Duality in Linear Programming Problem

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Duality in linear programming

- For every LP problem, there is a related unique LP problem involving the same data which also describes the original problem
- The given original problem is called the primal problem
- If the problem is rewritten by transposing the rows and columns of the algebraic statement of the problem, then we get another LP problem which we call as Dual of the problem.

- A solution to the dual problem may be found in a manner similar to that used for primal problem
- The optimal solution of the dual problem gives complete information about optimal solution of the primal problem and vice-versa

Some interesting features of Duality

- If the primal problem contains a large number of rows (constraints) and smaller number of columns (variables), then the computational procedure can be considerably reduced by converting it into dual and then solving it. Hence, it offers an advantage in many applications
- It gives additional information as to how the optimal solution changes as a result of the changes in the coefficients and the formulation of the problem.

Some interesting features (Cont…)

- It helps managers answer questions about alternative courses of actions and their relative values
- Calculation of the dual checks the accuracy of the primal solution
- Duality in linear programming shows that each linear program is equivalent to Two-person-Zero sum game. This indicates that fairly close relationship exist between linear programming and the theory of games
- Duality is not restricted to linear programming problem only but finds applications in many domains like Economics, physics and other fields.

Some interesting features (Cont…)

- Economics interpretations of the dual helps the management in making future decisions
- Duality is used to solve LP problems in which the initial solution is infeasible.

Rules for converting the primal problem into a Dual Problem

- If the primal contains n-variables and m-constraints, the dual will contain m variables and n constraints
- The maximization problem in the primal becomes the minimization problem in the dual and vice-versa
- The maximization problem has ≤ constraints while the minimization problem has ≥ constraints
- Constraints of ≤ type in the primal become ≥ type in the dual and vice-versa
- The coefficient matrix of the constraints of the dual is the transpose of the primal
- A new set of variables appear in the dual.

Rules for converting the primal problem into a Dual Problem

- The constants $c_1,c_2,c_3....,c_n$ in the objective function of the primal appear in the constraints of the dual
- The constants $b_1,b_2,b_3....,b_n$ in the constraints of the primal appear in the objective function of the dual
- The variables in both problems are non-negative.

Problem

- Construct the dual to the primal problem

Maximize $Z=3x_1+5x_2$

Subject to $2x_1+6x_2 \leq 50$

$3x_1+2x_2 \leq 35$

$5x_1-3x_2 \leq 10$

$x_2 \leq 20$ where $x_1 \geq 0$, $x_2 \geq 0$

- The dual of the given LPP is

$W=50y1+35y2+10y3+20y4$

Subject to $2y1+3y2+5y3\geq 3$

$6y1+2y2-3y3+y4\geq 5$

where $y1,y2,y3$ and $y4\geq 0$

As the dual problem has lesser number of constraints than the primal, it requires lesser work and effort to solve it.

Note: This follows from the fact that, the computational difficulty in the LPP is mainly associated with the number of constraints rather than number of variables.

- Construct the dual of the problem

  Minimize $Z=3x1-2x2+4x3$ subject to the constraints $3x1+5x2+4x3\geq 7$

  $$6x1+x2+3x3\geq 4$$

  $$7x1-2x2-x3 \leq 10$$

  $$x1-2x2+5x3\geq 3$$

  $$4x1+7x2-2x3\geq 2$$

  $$x1,x2,x3\geq 0$$

- Solution

As the given problem is minimization, All constraints should be of ≥ type. Multiplying the third constraint by -1 on both sides, we get

$$-7x1+2x2+x3\geq-10$$

So, the dual of the given problem will be

Maximize $W=7y1+4y2-10y3+3y4+2y3$

Subject to $3y1+6y2-7y3+y4+4y5\leq3$

$5y1+y2+2y3-2y4+7y5\leq-2$

$4y1+3y2+y3+5y4-2y5\leq4$ where

$y1,y2,y3,y4$ and $y5\geq0$

Assignment

- Construct the dual of the problem
- Maximize $Z=3x1+17x2+9x3$
- Subject to $x1-x2+x3\geq 3$
- $-3x1+2x3\leq 1$ where $x1,x2,x3 \geq 0$

2) Construct the dual of the problem

Minimize $Z = 3x_1 - 2x_2 + 4x_3$

Subject to the Constraints
$$3x_1 + 5x_2 + 4x_3 \geq 7$$
$$6x_1 + x_2 + 3x_3 \geq 4$$
$$7x_1 - 2x_2 - x_3 \leq 10$$
$$x_1 - 2x_2 + 5x_3 \geq 3$$
$$4x_1 + 7x_2 - 2x_3 \geq 2$$

and x_1, x_2 and $x_3 \geq 0$.

Sol: As the Given problem is of Minimization, all Constraints should be of $\geq$ type. Multiplying the third Constraint by -1 on both Sides, we get
$$-7x_1 + 2x_2 + x_3 \geq -10.$$

$\therefore$ The dual of the Given problem will be

Maximize $W = 7y_1 + 4y_2 - 10y_3 + 3y_4 + 2y_5$

Subject to the Constraints
$$3y_1 + 6y_2 + 7y_3 + y_4 + 4y_5 \leq 3$$
$$5y_1 + y_2 - 2y_3 - 2y_4 + 7y_5 \leq -2$$
$$4y_1 + 3y_2 - y_3 + 5y_4 - 2y_5 \leq 4$$

where y_1, y_2, y_3, y_4 and $y_5 \geq 0$.

Here y_1, y_2, y_3, y_4 and y_5 are the dual Variables in the 1st, 2nd, 3rd, 4th and 5th Constraints respectively.

Solution: The given problem can be written as

Maximization $Z = 5x_1 - 2x_2 + 3x_3$

Subject to $-2x_1 - 2x_2 + x_3 \leq -2$

$3x_1 - 4x_2 \leq 3$

$x_2 + 3x_3 \leq 5$

$x_1, x_2, x_3 \geq 0.$

The associated dual is given by

Minimize $W = -2y_1 + 3y_2 + 5y_3$

Subject to $-2y_1 + 3y_2 \geq 5$

$-2y_1 - 4y_2 + y_3 \geq -2$

$y_1 + 3y_3 \geq 3$

y_1, y_2 and $y_3 \geq 0.$

The Solution of the dual by Simplex method consists of the following steps.

Step 1: Express the problem in Standard form.

Multiplying the second constraint by -1, it can be written as

$2y_1 + 4y_2 - y_3 \leq 2$

Introducing the slack and surplus variables, we get an artificial system given by

minimize $W = -2y_1 + 3y_2 + 5y_3 + 0s_1 + 0s_2 + 0s_3 + MA_1 + MA_2$

Subject to $-2y_1 + 3y_2 - s_1 + A_1 = 5$ $\quad y_1 + 3y_3 - s_3 + A_2 = 3$

Dual problem when the primal is in standard form.

Problem: Construct the dual of the problem (18)

$$\text{Maximize } Z = 3x_1 + 10x_2 + 2x_3$$

$$\text{Subject to } \quad 2x_1 + 3x_2 + 2x_3 \leq 7$$

$$3x_1 - 2x_2 + 4x_3 = 3$$

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0.$$

Sol: Since the given problem is of Maximization, all constraints should be of type $\leq$. The equation $3x_1 - 2x_2 + 4x_3 = 3$ can be expressed as a pair of inequalities

$$3x_1 - 2x_2 + 4x_3 \leq 3 \text{ and } 3x_1 - 2x_2 + 4x_3 \geq 3.$$

or $\quad 3x_1 - 2x_2 + 4x_3 \leq 3$ and $-3x_1 + 2x_2 - 4x_3 \leq -3.$

$\therefore$ The primal problem becomes

$$\text{Maximize } Z = 3x_1 + 10x_2 + 2x_3$$

$$\text{Subject to } \quad 2x_1 + 3x_2 + 2x_3 \leq 7$$

$$3x_1 - 2x_2 + 4x_3 \leq 3$$

$$-3x_1 + 2x_2 - 4x_3 \leq -3.$$

where $x_1, x_2, x_3 \geq 0.$

Let y_1, y_2', y_2'' be the associated non-negative dual variables. Then the dual of the problem is,

$$\text{Minimize } W = 7y_1 + 3y_2' - 3y_2''$$

Subject to the constraint

$$2y_1 + 3y_2' - 3y_2'' \geq 3$$

$$3y_1 - 2y_2' + 2y_2'' \geq 10$$

$$2y_1 + 4y_2' - 4y_2'' \geq 2, \text{ where } y_1, y_2' \text{ and } y_2'' \geq 0.$$

Substituting $y_2' - y_2'' = y_2$, where y_2 is unrestricted in sign, the dual problem becomes,

$$\text{Minimize } W = 7y_1 + 3y_2$$

Subject to the Constraints given by

$$2y_1 + 3y_2 \geq 3$$
$$3y_1 - 2y_2 \geq 10$$
$$2y_1 + 4y_2 \geq 2, \text{ where } y_1 \geq 0, \text{ and } y_2 \text{ is unrestricted in Sign.}$$

<u>Assignment problem:</u>

Construct the dual of the problem

$$\text{Minimize } Z = x_2 + 3x_3$$

Subject to

$$2x_1 + x_2 \leq 3$$
$$x_1 + 2x_2 + 6x_3 \geq 5$$
$$-x_1 + x_2 + 2x_3 = 2$$
$$\text{and } x_1, x_2, x_3 \geq 0.$$

<u>Solving the LPP by using its dual:</u>

Solve the following LPP by using its dual

$$\text{Maximize } Z = 5x_1 - 2x_2 + 3x_3$$

Subject to

$$2x_1 + 2x_2 - x_3 \geq 2$$
$$3x_1 - 4x_2 \leq 3$$
$$x_2 + 3x_3 \leq 5$$
$$x_1, x_2, x_3 \geq 0.$$

<u>Solution</u>: The problem can be written as

$$\text{maximize } Z = 5x_1 - 2x_2 + 3x_3$$

Subject to $2x_1 + 2x_2 - x_3 \geq 2$ ... $+3x_3 \leq 5$, $x_1, x_2, x_3 \geq 0$

Example 34.26. *Using dual simplex method :*
maximize $-3x_1 - 2x_2$
subject to $x_1 + x_2 \geq 1, x_1 + x_2 \leq 7, x_1 + 2x_2 \geq 10, x_2 \geq 3, x_1 \geq 0, x_2 \geq 0.$ *(Mumbai, 2004)*

Solution. Consists of the following steps :

Step 1. (i) Convert the first and third constraints into (≤) type. These constraints become

$$-x_1 - x_2 \leq -1, -x_1 - 2x_2 \leq -10.$$

(ii) Express the problem in standard form

Introducing slack variables s_1, s_2, s_3, s_4 the given problem takes the form

Max. $Z = -3x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$

subject to $-x_1 - x_2 + s_1 = -1, x_1 + x_2 + s_2 = 7, -x_1 - 2x_2 + s_3 = -10,$

$x_2 + s_4 = 3, x_1, x_2, s_1, s_2, s_3, s_4 \geq 0.$

Step 2. Find the initial basic solution

Setting the decision variables x_1, x_2 each equal to zero, we get the basic solution

$$x_1 = x_2 = 0, s_1 = -1, s_2 = 7, s_3 = -10, s_4 = 3 \text{ and } Z = 0.$$

∴ Initial solution is given by the table below :

	c_j	-3	-2	0	0	0	0	
c_B	*Basis*	x_1	x_2	s_1	s_2	s_3	s_4	b
0	s_1	-1	-1	1	0	0	0	-1
0	s_2	1	1	0	1	0	0	7
0	s_3	-1	(-2)	0	0	1	0	$-10 \leftarrow$
0	s_4	0	1	0	0	0	1	3
	$Z_j = \Sigma c_B a_{ij}$	0	0	0	0	0	0	0
	$C_j = c_j - Z_j$	-3	-2	0	0	0	0	
			$\uparrow$					

Step 3. Test nature of C_j

Since all C_j values are ≤ 0 and $b_1 = -1$, $b_3 = -10$, the initial solution is optimal but infeasible. We therefore, proceed further.

Step 4. Mark the outgoing variable.

Since b_3 is negative and numerically largest, the third row is the key row and s_3 is the outgoing variable.

Step 5. Calculate ratios of elements in C_j-row to the corresponding negative elements of the key row.

These ratios are $-3/-1 = 3, -2/-2 = 1$ (neglecting ratios corresponding to +ve or zero elements of key row). Since the smaller ratio is 1, therefore, x_2-column is the key column and (-2) is the key element.

Step 6. Iterate towards optimal feasible solution.

(i) Drop s_3 and introduce x_2 alongwith its associated value -2 under c_B column. Convert the key element to unity and make all other elements of the key column zero. Then the second solution is given by the table below :

Example 34.26. *Using dual simplex method :*
maximize $-3x_1 - 2x_2$
subject to $x_1 + x_2 \geq 1, x_1 + x_2 \leq 7, x_1 + 2x_2 \geq 10, x_2 \geq 3, x_1 \geq 0, x_2 \geq 0.$ *(Mumbai, 2004)*

Solution. Consists of the following steps :

Step 1. (i) Convert the first and third constraints into (≤) type. These constraints become

$$-x_1 - x_2 \leq -1, -x_1 - 2x_2 \leq -10.$$

(ii) Express the problem in standard form

Introducing slack variables s_1, s_2, s_3, s_4 the given problem takes the form

Max. $Z = -3x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$

subject to $-x_1 - x_2 + s_1 = -1, x_1 + x_2 + s_2 = 7, -x_1 - 2x_2 + s_3 = -10,$

$x_2 + s_4 = 3, x_1, x_2, s_1, s_2, s_3, s_4 \geq 0.$

Step 2. Find the initial basic solution

Setting the decision variables x_1, x_2 each equal to zero, we get the basic solution

$$x_1 = x_2 = 0, s_1 = -1, s_2 = 7, s_3 = -10, s_4 = 3 \text{ and } Z = 0.$$

∴ Initial solution is given by the table below :

	c_j	-3	-2	0	0	0	0	
c_B	*Basis*	x_1	x_2	s_1	s_2	s_3	s_4	b

SESSINON-5 SENSITIVITY ANALYSIS

SESSION-5

Example 34.26. *Using dual simplex method :*
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subject to $x_1 + x_2 \geq 1, x_1 + x_2 \leq 7, x_1 + 2x_2 \geq 10, x_2 \geq 3, x_1 \geq 0, x_2 \geq 0.$ *(Mumbai, 2004)*

Solution. Consists of the following steps :

Step 1. (i) Convert the first and third constraints into ($\leq$) *type. These constraints become*

$$-x_1 - x_2 \leq -1, -x_1 - 2x_2 \leq -10.$$

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Introducing slack variables s_1, s_2, s_3, s_4 the given problem takes the form

Max. $Z = -3x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$

subject to $-x_1 - x_2 + s_1 = -1, x_1 + x_2 + s_2 = 7, -x_1 - 2x_2 + s_3 = -10,$
$x_2 + s_4 = 3, x_1, x_2, s_1, s_2, s_3, s_4 \geq 0.$

Step 2. Find the initial basic solution

Setting the decision variables x_1, x_2 each equal to zero, we get the basic solution

$$x_1 = x_2 = 0, s_1 = -1, s_2 = 7, s_3 = -10, s_4 = 3 \text{ and } Z = 0.$$

$\therefore$ Initial solution is given by the table below :

	c_j	-3	-2	0	0	0	0	
c_B	Basis	x_1	x_2	s_1	s_2	s_3	s_4	b
0	s_1	-1	-1	1	0	0	0	-1
0	s_2	1	1	0	1	0	0	7
0	s_3	-1	(-2)	0	0	1	0	$-10 \leftarrow$
0	s_4	0	1	0	0	0	1	3
	$Z_j = \Sigma c_B a_{ij}$	0	0	0	0	0	0	0
	$C_j = c_j - Z_j$	-3	-2	0	0	0	0	
			$\uparrow$					

Step 3. Test nature of C_j

Since all C_j values are ≤ 0 and $b_1 = -1$, $b_3 = -10$, the initial solution is optimal but infeasible. We therefore, proceed further.

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These ratios are $-3/-1 = 3, -2/-2 = 1$ (neglecting ratios corresponding to +ve or zero elements of key row). Since the smaller ratio is 1, therefore, x_2-column is the key column and (-2) is the key element.

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(i) Drop s_3 and introduce x_2 alongwith its associated value -2 under c_B column. Convert the key element to unity and make all other elements of the key column zero. Then the second solution is given by the table below :

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Solution. Consists of the following steps :

Step 1. (i) Convert the first and third constraints into ($\leq$) *type. These constraints become*

$$-x_1 - x_2 \leq -1, -x_1 - 2x_2 \leq -10.$$

(ii) Express the problem in standard form

Introducing slack variables s_1, s_2, s_3, s_4 the given problem takes the form

Max. $Z = -3x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$

subject to $-x_1 - x_2 + s_1 = -1, x_1 + x_2 + s_2 = 7, -x_1 - 2x_2 + s_3 = -10,$
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Step 2. Find the initial basic solution

Setting the decision variables x_1, x_2 each equal to zero, we get the basic solution

$$x_1 = x_2 = 0, s_1 = -1, s_2 = 7, s_3 = -10, s_4 = 3 \text{ and } Z = 0.$$

$\therefore$ Initial solution is given by the table below :

	c_j	-3	-2	0	0	0	0	
c_B	Basis	x_1	x_2	s_1	s_2	s_3	s_4	b

Sensitivity analysis is carried out after the optimum solution of the LP problem is obtained. The goal is to determine whether changes in the given model's coefficients will leave the current solution unchanged, and if not, how a new optimum (assuming one exists) can be obtained efficiently.

Definition of Sensitivity Analysis Sensitivity analysis in LPP refers to the sensitivity of the solution towards changes in the techno-economic composition, changes in resources, changes in profit composition, and addition of new constraints. If these changes have no effect on the optional solution, the solution is said to be insensitive.

The sensitivity analysis mainly focuses on:

1. Changes affecting feasibility
2. Changes affecting optimality

1. Changes affecting feasibility The feasibility of the current optimum solution may be affected only if

(a) The right hand side of the constraints b is changed or
(b) A new constraint is added to the model.

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maximize $-3x_1 - 2x_2$
subject to $x_1 + x_2 \geq 1, x_1 + x_2 \leq 7, x_1 + 2x_2 \geq 10, x_2 \geq 3, x_1 \geq 0, x_2 \geq 0.$ (*Mumbai, 2004*)

Solution. Consists of the following steps :

Step 1. (i) Convert the first and third constraints into (≤) type. These constraints become

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∴ Initial solution is given by the table below :

	c_j	-3	-2	0	0	0	0	
c_B	*Basis*	x_1	x_2	s_1	s_2	s_3	s_4	b
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0	s_2	1	1	0	1	0	0	7
0	s_3	-1	(-2)	0	0	1	0	$-10 \leftarrow$
0	s_4	0	1	0	0	0	1	3
	$Z_j = \Sigma c_B a_{ij}$	0	0	0	0	0	0	0
	$C_j = c_j - Z_j$	-3	-2	0	0	0	0	
			$\uparrow$					

Step 3. Test nature of C_j

Since all C_j values are ≤ 0 and $b_1 = -1$, $b_3 = -10$, the initial solution is optimal but infeasible. We therefore, proceed further.

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Solution. Consists of the following steps :

Step 1. (i) Convert the first and third constraints into (≤) type. These constraints become

$$-x_1 - x_2 \leq -1, -x_1 - 2x_2 \leq -10.$$

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Introducing slack variables s_1, s_2, s_3, s_4 the given problem takes the form

Max. $Z = -3x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$

subject to $-x_1 - x_2 + s_1 = -1, x_1 + x_2 + s_2 = 7, -x_1 - 2x_2 + s_3 = -10,$
$x_2 + s_4 = 3, x_1, x_2, s_1, s_2, s_3, s_4 \geq 0.$

Step 2. Find the initial basic solution

Setting the decision variables x_1, x_2 each equal to zero, we get the basic solution

$$x_1 = x_2 = 0, s_1 = -1, s_2 = 7, s_3 = -10, s_4 = 3 \text{ and } Z = 0.$$

∴ Initial solution is given by the table below :

	c_j	-3	-2	0	0	0	0	
c_B	*Basis*	x_1	x_2	s_1	s_2	s_3	s_4	b

The optimality of the solution can be affected only when we change the objective coefficients C_j (and, hence, C_B) or the unit resource usage vector P_j. The effect of making changes in C_j on optimality entails recomputing $Z_j - C_j$ for the non-basic variables only.

Computational Procedure

1. Compute the dual prices vector $Y = C_B B^{-1}$ using the new vector C_B, if it has been changed.
2. Compute $Z_j - C_j = YP_j - C_j$ for all the current non-basic X_j.
   Two cases will result.
   - If the optimality condition is satisfied the current solution will remain the same, but at a new optimum value of the objective function. However, if C_B is unchanged, the optimum objective value will remain the same.
   - If the optimality condition is not satisfied, we apply the (primal) simplex method to recover optimality.

Example 34.26. *Using dual simplex method :*
maximize $-3x_1 - 2x_2$
subject to $x_1 + x_2 \geq 1, x_1 + x_2 \leq 7, x_1 + 2x_2 \geq 10, x_2 \geq 3, x_1 \geq 0, x_2 \geq 0.$ *(Mumbai, 2004)*

Solution. Consists of the following steps :

Step 1. (i) Convert the first and third constraints into (≤) type. These constraints become

$$-x_1 - x_2 \leq -1, -x_1 - 2x_2 \leq -10.$$

(ii) Express the problem in standard form

Introducing slack variables s_1, s_2, s_3, s_4 the given problem takes the form

Max. $Z = -3x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$

subject to $-x_1 - x_2 + s_1 = -1, x_1 + x_2 + s_2 = 7, -x_1 - 2x_2 + s_3 = -10,$
$x_2 + s_4 = 3, x_1, x_2, s_1, s_2, s_3, s_4 \geq 0.$

Step 2. Find the initial basic solution

Setting the decision variables x_1, x_2 each equal to zero, we get the basic solution

$$x_1 = x_2 = 0, s_1 = -1, s_2 = 7, s_3 = -10, s_4 = 3 \text{ and } Z = 0.$$

∴ Initial solution is given by the table below :

	c_j	-3	-2	0	0	0	0	
c_B	*Basis*	x_1	x_2	s_1	s_2	s_3	s_4	b
0	s_1	-1	-1	1	0	0	0	-1
0	s_2	1	1	0	1	0	0	7
0	s_3	-1	(-2)	0	0	1	0	$-10 \leftarrow$
0	s_4	0	1	0	0	0	1	3
	$Z_j = \Sigma c_B a_{ij}$	0	0	0	0	0	0	0
	$C_j = c_j - Z_j$	-3	-2	0	0	0	0	
			↑					

Step 3. Test nature of C_j

Since all C_j values are ≤ 0 and $b_1 = -1$, $b_3 = -10$, the initial solution is optimal but infeasible. We therefore, proceed further.

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Example 34.26. *Using dual simplex method :*
maximize $-3x_1 - 2x_2$
subject to $x_1 + x_2 \geq 1, x_1 + x_2 \leq 7, x_1 + 2x_2 \geq 10, x_2 \geq 3, x_1 \geq 0, x_2 \geq 0.$ *(Mumbai, 2004)*

Solution. Consists of the following steps :

Step 1. (i) Convert the first and third constraints into (≤) type. These constraints become

$$-x_1 - x_2 \leq -1, -x_1 - 2x_2 \leq -10.$$

(ii) Express the problem in standard form

Introducing slack variables s_1, s_2, s_3, s_4 the given problem takes the form

Max. $Z = -3x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$

subject to $-x_1 - x_2 + s_1 = -1, x_1 + x_2 + s_2 = 7, -x_1 - 2x_2 + s_3 = -10,$
$x_2 + s_4 = 3, x_1, x_2, s_1, s_2, s_3, s_4 \geq 0.$

Step 2. Find the initial basic solution

Setting the decision variables x_1, x_2 each equal to zero, we get the basic solution

$$x_1 = x_2 = 0, s_1 = -1, s_2 = 7, s_3 = -10, s_4 = 3 \text{ and } Z = 0.$$

∴ Initial solution is given by the table below :

	c_j	-3	-2	0	0	0	0	
c_B	*Basis*	x_1	x_2	s_1	s_2	s_3	s_4	b

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0	s_3	-1	(-2)	0	0	1	0	$-10 \leftarrow$
0	s_4	0	1	0	0	0	1	3
	$Z_j = \Sigma c_B a_{ij}$	0	0	0	0	0	0	0
	$C_j = c_j - Z_j$	-3	-2	0	0	0	0	
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	c_j	-3	-2	0	0	0	0	
c_B	Basis	x_1	x_2	s_1	s_2	s_3	s_4	b

Changes in the cost coefficient C_j Suppose any changes are made in the cost co-efficient C_j or C_k, it does not affect the optimality condition but it affects basic feasible solution. If the optimum feasible solution (X_B) = Maximise $Z = CX$, subject to the constraints, $AX = b$ and $X \geq 0$, then, changes are made in C_k, that is, ΔC_k (amount added to k^{th} component). The new value of the k^{th} component is $C_K^* = C_K + \Delta C_K$.

Since $X_B = B^{-1}b$ is independent of C, and changes in C will not affect the value of X_B. Therefore, the current solution X_B remains the same.

If C_B is the cost vector associated with the optimum basic feasible solution X_B, then any change in C will affect the optimality condition.

The reason is that $Z_k - C_k = C_B y_k$ where, $k = 1, 2, ..., n$

There are two possibilities in case of variation in the cost vector

(i) C_k is not in C_B, and (ii) C_k is in C_B.

Case (I) *C_k is not in C_B*: It means that C_k is not the coefficient of basic variable in the objective function. The net evaluation, with respect to the non-basic variable X_k, is given below:

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Step 1. (i) Convert the first and third constraints into (≤) type. These constraints become

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Introducing slack variables s_1, s_2, s_3, s_4 the given problem takes the form

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0	s_3	-1	(-2)	0	0	1	0	$-10 \leftarrow$
0	s_4	0	1	0	0	0	1	3
	$Z_j = \Sigma c_B a_{ij}$	0	0	0	0	0	0	0
	$C_j = c_j - Z_j$	-3	-2	0	0	0	0	
			$\uparrow$					

Step 3. Test nature of C_j

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0	s_4	0	1	0	0	0	1	3
	$Z_j = \Sigma c_B a_{ij}$	0	0	0	0	0	0	0
	$C_j = c_j - Z_j$	-3	-2 ↑	0	0	0	0	

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Since all C_j values are ≤ 0 and $b_1 = -1$, $b_3 = -10$, the initial solution is optimal but infeasible. We therefore, proceed further.

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c_B	*Basis*	x_1	x_2	s_1	s_2	s_3	s_4	b

$$Z_j^* - C_j = \sum_{\substack{i=1 \\ i \neq k}}^{m} C_{Bi} Y_{ij} + (C_{Bk} + \Delta C_{Bk}) Y_{ij} - C_j$$

$$= \sum_{i=1}^{m} C_{Bi} Y_{ij} - C_j + \Delta C_{Bk} Y_{kj}$$

$$= (Z_j - C_j) + \Delta C_k Y_{kj} \qquad \because C_{Bk} = C_k$$

The current basic feasible solution will remain optimum when $Z_j^* - C_j \geq 0$.

That is, $(Z_j - C_j) + \Delta C_k Y_{kj} \geq 0$ or $\Delta C_k Y_{kj} \geq -(Z_j - C_j)$

$$\Delta C_k \geq \frac{-(Z_j - C_j)}{Y_{kj}} \text{ if } Y_{kj} > 0 \text{ and } \Delta C_k \leq \frac{-(Z_j - C_j)}{Y_{kj}} \text{ if } Y_{kj} < 0$$

That is, Maximise $\left\{\frac{-(Z_i - C_j)}{}\right\} \leq \Delta C_k \leq$ Minimise $\left\{\frac{-(Z_i - C_j)}{}\right\}$

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c_B	Basis	x_1	x_2	s_1	s_2	s_3	s_4	b

Illustration 1: Consider the following LPP.

Maximise $Z = 5x_1 + 3x_2$

Subject to,

$$3x_1 + 5x_2 \leq 15$$
$$5x_1 + 6x_2 \leq 10$$
$$x_1, x_2 \geq 0$$

(a) Solve the LPP.

(b) Find how far the component C_1 of C can be increased without affecting the optimality of the solution.

Solution:

(a) First we convert the inequalities into equalities by adding the non-negative slack variables s_1 and s_2. The optimal simplex table is given below:

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c_B	*Basis*	x_1	x_2	s_1	s_2	s_3	s_4	b
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0	s_2	1	1	0	1	0	0	7
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0	s_4	0	1	0	0	0	1	3
	$Z_j = \Sigma c_B a_{ij}$	0	0	0	0	0	0	0
	$C_j = c_j - Z_j$	-3	-2	0	0	0	0	
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subject to $x_1 + x_2 \geq 1, x_1 + x_2 \leq 7, x_1 + 2x_2 \geq 10, x_2 \geq 3, x_1 \geq 0, x_2 \geq 0.$ *(Mumbai, 2004)*

Solution. Consists of the following steps :

Step 1. (i) Convert the first and third constraints into (≤) type. These constraints become

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(ii) Express the problem in standard form

Introducing slack variables s_1, s_2, s_3, s_4 the given problem takes the form

Max. $Z = -3x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$

subject to $-x_1 - x_2 + s_1 = -1, x_1 + x_2 + s_2 = 7, -x_1 - 2x_2 + s_3 = -10,$
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Step 2. Find the initial basic solution

Setting the decision variables x_1, x_2 each equal to zero, we get the basic solution

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∴ Initial solution is given by the table below :

	c_j	-3	-2	0	0	0	0	
c_B	*Basis*	x_1	x_2	s_1	s_2	s_3	s_4	b

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0	s_3	-1	(-2)	0	0	1	0	$-10 \leftarrow$
0	s_4	0	1	0	0	0	1	3
	$Z_j = \Sigma c_B a_{ij}$	0	0	0	0	0	0	0
	$C_j = c_j - Z_j$	-3	-2	0	0	0	0	
			$\uparrow$					

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c_B	*Basis*	x_1	x_2	s_1	s_2	s_3	s_4	b

Optimal Simplex Table

C_j	5	3	0	0			
X_j	x_1	x_2	s_1	s_2	X_B	Y_B	C_B
	0	$\frac{7}{5}$	1	$-\frac{3}{5}$	9	s_1	0
	1	$\frac{6}{5}$	0	$\frac{1}{5}$	2.	x_1	5
Z_j	5	6	0	1	10		
$Z_j - C_j$	0	3	0	1	10		

Since all $Z_j - C_j \geq 0$, then optimum solution is obtained. Therefore the optimal solution is Maximise $Z = 10$; $x_1 = 2, x_2 = 0$.

(b) Here, $C_1 \in C_B$, i.e., C_1 is the co-efficient of the basic variable x_1 and $C_1 = 5$.

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Illustration 2: Given the linear programming problem:

Maximise $Z = 3x_1 + 5x_2$

subject to, $x_1 + x_2 \leq 1$

$2x_1 + 2x_2 \leq 1$

$x_1, x_2 \geq 0$

obtain variations in C_j ($j = 1, 2$), which are permitted without changing the optimum solutions.

Solution: First, convert the inequalities into equalities by adding slack variables $s_1 \geq 0$ and $s_2 \geq 0$ and then solving the LPP by the simplex method.

Initial iteration

C_j	3	5	0	0			
X_j	x_1	x_2	s_1	s_2	X_B	Y_B	C_B
	1/3	0	1	$-1/3$	2/3	s_1	0

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Case 1: Variation in C_1

When C_k is not in C_B ($C_2 \notin C_B$), the current solution results in the same optimum solution, i.e., $\Delta C_1 \leq Z_1 - C_1$ or $\Delta C_1 \leq 1/3$.

∴ the range of ΔC_1 is given by

$$-\infty \leq C_1 \leq C_1 + \Delta C_1$$

$$-\infty \leq C_1 \leq 3 + 1/3, \text{ i.e., } -\infty \leq C_1 \leq 10/3.$$

Case 2: Variation in C_2

When C_k is in C_B ($C_2 \in C_B$), the range of ΔC_2 is given by: $\underset{Y_{2j}>0}{\text{Max}} \left\{\dfrac{-(Z_j - C_j)}{Y_{2j}}\right\} \leq C_2 \underset{Y_{2j}>0}{\text{Min}} \left\{\dfrac{-(Z_j - C_j)}{Y_{2j}}\right\}$

$[-1/3 \;\; -5/3]$

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0	s_3	-1	(-2)	0	0	1	0	-10 ←
0	s_4	0	1	0	0	0	1	3
	$Z_j = \Sigma c_B a_{ij}$	0	0	0	0	0	0	0
	$C_j = c_j - Z_j$	-3	-2 ↑	0	0	0	0	

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PROBLEMS:

1. Solve the problem and discuss sensitivity analysis of changing c vector(cost values of decision variables

Maximize $z = 2x_1 + 3x_2$,

subject to $x_1 + 3x_2 \leq 6$,

$3x_1 + 2x_2 \leq 6$,

and $x_1, x_2 \geq 0$.

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subject to $x_1 + x_2 \geq 1, x_1 + x_2 \leq 7, x_1 + 2x_2 \geq 10, x_2 \geq 3, x_1 \geq 0, x_2 \geq 0.$ *(Mumbai, 2004)*

Solution. Consists of the following steps :

Step 1. (i) Convert the first and third constraints into (≤) type. These constraints become

$$-x_1 - x_2 \leq -1, -x_1 - 2x_2 \leq -10.$$

(ii) Express the problem in standard form

Introducing slack variables s_1, s_2, s_3, s_4 the given problem takes the form

Max. $Z = -3x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$

subject to $-x_1 - x_2 + s_1 = -1, x_1 + x_2 + s_2 = 7, -x_1 - 2x_2 + s_3 = -10,$
$x_2 + s_4 = 3, x_1, x_2, s_1, s_2, s_3, s_4 \geq 0.$

Step 2. Find the initial basic solution

Setting the decision variables x_1, x_2 each equal to zero, we get the basic solution

$$x_1 = x_2 = 0, s_1 = -1, s_2 = 7, s_3 = -10, s_4 = 3 \text{ and } Z = 0.$$

∴ Initial solution is given by the table below :

	c_j	-3	-2	0	0	0	0	
c_B	*Basis*	x_1	x_2	s_1	s_2	s_3	s_4	b
0	s_1	-1	-1	1	0	0	0	-1
0	s_2	1	1	0	1	0	0	7
0	s_3	-1	(-2)	0	0	1	0	-10 ←
0	s_4	0	1	0	0	0	1	3
	$Z_j = \Sigma c_B a_{ij}$	0	0	0	0	0	0	0
	$C_j = c_j - Z_j$	-3	-2	0	0	0	0	
			↑					

Step 3. Test nature of C_j

Since all C_j values are ≤ 0 and $b_1 = -1$, $b_3 = -10$, the initial solution is optimal but infeasible. We therefore, proceed further.

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These ratios are $-3/-1 = 3, -2/-2 = 1$ (neglecting ratios corresponding to +ve or zero elements of key row). Since the smaller ratio is 1, therefore, x_2-column is the key column and (-2) is the key element.

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c_B	*Basis*	x_1	x_2	s_1	s_2	s_3	s_4	b

2. Solve using the Simplex method and discuss sensitivity analysis of changing c vector(cost values 0f decision variables

Maximize $z = 16x_1 + 15x_2$,

Subject to: $40x_1 + 31x_2 \leq 124$, $-x_1 + x_2 \leq 1$, $x_1 \leq 3$, $x_1, x_2 \geq 0$.

SESSSIN -6

. Variation in the right side of constraints Since the optimality condition $z_j - c_j$ does not involve

Consider the LPP

Maximise $z = 5x_1 + 12x_2 + 4x_3$

subject to
$$x_1 + 2x_2 + x_3 \le 5$$
$$2x_1 - x_2 + 3x_3 = 2$$
$$x_1, x_2, x_3 \ge 0$$

(i) Discuss the effect of changing the requirement vector from $\begin{bmatrix}5\\2\end{bmatrix}$ to $\begin{bmatrix}7\\2\end{bmatrix}$ on the optimum solution.

(ii) Discuss the effect of changing the requirement vector from $\begin{bmatrix}5\\2\end{bmatrix}$ to $\begin{bmatrix}3\\9\end{bmatrix}$ on the optimum solution.

SOLUTION: Big- M method final table is as follows

		c_j	5	12	4	0	0
C_B	Y_B	X_B	x_1	x_2	x_3	s_1	A_1
12	x_2	8/5	0	1	- 1/5	2/5	- 1/5
5	x_1	9/5	1	0	7/5	1/5	2/5
$z_j - c_j$			0	0	3/5	29/5	M - 2/5

and the optimal solution is

$$\boxed{x_1 = 9/5,\ x_2 = 8/5,\ x_3 = 0 \text{ and maximum } z = 141/5}$$

(i) If the requirement vector changes to $\begin{bmatrix}7\\2\end{bmatrix}$ then the new values of the current basic variables are given by

$$\begin{bmatrix}x_2\\x_1\end{bmatrix} = B^{-1}b = \begin{bmatrix}2/5 & -1/5\\1/5 & 2/5\end{bmatrix}\begin{bmatrix}7\\2\end{bmatrix} = \begin{bmatrix}12/5\\11/5\end{bmatrix}$$

Hence, the new solution is optimal and feasible and the new optimum solution is

$$\boxed{x_1 = 11/5,\ x_2 = 12/5 \text{ with maximum of } z = 5 \times 11/5 + 12 \times 12/5 + 4 \times 0 = 199/5}$$

(ii) If the requirement vector changes to $\begin{bmatrix}3\\9\end{bmatrix}$ then the new values of the basic variables are given by

$$\begin{bmatrix}x_2\\x_1\end{bmatrix} = B^{-1}b = \begin{bmatrix}2/5 & -1/5\\1/5 & 2/5\end{bmatrix}\begin{bmatrix}3\\9\end{bmatrix} = \begin{bmatrix}-3/5\\21/5\end{bmatrix}$$

Hence, the current optimal solution becomes infeasible. Hence, we have to apply the dual simplex method.

			5	12	4	0	0
C_B	Y_B	X_B	x_1	x_2	$x_3\downarrow$	s_1	A_1
12	$\bar{x}_2$	- 3/5	0	1	$\boxed{-1/5}$	2/5	- 1/5
5	x_1	21/5	1	0	7/5	1/5	2/5
			0	0	3/5	29/5	M - 2/5

Introduce x_3 into the basis and drop x_2 from the basis then the next iteration is

		c_i	5	12	4	0	0
C_B	Y_B	X_B	x_1	x_2	x_3	s_1	A_1
4	x_3	3	0	- 5	1	- 2	1
5	x_1	0	1	7	0	3	- 1
$x_j - c_j$			0	3	0	7	M - 1

Since all $z_j - c_j \geq 0$ and all $X_{B_i} \geq 0$ the current solution is optimum. The optimal solution is

$$\boxed{x_1 = 0, x_2 = 0, x_3 = 3 \text{ with maximum of } z = 5(0) + 12(0) + 4 \times 3 = 12}$$

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	$Z_j = \Sigma c_B a_{ij}$	0	0	0	0	0	0	0
	$C_j = c_j - Z_j$	-3	-2	0	0	0	0	
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Step 3. Test nature of C_j

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c_B	*Basis*	x_1	x_2	s_1	s_2	s_3	s_4	b

SESSION -7

ADD ION OF NEW CONSTRAINT:

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Addition of a new constraint may or may not affect the feasibility of the current optimal solution. The addition of a new constraint can result in one of the two conditions.

(i) If the new constraint is satisfied by the current optimal solution, then this new constraint is said to be redundant and its addition will not change the solution. That is, the current solution remains feasible as well as optimal.

(ii) If the new constraint is not satisfied by the current optimal solution, the solution becomes infeasible. In this case the new solution is obtained by using the dual simplex method.

However, in general, whenever a new constraint is added to a linear programming problem, the old optimal value will always be better or at least equal to the new optimal value. In other words, the addition of a new constraint cannot improve the optimal value of any LPP.

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EX:

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$$x_1 = x_2 = 0, s_1 = -1, s_2 = 7, s_3 = -10, s_4 = 3 \text{ and } Z = 0.$$

$\therefore$ Initial solution is given by the table below :

	c_j	-3	-2	0	0	0	0	
c_B	*Basis*	x_1	x_2	s_1	s_2	s_3	s_4	b

Consider the LPP

Maximise $z = 4x_1 + 6x_2 + 2x_3,$

subject to

$$\left.\begin{aligned} x_1 + x_2 + x_3 &\leq 3 \\ x_1 + 4x_2 + 7x_3 &\leq 9 \\ x_1, x_2, x_3 &\leq 0 \end{aligned}\right\}$$

If we solve the problem by simplex method then the optimal simplex table will be

Example 34.26. *Using dual simplex method :*
maximize $-3x_1 - 2x_2$
subject to $x_1 + x_2 \geq 1, x_1 + x_2 \leq 7, x_1 + 2x_2 \geq 10, x_2 \geq 3, x_1 \geq 0, x_2 \geq 0.$ *(Mumbai, 2004)*

Solution. Consists of the following steps :

Step 1. (i) Convert the first and third constraints into (≤) type. These constraints become

$$-x_1 - x_2 \leq -1, -x_1 - 2x_2 \leq -10.$$

(ii) Express the problem in standard form

Introducing slack variables s_1, s_2, s_3, s_4 the given problem takes the form

Max. $Z = -3x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$

subject to $-x_1 - x_2 + s_1 = -1, x_1 + x_2 + s_2 = 7, -x_1 - 2x_2 + s_3 = -10,$

$x_2 + s_4 = 3, x_1, x_2, s_1, s_2, s_3, s_4 \geq 0.$

Step 2. Find the initial basic solution

Setting the decision variables x_1, x_2 each equal to zero, we get the basic solution

$$x_1 = x_2 = 0, s_1 = -1, s_2 = 7, s_3 = -10, s_4 = 3 \text{ and } Z = 0.$$

∴ Initial solution is given by the table below :

	c_j	-3	-2	0	0	0	0	
c_B	Basis	x_1	x_2	s_1	s_2	s_3	s_4	b
0	s_1	-1	-1	1	0	0	0	-1
0	s_2	1	1	0	1	0	0	7
0	s_3	-1	(-2)	0	0	1	0	-10 ←
0	s_4	0	1	0	0	0	1	3
	$Z_j = \Sigma c_B a_{ij}$	0	0	0	0	0	0	0
	$C_j = c_j - Z_j$	-3	-2 ↑	0	0	0	0	

Step 3. Test nature of C_j

Since all C_j values are ≤ 0 and $b_1 = -1$, $b_3 = -10$, the initial solution is optimal but infeasible. We therefore, proceed further.

Step 4. Mark the outgoing variable.

Since b_3 is negative and numerically largest, the third row is the key row and s_3 is the outgoing variable.

Step 5. Calculate ratios of elements in C_j-row to the corresponding negative elements of the key row.

These ratios are $-3/-1 = 3, -2/-2 = 1$ (neglecting ratios corresponding to +ve or zero elements of key row). Since the smaller ratio is 1, therefore, x_2-column is the key column and (-2) is the key element.

Step 6. Iterate towards optimal feasible solution.

(i) Drop s_3 and introduce x_2 alongwith its associated value -2 under c_B column. Convert the key element to unity and make all other elements of the key column zero. Then the second solution is given by the table below :

Example 34.26. *Using dual simplex method :*
maximize $-3x_1 - 2x_2$
subject to $x_1 + x_2 \geq 1, x_1 + x_2 \leq 7, x_1 + 2x_2 \geq 10, x_2 \geq 3, x_1 \geq 0, x_2 \geq 0.$ *(Mumbai, 2004)*

Solution. Consists of the following steps :

Step 1. (i) Convert the first and third constraints into (≤) type. These constraints become

$$-x_1 - x_2 \leq -1, -x_1 - 2x_2 \leq -10.$$

(ii) Express the problem in standard form

Introducing slack variables s_1, s_2, s_3, s_4 the given problem takes the form

Max. $Z = -3x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$

subject to $-x_1 - x_2 + s_1 = -1, x_1 + x_2 + s_2 = 7, -x_1 - 2x_2 + s_3 = -10,$

$x_2 + s_4 = 3, x_1, x_2, s_1, s_2, s_3, s_4 \geq 0.$

Step 2. Find the initial basic solution

Setting the decision variables x_1, x_2 each equal to zero, we get the basic solution

$$x_1 = x_2 = 0, s_1 = -1, s_2 = 7, s_3 = -10, s_4 = 3 \text{ and } Z = 0.$$

∴ Initial solution is given by the table below :

	c_j	-3	-2	0	0	0	0	
c_B	Basis	x_1	x_2	s_1	s_2	s_3	s_4	b

		C_i	4	6	2	0	0
C_B	Y_B	X_B	x_1	x_2	x_3	s_1	s_2
4	x_1	1	1	0	-1	4/3	$-1/3$
6	x_2	2	0	1	2	1/3	1/3
			0	0	6	10/3	2/3

and the optimal solution will be

Example 34.26. *Using dual simplex method :*
maximize $-3x_1 - 2x_2$
subject to $x_1 + x_2 \ge 1, x_1 + x_2 \le 7, x_1 + 2x_2 \ge 10, x_2 \ge 3, x_1 \ge 0, x_2 \ge 0.$ (*Mumbai, 2004*)

Solution. Consists of the following steps :

Step 1. (i) Convert the first and third constraints into (≤) type. These constraints become

$$-x_1 - x_2 \le -1, -x_1 - 2x_2 \le -10.$$

(ii) Express the problem in standard form

Introducing slack variables s_1, s_2, s_3, s_4 the given problem takes the form

Max. $Z = -3x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$

subject to $-x_1 - x_2 + s_1 = -1, x_1 + x_2 + s_2 = 7, -x_1 - 2x_2 + s_3 = -10,$
$x_2 + s_4 = 3, x_1, x_2, s_1, s_2, s_3, s_4 \ge 0.$

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∴ Initial solution is given by the table below :

	c_j	-3	-2	0	0	0	0	
c_B	Basis	x_1	x_2	s_1	s_2	s_3	s_4	b
0	s_1	-1	-1	1	0	0	0	-1
0	s_2	1	1	0	1	0	0	7
0	s_3	-1	(-2)	0	0	1	0	$-10 \leftarrow$
0	s_4	0	1	0	0	0	1	3
	$Z_j = \Sigma c_B a_{ij}$	0	0	0	0	0	0	0
	$C_j = c_j - Z_j$	-3	-2	0	0	0	0	
			$\uparrow$					

Step 3. Test nature of C_j

Since all C_j values are ≤ 0 and $b_1 = -1$, $b_3 = -10$, the initial solution is optimal but infeasible. We therefore, proceed further.

Step 4. Mark the outgoing variable.

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Step 6. Iterate towards optimal feasible solution.

(*i*) Drop s_3 and introduce x_2 alongwith its associated value -2 under c_B column. Convert the key element to unity and make all other elements of the key column zero. Then the second solution is given by the table below :

Example 34.26. *Using dual simplex method :*
maximize $-3x_1 - 2x_2$
subject to $x_1 + x_2 \ge 1, x_1 + x_2 \le 7, x_1 + 2x_2 \ge 10, x_2 \ge 3, x_1 \ge 0, x_2 \ge 0.$ (*Mumbai, 2004*)

Solution. Consists of the following steps :

Step 1. (i) Convert the first and third constraints into (≤) type. These constraints become

$$-x_1 - x_2 \le -1, -x_1 - 2x_2 \le -10.$$

(ii) Express the problem in standard form

Introducing slack variables s_1, s_2, s_3, s_4 the given problem takes the form

Max. $Z = -3x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$

subject to $-x_1 - x_2 + s_1 = -1, x_1 + x_2 + s_2 = 7, -x_1 - 2x_2 + s_3 = -10,$
$x_2 + s_4 = 3, x_1, x_2, s_1, s_2, s_3, s_4 \ge 0.$

Step 2. Find the initial basic solution

Setting the decision variables x_1, x_2 each equal to zero, we get the basic solution

$$x_1 = x_2 = 0, s_1 = -1, s_2 = 7, s_3 = -10, s_4 = 3 \text{ and } Z = 0.$$

∴ Initial solution is given by the table below :

	c_j	-3	-2	0	0	0	0	
c_B	Basis	x_1	x_2	s_1	s_2	s_3	s_4	b

$$\boxed{x_1 = 1, x_2 = 2, x_3 = 0 \text{ with maximum of } z = 4 \times 1 + 6 \times 2 + 2 \times 0 = 16}$$

If we add the additional constraint $2x_1 + 3x_2 + 2x_3 \le 4$ then the optimal solution

Example 34.26. *Using dual simplex method :*

maximize $-3x_1 - 2x_2$

subject to $x_1 + x_2 \geq 1, x_1 + x_2 \leq 7, x_1 + 2x_2 \geq 10, x_2 \geq 3, x_1 \geq 0, x_2 \geq 0.$ (*Mumbai, 2004*)

Solution. Consists of the following steps :

Step 1. (*i*) *Convert the first and third constraints into* ($\leq$) *type. These constraints become*

$$-x_1 - x_2 \leq -1, -x_1 - 2x_2 \leq -10.$$

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$x_2 + s_4 = 3, x_1, x_2, s_1, s_2, s_3, s_4 \geq 0.$

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Setting the decision variables x_1, x_2 each equal to zero, we get the basic solution

$$x_1 = x_2 = 0, s_1 = -1, s_2 = 7, s_3 = -10, s_4 = 3 \text{ and } Z = 0.$$

$\therefore$ Initial solution is given by the table below :

	c_j	-3	-2	0	0	0	0	
c_B	Basis	x_1	x_2	s_1	s_2	s_3	s_4	b
0	s_1	-1	-1	1	0	0	0	-1
0	s_2	1	1	0	1	0	0	7
0	s_3	-1	(-2)	0	0	1	0	$-10 \leftarrow$
0	s_4	0	1	0	0	0	1	3
	$Z_j = \Sigma c_B a_{ij}$	0	0	0	0	0	0	0
	$C_j = c_j - Z_j$	-3	-2	0	0	0	0	
			$\uparrow$					

Step 3. Test nature of C_j

Since all C_j values are ≤ 0 and $b_1 = -1$, $b_3 = -10$, the initial solution is optimal but infeasible. We therefore, proceed further.

Step 4. Mark the outgoing variable.

Since b_3 is negative and numerically largest, the third row is the key row and s_3 is the outgoing variable.

Step 5. Calculate ratios of elements in C_j*-row to the corresponding negative elements of the key row.*

These ratios are $-3/-1 = 3, -2/-2 = 1$ (neglecting ratios corresponding to +ve or zero elements of key row). Since the smaller ratio is 1, therefore, x_2-column is the key column and (-2) is the key element.

Step 6. Iterate towards optimal feasible solution.

(*i*) Drop s_3 and introduce x_2 alongwith its associated value -2 under c_B column. Convert the key element to unity and make all other elements of the key column zero. Then the second solution is given by the table below :

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subject to $x_1 + x_2 \geq 1, x_1 + x_2 \leq 7, x_1 + 2x_2 \geq 10, x_2 \geq 3, x_1 \geq 0, x_2 \geq 0.$ (*Mumbai, 2004*)

Solution. Consists of the following steps :

Step 1. (*i*) *Convert the first and third constraints into* ($\leq$) *type. These constraints become*

$$-x_1 - x_2 \leq -1, -x_1 - 2x_2 \leq -10.$$

(*ii*) *Express the problem in standard form*

Introducing slack variables s_1, s_2, s_3, s_4 the given problem takes the form

Max. $Z = -3x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$

subject to $-x_1 - x_2 + s_1 = -1, x_1 + x_2 + s_2 = 7, -x_1 - 2x_2 + s_3 = -10,$
$x_2 + s_4 = 3, x_1, x_2, s_1, s_2, s_3, s_4 \geq 0.$

Step 2. Find the initial basic solution

Setting the decision variables x_1, x_2 each equal to zero, we get the basic solution

$$x_1 = x_2 = 0, s_1 = -1, s_2 = 7, s_3 = -10, s_4 = 3 \text{ and } Z = 0.$$

$\therefore$ Initial solution is given by the table below :

	c_j	-3	-2	0	0	0	0	
c_B	Basis	x_1	x_2	s_1	s_2	s_3	s_4	b

Doesnot satisfy the additional constraint.The current optimal solution is not optimal for the modified problem. Let S_3 be the slack variable for the new constraint the modified table will be

Example 34.26. *Using dual simplex method :*
maximize $-3x_1 - 2x_2$
subject to $x_1 + x_2 \geq 1, x_1 + x_2 \leq 7, x_1 + 2x_2 \geq 10, x_2 \geq 3, x_1 \geq 0, x_2 \geq 0.$ *(Mumbai, 2004)*

Solution. Consists of the following steps :

Step 1. (i) Convert the first and third constraints into ($\leq$) *type. These constraints become*

$$-x_1 - x_2 \leq -1, -x_1 - 2x_2 \leq -10.$$

(ii) Express the problem in standard form

Introducing slack variables s_1, s_2, s_3, s_4 the given problem takes the form

Max. $Z = -3x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$

subject to $-x_1 - x_2 + s_1 = -1, x_1 + x_2 + s_2 = 7, -x_1 - 2x_2 + s_3 = -10,$
$x_2 + s_4 = 3, x_1, x_2, s_1, s_2, s_3, s_4 \geq 0.$

Step 2. Find the initial basic solution

Setting the decision variables x_1, x_2 each equal to zero, we get the basic solution

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$\therefore$ Initial solution is given by the table below :

	c_j	-3	-2	0	0	0	0	
c_B	Basis	x_1	x_2	s_1	s_2	s_3	s_4	b
0	s_1	-1	-1	1	0	0	0	-1
0	s_2	1	1	0	1	0	0	7
0	s_3	-1	(-2)	0	0	1	0	$-10 \leftarrow$
0	s_4	0	1	0	0	0	1	3
	$Z_j = \Sigma c_B a_{ij}$	0	0	0	0	0	0	0
	$C_j = c_j - Z_j$	-3	-2	0	0	0	0	
			$\uparrow$					

Step 3. Test nature of C_j

Since all C_j values are ≤ 0 and $b_1 = -1$, $b_3 = -10$, the initial solution is optimal but infeasible. We therefore, proceed further.

Step 4. Mark the outgoing variable.

Since b_3 is negative and numerically largest, the third row is the key row and s_3 is the outgoing variable.

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These ratios are $-3/-1 = 3, -2/-2 = 1$ (neglecting ratios corresponding to +ve or zero elements of key row). Since the smaller ratio is 1, therefore, x_2-column is the key column and (-2) is the key element.

Step 6. Iterate towards optimal feasible solution.

(i) Drop s_3 and introduce x_2 alongwith its associated value -2 under c_B column. Convert the key element to unity and make all other elements of the key column zero. Then the second solution is given by the table below :

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Solution. Consists of the following steps :

Step 1. (i) Convert the first and third constraints into ($\leq$) *type. These constraints become*

$$-x_1 - x_2 \leq -1, -x_1 - 2x_2 \leq -10.$$

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Max. $Z = -3x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$

subject to $-x_1 - x_2 + s_1 = -1, x_1 + x_2 + s_2 = 7, -x_1 - 2x_2 + s_3 = -10,$
$x_2 + s_4 = 3, x_1, x_2, s_1, s_2, s_3, s_4 \geq 0.$

Step 2. Find the initial basic solution

Setting the decision variables x_1, x_2 each equal to zero, we get the basic solution

$$x_1 = x_2 = 0, s_1 = -1, s_2 = 7, s_3 = -10, s_4 = 3 \text{ and } Z = 0.$$

$\therefore$ Initial solution is given by the table below :

	c_j	-3	-2	0	0	0	0	
c_B	Basis	x_1	x_2	s_1	s_2	s_3	s_4	b

		c_j	4	6	2	0	0	0
C_B	Y_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3
4	x_1	1	1	0	-1	4/3	$-1/3$	0
6	x_2	2	0	1	2	$-1/3$	1/3	0
0	x_3	4	2	3	2	0	0	1
$z_j - c_j$			0	0	6	10/3	2/3	0

Since x_1 and x_2 are basic variables the corresponding entries in the third column should be zero. To achieve this multiply the first row by -2 and second row by -3 and add them to the third row, the above Table becomes

Example 34.26. *Using dual simplex method :*
maximize $-3x_1 - 2x_2$
subject to $x_1 + x_2 \geq 1, x_1 + x_2 \leq 7, x_1 + 2x_2 \geq 10, x_2 \geq 3, x_1 \geq 0, x_2 \geq 0.$ (*Mumbai, 2004*)

Solution. Consists of the following steps :

Step 1. (i) Convert the first and third constraints into (≤) type. These constraints become

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subject to $-x_1 - x_2 + s_1 = -1, x_1 + x_2 + s_2 = 7, -x_1 - 2x_2 + s_3 = -10,$
$x_2 + s_4 = 3, x_1, x_2, s_1, s_2, s_3, s_4 \geq 0.$

Step 2. Find the initial basic solution

Setting the decision variables x_1, x_2 each equal to zero, we get the basic solution

$$x_1 = x_2 = 0, s_1 = -1, s_2 = 7, s_3 = -10, s_4 = 3 \text{ and } Z = 0.$$

∴ Initial solution is given by the table below :

	c_j	-3	-2	0	0	0	0	
c_B	Basis	x_1	x_2	s_1	s_2	s_3	s_4	b
0	s_1	-1	-1	1	0	0	0	-1
0	s_2	1	1	0	1	0	0	7
0	s_3	-1	(-2)	0	0	1	0	$-10 \leftarrow$
0	s_4	0	1	0	0	0	1	3
	$Z_j = \Sigma c_B a_{ij}$	0	0	0	0	0	0	0
	$C_j = c_j - Z_j$	-3	-2 ↑	0	0	0	0	

Step 3. Test nature of C_j

Since all C_j values are ≤ 0 and $b_1 = -1$, $b_3 = -10$, the initial solution is optimal but infeasible. We therefore, proceed further.

Step 4. Mark the outgoing variable.

Since b_3 is negative and numerically largest, the third row is the key row and s_3 is the outgoing variable.

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Solution. Consists of the following steps :

Step 1. (i) Convert the first and third constraints into (≤) type. These constraints become

$$-x_1 - x_2 \leq -1, -x_1 - 2x_2 \leq -10.$$

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Introducing slack variables s_1, s_2, s_3, s_4 the given problem takes the form

Max. $Z = -3x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$

subject to $-x_1 - x_2 + s_1 = -1, x_1 + x_2 + s_2 = 7, -x_1 - 2x_2 + s_3 = -10,$
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Setting the decision variables x_1, x_2 each equal to zero, we get the basic solution

$$x_1 = x_2 = 0, s_1 = -1, s_2 = 7, s_3 = -10, s_4 = 3 \text{ and } Z = 0.$$

∴ Initial solution is given by the table below :

	c_j	-3	-2	0	0	0	0	
c_B	Basis	x_1	x_2	s_1	s_2	s_3	s_4	b

		c_j	4	6	2	0	0	0
C_B	Y_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3
4	x_1	1	1	0	-1	4/3	$-1/3$	0
6	x_2	2	0	1	2	$-1/3$	1/3	0
0	s_3	-4	0	0	-6	$-5/3$	$-1/3$	1
$z_j - c_j$			0	0	6	10/3	2/3	0

Since all $z_j - c_j \geq 0$, the optimality condition is satisfied but X_B value for the third row is negative. Now, we apply the dual simplex method then the optimal solution becomes

$$\boxed{x_1 = 5/3, x_2 = 5/3 \text{ and } x_3 = 2/3 \text{ with maximum of } z = 12}$$

Example 34.26. *Using dual simplex method :*
maximize $-3x_1 - 2x_2$
subject to $x_1 + x_2 \geq 1, x_1 + x_2 \leq 7, x_1 + 2x_2 \geq 10, x_2 \geq 3, x_1 \geq 0, x_2 \geq 0.$ (Mumbai, 2004)

Solution. Consists of the following steps :

Step 1. (i) Convert the first and third constraints into (≤) type. These constraints become

$$-x_1 - x_2 \leq -1, -x_1 - 2x_2 \leq -10.$$

(ii) Express the problem in standard form

Introducing slack variables s_1, s_2, s_3, s_4 the given problem takes the form

Max. $Z = -3x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$

subject to $-x_1 - x_2 + s_1 = -1, x_1 + x_2 + s_2 = 7, -x_1 - 2x_2 + s_3 = -10,$
$x_2 + s_4 = 3, x_1, x_2, s_1, s_2, s_3, s_4 \geq 0.$

Step 2. Find the initial basic solution

Setting the decision variables x_1, x_2 each equal to zero, we get the basic solution

$$x_1 = x_2 = 0, s_1 = -1, s_2 = 7, s_3 = -10, s_4 = 3 \text{ and } Z = 0.$$

∴ Initial solution is given by the table below :

	c_j	-3	-2	0	0	0	0	
c_B	*Basis*	x_1	x_2	s_1	s_2	s_3	s_4	b
0	s_1	-1	-1	1	0	0	0	-1
0	s_2	1	1	0	1	0	0	7
0	s_3	-1	(-2)	0	0	1	0	$-10 \leftarrow$
0	s_4	0	1	0	0	0	1	3
	$Z_j = \Sigma c_B a_{ij}$	0	0	0	0	0	0	0
	$C_j = c_j - Z_j$	-3	-2	0	0	0	0	
			$\uparrow$					

Step 3. Test nature of C_j

Since all C_j values are ≤ 0 and $b_1 = -1$, $b_3 = -10$, the initial solution is optimal but infeasible. We therefore, proceed further.

Step 4. Mark the outgoing variable.

Since b_3 is negative and numerically largest, the third row is the key row and s_3 is the outgoing variable.

Step 5. Calculate ratios of elements in C_j-row to the corresponding negative elements of the key row.

These ratios are $-3/-1 = 3, -2/-2 = 1$ (neglecting ratios corresponding to +ve or zero elements of key row). Since the smaller ratio is 1, therefore, x_2-column is the key column and (-2) is the key element.

Step 6. Iterate towards optimal feasible solution.

(*i*) Drop s_3 and introduce x_2 alongwith its associated value -2 under c_B column. Convert the key element to unity and make all other elements of the key column zero. Then the second solution is given by the table below :

Example 34.26. *Using dual simplex method :*
maximize $-3x_1 - 2x_2$
subject to $x_1 + x_2 \geq 1, x_1 + x_2 \leq 7, x_1 + 2x_2 \geq 10, x_2 \geq 3, x_1 \geq 0, x_2 \geq 0.$ (Mumbai, 2004)

Solution. Consists of the following steps :

Step 1. (i) Convert the first and third constraints into (≤) type. These constraints become

$$-x_1 - x_2 \leq -1, -x_1 - 2x_2 \leq -10.$$

(ii) Express the problem in standard form

Introducing slack variables s_1, s_2, s_3, s_4 the given problem takes the form

Max. $Z = -3x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$

subject to $-x_1 - x_2 + s_1 = -1, x_1 + x_2 + s_2 = 7, -x_1 - 2x_2 + s_3 = -10,$
$x_2 + s_4 = 3, x_1, x_2, s_1, s_2, s_3, s_4 \geq 0.$

Step 2. Find the initial basic solution

Setting the decision variables x_1, x_2 each equal to zero, we get the basic solution

$$x_1 = x_2 = 0, s_1 = -1, s_2 = 7, s_3 = -10, s_4 = 3 \text{ and } Z = 0.$$

∴ Initial solution is given by the table below :

	c_j	-3	-2	0	0	0	0	
c_B	*Basis*	x_1	x_2	s_1	s_2	s_3	s_4	b