A NEW ANALYSIS OF INTRABEAM SCATTERING*

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Abstract

Beginning with the Fokker-Planck equation we present a new analysis of intrabeam scattering (IBS) in electron storage rings. Our approach is distinguished by having no ill-defined Coulomb logarithm, a fundamental drawback of previous approaches. We treat the case of linear $x_\beta y_\beta$ coupling in detail, deriving explicit expressions for the second moment invariants and their time evolution in the presence of IBS. We compare our results with those of Bjorken-Mtingwa, as well as with measurements performed at KEK's ATF damping ring. More details of our derivations will be published elsewhere.

EVOLUTION EQUATIONS

We consider a smooth focusing approximation Hamiltonian representing the symplectic part of the dynamics in the storage ring given by $H = (1/2)S_{ij}z_iz_j$ with $\vec{z} = (x, x', y, y', z, \delta)$. If we consider damping and diffusion processes (both radiation and IBS) as well as the Hamiltonian evolution, the beam distribution function $f(\vec{z}, t)$ evolves via the Fokker-Planck equation:

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial z_i}\left(-B_i f + \frac{1}{2}\frac{\partial}{\partial z_j} D_{ij} f\right) \quad (1)$$

with $B_i = C_{ij}z_j + (\langle \delta z_i \rangle_W / \delta t)_{\rm IBS}$, $D_{ij} = d_{ij} + (\langle \delta z_i \delta z_j \rangle_W / \delta t)_{\rm IBS}$. The various matrices are given by $C = JS - b$, b is the damping matrix, d is the diffusion matrix, and J is the symplectic inner product matrix $J = {\rm diag}(\{0\,1\}, \{-1\,0\})$. The IBS average $\langle\ \rangle_W$ is over the probability that a given particle with phase space position $\vec{z}$ will change by $\delta\vec{z}$ in a time δt. In a coordinate system where x, y, z are real positions, only $W(\delta p_a)$ $(a = 1 \ldots n)$ will be non-zero. For $2n$-D phase space, there exist n invariants of H: $g_a = \vec{z}^T G^a \vec{z}$ with $G^a = JU\tilde{G}^a U^T J$, where U is the symplectic matrix whose columns consist of pairs of eigenvectors of JS, $(v_a, -iv_a^*)$ normalized such that $v_a^T J v_b = -i\delta_{ab}$, and $\tilde{G}^a$ is given by having $-i\sigma_x = (\{0\ -i\}, \{-i\ 0\})$ in the $a^{\rm th}$ spot along the diagonal and the rest 0's. The RMS emittance is $\epsilon_a = \langle g_a \rangle / 2$.

If the damping and diffusion are slow compared to the Hamiltonian evolution, the distribution will approximately be a Gaussian function of the invariants:

$$f(\vec{z}, t) = \frac{N}{\Gamma} e^{-\frac{1}{2} z_i z_j \mathbb{M}_{ij}(t)} \approx \frac{N}{\Gamma} e^{-\frac{g_1}{\langle g_1 \rangle(t)} - \frac{g_2}{\langle g_2 \rangle(t)} - \frac{g_3}{\langle g_3 \rangle(t)}} \quad (2)$$

The distribution is normalized so that $\int d\vec{z} f(\vec{z}) = N$, and the phase space volume is given by $\Gamma = (\pi)^3 \langle g_1 \rangle \langle g_2 \rangle \langle g_3 \rangle$. IBS is most naturally analyzed in the beam frame which we notate by $\vec{Z} = (\vec{X}, \vec{P})$. With small x', y', and for large relativistic γ factor, the Lorentz transformation is simply $X = x, Y = y, Z = \gamma z, P_x = P_0 x', P_y = P_0 y', P_z = P_0\delta/\gamma, dt = \gamma d\bar{t}$ with P_0 the reference momentum. We introduce $\vec{z} = L\vec{Z}$. For the distribution matrices, we write

$$\bar{\mathbb{M}} = \sum_{a=1}^{3} \bar{\mathbb{M}}^{(a)} = \sum_{a=1}^{3} \begin{pmatrix} \mathbb{A}^{(a)} & \mathbb{B}^{(a)} \\ \mathbb{B}^{(a)T} & \mathbb{C}^{(a)} \end{pmatrix} \quad (3)$$

where $\bar{\mathbb{M}} = L^T \mathbb{M} L$ is $\mathbb{M}$ expressed in the beam frame and $\bar{\mathbb{M}}^{(a)} = 2L^T G^{(a)} L / \langle g_a \rangle$.

From (1), the evolution of the moments $\Sigma_{ij} = \langle z_i z_j \rangle_f$ is

$$\frac{d\Sigma_{ij}}{dt} = \left(\langle B_i z_j \rangle_f + \frac{1}{2}\langle D_{ij} \rangle_f\right) + (i \leftrightarrow j) \quad (4)$$

Note that $\Sigma^{-1} = \mathbb{M}$. We can also show that the evolution of the average values of the invariants is

$$\frac{d\langle g_a \rangle}{dt} = -2\alpha_a(\langle g_a \rangle - \langle g_a \rangle_0) + \left(\frac{d\langle g_a \rangle}{dt}\right)^{\rm IBS} \quad (5)$$

where the equilibrium values of the invariants without IBS are $\langle g_a \rangle_0 = d_a/(2\alpha_a)$, or $\epsilon_{a0} = d_a/(4\alpha_a)$ with $2\alpha_a = {\rm Tr}(\tilde{b}_a)$ and $d_a = {\rm Tr}\,(G^a d)$ where $\tilde{b}_a$ is the $a^{\rm th}$ 2×2 block along the diagonal of $\tilde{b} = U^{-1}bU$. To first order, the α_a are the real parts of the eigenvalues of the matrix C.

We define the IBS growth rate as $T_a^{-1} = (d\langle g_a \rangle/dt)^{\rm IBS} / \langle g_a \rangle$. For Eq. (4) there are 2 types of terms in $(d\Sigma/dt)^{\rm IBS}$, one in the form of $\langle x_a p_b \rangle$ and the other $\langle p_a p_b \rangle$. Using Eq. (3), we get

$$\frac{1}{T_a} = \frac{1}{2}{\rm Tr}\left(M^a \frac{d\Sigma}{dt}^{\rm IBS}\right) = \frac{1}{2}\mathcal{A}{\rm Tr}\left(\mathbb{B}^{(a)}\mathbb{Q} + \mathbb{C}^{(a)}\mathbb{K}\right) \quad (6)$$

where $\mathcal{A}\mathbb{K}_{ab} = d\langle P_a P_b \rangle / dt|_{\rm IBS}$, $\mathcal{A}\mathbb{Q}_{ab} = d\langle X_a P_b \rangle / dt|_{\rm IBS}$ and $\mathcal{A}$ contains overall constants. We've moved into the beam frame using ${\rm Tr}\,(M^a \Sigma) = {\rm Tr}(\bar{M}^a \bar{\Sigma})$ with $\Sigma = L\bar{\Sigma}L^T$.

IBS DAMPING AND DIFFUSION

IBS has been studied extensively [1-4,7], with Bjorken-Mtingwa (BM)[2] and Piwinski (P)[3] the main foundations of later derivations. We start from first principles, aiming for a clearer understanding of the subject. In addition to having no Coulomb log, our approach differs from [4] and [7] in that we follow invariants directly and give explicit expressions for them in the case of global coupling.

We compute $\bar{B}_a^{\rm IBS}(\vec{Z}) = \langle \delta P_a \rangle_W / \delta\bar{t}$ and $\bar{D}_{ab}^{\rm IBS}(\vec{Z}) = \langle \delta P_a \delta P_b \rangle_W / \delta\bar{t}$. Let the two particles have coordinates $\vec{Z}_1$ and $\vec{Z}_2$ and let $\vec{r} = \vec{X}_1 - \vec{X}_2$ and $\vec{\Delta} = \vec{P}_1 - \vec{P}_2$. If we consider them undergoing a scattering process, the impact parameter is $\vec{b} = \vec{r} - r(\hat{\Delta} \cdot \hat{r})\hat{\Delta}$, where hats designate unit vectors. In the small angle approximation, the scatter leads to a total momentum kick $\delta\vec{P}_1 \approx -4k^2/(\Delta^3 b^2)\hat{\Delta} + 2k/(\Delta b)\hat{b}$ where $k = (mc)^2 r_0$ with r_0 the classical particle radius. For δP_a, we discard the 2nd term which should rightfully be included in the space charge analysis, and then keep the term in $\delta P_a \delta P_b$ required for energy conservation. We then average these quantities over the portion of particle 2's phase space where the time to the distance of minimum approach

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$t_{\min} = -m(r/\Delta)\hat{\Delta}\cdot\hat{r}$ is less than $\delta\bar{t}$. Letting $\delta\bar{t}\to 0$ we arrive at

$$\frac{\langle\delta P_a\rangle_W}{\delta\bar{t}} = -\frac{4k^2}{m}\int d\vec{\Delta}\, d\vec{r}\;\frac{\hat{\Delta}_a}{\Delta^2 b^3}\bar{f}(\vec{X}_2,\vec{P}_2)\delta(\hat{\Delta}\cdot\hat{r}) \quad (7)$$

$$\frac{\langle\delta P_a\delta P_b\rangle_W}{\delta\bar{t}} = \frac{4k^2}{m}\int d\vec{\Delta}\, d\vec{r}\;\frac{\hat{b}_i\hat{b}_j}{\Delta b^3}\bar{f}(\vec{X}_2,\vec{P}_2)\delta(\hat{\Delta}\cdot\hat{r}) \quad (8)$$

After use of the delta function in the integral, b can be replaced by $r=|\vec{r}|$. $\bar{f}$ is f normalized so that $\int d\vec{Z}\bar{f}(\vec{Z}) = 1$. If we replace the spatial distribution with the local constant density, and absorb the spatial divergence into a Coulomb log, these reduce to the Rosenbluth Potentials[6].

REDUCE TO ANGULAR INTEGRALS

For the IBS contribution to the moment evolution equations, we can combine the damping and diffusion together using (4) [1]. For Gaussian distributions, the result is:

$$\mathbb{K}_{ab} = \int \frac{d^6\vec{\xi}}{\Delta r^3}\left[\hat{r}_a\hat{r}_b - \hat{\Delta}_a\hat{\Delta}_b\right]e^{-\frac{1}{2}\xi_i\xi_j\breve{\mathbb{M}}_{ij}}\delta(\hat{\Delta}\cdot\hat{r}) \quad (9)$$

where $\breve{\mathbb{M}}$ is the unitless $\mathbb{M}$ matrix expressed in the basis of $\vec{Z}$, i.e. we use[2] $\breve{\mathbb{A}} = r_m^2\mathbb{A}$, $\breve{\mathbb{B}} = (r_m P_0)\mathbb{B}$, $\breve{\mathbb{C}} = P_0^2\mathbb{C}$. Also, $\vec{\xi} = (\vec{r},\vec{\Delta})$. For the overall constant, $\mathcal{A} = Nk^2/(\gamma m\bar{\Gamma})$ with $\bar{\Gamma} = P_0^3\Gamma$. We also can compute:

$$\mathbb{Q}_{ab} = -\frac{1}{2}\int d^6\vec{\xi}\,\frac{\hat{r}_a\hat{\Delta}_b}{\Delta^2 r^2}e^{-\frac{1}{2}\xi_i\xi_j\breve{\mathbb{M}}_{ij}}\delta(\hat{\Delta}\cdot\hat{r}) \quad (10)$$

Ignore $\mathbb{Q}_{ab}$ for now (no "Coulomb log" behaviour). If we do the $|\vec{r}|$ and $|\vec{\Delta}|$ integrals and the integral from the δ function in $\mathbb{K}_{ab}$ we are left with

$$\mathbb{K}_{ab} = \frac{1}{2}\int d\Omega\,\frac{-h_{ab}}{h_3}[\log(\frac{h_1}{2}) + \gamma_E - q\tan^{-1}q] \quad (11)$$

with $h_{ab} = \hat{r}_a\hat{r}_b - \hat{\Delta}_a\hat{\Delta}_b$, $h_1 = \breve{\mathbb{A}}_{ab}\hat{r}_a\hat{r}_b$, $h_2 = \breve{\mathbb{B}}_{ab}\hat{r}_a\hat{\Delta}_b$, $h_3 = \breve{\mathbb{C}}_{ab}\hat{\Delta}_a\hat{\Delta}_b$, and $q = h_2/\sqrt{4h_1h_3 - h_2^2}$; $\gamma_E \approx 0.577$. The q term in (11) will often be small and we drop it here. Approximating the $\log() + \gamma_E$ term as a constant gives what we call the Coulomb log approximation. This can be shown to reduce exactly to the equivalent expression in BM. We denote it by $\mathbb{K}^{\mathrm{BM}}$. The quotient of these two expressions can be used to define the Coulomb log:

$$2\mathrm{Log}_{ab} = \frac{\int d\Omega\frac{h_{ab}}{h_3}(-\log(\frac{h_1}{2}) - \gamma_E)}{\int d\Omega\frac{h_{ab}}{h_3}} = \frac{\mathbb{K}_{ab}}{\mathbb{K}^{\mathrm{BM}}_{ab}} \quad (12)$$

This allows us to explore the range over which the usual approach of having a single Coulomb log makes sense.

THE CASE OF A COUPLED BEAM

With both x and y dispersion and $x_\beta y_\beta$ coupling parameter κ in the smooth approximation, we use the Hamiltonian $H = (\beta c/2)(k_x x_\beta^2 + x'^2 + 2\kappa x_\beta y_\beta + k_y y_\beta^2 + y'^2 - (k_z/\alpha_c)z_\beta^2 - \alpha_c\delta^2)$ where $x_\beta = x - \eta_x\delta$, $y_\beta = y - \eta_y\delta$, $z_\beta = z - \eta_x x' - \eta_y y'$ and $\alpha_c = k_x\eta_x^2 + k_y\eta_y^2$ is the momentum compaction factor. βc is the reference particle velocity. We use lab time as the independent variable. The smoothed frequencies are $k_x = 1/\beta_x^2 = (\nu_x/R)^2$, $k_y = 1/\beta_y^2 = (\nu_y/R)^2$, $k_z = (\nu_s/R)^2$, with $\nu_{x,y,s}$ the horizontal, vertical betatron and synchrotron tunes respectively. $R = \mathbf{C}/2\pi$ with $\mathbf{C}$ the storage ring circumference. We parametrize k_x, k_y and κ by $k_x = k_0 + \Lambda\cos\psi$, $k_y = k_0 - \Lambda\cos\psi$, $\kappa = \Lambda\sin\psi$. $\Lambda = \sqrt{(k_x-k_y)^2/4 + \kappa^2}$ and $\tan\psi = 2\kappa/(k_x - k_y)$. $\psi/2$ is the tilt angle in the x_β,y_β plane. The eigeninvariants are

$$\begin{aligned} g_{1,2} &= \frac{1}{2\sqrt{k_0\pm\Lambda}}([(k_0\pm\Lambda)x_\beta^2 + x'^2](1\pm\cos\psi) + \\ &\quad [(k_0\pm\Lambda)y_\beta^2 + y'^2](1\mp\cos\psi) \pm \\ &\quad 2[(k_0\pm\Lambda)x_\beta y_\beta + x'y']\sin\psi),\ \text{(1 upper sign,2 lower sign)} \\ g_3 &= (z_\beta^2/\beta_z) + \beta_z\delta^2 \end{aligned} \quad (13)$$

where $\beta_z = R\alpha_c/\nu_s$. Note that $H = \frac{\beta c}{2}(\sqrt{k_0+\Lambda}\,g_1 + \sqrt{k_0-\Lambda}\,g_2 - \sqrt{k_z}\,g_3)$. The IBS growth rates are

$$\begin{aligned} \frac{1}{T_{1,2}} &= \frac{\mathcal{A}}{4\epsilon_{1,2}\sqrt{k_0\pm\Lambda}}[(1\pm\cos\psi)\mathbb{K}_{11} \pm 2\sin\psi\mathbb{K}_{12} \\ &\quad + (1\mp\cos\psi)\mathbb{K}_{22} + \gamma^2(k_0\pm\Lambda)(\eta_x^2(1\pm\cos\psi) \\ &\quad + \eta_y^2(1\mp\cos\psi))\mathbb{K}_{33}] \\ \frac{1}{T_3} &= \frac{\mathcal{A}\gamma^2}{\sigma_\delta^2}\mathbb{K}_{33} \end{aligned} \quad (14)$$

The damping matrix has non-zero elements $b_{22} = 2\alpha_x$, $b_{44} = 2\alpha_y$, $b_{66} = 2\alpha_z$ and the rest of the elements are 0. The diffusion matrix d has just one non-zero element (ignoring intrinsic x', y' diffusion), $d_{66} = \mathcal{D}/E_0^2$, where $\mathcal{D} = 55r_0\hbar mc^4\gamma^7/(24\sqrt{3}\rho^3)$. These yield coupled damping constants $\alpha_{1,2} = \frac{1}{2}\alpha_x(1\pm\cos\psi) + \frac{1}{2}\alpha_y(1\mp\cos\psi)$, $\alpha_3 = \alpha_z$ and diffusion constants $d_{1,2} = \mathcal{D}\sqrt{k_0\pm\Lambda}(\eta_{x,y}\cos\frac{\psi}{2} \pm \eta_{y,x}\sin\frac{\psi}{2})^2$, $d_3 = \beta_z\mathcal{D}$ which yield equilibrium emittances without IBS of

$$\begin{aligned} \epsilon_{(1,2)0} &= \left\langle\frac{\mathcal{D}\sqrt{k_0\pm\Lambda}[\eta_{x,y}\cos\frac{\psi}{2} \pm \eta_{y,x}\sin\frac{\psi}{2}]^2}{2[\alpha_x + \alpha_y \pm (\alpha_x - \alpha_y)\cos\psi]}\right\rangle_b\frac{\rho}{R} \\ \epsilon_{30} &= \left\langle\frac{\mathcal{D}\beta_z}{4\alpha_z}\right\rangle_b\frac{\rho}{R}. \end{aligned} \quad (15)$$

We have assumed that the lattice is isomagnetic and hence the $1/\rho^3$ average in $\mathcal{D}$ causes the other parameters to be averaged only in the bends with an overall ρ/R normalization. In the zero coupling limit, we also replace $\langle\eta_{x,y}^2/\beta_{x,y}\rangle$ with $\langle\mathcal{H}_{x,y}\rangle$. The observable transverse beam sizes are related to the invariants by

$$\sigma_{x,y}^2 = \frac{\epsilon_{1,2}\cos^2\frac{\psi}{2}}{\sqrt{k_0\pm\Lambda}} + \frac{\epsilon_{2,1}(1-\cos\psi)}{\sqrt{k_0\mp\Lambda}} + \frac{\epsilon_3\sqrt{k_z}\eta_{x,y}^2}{\alpha_c} \quad (16)$$

When $\psi\to 0$, we recover $\sigma_{x,y}^2 = \beta_{x,y}\epsilon_{x,y} + \sigma_\delta^2\eta_{x,y}^2$.

APPLICATION TO THE ATF

As an application of our analysis, we compared with the data taken at the ATF in April 2000. Ref. [1] attempted this using a combination of the program SAD and IBS expressions based on BM with coupling added in a heuristic way. There was an apparent discrepancy between theory and experiment in the current dependence of the projected vertical emittance, $\epsilon_{y,\mathrm{pr}} = (\sigma_y^2\sigma_{y'}^2 - \sigma_{yy'}^2)^{\frac{1}{2}}$. Our analysis provides a solid base to explore this issue.

[1] This derivation of the BM results from the Rosenbluth potentials[6] was carried out by M. Venturini [5].

[2] The quantity r_m is the minimum impact parameter cut-off required because (9) diverges for small distances. We take it to be a typical distance of minimum approach, $r_m = r_0\beta_x/(\gamma^2\epsilon_x)$. This cut-off is also consistent with the small angle approximation used in the analysis.

We vary κ and compare equilibrium values for $\epsilon_x, \sigma_s, \sigma_\delta$, and $\epsilon_{y,\mathrm{pr}}$. We also (as in [1]) vary β_z with current computed from the σ_s, σ_δ data, as a linear model for bunch lengthening. We use $E_0 = 1.28$ GeV, $\eta_x = 0.052$ m, $\eta_y = 3\,\mathrm{mm}$, $\beta_x = 3.9$ m, $\beta_y = 4.5$ m, $\epsilon_{x0} = 1.05 \times 10^{-9}$ m, $\nu_s = .0049$, $\alpha_c = .0029$, $\mathbf{C} = 138.6\,\mathrm{m}$, $\rho/R = 0.260$. $\tau_a = 1/\alpha_a$ are $\tau_x = 18.2$ ms, $\tau_y = 29.2$ ms, and $\tau_p = 20.9$ ms.

Figure 1 shows the evolution of the invariants starting with injection values. We evolve the invariants using (5), (11), and (14). The parameters correspond to the 3.1 mA point on the middle curve of Figure 2. Note that $\epsilon_{y\,\mathrm{pr}}/\epsilon_x$ is not constant, and hence there is not an exact global coupling paameter κ' such that $\epsilon_{y,\,\mathrm{pr}}(t) = \kappa' \epsilon_x(t)$.

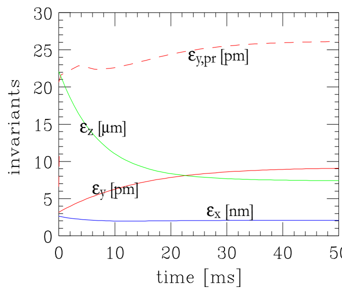

Figure 1: Time evolution of emittances.

Figure 2 shows a comparison of equilibrium projected vertical emittance (solid curves) to the data (diamonds). ϵ_x and σ_δ are not shown, but agreement is comparable to that in [1]. By adjusting the coupling, we can get the correct magnitude in $\epsilon_{y,pr}$, but the slope still does not agree. For $\eta_y = 3\mathrm{mm}$, we need κ/k_0 between .02 and .03 which corresponds to a tilt angle between 4 and 6 degrees. ϵ_y (dashed curves) is smaller than $\epsilon_{y,\mathrm{pr}}$. This gives an indication of the effect coupling has had on the measurements. In the

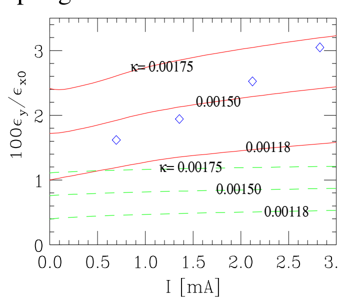

Figure 2: Comparison with ATF measurements. (κ in m^{-2})

ATF, the beam is coolest in δ and hottest in x. Thus, $\mathbb{K}_{33}$ is positive and $\mathbb{K}_{11}$ negative. $\mathbb{K}_{22}$ can be either positive or negative (energy conservation gives $\sum_a \mathbb{K}_{aa} = 0$.) We find the biggest difference between $\mathbb{K}_{ab}$ and $\mathbb{K}_{ab}^{\mathrm{BM}}$ for $\mathbb{K}_{22}$. In the small coupling limit, the $\mathbb{K}_{22}$ and the $\mathbb{K}_{33}$ contributions to $1/T_2$ have a relative coefficient of $\gamma^2 \eta_y^2/\beta_y^2$. In the "High Energy Approximation", one keeps only the $\mathbb{K}_{33}$ term. However, for the ATF parameters, if $\eta_y < 1.8$ mm, $\mathbb{K}_{22}$ can become important. In Figure 3 we plot the ratio of the Coulomb logs defined in (12) to the conventional Coulomb log $L_c = \log(\sigma_y/r_m)$ ($L_c \approx 16$ for ATF) for varying vertical dispersion and zero coupling (since BM dealt only with the uncoupled case). Effectively, we are varying the beam aspect ratio, with the right side of the plot approaching a round beam. Near $\eta_y = 28\,\mathrm{mm}$, the intrinsic vertical growth rates $\mathbb{K}_{22}$ and $\mathbb{K}_{22}^{\mathrm{BM}}$ have opposite signs. Finally note that both Log_{11}/L_c and Log_{33}/L_c are close to 1 over a wide range of beam aspect ratios.

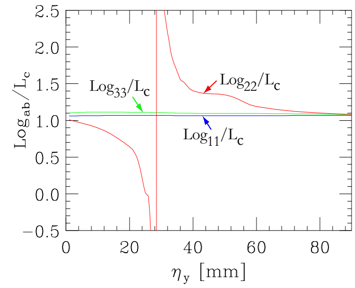

Figure 3: Ratios of computed to nominal Coulomb log.

CONCLUSIONS AND FUTURE WORK

We have given new expressions for the IBS damping and diffusion coefficients B_a and D_{ab} and the moment evolution quantities $\mathcal{A}\mathbb{K}_{ab}$. We include the position distribution in our approach (the matrix $\mathbb{A}$ for Gaussians), a necessary step in understanding the effect that the shape of the beam has on IBS. For Gaussian beams, we have reduced the expressions to 3-D angular integrals. In fact we can reduce them to 2-D integrals with some increase in complexity. We find that for flat beams with ATF parameters, Bjorken-Mtingwa with $b_{\max}$ equal to the vertical beam size gives excellent results for the horizontal and longitudinal growth rates, but can break down for the intrinsic vertical growth rate. We expect that for $\eta_y < 1.8\,\mathrm{mm}$, for some values of κ and $\nu_x - \nu_y$, there may be observable differences in the growth rates and/or equilibria in the ATF.

We have also included global $x_\beta y_\beta$ coupling explicitly for the first time and computed the evolution of the invariants for the case of the ATF damping ring. We find that the dependence of $\epsilon_{y,\mathrm{pr}}$ on beam current cannot be explained by our model which suggests that non-IBS physics and/or measurement error may be occurring. The offset can be explained, however, with a beam tilt angle of 4-6 degrees.

Future plans include exploration of full $\eta_y, \kappa, \nu_x - \nu_y$ parameter space in the ATF, application to protons or heavy ions, synchrobetatron coupling, non-Gaussian equilibria and extension beyond the smooth approximation. BN would like to acknowledge Marco Venturini and Ben Freivogel for many useful discussions.

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