Problem 1

The individual homogenous transformations are:

$$H_1^0(R_{x,\pi}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C\pi & -S\pi & 0 \\ 0 & +S\pi & C\pi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(fixed)

$$H_2^1(T_{z,5}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(current)

$$H(T_{y,-3}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(fixed)

a) fixed $\rightarrow$ premultiply
current $\rightarrow$ post multiply.

$\therefore$

Overall homogenous trans. matrix :-

$$T_{y,-3} \cdot R_{x,\pi} \cdot T_{z,5}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -3 \\ 0 & 0 & -1 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ Overall homogenous transf. matrix :

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -3 \\ 0 & 0 & -1 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

b) Co-ordinates of frame 3 origin wrt frame 0

Frame 3's origin is like any other point in frame 3. Hence, to get its cordinates in frame 0, we can do

$O_3^0 = H_3^0 \, O_3^3$ (where H_3^0 is the overalle homogenous transf. matrix)

$$\therefore \; O_3^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -3 \\ 0 & 0 & -1 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \text{(Augmentation)}$$

$$= \begin{bmatrix} 0 \\ -3 \\ -5 \\ 1 \end{bmatrix} \Rightarrow \text{Therefore, } O_3^0 = \begin{bmatrix} 0 & -3 & -5 \end{bmatrix}^T$$

c) $p^2 = \begin{bmatrix} -3 & 4 & 1 \end{bmatrix}^T$; Find p^0, p^3

$p^0 = H_2^0 \, p^2$; $H_2^0 = H_1^0 H_2^1$

We have H_1^0 and H_2^1 from part 'a'

$$\therefore \; H_2^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$p^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 4 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -4 \\ -6 \\ 1 \end{bmatrix}$$

Augmentatn.

$$\therefore \; p^0 = \begin{bmatrix} -3 & -4 & -6 \end{bmatrix}^T$$

$p^3 = ?$

We just calculated p^0 and we have H_3^0

$\therefore$ from equatn: $p^0 = H_3^0 p^3$

$$\therefore \quad p^3 = (H_3^0)^{-1} p^0$$

$$H_3^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -3 \\ 0 & 0 & -1 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Here $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

and $d = \begin{bmatrix} 0 & -3 & -5 \end{bmatrix}^T$

We know that,

$$H = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad H^{-1} = \begin{bmatrix} R^T & -R^T d \\ 0 & 1 \end{bmatrix}$$

$$-R^T d = (-1) \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ -3 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ -5 \end{bmatrix}$$

$$\therefore \quad H^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -3 \\ 0 & 0 & -1 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad p^3 = (H_3^0)^{-1} p^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -3 \\ 0 & 0 & -1 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ -4 \\ -6 \\ 1 \end{bmatrix} \rightarrow \text{Augmentatn}$$

$$= \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\therefore \quad p^3 = \begin{bmatrix} -3 & 1 & 1 \end{bmatrix}^T$$

Problem 2

Frame 1 is translated wrt frame 0 by 1 m in both +y &

+z directions

a) $\therefore \quad H_1^0 = T_{y,1} \cdot T_{z,1}$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_1^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

b) $H_2^0 = ?$

Frame 2 is translated wrt frame 1

- by 0.5m along the +y-axis.
- by 0.1m along the +z-axis.
- by 0.5m along the -x-axis.

$\therefore \quad H_2^1 = T_{x,-0.5} \cdot T_{y,0.5} \cdot T_{z,0.1}$

$$= \begin{bmatrix} 1 & 0 & 0 & -0.5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0.5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_2^1 = \begin{bmatrix} 1 & 0 & 0 & -0.5 \\ 0 & 1 & 0 & 0.5 \\ 0 & 0 & 1 & 0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$H_2^0 = H_1^0 H_2^1$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -0.5 \\ 0 & 1 & 0 & 0.5 \\ 0 & 0 & 1 & 0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -0.5 \\ 0 & 1 & 0 & 1.5 \\ 0 & 0 & 1 & 1.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

c) $H_3^0 = ?$

But before that, $H_3^2 = ?$

The rotation matrix can be calculated using:
(between frames 2 and 3)

$$R_3^2 = [\, x_3^2 \mid y_3^2 \mid z_3^2 \,]$$

$$= \begin{bmatrix} x_3 \cdot x_2 & y_3 \cdot x_2 & z_3 \cdot x_2 \\ x_3 \cdot y_2 & y_3 \cdot y_2 & z_3 \cdot y_2 \\ x_3 \cdot z_2 & y_3 \cdot z_2 & z_3 \cdot z_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$ (By observation)

The translation vector is 1.9m along z-axis (i.e)

$$T_3^2 = [0 \;\; 0 \;\; 1.9]^T$$

$$\therefore \; H_3^2 = \begin{bmatrix} R_3^2 & d_3^2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1.9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

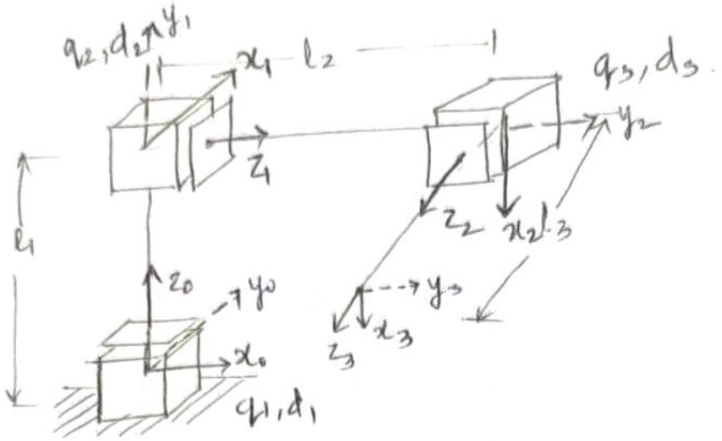
$$H_3^0 = H_1^0 H_2^1 H_3^2$$

$$= H_2^0 H_3^2$$

$$= \begin{bmatrix} 1 & 0 & 0 & -0.5 \\ 0 & 1 & 0 & 1.5 \\ 0 & 0 & 1 & 1.1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1.9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_3^0 = \begin{bmatrix} 0 & 1 & 0 & -0.5 \\ 1 & 0 & 0 & 1.5 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 3



i	θ	d	a	α	Notes
1	$\pi/2$	$l_1+d_1^*$	0	$\pi/2$	$x_0 \to x_1$ about z_0 $z_0 \to x_1$ along z_0 $z_0 \to x_1$ along x_1 $z_0 \to z_1$ about x_1
2	$-\pi/2$	$l_2+d_2^*$	0	$\pi/2$	$x_1 \to x_2$ about z_1 $z_1 \to x_2$ along z_1 $z_1 \to x_2$ along x_2 $z_1 \to z_2$ about x_2
3	0	$l_3+d_3^*$	0	0	$x_2 \to x_3$ about z_2 $z_2 \to x_3$ along z_2 $z_2 \to x_3$ along x_3 $z_2 \to z_3$ about x_3

Homogenous transformation matrix Ai is given by:

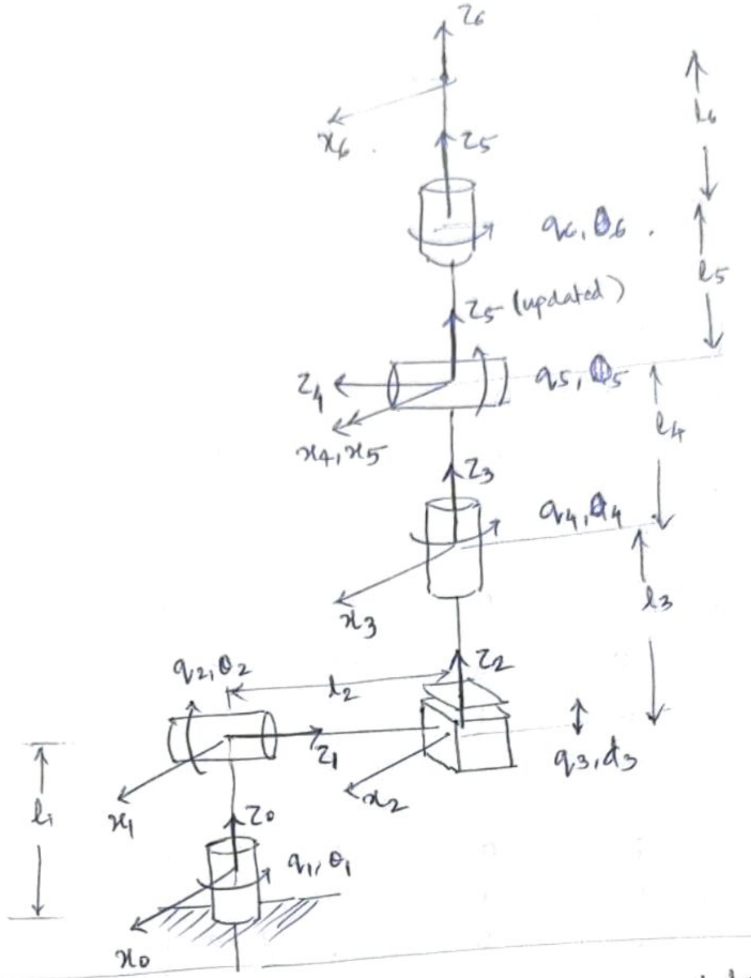
$$\begin{bmatrix} c\theta_i & -s\theta_i c\alpha_i & s\theta_i s\alpha_i & c\theta_i \cdot a_i \\ s\theta_i & c\theta_i c\alpha_i & -c\theta_i s\alpha_i & s\theta_i \cdot a_i \\ 0 & s\alpha_i & c\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore A_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & l_1+d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & l_2+d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & l_3+d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Forward kinematics transf. matrix: $T_3^0 = A_1 A_2 A_3$

$$\therefore T_3^0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & l_1+d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & l_2+d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & l_3+d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & l_2+d_2 \\ 0 & 0 & -1 & -(l_3+d_3) \\ -1 & 0 & 0 & l_1+d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 4



i	θ	d	a	α	Notes
1	θ_1^*	l_1	0	$-\pi/2$	$x_0 \to x_1$ ab z_0 $z_0 \to x_1$ al z_0 $z_0 \to x_1$ al x_1 $z_0 \to z_1$ ab x_1
2	θ_2^*	l_2	0	$\pi/2$	$x_1 \to x_2$ ab z_1 $z_1 \to x_2$ al z_1 $z_1 \to x_2$ al x_2 $z_1 \to z_2$ ab x_2
3	0	$l_3+d_3^*$	0	0	$x_2 \to x_3$ ab z_2 $z_2 \to x_3$ al z_2 $z_2 \to x_3$ al x_3 $z_2 \to z_3$ ab x_3
4	θ_4^*	l_4	0	$\pi/2$	$x_3 \to x_4$ ab z_3 $z_3 \to x_4$ al z_3 $z_3 \to x_4$ al x_4 $z_3 \to z_4$ ab x_4
5	θ_5^*	0	0	$-\pi/2$	$x_4 \to x_5$ ab z_4 $z_4 \to x_5$ al z_4 $z_4 \to x_5$ al x_5 $z_4 \to z_5$ ab x_5
6	θ_6^*	l_5+l_6	0	0	$x_5 \to x_6$ ab z_5 $z_5 \to x_6$ al z_5 $z_5 \to x_6$ al x_6 $z_5 \to z_6$ ab x_6

Homogenous transformation matrix Ai is given by:

$$A_i = \begin{bmatrix} C\theta_i & -S\theta_i C\alpha_i & S\theta_i S\alpha_i & a_i C\theta_i \\ S\theta_i & C\theta_i C\alpha_i & -C\theta_i S\alpha_i & a_i S\theta_i \\ 0 & S\alpha_i & C\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} C\theta_1 & 0 & -S\theta_1 & 0 \\ S\theta_1 & 0 & C\theta_1 & 0 \\ 0 & -1 & 0 & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} C\theta_2 & 0 & S\theta_2 & 0 \\ S\theta_2 & 0 & -C\theta_2 & 0 \\ 0 & 1 & 0 & l_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & l_3+d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} C\theta_4 & 0 & S\theta_4 & 0 \\ S\theta_4 & 0 & -C\theta_4 & 0 \\ 0 & 1 & 0 & l_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} C\theta_5 & 0 & -S\theta_5 & 0 \\ S\theta_5 & 0 & C\theta_5 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_6 = \begin{bmatrix} C\theta_6 & -S\theta_6 & 0 & 0 \\ S\theta_6 & C\theta_6 & 0 & 0 \\ 0 & 0 & 1 & l_5+l_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_6^0 = A_1 \cdot A_2 \cdot A_3 \cdot A_4 \cdot A_5 \cdot A_6$$

$$= \begin{bmatrix} C\theta_1 & 0 & -S\theta_1 & 0 \\ S\theta_1 & 0 & C\theta_1 & 0 \\ 0 & -1 & 0 & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} C\theta_2 & 0 & S\theta_2 & 0 \\ S\theta_2 & 0 & -C\theta_2 & 0 \\ 0 & 1 & 0 & l_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & l_3+d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot$$

(contd.)→

$$\begin{bmatrix} c\theta_4 & 0 & s\theta_4 & 0 \\ s\theta_4 & 0 & -c\theta_4 & 0 \\ 0 & 1 & 0 & l_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} c\theta_5 & 0 & -s\theta_5 & 0 \\ s\theta_5 & 0 & c\theta_5 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta_6 & -s\theta_6 & 0 & 0 \\ s\theta_6 & c\theta_6 & 0 & 0 \\ 0 & 0 & 1 & l_5+l_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$