INTRA-BEAM-SCBTTERING

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Summary

The excitation and damping of betatron oscillations and the energy spread due to intra-beam scattering is investigated. It is shown that below transition energy an equilibrium for the particle distribution exists which does not depend on the number of the particles. The rise times and damping times for betatron oscillations and energy spread are calculated. The investigation shows that this effect sets a limit to the intensity of stored proton beams at energies below a few GeV.

Introduction

The intra-beam scattering or multiple scattering, i.e. the scattering of the particles within the beam, was investigated in several reports[1-8]. The result of these investigations showed an increase of all beam or bunch dimensions and an increase of the energy spread. But in these investigations two facts have been disregarded, the energy spread within the beam and the influence of the coordinates of the scattered particle on the scattering angle.

The energy spread in the beam, i.e. the energy difference of two colliding particles before the collision, leads to an excitation of betatron oscillations which is, for small energies, the main contribution to the increase of the beam dimensions.

The second point, the linear dependence of the change of the coordinates on these same coordinates, leads to a damping of the betatron oscillations and to a reduction of the energy spread. Without this damping one obtains an infinite increase of the beam dimensions and the energy spread. This seems to be impossible for particles below transition energy.

Far below transition energy the influence of the dispersion can be neglected as will be shown later, and the particles in the beam behave like the particles of a gas in a closed box. Here the focusing forces play the same role as the walls of the box. Since the collisions within a gas cannot lead to an increase of the temperature the collisions within the beam cannot , below transition, lead to an increase of the total oscillation energy and the energy spread. One can only expect a transfer of oscillation energy from one direction into another. Thus, there must exist an equilibrium distribution where the intra-beam scattering does not change the beam dimensions. It is shown that the equilibrium distribution is a Gaussian distribution.

Above transition energy the situation is changed by the property of the particles that is often characterized by the so called "negative mass" behaviour. Here the comparison with a gas in a closed box is not valid, and the calculation shows that the total oscillation energy can increase. The behaviour of the beam can be described with help of an invariant which is given by

$$\frac{1}{n}\left(\frac{1}{\gamma^2}-\alpha_M\right)\left\langle\frac{\Delta p^2}{p^2}\right\rangle + \langle x'^2\rangle + \langle z'^2\rangle = \mathrm{const} \qquad (1)$$

with

p, Δp = momentum and momentum deviation, respectively
x', z' = betatron angles for horizontal and vertical direction, respectively
γ = particle energy divided by its rest energy
α_M = momentum compaction factor
n = 1 for bunched beams, 2 for unbunched beams

The momentum compaction factor α_M is, in Eq.(1), precise only for a weak focusing machine. For a strong focusing machine α_M is a good approximation for the mean value of D^2/β_x^2, where D denotes the dispersion and β_x the amplitude function.

If $1/\gamma^2-\alpha_M$ is positive, i.e. below transition, the three mean values in Eq.(1) are limited. But for negative $1/\gamma^2-\alpha_M$ the three mean values can increase so far as they do not exceed other limitations, and an equilibrium distribution does not exist.

In case of an equilibrium the three terms of the sum in Eq.(1) are equal and one can calculate the equilibrium dimensions starting from the initial dimensions. Eq.(1) further shows that the equilibrium dimensions do not depend on the number of particles, but only on the initial mean values. The number of particles determines the relaxation time for the equilibrium and the rise time for the increase above transition energy.

Kinematics

If s, x and z denote the longitudinal, horizontal and vertical coordinates, the momenta of two particles before a collision are given by (see fig.1)

$$\vec{p}_{1,2} = p_{1,2s}\,\{1,\ x'_{1,2}\ z'_{1,2}\}_{s,x,z} \qquad (2)$$

$x'_{1,2}$ and $z'_{1,2}$ are the betatron angles, and we consider in this investigation only linear and quadratic terms of $x'_{1,2}$ and $z'_{1,2}$. We define a coordinate system $\{u,v,w\}$ with help of the unit vectors

$$\vec{e}_u = (\vec{p}_2+\vec{p}_1)/|\vec{p}_2+\vec{p}_1|,\qquad \vec{e}_v = \vec{p}_1\times\vec{p}_2/|\vec{p}_1\times\vec{p}_2|,\qquad \vec{e}_w = \vec{e}_u\times\vec{e}_v \qquad (3)$$

The momenta can then be written in the form

$$\vec{p}_{1,2} = p_{1,2}\,\{\cos\alpha_{1,2},\ 0,\ \pm\sin\alpha_{1,2}\}_{u,v,w} \qquad (4)$$

The angles α_1 and α_2 are defined by

$$p_1\sin\alpha_1 = p_2\sin\alpha_2 \qquad (5)$$

and

$$\alpha_1+\alpha_2 = 2\alpha = \sqrt{(x_1'-x_2')^2+(z_1'-z_2')^2} \qquad (6)$$

A Lorentz transformation parallel to the u-axis gives the representation of the momenta in the center of mass system

$$\vec{\bar p}_{1,2} = \pm\{\bar p_{\bar u}, 0, \bar p_{\bar w}\}_{\bar u,\bar v,\bar w}$$

$$= p_{1,2}\{\gamma_{tr}(\cos\alpha_{1,2} - \beta_{tr}/\beta_{1,2}), 0, \pm\sin\alpha_{1,2}\}_{\bar u,\bar v,\bar w} \tag{7}$$

where β_{tr} and γ_{tr} are defined by

$$\beta_{tr} = \frac{\beta_1\gamma_1\cos\alpha_1 + \beta_2\gamma_2\cos\alpha_2}{\gamma_1 + \gamma_2} \tag{8}$$

$$\gamma_{tr} = \frac{1}{\sqrt{1-\beta_{tr}^2}}$$

The bar denotes all quantities in the center of **mass** system. After the collision the momenta are changed by the polar angle $\bar\psi$ and the azimuthal angle $\bar\phi$, and the momenta are

$$\vec{\bar p}'_{1,2} = \pm\{\bar p_{\bar w}\sin\bar\psi\cos\bar\phi + \bar p_{\bar u}\cos\bar\psi,\ \bar p\sin\bar\psi\sin\bar\phi,$$
$$\bar p_{\bar w}\cos\bar\psi - \bar p_{\bar u}\sin\bar\psi\cos\bar\phi\}_{\bar u,\bar v,\bar w} \tag{9}$$

where the prime denotes all quantities after the collision. The inverse Lorentz transformation gives the momenta in the $\{u,v,w\}$ system, and the change of the momenta has the form

$$\vec p\,'_{1,2} - \vec p_{1,2} = p_{1,2}\{\gamma_{tr}(\pm\sin\bar\psi\cos\bar\phi\sin\alpha_{1,2} + (\cos\bar\psi-1)\gamma_{tr}(\cos\alpha_{1,2}-\beta_{tr}/\beta_{1,2})),$$
$$\pm\sin\bar\psi\sin\bar\phi\,\bar p/p_{1,2},\ -\sin\bar\psi\cos\bar\phi\,\gamma_{tr}(\cos\alpha_{1,2}-\beta_{tr}/\beta_{1,2})$$
$$\pm(\cos\bar\psi-1)\sin\alpha_{1,2}\}_{u,v,w}$$

We now assume that the velocity of the particles in the center of mass system is non relativistic. Then the relation

$$\bar\beta^2 = \frac{1}{4}\beta^2\gamma^2((\eta_1-\eta_2)^2/\gamma^2 + (x_1'-x_2')^2 + (z_1'-z_2')^2) \ll 1$$

with

$$\eta_{1,2} = \frac{\Delta p_{1,2}}{p}$$

is satisfied, and one obtains

$$\gamma_{tr} = \gamma, \qquad \cos\alpha_{1,2} - \beta_{tr}/\beta_{1,2} = \pm\frac{\eta_1-\eta_2}{2\gamma^2}.$$

The change of the momenta can now be written in the form

$$\vec p\,'_{1,2} - \vec p_{1,2} = \pm\frac{p}{2}\{2\alpha\gamma\sin\bar\psi\cos\bar\phi + \gamma\xi(\cos\bar\psi-1),$$
$$(\zeta\sqrt{1+\xi^2/\alpha^2/4}\,\sin\bar\phi + \xi\Theta/\alpha/2\cos\bar\phi)\sin\bar\psi + \Theta(\cos\bar\psi-1),$$
$$(-\Theta\sqrt{1+\xi^2/\alpha^2/4}\,\sin\bar\phi + \xi\zeta/\alpha/2\cos\bar\phi)\sin\bar\psi + \zeta(\cos\bar\psi-1)\}_{s,x,z}$$

with

$$(\eta_1-\eta_2)/\gamma = \xi, \quad x_1' - x_2' = \Theta, \quad z_1' - z_2' = \zeta .$$

Change of the emittances and the momentum spread

The emittance ε is defined by

$$\beta_x\varepsilon_x = x_\beta^2 + \beta_x^2 x'^2$$

where we have neglected β_x'. The change of the emittance due to a scattering event is

$$\beta_x\delta\varepsilon_x = 2x_\beta\delta x_\beta + \delta^2 x_\beta + 2\beta_x^2 x'\delta x' + \beta_x^2\delta^2 x'$$
$$= -2x_\beta D\frac{\delta p}{p} + D^2\frac{\delta^2 p}{p^2} + 2\beta_x^2 x'\frac{\delta p_x}{p} + \beta_x^2\frac{\delta^2 p_x}{p^2} \tag{11}$$

For the vertical oscillations one obtains

$$\beta_z\delta\varepsilon_z = 2\beta_z^2 z'\frac{\delta p_z}{p} + \beta_z^2\frac{\delta^2 p_z}{p^2} \tag{12}$$

For the momentum spread one can define the following invariant

$$H = \begin{cases} \eta^2 + \dot\eta^2/\Omega^2 & \text{for a bunched beam} \\ \eta^2 & \text{for an unbunched beam} \end{cases}$$

where Ω is the synchrotron frequency and η the relative momentum deviation. The change of H due to a scattering event is in both cases

$$\delta H = 2\eta\frac{\delta p}{p} + \frac{\delta^2 p}{p^2} \tag{13}$$

Determination of an invariant

Since the particle velocity should be non relativistic in the center of mass system we may employ the Rutherford cross section which has the form

$$d\bar\sigma = \left(\frac{r_o}{4\bar\beta^2\sin^2\bar\psi/2}\right)^2\sin\bar\psi\, d\bar\psi\, d\bar\phi \tag{14}$$

with $\quad r$ = classical particle radius.

The impact parameter for the smallest scattering angle $\bar\psi_m$ is given by

$$\tan\frac{\bar\psi_m}{2} = \frac{r_o}{2\bar\beta^2\bar b} \tag{15}$$

We assume $\bar b$ to be equal one half of the average distance between the particles in the center of mass system

$$\bar b = \frac{1}{2}(\gamma/\rho)^{1/3} \tag{16}$$

To calculate the mean value of the change of the emittance or the momentum deviation for one particle we have to average with respect to all betatron angles and momentum deviations of the second particle. For the total mean value of the change of the emittance and momentum deviations of all particles we have to average additionally with respect to all betatron angles, momentum deviations and positions of the first particle. Thus, we have to integrate with the following density function

$$P = \rho_s^2(s)\rho_{x\beta}(x_{\beta 1})\rho_{x\beta}(x_{\beta 2})\rho_z^2(z)\rho_\eta(\eta_1)\rho_\eta(\eta_2)$$
$$\cdot\rho_{x'}(x_1')\rho_{x'}(x_2')\rho_{z'}(z_1')\rho_{z'}(z_2')$$

since

$$s_1 = s_2 = s, \qquad z_1 = z_2 = z$$

but

$$x_{1\beta} + \eta_1 D = x_{2\beta} + \eta_2 D$$

The functions ρ_i are normalized to one. We replace the variables $x_{\beta 1,2}$, $\eta_{1,2}$, $x'_{1,2}$, $z'_{1,2}$ by the variables x_β, η, x', z', ξ, Θ, ζ with help of the relations

$$x_{\beta 1,2} = x_\beta \mp D\gamma\xi/2 \;, \quad \eta_{1,2} = \eta \pm \gamma\xi/2$$

$$x'_{1,2} = x' \pm \Theta/2 \quad , \; z'_{1,2} = z' \pm \zeta/2$$

The function P is now symmetric with respect to ξ, Θ and ζ. The relative velocity between two colliding particles in the center of mass system is $2c\bar\beta$. The mean value of the change of the emittances and the momentum deviation per unit time is then given by

$$\int 2c\bar\beta\, P \int_{\bar\psi_m}^{\pi}\int_0^{2\pi} \frac{d\bar\sigma}{d\bar\omega}\begin{pmatrix}\delta H_1/\gamma^2\\ \delta\varepsilon_{x1}/\beta_x\\ \delta\varepsilon_{z1}/\beta_z\end{pmatrix} \sin\bar\psi\, d\bar\phi\, d\bar\psi\, d\tau \quad =$$

$$= \frac{\pi}{2}\, c\, r_o^2\gamma \int P \begin{pmatrix} \Theta^2 + \zeta^2 - 2\xi^2 \\ \xi^2 + \zeta^2 - 2\Theta^2 + \dfrac{D^2\gamma^2}{\beta_x^2}(\Theta^2+\zeta^2-2\xi^2) \\ \xi^2 + \Theta^2 - 2\zeta^2 \end{pmatrix}$$

$$\ln\left(\frac{2\beta^4 b}{-}\right) \frac{d\tau}{}$$

$$d\tau = ds\, dx_\beta dz\, d\eta dx'\, dz'\, d\xi d\Theta d\zeta$$

$$4\bar\beta^4\bar b^2/r^2 \gg 1$$

From Eq.(17) we get

$$\int\int_{\bar\psi_m}^{\pi}\int_0^{2\pi} P\left(\left(\frac{1}{\gamma^2} - \frac{D^2}{\beta_x^2}\right)\delta H_1 + \frac{\delta\varepsilon_{x1}}{\beta_x} + \frac{\delta\varepsilon_{z1}}{\beta_z}\right)\frac{d\bar\sigma}{d\bar\omega}\,\bar\beta\sin\bar\psi d\bar\phi d\bar\psi d\tau = 0 \tag{18}$$

and

$$\left(\frac{1}{\gamma^2} - \frac{D^2}{\beta_x^2}\right)\langle H\rangle + \langle\varepsilon_x\rangle/\beta_x + \langle\varepsilon_z\rangle/\beta_z = \text{const.}$$

where the brackets denote the average with respect to all particles of the beam. With the relations

$$\langle H\rangle = \begin{cases} 2\langle\dfrac{\Delta^2 p}{p^2}\rangle & \text{for bunched beams} \\ \langle\dfrac{\Delta^2 p}{p^2}\rangle & \text{for unbunched beams}\end{cases}$$

$$\langle\varepsilon_x/\beta_x\rangle = 2\langle x'^2\rangle \;, \; \langle\varepsilon_z/\beta_z\rangle = 2\langle z'^2\rangle \tag{19}$$

we obtain finally for the invariant the expression

$$\frac{1}{n}\left(\frac{1}{\gamma^2} - \frac{D^2}{\beta_x^2}\right)\langle\frac{\Delta^2 p}{p^2}\rangle \quad \langle x'^2\rangle + \langle z'^2\rangle = \text{const.}$$

Rise times and damping times

For the calculation of the rise time and damping time we assume a Gaussian distribution for x_β, z, η, x' and z'. The distribution in longitudinal direction may be uniform or Gaussian-like. With Eq.(17) one gets for the change of the mean value of the emittances and the momentum spread per unit time

$$\begin{pmatrix} \dfrac{d}{dt}\dfrac{\langle H\rangle}{\gamma^2} \\ \dfrac{d}{dt}\dfrac{\langle\varepsilon_x\rangle}{\beta_x} \\ \dfrac{d}{dt}\dfrac{\langle\varepsilon_z\rangle}{\beta_z}\end{pmatrix} = 2c\int P\int_{\bar\psi_m}^{\pi}\int_0^{2\pi}\bar\beta\,\frac{d\bar\sigma}{d\bar\omega}\begin{pmatrix}\dfrac{\delta H_1}{\gamma^2}\\ \dfrac{\delta\varepsilon_{x1}}{\beta_x}\\ \dfrac{\delta\varepsilon_{z1}}{\beta_z}\end{pmatrix}\sin\bar\psi d\bar\phi d\bar\psi d\tau$$

$$= 2A\int\int\int_{-\infty}^{\infty}\exp\left\{-\frac{\xi^2\gamma^2}{4}\left(\frac{D^2}{\sigma_{x\beta}^2}+\frac{1}{\sigma_\eta^2}\right)-\frac{\Theta^2}{4\sigma_{x'}^2}-\frac{\zeta^2}{4\sigma_{z'}^2}\right\}$$

$$\cdot\begin{pmatrix}\Theta^2 + \zeta^2 - 2\xi^2 \\ \xi^2 + \zeta^2 - 2\Theta^2 + \dfrac{D^2\gamma^2}{\beta_x^2}(\Theta^2 + \zeta^2 - 2\xi^2) \\ \xi^2 + \Theta^2 - 2\zeta^2\end{pmatrix}$$

$$\cdot\ln\left(\frac{q^2}{4}(\xi^2+\Theta^2+\zeta^2)\right)\frac{d\xi d\Theta d\zeta}{(\xi^2+\Theta^2+\zeta^2)^{3/2}} \tag{20}$$

with

$$A = \frac{c\, r_o^2\Lambda}{16\pi\sqrt{\pi}\,\sigma_{x\beta}\sigma_z\sigma_\eta\sigma_{x'}\sigma_{z'}\beta^3\gamma^4}$$

$$\Lambda = \begin{cases}\dfrac{N_b}{2\sqrt{\pi}\sigma_s} & \text{for bunched beams}\\ \dfrac{N}{C} & \text{for unbunched beams}\end{cases}$$

$\sigma_{s,x,z}$ = standard deviations for longitudinal, horizontal and vertical bunch dimensions

N_b, N = number of particles in a bunch and in the beam, respectively

C = circumference, $c\beta$ = particle velocity

With Eq.(19) one obtains

$$\frac{1}{\tau_\eta} = \frac{1}{2\sigma_\eta^2}\frac{d\sigma_\eta^2}{dt} = nA\left(1-\frac{D^2\sigma_\eta^2}{\sigma_x^2}\right) f\left(\frac{\sigma_y}{\sigma_{x'}}, \frac{\sigma_y}{\sigma_{z'}}, q\sigma_y\right)$$

$$\frac{1}{\tau_{x'}} = \frac{1}{2\sigma_{x'}^2}\frac{d\sigma_{x'}^2}{dt} = A\left\{f\left(\frac{\sigma_{x'}}{\sigma_y}, \frac{\sigma_{x'}}{\sigma_{z'}}, q\sigma_{x'}\right) + \frac{D^2\sigma_\eta^2}{\sigma_x^2} f\left(\frac{\sigma_y}{\sigma_{x'}}, \frac{\sigma_y}{\sigma_{z'}}, q\sigma_y\right)\right\}$$

$$\frac{1}{\tau_{z'}} = \frac{1}{2\sigma_{z'}^2}\frac{d\sigma_{z'}^2}{dt} = A\; f\left(\frac{\sigma_{z'}}{\sigma_y}, \frac{\sigma_{z'}}{\sigma_{x'}}, q\sigma_{z'}\right)$$

with

$$\sigma_y = \frac{\sigma_\eta\sigma_{x\beta}}{\gamma\sigma_x} \quad , \quad q = \beta\gamma\sqrt{2\frac{\bar b}{r_o}}$$

The function f is defined by

$$f(a,b,c) = 2\int_0^\infty\int_0^\pi\int_0^{2\pi}\exp\left\{-\rho\left(\cos^2\mu + (a^2\cos^2\nu + b^2\sin^2\nu)\cdot\sin^2\mu\right)\right\}\cdot\ln(c^2\rho)(1-3\cos^2\mu)\sin\mu d\nu d\mu d\rho \tag{21}$$

f cannot **be** evaluated analytically. **It** is calculated by a computer program and plotted in fig.2 and **3**. To reduce the range of numerical values for f one can use the relations

$$f(a, b, c) = f(b, a, c)$$

$$f(a,b,c) + \frac{1}{a^2} f\left(\frac{1}{a},\frac{b}{a},\frac{c}{a}\right) + \frac{1}{b^2} f\left(\frac{1}{b},\frac{a}{b},\frac{c}{b}\right) = 0$$

One can proof that

$$f(1, 1, c) = 0 \;.$$

Thus, with

$$\sigma_\eta\sqrt{\frac{1}{\gamma^2} - \frac{D^2}{\beta_x^2}} = \sigma_{x'} = \sigma_{z'}$$

an equilibrium is reached and the Gaussian distribution remains stable, This equilibrium can only be reached for

$$\gamma < \left(\left\langle \frac{D^2}{\beta_x^2} \right\rangle\right)^{-1/2} \approx \frac{1}{\sqrt{\alpha_M}}$$

i.e. below transition energy.

References

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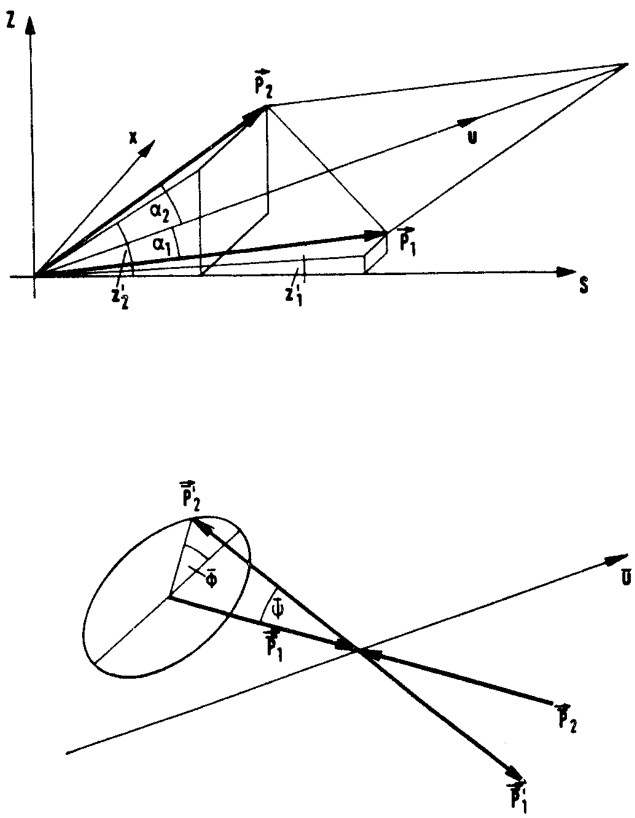


Fig. 1

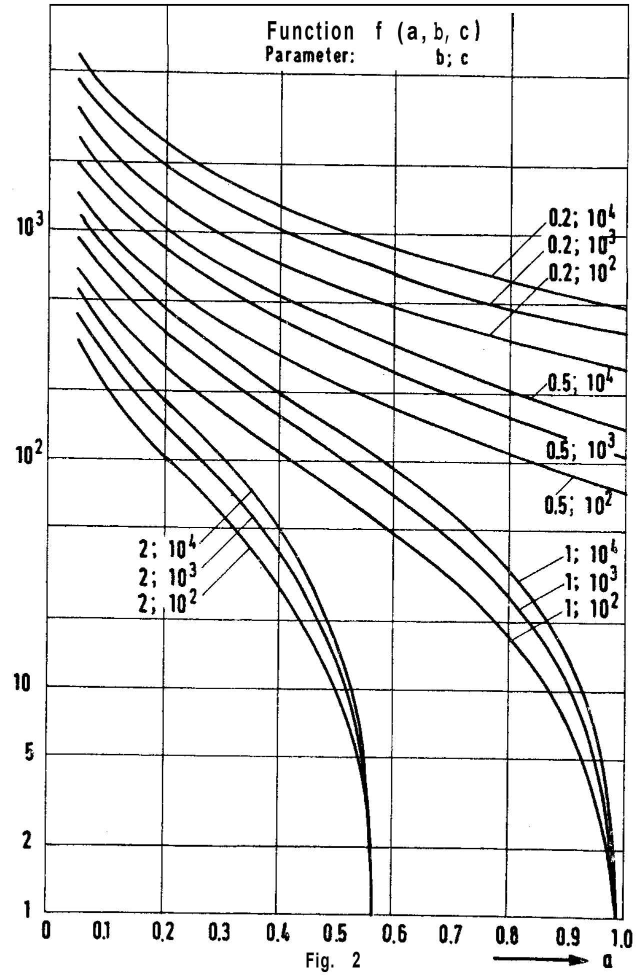
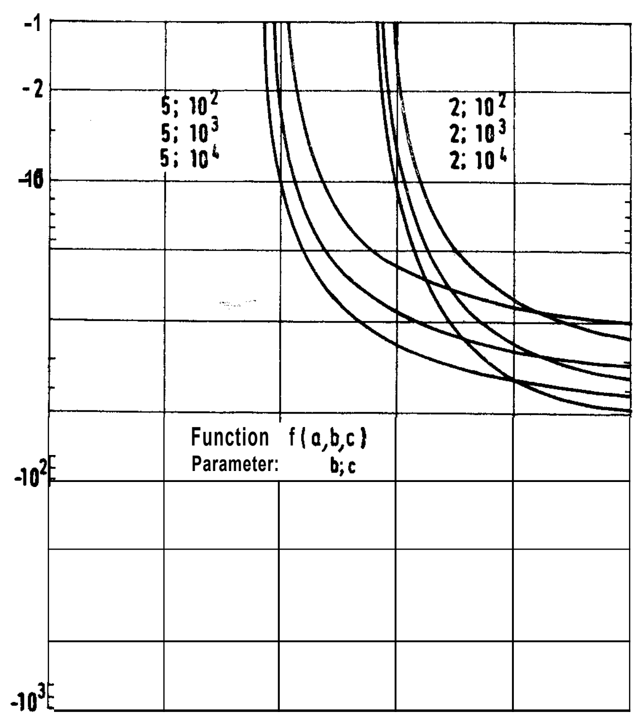


Fig. 2



Andrew Sessler (LBL): Could you remind me how the rise time goes with γ, as one goes to higher energy.

Anton Piwinski (DESY): The risetimes are proportional to 1/A, and A is proportional to $1/\gamma^4$, so the risetimes are proportional to γ^4. But, there is also a $\nu\gamma$ weaker γ dependence in the function f.

Andrei Kolomensky (Lebedev Institute): Are the risetimes independent or are they connected by some rule?

Piwinski: There is a linear relation between $1/\tau_n$ and $1/\tau_z$, but multiplied with certain factors.

Wolfgang K. H. Panofsky (SLAC): How is the numerical agreement with the 2 GeV tests made in the ISR?

Piwinski: Buchner has made an estimate and calculation, and he found that the values that come from measurement have the right order of magnitude, but I don't remember the exact measurement.