alpha \in J}$ of X such that $\overline{W_\alpha} \subseteq V_\alpha$ for all α.
PROOF: By the Shrinking Lemma and ⟨1⟩1.
⟨1⟩3. For $\alpha \in J$, PICK a continuous $\psi_\alpha : X \to [0, 1]$ such that $\psi_\alpha(\overline{W_\alpha}) = \{1\}$ and $\psi_\alpha(X \setminus V_\alpha) = \{0\}$.
⟨2⟩1. LET: $\alpha \in J$
⟨2⟩2. X is normal.
PROOF: Theorem 9.4.2.
⟨2⟩3. $\overline{W_\alpha}$ and $X \setminus V_\alpha$ are disjoint.
PROOF: From ⟨1⟩2.
⟨2⟩4. $\overline{W_\alpha}$ is closed.
PROOF: Proposition 3.12.3.
⟨2⟩5. $X \setminus V_\alpha$ is closed.
PROOF: Proposition 3.6.6, ⟨1⟩1.
⟨2⟩6. Q.E.D.
PROOF: By the Urysohn Lemma.
⟨1⟩4. For all $\alpha \in J$ we have $\operatorname{supp} \psi_\alpha \subseteq \overline{V_\alpha}$
⟨2⟩1. LET: $\alpha \in J$
⟨2⟩2. $\phi^{-1}(\mathbb{R} \setminus \{0\}) \subseteq V_\alpha$
PROOF: ⟨1⟩3, ⟨2⟩1
⟨2⟩3. Q.E.D.
PROOF: Proposition 3.12.5.
⟨1⟩5. $\{\overline{V_\alpha}\}_{\alpha \in J}$ is locally finite.
PROOF: Lemma 3.12.9, ⟨1⟩1.
⟨1⟩6. $\{\operatorname{supp} \psi_\alpha\}_{\alpha \in J}$ is locally finite.
PROOF: Proposition 3.8.2, ⟨1⟩4, ⟨1⟩5.
⟨1⟩7. For $x \in X$, there exists $\alpha \in J$ such thot $\psi_\alpha(x) > 0$.
PROOF: ⟨1⟩1, ⟨1⟩3.
⟨1⟩8. LET: $\Psi : X \to \mathbb{R}$ with $\Psi(x) = \sum_{\alpha \in J} \psi_\alpha(x)$
⟨2⟩1. For all $x \in X$ there are only finitely many α such that $\psi_\alpha(x) \neq 0$.
⟨3⟩1. LET: $x \in X$
⟨3⟩2. PICK a neighbourhood U of x that intersects only finitely many V_α, say $V_{\alpha_1}, \ldots, V_{\alpha_n}$
PROOF: ⟨1⟩1, ⟨3⟩1
⟨3⟩3. If $\psi_\alpha(x) \neq 0$ then α is one of $\alpha_1, \ldots, \alpha_n$.
⟨4⟩1. ASSUME: $\psi_\alpha(x) \neq 0$
⟨4⟩2. $x \in V_\alpha$

PROOF: ⟨1⟩3, ⟨4⟩1

⟨4⟩3. U intersects V_α

PROOF: ⟨3⟩2, ⟨4⟩2

⟨4⟩4. Q.E.D.

PROOF: By ⟨3⟩2

⟨1⟩9. Ψ is continuous.

⟨2⟩1. For $x \in X$, PICK an open neighbourhood W_x of x that intersects $\operatorname{supp}\psi_\alpha$ for only finitely many α.

PROOF: ⟨1⟩6

⟨2⟩2. For all $x \in X$ we have $\Psi \restriction W_x$ is continuous.

⟨3⟩1. LET: $x \in X$

⟨3⟩2. $\alpha_1, \ldots, \alpha_n$ be the values of α such that W_x intersects $\operatorname{supp}\psi_\alpha$

PROOF: ⟨2⟩1

⟨3⟩3. For $y \in W_x$ we have $\Psi(y) = \sum_{i=1}^n \psi_{\alpha_i}(y)$

⟨4⟩1. LET: $y \in W_x$

⟨4⟩2. For $\alpha \neq \alpha_1, \ldots, \alpha_n$ we have $\psi_\alpha(y) = 0$

⟨5⟩1. LET: $\alpha \in J \setminus \{\alpha_1, \ldots, \alpha_n\}$

⟨5⟩2. $y \notin \operatorname{supp}\psi_\alpha$

PROOF: ⟨3⟩2, ⟨4⟩1, ⟨5⟩1

⟨5⟩3. $\psi_\alpha(y) = 0$

PROOF: Proposition 3.12.2, ⟨5⟩2

⟨3⟩4. Q.E.D.

PROOF: Theorem 5.2.9, Lemma 10.2.21, ⟨1⟩3.

⟨2⟩3. Q.E.D.

PROOF: Theorem 5.2.13.

⟨1⟩10. $\Psi(x) > 0$ for all $x \in X$.

⟨2⟩1. LET: $x \in X$

⟨2⟩2. PICK $\alpha \in J$ such that $x \in W_\alpha$

PROOF: ⟨1⟩2, ⟨2⟩1

⟨2⟩3. $\psi_\alpha(x) = 1$

PROOF: ⟨1⟩3, ⟨2⟩2

⟨2⟩4. Q.E.D.

PROOF: ⟨1⟩3, ⟨1⟩8, ⟨2⟩3

⟨1⟩11. For $\alpha \in J$,

LET: $\phi_\alpha(x) = \psi_\alpha(x)/\Psi(x)$

PROOF: $\Psi(x) \neq 0$ by ⟨1⟩10

⟨1⟩12. $\{\phi_\alpha\}_{\alpha \in J}$ is a partition of unity dominated by $\{U_\alpha\}_{\alpha \in J}$.

⟨2⟩1. For all $\alpha \in J$ we have $\operatorname{supp}\phi_\alpha = \operatorname{supp}\psi_\alpha$

⟨3⟩1. LET: $\alpha \in J$

⟨3⟩2. For all $x \in X$ we have $\phi_\alpha(x) = 0$ iff $\psi_\alpha(x) = 0$

PROOF: From ⟨1⟩11

⟨2⟩2. For all $\alpha \in J$ we have $\operatorname{supp}\phi_\alpha \subseteq U_\alpha$.

⟨3⟩1. LET: $\alpha \in J$

⟨3⟩2. $\operatorname{supp}\phi_\alpha \subseteq U_\alpha$

PROOF:

$$\begin{aligned} \operatorname{supp} \phi_\alpha &= \operatorname{supp} \psi_\alpha && (\langle 2\rangle 1) \\ &\subseteq \overline{V_\alpha} && (\langle 1\rangle 4,\ \langle 3\rangle 1) \\ &\subseteq U_\alpha && (\langle 1\rangle 1,\ \langle 3\rangle 1) \end{aligned}$$

⟨2⟩3. $\{\text{supp } \phi_\alpha\}_{\alpha \in J}$ is locally finite.
PROOF: ⟨1⟩6, ⟨2⟩1
⟨2⟩4. For all $x \in X$ we have $\sum_{\alpha \in J} \phi_\alpha(x) = 1$
PROOF: ⟨1⟩8, ⟨1⟩11
□

Theorem 10.2.59 (Smirnov Metrization Theorem (AC))**.** *A space is metrizable if and only if it is locally metrizable, paracompact and Hausdorff.*

PROOF:
⟨1⟩1. Every metrizable space is locally metrizable.
PROOF: If x is a point in the metrizable space X, then X is a metrizable neighbourhood.
⟨1⟩2. Every metrizable space is paracompact.
PROOF: Theorem 10.2.56.
⟨1⟩3. Every metrizable space is Hausdorff.
PROOF: Lemma 10.2.9.
⟨1⟩4. Every locally metrizable, paracompact Hausdorff space is metrizable.
⟨2⟩1. LET: X be a locally metrizable, paracompact Hausdorff space.
⟨2⟩2. X is regular.
PROOF: Theorem 9.4.2.
⟨2⟩3. X has a countably locally finite basis.
⟨3⟩1. PICK a locally finite open cover $\mathcal{C}$ of X by metrizable sets.
⟨4⟩1. $\{U \text{ open in } X : U \text{ is metrizable}\}$ covers X.
PROOF: Because X is locally metrizable (⟨2⟩1).
⟨4⟩2. Q.E.D.
PROOF: Because X is paracompact (⟨2⟩1).
⟨3⟩2. For $C \in \mathcal{C}$, PICK a metric $d_C : C^2 \to \mathbb{R}$ that induces the topology on C.
⟨3⟩3. For $C \in \mathcal{C}$ and $x \in C$ and $\epsilon > 0$,
LET: $B_C(x, \epsilon) = \{y \in C : d_C(x, y) < \epsilon\}$
⟨3⟩4. For $n \geq 1$,
LET: $\mathcal{A}_n = \{B_C(x, 1/n) : C \in \mathcal{C}, x \in C\}$
⟨3⟩5. For $n \geq 1$, PICK a locally finite open refinement $\mathcal{D}_n$ of $\mathcal{A}_n$ that covers X.
PROOF: Because X is paracompact (⟨2⟩1).
⟨3⟩6. LET: $\mathcal{D} = \bigcup_{n=1}^\infty \mathcal{D}_n$.
PROVE: $\mathcal{D}$ is a basis for X.
⟨3⟩7. LET: U be open in X and $x \in U$.
⟨3⟩8. LET: $C_1, \ldots, C_k$ be the elements of $\mathcal{C}$ that U intersects.
PROOF: Because $\mathcal{C}$ is locally finite (⟨3⟩1).
⟨3⟩9. For $1 \leq i \leq k$, PICK $\epsilon_i > 0$ such that $B_{C_i}(x, \epsilon_i) \subseteq U \cap C_i$

$\langle 3\rangle 10$. Pick $m \geq 1$ such that $2/m < \epsilon_1, \ldots, \epsilon_k$

$\langle 3\rangle 11$. Pick $D \in \mathcal{D}_m$ such that $x \in D$

Proof: Since $\mathcal{D}_m$ covers X ($\langle 3\rangle 5$).

$\langle 3\rangle 12$. $D \subseteq U$

$\langle 4\rangle 1$. Pick $C \in \mathcal{C}$ and $y \in C$ such that $D \subseteq B_C(y, 1/m)$

Proof: $\langle 3\rangle 5$

$\langle 4\rangle 2$. Pick i such that $C = C_i$

Proof: $\langle 3\rangle 8$ since $x \in U \cap C$.

$\langle 4\rangle 3$. $B_C(y, 1/m) \subseteq B_C(x, 2/m)$

$\langle 5\rangle 1$. Let: $z \in B_C(y, 1/m)$

$\langle 5\rangle 2$. $d_C(x, z) < 2/m$

Proof:

$$\begin{aligned} d_C(x,z) &\leq d_C(x,y) + d_C(y,z) && \text{(Triangle inequality)} \\ &< 1/m + 1/m && (\langle 3\rangle 11, \langle 4\rangle 1, \langle 5\rangle 1) \\ &= 2/m \end{aligned}$$

$\langle 4\rangle 4$. $D \subseteq U$

Proof:

$$\begin{aligned} D &\subseteq B_{C_i}(y, 1/m) && (\langle 4\rangle 1) \\ &\subseteq B_{C_i}(x, 2/m) && (\langle 4\rangle 3) \\ &\subseteq B_{C_i}(x, \epsilon_i) && (\langle 3\rangle 10) \\ &\subseteq U && (\langle 3\rangle 9) \end{aligned}$$

$\langle 2\rangle 4$. Q.E.D.

Proof: By the Bing-Nagata-Smirnov Metrization Theorem.

□

Theorem 10.2.60. *Let X be a topological space and Y a complete metric space. Then the set $\mathcal{C}(X, Y)$ of all continuous functions from X to Y is closed in Y^X under the uniform topology.*

Proof:

$\langle 1\rangle 1$. Let: $f : X \to Y$ be a limit point of $\mathcal{C}(X, Y)$ in the uniform topology.

$\langle 1\rangle 2$. Pick a sequence (f_n) in Y^X that converges to f under the uniform topology.

Proof: By the Sequence Lemma.

$\langle 1\rangle 3$. f_n converges to f uniformly.

Proof: Lemma 10.2.27.

$\langle 1\rangle 4$. f is continuous.

Proof: By the Uniform Limit Theorem.

$\langle 1\rangle 5$. Q.E.D.

Proof: Corollary 3.15.3.1.

□

Theorem 10.2.61. *Let X be a topological space and Y a complete metric space. Then the set $\mathcal{B}(X, Y)$ of all bounded functions from X to Y is closed in Y^X under the uniform topology.*

Proof:

⟨1⟩1. LET: f be a limit point of $\mathcal{B}(X, Y)$
⟨1⟩2. PICK a sequence (f_n) of bounded functions that converges to f in the uniform topology.
⟨1⟩3. PICK N such that, for all $n \geq N$, we have $\overline{\rho}(f_n, f) < 1/2$
⟨1⟩4. For all $x \in X$ and $n \geq N$ we have $d(f_n(x), f(x)) < 1/2$
⟨1⟩5. LET: $M = \operatorname{diam} f_N(X)$
⟨1⟩6. $\operatorname{diam} f(X) \leq M + 1$
PROOF:For $x, y \in X$ we have

$$\begin{aligned} d(f(x), f(y)) &\leq d(f(x), f_N(x)) + d(f_N(x), f_N(y)) + d(f_N(y), f(y)) \\ &< 1/2 + M + 1/2 && (\langle 1\rangle 4,\ \langle 1\rangle 5) \\ &= M + 1 \end{aligned}$$

□

10.3 Isometries

Definition 10.3.1 (Isometry)**.** Let X be a metric space. An *isometry* of X is a function $f : X \to X$ such that, for all $x, y \in X$,

$$d(f(x), f(y)) = d(x, y) \ .$$

10.4 Lebesgue Numbers

Definition 10.4.1 (Lebesgue Number)**.** Let X be a metric space and $\mathcal{A}$ an open covering of X. A *Lebesgue number* for $\mathcal{A}$ is a real $\delta > 0$ such that, for every nonempty set $A \subseteq X$ of diameter $< \delta$, there exists $U \in \mathcal{A}$ such that $A \subseteq U$.

Lemma 10.4.2 (Lebesgue Number Lemma)**.** *In a compact metric space, every open covering has a Lebesgue number.*

PROOF:
⟨1⟩1. LET: X be a compact metric space and $\mathcal{A}$ an open covering of X
PROVE: There exists a Lebesgue number δ for $\mathcal{A}$.
⟨1⟩2. ASSUME: w.l.o.g. $X \notin \mathcal{A}$
PROOF: If $X \in \mathcal{A}$ then we can take $\delta = 1$.
⟨1⟩3. PICK a finite subcovering $\{U_1, \ldots, U_n\} \subseteq \mathcal{A}$ that covers X
⟨1⟩4. For $1 \leq i \leq n$,
LET: $C_i = X \setminus U_i$
⟨1⟩5. LET: $f : X \to \mathbb{R}$ be defined by

$$f(x) = 1/n \sum_{i=1}^{n} d(x, C_i) \ .$$

PROOF: Each C_i is nonempty by ⟨1⟩2.
⟨1⟩6. For all $x \in X$ we have $f(x) > 0$
⟨2⟩1. LET: $x \in X$

$\langle 2\rangle 2$. PICK i such that $x \in U_i$

PROOF: By $\langle 1\rangle 3$.

$\langle 2\rangle 3$. PICK $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U_i$

PROOF: By Lemma 10.2.2.

$\langle 2\rangle 4$. $d(x, C_i) \geq \epsilon$

$\langle 1\rangle 7$. f is continuous

PROOF: From Lemma 10.2.35.

$\langle 1\rangle 8$. LET: $\delta = \min f(X)$

PROVE: For every nonempty set $A \subseteq X$ with diameter $< \delta$, there exists $U \in \mathcal{A}$ such that $A \subseteq U$

PROOF: $f(X)$ has a minimum by the Extreme Value Theorem.

$\langle 1\rangle 9$. LET: $A \subseteq X$ be nonempty with $\operatorname{diam} A < \delta$

$\langle 1\rangle 10$. PICK $x_0 \in A$

$\langle 1\rangle 11$. LET: i be such that $d(x_0, C_i)$ is greatest among $d(x_0, C_1)$, $\ldots$, $d(x_0, C_n)$

$\langle 1\rangle 12$. $\delta \leq d(x_0, C_i)$

PROOF:

$$\begin{aligned}
\delta &\leq f(x_0) && (\langle 1\rangle 8) \\
&= 1/n \sum_{j=1}^{n} d(x_0, C_j) && (\langle 1\rangle 5) \\
&\leq 1/n \sum_{j=1}^{n} d(x_0, C_i) && (\langle 1\rangle 11) \\
&= d(x_0, C_i)
\end{aligned}$$

$\langle 1\rangle 13$. $x_0 \in U_i$

PROOF: $x_0 \notin C_i$ because $d(x_0, C_i) > 0$.

□

Theorem 10.4.3 (DC). *Let X be a metrizable space. Then the following are equivalent:*

1. *X is compact.*
2. *X is limit point compact.*
3. *X is sequentially compact.*

PROOF:

$\langle 1\rangle 1$. $1 \Rightarrow 2$

PROOF: Theorem 9.5.22.

$\langle 1\rangle 2$. $2 \Rightarrow 3$

$\langle 2\rangle 1$. ASSUME: X is limit point compact.

$\langle 2\rangle 2$. LET: (x_n) be a sequence in X

PROVE: (x_n) has a convergent subsequence.

$\langle 2\rangle 3$. CASE: $\{x_n : n \in \mathbb{Z}^+\}$ is finite.

PROOF: In this case, (x_n) has a constant subsequence.

$\langle 2\rangle 4$. CASE: $\{x_n : n \in \mathbb{Z}^+\}$ is infinite.

$\langle 3\rangle 1$. Pick a limit point l of $\{x_n : n \in \mathbb{Z}^+\}$

$\langle 3\rangle 2$. For every poisitive integer r, Pick n_r such that $n_r > n_{r-1}$ and $d(x_{n_r}, l) < 1/r$

Proof: There always exists such an n_r since $B(l, 1/r)$ intersects $\{x_n : n \in \mathbb{Z}^+\}$ in infinitely many points by Theorem 6.1.2.

$\langle 3\rangle 3$. $x_{n_r} \to l$ as $r \to \infty$

$\langle 1\rangle 3$. $3 \Rightarrow 1$

$\langle 2\rangle 1$. Assume: X is sequentially compact.

$\langle 2\rangle 2$. Every open covering of X has a Lebesgue number.

$\langle 3\rangle 1$. Let: $\mathcal{A}$ be an open covering of X.

$\langle 3\rangle 2$. Assume: for a contradiction that, for all $\delta > 0$, there exists a set $C \subseteq X$ with $\operatorname{diam} C < \delta$ such that there is no $U \in \mathcal{A}$ such that $C \subseteq U$

$\langle 3\rangle 3$. For $n \geq 1$, Pick $C_n \subseteq X$ with $\operatorname{diam} C_n < 1/n$ such that there is no $U \in \mathcal{A}$ such that $C_n \subseteq U$

$\langle 3\rangle 4$. For $n \geq 1$, Pick $x_n \in C_n$

$\langle 3\rangle 5$. Pick a convergent subsequence (x_{n_r}) of (x_n)

Proof: By $\langle 2\rangle 1$.

$\langle 3\rangle 6$. Let: $x_{n_r} \to l$ as $r \to \infty$

$\langle 3\rangle 7$. Pick $U \in \mathcal{A}$ with $l \in U$

Proof: By $\langle 3\rangle 1$

$\langle 3\rangle 8$. Pick $\epsilon > 0$ such that $B(l, \epsilon) \subseteq U$

Proof: By Lemma 10.2.2.

$\langle 3\rangle 9$. Pick R such that $1/n_R < \epsilon/2$ and $d(x_{n_R}, l) < \epsilon/2$

Proof: By $\langle 3\rangle 6$

$\langle 3\rangle 10$. $C_{n_R} \subseteq U$

Proof:

$$\begin{aligned} C_{n_R} &\subseteq B(x_{n_R}, 1/n_R) && (\langle 3\rangle 3,\ \langle 3\rangle 4) \\ &\subseteq B(x_{n_R}, \epsilon/2) && (\langle 3\rangle 9) \\ &\subseteq B(l, \epsilon) && (\langle 3\rangle 9) \\ &\subseteq U && (\langle 3\rangle 8) \end{aligned}$$

$\langle 3\rangle 11$. Q.E.D.

Proof: This contradicts $\langle 3\rangle 3$.

$\langle 2\rangle 3$. For all $\epsilon > 0$, there exists a finite covering of X by ϵ-balls.

$\langle 3\rangle 1$. Let: $\epsilon > 0$

$\langle 3\rangle 2$. Assume: for a contradiction there is no finite covering of X by ϵ-balls.

$\langle 3\rangle 3$. Pick a sequence (x_n) in X such that, for all n,

$$x_n \notin B(x_1, \epsilon) \cup \cdots \cup B(x_{n-1}, \epsilon) \ .$$

$\langle 3\rangle 4$. For all m, n with $m > n$ we have $d(x_m, x_n) \geq \epsilon$

$\langle 3\rangle 5$. Any $\epsilon/2$-ball contains at most one element of (x_n).

$\langle 3\rangle 6$. (x_n) has no convergent subsequence.

$\langle 3\rangle 7$. Q.E.D.

Proof: This contradicts $\langle 2\rangle 1$.

$\langle 2\rangle 4$. Let: $\mathcal{A}$ be an open covering of X

$\langle 2\rangle 5$. Pick a Lebesgue number δ for $\mathcal{A}$

PROOF: By $\langle 2\rangle 2$.
$\langle 2\rangle 6$. PICK a finite covering $\{B_1, \ldots, B_n\}$ of X by $\delta/3$-balls.
PROOF: By $\langle 2\rangle 3$.
$\langle 2\rangle 7$. For $1 \leq i \leq n$, PICK $U_i \in \mathcal{A}$ such that $B_i \subseteq U_i$
$\langle 2\rangle 8$. $\{U_1, \ldots, U_n\}$ covers X.
$\square$

Corollary 10.4.3.1. *S_Ω is not metrizable.*

PROOF: It is limit point compact (Corollary 9.5.19.2) but not compact (Proposition 9.5.2). $\square$

Corollary 10.4.3.2. *The space $\mathbb{R}^\omega$ is not limit point compact.*

10.5 Uniform Continuity

Definition 10.5.1 (Uniform Continuity)**.** Let X and Y be metric spaces and $f : X \to Y$. Then f is *uniformly continuous* iff, for all $\epsilon > 0$, there exists $\delta > 0$ such that, for all $x, y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$.

Theorem 10.5.2 (Uniform Continuity Theorem)**.** *Let X be a compact metric space, Y a metric space, and $f : X \to Y$ be continuous. Then f is uniformly continuous.*

PROOF:
$\langle 1\rangle 1$. LET: $\epsilon > 0$
PROVE: There exists $\delta > 0$ such that, for all $x, y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$.
$\langle 1\rangle 2$. LET: $\mathcal{A} = \{f^{-1}(B(y, \epsilon/2)) : y \in Y\}$
$\langle 1\rangle 3$. $\mathcal{A}$ is an open covering of X
$\langle 1\rangle 4$. PICK a Lebesgue number δ for $\mathcal{A}$.
PROVE: For all $x, y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$
PROOF: By the Lebesgue Number Lemma
$\langle 1\rangle 5$. LET: $x, y \in X$ with $d(x, y) < \delta$
$\langle 1\rangle 6$. $\operatorname{diam}\{x, y\} < \delta$
$\langle 1\rangle 7$. PICK $z \in Y$ such that $\{x, y\} \subseteq f^{-1}(B(z, \epsilon/2))$
$\langle 1\rangle 8$. $d(f(x), f(y)) < \epsilon$
$\square$

Definition 10.5.3 (Metrically Equivalent)**.** Let d and d' be two metrics on the same set X. Then d and d' are *metrically equivalent* iff the identity map $i : (X, d) \to (X, d')$ and its inverse are both uniformly continuous.

10.6 Locally Metrizable Spaces

Definition 10.6.1 (Locally Metrizable)**.** A space is *locally metrizable* iff every point has a metrizable neighbourhood.

Proposition 10.6.2. *Every metrizable space is locally metrizable.*

PROOF: Trivial. □

Corollary 10.6.2.1. *The space $\mathbb{R}^\omega$ is locally metrizable.*

Proposition 10.6.3. *A compact Hausdorff space is metrizable if and only if it is locally metrizable.*

PROOF:
⟨1⟩1. LET: X be a locally metrizable compact Hausdorff space
⟨1⟩2. X is regular
PROOF: Lemma 9.5.18
⟨1⟩3. X is second countable
⟨2⟩1. $\{U : U \text{ open in } X \text{ and metrizable}\}$ covers X
⟨2⟩2. PICK a finite subcover $U_1, \ldots, U_n$
⟨2⟩3. For $1 \leq i \leq n$, PICK a countable basis $\mathcal{B}_i$ of U_i
⟨2⟩4. $\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_n$ is a basis for X
⟨1⟩4. Q.E.D.
PROOF: By the Urysohn Metrization Theorem.
□

Corollary 10.6.3.1. $\overline{S_\Omega}$ *is not locally metrizable.*

Corollary 10.6.3.2. *The ordered square is not locally metrizable.*

Proposition 10.6.4. *Every subspace of a locally metrizable space is locally metrizable.*

PROOF:
⟨1⟩1. LET: X be locally metrizable and $Y \subseteq X$
⟨1⟩2. LET: $y \in Y$
⟨1⟩3. PICK a metrizable neighbourhood U of y in X
⟨1⟩4. $U \cap Y$ is a metrizable neighbourhood of y in Y
□

Corollary 10.6.4.1. $S_\Omega \times \overline{S_\Omega}$ *is not locally metrizable.*

PROOF: It has a subspace homeomorphic to $\overline{S_\Omega}$. □

Proposition 10.6.5 (CC)**.** *Every locally metrizable regular Lindelöf space is metrizable.*

PROOF:
⟨1⟩1. LET: X be a locally metrizable regular Lindelöf space.
⟨1⟩2. Every point in X has an open second countable neighbourhood.
⟨2⟩1. LET: $x \in X$
⟨2⟩2. PICK an open metrizable U containing x
PROOF: X is locally metrizable (⟨1⟩1)
⟨2⟩3. PICK an open V such that $x \in V \subseteq \overline{V} \subseteq U$

PROOF: Proposition 6.3.2

⟨2⟩4. $\overline{V}$ is Lindelöf

PROOF: Proposition 13.1.32

⟨2⟩5. $\overline{V}$ is second countable

PROOF: Proposition 10.2.39

⟨1⟩3. PICK a countable covering of secound countable open sets $\mathcal{U}$

PROOF: X is Lindelöf (⟨1⟩1)

⟨1⟩4. For $U \in \mathcal{U}$, PICK a countable basis $\mathcal{B}_U$

⟨1⟩5. $\bigcup_{U \in \mathcal{U}} \mathcal{B}_U$ is a countable basis for X

⟨2⟩1. LET: $x \in U$ where U is open in X

⟨2⟩2. PICK $V \in \mathcal{U}$ such that $x \in V$

⟨2⟩3. There exists $B \in \mathcal{B}_V$ such that $x \in B \subseteq U \cap V$

⟨1⟩6. Q.E.D.

PROOF: By the Urysohn Metrization Theorem.

□

Corollary 10.6.5.1. *$\mathbb{R}_l$ is not locally metrizable.*

Proposition 10.6.6. *The Sorgenfrey plane is not locally metrizable.*

PROOF:

⟨1⟩1. LET: U be any neighbourhood of $(0,0)$

PROVE: U is not Lindelöf

⟨1⟩2. PICK $a > 0$ such that $[0,a)^2 \subseteq U$

⟨1⟩3. LET: $L = \{(x, a-x) : 0 < x < a\}$

⟨1⟩4. L is closed in U

PROOF: By Lemma 6.5.16 since $(x,y) \mapsto (x, a+y)$ is a homeomorphism of $\mathbb{R}_l^2$ with itself.

⟨1⟩5. LET: $\mathcal{U} = \{U \setminus L\} \cup \{([x,b) \times [a-x,c)) \cap U : b > a, c > a-x\}$

⟨1⟩6. $\mathcal{U}$ covers U

⟨1⟩7. No countable subset of $\mathcal{U}$ covers U

PROOF: Every set of the for $[x,b) \times [a-x,c)$ intersects L in exactly one point.

□

Corollary 10.6.6.1. *The Sorgenfrey plane is not metrizable.*

Proposition 10.6.7. *The space $\mathbb{R}_K$ is locally metrizable.*

PROOF: The set $(-1,1) - K$ is a metrizable neighbourhood of 0. For any other point p, pick an open interval around p that does not contain 0. □

Proposition 10.6.8. *The product of two locally metrizable spaces is locally metrizable.*

PROOF:

⟨1⟩1. LET: X and Y be locally metrizable

⟨1⟩2. LET: $(a,b) \in X \times Y$

⟨1⟩3. PICK metrizable neighbourhoods U of a and V of b

⟨1⟩4. $U \times V$ is a metrizable neighbourhood of (a,b).

PROOF: By Lemma 10.2.15.
∎

Proposition 10.6.9. *The product of two locally metrizable spaces is locally metrizable.*

PROOF:
⟨1⟩1. LET: X and Y be locally metrizable
⟨1⟩2. LET: $(a, b) \in X \times Y$
⟨1⟩3. PICK metrizable neighbourhoods U of a and V of b
⟨1⟩4. $U \times V$ is a metrizable neighbourhood of (a, b).
PROOF: By Lemma 10.2.15.
∎

Proposition 10.6.10. *The space $\mathbb{R}_K^\omega$ is not locally metrizable.*

PROOF: If it were, then there would be a basic open set $\prod_n U_n$ that is metrizable, but then $\mathbb{R}_K$ would be metrizable as it is homeomorphic to a subspace of $\prod_n U_n$.
∎

Corollary 10.6.10.1. *The product of a countable family of locally metrizable spaces is not necessarily locally metrizable.*

Proposition 10.6.11. *The continuous image of a locally metrizable space is not necessarily locally metrizable.*

PROOF: The identity map from the discrete two-point space to the indiscrete two-point space is continuous. ∎

10.7 Completeness

Definition 10.7.1 (Cauchy Sequence)**.** Let X be a metric space. A sequence (x_n) of points in X is a *Cauchy sequence* iff, for every $\epsilon > 0$, there exists N such that, for all $m, n \geq N$,

$$d(x_m, x_n) < \epsilon \ .$$

Lemma 10.7.2. *Every convergent sequence is Cauchy.*

PROOF:
⟨1⟩1. LET: $x_n \to l$ as $n \to \infty$
⟨1⟩2. LET: $\epsilon > 0$
⟨1⟩3. PICK N such that, for all $n \geq N$, we have $d(x_n, l) < \epsilon/2$
⟨1⟩4. For all $m, n \geq N$, we have $d(x_m, x_n) < \epsilon$
∎

Definition 10.7.3 (Complete)**.** A metric space is *complete* iff every Cauchy sequence converges.

Definition 10.7.4 (Topologically Complete)**.** A topological space X is *topologically complete* iff there exists a metric that induces the topology on X under which X is complete.

Lemma 10.7.5. *A metric space is complete if and only if every Cauchy sequence has a convergent subsequence.*

PROOF:

$\langle 1\rangle 1$. In a complete metric space, every Cauchy sequence has a convergent subsequence.

PROOF: Trivial.

$\langle 1\rangle 2$. In a metric space, if every Cauchy sequence has a convergent subsequence, then the space is complete.

$\langle 2\rangle 1$. LET: X be a metric space in which every Cauchy sequence has a convergent subsequence.

$\langle 2\rangle 2$. LET: (x_n) be a Cauchy sequence in X.

$\langle 2\rangle 3$. PICK a convergent subsequence (x_{n_r}) with limit l.

$\langle 2\rangle 4$. $x_n \to l$ as $n \to \infty$

$\langle 3\rangle 1$. LET: $\epsilon > 0$

$\langle 3\rangle 2$. PICK N such that, for all $m, n \geq N$ we have $d(x_m, x_n) < \epsilon/2$ and for all $r \geq N$ we have $d(x_{n_r}, l) < \epsilon/2$

PROOF: $\langle 2\rangle 3$, $\langle 2\rangle 4$

$\langle 3\rangle 3$. For $n \geq N$ we have $d(x_n, l) < \epsilon$.

PROOF:

$$\begin{aligned} d(x_n, l) &\leq d(x_n, x_{n_n}) + d(x_{n_n}, l) && \text{(Triangle Inequality)} \\ &< \epsilon/2 + \epsilon/2 && (\langle 3\rangle 2) \\ &= \epsilon \end{aligned}$$

□

Theorem 10.7.6 (DC)**.** *For any k we have $\mathbb{R}^k$ is complete.*

PROOF:

$\langle 1\rangle 1$. LET: (x_n) be a Cauchy sequence in $\mathbb{R}^k$

$\langle 1\rangle 2$. $\{x_n : n \geq 1\}$ is bounded.

$\langle 2\rangle 1$. PICK N such that, for all $m, n \geq N$, we have $\rho(x_m, x_n) < 1$

PROOF: $\langle 1\rangle 1$

$\langle 2\rangle 2$. LET: $M = \max(\rho(x_1, 0), \ldots, \rho(x_{N-1}, 0), \rho(x_N, 0) + 1)$

$\langle 2\rangle 3$. For all n, we have $x_n \in [-M, M]^k$

$\langle 3\rangle 1$. LET: $n \geq 1$

PROVE: $x_n \in [-M, M]^k$

$\langle 3\rangle 2$. CASE: $n < N$

PROOF: For $1 \leq i \leq k$,

$$\begin{aligned} |\pi_i(x_n)| &\leq \rho(x_n, 0) && \text{(definition of Euclidean metric)} \\ &\leq M && (\langle 2\rangle 2) \end{aligned}$$

$\langle 3\rangle 3$. CASE: $n \geq N$

PROOF: For $1 \leq i \leq k$,

$$\begin{aligned} |\pi_i(x_n)| &\leq \rho(x_n, 0) && \text{(definition of Euclidean metric)} \\ &\leq \rho(x_n, x_N) + \rho(x_N, 0) && \text{(Triangle inequality)} \\ &< 1 + \rho(x_N, 0) && (\langle 2\rangle 1) \\ &\leq M && (\langle 2\rangle 2) \end{aligned}$$

$\langle 1\rangle 3$. PICK M such that $\{x_n : n \geq 1\} \subseteq [-M, M]^k$
PROOF: From $\langle 1\rangle 2$.
$\langle 1\rangle 4$. (x_n) has a convergent subsequence.
$\langle 2\rangle 1$. $[-M, M]^k$ is compact.
PROOF: Theorem 9.5.19, Proposition 9.5.14.
$\langle 2\rangle 2$. Q.E.D.
PROOF: Theorem 10.4.3.
$\langle 1\rangle 5$. Q.E.D.
PROOF: Lemma 10.7.5.
□

Theorem 10.7.7 (DC)**.** *For any k we have $\mathbb{R}^k$ is complete under the square metric.*

PROOF:
$\langle 1\rangle 1$. LET: (x_n) be a Cauchy sequence under the square metric.
$\langle 1\rangle 2$. (x_n) is Cauchy under the Euclidean metric.
$\langle 2\rangle 1$. LET: $\epsilon > 0$
$\langle 2\rangle 2$. PICK N such that, for all $m, n \geq N$, we have $\rho(x_m, x_n) < \epsilon/\sqrt{k}$
$\langle 2\rangle 3$. For $m, n \geq N$, we have $d(x_m, x_n) < \epsilon$
PROOF:
$$\begin{aligned} d(x_m, x_n) &= \sqrt{((x_m)_1 - (x_n)_1)^2 + \cdots + ((x_m)_k - (x_n)_k)^2} \\ &\leq \sqrt{\rho(x_m, x_n)^2 + \cdots + \rho(x_m, x_n)^2} \\ &= \sqrt{k}\rho(x_m, x_n) \\ &< \epsilon && (\langle 2\rangle 2) \end{aligned}$$
$\langle 1\rangle 3$. PICK a subsequence (x_{n_r}) that converges under the Euclidean metric.
PROOF: Theorem 10.7.6, $\langle 1\rangle 2$.
$\langle 1\rangle 4$. (x_{n_r}) converges under the square metric.
$\langle 2\rangle 1$. LET: $l = \lim_{r\to\infty} x_{n_r}$ under the Euclidean metric.
$\langle 2\rangle 2$. LET: $\epsilon > 0$
$\langle 2\rangle 3$. PICK R such that, for all $r \geq R$, we have $d(x_{n_r}, l) < \epsilon$
$\langle 2\rangle 4$. For all $r \geq R$ we have $\rho(x_{n_r}, l) < \epsilon$
PROOF: From $\langle 2\rangle 3$ since $\rho(x, y) \leq d(x, y)$ for all x, y.
□

Theorem 10.7.8. *There exists a metric that induces the product topology on $\mathbb{R}^\omega$ under which $\mathbb{R}^\omega$ is complete.*

PROOF:
$\langle 1\rangle 1$. LET: $\overline{d}$ be the standard bounded metric on $\mathbb{R}$.
$\langle 1\rangle 2$. LET: $D : (\mathbb{R}^\omega)^2 \to \mathbb{R}$ be defined by $D(x, y) = \sup_{i\geq 1} \overline{d}(x_i, y_i)/i$
$\langle 1\rangle 3$. D is a metric that induces the product topology on $\mathbb{R}^\omega$
$\langle 2\rangle 1$. D is a metric on $\mathbb{R}^\omega$
$\langle 3\rangle 1$. $D(\vec{x}, \vec{y}) \geq 0$
PROOF: Immediate from definitions.
$\langle 3\rangle 2$. $D(\vec{x}, \vec{y}) = 0$ iff $\vec{x} = \vec{y}$

PROOF: Immediate from definitions.
⟨3⟩3. $D(\vec{x}, \vec{y}) = D(\vec{y}, \vec{x})$
PROOF: Immediate from definitions.
⟨3⟩4. $D(\vec{x}, \vec{z}) \leq D(\vec{x}, \vec{y}) + D(\vec{y}, \vec{z})$
⟨4⟩1. For all n, we have $\frac{d(x_n,z_n)}{n} \leq \frac{d(x_n,y_n)}{n} + \frac{d(y_n,z_n)}{n}$
⟨4⟩2. For all n, we have $\frac{d(x_n,z_n)}{n} \leq D(\vec{x}, \vec{y}) + D(\vec{y}, \vec{z})$
⟨4⟩3. $D(\vec{x}, \vec{z}) \leq D(\vec{x}, \vec{y}) + D(\vec{y}, \vec{z})$
⟨2⟩2. LET: $\mathcal{T}_D$ be the topology induced by D and $\mathcal{T}_p$ the product topology.
⟨2⟩3. $\mathcal{T}_D \subseteq \mathcal{T}_p$
⟨3⟩1. LET: $U \in \mathcal{T}_D$
PROVE: $U \in \mathcal{T}_p$
⟨3⟩2. LET: $\vec{x} \in U$
⟨3⟩3. PICK $\epsilon > 0$ such that $B_D(\vec{x}, \epsilon) \subseteq U$
⟨3⟩4. PICK N such that $1/N < \epsilon$
⟨3⟩5. LET: $V = B(x_1, \epsilon) \times \cdots \times B(x_N, \epsilon) \times \mathbb{R} \times \mathbb{R} \times \cdots$
⟨3⟩6. $\vec{x} \in V \subseteq B_D(\vec{x}, \epsilon)$
⟨2⟩4. $\mathcal{T}_p \subseteq \mathcal{T}_D$
⟨3⟩1. LET: $U = \prod_{n=1}^{\infty} U_n$ be a basic open set in $\mathcal{T}_p$, where each U_n is open, and $U_n = \mathbb{R}$ for $n > N$.
⟨3⟩2. LET: $\vec{x} \in U$
PROVE: There exists $\epsilon > 0$ such that $B_D(\vec{x}, \epsilon) \subseteq U$.
⟨3⟩3. For $n \leq N$, PICK $\epsilon_n > 0$ such that $B(x_n, \epsilon_n) \subseteq U_n$
⟨3⟩4. LET: $\epsilon = \min(\epsilon_1, \epsilon_2/2, \ldots, \epsilon_n/n)$
⟨3⟩5. LET: $\vec{y} \in B_D(\vec{x}, \epsilon)$
⟨3⟩6. For $n \leq N$, $y_n \in U_n$
⟨4⟩1. $D(\vec{x}, \vec{y}) < \epsilon$
⟨4⟩2. $d(x_n, y_n)/n < \epsilon$
⟨4⟩3. $d(x_n, y_n)/n < \epsilon_n/n$
⟨4⟩4. Q.E.D.
PROOF: By ⟨3⟩3.
⟨1⟩4. $\mathbb{R}^\omega$ is complete under D.
⟨2⟩1. LET: (x_n) be a Cauchy sequence
⟨2⟩2. For all i we have $(\pi_i(x_n))$ is Cauchy.
⟨3⟩1. LET: $\epsilon > 0$
⟨3⟩2. PICK N such that, for all $m, n \geq N$, we have $D(x_m, x_n) < \epsilon/i$
⟨3⟩3. For all $m, n \geq N$ we have $d(\pi_i(x_m), \pi_i(x_n)) < \epsilon$
⟨2⟩3. For all i we have $(\pi_i(x_n))$ converges.
⟨2⟩4. Q.E.D.
pf Corollary 5.2.12.1.
□

Theorem 10.7.9. *Let X be a complete metric space and J a set. Then X^J is complete under the uniform metric.*

PROOF:
⟨1⟩1. LET: (f_n) be a Cauchy sequence in X^J.

$\langle 1 \rangle 2$. LET: $f : J \to X$ be given by: $f(\alpha) = \lim_{n\to\infty} f_n(\alpha)$
PROVE: $f_n \to f$ as $n \to \infty$
$\langle 2 \rangle 1$. For all $\alpha \in J$, we have $(f_n(\alpha))$ is Cauchy in X.
$\langle 3 \rangle 1$. LET: $\alpha \in J$
$\langle 3 \rangle 2$. LET: $\epsilon > 0$
$\langle 3 \rangle 3$. PICK N such that, for all $m, n \geq N$, we have $\overline{\rho}(f_m, f_n) < \epsilon$
$\langle 3 \rangle 4$. For all $m, n \geq N$ we have $d(f_m(\alpha), f_n(\alpha)) < \epsilon$
$\langle 2 \rangle 2$. For all $\alpha \in J$, we have $(f_n(\alpha))$ converges.
PROOF: Since X is complete.
$\langle 1 \rangle 3$. LET: $\epsilon > 0$
$\langle 1 \rangle 4$. PICK N such that, for all $m, n \geq N$, we have $\overline{\rho}(f_m, f_n) < \epsilon/2$
$\langle 1 \rangle 5$. For all $\alpha \in J$ and $m \geq N$ we have $\overline{d}(f_m(\alpha), f(\alpha)) \leq \epsilon/2$
$\langle 2 \rangle 1$. LET: $\alpha \in J$ and $m \geq N$
$\langle 2 \rangle 2$. For all $n \geq N$ we have $d(f_m(\alpha), f_n(\alpha)) < \epsilon/2$
$\langle 2 \rangle 3$. Q.E.D.
PROOF: Taking the limit as $n \to \infty$.
$\langle 1 \rangle 6$. For $n \geq N$ we have $\overline{\rho}(f_n, f) < \epsilon$
□

Proposition 10.7.10. *A closed subspace of a complete metric space is complete.*

PROOF:
$\langle 1 \rangle 1$. LET: X be a complete metric space and $A \subseteq X$ be closed.
$\langle 1 \rangle 2$. LET: (x_n) be a Cauchy sequence in A.
$\langle 1 \rangle 3$. LET: l be the limit of x_n in X
$\langle 1 \rangle 4$. $l \in A$
PROOF: Corollary 3.15.3.1.
□

Theorem 10.7.11. *Let X be a topological space and Y a metric space. Then the space $\mathcal{C}(X, Y)$ of all continuous functions under the uniform metric is complete.*

PROOF: From Theorem 10.2.60 and Proposition 10.7.10. □

Theorem 10.7.12. *Let X be a topological space and Y a metric space. Then the space $\mathcal{B}(X, Y)$ of all bounded functions under the uniform metric is complete.*

PROOF: From Theorem 10.2.61 and Proposition 10.7.10. □

Theorem 10.7.13. *Every metric space can be isometrically imbedded in a complete metric space.*

PROOF:
$\langle 1 \rangle 1$. LET: X be a metric space.
$\langle 1 \rangle 2$. ASSUME: w.l.o.g. X is nonempty
PROOF: Otherwise X is already complete.
$\langle 1 \rangle 3$. PICK $x_0 \in X$
PROOF: $\langle 1 \rangle 2$
$\langle 1 \rangle 4$. $\mathcal{B}(X, \mathbb{R})$ is complete.

PROOF: Theorem 10.7.12.

$\langle 1\rangle 5$. LET: $\Phi : X \to \mathcal{B}(X, \mathbb{R})$ be defined by

$$\Phi(x)(y) = d(x, y) - d(x_0, y)$$

PROOF: For all $x \in X$, $\Phi(x)$ is bounded because $\Phi(x)(y) \leq d(x, x_0)$ for all $y \in X$ by the triangle inequality.

$\langle 1\rangle 6$. Φ is an isometric imbedding.

$\langle 2\rangle 1$. For $x, y \in X$ we have $\sup_{z \in X} |d(x, z) - d(y, z)| = d(x, y)$

$\langle 3\rangle 1$. $\sup_{z \in X} |d(x, z) - d(y, z)| \leq d(x, y)$

PROOF: From the triangle inequality.

$\langle 3\rangle 2$. $\sup_{z \in X} |d(x, z) - d(y, z)| \geq d(x, y)$

PROOF: This holds because $|d(x, y) - d(y, y)| = d(x, y)$.

$\langle 2\rangle 2$. For $x, y \in X$ we have $\overline{\rho}(\Phi(x), \Phi(y)) = d(x, y)$

PROOF:

$$\begin{aligned}
\overline{\rho}(\Phi(x), \Phi(y)) &= \sup_{z \in X} |\Phi(x)(z) - \Phi(y)(z)| \\
&= \sup_{z \in X} |d(x, z) - d(x_0, z) - d(y, z) + d(y_0, z)| \\
&= \sup_{z \in X} |d(x, z) - d(y, z)| \\
&= d(x, y) && (\langle 2\rangle 1)
\end{aligned}$$

□

10.7.1 Completion of a Metric Space

Theorem 10.7.14. *For every metric space X, there exists a complete metric space $C(X)$ and an isometric imbedding $i : X \to C(X)$ such that, for every complete metric space Y and isometric imbedding $j : X \to Y$, there exists a unique isometric imbedding $\overline{j} : C(X) \to Y$ such that*

$$j = \overline{j} \circ i$$

PROOF:

$\langle 1\rangle 1$. PICK a complete metric space Z such that $X \subseteq Z$

PROOF: From Theorem 10.7.13.

$\langle 1\rangle 2$. LET: $C(X) = \overline{X}$ as a subspace of Z and i be the inclusion.

$\langle 1\rangle 3$. LET: Y be a complete metric space and $j : X \to Y$ an isometric imbedding

$\langle 1\rangle 4$. LET: $\overline{j} : C(X) \to Y$ be defined as follows: for $a \in \overline{X}$, pick a sequence (x_n) in X that converges to a. Then $\overline{j}(a) = \lim_{n \to \infty} j(x_n)$

$\langle 2\rangle 1$. For all $a \in \overline{X}$, there exists a sequence in X that converges to a.

PROOF: By the Sequence Lemma.

$\langle 2\rangle 2$. If (x_n) is a sequence in X that converges in $C(X)$ then $(j(x_n))$ converges in Y

$\langle 3\rangle 1$. LET: (x_n) be a convergent sequence in X.

$\langle 3\rangle 2$. (x_n) is Cauchy.

PROOF: Lemma 10.7.2

$\langle 3\rangle 3$. $(j(x_n))$ is Cauchy in Y.

PROOF: This holds because j is an isometry between X and $j(X)$.

⟨3⟩4. Q.E.D.

PROOF: Since Y is complete.

⟨2⟩3. If (x_n) and (y_n) are sequences in X that have the same limit in $C(X)$ then $\lim_{n\to\infty} j(x_n) = \lim_{n\to\infty} j(y_n)$

PROOF:

$$\begin{aligned} d(\lim_{n\to\infty} j(x_n), \lim_{n\to\infty} j(y_n)) &= \lim_{n\to\infty} d(j(x_n), j(y_n)) \text{(Theorem 5.2.12, Lemma 10.2.20)} \\ &= \lim_{n\to\infty} d(x_n, y_n) && (j \text{ is isometric}) \\ &= d(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n) \text{(Theorem 5.2.12, Lemma 10.2.20)} \\ &= 0 \end{aligned}$$

⟨1⟩5. $\overline{j}$ is an isometric imbedding

⟨2⟩1. LET: $a, b \in C(X)$

⟨2⟩2. PICK sequences (x_n), (y_n) in X that converge to a and b respectively.

PROOF: By the Sequence Lemma.

⟨2⟩3. $d(\overline{j}(a), \overline{j}(b)) = d(a, b)$

PROOF:

$$d(\overline{j}(a), \overline{j}(b)) = d(\lim_{n\to\infty} j(x_n), \lim_{n\to\infty} j(y_n))$$

$$\begin{aligned} d(\lim_{n\to\infty} j(x_n), \lim_{n\to\infty} j(y_n)) &= \lim_{n\to\infty} d(j(x_n), j(y_n)) \text{(Theorem 5.2.12, Lemma 10.2.20)} \\ &= \lim_{n\to\infty} d(x_n, y_n) && (j \text{ is isometric}) \\ &= d(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n) \text{(Theorem 5.2.12, Lemma 10.2.20)} \\ &= d(a, b) \end{aligned}$$

⟨1⟩6. $j = \overline{j} \circ i$

PROOF: For $a \in X$ we have

$$\begin{aligned} \overline{j}(i(a)) &= \overline{j}(a) \\ &= \overline{j}(\lim_{n\to\infty} a) \\ &= \lim_{n\to\infty} j(a) \\ &= j(a) \end{aligned}$$

⟨1⟩7. If $k : C(X) \to Y$ is an isometric imbedding and $j = k \circ i$ then $k = \overline{j}$

⟨2⟩1. LET: $a \in C(X)$

⟨2⟩2. PICK a sequence (x_n) in X that converges to a

PROOF: By the Sequence Lemma.

⟨2⟩3. $k(a) = \lim_{n\to\infty} j(x_n)$

PROOF:

$$\begin{aligned} k(a) &= k(\lim_{n\to\infty} x_n) \\ &= \lim_{n\to\infty} k(x_n) && (Theorem 5.2.12) \\ &= \lim_{n\to\infty} j(x_n) && (j = k \circ i) \\ &= \overline{j}(a) \end{aligned}$$

∎

Definition 10.7.15 (Completion)**.** The *completion* of a metric space X is the complete metric space $C(X)$ such that:

- X is a sub-metric space of $C(X)$
- For every complete metric space Y, every isometric imbedding $X \to Y$ extends uniquely to an isometric imbedding $C(X) \to Y$

Theorem 10.7.16 (Uniqueness of Completion)**.** *Suppose $C_1(X)$ and $C_2(X)$ are both completions of the metric space X. Then there exists a unique isometry $\phi : C_1(X) \cong C_2(X)$ that is the identity on X.*

PROOF:
⟨1⟩1. LET: $\phi : C_1(X) \to C_2(X)$ be the unique isometric imbedding that extends the inclusion $X \hookrightarrow C_2(X)$
⟨1⟩2. LET: $\phi^{-1} : C_2(X) \to C_1(X)$ be the unique isometric imbedding that extends the inclusion $X \hookrightarrow C_1(X)$
⟨1⟩3. $\phi \circ \phi^{-1} = \mathrm{id}_{C_2(X)}$
PROOF: This holds because $\mathrm{id}_{C_2(X)}$ is the unique isometric imbedding $C_2(X) \to C_2(X)$ that extends the inclusion $X \hookrightarrow C_2(X)$.
⟨1⟩4. $\phi^{-1} \circ \phi = \mathrm{id}_{C_1(X)}$
PROOF: Similar.
□

Definition 10.7.17 (Peano space)**.** A topological space is a *Peano space* iff it is Hausdorff and it is the continuous image of the unit interval $[0, 1]$.

Theorem 10.7.18. $[0, 1]^2$ *is a Peano space.*

PROOF:
⟨1⟩1. LET: $I = [0, 1]$
⟨1⟩2. Give I^2 the square metric and $\mathcal{C}(I, I^2)$ the sup-metric.
⟨1⟩3. Define the sequence (f_n) in $\mathcal{C}(I, I^2)$ by:

- f_0 is the path consisting of a straight line from $(0, 0)$ to $(1/2, 1/2)$ then a straight line from $(1/2, 1/2)$ to $(1, 0)$.
- Given f_n, f_{n+1} is the result of replacing:
  - Every path UR-DR with a path UR-UL-UR-DR-UR-DR-DL-DR
  - Every path UR-UL with a path UR-DR-UR-UL-UR-UL-DL-UL
  - Etc.

⟨1⟩4. $\rho(f_n, f_{n+1}) \leq 1/2^n$
⟨1⟩5. (f_n) is Cauchy
⟨1⟩6. LET: f be the limit of (f_n)
⟨1⟩7. $f(I)$ is dense in I^2
⟨2⟩1. LET: $x \in I^2$ and $\epsilon > 0$
⟨2⟩2. PICK N such that $\rho(f_N, f) < \epsilon/2$ and $1/2^N < \epsilon/2$
⟨2⟩3. PICK $t \in I$ such that $d(f_N(t), x) < 1/2^N$
⟨2⟩4. $d(f(t), x) < \epsilon$

$\langle 1 \rangle 8.$ $f(I) = I^2$

$\langle 2 \rangle 1.$ $f(I)$ is compact.

PROOF: Proposition 9.5.10.

$\langle 2 \rangle 2.$ $f(I)$ is closed.

PROOF: Proposition 9.5.9.

□

Theorem 10.7.19 (Hahn-Mazurkiewicz). *A space is a Peano space if and only if it is compact, connected, locally connected and metrizable.*

PROOF:

$\langle 1 \rangle 1.$ Every Peano space is compact, connected, locally connected and metrizable.

$\langle 2 \rangle 1.$ LET: X be a Peano space.

$\langle 2 \rangle 2.$ PICK a continuous surjection $p : [0,1] \twoheadrightarrow X$

$\langle 2 \rangle 3.$ p is a perfect map.

$\langle 3 \rangle 1.$ p is a closed map.

$\langle 4 \rangle 1.$ LET: $C \subseteq [0,1]$ be closed.

$\langle 4 \rangle 2.$ C is compact.

PROOF: Proposition 9.5.6.

$\langle 4 \rangle 3.$ $p(C)$ is compact.

PROOF: Proposition 9.5.10.

$\langle 4 \rangle 4.$ $p(C)$ is closed.

PROOF: Proposition 9.5.9.

$\langle 3 \rangle 2.$ For all $x \in X$ we have $p^{-1}(x)$ is compact.

$\langle 4 \rangle 1.$ LET: $x \in X$

$\langle 4 \rangle 2.$ $\{x\}$ is closed.

PROOF: Theorem 6.2.2

$\langle 4 \rangle 3.$ $p^{-1}(x)$ is closed.

PROOF: Theorem 5.2.2

$\langle 4 \rangle 4.$ $p^{-1}(x)$ is compact.

PROOF: Proposition 9.5.9.

$\langle 2 \rangle 4.$ X is compact.

PROOF: Proposition 9.5.10.

$\langle 2 \rangle 5.$ X is connected.

PROOF: Theorem 8.1.9.

$\langle 2 \rangle 6.$ X is locally connected.

PROOF: Proposition 8.6.15

$\langle 2 \rangle 7.$ X is metrizable.

$\langle 3 \rangle 1.$ X is second countable.

PROOF: Proposition 13.1.20

$\langle 3 \rangle 2.$ X is regular.

PROOF: Proposition 9.6.7.

$\langle 3 \rangle 3.$ Q.E.D.

PROOF: By the Urysohn Metrization Theorem.

$\langle 1 \rangle 2.$ Every compact, connected, locally connected, metrizable space is a Peano space.

PROOF: See J. G. Hocking and G. S. Young, *Topology* p. 129.
□

Theorem 10.7.20 (DC)**.** *A metric space is compact if and only if it is complete and totally bounded.*

PROOF:
⟨1⟩1. Every compact metric space is complete.
PROOF: Lemma 10.7.5 and Theorem 10.4.3.
⟨1⟩2. Every compact metric space is totally bounded.
PROOF: For every $\epsilon > 0$, the set of all ϵ-balls covers the space, hence has a finite subcover.
⟨1⟩3. Every complete, totally bounded metric space is compact.
⟨2⟩1. LET: X be a complete, totally bounded metric space.
PROVE: X is sequentially compact.
⟨2⟩2. LET: (x_n) be a sequence of points in X.
⟨2⟩3. PICK a sequence of infinite sets of integers $J_1 \supseteq J_2 \supseteq \cdots$ such that, for each k, there exists an open ball of radius $1/k$ that contains x_n for all $n \in J_k$
⟨3⟩1. LET: $J_0 = \mathbb{Z}^+$
⟨3⟩2. ASSUME: we have chosen $J_1 \supseteq \cdots \supseteq J_{k-1}$ satisfying the condition
⟨3⟩3. PICK finitely many balls $B_1, \ldots, B_r$ of radius $1/k$ that cover X.
⟨3⟩4. PICK i such that B_i contains x_n for infinitely many $n \in J_{k-1}$
⟨3⟩5. LET: $J_k = \{n \in J_{k-1} : x_n \in B_i\}$
⟨2⟩4. PICK a sequence $n_1 < n_2 < \cdots$ with $n_k \in J_k$ for all k.
⟨2⟩5. (x_{n_r}) is Cauchy.
PROOF: For all r, s with $r \leq s$ we have $d(x_{n_r}, x_{n_s}) \leq 2/r$.
⟨2⟩6. (x_{n_r}) converges.
PROOF: ⟨2⟩1, ⟨2⟩5
⟨2⟩7. Q.E.D.
PROOF: Theorem 10.4.3.
□

Chapter 11

Manifolds

11.1 Manifolds

Definition 11.1.1 (Manifold)**.** Let $m \geq 1$. An *m-manifold* is a second countable Hausdorff space such that each point has a neighbourhood that is homeomorphic to an open subset of $\mathbb{R}^m$.

A *curve* is a 1-manifold and a *surface* is a 2-manifold.

Theorem 11.1.2 (Existence of Finite Partitions of Unity)**.** *Let X be a normal space. Let $\{U_1, \ldots, U_n\}$ be a finite indexed open covering of X. Then there exists a partition of unity dominated by $\{U_1, \ldots, U_n\}$.*

PROOF:
$\langle 1 \rangle 1$. For every finite indexed open covering $\{U_1, \ldots, U_n\}$ of X, there exists a finite indexed open covering $\{V_1, \ldots, V_n\}$ such that $\overline{V_i} \subseteq U_i$
  $\langle 2 \rangle 1$. For $1 \leq k \leq n$, there exist open sets $V_1, \ldots, V_k$ such that $\overline{V_i} \subseteq U_i$ for all i and $\{V_1, \ldots, V_k, U_{k+1}, \ldots, U_n\}$ covers X
    $\langle 3 \rangle 1$. ASSUME: as an induction hypothesis that $0leqk < k$ and there exist open sets $V_1, \ldots, V_k$ such that $\overline{V_i} \subseteq U_i$ for all i and $\{V_1, \ldots, V_k, U_{k+1}, \ldots, U_n\}$ covers X
    $\langle 3 \rangle 2$. LET: $A = X \setminus (V_1 \cup \cdots \cup V_k) \setminus (U_{k+2} \cup \cdots \cup U_n)$
    $\langle 3 \rangle 3$. A is closed
    $\langle 3 \rangle 4$. $A \subseteq U_{k+1}$
      PROOF: Since $\{V_1, \ldots, V_k, U_{k+1}, \ldots, U_n\}$ covers X
    $\langle 3 \rangle 5$. PICK an open set V_{k+1} such that $A \subseteq V_{k+1}$ and $\overline{V_{k+1}} \subseteq U_{k+1}$
      PROOF: By Proposition 6.3.2
    $\langle 3 \rangle 6$. $\{V_1, \ldots, V_k, V_{k+1}, U_{k+2}, \ldots, U_n\}$ covers X
$\langle 1 \rangle 2$. PICK an open covering $\{V_1, \ldots, V_n\}$ with $\overline{V_i} \subseteq U_i$ for all i
  PROOF: By $\langle 1 \rangle 1$.
$\langle 1 \rangle 3$. PICK an open covering $\{W_1, \ldots, W_n\}$ with $\overline{W_i} \subseteq V_i$ for all i
  PROOF: By $\langle 1 \rangle 1$.
$\langle 1 \rangle 4$. For $1 \leq i \leq n$, PICK a continuous function $\psi_i : X \to [0, 1]$ such that $\psi_i(\overline{W_i}) = \{1\}$ and $\psi_i(X \setminus V_i) = \{0\}$

PROOF: By the Urysohn Lemma.
⟨1⟩5. LET: $\Psi : X \to \mathbb{R}$ where $\Psi(x) = \sum_{i=1}^{n} \psi_i(x)$
⟨1⟩6. $\Psi(x) > 0$ for all $x \in X$
⟨2⟩1. LET: $x \in X$
⟨2⟩2. PICK i such that $x \in W_i$
⟨2⟩3. $\psi_i(x) = 1$
⟨1⟩7. For $1 \leq j \leq n$,
LET: $\phi_j(x) = \frac{\psi_j(x)}{\Psi(x)}$
⟨1⟩8. $\psi_1, \ldots, \psi_n$ are a partition of unity dominated by $\{U_1, \ldots, U_n\}$
⟨2⟩1. $\operatorname{supp} \psi_i \subseteq U_i$
⟨3⟩1. ${\psi_i}^{-1}(\mathbb{R} \setminus \{0\}) \subseteq V_i$
PROOF: By ⟨1⟩4
⟨3⟩2. $\operatorname{supp} \psi_i \subseteq \overline{V_i}$
PROOF: Proposition 3.12.5
⟨2⟩2. $\sum_{i=1}^{n} \psi_i(x) = 1$ for all $x \in X$
□

Theorem 11.1.3. *Let X be a compact Hausdorff space. Suppose that, for every $x \in X$, there exists a neighbourhood U of x and a positive integer k such that U can be imbedded in $\mathbb{R}^k$. Then there exists a positive integer N such that X can be imbedded in $\mathbb{R}^N$.*

PROOF:
⟨1⟩1. PICK a finite open covering $\{U_1, \ldots, U_n\}$ of X such that each U_i can be imbedded in $\mathbb{R}^k$ for some k
PROOF: Since $\{U \text{ open in } X : U \text{ can be imbedded in } \mathbb{R}^k \text{ for some } k\}$ covers X.
⟨1⟩2. For $1 \leq i \leq n$, PICK a positive integer k_i and an imbedding $g_i : U_i \to \mathbb{R}^{k_i}$
⟨1⟩3. PICK a partition of unity $\phi_1, \ldots, \phi_n$ dominated by $\{U_1, \ldots, U_n\}$
⟨2⟩1. X is normal
PROOF: By Lemma 9.5.18.
⟨2⟩2. Q.E.D.
PROOF: Theorem 11.1.2
⟨1⟩4. For $1 \leq i \leq n$,
LET: $A_i = \operatorname{supp} \phi_i$
⟨1⟩5. For $1 \leq i \leq n$,
LET: $h_i : X \to \mathbb{R}^{k_i}$ be defined by
$$h_i(x) = \begin{cases} \phi_i(x) g_i(x) & \text{for } x \in U_i \\ \vec{0} & \text{for } x \in X \setminus A_i \end{cases}$$
PROOF: If $x \in U_i$ and $x \in X \setminus A_i$ then $x \notin \operatorname{supp} \phi_i$ so $\phi_i(x) = 0$
⟨1⟩6. LET: $N = n + k_1 + \cdots + k_n$
⟨1⟩7. LET: $F : X \to \mathbb{R}^N$ be the function
$$F(x) = (\phi_1(x), \ldots, \phi_n(x), h_1(x), \ldots, h_n(x))$$
⟨1⟩8. F is an imbedding
⟨2⟩1. F is continuous
PROOF: Each h_i is continuous by Theorem 5.2.13.

$\langle 2\rangle 2$. F is injective
$\langle 3\rangle 1$. ASSUME: $F(x) = F(y)$
$\langle 3\rangle 2$. PICK i such that $\phi_i(x) > 0$
PROOF: Since $\sum_i \phi_i(x) = 1$ ($\langle 1\rangle 3$)
$\langle 3\rangle 3$. $\phi_i(y) = 0$
PROOF: By $\langle 3\rangle 1$
$\langle 3\rangle 4$. $x, y \in U_i$
PROOF: Since $\operatorname{supp} \phi_i \subseteq U_i$
$\langle 3\rangle 5$. $h_i(x) = h_i(y)$
PROOF: By $\langle 3\rangle 1$
$\langle 3\rangle 6$. $g_i(x) = g_i(y)$
PROOF: By $\langle 1\rangle 5$
$\langle 3\rangle 7$. $x = y$
PROOF: By $\langle 1\rangle 2$
$\langle 2\rangle 3$. Q.E.D.
PROOF: By Theorem 9.5.11
□

Corollary 11.1.3.1. *Every compact manifold can be imbedded in $\mathbb{R}^N$ for some N.*

Proposition 11.1.4. *The line with two origins is a second countable T_1 space where every point has a neighbourhood that is homeomorphic to an open subset of $\mathbb{R}$, but it is not a 1-manifold.*

Chapter 12

Normed Spaces

12.1 The Norm on $\mathbb{R}^n$

Definition 12.1.1 (Norm)**.** Given $\vec{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, the *norm* $\|\vec{x}\|$ is defined by

$$\|\vec{x}\| = \sqrt{x_1^2 + \cdots + x_n^2} \enspace .$$

Definition 12.1.2 (Vector Sum)**.** Define the *sum* of $\vec{x}, \vec{y} \in \mathbb{R}^n$ by

$$\vec{x} + \vec{y} = (x_1 + y_1, \ldots, x_n + y_n) \enspace .$$

Definition 12.1.3 (Scalar Product)**.** Given $c \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^n$, define the *scalar product* $c\vec{x}$ to be

$$c\vec{x} = (cx_1, \ldots, cx_n) \enspace .$$

Definition 12.1.4 (Inner Product)**.** The *inner product* of $\vec{x}, \vec{y} \in \mathbb{R}^n$ is

$$\vec{x} \cdot \vec{y} = x_1 y_1 + \cdots + x_n y_n \enspace .$$

Lemma 12.1.5.

$$\vec{x} \cdot (\vec{y} + \vec{z}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z}$$

PROOF: Both are equal to $\sum_{i=1}^n (x_i y_i + x_i z_i)$. $\square$

Lemma 12.1.6.

$$|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \|\vec{y}\|$$

PROOF:
$\langle 1 \rangle 1$. CASE: $\vec{x} = \vec{0}$ or $\vec{y} = \vec{0}$
  PROOF: In this case, both sides are 0.
$\langle 1 \rangle 2$. CASE: $\vec{x} \neq \vec{0} \neq \vec{y}$
  $\langle 2 \rangle 1$. LET: $a = 1/\|\vec{x}\|$, $b = 1/\|\vec{y}\|$
  $\langle 2 \rangle 2$. $2 + 2ab\vec{x} \cdot \vec{y} \geq 0$
    $\langle 3 \rangle 1$. $\|a\vec{x} + b\vec{y}\|^2 \geq 0$

⟨3⟩2. $\sum_{i=1}^{n}(ax_i + by_i)^2 \geq 0$
⟨3⟩3. $a^2 \sum_{i=1}^{n} x_i^2 + b^2 \sum_{i=1}^{n} y_i^2 + 2ab \sum_{i=1}^{n} x_i y_i \geq 0$
⟨3⟩4. $a^2 \|\vec{x}\|^2 + b^2 \|\vec{y}\|^2 + 2ab\vec{x} \cdot \vec{y} \geq 0$
⟨2⟩3. $2 - 2ab\vec{x} \cdot \vec{y} \geq 0$
Proof: Similar.
⟨2⟩4. $2 - 2ab|\vec{x} \cdot \vec{y}| \geq 0$
Proof: From ⟨2⟩2 and ⟨2⟩3.
⟨2⟩5. $|\vec{x} \cdot \vec{y}| \leq 1/ab$
$\square$

Lemma 12.1.7.

$$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$$

Proof:

$$\begin{aligned}
\|\vec{x} + \vec{y}\|^2 &= (\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y}) \\
&= \|\vec{x}\|^2 + 2\vec{x} \cdot \vec{y} + \|\vec{y}\|^2 && \text{(Lemma 12.1.5)} \\
&\leq \|\vec{x}\|^2 + 2\|\vec{x}\|\|\vec{y}\| + \|\vec{y}\|^2 && \text{(Lemma 12.1.6)} \\
&= (\|\vec{x}\| + \|\vec{y}\|)^2 && \square
\end{aligned}$$

Definition 12.1.8 (Euclidean Metric)**.** The *euclidean metric* on $\mathbb{R}^n$ is given by

$$d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|$$

We prove this is a metric.

Proof:
⟨1⟩1. $d(\vec{x}, \vec{y}) \geq 0$
Proof: Immediate from definitions.
⟨1⟩2. $d(\vec{x}, \vec{y}) = 0$ iff $\vec{x} = \vec{y}$
Proof: Immediate from definitions.
⟨1⟩3. $d(\vec{x}, \vec{y}) = d(\vec{y}, \vec{x})$
Proof: Immediate from definitions.
⟨1⟩4. $d(\vec{x}, \vec{z}) \leq d(\vec{x}, \vec{y}) + d(\vec{y}, \vec{z})$
Proof: From Lemma 12.1.7.
$\square$

Lemma 12.1.9. *Let d be the euclidean topology on $\mathbb{R}^n$ and ρ the square topology. Then, for all $x, y \in \mathbb{R}^n$, we have*

$$\rho(x, y) \leq d(x, y) \leq \sqrt{n}\rho(x, y)$$

Proof:
⟨1⟩1. $\rho(x, y) \leq d(x, y)$
⟨2⟩1. For $1 \leq i \leq n$ we have $|x_i - y_i| \leq d(x, y)$
Proof: By the definition of the euclidean metric.
⟨2⟩2. Q.E.D.
Proof: By the definition of the square metric.

⟨1⟩2. $d(x, y) \leq \sqrt{n}\rho(x, y)$

PROOF:

$$\begin{aligned} d(x,y) &= \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2} \\ &\leq \sqrt{\rho(x,y)^2 + \cdots + \rho(x,y)^2} \\ &= \sqrt{n\rho(x,y)^2} \\ &= \sqrt{n}\rho(x,y) \end{aligned}$$

□

Corollary 12.1.9.1. *The euclidean metric induces the standard topology on* $\mathbb{R}^n$*.*

Definition 12.1.10. Let l_2 be the set of sequences $\vec{a} \in \mathbb{R}^\omega$ such that $\sum_{n=1}^\infty a_n^2 < \infty$.

Lemma 12.1.11. *If* $\vec{a}, \vec{b} \in l_2$ *then* $\sum_{n=1}^\infty |a_n b_n| < \infty$*.*

PROOF:

$$\begin{aligned} \sum_{n=1}^N |a_n b_n| &\leq \sqrt{(\sum_{n=1}^N a_n^2)(\sum_{n=1}^N b_n^2)} && \text{(Lemma 12.1.6)} \\ &\to \sqrt{\sum_{n=1}^\infty a_n^2)(\sum_{n=1}^\infty b_n^2)} \text{ as } n \to \infty \end{aligned}$$

□

Lemma 12.1.12. *If* $\vec{a}, \vec{b} \in l_2$ *then* $\vec{a} + \vec{b} \in l_2$*.*

PROOF:

$$\begin{aligned} \sum_{n=1}^\infty (a_n + b_n)^2 &= \sum_{n=1}^\infty a_n^2 + 2\sum_{n=1}^\infty a_n b_n + \sum_{n=1}^\infty b_n^2 \\ &\leq \sum_{n=1}^\infty a_n^2 + 2\sum_{n=1}^\infty |a_n b_n| + \sum_{n=1}^\infty b_n^2 \\ &< \infty && \text{(Lemma 12.1.11)} \end{aligned}$$

□

Lemma 12.1.13. *If* $c \in \mathbb{R}$ *and* $\vec{a} \in l_2$ *then* $c\vec{a} \in l_2$*.*

PROOF:$\sum_{n=1}^\infty (ca_n)^2 = c^2 \sum_{n=1}^\infty a_n^2$. □

Definition 12.1.14 (The l^2-metric)**.** The l^2*-metric* is defined on l_2 by

$$d(\vec{a}, \vec{b}) = \left[\sum_{n=1}^\infty (a_n - b_n)^2\right]^{\frac{1}{2}} .$$

The topology induced by this metric is the l^2*-topology*. We write l_2 for this set under the l^2-topology.

We prove this is a metric.

PROOF:
⟨1⟩1. $d(\vec{a}, \vec{b}) \geq 0$
PROOF: Immediate from definitions.
⟨1⟩2. $d(\vec{a}, \vec{b}) = 0$ iff $\vec{a} = \vec{b}$
PROOF: Immediate from definitions.
⟨1⟩3. $d(\vec{a}, \vec{b}) = d(\vec{b}, \vec{a})$
PROOF: Immediate from definitions.
⟨1⟩4. $d(\vec{a}, \vec{c}) \leq d(\vec{a}, \vec{b}) + d(\vec{b}, \vec{c})$
PROOF: $\sqrt{\sum_{i=1}^N (a_n - c_n)^2} \leq \sqrt{\sum_{i=1}^N (a_n - b_n)^2} + \sqrt{\sum_{i=1}^N (b_n - c_n)^2}$ since the euclidean metric on $\mathbb{R}^N$ is a metric.
□

Definition 12.1.15 (Hilbert Cube)**.** The *Hilbert cube* is $\prod_{n=1}^{\infty}[0, 1/n]$ as a subspace of the l_2.

Definition 12.1.16 (Isometric Imbedding)**.** Let X, Y be metric spaces and $f : X \to Y$. Then f is an *isometric imbedding* iff, for all $x, y \in X$, $d(f(x), f(y)) = d(x, y)$.

Lemma 12.1.17. *Every isometric imbedding is an imbedding.*

PROOF:
⟨1⟩1. LET: $f : X \to Y$ be an isometric imbedding.
⟨1⟩2. f is continuous.
PROOF: If $d(x, y) < \epsilon$ then $d(f(x), f(y)) < \epsilon$.
⟨1⟩3. f is injective.
PROOF: If $f(x) = f(y)$ then $d(f(x), f(y)) = 0$ so $d(x, y) = 0$ hence $x = y$.
⟨1⟩4. $f^{-1} : f(X) \to X$ is continuous.
PROOF: If $d(f^{-1}(x), f^{-1}(y)) < \epsilon$ then $d(x, y) < \epsilon$.
□

Chapter 13

Topological Groups

13.1 Topological Groups

Definition 13.1.1 (Topological Group)**.** A *topological group* G consists of a group G that is also a T_1 space such that $\cdot : G^2 \to G$ and $(\ )^{-1} : G \to G$ are continuous.

Proposition 13.1.2. *Every topological group is homogeneous.*

PROOF:
$\langle 1 \rangle$1. LET: G be a topological group.
$\langle 1 \rangle$2. LET: $x, y \in G$
$\langle 1 \rangle$3. LET: $f : G \to G$ be given by $f(g) = yx^{-1}z$
$\langle 1 \rangle$4. f is a homeomorphism
$\langle 1 \rangle$5. $f(x) = y$
□

Definition 13.1.3 (Symmetric)**.** Let G be a topological group. A neighbourhood U of e is *symmetric* iff $U = U^{-1}$.

Proposition 13.1.4. *For every neighbourhood U of e, there exists a symmetric neighbourhood V of e such that $VV \subseteq U$.*

PROOF:
$\langle 1 \rangle$1. LET: $m : G^2 \to G$ be the multiplication function
$\langle 1 \rangle$2. $ee \in U$
$\langle 1 \rangle$3. $(e, e) \in m^{-1}(U)$
$\langle 1 \rangle$4. PICK neighbourhoods U_1, U_2 of e such that $(e, e) \in U_1 \times U_2 \subseteq m^{-1}(U)$
$\langle 1 \rangle$5. LET: $V' = U_1 \cap U_2$
$\langle 1 \rangle$6. $V'V' \subseteq U$
$\langle 1 \rangle$7. LET: $f : G^2 \to G$ be the function $f(x, y) = xy^{-1}$
$\langle 1 \rangle$8. $(e, e) \in f^{-1}(V')$
$\langle 1 \rangle$9. PICK a neighbourhood W of e such that $WW^{-1} \subseteq V'$
$\langle 1 \rangle$10. LET: $V = WW^{-1}$

$\langle 1\rangle 11.$ V is a neighbourhood of e
PROOF: V is open because $V = \bigcup_{a \in W^{-1}} Wa$.
$\langle 1\rangle 12.$ V is symmetric
$\langle 1\rangle 13.$ $VV \subseteq U$
□

Proposition 13.1.5. *Every topological group is regular.*

PROOF:
$\langle 1\rangle 1.$ LET: G be a topological group
$\langle 1\rangle 2.$ LET: $A \subseteq G$ be closed and $a \notin A$
$\langle 1\rangle 3.$ $G \setminus Aa^{-1}$ is a neighbourhood of e
$\langle 1\rangle 4.$ PICK a symmetric neighbourhood V of e such that $VV \subseteq G \setminus Aa^{-1}$
PROOF: Proposition 13.1.4.
$\langle 1\rangle 5.$ VA and Va are disjoint neighbourhoods of A and a
□

Proposition 13.1.6. *The long line is not second countable.*

PROOF:Let $\mathcal{B}$ be a basis for L. Then, for every countable ordinal α, $\mathcal{B}$ mst contain a basic open set that contains $(\alpha, 1/2)$ but not $(\beta, 1/2)$ for any other β. Therefore, $\mathcal{B}$ is uncountable. □

Corollary 13.1.6.1. *The long line cannot be imbedded in* $\mathbb{R}$.

Theorem 13.1.7. *Let $f : X \to Y$. Let Y be compact Hausdorff. Then f is continuous if and only if the graph of f is closed in $X \times Y$.*

PROOF:
$\langle 1\rangle 1.$ LET: G_f be the graph of f.
$\langle 1\rangle 2.$ If f is continuous then the graph of f is closed.
$\langle 2\rangle 1.$ ASSUME: f is continuous.
$\langle 2\rangle 2.$ LET: $(x, y) \in (X \times Y) \setminus G_f$
$\langle 2\rangle 3.$ $y \neq f(x)$
$\langle 2\rangle 4.$ PICK disjoint open neighbourhoods U of $f(x)$ and V of y
PROOF: Y is Hausdorff.
$\langle 2\rangle 5.$ $(x, y) \in f^{-1}(U) \times V \subseteq (X \times Y) \setminus G_f$
$\langle 2\rangle 6.$ Q.E.D.
$\langle 1\rangle 3.$ If the graph of f is closed then f is continuous.
$\langle 2\rangle 1.$ ASSUME: G_f is closed.
$\langle 2\rangle 2.$ LET: $x_0 \in X$ and V be an open neighbourhood of $f(x_0)$
$\langle 2\rangle 3.$ $G_f \cap (X \times (Y \setminus V))$ is closed
$\langle 2\rangle 4.$ $\pi_1(G_f \cap (X \times (Y \setminus V)))$ is closed
PROOF: Lemma 9.5.16
$\langle 2\rangle 5.$ $x_0 \in X \setminus \pi_1(G_f \cap (X \times (Y \setminus V))) \subseteq f^{-1}(V)$
$\langle 2\rangle 6.$ Q.E.D.
□

Theorem 13.1.8. *Let X be a compact Hausdorff space. Let $\mathcal{A}$ be a set of closed connected subspaces of X that is linearly ordered by proper inclusion. Then*

$$Y = \bigcap \mathcal{A}$$

is connected.

PROOF:
⟨1⟩1. ASSUME: for a contradiction C and D form a separation of Y
⟨1⟩2. PICK disjoint U and V open in X such that $C = U \cap Y$ and $D = V \cap Y$
  ⟨2⟩1. C and D are compact
    ⟨3⟩1. Y is compact
      PROOF: Y is a closed subset of X, hence compact by Proposition 9.5.6.
    ⟨3⟩2. Q.E.D.
      PROOF: C and D are closed subsets of Y hence compact by Proposition 9.5.6.
  ⟨2⟩2. Q.E.D.
    PROOF: By Lemma 9.5.18.
⟨1⟩3. For all $A \in \mathcal{A}$, we have $A \setminus (U \cup V)$ is nonempty
  PROOF: Since A is connected.
⟨1⟩4. $\{A \setminus (U \cup V) : A \in \mathcal{A}\}$ has the finite intersection property
  PROOF: This holds because $\mathcal{A}$ is linearly ordered under proper inclusion.
⟨1⟩5. $\bigcap_{A \in \mathcal{A}} (A \setminus (U \cup V))$ is nonempty
  PROOF: By Proposition 9.5.15.
□

Theorem 13.1.9. *Let $A \subseteq \mathbb{R}^n$. Then the following are equivalent:*

1. *A is compact.*
2. *A is closed and bounded under the euclidean metric.*
3. *A is closed and bounded under the square metric.*

PROOF:
⟨1⟩1. $1 \Rightarrow 2$
  ⟨2⟩1. ASSUME: A is compact.
  ⟨2⟩2. A is closed.
    PROOF: By Proposition 9.5.9.
  ⟨2⟩3. $\{B(\vec{0}, n) : n \in \mathbb{Z}^+\}$ covers A
  ⟨2⟩4. PICK a finite subcover $\{B(\vec{0}, n_1), \ldots, B(\vec{0}, n_k)\}$
  ⟨2⟩5. LET: $N = \max(n_1, \ldots, n_k)$
  ⟨2⟩6. For all $x, y \in A$ we have $d(x, y) < 2N$
    PROOF: We have $d(x, y) \leq d(\vec{0}, x) + d(\vec{0}, y) < N + N$.
⟨1⟩2. $2 \Rightarrow 3$
  PROOF: If $d(x, y) < \epsilon$ for all $x, y \in A$ then $\rho(x, y) < \epsilon\sqrt{n}$ by Lemma 12.1.9.
⟨1⟩3. $3 \Rightarrow 1$
  ⟨2⟩1. ASSUME: A is closed and $\rho(x, y) < \epsilon$ for all $x, y \in A$

⟨2⟩2. PICK $x_0 \in A$
⟨2⟩3. LET: $b = \rho(\vec{0}, x_0)$
⟨2⟩4. LET: $P = \epsilon + b$
⟨2⟩5. $A \subseteq [-P, P]^n$
PROOF:For any $y \in A$ we have

$$\begin{aligned} \rho(\vec{0}, y) &\leq \rho(\vec{0}, x_0) + \rho(x_0, y) && \text{(Triangle Inequality)} \\ &< b + \epsilon && (\langle 2\rangle 3,\ \langle 2\rangle 1) \\ &= P && (\langle 2\rangle 4) \end{aligned}$$

⟨2⟩6. $[-P, P]^n$ is compact.
PROOF: By Corollary 9.5.19.1 and Proposition 9.5.14.
⟨2⟩7. Q.E.D.
PROOF: By Proposition 9.5.6.
□

Theorem 13.1.10 (AC). *Let X be a topological space. Then X is compact if and only if every nonempty net in X has a convergent subnet.*

PROOF:
⟨1⟩1. If X is compact then every nonempty net in X has a convergent subnet.
⟨2⟩1. ASSUME: X is compact.
⟨2⟩2. LET: $(x_\alpha)_{\alpha \in J}$ be a nonempty net in X
⟨2⟩3. For $\alpha \in J$,
LET: $B_\alpha = \{\beta \in J : \alpha \leq \beta\}$.
⟨2⟩4. $\{B_\alpha : \alpha \in J\}$ has the finite intersection property.
⟨3⟩1. LET: $\alpha_1, \ldots, \alpha_n \in J$
⟨3⟩2. PICK $\beta \in J$ such that $\alpha_1 \leq \beta$, ..., $\alpha_n \leq \beta$
⟨3⟩3. $x_\beta \in B_{\alpha_1} \cap \cdots \cap B_{\alpha_n}$
⟨2⟩5. PICK $l \in \bigcap_{\alpha \in J} B_\alpha$
PROOF: Proposition 9.5.15.
⟨2⟩6. LET: $K = \{\alpha \in J : x_\alpha = l\}$
⟨2⟩7. K is cofinal in J
⟨3⟩1. LET: $\alpha \in J$
⟨3⟩2. $l \in B_\alpha$
PROOF: By ⟨2⟩5.
⟨3⟩3. There exists $\beta \geq \alpha$ such that $x_\beta = l$.
⟨2⟩8. $(x_\alpha)_{\alpha \in K}$ is a subnet of $(x_\alpha)_{\alpha \in J}$ that converges to l.
⟨1⟩2. If every nonempty net in X has a convergent subnet then X is compact.
⟨2⟩1. ASSUME: Every nonempty net in X has a convergent subnet
⟨2⟩2. LET: $\mathcal{A}$ be a nonempty set of closed sets with the finite intersection property.
⟨2⟩3. LET: J be the poset of all finite intersections of elements of $\mathcal{A}$ under $\supseteq$
⟨2⟩4. PICK $x_C \in C$ for all $C \in J$
PROOF: These are all nonempty by ⟨2⟩2.
⟨2⟩5. PICK an accumulation point l of (x_C)
PROVE: $l \in \bigcap \mathcal{A}$
PROOF: One exists by Lemma 3.18.2.

⟨2⟩6. LET: $C \in \mathcal{A}$
PROVE: $l \in C$
⟨2⟩7. LET: U be a neighbourhood of l
PROVE: U intersects C
⟨2⟩8. PICK $D \subseteq C$ such that $x_D \in U$
PROOF: By ⟨2⟩5.
⟨2⟩9. U intersects C
⟨2⟩10. $l \in C$
PROOF: By Theorem 3.13.3 since C is closed (⟨2⟩2).
⟨2⟩11. Q.E.D.
PROOF: Proposition 9.5.15.
□

Corollary 13.1.10.1 (AC)**.** *Let G be a topological group. Let A and B be subsets of G. If A is closed in G and B is compact then AB is closed in G.*

PROOF:
⟨1⟩1. LET: $c \in \overline{AB}$
PROVE: $c \in AB$
⟨1⟩2. PICK a net $(x_\alpha)_{\alpha \in J}$ that converges to c
PROOF: By Theorem 3.17.3.
⟨1⟩3. For $\alpha \in J$, PICK $a_\alpha \in A$ and $b_\alpha \in B$ such that $x_\alpha = a_\alpha b_\alpha$
⟨1⟩4. PICK a convergent subnet $(b_{g(\beta)})_{\beta \in K}$ of $(b_\alpha)_{\alpha \in J}$
PROOF: By Theorem 13.1.10.
⟨1⟩5. LET: $b_{g(\beta)} \to b$
⟨1⟩6. $b \in B$
⟨2⟩1. B is closed
PROOF: By Proposition 9.5.9.
⟨2⟩2. Q.E.D.
PROOF: By Theorem 3.17.3
⟨1⟩7. $a_{g(\beta)} \to cb^{-1}$
PROOF: By Theorem 3.17.4
⟨1⟩8. $cb^{-1} \in A$
PROOF: By Theorem 3.17.3
⟨1⟩9. $c \in AB$
⟨1⟩10. Q.E.D.
PROOF: By Proposition 3.12.6.

Proposition 13.1.11. *Let $A_0 + A_1$ be the sum of A_0 and A_1 with injections $i_0 : A_0 \to A_0 + A_1$ and $i_1 : A_1 \to A_0 + A_1$.*

Let $g : B \to A_0 + A_1$ be a function.

Let B_0 be the pullback of i_0 and g with projections $j_0 : B_0 \to B$ and $k_0 : B_0 \to A_0$.

Let B_1 be the pullback of i_1 and g with projection $sj_1 : B_1 \to B$ and $k_1 : B_1 \to A_1$.

Then B is the sum of B_0 and B_1 with injections j_0 and j_1.

$$\begin{tikzcd}
B_0 \arrow[r, "j_0"] \arrow[d] & B \arrow[d, "g"] & B_1 \arrow[l, "j_1"'] \arrow[d] \\
A_0 \arrow[r, "i_0"'] & A_0 + A_1 & A_1 \arrow[l, "i_1"']
\end{tikzcd}$$

PROOF:
⟨1⟩1. LET: X be any set and $x : B_0 \to X$, $y : B_1 \to X$

Proposition 13.1.12 (CC)**.** *Let X be a space and $\mathcal{B}$ be a basis for X. Suppose that every subset of $\mathcal{B}$ that covers X has a countable subcover. Then X is Lindelöf.*

PROOF:
⟨1⟩1. LET: $\mathcal{A}$ be an open cover of X.
⟨1⟩2. $\{B \in \mathcal{B} : \exists U \in \mathcal{A}.B \subseteq U\}$ covers X.
⟨1⟩3. PICK a countable subcover $\mathcal{B}_0$
⟨1⟩4. For $B \in \mathcal{B}_0$, PICK $U_B \in \mathcal{A}$ such that $B \subseteq U_B$
⟨1⟩5. $\{U_B : B \in \mathcal{B}_0\}$ is a countable subcover of $\mathcal{A}$.
$\square$

Proposition 13.1.13 (CC)**.** *The space $\mathbb{R}_l$ is Lindelöf.*

PROOF:
⟨1⟩1. LET: $\mathcal{A}$ be a set of basis elements $[a, b)$ that covers X
PROVE: $\mathcal{A}$ has a countable subcover.
⟨1⟩2. LET: $C = \bigcup\{(a, b) : [a, b) \in \mathcal{A}\}$
⟨1⟩3. $\mathbb{R} \setminus C$ is countable.
⟨2⟩1. For all $x \in \mathbb{R} \setminus C$, PICK a rational q_x such that there exists b such that $q_x \in [x, b) \in \mathcal{A}$
⟨3⟩1. PICK $[a, b) \in \mathcal{A}$ such that $x \in [a, b)$
⟨3⟩2. $x = a$
PROOF: If not we would have $x \in C$
⟨3⟩3. There exists a rational in (a, b)
⟨2⟩2. For $x, y \in \mathbb{R} \setminus C$, if $x < y$ then $q_x < q_y$
⟨3⟩1. PICK b, c such that $q_x \in [x, b) \in \mathcal{A}$ and $q_y \in [y, c) \in \mathcal{A}$
PROOF: By ⟨2⟩1.
⟨3⟩2. $b \leq y$
PROOF: Otherwise we would have $y \in (x, b) \subseteq C$.
⟨3⟩3. $q_x < q_y$
PROOF: $q_x < b \leq y \leq q_y$
⟨2⟩3. The map $q_- : \mathbb{R} \setminus C \to \mathbb{Q}$ is injective.
⟨1⟩4. For $x \in \mathbb{R} \setminus C$, PICK $[a_x, b_x) \in \mathcal{A}$ such that $a_x \leq x < b_x$
⟨1⟩5. PICK a countable subset $((a_n, b_n))_{n \in \mathbb{Z}^+}$ of $\{(a, b) : [a, b) \in \mathcal{A}\}$ that covers C
⟨2⟩1. The set C as a subspace of $\mathbb{R}$ with the standard topology is second countable.

$\langle 2\rangle 2$. The set C as a subspace of $\mathbb{R}$ with the standard topology is Lindelöf.
PROOF: By Theorem 9.3.2.
$\langle 1\rangle 6$. $\{[a_x, b_x) : x \in \mathbb{R} \setminus C\} \cup \{[a_n, b_n) : n \in \mathbb{Z}^+\}$ is a countable subcover of $\mathcal{A}$.
$\langle 1\rangle 7$. Q.E.D.
PROOF: By Proposition 13.1.12.
□

Proposition 13.1.14 (AC). *The space $\mathbb{R}_l$ is not second countable.*

PROOF:
$\langle 1\rangle 1$. LET: $\mathcal{B}$ be any basis for $\mathbb{R}_l$
$\langle 1\rangle 2$. For $x \in \mathbb{R}$, PICK $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq [x, x+1)$
$\langle 1\rangle 3$. The mapping $B_{(-)}$ is an injective function $\mathbb{R} \to \mathcal{B}$
PROOF: For any x we have $x = \min B_x$.
$\langle 1\rangle 4$. $\mathcal{B}$ is uncountable.
□

Proposition 13.1.15. *The product of a Lindelöf space and a compact space is Lindelöf.*

PROOF:
$\langle 1\rangle 1$. LET: X be a Lindelöf space and Y a compact space.
$\langle 1\rangle 2$. LET: $\mathcal{A}$ be an open covering of $X \times Y$
$\langle 1\rangle 3$. For all $x \in X$, there exists a neighbourhood W of x such that $W \times Y$ is covered by finitely many elements of $\mathcal{A}$.
$\langle 2\rangle 1$. LET: $x \in X$
$\langle 2\rangle 2$. $\{x\} \times Y$ is compact.
PROOF: It is homeomorphic to Y.
$\langle 2\rangle 3$. PICK a finite subset $\{U_1, \ldots, U_m\}$ of $\mathcal{A}$ that covers $\{x\} \times Y$
PROOF: By Proposition 9.5.5.
$\langle 2\rangle 4$. There exists a neighbourhood W of x such that $W \times Y \subseteq U_1 \cup \cdots \cup U_m$
PROOF: By the Tube Lemma.
$\langle 1\rangle 4$. $\{W \text{ open in } X : W \times Y \text{ is covered by finitely many elements of } \mathcal{A}\}$ is an open covering of X.
$\langle 1\rangle 5$. PICK a countable subcovering $\{W_1, W_2, \ldots\}$
$\langle 1\rangle 6$. For $i \geq 1$, PICK a finite subset $\{U_{i1}, \ldots, U_{ir_i}\}$ of $\mathcal{A}$ that covers $W_i \times Y$
$\langle 1\rangle 7$. $\{U_{1j} : i \geq 1, 1 \leq j \leq r_i\}$ is a countable subcovering of $\mathcal{A}$.
□

Proposition 13.1.16. *Let X be a T_1 space. Then X is normal if and only if, for every closed set A and open set $U \supseteq A$, there exists an open set $V \supseteq A$ such that $\overline{V} \subseteq U$.*

PROOF:
$\langle 1\rangle 1$. If X is normal then, for every closed set A and open set $U \supseteq A$, there exists an open set $V \supseteq A$ such that $\overline{V} \subseteq U$.
$\langle 2\rangle 1$. ASSUME: X is normal.
$\langle 2\rangle 2$. LET: A be a closed set and U an open set with $A \subseteq U$

⟨2⟩3. Pick disjoint open sets V, W such that $A \subseteq V$ andn $X \setminus U \subseteq W$

⟨2⟩4. $\overline{V} \subseteq U$

Proof:

$$\begin{aligned}\overline{V} &\subseteq X \setminus W \\ &\subseteq U\end{aligned}$$

⟨1⟩2. If, for every closed set A and open set $U \supseteq A$, there exists an open set $V \supseteq A$ such that $\overline{V} \subseteq U$, then X is normal.

⟨2⟩1. Assume: for every closed set A and open set $U \supseteq A$, there exists an open set $V \supseteq A$ such that $\overline{V} \subseteq U$.

⟨2⟩2. Let: A, B be disjoint closed sets

⟨2⟩3. Pick an open set V such that $A \subseteq V$ and $\overline{V} \subseteq X \setminus B$

⟨2⟩4. $A \subseteq V$ and $B \subseteq X \setminus \overline{V}$

□

Definition 13.1.17 (Action)**.** Let G be a topological group and X a topological space. An *action* of G on X is a continuous function $\cdot : G \times X \to X$ such that, for all $g, h \in G$ and $x \in X$:

1. $e \cdot x = x$
2. $g \cdot (h \cdot x) = gh \cdot x$

Definition 13.1.18 (Orbit Space)**.** Let G be a topological group, X a topological space, and $\cdot : G \times X \to X$ an action of G on X. Then the *orbit space* X/G is the quotient space of X by the equivalence relation $\sim$ generated by $x \sim g \cdot x$ for all $x \in X$, $g \in G$.

Theorem 13.1.19. *Let G be a topological group. Let X be a topological space. Let $\cdot : G \times X \to X$ be an action of G on X. Then the canonical map $\pi : X \twoheadrightarrow X/G$ is perfect.*

⟨1⟩1. π is closed.

⟨2⟩1. Let: $A \subseteq X$ be closed.

⟨2⟩2. $GA = \{g \cdot a : g \in G, a \in A\}$ is closed

⟨3⟩1. Let: $z \notin GA$

⟨3⟩2. For all $g \in G$ we have $g \cdot z \notin A$

⟨3⟩3. For $g \in G$, there exist U an open neighbourhood of g and V an open neighbourhood of z such that UV does not intersect A

⟨3⟩4. $\{U \text{ open in } G : \exists V \text{ an open neighbourhood of } z. UV \cap A = \emptyset\}$ covers G

⟨3⟩5. Pick a finite subcover $\{U_1, \ldots, U_n\}$

⟨3⟩6. For $1 \leq i \leq n$, Pick V_i an open neighbourhood of z such that $U_i V_i \cap A = \emptyset$

⟨3⟩7. $z \in V_1 \cap \cdots \cap V_n \subseteq X \setminus GA$

⟨2⟩3. $\pi(A)$ is closed

$\pi^{-1}(\pi(A)) = GA$

⟨1⟩2. π is continuous.

Proof: By definition of the quotient topology.

$\langle 1 \rangle 3.$ π is surjective.
PROOF: By definition.
$\langle 1 \rangle 4.$ For all $a \in X/G$ we have $\pi^{-1}(a)$ is compact.
$\langle 2 \rangle 1.$ LET: $a \in X/G$
$\langle 2 \rangle 2.$ PICK $x \in X$ such that $a = \pi(x)$
$\langle 2 \rangle 3.$ $\pi^{-1}(a) = \{gx : g \in G\}$
$\langle 2 \rangle 4.$ $\pi^{-1}(a)$ is homeomorphic to G
$\square$

Corollary 13.1.19.1. *If X is Hausdorff then so is X/G.*

Corollary 13.1.19.2. *If X is regular then so is X/G.*

Corollary 13.1.19.3. *If X is normal then so is X/G.*

Corollary 13.1.19.4. *If X is locally compact then so is X/G.*

Corollary 13.1.19.5. *If X is second countable then so is X/G.*

Proposition 13.1.20. *Let $p : X \twoheadrightarrow Y$ be perfect. If X is second countable then so is Y.*

PROOF:
$\langle 1 \rangle 1.$ PICK a countable basis $\mathcal{B}$ for X
$\langle 1 \rangle 2.$ LET: $\mathcal{J} = \{J \subseteq^{\text{fin}} \mathcal{B} : \exists W \text{ open in } Y. p^{-1}(W) \subseteq \bigcup J\}$
$\langle 1 \rangle 3.$ For every $J \in \mathcal{J}$,
LET: $W_J = \bigcup \{W \text{ open in } Y : p^{-1}(W) \subseteq \bigcup J\}$.
PROVE: $\{W_J : J \in \mathcal{J}\}$ is a basis for Y.
$\langle 1 \rangle 4.$ $y \in V$ where V is open in Y
$\langle 1 \rangle 5.$ $\{B \in \mathcal{B} : x \in B \subseteq p^{-1}(V)\}$ covers $p^{-1}(y)$
$\langle 1 \rangle 6.$ PICK a countable subcover $J \subseteq^{\text{fin}} \mathcal{B}$
$\langle 1 \rangle 7.$ $y \in W_J \subseteq V$
$\langle 2 \rangle 1.$ $p^{-1}(y) \subseteq \bigcup J$
$\langle 2 \rangle 2.$ PICK an open neighbourhood W of y such that $p^{-1}(W) \subseteq \bigcup J$
PROOF: By Proposition 9.6.1.
$\langle 2 \rangle 3.$ $W \subseteq W_J$
$\square$

Proposition 13.1.21. *A subspace of a T_1 space is T_1.*

PROOF:
$\langle 1 \rangle 1.$ LET: X be T_1 and $Y \subseteq X$
$\langle 1 \rangle 2.$ LET: $a \in Y$
$\langle 1 \rangle 3.$ $\{a\}$ is closed in X
$\langle 1 \rangle 4.$ $\{a\}$ is closed in Y
PROOF: By Corollary 4.3.4.1.
$\square$

Proposition 13.1.22 (DC)**.** *Not every topological group is normal.*

PROOF: From Proposition 6.5.6. ∎

Theorem 13.1.23. *A subspace of a completely regular space is completely regular.*

PROOF:
⟨1⟩1. LET: X be completely regular and $Y \subseteq X$
⟨1⟩2. LET: $a \in Y$ and A be closed in Y such that $a \notin A$
⟨1⟩3. PICK C closed in X such that $A = X \cap C$
⟨1⟩4. PICK a continuous function $f : X \to [0, 1]$ such that $f(a) = 0$ and $f(C) = \{1\}$
⟨1⟩5. $f \restriction Y : Y \to [0, 1]$ is a continuous function such that $(f \restriction Y)(a) = 0$ and $(f \restriction Y)(A) = \{1\}$
∎

Proposition 13.1.24 (DC)**.** *Every topological group is completely regular.*

PROOF:
⟨1⟩1. LET: G be a topological group
⟨1⟩2. LET: $x \in G$ and $A \subseteq G$ be closed such that $x \notin A$
PROVE: There exists a continuous $f : G \to [0, 1]$ such that $f(x) = 0$ and $f(A) = \{1\}$
⟨1⟩3. ASSUME: w.l.o.g. $x = e$
PROOF: $\lambda y.x^{-1}y$ is an automorphism of G that maps x to e.
⟨1⟩4. PICK a sequence V_n $(n \geq 0)$ of symmetric neighbourhoods of e disjoint from A such that $V_n V_n \subseteq V_{n-1}$ for all n
⟨2⟩1. LET: $V_0 = X \setminus A$
⟨2⟩2. Given V_n, PICK a symmetric neighbourhood V_{n+1} of e such that $V_{n+1}V_{n+1} \subseteq V_n$
PROOF: By Proposition 13.1.4.
⟨1⟩5. For every dyadic rational p, define an open set $U(p)$ as follows:

$$\begin{aligned} U(1/2^n) &= V_n && (n \geq 0) \\ U((2k+1)/2^{n+1}) &= V_{n+1}U(k/2^n) && (0 < k < 2^n) \\ U(p) &= \emptyset && (p \leq 0) \\ U(p) &= G && (p > 1) \end{aligned}$$

⟨1⟩6. For all k and n, we have

$$V_n U(k/2^n) \subseteq U((k+1)/2^n)$$

⟨2⟩1. $k \leq 0$
PROOF: In this case, $V_n U(k/2^n) = \emptyset$
⟨2⟩2. $k = 1$ and $n > 0$
PROOF:

$$\begin{aligned} V_n U(1/2^n) &= V_n V_n \\ &\subseteq V_{n-1} \\ &= U(1/2^{n-1}) \end{aligned}$$

⟨2⟩3. $k = 2a$ for some $0 < a < 2^{n-1}$

PROOF:

$$\begin{aligned} V_n U(2a/2^n) &= V_n U(a/2^{n-1}) \\ &= U(2a+1/2^n) \end{aligned}$$

⟨2⟩4. $k = 2a+1$ for some $0 < a < 2^{n-1}$

PROOF:

$$\begin{aligned} V_n U((2a+1)/2^n) &= V_n V_n U(a/2^{n-1}) \\ &\subseteq V_{n-1} U(a/2^{n-1}) \\ &\subseteq U((a+1)/2^{n-1}) \end{aligned}$$

⟨2⟩5. $k \geq 2^n$

PROOF: In this case, $U((k+1)/2^n) = G$.

⟨1⟩7. Define $f : G \to [0,1]$ by

$$f(x) = \inf\{p : x \in U(p)\}$$

PROOF: This set is nonempty because $x \in U(1)$ and bounded below because if $x \in U(p)$ then $p > 0$.

⟨1⟩8. For $n > 0$ we have $\overline{U(k/2^n)} \subseteq V_n U(k/2^n)$

⟨2⟩1. LET: $x \in \overline{U(k/2^n)}$

⟨2⟩2. $V_n x$ is a neighbourhood of x

⟨2⟩3. PICK $y \in V_n x \cap U(k/2^n)$

⟨2⟩4. PICK $z \in V_n$ such that $y = zx$

⟨2⟩5. $x = z^{-1} y$

⟨1⟩9. For p and q dyadic rationals, if $p < q$ then $\overline{U(p)} \subseteq U(q)$

⟨1⟩10. If $x \in \overline{U(p)}$ then $f(x) \leq p$

⟨2⟩1. For all $q > p$ we have $x \in U(q)$

⟨2⟩2. For all $q > p$ we have $f(x) \leq q$

⟨1⟩11. If $x \notin U(p)$ then $f(x) \geq p$

PROOF: If $x \notin U(p)$ and $x \in U(q)$ then $q > p$.

⟨1⟩12. f is continuous

⟨2⟩1. LET: $x_0 \in X$

⟨2⟩2. LET: $c < f(x_0) < d$

PROVE: There exist a neighbourhood U of x_0 such that $f(U) \subseteq (c,d)$

⟨2⟩3. PICK rational numbers p, q such that $c < p < f(x_0) < q < d$

⟨2⟩4. $x \notin \overline{U(p)}$

⟨2⟩5. $x \in U(q)$

⟨2⟩6. Take $U = U(q) \setminus \overline{U(p)}$

⟨1⟩13. $f(e) = 0$

PROOF: We have $e \in U(1/2^n)$ for all n.

⟨1⟩14. $f(A) = \{1\}$

PROOF: If $x \in A$ and $x \in U(p)$ then $p > 1$.

□

Definition 13.1.25 (Bijection)**.** A function $f : A \to B$ is a *bijection*, $f : A \cong B$, iff there exists a function $f^{-1} : B \to A$, the *inverse* of f, such that $f^{-1} \circ f = \mathrm{id}_A$ and $f \circ f^{-1} = \mathrm{id}_B$.

Theorem 13.1.26. *Let Y be a normal space. Then Y is an absolute retract if and only if Y has the universal extension property.*

PROOF:
⟨1⟩1. If Y is an absolute retract then Y has the universal extension property.
⟨2⟩1. ASSUME: Y is an absolute retract.
⟨2⟩2. LET: X be a normal space, A a closed subspace of X and $f : A \to Y$ a continuous function.
⟨2⟩3. LET: Z_f be the quotient space of $X \cup Y$ under: $a \sim f(a)$ for all $a \in A$
⟨2⟩4. LET: $p : X \cup Y \twoheadrightarrow Z_f$ be the quotient map
⟨2⟩5. For all $x_1, x_2 \in X$ we have $p(x_1) = p(x_2)$ iff $x_1 = x_2$ or $x_1, x_2 in A$ and $f(x_1) = f(x_2)$; and for $x \in X$ and $y \in Y$ we have $p(x) = p(y)$ iff $f(x) = y$; and for $y_1, y_2 \in Y$ we have $p(y_1) = p(y_2)$ iff $y_1 = y_2$
⟨2⟩6. p imbeds Y into a closed subspace of Z_f
⟨3⟩1. p is injective on Y
⟨3⟩2. $p^{-1} : p(Y) \to Y$ is continuous
⟨4⟩1. LET: $U \subseteq Y$ be open
PROVE: $p(U)$ is open
⟨4⟩2. $p^{-1}(p(U)) = f^{-1}(U) \cup U$
⟨3⟩3. $p(Y)$ is closed
PROOF: $p^{-1}(p(Y)) = A \cup Y$
⟨2⟩7. Z_f is normal
⟨3⟩1. Z_f is T_1
PROOF: For $y \in Y$ we have $p^{-1}(y) = f^{-1}(y) \cup \{y\}$ which is closed.
⟨3⟩2. Any two disjoint closed sets in Z_f can be separated by a continuous function.
⟨4⟩1. LET: C and D be disjoint closed sets in Z_f
⟨4⟩2. PICK $g : Y \to [0, 1]$ such that $g(Y \cap p^{-1}(C)) = \{0\}$ and $g(Y \cap p^{-1}(D)) = \{1\}$
PROOF: By the Urysohn Lemma.
⟨4⟩3. PICK $h : X \to [0, 1]$ such that $h(X \cap p^{-1}(C)) = \{0\}$ and $h(X \cap p^{-1}(D)) = \{1\}$ and h agrees with $g \circ f$ on A
PROOF: By the Tietze Extension Theorem applied to $A \cup (X \cap p^{-1}(C)) \cup (X \cap p^{-1}(D))$.
⟨4⟩4. LET: $k : Z_f \to [0, 1]$ be the continuous function such that $k(p(x)) = h(x)$ for $x \in X$ and $k(p(y)) = g(y)$ for $y \in Y$
PROOF: By the Pasting Lemma
⟨4⟩5. $k(C) = \{0\}$
⟨4⟩6. $k(D) = \{1\}$
⟨3⟩3. Q.E.D.
PROOF: If g is such a continuous function then $g^{-1}([0, 1/2))$ and $g^{-1}((1/2, 1])$ are disjoint open sets that include A and B respectively.
⟨2⟩8. PICK a retraction $r : Z_f \to p(Y)$
⟨2⟩9. $p^{-1} \circ r \circ p : X \to Y$ extends f
⟨1⟩2. If Y has the universal extension property then Y is an absolute retract.
⟨2⟩1. ASSUME: Y has the universal extension property
⟨2⟩2. LET: Z be a normal space, Y_0 a closed subspace of Z, and $\phi : Y \cong Y_0$ a homeomorphism
⟨2⟩3. PICK a continuous extension $f : Z \to Y$ of ϕ^{-1}

⟨2⟩4. $\phi \circ f$ is a retraction
∎

Theorem 13.1.27. *Every manifold is metrizable.*

PROOF:
⟨1⟩1. LET: X be an m-manifold.
⟨1⟩2. X is regular.
⟨2⟩1. X is T_1
⟨2⟩2. LET: $x \in X$ and U be a neighbourhood of x
⟨2⟩3. PICK a neighbourhood V of x that is imbeddable in $\mathbb{R}^m$
⟨2⟩4. PICK a neighbourhood W of x such that $\overline{W} \subseteq U \cap V$
PROOF: One exists since V is regular (Proposition 6.3.4)
⟨2⟩5. $x \in W$ and $\overline{W} \subseteq U$
⟨2⟩6. Q.E.D.
PROOF: Proposition 6.3.2
⟨1⟩3. Q.E.D.
PROOF: By the Urysohn Metrization Theorem.
∎

Theorem 13.1.28. *Let X be a compact Hausdorff space in which every point has a neighbourhood that is imbeddable in $\mathbb{R}^m$. Then X is an m-manifold.*

PROOF:
⟨1⟩1. There exists N such that X is imbeddable in $\mathbb{R}^N$
PROOF: Theorem 11.1.3
⟨1⟩2. X is second countable.
PROOF: Proposition 7.3.3
∎

Proposition 13.1.29. *S_Ω is locally metrizable.*

PROOF: For any $\alpha \in S_\Omega$, the neighbourhood $[0, \alpha] = (-\infty, \alpha + 1)$ is imbeddable in $\mathbb{R}$. ∎

Proposition 13.1.30 (DC)**.** *$\overline{S_\Omega}$ is compact.*

PROOF:PROOF:
⟨1⟩1. LET: $\mathcal{A}$ be an open cover of $\overline{S_\Omega}$
⟨1⟩2. ASSUME: for a contradiction there is no finite subcover of $\mathcal{A}$
⟨1⟩3. There exists a sequence of sets $U_n \in \mathcal{A}$ and ordinals α_n such that $\alpha_{n+1} < \alpha_n$ for all n and $\alpha_n \in U_n$ for all n
⟨2⟩1. LET: $\alpha_1 = \Omega$
⟨2⟩2. Given $\alpha_1, \ldots, \alpha_n$ and $U_1, \ldots, U_{n-1}$ with $0 \neq \alpha_n < \alpha_{n-1} < \cdots < \alpha_1$ and $\alpha_i \in U_i$ for $i < n$, PICK $U_n \in \mathcal{A}$ with $\alpha_n \in U_n$
PROOF: By ⟨1⟩1.
⟨2⟩3. PICK $\alpha_{n+1} < \alpha_n$ such that $(\alpha_{n+1}, \alpha_n] \subseteq U_n$
PROOF: By Lemma 4.1.2.
⟨2⟩4. $\alpha_{n+1} \neq 0$

PROOF: If $\alpha_{n+1} = 0$ then $U_1, \ldots, U_n$ cover $\overline{S_\Omega}$, contradicting $\langle 1 \rangle 2$.
$\langle 1 \rangle 4$. Q.E.D.
PROOF: This is a contradiction because the ordinals are well-ordered.
$\square$

Proposition 13.1.31. $\mathbb{R}_l$ *is not limit point compact.*

PROOF: $\mathbb{Z}$ has no limit point. $\square$

Proposition 13.1.32. *Every closed subspace of a Lindelöf space is Lindelöf.*

PROOF:
$\langle 1 \rangle 1$. LET: X be Lindelöf and $A \subseteq X$ be closed
$\langle 1 \rangle 2$. LET: $\mathcal{U}$ be an open covering of A
$\langle 1 \rangle 3$. $\{U$ open in $X : U \cap A \in \mathcal{U}\} \cup \{X \setminus A\}$ covers X
$\langle 1 \rangle 4$. PICK a countable subcovering $\mathcal{V}$
$\langle 1 \rangle 5$. $\{U \cap A : U \in \mathcal{V}, U \neq X \setminus A\}$ is a countable subcover of $\mathcal{U}$
$\square$

Proposition 13.1.33. $\mathbb{R}^\omega$ *is locally connected.*

PROOF:This holds because every basic open set is connected, being the product of a family of connected spaces. $\square$

Proposition 13.1.34. *The space* $\mathbb{R}^\omega$ *under the box topology is not first countable.*

PROOF:
$\langle 1 \rangle 1$. ASSUME: for a contradiction $\{U_n\}_{n \geq 0}$ is a countable basis at 0.
$\langle 1 \rangle 2$. For $n \geq 1$, PICK a basic open set $B_n = \prod_{j=0}^{\infty}(a_{nj}, b_{nj})$ such that $0 \in B_n \subseteq U_n$
$\langle 1 \rangle 3$. $\prod_{n=0}^{\infty}(a_{nn}/2, b_{nn}/2)$ is a neighbourhood of 0 that does not include any U_n
$\square$

Proposition 13.1.35. *The space* $\mathbb{R}^\omega$ *under the box topology is not locally metrizable.*

PROOF:
$\langle 1 \rangle 1$. LET: U be any neighbourhood of 0
$\langle 1 \rangle 2$. LET: A be the set of all sequences in U with all coordinates positive
$\langle 1 \rangle 3$. $0 \in \overline{A}$
$\langle 1 \rangle 4$. There is no sequence of points of A converging to 0.
$\langle 1 \rangle 5$. U is not metrizable.
PROOF: By the Sequence Lemma.
$\square$

Proposition 13.1.36. *For any nonempty set* I*, the space* $\mathbb{R}^I$ *is not limit point compact.*

PROOF: $\mathbb{Z}^I$ is an infinite set with no limit point. $\square$

Proposition 13.1.37. *The space $\mathbb{R}^{[0,1]}$ is separable.*

PROOF: The set D is dense where D is the set of all functions $f : [0,1] \to \mathbb{Q}$ such that there exists a sequence of rationals $0 = q_0 < q_1 < \cdots < q_N = 1$ such that f is constant on $[q_i, q_{i+1})$ for $0 \leq i < N$. □

Proposition 13.1.38. *If J is uncountable then $\mathbb{R}^J$ is not locally metrizable.*

PROOF: Every point has a neighbourhood homeomorphic to $\mathbb{R}^J$. □

Proposition 13.1.39. *The space $\mathbb{R}_K$ is not limit point compact.*

PROOF: The set $\mathbb{Z}$ has no limit point. □

Proposition 13.1.40. *The topologist's sine curve is not locally connected.*

PROOF: There is no connected neighbourhood of $(0,0)$. □

Corollary 13.1.40.1. *Not every metric space is locally connected.*

Corollary 13.1.40.2. *Not every metric space is locally path connected.*

Proposition 13.1.41. *Not every metric space is compact.*

PROOF: The space $\mathbb{R}$ is not compact. □

Proposition 13.1.42. *Every closed subspace of a limit point compact space is limit point compact.*

PROOF:
⟨1⟩1. LET: X be a limit point compact space and $C \subseteq X$ be closed.
⟨1⟩2. LET: $A \subseteq C$ be infinite.
⟨1⟩3. PICK a limit point l of A in X
⟨1⟩4. $l \in C$
  ⟨2⟩1. l is a limt point of C
    PROOF: By Lemma 3.15.2.
  ⟨2⟩2. Q.E.D.
    PROOF: By Corollary 3.15.3.1.
⟨1⟩5. l is a limit point of A in C.
  PROOF: By Proposition 4.3.10.
□

Proposition 13.1.43. *For any part $i : S \hookrightarrow X$ of a set X, we have $\emptyset \subseteq_X i$.*

PROOF: We have $i \circ ¡_S = ¡_X$ by the uniqueness of $¡_X$. □

Theorem 13.1.44. *Let X be a completely regular space. Then there exists a compactification Y of X such that every bounded continuous map $X \to \mathbb{R}$ extends uniquely to a continuous map $Y \to \mathbb{R}$.*

PROOF:
⟨1⟩1. LET: J be the set of all bounded continuous functions $X \to \mathbb{R}$

$\langle 1\rangle 2.$ For $\alpha \in J$,
  LET: $I_\alpha = [\inf \alpha, \sup \alpha]$
$\langle 1\rangle 3.$ LET: $Z = \prod_{\alpha \in J} I_\alpha$
$\langle 1\rangle 4.$ LET: $h : X \to Z$ be defined by

$$h(x)_\alpha = \alpha(x)$$

$\langle 1\rangle 5.$ Z is compact Hausdorff
  $\langle 2\rangle 1.$ Z is compact
    PROOF: By Tychonoff's Theorem.
  $\langle 2\rangle 2.$ Z is Hausdorff
    PROOF: By Theorem 6.2.5
$\langle 1\rangle 6.$ h is an imbedding
  $\langle 2\rangle 1.$ The set J separates points from closed sets
    PROOF: This holds because X is completely regular.
  $\langle 2\rangle 2.$ Q.E.D.
    PROOF: By the Imbedding Theorem.
$\langle 1\rangle 7.$ LET: Y be the compactification of X such that $X \subseteq Y \to Z$ factors h
  PROOF: By Lemma 9.9.2
$\langle 1\rangle 8.$ Every bounded continuous map $X \to \mathbb{R}$ extends uniquely to a continuous map $Y \to \mathbb{R}$
  $\langle 2\rangle 1.$ LET: $\alpha : X \to \mathbb{R}$ be a bounded continuous function
  $\langle 2\rangle 2.$ LET: $k : Y \to Z$ be the imbedding from $\langle 1\rangle 7$
  $\langle 2\rangle 3.$ LET: $\overline{\alpha} = \pi_\alpha \circ k : Y \to \mathbb{R}$
  $\langle 2\rangle 4.$ $\overline{\alpha}$ extends α
    PROOF:For $x \in X$, we have

$$\begin{aligned} \overline{\alpha}(x) &= k(x)_\alpha \\ &= h(x)_\alpha \\ &= \alpha(x) \end{aligned}$$

  $\langle 2\rangle 5.$ If $f : Y \to Z$ is continuous and extends α then $f = \overline{\alpha}$
    PROOF: By Lemma 6.2.9.

$\square$

Lemma 13.1.45. *Every subfamily of a locally finite family is locally finite.*

PROOF: Immediate from the definition. $\square$