Checking of

Toepffer. Scattering of Magnetized Electrons with Ions

Phys. Rev., <u>A 66</u> (2002) 02274

$$\delta^{(1)}\vec{V}_{\beta}(t) = -\frac{ze^2}{m}\int_{-\infty}^{t}\frac{dt'}{\bar{r}^3(t')}\,T^{-1}(\Omega t')\cdot\vec{r}(t')$$

↑ using (3.34)

$$\bar{r}(t) = \left[r_0^2 + \bar{v}^2(t-t_0)^2\right]^{1/2} =$$

From (3.35)

$$= r_0\left[1 + \frac{\bar{v}^2(t-t_0)^2}{r_0^2}\right]^{1/2} =$$

$$= r_0\left[1 + (\tau-\tau_0)^2\right]^{1/2}$$

From (3.10) in the limit $R \to 0$: (1)

$$\vec{r}(t) = \begin{pmatrix} b\sin\theta - v_{i\perp}t \\ -b\cos\theta \\ (v_{e\parallel} - v_{i\parallel})t \end{pmatrix} =$$

$$= r_0\begin{pmatrix} \frac{b}{r_0}\sin\theta - \frac{v_{i\perp}}{\bar{v}}\frac{\bar{v}t}{r_0} \\ -\frac{b}{r_0}\cos\theta \\ \frac{v_{e\parallel}-v_{i\parallel}}{\bar{v}}\frac{\bar{v}t}{r_0} \end{pmatrix} =$$

$$\boxed{\tau = \frac{\bar{v}t}{r_0},\ \tau_0 = \frac{\bar{v}t_0}{r_0}}$$

Thus is (3.34)

$$= r_0\begin{pmatrix} \beta\sin\theta + \gamma_\perp\tau \\ -\beta\cos\theta \\ \gamma_\parallel\tau \end{pmatrix}$$

$$= -\frac{ze^2}{m}\frac{r_0}{\bar{v}}\int_{-\infty}^{\tau}\frac{d\tau'\quad r_0}{r_0^3\left[1+(\tau'-\tau_0)\right]^{3/2}}\begin{pmatrix} \cos\frac{\Omega r_0}{\bar{v}}\tau' & \sin\frac{\Omega r_0}{\bar{v}}\tau' & 0 \\ -\sin\frac{\Omega r_0}{\bar{v}}\tau' & \cos\frac{\Omega r_0}{\bar{v}}\tau' & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \beta\sin\theta + \gamma_\perp\tau' \\ -\beta\cos\theta \\ \gamma_\parallel\tau' \end{pmatrix}$$

$$\boxed{\begin{aligned}\sigma &= \tau' - \tau_0 \\ d\tau' &= d\sigma\end{aligned}}$$

$$= -\frac{ze^2}{m}\frac{1}{\bar{v}r_0}\int_{-\infty}^{\sigma}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\begin{pmatrix} \cos\omega\tau' & \sin\omega\tau' & 0 \\ -\sin\omega\tau' & \cos\omega\tau' & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \beta\sin\theta + \gamma_\perp\tau' \\ -\beta\cos\theta \\ \gamma_\parallel\tau' \end{pmatrix}$$

τ_0

3.36

(2)

„x": $\cos\omega\tau' (\beta \sin\theta + \gamma_\perp \tau') + \beta \cos\theta \sin\omega\tau' =$

$$= \beta [\sin\theta \cos\omega\tau' + \cos\theta \sin\omega\tau'] + \gamma_\perp \tau' \cos\omega\tau' = \beta \sin(\theta + \omega\tau') + \gamma_\perp \tau' \cos\omega\tau'$$

„y": $-\sin\omega\tau' (\beta \sin\theta + \gamma_\perp \tau') + \beta \cos\omega\tau' \cos\theta =$

$$= -\beta (\sin\theta \sin\omega\tau' + \cos\theta \cos\omega\tau') - \gamma_\perp \tau' \sin\omega\tau' = -\beta \cos(\omega\tau' - \theta) - \gamma_\perp \tau' \sin\omega\tau'$$

Смотрим z-компоненту от $\delta \vec{V}_B(t)$ при $t \to \infty$:

$$\left(\delta^{(1)} \vec{V}_B\right)_z \Big|_{t\to\infty} = -\frac{ze^2}{m} \frac{1}{\sqrt{\nu} r_0} \int_{-\infty}^{\infty} \frac{d\sigma'}{[1+\sigma'^2]^{3/2}} \gamma_\parallel \tau' = -\frac{ze^2}{m} \frac{\gamma_\parallel}{\sqrt{\nu} r_0} \left[\underbrace{\int_{-\infty}^{\infty} \frac{\sigma' d\sigma'}{(1+\sigma'^2)^{3/2}}}_{=0} + \tau_0 \underbrace{\int_{-\infty}^{\infty} \frac{d\sigma'}{(1+\sigma'^2)^{3/2}}}_{= \frac{\sigma'}{\sqrt{1+\sigma'^2}} \Big|_{-\infty}^{\infty} = 2} \right] =$$

$= 0$ из-за нечётности ф-ции

в силу чётности подынтегральной ф-ции

$$= -\frac{2ze^2}{m} \frac{\gamma_\parallel \tau_0}{\sqrt{\nu} r_0} \qquad \text{OK} \quad (3.37c)$$

Смотрим x-компоненту:

$$\left(\delta^{(1)} \vec{V}_B\right)_x \Big|_{t\to\infty} = -\frac{ze^2}{m} \frac{1}{\sqrt{\nu} r_0} \int_{-\infty}^{\infty} \frac{d\sigma'}{(1+\sigma'^2)^{3/2}} \Big[\underbrace{\beta \sin(\omega\tau' + \theta)}_{„1"} + \underbrace{\gamma_\perp \overbrace{\tau'}^{=\sigma'+\tau_0} \cos\omega\tau'}_{„2"} \Big]$$

$$„2" = \gamma_\perp \int_{-\infty}^{\infty} \frac{d\sigma'}{(1+\sigma'^2)^{3/2}} (\sigma' + \tau_0) \cos[\omega(\sigma' + \tau_0)] = \gamma_\perp \tau_0 \int_{-\infty}^{\infty} \frac{\cos\omega\sigma' \cos\omega\tau_0 - \sin\omega\sigma' \sin\tau_0}{(1+\sigma'^2)^{3/2}} d\sigma'$$

$$= \gamma_\perp \tau_0 \left[\cos\omega\tau_0 \int_{-\infty}^{\infty} \frac{\cos\omega x \, dx}{(1+x^2)^{3/2}} - \sin\omega\tau_0 \underbrace{\int_{-\infty}^{\infty} \frac{\sin\omega x \, dx}{(1+x^2)^{3/2}}}_{=0 \text{ из-за нечётности ф-ции}} \right] = 2\gamma_\perp \tau_0 \cos\omega\tau_0 \int_0^{\infty} \frac{\cos\omega x \, dx}{(1+x^2)^{3/2}}$$

$$+\gamma_\perp \int_{-\infty}^{\infty} \frac{\sigma'(\cos\omega\sigma'\cos\omega\tau_0 - \sin\omega\sigma'\sin\omega\tau_0)\,d\sigma'}{(1+\sigma'^2)^{3/2}} =$$

$$= \gamma_\perp\tau_0\left[\cos\omega\tau_0\int_{-\infty}^{\infty}\frac{\cos\omega x\,dx}{(1+x^2)^{3/2}} - \sin\omega\tau_0\underbrace{\int_{-\infty}^{\infty}\frac{\sin\omega x\,dx}{(1+x^2)^{3/2}}}_{=0 \text{ из-за нечётности}}\right] +$$

$$+\gamma_\perp\left[\cos\omega\tau_0\underbrace{\int_{-\infty}^{\infty}\frac{x\cos\omega x\,dx}{(1+x^2)^{3/2}}}_{=0 \text{ из-за нечётности}} - \sin\omega\tau_0\int_{-\infty}^{\infty}\frac{x\sin\omega x\,dx}{(1+x^2)^{3/2}}\right] =$$

$$= 2\gamma_\perp\left[\tau_0\cos\omega\tau_0\underbrace{\int_0^{\infty}\frac{\cos\omega x\,dx}{(1+x^2)^{3/2}}}_{I} - \sin\omega\tau_0\underbrace{\int_0^{\infty}\frac{x\sin\omega x\,dx}{(1+x^2)^{3/2}}}_{\text{это } 3.754.3 \,=\, \omega K_0(\omega)}\right]$$

чтобы найти I найдем сначала

$$\hat{I}(a) = \int_0^{\infty}\frac{\cos\omega x\,dx}{(1+ax^2)^{1/2}} = \frac{1}{\sqrt{a}}\underbrace{\int_0^{\infty}\frac{\cos\omega x\,dx}{\sqrt{\frac{1}{a}+x^2}}}_{\text{это } 3.754.2} = \frac{1}{\sqrt{a}}K_0\left(\frac{\omega}{\sqrt{a}}\right)$$

тогда

$$I = \int_0^{\infty}\frac{\cos\omega x\,dx}{(1+x^2)^{3/2}} = -2\frac{\partial}{\partial a}\int_0^{\infty}\frac{\cos\omega x\,dx}{(1+ax^2)^{1/2}}$$

$$I(\beta) = \underbrace{\int_0^{\infty}\frac{\cos\omega x\,dx}{\sqrt{\beta+x^2}}}_{\text{это } 3.754.2} = K_0(\sqrt{\beta}\,\omega), \quad \text{тогда}$$

$$\int_0^{\infty}\frac{\cos\omega x\,dx}{\sqrt{1+x^2}^{\,3}} = -2\left(\frac{\partial}{\partial\beta}\int_0^{\infty}\frac{\cos\omega x\,dx}{\sqrt{\beta+x^2}}\right)\Bigg|_{\beta=1} = -2\frac{\partial}{\partial\beta}\left(K_0\sqrt{\beta}\,\omega\right)\Big|_{\beta=1} =$$

$$= -2\,\frac{1}{2}\,\frac{\omega}{\sqrt{\beta}}\,K_0'(\sqrt{\beta}\,\omega)\Big|_{\beta=1} = -\omega \underset{\uparrow\, 8.486\,18}{K_0'(\omega)} = \omega K_1(\omega)$$

Итак,

$$„2" = 2\gamma_\perp\left[\omega\tau_0\cos(\omega\tau_0)\,K_1(\omega) - \omega\sin(\omega\tau_0)\,K_0(\omega)\right]$$

Теперь „1":

$$„1" = \beta\int_{-\infty}^{\infty}\frac{\sin(\omega\overset{=\sigma'+\tau_0}{\tau'}+\theta)\,d\sigma'}{(1+\sigma'^2)^{3/2}} = \beta\int_{-\infty}^{\infty}\frac{\sin(\omega\sigma'+\omega\tau_0+\theta)\,d\sigma'}{(1+\sigma'^2)^{3/2}} =$$

$$= \beta\left[\sin(\omega\tau_0+\theta)\int_{-\infty}^{\infty}\frac{\cos\omega x\,dx}{(1+x^2)^{3/2}} + \cos(\omega\tau_0+\theta)\underbrace{\int_{-\infty}^{\infty}\frac{\sin\omega x\,dx}{(1+x^2)^{3/2}}}_{=0\ \text{из-за нечётности}}\right] = 2\beta\sin(\omega\tau_0+\theta)\underbrace{\int_0^{\infty}\frac{\cos\omega x\,dx}{(1+x^2)^{3/2}}}_{\text{найдено выше и равно } \omega K_1(\omega)} =$$

$$= 2\beta\omega\sin(\omega\tau_0+\theta)\,K_1(\omega)$$

Т.о. „1"+„2":

$$„1"+„2" = 2\beta\omega\sin(\omega\tau_0+\theta)\,K_1(\omega) + 2\gamma_\perp\left[\omega\tau_0\cos(\omega\tau_0)K_1(\omega) - \omega\sin(\omega\tau_0)\,K_0(\omega)\right] =$$

$$= 2\gamma_\perp\omega K_0(\omega)\sin\omega\tau_0 + 2\omega K_1(\omega)\left[\beta\sin(\omega\tau_0+\theta) \mp \gamma_\perp\tau_0\cos(\omega\tau_0)\right]$$

Имеем

$$[\ \ ] = \underbrace{\beta\cos\theta}_{3.30}\sin(\omega\tau_0) + \beta\sin\theta\cos\omega\tau_0 \mp \gamma_\perp\tau_0\cos\omega\tau_0 = \sin(\omega\tau_0)\sin\psi +$$

$$+\cos(\omega\tau_0)\big[\underbrace{\beta\sin\theta + \gamma_\perp\tau_0}_{\text{это } 3.31}\big] = \sin(\omega\tau_0)\sin\psi \mp \gamma_\parallel\cos\psi\cos(\omega\tau_0)$$

$$\left(\delta^{(1)} \overset{\text{т.о.}}{\vec{V}_B}\right)_x\Big|_{t\to\infty} = -\frac{Ze^2}{mVr_0}\Big[2\omega K_1(\omega)\left(\gamma_\parallel \cos\psi\cos\omega\tau_0 + \sin\psi\sin\omega\tau_0\right) \mp 2\omega K_0(\omega)\gamma_\perp \sin\omega\tau_0\Big] \quad (3.37a)$$

совпадает с Toepffer

~~у Toepffer'а другой знак~~

Аналогично:

$$\left(\delta^{(1)} \vec{V}_B\right)_y\Big|_{t\to\infty} = -\frac{Ze^2}{mVr_0}\int_{-\infty}^{\infty}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\Big[\underbrace{\mp\beta\cos(\omega\tau'-\theta)}_{-\text{„}3\text{"}} \underbrace{\mp\gamma_\perp\tau'\sin\omega\tau'}_{-\text{„}4\text{"}}\Big]$$

Имеем:

$$\text{„}3\text{"} = \beta\int_{-\infty}^{\infty}\frac{\cos(\omega\overset{=\sigma'+\tau_0}{\tau'}-\theta)\,d\sigma'}{(1+\sigma'^2)^{3/2}} = \beta\Big[\cos(\omega\tau_0-\theta)\int_{-\infty}^{\infty}\frac{\cos\omega x\,dx}{(1+x^2)^{3/2}} + \sin(\omega\tau_0-\theta)\underbrace{\int_{-\infty}^{\infty}\frac{\sin\omega x\,dx}{(1+x^2)^{3/2}}}_{=0 \text{ из-за нечётности}}\Big] =$$

$$= 2\beta\cos(\omega\tau_0-\theta)\underbrace{\int_0^{\infty}\frac{\cos\omega x\,dx}{(1+x^2)^{3/2}}}_{\text{см. ранее } = \omega K_1(\omega)} = 2\beta\omega K_1(\omega)\cos(\omega\tau_0-\theta)$$

$$\text{„}4\text{"} = \gamma_\perp\int_{-\infty}^{\infty}\frac{\overset{=\sigma'+\tau_0}{\tau'}\sin\omega\tau'\,d\sigma'}{(1+\sigma'^2)^{3/2}} = \gamma_\perp\tau_0\int_{-\infty}^{\infty}\frac{\sin(\omega\sigma'+\omega\tau_0)\,d\sigma'}{(1+\sigma'^2)^{3/2}} + \gamma_\perp\int_{-\infty}^{\infty}\frac{\sigma'\sin(\omega\sigma'+\omega\tau_0)\,d\sigma'}{(1+\sigma'^2)^{3/2}} =$$

$$= \gamma_\perp\tau_0\Big[\cos\omega\tau_0\underbrace{\int_{-\infty}^{\infty}\frac{\sin\omega x\,dx}{(1+x^2)^{3/2}}}_{=0 \text{ из-за нечётности}} + \sin\omega\tau_0\int_{-\infty}^{\infty}\frac{\cos\omega x\,dx}{(1+x^2)^{3/2}}\Big] + \gamma_\perp\Big[\cos\omega\tau_0\int_{-\infty}^{\infty}\frac{x\sin\omega x\,dx}{(1+x^2)^{3/2}} +$$

$$+ \sin\omega\tau_0\underbrace{\int_{-\infty}^{\infty}\frac{x\cos\omega x\,dx}{(1+x^2)^{3/2}}}_{=0 \text{ из-за нечётности}}\Big] = 2\gamma_\perp\Big[\tau_0\sin\omega\tau_0\int_0^{\infty}\frac{\cos\omega x\,dx}{(1+x^2)^{3/2}} + \cos\omega\tau_0\int_0^{\infty}\frac{x\sin\omega x\,dx}{(1+x^2)^{3/2}}\Big] =$$

$$= 2\gamma_\perp\Big[\tau_0\,\omega K_1(\omega)\sin\omega\tau_0 + \gamma_\perp\cos\omega\tau_0\,\omega K_0(\omega)\Big] = 2\omega\gamma_\perp\Big[\tau_0 K_1(\omega)\sin\omega\tau_0 + K_0(\omega)\cos\omega\tau_0\Big]$$

Т.о.

$$\left(\delta^{(1)}\vec{V}_B\right)_y\Big|_{t\to\infty} = -\frac{ze^2}{mVr_0}\Big[-2\beta\omega K_1(\omega)\cos(\omega\tau_0-\theta) \mp 2\omega\gamma_\perp\tau_0 K_1(\omega)\sin\omega\tau_0 \mp -$$

$$\mp 2\omega\gamma_\perp K_0\cos\omega\tau_0\Big] =$$

$$= -\frac{ze^2}{mVr_0}\Big[-2\omega K_0(\omega)\gamma_\perp\cos\omega\tau_0 + 2\omega K_1(\omega)\underbrace{\left[-\beta\cos(\omega\tau_0-\theta)\mp\gamma_\perp\tau_0\sin\omega\tau_0\right]}_{\beta}\Big]$$

$$[\ ] = -\beta\cos\theta\cos\omega\tau_0 - \beta\sin\theta\sin\omega\tau_0 - \gamma_\perp\tau_0\sin\omega\tau_0 =$$

$$= -\underbrace{\beta\cos\theta}_{(3.30)}\cos\omega\tau_0 - \underbrace{(\beta\sin\theta + \gamma_\perp\tau_0)}_{3.31}\sin\omega\tau_0 = +\sin\Psi\cos\omega\tau_0 - \gamma_{\parallel}\cos\Psi\sin\omega\tau_0$$

Так что окончательно

$$\left(\delta^{(1)}\vec{V}_B\right)_y\Big|_{t\to\infty} = -\frac{ze^2}{mVr_0}\Big[+2\omega K_1(\omega)\left(-\gamma_{\parallel}\cos\Psi\sin\omega\tau_0+\sin\Psi\cos\omega\tau_0\right) - 2\omega K_0(\omega)\gamma_\perp\cos\omega\tau_0\Big] \quad (3.37b)$$

$$\delta^{(1)}r_{\parallel}(t) = -\frac{ze^2}{mV^2}\int_{-\infty}^{\sigma}d\sigma'\int_{-\infty}^{\sigma'}\frac{\gamma_{\parallel}(\sigma''+\tau_0)}{(1+\sigma''^2)^{3/2}}d\sigma'' = -\frac{ze^2}{mV^2}\gamma''\int_{-\infty}^{\sigma}d\sigma'\left[\int_{-\infty}^{\sigma'}\frac{\sigma''d\sigma''}{(1+\sigma''^2)^{3/2}} + \tau_0\int_{-\infty}^{\sigma'}\frac{d\sigma''}{(1+\sigma''^2)^{3/2}}\right.$$

это (3.41)

$$= -\frac{ze^2}{mV^2}\gamma''\int_{-\infty}^{\sigma}d\sigma'\left[-\frac{1}{\sqrt{1+\sigma''^2}}\Big|^{\sigma'} + \tau_0\frac{\sigma''}{\sqrt{1+\sigma''^2}}\Big|^{\sigma'}\right] = -\frac{ze^2}{mV^2}\gamma''\int^{\sigma}d\sigma'\left\{-\frac{\sigma'}{\sqrt{1+\sigma'^2}} + \tau_0\left(\frac{\sigma'}{\sqrt{1+\sigma'^2}}\right.\right.$$

Для будущего:

$$\int \frac{dx}{\sqrt{a+bx+cx^2}^{\,3}} = \frac{2(2cx+b)}{\Delta\sqrt{}} \quad (\Delta = 4ac-b^2) \longrightarrow \int \frac{dx}{\sqrt{1+x^2}^{\,3}} = \frac{x}{\sqrt{1+x^2}}$$

$$\int \frac{x\,dx}{\sqrt{}^{\,3}} = -\frac{2(2a+bx)}{\Delta\sqrt{}} \longrightarrow \int \frac{x\,dx}{\sqrt{1+x^2}^{\,3}} = -\frac{1}{\sqrt{1+x^2}}$$

$$\int_0^\infty \frac{x\sin\omega x\,dx}{\sqrt{1+x^2}^{\,3}} = \omega K_0(\omega) \qquad \int_0^\infty \frac{\cos\omega x\,dx}{\sqrt{\beta+x^2}} = K_0(\sqrt{\beta}\omega) \rightarrow \int_0^\infty \frac{\cos\omega x\,dx}{\sqrt{\beta+x^2}^{\,3}} = -\frac{\omega}{\sqrt{\beta}}K_0'(\sqrt{\beta}\omega) = \frac{\omega}{\sqrt{\beta}}K_1(\sqrt{\beta}\omega)$$

3.754.3 3.754.2

$$\delta\vec{r}_\perp^{(1)}(t) = -\frac{ze^2}{mv^2}\int_{-\infty}^{\sigma} d\sigma' \begin{pmatrix}\cos\omega\tau' & -\sin\omega\tau' \\ \sin\omega\tau' & \cos\omega\tau'\end{pmatrix} \int_{-\infty}^{\sigma'} \frac{d\sigma''}{(1+\sigma''^2)^{3/2}} \begin{pmatrix}\cos\omega\tau'' & \sin\omega\tau'' \\ -\sin\omega\tau'' & \cos\omega\tau''\end{pmatrix} \begin{pmatrix}\beta\sin\theta + \gamma_\perp\tau'' \\ -\beta\cos\theta\end{pmatrix} \quad (3.4?)$$

(3.22)

Это лишнее

$$\tau = \sigma + \tau_0$$

Тогда

$$\cos\omega\tau' = \cos\omega(\sigma'+\tau_0) = \cos\omega\tau_0\cos\omega\sigma' - \sin\omega\tau_0\sin\omega\sigma'$$ и аналогично для σ

$$\sin\omega\tau' = \sin(\sigma'+\tau_0) = \cos\omega\tau_0\sin\omega\sigma' + \sin\omega\tau_0\cos\omega\tau'$$

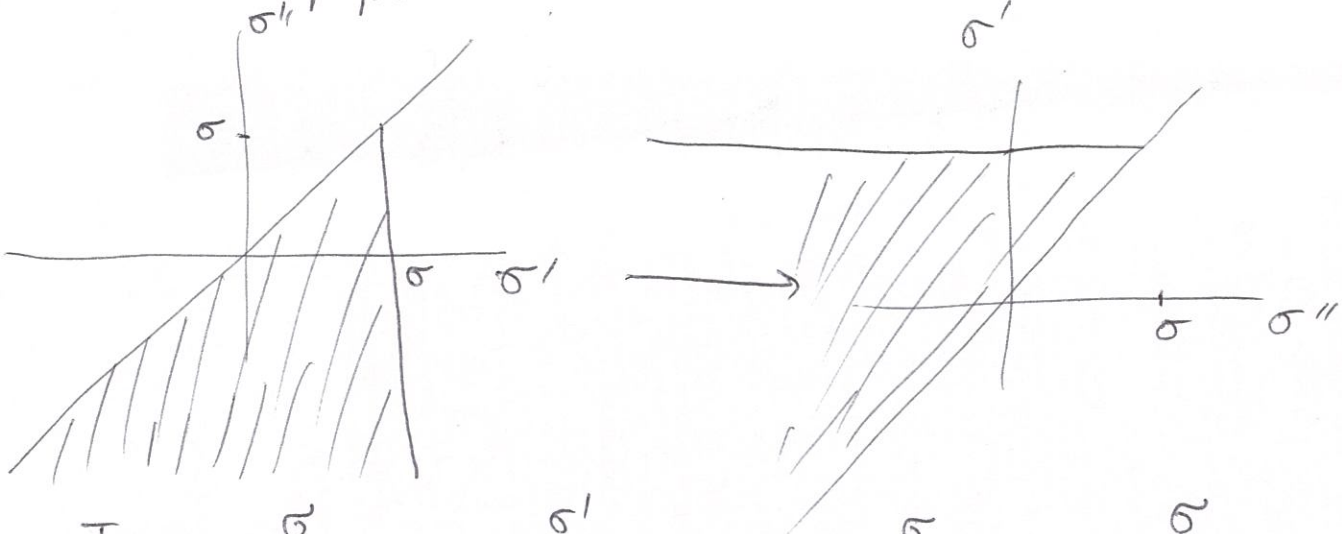
~~$\delta\vec{r}_\perp^{(1)}(t)$~~

$$\delta\vec{r}_\perp^{(1)}(t) = -\frac{ze^2}{mv^2}\int_{-\infty}^{\sigma} d\sigma' \int_{-\infty}^{\sigma'} \frac{d\sigma''}{(1+\sigma''^2)^{3/2}} \underbrace{\begin{pmatrix}\cos\omega\tau' & -\sin\omega\tau' \\ \sin\omega\tau' & \cos\omega\tau'\end{pmatrix}\begin{pmatrix}\cos\omega\tau'' & \sin\omega\tau'' \\ -\sin\omega\tau'' & \cos\omega\tau''\end{pmatrix}\begin{pmatrix}\beta\sin\theta + \gamma_\perp\tau'' \\ -\beta\cos\theta\end{pmatrix}}_{M} =$$

$$M = \begin{pmatrix}\cos\omega(\tau'-\tau'') & \sin\omega(\tau''-\tau') \\ \sin\omega(\tau'-\tau'') & \cos\omega(\tau'-\tau'')\end{pmatrix}\begin{pmatrix}\beta\sin\theta - \gamma_\perp\tau'' \\ -\beta\cos\theta\end{pmatrix} = \begin{pmatrix}\cos\omega(\sigma'-\sigma'')(\beta\sin\theta - \gamma_\perp\tau'') + \sin\omega(\sigma'-\sigma'')\beta\cos\theta \\ \sin\omega(\sigma'-\sigma'')(\beta\sin\theta - \gamma_\perp\tau'') - \cos\omega(\sigma'-\sigma'')\beta\cos\theta\end{pmatrix}$$

$$= \begin{pmatrix}\beta\sin[\omega(\sigma'-\sigma'')+\theta] - \gamma_\perp(\sigma''+\tau_0)\cos\omega(\sigma'-\sigma'') \\ -\beta\cos[\omega(\sigma'-\sigma'')+\theta] - \gamma_\perp(\sigma''+\tau_0)\sin\omega(\sigma'-\sigma'')\end{pmatrix}$$

Область интегрирования



Тогда $\int_{-\infty}^{\sigma} f_1(\sigma')d\sigma' \int_{-\infty}^{\sigma'} f_2(\sigma'')d\sigma'' = \int_{-\infty}^{\sigma} d\sigma'' f_2(\sigma'') \int_{\sigma''}^{\sigma} f_1(\sigma')d\sigma' \underset{\text{переобозначение}}{=} \int_{-\infty}^{\sigma} f_2(\sigma')d\sigma' \int_{\sigma'}^{\sigma} f_1(\sigma'')d\sigma''$

В частности, если $f_1(\sigma) = \frac{dg(\sigma)}{d\sigma}$

то $\int_{-\infty}^{\sigma} \frac{dg(\sigma')}{d\sigma'}d\sigma' \int_{-\infty}^{\sigma'} f_2(\sigma'')d\sigma'' = \int_{-\infty}^{\sigma} f_2(\sigma')d\sigma' \int_{\sigma'}^{\sigma} \frac{dg_1(\sigma'')}{d\sigma''}d\sigma'' =$

$= \int_{-\infty}^{\sigma} f_2(\sigma')d\sigma' \cdot g(\sigma'')\Big|_{\sigma'}^{\sigma} = g(\sigma)\int_{-\infty}^{\sigma} f_2(\sigma')d\sigma' - \int_{-\infty}^{\sigma} g(\sigma')f_2(\sigma')d\sigma'$ | Это из Appendix но

$$\begin{pmatrix}\cos\omega\tau' & -\sin\omega\tau'\\ \sin\omega\tau' & \cos\omega\tau'\end{pmatrix} = \frac{1}{\omega}\frac{d}{d\tau'}\begin{pmatrix}\sin\omega\tau' & \cos\omega\tau'\\ -\cos\omega\tau' & \sin\omega\tau'\end{pmatrix}$$

Тогда в соответствии с рзтом п-та со стр (8) при перестановке порядка интегрирования имеем

$$\delta^{(1)}\vec r_\perp(t) = -\frac{Ze^2}{m\bar v^2}\int_{-\infty}^{\sigma}\frac{d\sigma'}{\omega}\frac{d}{d\sigma'}\begin{pmatrix}\sin\omega\tau' & \cos\omega\tau'\\ -\cos\omega\tau' & \sin\omega\tau'\end{pmatrix}\int_{-\infty}^{\sigma'}\frac{d\sigma''}{(1+\sigma''^2)^{3/2}}\begin{pmatrix}\cos\omega\tau'' & \sin\omega\tau''\\ -\sin\omega\tau'' & \cos\omega\tau''\end{pmatrix}\begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau''\\ -\beta\cos\theta\end{pmatrix} =$$

$\boxed{\sigma = \frac{\bar v t}{r_0}}$ (3.21)

$$= -\frac{1}{\omega\bar v}\frac{Ze^2}{m\bar v^{\cancel{2}}}\begin{pmatrix}\sin\omega\tau & \cos\omega\tau\\ -\cos\omega\tau & \sin\omega\tau\end{pmatrix}\underbrace{\int_{-\infty}^{\sigma}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}T(\omega\tau')\begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau'\\ -\beta\cos\theta\end{pmatrix}}_{= r_0\,\delta^{(1)}\vec V_{B\perp}(t)\ \text{from (3.36)}} - \frac{1}{\omega}\left(-\frac{Ze^2}{m\bar v^2}\right)\times$$

$\boxed{\omega = \frac{r_0\Omega_\perp}{\bar v}}$ (3.23)

$$\times\int_{-\infty}^{\sigma}\begin{pmatrix}\sin\omega\tau' & \cos\omega\tau'\\ -\cos\omega\tau' & \sin\omega\tau'\end{pmatrix}d\sigma'\frac{1}{(1+\sigma'^2)^{3/2}}\begin{pmatrix}\cos\omega\tau' & \sin\omega\tau'\\ -\sin\omega\tau' & \cos\omega\tau'\end{pmatrix}\begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau'\\ -\beta\cos\theta\end{pmatrix}$$

↓ перемножение матриц!

$$\begin{pmatrix}\sin\omega\tau' & \cos\omega\tau'\\ -\cos\omega\tau' & \sin\omega\tau'\end{pmatrix}\begin{pmatrix}\cos\omega\tau' & \sin\omega\tau'\\ -\sin\omega\tau' & \cos\omega\tau'\end{pmatrix}\begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau'\\ -\beta\cos\theta\end{pmatrix} =$$

$$= \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau'\\ -\beta\cos\theta\end{pmatrix} = \begin{pmatrix}-\beta\cos\theta\\ -(\beta\sin\theta+\gamma_\perp\tau')\end{pmatrix} =$$

$$= -\begin{pmatrix}\beta\cos\theta\\ \beta\sin\theta+\gamma_\perp\tau'\end{pmatrix}$$

$\begin{pmatrix}\sin\omega\tau & \cos\omega\tau\\ -\cos\omega\tau & \sin\omega\tau\end{pmatrix}$

$$= \frac{r_0}{\omega\bar v}\delta^{(1)}\vec V_{B\perp}(t) - \frac{1}{\omega}\left(-\frac{Ze^2}{m\bar v^2}\right)\int_{-\infty}^{\sigma}-\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\begin{pmatrix}\beta\cos\theta\\ \beta\sin\theta+\gamma_\perp\tau'\end{pmatrix} =$$

$$= \underbrace{\frac{r_0}{\omega\bar v}}_{\begin{pmatrix}\sin\omega\tau & \cos\omega\tau\\ -\cos\omega\tau & \sin\omega\tau\end{pmatrix}}\delta^{(1)}\vec V_{B\perp}(t) - \frac{Ze^2}{m\bar v^2}\int_{-\infty}^{\sigma}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\cdot\begin{pmatrix}\beta\cos\theta\\ \beta\sin\theta+\gamma_\perp\tau'\end{pmatrix}$$

это (3.42)

So,

$$\delta^{(1)}\vec{r}_\perp(t) = \frac{r_0}{\omega\bar{v}}\,\delta^{(1)}\vec{V}_{B\perp}(t)\begin{pmatrix}\sin\omega\tau & \cos\omega\tau\\ -\cos\omega\tau & \sin\omega\tau\end{pmatrix} - \frac{Ze^2}{m\bar{v}^2}\int_{-\infty}^{\bar{v}t/r_0}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\begin{pmatrix}\beta\cos\theta\\ \beta\sin\theta+\gamma_\perp\tau'\end{pmatrix} \quad (3.22)$$

$$\boxed{\tau' = \frac{vt'}{r_0}\ (3.21);\quad \sigma = \tau' - \tau_0\ (3.22);\quad d\sigma = d\tau'}$$

Analogously: (from 3.40):

$$\delta^{(1)} r_\parallel(t) = -\frac{Ze^2}{m\bar{v}^2}\int_{-\infty}^{\bar{v}t/r_0} d\sigma'\, T(\omega\tau')\int_{-\infty}^{\sigma'}\frac{d\sigma''}{(1+\sigma''^2)^{3/2}}\, T^{-1}(\omega\tau'')\,\gamma_\parallel\tau'' \text{ (more accurasy!)} =$$

$$= -\frac{Ze^2}{m\bar{v}^2}\int_0^{\bar{v}t/r_0} d\sigma' \begin{pmatrix}\cos\omega\tau' & -\sin\omega\tau' & 0\\ \sin\omega\tau' & \cos\omega\tau' & 0\\ 0 & 0 & 1\end{pmatrix}\int^{\sigma'}\frac{d\sigma''}{(1+\sigma''^2)^{3/2}}\begin{pmatrix}\cos\omega\tau'' & \sin\omega\tau'' & 0\\ -\sin\omega\tau'' & \cos\omega\tau'' & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau''\\ -\beta\cos\theta\\ \gamma_\parallel\tau''\end{pmatrix}$$

But for $\delta^{(1)} r_\parallel(t)$ it is necessary to calculate only 3rd component of the product ~~of~~ the matrices:

$$\begin{pmatrix}\cos\omega\tau' & -\sin\omega\tau' & 0\\ \sin\omega\tau' & \cos\omega\tau' & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\cos\omega\tau'' & \sin\omega\tau'' & 0\\ -\sin\omega\tau'' & \cos\omega\tau'' & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau''\\ -\beta\cos\theta\\ \gamma_\parallel\tau''\end{pmatrix} = \begin{pmatrix}\cos\omega(\tau'+\tau'') & -\sin\omega(\tau'-\tau'') & 0\\ \sin\omega(\tau'-\tau'') & \cos\omega(\tau'+\tau'') & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$\begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau''\\ -\beta\cos\theta\\ \gamma_\parallel\tau''\end{pmatrix} = \begin{pmatrix}\dots & \dots\\ \dots & \dots\\ & \gamma_\parallel\tau''\end{pmatrix} \longrightarrow \delta r_\parallel^{(1)}(t) = -\frac{Ze^2}{m\bar{v}^2}\int_{-\infty}^{\sigma} d\sigma'\int_{-\infty}^{\sigma'}\frac{\gamma_\parallel\tau''}{(1+\sigma''^2)^{3/2}}\,d\sigma'' = \left|\tau'' = \sigma''+\tau_0\right. \ (3.22)$$

$$= -\frac{Ze^2}{m\bar{v}^2}\gamma_\parallel\int^{\sigma} d\sigma'\int_{-\infty}^{\sigma'}\frac{(\sigma''+\tau_0)\,d\sigma''}{(1+\sigma''^2)^{3/2}} =$$

$$\int\frac{\sigma''\,d\sigma''}{(1+\sigma''^2)^{3/2}} = (2.271.7) = -\frac{1}{\sqrt{1+\sigma''^2}}$$

$$\int\frac{d\sigma''}{(1+\sigma''^2)^{3/2}} = (2.271.5) = \frac{\sigma''}{\sqrt{1+\sigma''^2}}$$

$$= -\frac{ze^2}{m\bar v^2}\gamma_{\|}\int_{-\infty}^{\sigma} d\sigma'\left[-\frac{1}{\sqrt{1+\sigma''^2}}\Bigg|_{-\infty}^{\sigma'} + \tau_0 \frac{\sigma''}{\sqrt{1+\sigma''^2}}\Bigg|_{-\infty}^{\sigma'}\right] = -\frac{ze^2}{m\bar v^2}\gamma_{\|}\int_{-\infty}^{\sigma} d\sigma'\left[-\frac{1}{\sqrt{1+\sigma'^2}} + \frac{\tau_0\sigma'}{\sqrt{1+\sigma'^2}} - \tau_0(-1)\right] = \quad (9'')$$

so,

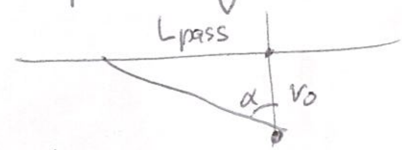
$$\delta v_{\|}(t) = -\frac{ze^2}{m\bar v^2}\gamma_{\|}\int_{-\infty}^{\sigma} d\sigma'\left[\frac{\tau_0\sigma'-1}{\sqrt{1+\sigma'^2}} + \tau_0\right] =$$

This is (3.41)

$$= -\frac{ze^2}{m\bar v^2}\gamma_{\|}\int_{-\infty}^{\sigma} d\sigma'\left(\underbrace{\frac{\tau_0\sigma'}{\sqrt{1+\sigma'^2}}}_{2.271.7} - \underbrace{\frac{1}{\sqrt{1+\sigma'^2}}}_{2.271.3} + \tau_0\right) = -\frac{ze^2}{m\bar v^2}\gamma_{\|}\left[\tau_0\sqrt{1+\sigma'^2}\Bigg|_{-\infty}^{\sigma} - \ln\left(\sigma'+\sqrt{1+\sigma'^2}\right)\Bigg|_{-\infty}^{\sigma} + \tau_0\sigma'\Bigg|_{-\infty}^{\sigma}\right] =$$

$$= -\frac{ze^2}{m\bar v^2}\gamma_{\|}\left(\tau_0\sqrt{1+\sigma^2} - \tau_0\sqrt{1+\sigma'^2}\Bigg|_{\sigma'\to-\infty} + \tau_0\sigma - \tau_0\sigma'\Bigg|_{\sigma'\to-\infty} - \ln\frac{\sigma+\sqrt{1+\sigma^2}}{\sigma'+\sqrt{1+\sigma'^2}}\Bigg|_{\sigma'\to-\infty}\right)$$

The divergence on the limit $t\to-\infty$ can be bypassed to to restriction on „length" of trajectory: $-\infty \to \frac{\bar v\Delta t_{pass}}{r_0}$, where $\Delta t_{pass} \simeq \frac{L_{pass}}{\bar V} \simeq \frac{r_0\, tg\,\alpha}{\bar V}$.



Lagrangian: $\mathcal{L} = \frac{m\vec{v}_e^{\,2}}{2} + \frac{M\vec{v}_i^{\,2}}{2} - e\vec{A}(\vec{r}_e)\vec{v}_e + Ze\vec{A}(\vec{r}_i)\vec{v}_i + \frac{Ze^2}{|\vec{r}_e - \vec{r}_i|}$ (3.3)

magnetic field along $\vec{e}_z$ $\Rightarrow$ $\vec{B} = B\vec{e}_z$ and vector-potential for that is

$$\vec{A}(\vec{r}) = \frac{1}{2}[\vec{B}\vec{r}] = \frac{1}{2}\begin{vmatrix}\vec{e}_x & \vec{e}_y & \vec{e}_z\\ B_x & B_y & B_z\\ x & y & z\end{vmatrix} = \frac{1}{2}\begin{vmatrix}\vec{e}_x & \vec{e}_y & \vec{e}_z\\ 0 & 0 & B\\ x & y & z\end{vmatrix} = \frac{1}{2}(yB\,\vec{e}_x - Bx\vec{e}_y) = \frac{B}{2}(y\vec{e}_x - x\vec{e}_y)$$

Checking:

$$\vec{B} = \mathrm{rot}\vec{A} = \begin{vmatrix}\vec{e}_x & \vec{e}_y & \vec{e}_z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ A_x & A_y & A_z\end{vmatrix} = \begin{vmatrix}\vec{e}_x & \vec{e}_y & \vec{e}_z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{yB}{2} & -\frac{xB}{2} & 0\end{vmatrix} = \vec{e}_x\left(0 - \frac{1}{2}\frac{\partial(xB)}{\partial z}\right) + \vec{e}_y\left(\frac{1}{2}\frac{\partial(yB)}{\partial z} - 0\right) + \vec{e}_z\left(-\frac{1}{2}\frac{\partial(xB)}{\partial x} - \frac{\partial(\dots)}{\partial y}\right)$$

$$= 0\cdot\vec{e}_x + 0\cdot\vec{e}_y + B\vec{e}_z \quad \text{OK!}$$

Let ion moves uniformly with velocity ~~$\vec{v}_i$~~

$$\vec{V}_i = \begin{pmatrix} V_{i\perp} \\ 0 \\ V_{i\parallel}\end{pmatrix} \quad (3.1) \longrightarrow \vec{r}_i(t) = \vec{V}_i\cdot t$$

Input relative coordinate of electron $\vec{r}(t) = \vec{r}_e(t) - \vec{r}_i(t) = \vec{r}_e(t) - \vec{V}_i\cdot t$ (3.2a)

So, $\vec{r}(t)$ — coordinate ~~of~~ electron in the ion's frame. Farther

(3.2b) $\vec{V}(t) = \vec{V}_e(t) - \vec{V}_i(t) = \vec{V}_e(t) - \vec{V}_i$ — relative electron's velocity in the ion's frame

Let's define the radius and velocity of the center of mass:

$$\begin{cases}\vec{r}_{c.m} = (m_e\vec{r}_e + M_i\vec{r}_i)/(m_e+M_i)\\ \vec{r} = \vec{r}_e - \vec{r}_i\end{cases} \text{ and } \begin{cases}\vec{V}_{c.m} = (m_e\vec{V}_e + M_i\vec{V}_i)/(m_e+M_i)\\ \vec{V} = \vec{V}_e - \vec{V}_i\end{cases}$$

$$\begin{cases} m\vec{r}_e + M\vec{r}_i = (m_e+M_i)\vec{r}_{cm}\\ \vec{r}_e - \vec{r}_i = \vec{r}\end{cases} \longrightarrow \Delta = \begin{vmatrix} m_e & M_i\\ 1 & -1\end{vmatrix} = -(m_e+M_i) \longrightarrow \vec{r}_e = \frac{1}{\Delta}\begin{vmatrix}(m_e+M_i)\vec{r}_{cm} & M_i\\ \vec{r} & -1\end{vmatrix} = \vec{r}_{cm} + \frac{M_i}{m_e+M_i}\vec{r}$$

and $\vec{r}_i = \frac{1}{\Delta}\begin{vmatrix} m_e & (m_e+M_i)\vec{r}_{cm}\\ 1 & \vec{r}\end{vmatrix} = \vec{r}_{cm} - \frac{m_e}{m_e+M_i}\vec{r} \simeq \vec{r}_{cm} - \frac{m}{M}\vec{r}$

$\vec{V}_i t$

$\vec{r}_{cm} \simeq \vec{V}_i t + \frac{m}{M}\vec{r}$

and analogusly

$$\vec{V}_e = \vec{V}_{cm} + \frac{M_i}{m_e+M_i}\vec{V}$$

$$\vec{V}_i = \vec{V}_{cm} - \frac{m_e}{m_e+M_i}\vec{V} \simeq \vec{V}_{cm} - \frac{m}{M}\vec{V}$$

$$\mu = 1\Big/\left(\frac{1}{m_e} + \frac{1}{M_i}\right) = \frac{m_e M_i}{m_e+M_i}$$

So

$$\begin{cases} \vec{r}_e = \vec{r}_{cm} + \frac{\mu}{m_e}\vec{r} \\ \vec{r}_i = \vec{r}_{cm} - \frac{\mu}{M_i}\vec{r} \end{cases} \quad \text{and} \quad \begin{cases} \vec{V}_e = \vec{V}_{cm} + \frac{\mu}{m_e}\vec{V} \\ \vec{V}_i = \vec{V}_{cm} - \frac{\mu}{M_i}\vec{V} \end{cases}$$

$$\vec{A}(\vec{r}_e) = \frac{1}{2}[\vec{B}\cdot\vec{r}_e] = \frac{1}{2}\left[\vec{B}\cdot\left(\vec{r}_{cm} + \frac{\mu}{m}\vec{r}\right)\right]$$

$$\vec{A}(\vec{r}_i) = \frac{1}{2}[\vec{B}\cdot\vec{r}_i] = \frac{1}{2}\left[\vec{B}\cdot\left(\vec{r}_{cm} - \frac{\mu}{M}\vec{r}\right)\right]$$

Then

$$\mathcal{L} = \frac{m}{2}\left(\vec{V}_{cm} + \frac{\mu}{m}\vec{V}\right)^2 + \frac{M}{2}\left(\vec{V}_{cm} - \frac{\mu}{M}\vec{V}\right)^2 - e\frac{1}{2}\left[\vec{B}\left(\vec{r}_{cm} + \frac{\mu}{m}\vec{r}\right)\right]\left(\vec{V}_{cm} + \frac{\mu}{m}\vec{V}\right) - \frac{Ze}{2}\left[\vec{B}\left(\vec{r}_{cm} - \frac{\mu}{M}\vec{r}\right)\right]\left(\vec{V}_{cm} - \frac{\mu}{M}\vec{V}\right) + \frac{Ze^2}{r} =$$

$$= \underset{(1)}{\frac{m}{2}\vec{V}_{cm}^2} + \cancel{2\mu\vec{V}\vec{V}_{cm}} + \underset{(2)}{\frac{\mu^2}{2m}\vec{V}^2} + \underset{(3)}{\frac{M}{2}\vec{V}_{cm}^2} - \cancel{\mu\vec{V}\vec{V}_{cm}} + \underset{(4)}{\frac{\mu^2}{2M}\vec{V}^2} - \frac{e}{2}\Big\{\underset{(5)}{([\vec{B}\vec{r}_{cm}]\vec{V}_{cm})} + \frac{\mu}{m}\underset{(6)}{([\vec{B}\vec{r}]\vec{V}_{cm})} + \underset{(7)}{([\vec{B}\vec{r}_{cm}]\frac{\mu}{m}\vec{V})} +$$

$$+ \frac{\mu^2}{m^2}\underset{(8)}{([\vec{B}\vec{r}]\vec{V})}\Big\} + \frac{Ze}{2}\Big\{\underset{(9)}{([\vec{B}\vec{r}_{cm}]\vec{V}_{cm})} - \frac{\mu}{M}\underset{(10)}{([\vec{B}\vec{r}]\vec{V}_{cm})} - \frac{\mu}{M}\underset{(11)}{([\vec{B}\vec{r}_{cm}]\vec{V})} + \frac{\mu^2}{M^2}\underset{(12)}{([\vec{B}\vec{r}]\vec{V})}\Big\} + \frac{Ze^2}{r}$$

$$= \underset{(1)+(3)\ \leftarrow \text{this is an constant}}{\frac{m+M}{2}\vec{V}_{cm}^2} + \underset{(2)\quad(4)}{\frac{\mu^2}{2}\vec{V}^2\left(\frac{1}{m} + \frac{1}{M}\right)} + \frac{Ze^2}{r} + \underset{(5)\quad(9)\ \leftarrow \text{this is an constant}}{\frac{(Z-1)e}{2}([\vec{B}\cdot\vec{r}_{cm}]\vec{V}_{cm})} + \underset{(12)\quad(8)}{\frac{\mu^2}{2}\left(\frac{Ze}{M^2} - \frac{e}{m^2}\right)([\vec{B}\vec{r}]\vec{V})} +$$

$$-\frac{\mu}{2}([\vec{B}\vec{r}]\vec{V}_{cm})\underset{(6)\quad(10)}{\left(\frac{e}{m} + \frac{Ze}{M}\right)} - \frac{\mu}{2}\underset{(7)\quad(11)}{\left(\frac{e}{m} + \frac{Ze}{M}\right)}([\vec{B}\vec{r}_{cm}]\vec{V}) \simeq$$

this is (3.5) and taking into account $\mu \simeq m \ll M$

$$\simeq \underset{(2)+(4)}{\frac{m\vec{V}^2}{2}} + \frac{Ze^2}{r} - \underset{(12)+(8)}{\frac{e}{2}([\vec{B}\vec{r}]\vec{V})} - \left(\frac{e}{m} + \frac{Ze}{M}\right)\frac{\mu}{2}\underset{(6)+(10)+(7)+(11)}{\left(([\vec{B}\vec{r}]\overset{\vec{V}_i + \frac{m}{M}\vec{V}}{\vec{V}_{cm}}) + ([\vec{B}\vec{r}_{cm}]\vec{V})\right)} =$$

$$\simeq \frac{m\vec{V}^2}{2} + \frac{Ze^2}{r} - \frac{e}{2}([\vec{B}\vec{r}]\vec{V}) - \frac{e}{2}\left\{\left([\vec{B}\vec{r}]\left(\vec{V}_i + \frac{m}{M}\vec{V}\right)\right) + ([\vec{B}\vec{r}_{cm}]\vec{V})\right\} = \frac{m\vec{V}^2}{2} + \frac{Ze^2}{r} - \frac{e}{2}([\vec{B}\vec{r}]\vec{V}) -$$

$$-\frac{e}{2}\left\{([\vec{B}\vec{r}]\vec{V}_i + \left[\vec{B}\left(\vec{r}_{cm} + \frac{m}{M}\vec{r}\right)\right]\vec{V}\right\} =$$

$$= \frac{m\vec{v}^2}{2} + \frac{ze^2}{r} - \frac{e}{2}([\vec{B}\vec{r}]\vec{v}) - \frac{e}{2}\left\{([\vec{B}\vec{r}]\vec{v}_i) + ([\vec{B}\cdot\vec{v}_i t]\vec{v}) + 2\frac{m}{M}([\vec{B}\vec{r}]v)\right\} =$$

$$= \frac{m\vec{v}^2}{2} + \frac{ze^2}{r} - \frac{e}{2}([\vec{B}\vec{r}]\vec{v}_i) - \frac{e}{2}([\vec{B}\cdot\vec{v}_i t]\vec{v}) - \frac{e}{2}(1 + \underbrace{2\frac{m}{M}}_{2\frac{m}{M} \ll 1})([\vec{B}\vec{r}]\vec{v}) \Longrightarrow$$

$$\mathcal{L} = \frac{m\vec{v}^2}{2} + \frac{ze^2}{r} - \frac{e}{2}([\vec{B}\vec{r}]\vec{v}) - \frac{e}{2}\left\{([\vec{B}\cdot\vec{v}_i t]\vec{v}) + ([\vec{B}\vec{r}]\vec{v}_i)\right\}$$ this is (3.6)

Equation of motion:

$$m\frac{d\vec{v}}{dt} = -\frac{\partial \mathcal{L}}{\partial \vec{r}} = -\vec{v}\,\frac{ze^2}{r}$$

From (3.14) for $R \to 0$ („Guiding" center approach):

$$\vec{r}^{\,2}(t) = b^2 + [(v_{e\parallel} - v_{i\parallel})^2 + v_{i\perp}^2]\,t^2 - 2v_{i\perp} b\, t \sin\theta \qquad (3.15)$$

Let's define the relative velocity $\vec{\bar{V}}$ of the guiding center and ion:

$$\vec{\bar{V}} = \vec{\bar{V}}_\perp + \vec{\bar{V}}_\parallel = \begin{pmatrix} 0 \\ 0 \\ v_{e\parallel} - v_{i\parallel} \end{pmatrix} + \begin{pmatrix} -v_{i\perp} \\ 0 \\ 0 \end{pmatrix} \qquad (3.16)$$

Then $(v_{e\parallel} - v_{i\parallel})^2 + v_{i\perp}^2 = \vec{\bar{V}}^2 \equiv \bar{V}^2$

Let's t_0 – time t_0 when electron reaches the minimal distance $\vec{r}_0$ to ion (this is impact parameter of collision); i.e.

$$r_0^2 = b^2 + \bar{V}^2 t_0^2 - 2v_{i\perp} b\, t_0 \sin\theta$$

Very important: $\vec{r}_0 \perp \vec{\bar{V}}$ and from picture one has

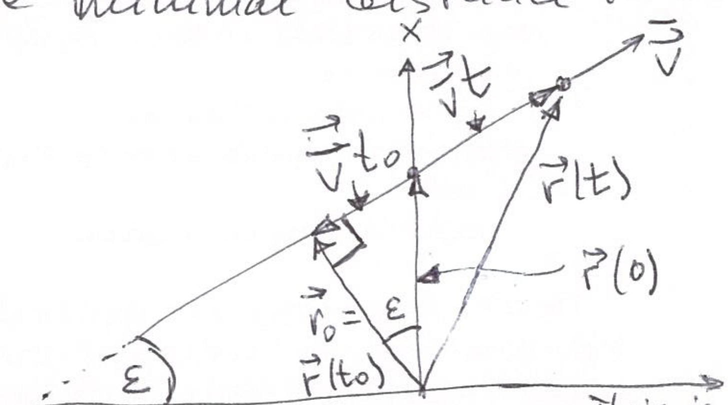


Fig. 2

This is direction of magnetic field !

$$\vec{r}(t) = \vec{r}_0 + \vec{\bar{V}}(t - t_0) \qquad (3.20)$$

then $$\vec{r}^{\,2} = r_0^2 + \underbrace{\vec{\bar{V}}^2}_{\equiv \bar{V}^2}(t-t_0)^2 + 2\underbrace{\vec{r}_0\vec{\bar{V}}}_{=0}(t-t_0) = r_0^2 + \bar{V}^2(t-t_0)^2$$

So
$$\vec{r}^{\,2} = b^2 + \bar{V}^2 t^2 - 2v_{i\perp} b\, t \sin\theta = r_0^2 + \bar{V}^2(t-t_0)^2 = b^2 + \bar{V}^2 t_0^2 - 2v_{i\perp} b\, t_0 \sin\theta + \bar{V}^2(t-t_0)^2$$

or

$$\cancel{b^2} + \cancel{\bar{V}^2 t^2} - 2v_{i\perp} b\, t\sin\theta = \cancel{b^2} + \underline{\bar{V}^2 t_0^2} - 2v_{i\perp} b\, t_0 \sin\theta + \cancel{\bar{V}^2 t^2} + \underline{\bar{V}^2 t_0^2} - 2\bar{V}^2 t\, t_0$$

$$\cancel{2\bar{V}^2 t_0^2 - 2t_0(v_{i\perp} b \sin\theta - \bar{V}^2 t) + 2v_{i\perp} b\, t \sin\theta = 0}$$

So

$$0 = 2v_{i\perp} b\sin\theta\cdot(t-t_0) + 2\bar{V}^2 t_0(t_0-t) \Longrightarrow t_0 = \frac{v_{i\perp} b\sin\theta}{\bar{V}^2} \quad (3.18) = -\frac{\bar{V}_\perp b\sin\theta}{\bar{V}^2} =$$

and for this reason

$$r_0^2 = b^2 + \bar{V}^2 t_0^2 - 2t_0\cdot \overbrace{\frac{v_{i\perp}\sin\theta}{\bar{V}^2}}^{=t_0}\bar{V}^2 = b^2 + \bar{V}^2 t_0^2 - 2\bar{V}^2 t_0^2 = b^2 - \bar{V}^2 t_0^2 \quad (3.19)$$

and

$$\vec{r}_0 = \vec{r}(t_0) = \begin{pmatrix} b\sin\theta - v_{i\perp}t_0 \\ -b\cos\theta \\ (v_{e\parallel}-v_{i\parallel})t_0 \end{pmatrix} = \begin{pmatrix} b\sin\theta + \bar{V}_\perp t_0 \\ -b\cos\theta \\ \bar{V}_\parallel t_0 \end{pmatrix}$$

this is from (3.10) with $R=0$

and then

$$\vec{r}(t) = \begin{pmatrix} -b\sin\theta - v_{i\perp}t \\ -b\cos\theta \\ (v_{e\parallel}-v_{i\parallel})t \end{pmatrix} = \begin{pmatrix} -b\sin\theta + \bar{V}_\perp t \\ -b\cos\theta \\ \bar{V}_\parallel t \end{pmatrix} = \begin{pmatrix} -b\sin\theta + \bar{V}_\perp(t_0+t-t_0) \\ -b\cos\theta \\ \bar{V}_\parallel(t_0+t-t_0) \end{pmatrix} =$$

$$= \begin{pmatrix} -b\sin\theta + \bar{V}_\perp t_0 \\ -b\cos\theta \\ \bar{V}_\parallel t_0 \end{pmatrix} + \begin{pmatrix} \bar{V}_\perp(t-t_0) \\ 0 \\ \bar{V}_\parallel(t-t_0) \end{pmatrix} = \vec{r}_0 + \vec{\bar{V}}(t-t_0)$$

this is (3.20) again

Let's introduce the dimentionless variables:

$$\tau = \frac{\bar{V}t}{r_0} \quad (3.21) \qquad \tau_0 = \frac{\bar{V}t_0}{r_0}, \quad \sigma = \tau - \tau_0 = \frac{\bar{V}(t-t_0)}{r_0} \quad (3.22)$$

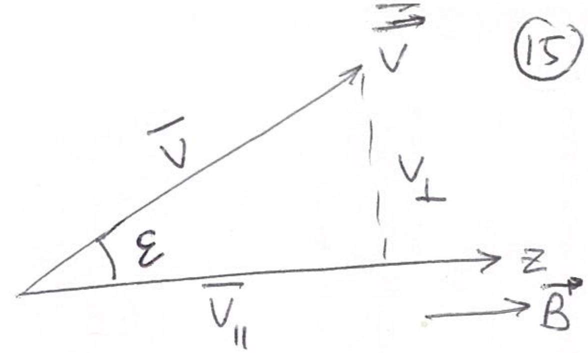
This is (3.29 left)

$$\gamma_\parallel = \frac{\bar{V}_\parallel}{\bar{V}}, \qquad \gamma_\perp = \frac{\bar{V}_\perp}{\bar{V}}, \qquad \beta = \frac{b}{r_0}$$

(3.24) (3.25) (3.26)

Then from (3.18) $\tau_0 = \frac{\bar{V}t_0}{r_0} = \frac{\bar{V}}{r_0}\cdot\frac{-\bar{V}_\perp b\sin\theta}{\bar{V}^2} =$

$$\frac{\vec r_0}{r_0} = \frac{1}{r_0}\begin{pmatrix} b\sin\theta + \bar v_\perp t_0 \\ -b\cos\theta \\ \bar v_\parallel t_0\end{pmatrix} = \begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau_0 \\ -\beta\cos\theta \\ \gamma_\parallel \tau_0\end{pmatrix}$$



Instead (3.19) one has

$$b^2 = r_0^2 + \bar v^2 t_0^2 \to \beta^2 = 1+\tau_0^2 \quad (3.32)$$

$$\cos\varepsilon = \frac{\bar v_\parallel}{\bar v} = \gamma_\parallel \quad (3.28)$$

$$\sin\varepsilon = \frac{\bar v_\perp}{\bar v} = \gamma_\perp \quad (3.28^*)$$

and, of course, $\gamma_\parallel^2+\gamma_\perp^2 = 1$

Let's input

$$\sin\psi = -\beta\cos\theta \quad (3.30)$$

Then

$$\beta\sin\theta + \gamma_\perp\tau_0 = \left(\text{from 3.29 left } \beta\sin\theta = -\frac{\tau_0}{\gamma_\perp}\right) = -\frac{\tau_0}{\gamma_\perp} + \gamma_\perp\tau_0 = \tau_0\frac{\gamma_\perp^2-1}{\gamma_\perp} = -\frac{\tau_0\gamma_\parallel^2}{\gamma_\perp}$$

But

$$\tau_0 = -\beta\gamma_\perp\sin\theta = -\gamma_\perp\beta\sqrt{1-\cos^2\theta} = -\gamma_\perp\beta\sqrt{1-\frac{\sin^2\psi}{\beta^2}} = -\gamma_\perp\sqrt{\beta^2-\sin^2\psi} =$$

$$= -\gamma_\perp\sqrt{\beta^2-1+\cos^2\psi} = -\gamma_\perp\sqrt{\tau_0^2+\cos^2\psi} \quad \text{or}$$

from (3.32)

$$\tau_0^2 = \gamma_\perp^2\tau_0^2 + \gamma_\perp^2\cos^2\psi \to \tau_0^2(1-\gamma_\perp^2) = \gamma_\perp^2\cos^2\psi \to$$

$$\tau_0^2\gamma_\parallel^2 = \gamma_\perp^2\cos^2\psi \to \tau_0 = \pm\frac{\gamma_\perp}{\gamma_\parallel}\cos\psi$$ and it is neccesary to select sign "−"!

from (3.28*)

$$\tau_0 = -\frac{\gamma_\perp}{\gamma_\parallel}\cos\psi \quad (3.29\text{ right}) \to \gamma_\parallel\tau_0 = -\gamma_\perp\cos\psi = -\sin\varepsilon\cos\psi \quad (3.29^*)$$

So

$$\to \beta\sin\theta + \gamma_\perp\tau_0 = -\frac{\tau_0\gamma_\parallel^2}{\gamma_\perp} = \frac{\gamma_\perp\cos\psi\cdot\gamma_\parallel^2}{\gamma_\parallel\gamma_\perp} = \gamma_\parallel\cos\psi \quad (3.31)$$

Therefore

$$\frac{\vec{r}_0}{r_0} = \begin{pmatrix} \beta \sin\theta + \gamma_\perp \tau_0 \\ -\beta\cos\theta \\ \gamma_{\parallel}\tau_0 \end{pmatrix} = \begin{pmatrix} \text{using } (3.31) \\ \text{using } (3.30) \\ \text{using } (3.29^*) \end{pmatrix} = \begin{pmatrix} \cos\varepsilon\cos\psi \\ \sin\psi \\ -\sin\varepsilon\cos\psi \end{pmatrix} \quad (3.27)$$

Finaly:

(3.21) $\tau = \dfrac{\bar{v}t}{r_0}$ $\quad \tau_0 = \dfrac{\bar{v}t_0}{r_0}$

(3.22) $\sigma = \tau - \tau_0 = \dfrac{\bar{v}(t-t_0)}{r_0}$

(3.24) $\gamma_{\parallel} = \dfrac{\bar{v}_{\parallel}}{\bar{v}_0}$

(3.25) $\gamma_\perp = \dfrac{\bar{v}_\perp}{\bar{v}_0}$

$$\gamma_{\parallel}^2 + \gamma_\perp^2 = 1$$

(3.26) $\beta = \dfrac{b}{r_0}$

(3.27) $\dfrac{\vec{r}_0}{r_0} = \begin{pmatrix} \cos\varepsilon\cos\psi \\ \sin\psi \\ -\sin\varepsilon\cos\psi \end{pmatrix}$

$\beta\sin\theta = \dfrac{\cos\psi}{\cancel{\cos}\,\gamma_{\parallel}} = \dfrac{\cos\psi}{\cos\varepsilon}$

$\beta\cos\theta = -\sin\psi$, ~~$\tau_0\gamma_\parallel$~~

so

$$\boxed{\operatorname{tg}\theta = -\frac{\cos\psi/\cos\varepsilon}{\sin\psi} = -\frac{\operatorname{ctg}\psi}{\cos\varepsilon}}$$

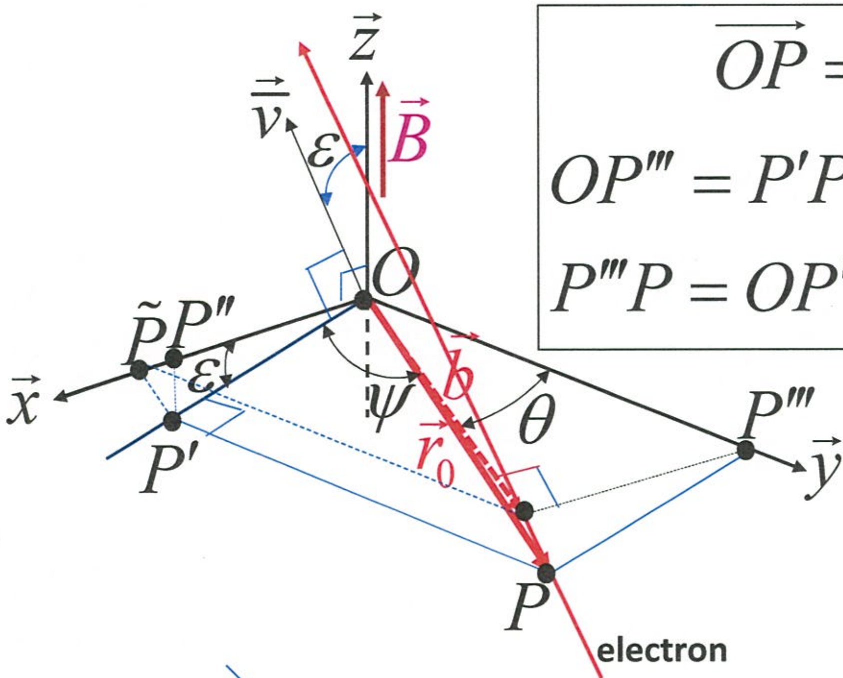
(3.28) $\cos\varepsilon = \gamma_{\parallel} \longrightarrow \sin\varepsilon = \sqrt{1-\cos^2\varepsilon} = \sqrt{1-\gamma_{\parallel}^2} = \gamma_\perp$ (3.28*)

(3.29) $\tau_0 = -\beta\gamma_\perp\sin\theta = -\dfrac{\gamma_\perp}{\gamma_{\parallel}}\cos\psi \longrightarrow \gamma_{\parallel}\tau_0 = -\sin\varepsilon\cos\psi$ (3.29*)

(3.30) $\beta\cos\theta = -\sin\psi$

(3.31) $\beta\sin\theta + \gamma_\perp\tau_0 = -\dfrac{\gamma_{\parallel}^2\tau_0}{\gamma_\perp} = \gamma_{\parallel}\cos\psi$

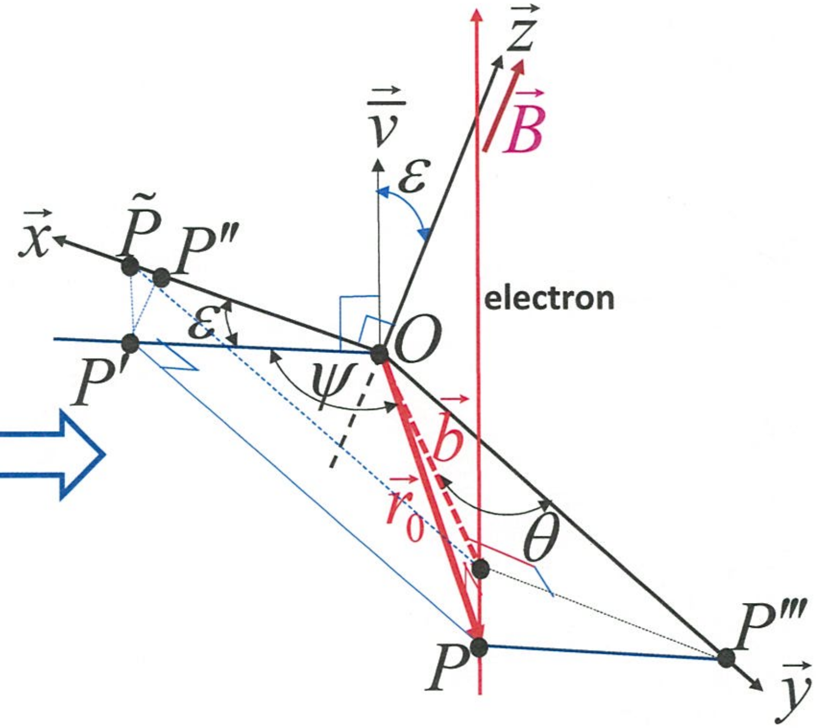
(3.32) $\beta^2 = 1 + \tau_0^2$



$$\overrightarrow{OP} = \vec{r}_0 \Rightarrow$$

$$\left.\begin{aligned} OP''' &= P'P = r_0 \cdot \sin\psi, \\ P'''P &= OP' = r_0 \cdot \cos\psi \end{aligned}\right\}$$

or



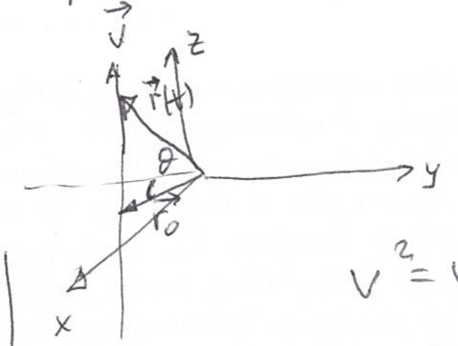
$$\vec{r}_0 = \begin{pmatrix} OP'' \\ PP' \\ P''P' \end{pmatrix} = r_0 \begin{pmatrix} \cos\varepsilon \cdot \cos\psi \\ \sin\psi \\ -\sin\varepsilon \cdot \cos\psi \end{pmatrix};$$

$$O\tilde{P} = OP' / \cos\varepsilon = r_0 \cdot \cos\psi / \cos\varepsilon \rightarrow$$
$$\rightarrow tg\theta = O\tilde{P} / OP''' = -\cot\psi / \cos\varepsilon$$

Sign from corresponding formulas

Unmagnetized electron scattering with ion in fixed point of origin

$$\frac{d\vec{v}}{dt} = -\frac{e}{m}\vec{E}(\vec{r}) \quad (2.3), \qquad \vec{E}(\vec{r}) = Ze\,\frac{\vec{r}(t)}{r^3(t)} \quad (2.1)$$



$$V^2 = V_\perp^2 + V_\parallel^2 \;!$$

r_0 – impact parameter is $\vec{r}(t=0)$, and

$$\vec{r}_0 = r_0\begin{pmatrix}\sin\theta\\ -\cos\theta\\ 0\end{pmatrix} \quad (2.5) \quad \text{then} \quad \vec{r}(t) = \begin{pmatrix} r_{ox}\\ r_{oy}\\ vt\end{pmatrix} = \begin{pmatrix} r_0\sin\theta\\ -r_0\cos\theta\\ vt\end{pmatrix}$$

Then first order:

$$(2.6)\quad \delta^{(1)}\vec{V}_c(t) = -\frac{Ze^2}{m}\int_{-\infty}^{t}\frac{dt'}{\sqrt{(r_0^2+V^2t'^2)^3}}\begin{pmatrix} r_0\sin\theta\\ -r_0\cos\theta\\ V_\parallel t'\end{pmatrix} \underset{x=vt}{=} -\frac{Ze^2}{mV}\int_{-\infty}^{vt}\frac{r_0\,dx/r_0}{(r_0^2+x^2)^{3/2}}\begin{pmatrix} r_0\sin\theta\\ -r_0\cos\theta\end{pmatrix} = -\frac{Ze^2}{mVr_0}\int_{-\infty}^{\frac{vt}{r_0}}\frac{dx}{(1+x^2)^{3/2}}\begin{pmatrix}\sin\theta\\ -\cos\theta\\ \frac{V_\parallel}{V}x\end{pmatrix}$$

(lower index „c" means „Coulomb").

Total $\delta^{(1)}\vec{V}_c(t)$ is reached for $t\to\infty$!

$$\left(\delta^{(1)}\vec{V}_c\right)_{total} = -\frac{Ze^2}{mVr_0}\int_{-\infty}^{\infty}\frac{dx}{(1+x^2)^{3/2}}\begin{pmatrix}\sin\theta\\ -\cos\theta\\ \frac{V_\parallel}{V}x\end{pmatrix} = -\frac{2Ze^2}{mVr_0}\underbrace{\int^{\infty}\frac{dx}{(1+x^2)^{3/2}}}_{=1}\begin{pmatrix}\sin\theta\\ -\cos\theta\\ 0\end{pmatrix}$$

integral $=0$ due to odd function!

2.271.5: $\int\frac{dx}{(a+cx^2)^{3/2}} = \frac{1}{a}\frac{x}{\sqrt{a+cx^2}}$

$$= -\frac{\cancel{2}Z^2e^2}{mVr_0}\begin{pmatrix}\sin\theta\\ -\cos\theta\\ 0\end{pmatrix} = -\frac{2Ze^2}{mVr_0^2}\,r_0\begin{pmatrix}\sin\theta\\ -\cos\theta\\ 0\end{pmatrix} = -\frac{2Ze^2\vec{r}_0}{mVr_0^2} \quad (2.7)$$

Now let calculate $\delta\vec{r}^{(1)}$: $\quad \frac{d\vec{r}}{dt} = \vec{V} \quad (2.2)$

~~then~~ But it is possible to write the solution for $\delta^{(1)}\vec{V}_c(t)$ from (2.6):

$$\delta^{(1)}\vec{V}_c(t) = -\frac{Ze^2}{mVr_0}\int_{-\infty}^{\frac{vt}{r_0}}\frac{dx}{(1+x^2)^{3/2}}\begin{pmatrix}\sin\theta\\ -\cos\theta\\ \frac{V_\parallel}{V}x\end{pmatrix} = \begin{pmatrix}\int 2.27.5\\ 2.27.7\\ \text{with } n=1\end{pmatrix} =$$

2.271.7: $\int\frac{x\,dx}{\sqrt{a+cx^2}^3} = -\frac{1}{c\sqrt{a+cx^2}}$

$$= -\frac{Ze^2}{mVr_0}\left(\frac{vt/r_0}{\sqrt{1+v^2t^2/r_0^2}}+1\right)\begin{pmatrix}\sin\theta\\ -\cos\theta\\ 0\end{pmatrix} - \frac{Ze^2}{mVr_0}\left(-\frac{\mathrm{sgn}(v)}{\sqrt{1+v^2t^2/r_0^2}}\right)\begin{pmatrix}0\\0\\1\end{pmatrix}$$

my opinion: factor $\frac{v_{\parallel}}{V} = \gamma_{\parallel}$!

and for $t \to \infty$ one will receive the result (2,7)

Now $\delta^{(1)}\vec{r}_c$ can be received from $\delta^{(1)}\vec{V}_c(t)$:

$$\delta^{(1)}\vec{r}_c(t) = \int_{-\infty}^{t} dt'\,\delta^{(1)}\vec{V}_c(t') = -\frac{Ze^2}{mVr_0}\int_{-\infty}^{t}\left(\frac{vt'/r_0}{\sqrt{1+v^2t'^2/r_0^2}}+1\right)dt'\begin{pmatrix}\sin\theta\\ -\cos\theta\\ 0\end{pmatrix} +$$

$$+\frac{Ze^2}{mVr_0}\int_{-\infty}^{t}\frac{dt'\,\mathrm{sgn}(v)}{\sqrt{1+v^2t'^2/r_0^2}}\begin{pmatrix}0\\0\\1\end{pmatrix} = -\frac{Ze^2}{mV^2}\int_{-\infty}^{t}\left(\frac{vt'/r_0}{\sqrt{1+v^2t'^2/r_0^2}}+1\right)d\left(\frac{vt'}{r_0}\right)\begin{pmatrix}\sin\theta\\ -\cos\theta\\ 0\end{pmatrix} + \frac{Ze^2}{mV^2}\int_{-\infty}^{t}\frac{\mathrm{sgn}(v)\,d\left(\frac{vt'}{r_0}\right)}{\sqrt{1+v^2t'^2/r_0^2}}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

must be $\gamma_{\parallel}$

$$= -\frac{Ze^2}{mV^2}\int_{-\infty}^{vt/r_0} d\tau'\left[\left(\frac{\tau'}{\sqrt{1+\tau'^2}}+1\right)\begin{pmatrix}\sin\theta\\ -\cos\theta\\ 0\end{pmatrix} - \frac{\mathrm{sgn}(v)}{\sqrt{1+\tau'^2}}\begin{pmatrix}0\\0\\1\end{pmatrix}\right]$$ this is (2,11)

Question: how from expessions for $\delta^{(1)}\vec{r}_{\perp}(t)$ and $\delta^{(1)}v_{\parallel}(t)$ ((3.41) and (3.42) correspondingly) to receive the expressions (2.11), (2.7) because, it seems, that

$$\boxed{(2.11) = \lim_{\omega \Rightarrow \frac{r_0}{V}\Omega_B \to 0} (3.41), (3.42)}$$

Differences between (2,7), (2,11) and (3.41), (3.42): first are in lab. system second — in ion frame system

Second-order approach

A) Field:
$$\vec{E}(\vec{r}(t)) = Ze\frac{\vec{r}(t)}{r^3(t)} \longrightarrow E_k(\vec{r}(t)) = Ze\frac{r_k(t)}{r_i r_i}$$

$$E_k = E_k(\vec{r}+\delta^{(1)}\vec{r}) - E_k(\vec{r}) = Ze\left[E_k(\vec{r}) + \delta^{(1)} r_j \frac{\partial}{\partial r_j}\left(\frac{r_k}{r_i r_i^{3/2}}\right) - E_k(\vec{r})\right] =$$

$$= Ze\,\delta^{(1)} r_j\left[\frac{1}{(r_i r_i)^{3/2}}\frac{\partial r_k}{\partial r_j} + r_k\frac{\partial}{\partial r_j}\left(\frac{1}{(r_i r_i)^{3/2}}\right)\right] = Ze\,\delta^{(1)} r_j\left(\frac{\delta_{kj}}{(r_i r_i)^{3/2}} + r_k\frac{-3}{2}\frac{2\delta_{ij} r_i}{(r_i r_i)^{5/2}}\right) =$$

$$= \frac{Ze}{(r_i r_i)^{3/2}}\left(r_k - 3r_k\frac{r_i\delta^{(1)} r_i}{r_i r_i}\right)$$

This is (2.13) in lab. system

In guiding center system
$$\delta^{(1)} E_k(t) = \frac{Ze}{(\bar{r}_j\bar{r}_j)^{3/2}}\left(\delta^{(1)} r_k - 3\bar{r}_k\frac{\bar{r}_i\delta^{(1)} r_i}{r_j r_j}\right), \qquad (3.47)$$

where
$$\vec{r}(t) = \begin{pmatrix} b\sin\theta + \bar{V}_\perp t_0 \\ -b\cos\theta \\ \bar{V}_\parallel t_0\end{pmatrix} + \begin{pmatrix}\bar{V}_\perp \\ 0 \\ V_\parallel\end{pmatrix}(t-t_0) = \vec{r}_0 + \vec{\bar{V}}(t-t_0) = (3.20)$$

$$= r_0\begin{pmatrix}\beta\sin\theta + \gamma_\perp\tau \\ -\beta\cos\theta \\ \gamma_\parallel\tau\end{pmatrix} = \begin{pmatrix}\bar{r}_\perp \\ \bar{r}_\parallel\end{pmatrix} \qquad \Omega_B t' = \frac{\bar{V}}{r_0}t' \;\; \frac{r_0}{V}\Omega_B = \omega\tau$$

Then
$$\delta^{(2)}\vec{V}_B(t) = -\frac{e}{m}\int_{-\infty}^{t} dt'\, T^{-1}(\Omega_B t')\,\vec{E}(t') \qquad (3.48)$$

and
$$\delta^{(1)} r_\parallel = -\frac{Ze^2}{m\bar{V}^2}\gamma_\parallel\int_{-\infty}^{\sigma} d\sigma'\left(\frac{\tau_0\sigma' - 1}{(1+\sigma'^2)^{1/2}} + \tau_0\right) \qquad (3.41)$$

$$\delta^{(1)} r_\perp = \frac{r_0}{\bar{V}\omega}\begin{pmatrix}\sin\omega\tau & \cos\omega\tau \\ -\cos\omega\tau & \sin\omega\tau\end{pmatrix}\delta^{(1)}\vec{V}_{B\perp}(t) = \frac{Ze^2}{m\bar{V}^2\omega}\int_{-\infty}^{\sigma}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\begin{pmatrix}\beta\cos\theta \\ \beta\sin\theta + \gamma_\perp\tau'\end{pmatrix} \qquad (3.42)$$

and

(21)

$$\delta^{(1)}\vec{V}_B(t) = -\frac{Ze^2}{m}\frac{1}{\bar{V}r_0}\int_{-\infty}^{\sigma}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\underbrace{\begin{pmatrix}\cos\omega\tau' & \sin\omega\tau' & 0\\ -\sin\omega\tau' & \cos\omega\tau' & 0\\ 0 & 0 & 1\end{pmatrix}}_{T^{-1}(\Omega_B t')}\begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau'\\ -\beta\cos\theta\\ \gamma_\parallel\tau'\end{pmatrix} \quad (3.36)$$

or

$$\delta^{(1)}V_{\perp B}(t) = -\frac{Ze^2}{m\bar{V}r_0}\int_{-\infty}^{\sigma}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\begin{pmatrix}\cos\omega\tau' & \sin\omega\tau'\\ -\sin\omega\tau' & \cos\omega\tau'\end{pmatrix}\begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau'\\ -\beta\cos\theta\end{pmatrix}$$

$$\Omega_B t' = \underbrace{\frac{r_0}{\bar{V}}\Omega_B}_{\omega}\cdot\underbrace{\frac{\bar{V}t'}{r_0}}_{\tau'} = \omega\tau'$$

Then

$$(\bar{r}_i\delta^{(1)}r_i)(t) = \bar{r}_{\perp k}\delta^{(1)}r_{\perp k} + \bar{r}_\parallel\delta^{(1)}r_\parallel = r_0\begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau\\ -\beta\cos\theta\end{pmatrix}^T\Bigg[\frac{r_0}{\bar{V}\omega}\begin{pmatrix}\sin\omega\tau & \cos\omega\tau\\ -\cos\omega\tau & \sin\omega\tau\end{pmatrix}\delta^{(1)}\vec{V}_{\perp B} -$$

$$-\frac{Ze^2}{m\bar{V}^2\omega}\int_{-\infty}^{\sigma}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\begin{pmatrix}\beta\cos\theta\\ \beta\sin\theta+\gamma_\perp\tau'\end{pmatrix}\Bigg] + \overbrace{r_0\gamma_\parallel\tau\cdot\left(-\frac{Ze^2}{m\bar{V}^2}\gamma_\parallel\right)\int_{-\infty}^{\sigma}d\sigma'\left(\frac{\tau_0\sigma'-1}{(1+\sigma'^2)^{1/2}}+\tau_0\right)}^{\bar{r}_\parallel\delta^{(1)}r_\parallel} \quad \Big|\ \tau'=\sigma'+\tau_0 \quad (3.22)$$

$$= r_0\begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau\\ -\beta\cos\theta\end{pmatrix}^T\Bigg[\frac{r_0}{\bar{V}\omega}\begin{pmatrix}\sin\omega\tau & \cos\omega\tau\\ -\cos\omega\tau & \sin\omega\tau\end{pmatrix}\left(-\frac{Ze^2}{m\bar{V}r_0}\right)\int_{-\infty}^{\sigma}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\begin{pmatrix}\cos\omega\tau' & \sin\omega\tau'\\ -\sin\omega\tau' & \cos\omega\tau'\end{pmatrix}\begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau'\\ -\beta\cos\theta\end{pmatrix} -$$

$$-\frac{Ze^2}{m\bar{V}^2\omega}\int_{-\infty}^{\sigma}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\begin{pmatrix}\beta\cos\theta\\ \beta\sin\theta+\gamma_\perp\tau'\end{pmatrix}\Bigg] - \frac{Ze^2 r_0}{m\bar{V}^2}\gamma_\parallel^2(\sigma+\tau_0)\int_{-\infty}^{\sigma}d\sigma'\left(\frac{\sigma'\tau_0-1}{(1+\sigma'^2)^{1/2}}+\tau_0\right) = \quad \Big|\ \tau=\sigma+\tau_0 \ (3.32)$$

$$= -\frac{Ze^2}{m}\frac{r_0}{\bar{V}^2}\Bigg[\frac{1}{\omega}\int_{-\infty}^{\sigma}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau\\ -\beta\cos\theta\end{pmatrix}^T\left\{\begin{pmatrix}\sin\omega\tau & \cos\omega\tau\\ -\cos\omega\tau & \sin\omega\tau\end{pmatrix}\begin{pmatrix}\cos\omega\tau' & \sin\omega\tau'\\ -\sin\omega\tau' & \cos\omega\tau'\end{pmatrix}\begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau'\\ -\beta\cos\theta\end{pmatrix}+\begin{pmatrix}\beta\cos\theta\\ \beta\sin\theta+\gamma_\perp\tau'\end{pmatrix}\right\}$$

$$+\gamma_\parallel^2(\sigma+\tau_0)\int_{-\infty}^{\sigma}\left[\frac{\sigma'\tau_0-1}{(1+\sigma'^2)^{1/2}}+\tau_0\right]d\sigma'\Bigg] =$$

But

$$\begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau\\ -\beta\cos\theta\end{pmatrix}^T \underbrace{\begin{pmatrix}\sin\omega\tau & \cos\omega\tau\\ -\cos\omega\tau & \sin\omega\tau\end{pmatrix}\begin{pmatrix}\cos\omega\tau' & \sin\omega\tau'\\ -\sin\omega\tau' & \cos\omega\tau'\end{pmatrix}}_{\begin{pmatrix}\sin\omega(\tau-\tau') & \cos\omega(\tau-\tau')\\ -\cos\omega(\tau-\tau') & \sin\omega(\tau-\tau')\end{pmatrix}}\begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau'\\ -\beta\cos\theta\end{pmatrix} + \underbrace{\begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau\\ -\beta\cos\theta\end{pmatrix}^T\begin{pmatrix}\beta\cos\theta\\ \beta\sin\theta+\gamma_\perp\tau'\end{pmatrix}} =$$

$$\beta\cos\theta(\cancel{\beta\sin\theta}+\gamma_\perp\tau-\cancel{\beta\sin\theta}-\gamma_\perp\tau') = \beta\gamma_\perp\underbrace{(\tau-\tau')}_{=\Delta\tau}\cos\theta$$

$$= (\quad)^T\begin{pmatrix}\sin\omega\Delta\tau & \cos\omega\Delta\tau\\ -\cos\omega\Delta\tau & \sin\omega\Delta\tau\end{pmatrix}\begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau'\\ -\beta\cos\theta\end{pmatrix} + \beta\gamma\Delta\tau\cos\theta =$$

$$= (\beta\sin\theta+\gamma_\perp\tau,\ -\beta\cos\theta)\begin{pmatrix}(\beta\sin\theta+\gamma_\perp\tau')\sin\omega\Delta\tau-\beta\cos\theta\cos\omega\Delta\tau\\ -(\beta\sin\theta+\gamma_\perp\tau')\cos\omega\Delta\tau-\beta\cos\theta\sin\omega\Delta\tau\end{pmatrix} + \beta\gamma_\perp\Delta\tau\cos\theta =$$

$$= (\beta\sin\theta+\gamma_\perp\tau)[(\beta\sin\theta+\gamma_\perp\tau')\sin\omega\Delta\tau-\beta\cos\theta\cos\omega\Delta\tau] +$$

$$+\beta\cos\theta[(\beta\sin\theta+\gamma_\perp\tau')\cos\omega\Delta\tau+\beta\cos\theta\sin\omega\Delta\tau] + \beta\gamma_\perp\Delta\tau\cos\theta =$$

$$= (\underline{\beta^2\sin^2\theta}+\beta\gamma_\perp\tau'\sin\theta+\beta\gamma_\perp\tau\sin\theta+\gamma_\perp^2\tau\tau')\sin\omega\Delta\tau + \beta\cos\theta[-\cancel{\beta\sin\theta}-\gamma_\perp\tau+\cancel{\beta\sin\theta}+\gamma_\perp\tau']\cos\omega\Delta\tau +$$

$$+\underline{\beta^2\cos^2\theta}\sin\omega\Delta\tau + \beta\gamma_\perp\Delta\tau\cos\theta = \beta^2\sin\omega\Delta\tau + \beta\gamma_\perp(\tau+\tau')\sin\theta\sin\omega\Delta\tau +$$

$$+\gamma_\perp^2\tau\tau'\sin\omega\Delta\tau + \beta\gamma_\perp(\tau'-\tau)\cos\theta\cos\omega\Delta\tau + \beta\gamma_\perp\Delta\tau\cos\theta =$$

$$= \beta^2\sin\omega\Delta\tau + \beta\gamma_\perp[(\tau+\tau')\sin\theta\sin\omega\Delta\tau-(\tau-\tau')\cos\theta\cos\omega\Delta\tau] + \gamma_\perp^2\tau\tau'\sin\omega\Delta\tau + \beta\gamma_\perp\Delta\tau\cos\theta$$

Then

$$(\bar r_i\delta^{(1)}\bar r_i)(t) = -\frac{Ze^2}{m}\frac{r_0}{v^2}\left\{\frac{1}{\omega}\int_{-\infty}^{\sigma}\frac{d\sigma'}{(1+\sigma'^2)^{3/2}}\Big\langle\beta^2\sin\omega(\tau-\tau')+\beta\gamma_\perp[(\tau+\tau')\sin\theta\sin\omega(\tau-\tau')-(\tau-\tau')\cos\theta\cos\omega(\tau-\tau')] + \gamma_\perp^2\tau\tau'\sin\omega(\tau-\tau')+\beta\gamma_\perp(\tau-\tau')\cos\theta\Big\rangle + \gamma_\parallel^2\tau\int_{-\infty}^{\sigma}\left[\frac{\sigma'\tau_0-1}{(1+\sigma'^2)^{1/2}}+\tau_0\right]d\sigma'\right\} \tag{3.49}$$

Two limits: a) $\omega \to \infty$ („tight" trajectory $\equiv$ magnetized electrons) and

b) $\omega \to 0$ („stretched" trajectory $\equiv$ nonmagnetized electrons)

a) Term $\frac{1}{\omega} \int_{-\infty}^{\sigma} \frac{d\sigma'}{(1+\sigma'^2)^{3/2}} < \quad > \to 0$ for $\sigma \to \infty$, i.e.

$$\left(\bar{r}_i \delta^{(1)} r_i\right)\Big|_{tight}(t) = -\frac{ze^2}{m} \frac{r_0}{\sqrt{2}} \gamma_{\parallel}^2 (\sigma + \tau_0) \int_{-\infty}^{\sigma} d\sigma' \left[\frac{\tau_0 \sigma' - 1}{(1+\sigma'^2)^{1/2}} + \tau_0 \right] \quad (3,50)$$

b) (How!?):

$$\left(\bar{r}_i \delta^{(1)} r_i\right)\Big|_{stretch}(t) = -\frac{ze^2}{m} \frac{r_0}{\sqrt{2}} \Big\{ \int_{-\infty}^{\sigma} d\sigma' \left(\frac{\tau_0 \sigma' - 1}{(1+\sigma'^2)^{1/2}} + \tau_0 \right) \left[1 - \gamma_{\parallel}^2 \tau_0^2 - \gamma_{\parallel}^2 \tau_0 \sigma - \gamma_{\parallel}^2 \tau_0 \sigma' + \gamma_{\perp}^2 \sigma \sigma' \right] +$$

$$+ \gamma_{\parallel}^2 (\sigma + \tau_0) \int_{-\infty}^{\sigma} d\sigma' \left(\frac{\tau_0 \sigma' - 1}{\underbrace{(1+\sigma'^2)^{1/2} + \tau_0}_{??}} \right) \Big\} \quad (3,51)$$