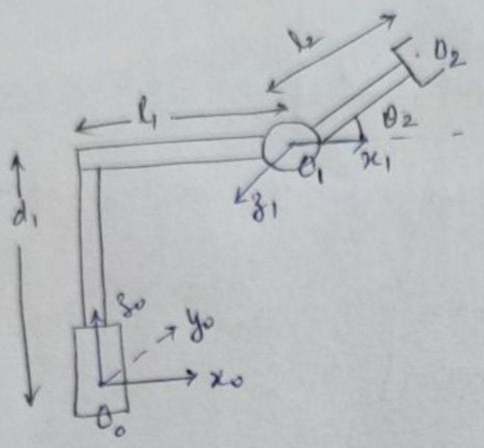
Problem 1



Alternatively, you can also take derivative of given O_2^0 matrix.

$$J = \begin{bmatrix} \frac{\partial f_x}{\partial q_1} & \frac{\partial f_x}{\partial q_2} \\ \frac{\partial f_z}{\partial q_1} & \frac{\partial f_z}{\partial q_2} \end{bmatrix} \qquad q_1 : d_1 \quad q_2 : \theta_2$$

$$J = \begin{bmatrix} 0 & -l_2 \sin\theta_2 \\ 1 & l_2 \cos\theta_2 \end{bmatrix}$$

$$O_2^0 = \begin{bmatrix} x_{ef} \\ z_{ef} \end{bmatrix} = \begin{bmatrix} l_1 + l_2\cos\theta_2 \\ d_1 + l_2 \sin\theta_2 \end{bmatrix}$$

a) Velocity kinematics

	Linear component	Angular component
Revolute	$J_{vi} = Z_{i-1}^0 \times (O_2 - O_1)$	$J_{wi} = Z_{i-1}^0$
Prismatic	$J_{vi} = Z_{i-1}^0$	$J_{wi} = 0$

$J_{v1} = Z_0$, $J_{w1} = 0$

$J_{v2} = Z_1 \times (O_2 - O_1)$, $J_{w2} = Z_1$

Z_1 and O_1 can be obtained from observation simply.

$Z_1 = [0 \ -1 \ 0]^T$ and $O_1 = [l_1 \ 0 \ d_1]^T$

$$\text{Jacobian} = \begin{bmatrix} J_{v1} & J_{v2} \\ J_{w1} & J_{w2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \times \begin{bmatrix} l_1 + l_2\cos\theta_2 - l_1 \\ 0 \\ d_1 + l_2 \sin\theta_2 - d_1 \end{bmatrix} \\ 0 & \\ 1 & \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -l_2 S\theta_2 \\ 0 & 0 \\ \cancel{0}\,1 & l_2 C\theta_2 \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}$$

2x2 jacobian that relates joint velocities to the linear velocity of EF

$$\begin{bmatrix} 0 & -l_2 \sin\theta_2 \\ 1 & l_2 \cos\theta_2 \end{bmatrix}$$

(b) Singular configurations ?

These exist when the inverse of the jacobian matrix doesn't exist.

When it is not possible to deduce the joint velocities corresponding to given EF velocities.

$$\text{determinant } (J) = 0$$

$$0(l_2 \cos\theta_2) - (-l_2 \sin\theta_2 (1)) = 0$$

$$\boxed{l_2 \sin\theta_2 = 0}$$

$$\begin{bmatrix} V_x \\ V_z \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & l_2 \cos\theta_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

We can see that if $\theta_2 = 0$ or $\theta_2 = \pi$, no matter what the joint velocities $\dot{q}_1$ & $\dot{q}_2$ are, we cannot obtain velocities V_x ~~and V_z~~ for the End effector.

$\therefore \theta_2 = 0, \pi$ restrict motion in x

Problem 2

(a)
$$J(\theta) = \begin{bmatrix} 0 & 0 & 0 & l_3 \\ 0 & -l_1 & -l_1-l_2 & -l_1-l_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$f = [10 \quad 10 \quad 0]^T, \quad n = [0 \quad 0 \quad 10]^T$$

we know that,
$$\tau = J^T F, \text{ where } F = [F_x \ F_y \ F_z \ n_x \ n_y \ n_z]^T$$

$$\therefore \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -l_1 & 0 & 0 & 0 & 1 \\ 0 & -(l_1+l_2) & 0 & 0 & 0 & 1 \\ l_3 & -(l_1+l_2) & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \\ 0 \\ 0 \\ 0 \\ 10 \end{bmatrix}$$

$$\tau = \begin{bmatrix} 10 \\ -10l_1+10 \\ -10(l_1+l_2)+10 \\ -10(l_1+l_2-l_3)+10 \end{bmatrix} \text{ OR } \begin{bmatrix} 10 \\ 10(1-l_1) \\ 10(1-l_1-l_2) \\ 10(1+l_3-l_1-l_2) \end{bmatrix}$$

b)
$$J(\theta) = \begin{bmatrix} l_3 s_4 + l_2 s_{34} + l_1 s_{234} & l_3 s_4 + l_2 s_{34} & l_3 s_4 & 0 \\ l_4 + l_3 c_4 + l_2 c_{34} + l_1 c_{234} & l_4 + l_3 c_4 + l_2 c_{34} & l_4 + l_3 c_4 & l_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

This jacobian matrix is equivalent to the following matrix after applying the column operations:

$$C_1 \rightarrow C_1 - C_2$$
$$C_2 \rightarrow C_2 - C_3$$
$$C_3 \rightarrow C_3 - C_4$$

$$
\begin{bmatrix}
l_1 s_{234} & l_2 s_{34} & l_3 s_4 & 0 \\
l_1 c_{234} & l_2 c_{34} & l_3 c_4 & l_4 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
$$

we can see that

(i) $c_2 = \frac{l_2 c_1}{l_1}$ when $\theta_2 = 0$

(ii) $c_3 = \frac{l_3 c_1}{l_1}$ when $\theta_2 = \theta_3 = 0$

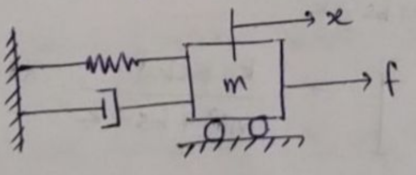
(iii) $c_3 = \frac{l_3 c_2}{l_2}$ when $\theta_3 = 0$

(iv) $c_3 = \frac{l_3}{2l_1} c_1 + \frac{l_3}{2l_2} c_2$ when $\theta_2 = \theta_3 = 0$.

Hence the singularities of the 4R Robot for the given configuration occur when $\theta_2 = 0$, $\theta_3 = 0$ and $\theta_2 = \theta_3 = 0$ together.

Problem 3

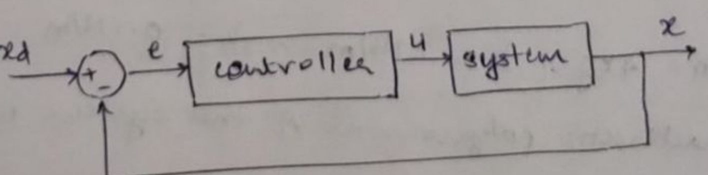
1DOF mass-spring damper system.



x: horizontal pos.
f: control force.

$$m\ddot{x} + b\dot{x} + kx = f$$

controller: $f = k_p(e) + k_d(\dot{e})$



(a) system model in the Laplace domain

$$m\ddot{x} + b\dot{x} + kx = f$$

$$ms^2 X(s) + bsX(s) + k(X(s)) = F(s) \quad \text{--- ①}$$

controller: $f = k_p e + k_d \cdot \dot{e}$

$$F(s) = k_p E(s) + k_d s E(s) \quad \text{--- ②}$$

(b) Transfer functions for $\frac{F(s)}{E(s)}$, $\frac{X(s)}{F(s)}$, $\frac{X(s)}{E(s)}$

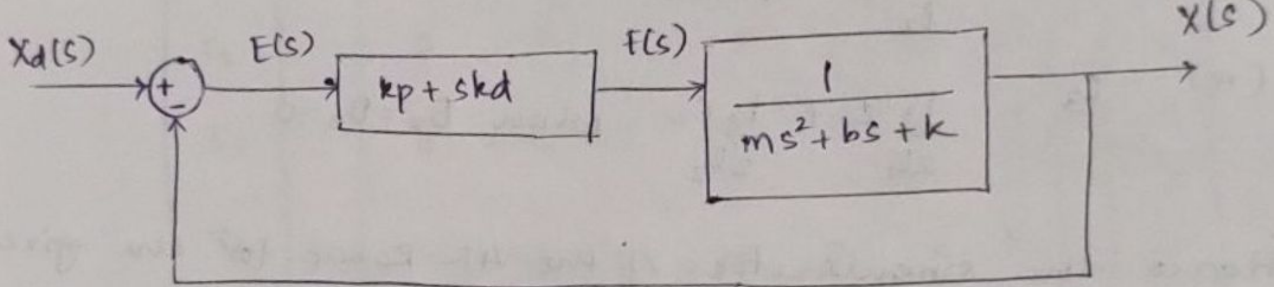
from ② we have $\frac{F(s)}{E(s)} = kp + s\,kd$ ——— ③

from ① we have $\frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k}$ ——— ④

from ③ & ④, we have $\frac{X(s)}{E(s)} = \frac{F(s)}{E(s)} \times \frac{X(s)}{F(s)}$

$$= \frac{kp + s\,kd}{ms^2 + bs + k}$$

(C) Closed-loop transfer function (i.e) $\frac{X(s)}{Xd(s)}$



$F(s) = (kp + s\,kd)\,E(s)$ & $E(s) = Xd(s) - X(s)$

$$X(s) = F(s) \cdot \frac{1}{ms^2 + bs + k}$$

$$X(s) = \frac{(kp + s\,kd)(Xd(s) - X(s))}{ms^2 + bs + k}$$

$$X(s)\left[1 + \frac{kp + s\,kd}{ms^2 + bs + k}\right] = \frac{kp + s\,kd}{ms^2 + bs + k} \cdot Xd(s)$$

$$\therefore \frac{X(s)}{Xd(s)} = \frac{kp + s\,kd}{ms^2 + (b + kd)s + (kp + k)}$$

Given $m = 4\,kg$, $b = 2\,Ns/m$, $k = 0.1\,N/m$

d) For the uncontrolled system,

natural freq $= w_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{0.1}{4}} = \frac{\sqrt{0.1}}{2} = 0.158$

damping ratio $= \zeta = \frac{b}{2\sqrt{km}} = \frac{2}{2\sqrt{0.1(4)}} = \frac{1}{2\sqrt{0.1}} = 1.58$

Since the damping ratio is > 1, the uncontrolled system is <u>overdamped</u>

2% settling time $T_s = \frac{4}{\zeta w_n} = \frac{4}{(1.58)(0.158)} = 16.02$ s

e) Overall closed loop transfer function for the system in question as calculated in (c) is

$$\text{CLTF} : \frac{k_p + s k_d}{ms^2 + (b + k_d)s + (k_p + k)}$$

for a P (proportional) controller, $k_d = 0$

$$\therefore \text{CLTF} = \frac{k_p}{ms^2 + bs + (k_p + k)}$$

$$CP = ms^2 + bs + (k_p + k)$$

$$CP \text{ (std. form)} : s^2 + \frac{b}{m}s + \left(\frac{k_p + k}{m}\right)$$

$$\therefore w_n = \sqrt{\frac{k_p + k}{m}} \quad \text{and} \quad \zeta = \frac{b}{2\sqrt{(k_p + k)m}}$$

for critical damping, $\zeta = 1$

$$\therefore 1 = \frac{b}{2\sqrt{(k_p + k)m}}$$

$$2\sqrt{(k_p + k)m} = b$$

$$2\sqrt{(k_p + 0.1)4} = 2$$

$$(k_p + 0.1)4 = 1$$

$$k_p = 0.25 - 0.1 = 0.15$$

f) Overall CLTF $= \dfrac{kp + skd}{ms^2 + (b+kd)s + (kp+k)}$

for a D (derivative) controller, $kp = 0$

$\therefore$ CLTF $= \dfrac{skd}{ms^2 + (b+kd)s + k}$

CP $= ms^2 + (b+kd)s + k$

CP (std. form) : $s^2 + \left(\dfrac{b+kd}{m}\right)s + \dfrac{k}{m}$

$w_n = \sqrt{\dfrac{k}{m}}$ and $\zeta = \dfrac{b+kd}{2\sqrt{km}}$

for critical damping, $\zeta = 1$

$2\sqrt{km} = b + kd$

$2\sqrt{0.1(4)} = 2 + kd$

$\cancel{4(0.1)(4) = kd^2 + 4kd + (kd+2)^2}$

$\sqrt{1.6} = kd + 2$

$\therefore kd = -2 + \sqrt{1.6} = -0.735$

g) overall CLTF for PD controller :

$$\frac{kp + skd}{ms^2 + (b+kd)s + (kp+k)}$$

CP $= ms^2 + (b+kd)s + (kp+k)$

CP (std. form) $= s^2 + \left(\dfrac{b+kd}{m}\right)s + \left(\dfrac{kp+k}{m}\right)$

$w_n = \sqrt{\dfrac{kp+k}{m}}$, $\zeta = \dfrac{b+kd}{2\sqrt{(kp+k)m}}$

for critical damping, $\zeta = 1$

$b + kd = 2\sqrt{(kp+k)m}$ — (1)

for 2% settling time of 0.01,

$\dfrac{4}{\zeta w_n} = 0.01$

$w_n = \dfrac{4}{0.01(1)} = 400$

$$w_n = \sqrt{\frac{k_p + k}{m}}$$

$$(400)^2 = \frac{k_p + 0.1}{4}$$

$$\therefore k_p = 640000 - 0.1$$

$$\boxed{k_p = 639999.9}$$

From ①, $b + k_d = 2\sqrt{(k_p+k)m}$

$$= 2\sqrt{\frac{k_p+k}{m}} \cdot m$$

$$= 2 w_n \cdot m$$

$$= 2(400)(4)$$

$$k_d = \cancel{1600 - 2}\ 3200 - 2$$

$$\boxed{k_d = 3198}$$

<u>Problem 4</u> :

(a) $M\ddot{\theta}(t) + b\dot{\theta} = u(t) + d(t)$

Given: disturbance $\to 0$, $M = 10$, $b = 1$

Taking Laplace Tramform for robot model :

$$Ms^2\theta(s) + bs\theta(s) = U(s)$$

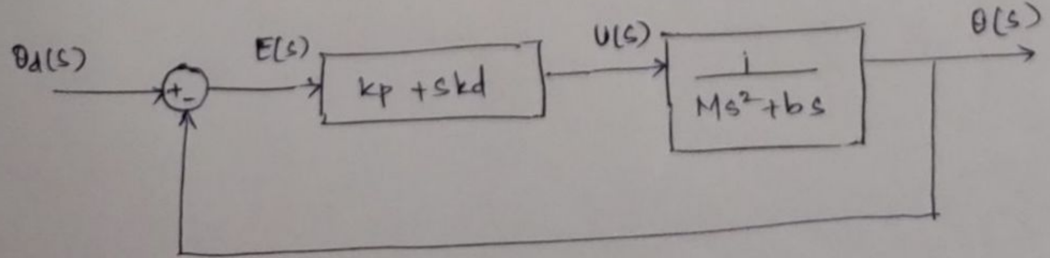
$$\therefore \frac{\theta(s)}{U(s)} = \frac{1}{Ms^2 + bs}$$

PD controller : $(e)k_p + k_d(\dot{e}) = u$ (Input to system)

Taking Laplace transform :

$$k_p E(s) + s k_d E(s) = U(s)$$

$$\frac{U(s)}{E(s)} = (k_p + s k_d)$$



$$E(s) = \theta_d(s) - \theta(s)$$

$$U(s) = E(s)\{k_p + s k_d\}$$

$$\theta(s) = U(s) \cdot \frac{1}{Ms^2 + bs}$$

$$\theta(s) = \frac{(\theta_d(s) - \theta(s))(k_p + sk_d)}{Ms^2 + bs}$$

$$CLTF = \frac{\theta(s)}{\theta_d(s)} = \frac{k_p + sk_d}{Ms^2 + (b + k_d)s + k_p}$$

standandized CP = $s^2 + \left(\frac{b + k_d}{M}\right)s + \left(\frac{k_p}{M}\right)$

$$\therefore \omega_n = \sqrt{\frac{k_p}{m}}, \quad \zeta = \frac{b + k_d}{2\sqrt{k_p m}}$$

Given that $T_s = 2$ and $\zeta = 1$ (critically damped)

$$\frac{4}{\zeta \omega_n} = 2$$

$$\omega_n = \frac{4}{2} = 2 \Rightarrow \frac{k_p}{M} = 4 \Rightarrow k_p = 4(10) = 40$$

$$b + k_d = 2\sqrt{k_p \cdot m}$$

$$1 + k_d = 2\sqrt{40(10)}$$

$$= 2\sqrt{400}$$

$$= 2(20) = 40$$

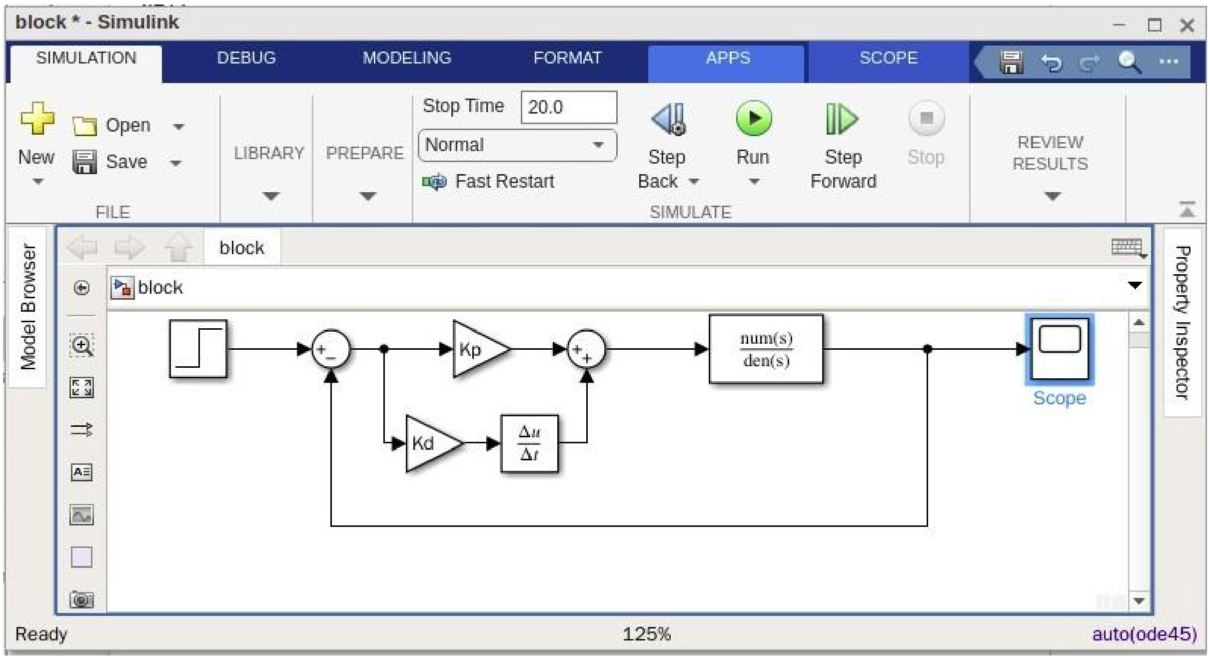
$$\therefore k_d = 40 - 1 = 39$$

RBE500- Foundations of Robotics

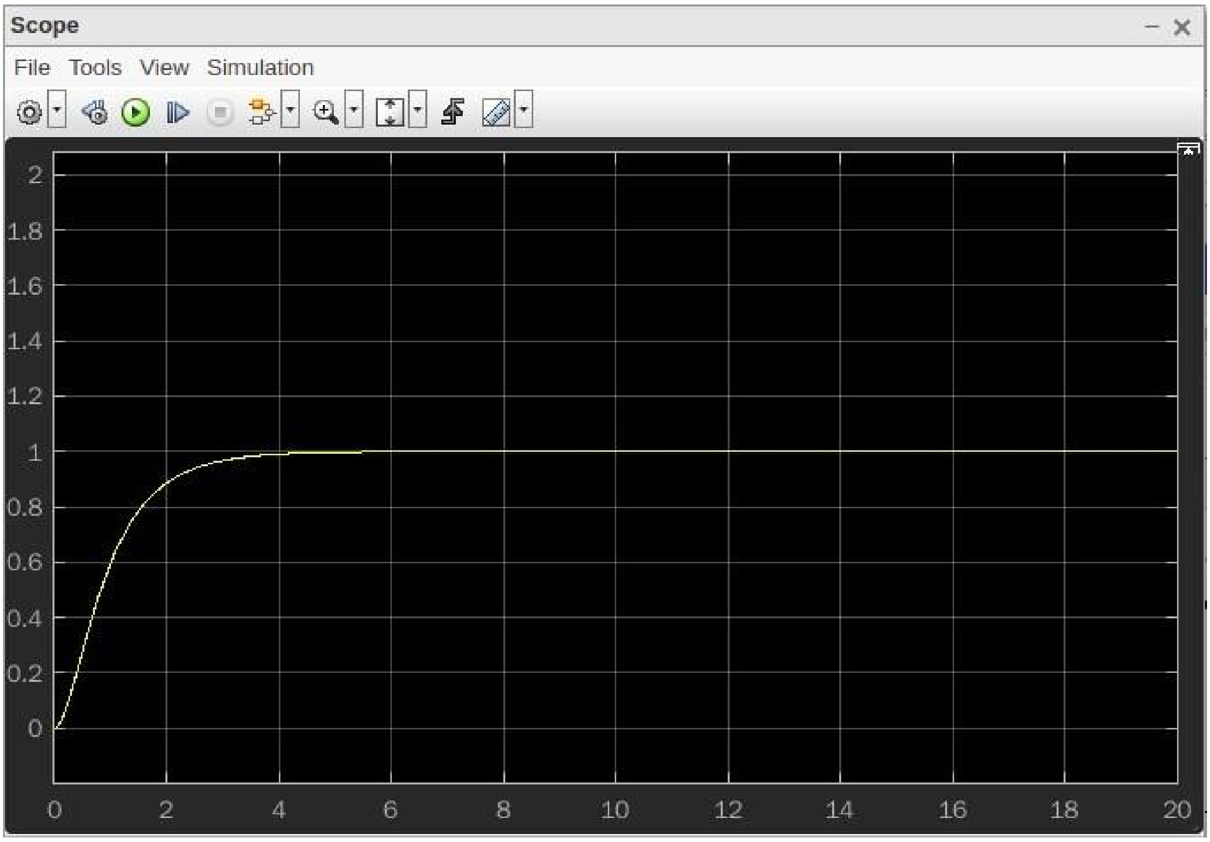
Problem Set 3 Report

Problem 4 (Part b): Given- M = 10, b = 1

Block diagram:



Scope output:



PD Controller: Kp = 40, Kd = 39

The controller gains Kp and Kd as calculated from the requirement of a critically damped system with settling time of 2s did not result in the expected system response.

The settling time was found to be 3.4s rather than its intended value of 2s. It did come close to representing a critically damped system, though.

The deviation from the expected response was because of the fact that the system was untuned and the expressions for Wn and zeta were only approximations.

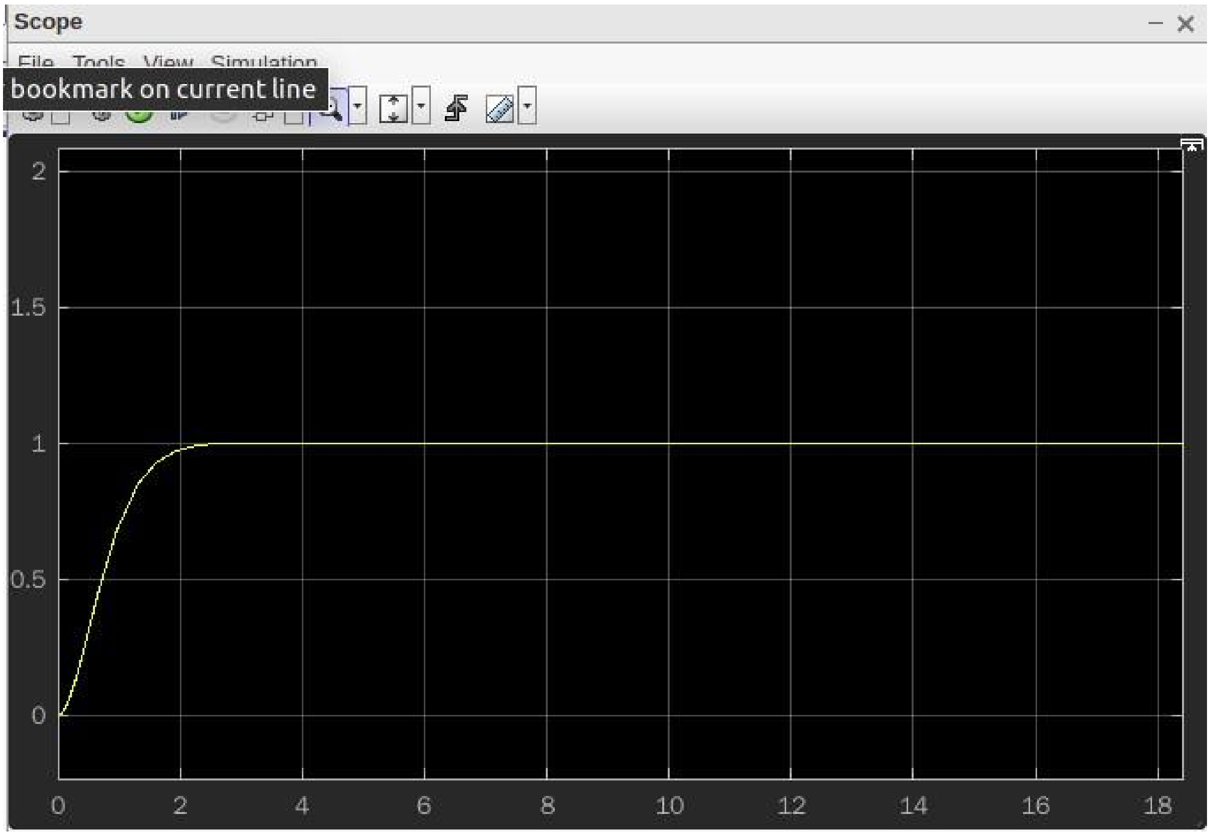
Tuning process(Part c**):**

Initially, Kp was kept constant and Kd was varied since derivative gain helps with reducing the settling time. On increasing Kd, it was found that the settling time was increasing still. Hence, now Kd was decreased from its original value of 39 to 35 to 30 and so on gradually to bring the settling time as close as possible to 2s. Decreasing it to 28 caused a slight overshoot which was undesirable. Finally, the sweet spot of 29.5 was set where the settling time was close to 2 without any overshoot.

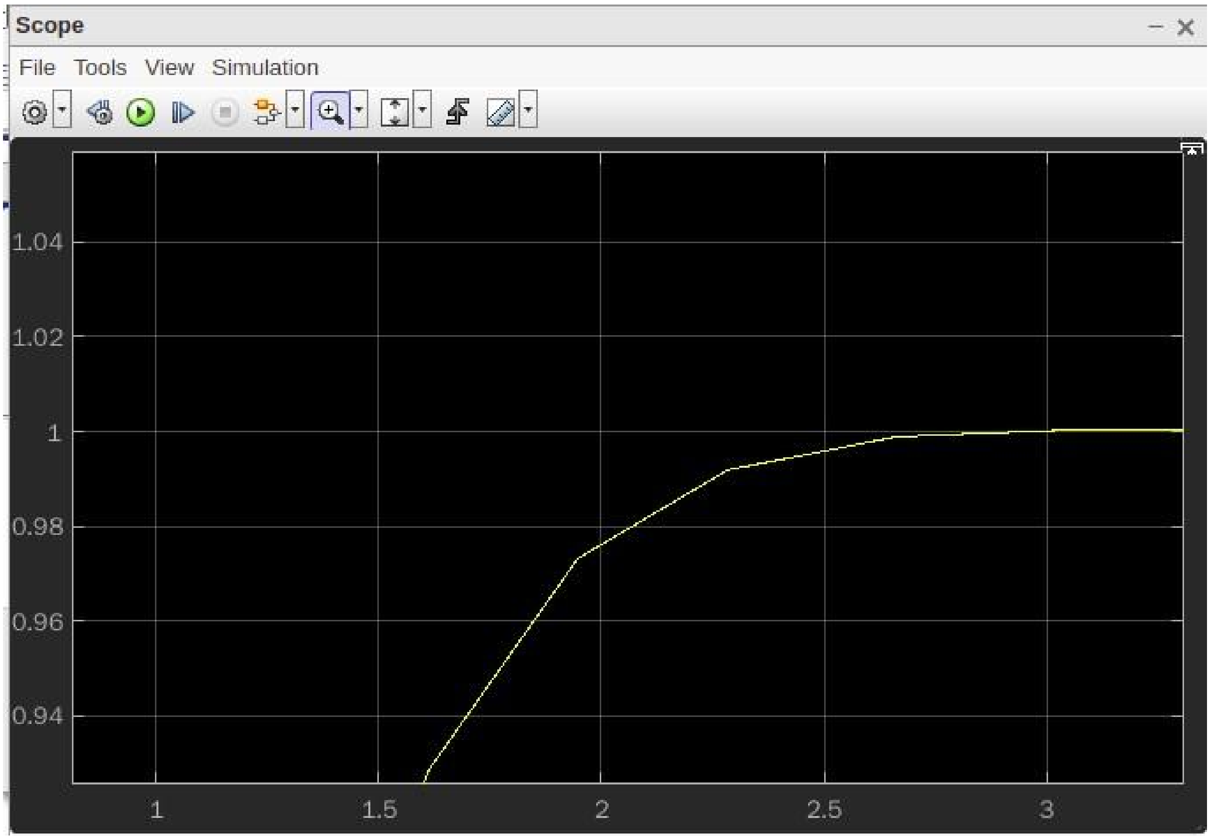
To check the effect of Kp on the system response, it was initially increased which resulted in increasing the overshoot and slight decrease of settling time. Reducing the Kp from the original value increased the settling time drastically.

Parameter changed	**Value**	**Overshoot**	**Settling time**	**Stability**
Kd	45	No effect	4.1s	Stable
Kd	50	No effect	4.7s	Stable
Kd	35	No effect	2.9s	Stable
Kd	30	No effect	2.15s	Stable
Kd	28	Slight overshoot	1.85s	Stable
Kd	29	Slight overshoot	~2s	Stable
Kd	**29.5**	No overshoot	2.05s	Stable
Kp	50	Slight overshoot	1.48s	Stable
Kp	60	Overshoot increases	1.17s	Stable
Kp	30	No overshoot	3.2s	Stable
Kp	15	No overshoot	7.3s	Stable

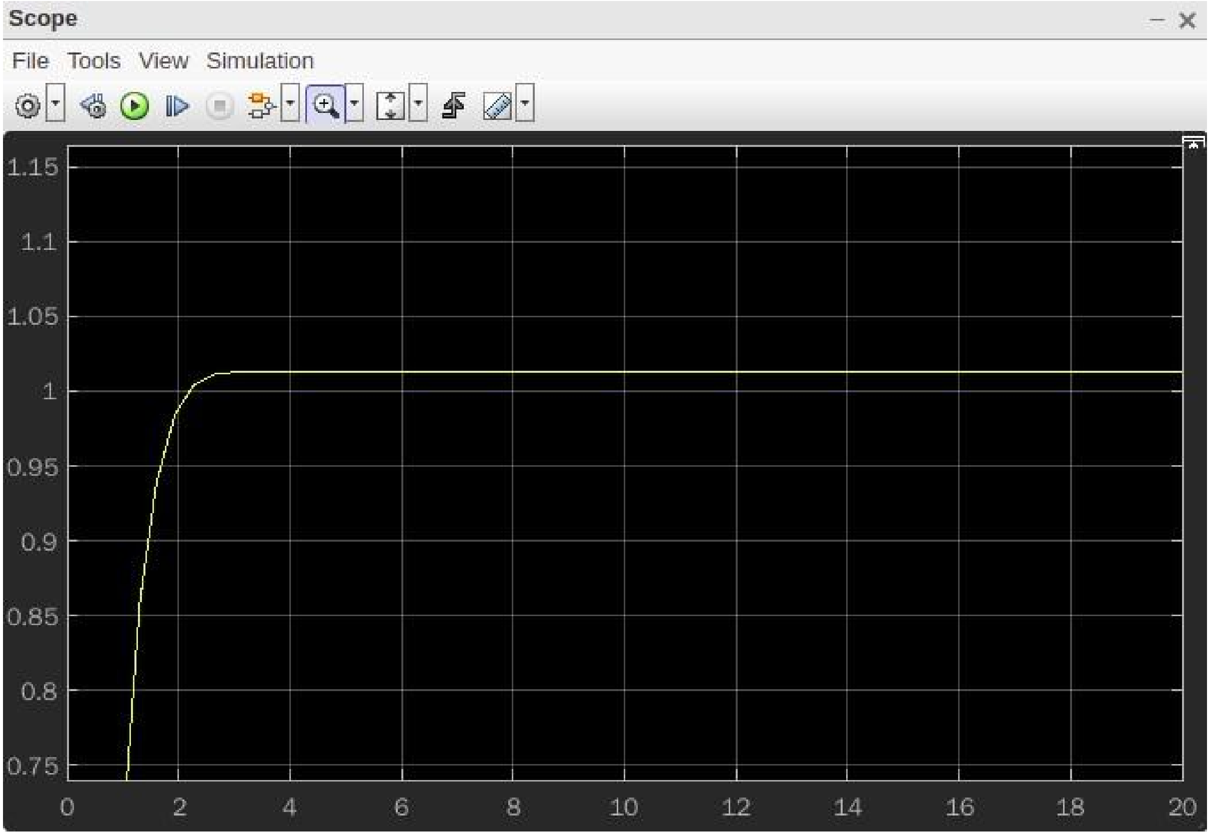
Results post tuning:



Settling time ~2s at Kd = 29.5:



Part d:
The steady state error can be seen in the response plot below. Its value is 0.0125.



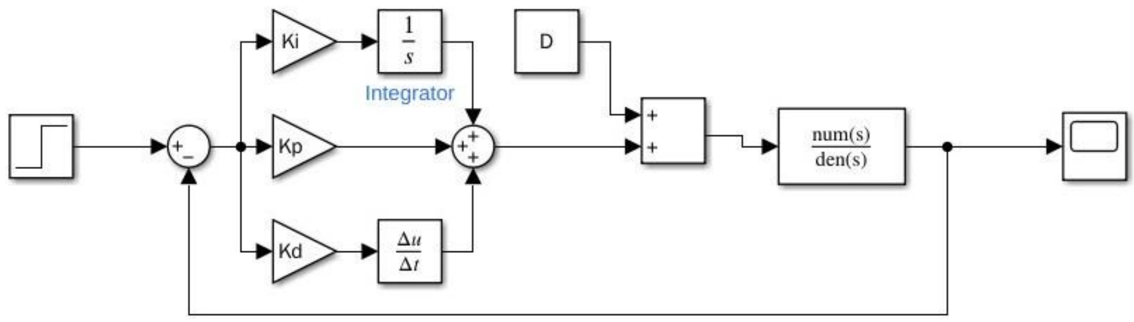
Part e:

Following table shows the tuning stages.

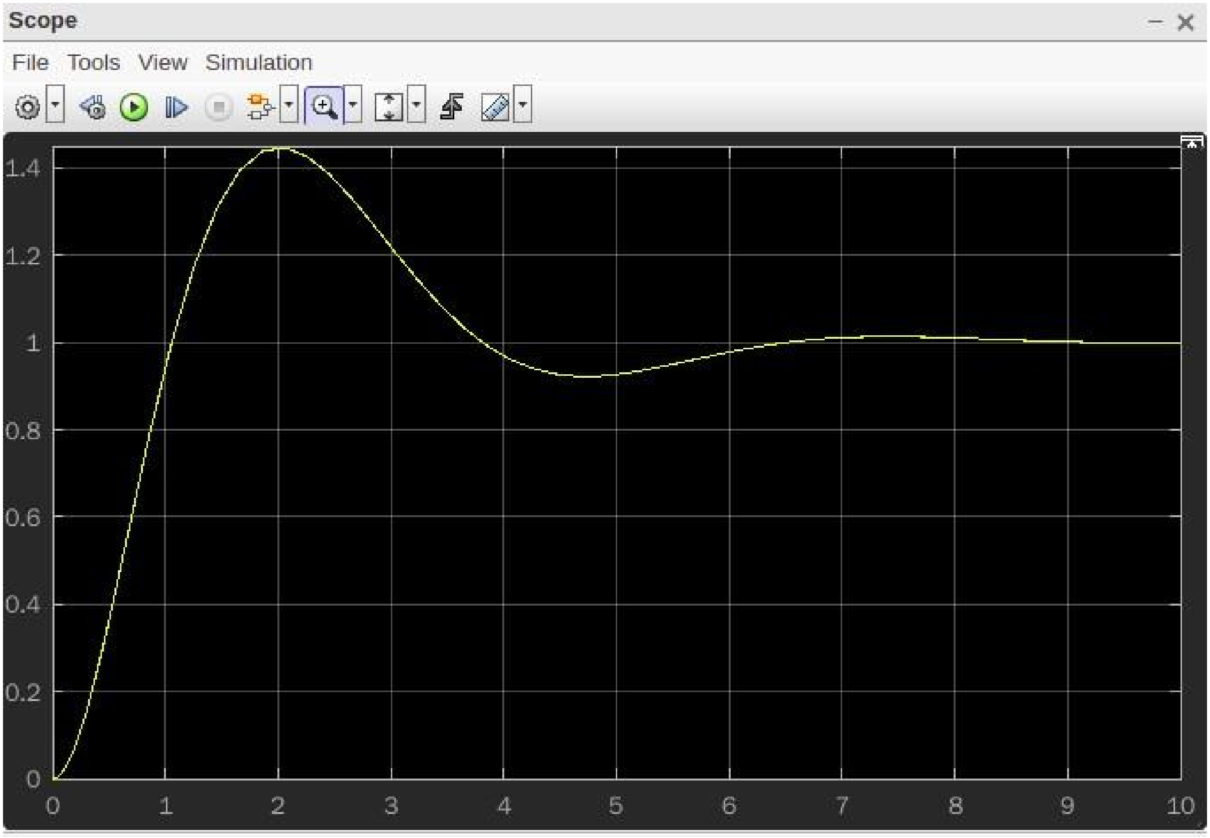
The parameters Kp and Kd were also varied but the performance was best for the already set values from previous tuning.

Parameter changed	**Value**	**Overshoot**	**Settling time**	**Steady-state error**
Ki	0	No effect	1.9s	0.0125
Ki	1	Increases	-	-
Ki	5	Increases	5.87s	0
Ki	15	Same as before	5.87s	0
Ki	35	Slight decrease	6.01s	0

Block Diagram



Response



The final values of Kp, Kd and Ki were found out to be **40, 29.5** and **35** respectively.