Training Course

Advanced Waveform Design and Optimization Techniques

An application to pulse compression in weather radar systems

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Duration: Half Day

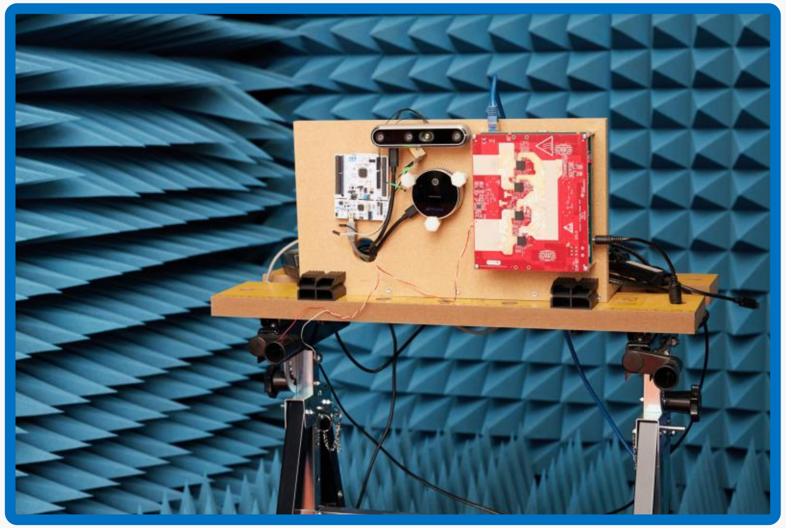
Goals of the course

- Getting familiar with advanced radar waveforms including LFM, binary phase, and polyphase sequences.
- Learn metrics for quantifying different waveforms, emphasizing their importance in weather radar applications.
- Understand basics of convex and nonconvex optimization techniques, including coordinate descent, gradient descent, and majorization minimization.
- Learn how to effectively apply nonconvex optimization techniques to design weather radar waveforms and improve sensitivity and accuracy in pulse compression.
- Understand the principles and benefits of waveform diversity in MIMO radar systems.

Expertise in radar signal processing, electromagnetics, systems modelling, wireless communications and hardware.

Signal Processing Applications in Radar and Communications (SPARC)

Research at SPARC



Projects

2 EC (ERC Advanced & ERC Proof-of-Concept)

6 National projects – Fundamental Research (1 bilateral with Germany), **2** National projects – Collaboration with Industry

1 US Airforce Overseas Research Lab Grant

Personnel

22 @ End of current hiring: 12 PhD, 3 Post Docs, 3 Research Scientists, 2 Developers, 1 Research Fellow, 1 Assistant Prof

Alumni : 4 Post Docs (Prof UIC USA, Director D-TA Canada, LiangDao), 5 PhD (Valeo, Barkhausen, Amphinicy, IEE, ApraNorm)

Members from academia, industry (Embraer, Continental, ..)

Dissemination

3 Books / > **20** Journals / > **5** Book Chapters / **>40** Conferences, **6** patents

Tutorials: 12

Prototype Demonstrations: 5

Interaction with industry

Actively leading the **strategic partnership with IEE** (www.iee.lu) **since 2015**

6 industry PhD students

2 National Projects

Distinctions

Best PhD Thesis Award from University of Luxembourg

Special mention : Barry Carlton Award from IEEE TAES

Student paper finalist IEEE Radar Conference 2019

Paper on Joint Radar Communications listed in the top 40 popular articles on IEEE TAES

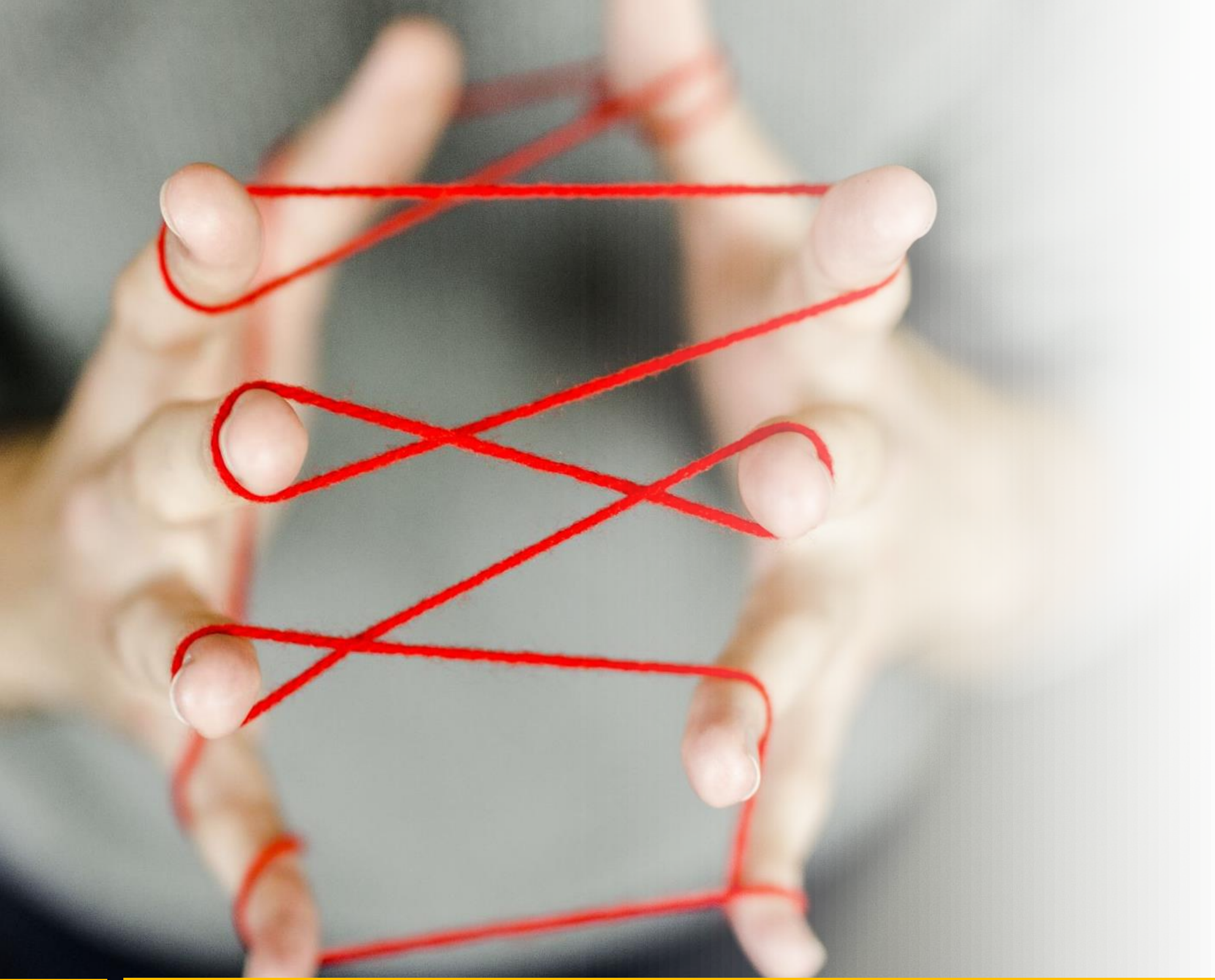
Paper in SP Magazine top 45 most cited (from 2019)

Luxembourg National Research Fund

Funded by the European Union

European Research Council
Established by the European Commission

Acknowledgement



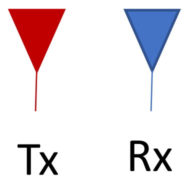
Radar Waveforms

Radar Classification

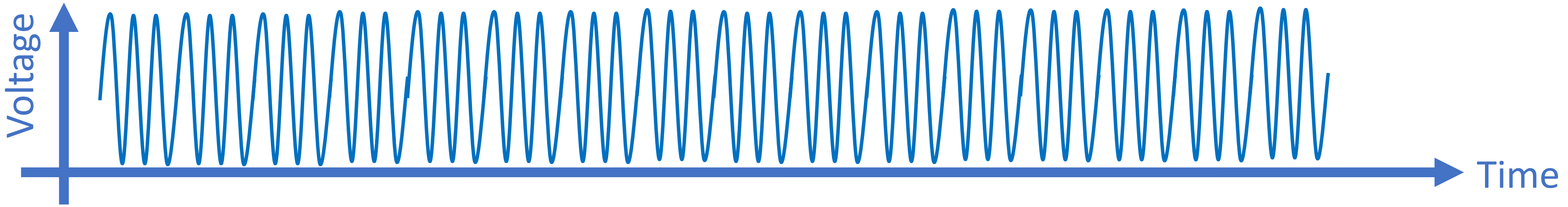
- **Frequency**
  - HF/ VHF Band
  - L Band
  - S Band
  - C Band
  - X Band
  - Ku, K, Ka Band
- **Waveform**
  - Un/modulated Pulse Radar
  - Un/modulated CW
  - OFDM/OTFS
  - Noise Radar
- **Applications**
  - Surveillance (Ground, Air, Sea) / Tracking
  - SAR/ Airborne
  - Weather
  - Automotive / Industry/ Health
  - Space
- **Type**
  - Primary
  - Secondary
- **Operation**
  - Active
  - Passive
- **Antenna Scan**
  - Mechanical
  - Phased Array / Digital Beamforming
  - MIMO
- **Geo Location**
  - Mono-static
  - Bi-static
  - Multi-static
- **Polarization**
  - Vertical
  - Horizontal
  - Circular
  - Dual Polarized

Transmit Signal, CW or Pulsed?

Unmodulated CW

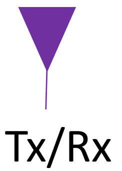


CW



Pulsed

Unmodulated Pulsed Radar

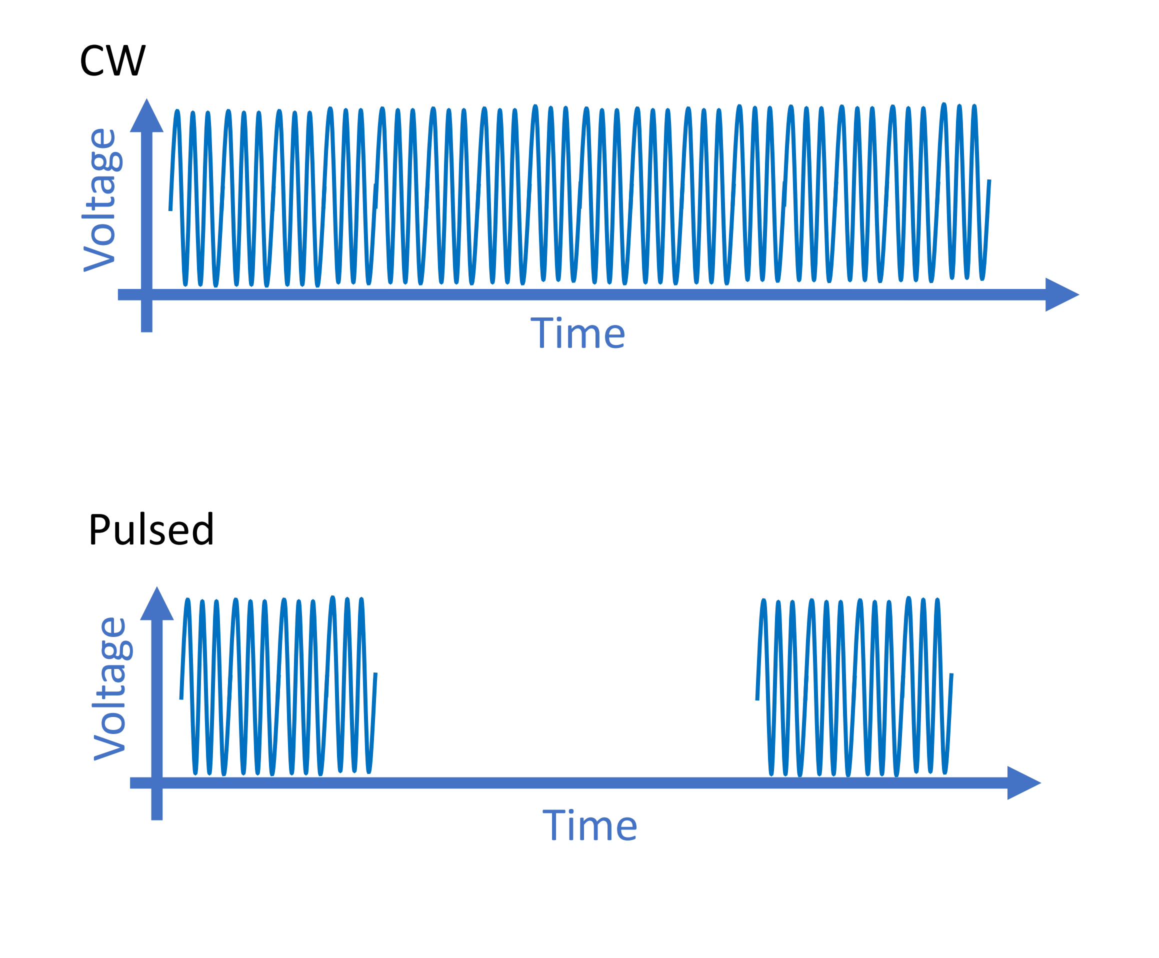


Interleave transmit and receive periods

Bandwidth $\approx \frac{1}{T_p}$ $\Rightarrow$ Time × Bandwidth ≈ 1

Transmit Signal, CW or Pulsed?

Continuous Wave	**Pulsed**
Requires separate transmit and receive antennas.	Same antenna is used for transmit and receive.
Isolation requirements limit transmit power.	Time-multiplexing relaxes isolation requirements to allow high power.
Radar has no blind ranges.	Radar has blind ranges due to "eclipsing" during transmit events.



What is a Pulse Compression?

- Higher average power is proportional to pulse width
- Better resolution is inversely proportional to pulse width

A long pulse can have the same bandwidth (resolution) as a short pulse if the long pulse be modulated with a "waveform"

energy of a long pulse + resolution of a short pulse

Time × Bandwidth ≫ 1

What is a Waveform?

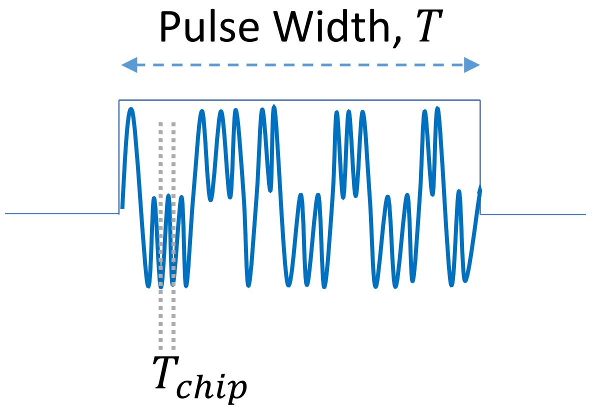
a waveform is a structured modulation of the pulse, typically in frequency/phase (FM/PM), and sometimes also in amplitude (AM).

AM waveform also necessitates linearity at the transmitter power amplifier (PA) to prevent waveform distortion

If a waveform has constant amplitude, the PA can be operated in saturation with much less distortion.

Pulse Compression (intra-pulse modulation)

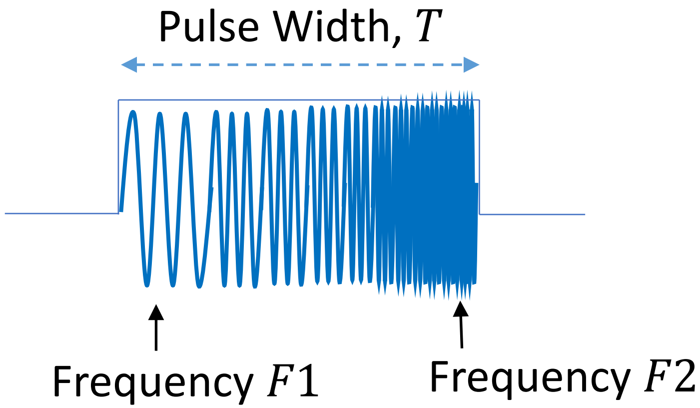
Phase Modulated Waveform



$$\text{Bandwidth} = \frac{1}{T_{chip}}$$

$$\text{Time} \times \text{Bandwidth} = \frac{\mathrm{T}}{T_{chip}}$$

Frequency Modulated Waveform



$$\text{Bandwidth} = \mathrm{B} = \Delta F = F2 - F1$$

$$\text{Time} \times \text{Bandwidth} = T \times B$$

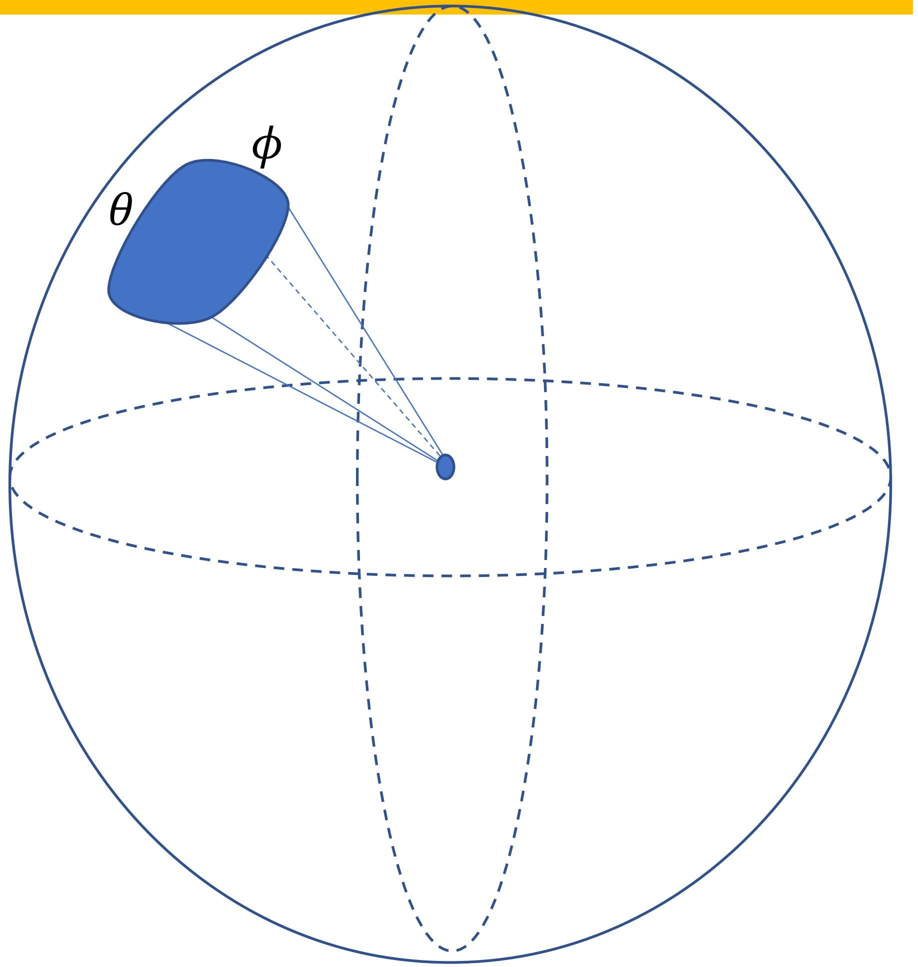
Why not using amplitude modulation?

Received Power in Radar

Assume, for simplicity, that the antenna is illuminating the interior of an imaginary sphere with equal power density in each unit of surface area

$$A_s = 4\,\pi R^2$$

A_s = area of a sphere
R = radius of the sphere



The power density is found by dividing the total transmit power, in watts, by the surface area of the sphere in square meters:

power density in watts per square meters

$$\rho = \frac{P_t}{A_s} = \frac{P_t}{4\pi R^2}$$

total transmitted power in watts

Received Power in Radar

Because radar systems use directive antennas to focus radiated energy on a target, the equation can be modified to account for the antenna's directive gain.

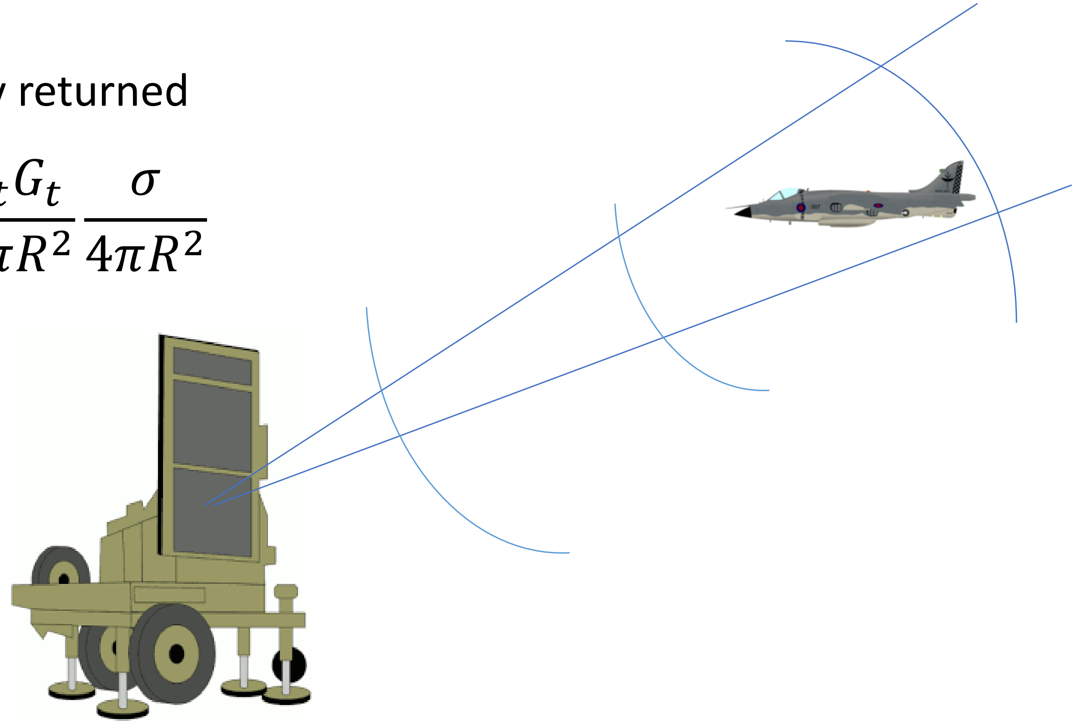
Power density at target

$$\rho_t = \frac{P_t G_t}{4\pi R^2}$$

Power density returned

$$\rho_R = \frac{P_t G_t}{4\pi R^2}\frac{\sigma}{4\pi R^2}$$

Some of that energy will be reflected in different directions, while others will be reradiated back to the radar system.



Received Power in Radar

The radar antenna will receive a portion of this signal reflected by the target. This signal power is equal to the return power density at the antenna multiplied by the effective area of the antenna

$$P_r = \frac{P_t\, G_t\, \sigma\, A_e}{(4\pi)^2 R^4}$$

P_r = signal power received at the receiver in watts
P_t = transmitted power in watts
G_t = gain of transmit antenna
σ = RCS in square meters
R = radius or distance to the target in meters
A_e = effective area of the receive antenna square meters

Antenna theory allows us to relate the gain of an antenna to its effective area as follows:

$$A_e = \frac{G_r \lambda^2}{4\pi}$$

G_r = gain of the receive antenna
λ = wavelength of the radar signal in meters

Received Power in Radar

For a monostatic radar the antenna gain G_t and G_r are equivalent. This is assumed to be the case for this derivation:

$$P_r = \frac{P_t\, G_t\, \sigma}{(4\pi)^2 R^4} \frac{G_r \lambda^2}{4\pi}$$

If $G_t = G_r$ $\longrightarrow$

$$\boxed{P_r = \frac{P_t\, G^2 \lambda^2 \sigma}{(4\pi)^3 R^4}}$$

P_r = signal power received at the receiver in watts
P_t = transmitted power in watts
G = antenna gain (assume same antenna for transmit and receive)
λ = wavelength of the radar signal in meters
σ = RCS of the target in square meters
R = radius or distance to the target in meters

What is the effect of transmit signal bandwidth on the received power/SNR ?

Noise Power in Radar

The theoretical limit of the noise power at the input of the receiver is described the thermal noise. It is a result of the random motion of electrons and is proportional to temperature:

$$P_n = k\,T_s B_n \approx k\,T_0 F_n B_n$$

System noise temperature

$$T_s = \underbrace{T_a}_{\text{Antenna temperature}} + \underbrace{T_r}_{\text{RF components temperature}} + F_n T_0$$

P_n = noise power in watts
k = Boltzmann's constant (1.38×10^{-23} J/K)
T_s = system noise temperature in Kelvin = $T_0 F_n$
T_0 = standard temperature in Kelvin (290 K)
B_n = system noise bandwidth in Hz
F_n = noise figure of the receiver subsystem (unitless)

At a room temperature of 290 K, the available noise power at the input of the receiver is 4 x 10-21 W/Hz, –203.98 dBW/Hz, or –173.98 dBm/Hz.

Signal to Noise Ratio in Radar

$$P_r = \frac{P_t\, G_t G_r \lambda^2 \sigma}{(4\pi)^3 R^4}$$

$$P_n = k\, T_0 F_n B_n$$

$$\Rightarrow \quad \boxed{SNR = \frac{P_t\, G_t G_r \lambda^2 \sigma}{(4\pi)^3 k\, T_0 F_n B_n R^4}}$$

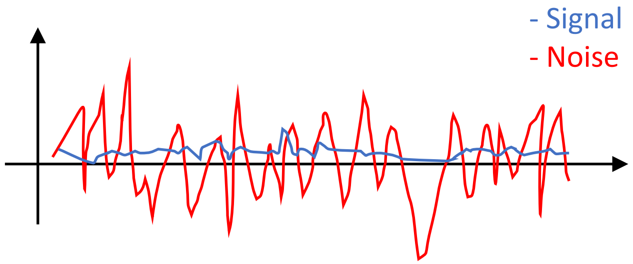
Signal to interference (clutter + jamming) Ratio

$$SINR = \frac{P_r}{P_n + P_c + P_J}$$

P_c = received power from clutter
P_J = received power from jammer

Signal Processing Loss and Gain

Integration gain (inter-pulse)



Signals are same each time; add "coherently"

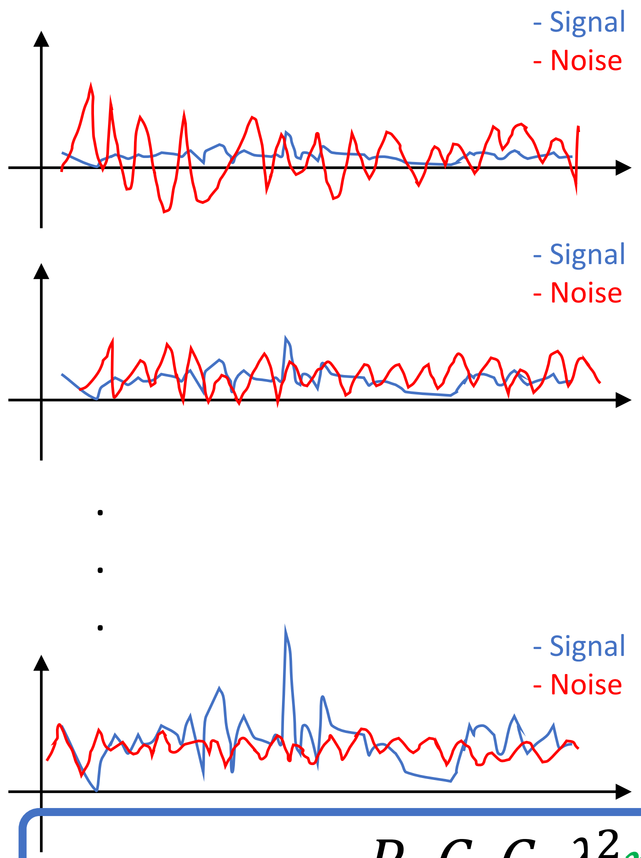
Noise is different each time; doesn't add coherently

1st Return

+ 2nd Return

.
.
.

+ n_p^{th} Return

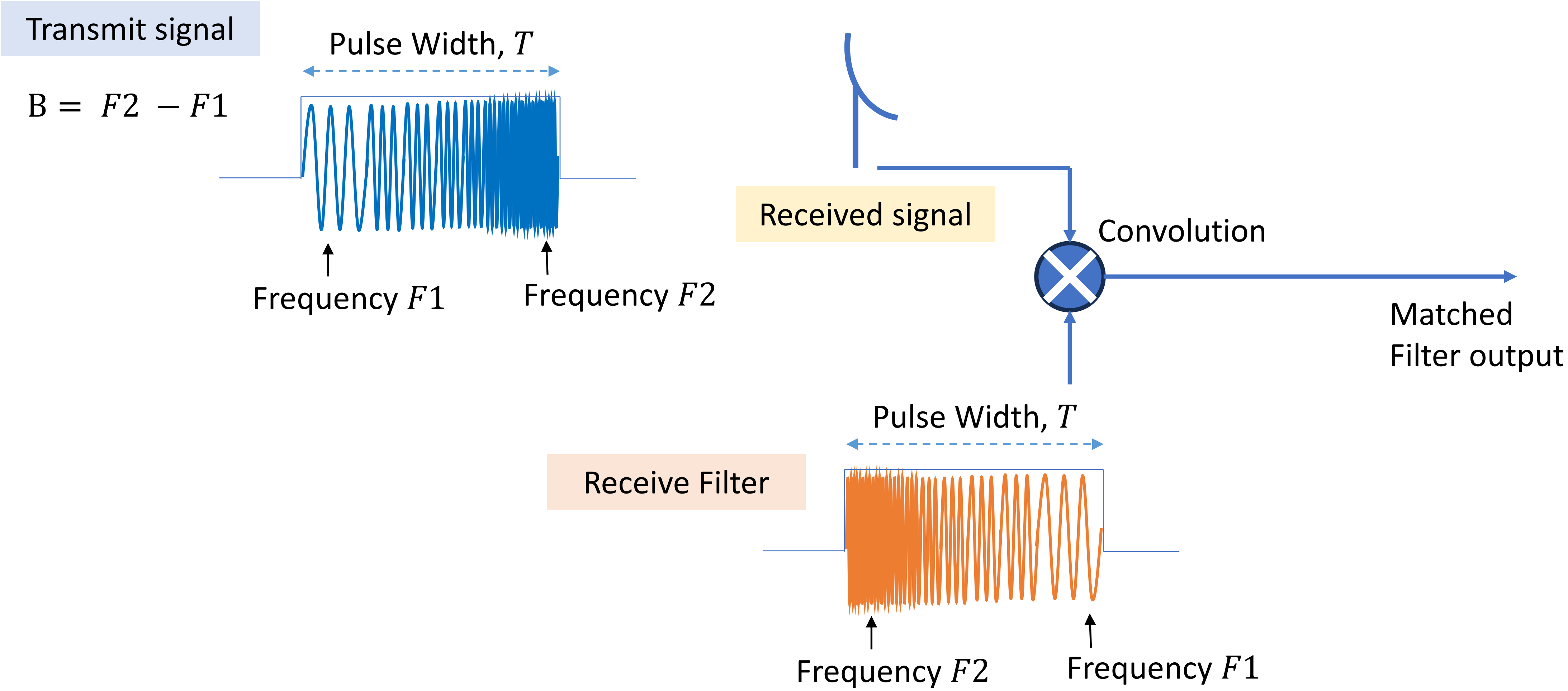


Signal integrated out of Noise (SNR increases by n_p)

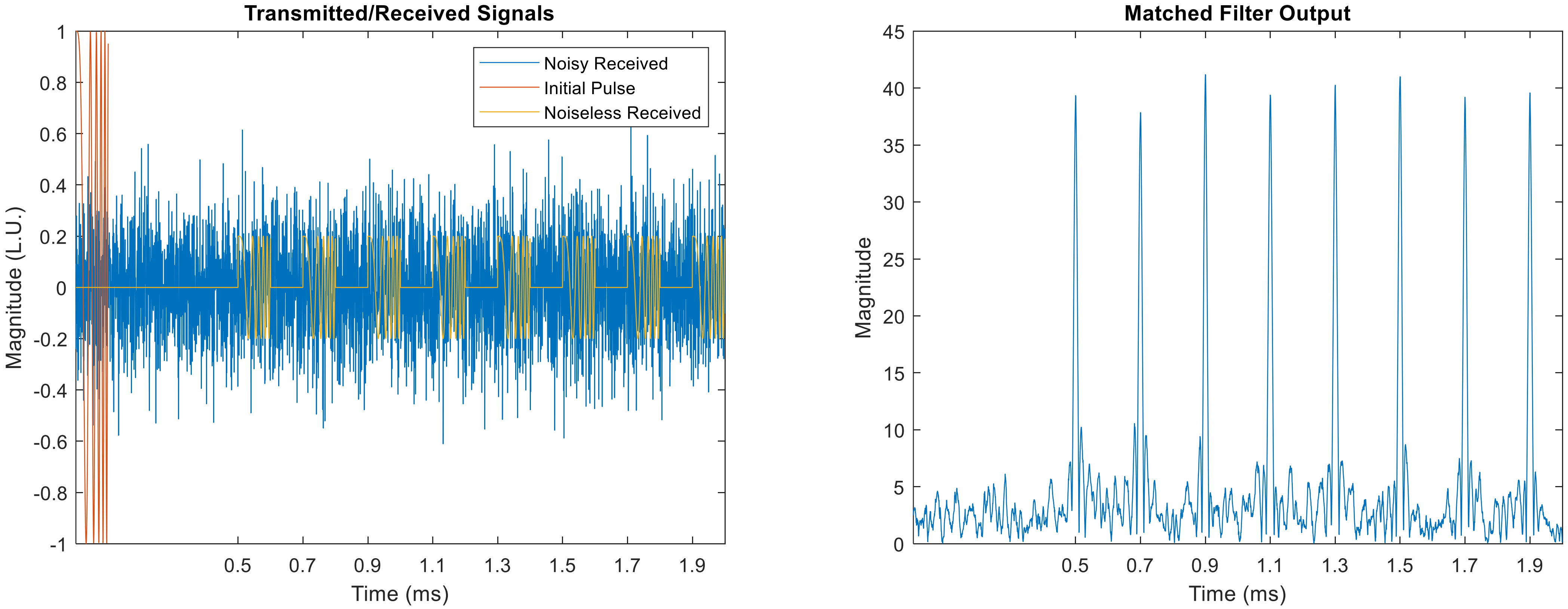
SNR_u is the signal-to-noise ratio for an unmodulated pulse

$$SNR_u = \frac{P_t\, G_t G_r \lambda^2 n_p \sigma}{(4\pi)^3 k\, T_0 F_n B_n L_s R^4}$$

Pulse Compression



Pulse Compression



Time × Bandwidth = $T \times B = N$

Signal to Noise Ratio in Radar with Pulse Compression

Time-Bandwidth Product

$$\boxed{SNR_{pc} = SNR_u \times \underbrace{(T \times B)}}$$

SNR_{pc} is the signal-to-noise ratio for a modulated (pulse compression) pulse.
SNR_u is the signal-to-noise ratio for an unmodulated pulse.
T is the pulse length.
B is the pulse modulation bandwidth

$$SNR_u = \frac{P_t\, G_t G_r \lambda^2 n_p \sigma}{(4\pi)^3 k\, T_0 F_n B_n L_s R^4}$$

If $T \times B = N \Rightarrow$

$$\boxed{SNR_{pc} = \frac{P_t\, G_t G_r \lambda^2 N n_p \sigma}{(4\pi)^3 k\, T_0 F_n B_n L_s R^4}}$$

How much is pulse compression loss?

Received Power in Weather Radar

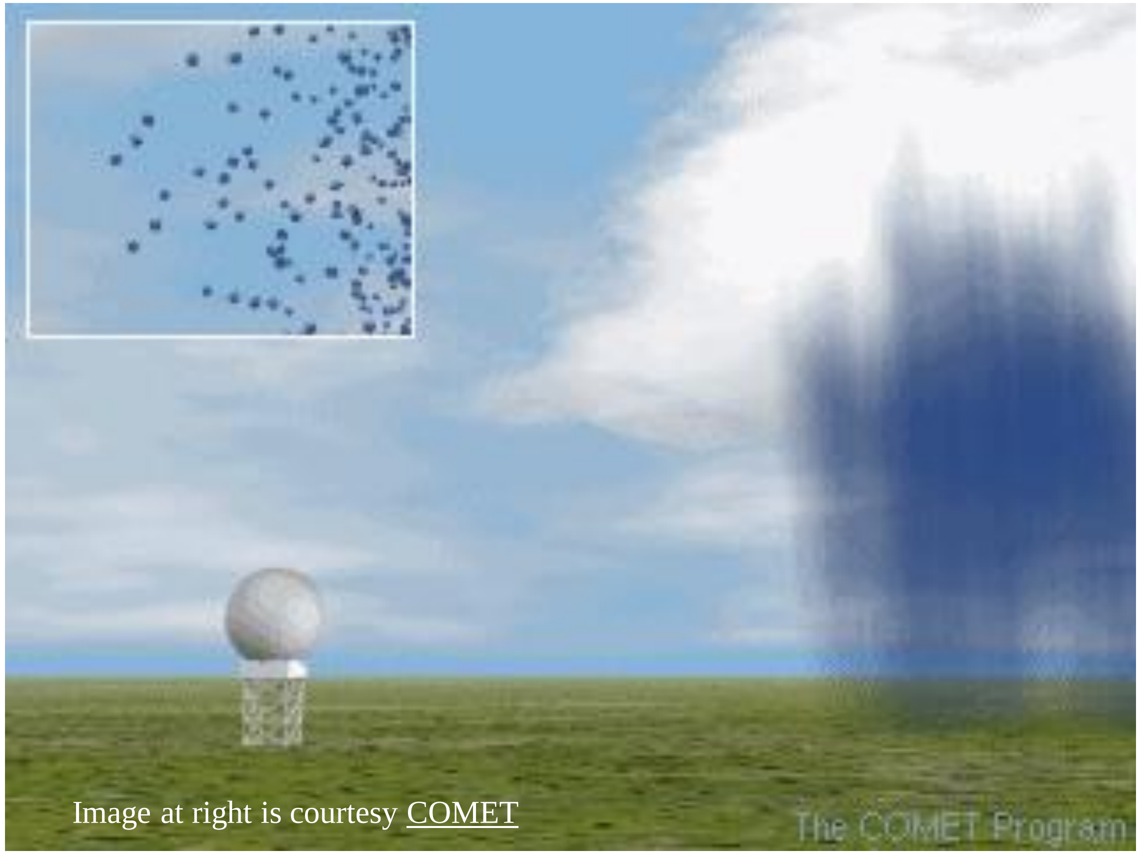


Image at right is courtesy COMET

Point Targets

$$P_r = \frac{P_t G^2 \lambda^2 \sigma}{(4\pi)^3 R^4}$$

Distributed Targets

$$P_r = \frac{P_t G^2 \lambda^2 \sum \sigma}{(4\pi)^3 R^4}$$

contributing volume $V = \pi \left(\frac{R\theta}{2}\right)^2 \frac{h}{2}$ $\quad h = \frac{c\tau}{2}$

$$P_r = \frac{P_t G^2 \lambda^2}{(4\pi)^3 R^4} \pi \left(\frac{R\theta}{2}\right)^2 \frac{c\tau}{4} \eta$$

η denotes the radar reflectivity per unit volume

Received Power in Weather Radar

$$P_r = \frac{P_t\, G^2 \lambda^2}{(4\pi)^3 R^4} \pi \left(\frac{R\theta}{2}\right)^2 \frac{c\tau}{4} \eta$$

sphere radius

$$\eta = 64 \frac{\pi^5}{\lambda^4} |K|^2 \sum D^6$$

Reflectivity

$$P_r = \frac{\pi^3 c}{(4)^3 16 \ln 2} \left[\frac{P_t\, \tau G^2 \theta^2}{\lambda^2}\right] \left[|K|^2 \frac{Z}{R^2}\right]$$

Signal to Noise Ratio in Weather Radar

$$P_r = \frac{\pi^3 c}{1024 \ln 2} \left[\frac{P_t \tau G^2 \theta^2}{\lambda^2} \right] \left[|K|^2 \frac{Z}{R^2} \right]$$

$$P_n = k\, T_0 F_n B_n$$

$$\Rightarrow \quad \boxed{SNR = \frac{\pi^3 c P_t \tau G^2 \theta^2 |K|^2 Z}{1024 \ln 2\, \lambda^2 k\, T_0 F_n B_n R^2}}$$

Signal to Noise Ratio in Weather Radar with Pulse Compression

$$\boxed{SNR_{pc} = SNR_u \times \underbrace{(T \times B)}_{\text{Time-Bandwidth Product}}}$$

SNR_{pc} is the signal-to-noise ratio for a modulated (pulse compression) pulse.
SNR_u is the signal-to-noise ratio for an unmodulated pulse.
T is the pulse length.
B is the pulse modulation bandwidth

$$SNR_u = \frac{\pi^3 c P_t \tau G^2 \theta^2 n_p |K|^2 Z}{1024 \ln 2\, \lambda^2 k\, T_0 F_n L_s B_n R^2}$$

If $T \times B = N \Rightarrow$

$$\boxed{SNR_{pc} = \frac{\pi^3 c P_t \tau G^2 \theta^2 N n_p |K|^2 Z}{1024 \ln 2\, \lambda^2 k\, T_0 F_n L_s B_n R^2}}$$

Pulse Compression in Weather Radar

- Instead of a short pulse, we transmit a modulated long pulse ($\Delta R = \frac{c}{2B}$)
- The energy of the transmitted pulse is now $P_t T$, so SNR will be increased by ***time-bandwidth product*** TB or $\frac{T}{\tau}$
- The maximum detection range for a given target is increased by a factor $\sqrt{TB}$
- For $BT = 100$, the processing gain is $10 \log 10(100) = 20\ dB$
- The maximum detection range for this target would be increased by a factor $\sqrt{100} = 10$

1. Blind Range [in pulsed radar]
2. Sidelobes (PSL and ISL)
3. Doppler tolerance
4. Receiver linearity

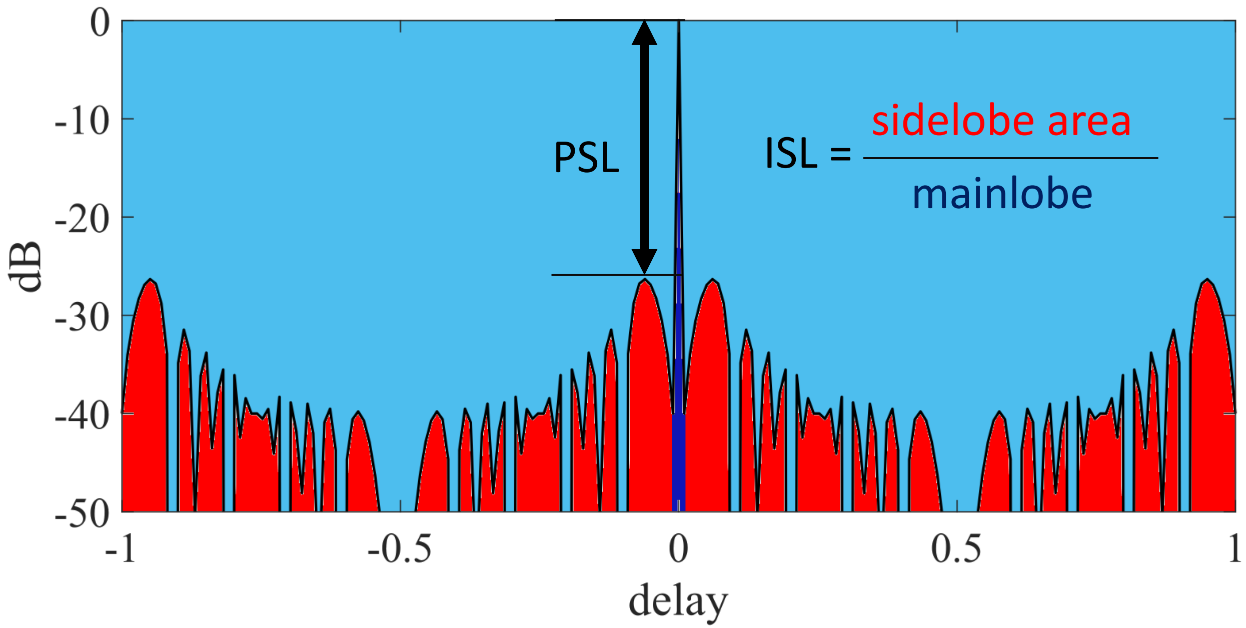
Mathematical optimization for Sidelobe Reduction in Weather Radar

Metrics for Goodness of the Waveforms

Low PSL
- avoid masking of weak targets

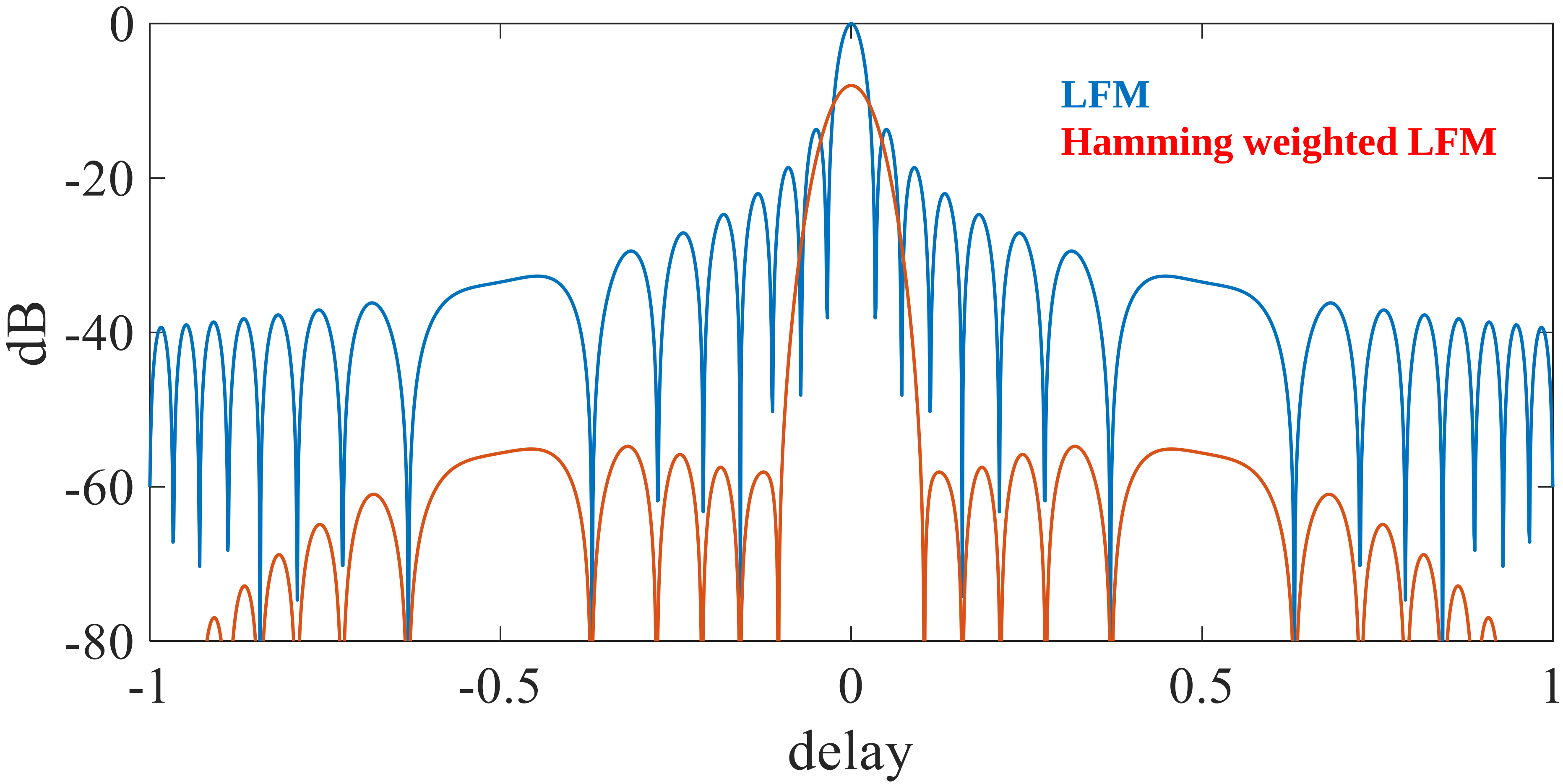
Low ISL
- mitigate deleterious effects of distributed clutter

Good Waveform

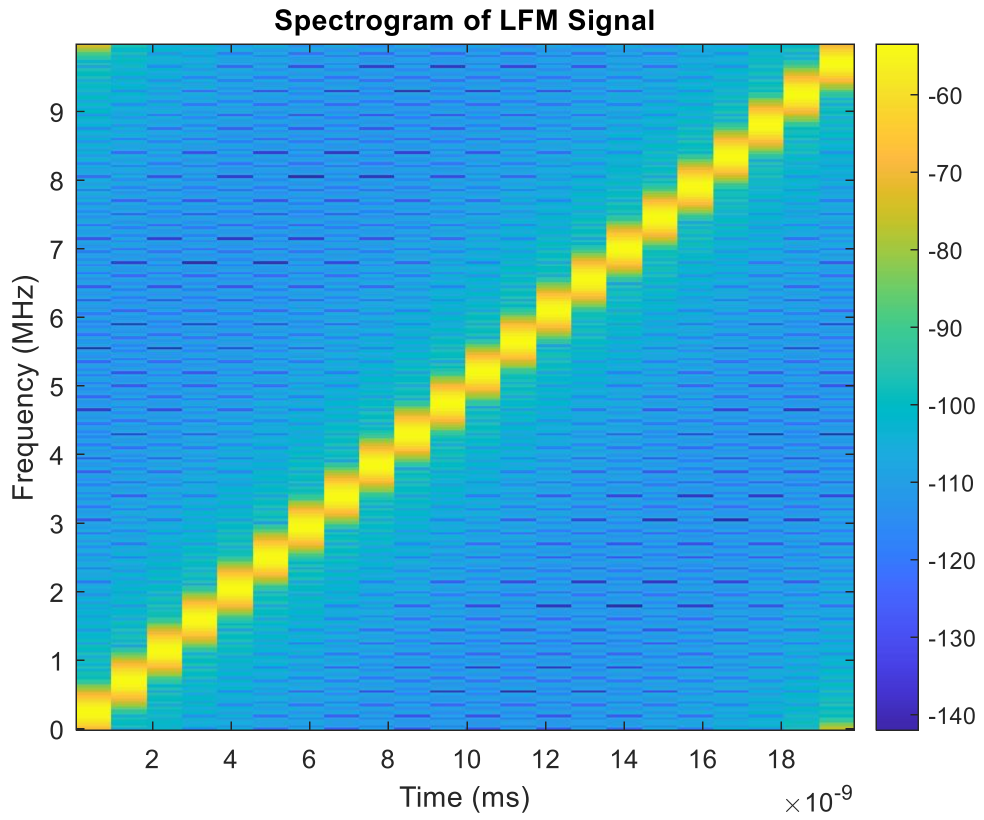


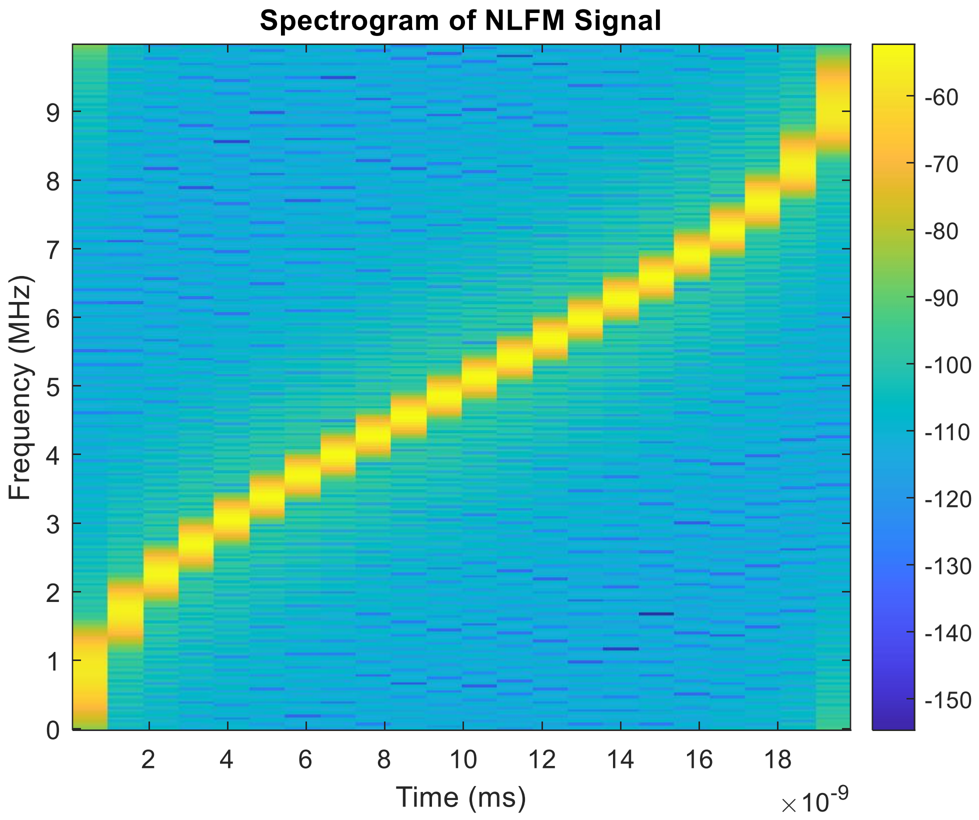
Conceptual definition of PSL and ISL measured on autocorrelation function response of a Golomb sequence

Weighted LFM

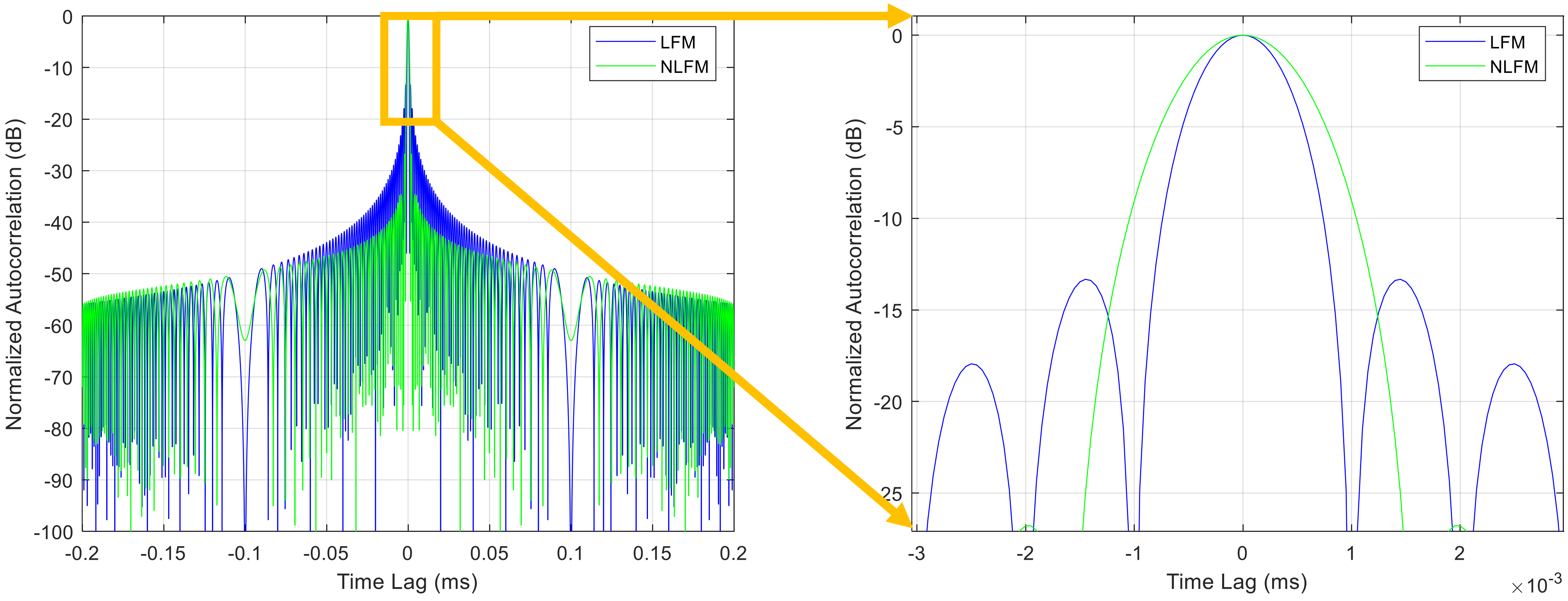


Nonlinear FM (NLFM)





NLFM



LFM
NLFM
Normalized Autocorrelation (dB)
Time Lag (ms)
0
-10
-20
-30
-40
-50
-60
-70
-80
-90
-100
-0.2
-0.15
-0.1
-0.05
0
0.05
0.1
0.15
0.2
LFM
NLFM
Normalized Autocorrelation (dB)
Time Lag (ms)
0
-5
-10
-15
-20
-25
-3
-2
-1
0
1
2
$\times 10^{-3}$

Searching for Good Phase Sequences

Some well-known binary sequences	Barker codes
	M-sequences (Maximum length sequences)
	Gold codes
	Kasami codes
	Minimum Peak Sidelobe (MPS) sequences

Limited in length

No constraint on aperiodic auto-correlation function

Hard to find

Mathematical optimization

The selection of a best element, with regard to some criterion, from some set of available alternatives

Simple example:

$$\begin{cases} \underset{x}{\text{minimize}} & x^2 + 1 \\ \text{subject to} & x \in [-1,1] \end{cases} \longrightarrow \quad x^\star = 0$$

Optimal solution:

If $x \in [1,2]$, then $x^\star = 1$

Optimization (disambiguation)

Article Talk

From Wikipedia, the free encyclopedia

Our scope!

Mathematical optimization is the theory and computation of extrema or stationary points of functions.

Optimization, **optimisation**, or **optimality** may also refer to:

- Engineering optimization
- Feedback-directed optimisation, in computing
- Optimality model in biology
- Optimality theory, in linguistics
- Optimization (role-playing games), a gaming play style
- *Optimize* (magazine)
- Process optimization, in business and engineering, methodologies for improving the efficiency c
- Product optimization, in business and marketing, methodologies for improving the quality and d concept
- Program optimization, in computing, methodologies for improving the efficiency of software
- Search engine optimization, in internet marketing
- Supply chain optimization, a methodology aiming to ensure the optimal operation of a manufact
- Social media optimization, in internet marketing, involves optimizing social media profiles

How to formulate the optimization problem for waveform design?

$$\begin{cases} \underset{\boldsymbol{x}}{\text{minimize}} & f(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in \boldsymbol{\mathcal{X}} \end{cases}$$

$\boldsymbol{x}$: optimization variable

$f(\boldsymbol{x})$: objective function

$\boldsymbol{x} \in \boldsymbol{\chi}$: constraint

$\Rightarrow$

$\boldsymbol{x}$: waveform

$f(\boldsymbol{x})$: performance metrics

$\boldsymbol{x} \in \boldsymbol{\chi}$: requirements on waveform

Mathematical Optimization

$$\boldsymbol{x} = [x_1, x_2, \dots, x_N]^T$$

Transmit waveform

Code length

$$r_k = \sum_{n=1}^{N-k} x_n^* x_{n+k}\,, \qquad k = 0, \dots, N-1$$

$$\boxed{\mathrm{PSL} = \max_{k \neq 0} |r_k|}$$

$$\boxed{\mathrm{ISL} = \sum_{k=1}^{N-1} r_k^2}$$

PSL Minimization Problem

$$\boldsymbol{x} = [x_1, x_2, \dots, x_N]^T$$

$$\mathcal{P}_{\boldsymbol{x}} \begin{cases} \underset{\boldsymbol{x}}{\text{minimize}} & \max_{k \neq 0} |r_k| \\ \text{subject to} & x_n \in \psi_n \end{cases}$$

ISL Minimization Problem

$$\boldsymbol{x} = [x_1, x_2, \dots., x_N]^T$$

$$\mathcal{P}_{\boldsymbol{x}} \begin{cases} \underset{\boldsymbol{x}}{\text{minimize}} & \sum_{k=1}^{N-1} r_k^2 \\ \text{subject to} & x_n \in \psi_n \end{cases}$$

Waveform Requirements

- Hardware perspective: costly, non-ideal
- Some factors need to consider when designing waveforms
- Two common waveform constraints: Unimodularity, finite phase value

Nonideal power amplifier:
limited linear region

Unimodular waveform: $|x_n| = 1, \forall n = 1, \ldots, N$

More general version

Peak to average power ration (PAPR):

$$\text{PAR}\left(\mathbf{x}\right) = \max_n \left\{ |x_n|^2 \right\} / \, \|\mathbf{x}\|_2^2 \leq \gamma$$

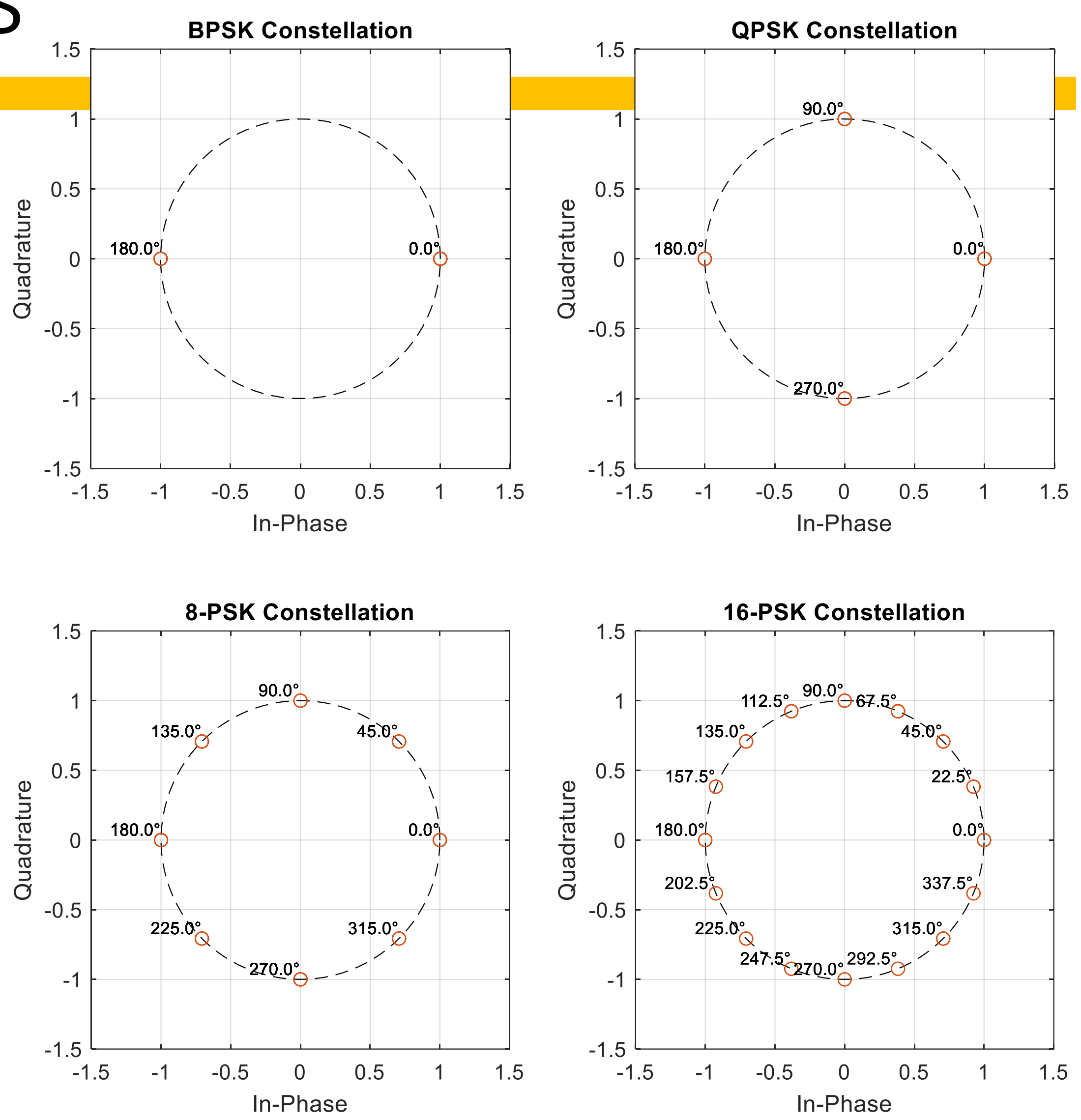
Waveform Requirements

- Generates finite phase values
- Phase quantization should be considered

Constraint on phase alphabet

$$x_n \in \Omega_M$$

$$\Omega_M = \left\{1, e^{\frac{j2\pi}{M}}, \ldots, e^{\frac{j2\pi(M-1)}{M}}\right\}$$



Constraints

- Constraints
  - Energy
  - Peak-to-Average Power Ratio (PAPR, PAR)
  - Unimodularity (being Constant-Modulus)
  - Finite or Discrete-Alphabet (integer, binary, m-ary constellation)
  - …
- Challenges
  - How to handles signal constraints?
  - How to do it fast?

- Many of these problems are shown to be **NP-hard**
- Many others are deemed to be difficult

How to solve the formulated waveform design problem?

objective function

Useful performance metrics:
PSL / ISL

Practical waveform requirement:
Unimodular, phase alphabet

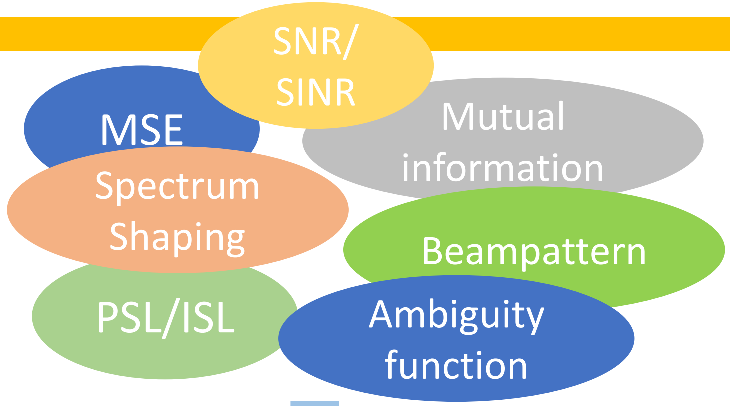
$$\begin{cases} \underset{\boldsymbol{x}}{\text{minimize}} & \text{ISL/PSL} \\ \text{subject to} & x_n \in \boldsymbol{\Omega}_{\boldsymbol{M}} \end{cases}$$

constraints

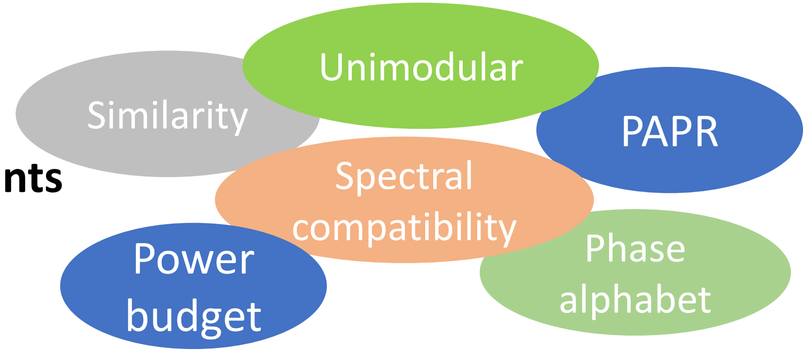
How to solve the formulated waveform design problem?

Beyond ISL/PSL

Metrics



Various waveform design problems



Constraints

Algebraic construction: Frank sequence, Golomb sequence…

Heuristic constriction: exhaustive search, evolutionary algorithm, simulated annealing…

- Still cannot cover all needs
- Many problems are nonconvex and NP-hard
- High dimension if long sequence is needed
- Time efficiency matters

We focus on Optimization-based approach

Waveform Design Techniques

Optimization Techniques for Waveform Design

- Gradient-Descent Based Methods (GD)
- Majorization-Minimization (MM)
- Coordinate Descent (CD)
- Block Successive Upper-bound Minimization (BSUM)
- Alternating Direction Method of Multipliers (ADMM)
- Several others ...

Recall ISL/PSL Problems

Waveform to be designed: $x = [x_1, x_2, \ldots, x_N]^T \in \mathbb{C}^N$

	Unimodular	Phase alphabet
PSL	$\begin{cases} \underset{x}{\text{minimize}} & \max\{\lvert r_k\rvert\}_{k=1}^{N-1} \\ \text{subject to} & \lvert x_n\rvert = 1 \end{cases}$	$\begin{cases} \underset{x}{\text{minimize}} & \max\{\lvert r_k\rvert\}_{k=1}^{N-1} \\ \text{subject to} & x_{\text{n}} \in \Omega_M \end{cases}$
ISL	$\begin{cases} \underset{\boldsymbol{x}}{\text{minimize}} & \sum_{k=1}^{N-1} \lvert r_k\rvert^2 \\ \text{subject to} & \lvert x_n\rvert = 1 \end{cases}$	$\begin{cases} \underset{\boldsymbol{x}}{\text{minimize}} & \sum_{k=1}^{N-1} \lvert r_k\rvert^2 \\ \text{subject to} & x_{\text{n}} \in \Omega_M \end{cases}$

Phase alphabet: $\Omega_M = \left\{1, e^{\frac{j2\pi}{M}}, \ldots, e^{\frac{j2\pi(M-1)}{M}}\right\}$

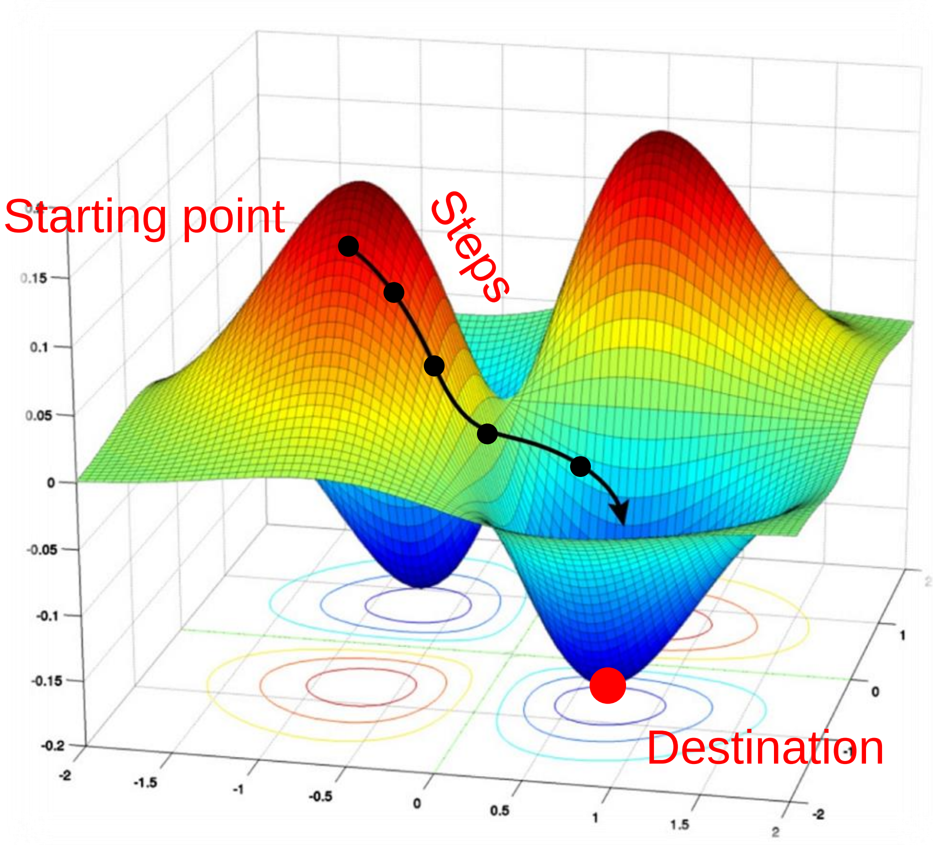
Gradient-Descent Based Methods (GD)

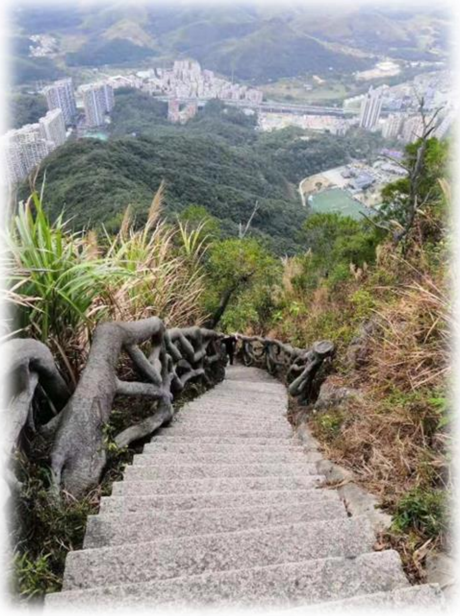
Unconstrained problem:

$$\underset{\boldsymbol{x}}{\text{minimize}} \quad f(\boldsymbol{x})$$

Gradient descent (GD) is well-known

$$\underbrace{\mathbf{x}_{k+1}}_{\text{Updated point}} = \underbrace{\mathbf{x}_k}_{\text{Current point}} - \underbrace{\alpha}_{\text{Step-size}} \underbrace{\nabla f(\mathbf{x}_k)}_{\text{Direction}}$$

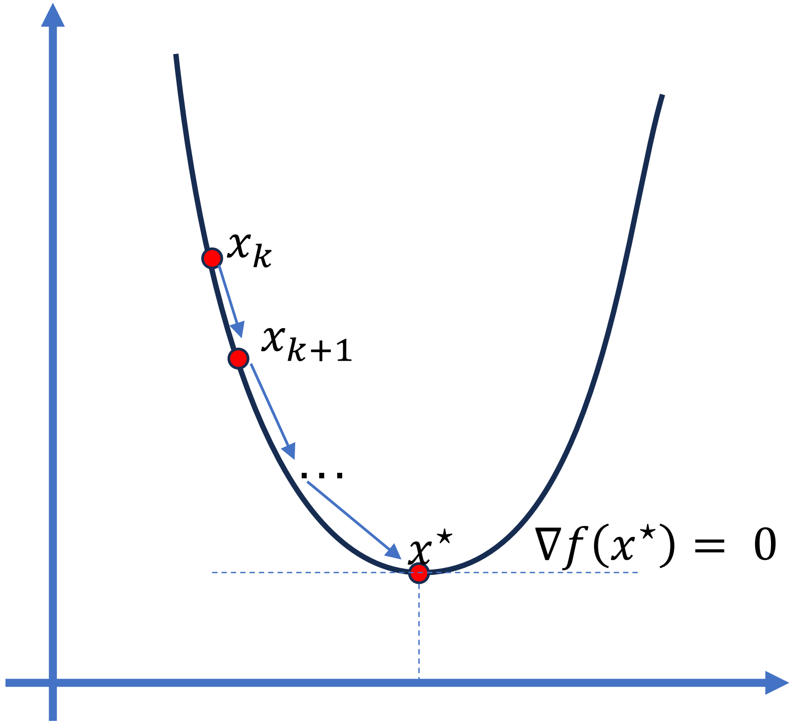


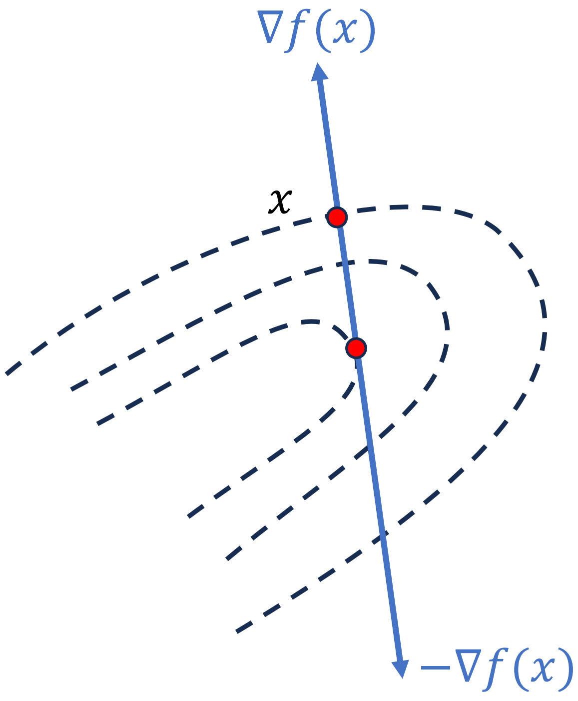


Iteratively repeat the update rule,
the sequence $\{\boldsymbol{x}_k\}$ **converge** at local optimum

GD Based Methods

$$\underset{\boldsymbol{x}}{\text{minimize}} \qquad f(\boldsymbol{x})$$





GD Based Methods – Algorithm

$$\underset{\boldsymbol{x}}{\text{minimize}} \qquad f(\boldsymbol{x})$$

1. Start with some guess $\boldsymbol{x}_0$
2. For each $k = 0, 1, \ldots$
   - $\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - \alpha \nabla f(\boldsymbol{x}_k)$
   - Check when to stop (e. g. if $\nabla f(\boldsymbol{x}_{k+1}) = 0$)

GD Based Methods

$$\underset{\boldsymbol{x}}{\text{minimize}} \qquad f(\boldsymbol{x})$$

$$\boxed{\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - \alpha \nabla f(\boldsymbol{x}_k), \qquad k = 0, 1, \ldots}$$

Stepsize $\alpha \geq 0$, usually ensures $f(\boldsymbol{x}_{k+1}) < f(\boldsymbol{x}_k)$

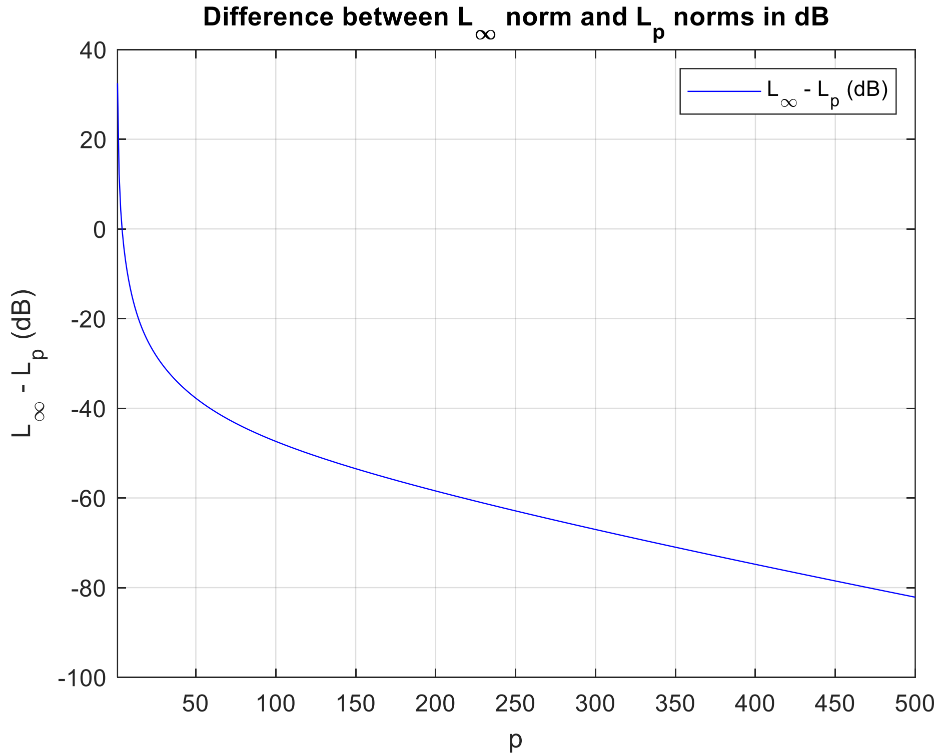
Numerous ways to select α

ℓ_p- Norm Minimization using GD

$$\ell_p \text{ norm: } \|\mathbf{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$$

$$\begin{cases} p = 2 : \|\mathbf{x}\|_2 = \sum_{n=1}^{N} |x_n|^2 & \text{ISL} \\ p = \infty : \|\mathbf{x}\|_\infty = \max_{1 \le i \le n} |x_i| & \text{PSL} \end{cases}$$

ℓ_∞ norm approximation: use a large value of p



A unified formulation:

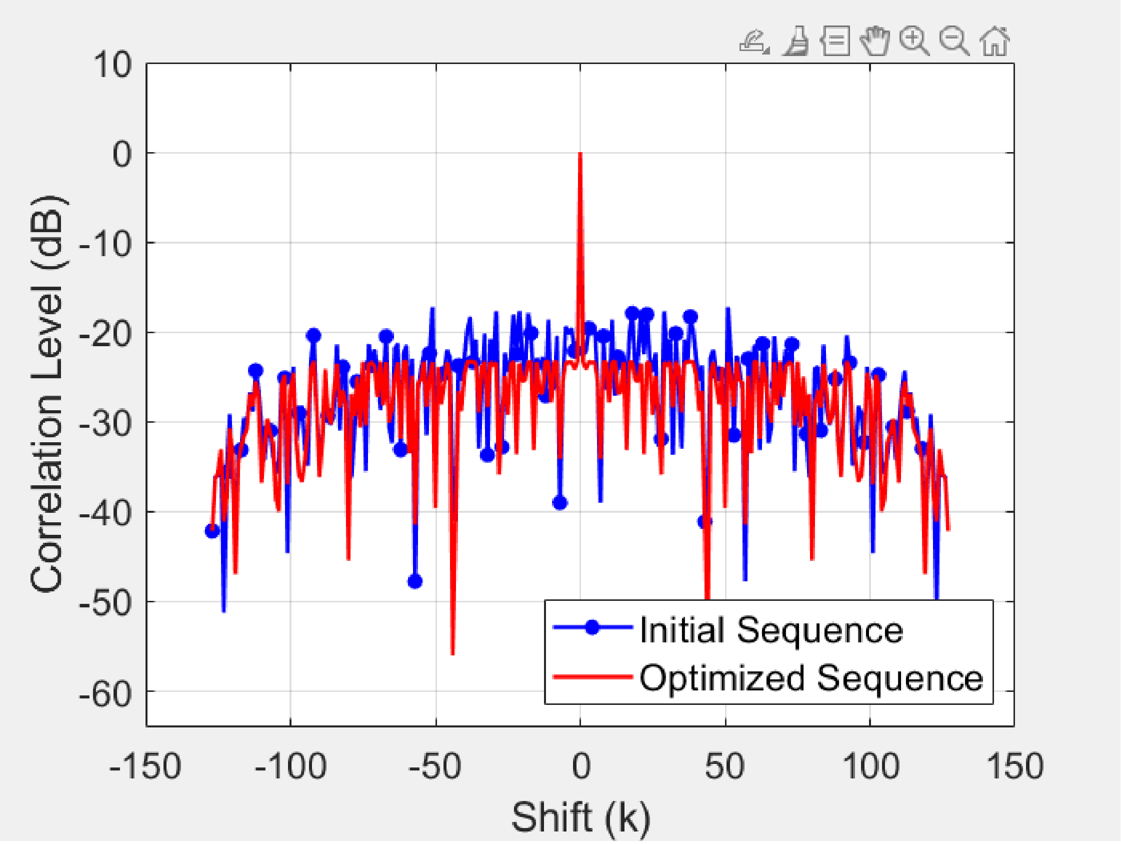
$$\begin{cases} \underset{\boldsymbol{x}}{\text{minimize}} & \sum_{k=1}^{N-1} |r_k|^p \\ \text{subject to} & x_n \in \Omega_M \end{cases}$$

How to solve this non-convex problem?

ℓ_p- Norm Minimization using GD

$$\begin{cases} \underset{\boldsymbol{x}}{\text{minimize}} & \|r_k\|_p \\ \text{subject to} & |x_n| = 1 \end{cases}$$

Scan the QR code to have download the code



J. M. Baden, B. O'Donnell and L. Schmieder, "Multiobjective Sequence Design via Gradient Descent Methods," in IEEE Transactions on Aerospace and Electronic Systems, vol. 54, no. 3, pp. 1237-1252, June 2018, doi: 10.1109/TAES.2017.2780538.

Why ℓ_p- Norm of autocorrelation?

For a real number $p \geq 1$, the ℓ_p- Norm of $\boldsymbol{x} = [x_1, x_2, \dots, x_N]^T$ is defined by

$$\|\boldsymbol{x}\|_p = (|x_1|^p + |x_2|^p + \dots + |x_N|^p)^{\frac{1}{p}}$$

The absolute value bars are unnecessary when p is a rational number, and, in reduced form, has an even numerator. The Euclidean norm from above falls into this class and is the 2-Norm, and the 1-Norm is the norm that corresponds to the rectilinear distance. The ℓ_∞-Norm or maximum norm (or uniform norm) is the limit of the ℓ_p-Norm for $p \mapsto \infty$. It turns out that this limit is equivalent to the following definition:

$$\|\boldsymbol{x}\|_\infty = \max\{|x_1|, |x_2|, \dots, |x_N|\}$$

Not Easy for Waveform Design

$$\begin{cases} \underset{\boldsymbol{x}}{\text{minimize}} & f(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in \boldsymbol{\mathcal{X}} \end{cases}$$

For waveform design problems,

- $f(\boldsymbol{x})$ can be complicated even non-differentiable
- $\boldsymbol{x}$ can be high-dimension → computational cost
- Some constraints to consider, i.e., $\boldsymbol{x} \in \boldsymbol{\mathcal{X}}$

We need more efficient optimization techniques

Majorization-Minimization (MM)

An MM algorithm operates by creating a surrogate function that minorizes or majorizes the objective function. When the surrogate function is optimized, the objective function is driven uphill or downhill as needed.

Majorization-Minimization (MM)

$$\begin{cases} \text{minimize} & f(x) \\ \text{subject to} & x \in X \end{cases}$$

First step: Majorization

Construct the majorizer satisfying

$$u\left(\mathbf{x}, \mathbf{x}^{(\ell)}\right) \geq f(\mathbf{x}), \quad \text{for all } \mathbf{x} \in \mathcal{X},$$

$$u\left(\mathbf{x}^{(\ell)}, \mathbf{x}^{(\ell)}\right) = f\left(\mathbf{x}^{(\ell)}\right).$$

Second step: Minimization

$$\mathbf{x}^{(\ell+1)} \in \underset{\mathbf{x}\in\mathcal{X}}{\arg\min} u\left(\mathbf{x}, \mathbf{x}^{(\ell)}\right)$$

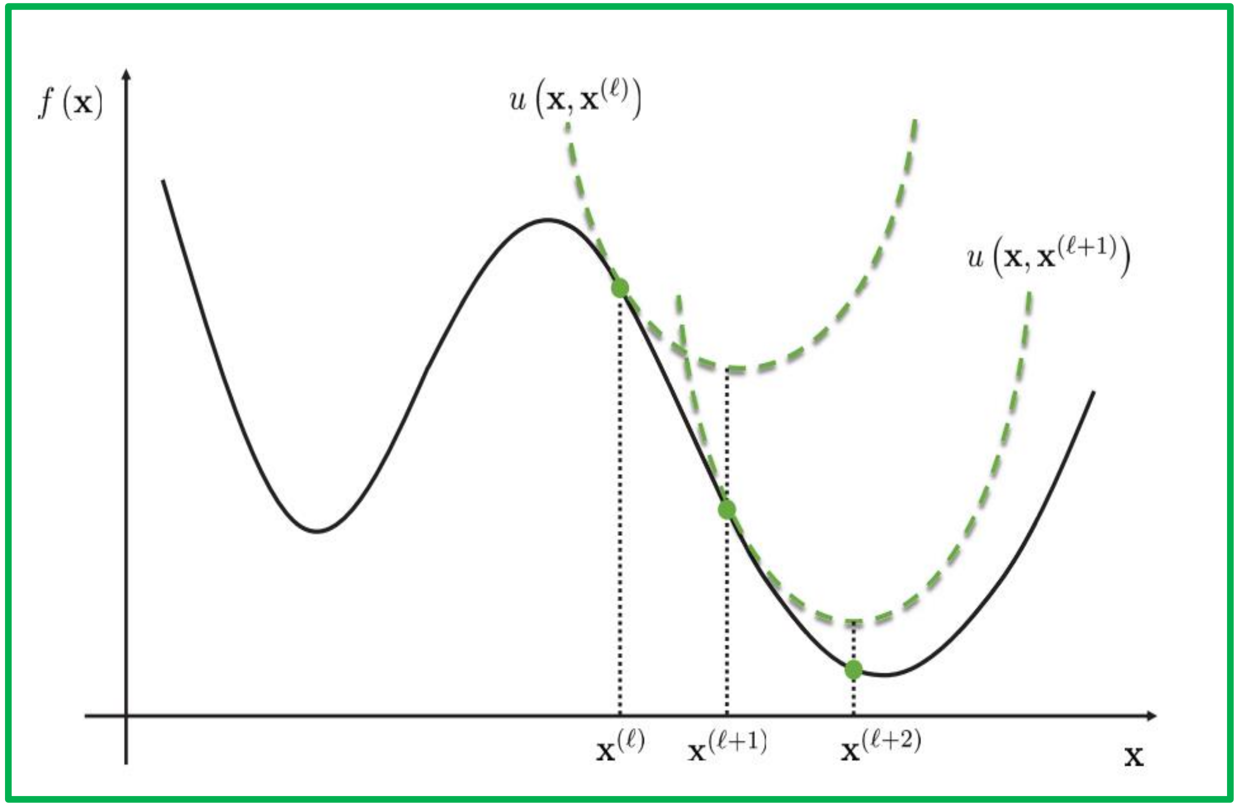
Algorithm 2: Sketch of the MM Method

Result: Optimized code vector $\mathbf{x}^{\star}$
initialization;
for $\ell = 0, 1, 2, \ldots$ **do**
  $\boxed{\mathbf{x}^{(\ell+1)} \in \underset{\mathbf{x}\in\mathcal{X}}{\operatorname{argmin}}\, u\left(\mathbf{x}, \mathbf{x}^{(\ell)}\right);}$
  Stop if convergence criterion is met;
  $\ell \leftarrow \ell + 1$
end

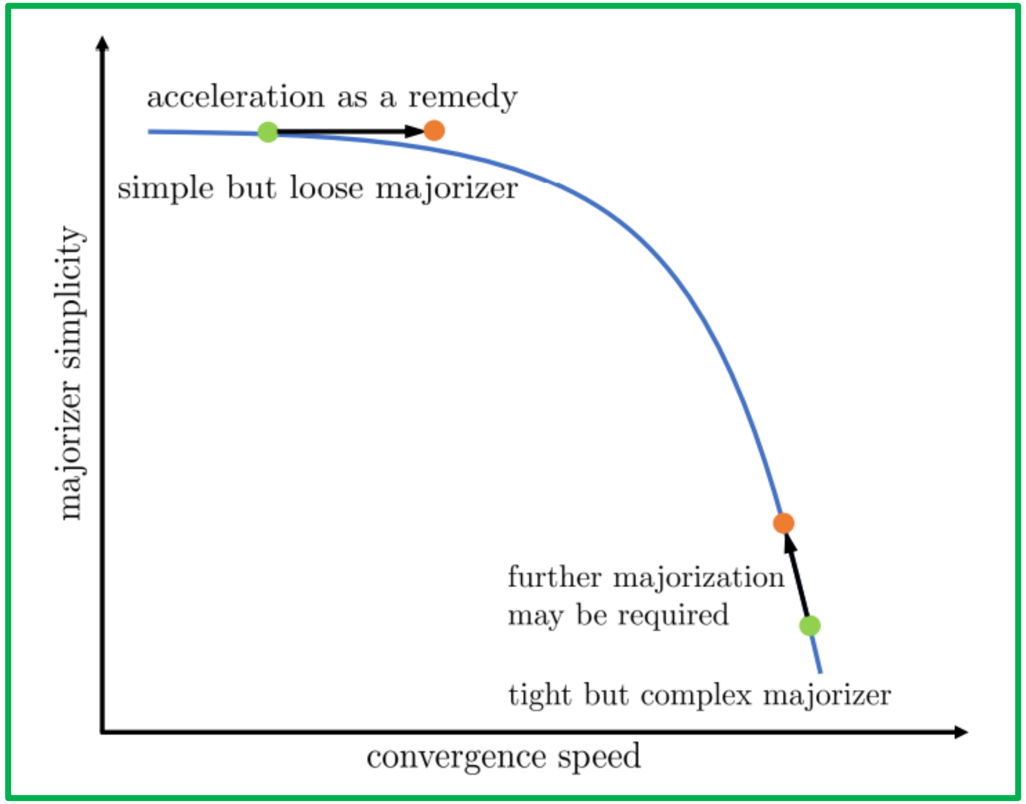
Note: For maximization problem, minorization maximization
Construct the minorizer and then maximize

MM Method

Graphic illustration of MM



Simplicity versus convergence



MM Example

Minimization of $\cos(x)$

Second order Taylor expansion

$$\cos(x) = \cos(x_n) - \sin(x_n)\,(x - x_n) - \frac{1}{2}\cos(z)\,(x - x_n)^2$$

Holds for some z between x and x_n

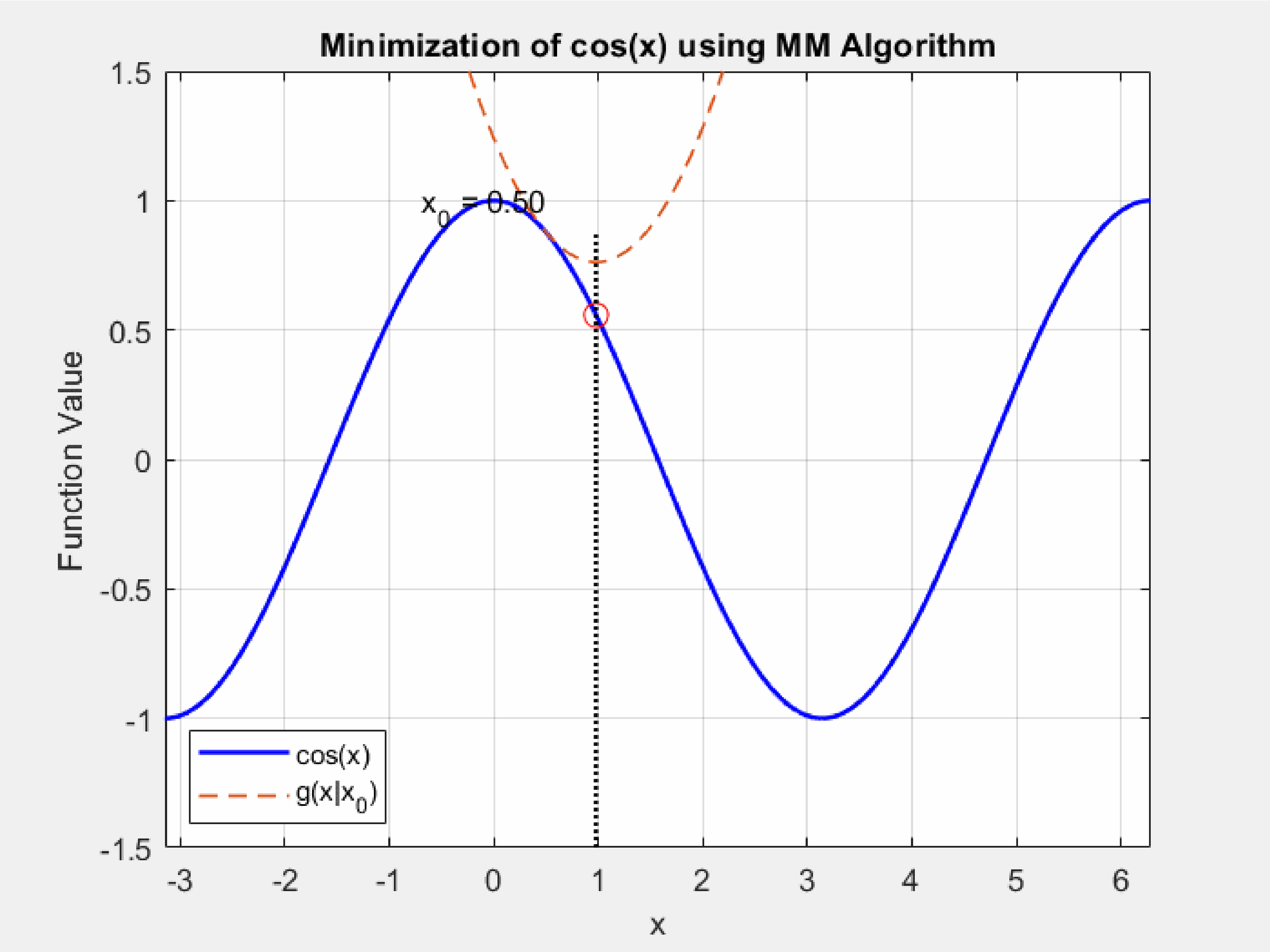
MM Example

$$g(x|x_n) = \cos(x_n) - \sin(x_n)\,(x - x_n) + \frac{1}{2}(x - x_n)\hat{}2$$

Can be selected as majorizer that majorizes $f(x)$

Solving $\frac{d}{dx} g(x|x_n) = 0$ gives the MM algorithm

$$x_{n+1} = x_n + \sin(x_n)$$



Minimum of $\cos(x)$ using MM

MM Algorithm

Input: $\boldsymbol{x}_0 \in \mathbb{C}^N$

1: **while** not converged **do**

2: Construct a surrogate function $g(\boldsymbol{x}|\boldsymbol{x}_n)$ of $f(\boldsymbol{x})$ at the current iteration

3: Minimize the surrogate to get the next iterate:

$$\boldsymbol{x}_{n+1}0 = \underset{\boldsymbol{x}}{\operatorname{argmin}}\, g(\boldsymbol{x}|\boldsymbol{x}_n)$$

4: $n \leftarrow n + 1$

5: **end while**

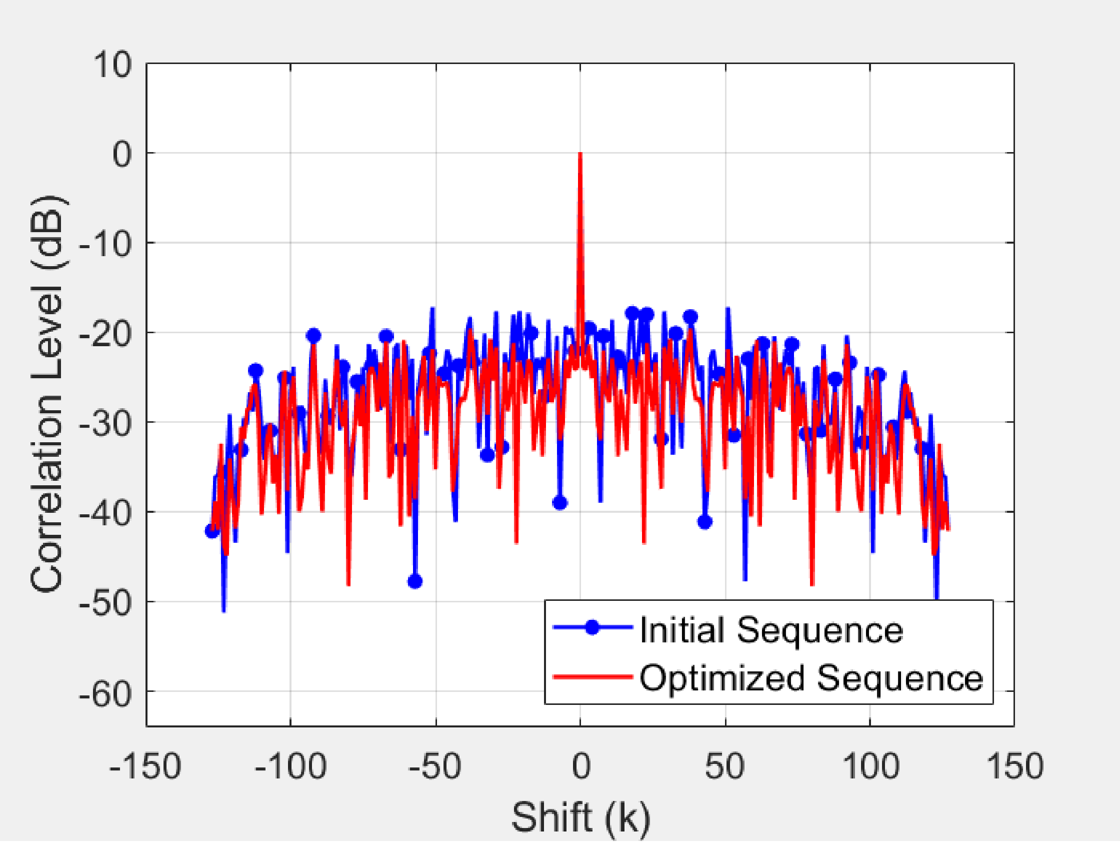
Output: The solution $\boldsymbol{x}_n$

ISL Minimization Problem using MM

$$\mathrm{WISL} = \sum_{k=1}^{N-1} w_k |r_k|^2,$$

$$\underset{x_n}{\text{minimize}} \quad \mathrm{WISL}$$
$$\text{subject to} \quad |x_n| = 1, n = 1, \ldots, N,$$

Access to code



J. Song, P. Babu and D. P. Palomar, "Sequence Design to Minimize the Weighted Integrated and Peak Sidelobe Levels," in IEEE Transactions on Signal Processing, vol. 64, no. 8, pp. 2051-2064, April15, 2016, doi: 10.1109/TSP.2015.2510982.

Coordinate Descent (CD)

Successively minimizes along coordinate directions

Optimize each parameter separately, holding all the others fixed.

- ✓ Very simple and easy to implement
- ✓ Careful implementations can attain state-of-the-art
- ✓ Scalable, don't need to keep data in memory, low memory requirements
- ✓ Faster than gradient descent if iterations are N times cheaper

CD idea

$$\boldsymbol{x} = [x_1, x_2, \ldots., x_N]^T$$

$$\mathcal{P}_{\boldsymbol{x}} \begin{cases} \underset{\boldsymbol{x}}{\text{minimize}} & f(\boldsymbol{x}) \\ \text{subject to} & x_n \in \psi_n \end{cases}$$

idea: optimize over individual coordinates

CD Algorithm

$$x_1^{(k)} \in \arg\min_{x_1} f(x_1, x_2^{(k-1)}, x_3^{(k-1)}, \dots, x_N^{(k-1)})$$

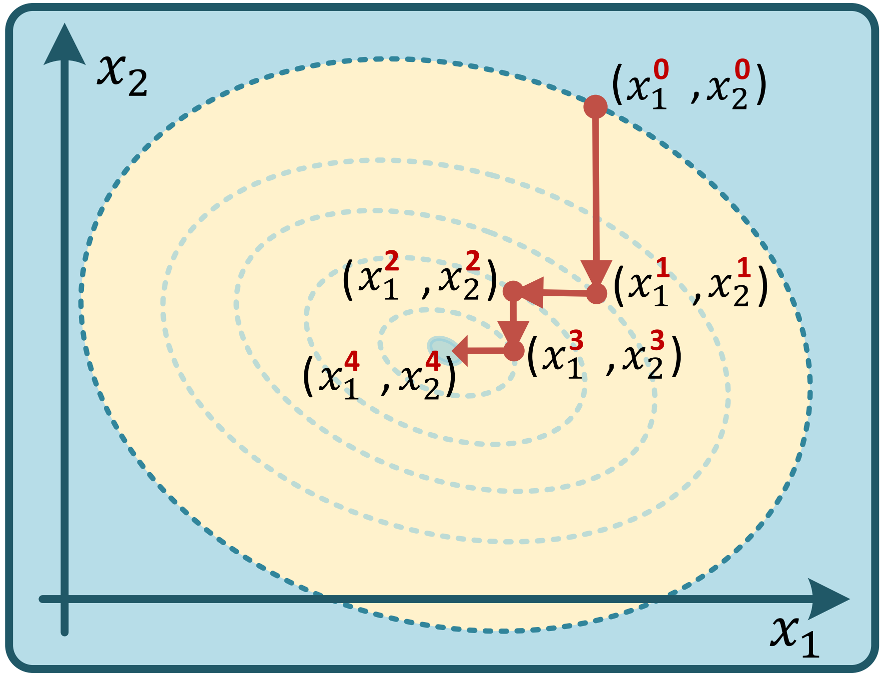
$$x_2^{(k)} \in \arg\min_{x_2} f(x_1^{(k)}, x_2, x_3^{(k-1)}, \dots, x_N^{(k-1)})$$

$$x_3^{(k)} \in \arg\min_{x_3} f(x_1^{(k)}, x_2^{(k)}, x_3, \dots, x_N^{(k-1)})$$

$$\vdots$$

$$x_N^{(k)} \in \arg\min_{x_N} f(x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, \dots, x_N)$$

Successively minimizes along coordinate directions



$$y = x_1^2 + 2\,x_2^2 - 9$$

No stepsize tuning!

Variable update rule

Gauss-Seidel style (One-at-a-time)

$$x_i^{(k+1)} \leftarrow \arg\min_{\zeta} f(x_1^{(k+1)}, \dots, x_{i-1}^{(k+1)}, \zeta, x_{i+1}^{(k)}, \dots, x_N^{(k)})$$

Jacobi style (all-at-once ; easy to parallelize)

$$x_i^{(k+1)} \leftarrow \arg\min_{\zeta} f(x_1^{(k)}, \dots, x_{i-1}^{(k)}, \zeta, x_{i+1}^{(k)}, \dots, x_N^{(k)})$$

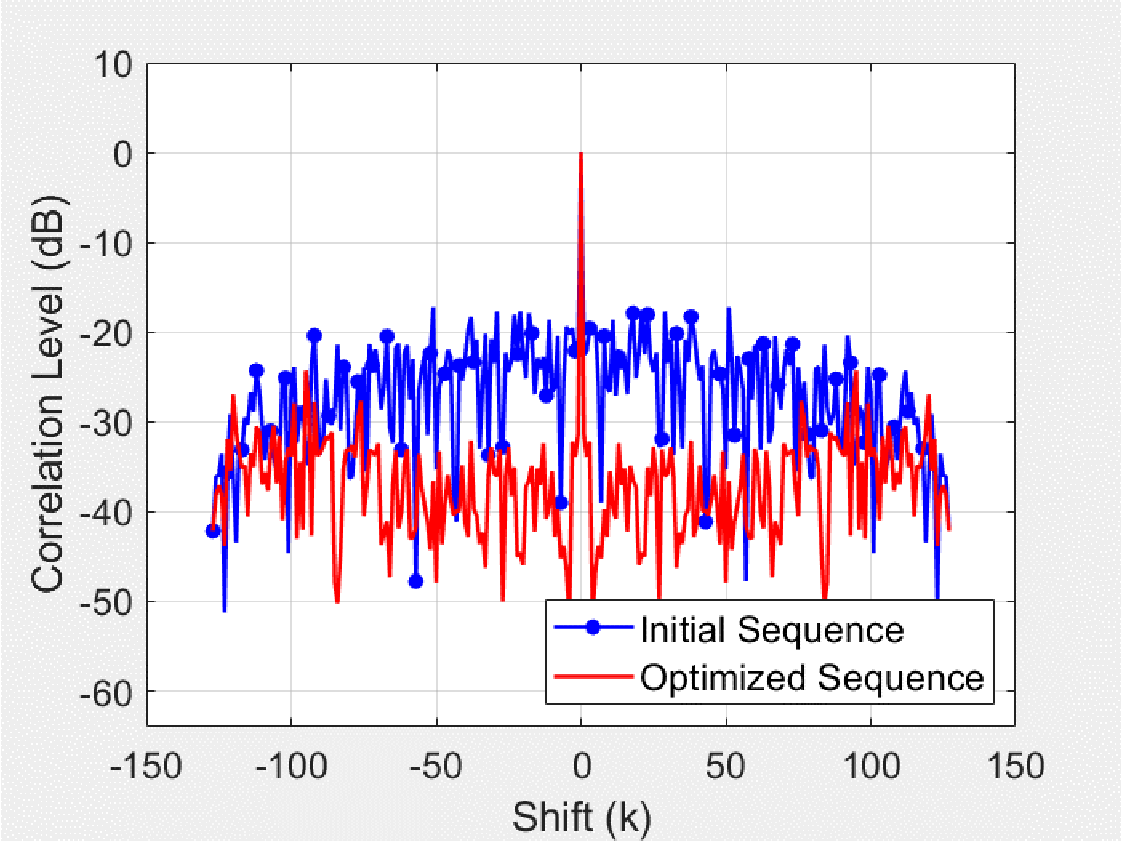
- Maximum Block Improvement
  - For differentiable f, pick the index that minimizes $\nabla f(x_i^k)$
- Various update order:
  - Cyclic order: $1, 2, \dots, N, 1, \dots$
  - Double sweep: $1, 2, \dots, N$, then $N-1, \dots, 1$, repeat
  - Cyclic with permutation: random order each cycle
  - Random sampling: pick random index at each iteration

ISL Minimization Problem using CD

$$\boldsymbol{x} = [x_1, x_2, \ldots, x_N]^T$$

$$\begin{cases} \underset{\boldsymbol{x}}{\text{minimize}} & \sum_{k=1}^{N-1} r_k^2 \\ \text{subject to} & x_n \in \psi_n \end{cases}$$

Access to code



M. Alaee-Kerahroodi, A. Aubry, A. De Maio, M. M. Naghsh and M. Modarres-Hashemi, "A Coordinate-Descent Framework to Design Low PSL/ISL Sequences," in IEEE Transactions on Signal Processing, vol. 65, no. 22, pp. 5942-5956, 15 Nov.15, 2017, doi: 10.1109/TSP.2017.2723354.

CD Advantages

- Each iteration is usually cheap (single variable optimization)
- No extra storage vectors needed
- No stepsize tuning
- No other parameters that must be tuned
- In general, “derivative free”
- Simple to implement
- Works well for large-scale problems
- Currently quite popular; parallel version exist

Alternative optimization

2 blocks CD is called alternative optimization

$$\boldsymbol{x} = [x_1, x_2]^T$$

$$x_1 \downarrow x \qquad x_2 \downarrow w$$

$$\mathcal{P}_{x,w} \begin{cases} \underset{x,w}{\text{minimize}} & f(x, w) \\ \text{subject to} & x \in \psi_1, w \in \psi_2 \end{cases}$$

Block MM/BSUM

$$\boldsymbol{x} = [x_1, x_2, \dots, x_N]^T$$

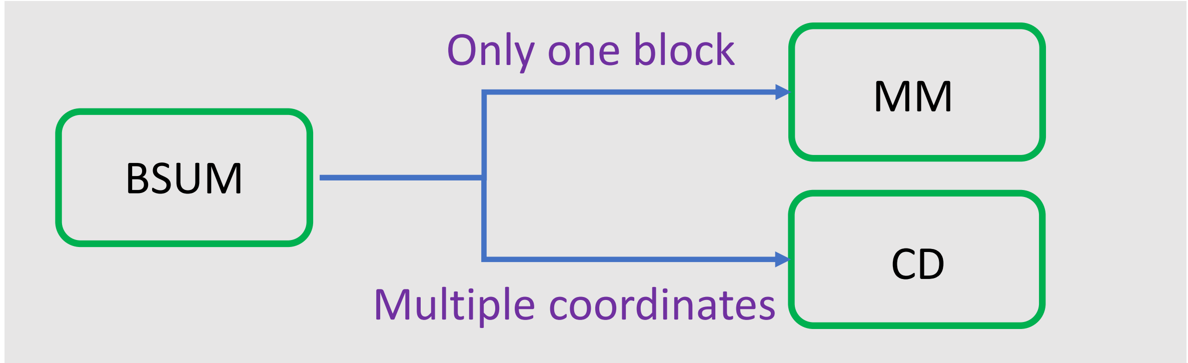
$$\mathcal{P}_{\boldsymbol{x}} \begin{cases} \underset{\boldsymbol{x}}{\text{minimize}} & f(\boldsymbol{x}) \\ \text{subject to} & x_n \in \psi_n \end{cases}$$

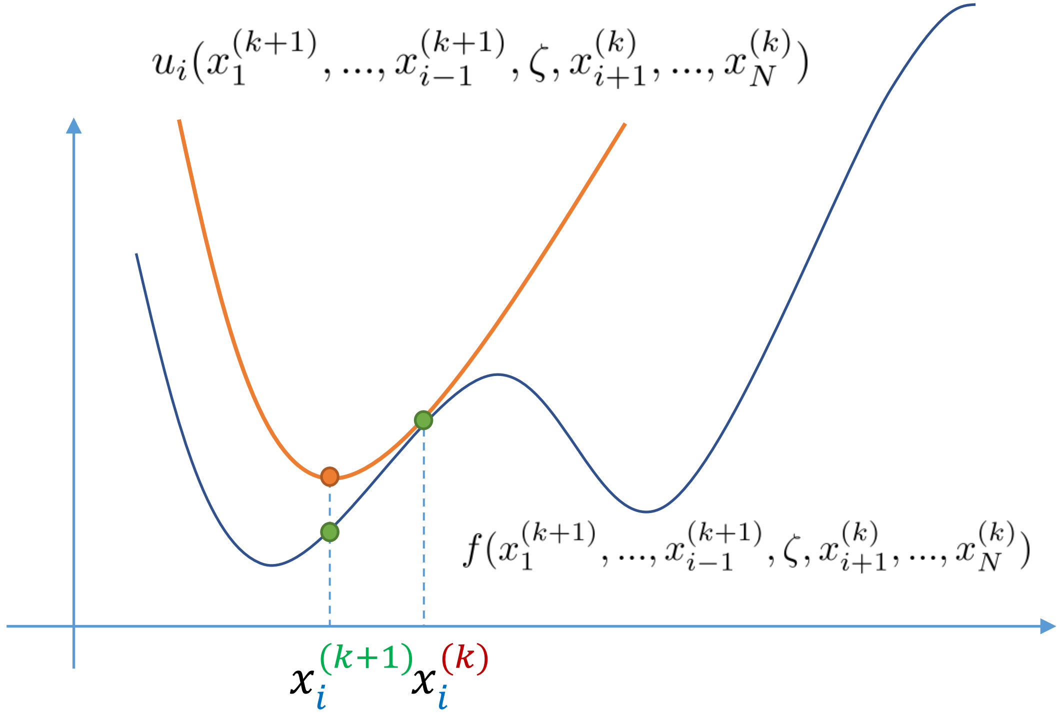
$$x_i^{(k+1)} \leftarrow \arg\min_{\zeta} u_i\left(x_1^{(k+1)}, \dots, x_{i-1}^{(k+1)}, \zeta, x_{i+1}^{(k)}, \dots, x_N^{(k)}\right)$$

Local approximation of the objective function

Block MM/BSUM

$$\begin{cases} \underset{\boldsymbol{x}_1,\dots,\boldsymbol{x}_N}{\text{minimize}} & f\left(\boldsymbol{x}_1,\dots,\boldsymbol{x}_N\right) \\ \text{subject to} & \boldsymbol{x}_n \in \mathcal{X}_n, n = 1,\dots,N \end{cases}$$





$$x_i^{(k+1)} \leftarrow \arg\min_{\zeta} u_i(x_1^{(k+1)},\dots,x_{i-1}^{(k+1)},\zeta,x_{i+1}^{(k)},\dots,x_N^{(k)})$$

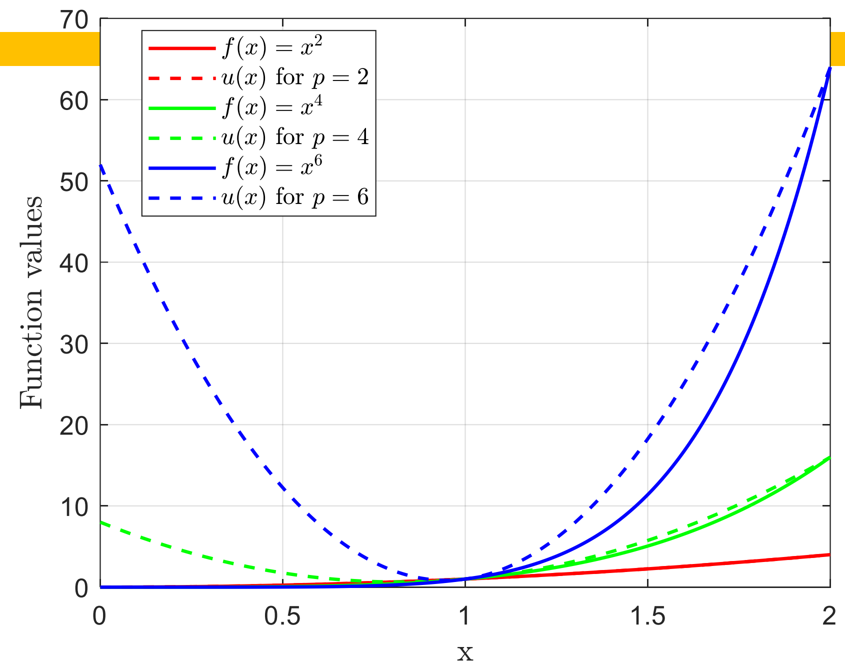
Majorizer/upper bound of the objective function

Block MM/BSUM for ℓ_p- Norm Minimization

Majorizer of $f(x) = x^p, \quad x \in [0, t]$ with $p \geq 2$

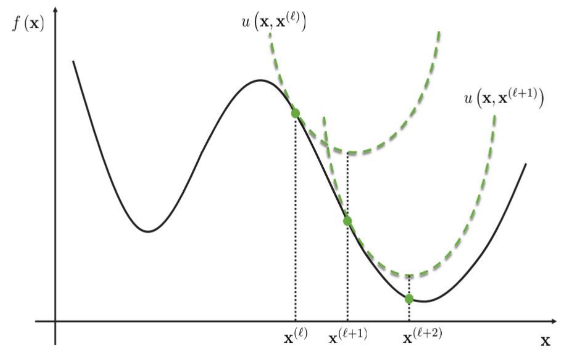
$$u(x) = ax^2 + \left(px_0^{p-1} - 2ax_0\right)x + ax_0^2 - (p-1)x_0^p$$

$$a = \frac{t^p - x_0^p - px_0^{p-1}(t - x_0)}{(t - x_0)^2}$$



At each iteration, we solve

$$\begin{cases} \underset{\boldsymbol{x}}{\text{minimize}} & \sum_{k=1}^{N-1} |r_k|^p \\ \text{subject to} & x_n \in \Omega_M \end{cases} \Rightarrow \begin{cases} \underset{\boldsymbol{x}}{\text{minimize}} & \sum_{k=1}^{N-1} a_k |r_k|^2 + \sum_{k=1}^{N-1} b_k |r_k| \\ \text{subject to} & x_n \in \Omega_M \end{cases}$$



Use CD for the Majorized Problem

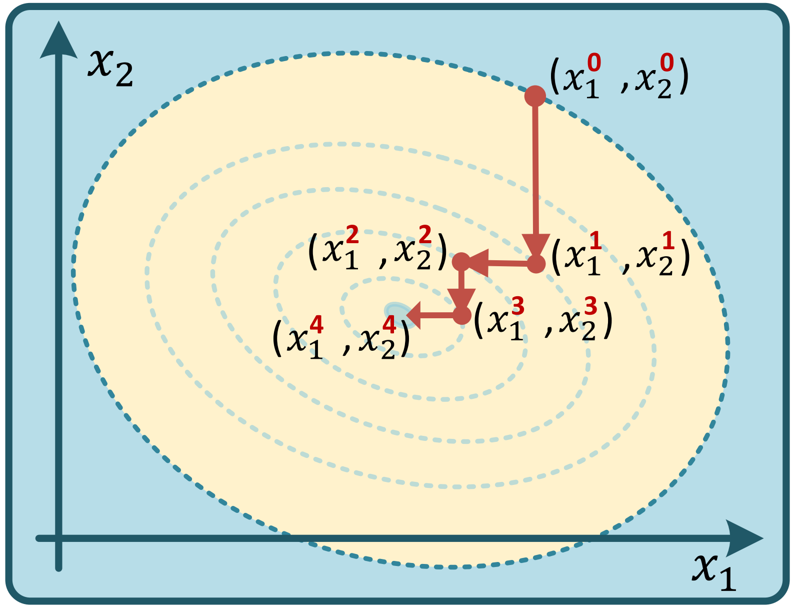
$$\begin{cases} \underset{\boldsymbol{x}}{\text{minimize}} & \sum_{k=1}^{N-1} a_k \left|r_k\right|^2 + \sum_{k=1}^{N-1} b_k \left|r_k\right| \\ \text{subject to} & x_n \in \Omega_M \end{cases}$$

$$x = [x_1, x_2, \ldots, x_N]^T \in \mathbb{C}^N$$

x_d → Only variable to optimize

$$\mathbf{x}_{-d} = \left[x_1^{(i+1)}, \ldots, x_{d-1}^{(i+1)}, 0, x_{d+1}^{(i)}, \ldots x_N^{(i+1)}\right]^T \in \mathbb{C}^N$$

$$r_k\left(x_d\right) = a_{1k} x_d + a_{2k} x_d^* + a_{3k}$$



Find the Optimal Phase

$x_d \in \Omega_M$

$\Omega_M = \left\{1, e^{\frac{j2\pi}{M}}, \dots, e^{\frac{j2\pi(M-1)}{M}}\right\}$ $\longrightarrow$

$x_d = e^{j\phi_d}$

$\widetilde{r}_k(\phi_d) = a_{1k}e^{j\phi_d} + a_{2k}e^{-j\phi_d} + a_{3k}$

$$\widetilde{\mathcal{H}}_h^{(i+1)} \begin{cases} \min_{\phi_d} & \sum_{k=1}^{N-1} a_k \left|\widetilde{r}_k(\phi_d)\right|^2 + \sum_{k=1}^{N-1} b_k \operatorname{Re}\left\{\widetilde{r}_k(\phi_d)^* \frac{r_k^{(\ell)}}{\left|r_k^{(\ell)}\right|}\right\} \\ \text{s.t.} & \phi_d \in \Phi_M = \left\{0, \frac{2\pi}{M}, \frac{4\pi}{M}, \dots, \frac{2\pi(M-1)}{M}\right\} \end{cases}$$

$\beta_d = \tan\left(\frac{\phi_d}{2}\right)$

$|\widetilde{r}_k(\phi_d)|^2 = \frac{\widetilde{p}_k(\beta_d)}{q(\beta_d)}$ $\quad$ $\operatorname{Re}\left\{\widetilde{r}_k^*(\beta_d)\frac{r_k^{(i)}}{\left|r_k^{(i)}\right|}\right\} = \frac{\bar{p}_k(\beta_d)}{q(\beta_d)}$

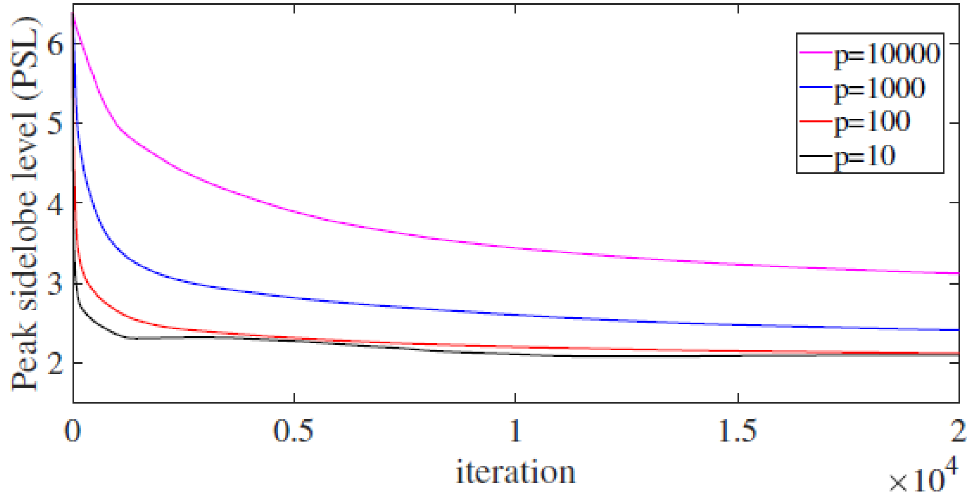
$$\begin{cases} \min\limits_{\beta_d} & \frac{1}{q(\beta_d)} \sum_{k=1}^{N-1} a_k\widetilde{p}_k(\beta_d) + b_k\bar{p}_k(\beta_d) \\ s.t. & \beta_d \in B \end{cases}$$

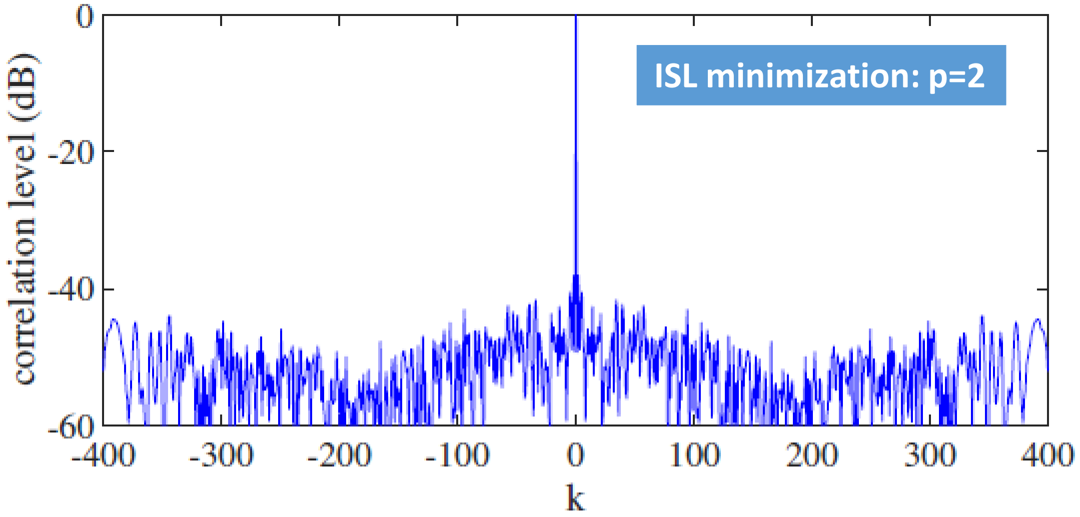
$\widetilde{p}_k(\beta_d) = \mu_{1k}\beta_d^4 + \mu_{2k}\beta_d^3 + \mu_{3k}\beta_d^2 + \mu_{4k}\beta_d + \mu_{5k}$

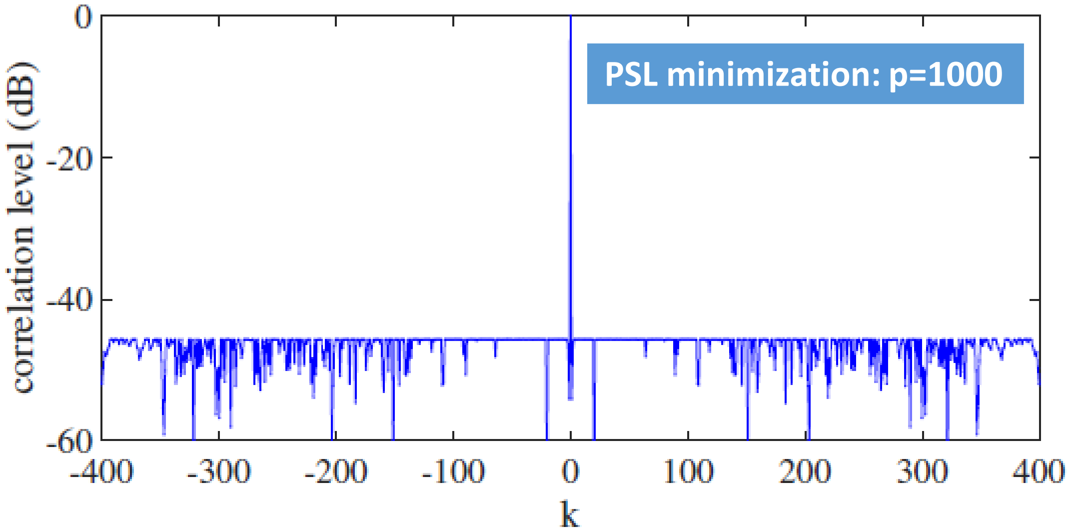
$\bar{p}_k(\beta_d) = \kappa_{1k}\beta_d^4 + \kappa_{2k}\beta_d^3 + \kappa_{3k}\beta_d^2 + \kappa_{4k}\beta_d + \kappa_{5k}$

$q(\beta_d) = (1 + \beta_d^2)^2$

Block MM/BSUM for ℓ_p- Norm Minimization

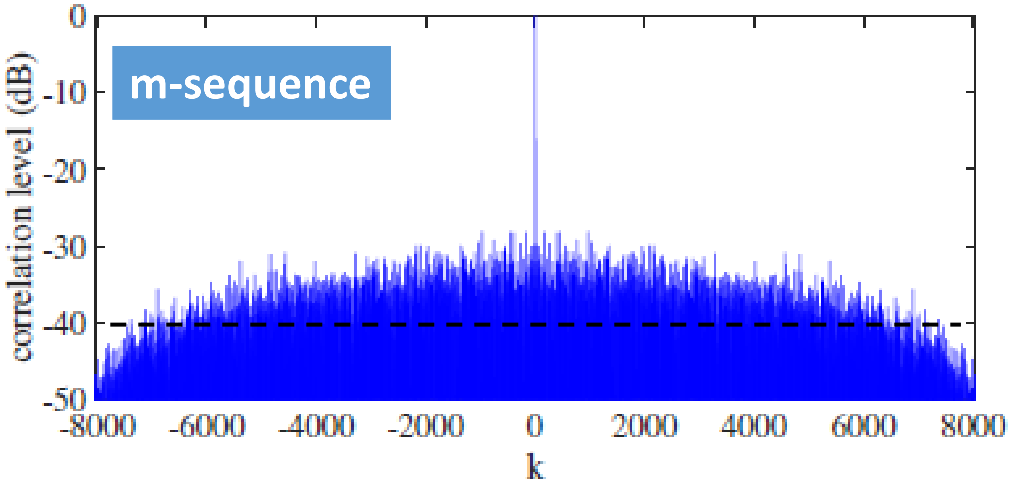




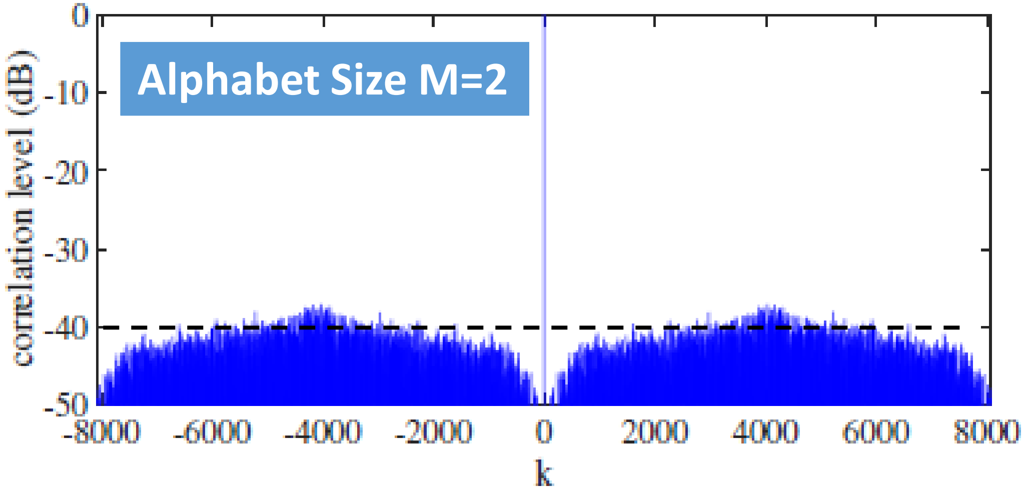


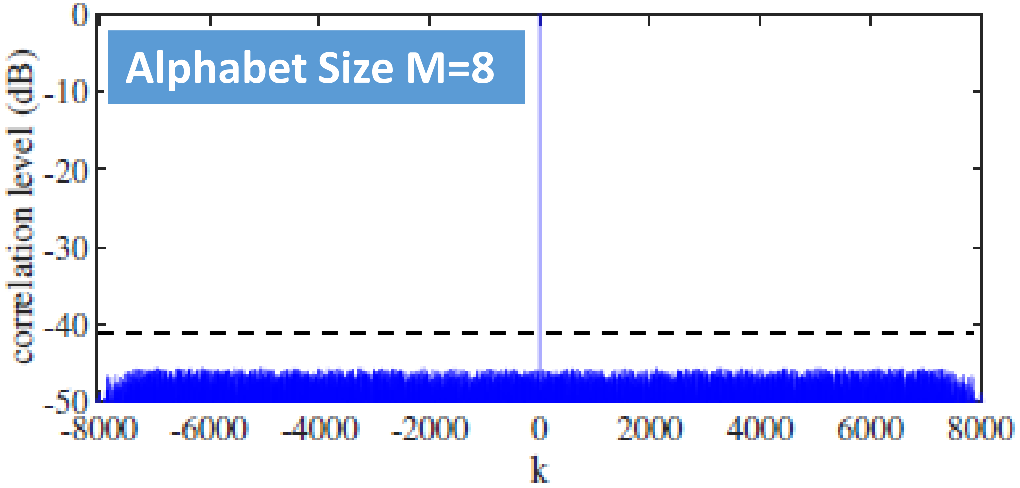
- Monotonicity is ensured → know when to stop
- Both ISL and PSL ensure a low sidelobe level
- Slight difference in sidelobes between ISL and PSL

Block MM/BSUM for ℓ_p- Norm Minimization



- Optimized sequence is better than m-sequence
- Alphabet size ↗, sidelobe level ↘





Joint Waveform and Receive Filter Design

for Pulse Compression in Weather Radar Systems

M. Alaee-Kerahroodi, L. Wu, E. Raei and M. R. B. Shankar, "Joint Waveform and Receive Filter Design for Pulse Compression in Weather Radar Systems," in IEEE Transactions on Radar Systems, vol. 1, pp. 212-229, 2023, doi: 10.1109/TRS.2023.3290846.

Signal Model

Let $\mathbf{J}_k$ be $N \times N$ shift matrix which its (m, n)-th entry $(m = 1, 2, \ldots, N,\ n = 1, 2, \ldots, N)$ is given by

$$\mathbf{J}_k(m,\ n) = \begin{cases} 1, & m - n = k \\ 0, & m - n \neq k. \end{cases}$$

Received Signal

$$\mathbf{y} = \alpha_0 \mathbf{x} + \underbrace{\sum_{\substack{k=-N+1 \\ k \neq 0}}^{N-1} \alpha_k \mathbf{J}_k \mathbf{x}}_{\text{interference caused by radar code}} + \boldsymbol{\nu},$$

Matched Filtering

$$\mathbf{x}^H\mathbf{y} = \alpha_0\mathbf{x}^H\mathbf{x} + \sum_{\substack{k=-N+1 \\ k\neq 0}}^{N-1} \alpha_k\mathbf{x}^H\mathbf{J}_k\mathbf{x} + \mathbf{x}^H\boldsymbol{\nu}.$$

Considering $||\mathbf{x}||^2 = 1$, the received Signal to Interference plus Noise Ratio (SINR) can be obtained by,

$$\text{SINR} = \frac{|\alpha_0|^2}{\zeta\sum_{\substack{k=-N+1 \\ k\neq 0}}^{N-1} |\mathbf{x}^H\mathbf{J}_k\mathbf{x}|^2 + \sigma_\nu^2}$$

$$\hat{\alpha}_0 = \mathbf{x}^H\mathbf{y},$$

the term $\sum_{\substack{k=-N+1 \\ k\neq 0}}^{N-1} |\mathbf{x}^H\mathbf{J}_k\mathbf{x}|^2$ is the ISL

Mismatched Filtering

$$\widetilde{\mathbf{x}} = [\mathbf{0}_M^T, \mathbf{x}^T, \mathbf{0}_M^T]^T,$$

and let $\widetilde{\mathbf{w}} \in \mathbb{C}^{\widetilde{N}}$ be the receive filter, with $\widetilde{N} = 2M + N$ where $M >> N$. In this case, the received signal after filtering can be obtained by

$$\widetilde{\mathbf{z}} = \alpha_0 \widetilde{\mathbf{w}}^H \widetilde{\mathbf{x}} + \sum_{\substack{k=-\widetilde{N}+1 \\ k \neq 0}}^{N-1} \alpha_k \widetilde{\mathbf{w}}^H \widetilde{\mathbf{J}}_k \widetilde{\mathbf{x}} + \widetilde{\mathbf{w}}^H \widetilde{\boldsymbol{\nu}},$$

$$\mathrm{SINR} = \frac{|\alpha_0|^2 |\widetilde{\mathbf{w}}^H \widetilde{\mathbf{x}}|^2}{\widetilde{\mathbf{w}}^H \mathbf{R} \widetilde{\mathbf{w}}},$$

$$\mathbf{R} = \zeta \sum_{\substack{k=-\widetilde{N}+1 \\ k \neq 0}}^{\widetilde{N}-1} \widetilde{\mathbf{J}}_k \widetilde{\mathbf{x}} \widetilde{\mathbf{x}}^H \widetilde{\mathbf{J}}_k^H + \sigma_{\widetilde{\nu}}^2 \mathbf{I}.$$

Mismatched Filtering

$$\hat{\alpha}_0 = \frac{\widetilde{\mathbf{z}}}{\widetilde{\mathbf{w}}^H \widetilde{\mathbf{x}}},$$

$$\mathrm{MSE}(\hat{\alpha}_0) = \mathbb{E}\left\{ \left| \frac{\widetilde{\mathbf{z}}}{\widetilde{\mathbf{w}}^H \widetilde{\mathbf{x}}} - \alpha_0 \right|^2 \right\} = \frac{\widetilde{\mathbf{w}}^H \mathbf{R} \widetilde{\mathbf{w}}}{|\widetilde{\mathbf{w}}^H \widetilde{\mathbf{x}}|^2}.$$

$$\mathcal{S}_{\widetilde{\mathbf{w}}} \left\{ \min_{\widetilde{\mathbf{w}}} \quad \frac{\widetilde{\mathbf{w}}^H \mathbf{R} \widetilde{\mathbf{w}}}{|\widetilde{\mathbf{w}}^H \widetilde{\mathbf{x}}|^2}, \right. \qquad \widetilde{\mathbf{w}}^\star = \mathbf{R}^{-1} \widetilde{\mathbf{x}}.$$

Waveform and Mismatched Filter Design

$$\hat{\alpha}_0 - \alpha_0 = \sum_{\substack{k=-\widetilde{N}+1 \\ k \neq 0}}^{\widetilde{N}-1} \alpha_k \frac{\widetilde{\mathbf{w}}^H \widetilde{\mathbf{J}}_k \widetilde{\mathbf{x}}}{\widetilde{\mathbf{w}}^H \widetilde{\mathbf{x}}} + \frac{\widetilde{\mathbf{w}}^H \widetilde{\boldsymbol{\nu}}}{\widetilde{\mathbf{w}}^H \widetilde{\mathbf{x}}}.$$

$$\boxed{\text{Mismatch ISL} = \sum_{\substack{k=-\widetilde{N}+1 \\ k \neq 0}}^{\widetilde{N}-1} \frac{|\widetilde{\mathbf{w}}^H \widetilde{\mathbf{J}}_k \widetilde{\mathbf{x}}|^2}{|\widetilde{\mathbf{w}}^H \widetilde{\mathbf{x}}|^2}.}$$

Waveform and Filter Design based on ISL Minimization

$$\mathcal{I}\begin{cases}\min\limits_{\widetilde{\mathbf{x}},\widetilde{\mathbf{w}}} & \dfrac{\sum_{\substack{k=-\widetilde{N}+1\\k\neq 0}}^{\widetilde{N}-1}|\widetilde{\mathbf{w}}^H\widetilde{\mathbf{J}}_k\widetilde{\mathbf{x}}|^2}{|\widetilde{\mathbf{w}}^H\widetilde{\mathbf{x}}|^2}\\ s.t. & \mathbf{x}\in\Omega\end{cases}$$

$$\boxed{\mathcal{S}_{\widetilde{\mathbf{x}},\widetilde{\mathbf{w}}}\begin{cases}\min\limits_{\widetilde{\mathbf{x}},\widetilde{\mathbf{w}}} & \dfrac{\widetilde{\mathbf{w}}^H\mathbf{R}\widetilde{\mathbf{w}}}{|\widetilde{\mathbf{w}}^H\widetilde{\mathbf{x}}|^2}\\ s.t. & \mathbf{x}\in\Omega\end{cases}}$$

Joint Waveform and Filter Design

$$\text{Mismatch PSL} = \max_{\substack{k=-\widetilde{N}+1 \\ k\neq 0}}^{\widetilde{N}-1} \frac{|\widetilde{\mathbf{w}}^H \widetilde{\mathbf{J}}_k \widetilde{\mathbf{x}}|^2}{|\widetilde{\mathbf{w}}^H \widetilde{\mathbf{x}}|^2}.$$

$$\mathcal{P}\begin{cases} \min\limits_{\widetilde{\mathbf{x}},\widetilde{\mathbf{w}}} & \max\limits_{k\neq 0} \dfrac{|\widetilde{\mathbf{w}}^H \widetilde{\mathbf{J}}_k \widetilde{\mathbf{x}}|^2}{|\widetilde{\mathbf{w}}^H \widetilde{\mathbf{x}}|^2} \\ s.t. & \mathbf{x} \in \Omega \end{cases}$$

Alternate between waveform and filter

Waveform optimization – use CD

$$\mathcal{P}_{\widetilde{\mathbf{x}}}\begin{cases}\underset{\widetilde{\mathbf{x}}}{\text{minimize}} & \displaystyle\sum_{\substack{k=-\widetilde{N}+1\\k\neq 0}}^{\widetilde{N}-1}\frac{|\widetilde{\mathbf{w}}^H\widetilde{\mathbf{J}}_k\widetilde{\mathbf{x}}|^p}{|\widetilde{\mathbf{w}}^H\widetilde{\mathbf{x}}|^2}\\ \text{subject to} & \mathbf{x}\in\Omega_h\end{cases}$$

$$|\widetilde{\mathbf{w}}^H\widetilde{\mathbf{J}}_k\widetilde{\mathbf{x}}|^p = |a_{1k}e^{j\phi_d} + a_{2k}|^p$$

$$|\widetilde{\mathbf{w}}^H\widetilde{\mathbf{x}}|^2 = |b_1 e^{j\phi_d} + b_2|^2,$$

Waveform optimization – use CD

$$\mathcal{P}_{\phi_d}\begin{cases}\min\limits_{\phi_d} & \sum\limits_{\substack{k=-\widetilde{N}+1\\k\neq 0}}^{\widetilde{N}-1}\dfrac{|a_{1k}e^{j\phi_d}+a_{2k}|^p}{|b_1e^{j\phi_d}+b_2|^2}\\ s.t. & \phi_d\in\{1,e^{\mathrm{j}\frac{2\pi}{L}},\ldots,e^{\mathrm{j}\frac{2\pi(L-1)}{L}}\}\end{cases}$$

$$l^\star=\arg\min_{l=1,\ldots,L}\sum_{\substack{k=-\widetilde{N}+1\\k\neq 0}}^{\widetilde{N}-1}\frac{|\mathcal{F}_L\{a_{1k},a_{2k}\}|^p}{|\mathcal{F}_L\{b_1,b_2\}|^2},$$

Filter optimization – use BSUM

$$\mathcal{P}_{\widetilde{\mathbf{w}}}\left\{\min_{\widetilde{\mathbf{w}}} \sum_{\substack{k=-\widetilde{N}+1\\k\neq 0}}^{\widetilde{N}-1} \frac{|\widetilde{\mathbf{w}}^H \widetilde{\mathbf{J}}_k \widetilde{\mathbf{x}}|^p}{|\widetilde{\mathbf{w}}^H \widetilde{\mathbf{x}}|^2}\right.$$

Lemma Let $f(x) = x^p$ with $p \geq 2$ and $x \in [0,t]$. Then for any given $x_0 \in [0,t)$, $f(x)$ is majorized at x_0 over the interval $[0,t]$ by

$$u(x) = ax^2 + \left(px_0^{p-1} - 2ax_0\right)x + ax_0^2 - (p-1)x_0^p$$

with

$$a = \frac{t^p - x_0^p - px_0^{p-1}(t-x_0)}{(t-x_0)^2}.$$

Filter optimization – use BSUM

$$\min_{\widetilde{\mathbf{w}}} \frac{\sum_{k\neq 0}\left(\tau_k \left|\widetilde{\mathbf{w}}^H \widetilde{\mathbf{J}}_k \widetilde{\mathbf{x}}\right|^2 + \lambda_k \mathrm{Re}\left\{(\widetilde{\mathbf{w}}^H \widetilde{\mathbf{J}}_k \widetilde{\mathbf{x}})^* \frac{\widetilde{r}_k^{(\ell)}}{|\widetilde{r}_k^{(\ell)}|}\right\} + \gamma_k\right)}{|\widetilde{\mathbf{w}}^H \widetilde{\mathbf{x}}|^2}$$

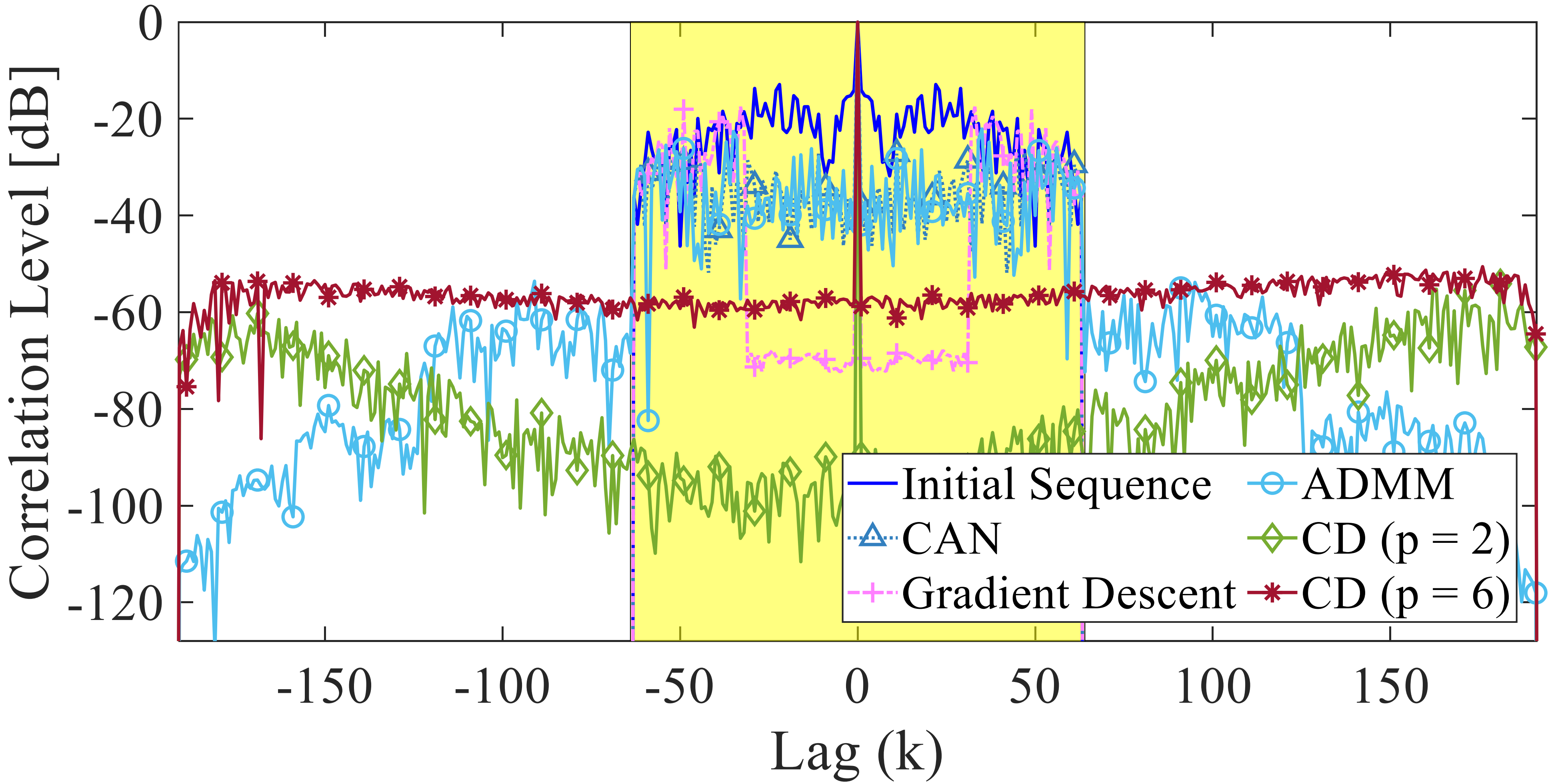
$$\sum_{k\neq 0}\left(\tau_k \left|\widetilde{\mathbf{w}}^H \widetilde{\mathbf{J}}_k \widetilde{\mathbf{x}}\right|^2 + \lambda_k \mathrm{Re}\left\{(\widetilde{\mathbf{w}}^H \widetilde{\mathbf{J}}_k \widetilde{\mathbf{x}})^* \frac{\widetilde{r}_k^{(\ell)}}{|\widetilde{r}_k^{(\ell)}|}\right\} + \gamma_k\right) = \widetilde{\eta}_1 \rho_d^2 + \widetilde{\eta}_2 \rho_d e^{j\theta_d} + \widetilde{\eta}_3 \rho_d e^{-j\theta_d} + \widetilde{\eta}_4,$$

$$|\widetilde{\mathbf{w}}^H \widetilde{\mathbf{x}}|^2 = \widetilde{\mu}_1 \rho_d^2 + \widetilde{\mu}_2 \rho_d e^{j\theta_d} + \widetilde{\mu}_3 \rho_d e^{-j\theta_d} + \widetilde{\mu}_4,$$

Filter optimization – use BSUM

$$\mathcal{P}_{\rho_d,\theta_d}\begin{cases}\underset{\rho_d,\theta_d}{\text{minimize}} & f(\rho_d,\theta_d)\\ \text{subject to} & \rho_d\in\mathbb{R}\\ & \theta_d\in[0,2\pi)\end{cases}$$

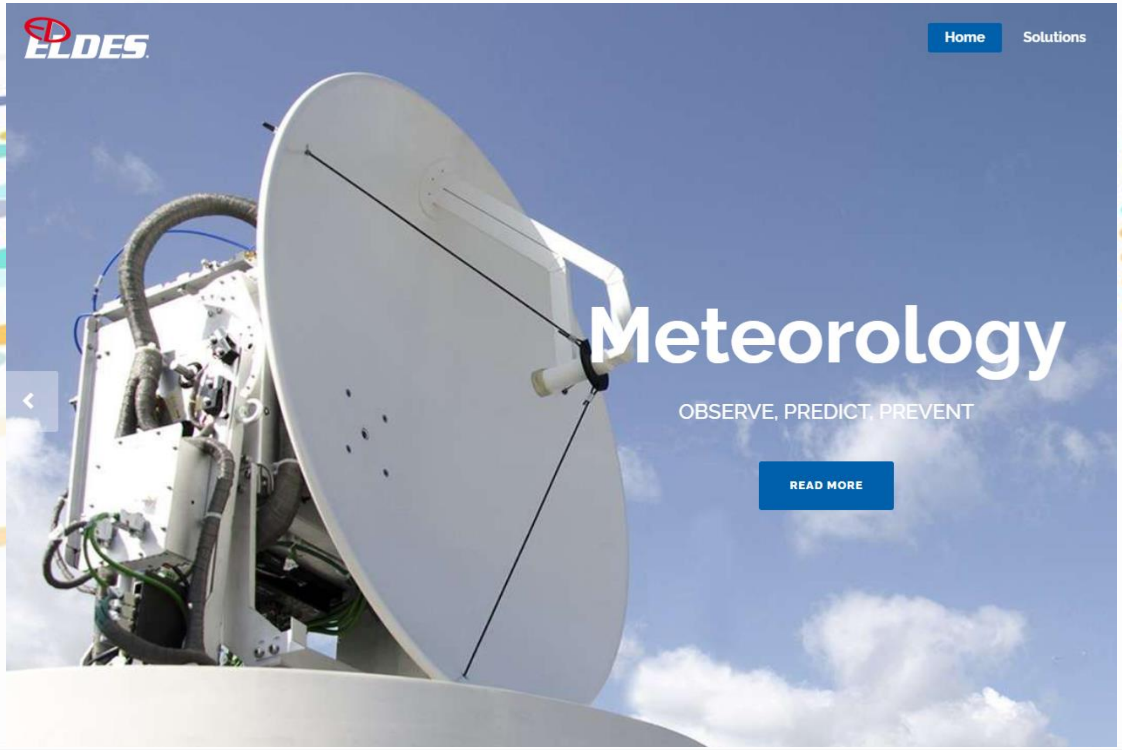
Optimized Waveform and Filter

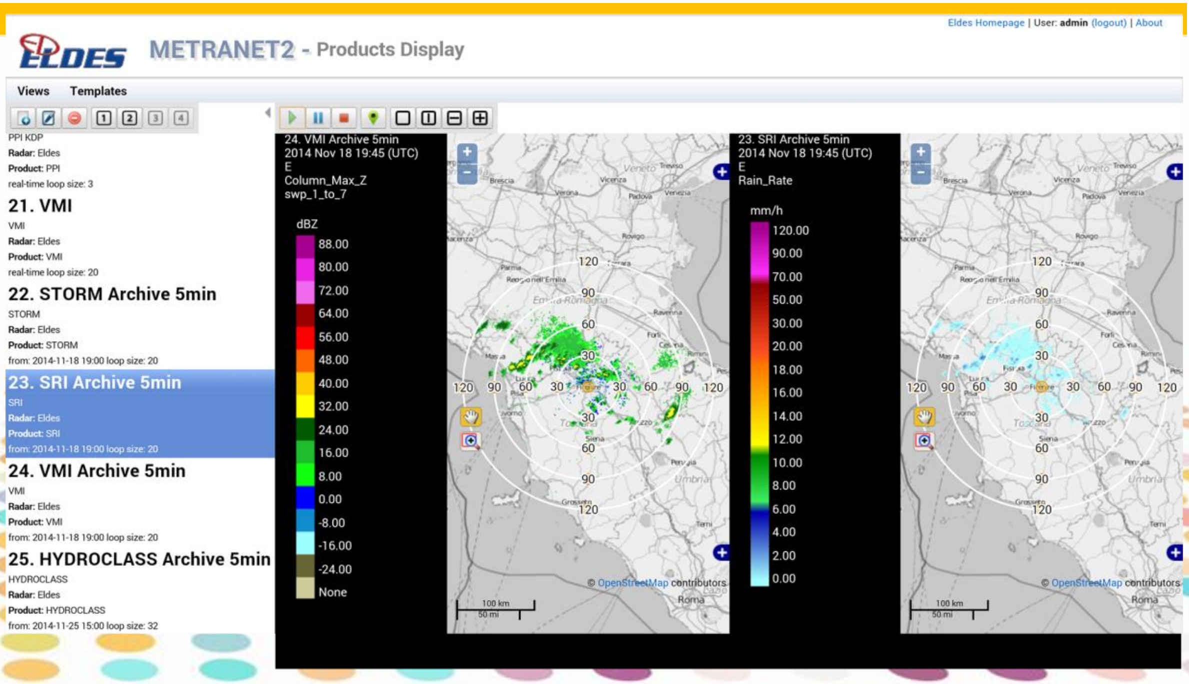


Real radar data

ELDES® S.r.l.

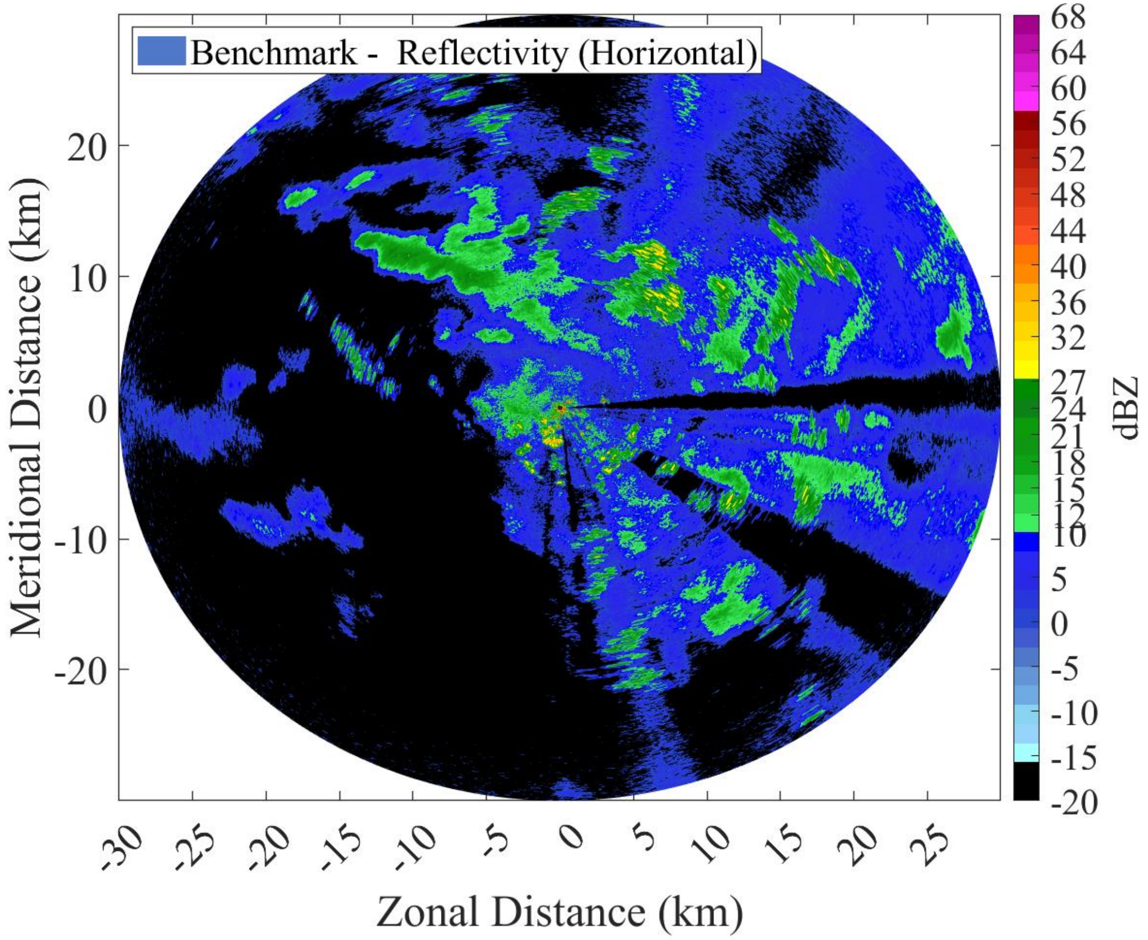
Via di Porto, 2/B – 50018 Scandicci, Florence, Italy
Tel. +39 055 3981100 Fax +39 055 790950

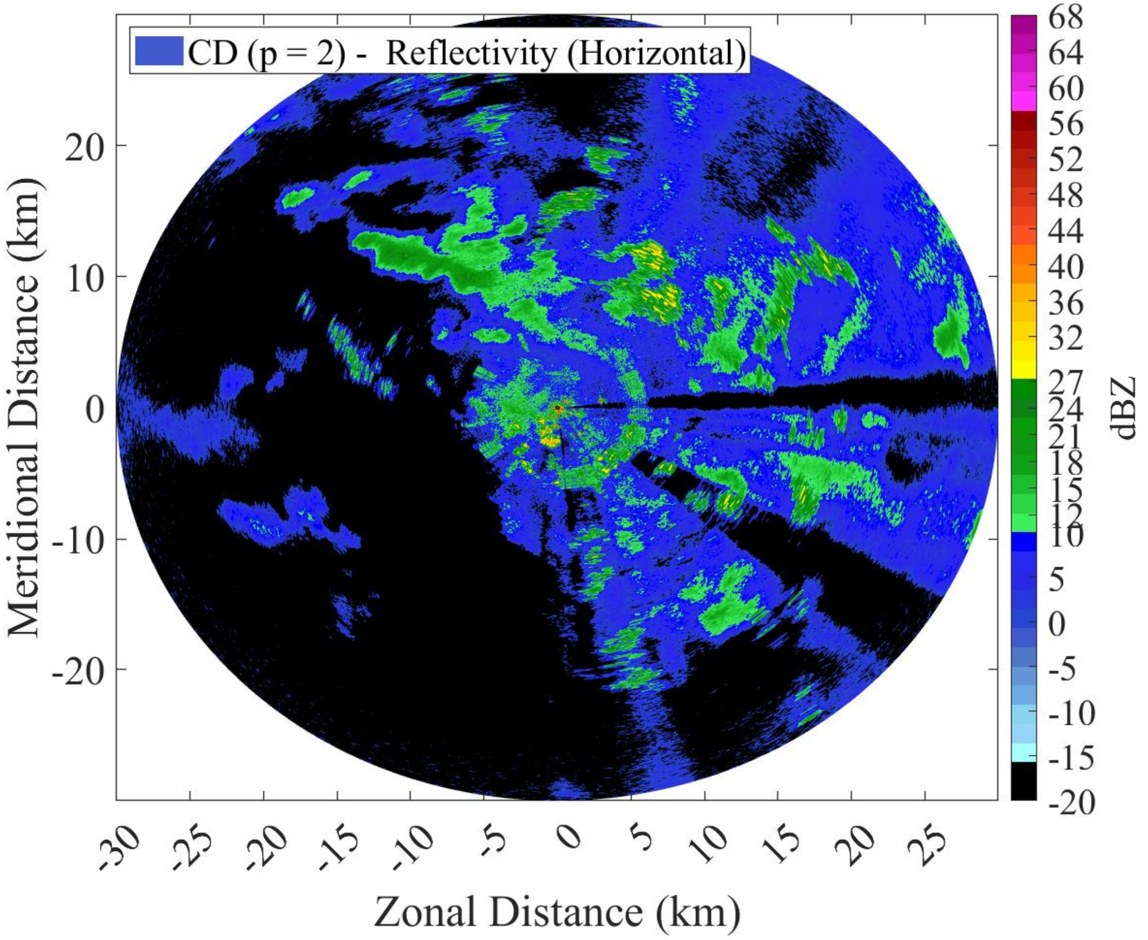




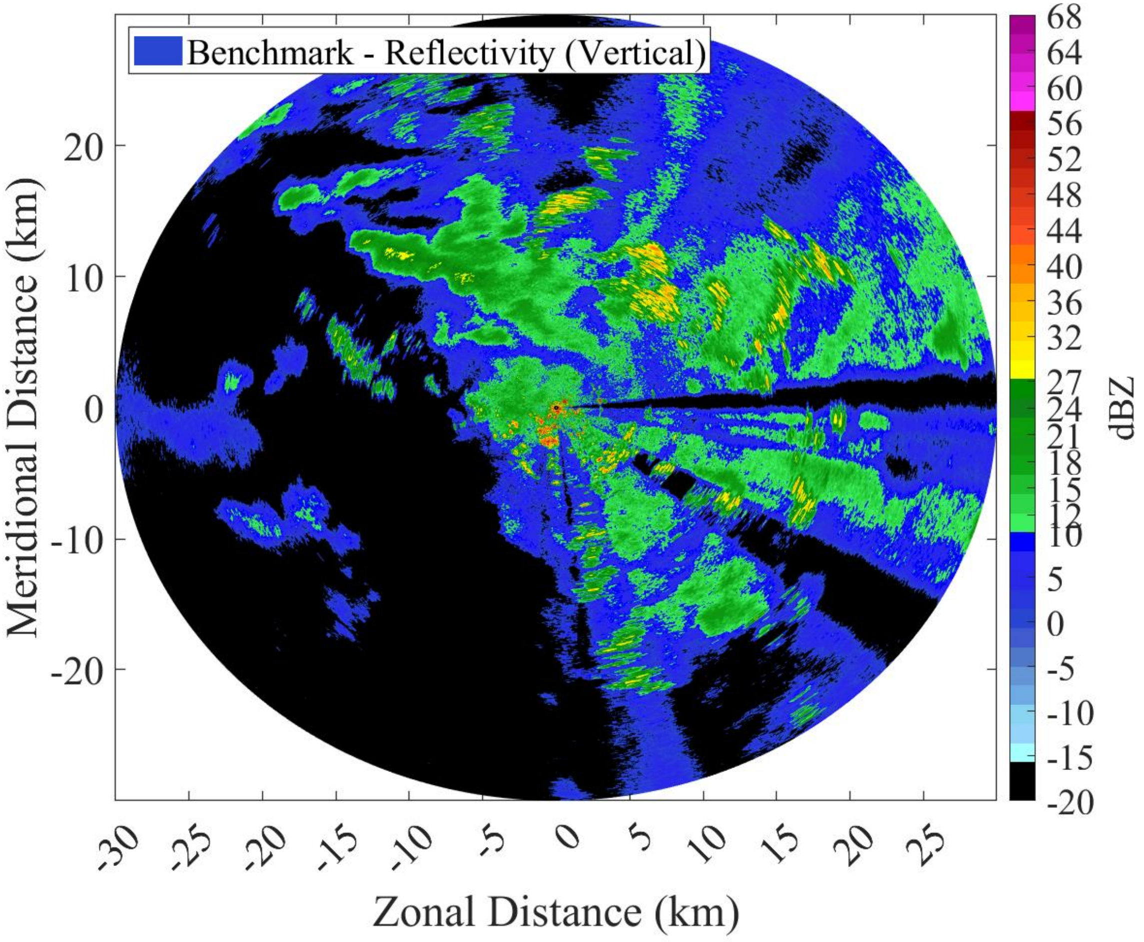
Specification:	Value
Type:	Dual Polarized Weather Radar System
Operating frequency:	9410MHz $\pm$30MHz
Peak power:	25Kw (12.5Kw Horizontal Channel - 12.5Kw Vertical Channel)
Pulse width:	0.2 - 0.4 - 0.8μs (typ.) User selectable
Repetition frequency (PRF):	1000 — 800 — 500 Hz User selectable
Receiver:	Linear Digital for simultaneous dual-polarization coherent on receive (Doppler)
Transmitter:	Magnetron
Dynamic range:	$>$ 90dB
Antenna Type:	Parabolic prime focus reflector
Horizontal and Vertical Beamwidth:	$\leq$ 2, 5 deg
Gain:	$\geq$ 36 dB
Range scale:	30Km - 40Km - 60Km - 120Km User selectable
Range resolution:	31.25m, 62.5m, 125m

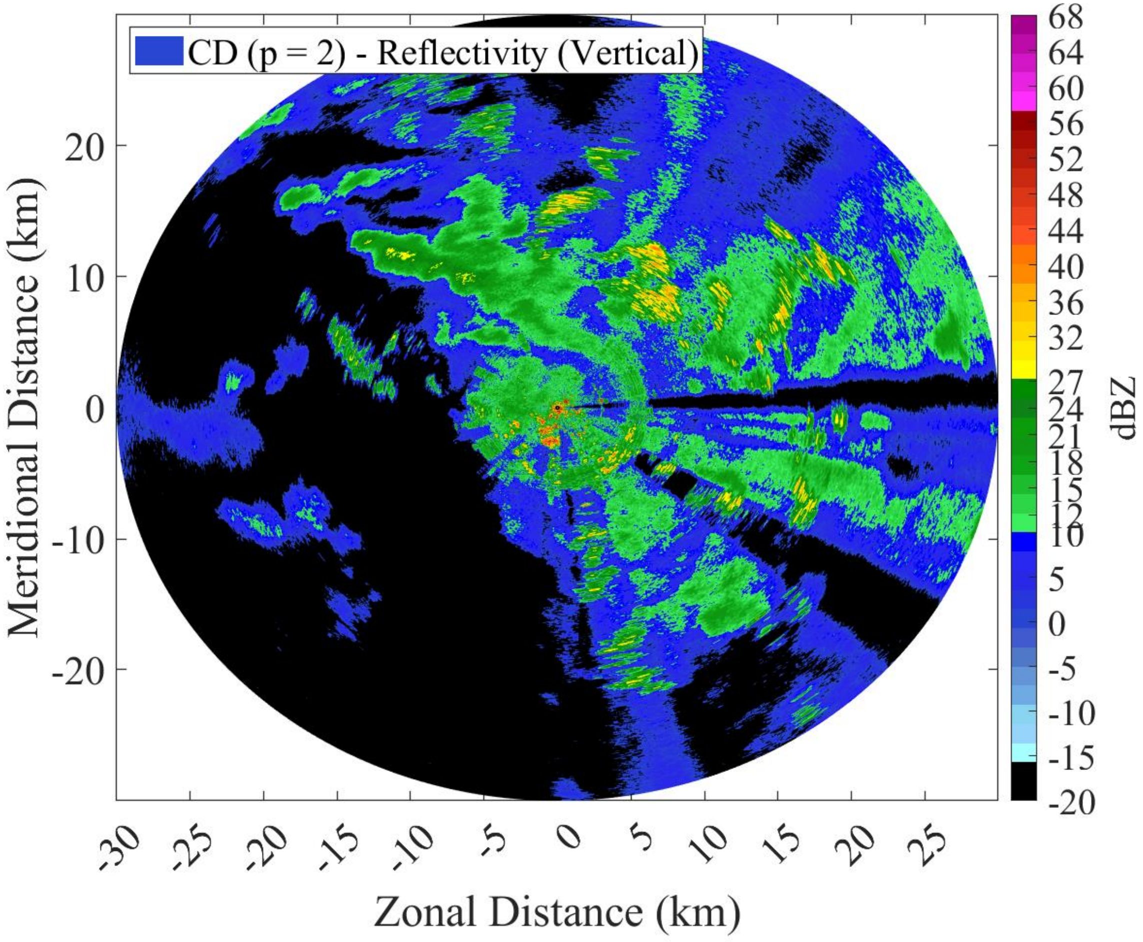
Reflectivity (Horizontal)





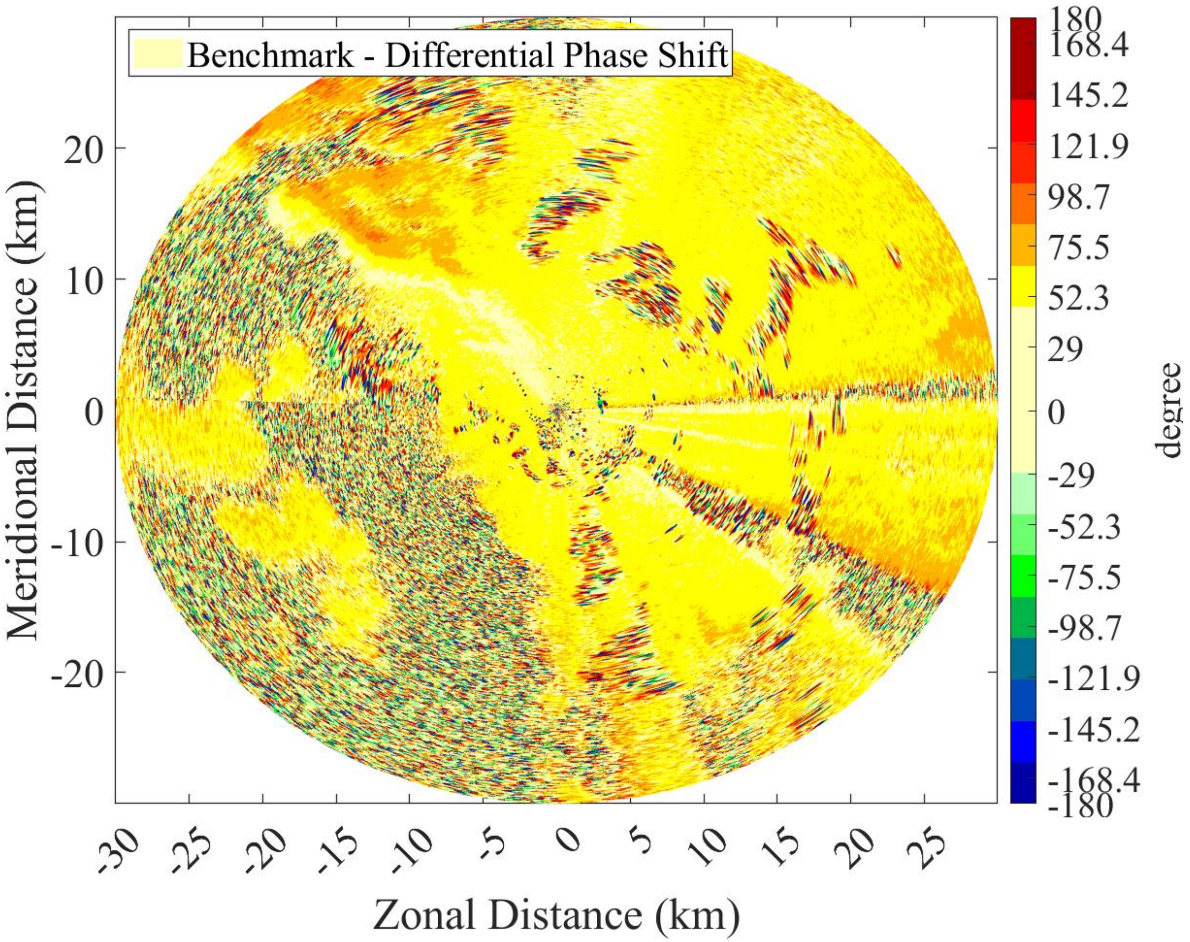
Reflectivity (Vertical)

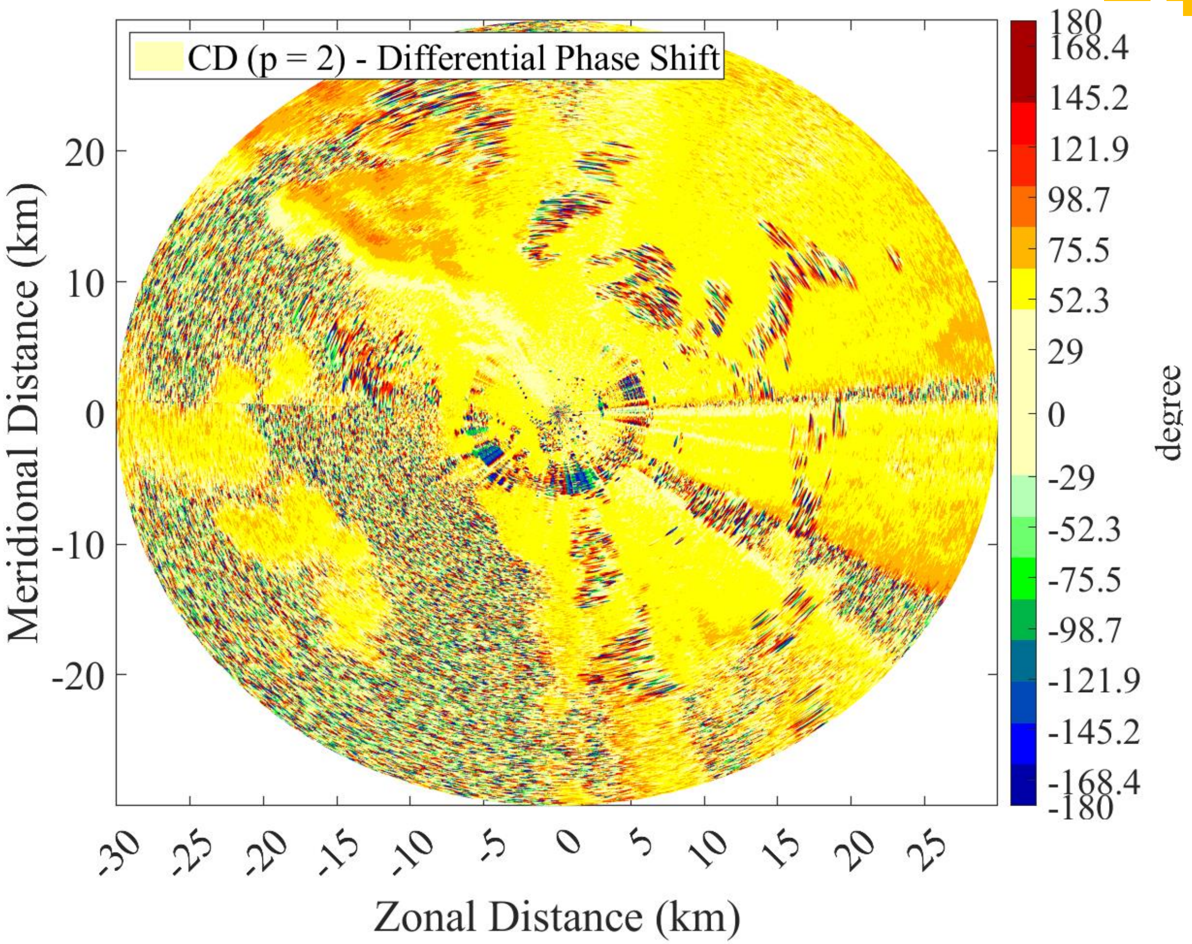




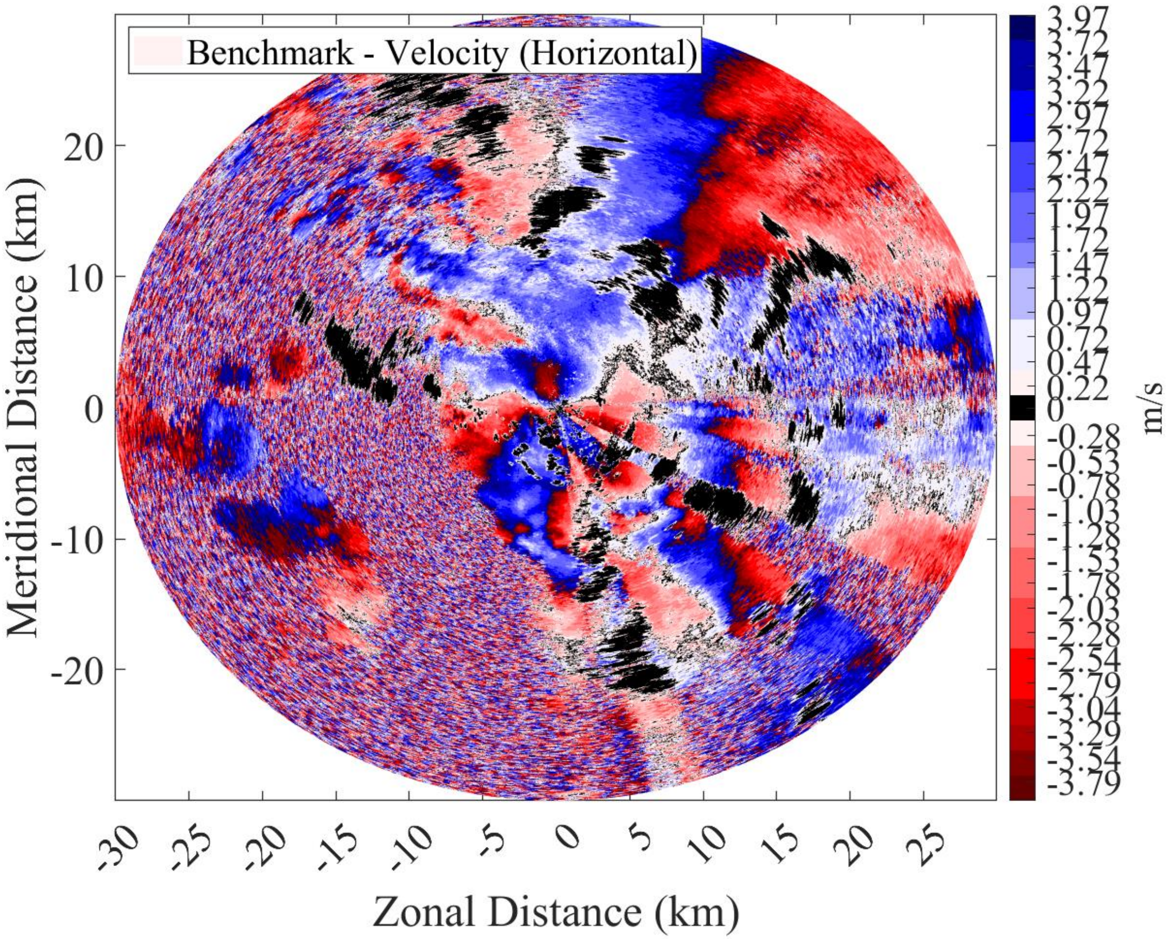
Differential Phase

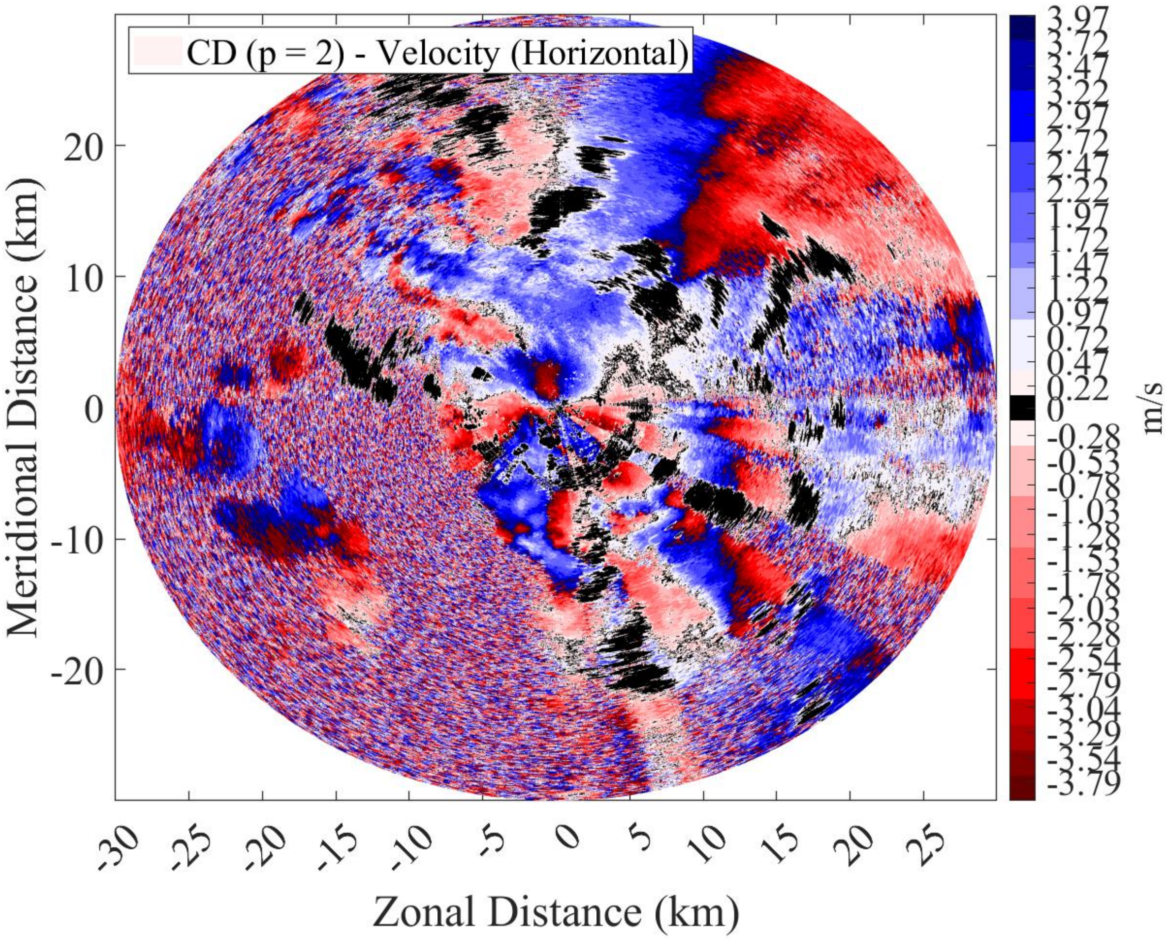
$$\hat{\phi}_{DP} = \angle\langle S_h S_v^* \rangle$$



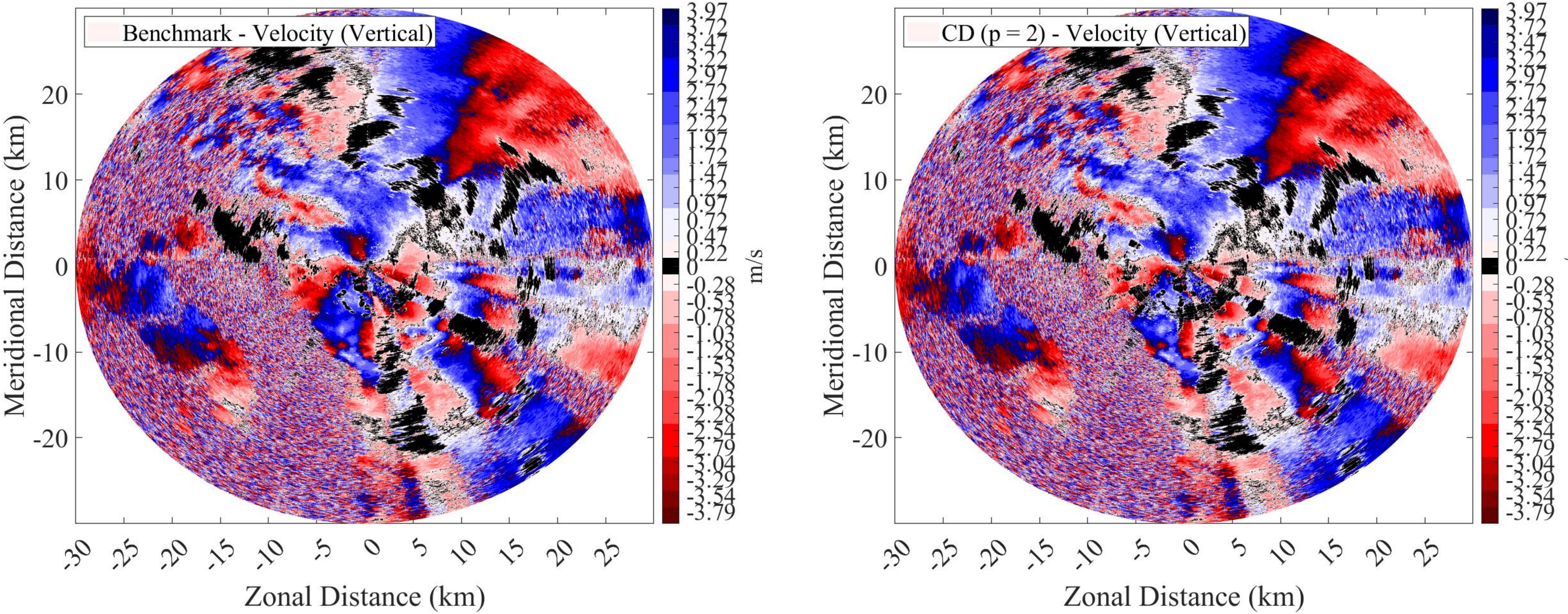


Velocity (Horizontal)



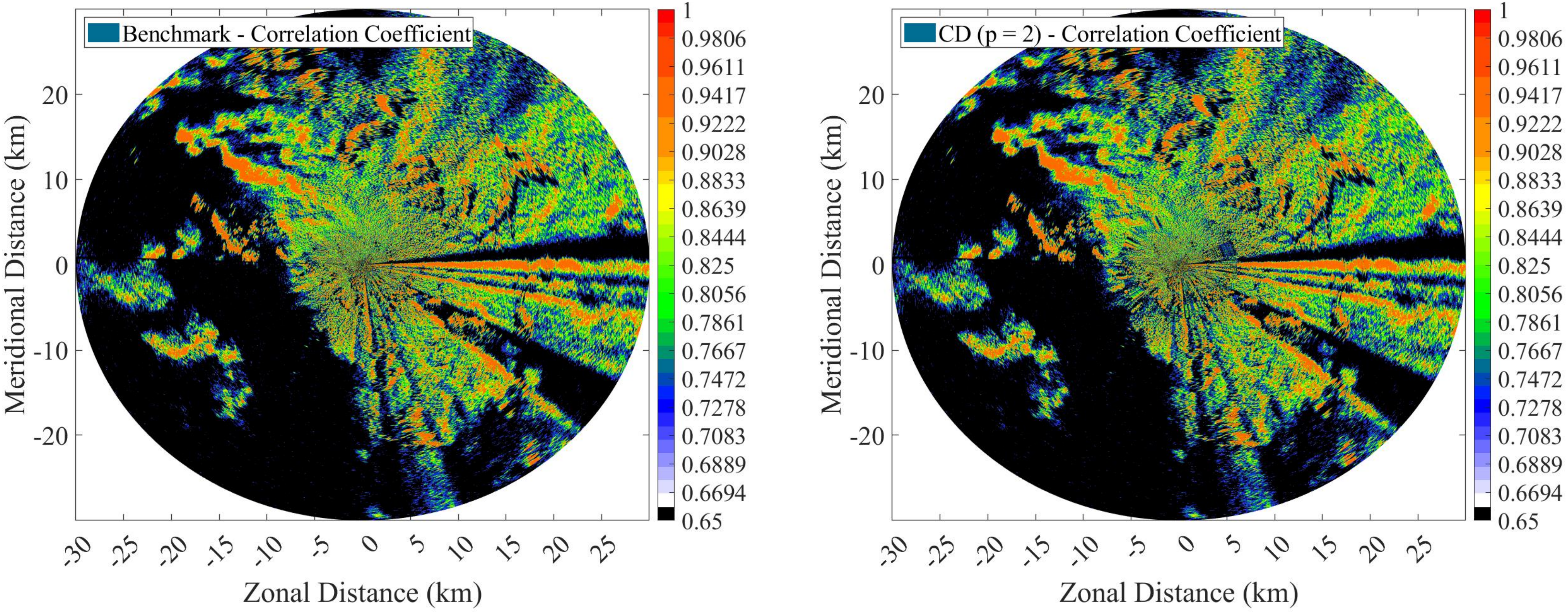


Velocity (Vertical)

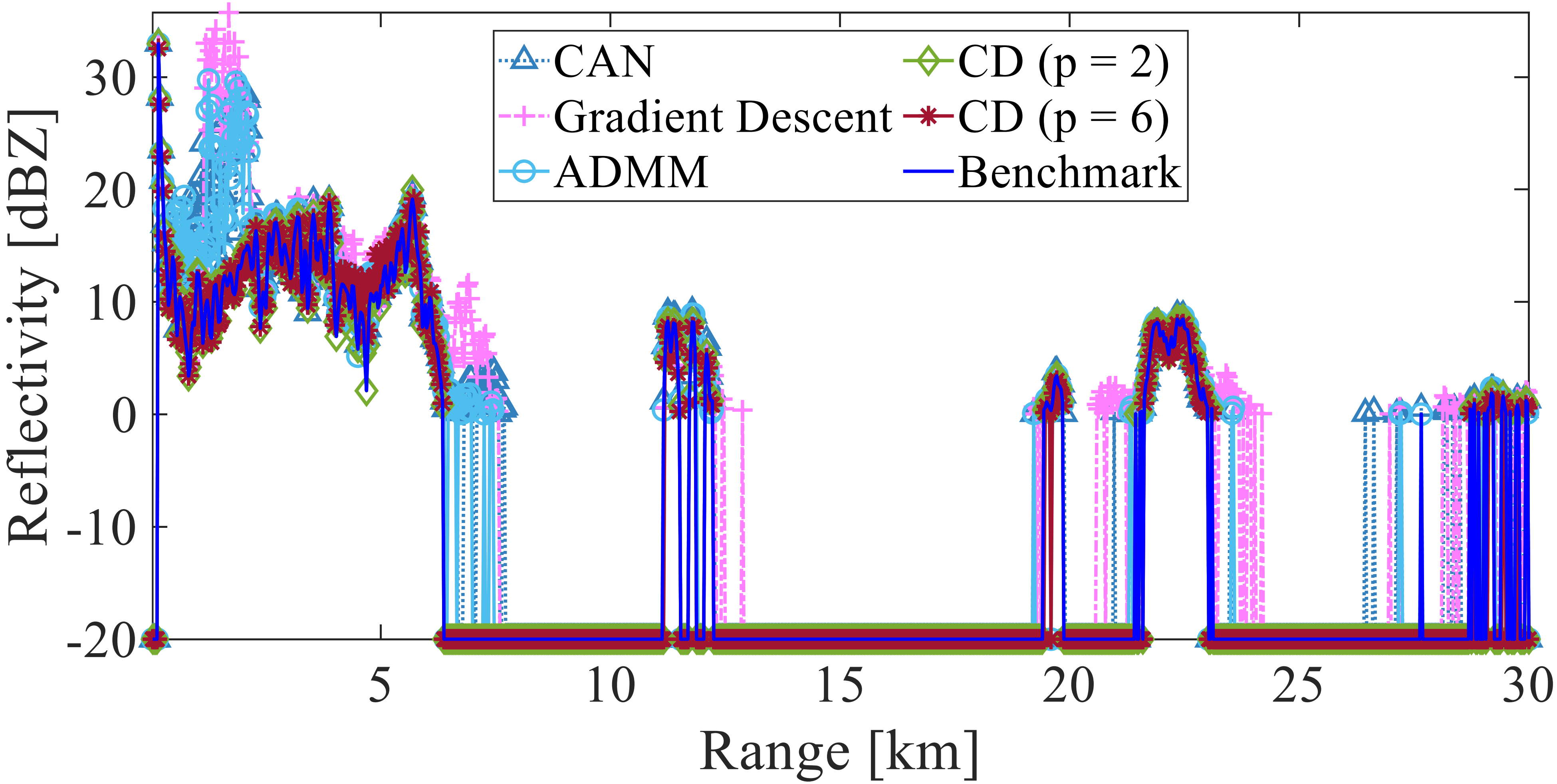


Correlation coefficient

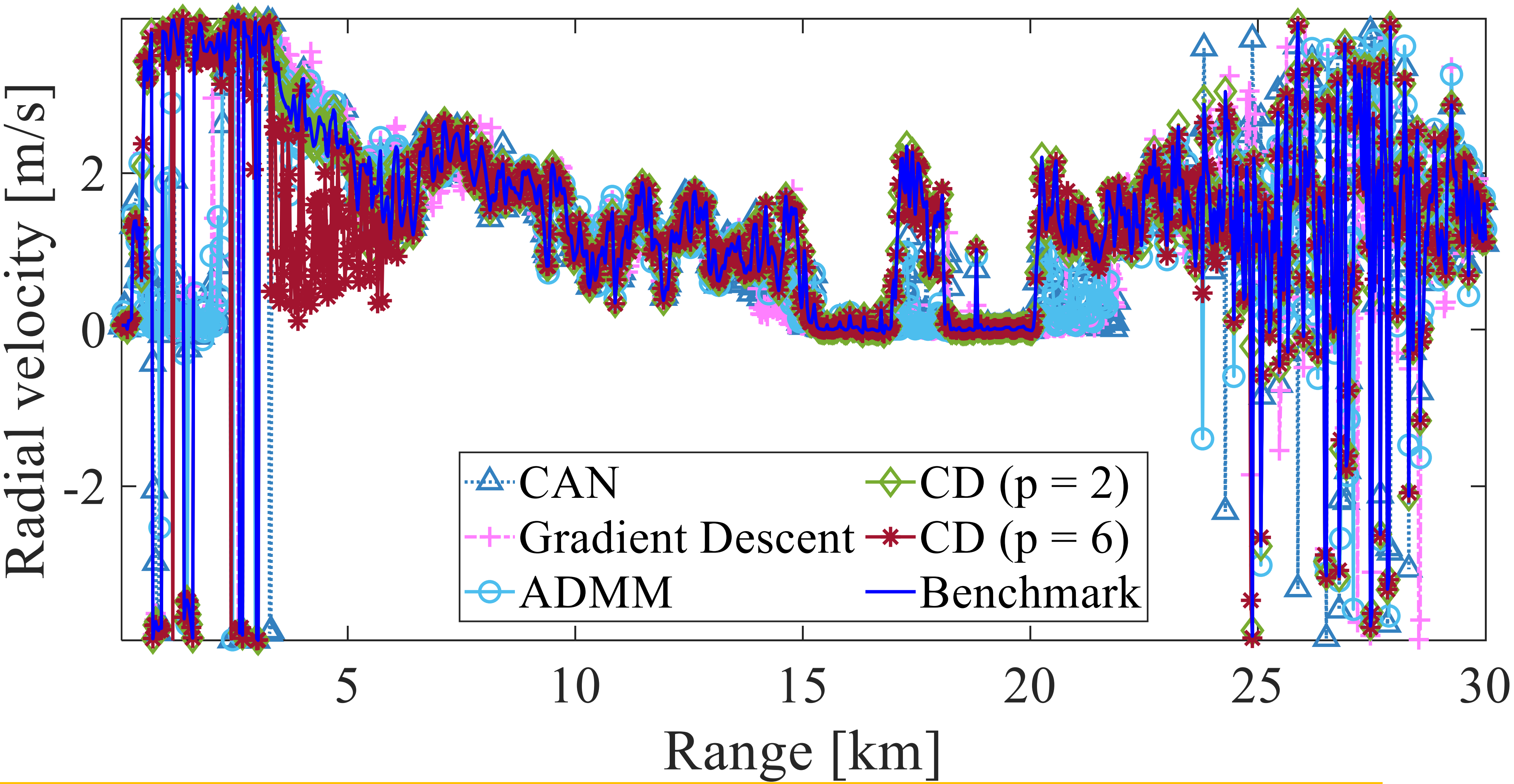
$$\hat{\rho}_{hv} = \frac{|\langle S_h S_v^* \rangle|}{\sqrt{\langle |S_h|^2 \rangle \langle |S_v|^2 \rangle}}$$



Reflectivity (Horizontal)



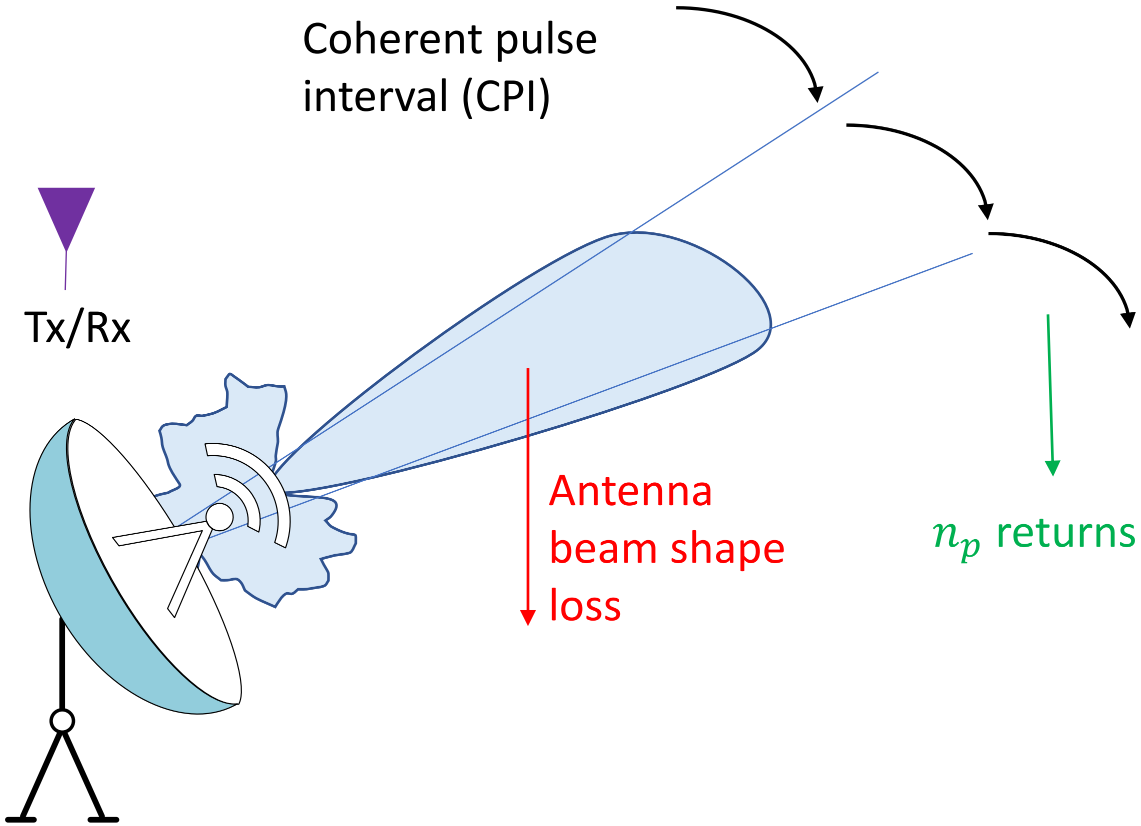
Radial Velocity

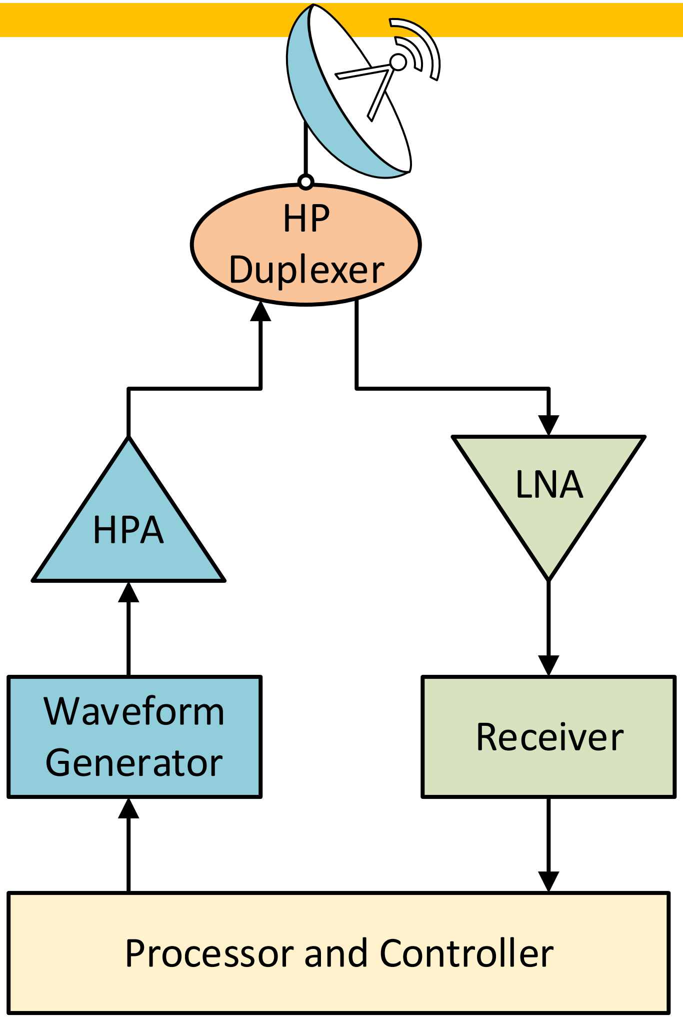


Waveform Design in

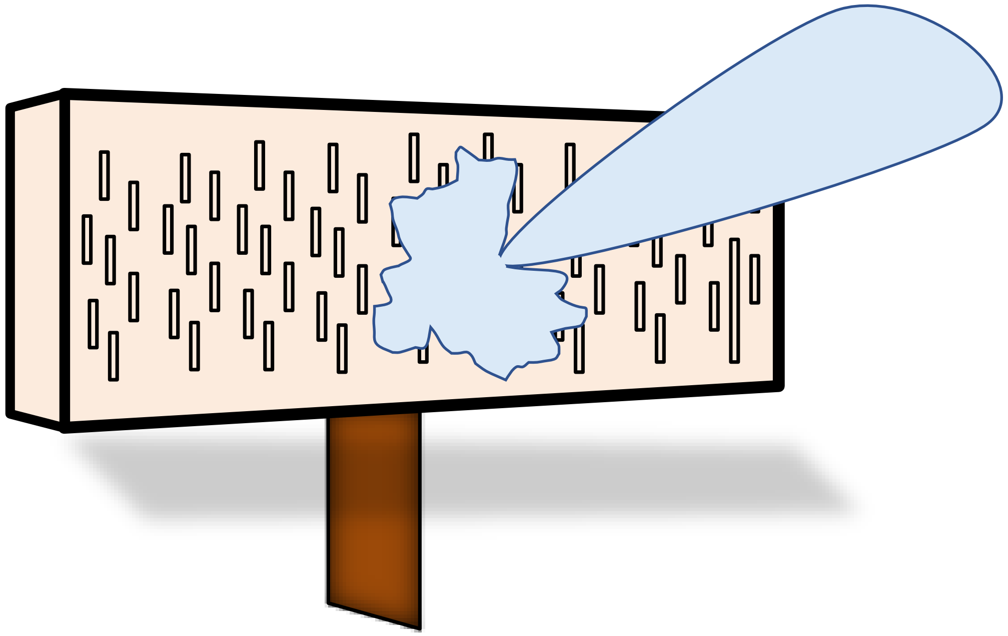
Phased Array and MIMO Radars

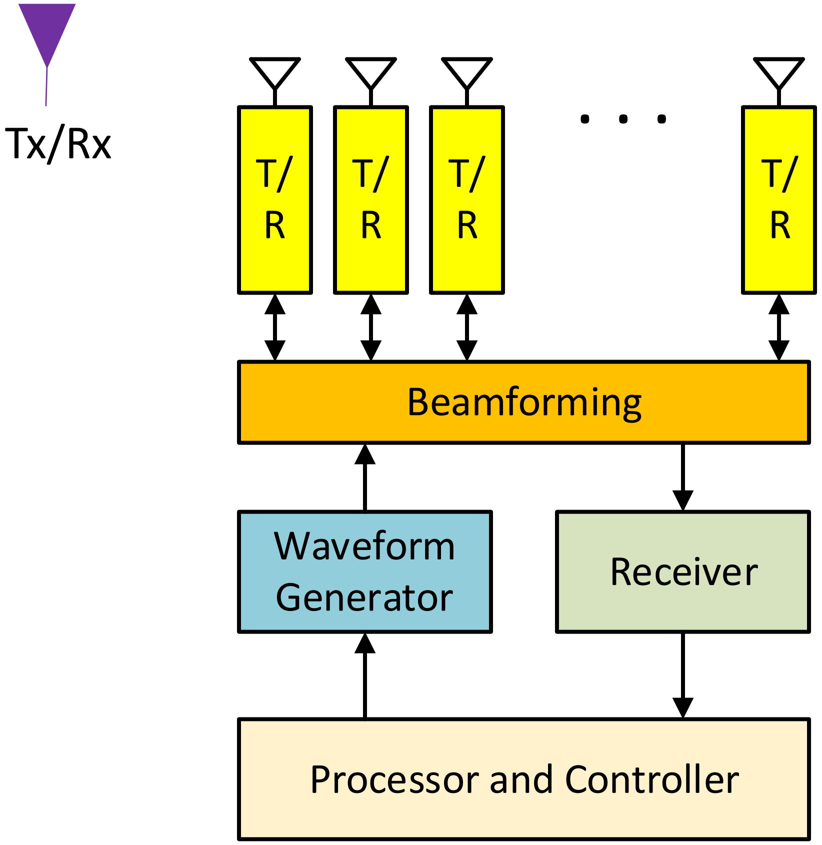
Antenna – Mechanical Scan





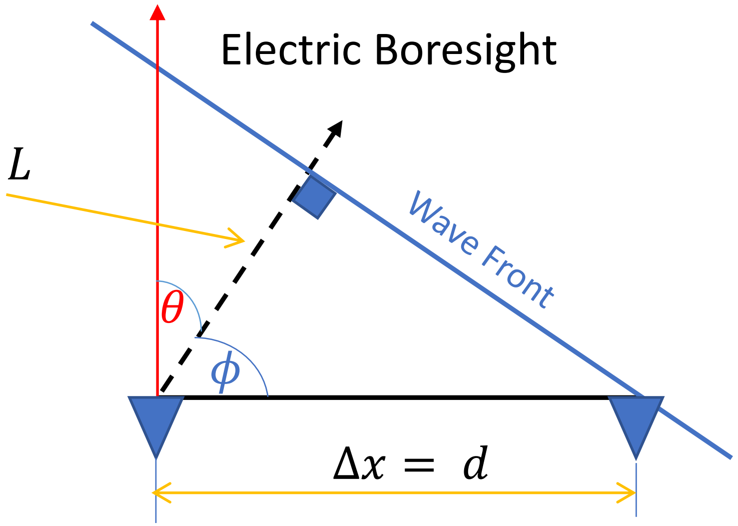
Antenna – Phased Array





Active Phase Array

Antenna – Phased Array



$$\cos\phi = \frac{L}{d}, \qquad \theta + \phi = 90 \qquad \cos\phi = \cos(90 - \theta) = \sin\theta$$

$$\sin\theta = \frac{L}{d} \Rightarrow \qquad L = d\sin\theta$$

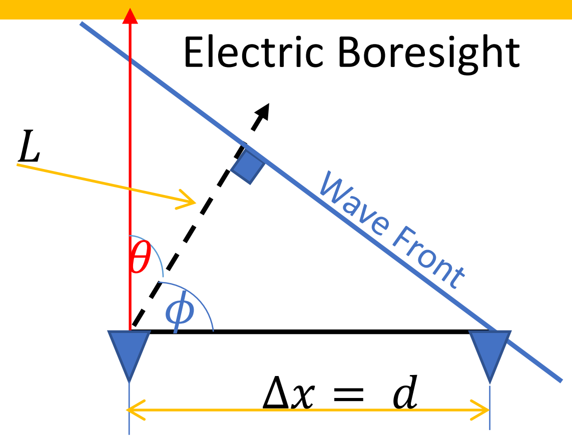
The phase variation across the array surface, or *aperture*, is the total path length variation times $\frac{2\pi}{\lambda}$

$$\boxed{\Delta\phi = \frac{2\pi d \sin\theta}{\lambda}}$$

If $d = \frac{\lambda}{2} \Rightarrow \quad \Delta\phi = \pi\sin\theta$

What happens if we increase d ?

Antenna – Phased Array



1

$$\exp -j\frac{2\pi(1)\Delta x \sin\theta}{\lambda}$$

$$\exp -j\frac{2\pi(2)\Delta x \sin\theta}{\lambda}$$

$\vdots$

$$\exp -j\frac{2\pi(\mathrm{N}-1)\Delta x \sin\theta}{\lambda}$$

(3dB) Beamwidth [rad] $\cong \dfrac{\alpha\lambda}{N\,\Delta x}$

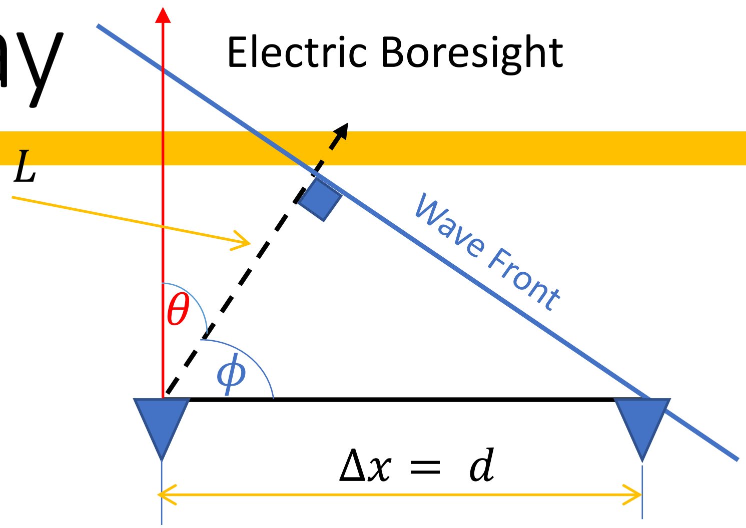
α is the beamwidth factor and is determined by the aperture taper function
N is number of antennas
Δx is the distance between two antenna elements

$$AF(\theta) = \frac{1}{N}\sum_{n=0}^{N-1}\exp\left(-j\frac{2\pi}{\lambda}\mathrm{n}\Delta x \sin\theta\right)$$

This expression is referred to as the array factor (AF)

If the element is assumed to be an isotropic radiator, which has no angular dependence, then the array factor and the phased array radiation pattern will be equal.

Antenna – Phased Array



$$1$$

$$\exp -j\frac{2\pi(1)\Delta x \sin\theta}{\lambda}$$

$$\exp -j\frac{2\pi(2)\Delta x \sin\theta}{\lambda}$$

$$\vdots$$

$$\exp -j\frac{2\pi(\mathrm{N}-1)\Delta x \sin\theta}{\lambda}$$

$$\boldsymbol{a}(\theta) = \left[1, \exp\left(-j\frac{2\pi\Delta x \sin\theta}{\lambda}\right), \ldots, \exp\left(-j\frac{2\pi(\mathrm{N}-1)\Delta x \sin\theta}{\lambda}\right)\right]^T \in \mathbb{C}^N$$

Steering vector

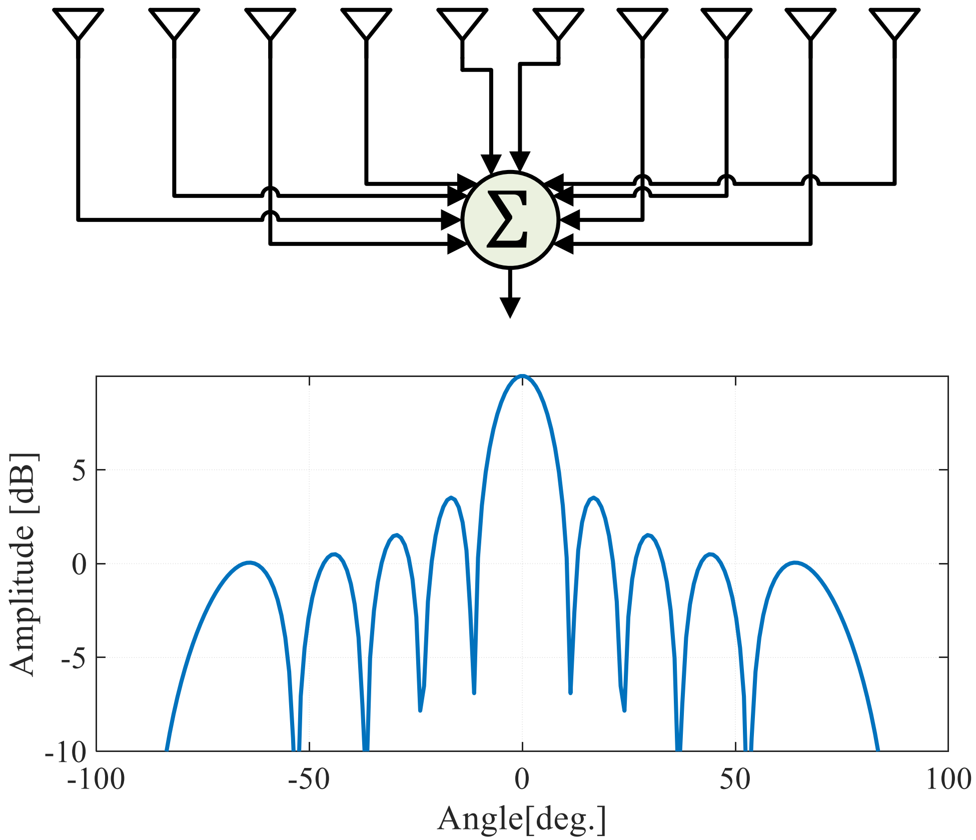
$$AF(\theta) = \boldsymbol{w}^H(\theta)\,\boldsymbol{a}(\theta)$$

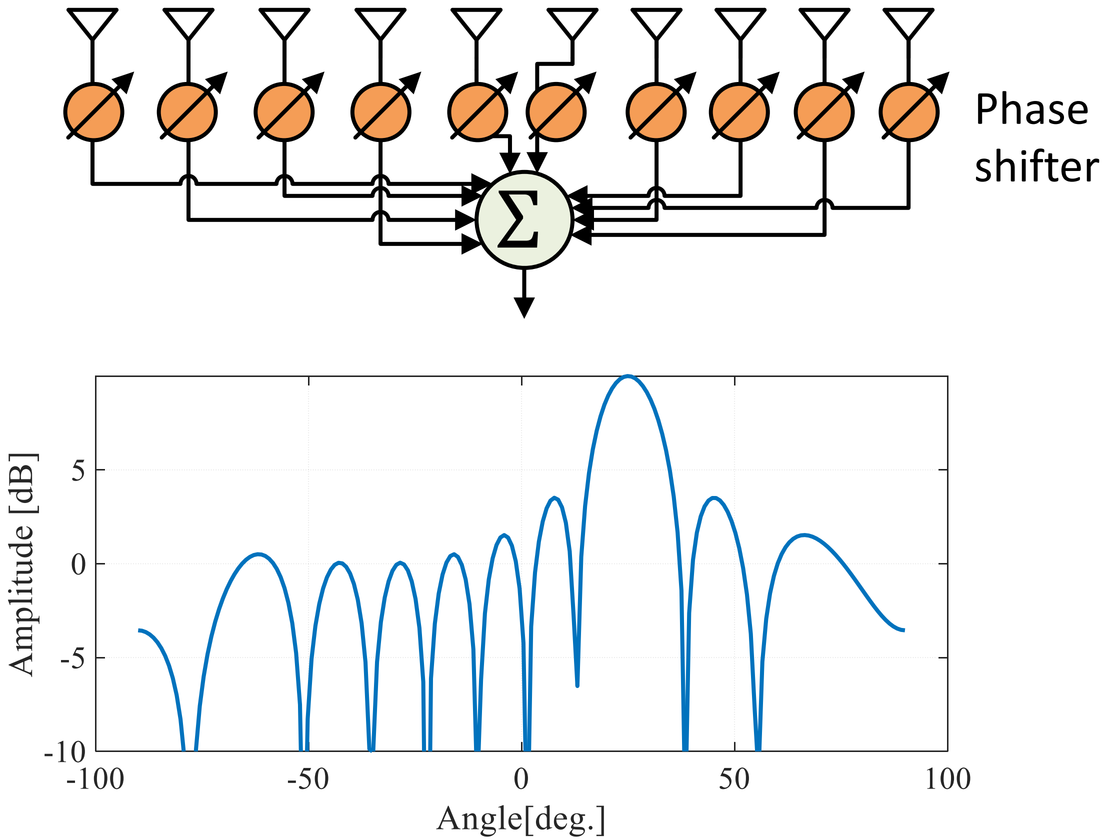
Weights

Can include
amplitude and phase

- ✓ steer beam to a desired angle
- ✓ control the sidelobe levels

Antenna – Phased Array

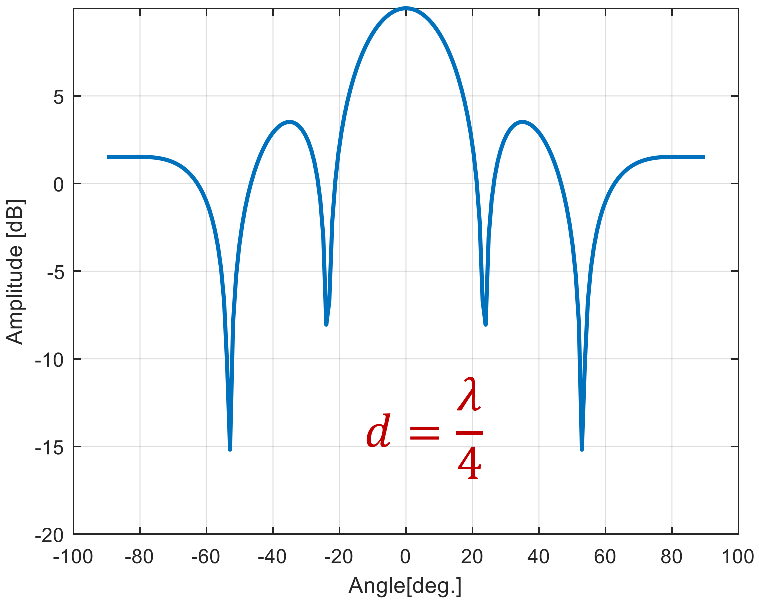


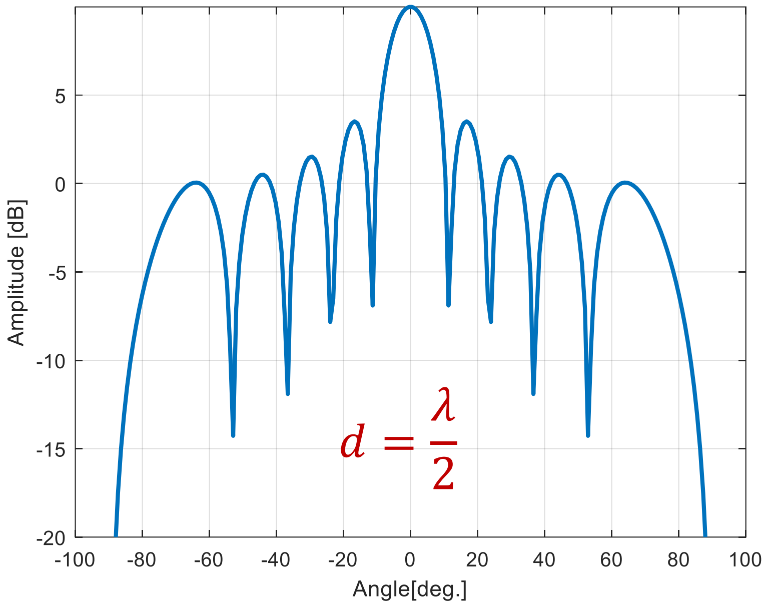


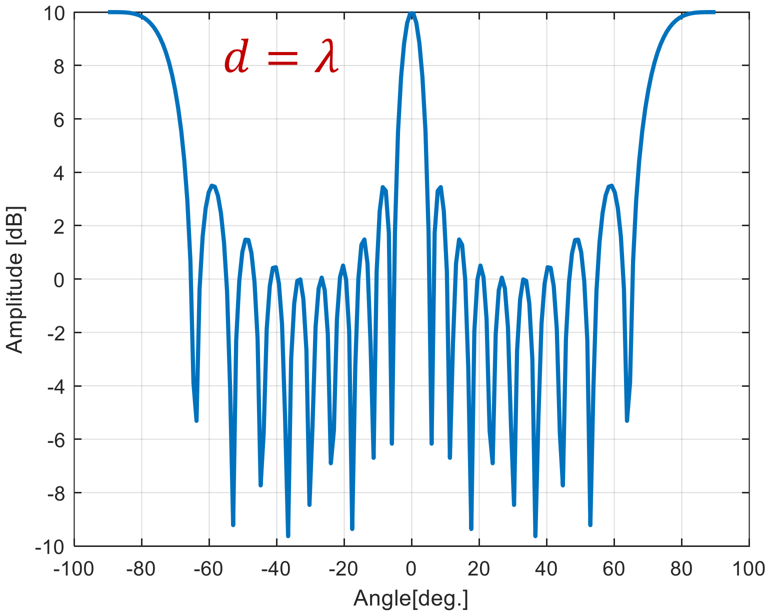
Antenna – Phased Array

Linear Array

$N = 10$ Isotropic Elements
No Phase Shifting







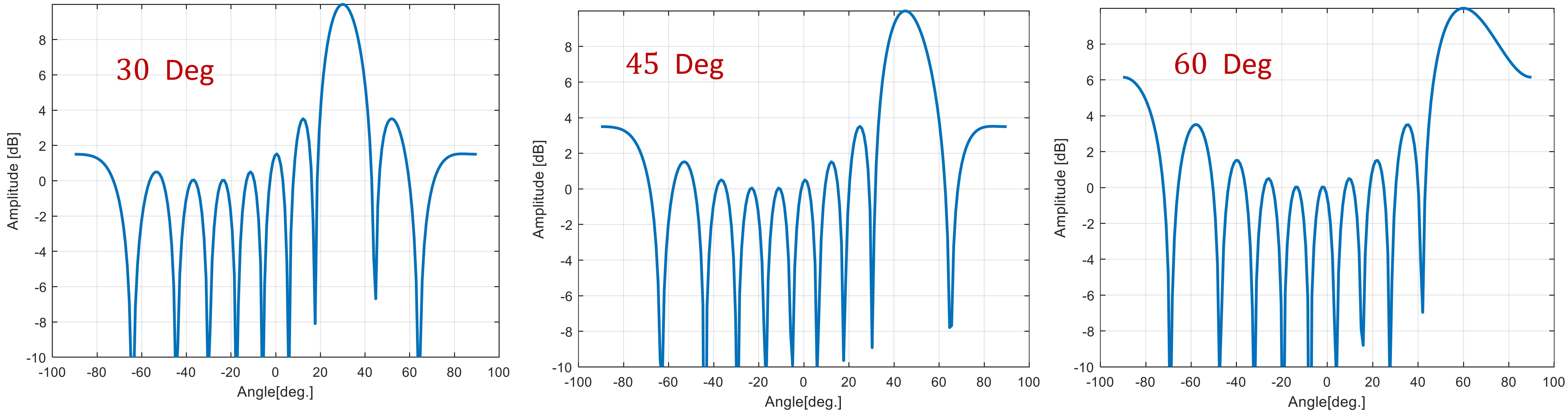
Limit element separation to $d < \lambda$ to prevent grating lobes for broadside array

Antenna – Phased Array

Linear Array

$N = 10$ Isotropic Elements

$d = \frac{\lambda}{2}$, Beam pointing direction = 30, 45 , 60



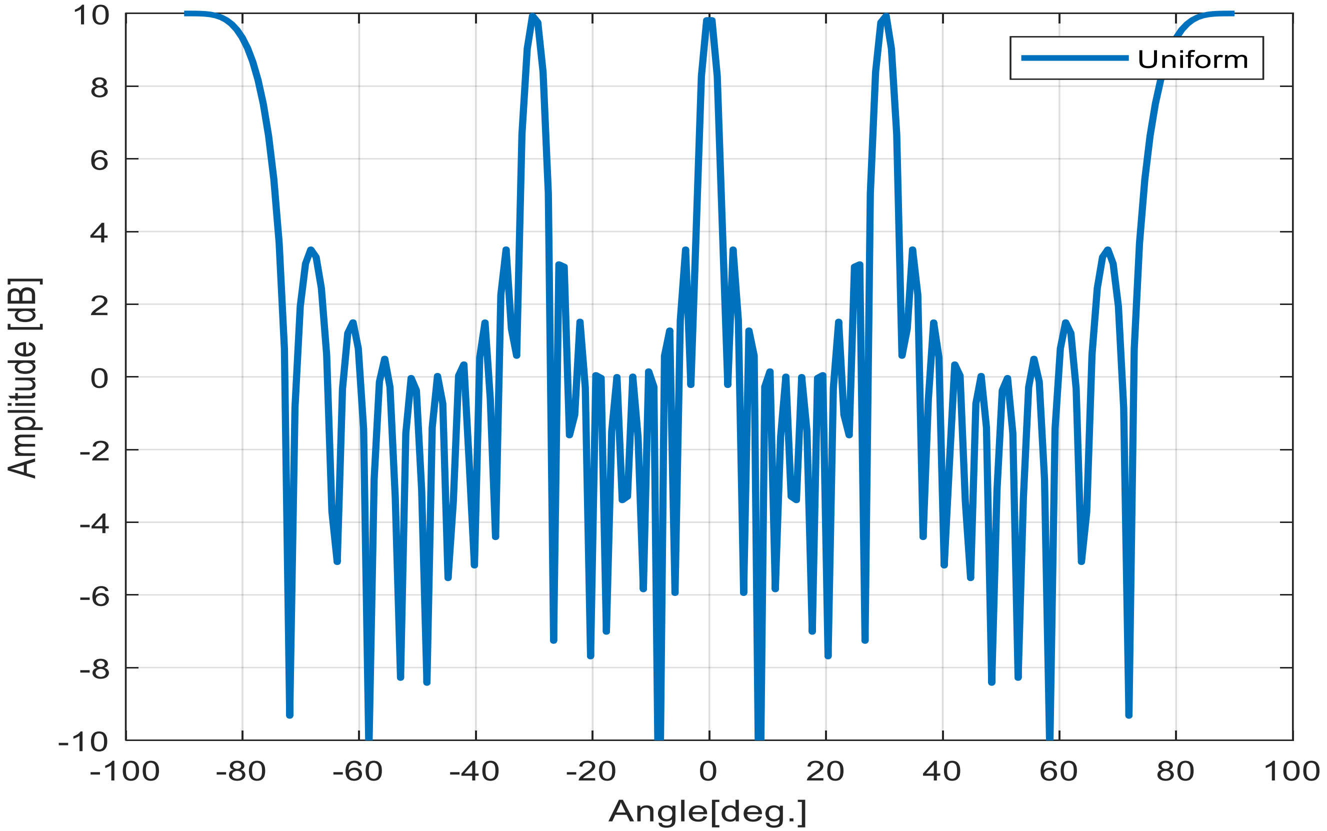
Antenna – Phased Array

Linear Array

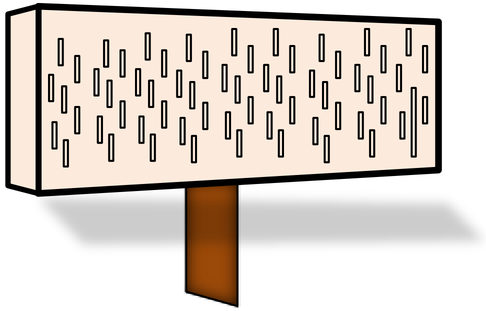
$N = 10$ Isotropic Elements
No Phase Shifting

$d = 2\lambda$

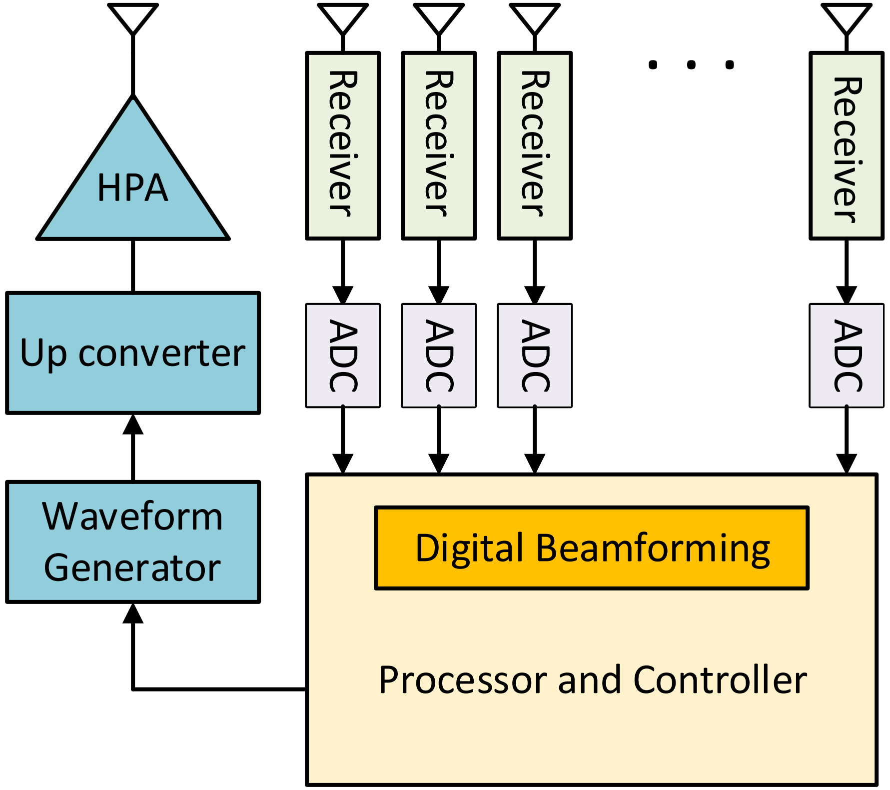
What are side effects of grating lobes ?



Antenna – Phased Array



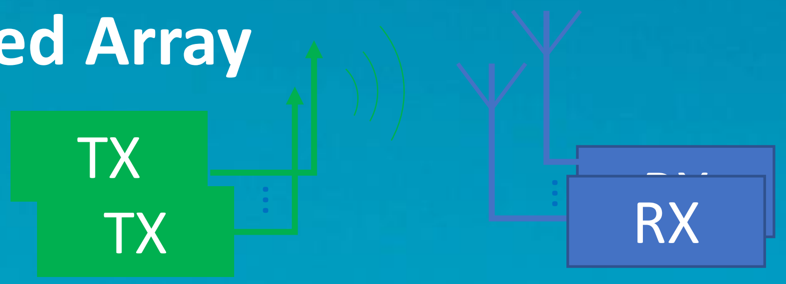
Tx



Phase Array Radars
with Digital Beamforming

Rx

Phased Array



Single Input Multi Output

MIMO

Multi Input Multi Output

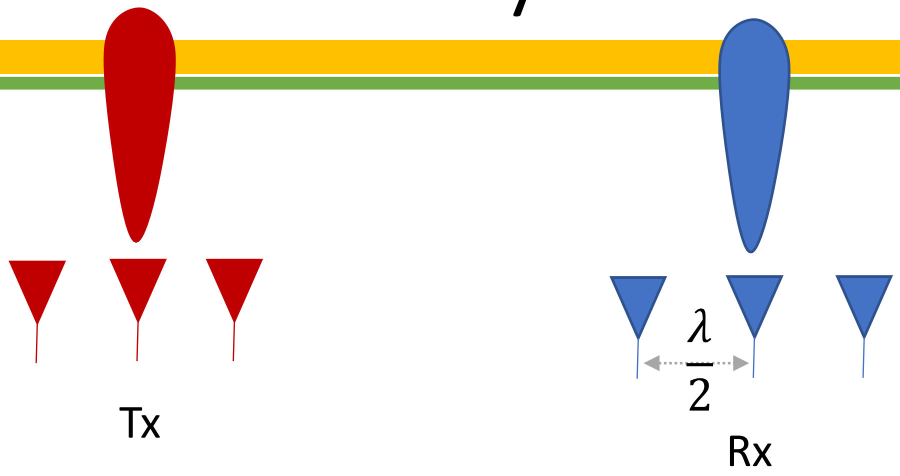
Waveform Diversity

Phased Array and MIMO Radars

SNR For Single Pulse

A

Phased Array

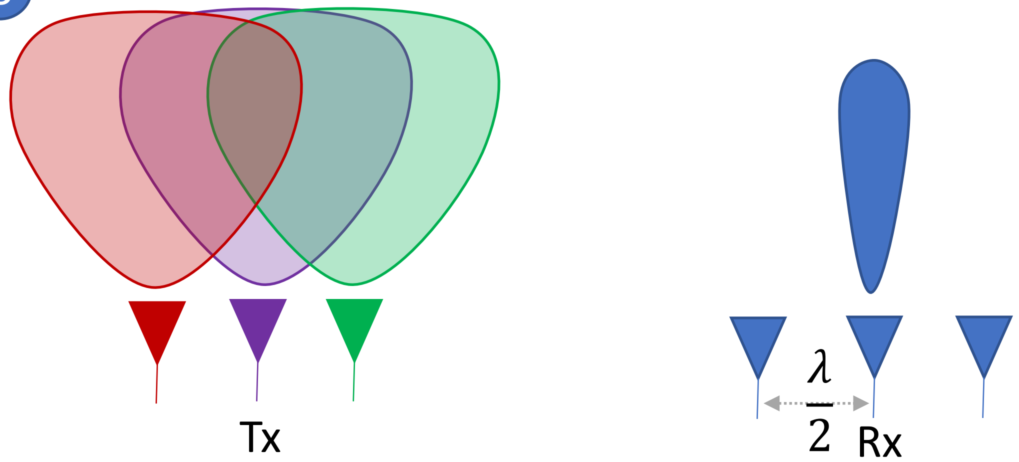


The Tx antennas transmit a single waveform.

$$\text{SNR} \propto G_t G_r \, \overbrace{N\tau}^{T_{pulse}}$$

B

MIMO



Every antenna element will emit a different waveform. Why?

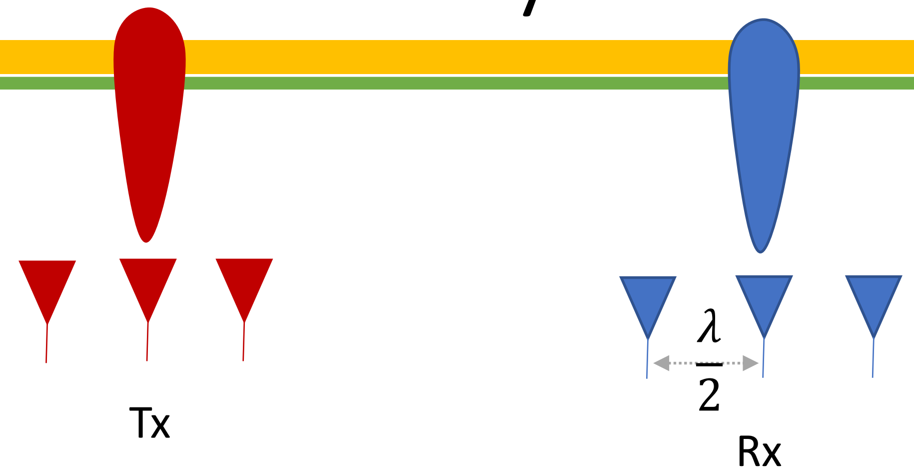
$$\text{SNR} \propto \frac{G_t}{N_t} G_r \, \overbrace{N\tau}^{T_{pulse}}$$

MIMO single pulse SNR that is N_t times lower

Phased Array and MIMO Radars

SNR For CPI

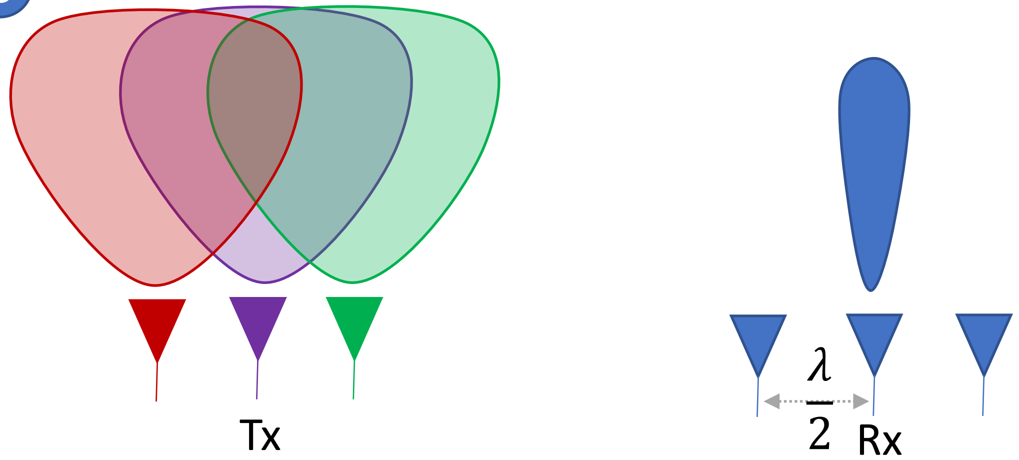
A



Phased Array

$$\text{SNR} \propto G_t G_r T_{PA}$$

B



MIMO

$$\text{SNR} \propto \frac{G_t}{N_t} G_r T_{MIMO}$$

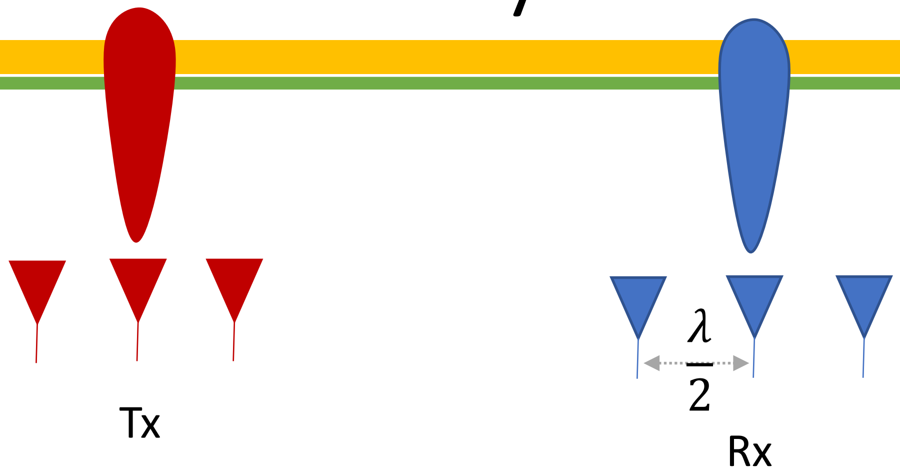
If $T_{MIMO} = N_t T_{PA}$, MIMO obtains same SNR over CPI as PA

Phased Array and MIMO Radars

Area Rate

A

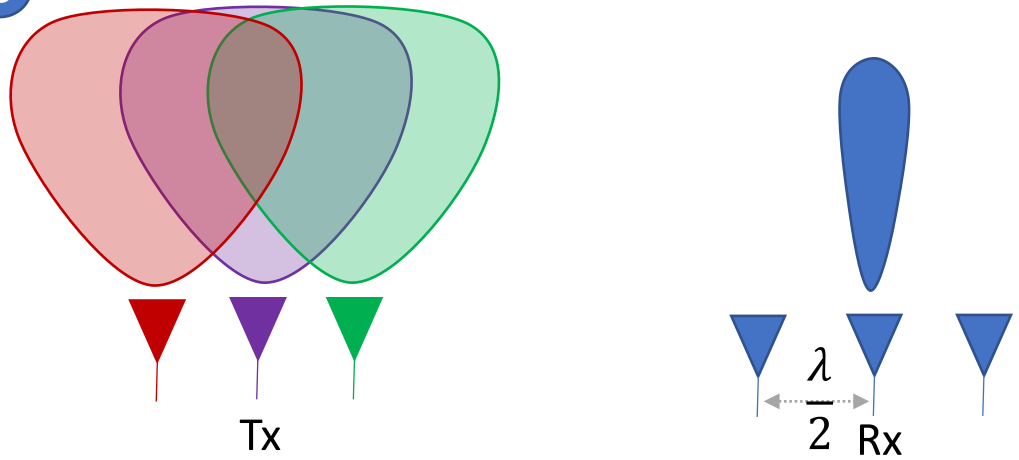
Phased Array



$$\text{Area rate} = \frac{4\pi R^2}{G_t T_{PA}}$$

B

MIMO



$$\text{Area rate} = \frac{4\pi R^2}{\frac{G_t}{N_t} T_{MIMO}}$$

If $T_{MIMO} = T_{PA}$, MIMO searches N_t times faster than PA

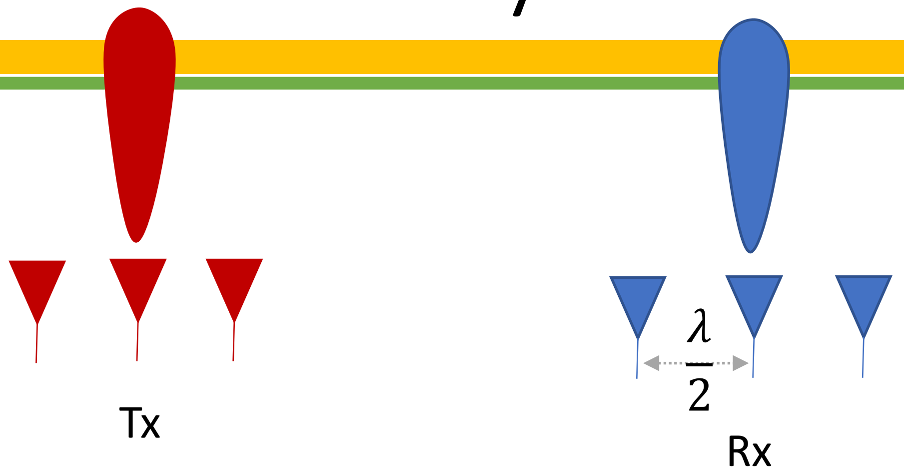
If $T_{MIMO} = N_t\, T_{PA}$, MIMO searches at the same speed as PA and provides same SNR with better Doppler resolution than PA

Phased Array and MIMO Radars

Area Rate and SNR For CPI

A

Phased Array

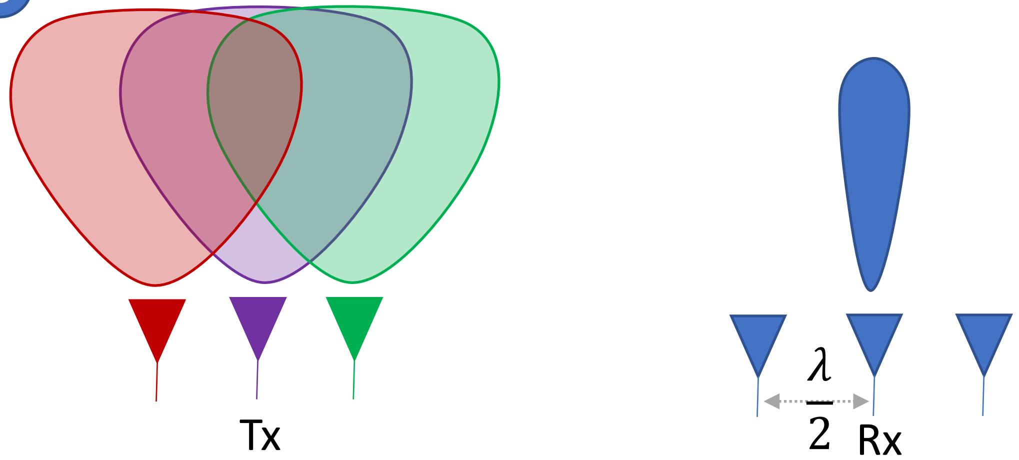


$$\text{SNR} \propto G_t G_r T_{PA}$$

$$\text{Area rate} = \frac{4\pi R^2}{G_t T_{PA}}$$

B

MIMO



$$\text{SNR} \propto \frac{G_t}{N_t} G_r T_{MIMO}$$

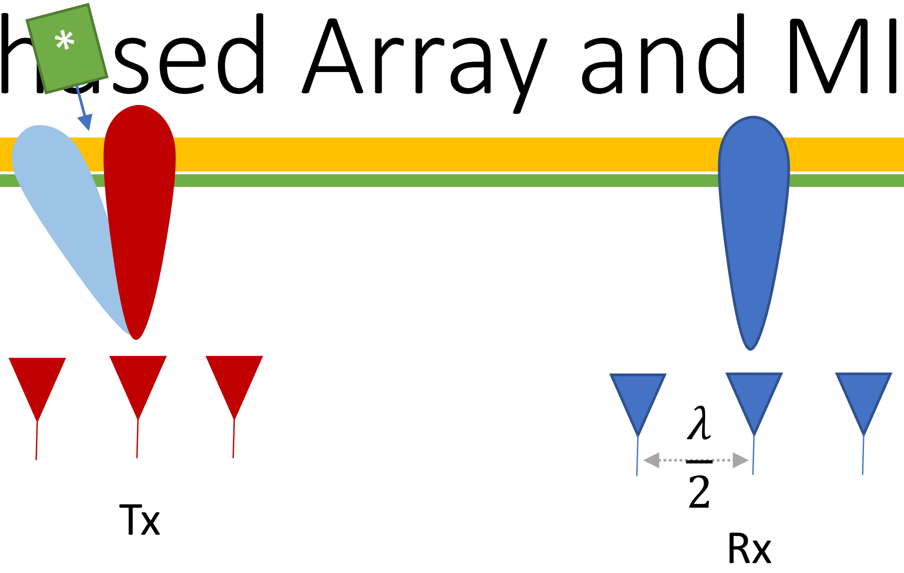
$$\text{Area rate} = \frac{4\pi R^2}{\frac{G_t}{N_t} T_{MIMO}}$$

Phased Array and MIMO Radars

Beam Shape Loss

A

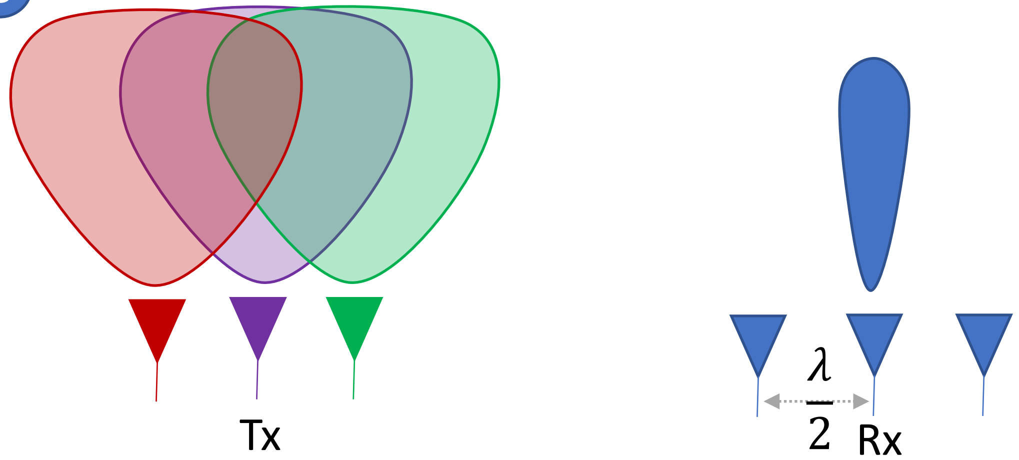
Phased Array



Analog phased array beam shape results in loss up to 6 dB

B

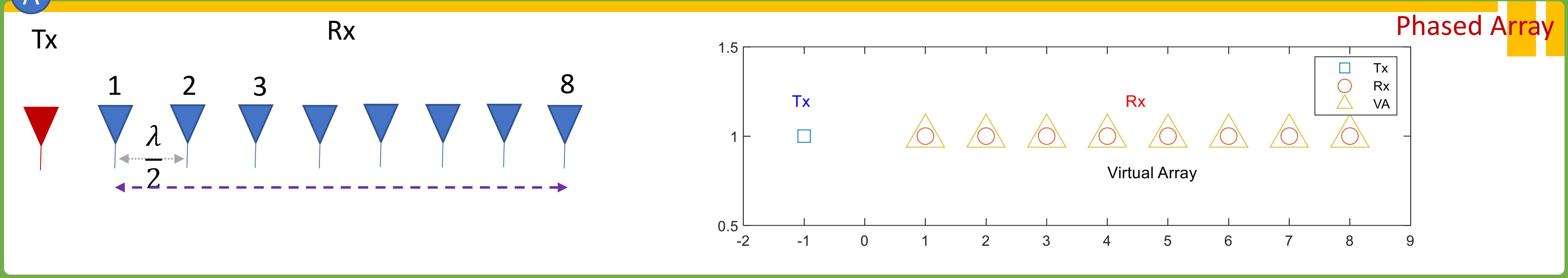
MIMO



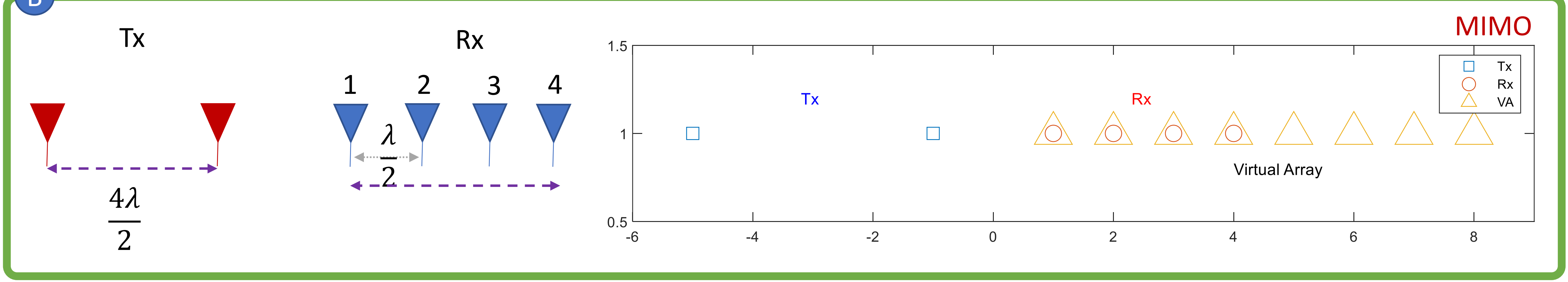
MIMO radar can eliminate Tx beam shape loss and reduce Rx beam shape loss

Phased Array and MIMO Radars

A



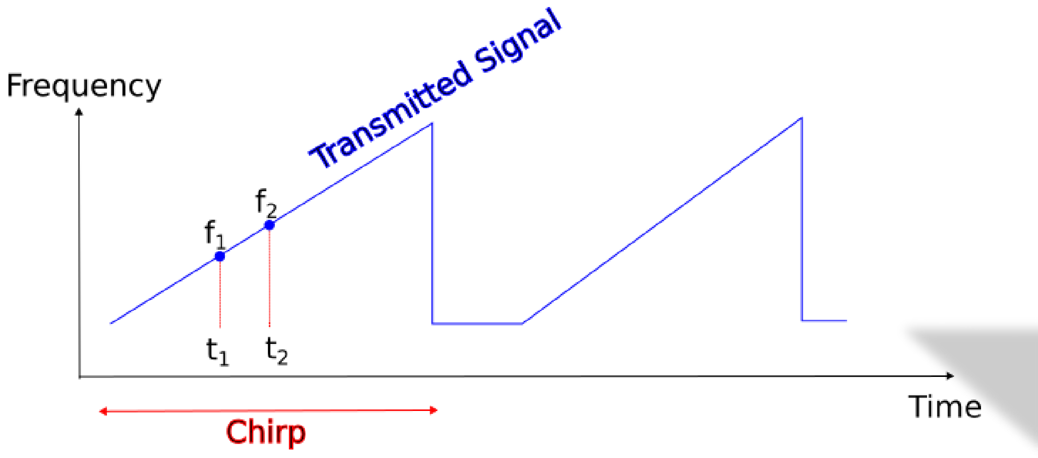
B



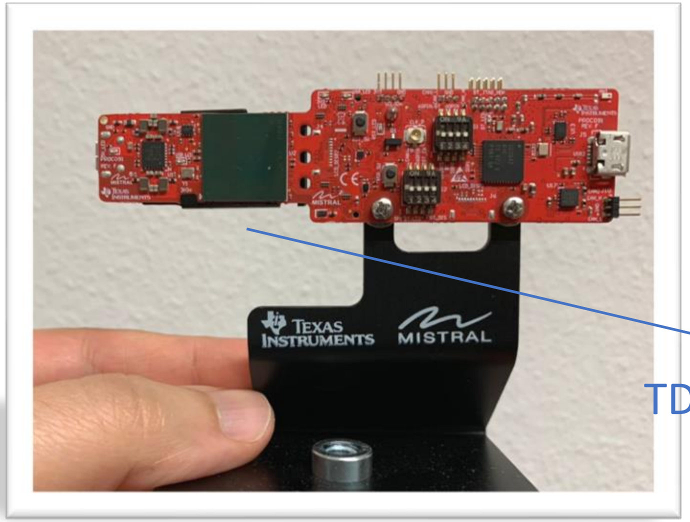
Compare (A) and (B) in terms of: Two-way antenna beampattern (angular resolution)

Example Application: mmWave MIMO Radars

- ✓ Operating in mmWave (60GHz, and 79 GHz)
- ✓ More than 5GHz bandwidth



- ✓ Radar on chip (ROC)
- ✓ MIMO capabilities



TDM-MIMO ROC

Today's mmWave radars :

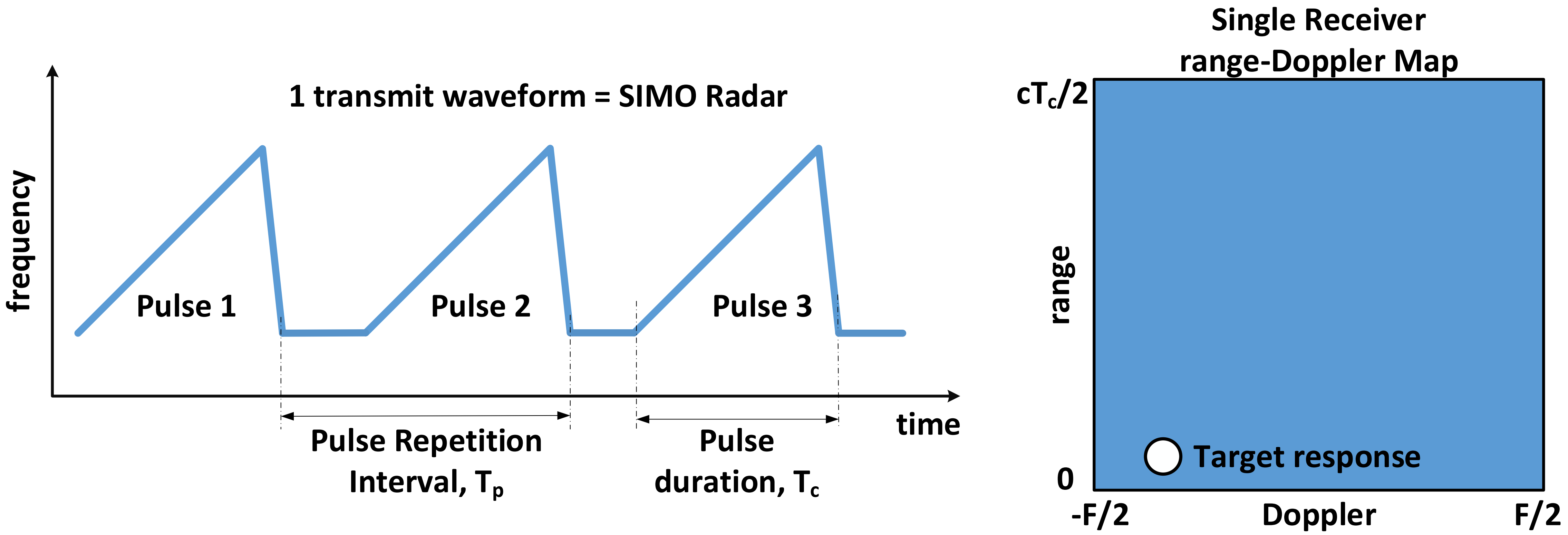
- ❑ Adaptive cruise control (ACC)
- ❑ Lane change assistance
- ❑ Collision avoidance
- ❑ Emergency braking
- ❑ Blind spot detection
- ❑ New-born baby monitoring
- ❑ Elderly fall and emergency detection
- ❑ Patient monitoring
- ❑ Drone navigation
- ❑ Drone swarm collision avoidance
- ❑ Occupancy sensing
- ❑ People flow management
- ❑ Public building security
- ❑ Smart street lighting
- ❑ Factories
- ❑ Robotics

Inter-Pulse Modulation Techniques

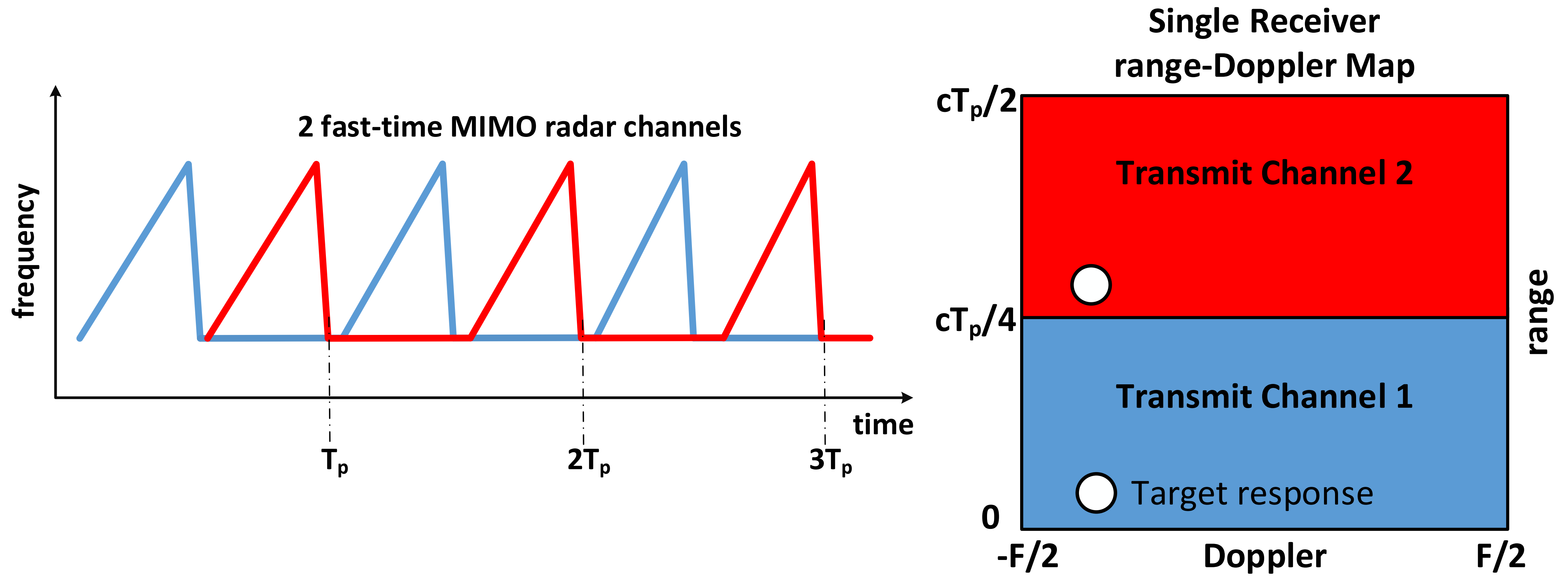
- Time Division Multiplexing (TDM)
- Frequency Division Multiplexing (FDM)
- Doppler Division Multiplexing (DDM)
- Binary Phase Modulation (BPM)

Time Division Multiplexing (TDM)

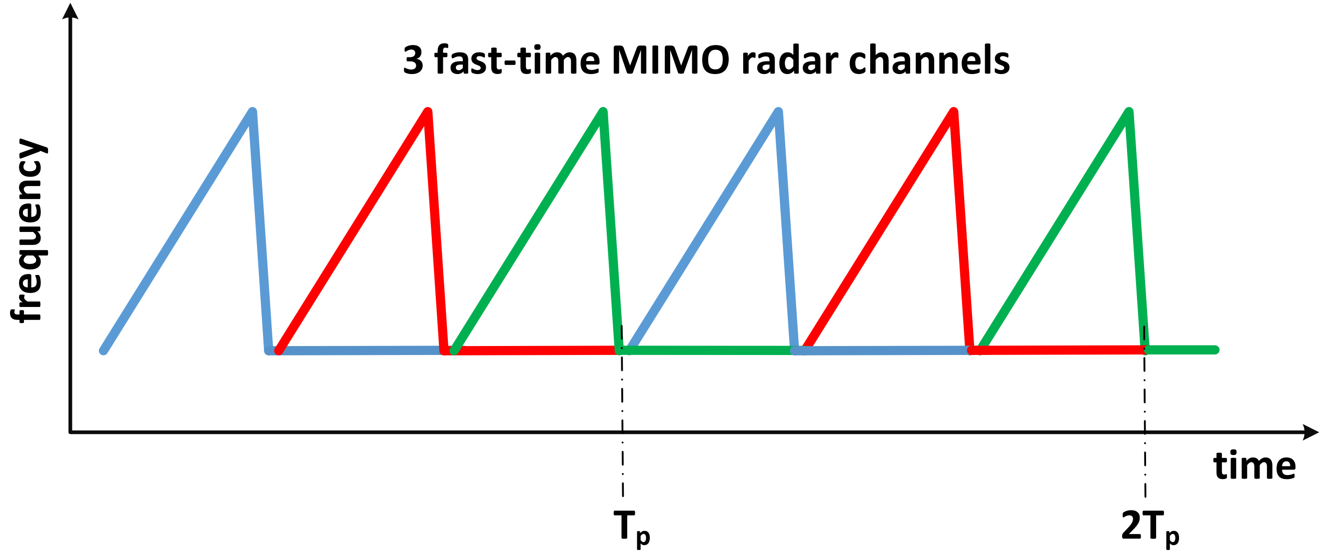
$$F = \frac{1}{T_R}$$

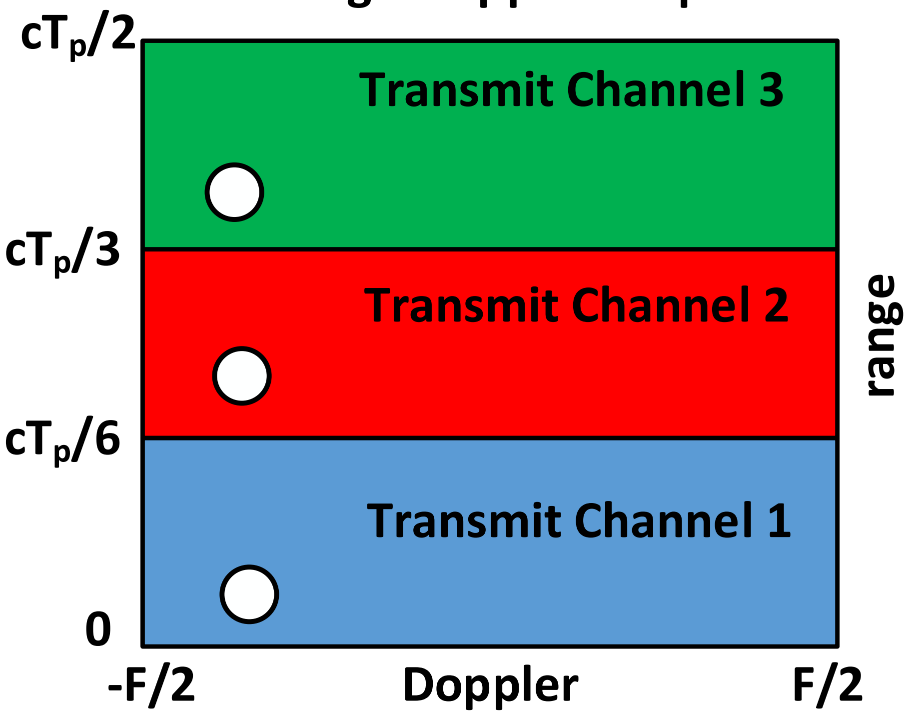


Time Division Multiplexing (TDM)



Time Division Multiplexing (TDM)



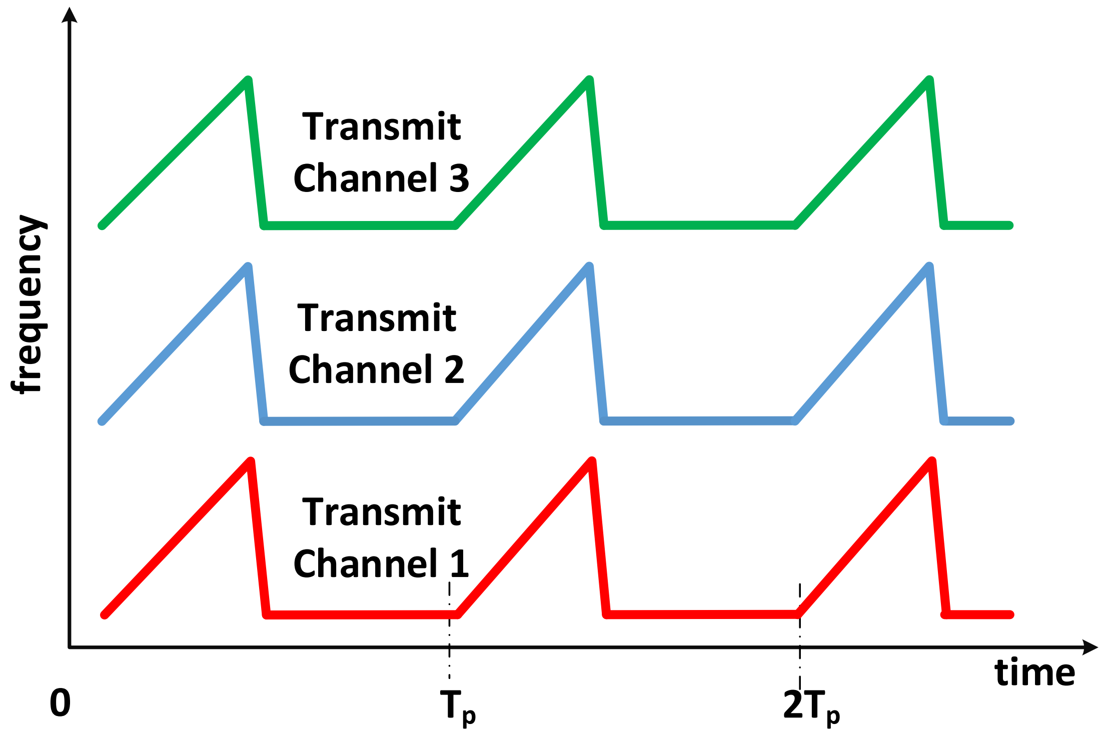


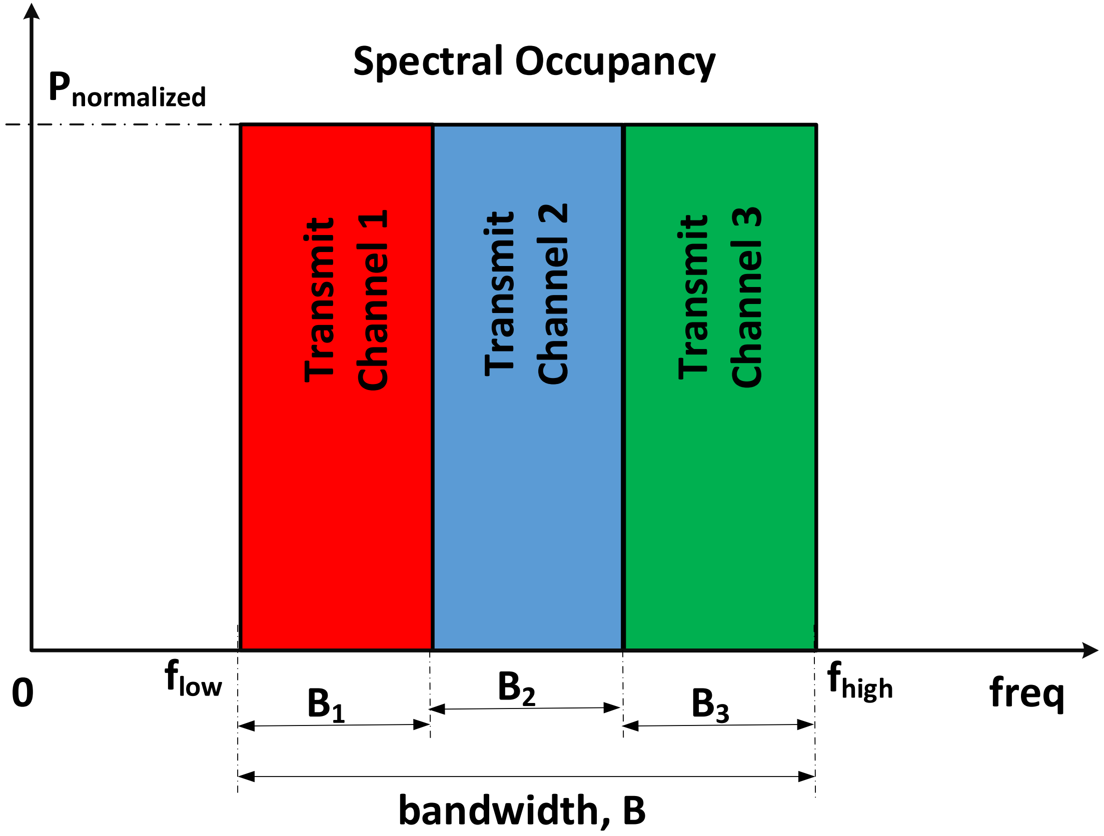
Frequency Division Multiplexing (FDM)

- Easy to implement with minimal hardware complexity
- range resolution compromised for more channels

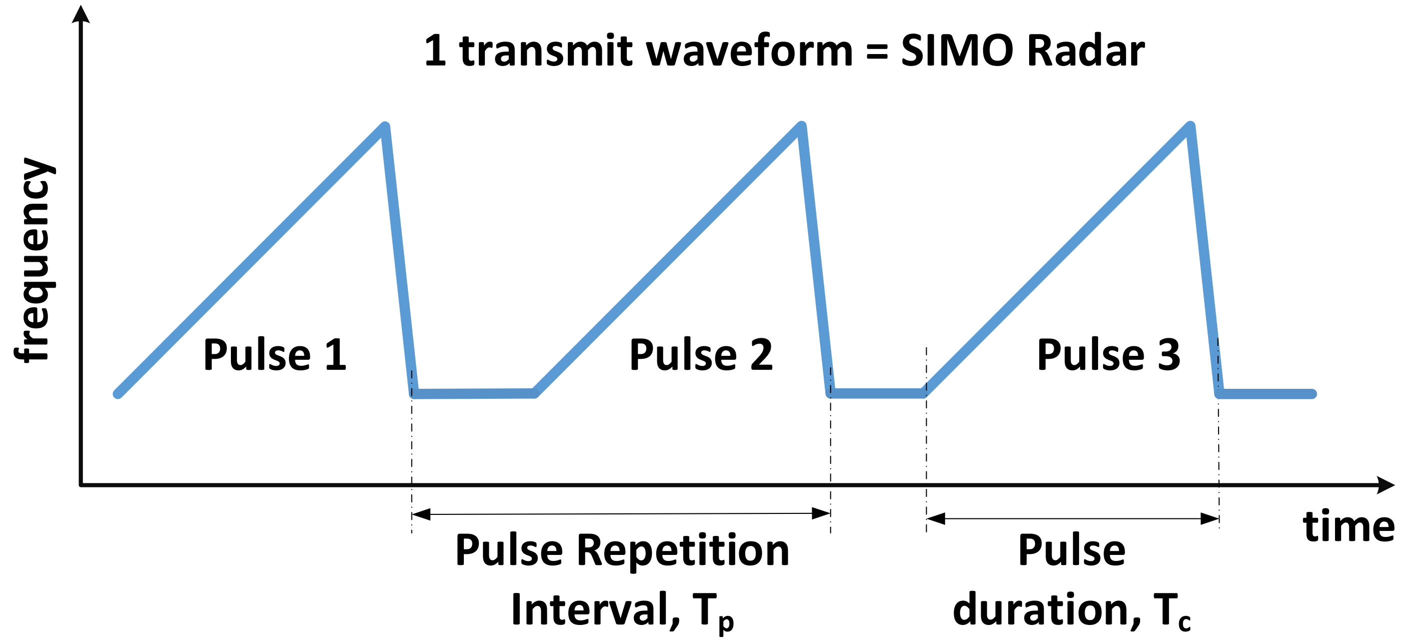
$$\text{range resolution} = \frac{c}{2B},$$

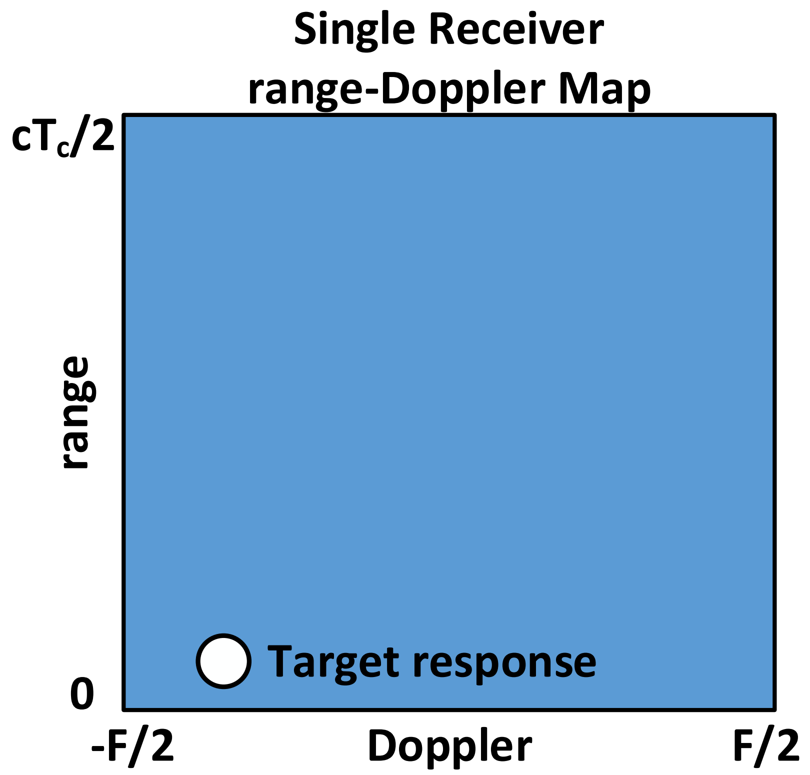
$$\text{where } c = \text{speed of light}, \ B = \text{bandwidth}.$$



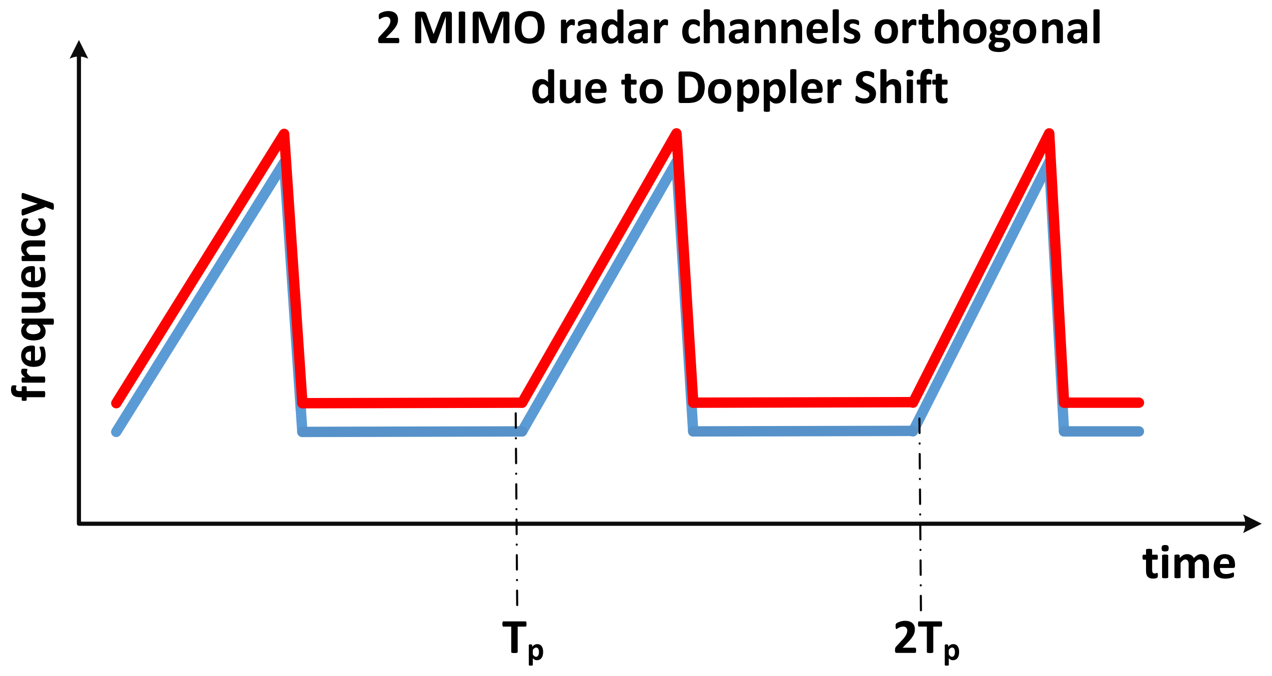


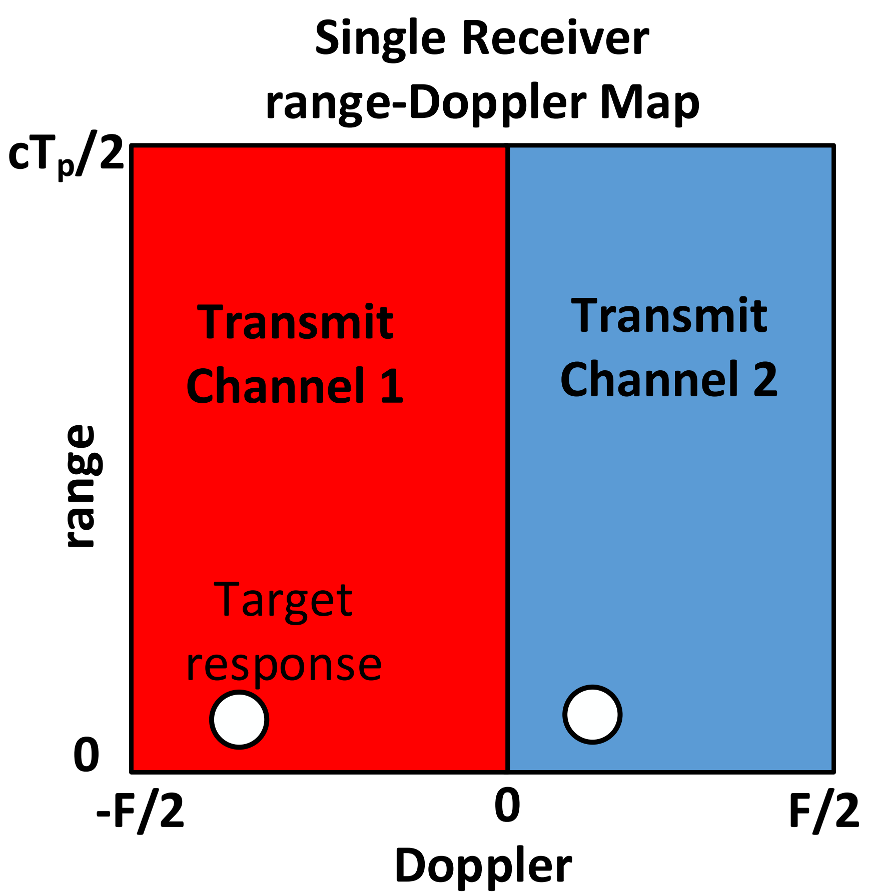
Doppler Division Multiplexing (DDM)



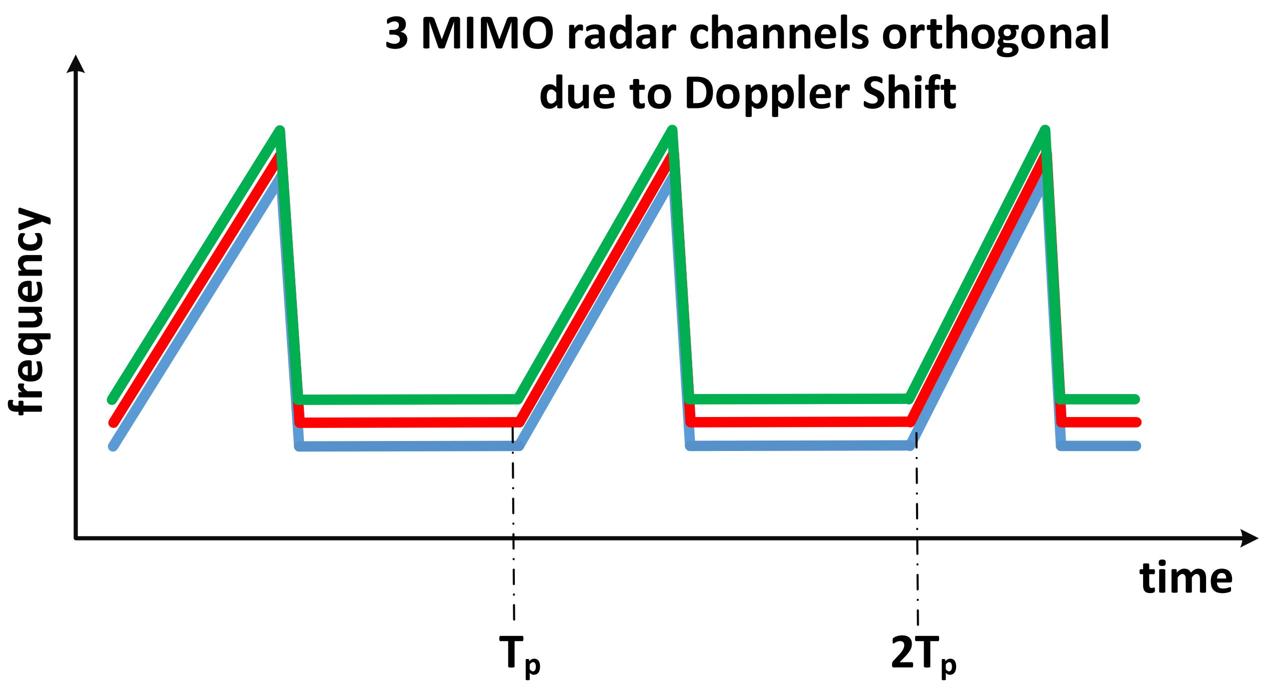


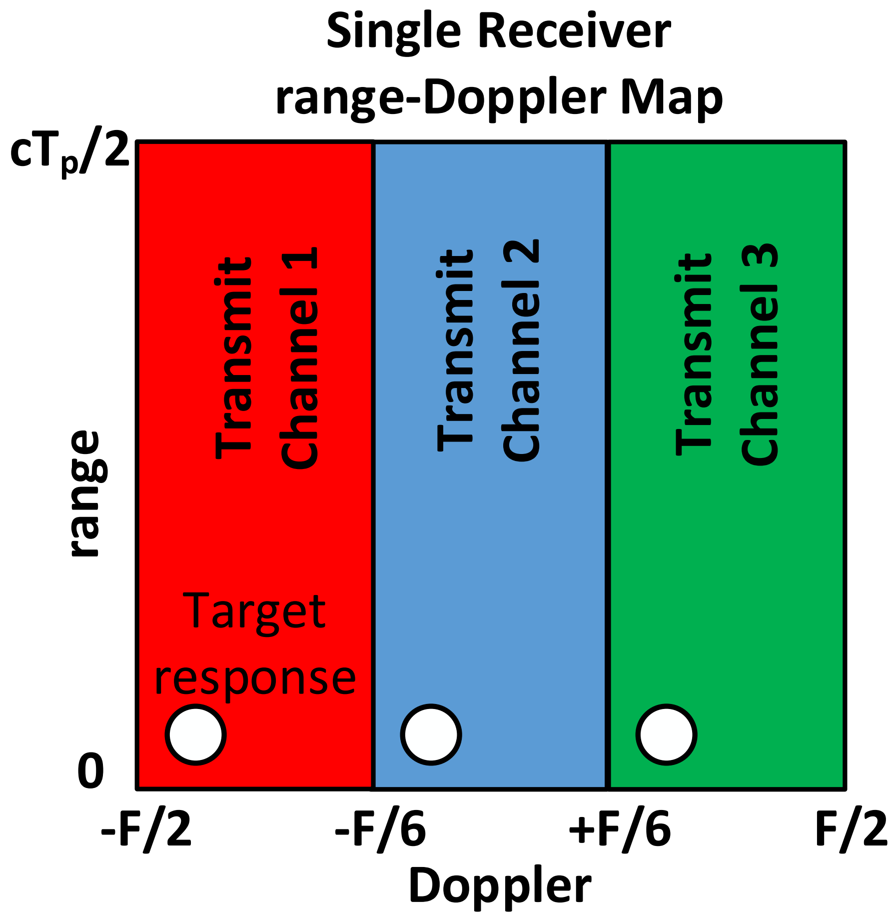
Doppler Division Multiplexing (DDM)



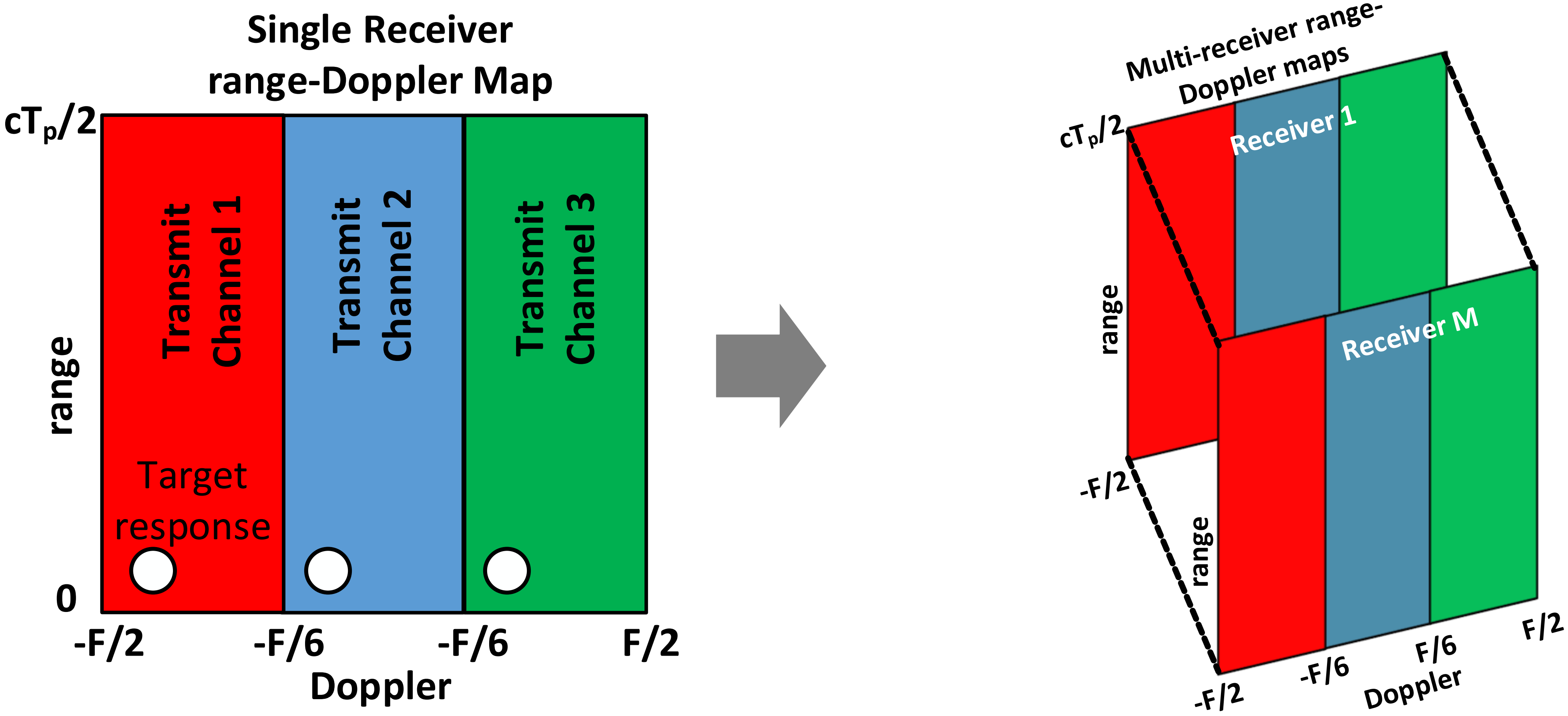


Doppler Division Multiplexing (DDM)

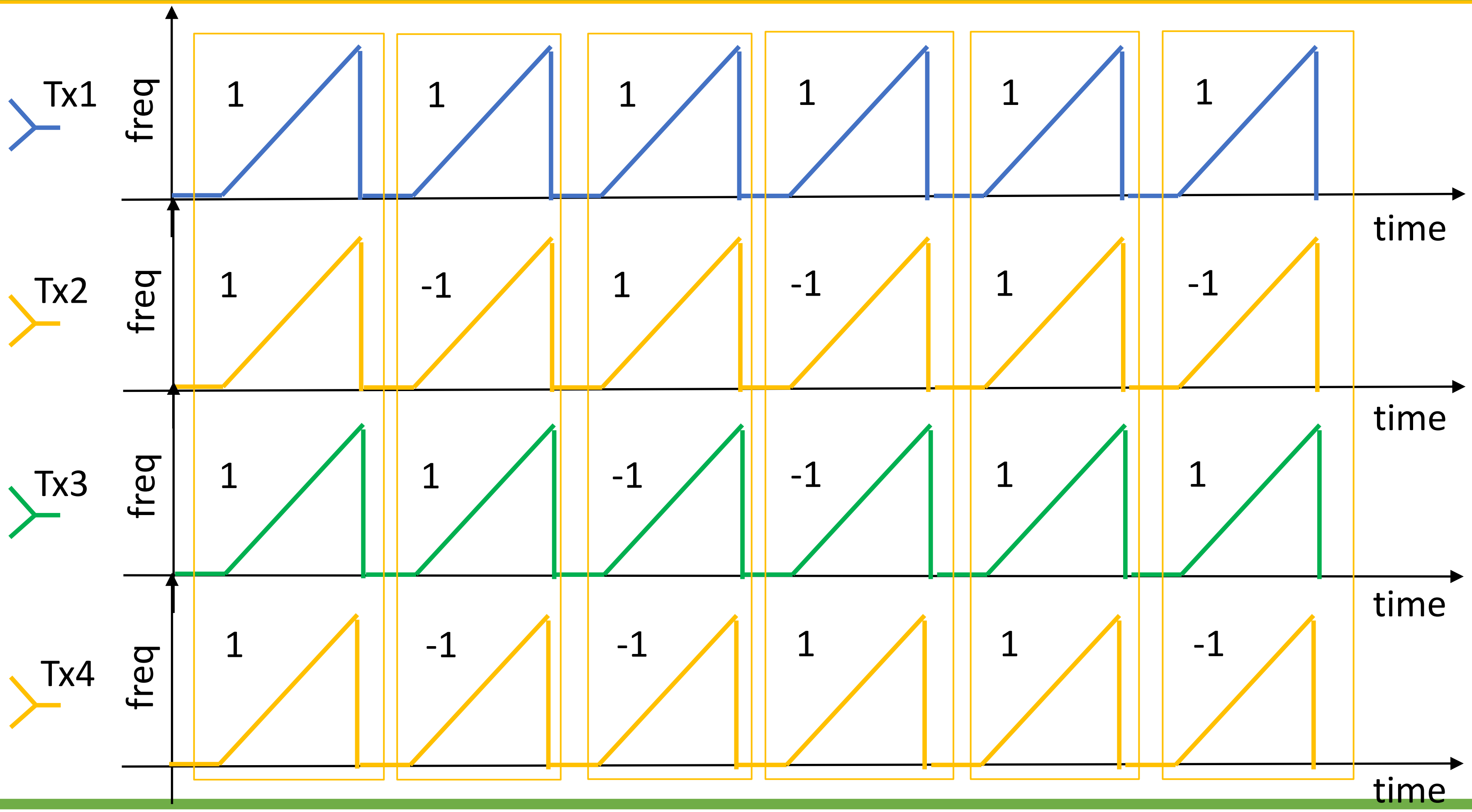




Doppler Division Multiplexing (DDM)



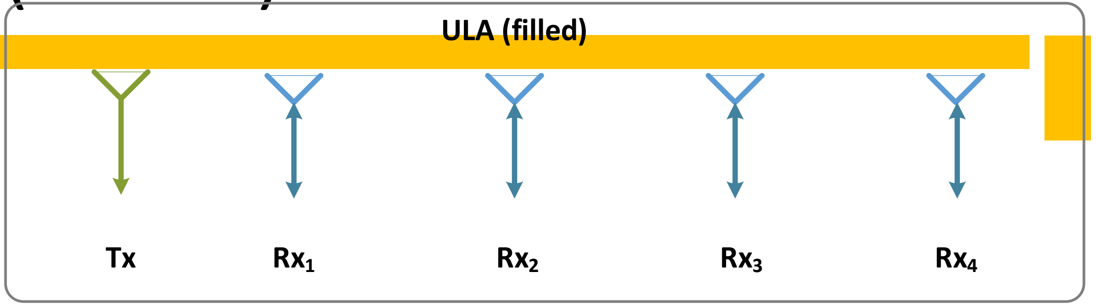
Binary Phase Modulation (BPM)



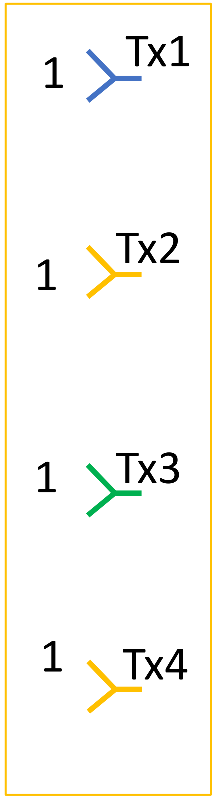
Hadamard Code

1	1	1	1
1	-1	1	-1
1	1	-1	-1
1	-1	-1	1

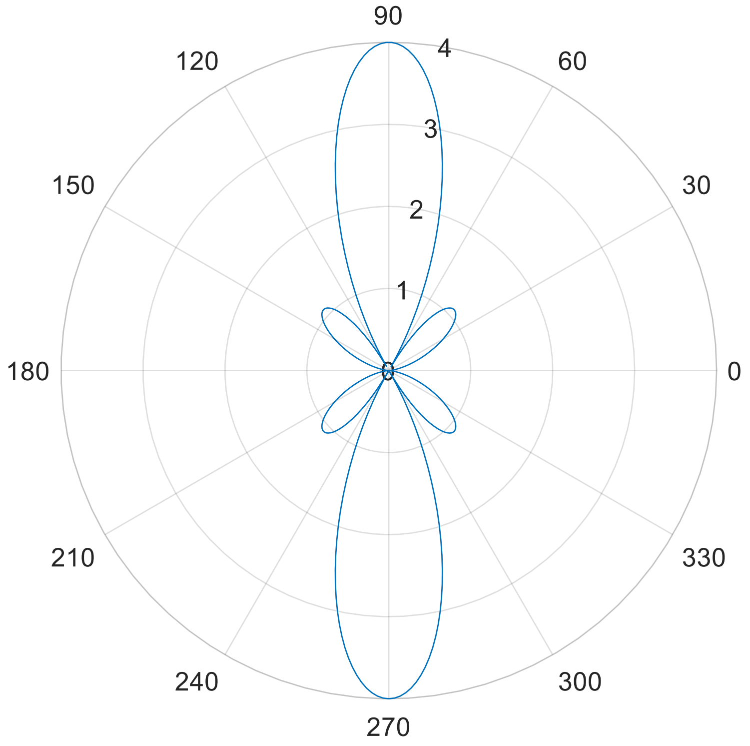
Binary Phase Modulation (BPM)

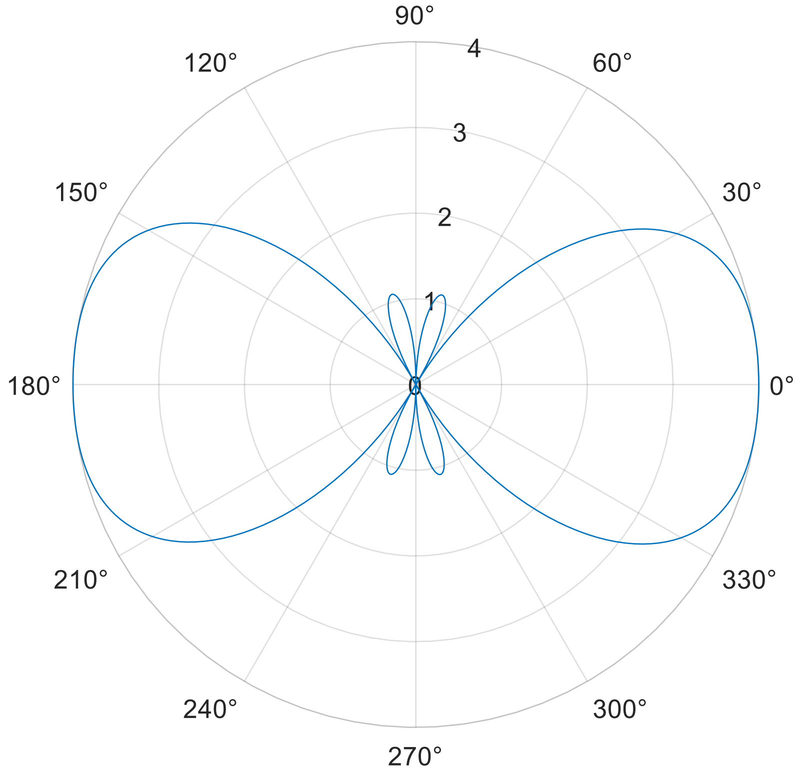


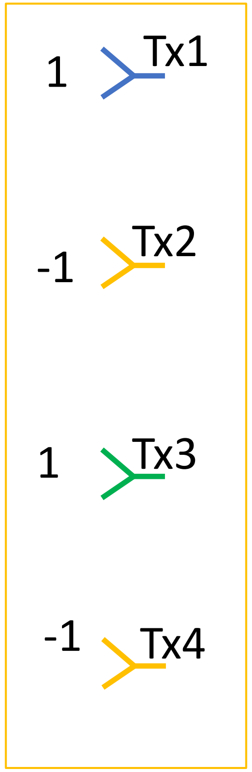
Filled Array



Tx 1 beampattern

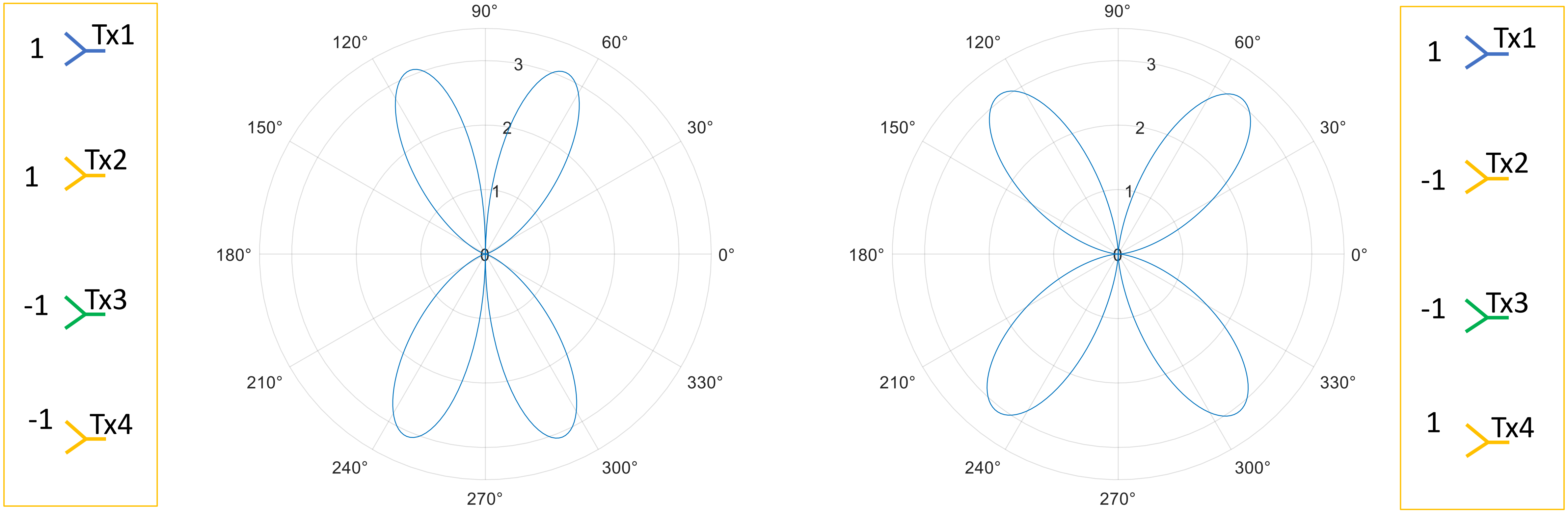




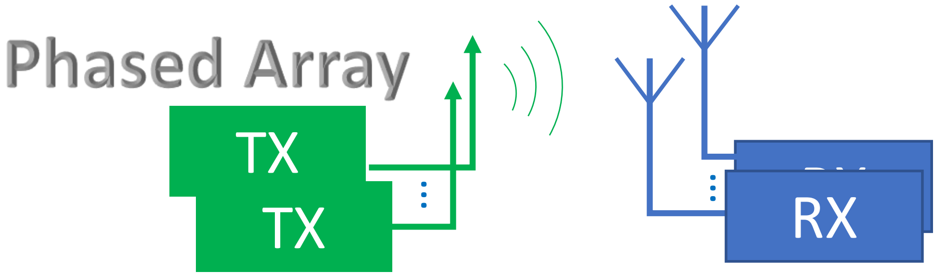


Binary Phase Modulation (BPM)

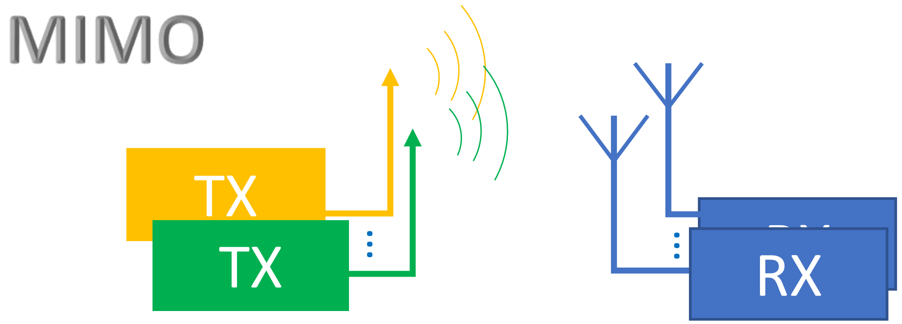
Filled Array



Intra-Pulse Code Division Multiplexing (CDM)



Single **I**nput **M**ulti **O**utput



Multi **I**nput **M**ulti **O**utput

$$x_m = [x_m(1), x_m(2), ..., x_m(N)]^T \in \mathbb{C}^N$$

$$\mathbf{X} = [x_1, \quad x_2, \quad \ldots, x_M] \in \mathbb{C}^{N \times M}$$

$$r_{ml}(k) = \sum_{n=1}^{N-k} x_m(n) x_l^*(n+k) = r_{lm}^*(-k)$$

$$\text{PSL} = \max \left\{ \max_m \max_{k \neq 0} |r_{mm}(k)|, \max_{m,l} \max_k |r_{ml}(k)| \right\}$$

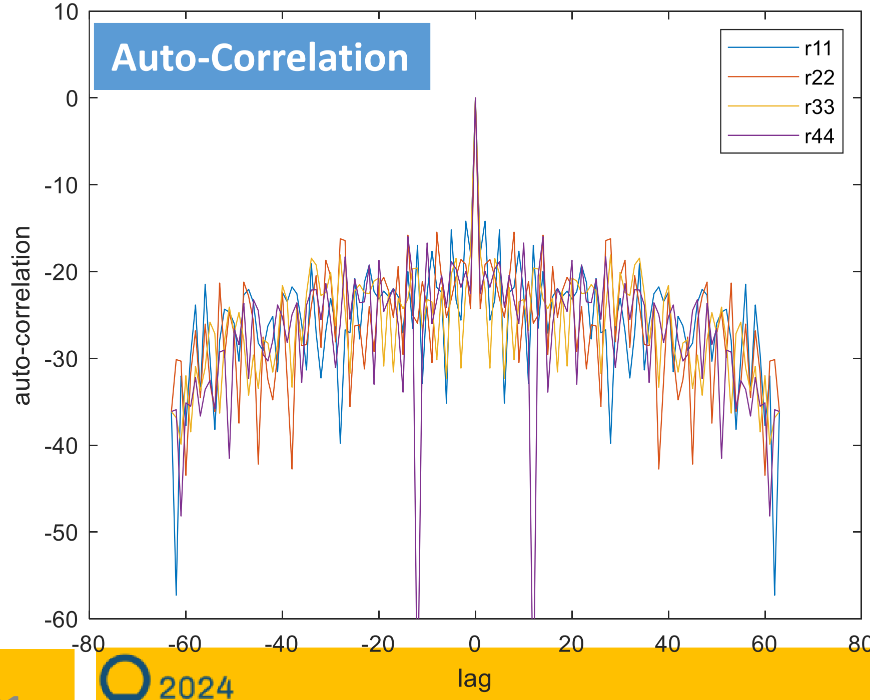
Intra Sequence (solved) | **Between Sequences**

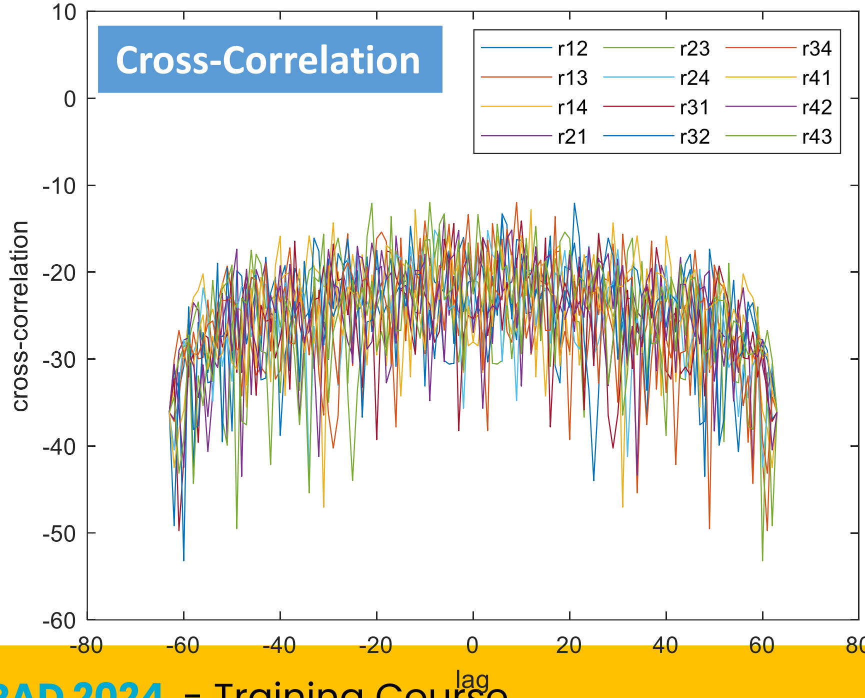
$$\text{ISL} = \sum_{m=1}^{N_T} \sum_{k=-N+1}^{N-1} |r_{mm}(k)|^2 + \sum_{m,l=1}^{N_T} \sum_{k=-N+1}^{N-1} |r_{ml}(k)|^2$$

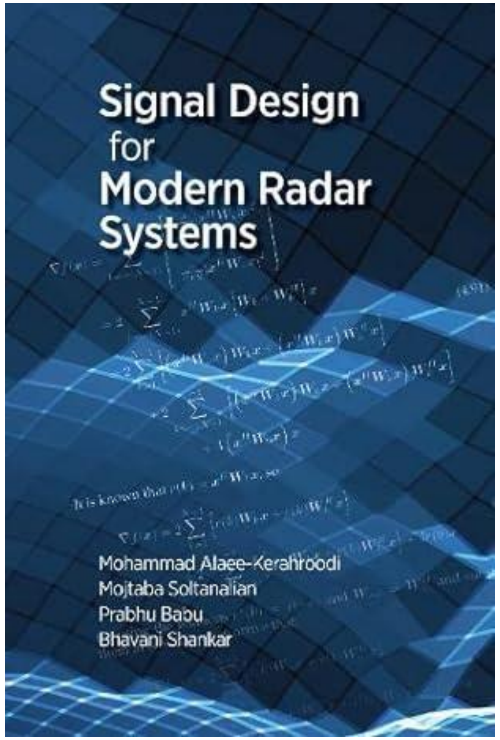
How to design set of sequences with small PSL / ISL ?

What could be waveform for MIMO Radars?

$$\begin{cases}\min\limits_{\mathbf{x}\in\mathbb{C}^{N\times M}} & \mathrm{ISL}=\sum_{m=1}^{N_T}\sum_{k=-N+1}^{N-1}|r_{mm}(k)|^2+\sum_{m,l=1}^{N_T}\sum_{k=-N+1}^{N-1}|r_{ml}(k)|^2\\ \text{subject to} & |x_{n,m}|=1,\forall\begin{cases}n & =1,\dots,N\\ m,m' & =1,\dots,M.\end{cases}\end{cases}$$







Alaee-Kerahroodi, Mohammad, et al. *Signal design for modern radar systems*. Artech House, 2022.

ROME

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https://radarmimo.com/

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Thank you

Questions?