Checking of

Toepffer. Scattering of Magnetized Electrons with Ions

Phys. Rev., A 66 (2002) 02274 ~~5~~

$$\delta^{(1)}\vec{V}_B(t) = -\frac{Ze^2}{m}\int_{-\infty}^{t}\frac{dt'}{\bar{r}^3(t')}\,\underbrace{T^{-1}(\Omega t')\cdot\vec{r}(t')}_{\text{using (3.34)}}$$

From (3.35)

$$\bar{r}(t) = \left[r_0^2 + \bar{v}^2(t-t_0)^2\right]^{1/2} =$$

$$= r_0\left[1 + \frac{\bar{v}^2(t-t_0)^2}{r_0^2}\right]^{1/2} =$$

$$= r_0\left[1 + (\tau-\tau_0)^2\right]^{1/2}$$

From (3.10) in the limit $R \to 0$: ①

$$\vec{r}(t) = \begin{pmatrix} b\sin\theta - v_{i\perp}t \\ -b\cos\theta \\ (v_{e\parallel} - v_{i\parallel})t \end{pmatrix} =$$

$$= r_0\begin{pmatrix} \frac{b}{r_0}\sin\theta - \frac{v_{i\perp}}{\bar{v}}\frac{\bar{v}t}{r_0} \\ -\frac{b}{r_0}\cos\theta \\ \frac{v_{e\parallel} - v_{i\parallel}}{\bar{v}}\frac{\bar{v}t}{r_0} \end{pmatrix} =$$

$$\boxed{\tau = \frac{\bar{v}t}{r_0},\ \tau_0 = \frac{\bar{v}t_0}{r_0}}$$

$$= r_0\begin{pmatrix} \beta\sin\theta + \gamma_\perp\tau \\ -\beta\cos\theta \\ \gamma_\parallel\tau \end{pmatrix}$$

This is (3.34)

$$= -\frac{Ze^2}{m}\frac{r_0}{\bar{v}}\int_{-\infty}^{\tau}\frac{d\tau'\quad r_0}{r_0^3\left[1+(\tau'-\tau_0)\right]^{3/2}}\begin{pmatrix} \cos\frac{\Omega r_0}{\bar{v}}\tau' & \sin\frac{\Omega r_0}{\bar{v}}\tau' & 0 \\ -\sin\frac{\Omega r_0}{\bar{v}}\tau' & \cos\frac{\Omega r_0}{\bar{v}}\tau' & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \beta\sin\theta + \gamma_\perp\tau' \\ -\beta\cos\theta \\ \gamma_\parallel\tau' \end{pmatrix}$$

$$\boxed{\begin{array}{c}\sigma = \tau' - \tau_0 \\ d\tau' = d\sigma\end{array}}$$

$$= -\frac{Ze^2}{m}\frac{1}{\bar{v}r_0}\int_{-\infty}^{\sigma}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\begin{pmatrix} \cos\omega\tau' & \sin\omega\tau' & 0 \\ -\sin\omega\tau' & \cos\omega\tau' & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \beta\sin\theta + \gamma_\perp\tau' \\ -\beta\cos\theta \\ \gamma_\parallel\tau' \end{pmatrix}$$

3.36 $\not\to 0$

„x": $\cos\omega\tau'(\beta\sin\theta+\gamma_\perp\tau') + \beta\cos\theta\sin\omega\tau' =$

$$= \beta[\sin\theta\cos\omega\tau' + \cos\theta\sin\omega\tau'] + \gamma_\perp\tau'\cos\omega\tau' = \beta\sin(\theta+\omega\tau') + \gamma_\perp\tau'\cos\omega\tau'$$

„y": $-\sin\omega\tau'(\beta\sin\theta+\gamma_\perp\tau') + \beta\cos\omega\tau'\cos\theta =$

$$= -\beta(-\sin\theta\sin\omega\tau' + \cos\theta\cos\omega\tau') - \gamma_\perp\tau'\sin\omega\tau' = -\beta\cos(\omega\tau'-\theta) - \gamma_\perp\tau'\sin\omega\tau'$$

Смотрим z-компоненту от $\delta\vec{V}_B(t)$ при $t\to\infty$:

$$(\delta^{(1)}\vec{V}_B)_z\Big|_{t\to\infty} = -\frac{Ze^2}{m}\frac{1}{\sqrt{r_0}}\int_{-\infty}^{\infty}\frac{d\sigma'}{[1+\sigma'^2]^{3/2}}\gamma_\parallel\tau' = -\frac{Ze^2}{m}\frac{\gamma_\parallel}{\sqrt{r_0}}\Bigg[\underbrace{\int_{-\infty}^{\infty}\frac{\sigma'd\sigma'}{(1+\sigma'^2)^{3/2}}}_{=0\ \text{из-за нечётности ф-ции}} + \tau_0\int_{-\infty}^{\infty}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\Bigg] =$$

$$= \frac{\sigma'}{\sqrt{1+\sigma'^2}}\Bigg|_{-\infty}^{\infty} = 2$$

$$= -\frac{2Ze^2}{m}\frac{\gamma_\parallel\tau_0}{\sqrt{r_0}} \qquad \text{OK (3.37c)}$$

Смотрим x-компоненту:

$$(\delta^{(1)}\vec{V}_B)_x\Big|_{t\to\infty} = -\frac{Ze^2}{m}\frac{1}{\sqrt{r_0}}\int_{-\infty}^{\infty}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\Big[\underbrace{\beta\sin(\omega\tau'+\theta)}_{\text{„1"}} + \underbrace{\gamma_\perp\tau'\cos\omega\tau'}_{\text{„2"}}\Big], \quad \tau' = \sigma'+\tau_0$$

$$\text{„2"} = \gamma_\perp\int_{-\infty}^{\infty}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}(\sigma'+\tau_0)\cos[\omega(\sigma'+\tau_0)] = \gamma_\perp\tau_0\int_{-\infty}^{\infty}\frac{\cos\omega\sigma'\cos\omega\tau_0 - \sin\omega\sigma'\sin\omega\tau_0}{(1+\sigma'^2)^{3/2}}d\sigma' +$$

$$= \gamma_\perp\tau_0\Bigg[\cos\omega\tau_0\int_{-\infty}^{\infty}\frac{\cos\omega x\,dx}{(1+x^2)^{3/2}} - \sin\omega\tau_0\underbrace{\int_{-\infty}^{\infty}\frac{\sin\omega x\,dx}{(1+x^2)^{3/2}}}_{=0\ \text{из-за нечётности ф-ции}}\Bigg] = 2\gamma_\perp\tau_0\cos\omega\tau_0\int_0^{\infty}\frac{\cos\omega x\,dx}{(1+x^2)^{3/2}}$$

$$+\gamma_\perp \int_{-\infty}^{\infty} \frac{\sigma'(\cos\omega\sigma'\cos\omega\tau_0 - \sin\omega\sigma'\sin\omega\tau_0)\,d\sigma'}{(1+\sigma'^2)^{3/2}} =$$

$$= \gamma_\perp\tau_0\left[\cos\omega\tau_0 \int_{-\infty}^{\infty}\frac{\cos\omega x\,dx}{(1+x^2)^{3/2}} - \sin\omega\tau_0 \underbrace{\int_{-\infty}^{\infty}\frac{\sin\omega x\,dx}{(1+x^2)^{3/2}}}_{=0 \text{ из-за нечётности}}\right] +$$

$$+\gamma_\perp\left[\cos\omega\tau_0 \underbrace{\int_{-\infty}^{\infty}\frac{x\cos\omega x\,dx}{(1+x^2)^{3/2}}}_{=0 \text{ из-за нечётности}} - \sin\omega\tau_0\int_{-\infty}^{\infty}\frac{x\sin\omega x\,dx}{(1+x^2)^{3/2}}\right] =$$

$$= 2\gamma_\perp\left[\tau_0\cos\omega\tau_0 \underbrace{\int_0^{\infty}\frac{\cos\omega x\,dx}{(1+x^2)^{3/2}}}_{I} - \sin\omega\tau_0 \underbrace{\int_0^{\infty}\frac{x\sin\omega x\,dx}{(1+x^2)^{3/2}}}_{\text{это } 3.754.3 \,=\, \omega K_0(\omega)}\right]$$

чтобы найти I найдем сначала

$$\hat{I}(a) = \int_0^{\infty}\frac{\cos\omega x\,dx}{(1+ax^2)^{1/2}} = \frac{1}{\sqrt{a}}\int_0^{\infty}\frac{\cos\omega x\,dx}{\sqrt{\frac{1}{a}+x^2}} = \frac{1}{\sqrt{a}}K_0\left(\frac{\omega}{\sqrt{a}}\right)$$

это 3.754.2

тогда

$$I = \int_0^{\infty}\frac{\cos\omega x\,dx}{(1+x^2)^{3/2}} = -2\frac{\partial}{\partial a}\int_0^{\infty}\frac{\cos\omega x\,dx}{(1+ax^2)^{1/2}}$$

$$I(\beta) = \int_0^{\infty}\frac{\cos\omega x\,dx}{\sqrt{\beta+x^2}} = K_0(\sqrt{\beta}\,\omega), \quad \text{тогда}$$

это 3.754.2

$$\int_0^{\infty}\frac{\cos\omega x\,dx}{\sqrt{1+x^2}^{\,3}} = -2\left(\frac{\partial}{\partial\beta}\int_0^{\infty}\frac{\cos\omega x\,dx}{\sqrt{\beta+x^2}}\right)\Bigg|_{\beta=1} = -2\frac{\partial}{\partial\beta}\left(K_0\sqrt{\beta}\,\omega\right)\Big|_{\beta=1} =$$

$$= -2\,\frac{1}{2}\,\frac{\omega}{\sqrt{\beta}}\,K_0'(\sqrt{\beta}\,\omega)\Big|_{\beta=1} = -\omega \underset{\uparrow\, 8.486\,18}{K_0'(\omega)} = \omega K_1(\omega)$$

Итак,

$$\text{„}2\text{"} = 2\gamma_\perp\left[\omega\tau_0\cos(\omega\tau_0)K_1(\omega) - \omega\sin(\omega\tau_0)K_0(\omega)\right]$$

Теперь „1":

$$\text{„}1\text{"} = \beta\int_{-\infty}^{\infty}\frac{\sin(\omega\overset{=\sigma'+\tau_0}{\tau'}+\theta)\,d\sigma'}{(1+\sigma'^2)^{3/2}} = \beta\int_{-\infty}^{\infty}\frac{\sin(\omega\sigma'+\omega\tau_0+\theta)\,d\sigma'}{(1+\sigma'^2)^{3/2}} =$$

$$= \beta\left[\sin(\omega\tau_0+\theta)\int_{-\infty}^{\infty}\frac{\cos\omega x\,dx}{(1+x^2)^{3/2}} + \cos(\omega\tau_0+\theta)\underbrace{\int_{-\infty}^{\infty}\frac{\sin\omega x\,dx}{(1+x^2)^{3/2}}}_{=0\text{ из-за нечётности}}\right] = 2\beta\sin(\omega\tau_0+\theta)\underbrace{\int_0^{\infty}\frac{\cos\omega x\,dx}{(1+x^2)^{3/2}}}_{\text{найдем выше и равно } \omega K_1(\omega)} =$$

$$= 2\beta\omega\sin(\omega\tau_0+\theta)K_1(\omega)$$

Т.о. „1"+„2":

$$\text{„}1\text{"}+\text{„}2\text{"} = 2\beta\omega\sin(\omega\tau_0+\theta)K_1(\omega) + 2\gamma_\perp\left[\omega\tau_0\cos(\omega\tau_0)K_1(\omega) - \omega\sin(\omega\tau_0)K_0(\omega)\right] =$$

$$= 2\gamma_\perp\omega K_0(\omega)\sin\omega\tau_0 + 2\omega K_1(\omega)\left[\beta\sin(\omega\tau_0+\theta) \mp \gamma_\perp\tau_0\cos(\omega\tau_0)\right]$$

Имеем

$$[\ \ ] = \underbrace{\beta\cos\theta}_{3.30}\sin(\omega\tau_0) + \beta\sin\theta\cos\omega\tau_0 + \gamma_\perp\tau_0\cos\omega\tau_0 = \sin(\omega\tau_0)\sin\psi +$$

$$+\cos(\omega\tau_0)\big[\underbrace{\beta\sin\theta + \gamma_\perp\tau_0}_{\text{это } 3.31}\big] = \sin(\omega\tau_0)\sin\psi \mp \gamma_\parallel\cos\psi\cos(\omega\tau_0)$$

$$\left(\delta^{(1)}\vec{V}_B\right)_x\Big|_{t\to\infty} = -\frac{Ze^2}{m v r_0}\left[2\omega K_1(\omega)\left(\gamma_\parallel \cos\psi \cos\omega\tau_0 + \sin\psi \sin\omega\tau_0\right) \mp 2\omega K_0(\omega)\,\gamma_\perp \sin\omega\tau_0\right] \qquad (3.37a)$$

совпадает с Toepffer

~~у Toepffer'а другой знак~~

Аналогично:

$$\left(\delta^{(1)}\vec{V}_B\right)_y\Big|_{t\to\infty} = -\frac{Ze^2}{m v r_0}\int_{-\infty}^{\infty}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\left[\underbrace{-\beta\cos(\omega\tau'-\theta)}_{-\text{„}3\text{"}} \mp \underbrace{\gamma_\perp\tau'\sin\omega\tau'}_{-\text{„}4\text{"}}\right]$$

Имеем:

$$\text{„}3\text{"} = \beta\int_{-\infty}^{\infty}\frac{\cos(\omega\tau'-\theta)\,d\sigma'}{(1+\sigma'^2)^{3/2}} = \beta\left[\cos(\omega\tau_0-\theta)\int_{-\infty}^{\infty}\frac{\cos\omega x\,dx}{(1+x^2)^{3/2}} + \sin(\omega\tau_0-\theta)\underbrace{\int_{-\infty}^{\infty}\frac{\sin\omega x\,dx}{(1+x^2)^{3/2}}}_{=0 \text{ из-за нечётности}}\right] =$$

$\tau' = \sigma' \pm \tau_0$

$$= 2\beta\cos(\omega\tau_0-\theta)\underbrace{\int_0^{\infty}\frac{\cos\omega x\,dx}{(1+x^2)^{3/2}}}_{\text{см. ранее } = \omega K_1(\omega)} = 2\beta\omega K_1(\omega)\cos(\omega\tau_0-\theta)$$

$$\text{„}4\text{"} = \gamma_\perp\int_{-\infty}^{\infty}\frac{\tau'\sin\omega\tau'\,d\sigma'}{(1+\sigma'^2)^{3/2}} = \gamma_\perp\tau_0\int_{-\infty}^{\infty}\frac{\sin(\omega\sigma'\pm\omega\tau_0)\,d\sigma'}{(1+\sigma'^2)^{3/2}} + \gamma_\perp\int_{-\infty}^{\infty}\frac{\sigma'\sin(\omega\sigma'\pm\omega\tau_0)\,d\sigma'}{(1+\sigma'^2)^{3/2}} =$$

$\tau' = \sigma' \pm \tau_0$

$$= \gamma_\perp\tau_0\left[\cos\omega\tau_0\underbrace{\int_{-\infty}^{\infty}\frac{\sin\omega x\,dx}{(1+x^2)^{3/2}}}_{=0 \text{ из-за нечётности}} + \sin\omega\tau_0\int_{-\infty}^{\infty}\frac{\cos\omega x\,dx}{(1+x^2)^{3/2}}\right] + \gamma_\perp\left[\cos\omega\tau_0\int_{-\infty}^{\infty}\frac{x\sin\omega x\,dx}{(1+x^2)^{3/2}} +\right.$$

$$\left. + \sin\omega\tau_0\underbrace{\int_{-\infty}^{\infty}\frac{x\cos\omega x\,dx}{(1+x^2)^{3/2}}}_{=0 \text{ из-за нечётности}}\right] = 2\gamma_\perp\left[\tau_0\sin\omega\tau_0\int_0^{\infty}\frac{\cos\omega x\,dx}{(1+x^2)^{3/2}} + \cos\omega\tau_0\int_0^{\infty}\frac{x\sin\omega x\,dx}{(1+x^2)^{3/2}}\right] =$$

$$= 2\gamma_\perp\left\{\tau_0\,\omega K_1(\omega)\sin\omega\tau_0 + \gamma_\perp\cos\omega\tau_0\,\omega K_0(\omega)\right\} = 2\omega\gamma_\perp\left[\tau_0 K_1(\omega)\sin\omega\tau_0 + K_0(\omega)\cos\omega\tau_0\right]$$

Т.о.

$$\left(\delta^{(1)}\vec{V}_B\right)_y\Big|_{t\to\infty} = -\frac{Ze^2}{m\bar{v}r_0}\Big[-2\beta\omega K_1(\omega)\cos(\omega\tau_0-\theta) \mp 2\omega\gamma_\perp\tau_0 K_1(\omega)\sin\omega\tau_0 \mp -$$

$$\mp 2\omega\gamma_\perp K_0\cos\omega\tau_0\Big] =$$

$$= -\frac{Ze^2}{m\bar{v}r_0}\Big[-2\omega K_0(\omega)\gamma_\perp\cos\omega\tau_0 + 2\omega K_1(\omega)\underbrace{\big[-\beta\cos(\omega\tau_0-\theta)\mp\gamma_\perp\tau_0\sin\omega\tau_0\big]}_{\beta}\Big]$$

$$[\ ] = -\beta\cos\theta\cos\omega\tau_0 - \beta\sin\theta\sin\omega\tau_0 - \gamma_\perp\tau_0\sin\omega\tau_0 =$$

$$= \underbrace{-\beta\cos\theta}_{(3.30)}\cos\omega\tau_0 - \underset{3.31}{(\beta\sin\theta+\gamma_\perp\tau_0)}\sin\omega\tau_0 = +\sin\Psi\cos\omega\tau_0 - \gamma_{\parallel}\cos\Psi\sin\omega\tau_0$$

Так что окончательно

$$\left(\delta^{(1)}\vec{V}_B\right)_y\Big|_{t\to\infty} = -\frac{Ze^2}{m\bar{v}r_0}\Big[+2\omega K_1(\omega)\left(-\gamma_{\parallel}\cos\Psi\sin\omega\tau_0+\sin\Psi\cos\omega\tau_0\right) - 2\omega K_0(\omega)\gamma_\perp\cos\omega\tau_0\Big] \quad (3.27b)$$

$$\delta^{(1)}r_{\parallel}(t) = -\frac{Ze^2}{m\bar{v}^2}\int_{-\infty}^{\sigma}d\sigma'\int_{-\infty}^{\sigma'}\frac{\gamma_{\parallel}(\sigma''+\tau_0)}{(1+\sigma''^2)^{3/2}}d\sigma'' = -\frac{Ze^2}{m\bar{v}^2}\gamma''\int_{-\infty}^{\sigma}d\sigma'\left[\int_{-\infty}^{\sigma'}\frac{\sigma''d\sigma''}{(1+\sigma''^2)^{3/2}} + \tau_0\int_{-\infty}^{\sigma'}\frac{d\sigma''}{(1+\sigma''^2)^{3/2}}\right] =$$

это (3.41)

$$= -\frac{Ze^2}{m\bar{v}^2}\gamma''\int_{-\infty}^{\sigma}d\sigma'\left[-\frac{1}{\sqrt{1+\sigma''^2}}\Big|^{\sigma'} + \tau_0\frac{\sigma''}{\sqrt{1+\sigma''^2}}\Big|^{\sigma'}\right] = -\frac{Ze^2}{m\bar{v}^2}\gamma''\int^{\sigma}d\sigma'\Big[-\frac{\sigma'}{\sqrt{1+\sigma'^2}} + \tau_0\Big(\frac{\sigma'}{\sqrt{1+\sigma'^2}} +$$

Для будущего: (7)

$$\int \frac{dx}{\sqrt{a+bx+cx^2}^{\,3}} = \frac{2(2cx+b)}{\Delta\sqrt{}} \quad (\Delta = 4ac-b^2) \longrightarrow \int \frac{dx}{\sqrt{1+x^2}^{\,3}} = \frac{x}{\sqrt{1+x^2}}$$

$$\int \frac{x\,dx}{\sqrt{}^{\,3}} = -\frac{2(2a+bx)}{\Delta\sqrt{}} \longrightarrow \int \frac{x\,dx}{\sqrt{1+x^2}^{\,3}} = -\frac{1}{\sqrt{1+x^2}}$$

$$\int_0^\infty \frac{x\sin\omega x\,dx}{\sqrt{1+x^2}^{\,3}} = \omega K_0(\omega) \qquad \int_0^\infty \frac{\cos\omega x\,dx}{\sqrt{\beta+x^2}} = K_0(\sqrt{\beta}\omega) \longrightarrow \int_0^\infty \frac{\cos\omega x\,dx}{\sqrt{\beta+x^2}^{\,3}} = -\frac{\omega}{\sqrt{\beta}}K_0'(\sqrt{\beta}\omega) = \frac{\omega}{\sqrt{\beta}}K_1(\sqrt{\beta}\omega)$$

3.754.3 3.754.2

$$\delta\vec{r}_\perp^{\,(1)}(t) = -\frac{Ze^2}{mv^2}\int_{-\infty}^{\sigma} d\sigma' \begin{pmatrix} \cos\omega\tau' & -\sin\omega\tau' \\ \sin\omega\tau' & \cos\omega\tau' \end{pmatrix} \int_{-\infty}^{\sigma'} \frac{d\sigma''}{(1+\sigma''^2)^{3/2}} \begin{pmatrix} \cos\omega\tau'' & \sin\omega\tau'' \\ -\sin\omega\tau'' & \cos\omega\tau'' \end{pmatrix} \begin{pmatrix} \beta\sin\theta + \gamma_\perp\tau'' \\ -\beta\cos\theta \end{pmatrix} \quad (3.40)$$

(3.22)

$$\tau = \sigma + \tau_0$$

это лишнее

Тогда

$$\cos\omega\tau' = \cos\omega(\sigma'+\tau_0) = \cos\omega\tau_0\cos\omega\sigma' - \sin\omega\tau_0\sin\omega\sigma' \quad \text{и аналогично для } \sigma''$$

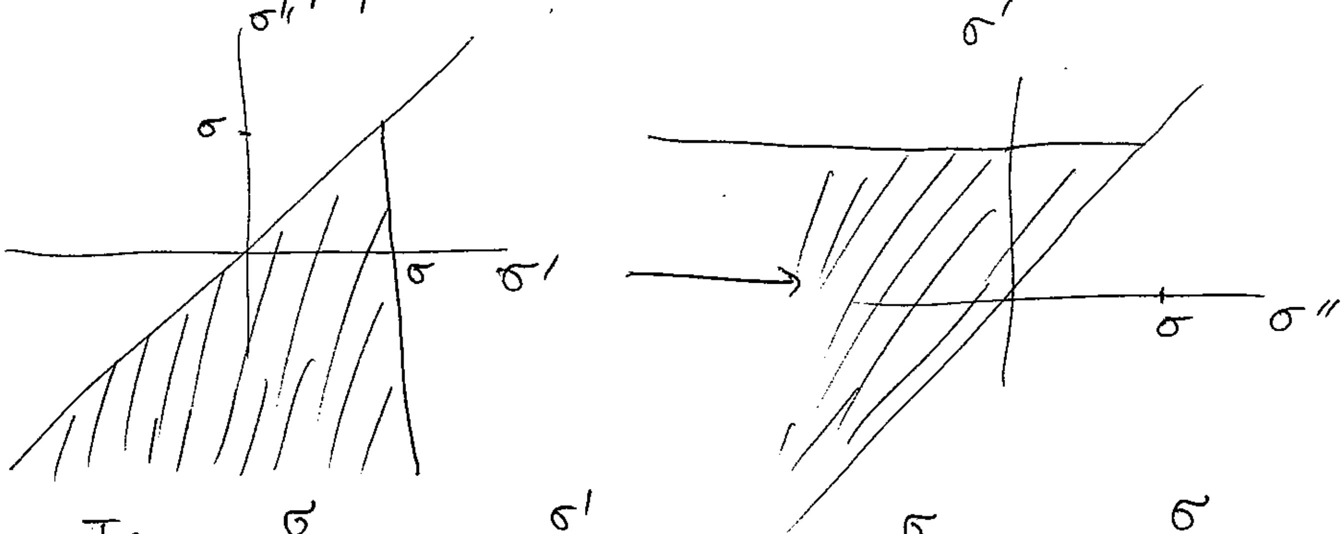
$$\sin\omega\tau' = \sin(\sigma'+\tau_0) = \cos\omega\tau_0\sin\omega\sigma' + \sin\omega\tau_0\cos\omega\tau'$$

$$\delta\vec{r}_\perp^{\,(1)}(t) = -\frac{Ze^2}{mv^2}\int_{-\infty}^{\sigma} d\sigma' \int_{-\infty}^{\sigma'} \frac{d\sigma''}{(1+\sigma''^2)^{3/2}} \underbrace{\begin{pmatrix} \cos\omega\tau' & -\sin\omega\tau' \\ \sin\omega\tau' & \cos\omega\tau' \end{pmatrix}\begin{pmatrix} \cos\omega\tau'' & \sin\omega\tau'' \\ -\sin\omega\tau'' & \cos\omega\tau'' \end{pmatrix}\begin{pmatrix} \beta\sin\theta + \gamma_\perp\tau'' \\ -\beta\cos\theta \end{pmatrix}}_{M} =$$

$$M = \begin{pmatrix} \cos\omega(\tau'-\tau'') & \sin\omega(\tau''-\tau') \\ \sin\omega(\tau'-\tau'') & \cos\omega(\tau'-\tau'') \end{pmatrix}\begin{pmatrix} \beta\sin\theta - \gamma_\perp\tau'' \\ -\beta\cos\theta \end{pmatrix} = \begin{pmatrix} \cos\omega(\sigma'-\sigma'')(\beta\sin\theta - \gamma_\perp\tau'') + \sin\omega(\sigma'-\sigma'')\beta\cos\theta \\ \sin\omega(\sigma'-\sigma'')(\beta\sin\theta - \gamma_\perp\tau'') - \cos\omega(\sigma'-\sigma'')\beta\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \beta\sin[\omega(\sigma'-\sigma'')+\theta] - \gamma_\perp(\sigma''+\tau_0)\cos\omega(\sigma'-\sigma'') \\ -\beta\cos[\omega(\sigma'-\sigma'')+\theta] - \gamma_\perp(\sigma''+\tau_0)\sin\omega(\sigma'-\sigma'') \end{pmatrix}$$

Область интегрирования



Тогда $\displaystyle\int_{-\infty}^{\sigma} f_1(\sigma')d\sigma' \int_{-\infty}^{\sigma'} f_2(\sigma'')d\sigma'' = \int_{-\infty}^{\sigma} d\sigma'' f_2(\sigma'') \int_{\sigma''}^{\sigma} f_1(\sigma')d\sigma' \underset{\text{переобозначение}}{=} \int_{-\infty}^{\sigma} f_2(\sigma')d\sigma' \int_{\sigma'}^{\sigma} f_1(\sigma'')d\sigma''$

В частности, если $f_1(\sigma) = \dfrac{dg(\sigma)}{d\sigma}$

то
$$\int_{-\infty}^{\sigma} \frac{dg(\sigma')}{d\sigma'}d\sigma' \int_{-\infty}^{\sigma'} f_2(\sigma'')d\sigma'' = \int_{-\infty}^{\sigma} f_2(\sigma')d\sigma' \int_{\sigma'}^{\sigma} \frac{dg_1(\sigma'')}{d\sigma''}d\sigma'' =$$

$$= \int_{-\infty}^{\sigma} f_2(\sigma')d\sigma' \cdot g(\sigma'')\Big|_{\sigma'}^{\sigma} = g(\sigma)\int_{-\infty}^{\sigma} f_2(\sigma')d\sigma' - \int_{-\infty}^{\sigma} g(\sigma')f_2(\sigma')d\sigma'$$

| Это из Appendix

но

$$\begin{pmatrix}\cos\omega\tau' & -\sin\omega\tau'\\ \sin\omega\tau' & \cos\omega\tau'\end{pmatrix} = \frac{1}{\omega}\frac{d}{d\tau'}\begin{pmatrix}\sin\omega\tau' & \cos\omega\tau'\\ -\cos\omega\tau' & \sin\omega\tau'\end{pmatrix}$$

Тогда в соответствии с р-том со стр (8) при перестановке порядка интегрирования имеем

$$\delta^{(1)}\vec r_\perp(t) = -\frac{Ze^2}{m\bar v^2}\int_{-\infty}^{\sigma}\frac{d\sigma'}{\omega}\frac{d}{d\sigma'}\begin{pmatrix}\sin\omega\tau' & \cos\omega\tau'\\ -\cos\omega\tau' & \sin\omega\tau'\end{pmatrix}\int_{-\infty}^{\sigma'}\frac{d\sigma''}{(1+\sigma''^2)^{3/2}}\begin{pmatrix}\cos\omega\tau'' & \sin\omega\tau''\\ -\sin\omega\tau'' & \cos\omega\tau''\end{pmatrix}\begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau''\\ -\beta\cos\theta\end{pmatrix} =$$

$\boxed{\sigma=\frac{\bar v t}{r_0}}$ (3.21)

$$= -\frac{1}{\omega v}\frac{Ze^2}{m\bar v^2}\begin{pmatrix}\sin\omega\tau & \cos\omega\tau\\ -\cos\omega\tau & \sin\omega\tau\end{pmatrix}\underbrace{\int_{-\infty}^{\sigma}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}T(\omega\tau')\begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau'\\ -\beta\cos\theta\end{pmatrix}}_{= r_0\delta^{(1)}\vec V_{B\perp}(t)\ \text{from (3.36)}} - \frac{1}{\omega}\left(-\frac{Ze^2}{m\bar v^2}\right)\times$$

$\boxed{\omega=\frac{r_0\Omega_\perp}{\bar v}}$ (3.23)

$$\times\int_{-\infty}^{\sigma}\begin{pmatrix}\sin\omega\tau' & \cos\omega\tau'\\ -\cos\omega\tau' & \sin\omega\tau'\end{pmatrix}d\sigma'\frac{1}{(1+\sigma'^2)^{3/2}}\begin{pmatrix}\cos\omega\tau' & \sin\omega\tau'\\ -\sin\omega\tau' & \cos\omega\tau'\end{pmatrix}\begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau'\\ -\beta\cos\theta\end{pmatrix}$$

↓ перемножение матриц!

$$\begin{pmatrix}\sin\omega\tau' & \cos\omega\tau'\\ -\cos\omega\tau' & \sin\omega\tau'\end{pmatrix}\begin{pmatrix}\cos\omega\tau' & \sin\omega\tau'\\ -\sin\omega\tau' & \cos\omega\tau'\end{pmatrix}\begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau'\\ -\beta\cos\theta\end{pmatrix} =$$

$$= \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau'\\ -\beta\cos\theta\end{pmatrix} = \begin{pmatrix}-\beta\cos\theta\\ -(\beta\sin\theta+\gamma_\perp\tau')\end{pmatrix} =$$

$$= -\begin{pmatrix}\beta\cos\theta\\ \beta\sin\theta+\gamma_\perp\tau'\end{pmatrix}$$

$$\begin{pmatrix}\sin\omega\tau & \cos\omega\tau\\ -\cos\omega\tau & \sin\omega\tau\end{pmatrix}$$

$$= \frac{r_0}{\omega v}\delta^{(1)}\vec V_{B\perp}(t) - \frac{1}{\omega}\left(-\frac{Ze^2}{m\bar v^2}\right)\int_{-\infty}^{\sigma}-\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\begin{pmatrix}\beta\cos\theta\\ \beta\sin\theta+\gamma_\perp\tau'\end{pmatrix} =$$

$$= \frac{r_0}{\omega\bar v}\delta^{(1)}\vec V_{B\perp}(t) - \frac{Ze^2}{m\bar v^2}\int_{-\infty}^{\sigma}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\begin{pmatrix}\beta\cos\theta\\ \beta\sin\theta+\gamma_\perp\tau'\end{pmatrix}$$

$$\begin{pmatrix}\sin\omega\tau & \cos\omega\tau\\ -\cos\omega\tau & \sin\omega\tau\end{pmatrix}$$

это (3.42)

So, (9')

$$\delta^{(1)}\vec{r}_{\perp}(t)=\frac{r_0}{\omega\bar{v}}\delta^{(1)}\vec{v}_{B\perp}(t)\begin{pmatrix}\sin\omega\tau & \cos\omega\tau\\ -\cos\omega\tau & \sin\omega\tau\end{pmatrix}-\frac{Ze^2}{m\bar{v}^2}\int_{-\infty}^{\bar{v}t/r_0}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\begin{pmatrix}\beta\cos\theta\\ \beta\sin\theta+\gamma_\perp\tau'\end{pmatrix} \quad (3.29)$$

$$\boxed{\tau'=\frac{vt'}{r_0}\ (3.21),\quad \sigma=\tau'-\tau_0\ (3.22),\quad d\sigma=d\tau'}$$

Analogously: (from 3.40):

$$\delta^{(1)}r_{\parallel}(t)=-\frac{Ze^2}{m\bar{v}^2}\int_{-\infty}^{\bar{v}t/r_0}d\sigma'\,T(\omega\tau')\int_{-\infty}^{\sigma'}\frac{d\sigma''}{(1+\sigma''^2)^{3/2}}T^{-1}(\omega\tau'')\gamma_\parallel\tau''\ \text{(more accuracy!)}=$$

$$=-\frac{Ze^2}{m\bar{v}^2}\int_0^{\bar{v}t/r_0}d\sigma'\begin{pmatrix}\cos\omega\tau' & -\sin\omega\tau' & 0\\ \sin\omega\tau' & \cos\omega\tau' & 0\\ 0 & 0 & 1\end{pmatrix}\int^{\sigma'}\frac{d\sigma''}{(1+\sigma''^2)^{3/2}}\begin{pmatrix}\cos\omega\tau'' & \sin\omega\tau'' & 0\\ -\sin\omega\tau'' & \cos\omega\tau'' & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau''\\ -\beta\cos\theta\\ \gamma_\parallel\tau''\end{pmatrix}=$$

But for $\delta^{(1)}r_{\parallel}(t)$ it is necessary to calculate only 3rd component of the product ~~of~~ the matrices:

$$\begin{pmatrix}\cos\omega\tau' & -\sin\omega\tau' & 0\\ \sin\omega\tau' & \cos\omega\tau' & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\cos\omega\tau'' & \sin\omega\tau'' & 0\\ -\sin\omega\tau'' & \cos\omega\tau'' & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau''\\ -\beta\cos\theta\\ \gamma_\parallel\tau''\end{pmatrix}=\begin{pmatrix}\cos\omega(\tau'+\tau'') & -\sin\omega(\tau'-\tau'') & 0\\ \sin\omega(\tau'-\tau'') & \cos\omega(\tau'+\tau'') & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$\begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau''\\ -\beta\cos\theta\\ \gamma_\parallel\tau''\end{pmatrix}=\begin{pmatrix}\cdots & \cdots\\ \cdots & \cdots\\ & \gamma_\parallel\tau''\end{pmatrix}\longrightarrow \delta r_{\parallel}^{(1)}(t)=-\frac{Ze^2}{m\bar{v}^2}\int_{-\infty}^{\sigma}d\sigma'\int_{-\infty}^{\sigma'}\frac{\gamma_\parallel\tau''}{(1+\sigma''^2)^{3/2}}d\sigma''=\left|\tau''=\sigma''+\tau_0\ (3.22)\right.$$

$$=-\frac{Ze^2}{m\bar{v}^2}\gamma_\parallel\int^{\sigma}d\sigma'\int_{-\infty}^{\sigma'}\frac{(\sigma''+\tau_0)\,d\sigma''}{(1+\sigma''^2)^{3/2}}=$$

$$\int\frac{\sigma''d\sigma''}{(1+\sigma''^2)^{3/2}}=(2.227)=-\frac{1}{\sqrt{1+\sigma''^2}}$$

$$\int\frac{d\sigma''}{(1+\sigma''^2)^{3/2}}=(2.271.5)=\frac{\sigma''}{\sqrt{1+\sigma''^2}}$$

(9'')

$$= -\frac{Ze^2}{m\bar{v}^2}\gamma_{11}\int_{-\infty}^{\sigma} d\sigma' \left[-\frac{1}{\sqrt{1+\sigma''^2}}\Bigg|_{-\infty}^{\sigma'} + \tau_0 \frac{\sigma''}{\sqrt{1+\sigma''^2}}\Bigg|_{-\infty}^{\sigma'} \right] = -\frac{Ze^2}{m\bar{v}^2}\gamma_{11}\int_{-\infty}^{\sigma} d\sigma'\left[-\frac{1}{\sqrt{1+\sigma'^2}} + \frac{\tau_0\sigma'}{\sqrt{1+\sigma'^2}} - \tau_0(-1)\right] =$$

So,

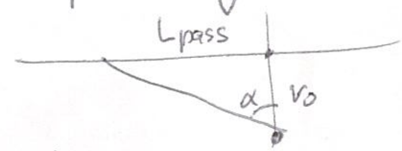
$$r_{\parallel}(t) = -\frac{Ze^2}{m\bar{v}^2}\gamma_{11}\int_{-\infty}^{\sigma} d\sigma'\left[\frac{\tau_0\sigma'-1}{\sqrt{1+\sigma'^2}}+\tau_0\right] =$$

| This is (3.41)

$$= -\frac{Ze^2}{m\bar{v}^2}\gamma_{11}\int_{-\infty}^{\sigma} d\sigma'\left(\underbrace{\frac{\tau_0\sigma'}{\sqrt{1+\sigma'^2}}}_{2.271.7} - \underbrace{\frac{1}{\sqrt{1+\sigma'^2}}}_{2.271.3} + \tau_0\right) = -\frac{Ze^2}{m\bar{v}^2}\gamma_{11}\left[\tau_0\sqrt{1+\sigma'^2}\Big|_{-\infty}^{\sigma} - \ln\left(\sigma'+\sqrt{1+\sigma'^2}\right)\Big|_{-\infty}^{\sigma} + \tau_0\sigma'\Big|_{-\infty}^{\sigma}\right] =$$

$$= -\frac{Ze^2}{m\bar{v}^2}\gamma_{11}\left(\tau_0\sqrt{1+\sigma^2} - \tau_0\sqrt{1+\sigma'^2}\Big|_{\sigma'\to-\infty} + \tau_0\sigma - \tau_0\sigma'\Big|_{\sigma'\to-\infty} - \ln\frac{\sigma+\sqrt{1+\sigma^2}}{\sigma'+\sqrt{1+\sigma'^2}\Big|_{\sigma'\to-\infty}}\right)$$

The divergence on the limit $t \to -\infty$ can be bypassed to to restriction on „length" of trajectory: $-\infty \to \frac{\bar{v}\,\Delta t_{pass}}{v_0}$, where $\Delta t_{pass} \simeq \frac{L_{pass}}{\bar{V}} \simeq \frac{v_0 \,\mathrm{tg}\,\alpha}{\bar{V}}$.



Lagrangian: $\mathcal{L} = \frac{m\vec{v}_e^{\,2}}{2} + \frac{M\vec{v}_i^{\,2}}{2} - e\vec{A}(\vec{r}_e)\vec{v}_e + Ze\vec{A}(\vec{r}_i)\vec{v}_i + \frac{Ze^2}{|\vec{r}_e - \vec{r}_i|}$ (3.3)

magnetic field along $\vec{e}_z$ из $\to$ $\vec{B} = B\vec{e}_z$ and vector-potential for that is

$$\vec{A}(\vec{r}) = \frac{1}{2}[\vec{B}\vec{r}] = \frac{1}{2}\begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ B_x & B_y & B_z \\ x & y & z \end{vmatrix} = \frac{1}{2}\begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ 0 & 0 & \\ x & y & z \end{vmatrix} = \frac{1}{2}(yB\,\vec{e}_x - Bx\vec{e}_y) = \frac{B}{2}(y\vec{e}_x - x\vec{e}_y)$$

Checking:

$$\vec{B} = \mathrm{rot}\vec{A} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{yB}{2} & -\frac{xB}{2} & 0 \end{vmatrix} = \vec{e}_x\left(0 - \frac{1}{2}\frac{\partial(xB)}{\partial z}\right) + \vec{e}_y\left(\frac{1}{2}\frac{\partial(yB)}{\partial z} - 0\right) + \vec{e}_z\left(\frac{1}{2}\frac{\partial(xB)}{\partial x} - \frac{\partial(-xy}{\partial y}\right.$$

$$= 0\cdot\vec{e}_x + 0\cdot\vec{e}_y + B\vec{e}_z \quad \text{OK!}$$

Let ion moves uniformly with velocity

$$\vec{V}_i = \begin{pmatrix} V_{i\perp} \\ 0 \\ V_{i\parallel} \end{pmatrix} \quad (3.1) \longrightarrow \vec{r}_i(t) = \vec{V}_i \cdot t$$

Input relative coordinate of electron $\vec{r}(t) = \vec{r}_e(t) - \vec{r}_i(t) = \vec{r}_e(t) - \vec{V}_i\cdot t$ (3.2a)

So, $\vec{r}(t)$ — coordinate ~~of~~ electron in the ion's frame. Farther

(3.2b) $\vec{V}(t) = \vec{V}_e(t) - \vec{V}_i(t) = \vec{V}_e(t) - \vec{V}_i$ — relative elecrton's velocity in the ion's frame

Let's define the radius and velocity of the center of mass:

$$\begin{cases} \vec{r}_{c.m} = (m_e\vec{r}_e + M_i\vec{r}_i)/(m_e + M_i) \\ \vec{r} = \vec{r}_e - \vec{r}_i \end{cases} \quad \text{and} \quad \begin{cases} \vec{V}_{c.m} = (m_e\vec{V}_e + M_i\vec{V}_i)/(m_e + M_i) \\ \vec{V} = \vec{V}_e - \vec{V}_i \end{cases}$$

$$\begin{cases} m\vec{r}_e + M\vec{r}_i = (m_e + M_i)\vec{r}_{cm} \\ \vec{r}_e - \vec{r}_i = \vec{r} \end{cases} \longrightarrow \Delta = \begin{vmatrix} m_e & M_i \\ 1 & -1 \end{vmatrix} = -(m_e + M_i) \longrightarrow \vec{r}_e = \frac{1}{\Delta}\begin{vmatrix} (m_e+M_i)\vec{r}_{cm} & M_i \\ \vec{r} & -1 \end{vmatrix} = \vec{r}_{cm} + \frac{M_i\vec{r}}{m_e + M_i}$$

and $\vec{r}_i = \frac{1}{\Delta}\begin{vmatrix} m_e & (m_e+M_i)\vec{r}_{cm} \\ 1 & \vec{r} \end{vmatrix} = \vec{r}_{cm} - \frac{m_e}{m_e+M_i}\vec{r} \simeq \vec{r}_{cm} - \frac{m}{M}\vec{r}$

$\vec{V}_i t$

$\vec{r}_{cm} \simeq \vec{V}_i t + \frac{m}{M}\vec{r}$

and analogusly

$$\vec{V}_e = \vec{V}_{cm} + \frac{M_i}{m_e + M_i}\vec{V}$$

$$\vec{V}_i = \vec{V}_{cm} - \frac{m_e}{m_e + M_i}\vec{V} \simeq \vec{V}_{cm} - \frac{m}{M}\vec{V}$$

$$\mu = 1\Big/\left(\frac{1}{m_e} + \frac{1}{M_i}\right) = \frac{m_e M_i}{m_e + M_i}$$

so

$$\begin{cases} \vec r_e = \vec r_{cm} + \frac{\mu}{m_e}\vec r \\ \vec r_i = \vec r_{cm} - \frac{\mu}{M_i}\vec r \end{cases} \quad \text{and} \quad \begin{cases} \vec V_e = \vec V_{cm} + \frac{\mu}{m_e}\vec V \\ \vec V_i = \vec V_{cm} - \frac{\mu}{M_i}\vec V \end{cases} \qquad \begin{matrix} \vec A(\vec r_e) = \frac{1}{2}[\vec B \cdot \vec r_e] = \frac{1}{2}\left[\vec B \cdot \left(\vec r_{cm} + \frac{\mu}{m}\vec r\right)\right] \\ \vec A(\vec r_i) = \frac{1}{2}[\vec B \cdot \vec r_i] = \frac{1}{2}\left[\vec B \cdot \left(\vec r_{cm} - \frac{\mu}{M}\vec r\right)\right] \end{matrix}$$

Then

$$\mathcal{L} = \frac{m}{2}\left(\vec V_{cm} + \frac{\mu}{m}\vec V\right)^2 + \frac{M}{2}\left(\vec V_{cm} - \frac{\mu}{M}\vec V\right)^2 - e\frac{1}{2}\left[\vec B\left(\vec r_{cm} + \frac{\mu}{m}\vec r\right)\right]\left(\vec V_{cm} + \frac{\mu}{m}\vec V\right) - \frac{Ze}{2}\left[\vec B\left(\vec r_{cm} - \frac{\mu}{M}\vec r\right)\right]\left(\vec V_{cm} - \frac{\mu}{M}\vec V\right) + \frac{Ze^2}{r} =$$

$$= \underbrace{\frac{m}{2}\vec V_{cm}^2}_{(1)} + \cancel{2\mu \vec V \vec V_{cm}} + \underbrace{\frac{\mu^2}{2m}\vec V^2}_{(2)} + \underbrace{\frac{M}{2}\vec V_{cm}^2}_{(3)} - \cancel{\mu \vec V \vec V_{cm}} + \underbrace{\frac{\mu^2}{2M}\vec V^2}_{(4)} - \frac{e}{2}\Big\{\underbrace{([\vec B \vec r_{cm}]\vec V_{cm})}_{(5)} + \frac{\mu}{m}\underbrace{([\vec B \vec r]\vec V_{cm})}_{(6)} + \underbrace{([\vec B \vec r_{cm}]\frac{\mu}{m}\vec V)}_{(7)} +$$

$$+ \underbrace{\frac{\mu^2}{m^2}([\vec B \vec r]\vec V)}_{(8)}\Big\} + \frac{Ze}{2}\Big\{\underbrace{([\vec B \vec r_{cm}]\vec V_{cm})}_{(9)} - \frac{\mu}{M}\underbrace{([\vec B \vec r]\vec V_{cm})}_{(10)} - \frac{\mu}{M}\underbrace{([\vec B \vec r_{cm}]\vec V)}_{(11)} + \underbrace{\frac{\mu^2}{M^2}([\vec B \vec r]\vec V)}_{(12)}\Big\} + \frac{Ze^2}{r} =$$

$$= \underbrace{\frac{m+M}{2}\vec V_{cm}^2}_{(1)+(3) \leftarrow \text{this is an constant}} + \frac{\mu^2}{2}\vec V^2\Big(\underbrace{\frac{1}{m}}_{(2)} + \underbrace{\frac{1}{M}}_{(4)}\Big) + \frac{Ze^2}{r} + \underbrace{\frac{(Z-1)e}{2}([\vec B \cdot \vec r_{cm}]\vec V_{cm})}_{(5)\ (9) \leftarrow \text{this is an constant}} + \frac{\mu^2}{2}\Big(\underbrace{\frac{Ze}{M^2}}_{(12)} - \underbrace{\frac{e}{m^2}}_{(8)}\Big)([\vec B \vec r]\vec V) +$$

$$-\frac{\mu}{2}([\vec B \vec r]\vec V_{cm})\Big(\underbrace{\frac{e}{m}}_{(6)} + \underbrace{\frac{Ze}{M}}_{(10)}\Big) - \frac{\mu}{2}\Big(\underbrace{\frac{e}{m}}_{(7)} + \underbrace{\frac{Ze}{M}}_{(11)}\Big)([\vec B \vec r_{cm}]\vec V) \simeq$$

this is (3.5) and taking into account $\mu \simeq m \ll M$

$$\simeq \underbrace{\frac{m\vec V^2}{2}}_{(2)+(4)} + \frac{Ze^2}{r} - \underbrace{\frac{e}{2}([\vec B \vec r]\vec V)}_{(12)+(8)} - \left(\frac{e}{m} + \frac{Ze}{M}\right)\underbrace{\frac{\mu}{2}\Big(([\vec B \vec r]\overset{\vec V_i + \frac{m}{M}\vec V}{\vec V_{cm}}) + ([\vec B \vec r_{cm}]\vec V)\Big)}_{(6)+(10)+(7)+(11)} =$$

$$\simeq \frac{m\vec V^2}{2} + \frac{Ze^2}{r} - \frac{e}{2}([\vec B \vec r]\vec V) - \frac{e}{2}\left\{\left([\vec B \vec r]\left(\vec V_i + \frac{m}{M}\vec V\right)\right) + ([\vec B \vec r_{cm}]\vec V)\right\} = \frac{m\vec V^2}{2} + \frac{Ze^2}{r} - \frac{e}{2}([\vec B \vec r]\vec V) -$$

$$-\frac{e}{2}\left\{([\vec B \vec r]\vec V_i + \left[\vec B\left(\vec r_{cm} + \frac{m}{M}\vec r\right)\right]\vec V\right\} =$$

$$= \frac{m\vec{V}^2}{2} + \frac{Ze^2}{r} - \frac{e}{2}([\vec{B}\vec{r}]\vec{V}) - \frac{e}{2}\left\{([\vec{B}\vec{r}]\vec{V}_i) + ([\vec{B}\cdot\vec{V}_i t]\vec{V}) + 2\frac{m}{M}([\vec{B}\vec{r}]\vec{V})\right\} =$$

$$= \frac{m\vec{V}^2}{2} + \frac{Ze^2}{r} - \frac{e}{2}([\vec{B}\vec{r}]\vec{V}_i) - \frac{e}{2}([\vec{B}\cdot\vec{V}_i t]\vec{V}) - \frac{e}{2}\Big(1 + \underbrace{2\frac{m}{M}}_{2\frac{m}{M} \ll 1}\Big)([\vec{B}\vec{r}]\vec{V}) \Longrightarrow$$

$$\mathcal{L} = \frac{m\vec{V}^2}{2} + \frac{Ze^2}{r} - \frac{e}{2}([\vec{B}\vec{r}]\vec{V}) - \frac{e}{2}\left\{([\vec{B}\cdot\vec{V}_i t]\vec{V}) + ([\vec{B}\vec{r}]\vec{V}_i)\right\}$$ this is (3.6)

Equation of motion:

$$m\frac{d\vec{V}}{dt} = -\frac{\partial \mathcal{L}}{\partial \vec{r}} = -\vec{\nabla}\frac{Ze^2}{r}$$

From (3.14) for $R \to 0$ („Guiding" center approach):

$$\vec r^{\,2}(t) = b^2 + [(V_{e\parallel}-V_{i\parallel})^2 + V_{i\perp}^2]\,t^2 - 2V_{i\perp}\, b\, t \sin\theta \qquad (3.15)$$

Let's define the relative velocity $\vec V$ of the guiding center and ion:

$$\vec V = \vec V_\perp + \vec V_\parallel = \begin{pmatrix}0\\0\\V_{e\parallel}-V_{i\parallel}\end{pmatrix} + \begin{pmatrix}-V_{i\perp}\\0\\0\end{pmatrix} \qquad (3.16)$$

Then $(V_{e\parallel}-V_{i\parallel})^2+V_{i\perp}^2 = \vec V^{\,2} \equiv \bar V^2$

Let's t_0 – time t_0 when electron reaches the minimal distance $\vec r_0$ to ion (this is impact parameter of collision); i.e.

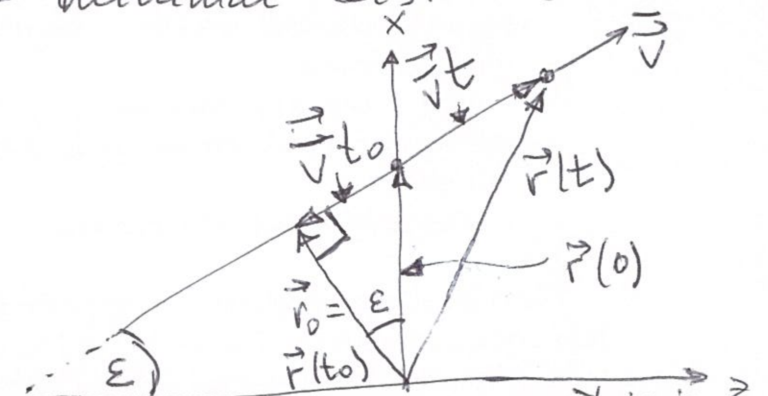


Fig. 2

This is z direction of magnetic field!

$$r_0^2 = b^2 + \bar V^2 t_0^2 - 2V_{i\perp}\, b\, t_0 \sin\theta$$

Very important: $\vec r_0 \perp \vec V$ and from picture one has

$$\vec r(t) = \vec r_0 + \vec V (t-t_0) \qquad (3.20)$$

then

$$\vec r^{\,2} = r_0^2 + \underbrace{\vec V^{\,2}}_{\equiv \bar V^2}(t-t_0)^2 + 2\underbrace{\vec r_0 \vec V}_{=0}(t-t_0) = r_0^2 + \bar V^2 (t-t_0)^2$$

So

$$\vec r^{\,2} = b^2 + \bar V^2 t^2 - 2V_{i\perp}\, b\, t \sin\theta = r_0^2 + \bar V^2 (t-t_0)^2 = b^2 + \bar V^2 t_0^2 - 2V_{i\perp}\, b\, t_0 \sin\theta + \bar V^2 (t-t_0)^2$$

or

$$\cancel{b^2} + \cancel{\bar V^2 t^2} - 2V_{i\perp}\, b\, t \sin\theta = \cancel{b^2} + \bar V^2 t_0^2 - 2V_{i\perp}\, b\, t_0 \sin\theta + \cancel{\bar V^2 t^2} + \bar V^2 t_0^2 - 2\bar V^2 t\, t_0$$

$$\cancel{\text{or}\ 2\bar V^2 t_0^2 - 2t_0 (V_{i\perp} b \sin\theta - \bar V^2 t) + 2V_{i\perp} b t \sin\theta = 0}$$

So $$0 = 2v_{i\perp} b\sin\theta\cdot(t-t_0) + 2\bar{V}^2 t_0(t_0-t) \Longrightarrow t_0 = \frac{v_{i\perp} b\sin\theta}{\bar{V}^2} \quad (3.18) = -\frac{\bar{V}_\perp b\sin\theta}{\bar{V}^2} =$$

and for this reason

$$r_0^2 = b^2 + \bar{V}^2 t_0^2 - 2t_0 \cdot \underbrace{\frac{v_{i\perp}\sin\theta}{\bar{V}^2}}_{=t_0} \bar{V}^2 = b^2 + \bar{V}^2 t_0^2 - 2\bar{V}^2 t_0^2 = b^2 - \bar{V}^2 t_0^2 \quad (3.19)$$

and

$$\vec{r}_0 = \vec{r}(t_0) = \begin{pmatrix} b\sin\theta - v_{i\perp} t_0 \\ -b\cos\theta \\ (v_{e\parallel} - v_{i\parallel}) t_0 \end{pmatrix} = \begin{pmatrix} b\sin\theta + \bar{V}_\perp t_0 \\ -b\cos\theta \\ \bar{V}_\parallel t_0 \end{pmatrix}$$

this is from (3.10) with $R=0$

and then

$$\vec{r}(t) = \begin{pmatrix} -b\sin\theta - v_{i\perp} t \\ -b\cos\theta \\ (v_{e\parallel} - v_{i\parallel}) t \end{pmatrix} = \begin{pmatrix} -b\sin\theta + \bar{V}_\perp t \\ -b\cos\theta \\ \bar{V}_\parallel t \end{pmatrix} = \begin{pmatrix} -b\sin\theta + \bar{V}_\perp (t_0 + t - t_0) \\ -b\cos\theta \\ \bar{V}_\parallel (t_0 + t - t_0) \end{pmatrix} =$$

$$= \begin{pmatrix} -b\sin\theta + \bar{V}_\perp t_0 \\ -b\cos\theta \\ \bar{V}_\parallel t_0 \end{pmatrix} + \begin{pmatrix} \bar{V}_\perp (t-t_0) \\ 0 \\ \bar{V}_\parallel (t-t_0) \end{pmatrix} = \vec{r}_0 + \vec{\bar{V}}(t-t_0)$$

this is (3.20) again

Let's introduce the dimentionless variables:

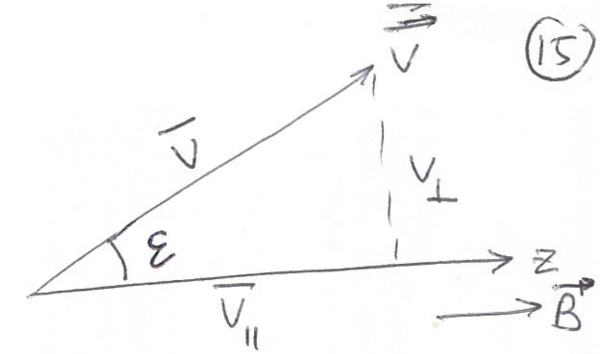
$$\tau = \frac{\bar{V} t}{r_0} \quad (3.21) \qquad \tau_0 = \frac{\bar{V} t_0}{r_0}, \quad \sigma = \tau - \tau_0 = \frac{\bar{V}(t-t_0)}{r_0} \quad (3.22)$$

This is (3.29 left)

$$\gamma_\parallel = \frac{\bar{V}_\parallel}{\bar{V}} \quad (3.24), \qquad \gamma_\perp = \frac{\bar{V}_\perp}{\bar{V}} \quad (3.25) \qquad \beta = \frac{b}{r_0} \quad (3.26)$$

Then from (3.18) $\tau_0 = \frac{\bar{V} t_0}{r_0} = \frac{\bar{V}}{r_0} \cdot \frac{-\bar{V}_\perp b\sin\theta}{\bar{V}^2} =$

$= -\beta\gamma_\perp \sin\theta$



$$\frac{\vec{r}_0}{r_0} = \frac{1}{r_0}\begin{pmatrix} b\sin\theta + \overline{V}_\perp t_0 \\ -b\cos\theta \\ \overline{V}_\parallel t_0 \end{pmatrix} = \begin{pmatrix} \beta\sin\theta + \gamma_\perp \tau_0 \\ -\beta\cos\theta \\ \gamma_\parallel \tau_0 \end{pmatrix}$$

$$\cos\varepsilon = \frac{\overline{V}_\parallel}{\overline{V}} = \gamma_\parallel \quad (3.28)$$

$$\sin\varepsilon = \frac{\overline{V}_\perp}{\overline{V}} = \gamma_\perp \quad (3.28^*)$$

Instead (3.19) one has

$$b^2 = r_0^2 + \overline{V}^2 t_0^2 \to \beta^2 = 1 + \tau_0^2 \quad (3.32)$$

and, of course, $\gamma_\parallel^2 + \gamma_\perp^2 = 1$

Let's input

$$\sin\psi = -\beta\cos\theta \quad (3.30)$$

Then

$$\beta\sin\theta + \gamma_\perp\tau_0 = \left(\text{from 3.29 left } \beta\sin\theta = -\frac{\tau_0}{\gamma_\perp}\right) = -\frac{\tau_0}{\gamma_\perp} + \gamma_\perp\tau_0 = \tau_0\frac{\gamma_\perp^2 - 1}{\gamma_\perp} = -\frac{\tau_0\gamma_\parallel^2}{\gamma_\perp}$$

But

$$\tau_0 = -\beta\gamma_\perp\sin\theta = -\gamma_\perp\beta\sqrt{1-\cos^2\theta} = -\gamma_\perp\beta\sqrt{1-\frac{\sin^2\psi}{\beta^2}} = -\gamma_\perp\sqrt{\beta^2 - \sin^2\psi} =$$

$$= -\gamma_\perp\sqrt{\beta^2 - 1 + \cos^2\psi} = -\gamma_\perp\sqrt{\tau_0^2 + \cos^2\psi} \quad \text{or}$$

from (3.32)

$$\tau_0^2 = \gamma_\perp^2\tau_0^2 + \gamma_\perp^2\cos^2\psi \to \tau_0^2(1-\gamma_\perp^2) = \gamma_\perp^2\cos^2\psi \to$$

$$\tau_0^2\gamma_\parallel^2 = \gamma_\perp^2\cos^2\psi \to \tau_0 = \pm\frac{\gamma_\perp}{\gamma_\parallel}\cos\psi$$ and it is neccesary to select sign "−"!

from (3.28*)

$$\tau_0 = -\frac{\gamma_\perp}{\gamma_\parallel}\cos\psi \;(3.29\text{ right}) \to \gamma_\parallel\tau_0 = -\gamma_\perp\cos\psi = -\sin\varepsilon\cos\psi \quad (3.29^*)$$

So

$$\beta\sin\theta + \gamma_\perp\tau_0 = -\frac{\tau_0\gamma_\parallel^2}{\gamma_\perp} = \frac{\gamma_\perp\cos\psi\cdot\gamma_\parallel^2}{\gamma_\parallel\gamma_\perp} = \gamma_\parallel\cos\psi \quad (3.31)$$

Therefore

$$\frac{\vec{r}_0}{r_0} = \begin{pmatrix} \beta \sin\theta + \gamma_\perp \tau_0 \\ -\beta\cos\theta \\ \gamma_{\parallel}\tau_0 \end{pmatrix} = \begin{pmatrix} \text{using } (3.31) \\ \text{using } (3.30) \\ \text{using } (3.29^*) \end{pmatrix} = \begin{pmatrix} \cos\varepsilon \cos\psi \\ \sin\psi \\ -\sin\varepsilon\cos\psi \end{pmatrix} \quad (3.27)$$

Finaly!

(3.21) $\tau = \frac{\bar{v}t}{r_0} \qquad \tau_0 = \frac{\bar{v}t_0}{r_0}$

(3.22) $\sigma = \tau - \tau_0 = \frac{\bar{v}(t-t_0)}{r_0}$

(3.24) $\gamma_{\parallel} = \frac{\overline{v_{\parallel}}}{v_0}$

(3.25) $\gamma_\perp = \frac{\overline{v_\perp}}{v_0}$

$\gamma_{\parallel}^2 + \gamma_\perp^2 = 1$

(3.26) $\beta = \frac{b}{r_0}$

(3.27) $\frac{\vec{r}_0}{r_0} = \begin{pmatrix} \cos\varepsilon\cos\psi \\ \sin\psi \\ -\sin\varepsilon\cos\psi \end{pmatrix}$

$\beta\sin\theta = \frac{\cos\psi}{\gamma_{\parallel}} = \frac{\cos\psi}{\cos\varepsilon}$

$\beta\cos\theta = -\sin\psi$,

so

$$\boxed{\begin{aligned} \operatorname{tg}\theta &= -\frac{\cos\psi/\cos\varepsilon}{\sin\psi} = \\ &= -\frac{\operatorname{ctg}\psi}{\cos\varepsilon} \end{aligned}}$$

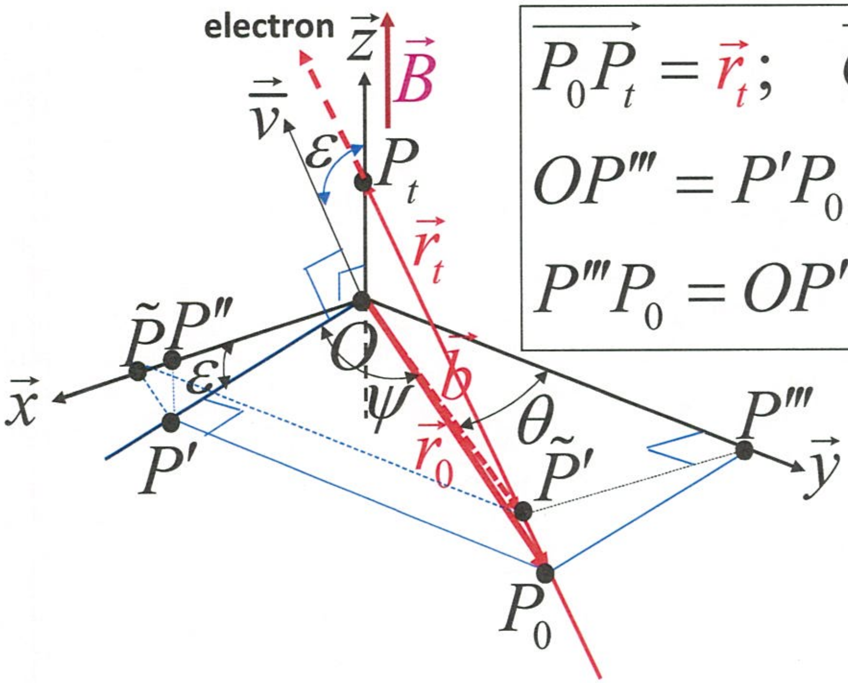
(3.28) $\cos\varepsilon = \gamma_{\parallel} \longrightarrow \sin\varepsilon = \sqrt{1-\cos^2\varepsilon} = \sqrt{1-\gamma_{\parallel}^2} = \gamma_\perp$ (3.28*)

(3.29) $\tau_0 = -\beta\gamma_\perp\sin\theta = -\frac{\gamma_\perp}{\gamma_{\parallel}}\cos\psi \longrightarrow \gamma_{\parallel}\tau_0 = -\sin\varepsilon\cos\psi$ (3.29*)

(3.30) $\beta\cos\theta = -\sin\psi$

(3.31) $\beta\sin\theta + \gamma_\perp\tau_0 = -\frac{\gamma_{\parallel}^2\tau_0}{\gamma_\perp} = \gamma_{\parallel}\cos\psi$

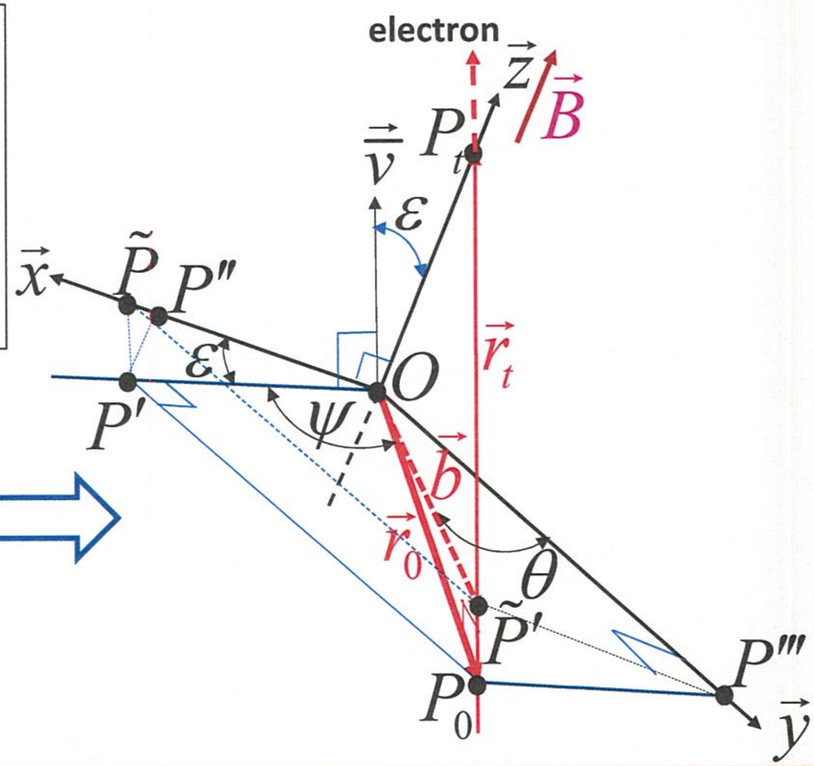
(3.32) $\beta^2 = 1+\tau_0^2$



$$\overrightarrow{P_0P_t} = \vec{r}_t; \quad \overrightarrow{OP_0} = \vec{r}_0 \;\to$$

$$\left.\begin{aligned} OP''' = P'P_0 = r_0\cdot\sin\psi, \\ P'''P_0 = OP' = r_0\cdot\cos\psi \end{aligned}\right\}$$

or



$$\vec{r}_0 = \begin{pmatrix} OP'' \\ P_0P' \\ P''P' \end{pmatrix} = r_0 \begin{pmatrix} \cos\varepsilon\cdot\cos\psi \\ \sin\psi \\ -\sin\varepsilon\cdot\cos\psi \end{pmatrix};$$

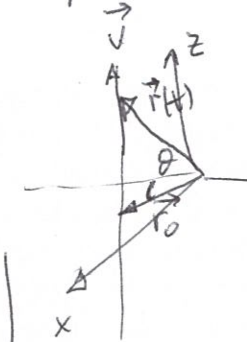
$$\tilde{P}'P''' = O\tilde{P} = \frac{OP'}{\cos\varepsilon} = r_0\cdot\frac{\cos\psi}{\cos\varepsilon} \to$$

$$\to tg\theta = \tilde{P}'P''' / OP''' = -\cot\psi / \cos\varepsilon$$

Sign from corresponding formulas

Unmagnetized electron scattering with ion in fixed point of origin:

$$\frac{d\vec v}{dt} = -\frac{e}{m}\vec E(\vec r) \quad (2.3), \qquad \vec E(\vec r) = Ze\,\frac{\vec r(t)}{r^3(t)} \quad (2.1)$$



r_0 – impact parameter is $\vec r(t=0)$, and

$$\vec r_0 = r_0\begin{pmatrix}\sin\theta\\ -\cos\theta\\ 0\end{pmatrix} \quad (2.5) \quad \text{then} \quad \vec r(t) = \begin{pmatrix} r_{0x}\\ r_{0y}\\ vt\end{pmatrix} = \begin{pmatrix} r_0\sin\theta\\ -r_0\cos\theta\\ vt\end{pmatrix}$$

$v^2 = v_\perp^2 + v_\parallel^2$!

Then first order:

$$(2.6)\quad \delta^{(1)}\vec v_c(t) = -\frac{Ze^2}{m}\int_{-\infty}^{t}\frac{dt'}{\sqrt{(r_0^2+v^2t'^2)^3}}\begin{pmatrix} r_0\sin\theta\\ -r_0\cos\theta\\ v_\parallel t'\end{pmatrix} \underset{x=vt}{=} -\frac{Ze^2}{mv}\int_{-\infty}^{vt}\frac{r_0\,dx/r_0}{(r_0^2+x^2)^{3/2}}\begin{pmatrix} r_0\sin\theta\\ -r_0\cos\theta\\ \end{pmatrix} = -\frac{Ze^2}{mvr_0}\int_{-\infty}^{\frac{vt}{r_0}}\frac{dx}{(1+x^2)^{3/2}}\begin{pmatrix}\sin\theta\\ -\cos\theta\\ \frac{v_\parallel}{v}x\end{pmatrix}$$

(lower index „c" means „Coulomb").

Total $\delta^{(1)}\vec v_c(t)$ is reached for $t\to\infty$!

$$(\delta^{(1)}\vec v_c)_{total} = -\frac{Ze^2}{mvr_0}\int_{-\infty}^{\infty}\frac{dx}{(1+x^2)^{3/2}}\begin{pmatrix}\sin\theta\\ -\cos\theta\\ \frac{v_\parallel}{v}x\end{pmatrix} = -\frac{2Ze^2}{mvr_0}\int^{\infty}\frac{dx}{(1+x^2)^{3/2}}\begin{pmatrix}\sin\theta\\ -\cos\theta\\ 0\end{pmatrix}$$

integral = 0 due to odd function!

2.271.5: $\underbrace{\int\frac{dx}{(a+cx^2)^{3/2}} = \frac{1}{a}\frac{x}{\sqrt{a+cx^2}}}_{=1}$

$$= -\frac{2Z^2e^2}{mvr_0}\begin{pmatrix}\sin\theta\\ -\cos\theta\\ 0\end{pmatrix} = -\frac{2Ze^2}{mvr_0^2}r_0\begin{pmatrix}\sin\theta\\ -\cos\theta\\ 0\end{pmatrix} = -\frac{2Ze^2\vec r_0}{mvr_0^2} \quad (2.7)$$

Now let calculate $\delta\vec r^{(1)}$: $\frac{d\vec r}{dt} = \vec v$ (2.2)

~~then~~ But it is possible to write the solution for $\delta^{(1)}\vec v_c(t)$ from (2.6):

$$\delta^{(1)}\vec v_c(t) = -\frac{Ze^2}{mvr_0}\int_{-\infty}^{\frac{vt}{r_0}}\frac{dx}{(1+x^2)^{3/2}}\begin{pmatrix}\sin\theta\\ -\cos\theta\\ \frac{v_\parallel}{v}x\end{pmatrix} = \begin{pmatrix}\} \ 2.27.5\\ 2.27.7\\ \text{with } n=1\end{pmatrix} =$$

2.271.7: $\int\frac{x\,dx}{\sqrt{a+cx^2}^3} = -\frac{1}{c\sqrt{a+cx^2}}$

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$$= -\frac{Ze^2}{mVr_0}\left(\frac{vt/r_0}{\sqrt{1+v^2t^2/r_0^2}}+1\right)\begin{pmatrix}\sin\theta\\-\cos\theta\\0\end{pmatrix} - \frac{Ze^2}{mVr_0}\left(-\frac{\mathrm{sgn}(v)}{\sqrt{1+v^2t^2/r_0^2}}\right)\begin{pmatrix}0\\0\\1\end{pmatrix}$$

my opinion: factor $\frac{v_{\parallel}}{V} = \gamma_{\parallel}$!

and for $t\to\infty$ one will receive the result (2,7)

Now $\delta^{(1)}\vec{r}_c$ can be received from $\delta^{(1)}\vec{V}_c(t)$:

$$\delta^{(1)}\vec{r}_c(t) = \int_{-\infty}^{t} dt'\,\delta^{(1)}\vec{V}_c(t') = -\frac{Ze^2}{mVr_0}\int_{-\infty}^{t}\left(\frac{vt'/r_0}{\sqrt{1+v^2t'^2/r_0^2}}+1\right)dt'\begin{pmatrix}\sin\theta\\-\cos\theta\\0\end{pmatrix} +$$

$$+\frac{Ze^2}{mVr_0}\int_{-\infty}^{t}\frac{dt'\,\mathrm{sgn}(v)}{\sqrt{1+v^2t'^2/r_0^2}}\begin{pmatrix}0\\0\\1\end{pmatrix} = -\frac{Ze^2}{mv^2}\int_{-\infty}^{t}\left(\frac{vt'/r_0}{\sqrt{1+v^2t'^2/r_0^2}}+1\right)d\left(\frac{vt'}{r_0}\right)\begin{pmatrix}\sin\theta\\-\cos\theta\\0\end{pmatrix} + \frac{Ze^2}{mv^2}\int_{-\infty}^{t}\frac{\mathrm{sgn}(v)\,d\left(\frac{vt'}{r_0}\right)}{\sqrt{1+v^2t'^2/r_0^2}}\begin{pmatrix}0\\0\\1\end{pmatrix} =$$

must be $\gamma_{\parallel}$

$$= -\frac{Ze^2}{mv^2}\int_{-\infty}^{vt/r_0} d\tau'\left[\left(\frac{\tau'}{\sqrt{1+\tau'^2}}+1\right)\begin{pmatrix}\sin\theta\\-\cos\theta\\0\end{pmatrix} - \frac{\mathrm{sgn}(v)}{\sqrt{1+\tau'^2}}\begin{pmatrix}0\\0\\1\end{pmatrix}\right]$$ this is (2,11)

Question: how from expessions for $\delta^{(1)}\vec{r}_{\perp}(t)$ and $\delta^{(1)}v_{\parallel}(t)$ ((3.41) and (3.42) correspondingly) to receive the expressions (2.11), (2.7) because, it seems, that

$$\boxed{(2.11) = \lim_{\omega = \frac{r_0}{v}\Omega_B \to 0} (3.41), (3.42)}$$

Differences between (2,7), (2,11) and (3.41), (3.42): first are in lab. system second — in ion frame system

Second-order approach

A) Field:
$$\vec{E}(\vec{r}(t)) = Ze\,\frac{\vec{r}(t)}{r^3(t)} \longrightarrow E_k(\vec{r}(t)) = Ze\,\frac{r_k(t)}{r_i r_i}$$

$$E_k = E_k(\vec{r}+\delta^{(1)}\vec{r}) - E_k(\vec{r}) = Ze\left[E_k(\vec{r}) + \delta^{(1)} r_j \frac{\partial}{\partial r_j}\left(\frac{r_k}{r_i r_i^{3/2}}\right) - E_k(\vec{r})\right] =$$

$$= Ze\,\delta^{(1)} r_j\left[\frac{1}{(r_i r_i)^{3/2}}\frac{\partial r_k}{\partial r_j} + r_k \frac{\partial}{\partial r_j}\left(\frac{1}{(r_i r_i)^{3/2}}\right)\right] = Ze\,\delta^{(1)} r_j\left(\frac{\delta_{kj}}{(r_i r_i)^{3/2}} + r_k \frac{-3}{2}\frac{2\delta_{ij} r_i}{(r_i r_i)^{5/2}}\right) =$$

$$= \frac{Ze}{(r_i r_i)^{3/2}}\left(r_k - 3 r_k \frac{r_i \delta^{(1)} r_i}{r_i r_i}\right)$$

This is (2.13) in lab. system

In guiding center system
$$\delta^{(1)} E_k(t) = \frac{Ze}{(\bar{r}_j \bar{r}_j)^{3/2}}\left(\delta^{(1)} r_k - 3\bar{r}_k \frac{\bar{r}_i \delta^{(1)} r_i}{\bar{r}_j \bar{r}_j}\right), \qquad (3.47)$$

where
$$\vec{\bar{r}}(t) = \begin{pmatrix} b\sin\theta + \bar{V}_\perp t_0 \\ -b\cos\theta \\ \bar{V}_\parallel t_0 \end{pmatrix} + \begin{pmatrix} \bar{V}_\perp \\ 0 \\ \bar{V}_\parallel \end{pmatrix}(t - t_0) = \vec{r}_0 + \vec{\bar{V}}(t-t_0) = (3.20)$$

$$= r_0 \begin{pmatrix} \beta\sin\theta + \gamma_\perp\tau \\ -\beta\cos\theta \\ \gamma_\parallel\tau \end{pmatrix} = \begin{pmatrix} \bar{r}_\perp \\ \bar{r}_\parallel \end{pmatrix} \quad \Big|\; \Omega_B t' = \frac{\bar{V}}{r_0} t' \;\frac{r_0}{\bar{V}}\Omega_B = \omega\tau$$

Then
$$\delta^{(2)}\vec{V}_B(t) = -\frac{e}{m}\int_{-\infty}^{t} dt'\, T^{-1}(\Omega_B t')\,\delta^{(1)}\vec{E}(t') \qquad (3.48)$$

and
$$\delta^{(1)} r_\parallel = -\frac{Ze^2}{m\bar{V}^2}\gamma_\parallel \int_{-\infty}^{\sigma} d\sigma' \left(\frac{\tau_0\sigma' - 1}{(1+\sigma'^2)^{1/2}} + \tau_0\right) \qquad (3.41)$$

$$\delta^{(1)} r_\perp = \frac{r_0}{\bar{V}\omega}\begin{pmatrix} \sin\omega\tau & \cos\omega\tau \\ -\cos\omega\tau & \sin\omega\tau \end{pmatrix}\delta^{(1)}\vec{V}_{B\perp}(t) = \frac{Ze^2}{m\bar{V}^2\omega}\int_{-\infty}^{\sigma}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\begin{pmatrix} \beta\cos\theta \\ \beta\sin\theta + \gamma_\perp\tau' \end{pmatrix} \qquad (3.42)$$

and

$$\delta^{(1)}\vec{V}_B(t) = -\frac{Ze^2}{m}\frac{1}{\bar{v}r_0}\int_{-\infty}^{\sigma}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\underbrace{\begin{pmatrix}\cos\omega\tau' & \sin\omega\tau' & 0\\ -\sin\omega\tau' & \cos\omega\tau' & 0\\ 0 & 0 & 1\end{pmatrix}}_{T^{-1}(\Omega_B t')}\begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau'\\ -\beta\cos\theta\\ \gamma_\parallel\tau'\end{pmatrix} \qquad (3.36)$$

or

$$\delta^{(1)}\vec{V}_{\perp B}(t) = -\frac{Ze^2}{m\bar{v}r_0}\int_{-\infty}^{\sigma}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\begin{pmatrix}\cos\omega\tau' & \sin\omega\tau'\\ -\sin\omega\tau' & \cos\omega\tau'\end{pmatrix}\begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau'\\ -\beta\cos\theta\end{pmatrix}$$

$$\Big|\; \Omega_B t' = \underbrace{\frac{r_0}{\bar{v}}\Omega_B}_{\omega}\cdot\underbrace{\frac{\bar{v}t'}{r_0}}_{\tau'} = \omega\tau'$$

Then

$$\left(\bar{r}_i\,\delta^{(1)}r_i\right)(t) = \bar{r}_{\perp k}\,\delta^{(1)}r_{\perp k} + \bar{r}_\parallel\,\delta^{(1)}r_\parallel = r_0\begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau\\ -\beta\cos\theta\end{pmatrix}^T\left[\frac{r_0}{\bar{v}\omega}\begin{pmatrix}\sin\omega\tau & \cos\omega\tau\\ -\cos\omega\tau & \sin\omega\tau\end{pmatrix}\delta^{(1)}\vec{V}_{\perp B} -\right.$$

$$\left.-\frac{Ze^2}{m\bar{v}^2\omega}\int_{-\infty}^{\sigma}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\begin{pmatrix}\beta\cos\theta\\ \beta\sin\theta+\gamma_\perp\tau'\end{pmatrix}\right] + \underbrace{r_0\gamma_\parallel\tau\cdot\left(-\frac{Ze^2}{m\bar{v}^2}\gamma_\parallel\right)\int_{-\infty}^{\sigma}d\sigma'\left(\frac{\tau_0\sigma'-1}{(1+\sigma'^2)^{1/2}}+\tau_0\right)}_{\bar{r}_\parallel\,\delta^{(1)}r_\parallel} \qquad \Big|\; \tau'=\sigma'+\tau_0 \quad (3.22)$$

$$= r_0\begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau\\ -\beta\cos\theta\end{pmatrix}^T\left[\frac{r_0}{\bar{v}\omega}\begin{pmatrix}\sin\omega\tau & \cos\omega\tau\\ -\cos\omega\tau & \sin\omega\tau\end{pmatrix}\left(-\frac{Ze^2}{m\bar{v}r_0}\right)\int_{-\infty}^{\sigma}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\begin{pmatrix}\cos\omega\tau' & \sin\omega\tau'\\ -\sin\omega\tau' & \cos\omega\tau'\end{pmatrix}\begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau'\\ -\beta\cos\theta\end{pmatrix} -\right.$$

$$\left.-\frac{Ze^2}{m\bar{v}^2\omega}\int_{-\infty}^{\sigma}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\begin{pmatrix}\beta\cos\theta\\ \beta\sin\theta+\gamma_\perp\tau'\end{pmatrix}\right] - \frac{Ze^2 r_0}{m\bar{v}^2}\gamma_\parallel^2(\sigma+\tau_0)\int_{-\infty}^{\sigma}d\sigma'\left(\frac{\sigma'\tau_0-1}{(1+\sigma'^2)^{1/2}}+\tau_0\right) = \qquad \Big|\; \tau=\sigma+\tau_0 \;(3.32)$$

$$= -\frac{Ze^2}{m}\frac{r_0}{\bar{v}^2}\left[\frac{1}{\omega}\int_{-\infty}^{\sigma}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau\\ -\beta\cos\theta\end{pmatrix}^T\left\{\begin{pmatrix}\sin\omega\tau & \cos\omega\tau\\ -\cos\omega\tau & \sin\omega\tau\end{pmatrix}\begin{pmatrix}\cos\omega\tau' & \sin\omega\tau'\\ -\sin\omega\tau' & \cos\omega\tau'\end{pmatrix}\begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau'\\ -\beta\cos\theta\end{pmatrix} + \begin{pmatrix}\beta\cos\theta\\ \beta\sin\theta+\gamma_\perp\tau'\end{pmatrix}\right\}\right.$$

$$\left.+\gamma_\parallel^2(\sigma+\tau_0)\int_{-\infty}^{\sigma}\left[\frac{\sigma'\tau_0-1}{(1+\sigma'^2)^{1/2}}+\tau_0\right]d\sigma'\right] =$$

But

$$\begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau\\ -\beta\cos\theta\end{pmatrix}^T \underbrace{\begin{pmatrix}\sin\omega\tau & \cos\omega\tau\\ -\cos\omega\tau & \sin\omega\tau\end{pmatrix}\begin{pmatrix}\cos\omega\tau' & \sin\omega\tau'\\ -\sin\omega\tau' & \cos\omega\tau'\end{pmatrix}}_{\begin{pmatrix}\sin\omega(\tau-\tau') & \cos\omega(\tau-\tau')\\ -\cos\omega(\tau-\tau') & \sin\omega(\tau-\tau')\end{pmatrix}}\begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau'\\ -\beta\cos\theta\end{pmatrix} + \underbrace{\begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau\\ -\beta\cos\theta\end{pmatrix}^T\begin{pmatrix}\beta\cos\theta\\ \beta\sin\theta+\gamma_\perp\tau'\end{pmatrix}}_{} =$$

$$\beta\cos\theta(\cancel{\beta\sin\theta}+\gamma_\perp\tau-\cancel{\beta\sin\theta}-\gamma_\perp\tau') = \beta\gamma_\perp\underbrace{(\tau-\tau')}_{=\Delta\tau}\cos\theta$$

$$= (\quad)^T\begin{pmatrix}\sin\omega\Delta\tau & \cos\omega\Delta\tau\\ -\cos\omega\Delta\tau & \sin\omega\Delta\tau\end{pmatrix}\begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau'\\ -\beta\cos\theta\end{pmatrix} + \beta\gamma\Delta\tau\cos\theta =$$

$$= (\beta\sin\theta+\gamma_\perp\tau,\ -\beta\cos\theta)\begin{pmatrix}(\beta\sin\theta+\gamma_\perp\tau')\sin\omega\Delta\tau - \beta\cos\theta\cos\omega\Delta\tau\\ -(\beta\sin\theta+\gamma_\perp\tau')\cos\omega\Delta\tau - \beta\cos\theta\sin\omega\Delta\tau\end{pmatrix} + \beta\gamma_\perp\Delta\tau\cos\theta =$$

$$= (\beta\sin\theta+\gamma_\perp\tau)\left[(\beta\sin\theta+\gamma_\perp\tau')\sin\omega\Delta\tau - \beta\cos\theta\cos\omega\Delta\tau\right] +$$

$$+ \beta\cos\theta\left[(\beta\sin\theta+\gamma_\perp\tau')\cos\omega\Delta\tau + \beta\cos\theta\sin\omega\Delta\tau\right] + \beta\gamma_\perp\Delta\tau\cos\theta =$$

$$= (\beta^2\sin^2\theta + \beta\gamma_\perp\tau\overset{\sin\theta}{} + \beta\gamma_\perp\tau'\overset{\sin\theta}{} + \gamma_\perp^2\tau\tau')\sin\omega\Delta\tau + \beta\cos\theta\left[-\cancel{\beta\sin\theta}-\gamma_\perp\tau+\cancel{\beta\sin\theta}+\gamma_\perp\tau'\right]\cos\omega\Delta\tau +$$

$$+\beta^2\cos^2\theta\sin\omega\Delta\tau + \beta\gamma_\perp\Delta\tau\cos\theta = \beta^2\sin\omega\Delta\tau + \beta\gamma_\perp(\tau+\tau')\sin\theta\sin\omega\Delta\tau +$$

$$+ \beta\gamma_\perp(\tau'-\tau)\cos\theta\cos\omega\Delta\tau + \beta\gamma_\perp\Delta\tau\cos\theta =$$

$$+\gamma_\perp^2\tau\tau'\sin\omega\Delta\tau$$

$$= \beta^2\sin\omega\Delta\tau + \beta\gamma_\perp\left[(\tau+\tau')\sin\theta\sin\omega\Delta\tau - (\tau-\tau')\cos\theta\cos\omega\Delta\tau\right] + \gamma_\perp^2\tau\tau'\sin\omega\Delta\tau + \beta\gamma_\perp\Delta\tau\cos\theta$$

Then $= \beta^2\sin\omega\Delta\tau + \beta\gamma_\perp\left[(\tau+\tau')\sin\theta\sin\omega(\tau-\tau') - (\tau-\tau')\cos\theta\cos\omega(\tau-\tau')\right] +$

$$\overline{(r_i\delta^{(1)}r_i)}(t) = -\frac{Ze^2}{m}\frac{r_0}{v^2}\left\{\frac{1}{\omega}\int_{-\infty}^{\sigma}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\left\langle\beta^2\sin\omega(\tau-\tau') + \beta\gamma_\perp\left[(\tau+\tau')\sin\theta\sin\omega(\tau-\tau') - (\tau-\tau')\cos\theta\cos\omega(\tau-\tau')\right] + \gamma_\perp^2\tau\tau'\sin\omega(\tau-\tau') + \beta\gamma_\perp(\tau-\tau')\cos\theta\right\rangle + \gamma_\parallel^2\tau\int_{-\infty}^{\sigma}\left[\frac{\sigma'\tau_0-1}{(1+\sigma'^2)^{1/2}}+\tau_0\right]d\sigma'\right\} \quad (3.49)$$

Two limits: a) $\omega \to \infty$ („tight" trajectory $\equiv$ magnetized electrons) and

b) $\omega \to 0$ („stretched" trajectory $\equiv$ non magnetized electrons)

a) Term $\frac{1}{\omega}\int\limits_{-\infty}^{\sigma}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}<\quad> \to 0$ for $\sigma \to \infty$, i.e.

$$\left(\bar{r}_i \delta^{(1)} r_i\right)\Big|_{tight}(t) = -\frac{ze^2}{m}\frac{r_0}{\bar{V}^2}\gamma_{\parallel}^2(\sigma+\tau_0)\int\limits_{-\infty}^{\sigma} d\sigma'\left[\frac{\tau_0\sigma'-1}{(1+\sigma'^2)^{1/2}}+\tau_0\right] \qquad (3.50)$$

b) (How!?):

$$\left(\bar{r}_i \delta^{(1)} r_i\right)\Big|_{stretch}(t) = -\frac{ze^2}{m}\frac{r_0}{\bar{V}^2}\left\{\int\limits_{-\infty}^{\sigma} d\sigma'\left(\frac{\tau_0\sigma'-1}{(1+\sigma'^2)^{1/2}}+\tau_0\right)\left[1-\gamma_{\parallel}^2\tau_0^2-\gamma_{\parallel}^2\tau_0\sigma-\gamma_{\parallel}^2\tau_0\sigma'+\gamma_{\perp}^2\sigma\sigma'\right]+\right.$$

$$\left.+\gamma_{\parallel}^2(\sigma+\tau_0)\int\limits_{-\infty}^{\sigma} d\sigma'\left(\frac{\tau_0\sigma'-1}{(1+\sigma'^2)^{1/2}+\tau_0}\right)\right\} \qquad (3.51)$$

Answer: below I found correct expression for this value! (see pages (24)(25))

$$\left(r_i \delta^{(1)} r_i\right)\Big|_{stretch}(t) = -\frac{ze^2 r_0}{m\bar{V}^2}\left[I_0(\tau)+\sigma I_1(\tau)\right]$$ where as usual $\tau=\sigma+\tau_0$; an 3.51 Corrected.

Expessions for $I_0(\tau)$ and $I_1(\tau)$ see on page (25)

Correct expression for this value gives correct expression for $\langle \delta^{(2)} V_z^{(s)} \rangle$ (see pages (25) – (28)).

Let's investigate „stretched case ($\omega \to 0$; index (s)).

$$\delta^{(1)}\vec{r} = -\frac{Ze^2}{m\bar{v}^2}\int_{-\infty}^{\sigma} d\sigma' T(\omega\tau') \int_{-\infty}^{\sigma'} \frac{d\sigma''}{(1+\sigma''^2)^{3/2}} T^{-1}(\omega\tau'') \frac{\vec{F}(\tau'')}{r_0} \qquad (3.40)$$

where

$$\vec{r}(\tau) = r_0(\beta\sin\theta + \gamma_\perp\tau, -\beta\cos\theta, \gamma_{\parallel}\tau)^T \qquad | \ \tau = \sigma + \tau_0 \ (3.22)$$

and

$$T(\omega\tau) = \begin{pmatrix} \cos\omega\tau & \sin\omega\tau & 0 \\ -\sin\omega\tau & \cos\omega\tau & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[\omega\to 0]{} \begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}; \quad T^{-1}(\omega\tau) = \begin{pmatrix} \cos\omega\tau & -\sin\omega\tau & 0 \\ \sin\omega\tau & \cos\omega\tau & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[\omega\to 0]{} \begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}$$

Then

$$\delta^{(1)}\vec{r}^{(s)} = -\frac{Ze^2}{m\bar{v}^2}\int_{-\infty}^{\sigma} d\sigma' \int_{-\infty}^{\sigma'} \frac{d\sigma''}{(1+\sigma''^2)^{3/2}} \begin{pmatrix} \beta\sin\theta + \gamma_\perp(\sigma''+\tau_0) \\ -\beta\cos\theta \\ \gamma_{\parallel}(\sigma''+\tau_0) \end{pmatrix}$$

or

$$\delta^{(1)} r_x^{(s)} = -\frac{Ze^2}{m\bar{v}^2}\int_{-\infty}^{\sigma} d\sigma' \int_{-\infty}^{\sigma'} \frac{\beta\sin\theta + \gamma_\perp(\sigma''+\tau_0)}{(1+\sigma''^2)^{3/2}} d\sigma'' = -\frac{Ze^2}{m\bar{v}^2}\Big[(\beta\sin\theta + \gamma_\perp\tau_0)\underbrace{\int_{-\infty}^{\sigma} d\sigma' \int_{-\infty}^{\sigma'} \frac{d\sigma''}{(1+\sigma''^2)^{3/2}}}_{I_0(\sigma)} +$$

$$+\gamma_\perp \underbrace{\int_{-\infty}^{\sigma} d\sigma' \int_{-\infty}^{\sigma'} \frac{\sigma'' d\sigma''}{(1+\sigma''^2)^{3/2}}}_{I_1(\sigma)}\Big] = -\frac{Ze^2}{m\bar{v}^2}\Big[(\beta\sin\theta + \gamma_\perp\tau_0) I_0(\sigma) + \gamma_\perp I_1(\sigma)\Big]$$

$$\delta^{(1)} r_y^{(s)} = -\frac{Ze^2}{m\bar{v}^2}\int_{-\infty}^{\sigma} d\sigma' \int_{-\infty}^{\sigma'} \frac{-\beta\cos\theta}{(1+\sigma''^2)^{3/2}} d\sigma'' = -\frac{Ze^2}{m\bar{v}^2}(-\beta\cos\theta) I_0(\sigma)$$

$$\delta^{(1)} r_z^{(s)} = -\frac{Ze^2}{m\bar{v}^2}\int_{-\infty}^{\sigma} d\sigma' \int_{-\infty}^{\sigma'} \frac{\gamma_{\parallel}(\sigma''+\tau_0)}{(1+\sigma''^2)^{3/2}} d\sigma'' = -\frac{Ze^2}{m\bar{v}^2}\gamma_{\parallel}\Big[I_1(\sigma) + \tau_0 I_0(\sigma)\Big] \longrightarrow (3.41)$$

For this reason

$$\bar{r}_i \delta^{(1)} r_i^{(s)} = \bar{r}_x \delta^{(1)} r_x^{(s)} + \bar{r}_y \delta^{(1)} r_y^{(s)} + \bar{r}_z \delta^{(1)} r_z^{(s)} =$$

$$= r_0(\beta\sin\theta+\gamma_\perp\tau)\left(-\frac{Ze^2}{m\bar{v}^2}\right)\left[(\beta\sin\theta+\gamma_\perp\tau_0)I_0+\gamma_\perp I_1\right]+r_0(-\beta\cos\theta)\left(-\frac{Ze^2}{m\bar{v}^2}\right)(-\beta\cos\theta)I_0+r_0\gamma_\parallel^2\tau\left(-\frac{Ze^2}{m\bar{v}^2}\right)\left[I_1+\tau_0 I_0\right]=$$

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$$=-\frac{Ze^2 r_0}{m\bar{v}^2}\left\{\left[(\beta\sin\theta+\gamma_\perp\tau)(\beta\sin\theta+\gamma_\perp\tau_0)+\beta^2\cos^2\theta+\gamma_\parallel^2\tau\tau_0\right]I_0+\left[\gamma_\perp(\beta\sin\theta+\gamma_\perp\tau)+\gamma_\parallel^2\tau\right]I_1\right\}=$$

$$=-\frac{Ze^2 r_0}{m\bar{v}^2}\left\{\left[\beta^2\sin^2\theta+\underbrace{\beta\gamma_\perp\sin\theta}_{-\tau_0}\cdot\tau_0+\underbrace{\beta\gamma_\perp\sin\theta}_{-\tau_0}\cdot\tau+\gamma_\perp^2\tau\tau_0+\beta^2\cos^2\theta+\gamma_\parallel^2\tau\tau_0\right]I_0+\left[\underbrace{\beta\gamma_\perp\sin\theta}_{-\tau_0}+\gamma_\perp^2\tau+\gamma_\parallel^2\tau\right]I_1\right\}=$$

$-\beta\gamma_\perp\sin\theta=\tau_0$ (3.29)

$\beta^2=1+\tau_0^2$ (3.32)

$\gamma_\perp^2+\gamma_\parallel^2=1$

$$=-\frac{Ze^2 r_0}{m\bar{v}^2}\left\{\left[1+\cancel{\tau_0^2}-\cancel{\tau_0^2}-\cancel{\tau_0\tau}+\underbrace{(\gamma_\perp^2+\gamma_\parallel^2)}_{=1}\cancel{\tau\tau_0}\right]I_0+(\tau-\tau_0)I_1\right\}\text{ ~~or~~}$$

$$\delta^{(1)}r_i^{(s)}=-\frac{Ze^2 r_0}{m\bar{v}^2}\left[I_0(\tau)+\underbrace{(\tau-\tau_0)}_{=\sigma\ (3.32)}I_1(\tau)\right]\ (3.51),\text{ where } I_n(\tau)=\int_{-\infty}^{\sigma}d\sigma'\int_{-\infty}^{\sigma'}\frac{\sigma''^n\,d\sigma''}{(1+\sigma''^2)^{3/2}}\text{ and}$$

Attention! Toepffer has another expression for (3.51) and more complicated

$$I_0(\sigma)=\int_{-\infty}^{\sigma}d\sigma'\int_{-\infty}^{\sigma'}\frac{d\sigma''}{(1+\sigma''^2)^{3/2}}=\int_{-\infty}^{\sigma}d\sigma'\left.\frac{\sigma''}{\sqrt{1+\sigma''^2}}\right|_{-\infty}^{\sigma'}=\int_{-\infty}^{\sigma}d\sigma'\left(\frac{\sigma'}{\sqrt{1+\sigma'^2}}+1\right)$$

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$$I_1(\sigma)=\int_{-\infty}^{\sigma}d\sigma'\int_{-\infty}^{\sigma'}\frac{\sigma''d\sigma''}{(1+\sigma''^2)^{3/2}}=\int_{-\infty}^{\sigma}d\sigma'\left.\left(-\frac{1}{\sqrt{1+\sigma''^2}}\right)\right|_{-\infty}^{\sigma'}=-\int_{-\infty}^{\sigma}\frac{d\sigma'}{\sqrt{1+\sigma'^2}}$$

2.271.7

and then

$$\delta^{(1)}E_z^{(s)}=\frac{Ze}{(\bar{r}_i\bar{r}_i)^{3/2}}\left[\delta^{(1)}r_z^{(s)}-3\bar{r}_z\frac{\bar{r}_j\delta^{(1)}r_j^{(s)}}{(\bar{r}_i\bar{r}_i)}\right]$$

but $\bar{r}_i\bar{r}_i=\left[(\beta\sin\theta+\gamma_\perp\tau)^2+\beta^2\cos^2\theta+\gamma_\parallel^2\tau^2\right]r_0^2=r_0^2\left[\beta^2\sin^2\theta+\underbrace{2\beta\sin\theta\gamma_\perp}_{-\tau_0}\tau+\gamma_\perp^2\tau^2+\beta^2\cos^2\theta+\right.$

$\left.+\gamma_\parallel^2\tau^2\right]=r_0^2\left[\beta^2-2\tau\tau_0+\tau^2\right]=r_0^2\left[1+\tau_0^2-2\tau\tau_0+\tau^2\right]=r_0^2\left[1+(\tau-\tau_0)^2\right]=r_0^2(1+\sigma^2)$ (3.22)

So,

$$\delta^{(1)} E_3^{(s)} = \frac{ze}{r_0^3(1+\sigma^2)^{3/2}}\left\{\left(-\frac{ze^2}{m\bar{v}^2}\right)\gamma_{\parallel}\left[I_1+\tau_0 I_0\right] - 3\cancel{r_0}\gamma_{\parallel}\tau \frac{\left(-\frac{ze^2 \cancel{r_0}}{m\bar{v}^2}\right)(I_0+\sigma I_1)}{\cancel{r_0^2}(1+\sigma^2)}\right\} =$$

$$= -\frac{z^2e^3}{mr_0^3\bar{v}^2}\gamma_{\parallel}\frac{1}{(1+\sigma^2)^{3/2}}\left\{I_1+\tau_0 I_0 - 3\tau\frac{I_0+\sigma I_1}{1+\sigma^2}\right\} = -\frac{z^2e^3}{mr_0^3\bar{v}^2}\frac{\gamma_{\parallel}}{(1+\sigma^2)^{3/2}}\left[I_1+\tau_0 I_0 - 3(\sigma+\tau_0)\frac{I_0+\sigma I_1}{1+\sigma^2}\right]$$

Then

$$\delta^{(2)}\vec{V}^{(s)}(t) = -\frac{e}{m}\int_{-\infty}^{t} dt' \, T^{-1}(\Omega t')\,\delta^{(1)}\vec{E}^{(s)} \xrightarrow[\omega=0]{} -\frac{er_0}{m\bar{v}}\int^{\sigma} d\sigma' \delta^{(1)}\vec{E}^{(s)}$$

and for z-component one has

$$\delta^{(3)} V_z^{(s)}(t) = \left(\frac{ze^2}{mr_0}\right)^2\frac{\gamma_{\parallel}}{\bar{v}^3}\int_{-\infty}^{\sigma} d\sigma' \frac{1}{(1+\sigma'^2)^{3/2}}\left[I_1(\sigma') + \tau_0 I_0(\sigma') - 3(\sigma'+\tau_0)\frac{I_0(\sigma')+\sigma' I_1(\sigma')}{1+\sigma'^2}\right]$$

The averaging over $\tau_0 = -\frac{\gamma_\perp}{\gamma_{\parallel}}\cos\psi$ $\left(\langle\,\rangle = \frac{1}{\pi}\int_0^{\pi} f(\psi)\,d\psi\right)$ means that odd powers of $\cos\psi$ give 0. So

$$\langle\delta^{(3)} V_z^{(s)}(t)\rangle = \left(\frac{ze^2}{mr_0}\right)^2\frac{\gamma_{\parallel}}{\bar{v}^3}\int_{-\infty}^{\sigma}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\left[I_1(\sigma') - 3\sigma'\frac{I_0(\sigma')+\sigma' I_1(\sigma')}{1+\sigma'^2}\right] = \left(\frac{ze^2}{mr_0}\right)^2\frac{\gamma_{\parallel}}{\bar{v}}\int_{-\infty}^{\sigma}\frac{(1+\sigma'^2)I_1 - 3\sigma'(I_0+\sigma' I_1)}{(1+\sigma'^2)^{5/2}}d\sigma'$$

$$= \left(\frac{ze^2}{mr_0}\right)^2\frac{\gamma_{\parallel}}{\bar{v}^3}\int_{-\infty}^{\sigma}\frac{d\sigma'}{(1+\sigma'^2)^{5/2}}\left[I_1(1+\sigma'^2-3\sigma'^2) - 3\sigma' I_0\right] = \left(\frac{ze^2}{mr_0}\right)^2\frac{\gamma_{\parallel}}{\bar{v}^3}\left[\int_{-\infty}^{\sigma}\frac{(1-2\sigma'^2)}{(1+\sigma'^2)^{5/2}}I_1(\sigma')\,d\sigma' - 3\int_{-\infty}^{\sigma}\frac{\sigma'}{(1+\sigma'^2)^{5/2}}I_0(\sigma')\,d\sigma'\right]$$

Total transferred velocity:

$$\langle \delta^{(3)} V_z^{(3)} \rangle = \langle \delta^3 V_z(t) \rangle \Big|_{t\to\infty} = \left(\frac{Ze^2}{mv_0'}\right)^2 \frac{\gamma_{\parallel}}{v^3} \Bigg[\underbrace{\int_{-\infty}^{\infty} \frac{1-\cancel{2}\sigma'^2}{(1+\sigma'^2)^{5/2}} I_1(\sigma') d\sigma'}_{J_1} - \underbrace{3\int_{-\infty}^{\infty} \frac{\sigma'}{(1+\sigma'^2)^{5/2}} I_0(\sigma') d\sigma'}_{J_0} \Bigg]$$

$$J_1 = \int_{-\infty}^{\infty} \frac{1-2\sigma'^2}{(1+\sigma'^2)^{5/2}} I_1(\sigma') d\sigma' = \int_{-\infty}^{\infty} \frac{1-2\sigma'^2}{(1+\sigma'^2)^{5/2}} \left(-\int_{-\infty}^{\sigma'} \frac{d\sigma''}{\sqrt{1+\sigma''^2}}\right) d\sigma' =$$

from page 25

$$= -\int_{-\infty}^{\infty} \frac{-2(1+\sigma'^2)+3}{(1+\sigma'^2)^{5/2}} d\sigma' \int_{-\infty}^{\sigma'} \frac{d\sigma''}{\sqrt{1+\sigma''^2}} = -2 \underbrace{\int_{-\infty}^{\infty} \frac{d\sigma'}{(1+\sigma'^2)^{3/2}} \int_{-\infty}^{\sigma'} \frac{d\sigma''}{\sqrt{1+\sigma'^2}}}_{J_{13}} + 3 \underbrace{\int_{-\infty}^{\infty} \frac{d\sigma'}{(1+\sigma'^2)^{5/2}} \int_{-\infty}^{\sigma'} \frac{d\sigma''}{\sqrt{1+\sigma'^2}}}_{J_{15}}$$

Will use the rule from page 8:

$$J_{13} = \int_{-\infty}^{\infty} \frac{d\sigma'}{(1+\sigma'^2)^{3/2}} \int_{-\infty}^{\sigma'} \frac{d\sigma''}{\sqrt{1+\sigma''^2}} = \int_{-\infty}^{\infty} d\left(\frac{\sigma'}{\sqrt{1+\sigma'^2}}\right) \int_{-\infty}^{\sigma'} \frac{d\sigma''}{\sqrt{1+\sigma''^2}} = \frac{\sigma}{\sqrt{1+\sigma^2}}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{\sigma'}{\sqrt{1+\sigma'^2}} \frac{1}{\sqrt{1+\sigma'^2}} d\sigma' =$$

$$= 2 - \int_{-\infty}^{\infty} \frac{\sigma' d\sigma'}{1+\sigma'^2} = 2 - \frac{1}{2} \ln(1+\sigma'^2)\Big|_{-\infty}^{\infty} = 2 - \frac{1}{2} \ln \frac{1+\infty^2}{1+(-\infty^2)} = 2$$

$$J_{15} = \int_{-\infty}^{\infty} \frac{d\sigma'}{(1+\sigma'^2)^{5/2}} \int_{-\infty}^{\sigma'} \frac{d\sigma''}{\sqrt{1+\sigma''^2}} = \int_{-\infty}^{\infty} d\left(\frac{\sigma'}{\sqrt{1+\sigma'^2}} - \frac{1}{3}\frac{\sigma'^3}{(1+\sigma'^2)^{3/2}}\right) \int_{-\infty}^{\sigma'} \frac{d\sigma''}{\sqrt{1+\sigma''^2}} =$$

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$$= \frac{\sigma}{\sqrt{1+\sigma^2}}\Big|_{-\infty}^{\infty} - \frac{1}{3}\frac{\sigma^3}{\sqrt{1+\sigma^2}^3}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \left(\frac{\sigma'}{\sqrt{1+\sigma'^2}} - \frac{1}{3}\frac{\sigma'^3}{(1+\sigma'^2)^{3/2}}\right) \frac{1}{\sqrt{1+\sigma'^2}} d\sigma' = 2 - \frac{2}{3} -$$

$$- \int_{-\infty}^{\infty} \frac{\sigma' d\sigma'}{1+\sigma'^2} + \frac{1}{3} \int_{-\infty}^{\infty} \frac{\sigma'^3 d\sigma'}{(1+\sigma'^2)^2} =$$

$$= \frac{4}{3} - \underbrace{\frac{1}{2}\ln\frac{1+\infty^2}{1+(-\infty)^2}}_{\approx 0} + \frac{1}{3}\frac{1}{2}\underbrace{\int_{-\infty}^{\infty}\frac{x\,dx}{(1+x)^2}}_{=0 \text{ because odd function}} = \frac{4}{3}$$

So,

$$\mathcal{J}_1 = -2\mathcal{J}_{13} + 3\mathcal{J}_{15} = -2\cdot 2 + 3\cdot\frac{4}{3} = 0$$

now

$$\mathcal{J}_0 = \int_{-\infty}^{\infty}\frac{\sigma'}{(1+\sigma'^2)^{5/2}}\underbrace{I_0(\sigma')}_{\text{from page (25)}}d\sigma' = \int_{-\infty}^{\infty}\frac{\sigma' d\sigma'}{(1+\sigma'^2)^{5/2}}\int_{-\infty}^{\sigma'}\left(\frac{\sigma''}{\sqrt{1+\sigma''^2}}+1\right)d\sigma'' = \text{again rule from page (8)} =$$

$$= \int_{-\infty}^{\infty} d\left(-\frac{1}{3}\frac{1}{\sqrt{1+\sigma'^2}^3}\right)\int_{-\infty}^{\sigma'}\left(\frac{\sigma''}{\sqrt{1+\sigma''^2}}+1\right)d\sigma' = \underbrace{-\frac{1}{3}\frac{1}{(1+\sigma^2)^{3/2}}\Big|_{-\infty}^{\infty}}_{\approx 0} + \frac{1}{3}\int_{-\infty}^{\infty}\frac{1}{\sqrt{1+\sigma'^2}^3}\left(\frac{\sigma'}{\sqrt{1+\sigma'^2}}+1\right)d\sigma' =$$

$$= \frac{1}{3}\left[\underbrace{\int_{-\infty}^{\infty}\frac{\sigma' d\sigma'}{(1+\sigma'^2)^2}}_{= 0 \text{ because odd function}} + \int_{-\infty}^{\infty}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\right] = \frac{1}{3}\frac{\sigma}{\sqrt{1+\sigma^2}}\Big|_{-\infty}^{\infty} = \frac{2}{3}$$

So,

$$\langle \delta^3 V_z^{(s)}\rangle = \left(\frac{Ze^2}{mv_0}\right)^2\frac{\gamma_{\parallel}}{v^3}\left(\mathcal{J}_1 - 3\mathcal{J}_0\right) = \left(\frac{Ze^2}{mv_0}\right)^2\frac{\gamma_{\parallel}}{v^3}\left(0 - 3\cdot\frac{2}{3}\right) = -\left(\frac{Ze^2}{mv_0}\right)^2\frac{2\gamma_{\parallel}}{v^3} \quad (3.54)$$

Let's investigate „tight" case ($\omega \to \infty$, index „t") for longitudinal component only!

$$\delta^{(1)} r_z(t) = -\frac{ze^2}{m\bar{v}^2}\gamma_{\parallel}\int_{-\infty}^{\sigma}\left(\frac{\tau_0\sigma' - 1}{\sqrt{1+\sigma'^2}} + \tau_0\right)d\sigma' \qquad (3.41)$$

and

$$(\bar{r}_i\,\delta^{(1)} r_i)(t) \xrightarrow[\omega\to\infty]{} = -\frac{ze^2 r_0}{m\bar{v}^2}\gamma_{\parallel}^2(\sigma+\tau_0)\int_{-\infty}^{\sigma}\left(\frac{\tau_0\sigma' - 1}{\sqrt{1+\sigma'^2}} + \tau_0\right)d\sigma' \qquad (3.50)$$

Then

$$\delta^{(2)} E_z = \frac{ze}{(\bar{r}_i\bar{r}_i)^{3/2}}\left[\delta^{(1)} r_z - 3 r_z\frac{(\bar{r}_i\,\delta^{(1)} r_i)}{\bar{r}_i\bar{r}_i}\right] \qquad (3.47)$$

where (see page 25) $\quad \bar{r}_i\bar{r}_i = r_0^2(1+\sigma^2)$, and $\quad r_z = \gamma_{\parallel} r_0\tau = r_0\gamma_{\parallel}(\sigma+\tau_0)$

So

$$\delta^{(2)} E_z(t) = \frac{ze^2}{r_0^3(1+\sigma^2)^{3/2}}\Bigg[\left(-\frac{ze^2}{m\bar{v}^2}\right)\gamma_{\parallel}\int_{-\infty}^{\sigma}\left(\frac{\tau_0\sigma' - 1}{\sqrt{1+\sigma'^2}} + \tau_0\right)d\sigma' -$$

$$-3\frac{r_0\gamma_{\parallel}(\sigma+\tau_0)}{\cancel{r_0^2}(1+\sigma^2)}\left(-\frac{ze^2\cancel{r_0}}{m\bar{v}^2}\right)\gamma_{\parallel}^2(\sigma+\tau_0)\int_{-\infty}^{\sigma}\left(\frac{\tau_0\sigma' - 1}{\sqrt{1+\sigma'^2}} + \tau_0\right)d\sigma' =$$

$$= -\frac{ze}{r_0^3(1+\sigma^2)^{3/2}}\cdot\frac{ze^2}{m\bar{v}^2}\gamma_{\parallel}\Bigg[\underbrace{\int_{-\infty}^{\sigma}\left(\frac{\tau_0\sigma' - 1}{\sqrt{1+\sigma'^2}} + \tau_0\right)d\sigma'}_{\mathcal{F}(\sigma,\tau_0)} - 3\gamma_{\parallel}^2\frac{(\sigma+\tau_0)^2}{1+\sigma^2}\underbrace{\int_{-\infty}^{\sigma}\left(\frac{\tau_0\sigma' - 1}{\sqrt{1+\sigma'^2}} + \tau_0\right)d\sigma'}_{\mathcal{F}(\sigma,\tau_0)} =$$

$$= -\frac{z^2e^3}{r_0^3(1+\sigma^2)^{3/2}}\frac{\gamma_{\parallel}}{m\bar{v}^2}\left[\mathcal{F}(\sigma,\tau_0) - 3\gamma_{\parallel}^2\frac{(\sigma+\tau_0)^2}{1+\sigma^2}\mathcal{F}(\sigma,\tau_0)\right]$$

$$\langle \tau_0^2 \mathcal{F}(\sigma,\tau_0)\rangle = \langle \int_{-\infty}^{0} \left(\frac{\tau_0^3\sigma' - \tau_0^2}{\sqrt{1+\sigma'^2}} + \tau_0^3\right) d\sigma' \rangle = -\frac{1}{2}\frac{\gamma_\perp^2}{\gamma_\parallel^2}\int_{-\infty}^{0}\frac{d\sigma'}{\sqrt{1+\sigma'^2}} \qquad (31)$$

then

$$\langle \delta^2 V_z^{(t)}(t)\rangle = \left(\frac{ze^2}{mr_0}\right)^2 \frac{\gamma_\parallel}{V^2}\int_{-\infty}^{\sigma}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\left\{-\underline{\int_{-\infty}^{\sigma'}\frac{d\sigma''}{\sqrt{1+\sigma''^2}}} - \underline{3\gamma_\parallel^2\frac{\sigma'^2}{1+\sigma'^2}\left(-\int_{-\infty}^{\sigma'}\frac{d\sigma''}{\sqrt{1+\sigma''^2}}\right)} - \right.$$

$$\left. -6\frac{\sigma'\gamma_\parallel^2}{1+\sigma'^2}\left(\frac{1}{2}\frac{\gamma_\perp^2}{\gamma_\parallel^2}\right)\left[\int_{-\infty}^{\sigma'}\frac{\sigma''d\sigma''}{\sqrt{1+\sigma''^2}} + \int_{-\infty}^{\sigma'}d\sigma''\right] - \underline{3\left(-\frac{1}{2}\frac{\gamma_\perp^2}{\gamma_\parallel^2}\right)\frac{\gamma_\parallel^2}{1+\sigma'^2}\int_{-\infty}^{\sigma'}\frac{d\sigma''}{\sqrt{1+\sigma''^2}}}\right\} =$$

$$= \left(\frac{ze^2}{mr_0}\right)^2 \frac{\gamma_\parallel}{V^3}\left\{\int_{-\infty}^{\sigma}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\left[-1 + 3\gamma_\parallel^2\frac{\sigma'^2}{1+\sigma'^2} + \frac{3\gamma_\parallel^2}{2}\frac{\gamma_\perp^2}{\gamma_\parallel^2}\frac{1}{1+\sigma'^2}\right]\int_{-\infty}^{\sigma'}\frac{d\sigma''}{\sqrt{1+\sigma''^2}} - \right.$$

$$\left. -3\gamma_\parallel^2\frac{\gamma_\perp^2}{\gamma_\parallel^2}\int_{-\infty}^{\sigma}\frac{\sigma'd\sigma'}{(1+\sigma'^2)^{5/2}}\left[\int_{-\infty}^{\sigma'}\frac{\sigma''d\sigma''}{\sqrt{1+\sigma''^2}} + \int_{-\infty}^{\sigma'}d\sigma''\right]\right\} =$$

$$= \left(\frac{ze^2}{mr_0}\right)^2 \frac{\gamma_\parallel}{V^3}\left\{\int_{-\infty}^{\sigma}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\left[-1+3\gamma_\parallel^2\frac{\sigma'^2+1-1}{1+\sigma'^2} + \frac{3}{2}\gamma_\perp^2\frac{1}{1+\sigma'^2}\right]\int_{-\infty}^{\sigma'}\frac{d\sigma''}{\sqrt{1+\sigma''^2}} - \right.$$

$$\left. -3\gamma_\perp^2\int_{-\infty}^{\sigma}\frac{\sigma'd\sigma'}{(1+\sigma'^2)^{5/2}}\left[\int_{-\infty}^{\sigma'}\frac{\sigma''d\sigma''}{\sqrt{1+\sigma''^2}} + \int_{-\infty}^{\sigma'}d\sigma''\right]\right\} =$$

$$= \left(\frac{ze^2}{mr_0}\right)^2 \frac{\gamma_\parallel}{V^3}\left\{\int_{-\infty}^{\sigma}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\left[\underbrace{(3\gamma_\parallel^2-1)}_{-3\gamma_\parallel^2\frac{1}{1+\sigma'^2}} + \frac{3}{2}\frac{\gamma_\perp^2}{1+\sigma'^2}\right]\int_{-\infty}^{\sigma'}\frac{d\sigma''}{\sqrt{1+\sigma''^2}} - 3\gamma_\perp^2\int_{-\infty}^{\sigma}\frac{\sigma'd\sigma'}{(1+\sigma'^2)^{5/2}}\left[\int_{-\infty}^{\sigma'}\frac{\sigma''d\sigma''}{\sqrt{1+\sigma''^2}} + \int_{-\infty}^{\sigma'}d\sigma''\right]\right.$$

Next step: $t = \sigma\frac{r_0}{V} \to \infty$ (it means that $\langle \delta^2 V_z^{(t)}(t\to\infty)\rangle$ is a total transferred velocity and then

$$\delta^{(2)} V_z^{(t)}(t) = -\frac{e}{m}\int_{-\infty}^{t} dt'\, \delta^{(2)} E_z(t') = \left(\frac{Ze^2}{m}\right)^2 \frac{\gamma_{\parallel}}{r_0^3 V^2}\frac{r_0}{V}\int_{-\infty}^{\sigma} \frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\Big[\mathcal{F}(\sigma',\tau_0) - \quad \Big\| \, t = \frac{r_0}{V}\sigma$$

$$-3\gamma_{\parallel}^2 \frac{(\sigma'+\tau_0)^2}{1+\sigma'^2}\mathcal{F}(\sigma',\tau_0)\Big] =$$

$$= \left(\frac{Ze^2}{m r_0}\right)^2 \frac{\gamma_{\parallel}}{V^3}\int_{-\infty}^{\sigma} \frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\left[1 - 3\gamma_{\parallel}^2\frac{(\sigma'+\tau_0)^2}{1+\sigma'^2}\right]\mathcal{F}(\sigma',\tau_0)$$

First step —

Averaging over $\tau_0 = -\frac{\gamma_\perp}{\gamma_\parallel}\cos\psi$:

$$\left.\begin{aligned}
\langle \tau_0 \rangle &= \frac{1}{\pi}\int_0^{\pi} \tau_0\, d\psi = 0 \\
\langle \tau_0^2 \rangle &= \frac{1}{\pi}\int_0^{\pi} \tau_0^2\, d\psi = \frac{\gamma_\perp^2}{\gamma_\parallel^2}\frac{1}{\pi}\int_0^{\pi}\cos^2\psi\, d\psi = \frac{1}{2}\frac{\gamma_\perp^2}{\gamma_\parallel^2} \\
\langle \tau_0^3 \rangle &= \frac{1}{\pi}\int_0^{\pi} \tau_0^3\, d\psi = -\frac{\gamma_\perp^3}{\gamma_\parallel^3}\frac{1}{\pi}\int_0^{\pi}\cos^3\psi\, d\psi = 0
\end{aligned}\right\} \quad (3.52)$$

Then

$$\langle \delta^2 v_z^{(t)}(t)\rangle = \left(\frac{Ze^2}{m r_0}\right)^2\frac{\gamma_\parallel}{V^3}\int_{-\infty}^{\sigma}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\Big[\langle \mathcal{F}(\sigma',\tau_0)\rangle - 3\gamma_\parallel^2 \frac{\sigma'^2\langle\mathcal{F}(\sigma',\tau_0)\rangle + 2\sigma'\langle\tau_0\mathcal{F}(\sigma',\tau_0)\rangle + \langle\tau_0^2\mathcal{F}(\sigma',\tau_0)\rangle}{1+\sigma'^2}\Big] =$$

but

$$\langle \mathcal{F}(\sigma,\tau_0)\rangle = \left\langle \int_{-\infty}^{\sigma}\left(\frac{\tau_0\sigma'-1}{\sqrt{1+\sigma'^2}} + \tau_0\right)d\sigma'\right\rangle = -\int_{-\infty}^{\sigma}\frac{d\sigma'}{\sqrt{1+\sigma'^2}}$$

$$\langle \tau_0\mathcal{F}(\sigma,\tau_0)\rangle = \left\langle \int^{\sigma}\left(\frac{\tau_0^2\sigma'-\tau_0}{\sqrt{1+\sigma'^2}} + \tau_0^2\right)d\sigma'\right\rangle = \frac{1}{2}\frac{\gamma_\perp^2}{\gamma^2}\int^{\sigma}\left(\frac{\sigma'}{\sqrt{1+\sigma'^2}} + 1\right)d\sigma'$$

$$\langle \delta^2 V_z^{(t)} \rangle = \left(\frac{Ze^2}{m v_0}\right)^2 \frac{\gamma_{\parallel}}{v^3}\Bigg[(3\gamma_\parallel^2-1)\underbrace{\int_{-\infty}^{\infty}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\int_{-\infty}^{\sigma'}\frac{d\sigma''}{\sqrt{1+\sigma''^2}}}_{J_1} + \frac{3}{2}\gamma_\perp^2 \underbrace{\int_{-\infty}^{\infty}\frac{d\sigma'}{(1+\sigma'^2)^{5/2}}\int_{-\infty}^{\sigma'}\frac{d\sigma''}{\sqrt{1+\sigma''^2}}}_{J_2} -$$

$$- 3\gamma_\parallel^2 \underbrace{\int_{-\infty}^{\infty}\frac{d\sigma'}{(1+\sigma'^2)^{5/2}}\int_{-\infty}^{\sigma'}\frac{d\sigma''}{\sqrt{1+\sigma''^2}}}_{J_2} - 3\gamma_\perp^2\underbrace{\int_{-\infty}^{\infty}\frac{\sigma' d\sigma'}{(1+\sigma'^2)^{5/2}}\int_{-\infty}^{\sigma'}\frac{\sigma'' d\sigma''}{\sqrt{1+\sigma''^2}}}_{J_3} - 3\gamma_\perp^2 \underbrace{\int_{-\infty}^{\infty}\frac{\sigma' d\sigma'}{(1+\sigma'^2)^{5/2}}\int_{-\infty}^{\sigma'} d\sigma''}_{J_4}\Bigg]$$

$$J_1 = \int_{-\infty}^{\infty}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\int_{-\infty}^{\sigma'}\frac{d\sigma''}{\sqrt{1+\sigma''^2}} = (\text{see page } 27) = J_{13} = 2$$

$$J_2 = \int_{-\infty}^{\infty}\frac{d\sigma'}{(1+\sigma'^2)^{5/2}}\int_{-\infty}^{\sigma'}\frac{d\sigma''}{\sqrt{1+\sigma''^2}} = (\text{see page } 28) = J_{15} = \frac{4}{3}$$

$$J_3 = \int_{-\infty}^{\infty}\frac{\sigma' d\sigma'}{(1+\sigma'^2)^{5/2}}\int_{-\infty}^{\sigma'}\frac{\sigma'' d\sigma''}{\sqrt{1+\sigma''^2}} = \int_{-\infty}^{\infty} d\left(\frac{-1}{3(1+\sigma'^2)^{3/2}}\right)\int_{-\infty}^{\sigma'}\frac{\sigma'' d\sigma''}{\sqrt{1+\sigma''^2}} = (\text{rule from page } 8) =$$

$$= -\frac{1}{3(1+\sigma^2)^{3/2}}\Bigg|_{-\infty}^{\infty} + \frac{1}{3}\underbrace{\int_{-\infty}^{\infty}\frac{\sigma' d\sigma'}{(1+\sigma'^2)^2}}_{=0 \text{ because odd function}} = -\frac{1}{3}\left[\frac{1}{(1+\infty^2)^{3/2}} - \frac{1}{(1+(-\infty)^2)^{3/2}}\right] = 0$$

$$J_4 = \int_{-\infty}^{\infty}\frac{\sigma' d\sigma'}{(1+\sigma'^2)^{5/2}}\int_{-\infty}^{\sigma'} d\sigma'' = \underbrace{-\frac{1}{3(1+\sigma^2)^{3/2}}\Bigg|_{-\infty}^{\infty}}_{=0} + \frac{1}{3}\int_{-\infty}^{\infty}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}} = \frac{1}{3}\frac{\sigma}{\sqrt{1+\sigma^2}}\Bigg|_{-\infty}^{\infty} = \frac{2}{3}$$

So, finally:

$$\langle \delta^2 V_z^{(t)} \rangle = \left(\frac{Ze^2}{m v_0}\right)^2 \frac{\gamma_\parallel}{v^3}\Big[\underbrace{(3\gamma_\parallel^2 - 1)\cdot 2 + \cancel{\frac{3}{2}\gamma_\perp^2\cdot\frac{4}{3}} - 0 - \cancel{3\gamma_\perp^2\,\frac{2}{3}} - 3\gamma_\parallel^2\cdot\frac{4}{3}}_{= 6\gamma_\parallel^2 - 2 - 4\gamma_\parallel^2 = 2\gamma_\parallel^2 - 2 = 2(\gamma_\parallel^2 - 1) = -2\gamma_\perp^2}\Big] = -\left(\frac{Ze^2}{m v_0}\right)^2 \frac{2\gamma_\perp^2\gamma_\parallel}{v^3} \qquad (3.53)$$