Numerical Optimization Methods in Imaging

Part I: Subdifferential and proximity operator

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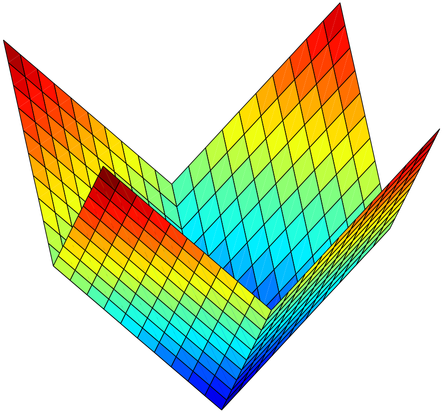
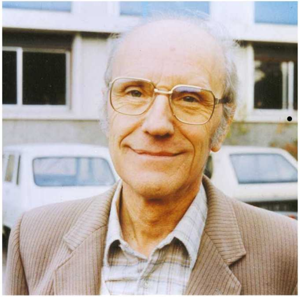
PhD Summer School MMLIA – Bologna

Non-smooth convex optimization



A pioneer



Jean-Jacques Moreau
(1923–2014)

Subdifferential of function: definition

The (Moreau) subdifferential of f, denoted by ∂f,

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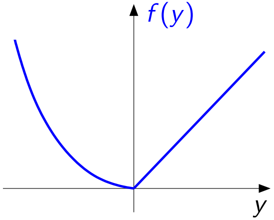

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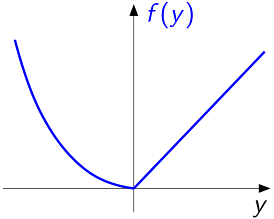

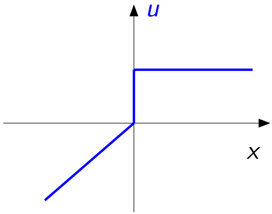

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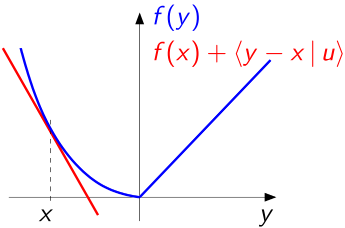

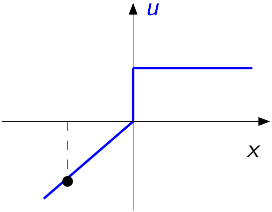

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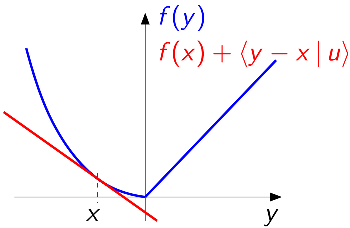

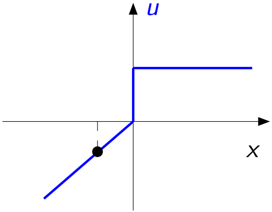

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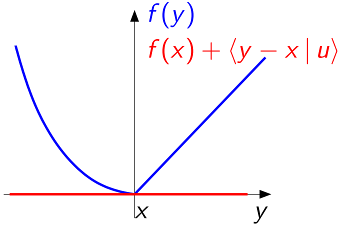

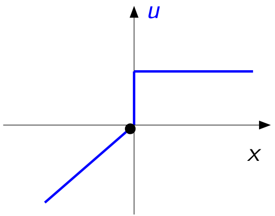

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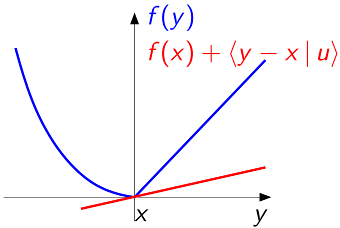

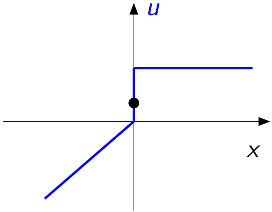

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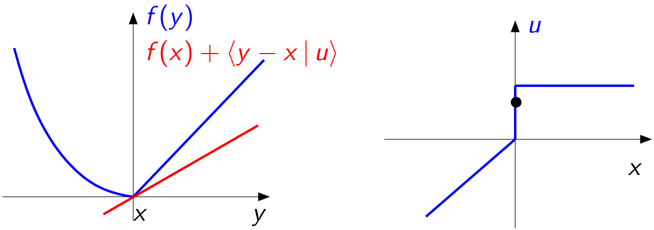

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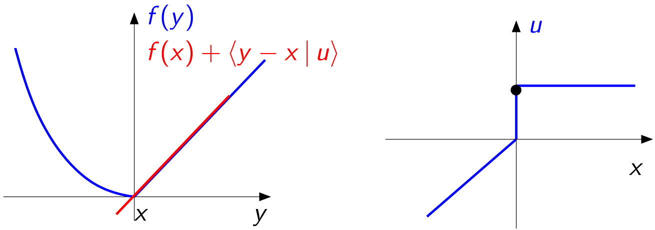

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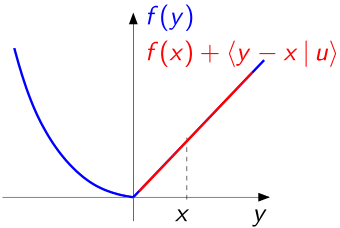

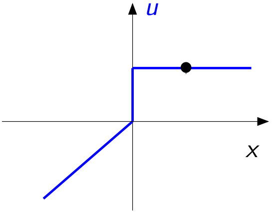

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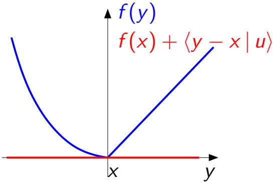

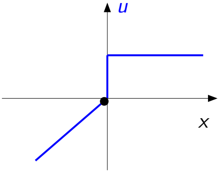

Fermat's rule: $0 \in \partial f(x) \Leftrightarrow x \in \operatorname{Argmin} f$

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- $u \in \partial f(x)$ is a subgradient of f at x.

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- If $x \notin \operatorname{dom} f = \left\{x \in \mathcal{H} \;\middle|\; f(x) < +\infty\right\}$, then $\partial f(x) = \varnothing$.
- For every $x \in \operatorname{dom} f$, $\partial f(x)$ is a closed and convex set.

Subdifferential of a function: properties

Let $f\colon \mathcal{H} \to \,]-\infty,+\infty]$ be a proper function.
Its subdifferential is a monotone operator, i.e.

$$\big(\forall (x_1,x_2)\in\mathcal{H}^2\big)\big(\forall u_1\in\partial f(x_1)\big)\big(\forall u_2\in\partial f(x_2)\big)\quad \langle u_1-u_2 \mid x_1-x_2\rangle \geq 0.$$

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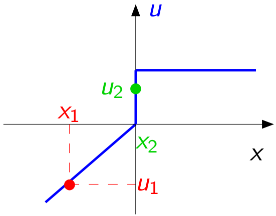
- Proof:
  By definition:
  $$\langle x_2 - x_1 | u_1\rangle + f(x_1) \leq f(x_2)$$
  $$\langle x_1 - x_2 | u_2\rangle + f(x_2) \leq f(x_1)$$
- It results that $\boxed{\langle x_1 - x_2 \mid u_1 - u_2\rangle \geq 0}$.



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If $f : \mathcal{H} \to \,]-\infty, +\infty]$ is convex and it is Gâteaux differentiable at x, then

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$$(\forall y \in \mathcal{H}) \qquad \langle \nabla f(x) \mid y \rangle = \lim_{\substack{\alpha \to 0 \\ \alpha \neq 0}} \frac{f(x+\alpha y) - f(x)}{\alpha}.$$

<u>Proof</u>:
For every $\alpha \in [0,1]$ and $y \in \mathcal{H}$,

$$f\big(x + \alpha(y-x)\big) \le (1-\alpha) f(x) + \alpha f(y)$$

$$\Rightarrow \quad \langle \nabla f(x) \mid y - x \rangle = \lim_{\substack{\alpha \to 0 \\ \alpha > 0}} \frac{f\big(x+\alpha(y-x)\big) - f(x)}{\alpha} \le f(y) - f(x)$$

Then $\nabla f(x) \in \partial f(x)$.

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Proof:
Conversely, if $u \in \partial f(x)$, then, for every $\alpha \in [0, +\infty[$ and $y \in \mathcal{H}$,

$$\begin{aligned} & f(x+\alpha y) \geq f(x) + \langle u \mid x + \alpha y - x \rangle \\ \Rightarrow \quad & \langle \nabla f(x) \mid y \rangle = \lim_{\substack{\alpha \to 0 \\ \alpha > 0}} \frac{f(x+\alpha y) - f(x)}{\alpha} \geq \langle u \mid y \rangle . \end{aligned}$$

By selecting $y = u - \nabla f(x)$, it results that $\|u - \nabla f(x)\|^2 \leq 0$ and then $u = \nabla f(x)$.

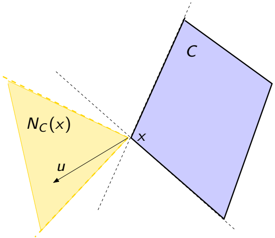
Subdifferential of a convex function: example

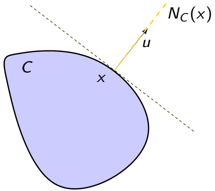
Let C be a nonempty subset of $\mathcal{H}$.
Its indicator function is

$$(\forall x \in \mathcal{H}) \quad \iota_C(x) = \begin{cases} 0 & \text{if } x \in C \\ +\infty & \text{otherwise.} \end{cases}$$

For every $x \in \mathcal{H}$, $\partial \iota_C(x)$ is the normal cone to C at x defined by

$$N_C(x) = \begin{cases} \{u \in \mathcal{H} \mid (\forall y \in C) \ \ \langle u \mid y - x \rangle \leq 0\} & \text{if } x \in C \\ \varnothing & \text{otherwise.} \end{cases}$$





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- If $x \in \operatorname{int} C$, then $N_C(x) = \{0\}$.
- If C is a vector space, then for every $x \in C$, $N_C(x) = C^\perp$.

Subdifferential calculus

Let $\mathcal{H}$ and $\mathcal{G}$ be two real Hilbert spaces.

- Let $f\colon \mathcal{H} \to \left]-\infty,+\infty\right]$ be proper, i.e., $\operatorname{dom} f \neq \varnothing$, then for every $\lambda \in \left]0,+\infty\right[$ $\partial(\lambda f) = \lambda \partial f$.
- Let $f\colon \mathcal{H} \to \left]-\infty,+\infty\right]$, $g\colon \mathcal{G} \to \left]-\infty,+\infty\right]$, and $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$.
  Let L^* denote the adjoint of L.
  If $\operatorname{dom} g \cap L(\operatorname{dom} f) \neq \varnothing$, then

$$(\forall x \in \mathcal{H}) \qquad \partial f(x) + L^* \partial g(Lx) \subset \partial(f + g \circ L)(x).$$

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<u>Proof</u>: Let $x \in \mathcal{H}$

$$\partial f(x) + L^*\partial g(Lx) = \left\{u + L^* v \;\middle|\; u \in \partial f(x), v \in \partial g(Lx)\right\}$$

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Proof: Let $x \in \mathcal{H}$ Let $x \in \mathcal{H}$, $u \in \partial f(x)$ and $v \in \partial g(Lx)$. We have: $u + L^*v \in \partial f(x) + L^*\partial g(Lx)$ and

$$(\forall y \in \mathcal{H}) \qquad \begin{aligned} f(y) &\geq f(x) + \langle y - x \mid u \rangle \\ g(Ly) &\geq g(Lx) + \langle L(y - x) \mid v \rangle \,. \end{aligned}$$

Therefore, by summing,

$$f(y) + g(Ly) \geq f(x) + g(Lx) + \langle y - x \mid u + L^*v \rangle \,.$$

We deduce that $u + L^*v \in \partial(f + g \circ L)(x)$.

Subdifferential calculus

Let $\mathcal{H}$ and $\mathcal{G}$ be two real Hilbert spaces.
Let $f \in \Gamma_0(\mathcal{H})$, $g \in \Gamma_0(\mathcal{G})$, and $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$.
If $\operatorname{int}(\operatorname{dom} g - L(\operatorname{dom} f)) \neq \varnothing$, then

$$\partial f + L^* \circ \partial g \circ L = \partial(f + g \circ L).$$

Particular case:

- If $f \in \Gamma_0(\mathcal{H})$, $g \in \Gamma_0(\mathcal{H})$, and g is finite valued, then $\partial f + \partial g = \partial(f + g)$.
- If $g \in \Gamma_0(\mathcal{G})$, $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$, and $\operatorname{int}(\operatorname{dom} g) \cap \operatorname{ran} L \neq \varnothing$ or $\operatorname{ran} L = \mathcal{H}$, then $L^* \circ \partial g \circ L = \partial(g \circ L)$.

Subdifferential calculus

Let I be a finite subset of $\mathbb{N}$.
Let $(\mathcal{H}_i)_{i\in I}$ be Hilbert spaces and let $\mathcal{H} = \bigtimes_{i\in I} \mathcal{H}_i$.
For every $i \in I$, let $f_i \colon \mathcal{H}_i \to \left]-\infty,+\infty\right]$ be a proper function. Let

$$f \colon \mathcal{H} \to \left]-\infty,+\infty\right] : x = (x_i)_{i\in I} \mapsto \sum_{i\in I} f_i(x_i)$$

Then,

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<u>Proof</u>: Let $x = (x_i)_{i\in I} \in \mathcal{H}$. We have

$$\begin{aligned}
& t = (t_i)_{i\in I} \in \bigtimes_{i\in I} \partial f_i(x_i) \\
\Leftrightarrow\quad & (\forall i \in I)(\forall y_i \in \mathcal{H}_i) \;\; f_i(y_i) \geq f_i(x_i) + \langle t_i \mid y_i - x_i \rangle \\
\Rightarrow\quad & (\forall y = (y_i)_{i\in I} \in \mathcal{H}) \;\; \sum_{i\in I} f_i(y_i) \geq \sum_{i\in I} f_i(x_i) + \sum_{i\in I} \langle t_i \mid y_i - x_i \rangle \\
\Leftrightarrow\quad & (\forall y \in \mathcal{H}) \;\; f(y) \geq f(x) + \langle t \mid y - x \rangle .
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Then,

$$(\forall x = (x_i)_{i\in I} \in \mathcal{H}) \qquad \partial f(x) = \bigtimes_{i\in I} \partial f_i(x_i).$$

Proof: Conversely,

$$\begin{aligned} & t = (t_i)_{i\in I} \in \partial f(x) \\ \Leftrightarrow \quad & (\forall y = (y_i)_{i\in I} \in \mathcal{H}) \ \ \sum_{i\in I} f_i(y_i) \geq \sum_{i\in I} f_i(x_i) + \sum_{i\in I} \langle t_i \mid y_i - x_i \rangle . \end{aligned}$$

Let $j \in I$. By setting $(\forall i \in I \setminus \{j\})$ $y_i = x_i \in \operatorname{dom} f_i$, we get

$$(\forall y_j \in \mathcal{H}_j) \ \ f_j(y_j) \geq f_j(x_j) + \langle t_j \mid y_j - x_j \rangle .$$

Conjugate

Adrien-Marie Legendre
(1752–1833)

Werner Fenchel
(1905–1988)

Conjugate

Adrien-Marie Legendre
(1752–1833)

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Conjugate: reminders

Let $\mathcal{H}$ be a Hilbert space and $f\colon \mathcal{H} \to \,]-\infty,+\infty]$.
The conjugate of f is the function $f^*\colon \mathcal{H} \to [-\infty,+\infty]$ such that

$$(\forall u \in \mathcal{H}) \qquad f^*(u) = \sup_{x\in\mathcal{H}} \big(\langle x \mid u\rangle - f(x)\big).$$

Moreau-Fenchel theorem

Let $\mathcal{H}$ be a Hilbert space and $f\colon \mathcal{H} \to \,]-\infty,+\infty]$ be a proper function.

$$f \text{ is l.s.c. and convex} \Leftrightarrow f^{**} = f.$$

Conjugate: properties

Fenchel-Young inequality : If f is proper, then

1. $\big(\forall (x,u) \in \mathcal{H}^2\big) \qquad f(x) + f^*(u) \geq \langle x \mid u \rangle$

2. $\big(\forall (x,u) \in \mathcal{H}^2\big) \qquad u \in \partial f(x) \quad \Leftrightarrow \quad f(x) + f^*(u) = \langle x \mid u \rangle.$

Conjugate: properties

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Proof: Let $(x,u) \in \mathcal{H}^2$. We have

$$
\begin{aligned}
& f(x) + f^*(u) = \langle x \mid u \rangle \\
\Leftrightarrow\ & f(x) + f^*(u) \leq \langle x \mid u \rangle \\
\Leftrightarrow\ & \sup_{y \in \mathcal{H}} \langle y \mid u \rangle - f(y) \leq \langle x \mid u \rangle - f(x) \\
\Leftrightarrow\ & (\forall y \in \mathcal{H})\ \langle y \mid u \rangle - f(y) \leq \langle x \mid u \rangle - f(x) \\
\Leftrightarrow\ & (\forall y \in \mathcal{H})\ f(y) \geq f(x) + \langle y - x \mid u \rangle \\
\Leftrightarrow\ & u \in \partial f(x).
\end{aligned}
$$

Conjugate: properties

Fenchel-Young inequality : If f is proper, then

1. $\big(\forall (x,u) \in \mathcal{H}^2\big) \qquad f(x) + f^*(u) \geq \langle x \mid u \rangle$
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If $f \in \Gamma_0(\mathcal{H})$, then

$$\big(\forall (x,u) \in \mathcal{H}^2\big) \qquad u \in \partial f(x) \ \Leftrightarrow \ x \in \partial f^*(u).$$

Conjugate: properties

Fenchel-Young inequality: If f is proper, then

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If $f\in\Gamma_0(\mathcal{H})$, then

$$\big(\forall(x,u)\in\mathcal{H}^2\big) \qquad u\in\partial f(x)\ \Leftrightarrow\ x\in\partial f^*(u).$$

Proof: Since $f\in\Gamma_0(\mathcal{H})$, we have

$$f^*(u)+f^{**}(x)=\langle x\mid u\rangle\,.$$

which is equivalent to $x\in\partial f^*(u)$.

Proximity operator: definition

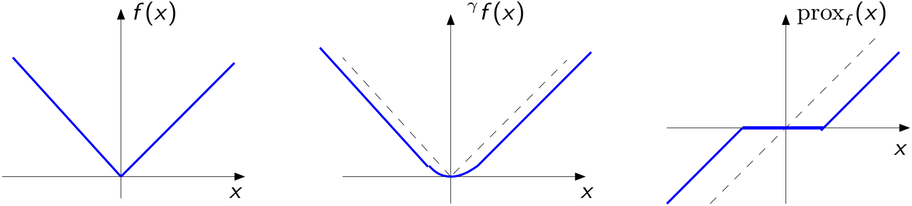
Let $\mathcal{H}$ be a Hilbert space. Let $f \in \Gamma_0(\mathcal{H})$.

- The **Moreau envelope** of f of parameter $\gamma \in\,]0,+\infty[$ is

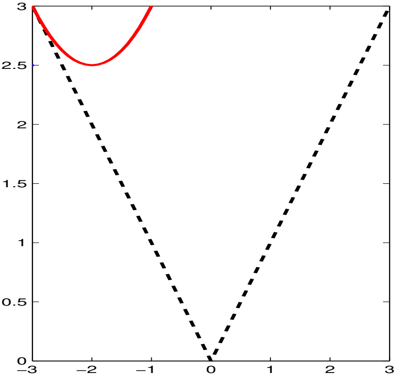
$$^{\gamma}f\colon \mathcal{H} \to \mathbb{R}\colon x \mapsto \inf_{y\in\mathcal{H}} f(y) + \frac{1}{2\gamma}\|y-x\|^2.$$

- The **proximity operator** of f is

$$\mathrm{prox}_f\colon \mathcal{H} \to \mathcal{H}\colon x \mapsto \underset{y\in\mathcal{H}}{\mathrm{argmin}}\; f(y) + \frac{1}{2}\|y-x\|^2.$$

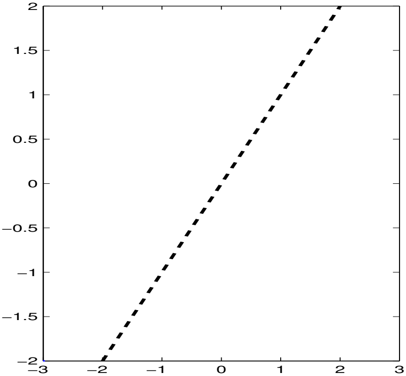


Proximity operator: definition



Moreau envelope

$$^{\gamma}f(x) = \inf_{y \in \mathcal{H}} f(y) + \frac{1}{2\gamma}\|y - x\|^2$$



Proximity operator

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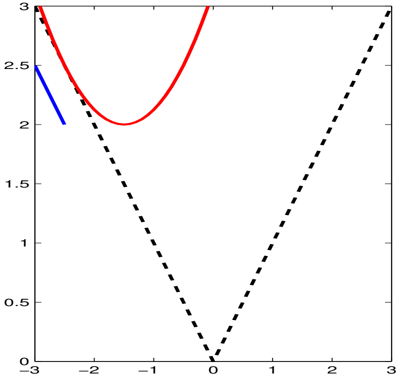
Proximity operator: definition



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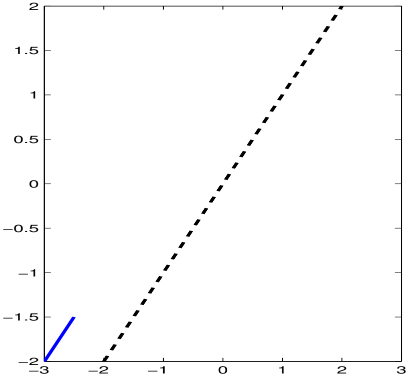

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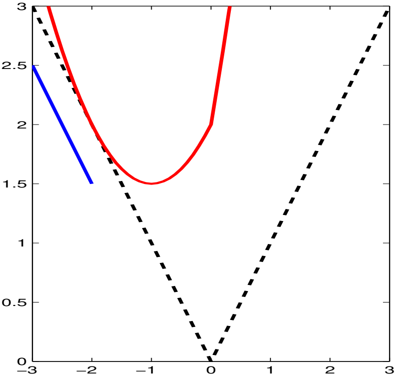

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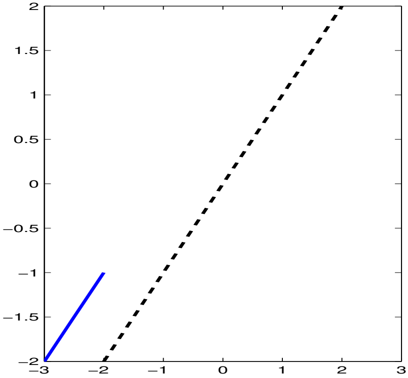

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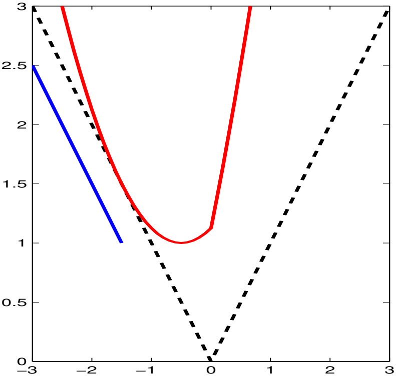

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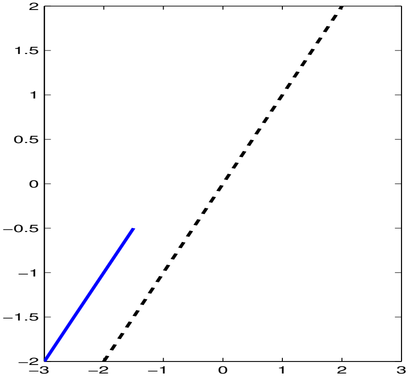

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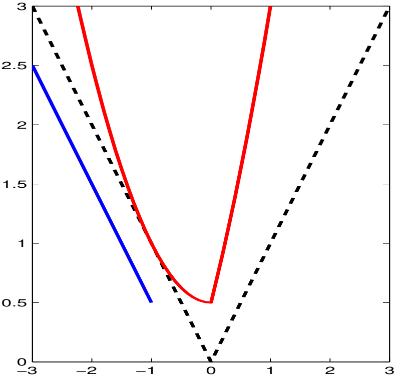

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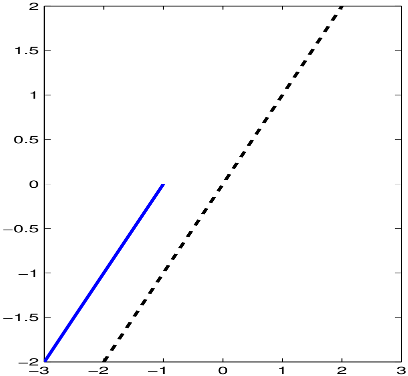

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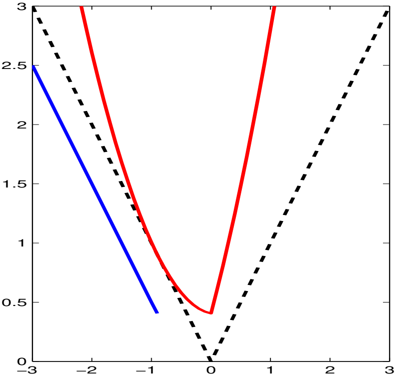

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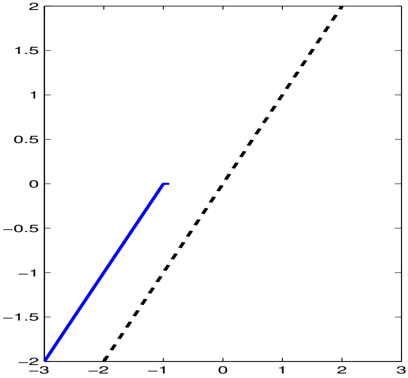

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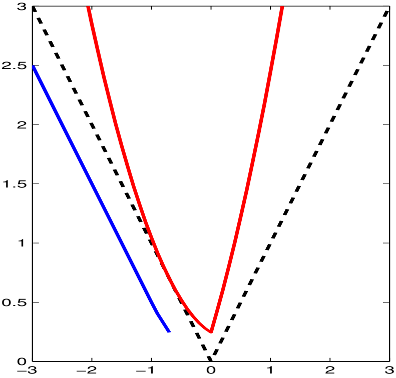

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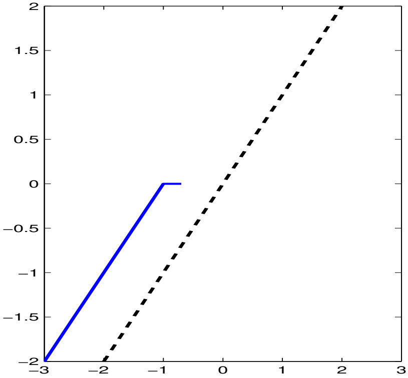

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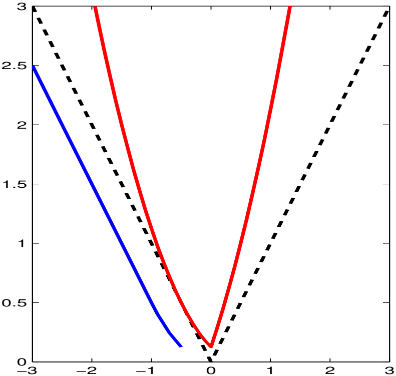

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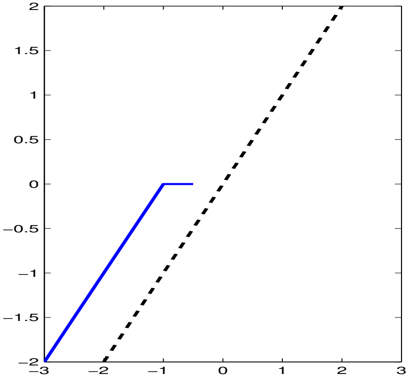

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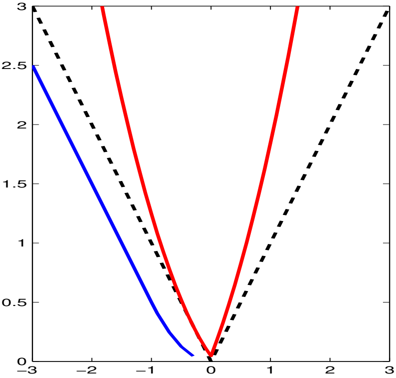

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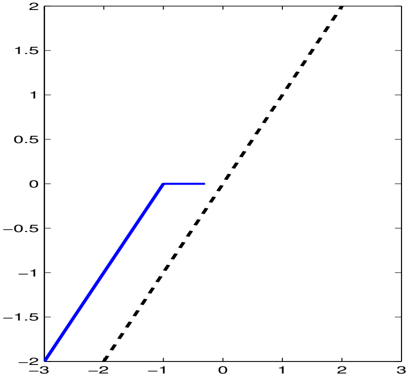

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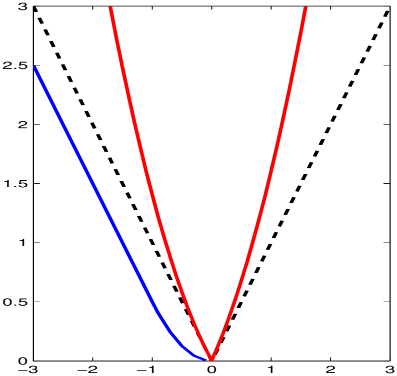

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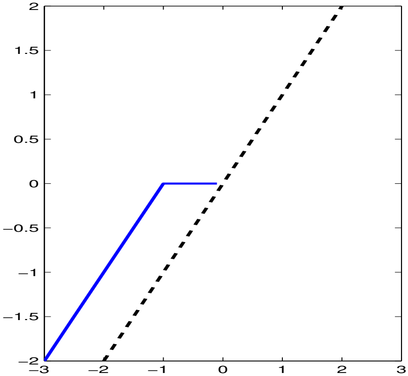

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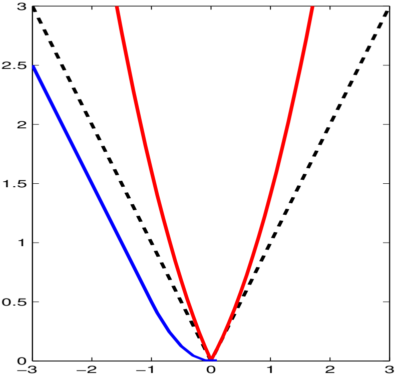

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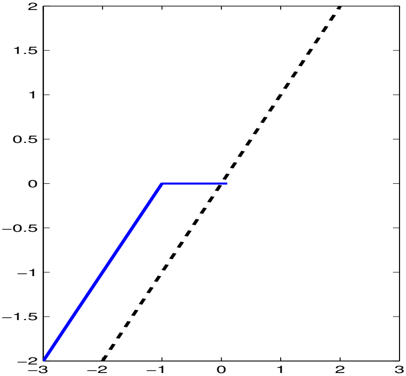

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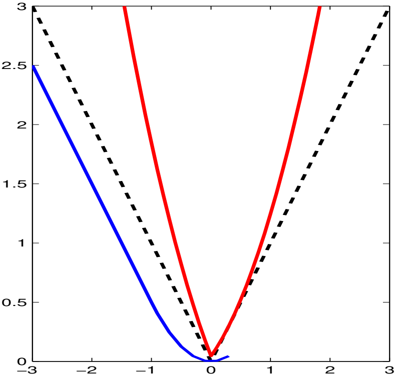

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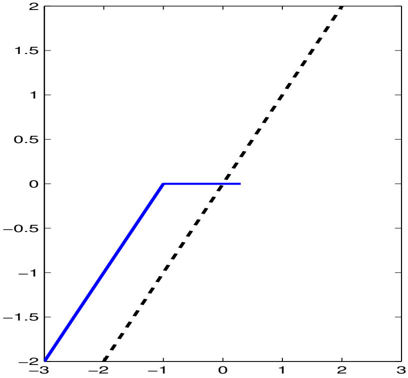

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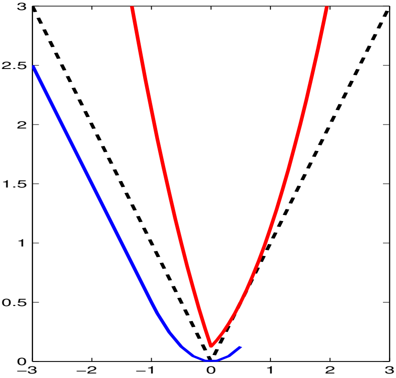

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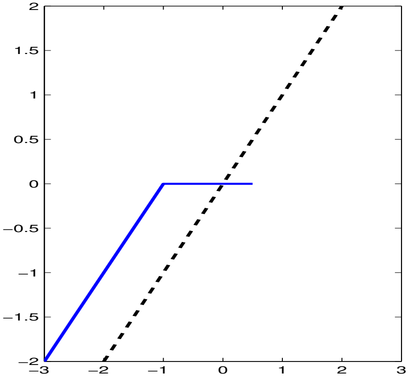

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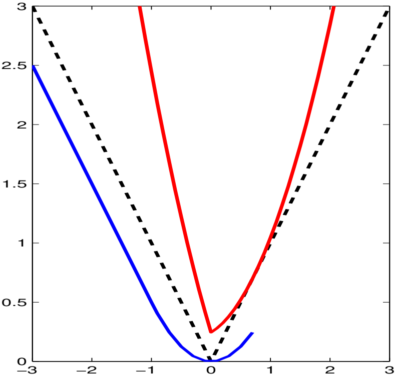

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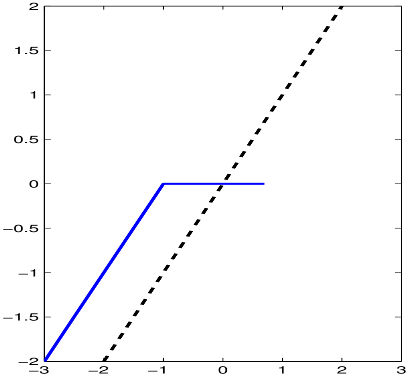

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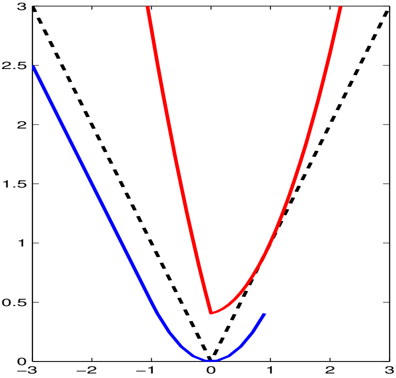

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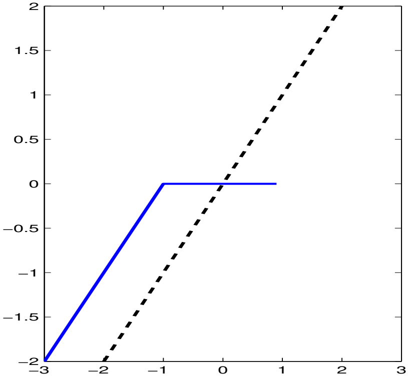

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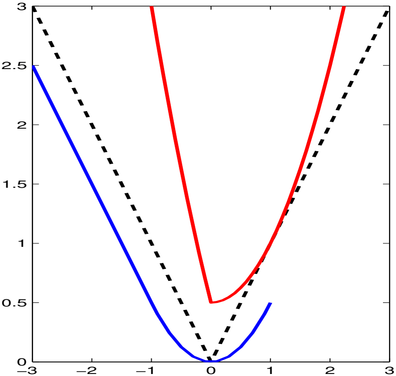

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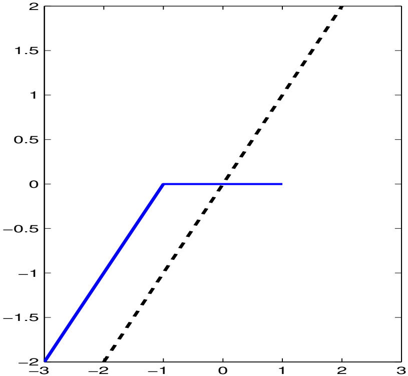

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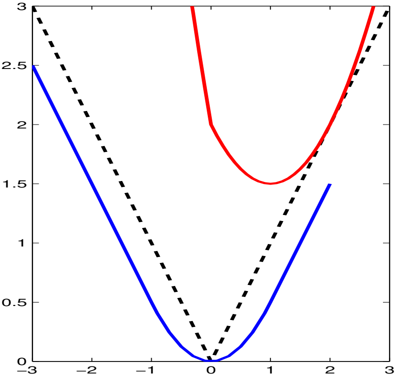

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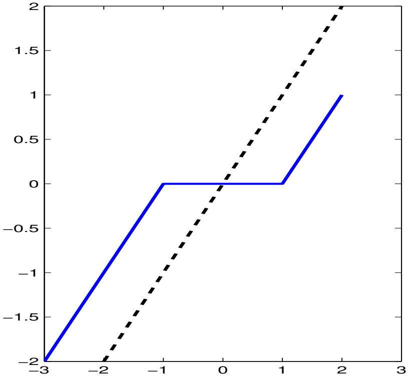

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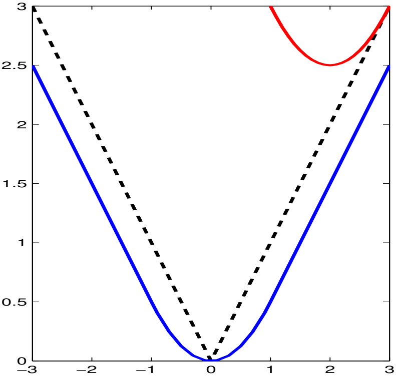

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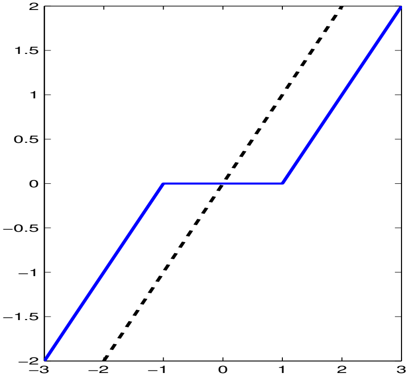

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$\mathrm{prox}_f(x) = \underset{y\in\mathcal{H}}{\mathrm{argmin}} \; f(y) + \frac{1}{2}\|y-x\|^2$

Proximity operator: existence and uniqueness

Let $f \in \Gamma_0(\mathcal{H})$ and $\gamma \in \,]0,+\infty[$.
For every $x \in \mathcal{H}$, there exists a unique vector $p \in \mathcal{H}$ such that

$$f(p) + \frac{1}{2\gamma}\|p - x\|^2 = \inf_{y \in \mathcal{H}} f(y) + \frac{1}{2\gamma}\|y - x\|^2 .$$

Proof: $f \in \Gamma_0(\mathcal{H}) \Rightarrow f^* \in \Gamma_0(\mathcal{H})$. Thus, there exists $u \in \mathcal{H}$ such that $f^*(u) \in \mathbb{R}$. According to Fenchel-Young inequality, we have

$$(\forall y \in \mathcal{H}) \qquad f(y) \geq \langle u \mid y \rangle - f^*(u).$$

Then, $f(y) + (2\gamma)^{-1}\|y - x\|^2 \to +\infty$ when $\|y\| \to +\infty$.
Furthermore $(2\gamma)^{-1}\| \cdot - x\|^2$ being strictly convex, $f + (2\gamma)^{-1}\| \cdot - x\|^2$ is a strictly convex coercive function.

Proximity operator: characterization

Let $\mathcal{H}$ be a Hilbert space and $f \in \Gamma_0(\mathcal{H})$.

$$(\forall x \in \mathcal{H}) \quad p = \mathrm{prox}_f(x) \quad \Leftrightarrow \quad x - p \in \partial f(p).$$

Proximity operator: characterization

Let $\mathcal{H}$ be a Hilbert space and $f \in \Gamma_0(\mathcal{H})$.

$$(\forall x \in \mathcal{H}) \quad \boxed{p = \mathrm{prox}_f(x) \quad \Leftrightarrow \quad x - p \in \partial f(p)}.$$

Proof: By using Fermat's rule, for every $x \in \mathcal{H}$, $p = \mathrm{prox}_f(x)$ if and only if

$$\begin{aligned}
& p = \underset{y \in \mathcal{H}}{\arg\min} \;\; f(y) + \frac{1}{2}\|y - x\|^2 \\
\Leftrightarrow \quad & 0 \in \partial\Big(f + \frac{1}{2}\|\cdot - x\|^2\Big)(p) \\
\Leftrightarrow \quad & 0 \in \partial f(p) + p - x \\
\Leftrightarrow \quad & x \in (\mathrm{Id} + \partial f)(p).
\end{aligned}$$

Remark:
prox_f is the resolvent of ∂f:

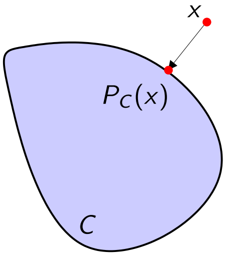
$$\mathrm{prox}_f = (I + \partial f)^{-1} = J_{\partial f}$$

Proximity operator: examples

Projection :

Let $\mathcal{H}$ be a Hilbert space. Let C be a nonempty closed convex subset of $\mathcal{H}$.

$$(\forall x \in \mathcal{H}) \qquad \mathrm{prox}_{\iota_C}(x) = \underset{y \in C}{\operatorname{argmin}} \frac{1}{2}\|y - x\|^2 = P_C(x).$$



Proximity operator: examples

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Remark :

- $p = P_C(x) \Leftrightarrow x - p \in \partial\iota_C(p) = N_C(p)$
  $\Leftrightarrow$ $p \in C$ and $(\forall y \in C)\ \langle y - p \mid x - p\rangle \leq 0$.

  Particular case: if C is a vector space: $p = P_C(x) \Leftrightarrow \begin{cases} p \in C \\ x - p \in C^\perp \end{cases}$.
- ${}^\gamma\iota_C = (2\gamma)^{-1} d_C^2$ where d_C distance to the convex set C is defined by $d_C \colon x \mapsto \inf_{y \in C} \|y - x\| = \|x - P_C x\|$.

Proximity operator: examples

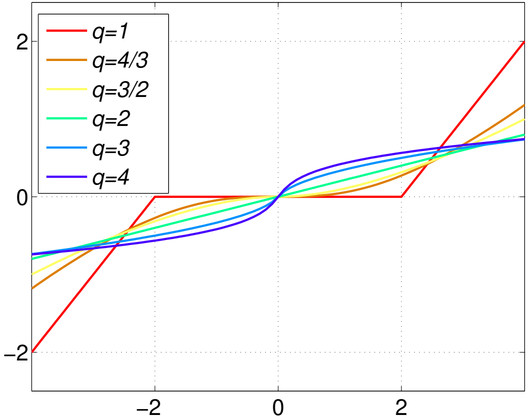
Power q function with $q \geq 1$:

Let $\chi > 0$, $q \in [1, +\infty[$ and $\varphi \colon \mathbb{R} \to]-\infty, +\infty] : \xi \mapsto \chi|\xi|^q$.
Then, for every $\xi \in \mathbb{R}$,

$$\operatorname{prox}_\varphi \xi = \begin{cases} \operatorname{sign}(\xi) \max\{|\xi| - \chi, 0\} & \text{if } q = 1 \\ \xi + \frac{4\chi}{3 \,.\, 2^{1/3}}\left((\epsilon - \xi)^{1/3} - (\epsilon + \xi)^{1/3}\right) & \\ \qquad\qquad \text{where} \quad \epsilon = \sqrt{\xi^2 + 256\chi^3/729} & \text{if } q = \frac{4}{3} \\ \xi + \frac{9\chi^2 \operatorname{sign}(\xi)}{8}\left(1 - \sqrt{1 + \frac{16|\xi|}{9\chi^2}}\right) & \text{if } q = \frac{3}{2} \\ \frac{\xi}{1+2\chi} & \text{if } q = 2 \\ \operatorname{sign}(\xi) \frac{\sqrt{1+12\chi|\xi|}-1}{6\chi} & \text{if } q = 3 \\ \left(\frac{\epsilon+\xi}{8\chi}\right)^{1/3} - \left(\frac{\epsilon-\xi}{8\chi}\right)^{1/3} \quad \text{where} \quad \epsilon = \sqrt{\xi^2 + 1/(27\chi)} & \text{if } q = 4 \end{cases}$$

Proximity operator: examples

Plot of the graphs of these proximity operator on the same figure.
Power q function with $q \geq 1$ and $\chi = 2$.



Proximity operator: examples

Quadratic function :
Let $\mathcal{H}$ and $\mathcal{G}$ be two Hilbert spaces.
Let $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$, $\gamma \in \;]0, +\infty[$ and $z \in \mathcal{G}$.

$$f = \gamma \left\| L \cdot - z \right\|^2 / 2 \quad \Rightarrow \quad \mathrm{prox}_f = (\mathrm{Id} + \gamma L^* L)^{-1} (\cdot + \gamma L^* z).$$

- Exercise : Prove this property.

Proximity operator: examples

Quadratic function :
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- Proof: We have, for every $x \in \mathcal{H}$,

$$p = \mathrm{prox}_f x \quad \Leftrightarrow \quad x - p \in \partial f(p).$$

In addition, f is Gâteaux differentiable and its gradient at p is

$$\nabla f(p) = \gamma L^*(Lp - z).$$

Therefore,

$$x - p = \gamma L^*(Lp - z) \quad \Leftrightarrow \quad p = (\mathrm{Id} + \gamma L^* L)^{-1}(x + \gamma L^* z).$$

Proximity operator: properties

Let $\mathcal{H}$ be a Hilbert space, $x \in \mathcal{H}$ and $f \in \Gamma_0(\mathcal{H})$.

Properties	$g(x)$	$\mathrm{prox}_g x$
Translation	$f(x-z), z \in \mathcal{H}$	$z + \mathrm{prox}_f(x-z)$
Quadratic perturbation	$f(x) + \alpha \parallel x \parallel^2 /2 + \langle z \mid x \rangle + \gamma$ $z \in \mathcal{H}, \alpha > 0, \gamma \in \mathbb{R}$	$\mathrm{prox}_{\frac{f}{\alpha+1}}(\frac{x-z}{\alpha+1})$
Scaling	$f(\rho x), \rho \in \mathbb{R}^*$	$\frac{1}{\rho}\mathrm{prox}_{\rho^2 f}(\rho x)$
Reflection	$f(-x)$	$-\mathrm{prox}_f(-x)$
Moreau enveloppe	${}^{\gamma}f(x) = \inf_{y \in \mathcal{H}} f(y) + \frac{1}{2\gamma}\|x-y\|^2$ $\gamma > 0$	$\frac{1}{1+\gamma}\left(\gamma x + \mathrm{prox}_{(1+\gamma)f}(x)\right)$

Proximity operator: properties

For every $i \in \{1, \ldots, n\}$, let $\mathcal{H}_i$ be a Hilbert space and let $f_i \in \Gamma_0(\mathcal{H}_i)$.
If

$$\big(\forall x = (x_1, \ldots, x_n) \in \mathcal{H}_1 \times \cdots \times \mathcal{H}_n\big) \quad f(x) = \sum_{i=1}^{n} f_i(x_i),$$

then

$$\big(\forall x = (x_1, \ldots, x_n) \in \mathcal{H}_1 \times \cdots \times \mathcal{H}_n\big) \quad \mathrm{prox}_f(x) = \big(\mathrm{prox}_{f_i}(x_i)\big)_{1 \leq i \leq n}.$$

Proximity operator: properties

Let $\mathcal{H}$ be a separable Hilbert space.
Let $(b_i)_{i\in I}$ be an orthonormal basis of $\mathcal{H}$.
For every $i \in I$, let $\varphi_i \in \Gamma_0(\mathbb{R})$ such that $\varphi_i \geq 0$. For every $x \in \mathcal{H}$, if

$$f(x) = \sum_{i\in I} \varphi_i(\langle x \mid b_i\rangle)$$

then

$$\mathrm{prox}_f(x) = \sum_{i\in I} \mathrm{prox}_{\varphi_i}(\langle x \mid b_i\rangle) b_i.$$

Remark: The assumption $(\forall i \in I)\ \varphi_i \geq 0$ can be relaxed if $\mathcal{H}$ is finite dimensional.

Proximity operator: properties

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Example: $\mathcal{H} = \mathbb{R}^N$, $(b_i)_{1\leq i\leq N}$ canonical basis of $\mathbb{R}^N$, $f = \lambda\|\cdot\|_1$ with $\lambda \in [0, +\infty[$.

$$(\forall x = (x^{(i)})_{1\leq i\leq N} \in \mathbb{R}^N) \qquad \mathrm{prox}_{\lambda\|\cdot\|_1}(x) = \big(\mathrm{prox}_{\lambda|\cdot|}(x^{(i)})\big)_{1\leq i\leq N}$$

Proximity operator: properties

Moreau decomposition formula

Let $\mathcal{H}$ be a Hilbert space, $f \in \Gamma_0(\mathcal{H})$ and $\gamma \in \,]0, +\infty[$.

$$(\forall x \in \mathcal{H}) \qquad \boxed{\operatorname{prox}_{\gamma f^*} x = x - \gamma \operatorname{prox}_{\gamma^{-1} f}(\gamma^{-1} x)}\,.$$

Proof:

$$\begin{aligned}
p = \operatorname{prox}_{\gamma f^*} x &\Leftrightarrow x - p \in \gamma \partial f^*(p) \\
&\Leftrightarrow p \in \partial f\left(\frac{x-p}{\gamma}\right) \\
&\Leftrightarrow \frac{x}{\gamma} - \frac{x-p}{\gamma} \in \frac{1}{\gamma}\partial f\left(\frac{x-p}{\gamma}\right) \\
&\Leftrightarrow \frac{x-p}{\gamma} = \operatorname{prox}_{\gamma^{-1} f}(\gamma^{-1} x) \\
&\Leftrightarrow p = x - \gamma \operatorname{prox}_{\gamma^{-1} f}(\gamma^{-1} x).
\end{aligned}$$

Proximity operator: properties

Moreau decomposition formula

Let $\mathcal{H}$ be a Hilbert space, $f \in \Gamma_0(\mathcal{H})$ and $\gamma \in \,]0, +\infty[$.

$$(\forall x \in \mathcal{H}) \qquad \boxed{\mathrm{prox}_{\gamma f^*} x = x - \gamma \mathrm{prox}_{\gamma^{-1} f}(\gamma^{-1} x)}.$$

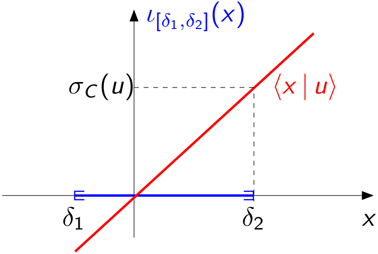
Example: If $\mathcal{H} = \mathbb{R}^N$, $f = \frac{1}{q}\|\cdot\|_q^q$ with $q \in]1, +\infty[$, then $f^* = \frac{1}{q^*}\|\cdot\|_{q^*}^{q^*}$ with $1/q + 1/q^* = 1$, and

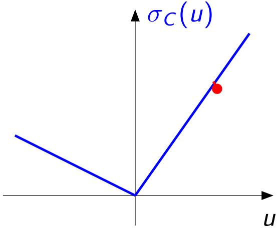
$$(\forall x \in \mathbb{R}^N) \qquad \mathrm{prox}_{\frac{\gamma}{q^*}\|\cdot\|_{q^*}^{q^*}} x = x - \gamma \mathrm{prox}_{\frac{1}{\gamma q}\|\cdot\|_q^q}(\gamma^{-1} x).$$

Support function: reminders

Let $\mathcal{H}$ be a Hilbert space and $C \subset \mathcal{H}$.
σ_C is the support function of C if

$$(\forall u \in \mathcal{H}) \qquad \sigma_C(u) = \sup_{x \in C} \langle x \mid u \rangle = \iota_C^*(u).$$





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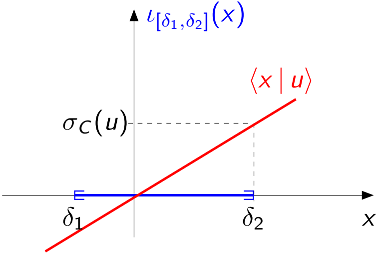

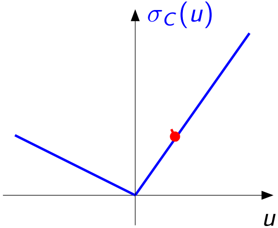

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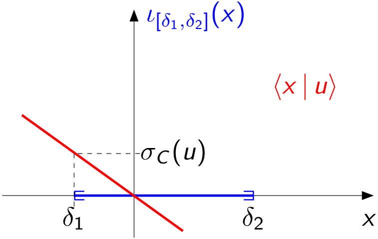
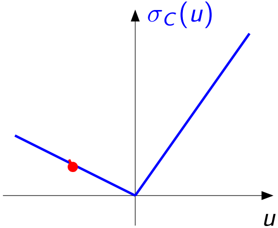
$$= \iota_C^*(u).$$





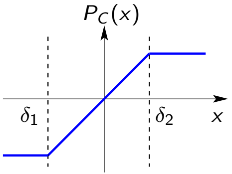
Proximity operator: examples

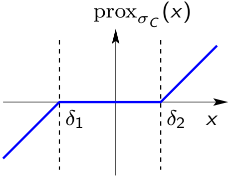
Support function :
Let $\mathcal{H}$ be a Hilbert space and $C \subset \mathcal{H}$ be nonempty closed convex.

$$(\forall x \in \mathcal{H}) \qquad \mathrm{prox}_{\sigma_C} = \mathrm{Id} - P_C.$$

Soft-thresholding : $\mathcal{H} = \mathbb{R}$, $\delta_1 = \inf C$ and $\delta_2 = \sup C$. For every $x \in \mathbb{R}$,

$$\sigma_C(x) = \begin{cases} \delta_1 x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ \delta_2 x & \text{if } x > 0 \end{cases} \Rightarrow \mathrm{prox}_{\sigma_C}(x) = \mathrm{soft}_C(x) = \begin{cases} x - \delta_1 & \text{if } x < \delta_1 \\ 0 & \text{if } x \in C \\ x - \delta_2 & \text{if } x > \delta_2. \end{cases}$$





Proximity operator: properties

Let $\mathcal{H}$ be a Hilbert spaces. Let $f \in \Gamma_0(\mathcal{H})$ and $L \in \mathcal{B}(\mathcal{H}, \mathcal{H})$ be unitary. Then

$$\mathrm{prox}_{f \circ L} = L^* \circ \mathrm{prox}_f \circ L.$$

Proximity operator: properties

Let $\mathcal{H}$ be a Hilbert spaces. Let $f \in \Gamma_0(\mathcal{H})$ and $L \in \mathcal{B}(\mathcal{H},\mathcal{H})$ be unitary. Then

$$\mathrm{prox}_{f\circ L} = L^* \circ \mathrm{prox}_f \circ L.$$

Proof: $LL^* = \mathrm{Id} \Rightarrow \mathrm{ran}\, L = \mathcal{H}$.
Thus $(\forall x \in \mathcal{H})\ p = \mathrm{prox}_{f\circ L} x \Leftrightarrow x - p \in \partial(f\circ L)(p) = L^*\partial f(Lp)$.
This yields

$$\begin{aligned}
& Lx - Lp \in \partial f(Lp) \\
\Leftrightarrow \quad & Lp = \mathrm{prox}_f(Lx) \\
\Rightarrow \quad & p = L^* Lp = L^*\mathrm{prox}_f(Lx).
\end{aligned}$$

Proximity operator: properties

Let $\mathcal{H}$ be a Hilbert spaces. Let $f \in \Gamma_0(\mathcal{H})$ and $L \in \mathcal{B}(\mathcal{H}, \mathcal{H})$ be unitary. Then

$$\mathrm{prox}_{f \circ L} = L^* \circ \mathrm{prox}_f \circ L.$$

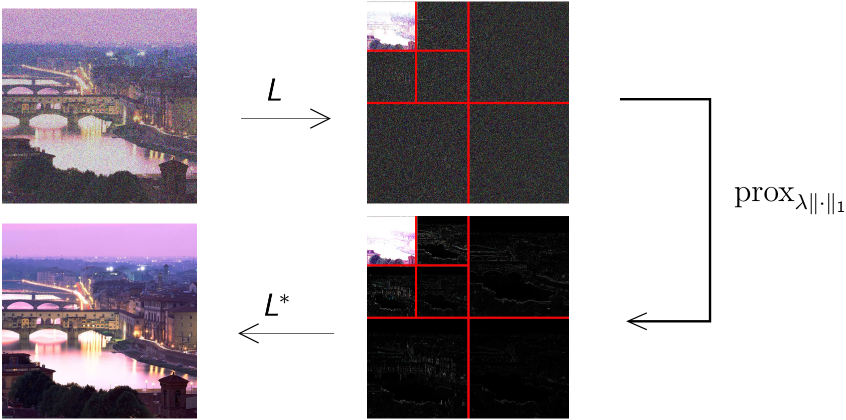
Let $\mathcal{H}$ and $\mathcal{G}$ be two Hilbert spaces. Let $f \in \Gamma_0(\mathcal{H})$ and $L \in \mathcal{B}(\mathcal{G}, \mathcal{H})$ such that $LL^* = \mu \mathrm{Id}$ where $\mu \in \,]0, +\infty[$. Then

$$\mathrm{prox}_{f \circ L} = \mathrm{Id} - \mu^{-1} L^* \circ (\mathrm{Id} - \mathrm{prox}_{\mu f}) \circ L.$$

Proximity operator: properties

<u>Illustration</u> : $L \in \mathcal{B}(\mathcal{H},\mathcal{H})$ unitary, $\mathrm{prox}_{f \circ L} = L^* \mathrm{prox}_f L$.

- Application: denoising using an ℓ_1 penalty on the coefficients resulting from an orthogonal wavelet transform L.



Proximity operator: Bayesian interpretation

- If $\mathcal{H} = \mathbb{R}^N$ and

$$x = \overline{y} + w$$

  where $\overline{y}$ is a realization of a random vector with probability density function $\exp(-f)$ and w is a realization of a $\mathcal{N}(0, \mathrm{I})$ noise, then $\mathrm{prox}_f(x)$ is a Maximum A Posteriori estimate of $\overline{y}$.
- Explicit form for objective functions associated with usual log-concave probability densities

  - Laplace
  - Generalized Gaussian
  - maximum entropy
  - gamma
  - uniform
  - Weibull
  - Generalized inverse Gaussian
  - Gaussian
  - Huber
  - Smoothed Laplace
  - chi
  - triangular
  - Pearson type I

  ...

- And many other functions ! http://proximity-operator.net