Mathematical Programming-1
CO3-Network Models

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Session: 16 - Network Model

Network Model

Introduction

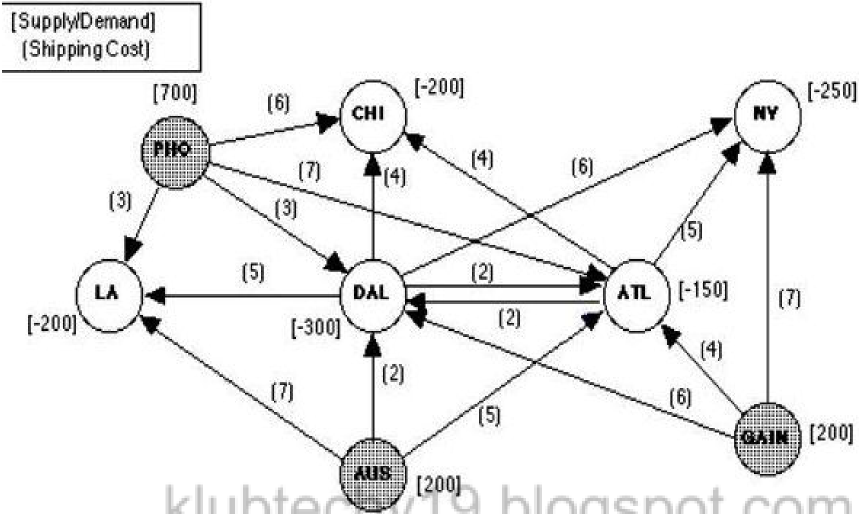
A common scenario of a network-flow problem arising in industrial logistics concerns the distribution of a single homogeneous product from plants (origins) to consumer markets (destinations). The total number of units produced at each plant and the total number of units required at each market are assumed to be known. The product need not be sent directly from source to destination, but may be routed through intermediary points reflecting warehouses or distribution centers. Further, there may be capacity restrictions that limit some of the shipping links. The objective is to minimize the variable cost of producing and shipping the products to meet the consumer demand.

Network Model

Terminology

- The centres from where the goods need to be shipped are called **origins**.
- The places to where the goods need to be shipped from origins are called **destinations**.
- The sources, destinations, and intermediate points are collectively called **nodes** of the network
- The transportation links connecting nodes are termed **arcs**.

Example


[Supply/Demand]
(Shipping Cost)
[700]
PHO
(6)
CHI
[-200]
NY
[-250]
(7)
(4)
(4)
(6)
(3)
(3)
(5)
LA
[-200]
(5)
DAL
[-300]
(2)
(2)
ATL
[-150]
(7)
(7)
(2)
(4)
(5)
(6)
AUS
[200]
GAIN
[200]

Types of Network Models

The class of network flow programs includes such problems as

- Transportation problem
- Assignment problem
- Shortest path problem
- Maximum flow problem
- Pure minimum cost flow problem
- Generalized minimum cost flow problem

What is Transportation Problem

The transportation problem is to transport various amounts of a single homogeneous commodity, that are initially stored at various origins, to different destinations in such a way that the total transportation cost is a minimum.

- The transportation problem is a special type of linear programming problem where the objective is to minimize the cost of distributing a product from a number of sources or origins to a number of destinations.
- Because of its special structure, the usual simplex method is not suitable for solving transportation problems. These problems require a special method of solution

Objective of Transportation Problem

- To find out the **optimum** transportation schedule keeping in mind cost of Transportation to be **minimized**.
- The transportation problem indicates the amount of consignment to be transported from various origins to different destinations so that the total transportation cost is minimized without violating the **availability constraints** and the **requirement constraints**.
- The **unit transportation cost** is the cost of transporting one unit of the consignment from an origin to a destination.

Applications of Transportation Problem

- It is used to compute transportation routes in such a way as to minimize transportation cost for finding out locations of warehouses.
- It is used to find out locations of transportation corporations depots where insignificant total cost difference may not matter.
- Minimize shipping costs from factories to warehouses(or from warehouses to retail outlets).
- Determine lowest cost location for new factory, warehouse, office ,or other outlet facility.
- Find minimum cost production schedule that satisfies firms demand and production limitations.

Mathematical formulation

Let m be no. of origins, n be no. of destinations $m = n$ or $m \neq n$
a_i be the no. of units that i^{th} origin possesses,
b_j be the no. of units that j^{th} destination requires,
c_{ij} be the cost of shipping from i^{th} origin to j^{th} destination,
x_{ij} be the allocation from i^{th} origin to j^{th} destination.
Assumed that $\sum a_i = \sum b_j$

Now the transportation problem is to determine $x_{ij} (\geq 0)$ satisfying

$\sum_{j=1}^{n} x_{ij} = a_i$ (known as Availability constraints)

$\sum_{i=1}^{m} x_{ij} = b_j$ (known as Requirement constraints)

and *minimizing* the total cost $z = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} c_{ij}$ (known as Objective function)

Types of Transportation problem

Balanced Transportation Problem

Total availability=Total Requirement

Unbalanced Transportation Problem

Total availability$\neq$ Total Requirement

Phases of Solution and Methods

Two Phases

1. Obtaining Initial Basic Feasible Solution.(Any one method is applied)
   - North-West Corner Rule
   - Row Minima Method
   - Column Minima Method
   - Lowest Cost Entry Method(Matrix Minima Method)
   - Vogel's Approximation Method (VAM)
2. Obtaining Optimal Basic Solution.
   - Modified Distribution Method or MODI Method or U-V Method

North West Corner Rule

Steps

1. Select the upper left (north-west) cell of the transportation matrix and allocate minimum of supply and demand, i.e min(A1,B1) value in that cell.
2. Obtaining Optimal Basic Solution.
   - If A1<B1, then allocation made is equal to the supply available at the first source(A1 in first row), then move vertically down to the cell (2,1).
   - If A1>B1, then allocation made is equal to demand of the first destination (B1 in first column), then move horizontally to the cell (1,2).
   - If A1=B1, then allocate the value of A1 or B1and then move to cell (2,2).
3. Continue the process until an allocation is made in the south-east corner cell of the transportation table.

Session 17: Transportation Problem - Balanced Case - Initial Basic Feasible Solution.

Problem - Find IBFS using NWCR.

In the table, three sources A,B and C with the Production Capacity of 50units, 40units, 60 units of product respectively is given. Every day the demand of three retailers P,Q,R is to be furnished with at least 20units, 95units and 35units of product respectively. The transportation costs are also given in the matrix.

Source/Destination	P	Q	R	Supply
A	5	8	4	50
B	6	6	3	40
C	3	9	6	60
Demand	20	95	35	150

Step 1

Check whether Total Demand is equal to Total Supply. In case the demand is more then supply, then dummy origin is added to the table. The cost associated with the dummy origin will be zero.

Problem - Find IBFS using NWCR.

Step 2: North-west cell is (1,1). Allocate min{20,50}to North-west cell (1,1). Update the demand and supply.

Source/Destination	P		Q	R	Supply	
A	$\boxed{20}$	5	8	4	~~50~~	30
B	6		6	3	40	
C	3		9	6	60	
Demand	~~20~~		95	35	150	
	0					

Problem - Find IBFS using NWCR.

Step 3: Next north-west cell is (1,2). Allocate min{95,30}to the cell (1,2). Update the demand and supply.

Source/Destination	P	Q	R	Supply		
A	20 5	30 8	4	~~50~~	~~30~~	0
B	6	6	3	40		
C	3	9	6	60		
Demand	~~20~~	~~95~~	35	150		
	0	65				

Problem - Find IBFS using NWCR.

Step 4: Next north-west cell is (2,2). Allocate min{65,40}to the cell (2,2). Update the demand and supply.

Source/Destination	P	Q	R	Supply
A	20 5	30 8	4	~~50~~ ~~30~~ 0
B	6	40 6	3	~~40~~ 0
C	3	9	6	60
Demand	~~20~~	~~95~~	35	150
	0	~~65~~ 25		

Step 5: Problem - Find IBFS using NWCR.

Next north-west cell is (3,2). Allocate min{25,60}to the cell (3,2). Update the demand and supply.

Source/Destination	P	Q	R	Supply	
A	20 5	30 8	4	~~50~~	~~30~~ 0
B	6	40 6	3	~~40~~	0
C	3	25 9	6	~~60~~	35
Demand	~~20~~	~~95~~	35	150	
	0	~~65~~ ~~25~~ 0			

Step :6 Problem - Find IBFS using NWCR.

Next north-west cell is (3,3). Allocate min{35,35}to the cell (3,3). Update the demand and supply.

Source/Destination	P	Q	R	Supply	
A	20 5	30 8	4	~~50~~	~~30~~ 0
B	6	40 6	3	~~40~~	0
C	3	25 9	35 6	~~60~~	~~35~~ 0
Demand	~~20~~	~~95~~	~~35~~	150	
	0	~~65~~ ~~25~~ 0	0		

Problem - Find IBFS by NWCR.

Step 6

Source/Destination	P	Q	R	Supply
A	20 5	30 8	4	50
B	6	40 6	3	40
C	3	35 9	35 6	60
Demand	20	95	35	150

Total Cost can be computed by multiplying the units assigned to each cell with the concerned transportation cost.
Total
Cost= $(20 * 5) + (30 * 8) + (40 * 6) + (25 * 9) + (35 * 6) = Rs.1015.$

Lowest Cost Entry Method

Steps

1. Start with the lowest cost entry and allocate as much as possible.
2. Repeat the same with next lowest cost. Continue the process till the requirement/availability exhausts.

Problem - Find IBFS by LCEM.

Step 1

The minimum cost in the matrix is Rs 3, but there is a tie in the cell BR, and CP, now the question arises in which cell we shall allocate. Generally, the cost where maximum quantity can be assigned should be chosen to obtain the better initial solution. Therefore, 35 units shall be assigned to the cell BR.

Source/Destination	P	Q	R	Supply
A	5	8	4	50
B	6	6	$\boxed{35}$ 3	$\cancel{40}$ 5
C	3	9	6	60
Demand	20	95	$\cancel{35}$ 0	150

Problem - Find IBFS by LCEM.

Step 2

Again the minimum cost in the matrix is Rs 3. Therefore, 20 units shall be assigned to the cell CD. With this, the demand of retailer P gets fulfilled. Only 40 units are left with the source C.

Source/Destination	D	E	F	Supply
A	5	8	4	50
B	6	6	35 3	~~40~~ 5
C	20 3	9	6	~~60~~ 40
Demand	~~20~~ 0	95	~~35~~ 0	150

Problem - Find IBFS by LCEM.

Step 3

The next minimum cost is 8, assign 50 units to the cell AQ. The supply of source A gets saturated. The next minimum cost is Rs 9; we shall assign 40 units to the cell CE. With his both the demand and supply of all the sources and origins gets saturated.

Source/Destination	D	E	F	Supply
A	5	50 8	4	~~50~~ 0
B	6	5 6	3 35	~~40~~ ~~5~~ 0
C	20 3	40 9	6	~~60~~ ~~40~~ 0
Demand	~~20~~ 0	~~95~~ ~~90~~ ~~40~~ 0	~~35~~ 0	150

Total Cost $= 50 * 8 + 5 * 6 + 35 * 3 + 20 * 3 + 40 * 9 = Rs.955.$
Any Observation?

Session 18: Transportation Problems- Un Balanced Case- Optimal solution by UV Method

Find IBFS by Matrix Minima Method.

The total demand is 1000, whereas the total supply is 800.
Total supply $<$ total demand.

Plant	Warehouse			Supply
	W1	W2	W3	
A	28	17	26	500
B	19	12	16	300
Demand	250	250	500	

Here dummy origin is added to the table. The cost associated with the dummy origin will be zero.

Unbalanced Transport Problem - Find IBFS by Matrix Minima Method.

Plant	Warehouse			Supply
	W1	W2	W3	
A	28	17	26	500
B	19	12	16	300
Unsatisfied demand	0	0	0	200
Demand	250	250	500	1000

Now proceed with Lowest cost entry method.

Unbalanced Transport Problem - Find IBFS by Matrix Minima Method.

Plant	Warehouse			Supply
	W1	W2	W3	
A	28 (50)	17	26 450	~~500~~
B	19	12 250	16 50	~~300~~
Unsatisfied demand	0 200	0	0	~~200~~
Demand	~~250~~	~~250~~	~~500~~	1000

Total Cost$= 28(50) + 26(450) + 12(150) + 16(50) + 0(200) = 15700$

Vogel's Approximation Method (VAM Method)

The Vogel Approximation Method is an improved version of the Minimum Cell Cost Method and the Northwest Corner Method that in general produces better initial basic feasible solution, that report a smaller value in the objective (minimization) function of a balanced Transportation Problem. It consists the following steps

- **Step1:** Determine a penalty cost for each row (column) by subtracting the lowest unit cell cost in the row (column) from the next lowest unit cell cost in the same row (column).
- **Step2:** Identify the row or column with the greatest penalty cost. Break the ties arbitrarily (if there are any). Allocate as much as possible to the variable with the lowest unit cost in the selected row or column. Adjust the supply and demand and cross out the row or column that is already satisfied. If a row and column are satisfied simultaneously, only cross out one of the two and allocate a supply or demand of zero to the one that remains.

Vogel's Approximation Method (VAM Method)

steps Cont..

- **Step3:**
  - If there is exactly one row or column left with a supply or demand of zero, stop.
  - If there is one row (column) left with a positive supply (demand), determine the basic variables in the row (column) using the Minimum Cell Cost Method. Stop.
  - If all of the rows and columns that were not crossed out have zero supply and demand (remaining), determine the basic zero variables using the Minimum Cell Cost Method. Stop.
  - In any other case, continue with Step 1.

Problem - Find IBFS using VAM

From \ To	D	E	F	Supply	Iteration-I
A	6	4	1	50	3
B	3 20	8	7	40	4
C	4	4	2	60	2
Demand	20	95	35	150	
Iteration-I	1	0	1		

Step 1: First of all the difference between two least cost cells are calculated for each row and column, which can be seen in the iteration given for each row and column. Then the largest difference is selected, which is 4 in this case. So, allocate 20 units to cell BD, since the minimum cost is to be chosen for the allocation. Now, only 20 units are left with the source B.

Problem - Find IBFS using VAM

From \ To	E	F	Supply	Iteration-II
A	4 15	1 35	50	3
B	8	7	20	1
C	4	2	60	2
Demand	95	35	150	
Iteration-II	0	1		

Step 2: Column D is deleted, again the difference between the least cost cells is calculated for each row and column, as seen in the iteration below. The largest difference value comes to be 3, so allocate 35 units to cell AF and 15 units to the cell AE. With this, the Supply and demand of source A and origin F gets saturated, so delete both the row A and Column F.

Problem - Find IBFS using VAM

From \ To	E	Supply
B	8 (20)	20
C	4 (60)	60
Demand	80	150

Step 3: Now, single column E is left, since no difference can be found out, so allocate 60 units to the cell CE and 20 units to cell BE, as only 20 units are left with source B. Hence the demand and supply are completely met.

Problem - Find IBFS using VAM

- Now the total cost can be computed, by multiplying the units assigned to each cell with the cost concerned.
- Therefore, Total Cost = 20*3 + 35*1 + 15*4 + 60*4 + 20*8 = Rs. 555.
- Note: Vogels Approximation Method is also called as Penalty Method because the difference costs chosen are nothing but the penalties of not choosing the least cost routes.

Problem 2- VAM

Obtain an initial basic feasible solution to the transport problem, using VAM

Factory/Warehouse	W_1	W_2	W_3	W_4	Factory Capacity
F_1	19	30	50	10	7
F_2	70	30	40	60	9
F_3	40	8	70	20	18
Warehouse Requirement	5	8	7	14	34

Problem2-VAM - Solution

Step 1

	W_1	W_2	W_3	W_4	Available	Row difference
F_1	19	30	50	10	7	9
F_2	70	30	40	60	9	10
F_3	40	$\boxed{8}$ 8	70	20	$\cancel{18}$ 10	12
Required	5	$\cancel{8}$ 0	7	14	34	
Column Difference	21	22	10	10		

Problem2-VAM - Solution

Step 2

	W_1	W_2	W_3	W_4	Available	Row Difference	
F_1	$\boxed{5}$ 19	30	50	10	$\cancel{7}$ 2	9	9
F_2	70	30	40	60	9	10	20
F_3	40	$\boxed{8}$ 8	70	20	$\cancel{18}$ 10	12	20
Required	$\cancel{5}$ 0	$\cancel{8}$ 0	7	14	34		
Column Difference	21 21	22 -	10 10	10 10			

Step 3

	W_1	W_2	W_3	W_4	Available	Row Diff.		
F_1	$\boxed{5}$ 19	30	50	10	$\cancel{7}$ 2	9	9	40
F_2	70	30	40	60	9	10	20	20
F_3	40	$\boxed{8}$ 8	70	$\boxed{10}$ 20	$\cancel{18}$ $\cancel{10}$ 0	12	20	50
Required	$\cancel{5}$ 0	$\cancel{8}$ 0	7	$\cancel{14}$ 4				
Column Difference	21	22	10	10				
	21	-	10	10				
	-	-	10	10				

Step 4

	W_1	W_2	W_3	W_4	Available	Row Diff.			
F_1	$\boxed{5}$ 19	30	50	$\boxed{2}$ 10	~~7~~ ~~2~~ 0	9	9	40	40
F_2	70	30	40	60	9	10	20	20	20
F_3	40	$\boxed{8}$ 8	70	$\boxed{10}$ 20	~~18~~ ~~10~~ 0	12	20	50	-
Req.	~~5~~ 0 0	~~8~~ 0 0	7	~~14~~ ~~4~~ 2					
Column Diff.	21	22	10	10					
	21	-	10	10					
	-	-	10	10					
	-	-	10	50					

Step 5

W_1	W_2	W_3	W_4	Available	Row Diff.				
$\boxed{5}$ 19	30	50	$\boxed{2}$ 10	~~7~~ ~~2~~ 0	9	9	40	40	-
70	30	$\boxed{7}$ 40	60	~~9~~ 2	10	20	20	20	20
40	$\boxed{8}$ 8	70	$\boxed{10}$ 20	~~18~~ ~~10~~ 0	12	20	50	-	
~~5~~ 0	~~8~~ 0	~~7~~ 0	~~14~~ ~~4~~ 2						
21	22	10	10						
21	-	10	10						
-	-	10	10						
-	-	10	50						

Problem2 VAM - Solution

Step 6

W_1	W_2	W_3	W_4	Available	Row Diff.				
5 19	30	50	2 10	~~7~~ ~~2~~ 0	9	9	40	40	-
70	30	7 40	2 60	~~9~~ ~~2~~ 0	10	20	20	20	20
40	8 8	70	10 20	~~18~~ ~~10~~ 0	12	20	50	-	
~~5~~ 0	~~8~~ 0	~~7~~ 0	~~14~~ ~~4~~ ~~2~~ 0						
21	22	10	10						
21	-	10	10						
-	-	10	10						
-	-	10	50						

The total cost$=5(19)+8(8)+2(10)+2(60)+10(20)+7(40)=$Rs.779.

Algorithm of MODI Method or U-V Method

- **Step1:** Determine an initial basic feasible solution using any one of the three methods given below:
  - North West Corner Method
  - Matrix Minimum Method(Lowest Cost Entry Method)
  - Vogel's Approximation Method
- **Step2:** Start with the basic feasible solution consisting of $m+n-1$ allocations in independent positions.
- **Step3:** Determine the $(m+n)$ values of u_i and v_j, for allocated cells, using $u_i+v_j=c_{ij}$. Starting initially with some $u_i=0$.
- **Step4:** Compute the cell evaluations for unallocated cells(empty cells), using $c_{ij}-(u_i+v_j)$.
- **Step5:** If the cell evaluations of all the unallocated cells are either positive or zero, the given solution is the optimal solution. On the other hand, if one or more unallocated cell has negative value, the given solution is not an optimal solution.

Algorithm of MODI Method or U-V Method

- **Step6:** Select the unoccupied cell with the most negative cell evaluation, as the cell to be entered in the next solution. Let the variable x_{rs} enter the basis.
- **Step7:** Allocate an unknown quantity θ, to the cell (r, s). Then construct a loop that starts and ends at the cell (r, s) and connects some of the basic cells. The amount θ is added to and subtracted from the transition cells of the loop in such a manner that the availabilities and requirements remains satisfied.
- **Step8:** Assign the largest possible value to θ in such a way that the value of at least one basic variable becomes zero and other basic variables remain non-negative. The basic cell whose allocation has been made zero will leave the basis.
- **Step9:** Now return to step 3 and repeat the process until an optimum basic feasible solution is obtained.

Session 19: Transportation Problems- Balanced Case- Obtaining Optimal solution by using U-V Method

Problem - MODI Method

Obtain an initial basic feasible solution to the transport problem. Is this solution an optimal solution? If not, obtain the optimal solution

Factory/Warehouse	W_1	W_2	W_3	W_4	Factory Capacity
F_1	19	30	50	10	7
F_2	70	30	40	60	9
F_3	40	8	70	20	18
Warehouse Requirement	5	8	7	14	34

Problem MODI Method - Solution

Step1: On applying Vogel's Approximation Method (see Problem2-VAM-Solution)

Factory/Warehouse	W_1	W_2	W_3	W_4	Available
F_1	5 19	30	50	2 10	7
F_2	70	30	7 40	2 60	9
F_3	40	8 8	70	10 20	18
Required	5	8	7	14	

The total cost=5(19)+8(8)+2(10)+2(60)+10(20)+7(40)=Rs.779.

Step2: This initial basic feasible solution has m+n-1=3+4-1=6 allocation in independent positions.

Step 3: Calculating u_i, v_j for allocated cells

					u_i
	$\boxed{5}$ 19	30	50	$\boxed{2}$ 10	
	70	30	$\boxed{7}$ 40	$\boxed{2}$ 60	
	40	$\boxed{8}$ 8	70	$\boxed{10}$ 20	
v_j					

Suppose $u_1 = 0$ then

cell(1,1): $u_1 + v_1 = 19 \Rightarrow v_1 = 19$

cell(1,4): $u_1 + v_4 = 10 \Rightarrow v_4 = 10$

cell(2,4): $u_2 + v_4 = 60 \Rightarrow u_2 = 50$

cell(2,3): $u_2 + v_3 = 40 \Rightarrow v_3 = -10$

cell(3,4): $u_3 + v_4 = 20 \Rightarrow u_3 = 10$

cell(3,2): $u_3 + v_2 = 8 \Rightarrow v_2 = -2$

Step 3: Calculating u_i, v_j for allocated cells

					u_i
	5 19	30	50	2 10	0
	70	30	7 40	2 60	50
	40	8 8	70	10 20	10
v_j	19	-2	-10	10	

Suppose $u_1 = 0$ then

cell(1,1): $u_1 + v_1 = 19 \Rightarrow v_1 = 19$

cell(1,4): $u_1 + v_4 = 10 \Rightarrow v_4 = 10$

cell(2,4): $u_2 + v_4 = 60 \Rightarrow u_2 = 50$

cell(2,3): $u_2 + v_3 = 40 \Rightarrow v_3 = -10$

cell(3,4): $u_3 + v_4 = 20 \Rightarrow u_3 = 10$

cell(3,2): $u_3 + v_2 = 8 \Rightarrow v_2 = -2$

Problem MODI Method - Solution

Step 4: Calculating cell evaluations for unallocated cells

					u_i
	$\boxed{5}$ 19	30	50	$\boxed{2}$ 10	0
	70	30	$\boxed{7}$ 40	$\boxed{2}$ 60	50
	40	$\boxed{8}$ 8	70	$\boxed{10}$ 20	10
v_j	19	-2	-10	10	

cell(1,2): $30 - (u_1 + v_2) = 30 - (0 - 2) = 32$
cell(1,3): $50 - (u_1 + v_3) = 50 - (0 - 10) = 60$
cell(2,1): $70 - (u_2 + v_1) = 70 - (50 + 19) = 1$
cell(2,2): $30 - (u_2 + v_2) = 30 - (50 - 20) = -18$
cell(3,1): $40 - (u_3 + v_1) = 40 - (10 + 19) = 11$
cell(3,3): $70 - (u_3 + v_3) = 70 - (10 - 10) = 70$

Step 4: Calculating cell evaluations for unallocated cells

5 19	(32) 30	(60) 50	2 10
(1) 70	(-18) 30	7 40	2 60
(11) 40	8 8	(70) 70	10 20

Step 5: Check for optimum solution, if not, identify the entering cell

Since cell(2,2) has largest negative cell evaluation -18, it must be entered into solution. So form a loop from this cell and allocate θ to the cell, followed by alternatively subtracting and adding the amount of this allocation to other corners of the loop in order to restore feasibility.

Problem MODI Method - Solution

Step 6: Allocating θ for entering cell

$\boxed{5}$ 19	(32) 30	(60) 50	$\boxed{2}$ 10
(1) 70	θ 30	$\boxed{7}$ 40	$(2-\theta)$ 60
(11) 40	$(8-\theta)$ 8	(70) 70	$(10+\theta)$ 20

$$\text{Min}\{8-\theta, 2-\theta\} = 0 \Rightarrow \theta = 2$$

Problem MODI Method - Solution

Step 7: Identifying the leaving cell from allocation

5 19	(32) 30	(60) 50	2 10
(1) 70	2 30	7 40	0 60
(11) 40	6 8	(70) 70	12 20

The cell with 0 value i.e., cell (2,4)is removed from allocaion.

Step 8: Update the allocations and Repeat steps 3-5 : Calculating u_i, v_j

					u_i
	5 19	30	50	2 10	
	70	2 30	7 40	60	
	40	6 8	70	12 20	
v_j					

Suppose $u_1 = 0$ then

cell(1,1): $u_1 + v_1 = 19 \Rightarrow v_1 = 19$

cell(1,4): $u_1 + v_4 = 10 \Rightarrow v_4 = 10$

cell(3,4): $u_3 + v_4 = 20 \Rightarrow u_3 = 10$

cell(3,2): $u_3 + v_2 = 8 \Rightarrow v_2 = -2$

cell(2,2): $u_2 + v_2 = 30 \Rightarrow u_2 = 32$

cell(2,3): $u_2 + v_3 = 40 \Rightarrow v_3 = 8$

Problem MODI Method - Solution

Calculating cell evaluations for unallocated cells

					u_i
	$\boxed{5}$ 19	30	50	$\boxed{2}$ 10	0
	70	$\boxed{2}$ 30	$\boxed{7}$ 40	60	32
	40	$\boxed{6}$ 8	70	$\boxed{12}$ 20	10
v_j	19	-2	8	10	

cell(1,2): $30 - (u_1 + v_2) = 30 - (0 - 2) = 32$
cell(1,3): $50 - (u_1 + v_3) = 50 - (0 - 8) = 42$
cell(2,1): $70 - (u_2 + v_1) = 70 - (32 + 19) = 19$
cell(2,4): $60 - (u_2 + v_4) = 60 - (32 + 10) = 18$
cell(3,1): $40 - (u_3 + v_1) = 40 - (10 + 19) = 11$
cell(3,3): $70 - (u_3 + v_3) = 70 - (10 - 8) = 52$

Problem MODI Method - Solution

Check for optimum solution

$\boxed{5}$ 19	(32) 30	(42) 50	$\boxed{2}$ 10
(19) 70	$\boxed{2}$ 30	$\boxed{7}$ 40	(18) 60
(11) 40	$\boxed{6}$ 8	(52) 70	$\boxed{12}$ 20

Since all cell evaluations are non-negative, the solution of above table is optimal with minimum cost.

The minimum cost$= 5.19 + 2.10 + 2.30 + 7.40 + 6.8 + 12.20 = 743$
It is less than the total cost obtained by Vogel's approximation method.

Session: 20 - Assignment Problems - Balanced and Unbalanced Cases

Assignment Problem

Definition

Suppose there are n jobs to be performed and n persons are available for doing these jobs. Assume that each person can do each job at a time, though with varying degree of efficiency. Let C_{ij} be the cost (payment) if the i^{th} person is assigned the j^{th} job, the problem is to find an assignment (which job should be assigned to which person)so that the total cost for performing all jobs is minimum.

Mathematical Formulation of Assignment Problem

Minimize the total cost $z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$

subject to the restrictions of the form

$$x_{ij} = \begin{cases} 1 & \text{if job } j \text{ assigned to person } i \\ 0 & \text{if not} \end{cases}$$

$$\sum_{j=1}^{n} x_{ij} = 1 \text{ and } \sum_{i=1}^{n} x_{ij} = 1$$

Hungarian Algorithm

Steps

1. Subtract the minimum of each row from all the elements of the respective rows.
2. Further,subtract the minimum of each column from all the elements of the respective columns.
3. If a row contains more than one zero then move to next row. If a row contains single 0 then assignment is marked to that cell(enrectangle the zero). And cross all other zeros in the column in which the assignment is made. Repeat this for all the rows.
4. If a column is already assigned then move to the next column. If a column contains single 0 then assignment is marked to that cell. And cross all other zeros in the row in which the assignment is made. Repeat this for all the columns.

Steps

5. Draw the minimum number of horizontal and vertical lines to cover all the zeros in the resulting matrix. Let the minimum number of line be N. Now there two possibilities.
   - If $N = n$, then an optimal assignment can be made. We are finished.
   - If $N < n$, then repeat the steps $1 - 5$ until $N = n$.

Rule to draw minimum number of lines

1. mark the rows in which the assignment is not done.
2. In the marked row, mark the column which has zero.
3. In the marked column, mark the row which has assignment.
4. Now draw the lines through unmarked rows and marked columns.
5. If there are still uncovered zeros, draw horizontal/vertical lines to cover the zeros.

Problem1

A department head has four subordinates, and four tasks have to be performed. Subordinates differ in efficiency and tasks differ in their intrinsic difficulty. Time each man would take to perform each task is given in the effectiveness matrix. How the tasks should be allocated to each person so as to minimize the total man hours?

		Subordinates			
		I	**II**	**III**	**IV**
Tasks	A	8	26	17	11
	B	13	28	4	26
	C	38	19	18	15
	D	19	26	24	10

Problem1 - Solution

Step 1: Choose the least number in row and subtract from all the elements of the row.

0	18	9	3
9	24	0	22
23	4	3	0
9	16	14	0

Step 2: Choose the least number in column and subtract from all the elements of the column.

0	14	9	3
9	20	0	22
23	0	3	0
9	12	14	0

Problem1 - Solution

Step 3: Assignment of cells in rows

$\boxed{0}$	14	9	3
9	20	$\boxed{0}$	22
23	0	3	0
9	12	14	$\boxed{0}$

Step 4: Assignment of cells in columns.

$\boxed{0}$	14	9	3
9	20	$\boxed{0}$	22
23	$\boxed{0}$	3	$\cancel{0}$
9	12	14	$\boxed{0}$

Problem1 - Solution

Step 5: Assignment is optimum

Observe that all the zeros are either assigned or crossed-out. It is found that no additional assignments are possible. The optimal assignment is given by
$A \to I, B \to III, C \to II, D \to IV$
The man hours= 8+4+19+10=41 hours

Problem2

A car hire company has one car at each of five depots a, b, c, d and e. A customer requires a car in each town, namely A, B, C, D and E. Distance (in Km) between depots (origins) and towns (destinations) are given in the following distance matrix:

	a	b	c	d	e
A	160	130	175	190	200
B	135	120	130	160	175
C	140	110	155	170	185
D	50	50	80	80	110
E	55	35	70	80	105

Problem2 - Solution

Step 1: Minimum value of the row is subtracted from all the elements of the row.(repeat for all rows)

30	0	45	60	70
15	0	10	40	55
30	0	45	60	75
0	0	30	30	60
20	0	35	45	70

Problem2 - Solution

Step 2: Minimum value of the column is subtracted from all the elements of the column.(repeat for all columns)

30	0	35	30	15
15	0	0	10	0
30	0	35	30	20
0	0	20	0	5
20	0	25	15	15

Problem2 - Solution

Step 3: Assignment in rows

30	$\boxed{0}$	35	30	15
15	$\cancel{0}$	0	10	0
30	$\cancel{0}$	35	30	20
0	$\cancel{0}$	20	0	5
20	$\cancel{0}$	25	15	15

Problem2 - Solution

Step 4: Assignment in columns

30	$\boxed{0}$	35	30	15
15	$\cancel{0}$	$\boxed{0}$	10	$\cancel{0}$
30	$\cancel{0}$	35	30	20
$\boxed{0}$	$\cancel{0}$	20	$\cancel{0}$	5
20	$\cancel{0}$	25	15	15

Problem2 - Solution

Step 5: (i)Mark unassigned rows, (ii)then mark the column which has crossed zero in the marked rows. (iii)Mark assigned rows in the marked columns.

(i)

30	$\boxed{0}$	35	30	15	
15	$\cancel{0}$	$\boxed{0}$	10	$\cancel{0}$	
30	$\cancel{0}$	35	30	20	✓
$\boxed{0}$	$\cancel{0}$	20	$\cancel{0}$	5	
20	$\cancel{0}$	25	15	15	✓

(ii)

30	$\boxed{0}$	35	30	15	
15	$\cancel{0}$	$\boxed{0}$	10	$\cancel{0}$	
30	$\cancel{0}$	35	30	20	
$\boxed{0}$	$\cancel{0}$	20	$\cancel{0}$	5	
20	$\cancel{0}$	25	15	15	✓
	✓				

(iii)

30	$\boxed{0}$	35	30	15	✓
15	$\cancel{0}$	$\boxed{0}$	10	$\cancel{0}$	
30	$\cancel{0}$	35	30	20	✓
$\boxed{0}$	$\cancel{0}$	20	$\cancel{0}$	5	
20	$\cancel{0}$	25	15	15	✓
	✓				

Problem2 - Solution

Step 6: Draw line through unmarked rows and marked columns.

30	$\boxed{0}$	35	30	15	✓
15	$\cancel{0}$	$\boxed{0}$	10	$\cancel{0}$	
30	$\cancel{0}$	35	30	20	✓
$\boxed{0}$	$\cancel{0}$	20	$\cancel{0}$	5	
20	$\cancel{0}$	25	15	15	✓
	✓				

Step 7: Identify the minimum from uncovered elements(15). Subtract from uncovered elements. Add to the intersection points.

15	0	20	15	0
15	15	0	10	0
15	0	20	15	5
0	15	20	0	5
5	0	10	0	0

Problem2 - Solution

Step 8: (i) Assignment in rows (ii) Assignment in columns

(i)

15	$\cancel{0}$	20	15	0
15	15	0	10	0
15	$\boxed{0}$	20	15	5
0	15	20	0	5
5	$\cancel{0}$	10	0	0

(ii)

15	$\cancel{0}$	20	15	$\boxed{0}$
15	15	$\boxed{0}$	10	$\cancel{0}$
15	$\boxed{0}$	20	15	5
$\boxed{0}$	15	20	$\cancel{0}$	5
5	$\cancel{0}$	10	$\boxed{0}$	$\cancel{0}$

Step 9: Assignment is optimum

Observe that all the zeros are either assigned or crossed-out. Every row(column) has an assignment. The optimal assignment is given by

$A \rightarrow e, B \rightarrow c, C \rightarrow b, D \rightarrow a, E \rightarrow d$

The man hours= 200+130+110+50+80=570 Kms.

Session: 21 - Assignment Problems - Balanced and Unbalanced Cases

Problem3

Solve the assignment problem represented by the following matrix.

	a	b	c	d	e	f
A	9	22	58	11	19	27
B	43	78	72	50	63	48
C	41	28	91	37	45	33
D	74	42	27	49	39	32
E	36	11	57	22	25	18
F	3	56	53	31	17	28

Problem3 - Solution

Step 1: (i)Identify minimum of the row and subtract from each element of the row.(repeat for all rows) (ii)Identify minimum of the column and subtract from each element of the column(repeat for all columns)

(i)

0	13	49	2	10	18
0	35	29	7	20	5
13	0	63	9	17	5
47	15	0	22	12	5
25	0	46	11	14	7
0	53	50	28	14	25

(ii)

0	13	49	0	0	13
0	35	29	5	10	0
13	0	63	7	7	0
47	15	0	20	2	0
25	0	46	9	4	2
0	53	50	26	4	20

Step 2: (i)Assignment in rows (ii)Assignment in columns

(i)

$\cancel{0}$	13	49	0	0	13
$\cancel{0}$	35	29	5	10	0
13	$\cancel{0}$	63	7	7	0
47	15	0	20	2	0
25	$\boxed{0}$	46	9	4	2
$\boxed{0}$	53	50	26	4	20

(ii)

$\cancel{0}$	13	49	$\boxed{0}$	$\cancel{0}$	13
$\cancel{0}$	35	29	5	10	0
13	$\cancel{0}$	63	7	7	0
47	15	$\boxed{0}$	20	2	$\cancel{0}$
25	$\boxed{0}$	46	9	4	2
$\boxed{0}$	53	50	26	4	20

Problem3 - Solution

Step 2: All zeros must be assigned or crossed out. So assign the zero of row 2 column 6.

~~0~~	13	49	$\boxed{0}$	~~0~~	13
~~0~~	35	29	5	10	$\boxed{0}$
13	~~0~~	63	7	7	~~0~~
47	15	$\boxed{0}$	20	2	~~0~~
25	$\boxed{0}$	46	9	4	2
$\boxed{0}$	53	50	26	4	20

Step 3: (i)Mark unassigned rows. (ii) In marked rows, mark the columns which has zeros.

(i)

$\cancel{0}$	13	49	$\boxed{0}$	$\cancel{0}$	13	
$\cancel{0}$	35	29	5	10	$\boxed{0}$	
13	$\cancel{0}$	63	7	7	$\cancel{0}$	✓
47	15	$\boxed{0}$	20	2	$\cancel{0}$	
25	$\boxed{0}$	46	9	4	2	
$\boxed{0}$	53	50	26	4	20	

(ii)

$\cancel{0}$	13	49	$\boxed{0}$	$\cancel{0}$	13	
$\cancel{0}$	35	29	5	10	$\boxed{0}$	
13	$\cancel{0}$	63	7	7	$\cancel{0}$	✓
47	15	$\boxed{0}$	20	2	$\cancel{0}$	
25	$\boxed{0}$	46	9	4	2	
$\boxed{0}$	53	50	26	4	20	
	✓				✓	

Problem3 - Solution

Step 3: (iii) In marked columns, mark the rows which have assigned zeros.

(iii)

$\cancel{0}$	13	49	$\boxed{0}$	$\cancel{0}$	13	
$\cancel{0}$	35	29	5	10	$\boxed{0}$	✓
13	$\cancel{0}$	63	7	7	$\cancel{0}$	✓
47	15	$\boxed{0}$	20	2	$\cancel{0}$	
25	$\boxed{0}$	46	9	4	2	✓
$\boxed{0}$	53	50	26	4	20	
	✓				✓	

Problem3 - Solution

Step 4: Draw minimum no. of lines from unmarked rows and marked columns. Draw lines L_1, L_2 through unmarked rows 1 and 4. Draw lines L_3, L_4 through marked columns 2 and 6(because both having two uncovered zeros). Now draw line L_5 through unmarked column 1

$\cancel{0}$	13	49	$\boxed{0}$	$\cancel{0}$	13		L_1
$\cancel{0}$	35	29	5	10	$\boxed{0}$	✓	
13	$\cancel{0}$	63	7	7	$\cancel{0}$	✓	
47	15	$\boxed{0}$	20	2	$\cancel{0}$		L_2
25	$\boxed{0}$	46	9	4	2	✓	
$\boxed{0}$	53	50	26	4	20		
L_5	✓ L_3				✓ L_4		

Step 5: Minimum of uncovered elements is to subtracted from all uncovered elements and added to intersection points.

4	17	49	0	0	17
0	35	25	1	6	0
13	0	59	3	3	0
51	19	0	20	2	4
25	0	42	5	0	2
0	53	46	22	0	0

Problem3 - Solution

Step 6: Repeat steps 1 and 2

(i)

4	17	49	$\boxed{0}$	$\cancel{0}$	17
0	35	25	1	6	0
13	0	59	3	3	0
51	19	$\boxed{0}$	20	2	4
25	0	42	5	0	2
0	53	46	22	0	20

(ii)

4	17	49	$\boxed{0}$	$\cancel{0}$	17
0	35	25	1	6	$\cancel{0}$
13	$\cancel{0}$	59	3	3	$\boxed{0}$
51	19	$\boxed{0}$	20	2	4
25	0	42	5	0	2
0	53	46	22	0	20

(iii)

4	17	49	$\boxed{0}$	$\cancel{0}$	17
$\boxed{0}$	35	25	1	6	$\cancel{0}$
13	$\cancel{0}$	59	3	3	$\boxed{0}$
51	19	$\boxed{0}$	20	2	4
25	$\boxed{0}$	42	5	$\cancel{0}$	2
$\cancel{0}$	53	46	22	$\boxed{0}$	20

Step 7: Assignment is optimum

Observe that all the zeros are either assigned or crossed-out. Every row(column) has an assignment. The optimal assignment is given by

$A \to d, B \to a, C \to f, D \to c, E \to b, F \to e$

The Minimum cost= 11+43+33+27+11+17=Rs.142.

Problem4

Solve the minimal assignment problem whose effectiveness matrix is:

	1	**2**	**3**	**4**
I	2	3	4	5
II	4	5	6	7
III	7	8	9	8
IV	3	5	8	4

Problem5

A certain equipment needs five repair jobs which have to be assigned to five machines. The estimated time (in hours) that each mechanic requires to complete the repair job is given in the following table: Assuming that each mechanic can be assigned to only one job, determine the minimum time assignment.

	1	**2**	**3**	**4**
I	2	3	4	5
II	4	5	6	7
III	7	8	9	8
IV	3	5	8	4

Session: 22 - Interior point methods, applications and algorithms

Interior Point method, Applications

Introduction

Modern interior point methods originated from an algorithm introduced by Karmarkar in 1984 for linear programming. In the years since then, algorithms and software for linear programming have become quite popular, while extensions to more general classes of problems, such as convex quadratic programming, linear complementarity problem, semi-definite programming, second order cone programming and nonconvex and nonlinear problems, have reached varying levels of maturity. The interior point algorithms are applied in some optimization problems, such as linear programming, linear complementarity problem, semi-definite programming and some convex programming.

Development

In 1984, Karmarkar developed a polynomial time algorithm that cuts across the interior of the solution space. This algorithm was more effective for solving extremely large linear programming problems. His algorithm was theoretically faster than the ellipsoid method and Karmarkar made some strong claims about its performance in practice. The algorithm was controversial at the time of its introduction, but there have been many improvements both in theory and practice since then. The method is now considered to be better than simplex, especially on large LPs.

Conversion of General LPP to Standard Homogeneous form

General LPP

- Min $Z = CX$
- subject to $AX \leq b, X \geq 0$

Standard Homogeneous LPP

- Min $Z = CX$
- subject to $AX = 0, IX = 1, X \geq 0$

Assumptions

- $X = \left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right)$ satisfies $AX = 0$ is a feasible solution
- Min $Z = 0$
- Using an Algebraic transformation we can convert the general LPP to Homogeneous form.

Karmarkar's Interior Point Algorithm

Step 1

Start with the solution point $X_0 = \left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right)$ and compute $r = \sqrt{\frac{1}{n(n-1)}}$ and $\alpha = \frac{n-1}{3n}$

Step 2

Define $D_k = diag(x_{k1}, x_{k2}, ..., x_{kn})$

$P = \begin{pmatrix} AD_k \\ \mathbf{1} \end{pmatrix}$ where **1** is row matrix of one's

$c_p = [I - P^T(PP^T)^{-1}P](cD_k)^T$

$Y_{new} = \left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right)^T + \alpha r \frac{c_p}{\|c_p\|}$

k^{th} iteration solution is $X_{k+1} = \frac{D_k Y_{new}}{\mathbf{1} D_k Y_{new}}$

Session: 23 - Karmarkar's interior point method and algorithm

Problem 1

Solve the following linear programming problem by interior point algorithm

Minimize $Z = x_1 - 2x_2$

subject to $x_1 - 2x_2 + x_3 = 0$

$x_1 + x_2 + x_3 = 1$

$x_1, x_2, x_3 \geq 0$

Iteration 0

Given LPP satisfies all the conditions of interior point algorithm.

i.e., $X = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ satisfies $x_1 - 2x_2 + x_3 = 0$ and optimum values of Z is zero

$\therefore X_0 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, $r = \frac{1}{\sqrt{6}}$, $\alpha = \frac{2}{9}, z = \frac{-1}{3} = -0.33333$

Iteration 1

$$D_0 = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$$

$$cD_0 = \begin{pmatrix} 1 & -2 & 0 \end{pmatrix} \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{-2}{3} & 0 \end{pmatrix}$$

$$AD_0 = \begin{pmatrix} 1 & -2 & 1 \end{pmatrix} \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{-2}{3} & \frac{1}{3} \end{pmatrix}$$

$$P = \begin{pmatrix} AD_0 \\ \mathbf{1} \end{pmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{-2}{3} & \frac{1}{3} \\ 1 & 1 & 1 \end{bmatrix}$$

$$(PP^T)^{-1} = \begin{bmatrix} 2/3 & 0 \\ 0 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3/2 & 0 \\ 0 & 1/3 \end{bmatrix}$$

Solution

Iteration 1 conti...

$$c_p = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1/3 & 1 \\ -2/3 & 1 \\ 1/3 & 1 \end{bmatrix} \begin{bmatrix} 3/2 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 1/3 & -2/3 & 1/3 \\ 1 & 1 & 1 \end{bmatrix} \right\} \begin{bmatrix} 1/3 \\ -2/3 \\ 0 \end{bmatrix}$$

$$\therefore c_p = \begin{bmatrix} \frac{1}{6} \\ 0 \\ -\frac{1}{6} \end{bmatrix} \text{ and } \|c_p\| = 0.2357$$

$$\frac{\alpha r}{\|c_p\|} = \frac{\frac{2}{9}\frac{1}{\sqrt{6}}}{0.2357} = 0.384901$$

$$Y_{new} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} + 0.384901 \begin{bmatrix} \frac{1}{6} \\ 0 \\ \frac{-1}{6} \end{bmatrix} = \begin{bmatrix} 0.397483 \\ 0.33333 \\ 0.269183 \end{bmatrix}$$

Solution

Iteration 1 conti...

Now we compute X_1 using the formula, $X_1 = \frac{D_0 Y_{new}}{\mathbf{1} D_0 Y_{new}}$

we obtain $X_1 = \begin{bmatrix} 0.397483 \\ 0.33333 \\ 0.269183 \end{bmatrix}$

Corresponding value of z is $z = -0.269183$ which is better approximation than the preceeding

Iteration 2

$$D_1 = \begin{bmatrix} 0.397483 & 0 & 0 \\ 0 & 0.33333 & 0 \\ 0 & 0 & 0.269183 \end{bmatrix}$$

$$cD_1 = \begin{pmatrix} 1 & -2 & 0 \end{pmatrix} \begin{bmatrix} 0.397483 & 0 & 0 \\ 0 & 0.33333 & 0 \\ 0 & 0 & 0.269183 \end{bmatrix} =$$

$$\begin{pmatrix} 0.397485 & -0.66666 & 0 \end{pmatrix}$$

Iteration 2 conti...

$$AD_1 = \begin{pmatrix} 1 & -2 & 1 \end{pmatrix} \begin{bmatrix} 0.397483 & 0 & 0 \\ 0 & 0.33333 & 0 \\ 0 & 0 & 0.269183 \end{bmatrix} =$$

$$\begin{pmatrix} 0.397485 & -0.66666 & 0.269182 \end{pmatrix}$$

$$P = \begin{pmatrix} AD_1 \\ \mathbf{1} \end{pmatrix} = \begin{bmatrix} 0.397485 & -0.66666 & 0.269182 \\ 1 & 1 & 1 \end{bmatrix}$$

$$c_p = \begin{bmatrix} 0.132402 \\ 0.018152 \\ -0.150555 \end{bmatrix} \text{ and } \|c_p\| = 0.201312$$

$$\frac{\alpha r}{\|c_p\|} = \frac{\frac{2}{9}\frac{1}{\sqrt{6}}}{0.201312} = 0.450653$$

$$Y_{new} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} + 0.450653 \begin{bmatrix} 0.132402 \\ 0.018152 \\ -0.150555 \end{bmatrix} = \begin{bmatrix} 0.393001 \\ 0.341514 \\ 0.265486 \end{bmatrix}$$

Solution

Iteration 2 conti...

Now we compute X_2 using the formula, $X_2 = \frac{D_1 Y_{new}}{\mathbf{1} D_1 Y_{new}}$

we obtain $X_2 = \begin{bmatrix} 0.457411 \\ 0.33333 \\ 0.209256 \end{bmatrix}$

Corresponding value of Z is $Z = 0.20934$ which is better approximation than the preceeding.

Iteration 3

The steps of algorithm can be repeated to move the solution closer to the optimum point $(\frac{2}{3}, \frac{1}{3}, 0)$. And the value of $Z = 0$

```
x1 = [0.397483 0.333333  0.269183]    z=-0.269000
x2 = [0.457409 0.333333  0.209258]    z=-0.209000
x3 = [0.509027 0.333333  0.157639]    z=-0.158000
x4 = [0.550656 0.333333  0.116011]    z=-0.116000
x5 = [0.582667 0.333333  0.083999]    z=-0.084000
x6 = [0.606509 0.333333  0.060158]    z=-0.060200
x7 = [0.623898 0.333333  0.042769]    z=-0.042800
x8 = [0.636409 0.333333  0.030258]    z=-0.030300
x9 = [0.645331 0.333333  0.021336]    z=-0.021300
x10 = [0.651656 0.333333  0.015010]        z=-0.015000
x11 = [0.656123 0.333333  0.010544]        z=-0.010500
x12 = [0.659268 0.333333  0.007399]        z=-0.007400
x13 = [0.661479 0.333333  0.005188]        z=-0.005190
x14 = [0.663031 0.333333  0.003636]        z=-0.003640
x15 = [0.664119 0.333333  0.002547]        z=-0.002550
x16 = [0.664883 0.333333  0.001784]        z=-0.001780
x17 = [0.665417 0.333333  0.001249]        z=-0.001250
```

$X_{17} \approx (\frac{2}{3}, \frac{1}{3}, 0)$ and $Z \approx 0$ upto 3 decimal places.

Problems for practice

Problem 2

Carry out three iterations of interior point algorithm for following problem

Maximize $Z = -4x_1 + x_3 + x_4$

subject to $-2x_1 + 2x_2 + x_3 - x_4 = 0$

$x_1 + x_2 + x_3 + x_4 = 1$

$x_1, x_2, x_3, x_4 \geq 0$

Course Code: 19CS2104

Course Name: Mathematical Programming-1

CO3-Network Models

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Session: 16 - Network Model

Network Model

Introduction

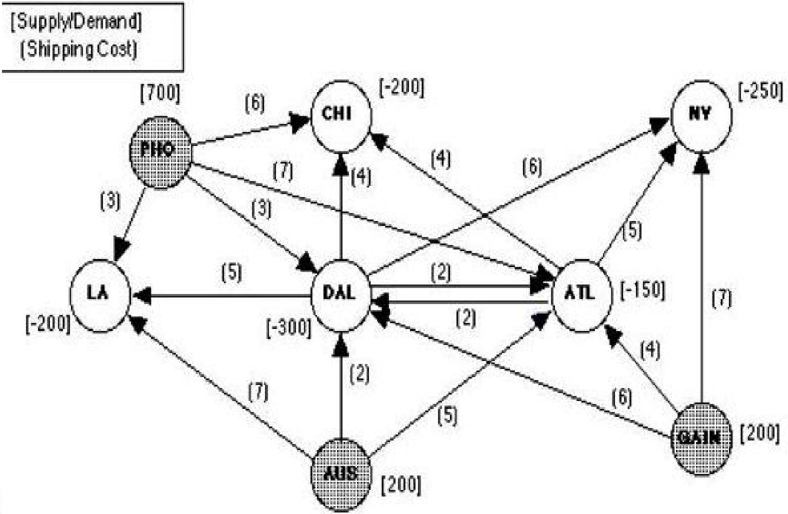
A common scenario of a network-flow problem arising in industrial logistics concerns the distribution of a single homogeneous product from plants (origins) to consumer markets (destinations). The total number of units produced at each plant and the total number of units required at each market are assumed to be known. The product need not be sent directly from source to destination, but may be routed through intermediary points reflecting warehouses or distribution centers. Further, there may be capacity restrictions that limit some of the shipping links. The objective is to minimize the variable cost of producing and shipping the products to meet the consumer demand.

Network Model

Terminology

- The centres from where the goods need to be shipped are called **origins**.
- The places to where the goods need to be shipped from origins are called **destinations**.
- The sources, destinations, and intermediate points are collectively called **nodes** of the network
- The transportation links connecting nodes are termed **arcs**.

Example



[Supply/Demand]
(Shipping Cost)
[700]
PHO
(6)
CHI
[-200]
NY
[-250]
(7)
(4)
(4)
(6)
(3)
(3)
(5)
LA
[-200]
(5)
DAL
[-300]
(2)
(2)
ATL
[-150]
(7)
(4)
(2)
(7)
(5)
(6)
GAIN
[200]
AUS
[200]

Types of Network Models

The class of network flow problems includes such as

- Transportation problem
- Assignment problem
- Shortest path problem
- Maximum flow problem
- Pure minimum cost flow problem
- Generalized minimum cost flow problem

What is Transportation Problem

The transportation problem is to transport various amounts of a single homogeneous commodity, that are initially stored at various origins, to different destinations in such a way that the total transportation cost is a minimum.

- The transportation problem is a special type of linear programming problem where the objective is to minimize the cost of distributing a product from a number of sources or origins to a number of destinations.
- Because of its special structure, the usual simplex method is not suitable for solving transportation problems. These problems require a special method of solution

Objective of Transportation Problem

- To find out the **optimum** transportation schedule keeping in mind cost of Transportation to be **minimized**.
- The transportation problem indicates the amount of consignment to be transported from various origins to different destinations so that the total transportation cost is minimized without violating the **availability constraints** and the **requirement constraints**.
- The **unit transportation cost** is the cost of transporting one unit of the consignment from an origin to a destination.

Applications of Transportation Problem

- It is used to compute transportation routes in such a way as to minimize transportation cost for finding out locations of warehouses.
- It is used to find out locations of transportation corporations depots where insignificant total cost difference may not matter.
- Minimize shipping costs from factories to warehouses(or from warehouses to retail outlets).
- Determine lowest cost location for new factory, warehouse, office ,or other outlet facility.
- Find minimum cost production schedule that satisfies firms demand and production limitations.

Mathematical formulation

Let m be no. of origins, n be no. of destinations $m = n$ or $m \neq n$
a_i be the no. of units that i^{th} origin possesses,
b_j be the no. of units that j^{th} destination requires,
c_{ij} be the cost of shipping from i^{th} origin to j^{th} destination,
x_{ij} be the allocation from i^{th} origin to j^{th} destination.
Assumed that $\sum a_i = \sum b_j$

Now the transportation problem is to determine $x_{ij} (\geq 0)$ satisfying

$$\sum_{j=1}^{n} x_{ij} = a_i \text{ (known as Availability constraints)}$$

$$\sum_{i=1}^{m} x_{ij} = b_j \text{ (known as Requirement constraints)}$$

and *minimizing* the total cost $z = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} c_{ij}$ (known as Objective function)

Types of Transportation problem

Balanced Transportation Problem

Total availability$=$Total Requirement

Unbalanced Transportation Problem

Total availability$\neq$ Total Requirement

Phases of Solution and Methods

Two Phases

1. Obtaining Initial Basic Feasible Solution.(Any one method is applied)
   - North-West Corner Rule
   - Row Minima Method
   - Column Minima Method
   - Lowest Cost Entry Method(Matrix Minima Method)
   - Vogel's Approximation Method (VAM)
2. Obtaining Optimal Basic Solution.
   - Modified Distribution Method or MODI Method or U-V Method

North West Corner Rule

Steps

1. Select the upper left (north-west) cell of the transportation matrix and allocate minimum of supply and demand, i.e min(A1,B1) value in that cell.
2. Obtaining Optimal Basic Solution.
   - If A1<B1, then allocation made is equal to the supply available at the first source(A1 in first row), then move vertically down to the cell (2,1).
   - If A1>B1, then allocation made is equal to demand of the first destination (B1 in first column), then move horizontally to the cell (1,2).
   - If A1=B1, then allocate the value of A1 or B1and then move to cell (2,2).
3. Continue the process until an allocation is made in the south-east corner cell of the transportation table.

Session 17: Transportation Problem - Balanced Case - Initial Basic Feasible Solution.

Problem - Find IBFS using NWCR.

In the table, three sources A,B and C with the Production Capacity of 50units, 40units, 60 units of product respectively is given. Every day the demand of three retailers P,Q,R is to be furnished with at least 20units, 95units and 35units of product respectively. The transportation costs are also given in the matrix.

Source/Destination	P	Q	R	Supply
A	5	8	4	50
B	6	6	3	40
C	3	9	6	60
Demand	20	95	35	150

Step 1

Check whether Total Demand is equal to Total Supply. In case the demand is more then supply, then dummy origin is added to the table. The cost associated with the dummy origin will be zero.

Problem - Find IBFS using NWCR.

Step 2: North-west cell is (1,1). Allocate min{20,50}to North-west cell (1,1). Update the demand and supply.

Source/Destination	P	Q	R	Supply	
A	20 5	8	4	~~50~~	30
B	6	6	3	40	
C	3	9	6	60	
Demand	~~20~~	95	35	150	
	0				

Problem - Find IBFS using NWCR.

Step 3: Next north-west cell is (1,2). Allocate min{95,30}to the cell (1,2). Update the demand and supply.

Source/Destination	P	Q	R	Supply
A	$\boxed{20}$ 5	$\boxed{30}$ 8	4	~~50~~ ~~30~~ 0
B	6	6	3	40
C	3	9	6	60
Demand	~~20~~	~~95~~	35	150
	0	65		

Problem - Find IBFS using NWCR.

Step 4: Next north-west cell is (2,2). Allocate min{65,40}to the cell (2,2). Update the demand and supply.

Source/Destination	P	Q	R	Supply
A	**20** 5	**30** 8	4	~~50~~ ~~30~~ 0
B	6	**40** 6	3	~~40~~ 0
C	3	9	6	60
Demand	~~20~~	~~95~~	35	150
	0	~~65~~ 25		

Step 5: Problem - Find IBFS using NWCR.

Next north-west cell is (3,2). Allocate min{25,60}to the cell (3,2). Update the demand and supply.

Source/Destination	P	Q	R	Supply	
A	20 5	30 8	4	~~50~~	~~30~~ 0
B	6	40 6	3	~~40~~	0
C	3	25 9	6	~~60~~	35
Demand	~~20~~	~~95~~	35	150	
	0	~~65~~ ~~25~~ 0			

Step :6 Problem - Find IBFS using NWCR.

Next north-west cell is (3,3). Allocate min{35,35}to the cell (3,3). Update the demand and supply.

Source/Destination	P	Q	R	Supply	
A	20 5	30 8	4	~~50~~	~~30~~ 0
B	6	40 6	3	~~40~~	0
C	3	25 9	35 6	~~60~~	~~35~~ 0
Demand	~~20~~	~~95~~	~~35~~	150	
	0	~~65~~ ~~25~~ 0	0		

Problem - Find IBFS by NWCR.

Step 6

Source/Destination	P	Q	R	Supply
A	20 5	30 8	4	50
B	6	40 6	3	40
C	3	35 9	35 6	60
Demand	20	95	35	150

Total Cost can be computed by multiplying the units assigned to each cell with the concerned transportation cost.
Total
Cost= $(20 * 5) + (30 * 8) + (40 * 6) + (25 * 9) + (35 * 6) = Rs.1015.$

Steps

1. Start with the lowest cost entry and allocate as much as possible.
2. Repeat the same with next lowest cost. Continue the process till the requirement/availability exhausts.

Problem - Find IBFS by LCEM.

Step 1

The minimum cost in the matrix is Rs 3, but there is a tie in the cell BR, and CP, now the question arises in which cell we shall allocate. Generally, the cost where maximum quantity can be assigned should be chosen to obtain the better initial solution. Therefore, 35 units shall be assigned to the cell BR.

Source/Destination	P	Q	R	Supply
A	5	8	4	50
B	6	6	$\boxed{35}$ 3	~~40~~ 5
C	3	9	6	60
Demand	20	95	~~35~~ 0	150

Problem - Find IBFS by LCEM.

Step 2

Again the minimum cost in the matrix is Rs 3. Therefore, 20 units shall be assigned to the cell CD. With this, the demand of retailer P gets fulfilled. Only 40 units are left with the source C.

Source/Destination	D	E	F	Supply
A	5	8	4	50
B	6	6	35 3	~~40~~ 5
C	20 3	9	6	~~60~~ 40
Demand	~~20~~ 0	95	~~35~~ 0	150

Problem - Find IBFS by LCEM.

Step 3

The next minimum cost is 8, assign 50 units to the cell AQ. The supply of source A gets saturated. The next minimum cost is Rs 9; we shall assign 40 units to the cell CE. With his both the demand and supply of all the sources and origins gets saturated.

Source/Destination	D	E	F	Supply
A	5	50 8	4	~~50~~ 0
B	6	5 6	3 35	~~40~~ ~~5~~ 0
C	20 3	40 9	6	~~60~~ ~~40~~ 0
Demand	~~20~~ 0	~~95~~ ~~90~~ ~~40~~ 0	~~35~~ 0	150

Total Cost $= 50 * 8 + 5 * 6 + 35 * 3 + 20 * 3 + 40 * 9 = Rs.955.$
Any Observation?

Session 18: Transportation Problems- Un Balanced Case-

Find IBFS by Matrix Minima Method.

The total demand is 1000, whereas the total supply is 800.
Total supply $<$ total demand.

Plant	Warehouse			Supply
	W1	W2	W3	
A	28	17	26	500
B	19	12	16	300
Demand	250	250	500	

Here dummy origin is added to the table. The cost associated with the dummy origin will be zero.

Unbalanced Transport Problem - Find IBFS by Matrix Minima Method.

Plant	Warehouse			Supply
	W1	W2	W3	
A	28	17	26	500
B	19	12	16	300
Unsatisfied demand	0	0	0	200
Demand	250	250	500	1000

Now proceed with Lowest cost entry method.

Plant	Warehouse			Supply
	W1	W2	W3	
A	28 (50)	17	26 450	~~500~~
B	19	12 250	16 50	~~300~~
Unsatisfied demand	0 200	0	0	~~200~~
Demand	~~250~~	~~250~~	~~500~~	1000

Total Cost$= 28(50) + 26(450) + 12(250) + 16(50) + 0(200) = 16900$

Exercise Problem 2

Obtain an initial basic feasible solution to the transport problem, using NWCR, Row Minima Method, Column minima method, Lowest Cost Entry Method

Factory/Warehouse	W_1	W_2	W_3	W_4	Factory Capacity
F_1	19	30	50	10	7
F_2	70	30	40	60	9
F_3	40	8	70	20	18
Warehouse Requirement	5	8	7	14	

Exercise Problem 3

Obtain an initial basic feasible solution to the transport problem, using Matrix minima method.

	D_1	D_2	D_3	D_4	Capacity
O_1	1	2	3	4	6
O_2	4	3	2	0	8
O_3	0	2	2	1	10
Demand	4	6	8	6	

Is the T.P. non-degenerate or degenerate?

Vogel's Approximation Method (VAM Method)

The Vogel Approximation Method is an improved version of the Minimum Cell Cost Method and the Northwest Corner Method that in general produces better initial basic feasible solution, that report a smaller value in the objective (minimization) function of a balanced Transportation Problem. It consists the following steps

- **Step1:** Determine a penalty cost for each row (column) by subtracting the lowest unit cell cost in the row (column) from the next lowest unit cell cost in the same row (column).
- **Step2:** Identify the row or column with the greatest penalty cost. Break the ties arbitrarily (if there are any). Allocate as much as possible to the variable with the lowest unit cost in the selected row or column. Adjust the supply and demand and cross out the row or column that is already satisfied. If a row and column are satisfied simultaneously, only cross out one of the two and allocate a supply or demand of zero to the one that remains.

steps Cont..

- **Step3:**
  - If there is exactly one row or column left with a supply or demand of zero, stop.
  - If there is one row (column) left with a positive supply (demand), determine the basic variables in the row (column) using the Minimum Cell Cost Method. Stop.
  - If all of the rows and columns that were not crossed out have zero supply and demand (remaining), determine the basic zero variables using the Minimum Cell Cost Method. Stop.
  - In any other case, continue with Step 1.

Problem - Find IBFS using VAM

From \ To	D	E	F	Supply	Iteration-I
A	6	4	1	50	3
B	3 20	8	7	40	4
C	4	4	2	60	2
Demand	20	95	35	150	
Iteration-I	1	0	1		

Step 1: First of all the difference between two least cost cells are calculated for each row and column, which can be seen in the iteration given for each row and column. Then the largest difference is selected, which is 4 in this case. So, allocate 20 units to cell BD, since the minimum cost is to be chosen for the allocation. Now, only 20 units are left with the source B.

Problem - Find IBFS using VAM

From \ To	E	F	Supply	Iteration-II
A	4 (15)	1 (35)	50	(3)
B	8	7	20	1
C	4	2	60	2
Demand	95	35	150	
Iteration-II	0	1		

Step 2: Column D is deleted, again the difference between the least cost cells is calculated for each row and column, as seen in the iteration below. The largest difference value comes to be 3, so allocate 35 units to cell AF and 15 units to the cell AE. With this, the Supply and demand of source A and origin F gets saturated, so delete both the row A and Column F.

Problem - Find IBFS using VAM

From \ To	E	Supply
B	8 (20)	20
C	4 (60)	60
Demand	80	150

Step 3: Now, single column E is left, since no difference can be found out, so allocate 60 units to the cell CE and 20 units to cell BE, as only 20 units are left with source B. Hence the demand and supply are completely met.

Problem - Find IBFS using VAM

- Now the total cost can be computed, by multiplying the units assigned to each cell with the cost concerned.
- Therefore, Total Cost $= 20*3 + 35*1 + 15*4 + 60*4 + 20*8$ $=$ Rs. 555.
- Note: Vogels Approximation Method is also called as Penalty Method because the difference costs chosen are nothing but the penalties of not choosing the least cost routes.

Problem 2- VAM

Obtain an initial basic feasible solution to the transport problem, using VAM

Factory/Warehouse	W_1	W_2	W_3	W_4	Factory Capacity
F_1	19	30	50	10	7
F_2	70	30	40	60	9
F_3	40	8	70	20	18
Warehouse Requirement	5	8	7	14	34

Step 1

	W_1	W_2	W_3	W_4	Available	Row difference
F_1	19	30	50	10	7	9
F_2	70	30	40	60	9	10
F_3	40	$\boxed{8}$ 8	70	20	~~18~~ 10	12
Required	5	~~8~~ 0	7	14	34	
Column Difference	21	22	10	10		

Problem2-VAM - Solution

Step 2

	W_1	W_2	W_3	W_4	Available	Row Difference	
F_1	5 19	30	50	10	~~7~~ 2	9	9
F_2	70	30	40	60	9	10	20
F_3	40	8 8	70	20	~~18~~ 10	12	20
Required	~~5~~ 0	~~8~~ 0	7	14	34		
Column Difference	21	22	10	10			
	21	-	10	10			

Problem2-VAM - Solution

Step 3

	W_1	W_2	W_3	W_4	Available	Row Diff.		
F_1	$\boxed{5}$ 19	30	50	10	~~7~~ 2	9	9	40
F_2	70	30	40	60	9	10	20	20
F_3	40	$\boxed{8}$ 8	70	$\boxed{10}$ 20	~~18~~ ~~10~~ 0	12	20	50
Required	~~5~~ 0	~~8~~ 0	7	~~14~~ 4				
Column Difference	21	22	10	10				
	21	-	10	10				
	-	-	10	10				

Problem2-VAM - Solution

Step 4

	W_1	W_2	W_3	W_4	Available	Row Diff.			
F_1	$\boxed{5}$ 19	30	50	$\boxed{2}$ 10	~~7~~ ~~2~~ 0	9	9	40	40
F_2	70	30	40	60	9	10	20	20	20
F_3	40	$\boxed{8}$ 8	70	$\boxed{10}$ 20	~~18~~ ~~10~~ 0	12	20	50	-
Req.	~~5~~ 0 0	~~8~~ 0 0	7	~~14~~ ~~4~~ 2					
Column Diff.	21 21 - -	22 - - -	10 10 10 10	10 10 10 50					

Problem2-VAM - Solution

Step 5

W_1	W_2	W_3	W_4	Available	Row Diff.				
5 19	30	50	2 10	~~7~~ ~~2~~ 0	9	9	40	40	-
70	30	7 40	60	~~9~~ 2	10	20	20	20	20
40	8 8	70	10 20	~~18~~ ~~10~~ 0	12	20	50	-	
~~5~~ 0	~~8~~ 0	~~7~~ 0	~~14~~ ~~4~~ 2						
21	22	10	10						
21	-	10	10						
-	-	10	10						
-	-	10	50						

Problem2 VAM - Solution

Step 6

W_1	W_2	W_3	W_4	Available	Row Diff.				
5 19	30	50	2 10	~~7~~ ~~2~~ 0	9	9	40	40	-
70	30	7 40	2 60	~~9~~ ~~2~~ 0	10	20	20	20	20
40	8 8	70	10 20	~~18~~ ~~10~~ 0	12	20	50	-	
~~5~~ 0	~~8~~ 0	~~7~~ 0	~~14~~ ~~4~~ ~~2~~ 0						
21	22	10	10						
21	-	10	10						
-	-	10	10						
-	-	10	50						

The total cost=5(19)+8(8)+2(10)+2(60)+10(20)+7(40)=Rs.779.

Algorithm of MODI Method or U-V Method

- **Step1:** Determine an initial basic feasible solution using any one of the three methods given below:
  - North West Corner Method
  - Matrix Minimum Method(Lowest Cost Entry Method)
  - Vogel's Approximation Method
- **Step2:** Start with the basic feasible solution consisting of $m + n - 1$ allocations in independent positions.
- **Step3:** Determine the $(m + n)$ values of u_i and v_j, for allocated cells, using $u_i + v_j = c_{ij}$. Starting initially with some $u_i = 0$.
- **Step4:** Compute the cell evaluations for unallocated cells(empty cells), using $c_{ij} - (u_i + v_j)$.
- **Step5:** If the cell evaluations of all the unallocated cells are either positive or zero, the given solution is the optimal solution. On the other hand, if one or more unallocated cell has negative value, the given solution is not an optimal solution.

Algorithm of MODI Method or U-V Method

- **Step6:** Select the unoccupied cell with the most negative cell evaluation, as the cell to be entered in the next solution. Let the variable x_{rs} enter the basis.
- **Step7:** Allocate an unknown quantity θ, to the cell (r, s). Then construct a loop that starts and ends at the cell (r, s) and connects some of the basic cells. The amount θ is added to and subtracted from the transition cells of the loop in such a manner that the availabilities and requirements remains satisfied.
- **Step8:** Assign the largest possible value to θ in such a way that the value of at least one basic variable becomes zero and other basic variables remain non-negative. The basic cell whose allocation has been made zero will leave the basis.
- **Step9:** Now return to step 3 and repeat the process until an optimum basic feasible solution is obtained.

Session 19: Transportation Problems- Balanced Case- Obtaining Optimal solution by using U-V Method

Problem - MODI Method

Obtain an initial basic feasible solution to the transport problem. Is this solution an optimal solution? If not, obtain the optimal solution

Factory/Warehouse	W_1	W_2	W_3	W_4	Factory Capacity
F_1	19	30	50	10	7
F_2	70	30	40	60	9
F_3	40	8	70	20	18
Warehouse Requirement	5	8	7	14	34

Problem MODI Method - Solution

Step1: On applying Vogel's Approximation Method (see Problem2-VAM-Solution)

Factory/Warehouse	W_1	W_2	W_3	W_4	Available
F_1	$\boxed{5}$ 19	30	50	$\boxed{2}$ 10	7
F_2	70	30	$\boxed{7}$ 40	$\boxed{2}$ 60	9
F_3	40	$\boxed{8}$ 8	70	$\boxed{10}$ 20	18
Required	5	8	7	14	

The total cost=5(19)+8(8)+2(10)+2(60)+10(20)+7(40)=Rs.779.

Step2: This initial basic feasible solution has m+n-1=3+4-1=6 allocation in independent positions.

Step 3: Calculating u_i, v_j for allocated cells

					u_i
	[5] 19	30	50	[2] 10	
	70	30	[7] 40	[2] 60	
	40	[8] 8	70	[10] 20	
v_j					

Suppose $u_1 = 0$ then

cell(1,1): $u_1 + v_1 = 19 \Rightarrow v_1 = 19$

cell(1,4): $u_1 + v_4 = 10 \Rightarrow v_4 = 10$

cell(2,4): $u_2 + v_4 = 60 \Rightarrow u_2 = 50$

cell(2,3): $u_2 + v_3 = 40 \Rightarrow v_3 = -10$

cell(3,4): $u_3 + v_4 = 20 \Rightarrow u_3 = 10$

cell(3,2): $u_3 + v_2 = 8 \Rightarrow v_2 = -2$

Problem MODI Method - Solution

Step 3: Calculating u_i, v_j for allocated cells

					u_i
	$\boxed{5}$ 19	30	50	$\boxed{2}$ 10	0
	70	30	$\boxed{7}$ 40	$\boxed{2}$ 60	50
	40	$\boxed{8}$ 8	70	$\boxed{10}$ 20	10
v_j	19	-2	-10	10	

Suppose $u_1 = 0$ then

cell(1,1): $u_1 + v_1 = 19 \Rightarrow v_1 = 19$

cell(1,4): $u_1 + v_4 = 10 \Rightarrow v_4 = 10$

cell(2,4): $u_2 + v_4 = 60 \Rightarrow u_2 = 50$

cell(2,3): $u_2 + v_3 = 40 \Rightarrow v_3 = -10$

cell(3,4): $u_3 + v_4 = 20 \Rightarrow u_3 = 10$

cell(3,2): $u_3 + v_2 = 8 \Rightarrow v_2 = -2$

Problem MODI Method - Solution

Step 4: Calculating cell evaluations for unallocated cells

					u_i
	$\boxed{5}$ 19	30	50	$\boxed{2}$ 10	0
	70	30	$\boxed{7}$ 40	$\boxed{2}$ 60	50
	40	$\boxed{8}$ 8	70	$\boxed{10}$ 20	10
v_j	19	-2	-10	10	

cell(1,2): $30-(u_1+v_2)=30-(0-2)=32$
cell(1,3): $50-(u_1+v_3)=50-(0-10)=60$
cell(2,1): $70-(u_2+v_1)=70-(50+19)=1$
cell(2,2): $30-(u_2+v_2)=30-(50-20)=-18$
cell(3,1): $40-(u_3+v_1)=40-(10+19)=11$
cell(3,3): $70-(u_3+v_3)=70-(10-10)=70$

Problem MODI Method - Solution

Step 4: Calculating cell evaluations for unallocated cells

5 19	(32) 30	(60) 50	2 10
(1) 70	(-18) 30	7 40	2 60
(11) 40	8 8	(70) 70	10 20

Step 5: Check for optimum solution, if not, identify the entering cell

Since cell(2,2) has largest negative cell evaluation -18, it must be entered into solution. So form a loop from this cell and allocate θ to the cell, followed by alternatively subtracting and adding the amount of this allocation to other corners of the loop in order to restore feasibility.

Problem MODI Method - Solution

Step 6: Allocating θ for entering cell

$\boxed{5}$ 19	(32) 30	(60) 50	$\boxed{2}$ 10
(1) 70	θ 30	$\boxed{7}$ 40	$(2-\theta)$ 60
(11) 40	$(8-\theta)$ 8	(70) 70	$(10+\theta)$ 20

$\text{Min}\{8-\theta, 2-\theta\} = 0 \Rightarrow \theta = 2$

Step 7: Identifying the leaving cell from allocation

$\boxed{5}$ 19	(32) 30	(60) 50	$\boxed{2}$ 10
(1) 70	2 30	$\boxed{7}$ 40	0 60
(11) 40	6 8	(70) 70	12 20

The cell with 0 value i.e., cell (2,4)is removed from allocaion.

Problem MODI Method - Solution

Step 8: Update the allocations and Repeat steps 3-5 : Calculating u_i, v_j

					u_i
	$\boxed{5}$ 19	30	50	$\boxed{2}$ 10	
	70	$\boxed{2}$ 30	$\boxed{7}$ 40	60	
	40	$\boxed{6}$ 8	70	$\boxed{12}$ 20	
v_j					

Suppose $u_1 = 0$ then

cell(1,1): $u_1 + v_1 = 19 \Rightarrow v_1 = 19$

cell(1,4): $u_1 + v_4 = 10 \Rightarrow v_4 = 10$

cell(3,4): $u_3 + v_4 = 20 \Rightarrow u_3 = 10$

cell(3,2): $u_3 + v_2 = 8 \Rightarrow v_2 = -2$

cell(2,2): $u_2 + v_2 = 30 \Rightarrow u_2 = 32$

cell(2,3): $u_2 + v_3 = 40 \Rightarrow v_3 = 8$

Problem MODI Method - Solution

Calculating cell evaluations for unallocated cells

					u_i
	$\boxed{5}$ 19	30	50	$\boxed{2}$ 10	0
	70	$\boxed{2}$ 30	$\boxed{7}$ 40	60	32
	40	$\boxed{6}$ 8	70	$\boxed{12}$ 20	10
v_j	19	-2	8	10	

cell(1,2): $30 - (u_1 + v_2) = 30 - (0 - 2) = 32$
cell(1,3): $50 - (u_1 + v_3) = 50 - (0 - 8) = 42$
cell(2,1): $70 - (u_2 + v_1) = 70 - (32 + 19) = 19$
cell(2,4): $60 - (u_2 + v_4) = 60 - (32 + 10) = 18$
cell(3,1): $40 - (u_3 + v_1) = 40 - (10 + 19) = 11$
cell(3,3): $70 - (u_3 + v_3) = 70 - (10 - 8) = 52$

Problem MODI Method - Solution

Check for optimum solution

$\boxed{5}$ 19	(32) 30	(42) 50	$\boxed{2}$ 10
(19) 70	$\boxed{2}$ 30	$\boxed{7}$ 40	(18) 60
(11) 40	$\boxed{6}$ 8	(52) 70	$\boxed{12}$ 20

Since all cell evaluations are non-negative, the solution of above table is optimal with minimum cost.

The minimum cost$= 5.19 + 2.10 + 2.30 + 7.40 + 6.8 + 12.20 = 743$
It is less than the total cost obtained by Vogel's approximation method.

A company has three plants A, B and C and three warehouses X, Y and Z. Number of units available at the plants is 60, 70, 80, respectively. Demands at X, Y and Z are 50, 80 and 80, respectively. Unit costs of transportation are as follows:

To

	X	Y	Z
A	8	7	3
B	3	8	9
C	11	3	5

What would be your plan of transportation? Give minimum distribution cost.

Exercise Problem

A firm manufacturing a single product has plants I, II and III. The three plants have produced 60, 35 and 40 units respectively during this month. The firm had made a commitment to sell 22 units to customer A, 45 units to customer B, 20 units to customer C, 18 units to customer D and 30 units to customer E. Find the minimum possible transportation cost of shipping the manufactured product to five customers. The net per unit cost of transporting from the three plants to five customers is given in the table(Ans:290)

		Customer				
		A	B	C	D	E
Plant	I	4	1	3	4	4
	II	2	3	2	2	3
	III	3	5	2	4	4

Hindustan Construction Company needs 3, 3, 4 and 5 million cubic feet of fill at four earthen dam-sites in Pubjab. It can transfer the fill from three mounds A, B and C where 2, 6 and 7 million cubic feet of fill is available respectively. Costs of transporting one million cubic feet of fill from mounds to the four sites in lakhs are given in the table.
(a) Solve the problem using transportation algorithm for minimum cost.
(b) Formulate the problem as L.P.P.(Ans:174)

		To			
		I	II	III	IV
From	A	15	10	17	18
	B	16	13	12	13
	C	12	17	20	11

Transportation problem of Maximization

If the problem is to maximize the profit then identify the maximum cost value in the whole matrix and subtract each element from the maximum value. Now proceed with the usual methods.

Maximize the profit using Matrix Minima Method

4	5	1	40
3	4	3	60
6	2	8	70
70	40	60	

Solution

Since it is maximization problem, subtract each element from maximum value. So subtract each element from 8.

4	3	7	40
5	4	5	60
2	6	0	70
70	40	60	

Now apply the Matrix minima method

Maximize the profit using Matrix Minima Method

we obtain the following allocations(final step is given here).

4	40 3	7
60 5	4	5
10 2	6	60 0

To find the maximum profit multiply the allocations with original cost values.

Hence the maximum profit=40*5+60*3+10*6+60*8=920.

Maximize the profit using Column Minima Method

4	5	1	40
3	4	3	60
6	2	8	70
70	40	60	

Solution

Since it is maximization problem, subtract each element from maximum value. So subtract each element from 8.

4	3	7	40
5	4	5	60
2	6	0	70
70	40	60	

Now apply the Column minima method

we obtain the following allocations(final step is given here).

4	40 3	7
5	4	60 5
70 2	6	0

To find the maximum profit multiply the allocations with original cost values.

Hence the maximum profit=40*5+60*3+70*6=800.

Session: 20 - Assignment Problems - Balanced and Unbalanced Cases

Definition

Suppose there are n jobs to be performed and n persons are available for doing these jobs. Assume that each person can do each job at a time, though with varying degree of efficiency. Let C_{ij} be the cost (payment) if the i^{th} person is assigned the j^{th} job, the problem is to find an assignment (which job should be assigned to which person)so that the total cost for performing all jobs is minimum.

Mathematical Formulation of Assignment Problem

Minimize the total cost $z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$

subject to the restrictions of the form

$$x_{ij} = \begin{cases} 1 & \text{if job } j \text{ assigned to person } i \\ 0 & \text{if not} \end{cases}$$

$\sum_{j=1}^{n} x_{ij} = 1$ and $\sum_{i=1}^{n} x_{ij} = 1$

Steps

1. Subtract the minimum of each row from all the elements of the respective rows.
2. Further,subtract the minimum of each column from all the elements of the respective columns.
3. If a row contains more than one zero then move to next row. If a row contains single 0 then assignment is marked to that cell(enrectangle the zero). And cross all other zeros in the column in which the assignment is made. Repeat this for all the rows.
4. If a column is already assigned then move to the next column. If a column contains single 0 then assignment is marked to that cell. And cross all other zeros in the row in which the assignment is made. Repeat this for all the columns.

Steps

5. Draw the minimum number of horizontal and vertical lines to cover all the zeros in the resulting matrix. Let the minimum number of line be N. Now there two possibilities.
   - If $N = n$, then an optimal assignment can be made. We are finished.
   - If $N < n$, then repeat the steps $1 - 5$ until $N = n$.

Rule to draw minimum number of lines

1. mark the rows in which the assignment is not done.
2. In the marked row, mark the column which has zero.
3. In the marked column, mark the row which has assignment.
4. Now draw the lines through unmarked rows and marked columns.
5. If there are still uncovered zeros, draw horizontal/vertical lines to cover the zeros.

Problem1

A department head has four subordinates, and four tasks have to be performed. Subordinates differ in efficiency and tasks differ in their intrinsic difficulty. Time each man would take to perform each task is given in the effectiveness matrix. How the tasks should be allocated to each person so as to minimize the total man hours?

		Subordinates			
		I	**II**	**III**	**IV**
Tasks	A	8	26	17	11
	B	13	28	4	26
	C	38	19	18	15
	D	19	26	24	10

Problem1 - Solution

Step 1: Choose the least number in row and subtract from all the elements of the row.

0	18	9	3
9	24	0	22
23	4	3	0
9	16	14	0

Step 2: Choose the least number in column and subtract from all the elements of the column.

0	14	9	3
9	20	0	22
23	0	3	0
9	12	14	0

Problem1 - Solution

Step 3: Assignment of cells in rows

$\boxed{0}$	14	9	3
9	20	$\boxed{0}$	22
23	0	3	0
9	12	14	$\boxed{0}$

Step 4: Assignment of cells in columns.

$\boxed{0}$	14	9	3
9	20	$\boxed{0}$	22
23	$\boxed{0}$	3	$\cancel{0}$
9	12	14	$\boxed{0}$

Problem1 - Solution

Step 5: Assignment is optimum

Observe that all the zeros are either assigned or crossed-out. It is found that no additional assignments are possible. The optimal assignment is given by

$A \to I, B \to III, C \to II, D \to IV$

The man hours= 8+4+19+10=41 hours

Problem2

A car hire company has one car at each of five depots a, b, c, d and e. A customer requires a car in each town, namely A, B, C, D and E. Distance (in Km) between depots (origins) and towns (destinations) are given in the following distance matrix:

	a	b	c	d	e
A	160	130	175	190	200
B	135	120	130	160	175
C	140	110	155	170	185
D	50	50	80	80	110
E	55	35	70	80	105

Problem2 - Solution

Step 1: Minimum value of the row is subtracted from all the elements of the row.(repeat for all rows)

30	0	45	60	70
15	0	10	40	55
30	0	45	60	75
0	0	30	30	60
20	0	35	45	70

Problem2 - Solution

Step 2: Minimum value of the column is subtracted from all the elements of the column.(repeat for all columns)

30	0	35	30	15
15	0	0	10	0
30	0	35	30	20
0	0	20	0	5
20	0	25	15	15

Problem2 - Solution

Step 3: Assignment in rows

30	$\boxed{0}$	35	30	15
15	$\cancel{0}$	0	10	0
30	$\cancel{0}$	35	30	20
0	$\cancel{0}$	20	0	5
20	$\cancel{0}$	25	15	15

Problem2 - Solution

Step 4: Assignment in columns

30	$\boxed{0}$	35	30	15
15	~~0~~	$\boxed{0}$	10	~~0~~
30	~~0~~	35	30	20
$\boxed{0}$	~~0~~	20	~~0~~	5
20	~~0~~	25	15	15

Problem2 - Solution

Step 5: (i)Mark unassigned rows, (ii)then mark the column which has crossed zero in the marked rows. (iii)Mark assigned rows in the marked columns.

(i)

30	$\boxed{0}$	35	30	15	
15	$\cancel{0}$	$\boxed{0}$	10	$\cancel{0}$	
30	$\cancel{0}$	35	30	20	✓
$\boxed{0}$	$\cancel{0}$	20	$\cancel{0}$	5	
20	$\cancel{0}$	25	15	15	✓

(ii)

30	$\boxed{0}$	35	30	15	
15	$\cancel{0}$	$\boxed{0}$	10	$\cancel{0}$	
30	$\cancel{0}$	35	30	20	
$\boxed{0}$	$\cancel{0}$	20	$\cancel{0}$	5	
20	$\cancel{0}$	25	15	15	✓
	✓				

(iii)

30	$\boxed{0}$	35	30	15	✓
15	$\cancel{0}$	$\boxed{0}$	10	$\cancel{0}$	
30	$\cancel{0}$	35	30	20	✓
$\boxed{0}$	$\cancel{0}$	20	$\cancel{0}$	5	
20	$\cancel{0}$	25	15	15	✓
	✓				

Problem2 - Solution

Step 6: Draw line through unmarked rows and marked columns.

30	$\boxed{0}$	35	30	15	✓
15	$\cancel{0}$	$\boxed{0}$	10	$\cancel{0}$	
30	$\cancel{0}$	35	30	20	✓
$\boxed{0}$	$\cancel{0}$	20	$\cancel{0}$	5	
20	$\cancel{0}$	25	15	15	✓
	✓				

Step 7: Identify the minimum from uncovered elements(15). Subtract from uncovered elements. Add to the intersection points.

15	0	20	15	0
15	15	0	10	0
15	0	20	15	5
0	15	20	0	5
5	0	10	0	0

Problem2 - Solution

Step 8: (i) Assignment in rows (ii) Assignment in columns

(i)

15	$\cancel{0}$	20	15	0
15	15	0	10	0
15	$\boxed{0}$	20	15	5
0	15	20	0	5
5	$\cancel{0}$	10	0	0

(ii)

15	$\cancel{0}$	20	15	$\boxed{0}$
15	15	$\boxed{0}$	10	$\cancel{0}$
15	$\boxed{0}$	20	15	5
$\boxed{0}$	15	20	$\cancel{0}$	5
5	$\cancel{0}$	10	$\boxed{0}$	$\cancel{0}$

Step 9: Assignment is optimum

Observe that all the zeros are either assigned or crossed-out. Every row(column) has an assignment. The optimal assignment is given by

$A \rightarrow e, B \rightarrow c, C \rightarrow b, D \rightarrow a, E \rightarrow d$

The man hours= 200+130+110+50+80=570 Kms.

Session: 21 - Assignment Problems - Balanced and Unbalanced Cases

Problem3

Solve the assignment problem represented by the following matrix.

	a	b	c	d	e	f
A	9	22	58	11	19	27
B	43	78	72	50	63	48
C	41	28	91	37	45	33
D	74	42	27	49	39	32
E	36	11	57	22	25	18
F	3	56	53	31	17	28

Problem3 - Solution

Step 1: (i)Identify minimum of the row and subtract from each element of the row.(repeat for all rows) (ii)Identify minimum of the column and subtract from each element of the column(repeat for all columns)

(i)

0	13	49	2	10	18
0	35	29	7	20	5
13	0	63	9	17	5
47	15	0	22	12	5
25	0	46	11	14	7
0	53	50	28	14	25

(ii)

0	13	49	0	0	13
0	35	29	5	10	0
13	0	63	7	7	0
47	15	0	20	2	0
25	0	46	9	4	2
0	53	50	26	4	20

Problem3 - Solution

Step 2: (i)Assignment in rows (ii)Assignment in columns

(i)

$\cancel{0}$	13	49	0	0	13
$\cancel{0}$	35	29	5	10	0
13	$\cancel{0}$	63	7	7	0
47	15	0	20	2	0
25	$\boxed{0}$	46	9	4	2
$\boxed{0}$	53	50	26	4	20

(ii)

$\cancel{0}$	13	49	$\boxed{0}$	$\cancel{0}$	13
$\cancel{0}$	35	29	5	10	0
13	$\cancel{0}$	63	7	7	0
47	15	$\boxed{0}$	20	2	$\cancel{0}$
25	$\boxed{0}$	46	9	4	2
$\boxed{0}$	53	50	26	4	20

Problem3 - Solution

Step 2: All zeros must be assigned or crossed out. So assign the zero of row 2 column 6.

$\cancel{0}$	13	49	$\boxed{0}$	$\cancel{0}$	13
$\cancel{0}$	35	29	5	10	$\boxed{0}$
13	$\cancel{0}$	63	7	7	$\cancel{0}$
47	15	$\boxed{0}$	20	2	$\cancel{0}$
25	$\boxed{0}$	46	9	4	2
$\boxed{0}$	53	50	26	4	20

Step 3: (i)Mark unassigned rows. (ii) In marked rows, mark the columns which has zeros.

(i)

$\cancel{0}$	13	49	$\boxed{0}$	$\cancel{0}$	13	
$\cancel{0}$	35	29	5	10	$\boxed{0}$	
13	$\cancel{0}$	63	7	7	$\cancel{0}$	✓
47	15	$\boxed{0}$	20	2	$\cancel{0}$	
25	$\boxed{0}$	46	9	4	2	
$\boxed{0}$	53	50	26	4	20	

(ii)

$\cancel{0}$	13	49	$\boxed{0}$	$\cancel{0}$	13	
$\cancel{0}$	35	29	5	10	$\boxed{0}$	
13	$\cancel{0}$	63	7	7	$\cancel{0}$	✓
47	15	$\boxed{0}$	20	2	$\cancel{0}$	
25	$\boxed{0}$	46	9	4	2	
$\boxed{0}$	53	50	26	4	20	
	✓				✓	

Problem3 - Solution

Step 3: (iii) In marked columns, mark the rows which have assigned zeros.

(iii)

$\cancel{0}$	13	49	$\boxed{0}$	$\cancel{0}$	13	
$\cancel{0}$	35	29	5	10	$\boxed{0}$	✓
13	$\cancel{0}$	63	7	7	$\cancel{0}$	✓
47	15	$\boxed{0}$	20	2	$\cancel{0}$	
25	$\boxed{0}$	46	9	4	2	✓
$\boxed{0}$	53	50	26	4	20	
	✓				✓	

Problem3 - Solution

Step 4: Draw minimum no. of lines from unmarked rows and marked columns. Draw lines L_1, L_2 through unmarked rows 1 and 4. Draw lines L_3, L_4 through marked columns 2 and 6(because both having two uncovered zeros). Now draw line L_5 through unmarked column 1

~~0~~	13	49	0	~~0~~	13		L_1
~~0~~	35	29	5	10	0	✓	
13	~~0~~	63	7	7	~~0~~	✓	
47	15	0	20	2	~~0~~		L_2
25	0	46	9	4	2	✓	
0	53	50	26	4	20		
L_5	✓ L_3				✓ L_4		

Problem3 - Solution

Step 5: Minimum of uncovered elements is to subtracted from all uncovered elements and added to intersection points.

4	17	49	0	0	17
0	35	25	1	6	0
13	0	59	3	3	0
51	19	0	20	2	4
25	0	42	5	0	2
0	53	46	22	0	0

Step 6: Repeat steps 1 and 2

(i)

4	17	49	$\boxed{0}$	$\cancel{0}$	17
0	35	25	1	6	0
13	0	59	3	3	0
51	19	$\boxed{0}$	20	2	4
25	0	42	5	0	2
0	53	46	22	0	20

(ii)

4	17	49	$\boxed{0}$	$\cancel{0}$	17
0	35	25	1	6	$\cancel{0}$
13	$\cancel{0}$	59	3	3	$\boxed{0}$
51	19	$\boxed{0}$	20	2	4
25	0	42	5	0	2
0	53	46	22	0	20

(iii)

4	17	49	$\boxed{0}$	$\cancel{0}$	17
$\boxed{0}$	35	25	1	6	$\cancel{0}$
13	$\cancel{0}$	59	3	3	$\boxed{0}$
51	19	$\boxed{0}$	20	2	4
25	$\boxed{0}$	42	5	$\cancel{0}$	2
$\cancel{0}$	53	46	22	$\boxed{0}$	20

Step 7: Assignment is optimum

Observe that all the zeros are either assigned or crossed-out. Every row(column) has an assignment. The optimal assignment is given by

$A \to d, B \to a, C \to f, D \to c, E \to b, F \to e$

The Minimum cost$=$ 11$+$43$+$33$+$27$+$11$+$17$=$Rs.142.

Problem4

Solve the minimal assignment problem whose effectiveness matrix is:(Ans:20)

	1	2	3	4
I	2	3	4	5
II	4	5	6	7
III	7	8	9	8
IV	3	5	8	4

Problem5

A certain equipment needs five repair jobs which have to be assigned to five machines. The estimated time (in hours) that each mechanic requires to complete the repair job is given in the following table: Assuming that each mechanic can be assigned to only one job, determine the minimum time assignment.(Ans:27)

		Job				
		J1	J2	J3	J4	J5
Machine	M1	7	5	9	8	11
	M2	9	12	7	11	10
	M3	8	5	4	6	9
	M4	7	3	6	9	5
	M5	4	6	7	5	11

If the problem is to maximize the profit then identify the maximum cost value in the whole matrix and subtract each element from the maximum value. Now proceed with the usual Hungarian method.

Maximize the profit for the following assignment problem

17	10	12
9	8	10
6	2	8

Solution

Since it is maximization problem, subtract each element from maximum value. So subtract each element from 17.

0	7	5
8	9	7
11	15	9

Now apply the Hungarian method

Maximize the profit for the following assignment problem

we obtain the following assignment(final step is given here).

$\boxed{0}$	5	5
1	$\boxed{0}$	$\cancel{0}$
2	4	$\boxed{0}$

To find the maximum profit add the assignments with original cost values.

Hence the maximum profit=17+8+8=33.

Session: 22 - Interior point methods, applications and algorithms

Introduction

Modern interior point methods originated from an algorithm introduced by Karmarkar in 1984 for linear programming. In the years since then, algorithms and software for linear programming have become quite popular, while extensions to more general classes of problems, such as convex quadratic programming, linear complementarity problem, semi-definite programming, second order cone programming and nonconvex and nonlinear problems, have reached varying levels of maturity. The interior point algorithms are applied in some optimization problems, such as linear programming, linear complementarity problem, semi-definite programming and some convex programming.

Development

In 1984, Karmarkar developed a polynomial time algorithm that cuts across the interior of the solution space. This algorithm was more effective for solving extremely large linear programming problems. His algorithm was theoretically faster than the ellipsoid method and Karmarkar made some strong claims about its performance in practice. The algorithm was controversial at the time of its introduction, but there have been many improvements both in theory and practice since then. The method is now considered to be better than simplex, especially on large LPs.

Conversion of General LPP to Standard Homogeneous form

General LPP

- Min $Z = CX$
- subject to $AX \leq b, X \geq 0$

Standard Homogeneous LPP

- Min $Z = CX$
- subject to $AX = 0, IX = 1, X \geq 0$

Assumptions

- $X = \left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right)$ satisfies $AX = 0$ is a feasible solution
- Min $Z = 0$
- Using an Algebraic transformation we can convert the general LPP to Homogeneous form.

Karmarkar's Interior Point Algorithm

Step 1

Start with the solution point $X_0 = \left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right)$ and compute $r = \sqrt{\frac{1}{n(n-1)}}$ and $\alpha = \frac{n-1}{3n}$

Step 2

Define $D_k = diag(x_{k1}, x_{k2}, ..., x_{kn})$

$P = \begin{pmatrix} AD_k \\ \mathbf{1} \end{pmatrix}$ where **1** is row matrix of one's

$c_p = [I - P^T(PP^T)^{-1}P](cD_k)^T$

$Y_{new} = \left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right)^T + \alpha r \frac{c_p}{\|c_p\|}$

k^{th} iteration solution is $X_{k+1} = \frac{D_k Y_{new}}{\mathbf{1} D_k Y_{new}}$

Session: 23 - Karmarkar's interior point method and algorithm

Problem 1

Solve the following linear programming problem by interior point algorithm

Minimize $Z = x_1 - 2x_2$

subject to $x_1 - 2x_2 + x_3 = 0$

$x_1 + x_2 + x_3 = 1$

$x_1, x_2, x_3 \geq 0$

Iteration 0

Given LPP satisfies all the conditions of interior point algorithm.

i.e., $X = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ satisfies $x_1 - 2x_2 + x_3 = 0$ and optimum values of Z is zero

$\therefore X_0 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, $r = \frac{1}{\sqrt{6}}$, $\alpha = \frac{2}{9}, z = \frac{-1}{3} = -0.33333$

Solution

Iteration 1

$$D_0 = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$$

$$cD_0 = \begin{pmatrix} 1 & -2 & 0 \end{pmatrix} \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{-2}{3} & 0 \end{pmatrix}$$

$$AD_0 = \begin{pmatrix} 1 & -2 & 1 \end{pmatrix} \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{-2}{3} & \frac{1}{3} \end{pmatrix}$$

$$P = \begin{pmatrix} AD_0 \\ \mathbf{1} \end{pmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{-2}{3} & \frac{1}{3} \\ 1 & 1 & 1 \end{bmatrix}$$

$$(PP^T)^{-1} = \begin{bmatrix} 2/3 & 0 \\ 0 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3/2 & 0 \\ 0 & 1/3 \end{bmatrix}$$

Solution

Iteration 1 conti...

$c_p =$

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1/3 & 1 \\ -2/3 & 1 \\ 1/3 & 1 \end{bmatrix} \begin{bmatrix} 3/2 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 1/3 & -2/3 & 1/3 \\ 1 & 1 & 1 \end{bmatrix} \right\} \begin{bmatrix} 1/3 \\ -2/3 \\ 0 \end{bmatrix}$$

$$\therefore c_p = \begin{bmatrix} \frac{1}{6} \\ 0 \\ -\frac{1}{6} \end{bmatrix} \text{ and } \|c_p\| = 0.2357$$

$$\frac{\alpha r}{\|c_p\|} = \frac{\frac{2}{9}\frac{1}{\sqrt{6}}}{0.2357} = 0.384901$$

$$Y_{new} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} + 0.384901 \begin{bmatrix} \frac{1}{6} \\ 0 \\ \frac{-1}{6} \end{bmatrix} = \begin{bmatrix} 0.397483 \\ 0.33333 \\ 0.269183 \end{bmatrix}$$

Iteration 1 conti...

Now we compute X_1 using the formula, $X_1 = \frac{D_0 Y_{new}}{\mathbf{1} D_0 Y_{new}}$

we obtain $X_1 = \begin{bmatrix} 0.397483 \\ 0.33333 \\ 0.269183 \end{bmatrix}$

Corresponding value of z is $z = -0.269183$ which is better approximation than the preceeding

Iteration 2

$$D_1 = \begin{bmatrix} 0.397483 & 0 & 0 \\ 0 & 0.33333 & 0 \\ 0 & 0 & 0.269183 \end{bmatrix}$$

$$cD_1 = \begin{pmatrix} 1 & -2 & 0 \end{pmatrix} \begin{bmatrix} 0.397483 & 0 & 0 \\ 0 & 0.33333 & 0 \\ 0 & 0 & 0.269183 \end{bmatrix} =$$

$$\begin{pmatrix} 0.397485 & -0.66666 & 0 \end{pmatrix}$$

Iteration 2 conti...

$$AD_1 = \begin{pmatrix} 1 & -2 & 1 \end{pmatrix} \begin{bmatrix} 0.397483 & 0 & 0 \\ 0 & 0.33333 & 0 \\ 0 & 0 & 0.269183 \end{bmatrix} =$$

$$\begin{pmatrix} 0.397485 & -0.66666 & 0.269182 \end{pmatrix}$$

$$P = \begin{pmatrix} AD_1 \\ \mathbf{1} \end{pmatrix} = \begin{bmatrix} 0.397485 & -0.66666 & 0.269182 \\ 1 & 1 & 1 \end{bmatrix}$$

$$c_p = \begin{bmatrix} 0.132402 \\ 0.018152 \\ -0.150555 \end{bmatrix} \text{ and } \|c_p\| = 0.201312$$

$$\frac{\alpha r}{\|c_p\|} = \frac{\frac{2}{9}\frac{1}{\sqrt{6}}}{0.201312} = 0.450653$$

$$Y_{new} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} + 0.450653 \begin{bmatrix} 0.132402 \\ 0.018152 \\ -0.150555 \end{bmatrix} = \begin{bmatrix} 0.393001 \\ 0.341514 \\ 0.265486 \end{bmatrix}$$

Solution

Iteration 2 conti...

Now we compute X_2 using the formula, $X_2 = \frac{D_1 Y_{new}}{\mathbf{1} D_1 Y_{new}}$

we obtain $X_2 = \begin{bmatrix} 0.457411 \\ 0.33333 \\ 0.209256 \end{bmatrix}$

Corresponding value of Z is $Z = -0.20934$ which is better approximation than the preceeding.

Iteration 3

The steps of algorithm can be repeated to move the solution closer to the optimum point $(\frac{2}{3}, \frac{1}{3}, 0)$. And the value of $Z = 0$

```
X1 = [0.397483 0.333333  0.269183]    Z=-0.269000
X2 = [0.457409 0.333333  0.209258]    Z=-0.209000
X3 = [0.509027 0.333333  0.157639]    Z=-0.158000
X4 = [0.550656 0.333333  0.116011]    Z=-0.116000
X5 = [0.582667 0.333333  0.083999]    Z=-0.084000
X6 = [0.606509 0.333333  0.060158]    Z=-0.060200
X7 = [0.623898 0.333333  0.042769]    Z=-0.042800
X8 = [0.636409 0.333333  0.030258]    Z=-0.030300
X9 = [0.645331 0.333333  0.021336]    Z=-0.021300
X10 = [0.651656 0.333333  0.015010]        Z=-0.015000
X11 = [0.656123 0.333333  0.010544]        Z=-0.010500
X12 = [0.659268 0.333333  0.007399]        Z=-0.007400
X13 = [0.661479 0.333333  0.005188]        Z=-0.005190
X14 = [0.663031 0.333333  0.003636]        Z=-0.003640
X15 = [0.664119 0.333333  0.002547]        Z=-0.002550
X16 = [0.664883 0.333333  0.001784]        Z=-0.001780
X17 = [0.665417 0.333333  0.001249]        Z=-0.001250
```

$X_{17} \approx (\frac{2}{3}, \frac{1}{3}, 0)$ and $Z \approx 0$ upto 3 decimal places.

Problem 2

Carry out three iterations of interior point algorithm for following problem

Maximize $Z = -4x_1 + x_3 + x_4$

subject to $-2x_1 + 2x_2 + x_3 - x_4 = 0$

$x_1 + x_2 + x_3 + x_4 = 1$

$x_1, x_2, x_3, x_4 \geq 0$