Contact Geometry

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A Very Small Introduction

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1 Motivation?

"Contact geometry is all of geometry" the Russian mathematican V.I. Arnold is reputed to have said.

He was one of the greatest geometers of the last century, so what other motivation do we need? :)

These notes and the accompanying presentation are an artifact of my trying to understand what *contact geometry* is, so forgive me for not having figured out how to explain it just yet (we are only in Section 1!).

But I can say the subject is something like how it sounds: understanding how two (or more!) geometric objects touch, and the structure that arises from their contact.

Tangent lines and planes will be central. When two spheres touch, for example, then at the point of contact derivatives and tangent lines coincide. These tangent lines are examples of **contact structures**.

Also the n-truncated Taylor expansion of a function f is said to be *in contact* with f.

So if we aren't inclined to agree immediately with Arnold, perhaps we can nonethless see that the notion of *contact* isn't so foreign.

We will try to understand how **contact geometry** is more concerned with **global** aspects of geometry, and how **locally** all contact structures look the same.

To understand this we will develop a good bit of algebra! I hope we will find the geometric payoff worth it!

2 Algebra

If calculus is in some sense about understanding nonlinear things (curves and surfaces) in terms of linear approximations (tangent lines and spaces) then it's not too surprising that linear algebra has a large role to play.

We should remember from calculus that the derivative is linear - indeed it is a linear transformation of tangent spaces!

It turns out once you have this initial vector space structure you can start unfolding even more:

· **duality** The dual space V^* to a given vector space V

· **multilinearity** Generalize from linear to *multilinear* transformations

Understanding something about multilinear functions defined on the dual space V^* will be a central goal of these notes. These objects are called *k-forms* and will let us define families of lines and planes which form the basis for **contact structures**.

We will also see how the pairing of vectors and k-forms define *invariants* that are independent of coordinates.

Let's get started!

2.1 Co-vectors and the dual space

We are going to assume knowledge of (finite) dimensional vector spaces over $\mathbb{R}$ but repeat a few relevant definitions. We'll denote a typical example by V but very shortly we'll turn to a discussion of tangent spaces like $T_p\mathbb{R}^N$.

Such spaces have *bases*: linearly independent subsets which span the entire space. This means any element $v \in V$ can be written uniquely as the sum of elements in the basis. The vector space $\mathbb{R}^N$ has the standard basis e_i which are vectors with a 1 in the i'th component and 0 everywhere else.

It is a fact that a linear transformation $T : V \to W$ of vector spaces maps a basis v_i of V to a basis w_j of W.

Elements of V, which we usually just call vectors, could more precisely be called *contravariant vectors*. Why? To distinguish them from *covariant vectors*.

How is a covariant vector different from a contravariant vector? Covariant vectors live inside the **dual space** V^*. We are going to define that right now!

$$V^* = \{\epsilon | \epsilon : V \to \mathbb{R}\}$$

where the ϵ's are linear. So V^* is the set of functions that map vectors in V to real numbers in $\mathbb{R}$. In fact it turns out that V^* is itself a vector space. So that's nice, it's like you pay for one vector space and get one free.

Elements of V^* are also called *linear forms* or *one-forms*.

(We haven't talked about multilinearity yet but you already know some **fundamental** (*hint, hint*) examples of these!)

How can we tell if a given vector is secretly a covariant vector living in some other vector space's dual space? It turns out that $V \cong (V^*)^*$ so in a sense whether or not a vector is co- or contravariant is a matter of perspective.

What does seem to be important is this: if $v \in V$ and $\epsilon \in V^*$ then $\epsilon(v) \in \mathbb{R}$ is **invariant**. By that I mean it *does not depend* on which basis you may have used to write down v or ϵ in. Coordinates don't matter.

Oh I've gotten a bit ahead of myself: since V^* is a vector space that means it has a basis. What does that look like? And are they related to bases for V?

We can associate to each basis v_i in V a **dual basis** ϵ_i in V^*.

How does ϵ_i act on v_i? Quite simply! Each ϵ_i projects off the i'th component of the vector it acts on. If $V = \mathbb{R}^{\mathbb{N}}$ so that $v_i = e_i$ then

$$\epsilon_i(e_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

and more generally:

$$\epsilon_i(v) = \epsilon_i(a_1 * v_1 + ... + a_n * v_n) = a_1 * \epsilon_i(v_1) + ... + a_n * \epsilon_i(v_n) = a_j * \epsilon_i(v_j) = a_i$$

TODO: show that the ϵ_i are linearly independent and span V^*