**Waves in a periodical structure (disk-loaded waveguide)**

A disk-loaded waveguide (DLW) is the most common structure for linear accelerators and is shown in Figure 1.

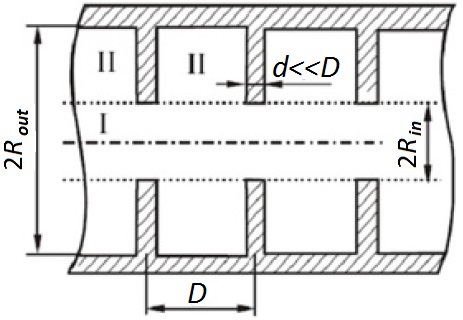


Figure 1. Disk-loaded waveguide.

1. *Floquet theorem*

Electromagnetic fields in the longitudinal periodical structure with period  for each cell obey to the Floquet theorem:

 (1)

Here  is a phase change for a spatial shift along one cell of the structure. These relations mean that fields can be described using periodical over  functions :

 (2a)

or

 (2b)

These periodical functions can be expanding to the Fourier series:

 (2c)

where Fourier amplitudes are



The verifying of these relations is as follows (for example, for electric field):



So,

(3a)

And similarly

 (3b)

These expressions mean that any wave processes like  in the periodical structure with phase shift  during shift on one spatial period  is equivalent to the sum of the infinite number of plain inhomogeneous waves  correspondingly, i.e. these processes consist from simple harmonics with wave constant

 (4a)

and

 (4b)

Each of these harmonics has the phase velocity

 (4c)

and the same group velocity

 (4d)

1. *Electromagnetic waves and Hertz vectors*

In the structure under consideration it is advantageous to derive the six electromagnetic field components from the longitudinal electric vector  and magnetic vector . The fields with the harmonic dependence from time, i.e. for , can then be found using the following expressions:

 (5)

and Hertz vectors obey the wave equations

 (6)

Because of the periodicity of all the values from the azimuthal angle  the longitudinal components  and of the Hertz vectors can be are represented as

 (7)

In this case the equations (6) mean that

 (8)

and equations (5) mean that for each  variation of angle 

 (9)

Floque theorem for each  variation of the field components means that instead expressions (2) and (3) one can write the following:

 (10)

It is natural to call the values  as the  variations of the periodical parts of the fields, and values  as Fourier components of these variations.

Figure 1 shows that it is possible to divide the geometry into two subdivisions: area I (“channel”) with uniform cross-section in the direction, one consisting of the infinite circular cylinder , which can support traveling waves and the area II (“cells”) with the annular cylinders   in which only standing waves are possible.

*3. Electric field in “channel” (area I)*

On the boundary  of the “channel” area the expressions (10) mean that the longitudinal field  on this surface can be described by set of coefficients :

 (11)

Since each  variations of the Hertz vectors  can be presented inside of “channel” area as the sum of traveling in the bothdirection «simple» waves, i.e.

 (12)

Then the equations (8) mean the following equations for amplitudes  of these waves:

 (13)

This is a Bessel equation and its finite solutions are the modified Bessel functions  of first kind with

 (14)

The solution in this area must be finite, so that



More convenient override the coefficients , so that

 (15)

The expressions (9b) and (9c) give the following longitudinal components of the electric field:

 (16a)

and



or

(16b)

Expression (16) allow find the longitudinal components of the electric field on the boundary surface () of the “channel” area:

 (17)

Comparison of these expressions with formulas (11) shows that (???)

 (18)

where

 (19)

As already mentioned, each term in the expressions (16) describes the corresponding spatial harmonic of the field, characterized by the phase velocity of propagation  and the group velocity which is the same for all harmonics (see expressions (4)). Effective interaction of particles of the beam with the field occurs in this case only with synchronous harmonics for which the phase velocity close to the velocity of the beam. The action of the remaining harmonics of the field on the particle beam has an oscillating character and its integral effect when particles pass the period of the structure is close to zero.

*4. Electric field in “cell” (area II)*

In this area the field for each  azimuthal variation can be represented in the form of standing waves in direction corresponding to each individual cell.

Let us present electrical Hertz vector in the following view:

 (20a)

where

 (21)

This dependence on  give the vanishing of the electric field on the surfaces of the diaphragms of each cell of the structure.

Expressions (20) and (21) mean that the equation (8) transform to



i.e. dependence on radius for each  spatial harmonic obeys to the following equation:

 (22)

with

 (23)

The solution of the Bessel equation (22) are the Bessel functions  of the first and second kind correspondingly. So, expression (20a) take a form

 (20b)

Then electrical field equals to

 (24a)

The vanishing of the electric field on the outer () wall of the cell means that



i.e.



For this reason, expression (20b) transform to



Let introduce sets of functions

 (25a)

with the properties

 (25b)

and redefine the constants  in the last expression for electric Hertz vector:  (26a)

where  Kronecker symbol.

Quite the same way it is possible to define the magnetic Hertz vector :

 (26b)

Expressions (26) give the following expressions for longitudinal components of the electric field:

 (27a)

and





or

 (27b)

Due to properties (25b) the expressions (27) are vanished on the outer () surface of the “cell” area.

Beside that the “longitudinal” electrical field (components  and ) on the outer surfaces of the diaphragms, i.e. for  and  must vanished. The expression (27b) obeys to this condition due to value of the factor  for . To verify the condition  let calculate the radial component of the electric field:

(27c)

and this expression is vanished also due to value of the factor  for .

Expressions (27) allow find the longitudinal components of the electric field on the inner () surface of the “channel” area:

 (28)

Comparison of these expressions with formulas (???) shows that (???)

 (29)

where

 (30a)

and

 (30b)

*5. Matching of the magnetic fields on the boundary of the both areas*

Boundary conditions on the surface  are as follows:

 (31)

Let us find the longitudinal components of the magnetic fields in the both areas.

 (32a)

 (32b)



or

 (32c)





or

(32d)

Then the boundary conditions (31) mean that

(33a)

and

(33b)

*6. Dispersion equation*

Substitution of the expressions (18), (19) and (29), (30) into expressions (33) and expanding of the exponential functions  (in the left parts of expressions (33)) over orthogonal sets of the functions  and gives the infinite set of homogeneous linear equations for Fourier amplitudes  and:

 (34)

Where new coefficients are as follows:

 (35a)

 (35b)

 (35c)

 (35d)

Let present system (34) in the more understandable view:

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Index «t» | | | … | -2 | -1 | 0 | 1 | 2 | … | -2 | -1 | 0 | 1 | 2 | … |
| Unknown  variables | | | … |  |  |  |  |  | … |  |  |  |  |  | … |
| Equation (34a) | Index «s» in equations (34) | 1 | … |  |  |  |  |  | … |  |  |  |  |  | … |
| 2 | … |  |  |  |  |  | … |  |  |  |  |  | … |
| 3 | … |  |  |  |  |  | … |  |  |  |  |  | … |
|  | … … … … … … … …. … … … … … | | | | | | | | | | | | | |
| Equation (34b) | 0 | … |  |  |  |  |  | … |  |  |  |  |  | … |
| 1 | … |  |  |  |  |  | … |  |  |  |  |  | … |
| 2 | … |  |  |  |  |  | … |  |  |  |  |  | … |
| 3 | … |  |  |  |  |  | … |  |  |  |  |  | … |
| … … … … … … … …. … … … … … | | | | | | | | | | | | | |

System (34) is homogeneous and it is means that nontrivial solution exists if

 (36)

This is the desired dispersion equation. It determines the dependence of the wave number (wave frequency ) from the predetermined phase shift  for each cell of the structure. When the relationship  is found, the equations (34) give all coefficients  (up to a common factor) and, hence, all amplitudes ; then expressions (27), (28), (32) give the desired fields in the system.

*7. Analysis of the dispersion equation*

The dispersion equation (36) takes into account, in general, an infinite number of Fourier harmonics of the fields. This is necessary for a rigorous description of the fields on the sharp corners of the iris diaphragms. "Quality" of this description is defined by the view of the coefficients (see expressions (10) and (30) also. It can be shown that if select these coefficients as

 (37)

then number of Fourier-harmonics, which necessary take into account, may be decreased significantly.

Expressions (35), (37) together with (4a), (14), (21) and (23) show that for large values of  the asymptotic behavior of the amplitudes is as follows:



*8. Symmetrical modes*

For symmetrical modes , so that  and equation (36) moves to

 (38)

Let us restrict the further analysis by determinant of 5-th order and taking into account three terms in summations in formulas (35c). Then formulae (38) gives

 (39)

where

(40)

with

 (41)

and

 (42)

*9. First nonsymmetrical mode*