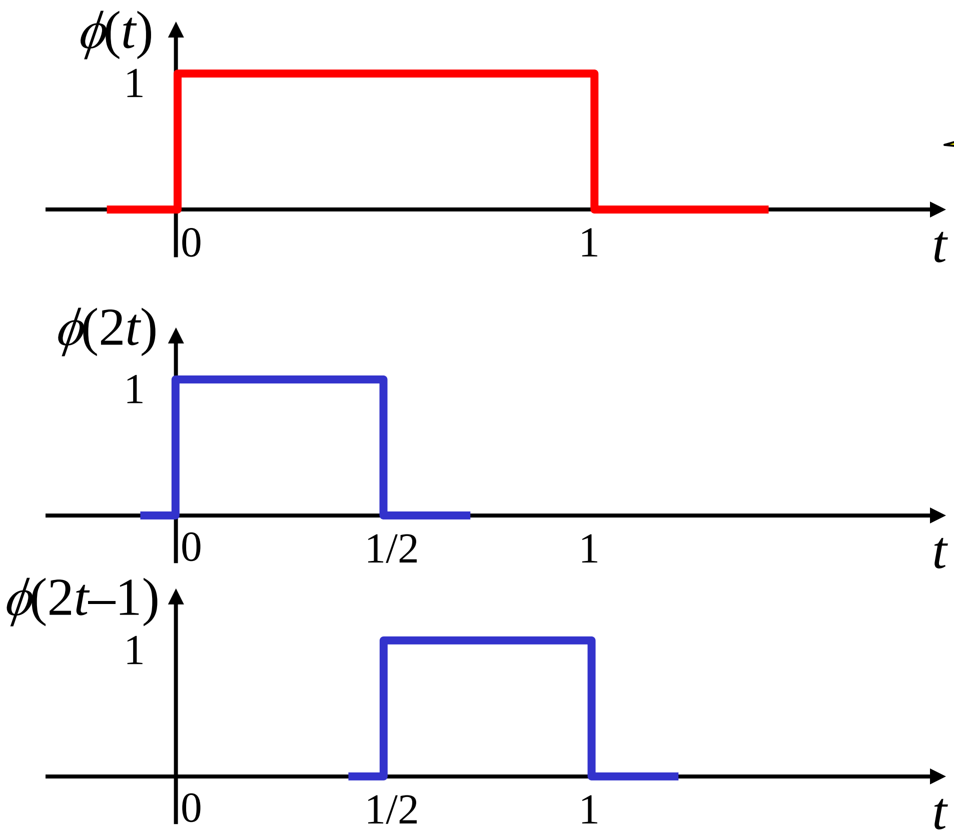
Wavelet Example: Haar Wavelet

Suppose we specify the MRE coefficients to be $h[n] = \left\{ \frac{1}{\sqrt{2}} \quad , \quad \frac{1}{\sqrt{2}} \right\}$

Then the MRE becomes $\phi(t) = \sum_n h(n)\sqrt{2}\phi(2t - n)$ $\Rightarrow$ $\varphi(t) = \varphi(2t) + \varphi(2t - 1)$

Clearly the scaling function $\phi(t)$ as shown below satisfies this MRE



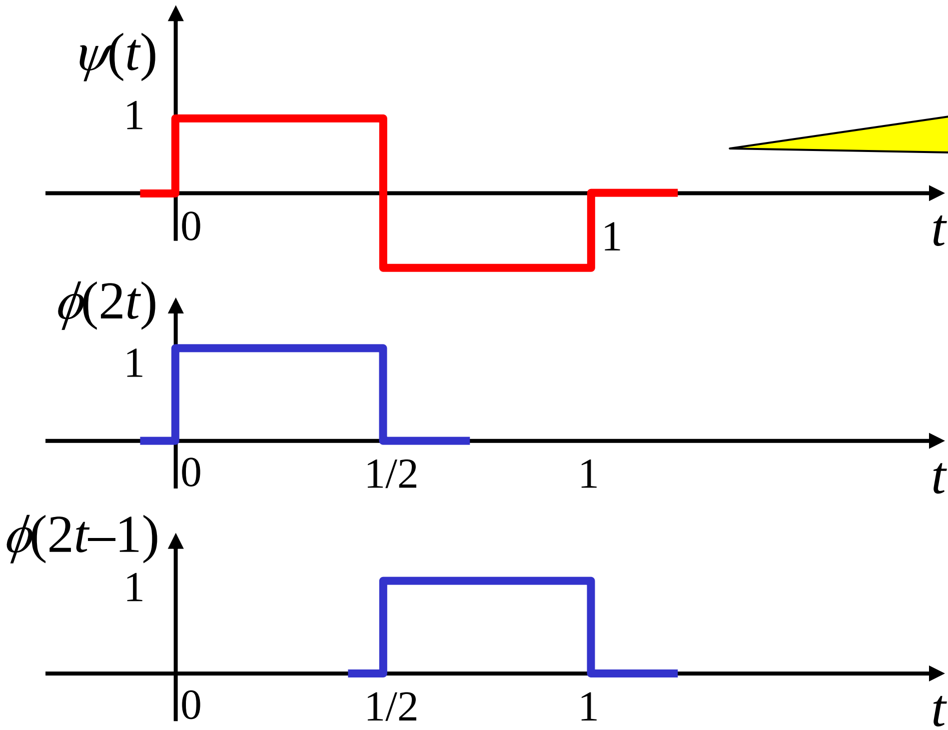
This is the Haar Scaling Function

- Special case: finite number N of nonzero $h(n)$ and ON wavelets & scaling functions
- Given the $h(n)$ for the scaling function, then the $h_1(n)$ that define the wavelet function are given by $h_1[n] = (-1)^n h(N-1-n)$ where N is the length of the filter

Thus the WE coefficients are $$h_1[n] = \left\{ \frac{1}{\sqrt{2}} \quad , \quad -\frac{1}{\sqrt{2}} \right\}$$

Then the WE becomes $$\psi(t) = \sum_n h_1(n)\sqrt{2}\varphi(2t-n) \Rightarrow \psi(t) = \varphi(2t) - \varphi(2t-1)$$

Clearly the scaling function $\phi(t)$ as shown below satisfies this MRE



This is the Haar Wavelet Function

Next

Define a nested set of signal spaces

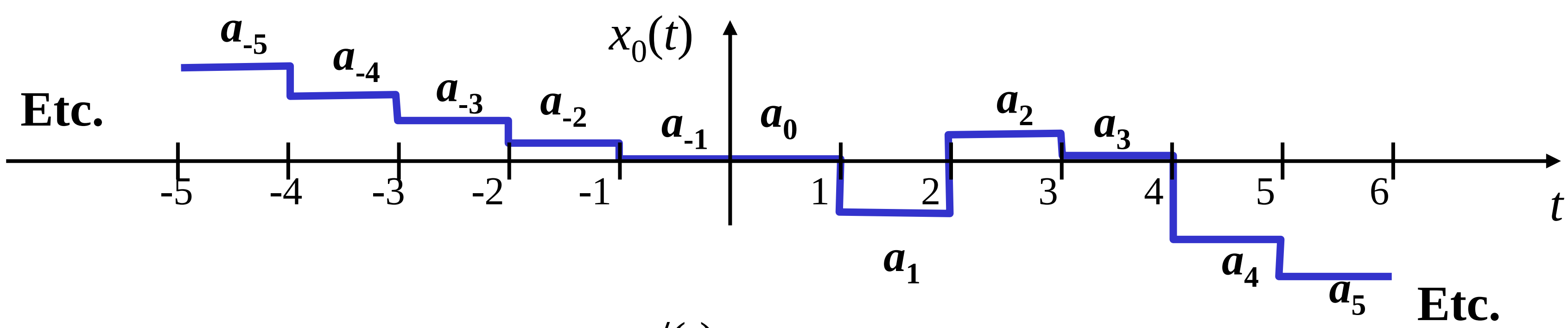
$$\cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots \subset L^2$$

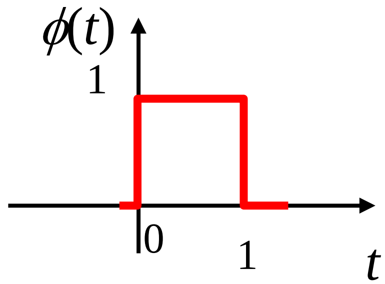
Let V_0 be the space spanned by the integer translations of scaling function $\phi(t)$ so that ***if*** $x_0(t)$ is in V_0 ***then*** it can be represented by:

$$x_0(t) = \sum_k a_k \varphi(t-k)$$

Q: For the Haar scaling function what kind of functions are in V_0??

A: Those that are "piece-wise" constant on the intervals $[k,k+1]$ for integer k…



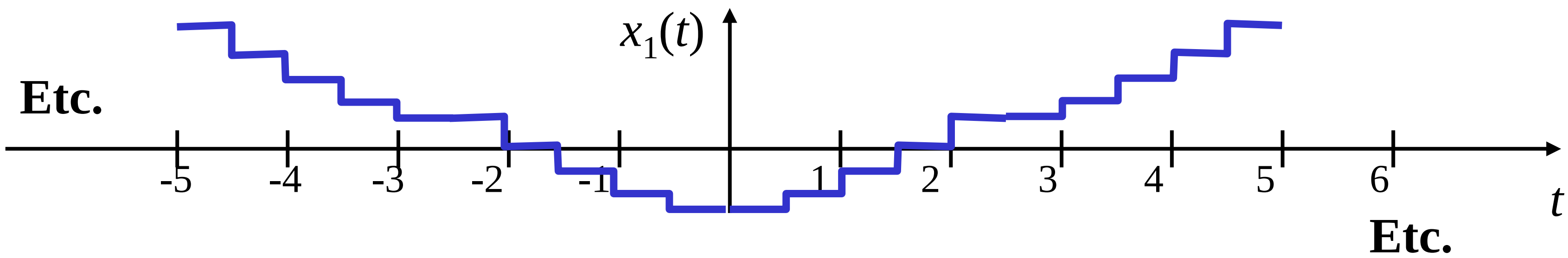


Next

If we let V_1 be the space spanned by integer translates of $\phi(2t)$ then V_1 is indeed a space of functions having higher resolution.

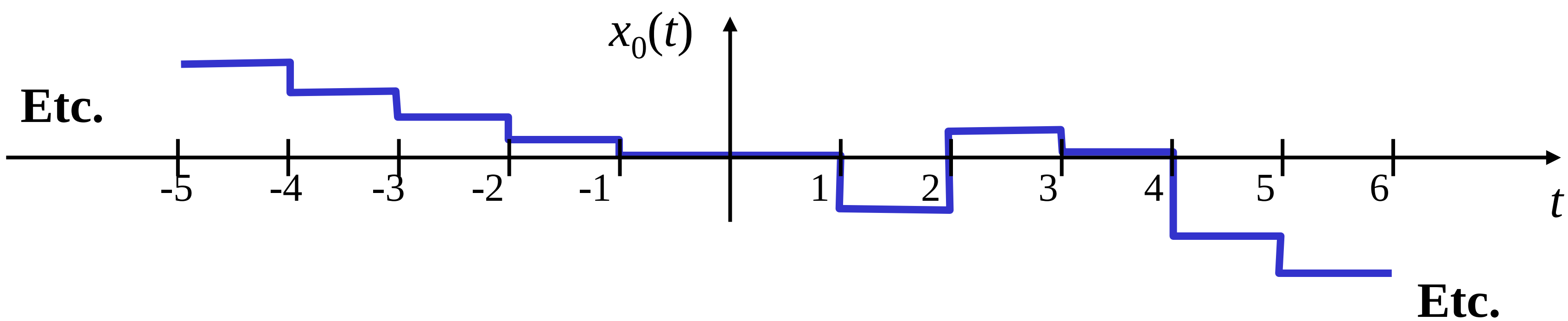
Q: For the Haar scaling function what kind of functions are in V_1??

A: Those that are "piece-wise" constant on the intervals $[k/2, k/2+ ½]$ for integer k



Note: $x_0(t)$ is also in V_1 because it is also "piece-wise" constant on $[k/2, k/2+ ½]$

In fact, $x_0(t)$ is also in every V_j for $j \geq 0$ … that is the nesting!!!



Next

If we keep going to higher j values we get finer and finer resolution and can ultimately express (in the limit of j) any finite energy signal

Increasing j

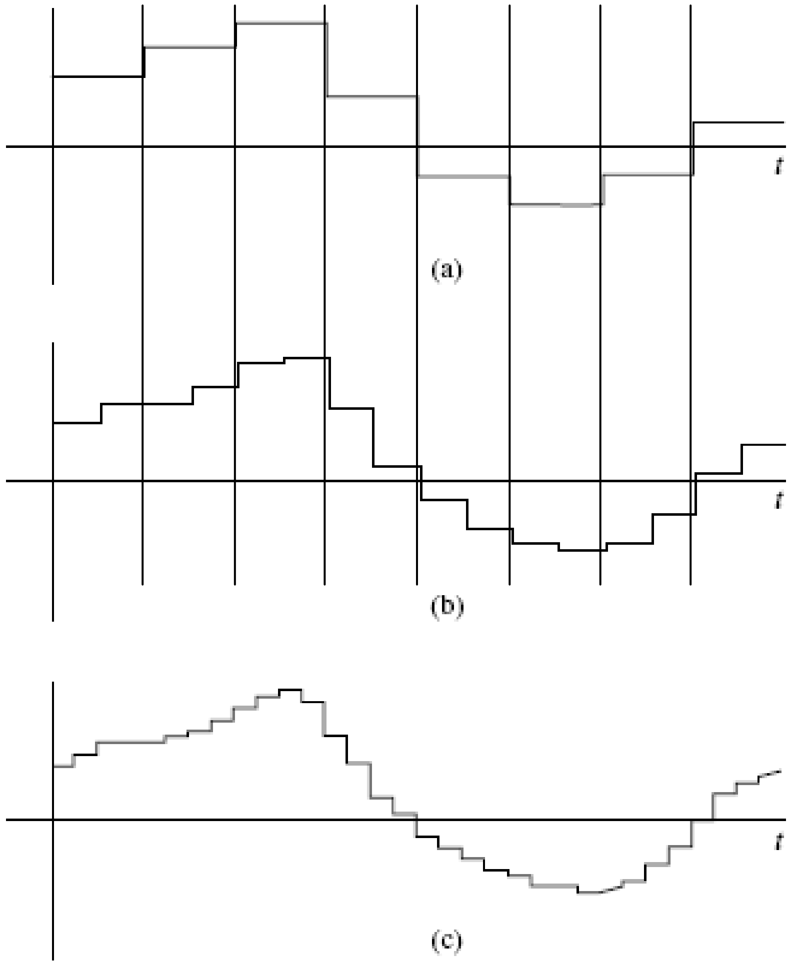


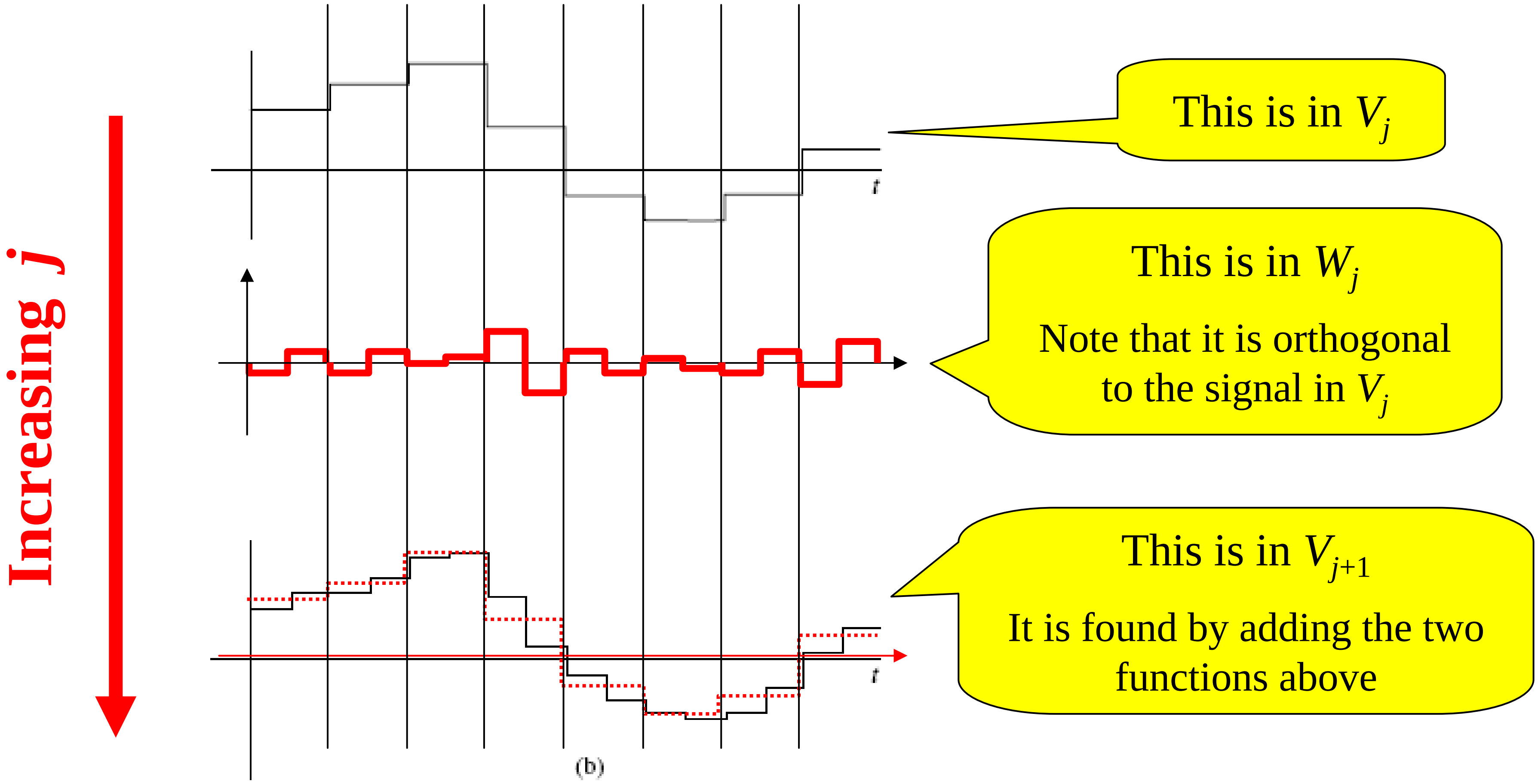
Figure 15.8 from Textbook

This MRA development started at V_0 and worked its way up to higher resolutions…

Next

How do the wavelets enter into this?

- To go from V_j to higher resolution V_{j+1} requires the addition of “details”
  - These details are the part of V_{j+1} not able to be represented in V_j
  - This is captured through W_j the “orthogonal complement” of V_j w.r.t V_{j+1}



Next

EE269
Signal Processing for Machine Learning

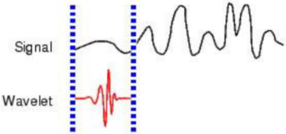
Lecture 17

Instructor : Mert Pilanci

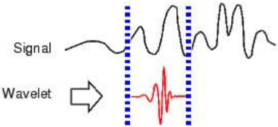
Stanford University

March 13, 2019

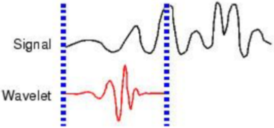
Wavelets



Low value for $W_\psi(s,\tau)$



Higher value of $W_\psi(s,\tau_2)$



Different scale

Discrete Wavelet Transform

- Discrete shifts and scales $\psi(\frac{t-\tau}{s})$
- Suppose we have a signal of length N

$$x = [x_1, x_2, ...x_N]$$

- Consider a length $N/2$ approximation of x, e.g., for transmission

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- example

$$x = [6, 12, 15, 15, 14, 12, 120, 116] \rightarrow s = [9, 15, 13, 118]$$

- suppose that we are allowed to send $N/2$ more numbers
- differences

$$d_k = \frac{x_{2k-1} - x_{2k}}{2}, \quad k = 1, ..., N/2$$

- we can recover x

$$x = [6, 12, 15, 15, 14, 12, 120, 116] \rightarrow$$
$$[s \mid d] = [9, 15, 13, 118 \mid 3, 0, -1, -2]$$

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$$x = [6, 12, 15, 15, 14, 12, 120, 116] \to$$
$$[s \mid d] = [9, 15, 13, 118 \mid 3, 0, -1, -2]$$

- One step Haar Transformation $x \to [s|d]$

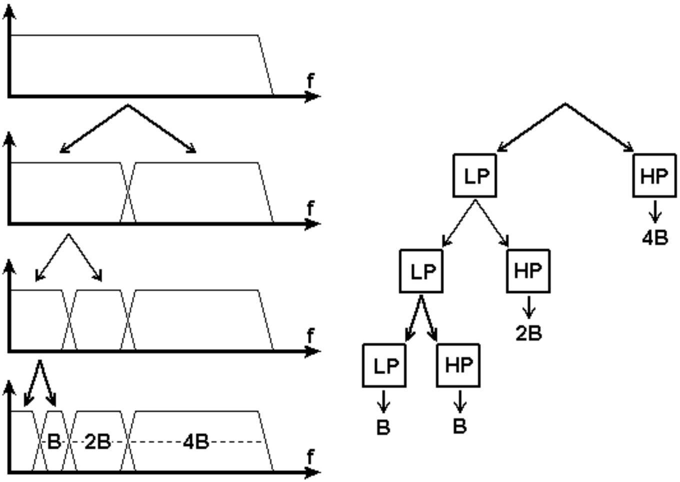
One Step Haar Transformation

$$\left[\begin{array}{cccccccc} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \hline -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{array}\right] \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \frac{1}{2} \left[\begin{array}{c} x_1 + x_2 \\ x_3 + x_4 \\ x_5 + x_6 \\ x_7 + x_8 \\ \hline x_2 - x_1 \\ x_4 - x_3 \\ x_6 - x_5 \\ x_8 - x_7 \end{array}\right]$$

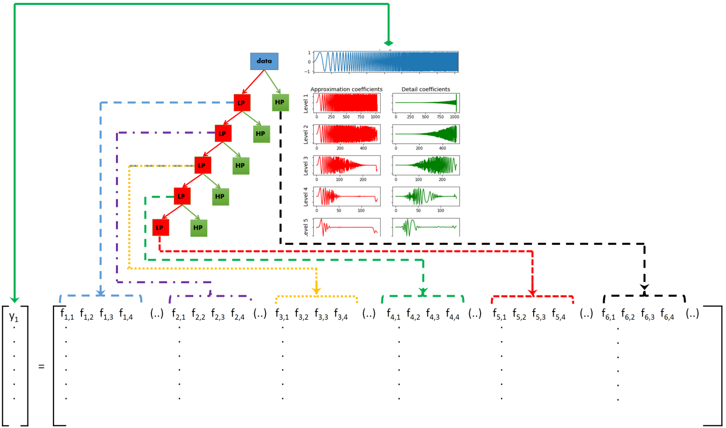
Discrete Haar Transform Matrix

- repeat the computation on the **means**
- keep differences in each step

Discrete Haar Wavelet Transform



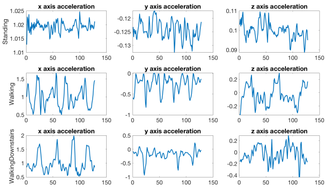
Wavelet Transform Features



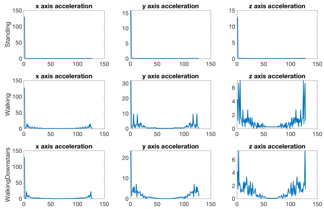
- mean, median
- variance
- zero crossing rate, mean crossing rate
- entropy

slide credit: A. Taspinar

Results: training set: 7724 signals, test set: 2575 signals



3-Nearest Neighbors, ℓ_2-norm distance on $x[n]$. **Accuracy** : 0.77



3-Nearest Neighbors, ℓ_2-norm distance on $|X[k]|$. **Accuracy** : 0.85

Human Activity Recognition dataset

- 3-Nearest Neighbors, ℓ_2-norm distance on $x[n]$.

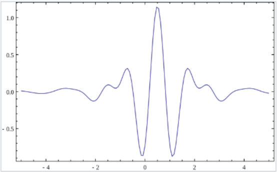
  accuracy : 77%
- 3-Nearest Neighbors, ℓ_2-norm distance on $|X[k]|$.

  accuracy : 85%
- 1D Convolutional Net (4 layers)

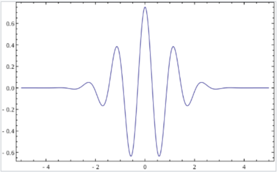
  accuracy : 91%
- Wavelet Transform Features (entropy, zero crossing, simple statistics) $+$ linear classifier

  accuracy : 95%

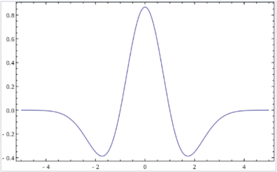
Other Wavelets



Meyer



Morlet

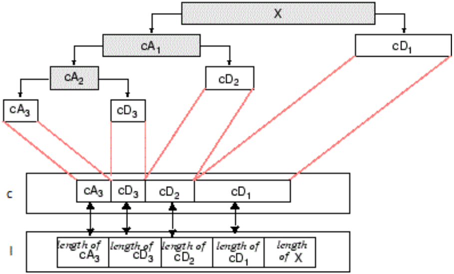


Mexican hat

Other Wavelets

- In MATLAB

  `[c,l] = wavedec(x,n,wname)` returns the wavelet decomposition of the signal x at level n using the wavelet *wname*



What makes a good wavelet

Application specific

- Compact time support vs frequency support
- Smoothness
- Orthogonality