Contact Geometry

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A Very Small Introduction

Contents

Emphatic note: none of the mathematical ideas here represent original work in either content or presentation! The main source for the material is McInerney [1] but see the full list of references for other very nice resources.

1 Motivation?

1.1 Roots

"Contact geometry is all of geometry" the Russian mathematican V.I. Arnold is reputed to have said.

He was one of the greatest geometers of the last century, so what other motivation do we need? :)

These notes and the accompanying presentation are an artifact of my trying to understand what *contact geometry* is, so forgive me for not having figured out how to explain it just yet (we are only in Section 1!).

But I can say the subject is something like how it sounds: understanding how two (or more!) geometric objects touch, and the structure that arises from their contact.

Tangent lines and planes will be central. When two spheres touch, for example, then at the point of contact derivatives and tangent lines coincide. These tangent lines are examples of **contact structures**.

Also the n-truncated Taylor expansion of a function f is said to be *in contact* with f.

So if we aren't inclined to agree immediately with Arnold, perhaps we can nonethless see that the notion of *contact* isn't so foreign.

We will try to understand how **contact geometry** is more concerned with **global** aspects of geometry, and how **locally** all contact structures look the same.

To understand this we will develop a good bit of algebra! I hope we will find the geometric payoff worth it!

1.2 The Destination

I want to give a very brief glimpse of what it is we're going to be defining and maybe eventually understanding.

1.2.1 Vector Fields

In our main text [2] one definition of a vector field (along a curve) on a surface S is as the derivative $\frac{d}{dt}\gamma$ where $\gamma : I \in \mathbb{R} \to S$ is the image of a curve on the u, v plane but I have got the types a bit wrong there.

We are going to need (or at least *use*) a slightly more abstract defintion that says: a vector field on a surface M is a linear derivation on the set of smooth functions on M. This set is denoted $C^\infty(M, R)$.

So a vector field has the type: $C^\infty(M, R) \to C^\infty(M, R)$.

How does this relate to the earlier definition? And what does that definition of a vector field imply about the definition of a *tangent vector*?

It implies a slightly different definition of that as well. It turns out that thanks to the idea of the *directional derivative* there's a one-to-one correspondence between tangent vectors and differential operators.

We are going to lean in to that correspondence and *define* (because who will stop us?) tangent vectors to be linear functions from $C^\infty(M, R) \to \mathbb{R}$.

1.2.2 Integral Curves and Flows

1.2.3 Differential Forms

1.2.4 Contact

2 Algebra

If calculus is in some sense about understanding nonlinear things (curves and surfaces) in terms of linear approximations (tangent lines and spaces) then it's not too surprising that linear algebra has a large role to play.

We should remember from calculus that the derivative is linear - indeed it is a linear transformation of tangent spaces!

It turns out once you have this initial vector space structure you can start unfolding even more:

· **duality** The dual space V^* to a given vector space V

· **multilinearity** Generalize from linear to *multilinear* transformations

Understanding something about multilinear functions defined on the dual space V^* will be a central goal of these notes. These objects are called *k-forms* and will let us define families of lines and planes which form the basis for **contact structures**.

We will also see how the pairing of vectors and k-forms define *invariants* that are independent of coordinates.

Let's get started!

2.1 Co-vectors and the dual space

We are going to assume knowledge of (finite) dimensional vector spaces over $\mathbb{R}$ but repeat a few relevant definitions. We'll denote a typical example by V but very shortly we'll turn to a discussion of tangent spaces like $T_p\mathbb{R}^N$.

Such spaces have *bases*: linearly independent subsets which span the entire space. This means any element $v \in V$ can be written uniquely as the sum of elements in the basis. The vector space $\mathbb{R}^N$ has the standard basis e_i which are vectors with a 1 in the i'th component and 0 everywhere else.

It is a fact that a linear transformation $T : V \to W$ of vector spaces maps a basis v_i of V to a basis w_j of W.

Elements of V, which we usually just call vectors, could more precisely be called *contravariant vectors*. Why? To distinguish them from *covariant vectors*.

How is a covariant vector different from a contravariant vector? Covariant vectors live inside the **dual space** V^*. We are going to define that right now!

$$V^* = \{\epsilon | \epsilon : V \to \mathbb{R}\}$$

where the ϵ's are linear. So V^* is the set of functions that map vectors in V to real numbers in $\mathbb{R}$. In fact it turns out that V^* is itself a vector space. So that's nice, it's like you pay for one vector space and get one free.

Elements of V^* are also called *linear forms* or *one-forms*.

(We haven't talked about multilinearity yet but you already know some **fundamental** (*hint, hint*) examples of these!)

How can we tell if a given vector is secretly a covariant vector living in some other vector space's dual space? It turns out that $V \cong (V^*)^*$ so in a sense whether or not a vector is co- or contravariant is a matter of perspective.

What does seem to be important is this: if $v \in V$ and $\epsilon \in V^*$ then $\epsilon(v) \in \mathbb{R}$ is **invariant**. By that I mean it *does not depend* on which basis you may have used to write down v or ϵ in. Coordinates don't matter.

Oh I've gotten a bit ahead of myself: since V^* is a vector space that means it has a basis. What does that look like? And are they related to bases for V?

We can associate to each basis v_i in V a **dual basis** ϵ_i in V^*.

How does ϵ_i act on v_i? Quite simply! Each ϵ_i projects off the i'th component of the vector it acts on. If $V = \mathbb{R}^{\mathbb{N}}$ so that $v_i = e_i$ then

$$\epsilon_i(e_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

and more generally:

$$\epsilon_i(v) = \epsilon_i(a_1 * v_1 + ... + a_n * v_n) = a_1 * \epsilon_i(v_1) + ... + a_n * \epsilon_i(v_n) = a_j * \epsilon_i(v_j) = a_i$$

TODO: show that the ϵ_i are linearly independent and span V^*

2.2 Forms

A covector is the smallest example of what we will call a form: it is a 1-form.

We know at least the following forms: dx, dy, dz and we know how they act on vectors since we know they are the *dual basies*.

So given any vector $v \in \mathbb{R}^3$ we can write v as a linear combination $xe_1 + ye_2 + ze_3$ and we have the following relationships:

TODO: talk about how $e_k = \frac{\partial}{\partial k}$ and use that notation everywhere!

$$\begin{aligned} dx(v) &= dx(xe_i + ye_2 + ze_3) = dx(xe_1) + dx(ye_2) + dz(ze_3) = x \\ dy(v) &= dy(xe_i + ye_2 + ze_3) = dy(xe_1) + dy(ye_2) + dz(ze_3) = y \\ dz(v) &= dz(xe_i + ye_2 + ze_3) = dz(xe_1) + dz(ye_2) + dz(ze_3) = z \end{aligned}$$

TODO: this is pretty bad notation:

Which uses the fact that $di * \frac{\partial}{\partial j = \delta_{ij}}$ and the linearity of the di.

You may have heard the term *bilinear* form, which is a function that maps two arguments (vectors in our case) to a real number. Two examples are the first and second fundamental forms - these are also *skew-symmetric* which is another important property of forms.

So if a covector is a 1-form how do we get to (bilinear) 2-forms? 3-forms? n-forms?

TODO: give some notation for the spaces of forms of various degree so we can write down the type of the exterior product

We define an operation called the **exterior product** (or sometimes the *wedge product*), which is denoted $\wedge$, and somehow turns two 1-forms into a single 2-form.

We do this by relying on the *determinant* since that turns out to already be what we want: it *is* an antisymmetric bilinear form!

The recipe goes like this. We have two 1-forms that we will call α and β. We will thus need two vectors which we will call a and b.

We combine them into a $2x2$ matrix and then take the determinant:
$\begin{vmatrix} \alpha(a) & \alpha(b) \\ \beta(a) & \beta(b) \end{vmatrix}$
So the 2-form $\gamma = \alpha \wedge \beta$ takes two vectors, applies the 1-forms α and β, and takes the determinant.

There is a nice geometric interpretation of the wedge product that I won't go into right now!

But here's the TODO for this section:

1. Talk about that geometric interpretation 2. Do some concrete examples 3. Talk about the space of 3-forms and what sort of basis it has 4. Talk about integrating them, at least mention pullback! 5. Talk about volume forms!

2.3 Differential Forms

These are to forms (or k-forms) as vector fields are to vector fields

1. Do an example 2. Talk about how they relate to Jacobians 3. Give an example of a contact form that generates a volume form

3 Geometry

3.1 Contact Elements

4 Generalizations

References

[1] Andrew McInerney *First Steps in Differential Geometry: Riemannian, Contact, Symplectic*. Springer-Verlag New York, 2013.

[2] Victor Andreevich Toponogov *Differential Geometry of Curves and Surfaces: A Concise Guide*. Birkhuser, 2006