Checking of

Toepffer. Scattering of Magnetized Electrons with Ions

Phys. Rev., A 66 (2002) 02274

$$\delta^{(1)}\vec{V}_{\beta}(t) = -\frac{ze^2}{m}\int_{-\infty}^{t}\frac{dt'}{\bar{r}^3(t')}\,T^{-1}(\Omega t')\cdot\vec{r}(t')$$

using (3.34)

From (3.35)

$$\bar{r}(t) = \left[r_0^2 + \bar{v}^2(t-t_0)^2\right]^{1/2} =$$

$$= r_0\left[1 + \frac{\bar{v}^2(t-t_0)^2}{r_0^2}\right]^{1/2} =$$

$$= r_0\left[1 + (\tau-\tau_0)^2\right]^{1/2}$$

From (3.10) in the limit $R \to 0$: (1)

$$\vec{r}(t) = \begin{pmatrix} b\sin\theta - v_{i\perp}t \\ -b\cos\theta \\ (v_{e\parallel} - v_{i\parallel})t \end{pmatrix} = r_0\begin{pmatrix} \frac{b}{r_0}\sin\theta - \frac{v_{i\perp}}{\bar{v}}\frac{\bar{v}t}{r_0} \\ -\frac{b}{r_0}\cos\theta \\ \frac{v_{e\parallel}-v_{i\parallel}}{\bar{v}}\frac{\bar{v}t}{r_0} \end{pmatrix}.$$

$$\boxed{\tau = \frac{\bar{v}t}{r_0},\ \tau_0 = \frac{\bar{v}t_0}{r_0}}$$

Thus is (3.34)

$$= r_0\begin{pmatrix} \beta\sin\theta + \gamma_\perp\tau \\ -\beta\cos\theta \\ \gamma_\parallel\tau \end{pmatrix}$$

$$= -\frac{ze^2}{m}\frac{r_0}{\bar{v}}\int_{-\infty}^{\tau}\frac{d\tau'\ r_0}{r_0^3\left[1+(\tau'-\tau_0)\right]^{3/2}}\begin{pmatrix} \cos\frac{\Omega r_0}{\bar{v}}\tau' & \sin\frac{\Omega r_0}{\bar{v}}\tau' & 0 \\ -\sin\frac{\Omega r_0}{\bar{v}}\tau' & \cos\frac{\Omega r_0}{\bar{v}}\tau' & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \beta\sin\theta + \gamma_\perp\tau' \\ -\beta\cos\theta \\ \gamma_\parallel\tau' \end{pmatrix}$$

$$\boxed{\begin{gathered}\sigma = \tau' - \tau_0 \\ d\tau' = d\sigma\end{gathered}}$$

$$= -\frac{ze^2}{m}\frac{1}{\bar{v}r_0}\int_{-\infty}^{\sigma}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\begin{pmatrix} \cos\omega\tau' & \sin\omega\tau' & 0 \\ -\sin\omega\tau' & \cos\omega\tau' & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \beta\sin\theta + \gamma_\perp\tau' \\ -\beta\cos\theta \\ \gamma_\parallel\tau' \end{pmatrix}$$

3.36

(2)

„x": $\cos\omega\tau'(\beta\sin\theta+\gamma_\perp\tau')+\beta\cos\theta\sin\omega\tau'=$

$$=\beta[\sin\theta\cos\omega\tau'+\cos\theta\sin\omega\tau']+\gamma_\perp\tau'\cos\omega\tau'=\beta\sin(\theta+\omega\tau')+\gamma_\perp\tau'\cos\omega\tau'$$

„y": $-\beta\sin\omega\tau'(\beta\sin\theta+\gamma_\perp\tau')+\beta\cos\omega\tau'\cos\theta=$

$$=-\beta(\sin\theta\sin\omega\tau'+\cos\theta\cos\omega\tau')-\gamma_\perp\tau'\sin\omega\tau'=-\beta\cos(\omega\tau'-\theta)-\gamma_\perp\tau'\sin\omega\tau'$$

Смотрим z-компоненту от $\delta\vec{V}_B(t)$ при $t\to\infty$:

$$\left(\delta^{(1)}\vec{V}_B\right)_z\Big|_{t\to\infty}=-\frac{ze^2}{m}\frac{1}{\sqrt{v}r_0}\int_{-\infty}^{\infty}\frac{d\sigma'}{[1+\sigma'^2]^{3/2}}\gamma_\parallel\tau'=-\frac{ze^2}{m}\frac{\gamma_\parallel}{\sqrt{v}r_0}\Bigg[\underbrace{\int_{-\infty}^{\infty}\frac{\sigma'd\sigma'}{(1+\sigma'^2)^{3/2}}}_{=0\text{ из-за нечётности ф-ции}}+\tau_0\underbrace{\int_{-\infty}^{\infty}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}}_{=\frac{\sigma'}{\sqrt{1+\sigma'^2}}\Big|_{-\infty}^{\infty}=2}\Bigg]=$$

в силу чётности подынтегральной ф-ции

$$=-\frac{2ze^2}{m}\frac{\gamma_\parallel\tau_0}{\sqrt{v}r_0}\qquad \text{OK}\quad (3.37c)$$

Смотрим x-компоненту:

$$\left(\delta^{(1)}\vec{V}_B\right)_x\Big|_{t\to\infty}=-\frac{ze^2}{m}\frac{1}{\sqrt{v}r_0}\int_{-\infty}^{\infty}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\Big[\underbrace{\beta\sin(\omega\tau'+\theta)}_{\text{„1"}}+\underbrace{\gamma_\perp\tau'\cos\omega\tau'}_{\text{„2"}}\Big],\quad \tau'=\sigma'+\tau_0$$

$$\text{„2"}=\gamma_\perp\int_{-\infty}^{\infty}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}(\sigma'+\tau_0)\cos[\omega(\sigma'+\tau_0)]=\gamma_\perp\tau_0\int_{-\infty}^{\infty}\frac{\cos\omega\sigma'\cos\omega\tau_0-\sin\omega\sigma'\sin\omega\tau_0}{(1+\sigma'^2)^{3/2}}d\sigma'$$

$$=\gamma_\perp\tau_0\Bigg[\cos\omega\tau_0\int_{-\infty}^{\infty}\frac{\cos\omega x\,dx}{(1+x^2)^{3/2}}-\sin\omega\tau_0\underbrace{\int_{-\infty}^{\infty}\frac{\sin\omega x\,dx}{(1+x^2)^{3/2}}}_{=0\text{ из-за нечётности ф-ции}}\Bigg]=2\gamma_\perp\tau_0\cos\omega\tau_0\int_0^{\infty}\frac{\cos\omega x\,dx}{(1+x^2)^{3/2}}$$

(3)

$$+\gamma_\perp \int_{-\infty}^{\infty} \frac{\sigma'(\cos\omega\sigma'\cos\omega\tau_0 - \sin\omega\sigma'\sin\omega\tau_0)\,d\sigma'}{(1+\sigma'^2)^{3/2}} =$$

$$= \gamma_\perp\tau_0 \left[\cos\omega\tau_0 \int_{-\infty}^{\infty} \frac{\cos\omega x\,dx}{(1+x^2)^{3/2}} - \sin\omega\tau_0 \underbrace{\int_{-\infty}^{\infty} \frac{\sin\omega x\,dx}{(1+x^2)^{3/2}}}_{=0 \text{ из-за нечётности}}\right] +$$

$$+\gamma_\perp\left[\cos\omega\tau_0 \underbrace{\int_{-\infty}^{\infty} \frac{x\cos\omega x\,dx}{(1+x^2)^{3/2}}}_{=0 \text{ из-за нечётности}} - \sin\omega\tau_0 \int_{-\infty}^{\infty} \frac{x\sin\omega x\,dx}{(1+x^2)^{3/2}}\right] =$$

$$= 2\gamma_\perp\left[\tau_0\cos\omega\tau_0 \underbrace{\int_0^{\infty} \frac{\cos\omega x\,dx}{(1+x^2)^{3/2}}}_{I} - \sin\omega\tau_0 \underbrace{\int_0^{\infty} \frac{x\sin\omega x\,dx}{(1+x^2)^{3/2}}}_{\text{это } 3.754.3 \;=\; \omega K_0(\omega)}\right]$$

чтобы найти I найдем сначала

$$\hat{I}(a) = \int_0^{\infty} \frac{\cos\omega x\,dx}{(1+ax^2)^{1/2}} = \frac{1}{\sqrt{a}} \underbrace{\int_0^{\infty} \frac{\cos\omega x\,dx}{\sqrt{\frac{1}{a}+x^2}}}_{\text{это } 3.754.2} = \frac{1}{\sqrt{a}} K_0\left(\frac{\omega}{\sqrt{a}}\right)$$

тогда

$$I = \int_0^{\infty} \frac{\cos\omega x\,dx}{(1+x^2)^{3/2}} = -2\frac{\partial}{\partial a}\int_0^{\infty} \frac{\cos\omega x\,dx}{(1+ax^2)^{1/2}}$$

$$I(\beta) = \underbrace{\int_0^{\infty} \frac{\cos\omega x\,dx}{\sqrt{\beta+x^2}}}_{\text{это } 3.754.2} = K_0(\sqrt{\beta}\,\omega), \quad \text{тогда}$$

$$\int_0^{\infty} \frac{\cos\omega x\,dx}{\sqrt{1+x^2}^{\,3}} = -2\left(\frac{\partial}{\partial\beta}\int_0^{\infty} \frac{\cos\omega x\,dx}{\sqrt{\beta+x^2}}\right)\Bigg|_{\beta=1} = -2\frac{\partial}{\partial\beta}\left(K_0\sqrt{\beta}\,\omega\right)\Bigg|_{\beta=1} =$$

$$= -2\,\frac{1}{2}\,\frac{\omega}{\sqrt{\beta}}\,K_0'(\sqrt{\beta}\,\omega)\Big|_{\beta=1} = \underset{\uparrow\,8.486\,18}{-\omega K_0'(\omega)} = \omega K_1(\omega)$$

Итак,

$$„2" = 2\gamma_\perp\left[\omega\tau_0\cos(\omega\tau_0)\,K_1(\omega) - \omega\sin(\omega\tau_0)\,K_0(\omega)\right]$$

Теперь „1":

$$„1" = \beta\int_{-\infty}^{\infty}\frac{\sin(\omega\overset{=\sigma'+\tau_0}{\tau'}+\theta)\,d\sigma'}{(1+\sigma'^2)^{3/2}} = \beta\int_{-\infty}^{\infty}\frac{\sin(\omega\sigma'+\omega\tau_0+\theta)\,d\sigma'}{(1+\sigma'^2)^{3/2}} =$$

$$= \beta\left[\sin(\omega\tau_0+\theta)\int_{-\infty}^{\infty}\frac{\cos\omega x\,dx}{(1+x^2)^{3/2}} + \cos(\omega\tau_0+\theta)\underbrace{\int_{-\infty}^{\infty}\frac{\sin\omega x\,dx}{(1+x^2)^{3/2}}}_{=0\text{ из-за нечётности}}\right] = 2\beta\sin(\omega\tau_0+\theta)\underbrace{\int_0^{\infty}\frac{\cos\omega x\,dx}{(1+x^2)^{3/2}}}_{\text{найдено выше и равно }\omega K_1(\omega)} =$$

$$= 2\beta\omega\sin(\omega\tau_0+\theta)\,K_1(\omega)$$

Т.о. „1"+„2":

$$„1"+„2" = 2\beta\omega\sin(\omega\tau_0+\theta)\,K_1(\omega) + 2\gamma_\perp\left[\omega\tau_0\cos(\omega\tau_0)K_1(\omega) - \omega\sin(\omega\tau_0)\,K_0(\omega)\right] =$$

$$= 2\gamma_\perp\omega K_0(\omega)\sin\omega\tau_0 + 2\omega K_1(\omega)\left[\beta\sin(\omega\tau_0+\theta) \mp \gamma_\perp\tau_0\cos(\omega\tau_0)\right]$$

Имеем

$$[\;\;] = \underbrace{\beta\cos\theta}_{3.30}\sin(\omega\tau_0) + \beta\sin\theta\cos\omega\tau_0 \mp \gamma_\perp\tau_0\cos\omega\tau_0 = \sin(\omega\tau_0)\sin\psi +$$

$$+\cos(\omega\tau_0)\Big[\underbrace{\beta\sin\theta + \gamma_\perp\tau_0}_{\text{это }3.31}\Big] = \sin(\omega\tau_0)\sin\psi \mp \gamma_\parallel\cos\psi\cos(\omega\tau_0)$$

$$\left(\overset{T.o.\to}{\delta^{(1)}} \vec{V}_B\right)_x\Big|_{t\to\infty} = -\frac{Ze^2}{mVr_0}\Big[2\omega K_1(\omega)\left(\gamma_\parallel \cos\psi \cos\omega\tau_0 + \sin\psi \sin\omega\tau_0\right) \mp 2\omega K_0(\omega)\gamma_\perp \sin\omega\tau_0\Big] \quad (3.37a)$$

совпадает с Toepffer

~~$\pm$, у Toepffer'а другой знак~~

аналогично:

$$\left(\delta^{(1)} \vec{V}_B\right)_y\Big|_{t\to\infty} = -\frac{Ze^2}{mVr_0}\int_{-\infty}^{\infty}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\Big[\underbrace{-\beta\cos(\omega\tau'-\theta)}_{\text{–„3"}} \underbrace{\mp\gamma_\perp\tau'\sin\omega\tau'}_{\text{–„4"}}\Big]$$

имеем:

$$\text{„3"} = \beta\int_{-\infty}^{\infty}\frac{\cos(\omega\overset{=\sigma'\pm\tau_0}{\tau'}-\theta)\,d\sigma'}{(1+\sigma'^2)^{3/2}} = \beta\Big[\cos(\omega\tau_0-\theta)\int_{-\infty}^{\infty}\frac{\cos\omega x\,dx}{(1+x^2)^{3/2}} + \sin(\omega\tau_0-\theta)\underbrace{\int_{-\infty}^{\infty}\frac{\sin\omega x\,dx}{(1+x^2)^{3/2}}}_{=0 \text{ из-за нечётности}}\Big] =$$

$$= 2\beta\cos(\omega\tau_0-\theta)\underbrace{\int_0^{\infty}\frac{\cos\omega x\,dx}{(1+x^2)^{3/2}}}_{\text{см. ранее } =\omega K_1(\omega)} = \; = 2\beta\omega K_1(\omega)\cos(\omega\tau_0-\theta)$$

$$\text{„4"} = \gamma_\perp\int_{-\infty}^{\infty}\frac{\overset{=\sigma'\pm\tau_0}{\tau'}\sin\omega\tau'\,d\sigma'}{(1+\sigma'^2)^{3/2}} = \gamma_\perp\tau_0\int_{-\infty}^{\infty}\frac{\sin(\omega\sigma'\pm\omega\tau_0)\,d\sigma'}{(1+\sigma'^2)^{3/2}} + \gamma_\perp\int_{-\infty}^{\infty}\frac{\sigma'\sin(\omega\sigma'\pm\omega\tau_0)\,d\sigma'}{(1+\sigma'^2)^{3/2}} =$$

$$= \gamma_\perp\tau_0\Big[\cos\omega\tau_0\underbrace{\int_{-\infty}^{\infty}\frac{\sin\omega x\,dx}{(1+x^2)^{3/2}}}_{=0 \text{ из-за нечётности}} + \sin\omega\tau_0\int_{-\infty}^{\infty}\frac{\cos\omega x\,dx}{(1+x^2)^{3/2}}\Big] + \gamma_\perp\Big[\cos\omega\tau_0\int_{-\infty}^{\infty}\frac{x\sin\omega x\,dx}{(1+x^2)^{3/2}} +$$

$$+ \sin\omega\tau_0\underbrace{\int_{-\infty}^{\infty}\frac{x\cos\omega x\,dx}{(1+x^2)^{3/2}}}_{=0 \text{ из-за нечётности}}\Big] = 2\gamma_\perp\Big[\tau_0\sin\omega\tau_0\int_0^{\infty}\frac{\cos\omega x\,dx}{(1+x^2)^{3/2}} + \cos\omega\tau_0\int_0^{\infty}\frac{x\sin\omega x\,dx}{(1+x^2)^{3/2}}\Big] =$$

$$= 2\gamma_\perp\Big[\tau_0\,\omega K_1(\omega)\sin\omega\tau_0 + \cos\omega\tau_0\,\omega K_0(\omega)\Big] = 2\omega\gamma_\perp\Big[\tau_0 K_1(\omega)\sin\omega\tau_0 + K_0(\omega)\cos\omega\tau_0\Big]$$

Т.о.

$$\left(\delta^{(1)}\vec{V}_B\right)_y\Big|_{t\to\infty} = -\frac{Ze^2}{mVr_0}\Big[-2\beta\omega K_1(\omega)\cos(\omega\tau_0-\theta) \mp 2\omega\gamma_\perp\tau_0 K_1(\omega)\sin\omega\tau_0 \mp$$

$$\mp 2\omega \gamma_\perp K_0 \cos\omega\tau_0\Big] =$$

$$= -\frac{Ze^2}{mVr_0}\Big[-2\omega K_0(\omega)\gamma_\perp\cos\omega\tau_0 + 2\omega K_1(\omega)\underbrace{\big[-\beta\cos(\omega\tau_0-\theta)\mp\gamma_\perp\tau_0\sin\omega\tau_0\big]}_{\beta}\Big]$$

$$[\ ] = -\beta\cos\theta\cos\omega\tau_0 - \beta\sin\theta\sin\omega\tau_0 - \gamma_\perp\tau_0\sin\omega\tau_0 =$$

$$= \underbrace{-\beta\cos\theta}_{(3.30)}\cos\omega\tau_0 - \underset{3.31}{(\beta\sin\theta+\gamma_\perp\tau_0)}\sin\omega\tau_0 = +\sin\Psi\cos\omega\tau_0 - \gamma_\parallel\cos\Psi\sin\omega\tau_0$$

Так что окончательно

$$\left(\delta^{(1)}\vec{V}_B\right)_y\Big|_{t\to\infty} = -\frac{Ze^2}{mVr_0}\Big[+2\omega K_1(\omega)\left(-\gamma_\parallel\cos\Psi\sin\omega\tau_0+\sin\Psi\cos\omega\tau_0\right) - 2\omega K_0(\omega)\gamma_\perp\cos\omega\tau_0\Big] \quad (3.37b)$$

$$\delta^{(1)}r_\parallel(t) = -\frac{Ze^2}{mV^2}\int_{-\infty}^{\sigma}d\sigma'\int_{-\infty}^{\sigma'}\frac{\gamma_\parallel(\sigma''+\tau_0)}{(1+\sigma''^2)^{3/2}}d\sigma'' = -\frac{Ze^2}{mV^2}\gamma''\int_{-\infty}^{\sigma}d\sigma'\Big[\int_{-\infty}^{\sigma'}\frac{\sigma''d\sigma''}{(1+\sigma''^2)^{3/2}}+\tau_0\int_{-\infty}^{\sigma'}\frac{d\sigma''}{(1+\sigma''^2)^{3/2}}\Big]$$

это (3.41)

$$= -\frac{Ze^2}{mV^2}\gamma''\int_{-\infty}^{\sigma}d\sigma'\Big[-\frac{1}{\sqrt{1+\sigma''^2}}\Big|_{-\infty}^{\sigma'}+\tau_0\frac{\sigma''}{\sqrt{1+\sigma''^2}}\Big|_{-\infty}^{\sigma'}\Big] = -\frac{Ze^2}{mV^2}\gamma''\int^{\sigma}d\sigma'\Big\{-\frac{\sigma'}{\sqrt{1+\sigma'^2}}+\tau_0\Big(\frac{\sigma'}{\sqrt{1+\sigma'^2}}+$$

Для будущего:

$$\int \frac{dx}{\sqrt{a+bx+cx^2}^3} = \frac{2(2cx+b)}{\Delta\sqrt{}} \quad (\Delta = 4ac-b^2) \longrightarrow \int \frac{dx}{\sqrt{1+x^2}^3} = \frac{x}{\sqrt{1+x^2}}$$

$$\int \frac{x\,dx}{\sqrt{}^3} = -\frac{2(2a+bx)}{\Delta\sqrt{}} \longrightarrow \int \frac{x\,dx}{\sqrt{1+x^2}^3} = -\frac{1}{\sqrt{1+x^2}}$$

$$\int_0^\infty \frac{x\sin\omega x\,dx}{\sqrt{1+x^2}^3} = \omega K_0(\omega) \qquad \int_0^\infty \frac{\cos\omega x\,dx}{\sqrt{\beta+x^2}} = K_0(\sqrt{\beta}\omega) \longrightarrow \int_0^\infty \frac{\cos\omega x\,dx}{\sqrt{\beta+x^2}^3} = -\frac{\omega}{\sqrt{\beta}}K_0'(\sqrt{\beta}\omega) = \frac{\omega}{\sqrt{\beta}}K_1(\sqrt{\beta}\omega)$$

3.754.3 3.754.2

$$\delta\vec{r}_\perp^{(1)}(t) = -\frac{ze^2}{mv^2}\int_{-\infty}^{\sigma} d\sigma' \begin{pmatrix} \cos\omega\tau' & -\sin\omega\tau' \\ \sin\omega\tau' & \cos\omega\tau' \end{pmatrix} \int_{-\infty}^{\sigma'} \frac{d\sigma''}{(1+\sigma''^2)^{3/2}} \begin{pmatrix} \cos\omega\tau'' & \sin\omega\tau'' \\ -\sin\omega\tau'' & \cos\omega\tau'' \end{pmatrix} \begin{pmatrix} \beta\sin\theta + \gamma_\perp\tau'' \\ -\beta\cos\theta \end{pmatrix} \quad (3.40)$$

(3.22)

Это лишнее

$$\tau = \sigma + \tau_0$$

Тогда

$$\cos\omega\tau' = \cos\omega(\sigma'+\tau_0) = \cos\omega\tau_0\cos\omega\sigma' - \sin\omega\tau_0\sin\omega\sigma'$$ и аналогично для σ

$$\sin\omega\tau' = \sin(\sigma'+\tau_0) = \cos\omega\tau_0\sin\omega\sigma' + \sin\omega\tau_0\cos\omega\tau'$$

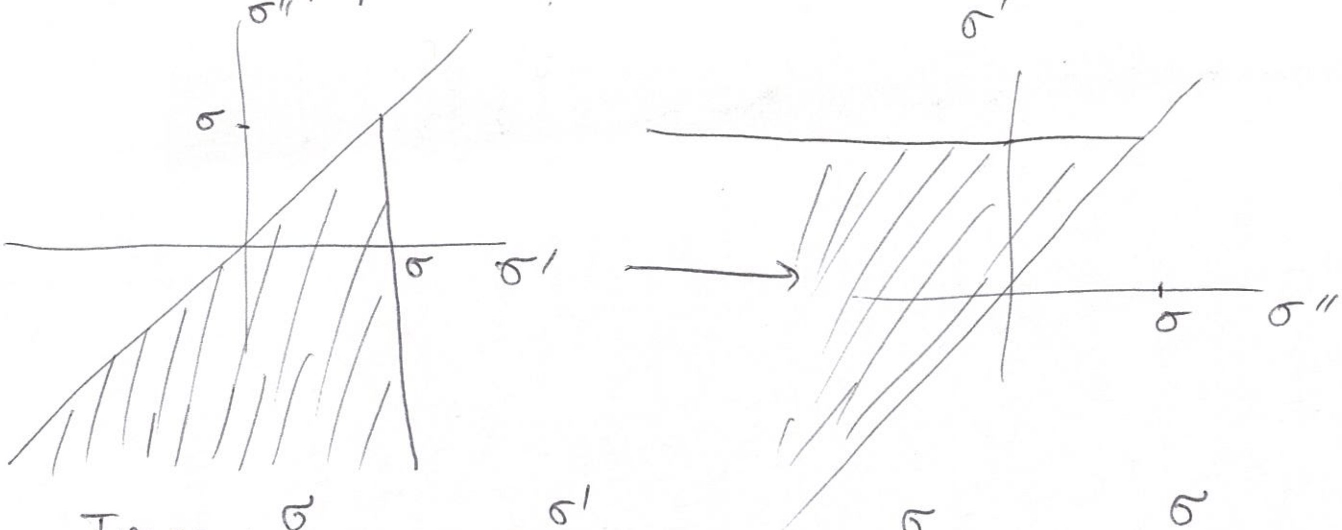
$\cancel{\delta\vec{r}_\perp^{(1)}(t)}$

$$\delta\vec{r}_\perp^{(1)}(t) = -\frac{ze^2}{mv^2}\int_{-\infty}^{\sigma} d\sigma' \int_{-\infty}^{\sigma'} \frac{d\sigma''}{(1+\sigma''^2)^{3/2}} \underbrace{\begin{pmatrix} \cos\omega\tau' & -\sin\omega\tau' \\ \sin\omega\tau' & \cos\omega\tau' \end{pmatrix}\begin{pmatrix} \cos\omega\tau'' & \sin\omega\tau'' \\ -\sin\omega\tau'' & \cos\omega\tau'' \end{pmatrix}\begin{pmatrix} \beta\sin\theta + \gamma_\perp\tau'' \\ -\beta\cos\theta \end{pmatrix}}_{M} =$$

$$M = \begin{pmatrix} \cos\omega(\tau'-\tau'') & \sin\omega(\tau''-\tau') \\ \sin\omega(\tau'-\tau'') & \cos\omega(\tau'-\tau'') \end{pmatrix}\begin{pmatrix} \beta\sin\theta - \gamma_\perp\tau'' \\ -\beta\cos\theta \end{pmatrix} = \begin{pmatrix} \cos\omega(\sigma'-\sigma'')(\beta\sin\theta - \gamma_\perp\tau'') + \sin\omega(\sigma'-\sigma'')\beta\cos\theta \\ \sin\omega(\sigma'-\sigma'')(\beta\sin\theta - \gamma_\perp\tau'') - \cos\omega(\sigma'-\sigma'')\beta\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \beta\sin[\omega(\sigma'-\sigma'')+\theta] - \gamma_\perp(\sigma''+\tau_0)\cos\omega(\sigma'-\sigma'') \\ -\beta\cos[\omega(\sigma'-\sigma'')+\theta] - \gamma_\perp(\sigma''+\tau_0)\sin\omega(\sigma'-\sigma'') \end{pmatrix}$$

Область интегрирования



Тогда
$$\int_{-\infty}^{\sigma} f_1(\sigma')d\sigma' \int_{-\infty}^{\sigma'} f_2(\sigma'')d\sigma'' = \int_{-\infty}^{\sigma} d\sigma'' f_2(\sigma'') \int_{\sigma''}^{\sigma} f_1(\sigma')d\sigma' \underset{\text{переобозначение}}{=} \int_{-\infty}^{\sigma} f_2(\sigma')d\sigma' \int_{\sigma'}^{\sigma} f_1(\sigma'')d\sigma''$$

В частности, если $f_1(\sigma) = \frac{dg(\sigma)}{d\sigma}$

то
$$\int_{-\infty}^{\sigma} \frac{dg(\sigma')}{d\sigma'}d\sigma' \int_{-\infty}^{\sigma'} f_2(\sigma'')d\sigma'' = \int_{-\infty}^{\sigma} f_2(\sigma')d\sigma' \int_{\sigma'}^{\sigma} \frac{dg_1(\sigma'')}{d\sigma''}d\sigma'' =$$

$$= \int_{-\infty}^{\sigma} f_2(\sigma')d\sigma' \cdot g(\sigma'')\Big|_{\sigma'}^{\sigma} = g(\sigma)\int_{-\infty}^{\sigma} f_2(\sigma')d\sigma' - \int_{-\infty}^{\sigma} g(\sigma')f_2(\sigma')d\sigma'$$

| Это из Appendix

но

$$\begin{pmatrix} \cos\omega\tau' & -\sin\omega\tau' \\ \sin\omega\tau' & \cos\omega\tau' \end{pmatrix} = \frac{1}{\omega}\frac{d}{d\tau'}\begin{pmatrix} \sin\omega\tau' & \cos\omega\tau' \\ -\cos\omega\tau' & \sin\omega\tau' \end{pmatrix}$$

Тогда в соответствии с расчетом из со стр 8 при перестановке порядка интегрирования имеем

$$\delta^{(1)}\vec{r}_\perp(t) = -\frac{ze^2}{m\bar{v}^2}\int_{-\infty}^{\sigma}\frac{d\sigma'}{\omega}\frac{d}{d\sigma'}\begin{pmatrix} \sin\omega\tau' & \cos\omega\tau' \\ -\cos\omega\tau' & \sin\omega\tau' \end{pmatrix}\int_{-\infty}^{\sigma'}\frac{d\sigma''}{(1+\sigma''^2)^{3/2}}\begin{pmatrix} \cos\omega\tau'' & \sin\omega\tau'' \\ -\sin\omega\tau'' & \cos\omega\tau'' \end{pmatrix}\begin{pmatrix} \beta\sin\theta+\gamma_\perp\tau'' \\ -\beta\cos\theta \end{pmatrix} =$$

$\boxed{\sigma = \frac{\bar{v}t}{r_0}}$ (3.21)

$$= -\frac{1}{\omega}\frac{ze^2}{m\bar{v}^2}\begin{pmatrix} \sin\omega\tau & \cos\omega\tau \\ -\cos\omega\tau & \sin\omega\tau \end{pmatrix}\underbrace{\int_{-\infty}^{\sigma}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}T(\omega\tau')\begin{pmatrix} \beta\sin\theta+\gamma_\perp\tau' \\ -\beta\cos\theta \end{pmatrix}}_{= r_0\delta^{(1)}\vec{V}_{B\perp}(t) \text{ from } (3.36)} - \frac{1}{\omega}\left(-\frac{ze^2}{m\bar{v}^2}\right)\times$$

$\boxed{\omega = \frac{r_0\Omega_\perp}{\bar{v}}}$ (3.23)

$$\times\cancel{\int_{-\infty}^{\sigma'}\begin{pmatrix} \cos\omega\tau' & \sin\omega\tau' \\ -\sin\omega\tau' & \cos\omega\tau' \end{pmatrix}} \times \int_{-\infty}^{\sigma}\underbrace{\begin{pmatrix} \sin\omega\tau' & \cos\omega\tau' \\ -\cos\omega\tau' & \sin\omega\tau' \end{pmatrix}d\sigma'\frac{1}{(1+\sigma'^2)^{3/2}}\begin{pmatrix} \cos\omega\tau' & \sin\omega\tau' \\ -\sin\omega\tau' & \cos\omega\tau' \end{pmatrix}}\begin{pmatrix} \beta\sin\theta+\gamma_\perp\tau' \\ -\beta\cos\theta \end{pmatrix}$$

↓ ~~перемножение матриц!~~

$$\begin{pmatrix} \sin\omega\tau' & \cos\omega\tau' \\ -\cos\omega\tau' & \sin\omega\tau' \end{pmatrix}\begin{pmatrix} \cos\omega\tau' & \sin\omega\tau' \\ -\sin\omega\tau' & \cos\omega\tau' \end{pmatrix}\begin{pmatrix} \beta\sin\theta+\gamma_\perp\tau' \\ -\beta\cos\theta \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \beta\sin\theta+\gamma_\perp\tau' \\ -\beta\cos\theta \end{pmatrix} = \begin{pmatrix} -\beta\cos\theta \\ -(\beta\sin\theta+\gamma_\perp\tau') \end{pmatrix} =$$

$$= -\begin{pmatrix} \beta\cos\theta \\ \beta\sin\theta+\gamma_\perp\tau' \end{pmatrix}$$

$$= \frac{r_0}{\omega\bar{v}}\begin{pmatrix} \sin\omega\tau & \cos\omega\tau \\ -\cos\omega\tau & \sin\omega\tau \end{pmatrix}\delta^{(1)}\vec{V}_{B\perp}(t) - \frac{1}{\omega}\left(-\frac{ze^2}{m\bar{v}^2}\right)\int_{-\infty}^{\sigma} -\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\begin{pmatrix} \beta\cos\theta \\ \beta\sin\theta+\gamma_\perp\tau' \end{pmatrix} =$$

$$= \frac{r_0}{\omega\bar{v}}\begin{pmatrix} \sin\omega\tau & \cos\omega\tau \\ -\cos\omega\tau & \sin\omega\tau \end{pmatrix}\delta^{(1)}\vec{V}_{B\perp}(t) - \frac{ze^2}{m\bar{v}^2}\int_{-\infty}^{\sigma}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\begin{pmatrix} \beta\cos\theta \\ \beta\sin\theta+\gamma_\perp\tau' \end{pmatrix}$$

это (3.42)

So,

$$\delta^{(1)}\vec{r}_{\perp}(t) = \frac{r_0}{\omega \bar{v}}\,\delta^{(1)}\vec{v}_{B\perp}(t)\begin{pmatrix} \sin\omega\tau & \cos\omega\tau \\ -\cos\omega\tau & \sin\omega\tau \end{pmatrix} - \frac{Ze^2}{m\bar{v}^2}\int_{-\infty}^{\bar{v}t/r_0} \frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\begin{pmatrix} \beta\cos\theta \\ \beta\sin\theta + \gamma_\perp \tau' \end{pmatrix} \qquad (3.29)$$

$$\boxed{\tau' = \frac{vt'}{r_0}\ (3.21)\ ; \quad \sigma = \tau' - \tau_0\ (3.22) \qquad d\sigma = d\tau'}$$

Analogously: (from 3.40):

$$\delta^{(1)} r_{\parallel}(t) = -\frac{Ze^2}{m\bar{v}^2}\int_{-\infty}^{\bar{v}t/r_0} d\sigma'\, T(\omega\tau')\int_{-\infty}^{\sigma'} \frac{d\sigma''}{(1+\sigma''^2)^{3/2}}\, T^{-1}(\omega\tau'')\,\gamma_\parallel \tau'' \text{ (more accuracy!)} =$$

$$= -\frac{Ze^2}{m\bar{v}^2}\int_0^{\bar{v}t/r_0} d\sigma' \begin{pmatrix} \cos\omega\tau' & -\sin\omega\tau' & 0 \\ \sin\omega\tau' & \cos\omega\tau' & 0 \\ 0 & 0 & 1 \end{pmatrix}\int^{\sigma'} \frac{d\sigma''}{(1+\sigma''^2)^{3/2}}\begin{pmatrix} \cos\omega\tau'' & \sin\omega\tau'' & 0 \\ -\sin\omega\tau'' & \cos\omega\tau'' & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \beta\sin\theta + \gamma_\perp\tau'' \\ -\beta\cos\theta \\ \gamma_\parallel \tau'' \end{pmatrix} =$$

But for $\delta^{(1)} r_{\parallel}(t)$ it is necessary to calculate only 3rd component of the product ~~of~~ the matrices:

$$\begin{pmatrix} \cos\omega\tau' & -\sin\omega\tau' & 0 \\ \sin\omega\tau' & \cos\omega\tau' & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\omega\tau'' & \sin\omega\tau'' & 0 \\ -\sin\omega\tau'' & \cos\omega\tau'' & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \beta\sin\theta + \gamma_\perp\tau'' \\ -\beta\cos\theta \\ \gamma_\parallel\tau'' \end{pmatrix} = \begin{pmatrix} \cos\omega(\tau'+\tau'') & -\sin\omega(\tau'-\tau'') & 0 \\ \sin\omega(\tau'-\tau'') & \cos\omega(\tau'+\tau'') & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \beta\sin\theta + \gamma_\perp\tau'' \\ -\beta\cos\theta \\ \gamma_\parallel\tau'' \end{pmatrix} = \begin{pmatrix} \dots & \dots \\ \dots & \dots \\ & \gamma_\parallel\tau'' \end{pmatrix} \longrightarrow \delta r^{(1)}_{\parallel}(t) = -\frac{Ze^2}{m\bar{v}^2}\int_{-\infty}^{\sigma} d\sigma' \int_{-\infty}^{\sigma'} \frac{\gamma_\parallel \tau''}{(1+\sigma''^2)^{3/2}}\, d\sigma'' = \Big|\ \tau'' = \sigma'' + \tau_0\ (3.22)$$

$$= -\frac{Ze^2}{m\bar{v}^2}\gamma_\parallel \int^{\sigma} d\sigma' \int_{-\infty}^{\sigma'} \frac{(\sigma''+\tau_0)\, d\sigma''}{(1+\sigma''^2)^{3/2}} =$$

$$\int \frac{\sigma'' d\sigma''}{(1+\sigma''^2)^{3/2}} = (2.271.7) = -\frac{1}{\sqrt{1+\sigma''^2}}$$

$$\int \frac{d\sigma''}{(1+\sigma''^2)^{3/2}} = (2.271.5) = \frac{\sigma''}{\sqrt{1+\sigma''^2}}$$

9''

$$= -\frac{Ze^2}{m\bar{v}^2}\gamma_{\parallel}\int_{-\infty}^{\sigma} d\sigma'\left[-\frac{1}{\sqrt{1+\sigma''^2}}\Bigg|_{-\infty}^{\sigma'} + \tau_0\frac{\sigma''}{\sqrt{1+\sigma''^2}}\Bigg|_{-\infty}^{\sigma'}\right] = -\frac{Ze^2}{m\bar{v}^2}\gamma_{\parallel}\int_{-\infty}^{\sigma} d\sigma'\left[-\frac{1}{\sqrt{1+\sigma'^2}} + \frac{\tau_0\sigma'}{\sqrt{1+\sigma'^2}} - \tau_0(-1)\right] =$$

so,

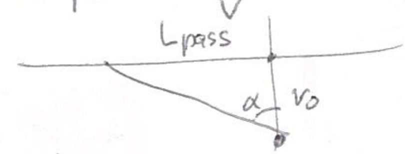
$$\gamma_{\parallel} v_{\parallel}(t) = -\frac{Ze^2}{m\bar{v}^2}\gamma_{\parallel}\int_{-\infty}^{\sigma} d\sigma'\left[\frac{\tau_0\sigma'-1}{\sqrt{1+\sigma'^2}} + \tau_0\right] =$$

| This is (3.41)

$$= -\frac{Ze^2}{m\bar{v}^2}\gamma_{\parallel}\int_{-\infty}^{\sigma} d\sigma'\left(\underset{2.271.7}{\frac{\tau_0\sigma'}{\sqrt{1+\sigma'^2}}} - \underset{2.271.3}{\frac{1}{\sqrt{1+\sigma'^2}}} + \tau_0\right) = -\frac{Ze^2}{m\bar{v}^2}\gamma_{\parallel}\left[\tau_0\sqrt{1+\sigma'^2}\Big|_{-\infty}^{\sigma} - \ln\left(\sigma'+\sqrt{1+\sigma'^2}\right)\Big|_{-\infty}^{\sigma} + \tau_0\sigma'\Big|_{-\infty}^{\sigma}\right] =$$

$$= -\frac{Ze^2}{m\bar{v}^2}\gamma_{\parallel}\left(\tau_0\sqrt{1+\sigma^2} - \tau_0\sqrt{1+\sigma'^2}\Big|_{\sigma'\to-\infty} + \tau_0\sigma - \tau_0\sigma'\Big|_{\sigma'\to-\infty} - \ln\frac{\sigma+\sqrt{1+\sigma^2}}{\sigma'+\sqrt{1+\sigma'^2}}\Big|_{\sigma'\to-\infty}\right)$$

The divergence on the limit $t\to-\infty$ can be bypassed to to restriction on „length" of trajectory: $-\infty \to \frac{\bar{V}\Delta t_{pass}}{v_0}$, where $\Delta t_{pass} \simeq \frac{L_{pass}}{\bar{V}} \simeq \frac{v_0 \, tg\,\alpha}{\bar{V}}$.



Lagrangian: $\mathcal{L} = \frac{m\vec{v}_e^{\,2}}{2} + \frac{M\vec{v}_i^{\,2}}{2} - e\vec{A}(\vec{r}_e)\vec{v}_e + Ze\vec{A}(\vec{r}_i)\vec{v}_i + \frac{Ze^2}{|\vec{r}_e - \vec{r}_i|}$ (3.3)

magnetic field along $\vec{e}_z$ $\Rightarrow$ $\vec{B} = B\vec{e}_z$ and vector-potential for that is

$$\vec{A}(\vec{r}) = \frac{1}{2}[\vec{B}\vec{r}] = \frac{1}{2}\begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ B_x & B_y & B_z \\ x & y & z \end{vmatrix} = \frac{1}{2}\begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ 0 & 0 & B \\ x & y & z \end{vmatrix} = \frac{1}{2}(yB\,\vec{e}_x - Bx\vec{e}_y) = \frac{B}{2}(y\vec{e}_x - x\vec{e}_y)$$

Checking: $\vec{B} = \mathrm{rot}\vec{A} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{yB}{2} & -\frac{xB}{2} & 0 \end{vmatrix} = \vec{e}_x\left(0 - \frac{\partial(xB)}{2\,\partial z}\right) + \vec{e}_y\left(\frac{1}{2}\frac{\partial(yB)}{\partial z} - 0\right) + \vec{e}_z\left(\frac{1}{2}\frac{\partial(xB)}{\partial x} - \frac{\partial(-x\ldots)}{\partial y}\ldots\right)$

$= 0\cdot\vec{e}_x + 0\cdot\vec{e}_y + B\vec{e}_z$ OK!

Let ion moves uniformly with velocity

$$\vec{v}_i = \begin{pmatrix} v_{i\perp} \\ 0 \\ v_{i\parallel} \end{pmatrix} \quad (3.1) \longrightarrow \vec{r}_i(t) = \vec{v}_i \cdot t$$

Input relative coordinate of electron $\vec{r}(t) = \vec{r}_e(t) - \vec{r}_i(t) = \vec{r}_e(t) - \vec{v}_i\cdot t$ (3.2a)

So, $\vec{r}(t)$ — coordinate ~~of~~ electron in the ion's frame. Farther

(3.2b) $\vec{v}(t) = \vec{v}_e(t) - \vec{v}_i(t) = \vec{v}_e(t) - \vec{v}_i$ — relative electron's velocity in the ion's frame

Let's define the radius and velocity of the center of mass:

$$\begin{cases} \vec{r}_{c.m} = (m_e\vec{r}_e + M_i\vec{r}_i)/(m_e + M_i) \\ \vec{r} = \vec{r}_e - \vec{r}_i \end{cases} \text{ and } \begin{cases} \vec{V}_{c.m} = (m_e\vec{v}_e + M_i\vec{v}_i)/(m_e + M_i) \\ \vec{V} = \vec{v}_e - \vec{v}_i \end{cases}$$

$$\begin{cases} m\vec{r}_e + M\vec{r}_i = (m_e + M_i)\vec{r}_{cm} \\ \vec{r}_e - \vec{r}_i = \vec{r} \end{cases} \longrightarrow \Delta = \begin{vmatrix} m_e & M_i \\ 1 & -1 \end{vmatrix} = -(m_e + M_i) \longrightarrow \vec{r}_e = \frac{1}{\Delta}\begin{vmatrix} (m_e + M_i)\vec{r}_{cm} & M_i \\ \vec{r} & -1 \end{vmatrix} = \vec{r}_{cm} + \frac{M_i\vec{r}}{m_e + M_i}$$

and $\vec{r}_i = \frac{1}{\Delta}\begin{vmatrix} m_e & (m_e + M_i)\vec{r}_{cm} \\ 1 & \vec{r} \end{vmatrix} = \vec{r}_{cm} - \frac{m_e}{m_e + M_i}\vec{r} \simeq \vec{r}_{cm} - \frac{m}{M}\vec{r}$

$\vec{v}_i t$

$\vec{r}_{cm} \simeq \vec{v}_i t + \frac{m}{M}\vec{r}$

and analogusly

$$\vec{v}_e = \vec{V}_{cm} + \frac{M_i}{m_e + M_i}\vec{V}$$

$$\vec{v}_i = \vec{V}_{cm} - \frac{m_e}{m_e + M_i}\vec{V} \simeq \vec{V}_{cm} - \frac{m}{M}\vec{V}$$

$$\mu = 1/\left(\frac{1}{m_e} + \frac{1}{M_i}\right) = \frac{m_e M_i}{m_e + M_i}$$

so

$$\begin{cases} \vec{r}_e = \vec{r}_{cm} + \frac{\mu}{m_e}\vec{r} \\ \vec{r}_i = \vec{r}_{cm} - \frac{\mu}{M_i}\vec{r} \end{cases} \quad \text{and} \quad \begin{cases} \vec{V}_e = \vec{V}_{cm} + \frac{\mu}{m_e}\vec{V} \\ \vec{V}_i = \vec{V}_{cm} - \frac{\mu}{M_i}\vec{V} \end{cases} \quad \begin{aligned} \vec{A}(\vec{r}_e) &= \frac{1}{2}[\vec{B}\cdot\vec{r}_e] = \frac{1}{2}\left[\vec{B}\cdot\left(\vec{r}_{cm} + \frac{\mu}{m}\vec{r}\right)\right] \\ \vec{A}(\vec{r}_i) &= \frac{1}{2}[\vec{B}\cdot\vec{r}_i] = \frac{1}{2}\left[\vec{B}\cdot\left(\vec{r}_{cm} - \frac{\mu}{M}\vec{r}\right)\right] \end{aligned}$$

then

$$\mathcal{L} = \frac{m}{2}\left(\vec{V}_{cm} + \frac{\mu}{m}\vec{V}\right)^2 + \frac{M}{2}\left(\vec{V}_{cm} - \frac{\mu}{M}\vec{V}\right)^2 - e\frac{1}{2}\left[\vec{B}\left(\vec{r}_{cm} + \frac{\mu}{m}\vec{r}\right)\right]\left(\vec{V}_{cm} + \frac{\mu}{m}\vec{V}\right) - \frac{Ze}{2}\left[\vec{B}\left(\vec{r}_{cm} - \frac{\mu}{M}\vec{r}\right)\right]\left(\vec{V}_{cm} - \frac{\mu}{M}\vec{V}\right) + \frac{Ze^2}{r} =$$

$$= \underset{(1)}{\frac{m}{2}\vec{V}_{cm}^2} + \cancel{2\mu\vec{V}\vec{V}_{cm}} + \underset{(2)}{\frac{\mu^2}{2m}\vec{V}^2} + \underset{(3)}{\frac{M}{2}\vec{V}_{cm}^2} \cancel{- \mu\vec{V}\vec{V}_{cm}} + \underset{(4)}{\frac{\mu^2}{2M}\vec{V}^2} - \frac{e}{2}\Big\{\underset{(5)}{([\vec{B}\vec{r}_{cm}]\vec{V}_{cm})} + \underset{(6)}{\frac{\mu}{m}([\vec{B}\vec{r}]\vec{V}_{cm})} + \underset{(7)}{([\vec{B}\vec{r}_{cm}]\frac{\mu}{m}\vec{V})} +$$

$$+ \underset{(8)}{\frac{\mu^2}{m^2}([\vec{B}\vec{r}]\vec{V})}\Big\} + \frac{Ze}{2}\Big\{\underset{(9)}{([\vec{B}\vec{r}_{cm}]\vec{V}_{cm})} - \underset{(10)}{\frac{\mu}{M}([\vec{B}\vec{r}]\vec{V}_{cm})} - \underset{(11)}{\frac{\mu}{M}([\vec{B}\vec{r}_{cm}]\vec{V})} + \underset{(12)}{\frac{\mu^2}{M^2}([\vec{B}\vec{r}]\vec{V})}\Big\} + \frac{Ze^2}{r}$$

$$= \underbrace{\frac{m+M}{2}\vec{V}_{cm}^2}_{(1)+(3)\ \leftarrow\ \text{this is an constant}} + \frac{\mu^2}{2}\vec{V}^2\left(\underset{(2)}{\frac{1}{m}} + \underset{(4)}{\frac{1}{M}}\right) + \frac{Ze^2}{r} + \underbrace{\frac{(Z-1)e}{2}([\vec{B}\cdot\vec{r}_{cm}]\vec{V}_{cm})}_{(5)\ (9)\ \leftarrow\ \text{this is an constant}} + \frac{\mu^2}{2}\left(\underset{(12)}{\frac{Ze}{M^2}} - \underset{(8)}{\frac{e}{m^2}}\right)([\vec{B}\vec{r}]\vec{V}) +$$

$$-\frac{\mu}{2}([\vec{B}\vec{r}]\vec{V}_{cm})\left(\underset{(6)}{\frac{e}{m}} + \underset{(10)}{\frac{Ze}{M}}\right) - \frac{\mu}{2}\left(\underset{(7)}{\frac{e}{m}} + \underset{(11)}{\frac{Ze}{M}}\right)([\vec{B}\vec{r}_{cm}]\vec{V}) \simeq$$

this is (3.5) and taking into account $\mu \simeq m \ll M$

$$\simeq \underbrace{\frac{m\vec{V}^2}{2}}_{(2)+(4)} + \frac{Ze^2}{r} - \underbrace{\frac{e}{2}([\vec{B}\vec{r}]\vec{V})}_{(12)+(8)} - \left(\frac{e}{m} + \frac{Ze}{M}\right)\underbrace{\frac{\mu}{2}\left(([\vec{B}\vec{r}]\overset{\vec{V}_i + \frac{m}{M}\vec{V}}{\vec{V}_{cm}}) + ([\vec{B}\vec{r}_{cm}]\vec{V})\right)}_{(6)+(10)+(7)+(11)} =$$

$$\simeq \frac{m\vec{V}^2}{2} + \frac{Ze^2}{r} - \frac{e}{2}([\vec{B}\vec{r}]\vec{V}) - \frac{e}{2}\left\{([\vec{B}\vec{r}](\vec{V}_i + \frac{m}{M}\vec{V})) + ([\vec{B}\vec{r}_{cm}]\vec{V})\right\} = \frac{m\vec{V}^2}{2} + \frac{Ze^2}{r} - \frac{e}{2}([\vec{B}\vec{r}]\vec{V}) -$$

$$-\frac{e}{2}\left\{([\vec{B}\vec{r}]\vec{V}_i) + \left[\vec{B}\left(\vec{r}_{cm} + \frac{m}{M}\vec{r}\right)\right]\vec{V}\right\} =$$

$$= \frac{m\vec{V}^2}{2} + \frac{ze^2}{r} - \frac{e}{2}([\vec{B}\vec{r}]\vec{V}) - \frac{e}{2}\left\{([\vec{B}\vec{r}]\vec{V}_i) + ([\vec{B}\cdot\vec{V}_i t]\vec{V}) + 2\frac{m}{M}([\vec{B}\vec{r}]V)\right\} =$$

$$= \frac{m\vec{V}^2}{2} + \frac{ze^2}{r} - \frac{e}{2}([\vec{B}\vec{r}]\vec{V}_i) - \frac{e}{2}([\vec{B}\cdot\vec{V}_i t]\vec{V}) - \frac{e}{2}\left(1 + \underbrace{2\frac{m}{M}}_{2\frac{m}{M} \ll 1}\right)([\vec{B}\vec{r}]\vec{V}) \Longrightarrow$$

$$\mathcal{L} = \frac{m\vec{V}^2}{2} + \frac{ze^2}{r} - \frac{e}{2}([\vec{B}\vec{r}]\vec{V}) - \frac{e}{2}\left\{([\vec{B}\cdot\vec{V}_i t]\vec{V}) + ([\vec{B}\vec{r}]\vec{V}_i)\right\}$$ this is (3.6)

Equation of motion:

$$m\frac{d\vec{V}}{dt} = -\frac{\partial \mathcal{L}}{\partial \vec{r}} = -\vec{V}\frac{ze^2}{r}$$

From (3.14) for $R \to 0$ („Guiding" center approach):

$$\vec{r}^{\,2}(t) = b^2 + [(V_{e\parallel} - V_{i\parallel})^2 + V_{i\perp}^2]\,t^2 - 2V_{i\perp} b\, t \sin\theta \qquad (3.15)$$

Let's define the relative velocity $\vec{V}$ of the guiding center and ion:

$$\vec{V} = \vec{V}_\perp + \vec{V}_\parallel = \begin{pmatrix} 0 \\ 0 \\ V_{e\parallel} - V_{i\parallel} \end{pmatrix} + \begin{pmatrix} -V_{i\perp} \\ 0 \\ 0 \end{pmatrix} \qquad (3.16)$$

Then $(V_{e\parallel} - V_{i\parallel})^2 + V_{i\perp}^2 = \vec{V}^{\,2} \equiv \bar{V}^2$

Let's t_0 - time t_0 when electron reaches the minimal distance $\vec{r}_0$ to ion (this is impact parameter of collision); i.e.

$$r_0^2 = b^2 + \bar{V}^2 t_0^2 - 2V_{i\perp} b\, t_0 \sin\theta$$

Very important: $\vec{r}_0 \perp \vec{V}$ and from picture one has

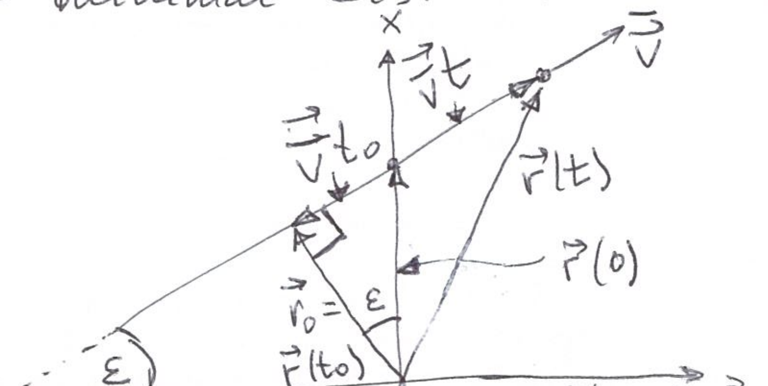


Fig. 2

$$\vec{r}(t) = \vec{r}_0 + \vec{V}(t - t_0) \qquad (3.20)$$

then $$\vec{r}^{\,2} = r_0^2 + \underbrace{\vec{V}^{\,2}}_{\equiv \bar{V}^2}(t-t_0)^2 + 2\underbrace{\vec{r}_0\vec{V}}_{=0}(t-t_0) = r_0^2 + \bar{V}^2(t-t_0)^2$$

So
$$\vec{r}^{\,2} = b^2 + \bar{V}^2 t^2 - 2V_{i\perp} b\, t \sin\theta = r_0^2 + \bar{V}^2(t-t_0)^2 = b^2 + \bar{V}^2 t_0^2 - 2V_{i\perp} b\, t_0 \sin\theta + \bar{V}^2(t-t_0)^2$$

or

$$\cancel{b^2} + \cancel{\bar{V}^2 t^2} - 2V_{i\perp} b\, t \sin\theta = \cancel{b^2} + \bar{V}^2 t_0^2 - 2V_{i\perp} b\, t_0 \sin\theta + \cancel{\bar{V}^2 t^2} + \bar{V}^2 t_0^2 - 2\bar{V}^2 t\, t_0$$

$$\cancel{2\bar{V}^2 t_0^2 - 2t_0(V_{i\perp} b \sin\theta - \bar{V}^2 t) + 2V_{i\perp} b\, t \sin\theta = 0}$$

So

$$0 = 2v_{i\perp} b\sin\theta\cdot(t-t_0) + 2\bar{v}^2 t_0(t_0-t) \Longrightarrow t_0 = \frac{v_{i\perp} b\sin\theta}{\bar{v}^2} \quad (3.18) = -\frac{\bar{v}_\perp b\sin\theta}{\bar{v}^2}$$

and for this reason

$$r_0^2 = b^2 + \bar{v}^2 t_0^2 - 2t_0\cdot \overbrace{\frac{v_{i\perp}\sin\theta}{\bar{v}^2}}^{=t_0}\bar{v}^2 = b^2 + \bar{v}^2 t_0^2 - 2\bar{v}^2 t_0^2 = b^2 - \bar{v}^2 t_0^2 \quad (3.19)$$

and

$$\vec{r}_0 = \vec{r}(t_0) = \underbrace{\begin{pmatrix} b\sin\theta - v_{i\perp} t_0 \\ -b\cos\theta \\ (v_{e\parallel} - v_{i\parallel})t_0 \end{pmatrix}}_{\text{this is from (3.10) with } R=0} = \begin{pmatrix} b\sin\theta + \bar{v}_\perp t_0 \\ -b\cos\theta \\ \bar{v}_\parallel t_0 \end{pmatrix}$$

and then

$$\vec{r}(t) = \begin{pmatrix} -b\sin\theta - v_{i\perp} t \\ -b\cos\theta \\ (v_{e\parallel} - v_{i\parallel})t \end{pmatrix} = \begin{pmatrix} -b\sin\theta + \bar{v}_\perp t \\ -b\cos\theta \\ \bar{v}_\parallel t \end{pmatrix} = \begin{pmatrix} -b\sin\theta + \bar{v}_\perp (t_0 + t - t_0) \\ -b\cos\theta \\ \bar{v}_\parallel (t_0 + t - t_0) \end{pmatrix} =$$

$$= \begin{pmatrix} -b\sin\theta + \bar{v}_\perp t_0 \\ -b\cos\theta \\ \bar{v}_\parallel t_0 \end{pmatrix} + \begin{pmatrix} \bar{v}_\perp (t-t_0) \\ 0 \\ \bar{v}_\parallel (t-t_0) \end{pmatrix} = \vec{r}_0 + \vec{\bar{v}}(t-t_0)$$ this is (3.20) again

Let's introduce the dimentionless variables:

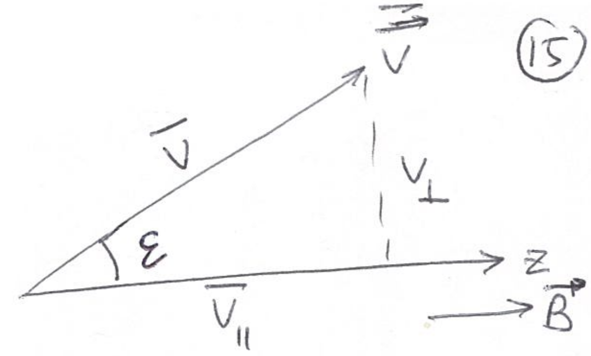
$$\tau = \frac{\bar{v}t}{r_0} \quad (3.21) \qquad \tau_0 = \frac{\bar{v}t_0}{r_0}, \quad \sigma = \tau - \tau_0 = \frac{\bar{v}(t-t_0)}{r_0} \quad (3.22)$$

This is (3.29 left)

$$\gamma_\parallel = \frac{\bar{v}_\parallel}{\bar{v}} \quad (3.24), \qquad \gamma_\perp = \frac{\bar{v}_\perp}{\bar{v}} \quad (3.25) \qquad \beta = \frac{b}{r_0} \quad (3.26)$$

Then from (3.18) $\tau_0 = \frac{\bar{v}t_0}{r_0} = \frac{\bar{v}}{r_0}\,\frac{-\bar{v}_\perp b\sin\theta}{\bar{v}^2} =$

$= -\beta\gamma_\perp\sin\theta$



$$\frac{\vec r_0}{r_0} = \frac{1}{r_0}\begin{pmatrix} b\sin\theta + \bar V_\perp t_0 \\ -b\cos\theta \\ \bar V_\parallel t_0\end{pmatrix} = \begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau_0\\ -\beta\cos\theta\\ \gamma_\parallel\tau_0\end{pmatrix}$$

Instead (3.19) one has

$$b^2 = r_0^2 + \bar V^2 t_0^2 \to \beta^2 = 1+\tau_0^2 \qquad (3.32)$$

$$\cos\varepsilon = \frac{\bar V_\parallel}{\bar V} = \gamma_\parallel \qquad (3.28)$$

$$\sin\varepsilon = \frac{\bar V_\perp}{\bar V} = \gamma_\perp \qquad (3.28^*)$$

and, of course, $\gamma_\parallel^2+\gamma_\perp^2 = 1$

Let's input

$$\sin\psi = -\beta\cos\theta \qquad (3.30)$$

Then

$$\beta\sin\theta + \gamma_\perp\tau_0 = \left(\text{from 3.29 left } \beta\sin\theta = -\frac{\tau_0}{\gamma_\perp}\right) = -\frac{\tau_0}{\gamma_\perp} + \gamma_\perp\tau_0 = \tau_0\frac{\gamma_\perp^2-1}{\gamma_\perp} = -\frac{\tau_0\gamma_\parallel^2}{\gamma_\perp}$$

But

$$\tau_0 = -\beta\gamma_\perp\sin\theta = -\gamma_\perp\beta\sqrt{1-\cos^2\theta} = -\gamma_\perp\beta\sqrt{1-\frac{\sin^2\psi}{\beta^2}} = -\gamma_\perp\sqrt{\beta^2-\sin^2\psi} =$$

$$= -\gamma_\perp\sqrt{\beta^2-1+\cos^2\psi} = -\gamma_\perp\sqrt{\tau_0^2+\cos^2\psi} \quad \text{or}$$

from (3.32)

$$\tau_0^2 = \gamma_\perp^2\tau_0^2 + \gamma_\perp^2\cos^2\psi \to \tau_0^2(1-\gamma_\perp^2) = \gamma_\perp^2\cos^2\psi \to$$

$$\tau_0^2\gamma_\parallel^2 = \gamma_\perp^2\cos^2\psi \to \tau_0 = \pm\frac{\gamma_\perp}{\gamma_\parallel}\cos\psi$$ and it is neccesary to select sign "−"!

$$\tau_0 = -\frac{\gamma_\perp}{\gamma_\parallel}\cos\psi \;(3.29\text{ right}) \to \gamma_\parallel\tau_0 = -\gamma_\perp\cos\psi = -\sin\varepsilon\cos\psi \qquad (3.29^*)$$

from (3.28*)

So

$$\to \beta\sin\theta + \gamma_\perp\tau_0 = -\frac{\tau_0\gamma_\parallel^2}{\gamma_\perp} = \frac{\gamma_\perp\cos\psi\cdot\gamma_\parallel^2}{\gamma_\parallel\gamma_\perp} = \gamma_\parallel\cos\psi \qquad (3.31)$$

Therefore

$$\frac{\vec{r}_0}{r_0} = \begin{pmatrix} \beta \sin\theta + \gamma_\perp \tau_0 \\ -\beta\cos\theta \\ \gamma_{\parallel}\tau_0 \end{pmatrix} = \begin{pmatrix} \text{using } (3.31) \\ \text{using } (3.30) \\ \text{using } (3.29^*) \end{pmatrix} = \begin{pmatrix} \cos\varepsilon\cos\psi \\ \sin\psi \\ -\sin\varepsilon\cos\psi \end{pmatrix} \quad (3.27)$$

Finaly:

(3.21) $\tau = \frac{\bar{v}t}{r_0} \qquad \tau_0 = \frac{\bar{v}t_0}{r_0}$

(3.22) $\sigma = \tau - \tau_0 = \frac{\bar{v}(t-t_0)}{r_0}$

(3.24) $\gamma_{\parallel} = \frac{\overline{v_{\parallel}}}{v_0}$

(3.25) $\gamma_\perp = \frac{\overline{v_\perp}}{v_0}$

$\gamma_{\parallel}^2 + \gamma_\perp^2 = 1$

(3.26) $\beta = \frac{b}{r_0}$

(3.27) $\frac{\vec{r}_0}{r_0} = \begin{pmatrix} \cos\varepsilon\cos\psi \\ \sin\psi \\ -\sin\varepsilon\cos\psi \end{pmatrix}$

$\beta\sin\theta = \frac{\cos\psi}{\gamma_{\parallel}} = \frac{\cos\psi}{\cos\varepsilon}$

$\beta\cos\theta = -\sin\psi$,

so

$$\boxed{\operatorname{tg}\theta = -\frac{\cos\psi/\cos\varepsilon}{\sin\psi} = -\frac{\operatorname{ctg}\psi}{\cos\varepsilon}}$$

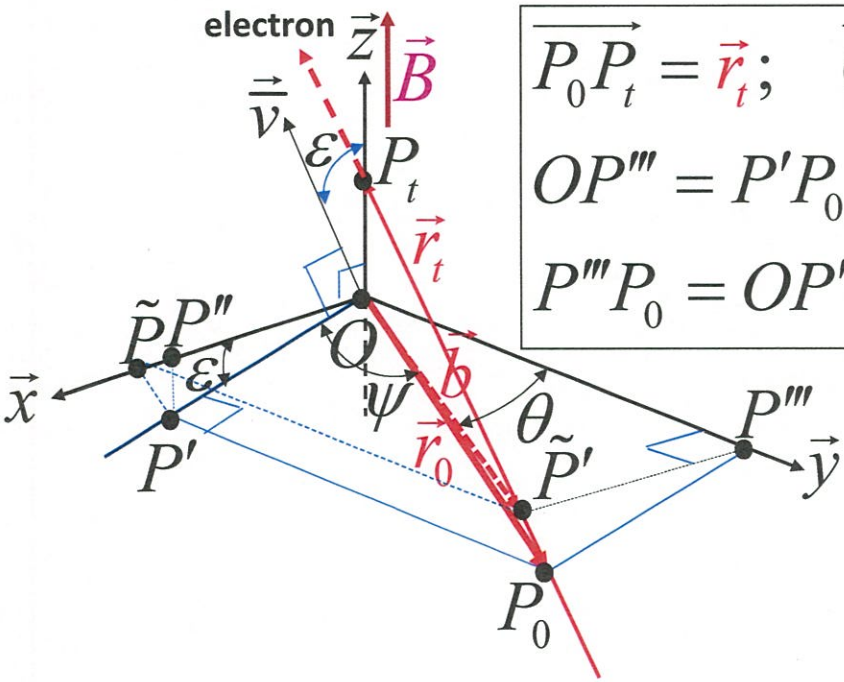
(3.28) $\cos\varepsilon = \gamma_{\parallel} \longrightarrow \sin\varepsilon = \sqrt{1-\cos^2\varepsilon} = \sqrt{1-\gamma_{\parallel}^2} = \gamma_\perp$ (3.28*)

(3.29) $\tau_0 = -\beta\gamma_\perp \sin\theta = -\frac{\gamma_\perp}{\gamma_{\parallel}}\cos\psi \longrightarrow \gamma_{\parallel}\tau_0 = -\sin\varepsilon\cos\psi$ (3.29*)

(3.30) $\beta\cos\theta = -\sin\psi$

(3.31) $\beta\sin\theta + \gamma_\perp\tau_0 = -\frac{\gamma_{\parallel}^2\tau_0}{\gamma_\perp} = \gamma_{\parallel}\cos\psi$

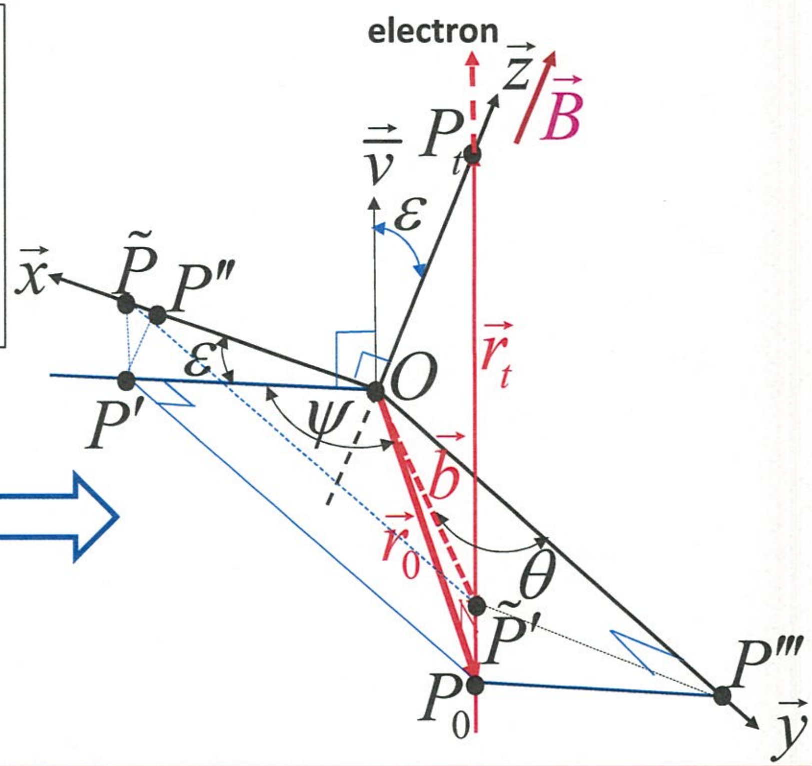
(3.32) $\beta^2 = 1 + \tau_0^2$



$$\overrightarrow{P_0P_t} = \vec{r}_t; \quad \overrightarrow{OP_0} = \vec{r}_0 \;\rightarrow$$

$$\left.\begin{aligned} OP''' &= P'P_0 = r_0 \cdot \sin\psi, \\ P'''P_0 &= OP' = r_0 \cdot \cos\psi \end{aligned}\right\}$$

or



$$\vec{r}_0 = \begin{pmatrix} OP'' \\ P_0P' \\ P''P' \end{pmatrix} = r_0 \begin{pmatrix} \cos\varepsilon \cdot \cos\psi \\ \sin\psi \\ -\sin\varepsilon \cdot \cos\psi \end{pmatrix};$$

$$\tilde{P}'P''' = O\tilde{P} = \frac{OP'}{\cos\varepsilon} = r_0 \cdot \frac{\cos\psi}{\cos\varepsilon} \rightarrow$$

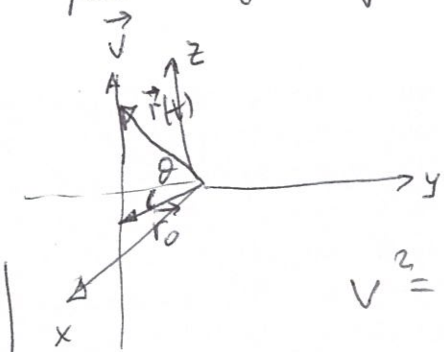
$$\rightarrow tg\theta = \tilde{P}'P''' \,/\, OP''' = -\cot\psi \,/\, \cos\varepsilon$$

Sign from corresponding formulas

Unmagnetized electron scattering with ion in fixed point of origin

(18)

$$\frac{d\vec v}{dt} = -\frac{e}{m}\vec E(\vec r) \quad (2.3), \qquad \vec E(\vec r) = Ze\,\frac{\vec r(t)}{r^3(t)} \quad (2.1)$$



r_0 – impact parameter is $\vec r(t=0)$, and

$V^2 = V_\perp^2 + V_\parallel^2$!

$$\vec r_0 = r_0\begin{pmatrix}\sin\theta\\ -\cos\theta\\ 0\end{pmatrix} \quad (2.5) \quad \text{then} \quad \vec r(t) = \begin{pmatrix}r_{0x}\\ r_{0y}\\ vt\end{pmatrix} = \begin{pmatrix}r_0\sin\theta\\ -r_0\cos\theta\\ vt\end{pmatrix}$$

Then first order:

$$(2.6) \quad \delta^{(1)}\vec V_c(t) = -\frac{Ze^2}{m}\int_{-\infty}^{t}\frac{dt'}{\sqrt{(r_0^2+V^2t'^2)^3}}\begin{pmatrix}r_0\sin\theta\\ -r_0\cos\theta\\ V_\parallel t'\end{pmatrix} \underset{x=vt}{=} -\frac{Ze^2}{mV}\int_{-\infty}^{vt}\frac{r_0\,dx/r_0}{(r_0^2+x^2)^{3/2}}\begin{pmatrix}r_0\sin\theta\\ -r_0\cos\theta\end{pmatrix} = -\frac{Ze^2}{mVr_0}\int_{-\infty}^{vt/r_0}\frac{dx}{(1+x^2)^{3/2}}\begin{pmatrix}\sin\theta\\ -\cos\theta\\ \frac{V_\parallel}{V}x\end{pmatrix}$$

(lower index „c" means „Coulomb").

Total $\delta^{(1)}\vec V_c(t)$ is reached for $t\to\infty$!

2.27.5: $\int\frac{dx}{(a+cx^2)^{3/2}} = \frac{1}{a}\frac{x}{\sqrt{a+cx^2}}$

$$(\delta^{(1)}\vec V_c)_{total} = -\frac{Ze^2}{mVr_0}\int_{-\infty}^{\infty}\frac{dx}{(1+x^2)^{3/2}}\begin{pmatrix}\sin\theta\\ -\cos\theta\\ \frac{V_\parallel}{V}x\end{pmatrix} = -\frac{2Ze^2}{mVr_0}\underbrace{\int^{\infty}\frac{dx}{(1+x^2)^{3/2}}}_{=1}\begin{pmatrix}\sin\theta\\ -\cos\theta\\ 0\end{pmatrix}$$

integral $=0$ due to odd function!

$$= -\frac{2Ze^2}{mVr_0}\begin{pmatrix}\sin\theta\\ -\cos\theta\\ 0\end{pmatrix} = -\frac{2Ze^2}{mVr_0^2}\,r_0\begin{pmatrix}\sin\theta\\ -\cos\theta\\ 0\end{pmatrix} = -\frac{2Ze^2\vec r_0}{mVr_0^2} \quad (2.7)$$

Now let calculate $\delta\vec r^{(1)}$: $\frac{d\vec r}{dt} = \vec V$ (2.2)

~~Then~~ But it is possible to write the solution for $\delta^{(1)}\vec V_c(t)$ from (2.6):

$$\delta^{(1)}\vec V_c(t) = -\frac{Ze^2}{mVr_0}\int_{-\infty}^{vt/r_0}\frac{dx}{(1+x^2)^{3/2}}\begin{pmatrix}\sin\theta\\ -\cos\theta\\ \frac{V_\parallel}{V}x\end{pmatrix} = \begin{pmatrix}\int 2.27.5\\ 2.27.7\\ \text{with } n=1\end{pmatrix} =$$

2.27.7: $\int\frac{x\,dx}{\sqrt{a+cx^2}^{\,3}} = -\frac{1}{c\sqrt{a+cx^2}}$

$$= -\frac{Ze^2}{mVr_0}\left(\frac{vt/r_0}{\sqrt{1+v^2t^2/r_0^2}}+1\right)\begin{pmatrix}\sin\theta\\ -\cos\theta\\ 0\end{pmatrix} - \frac{Ze^2}{mVr_0}\left(-\frac{\mathrm{sgn}(v)}{\sqrt{1+v^2t^2/r_0^2}}\right)\begin{pmatrix}0\\0\\1\end{pmatrix}$$

my opinion: factor $\frac{v_{\parallel}}{V}=\gamma_{\parallel}$!

and for $t\to\infty$ one will receive the result (2.7)

Now $\delta^{(1)}\vec{r}_c$ can be received from $\delta^{(1)}\vec{V}_c(t)$:

$$\delta^{(1)}\vec{r}_c(t) = \int_{-\infty}^{t} dt'\,\delta^{(1)}\vec{V}_c(t') = -\frac{Ze^2}{mVr_0}\int_{-\infty}^{t}\left(\frac{vt'/r_0}{\sqrt{1+v^2t'^2/r_0^2}}+1\right)dt'\begin{pmatrix}\sin\theta\\ -\cos\theta\\ 0\end{pmatrix} +$$

$$+\frac{Ze^2}{mVr_0}\int_{-\infty}^{t}\frac{dt'\,\mathrm{sgn}(v)}{\sqrt{1+v^2t'^2/r_0^2}}\begin{pmatrix}0\\0\\1\end{pmatrix} = -\frac{Ze^2}{mV^2}\int_{-\infty}^{t}\left(\frac{vt'/r_0}{\sqrt{1+v^2t'^2/r_0^2}}+1\right)d\left(\frac{vt'}{r_0}\right)\begin{pmatrix}\sin\theta\\ -\cos\theta\\ 0\end{pmatrix} + \frac{Ze^2}{mV^2}\int_{-\infty}^{t}\frac{\mathrm{sgn}(v)\,d\left(\frac{vt'}{r_0}\right)}{\sqrt{1+v^2t'^2/r_0^2}}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

must be $\gamma_{\parallel}$

$$= -\frac{Ze^2}{mV^2}\int_{-\infty}^{vt/r_0} d\tau'\left[\left(\frac{\tau'}{\sqrt{1+\tau'^2}}+1\right)\begin{pmatrix}\sin\theta\\ -\cos\theta\\ 0\end{pmatrix} - \frac{\mathrm{sgn}(v)}{\sqrt{1+\tau'^2}}\begin{pmatrix}0\\0\\1\end{pmatrix}\right]$$ this is (2.11)

Question: how from expessions for $\delta^{(1)}\vec{r}_{\perp}(t)$ and $\delta^{(1)}v_{\parallel}(t)$ ((3.41) and (3.42) correspondlugly) to receive the expressions (2.11), (2.7) because, it seems, that

$$\boxed{(2.11) = \lim_{\omega = \frac{r_0}{v}\Omega_B \to 0} (3.41),(3.42)}$$

Differences between (2.7), (2.11) and (3.41), (3.42): first are in lab. system second — in ion frame system

Second-order approach

A) Field:
$$\vec{E}(\vec{r}(t)) = Ze\frac{\vec{r}(t)}{r^3(t)} \longrightarrow E_k(\vec{r}(t)) = Ze\frac{r_k(t)}{r_i r_i}$$

$$E_k = E_k(\vec{r}+\delta^{(1)}\vec{r}) - E_k(\vec{r}) = Ze\left[E_k(\vec{r}) + \delta^{(1)} r_j \frac{\partial}{\partial r_j}\left(\frac{r_k}{r_i r_i^{3/2}}\right) - E_k(\vec{r})\right] =$$

$$= Ze\,\delta^{(1)} r_j\left[\frac{1}{(r_i r_i)^{3/2}}\frac{\partial r_k}{\partial r_j} + r_k\frac{\partial}{\partial r_j}\left(\frac{1}{(r_i r_i)^{3/2}}\right)\right] = Ze\,\delta^{(1)} r_j\left(\frac{\delta_{kj}}{(r_i r_i)^{3/2}} + r_k\frac{-3}{2}\frac{2\delta_{ij} r_i}{(r_i r_i)^{5/2}}\right) =$$

$$= \frac{Ze}{(r_i r_i)^{3/2}}\left(r_k - 3 r_k\frac{r_i\delta^{(1)} r_i}{r_i r_i}\right)$$ This is (2.13) in lab. system

In guiding center system
$$\delta^{(1)} E_k(t) = \frac{Ze}{(\bar{r}_j\bar{r}_j)^{3/2}}\left(\delta^{(1)} r_k - 3\bar{r}_k\frac{\bar{r}_i\delta^{(1)} r_i}{\bar{r}_j\bar{r}_j}\right), \quad (3.47)$$

where
$$\vec{\bar{r}}(t) = \begin{pmatrix} b\sin\theta + \bar{v}_\perp t_0 \\ -b\cos\theta \\ \bar{v}_{\parallel} t_0\end{pmatrix} + \begin{pmatrix}\bar{v}_\perp \\ 0 \\ v_{\parallel}\end{pmatrix}(t-t_0) = \vec{r}_0 + \vec{\bar{v}}(t-t_0) = (3.20)$$

$$= r_0\begin{pmatrix}\beta\sin\theta + \gamma_\perp\tau \\ -\beta\cos\theta \\ \gamma_{\parallel}\tau\end{pmatrix} = \begin{pmatrix}\bar{r}_\perp \\ \bar{r}_{\parallel}\end{pmatrix} \quad \Big| \; \Omega_B t' = \frac{\bar{v}}{r_0}t' \; \frac{r_0}{\bar{v}}\Omega_B = \omega\tau$$

Then
$$\delta^{(2)}\vec{v}_B(t) = -\frac{e}{m}\int_{-\infty}^{t} dt'\, T^{-1}(\Omega_B t')\,\delta^{(1)}\vec{E}(t') \quad (3.48)$$

and
$$\delta^{(1)} r_{\parallel} = -\frac{Ze^2}{m\bar{v}^2}\gamma_{\parallel}\int_{-\infty}^{\sigma} d\sigma'\left(\frac{\tau_0\sigma' - 1}{(1+\sigma'^2)^{1/2}} + \tau_0\right) \quad (3.41)$$

$$\delta^{(1)} r_\perp = \frac{r_0}{\bar{v}\omega}\begin{pmatrix}\sin\omega\tau & \cos\omega\tau \\ -\cos\omega\tau & \sin\omega\tau\end{pmatrix}\delta^{(1)}\vec{v}_{B\perp}(t) = \frac{Ze^2}{m\bar{v}^2\omega}\int_{-\infty}^{\sigma}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\begin{pmatrix}\beta\cos\theta \\ \beta\sin\theta + \gamma_\perp\tau'\end{pmatrix} \quad (3.42)$$

and

$$\delta^{(1)}\vec{V}_{B}(t) = -\frac{Ze^2}{m}\frac{1}{\bar{v} r_0}\int_{-\infty}^{\sigma}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\underbrace{\begin{pmatrix}\cos\omega\tau' & \sin\omega\tau' & 0\\ -\sin\omega\tau' & \cos\omega\tau' & 0\\ 0 & 0 & 1\end{pmatrix}}_{T^{-1}(\Omega_B t')}\begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau'\\ -\beta\cos\theta\\ \gamma_\parallel\tau'\end{pmatrix} \qquad (3.36)$$

or

$$\delta^{(1)}V_{\perp B}(t) = -\frac{Ze^2}{m\bar{v} r_0}\int_{-\infty}^{\sigma}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\begin{pmatrix}\cos\omega\tau' & \sin\omega\tau'\\ -\sin\omega\tau' & \cos\omega\tau'\end{pmatrix}\begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau'\\ -\beta\cos\theta\end{pmatrix}$$

$$\Big|\ \Omega_B t' = \underbrace{\frac{r_0}{\bar{v}}\Omega_B}_{\omega}\cdot\underbrace{\frac{\bar{v}t'}{r_0}}_{\tau'} = \omega\tau'$$

Then

$$(\bar{r}_i\delta^{(1)} r_i)(t) = \bar{r}_{\perp k}\delta^{(1)} r_{\perp k} + \bar{r}_\parallel\delta^{(1)} r_\parallel = r_0\begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau\\ -\beta\cos\theta\end{pmatrix}^T\Bigg[\frac{r_0}{\bar{v}\omega}\begin{pmatrix}\sin\omega\tau & \cos\omega\tau\\ -\cos\omega\tau & \sin\omega\tau\end{pmatrix}\delta^{(1)}\vec{V}_{\perp B} -$$

$$-\frac{Ze^2}{m\bar{v}^2\omega}\int_{-\infty}^{\sigma}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\begin{pmatrix}\beta\cos\theta\\ \beta\sin\theta+\gamma_\perp\tau'\end{pmatrix}\Bigg] + \overbrace{r_0\gamma_\parallel\tau\cdot\left(-\frac{Ze^2}{m\bar{v}^2}\gamma_\parallel\right)\int_{-\infty}^{\sigma}d\sigma'\left(\frac{\tau_0\sigma'-1}{(1+\sigma'^2)^{1/2}}+\tau_0\right)}^{\bar{r}_\parallel\delta^{(1)} r_\parallel} \quad \Big|\ \tau'=\sigma'+\tau_0 \quad (3.2$$

$$= r_0\begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau\\ -\beta\cos\theta\end{pmatrix}^T\Bigg[\frac{r_0}{\bar{v}\omega}\begin{pmatrix}\sin\omega\tau & \cos\omega\tau\\ -\cos\omega\tau & \sin\omega\tau\end{pmatrix}\left(-\frac{Ze^2}{m\bar{v} r_0}\right)\int_{-\infty}^{\sigma}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\begin{pmatrix}\cos\omega\tau' & \sin\omega\tau'\\ -\sin\omega\tau' & \cos\omega\tau'\end{pmatrix}\begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau'\\ -\beta\cos\theta\end{pmatrix} -$$

$$-\frac{Ze^2}{m\bar{v}^2\omega}\int_{-\infty}^{\sigma}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\begin{pmatrix}\beta\cos\theta\\ \beta\sin\theta+\gamma_\perp\tau'\end{pmatrix}\Bigg] - \frac{Ze^2 r_0}{m\bar{v}^2}\gamma_\parallel^2(\sigma+\tau_0)\int_{-\infty}^{\sigma}d\sigma'\left(\frac{\sigma'\tau_0-1}{(1+\sigma'^2)^{1/2}}+\tau_0\right) = \quad \Big|\ \tau=\sigma+\tau_0 \ (3.32$$

$$= -\frac{Ze^2}{m}\frac{r_0}{\bar{v}^2}\Bigg[\frac{1}{\omega}\int_{-\infty}^{\sigma}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau\\ -\beta\cos\theta\end{pmatrix}^T\left\{\begin{pmatrix}\sin\omega\tau & \cos\omega\tau\\ -\cos\omega\tau & \sin\omega\tau\end{pmatrix}\begin{pmatrix}\cos\omega\tau' & \sin\omega\tau'\\ -\sin\omega\tau' & \cos\omega\tau'\end{pmatrix}\begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau'\\ -\beta\cos\theta\end{pmatrix} + \begin{pmatrix}\beta\cos\theta\\ \beta\sin\theta+\gamma_\perp\tau'\end{pmatrix}\right\}$$

$$+\gamma_\parallel^2(\sigma+\tau_0)\int_{-\infty}^{\sigma}\left[\frac{\sigma'\tau_0-1}{(1+\sigma'^2)^{1/2}}+\tau_0\right]d\sigma'\Bigg] =$$

But

$$\begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau \\ -\beta\cos\theta\end{pmatrix}^T \underbrace{\begin{pmatrix}\sin\omega\tau & \cos\omega\tau \\ -\cos\omega\tau & \sin\omega\tau\end{pmatrix}\begin{pmatrix}\cos\omega\tau' & \sin\omega\tau' \\ -\sin\omega\tau' & \cos\omega\tau'\end{pmatrix}}_{\begin{pmatrix}\sin\omega(\tau-\tau') & \cos\omega(\tau-\tau')\\ -\cos\omega(\tau-\tau') & \sin\omega(\tau-\tau')\end{pmatrix}} \begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau' \\ -\beta\cos\theta\end{pmatrix} + \underbrace{\begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau\\ -\beta\cos\theta\end{pmatrix}^T\begin{pmatrix}\beta\cos\theta\\ \beta\sin\theta+\gamma_\perp\tau'\end{pmatrix}}_{\beta\cos\theta(\cancel{\beta\sin\theta}+\gamma_\perp\tau-\cancel{\beta\sin\theta}-\gamma_\perp\tau')=} =$$

$$= \beta\gamma_\perp\underbrace{(\tau-\tau')}_{=\Delta\tau}\cos\theta$$

$$= (\quad)^T \begin{pmatrix}\sin\omega\Delta\tau & \cos\omega\Delta\tau\\ -\cos\omega\Delta\tau & \sin\omega\Delta\tau\end{pmatrix}\begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau'\\ -\beta\cos\theta\end{pmatrix} + \beta\gamma\Delta\tau\cos\theta =$$

$$= (\beta\sin\theta+\gamma_\perp\tau, -\beta\cos\theta)\begin{pmatrix}(\beta\sin\theta+\gamma_\perp\tau')\sin\omega\Delta\tau - \beta\cos\theta\cos\omega\Delta\tau \\ -(\beta\sin\theta+\gamma_\perp\tau')\cos\omega\Delta\tau - \beta\cos\theta\sin\omega\Delta\tau\end{pmatrix} + \beta\gamma_\perp\Delta\tau\cos\theta =$$

$$= (\beta\sin\theta+\gamma_\perp\tau)\left[(\beta\sin\theta+\gamma_\perp\tau')\sin\omega\Delta\tau - \beta\cos\theta\cos\omega\Delta\tau\right] +$$

$$+ \beta\cos\theta\left[(\beta\sin\theta+\gamma_\perp\tau')\cos\omega\Delta\tau + \beta\cos\theta\sin\omega\Delta\tau\right] + \beta\gamma_\perp\Delta\tau\cos\theta =$$

$$= (\beta^2\sin^2\theta + \beta\gamma_\perp\tau\sin\theta + \beta\gamma_\perp\tau'\sin\theta + \gamma_\perp^2\tau\tau')\sin\omega\Delta\tau + \beta\cos\theta\left[-\cancel{\beta\sin\theta} - \gamma_\perp\tau + \cancel{\beta\sin\theta} + \gamma_\perp\tau'\right]\cos\omega\Delta\tau +$$

$$+\beta^2\cos^2\theta\sin\omega\Delta\tau + \beta\gamma_\perp\Delta\tau\cos\theta = \beta^2\sin\omega\Delta\tau + \beta\gamma_\perp(\tau+\tau')\sin\theta\sin\omega\Delta\tau +$$

$$+\beta\gamma_\perp(\tau'-\tau)\cos\theta\cos\omega\Delta\tau + \beta\gamma_\perp\Delta\tau\cos\theta =$$

$$+\gamma_\perp^2\tau\tau'\sin\omega\Delta\tau$$

$$= \beta^2\sin\omega\Delta\tau + \beta\gamma_\perp\left[(\tau+\tau')\sin\theta\sin\omega\Delta\tau - (\tau-\tau')\cos\theta\cos\omega\Delta\tau\right] + \gamma_\perp^2\tau\tau'\sin\omega\Delta\tau + \beta\gamma_\perp\Delta\tau\cos\theta$$

Then

$$(\mathbf{r}_i\,\delta^{(1)}\mathbf{r}_i)(t) = -\frac{Ze^2}{m}\frac{r_0}{v^2}\left\{\frac{1}{\omega}\int_{-\infty}^{\sigma}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\Big\langle\beta^2\sin\omega(\tau-\tau') + \beta\gamma_\perp\left[(\tau+\tau')\sin\theta\sin\omega(\tau-\tau') - (\tau-\tau')\cos\theta\cos\omega(\tau-\tau')\right] + \gamma_\perp^2\tau\tau'\sin\omega(\tau-\tau') + \beta\gamma_\perp(\tau-\tau')\cos\theta\Big\rangle + \gamma_\parallel^2\tau\int_{-\infty}^{\sigma}\left[\frac{\sigma'\tau_0-1}{(1+\sigma'^2)^{1/2}}+\tau_0\right]d\sigma'\right\} \quad (3.49)$$

Two limits: a) $\omega \to \infty$ („tight" trajectory $\equiv$ magnetized electrons) and

b) $\omega \to 0$ („stretched" trajectory $\equiv$ nonmagnetized electrons)

a) Term $\frac{1}{\omega}\int_{-\infty}^{\sigma}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}<\quad> \to 0$ for $\sigma \to \infty$, i.e.

$$\left(\bar{r}_i\delta^{(1)}r_i\right)\Big|_{tight}(t) = -\frac{ze^2}{m}\frac{r_0}{\sqrt{2}}\gamma_{\parallel}^2(\sigma+\tau_0)\int_{-\infty}^{\sigma}d\sigma'\left[\frac{\tau_0\sigma'-1}{(1+\sigma'^2)^{1/2}}+\tau_0\right] \qquad (3.50)$$

b) (How!?):

$$\left(\bar{r}_i\delta^{(1)}r_i\right)\Big|_{stretch}(t) = -\frac{ze^2}{m}\frac{r_0}{\sqrt{2}}\left\{\int_{-\infty}^{\sigma}d\sigma'\left(\frac{\tau_0\sigma'-1}{(1+\sigma'^2)^{1/2}}+\tau_0\right)\left[1-\gamma_{\parallel}^2\tau_0^2-\gamma_{\parallel}^2\tau_0\sigma-\gamma_{\parallel}^2\tau_0\sigma'+\gamma_{\perp}^2\sigma\sigma'\right]+\right.$$

$$\left.+\gamma_{\parallel}^2(\sigma+\tau_0)\int_{-\infty}^{\sigma}d\sigma'\left(\underbrace{\frac{\tau_0\sigma'-1}{(1+\sigma'^2)^{1/2}}+\tau_0}_{??}\right)\right\} \qquad (3.51)$$

Let's investigate „stretched case ($\omega \to 0$; index (s)).

$$\delta^{(1)}\vec{r} = -\frac{Ze^2}{m\bar{v}^2}\int_{-\infty}^{\sigma} d\sigma' T(\omega\tau')\int_{-\infty}^{\sigma'} \frac{d\sigma''}{(1+\sigma''^2)^{3/2}} T^{-1}(\omega\tau'')\frac{\vec{r}(\tau'')}{r_0} \qquad (3.40)$$

where

$$\vec{r}(\tau) = r_0\,(\beta\sin\theta + \gamma_\perp\tau,\ -\beta\cos\theta,\ \gamma_\parallel\tau)^T \qquad \Big|\ \tau = \sigma + \tau_0 \quad (3.22)$$

and

$$T(\omega\tau) = \begin{pmatrix} \cos\omega\tau & \sin\omega\tau & 0 \\ -\sin\omega\tau & \cos\omega\tau & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[\omega\to 0]{} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad T^{-1}(\omega\tau) = \begin{pmatrix} \cos\omega\tau & -\sin\omega\tau & 0 \\ \sin\omega\tau & \cos\omega\tau & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[\omega\to 0]{} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then

$$\delta^{(1)}\vec{r}^{(s)} = -\frac{Ze^2}{m\bar{v}^2}\int_{-\infty}^{\sigma} d\sigma'\int_{-\infty}^{\sigma'} \frac{d\sigma''}{(1+\sigma''^2)^{3/2}} \begin{pmatrix} \beta\sin\theta + \gamma_\perp(\sigma''+\tau_0) \\ -\beta\cos\theta \\ \gamma_\parallel(\sigma''+\tau_0) \end{pmatrix}$$

or

$$\delta^{(1)} r_x^{(s)} = -\frac{Ze^2}{m\bar{v}^2}\int_{-\infty}^{\sigma} d\sigma'\int_{-\infty}^{\sigma'} \frac{\beta\sin\theta + \gamma_\perp(\sigma''+\tau_0)}{(1+\sigma''^2)^{3/2}}\, d\sigma'' = -\frac{Ze^2}{m\bar{v}^2}\Big[(\beta\sin\theta + \gamma_\perp\tau_0)\underbrace{\int_{-\infty}^{\sigma} d\sigma'\int_{-\infty}^{\sigma'} \frac{d\sigma''}{(1+\sigma''^2)^{3/2}}}_{I_0(\sigma)} +$$

$$+\gamma_\perp \underbrace{\int_{-\infty}^{\sigma} d\sigma'\int_{-\infty}^{\sigma'} \frac{\sigma''\, d\sigma''}{(1+\sigma''^2)^{3/2}}}_{I_1(\sigma)}\Big] = -\frac{Ze^2}{m\bar{v}^2}\Big[(\beta\sin\theta + \gamma_\perp\tau_0) I_0(\sigma) + \gamma_\perp I_1(\sigma)\Big]$$

$$\delta^{(1)} r_y^{(s)} = -\frac{Ze^2}{m\bar{v}^2}\int_{-\infty}^{\sigma} d\sigma'\int_{-\infty}^{\sigma'} \frac{-\beta\cos\theta}{(1+\sigma''^2)^{3/2}}\, d\sigma'' = -\frac{Ze^2}{m\bar{v}^2}(-\beta\cos\theta)\, I_0(\sigma)$$

$$\delta^{(1)} r_z^{(s)} = -\frac{Ze^2}{m\bar{v}^2}\int_{-\infty}^{\sigma} d\sigma'\int_{-\infty}^{\sigma'} \frac{\gamma_\parallel(\sigma''+\tau_0)}{(1+\sigma''^2)^{3/2}}\, d\sigma'' = -\frac{Ze^2}{m\bar{v}^2}\gamma_\parallel\Big[I_1(\sigma) + \tau_0 I_0(\sigma)\Big] \longrightarrow (3.41)$$

For this reason

$$\bar{r}_i\,\delta^{(1)} r_i^{(s)} = \bar{r}_x\,\delta^{(1)} r_x^{(s)} + \bar{r}_y\,\delta^{(1)} r_y^{(s)} + \bar{r}_z\,\delta^{(1)} r_z^{(s)} =$$

$$= r_0(\beta\sin\theta+\gamma_\perp\tau)\left(-\frac{Ze^2}{m\bar v^2}\right)\left[(\beta\sin\theta+\gamma_\perp\tau_0)I_0+\gamma_\perp I_1\right]+r_0(-\beta\cos\theta)\left(-\frac{Ze^2}{m\bar v^2}\right)(-\beta\cos\theta)I_0+r_0\gamma_\parallel^2\tau\left(-\frac{Ze^2}{m\bar v^2}\right)\left(I_1+\tau_0 I_0\right)=$$

$$=-\frac{Ze^2 r_0}{m\bar v^2}\left\{\left[(\beta\sin\theta+\gamma_\perp\tau)(\beta\sin\theta+\gamma_\perp\tau_0)+\beta^2\cos^2\theta+\gamma_\parallel^2\tau\tau_0\right]I_0+\left[\gamma_\perp(\beta\sin\theta+\gamma_\perp\tau)+\gamma_\parallel^2\tau\right]I_1\right\}=$$

$$=-\frac{Ze^2 r_0}{m\bar v^2}\left\{\left[\beta^2\sin^2\theta+\underbrace{\beta\gamma_\perp\sin\theta\cdot\tau_0}_{-\tau_0}+\underbrace{\beta\gamma_\perp\sin\theta\cdot\tau}_{-\tau_0}+\gamma_\perp^2\tau\tau_0+\beta^2\cos^2\theta+\gamma_\parallel^2\tau\tau_0\right]I_0+\left[\underbrace{\beta\gamma_\perp\sin\theta}_{-\tau_0}+\gamma_\perp^2\tau+\gamma_\parallel^2\tau\right]I_1\right\}=$$

$-\beta\gamma_\perp\sin\theta=\tau_0$ (3.2

$\beta^2=1+\tau_0^2$ (3.32

$\gamma_\perp^2+\gamma_\parallel^2=1$

$$=-\frac{Ze^2 r_0}{m\bar v^2}\left\{\left[1+\cancel{\tau_0^2}-\cancel{\tau_0^2}-\cancel{\tau_0\tau}+\underbrace{(\gamma_\perp^2+\gamma_\parallel^2)}_{=1}\cancel{\tau\tau_0}\right]I_0+(\tau-\tau_0)I_1\right\}$$ ~~or~~

Attention! Toepffer has another expre for (3.41) an more complicate

$$\bar r_i\delta^{(1)}r_i^{(s)}=-\frac{Ze^2 r_0}{m\bar v^2}\left[I_0(\tau)+\underbrace{(\tau-\tau_0)}_{=\sigma\ (3.32)}I_1(\tau)\right]\quad (3.41)$$, where $I_n(\tau)=\int_{-\infty}^{\sigma}d\sigma'\int_{-\infty}^{\sigma'}\frac{\sigma''^n\,d\sigma''}{(1+\sigma''^2)^{3/2}}$ and

$$I_0(\sigma)=\int_{-\infty}^{\sigma}d\sigma'\int_{-\infty}^{\sigma'}\underbrace{\frac{d\sigma''}{(1+\sigma''^2)^{3/2}}}_{2.271.5}=\int_{-\infty}^{\sigma}d\sigma'\frac{\sigma''}{\sqrt{1+\sigma''^2}}\Bigg|_{-\infty}^{\sigma'}=\int_{-\infty}^{\sigma}d\sigma'\left(\frac{\sigma'}{\sqrt{1+\sigma'^2}}+1\right)$$

$$I_1(\sigma)=\int_{-\infty}^{\sigma}d\sigma'\int_{-\infty}^{\sigma'}\underbrace{\frac{\sigma''d\sigma''}{(1+\sigma''^2)^{3/2}}}_{2.271.7}=\int_{-\infty}^{\sigma}d\sigma'\left(-\frac{1}{\sqrt{1+\sigma''^2}}\right)\Bigg|_{-\infty}^{\sigma'}=-\int_{-\infty}^{\sigma}\frac{d\sigma'}{\sqrt{1+\sigma''^2}}$$

and then

$$\delta^{(1)}E_z^{(s)}=\frac{Ze}{(\bar r_i\bar r_i)^{3/2}}\left[\delta^{(1)}r_z^{(s)}-3\bar r_z\frac{\bar r_j\delta^{(1)}r_j^{(s)}}{(\bar r_i\bar r_i)}\right]$$

but $$\bar r_i\bar r_i=\left[(\beta\sin\theta+\gamma_\perp\tau)^2+\beta^2\cos^2\theta+\gamma_\parallel^2\tau^2\right]r_0^2=r_0^2\left[\beta^2\sin^2\theta+\underbrace{2\beta\sin\theta\gamma_\perp\tau}_{-\tau_0}+\gamma_\perp^2\tau^2+\beta^2\cos^2\theta+\gamma_\parallel^2\tau^2\right]=r_0^2\left[\beta^2-2\tau\tau_0+\tau^2\right]=r_0^2\left[1+\tau_0^2-2\tau\tau_0+\tau^2\right]=r_0^2\left[1+(\tau-\tau_0)^2\right]=r_0^2(1+\sigma^2)$$ (3.22)

So,

$$\delta^{(1)} E_3^{(s)} = \frac{Ze}{r_0^3 (1+\sigma^2)^{3/2}} \left\{ \left(-\frac{Ze^2}{m\bar{v}^2}\right) \gamma_{\parallel} \left[I_1 + \tau_0 I_0 \right] - 3 \cancel{r_0} \gamma_{\parallel} \tau \frac{\left(-\frac{Ze^2 \cancel{r_0}}{m\bar{v}^2}\right)(I_0 + \sigma I_1)}{\cancel{r_0^2}(1+\sigma^2)} \right\} =$$

$$= -\frac{Z^2 e^3}{m r_0^3 \bar{v}^2} \gamma_{\parallel} \frac{1}{(1+\sigma^2)^{3/2}} \left\{ I_1 + \tau_0 I_0 - 3\tau \frac{I_0 + \sigma I_1}{1+\sigma^2} \right\} = -\frac{Z^2 e^3}{m r_0^3 \bar{v}^2} \frac{\gamma_{\parallel}}{(1+\sigma^2)^{3/2}} \left[I_1 + \tau_0 I_0 - 3(\sigma+\tau_0) \frac{I_0 + \sigma I_1}{1+\sigma^2} \right]$$

Then

$$\delta^{(2)} \vec{v}^{(s)}(t) = -\frac{e}{m} \int_{-\infty}^{t} dt' \, T^{-1}(\Omega t') \, \delta^{(1)} \vec{E}^{(s)} \xrightarrow[\omega=0]{} -\frac{e r_0}{m \bar{v}} \int^{\sigma} d\sigma' \delta^{(1)} \vec{E}^{(s)}$$

and for z-component one has

$$\delta^{(3)} v_z^{(s)}(t) = \left(\frac{Ze^2}{m r_0}\right)^2 \frac{\gamma_{\parallel}}{\bar{v}^3} \int_{-\infty}^{\sigma} d\sigma' \frac{1}{(1+\sigma'^2)^{3/2}} \left[I_1(\sigma') + \tau_0 I_0(\sigma') - 3(\sigma' + \tau_0) \frac{I_0(\sigma') + \sigma' I_1(\sigma')}{1+\sigma'^2} \right]$$

The averaging over $\tau_0 = -\frac{\gamma_\perp}{\gamma_{\parallel}} \cos\psi$ $\left(\langle \rangle = \frac{1}{\pi} \int_0^{\pi} f(\psi)\, d\psi \right)$ means that odd powers of $\cos\psi$ give 0. So

$$\langle \delta^{(3)} v_z(t) \rangle = \left(\frac{Ze^2}{m r_0}\right)^2 \frac{\gamma_{\parallel}}{\bar{v}^3} \int_{-\infty}^{\sigma} \frac{d\sigma'}{(1+\sigma'^2)^{3/2}} \left[I_1(\sigma') - 3\sigma' \frac{I_0(\sigma') + \sigma' I_1(\sigma')}{1+\sigma'^2} \right] = \left(\frac{Ze^2}{m r_0}\right)^2 \frac{\gamma_{\parallel}}{\bar{v}} \int_{-\infty}^{\sigma} \frac{(1+\sigma'^2) I_1 - 3\sigma'(I_0 + \sigma' I_1)}{(1+\sigma'^2)^{5/2}}$$

$$= \left(\frac{Ze^2}{m r_0}\right)^2 \frac{\gamma_{\parallel}}{\bar{v}^3} \int_{-\infty}^{\sigma} \frac{d\sigma'}{(1+\sigma'^2)^{5/2}} \left[I_1 (1+\sigma'^2 - 3\sigma'^2) - 3\sigma' I_0 \right] = \left(\frac{Ze^2}{m r_0}\right)^2 \frac{\gamma_{\parallel}}{\bar{v}^3} \Bigg[\underbrace{\int_{-\infty}^{\sigma} \frac{(1-2\sigma'^2)}{(1+\sigma'^2)^{5/2}} I_1(\sigma') d\sigma'}_{J_1} - \underbrace{3 \int_{-\infty}^{\sigma} \frac{\sigma'}{(1+\sigma'^2)^{5/2}} I_0(\sigma') d\sigma'}_{J_0} \Bigg]$$

$$\mathcal{J}_1 = \int_{-\infty}^{\sigma} \frac{1-2\sigma'^2}{(1+\sigma'^2)^{5/2}} \underbrace{I_1(\sigma')}_{\text{from page (25)}} d\sigma' = \int_{-\infty}^{\sigma} \frac{-2(1+\sigma'^2)+3}{(1+\sigma'^2)^{5/2}} \left(-\int_{-\infty}^{\sigma'} \frac{d\sigma''}{\sqrt{1+\sigma''^2}}\right) d\sigma' =$$

$$= 2 \underbrace{\int_{-\infty}^{\sigma} \frac{d\sigma'}{(1+\sigma'^2)^{3/2}} \int_{-\infty}^{\sigma'} \frac{d\sigma''}{\sqrt{1+\sigma''^2}}}_{\mathcal{J}_{13}} - 3 \underbrace{\int_{-\infty}^{\sigma} \frac{d\sigma'}{(1+\sigma'^2)^{5/2}} \int_{-\infty}^{\sigma'} \frac{d\sigma''}{\sqrt{1+\sigma''^2}}}_{\mathcal{J}_{15}}.$$

and with using of rule of multiple integration from page (8):

$$\mathcal{J}_{13} = \int_{-\infty}^{\sigma} \frac{d\sigma'}{(1+\sigma'^2)^{3/2}} \int_{-\infty}^{\sigma'} \frac{d\sigma''}{\sqrt{1+\sigma''^2}} = \left(\frac{1}{(1+\sigma'^2)^{3/2}} = \left(\frac{\sigma'}{(1+\sigma'^2)^{1/2}}\right)' \right) =$$

$$= \frac{\sigma}{\sqrt{1+\sigma^2}} \int_{-\infty}^{\sigma} \frac{d\sigma'}{\sqrt{1+\sigma'^2}} - \int_{-\infty}^{\sigma} \frac{\sigma'}{(1+\sigma'^2)^{1/2}} \cdot \frac{1}{(1+\sigma'^2)^{1/2}} d\sigma' = \frac{\sigma}{\sqrt{1+\sigma^2}} \ln\left(\sigma'+\sqrt{1+\sigma'^2}\right)\Big|_{-\infty}^{\sigma} - \int_{-\infty}^{\sigma} \frac{\sigma' d\sigma'}{1+\sigma'^2} =$$

$$= \frac{\sigma}{\sqrt{1+\sigma^2}}\left[\ln\left(\sigma+\sqrt{1+\sigma^2}\right) - \overline{\ln}_1\right] - \frac{1}{2}\ln(1+\sigma'^2)\Big|_{-\infty}^{\sigma} = \frac{\sigma}{\sqrt{1+\sigma^2}}\left[\ln\left(\sigma+\sqrt{1+\sigma^2}\right) - \overline{\ln}_1\right] - \frac{1}{2}\left[\ln(1+\sigma^2) - \overline{\ln}\right.$$

$$\mathcal{J}_{15} = \int_{-\infty}^{\sigma} \frac{d\sigma'}{(1+\sigma'^2)^{5/2}} \int_{-\infty}^{\sigma'} \frac{d\sigma''}{\sqrt{1+\sigma''^2}} = \left(\frac{1}{(1+\sigma'^2)^{5/2}} = \left(\frac{\sigma'}{\sqrt{1+\sigma'^2}} - \frac{1}{3}\frac{\sigma'^3}{(1+\sigma'^2)^{3/2}} \right)' \right) =$$

$$= \left(\frac{\sigma}{\sqrt{1+\sigma^2}} - \frac{\sigma^3}{3(1+\sigma^2)^{3/2}}\right) \int_{-\infty}^{\sigma} \frac{d\sigma'}{\sqrt{1+\sigma''^2}} - \int_{-\infty}^{\sigma} \left[\frac{\sigma'}{\sqrt{1+\sigma'^2}} - \frac{\sigma'^3}{3(1+\sigma'^2)^{3/2}} \right] \frac{d\sigma'}{\sqrt{1+\sigma'^2}}$$

$$= \left[\frac{\sigma}{\sqrt{1+\sigma^2}} - \frac{\sigma^3}{3(1+\sigma^2)^{3/2}}\right]\left[\ln\left(\sigma+\sqrt{1+\sigma^2}\right) - \overline{\ln}_1\right] - \int_{-\infty}^{\sigma} \frac{\sigma' d\sigma'}{1+\sigma'^2} + \frac{1}{3}\int_{-\infty}^{\sigma} \frac{\sigma'^3 d\sigma'}{(1+\sigma'^2)^2} =$$