

# Topology

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# Chapter 1

## Set Theory

### 1.1 Primitive Notions

Let there be *sets*.

Given sets  $A$  and  $B$ , let there be *functions* from  $A$  to  $B$ . We write  $f : A \rightarrow B$  iff  $f$  is a function from  $A$  to  $B$ .

Given functions  $f : A \rightarrow B$  and  $g : B \rightarrow C$ , let there be a function  $g \circ f : A \rightarrow C$ , the *composite* of  $f$  and  $g$ .

### 1.2 The Axiom of Associativity

**Axiom 1.2.1** (Associativity). Let  $f : A \rightarrow B$ ,  $g : B \rightarrow C$  and  $h : C \rightarrow D$ . Then  $h \circ (g \circ f) = (h \circ g) \circ f : A \rightarrow D$ .

From now on we write  $h \circ g \circ f$  for the composite of  $f$ ,  $g$  and  $h$ , and similarly for more than three functions.

### 1.3 Injective Functions

**Definition 1.3.1** (Injective). A function  $f : A \rightarrow B$  is *injective*,  $f : A \rightarrowtail B$ , iff, for every set  $X$  and functions  $g, h : X \rightarrow A$ , if  $f \circ g = f \circ h$  then  $g = h$ .

**Lemma 1.3.2.** Let  $f : A \rightarrow B$  and  $g : B \rightarrow C$ . If  $f$  and  $g$  are injective then  $g \circ f$  is injective.

PROOF:

$\langle 1 \rangle 1$ . ASSUME:  $f$  and  $g$  are injective.

$\langle 1 \rangle 2$ . LET:  $X$  be a set and  $x, y : X \rightarrow A$

$\langle 1 \rangle 3$ . ASSUME:  $g \circ f \circ x = g \circ f \circ y$

$\langle 1 \rangle 4$ .  $f \circ x = f \circ y$

PROOF:  $g$  is injective ( $\langle 1 \rangle 1$ )

$\langle 1 \rangle 5$ .  $x = y$

PROOF:  $f$  is injective ( $\langle 1 \rangle 1$ )  
 $\square$

**Lemma 1.3.3.** *Let  $f : A \rightarrow B$  and  $g : B \rightarrow C$ . If  $g \circ f$  is injective then  $f$  is injective.*

PROOF:  
 $\langle 1 \rangle 1$ . ASSUME:  $g \circ f$  is injective.  
 $\langle 1 \rangle 2$ . LET:  $X$  be any set and  $x, y : X \rightarrow A$   
 $\langle 1 \rangle 3$ . ASSUME:  $f \circ x = f \circ y$   
 $\langle 1 \rangle 4$ .  $g \circ f \circ x = g \circ f \circ y$   
 $\langle 1 \rangle 5$ .  $x = y$   
 PROOF: Using  $\langle 1 \rangle 1$ .  
 $\square$

## 1.4 Surjective Functions

**Definition 1.4.1** (Surjective). Let  $f : A \rightarrow B$ . Then  $f$  is *surjective*,  $f : A \twoheadrightarrow B$ , iff, for any set  $X$  and functions  $g, h : B \rightarrow X$ , if  $g \circ f = h \circ f$  then  $g = h$ .

**Lemma 1.4.2.** *Let  $f : A \rightarrow B$  and  $g : B \rightarrow C$ . If  $f$  and  $g$  are surjective then  $g \circ f$  is surjective.*

PROOF: Dual to Lemma 1.3.2.  $\square$

**Lemma 1.4.3.** *Let  $f : A \rightarrow B$  and  $g : B \rightarrow C$ . If  $g \circ f$  is surjective then  $g$  is surjective.*

PROOF: Dual to Lemma 1.3.3.  $\square$

## 1.5 Identity Functions

**Axiom 1.5.1** (Identity Function). *For any set  $A$ , there exists a function  $\text{id}_A : A \rightarrow A$ , the identity function on  $A$ , such that:*

**Left Unit Law** *for every set  $B$  and function  $f : B \rightarrow A$  we have  $\text{id}_A \circ f = f : B \rightarrow A$ ;*

**Right Unit Law** *for every set  $B$  and function  $f : A \rightarrow B$  we have  $f \circ \text{id}_A = f : A \rightarrow B$ .*

**Proposition 1.5.2.** *The identity function on a set is unique.*

PROOF: If  $i, j : A \rightarrow A$  are both identity functions, then  
 $i = i \circ j$  (Right Unit Law for  $j$ )  
 $= j$  (Left Unit Law for  $i$ )  
 $: A \rightarrow A$   $\square$

**Proposition 1.5.3.** *Every identity function is injective.*

PROOF: If  $\text{id}_B \circ x = \text{id}_B \circ y$  then  $x = y$  by the Left Unit Law.  $\square$

**Proposition 1.5.4.** *Every identity function is surjective.*

PROOF: Dual.  $\square$

**Definition 1.5.5** (Retraction, Section). Let  $r : A \rightarrow B$  and  $s : B \rightarrow A$ . Then  $r$  is a *retraction* of  $s$ , and  $s$  is a *section* of  $r$ , iff  $r \circ s = \text{id}_B$ .

**Proposition 1.5.6.** *If  $r_1 : A \rightarrow B$  is a retraction of  $s_1 : B \rightarrow A$  and  $r_2 : B \rightarrow C$  is a retraction of  $s_2 : C \rightarrow B$  then  $r_2 \circ r_1$  is a retraction of  $s_1 \circ s_2$ .*

PROOF:

$$\begin{aligned} r_2 \circ r_1 \circ s_1 \circ s_2 &= r_2 \circ \text{id}_B \circ s_2 && (r_1 \text{ is a retraction of } s_1) \\ &= r_2 \circ s_2 && (\text{Unit Laws}) \\ &= \text{id}_C && (r_2 \text{ is a retraction of } s_2) \end{aligned}$$

$\square$

**Proposition 1.5.7.** *Every retraction is surjective.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $r : A \rightarrow B$  be a retraction of  $s$
- $\langle 1 \rangle 2$ . LET:  $X$  be a set and  $x, y : B \rightarrow X$  with  $x \circ r = y \circ r$
- $\langle 1 \rangle 3$ .  $x \circ r \circ s = y \circ r \circ s$
- $\langle 1 \rangle 4$ .  $x = y$

$\square$

**Proposition 1.5.8.** *Every section is injective.*

PROOF: Dual.  $\square$

**Proposition 1.5.9.** *Every identity function is a retraction of itself.*

PROOF: Immediate from the Unit Laws.  $\square$

**Proposition 1.5.10.** *If  $r : B \rightarrow A$  is a retraction of  $f : A \rightarrow B$  and  $s$  is a section of  $f$  then  $r = s$ .*

PROOF:

$$\begin{aligned} r &= r \circ \text{id}_B && (\text{Right Unit Law}) \\ &= r \circ f \circ s && (s \text{ is a section of } f) \\ &= \text{id}_A \circ s && (r \text{ is a retraction of } f) \\ &= s && (\text{Left Unit Law}) \end{aligned}$$

### 1.5.1 Isomorphisms

**Definition 1.5.11** (Isomorphism). Let  $A$  and  $B$  be sets. A function  $i : A \rightarrow B$  is an *isomorphism* between  $A$  and  $B$ ,  $i : A \cong B$ , iff there exists a function  $i^{-1} : B \rightarrow A$ , the *inverse* to  $i$ , that is a section and a retraction of  $i$ .

**Proposition 1.5.12.** *The inverse of an isomorphism is unique.*

PROOF: Immediate from Proposition 1.5.10.  $\square$

**Proposition 1.5.13.** *Every isomorphism is injective.*

PROOF: Immediate from Proposition 1.5.8.  $\square$

**Proposition 1.5.14.** *Every isomorphism is surjective.*

PROOF: Immediate from Proposition 1.5.7.  $\square$

**Proposition 1.5.15.** *Every identity function is an isomorphism and is its own inverse.*

PROOF: Immediate from Proposition 1.5.9.  $\square$

**Proposition 1.5.16.** *If  $i : A \cong B$  is an isomorphism then  $i^{-1} : B \cong A$  is an isomorphism and  $(i^{-1})^{-1} = i$ .*

PROOF: Immediate from the definition of isomorphism.  $\square$

**Proposition 1.5.17.** *If  $i : A \cong B$  and  $j : B \cong C$  then  $j \circ i : A \cong C$  and  $(j \circ i)^{-1} = i^{-1} \circ j^{-1}$ .*

PROOF: Immediate from Proposition 1.5.6.  $\square$

### 1.5.2 Parts of a Set

**Definition 1.5.18** (Part). A *part*  $S$  of a set  $A$  consists of:

- a set  $\text{dom } S$ ;
- an injective function  $i : S \hookrightarrow A$

**Definition 1.5.19.** Two parts  $i : S \hookrightarrow A$ ,  $j : T \hookrightarrow A$  are *equivalent*,  $i \equiv_A j$ , iff there exists an isomorphism  $\phi : S \cong T$  such that  $i = j \circ \phi$ .

**Proposition 1.5.20.** *Any part of a set is equivalent to itself.*

PROOF:

$\langle 1 \rangle 1.$  LET:  $i : S \hookrightarrow A$  be a part of  $A$ .

PROVE:  $i \equiv_A i$

$\langle 1 \rangle 2.$   $\text{id}_S : S \cong S$

PROOF: By Proposition 1.5.15

$\langle 1 \rangle 3.$   $i = i \circ \text{id}_S$

PROOF: By the Right Unit Law.



□

**Proposition 1.5.21.** *If  $i \equiv_A j$  then  $j \equiv_A i$ .*

PROOF:

⟨1⟩1. LET:  $i : S \hookrightarrow A$  and  $j : T \hookrightarrow A$

⟨1⟩2. ASSUME:  $i \equiv_A j$

⟨1⟩3. PICK an isomorphism  $\phi : S \cong T$  such that  $i = j \circ \phi$

⟨1⟩4.  $\phi^{-1} : T \cong S$

PROOF: By Proposition 1.5.16.

⟨1⟩5.  $j = i \circ \phi^{-1}$

PROOF: Compose both sides of ⟨1⟩3 with  $\phi^{-1}$ .

□

**Proposition 1.5.22.** *If  $i \equiv_A j$  and  $j \equiv_A k$  then  $i \equiv_A k$ .*

PROOF:

⟨1⟩1. LET:  $i : R \hookrightarrow A$ ,  $j : S \hookrightarrow A$  and  $k : T \hookrightarrow A$

⟨1⟩2. PICK isomorphisms  $\phi : R \cong S$  and  $\psi : S \cong T$  such that  $i = j \circ \phi$  and  $j = k \circ \psi$

⟨1⟩3.  $\psi \circ \phi : R \cong T$

PROOF: By Proposition 1.5.17.

⟨1⟩4.  $i = k \circ \psi \circ \phi$

□

**Definition 1.5.23** (Inclusion). Let  $i : U \hookrightarrow A$  and  $j : V \hookrightarrow A$  be parts of  $A$ . Then  $i$  is *included* in  $j$ ,  $i \subseteq_A j$ , iff there exists a function  $\phi : U \rightarrow V$  such that  $i = j \circ \phi$ .

**Proposition 1.5.24.** *If  $i \equiv_A i'$  and  $j \equiv_A j'$  and  $i \subseteq_A j$  then  $i' \subseteq_A j'$ .*

PROOF:

⟨1⟩1. LET:  $i : S \hookrightarrow A$ ,  $i' : S' \hookrightarrow A$ ,  $j : T \hookrightarrow A$ ,  $j' : T' \hookrightarrow A$

⟨1⟩2. PICK  $\phi : S \cong S'$ ,  $\psi : T \cong T'$  and  $\chi : S \rightarrow T$  such that  $i = i' \circ \phi$ ,  $j = j' \circ \psi$  and  $i = j \circ \chi$

⟨1⟩3.  $\psi \circ \chi \circ \phi^{-1} : S' \rightarrow T'$

⟨1⟩4.  $i' = j' \circ \psi \circ \chi \circ \phi^{-1}$

□

**Proposition 1.5.25.** *For any part  $i$  of  $A$  we have  $i \subseteq_A i$ .*

PROOF:

⟨1⟩1. LET:  $i : S \hookrightarrow A$

⟨1⟩2.  $\text{id}_S : S \rightarrow S$

⟨1⟩3.  $i = i \circ \text{id}_S$

□

**Proposition 1.5.26.** *If  $i \subseteq_A j$  and  $j \subseteq_A k$  then  $i \subseteq_A k$ .*

PROOF:

- ⟨1⟩1. LET:  $i : R \hookrightarrow A$ ,  $j : S \hookrightarrow A$  and  $k : T \hookrightarrow A$
  - ⟨1⟩2. PICK  $\phi : R \rightarrow S$  and  $\psi : S \rightarrow T$  such that  $i = j \circ \phi$  and  $j = k \circ \psi$
  - ⟨1⟩3.  $\psi \circ \phi : R \rightarrow T$
  - ⟨1⟩4.  $i = k \circ \psi \circ \phi$
- 

**Proposition 1.5.27.** *If  $i \subseteq_A j$  and  $j \subseteq_A i$  then  $i \equiv_A j$ .*

PROOF:

- ⟨1⟩1. LET:  $i : R \hookrightarrow A$ ,  $j : S \hookrightarrow A$
- ⟨1⟩2. PICK  $\phi : R \rightarrow S$  and  $\phi^{-1} : S \rightarrow R$  such that  $i = j \circ \phi$  and  $j = i \circ \phi^{-1}$
- ⟨1⟩3.  $\phi \circ \phi^{-1} = \text{id}_S$ 
  - ⟨2⟩1.  $j \circ \phi \circ \phi^{-1} = j$
  - ⟨2⟩2. Q.E.D.

PROOF: The result follows because  $j$  is injective.

- ⟨1⟩4.  $\phi^{-1} \circ \phi = \text{id}_R$

PROOF: Similar.

□

### 1.5.3 The Empty Set

**Axiom 1.5.28** (Empty Set). *There exists a set  $\emptyset$ , the empty set, such that, for every set  $X$ , there exists a unique function  $\text{id}_X : \emptyset \rightarrow X$ .*

**Proposition 1.5.29** (Uniqueness of Empty Set). *Let  $E$  be any set. Then  $E$  is empty if and only if there exists an isomorphism  $E \cong \emptyset$ , in which case the isomorphism is unique.*

PROOF:

- ⟨1⟩1. If  $E$  is empty then  $E \cong \emptyset$ 
  - ⟨2⟩1. ASSUME:  $E$  is empty
  - ⟨2⟩2. LET:  $\phi$  be the unique function  $E \rightarrow \emptyset$
  - ⟨2⟩3.  $\text{id}_E \circ \phi = \text{id}_E$ 
    - PROOF: There is only one function  $E \rightarrow E$ .
  - ⟨2⟩4.  $\phi \circ \text{id}_E = \text{id}_\emptyset$ 
    - PROOF: There is only one function  $\emptyset \rightarrow \emptyset$ .
- ⟨1⟩2. If  $E \cong \emptyset$  then  $E$  is empty
  - ⟨2⟩1. LET:  $\phi : E \cong \emptyset$
  - ⟨2⟩2. LET:  $X$  be a set
    - PROVE: There is a unique function  $E \rightarrow X$
  - ⟨2⟩3.  $\text{id}_X \circ \phi : E \rightarrow X$
  - ⟨2⟩4. If  $f : E \rightarrow X$  then  $f = \text{id}_X \circ \phi$ 
    - ⟨3⟩1. LET:  $f : E \rightarrow X$
    - ⟨3⟩2.  $f \circ \phi^{-1} : \emptyset \rightarrow X$
    - ⟨3⟩3.  $f \circ \phi^{-1} = \text{id}_X$ 
      - PROOF: Uniqueness of  $\text{id}_X$ .
    - ⟨3⟩4. Q.E.D.

⟨1⟩3. There is at most one isomorphism  $E \cong \emptyset$

PROOF: This holds because there is at most one function  $E \rightarrow \emptyset$ .

□

**Proposition 1.5.30.**

$$i_{\emptyset} = \text{id}_{\emptyset}$$

PROOF: By the uniqueness of  $i_{\emptyset}$ . □

#### 1.5.4 The Terminal Set

**Axiom 1.5.31** (Terminal Set). *There exists a set  $1$ , the terminal set, such that, for every set  $X$ , there exists a unique function  $!_X : X \rightarrow 1$ .*

**Proposition 1.5.32** (Uniqueness of Terminal Set). *Let  $T$  be any set. Then  $T$  is terminal if and only if there exists an isomorphism  $T \cong 1$ , in which case the isomorphism is unique.*

PROOF: Dual to Proposition 1.5.29.

**Proposition 1.5.33.**

$$!_1 = \text{id}_1$$

PROOF: From the uniqueness of  $!_1$ . □

**Definition 1.5.34** (Element). An *element* of a set  $A$  is a function  $1 \rightarrow A$ . We write  $a \in A$  for  $a : 1 \rightarrow A$ . We write  $f(a)$  for  $f \circ a$  when  $f : A \rightarrow B$  and  $a \in A$ .

**Axiom 1.5.35** (Extensionality). *Let  $A$  and  $B$  be sets and  $f, g : A \rightarrow B$  be functions. If, for all  $a \in A$ , we have  $f(a) = g(a) \in B$ , then  $f = g$ .*

**Proposition 1.5.36.** *Let  $f : A \rightarrow B$ . Then  $f$  is injective if and only if, for all  $x, y \in A$ , if  $f(x) = f(y) \in B$  then  $x = y \in A$ .*

PROOF:

⟨1⟩1. If  $f$  is injective and  $f(x) = f(y) \in B$  then  $x = y \in A$

PROOF: Immediate from the definition of injective.

⟨1⟩2. If, for all  $x, y \in A$ , if  $f(x) = f(y) \in B$  then  $x = y \in A$

⟨2⟩1. ASSUME: For all  $x, y \in A$ , if  $f(x) = f(y)$ , then  $x = y$

⟨2⟩2. LET:  $X$  be any set and  $g, h : X \rightarrow A$  with  $f \circ g = f \circ h$

PROVE:  $g = h$

⟨2⟩3. LET:  $x \in X$

PROVE:  $g(x) = h(x)$

⟨2⟩4.  $f(g(x)) = f(h(x))$

PROOF: From ⟨2⟩2.

⟨2⟩5.  $g(x) = h(x)$

PROOF: By ⟨2⟩1

□

**Proposition 1.5.37.** *Any element  $e \in X$  is a section of the unique function  $!_X : X \rightarrow 1$ .*

PROOF:  $!_X \circ e = \text{id}_1$  because there is only one function  $1 \rightarrow 1$ .  $\square$

**Axiom 1.5.38** (Non-degeneracy). *The empty set  $\emptyset$  has no elements.*

**Proposition 1.5.39.** *For any set  $X$ , the function  $!_X : \emptyset \rightarrow X$  is injective.*

PROOF: From Proposition 1.5.36.  $\square$

**Definition 1.5.40** (Empty Part). For any set  $X$ , the *empty part* of  $X$  is  $\emptyset = !_X : \emptyset \hookrightarrow X$ .

**Definition 1.5.41** (Constant Function). A function  $f : A \rightarrow B$  is *constant* iff there exists  $b \in B$  such that  $f = b \circ !_A$ .

**Definition 1.5.42** (Membership). Let  $i : U \hookrightarrow A$  be a part of  $A$  and  $a \in A$ . Then  $a$  is a *member* of  $i$ ,  $a \in_A i$ , iff there exists  $\bar{a} \in U$  such that  $i(\bar{a}) = a$ .

**Proposition 1.5.43.** *Let  $A$  be a set. Let  $i, j$  be parts of  $A$  and  $a \in A$ . If  $a \in_A i$  and  $i \subseteq_A j$  then  $a \in_A j$ .*

PROOF:

$\langle 1 \rangle 1$ . PICK  $\bar{a} \in \text{dom } i$  such that  $a = i(\bar{a})$ .

$\langle 1 \rangle 2$ . PICK  $\phi : \text{dom } i \rightarrow \text{dom } j$  such that  $i = j \circ \phi$

$\langle 1 \rangle 3$ .  $a = j(\phi(\bar{a}))$

$\square$

## 1.5.5 Products

**Axiom 1.5.44** (Products). *For any sets  $A$  and  $B$ , there exists a set  $A \times B$ , the product of  $A$  and  $B$ , and functions  $\pi_1 : A \times B \rightarrow A$ ,  $\pi_2 : A \times B \rightarrow B$ , the projections, such that, for any set  $C$  and functions  $f : C \rightarrow A$ ,  $g : C \rightarrow B$ , there exists a unique function  $\langle f, g \rangle : C \rightarrow A \times B$  such that*

$$\pi_1 \circ \langle f, g \rangle = f; \quad \pi_2 \circ \langle f, g \rangle = g .$$

**Definition 1.5.45.** Given functions  $f : A \rightarrow B$  and  $g : C \rightarrow D$ , define  $f \times g : A \times C \rightarrow B \times D$  by

$$f \times g = \langle f \circ \pi_1, g \circ \pi_2 \rangle$$

## 1.5.6 Coproducts

**Axiom 1.5.46** (Coproducts). *For any sets  $A$  and  $B$ , there exists a set  $A \uplus B$ , the coproduct or sum of  $A$  and  $B$ , and functions  $\kappa_1 : A \rightarrow A \uplus B$ ,  $\kappa_2 : B \rightarrow A \uplus B$ , the injections, such that, for any set  $C$  and functions  $f : A \rightarrow C$ ,  $g : B \rightarrow C$ , there exists a unique function  $[f, g] : A \uplus B \rightarrow C$  such that*

$$[f, g] \circ \kappa_1 = f; \quad [f, g] \circ \kappa_2 = g .$$

**Definition 1.5.47** (Complement). Let  $i : I \hookrightarrow J$  and  $i' : I' \hookrightarrow J$  be parts of  $J$ . Then  $i'$  is the *complement* of  $i$  iff  $J$  is the sum of  $I$  and  $I'$  with injections  $i$  and  $i'$ .

### 1.5.7 Equalizers

**Axiom 1.5.48** (Equalizers). *For any sets  $A$  and  $B$  and functions  $f, g : A \rightarrow B$ , there exists a set  $E$  and function  $e : E \rightarrow A$ , the equalizer of  $A$  and  $B$ , such that:*

- $f \circ e = g \circ e : E \rightarrow B$ ;
- For any set  $C$  and function  $h : C \rightarrow A$  such that  $f \circ h = g \circ h$ , there exists a unique function  $\bar{h} : C \rightarrow E$  such that  $h = e \circ \bar{h}$ .

**Proposition 1.5.49.** *All equalizers are injective.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $e : E \rightarrow A$  be the equalizer of  $f, g : A \rightarrow B$

$\langle 1 \rangle 2$ . LET:  $x, y : X \rightarrow E$  with  $e \circ x = e \circ y$

$\langle 1 \rangle 3$ .  $f \circ e \circ x = g \circ e \circ x$

PROOF:  $f \circ e = g \circ e$  by  $\langle 1 \rangle 1$ .

$\langle 1 \rangle 4$ .  $x = y$

PROOF:  $x$  and  $y$  are both the unique  $z : X \rightarrow E$  such that  $e \circ z = e \circ x$ .

□

### 1.5.8 Coequalizers

**Axiom 1.5.50** (Coequalizers). *For any sets  $A$  and  $B$  and functions  $f, g : A \rightarrow B$ , there exists a set  $C$  and function  $c : B \rightarrow C$ , the coequalizer of  $f$  and  $g$ , such that:*

- $c \circ f = c \circ g : A \rightarrow C$
- For any set  $X$  and function  $h : B \rightarrow X$  such that  $h \circ f = h \circ g$ , there exists a unique function  $\bar{h} : C \rightarrow X$  such that  $\bar{h} \circ c = h$ .

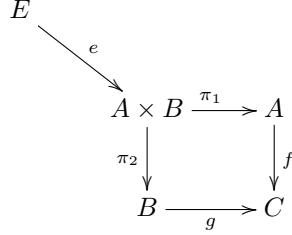
### 1.5.9 Pullbacks

**Definition 1.5.51** (Pullback). The diagram below is a *pullback diagram* iff:

- $f \circ p = g \circ q$
- for every set  $X$  and functions  $x : X \rightarrow B$  and  $y : X \rightarrow C$  such that  $f \circ x = g \circ y$ , there exists a unique function  $\langle x, y \rangle : X \rightarrow A$  such that  $p \circ \langle x, y \rangle = x$  and  $q \circ \langle x, y \rangle = y$ .

$$\begin{array}{ccc} A & \xrightarrow{p} & B \\ q \downarrow & & \downarrow f \\ C & \xrightarrow{g} & D \end{array}$$

**Proposition 1.5.52.** *Let  $f : A \rightarrow C$  and  $g : B \rightarrow C$ . Then  $f$  and  $g$  have a pullback.*



PROOF:

- <1>1. Construct the product  $\pi_1 : A \times B \rightarrow A$ ,  $\pi_2 : A \times B \rightarrow B$ .
- <1>2. Construct the equalizer  $e : E \rightarrow A \times B$  of  $f \circ \pi_1$  and  $g \circ \pi_2$ .  
PROVE:  $\pi_1 \circ e$  and  $\pi_2 \circ e$  form a pullback of  $f$  and  $g$
- <1>3.  $f \circ \pi_1 \circ e = g \circ \pi_2 \circ e$
- <1>4. LET:  $X$  be a set and  $x : X \rightarrow A$ ,  $y : X \rightarrow B$  satisfy  $f \circ x = g \circ y$
- <1>5.  $f \circ \pi_1 \circ \langle x, y \rangle = g \circ \pi_2 \circ \langle x, y \rangle$
- <1>6. LET:  $m : X \rightarrow E$  be the function such that  $e \circ m = \langle x, y \rangle$
- <1>7.  $\pi_1 \circ e \circ m = x$  and  $\pi_2 \circ e \circ m = y$
- <1>8.  $m$  is unique.

PROOF:

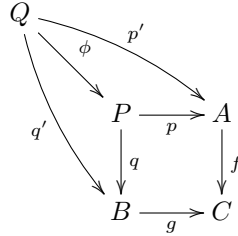
- <2>1. LET:  $n : X \rightarrow E$  be such that  $\pi_1 \circ e \circ n = x$  and  $\pi_2 \circ e \circ n = y$
- <2>2.  $e \circ n = \langle x, y \rangle$
- <2>3.  $n = m$

PROOF: By <1>6

□

**Proposition 1.5.53.** *Pullbacks are unique up to isomorphism.*

That is, let  $P$  be a pullback of  $f : A \rightarrow C$  and  $g : B \rightarrow C$  with projections  $p : P \rightarrow A$  and  $q : P \rightarrow B$ . Let  $Q$  be a set and  $p' : Q \rightarrow A$ ,  $q' : Q \rightarrow B$ . Then  $Q$  is a pullback of  $f$  and  $g$  with projections  $p'$  and  $q'$  if and only if there exists a bijection  $\phi : Q \cong P$  such that  $p \circ \phi = p'$  and  $q \circ \phi = q'$ , in which case  $\phi$  is unique.



PROOF:

- <1>1. If  $Q$  is a pullback then there exists a bijection  $\phi : Q \cong P$  such that  $p \circ \phi = p'$  and  $q \circ \phi = q'$
- <2>1. ASSUME:  $Q$  is a pullback with projections  $p'$  and  $q'$

- (2)2. LET:  $\phi : Q \rightarrow P$  be the unique function such that  $p \circ \phi = p'$  and  $q \circ \phi = q'$   
PROOF: Such a  $\phi$  exists because  $f \circ p' = g \circ q'$ .
- (2)3. LET:  $\phi^{-1} : P \rightarrow Q$  be the unique function such that  $p' \circ \phi^{-1} = p$  and  $q' \circ \phi^{-1} = q$   
PROOF: Such a function exists because  $f \circ p = g \circ q$ .
- (2)4.  $\phi \circ \phi^{-1} = \text{id}_P$   
PROOF: Each is the unique function  $x$  such that  $p \circ x = p$  and  $q \circ x = q$ .
- (2)5.  $\phi^{-1} \circ \phi = \text{id}_Q$   
PROOF: Similar.
- (1)2. If  $\phi : Q \cong P$  is a bijection then  $Q$  is a pullback with projections  $p \circ \phi$  and  $q \circ \phi$   
(2)1.  $f \circ p \circ \phi = g \circ q \circ \phi$   
PROOF: This holds because  $f \circ p = g \circ q$
- (2)2. For any set  $X$  and functions  $x : X \rightarrow A$ ,  $y : X \rightarrow B$  such that  $f \circ x = g \circ y$ , there exists a unique function  $m : X \rightarrow Q$  such that  $p \circ \phi \circ m = x$  and  $q \circ \phi \circ m = y$   
PROOF:

$$\begin{aligned} p \circ \phi \circ m &= x \text{ and } q \circ \phi \circ m = y \\ \Leftrightarrow \phi \circ m &= \langle x, y \rangle \\ \Leftrightarrow m &= \phi^{-1} \circ \langle x, y \rangle \end{aligned}$$

- (1)3. If  $\phi, \phi' : P \cong Q$  are bijections such that  $p \circ \phi = p \circ \phi'$  and  $q \circ \phi = q \circ \phi'$   
PROOF: This follows from the definition of pullback.

□

**Proposition 1.5.54.** *The pullback of an injective function is injective.*

*That is, if the diagram below is a pullback diagram and  $f$  is injective then  $q$  is injective.*

$$\begin{array}{ccc} A & \xrightarrow{p} & B \\ q \downarrow & & \downarrow f \\ C & \xrightarrow{g} & D \end{array}$$

PROOF:

- (1)1. LET:  $X$  be a set and  $x, y : X \rightarrow A$  with  $q \circ x = q \circ y$   
(1)2.  $f \circ p \circ x = g \circ q \circ x$   
(1)3. LET:  
 $z : X \rightarrow A$  be the function such that  $p \circ z = p \circ x$  and  $q \circ z = q \circ x$   
(1)4.  $z = x$   
(1)5.  $z = y$   
(2)1.  $q \circ x = q \circ y$   
PROOF: By (1)1.  
(2)2.  $f \circ p \circ x = f \circ p \circ y$

PROOF:

$$\begin{aligned}
 f \circ p \circ x &= g \circ q \circ x & (\langle 1 \rangle 2) \\
 &= g \circ q \circ y & (\langle 1 \rangle 1) \\
 &= f \circ p \circ y & (\text{the diagram is a pullback})
 \end{aligned}$$

$\langle 2 \rangle 3$ .  $p \circ x = p \circ y$

PROOF:  $f$  is injective.

□

### 1.5.10 Function Sets

**Axiom 1.5.55** (Function Sets). *For any sets  $A$  and  $B$ , there exists a set  $A^B$  and a function  $\epsilon : A^B \times B \rightarrow A$ , the evaluation function, such that, for any set  $C$  and function  $f : C \times B \rightarrow A$ , there exists a unique function  $\lambda f : C \rightarrow A^B$  such that*

$$\epsilon \circ (\lambda f \times \text{id}_B) = f .$$

### 1.5.11 The Subset Classifier

**Definition 1.5.56.** The set  $2$  is  $1 + 1$ . We write  $\top$  (*truth*) for  $\kappa_1 : 1 \rightarrow 2$ , and  $\perp$  (*falsehood*) for  $\kappa_2 : 1 \rightarrow 2$ .

**Axiom 1.5.57** (Subset Classifier). *For every injective function  $m : A \rightarrowtail B$ , there exists a unique function  $\chi_m : B \rightarrow 2$ , the characteristic function of  $m$ , such that the following diagram is a pullback diagram:*

$$\begin{array}{ccc}
 A & \xrightarrow{!} & 1 \\
 m \downarrow & & \downarrow \top \\
 B & \xrightarrow{\chi_m} & 2
 \end{array}$$

**Proposition 1.5.58.** *Every function  $\phi : A \rightarrow 2$  is the characteristic function of a part of  $A$ .*

PROOF:

$\langle 1 \rangle 1$ . Construct a pullback

$$\begin{array}{ccc}
 I & \xrightarrow{\quad} & 1 \\
 q \downarrow & & \downarrow \top \\
 A & \xrightarrow[\phi]{} & 2
 \end{array}$$

PROOF: By Proposition 1.5.52.

$\langle 1 \rangle 2$ .  $q$  is injective

PROOF: By Proposition 1.5.54.

□

**Axiom 1.5.59** (Boolean). *For any  $p \in 2$  we have  $p = \top$  or  $p = \perp$ .*



**Proposition 1.5.60.** *Let  $i : U \hookrightarrow A$  and  $j : V \hookrightarrow A$  be parts of  $A$ . Then the following are equivalent:*

1.  $i \subseteq_A j$  and  $j \subseteq_A i$
2. There exist  $h : U \rightarrow V$  and  $k : V \rightarrow U$  such that  $i = j \circ h$ ,  $j = i \circ k$ ,  $k \circ h = \text{id}_U$  and  $h \circ k = \text{id}_V$ .
3. The characteristic function of  $i$  is the characteristic function of  $j$ .

PROOF:

$\langle 1 \rangle 1. 1 \Rightarrow 2$

$\langle 2 \rangle 1.$  ASSUME:  $i \subseteq_A j$  and  $j \subseteq_A i$

$\langle 2 \rangle 2.$  LET:  $h : U \rightarrow V$  be such that  $i = j \circ h$

$\langle 2 \rangle 3.$  LET:  $k : V \rightarrow U$  be such that  $j = i \circ k$

$\langle 2 \rangle 4.$   $k \circ h = \text{id}_U$

$\langle 3 \rangle 1.$   $i \circ k \circ h = i$

PROOF: From  $\langle 2 \rangle 2$  and  $\langle 2 \rangle 3$ .

$\langle 3 \rangle 2.$  Q.E.D.

PROOF: Since  $i$  is injective.

$\langle 2 \rangle 5.$   $h \circ k = \text{id}_V$

PROOF: Similar.

$\langle 1 \rangle 2. 2 \Rightarrow 1$

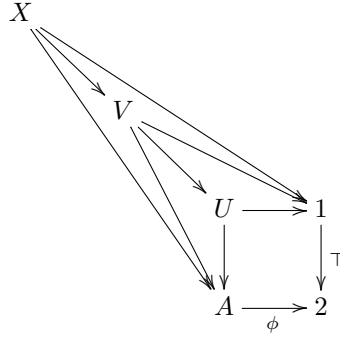
PROOF: Trivial.

$\langle 1 \rangle 3. 2 \Rightarrow 3$

$\langle 2 \rangle 1.$  ASSUME: 2

$\langle 2 \rangle 2.$  LET:  $\phi : A \rightarrow 2$  be the characteristic function of  $i$

PROVE:  $\phi$  is the characteristic function of  $j$



PROOF: By the Subset Classifier Axiom.

$\langle 2 \rangle 3.$  LET:  $X$  be a set and  $x : X \rightarrow 1$ ,  $y : X \rightarrow A$  satisfy  $\phi \circ y = \top \circ x$

$\langle 2 \rangle 4.$  LET:  $\langle x, y \rangle : X \rightarrow U$  be the unique function such that  $! \circ \langle x, y \rangle = x$  and

$$i \circ \langle x, y \rangle = y$$

PROOF: By  $\langle 2 \rangle 2$ .

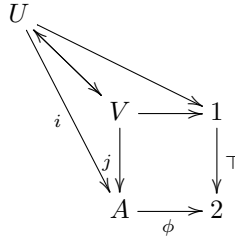
$\langle 2 \rangle 5.$   $h \circ \langle x, y \rangle$  is the unique function  $X \rightarrow V$  such that  $! \circ h \circ \langle x, y \rangle = x$  and

$$j \circ h \circ \langle x, y \rangle = y$$

$\langle 3 \rangle 1.$   $! \circ h \circ \langle x, y \rangle = x$

PROOF: Since 1 is terminal.

$\langle 3 \rangle 2. j \circ h \circ \langle x, y \rangle = y$   
 PROOF: From  $\langle 2 \rangle 1$  and  $\langle 2 \rangle 4$ .  
 $\langle 3 \rangle 3.$  If  $! \circ f = x$  and  $j \circ f = y$  then  $f = h \circ \langle x, y \rangle$   
 $\langle 4 \rangle 1.$  LET:  $f : X \rightarrow V$  satisfy  $! \circ f = x$  and  $j \circ f = y$   
 $\langle 4 \rangle 2. ! \circ k \circ f = x$   
 PROOF: As 1 is terminal.  
 $\langle 4 \rangle 3. i \circ k \circ f = y$   
 PROOF: From  $\langle 2 \rangle 1$  and  $\langle 4 \rangle 1$ .  
 $\langle 4 \rangle 4. k \circ f = \langle x, y \rangle$   
 PROOF: From  $\langle 2 \rangle 4$ ,  $\langle 4 \rangle 2$  and  $\langle 4 \rangle 3$ .  
 $\langle 4 \rangle 5. f = h \circ \langle x, y \rangle$   
 PROOF: From  $\langle 2 \rangle 1$  and  $\langle 4 \rangle 4$ .  
 $\langle 1 \rangle 4. 3 \Rightarrow 2$



$\langle 2 \rangle 1.$  ASSUME: 3  
 $\langle 2 \rangle 2.$  LET:  $\phi$  be the characteristic function of  $i$  and  $j$   
 $\langle 2 \rangle 3.$  LET:  $h : U \rightarrow V$  be the unique function such that  $! \circ h = !$  and  $j \circ h = i$   
 $\langle 3 \rangle 1. \top \circ ! = \phi \circ i$   
 PROOF: This holds because  $\phi$  is the characteristic function of  $i$ .  
 $\langle 3 \rangle 2.$  Q.E.D.  
 PROOF: Since  $\phi$  is the characteristic function of  $j$ .  
 $\langle 2 \rangle 4.$  LET:  $k : V \rightarrow U$  be the unique function such that  $! \circ k = !$  and  $i \circ k = j$   
 PROOF: Similar.  
 $\langle 2 \rangle 5. k \circ h = \text{id}_U$   
 PROOF: Each is the unique function  $f$  such that  $! \circ f = !$  and  $i \circ f = i$   
 $\langle 2 \rangle 6. h \circ k = \text{id}_V$   
 PROOF: Each is the unique function  $f$  such that  $! \circ f = !$  and  $j \circ f = j$   
 $\square$

## 1.6 The Basics

**Lemma 1.6.1.** *Let  $X$  be a set,  $\mathcal{B} \subseteq \mathcal{P}X$  and  $U \subseteq X$ . Then the following are equivalent:*

1. *For all  $x \in U$  there exists  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$ .*
2. *There exists  $\mathcal{B}_0 \subseteq \mathcal{B}$  such that  $U = \bigcup \mathcal{B}_0$ .*

PROOF:

$\langle 1 \rangle 1. 1 \Rightarrow 2$

PROOF: If 1 is true then  $U = \bigcup \{B \in \mathcal{B} : B \subseteq U\}$ .

$\langle 1 \rangle 2. 2 \Rightarrow 1$

PROOF: Trivial.

□

**Definition 1.6.2** (Fixed Point). Let  $X$  be a set,  $f : X \rightarrow X$ , and  $x \in X$ . Then  $x$  is a *fixed point* of  $f$  iff  $f(x) = x$ .

**Definition 1.6.3** (Saturated). Let  $X, Y$  be sets and  $p : X \rightarrow Y$  be a surjective function. Let  $C \subseteq X$ . Then  $C$  is *saturated* with respect to  $p$  iff, for all  $x, x' \in X$ , if  $x \in C$  and  $p(x) = p(x')$  then  $x' \in C$ .

**Definition 1.6.4** (Cover). Let  $A$  be a set and  $\mathcal{C} \subseteq \mathcal{P}A$ . Then  $\mathcal{C}$  *covers*  $A$  iff  $\bigcup \mathcal{C} = A$ .

**Definition 1.6.5** (Finite Intersection Property). Let  $X$  be a set and  $\mathcal{C} \subseteq \mathcal{P}X$ . Then  $\mathcal{C}$  has the *finite intersection property* if and only if every finite nonempty subset of  $\mathcal{C}$  has nonempty intersection.

**Lemma 1.6.6** (AC). *Let  $X$  be a set and  $\mathcal{A} \subseteq \mathcal{P}X$  have the finite intersection property. Then there exists a maximal  $\mathcal{D} \subseteq \mathcal{P}X$  that has the finite intersection property and includes  $\mathcal{A}$ .*

PROOF: A straightforward application of Zorn's lemma, since the union of a chain of sets that has the finite intersection property has the finite intersection property. □

**Lemma 1.6.7.** *Let  $X$  be a set and  $\mathcal{D} \subseteq \mathcal{P}X$  be maximal with respect to the finite intersection property. Then any finite intersection of elements of  $\mathcal{D}$  is an element of  $\mathcal{D}$ .*

PROOF:

$\langle 1 \rangle 1.$  LET:  $A$  be a finite intersection of elements of  $\mathcal{D}$

$\langle 1 \rangle 2.$   $\mathcal{D} \cup \{A\}$  has the finite intersection property.

$\langle 1 \rangle 3.$   $\mathcal{D} \cup \{A\} = \mathcal{D}$

□

**Lemma 1.6.8.** *Let  $X$  be a set and  $\mathcal{D} \subseteq \mathcal{P}X$  be maximal with respect to the finite intersection property. If  $A \subseteq X$  intersects every element of  $\mathcal{D}$  then  $A \in \mathcal{D}$ .*

PROOF: This holds because  $\mathcal{D} \cup \{A\}$  satisfies the finite intersection property. □

**Definition 1.6.9** (Graph). Let  $f : A \rightarrow B$ . The *graph* of  $f$  is the set  $\{(x, f(x)) : x \in A\} \subseteq A \times B$ .

**Definition 1.6.10** (Point-Finite). Let  $X$  be a set and  $\{A_\alpha\}_{\alpha \in J}$  be a family of subsets of  $X$ . Then  $\{A_\alpha\}_{\alpha \in J}$  is *point-finite* iff, for all  $x \in X$ , there are only finitely many  $\alpha \in J$  such that  $x \in A_\alpha$ .

**Definition 1.6.11** (Countable Intersection Property). A family of parts of a set  $X$  has the *countable intersection property* iff every countable subfamily has nonempty intersection.

## 1.7 Order Theory

**Definition 1.7.1** (Cofinal). Let  $J$  be a poset and  $K \subseteq J$ . Then  $K$  is *cofinal* iff, for all  $x \in J$ , there exists  $y \in K$  such that  $x \leq y$ .

**Definition 1.7.2** (Directed Set). A *directed set* is a poset  $J$  such that, for all  $x, y \in J$ , there exists  $z \in J$  such that  $x \leq z$  and  $y \leq z$ .

**Definition 1.7.3** (Linear Order). Let  $X$  be a set. A *linear order* on  $X$  is a relation  $\leq \subseteq X^2$  such that:

- For all  $x \in X$ ,  $x \leq x$
- For all  $x, y, z \in X$ , if  $x \leq y$  and  $y \leq z$  then  $x \leq z$
- For all  $x, y \in X$ , if  $x \leq y$  and  $y \leq x$  then  $x = y$
- For all  $x, y \in X$ , we have  $x \leq y$  or  $y \leq x$

We write  $x < y$  iff  $x \leq y$  and  $x \neq y$ .

A *linearly ordered set* consists of a set and a linear order on the set.

**Definition 1.7.4** (Convex). Let  $L$  be a linearly ordered set and  $A \subseteq L$ . Then  $A$  is *convex* iff, for all  $x, y \in A$  and  $z \in L$ , if  $x < z < y$  then  $z \in A$ .

**Definition 1.7.5** (Least Upper Bound Property). A linearly ordered set  $L$  has the *least upper bound property* iff every subset of  $L$  bounded above has a least upper bound.

**Definition 1.7.6** (Linear Continuum). A *linear continuum* is a linearly ordered set  $L$  such that:

- $L$  has the least upper bound property.
- For all  $x, y \in L$  with  $x < y$ , there exists  $z \in L$  such that  $x < z < y$ .

**Proposition 1.7.7.** *If  $L$  is a linear continuum then every convex subset of  $L$  is a linear continuum.*

PROOF:

<1>1. LET:  $L$  be a linear continuum and  $C \subseteq L$  be convex

<1>2.  $C$  satisfies the least upper bound property.

<2>1. LET:  $S \subseteq C$  be nonempty and bounded above by  $u$  in  $C$ .

<2>2. LET:  $s$  be the supremum of  $S$  in  $L$

<2>3. PICK  $x \in S$

<2>4.  $x \leq s \leq u$

<2>5.  $s \in C$

PROOF:  $C$  is convex.

<2>6.  $s$  is the supremum of  $S$  in  $C$

<1>3.  $C$  is dense.

PROOF:

- (2)1. LET:  $x, y \in C$  satisfy  $x < y$
- (2)2. PICK  $z \in L$  such that  $x < z < y$
- (2)3.  $z \in C$

PROOF:  $C$  is convex.

□

**Lemma 1.7.8.** *For any real numbers  $a, b$  with  $a < b$  we have  $[a, b] \cong [0, 1]$ .*

PROOF: The map  $\phi : [a, b] \cong [0, 1]$  where  $\phi(x) = (x - a)/(b - a)$  is an order isomorphism. □

**Proposition 1.7.9.** *Let  $X$  be a linearly ordered set. Let  $a, b, c \in X$  with  $a < c < b$ . Then  $[a, b] \cong [0, 1]$  if and only if  $[a, c] \cong [c, b] \cong [0, 1]$ .*

PROOF:

- (1)1. If  $[a, b] \cong [0, 1]$  then  $[a, c] \cong [c, b] \cong [0, 1]$ .
- (2)1. ASSUME:  $\phi : [a, b] \cong [0, 1]$  is an order isomorphism.
- (2)2.  $[a, c] \cong [0, 1]$

PROOF:

$$\begin{aligned} [a, c] &\cong [0, \phi(c)] && \text{(under } \phi) \\ &\cong [0, 1] && \text{(Lemma 1.7.8 )} \end{aligned}$$

- (2)3.  $[c, b] \cong [0, 1]$

PROOF: Similar.

- (1)2. If  $[a, c] \cong [c, b] \cong [0, 1]$  then  $[a, b] \cong [0, 1]$ .
- (2)1. ASSUME:  $[a, c] \cong [c, b] \cong [0, 1]$
- (2)2. LET:  $\phi : [a, c] \cong [0, 1/2]$  and  $\psi : [c, b] \cong [1/2, 1]$

- (2)3. LET:  $\chi : [a, b] \rightarrow [0, 1]$  be given by  $\chi(x) = \begin{cases} \phi(x) & \text{if } x < c \\ \psi(x) & \text{if } x \geq c \end{cases}$

- (2)4.  $\chi : [a, b] \cong [0, 1]$

PROOF: Easy to check.

□

**Proposition 1.7.10 (CC).** *Let  $X$  be a linearly ordered set. Let  $\{x_n\}_{n \geq 0}$  be an increasing sequence of points of  $X$ . Suppose  $b$  is the supremum of  $\{x_n : n \geq 0\}$ . Then  $[x_0, b] \cong [0, 1]$  if and only if  $[x_i, x_{i+1}] \cong [0, 1]$  for all  $i$ .*

PROOF:

- (1)1. If  $[x_0, b] \cong [0, 1]$  then for all  $i$   $[x_i, x_{i+1}] \cong [0, 1]$ .

PROOF: If  $\phi : [x_0, b] \cong [0, 1]$  then  $[x_i, x_{i+1}] \cong [\phi(x_i), \phi(x_{i+1})] \cong [0, 1]$  by Lemma 1.7.8.

- (1)2. If for all  $i$   $[x_i, x_{i+1}] \cong [0, 1]$  then  $[x_0, b] \cong [0, 1]$ .

PROOF:

- (2)1. LET:  $\phi_i : [x_i, x_{i+1}] \cong [0, 1]$  for all  $i$

- (2)2. Define  $\phi : [x_0, b] \cong [0, 1]$  by:  $\phi(y) = \phi_i(y)$  ( $x_0 \leq y < b$ ) where  $i$  is least such that  $y < x_{i+1}$

PROOF: There exists such an  $i$  because  $y$  is not an upper bound for  $\{x_n : n \geq 0\}$ .

$\langle 2 \rangle 3$ .  $\phi$  is an order isomorphism.

PROOF: Easy to check.

□

**Proposition 1.7.11 (CC).** *For all  $0 < \alpha < \Omega$ , the interval  $[(0, 0), (\alpha, 0))$  in  $S_\Omega \times [0, 1)$  is order isomorphic to  $[0, 1)$  in  $\mathbb{R}$ .*

PROOF:

$\langle 1 \rangle 1$ . If  $[(0, 0), (\alpha, 0)) \cong [0, 1)$  then  $[(0, 0), (\alpha + 1, 0)) \cong [0, 1)$

PROOF: By Proposition 1.7.9.

$\langle 1 \rangle 2$ . Let  $\lambda$  be a limit ordinal,  $0 < \lambda < \Omega$ . If, for all  $\alpha$  with  $0 < \alpha < \lambda$ , we have  $[(0, 0), (\alpha, 0)) \cong [0, 1)$ , then  $[(0, 0), (\lambda, 0)) \cong [0, 1)$ .

PROOF: By Proposition 1.7.10.

$\langle 1 \rangle 3$ . Q.E.D.

PROOF: By transfinite induction.

□

## Chapter 2

# Real Analysis

**Definition 2.0.1** (Cantor Set). Define a sequence of sets  $A_n \subseteq [0, 1]$  by:

$$A_0 = [0, 1]$$
$$A_{n+1} = A_n \setminus \bigcup_{k=0}^{3^{n-1}-1} \left( \frac{3k+1}{3^n}, \frac{3k+2}{3^n} \right)$$

The *Cantor set* is  $\bigcap_{n=0}^{\infty} A_n$ .

## Chapter 3

# Topological Spaces

### 3.1 Topologies

**Definition 3.1.1** (Topology). A *topology* on a set  $X$  is a set  $\mathcal{T} \subseteq \mathcal{P}X$  such that:

1.  $X \in \mathcal{T}$ ;
2. for all  $U, V \in \mathcal{T}$ , we have  $U \cap V \in \mathcal{T}$ ;
3. For all  $\mathcal{A} \subseteq \mathcal{T}$ , we have  $\bigcup \mathcal{A} \in \mathcal{T}$ .

A *topological space*  $X$  consists of a set  $X$  and a topology on  $X$ . The elements of  $X$  are called *points* and the elements of  $\mathcal{T}$  are called *open sets*.

**Proposition 3.1.2.** *In any topological space, the empty set is open.*

PROOF: Immediate from axiom 3.  $\square$

**Definition 3.1.3** (Discrete Topology). The *discrete* topology on a set  $X$  is  $\mathcal{P}X$ .

**Definition 3.1.4** (Indiscrete Topology). The *indiscrete* topology on a set  $X$  is  $\{\emptyset, X\}$ .

**Definition 3.1.5** (Open Cover). Let  $X$  be a topological space. A cover  $\mathcal{C} \subseteq \mathcal{P}X$  of  $X$  is an *open cover* iff every member of  $\mathcal{C}$  is open.

**Definition 3.1.6** (Finer, Coarser). Let  $\mathcal{T}, \mathcal{T}'$  be topologies on a set  $X$ . Then  $\mathcal{T}$  is *finer* than  $\mathcal{T}'$ , and  $\mathcal{T}'$  is *coarser* than  $\mathcal{T}$ , iff  $\mathcal{T}' \subseteq \mathcal{T}$ .

The topology  $\mathcal{T}$  is *strictly finer* than  $\mathcal{T}'$ , and  $\mathcal{T}'$  is *strictly coarser* than  $\mathcal{T}$ , iff  $\mathcal{T} \subset \mathcal{T}'$ .

The topologies  $\mathcal{T}$  and  $\mathcal{T}'$  are *comparable* iff  $\mathcal{T} \subseteq \mathcal{T}'$  or  $\mathcal{T}' \subseteq \mathcal{T}$ .

**Definition 3.1.7** (Finite Complement Topology). The *finite complement topology* on a set  $X$  is  $\{U : X \setminus U \text{ is finite}\} \cup \{X\}$ .

**Definition 3.1.8** (Isolated Point). Let  $X$  be a topological space and  $a \in X$ . Then  $a$  is an *isolated point* iff  $\{a\}$  is open.



## 3.2 Neighbourhoods

**Definition 3.2.1** (Neighbourhood). Let  $X$  be a topological space and  $A \subseteq X$ . A *neighbourhood* of  $A$  is a set that includes an open set that includes  $A$ .

A *neighbourhood* of a point  $a$  is a neighbourhood of  $\{a\}$ .

**Proposition 3.2.2.** *If  $N$  is a neighbourhood of  $A$  and  $B \subseteq A$  then  $N$  is a neighbourhood of  $B$ .*

PROOF: Immediate from definitions.  $\square$

**Proposition 3.2.3.** *A set  $U$  is open if and only if it is a neighbourhood of each of its points.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $X$  be a topological space and  $A \subseteq X$

$\langle 1 \rangle 2$ . If  $U$  is a neighbourhood of each of its points then  $A$  is open.

$\langle 2 \rangle 1$ . ASSUME:  $U$  includes a neighbourhood of each of its points

PROVE:  $U = \bigcup \{V \subseteq U : V \text{ is open}\}$

$\langle 2 \rangle 2$ .  $\bigcup \{V \subseteq U : V \text{ is open}\} \subseteq U$

PROOF: Set theory.

$\langle 2 \rangle 3$ .  $U \subseteq \bigcup \{V \subseteq U : V \text{ is open}\}$

PROOF: Immediate from  $\langle 2 \rangle 1$ .

$\langle 1 \rangle 3$ . If  $U$  is open then  $U$  is a neighbourhood of each of its points.

PROOF: Immediate from definitions.

$\square$

**Proposition 3.2.4.** *If  $M$  is a neighbourhood of  $A$  and  $M \subseteq N$  then  $N$  is a neighbourhood of  $A$ .*

PROOF: Immediate from definitions.  $\square$

**Proposition 3.2.5.** *If  $M$  and  $N$  are neighbourhoods of  $A$  then  $M \cap N$  is a neighbourhood of  $A$ .*

PROOF: Pick open sets  $U$  and  $V$  such that  $A \subseteq U \subseteq M$  and  $A \subseteq N \subseteq V$ . Then  $A \subseteq U \cap V \subseteq M \cap N$ .

**Proposition 3.2.6.** *If  $N$  is a neighbourhood of  $x$  then  $x \in N$ .*

PROOF: Immediate from definitions.  $\square$

**Proposition 3.2.7.** *If  $N$  is a neighbourhood of  $x$  then there exists a neighbourhood  $U$  of  $x$  such that, for all  $y \in U$ ,  $M$  is a neighbourhood of  $y$ .*

PROOF: Pick an open set  $U$  such that  $x \in U \subseteq N$ .  $\square$

**Theorem 3.2.8.** *Let  $X$  be a set and  $\triangleright \subseteq \mathcal{P}X \times X$  a relation such that:*

1. *If  $M \triangleright x$  and  $M \subseteq N$  then  $N \triangleright x$*
2.  *$X \triangleright x$  for all  $x \in X$*

3. If  $M \triangleright x$  and  $N \triangleright x$  then  $M \cap N \triangleright x$
4. If  $N \triangleright x$  then  $x \in N$
5. If  $M \triangleright x$  then there exists  $N \triangleright x$  such that, for all  $y \in N$ ,  $M \triangleright y$ .

Then there exists a unique topology  $\mathcal{T}$  such that  $N \triangleright x$  iff  $N$  is a neighbourhood of  $x$ .

PROOF:

- $\langle 1 \rangle 1$ . LET:  $\triangleright$  be a relation satisfying 1–3
- $\langle 1 \rangle 2$ . LET:  $\mathcal{T} = \{U \in \mathcal{P}X : \forall x \in U. U \triangleright x\}$
- $\langle 1 \rangle 3$ .  $\mathcal{T}$  is a topology.
  - $\langle 2 \rangle 1$ .  $X \in \mathcal{T}$   
PROOF: By axiom 2
  - $\langle 2 \rangle 2$ . For all  $U, V \in \mathcal{T}$  we have  $U \cap V \in \mathcal{T}$   
PROOF: By axiom 3
  - $\langle 2 \rangle 3$ . For all  $\mathcal{A} \subseteq \mathcal{T}$  we have  $\bigcup \mathcal{A} \in \mathcal{T}$ 
    - $\langle 3 \rangle 1$ . LET:  $x \in \bigcup \mathcal{A}$
    - $\langle 3 \rangle 2$ . PICK  $U \in \mathcal{A}$  such that  $x \in U$
    - $\langle 3 \rangle 3$ .  $U \triangleright x$
    - $\langle 3 \rangle 4$ .  $\bigcup \mathcal{A} \triangleright x$   
PROOF: By axiom 1
- $\langle 1 \rangle 4$ . In  $\mathcal{T}$ ,  $N \triangleright x$  iff  $N$  is a neighbourhood of  $x$ .
  - $\langle 2 \rangle 1$ . If  $N \triangleright x$  then  $N$  is a neighbourhood of  $x$ 
    - $\langle 3 \rangle 1$ . ASSUME:  $N \triangleright x$
    - $\langle 3 \rangle 2$ .  $x \in N$   
PROOF: By axiom 4
    - $\langle 3 \rangle 3$ . LET:  $U = \{y \in N : N \triangleright y\}$
    - $\langle 3 \rangle 4$ .  $U$  is open
      - $\langle 4 \rangle 1$ . LET:  $y \in U$   
PROVE:  $U \triangleright y$
      - $\langle 4 \rangle 2$ .  $N \triangleright y$
      - $\langle 4 \rangle 3$ . PICK  $W \triangleright y$  such that, for all  $z \in W$ ,  $N \triangleright z$   
PROOF: By axiom 5
      - $\langle 4 \rangle 4$ .  $W \subseteq U$
      - $\langle 4 \rangle 5$ .  $U \triangleright y$   
PROOF: By axiom 1
    - $\langle 3 \rangle 5$ .  $x \in U \subseteq N$
  - $\langle 2 \rangle 2$ . If  $N$  is a neighbourhood of  $x$  then  $N \triangleright x$ 
    - $\langle 3 \rangle 1$ . LET:  $N$  be a neighbourhood of  $x$
    - $\langle 3 \rangle 2$ . PICK  $U$  open such that  $x \in U \subseteq N$
    - $\langle 3 \rangle 3$ .  $U \triangleright x$   
PROOF: By  $\langle 1 \rangle 2$
    - $\langle 3 \rangle 4$ .  $N \triangleright x$   
PROOF: By axiom 1
- $\langle 1 \rangle 5$ .  $\mathcal{T}$  is unique.

PROOF: By Proposition 3.2.3.

□

**Definition 3.2.9** (Sufficiently Close). Let  $X$  be a topological space,  $a \in X$ , and  $P$  be a property of points of  $X$ . We write “For all  $x$  sufficiently close to  $a$ ,  $P(x)$ ” to mean “There exists a neighbourhood  $N$  of  $a$  such that, for all  $x \in N$ ,  $P(x)$ .”

### 3.3 Local Bases

**Definition 3.3.1** (Local Basis). Let  $X$  be a topological space and  $x \in X$ . A *local basis* at  $x$  is a set  $\mathcal{B}$  of open neighbourhoods of  $x$  such that every neighbourhood of  $x$  includes a member of  $\mathcal{B}$ . We call the elements of  $\mathcal{B}$  *basic open neighbourhoods*.

**Proposition 3.3.2.** Let  $\mathcal{B}$  be a local basis at  $x$  and  $M, N \in \mathcal{B}$ . Then there exists  $P \in \mathcal{B}$  such that  $P \subseteq M \cap N$ .

PROOF: This holds because  $M \cap N$  is a neighbourhood of  $x$  (Proposition 3.2.5).

□

**Proposition 3.3.3.** Let  $X$  be a topological space,  $x \in X$  and  $\mathcal{B} \subseteq \mathcal{P}X$ . Then  $\mathcal{B}$  is a local basis at  $x$  iff  $\mathcal{B}$  is a set of open neighbourhoods of  $x$  such that every open neighbourhood of  $x$  includes a member of  $\mathcal{B}$ .

PROOF:

⟨1⟩1. If  $\mathcal{B}$  is a local basis at  $x$  then  $\mathcal{B}$  is a set of open neighbourhoods of  $x$  such that every open neighbourhood of  $x$  includes a member of  $\mathcal{B}$

PROOF: Trivial.

⟨1⟩2. If  $\mathcal{B}$  is a set of open neighbourhoods of  $x$  such that every open neighbourhood of  $x$  includes a member of  $\mathcal{B}$  then  $\mathcal{B}$  is a local basis at  $x$ .

PROOF: Every neighbourhood of  $x$  includes an open neighbourhood of  $x$ , which therefore includes an element of  $\mathcal{B}$ .

□

### 3.4 Bases

**Definition 3.4.1** (Basis for a Topology). Let  $(X, \mathcal{T})$  be a topological space. A *basis* for the topology on  $X$  is a set of open sets  $\mathcal{B}$  such that every open set is a union of members of  $\mathcal{B}$ . The members of  $\mathcal{B}$  are called *basic open sets*, and  $\mathcal{T}$  is called the topology *generated* by  $\mathcal{B}$ .

**Proposition 3.4.2.** Let  $(X, \mathcal{T})$  be a topological space and  $\mathcal{B} \subseteq \mathcal{P}X$ . Then the following are equivalent:

1.  $\mathcal{B}$  is a basis for  $\mathcal{T}$ .

2. A set  $U$  is open if and only if, for all  $x \in U$ , there exists  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$ .
3.  $\mathcal{T}$  is the set of all unions of subsets of  $\mathcal{B}$ .
4. Every member of  $\mathcal{B}$  is open and, for all  $x \in X$  and every open neighbourhood  $U$  of  $x$ , there exists  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$ .
5. For all  $x \in X$ , the set  $\{B \in \mathcal{B} : x \in B\}$  is a local basis at  $x$ .

PROOF:

$\langle 1 \rangle 1. 1 \Rightarrow 2$

$\langle 2 \rangle 1.$  ASSUME:  $\mathcal{B}$  is a basis for the topology  $\mathcal{T}$ .

$\langle 2 \rangle 2.$  For all  $U \in \mathcal{T}$  and  $x \in U$ , there exists  $B \in \mathcal{B}$  such that  $x \in B$

PROOF: Immediate from the definition of basis ( $\langle 2 \rangle 1$ ).

$\langle 2 \rangle 3.$  For all  $U \subseteq X$ , if  $\forall x \in U. \exists B \in \mathcal{B}. x \in B \subseteq U$  then  $U \in \mathcal{T}$

PROOF: By Proposition 3.2.3.

$\langle 1 \rangle 2. 2 \Leftrightarrow 3$

PROOF: From Lemma 1.6.1.

$\langle 1 \rangle 3. 3 \Rightarrow 1$

PROOF: Trivial.

$\langle 1 \rangle 4. 2 \Rightarrow 4$

PROOF: Trivial.

$\langle 1 \rangle 5. 4 \Rightarrow 2$

PROOF:

$\langle 2 \rangle 1.$  ASSUME: 4

$\langle 2 \rangle 2.$  If  $U$  is open then, for all  $x \in U$ , there exists  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$

PROOF: Immediate from  $\langle 2 \rangle 1$ .

$\langle 2 \rangle 3.$  If, for all  $x \in U$ , there exists  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$ , then  $U$  is open.

PROOF: By Proposition 3.2.3 using the fact that every member of  $\mathcal{B}$  is open ( $\langle 2 \rangle 1$ ).

$\langle 1 \rangle 6. 4 \Leftrightarrow 5$

PROOF: From Proposition 3.3.3.

□

**Corollary 3.4.2.1.** *If  $\mathcal{B}$  is a basis for the topology  $\mathcal{T}$ , then  $\mathcal{T}$  is the coarsest topology in which every element of  $\mathcal{B}$  is open.*

**Lemma 3.4.3.** *Let  $X$  be a set and  $\mathcal{B} \subseteq \mathcal{P}X$ . Then  $\mathcal{B}$  is a basis for a topology  $\mathcal{T}$  on  $X$  if and only if:*

1.  $\bigcup \mathcal{B} = X$

2. for all  $B_1, B_2 \in \mathcal{B}$  and  $x \in B_1 \cap B_2$ , there exists  $B_3 \in \mathcal{B}$  such that  $x \in B_3 \subseteq B_1 \cap B_2$ .

*In this case,  $\mathcal{T}$  is unique.*

PROOF:

- (1)1. If  $\mathcal{B}$  is a basis for a topology then  $\bigcup \mathcal{B} = X$
  - (2)1. ASSUME:  $\mathcal{B}$  is a basis for the topology  $\mathcal{T}$
  - (2)2. LET:  $x \in X$
  - (2)3. There exists  $B \in \mathcal{B}$  such that  $x \in B$
  - PROOF: From the definition of basis, since  $X \in \mathcal{T}$ . ( $\langle 2 \rangle 1$ ,  $\langle 2 \rangle 2$ ).
- (1)2. If  $\mathcal{B}$  is a basis for a topology then it satisfies condition 2
  - (2)1. ASSUME:  $\mathcal{B}$  is a basis for the topology  $\mathcal{T}$
  - (2)2. LET:  $B_1, B_2 \in \mathcal{B}$
  - (2)3.  $B_1, B_2 \in \mathcal{T}$
  - PROOF: From the definition of basis ( $\langle 2 \rangle 1$ ,  $\langle 2 \rangle 2$ ).
  - (2)4.  $B_1 \cap B_2 \in \mathcal{T}$
  - PROOF: By the definition of topology, the open sets in  $\mathcal{T}$  are closed under binary intersection ( $\langle 2 \rangle 1$ ,  $\langle 2 \rangle 3$ )
  - (2)5. For all  $x \in B_1 \cap B_2$ , there exists  $B_3 \in \mathcal{B}$  such that  $x \in B_3 \subseteq B_1 \cap B_2$
  - PROOF: From the definition of basis ( $\langle 2 \rangle 1$ ,  $\langle 2 \rangle 4$ )
- (1)3. If  $\mathcal{B}$  satisfies conditions 1 and 2 then  $\mathcal{T} = \{U \subseteq X : \forall x \in U. \exists B \in \mathcal{B}. x \in B \subseteq U\}$  is a topology and  $\mathcal{B}$  is a basis for  $\mathcal{T}$ .
  - (2)1. ASSUME:  $\mathcal{B}$  satisfies conditions 1 and 2
  - (2)2.  $X \in \mathcal{T}$
  - PROOF: For all  $x \in X$ , there exists  $B \in \mathcal{B}$  such that  $x \in B \subseteq X$  by condition 1 ( $\langle 2 \rangle 1$ ).
  - (2)3. For all  $\mathcal{A} \subseteq \mathcal{T}$ , we have  $\bigcup \mathcal{A} \in \mathcal{T}$ 
    - (3)1. LET:  $\mathcal{A} \subseteq \mathcal{T}$
    - (3)2. LET:  $x \in \bigcup \mathcal{A}$
    - (3)3. PICK  $U \in \mathcal{A}$  such that  $x \in U$
    - PROOF: From (3)2.
    - (3)4. PICK  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$
    - PROOF: Since  $U \in \mathcal{T}$ , using the definition of  $\mathcal{T}$  ( $\langle 3 \rangle 1$ ,  $\langle 3 \rangle 3$ )
    - (3)5.  $x \in B \subseteq \bigcup \mathcal{A}$
    - PROOF: From (3)3 and (3)4.
- (2)4. For all  $U, V \in \mathcal{T}$ , we have  $U \cap V \in \mathcal{T}$ 
  - (3)1. LET:  $U, V \in \mathcal{T}$
  - (3)2. LET:  $x \in U \cap V$
  - (3)3. PICK  $B_1, B_2 \in \mathcal{B}$  such that  $x \in B_1 \subseteq U$  and  $x \in B_2 \subseteq V$
  - PROOF: From (3)1, (3)2 and the definition of  $\mathcal{T}$ .
  - (3)4. PICK  $B_3 \in \mathcal{B}$  such that  $x \in B_3 \subseteq B_1 \cap B_2$
  - PROOF: Using condition 2 ( $\langle 2 \rangle 1$ ,  $\langle 3 \rangle 3$ ).
  - (3)5.  $x \in B_3 \subseteq U \cap V$
  - PROOF: From (3)3 and (3)4.
- (2)5.  $\bigcup \mathcal{B} = X$ 
  - PROOF: This is condition 1 ( $\langle 2 \rangle 1$ ).
- (2)6. For all  $U \in \mathcal{T}$  and  $x \in U$ , there exists  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$ 
  - PROOF: Immediate from the definition of  $\mathcal{T}$ .
- (1)4.  $\mathcal{T}$  is unique.
  - PROOF: From Proposition 3.4.2.

□

**Corollary 3.4.3.1.** *Let  $X$  be a set and  $\mathcal{B} \subseteq \mathcal{P}X$  be such that  $\bigcup \mathcal{B} = X$  and  $\mathcal{B}$  is closed under binary intersection. Then  $\mathcal{B}$  is a basis for a unique topology on  $X$ .*

**Lemma 3.4.4.** *Let  $\mathcal{B}$  and  $\mathcal{B}'$  be bases for the topologies  $\mathcal{T}$  and  $\mathcal{T}'$  on  $X$  respectively. Then  $\mathcal{T} \subseteq \mathcal{T}'$  if and only if, for all  $B \in \mathcal{B}$  and  $x \in B$ , there exists  $B' \in \mathcal{B}'$  such that  $x \in B' \subseteq B$ .*

PROOF:

$\langle 1 \rangle 1$ . If  $\mathcal{T} \subseteq \mathcal{T}'$  then, for all  $B \in \mathcal{B}$  and  $x \in B$ , there exists  $B' \in \mathcal{B}'$  such that  $x \in B' \subseteq B$ .

$\langle 2 \rangle 1$ . ASSUME:  $\mathcal{T} \subseteq \mathcal{T}'$

$\langle 2 \rangle 2$ . LET:  $B \in \mathcal{B}$  and  $x \in B$

$\langle 2 \rangle 3$ .  $B \in \mathcal{T}$

PROOF: This holds because  $\mathcal{B} \subseteq \mathcal{T}$  by the definition of basis. ( $\langle 2 \rangle 2$ )

$\langle 2 \rangle 4$ .  $B \in \mathcal{T}'$

PROOF: From  $\langle 2 \rangle 1$  and  $\langle 2 \rangle 3$ .

$\langle 2 \rangle 5$ . There exists  $B' \in \mathcal{B}'$  such that  $x \in B' \subseteq B$ .

$\langle 1 \rangle 2$ . If, for all  $B \in \mathcal{B}$  and  $x \in B$ , there exists  $B' \in \mathcal{B}'$  such that  $x \in B' \subseteq B$ , then  $\mathcal{T} \subseteq \mathcal{T}'$ .

$\langle 2 \rangle 1$ . ASSUME: For all  $B \in \mathcal{B}$  and  $x \in B$ , there exists  $B' \in \mathcal{B}'$  such that  $x \in B' \subseteq B$

$\langle 2 \rangle 2$ . LET:  $U \in \mathcal{T}$

PROVE:  $U \in \mathcal{T}'$

$\langle 2 \rangle 3$ . LET:  $x \in U$

$\langle 2 \rangle 4$ . PICK  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$

PROOF: Since  $\mathcal{B}$  is a basis for  $\mathcal{T}$  ( $\langle 2 \rangle 2$ ,  $\langle 2 \rangle 3$ ).

$\langle 2 \rangle 5$ . PICK  $B' \in \mathcal{B}'$  such that  $x \in B' \subseteq B$

PROOF: From  $\langle 2 \rangle 1$  and  $\langle 2 \rangle 4$ .

$\langle 2 \rangle 6$ .  $x \in B' \subseteq U$

PROOF: From  $\langle 2 \rangle 4$  and  $\langle 2 \rangle 5$ .

$\langle 2 \rangle 7$ . Q.E.D.

PROOF: By Proposition 3.4.2.

□

**Definition 3.4.5** (Lower Limit Topology). The *lower limit topology* on  $\mathbb{R}$  is the one generated by the set of all half-open intervals of the form  $[a, b)$ . We write  $\mathbb{R}_l$  for the topological space consisting of  $\mathbb{R}$  under this topology.

We prove this is a topology.

PROOF:

$\langle 1 \rangle 1$ . LET:  $\mathcal{B}$  be the set of all half-open intervals of the form  $[a, b)$ .

$\langle 1 \rangle 2$ .  $\bigcup \mathcal{B} = \mathbb{R}$

PROOF: For all  $x \in \mathbb{R}$ , we have  $x \in [x, x+1) \in \mathcal{B}$ .

$\langle 1 \rangle 3$ . For all  $B_1, B_2 \in \mathcal{B}$  and  $x \in B_1 \cap B_2$ , there exists  $B_3 \in \mathcal{B}$  such that  $x \in B_3 \subseteq B_1 \cap B_2$ .

PROOF: If  $x \in [a, b) \cap [c, d)$  then  $x \in [\max(a, c), \min(b, d)) \subseteq [a, b) \cap [c, d)$ .

⟨1⟩4. Q.E.D.

PROOF: By Lemma 3.4.3.

□

**Definition 3.4.6** ( $K$ -topology). The  $K$ -topology on  $\mathbb{R}$  is the one generated by the set of all open intervals  $(a, b)$  and all sets of the form  $(a, b) \setminus K$ , where  $K = \{1/n : n \in \mathbb{Z}^+\}$ . We write  $\mathbb{R}_K$  for the topological space consisting of  $\mathbb{R}$  under this topology.

We prove this is a topology.

PROOF:

⟨1⟩1. LET:  $\mathcal{B} = \{(a, b) : a, b \in \mathbb{R}, a < b\} \cup \{(a, b) \setminus K : a, b \in \mathbb{R}, a < b\}$

⟨1⟩2.  $\bigcup \mathcal{B} = \mathbb{R}$

PROOF: For all  $x \in \mathbb{R}$ , we have  $x \in (x-1, x+1) \in \mathcal{B}$ .

⟨1⟩3. For all  $B_1, B_2 \in \mathcal{B}$  and  $x \in B_1 \cap B_2$ , there exists  $B_3 \in \mathcal{B}$  such that  $x \in B_3 \subseteq B_1 \cap B_2$ .

⟨2⟩1. LET:  $B_1, B_2 \in \mathcal{B}$  and  $x \in B_1 \cap B_2$

PROVE: There exists  $B_3 \in \mathcal{B}$  such that  $x \in B_3 \subseteq B_1 \cap B_2$

⟨2⟩2. CASE:  $B_1 = (a, b)$ ,  $B_2 = (c, d)$

PROOF: Take  $B_3 = (\max(a, c), \min(b, d))$

⟨2⟩3. CASE:  $B_1 = (a, b)$ ,  $B_2 = (c, d) \setminus K$

PROOF: Take  $B_3 = (\max(a, c), \min(b, d)) \setminus K$

⟨2⟩4. CASE:  $B_1 = (a, b) \setminus K$ ,  $B_2 = (c, d)$

PROOF: Take  $B_3 = (\max(a, c), \min(b, d)) \setminus K$

⟨2⟩5. CASE:  $B_1 = (a, b) \setminus K$ ,  $B_2 = (c, d) \setminus K$

PROOF: Take  $B_3 = (\max(a, c), \min(b, d)) \setminus K$

⟨1⟩4. Q.E.D.

PROOF: By Lemma 3.4.3.

□

**Lemma 3.4.7.** The lower limit topology and the  $K$ -topology are incomparable.

PROOF:  $[0, 1)$  is not open in the  $K$ -topology.  $(-1, 1) \setminus K$  is not open in the lower limit topology, because there is no half-open interval  $[a, b)$  such that  $0 \in [a, b) \subseteq (-1, 1) \setminus K$ . □

**Proposition 3.4.8.** The set of all singletons is a basis for any discrete space.

PROOF: Easy. □

**Definition 3.4.9** (Line with Two Origins). The *line with two origins* is the set  $\mathbb{R} \setminus \{0\} \cup \{p, q\}$  under the topology generated by the basis consisting of:

- all open intervals in  $\mathbb{R}$  that do not contain 0;
- all sets of the form  $(-a, 0) \cup \{p\} \cup (0, a)$  where  $a > 0$ ;
- all sets of the form  $(-a, 0) \cup \{q\} \cup (0, a)$  where  $a > 0$

### 3.5 Closed Sets

**Definition 3.5.1** (Closed). Let  $X$  be a topological space and  $A \subseteq X$ . Then  $A$  is *closed* iff  $X \setminus A$  is open.

**Proposition 3.5.2.** *In any topological space  $X$ , the empty set  $\emptyset$  is closed.*

PROOF: This holds because  $X \setminus \emptyset = X$  is open.  $\square$

**Proposition 3.5.3.** *In any topological space  $X$ , the set  $X$  is closed.*

PROOF: This holds because  $X \setminus X = \emptyset$  is open.  $\square$

**Proposition 3.5.4.** *The union of two closed sets is closed.*

PROOF: If  $C$  and  $D$  are closed then  $X \setminus (C \cup D) = (X \setminus C) \cup (X \setminus D)$  is open.  $\square$

**Proposition 3.5.5.** *In any topological space, the intersection of a nonempty set of closed sets is closed.*

PROOF: Let  $\mathcal{C}$  be a nonempty set of closed sets. Then  $X \setminus \bigcap \mathcal{C} = \bigcup \{X \setminus C : C \in \mathcal{C}\}$  is open.  $\square$

**Proposition 3.5.6.** *Let  $X$  be a topological space and  $U \subseteq X$ . Then  $U$  is open if and only if  $X \setminus U$  is closed.*

PROOF: Immediate from definitions.

**Theorem 3.5.7.** *Let  $X$  be a set and  $\mathcal{C} \subseteq \mathcal{P}X$ . Suppose:*

1.  $\emptyset, X \in \mathcal{C}$ ;
2. for all nonempty  $\mathcal{A} \subseteq \mathcal{C}$ , we have  $\bigcap \mathcal{A} \in \mathcal{C}$ ;
3. for all  $C, D \in \mathcal{C}$ , we have  $C \cup D \in \mathcal{C}$ .

*Then there exists a unique topology on  $X$  under which  $\mathcal{C}$  is the set of all closed sets, namely*

$$\mathcal{T} = \{U \subseteq X : X \setminus U \in \mathcal{C}\}$$

PROOF:

$\langle 1 \rangle 1$ . LET:  $\mathcal{C}$  be a set satisfying 1–3

$\langle 1 \rangle 2$ . LET:  $\mathcal{T} = \{X \setminus C : C \in \mathcal{C}\}$

$\langle 1 \rangle 3$ .  $\mathcal{T}$  is a topology

$\langle 2 \rangle 1$ .  $X \in \mathcal{T}$

PROOF:  $X \setminus X = \emptyset \in \mathcal{C}$  by condition 1.

$\langle 2 \rangle 2$ . For all  $\mathcal{A} \subseteq \mathcal{T}$  we have  $\bigcup \mathcal{A} \in \mathcal{T}$ .

$\langle 3 \rangle 1$ . LET:  $\mathcal{A} \subseteq \mathcal{T}$

$\langle 3 \rangle 2$ . CASE:  $\mathcal{A} = \emptyset$

PROOF: In this case,  $X \setminus \bigcup \mathcal{A} = X \in \mathcal{C}$  by condition 1.

$\langle 3 \rangle 3$ . CASE:  $\mathcal{A}$  is nonempty



PROOF: In this case, we have  $X \setminus \bigcup \mathcal{A} = \bigcap \{X \setminus U : U \in \mathcal{A}\} \in \mathcal{C}$  by condition 2.

(2)3. For all  $U, V \in \mathcal{T}$  we have  $U \cap V \in \mathcal{T}$

PROOF:  $X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V) \in \mathcal{C}$  by condition 3.

(1)4.  $\mathcal{C}$  is the set of closed sets.

PROOF:

$$\begin{aligned} C \text{ is closed} &\Leftrightarrow X \setminus C \in \mathcal{T} \\ &\Leftrightarrow X \setminus (X \setminus C) \in \mathcal{C} \\ &\Leftrightarrow C \in \mathcal{C} \end{aligned}$$

(1)5.  $\mathcal{T}$  is unique.

PROOF: By Proposition 3.5.6.

□

**Definition 3.5.8** (Closed Covering). A *closed covering* of a topological space is a covering in which every member is a closed set.

## 3.6 Locally Finite Families

**Definition 3.6.1** (Locally Finite). Let  $X$  be a topological space and  $\{A_i\}_{i \in I}$  a family of subsets of  $X$ . Then  $\{A_i\}_{i \in I}$  is *locally finite* iff, for all  $x \in X$ , there exists a neighbourhood  $N$  of  $x$  such that there are only finitely many  $i \in I$  such that  $N$  intersects  $A_i$ .

**Proposition 3.6.2.** If  $\{A_i\}_{i \in I}$  is locally finite and  $B_i \subseteq A_i$  for all  $i$  then  $\{B_i\}_{i \in I}$  is locally finite.

PROOF: Immediate from definitions. □

**Proposition 3.6.3.** Every finite family of open sets is locally finite.

PROOF: Trivial. □

## 3.7 Closure of a Set

**Definition 3.7.1** (Closure). Let  $X$  be a topological space and  $A \subseteq X$ . The *closure* of  $A$ ,  $\text{Cl } A$  or  $\overline{A}$ , is the intersection of all closed sets that include  $A$ .

PROOF: This intersection always exists because  $X$  is a closed set that includes  $A$ . □

**Proposition 3.7.2.** Let  $X$  be a topological space and  $A \subseteq X$ . Then  $A \subseteq \overline{A}$ .

PROOF: Immediate from definitions. □

**Proposition 3.7.3.** Let  $X$  be a topological space and  $A \subseteq X$ . Then  $\overline{A}$  is closed.

PROOF: This follows from Proposition 3.5.5. □

**Proposition 3.7.4.** *Let  $X$  be a topological space and  $A, C \subseteq X$ . If  $A \subseteq C$  and  $C$  is closed then  $\overline{A} \subseteq C$ .*

PROOF: Immediate from definitions.  $\square$

**Proposition 3.7.5.** *Let  $X$  be a topological space and  $A, B \subseteq X$ . If  $A \subseteq B$  then  $\overline{A} \subseteq \overline{B}$ .*

PROOF:

$\langle 1 \rangle 1.$  ASSUME:  $A \subseteq B$

$\langle 1 \rangle 2.$   $A \subseteq \overline{B}$

PROOF: Proposition 3.7.2.

$\langle 1 \rangle 3.$   $\overline{A} \subseteq \overline{B}$

PROOF: Propositions 3.7.3, 3.7.4.

$\square$

**Proposition 3.7.6.** *Let  $X$  be a set and  $A \subseteq X$ . Then  $A$  is closed if and only if  $A = \overline{A}$ .*

PROOF:

$\langle 1 \rangle 1.$  If  $A$  is closed then  $A = \overline{A}$

$\langle 2 \rangle 1.$  ASSUME:  $A$  is closed

$\langle 2 \rangle 2.$   $A \subseteq \overline{A}$

PROOF: By Proposition 3.7.2.

$\langle 2 \rangle 3.$   $\overline{A} \subseteq A$

PROOF: By Proposition 3.7.4 since  $A \subseteq A$ .

$\langle 1 \rangle 2.$  If  $A = \overline{A}$  then  $A$  is closed.

PROOF: By Proposition 3.7.3.

$\square$

**Corollary 3.7.6.1.**

$$\overline{\emptyset} = \emptyset$$

**Theorem 3.7.7** (Kuratowski Closure Axioms). *Let  $X$  be a set and  $(-) : \mathcal{P}X \rightarrow \mathcal{P}X$  be a function such that:*

1.  $\overline{\emptyset} = \emptyset$
2. For all  $A \subseteq X$ ,  $A \subseteq \overline{A}$
3. For all  $A \subseteq X$ ,  $\overline{A} = \overline{\overline{A}}$
4. For all  $A, B \subseteq X$ ,  $\overline{A \cup B} = \overline{A} \cup \overline{B}$

*Then there exists a unique topology  $\mathcal{T}$  on  $X$  such that  $\overline{A}$  is the closure of  $A$  for all  $A \in \mathcal{P}X$ .*

PROOF:

$\langle 1 \rangle 1.$  For all  $C, D \subseteq X$ , if  $C \subseteq D$  then  $\overline{C} \subseteq \overline{D}$

$\langle 2 \rangle 1.$  ASSUME:  $C \subseteq D$

⟨2⟩2.  $\overline{C} = \overline{D}$

PROOF:

$$\begin{aligned}\overline{D} &= \overline{C \cup D} && (\langle 2 \rangle 1) \\ &= \overline{C} \cup \overline{D} && (\text{axiom 4})\end{aligned}$$

⟨1⟩2. LET:  $\mathcal{T}$  be the topology in which a set  $C$  is closed iff  $\overline{C} = C$ .

⟨2⟩1.  $\overline{\emptyset} = \emptyset$

PROOF: This is axiom 1.

⟨2⟩2.  $\overline{X} = X$

PROOF: By axiom 2.

⟨2⟩3. For any set  $\mathcal{A}$  of sets  $C$  such that  $\overline{C} = C$ , we have  $\overline{\bigcap \mathcal{A}} = \bigcap \mathcal{A}$

⟨3⟩1.  $\bigcap \mathcal{A} \subseteq \bigcap \mathcal{A}$

⟨4⟩1. LET:  $C \in \mathcal{A}$

⟨4⟩2.  $\bigcap \mathcal{A} \subseteq C$

PROOF:

$$\begin{aligned}\overline{\bigcap \mathcal{A}} &\subseteq \overline{C} && (\langle 1 \rangle 1) \\ &= C && (\langle 4 \rangle 1)\end{aligned}$$

⟨3⟩2. Q.E.D.

⟨2⟩4. If  $\overline{C} = C$  and  $\overline{D} = D$  then  $\overline{C \cup D} = C \cup D$

PROOF: By axiom 4.

⟨2⟩5. Q.E.D.

PROOF: By Theorem 3.5.7.

⟨1⟩3. For all  $A \subseteq X$ , the closure of  $A$  in  $\mathcal{T}$  is  $\overline{A}$

⟨2⟩1.  $\overline{A}$  is closed

PROOF: From axiom 3.

⟨2⟩2. If  $A \subseteq C$  and  $C$  is closed then  $\overline{A} \subseteq C$

PROOF:

$$\begin{aligned}C &= \overline{C} && (C \text{ is closed}) \\ &= \overline{A \cup C} && (A \subseteq C) \\ &= \overline{A} \cup \overline{C} && (\text{axiom 4})\end{aligned}$$

□

**Theorem 3.7.8.** *Let  $A$  be a subset of the topological space  $X$  and  $\mathcal{B}$  a basis for  $X$ . Then  $x \in \overline{A}$  if and only if, for all  $B \in \mathcal{B}$ , if  $x \in B$  then  $B$  intersects  $A$ .*

PROOF:

⟨1⟩1. If  $x \in \overline{A}$  then, for all  $B \in \mathcal{B}$ , if  $x \in B$  then  $B$  intersects  $A$ .

PROOF: Immediate from Theorem 3.8.3.

⟨1⟩2. If, for all  $B \in \mathcal{B}$ , if  $x \in B$  then  $B$  intersects  $A$ , then  $x \in \overline{A}$ .

⟨2⟩1. ASSUME: for all  $B \in \mathcal{B}$ , if  $x \in B$  then  $B$  intersects  $A$ .

⟨2⟩2. LET:  $U$  be a neighbourhood of  $x$

⟨2⟩3. PICK  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$

PROOF:  $\mathcal{B}$  is a basis.

⟨2⟩4.  $B$  intersects  $A$ .

PROOF: By ⟨2⟩1.

$\langle 2 \rangle 5$ .  $U$  intersects  $A$ .

$\langle 2 \rangle 6$ . Q.E.D.

PROOF: By Theorem 3.8.3.

□

### 3.8 Interior of a Set

**Definition 3.8.1** (Interior). Let  $X$  be a topological space and  $A \subseteq X$ . The *interior* of  $A$ ,  $\text{Int } A$ , is the union of all open sets included in  $A$ .

**Lemma 3.8.2.** If  $A \subseteq B$  then  $\overline{A} \subseteq \overline{B}$ .

PROOF:  $\overline{B}$  is a closed set that includes  $B$ , hence includes  $A$ . □

**Theorem 3.8.3.** Let  $A$  be a subset of the topological space  $X$  and  $x \in X$ . Then  $x \in \overline{A}$  if and only if every neighbourhood of  $x$  intersects  $A$ .

PROOF:

$$\begin{aligned} x \notin \overline{A} &\Leftrightarrow \exists C \text{ closed } (A \subseteq C \wedge x \notin C) \\ &\Leftrightarrow \exists U \text{ open } (A \subseteq X \setminus U \wedge x \in U) \\ &\Leftrightarrow \exists U \text{ open } (A \text{ intersects } U \wedge x \in U) \end{aligned} \quad \square$$

**Lemma 3.8.4.**

$$X \setminus \text{Int } A = \overline{X \setminus A}$$

PROOF:

$$\begin{aligned} \langle 1 \rangle 1. & X \setminus \text{Int } A \subseteq \overline{X \setminus A} \\ \langle 2 \rangle 1. & X \setminus A \subseteq \overline{X \setminus A} \\ \langle 2 \rangle 2. & X \setminus \overline{X \setminus A} \subseteq A \\ \langle 2 \rangle 3. & X \setminus \overline{X \setminus A} \subseteq \text{Int } A \\ \langle 1 \rangle 2. & \overline{X \setminus A} \subseteq X \setminus \text{Int } A \\ \langle 2 \rangle 1. & \text{Int } A \subseteq A \\ \langle 2 \rangle 2. & X \setminus A \subseteq X \setminus \text{Int } A \\ \langle 2 \rangle 3. & \overline{X \setminus A} \subseteq X \setminus \text{Int } A \end{aligned}$$

□

### 3.9 Boundary

**Definition 3.9.1** (Boundary). Let  $X$  be a topological space and  $A \subseteq X$ . The *boundary* of  $A$ ,  $\text{Bd } A$ , is  $\overline{A} \cap \overline{X \setminus A}$ .

**Lemma 3.9.2.**

$$\text{Bd } A = \overline{A} \setminus \text{Int } A$$

PROOF: From Lemma 3.8.4. □

**Lemma 3.9.3.**  $\overline{A} = \text{Int } A \cup \text{Bd } A$

PROOF:

$$\begin{aligned}
\text{Int } A \cup \text{Bd } A &= \text{Int } A \cup (\overline{A} \cap (X \setminus \text{Int } A)) \\
&= \text{Int } A \cup \overline{A} \\
&= \overline{A}
\end{aligned}
\quad \square$$

**Corollary 3.9.3.1.**  $\text{Bd } A = \emptyset$  iff  $A$  is open and closed.

**Lemma 3.9.4.** For any set  $U$ , the following are equivalent:

1.  $U$  is open.
2.  $\text{Bd } U \cap U = \emptyset$
3.  $\text{Bd } U = \overline{U} \setminus U$

PROOF:

$\langle 1 \rangle 1. 1 \Rightarrow 3$

PROOF: From Lemma 3.9.2.

$\langle 1 \rangle 2. 3 \Rightarrow 2$

PROOF: Set theory.

$\langle 1 \rangle 3. 2 \Rightarrow 1$

PROOF:

$$\begin{aligned}
U &\subseteq \overline{U} \\
&= \text{Int } U \cup \text{Bd } U && (\text{Lemma 3.9.3}) \\
\therefore U &\subseteq \text{Int } U
\end{aligned}$$

$\square$

## 3.10 Limit Points

**Definition 3.10.1** (Limit Point). Let  $X$  be a topological space,  $A \subseteq X$ , and  $x \in X$ . Then  $x$  is a *limit point*, *cluster point* or *point of accumulation* of  $A$  iff every neighbourhood of  $x$  intersects  $A$  in a point other than  $x$ .

**Lemma 3.10.2.** If  $A \subseteq B$  then every limit point of  $A$  is a limit point of  $B$ .

PROOF: Immediate from the definition.  $\square$

**Theorem 3.10.3.** Let  $A$  be a subset of the topological space  $X$ . Let  $A'$  be the set of all limit points of  $A$ . Then  $\overline{A} = A \cup A'$ .

PROOF:

$\langle 1 \rangle 1.$  If  $x \in \overline{A}$  and  $x \notin A$  then  $x \in A'$

PROOF: in this case, every neighbourhood of  $x$  intersects  $A$  in a point other than  $x$ .

$\langle 1 \rangle 2. A \subseteq \overline{A}$

PROOF: From the definition of  $\overline{A}$ .

$\langle 1 \rangle 3. A' \subseteq \overline{A}$

PROOF: By Theorem 3.8.3.

$\square$

**Corollary 3.10.3.1.** A set is closed if and only if it contains all its limit points.

### 3.11 Subbases

**Definition 3.11.1** (Subbasis). Let  $X$  be a topological space. A *subbasis* for the topology on  $X$  is a set  $\mathcal{S} \subseteq \mathcal{P}X$  such that, for every open set  $U$  and  $x \in U$ , there exist  $S_1, \dots, S_n \in \mathcal{S}$  such that  $x \in S_1 \cap \dots \cap S_n \subseteq U$ . We say the topology is *generated* by  $\mathcal{S}$ .

**Lemma 3.11.2.** Let  $\mathcal{T}$  be a topology on  $X$  and  $\mathcal{S} \subseteq \mathcal{P}X$ . Then the following are equivalent:

1.  $\mathcal{S}$  is a subbasis for  $\mathcal{T}$ .
2. The set of all finite intersections of members of  $\mathcal{S}$  is a basis for  $\mathcal{T}$
3.  $\mathcal{T}$  is the set of all unions of finite intersections of members of  $\mathcal{S}$ .

PROOF:  $1 \Leftrightarrow 2$  holds immediately from the definitions.  $2 \Leftrightarrow 3$  holds by Proposition 3.4.2.  $\square$

**Corollary 3.11.2.1.** If  $\mathcal{S}$  is a subbasis for the topology  $\mathcal{T}$ , then  $\mathcal{T}$  is the coarsest topology in which every element of  $\mathcal{S}$  is open.

**Lemma 3.11.3.** Let  $X$  be a set and  $\mathcal{S} \subseteq \mathcal{P}X$ . Then  $\mathcal{S}$  is a subbasis for a topology on  $X$  if and only if  $\bigcup \mathcal{S} = X$ .

PROOF:

$\langle 1 \rangle 1$ . If  $\mathcal{S}$  is a subbasis for a topology on  $X$  then  $\bigcup \mathcal{S} = X$

$\langle 2 \rangle 1$ . ASSUME:  $\mathcal{S}$  is a subbasis for a topology  $\mathcal{T}$  on  $X$ .

$\langle 2 \rangle 2$ . LET:  $x \in X$

$\langle 2 \rangle 3$ . PICK  $S_1, \dots, S_n \in \mathcal{S}$  such that  $x \in S_1 \cap \dots \cap S_n \subseteq X$

PROOF: From the definition of subbasis ( $\langle 2 \rangle 1$ ,  $\langle 2 \rangle 2$ ).

$\langle 2 \rangle 4$ .  $x \in \bigcup \mathcal{S}$

PROOF: Immediate from  $\langle 2 \rangle 3$ .

$\langle 1 \rangle 2$ . If  $\bigcup \mathcal{S} = X$  then  $\mathcal{S}$  is a subbasis for a topology on  $X$

$\langle 2 \rangle 1$ . ASSUME:  $\bigcup \mathcal{S} = X$

PROVE: The set of all finite intersections of elements of  $\mathcal{S}$  is a basis for a topology on  $X$ .

$\langle 2 \rangle 2$ . LET:  $\mathcal{B}$  be the set of all finite intersections of elements of  $\mathcal{S}$ .

$\langle 2 \rangle 3$ .  $\bigcup \mathcal{B} = X$

PROOF: From  $\langle 2 \rangle 1$  and  $\langle 2 \rangle 2$ .

$\langle 2 \rangle 4$ . For all  $B_1, B_2 \in \mathcal{B}$  and  $x \in B_1 \cap B_2$ , there exists  $B_3 \in \mathcal{B}$  such that

$x \in B_3 \subseteq B_1 \cap B_2$

PROOF: Take  $B_3 = B_1 \cap B_2$  ( $\langle 2 \rangle 2$ ).

$\langle 2 \rangle 5$ .  $\mathcal{B}$  is a basis for a topology on  $X$ .

PROOF: By Lemma 3.4.3.

$\langle 2 \rangle 6$ . Q.E.D.

PROOF: By Lemma 3.11.2.

$\square$

## 3.12 Convergence

**Definition 3.12.1** (Net). Let  $X$  be a topological space. A *net*  $(x_\alpha)_{\alpha \in J}$  in  $X$  consists of a directed set  $J$  and a function  $x : J \rightarrow X$ .

**Definition 3.12.2** (Convergence). Let  $(x_\alpha)_{\alpha \in J}$  be a net in the topological space  $X$ , and  $l \in X$ . Then the net *converges* to  $l$ ,  $x_\alpha \rightarrow l$ , if and only if, for every neighbourhood  $U$  of  $l$ , there exists  $\alpha \in J$  such that, for all  $\beta \geq \alpha$ , we have  $x_\beta \in U$ .

**Theorem 3.12.3** (AC). Let  $X$  be a topological space and  $A \subseteq X$ . Then  $x \in \overline{A}$  if and only if there exists a net of points of  $A$  converging to  $x$ .

PROOF:

$\langle 1 \rangle 1$ . If  $x \in \overline{A}$  then there exists a net of points of  $A$  converging to  $x$ .

$\langle 2 \rangle 1$ . LET:  $x \in \overline{A}$

$\langle 2 \rangle 2$ . LET:  $J$  be the poset of neighbourhoods of  $x$  under  $\supseteq$ .

$\langle 2 \rangle 3$ . For  $U \in J$  PICK a point  $x_U \in U \cap A$

PROOF: By Theorem 3.8.3

$\langle 2 \rangle 4$ .  $(x_U)_{U \in J}$  is a net

PROOF: Given  $U, V \in J$  we have  $U \cap V \in J$  and  $U \supseteq U \cup V, V \supseteq U \cup V$ .

$\langle 2 \rangle 5$ .  $x_U \rightarrow x$

PROOF: For any neighbourhood  $U$  of  $x$  we have  $U \in J$  and if  $U \supseteq V$  then  $x_V \in U$ .

$\langle 1 \rangle 2$ . If there exists a net of points of  $A$  converging to  $x$  then  $x \in \overline{A}$ .

$\langle 2 \rangle 1$ . LET:  $(x_\alpha)_{\alpha \in J}$  be a net of points in  $A$  that converges to  $x$ .

$\langle 2 \rangle 2$ . LET:  $U$  be a neighbourhood of  $x$

$\langle 2 \rangle 3$ . PICK  $\alpha \in J$  such that, for all  $\beta \geq \alpha$ , we have  $x_\beta \in U$

$\langle 2 \rangle 4$ .  $x_\alpha \in U \cap A$

$\langle 2 \rangle 5$ . Q.E.D.

PROOF: By Theorem 3.8.3

□

**Theorem 3.12.4.** Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Then  $f$  is continuous if and only if, for every net  $(x_\alpha)_{\alpha \in J}$  in  $X$ , if  $x_\alpha \rightarrow x$  then  $f(x_\alpha) \rightarrow f(x)$ .

PROOF:

$\langle 1 \rangle 1$ . If  $f$  is continuous and  $x_\alpha \rightarrow x$  then  $f(x_\alpha) \rightarrow f(x)$

$\langle 2 \rangle 1$ . ASSUME:  $f$  is continuous.

$\langle 2 \rangle 2$ . ASSUME:  $x_\alpha \rightarrow x$

$\langle 2 \rangle 3$ . LET:  $V$  be a neighbourhood of  $f(x)$

$\langle 2 \rangle 4$ .  $f^{-1}(V)$  is a neighbourhood of  $x$

$\langle 2 \rangle 5$ . PICK  $\alpha$  such that, for all  $\beta \geq \alpha$ , we have  $x_\beta \in f^{-1}(V)$

$\langle 2 \rangle 6$ . For all  $\beta \geq \alpha$  we have  $f(x_\beta) \in V$

$\langle 1 \rangle 2$ . If, for every net  $(x_\alpha)$  in  $X$ , if  $x_\alpha \rightarrow x$  then  $f(x_\alpha) \rightarrow f(x)$ , then  $f$  is continuous.

$\langle 2 \rangle 1$ . ASSUME: for every net  $(x_\alpha)$  in  $X$ , if  $x_\alpha \rightarrow x$  then  $f(x_\alpha) \rightarrow f(x)$

- (2)2. LET:  $A \subseteq X$   
 PROVE:  $\overline{f(A)} \subseteq f(\overline{A})$   
 (2)3. LET:  $x \in \overline{A}$   
 (2)4. PICK a net  $(x_\alpha)$  in  $A$  such that  $x_\alpha \rightarrow x$   
 PROOF: Theorem 3.12.3  
 (2)5.  $f(x_\alpha) \rightarrow f(x)$   
 PROOF: By (2)1  
 (2)6.  $f(x) \in \overline{f(A)}$   
 PROOF: Theorem 3.12.3  
 (2)7. Q.E.D.  
 PROOF: By Theorem 5.2.2.

□

**Definition 3.12.5** (Subnet). Let  $(x_\alpha)_{\alpha \in J}$  be a net in  $X$ . Let  $K$  be a directed set and  $g : K \rightarrow J$  be a monotone function such that  $g(K)$  is cofinal in  $J$ . Then the net  $(x_{g(\beta)})_{\beta \in K}$  is called a *subnet* of  $(x_\alpha)$ .

### 3.13 Accumulation Points

**Definition 3.13.1** (Accumulation Point). Let  $X$  be a topological space, and  $(x_\alpha)_{\alpha \in J}$  a net in  $X$ , and  $a \in X$ . Then  $a$  is an *accumulation point* of  $(x_\alpha)$  iff, for every neighbourhood  $U$  of  $x$ , the set  $\{\alpha \in J : x_\alpha \in U\}$  is cofinal in  $J$ .

**Lemma 3.13.2.** Let  $X$  be a topological space,  $(x_\alpha)_{\alpha \in J}$  be a nonempty net in  $X$  and  $a \in X$ . Then  $a$  is an accumulation point of  $(x_\alpha)$  if and only if there exists a subnet of  $(x_\alpha)$  that converges to  $a$ .

PROOF:

- (1)1. If  $a$  is an accumulation point of  $(x_\alpha)$  then there exists a subnet of  $(x_\alpha)$  that converges to  $a$ .  
 (2)1. ASSUME:  $a$  is an accumulation point of  $(x_\alpha)$ .  
 (2)2. LET:  $K$  be the poset  $\{(\alpha, U) : \alpha \in J, U \text{ is a neighbourhood of } a, x_\alpha \in U\}$  under:  $(\alpha, U) \leq (\beta, V)$  iff  $\alpha \leq \beta$  and  $U \subseteq V$ .  
 (2)3.  $(x_\alpha)_{(\alpha, U) \in K}$  is a subnet of  $(x_\alpha)_{\alpha \in J}$   
 (3)1.  $K$  is directed.  
 (4)1. LET:  $(\alpha, U), (\beta, V) \in K$   
 (4)2. PICK  $\gamma \in J$  such that  $\alpha \leq \gamma$  and  $\beta \leq \gamma$ .  
 (4)3. PICK  $\delta \in J$  such that  $\gamma \leq \delta$  and  $x_\delta \in U \cap V$   
 PROOF: By (2)1.  
 (4)4.  $(\delta, U \cap V) \in K$  and  $(\alpha, U) \leq (\delta, U \cap V), (\beta, V) \leq (\delta, U \cap V)$   
 (3)2. If  $(\alpha, U) \leq (\beta, V)$  then  $\alpha \leq \beta$   
 PROOF: From (2)2.  
 (3)3.  $\{\alpha : \exists U. (\alpha, U) \in K\}$  is cofinal in  $J$   
 PROOF: For  $\alpha \in J$  we have  $(\alpha, X) \in K$ , so in fact  $\{\alpha : \exists U. (\alpha, U) \in K\} = J$ .  
 (2)4. The subnet converges to  $a$ .





## Chapter 4

# Constructions of Topological Spaces

### 4.1 The Order Topology

**Definition 4.1.1** (Order Topology). Let  $X$  be a linearly ordered set with more than one element. The *order topology* on  $X$  is the topology generated by the basis consisting of:

- all open intervals  $(a, b)$
- all half-open intervals  $(a, \top]$  where  $\top$  is the greatest element of  $X$ , if there is one;
- all half-open intervals  $[\perp, a)$  where  $\perp$  is the least element of  $X$ , if there is one.

We prove this is a basis for a topology.

PROOF:

$\langle 1 \rangle 1$ . LET:  $\mathcal{B}$  be the set of all sets of these three forms.

$\langle 1 \rangle 2$ .  $\bigcup \mathcal{B} = X$

$\langle 2 \rangle 1$ . LET:  $x \in X$

PROVE: There exists  $B \in \mathcal{B}$  such that  $x \in B$

$\langle 2 \rangle 2$ . CASE:  $x$  is least in  $X$

$\langle 3 \rangle 1$ . PICK  $a \in X$  such that  $a > x$

PROOF:  $X$  has more than one element.

$\langle 3 \rangle 2$ .  $x \in [x, a) \in \mathcal{B}$

$\langle 2 \rangle 3$ . CASE:  $x$  is greatest in  $X$

$\langle 3 \rangle 1$ . PICK  $a \in X$  such that  $a < x$

PROOF:  $X$  has more than one element.

$\langle 3 \rangle 2$ .  $x \in (a, x] \in \mathcal{B}$

$\langle 2 \rangle 4$ . CASE:  $x$  is neither least nor greatest in  $X$



**Lemma 4.1.3.** *The open rays form a subbasis for the order topology.*

- ⟨1⟩1. LET:  $X$  be a linearly ordered set with more than one element.
- ⟨1⟩2. The open rays form a subbasis for a topology.
  - ⟨2⟩1. LET:  $x \in X$ 
    - PROVE:  $x$  is an element of an open ray.
  - ⟨2⟩2. CASE:  $x$  is greatest in  $X$ 
    - ⟨3⟩1. PICK  $a \in X$  such that  $a < x$ 
      - PROOF:  $X$  has more than one element (⟨1⟩1).
    - ⟨3⟩2.  $x \in (a, +\infty)$
  - ⟨2⟩3. CASE:  $x$  is not greatest in  $X$ 
    - ⟨3⟩1. PICK  $a \in X$  such that  $x < a$ 
      - ⟨3⟩2.  $x \in (-\infty, a)$
  - ⟨2⟩4. Q.E.D.
    - PROOF: By Lemma 3.11.2.
- ⟨1⟩3. LET:  $\mathcal{T}_o$  be the order topology and  $\mathcal{T}_S$  be the topology generated by the open rays.
  - ⟨1⟩4.  $\mathcal{T}_o \subseteq \mathcal{T}_S$ 
    - ⟨2⟩1. Every open interval  $(a, b)$  is open in  $\mathcal{T}_S$ 
      - PROOF:  $(a, b) = (a, +\infty) \cap (-\infty, b)$ .
    - ⟨2⟩2. If  $\top$  is greatest then  $(a, \top]$  is open in  $\mathcal{T}_S$ 
      - PROOF:  $(a, \top] = (a, +\infty)$ .
    - ⟨2⟩3. If  $\perp$  is least then  $[\perp, b)$  is open in  $\mathcal{T}_S$ 
      - PROOF:  $[\perp, b) = [\perp, +\infty)$ .
    - ⟨2⟩4. Q.E.D.
      - PROOF: By Corollary 3.4.2.1.
  - ⟨1⟩5.  $\mathcal{T}_S \subseteq \mathcal{T}_o$ 
    - ⟨2⟩1. For all  $a \in X$ , we have  $(a, +\infty)$  is open in  $\mathcal{T}_o$ 
      - ⟨3⟩1. LET:  $x \in (a, +\infty)$ 
        - PROVE: There exists a basis element  $B$  such that  $x \in B \subseteq (a, +\infty)$
      - ⟨3⟩2. CASE:  $x$  is greatest
        - PROOF: Take  $B = (a, x]$
      - ⟨3⟩3. CASE:  $x$  is not greatest
        - ⟨4⟩1. PICK  $b > x$ 
          - ⟨4⟩2.  $x \in (a, b) \subseteq (a, +\infty)$
    - ⟨2⟩2. For all  $a \in X$ , we have  $(-\infty, a)$  is open in  $\mathcal{T}_o$ 
      - PROOF: Similar.
    - ⟨2⟩3. Q.E.D.
      - PROOF: By Corollary 3.11.2.1.

□

**Lemma 4.1.4.** *In a linearly ordered set  $X$  under the order topology, the closed intervals and closed rays are closed.*

PROOF:

$$\begin{aligned} X \setminus [a, b] &= (-\infty, a) \cup (b, +\infty) \\ X \setminus (-\infty, a] &= (a, +\infty) \\ X \setminus [a, +\infty) &= (-\infty, a) \end{aligned} \quad \square$$

**Definition 4.1.5** (Standard Topology on  $\mathbb{R}$ ). The *standard topology* on  $\mathbb{R}$  is the order topology.

**Lemma 4.1.6.** *The standard topology is strictly coarser than the lower limit topology.*

PROOF:

- $\langle 1 \rangle 1$ . The standard topology is coarser than the lower limit topology.
  - $\langle 2 \rangle 1$ . For every open interval  $(a, b)$  and  $x \in (a, b)$ , there exists a half-open interval  $[c, d)$  such that  $x \in [c, d) \subseteq (a, b)$ .

PROOF: Take  $[c, d) = [x, b)$ .
  - $\langle 2 \rangle 2$ . Q.E.D.

PROOF: By Lemma 3.4.4.
- $\langle 1 \rangle 2$ . There exists a set  $U$  open in the lower limit topology that is not open in the standard topology.

PROOF: Take  $U = [0, 1)$ .

$\square$

**Lemma 4.1.7.** *The standard topology is strictly coarser than the  $K$ -topology.*

PROOF:

- $\langle 1 \rangle 1$ . The standard topology is coarser than the  $K$ -topology.

PROOF: Every open interval is open in the  $K$ -topology.
- $\langle 1 \rangle 2$ . There exists a set  $U$  open in the  $K$ -topology that is not open in the standard topology.

PROOF: Take  $U = (-1, 1) \setminus K$ . Then  $0 \in U$  but there is no open interval  $(a, b)$  such that  $0 \in (a, b) \subseteq U$ .

$\square$

**Definition 4.1.8** (Ordered Square). The *ordered square*  $I_o^2$  is the topological space  $[0, 1]^2$  under the order topology induced by the lexicographic order.

**Lemma 4.1.9.** *Let  $L$  be a linear continuum with a greatest element. Then every non-empty closed set in  $L$  has a greatest element.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $C$  be a non-empty closed set in  $L$
- $\langle 1 \rangle 2$ . LET:  $u$  be the supremum of  $C$
- $\langle 1 \rangle 3$ .  $u \in C$ 
  - $\langle 2 \rangle 1$ . ASSUME: w.l.o.g  $u$  is not least in  $L$ 

PROOF: If  $u$  is least then  $C = \{u\}$ .
  - $\langle 2 \rangle 2$ . LET:  $U$  be any open neighbourhood of  $u$
  - $\langle 2 \rangle 3$ . PICK  $v < u$  such that  $(v, u] \subseteq U$

PROOF: By Lemma 4.1.2.

(2)4. PICK  $x \in C$  such that  $v < x$

PROOF:  $v$  is not an upper bound for  $C$  ( $\langle 1 \rangle 2$ ).

(2)5.  $U$  intersects  $C$  in  $v$

(2)6. Q.E.D.

PROOF: By Theorem 3.8.3.

□

**Definition 4.1.10** (Long Line). The *long line* is  $(S_\Omega \times [0, 1)) \setminus \{(0, 0)\}$  under the dictionary order, where  $S_\Omega$  is the first uncountable ordinal under the order topology.

## 4.2 The Product Topology

**Definition 4.2.1** (Product Topology). Let  $\{X_\alpha\}_{\alpha \in J}$  be a family of topological spaces. The *product topology* on  $\prod_{\alpha \in J} X_\alpha$  is the topology generated by the subbasis consisting of all sets of the form  $\pi_\alpha^{-1}(U)$  where  $\alpha \in J$  and  $U$  is open in  $X_\alpha$ . The *product space* of  $\{X_\alpha\}_{\alpha \in J}$  is  $\prod_{\alpha \in J} X_\alpha$  under the product topology.

**Lemma 4.2.2.** Let  $\{X_\alpha\}_{\alpha \in J}$  be a family of topological spaces and  $A_\alpha$  be closed in  $X_\alpha$  for all  $\alpha$ . Then  $\prod_{\alpha \in J} A_\alpha$  is closed in  $\prod_{\alpha \in J} X_\alpha$ .

PROOF: This holds because  $\prod_{\alpha \in J} X_\alpha \setminus \prod_{\alpha \in J} A_\alpha = \bigcup_{\alpha \in J} \pi_\alpha^{-1}(X_\alpha \setminus A_\alpha)$ . □

**Theorem 4.2.3.** Let  $\{X_\alpha\}_{\alpha \in J}$  be a family of topological spaces. The set of all sets of the form  $\prod_{\alpha \in J} U_\alpha$  where each  $U_\alpha$  is open in  $X_\alpha$ , and  $U_\alpha = X_\alpha$  for all but finitely many  $\alpha$ , is a basis for the product topology on  $\prod_{\alpha \in J} X_\alpha$ .

PROOF: By Lemma 3.11.2. □

**Theorem 4.2.4.** Let  $\{X_\alpha\}_{\alpha \in J}$  be a family of topological spaces, and let  $\mathcal{B}_\alpha$  be a basis for the topology on  $X_\alpha$  for each  $\alpha$ . Then

$$\mathcal{B} = \left\{ \prod_{\alpha \in J} U_\alpha : \text{for finitely many } \alpha \in J, U_\alpha \in \mathcal{B}_\alpha, \right. \\ \left. \text{and } U_\alpha = X_\alpha \text{ for all other values of } \alpha \right\}$$

is a basis for the product topology on  $\prod_{\alpha \in J} X_\alpha$ .

PROOF:

(1)1. Every member of  $\mathcal{B}$  is open in the product topology.

PROOF: Immediate from definitions.

(1)2. For every open set  $U$  and  $\{x_\alpha\}_{\alpha \in J} \in U$ , there exists  $B \in \mathcal{B}$  such that  $\{x_\alpha\}_{\alpha \in J} \in B \subseteq U$ .

(2)1. LET:  $U$  be open and  $\{x_\alpha\}_{\alpha \in J} \in U$

(2)2. PICK  $U_\alpha$  open in  $X_\alpha$  for each  $\alpha$  such that  $\{x_\alpha\}_{\alpha \in J} \in \prod_{\alpha \in J} U_\alpha \subseteq U$  and  $U_\alpha = X_\alpha$  for all  $\alpha$  except  $\alpha_1, \dots, \alpha_n$ .

PROOF: By Theorem 4.2.3.

- (2)3. PICK  $B_{\alpha_i} \in \mathcal{B}_{\alpha_i}$  such that  $x_\alpha \in B_{\alpha_i} \subseteq U_{\alpha_i}$  for  $i = 1, \dots, n$   
 (2)4.  $\{x_\alpha\}_{\alpha \in J} \in \prod_{\alpha \in J} V_\alpha \subseteq U$  where  $V_{\alpha_i} = B_{\alpha_i}$  for  $i = 1, \dots, n$ , and  $V_\alpha = X_\alpha$  for all other  $\alpha$ .

□

**Theorem 4.2.5 (AC).** *Let  $\{X_\alpha\}_{\alpha \in J}$  be a family of topological spaces and  $A_\alpha \subseteq X_\alpha$  for all  $\alpha$ . If  $\prod_{\alpha \in J} X_\alpha$  is given the product topology, then*

$$\prod_{\alpha \in J} \overline{A_\alpha} = \overline{\prod_{\alpha \in J} A_\alpha} .$$

PROOF:

- (1)1.  $\prod_{\alpha \in J} \overline{A_\alpha} \subseteq \overline{\prod_{\alpha \in J} A_\alpha}$   
 (2)1. LET:  $\{x_\alpha\}_{\alpha \in J} \in \prod_{\alpha \in J} \overline{A_\alpha}$   
 (2)2. LET:  $\prod_{\alpha \in J} U_\alpha$  be a basic neighbourhood of  $\{x_\alpha\}_{\alpha \in J}$ , where each  $U_\alpha$  is open in  $X_\alpha$ , and  $U_\alpha = X_\alpha$  except for  $\alpha = \alpha_1, \dots, \alpha_n$ .  
 (2)3. For  $\alpha \in J$ , PICK  $a_\alpha \in A_\alpha \cap U_\alpha$ .  
 PROOF: By Theorem 3.8.3, using the Axiom of Choice.  
 (2)4.  $\{a_\alpha\}_{\alpha \in J} \in \prod_{\alpha \in J} A_\alpha \cap \prod_{\alpha \in J} U_\alpha$   
 (2)5. Q.E.D.

PROOF: By Theorem 3.8.3.

- (1)2.  $\overline{\prod_{\alpha \in J} A_\alpha} \subseteq \prod_{\alpha \in J} \overline{A_\alpha}$   
 (2)1. LET:  $\{x_\alpha\}_{\alpha \in J} \in \overline{\prod_{\alpha \in J} A_\alpha}$   
 (2)2. LET:  $\alpha \in J$   
 PROVE:  $x_\alpha \in \overline{A_\alpha}$   
 (2)3. LET:  $U$  be a neighbourhood of  $x_\alpha$  in  $X_\alpha$   
 (2)4.  $\pi_\alpha^{-1}(U)$  is a neighbourhood of  $\{x_\alpha\}_{\alpha \in J}$   
 (2)5. PICK  $\{a_\alpha\}_{\alpha \in J} \in \pi_\alpha^{-1}(U) \cap \prod_{\alpha \in J} A_\alpha$   
 PROOF: By Theorem 3.8.3.  
 (2)6.  $a_\alpha \in U \cap A_\alpha$   
 (2)7. Q.E.D.

PROOF: By Theorem 3.8.3.

□

**Definition 4.2.6** (Standard Topology on  $\mathbb{R}^J$ ). For  $J$  a set, the *standard topology* on  $\mathbb{R}^J$  is the product topology where  $\mathbb{R}$  is given the standard topology.

**Definition 4.2.7** (Closed Unit Ball). The *closed unit ball*  $B^2$  is  $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$  as a subset of  $\mathbb{R}^2$ .

**Definition 4.2.8** (Sorgenfrey Plane). The *Sorgenfrey plane* is  $\mathbb{R}_l^2$ .

## 4.3 The Subspace Topology

**Definition 4.3.1** (Subspace Topology). Let  $X$  be a topological space and  $Y \subseteq X$ . The *subspace topology* on  $Y$  is  $\{Y \cap U : U \text{ open in } X\}$ . With this topology,  $Y$  is a *subspace* of  $X$ .

We prove this is a topology.

PROOF:

$\langle 1 \rangle 1$ . LET:  $\mathcal{T} = \{Y \cap U : U \text{ open in } X\}$

$\langle 1 \rangle 2$ .  $Y \in \mathcal{T}$

PROOF:  $Y = Y \cap X$

$\langle 1 \rangle 3$ .  $\mathcal{T}$  is closed under union.

$\langle 2 \rangle 1$ . LET:  $\mathcal{A} \subseteq \mathcal{T}$

PROVE:  $\bigcup \mathcal{A} = Y \cap \bigcup \{U \text{ open in } X : Y \cap U \in \mathcal{A}\}$

$\langle 2 \rangle 2$ .  $\bigcup \mathcal{A} \subseteq Y \cap \bigcup \{U \text{ open in } X : Y \cap U \in \mathcal{A}\}$

$\langle 3 \rangle 1$ . LET:  $x \in \bigcup \mathcal{A}$

$\langle 3 \rangle 2$ . PICK  $V \in \mathcal{A}$  such that  $x \in V$

$\langle 3 \rangle 3$ . PICK  $U$  open in  $X$  such that  $V = Y \cap U$

PROOF: By the definition of  $\mathcal{T}$  ( $\langle 1 \rangle 1$ ,  $\langle 2 \rangle 1$ ,  $\langle 3 \rangle 2$ )

$\langle 3 \rangle 4$ .  $x \in Y \cap \bigcup \{U \text{ open in } X : Y \cap U \in \mathcal{A}\}$

$\langle 2 \rangle 3$ .  $Y \cap \bigcup \{U \text{ open in } X : Y \cap U \in \mathcal{A}\} \subseteq \bigcup \mathcal{A}$

PROOF: Set theory.

$\langle 1 \rangle 4$ .  $\mathcal{T}$  is closed under binary intersection.

PROOF: This holds because  $(Y \cap U) \cap (Y \cap V) = Y \cap (U \cap V)$ .

□

**Lemma 4.3.2.** *Let  $X$  be a topological space,  $Y \subseteq X$ , and  $A \subseteq Y$ . Then the topology  $A$  inherits as a subspace of  $X$  is the same as the topology  $A$  inherits as a subspace of  $Y$ .*

PROOF:

topology as a subspace of  $Y$

$= \{V \cap A : V \text{ open in } Y\}$

$= \{V \cap A : \exists U \text{ open in } X. V = U \cap Y\}$

$= \{U \cap Y \cap A : U \text{ open in } X\}$

$= \{U \cap A : U \text{ open in } X\}$

$= \text{topology as a subspace of } X$  □

**Lemma 4.3.3.** *Let  $Y$  be a subspace of  $X$ . If  $U$  is open in  $Y$  and  $Y$  is open in  $X$  then  $U$  is open in  $X$ .*

PROOF:

$\langle 1 \rangle 1$ . PICK  $V$  open in  $X$  such that  $U = Y \cap V$

$\langle 1 \rangle 2$ .  $U$  is open in  $X$

PROOF: The open sets in  $X$  are closed under binary intersection.

□

**Theorem 4.3.4.** *Let  $Y$  be a subspace of  $X$ . Let  $A \subseteq Y$ . Let  $\bar{A}$  be the closure of  $A$  in  $X$ . Then the closure of  $A$  in  $Y$  is  $\bar{A} \cap Y$ .*

PROOF:

$\langle 1 \rangle 1$ .  $\bar{A} \cap Y$  is a closed set in  $Y$  that includes  $A$ .

$\langle 2 \rangle 1$ .  $\bar{A} \cap Y$  is closed in  $Y$ .

PROOF: By Lemma 4.3.4.1.



- (2)2.  $A \subseteq \overline{A} \cap Y$ .
- (1)2. If  $C$  is any closed set in  $Y$  that includes  $A$  then  $\overline{A} \cap Y \subseteq C$ .
- (2)1. LET:  $C$  be a closed set in  $Y$  that includes  $A$ .
- (2)2. PICK  $D$  closed in  $X$  such that  $C = D \cap Y$ .
- PROOF: By Lemma 4.3.4.1.
- (2)3.  $\overline{A} \subseteq D$
- (2)4.  $\overline{A} \subseteq C$

□

**Corollary 4.3.4.1.** *Let  $Y$  be a subspace of  $X$ . Then a set  $A \subseteq Y$  is closed in  $Y$  if and only if it is the intersection of a closed set in  $X$  with  $Y$ .*

**Corollary 4.3.4.2.** *Let  $Y$  be a subspace of  $X$ . If  $A$  is closed in  $Y$  and  $Y$  is closed in  $X$  then  $A$  is closed in  $X$ .*

**Lemma 4.3.5.** *Let  $X$  be a topological space and  $Y \subseteq X$ . If  $\mathcal{B}$  is a basis for the topology on  $X$  then  $\{B \cap Y : B \in \mathcal{B}\}$  is a basis for the subspace topology on  $Y$ .*

PROOF:

- (1)1. For all  $B \in \mathcal{B}$ , we have  $B \cap Y$  is open in  $Y$ .
- PROOF: Immediate from definitions.
- (1)2. For every  $V$  open in  $Y$  and  $y \in V$ , there exists  $B \in \mathcal{B}$  such that  $y \in B \cap Y \subseteq V$ .
- (2)1. LET:  $V$  be open in  $Y$  and  $y \in V$
- (2)2. PICK  $U$  open in  $X$  such that  $V = Y \cap U$
- (2)3. PICK  $B \in \mathcal{B}$  such that  $y \in B \subseteq U$
- (2)4.  $y \in B \cap Y \subseteq V$

□

**Lemma 4.3.6.** *Let  $X$  be a topological space and  $Y \subseteq X$ . If  $\mathcal{S}$  is a subbasis for the topology on  $X$  then  $\{S \cap Y : S \in \mathcal{S}\}$  is a subbasis for the subspace topology on  $Y$ .*

PROOF:

- (1)1. For all  $S \in \mathcal{S}$ , we have  $S \cap Y$  is open in  $Y$ .
- PROOF: Immediate from definitions.
- (1)2. For every  $V$  open in  $Y$  and  $y \in V$ , there exist  $S_1, \dots, S_n \in \mathcal{S}$  such that  $y \in (S_1 \cap Y) \cap \dots \cap (S_n \cap Y) \subseteq V$
- (2)1. LET:  $V$  be open in  $Y$  and  $y \in V$
- (2)2. PICK  $U$  open in  $X$  such that  $V = U \cap Y$
- (2)3. PICK  $S_1, \dots, S_n \in \mathcal{S}$  such that  $y \in S_1 \cap \dots \cap S_n \subseteq U$
- (2)4.  $y \in (S_1 \cap Y) \cap \dots \cap (S_n \cap Y) \subseteq V$

□

**Theorem 4.3.7.** *Let  $X$  be a linearly ordered set in the order topology. Let  $Y \subseteq X$  be convex. Then the order topology on  $Y$  is the same as the subspace topology.*

PROOF:

- (1)1. LET:  $\mathcal{T}_o$  be the order topology and  $\mathcal{T}_s$  be the subspace topology.  
 (1)2.  $\mathcal{T}_o \subseteq \mathcal{T}_s$   
 (2)1. For all  $a \in Y$ , we have  $\{y \in Y : a < y\} \in \mathcal{T}_s$   
 PROOF:  $\{y \in Y : a < y\} = \{x \in X : a < x\} \cap Y$   
 (2)2. For all  $a \in Y$ , we have  $\{y \in Y : y < a\} \in \mathcal{T}_s$   
 PROOF: Similar.  
 (2)3. Q.E.D.  
 PROOF: Lemma 4.1.3 and Corollary 3.11.2.1.  
 (1)3.  $\mathcal{T}_s \subseteq \mathcal{T}_o$   
 (2)1. The sets  $(a, +\infty) \cap Y$  and  $(-\infty, a) \cap Y$  for  $a \in X$  form a subbasis for  $\mathcal{T}_s$   
 PROOF: Lemma 4.3.6, Lemma 4.1.3.  
 (2)2. For all  $a \in X$ , we have  $(a, +\infty) \cap Y \in \mathcal{T}_o$   
 (3)1. LET:  $a \in X$   
 (3)2. CASE:  $a \in Y$   
 PROOF: In this case,  $(a, +\infty) \cap Y$  is an open ray in  $Y$ .  
 (3)3. CASE: For all  $y \in Y$  we have  $a < y$   
 PROOF: In this case,  $(a, +\infty) \cap Y = Y$ .  
 (3)4. CASE: For all  $y \in Y$  we have  $y < a$   
 PROOF: In this case,  $(a, +\infty) \cap Y = \emptyset$ .  
 (3)5. Q.E.D.  
 PROOF: These are the only cases because  $Y$  is convex.  
 (2)3. For all  $a \in X$ , we have  $(-\infty, a) \cap Y \in \mathcal{T}_o$   
 PROOF: Similar.  
 (2)4. Q.E.D.  
 PROOF: Corollary 3.11.2.1.

□

**Theorem 4.3.8.** *Let  $\{X_\alpha\}_{\alpha \in J}$  be a family of topological spaces, and let  $A_\alpha$  be a subspace of  $X_\alpha$  for all  $\alpha$ . Then the product topology on  $\prod_{\alpha \in J} A_\alpha$  is the same as the topology it inherits as a subspace of  $\prod_{\alpha \in J} X_\alpha$ .*

PROOF: Each is the topology generated by the subbasis consisting of  $\pi_\alpha^{-1}(U) \cap \prod_{\alpha \in J} A_\alpha = \pi_\alpha^{-1}(U \cap A_\alpha)$  where  $\alpha \in J$  and  $U$  is open in  $X_\alpha$ , using Lemma 4.3.6.

□

**Definition 4.3.9** (Unit Circle). The *unit circle*  $S^1$  is  $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$  as a subspace of  $\mathbb{R}^2$ .

**Proposition 4.3.10.** *Let  $Y$  be a subspace of  $X$ ,  $A \subseteq Y$ , and  $a \in Y$ . Then  $a$  is a limit point of  $A$  in the subspace topology on  $Y$  if and only if  $a$  is a limit point of  $A$  in the topology of  $X$ .*

PROOF:

$$\begin{aligned}
& a \text{ is a limit point of } A \text{ in } Y \\
& \Leftrightarrow \forall U \text{ open in } Y (a \in U \Rightarrow U \text{ intersects } A \text{ outside } a) \\
& \Leftrightarrow \forall V \text{ open in } X (a \in V \cap Y \Rightarrow V \cap Y \text{ intersects } A \text{ outside } a) \\
& \Leftrightarrow \forall V \text{ open in } X (a \in V \Rightarrow V \text{ intersects } A \text{ outside } a) \\
& \quad (a \in Y, A \subseteq Y) \\
& \Leftrightarrow a \text{ is a limit point of } A \text{ in } X
\end{aligned}$$

□

## 4.4 The Box Topology

**Definition 4.4.1** (Box Topology). Let  $\{X_\alpha\}_{\alpha \in J}$  be a family of topological spaces. The *box topology* on  $\prod_{\alpha \in J} X_\alpha$  is the topology generated by the basis consisting of all sets of the form  $\prod_{\alpha \in J} U_\alpha$ , where each  $U_\alpha$  is open in  $X_\alpha$ .

We prove this is a basis.

PROOF:

- ⟨1⟩1. LET:  $\mathcal{B}$  be the set of all sets of the form  $\prod_{\alpha \in J} U_\alpha$ , where each  $U_\alpha$  is open in  $X_\alpha$ .
- ⟨1⟩2.  $\bigcup \mathcal{B} = \prod_{\alpha \in J} X_\alpha$   
PROOF: This holds because  $\prod_{\alpha \in J} X_\alpha \in \mathcal{B}$ .
- ⟨1⟩3.  $\mathcal{B}$  is closed under binary intersection.  
PROOF:  $\prod_{\alpha \in J} U_\alpha \cap \prod_{\alpha \in J} V_\alpha = \prod_{\alpha \in J} (U_\alpha \cap V_\alpha)$ .
- ⟨1⟩4. Q.E.D.  
PROOF: Corollary 3.4.3.1.

**Theorem 4.4.2** (AC). Let  $\{X_\alpha\}_{\alpha \in J}$  be a family of topological spaces, and let  $\mathcal{B}_\alpha$  be a basis for the topology on  $X_\alpha$  for each  $\alpha$ . Then

$$\mathcal{B} = \left\{ \prod_{\alpha \in J} B_\alpha : \forall \alpha \in J, B_\alpha \in \mathcal{B}_\alpha \right\}$$

is a basis for the box topology on  $\prod_{\alpha \in J} X_\alpha$ .

PROOF:

- ⟨1⟩1. Every member of  $\mathcal{B}$  is open in the box topology.  
PROOF: Immediate from definitions.
- ⟨1⟩2. For every open set  $U$  and  $\{x_\alpha\}_{\alpha \in J} \in U$ , there exists  $B \in \mathcal{B}$  such that  $\{x_\alpha\}_{\alpha \in J} \in B \subseteq U$ .  
  - ⟨2⟩1. LET:  $U$  be open and  $\{x_\alpha\}_{\alpha \in J} \in U$
  - ⟨2⟩2. PICK  $U_\alpha$  open in  $X_\alpha$  for each  $\alpha$  such that  $\{x_\alpha\}_{\alpha \in J} \in \prod_{\alpha \in J} U_\alpha \subseteq U$ .
  - ⟨2⟩3. PICK  $B_\alpha \in \mathcal{B}_\alpha$  such that  $x_\alpha \in B_\alpha \subseteq U_\alpha$  for each  $\alpha$   
PROOF: Using the Axiom of Choice.
  - ⟨2⟩4.  $\{x_\alpha\}_{\alpha \in J} \in \prod_{\alpha \in J} B_\alpha \subseteq U$

□

**Theorem 4.4.3.** *Let  $\{X_\alpha\}_{\alpha \in J}$  be a family of topological spaces, and let  $A_\alpha$  be a subspace of  $X_\alpha$  for all  $\alpha$ . Let  $\prod_{\alpha \in J} X_\alpha$  be given the box topology. Then the box topology on  $\prod_{\alpha \in J} A_\alpha$  is the same as the topology it inherits as a subspace of  $\prod_{\alpha \in J} X_\alpha$ .*

PROOF: Each is the topology generated by the basis  $\{\prod_{\alpha \in J} (U_\alpha \cap A_\alpha) : U_\alpha \text{ is open in } X_\alpha\}$ , using Lemma 4.3.5.  $\square$

**Theorem 4.4.4.** *Let  $\{X_\alpha\}_{\alpha \in J}$  be a family of Hausdorff spaces. Then  $\prod_{\alpha \in J} X_\alpha$  is Hausdorff under the box topology.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $\{x_\alpha\}_{\alpha \in J}, \{y_\alpha\}_{\alpha \in J} \in \prod_{\alpha \in J} X_\alpha$  with  $\{x_\alpha\}_{\alpha \in J} \neq \{y_\alpha\}_{\alpha \in J}$
  - $\langle 1 \rangle 2$ . PICK  $\alpha \in J$  such that  $x_\alpha \neq y_\alpha$
  - $\langle 1 \rangle 3$ . PICK disjoint neighbourhoods  $U$  of  $x_\alpha$  and  $V$  of  $y_\alpha$ .
  - $\langle 1 \rangle 4$ .  $\pi_\alpha^{-1}(U)$  and  $\pi_\alpha^{-1}(V)$  are disjoint neighbourhoods of  $\{x_\alpha\}_{\alpha \in J}$  and  $\{y_\alpha\}_{\alpha \in J}$
- $\square$

**Corollary 4.4.4.1.** *The space  $\mathbb{R}^\omega$  under the box topology is Hausdorff.*

**Theorem 4.4.5 (AC).** *Let  $\{X_\alpha\}_{\alpha \in J}$  be a family of topological spaces and  $A_\alpha \subseteq X_\alpha$  for all  $\alpha$ . If  $\prod_{\alpha \in J} X_\alpha$  is given the box topology, then*

$$\prod_{\alpha \in J} \overline{A_\alpha} = \overline{\prod_{\alpha \in J} A_\alpha}.$$

PROOF:

- $\langle 1 \rangle 1$ .  $\prod_{\alpha \in J} \overline{A_\alpha} \subseteq \overline{\prod_{\alpha \in J} A_\alpha}$
  - $\langle 2 \rangle 1$ . LET:  $\{x_\alpha\}_{\alpha \in J} \in \prod_{\alpha \in J} \overline{A_\alpha}$
  - $\langle 2 \rangle 2$ . LET:  $\prod_{\alpha \in J} U_\alpha$  be a basic neighbourhood of  $\{x_\alpha\}_{\alpha \in J}$ , where each  $U_\alpha$  is open in  $X_\alpha$ .
  - $\langle 2 \rangle 3$ . For  $\alpha \in J$ , PICK  $a_\alpha \in A_\alpha \cap U_\alpha$ .
- PROOF: By Theorem 3.8.3, using the Axiom of Choice.
- $\langle 2 \rangle 4$ .  $\{a_\alpha\}_{\alpha \in J} \in \prod_{\alpha \in J} A_\alpha \cap \prod_{\alpha \in J} U_\alpha$
  - $\langle 2 \rangle 5$ . Q.E.D.

PROOF: By Theorem 3.8.3.

- $\langle 1 \rangle 2$ .  $\overline{\prod_{\alpha \in J} A_\alpha} \subseteq \prod_{\alpha \in J} \overline{A_\alpha}$
  - $\langle 2 \rangle 1$ . LET:  $\{x_\alpha\}_{\alpha \in J} \in \overline{\prod_{\alpha \in J} A_\alpha}$
  - $\langle 2 \rangle 2$ . LET:  $\alpha \in J$
- PROVE:  $x_\alpha \in \overline{A_\alpha}$
- $\langle 2 \rangle 3$ . LET:  $U$  be a neighbourhood of  $x_\alpha$  in  $X_\alpha$
  - $\langle 2 \rangle 4$ .  $\pi_\alpha^{-1}(U)$  is a neighbourhood of  $\{x_\alpha\}_{\alpha \in J}$
  - $\langle 2 \rangle 5$ . PICK  $\{a_\alpha\}_{\alpha \in J} \in \pi_\alpha^{-1}(U) \cap \prod_{\alpha \in J} A_\alpha$
- PROOF: By Theorem 3.8.3.
- $\langle 2 \rangle 6$ .  $a_\alpha \in U \cap A_\alpha$
  - $\langle 2 \rangle 7$ . Q.E.D.

PROOF: By Theorem 3.8.3.

$\square$

## 4.5 The Quotient Topology

**Definition 4.5.1** (Quotient Map). Let  $X$  and  $Y$  be topological spaces. Let  $p : X \twoheadrightarrow Y$  be a surjective map. Then  $p$  is a *quotient map* iff, for all  $U \subseteq Y$ , we have  $U$  is open in  $Y$  iff  $p^{-1}(U)$  is open in  $X$ .

**Lemma 4.5.2.** Let  $X$  and  $Y$  be topological spaces and  $p : X \twoheadrightarrow Y$  be surjective and continuous. Then the following are equivalent.

1.  $p$  is a quotient map.
2.  $p$  maps saturated open sets to open sets.
3.  $p$  maps saturated closed sets to closed sets.

PROOF:

$\langle 1 \rangle 1. 1 \Rightarrow 2$

$\langle 2 \rangle 1.$  ASSUME:  $p$  is a quotient map.

$\langle 2 \rangle 2.$  LET:  $U \subseteq Y$  be a saturated open set.

$\langle 2 \rangle 3.$   $U = p^{-1}(p(U))$

$\langle 3 \rangle 1.$   $U \subseteq p^{-1}(p(U))$

PROOF: Set theory.

$\langle 3 \rangle 2.$   $p^{-1}(p(U)) \subseteq U$

$\langle 4 \rangle 1.$  LET:  $x \in p^{-1}(p(U))$

$\langle 4 \rangle 2.$  PICK  $y \in U$  such that  $p(x) = p(y)$

$\langle 4 \rangle 3.$   $x \in U$

PROOF:  $\langle 2 \rangle 2, \langle 4 \rangle 2.$

$\langle 2 \rangle 4.$   $p(U)$  is open

PROOF:  $\langle 2 \rangle 1, \langle 2 \rangle 3.$

$\langle 1 \rangle 2. 2 \Rightarrow 3$

$\langle 2 \rangle 1.$  ASSUME:  $p$  maps saturated open sets to open sets

$\langle 2 \rangle 2.$  LET:  $C \subseteq X$  be a saturated closed set.

$\langle 2 \rangle 3.$   $X \setminus C$  is a saturated open set.

$\langle 3 \rangle 1.$  LET:  $x \in X \setminus C$  and  $x' \in X$  be such that  $p(x) = p(x')$

$\langle 3 \rangle 2.$   $x' \notin C$

PROOF: If  $x' \in C$  then  $x \in C$  since  $C$  is saturated.

$\langle 2 \rangle 4.$   $p(X \setminus C)$  is open.

PROOF: By  $\langle 2 \rangle 1$  and  $\langle 2 \rangle 3.$

$\langle 2 \rangle 5.$   $p(X \setminus C) = Y \setminus p(C)$

$\langle 3 \rangle 1.$   $p(X \setminus C) \subseteq Y \setminus p(C)$

$\langle 4 \rangle 1.$  LET:  $x \in X \setminus C$

$\langle 4 \rangle 2.$  ASSUME: for a contradiction  $p(x) \in p(C)$

$\langle 4 \rangle 3.$  PICK  $x' \in C$  such that  $p(x) = p(x')$

$\langle 4 \rangle 4.$  Q.E.D.

PROOF: We have  $x \notin C, x' \in C$  and  $p(x) = p(x')$ , contradicting  $\langle 2 \rangle 2.$

$\langle 3 \rangle 2.$   $Y \setminus p(C) \subseteq p(X \setminus C)$

$\langle 4 \rangle 1.$  LET:  $y \notin p(C)$

$\langle 4 \rangle 2.$  PICK  $x \in X$  such that  $p(x) = y$

PROOF:  $p$  is surjective.

$\langle 4 \rangle 3. x \notin C$

$\langle 1 \rangle 3. 3 \Rightarrow 1$

$\langle 2 \rangle 1.$  ASSUME:  $p$  maps saturated closed sets to closed sets

$\langle 2 \rangle 2.$  LET:  $C \subseteq Y$  be such that  $p^{-1}(Y)$  is closed

$\langle 2 \rangle 3. p^{-1}(C)$  is saturated

$\langle 3 \rangle 1.$  LET:  $x \in p^{-1}(C)$ ,  $x' \in X$  and  $p(x) = p(x')$

$\langle 3 \rangle 2. x' \in p^{-1}(C)$

$\langle 2 \rangle 4. p(p^{-1}(C))$  is closed

PROOF: By  $\langle 2 \rangle 1$  and  $\langle 2 \rangle 3.$

$\langle 2 \rangle 5. C = p(p^{-1}(C))$

PROOF: By set theory, since  $p$  is surjective.

□

**Corollary 4.5.2.1.** *If  $p : X \rightarrow Y$  is a surjective continuous map that is either an open map or a closed map, then  $p$  is a quotient map.*

**Definition 4.5.3** (Quotient Topology). Let  $X$  be a topological space,  $A$  a set, and  $p : X \rightarrow A$  a surjective map. Then the *quotient topology* on  $A$  induced by  $p$  is

$$\{U \subseteq A : p^{-1}(U) \text{ is open in } X\} .$$

It is easy to check this is a topology.

**Lemma 4.5.4.** *Let  $X$  be a topological space,  $A$  a set, and  $p : X \rightarrow A$  a surjective map. Then the quotient topology induced by  $p$  is the unique topology on  $A$  such that  $p$  is a quotient map.*

PROOF: Immediate from definitions. □

**Definition 4.5.5** (Quotient Space). Let  $X$  be a topological space and  $X^*$  a partition of  $X$ . Let  $p : X \rightarrow X^*$  be the canonical map. Then  $X^*$  under the quotient topology induced by  $p$  is called a *quotient space* of  $X$ .

**Proposition 4.5.6.** *Let  $p : X \rightarrow Y$  be a quotient map. Let  $A \subseteq X$  be open and saturated. Then  $p \upharpoonright_A : A \rightarrow p(A)$  is a quotient map.*

PROOF:

$\langle 1 \rangle 1.$  LET:  $q = p \upharpoonright_A : A \rightarrow p(A)$

$\langle 1 \rangle 2.$  For all  $V \subseteq p(A)$ , we have  $q^{-1}(V) = p^{-1}(V)$

$\langle 2 \rangle 1. q^{-1}(V) \subseteq p^{-1}(V)$

PROOF: Trivial.

$\langle 2 \rangle 2. p^{-1}(V) \subseteq q^{-1}(V)$

$\langle 3 \rangle 1.$  LET:  $x \in p^{-1}(V)$

$\langle 3 \rangle 2.$  PICK  $x' \in A$  such that  $p(x') = p(x)$

PROOF: One exists because  $p(x) \in V \subseteq p(A)$ .

$\langle 3 \rangle 3. x \in A$

PROOF: This holds because  $A$  is saturated.

$\langle 3 \rangle 4. x \in q^{-1}(V)$

PROOF: From  $\langle 3 \rangle 1$  and  $\langle 3 \rangle 3$ .

- $\langle 1 \rangle 3$ . For all  $U \subseteq X$ , we have  $p(U \cap A) = p(U) \cap p(A)$
- $\langle 1 \rangle 4$ . LET:  $V \subseteq p(A)$  be such that  $q^{-1}(V)$  is open in  $A$ .  
PROVE:  $V$  is open in  $p(A)$ .
- $\langle 1 \rangle 5$ .  $q^{-1}(V)$  is open in  $X$
- $\langle 1 \rangle 6$ .  $p^{-1}(V)$  is open in  $X$
- $\langle 1 \rangle 7$ .  $V$  is open in  $Y$
- $\langle 1 \rangle 8$ .  $V$  is open in  $p(A)$

□

**Proposition 4.5.7.** *Let  $p : X \twoheadrightarrow Y$  be a quotient map. Let  $A \subseteq X$  be closed and saturated. Then  $p \upharpoonright_A : A \twoheadrightarrow p(A)$  is a quotient map.*

PROOF: Similar. □

**Proposition 4.5.8.** *Let  $p : X \twoheadrightarrow Y$  be an open quotient map. Let  $A \subseteq X$  be saturated. Then  $p \upharpoonright_A : A \twoheadrightarrow p(A)$  is a quotient map.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $q = p \upharpoonright_A : A \twoheadrightarrow p(A)$
- $\langle 1 \rangle 2$ . For all  $V \subseteq p(A)$ , we have  $q^{-1}(V) = p^{-1}(V)$ 
  - $\langle 2 \rangle 1$ .  $q^{-1}(V) \subseteq p^{-1}(V)$   
PROOF: Trivial.
  - $\langle 2 \rangle 2$ .  $p^{-1}(V) \subseteq q^{-1}(V)$ 
    - $\langle 3 \rangle 1$ . LET:  $x \in p^{-1}(V)$
    - $\langle 3 \rangle 2$ . PICK  $x' \in A$  such that  $p(x') = p(x)$   
PROOF: One exists because  $p(x) \in V \subseteq p(A)$ .
    - $\langle 3 \rangle 3$ .  $x \in A$   
PROOF: This holds because  $A$  is saturated.
    - $\langle 3 \rangle 4$ .  $x \in q^{-1}(V)$   
PROOF: From  $\langle 3 \rangle 1$  and  $\langle 3 \rangle 3$ .
- $\langle 1 \rangle 3$ . For all  $U \subseteq X$ , we have  $p(U \cap A) = p(U) \cap p(A)$ 
  - $\langle 2 \rangle 1$ .  $p(U \cap A) \subseteq p(U) \cap p(A)$   
PROOF: Set theory.
  - $\langle 2 \rangle 2$ .  $p(U) \cap p(A) \subseteq p(U \cap A)$ 
    - $\langle 3 \rangle 1$ . LET:  $x \in U, y \in A, p(x) = p(y)$   
PROVE:  $p(x) \in p(U \cap A)$
    - $\langle 3 \rangle 2$ .  $x \in A$   
PROOF:  $A$  is saturated.
    - $\langle 3 \rangle 3$ .  $x \in U \cap A$
- $\langle 1 \rangle 4$ . LET:  $V \subseteq p(A)$  be such that  $q^{-1}(V)$  is open in  $A$ .  
PROVE:  $V$  is open in  $p(A)$ .
- $\langle 1 \rangle 5$ .  $p^{-1}(V)$  is open in  $A$   
PROOF: By  $\langle 1 \rangle 2$
- $\langle 1 \rangle 6$ . PICK  $U$  open in  $X$  such that  $p^{-1}(V) = U \cap A$
- $\langle 1 \rangle 7$ .  $V = p(U) \cap p(A)$

PROOF:

$$\begin{aligned}
 V &= p(p^{-1}(V)) && (p \text{ is surjective}) \\
 &= p(U \cap A) && (\langle 1 \rangle 6) \\
 &= p(U) \cap p(A) && (\langle 1 \rangle 3)
 \end{aligned}$$

$\langle 1 \rangle 8$ .  $p(U)$  is open in  $Y$

PROOF:  $\langle 1 \rangle 6$ ,  $p$  is an open map.

$\langle 1 \rangle 9$ .  $V$  is open in  $p(A)$

PROOF:  $\langle 1 \rangle 7$ ,  $\langle 1 \rangle 8$

□

**Proposition 4.5.9.** *Let  $p : X \twoheadrightarrow Y$  be a closed quotient map. Let  $A \subseteq X$  be saturated. Then  $p \restriction_A : A \twoheadrightarrow p(A)$  is a quotient map.*

PROOF: Similar. □

**Proposition 4.5.10.** *The composite of two quotient maps is a quotient map.*

PROOF: From Proposition 5.2.22. □

**Proposition 4.5.11.** *Let  $X^*$  be a quotient space of  $X$ . If every element of  $X^*$  is closed in  $X$ , then  $X^*$  is  $T_1$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $C \in X^*$

$\langle 1 \rangle 2$ .  $p^{-1}(\{C\}) = C$

PROOF: Definition of  $p$ .

$\langle 1 \rangle 3$ .  $p^{-1}(\{C\})$  is closed in  $X$

PROOF: By hypothesis.

$\langle 1 \rangle 4$ .  $\{C\}$  is closed in  $X^*$ .

PROOF: By Proposition 5.2.21.

□



## Chapter 5

# Functions Between Topological Spaces

### 5.1 Open Maps

**Definition 5.1.1.** Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Then  $f$  is an *open map* iff, for all  $U$  open in  $X$ ,  $f(U)$  is open in  $Y$ .

**Lemma 5.1.2.** Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Let  $\mathcal{B}$  be a basis for the topology on  $X$ . Then  $f$  is an open map if and only if, for all  $B \in \mathcal{B}$ ,  $f(B)$  is open in  $Y$ .

PROOF:

$\langle 1 \rangle 1$ . If  $f$  is an open map then, for all  $B \in \mathcal{B}$ ,  $f(B)$  is open in  $Y$ .

PROOF: Immediate from definitions.

$\langle 1 \rangle 2$ . If, for all  $B \in \mathcal{B}$ ,  $f(B)$  is open in  $Y$ , then  $f$  is an open map.

$\langle 2 \rangle 1$ . ASSUME: For all  $B \in \mathcal{B}$ ,  $f(B)$  is open in  $Y$ .

$\langle 2 \rangle 2$ . LET:  $U$  be open in  $X$

PROVE:  $f(U)$  is open in  $Y$

$\langle 2 \rangle 3$ . LET:  $\mathcal{B}_0 \subseteq \mathcal{B}$  be such that  $U = \bigcup \mathcal{B}_0$

$\langle 2 \rangle 4$ .  $f(U) = \bigcup_{B \in \mathcal{B}_0} f(B)$

PROOF: Set theory.

$\langle 2 \rangle 5$ .  $f(U)$  is open in  $Y$ .

PROOF: From  $\langle 2 \rangle 1$ ,  $\langle 2 \rangle 4$  and the fact that the open sets are closed under union.

□

**Corollary 5.1.2.1.** Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Let  $\mathcal{S}$  be a subbasis for the topology on  $X$ . Then  $f$  is an open map if and only if, for all  $S \in \mathcal{S}$ ,  $f(S)$  is open in  $Y$ .

**Lemma 5.1.3 (AC).** Let  $\{X_\alpha\}_{\alpha \in J}$  be a family of topological spaces. Then the projection  $\pi_\alpha : \prod_{\alpha \in J} X_\alpha \rightarrow X_\alpha$  is an open map.

PROOF:

$\langle 1 \rangle 1$ . For  $U$  open in  $X_\alpha$ , we have  $\pi_\alpha(\pi_\alpha^{-1}(U))$  is open in  $X_\alpha$

PROOF:  $\pi_\alpha(\pi_\alpha^{-1}(U)) = U$  if all the other  $X_\alpha$  are nonempty,  $\emptyset$  otherwise.

$\langle 1 \rangle 2$ . For  $\beta \neq \alpha$  and  $U$  open in  $X_\beta$ , we have  $\pi_\alpha(\pi_\beta^{-1}(U))$  is open in  $X_\alpha$

PROOF:  $\pi_\alpha(\pi_\beta^{-1}(U)) = X_\alpha$  if all the  $X_\gamma$  are nonempty for  $\gamma \neq \alpha$ ,  $\emptyset$  otherwise.

$\langle 1 \rangle 3$ . Q.E.D.

PROOF: By Corollary 5.1.2.1.

## 5.2 Continuous Functions

**Definition 5.2.1** (Continuous). Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$  a function. Then  $f$  is *continuous* if and only if, for every open set  $U$  in  $Y$ , the set  $f^{-1}(U)$  is open in  $X$ .

**Theorem 5.2.2.** Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Then the following are equivalent.

1.  $f$  is continuous.
2. For every closed set  $C$  in  $Y$ , the set  $f^{-1}(C)$  is closed in  $X$ .
3. For every set  $A \subseteq X$ , we have  $f(\overline{A}) \subseteq \overline{f(A)}$ .

PROOF:

$\langle 1 \rangle 1$ .  $1 \Rightarrow 3$

$\langle 2 \rangle 1$ . ASSUME:  $f$  is continuous.

$\langle 2 \rangle 2$ . LET:  $A \subseteq X$

$\langle 2 \rangle 3$ . LET:  $x \in \overline{A}$

PROVE:  $f(x) \in \overline{f(A)}$

$\langle 2 \rangle 4$ . LET:  $V$  be a neighbourhood of  $f(x)$

$\langle 2 \rangle 5$ .  $f^{-1}(V)$  is a neighbourhood of  $x$

PROOF:  $\langle 2 \rangle 1$ ,  $\langle 2 \rangle 3$ ,  $\langle 2 \rangle 4$

$\langle 2 \rangle 6$ .  $f^{-1}(V)$  intersects  $A$  in  $a$ , say.

PROOF:  $\langle 2 \rangle 3$ ,  $\langle 2 \rangle 5$ , Theorem 3.8.3.

$\langle 2 \rangle 7$ .  $V$  intersects  $f(A)$  in  $f(a)$ .

$\langle 2 \rangle 8$ . Q.E.D.

PROOF: Theorem 3.8.3.

$\langle 1 \rangle 2$ .  $3 \Rightarrow 2$

$\langle 2 \rangle 1$ . ASSUME: 3

$\langle 2 \rangle 2$ . LET:  $C$  be a closed set in  $Y$

$\langle 2 \rangle 3$ .  $f^{-1}(C) = \overline{f^{-1}(C)}$

PROOF:

$$\begin{aligned} f(\overline{f^{-1}(C)}) &\subseteq \overline{f(f^{-1}(C))} & (\langle 2 \rangle 1) \\ &\subseteq \overline{C} \end{aligned}$$

$\langle 1 \rangle 3$ .  $2 \Rightarrow 1$

$\langle 2 \rangle 1$ . ASSUME: 2

- ⟨2⟩2. LET:  $V$  be open in  $Y$
- ⟨2⟩3.  $f^{-1}(Y \setminus V)$  is closed in  $X$   
PROOF: By ⟨2⟩1.
- ⟨2⟩4.  $f^{-1}(V)$  is open in  $X$ .  
PROOF:  $f^{-1}(V) = X \setminus f^{-1}(Y \setminus V)$ .

□

**Lemma 5.2.3.** *If  $f : X \rightarrow Y$  maps all of  $X$  to the single point  $y_0$  of  $Y$ , then  $f$  is continuous.*

PROOF: For  $V$  open in  $Y$ , the set  $f^{-1}(V)$  is either  $X$  (if  $y_0 \in V$ ) or  $\emptyset$  (if  $y_0 \notin V$ ).

**Definition 5.2.4** (Continuity at a Point). Let  $X$  and  $Y$  be topological spaces,  $f : X \rightarrow Y$  a function, and  $x \in X$ . Then  $f$  is *continuous at  $x$*  if and only if, for every neighbourhood  $V$  of  $f(x)$ , there exists a neighbourhood  $U$  of  $x$  such that  $f(U) \subseteq V$ .

**Theorem 5.2.5.** *Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Then  $f$  is continuous if and only if  $f$  is continuous at every point of  $X$ .*

PROOF:

- ⟨1⟩1. If  $f$  is continuous then  $f$  is continuous at every point of  $X$ .
  - ⟨2⟩1. ASSUME:  $f$  is continuous
  - ⟨2⟩2. LET:  $x \in X$
  - ⟨2⟩3. LET:  $V$  be a neighbourhood of  $f(x)$
  - ⟨2⟩4.  $f^{-1}(V)$  is a neighbourhood of  $x$
  - ⟨2⟩5.  $f(f^{-1}(V)) \subseteq V$
- ⟨1⟩2. If  $f$  is continuous at every point of  $X$  then  $f$  is continuous.
  - ⟨2⟩1. ASSUME:  $f$  is continuous at every point of  $X$ .
  - ⟨2⟩2. LET:  $V$  be open in  $Y$   
PROVE:  $f^{-1}(V)$  is open in  $X$ .
  - ⟨2⟩3. LET:  $x \in f^{-1}(V)$
  - ⟨2⟩4.  $V$  is a neighbourhood of  $f(x)$
  - ⟨2⟩5. PICK a neighbourhood  $U$  of  $x$  such that  $f(U) \subseteq V$   
PROOF: By ⟨2⟩1.
  - ⟨2⟩6.  $x \in U \subseteq f^{-1}(V)$
  - ⟨2⟩7. Q.E.D.

□

**Lemma 5.2.6.** *Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Let  $\mathcal{B}$  be a basis for the topology on  $Y$ . Then  $f$  is continuous if and only if, for all  $B \in \mathcal{B}$ , the set  $f^{-1}(B)$  is open in  $X$ .*

PROOF:

- ⟨1⟩1. If  $f$  is continuous then, for all  $B \in \mathcal{B}$ , the set  $f^{-1}(B)$  is open in  $X$ .  
PROOF: Immediate from definitions.
- ⟨1⟩2. If, for all  $B \in \mathcal{B}$ , the set  $f^{-1}(B)$  is open in  $X$ , then  $f$  is continuous.

⟨2⟩1. ASSUME: For all  $B \in \mathcal{B}$ , the set  $f^{-1}(B)$  is open in  $X$ .

⟨2⟩2. LET:  $x \in X$

⟨2⟩3. LET:  $V$  be a neighbourhood of  $f(x)$

⟨2⟩4. PICK  $B \in \mathcal{B}$  such that  $f(x) \in B \subseteq V$

⟨2⟩5.  $f^{-1}(B)$  is a neighbourhood of  $x$

PROOF: By ⟨2⟩1.

⟨2⟩6.  $f(f^{-1}(B)) \subseteq B$

PROOF: Set theory.

⟨2⟩7. Q.E.D.

PROOF: Theorem 5.2.5.

□

**Lemma 5.2.7.** *The projections  $\pi_1 : X \times Y \rightarrow X$  and  $\pi_2 : X \times Y \rightarrow Y$  are continuous.*

PROOF: Immediate from definitions. □

**Theorem 5.2.8.** *If  $A$  is a subspace of  $X$ , the inclusion function  $j : A \rightarrow X$  is continuous.*

PROOF: For  $V$  open in  $X$ , the set  $j^{-1}(V) = V \cap A$  is open in  $A$ .

**Theorem 5.2.9.** *If  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  are continuous, then the map  $g \circ f : X \rightarrow Z$  is continuous.*

PROOF:

⟨1⟩1. LET:  $V$  be open in  $Z$

⟨1⟩2.  $g^{-1}(V)$  is open in  $Y$

⟨1⟩3.  $f^{-1}(g^{-1}(V))$  is open in  $X$

□

**Theorem 5.2.10.** *If  $f : X \rightarrow Y$  is continuous and if  $A$  is a subspace of  $X$ , then the restricted function  $f \upharpoonright A : A \rightarrow Y$  is continuous.*

PROOF: For  $V$  open in  $Y$ , the set  $(f \upharpoonright A)^{-1}(V) = f^{-1}(V) \cap A$  is open in  $A$ . □

**Theorem 5.2.11.** *Let  $f : X \rightarrow Y$  be continuous. If  $Z$  is a subspace of  $Y$  that includes the range of  $f$ , then the function  $g : X \rightarrow Z$  obtained by restricting the codomain of  $f$  is continuous. If  $Z$  is a space having  $Y$  as a subspace, then the function  $h : X \rightarrow Z$  obtained by expanding the codomain of  $f$  is continuous.*

PROOF:

⟨1⟩1. If  $Z$  is a subspace of  $Y$  that includes the range of  $f$ , then the function  $g : X \rightarrow Z$  obtained by restricting the codomain of  $f$  is continuous.

⟨2⟩1. LET:  $V$  be open in  $Z$

⟨2⟩2. PICK  $W$  open in  $Y$  such that  $V = W \cap Z$

⟨2⟩3.  $f^{-1}(W)$  is open in  $X$ .

⟨2⟩4.  $g^{-1}(V)$  is open in  $X$ .

PROOF:  $g^{-1}(V) = f^{-1}(W)$ .

⟨1⟩2. If  $Z$  is a space having  $Y$  as a subspace, then the function  $h : X \rightarrow Z$  obtained by expanding the codomain of  $f$  is continuous.

PROOF: For  $V$  open in  $Z$ , we have  $h^{-1}(V) = f^{-1}(V \cap Y)$  is open in  $X$ .  
□

**Theorem 5.2.12.** *Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . If  $x_n \rightarrow x$  as  $n \rightarrow \infty$  in  $X$  and  $f$  is continuous at  $x$ , then  $f(x_n) \rightarrow f(x)$  as  $n \rightarrow \infty$  in  $Y$ .*

PROOF:

⟨1⟩1. ASSUME:  $x_n \rightarrow x$  as  $n \rightarrow \infty$

⟨1⟩2. ASSUME:  $f$  is continuous at  $x$

⟨1⟩3. LET:  $V$  be a neighbourhood of  $f(x)$

⟨1⟩4. PICK a neighbourhood  $U$  of  $x$  such that  $f(U) \subseteq V$

PROOF: By ⟨1⟩2.

⟨1⟩5. PICK  $N$  such that, for all  $n \geq N$ ,  $x_n \in U$

PROOF: By ⟨1⟩1

⟨1⟩6. For  $n \geq N$ ,  $f(x_n) \in V$

PROOF: By ⟨1⟩4.

□

**Theorem 5.2.13.** *Let  $X$ ,  $Y$  and  $Z$  be topological spaces. Let  $f : X \rightarrow Y$ . If there exists a set  $\mathcal{A}$  of open sets in  $X$  such that:*

- $\bigcup \mathcal{A} = X$ ;
- for all  $U \in \mathcal{A}$ , the function  $f \upharpoonright U : U \rightarrow Y$  is continuous;

*then  $f$  is continuous.*

PROOF:

⟨1⟩1. LET:  $V$  be open in  $Y$

⟨1⟩2. For all  $U \in \mathcal{A}$ , the set  $(f \upharpoonright U)^{-1}(V)$  is open in  $X$ .

⟨2⟩1. LET:  $U \in \mathcal{A}$

⟨2⟩2.  $(f \upharpoonright U)^{-1}(V)$  is open in  $U$

PROOF: Since  $f \upharpoonright U : U \rightarrow Y$  is continuous.

⟨2⟩3. Q.E.D.

PROOF: By Lemma 4.3.3.

⟨1⟩3. Q.E.D.

PROOF: Since  $f^{-1}(V) = \bigcup_{U \in \mathcal{A}} (f \upharpoonright U)^{-1}(V)$ .

**Theorem 5.2.14** (The Pasting Lemma). *Let  $X = A \cup B$  where  $A$  and  $B$  are closed in  $X$ . Let  $f : A \rightarrow Y$  and  $g : B \rightarrow Y$  be continuous. If  $f(x) = g(x)$  for every  $x \in A \cap B$ , then the function  $h : X \rightarrow Y$  defined by*

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases}$$

*is continuous.*

PROOF:

- ⟨1⟩1. LET:  $C$  be closed in  $Y$
- ⟨1⟩2.  $f^{-1}(C)$  is closed in  $A$   
PROOF: Theorem 5.2.2.
- ⟨1⟩3.  $f^{-1}(C)$  is closed in  $X$   
PROOF: Lemma 4.3.4.1.
- ⟨1⟩4.  $g^{-1}(C)$  is closed in  $B$   
PROOF: Theorem 5.2.2.
- ⟨1⟩5.  $g^{-1}(C)$  is closed in  $X$   
PROOF: Lemma 4.3.4.1.
- ⟨1⟩6.  $h^{-1}(C)$  is closed in  $X$   
PROOF:  $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$
- ⟨1⟩7. Q.E.D.  
PROOF: Theorem 5.2.2.

□

**Theorem 5.2.15.** Let  $f : A \rightarrow \prod_{\alpha \in J} X_\alpha$  be given by the equation

$$f(a) = \{f_\alpha(a)\}_{\alpha \in J} ,$$

where  $f_\alpha : A \rightarrow X_\alpha$  for each  $\alpha$ . Let  $\prod_{\alpha \in J} X_\alpha$  have the product topology. Then the function  $f$  is continuous if and only if each function  $f_\alpha$  is continuous.

PROOF:

- ⟨1⟩1. If  $f$  is continuous then each  $f_\alpha$  is continuous.  
PROOF: This holds because  $f_\alpha = \pi_\alpha \circ f$ .
- ⟨1⟩2. If every  $f_\alpha$  is continuous then  $f$  is continuous.  
⟨2⟩1. ASSUME: Every  $f_\alpha$  is continuous.  
⟨2⟩2. LET:  $\alpha \in J$  and  $U$  be open in  $X_\alpha$   
⟨2⟩3.  $f^{-1}(\pi_\alpha^{-1}(U))$  is open in  $A$   
PROOF:  $f^{-1}(\pi_\alpha^{-1}(U)) = f_\alpha^{-1}(U)$ .

□

### 5.2.1 Homeomorphisms

**Definition 5.2.16** (Homeomorphism). Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Then  $f$  is a *homeomorphism* between  $X$  and  $Y$  iff  $f$  is a bijection, and  $f$  and  $f^{-1}$  are both continuous.

**Definition 5.2.17** (Topological Property). A property  $P$  of topological spaces is a *topological property* iff, for any spaces  $X$  and  $Y$ , if  $X$  is homeomorphic to  $Y$  then  $P$  holds of  $X$  if and only if  $P$  holds of  $Y$ .

**Definition 5.2.18** ((Topological) Imbedding). Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Then  $f$  is a *(topological) imbedding* iff  $f$  is a homeomorphism between  $X$  and  $\text{im } f$ .

**Definition 5.2.19** (Homogeneous). A topological space  $X$  is *homogeneous* iff, for all  $x, y \in X$ , there exists a homeomorphism  $f : X \cong X$  such that  $f(x) = y$ .

### 5.2.2 Strongly Continuous Functions

**Definition 5.2.20** (Strongly Continuous). Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Then  $f$  is *strongly continuous* iff, for all  $V \subseteq Y$ , we have  $V$  is open in  $Y$  if and only if  $f^{-1}(V)$  is open in  $X$ .

**Proposition 5.2.21.** *Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Then  $f$  is strongly continuous if and only if, for all  $C \subseteq Y$ ,  $C$  is closed in  $Y$  if and only if  $f^{-1}(C)$  is closed in  $X$ .*

PROOF:

(1)1. If  $f$  is strongly continuous then, for all  $C \subseteq Y$ , we have  $C$  is closed in  $Y$  if and only if  $f^{-1}(C)$  is closed in  $X$ .

PROOF:

$$\begin{aligned} C \text{ is closed in } Y &\Leftrightarrow Y \setminus C \text{ is open in } Y \\ &\Leftrightarrow f^{-1}(Y \setminus C) \text{ is open in } X \\ &\Leftrightarrow X \setminus f^{-1}(C) \text{ is open in } X \\ &\Leftrightarrow f^{-1}(C) \text{ is closed in } X \end{aligned}$$

(1)2. If, for all  $C \subseteq Y$ , we have  $C$  is closed in  $Y$  if and only if  $f^{-1}(C)$  is closed in  $X$ , then  $f$  is strongly continuous.

PROOF: Similar.

□

**Proposition 5.2.22.** *The composite of two strongly continuous functions is strongly continuous.*

PROOF:

(1)1. LET:  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  be strongly continuous.

(1)2. LET:  $V \subseteq Z$

(1)3.  $V$  is open iff  $f^{-1}(g^{-1}(V))$  is open

PROOF:

$$\begin{aligned} V \text{ is open} &\Leftrightarrow g^{-1}(V) \text{ is open} && ((1)1) \\ &\Leftrightarrow f^{-1}(g^{-1}(V)) \text{ is open} && ((1)1) \end{aligned}$$

□

**Proposition 5.2.23.** *Let  $X$ ,  $Y$  and  $Z$  be topological spaces. Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$ . If  $f$  is strongly continuous and  $g \circ f$  is continuous, then  $g$  is continuous.*

PROOF:

(1)1. LET:  $V \subseteq Z$  be open in  $Z$ .

(1)2.  $f^{-1}(g^{-1}(V))$  is open in  $X$ .

PROOF:  $g \circ f$  is continuous.

(1)3.  $g^{-1}(V)$  is open in  $Y$ .

PROOF:  $f$  is strongly continuous.

□

**Proposition 5.2.24.** *Let  $X, Y$  and  $Z$  be topological spaces. Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$ . If  $f$  and  $g \circ f$  are strongly continuous, then  $g$  is strongly continuous.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $U \subseteq Z$

$\langle 1 \rangle 2$ .  $U$  is open in  $Z$  iff  $g^{-1}(U)$  is open in  $Y$

PROOF:

$U$  is open in  $Z \Leftrightarrow f^{-1}(g^{-1}(U))$  is open in  $X$  ( $g \circ f$  is strongly continuous)

$\Leftrightarrow g^{-1}(U)$  is open in  $Y$  ( $f$  is strongly continuous)

□

### 5.3 Closed Maps

**Definition 5.3.1** (Closed Map). Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . Then  $f$  is a *closed map* iff, for every closed set  $C \subseteq X$ , the set  $f(C)$  is closed in  $Y$ .

### 5.4 Local Homeomorphism

**Definition 5.4.1** (Locally Homeomorphic). Let  $X$  and  $Y$  be topological spaces. Then  $X$  is *locally homeomorphic* to  $Y$  iff every point in  $X$  has an open neighborhood that is homeomorphic with an open set in  $Y$ .

**Proposition 5.4.2.** *The long line is locally homeomorphic with  $\mathbb{R}$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $x \in L$

$\langle 1 \rangle 2$ . PICK an ordinal  $\alpha$  such that  $x < (\alpha, 0)$ .

$\langle 1 \rangle 3$ .  $(-\infty, (\alpha, 0))$  is an open neighbourhood of  $x$  that is homeomorphic to  $(0, 1)$ .

□

### 5.5 Retracts

**Definition 5.5.1** (Retract). Let  $Z$  be a topological space. If  $Y$  is a subspace of  $Z$ , we say that  $Y$  is a *retract* of  $Z$  iff there exists a continuous function  $r : Z \rightarrow Y$  such that  $r(y) = y$  for all  $y \in Y$ .



## Chapter 6

# Separation Axioms

### 6.1 $T_1$ Spaces

**Definition 6.1.1** ( $T_1$  Space). A topological space  $X$  is a  $T_1$  space iff every finite set is closed.

**Theorem 6.1.2.** Let  $X$  be a  $T_1$  space and  $A \subseteq X$ . Then  $x$  is a limit point of  $A$  if and only if every neighbourhood of  $x$  contains infinitely many points of  $A$ .

PROOF:

$\langle 1 \rangle 1$ . If some neighbourhood of  $x$  contains only finitely many points of  $A$  then  $x$  is not a limit point of  $A$ .

$\langle 2 \rangle 1$ . ASSUME: Some neighbourhood  $U$  of  $x$  contains only finite many points  $a_1, \dots, a_n$  of  $A$ .

$\langle 2 \rangle 2$ .  $X \setminus \{a_1, \dots, a_n\}$  is open.

PROOF:  $X$  is  $T_1$ .

$\langle 2 \rangle 3$ .  $U \setminus \{a_1, \dots, a_n\}$  is a neighbourhood of  $x$  that does not intersect  $A$ .

$\langle 1 \rangle 2$ . If every neighbourhood of  $x$  contains infinitely many points of  $A$  then  $x$  is a limit point of  $A$ .

PROOF: From the definition of limit point.

□

**Proposition 6.1.3.** A subspace of a  $T_1$  space is  $T_1$ .

PROOF:

$\langle 1 \rangle 1$ . LET:  $X$  be a  $T_1$  space and  $Y \subseteq X$

$\langle 1 \rangle 2$ . LET:  $a \in Y$

$\langle 1 \rangle 3$ .  $\{a\}$  is closed in  $X$

PROOF: By  $\langle 1 \rangle 1$ .

$\langle 1 \rangle 4$ .  $\{a\}$  is closed in  $Y$

PROOF: By Corollary 4.3.4.1.

□

**Definition 6.1.4** (Separate Points from Closed Sets). Let  $X$  be a space and  $\{f_\alpha\}_{\alpha \in J}$  be a family of continuous functions  $f_\alpha : X \rightarrow \mathbb{R}$ . Then  $\{f_\alpha\}$  *separates points from closed sets* in  $X$  iff, for every point  $x_0 \in X$  and every neighbourhood  $U$  of  $x_0$ , there exists  $\alpha \in J$  such that  $f_\alpha$  is positive at  $x_0$  and vanishes outside  $U$ .

**Theorem 6.1.5** (Imbedding Theorem). Let  $X$  be a  $T_1$  space and  $\{f_\alpha\}_{\alpha \in J}$  be a family of functions  $X \rightarrow \mathbb{R}$  that separates points from closed sets. Then the function  $F : X \rightarrow \mathbb{R}^J$  defined by

$$F(x)_\alpha = f_\alpha(x)$$

is an imbedding. If each  $f_\alpha$  maps  $X$  into  $[0, 1]$  then  $F$  is an imbedding  $X \rightarrow [0, 1]^J$ .

PROOF:

(1)1.  $F$  is continuous

PROOF: By Theorem 5.2.15.

(1)2.  $F$  is injective

(2)1. LET:  $x, y \in X$  with  $x \neq y$

(2)2. PICK a neighbourhood  $U$  of  $x$  such that  $y \notin U$

PROOF:  $X$  is  $T_1$

(2)3. PICK  $\alpha \in J$  such that  $f_\alpha$  is positive at  $x$  and vanishes outside  $U$

(2)4.  $f_\alpha(x) \neq f_\alpha(y)$

(2)5.  $F(x) \neq F(y)$

(1)3.  $F$  is open as a map  $X \rightarrow F(U)$

(2)1. LET:  $U$  be open

(2)2. LET:  $z \in F(U)$

(2)3. PICK  $x \in U$  such that  $F(x) = z$

(2)4. PICK  $\alpha \in J$  such that  $f_\alpha$  is positive at  $x$  and vanishes outside  $U$

(2)5.  $z \in \pi_\alpha^{-1}((0, +\infty)) \cap F(U) \subseteq F(U)$

□

## 6.2 Hausdorff Spaces

**Definition 6.2.1** (Hausdorff Space). A topological space  $X$  is a *Hausdorff space* iff, for any points  $x, y \in X$  with  $x \neq y$ , there exist disjoint neighbourhoods  $U$  of  $x$  and  $V$  of  $y$ .

**Theorem 6.2.2.** Every Hausdorff space is  $T_1$ .

PROOF:

(1)1. LET:  $X$  be a Hausdorff space

(1)2. LET:  $a \in X$

PROVE:  $\{a\}$  is closed.

(1)3. LET:  $b \in X \setminus \{a\}$

(1)4. PICK disjoint neighbourhoods  $U$  of  $a$  and  $V$  of  $b$

⟨1⟩5.  $b \in V \subseteq X \setminus \{a\}$

⟨1⟩6. Q.E.D.

PROOF: By Proposition 3.2.3.

□

**Theorem 6.2.3.** *In a Hausdorff space, a sequence has at most one limit.*

PROOF:

⟨1⟩1. ASSUME: for a contradiction  $x_n \rightarrow l$  and  $x_n \rightarrow m$  as  $n \rightarrow \infty$ , and  $l \neq m$

⟨1⟩2. PICK disjoint neighbourhoods  $U$  of  $l$  and  $V$  of  $m$

⟨1⟩3. PICK  $N$  such that, for all  $n \geq N$ ,  $x_n \in U$  and  $x_n \in V$

⟨1⟩4.  $x_N \in U \cap V$

□

**Theorem 6.2.4.** *Every linearly ordered set is Hausdorff under the order topology.*

PROOF:

⟨1⟩1. LET:  $X$  be a linearly ordered set under the order topology.

⟨1⟩2. LET:  $x, y \in X$  with  $x \neq y$

⟨1⟩3. ASSUME: w.l.o.g.  $x < y$

PROVE: There exist disjoint neighbourhoods  $U$  of  $x$  and  $V$  of  $y$ .

⟨1⟩4. CASE: There exists  $z$  such that  $x < z < y$

PROOF: In this case, take  $U = (-\infty, z)$  and  $V = (z, +\infty)$ .

⟨1⟩5. CASE: There does not exist  $z$  such that  $x < z < y$

PROOF: In this case, take  $U = (-\infty, y)$  and  $V = (x, +\infty)$ .

□

**Theorem 6.2.5.** *Let  $\{X_\alpha\}_{\alpha \in J}$  be a family of Hausdorff spaces. Then  $\prod_{\alpha \in J} X_\alpha$  is Hausdorff under the product topology.*

PROOF:

⟨1⟩1. LET:  $\{x_\alpha\}_{\alpha \in J}, \{y_\alpha\}_{\alpha \in J} \in \prod_{\alpha \in J} X_\alpha$  with  $\{x_\alpha\}_{\alpha \in J} \neq \{y_\alpha\}_{\alpha \in J}$

⟨1⟩2. PICK  $\alpha \in J$  such that  $x_\alpha \neq y_\alpha$

⟨1⟩3. PICK disjoint neighbourhoods  $U$  of  $x_\alpha$  and  $V$  of  $y_\alpha$ .

⟨1⟩4.  $\pi_\alpha^{-1}(U)$  and  $\pi_\alpha^{-1}(V)$  are disjoint neighbourhoods of  $\{x_\alpha\}_{\alpha \in J}$  and  $\{y_\alpha\}_{\alpha \in J}$

□

**Corollary 6.2.5.1.** *The Sorgenfrey plane is Hausdorff.*

**Corollary 6.2.5.2.** *For any set  $I$ , the space  $\mathbb{R}^I$  is Hausdorff.*

**Proposition 6.2.6.** *Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$ . If  $f$  is continuous and injective and  $Y$  is Hausdorff then  $X$  is Hausdorff.*

PROOF:

⟨1⟩1. LET:  $x, y \in X$  with  $x \neq y$

⟨1⟩2.  $f(x) \neq f(y)$

PROOF:  $f$  is injective.

⟨1⟩3. PICK disjoint neighbourhoods  $U, V$  of  $f(x)$  and  $f(y)$

PROOF:  $Y$  is Hausdorff.

⟨1⟩4.  $f^{-1}(U)$  and  $f^{-1}(V)$  are disjoint neighbourhoods of  $x$  and  $y$ .

□

**Corollary 6.2.6.1.** *A subspace of a Hausdorff space is Hausdorff.*

**Corollary 6.2.6.2.** *Let  $\{X_\alpha\}_{\alpha \in J}$  be a family of nonempty spaces. If  $\prod_{\alpha \in J} X_\alpha$  is Hausdorff then so is each  $X_\alpha$ .*

**Corollary 6.2.6.3.** *Let  $\mathcal{T}$  and  $\mathcal{T}'$  be topologies on the same set  $X$ . If  $\mathcal{T} \subseteq \mathcal{T}'$  and  $X$  is Hausdorff under  $\mathcal{T}$  then  $X$  is Hausdorff under  $\mathcal{T}'$ .*

**Corollary 6.2.6.4.** *The space  $\mathbb{R}_K$  is Hausdorff.*

**Proposition 6.2.7.**  *$\mathbb{R}_l$  is Hausdorff.*

PROOF: Let  $a, b \in \mathbb{R}_l$  with  $a < b$ . Then  $(-\infty, b)$  and  $[b, +\infty)$  are disjoint open sets containing  $a$  and  $b$  respectively. □

**Proposition 6.2.8.** *The continuous image of a Hausdorff space is not necessarily Hausdorff.*

PROOF: The identity map from the discrete two-point space to the indiscrete two-point space is continuous. □

**Lemma 6.2.9.** *Let  $A$  be a subspace of  $X$  and  $Z$  be Hausdorff. Let  $f : A \rightarrow Z$  be continuous. Then there is at most one extension of  $f$  to a continuous function  $\bar{A} \rightarrow Z$ .*

PROOF:

⟨1⟩1. ASSUME: for a contradiction  $g, h : \bar{A} \rightarrow Z$  are continuous extensions of  $f$  with  $g(x) \neq h(x)$

⟨1⟩2. PICK disjoint open neighbourhoods  $U$  of  $g(x)$  and  $V$  of  $h(x)$

⟨1⟩3. PICK a point  $a \in A \cap g^{-1}(U) \cap h^{-1}(V)$

PROOF: One exists because  $g^{-1}(U) \cap h^{-1}(V)$  is a neighbourhood of  $x \in \bar{A}$ .

⟨1⟩4.  $g(a) \in U \cap V$

□

## 6.3 Regular Spaces

**Definition 6.3.1** (Regular). A topological space  $X$  is *regular* iff, for every closed set  $A$  and point  $a \notin A$ , there exist disjoint neighbourhoods  $U$  of  $A$  and  $V$  of  $a$ .

**Proposition 6.3.2.** *Let  $X$  be a  $T_1$  space. Then  $X$  is regular if and only if, for every point  $x$  and neighbourhood  $U$  of  $x$ , there exists a neighbourhood  $V$  of  $x$  such that  $\bar{V} \subseteq U$ .*

PROOF:

- ⟨1⟩1. If  $X$  is regular then, for every point  $x$  and neighbourhood  $N$  of  $x$ , there exists a neighbourhood  $V$  of  $x$  such that  $\overline{V} \subseteq N$ .
- ⟨2⟩1. ASSUME:  $X$  is regular.
- ⟨2⟩2. LET:  $x \in X$  and  $N$  be a neighbourhood of  $x$
- ⟨2⟩3. PICK an open set  $U$  such that  $x \in U \subseteq N$
- ⟨2⟩4. PICK disjoint open sets  $V, W$  such that  $x \in V$  and  $X \setminus U \subseteq W$
- ⟨2⟩5.  $\overline{V} \subseteq N$

PROOF:

$$\begin{aligned}\overline{V} &\subseteq X \setminus W \\ &\subseteq U \\ &\subseteq N\end{aligned}$$

- ⟨1⟩2. If, for every point  $x$  and neighbourhood  $U$  of  $x$ , there exists a neighbourhood  $V$  of  $x$  such that  $\overline{V} \subseteq U$ , then  $X$  is regular.
- ⟨2⟩1. ASSUME: For every point  $x$  and neighbourhood  $U$  of  $x$ , there exists a neighbourhood  $V$  of  $x$  such that  $\overline{V} \subseteq U$ .
- ⟨2⟩2. LET:  $x \in X$  and  $A$  be a closed set with  $x \notin A$
- ⟨2⟩3. PICK a neighbourhood  $V$  of  $x$  such that  $\overline{V} \subseteq X \setminus A$
- ⟨2⟩4.  $x \in V$  and  $A \subseteq X \setminus \overline{V}$

□

**Proposition 6.3.3.** *Every linearly ordered set under the order topology is regular.*

PROOF:

- ⟨1⟩1. LET:  $X$  be a linearly ordered set under the order topology.
- ⟨1⟩2. LET:  $x \in X$  and  $U$  be a neighbourhood of  $x$   
PROVE: There exists a neighbourhood  $V$  of  $x$  with  $\overline{V} \subseteq U$
- ⟨1⟩3. CASE:  $x$  is greatest and least in  $X$   
PROOF: Take  $V = U = X = \{x\}$
- ⟨1⟩4. CASE:  $x$  is greatest in  $X$  and there exists  $a < x$  such that  $(a, x] \subseteq U$ 
  - ⟨2⟩1. CASE: There exists  $b$  such that  $a < b < x$   
PROOF: Take  $V = (b, x]$ .
  - ⟨2⟩2. CASE: There is no  $b$  such that  $a < b < x$ 
    - ⟨3⟩1. LET:  $V = U = \{x\}$
    - ⟨3⟩2.  $\overline{V} = V$   
PROOF: For any  $y \neq x$ , we have  $(-\infty, x)$  is a neighbourhood of  $y$  that does not intersect  $V$ .
- ⟨1⟩5. CASE:  $x$  is least in  $X$  and there exists  $b > x$  such that  $[x, b) \subseteq U$   
PROOF: Similar.
- ⟨1⟩6. CASE: There exist  $a < x < b$  such that  $(a, b) \subseteq U$ 
  - ⟨2⟩1. PICK a point  $c$  such that  $a < c < x$  if there is one, otherwise  
LET:  $c = a$
  - ⟨2⟩2. PICK a point  $d$  such that  $x < d < b$  if there is one, otherwise  
LET:  $d = b$
  - ⟨2⟩3. LET:  $V = (c, d)$
  - ⟨2⟩4.  $\overline{V} \subseteq U$

PROOF:

$$\begin{aligned}\bar{V} &\subseteq [c, d] \\ &\subseteq (a, b) \\ &\subseteq U\end{aligned}$$

⟨1⟩7. Q.E.D.

PROOF: These cases are exhaustive by Lemma 4.1.2. They prove  $X$  is regular by Proposition 6.3.2.

□

**Proposition 6.3.4.** *A subspace of a regular space is regular.*

PROOF:

- ⟨1⟩1. LET:  $X$  be a regular space and  $Y \subseteq X$
- ⟨1⟩2. LET:  $A \subseteq Y$  be closed in  $Y$  and  $a \in Y \setminus A$
- ⟨1⟩3. PICK  $C$  closed in  $X$  such that  $A = C \cap Y$

PROOF: By Corollary 4.3.4.1.

- ⟨1⟩4. PICK disjoint open sets  $U, V$  in  $X$  such that  $C \subseteq U$  and  $a \in V$
- ⟨1⟩5.  $U \cap Y$  and  $V \cap Y$  are disjoint open sets in  $Y$  such that  $A \subseteq U \cap Y$  and  $a \in V \cap Y$

□

**Corollary 6.3.4.1.** *Let  $\{X_\alpha\}_{\alpha \in J}$  be a family of nonempty spaces. If  $\prod_{\alpha \in J} X_\alpha$  is regular then so is each  $X_\alpha$ .*

**Proposition 6.3.5 (AC).** *The product of a family of regular spaces is regular.*

PROOF:

- ⟨1⟩1. LET:  $\{X_\alpha\}_{\alpha \in J}$  be a family of regular spaces.
- ⟨1⟩2.  $\prod_{\alpha \in J} X_\alpha$  is  $T_1$
- ⟨1⟩3. LET:  $\vec{a} \in U$  where  $U$  is open in  $\prod_{\alpha \in J} X_\alpha$
- ⟨1⟩4. PICK  $\prod_{\alpha \in J} U_\alpha$  such that each  $U_\alpha$  is open in  $X_\alpha$ ,  $U_\alpha = X_\alpha$  except at  $\alpha_1, \dots, \alpha_n$ , and  $\vec{a} \in \prod_{\alpha \in J} U_\alpha \subseteq U$
- ⟨1⟩5. For  $1 \leq i \leq n$ , PICK  $V_{\alpha_i}$  open in  $X_{\alpha_i}$  such that  $a_{\alpha_i} \in V_{\alpha_i}$  and  $\overline{V_{\alpha_i}} \subseteq U_{\alpha_i}$
- ⟨1⟩6. For  $\alpha \neq \alpha_1, \dots, \alpha_n$ ,  
LET:  $V_\alpha = X_\alpha$
- ⟨1⟩7.  $\vec{a} \in \prod_{\alpha \in J} V_\alpha$
- ⟨1⟩8.  $\overline{\prod_{\alpha \in J} V_\alpha} \subseteq \prod_{\alpha \in J} U_\alpha$

PROOF: By Theorem 4.2.5.

□

**Corollary 6.3.5.1.** *The Sorgenfrey plane is regular.*

**Corollary 6.3.5.2.** *For any set  $I$ , the space  $\mathbb{R}^I$  is regular.*

**Proposition 6.3.6.** *The space  $\mathbb{R}_K$  is not regular.*

PROOF: There do not exist disjoint neighbourhoods of 0 and  $K$ . □

**Proposition 6.3.7.** *The continuous image of a regular space is not necessarily regular.*

PROOF: The identity map from the discrete two-point space to the indiscrete two-point space is continuous.  $\square$

## 6.4 Completely Regular Spaces

**Definition 6.4.1** (Separated by a Continuous Function). Let  $A$  and  $B$  be subsets of a topological space  $X$ . Then  $A$  and  $B$  can be *separated by a continuous function* iff there exists a continuous  $f : X \rightarrow [0, 1]$  such that  $f(A) = \{0\}$  and  $f(B) = \{1\}$ .

**Definition 6.4.2** (Completely Regular). A space  $X$  is *completely regular* iff  $X$  is  $T_1$  and, for every point  $a$  and closed set  $A$  not containing  $a$ , we have that  $\{a\}$  and  $A$  can be separated by a continuous function.

**Theorem 6.4.3.** *The product of a family of completely regular spaces is completely regular.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $\{X_\alpha\}_{\alpha \in J}$  be a family of completely regular spaces.
- $\langle 1 \rangle 2$ . LET:  $a \in \prod_{\alpha \in J} X_\alpha$  and  $A$  be closed in  $\prod_{\alpha \in J} X_\alpha$  such that  $a \notin A$
- $\langle 1 \rangle 3$ . PICK a basic open neighbourhood  $\prod_{\alpha \in J} U_\alpha \subseteq \prod_{\alpha \in J} X_\alpha \setminus A$  of  $a$  such that  $U_\alpha = X_\alpha$  except for  $\alpha = \alpha_1, \dots, \alpha_n$
- $\langle 1 \rangle 4$ . For  $1 \leq i \leq n$ , PICK a continuous  $f_i : X_{\alpha_i} \rightarrow [0, 1]$  that is 0 at  $a_{\alpha_i}$  and 1 on  $X_{\alpha_i} \setminus U_{\alpha_i}$
- $\langle 1 \rangle 5$ . LET:  $f : \prod_{\alpha \in J} X_\alpha \rightarrow [0, 1]$  be given by  $f(x) = \prod_{i=1}^n f_i(x_{\alpha_i})$
- $\langle 1 \rangle 6$ .  $f(a) = 0$
- $\langle 1 \rangle 7$ .  $f(x) = 1$  for  $x \in A$
- $\langle 1 \rangle 8$ .  $f$  is continuous

$\square$

**Corollary 6.4.3.1.** *The Sorgenfrey plane is completely regular.*

**Corollary 6.4.3.2.** *For any set  $I$ , the space  $\mathbb{R}^I$  is completely regular.*

**Proposition 6.4.4.** *For any set  $J$ , the space  $\mathbb{R}^J$  in the box topology is completely regular.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $a \in \mathbb{R}^J$  and  $A \subseteq \mathbb{R}^J$  be closed with  $a \notin A$   
 PROVE: There exists  $f : \mathbb{R}_{\text{box}}^J \rightarrow [0, 1]$  continuous such that  $f(a) = 1$  and  $f(A) = \{0\}$
- $\langle 1 \rangle 2$ . ASSUME: w.l.o.g.  $A \cap (-1, 1)^J = \emptyset$  and  $a = \vec{0}$ 
  - $\langle 2 \rangle 1$ . PICK a basic open set  $\prod_{\alpha \in J} U_\alpha$  such that  $a \in \prod_{\alpha \in J} U_\alpha \subseteq \mathbb{R}^J \setminus A$
  - $\langle 2 \rangle 2$ . For  $\alpha \in J$ , PICK  $b_\alpha, c_\alpha$  such that  $a_\alpha \in (b_\alpha, c_\alpha) \subseteq U_\alpha$
  - $\langle 2 \rangle 3$ . For  $\alpha \in J$ , PICK a homeomorphism  $f_\alpha : \mathbb{R} \rightarrow \mathbb{R}$  that maps  $b_\alpha$  to  $-1$ ,  $a_\alpha$  to 0 and  $c_\alpha$  to 1
  - $\langle 2 \rangle 4$ .  $\prod_{\alpha \in J} f_\alpha$  is an automorphism  $\mathbb{R}_{\text{box}}^J$  that maps  $a$  to  $\vec{0}$  and  $A$  to a closed set disjoint from  $(-1, 1)^J$

- (1)3. PICK a continuous function  $f : \mathbb{R}_{\text{uniform}}^J \rightarrow [0, 1]$  such that  $f(\vec{0}) = 1$  and  $f(\mathbb{R}^J \setminus (-1, 1)^J) = \{0\}$
- (1)4.  $f$  is continuous w.r.t. the box topology
- 

**Proposition 6.4.5.** *Not every regular space is completely regular.*

PROOF:

- (1)1. For  $m \in \mathbb{Z}$ ,  
LET:  $L_m = \{m\} \times [-1, 0]$
- (1)2. For each odd integer  $n$  and each integer  $k \geq 2$ ,  
LET:  $C_{nk} = (\{n+1-1/k\} \text{home/robin/fun/RogOMatic/src/actuatortimes}[-1, 0]) \cup (\{n-1+1/k\} \times [-1, 0]) \cup \{(x, y) : (x-n)^2 + y^2 = (1-1/k)^2, y \geq 0\}$
- (1)3. For each odd integer  $n$  and each integer  $k \geq 2$ ,  
LET:  $p_{nk} = (n, 1-1/k)$
- (1)4. PICK two points  $a, b$  not in any  $L_m$  or  $C_{nk}$
- (1)5. LET:  $X = \bigcup_{m \in \mathbb{Z}} L_m \cup \bigcup_{n,k} C_{nk} \cup \{a, b\}$
- (1)6. LET:  $\mathcal{B}$  be the set consisting of all subsets of  $\mathbb{R}^2$  of the following forms:
1. The intersection of  $X$  with a horizontal open line segment that contains none of the points  $p_{nk}$
  2. A set formed from one of the sets  $C_{nk}$  by deleting finitely many points.
  3. For each even integer  $m$ , the set  $\{a\} \cup \{(x, y) \in X : x < m\}$
  4. For each even integer  $m$ , the set  $\{b\} \cup \{(x, y) \in X : x > m\}$
- (1)7.  $\mathcal{B}$  is a basis for a topology on  $X$
- (2)1. For all  $x \in X$ , there exists  $B \in \mathcal{B}$  such that  $x \in B$
- (2)2. For all  $B_1, B_2 \in \mathcal{B}$  and  $x \in B_1 \cap B_2$ , there exists  $B_3 \in \mathcal{B}$  such that  $x \in B_3 \subseteq B_1 \cap B_2$
- (3)1. CASE:  $B_1, B_2$  are both of type 1  
PROOF: Their intersection is of type 1.
- (3)2. CASE:  $B_1$  is of type 1 and  $B_2$  is of type 2  
PROOF: Their intersection is of type 2, since a horizontal line segment intersects  $C_{nk}$  in at most two points.
- (3)3. CASE:  $B_1$  is of type 1 and  $B_2$  is of type 3  
PROOF: Their intersection is of type 1
- (3)4. CASE:  $B_1$  is of type 1 and  $B_2$  is of type 4  
PROOF: Their intersection is of type 1
- (3)5. CASE:  $B_1$  is of type 2 and  $B_2$  is of type 2  
PROOF: Their intersection is of type 2
- (3)6. CASE:  $B_1$  is of type 2 and  $B_2$  is of type 3  
PROOF: Their intersection is  $B_1$
- (3)7. CASE:  $B_1$  is of type 2 and  $B_2$  is of type 4  
PROOF: Their intersection is  $B_1$
- (3)8. CASE:  $B_1$  is of type 3 and  $B_2$  is of type 3  
PROOF: Their intersection is of type 3
- (3)9. CASE:  $B_1$  is of type 3 and  $B_2$  is of type 4



- (4)1. LET:  $B_1 = \{a\} \cup \{(x, y) \in X : x < m\}$  and  $B_2 = \{b\} \cup \{(x, y) \in X : x > n\}$   
 (4)2. CASE:  $x = (s, 1 - 1/k)$  for some  $s$  and integer  $x \geq 2$   
 PROOF: In this case,  $x \in C_{nk}$  for some  $n$  and  $C_{nk} \subseteq B_1 \cap B_2$ .  
 (4)3. CASE:  $x = (s, t)$  and  $t \neq 1 - 1/k$  for any integer  $k \geq 2$   
 PROOF: In this case,  $x \in ((n, m) \times \{t\}) \cap X \subseteq B_1 \cap B_2$   
 (3)10. CASE:  $B_1$  is of type 4 and  $B_2$  is of type 4  
 PROOF: Their intersection is of type 4  
 (2)8. For any continuous function  $f : X \rightarrow \mathbb{R}$ , we have  $f(a) = f(b)$   
 (2)1. LET:  $f : X \rightarrow \mathbb{R}$  be continuous  
 (2)2. For any  $c \in \mathbb{R}$ , we have  $f^{-1}(c)$  is  $G_\delta$   
 PROOF:  $f^{-1}(c) = \bigcap_{q \in \mathbb{Q}^+} f^{-1}(c - q, c + q)$   
 (2)3. LET:  $S_{nk} = \{p \in C_{nk} : f(p) \neq f(p_{nk})\}$   
 (2)4. For all  $n, k$ , we have  $S_{nk}$  is countable.  
 (3)1. LET:  $f^{-1}(p_{nk}) = \bigcap_{m=1}^{\infty} U_m$  where  $U_m$  is open in  $X$   
 (3)2. For each  $m$ , PICK  $B_m \in \mathcal{B}$  such that  $p_{nk} \in B_m \subseteq U_m$   
 (3)3.  $S_{nk} \subseteq \bigcup_{m=1}^{\infty} (C_{nk} \setminus B_m)$   
 (3)4. Each  $C_{nk} \setminus B_m$  is countable  
 (4)1. LET:  $m \in \mathbb{Z}$   
 (4)2.  $B_m$  cannot be of type 1  
 (4)3. If  $B_m$  is of type 2 then  $C_{nk} \setminus B_m$  is finite.  
 (4)4. If  $B_m$  is of type 3 or 4 then  $C_{nk} \setminus B_m$  is empty.  
 (2)5. PICK  $d \in [-1, 0]$  such that  $\mathbb{R} \times \{d\}$  intersects none of the sets  $S_{nk}$   
 (2)6. For  $n$  odd, we have  

$$f(n-1, d) = \lim_{k \rightarrow \infty} f(p_{nk}) .$$
  
 (3)1. LET:  $\epsilon > 0$   
 (3)2. PICK  $B \in \mathcal{B}$  such that  $(n-1, d) \in B \subseteq f^{-1}(f(n-1, d) - \epsilon, f(n-1, d) + \epsilon)$   
 (3)3. There exists  $\delta > 0$  such that, for  $x \in (n-1-\delta, n-1+\delta)$ , we have  $(x, d) \in B$   
 (3)4. PICK  $K$  such that  $1/K < \delta$   
 (3)5. LET:  $k \geq K$   
 (3)6.  $f(n-1+1/k, d) = f(p_{nk})$   
 (3)7.  $|f(n-1, d) - f(n-1+1/k, d)| < \epsilon$   
 (3)8.  $|f(n-1, d) - f(p_{nk})| < \epsilon$   
 (2)7. For  $n$  odd, we have  

$$f(n+1, d) = \lim_{k \rightarrow \infty} f(p_{nk}) .$$
  
 PROOF: Similar.  
 (2)8. Q.E.D.  
 (3)1. ASSUME:  $f(a) \neq f(b)$   
 (3)2. ASSUME: w.l.o.g.  $f(a) < f(b)$   
 (3)3. PICK  $B \in \mathcal{B}$  such that  $a \in B \subseteq f^{-1}(-\infty, (f(a) + f(b))/2)$   
 (3)4. LET:  $m$  be even such that  $B = \{a\} \cup \{(x, y) \in X : x < m\}$   
 (3)5. PICK  $B \in \mathcal{B}$  such that  $b \in B \subseteq f^{-1}((f(a) + f(b))/2, +\infty)$   
 (3)6. LET:  $m'$  be even such that  $B = \{b\} \cup \{(x, y) \in X : x > m'\}$

⟨3⟩7.  $f(m, d) = f(m', d)$

⟨3⟩8. Q.E.D.

⟨1⟩9.  $X$  is regular.

⟨1⟩10.  $X$  is not completely regular.

PROOF:  $a$  and  $b$  cannot be separated by a continuous function.

□

**Theorem 6.4.6 (AC).** *A space is completely regular iff it is homeomorphic to a subspace of  $[0, 1]^J$  for some  $J$ .*

PROOF:

⟨1⟩1. Every completely regular space is homeomorphic to a subspace of  $[0, 1]^J$  for some  $J$ .

⟨2⟩1. LET:  $X$  be completely regular

⟨2⟩2. For every point  $a$  and open set  $U$  that contains  $a$ , PICK a continuous function  $f_{aU}$  that is positive on  $a$  and vanishes outside  $U$

⟨2⟩3. The family  $\{f_{aU}\}$  separates points from closed sets

⟨2⟩4. Q.E.D.

PROOF: By the Imbedding Theorem.

⟨1⟩2. Every subspace of  $[0, 1]^J$  is completely regular.

PROOF: By Theorem 6.4.3 and Proposition 6.3.4.

□

**Proposition 6.4.7.** *The continuous image of a completely regular space is not necessarily completely regular.*

PROOF: The identity map from the discrete two-point space to the indiscrete two-point space is continuous. □

## 6.5 Normal Spaces

**Definition 6.5.1 (Normal Space).** A *normal* space is a  $T_1$  space such that, for any disjoint closed sets  $A, B$ , there exist disjoint open sets  $U, V$  such that  $A \subseteq U$  and  $B \subseteq V$ .

**Theorem 6.5.2.** *Every linearly ordered set is normal under the order topology.*

PROOF: See Steen and Steerbach *Counterexamples in Topology* Example 39. □

**Proposition 6.5.3.** *The product space  $S_\Omega \times \overline{S_\Omega}$  is not normal.*

PROOF:

⟨1⟩1. LET:  $\Delta = \{(x, x) : x \in \overline{S_\Omega}\} \subseteq \overline{S_\Omega} \times \overline{S_\Omega}$

⟨1⟩2.  $\Delta$  is closed in  $\overline{S_\Omega} \times \overline{S_\Omega}$

⟨1⟩3. LET:  $A = \Delta \cap (S_\Omega \times \overline{S_\Omega})$

⟨1⟩4.  $A$  is closed in  $S_\Omega \times \overline{S_\Omega}$

⟨1⟩5. LET:  $B = S_\Omega \times \{\Omega\}$

⟨1⟩6.  $B$  is closed

- (1)7.  $A \cap B = \emptyset$   
 (1)8. ASSUME: for a contradiction  $U$  and  $V$  are disjoint open sets including  $A$  and  $B$  respectively  
 (1)9. For all  $x \in S_\Omega$  there exists  $\beta \in (x, \Omega)$  such that  $(x, \beta) \notin U$   
 (2)1. LET:  $x \in S_\Omega$   
 (2)2.  $(x, \Omega) \in V$   
 PROOF:  $(x, \Omega) \in B \subseteq V$   
 (2)3. PICK  $y < \Omega$  such that  $\{x\} \times (y, \Omega] \subseteq V$   
 PROOF: By Lemma 4.1.2.  
 (2)4. PICK  $\beta$  such that  $x, y < \beta < \Omega$   
 PROOF: Such a  $\beta$  exists because  $\Omega$  is a limit ordinal.  
 (1)10. For  $x \in S_\Omega$ ,  
 LET:  $\beta(x)$  be the least element of  $(x, \Omega)$  such that  $(x, \beta(x)) \notin U$   
 (1)11. LET:  $b = \sup_{n=1}^\infty \beta^n(0)$   
 (1)12.  $\beta^n(0) \rightarrow b$  as  $n \rightarrow \infty$   
 (1)13.  $(\beta^n(0), \beta^{n+1}(0)) \rightarrow (b, b)$  as  $n \rightarrow \infty$   
 (1)14.  $(b, b) \in A$   
 (1)15.  $(b, b) \in U$   
 (1)16. For all  $n$  we have  $(\beta^n(0), \beta^{n+1}(0)) \notin U$   
 PROOF: By (1)10.  
 (1)17. Q.E.D.  
 PROOF: Steps (1)12, (1)15 and (1)16 form a contradiction.

□

**Corollary 6.5.3.1.** *Not every completely regular space is normal.*

**Corollary 6.5.3.2.** *An open subspace of a normal space is not necessarily normal.*

**Corollary 6.5.3.3.** *The product of two normal spaces is not necessarily normal.*

**Proposition 6.5.4.** *A closed subspace of a normal space is normal.*

PROOF:

- (1)1. LET:  $X$  be normal and  $C \subseteq X$  be closed.  
 (1)2. LET:  $A$  and  $B$  be closed in  $C$   
 (1)3.  $A$  and  $B$  are closed in  $X$   
 PROOF: By Corollary 4.3.4.2.  
 (1)4. PICK disjoint open neighbourhoods  $U$  and  $V$  of  $A$  and  $B$  in  $X$   
 (1)5.  $U \cap C$  and  $V \cap C$  are disjoint open neighbourhoods of  $A$  and  $B$  in  $C$

□

**Corollary 6.5.4.1.** *Let  $\{X_\alpha\}_{\alpha \in J}$  be a family of nonempty spaces. If  $\prod_{\alpha \in J} X_\alpha$  is normal then each  $X_\alpha$  is normal.*

**Proposition 6.5.5.** *If the Continuum Hypothesis then  $\mathbb{R}^\omega$  under the box topology is normal.*

PROOF: See Rudin. The box product of countably many compact metric spaces. *General Topology and Its Applications*, 2:293–298, 1972. □

**Proposition 6.5.6** (Stone (DC)). *If  $J$  is uncountable then  $\mathbb{R}^J$  is not normal.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $X = (\mathbb{Z}^+)^J$

PROVE:  $X$  is not normal.

$\langle 1 \rangle 2$ . For  $x \in X$  and  $B \subseteq^{\text{fin}} J$ ,

LET:

$$U(x, B) = \{y \in X : \forall \alpha \in B. y_\alpha = x_\alpha\} .$$

$\langle 1 \rangle 3$ .  $\{U(x, B) : x \in X, B \subseteq^{\text{fin}} J\}$  is a basis for  $X$

$\langle 2 \rangle 1$ . LET:  $x \in X$  and  $\prod_{\alpha \in J} U_\alpha$  be a basic open set including  $x$ , where  $U_\alpha = \mathbb{Z}^+$  for all  $\alpha$  except  $\alpha_1, \dots, \alpha_n$

$\langle 2 \rangle 2$ .  $x \in U(x, \{\alpha_1, \dots, \alpha_n\}) \subseteq \prod_{\alpha \in J} U_\alpha$

$\langle 1 \rangle 4$ . For  $n \in \mathbb{Z}^+$ ,

LET:  $P_n = \{x \in X : x \text{ is injective on } J \setminus x^{-1}(n)\}$

$\langle 1 \rangle 5$ .  $P_1$  and  $P_2$  are closed and disjoint.

$\langle 2 \rangle 1$ .  $P_1$  is closed

$\langle 3 \rangle 1$ . LET:  $x \in X \setminus P_1$

$\langle 3 \rangle 2$ . PICK  $\alpha, \beta \in J$  such that  $x_\alpha = x_\beta \neq 1$

$\langle 3 \rangle 3$ . LET:  $U_\gamma = \{x_\alpha\}$  if  $\gamma = \alpha$  or  $\gamma = \beta$ ,  $\mathbb{Z}^+$  for all other  $\gamma \in J$

$\langle 3 \rangle 4$ .  $x \in \prod_{\gamma \in J} U_\gamma \subseteq X \setminus P_1$

$\langle 2 \rangle 2$ .  $P_2$  is closed

PROOF: Similar.

$\langle 2 \rangle 3$ .  $P_1 \cap P_2 = \emptyset$

PROOF: If  $x \in P_1 \cap P_2$  then  $x$  is injective on  $J$ , contradicting the fact that  $J$  is uncountable.

$\langle 1 \rangle 6$ . ASSUME: for a contradiction  $U$  and  $V$  are disjoint open sets including  $P_1$  and  $P_2$

$\langle 1 \rangle 7$ . Given a sequence  $(\alpha_i)$  of distinct elements of  $J$  and a strictly increasing sequence  $(n_i)$  of positive integers,

LET:

$$B_i^{\alpha, n} = \{\alpha_1, \dots, \alpha_{n_i}\}$$

$$x_i^{\alpha, n} \in X$$

$$(x_i^{\alpha, n})_\beta = \begin{cases} j & \text{if } \beta = \alpha_j, 1 \leq j \leq n_{i-1} \\ 1 & \text{for all other values of } \beta \end{cases}$$

for  $i \geq 1$

$\langle 1 \rangle 8$ . PICK sequences  $(\alpha_i)$ ,  $(n_i)$  such that, for all  $i \geq 1$ , we have  $U(x_i^{\alpha, n}, B_i^{\alpha, n}) \subseteq U$

$\langle 2 \rangle 1$ . LET:  $x_1 \in X$  be given by  $(x_1)_\alpha = 1$  for all  $\alpha \in J$

$\langle 2 \rangle 2$ .  $x_1 \in U$

PROOF:  $x_1 \in P_1 \subseteq U$

$\langle 2 \rangle 3$ . PICK  $B_1 \subseteq^{\text{fin}} J$  such that  $U(x_1, B_1) \subseteq U$

PROOF: By  $\langle 1 \rangle 3$ .

$\langle 2 \rangle 4$ . LET:  $n_1 = |B_1|$  and  $B_1 = \{\alpha_1, \dots, \alpha_{n_1}\}$

$\langle 2 \rangle 5$ . ASSUME: We have chosen  $n_1, \dots, n_k$  strictly increasing and  $\alpha_1, \dots, \alpha_{n_k}$  such that, for  $1 \leq i \leq k$ , we have  $U(x_i^{\alpha, n}, B_i^{\alpha, n}) \subseteq U$

- (2)6.  $x_{i+1}^{\alpha,n} \in U$   
 PROOF:  $x_{i+1}^{\alpha,n} \in P_1 \subseteq U$   
 (2)7. PICK  $C \subseteq^{\text{fin}} J$  such that  $U(x_{i+1}^{\alpha,n}, C) \subseteq U$   
 (2)8. LET:  $n_{i+1}$  and  $\alpha_{n_{i+1}+1}, \dots, \alpha_{n_{i+1}}$  be such that  $B_i^{\alpha,n} \cup C = B_{i+1}^{\alpha,n}$   
 (2)9.  $U(x_{i+1}^{\alpha,n}, B_{i+1}^{\alpha,n}) \subseteq U$   
 (1)9. LET:  $A = \{\alpha_i : i \geq 1\}$   
 (1)10. LET:  $y \in X$ ,  $y_\beta = j$  if  $\beta = \alpha_j$ ,  $y_\beta = 2$  for  $\beta \notin A$   
 (1)11. PICK  $B$  such that  $U(y, B) \subseteq V$   
 (1)12. PICK  $i$  such that  $A \cap B \subseteq B_i^{\alpha,n}$   
 (1)13.  $U(x_{i+1}^{\alpha,n}, B_{i+1}^{\alpha,n}) \cap U(y, B) \neq \emptyset$   
 PROOF:  $x_{i+1}^{\alpha,n} \in U(x_{i+1}^{\alpha,n}, B_{i+1}^{\alpha,n}) \cap U(y, B)$   
 (1)14. Q.E.D.  
 PROOF: This contradicts the fact that  $U$  and  $V$  are disjoint ((1)6).  
 □

**Theorem 6.5.7** (Urysohn Lemma). *Let  $X$  be a normal space. Let  $A$  and  $B$  be disjoint closed subsets of  $X$ . Then there exists a continuous map  $f : X \rightarrow [0, 1]$  such that  $f(x) = 0$  for all  $x \in A$  and  $f(x) = 1$  for all  $x \in B$ .*

PROOF:

- (1)1. LET:  $P$  be the set of all rational numbers in  $[0, 1]$   
 (1)2. For all  $q \in P$ , PICK an open set  $U_q$  in  $X$  such that  $A \subseteq U_0$ ,  $U_1 \subseteq X \setminus B$ ,  
 and whenever  $p < q$  then  $\overline{U_p} \subseteq U_q$   
 (2)1. PICK an enumeration  $(q_n)$  of  $P$  such that  $q_1 = 1$  and  $q_2 = 0$   
 (2)2. LET:  $U_1 = X \setminus B$   
 (2)3. PICK an open set  $U_0$  such that  $A \subseteq U_0$  and  $\overline{U_0} \subseteq U_1$   
 (2)4. ASSUME: we have open sets  $U_1, U_0, \dots, U_{q_n}$  such that whenever  $p < q$   
 then  $\overline{U_p} \subseteq U_q$   
 (2)5.  $q_2 < q_{n+1} < q_1$   
 (2)6. LET:  $q_k$  be greatest among  $q_1, \dots, q_n$  such that  $q_k < q_{n+1}$ , and  $q_l$  be  
 least such that  $q_{n+1} < q_l$   
 (2)7. PICK an open set  $U_{q_{n+1}}$  such that  $\overline{U_{q_k}} \subseteq U_{q_{n+1}}$  and  $\overline{U_{q_{n+1}}} \subseteq U_{q_l}$   
 (2)8. For all  $p, q \in \{q_1, \dots, q_{n+1}\}$ , if  $p < q$  then  $\overline{U_p} \subseteq U_q$   
 (1)3. Extend the family  $(U_q)$  to  $\mathbb{Q}$  by defining:  $U_q = \emptyset$  if  $q < 0$  and  $U_q = X$  if  
 $q > 1$   
 (1)4. For all rationals  $p, q$  with  $p < q$  we have  $\overline{U_p} \subseteq U_q$   
 (1)5. Define  $f : X \rightarrow [0, 1]$  by  $f(x) = \inf\{q \in \mathbb{Q} : x \in U_q\}$   
 PROOF: This set is nonempty since  $x \in U_1$  and bounded below since if  $x \in U_q$   
 then  $q \geq 0$ .  
 (1)6. For all  $x \in A$  we have  $f(x) = 0$   
 (1)7. For all  $x \in B$  we have  $f(x) = 1$   
 (1)8. If  $x \in \overline{U_r}$  then  $f(x) \leq r$   
 (1)9. If  $x \notin U_r$  then  $f(x) \geq r$   
 (1)10.  $f$  is continuous  
 (2)1. LET:  $x_0 \in X$   
 (2)2. LET:  $(c, d)$  be an open interval containing  $f(x_0)$   
 PROVE: There exists a neighbourhood  $U$  of  $x_0$  such that  $f(U) \subseteq (c, d)$

- (2)3. PICK rationals  $p, q$  such that  $c < p < f(x_0) < q < d$   
 (2)4.  $x \notin \overline{U_p}$   
 PROOF: By (1)8  
 (2)5.  $x \in U_q$   
 PROOF: By (1)9  
 (2)6. LET:  $U = U_q \setminus \overline{U_p}$

□

**Definition 6.5.8** (Vanish Precisely). Let  $X$  be a set and  $A \subseteq X$ . Let  $f : X \rightarrow [0, 1]$ . Then  $f$  *vanishes precisely* on  $A$  iff  $f^{-1}(0) = A$ .

**Theorem 6.5.9** (CC). Let  $X$  be a normal space and  $A \subseteq X$ . Then there exists a continuous function  $f : X \rightarrow [0, 1]$  such that  $f$  vanishes precisely on  $A$  if and only if  $A$  is a closed  $G_\delta$  set.

PROOF:

- (1)1. If there exists  $f$  such that  $f$  vanishes precisely on  $A$  then  $A$  is closed.  
 PROOF: This holds because  $A = f^{-1}(0)$ .  
 (1)2. If there exists  $f$  such that  $f$  vanishes precisely on  $A$  then  $A$  is  $G_\delta$ .  
 PROOF: This holds because  $A = \bigcap_{q \in \mathbb{Q}^+} f^{-1}([0, q))$ .  
 (1)3. If  $A$  is closed and  $G_\delta$  then there exists  $f$  that vanishes precisely on  $A$ .  
 (2)1. LET:  $A = \bigcap_{n=1}^{\infty} U_n$   
 (2)2. For  $n \geq 1$ , PICK  $f_n : X \rightarrow [0, 1/2^n]$  such that  $f_n(x) = 0$  for  $x \in A$  and  $f_n(x) = 1/2^n$  for  $x \in X \setminus U_n$   
 PROOF: By the Urysohn Lemma.  
 (2)3. LET:  $f : X \rightarrow [0, 1]$  be given by  $f(x) = \sum_{n=1}^{\infty} f_n(x)$   
 PROOF: The series converges for every  $x$  by the Comparison Test.  
 (2)4.  $f$  is continuous  
 (3)1.  $f_n$  converges uniformly to  $f$   
 PROOF: By the Weierstrass M-test.  
 (3)2. Q.E.D.  
 PROOF: By the Uniform Limit Theorem.  
 (2)5.  $f(x) = 0$  for  $x \in A$   
 PROOF: From (2)2.  
 (2)6.  $f(x) > 0$  for  $x \notin A$   
 (3)1. LET:  $x \notin A$   
 (3)2. PICK  $N$  such that  $x \notin U_N$   
 (3)3. Q.E.D.  
 PROOF:

$$\begin{aligned}
 f(x) &= \sum_{n=1}^{\infty} f_n(x) & ((2)3) \\
 &\geq f_N(x) \\
 &> 0 & ((2)2)
 \end{aligned}$$

□

**Theorem 6.5.10** (Strong Form of Urysohn Lemma). Let  $X$  be a normal space.

Then there exists a continuous function  $f : X \rightarrow [0, 1]$  such that  $f^{-1}(0) = A$  and  $f^{-1}(1) = B$  if and only if  $A$  and  $B$  are disjoint, closed and  $G_\delta$ .

PROOF:

- (1)1. If there exists a continuous function  $f : X \rightarrow [0, 1]$  such that  $f^{-1}(0) = A$  and  $f^{-1}(1) = B$  then  $A$  and  $B$  are disjoint, closed and  $G_\delta$
- (2)1. ASSUME: there exists a continuous function  $f : X \rightarrow [0, 1]$  such that  $f^{-1}(0) = A$  and  $f^{-1}(1) = B$
- (2)2.  $A$  and  $B$  are disjoint
- (2)3.  $A$  is closed and  $G_\delta$   
PROOF: By Theorem 6.5.9.
- (2)4.  $B$  is closed and  $G_\delta$   
PROOF: Apply Theorem 6.5.9 to  $1 - f$ .
- (1)2. If  $A$  and  $B$  are disjoint, closed and  $G_\delta$  then there exists a continuous function  $f : X \rightarrow [0, 1]$  such that  $f^{-1}(0) = A$  and  $f^{-1}(1) = B$
- (2)1. ASSUME:  $A$  and  $B$  are disjoint, closed and  $G_\delta$
- (2)2. PICK  $g : X \rightarrow [0, 1]$  that vanishes precisely on  $A$  and  $h : X \rightarrow [0, 1]$  that vanishes precisely on  $B$
- (2)3. LET:  $f = g/(g + h)$

□

**Definition 6.5.11** (Universal Extension Property). A topological space  $Y$  has the *universal extension property* iff, for every normal space  $X$  and closed subspace  $A$  of  $X$ , every continuous function  $A \rightarrow Y$  can be extended to a continuous function  $X \rightarrow Y$ .

**Theorem 6.5.12** (Tietze Extension Theorem (DC)). Let  $X$  be a normal space. Let  $A$  be closed subspace of  $X$ .

- 1. Any continuous function  $A \rightarrow [a, b]$  can be extended to a continuous function  $X \rightarrow [a, b]$ .
- 2. Any continuous function  $A \rightarrow \mathbb{R}$  can be extended to a continuous function  $X \rightarrow \mathbb{R}$ .

PROOF:

- (1)1. Any continuous function  $A \rightarrow [-1, 1]$  can be extended to a continuous function  $X \rightarrow [-1, 1]$
- (2)1. For every continuous function  $f : A \rightarrow [-r, r]$ , there exists a continuous  $g : X \rightarrow \mathbb{R}$  such that

$$|g(x)| \leq \frac{1}{3}r \quad (x \in X)$$

$$|g(x) - f(x)| \leq \frac{2}{3}r \quad (x \in A)$$

- (3)1. LET:  $f : A \rightarrow [-r, r]$  be continuous
- (3)2. LET:  $I_1 = [-r, -\frac{1}{3}r]$
- (3)3. LET:  $I_2 = [-\frac{1}{3}r, \frac{1}{3}r]$
- (3)4. LET:  $I_3 = [\frac{1}{3}r, r]$

⟨3⟩5. LET:  $B = f^{-1}(I_1)$

⟨3⟩6. LET:  $C = f^{-1}(I_3)$

⟨3⟩7. PICK a continuous  $g : X \rightarrow [-\frac{1}{3}r, \frac{1}{3}r]$  such that  $g(x) = -\frac{1}{3}r$  for  $x \in B$  and  $g(x) = \frac{1}{3}r$  for  $x \in C$

PROOF: By the Urysohn Lemma, since  $B$  and  $C$  are closed disjoint subsets of  $X$ .

⟨3⟩8. For all  $x \in A$  we have  $|g(x) - f(x)| \leq \frac{2}{3}r$

⟨4⟩1. LET:  $x \in A$

⟨4⟩2. CASE:  $f(x) \in I_1$

PROOF:

$$\begin{aligned} |g(x) - f(x)| &= \left| -\frac{1}{3}r - f(x) \right| & (x \in B) \\ &\leq \frac{2}{3}r & (f(x) \in I_1) \end{aligned}$$

⟨4⟩3. CASE:  $f(x) \in I_2$

PROOF: In this case,  $|g(x) - f(x)| \leq \frac{2}{3}r$  since  $f(x), g(x) \in I_2$ .

⟨4⟩4. CASE:  $f(x) \in I_3$

PROOF:

$$\begin{aligned} |g(x) - f(x)| &= \left| \frac{1}{3}r - f(x) \right| & (x \in C) \\ &\leq \frac{2}{3}r & (f(x) \in I_3) \end{aligned}$$

⟨2⟩2. LET:  $f : A \rightarrow [-1, 1]$  be continuous.

⟨2⟩3. PICK a sequence of functions  $(g_n)$  such that

$$|g_n(x)| \leq \frac{1}{3} \left( \frac{2}{3} \right)^{n-1} \quad (x \in X)$$

$$|f(x) - g_1(x) - \cdots - g_n(x)| \leq (2/3)^n \quad (x \in A)$$

PROOF: Given  $g_1, \dots, g_n$ , we apply ⟨2⟩1 with  $f = f - g_1 - \cdots - g_n$  and  $r = (2/3)^n$ .

⟨2⟩4. LET:  $g(x) = \sum_{n=1}^{\infty} g_n(x)$  for  $x \in X$

PROOF: This series converges by the Comparison Test since  $\sum_{n=1}^{\infty} (2/3)^n$  converges.

⟨2⟩5.  $g$  is continuous.

⟨3⟩1.  $\sum_{n=1}^N g_n$  converges to  $g$  uniformly

PROOF: By the Weierstrass  $M$ -test.

⟨3⟩2. Q.E.D.

PROOF: By the Uniform Limit Theory.

⟨2⟩6. For all  $x \in A$  we have  $g(x) = f(x)$

PROOF:  $|\sum_{n=1}^N g_n(x) - f(x)| \leq (2/3)^N \rightarrow 0$  as  $N \rightarrow \infty$ .

⟨2⟩7. For all  $x \in X$  we have  $-1 \leq g(x) \leq 1$



PROOF:

$$\begin{aligned} \left| \sum_{n=1}^N g_n(x) \right| &\leq \sum_{n=1}^N |g_n(x)| \\ &\leq 1/3 \sum_{n=1}^N (2/3)^{n-1} \\ &\rightarrow 2/3 \quad \text{as } n \rightarrow \infty \end{aligned}$$

⟨1⟩2. Any continuous function  $A \rightarrow (-1, 1)$  can be extended to a continuous function  $X \rightarrow (-1, 1)$

⟨2⟩1. LET:  $f : A \rightarrow (-1, 1)$  be continuous

⟨2⟩2. PICK a continuous  $g : X \rightarrow [-1, 1]$  that extends  $f$

PROOF: By ⟨1⟩1.

⟨2⟩3. LET:  $D = g^{-1}(-1) \cup g^{-1}(1)$

⟨2⟩4.  $D$  is closed in  $X$

PROOF: Since  $g$  is continuous and  $\{-1\}, \{1\}$  are closed in  $[-1, 1]$ .

⟨2⟩5.  $D \cap A = \emptyset$

PROOF: Since  $g(A) = f(A) \subseteq (-1, 1)$ .

⟨2⟩6. PICK a continuous  $\phi : X \rightarrow [0, 1]$  such that  $\phi(D) = \{0\}$  and  $\phi(A) = \{1\}$

PROOF: By the Urysohn Lemma.

⟨2⟩7. LET:  $h = g\phi$

⟨2⟩8.  $h$  is continuous

⟨2⟩9.  $h$  extends  $f$

⟨2⟩10.  $\text{im } h \subseteq (-1, 1)$

⟨1⟩3. Q.E.D.

PROOF: The result follows because any closed interval in  $\mathbb{R}$  is homeomorphic to  $[-1, 1]$  and  $\mathbb{R} \cong (-1, 1)$ .

□

**Lemma 6.5.13** (Shrinking Lemma (AC)). *Let  $X$  be a normal space. Let  $\{U_\alpha\}_{\alpha \in J}$  be a point-finite indexed open covering of  $X$ . Then there exists an indexed open covering  $\{V_\alpha\}_{\alpha \in J}$  such that  $\overline{V_\alpha} \subseteq U_\alpha$  for all  $\alpha \in J$ .*

PROOF:

⟨1⟩1. PICK a well-ordering  $\prec$  on  $J$

⟨1⟩2. PICK open sets  $V_\alpha$  for  $\alpha \in J$  such that  $A_\alpha \subseteq V_\alpha$  and  $\overline{V_\alpha} \subseteq U_\alpha$ , where

$$A_\alpha = X \setminus \bigcup_{\beta \prec \alpha} V_\beta \cup \bigcup_{\alpha \prec \beta} U_\beta$$

PROOF: Apply transfinite induction to Proposition 13.1.16.

⟨1⟩3.  $\{V_\alpha\}_{\alpha \in J}$  covers  $X$

⟨2⟩1. LET:  $x \in X$

⟨2⟩2. LET:  $\alpha_1, \dots, \alpha_n$  be the elements of  $J$  such that  $x \in U_{\alpha_i}$ , where  $\alpha_1 \prec \dots \prec \alpha_n$

PROVE:  $x \in V_{\alpha_i}$  for some  $i$

⟨2⟩3. ASSUME:  $x \notin V_{\alpha_1}, \dots, V_{\alpha_{n-1}}$

⟨2⟩4.  $x \in A_{\alpha_n}$

⟨2⟩5.  $x \in V_{\alpha_n}$

□

**Proposition 6.5.14 (DC).**  $S_\Omega \times \overline{S_\Omega}$  is not normal.

PROOF:

- ⟨1⟩1. LET:  $\Delta = \{(x, x) : x \in \overline{S_\Omega}\}$
- ⟨1⟩2.  $\Delta$  is closed in  $\overline{S_\Omega}^2$ 
  - ⟨2⟩1. LET:  $(x, y) \in \overline{S_\Omega}^2 \setminus \Delta$
  - ⟨2⟩2. PICK disjoint open sets  $U, V$  such that  $x \in U$  and  $y \in V$
  - ⟨2⟩3.  $(x, y) \in U \times V \subseteq \overline{S_\Omega}^2 \setminus \Delta$
- ⟨1⟩3. LET:  $A = \Delta \cap (S_\Omega \times \overline{S_\Omega})$
- ⟨1⟩4.  $A$  is closed in  $S_\Omega \times \overline{S_\Omega}$
- ⟨1⟩5. LET:  $B = S_\Omega \times \{\Omega\}$
- ⟨1⟩6.  $B$  is closed in  $S_\Omega \times \overline{S_\Omega}$
- ⟨1⟩7.  $A \cap B = \emptyset$
- ⟨1⟩8. ASSUME: for a contradiction  $U$  and  $V$  are disjoint open sets including  $A$  and  $B$  respectively
- ⟨1⟩9. PICK a sequence  $x_n$  in  $S_\Omega$  such that  $x_n < x_{n+1} < \Omega$  and  $(x_n, x_{n+1}) \notin U$  for all  $n$ 
  - ⟨2⟩1. LET:  $x_n \in S_\Omega$
  - ⟨2⟩2.  $(x_n, \Omega) \in V$
  - ⟨2⟩3. PICK open sets  $W \subseteq S_\Omega, X \subseteq \overline{S_\Omega}$  such that  $x_n \in W, \Omega \in X$  and  $W \times X \subseteq V$
  - ⟨2⟩4. PICK  $y < \Omega$  such that  $(x_{n+1}, \Omega] \subseteq X$
  - ⟨2⟩5. LET:  $x_{n+1} = y + 1$
- ⟨1⟩10. LET:  $b$  be the supremum of  $\{x_n : n \geq 1\}$
- ⟨1⟩11.  $(x_n, x_{n+1}) \rightarrow (b, b)$  as  $n \rightarrow \infty$
- ⟨1⟩12.  $(b, b) \in A$
- ⟨1⟩13.  $(b, b) \in U$
- ⟨1⟩14. For all  $n$  we have  $(x_n, x_{n+1}) \notin U$

□

**Proposition 6.5.15 (AC).**  $\mathbb{R}_I$  is normal.

PROOF:

- ⟨1⟩1. LET:  $A$  and  $B$  be disjoint closed sets in  $\mathbb{R}_I$
- ⟨1⟩2. For  $a \in A$ , PICK  $x_a > a$  such that  $[a, x_a)$  not intersecting  $B$
- ⟨1⟩3. For  $b \in B$ , PICK  $x_b > b$  such that  $[b, x_b)$  does not intersect  $A$
- ⟨1⟩4. LET:  $U = \bigcup_{a \in A} [a, x_a)$  and  $V = \bigcup_{b \in B} [b, x_b)$
- ⟨1⟩5.  $U$  and  $V$  are disjoint open sets including  $A$  and  $B$  respectively.

□

**Lemma 6.5.16.** The set  $L = \{(x, -x) : x \in \mathbb{R}\}$  as a subspace of  $\mathbb{R}_I^2$  is closed

- ⟨1⟩1. LET:  $(x, y) \notin L$ , so  $y \neq -x$   
 PROVE: There exists a neighbourhood  $U$  of  $(x, y)$  that does not intersect  $L$
- ⟨1⟩2. CASE:  $y > -x$

PROOF: In this case, take  $U = [x, +\infty) \times [y, +\infty)$   
 (1)3. CASE:  $y < -x$   
 PROOF: In this case, take  $U = [x, (x - y)/2) \times [y, (y - x)/2)$ .

**Proposition 6.5.17 (AC).** *The Sorgenfrey plane is not normal.*

PROOF:

- (1)1. ASSUME: for a contradiction the Sorgenfrey plane is normal.
- (1)2. LET:  $L = \{(x, -x); x \in \mathbb{R}\}$  as a subspace of  $\mathbb{R}_l^2$
- (1)3.  $L$  has the discrete topology.
  - (2)1. LET:  $(x, -x) \in L$ 
    - PROVE:  $\{(x, -x)\}$  is open in  $L$
  - (2)2.  $\{(x, -x)\} = ([x, +\infty) \times [-x, +\infty)) \cap L$
- (1)4. Every subset of  $L$  is closed in  $\mathbb{R}_l^2$ 
  - PROOF: By Corollary 4.3.4.2.
- (1)5. For every nonempty proper subset  $A$  of  $L$ , PICK disjoint open sets  $U_A$ ,  $V_A$  containing  $A$  and  $L \setminus A$ 
  - PROOF: By (1)1 and (1)4.
- (1)6. LET:  $D = \mathbb{Q}^2$
- (1)7.  $D$  is dense in  $\mathbb{R}_l^2$ 
  - PROOF: Given any basic open set  $[a, b) \times [c, d)$ , pick rationals  $q, r$  such that  $a \leq q < b$  and  $c \leq r < d$ . Then  $(q, r) \in ([a, b) \times [c, d)) \cap D$
- (1)8. LET:  $\theta : \mathcal{P}L \rightarrow \mathcal{P}D$  be the function
 

$$\begin{aligned}\theta(A) &= U_A \cap D & (\emptyset \neq A \neq L) \\ \theta(\emptyset) &= \emptyset \\ \theta(L) &= D\end{aligned}$$
- (1)9.  $\theta$  is injective
  - (2)1. LET:  $A, B \subseteq L$  with  $\theta(A) = \theta(B)$ 
    - PROVE:  $A = B$
  - (2)2. CASE:  $\emptyset \neq A \neq L$  and  $\emptyset \neq B \neq L$ 
    - (3)1.  $A \subseteq B$ 
      - (4)1. LET:  $x \in A$
      - (4)2.  $x \in U_A$ 
        - PROOF: By (1)5
      - (4)3.  $x \in U_B$ 
        - PROOF: By (2)1
      - (4)4.  $x \notin L \setminus B$ 
        - PROOF: By (1)5
      - (4)5.  $x \in B$ 
        - PROOF: Since  $x \in L$  by (4)1
    - (3)2.  $B \subseteq A$ 
      - PROOF: Similar.
  - (2)3. CASE:  $\emptyset \neq A \neq L$  and  $B = \emptyset$ 
    - PROOF: This implies  $U_A \cap D = \emptyset$  which contradicts the fact that  $D$  is dense.
  - (2)4. CASE:  $\emptyset \neq A \neq L$  and  $B = L$ 
    - PROOF: This implies  $V_A \cap D = \emptyset$  which contradicts the fact that  $D$  is dense.

- (2)5. CASE:  $A = B = \emptyset$   
 PROOF: Trivial  
 (2)6. CASE:  $A = \emptyset$  and  $B = L$   
 PROOF: This implies  $D = \emptyset$  which is a contradiction.  
 (2)7. CASE:  $A = B = L$   
 PROOF: Trivial  
 (1)10. Q.E.D.  
 PROOF: This is a contradiction since  $D$  is countable and  $L$  is uncountable.  
 $\square$

**Proposition 6.5.18.** *The continuous image of a normal space is not necessarily normal.*

PROOF: The identity map from the discrete two-point space to the indiscrete two-point space is continuous.  $\square$

## 6.6 Completely Normal Spaces

**Definition 6.6.1** (Completely Normal). A space  $X$  is *completely normal* iff every subspace is normal.

**Proposition 6.6.2.** *A subspace of a completely normal space is completely normal.*

PROOF: Immediate from definitions.  $\square$

**Proposition 6.6.3.** *Let  $X$  be a topological space. Then  $X$  is completely normal iff  $X$  is  $T_1$  and, for any pair of separated sets  $A, B$  in  $X$ , there exist disjoint open sets including them.*

PROOF:

- (1)1. If  $X$  is completely normal then  $X$  is  $T_1$  and, for any pair of separated sets  $A, B$  in  $X$ , there exist disjoint open sets including them.  
 (2)1. ASSUME:  $X$  is completely normal.  
 (2)2.  $X$  is  $T_1$   
 PROOF: Holds because  $X$  is normal.  
 (2)3. For any pair of separated sets  $A, B$  in  $X$ , there exist disjoint open sets including them.  
 (3)1. LET:  $A$  and  $B$  be separated in  $X$   
 (3)2. LET:  $Y = X \setminus (\bar{A} \cap \bar{B})$   
 (3)3. PICK disjoint open sets  $U, V$  in  $Y$  such that  $\bar{A} \cap Y \subseteq U$  and  $\bar{B} \cap Y \subseteq V$   
 PROOF:  $Y$  is normal by (2)1.  
 (3)4. PICK open sets  $U_0, V_0$  in  $X$  such that  $U = U_0 \cap Y, V = V_0 \cap Y$   
 (3)5.  $A \subseteq U_0 \setminus \bar{B}$  and  $B \subseteq V_0 \setminus \bar{A}$   
 PROOF: Using (3)1.  
 (1)2. If  $X$  is  $T_1$  and, for any pair of separated sets  $A, B$  in  $X$ , there exist disjoint open sets including them, then  $X$  is completely normal.

- (2)1. ASSUME:  $X$  is  $T_1$  and, for any pair of separated sets  $A, B$  in  $X$ , there exist disjoint open sets including them  
 (2)2. LET:  $Y \subseteq X$   
 (2)3.  $Y$  is  $T_1$   
 PROOF: By Proposition 6.1.3.  
 (2)4. LET:  $A$  and  $B$  be disjoint closed sets in  $Y$   
 (2)5.  $A$  and  $B$  are separated in  $X$   
 (3)1.  $\overline{A} \cap Y = A$  and  $\overline{B} \cap Y = B$   
 PROOF: By Proposition 3.7.6 and Theorem 4.3.4.  
 (3)2.  $\overline{A} \cap B = \emptyset$
- $$\begin{aligned}
 \overline{A} \cap B &= \overline{A} \cap \overline{B} \cap Y & ((3)1) \\
 &= A \cap B & ((3)1) \\
 &= \emptyset & ((2)4)
 \end{aligned}$$
- (3)3.  $A \cap \overline{B} = \emptyset$   
 PROOF: Similar.  
 (2)6. PICK disjoint open sets  $U$  and  $V$  that include  $A$  and  $B$  respectively.  
 PROOF: By (2)1.  
 (2)7.  $U \cap Y$  and  $V \cap Y$  are disjoint open sets in  $Y$  that include  $A$  and  $B$  respectively.

□

**Proposition 6.6.4.** *A well-ordered set in the order topology is completely normal.*

PROOF:

- (1)1. LET:  $X$  be a well-ordered set.  
 (1)2. For all  $a, b \in X$  with  $a < b$ , we have  $(a, b]$  is open.  
 (2)1. CASE:  $b$  is greatest in  $X$   
 PROOF: This case holds by the definition of the order topology.  
 (2)2. CASE:  $b$  is not greatest in  $X$   
 PROOF: In this case,  $(a, b] = (a, c)$  where  $c$  is the successor of  $b$ .  
 (1)3. LET:  $A$  and  $B$  be separated sets in  $X$   
 PROVE: There exist disjoint open sets  $U, V$  including  $A$  and  $B$   
 (1)4. CASE: The least element of  $X$  is not in  $A$  or  $B$   
 (2)1. LET:  $U = \bigcup \{(x, a] : a \in A, x < a, (x, a] \cap B = \emptyset\}$   
 (2)2. LET:  $V = \bigcup \{(y, b] : b \in B, y < b, (y, b] \cap A = \emptyset\}$   
 (2)3.  $U$  is open  
 PROOF: From (1)2.  
 (2)4.  $V$  is open  
 PROOF: From (1)2.  
 (2)5.  $A \subseteq U$   
 (3)1. LET:  $a \in A$   
 (3)2. PICK  $W$  a neighbourhood of  $a$  such that  $W \cap B = \emptyset$   
 PROOF: By (1)3.  
 (3)3. PICK  $x < a$  such that  $(x, a] \subseteq W$

PROOF: By Lemma 4.1.2  
 ⟨3⟩4.  $a \in (x, a] \subseteq U$   
 ⟨2⟩6.  $B \subseteq V$   
 PROOF: Similar.  
 ⟨2⟩7.  $U \cap V = \emptyset$   
 ⟨1⟩5. CASE:  $\perp \in A$   
 ⟨2⟩1. PICK disjoint open sets  $U$  and  $V$  that include  $A \setminus \{\perp\}$  and  $B$   
 PROOF: From ⟨1⟩4.  
 ⟨2⟩2.  $U \cup \{\perp\}$  and  $V$  are disjoint open sets that include  $A$  and  $B$   
 PROOF:  $\{\perp\}$  is open because it is  $(-\infty, a)$  where  $a$  is the successor of  $\perp$ .  
 ⟨1⟩6. Q.E.D.  
 PROOF: By Proposition 6.6.3.  
 □

**Proposition 6.6.5.** *The product of two completely normal spaces is not necessarily completely normal.*

PROOF:  
 ⟨1⟩1.  $S_\Omega$  is completely normal.  
 PROOF: By Proposition 6.6.4  
 ⟨1⟩2.  $\overline{S_\Omega}$  is completely normal.  
 PROOF: By Proposition 6.6.4  
 ⟨1⟩3.  $S_\Omega \times \overline{S_\Omega}$  is not completely normal.  
 PROOF: By Proposition 6.5.3.  
 □

**Proposition 6.6.6.** *A compact Hausdorff space is not necessarily completely normal.*

PROOF:  
 ⟨1⟩1. PICK an uncountable set  $J$   
 ⟨1⟩2.  $[0, 1]^J$  is compact Hausdorff  
 PROOF: By Tychonoff's Theorem and Theorem 6.2.5.  
 ⟨1⟩3.  $(0, 1)^J$  is not normal.  
 PROOF: By Proposition 6.5.6, since  $(0, 1) \cong \mathbb{R}$ .  
 □

**Proposition 6.6.7.** *The space  $\mathbb{R}_l$  is completely normal.*

PROOF:  
 ⟨1⟩1. LET:  $X \subseteq \mathbb{R}_l$   
 ⟨1⟩2. LET:  $A$  and  $B$  be disjoint closed sets in  $X$ .  
 ⟨1⟩3. PICK closed sets  $C$  and  $D$  such that  $A = C \cap X$  and  $B = D \cap X$   
 ⟨1⟩4. For  $a \in A$ , PICK  $x_a > a$  such that  $[a, x_a) \cap D = \emptyset$   
 ⟨1⟩5. For  $b \in B$ , PICK  $x_b > b$  such that  $[b, x_b) \cap C = \emptyset$   
 ⟨1⟩6.  $\bigcup_{a \in A} [a, x_a) \cap X$  and  $\bigcup_{b \in B} [b, x_b) \cap X$  are disjoint open sets in  $X$  that include  $A$  and  $B$   
 □

## 6.7 Perfectly Normal Spaces

**Definition 6.7.1** (Perfectly Normal). A space is *perfectly normal* iff it is normal and every closed set is  $G_\delta$ .

**Proposition 6.7.2.** *Every perfectly normal space is completely normal.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $X$  be perfectly normal.

$\langle 1 \rangle 2$ . LET:  $A$  and  $B$  be separated sets in  $X$

$\langle 1 \rangle 3$ . PICK continuous functions  $f, g : X \rightarrow [0, 1]$  that vanish precisely on  $\overline{A}$  and  $\overline{B}$ , respectively.

PROOF: By Theorem 6.5.9.

$\langle 1 \rangle 4$ . LET:  $h = f - g$

$\langle 1 \rangle 5$ .  $B \subseteq h^{-1}((0, +\infty))$  and  $A \subseteq h^{-1}((-\infty, 0))$

$\langle 1 \rangle 6$ . Q.E.D.

PROOF: By Proposition 6.6.3.

□

**Proposition 6.7.3.** *The space  $\overline{S_\Omega}$  is not perfectly normal.*

PROOF: The set  $\{\Omega\}$  is not  $G_\delta$ . □

# Chapter 7

## Countability Axioms

### 7.1 The First Countability Axiom

**Definition 7.1.1** (First Countability Axiom). A topological space  $X$  satisfies the *first countability axiom*, or is *first countable*, iff every point has a countable local basis.

**Proposition 7.1.2.**  $S_\Omega$  is first countable.

PROOF: For every countable ordinal  $\alpha > 0$ , the set  $\{(\beta, \alpha + 1) : \beta < \alpha\}$  is a local basis at  $\alpha$ . The set  $\{\{0\}\}$  is a local basis at 0.  $\square$

**Theorem 7.1.3** (The Sequence Lemma (CC)). *Let  $X$  be a first countable space and  $A \subseteq X$ . If  $x \in \overline{A}$ , then there exists a sequence of points of  $A$  that converges to  $x$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $x \in \overline{A}$

$\langle 1 \rangle 2$ . PICK a countable basis  $\{B_n\}_{n \in \mathbb{Z}^+}$  at  $x$ .

$\langle 1 \rangle 3$ . For  $n \geq 1$ , PICK a point  $a_n \in B_1 \cap \cdots \cap B_n \cap A$

PROVE:  $a_n \rightarrow x$  as  $n \rightarrow \infty$

PROOF: Using Countable Choice. Such an  $a_n$  exists because  $B_1 \cap \cdots \cap B_n$  is a neighbourhood of  $x$ . Apply Theorem 3.8.3.

$\langle 1 \rangle 4$ . LET:  $U$  be a neighbourhood of  $x$

$\langle 1 \rangle 5$ . PICK  $N$  such that  $B_N \subseteq U$

PROOF: From  $\langle 1 \rangle 2$ .

$\langle 1 \rangle 6$ . For  $n \geq N$ , we have  $a_n \in U$

PROOF:

$$\begin{aligned} a_n &\in B_1 \cap \cdots \cap B_n && (\langle 1 \rangle 3) \\ &\subseteq B_N && (n \geq N) \\ &\subseteq U && (\langle 1 \rangle 5) \end{aligned}$$

$\square$



**Theorem 7.1.4** (CC). *Let  $X$  and  $Y$  be topological spaces where  $X$  is first countable. Let  $x \in X$ . Suppose that, for every sequence  $\{x_n\}_{n \geq 1}$  such that  $x_n \rightarrow x$  as  $n \rightarrow \infty$ , we have  $f(x_n) \rightarrow f(x)$  as  $n \rightarrow \infty$ . Then  $f$  is continuous at  $x$ .*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $V$  be a neighbourhood of  $f(x)$
- $\langle 1 \rangle 2$ . ASSUME: for a contradiction that, for every neighbourhood  $U$  of  $x$ ,  $f(U) \not\subseteq V$
- $\langle 1 \rangle 3$ . PICK a countable local basis  $\{B_n\}_{n \geq 1}$
- $\langle 1 \rangle 4$ . For  $n \geq 1$ , PICK  $a_n \in B_1 \cap \dots \cap B_n$  such that  $f(a_n) \notin V$
- $\langle 1 \rangle 5$ .  $a_n \rightarrow x$  as  $n \rightarrow \infty$

PROOF:

- $\langle 2 \rangle 1$ . LET:  $U$  be a neighbourhood of  $x$
- $\langle 2 \rangle 2$ . PICK  $N$  such that  $B_N \subseteq U$
- $\langle 2 \rangle 3$ . For all  $n \geq N$ ,  $a_n \in U$

PROOF:

$$\begin{aligned} a_n &\in B_1 \cap \dots \cap B_n && (\langle 1 \rangle 4) \\ &\subseteq B_N && (n \geq N) \\ &\subseteq U && (\langle 2 \rangle 2) \end{aligned}$$

- $\langle 1 \rangle 6$ .  $f(a_n) \rightarrow f(x)$  as  $n \rightarrow \infty$
- $\langle 1 \rangle 7$ . There exists  $N$  such that, for all  $n \geq N$ , we have  $f(a_n) \in V$
- $\langle 1 \rangle 8$ . Q.E.D.

**Lemma 7.1.5** (CC).  $\mathbb{R}^\omega$  under the box topology is not first countable.

PROOF:

- $\langle 1 \rangle 1$ . LET:  $\{B_n\}_{n \geq 1}$  be any countable set of neighbourhoods of  $\vec{0}$
  - $\langle 1 \rangle 2$ . For  $n \geq 1$ , PICK  $U_{nm}$  for  $m \geq 1$  such that  $\vec{0} \in \prod_{m=1}^\infty U_{nm} \subseteq B_n$
  - $\langle 1 \rangle 3$ . For  $n \geq 1$ , PICK  $a_n, b_n$  such that  $0 \in (a_n, b_n) \subseteq U_{nn}$
  - $\langle 1 \rangle 4$ . LET:  $U = \prod_{n=1}^\infty (a_n/2, b_n/2)$
  - $\langle 1 \rangle 5$ .  $\vec{0} \in U$
  - $\langle 1 \rangle 6$ . For all  $n$ ,  $B_n \not\subseteq U$
- 

**Lemma 7.1.6** (CC). *If  $J$  is uncountable then  $\mathbb{R}^J$  is not first countable.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $\{B_n\}_{n \geq 1}$  be a countable family of neighbourhoods of  $\vec{0}$
  - $\langle 1 \rangle 2$ . For  $n \geq 1$ , PICK  $U_{n\alpha}$  such that  $\vec{0} \in \prod_{\alpha \in J} U_{n\alpha} \subseteq B_n$  where  $U_{n\alpha}$  is open in  $\mathbb{R}$  and  $U_{n\alpha} = \mathbb{R}$  except for  $\alpha = \alpha_{n1}, \dots, \alpha_{nr_n}$
  - $\langle 1 \rangle 3$ . PICK  $\beta$  such that  $\beta$  is different from  $\alpha_{ni}$  for all  $n, i$
  - $\langle 1 \rangle 4$ . LET:  $V = \pi_\beta^{-1}((-1, 1))$
  - $\langle 1 \rangle 5$ .  $\vec{0} \in V$
  - $\langle 1 \rangle 6$ .  $V \not\subseteq B_n$  for all  $n$
-

**Lemma 7.1.7.**  $\mathbb{R}_l$  is first countable.

PROOF: For all  $x \in \mathbb{R}$ ,  $\{[x, q) : q \in \mathbb{Q}, q > x\}$  is a basis at  $x$ .  $\square$

**Lemma 7.1.8.** The ordered square is first countable.

PROOF:

(1)1. LET:  $(x, y) \in I_o^2$

PROVE: There exists a countable local basis  $\mathcal{B}$  at  $(x, y)$

(1)2. CASE:  $(x, y) = (0, 0)$

PROOF: Take  $\mathcal{B} = \{[(0, 0), (0, q)) : q \in \mathbb{Q}, 0 < q < 1\}$ .

(1)3. CASE:  $0 < y < 1$

PROOF: Take  $\mathcal{B} = \{((x, q), (x, q')) : q, q' \in \mathbb{Q}, q < y < q'\}$ .

(1)4. CASE:  $x < 1, y = 1$

PROOF: Take  $\mathcal{B} = \{((x, q), (q', 0)) : q, q' \in \mathbb{Q}, 0 < q < 1, x < q' < 1\}$ .

(1)5. CASE:  $x > 0, y = 0$

PROOF: Take  $\mathcal{B} = \{((q, 1), (x, q')) : q, q' \in \mathbb{Q}, 0 < q < x, 0 < q' < 1\}$

(1)6. CASE:  $(x, y) = (1, 1)$

PROOF: Take  $\mathcal{B} = \{((1, q), (1, 1)) : q \in \mathbb{Q}, 0 < q < 1\}$ .

$\square$

**Proposition 7.1.9.** A subspace of a first countable space is first countable.

PROOF:

(1)1. LET:  $X$  be a first countable space and  $A \subseteq X$

(1)2. LET:  $a \in A$

(1)3. PICK a countable basis  $\mathcal{B}$  at  $a$  in  $X$

(1)4.  $\{B \cap A : B \in \mathcal{B} \text{ is a countable basis at } a \text{ in } X\}$

$\square$

**Proposition 7.1.10 (CC).** A countable product of first countable spaces is first countable.

PROOF:

(1)1. LET:  $\{X_n\}_{n \in \mathbb{Z}^+}$  be a countable family of first countable spaces.

(1)2. LET:  $\vec{x} \in \prod_{n=1}^{\infty} X_n$

(1)3. PICK a countable basis  $\mathcal{B}_n$  at  $x_n$  in  $X_n$  for all  $n$

(1)4. LET:  $\mathcal{B}$  be the set of all sets  $\prod_{i=1}^n U_i$  where  $U_i \in \mathcal{B}_i$  for finitely many  $i$  and  $U_i = X_i$  for all other  $i$ .

(1)5.  $\mathcal{B}$  is a countable basis at  $\vec{x}$  in  $\prod_{n=1}^{\infty} X_n$

$\square$

**Corollary 7.1.10.1.** The space  $\mathbb{R}^{\omega}$  is first countable.

**Proposition 7.1.11.** The space  $S_{\Omega}$  is first countable.

PROOF:

(1)1. LET:  $\alpha \in S_{\Omega}$

PROVE:  $\alpha$  has a countable local basis.

- (1)2. CASE:  $\alpha$  is zero or a successor ordinal.  
 PROOF: In this case,  $\{\{\alpha\}\}$  is a local basis.  
 (1)3. CASE:  $\alpha$  is a limit ordinal.  
 (2)1. PICK a countable sequence  $(\beta_n)$  with supremum  $\alpha$   
 (2)2.  $\{(\beta_n, \alpha + 1) : n \in \mathbb{Z}^+\}$  is a local basis.

□

**Proposition 7.1.12.** *The space  $\overline{S_\Omega}$  is not first countable.*

PROOF:

- (1)1. ASSUME: for a contradiction  $\mathcal{B}$  is a countable local basis at  $\Omega$   
 (1)2. LET:  $\alpha = \sup\{\inf B : B \in \mathcal{B}\}$   
 (1)3.  $\alpha < \Omega$   
 (1)4. There is no  $B \in \mathcal{B}$  such that  $B \subseteq (\alpha, +\infty)$

□

**Proposition 7.1.13.** *The continuous image of a first countable space is first countable.*

PROOF:

- (1)1. LET:  $X$  be a first countable space,  $Y$  a space and  $f : X \rightarrow Y$  continuous.  
 (1)2. LET:  $y \in f(X)$   
 (1)3. PICK  $x \in X$  such that  $y = f(x)$   
 (1)4. PICK a countable local basis  $\mathcal{B}$  at  $x$   
 (1)5.  $\{f(B) : B \in \mathcal{B}\}$  is a countable local basis at  $y$ .

□

**Proposition 7.1.14.**  *$S_\Omega \times \overline{S_\Omega}$  is not first countable.*

PROOF:  $(0, \Omega)$  has no countable basis. □

**Proposition 7.1.15.** *The Sorgenfrey plane is first countable.*

PROOF: For any point  $(a, b)$ , the set  $\{[a, a + q) \times [b, b + r) : q, r \in \mathbb{Q}\}$  is a countable local basis at  $(a, b)$ . □

## 7.2 Separable Spaces

**Definition 7.2.1** (Separable Space). A topological space  $X$  is *separable* iff it has a countable dense subset.

**Proposition 7.2.2.** *The space  $S_\Omega$  is not separable.*

PROOF:

- (1)1. LET:  $D \subseteq S_\Omega$  be countable.  
 (1)2. LET:  $\alpha = \sup D$   
 (1)3.  $\overline{D} \subseteq (-\infty, \alpha]$

□

**Proposition 7.2.3.** *The space  $\overline{S_\Omega}$  is not separable.*

PROOF:

- ⟨1⟩1. LET:  $D \subseteq S_\Omega$  be countable.
- ⟨1⟩2. LET:  $\alpha = \sup\{\beta \in D : \beta < \Omega\}$
- ⟨1⟩3.  $\alpha < \Omega$

PROOF:  $\alpha$  is the supremum of countably many countable ordinals.

- ⟨1⟩4.  $\overline{D} \subseteq (-\infty, \alpha] \cup \{\Omega\}$

□

**Corollary 7.2.3.1.** *Not every compact Hausdorff space is separable.*

**Proposition 7.2.4.** *Every open subspace of a separable space is separable.*

PROOF:

- ⟨1⟩1. LET:  $X$  be a separable space with countable dense subset  $D$ .
- ⟨1⟩2. LET:  $U$  be an open subspace of  $X$   
PROVE:  $D \cap U$  is a countable dense subset of  $U$ .
- ⟨1⟩3.  $D \cap U$  is countable.
- ⟨1⟩4. LET:  $V$  be an open set in  $U$ .
- ⟨1⟩5.  $V$  is open in  $X$

PROOF: Lemma 4.3.3

- ⟨1⟩6.  $V$  intersects  $D$
- ⟨1⟩7.  $V$  intersects  $D \cap U$

□

**Proposition 7.2.5 (CC).** *The product of a countable family of separable spaces is separable.*

PROOF:

- ⟨1⟩1. LET:  $(X_n)$  be a countable family of separable spaces.
- ⟨1⟩2. For  $n \geq 1$ , PICK a dense set  $D_n$  in  $X_n$
- ⟨1⟩3.  $\prod_{n=1}^{\infty} D_n$  is dense in  $\prod_{n=1}^{\infty} X_n$ .

□

**Proposition 7.2.6.** *The continuous image of a separable space is separable.*

PROOF:

- ⟨1⟩1. LET:  $X$  be a separable space,  $Y$  a space and  $f : X \rightarrow Y$  be continuous.
- ⟨1⟩2. PICK a countable dense set  $D$  in  $X$
- ⟨1⟩3.  $f(D)$  is dense in  $f(X)$ .

□

**Corollary 7.2.6.1.** *Let  $\{X_\alpha\}_{\alpha \in J}$  be a family of nonempty topological spaces. If  $\prod_{\alpha \in J} X_\alpha$  is separable then each  $X_\alpha$  is separable.*

**Corollary 7.2.6.2.**  $S_\Omega \times \overline{S_\Omega}$  is not separable.

**Proposition 7.2.7.** *The ordered square is not separable.*

PROOF:  $\{\{x\} \times (0, 1) : x \in [0, 1]\}$  is an uncountable set of disjoint open sets. □

**Proposition 7.2.8.**  $\mathbb{R}_l$  is separable.

PROOF:  $\mathbb{Q}$  is dense.  $\square$

**Proposition 7.2.9.** The Sorgenfrey plane is separable.

PROOF:  $\mathbb{Q}^2$  is dense.  $\square$

**Proposition 7.2.10.** Not every closed subspace of a separable space is separable.

PROOF:  $\mathbb{R}_l^2$  is separable but the subspace  $\{(x, -x) : x \in \mathbb{R}\}$  is not.  $\square$

### 7.3 The Second Countability Axiom

**Definition 7.3.1** (Second Countability Axiom). A topological space satisfies the *second countability axiom*, or is *second countable*, iff it has a countable basis.

**Proposition 7.3.2.**  $S_\Omega$  is not second countable.

PROOF:  $\{\{\alpha\} : \alpha \text{ is a countable successor ordinal}\}$  is an uncountable set of disjoint open sets.  $\square$

**Proposition 7.3.3.** A subspace of a second countable space is second countable.

PROOF:

$\langle 1 \rangle 1.$  LET:  $X$  be a second countable space and  $A \subseteq X$

$\langle 1 \rangle 2.$  PICK a countable basis  $\mathcal{B}$  for  $X$

$\langle 1 \rangle 3.$   $\{B \cap A : B \in \mathcal{B}\}$  is a countable basis for  $A$

$\square$

**Proposition 7.3.4** (CC). The product of countably many second countable spaces is second countable.

PROOF:

$\langle 1 \rangle 1.$  LET:  $\{X_n\}_{n \in \mathbb{Z}^+}$  be a countable family of second countable spaces.

$\langle 1 \rangle 2.$  For  $n \in \mathbb{Z}^+$ , PICK a countable basis  $\mathcal{B}_n$  for  $X_n$ .

$\langle 1 \rangle 3.$  LET:  $\mathcal{B}$  be the set of all sets of the form  $\prod_{n=1}^{\infty} U_n$ , where  $U_n \in \mathcal{B}_n$  for finitely many  $n$ , and  $U_n = X_n$  for all other  $n$ .

$\langle 1 \rangle 4.$   $\mathcal{B}$  is a countable basis for  $\prod_{n=1}^{\infty} X_n$

$\square$

**Theorem 7.3.5** (CC). Every second countable space is separable.

PROOF:

$\langle 1 \rangle 1.$  LET:  $X$  be a second countable space.

$\langle 1 \rangle 2.$  PICK a countable basis  $\mathcal{B}$  for  $X$

$\langle 1 \rangle 3.$  For  $B \in \mathcal{B}$  nonempty, PICK a point  $x_B \in B$

$\langle 1 \rangle 4.$   $D = \{x_B : B \in \mathcal{B} \setminus \{\emptyset\}\}$  is dense.

$\langle 2 \rangle 1.$  LET:  $l \in X$

PROVE:  $l \in \overline{D}$

(2)2. LET:  $B \in \mathcal{B}$  such that  $l \in B$

(2)3.  $x_B \in B \cap D$

(2)4. Q.E.D.

PROOF: By Theorem 3.7.8

**Corollary 7.3.5.1.**  $S_\Omega \times \overline{S_\Omega}$  is not second countable.

**Corollary 7.3.5.2.** The space  $\mathbb{R}^\omega$  is separable.

**Corollary 7.3.5.3.** If  $J$  is uncountable then  $\mathbb{R}^J$  is not second countable.

**Proposition 7.3.6.** The ordered square is not second countable.

PROOF:

(1)1. LET:  $\mathcal{B}$  be any basis

(1)2. For  $x \in [0, 1]$ , PICK  $B_x$  such that  $x \in B_x \subseteq ((x, 0), (x, 1))$

(1)3. The function  $B_{(-)}$  is an injective function  $[0, 1] \rightarrow \mathcal{B}$

(1)4.  $\mathcal{B}$  is uncountable.

□

**Proposition 7.3.7.** The space  $\overline{S_\Omega}$  is not second countable.

PROOF: It is not first countable (Proposition 7.1.12). □

**Proposition 7.3.8.** The continuous image of a second countable space is second countable.

PROOF:

(1)1. LET:  $X$  be a second countable space,  $Y$  a space and  $f : X \rightarrow Y$  be continuous.

(1)2. PICK a countable basis  $\mathcal{B}$  for  $X$ .

(1)3.  $\{f(B) : B \in \mathcal{B}\}$  is a countable basis for  $f(X)$

□

**Theorem 7.3.9.** Every regular Lindelöf space is normal.

PROOF:

(1)1. LET:  $X$  be a regular Lindelöf space.

(1)2. LET:  $A$  and  $B$  be disjoint closed sets in  $X$ .

(1)3.  $\{U \text{ open in } X : \overline{U} \cap B = \emptyset\}$  covers  $A$

PROOF: Proposition 6.3.2.

(1)4. PICK a countable open covering  $\{U_n : n \in \mathbb{Z}^+\}$  of  $A$  such that  $\overline{U_n} \cap B = \emptyset$  for all  $n$

(1)5. PICK a countable open covering  $\{V_n : n \in \mathbb{Z}^+\}$  of  $B$  such that  $\overline{V_n} \cap A = \emptyset$  for all  $n$

PROOF: Similar.

(1)6. For  $n \in \mathbb{Z}^+$ ,

LET:  $U'_n = U_n \setminus \bigcup_{i=1}^n \overline{V_i}$  and  $V'_n = V_n \setminus \bigcup_{i=1}^n \overline{U_i}$

(1)7. LET:  $U' = \bigcup_{n=1}^\infty U'_n$  and  $V = \bigcup_{n=1}^\infty V'_n$

⟨1⟩8.  $A \subseteq U'$  and  $B \subseteq V'$

⟨1⟩9.  $U' \cap V' = \emptyset$

□

**Corollary 7.3.9.1.** *If  $J$  is uncountable then  $\mathbb{R}^J$  is not Lindelöf.*

**Proposition 7.3.10.** *Every second countable regular space is completely normal.*

PROOF:

⟨1⟩1. LET:  $X$  be second countable and regular and  $Y \subseteq X$

⟨1⟩2.  $Y$  is second countable

PROOF: Proposition 7.3.3.

⟨1⟩3.  $Y$  is regular

PROOF: Proposition 6.3.4

⟨1⟩4.  $Y$  is normal

PROOF: Theorem 7.3.9

□

**Proposition 7.3.11.** *The space  $\mathbb{R}^\omega$  is second countable.*

PROOF: The sets  $\prod_{n=0}^\infty U_n$  form a basis, where  $U_n$  is an interval of the form  $(q, r)$  for  $q, r \in \mathbb{Q}$  for finitely many  $n$ , and  $U_n = \mathbb{R}$  for all other  $n$ . □

**Proposition 7.3.12 (CC).** *In a second countable space, every discrete subspace is countable.*

PROOF:

⟨1⟩1. LET:  $X$  be a second countable space

⟨1⟩2. PICK a countable basis  $\mathcal{B}$

⟨1⟩3. LET:  $D \subseteq X$  be discrete

⟨1⟩4. For  $a \in D$ , PICK  $B_a \in \mathcal{B}$  such that  $B_a \cap D = \{a\}$

⟨1⟩5.  $a \mapsto B_a$  is injective

□

**Proposition 7.3.13.** *The space  $\mathbb{R}_K$  is second countable.*

PROOF:  $\{(a, b) : a, b \in \mathbb{R}\} \cup \{(a, b) - K : a, b \in \mathbb{Q}\}$  is a basis. □

**Corollary 7.3.13.1.** *The space  $\mathbb{R}_K$  is first countable.*

**Corollary 7.3.13.2.** *The space  $\mathbb{R}_K$  is separable.*

**Proposition 7.3.14.** *Let  $J$  be a set with  $|J| > |\mathbb{R}|$ . Then  $\mathbb{R}^J$  is not separable.*

PROOF:

⟨1⟩1. ASSUME:  $D$  is countable and dense in  $\mathbb{R}^J$

PROVE:  $|J| \leq |\mathbb{R}|$

⟨1⟩2. Define  $f : J \rightarrow \mathcal{P}D$  by  $f(\alpha) = D \cap \pi_\alpha^{-1}((0, 1))$

⟨1⟩3.  $f$  is injective

- $\langle 2 \rangle 1$ . LET:  $\alpha, \beta \in J$  with  $\alpha \neq \beta$
  - $\langle 2 \rangle 2$ . PICK  $x \in D \cap \pi_\alpha^{-1}((0, 1)) \cap \pi_\beta^{-1}((2, 3))$
  - $\langle 2 \rangle 3$ .  $x \in f(\alpha)$  but  $x \notin f(\beta)$
- 

**Corollary 7.3.14.1.** *The product of a family of separable spaces is not necessarily separable.*



## Chapter 8

# Connectedness

### 8.1 Connected Spaces

**Definition 8.1.1** (Separation). Let  $X$  be a topological space. A *separation* of  $X$  is a pair of disjoint nonempty subsets whose union is  $X$ .

**Definition 8.1.2** (Connected). A topological space is *connected* iff it has no separation.

**Proposition 8.1.3.**  $S_\Omega$  is not connected.

PROOF:  $\{0\}$  and  $S_\Omega \setminus \{0\}$  form a separation.  $\square$

**Proposition 8.1.4.** A space  $X$  is connected if and only if the only sets that are both closed and open are  $\emptyset$  and  $X$ .

PROOF: Immediate from definitions.  $\square$

**Proposition 8.1.5.** Let  $Y$  be a subspace of  $X$ . Then a separation of  $Y$  is a pair of disjoint nonempty sets  $A, B$  such that  $A \cup B = Y$  and neither of  $A, B$  contains a limit point of the other.

PROOF:

$\langle 1 \rangle 1$ . If  $A$  and  $B$  form a separation of  $Y$  then  $A$  and  $B$  are disjoint and nonempty,  $A \cup B = Y$ , and neither of  $A, B$  contains a limit point of the other.

$\langle 2 \rangle 1$ . LET:  $A$  and  $B$  be a separation of  $Y$

$\langle 2 \rangle 2$ .  $A$  and  $B$  are disjoint and nonempty and  $A \cup B = Y$

PROOF: Immediate from the definition of separation.

$\langle 2 \rangle 3$ .  $A$  does not contain a limit point of  $B$

PROOF:  $B$  is closed in  $Y$ , hence contains all its limit points (Corollary 3.10.3.1), and so the result follows because  $A$  and  $B$  are disjoint.

$\langle 2 \rangle 4$ .  $B$  does not contain a limit point of  $A$

PROOF: Similar.

$\langle 1 \rangle 2$ . If  $A$  and  $B$  are disjoint and nonempty,  $A \cup B = Y$ , and neither of  $A, B$  contains a limit point of the other, then  $A$  and  $B$  are a separation of  $Y$ .

- (2)1. ASSUME:  $A$  and  $B$  are disjoint and nonempty,  $A \cup B = Y$ , and neither of  $A, B$  contains a limit point of the other  
 (2)2.  $A$  is closed in  $Y$   
 PROOF: Every limit point of  $A$  is not in  $B$ , so is in  $A$ . Apply Corollary 3.10.3.1.  
 (2)3.  $B$  is open in  $Y$   
 PROOF:  $B = Y \setminus A$   
 (2)4.  $A$  is open in  $Y$   
 PROOF: Similar.
- 

**Proposition 8.1.6.** *If the sets  $C$  and  $D$  form a separation of  $X$ , and  $Y$  is a connected subspace of  $X$ , then  $Y \subseteq C$  or  $Y \subseteq D$ .*

PROOF: Otherwise,  $Y \cap C$  and  $Y \cap D$  would be a separation of  $Y$ . □

**Proposition 8.1.7.** *The union of a set of connected subspaces of  $X$  that have a point in common is connected.*

PROOF:

- (1)1. LET:  $\mathcal{S}$  be a set of connected subspaces that have the point  $a$  in common.  
 (1)2. ASSUME: for a contradiction  $U$  and  $V$  form a separation of  $\bigcup \mathcal{S}$   
 (1)3. ASSUME: w.l.o.g.  $a \in U$   
 (1)4. For all  $Y \in \mathcal{S}$  we have  $Y \subseteq U$   
 PROOF: By Proposition 8.1.6.  
 (1)5.  $V = \emptyset$   
 (1)6. Q.E.D.  
 PROOF: This contradicts (1)2.
- 

**Theorem 8.1.8.** *Let  $A$  be a connected subspace of  $X$ . If  $A \subseteq B \subseteq \overline{A}$  then  $B$  is connected.*

PROOF:

- (1)1. ASSUME: for a contradiction  $U$  and  $V$  are a separation of  $B$   
 (1)2.  $A \subseteq U$  or  $A \subseteq V$   
 PROOF: By Proposition 8.1.6.  
 (1)3. ASSUME: w.l.o.g.  $A \subseteq U$   
 (1)4.  $\overline{A} \subseteq \overline{U}$   
 PROOF: By Proposition 3.7.5.  
 (1)5.  $B \subseteq \overline{U}$   
 PROOF: Since  $B \subseteq \overline{A}$ .  
 (1)6. The closure of  $U$  in  $B$  is  $B$   
 PROOF: By Theorem 4.3.4.  
 (1)7.  $U = B$   
 PROOF: Since  $U$  is closed in  $B$ .  
 (1)8. Q.E.D.  
 PROOF: This contradicts (1)1.

□

**Theorem 8.1.9.** *The image of a connected space under a continuous map is connected.*

PROOF: Let  $X$  be a connected space,  $Y$  a topological space, and  $f : X \rightarrow Y$  be surjective. If  $U$  and  $V$  form a separation of  $Y$ , then  $f^{-1}(U)$  and  $f^{-1}(V)$  form a separation of  $X$ . □

**Corollary 8.1.9.1.** *Let  $\mathcal{T}$  and  $\mathcal{T}'$  be topologies on the same set  $X$ . If  $\mathcal{T} \subseteq \mathcal{T}'$  and  $X$  is connected under  $\mathcal{T}'$  then  $X$  is connected under  $\mathcal{T}$ .*

**Corollary 8.1.9.2.** *Let  $\{X_\alpha\}_{\alpha \in J}$  be a family of topological spaces. If  $\prod_{\alpha \in J} X_\alpha$  is connected then each  $X_\alpha$  is connected.*

**Corollary 8.1.9.3.** *The Sorgenfrey plane is disconnected.*

**Proposition 8.1.10.** *The product of a family of connected spaces is connected.*

PROOF:

⟨1⟩1. The product of two connected spaces is connected.

PROOF:

⟨2⟩1. LET:  $X$  and  $Y$  be connected spaces.

⟨2⟩2. ASSUME: w.l.o.g.  $X$  and  $Y$  are nonempty.

PROOF: If either is empty then  $X \times Y = \emptyset$  is connected.

⟨2⟩3. ASSUME: for a contradiction  $U$  and  $V$  are a separation of  $X \times Y$ .

⟨2⟩4. PICK  $b \in Y$

PROOF: By ⟨2⟩2.

⟨2⟩5. For all  $x \in X$ ,

LET:  $T_x = (X \times \{b\}) \cup (\{x\} \times Y)$

⟨2⟩6. For all  $x \in X$ ,  $T_x$  is connected

⟨3⟩1.  $X \times \{b\}$  is connected

PROOF: It is homeomorphic to  $X$ .

⟨3⟩2.  $\{x\} \times Y$  is connected

PROOF: It is homeomorphic to  $Y$ .

⟨3⟩3. Q.E.D.

PROOF: By Proposition 8.1.7.

⟨2⟩7.  $X \times Y = \bigcup_{x \in X} T_x$

⟨2⟩8. Q.E.D.

⟨3⟩1. PICK  $a \in X$

PROOF: By ⟨2⟩2.

⟨3⟩2.  $(a, b) \in T_x$  for all  $x \in X$

⟨3⟩3. Q.E.D.

PROOF: By Proposition 8.1.7.

⟨1⟩2. LET:  $\{X_\alpha\}_{\alpha \in J}$  be a family of connected spaces.

⟨1⟩3. ASSUME: w.l.o.g.  $\prod_{\alpha \in J} X_\alpha$  is nonempty

⟨1⟩4. PICK  $\vec{a} \in \prod_{\alpha \in J} X_\alpha$

⟨1⟩5. For  $K$  a finite subset of  $J$ ,

- LET:  $X_K = \{\vec{x} \in \prod_{\alpha \in J} X_\alpha : x_\alpha = a_\alpha \text{ for all } \alpha \in J \setminus K\}$
- $\langle 1 \rangle 6$ . For all  $K$ ,  $X_K$  is connected.
- PROOF: It is homeomorphic to  $\prod_{\alpha \in K} X_\alpha$ , so it is connected by  $\langle 1 \rangle 1$ .
- $\langle 1 \rangle 7$ .  $\bigcup_{K \subseteq \text{fin } J} X_K$  is connected.
- PROOF: By Proposition 8.1.7 since  $\vec{a} \in X_K$  for all  $K$ .
- $\langle 1 \rangle 8$ .  $\prod_{\alpha \in J} X_\alpha = \bigcup_{K \subseteq \text{fin } J} X_K$
- $\langle 2 \rangle 1$ . LET:  $\vec{x} \in \prod_{\alpha \in J} X_\alpha$
- $\langle 2 \rangle 2$ . LET:  $U$  be an open neighbourhood of  $\vec{x}$
- $\langle 2 \rangle 3$ . PICK a basic open set  $\prod_{\alpha \in J} V_\alpha$  such that  $\vec{x} \in \prod_{\alpha \in J} V_\alpha \subseteq U$ , where each  $V_\alpha$  is open in  $X_\alpha$ , and  $V_\alpha = X_\alpha$  except for  $\alpha \in K$  for some finite  $K \subseteq J$
- PROVE:  $U$  intersects  $X_K$
- $\langle 2 \rangle 4$ . LET:  $\vec{y} \in \prod_{\alpha \in J} X_\alpha$  with  $y_\alpha = x_\alpha$  for  $\alpha \in K$ ,  $y_\alpha = a_\alpha$  for  $\alpha \notin K$
- $\langle 2 \rangle 5$ .  $\vec{y} \in U \cap X_K$
- $\langle 1 \rangle 9$ . Q.E.D.

**Corollary 8.1.10.1.** *For any set  $I$ , the space  $\mathbb{R}^I$  under the product topology is connected.*

**Proposition 8.1.11.**  $\mathbb{R}^\omega$  under the box topology is disconnected.

PROOF: The set of all bounded sequences and the set of all unbounded sequences form a separation.  $\square$

**Definition 8.1.12** (Totally Disconnected). A space is *totally disconnected* iff the only connected subspaces are the singletons.

**Theorem 8.1.13.** *Let  $L$  be a linearly ordered set under the order topology. Then  $L$  is connected if and only if  $L$  is a linear continuum.*

PROOF:

- $\langle 1 \rangle 1$ . If  $L$  is a linear continuum then  $L$  is connected.
- $\langle 2 \rangle 1$ . LET:  $L$  be a linear continuum.
- $\langle 2 \rangle 2$ . ASSUME: for a contradiction  $U$  and  $V$  are a separation of  $L$ .
- $\langle 2 \rangle 3$ . PICK  $a \in U$  and  $b \in V$
- $\langle 2 \rangle 4$ . ASSUME: w.l.o.g.  $a < b$
- $\langle 2 \rangle 5$ . LET:  $l = \sup\{x \in A : x < b\}$
- $\langle 2 \rangle 6$ . CASE:  $l \in A$
- $\langle 3 \rangle 1$ . PICK  $a' > l$  such that  $[l, a') \subseteq A$
- PROOF: By Lemma 4.1.2. We know  $l$  is not greatest in  $X$  because  $l < b$ .
- $\langle 3 \rangle 2$ . PICK  $a^*$  such that  $l < a^* < a'$
- PROOF:  $L$  is dense.
- $\langle 3 \rangle 3$ .  $l < a^*$ ,  $a^* \in A$ ,  $a^* < b$
- PROOF: If  $b < a^*$  then  $b \in A$  by  $\langle 3 \rangle 1$ .
- $\langle 3 \rangle 4$ . Q.E.D.
- PROOF: This contradicts  $\langle 2 \rangle 5$ .
- $\langle 2 \rangle 7$ . CASE:  $l \in B$
- $\langle 3 \rangle 1$ . PICK  $b' < l$  such that  $(b', l] \subseteq B$

PROOF: By Lemma 4.1.2. We know  $l$  is not least in  $X$  because  $a < l$ .

⟨3⟩2. PICK  $b^*$  such that  $b' < b^* < l$

PROVE:  $b^*$  is an upper bound for  $\{x \in A : x < b\}$

⟨3⟩3. LET:  $x \in A$  and  $x < b$

⟨3⟩4.  $x \leq b^*$

PROOF: If  $b^* < x$  then  $b^* < x \leq l$  and so  $x \in B$  by ⟨3⟩1.

⟨3⟩5. Q.E.D.

PROOF: This contradicts ⟨2⟩5.

⟨1⟩2. If  $L$  is connected then  $L$  is a linear continuum.

⟨2⟩1. ASSUME:  $L$  is connected

⟨2⟩2.  $L$  has the least upper bound property

⟨3⟩1. ASSUME: for a contradiction  $A \subseteq L$  is bounded above with no least upper bound

⟨3⟩2. LET:  $U$  be the set of upper bounds of  $A$

⟨3⟩3.  $U$  is open

⟨4⟩1. LET:  $u \in U$

⟨4⟩2. PICK an upper bound  $v$  for  $A$  with  $v < u$

PROOF:  $u$  is not the least upper bound for  $A$  (⟨3⟩1)

⟨4⟩3.  $u \in (v, +\infty) \subseteq U$

⟨3⟩4. LET:  $V$  be the set of lower bounds of  $U$

⟨3⟩5.  $U$  and  $V$  form a separation of  $L$

⟨4⟩1.  $V$  is open

PROOF: Similar to ⟨3⟩3.

⟨4⟩2.  $U$  and  $V$  are disjoint

⟨5⟩1. ASSUME: for a contradiction  $x \in U \cap V$

⟨5⟩2. PICK  $u \in U$  such that  $u < x$

PROOF:  $x$  is not the lowest upper bound of  $A$

⟨5⟩3.  $x \leq u < x$

⟨4⟩3.  $U \cup V = L$

⟨5⟩1. LET:  $x \in L \setminus U$

⟨5⟩2. PICK  $a \in A$  such that  $x < a$

⟨5⟩3.  $a \in V$

⟨5⟩4.  $x \in V$

⟨2⟩3. For all  $x, y \in L$ , there exists  $z \in L$  such that  $x < z < y$

PROOF: Otherwise  $(-\infty, y)$  and  $(x, +\infty)$  would form a separation of  $L$ .

□

**Corollary 8.1.13.1.** *The real line  $\mathbb{R}$  is connected, and so is every ray and interval in  $\mathbb{R}$ .*

**Corollary 8.1.13.2.** *The ordered square is connected.*

**Corollary 8.1.13.3.** *Not every closed subspace of a connected space is connected.*

PROOF: The set  $\{0, 1\}$  is disconnected as a subspace of  $\mathbb{R}$ .

**Corollary 8.1.13.4.** *Not every open subspace of a connected space is connected.*

PROOF: The space  $\mathbb{R} \setminus \{0\}$  is a disconnected open subspace of  $\mathbb{R}$ .  $\square$

**Theorem 8.1.14** (Intermediate Value Theorem). *Let  $X$  be a connected space and  $Y$  a linearly ordered set under the order topology. Let  $f : X \rightarrow Y$  be continuous. Let  $a, b \in X$  and  $r \in Y$ . If  $f(a) < r < f(b)$ , then there exists  $c \in X$  such that  $f(c) = r$ .*

PROOF: If not, then  $f^{-1}((-\infty, r))$  and  $f^{-1}((r, +\infty))$  would be a separation of  $X$ .  $\square$

**Proposition 8.1.15.** *Every connected regular space with more than one point is uncountable.*

PROOF:

$\langle 1 \rangle$ 1. Every connected completely regular space with more than one point is uncountable.

$\langle 2 \rangle$ 1. LET:  $X$  be connected and completely regular and  $a, b \in X$  with  $a \neq b$

$\langle 2 \rangle$ 2. PICK a continuous  $f : X \rightarrow [0, 1]$  such that  $f(a) = 0$  and  $f(b) = 1$

$\langle 2 \rangle$ 3.  $f$  is surjective.

PROOF: By the Intermediate Value Theorem.

$\langle 1 \rangle$ 2. Every connected regular space with more than one point is uncountable.

$\langle 2 \rangle$ 1. ASSUME: for a contradiction  $X$  is connected, regular and countable with more than one point.

$\langle 2 \rangle$ 2.  $X$  is Lindelöf

$\langle 2 \rangle$ 3.  $X$  is normal

PROOF: By Theorem 7.3.9

$\langle 2 \rangle$ 4. Q.E.D.

PROOF: Contradicting  $\langle 1 \rangle$ 1.

$\square$

**Proposition 8.1.16.**  $\overline{S_\Omega}$  is not connected.

PROOF:  $\{0\}$  is clopen.  $\square$

**Proposition 8.1.17.**  $\mathbb{R}_l$  is not connected.

PROOF: The set  $[0, +\infty)$  is clopen.  $\square$

**Proposition 8.1.18.** *The space  $\mathbb{R}^\omega$  under the uniform topology is not connected.*

PROOF: The set of all bounded sequences and the set of all unbounded sequences form a separation.  $\square$

**Proposition 8.1.19.** *The space  $\mathbb{R}_K$  is connected.*

PROOF: Easy.  $\square$

## 8.2 Components and Local Connectedness

**Definition 8.2.1** ((Connected) Component). Let  $X$  be a topological space. Define an equivalence relation  $\sim$  on  $X$  by:  $x \sim y$  iff there exists a connected subspace  $U \subseteq X$  such that  $x \in U$  and  $y \in U$ . The *(connected) components* of  $X$  are the equivalence classes under  $\sim$ .

We prove this is an equivalence relation.

PROOF:

$\langle 1 \rangle 1$ . For all  $x \in X$  we have  $x \sim x$ .

PROOF: The subspace  $\{x\} \subseteq X$  is connected.

$\langle 1 \rangle 2$ . For all  $x, y \in X$ , if  $x \sim y$  then  $y \sim x$ .

PROOF: Immediate from definitions.

$\langle 1 \rangle 3$ . For all  $x, y, z \in X$ , if  $x \sim y$  and  $y \sim z$  then  $x \sim z$ .

PROOF: By Proposition 8.1.7.

□

**Proposition 8.2.2.** *Let  $X$  be a topological space. If  $C \subseteq X$  is connected and nonempty, then there exists a unique component  $D$  of  $X$  such that  $C \subseteq D$ .*

PROOF:

$\langle 1 \rangle 1$ . PICK  $a \in C$

$\langle 1 \rangle 2$ . LET:  $D$  be the  $\sim$ -equivalence class of  $A$

$\langle 1 \rangle 3$ .  $C \subseteq D$

PROOF: For all  $x \in C$  we have  $a \sim x$  by definition.

$\langle 1 \rangle 4$ .  $D$  is unique

PROOF: This holds because the components are disjoint.

□

**Proposition 8.2.3** (AC). *Every component is connected.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $C$  be a component of the topological space  $X$

$\langle 1 \rangle 2$ . PICK  $a \in C$

$\langle 1 \rangle 3$ . For all  $x \in C$ , PICK a connected subspace  $C_x$  of  $X$  containing both  $a$  and  $x$ .

PROOF: Such a  $C_x$  exists since  $a \sim x$ .

$\langle 1 \rangle 4$ .  $C = \bigcup_{x \in C} C_x$

PROOF: This holds because  $C_x \subseteq C$  by Proposition 8.2.2.

$\langle 1 \rangle 5$ . Q.E.D.

PROOF: It follows that  $C$  is connected by Proposition 8.1.7.

□

**Proposition 8.2.4.** *Every component is closed.*

PROOF: From Theorem 8.1.8. □

**Proposition 8.2.5.** *The component of  $\vec{a}$  in  $\mathbb{R}^\omega$  under the uniform topology is  $\{\vec{b} : \vec{b} - \vec{a} \text{ is bounded}\}$ .*

PROOF:

- ⟨1⟩1.  $C = \{\vec{b} : \vec{b} - \vec{a} \text{ is bounded}\}$  is connected.
  - ⟨2⟩1. ASSUME:  $C = U \cup V$  is a separation of  $C$  with  $\vec{a} \in U$
  - ⟨2⟩2. PICK  $\vec{b} \in V$
  - ⟨2⟩3.  $\{\epsilon : \epsilon\vec{b} + (1 - \epsilon)\vec{a} \in U\}$  and  $\{\epsilon : \epsilon\vec{b} + (1 - \epsilon)\vec{a} \in V\}$  form a separation of  $[0, 1]$
  - ⟨1⟩2. If  $\vec{a}, \vec{b} \in C$  and  $\vec{b} - \vec{a}$  is unbounded then  $C$  is disconnected.
- PROOF:  $\{\vec{c} : \vec{c} - \vec{a} \text{ is bounded}\}$  and  $\{\vec{c} : \vec{c} - \vec{a} \text{ is unbounded}\}$
- 

**Proposition 8.2.6.** *Let  $x, y \in \mathbb{R}^\omega$  under the box topology. Then  $x$  and  $y$  are in the same component iff  $x - y$  is eventually zero.*

PROOF:

- ⟨1⟩1. For all  $x \in \mathbb{R}^\omega$  the set  $\{y : x - y \text{ is eventually zero}\}$  is connected
- PROOF: It is the union of the sets  $C_N = \{y : \forall n \geq N. y_n = 0\}$ , each of which is connected because it is homeomorphic to  $\mathbb{R}^{N-1}$ .
- ⟨1⟩2. If  $x - y$  is not eventually zero then  $x$  and  $y$  are in different components
  - ⟨2⟩1. ASSUME:  $x - y$  is not eventually zero
  - ⟨2⟩2. Define  $h : \mathbb{R}^\omega \rightarrow \mathbb{R}^\omega$  by:  $h(z)_n = \begin{cases} z_n - x_n & \text{if } x_n = y_n \\ n(z_n - x_n)/(y_n - x_n) & \text{if } x_n \neq y_n \end{cases}$
  - ⟨2⟩3.  $h$  is an automorphism of  $\mathbb{R}^\omega$  under the box topology
  - ⟨2⟩4.  $h(x) = 0$
  - ⟨2⟩5.  $h(y)$  is unbounded
  - ⟨2⟩6. Q.E.D.
- PROOF: The inverse image under  $h$  of the set of bounded sequences and the set of unbounded sequences form a separation of  $\mathbb{R}^\omega$  with  $x$  and  $y$  in different sets.

□

□

### 8.3 Path Connectedness

**Definition 8.3.1** (Path). Let  $X$  be a topological space and  $a, b \in X$ . A *path* from  $a$  to  $b$  is a continuous function  $p : [0, 1] \rightarrow X$  such that  $p(0) = a$  and  $p(1) = b$ .

**Definition 8.3.2** (Path Connected). A topological space is *path connected* iff there exists a path between any two points.

**Proposition 8.3.3.** *Every path connected space is connected.*

PROOF:

- ⟨1⟩1. LET:  $X$  be a path connected space
- ⟨1⟩2. ASSUME: for a contradiction  $U$  and  $V$  are a separation of  $X$ .
- ⟨1⟩3. PICK  $a \in U$  and  $b \in V$



- ⟨1⟩4. PICK a path  $p : [0, 1] \rightarrow X$  from  $a$  to  $b$
- ⟨1⟩5.  $p^{-1}(U)$  and  $p^{-1}(V)$  form a separation of  $[0, 1]$ .
- ⟨1⟩6. Q.E.D.

PROOF: This contradicts Corollary 8.1.13.1.

**Corollary 8.3.3.1.**  $S_\Omega$  is not path connected.

**Corollary 8.3.3.2.**  $\overline{S_\Omega}$  is not path connected.

**Corollary 8.3.3.3.**  $\mathbb{R}_l$  is not path connected.

**Corollary 8.3.3.4.** The Sorgenfrey plane is not path connected.

**Corollary 8.3.3.5.** The space  $\mathbb{R}^\omega$  under the uniform topology is not path connected. connected.

**Corollary 8.3.3.6.** The space  $\mathbb{R}^\omega$  under the box topology is not path connected.

**Proposition 8.3.4.** The long line is path connected.

PROOF:

- ⟨1⟩1. LET:  $a, b \in L$
- ⟨1⟩2. PICK an ordinal  $\alpha$  such that  $a, b < (\alpha, 0)$
- ⟨1⟩3. There exists a path from  $a$  to  $b$

PROOF: This holds because  $[(0, 0), (\alpha, 0))$  is homeomorphic to  $[0, 1)$  by Proposition 1.7.11.

□

**Corollary 8.3.4.1.** Not every closed subspace of a path connected space is path connected.

PROOF: Take any two-element subspace of the long line.

**Corollary 8.3.4.2.** Not every open subspace of a path connected space is path connected.

PROOF: The space  $\mathbb{R} \setminus \{0\}$  is not path connected as a subspace of  $\mathbb{R}$ . □

**Definition 8.3.5** (Path Component). Let  $X$  be a topological space. Define an equivalence relation  $\sim$  on  $X$  by:  $x \sim y$  iff there exists a path from  $x$  to  $y$ . The equivalence classes are called the *path components* of  $X$ .

We prove this is an equivalence relation.

PROOF:

- ⟨1⟩1. For all  $x \in X$  we have  $x \sim x$

PROOF: The constant path  $p : [0, 1] \rightarrow X$  where  $p(t) = x$  is a path from  $x$  to  $x$ .

- ⟨1⟩2. If  $x \sim y$  then  $y \sim x$

PROOF: If  $p : [0, 1] \rightarrow X$  is a path from  $x$  to  $y$  then  $\lambda t.p(1 - t)$  is a path from  $y$  to  $x$ .

- ⟨1⟩3. If  $x \sim y$  and  $y \sim z$  then  $x \sim z$

(2)1. LET:  $p$  be a path from  $x$  to  $y$  and  $q$  be a path from  $y$  to  $z$ .

(2)2. LET:  $r : [0, 1] \rightarrow X$  where

$$r(t) = \begin{cases} p(2t) & \text{if } 0 \leq t \leq 1/2 \\ q(2t - 1) & \text{if } 1/2 \leq t \leq 1 \end{cases}$$

(2)3.  $r$  is a path from  $x$  to  $z$ .

PROOF:  $r$  is continuous by the Pasting Lemma.

□

**Proposition 8.3.6.** *Every path component is path connected.*

PROOF: By definition, if  $x$  and  $y$  are in the same path component then there is a path from  $x$  to  $y$ . □

**Proposition 8.3.7.** *If  $A$  is a nonempty path connected subspace of the space  $X$ , then  $A$  is included in a unique path component.*

PROOF:

(1)1. PICK  $a \in A$

(1)2. LET:  $C$  be the equivalence class of  $a$  under  $\sim$

(1)3.  $A \subseteq C$

PROOF: For all  $x \in A$ , there exists a path from  $a$  to  $x$ .

(1)4.  $C$  is unique

PROOF:  $C$  is the unique path component such that  $a \in C$ .

□

**Proposition 8.3.8.** *Every path component is included in a component.*

PROOF: From Propositions 8.3.3 and 8.2.2. □

**Proposition 8.3.9.** *The ordered square is not path connected.*

PROOF:

(1)1. ASSUME: for a contradiction  $p : [0, 1] \rightarrow I_o^2$  is a path from  $(0, 0)$  to  $(1, 1)$ .

(1)2. For all  $x \in [0, 1]$ ,  $p^{-1}(\{x\} \times (0, 1))$  is open in  $[0, 1]$

(1)3. For all  $x \in [0, 1]$ , PICK a rational  $q_x \in p^{-1}(\{x\} \times (0, 1))$

(1)4.  $\{q_x : x \in [0, 1]\}$  is an uncountable set of rationals.

□

**Proposition 8.3.10 (AC).** *The product of a family of path connected spaces is path connected.*

PROOF:

(1)1. LET:  $\{X_\alpha\}_{\alpha \in J}$  be a family of path connected spaces and  $a, b \in \prod_{\alpha \in J} X_\alpha$

(1)2. For  $\alpha \in J$ , PICK a path  $p_\alpha : [0, 1] \rightarrow X_\alpha$  from  $a_\alpha$  to  $b_\alpha$

(1)3. Define  $p : [0, 1] \rightarrow \prod_{\alpha \in J} X_\alpha$  by  $p(t)_\alpha = p_\alpha(t)$

(1)4.  $p$  is a path from  $a$  to  $b$

PROOF: By Theorem 5.2.15.

□

**Corollary 8.3.10.1.** *For any set  $I$ , the space  $\mathbb{R}^I$  in the product topology is path connected.*

**Proposition 8.3.11.** *The space  $\mathbb{R}_K$  is not path connected.*

PROOF:

<1>1. ASSUME: for a contradiction  $p : [0, 1] \rightarrow \mathbb{R}_K$  is a path from 0 to 1

<1>2. LET:  $p : [0, 1] \rightarrow \mathbb{R}_K$  be a path from 0 to 1

<1>3.  $p([0, 1])$  is compact and connected in  $\mathbb{R}_K$ .

PROOF: Theorem 8.1.9 and Proposition 9.4.10.

<1>4.  $p([0, 1])$  is connected in  $\mathbb{R}$ .

PROOF: Corollary 8.1.9.1

<1>5.  $[0, 1] \subseteq p([0, 1])$

PROOF: For any  $x \in [0, 1]$ , if  $x \notin p([0, 1])$  then  $p([0, 1]) \cap (-\infty, x)$  and  $p([0, 1]) \cap (x, +\infty)$  form a separation of  $p([0, 1])$ .

<1>6.  $[0, 1]$  is compact in  $\mathbb{R}_K$

PROOF: Proposition 9.4.6.

<1>7. Q.E.D.

PROOF: This contradicts Corollary 9.4.11.2.

□

**Proposition 8.3.12.** *Let  $f : X \rightarrow Y$  be continuous and surjective. If  $X$  is path connected then  $Y$  is path connected.*

PROOF:

<1>1. LET:  $a, b \in Y$

<1>2. PICK  $x, y \in X$  such that  $f(x) = a$  and  $f(y) = b$

<1>3. PICK a path  $p : [0, 1] \rightarrow X$  such that  $p(0) = x$  and  $p(1) = y$

<1>4.  $f \circ p$  is a path from  $a$  to  $b$

□

**Corollary 8.3.12.1.** *Let  $\{X_\alpha\}_{\alpha \in J}$  be a family of non-empty topological spaces. If  $\prod_{\alpha \in J} X_\alpha$  is path connected then each  $X_\alpha$  is path connected.*

## 8.4 Connected Subspaces of Euclidean Space

**Definition 8.4.1** (Unit 2-Sphere). The *unit 2-sphere* is  $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$  as a subspace of  $\mathbb{R}^3$ .

**Definition 8.4.2** (Unit Ball). For any  $n \geq 1$ , the *closed unit ball* in  $\mathbb{R}^n$  is

$$B^n = \{\vec{x} \in \mathbb{R}^n : \|\vec{x}\| \leq 1\} .$$

**Proposition 8.4.3.** *Every open unit ball and closed unit ball in  $\mathbb{R}^n$  is path connected.*

PROOF: The straight line between any two points is a path in the ball. □

**Definition 8.4.4** (Punctured Euclidean Space). For  $n \geq 1$ , *punctured Euclidean space* is  $\mathbb{R}^n \setminus \{\vec{0}\}$ .

**Proposition 8.4.5.** *Punctured Euclidean space in  $\mathbb{R}^n$  is path connected iff  $n > 1$ .*

PROOF: Easy.  $\square$

**Definition 8.4.6** (Unit Sphere). For  $n \geq 1$ , the *unit sphere*  $S^n$  is  $\{\vec{x} \in \mathbb{R}^{n+1} : \|\vec{x}\| = 1\}$ .

**Proposition 8.4.7.** *In any number of dimensions, the unit sphere is path connected.*

PROOF: Easy.  $\square$

**Definition 8.4.8** (Topologist's Sine Curve). The *topologist's sine curve* is the closure of

$$S = \{(x, \sin 1/x) : x \in \mathbb{R}\}$$

in  $\mathbb{R}^2$ .

**Proposition 8.4.9.** *The topologist's sine curve is connected.*

PROOF:

$\langle 1 \rangle 1$ .  $S = \{(x, \sin 1/x) : x \in \mathbb{R}\}$  is connected.

$\langle 2 \rangle 1$ . The function  $f : \mathbb{R} \rightarrow \mathbb{R}^2$  given by  $f(x) = (x, \sin 1/x)$  is continuous.

PROOF: By Theorem 5.2.15.

$\langle 2 \rangle 2$ . Q.E.D.

PROOF: By Theorem 8.1.9.

$\langle 1 \rangle 2$ . Q.E.D.

PROOF: By Theorem 8.1.8.

$\square$

**Proposition 8.4.10** (CC). *The topologist's sine curve is not path connected.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $S = \{(x, \sin 1/x) : x \in \mathbb{R}\}$

$\langle 1 \rangle 2$ . ASSUME: for a contradiction  $p : [0, 1] \rightarrow \bar{S}$  is a path from  $(0, 0)$  to  $(1, \sin 1)$ .

$\langle 1 \rangle 3$ .  $p^{-1}(\{0\} \times [-1, 1])$  is closed.

$\langle 1 \rangle 4$ .  $p^{-1}(\{0\} \times [-1, 1])$  has a greatest element.

PROOF: By Lemma 4.1.9.

$\langle 1 \rangle 5$ . LET:  $q : [0, 1] \rightarrow \bar{S}$  be a path such that:

- $q(0) \in \{0\} \times [-1, 1]$
- $q(x) \in S$  for  $x > 0$

PROOF: Let  $b$  be greatest in  $p^{-1}(\{0\} \times [-1, 1])$ . Then  $q$  is obtained by rescaling  $p$  restricted to  $[b, 1]$ .

$\langle 1 \rangle 6$ . LET:  $q(t) = (x(t), y(t))$  for  $0 \leq t \leq 1$

$\langle 1 \rangle 7$ .  $x(0) = 0$

- ⟨1⟩8.  $x(t) > 0$  for  $t > 0$   
 ⟨1⟩9.  $y(t) = \sin 1/x(t)$  for  $t > 0$   
 ⟨1⟩10. There exists a sequence  $t_n \in [0, 1]$  such that  $t_n \rightarrow 0$  as  $n \rightarrow \infty$  and  $y(t_n) = (-1)^n$  for all  $n$ .  
 ⟨2⟩1. For each  $n$ , PICK  $u_n$  such that  $0 < u_n < x(1/n)$  and  $\sin 1/u_n = (-1)^n$ .  
 PROOF: Such a  $u_n$  exists because  $\sin 1/x$  takes values 1 and -1 infinitely often in  $(0, x(1/n))$ .  
 ⟨2⟩2. For each  $n$ , PICK  $t_n$  such that  $0 < t_n < 1/n$  and  $x(t_n) = u$   
 PROOF: By the Intermediate Value Theorem.  
 ⟨1⟩11. Q.E.D.  
 PROOF: This is a contradiction as  $y(t_n) \rightarrow y(0)$  as  $n \rightarrow \infty$  because  $y$  is continuous.

□

## 8.5 Local Connectedness

**Definition 8.5.1** (Locally Connected). Let  $X$  be a topological space and  $x \in X$ . Then  $X$  is *locally connected* at  $x$  iff every neighbourhood of  $x$  includes a connected neighbourhood of  $x$ .

The space  $X$  is *locally connected* iff it is locally connected at every point.

**Proposition 8.5.2.**  $S_\Omega$  is not locally connected.

PROOF: There is no connected neighbourhood of  $\omega$ . □

**Proposition 8.5.3.**  $\overline{S_\Omega}$  is not locally connected.

PROOF: There is no connected neighbourhood of  $\omega$ . □

**Proposition 8.5.4.** For any set  $I$ , the space  $\mathbb{R}^I$  is locally connected.

PROOF: Every basic open set is the product of connected spaces, hence connected. □

**Proposition 8.5.5.** Let  $X$  be a topological space. Then  $X$  is locally connected if and only if, for every open set  $U$  in  $X$ , every component of  $U$  is open in  $X$ .

PROOF:

- ⟨1⟩1. If  $X$  is locally connected then, for every open set  $U$  in  $X$ , every component of  $U$  is open in  $X$ .  
 ⟨2⟩1. ASSUME:  $X$  is locally connected.  
 ⟨2⟩2. LET:  $U$  be open in  $X$ .  
 ⟨2⟩3. LET:  $C$  be a component of  $U$ .  
 ⟨2⟩4. LET:  $x \in C$   
 PROVE:  $C$  is a neighbourhood of  $x$   
 ⟨2⟩5.  $U$  is a neighbourhood of  $x$  in  $X$ .  
 PROOF: From ⟨2⟩2, ⟨2⟩3 and ⟨2⟩4.  
 ⟨2⟩6. PICK a connected neighbourhood  $V$  of  $x$  such that  $V \subseteq U$ .

PROOF: Using  $\langle 2 \rangle 1$ .

$\langle 2 \rangle 7$ .  $V \subseteq C$

PROOF: By Proposition 8.2.2.

$\langle 2 \rangle 8$ .  $C$  is a neighbourhood of  $x$

PROOF: By Proposition 3.2.4.

$\langle 2 \rangle 9$ . Q.E.D.

PROOF: By Proposition 3.2.3.

$\langle 1 \rangle 2$ . If, for every open set  $U$  in  $X$ , every component of  $U$  is open in  $X$ , then  $X$  is locally connected.

$\langle 2 \rangle 1$ . ASSUME: For every open set  $U$  in  $X$ , every component of  $U$  is open in  $X$ .

$\langle 2 \rangle 2$ . LET:  $x \in X$  and  $N$  be a neighbourhood of  $x$

$\langle 2 \rangle 3$ . PICK  $U$  open such that  $x \in U \subseteq N$

$\langle 2 \rangle 4$ . LET:  $C$  be the component of  $U$  that contains  $x$

$\langle 2 \rangle 5$ .  $C$  is open in  $X$

PROOF: By  $\langle 2 \rangle 1$ .

$\langle 2 \rangle 6$ .  $C$  is a connected neighbourhood of  $x$  that is included in  $N$

□

**Corollary 8.5.5.1.** *In a locally connected space, every component is open.*

**Corollary 8.5.5.2.** *The space  $\mathbb{R}^\omega$  under the box topology is not locally connected.*

**Corollary 8.5.5.3.** *Not every closed subspace of a locally connected space is locally connected.*

PROOF: The topologist's sine curve is not locally connected. □

**Proposition 8.5.6.**  *$S_\Omega \times \overline{S_\Omega}$  is not locally connected.*

$(\omega, \omega)$  has no connected neighbourhood. □

**Proposition 8.5.7.**  *$\mathbb{R}_l$  is not locally connected.*

PROOF: 0 has no connected neighbourhood. □

**Proposition 8.5.8.** *The Sorgenfrey plane is not locally connected.*

PROOF: Any basic open set  $[a, b) \times [c, d)$  can be separated into  $[a, b) \times [c, e)$  and  $[a, b) \times [e, d)$  for some  $c < e < d$ . □

**Proposition 8.5.9.** *The space  $\mathbb{R}^\omega$  under the uniform topology is locally connected.*

PROOF: For any neighbourhood  $U$  of a point  $x$ , the neighbourhood  $U \cap \{y : y - x \text{ is bounded}\}$  is connected. □

**Proposition 8.5.10.** *The space  $\mathbb{R}_K$  is not locally connected.*

PROOF: The open set  $(-1, 1) - K$  does not include a connected neighbourhood of 0. □

**Proposition 8.5.11.** *Every open subspace of a locally connected space is locally connected.*

PROOF: Follows easily from definition.  $\square$

**Proposition 8.5.12 (AC).** *The product of a family of locally connected spaces is locally connected.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $\{X_\alpha\}_{\alpha \in J}$  be a family of locally connected spaces and  $\vec{x} \in \prod_{\alpha \in J} X_\alpha$

$\langle 1 \rangle 2$ . LET:  $\prod_{\alpha \in J} U_\alpha$  be any basic neighbourhood of  $\vec{x}$ , where each  $U_\alpha$  is open in  $X_\alpha$ , and  $U_\alpha = X_\alpha$  except for  $\alpha = \alpha_1, \dots, \alpha_n$

$\langle 1 \rangle 3$ . For  $\alpha \in J$ , PICK a connected neighbourhood  $C_\alpha$  of  $x_\alpha$  with  $C_\alpha \subseteq U_\alpha$

$\langle 1 \rangle 4$ .  $\prod_{\alpha \in J} C_\alpha$  is connected

PROOF: Proposition 8.1.10

$\square$

**Proposition 8.5.13.** *Every discrete space is locally connected.*

PROOF: For any point  $x$ , the set  $\{x\}$  is a connected neighbourhood of  $x$ .  $\square$

**Corollary 8.5.13.1.** *The continuous image of a locally connected space is not necessarily locally connected.*

## 8.6 Local Path Connectedness

**Definition 8.6.1** (Locally Path Connected). Let  $X$  be a topological space and  $x \in X$ . Then  $X$  is *locally path connected at  $x$*  iff every neighbourhood of  $x$  includes a path connected neighbourhood of  $x$ .

The space  $X$  is *locally path connected* iff it is locally path connected at every point.

**Proposition 8.6.2.**  $S_\Omega$  is not locally path connected.

PROOF: There is no path connected neighbourhood of  $\omega$ .  $\square$

**Proposition 8.6.3.**  $\overline{S_\Omega}$  is not locally path connected.

PROOF: There is no path connected neighbourhood of  $\omega$ .  $\square$

**Proposition 8.6.4.** *Not every closed subspace of a locally path connected space is locally path connected.*

PROOF: The topologist's sine curve is not locally path connected.  $\square$

**Proposition 8.6.5.** *Every open subspace of a locally path connected space is locally path connected.*

PROOF: Follows easily from definition.  $\square$

**Proposition 8.6.6.** *Every locally path connected space is locally connected.*

PROOF: From Proposition 8.3.3.  $\square$

**Corollary 8.6.6.1.**  $\mathbb{R}_l$  is not locally path connected.

**Corollary 8.6.6.2.** The Sorgenfrey plane is not locally path connected.

**Corollary 8.6.6.3.** The space  $\mathbb{R}^\omega$  under the box topology is not locally path connected.

**Corollary 8.6.6.4.** The space  $\mathbb{R}_K$  is not locally path connected.

**Corollary 8.6.6.5.** The topologist's sine curve is not locally path connected.

**Proposition 8.6.7 (AC).** The product of a family of locally path connected spaces is locally path connected.

PROOF:

- $\langle 1 \rangle 1$ . LET:  $\{X_\alpha\}_{\alpha \in J}$  be a family of locally connected spaces and  $\vec{x} \in \prod_{\alpha \in J} X_\alpha$
- $\langle 1 \rangle 2$ . LET:  $\prod_{\alpha \in J} U_\alpha$  be any basic neighbourhood of  $\vec{x}$ , where each  $U_\alpha$  is open in  $X_\alpha$ , and  $U_\alpha = X_\alpha$  except for  $\alpha = \alpha_1, \dots, \alpha_n$
- $\langle 1 \rangle 3$ . For  $\alpha \in J$ , PICK a path connected neighbourhood  $C_\alpha$  of  $x_\alpha$  with  $C_\alpha \subseteq U_\alpha$
- $\langle 1 \rangle 4$ .  $\prod_{\alpha \in J} C_\alpha$  is path connected

PROOF: Proposition ??

$\square$

**Proposition 8.6.8.** Let  $X$  be a topological space. Then  $X$  is locally path connected if and only if, for every open set  $U$  in  $X$ , every path component of  $U$  is open in  $X$ .

PROOF:

- $\langle 1 \rangle 1$ . If  $X$  is locally path connected then, for every open set  $U$  in  $X$ , every path component of  $U$  is open in  $X$ .
- $\langle 2 \rangle 1$ . ASSUME:  $X$  is locally path connected.
- $\langle 2 \rangle 2$ . LET:  $U$  be open in  $X$ .
- $\langle 2 \rangle 3$ . LET:  $C$  be a path component of  $U$ .
- $\langle 2 \rangle 4$ . LET:  $x \in C$   
PROVE:  $C$  is a neighbourhood of  $x$
- $\langle 2 \rangle 5$ .  $U$  is a neighbourhood of  $x$  in  $X$ .  
PROOF: From  $\langle 2 \rangle 2$ ,  $\langle 2 \rangle 3$  and  $\langle 2 \rangle 4$ .
- $\langle 2 \rangle 6$ . PICK a path connected neighbourhood  $V$  of  $x$  such that  $V \subseteq U$ .  
PROOF: Using  $\langle 2 \rangle 1$ .
- $\langle 2 \rangle 7$ .  $V \subseteq C$   
PROOF: By Proposition 8.3.7.
- $\langle 2 \rangle 8$ .  $C$  is a neighbourhood of  $x$   
PROOF: By Proposition 3.2.4.
- $\langle 2 \rangle 9$ . Q.E.D.  
PROOF: By Proposition 3.2.3.
- $\langle 1 \rangle 2$ . If, for every open set  $U$  in  $X$ , every path component of  $U$  is open in  $X$ , then  $X$  is locally path connected.



- (2)1. ASSUME: For every open set  $U$  in  $X$ , every path component of  $U$  is open in  $X$ .  
 (2)2. LET:  $x \in X$  and  $N$  be a neighbourhood of  $x$   
 (2)3. PICK  $U$  open such that  $x \in U \subseteq N$   
 (2)4. LET:  $C$  be the path component of  $U$  that contains  $x$   
 (2)5.  $C$  is open in  $X$   
 PROOF: By (2)1.  
 (2)6.  $C$  is a path connected neighbourhood of  $x$  that is included in  $N$

□

**Theorem 8.6.9 (AC).** *Let  $X$  be a topological space. If  $X$  is locally path connected, then its components and its path components are the same.*

PROOF:

- (1)1. LET:  $P$  be a path component of  $X$   
 (1)2. LET:  $C$  be the component such that  $P \subseteq C$   
 PROVE:  $P = C$   
 (1)3. LET:  $Q = C \setminus P$   
 (1)4.  $P$  is open in  $X$   
 PROOF: By Proposition 8.6.8.  
 (1)5.  $Q$  is open in  $X$   
 PROOF: By Proposition 8.6.8 since  $Q$  is the union of the path components included in  $C$  other than  $P$ .  
 (1)6.  $Q = \emptyset$   
 PROOF: Otherwise  $P$  and  $Q$  would form a separation of  $C$ , contradicting 8.2.3.

□

**Proposition 8.6.10.**  $S_\Omega \times \overline{S_\Omega}$  is not locally path connected.

PROOF:  $(\omega, \omega)$  has no path connected neighbourhood. □

**Proposition 8.6.11.** *The ordered square is not locally path connected.*

PROOF:

- (1)1. ASSUME: for a contradiction  $(1/2, 0)$  has a path connected neighbourhood  $U$   
 (1)2. PICK  $a < 1/2$  such that  $((a, 1), (1/2, 0)) \subseteq U$   
 (1)3. LET:  $p : [0, 1] \rightarrow I_o^2$  be a path from  $(a, 1)$  to  $(1/2, 0)$   
 (1)4. For every  $x$  such that  $a < x < 1/2$ , PICK a rational  $q_x$  such that  $p(q_x) \in ((x, 0), (x, 1))$   
 (1)5.  $\{q_x : a < x < 1/2\}$  is an uncountable set of rationals.

□

**Proposition 8.6.12.** *For any set  $I$ , the space  $\mathbb{R}^I$  is locally path connected.*

PROOF: Every basic open set is the product of path connected spaces, hence path connected. □

**Proposition 8.6.13.** *The space  $\mathbb{R}^\omega$  under the uniform topology is locally path connected.*

PROOF: Its components and path components are the same.  $\square$

**Proposition 8.6.14.** *Every discrete space is locally path connected.*

PROOF: For any point  $x$ , the set  $\{x\}$  is a path connected neighbourhood of  $x$ .  $\square$

**Corollary 8.6.14.1.** *The continuous image of a locally path connected space is not necessarily locally path connected.*

## 8.7 Weak Local Connectedness

**Definition 8.7.1** (Weakly Locally Connected). Let  $X$  be a topological space and  $x \in X$ . Then  $X$  is *weakly locally connected at  $x$*  iff every neighbourhood of  $x$  contains a connected subspace that contains a neighbourhood of  $x$ .

## Chapter 9

# Compact Spaces

### 9.1 Countable Compactness

**Definition 9.1.1** (Countably Compact). A topological space is *countably compact* iff every countable open covering has a finite subcovering.

### 9.2 Limit Point Compactness

**Definition 9.2.1** (Limit Point Compact). A space is *limit point compact* iff every infinite set has a limit point.

**Proposition 9.2.2** (CC).  $S_\Omega \times \overline{S_\Omega}$  is limit point compact.

PROOF:

- $\langle 1 \rangle 1$ . LET:  $A \subseteq S_\Omega \times \overline{S_\Omega}$  be infinite
- $\langle 1 \rangle 2$ . CASE:  $\pi_1(A)$  is finite.
  - $\langle 2 \rangle 1$ . PICK  $x$  such that there are infinitely many  $y$  such that  $(x, y) \in A$
  - $\langle 2 \rangle 2$ . PICK a limit point  $l$  of  $\{y : (x, y) \in A\}$
  - $\langle 2 \rangle 3$ .  $(x, l)$  is a limit point of  $A$
- $\langle 1 \rangle 3$ . CASE:  $\pi_1(A)$  is infinite.
  - $\langle 2 \rangle 1$ . PICK a limit point  $l$  of  $\pi_1(A)$ .
  - $\langle 2 \rangle 2$ .  $l$  is a limit ordinal
  - $\langle 2 \rangle 3$ . PICK a countable sequence  $x_n$  with limit  $l$
  - $\langle 2 \rangle 4$ . For  $n \geq 1$ , PICK  $a_n > x_n$  and  $y_n$  such that  $(a_n, y_n) \in A$
  - $\langle 2 \rangle 5$ . CASE:  $\{y_n : n \geq 1\}$  is finite
    - $\langle 3 \rangle 1$ . PICK  $y$  such that  $y = y_n$  for infinitely many  $n$
    - $\langle 3 \rangle 2$ .  $(l, y)$  is a limit point for  $A$
  - $\langle 2 \rangle 6$ . CASE:  $\{y_n : n \geq 1\}$  is infinite
    - $\langle 3 \rangle 1$ . PICK a limit point  $m$  for  $\{y_n : n \geq 1\}$
    - $\langle 3 \rangle 2$ .  $(l, m)$  is a limit point for  $A$

□

**Proposition 9.2.3.** *The Sorgenfrey plane is not limit point compact.*

PROOF:  $\mathbb{Z}^2$  has no limit point.  $\square$

**Proposition 9.2.4.** *The space  $\mathbb{R}^\omega$  under the box topology is not limit point compact.*

PROOF: The set of all constant sequences of integers is an infinite set with no limit point.  $\square$

**Proposition 9.2.5.** *Not every open subspace of a limit point compact space is limit point compact.*

PROOF: The space  $[0, 1]$  is limit point compact but  $(0, 1)$  is not.  $\square$

**Proposition 9.2.6.** *The product of two limit point compact spaces is not necessarily limit point compact.*

PROOF: See Steen and Seebach *Counterexamples in Topology* Example 112.  $\square$

**Proposition 9.2.7.** *The continuous image of a limit point compact space is not necessarily limit point compact.*

PROOF: Let  $Y$  be a two-point set under the indiscrete topology. Then  $\mathbb{N} \times Y$  is limit point compact, but  $\mathbb{N}$  is not.  $\square$

### 9.3 Lindelöf Spaces

**Definition 9.3.1** (Lindelöf Space). A topological space  $X$  is *Lindelöf* iff every open covering has a countable subcovering.

**Theorem 9.3.2** (CC). *Every second countable space is Lindelöf.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $X$  be a second countable space
- $\langle 1 \rangle 2$ . PICK a countable basis  $\mathcal{B}$  for  $X$ .
- $\langle 1 \rangle 3$ . LET:  $\mathcal{A}$  be an open cover of  $X$
- $\langle 1 \rangle 4$ . For every  $B \in \mathcal{B}$  such that there exists  $U \in \mathcal{A}$  such that  $B \subseteq U$ , PICK  $U_B \in \mathcal{A}$  such that  $B \subseteq U_B$
- $\langle 1 \rangle 5$ .  $\{U_B : B \in \mathcal{B}, \exists U \in \mathcal{A}. B \subseteq U\}$  covers  $X$ .
- $\langle 2 \rangle 1$ . LET:  $x \in X$
- $\langle 2 \rangle 2$ . PICK  $U \in \mathcal{A}$  such that  $x \in U$
- $\langle 2 \rangle 3$ . PICK  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$
- $\langle 2 \rangle 4$ .  $x \in U_B$

$\square$

**Corollary 9.3.2.1.** *The space  $\mathbb{R}^\omega$  is Lindelöf.*

**Corollary 9.3.2.2.** *The space  $\mathbb{R}_K$  is Lindelöf.*

**Proposition 9.3.3.** *The space  $S_\Omega$  is not Lindelöf.*

PROOF:  $\{(-\infty, \alpha) : \alpha \in S_\Omega\}$  is an open cover that has no countable subcover.  $\square$

**Proposition 9.3.4 (CC).** *The space  $\overline{S_\Omega}$  is Lindelöf.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $\mathcal{A}$  be an open cover of  $\overline{S_\Omega}$
  - $\langle 1 \rangle 2$ . PICK  $U \in \mathcal{A}$  such that  $\Omega \in U$
  - $\langle 1 \rangle 3$ . PICK  $\alpha < \Omega$  such that  $(\alpha, \Omega] \subseteq U$
  - $\langle 1 \rangle 4$ . For  $\beta \leq \alpha$ , PICK  $U_\beta \in \mathcal{A}$  such that  $\beta \in U_\beta$
  - $\langle 1 \rangle 5$ .  $\{U\} \cup \{U_\beta : \beta \leq \alpha\}$  is a countable subcover of  $\mathcal{A}$ .
- $\square$

**Proposition 9.3.5 (CC).** *The continuous image of a Lindelöf space is Lindelöf.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $X$  be a Lindelöf space,  $Y$  a space and  $f : X \rightarrow Y$  continuous.
  - $\langle 1 \rangle 2$ . LET:  $\mathcal{A}$  be an open covering of  $Y$
  - $\langle 1 \rangle 3$ .  $\{f^{-1}(V) : V \in \mathcal{A}\}$  is an open covering of  $X$
  - $\langle 1 \rangle 4$ . PICK a countable subcovering  $\{f^{-1}(V_1), f^{-1}(V_2), \dots\}$  of  $\{f^{-1}(V) : V \in \mathcal{A}\}$
  - $\langle 1 \rangle 5$ .  $\{V_1, V_2, \dots\}$  is a countable subcovering of  $\mathcal{A}$
- $\square$

**Proposition 9.3.6.** *The Sorgenfrey plane is not Lindelöf.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $L = \{(x, -x) : x \in \mathbb{R}\}$
  - $\langle 1 \rangle 2$ .  $L$  is closed in  $\mathbb{R}_l^2$ 
    - $\langle 2 \rangle 1$ . LET:  $(x, y) \notin L$ , so  $y \neq -x$ 

PROVE: There exists a neighbourhood  $U$  of  $(x, y)$  that does not intersect  $L$
    - $\langle 2 \rangle 2$ . CASE:  $y > -x$ 

PROOF: In this case, take  $U = [x, +\infty) \times [y, +\infty)$
    - $\langle 2 \rangle 3$ . CASE:  $y < -x$ 

PROOF: In this case, take  $U = [x, (x - y)/2) \times [y, (y - x)/2)$ .
  - $\langle 1 \rangle 3$ . LET:  $\mathcal{U} = \{\mathbb{R}_l^2 \setminus L\} \cup \{[a, b) \times [-a, d) : a, b, d \in \mathbb{R}\}$
  - $\langle 1 \rangle 4$ .  $\mathcal{U}$  is an open covering of  $\mathbb{R}_l^2$
  - $\langle 1 \rangle 5$ . No countable subset of  $\mathcal{U}$  covers  $\mathbb{R}_l^2$ 

PROOF: Every set  $[a, b) \times [-a, d)$  intersects  $L$  in exactly one point, namely  $(a, -a)$ .
- $\square$

**Corollary 9.3.6.1.** *The Sorgenfrey plane is not second countable.*

**Corollary 9.3.6.2.** *The product of two Lindelöf spaces is not necessarily Lindelöf.*

**Proposition 9.3.7.** *The space  $\mathbb{R}^\omega$  under the box topology is not Lindelöf.*

PROOF: The set  $\{\prod_{n=0}^{\infty}(a_n, a_n + 1) : \forall n. a_n \in \mathbb{Z}\}$  covers the space but has no countable subcover.  $\square$

**Proposition 9.3.8.** *Not every open subspace of a Lindelöf space is Lindelöf.*

PROOF: The ordered square is Lindelöf but the subspace  $[0, 1] \text{ times } (0, 1)$  is not.  $\square$

## 9.4 Compactness

**Definition 9.4.1** (Compact). A topological space is *compact* iff every open cover has a finite subcover.

**Proposition 9.4.2.**  $S_{\Omega}$  is not compact.

PROOF: The open covering  $\{(-\infty, \alpha) : \alpha \in S_{\Omega}\}$  has no finite subcovering.  $\square$

**Proposition 9.4.3.**  $\mathbb{R}_l$  is not compact.

PROOF:  $\{[n, n + 1) : n \in \mathbb{Z}\}$  has no finite subcover.  $\square$

**Proposition 9.4.4.** The space  $\mathbb{R}^{\omega}$  under the box topology is not compact.

PROOF: The set  $\{\prod_{n=0}^{\infty}(a_n, a_n + 1) : n \in \mathbb{Z}\}$  is a cover that has no finite subcover.  $\square$

**Proposition 9.4.5.** *Let  $Y$  be a subspace of  $X$ . Then  $Y$  is compact if and only if every covering of  $Y$  by sets open in  $X$  contains a finite subcollection covering  $Y$ .*

PROOF:

- (1)1. If  $Y$  is compact then every covering of  $Y$  by sets open in  $X$  contains a finite subcollection covering  $Y$ .
- (2)1. ASSUME:  $Y$  is compact.
- (2)2. LET:  $\mathcal{A}$  be a covering of  $Y$  by sets open in  $X$ .
- (2)3.  $\{U \cap Y : U \in \mathcal{A}\}$  is an open covering of  $Y$ .
- (2)4. PICK a finite subcovering  $V_1, \dots, V_n$  of  $\{U \cap Y : U \in \mathcal{A}\}$
- (2)5. For  $1 \leq i \leq n$ , PICK  $U_i \in \mathcal{A}$  such that  $V_i = U_i \cap Y$ .
- (2)6.  $\{U_1, \dots, U_n\}$  is a finite subset of  $\mathcal{A}$  that covers  $Y$ .
- (1)2. If every covering of  $Y$  by sets open in  $X$  contains a finite subcollection covering  $Y$  then  $Y$  is compact.
- (2)1. ASSUME: Every covering of  $Y$  by sets open in  $X$  contains a finite subcollection covering  $Y$ .
- (2)2. LET:  $\mathcal{A}$  be an open covering of  $Y$
- (2)3. LET:  $\mathcal{B} = \{U \text{ open in } X : U \cap Y \in \mathcal{A}\}$
- (2)4.  $\mathcal{B}$  covers  $Y$
- (2)5. PICK a finite subcollection  $\{U_1, \dots, U_n\} \subseteq \mathcal{B}$  that covers  $Y$
- (2)6.  $\{U_1 \cap Y, \dots, U_n \cap Y\}$  is a finite subcover of  $\mathcal{A}$ .

$\square$

**Proposition 9.4.6.** *Every closed subspace of a compact space is compact.*

PROOF:

- ⟨1⟩1. LET:  $X$  be a compact space and  $Y \subseteq X$  be closed.
- ⟨1⟩2. LET:  $\mathcal{A}$  be a covering of  $Y$  by spaces open in  $X$
- ⟨1⟩3.  $\mathcal{A} \cup \{X \setminus Y\}$  is an open covering of  $X$ .
- ⟨1⟩4. PICK a finite subcovering  $\{U_1, \dots, U_n\}$  or  $\{U_1, \dots, U_n, X \setminus Y\}$
- ⟨1⟩5.  $\{U_1, \dots, U_n\}$  is a finite subset of  $\mathcal{A}$  that covers  $Y$ .
- ⟨1⟩6. Q.E.D.

PROOF: Proposition 9.4.5.

□

**Corollary 9.4.6.1.** *Not every compact Hausdorff space is connected.*

PROOF: The space  $[0, 1] \cup [2, 3]$  is compact Hausdorff and disconnected. □

**Corollary 9.4.6.2.** *Not every compact Hausdorff space is path connected.*

**Corollary 9.4.6.3.** *Not every compact Hausdorff space is locally connected.*

The space  $[0, 1] \cap \mathbb{Q}$  is not locally connected.

**Corollary 9.4.6.4.** *Not every compact Hausdorff space is locally path connected.*

**Proposition 9.4.7.** *Not every open subspace of a compact space is compact.*

PROOF: The space  $[0, 1]$  is compact but  $(0, 1)$  is not. □

**Lemma 9.4.8.** *If  $Y$  is a compact subspace of the Hausdorff space  $X$  and  $a \notin Y$ , then there exist disjoint open sets  $U$  and  $V$  of  $X$  containing  $a$  and  $Y$ , respectively.*

PROOF:

- ⟨1⟩1. For  $y \in Y$ , there exist disjoint open sets  $U$  and  $V$  such that  $a \in U$  and  $y \in V$ .
- ⟨1⟩2.  $\{V \text{ open in } X : \exists U \text{ open and disjoint from } V, a \in U\}$  is a covering of  $Y$  by open sets in  $X$ .
- ⟨1⟩3. PICK a finite subset  $\{V_1, \dots, V_n\}$  that covers  $Y$ .
- ⟨1⟩4. For  $1 \leq i \leq n$ , PICK  $U_i$  disjoint from  $V_i$  such that  $a \in U_i$
- ⟨1⟩5. LET:  $U = U_1 \cap \dots \cap U_n$  and  $V = V_1 \cup \dots \cup V_n$

□

**Proposition 9.4.9.** *Every compact subspace of a Hausdorff space is closed.*

PROOF:

- ⟨1⟩1. LET:  $X$  be a Hausdorff space and  $Y \subseteq X$  be compact.
- ⟨1⟩2. Every point  $a \notin Y$  has an open neighbourhood disjoint from  $Y$ .

PROOF: By Lemma 9.4.8.

- ⟨1⟩3. Q.E.D.

PROOF: By Proposition 3.2.3.

**Proposition 9.4.10.** *The image of a compact space under a continuous map is compact.*

PROOF:

- ⟨1⟩1. LET:  $f : X \rightarrow Y$  be continuous where  $X$  is compact.
- ⟨1⟩2. LET:  $\mathcal{A}$  be a covering of  $f(X)$  by open sets in  $Y$ .
- ⟨1⟩3.  $\{f^{-1}(U) : U \in \mathcal{A}\}$  is an open covering of  $X$ .
- ⟨1⟩4. PICK a finite subcovering  $\{f^{-1}(U_1), \dots, f^{-1}(U_n)\}$
- ⟨1⟩5.  $\{U_1, \dots, U_n\}$  is a finite subset of  $\mathcal{A}$  that covers  $f(X)$ .
- ⟨1⟩6. Q.E.D.

PROOF: By Proposition 9.4.5.

□

**Corollary 9.4.10.1.** *Let  $\{X_\alpha\}_{\alpha \in J}$  be a family of topological spaces. If  $\prod_{\alpha \in J} X_\alpha$  is compact then each  $X_\alpha$  is compact.*

**Corollary 9.4.10.2.**  $S_\Omega \times \overline{S_\Omega}$  is compact.

**Corollary 9.4.10.3.** *The Sorgenfrey plane is not compact.*

**Corollary 9.4.10.4.** *For any nonempty set  $I$ , the space  $\mathbb{R}^I$  is not compact.*

**Corollary 9.4.10.5.** *Let  $\mathcal{T}$  and  $\mathcal{T}'$  be topologies on the same set  $X$ . If  $\mathcal{T} \subseteq \mathcal{T}'$  and  $\mathcal{T}'$  is compact then  $\mathcal{T}$  is compact.*

**Corollary 9.4.10.6.** *The space  $\mathbb{R}_K$  is not compact.*

**Theorem 9.4.11.** *Let  $f : X \rightarrow Y$  be a bijective continuous function. If  $X$  is compact and  $Y$  is Hausdorff then  $f$  is a homeomorphism.*

PROOF:

- ⟨1⟩1. LET:  $C$  be closed in  $X$
- ⟨1⟩2.  $C$  is compact  
PROOF: Proposition 9.4.6.
- ⟨1⟩3.  $f(C)$  is compact  
PROOF: Proposition 9.4.10
- ⟨1⟩4.  $f(C)$  is closed  
PROOF: Proposition 9.4.9.
- ⟨1⟩5. Q.E.D.

PROOF: By Theorem 5.2.2 we have that  $f^{-1}$  is continuous.

□

**Corollary 9.4.11.1.** *Let  $\mathcal{T}$  and  $\mathcal{T}'$  be topologies on the same set  $X$ . If  $\mathcal{T} \subseteq \mathcal{T}'$ ,  $\mathcal{T}$  is Hausdorff and  $\mathcal{T}'$  is compact then  $\mathcal{T} = \mathcal{T}'$ .*

**Corollary 9.4.11.2.** *The space  $[0, 1]$  is not compact as a subspace of  $\mathbb{R}_K$ .*

**Theorem 9.4.12** (Tube Lemma). *Let  $A$  and  $B$  be subspaces of  $X$  and  $Y$ , respectively; let  $N$  be an open set in  $X \times Y$  including  $A \times B$ . If  $A$  and  $B$  are compact, then there exist open sets  $U$  and  $V$  in  $X$  and  $Y$ , respectively, such that*

$$A \times B \subseteq U \times V \subseteq N .$$



PROOF:

- (1)1. For all  $a \in A$ , there exist open sets  $U$  and  $V$  in  $X$  and  $Y$ , respectively, such that

$$\{a\} \times B \subseteq U \times V \subseteq N.$$

- (2)1. LET:  $a \in A$   
 (2)2. For all  $b \in B$ , there exist open sets  $U$  and  $V$  in  $X$  and  $Y$ , respectively, such that  $(a, b) \in U \times V \subseteq N$ .  
 (2)3.  $\{V \text{ open in } Y : \exists U \text{ open in } X. a \in U, U \times V \subseteq N\}$  covers  $B$   
 (2)4. PICK a finite subset  $\{V_1, \dots, V_n\}$  that covers  $B$ .  
 (2)5. For  $1 \leq i \leq n$ , PICK  $U_i$  open in  $X$  such that  $a \in U_i$  and  $U_i \times V_i \subseteq N$   
 (2)6. LET:  $U = U_1 \cap \dots \cap U_n$  and  $V = V_1 \cup \dots \cup V_n$   
 (1)2.  $\{U \text{ open in } X : \exists V \text{ open in } Y. B \subseteq V \text{ and } U \times V \subseteq N\}$  covers  $A$ .  
 (1)3. PICK a finite subset  $\{U_1, \dots, U_n\}$  that covers  $A$ .  
 (1)4. For  $1 \leq i \leq n$ , PICK  $V_i$  open in  $B$  such that  $B \subseteq V_i$  and  $U_i \times V_i \subseteq N$ .  
 (1)5. LET:  $U = U_1 \cup \dots \cup U_n$  and  $V = V_1 \cap \dots \cap V_n$   
 (1)6.  $A \times B \subseteq U \times V \subseteq N$

□

**Lemma 9.4.13.** *Let  $\mathcal{A}$  be a set of basis elements for  $X \times Y$  such that no finite subset of  $\mathcal{A}$  covers  $X \times Y$ . If  $X$  is compact, then there exists a point  $x \in X$  such that no finite subset of  $\mathcal{A}$  covers  $\{x\} \times Y$ .*

PROOF:

- (1)1. ASSUME:  $X$  is compact.  
 (1)2. ASSUME: For all  $x \in X$ , there is a finite subset of  $\mathcal{A}$  that covers  $\{x\} \times Y$   
 PROVE: A finite subset of  $\mathcal{A}$  covers  $X \times Y$   
 (1)3.  $\{U \text{ open in } X : \exists U_1 \times V_1, \dots, U_r \times V_r \in \mathcal{A}. U = U_1 \cap \dots \cap U_r, Y = V_1 \cup \dots \cup V_r\}$  covers  $X$ .  
 (1)4. PICK a finite subcover  $\{U_1, \dots, U_n\}$   
 (1)5. For  $1 \leq i \leq n$ , PICK  $U_{i1} \times V_{i1}, \dots, U_{ir_i} \times V_{ir_i} \in \mathcal{A}$  such that  $U_i = U_{i1} \cap \dots \cap U_{ir_i}$  and  $Y = V_{i1} \cup \dots \cup V_{ir_i}$   
 (1)6.  $\{U_{ij} : 1 \leq i \leq n, 1 \leq j \leq r_i\}$  covers  $X \times Y$

□

**Proposition 9.4.14.** *The product of two compact spaces is compact.*

PROOF:

- (1)1. LET:  $X$  and  $Y$  be compact spaces.  
 (1)2. LET:  $\mathcal{A}$  be an open covering of  $X \times Y$   
 (1)3. For all  $x \in X$ , there exists a neighbourhood  $W$  of  $x$  such that  $W \times Y$  is covered by finitely many elements of  $\mathcal{A}$ .  
 (2)1. LET:  $x \in X$   
 (2)2.  $\{x\} \times Y$  is compact.  
 PROOF: It is homeomorphic to  $Y$ .  
 (2)3. PICK a finite subset  $\{U_1, \dots, U_m\}$  of  $\mathcal{A}$  that covers  $\{x\} \times Y$   
 PROOF: By Proposition 9.4.5.  
 (2)4. There exists a neighbourhood  $W$  of  $x$  such that  $W \times Y \subseteq U_1 \cup \dots \cup U_m$

PROOF: By the Tube Lemma.

- ⟨1⟩4.  $\{W \text{ open in } X : W \times Y \text{ is covered by finitely many elements of } \mathcal{A}\}$  is an open covering of  $X$ .
- ⟨1⟩5. PICK a finite subcovering  $\{W_1, \dots, W_n\}$
- ⟨1⟩6. For  $1 \leq i \leq n$ , PICK a finite subset  $\{U_{i1}, \dots, U_{ir_i}\}$  of  $\mathcal{A}$  that covers  $W_i \times Y$
- ⟨1⟩7.  $\{U_{11}, \dots, U_{nr_n}\}$  is a finite subcovering of  $\mathcal{A}$ .

□

**Proposition 9.4.15.** *A topological space is compact if and only if every nonempty set of closed sets that has the finite intersection property has nonempty intersection.*

PROOF: Immediate from definitions. □

**Lemma 9.4.16.** *If  $Y$  is compact then  $\pi_1 : X \times Y \rightarrow X$  is a closed map.*

PROOF:

- ⟨1⟩1. LET:  $C \subseteq X \times Y$  be closed
- ⟨1⟩2. LET:  $x \in X \setminus \pi_1(C)$
- ⟨1⟩3. For all  $y \in Y$ , we have  $(x, y) \notin C$
- ⟨1⟩4. For all  $y \in Y$ , there exist open neighbourhoods  $U$  of  $x$  and  $V$  of  $y$  such that  $U \times V \subseteq (X \times Y) \setminus C$
- ⟨1⟩5.  $\{V \text{ open in } Y : \exists U \text{ an open neighbourhood of } x \text{ such that } U \times V \subseteq (X \times Y) \setminus C\}$  is an open covering of  $Y$ .
- ⟨1⟩6. PICK a finite subcovering  $\{V_1, \dots, V_n\}$
- ⟨1⟩7. For  $1 \leq i \leq n$ , PICK an open neighbourhood  $U_i$  of  $x$  such that  $U_i \times V_i \subseteq (X \times Y) \setminus C$
- ⟨1⟩8.  $x \in U_1 \cap \dots \cap U_n \subseteq X \setminus \pi_1(C)$

□

**Theorem 9.4.17.** *Let  $X$  be a compact space. Let  $f_n : X \rightarrow \mathbb{R}$  be a sequence of continuous functions such that, for all  $x \in X$ ,  $f_n(x) \rightarrow f(x)$  as  $n \rightarrow \infty$ . If  $f$  is continuous, and if the sequence  $(f_n)_n$  is monotone increasing, and if  $X$  is compact, then the convergence is uniform.*

PROOF:

- ⟨1⟩1. LET:  $\epsilon > 0$   
 PROVE: There exists  $N$  such that, for all  $n \geq N$ , we have  $|f_n(x) - f(x)| < \epsilon$
- ⟨1⟩2. For  $n \in \mathbb{Z}^+$ ,  
 LET:  $U_n = \{x \in X : f(x) - f_n(x) < \epsilon\}$
- ⟨1⟩3. Each  $U_n$  is open  
 PROOF: Let  $g(x) = f(x) - f_n(x)$ . Then  $g$  is continuous and  $U_n = g^{-1}((-\infty, \epsilon))$ .
- ⟨1⟩4.  $\{U_n : n \geq 1\}$  is an open covering of  $X$
- ⟨2⟩1. LET:  $x \in X$
- ⟨2⟩2. PICK  $N$  such that, for all  $n \geq N$ ,  $|f(x) - f_n(x)| < \epsilon$   
 PROOF:  $f_n(x) \rightarrow f(x)$  as  $n \rightarrow \infty$
- ⟨2⟩3.  $f(x) - f_N(x) < \epsilon$

PROOF: This holds since the sequence  $(f_n)_n$  is monotone.

- ⟨1⟩5. PICK a finite subcovering  $\{U_{n_1}, \dots, U_{n_k}\}$
  - ⟨1⟩6. LET:  $N = \max(n_1, \dots, n_k)$
  - ⟨1⟩7. For all  $n \geq N$  we have  $|f_n(x) - f(x)| < \epsilon$
- 

**Lemma 9.4.18.** *Every compact Hausdorff space is normal.*

PROOF:

- ⟨1⟩1. LET:  $A$  and  $B$  be disjoint closed sets in the compact Hausdorff space  $X$ .
- ⟨1⟩2. For all  $a \in A$ , there exist disjoint open sets  $U$  and  $V$  such that  $a \in U$  and  $B \subseteq V$ .

PROOF: By Lemma 9.4.8.

- ⟨1⟩3.  $\{U \text{ open in } X : \exists V \text{ open in } Y, U \cap V = \emptyset, B \subseteq V\}$  is an open covering of  $A$
  - ⟨1⟩4. PICK a finite subcovering  $\{U_1, \dots, U_n\}$
  - ⟨1⟩5. For  $1 \leq i \leq n$ , PICK  $V_i$  open in  $Y$  such that  $U_i \cap V_i = \emptyset$  and  $B \subseteq V_i$
  - ⟨1⟩6. LET:  $U = U_1 \cup \dots \cup U_n$  and  $V = V_1 \cap \dots \cap V_n$
- 

**Theorem 9.4.19.** *Let  $X$  be a complete linearly ordered set under the order topology. Then every closed interval in  $X$  is compact.*

PROOF:

- ⟨1⟩1. LET:  $X$  be a complete linearly ordered set in the order topology
- ⟨1⟩2. LET:  $a, b \in X, a < b$   
PROVE:  $[a, b]$  is compact
- ⟨1⟩3. LET:  $\mathcal{A}$  be a set of open sets that covers  $[a, b]$
- ⟨1⟩4. For all  $x \in [a, b)$ , there exists  $y \in (x, b]$  such that  $[x, y]$  is covered by at most two points of  $\mathcal{A}$ 
  - ⟨2⟩1. LET:  $x \in [a, b]$
  - ⟨2⟩2. PICK  $U \in \mathcal{A}$  such that  $x \in U$   
PROOF: By ⟨1⟩3 and ⟨2⟩1
  - ⟨2⟩3. PICK  $y \in (x, b]$  such that  $[x, y] \subseteq U$   
PROOF: By Lemma 4.1.2.
  - ⟨2⟩4. PICK  $V \in \mathcal{A}$  such that  $y \in V$   
PROOF: By ⟨1⟩3 and ⟨2⟩3.
  - ⟨2⟩5.  $[x, y]$  is covered by  $\{U, V\}$   
PROOF: By ⟨2⟩3 and ⟨2⟩4.
- ⟨1⟩5. LET:  $C = \{y \in (a, b] : [a, y] \text{ is covered by a finite subset of } \mathcal{A}\}$
- ⟨1⟩6.  $C$  is nonempty  
PROOF: By ⟨1⟩4.
- ⟨1⟩7. LET:  $c = \sup C$   
PROOF: By ⟨1⟩1.
- ⟨1⟩8.  $c \in C$ 
  - ⟨2⟩1. PICK  $U \in \mathcal{A}$  such that  $c \in U$
  - ⟨2⟩2. PICK  $y \in [a, c)$  such that  $(y, c] \subseteq U$

PROOF: By Lemma 4.1.2

⟨2⟩3. PICK  $z$  such that  $y < z$  and  $z \in C$

PROOF: This exists because  $y$  is not an upper bound for  $C$ .

⟨2⟩4. PICK a finite  $\mathcal{A}_0 \subseteq \mathcal{A}$  such that  $[a, z]$  is covered by  $\mathcal{A}_0$

⟨2⟩5.  $[a, c]$  is covered by  $\mathcal{A}_0 \cup \{U\}$

⟨1⟩9.  $c = b$

⟨2⟩1. ASSUME: for a contradiction  $c < b$

⟨2⟩2. PICK  $y \in (c, b]$  such that  $[c, y]$  is covered by at most two elements of  $\mathcal{A}$ .

PROOF: By ⟨1⟩4

⟨2⟩3.  $y > c$  and  $y \in C$

⟨2⟩4. Q.E.D.

PROOF: This contradicts ⟨1⟩7.

⟨1⟩10. Q.E.D.

**Corollary 9.4.19.1.** *Every closed interval in  $\mathbb{R}$  is compact.*

**Corollary 9.4.19.2 (CC).**  *$S_\Omega$  is limit point compact.*

PROOF:

⟨1⟩1. LET:  $A$  be an infinite subset of  $S_\Omega$

⟨1⟩2. PICK a countably infinite subset  $B \subseteq A$

⟨1⟩3. LET:  $b = \sup B$

⟨1⟩4.  $B \subseteq [0, b]$

⟨1⟩5.  $[0, b]$  is compact

PROOF: By the theorem.

⟨1⟩6.  $B$  has a limit point in  $[0, b]$

⟨1⟩7.  $A$  has a limit point in  $[0, b]$

□

**Corollary 9.4.19.3.** *The ordered square is compact.*

**Corollary 9.4.19.4.** *The ordered square is limit point compact.*

**Corollary 9.4.19.5.** *Not every subspace of a compact space is compact.*

PROOF:  $[0, 1]$  is compact but  $(0, 1)$  is not. □

**Theorem 9.4.20 (Extreme Value Theorem).** *Let  $f : X \rightarrow Y$  be continuous where  $Y$  is a linearly ordered set in the order topology. If  $X$  is compact, then there exist  $c, d \in X$  such that, for all  $x \in X$ , we have  $f(c) \leq f(x) \leq f(d)$ .*

PROOF:

⟨1⟩1.  $f(X)$  is compact.

PROOF: By Proposition 9.4.10.

⟨1⟩2.  $f(X)$  has a greatest element.

⟨2⟩1. ASSUME: for a contradiction  $f(X)$  has no greatest element.

⟨2⟩2.  $\{(-\infty, f(x)) : x \in X\}$  is a set of open sets that covers  $f(X)$ .

⟨2⟩3. PICK a finite subset  $\{(-\infty, f(x_1)), \dots, (-\infty, f(x_n))\}$  that covers  $f(X)$ .

PROOF: By Proposition 9.4.5

⟨2⟩4. LET:  $f(x_N)$  be largest out of  $f(x_1), \dots, f(x_n)$

⟨2⟩5.  $f(x_N) < f(x_N)$

⟨2⟩6. Q.E.D.

PROOF: This is a contradiction.

⟨1⟩3.  $f(X)$  has a least element.

PROOF: Similar.

□

**Theorem 9.4.21 (DC).** *A nonempty compact Hausdorff space with no isolated points is uncountable.*

PROOF:

⟨1⟩1. LET:  $X$  be a nonempty compact Hausdorff space with no isolated points.

⟨1⟩2. For every nonempty open  $U \subseteq X$  and point  $x \in X$ , there exists a nonempty open  $V \subseteq U$  such that  $x \notin \overline{V}$

⟨2⟩1. LET:  $U \subseteq X$  be nonempty and open and  $x \in X$

⟨2⟩2. PICK  $y \in U$  such that  $y \neq x$

PROOF: This is possible because  $U \neq \{x\}$  since  $x$  is not an isolated point.

⟨2⟩3. PICK disjoint open neighbourhoods  $W_1$  and  $W_2$  of  $x$  and  $y$

PROOF: Since  $X$  is Hausdorff

⟨2⟩4. LET:  $V = U \cap W_2$

⟨2⟩5.  $x \notin \overline{V}$

PROOF: We have  $\overline{V} \subseteq \overline{W_2} \subseteq X \setminus W_1$ .

⟨1⟩3. LET:  $f : \mathbb{Z}^+ \rightarrow X$

PROVE:  $f$  is not surjective

⟨1⟩4. PICK a sequence of open sets  $V_1 \supseteq V_2 \supseteq \dots$  such that  $f(n) \notin \overline{V_n}$

PROOF: By ⟨1⟩2 and Dependent Choice.

⟨1⟩5. PICK a point  $b \in \bigcap_{i=1}^{\infty} \overline{V_i}$

PROOF: By Proposition 9.4.15.

⟨1⟩6.  $b \neq f(n)$  for all  $n$

PROOF: For each  $n$  we have  $b \in \overline{V_n}$  (⟨1⟩5) and  $f(n) \notin \overline{V_n}$  (⟨1⟩4).

□

**Corollary 9.4.21.1.** *Every closed interval in  $\mathbb{R}$  is uncountable.*

**Theorem 9.4.22.** *Every compact space is limit point compact.*

PROOF:

⟨1⟩1. LET:  $X$  be a compact space.

⟨1⟩2. LET:  $A \subseteq X$  be a set with no limit points.

PROVE:  $A$  is finite.

⟨1⟩3.  $A$  is closed.

PROOF: By Corollary 3.10.3.1.

⟨1⟩4.  $A$  is compact.

PROOF: By Proposition 9.4.6.

⟨1⟩5.  $\{U \text{ open in } X : U \cap A \text{ is a singleton}\}$  covers  $A$

⟨2⟩1. LET:  $a \in A$

(2)2. PICK an open neighbourhood  $U$  of  $a$  such that  $U$  does not intersect  $A$  at a point other than  $a$   
 PROOF: One must exist because  $a$  is not a limit point of  $A$  ((1)2).  
 (2)3.  $U \cap A = \{a\}$   
 (1)6. PICK a finite subcover  $\{U_1, \dots, U_n\}$   
 PROOF: By (1)4 using Proposition 9.4.5.  
 (1)7. For  $1 \leq i \leq n$ ,  
 LET:  $U_i \cap A = \{a_i\}$   
 (1)8.  $A = \{a_1, \dots, a_n\}$   
 $\square$

**Proposition 9.4.23.** *Let  $X$  be a space and  $C, D \subseteq X$  be compact. Then  $C \cup D$  is compact.*

PROOF:  
 (1)1. LET:  $\mathcal{A}$  be a set of open sets that covers  $C \cup D$   
 (1)2. PICK a finite subset  $\mathcal{A}_1$  that covers  $C$  and a finite subset  $\mathcal{A}_2$  that covers  $D$ .  
 (1)3.  $\mathcal{A}_1 \cup \mathcal{A}_2$  is a finite subset of  $\mathcal{A}$  that covers  $C \cup D$ .  
 (1)4. Q.E.D.

**Theorem 9.4.24.** *Every compact Hausdorff space is normal.*

PROOF:  
 (1)1. LET:  $X$  be a compact Hausdorff space.  
 (1)2. LET:  $A$  and  $B$  be disjoint closed sets in  $X$ .  
 (1)3.  $\{U \text{ open in } X : \exists V \text{ open in } X. B \subseteq V \wedge U \cap V = \emptyset\}$  covers  $A$   
 (2)1.  $B$  is compact  
 PROOF: By Proposition 9.4.6.  
 (2)2. Q.E.D.  
 PROOF: By Lemma 9.4.8.  
 (1)4. PICK a finite subcover  $\{U_1, \dots, U_n\}$   
 PROOF:  $A$  is compact by Proposition 9.4.6.  
 (1)5. For  $1 \leq i \leq n$ , PICK  $V_i$  open in  $X$  such that  $B \subseteq V_i$  and  $U_i \cap V_i = \emptyset$   
 (1)6. LET:  $U = U_1 \cup \dots \cup U_n$  and  $V = V_1 \cap \dots \cap V_n$   
 (1)7.  $U$  and  $V$  are disjoint open sets,  $A \subseteq U$  and  $B \subseteq V$   
 $\square$

**Corollary 9.4.24.1.** *The ordered square is normal.*

**Proposition 9.4.25.** *Not every compact Hausdorff space is first countable.*

PROOF: The space  $\overline{S_\Omega}$  is compact Hausdorff but not first countable.  $\square$

**Corollary 9.4.25.1.** *Not every compact Hausdorff space is second countable.*

**Theorem 9.4.26** (Tychonoff (AC)). *The product of a family of compact spaces is compact.*

PROOF:

- (1)1. LET:  $\{X_\alpha\}_{\alpha \in J}$  be a family of compact spaces.  
 LET:  $X = \prod_{\alpha \in J} X_\alpha$
- (1)2. LET:  $\mathcal{A} \subseteq \mathcal{P}X$  satisfy the finite intersection property.  
 PROVE:  $\bigcap_{A \in \mathcal{A}} \overline{A}$  is nonempty.
- (1)3. PICK a set  $\mathcal{D} \subseteq \mathcal{P}X$  that includes  $\mathcal{A}$  and is maximal with respect to the finite intersection property.  
 PROOF: By Lemma 1.6.6.
- (1)4. For  $\alpha \in J$ , PICK  $x_\alpha \in \bigcap_{D \in \mathcal{D}} \overline{\pi_\alpha(D)}$
- (2)1. LET:  $\alpha \in J$
- (2)2.  $\{\pi_\alpha(D) : D \in \mathcal{D}\}$  satisfies the finite intersection property.
- (2)3. Q.E.D.  
 PROOF: By Proposition 9.4.15
- (1)5. LET:  $x = (x_\alpha)_{\alpha \in J}$
- (1)6. For all  $D \in \mathcal{D}$  we have  $(x_\alpha)_{\alpha \in J} \in \overline{D}$   
 PROOF:
- (2)1. Every subbasis element containing  $x$  intersects every member of  $\mathcal{D}$
- (3)1. LET:  $\pi_\alpha(U)^{-1}$  be a subbasis element containing  $x$  where  $U$  is open in  $X_\alpha$
- (3)2. LET:  $D \in \mathcal{D}$
- (3)3.  $U$  intersects  $\pi_\alpha(D)$
- (2)2. Every subbasis element containing  $x$  is a member of  $\mathcal{D}$   
 PROOF: By Lemma 1.6.8
- (2)3. Every basis element containing  $x$  is a member of  $\mathcal{D}$   
 PROOF: By Lemma 1.6.7
- (2)4. Every basis element containing  $x$  intersects every member of  $\mathcal{D}$   
 PROOF: This follows because  $\mathcal{D}$  satisfies the finite intersection property.
- (1)7. Q.E.D.  
 PROOF: By Proposition 9.4.15
- 
- PROOF:
- (1)1. LET:  $\{X_\alpha\}_{\alpha \in J}$  be a family of compact spaces and  $X = \prod_{\alpha \in J} X_\alpha$ .
- (1)2. PICK a well-ordering  $<$  of  $J$  such that  $J$  has a greatest element  $\top$
- (1)3. For all  $\alpha \in J$  and every family of points  $p = \{p_i \in X_i\}_{i \leq \alpha}$ ,  
 LET:  $Y_\alpha(p) = \{x \in X : \forall i \leq \alpha. x_i = p_i\}$
- (1)4. For all  $\beta \in J$  and every family of points  $p = \{p_i \in X_i\}_{i < \beta}$ ,  
 LET:  $Z_\beta(p) = \bigcap_{\alpha < \beta} Y_\alpha = \{x \in X : \forall i < \beta. x_i = p_i\}$
- (1)5. Given  $\beta \in J$ , a family of points  $\{p_i \in X_i\}_{i < \beta}$ , and a finite set  $\mathcal{A}$  of basis elements that covers  $Z_\beta(p)$ , there exists  $\alpha < \beta$  such that  $\mathcal{A}$  covers  $Y_\alpha(p)$
- (2)1. ASSUME: (  
 w.l.o.g.  $\beta$  has no immediate predecessor)
- (2)2. For  $A \in \mathcal{A}$ ,  
 LET:  $J_A = \{i < \beta : \pi_i(A) \neq X_i\}$
- (2)3. LET:  $\alpha$  be the largest element of  $\bigcup_{A \in \mathcal{A}} J_A$   
 PROOF: The set has a greatest element because each  $J_A$  is finite and  $\mathcal{A}$  is finite.





quasicomponent.

PROVE:  $x$  and  $y$  are in the same component.

$\langle 1 \rangle 2$ . LET:  $\mathcal{A}$  be the set of all closed subspaces  $A$  of  $X$  such that  $x$  and  $y$  lie in the same quasicomponent of  $A$ .

$\langle 1 \rangle 3$ . Every chain in  $\mathcal{A}$  has a lower bound.

$\langle 2 \rangle 1$ . LET:  $\mathcal{B} \subseteq \mathcal{A}$  be a chain

PROVE:  $Y = \bigcap \mathcal{B} \in \mathcal{A}$

$\langle 2 \rangle 2$ . ASSUME: for a contradiction  $Y = C \cup D$  where  $C$  and  $D$  are disjoint and open in  $Y$ ,  $x \in C$  and  $y \in D$

$\langle 2 \rangle 3$ . PICK disjoint open sets  $U$  and  $V$  in  $X$  such that  $C \subseteq U$  and  $D \subseteq V$

PROOF: By Lemma 9.4.18.

$\langle 2 \rangle 4$ .  $\{B \setminus (U \cup V) : B \in \mathcal{B}\}$  satisfies the finite intersection property.

$\langle 3 \rangle 1$ . LET:  $B_1, \dots, B_n \in \mathcal{B}$

$\langle 3 \rangle 2$ .  $B_1 \cap \dots \cap B_n \in \mathcal{B}$

PROOF: By  $\langle 2 \rangle 1$ .

$\langle 3 \rangle 3$ .  $B_1 \cap \dots \cap B_n \setminus (U \cup V)$  is nonempty

PROOF:  $B_1 \cap \dots \cap B_n \cap U$  and  $B_1 \cap \dots \cap B_n \cap V$  cannot be disjoint, because  $x$  and  $y$  are in the same quasicomponent of  $B_1 \cap \dots \cap B_n$ .

$\langle 2 \rangle 5$ .  $Y \setminus (U \cup V)$  is nonempty.

PROOF: By Proposition 9.4.15.

$\langle 2 \rangle 6$ . Q.E.D.

PROOF: This is a contradiction since  $Y \setminus (U \cup V) = Y \setminus (C \cup D)$ .

$\langle 1 \rangle 4$ . PICK a minimal element  $D \in \mathcal{A}$

PROOF: One exists by Zorn's Lemma.

$\langle 1 \rangle 5$ .  $D$  is connected.

$\langle 2 \rangle 1$ . ASSUME: [

for a contradiction  $D = U \uplus V$  is a separation of  $D$ ]

$\langle 2 \rangle 2$ . CASE:  $x, y \in U$

PROOF: In this case we have  $U \in \mathcal{A}$  contradicting the minimality of  $D$ .

$\langle 2 \rangle 3$ . CASE:  $x \in U, y \in V$

PROOF: This is a contradiction because  $x$  and  $y$  are in the same quasicomponent of  $D$ .

$\langle 2 \rangle 4$ . CASE:  $x \in V, y \in U$

PROOF: Similar to  $\langle 2 \rangle 3$ .

$\langle 2 \rangle 5$ . CASE:  $x, y \in V$

PROOF: Similar to  $\langle 2 \rangle 2$ .

□

## 9.5 Perfect Maps

**Proposition 9.5.1.** *Let  $p : X \twoheadrightarrow Y$  be a closed continuous surjective map. For all  $y \in Y$  and  $U$  an open neighbourhood of  $p^{-1}(y)$ , there exists an open neighbourhood  $W$  of  $y$  such that  $p^{-1}(W) \subseteq U$ .*

PROOF: Take  $W = Y \setminus p(X \setminus U)$ . □

**Proposition 9.5.2 (AC).** *Let  $p : X \rightarrow Y$  be a closed continuous surjective map. If  $X$  is normal then  $Y$  is normal.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $A, B \subseteq Y$  be closed
- $\langle 1 \rangle 2$ .  $p^{-1}(A), p^{-1}(B)$  are closed in  $X$ .
- $\langle 1 \rangle 3$ . PICK disjoint open sets  $U, V$  of  $p^{-1}(A), p^{-1}(B)$  respectively.
- $\langle 1 \rangle 4$ . For all  $a \in A$ , PICK an open neighbourhood  $W_a$  of  $a$  such that  $p^{-1}(W_a) \subseteq U$

PROOF: By Proposition 9.5.1.

- $\langle 1 \rangle 5$ . For all  $b \in B$ , PICK an open neighbourhood  $W'_b$  of  $b$  such that  $p^{-1}(W'_b) \subseteq V$

PROOF: By Proposition 9.5.1.

- $\langle 1 \rangle 6$ . LET:  $W = \bigcup_{a \in A} W_a$  and  $W' = \bigcup_{b \in B} W'_b$
- $\langle 1 \rangle 7$ .  $W \cap W' = \emptyset$

PROOF: This holds because  $p^{-1}(W) \subseteq U, p^{-1}(W') \subseteq V$ , and  $p$  is surjective.

□

**Definition 9.5.3 (Perfect Map).** Let  $X$  and  $Y$  be topological spaces and  $p : X \rightarrow Y$ . Then  $p$  is *perfect* iff  $p$  is closed, continuous, surjective, and  $p^{-1}(y)$  is compact for all  $y \in Y$ .

**Proposition 9.5.4.** *Let  $p : X \rightarrow Y$  be a perfect map. If  $X$  is Hausdorff then so is  $Y$ .*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $a, b \in Y$  with  $a \neq b$
- $\langle 1 \rangle 2$ . PICK disjoint open neighbourhoods  $U$  and  $V$  of  $\pi^{-1}(a)$  and  $\pi^{-1}(b)$ , respectively.

PROOF: By Lemma 9.4.18.

- $\langle 1 \rangle 3$ . PICK open neighbourhoods  $W$  and  $W'$  of  $a$  and  $b$  such that  $\pi^{-1}(W) \subseteq U$  and  $\pi^{-1}(W') \subseteq V$

PROOF: By Proposition 9.5.1.

- $\langle 1 \rangle 4$ .  $W$  and  $W'$  are disjoint.

□

**Proposition 9.5.5.** *Let  $p : X \rightarrow Y$  be perfect. If  $X$  is regular then so is  $Y$ .*

PROOF:

- $\langle 1 \rangle 1$ .  $Y$  is  $T_1$

PROOF: By Proposition 9.5.4.

- $\langle 1 \rangle 2$ . LET:  $C \subseteq Y$  be closed and  $a \in Y \setminus C$
- $\langle 1 \rangle 3$ .  $p^{-1}(C)$  is closed and  $p^{-1}(a)$  is disjoint from  $p^{-1}(C)$ .
- $\langle 1 \rangle 4$ . PICK disjoint open neighbourhoods  $U, V$  of  $p^{-1}(C), p^{-1}(a)$  respectively.

PROOF: By Lemma 9.4.8.

- $\langle 1 \rangle 5$ . PICK an open neighbourhood  $W'$  of  $a$  such that  $p^{-1}(W') \subseteq V$

PROOF: By Proposition 9.5.1.

- $\langle 1 \rangle 6$ . For  $c \in C$ , PICK an open neighbourhood  $W_c$  such that  $p^{-1}(W_c) \subseteq U$

PROOF: By Proposition 9.5.1.

(1)7.  $W = \bigcup_{c \in C} W_c$  is an open neighbourhood of  $C$  disjoint from  $W'$

□

**Proposition 9.5.6** (AC). *Let  $p : X \rightarrow Y$  be perfect. If  $X$  is locally compact then so is  $Y$ .*

PROOF:

(1)1. LET:  $b \in Y$

(1)2.  $\{U \text{ open in } X : \exists C \subseteq X \text{ compact. } U \subseteq C\}$  covers  $p^{-1}(b)$

(1)3. PICK a finite subcover  $\{U_1, \dots, U_n\}$

(1)4. For  $1 \leq i \leq n$ , PICK a compact  $C_i \subseteq X$  such that  $U_i \subseteq C_i$

(1)5. For  $1 \leq i \leq n$ , PICK a neighbourhood  $W_i$  of  $b$  such that  $p^{-1}(W_i) \subseteq U_i$

PROOF: By Proposition 9.5.1

(1)6.  $b \in W_1 \cup \dots \cup W_n \subseteq p(C_1) \cup \dots \cup p(C_n)$

(1)7.  $p(C_1) \cup \dots \cup p(C_n)$  is compact.

(2)1. Each  $p(C_i)$  is compact.

PROOF: By Proposition 9.4.10.

(2)2. Q.E.D.

PROOF: By Proposition 9.4.23.

□

## 9.6 Sequential Compactness

**Definition 9.6.1** (Sequentially Compact). A space is *sequentially compact* iff every sequence has a convergent subsequence.

**Proposition 9.6.2.**  $\overline{S_\Omega}$  is not sequentially compact.

PROOF:  $\Omega$  is a limit point of  $S_\Omega$  but is not the limit of any sequence of points in  $S_\Omega$ . □

## 9.7 Local Compactness

**Definition 9.7.1** (Local Compactness). Let  $X$  be a topological space.

For  $x \in X$ , the space  $X$  is *locally compact* at  $x$  iff there exists a compact subspace  $C \subseteq X$  that includes a neighbourhood of  $x$ .

The space  $X$  is *locally compact* iff it is locally compact at every point.

**Proposition 9.7.2.** *Every complete linearly ordered set is locally compact under the order topology.*

PROOF:

(1)1. LET:  $L$  be a complete linearly ordered set and  $x \in L$

PROVE: There exists a compact subspace  $C \subseteq L$  that includes a neighbourhood  $U$  of  $x$

(1)2. CASE:  $x$  is least and greatest in  $L$

- PROOF: In this case,  $L = \{x\}$  is compact.
- ⟨1⟩3. CASE:  $x$  is least in  $L$  but not greatest
    - ⟨2⟩1. PICK  $a < x$
    - ⟨2⟩2. Take  $C = [a, x]$  and  $U = (a, x]$
  - ⟨1⟩4. CASE:  $x$  is greatest in  $L$  but not least
 

PROOF: Similar.
  - ⟨1⟩5. CASE:  $x$  is neither least nor greatest
    - ⟨2⟩1. PICK  $a < x$  and  $b > x$
    - ⟨2⟩2. Take  $C = [a, b]$  and  $U = (a, b)$

□

**Corollary 9.7.2.1.** *For every ordinal  $\alpha$ , the space  $S_\alpha$  is locally compact.*

**Theorem 9.7.3.** *Every closed subspace of a locally compact Hausdorff space is locally compact.*

PROOF:

- ⟨1⟩1. LET:  $X$  be locally compact Hausdorff and  $C \subseteq X$  be closed.
- ⟨1⟩2. LET:  $x \in C$
- ⟨1⟩3. PICK  $D \subseteq X$  compact and  $U \subseteq D$  open such that  $x \in U$
- ⟨1⟩4.  $D$  is closed.
 

PROOF: Proposition 9.4.9.
- ⟨1⟩5.  $C \cap D$  is closed
 

PROOF: Proposition 3.5.5.
- ⟨1⟩6.  $C \cap D$  is compact
 

PROOF: Proposition 9.4.6.
- ⟨1⟩7. Q.E.D.
 

PROOF:  $C \cap D \subseteq C$  is compact and includes the open neighbourhood  $U \cap C$  of  $x$ .

□

**Proposition 9.7.4.** *Let  $\{X_\alpha\}_{\alpha \in J}$  be a family of nonempty topological spaces. If  $\prod_{\alpha \in J} X_\alpha$  is locally compact, then each  $X_\alpha$  is locally compact.*

PROOF:

- ⟨1⟩1. LET:  $\alpha \in J$  and  $x_\alpha \in X_\alpha$
- ⟨1⟩2. PICK  $x_\beta \in X_\beta$  for all  $\beta \in J \setminus \{\alpha\}$
- ⟨1⟩3. PICK a compact subspace  $C \subseteq \prod_{\alpha \in J} X_\alpha$  that a neighbourhood  $U$  of  $x$  included in  $C$
- ⟨1⟩4. PICK a basic open set  $\prod_{\alpha \in J} U_\alpha$  such that  $x \in \prod_{\alpha \in J} U_\alpha \subseteq U$
- ⟨1⟩5.  $x_\alpha \in U_\alpha \subseteq \pi_\alpha(C)$
- ⟨1⟩6.  $\pi_\alpha(C)$  is compact.
 

PROOF: By Proposition 9.4.10.

□

**Corollary 9.7.4.1.** *The Sorgenfrey plane is not locally compact.*

**Proposition 9.7.5.** *Let  $\{X_\alpha\}_{\alpha \in J}$  be a family of locally compact spaces such that  $X_\alpha$  is compact for all but finitely many values of  $\alpha$ . Then  $\prod_{\alpha \in J} X_\alpha$  is locally compact.*

PROOF:

- ⟨1⟩1. ASSUME:  $X_\alpha$  is compact if  $\alpha \neq \alpha_1, \dots, \alpha_n$
  - ⟨1⟩2. LET:  $\vec{x} \in \prod_{\alpha \in J} X_\alpha$
  - ⟨1⟩3. For  $1 \leq i \leq n$ , PICK  $C_{\alpha_i} \subseteq X_{\alpha_i}$  compact and  $U_{\alpha_i}$  open such that  $x_{\alpha_i} \in U_{\alpha_i} \subseteq C_{\alpha_i}$
  - ⟨1⟩4. For  $\alpha \neq \alpha_1, \dots, \alpha_n$ ,  
LET:  $C_\alpha = U_\alpha = X_\alpha$
  - ⟨1⟩5.  $\vec{x} \in \prod_{\alpha \in J} U_\alpha \subseteq \prod_{\alpha \in J} C_\alpha$
  - ⟨1⟩6.  $\prod_{\alpha \in J} C_\alpha$  is compact
- PROOF: By Tychonoff's Theorem.

□

**Proposition 9.7.6.**  $\mathbb{R}_l$  is not locally compact.

PROOF:  $[0, +\infty)$  can be partitioned into infinitely many disjoint open sets, which therefore do not have a finite subcover. □

**Proposition 9.7.7.** Let  $\{X_\alpha\}_{\alpha \in J}$  be a family of nonempty topological spaces. If  $\prod_{\alpha \in J} X_\alpha$  is locally compact, then all but finitely many of the  $X_\alpha$  are compact.

PROOF:

- ⟨1⟩1. PICK a point  $a = (a_\alpha)_{\alpha \in J} \in \prod_{\alpha \in J} X_\alpha$
- ⟨1⟩2. PICK a compact  $C \subseteq \prod_{\alpha \in J} X_\alpha$  that includes the basic neighbourhood  $\prod_{\alpha \in J} U_\alpha$  of  $a$ , where  $U_\alpha = X_\alpha$  for all  $\alpha$  except  $\alpha = \alpha_1, \dots, \alpha_n$
- ⟨1⟩3. For  $\alpha \neq \alpha_1, \dots, \alpha_n$ , we have  $X_\alpha$  is compact.

PROOF:  $X_\alpha$  is homeomorphic to a closed subspace of  $C$ .

□

**Corollary 9.7.7.1.** For any infinite set  $I$ , the space  $\mathbb{R}^I$  is not locally compact.

**Proposition 9.7.8.**  $[0, 1]^\omega$  is not compact under the uniform topology.

PROOF:  $\{a_i : i \geq 0\}$  is an infinite set with no limit point, where  $a_i$  is the point with  $i$ th component 1 and all other components 0. □

**Corollary 9.7.8.1.**  $\mathbb{R}^\omega$  under the uniform topology is not locally compact.

PROOF:

- ⟨1⟩1. ASSUME:  $\mathbb{R}^\omega$  is locally compact
- ⟨1⟩2. LET:  $C$  be a compact subspace such that  $B(\vec{0}, \epsilon) \subseteq C$
- ⟨1⟩3.  $\overline{B(\vec{0}, \epsilon)}$  is compact.
- ⟨1⟩4. Q.E.D.

PROOF: This contradicts the proposition.

□

**Proposition 9.7.9.** Not every subspace of a locally compact Hausdorff space is locally compact.

PROOF:  $\mathbb{R}$  is locally compact Hausdorff,  $\mathbb{Q}$  is not locally compact. □

**Proposition 9.7.10.** *The continuous image of a locally compact Hausdorff space is not necessarily locally compact.*

PROOF:

- ⟨1⟩1. LET:  $\{q_0, q_1, \dots\}$  be an enumeration of  $[0, 1] \cap \mathbb{Q}$ .
- ⟨1⟩2. Define  $f : (0, +\infty) \setminus \mathbb{Z} \rightarrow [0, 1] \cap \mathbb{Q}$  by:  $f(x) = q_n$  for  $x \in (n, n+1)$
- ⟨1⟩3.  $f$  is continuous.

PROOF: The inverse image of any set is a union of open intervals.

□

## 9.8 Compactifications

**Definition 9.8.1** (Compactification). Let  $X$  and  $Y$  be spaces. Then  $Y$  is a *compactification* of  $X$  iff  $Y$  is a compact Hausdorff space and  $X$  is a subspace of  $Y$  with  $\overline{X} = Y$ .

Two compactifications  $Y_1, Y_2$  of  $X$  are *equivalent* iff there exists a homeomorphism between  $Y_1$  and  $Y_2$  that is the identity on  $X$ .

**Lemma 9.8.2.** *Let  $h : X \rightarrow Z$  be an imbedding. Then there exists a compactification  $c : X \rightarrow Y$  of  $X$ , unique up to equivalence, and an imbedding  $i : Y \rightarrow Z$  such that  $h = i \circ c$ .*

PROOF: Simply take  $Y$  to be the closure of  $X$  in  $Z$ . □

**Definition 9.8.3** (One-Point Compactification). A *one-point compactification* of  $X$  is a compactification  $Y$  of  $X$  such that  $Y \setminus X$  is a singleton.

**Theorem 9.8.4.** *Let  $X$  be a topological space. Then  $X$  is locally compact Hausdorff if and only if there exists a space  $Y$  such that:*

1.  $X$  is a subspace of  $Y$
2. The set  $Y \setminus X$  is a singleton.
3.  $Y$  is a compact Hausdorff space.

*If  $Y$  and  $Y'$  are two spaces satisfying these conditions, then there exists a unique homeomorphism between  $Y$  and  $Y'$  that is the identity on  $X$ .*

PROOF:

- ⟨1⟩1. If  $X$  is locally compact Hausdorff then there exists a space  $Y$  satisfying 1–3.
- ⟨2⟩1. LET:  $Y = X \cup \{\infty\}$  under the topology  $\mathcal{T} = \{U \subseteq X : U \text{ is open in } X\} \cup \{Y \setminus C : C \subseteq X \text{ is compact}\}$ .
- ⟨3⟩1.  $Y \in \mathcal{T}$   
PROOF: This holds because  $Y = Y \setminus \emptyset$ .
- ⟨3⟩2. For all  $U, V \in \mathcal{T}$  we have  $U \cap V \in \mathcal{T}$ .
- ⟨4⟩1. LET:  $U, V \in \mathcal{T}$
- ⟨4⟩2. CASE:  $U, V$  are open in  $X$

- PROOF: In this case,  $U \cap V$  is open in  $X$ .
- ⟨4⟩3. CASE:  $U$  is open in  $X$ ,  $V = Y \setminus C$  where  $C \subseteq X$  is compact.
- ⟨5⟩1.  $U \cap V = U \setminus C$
- ⟨5⟩2.  $C$  is closed in  $X$
- PROOF: Proposition 9.4.9.
- ⟨5⟩3.  $U \cap V$  is open in  $X$
- ⟨4⟩4. CASE:  $U = Y \setminus C$  where  $C \subseteq X$  is compact,  $V$  is open in  $X$ .
- PROOF: Similar.
- ⟨4⟩5. CASE:  $U = Y \setminus C$ ,  $V = Y \setminus D$  where  $C, D \subseteq X$  are compact.
- ⟨5⟩1.  $U \cap V = Y \setminus (C \cup D)$
- ⟨5⟩2.  $C$  and  $D$  are closed in  $X$
- PROOF: Proposition 9.4.9.
- ⟨5⟩3.  $C \cup D$  is closed in  $X$
- PROOF: Proposition 3.5.4.
- ⟨5⟩4.  $C \cup D$  is compact.
- PROOF: By Proposition 9.4.23.  $\square$
- ⟨3⟩3. For all  $\mathcal{A} \subseteq \mathcal{T}$  we have  $\bigcup \mathcal{A} \in \mathcal{T}$ .
- ⟨4⟩1. LET:  $\mathcal{A} \subseteq \mathcal{T}$
- ⟨4⟩2. CASE: Every element of  $\mathcal{A}$  is an open set in  $X$ .
- PROOF: In this case,  $\bigcup \mathcal{A}$  is open in  $X$ .
- ⟨4⟩3. CASE: There exists  $C$  compact in  $X$  such that  $Y \setminus C \in \mathcal{A}$
- ⟨5⟩1.  $\bigcup \mathcal{A} = Y \setminus (\bigcap \{D \subseteq X : D \text{ compact}, Y \setminus D \in \mathcal{A}\} \setminus \bigcup \{U \text{ open in } X : U \in \mathcal{A}\})$
- PROOF: Set theory.
- ⟨5⟩2.  $\bigcap \{D \subseteq X : D \text{ compact}, Y \setminus D \in \mathcal{A}\} \setminus \bigcup \{U \text{ open in } X : U \in \mathcal{A}\}$  is compact.
- PROOF: It is a closed subset of the compact set  $C$ .
- ⟨2⟩2.  $X$  is a subspace of  $Y$
- ⟨3⟩1. For every open set  $U$  of  $X$ , there exists  $V$  open in  $Y$  such that  $U = V \cap X$
- PROOF: Take  $V = U$ .
- ⟨3⟩2. For every open set  $V$  in  $Y$ , we have  $V \cap X$  is open in  $X$ .
- ⟨4⟩1. LET:  $V$  be open in  $Y$
- ⟨4⟩2. CASE:  $V$  is open in  $X$
- PROOF: In this case,  $V \cap X = V$ .
- ⟨4⟩3. CASE:  $V = Y \setminus C$  where  $C \subseteq X$  is compact.
- ⟨5⟩1.  $C$  is closed in  $X$ .
- PROOF: By Proposition 9.4.9.
- ⟨5⟩2.  $V \cap X = X \setminus C$
- ⟨2⟩3.  $Y \setminus X = \{\infty\}$
- ⟨2⟩4.  $Y$  is compact.
- ⟨3⟩1. LET:  $\mathcal{A}$  be an open covering of  $Y$
- ⟨3⟩2. PICK  $U \in \mathcal{A}$  such that  $\infty \in U$
- ⟨3⟩3. PICK  $C \subseteq X$  compact such that  $U = Y \setminus C$ .
- ⟨3⟩4.  $\{V \cap X : V \in \mathcal{A}\}$  is set of open sets that covers  $C$
- ⟨3⟩5. PICK a finite subset  $\{V_1, \dots, V_n\}$  such that  $\{V_1 \cap X, \dots, V_n \cap X\}$

- covers  $C$ .
- ⟨3⟩6.  $\{U, V_1, \dots, V_n\}$  is a finite subcover of  $Y$ .
- ⟨2⟩5.  $Y$  is Hausdorff.
- ⟨3⟩1. LET:  $x, y \in Y$  with  $x \neq y$   
PROVE: There exist disjoint open neighbourhoods  $U, V$  of  $x$  and  $y$ .
- ⟨3⟩2. CASE:  $x, y \in X$   
PROOF: In this case, we just use the fact that  $X$  is Hausdorff.
- ⟨3⟩3. CASE:  $x = \infty, y \in X$   
⟨4⟩1. PICK  $C \subseteq X$  compact such that  $C$  includes an open neighbourhood  $V$  of  $y$   
⟨4⟩2. LET:  $U = Y \setminus C$
- ⟨3⟩4. CASE:  $x \in X, y = \infty$   
PROOF: Similar.
- ⟨1⟩2. If there exists a space  $Y$  satisfying 1–3 then  $X$  is locally compact Hausdorff.
- ⟨2⟩1. LET:  $Y$  be a space satisfying 1–3  
⟨2⟩2. LET:  $\infty$  be the point in  $Y \setminus X$   
⟨2⟩3.  $X$  is locally compact  
⟨3⟩1. LET:  $x \in X$   
⟨3⟩2. PICK disjoint open neighbourhoods  $U$  of  $x$  and  $V$  of  $\infty$   
⟨3⟩3.  $X \setminus V$  is compact and includes  $U$   
PROOF:  $X \setminus V = Y \setminus V$  is compact because it is a closed subset of  $Y$  (Proposition 9.4.6).  
⟨2⟩4.  $X$  is Hausdorff.  
PROOF: By Corollary 6.2.6.1.
- ⟨1⟩3. If  $Y$  and  $Y'$  are two spaces satisfying 1–3 then there exists a unique homeomorphism between  $Y$  and  $Y'$  that is the identity on  $X$ .
- ⟨2⟩1. LET:  $Y$  and  $Y'$  be two spaces that satisfy 1–3.  
⟨2⟩2. LET:  $Y \setminus X = \{p\}$  and  $Y' \setminus X = \{q\}$   
⟨2⟩3. LET:  $h : Y \rightarrow Y'$  be given by
- $$\begin{aligned} h(x) &= x & (x \in X) \\ h(p) &= q \end{aligned}$$
- ⟨2⟩4.  $h$  is a homeomorphism  
⟨3⟩1.  $h$  is bijective.  
⟨3⟩2.  $h$  is continuous.  
⟨4⟩1. LET:  $V \subseteq Y'$  be open.  
PROVE:  $h^{-1}(V)$  is open.  
⟨4⟩2. CASE:  $V \subseteq X$   
⟨5⟩1.  $h^{-1}(V) = V$   
⟨5⟩2.  $V$  is open in  $X$   
PROOF: Condition 1 for  $Y'$ .  
⟨5⟩3.  $V$  is open in  $Y$   
PROOF: Condition 1 for  $Y$ .  
⟨4⟩3. CASE:  $q \in V$   
⟨5⟩1.  $Y' \setminus V$  is compact.



PROOF: Proposition 9.4.6.

(5)2.  $Y' \setminus V$  is closed in  $Y$ .

PROOF: Proposition 9.4.9.

(5)3.  $h^{-1}(V) = Y \setminus (Y' \setminus V)$

(3)3.  $h^{-1}$  is continuous.

PROOF: Similar.

(2)5. If  $h' : Y \rightarrow Y'$  is a homeomorphism such that  $h' \upharpoonright_X = \text{id}_X$  then  $h' = h$

□

**Theorem 9.8.5.** *Let  $X$  be a Hausdorff space. Then  $X$  is locally compact if and only if, for all  $x \in X$  and any neighbourhood  $U$  of  $x$ , there exists an open neighbourhood  $V$  of  $x$  such that  $\overline{V}$  is compact and  $\overline{V} \subseteq U$ .*

PROOF:

(1)1. If  $X$  is locally compact then, for all  $x \in X$  and any neighbourhood  $U$  of  $x$ , there exists an open neighbourhood  $V$  of  $x$  such that  $\overline{V}$  is compact and  $\overline{V} \subseteq U$ .

(2)1. ASSUME:  $X$  is locally compact.

(2)2. LET:  $x \in X$  and  $U$  be a neighbourhood of  $x$ .

(2)3. LET:  $Y$  be the one-point compactification of  $X$ .

PROOF: By Theorem 9.8.4.

(2)4. LET:  $C = Y \setminus U$

(2)5.  $C$  is compact

PROOF: By Proposition 9.4.6.

(2)6. PICK disjoint open sets  $V, W$  containing  $x$  and  $C$

PROOF: Lemma 9.4.8

(2)7.  $V$  is open in  $X$

PROOF:  $V \subseteq X$  since  $\infty \in W$ .

(2)8. The closure of  $V$  in  $X$  is compact

(3)1. The closure of  $V$  in  $X$  is the same as the closure of  $V$  in  $Y$ .

PROOF: The point  $\infty$  cannot be a limit point of  $V$  since  $W$  is a neighbourhood disjoint from  $V$ .

(3)2. The closure of  $V$  in  $Y$  is compact.

PROOF: By Proposition 9.4.6.

(2)9.  $\overline{V} \subseteq U$

PROOF:

$$\begin{aligned}\overline{V} &\subseteq Y \setminus W \\ &\subseteq Y \setminus C \\ &= U\end{aligned}$$

(1)2. If, for all  $x \in X$  and any neighbourhood  $U$  of  $x$ , there exists an open neighbourhood  $V$  of  $x$  such that  $\overline{V}$  is compact and  $\overline{V} \subseteq U$ , then  $X$  is locally compact.

(2)1. ASSUME: for all  $x \in X$  and any neighbourhood  $U$  of  $x$ , there exists an open neighbourhood  $V$  of  $x$  such that  $\overline{V}$  is compact and  $\overline{V} \subseteq U$

(2)2. LET:  $x \in X$

PROVE: There exists  $C \subseteq X$  compact such that  $C$  includes a neighbourhood  $U$  of  $x$

(2)3. PICK an open neighbourhood  $V$  of  $x$  such that  $\bar{V}$  is compact and  $\bar{V} \subseteq X$

(2)4. Take  $C = \bar{V}$  and  $U = V$

□

**Corollary 9.8.5.1.** *Every open subspace of a locally compact Hausdorff space is locally compact.*

**Corollary 9.8.5.2.** *A space is locally compact Hausdorff if and only if it is an open subspace of a compact Hausdorff space.*

**Corollary 9.8.5.3.** *Every locally compact Hausdorff space is completely regular.*

**Corollary 9.8.5.4.** *The space  $\mathbb{R}_K$  is not locally compact.*

**Lemma 9.8.6 (AC).** *If  $p : X \rightarrow Y$  is a quotient map and  $Z$  is a locally compact Hausdorff space, then the map*

$$\pi = p \times \text{id}_Z : X \times Z \rightarrow Y \times Z$$

*is a quotient map.*

PROOF:

(1)1.  $\pi$  is surjective.

PROOF: This holds because  $p$  is surjective.

(1)2.  $\pi$  is continuous.

PROOF: By Theorem 5.2.15.

(1)3. For  $A \subseteq Y \times Z$ , if  $\pi^{-1}(A)$  is open in  $X \times Z$  then  $A$  is open in  $Y \times Z$ .

(2)1. LET:  $A \subseteq Y \times Z$

(2)2. ASSUME:  $\pi^{-1}(A)$  is open in  $X \times Z$

(2)3. LET:  $(y, z) \in A$

(2)4. PICK  $x \in X$  such that  $p(x) = y$

PROOF: Since  $p$  is surjective.

(2)5. PICK open sets  $U_1, V$  with  $\bar{V}$  compact such that  $(x, y) \in U_1 \times V$  and  $U_1 \times \bar{V} \subseteq \pi^{-1}(A)$

PROOF: Using Theorem 9.8.5

(2)6. PICK a sequence of open sets  $U_1, U_2, \dots$  in  $X$  such that  $p^{-1}(p(U_n)) \subseteq U_{n+1}$  and  $U_n \times \bar{V} \subseteq \pi^{-1}(A)$  for all  $n$

(3)1. LET:  $U$  be open with  $U \times \bar{V} \subseteq \pi^{-1}(A)$

PROVE: There exists  $W$  open with  $p^{-1}(p(U)) \subseteq W$  and  $W \times \bar{V} \subseteq \pi^{-1}(A)$

(3)2. For all  $x \in p^{-1}(p(U))$ , PICK open sets  $U_x, V_x$  such that  $x \in U_x$ ,  $\bar{V} \subseteq V_x$  and  $U_x \times V_x \subseteq \pi^{-1}(A)$

PROOF: By the Tube Lemma.

(3)3. LET:  $W = \bigcup_{x \in p^{-1}(p(U))} U_x$

(2)7. LET:  $U = \bigcup_{n=1}^{\infty} U_n$

(2)8.  $U$  is saturated with respect to  $p$

- ⟨3⟩1. LET:  $a \in U, b \in X, p(a) = p(b)$
- ⟨3⟩2. PICK  $n$  such that  $a \in U_n$
- ⟨3⟩3.  $b \in p^{-1}(p(U_n))$
- ⟨3⟩4.  $b \in U_{n+1}$
- ⟨3⟩5.  $b \in U$
- ⟨2⟩9.  $p(U)$  is open in  $Y$
- PROOF: By Lemma 4.5.2.
- ⟨2⟩10.  $(y, z) \in p(U) \times V \subseteq A$
- ⟨2⟩11. Q.E.D.
- PROOF: By Proposition 3.2.3.

□

**Theorem 9.8.7.** *Let  $p : A \rightarrow B$  and  $q : C \rightarrow D$  be quotient maps. If  $B$  and  $C$  are locally compact Hausdorff spaces, then  $p \times q : A \times C \rightarrow B \times D$  is a quotient map.*

PROOF: This holds by Lemma 9.8.6 and Proposition 4.5.10 because  $p \times q = (\text{id}_B \times q) \circ (p \times \text{id}_C)$ . □

**Theorem 9.8.8.** *Let  $X$  be a completely regular space. Let  $Y$  be a compactification of  $X$  such that every bounded continuous map  $X \rightarrow \mathbb{R}$  extends uniquely to a continuous map  $Y \rightarrow \mathbb{R}$ . Then, for every compact Hausdorff space  $C$ , every continuous map  $X \rightarrow C$  extends uniquely to a continuous map  $Y \rightarrow C$ .*

PROOF:

- ⟨1⟩1. LET:  $C$  be a compact Hausdorff space and  $f : X \rightarrow C$  a continuous function
- ⟨1⟩2. PICK a set  $J$  and an imedding  $C \subseteq [0, 1]^J$
- ⟨2⟩1.  $C$  is normal
- PROOF: By Lemma 9.4.18
- ⟨2⟩2. Q.E.D.
- PROOF: By Theorem 6.4.6.
- ⟨1⟩3. For  $\alpha \in J$ ,
- LET:  $g_\alpha : Y \rightarrow \mathbb{R}$  be the unique continuous extension of  $\pi_\alpha \circ f$
- ⟨1⟩4. Define  $g : Y \rightarrow \mathbb{R}^J$  by  $g(y)_\alpha = g_\alpha(y)$
- ⟨1⟩5.  $g$  is continuous
- PROOF: By Theorem 5.2.15.
- ⟨1⟩6.  $g$  extends  $f$
- ⟨1⟩7. We have  $g : Y \rightarrow C$

PROOF:

$$\begin{aligned}
 g(Y) &= g(\overline{X}) \\
 &\subseteq \overline{g(X)} && \text{(Theorem 5.2.2)} \\
 &= \overline{f(X)} && (\langle 1 \rangle 6) \\
 &\subseteq \overline{C} \\
 &= C && \text{(Proposition 9.4.9)}
 \end{aligned}$$

- ⟨1⟩8.  $g$  is unique

⟨2⟩1. LET:  $h : Y \rightarrow C$  be a continuous extension of  $f$

⟨2⟩2. For all  $\alpha \in J$ ,  $\pi_\alpha \circ h$  extends  $\pi_\alpha \circ f$

⟨2⟩3. For all  $\alpha \in J$ ,  $\pi_\alpha \circ h = g_\alpha$

PROOF: By ⟨1⟩3

⟨2⟩4.  $h = g$

PROOF: By ⟨1⟩4

□

**Corollary 9.8.8.1.** *Let  $X$  be a completely regular space. Let  $Y_1$  and  $Y_2$  be compactifications of  $X$  such that every bounded continuous map  $X \rightarrow \mathbb{R}$  extends uniquely to a continuous map  $Y_i \rightarrow \mathbb{R}$ . Then  $Y_1$  and  $Y_2$  are equivalent.*

**Definition 9.8.9** (Stone-Čech Compactification). Let  $X$  be a completely regular space. The *Stone-Čech compactification* of  $X$ ,  $\beta(X)$ , is the compactification of  $X$  such that, for every compact Hausdorff space  $C$ , every continuous function  $X \rightarrow C$  extends uniquely to a continuous function  $\beta(X) \rightarrow C$ .

## Chapter 10

# Metric Spaces

### 10.1 The Metric Topology

**Definition 10.1.1** (Metric). A *metric* on a set  $X$  is a function  $d : X \times X \rightarrow \mathbb{R}$  such that, for all  $x, y, z \in X$ :

1.  $d(x, y) \geq 0$ ;
2.  $d(x, y) = 0$  if and only if  $x = y$ ;
3.  $d(x, y) = d(y, x)$ ;
4. **Triangle Inequality**

$$d(x, z) \leq d(x, y) + d(y, z)$$

A *metric space*  $X$  consists of a set  $X$  and a metric on  $X$ . We call  $d(x, y)$  the *distance* between  $x$  and  $y$ .

**Definition 10.1.2** (Open Ball). Let  $X$  be a metric space with metric  $d$ ,  $x \in X$  and  $\epsilon > 0$ . The *open ball* with *centre*  $x$  and *radius*  $\epsilon$  is

$$B_d(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\} .$$

**Lemma 10.1.3.** Let  $X$  be a metric space,  $x, y \in X$  and  $\epsilon > 0$ . If  $y \in B(x, \epsilon)$ , then there exists  $\delta$  such that  $0 < \delta < \epsilon$  and

$$B(y, \delta) \subseteq B(x, \epsilon) .$$

PROOF:

- ⟨1⟩1. LET:  $\delta = \epsilon - d(x, y)$   
⟨1⟩2. LET:  $z \in B(y, \delta)$   
⟨1⟩3.  $d(x, z) < \epsilon$

PROOF:

$$\begin{aligned}
 d(x, z) &\leq d(x, y) + d(y, z) && \text{(Triangle Inequality)} \\
 &< d(x, y) + \delta && ((1)2) \\
 &= \epsilon && ((1)1)
 \end{aligned}$$

□

**Definition 10.1.4** (Metric Topology). Let  $d$  be a metric on  $X$ . The *metric topology* on  $X$  induced by  $d$  is the topology generated by the basis consisting of the open balls.

We prove this is a topology.

PROOF:

⟨1⟩1. Every point is in an open ball.

PROOF:  $x \in B(x, 1)$

⟨1⟩2. If  $B_1, B_2$  are open balls and  $x \in B_1 \cap B_2$ , then there exists an open ball  $B_3$  such that  $x \in B_3 \subseteq B_1 \cap B_2$ .

⟨2⟩1. LET:  $x \in B(y, \epsilon_1) \cap B(z, \epsilon_2)$

⟨2⟩2. PICK  $\delta_1, \delta_2$  such that  $0 < \delta_1 < \epsilon_1, 0 < \delta_2 < \epsilon_2, B(x, \delta_1) \subseteq B(y, \epsilon_1)$  and  $B(x, \delta_2) \subseteq B(z, \epsilon_2)$ .

PROOF: Lemma 10.1.3.

⟨2⟩3. LET:  $\delta = \min(\delta_1, \delta_2)$

⟨2⟩4.  $x \in B(x, \delta) \subseteq B(y, \epsilon_1) \cap B(z, \epsilon_2)$

⟨1⟩3. Q.E.D.

PROOF: Lemma 3.4.3.

**Lemma 10.1.5.** A set  $U$  is open in the metric topology induced by  $d$  if and only if, for all  $x \in U$ , there exists  $\epsilon > 0$  such that  $B(x, \epsilon) \subseteq U$ .

PROOF:

⟨1⟩1. If  $U$  is open then, for all  $x \in U$ , there exists  $\epsilon > 0$  such that  $B(x, \epsilon) \subseteq U$ .

⟨2⟩1. ASSUME:  $U$  is open.

⟨2⟩2. LET:  $x \in U$

⟨2⟩3. PICK  $B(y, \delta)$  such that  $x \in B(y, \delta) \subseteq U$

⟨2⟩4. PICK  $\epsilon$  such that  $0 < \epsilon < \delta$  and  $B(x, \epsilon) \subseteq B(y, \delta)$

PROOF: Lemma 10.1.3.

⟨2⟩5.  $B(x, \epsilon) \subseteq U$

PROOF: From ⟨2⟩3 and ⟨2⟩4.

⟨1⟩2. If, for all  $x \in U$ , there exists  $\epsilon > 0$  such that  $B(x, \epsilon) \subseteq U$ , then  $U$  is open.

PROOF: Immediate from definition of metric topology.

□

**Lemma 10.1.6.** Let  $d$  and  $d'$  be two metrics on the set  $X$ . Let  $\mathcal{T}$  and  $\mathcal{T}'$  be the topologies they induce, respectively. Then  $\mathcal{T} \subseteq \mathcal{T}'$  if and only if, for all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$ .

PROOF:

⟨1⟩1. If  $\mathcal{T} \subseteq \mathcal{T}'$  then, for all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$ .

(2)1. ASSUME:  $\mathcal{T} \subseteq \mathcal{T}'$   
 (2)2. LET:  $x \in X$  and  $\epsilon > 0$   
 (2)3.  $B_d(x, \epsilon) \in \mathcal{T}'$   
 PROOF: From (2)1.  
 (2)4. There exists  $\delta > 0$  such that  $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$   
 PROOF: By Lemma 10.1.5.  
 (1)2. If, for all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$ , then  $\mathcal{T} \subseteq \mathcal{T}'$   
 (2)1. ASSUME: For all  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$ .  
 (2)2. LET:  $U \in \mathcal{T}$   
 PROVE:  $U \in \mathcal{T}'$   
 (2)3. LET:  $x \in U$   
 (2)4. PICK  $\epsilon > 0$  be such that  $B_d(x, \epsilon) \subseteq U$   
 PROOF: By Lemma 10.1.5.  
 (2)5. PICK  $\delta > 0$  such that  $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$   
 PROOF: By (2)1.  
 (2)6.  $B_{d'}(x, \delta) \subseteq U$   
 PROOF: By (2)4 and (2)5.  
 (2)7. Q.E.D.  
 PROOF: By Lemma 10.1.5.

□

**Definition 10.1.7** (Metriizable). A topological space is *metrizable* if and only if there exists a metric that induces its topology.

**Lemma 10.1.8.** *Every discrete space is metrizable.*

PROOF: The discrete topology is induced by the metric  $d(x, y) = 1$  if  $x \neq y$ , 0 if  $x = y$ . □

**Proposition 10.1.9.** *The continuous image of a metrizable space is not necessarily metrizable.*

PROOF: The identity map from the discrete two-point space to the indiscrete two-point space is continuous. □

**Lemma 10.1.10.**  $\mathbb{R}$  is metrizable.

PROOF: The standard topology is induced by the metric  $d(x, y) = |x - y|$ . □

**Definition 10.1.11** (Bounded). Let  $X$  be a metric space and  $A \subseteq X$ . Then  $A$  is *bounded* iff  $\{d(x, y) : x, y \in A\}$  is bounded above, in which case its *diameter* is

$$\text{diam } A = \sup_{x, y \in A} d(x, y) .$$

**Lemma 10.1.12.** *Let  $(X, d)$  be a metric space and  $A \subseteq X$ . Then  $d \upharpoonright_{A \times A}$  is a metric on  $A$  that induces the subspace topology.*

PROOF:

⟨1⟩1.  $d \upharpoonright_{A \times A}$  is a metric on  $A$ .

PROOF: Each of the axioms for a metric follows immediately from the same axiom for  $d$ .

⟨1⟩2. The topology induced by  $d \upharpoonright_{A \times A}$  is the product topology.

PROOF: Both are the topology generated by the basis consisting of all the open balls  $B_{d \upharpoonright_{A \times A}}(a, \epsilon) = B_d(a, \epsilon) \cap A$ .

□

**Lemma 10.1.13.** *Every metric space is Hausdorff.*

PROOF:

⟨1⟩1. LET:  $X$  be a metric space and  $x, y \in X$  with  $x \neq y$ .

⟨1⟩2. LET:  $\epsilon = d(x, y)$

⟨1⟩3.  $B(x, \epsilon/2)$  and  $B(y, \epsilon/2)$  are disjoint neighbourhoods of  $x$  and  $y$ .

□

**Theorem 10.1.14.** *Every metric space is first countable.*

PROOF:  $\{B(x, q) : q \in \mathbb{Q}^+\}$  is a local basis at  $x$ . □

**Corollary 10.1.14.1.** *If  $J$  is infinite then the space  $\mathbb{R}^J$  is not metrizable.*

**Definition 10.1.15** (Standard Bounded Metric). Let  $d$  be a metric on  $X$ . The *standard bounded metric* corresponding to  $d$  is

$$\bar{d}(x, y) = \min(d(x, y), 1) .$$

We prove this is a metric.

PROOF:

⟨1⟩1.  $\bar{d}(x, y) \geq 0$

PROOF: This holds because  $d(x, y) \geq 0$  ( $d$  is a metric) and  $1 > 0$ .

⟨1⟩2.  $\bar{d}(x, y) = 0$  iff  $x = y$

PROOF: Immediate from definition.

⟨1⟩3.  $\bar{d}(x, y) = \bar{d}(y, x)$

PROOF: Immediate from definition.

⟨1⟩4.  $\bar{d}(x, z) \leq \bar{d}(x, y) + \bar{d}(y, z)$

⟨2⟩1. CASE:  $d(x, y) \leq 1, d(y, z) \leq 1$

PROOF:

$$\begin{aligned} \bar{d}(x, z) &\leq d(x, z) \\ &\leq d(x, y) + d(y, z) \\ &= \bar{d}(x, y) + \bar{d}(y, z) \end{aligned}$$

⟨2⟩2. CASE:  $d(y, z) > 1$

PROOF:

$$\begin{aligned} \bar{d}(x, z) &\leq 1 \\ &\leq \bar{d}(x, y) + 1 \\ &= \bar{d}(x, y) + \bar{d}(y, z) \end{aligned}$$



⟨2⟩3. CASE:  $d(x, y) > 1$

PROOF: Similar.

□

**Theorem 10.1.16.** *Let  $d$  be a metric on  $X$ . Then the standard bounded metric  $\bar{d}$  corresponding to  $d$  induces the same topology as  $d$ .*

PROOF:

⟨1⟩1. LET:  $\mathcal{T}$  be the topology induced by  $d$  and  $\mathcal{T}'$  be the topology induced by  $\bar{d}$ .

⟨1⟩2.  $\mathcal{T} \subseteq \mathcal{T}'$

⟨2⟩1. LET:  $x \in X$  and  $\epsilon > 0$

⟨2⟩2. LET:  $\delta = \min(\epsilon, 1/2)$

⟨2⟩3.  $B_{\bar{d}}(x, \delta) \subseteq B_d(x, \epsilon)$

⟨3⟩1. LET:  $y \in B_{\bar{d}}(x, \delta)$

⟨3⟩2.  $\bar{d}(x, y) < \delta$

⟨3⟩3.  $\bar{d}(x, y) < 1$

PROOF: From ⟨2⟩2 and ⟨3⟩2.

⟨3⟩4.  $\bar{d}(x, y) = d(x, y)$

PROOF: From ⟨3⟩3 and the definition of  $\bar{d}$ .

⟨3⟩5.  $d(x, y) < \epsilon$

PROOF: By ⟨2⟩2 and ⟨3⟩2 and ⟨3⟩4.

⟨1⟩3.  $\mathcal{T}' \subseteq \mathcal{T}$

⟨2⟩1. LET:  $x \in X$  and  $\epsilon > 0$

⟨2⟩2.  $B_d(x, \epsilon) \subseteq B_{\bar{d}}(x, \epsilon)$

PROOF: This holds because  $\bar{d}(x, y) \leq d(x, y)$ .

□

**Definition 10.1.17** (Square Metric). The *square metric* on  $\mathbb{R}^n$  is defined by

$$\rho(\vec{x}, \vec{y}) = \max(|x_1 - y_1|, \dots, |x_n - y_n|) .$$

We prove this is a metric.

PROOF:

⟨1⟩1.  $\rho(\vec{x}, \vec{y}) \geq 0$

PROOF: Immediate from definitions.

⟨1⟩2.  $\rho(\vec{x}, \vec{y}) = 0$  iff  $\vec{x} = \vec{y}$

PROOF: Immediate from definitions.

⟨1⟩3.  $\rho(\vec{x}, \vec{y}) = \rho(\vec{y}, \vec{x})$

PROOF: Immediate from definitions.

⟨1⟩4.  $\rho(\vec{x}, \vec{z}) \leq \rho(\vec{x}, \vec{y}) + \rho(\vec{y}, \vec{z})$

⟨2⟩1. For all  $i$ , we have  $|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$

⟨2⟩2. For all  $i$ ,  $|x_i - z_i| \leq \rho(\vec{x}, \vec{y}) + \rho(\vec{y}, \vec{z})$

⟨2⟩3.  $\rho(\vec{x}, \vec{z}) \leq \rho(\vec{x}, \vec{y}) + \rho(\vec{y}, \vec{z})$

□

**Theorem 10.1.18.** *The square metric induces the standard topology on  $\mathbb{R}^n$ .*

PROOF:

⟨1⟩1. LET:  $\mathcal{T}_\rho$  be the topology induced by the square metric and  $\mathcal{T}_s$  the standard topology.

⟨1⟩2.  $\mathcal{T}_\rho \subseteq \mathcal{T}_s$

PROOF: This holds because  $B_\rho(\vec{x}, \epsilon) = (x_1 - \epsilon, x_1 + \epsilon) \times \cdots \times (x_n - \epsilon, x_n + \epsilon)$ .

⟨1⟩3.  $\mathcal{T}_s \subseteq \mathcal{T}_\rho$

⟨2⟩1. LET:  $B = U_1 \times \cdots \times U_n$  be a basic open set in  $\mathcal{T}_s$ , where each  $U_i$  is open in  $\mathbb{R}$ .

⟨2⟩2. LET:  $\vec{x} \in B$

⟨2⟩3. For  $1 \leq i \leq n$ , PICK  $\epsilon_i > 0$  such that  $(x_i - \epsilon_i, x_i + \epsilon_i) \subseteq U_i$

⟨2⟩4. LET:  $\epsilon = \min(\epsilon_1, \dots, \epsilon_n)$

⟨2⟩5.  $B_\rho(\vec{x}, \epsilon) \subseteq B$

□

**Lemma 10.1.19.** *The product of a countable family of metrizable spaces is metrizable.*

PROOF:

⟨1⟩1. LET:  $\{X_n\}_{n \in \mathbb{Z}^+}$  be a family of metric spaces with metrics bounded by 1,  
 $X = \prod_{n=1}^{\infty} X_n$ .

⟨1⟩2. LET:  $D : X \times X \rightarrow \mathbb{R}$  be given by

$$D(\vec{x}, \vec{y}) = \sup_{n \geq 1} \frac{d(x_n, y_n)}{n}.$$

⟨1⟩3.  $D$  is a metric on  $X$ .

⟨2⟩1.  $D(\vec{x}, \vec{y}) \geq 0$

PROOF: Immediate from definitions.

⟨2⟩2.  $D(\vec{x}, \vec{y}) = 0$  iff  $\vec{x} = \vec{y}$

PROOF: Immediate from definitions.

⟨2⟩3.  $D(\vec{x}, \vec{y}) = D(\vec{y}, \vec{x})$

PROOF: Immediate from definitions.

⟨2⟩4.  $D(\vec{x}, \vec{z}) \leq D(\vec{x}, \vec{y}) + D(\vec{y}, \vec{z})$

⟨3⟩1. For all  $n$ , we have  $\frac{d(x_n, z_n)}{n} \leq \frac{d(x_n, y_n)}{n} + \frac{d(y_n, z_n)}{n}$

⟨3⟩2. For all  $n$ , we have  $\frac{d(x_n, z_n)}{n} \leq D(\vec{x}, \vec{y}) + D(\vec{y}, \vec{z})$

⟨3⟩3.  $D(\vec{x}, \vec{z}) \leq D(\vec{x}, \vec{y}) + D(\vec{y}, \vec{z})$

⟨1⟩4. LET:  $\mathcal{T}_D$  be the topology induced by  $D$  and  $\mathcal{T}_p$  the product topology.

⟨1⟩5.  $\mathcal{T}_D \subseteq \mathcal{T}_p$

⟨2⟩1. LET:  $U \in \mathcal{T}_D$

PROVE:  $U \in \mathcal{T}_p$

⟨2⟩2. LET:  $\vec{x} \in U$

⟨2⟩3. PICK  $\epsilon > 0$  such that  $B_D(\vec{x}, \epsilon) \subseteq U$

⟨2⟩4. PICK  $N$  such that  $1/N < \epsilon$

⟨2⟩5. LET:  $V = B(x_1, \epsilon) \times \cdots \times B(x_N, \epsilon) \times X_{N+1} \times X_{N+2} \times \cdots$

⟨2⟩6.  $\vec{x} \in V \subseteq B_D(\vec{x}, \epsilon)$

⟨1⟩6.  $\mathcal{T}_p \subseteq \mathcal{T}_D$

⟨2⟩1. LET:  $U = \prod_{n=1}^{\infty} U_n$  be a basic open set in  $\mathcal{T}_p$ , where each  $U_n$  is open in  $X_n$ , and  $U_n = X_n$  for  $n > N$ .

- (2)2. LET:  $\vec{x} \in U$   
 PROVE: There exists  $\epsilon > 0$  such that  $B_D(\vec{x}, \epsilon) \subseteq U$ .  
 (2)3. For  $n \leq N$ , PICK  $\epsilon_n > 0$  such that  $B(x_n, \epsilon_n) \subseteq U_n$   
 (2)4. LET:  $\epsilon = \min(\epsilon_1, \epsilon_2/2, \dots, \epsilon_n/n)$   
 (2)5. LET:  $\vec{y} \in B_D(\vec{x}, \epsilon)$   
 (2)6. For  $n \leq N$ ,  $y_n \in U_n$   
 (3)1.  $D(\vec{x}, \vec{y}) < \epsilon$   
 (3)2.  $d(x_n, y_n)/n < \epsilon$   
 (3)3.  $d(x_n, y_n)/n < \epsilon_n/n$   
 (3)4. Q.E.D.  
 PROOF: By (2)3.

□

**Corollary 10.1.19.1.** *The space  $\mathbb{R}^\omega$  is metrizable.*

**Definition 10.1.20** (Uniform Metric). Let  $J$  be a set. The *uniform metric*  $\bar{\rho}$  on  $\mathbb{R}^J$  is defined by

$$\bar{\rho}(\vec{x}, \vec{y}) = \sup_{\alpha \in J} \bar{d}(x_\alpha, y_\alpha) .$$

where  $\bar{d}$  is the standard bounded metric on  $\mathbb{R}$ . The *uniform topology* is the topology induced by the uniform metric.

We prove this is a metric.

PROOF:

- (1)1.  $\bar{\rho}(\vec{x}, \vec{y}) \geq 0$   
 PROOF: Immediate from definitions.  
 (1)2.  $\bar{\rho}(\vec{x}, \vec{y}) = 0$  iff  $\vec{x} = \vec{y}$   
 PROOF: Immediate from definitions.  
 (1)3.  $\bar{\rho}(\vec{x}, \vec{y}) = \bar{\rho}(\vec{y}, \vec{x})$   
 PROOF: Immediate from definitions.  
 (1)4.  $\bar{\rho}(\vec{x}, \vec{z}) \leq \bar{\rho}(\vec{x}, \vec{y}) + \bar{\rho}(\vec{y}, \vec{z})$   
 PROOF:  
 (2)1. For all  $\alpha \in J$ ,  $\bar{d}(x_\alpha, z_\alpha) \leq \bar{d}(x_\alpha, y_\alpha) + \bar{d}(y_\alpha, z_\alpha)$   
 (2)2. For all  $\alpha \in J$ ,  $\bar{d}(x_\alpha, z_\alpha) \leq \bar{\rho}(\vec{x}, \vec{y}) + \bar{\rho}(\vec{y}, \vec{z})$   
 (2)3.  $\bar{\rho}(\vec{x}, \vec{z}) \leq \bar{\rho}(\vec{x}, \vec{y}) + \bar{\rho}(\vec{y}, \vec{z})$

□

**Theorem 10.1.21** (DC). *The uniform topology on  $\mathbb{R}^J$  is finer than the product topology and coarser than the box topology. These three topologies are different iff  $J$  is infinite.*

PROOF:

- (1)1. The uniform topology is finer than the product topology.  
 (2)1. LET:  $B = \prod_{\alpha \in J} U_\alpha$  be a basic open set in the product topology, where each  $U_\alpha$  is open in  $\mathbb{R}$ , and  $U_\alpha = \mathbb{R}$  except for  $\alpha = \alpha_1, \dots, \alpha_n$ .  
 (2)2. LET:  $\vec{x} \in U$   
 (2)3. For  $1 \leq i \leq n$ , PICK  $0 < \epsilon_i < 1$  such that  $(x_{\alpha_i} - \epsilon_i, x_{\alpha_i} + \epsilon_i) \subseteq U_{\alpha_i}$ .



PROOF: The set of all sequences of 0s and 1s is discrete but uncountable.  $\square$

**Corollary 10.1.22.1.** *Not every metric space is second countable.*

**Theorem 10.1.23.** *Let  $X$  and  $Y$  be metric spaces. Let  $f : X \rightarrow Y$  and  $x \in X$ . Then  $f$  is continuous at  $x$  if and only if, for every  $\epsilon > 0$ , there exists  $\delta > 0$  such that, for all  $x' \in X$ , if  $d(x, x') < \delta$  then  $d(f(x), f(x')) < \epsilon$ .*

PROOF:

$\langle 1 \rangle 1$ . If  $f$  is continuous at  $x$  then, for every  $\epsilon > 0$ , there exists  $\delta > 0$  such that, for all  $x' \in X$ , if  $d(x, x') < \delta$  then  $d(f(x), f(x')) < \epsilon$ .

$\langle 2 \rangle 1$ . ASSUME:  $f$  is continuous at  $x$ .

$\langle 2 \rangle 2$ . LET:  $\epsilon > 0$

$\langle 2 \rangle 3$ . PICK a neighbourhood  $U$  of  $x$  such that  $f(U) \subseteq B(f(x), \epsilon)$

PROOF: One exists by  $\langle 2 \rangle 1$ , since  $B(f(x), \epsilon)$  is a neighbourhood of  $f(x)$ .

$\langle 2 \rangle 4$ . PICK  $\delta > 0$  such that  $B(x, \delta) \subseteq U$

PROOF: By  $\langle 2 \rangle 3$  and Lemma 10.1.5.

$\langle 2 \rangle 5$ . LET:  $x' \in X$  with  $d(x, x') < \delta$

$\langle 2 \rangle 6$ .  $x' \in U$

PROOF: From  $\langle 2 \rangle 4$  and  $\langle 2 \rangle 5$ .

$\langle 2 \rangle 7$ .  $f(x') \in B(f(x), \epsilon)$

PROOF: From  $\langle 2 \rangle 3$  and  $\langle 2 \rangle 6$ .

$\langle 1 \rangle 2$ . If, for all  $\epsilon > 0$ , there exists  $\delta > 0$  such that, for all  $x' \in X$ , if  $d(x, x') < \delta$  then  $d(f(x), f(x')) < \epsilon$ , then  $f$  is continuous at  $x$ .

$\langle 2 \rangle 1$ . ASSUME: For all  $\epsilon > 0$  there exists  $\delta > 0$  such that, for all  $x' \in X$ , if  $d(x, x') < \delta$  then  $d(f(x), f(x')) < \epsilon$ .

$\langle 2 \rangle 2$ . LET:  $V$  be a neighbourhood of  $f(x)$

$\langle 2 \rangle 3$ . PICK  $\epsilon > 0$  such that  $B(f(x), \epsilon) \subseteq V$

PROOF: By Lemma 10.1.5.

$\langle 2 \rangle 4$ . PICK  $\delta > 0$  such that, for all  $x' \in X$ , if  $d(x, x') < \delta$  then  $d(f(x), f(x')) < \epsilon$ .

PROOF: By  $\langle 2 \rangle 1$  and  $\langle 2 \rangle 3$ .

$\langle 2 \rangle 5$ .  $B(x, \delta)$  is a neighbourhood of  $x$

PROOF: By the definition of the metric topology.

$\langle 2 \rangle 6$ .  $f(B(x, \delta)) \subseteq V$

$\langle 3 \rangle 1$ . LET:  $x' \in B(x, \delta)$

$\langle 3 \rangle 2$ .  $d(f(x), f(x')) < \epsilon$

PROOF: From  $\langle 2 \rangle 4$ .

$\langle 3 \rangle 3$ .  $x' \in V$

PROOF: From  $\langle 2 \rangle 3$ .

$\square$

**Lemma 10.1.24.** *Addition is a continuous function  $\mathbb{R}^2 \rightarrow \mathbb{R}$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $(x, y) \in \mathbb{R}^2$  and  $\epsilon > 0$

$\langle 1 \rangle 2$ . LET:  $\delta = \epsilon/2$

⟨1⟩3. LET:  $(x', y') \in \mathbb{R}^2$  be such that  $\rho((x, y), (x', y')) < \delta$ , where  $\rho$  is the square metric

⟨1⟩4.  $|x - x'| < \delta$  and  $|y - y'| < \delta$

⟨1⟩5.  $|(x + y) - (x' + y')| < \epsilon$

PROOF:

$$\begin{aligned} |(x + y) - (x' + y')| &\leq |x - x'| + |y - y'| \\ &< 2\delta && \langle 1 \rangle 4 \\ &= \epsilon && \langle 1 \rangle 2 \end{aligned}$$

⟨1⟩6. Q.E.D.

PROOF: By Theorem 10.1.23.

□

**Lemma 10.1.25.** *Additive inverse is a continuous function  $- : \mathbb{R} \rightarrow \mathbb{R}$ .*

PROOF: If  $|x - y| < \epsilon$  then  $|(-x) - (-y)| < \epsilon$ . □

**Lemma 10.1.26.** *Multiplication is a continuous function  $\cdot : \mathbb{R}^2 \rightarrow \mathbb{R}$ .*

PROOF:

⟨1⟩1. LET:  $(x, y) \in \mathbb{R}^2$  and  $\epsilon > 0$

⟨1⟩2. LET:  $\delta = \min(1, \epsilon/(|x| + |y| + 1))$

⟨1⟩3. LET:  $(x', y') \in \mathbb{R}^2$  and  $\rho((x, y), (x', y')) < \delta$

⟨1⟩4.  $|xy - x'y'| < \epsilon$

PROOF:

$$\begin{aligned} |xy - x'y'| &= |x(y' - y) + y(x' - x) + (x - x')(y - y')| \\ &\leq |x||y' - y| + |y||x' - x| + |x - x'||y - y'| \\ &< |x|\delta + |y|\delta + \delta^2 && \langle 1 \rangle 3 \\ &= \delta(|x| + |y| + \delta) \\ &\leq \delta(|x| + |y| + 1) && \langle 1 \rangle 2 \\ &\leq \epsilon && \langle 1 \rangle 2 \end{aligned}$$

□

**Lemma 10.1.27.** *Multiplicative inverse is a continuous function  $(\ )^{-1} : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ .*

PROOF:

⟨1⟩1. LET:  $f : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$  be defined by  $f(x) = x^{-1}$ .

⟨1⟩2. LET:  $a, b \in \mathbb{R}$  with  $a < b$

PROVE:  $f^{-1}((a, b))$  is open

⟨1⟩3. CASE:  $0 < a < b$

PROOF:  $f^{-1}((a, b)) = (b^{-1}, a^{-1})$

⟨1⟩4. CASE:  $a < 0 < b$

PROOF:  $f^{-1}((a, b)) = (-\infty, a^{-1}) \cup (b^{-1}, +\infty)$

⟨1⟩5. CASE:  $a < b < 0$

PROOF:  $f^{-1}((a, b)) = (b^{-1}, a^{-1})$

□

**Definition 10.1.28** (Uniform Convergence). Let  $X$  be a set and  $Y$  a metric space. Let  $f_n : X \rightarrow Y$  for  $n \geq 1$ , and  $f : X \rightarrow Y$ . Then  $f_n$  converges uniformly to  $f$  as  $n \rightarrow \infty$  iff, for all  $\epsilon > 0$ , there exists  $N$  such that, for all  $x \in X$  and  $n \geq N$ ,  $d(f_n(x), f(x)) < \epsilon$ .

**Theorem 10.1.29** (Uniform Limit Theorem). Let  $X$  be a topological space and  $Y$  a metric space. Let  $f_n : X \rightarrow Y$  for  $n \geq 1$  and  $f : X \rightarrow Y$ . If  $f_n$  converges uniformly to  $f$  and each  $f_n$  is continuous, then  $f$  is continuous.

PROOF:

- (1)1. LET:  $x \in X$  and  $\epsilon > 0$
- (1)2. PICK  $N$  such that, for all  $x' \in X$  and  $\delta > 0$ ,  $d(f_n(x'), f(x')) < \epsilon/3$
- (1)3. PICK  $\delta > 0$  such that, for all  $x' \in X$ , if  $d(x, x') < \delta$  then  $d(f_N(x), f_N(x')) < \epsilon/3$
- (1)4. For all  $x' \in X$ , if  $d(x, x') < \delta$  then  $d(f(x), f(x')) < \epsilon$

PROOF:

$$\begin{aligned} d(f(x), f(x')) &\leq d(f(x), f_N(x)) + d(f_N(x), f_N(x')) + d(f_N(x'), f(x')) \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 \\ &= \epsilon \end{aligned}$$

□

**Lemma 10.1.30.** Let  $X$  be a set. Let  $f_n : X \rightarrow \mathbb{R}$  for  $n \geq 1$  and  $f : X \rightarrow \mathbb{R}$ . Then  $f_n$  converges uniformly to  $f$  if and only if  $f_n$  converges to  $f$  in  $\mathbb{R}^X$  under the uniform topology.

PROOF:

- (1)1. If  $f_n$  converges uniformly to  $f$  then  $f_n$  converges to  $f$  under the uniform topology.
  - (2)1. ASSUME:  $f_n$  converges uniformly to  $f$
  - (2)2. LET:  $\epsilon > 0$
  - (2)3. PICK  $N$  such that, for all  $x \in X$  and  $n \geq N$ ,  $d(f_n(x), f(x)) < \epsilon/2$
  - (2)4.  $\bar{\rho}(f_n, f) \leq \epsilon/2$
  - (2)5.  $\bar{\rho}(f_n, f) < \epsilon$
- (1)2. If  $f_n$  converges to  $f$  under the uniform topology then  $f_n$  converges uniformly to  $f$ .
  - (2)1. ASSUME:  $f_n$  converges to  $f$  under the uniform topology.
  - (2)2. LET:  $\epsilon > 0$
  - (2)3. PICK  $N$  such that, for all  $n \geq N$ ,  $\bar{\rho}(f_n, f) < \epsilon$
  - (2)4. For all  $n \geq N$  and  $x \in X$ ,  $d(f_n(x), f(x)) < \epsilon$

□

**Theorem 10.1.31.** Every monotone increasing sequence of real numbers that is bounded above converges to its supremum.

PROOF:

- (1)1. LET:  $\{s_n\}_{n \geq 1}$  be a monotone increasing sequence of real numbers bounded above with supremum  $l$ .

- (1)2. LET:  $\epsilon > 0$   
 (1)3.  $l - \epsilon$  is not an upper bound for  $\{s_n : n \geq 1\}$ .  
 (1)4. PICK  $N$  such that  $x_N > l - \epsilon$   
 (1)5. For all  $n \geq N$ , we have  $l - \epsilon < x_n \leq l$   
 (1)6. For all  $n \geq N$ , we have  $|x_n - l| < \epsilon$   
 $\square$

**Definition 10.1.32** (Infinite Series). Let  $\{a_n\}_{n \geq 1}$  be a sequence of real numbers. The *infinite series*  $\sum_{n=1}^{\infty} a_n$  *converges* to  $s$  iff  $\sum_{n=1}^N a_n \rightarrow s$  as  $N \rightarrow \infty$ .

**Proposition 10.1.33.** If  $\sum_{n=1}^{\infty} a_n = s$  and  $\sum_{n=1}^{\infty} b_n = t$  then  $\sum_{n=1}^{\infty} (ca_n + b_n) = cs + t$ .

PROOF: This holds because  $\sum_{n=1}^N (ca_n + b_n) = c \sum_{n=1}^N a_n + \sum_{n=1}^N b_n \rightarrow cs + t$  as  $N \rightarrow \infty$ .  $\square$

**Theorem 10.1.34** (Comparison Test). If  $|a_i| \leq b_i$  for all  $i$  and  $\sum_{i=1}^{\infty} b_i$  converges, then  $\sum_{i=1}^{\infty} a_i$  converges.

PROOF:

- (1)1.  $\sum_{i=1}^{\infty} |a_i|$  converges

PROOF:  $\sum_{i=1}^N |a_i|$  is a monotone increasing sequence bounded above by  $\sum_{i=1}^{\infty} b_i$ .

- (1)2. LET:  $c_i = |a_i| + a_i$

- (1)3.  $\sum_{i=1}^{\infty} c_i$  converges

PROOF:  $\sum_{i=1}^N c_i$  is a monotone increasing sequence bounded above by  $2 \sum_{i=1}^{\infty} |a_i|$ .

- (1)4. Q.E.D.

PROOF: Since  $a_i = c_i - |a_i|$ .

**Lemma 10.1.35.** If  $\sum_{n=1}^{\infty} a_n$  converges then  $\sum_{n=N}^{\infty} a_n \rightarrow 0$  as  $N \rightarrow \infty$ .

PROOF:

$$\begin{aligned}
 \sum_{n=N}^{\infty} a_n &= \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N-1} a_n \\
 &\rightarrow \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} a_n \\
 &= 0
 \end{aligned}$$

as  $N \rightarrow \infty$ .  $\square$

**Theorem 10.1.36** (Weierstrass M-Test). Let  $X$  be a set and  $f_n : X \rightarrow \mathbb{R}$  for  $n \geq 1$ . If  $|f_n(x)| \leq M_n$  for all  $n \geq 1$  and all  $x \in X$ , and if  $\sum_{n=1}^{\infty} M_n$  converges, then

$$\sum_{n=1}^N f_n(x) \rightarrow \sum_{n=1}^{\infty} f_n(x)$$

uniformly in  $x$  as  $N \rightarrow \infty$ .

PROOF:



⟨1⟩1. For  $N \geq 1$ ,

LET:  $s_N : X \rightarrow \mathbb{R}$ ,  $s_N(x) = \sum_{n=1}^N f_n(x)$

⟨1⟩2. For all  $x \in X$ ,  $\sum_{n=1}^{\infty} f_n(x)$  converges.

PROOF: By the Comparison Test.

⟨1⟩3. LET:  $s : X \rightarrow \mathbb{R}$ ,  $s(x) = \sum_{n=1}^{\infty} f_n(x)$ .

⟨1⟩4. For  $N \geq 1$ ,

LET:  $r_N = \sum_{n=N+1}^{\infty} M_n$

⟨1⟩5. For  $1 \leq N < K$ , we have  $|s_K(x) - s_N(x)| \leq r_N$  for all  $x \in X$

PROOF:

$$\begin{aligned} |s_K(x) - s_N(x)| &= \left| \sum_{n=N+1}^K f_n(x) \right| \\ &\leq \sum_{n=N+1}^K |f_n(x)| \\ &\leq \sum_{n=N+1}^K M_n \\ &\leq \sum_{n=N+1}^{\infty} M_n \\ &= r_N \end{aligned}$$

⟨1⟩6. For  $N \geq 1$  and  $x \in X$  we have  $|s(x) - s_N(x)| \leq r_N$

PROOF: Let  $K \rightarrow \infty$  in ⟨1⟩5.

⟨1⟩7. LET:  $\epsilon > 0$

⟨1⟩8. PICK  $N$  such that, for all  $N' \geq N$ , we have  $r_{N'} < \epsilon$

PROOF: Such an  $N$  exists by Lemma 10.1.35.

⟨1⟩9. For all  $N' \geq N$  and  $x \in X$  we have  $|s_{N'}(x) - s(x)| < \epsilon$

□

**Definition 10.1.37.** Let  $X$  be a metric space. Let  $x \in X$  and  $A \subseteq X$  be nonempty. The *distance* from  $x$  to  $A$  is

$$d(x, A) = \inf_{a \in A} d(x, a) .$$

**Lemma 10.1.38.** Let  $X$  be a metric space and  $A \subseteq X$  be nonempty. Then the function  $d(-, A) : X \rightarrow \mathbb{R}$  is continuous.

PROOF:

⟨1⟩1. LET:  $x \in X$  and  $\epsilon > 0$

⟨1⟩2. LET:  $y \in X$  with  $d(x, y) < \epsilon$

⟨1⟩3.  $|d(x, A) - d(y, A)| < \epsilon$

PROOF:

⟨2⟩1.  $d(x, A) - d(y, A) < \epsilon$

PROOF:

$$\begin{aligned}
d(x, A) &= \inf_{a \in A} d(x, a) \\
&\leq \inf_{a \in A} (d(x, y) + d(y, a)) \\
&= d(x, y) + \inf_{a \in A} d(y, a) \\
&= d(x, y) + d(y, A) \\
&< \epsilon + d(y, A)
\end{aligned}$$

(2)2.  $d(y, A) - d(x, A) < \epsilon$

PROOF: Similar.

(1)4. Q.E.D.

PROOF: By Theorem 10.1.23.

□

**Definition 10.1.39** (Shrinking Map). Let  $X$  be a metric space and  $f : X \rightarrow X$ . Then  $f$  is a *shrinking map* iff, for all  $x, y \in X$  with  $x \neq y$ , we have  $d(f(x), f(y)) < d(x, y)$ .

**Definition 10.1.40** (Contraction). Let  $X$  be a metric space and  $f : X \rightarrow X$ . Then  $f$  is a *contraction* iff there exists  $\alpha < 1$  such that, for all  $x, y \in X$ ,

$$d(f(x), f(y)) \leq \alpha d(x, y) .$$

**Proposition 10.1.41.** *Every separable metric space is second countable.*

PROOF:

(1)1. LET:  $X$  be a separable metric space.

(1)2. PICK a countable dense set  $D$

(1)3. LET:  $\mathcal{B} = \{B(d, q) : d \in D, q \in \mathbb{Q}^+\}$

(1)4.  $\mathcal{B}$  is a countable basis for  $X$

□

**Corollary 10.1.41.1.** *The space  $\mathbb{R}^\omega$  under the uniform topology is not separable.*

**Corollary 10.1.41.2.** *Not every metric space is separable.*

**Corollary 10.1.41.3.** *The space  $\mathbb{R}^\omega$  under the box topology is not separable.*

**Proposition 10.1.42** (CC). *Every Lindelöf metric space is second countable.*

PROOF:

(1)1. LET:  $X$  be a Lindelöf metric space.

(1)2. For all  $n \in \mathbb{Z}^+$ , PICK a countable covering  $\mathcal{A}_n$  of  $X$  by  $1/n$ -balls

PROOF: One exists by the Lindelöf condition, since the set of all  $1/n$ -balls covers  $X$ .

(1)3.  $\bigcup_{n=1}^{\infty} \mathcal{A}_n$  is a countable basis.

□

**Corollary 10.1.42.1.** *The space  $\mathbb{R}^\omega$  under the uniform topology is not Lindelöf.*

**Corollary 10.1.42.2.** *Not every metric space is Lindelöf.*

**Proposition 10.1.43.** *The space  $\mathbb{R}_l$  is not metrizable.*

PROOF: It is Lindelöf but not second countable.  $\square$

**Proposition 10.1.44.** *The ordered square is not metrizable.*

PROOF: It is compact but not second countable.  $\square$

**Proposition 10.1.45.** *The space  $\mathbb{R}^\omega$  under the uniform topology is not second countable.*

PROOF: It contains a subspace homeomorphic to  $\mathbb{R}$ .  $\square$

**Theorem 10.1.46 (AC).** *Every metrizable space is normal.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $X$  be a metric space.

$\langle 1 \rangle 2$ . LET:  $A$  and  $B$  be disjoint closed subspaces of  $X$ .

$\langle 1 \rangle 3$ . For  $a \in A$ , PICK  $\epsilon_a > 0$  such that  $B(a, \epsilon_a)$  does not intersect  $B$ .

$\langle 1 \rangle 4$ . For  $b \in B$ , PICK  $\epsilon_b > 0$  such that  $B(b, \epsilon_b)$  does not intersect  $A$ .

$\langle 1 \rangle 5$ . LET:  $U = \bigcup_{a \in A} B(a, \epsilon_a/2)$

$\langle 1 \rangle 6$ . LET:  $V = \bigcup_{b \in B} B(b, \epsilon_b/2)$

$\langle 1 \rangle 7$ .  $U \cap V = \emptyset$

$\langle 2 \rangle 1$ . LET:  $z \in U \cap V$

$\langle 2 \rangle 2$ . PICK  $a \in A$  and  $b \in B$  such that  $z \in B(a, \epsilon_a/2)$  and  $z \in B(b, \epsilon_b/2)$

$\langle 2 \rangle 3$ . ASSUME: w.l.o.g.  $\epsilon_a \leq \epsilon_b$

$\langle 2 \rangle 4$ .  $a \in B(b, \epsilon_b)$

PROOF:

$$d(a, b) \leq d(a, z) + d(b, z) \quad (\text{Triangle Inequality})$$

$$< \epsilon_a/2 + \epsilon_b/2 \quad (\langle 2 \rangle 2)$$

$$\leq \epsilon_b \quad (\langle 2 \rangle 3)$$

$\langle 2 \rangle 5$ . Q.E.D.

PROOF: This contradicts  $\langle 1 \rangle 4$ .

$\square$

**Corollary 10.1.46.1.** *The space  $\mathbb{R}^\omega$  is normal.*

**Corollary 10.1.46.2.** *The space  $\mathbb{R}_K$  is not metrizable.*

**Proposition 10.1.47.** *Every metrizable space is completely normal.*

PROOF: Every subspace is metrizable (Lemma 10.1.12) hence normal (Theorem 10.1.46).  $\square$

**Proposition 10.1.48.** *Every metrizable space is perfectly normal.*

PROOF:

⟨1⟩1. LET:  $X$  be a metric space.

⟨1⟩2.  $X$  is normal.

PROOF: Theorem 10.1.46

⟨1⟩3. Every closed set is  $G_\delta$ .

PROOF: If  $A$  is closed then  $A = \bigcap_{q \in \mathbb{Q}^+} \{x \in X : d(A, x) < q\}$ .

□

**Theorem 10.1.49** (Urysohn Metrization Theorem (CC)). *Every second countable regular space is metrizable.*

PROOF:

⟨1⟩1. LET:  $X$  be a second countable regular space.

⟨1⟩2.  $X$  is normal.

⟨1⟩3. PICK a countable basis  $\mathcal{B} = \{B_n : n \in \mathbb{Z}^+\}$

⟨1⟩4. For every pair of integers  $m, n$  with  $\overline{B_m} \subseteq B_n$ , PICK a continuous function  $g_{mn} : X \rightarrow [0, 1]$  such that  $g_{mn}(\overline{B_m}) = \{1\}$  and  $g_{mn}(X \setminus B_n) = \{0\}$

PROOF: By the Urysohn Lemma.

⟨1⟩5. The set  $\{g_{mn} : \overline{B_m} \subseteq B_n\}$  separates points from closed sets in  $X$

⟨2⟩1. LET:  $x \in X$  and  $U$  be a neighbourhood of  $x$

⟨2⟩2. PICK  $B_n \in \mathcal{B}$  such that  $x \in B_n \subseteq U$

⟨2⟩3. PICK  $V$  open such that  $x \in V$  and  $\overline{V} \subseteq B_n$

⟨2⟩4. PICK  $B_m \in \mathcal{B}$  such that  $x \in B_m \subseteq V$

⟨2⟩5.  $g_{mn}(x) = 1$  and  $g_{mn}$  vanishes outside  $U$

⟨1⟩6.  $X$  is imbeddable in  $[0, 1]^\omega$

PROOF: By the Imbedding Theorem.

⟨1⟩7. Q.E.D.

**Corollary 10.1.49.1.** *The space  $\mathbb{R}^\omega$  under the box topology is not second countable.*

**Proposition 10.1.50.** *Not every second countable Hausdorff space is metrizable.*

PROOF:  $\mathbb{R}_K$  is second countable and Hausdorff but not metrizable (because it is not regular). □

**Proposition 10.1.51.** *There exists a space that is completely normal, first countable, Lindelöf and separable but not metrizable.*

PROOF: The space  $\mathbb{R}_l$  is all of these. □

**Proposition 10.1.52.**  *$\overline{S_\Omega}$  is not metrizable.*

PROOF: It is compact but not sequentially compact. □

**Proposition 10.1.53.** *Every compact metric space is second countable.*

PROOF:

⟨1⟩1. LET:  $X$  be a compact metric space

(1)2. For every  $n \geq 1$ , PICK a finite covering  $\mathcal{A}_n$  of  $X$  by open balls of radius  $1/n$

PROOF: Such a covering exists because  $\{B_{1/n}(x) : x \in X\}$  covers  $X$ .

(1)3.  $\bigcup_{n=1}^{\infty} \mathcal{A}_n$  is a countable basis for  $X$

□

**Corollary 10.1.53.1.** *The space  $\mathbb{R}^{\omega}$  under the uniform topology is not compact.*

**Corollary 10.1.53.2.** *The space  $\mathbb{R}^{\omega}$  under the uniform topology is not limit point compact.*

**Proposition 10.1.54.** *The space  $\mathbb{R}^{\omega}$  under the box topology is not locally compact.*

PROOF:

(1)1. ASSUME:  $\mathbb{R}^{\omega}$  under the box topology is locally compact.

(1)2. For every point  $x$ , there exists a basic open set  $B = \prod_{i=0}^{\infty} U_i$  such that  $x \in B$  and  $\overline{B}$  is compact.

(1)3. The box topology on  $\overline{B}$  is the same as the product topology on  $\overline{B}$

PROOF: By Corollary 9.4.11.1.

(1)4. The box topology on  $\overline{B}$  is strictly finer than the product topology.

PROOF: By Theorem 10.1.21.

□

**Proposition 10.1.55.** *Not every metrizable space is connected.*

PROOF: The discrete space with two points is metrizable but not connected. □

**Corollary 10.1.55.1.** *Not every metrizable space is path connected.*

**Proposition 10.1.56.** *Not every metric space is limit point compact.*

PROOF: The space  $\mathbb{R}$  is not limit point compact. □

**Proposition 10.1.57.** *Not every metric space is locally compact.*

The space  $\mathbb{R}^{\omega}$  in the uniform topology is not locally compact.

## 10.2 Isometries

**Definition 10.2.1** (Isometry). Let  $X$  be a metric space. An *isometry* of  $X$  is a function  $f : X \rightarrow X$  such that, for all  $x, y \in X$ ,

$$d(f(x), f(y)) = d(x, y) .$$

## 10.3 Lebesgue Numbers

**Definition 10.3.1** (Lebesgue Number). Let  $X$  be a metric space and  $\mathcal{A}$  an open covering of  $X$ . A *Lebesgue number* for  $\mathcal{A}$  is a real  $\delta > 0$  such that, for every nonempty set  $A \subseteq X$  of diameter  $< \delta$ , there exists  $U \in \mathcal{A}$  such that  $A \subseteq U$ .

**Lemma 10.3.2** (Lebesgue Number Lemma). *In a compact metric space, every open covering has a Lebesgue number.*

PROOF:

⟨1⟩1. LET:  $X$  be a compact metric space and  $\mathcal{A}$  an open covering of  $X$

PROVE: There exists a Lebesgue number  $\delta$  for  $\mathcal{A}$ .

⟨1⟩2. ASSUME: w.l.o.g.  $X \notin \mathcal{A}$

PROOF: If  $X \in \mathcal{A}$  then we can take  $\delta = 1$ .

⟨1⟩3. PICK a finite subcovering  $\{U_1, \dots, U_n\} \subseteq \mathcal{A}$  that covers  $X$

⟨1⟩4. For  $1 \leq i \leq n$ ,

LET:  $C_i = X \setminus U_i$

⟨1⟩5. LET:  $f : X \rightarrow \mathbb{R}$  be defined by

$$f(x) = 1/n \sum_{i=1}^n d(x, C_i) .$$

PROOF: Each  $C_i$  is nonempty by ⟨1⟩2.

⟨1⟩6. For all  $x \in X$  we have  $f(x) > 0$

⟨2⟩1. LET:  $x \in X$

⟨2⟩2. PICK  $i$  such that  $x \in U_i$

PROOF: By ⟨1⟩3.

⟨2⟩3. PICK  $\epsilon > 0$  such that  $B(x, \epsilon) \subseteq U_i$

PROOF: By Lemma 10.1.5.

⟨2⟩4.  $d(x, C_i) \geq \epsilon$

⟨1⟩7.  $f$  is continuous

PROOF: From Lemma 10.1.38.

⟨1⟩8. LET:  $\delta = \min f(X)$

PROVE: For every nonempty set  $A \subseteq X$  with diameter  $< \delta$ , there exists  $U \in \mathcal{A}$  such that  $A \subseteq U$

PROOF:  $f(X)$  has a minimum by the Extreme Value Theorem.

⟨1⟩9. LET:  $A \subseteq X$  be nonempty with  $\text{diam } A < \delta$

⟨1⟩10. PICK  $x_0 \in A$

⟨1⟩11. LET:  $i$  be such that  $d(x_0, C_i)$  is greatest among  $d(x_0, C_1), \dots, d(x_0, C_n)$

⟨1⟩12.  $\delta \leq d(x_0, C_i)$

PROOF:

$$\delta \leq f(x_0) \tag{⟨1⟩8}$$

$$= 1/n \sum_{j=1}^n d(x_0, C_j) \tag{⟨1⟩5}$$

$$\leq 1/n \sum_{j=1}^n d(x_0, C_i) \tag{⟨1⟩11}$$

$$= d(x_0, C_i)$$

⟨1⟩13.  $x_0 \in U_i$

PROOF:  $x_0 \notin C_i$  because  $d(x_0, C_i) > 0$ .

□

**Theorem 10.3.3** (DC). *Let  $X$  be a metrizable space. Then the following are*

equivalent:

1.  $X$  is compact.
2.  $X$  is limit point compact.
3.  $X$  is sequentially compact.

PROOF:

$\langle 1 \rangle 1. 1 \Rightarrow 2$

PROOF: Theorem 9.4.22.

$\langle 1 \rangle 2. 2 \Rightarrow 3$

$\langle 2 \rangle 1.$  ASSUME:  $X$  is limit point compact.

$\langle 2 \rangle 2.$  LET:  $(x_n)$  be a sequence in  $X$

PROVE:  $(x_n)$  has a convergent subsequence.

$\langle 2 \rangle 3.$  CASE:  $\{x_n : n \in \mathbb{Z}^+\}$  is finite.

PROOF: In this case,  $(x_n)$  has a constant subsequence.

$\langle 2 \rangle 4.$  CASE:  $\{x_n : n \in \mathbb{Z}^+\}$  is infinite.

$\langle 3 \rangle 1.$  PICK a limit point  $l$  of  $\{x_n : n \in \mathbb{Z}^+\}$

$\langle 3 \rangle 2.$  For every positive integer  $r$ , PICK  $n_r$  such that  $n_r > n_{r-1}$  and  $d(x_{n_r}, l) < 1/r$

PROOF: There always exists such an  $n_r$  since  $B(l, 1/r)$  intersects  $\{x_n : n \in \mathbb{Z}^+\}$  in infinitely many points by Theorem 6.1.2.

$\langle 3 \rangle 3.$   $x_{n_r} \rightarrow l$  as  $r \rightarrow \infty$

$\langle 1 \rangle 3. 3 \Rightarrow 1$

$\langle 2 \rangle 1.$  ASSUME:  $X$  is sequentially compact.

$\langle 2 \rangle 2.$  Every open covering of  $X$  has a Lebesgue number.

$\langle 3 \rangle 1.$  LET:  $\mathcal{A}$  be an open covering of  $X$ .

$\langle 3 \rangle 2.$  ASSUME: for a contradiction that, for all  $\delta > 0$ , there exists a set  $C \subseteq X$  with  $\text{diam } C < \delta$  such that there is no  $U \in \mathcal{A}$  such that  $C \subseteq U$

$\langle 3 \rangle 3.$  For  $n \geq 1$ , PICK  $C_n \subseteq X$  with  $\text{diam } C_n < 1/n$  such that there is no  $U \in \mathcal{A}$  such that  $C_n \subseteq U$

$\langle 3 \rangle 4.$  For  $n \geq 1$ , PICK  $x_n \in C_n$

$\langle 3 \rangle 5.$  PICK a convergent subsequence  $(x_{n_r})$  of  $(x_n)$

PROOF: By  $\langle 2 \rangle 1.$

$\langle 3 \rangle 6.$  LET:  $x_{n_r} \rightarrow l$  as  $r \rightarrow \infty$

$\langle 3 \rangle 7.$  PICK  $U \in \mathcal{A}$  with  $l \in U$

PROOF: By  $\langle 3 \rangle 1$

$\langle 3 \rangle 8.$  PICK  $\epsilon > 0$  such that  $B(l, \epsilon) \subseteq U$

PROOF: By Lemma 10.1.5.

$\langle 3 \rangle 9.$  PICK  $R$  such that  $1/n_R < \epsilon/2$  and  $d(x_{n_R}, l) < \epsilon/2$

PROOF: By  $\langle 3 \rangle 6$

$\langle 3 \rangle 10.$   $C_{n_R} \subseteq U$

PROOF:

$$\begin{aligned}
C_{n_R} &\subseteq B(x_{n_R}, 1/n_R) & (\langle 3 \rangle 3, \langle 3 \rangle 4) \\
&\subseteq B(x_{n_R}, \epsilon/2) & (\langle 3 \rangle 9) \\
&\subseteq B(l, \epsilon) & (\langle 3 \rangle 9) \\
&\subseteq U & (\langle 3 \rangle 8)
\end{aligned}$$

$\langle 3 \rangle 11$ . Q.E.D.

PROOF: This contradicts  $\langle 3 \rangle 3$ .

$\langle 2 \rangle 3$ . For all  $\epsilon > 0$ , there exists a finite covering of  $X$  by  $\epsilon$ -balls.

$\langle 3 \rangle 1$ . LET:  $\epsilon > 0$

$\langle 3 \rangle 2$ . ASSUME: for a contradiction there is no finite covering of  $X$  by  $\epsilon$ -balls.

$\langle 3 \rangle 3$ . PICK a sequence  $(x_n)$  in  $X$  such that, for all  $n$ ,  

$$x_n \notin B(x_1, \epsilon) \cup \dots \cup B(x_{n-1}, \epsilon) .$$

$\langle 3 \rangle 4$ . For all  $m, n$  with  $m > n$  we have  $d(x_m, x_n) \geq \epsilon$

$\langle 3 \rangle 5$ . Any  $\epsilon/2$ -ball contains at most one element of  $(x_n)$ .

$\langle 3 \rangle 6$ .  $(x_n)$  has no convergent subsequence.

$\langle 3 \rangle 7$ . Q.E.D.

PROOF: This contradicts  $\langle 2 \rangle 1$ .

$\langle 2 \rangle 4$ . LET:  $\mathcal{A}$  be an open covering of  $X$

$\langle 2 \rangle 5$ . PICK a Lebesgue number  $\delta$  for  $\mathcal{A}$

PROOF: By  $\langle 2 \rangle 2$ .

$\langle 2 \rangle 6$ . PICK a finite covering  $\{B_1, \dots, B_n\}$  of  $X$  by  $\delta/3$ -balls.

PROOF: By  $\langle 2 \rangle 3$ .

$\langle 2 \rangle 7$ . For  $1 \leq i \leq n$ , PICK  $U_i \in \mathcal{A}$  such that  $B_i \subseteq U_i$

$\langle 2 \rangle 8$ .  $\{U_1, \dots, U_n\}$  covers  $X$ .

□

**Corollary 10.3.3.1.**  $S_\Omega$  is not metrizable.

PROOF: It is limit point compact (Corollary 9.4.19.2) but not compact (Proposition 9.4.2). □

**Corollary 10.3.3.2.** The space  $\mathbb{R}^\omega$  is not limit point compact.

## 10.4 Uniform Continuity

**Definition 10.4.1** (Uniform Continuity). Let  $X$  and  $Y$  be metric spaces and  $f : X \rightarrow Y$ . Then  $f$  is *uniformly continuous* iff, for all  $\epsilon > 0$ , there exists  $\delta > 0$  such that, for all  $x, y \in X$ , if  $d(x, y) < \delta$  then  $d(f(x), f(y)) < \epsilon$ .

**Theorem 10.4.2** (Uniform Continuity Theorem). *Let  $X$  be a compact metric space,  $Y$  a metric space, and  $f : X \rightarrow Y$  be continuous. Then  $f$  is uniformly continuous.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $\epsilon > 0$

PROVE: There exists  $\delta > 0$  such that, for all  $x, y \in X$ , if  $d(x, y) < \delta$  then  $d(f(x), f(y)) < \epsilon$ .



- ⟨1⟩2. LET:  $\mathcal{A} = \{f^{-1}(B(y, \epsilon/2)) : y \in Y\}$
  - ⟨1⟩3.  $\mathcal{A}$  is an open covering of  $X$
  - ⟨1⟩4. PICK a Lebesgue number  $\delta$  for  $\mathcal{A}$ .  
PROVE: For all  $x, y \in X$ , if  $d(x, y) < \delta$  then  $d(f(x), f(y)) < \epsilon$   
PROOF: By the Lebesgue Number Lemma
  - ⟨1⟩5. LET:  $x, y \in X$  with  $d(x, y) < \delta$
  - ⟨1⟩6.  $\text{diam}\{x, y\} < \delta$
  - ⟨1⟩7. PICK  $z \in Y$  such that  $\{x, y\} \subseteq f^{-1}(B(z, \epsilon/2))$
  - ⟨1⟩8.  $d(f(x), f(y)) < \epsilon$
- 

## 10.5 Locally Metrizable Spaces

**Definition 10.5.1** (Locally Metrizable). A space is *locally metrizable* iff every point has a metrizable neighbourhood.

**Proposition 10.5.2.** *Every metrizable space is locally metrizable.*

PROOF: Trivial. □

**Corollary 10.5.2.1.** *The space  $\mathbb{R}^\omega$  is locally metrizable.*

**Proposition 10.5.3.** *A compact Hausdorff space is metrizable if and only if it is locally metrizable.*

PROOF:

- ⟨1⟩1. LET:  $X$  be a locally metrizable compact Hausdorff space
- ⟨1⟩2.  $X$  is regular  
PROOF: Lemma 9.4.18
- ⟨1⟩3.  $X$  is second countable
  - ⟨2⟩1.  $\{U : U \text{ open in } X \text{ and metrizable}\}$  covers  $X$
  - ⟨2⟩2. PICK a finite subcover  $U_1, \dots, U_n$
  - ⟨2⟩3. For  $1 \leq i \leq n$ , PICK a countable basis  $\mathcal{B}_i$  of  $U_i$
  - ⟨2⟩4.  $\mathcal{B}_1 \cup \dots \cup \mathcal{B}_n$  is a basis for  $X$
- ⟨1⟩4. Q.E.D.

PROOF: By the Urysohn Metrization Theorem.  
□

**Corollary 10.5.3.1.**  $\overline{S_\Omega}$  is not locally metrizable.

**Corollary 10.5.3.2.** *The ordered square is not locally metrizable.*

**Proposition 10.5.4.** *Every subspace of a locally metrizable space is locally metrizable.*

PROOF:

- ⟨1⟩1. LET:  $X$  be locally metrizable and  $Y \subseteq X$
- ⟨1⟩2. LET:  $y \in Y$

- ⟨1⟩3. PICK a metrizable neighbourhood  $U$  of  $y$  in  $X$
- ⟨1⟩4.  $U \cap Y$  is a metrizable neighbourhood of  $y$  in  $Y$

□

**Corollary 10.5.4.1.**  $S_\Omega \times \overline{S_\Omega}$  is not locally metrizable.

PROOF: It has a subspace homeomorphic to  $\overline{S_\Omega}$ . □

**Proposition 10.5.5 (CC).** Every locally metrizable regular Lindelöf space is metrizable.

PROOF:

- ⟨1⟩1. LET:  $X$  be a locally metrizable regular Lindelöf space.
- ⟨1⟩2. Every point in  $X$  has an open second countable neighbourhood.
  - ⟨2⟩1. LET:  $x \in X$
  - ⟨2⟩2. PICK an open metrizable  $U$  containing  $x$ 
    - PROOF:  $X$  is locally metrizable (⟨1⟩1)
  - ⟨2⟩3. PICK an open  $V$  such that  $x \in V \subseteq \overline{V} \subseteq U$ 
    - PROOF: Proposition 6.3.2
  - ⟨2⟩4.  $\overline{V}$  is Lindelöf
    - PROOF: Proposition 13.1.32
  - ⟨2⟩5.  $\overline{V}$  is second countable
    - PROOF: Proposition 10.1.42
- ⟨1⟩3. PICK a countable covering of second countable open sets  $\mathcal{U}$ 
  - PROOF:  $X$  is Lindelöf (⟨1⟩1)
- ⟨1⟩4. For  $U \in \mathcal{U}$ , PICK a countable basis  $\mathcal{B}_U$
- ⟨1⟩5.  $\bigcup_{U \in \mathcal{U}} \mathcal{B}_U$  is a countable basis for  $X$ 
  - ⟨2⟩1. LET:  $x \in U$  where  $U$  is open in  $X$
  - ⟨2⟩2. PICK  $V \in \mathcal{U}$  such that  $x \in V$
  - ⟨2⟩3. There exists  $B \in \mathcal{B}_V$  such that  $x \in B \subseteq U \cap V$
- ⟨1⟩6. Q.E.D.
  - PROOF: By the Urysohn Metrization Theorem.

□

**Corollary 10.5.5.1.**  $\mathbb{R}_l$  is not locally metrizable.

**Proposition 10.5.6.** The Sorgenfrey plane is not locally metrizable.

PROOF:

- ⟨1⟩1. LET:  $U$  be any neighbourhood of  $(0, 0)$ 
  - PROVE:  $U$  is not Lindelöf
- ⟨1⟩2. PICK  $a > 0$  such that  $[0, a]^2 \subseteq U$
- ⟨1⟩3. LET:  $L = \{(x, a - x) : 0 < x < a\}$
- ⟨1⟩4.  $L$  is closed in  $U$ 
  - PROOF: By Lemma 6.5.16 since  $(x, y) \mapsto (x, a + y)$  is a homeomorphism of  $\mathbb{R}_l^2$  with itself.
- ⟨1⟩5. LET:  $\mathcal{U} = \{U \setminus L\} \cup \{([x, b] \times [a - x, c]) \cap U : b > a, c > a - x\}$
- ⟨1⟩6.  $\mathcal{U}$  covers  $U$

⟨1⟩7. No countable subset of  $\mathcal{U}$  covers  $U$

PROOF: Every set of the form  $[x, b) \times [a - x, c)$  intersects  $L$  in exactly one point.  
□

**Corollary 10.5.6.1.** *The Sorgenfrey plane is not metrizable.*

**Proposition 10.5.7.** *The space  $\mathbb{R}_K$  is locally metrizable.*

PROOF: The set  $(-1, 1) - K$  is a metrizable neighbourhood of 0. For any other point  $p$ , pick an open interval around  $p$  that does not contain 0. □

**Proposition 10.5.8.** *The product of two locally metrizable spaces is locally metrizable.*

PROOF:

⟨1⟩1. LET:  $X$  and  $Y$  be locally metrizable

⟨1⟩2. LET:  $(a, b) \in X \times Y$

⟨1⟩3. PICK metrizable neighbourhoods  $U$  of  $a$  and  $V$  of  $b$

⟨1⟩4.  $U \times V$  is a metrizable neighbourhood of  $(a, b)$ .

PROOF: By Lemma 10.1.19.  
□

**Proposition 10.5.9.** *The product of two locally metrizable spaces is locally metrizable.*

PROOF:

⟨1⟩1. LET:  $X$  and  $Y$  be locally metrizable

⟨1⟩2. LET:  $(a, b) \in X \times Y$

⟨1⟩3. PICK metrizable neighbourhoods  $U$  of  $a$  and  $V$  of  $b$

⟨1⟩4.  $U \times V$  is a metrizable neighbourhood of  $(a, b)$ .

PROOF: By Lemma 10.1.19.  
□

**Proposition 10.5.10.** *The space  $\mathbb{R}_K^\omega$  is not locally metrizable.*

PROOF: If it were, then there would be a basic open set  $\prod_n U_n$  that is metrizable, but then  $\mathbb{R}_K$  would be metrizable as it is homeomorphic to a subspace of  $\prod_n U_n$ .  
□

**Corollary 10.5.10.1.** *The product of a countable family of locally metrizable spaces is not necessarily locally metrizable.*

**Proposition 10.5.11.** *The continuous image of a locally metrizable space is not necessarily locally metrizable.*

PROOF: The identity map from the discrete two-point space to the indiscrete two-point space is continuous. □

# Chapter 11

## Manifolds

### 11.1 Manifolds

**Definition 11.1.1** (Manifold). Let  $m \geq 1$ . An  $m$ -manifold is a second countable Hausdorff space such that each point has a neighbourhood that is homeomorphic to an open subset of  $\mathbb{R}^m$ .

A *curve* is a 1-manifold and a *surface* is a 2-manifold.

**Definition 11.1.2** (Support). Let  $X$  be a topological space and  $\phi : X \rightarrow \mathbb{R}$  be a function. Then the *support* of  $\phi$  is the closure of  $\phi^{-1}(\mathbb{R} \setminus \{0\})$ .

**Definition 11.1.3** (Partition of Unity). Let  $X$  be a topological space. Let  $\{U_1, \dots, U_n\}$  be a finite indexed open covering of  $X$ . An indexed family of continuous functions  $\phi_1, \dots, \phi_n : X \rightarrow [0, 1]$  is a *partition of unity* dominated by  $\{U_1, \dots, U_n\}$  iff:

1.  $\text{supp } \phi_i \subseteq U_i$  for all  $i$ ;
2.  $\sum_{i=1}^n \phi_i(x) = 1$  for all  $x \in X$ .

**Theorem 11.1.4** (Existence of Finite Partitions of Unity). *Let  $X$  be a normal space. Let  $\{U_1, \dots, U_n\}$  be a finite indexed open covering of  $X$ . Then there exists a partition of unity dominated by  $\{U_1, \dots, U_n\}$ .*

PROOF:

- (1)1. For every finite indexed open covering  $\{U_1, \dots, U_n\}$  of  $X$ , there exists a finite indexed open covering  $\{V_1, \dots, V_n\}$  such that  $\overline{V_i} \subseteq U_i$
- (2)1. For  $1 \leq k \leq n$ , there exist open sets  $V_1, \dots, V_k$  such that  $\overline{V_i} \subseteq U_i$  for all  $i$  and  $\{V_1, \dots, V_k, U_{k+1}, \dots, U_n\}$  covers  $X$
- (3)1. ASSUME: as an induction hypothesis that  $0 \leq k < n$  and there exist open sets  $V_1, \dots, V_k$  such that  $\overline{V_i} \subseteq U_i$  for all  $i$  and  $\{V_1, \dots, V_k, U_{k+1}, \dots, U_n\}$  covers  $X$
- (3)2. LET:  $A = X \setminus (V_1 \cup \dots \cup V_k) \setminus (U_{k+2} \cup \dots \cup U_n)$
- (3)3.  $A$  is closed

PROOF: Since  $\{V_1, \dots, V_k, U_{k+1}, \dots, U_n\}$  covers  $X$

PROOF: By Proposition 6.3.2

**⟨1⟩2.** PICK an open covering  $\{V_1, \dots, V_n\}$  with  $\overline{V_i} \subseteq U_i$  for all  $i$

$\langle 1 \rangle 3$ . PICK an open covering  $\{W_1, \dots, W_n\}$  with  $\overline{W_i} \subseteq V_i$  for all  $i$

⟨1⟩4. For  $1 \leq i \leq n$ , PICK a continuous function  $\psi_i : X \rightarrow [0, 1]$  such that  $\psi_i(\overline{W_i}) = \{1\}$  and  $\psi_i(X \setminus V_i) = \{0\}$

(1)5. LET:  $\Psi : X \rightarrow \mathbb{R}$  where  $\Psi(x) = \sum_{i=1}^n \psi_i(x)$

⟨2⟩1. LET:  $x \in X$

$\langle 2 \rangle 3.$   $\psi_i(x) = 1$

LET:  $\phi_j(x) = \frac{\psi_j(x)}{\Psi(x)}$

$\langle 2 \rangle 1.$   $\text{supp } \psi_i \subseteq U_i$

PROOF: By  $\langle 1 \rangle_4$

PROOF: Proposition 3.7.5

$$\langle 2 \rangle 2. \sum_{i=1}^n \psi_i(x) = 1 \text{ for all } x \in X$$

PROOF:

PROOF: Since  $\{U \text{ open in } X : U \text{ can be imbedded in } \mathbb{R}^k \text{ for some } k\}$  covers  $X$ .

(1)3. PICK a partition of unity  $\phi_1, \dots, \phi_n$  dominated by  $\{U_1, \dots, U_n\}$

PROOF: By Lemma 9.4.18.

PROOF: Theorem 11.1.4

LET:  $A_i = \text{supp } \phi_i$

⟨1⟩5. For  $1 \leq i \leq n$ ,

LET:  $h_i : X \rightarrow \mathbb{R}^{k_i}$  be defined by

$$h_i(x) = \begin{cases} \phi_i(x)g_i(x) & \text{for } x \in U_i \\ \vec{0} & \text{for } x \in X \setminus A_i \end{cases}$$

PROOF: If  $x \in U_i$  and  $x \in X \setminus A_i$  then  $x \notin \text{supp } \phi_i$  so  $\phi_i(x) = 0$

⟨1⟩6. LET:  $N = n + k_1 + \cdots + k_n$

⟨1⟩7. LET:  $F : X \rightarrow \mathbb{R}^N$  be the function

$$F(x) = (\phi_1(x), \dots, \phi_n(x), h_1(x), \dots, h_n(x))$$

⟨1⟩8.  $F$  is an imbedding

⟨2⟩1.  $F$  is continuous

PROOF: Each  $h_i$  is continuous by Theorem 5.2.13.

⟨2⟩2.  $F$  is injective

⟨3⟩1. ASSUME:  $F(x) = F(y)$

⟨3⟩2. PICK  $i$  such that  $\phi_i(x) > 0$

PROOF: Since  $\sum_i \phi_i(x) = 1$  (⟨1⟩3)

⟨3⟩3.  $\phi_i(y) = 0$

PROOF: By ⟨3⟩1

⟨3⟩4.  $x, y \in U_i$

PROOF: Since  $\text{supp } \phi_i \subseteq U_i$

⟨3⟩5.  $h_i(x) = h_i(y)$

PROOF: By ⟨3⟩1

⟨3⟩6.  $g_i(x) = g_i(y)$

PROOF: By ⟨1⟩5

⟨3⟩7.  $x = y$

PROOF: By ⟨1⟩2

⟨2⟩3. Q.E.D.

PROOF: By Theorem 9.4.11

□

**Corollary 11.1.5.1.** *Every compact manifold can be imbedded in  $\mathbb{R}^N$  for some  $N$ .*

**Proposition 11.1.6.** *The line with two origins is a second countable  $T_1$  space where every point has a neighbourhood that is homeomorphic to an open subset of  $\mathbb{R}$ , but it is not a 1-manifold.*

# Chapter 12

## Normed Spaces

### 12.1 The Norm on $\mathbb{R}^n$

**Definition 12.1.1** (Norm). Given  $\vec{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ , the *norm*  $\|\vec{x}\|$  is defined by

$$\|\vec{x}\| = \sqrt{x_1^2 + \dots + x_n^2} .$$

**Definition 12.1.2** (Vector Sum). Define the *sum* of  $\vec{x}, \vec{y} \in \mathbb{R}^n$  by

$$\vec{x} + \vec{y} = (x_1 + y_1, \dots, x_n + y_n) .$$

**Definition 12.1.3** (Scalar Product). Given  $c \in \mathbb{R}$  and  $\vec{x} \in \mathbb{R}^n$ , define the *scalar product*  $c\vec{x}$  to be

$$c\vec{x} = (cx_1, \dots, cx_n) .$$

**Definition 12.1.4** (Inner Product). The *inner product* of  $\vec{x}, \vec{y} \in \mathbb{R}^n$  is

$$\vec{x} \cdot \vec{y} = x_1 y_1 + \dots + x_n y_n .$$

**Lemma 12.1.5.**

$$\vec{x} \cdot (\vec{y} + \vec{z}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z}$$

PROOF: Both are equal to  $\sum_{i=1}^n (x_i y_i + x_i z_i)$ .  $\square$

**Lemma 12.1.6.**

$$|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \|\vec{y}\|$$

PROOF:

$\langle 1 \rangle 1$ . CASE:  $\vec{x} = \vec{0}$  or  $\vec{y} = \vec{0}$

PROOF: In this case, both sides are 0.

$\langle 1 \rangle 2$ . CASE:  $\vec{x} \neq \vec{0} \neq \vec{y}$

$\langle 2 \rangle 1$ . LET:  $a = 1/\|\vec{x}\|$ ,  $b = 1/\|\vec{y}\|$

$\langle 2 \rangle 2$ .  $2 + 2ab\vec{x} \cdot \vec{y} \geq 0$

$\langle 3 \rangle 1$ .  $\|a\vec{x} + b\vec{y}\|^2 \geq 0$

- $\langle 3 \rangle 2. \sum_{i=1}^n (ax_i + by_i)^2 \geq 0$   
 $\langle 3 \rangle 3. a^2 \sum_{i=1}^n x_i^2 + b^2 \sum_{i=1}^n y_i^2 + 2ab \sum_{i=1}^n x_i y_i \geq 0$   
 $\langle 3 \rangle 4. a^2 \|\vec{x}\|^2 + b^2 \|\vec{y}\|^2 + 2ab \vec{x} \cdot \vec{y} \geq 0$   
 $\langle 2 \rangle 3. 2 - 2ab \vec{x} \cdot \vec{y} \geq 0$   
 PROOF: Similar.  
 $\langle 2 \rangle 4. 2 - 2ab |\vec{x} \cdot \vec{y}| \geq 0$   
 PROOF: From  $\langle 2 \rangle 2$  and  $\langle 2 \rangle 3$ .  
 $\langle 2 \rangle 5. |\vec{x} \cdot \vec{y}| \leq 1/ab$   
 $\square$

**Lemma 12.1.7.**

$$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$$

PROOF:

$$\begin{aligned}
 \|\vec{x} + \vec{y}\|^2 &= (\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y}) \\
 &= \|\vec{x}\|^2 + 2\vec{x} \cdot \vec{y} + \|\vec{y}\|^2 && \text{(Lemma 12.1.5)} \\
 &\leq \|\vec{x}\|^2 + 2\|\vec{x}\|\|\vec{y}\| + \|\vec{y}\|^2 && \text{(Lemma 12.1.6)} \\
 &= (\|\vec{x}\| + \|\vec{y}\|)^2 && \square
 \end{aligned}$$

**Definition 12.1.8** (Euclidean Metric). The *euclidean metric* on  $\mathbb{R}^n$  is given by

$$d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|$$

We prove this is a metric.

PROOF:

- $\langle 1 \rangle 1. d(\vec{x}, \vec{y}) \geq 0$   
 PROOF: Immediate from definitions.  
 $\langle 1 \rangle 2. d(\vec{x}, \vec{y}) = 0$  iff  $\vec{x} = \vec{y}$   
 PROOF: Immediate from definitions.  
 $\langle 1 \rangle 3. d(\vec{x}, \vec{y}) = d(\vec{y}, \vec{x})$   
 PROOF: Immediate from definitions.  
 $\langle 1 \rangle 4. d(\vec{x}, \vec{z}) \leq d(\vec{x}, \vec{y}) + d(\vec{y}, \vec{z})$   
 PROOF: From Lemma 12.1.7.  
 $\square$

**Lemma 12.1.9.** Let  $d$  be the euclidean topology on  $\mathbb{R}^n$  and  $\rho$  the square topology. Then, for all  $x, y \in \mathbb{R}^n$ , we have

$$\rho(x, y) \leq d(x, y) \leq \sqrt{n}\rho(x, y)$$

PROOF:

- $\langle 1 \rangle 1. \rho(x, y) \leq d(x, y)$   
 $\langle 2 \rangle 1.$  For  $1 \leq i \leq n$  we have  $|x_i - y_i| \leq d(x, y)$   
 PROOF: By the definition of the euclidean metric.  
 $\langle 2 \rangle 2.$  Q.E.D.  
 PROOF: By the definition of the square metric.



(1)2.  $d(x, y) \leq \sqrt{n}\rho(x, y)$

PROOF:

$$\begin{aligned} d(x, y) &= \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2} \\ &\leq \sqrt{\rho(x, y)^2 + \cdots + \rho(x, y)^2} \\ &= \sqrt{n\rho(x, y)^2} \\ &= \sqrt{n}\rho(x, y) \end{aligned}$$

□

**Corollary 12.1.9.1.** *The euclidean metric induces the standard topology on  $\mathbb{R}^n$ .*

**Definition 12.1.10.** Let  $l_2$  be the set of sequences  $\vec{a} \in \mathbb{R}^\omega$  such that  $\sum_{n=1}^\infty a_n^2 < \infty$ .

**Lemma 12.1.11.** *If  $\vec{a}, \vec{b} \in l_2$  then  $\sum_{n=1}^\infty |a_n b_n| < \infty$ .*

PROOF:

$$\begin{aligned} \sum_{n=1}^N |a_n b_n| &\leq \sqrt{\left(\sum_{n=1}^N a_n^2\right)\left(\sum_{n=1}^N b_n^2\right)} && \text{(Lemma 12.1.6)} \\ &\rightarrow \sqrt{\sum_{n=1}^\infty a_n^2 \sum_{n=1}^\infty b_n^2} \text{ as } n \rightarrow \infty \end{aligned}$$

□

**Lemma 12.1.12.** *If  $\vec{a}, \vec{b} \in l_2$  then  $\vec{a} + \vec{b} \in l_2$ .*

PROOF:

$$\begin{aligned} \sum_{n=1}^\infty (a_n + b_n)^2 &= \sum_{n=1}^\infty a_n^2 + 2 \sum_{n=1}^\infty a_n b_n + \sum_{n=1}^\infty b_n^2 \\ &\leq \sum_{n=1}^\infty a_n^2 + 2 \sum_{n=1}^\infty |a_n b_n| + \sum_{n=1}^\infty b_n^2 \\ &< \infty && \text{(Lemma 12.1.11)} \end{aligned}$$

□

**Lemma 12.1.13.** *If  $c \in \mathbb{R}$  and  $\vec{a} \in l_2$  then  $c\vec{a} \in l_2$ .*

PROOF:  $\sum_{n=1}^\infty (ca_n)^2 = c^2 \sum_{n=1}^\infty a_n^2$ . □

**Definition 12.1.14** (The  $l^2$ -metric). The  $l^2$ -metric is defined on  $l_2$  by

$$d(\vec{a}, \vec{b}) = \left[ \sum_{n=1}^\infty (a_n - b_n)^2 \right]^{\frac{1}{2}}.$$

The topology induced by this metric is the  $l^2$ -topology. We write  $l_2$  for this set under the  $l^2$ -topology.

We prove this is a metric.

PROOF:

⟨1⟩1.  $d(\vec{a}, \vec{b}) \geq 0$

PROOF: Immediate from definitions.

⟨1⟩2.  $d(\vec{a}, \vec{b}) = 0$  iff  $\vec{a} = \vec{b}$

PROOF: Immediate from definitions.

⟨1⟩3.  $d(\vec{a}, \vec{b}) = d(\vec{b}, \vec{a})$

PROOF: Immediate from definitions.

⟨1⟩4.  $d(\vec{a}, \vec{c}) \leq d(\vec{a}, \vec{b}) + d(\vec{b}, \vec{c})$

PROOF:  $\sqrt{\sum_{i=1}^N (a_i - c_i)^2} \leq \sqrt{\sum_{i=1}^N (a_i - b_i)^2} + \sqrt{\sum_{i=1}^N (b_i - c_i)^2}$  since the euclidean metric on  $\mathbb{R}^N$  is a metric.

□

**Definition 12.1.15** (Hilbert Cube). The *Hilbert cube* is  $\prod_{n=1}^{\infty} [0, 1/n]$  as a subspace of the  $l_2$ .

**Definition 12.1.16** (Isometric Imbedding). Let  $X, Y$  be metric spaces and  $f : X \rightarrow Y$ . Then  $f$  is an *isometric imbedding* iff, for all  $x, y \in X$ ,  $d(f(x), f(y)) = d(x, y)$ .

**Lemma 12.1.17.** *Every isometric imbedding is an imbedding.*

PROOF:

⟨1⟩1. LET:  $f : X \rightarrow Y$  be an isometric imbedding.

⟨1⟩2.  $f$  is continuous.

PROOF: If  $d(x, y) < \epsilon$  then  $d(f(x), f(y)) < \epsilon$ .

⟨1⟩3.  $f$  is injective.

PROOF: If  $f(x) = f(y)$  then  $d(f(x), f(y)) = 0$  so  $d(x, y) = 0$  hence  $x = y$ .

⟨1⟩4.  $f^{-1} : f(X) \rightarrow X$  is continuous.

PROOF: If  $d(f^{-1}(x), f^{-1}(y)) < \epsilon$  then  $d(x, y) < \epsilon$ .

□

# Chapter 13

## Topological Groups

### 13.1 Topological Groups

**Definition 13.1.1** (Topological Group). A *topological group*  $G$  consists of a group  $G$  that is also a  $T_1$  space such that  $\cdot : G^2 \rightarrow G$  and  $(\ )^{-1} : G \rightarrow G$  are continuous.

**Proposition 13.1.2.** *Every topological group is homogeneous.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $G$  be a topological group.
- $\langle 1 \rangle 2$ . LET:  $x, y \in G$
- $\langle 1 \rangle 3$ . LET:  $f : G \rightarrow G$  be given by  $f(g) = yx^{-1}g$
- $\langle 1 \rangle 4$ .  $f$  is a homeomorphism
- $\langle 1 \rangle 5$ .  $f(x) = y$

□

**Definition 13.1.3** (Symmetric). Let  $G$  be a topological group. A neighbourhood  $U$  of  $e$  is *symmetric* iff  $U = U^{-1}$ .

**Proposition 13.1.4.** *For every neighbourhood  $U$  of  $e$ , there exists a symmetric neighbourhood  $V$  of  $e$  such that  $VV \subseteq U$ .*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $m : G^2 \rightarrow G$  be the multiplication function
- $\langle 1 \rangle 2$ .  $ee \in U$
- $\langle 1 \rangle 3$ .  $(e, e) \in m^{-1}(U)$
- $\langle 1 \rangle 4$ . PICK neighbourhoods  $U_1, U_2$  of  $e$  such that  $(e, e) \in U_1 \times U_2 \subseteq m^{-1}(U)$
- $\langle 1 \rangle 5$ . LET:  $V' = U_1 \cap U_2$
- $\langle 1 \rangle 6$ .  $V'V' \subseteq U$
- $\langle 1 \rangle 7$ . LET:  $f : G^2 \rightarrow G$  be the function  $f(x, y) = xy^{-1}$
- $\langle 1 \rangle 8$ .  $(e, e) \in f^{-1}(V')$
- $\langle 1 \rangle 9$ . PICK a neighbourhood  $W$  of  $e$  such that  $WW^{-1} \subseteq V'$
- $\langle 1 \rangle 10$ . LET:  $V = WW^{-1}$

- ⟨1⟩11.  $V$  is a neighbourhood of  $e$   
 PROOF:  $V$  is open because  $V = \bigcup_{a \in W^{-1}} Wa$ .  
 ⟨1⟩12.  $V$  is symmetric  
 ⟨1⟩13.  $VV \subseteq U$   
 □

**Proposition 13.1.5.** *Every topological group is regular.*

PROOF:

- ⟨1⟩1. LET:  $G$  be a topological group  
 ⟨1⟩2. LET:  $A \subseteq G$  be closed and  $a \notin A$   
 ⟨1⟩3.  $G \setminus Aa^{-1}$  is a neighbourhood of  $e$   
 ⟨1⟩4. PICK a symmetric neighbourhood  $V$  of  $e$  such that  $VV \subseteq G \setminus Aa^{-1}$   
 PROOF: Proposition 13.1.4.  
 ⟨1⟩5.  $VA$  and  $Va$  are disjoint neighbourhoods of  $A$  and  $a$   
 □

**Proposition 13.1.6.** *The long line is not second countable.*

PROOF: Let  $\mathcal{B}$  be a basis for  $L$ . Then, for every countable ordinal  $\alpha$ ,  $\mathcal{B}$  must contain a basic open set that contains  $(\alpha, 1/2)$  but not  $(\beta, 1/2)$  for any other  $\beta$ . Therefore,  $\mathcal{B}$  is uncountable. □

**Corollary 13.1.6.1.** *The long line cannot be imbedded in  $\mathbb{R}$ .*

**Theorem 13.1.7.** *Let  $f : X \rightarrow Y$ . Let  $Y$  be compact Hausdorff. Then  $f$  is continuous if and only if the graph of  $f$  is closed in  $X \times Y$ .*

PROOF:

- ⟨1⟩1. LET:  $G_f$  be the graph of  $f$ .  
 ⟨1⟩2. If  $f$  is continuous then the graph of  $f$  is closed.  
 ⟨2⟩1. ASSUME:  $f$  is continuous.  
 ⟨2⟩2. LET:  $(x, y) \in (X \times Y) \setminus G_f$   
 ⟨2⟩3.  $y \neq f(x)$   
 ⟨2⟩4. PICK disjoint open neighbourhoods  $U$  of  $f(x)$  and  $V$  of  $y$   
 PROOF:  $Y$  is Hausdorff.  
 ⟨2⟩5.  $(x, y) \in f^{-1}(U) \times V \subseteq (X \times Y) \setminus G_f$   
 ⟨2⟩6. Q.E.D.  
 ⟨1⟩3. If the graph of  $f$  is closed then  $f$  is continuous.  
 ⟨2⟩1. ASSUME:  $G_f$  is closed.  
 ⟨2⟩2. LET:  $x_0 \in X$  and  $V$  be an open neighbourhood of  $f(x_0)$   
 ⟨2⟩3.  $G_f \cap (X \times (Y \setminus V))$  is closed  
 ⟨2⟩4.  $\pi_1(G_f \cap (X \times (Y \setminus V)))$  is closed  
 PROOF: Lemma 9.4.16  
 ⟨2⟩5.  $x_0 \in X \setminus \pi_1(G_f \cap (X \times (Y \setminus V))) \subseteq f^{-1}(V)$   
 ⟨2⟩6. Q.E.D.  
 □

**Theorem 13.1.8.** *Let  $X$  be a compact Hausdorff space. Let  $\mathcal{A}$  be a set of closed connected subspaces of  $X$  that is linearly ordered by proper inclusion. Then*

$$Y = \bigcap \mathcal{A}$$

*is connected.*

PROOF:

⟨1⟩1. ASSUME: for a contradiction  $C$  and  $D$  form a separation of  $Y$

⟨1⟩2. PICK disjoint  $U$  and  $V$  open in  $X$  such that  $C = U \cap Y$  and  $D = V \cap Y$

⟨2⟩1.  $C$  and  $D$  are compact

⟨3⟩1.  $Y$  is compact

PROOF:  $Y$  is a closed subset of  $X$ , hence compact by Proposition 9.4.6.

⟨3⟩2. Q.E.D.

PROOF:  $C$  and  $D$  are closed subsets of  $Y$  hence compact by Proposition 9.4.6.

⟨2⟩2. Q.E.D.

PROOF: By Lemma 9.4.18.

⟨1⟩3. For all  $A \in \mathcal{A}$ , we have  $A \setminus (U \cup V)$  is nonempty

PROOF: Since  $A$  is connected.

⟨1⟩4.  $\{A \setminus (U \cup V) : A \in \mathcal{A}\}$  has the finite intersection property

PROOF: This holds because  $\mathcal{A}$  is linearly ordered under proper inclusion.

⟨1⟩5.  $\bigcap_{A \in \mathcal{A}} (A \setminus (U \cup V))$  is nonempty

PROOF: By Proposition 9.4.15.

□

**Theorem 13.1.9.** *Let  $A \subseteq \mathbb{R}^n$ . Then the following are equivalent:*

1.  $A$  is compact.
2.  $A$  is closed and bounded under the euclidean metric.
3.  $A$  is closed and bounded under the square metric.

PROOF:

⟨1⟩1.  $1 \Rightarrow 2$

⟨2⟩1. ASSUME:  $A$  is compact.

⟨2⟩2.  $A$  is closed.

PROOF: By Proposition 9.4.9.

⟨2⟩3.  $\{B(\vec{0}, n) : n \in \mathbb{Z}^+\}$  covers  $A$

⟨2⟩4. PICK a finite subcover  $\{B(\vec{0}, n_1), \dots, B(\vec{0}, n_k)\}$

⟨2⟩5. LET:  $N = \max(n_1, \dots, n_k)$

⟨2⟩6. For all  $x, y \in A$  we have  $d(x, y) < 2N$

PROOF: We have  $d(x, y) \leq d(\vec{0}, x) + d(\vec{0}, y) < N + N$ .

⟨1⟩2.  $2 \Rightarrow 3$

PROOF: If  $d(x, y) < \epsilon$  for all  $x, y \in A$  then  $\rho(x, y) < \epsilon\sqrt{n}$  by Lemma 12.1.9.

⟨1⟩3.  $3 \Rightarrow 1$

⟨2⟩1. ASSUME:  $A$  is closed and  $\rho(x, y) < \epsilon$  for all  $x, y \in A$

- (2)2. PICK  $x_0 \in A$   
 (2)3. LET:  $b = \rho(\vec{0}, x_0)$   
 (2)4. LET:  $P = \epsilon + b$   
 (2)5.  $A \subseteq [-P, P]^n$   
 PROOF: For any  $y \in A$  we have  

$$\begin{aligned} \rho(\vec{0}, y) &\leq \rho(\vec{0}, x_0) + \rho(x_0, y) && \text{(Triangle Inequality)} \\ &< b + \epsilon && ((2)3, (2)1) \\ &= P && ((2)4) \end{aligned}$$
  
 (2)6.  $[-P, P]^n$  is compact.  
 PROOF: By Corollary 9.4.19.1 and Proposition 9.4.14.  
 (2)7. Q.E.D.  
 PROOF: By Proposition 9.4.6.

□

**Theorem 13.1.10 (AC).** *Let  $X$  be a topological space. Then  $X$  is compact if and only if every nonempty net in  $X$  has a convergent subnet.*

PROOF:

- (1)1. If  $X$  is compact then every nonempty net in  $X$  has a convergent subnet.  
 (2)1. ASSUME:  $X$  is compact.  
 (2)2. LET:  $(x_\alpha)_{\alpha \in J}$  be a nonempty net in  $X$   
 (2)3. For  $\alpha \in J$ ,  
     LET:  $B_\alpha = \{\beta \in J : \alpha \leq \beta\}$ .  
 (2)4.  $\{B_\alpha : \alpha \in J\}$  has the finite intersection property.  
     (3)1. LET:  $\alpha_1, \dots, \alpha_n \in J$   
     (3)2. PICK  $\beta \in J$  such that  $\alpha_1 \leq \beta, \dots, \alpha_n \leq \beta$   
     (3)3.  $x_\beta \in B_{\alpha_1} \cap \dots \cap B_{\alpha_n}$   
 (2)5. PICK  $l \in \bigcap_{\alpha \in J} B_\alpha$   
 PROOF: Proposition 9.4.15.  
 (2)6. LET:  $K = \{\alpha \in J : x_\alpha = l\}$   
 (2)7.  $K$  is cofinal in  $J$   
     (3)1. LET:  $\alpha \in J$   
     (3)2.  $l \in B_\alpha$   
 PROOF: By (2)5.  
     (3)3. There exists  $\beta \geq \alpha$  such that  $x_\beta = l$ .  
 (2)8.  $(x_\alpha)_{\alpha \in K}$  is a subnet of  $(x_\alpha)_{\alpha \in J}$  that converges to  $l$ .  
 (1)2. If every nonempty net in  $X$  has a convergent subnet then  $X$  is compact.  
 (2)1. ASSUME: Every nonempty net in  $X$  has a convergent subnet  
 (2)2. LET:  $\mathcal{A}$  be a nonempty set of closed sets with the finite intersection property.  
 (2)3. LET:  $J$  be the poset of all finite intersections of elements of  $\mathcal{A}$  under  $\supseteq$   
 (2)4. PICK  $x_C \in C$  for all  $C \in J$   
 PROOF: These are all nonempty by (2)2.  
 (2)5. PICK an accumulation point  $l$  of  $(x_C)$   
 PROVE:  $l \in \bigcap \mathcal{A}$   
 PROOF: One exists by Lemma 3.13.2.

- (2)6. LET:  $C \in \mathcal{A}$   
 PROVE:  $l \in C$   
 (2)7. LET:  $U$  be a neighbourhood of  $l$   
 PROVE:  $U$  intersects  $C$   
 (2)8. PICK  $D \subseteq C$  such that  $x_D \in U$   
 PROOF: By (2)5.  
 (2)9.  $U$  intersects  $C$   
 (2)10.  $l \in C$   
 PROOF: By Theorem 3.8.3 since  $C$  is closed ((2)2).  
 (2)11. Q.E.D.  
 PROOF: Proposition 9.4.15.

□

**Corollary 13.1.10.1 (AC).** *Let  $G$  be a topological group. Let  $A$  and  $B$  be subsets of  $G$ . If  $A$  is closed in  $G$  and  $B$  is compact then  $AB$  is closed in  $G$ .*

PROOF:

- (1)1. LET:  $c \in \overline{AB}$   
 PROVE:  $c \in AB$   
 (1)2. PICK a net  $(x_\alpha)_{\alpha \in J}$  that converges to  $c$   
 PROOF: By Theorem 3.12.3.  
 (1)3. For  $\alpha \in J$ , PICK  $a_\alpha \in A$  and  $b_\alpha \in B$  such that  $x_\alpha = a_\alpha b_\alpha$   
 (1)4. PICK a convergent subnet  $(b_{g(\beta)})_{\beta \in K}$  of  $(b_\alpha)_{\alpha \in J}$   
 PROOF: By Theorem 13.1.10.  
 (1)5. LET:  $b_{g(\beta)} \rightarrow b$   
 (1)6.  $b \in B$   
 (2)1.  $B$  is closed  
 PROOF: By Proposition 9.4.9.  
 (2)2. Q.E.D.  
 PROOF: By Theorem 3.12.3  
 (1)7.  $a_{g(\beta)} \rightarrow cb^{-1}$   
 PROOF: By Theorem 3.12.4  
 (1)8.  $cb^{-1} \in A$   
 PROOF: By Theorem 3.12.3  
 (1)9.  $c \in AB$   
 (1)10. Q.E.D.  
 PROOF: By Proposition 3.7.6.

**Proposition 13.1.11.** *Let  $A_0 + A_1$  be the sum of  $A_0$  and  $A_1$  with injections  $i_0 : A_0 \rightarrow A_0 + A_1$  and  $i_1 : A_1 \rightarrow A_0 + A_1$ .*

*Let  $g : B \rightarrow A_0 + A_1$  be a function.*

*Let  $B_0$  be the pullback of  $i_0$  and  $g$  with projections  $j_0 : B_0 \rightarrow B$  and  $k_0 : B_0 \rightarrow A_0$ .*

*Let  $B_1$  be the pullback of  $i_1$  and  $g$  with projection  $sj_1 : B_1 \rightarrow B$  and  $k_1 : B_1 \rightarrow A_1$ .*

*Then  $B$  is the sum of  $B_0$  and  $B_1$  with injections  $j_0$  and  $j_1$ .*

$$\begin{array}{ccccc}
B_0 & \xrightarrow{j_0} & B & \xleftarrow{j_1} & B_1 \\
\downarrow & & \downarrow g & & \downarrow \\
A_0 & \xrightarrow{i_0} & A_0 + A_1 & \xleftarrow{i_1} & A_1
\end{array}$$

PROOF:

(1)1. LET:  $X$  be any set and  $x : B_0 \rightarrow X, y : B_1 \rightarrow X$

**Proposition 13.1.12 (CC).** *Let  $X$  be a space and  $\mathcal{B}$  be a basis for  $X$ . Suppose that every subset of  $\mathcal{B}$  that covers  $X$  has a countable subcover. Then  $X$  is Lindelöf.*

PROOF:

- (1)1. LET:  $\mathcal{A}$  be an open cover of  $X$ .
- (1)2.  $\{B \in \mathcal{B} : \exists U \in \mathcal{A}. B \subseteq U\}$  covers  $X$ .
- (1)3. PICK a countable subcover  $\mathcal{B}_0$
- (1)4. For  $B \in \mathcal{B}_0$ , PICK  $U_B \in \mathcal{A}$  such that  $B \subseteq U_B$
- (1)5.  $\{U_B : B \in \mathcal{B}_0\}$  is a countable subcover of  $\mathcal{A}$ .

□

**Proposition 13.1.13 (CC).** *The space  $\mathbb{R}_l$  is Lindelöf.*

PROOF:

- (1)1. LET:  $\mathcal{A}$  be a set of basis elements  $[a, b)$  that covers  $X$   
 PROVE:  $\mathcal{A}$  has a countable subcover.
- (1)2. LET:  $C = \bigcup \{(a, b) : [a, b) \in \mathcal{A}\}$
- (1)3.  $\mathbb{R} \setminus C$  is countable.
  - (2)1. For all  $x \in \mathbb{R} \setminus C$ , PICK a rational  $q_x$  such that there exists  $b$  such that  $q_x \in [x, b) \in \mathcal{A}$ 
    - (3)1. PICK  $[a, b) \in \mathcal{A}$  such that  $x \in [a, b)$
    - (3)2.  $x = a$   
 PROOF: If not we would have  $x \in C$
    - (3)3. There exists a rational in  $(a, b)$
  - (2)2. For  $x, y \in \mathbb{R} \setminus C$ , if  $x < y$  then  $q_x < q_y$ 
    - (3)1. PICK  $b, c$  such that  $q_x \in [x, b) \in \mathcal{A}$  and  $q_y \in [y, c) \in \mathcal{A}$   
 PROOF: By (2)1.
    - (3)2.  $b \leq y$   
 PROOF: Otherwise we would have  $y \in (x, b) \subseteq C$ .
    - (3)3.  $q_x < q_y$   
 PROOF:  $q_x < b \leq y \leq q_y$
  - (2)3. The map  $q_- : \mathbb{R} \setminus C \rightarrow \mathbb{Q}$  is injective.
- (1)4. For  $x \in \mathbb{R} \setminus C$ , PICK  $[a_x, b_x) \in \mathcal{A}$  such that  $a_x \leq x < b_x$
- (1)5. PICK a countable subset  $((a_n, b_n))_{n \in \mathbb{Z}^+}$  of  $\{(a, b) : [a, b) \in \mathcal{A}\}$  that covers  $C$ 
  - (2)1. The set  $C$  as a subspace of  $\mathbb{R}$  with the standard topology is second countable.



⟨2⟩2. The set  $C$  as a subspace of  $\mathbb{R}$  with the standard topology is Lindelöf.

PROOF: By Theorem 9.3.2.

⟨1⟩6.  $\{[a_x, b_x) : x \in \mathbb{R} \setminus C\} \cup \{[a_n, b_n) : n \in \mathbb{Z}^+\}$  is a countable subcover of  $\mathcal{A}$ .

⟨1⟩7. Q.E.D.

PROOF: By Proposition 13.1.12.

□

**Proposition 13.1.14 (AC).** *The space  $\mathbb{R}_l$  is not second countable.*

PROOF:

⟨1⟩1. LET:  $\mathcal{B}$  be any basis for  $\mathbb{R}_l$

⟨1⟩2. For  $x \in \mathbb{R}$ , PICK  $B_x \in \mathcal{B}$  such that  $x \in B_x \subseteq [x, x+1)$

⟨1⟩3. The mapping  $B_{(-)}$  is an injective function  $\mathbb{R} \rightarrow \mathcal{B}$

PROOF: For any  $x$  we have  $x = \min B_x$ .

⟨1⟩4.  $\mathcal{B}$  is uncountable.

□

**Proposition 13.1.15.** *The product of a Lindelöf space and a compact space is Lindelöf.*

PROOF:

⟨1⟩1. LET:  $X$  be a Lindelöf space and  $Y$  a compact space.

⟨1⟩2. LET:  $\mathcal{A}$  be an open covering of  $X \times Y$

⟨1⟩3. For all  $x \in X$ , there exists a neighbourhood  $W$  of  $x$  such that  $W \times Y$  is covered by finitely many elements of  $\mathcal{A}$ .

⟨2⟩1. LET:  $x \in X$

⟨2⟩2.  $\{x\} \times Y$  is compact.

PROOF: It is homeomorphic to  $Y$ .

⟨2⟩3. PICK a finite subset  $\{U_1, \dots, U_m\}$  of  $\mathcal{A}$  that covers  $\{x\} \times Y$

PROOF: By Proposition 9.4.5.

⟨2⟩4. There exists a neighbourhood  $W$  of  $x$  such that  $W \times Y \subseteq U_1 \cup \dots \cup U_m$

PROOF: By the Tube Lemma.

⟨1⟩4.  $\{W \text{ open in } X : W \times Y \text{ is covered by finitely many elements of } \mathcal{A}\}$  is an open covering of  $X$ .

⟨1⟩5. PICK a countable subcovering  $\{W_1, W_2, \dots\}$

⟨1⟩6. For  $i \geq 1$ , PICK a finite subset  $\{U_{i1}, \dots, U_{ir_i}\}$  of  $\mathcal{A}$  that covers  $W_i \times Y$

⟨1⟩7.  $\{U_{1j} : i \geq 1, 1 \leq j \leq r_i\}$  is a countable subcovering of  $\mathcal{A}$ .

□

**Proposition 13.1.16.** *Let  $X$  be a  $T_1$  space. Then  $X$  is normal if and only if, for every closed set  $A$  and open set  $U \supseteq A$ , there exists an open set  $V \supseteq A$  such that  $\overline{V} \subseteq U$ .*

PROOF:

⟨1⟩1. If  $X$  is normal then, for every closed set  $A$  and open set  $U \supseteq A$ , there exists an open set  $V \supseteq A$  such that  $\overline{V} \subseteq U$ .

⟨2⟩1. ASSUME:  $X$  is normal.

⟨2⟩2. LET:  $A$  be a closed set and  $U$  an open set with  $A \subseteq U$

- (2)3. PICK disjoint open sets  $V, W$  such that  $A \subseteq V$  and  $X \setminus U \subseteq W$   
 (2)4.  $\overline{V} \subseteq U$

PROOF:

$$\begin{aligned}\overline{V} &\subseteq X \setminus W \\ &\subseteq U\end{aligned}$$

- (1)2. If, for every closed set  $A$  and open set  $U \supseteq A$ , there exists an open set  $V \supseteq A$  such that  $\overline{V} \subseteq U$ , then  $X$  is normal.  
 (2)1. ASSUME: for every closed set  $A$  and open set  $U \supseteq A$ , there exists an open set  $V \supseteq A$  such that  $\overline{V} \subseteq U$ .  
 (2)2. LET:  $A, B$  be disjoint closed sets  
 (2)3. PICK an open set  $V$  such that  $A \subseteq V$  and  $\overline{V} \subseteq X \setminus B$   
 (2)4.  $A \subseteq V$  and  $B \subseteq X \setminus \overline{V}$

□

**Definition 13.1.17** (Action). Let  $G$  be a topological group and  $X$  a topological space. An *action* of  $G$  on  $X$  is a continuous function  $\cdot : G \times X \rightarrow X$  such that, for all  $g, h \in G$  and  $x \in X$ :

1.  $e \cdot x = x$
2.  $g \cdot (h \cdot x) = gh \cdot x$

**Definition 13.1.18** (Orbit Space). Let  $G$  be a topological group,  $X$  a topological space, and  $\cdot : G \times X \rightarrow X$  an action of  $G$  on  $X$ . Then the *orbit space*  $X/G$  is the quotient space of  $X$  by the equivalence relation  $\sim$  generated by  $x \sim g \cdot x$  for all  $x \in X, g \in G$ .

**Theorem 13.1.19.** Let  $G$  be a topological group. Let  $X$  be a topological space. Let  $\cdot : G \times X \rightarrow X$  be an action of  $G$  on  $X$ . Then the canonical map  $\pi : X \rightarrow X/G$  is perfect.

- (1)1.  $\pi$  is closed.  
 (2)1. LET:  $A \subseteq X$  be closed.  
 (2)2.  $GA = \{g \cdot a : g \in G, a \in A\}$  is closed  
 (3)1. LET:  $z \notin GA$   
 (3)2. For all  $g \in G$  we have  $g \cdot z \notin A$   
 (3)3. For  $g \in G$ , there exist  $U$  an open neighbourhood of  $g$  and  $V$  an open neighbourhood of  $z$  such that  $UV$  does not intersect  $A$   
 (3)4.  $\{U \text{ open in } G : \exists V \text{ an open neighbourhood of } z. UV \cap A = \emptyset\}$  covers  $G$   
 (3)5. PICK a finite subcover  $\{U_1, \dots, U_n\}$   
 (3)6. For  $1 \leq i \leq n$ , PICK  $V_i$  an open neighbourhood of  $z$  such that  $U_i V_i \cap A = \emptyset$   
 (3)7.  $z \in V_1 \cap \dots \cap V_n \subseteq X \setminus GA$   
 (2)3.  $\pi(A)$  is closed  
 $\pi^{-1}(\pi(A)) = GA$   
 (1)2.  $\pi$  is continuous.  
 PROOF: By definition of the quotient topology.

⟨1⟩3.  $\pi$  is surjective.

PROOF: By definition.

⟨1⟩4. For all  $a \in X/G$  we have  $\pi^{-1}(a)$  is compact.

⟨2⟩1. LET:  $a \in X/G$

⟨2⟩2. PICK  $x \in X$  such that  $a = \pi(x)$

⟨2⟩3.  $\pi^{-1}(a) = \{gx : g \in G\}$

⟨2⟩4.  $\pi^{-1}(a)$  is homeomorphic to  $G$

□

**Corollary 13.1.19.1.** *If  $X$  is Hausdorff then so is  $X/G$ .*

**Corollary 13.1.19.2.** *If  $X$  is regular then so is  $X/G$ .*

**Corollary 13.1.19.3.** *If  $X$  is normal then so is  $X/G$ .*

**Corollary 13.1.19.4.** *If  $X$  is locally compact then so is  $X/G$ .*

**Corollary 13.1.19.5.** *If  $X$  is second countable then so is  $X/G$ .*

**Proposition 13.1.20.** *Let  $p : X \twoheadrightarrow Y$  be perfect. If  $X$  is second countable then so is  $Y$ .*

PROOF:

⟨1⟩1. PICK a countable basis  $\mathcal{B}$  for  $X$

⟨1⟩2. LET:  $\mathcal{J} = \{J \subseteq^{\text{fin}} \mathcal{B} : \exists W \text{ open in } Y. p^{-1}(W) \subseteq \bigcup J\}$

⟨1⟩3. For every  $J \in \mathcal{J}$ ,

LET:  $W_J = \bigcup \{W \text{ open in } Y : p^{-1}(W) \subseteq \bigcup J\}$ .

PROVE:  $\{W_J : J \in \mathcal{J}\}$  is a basis for  $Y$ .

⟨1⟩4.  $y \in V$  where  $V$  is open in  $Y$

⟨1⟩5.  $\{B \in \mathcal{B} : x \in B \subseteq p^{-1}(V)\}$  covers  $p^{-1}(y)$

⟨1⟩6. PICK a countable subcover  $J \subseteq^{\text{fin}} \mathcal{B}$

⟨1⟩7.  $y \in W_J \subseteq V$

⟨2⟩1.  $p^{-1}(y) \subseteq \bigcup J$

⟨2⟩2. PICK an open neighbourhood  $W$  of  $y$  such that  $p^{-1}(W) \subseteq \bigcup J$

PROOF: By Proposition 9.5.1.

⟨2⟩3.  $W \subseteq W_J$

□

**Proposition 13.1.21.** *A subspace of a  $T_1$  space is  $T_1$ .*

PROOF:

⟨1⟩1. LET:  $X$  be  $T_1$  and  $Y \subseteq X$

⟨1⟩2. LET:  $a \in Y$

⟨1⟩3.  $\{a\}$  is closed in  $X$

⟨1⟩4.  $\{a\}$  is closed in  $Y$

PROOF: By Corollary 4.3.4.1.

□

**Proposition 13.1.22 (DC).** *Not every topological group is normal.*

PROOF: From Proposition 6.5.6.  $\square$

**Theorem 13.1.23.** *A subspace of a completely regular space is completely regular.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $X$  be completely regular and  $Y \subseteq X$
- $\langle 1 \rangle 2$ . LET:  $a \in Y$  and  $A$  be closed in  $Y$  such that  $a \notin A$
- $\langle 1 \rangle 3$ . PICK  $C$  closed in  $X$  such that  $A = X \cap C$
- $\langle 1 \rangle 4$ . PICK a continuous function  $f : X \rightarrow [0, 1]$  such that  $f(a) = 0$  and  $f(C) = \{1\}$
- $\langle 1 \rangle 5$ .  $f \upharpoonright Y : Y \rightarrow [0, 1]$  is a continuous function such that  $(f \upharpoonright Y)(a) = 0$  and  $(f \upharpoonright Y)(A) = \{1\}$

$\square$

**Proposition 13.1.24 (DC).** *Every topological group is completely regular.*

PROOF:

- $\langle 1 \rangle 1$ . LET:  $G$  be a topological group
- $\langle 1 \rangle 2$ . LET:  $x \in G$  and  $A \subseteq G$  be closed such that  $x \notin A$   
 PROVE: There exists a continuous  $f : G \rightarrow [0, 1]$  such that  $f(x) = 0$  and  $f(A) = \{1\}$
- $\langle 1 \rangle 3$ . ASSUME: w.l.o.g.  $x = e$   
 PROOF:  $\lambda y.x^{-1}y$  is an automorphism of  $G$  that maps  $x$  to  $e$ .
- $\langle 1 \rangle 4$ . PICK a sequence  $V_n$  ( $n \geq 0$ ) of symmetric neighbourhoods of  $e$  disjoint from  $A$  such that  $V_n V_n \subseteq V_{n-1}$  for all  $n$
- $\langle 2 \rangle 1$ . LET:  $V_0 = X \setminus A$
- $\langle 2 \rangle 2$ . Given  $V_n$ , PICK a symmetric neighbourhood  $V_{n+1}$  of  $e$  such that  $V_{n+1} V_{n+1} \subseteq V_n$

PROOF: By Proposition 13.1.4.

- $\langle 1 \rangle 5$ . For every dyadic rational  $p$ , define an open set  $U(p)$  as follows:

$$\begin{aligned} U(1/2^n) &= V_n & (n \geq 0) \\ U((2k+1)/2^{n+1}) &= V_{n+1}U(k/2^n) & (0 < k < 2^n) \\ U(p) &= \emptyset & (p \leq 0) \\ U(p) &= G & (p > 1) \end{aligned}$$

- $\langle 1 \rangle 6$ . For all  $k$  and  $n$ , we have

$$V_n U(k/2^n) \subseteq U((k+1)/2^n)$$

- $\langle 2 \rangle 1$ .  $k \leq 0$

PROOF: In this case,  $V_n U(k/2^n) = \emptyset$

- $\langle 2 \rangle 2$ .  $k = 1$  and  $n > 0$

PROOF:

$$\begin{aligned} V_n U(1/2^n) &= V_n V_n \\ &\subseteq V_{n-1} \\ &= U(1/2^{n-1}) \end{aligned}$$

- $\langle 2 \rangle 3$ .  $k = 2a$  for some  $0 < a < 2^{n-1}$

PROOF:

$$\begin{aligned} V_n U(2a/2^n) &= V_n U(a/2^{n-1}) \\ &= U(2a+1/2^n) \end{aligned}$$

(2)4.  $k = 2a + 1$  for some  $0 < a < 2^{n-1}$

PROOF:

$$\begin{aligned} V_n U((2a+1)/2^n) &= V_n V_n U(a/2^{n-1}) \\ &\subseteq V_{n-1} U(a/2^{n-1}) \\ &\subseteq U((a+1)/2^{n-1}) \end{aligned}$$

(2)5.  $k \geq 2^n$

PROOF: In this case,  $U((k+1)/2^n) = G$ .

(1)7. Define  $f : G \rightarrow [0, 1]$  by

$$f(x) = \inf\{p : x \in U(p)\}$$

PROOF: This set is nonempty because  $x \in U(1)$  and bounded below because if  $x \in U(p)$  then  $p > 0$ .

(1)8. For  $n > 0$  we have  $\overline{U(k/2^n)} \subseteq V_n U(k/2^n)$

(2)1. LET:  $x \in \overline{U(k/2^n)}$

(2)2.  $V_n x$  is a neighbourhood of  $x$

(2)3. PICK  $y \in V_n x \cap U(k/2^n)$

(2)4. PICK  $z \in V_n$  such that  $y = zx$

(2)5.  $x = z^{-1}y$

(1)9. For  $p$  and  $q$  dyadic rationals, if  $p < q$  then  $\overline{U(p)} \subseteq U(q)$

(1)10. If  $x \in \overline{U(p)}$  then  $f(x) \leq p$

(2)1. For all  $q > p$  we have  $x \in U(q)$

(2)2. For all  $q > p$  we have  $f(x) \leq q$

(1)11. If  $x \notin U(p)$  then  $f(x) \geq p$

PROOF: If  $x \notin U(p)$  and  $x \in U(q)$  then  $q > p$ .

(1)12.  $f$  is continuous

(2)1. LET:  $x_0 \in X$

(2)2. LET:  $c < f(x_0) < d$

PROVE: There exist a neighbourhood  $U$  of  $x_0$  such that  $f(U) \subseteq (c, d)$

(2)3. PICK rational numbers  $p, q$  such that  $c < p < f(x_0) < q < d$

(2)4.  $x \notin \overline{U(p)}$

(2)5.  $x \in U(q)$

(2)6. Take  $U = U(q) \setminus \overline{U(p)}$

(1)13.  $f(e) = 0$

PROOF: We have  $e \in U(1/2^n)$  for all  $n$ .

(1)14.  $f(A) = \{1\}$

PROOF: If  $x \in A$  and  $x \in U(p)$  then  $p > 1$ .

□

**Definition 13.1.25** (Bijection). A function  $f : A \rightarrow B$  is a *bijection*,  $f : A \cong B$ , iff there exists a function  $f^{-1} : B \rightarrow A$ , the *inverse* of  $f$ , such that  $f^{-1} \circ f = \text{id}_A$  and  $f \circ f^{-1} = \text{id}_B$ .

**Theorem 13.1.26.** *Let  $Y$  be a normal space. Then  $Y$  is an absolute retract if and only if  $Y$  has the universal extension property.*

PROOF:

- ⟨1⟩1. If  $Y$  is an absolute retract then  $Y$  has the universal extension property.
- ⟨2⟩1. ASSUME:  $Y$  is an absolute retract.
- ⟨2⟩2. LET:  $X$  be a normal space,  $A$  a closed subspace of  $X$  and  $f : A \rightarrow Y$  a continuous function.
- ⟨2⟩3. LET:  $Z_f$  be the quotient space of  $X \cup Y$  under:  $a \sim f(a)$  for all  $a \in A$
- ⟨2⟩4. LET:  $p : X \cup Y \rightarrow Z_f$  be the quotient map
- ⟨2⟩5. For all  $x_1, x_2 \in X$  we have  $p(x_1) = p(x_2)$  iff  $x_1 = x_2$  or  $x_1, x_2 \in A$  and  $f(x_1) = f(x_2)$ ; and for  $x \in X$  and  $y \in Y$  we have  $p(x) = p(y)$  iff  $f(x) = y$ ; and for  $y_1, y_2 \in Y$  we have  $p(y_1) = p(y_2)$  iff  $y_1 = y_2$
- ⟨2⟩6.  $p$  imbeds  $Y$  into a closed subspace of  $Z_f$ 
  - ⟨3⟩1.  $p$  is injective on  $Y$
  - ⟨3⟩2.  $p^{-1} : p(Y) \rightarrow Y$  is continuous
    - ⟨4⟩1. LET:  $U \subseteq Y$  be open  
PROVE:  $p(U)$  is open
    - ⟨4⟩2.  $p^{-1}(p(U)) = f^{-1}(U) \cup U$
  - ⟨3⟩3.  $p(Y)$  is closed  
PROOF:  $p^{-1}(p(Y)) = A \cup Y$
- ⟨2⟩7.  $Z_f$  is normal
  - ⟨3⟩1.  $Z_f$  is  $T_1$   
PROOF: For  $y \in Y$  we have  $p^{-1}(y) = f^{-1}(y) \cup \{y\}$  which is closed.
  - ⟨3⟩2. Any two disjoint closed sets in  $Z_f$  can be separated by a continuous function.
    - ⟨4⟩1. LET:  $C$  and  $D$  be disjoint closed sets in  $Z_f$
    - ⟨4⟩2. PICK  $g : Y \rightarrow [0, 1]$  such that  $g(Y \cap p^{-1}(C)) = \{0\}$  and  $g(Y \cap p^{-1}(D)) = \{1\}$   
PROOF: By the Urysohn Lemma.
    - ⟨4⟩3. PICK  $h : X \rightarrow [0, 1]$  such that  $h(X \cap p^{-1}(C)) = \{0\}$  and  $h(X \cap p^{-1}(D)) = \{1\}$  and  $h$  agrees with  $g \circ f$  on  $A$   
PROOF: By the Tietze Extension Theorem applied to  $A \cup (X \cap p^{-1}(C)) \cup (X \cap p^{-1}(D))$ .
    - ⟨4⟩4. LET:  $k : Z_f \rightarrow [0, 1]$  be the continuous function such that  $k(p(x)) = h(x)$  for  $x \in X$  and  $k(p(y)) = g(y)$  for  $y \in Y$   
PROOF: By the Pasting Lemma
    - ⟨4⟩5.  $k(C) = \{0\}$
    - ⟨4⟩6.  $k(D) = \{1\}$
  - ⟨3⟩3. Q.E.D.  
PROOF: If  $g$  is such a continuous function then  $g^{-1}([0, 1/2])$  and  $g^{-1}((1/2, 1])$  are disjoint open sets that include  $A$  and  $B$  respectively.
- ⟨2⟩8. PICK a retraction  $r : Z_f \rightarrow p(Y)$
- ⟨2⟩9.  $p^{-1} \circ r \circ p : X \rightarrow Y$  extends  $f$
- ⟨1⟩2. If  $Y$  has the universal extension property then  $Y$  is an absolute retract.
  - ⟨2⟩1. ASSUME:  $Y$  has the universal extension property
  - ⟨2⟩2. LET:  $Z$  be a normal space,  $Y_0$  a closed subspace of  $Z$ , and  $\phi : Y \cong Y_0$  a homeomorphism
  - ⟨2⟩3. PICK a continuous extension  $f : Z \rightarrow Y$  of  $\phi^{-1}$

□  
 (2)4.  $\phi \circ f$  is a retraction

**Theorem 13.1.27.** *Every manifold is metrizable.*

PROOF:

(1)1. LET:  $X$  be an  $m$ -manifold.

(1)2.  $X$  is regular.

(2)1.  $X$  is  $T_1$

(2)2. LET:  $x \in X$  and  $U$  be a neighbourhood of  $x$

(2)3. PICK a neighbourhood  $V$  of  $x$  that is imbeddable in  $\mathbb{R}^m$

(2)4. PICK a neighbourhood  $W$  of  $x$  such that  $\overline{W} \subseteq U \cap V$

PROOF: One exists since  $V$  is regular (Proposition 6.3.4)

(2)5.  $x \in W$  and  $\overline{W} \subseteq U$

(2)6. Q.E.D.

PROOF: Proposition 6.3.2

(1)3. Q.E.D.

PROOF: By the Urysohn Metrization Theorem.

□

**Theorem 13.1.28.** *Let  $X$  be a compact Hausdorff space in which every point has a neighbourhood that is imbeddable in  $\mathbb{R}^m$ . Then  $X$  is an  $m$ -manifold.*

PROOF:

(1)1. There exists  $N$  such that  $X$  is imbeddable in  $\mathbb{R}^N$

PROOF: Theorem 11.1.5

(1)2.  $X$  is second countable.

PROOF: Proposition 7.3.3

□

**Proposition 13.1.29.**  *$S_\Omega$  is locally metrizable.*

PROOF: For any  $\alpha \in S_\Omega$ , the neighbourhood  $[0, \alpha] = (-\infty, \alpha + 1)$  is imbeddable in  $\mathbb{R}$ . □

**Proposition 13.1.30 (DC).**  *$\overline{S_\Omega}$  is compact.*

PROOF:PROOF:

(1)1. LET:  $\mathcal{A}$  be an open cover of  $\overline{S_\Omega}$

(1)2. ASSUME: for a contradiction there is no finite subcover of  $\mathcal{A}$

(1)3. There exists a sequence of sets  $U_n \in \mathcal{A}$  and ordinals  $\alpha_n$  such that  $\alpha_{n+1} < \alpha_n$  for all  $n$  and  $\alpha_n \in U_n$  for all  $n$

(2)1. LET:  $\alpha_1 = \Omega$

(2)2. Given  $\alpha_1, \dots, \alpha_n$  and  $U_1, \dots, U_{n-1}$  with  $0 \neq \alpha_n < \alpha_{n-1} < \dots < \alpha_1$  and  $\alpha_i \in U_i$  for  $i < n$ , PICK  $U_n \in \mathcal{A}$  with  $\alpha_n \in U_n$

PROOF: By (1)1.

(2)3. PICK  $\alpha_{n+1} < \alpha_n$  such that  $(\alpha_{n+1}, \alpha_n] \subseteq U_n$

PROOF: By Lemma 4.1.2.

(2)4.  $\alpha_{n+1} \neq 0$

PROOF: If  $\alpha_{n+1} = 0$  then  $U_1, \dots, U_n$  cover  $\overline{S_\Omega}$ , contradicting  $\langle 1 \rangle 2$ .  
 $\langle 1 \rangle 4$ . Q.E.D.

PROOF: This is a contradiction because the ordinals are well-ordered.  
 $\square$

**Proposition 13.1.31.**  $\mathbb{R}_l$  is not limit point compact.

PROOF:  $\mathbb{Z}$  has no limit point.  $\square$

**Proposition 13.1.32.** Every closed subspace of a Lindelöf space is Lindelöf.

PROOF:

- $\langle 1 \rangle 1$ . LET:  $X$  be Lindelöf and  $A \subseteq X$  be closed
  - $\langle 1 \rangle 2$ . LET:  $\mathcal{U}$  be an open covering of  $A$
  - $\langle 1 \rangle 3$ .  $\{U \text{ open in } X : U \cap A \in \mathcal{U}\} \cup \{X \setminus A\}$  covers  $X$
  - $\langle 1 \rangle 4$ . PICK a countable subcovering  $\mathcal{V}$
  - $\langle 1 \rangle 5$ .  $\{U \cap A : U \in \mathcal{V}, U \neq X \setminus A\}$  is a countable subcover of  $\mathcal{U}$
- $\square$

**Proposition 13.1.33.**  $\mathbb{R}^\omega$  is locally connected.

PROOF: This holds because every basic open set is connected, being the product of a family of connected spaces.  $\square$

**Proposition 13.1.34.** The space  $\mathbb{R}^\omega$  under the box topology is not first countable.

PROOF:

- $\langle 1 \rangle 1$ . ASSUME: for a contradiction  $\{U_n\}_{n \geq 0}$  is a countable basis at 0.
  - $\langle 1 \rangle 2$ . For  $n \geq 1$ , PICK a basic open set  $B_n = \prod_{j=0}^\infty (a_{nj}, b_{nj})$  such that  $0 \in B_n \subseteq U_n$
  - $\langle 1 \rangle 3$ .  $\prod_{n=0}^\infty (a_{nn}/2, b_{nn}/2)$  is a neighbourhood of 0 that does not include any  $U_n$
- $\square$

**Proposition 13.1.35.** The space  $\mathbb{R}^\omega$  under the box topology is not locally metrizable.

PROOF:

- $\langle 1 \rangle 1$ . LET:  $U$  be any neighbourhood of 0
- $\langle 1 \rangle 2$ . LET:  $A$  be the set of all sequences in  $U$  with all coordinates positive
- $\langle 1 \rangle 3$ .  $0 \in \overline{A}$
- $\langle 1 \rangle 4$ . There is no sequence of points of  $A$  converging to 0.
- $\langle 1 \rangle 5$ .  $U$  is not metrizable.

PROOF: By the Sequence Lemma.  
 $\square$

**Proposition 13.1.36.** For any nonempty set  $I$ , the space  $\mathbb{R}^I$  is not limit point compact.

PROOF:  $\mathbb{Z}^I$  is an infinite set with no limit point.  $\square$



**Proposition 13.1.37.** *The space  $\mathbb{R}^{[0,1]}$  is separable.*

PROOF: The set  $D$  is dense where  $D$  is the set of all functions  $f : [0, 1] \rightarrow \mathbb{Q}$  such that there exists a sequence of rationals  $0 = q_0 < q_1 < \cdots < q_N = 1$  such that  $f$  is constant on  $[q_i, q_{i+1})$  for  $0 \leq i < N$ .  $\square$

**Proposition 13.1.38.** *If  $J$  is uncountable then  $\mathbb{R}^J$  is not locally metrizable.*

PROOF: Every point has a neighbourhood homeomorphic to  $\mathbb{R}^J$ .  $\square$

**Proposition 13.1.39.** *The space  $\mathbb{R}_K$  is not limit point compact.*

PROOF: The set  $\mathbb{Z}$  has no limit point.  $\square$

**Proposition 13.1.40.** *The topologist's sine curve is not locally connected.*

PROOF: There is no connected neighbourhood of  $(0, 0)$ .  $\square$

**Corollary 13.1.40.1.** *Not every metric space is locally connected.*

**Corollary 13.1.40.2.** *Not every metric space is locally path connected.*

**Proposition 13.1.41.** *Not every metric space is compact.*

PROOF: The space  $\mathbb{R}$  is not compact.  $\square$

**Proposition 13.1.42.** *Every closed subspace of a limit point compact space is limit point compact.*

PROOF:

$\langle 1 \rangle 1$ . LET:  $X$  be a limit point compact space and  $C \subseteq X$  be closed.

$\langle 1 \rangle 2$ . LET:  $A \subseteq C$  be infinite.

$\langle 1 \rangle 3$ . PICK a limit point  $l$  of  $A$  in  $X$

$\langle 1 \rangle 4$ .  $l \in C$

$\langle 2 \rangle 1$ .  $l$  is a limit point of  $C$

PROOF: By Lemma 3.10.2.

$\langle 2 \rangle 2$ . Q.E.D.

PROOF: By Corollary 3.10.3.1.

$\langle 1 \rangle 5$ .  $l$  is a limit point of  $A$  in  $C$ .

PROOF: By Proposition 4.3.10.

$\square$

**Proposition 13.1.43.** *For any part  $i : S \hookrightarrow X$  of a set  $X$ , we have  $\emptyset \subseteq_X i$ .*

PROOF: We have  $i \circ j_S = j_X$  by the uniqueness of  $j_X$ .  $\square$

**Theorem 13.1.44.** *Let  $X$  be a completely regular space. Then there exists a compactification  $Y$  of  $X$  such that every bounded continuous map  $X \rightarrow \mathbb{R}$  extends uniquely to a continuous map  $Y \rightarrow \mathbb{R}$ .*

PROOF:

$\langle 1 \rangle 1$ . LET:  $J$  be the set of all bounded continuous functions  $X \rightarrow \mathbb{R}$

- (1)2. For  $\alpha \in J$ ,  
 LET:  $I_\alpha = [\inf \alpha, \sup \alpha]$   
 (1)3. LET:  $Z = \prod_{\alpha \in J} I_\alpha$   
 (1)4. LET:  $h : X \rightarrow Z$  be defined by  $h(x)_\alpha = \alpha(x)$   
 (1)5.  $Z$  is compact Hausdorff  
 (2)1.  $Z$  is compact  
 PROOF: By Tychonoff's Theorem.  
 (2)2.  $Z$  is Hausdorff  
 PROOF: By Theorem 6.2.5  
 (1)6.  $h$  is an imbedding  
 (2)1. The set  $J$  separates points from closed sets  
 PROOF: This holds because  $X$  is completely regular.  
 (2)2. Q.E.D.  
 PROOF: By the Imbedding Theorem.  
 (1)7. LET:  $Y$  be the compactification of  $X$  such that  $X \subseteq Y \rightarrow Z$  factors  $h$   
 PROOF: By Lemma 9.8.2  
 (1)8. Every bounded continuous map  $X \rightarrow \mathbb{R}$  extends uniquely to a continuous map  $Y \rightarrow \mathbb{R}$   
 (2)1. LET:  $\alpha : X \rightarrow \mathbb{R}$  be a bounded continuous function  
 (2)2. LET:  $k : Y \rightarrow Z$  be the imbedding from (1)7  
 (2)3. LET:  $\bar{\alpha} = \pi_\alpha \circ k : Y \rightarrow \mathbb{R}$   
 (2)4.  $\bar{\alpha}$  extends  $\alpha$   
 PROOF: For  $x \in X$ , we have

$$\begin{aligned}
 \bar{\alpha}(x) &= k(x)_\alpha \\
 &= h(x)_\alpha \\
 &= \alpha(x)
 \end{aligned}$$

- (2)5. If  $f : Y \rightarrow Z$  is continuous and extends  $\alpha$  then  $f = \bar{\alpha}$   
 PROOF: By Lemma 6.2.9.

□