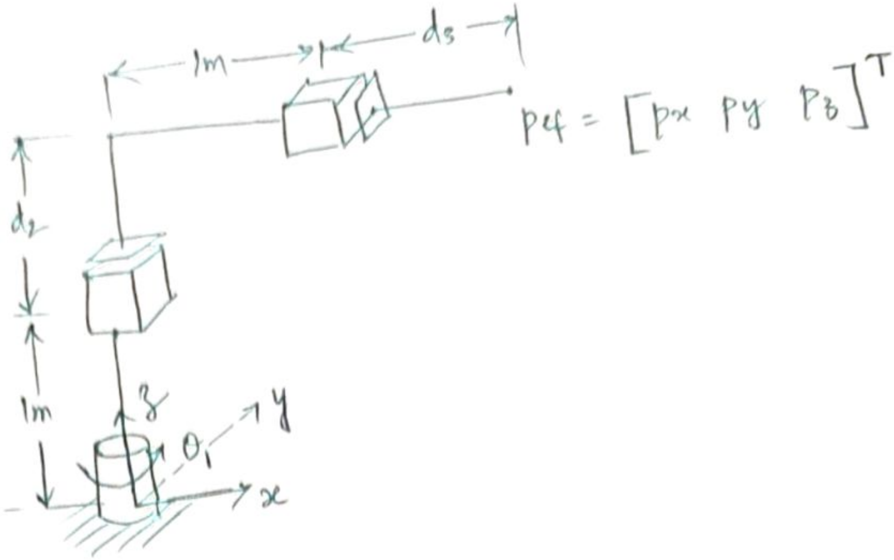
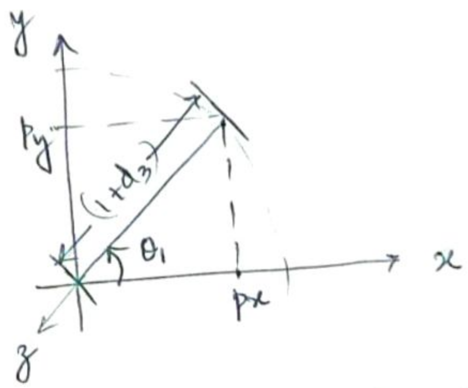
Problem 1 :



From side view, we can notice that :

$$p_z = 1 + d_2 \Rightarrow \boxed{d_2 = p_z - 1} \quad \text{—} \; (1)$$

From top view as shown below :



we can deduce from geometry that

$$p_x = (1+d_3)\cos\theta_1$$
$$p_y = (1+d_3)\sin\theta_1$$

$$\Rightarrow \boxed{d_3 = \frac{p_x}{\cos\theta_1} - 1 = \frac{p_y}{\sin\theta_1} - 1} \quad \text{—} \; (2)$$

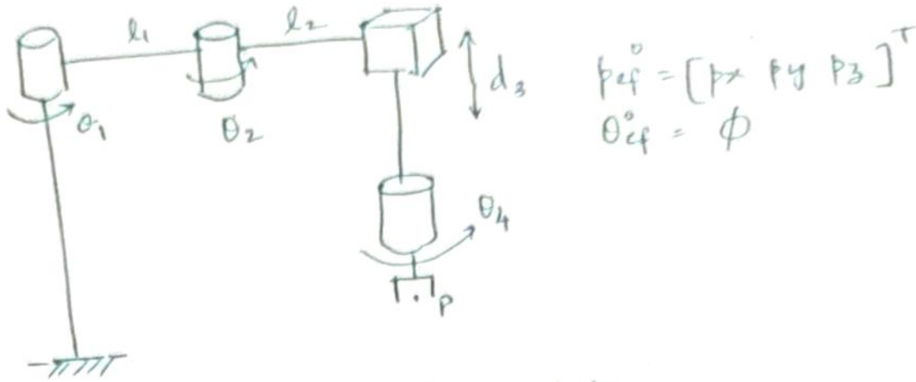
Also, $\tan\theta_1 = \dfrac{p_y}{p_x}$

$$\therefore \boxed{\theta_1 = \tan^{-1}\left(\frac{p_y}{p_x}\right)} \quad \text{—} \; (3)$$

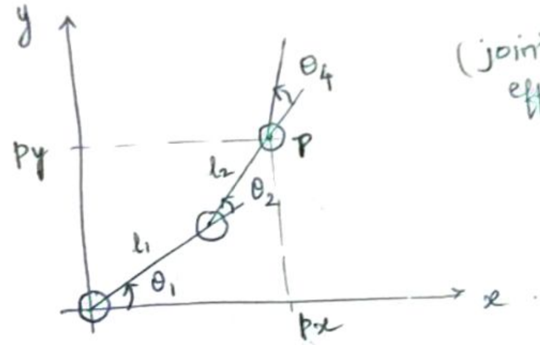
(1), (2) & (3) give the closed form solution for the inverse kinematic positions of the robot.

Problem 2 :

RRPR robot manipulator.



The top view of the robot is shown below.



(joint 3, joint 4 and the End-effector coincide in this top view)

Given:

$$p_x = l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2)$$
$$p_y = l_1 \sin\theta_1 + l_2 \sin(\theta_1+\theta_2)$$

$$p_x^2 + p_y^2 = l_1^2 + l_2^2 + 2l_1 l_2 \left(c\theta_1 c(\theta_1+\theta_2) + s\theta_1 s(\theta_1+\theta_2) \right)$$
$$= l_1^2 + l_2^2 + 2 l_1 l_2 \, c(\cancel{\theta_1} + \theta_2 - \cancel{\theta_1})$$
$$= l_1^2 + l_2^2 + 2 l_1 l_2 \cos\theta_2$$

$$\therefore \cos\theta_2 = \frac{p_x^2 + p_y^2 - l_1^2 - l_2^2}{2 l_1 l_2}$$

$$\sin\theta_2 = \pm\sqrt{1 - \cos^2\theta_2}$$

$$\therefore \theta_2 = \tan^{-1}\left(\frac{\sin\theta_2}{\cos\theta_2} \right) \quad \text{———} \; (1)$$

Expanding ~~px~~ & py :

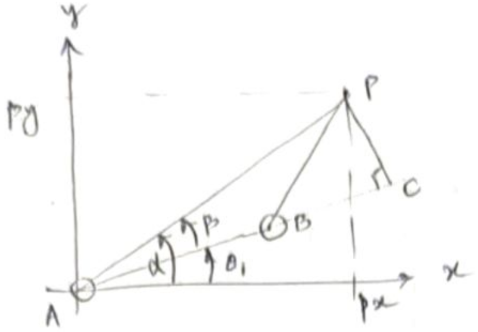
$p_x = \cancel{l_1 (\cos\theta_1) +} \; l_2 \cos\cancel{\theta_1} \cos\theta_2 - l_2 \sin\theta_1 \cancel{\sin\theta_2}$

$p_y \cancel{=} \; l_1 \sin\theta_1 + \; l_2 \cancel{\sin\theta_1} \cos\theta_2 + l_2 \cos\cancel{\theta_1} \sin\theta_2$

~~let~~ $\cos\theta_1 = m$

Geometric approach would be rather simpler.

PTO



$\alpha = \beta + \theta_1$

$\therefore \boxed{\theta_1 = \alpha - \beta}$

Let's find α & β.

$\alpha = \tan^{-1}(py/px)$

Consider ΔAPC

$AP = (px^2 + py^2)^{1/2}$

$AC = l_1 + l_2 \cos\theta_2$

also $AC = AP \cos\beta$

$\therefore l_1 + l_2 \cos\theta_2 = (px^2 + py^2)^{1/2} \cos\beta$

$$\therefore \beta = \cos^{-1}\left(\frac{l_1 + l_2\cos\theta_2}{(px^2 + py^2)^{1/2}}\right)$$

θ_2 is known from ①, so we can find β, and then subsequently θ_1 as $\boxed{\theta_1 = \alpha - \beta}$ — ②

It is also given that EF orientation is ϕ

so, we have $\theta_1 + \theta_2 + \theta_4 = \phi$

since we have θ_1, θ_2

$\boxed{\theta_4 = \phi - \theta_1 - \theta_2}$ — ③

Also since $p_z = d_3$

we have $\boxed{d_3 = p_z}$ — ④

(Can be negative too!)

Thus;

①, ②, ③ & ④ form the closed form solution for the inverse kinematics of the RRPR robot.

Problem 3 :

$a = [2 \;\; -1 \;\; 1]^T$

$R = R_{y,\pi/2}$

$LHS = R\,S(a)\,R^T$

$$R_{y\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} = \begin{bmatrix} c\pi/2 & 0 & s\pi/2 \\ 0 & 1 & 0 \\ -s\pi/2 & 0 & c\pi/2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

if $a = (a_x \; a_y \; a_z)^T$

$$S(a) = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

$$\therefore S(a) = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & -2 \\ 1 & 2 & 0 \end{bmatrix}$$

$$\therefore LHS = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & -2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \quad (1)$$

$RHS = S(Ra)$

$$Ra = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$$

$$S(Ra) = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \quad (2)$$

(1) & (2) show that $R\,S(a)\,R^T = S(Ra)$

Problem 4 :

Rotation of frame 1 wrt frame 0 : $R_1^0 = R_{z,\theta}$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Point P^1 : $[1\ 2\ 0]^T$

w_1^0 : $1k$ rad/s $\Rightarrow 1 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = [0\ 0\ 1]^T$

v_1^0 : $[v_{1x}^0\ \ v_{1y}^0\ \ v_{1z}^0]^T \Rightarrow [3\ \ 0\ \ 0]^T$

Linear velocity here is a function of both angular & linear components,

(ii) $\dot{p}^0 = w \times r + v$ where $r = R_1^0 p^1$

$$\Rightarrow w_1^0 \times \begin{bmatrix} C\theta & -S\theta & 0 \\ S\theta & C\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

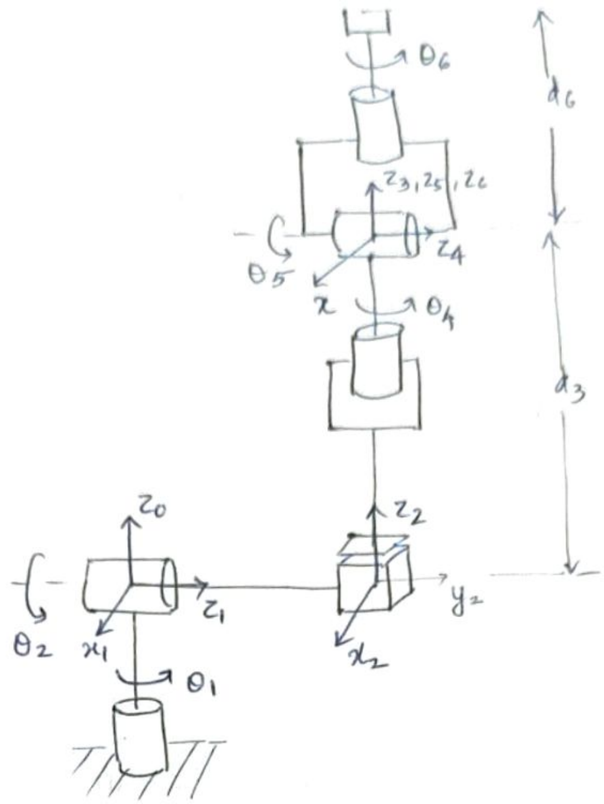
$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} C\theta - 2S\theta \\ S\theta + 2C\theta \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -(S\theta + 2C\theta) \\ C\theta - 2S\theta \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

since cross product of vectors U & V = $\begin{bmatrix} U_2V_3 - V_2U_3 \\ V_1U_3 - U_1V_3 \\ U_1V_2 - V_1U_2 \end{bmatrix}$

$$\therefore \dot{p}^0 = \begin{bmatrix} 3 - S\theta - 2C\theta \\ C\theta - 2S\theta \\ 0 \end{bmatrix}$$

Problem 5 :



Velocity kinematics :

	Linear Component	Angular component
Revolute joint.	$J_{vi} = z_{i-1}^0 \times (O_6^0 - O_{i-1}^0)$	$J_{\omega i} = z_{i-1}$
Prismatic joint.	$J_{vi} = z_{i-1}$	$J_{\omega i} = 0$

$J_{v1} = z_0^0 \times (O_6^0 - O_0^0)$

$J_{\omega 1} = z_0^0$

$J_{v2} = z_1^0 \times (O_6^0 - O_1^0)$

$J_{\omega 2} = z_1^0$

$J_{v3} = z_2^0 \cancel{\times (O_6^0 - O_2^0)}$

$J_{\omega 3} = 0$

$J_{v4} = z_3^0 \times (O_6^0 - O_3^0)$

$J_{\omega 4} = z_3^0$

$J_{v5} = z_4^0 \times (O_6^0 - O_4^0)$

$J_{\omega 5} = z_4^0$

$J_{v6} = z_5^0 \times (O_6^0 - O_5^0)$

$J_{\omega 6} = z_5^0$

The forward kinematics DH table is as follows:

i	θ	d	a	α	Notes.
1	θ_1^*	0	0	$-\pi/2$	$x_0 \to x_1$ abt z_0 $z_0 \to x_1$ al z_0 $z_0 \to x_1$ al x_1 $z_0 \to z_1$ abt x_1
2	θ_2^*	l_2	0	$\pi/2$	$x_1 \to x_2$ abt z_1 $z_1 \to x_2$ abt z_1 $z_1 \to x_2$ al x_2 $z_1 \to z_2$ abt x_2
3	0	d_3^*	0	0	$x_2 \to x_3$ abt z_2 $z_2 \to x_3$ al z_2 $z_2 \to x_3$ al x_3 $z_2 \to z_3$ abt x_3
4.	θ_4^*	0	0	$\cancel{0}\ -\pi/2$	$x_3 \to x_4$ abt z_3 $z_3 \to x_4$ al z_3 $z_3 \to x_4$ al x_4 $z_3 \to z_4$ abt x_4
5	θ_5^*	0	0	$\pi/2$	$x_4 \to x_5$ abt z_4 $z_4 \to x_5$ al z_4 $z_4 \to x_5$ al x_5 $z_4 \to z_5$ abt x_5
6	θ_6^*	0	0	0	$x_5 \to x_6$ abt z_5 $z_5 \to x_6$ al z_5 $z_5 \to x_6$ al x_6 $z_5 \to z_6$ abt x_6

Manipulator Jacobian Matrix:

$$\begin{bmatrix} z_0^0 \times (O_6^0 - O_0^0) & z_1^0 \times (O_6^0 - O_1^0) & z_2^0 & z_3^0 \times (O_6^0 - O_3^0) & z_4^0 \times (O_6^0 - O_4^0) & z_5^0 \times (O_6^0 - O_5^0) \\ z_0^0 & z_1^0 & 0 & z_3^0 & z_4^0 & z_5^0 \end{bmatrix}$$

$z_0^0 = [0 \ 0 \ 1]^T$

$z_1^0 = R_1^0 \cdot [0 \ 0 \ 1]^T$ where R_1^0 is obtained from A_1

$z_2^0 = R_2^0 \cdot [0 \ 0 \ 1]^T$ where R_2^0 is obtained from $A_1 A_2$

$z_3^0 = R_3^0 \cdot [0 \ 0 \ 1]^T$ where R_3^0 is obtained from $A_1 A_2 A_3$

$\vdots$

$z_5^0 = R_5^0 \cdot [0 \ 0 \ 1]^T$ where R_5^0 is obtained from $A_1 A_2 A_3 A_4 A_5$

$O_0^0 = [0 \quad 0 \quad 0]^T$

O_1^0 is obtained from the translation component of A_1

O_3^0 is — " — $A_1 A_2 A_3$

O_4^0 is — " — $A_1 A_2 A_3 A_4$

$\vdots$

O_6^0 is — " — $A_1 A_2 A_3 A_4 A_5 A_6$