Module-4

Counting

Syllabus

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Basic Principles of Counting

- Product Rule
- Sum Rule

Product Rule

- Suppose that a procedure can be broken down into a sequence of two tasks. If there are n1 ways to do the first task and for each of these ways of doing the first task, there are n2 ways to do the second task, then there are n1x n2 ways to do the procedure.

Product Rule

Example:

- The chairs of an auditorium are to be labeled with an uppercase English letter followed by a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?

Solution:

- The procedure of labeling a chair consists of two tasks:
- assigning one of the 26 uppercase English letters,and then
- assigning one of the 100 possible integers.
- The product rule shows that there are $26 \cdot 100 = 2600$ different ways that a chair can be labeled.
- Therefore, the largest number of chairs that can be labeled differently is 2600.

Product Rule

Example:

- How many functions are there from a set with m elements to a set with n elements?

Solution:

- A function corresponds to a choice of one of the n elements in the codomain for each of the m elements in the domain.

- Hence, by the product rule there are $n \cdot n \cdots n = n^m$ functions from a set with m elements to one with n elements.

- For example, from a set with three elements to a set with five elements there are $5^3 = 125$ different functions

Product Rule

The product rule is often phrased in terms of sets in this way:

- If $A_1, A_2, \ldots, A_m$ are finite sets, then the number of elements in the Cartesian product of these sets is the product of the number of elements in each set.

- The task of choosing an element in the Cartesian product $A_1 \times A_2 \times \cdots \times A_m$ is done by choosing an element in A_1, an element in A_2, … , and an element in A_m.

- By the product rule it follows that:

$$|A_1 \times A_2 \times \cdots \times A_m| = |A_1| \cdot |A_2| \cdot \cdots \cdot |A_m|$$

Sum Rule

- If a task can be done either in one of n1 ways or in one of n2 ways, where none of the set of n1 ways is the same as any of the set of n2 ways, then there are n1 + n2 ways to do the task.

Sum Rule

Example:

- A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from?

Solution:

- The student can choose a project by selecting a project from the first list, the second list, or the third list.
- Because no project is on more than one list, by the sum rule there are $23 + 15 + 19 = 57$ ways to choose a project.

Sum Rule

The sum rule is often phrased in terms of sets in this way:

- If $A_1, A_2, \ldots, A_m$ are pairwise disjoint finite sets, then the number of elements in the union of these sets is the sum of the numbers of elements in the sets.

- There are $|A_i|$ ways to choose an element from A_i for $i = 1, 2, \ldots, m$.

- By the sum rule, because we cannot select an element from two of these sets at the same time, the number of ways to choose an element from one of the sets, which is the number of elements in the union, is :

$|A_1 \cup A_2 \cup \cdots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m|$ when $A_i \cap A_j =$ for all i, j

Questions

There are 18 mathematics majors and 325 computer science majors at a college.

a) In how many ways can two representatives be picked so that one is a mathematics major and the other is a computer science major?

b) In how many ways can one representative be picked who is either a mathematics major or a computer science major?

Sol. a)

The product rule applies here, since we want to do two things, one after the other.

First, since there are 18 mathematics majors, and we need to choose one of them, there are 18 ways to choose the mathematics major

Then we must choose the computer science major from among the 325 computer science majors, and that can clearly be done in 325 ways.

Therefore, by the product rule, there are $18 \cdot 325 = 5850$ ways to pick the two representatives.

Questions

b) In how many ways can one representative be picked who is either a mathematics major or a computer science major?

Sol.

b) The sum rule applies here, since we want to do one of two mutually exclusive things.

Either we can choose a mathematics major to be the representative, or we can choose a computer science major.

There are 18 ways to choose a mathematics major, and there are 325 ways to choose a computer science major.

Since these two actions are mutually exclusive (no one is both a mathematics major and a computer science major), and since we want to select one of them or the other, there are 18 + 325 = 343 ways to pick the representative.

Questions

A committee is formed consisting of one representative from each of the 50 states in the United States, where the representative from a state is either the governor or one of the two senators from that state. How many ways are there to form this committee?

Sol:

There are 50 choices to make, each of which can be done in 3 ways:
by choosing the governor or
by choosing the senior senator or
by choosing the junior senator.
By the product rule the answer is therefore 3^{50}

Questions

How many license plates can be made using either two uppercase English letters followed by four digits or two digits followed by four uppercase English letters?

A B1234 or 12ABCD

Sol:

By the sum rule we need to add the number of license plates of the first type and the number of license plates of the second type.

By the product rule:

a) there are $26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 6,760,000$ license plates consisting of 2 letters followed by 4 digits;

OR

b) there are $10 \cdot 10 \cdot 26 \cdot 26 \cdot 26 \cdot 26 = 45,697,600$ license plates consisting of 2 digits followed by 4 letters.

Therefore the answer is $6,760,000 + 45,697,600 = 52,457,600$.

Inclusion-Exclusion Principle

If A and B are two finite sets and n(A), n(B) denote the number of elements in A and B then

$n(A \cup B) = n(A)+n(B)-n(A \cap B)$

For 3 sets:

If A,B and C are three finite sets and n(A), n(B), n(c) denote the number of elements in A, B and C then

$$n(A \cup B \cup C) = n(A)+n(B))+n(C)-n(A \cap B)- n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$

Rules (for 2 sets)

1. No. of elements in A only

$$n(A - B) = n(A \cap B`) = n(A) - n(A \cap B)$$

2. No. of elements in B only

$$n(B - A) = n(B \cap A`) = n(B) - n(A \cap B)$$

3. No. of elements in either A or in B

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

4. No. of elements neither in A nor in B

$$n(A \cup B)` = n(A` \cap B`) = n(S) - n(A \cup B)$$

n(s) is no. of elements in the set

Rules (for 3 sets)

1. No. of elements in A only

$$n(A – B\text{-}C) = n(A)\text{-}\ n(A \cap B)\text{-}\ n(A \cap C)+n\ (A \cap B \cap C)$$

2. No. of elements in B only

$$n(A – B\text{-}C) = n(B)\text{-}\ n(B \cap A)\text{-}\ n(B \cap C)+n\ (A \cap B \cap C)$$

3. No. of elements in C only

$$n(A – B\text{-}C) = n(C)\text{-}\ n(C \cap A)\text{-}\ n(C \cap B)+n\ (A \cap B \cap C)$$

4. No. of elements in either A or in B or in C

$$n(A \cup B \cup C) = n(A)+n(B)+n(C)\text{-}n(A \cap B)\ )\text{-}\ n(B \cap C)\text{-}\ n(C \cap B\ ) +n\ (A \cap B \cap C)$$

5. No. of elements neither in A nor in B nor in C

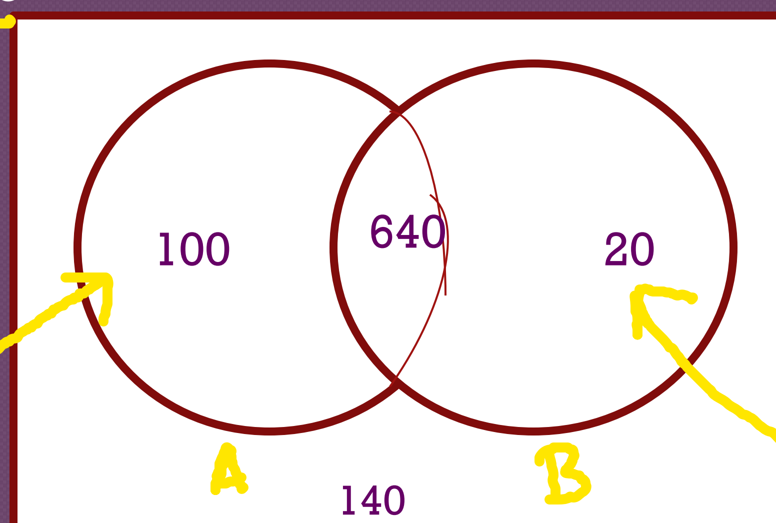
$$n(A \cup B \cup C)\ ` = n\ (S) – n(A \cup B \cup C)$$

Question

In an exam there are two papers A and B. 900 students appear for the exam. Exactly 740 and 660 passed in papers A and B respectively. 640 passed in both. Find the no. of students who failed in both papers.

Sol. n(S)=900

A: 740
− 640
100



Venn Diagram

B = 660 − 640 = 20

Question

given

n(s) = no. of students who appeared for the exam = 900

n(A) = no. of students who passed in paper A= 740

n(B) = no. of students who passed in paper B= 660

$n(A \cap B)$ = no. of students who passed in both= 640

Therefore, the no. of students who passed in either A or B

$$n(A \cup B) = n(A)+n(B)-n(A \cap B)$$

$$= 740+660-640 = 760$$

Therefore, the no. of students who failed in both

$$n(A \cup B)` = n(S)-n(A \cup B)$$

$$= 900-760$$

$$= 140$$

Question

- In a survey of 260 students, 64 had taken maths, 94 had taken computers, 58 had taken science, 28 had taken both maths and science, 26 had taken both maths and computers, 22 had taken both computers and science. 14 had taken all three. Find: a) no. of students who had not taken any one of the above course b) no. of students who had taken only computers

Soln : $n(U) = 260$ → No. of students surveyed

$n(M)$: No. of students taken Maths $= 64$.

$n(C)$ = No. of students taken Comp. $= 94$.

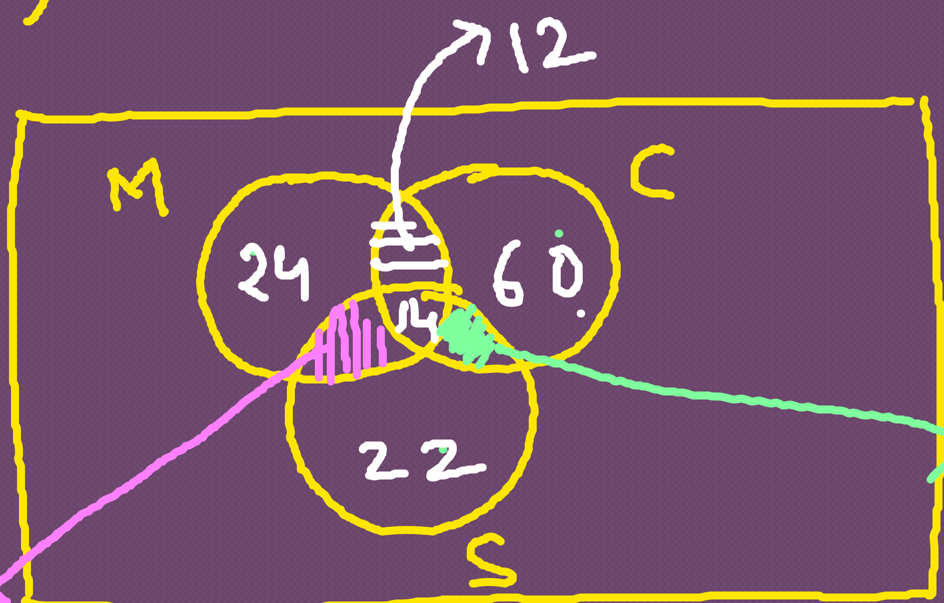
$n(S)$ = No. of students taken Science = 58

$n(M \cap S) = 28$: Both maths and sci

$n(M \cap C) = 26$: Both maths and comp.

$n(C \cap S) = 22$: Both comp. and sci

$n(M \cap C \cap S) = 14$: All three



$n(D) = 260$

$M = 64$

$26 - 14 = 12$

$28 - 14 = 14$

$22 - 14 = 8$

$M = 64 - (12 + 14 + 14) = 24 \leftarrow$ only maths

$\rightarrow C = 94 - (12 + 14 + 8) = 60 \leftarrow$ only com

$S = 58 - (14 + 14 + 8) = 22 \leftarrow$ only Sc.

No. of students who have taken at least one of the subj.

$$n(M \cup C \cup S) = n(M) + n(C) + n(S) - n(M \cap C) - n(C \cap S) - n(M \cap S) + n(M \cap C \cap S)$$

$$= 64 + 94 + 58 - 28 - 26 - 22 + 14$$
$$= 154$$

a) No. of students who have not taken any of the above course

$$n(\overline{M \cup C \cup S}) = n(D) - n(M \cup C \cup S)$$

$$= 260 - 154 = 106$$

b) No. of students who have taken only comp.

$\underline{60}$

$$n(C - M - S) = n(C) - n(C \cap M) - n(C \cap S) + n(C \cap M \cap S)$$

$$= 94 - 26 - 22 + 14 = 60$$

Question

Out of 100 students, 12 drink milk only. 8 drink tea only, 5 drink coffee only. 20 drink milk and coffee, 30 drink coffee and tea, 25 drink milk and tea while 10 drink all three drinks. Find

i) How many do not take any of the three drinks?

ii) How many drink milk but not coffee?

iii) How many drink tea and coffee but not milk ?

Sol^n : $n(S) = 100 \rightarrow$ Total no. of students

$n(M \cap C) = 20 \rightarrow$ The no. of students who drink milk and coffee

$n(T \cap C) = 30 \rightarrow$ The no. of students who drink tea and coffee

$n(M \cap T) = 25 \rightarrow$ The no. of students who drink milk and tea

$n(M \cap T \cap C) = 10 \rightarrow$ The no. of students who take all three

$M \cap C = 20$

$20 - 10 = 10$

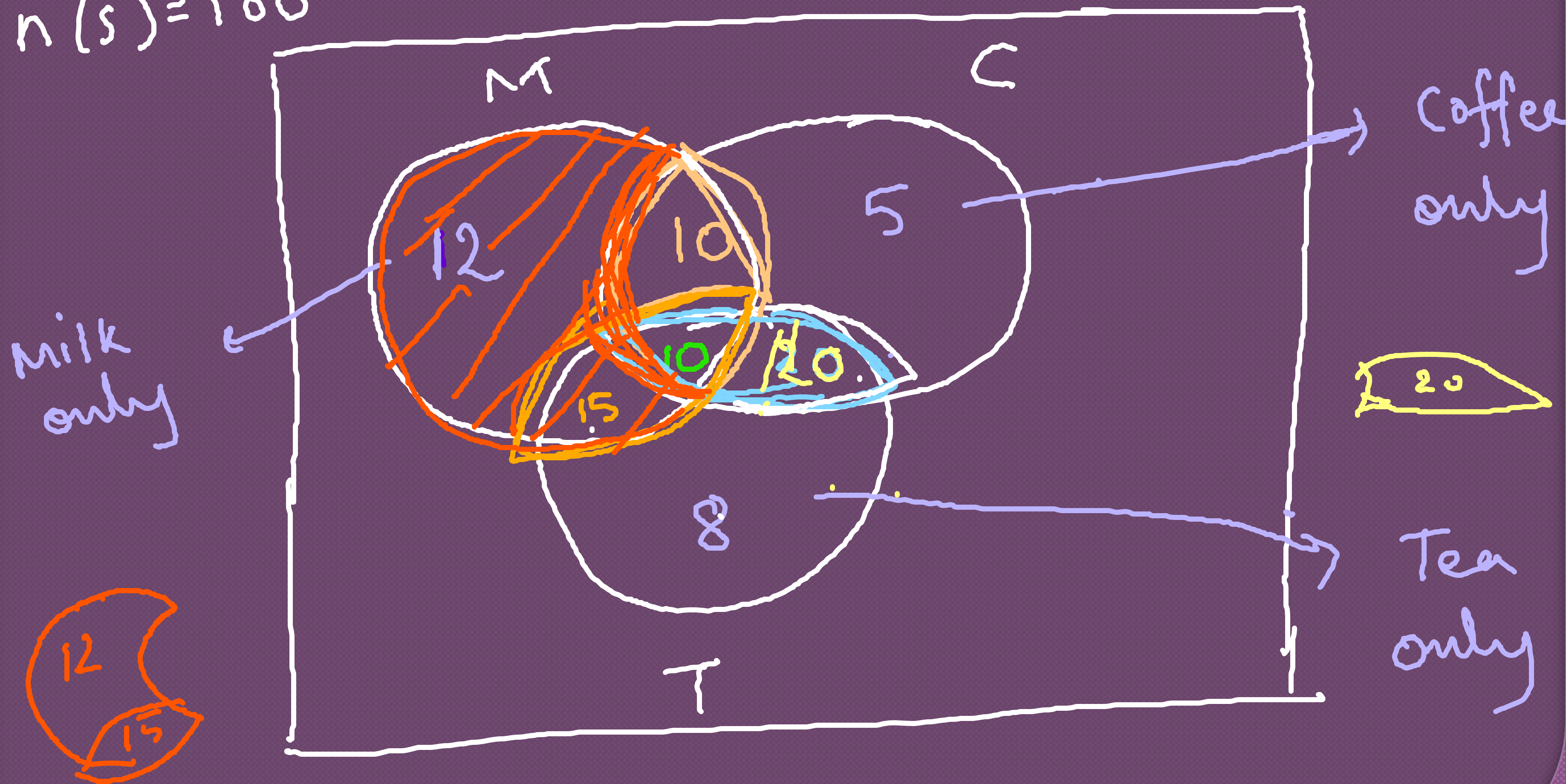
$T \cap C = 30$

$30 - 10 = 20$

$M \cap T = 25$

$25 - 10 = 15$

$n(s) = 100$



i) The no. of students who do not take any of the three drinks

$= n(S) - n(\text{No. of students who take drinks})$

$= 100 - (12 + 5 + 8 + 10 + 15 + 20 + 10)$

$= 100 - 80 = 20$

ii) The no. of students who drink milk but not coffee $= 12 + 15 = 27$

iii) The no. of students who drink tea and coffee but not milk

= 20

Question

In a class, 42% students passed in Maths, 45% students passed in Physics, 41% passed in Chemistry, 16% passed in both Maths and Physics, 19% passed in Physics and Chemistry, 18% passed in Chemistry and Maths. Find the no. of students who passed in all three subjects if there were 260 students in the class and 15% students failed in all subjects.

Soln.

$n(S)$ = The percentage of students in the class = 100

$n(M)$ = The percentage of students who passed in Maths = 42

$n(P)$ = The percentage of students who passed in Physics = 45

$n(C)$ = The percentage of students who passed in Chemistry = 41

$n(M \cap P)$ = The percentage of students who passed in both Maths and Physics

= 16

$n(P \cap C)$ = The percentage of students who passed in both Physics and Chemistry

= 19

$n(C \cap M)$ = The percentage of students who passed in both Chemistry and Maths

= 18

Percentage of students who failed in all subjects = 15

$\therefore$ Percentage of students who passed in all subjects = $n(S)$ − Percentage of students who failed in all subjects

= 100 − 15

= 85

$\therefore n(M \cup P \cup C) = 85$

Using inclusion exclusion principle:

$$n(M \cup P \cup C) = n(M) + n(P) + n(C) - n(M \cap P) - n(P \cap C) - n(C \cap M) + n(M \cap P \cap C)$$

$$\Rightarrow 85 = 42 + 45 + 41 - 16 - 18 - 19 + n(M \cap P \cap C)$$

$$\Rightarrow n(M \cap P \cap C) = 10$$

$\therefore$ The percentage of students who passed in all three subjects = 10

Now draw the venn diagram

$M \cap P = 16$

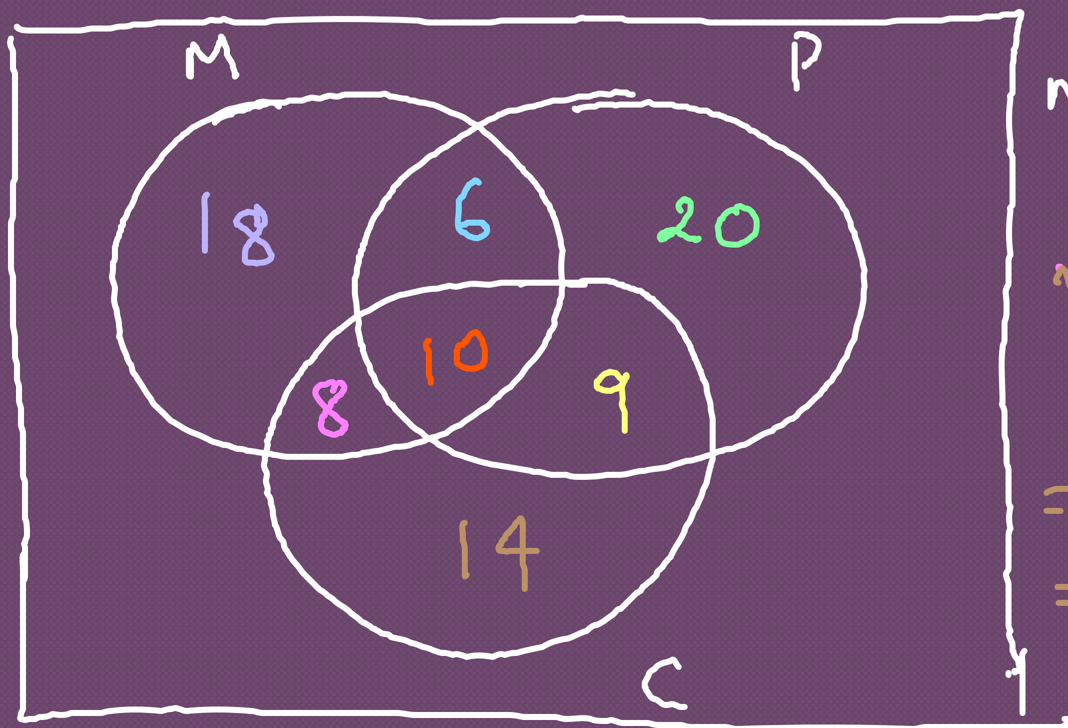
$16 - 10 = 6$

$P \cap C = 19$

$19 - 10 = 9$

$M \cap C = 18$

$18 - 10 = 8$



$n(S) = 100$

$n(C) = 41$

$41 - (8 + 10 + 9)$

$= 41 - 27$

$= 14$

$n(M) = 42$

$42 - (6 + 10 + 8)$

$= 42 - 24 = 18$

$n(P) = 45$

$45 - (6 + 10 + 9)$

$= 45 - 25 = 20.$

$\therefore$ The number of students who passed in all subjects

$$= \frac{10}{100} \times 260$$

$$= 26$$

Note: The complete calculation is done considering 100 percent students but final answer needs to be calculated out of 260 students. So, the no. of students who passed in all subjects is $n(P \cap C \cap M) = 10$ out of 260 it is 26

Questions on Pigeonhole Principle

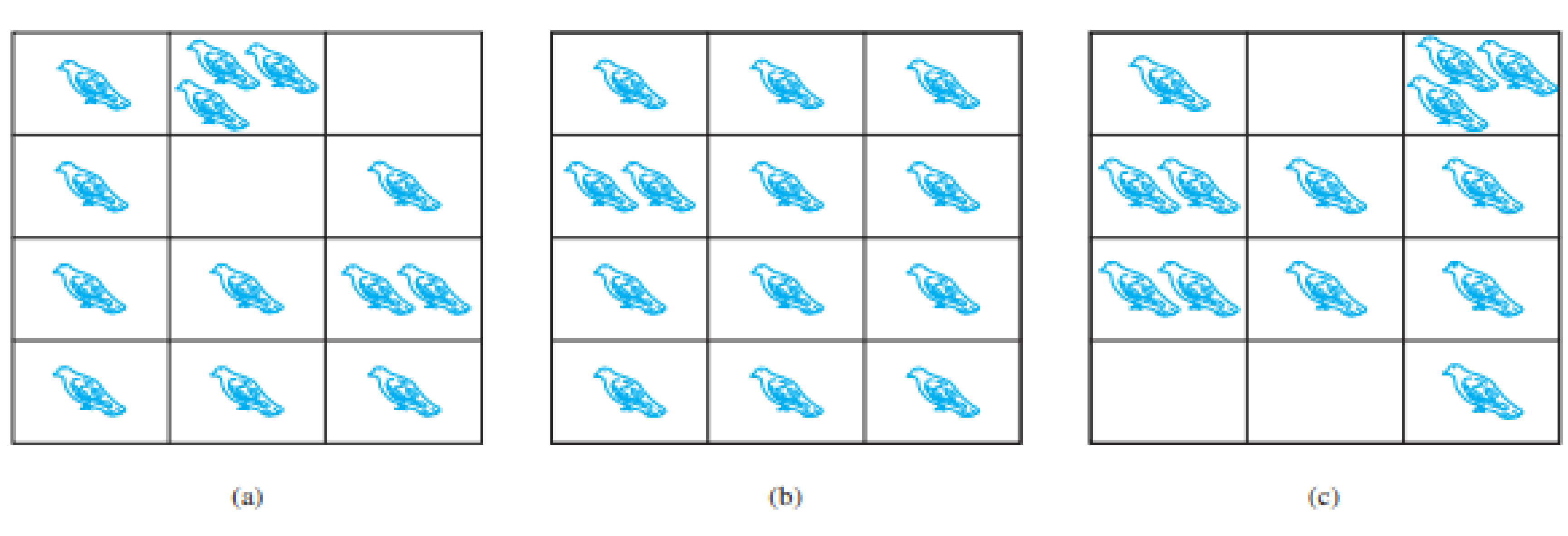
Q.1. If any 5 numbers are chosen from 1 to 8. Show that sum of two of them will be 9

Sol^n:

Construct 4 sets containing two numbers in each set. such that sum of these two no.s will be 9

Pigeonhole Principle

Suppose that a flock of 13 pigeons flies into a set of 12 pigeonholes. Because there are 13 pigeons but only 12 pigeonholes, at least one of these pigeonholes must have at least two pigeons in it.



This illustrates a general principle called the pigeonhole principle, which states that if there are more pigeons than pigeonholes, then there must be at least one pigeonhole with at least two pigeons in it

Pigeonhole Principle

If k is a positive integer and k + 1 or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

Pigeonhole Principle

Example:

How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points?

There are 101 possible scores on the final. The pigeonhole principle shows that among any 102 students there must be at least 2 students with the same score.

Generalized Pigeonhole Principle

If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Generalized Pigeonhole Principle

Example:

What is the minimum number of students required in a discrete mathematics class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D, and F?

Sol:

The minimum number of students needed to ensure that at least six students receive the same grade is the smallest integer N such that $\lceil N/5 \rceil = 6$.

The smallest such integer is $N = 5 \cdot 5 + 1 = 26$.

If we have only 25 students, it is possible for there to be five who have received each grade so that no six students have received the same grade.

Thus, 26 is the minimum number of students needed to ensure that at least six students will receive the same grade.

Extended Pigeonhole Principle

If there are x pigeons and y pigeonholes then one pigeonhole must be occupied by at least [(x-1)/y] +1 pigeons

Extended pigeonhole principle is used if the number of pigeons is much larger than the number of pigeonholes

(mostly in cases where no. of pigeons, x are more than double of the no. of pigeonholes, y)

Question

a) How many friends you must have to guarantee that at least five of them have birthday in the same month?

Sol.

By the extended pigeonhole principle:

$[(x-1)/y] + 1 = 5$

x: no of pigeons

y : no. of pigeonholes =no. of months in a year=12

$[(x-1)/12] + 1 = 5$

$x = 49$

Hence , if there are 49 friends then at least five of them will have birthday in the same month

Questions on Pigeonhole Principle

Q.1. If any 5 numbers are chosen from 1 to 8. Show that sum of two of them will be 9

Sol^n:

Construct 4 sets containing two numbers in each set. such that sum of these two no.s will be 9

Set 1 : $\{1, 8\}$ Set 2 : $\{2, 7\}$

Set 3: $\{3, 6\}$ Set 4: $\{4, 5\}$

Each of the five no.s chosen from 1 to 8 must belong to these 4 sets. e.g → 2, 5, 6, 1, 3 or 8, 5, 4, 7, 3

Since there are five no.s and 4 sets, the pigeonhole principle states that: two of the chosen no.s must belong to same set.

Here if we choose 2, 5, 6, 1, 3

then 3, 6 $\in$ set 3 and their sum is equal to 9

Similarly if we choose 8, 5, 4, 7, 3 then 5, 4 $\in$ set 4 and their sum is equal to 9

Q.2. How many bicycles will be reqd. to paint using 7 colours so that at least 8 bicycles will have same colour?

Sol^n: By extended pigeonhole principle.

$$\left\lceil \frac{n-1}{m} \right\rceil + 1 = 8$$

$n \to$ no. of bicycles
$m \to$ no. of colours
$n > m$

$$\Rightarrow \left\lceil \frac{n-1}{7} \right\rceil + 1 = 8 \Rightarrow \frac{n-1}{7} = 7 \Rightarrow n = 50$$

So, total 50 bicycles will be required which can be painted using 7 colours such that at least 8 of them will have the same colour.