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INTRABEAM SCATTERING IN THE ACOL-AA MACHINES

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Abstract

The ACOL machine is designed to achieve in the AA a daily production of about 10^{12} antiprotons. At this expected high intensity, intrabeam scattering effect might be a severe restriction for the accumulation process. In the present report the beam enlargement due to intrabeam scattering is evaluated using an extended version of the Piwinski's model, which allows for the variation of lattice parameters. Experiments made in the AA in which convincing intrabeam scattering evidence has been seen are reported. The agreement between the observed and the calculated beam growth times was found to be satisfactory.

CONTENTS

1. INTRODUCTION

The aim of this paper is to provide the reader with a comprehensive presentation of the multiple Coulomb interactions of the particles within the beam commonly known as intrabeam scattering. Today there is at CERN a revival of interest for that subject, mainly due to the approval of the Antiproton Collector (ACOL). Its construction will increase by a factor of ten the daily production of antiprotons in the Antiproton Accumulator (AA), so that an accumulation rate of about 10^{12} antiprotons a day is to be expected. At this high intensity, intrabeam scattering phenomenon becomes noticeable, and careful analysis must be undertaken to get reliable estimation of that effect.

The original Piwinski's model[1] is outlined along with some refinements worked out subsequently by F. Sacherer and D. Möhl in collaboration with A. Piwinski. These improvements, described in details below, take into account the variation of lattice parameters around the ring, which are significant for a strong-focusing machine such as the AA. The theory presented here is then compared with another formulation designed at Fermilab, which makes essentially use of the same initial assumptions as ours, but uses a rather different approach (S-matrix formalism).The beam enlargement calculated for various intensities, nominal emittances and momentum spread, using both methods are found to be in fairly good agreement.

Afterwards, the validity of the theory is checked for a coasting beam with a number of experiments made in the AA. Stack transverse emittances and momentum dispersion blow-ups were observed when all cooling systems have been switched off and then compared with the values predicted by the generalized model. As shown in the subsequent paragraphs, the measurements obtained agree rather well with the theory.

2. EMITTANCE ENLARGEMENT IN MULTIPLE PARTICLE COLLISIONS

In his paper[1], Piwinski derives the formulae for the variations of the mean radial and vertical emittances and the mean momentum spread per unit time due to a scattering event, neglecting the derivative of the beta and dispersion functions with respect to the longitudinal beam axis. Taking these derivatives into account, we may still use the last two Piwinski's formulae unchanged, but the derivation of the expression for the change of the mean radial emittance must be modified as follows.

To begin with, let us recall that the radial displacement of the particle from the closed orbit is the sum of the betatron and momentum deviation contributions, that is, to the first order in $\Delta p/p$:

$$x = x_\beta + D(\Delta p/p) \tag{1}$$

where D(s) is the dispersion function.

The derivative of x with respect to the distance s along the orbit is equal to the small angle the particle makes with s:

$$x' = x'_\beta + D'(\Delta p/p) \quad \text{and equivalently} \quad x' = p_x/p \tag{2}$$

p_x being the momentum in the x-direction.

The phase space betatron oscillations of a particle satisfy the Courant and Snyder invariant:

$$\varepsilon = \gamma_x x_\beta^2 + 2\alpha_x x_\beta x'_\beta + \beta_x x'^2_\beta \tag{3}$$

where $\beta_x \gamma_x - \alpha_x^2 = 1$ and $\beta'_x = -2\alpha_x$.

The change of the emittance ε is given by: (4)

$$\beta_x \delta\varepsilon = (1 + \alpha_x^2)(2x_\beta \delta x_\beta + \delta x_\beta^2) + 2\alpha_x\beta_x(x'_\beta\delta x_\beta + x_\beta\delta x'_\beta + \delta x_\beta \delta x'_\beta) + \beta_x^2(2x'_\beta\delta x'_\beta + \delta x'^2_\beta) .$$

We can assume that the radial position of the particle does not change during the short time of interaction[2], so that:

$$x = \text{constant} \quad \text{and} \quad \delta x_\beta = -D(\delta p/p) \quad \text{since} \quad \delta\Delta p/p = \delta p/p . \tag{5}$$

Furthermore, $\delta x'_\beta = (\delta p_x/p) - D'(\delta p/p)$.

Let us define $\tilde{D} = \alpha_x D + \beta_x D'$. Whence we find:

$$\frac{\delta\varepsilon}{\beta_x} = -\frac{2}{\beta_x}\left(x_\beta\left(\frac{(1+\alpha_x^2)}{\beta_x} D + \alpha_x D'\right) + x'_\beta \tilde{D}\right)\frac{\delta p}{p} + \frac{(D^2 + \tilde{D}^2)}{\beta_x^2}\left(\frac{\delta p}{p}\right)^2 + 2\left(x'_\beta + \frac{\alpha_x}{\beta_x} x_\beta\right)\frac{\delta p_x}{p}$$
$$+ \left(\frac{\delta p_x}{p}\right)^2 - \frac{2\tilde{D}}{\beta_x}\frac{\delta p}{p}\frac{\delta p_x}{p} . \tag{6}$$

According to Piwinski[1], the relative change of momenta after a collision between two particles labelled 1 and 2 is (for the longitudinal and radial axis):

$$\frac{\delta p}{p} = \frac{1}{2}\left(2\alpha\gamma \sin\bar{\Psi}\sin\bar{\phi} + \gamma\xi(\cos\bar{\Psi} - 1)\right)$$
$$\frac{\delta p_x}{p} = \frac{1}{2}\left(\left[\zeta\sqrt{1 + \frac{\xi^2}{4\alpha^2}}\cos\bar{\phi} - \frac{\xi\theta}{2\alpha}\sin\bar{\phi}\right]\sin\bar{\Psi} + \theta(\cos\bar{\Psi} - 1)\right) , \tag{7}$$

with the notation

$$\gamma\xi = \frac{\Delta p_1}{p} - \frac{\Delta p_2}{p} , \quad \theta = x'_1 - x'_2 , \quad \zeta = z'_1 - z'_2 \tag{8}$$

γ is the particle energy factor.

$\bar{\Psi}$ and $\bar{\phi}$ denote the scattering and azimuthal angles in the centre of momentum (C.M) between the particles after the collision.

2α is the the angle between the incident particles in the laboratory frame. This angle can be evaluated by taking the scalar product of the particle momentum vectors.

We approximately find under small angle assumption that:

$$2\alpha \approx ((x'_2 - x'_1)^2 + (z'_1 - z'_2)^2)^{1/2} . \tag{9}$$

Henceforth the bar indicates the values in the C.M frame.

We shall now consider the distribution of scattering angles $\bar{\Psi}$ which would result from the collision of charged particles in a repulsive Coulomb field. For the case in which the particle velocities in the C.M system are non-relativistic, the Rutherford scattering cross-section formula applies:

$$\frac{d\bar{\sigma}(\bar{\Psi})}{d\bar{\Omega}} = \frac{1}{4}\left(\frac{e^2}{8\pi\varepsilon_0 \bar{T}}\right)^2 \frac{1}{\sin^4(\bar{\Psi}/2)} \quad , \quad d\bar{\Omega} = \sin\bar{\Psi}\, d\bar{\Psi} d\bar{\phi} \tag{10}$$

where $d\bar{\sigma}/d\bar{\Omega}$ is the differential cross-section in the C.M system for the scattering into the element of solid angle $d\bar{\Omega}$ at given angles $\bar{\Psi}$ and $\bar{\phi}$. $\bar{T}$ is the kinetic energy of both particles in the C.M system.

Let $\bar{\Psi}_m$ be the smallest scattering angle, its corresponding impact parameter $\bar{b}$ is given by the equation:

$$\bar{b} = \frac{e^2}{2\bar{T}} \operatorname{ctg} \frac{\bar{\Psi}_m}{2} \quad . \tag{11}$$

Alternatively, equation (10) may be expressed as follows:

$$\frac{d\bar{\sigma}(\bar{\Psi})}{d\bar{\Omega}} = \left(\frac{r_0}{4\bar{\beta}^2}\right)^2 \frac{1}{\sin^4(\bar{\Psi}/2)} \quad , \quad r_0 = \frac{e^2}{4\pi\varepsilon_0 m_0 c^2} = 1.53 \cdot 10^{-18} \; [m] \; . \tag{12}$$

$\bar{\beta}c$ is the particle velocity in the C.M system and $\bar{\beta} \ll 1$ by hypothesis, r_0 is the classical proton radius. In a similar fashion one gets:

$$\operatorname{tg}\bar{\Psi}_m = \frac{r_0}{2\bar{\beta}^2\bar{b}} \quad . \tag{13}$$

In order to evaluate $\bar{\beta}$ we express the momentum difference between the two particles, which is roughly equal to $2m_0\bar{\beta}c$, in terms of their components in the C.M frame. Then, using the Lorentz transformation for the momenta, we come back to the laboratory frame. All calculations done, we find in a first approximation that:

$$\bar{\beta} \approx \frac{1}{2}\, \beta\gamma \left(\frac{1}{\gamma^2}\left(\frac{\Delta p_1}{p} - \frac{\Delta p_2}{p}\right)^2 + (x'_1 - x'_2)^2 + (z'_1 - z'_2)^2\right)^{1/2} \tag{14}$$

where βc is the particle velocity in the rest system and γ is the associated energy factor.

We shall now evaluate an integral which will be considered in the next paragraph, namely:

$$I = \int_{\bar{\Psi}_m}^{\pi} d\bar{\Psi} \int_0^{2\pi} d\bar{\phi}\, \frac{\delta\varepsilon}{\beta_x} \frac{d\bar{\sigma}}{d\bar{\Omega}} \sin\bar{\Psi} \; . \tag{15}$$

Substituting the integral by the expressions given in equations (6, 7, 8, 12) and then performing the trivial integration over the azimuthal angle we find:

$$I = \frac{\pi r_0^2}{16\bar{\beta}^4} \int_{\bar{\Psi}_m}^{\pi} d\bar{\Psi} \frac{\sin\bar{\Psi}}{\sin^4(\bar{\Psi}/2)} \left\{ (1 - \cos\bar{\Psi}) \left(\frac{2x_\beta}{\beta_x} \left(\frac{(1+\alpha_x^2)}{\beta_x} D\gamma\xi + \alpha_x (D'\gamma\xi - \theta) \right) \right. \right.$$

$$\left. + 2x'_\beta \left(\frac{\tilde{D}\gamma\xi}{\beta_x} - \theta \right) + \frac{(D^2 + \tilde{D}^2)}{\beta_x^2} \gamma^2\xi^2 + \theta^2 - \frac{2\tilde{D}}{\beta_x} \gamma\xi\theta \right) \tag{16}$$

$$\left. + \sin^2\bar{\Psi} \left(\frac{(D^2+\tilde{D}^2)}{4\beta_x^2} \gamma^2(\theta^2 + \zeta^2 - 2\xi^2) + \frac{1}{4}(\xi^2 + \zeta^2 - 2\theta^2) + \frac{3\tilde{D}}{2\beta_x} \gamma\xi\theta \right) \right\}.$$

The indefinite integrals over the scattering angle $\bar{\Psi}$ can be read from the table of integrals[3]. Thus:

$$\int_{\bar{\Psi}_m}^{\pi} \frac{\sin\bar{\Psi}(1 - \cos\bar{\Psi})}{\sin^4(\bar{\Psi}/2)} d\bar{\Psi} = -4\ln\left(\frac{1 - \cos\bar{\Psi}}{2}\right)\Bigg|_{\bar{\Psi}_m}^{\pi} = 4\ln\left(1 + \frac{4\bar{\beta}^4\bar{b}^2}{r_0^2}\right) \tag{17}$$

since $1 - \cos\bar{\Psi} = 2\mathrm{tg}^2(\bar{\Psi}/2)/(1 + \mathrm{tg}^2(\bar{\Psi}/2))$ and with the help of formula (13).

In a similar way one find:

$$\int_{\bar{\Psi}_m}^{\pi} \frac{\sin^3\bar{\Psi}}{\sin^4(\bar{\Psi}/2)} d\bar{\Psi} = -8\left(\cos^2(\bar{\Psi}/2) + \ln\left(\frac{1 - \cos\bar{\Psi}}{2}\right)\right)\Bigg|_{\bar{\Psi}_m}^{\pi} = 8\ln\left(1 + \frac{4\bar{\beta}^4\bar{b}^2}{r_0^2}\right) \tag{18}$$

under the assumption $4\bar{\beta}^4\bar{b}^2/r_0^2 \gg 1$.

Taking advantage of these results we arrive, after simple formal transformations, at the expression:

$$I = \frac{\pi r_0^2}{8\bar{\beta}^4} \left\{ \frac{4x_\beta}{\beta_x} \left(\frac{(1+\alpha_x^2)}{\beta_x} D\gamma\xi + \alpha_x(D'\gamma\xi - \theta) \right) + 4x'_\beta \left(\frac{\tilde{D}}{\beta_x}\gamma\xi - \theta \right) + \xi^2 + \zeta^2 \right.$$

$$\left. + \frac{(D^2 + \tilde{D}^2)}{\beta_x^2} \gamma^2(\theta^2 + \zeta^2) + \frac{2\tilde{D}}{\beta_x}\gamma\xi\theta \right\} \ln\left(1 + \frac{4\bar{\beta}^4\bar{b}^2}{r_0^2}\right). \tag{19}$$

3. BETATRON ANGLES AND MOMENTUM SPREAD GROWTH RATES

A. Piwinski introduces[1] the time derivative in the C.M system of the average radial emittance for all particles (denoted $\langle\varepsilon\rangle$) by means of the integral:

$$\left\langle \frac{d}{d\bar{t}} \frac{\langle\varepsilon\rangle}{\beta_x} \right\rangle = \int 2c\bar{\beta}\, P \int_{\bar{\Psi}_m}^{\pi} d\bar{\Psi} \int_0^{2\pi} d\bar{\phi} \frac{d\bar{\sigma}}{d\bar{\Omega}} \frac{\delta\varepsilon_1}{\beta_x} \sin\bar{\Psi}\, d\bar{\tau} \tag{20}$$

where the outer brackets <...> indicate the average value around the ring. The first integral extends over all phase space betatron coordinates, momentum spread values and azimuthal location of two interacting particles (denoted 1 and 2). P is the probability density function for the betatron amplitudes and angles, the momentum errors and the azimuthal positions of the interacting particles. $\bar{P}$ is the probability written in terms of the C.M variables, while P is the same probability written for the laboratory frame variables. $d\bar{t}$ is the C.M time interval which is related to the laboratory time by the equation $dt = \gamma d\bar{t}$. P may be written as a product of independent probability density laws, namely:

$$P = \rho_{x_\beta x'_\beta}(x_{\beta_1} x'_{\beta_1})\rho_{x_\beta x'_\beta}(x_{\beta_2} x'_{\beta_2})\rho_{zz'}(z_1 z'_1)\rho_{zz'}(z_2 z'_2)\rho_\eta\left(\frac{\Delta p_1}{p}\right)\rho_\eta\left(\frac{\Delta p_2}{p}\right)\rho_s(s_1)\rho_s(s_2)$$

$d\tau$ is the infinitesimal element $dx_{\beta_1}\, dx'_{\beta_1} \dots ds_1 ds_2$.

At the time when the interaction takes place we have $s_1 = s_2 = s$, $z_1 = z_2$ and $x_1 = x_2$ where $x_{1,2}$ satisfies equation (1).

Let us perform the substitutions (in full accordance with formula (8)):

$$x_{\beta_{1,2}} = x_\beta \mp \frac{D\gamma\xi}{2}, \quad x'_{\beta_{1,2}} = x'_\beta \pm \frac{\theta - D'\gamma\xi}{2}, \quad z'_{1,2} = z' \pm \frac{\zeta}{2}, \quad \left(\frac{\Delta p}{p}\right)_{1,2} = \eta \pm \frac{\gamma\xi}{2} \qquad (21)$$

(Note that η does not mean here the usual eta function $-(\Delta f/f)/(\Delta p/p)$).

Hence, by virtue of the theorem on functions of random variables[4], the joint probability density law P expressed in terms of these new variables takes on the following form:

$$P(x_\beta, x'_\beta, z, z', \eta, s, \xi, \theta, \zeta) = |J|\; P(x_{\beta_1}, x'_{\beta_1}, x'_{\beta_2}, z_1, z'_1, z'_2, \frac{\Delta p_1}{p}, \frac{\Delta p_2}{p}, s) \qquad (22)$$

and $|J| = \gamma$ where J is the Jacobian of the transformation (21).

In accordance with calculations carried out earlier (eq. 19), one has:

$$\frac{d}{d\bar{t}}\frac{\langle\varepsilon\rangle}{\beta_x} = \frac{\pi}{4} c r_0^2 \gamma \int \bar{P} \left\{ \frac{4x_{\beta_1}}{\beta_x}\left(\frac{(1+\alpha_x^2)}{\beta_x} D\gamma\xi + \alpha_x(D'\gamma\xi - \theta)\right) + 4x'_{\beta_1}(D'\gamma\xi - \theta) + \xi^2 + \zeta^2 - 2\theta^2 \right.$$
$$\left. + \frac{(D^2 + \tilde{D}^2)}{\beta_x^2}\gamma^2(\zeta^2 + \theta^2 - 2\xi^2) + \frac{6\gamma}{\beta_x}\tilde{D}\xi\theta \right\} \ln\left(1 + \frac{4\bar{\beta}^4\bar{b}^2}{r_0^2}\right)\frac{d\bar{\tau}}{\bar{\beta}^3} \qquad (23)$$

where $d\tau = dx_\beta dx'_\beta dz dz' d\eta ds d\xi d\theta d\zeta$.

Clearly, P is symmetrical with respect to ξ, θ and ζ. Therefore, the integral vanishes for the linear terms in ξ, θ, ζ of the integrand. Consequently, we are left with:

$$\frac{d}{d\bar{t}}\frac{\langle\varepsilon\rangle}{\beta_x} = \frac{\pi}{4} c r_0^2 \gamma \int \bar{P} \left\{ \xi^2 + \zeta^2 - 2\theta^2 + \frac{(D^2 + \tilde{D}^2)}{\beta_x^2}\gamma^2(\xi^2 + \theta^2 - 2\zeta^2) + \frac{6\gamma\tilde{D}}{\beta_x}\xi\theta \right\}$$
$$\ln\left(1 + \frac{4\bar{\beta}^4\bar{b}^2}{r_0^2}\right)\frac{d\bar{\tau}}{\bar{\beta}^3}. \qquad (24)$$

At this stage, we shall assume that all betatron amplitudes and angles as well as the momentum deviations obey a Gaussian probability law, whilst the particle distribution in the longitudinal direction is assumed to be uniform if the beam is unbunched, otherwise it is assumed to be Gaussian. More precisely: (see also Appendix A)

$$\rho_{x_\beta x'_\beta}(x_{\beta_{1,2}} x'_{\beta_{1,2}}) = \frac{\sqrt{1+\alpha_x^2}}{2\pi\, \sigma_{x_{\beta_{1,2}}} \sigma_{x'_{\beta_{1,2}}}} \exp\left[-Q(x_{\beta_{1,2}} x'_{\beta_{1,2}})\right] \quad \text{ditto for } \rho_{zz'}(z_{1,2} z'_{1,2})$$

$$Q(x_\beta x'_\beta) = \frac{(1+\alpha_x^2)}{2}\left(\frac{x_\beta^2}{\sigma_{x_\beta}^2} + \frac{2\alpha_x x_\beta x'_\beta}{\sqrt{1+\alpha_x^2}\, \sigma_{x_\beta}\sigma'_{x_\beta}} + \frac{x'^2_\beta}{\sigma^2_{x'_\beta}}\right) \tag{25}$$

$$\rho_\eta\left(\frac{\Delta p_{1,2}}{p}\right) = \frac{1}{\sqrt{2\pi}\, \Delta p_{1,2}/p} \exp\left(-\left(\frac{\Delta p_{1,2}}{p}\right)^2 / 2\sigma_\eta^2\right)$$

where σ_{x_β}, $\sigma_{x'_\beta}$, σ_z and $\sigma_{z'}$ are the standard deviations (or rms fluctuations) of betatron amplitude and angle in the radial and vertical plane. σ_η is the standard deviation of the relative momentum spread.

$$\rho_s(s) = \begin{cases} 1/L & \text{for an unbunched beam} \\ (1/\sqrt{2\pi}\, \sigma_s) \exp(-(s-s_0)^2/2\sigma_s^2) & \text{for a bunched beam ,} \end{cases}$$

the additional azimuthal variable s_0 being distributed uniformly around the ring. L denotes the ring circumference and σ_s the rms bunch length value.

Due to the Lorentz length contraction along the longitudinal beam axis, all the derivatives with respect to s in the C.M frame are reduced by a factor γ (e.g. $\bar{\sigma}_{z'} = \sigma_{z'}/\gamma$), the bunch length becomes $\bar{\sigma}_s = \gamma\sigma_s$, whereas the transverse quantities are unchanged (e.g. $\bar{\sigma}_z = \sigma_z$) as well as the rms relative momentum dispersion value σ_η.

Let us introduce $q = \beta\gamma\sqrt{2\bar{b}/r_0}$.

Hence, representing all variables in the C.M. system, equation (24) becomes:

$$\left\langle \frac{d}{d\bar{t}} \frac{\langle\varepsilon\rangle}{\beta_x} \right\rangle = \frac{2\pi c r_0^2}{\beta^3\gamma^2} \int d\bar{\tau}\, \ln\left[1 + \frac{q^4}{16}\left(\xi^2 + \gamma^2(\bar{\theta}^2 + \bar{\zeta}^2)\right)^2\right] \left(\xi^2 + \gamma^2(\bar{\theta}^2 + \bar{\zeta}^2)\right)^{-3/2}$$

$$\left[\xi^2 + \gamma^2(\bar{\zeta}^2 - 2\bar{\theta}^2) + \frac{(D^2 + \gamma^2\bar{\tilde{D}}^2)}{\beta_x^2}\gamma^2(\xi^2 + \gamma^2(\bar{\theta}^2 - 2\bar{\zeta}^2)) + \frac{6\gamma^3\bar{\tilde{D}}}{\beta_x}\xi\bar{\theta}\right]$$

$$\bar{\rho}_{x_\beta x'_\beta}\left(x_\beta - \frac{D\gamma\xi}{2}, \bar{x}'_\beta + \frac{\gamma(\bar{\theta} - \bar{D}'\gamma\xi)}{2}\right) \bar{\rho}_{x_\beta x'_\beta}\left(x_\beta + \frac{D\gamma\xi}{2}, \bar{x}'_\beta - \frac{\gamma(\bar{\theta} - \bar{D}'\gamma\xi)}{2}\right)$$

$$\bar{\rho}_{zz'}\left(z, \bar{z}' - \frac{\bar{\zeta}}{2}\right) \bar{\rho}_{zz'}\left(z, \bar{z}' + \frac{\bar{\zeta}}{2}\right) \rho_\eta\left(\eta - \frac{\gamma\xi}{2}\right) \rho_\eta\left(\eta + \frac{\gamma\xi}{2}\quad \bar{\rho}_s^2(\bar{s})\right) \tag{26}$$

where the probability density function $\bar{\rho}_{zz'}$ for instance is obtained from $\rho_{zz'}$ by setting $\sigma_z = \bar{\sigma}_z$, $\sigma_{z'} = \gamma\bar{\sigma}_{z'}$, $\alpha_z = \gamma\bar{\alpha}_z$ and $\zeta = \gamma\bar{\zeta}$, such that:

$$\bar{\rho}_{zz'}\left(z, \bar{z}' - \frac{\bar{\zeta}}{2}\right)\bar{\rho}_{zz'}\left(z, \bar{z}' + \frac{\bar{\zeta}}{2}\right) dz\, d\bar{z}'\, d\bar{\zeta} = \rho_{zz'}\left(z, z' - \frac{\zeta}{2}\right)\rho_{zz'}\left(z, z' + \frac{\zeta}{2}\right) dz\, dz'\, d\zeta .$$

For a bunched beam, in addition to these contributions, equation (26) is to be multiplied by a uniform probability law and averaged over one revolution period to take into account the movement of the bunch over the ring.

We solve equation (26) by replacing the probabilities on the right-hand side of (26) by the expressions in (25) and then performing the integrations over the variables x_β, $\bar{x}'_\beta$, z, $\bar{z}'$, η and $\bar{s}$. The integrals are routinely performed with the help of the basic formula[3]:

$$\int_{-\infty}^{+\infty} \exp(-ay^2 - 2by)dy = \sqrt{\pi/a}\, \exp(b^2/a) \text{ for any b and } a > 0.$$

Finally, transforming the result relativistically back into the laboratory frame according to the length contraction and time dilatation formulae, one finds, taking into account all particles in the beam:

$$\left\langle \frac{d}{dt} \frac{\langle\varepsilon\rangle}{\beta_x} \right\rangle = \left\langle \frac{c r_0^2 \sqrt{1+\alpha_x^2}\, \sqrt{1+\alpha_z^2}\, \lambda}{16\pi\sqrt{\pi}\, \sigma_{x_\beta} \sigma_{x'_\beta} \sigma_z \sigma_{z'} \sigma_\eta \beta^3 \gamma^4} \iiint_{-\infty}^{+\infty} \frac{d\xi d\theta d\zeta}{(\xi^2+\theta^2+\zeta^2)^{3/2}} \ln\left[1 + \frac{q^4}{16}(\xi^2+\theta^2+\zeta^2)^2\right] \right.$$

$$\left[\xi^2 + \zeta^2 - 2\theta^2 + \frac{\gamma^2}{\beta_x^2}(D^2+\tilde{D}^2)(\xi^2+\theta^2-2\zeta^2) + \frac{6\gamma}{\beta_x}\tilde{D}\xi\theta\right] \tag{27}$$

$$\left. \exp\left\{ -\frac{\gamma^2\xi^2}{4}\left(\frac{1}{\sigma_\eta^2} + \frac{D^2+\tilde{D}^2}{\sigma_{x_\beta}^2}\right) - \frac{(1+\alpha_x^2)\theta^2}{4\sigma_{x'_\beta}^2} + \frac{\sqrt{1+\alpha_x^2}\,\tilde{D}\gamma\xi\theta}{2\sigma_{x_\beta}\sigma_{x'_\beta}} - \frac{(1+\alpha_z^2)\zeta^2}{4\sigma_{z'}^2} \right\} \right\rangle$$

$$\text{where } \lambda = \begin{cases} N/L & \text{for an unbunched beam} \\ N_b/(2\sqrt{\pi}\,\sigma_s) & \text{for a bunched beam} \end{cases}$$

N is the number of particles in the beam, N_b is the number of particles per bunch. Remember that the outer brackets <...> denote an average around the ring, using a uniform probability law.

As mentioned at the beginning of the previous section, we may include Piwinski's expressions[1] for the derivatives of the mean vertical emittance and mean momentum spread, thus we get the full set of equations:

$$\begin{bmatrix} \left\langle \frac{d}{dt} \frac{\langle H\rangle}{\gamma^2} \right\rangle \\ \left\langle \frac{d}{dt} \frac{\langle\varepsilon_x\rangle}{\beta_x} \right\rangle \\ \left\langle \frac{d}{dt} \frac{\langle\varepsilon_z\rangle}{\beta_z} \right\rangle \end{bmatrix} = \left\langle A \iiint_{-\infty}^{+\infty} \exp\left\{ -\frac{\gamma^2\xi^2}{4}\left(\frac{1}{\sigma_\eta^2} + \frac{D^2+\tilde{D}^2}{\sigma_{x_\beta}^2}\right) - \frac{(1+\alpha_x^2)\theta^2}{4\sigma_{x'_\beta}^2} + \frac{\sqrt{1+\alpha_x^2}\,\tilde{D}\gamma\xi\theta}{2\sigma_{x'_\beta}\sigma_{x_\beta}} + \frac{(1+\alpha_z^2)\zeta^2}{4\sigma_{z'}^2} \right\} \right.$$

$$\begin{bmatrix} \theta^2 + \zeta^2 - 2\xi^2 \\ \xi^2 + \zeta^2 - 2\theta^2 + \frac{\gamma^2}{\beta_x^2}(D^2+\tilde{D}^2)(\theta^2+\zeta^2-2\xi^2) + \frac{6\gamma}{\beta_x}\tilde{D}\xi\theta \\ \xi^2 + \theta^2 - 2\zeta^2 \end{bmatrix} \tag{28}$$

$$\left. \ln\left[1 + \frac{q^4}{16}(\xi^2+\theta^2+\zeta^2)^2\right] \frac{d\xi d\theta d\zeta}{(\xi^2+\theta^2+\zeta^2)^{3/2}} \right\rangle$$

with $A = (\sqrt{1+\alpha_x^2}\sqrt{1+\alpha_z^2}\, c r_0^2 \lambda)/(16\pi\sqrt{\pi}\; \sigma_{x_\beta}\sigma_{x'_\beta}\sigma_z\sigma_{z'}\sigma_\eta \beta^3\gamma^4)$ ans where H is defined by[1]:

$$H = \begin{cases} (\Delta p/p)^2 + 1/\Omega\, d/dt(\Delta p/p)^2 & \text{for a bunched beam, } \Omega \text{ being the synchrotron frequency,} \\ (\Delta p/p)^2 & \text{for an unbunched beam.} \end{cases}$$

One sees that the factor A is proportional to the number of circulating particles.

Let us define

$$\sigma_x^2 = \sigma_{x_\beta}^2 + D^2\sigma_\eta^2 , \quad \sigma_y = \frac{\sigma_\eta \sigma_{x_\beta}}{\gamma\sigma_x} , \quad d = \frac{\sigma_\eta}{\sigma_x} D , \quad \tilde{d} = \frac{\sigma_\eta}{\sigma_x}\tilde{D}$$

and

$$a = \frac{\sigma_y}{\sigma_{x'_\beta}}\sqrt{1+\alpha_x^2} , \quad b = \frac{\sigma_y}{\sigma_{z'}}\sqrt{1+\alpha_z^2} , \quad c = q\sigma_y . \tag{29}$$

Assuming that the impact parameter $\bar{b}$ is equal to the beam height $2\sigma_z$[5,6], we have consequently:

$$q = 2\beta\gamma\sqrt{\sigma_z/r_0} .$$

Then, with equations (A11) and taking the mean value of H[1], one can write:

$$\sigma_{x'_\beta}^2 = \frac{(1+\alpha_x^2)}{\beta_x}\frac{\langle\varepsilon_x\rangle}{2} , \quad \sigma_{z'}^2 = \frac{(1+\alpha_{z'}^2)}{\beta_z}\frac{\langle\varepsilon_z\rangle}{2} , \quad \langle H\rangle = m\,\sigma_\eta^2 \tag{30}$$

where m = 1 if the beam is unbunched, m = 2 otherwise.

From these formulae it follows that the rms fluctuation of relative momentum spread and betatron angles growth rates can be put in the form:

$$\begin{bmatrix} \dfrac{1}{\tau_\eta} \\[2ex] \dfrac{1}{\tau_{x'}} \\[2ex] \dfrac{1}{\tau_{z'}} \end{bmatrix} = \begin{bmatrix} \left\langle \dfrac{1}{\sigma_\eta}\dfrac{d\sigma_\eta}{dt} \right\rangle \\[2ex] \left\langle \dfrac{1}{\sigma_{x'_\beta}}\dfrac{d\sigma_{x'_\beta}}{dt} \right\rangle \\[2ex] \left\langle \dfrac{1}{\sigma_{z'}}\dfrac{d\sigma_{z'}}{dt} \right\rangle \end{bmatrix} = \begin{bmatrix} \left\langle \dfrac{n(1-d^2)q^2}{4c^2}\dfrac{d}{dt}\dfrac{\langle H\rangle}{\gamma^2} \right\rangle \\[2ex] \left\langle \dfrac{a^2q^2}{c^2}\dfrac{d}{dt}\dfrac{\langle\varepsilon_x\rangle}{\beta_x} \right\rangle \\[2ex] \left\langle \dfrac{b^2q^2}{c^2}\dfrac{d}{dt}\dfrac{\langle\varepsilon_z\rangle}{\beta_z} \right\rangle \end{bmatrix} \qquad n = \begin{cases} 1 & \text{for a bunched beam} \\ 2 & \text{for an unbunched beam} \end{cases} \tag{31}$$

The quantities τ_η, $\tau_{x'}$ and $\tau_{z'}$ are time constants which determine how rapidly σ_η, $\sigma_{x'_\beta}$ and $\sigma_{z'}$ increase.

Substituting $2u = q\xi$, $2v = q\theta$, $2w = q\zeta$ and $z^2 = u^2 + v^2 + w^2$ leads to:

$$\begin{bmatrix} \frac{1}{\tau_\eta} \\ \frac{1}{\tau_{x'}} \\ \frac{1}{\tau_{z'}} \end{bmatrix} = \Big\langle \frac{A}{c^2} \iiint_{-\infty}^{+\infty} \exp\left\{ -\frac{1}{c^2} \left[u^2 + b^2 w^2 + (av - \tilde{d}u)^2\right] \right\} \frac{\ln(1+z^2)}{z^{3/2}} \tag{32}$$

$$\begin{bmatrix} n(1-d^2)(-2u^2+v^2+w^2) \\ a^2(u^2-2v^2+w^2) + (d^2+\tilde{d}^2)(-2u^2+v^2+w^2) + 6a\tilde{d}uv \\ b^2(u^2+v^2-2w^2) \end{bmatrix} du\, dv\, dw \Big\rangle$$

We transform this later expression to an integration in Polar coordinates with:

$$u = \sqrt{z}\,\sin\mu\,\cos\nu, \quad v = \sqrt{z}\,\sin\mu\,\sin\nu, \quad w = \sqrt{z}\,\cos\mu,$$

then we get:

$$\begin{bmatrix} \frac{1}{\tau_\eta} \\ \frac{1}{\tau_{x'}} \\ \frac{1}{\tau_{z'}} \end{bmatrix} = \Big\langle \frac{A}{c^2} \int_0^\infty dz \int_0^\pi d\mu \int_0^{2\pi} d\nu\, \sin\mu \begin{vmatrix} n(1-d^2)g_1(\mu\nu) \\ d^2 g_2(\mu\nu) + (d^2+\tilde{d}^2)g_1(\mu\nu) \\ b^2 g_3(\mu\nu) \end{vmatrix} \exp[-D(\mu\nu)z]\ln(1+z^2) \Big\rangle \tag{33}$$

where $D(\mu\nu) = [\sin^2\mu\,\cos^2\nu + \sin^2\mu(a\,\sin\nu - \tilde{d}\,\cos\nu)^2 + b^2\cos^2\mu]/c^2$,

$g_1(\mu\nu) = 1 - 3\sin^2\mu\,\cos^2\nu$,

$g_2(\mu\nu) = 1 - 3\sin^2\mu\,\sin^2\nu + 6\tilde{d}\,\sin\mu\,\sin\nu\,\cos\nu/a$,

$g_3(\mu\nu) = 1 - 3\cos^2\mu$.

Let us introduce the scattering function by means of the triple integrals:

$$f_i = k_i \int_0^\infty dz \int_0^\pi d\mu \int_0^{2\pi} d\nu\, \sin\mu\, g_i(\mu\nu)\, \exp[-D(\mu\nu)z]\, \ln(1+z^2) \tag{34}$$

with $k_1 = 1/c^2$, $k_2 = a^2/c^2$ and $k_3 = b^2/c^2$.

The growth rates are then given by three expressions:

$$\frac{1}{\tau_\eta} = \Big\langle \frac{nA}{2}(1-d^2)f_1 \Big\rangle$$

$$\frac{1}{\tau_{x'}} = \Big\langle \frac{A}{2}\left[f_2 + (d^2+\tilde{d}^2)f_1\right] \Big\rangle \tag{35}$$

$$\frac{1}{\tau_{z'}} = \Big\langle \frac{A}{2} f_3 \Big\rangle$$

It should be noted that the growth rates are generally varying with time, and are to be considered as instantaneous values.

On the other hand, since the analytical variation of the lattice parameters $\alpha_{x,z}$, $\beta_{x,z}$, D and D' around the ring are not known, the expressions (35) will be numerically calculated at M different lattice locations and their average values will be considered as the mean growth rates. That is:

$$\frac{1}{\tau_\eta} = \frac{1}{M}\sum_i \frac{nA_i}{2}(1 - d_i^2)f_{1_i}$$

$$\frac{1}{\tau_{x'}} = \frac{1}{M}\sum_i \frac{A_i}{2}[f_{2_i} + (d_i^2 + \tilde{d}_i^2)f_{1_i}] \tag{36}$$

$$\frac{1}{\tau_{z'}} = \frac{1}{M}\sum_i \frac{A_i}{2} f_{3_i} \quad .$$

A_i and $f_{\ell i}$ denote the values of A and f_ℓ computed at the azimuthal location ($i = 1...M$, $\ell = 1...3$).

In the case where the derivatives of the beta functions and dispersion are neglected (i.e. $\beta_x' = \beta_z'$ and $D' = 0$) the scattering function can be reduced to the following form:

$$f_1 = f(a,b,c), \qquad f_2 = f\left(\frac{1}{a}, \frac{b}{a}, \frac{c}{a}\right), \qquad f_3 = f\left(\frac{1}{b}, \frac{a}{b}, \frac{c}{b}\right)$$

with

$$f(a,b,c) = 2\int_0^\infty d\rho \int_0^\pi d\mu \int_0^{2\pi} d\nu \, \sin\mu (1 - 3\cos^2\mu) \exp[-D_0(\mu\nu)\rho] \ln(c^2\rho) \tag{37}$$

and

$$D_0(\mu\nu) = \sin^2\mu(a^2\cos^2\nu + b^2\sin^2\nu) + \cos^2\mu, \qquad z = c^2\rho \; .$$

(See derivation in Appendix B).

The radial betatron angle growth rate changes to $1/\tau_{x'} = (A/2)(f_2 + d^2 f_1)$.

The others two growth rate formulae stay unchanged (but the averaging process over beam positions around the ring vanishes since the beta and dispersion functions are assumed to be constant).

Formulae (37) are the original Piwinski's results[1,5]. With these approximations, the CERN computer program BBI[7] computes the three growth-rates for bunched beams.

4. EVALUATION OF THE SCATTERING FUNCTION

Let us come back to the relation (33). The integration over z is found in the integral' s tables[3].

$$\int_0^\infty \exp(-Dz)\ \ln(1+z^2)dz = \frac{2}{D}\left[\left(\frac{\pi}{2} - \mathrm{Si}(D)\right)\sin D - \mathrm{Ci}(D)\ \cos D\right] \tag{38}$$

where

$$\mathrm{Si}(D) = \sum_{i=1}^{\infty} (-1)^{i-1} \frac{D^{2i-1}}{(2i-1)(2i-1)!} \qquad \text{sine integral,}$$

$$\mathrm{Ci}(D) = E + \ln D + \sum_{i=1}^{\infty} (-1)^{i} \frac{D^{2i}}{2i(2i)!} \qquad \text{cosine integral,}$$

$$E = 0.577216 \text{ (Euler's constant)}$$

Substituting $x = \mu$ and $y = \mu\nu$ leads to:

$$f_i = 2k_i \int_0^\pi dx \int_0^{2\pi x} dy\ \frac{\sin x}{x}\ \frac{gi(xy)}{D(xy)} \left[\sin D(xy)\left(\frac{\pi}{2} - \mathrm{Si}\left(D(xy)\right)\right) - \cos D(xy)\ \mathrm{Ci}\left(D(xy)\right)\right].$$

Retaining only a few terms of each series expansion, we get approximately:

$$f_i \approx 2k_i \int_0^\pi dx\ \frac{\sin x}{x} \int_0^{2\pi x} dy\ \frac{gi}{D}\left[\sin D\left(\frac{\pi}{2} - D + \frac{D^3}{18} - \frac{D^5}{600}\right) - \cos D\left(E + \ln D - \frac{D^2}{4} + \frac{D^4}{96}\right)\right] \tag{39}$$

This is a double integral over a triangle of area π^3 which cannot be expressed in closed form. However, it can be numerically evaluated by computer[8,9,22] using for instance the CERN integration routine TRIINT. In order to reduce the calculation time and to increase the accuracy, the integration's domain can be made smaller by splitting the integral in two parts, using the periodic and symmetric properties of the integrand.

Clearly, we obtain:

$$f_i = 8k_i \int_0^{\pi/2} dx \int_0^{\pi x/2} dy\ \frac{\sin x}{x}\ \frac{gi^+}{D^+}\left[\sin D^+\left(\frac{\pi}{2} - \mathrm{Si}(D^+)\right) - \cos D^+\ \mathrm{Ci}(D^+)\right]$$

$$+ 8k_i \int_0^{\pi/2} dx \int_0^{\pi x/2} dy\ \frac{\sin x}{x}\ \frac{gi^-}{D^-}\left[\sin D^-\left(\frac{\pi}{2} - \mathrm{Si}(D^-)\right) - \cos D^-\ \mathrm{Ci}(D^-)\right] \tag{40}$$

with

$$D^{\pm}(xy) = \frac{1}{c^2}\left[\sin^2 x\ \cos^2\frac{y}{x} + \sin^2 x\left(a\ \sin\frac{y}{x} \mp \tilde{d}\ \cos\frac{y}{x}\right)^2 + b^2\cos^2 x\right],$$

$$g_1^{\pm}(xy) = 1 - 3 \sin^2 x \cos^2 \frac{y}{x}$$

$$g_2^{\pm}(xy) = 1 - 3 \sin^2 x \sin^2 \frac{y}{x} \mp \frac{6\tilde{d}}{a} \sin^2 x \sin \frac{y}{x} \cos \frac{y}{x}$$

$$g_3^{\pm}(xy) = 1 - 3 \cos^2 x \ .$$

So, for each integral the triangular integration domain area is now equal to $\pi^3/8$.

For practical convenience, it might be better to express the factor A (equ. 28) in terms of the transverse emittances and the relative momentum spread.

Remembering that if the emittances are defined at 95% of particles in the phase space, the following formulae yield for Gaussian beam (equ. A16).

$$\varepsilon_{x,z} = \frac{6\sigma_{x,z}^2}{\beta_{x,z}} = \frac{6\sigma_{x',z'}^2}{\gamma_{x,z}} \quad \text{and} \quad \frac{\Delta p}{p} = 1.96\ \sigma_\eta \ . \tag{41}$$

Thus, for a coasting beam, the factor A can be rewritten as:

$$A = \frac{4.41\ Nc r_0^2}{\pi\sqrt{\pi}\ L\ \varepsilon_x \varepsilon_z\ \Delta p/p\ \beta^3\gamma^4} \ . \tag{42}$$

5. APPLICATION

As an example, let us now evaluate the enlargement of an antiproton beam in the AA, due to intrabeam scattering effect, using formulae (36.40.42). Following the AA Design Report[12] the nominal stack parameters are:

$N = 6 \cdot 10^{11}\ \bar{p}$ (at 3.41 GeV/c in the stack center, $\gamma = 3.7722$),

$\varepsilon_x = 1.4\pi$ mm.mrad,

$\varepsilon_z = 1.0\pi$ mm.mrad,

$\Delta p/p = \pm 1.5 \cdot 10^{-3}$.

The ring circumference is L = 155.84 m.

The theoretical betatron angles and relative momentum spread instantaneous growth rates are calculated by averaging the local growth rates at 20 locations of half an AA lattice super-period[13]. To illustrate the nature of the improved theory relative to the original Piwinski's work, we have computed the growth rates, neglecting the variation of the lattice parameters around the ring (eq. 37). Furthermore, as a matter of comparison, we have also evaluated the theoretical growth rates using the Fermilab model [14-15]. All the results are gathered in Table 1.

Fig. 1 to Fig. 4 show the various growth times versus radial emittance for the quoted vertical emittances and momentum spread. For the vertical plane, instead of the growth times, the growth rates are plotted to emphasize the tendency for $1/\tau_{z'}$ to become negative (beam damped) when the transverse emittances or the momentum spread decrease.

Table 1

	Radial $\tau_{x'}$ [h]	Vertical $\tau_{z'}$ [h]	Longitudinal τ_η [h]
Extended Piwinski's model	0.15	1.62	1.70
Original Piwinski's model ($\beta'_{x,z}$ = D' = 0)	0.26	1.21	3.30
Fermilab model	0.16	1.88	1.73

6. EXPERIMENTAL RESULTS

We now turn our attention to the experiments in which intrabeam scattering effect has been observed. An intense proton stack was made in the AA with only the stack core cooling systems switched on. The stack was then left cooling down until it no longer seemed to get cooler. Thereafter, the cooling was switched off and the proton beam was left to blow-up in all planes. The evolution of the transverse emittances $\varepsilon_{x,z}$ and the rms frequency (half) stack width Δf_{rms} was observed every 5 minutes over an interval of time of 8 hours. The acquired data are shown on a semi-log plot in Fig. 5. Zero time corresponds to when the cooling was turned off. The corresponding stable beam parameters were:

$$N = 6.01 \cdot 10^{11}\ p \quad (\gamma = 3.7722 \text{ at } 3.41 \text{ GeV/c}),$$

$$\varepsilon_x = 3.9\pi \text{ mm.mrad},$$

$$\varepsilon_z = 7.6\pi \text{ mm.mrad},$$

$$\Delta p/p = \pm 1.9 \cdot 10^{-3} \quad (\text{or } \Delta f_{rms} = 154 \text{ Hz}).$$

It is worth noting that the emittances were measured by means of a Schottky pick-up and a spectrum analyser, where the beam sidebands were analysed using a calibration factor. This empirical factor was settled once and for all by doing an emittance measurement with scapers at 95% of particles.

By definition, the growth rates obey the following first order differential equations:

$$\frac{1}{\tau_{\varepsilon_{x,z}}} = \frac{1}{\varepsilon_{x,z}} \frac{d\varepsilon_{x,z}}{dt} \quad \text{and} \quad \frac{1}{\tau_\eta} = \frac{1}{\sigma_\eta} \frac{d\sigma_\eta}{dt} \tag{43}$$

whose general solution is:

$$\varepsilon_{x,z}(t) = \varepsilon_{0,x,z} \exp\left(\int \frac{dt}{\tau_{x,z}(t)}\right), \quad \text{ditto for } \sigma_\eta . \tag{44}$$

To describe the correspondence between the foregoing theory and the measured data, the emittance growth rates are related to the betatron angle growth rates by means of the expression (see eq. 31 and 43):

$$\frac{1}{\tau_{\varepsilon_{x,z}}} = \frac{2}{\tau_{x'_\beta,z'}} \quad \text{for any fixed lattice location s.} \tag{45}$$

One has also:

$$\sigma_\eta = \frac{1}{|\eta_0|} \frac{\Delta f_{rms}}{f}$$

where η_0 = -0.0853 is the eta function and f ≈ 1.855 MHz is the stack centre frequency [13].

Considering equation (43) one sees that the heating rates are nothing else than the logarithm derivatives of the emittances with respect to the time. Thus, if one can find analytic functions which are in some sense the best approximation of the curves shown in Fig. 5, then the instantaneous growth rates will be just equal to the first derivative of these functions. A solution to that problem is given by what is being referred to as spline smoothing techniques [18].

Let us describe briefly the method. For the sake of brevity, let ε_a denotes ε_x, ε_z or σ_η.

Given the support points $\{t_i, \ln \varepsilon_{ai}\}$ i = 0...N, the ε_{ai} being known only to a certain error, an approximating-smoothing cubic spline function $s_a(t)$ consists of cubic polynomials pieced together. The piecewise function $s_a(t)$ is required to have the properties:

1) $s_a(t)$ is twice continuously differentiable on $[t_0, t_N]$,

2) on each subinterval $[t_{i-1}, t_i]$ $s_a(t)$ coincides with the cubic polynomial:

$$s_a(t) = s_{ai}(t) = \alpha_{ai} + \beta_{ai}(t_i - t) + \gamma_{ai}(t_i - t)^2 + \delta_{ai}(t_i - t)^3 \quad \text{for } i = 1 \ldots N, \tag{46}$$

3) end-conditions: $s_a''(t_0) = s_a''(t_N) = 0$,

4) smoothing-conditions: $\phi(s_y) = \min \phi(u)$ for all functions u(t) such as:

$$\phi(u) = \int_{t_0}^{t_N} u''^2(t)dt + \sum_{i=0}^{N} p_i (u(t_i) - \ln \varepsilon_{ai})^2$$

where p_i are positive factors used to control the smoothing process. The more accurate the ε_{ai}, the larger the p_i. Empirical test computations may help to fix these factors.

The 4N coefficients α_{ai}, β_{ai}, γ_{ai}, δ_{ai} are obtained by solving a (N × N) five-diagonal matrix equation. See Reference 18 for further information. It should be pointed out that cubic splines yield smooth approximating curves which are less likely to exhibit the large oscillations characteristic of high-degree polynomial least-squares fit.

In regard to the growth rates, since ln $\varepsilon_a(t)$ may be approximated by the spline functions $s_a(t)$, one gets in virtue of equations (43-46):

$$\frac{1}{\tau_{ai}(t)} \simeq -\beta_{ai} - 2\gamma_{ai}(t_i - t) - 3\delta_{ai}(t_i - t)^2 \quad \text{for } i = 1 \ldots N . \tag{47}$$

In particular, at the knots t_i one obtains $1/\tau_{ai} \simeq -\beta_{ai}$. After having themselves been smoothed, the experimental growth times τ_{ai} are plotted in Fig. 6 and Fig. 7, where they are compared with their corresponding theoretical ones. As before the theoretical growth rates are calculated for each of the 20 AA lattice locations, and then the average values are computed.

From Fig. 6 one notices that the measured radial emittance growth times are fairly close to the calculated values, whilst the measured momentum spread growth times are slightly shorter than those calculated (the mean deviation is approximately 30%). On the other hand, the observed vertical emittance growth times are somewhat erratic (Fig. 7), even after smoothing, and do not agree with the values predicted by the theory. For this case, other high intensity-dependent effects might contribute to the beam blow-up[12]. However, both the measured and calculated growth rates $1/\tau_{\varepsilon V}$ are small enough to be negligible in practice when considering the growth of the beam.

A similar experiment has been performed in the AA during a $\bar{p}$ accumulation period, using an antiproton stack whose characteristics after cooling were:

$$N = 7.82 \cdot 10^{10}\ \bar{p} \qquad (\gamma = 3.7722)$$

$$\varepsilon_x = 1.0\pi \text{ mm.mrad}$$

$$\varepsilon_z = 1.7\pi \text{ mm.mrad}$$

$$\Delta p/p = \pm 1.3 \,.\, 10^{-3} \quad (\text{or } \Delta f_{rms} = 108 \text{ Hz})$$

After having switched the cooling system off, the evolution of the transverse emittances and momentum spread was observed for about 1 hour. The data are shown in Fig. 8 in a semi-log plot. Figures 9 and 10 show the measured growth times obtained from the data after smoothing by the spline method. Not surprisingly, the experimental radial emittance and momentum spread rise times are found to agree satisfactorily with the values expected from the theory, but like for the previous experiment, there is a lack of agreement about the vertical emittance growth times.

7. DISCUSSION

When computing the diffusion rates of a coasting beam, pure intrabeam scattering effect was considered. In reality, other effects should be taken into account to explain the difference between the observed and the calculated growth rates [12,19]. One evident cause of this discrepancy comes because the real betatron and momentum spread distributions of the beam do not obey Gaussian probability laws, as it is assumed by the theory. A second reason may be attributed to the coupling between the radial and vertical motions which can give rise to an instability inducing a growth of betatron oscillations. Nothing quantitative about these two later effects will be given here.

Another reason for the transverse emittance growth could be due to multiple scattering in the rest gas, one has for $\varepsilon_{x,z} = 2\pi$ mm.mrad[19]:

$$\frac{1}{\tau_{gaz}} \approx 2.5 \cdot 10^{-4}\, P_G$$

where $P_G \approx 5 \cdot 10^{-11}$ Torr is the AA gauge pressure.

In addition to this, the residual gas ionization by the circulating antiprotons produces heavy positive ions trapped by the beam, causing a slow blow-up of the beam emittances, given for the AA by[19]:

$$\frac{1}{\tau_{ions}} \approx 1.5 \cdot 10^{-21} \frac{N\eta}{\varepsilon_{x,z}}$$

where $\eta \approx 0.03$[12] is the neutralization factor when the ion clearing electrodes are switched on. Thus, for $N = 6 \cdot 10^{11}$ $\bar{p}$ and $\varepsilon_{x,z} = 2\pi$ mm.mrad one gets $\tau_{gas} \approx 200$ h and $\tau_{ions} \approx 580$ h. Clearly these two effects are negligible with respect to the intrabeam scattering beam enlargement.

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APPENDIX A

BIVARIATE GAUSSIAN DISTRIBUTION FOR REPRESENTING BEAM DENSITY

We assume that the beam density in the transverse phase space is specified by a jointly normal distribution. In practice we know that the beam distribution does not obey exactly a normal probability law. For instance, the real distribution must be a truncated one due to aperture limits. Nevertheless, for mathematical convenience, we insist on regarding the joint betatron amplitude and angle probability distribution of the particles as obeying approximately a normal probability law, provided the justification of this approximation is kept clearly in mind. More precisely, either for radial and vertical betatron plane:

$$\rho(zz') = \frac{1}{2\pi\,\sigma_z\sigma_{z'}\sqrt{1-r^2}}\exp[-Q(zz')] \qquad -1 < r < +1 \qquad \text{(A1)}$$

in which the function Q(zz') is defined by:

$$Q(zz') = \frac{1}{2(1-r^2)}\left(\frac{z^2}{\sigma_z^2} - \frac{2rzz'}{\sigma_z\sigma_{z'}} + \frac{z'^2}{\sigma_{z'}^2}\right).$$

$\rho(zz')$ is normalized to one and $\langle z^2\rangle = \sigma_z^2$, $\langle z'^2\rangle = \sigma_{z'}^2$, $\langle z\rangle = \langle z'\rangle = 0$,
$r = \langle zz'\rangle/\sigma_z\sigma_{z'}$ is the correlation coefficient.

The curve $Q(zz')$ = constant is an ellipse. Furthermore, according to the betatron motion description in the transverse phase space one might link the quadratic form $Q(zz')$ with the Courant and Snyder invariant thanks to the relationship.

$$Q(zz') = \lambda\varepsilon \qquad \text{(A2)}$$

where $\varepsilon = \gamma_z z^2 + 2\alpha_z zz' + \beta_z z'^2$.

The Twiss parameters are related by $\beta_z\gamma_z - \alpha_z^2 = 1$, ε is the emittance and λ is a proportionality constant.

Equating coefficients of z^2, zz' and z'^2 on both sides of $Q = \lambda\varepsilon$ gives:

$$r = \frac{-\alpha_z}{\sqrt{1+\alpha_z^2}}, \quad \sigma_z^2 = \frac{\beta_z}{2\lambda}, \quad \sigma_{z'}^2 = \frac{\gamma_z}{2\lambda}. \qquad \text{(A3)}$$

Let us find now the distribution function $F(\varepsilon)$ which is equal to the probability that a phase space particle coordinates fall within the ellipse bounded by Q.

$$F(\varepsilon) = \iint_{D(\varepsilon)} \rho(zz')\,dz\,dz' \qquad \text{(A4)}$$

where $D(\varepsilon)$ is the domain bounded by Q. We introduce the Polar coordinates:

$$z = y\cos\mu, \quad z' = y\sin\mu .$$

Thus, we get[4].

$$F(\varepsilon) = \frac{1}{2\pi\,\sigma_z\sigma_{z'}\sqrt{1-r^2}} \int_0^{2\pi} d\mu \int_0^{\sqrt{2\lambda\varepsilon/u}} y \exp\left[-\frac{y^2 u(\mu)^2}{2}\right] dy \tag{A5}$$

with

$$u(\mu)^2 = \frac{1}{(1-r^2)}\ \frac{\cos^2\mu}{\sigma_z^2} - \frac{2r\cos\mu\sin\mu}{\sigma_z\sigma_{z'}} + \frac{\sin^2\mu}{\sigma_{z'}^2}\ .$$

Integrating over y we deduce:

$$F(\varepsilon) = \frac{1-\exp(-\lambda\varepsilon)}{2\pi\,\sigma_z\sigma_{z'}\sqrt{1-r^2}} \int_0^{2\pi} \frac{d\mu}{u(\mu)^2}\ . \tag{A6}$$

By definition we know that $F(\infty)$ must be equal to one, thus we must have:

$$\int_0^{2\pi} \frac{d\mu}{u(\mu)^2} = 2\pi\,\sigma_z\sigma_{z'}\sqrt{1-r^2}\ . \tag{A7}$$

It follows immediately that:

$$F(\varepsilon) = 1 - \exp(-\lambda\varepsilon)\ . \tag{A8}$$

The probability density function $\rho_\varepsilon(\varepsilon)$ is then given by the derivative of $F(\varepsilon)$:

$$\rho_\varepsilon(\varepsilon) = \lambda\exp(-\lambda\varepsilon) \quad \text{for } \varepsilon > 0\ . \tag{A9}$$

Taking the average with respect to all particles gives:

$$\langle\varepsilon\rangle = \lambda^{-1} \quad \text{and} \quad \rho_\varepsilon(\varepsilon) = \frac{1}{\langle\varepsilon\rangle}\exp(-\varepsilon/\langle\varepsilon\rangle)\ . \tag{A10}$$

Hence we have the interesting results:

$$\sigma_z^2 = \frac{\beta_z\langle\varepsilon\rangle}{2} \quad \text{and} \quad \sigma_{z'}^2 = \frac{\gamma_z\langle\varepsilon\rangle}{2}\ . \tag{A11}$$

Now let us define p as the ratio of the number of particles whose the emittance is less than ε_p to the total number of particles in the beam. Thus, p can be identified with $F(\varepsilon_p)$ and the following formula holds:

$$p = 1 - \exp(-\varepsilon_p/\langle\varepsilon\rangle) \qquad 0 < p < 1 \tag{A12}$$

In words, a beam whose particles have $\varepsilon < \varepsilon_p$ is said to have an emittance ε_p at 100 p per cent of particles in the phase space.

Let us introduce again the well known expression for the emittance ε_p in function of the half-beam width e containing 100 p per cent of particles:

$$\varepsilon_p = \frac{(\delta\sigma_z)^2}{\beta_z} \qquad \text{where } e = \delta\sigma_z. \text{ We have also its adjoin formula:}$$

$$\varepsilon_p = \frac{(\delta\sigma_{z'})^2}{1 + \alpha_z^2}\,\beta_z \qquad \text{where } e' = \delta\sigma_{z'} \text{ is the half angular divergence of the beam.} \tag{A13}$$

Using equations (A11-A12) we find that:

$$\delta = [-2\ln(1 - p)]^{1/2} \; . \tag{A14}$$

To illustrate the use of this expression, the half-width of the beam comprising 95% of particles in the (radial or vertical) phase space is $\sqrt{6}\ \sigma_z$.

In connection with this example, it should be pointed out that the half-beam width comprising 95% of particles projected onto the one-dimensional betatron amplitude axis is equal to 1.96 σ_z. This value results from the fact that:

$$0.95 = \int_{-1.96\sigma_z}^{+1.96\sigma_z} \rho_z(z)dz \qquad \text{with} \quad \rho_z(z) = \frac{1}{\sqrt{2\pi}\ \sigma_z} \exp(-z^2/2\sigma_z^2) \; . \tag{A15}$$

The function $\rho_z(z)$ is the marginal probability density function of z corresponding to the joint probability law $\rho(zz')$.

Consequently, the rms values of betatron amplitude and angle are related to the beam emittance (at 95% of particles in the phase space) by means of:

$$\sigma_z^2 = \frac{\beta_z\ \varepsilon_{0.95}}{6} \quad \text{and} \quad \sigma_{z'}^2 = \frac{\gamma_z\ \varepsilon_{0.95}}{6} \; . \tag{A16}$$

APPENDIX B

SCATTERNG FUNCTION WHEN THE LATTICE PARAMETERS DO NOT VARY AROUND THE RING

Let us consider here the case where the variation of the lattice parameters $\beta_{x,z}$ and D around the machine is neglected. Thus, for $\beta'_{x,z} = 0$ and $D' = 0$, the scattering function (equ. 34) is reduced to the form

$$f_i = k_i \int_0^\infty dz \int_0^\pi d\mu \int_0^{2\pi} d\nu \sin\mu \; g_i(\mu\nu) \exp\left[-D(\mu\nu)z\right] \ln(1 + z^2) \qquad \text{(B1)}$$

with $D(\mu\nu) = 1/c^2 \left[\sin^2\mu\,(\cos^2\nu + a^2\sin^2\nu) + b^2\cos^2\mu\right]$ since $\tilde{d} = 0$

$g_2(\mu\nu) = 1 - 3\sin^2\mu\,\sin^2\nu$, g_1 and g_3 are unchanged.

Passing to the Cartesian coordinates with

$$u = \sqrt{z}\,\sin\mu\cos\nu \qquad v = \sqrt{z}\,\sin\mu\sin\nu \qquad w = \sqrt{z}\,\cos\mu\,.$$

Thus one gets:

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = 2k_i \iiint_{-\infty}^{+\infty} \exp\left[-\frac{1}{c^2}(u^2 + a^2v^2 + b^2w^2)\right] \frac{\ln(1 + z^2)}{z^{3/2}} \begin{bmatrix} -2u^2 + v^2 + w^2 \\ u^2 - 2v^2 + w^2 \\ u^2 + v^2 - 2w^2 \end{bmatrix} du\,dv\,dw\,. \qquad \text{(B2)}$$

Now in f_1 let $u' = w$ in f_2 let $u' = u$ f_3 does not change

$v' = u$ $v' = w$

$w' = v$ $w' = v$

Dropping out the primes, the f_i become:

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = 2k_i \iiint_{-\infty}^{+\infty} (u^2 + v^2 - 2w^2) \frac{\ln(1 + z^2)}{z^{3/2}} \begin{bmatrix} \exp[-1/c^2(a^2u^2 + b^2v^2 + w^2)] \\ \exp[-1/c^2(u^2 + b^2v^2 + a^2w^2)] \\ \exp[-1/c^2(u^2 + a^2v^2 + b^2w^2)] \end{bmatrix} du\,dv\,dw\,. \qquad \text{(B3)}$$

Coming back to the Polar coordinates and introducing $z = c^2\rho$ leads to:

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \int_0^\infty d\rho \int_0^\pi d\mu \int_0^{2\pi} d\nu \sin\mu\,(1 - 3\cos^2\mu) \ln(1 + c^4\rho^2) \qquad \text{(B4)}$$

$$\begin{bmatrix} \exp\left\{-\rho\left[\sin^2\mu\,(a^2\cos^2\nu + b^2\sin^2\nu) + \cos^2\mu\right]\right\} \\ a^2\exp\left\{-\rho\left[\sin^2\mu\,(\cos^2\nu + b^2\sin^2\nu) + a^2\cos^2\mu\right]\right\} \\ b^2\exp\left\{-\rho\left[\sin^2\mu\,(\cos^2\nu + a^2\sin^2\nu) + b^2\cos^2\mu\right]\right\} \end{bmatrix}$$

Hence we write the scattering function in a more condensed form

$$f_1 = f\,(a,b,c) \qquad f_2 = f\,\left(\frac{1}{a}, \frac{b}{a}, \frac{c}{a}\right) \qquad \text{and } f_3 = f\,\left(\frac{1}{b}, \frac{a}{b}, \frac{c}{b}\right) \quad \text{where}$$

$$f\,(a,b,c) = \int_0^{\infty} d\rho \int_0^{\pi} d\mu \int_0^{2\pi} d\nu \, \sin\mu \, (1 - 3\cos^2\mu) \, \exp[-\rho D_0(\mu\nu)] \ln(1+c^4\rho^2) \qquad \text{(B5)}$$

$$D_0(\mu\nu) = \sin^2\mu \, (a^2 \cos^2 \nu + b^2 \sin^2\nu) + \cos^2\mu.$$

Now, assuming that $c^4\rho^2 \gg 1$ (see hypothesis in Eq. 18), one finally obtains.

$$f\,(a,b,c) = 2 \int_0^{\infty} d\rho \int_0^{\pi} d\mu \int_0^{2\pi} d\nu \, \sin\mu \, (1-3\cos^2\mu) \, \exp[-D_0(\mu\nu)\rho] \, \ln\,(c^2\rho) \qquad \text{(B6)}$$

It worth noting that this triple integral may be reduced analytically to the single integral[5].

$$f\,(a,b,c) = 8\pi \int_0^1 dx \, \frac{1-3\,x^2}{\sqrt{PQ}} \left(2 \left[\ln \frac{c}{2} \;\; \frac{1}{\sqrt{P}} + \frac{1}{\sqrt{Q}} \right] - E \right) \qquad \text{(B7)}$$

Where $\qquad P(x) = a^2 + (1-a^2)x^2$

$\qquad Q(x) = b^2 + (1-b^2)x^2$

$\qquad E \quad = 0.577216$ (Euler's constant).

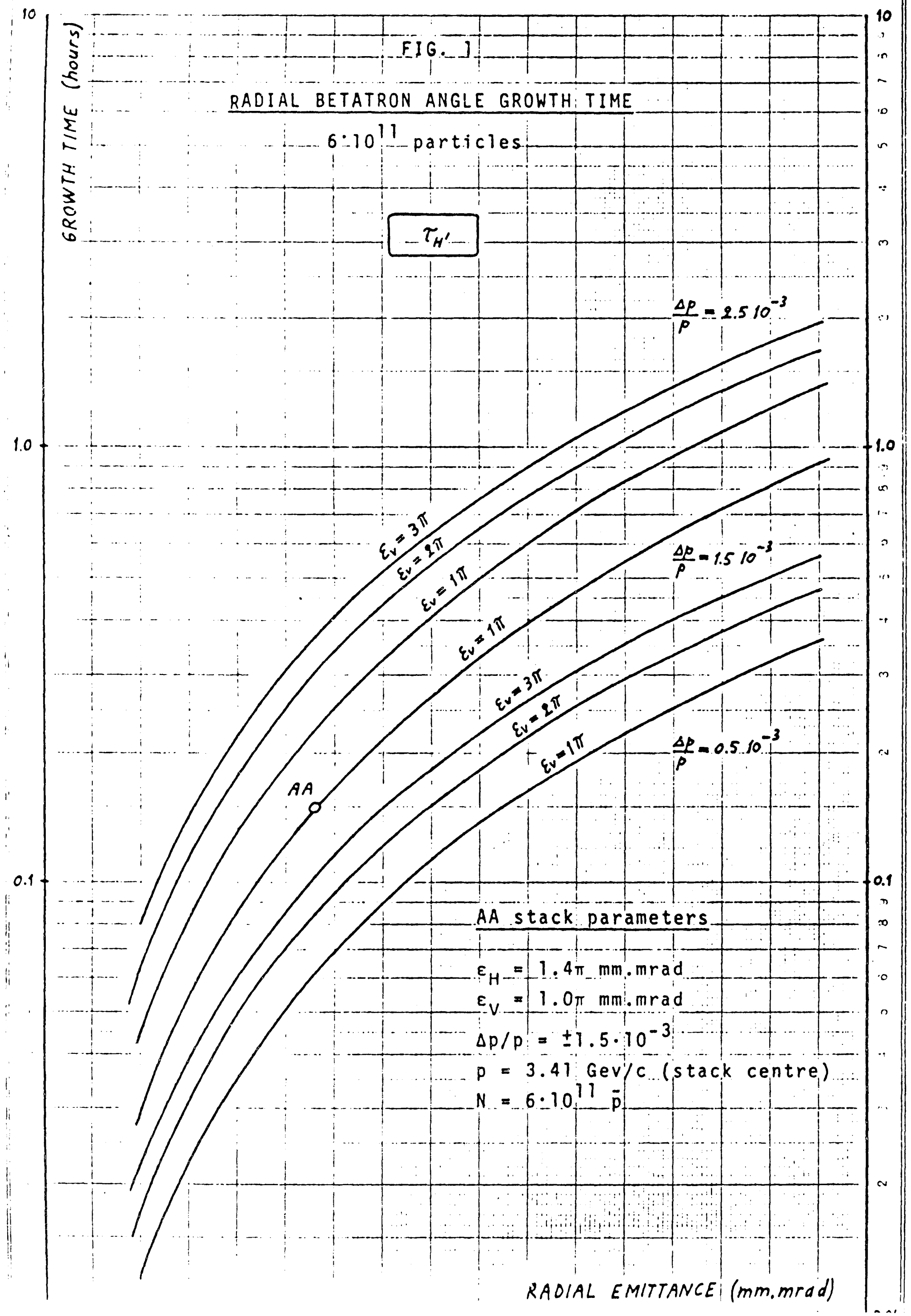
FIG. 1
RADIAL BETATRON ANGLE GROWTH TIME
$6 \cdot 10^{11}$ particles
$\tau_{H'}$
GROWTH TIME (hours)
RADIAL EMITTANCE (mm.mrad)
$\frac{\Delta p}{p} = 2.5\ 10^{-3}$
$\varepsilon_v = 3\pi$
$\varepsilon_v = 2\pi$
$\varepsilon_v = 1\pi$
$\frac{\Delta p}{p} = 1.5\ 10^{-3}$
$\varepsilon_v = 1\pi$
$\varepsilon_v = 3\pi$
$\varepsilon_v = 2\pi$
$\varepsilon_v = 1\pi$
$\frac{\Delta p}{p} = 0.5\ 10^{-3}$
AA
AA stack parameters
$\varepsilon_H = 1.4\pi$ mm.mrad
$\varepsilon_V = 1.0\pi$ mm.mrad
$\Delta p/p = \pm 1.5 \cdot 10^{-3}$
p = 3.41 Gev/c (stack centre)
$N = 6 \cdot 10^{11}\ \bar{p}$
10
1.0
0.1

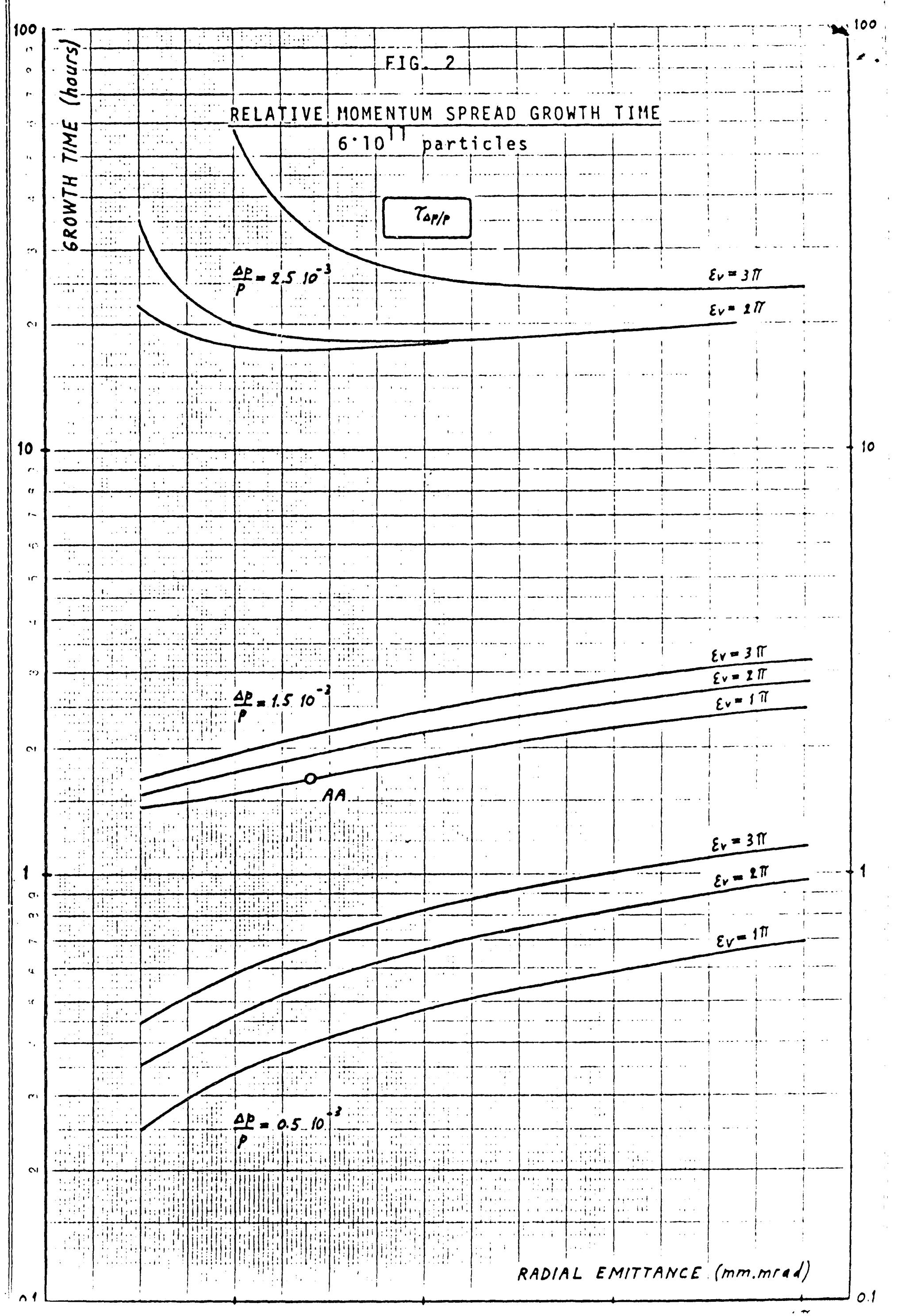
FIG. 2
RELATIVE MOMENTUM SPREAD GROWTH TIME
$6 \cdot 10^{11}$ particles
$\tau_{\Delta p/p}$
GROWTH TIME (hours)
RADIAL EMITTANCE (mm.mrad)
$\frac{\Delta p}{p} = 2.5\ 10^{-3}$
$\varepsilon_v = 3\pi$
$\varepsilon_v = 2\pi$
$\frac{\Delta p}{p} = 1.5\ 10^{-3}$
$\varepsilon_v = 3\pi$
$\varepsilon_v = 2\pi$
$\varepsilon_v = 1\pi$
AA
$\varepsilon_v = 3\pi$
$\varepsilon_v = 2\pi$
$\varepsilon_v = 1\pi$
$\frac{\Delta p}{p} = 0.5\ 10^{-3}$
100
10
1
0.1

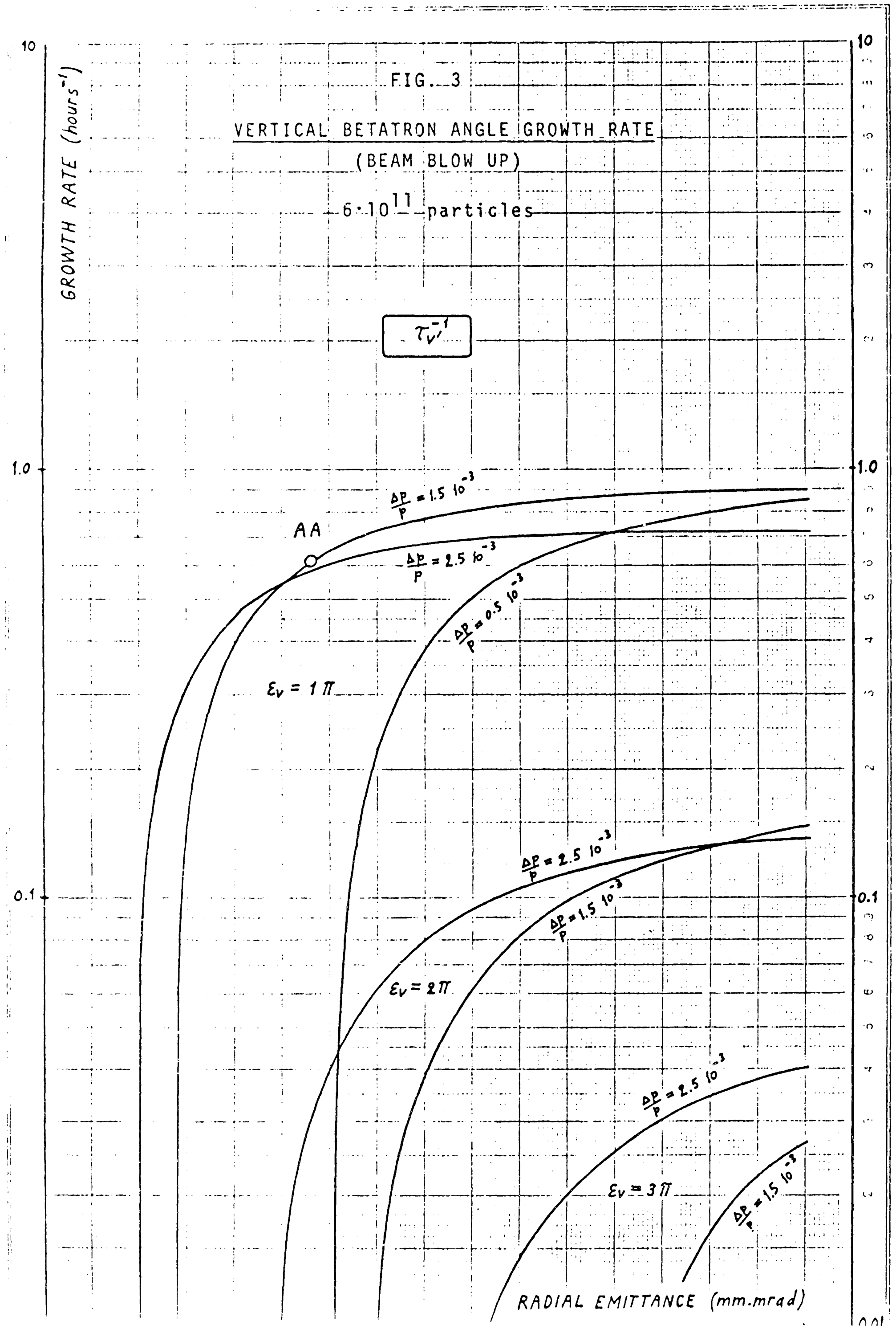
FIG. 3
VERTICAL BETATRON ANGLE GROWTH RATE
(BEAM BLOW UP)
6·10^11 particles
$\tau_{v'}^{-1}$
GROWTH RATE (hours^{-1})
RADIAL EMITTANCE (mm.mrad)
AA
$\varepsilon_v = 1\pi$
$\varepsilon_v = 2\pi$
$\varepsilon_v = 3\pi$
$\frac{\Delta p}{p} = 1.5\ 10^{-3}$
$\frac{\Delta p}{p} = 2.5\ 10^{-3}$
$\frac{\Delta p}{p} = 0.5\ 10^{-3}$
10
1.0
0.1
0.01

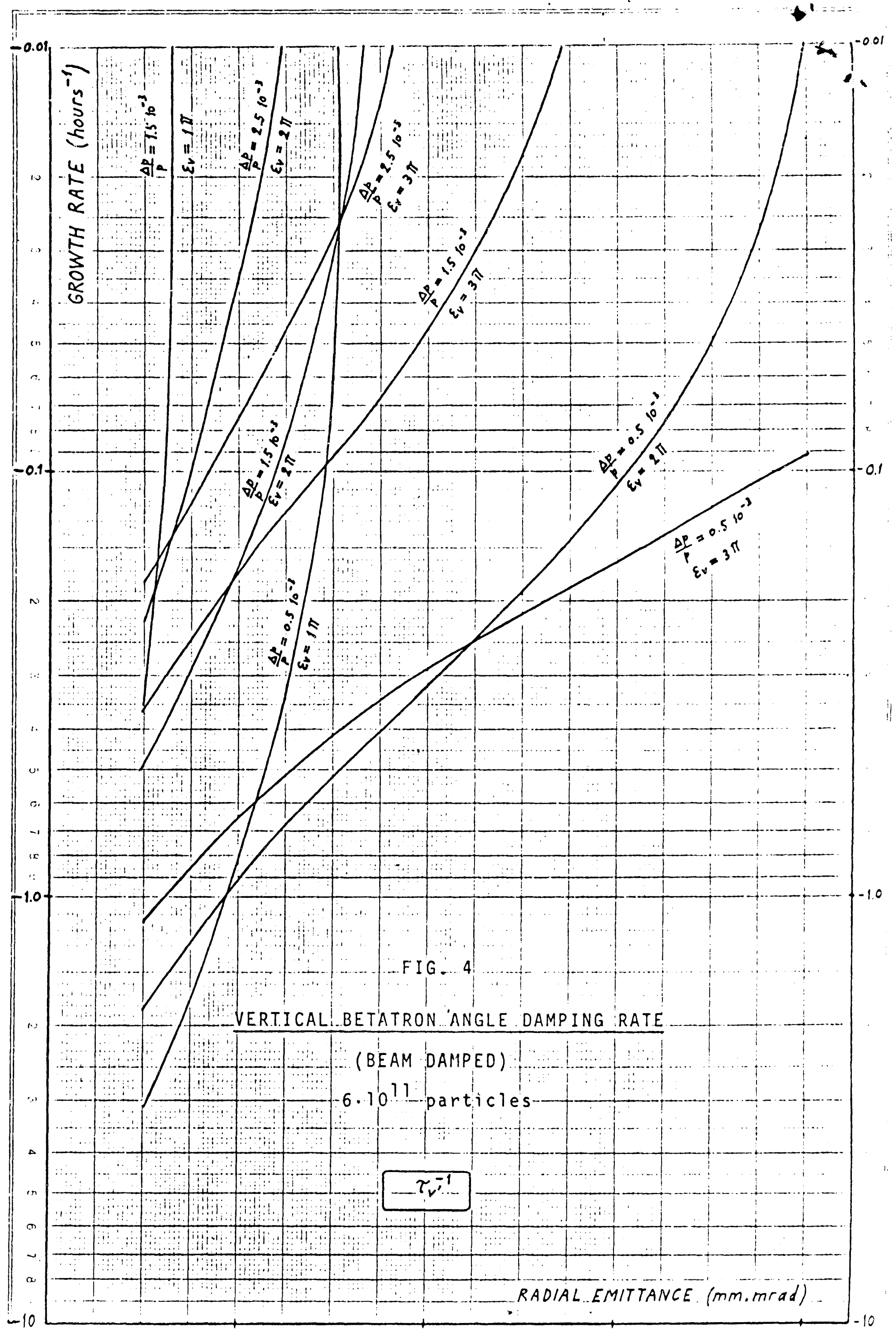
FIG. 4
VERTICAL BETATRON ANGLE DAMPING RATE
(BEAM DAMPED)
6·10^11 particles
$\tau_{v'}^{-1}$
GROWTH RATE (hours^{-1})
RADIAL EMITTANCE (mm.mrad)
$\frac{\Delta p}{p} = 1.5\ 10^{-3}$ $\varepsilon_v = 1\pi$
$\frac{\Delta p}{p} = 2.5\ 10^{-3}$ $\varepsilon_v = 2\pi$
$\frac{\Delta p}{p} = 2.5\ 10^{-3}$ $\varepsilon_v = 3\pi$
$\frac{\Delta p}{p} = 1.5\ 10^{-3}$ $\varepsilon_v = 3\pi$
$\frac{\Delta p}{p} = 1.5\ 10^{-3}$ $\varepsilon_v = 2\pi$
$\frac{\Delta p}{p} = 0.5\ 10^{-3}$ $\varepsilon_v = 1\pi$
$\frac{\Delta p}{p} = 0.5\ 10^{-3}$ $\varepsilon_v = 2\pi$
$\frac{\Delta p}{p} = 0.5\ 10^{-3}$ $\varepsilon_v = 3\pi$
0.01
0.1
1.0
10

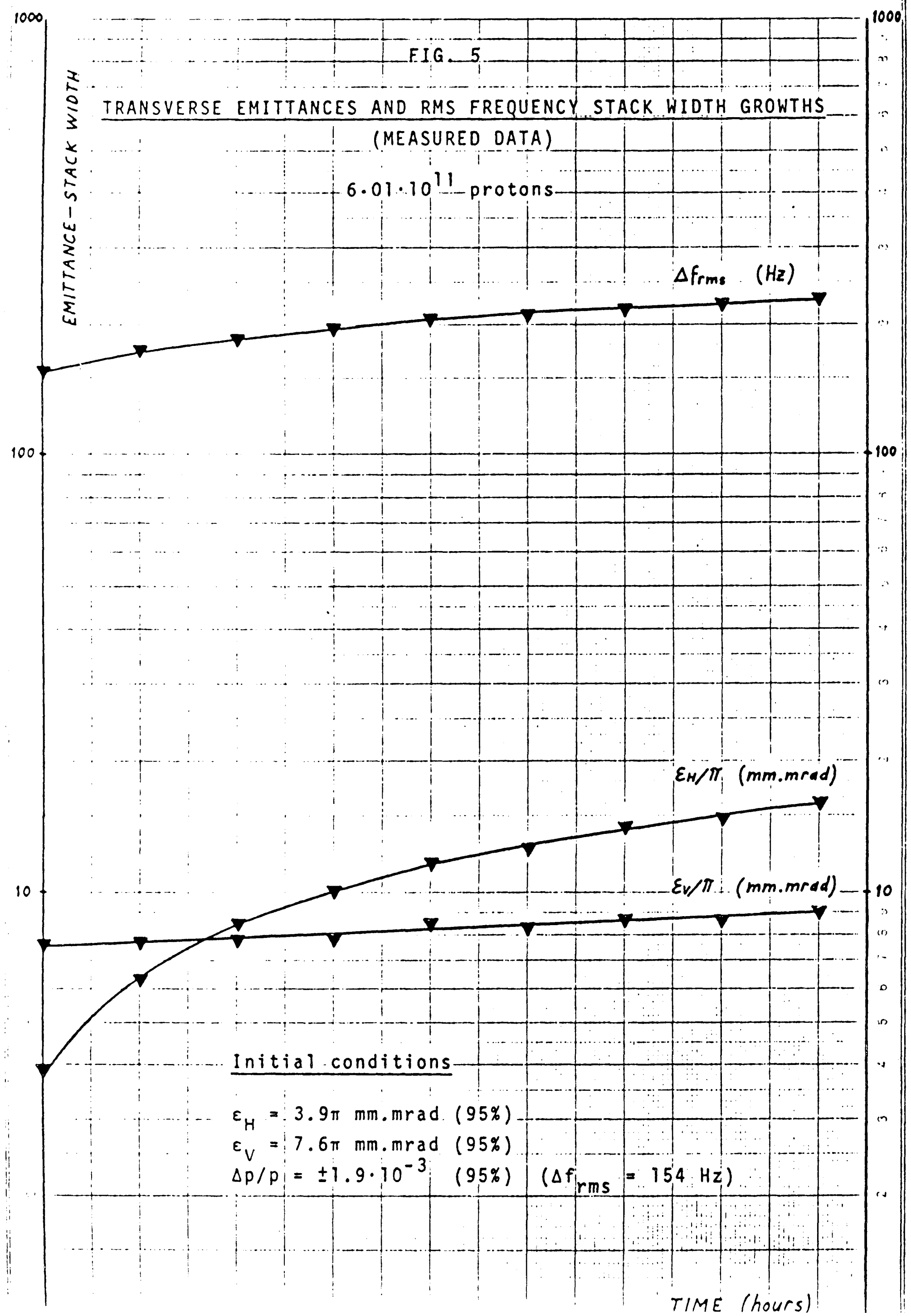
FIG. 5
TRANSVERSE EMITTANCES AND RMS FREQUENCY STACK WIDTH GROWTHS
(MEASURED DATA)
6·01·10^11 protons
EMITTANCE-STACK WIDTH
Δf_rms (Hz)
ε_H/π (mm.mrad)
ε_V/π (mm.mrad)
Initial conditions
ε_H = 3.9π mm.mrad (95%)
ε_V = 7.6π mm.mrad (95%)
Δp/p = ±1.9·10^-3 (95%) (Δf_rms = 154 Hz)
TIME (hours)
1000
100
10

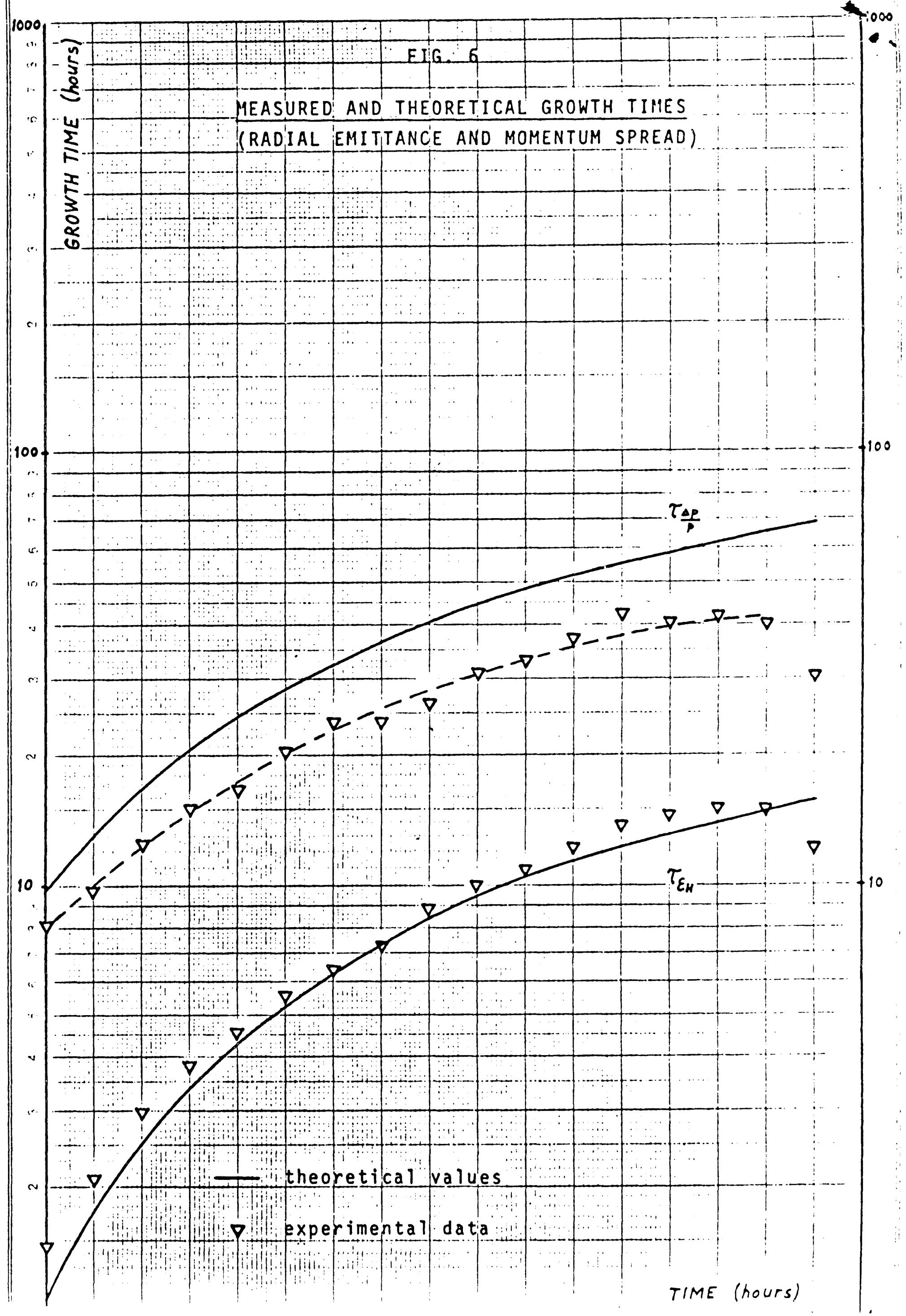
FIG. 6
MEASURED AND THEORETICAL GROWTH TIMES
(RADIAL EMITTANCE AND MOMENTUM SPREAD)
GROWTH TIME (hours)
τ_Δp/p
τ_εH
theoretical values
experimental data
TIME (hours)
1000
100
10

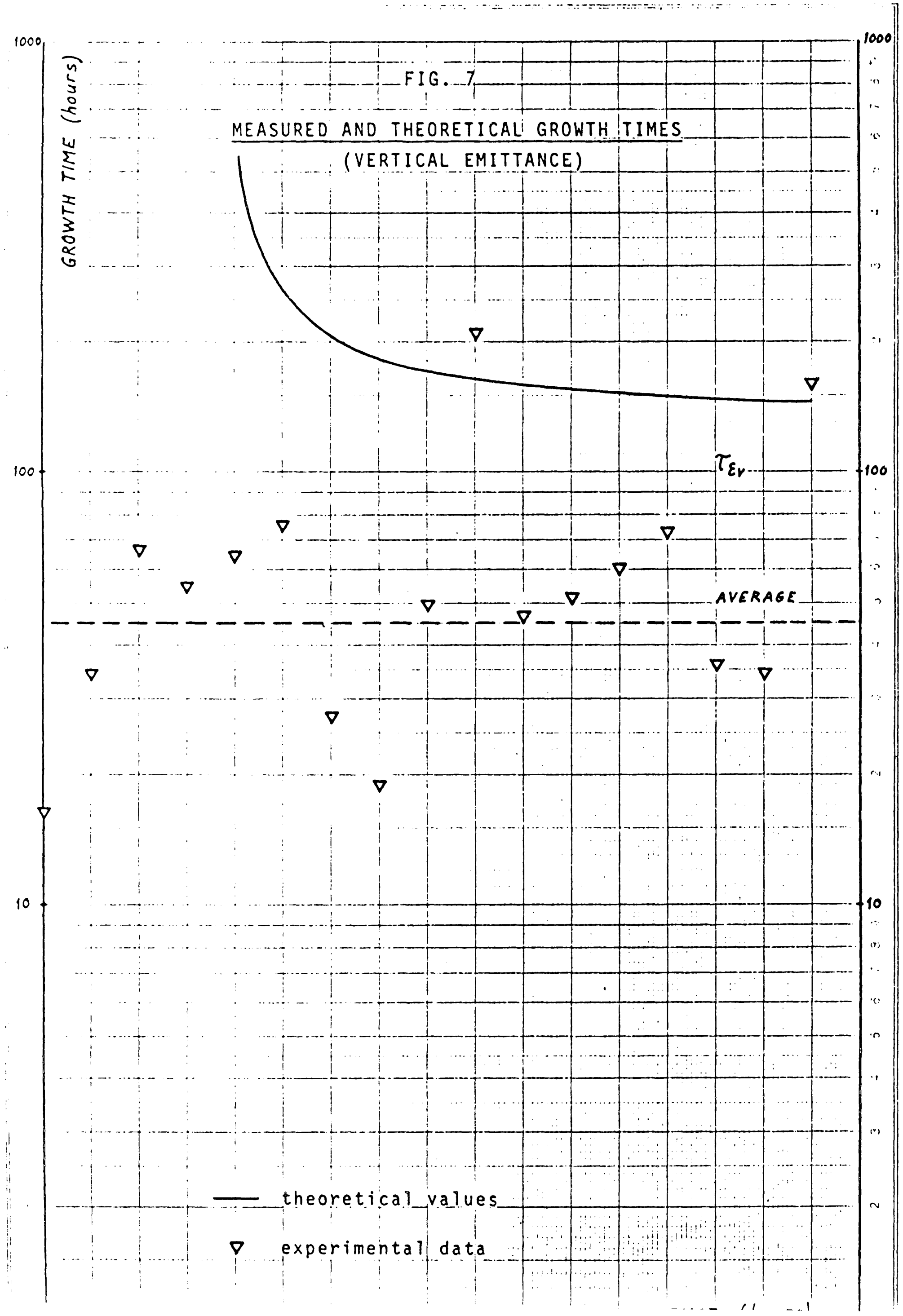

FIG. 7
MEASURED AND THEORETICAL GROWTH TIMES
(VERTICAL EMITTANCE)
GROWTH TIME (hours)
1000
100
10
τ_{ε_V}
AVERAGE
theoretical values
experimental data

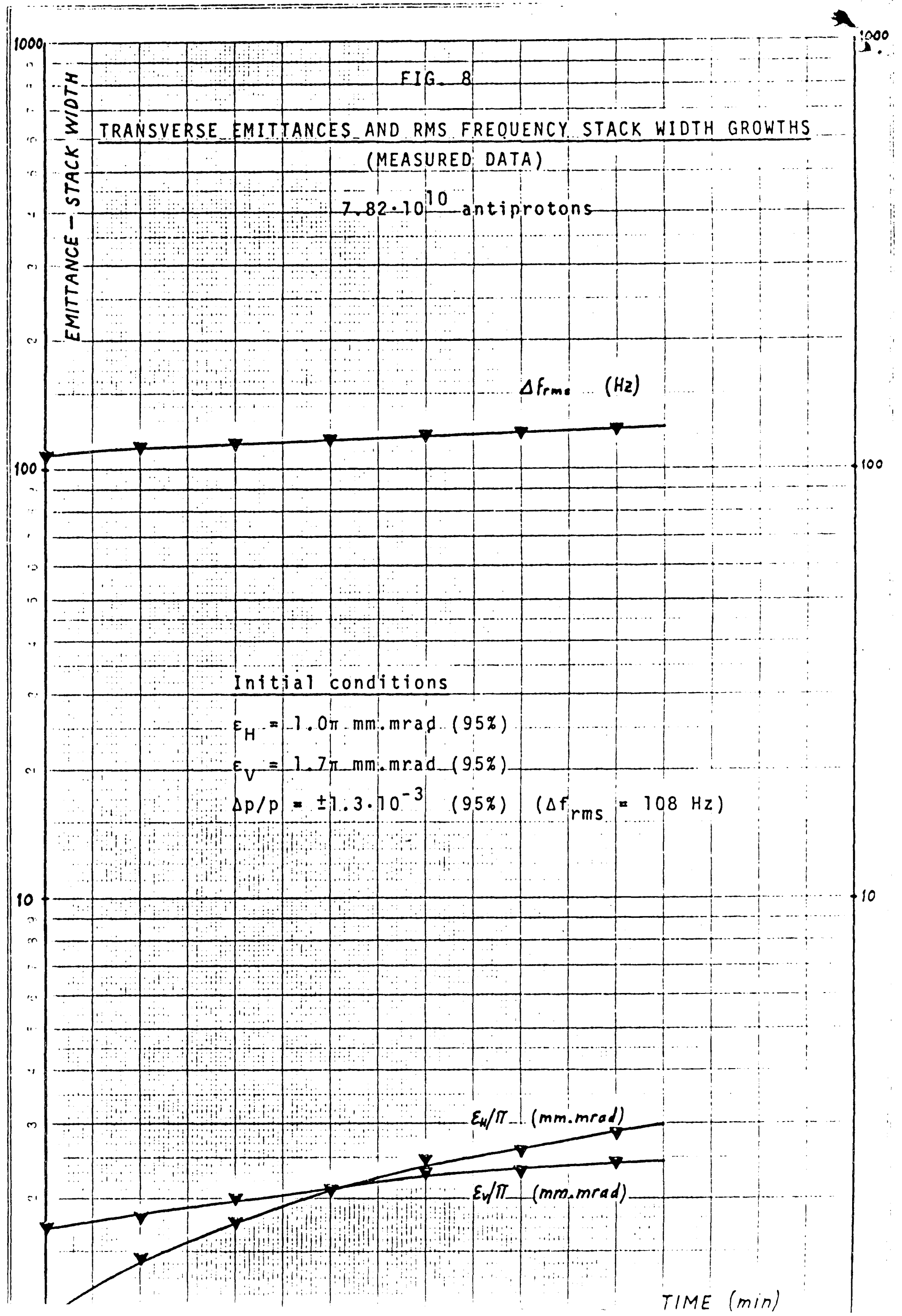

FIG. 8
TRANSVERSE EMITTANCES AND RMS FREQUENCY STACK WIDTH GROWTHS
(MEASURED DATA)
$7.82 \cdot 10^{10}$ antiprotons
EMITTANCE – STACK WIDTH
1000
100
10
Δf_{rms} (Hz)
Initial conditions
$\varepsilon_H = 1.0\pi$ mm.mrad (95%)
$\varepsilon_V = 1.7\pi$ mm.mrad (95%)
$\Delta p/p = \pm 1.3 \cdot 10^{-3}$ (95%) ($\Delta f_{rms} = 108$ Hz)
ε_H/π (mm.mrad)
ε_V/π (mm.mrad)
TIME (min)

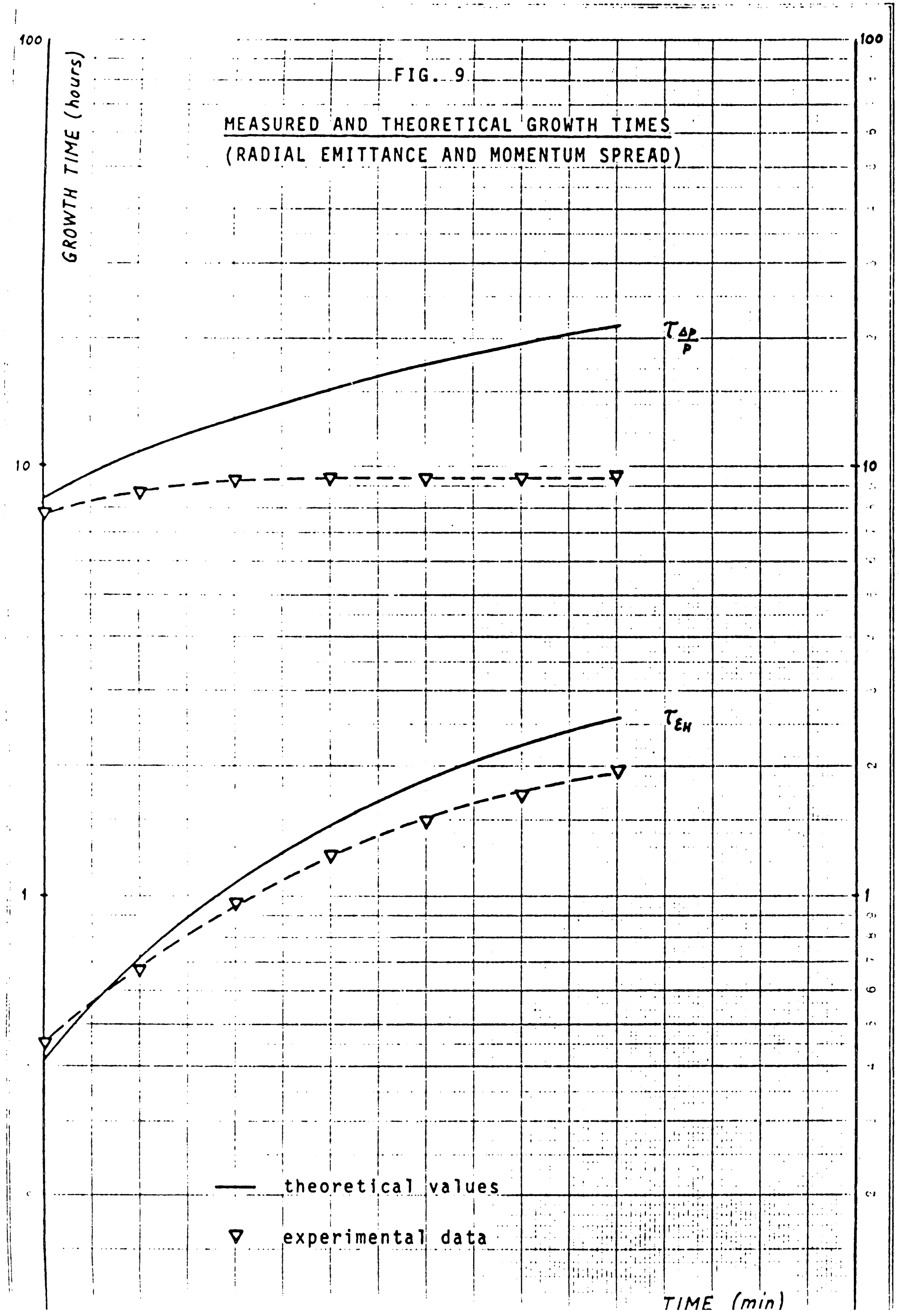
FIG. 9
MEASURED AND THEORETICAL GROWTH TIMES
(RADIAL EMITTANCE AND MOMENTUM SPREAD)
GROWTH TIME (hours)
TIME (min)
$\tau_{\frac{\Delta p}{p}}$
τ_{ε_H}
theoretical values
experimental data
100
10
1

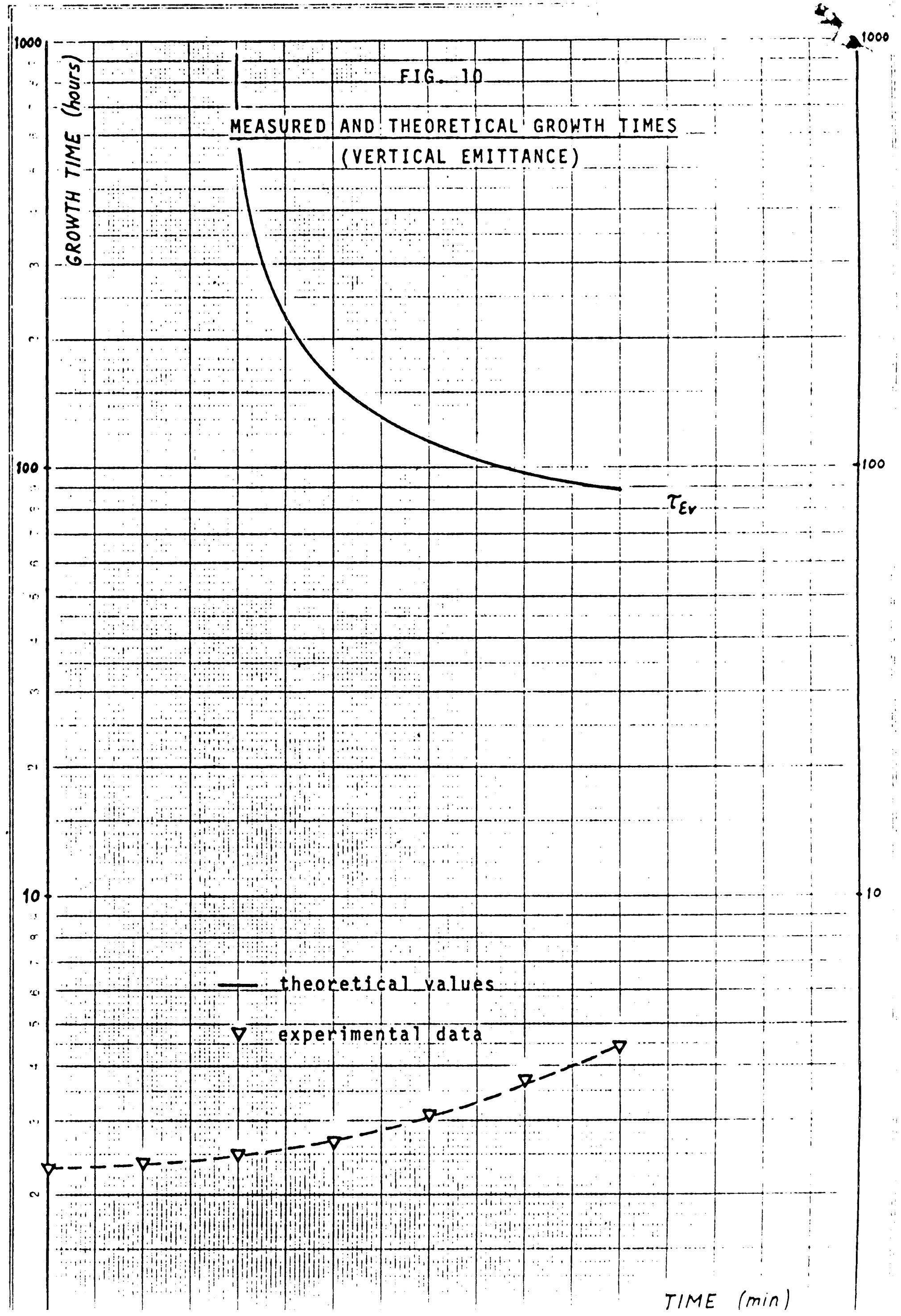
FIG. 10
MEASURED AND THEORETICAL GROWTH TIMES
(VERTICAL EMITTANCE)
GROWTH TIME (hours)
TIME (min)
τ_{ε_V}
theoretical values
experimental data
1000
100
10