Checking of

Toepffer. Scattering of Magnetized Electrons with Ions

Phys. Rev., A 66 (2002) 02274

$$\delta \vec{V}^{(n)}_{\beta}(t) = -\frac{Ze^2}{m} \int_{-\infty}^{t} \frac{dt'}{\bar{r}^3(t')} T^{-1}(\Omega t') \cdot \vec{r}(t')$$

using (3.34)

From (3.35)

$$\bar{r}(t) = \left[r_0^2 + \bar{v}^2 (t-t_0)^2\right]^{1/2} =$$

$$= r_0 \left[1 + \frac{\bar{v}^2 (t-t_0)^2}{r_0^2}\right]^{1/2} =$$

$$= r_0 \left[1 + (\tau - \tau_0)^2\right]^{1/2}$$

From (3.10) in the limit $R \to 0$: ①

$$\vec{r}(t) = \begin{pmatrix} b \sin\theta - v_{i\perp} t \\ -b\cos\theta \\ (v_{e\parallel} - v_{i\parallel}) t \end{pmatrix} =$$

$$= r_0 \begin{pmatrix} \frac{b}{r_0} \sin\theta - \frac{v_{i\perp}}{\bar{v}} \frac{\bar{v} t}{r_0} \\ -\frac{b}{r_0} \cos\theta \\ \frac{v_{e\parallel} - v_{i\parallel}}{\bar{v}} \frac{\bar{v} t}{r_0} \end{pmatrix} .$$

$$\boxed{\tau = \frac{\bar{v} t}{r_0}, \quad \tau_0 = \frac{\bar{v} t_0}{r_0}}$$

Thus is (3.34)

$$= r_0 \begin{pmatrix} \beta \sin\theta + \gamma_\perp \tau \\ -\beta \cos\theta \\ \gamma_\parallel \tau \end{pmatrix}$$

$$= -\frac{Ze^2}{m} \frac{r_0}{\bar{v}} \int_{-\infty}^{\tau} \frac{d\tau' \quad r_0}{r_0^3 \left[1 + (\tau' - \tau_0)\right]^{3/2}} \begin{pmatrix} \cos \frac{\Omega r_0}{\bar{v}} \tau' & \sin \frac{\Omega r_0}{\bar{v}} \tau' & 0 \\ -\sin \frac{\Omega r_0}{\bar{v}} \tau' & \cos \frac{\Omega r_0}{\bar{v}} \tau' & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \beta \sin\theta + \gamma_\perp \tau' \\ -\beta\cos\theta \\ \gamma_\parallel \tau' \end{pmatrix}$$

$$\boxed{\begin{matrix} \sigma = \tau' - \tau_0 \\ d\tau' = d\sigma \end{matrix}}$$

$$= -\frac{Ze^2}{m} \frac{1}{\bar{v} r_0} \int_{-\infty}^{\sigma} \frac{d\sigma'}{(1+\sigma'^2)^{3/2}} \begin{pmatrix} \cos\omega\tau' & \sin\omega\tau' & 0 \\ -\sin\omega\tau' & \cos\omega\tau' & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \beta \sin\theta + \gamma_\perp \tau' \\ -\beta\cos\theta \\ \gamma_\parallel \tau' \end{pmatrix}$$

710

3.36

„x": $\cos\omega\tau'(\beta\sin\theta+\gamma_\perp\tau')+\beta\cos\theta\sin\omega\tau' =$

$$=\beta[\sin\theta\cos\omega\tau'+\cos\theta\sin\omega\tau'] + \gamma_\perp\tau'\cos\omega\tau' = \beta\sin(\theta+\omega\tau')+\gamma_\perp\tau'\cos\omega\tau'$$

„y": $-\cancel{\beta}\sin\omega\tau'(\beta\sin\theta+\gamma_\perp\tau') - \beta\cos\omega\tau'\cos\theta =$

$$= -\beta(\sin\theta\sin\omega\tau' + \cos\theta\cos\omega\tau'] - \gamma_\perp\tau'\sin\omega\tau' = -\beta\cos(\omega\tau'-\theta) - \gamma_\perp\tau'\sin\omega\tau'$$

Смотрим z-компоненту от $\delta\vec{V}_B(t)$ при $t\to\infty$:

$$(\delta\vec{V}_B^{(1)})_z\Big|_{t\to\infty} = -\frac{ze^2}{m}\frac{1}{\sqrt{r_0}}\int_{-\infty}^{\infty}\frac{d\sigma'}{[1+\sigma'^2]^{3/2}}\gamma_\parallel\tau' = -\frac{ze^2}{m}\frac{\gamma_\parallel}{\sqrt{r_0}}\Big[\underbrace{\int_{-\infty}^{\infty}\frac{\sigma'd\sigma'}{(1+\sigma'^2)^{3/2}}}_{=0\ \text{из-за нечётности ф-ции}} + \tau_0\underbrace{\int_{-\infty}^{\infty}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}}_{=\frac{\sigma'}{\sqrt{1+\sigma'^2}}\Big|_{-\infty}^{\infty}=2}\Big] =$$

$$= -\frac{2ze^2}{m}\frac{\gamma_\parallel\tau_0}{\sqrt{r_0}}$$ ok (3.37c)

Смотрим x-компоненту:

$$(\delta\vec{V}_B^{(1)})_x\Big|_{t\to\infty} = -\frac{ze^2}{m}\frac{1}{\sqrt{r_0}}\int_{-\infty}^{\infty}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\Big[\underbrace{\beta\sin(\omega\tau'+\theta)}_{„1"} + \underbrace{\gamma_\perp\overbrace{\tau'}^{=\sigma'+\tau_0}\cos\omega\tau'}_{„2"}\Big]$$

$$„2" = \gamma_\perp\int_{-\infty}^{\infty}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}(\sigma'+\tau_0)\cos[\omega(\sigma'+\tau_0)] = \gamma_\perp\tau_0\int_{-\infty}^{\infty}\frac{\cos\omega\sigma'\cos\omega\tau_0-\sin\omega\sigma'\sin\omega\tau_0}{(1+\sigma'^2)^{3/2}}d\sigma'$$

$$=\gamma_\perp\tau_0\Big[\cos\omega\tau_0\int_{-\infty}^{\infty}\frac{\cos\omega x\,dx}{(1+x^2)^{3/2}} - \sin\omega\tau_0\underbrace{\int_{-\infty}^{\infty}\frac{\sin\omega x\,dx}{(1+x^2)^{3/2}}}_{=0\ \text{из-за нечётности ф-ции}}\Big] = 2\gamma_\perp\tau_0\cos\omega\tau_0\int_{0}^{\infty}\frac{\cos\omega x\,dx}{(1+x^2)^{3/2}}$$

$$+\gamma_\perp \int_{-\infty}^{\infty} \frac{\sigma'(\cos\omega\sigma'\cos\omega\tau_0 - \sin\omega\sigma'\sin\omega\tau_0)\,d\sigma'}{(1+\sigma'^2)^{3/2}} =$$

$$= \gamma_\perp\tau_0 \left[\cos\omega\tau_0 \int_{-\infty}^{\infty} \frac{\cos\omega x\,dx}{(1+x^2)^{3/2}} - \sin\omega\tau_0 \underbrace{\int_{-\infty}^{\infty} \frac{\sin\omega x\,dx}{(1+x^2)^{3/2}}}_{=0 \text{ из-за нечётности}}\right] +$$

$$+\gamma_\perp \left[\cos\omega\tau_0 \underbrace{\int_{-\infty}^{\infty} \frac{x\cos\omega x\,dx}{(1+x^2)^{3/2}}}_{=0 \text{ из-за нечётности}} - \sin\omega\tau_0 \int_{-\infty}^{\infty} \frac{x\sin\omega x\,dx}{(1+x^2)^{3/2}}\right] =$$

$$= 2\gamma_\perp\left[\tau_0\cos\omega\tau_0 \underbrace{\int_0^{\infty} \frac{\cos\omega x\,dx}{(1+x^2)^{3/2}}}_{I} - \sin\omega\tau_0 \underbrace{\int_0^{\infty} \frac{x\sin\omega x\,dx}{(1+x^2)^{3/2}}}_{\text{это } 3.754.3 \,=\, \omega K_0(\omega)}\right]$$

чтобы найти $I$ найдем сначала

$$\hat{I}(a) = \int_0^{\infty} \frac{\cos\omega x\,dx}{(1+ax^2)^{1/2}} = \frac{1}{\sqrt{a}} \int_0^{\infty} \frac{\cos\omega x\,dx}{\sqrt{\frac{1}{a}+x^2}} = \frac{1}{\sqrt{a}} K_0\left(\frac{\omega}{\sqrt{a}}\right)$$

это 3.754.2

тогда

$$I = \int_0^{\infty} \frac{\cos\omega x\,dx}{(1+x^2)^{3/2}} = -2\frac{\partial}{\partial a}\int_0^{\infty} \frac{\cos\omega x\,dx}{(1+ax^2)^{1/2}}$$

$$I(\beta) = \int_0^{\infty} \frac{\cos\omega x\,dx}{\sqrt{\beta+x^2}} = K_0(\sqrt{\beta}\,\omega), \quad \text{тогда}$$

это 3.754.2

$$\int_0^{\infty} \frac{\cos\omega x\,dx}{\sqrt{1+x^2}^{\,3}} = -2\left(\frac{\partial}{\partial\beta}\int_0^{\infty} \frac{\cos\omega x\,dx}{\sqrt{\beta+x^2}}\right)\Bigg|_{\beta=1} = -2\frac{\partial}{\partial\beta}\left(K_0\sqrt{\beta}\,\omega\right)\Bigg|_{\beta=1} =$$

$$= -2\,\frac{1}{2}\,\frac{\omega}{\sqrt{\beta}}\,K_0'(\sqrt{\beta}\,\omega)\Big|_{\beta=1} = -\omega\, \underset{\uparrow\, 8.486.18}{K_0'(\omega)} = \omega K_1(\omega)$$

Итак,

$$\text{„}2\text{"} = 2\gamma_\perp\left[\omega\tau_0\cos(\omega\tau_0)\,K_1(\omega) - \omega\sin(\omega\tau_0)\,K_0(\omega)\right]$$

Теперь „1":

$$\text{„}1\text{"} = \beta\int_{-\infty}^{\infty}\frac{\sin(\omega\overset{=\sigma'+\tau_0}{\tau'}+\theta)\,d\sigma'}{(1+\sigma'^2)^{3/2}} = \beta\int_{-\infty}^{\infty}\frac{\sin(\omega\sigma'+\omega\tau_0+\theta)\,d\sigma'}{(1+\sigma'^2)^{3/2}} =$$

$$= \beta\left[\sin(\omega\tau_0+\theta)\int_{-\infty}^{\infty}\frac{\cos\omega x\,dx}{(1+x^2)^{3/2}} + \cos(\omega\tau_0+\theta)\underbrace{\int_{-\infty}^{\infty}\frac{\sin\omega x\,dx}{(1+x^2)^{3/2}}}_{=0\text{ из-за нечётности}}\right] = 2\beta\sin(\omega\tau_0+\theta)\underbrace{\int_0^{\infty}\frac{\cos\omega x\,dx}{(1+x^2)^{3/2}}}_{\text{найдено выше и равно } \omega K_1(\omega)} =$$

$$= 2\beta\omega\sin(\omega\tau_0+\theta)\,K_1(\omega)$$

Т.о. „1"+„2":

$$\text{„}1\text{"}+\text{„}2\text{"} = 2\beta\omega\sin(\omega\tau_0+\theta)\,K_1(\omega) + 2\gamma_\perp\left[\omega\tau_0\cos(\omega\tau_0)K_1(\omega) - \omega\sin(\omega\tau_0)\,K_0(\omega)\right] =$$

$$= 2\gamma_\perp\,\omega\, K_0(\omega)\sin\omega\tau_0 + 2\omega\, K_1(\omega)\left[\beta\sin(\omega\tau_0+\theta) + \gamma_\perp\tau_0\cos(\omega\tau_0)\right]$$

Имеем

$$[\ \ ] = \underbrace{\beta\cos\theta}_{3.30}\sin(\omega\tau_0) + \beta\sin\theta\cos\omega\tau_0 + \gamma_\perp\tau_0\cos\omega\tau_0 = \sin(\omega\tau_0)\sin\psi +$$

$$+ \cos(\omega\tau_0)\Big[\underbrace{\beta\sin\theta + \gamma_\perp\tau_0}_{\text{это } 3.31}\Big] = \sin(\omega\tau_0)\sin\psi + \gamma_\parallel\cos\psi\cos(\omega\tau_0)$$

т.о. $$(\delta \vec{V}_B^{(1)})_x\Big|_{t\to\infty} = -\frac{Ze^2}{mVr_0}\Big[2\omega K_1(\omega)\left(\gamma_\parallel \cos\psi \cos\omega\tau_0 + \sin\psi \sin\omega\tau_0\right) \mp 2\omega K_0(\omega)\,\gamma_\perp \sin\omega\tau_0\Big]$$ (3.37а)

совпадает с Toepffer

~~у Toepffer'а другой знак~~

Аналогично:

$$(\delta \vec{V}_B^{(1)})_y\Big|_{t\to\infty} = -\frac{Ze^2}{mVr_0}\int_{-\infty}^{\infty}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\Big[\underbrace{-\beta\cos(\omega\tau'-\theta)}_{-\,\text{„3"}} \mp \underbrace{\gamma_\perp\tau'\sin\omega\tau'}_{-\,\text{„4"}}\Big]$$

Имеем:

$$\text{„3"} = \beta\int_{-\infty}^{\infty}\frac{\cos(\omega\overbrace{\tau'}^{=\sigma'+\tau_0}-\theta)\,d\sigma'}{(1+\sigma'^2)^{3/2}} = \beta\Big[\cos(\omega\tau_0-\theta)\int_{-\infty}^{\infty}\frac{\cos\omega x\,dx}{(1+x^2)^{3/2}} + \sin(\omega\tau_0-\theta)\underbrace{\int_{-\infty}^{\infty}\frac{\sin\omega x\,dx}{(1+x^2)^{3/2}}}_{=0\text{ из-за нечётности}}\Big] =$$

$$= 2\beta\cos(\omega\tau_0-\theta)\underbrace{\int_0^{\infty}\frac{\cos\omega x\,dx}{(1+x^2)^{3/2}}}_{\text{см. ранее } =\omega K_1(\omega)} = 2\beta\omega K_1(\omega)\cos(\omega\tau_0-\theta)$$

$$\text{„4"} = \gamma_\perp\int_{-\infty}^{\infty}\frac{\overbrace{\tau'}^{=\sigma'+\tau_0}\sin\omega\tau'\,d\sigma'}{(1+\sigma'^2)^{3/2}} = \gamma_\perp\tau_0\int_{-\infty}^{\infty}\frac{\sin(\omega\sigma'+\omega\tau_0)\,d\sigma'}{(1+\sigma'^2)^{3/2}} + \gamma_\perp\int_{-\infty}^{\infty}\frac{\sigma'\sin(\omega\sigma'+\omega\tau_0)\,d\sigma'}{(1+\sigma'^2)^{3/2}} =$$

$$= \gamma_\perp\tau_0\Big[\cos\omega\tau_0\underbrace{\int_{-\infty}^{\infty}\frac{\sin\omega x\,dx}{(1+x^2)^{3/2}}}_{=0\text{ из-за нечётности}} + \sin\omega\tau_0\int_{-\infty}^{\infty}\frac{\cos\omega x\,dx}{(1+x^2)^{3/2}}\Big] + \gamma_\perp\Big[\cos\omega\tau_0\int_{-\infty}^{\infty}\frac{x\sin\omega x\,dx}{(1+x^2)^{3/2}} +$$

$$+ \sin\omega\tau_0\underbrace{\int_{-\infty}^{\infty}\frac{x\cos\omega x\,dx}{(1+x^2)^{3/2}}}_{=0\text{ из-за нечётности}}\Big] = 2\gamma_\perp\Big[\tau_0\sin\omega\tau_0\int_0^{\infty}\frac{\cos\omega x\,dx}{(1+x^2)^{3/2}} + \cos\omega\tau_0\int_0^{\infty}\frac{x\sin\omega x\,dx}{(1+x^2)^{3/2}}\Big] =$$

$$= 2\gamma_\perp\Big\{\tau_0\,\omega K_1(\omega)\sin\omega\tau_0 + \cos\omega\tau_0\,\omega K_0(\omega)\Big\} = 2\omega\gamma_\perp\Big[\tau_0 K_1(\omega)\sin\omega\tau_0 + K_0(\omega)\cos\omega\tau_0\Big]$$

Т.о.

$$\left(\delta \vec{V}_B^{(1)}\right)_y \Big|_{t\to\infty} = -\frac{Ze^2}{m\bar{v}r_0}\Big[-2\beta\omega K_1(\omega)\cos(\omega\tau_0-\theta) \mp 2\omega\gamma_\perp\tau_0 K_1(\omega)\sin\omega\tau_0 \mp -$$

$$\mp 2\omega\gamma_\perp K_0 \cos\omega\tau_0\Big] =$$

$$= -\frac{Ze^2}{m\bar{v}r_0}\Big[-2\omega K_0(\omega)\gamma_\perp\cos\omega\tau_0 + 2\omega K_1(\omega)\underbrace{\big[-\beta\cos(\omega\tau_0-\theta) \mp \gamma_\perp\tau_0\sin\omega\tau_0\big]}_{\text{⑧}}\Big]$$

$$[\ ] = \mp\beta\cos\theta\cos\omega\tau_0 - \beta\sin\theta\sin\omega\tau_0 - \gamma_\perp\tau_0\sin\omega\tau_0 =$$

$$= \underbrace{-\beta\cos\theta}_{3.30}\cos\omega\tau_0 - (\beta\sin\theta + \gamma_\perp\tau_0)\sin\omega\tau_0 = + \sin\Psi\cos\omega\tau_0 - \gamma_\parallel\cos\Psi\sin\omega\tau_0$$

3.31

Так что окончательно

$$\left(\delta \vec{V}_B^{(1)}\right)_y \Big|_{t\to\infty} = -\frac{Ze^2}{m\bar{v}r_0}\Big[+2\omega K_1(\omega)\left(-\gamma_\parallel\cos\Psi\sin\omega\tau_0 + \sin\Psi\cos\omega\tau_0\right) - 2\omega K_0(\omega)\gamma_\perp\cos\omega\tau_0\Big]$$

у Toepffer'a другой знак

? совпадает с Toepffer (3.37b)

~~у Toepffer'a другой знак!~~

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$$\delta r_\parallel^{(1)}(t) = -\frac{Ze^2}{m\bar{v}^2}\int_{-\infty}^{\sigma} d\sigma' \int_{-\infty}^{\sigma'} \frac{\gamma_\parallel(\sigma''+\tau_0)}{(1+\sigma''^2)^{3/2}}d\sigma'' = -\frac{Ze^2}{m\bar{v}^2}\gamma''\int_{-\infty}^{\sigma} d\sigma'\left[\int_{-\infty}^{\sigma'}\frac{\sigma''d\sigma''}{(1+\sigma''^2)^{3/2}} + \tau_0\int_{-\infty}^{\sigma'}\frac{d\sigma''}{(1+\sigma''^2)^{3/2}}\right] =$$

это (3.41)

$$= -\frac{Ze^2}{m\bar{v}^2}\gamma''\int_{-\infty}^{\sigma}d\sigma'\left[-\frac{1}{\sqrt{1+\sigma''^2}}\Big|^{\sigma'} + \tau_0\frac{\sigma''}{\sqrt{1+\sigma''^2}}\Big|^{\sigma'}\right] = -\frac{Ze^2}{m\bar{v}^2}\gamma''\int^{\sigma}d\sigma'\left\{-\frac{\sigma'}{\sqrt{1+\sigma'^2}} + \tau_0\left(\frac{\sigma'}{\sqrt{1+\sigma'^2}} + \right.\right.$$

Для будущего:

(7)

$$\int \frac{dx}{\sqrt{a+bx+cx^2}^3} = \frac{2(2cx+b)}{\Delta\sqrt{\phantom{x}}} \quad (\Delta = 4ac-b^2) \longrightarrow \int \frac{dx}{\sqrt{1+x^2}^3} = \frac{x}{\sqrt{1+x^2}}$$

$$\int \frac{x\,dx}{\sqrt{\phantom{x}}^3} = -\frac{2(2a+bx)}{\Delta\sqrt{\phantom{x}}} \longrightarrow \int \frac{x\,dx}{\sqrt{1+x^2}^3} = -\frac{1}{\sqrt{1+x^2}}$$

$$\int_0^\infty \frac{x\sin\omega x\,dx}{\sqrt{1+x^2}^3} = \omega K_0(\omega) \qquad \int_0^\infty \frac{\cos\omega x\,dx}{\sqrt{\beta+x^2}} = K_0(\sqrt{\beta}\omega) \rightarrow \int_0^\infty \frac{\cos\omega x\,dx}{\sqrt{\beta+x^2}^3} = -\frac{\omega}{\sqrt{\beta}}K_0'(\sqrt{\beta}\omega) = \frac{\omega}{\sqrt{\beta}}K_1(\sqrt{\beta}\omega)$$

3.754.3 3.754.2

$$\delta\vec{r}_\perp^{(1)}(t) = -\frac{Ze^2}{mv^2}\int_{-\infty}^{\sigma} d\sigma' \begin{pmatrix} \cos\omega\tau' & -\sin\omega\tau' \\ \sin\omega\tau' & \cos\omega\tau' \end{pmatrix} \int_{-\infty}^{\sigma'} \frac{d\sigma''}{(1+\sigma''^2)^{3/2}} \begin{pmatrix} \cos\omega\tau'' & \sin\omega\tau'' \\ -\sin\omega\tau'' & \cos\omega\tau'' \end{pmatrix} \begin{pmatrix} \beta\sin\theta - \gamma_\perp\tau'' \\ -\beta\cos\theta \end{pmatrix}$$

это лишнее

$$\tau = \sigma + \tau_0$$

Тогда

$$\cos\omega\tau' = \cos\omega(\sigma'+\tau_0) = \cos\omega\tau_0\cos\omega\sigma' - \sin\omega\tau_0\sin\omega\sigma'$$ и аналогично для $\sigma''$

$$\sin\omega\tau' = \sin(\sigma'+\tau_0) = \cos\omega\tau_0\sin\omega\sigma' + \cancel{\sin}\,\omega\tau_0\cos\omega\tau'$$

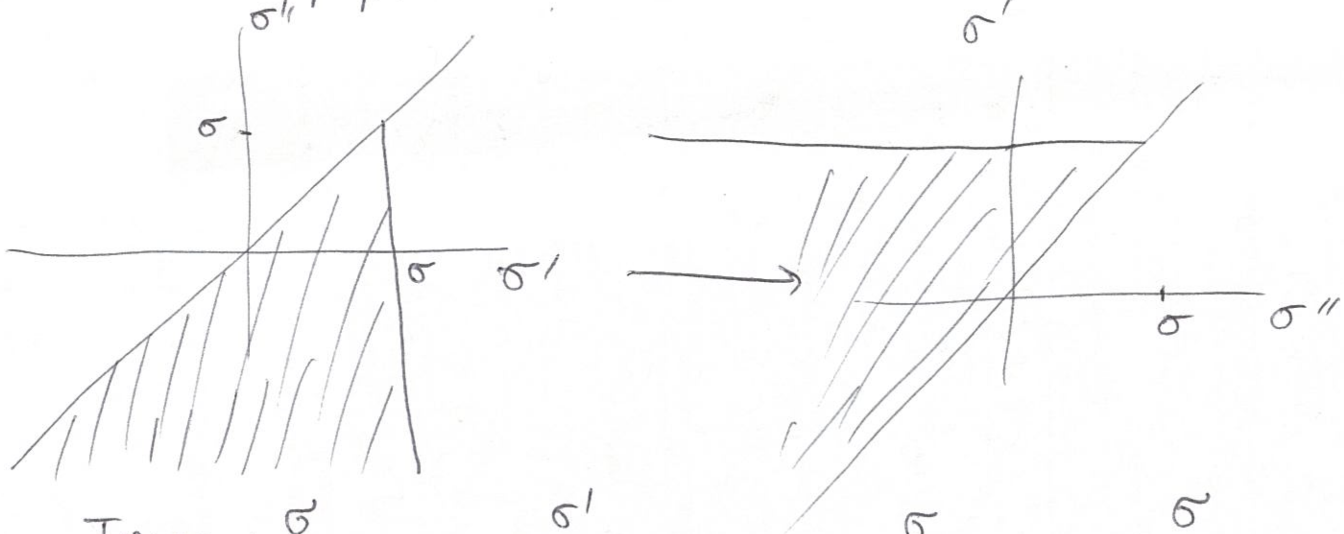
$\cancel{\delta\vec{r}_\perp^{(1)}(t)}$

$$\delta\vec{r}_\perp^{(1)}(t) = -\frac{Ze^2}{mv^2}\int_{-\infty}^{\sigma} d\sigma' \int_{-\infty}^{\sigma'} \frac{d\sigma''}{(1+\sigma''^2)^{3/2}} \underbrace{\begin{pmatrix} \cos\omega\tau' & -\sin\omega\tau' \\ \sin\omega\tau' & \cos\omega\tau' \end{pmatrix} \begin{pmatrix} \cos\omega\tau'' & \sin\omega\tau'' \\ -\sin\omega\tau'' & \cos\omega\tau'' \end{pmatrix} \begin{pmatrix} \beta\sin\theta - \gamma_\perp\tau'' \\ -\beta\cos\theta \end{pmatrix}}_{M} =$$

$$M = \begin{pmatrix} \cos\omega(\tau'-\tau'') & \sin\omega(\tau''-\tau') \\ \sin\omega(\tau'-\tau'') & \cos\omega(\tau'-\tau'') \end{pmatrix} \begin{pmatrix} \beta\sin\theta - \gamma_\perp\tau'' \\ -\beta\cos\theta \end{pmatrix} = \begin{pmatrix} \cos\omega(\sigma'-\sigma'')(\beta\sin\theta - \gamma_\perp\tau'') + \sin\omega(\sigma'-\sigma'')\beta\cos\theta \\ \sin\omega(\sigma'-\sigma'')(\beta\sin\theta - \gamma_\perp\tau'') - \cos\omega(\sigma'-\sigma'')\beta\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \beta\sin[\omega(\sigma'-\sigma'')+\theta] - \gamma_\perp(\sigma''+\tau_0)\cos\omega(\sigma'-\sigma'') \\ -\beta\cos[\omega(\sigma'-\sigma'')+\theta] - \gamma_\perp(\sigma''+\tau_0)\sin\omega(\sigma'-\sigma'') \end{pmatrix}$$

Область интегрирования




Тогда $\int\limits_{-\infty}^{\sigma} f_1(\sigma')d\sigma' \int\limits_{-\infty}^{\sigma'} f_2(\sigma'')d\sigma'' = \int\limits_{-\infty}^{\sigma} d\sigma'' f_2(\sigma'') \int\limits_{\sigma''}^{\sigma} f_1(\sigma')d\sigma' \underset{\text{переобозначение}}{=} \int\limits_{-\infty}^{\sigma} f_2(\sigma')d\sigma' \int\limits_{\sigma'}^{\sigma} f_1(\sigma'')d\sigma''$

В частности, если $f_1(\sigma) = \dfrac{dg(\sigma)}{d\sigma}$

то
$$\int\limits_{-\infty}^{\sigma} \frac{dg(\sigma')}{d\sigma'}d\sigma' \int\limits_{-\infty}^{\sigma'} f_2(\sigma'')\,d\sigma'' = \int\limits_{-\infty}^{\sigma} f_2(\sigma')d\sigma' \int\limits_{\sigma'}^{\sigma} \frac{dg_1(\sigma'')}{d\sigma''}d\sigma'' =$$

$$= \int\limits_{-\infty}^{\sigma} f_2(\sigma')d\sigma' \; g(\sigma'')\Big|_{\sigma'}^{\sigma} = g(\sigma)\int\limits_{-\infty}^{\sigma} f_2(\sigma')d\sigma' - \int\limits_{-\infty}^{\sigma} g(\sigma')f_2(\sigma')d\sigma'$$

Это из Appendix

но

$$\begin{pmatrix}\cos\omega\tau' & -\sin\omega\tau' \\ \sin\omega\tau' & \cos\omega\tau'\end{pmatrix} = -\frac{1}{\omega}\frac{d}{d\tau'}\begin{pmatrix}\sin\omega\tau' & \cos\omega\tau' \\ -\cos\omega\tau' & \sin\omega\tau'\end{pmatrix}$$

Тогда в соответствии с р-том пр-а со стр 8 при перестановке порядка интегрирования имеем

$$\delta\vec{r}_\perp^{\,(1)}(t) = -\frac{ze^2}{m\bar{v}^2}\int_{-\infty}^{\sigma}\frac{d\sigma'}{\omega}\frac{d}{d\sigma'}\begin{pmatrix}\sin\omega\tau' & \cos\omega\tau' \\ -\cos\omega\tau' & \sin\omega\tau'\end{pmatrix}\int_{-\infty}^{\sigma'}\frac{d\sigma''}{(1+\sigma''^2)^{3/2}}\begin{pmatrix}\cos\omega\tau'' & \sin\omega\tau'' \\ -\sin\omega\tau'' & \cos\omega\tau''\end{pmatrix}\begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau'' \\ -\beta\cos\theta\end{pmatrix} =$$

$\boxed{\sigma = \frac{\bar{v}t}{r_0}}$ (3.21)

$$= -\frac{1}{\omega\bar{v}}\frac{ze^2}{m\bar{v}^2}\begin{pmatrix}\sin\omega\tau & \cos\omega\tau \\ -\cos\omega\tau & \sin\omega\tau\end{pmatrix}\underbrace{\int_{-\infty}^{\sigma}\frac{d\sigma''}{(1+\sigma'^2)^{3/2}}T(\omega\tau')\begin{pmatrix}\beta\sin\theta-\gamma_\perp\tau' \\ -\beta\cos\theta\end{pmatrix}}_{=r_0\delta\vec{V}_{B\perp}(t)\ \text{from (3.36)}} - \frac{1}{\omega}\left(-\frac{ze^2}{m\bar{v}^2}\right)\times$$

$\boxed{\omega = \frac{r_0\Omega_\perp}{\bar{v}}}$ (3.23)

$$\cancel{\times\int_{-\infty}^{\sigma}\begin{pmatrix}\cos\omega\tau' & \sin\omega\tau' \\ -\sin\omega\tau' & \cos\omega\end{pmatrix}}$$

$$\times\underbrace{\int_{-\infty}^{\sigma}\begin{pmatrix}\sin\omega\tau' & \cos\omega\tau' \\ -\cos\omega\tau' & \sin\omega\tau'\end{pmatrix}d\sigma'\frac{1}{(1+\sigma'^2)^{3/2}}\begin{pmatrix}\cos\omega\tau' & \sin\omega\tau' \\ -\sin\omega\tau' & \cos\omega\tau'\end{pmatrix}\begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau' \\ -\beta\cos\theta\end{pmatrix}}$$

↓ перемножение матриц!

$$\begin{pmatrix}\sin\omega\tau' & \cos\omega\tau' \\ -\cos\omega\tau' & \sin\omega\tau\end{pmatrix}\begin{pmatrix}\cos\omega\tau' & \sin\omega\tau' \\ -\sin\omega\tau' & \cos\omega\tau'\end{pmatrix}\begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau' \\ -\beta\cos\theta\end{pmatrix} =$$

$$= \begin{pmatrix}0 & 1 \\ - & 1\end{pmatrix}\begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau' \\ -\beta\cos\theta\end{pmatrix} = \begin{pmatrix}-\beta\cos\theta \\ -(\beta\sin\theta+\gamma_\perp\tau')\end{pmatrix} =$$

$$= -\begin{pmatrix}\beta\cos\theta \\ \beta\sin\theta+\gamma_\perp\tau'\end{pmatrix}$$

$$\begin{pmatrix}\sin\omega\tau & \cos\omega\tau \\ -\cos\omega\tau & \sin\omega\tau\end{pmatrix}$$

$$= \frac{r_0}{\omega\bar{v}}\delta\vec{V}_{B\perp}(t) - \frac{1}{\omega}\left(-\frac{ze^2}{m\bar{v}^2}\right)\int_{-\infty}^{\sigma}-\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\begin{pmatrix}\beta\cos\theta \\ \beta\sin\theta-\gamma_\perp\tau'\end{pmatrix} =$$

$$= \frac{r_0}{\omega\bar{v}}\delta\vec{V}_{B\perp}(t) - \frac{ze^2}{m\bar{v}^2}\int_{-\infty}^{\sigma}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\begin{pmatrix}\beta\cos\theta \\ \beta\sin\theta+\gamma_\perp\tau'\end{pmatrix}$$

это (3.42)

$$\begin{pmatrix}\sin\omega\tau & \cos\omega\tau \\ -\cos\omega\tau & \sin\omega\tau\end{pmatrix}$$

So,



$$\delta \vec{r}_{\perp}^{(1)}(t) = \frac{r_0}{\omega \bar{v}}\, \delta \vec{V}_{B\perp}^{(1)}(t) \begin{pmatrix} \sin\omega\tau & \cos\omega\tau \\ -\cos\omega\tau & \sin\omega\tau \end{pmatrix} - \frac{Ze^2}{m\bar{v}^2} \int_{-\infty}^{\bar{v}t/r_0} \frac{d\sigma'}{(1+\sigma'^2)^{3/2}} \begin{pmatrix} \beta\cos\theta \\ \beta\sin\theta + \gamma_\perp \tau' \end{pmatrix} \qquad (3.22)$$

$$\boxed{\tau' = \frac{vt'}{r_0} \;(3.21)\,; \quad \sigma = \tau' - \tau_0 \;(3.22) \quad d\sigma = d\tau'}$$

Analogously: (from 3.40):

$$\delta r_{\parallel}^{(1)}(t) = -\frac{Ze^2}{m\bar{v}^2} \int_{-\infty}^{\bar{v}t/r_0} d\sigma'\, T(\omega\tau') \int_{-\infty}^{\sigma'} \frac{d\sigma''}{(1+\sigma''^2)^{3/2}}\, T^{-1}(\omega\tau'')\, \gamma_\parallel \tau'' \text{ (more accuracy!)} =$$

$$= -\frac{Ze^2}{m\bar{v}^2} \int_0^{\bar{v}t/r_0} d\sigma' \begin{pmatrix} \cos\omega\tau' & -\sin\omega\tau' & 0 \\ \sin\omega\tau' & \cos\omega\tau' & 0 \\ 0 & 0 & 1 \end{pmatrix} \int^{\sigma'} \frac{d\sigma''}{(1+\sigma''^2)^{3/2}} \begin{pmatrix} \cos\omega\tau'' & \sin\omega\tau'' & 0 \\ -\sin\omega\tau'' & \cos\omega\tau'' & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \beta\sin\theta + \gamma_\perp\tau'' \\ -\beta\cos\theta \\ \gamma_\parallel \tau'' \end{pmatrix} =$$

But for $\delta r_{\parallel}^{(1)}(t)$ it is necessary to calculate only 3rd component of the product of the matrices:

$$\begin{pmatrix} \cos\omega\tau' & -\sin\omega\tau' & 0 \\ \sin\omega\tau' & \cos\omega\tau' & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\omega\tau'' & \sin\omega\tau'' & 0 \\ -\sin\omega\tau'' & \cos\omega\tau'' & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \beta\sin\theta + \gamma_\perp\tau'' \\ -\beta\cos\theta \\ \gamma_\parallel\tau'' \end{pmatrix} = \begin{pmatrix} \cos\omega(\tau'+\tau'') & -\sin\omega(\tau'-\tau'') & 0 \\ \sin\omega(\tau'-\tau'') & \cos\omega(\tau'+\tau'') & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot$$

$$\cdot \begin{pmatrix} \beta\sin\theta + \gamma_\perp\tau'' \\ -\beta\cos\theta \\ \gamma_\parallel\tau'' \end{pmatrix} = \begin{pmatrix} \cdots & \cdots \\ \cdots & \cdots \\ & \gamma_\parallel\tau'' \end{pmatrix} \longrightarrow \delta r_{\parallel}^{(1)}(t) = -\frac{Ze^2}{m\bar{v}^2} \int_{-\infty}^{\sigma} d\sigma' \int_{-\infty}^{\sigma'} \frac{\gamma_\parallel \tau''}{(1+\sigma''^2)^{3/2}}\, d\sigma'' = \left| \tau'' = \sigma'' + \tau_0 \right. \;(3.22)$$

$$= -\frac{Ze^2}{m\bar{v}^2}\gamma_\parallel \int^{\sigma} d\sigma' \int_{-\infty}^{\sigma'} \frac{(\sigma''+\tau_0)\, d\sigma''}{(1+\sigma''^2)^{3/2}} =$$

$$\int \frac{\sigma'' d\sigma''}{(1+\sigma''^2)^{3/2}} = (2.227) = -\frac{1}{\sqrt{1+\sigma''^2}}$$

$$\int \frac{d\sigma''}{(1+\sigma''^2)^{3/2}} = (2.271.5) = \frac{\sigma''}{\sqrt{1+\sigma''^2}}$$

$$= -\frac{Ze^2}{m\bar{v}^2}\gamma_{\parallel}\int_{-\infty}^{\sigma} d\sigma'\left[-\frac{1}{\sqrt{1+\sigma''^2}}\Bigg|_{-\infty}^{\sigma'} + \tau_0\frac{\sigma''}{\sqrt{1+\sigma''^2}}\Bigg|_{-\infty}^{\sigma'}\right] = -\frac{Ze^2}{m\bar{v}^2}\gamma_{\parallel}\int_{-\infty}^{\sigma} d\sigma'\left[-\frac{1}{\sqrt{1+\sigma'^2}} + \frac{\tau_0\sigma'}{\sqrt{1+\sigma'^2}} - \tau_0(-1)\right] =$$

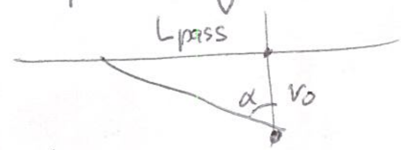
so,

$$\delta r_{\parallel}^{(1)}(t) = -\frac{Ze^2}{m\bar{v}^2}\gamma_{\parallel}\int_{-\infty}^{\sigma} d\sigma'\left[\frac{\tau_0\sigma'-1}{\sqrt{1+\sigma'^2}} + \tau_0\right] =$$ | This is (3.41)

$$= -\frac{Ze^2}{m\bar{v}^2}\gamma_{\parallel}\int_{-\infty}^{\sigma} d\sigma'\left(\underbrace{\frac{\tau_0\sigma'}{\sqrt{1+\sigma'^2}}}_{2.271.7} - \underbrace{\frac{1}{\sqrt{1+\sigma'^2}}}_{2.271.3} + \tau_0\right) = -\frac{Ze^2}{m\bar{v}^2}\gamma_{\parallel}\left[\tau_0\sqrt{1+\sigma'^2}\Big|_{-\infty}^{\sigma} - \ln\left(\sigma'+\sqrt{1+\sigma'^2}\right)\Big|_{-\infty}^{\sigma} + \tau_0\sigma'\Big|_{-\infty}^{\sigma}\right] =$$

$$= -\frac{Ze^2}{m\bar{v}^2}\gamma_{\parallel}\left(\tau_0\sqrt{1+\sigma^2} - \tau_0\sqrt{1+\sigma'^2}\Big|_{\sigma'\to-\infty} + \tau_0\sigma - \tau_0\sigma'\Big|_{\sigma'\to-\infty} - \ln\frac{\sigma+\sqrt{1+\sigma^2}}{\sigma'+\sqrt{1+\sigma'^2}}\Big|_{\sigma'\to-\infty}\right)$$

The divergence on the limit $t\to-\infty$ can be bypassed to to restriction on „length" of trajectory: $-\infty \to \frac{\bar{v}\Delta t_{pass}}{v_0}$, where $\Delta t_{pass} \simeq \frac{L_{pass}}{\bar{V}} \simeq \frac{v_0\, tg\,\alpha}{\bar{V}}$.



Lagrangian: $\mathcal{L} = \frac{m\vec v_e^{\,2}}{2} + \frac{M\vec v_i^{\,2}}{2} - e\vec A(\vec r_e)\vec v_e + Ze\vec A(\vec r_i)\vec v_i + \frac{Ze^2}{|\vec v_e - \vec r_i|}$ (3.3)

magnetic field along $\vec e_z$ из $\to$ $\vec B = B\vec e_z$ and vector-potential for that is

$$\vec A(\vec r) = \frac{1}{2}[\vec B\vec r] = \frac{1}{2}\begin{vmatrix} \vec e_x & \vec e_y & \vec e_z \\ B_x & B_y & B_z \\ x & y & z \end{vmatrix} = \frac{1}{2}\begin{vmatrix} \vec e_x & \vec e_y & \vec e_z \\ 0 & 0 & \\ x & y & z \end{vmatrix} = \frac{1}{2}(yB\,\vec e_x - Bx\vec e_y) = \frac{B}{2}(y\vec e_x - x\vec e_y)$$

Checking:

$$\vec B = \mathrm{rot}\vec A = \begin{vmatrix} \vec e_x & \vec e_y & \vec e_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \begin{vmatrix} \vec e_x & \vec e_y & \vec e_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{yB}{2} & -\frac{xB}{2} & 0 \end{vmatrix} = \vec e_x\left(0 - \frac{\partial(xB)}{2\,\partial z}\right) + \vec e_y\left(\frac{1}{2}\frac{\partial(yB)}{\partial z} - 0\right) + \vec e_z\left(\frac{1}{2}\frac{\partial(xB)}{\partial x} - \frac{\partial(-x\ldots)}{\partial y}\ldots\right.$$

$$= 0\cdot\vec e_x + 0\cdot\vec e_y + B\vec e_z \quad \text{OK!}$$

Let ion moves uniformly with velocity

$$\vec v_i = \begin{pmatrix} v_{i\perp} \\ 0 \\ v_{i\parallel} \end{pmatrix} \quad (3.1) \longrightarrow \vec r_i(t) = \vec v_i \cdot t$$

Input relative coordinate of electron $\vec r(t) = \vec r_e(t) - \vec r_i(t) = \vec r_e(t) - \vec v_i\cdot t$ (3.2a)

So, $\vec r(t)$ — coordinate ~~of~~ electron in the ion's frame. Farther

(3.2b) $\vec V(t) = \vec v_e(t) - \vec v_i(t) = \vec v_e(t) - \vec v_i$ — relative electron's velocity in the ion's frame

Let's define the radius and velocity of the center of mass:

$$\begin{cases} \vec r_{c.m} = (m_e\vec r_e + M_i\vec r_i)/(m_e+M_i) \\ \vec r = \vec r_e - \vec r_i \end{cases} \quad \text{and} \quad \begin{cases} \vec V_{c.m} = (m_e\vec v_e + M_i\vec v_i)/(m_e+M_i) \\ \vec V = \vec v_e - \vec v_i \end{cases}$$

$$\begin{cases} m\vec r_e + M\vec r_i = (m_e+M_i)\vec r_{cm} \\ \vec r_e - \vec r_i = \vec r \end{cases} \to \Delta = \begin{vmatrix} m_e & M_i \\ 1 & -1 \end{vmatrix} = -(m_e+M_i) \longrightarrow \vec r_e = \frac{1}{\Delta}\begin{vmatrix} (m_e+M_i)\vec r_{cm} & M_i \\ \vec r & -1 \end{vmatrix} = \vec r_{cm} + \frac{M_i\vec r}{m_e+M_i}$$

and $\vec r_i = \frac{1}{\Delta}\begin{vmatrix} m_e & (m_e+M_i)\vec r_{cm} \\ 1 & \vec r \end{vmatrix} = \vec r_{cm} - \frac{m_e}{m_e+M_i}\vec r \simeq \vec r_{cm} - \frac{m}{M}\vec r$

$\vec v_i t$

$\vec r_{cm} \simeq \vec v_i t + \frac{m}{M}\vec r$

and analogusly

$$\vec v_e = \vec V_{cm} + \frac{M_i}{m_e+M_i}\vec V$$

$$\vec v_i = \vec V_{cm} - \frac{m_e}{m_e+M_i}\vec V \simeq \vec V_{cm} - \frac{m}{M}\vec V$$

$$\mu = 1\Big/\left(\frac{1}{m_e} + \frac{1}{M_i}\right) = \frac{m_e M_i}{m_e+M_i}$$

So

$$\begin{cases} \vec{r}_e = \vec{r}_{cm} + \frac{\mu}{m_e}\vec{r} \\ \vec{r}_i = \vec{r}_{cm} - \frac{\mu}{M_i}\vec{r} \end{cases} \quad \text{and} \quad \begin{cases} \vec{V}_e = \vec{V}_{cm} + \frac{\mu}{m_e}\vec{V} \\ \vec{V}_i = \vec{V}_{cm} - \frac{\mu}{M_i}\vec{V} \end{cases} \qquad \begin{aligned} \vec{A}(\vec{r}_e) &= \frac{1}{2}[\vec{B}\cdot\vec{r}_e] = \frac{1}{2}\left[\vec{B}\cdot\left(\vec{r}_{cm} + \frac{\mu}{m}\vec{r}\right)\right] \\ \vec{A}(\vec{r}_i) &= \frac{1}{2}[\vec{B}\cdot\vec{r}_i] = \frac{1}{2}\left[\vec{B}\cdot\left(\vec{r}_{cm} - \frac{\mu}{M}\vec{r}\right)\right] \end{aligned}$$

Then

$$\mathcal{L} = \frac{m}{2}\left(\vec{V}_{cm} + \frac{\mu}{m}\vec{V}\right)^2 + \frac{M}{2}\left(\vec{V}_{cm} - \frac{\mu}{M}\vec{V}\right)^2 - e\frac{1}{2}\left[\vec{B}\left(\vec{r}_{cm} + \frac{\mu}{m}\vec{r}\right)\right]\left(\vec{V}_{cm} + \frac{\mu}{m}\vec{V}\right) - \frac{Ze}{2}\left[\vec{B}\left(\vec{r}_{cm} - \frac{\mu}{M}\vec{r}\right)\right]\left(\vec{V}_{cm} - \frac{\mu}{M}\vec{V}\right) + \frac{Ze^2}{r} =$$

$$= \underset{(1)}{\frac{m}{2}\vec{V}_{cm}^2} + \cancel{2\mu\vec{V}\vec{V}_{cm}} + \underset{(2)}{\frac{\mu^2}{2m}\vec{V}^2} + \underset{(3)}{\frac{M}{2}\vec{V}_{cm}^2} - \cancel{\mu\vec{V}\vec{V}_{cm}} + \underset{(4)}{\frac{\mu^2}{2M}\vec{V}^2} - \frac{e}{2}\left\{\underset{(5)}{([\vec{B}\vec{r}_{cm}]\vec{V}_{cm})} + \frac{\mu}{m}\underset{(6)}{([\vec{B}\vec{r}]\vec{V}_{cm})} + \underset{(7)}{([\vec{B}\vec{r}_{cm}]\frac{\mu}{m}\vec{V})} +\right.$$

$$\left. + \frac{\mu^2}{m^2}\underset{(8)}{([\vec{B}\vec{r}]\vec{V})}\right\} + \frac{Ze}{2}\left\{\underset{(9)}{([\vec{B}\vec{r}_{cm}]\vec{V}_{cm})} - \frac{\mu}{M}\underset{(10)}{([\vec{B}\vec{r}]\vec{V}_{cm})} - \frac{\mu}{M}\underset{(11)}{([\vec{B}\vec{r}_{cm}]\vec{V})} + \frac{\mu^2}{M^2}\underset{(12)}{([\vec{B}\vec{r}]\vec{V})}\right\} + \frac{Ze^2}{r} =$$

$$= \underset{(1)+(3)\ \leftarrow \text{ this is an constant}}{\frac{m+M}{2}\vec{V}_{cm}^2} + \frac{\mu^2}{2}\vec{V}^2\left(\underset{(2)}{\frac{1}{m}} + \underset{(4)}{\frac{1}{M}}\right) + \frac{Ze^2}{r} + \underset{(5)\ (9)\ \leftarrow \text{ this is an constant}}{\frac{(Z-1)e}{2}([\vec{B}\cdot\vec{r}_{cm}]\vec{V}_{cm})} + \frac{\mu^2}{2}\left(\underset{(12)}{\frac{Ze}{M^2}} - \underset{(8)}{\frac{e}{m^2}}\right)([\vec{B}\vec{r}]\vec{V}) +$$

$$-\frac{\mu}{2}([\vec{B}\vec{r}]\vec{V}_{cm})\left(\underset{(6)}{\frac{e}{m}} + \underset{(10)}{\frac{Ze}{M}}\right) - \frac{\mu}{2}\left(\underset{(7)}{\frac{e}{m}} + \underset{(11)}{\frac{Ze}{M}}\right)([\vec{B}\vec{r}_{cm}]\vec{V}) \simeq$$

this is (3.5) and taking into account $\mu \simeq m \ll M$

$$\simeq \underset{(2)+(4)}{\frac{m\vec{V}^2}{2}} + \frac{Ze^2}{r} - \underset{(12)+(8)}{\frac{e}{2}([\vec{B}\vec{r}]\vec{V})} - \left(\frac{e}{m} + \frac{Ze}{M}\right)\frac{\mu}{2}\underset{(6)+(10)+(7)+(11)}{\left(([\vec{B}\vec{r}]\overset{\vec{V}_i + \frac{m}{M}\vec{V}}{\vec{V}_{cm}}) + ([\vec{B}\vec{r}_{cm}]\vec{V})\right)} =$$

$$\simeq \frac{m\vec{V}^2}{2} + \frac{Ze^2}{r} - \frac{e}{2}([\vec{B}\vec{r}]\vec{V}) - \frac{e}{2}\left\{([\vec{B}\vec{r}](\vec{V}_i + \frac{m}{M}\vec{V})) + ([\vec{B}\vec{r}_{cm}]\vec{V})\right\} = \frac{m\vec{V}^2}{2} + \frac{Ze^2}{r} - \frac{e}{2}([\vec{B}\vec{r}]\vec{V}) -$$

$$-\frac{e}{2}\left\{([\vec{B}\vec{r}]\vec{V}_i) + \left[\vec{B}\left(\vec{r}_{cm} + \frac{m}{M}\vec{r}\right)\right]\vec{V}\right\} =$$

$$= \frac{m\vec{V}^2}{2} + \frac{ze^2}{r} - \frac{e}{2}([\vec{B}\vec{r}]\vec{V}) - \frac{e}{2}\left\{([\vec{B}\vec{r}]\vec{V}_i) + ([\vec{B}\cdot\vec{V}_i t]\vec{V}) + 2\frac{m}{M}([\vec{B}\vec{r}]\vec{V})\right\} =$$

$$= \frac{m\vec{V}^2}{2} + \frac{ze^2}{r} - \frac{e}{2}([\vec{B}\vec{r}]\vec{V}_i) - \frac{e}{2}([\vec{B}\cdot\vec{V}_i t]\vec{V}) - \frac{e}{2}\Big(1 + \underbrace{2\frac{m}{M}}_{2\frac{m}{M} \ll 1}\Big)([\vec{B}\vec{r}]\vec{V}) \Longrightarrow$$

$$\mathcal{L} = \frac{m\vec{V}^2}{2} + \frac{ze^2}{r} - \frac{e}{2}([\vec{B}\vec{r}]\vec{V}) - \frac{e}{2}\left\{([\vec{B}\cdot\vec{V}_i t]\vec{V}) + ([\vec{B}\vec{r}]\vec{V}_i)\right\}$$ this is (3.6)

Equation of motion:

$$m\frac{d\vec{V}}{dt} = -\frac{\partial \mathcal{L}}{\partial \vec{r}} = -\vec{\nabla}\frac{ze^2}{r}$$

From (3.14) for $R \to 0$ („Guiding" center approach):

$$\vec r^{\,2}(t) = b^2 + [(v_{e\parallel}-v_{i\parallel})^2 + v_{i\perp}^2]\,t^2 - 2v_{i\perp} b t \sin\theta \qquad (3.15)$$

Let's define the relative velocity $\vec V$ of the guiding center and ion:

$$\vec V = \vec V_\perp + \vec V_\parallel = \begin{pmatrix}0\\0\\v_{e\parallel}-v_{i\parallel}\end{pmatrix} + \begin{pmatrix}-v_{i\perp}\\0\\0\end{pmatrix} \qquad (3.16)$$

Then $(v_{e\parallel}-v_{i\parallel})^2 + v_{i\perp}^2 = \vec V^2 \equiv \bar V^2$

Let's $t_0$ – time $t_0$ when electron reaches the minimal distance $\vec r_0$ to ion (this is impact parameter of collision); i.e.

$$r_0^2 = b^2 + \bar V^2 t_0^2 - 2v_{i\perp} b t_0 \sin\theta$$

Very important: $\vec r_0 \perp \vec V$ and from picture one has

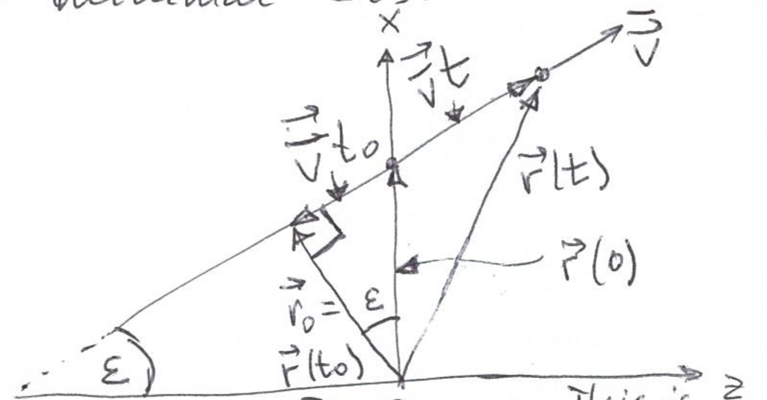



Fig. 2

$$\vec r(t) = \vec r_0 + \vec V(t-t_0) \qquad (3.20)$$

then $\vec r^{\,2} = r_0^2 + \underbrace{\vec V^2}_{\equiv \bar V^2}(t-t_0)^2 + 2\underbrace{\vec r_0 \vec V}_{=0}(t-t_0) = r_0^2 + \bar V^2 (t-t_0)^2$

So

$$\vec r^{\,2} = b^2 + \bar V^2 t^2 - 2v_{i\perp} b t \sin\theta = r_0^2 + \bar V^2 (t-t_0)^2 = b^2 + \bar V^2 t_0^2 - 2v_{i\perp} b t_0 \sin\theta + \bar V^2 (t-t_0)^2$$

or

$$\cancel{b^2} + \cancel{\bar V^2 t^2} - 2v_{i\perp} b t \sin\theta = \cancel{b^2} + \bar V^2 t_0^2 - 2v_{i\perp} b t_0 \sin\theta + \cancel{\bar V^2 t^2} + \bar V^2 t_0^2 - 2\bar V^2 t t_0$$

$$\cancel{2\bar V^2 t_0^2 - 2t_0(v_{i\perp} b \sin\theta - \bar V^2 t) + 2v_{i\perp} b t \sin\theta = 0}$$

So $$0 = 2v_{i\perp} b\sin\theta\cdot(t-t_0) + 2\bar{V}^2 t_0(t_0-t) \Longrightarrow t_0 = \frac{v_{i\perp} b\sin\theta}{\bar{V}^2} \quad (3.18) = -\frac{\bar{V}_\perp b\sin\theta}{\bar{V}^2}$$ ~~opposite sign with respect~~ =

and for this reason

$$r_0^2 = b^2 + \bar{V}^2 t_0^2 - 2t_0\cdot \underbrace{\frac{v_{i\perp}\sin\theta}{\bar{V}^2}}_{=t_0}\bar{V}^2 = b^2+\bar{V}^2t_0^2 - 2\bar{V}^2t_0^2 = b^2 - \bar{V}^2 t_0^2 \quad (3.19)$$

and

$$\vec{r}_0 = \vec{r}(t_0) = \underbrace{\begin{pmatrix} b\sin\theta - v_{i\perp}t_0 \\ -b\cos\theta \\ (v_{e\parallel}-v_{i\parallel})t_0 \end{pmatrix}}_{\text{this is from (3.10) with } R=0} = \begin{pmatrix} b\sin\theta + \bar{V}_\perp t_0 \\ -b\cos\theta \\ \bar{V}_\parallel t_0 \end{pmatrix}$$

and then

$$\vec{r}(t) = \begin{pmatrix} -b\sin\theta - v_{i\perp}t \\ -b\cos\theta \\ (v_{e\parallel}-v_{i\parallel})t \end{pmatrix} = \begin{pmatrix} -b\sin\theta + \bar{V}_\perp t \\ -b\cos\theta \\ \bar{V}_\parallel t \end{pmatrix} = \begin{pmatrix} -b\sin\theta + \bar{V}_\perp (t_0+t-t_0) \\ -b\cos\theta \\ \bar{V}_\parallel (t_0+t-t_0) \end{pmatrix} =$$

$$= \begin{pmatrix} -b\sin\theta + \bar{V}_\perp t_0 \\ -b\cos\theta \\ \bar{V}_\parallel t_0 \end{pmatrix} + \begin{pmatrix} \bar{V}_\perp (t-t_0) \\ 0 \\ \bar{V}_\parallel (t-t_0) \end{pmatrix} = \vec{r}_0 + \vec{\bar{V}}(t-t_0)$$ this is (3.20) again

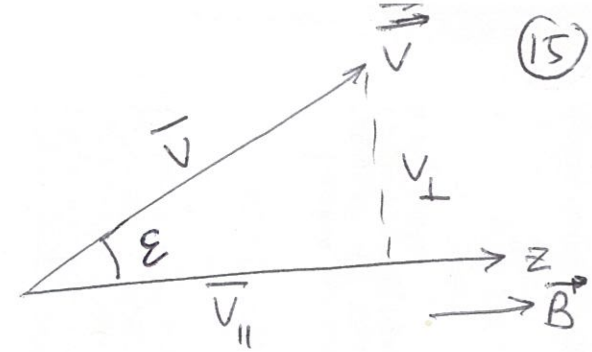
Let's introduce the dimentionless variables:

$$\tau = \frac{\bar{V}t}{r_0} \quad (3.21) \qquad \tau_0 = \frac{\bar{V}t_0}{r_0}, \quad \sigma = \tau - \tau_0 = \frac{\bar{V}(t-t_0)}{r_0} \quad (3.22)$$ This is (3.29 left)

$$\gamma_\parallel = \frac{\bar{V}_\parallel}{\bar{V}} \quad (3.24), \qquad \gamma_\perp = \frac{\bar{V}_\perp}{\bar{V}} \quad (3.25) \qquad \beta = \frac{b}{r_0} \quad (3.26)$$

Then from (3.18) $\tau_0 = \frac{\bar{V}t_0}{r_0} = \frac{\bar{V}}{r_0}\,\frac{-\bar{V}_\perp b\sin\theta}{\bar{V}^2} =$

$$\frac{\vec r_0}{r_0} = \frac{1}{r_0}\begin{pmatrix} b\sin\theta + \bar V_\perp t_0 \\ -b\cos\theta \\ \bar V_{\parallel} t_0\end{pmatrix} = \begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau_0 \\ -\beta\cos\theta \\ \gamma_\parallel \tau_0\end{pmatrix}$$




$$\cos\varepsilon = \frac{\bar V_\parallel}{\bar V} = \gamma_\parallel \quad (3.28)$$

$$\sin\varepsilon = \frac{\bar V_\perp}{\bar V} = \gamma_\perp \quad (3.28^*)$$

Instead (3.19) one has

$$b^2 = r_0^2 + \bar V^2 t_0^2 \to \beta^2 = 1+\tau_0^2 \quad (3.32)$$

and, of course, $\gamma_\parallel^2+\gamma_\perp^2=1$

Let's input

$$\sin\psi = -\beta\cos\theta \quad (3.30)$$

Then

$$\beta\sin\theta+\gamma_\perp\tau_0 = \left(\text{from 3.29 left } \beta\sin\theta = -\frac{\tau_0}{\gamma_\perp}\right) = -\frac{\tau_0}{\gamma_\perp}+\gamma_\perp\tau_0 = \tau_0\frac{\gamma_\perp^2-1}{\gamma_\perp} = -\frac{\tau_0\gamma_\parallel^2}{\gamma_\perp}$$

But

$$\tau_0 = -\beta\gamma_\perp\sin\theta = -\gamma_\perp\beta\sqrt{1-\cos^2\theta} = -\gamma_\perp\beta\sqrt{1-\frac{\sin^2\psi}{\beta^2}} = -\gamma_\perp\sqrt{\beta^2-\sin^2\psi} =$$

$$= -\gamma_\perp\sqrt{\beta^2-1+\cos^2\psi} = -\gamma_\perp\sqrt{\tau_0^2+\cos^2\psi}$$ or

from (3.32)

$$\tau_0^2 = \gamma_\perp^2\tau_0^2+\gamma_\perp^2\cos^2\psi \to \tau_0^2(1-\gamma_\perp^2) = \gamma_\perp^2\cos^2\psi \to$$

$$\tau_0^2\gamma_\parallel^2 = \gamma_\perp^2\cos^2\psi \to \tau_0 = \pm\frac{\gamma_\perp}{\gamma_\parallel}\cos\psi$$ and it is neccesary to select sign "−"!

$$\tau_0 = -\frac{\gamma_\perp}{\gamma_\parallel}\cos\psi \text{ (3.29 right)} \to \underbrace{\gamma_\parallel\tau_0 = -\overset{\text{from }(3.28^*)}{\gamma_\perp}\cos\psi = -\sin\varepsilon\cos\psi}_{(3.29^*)}$$

So

$$\beta\sin\theta+\gamma_\perp\tau_0 = -\frac{\tau_0\gamma_\parallel^2}{\gamma_\perp} = \frac{\gamma_\perp\cos\psi\cdot\gamma_\parallel^2}{\gamma_\parallel\gamma_\perp} = \gamma_\parallel\cos\psi \quad (3.31)$$

Therefore

$$\frac{\vec{r}_0}{r_0} = \begin{pmatrix} \beta \sin\theta + \gamma_\perp \tau_0 \\ -\beta\cos\theta \\ \gamma_\parallel \tau_0 \end{pmatrix} = \begin{pmatrix} \text{using } (3.31) \\ \text{using } (3.30) \\ \text{using } (3.29^*) \end{pmatrix} = \begin{pmatrix} \cos\varepsilon \cos\psi \\ \sin\psi \\ -\sin\varepsilon \cos\psi \end{pmatrix} \qquad (3.27)$$

Finaly:

(3.21) $\tau = \frac{\bar{v}t}{r_0} \qquad \tau_0 = \frac{\bar{v}t_0}{r_0}$

(3.22) $\sigma = \tau - \tau_0 = \frac{\bar{v}(t-t_0)}{r_0}$

(3.24) $\gamma_\parallel = \frac{\overline{v_\parallel}}{v_0}$

(3.25) $\gamma_\perp = \frac{\overline{v_\perp}}{v_0}$

$\gamma_\parallel^2 + \gamma_\perp^2 = 1$

(3.26) $\beta = \frac{b}{r_0}$

(3.27) $\frac{\vec{r}_0}{r_0} = \begin{pmatrix} \cos\varepsilon \cos\psi \\ \sin\psi \\ -\sin\varepsilon \cos\psi \end{pmatrix}$

(3.28) $\cos\varepsilon = \gamma_\parallel \longrightarrow \sin\varepsilon = \sqrt{1-\cos^2\varepsilon} = \sqrt{1-\gamma_\parallel^2} = \gamma_\perp$ (3.28*)

(3.29) $\tau_0 = -\beta\gamma_\perp \sin\theta = -\frac{\gamma_\perp}{\gamma_\parallel}\cos\psi \longrightarrow \gamma_\parallel \tau_0 = -\sin\varepsilon \cos\psi$ (3.29*)

(3.30) $\beta\cos\theta = -\sin\psi$

(3.31) $\beta\sin\theta + \gamma_\perp \tau_0 = -\frac{\gamma_\parallel^2 \tau_0}{\gamma_\perp} = \gamma_\parallel \cos\psi$

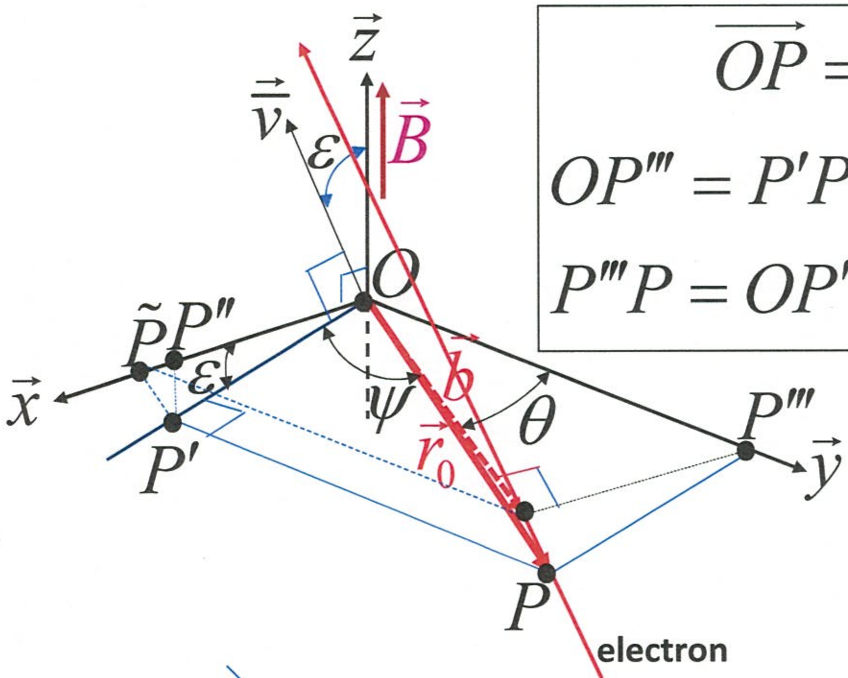
(3.32) $\beta^2 = 1 + \tau_0^2$

$\beta \sin\theta = \frac{\cos\psi}{\gamma_\parallel} = \frac{\cos\psi}{\cos\varepsilon}$

$\beta\cos\theta = -\sin\psi$,

so

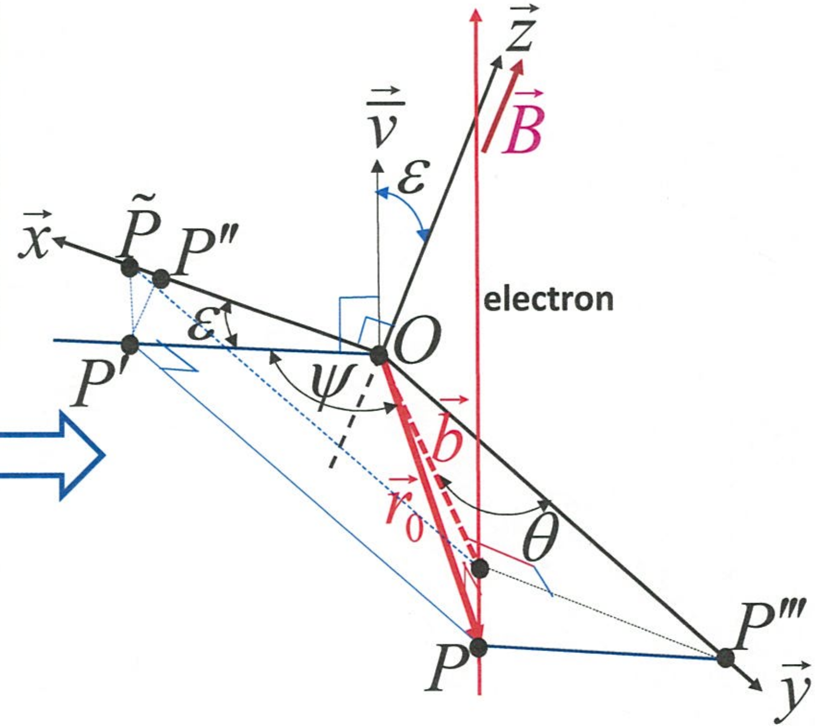
$$\boxed{\operatorname{tg}\theta = -\frac{\cos\psi/\cos\varepsilon}{\sin\psi} = -\frac{\operatorname{ctg}\psi}{\cos\varepsilon}}$$




$$\overrightarrow{OP} = \vec{r}_0 \Rightarrow$$

$$\left. \begin{aligned} OP''' &= P'P = r_0 \cdot \sin\psi, \\ P'''P &= OP' = r_0 \cdot \cos\psi \end{aligned} \right\}$$

or




$$\vec{r}_0 = \begin{pmatrix} OP'' \\ PP' \\ P''P' \end{pmatrix} = r_0 \begin{pmatrix} \cos\varepsilon \cdot \cos\psi \\ \sin\psi \\ -\sin\varepsilon \cdot \cos\psi \end{pmatrix};$$

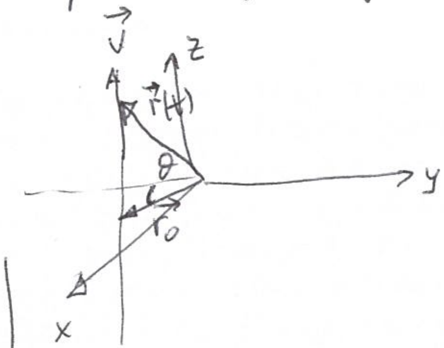
$$O\tilde{P} = OP' / \cos\varepsilon = r_0 \cdot \cos\psi / \cos\varepsilon \rightarrow$$

$$\rightarrow tg\theta = O\tilde{P} / OP''' = -\cot\psi / \cos\varepsilon$$

Sign from corresponding formulas

Unmagnetized electron scattering with ion in fixed point of origin.

$$\frac{d\vec{v}}{dt} = -\frac{e}{m}\vec{E}(\vec{r}) \quad (2.3), \qquad \vec{E}(\vec{r}) = ze\frac{\vec{r}(t)}{r^3(t)} \quad (2.1)$$




$r_0$ – impact parameter is $\vec{r}(t=0)$, and

$$\vec{r}_0 = r_0\begin{pmatrix}\sin\theta\\ -\cos\theta\\ 0\end{pmatrix} \quad (2.5) \quad \text{then} \quad \vec{r}(t) = \begin{pmatrix} r_{ox}\\ r_{oy}\\ vt\end{pmatrix} = \begin{pmatrix} r_0\sin\theta\\ -r_0\cos\theta\\ vt\end{pmatrix}$$

Then first order:

$$(2.6) \quad \delta\vec{V}^{(1)}_c(t) = -\frac{ze^2}{m}\int_{-\infty}^{t}\frac{dt'}{\sqrt{(r_0^2+v^2t'^2)^3}}\begin{pmatrix} r_0\sin\theta\\ -r_0\cos\theta\\ vt'\end{pmatrix} = -\frac{ze^2}{mV}\int_{-\infty}^{vt}\frac{r_0\,dx/r_0}{(r_0^2+x^2)^{3/2}}\begin{pmatrix} r_0\sin\theta\\ -r_0\cos\theta\\ r_0 x/r_0\end{pmatrix} = -\frac{ze^2}{mVr_0}\int_{-\infty}^{\frac{vt}{r_0}}\frac{dx}{(1+x^2)^{3/2}}\begin{pmatrix}\sin\theta\\ -\cos\theta\\ x\end{pmatrix}$$

(lower index „c" means „Coulomb").

Total $\delta\vec{V}^{(1)}_c(t)$ is reached for $t\to\infty$!

2.27.5: $\int\frac{dx}{(a+cx^2)^{3/2}} = \frac{1}{a}\frac{x}{\sqrt{a+cx^2}}$

$$\left(\delta\vec{V}^{(1)}_c\right)_{total} = -\frac{ze^2}{mVr_0}\int_{-\infty}^{\infty}\frac{dx}{(1+x^2)^{3/2}}\begin{pmatrix}\sin\theta\\ -\cos\theta\\ x\end{pmatrix} = -\frac{2ze^2}{mVr_0}\underbrace{\int^{\infty}\frac{dx}{(1+x^2)^{3/2}}}_{=1}\begin{pmatrix}\sin\theta\\ -\cos\theta\\ 0\end{pmatrix}$$

integral $=0$ due to odd function!

$$= -\frac{\cancel{2}z^2e^2}{mVr_0}\begin{pmatrix}\sin\theta\\ -\cos\theta\\ 0\end{pmatrix} = -\frac{2ze^2}{mVr_0^2}r_0\begin{pmatrix}\sin\theta\\ -\cos\theta\\ 0\end{pmatrix} = -\frac{2ze^2\vec{r}_0}{mVr_0^2} \quad (2.7)$$

Now let calculate $\delta\vec{r}^{(1)}$: $\frac{d\vec{r}}{dt} = \vec{v}$ (2.2)

~~then~~ But it is possible to write the solution for $\delta\vec{V}^{(1)}_c(t)$ from (2.6):

$$\delta\vec{V}^{(1)}_c(t) = -\frac{ze^2}{mVr_0}\int_{-\infty}^{\frac{vt}{r_0}}\frac{dx}{(1+x^2)^{3/2}}\begin{pmatrix}\sin\theta\\ -\cos\theta\\ x\end{pmatrix} = \begin{pmatrix} \int 2.27.5\\ 2.27.7\\ \text{with } n=1\end{pmatrix} =$$

2.27.7: $\int\frac{x\,dx}{\sqrt{a+cx^2}^3} = -\frac{1}{c\sqrt{a+cx^2}}$

$$= -\frac{Ze^2}{mVr_0}\left(\frac{vt/r_0}{\sqrt{1+v^2t^2/r_0^2}}+1\right)\begin{pmatrix}\sin\theta\\-\cos\theta\\0\end{pmatrix} - \frac{Ze^2}{mVr_0}\left(-\frac{\mathrm{sgn}(v)}{\sqrt{1+v^2t^2/r_0^2}}\right)\begin{pmatrix}0\\0\\1\end{pmatrix}$$

and for $t\to\infty$ one will receive the result (2,7)

Now $\delta\vec{r}_c^{\,(1)}$ can be received from $\delta\vec{V}_c^{(1)}(t)$:

$$\delta\vec{r}_c^{\,(1)}(t) = \int_{-\infty}^{t} dt'\,\delta\vec{V}_c^{(1)}(t') = -\frac{Ze^2}{mVr_0}\int_{-\infty}^{t}\left(\frac{vt'/r_0}{\sqrt{1+v^2t'^2/r_0^2}}+1\right)dt'\begin{pmatrix}\sin\theta\\-\cos\theta\\0\end{pmatrix} +$$

$$+\frac{Ze^2}{mVr_0}\int_{-\infty}^{t}\frac{dt'\,\mathrm{sgn}(v)}{\sqrt{1+v^2t'^2/r_0^2}}\begin{pmatrix}0\\0\\1\end{pmatrix} = -\frac{Ze^2}{mV^2}\int_{-\infty}^{t}\left(\frac{vt'/r_0}{\sqrt{1+v^2t'^2/r_0^2}}+1\right)d\left(\frac{vt'}{r_0}\right)\begin{pmatrix}\sin\theta\\-\cos\theta\\0\end{pmatrix} + \frac{Ze^2}{mV^2}\int_{-\infty}^{t}\frac{\mathrm{sgn}(v)\,d\left(\frac{vt'}{r_0}\right)}{\sqrt{1+v^2t'^2/r_0^2}}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$= -\frac{Ze^2}{mV^2}\int_{-\infty}^{vt/r_0} d\tau'\left[\left(\frac{\tau'}{\sqrt{1+\tau'^2}}+1\right)\begin{pmatrix}\sin\theta\\-\cos\theta\\0\end{pmatrix} - \frac{\mathrm{sgn}(v)}{\sqrt{1+\tau'^2}}\begin{pmatrix}0\\0\\1\end{pmatrix}\right]$$ this is (2,11)