Contact Geometry

April 25, 2019

The Presentation Notes

Contents

Emphatic note: none of the mathematical ideas here represent original work in either content or presentation! The main sources for the material are McInerney [1] and Bachman [2] but see the full list of references for other very nice resources.

1 Overview

"Contact geometry is all of geometry." VI Arnold.

Contact Geometry has a history going back centuries and features contributions from Christiaan Huygens, Sophus Lie and others.

Huygens developed an approach to geometric optics in terms of wavefronts. The way that wavefronts interact with each other leads to the notion of *contact element*. A kind of geodesic flow associated to the contact elements via the *Reeb field* relates Huygen's principle to Fermat's principle of least time.

Lie was interested in solutions to differential equations and developed ideas related to *contact transformations*. The Legendre transformation is a contact transformation. Contact transformations turn out to (preserve? something?) the Reeb field. Maybe.

Physics appears to still be a central source of insight and applications - especially to thermodynamics and the Hamiltonian formulation of classical mechanics.

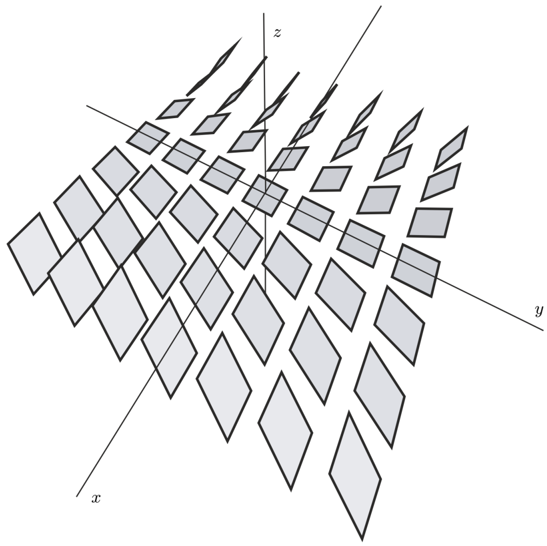
This small text and accompanying presentation will mostly focus on a few of the underlying geometric ideas, excluding contact transformations almost entirely.

Here's what we'll be talking about:

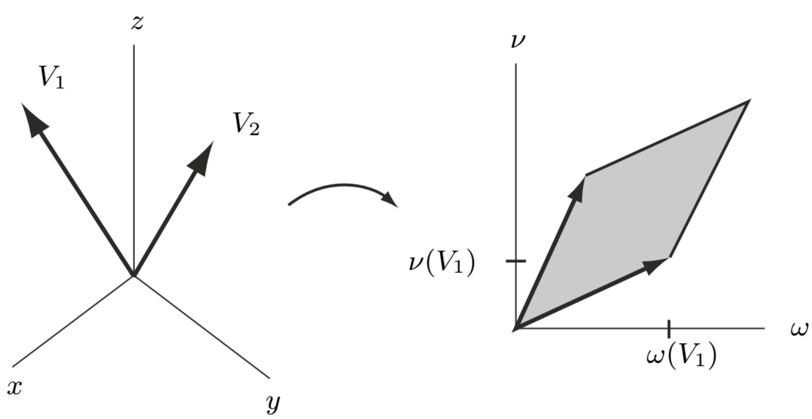
- contact elements
- fields of planes and a bit about foliations
- differential forms and the contact one-form
- the non-integrability condition
- the Reeb field associated to a contact one-form

1.1 Pictures

TODO wavefronts TODO field of planes TODO form / vector visualization



The contact structure generated by $xdy + dz$



Visualizing forms

2 Contact Elements

Let's work in the plane $\mathbb{R}^2$. Then a contact element is a point $p \in \mathbb{R}^2$ and a non-vertical line-segment centered at p. We could call a collection of these at every point of $\mathbb{R}^2$ a field of line-segments.

If we imagine a sequence of these points as being sampled from a regular curve, and the line-segments the associated tangents, then we can integrate the tangents to recover the curve.

Continuing in this kind of reverse direction we might call the curve an *integral curve*. We have seen such things a few times in class when talking about extending some local piece of a curve on a patch to the entire surface.

Looking at the contact elements from a slightly different angle, we might consider the line-segments to be a vector field and then the question of integrating the field is the question of the existence of these integral curves.

If we bump up a dimension and imagine a field of planes we are led to consider integrating those planes into an *integral surface*.

Integral curves and surfaces are examples of foliations. They decompose a space into a disjoint union of subspaces and seem pretty cool!

But here's the punch line - the fields of planes we talk about in contact geometry are **maximally** non-integrable!

3 Differential Forms

Differential forms give us a way to talk about directed linear subspaces. In particular we can use a one-form (the so-called **contact one-form**) to characterize contact elements.

3.1 Covectors and the Dual Space

We've talked about at least three different forms in class: the first, second, and third fundamental forms.

They are called *forms* because they are functions that map vectors to a number.

The fundamental forms are also *bilinear*, but there are also simpler *linear* forms. We will call these **one-forms**.

If you've taken enough linear algebra you may recognize one-forms as living in the *dual space* $\mathbb{V}^*$ associated to a vector space $\mathbb{V}$.

If you've taken enough physics you may recognize these as *covectors*.

From a geometrical perspective a very important fact about covectors is that they have a natural pairing with vectors.

If $v \in \mathbb{V}$ and $w \in \mathbb{V}^*$ then the number $w(v) \in \mathbb{R}$ is **invariant**: it does not depend on what basis v or w may be expressed in.

Said another way the pairing does not depend on coordinates!

Avoiding coordinate representations as much as possible is a big goal of this formalism. Nonetheless I am about to introduce bases for both $\mathbb{V}$ and $\mathbb{V}^*$!

For a given basis e_i of $\mathbb{V}$ there is an associated basis e^j of $\mathbb{V}^*$ called the **dual basis**.

The dual basis has the property that under the pairing we have $e^j(e_i = \delta_i^j)$.

That is the **Kronecker delta** which is 1 when $i = j$ and 0 otherwise.

If we then write a $v \in \mathbb{R}^2$ as $v = ae_1 + be_2$ and apply the dual form e^1 we get $e^1(v) = e^1(ae_1+be_2) = e^1(ae_1)+e^1(be_2) = a(e^1(e_1))+b(e^1(e_2)) = a(1)+b(0) = a$

Which shows that elements of the dual basis are projection operators.

If we specialize to the context of differential geometry then our $\mathbb{V}$ becomes a tangent plane or space, maybe $T_p\mathbb{R}^3$.

The dual vector space $\mathbb{V}^*$ then becomes the **cotangent space** (TODO what's notation for this, do I still use T?).

In class we're used to writing ρ_u, ρ_v or maybe r_u, r_v as a basis for the tanget space.

In these notes we're going to try to use $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$.

For the cotangent space we'll use dx, dy.

3.2 One-forms, Two-forms, Three-forms

Differential forms are grounded in *multilinear* algebra, so we'll be interested in *multilinear* functions. But a very specific kind of multilinear function - antisymmetric (or sometimes skew-symmetric) ones.

We actually already know an antisymmetric multilinear function and that is the cross product!

That's going to be handy because one way to motivate the idea of (not yet differential) forms is to try to generalize the cross product to higher dimensions.

It turns out you can't, exactly, but you can get something even cooler: the **wedge** (or exterior) product.

Why is it cooler? It exists for every dimension and it lets you build linear subspaces of arbitrary rank in the same way that a cross product kind of represents a plane.

In fact the cross product of two vectors in $\mathbb{R}^3$ is dual in a sense to the wedge product of two vectors. The cross product being the normal associated to the plane while the wedge product represents the subset of the plane spanned by the parallelogram formed by the vectors.

That was a mouthful but the picture looks like this:

TODO INSERT PICTURE

Before we go any further we want to emphasize that in this section we are usually only working with covectors but the wedge product can operate on general vector spaces. The point is, it makes sense to take the exterior product of any sort of vector, not just the ones that hang out in tangent or cotangent spaces.

However since we do care mostly about dual spaces, I'll tell you that if you take the wedge product of two one-forms you get a **two-form**. We'll look at a calculation in a second, but in words a two-form is an antisymmetric bilinear form.

That means it's a bilinear map $w : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ that satisfies $w(a, b) = -w(b, a)$.

Secretly you already know an antisymmetric bilinear form: the 2 x 2 determinant! And in fact this is what we'll use to define the wedge product.

So, given two one forms v, w we want to create a two-form that will map two vectors a, b to a real number. The recipe goes like this:

$$(v \wedge w)(a, b) = \begin{vmatrix} v(a) & w(a) \\ v(b) & w(b) \end{vmatrix}$$

TODO a 2-form example

We interpret $v \wedge w$ geometrically as the parallelipiped spanned by v and w. I think. Does that picture make sense for two-forms or just for two-vectors in $\mathbb{R}^3$? Yeah I should only talk about the geometric picture in $\mathbb{R}^3$. TODO this.

An interesting consequence is that $v \wedge v = 0$ for every one-form v.

I should also emphasize that the antisymmetry gives the wedge product an orientation! So these are oriented linear subspaces!

We can keep going and given three one-forms build the three-form

$$(u \wedge v \wedge \wedge w)(a, b, c) = \begin{vmatrix} u(a) & v(a) & w(a) \\ u(b) & v(b) & w(b) \\ u(c) & v(c) & w(c) \end{vmatrix}$$

for arbitrary vectors a, b, c.

We can keep building forms of higher degree if we keep going up a dimension but we're going to stop here and summarize what we have.

These are:

- three one-forms dx, dy, dz that act on vectors a, b, c
- three two-forms $dx \wedge dy, dy \wedge dz, dz \wedge dx$ that act on pairs of vectors (a, b)
- one three form $dx \wedge dy \wedge dz$ that act on triples of vectors (a, b, c)

And collectively we will call these *k-forms*.

In the same way that we can let a vector's coefficients vary and get a vector field we can also let the coefficients of a k-form vary and it is *this* that we call a **differential form**.

So a differential two-form looks like $u = f dx \wedge dy$ where f is a differential function.

3.3 Exterior differentiation

I won't keep you in suspense any longer! *Yes*, there is a way to differentiate a differential form!

It behaves a lot like the usual derivative, as we'll see shortly, and also has the curious effect of bumping up the degree of a k-form. So the derivative of a zero-form is a one-form, of a one-form is a two-form, and so on.

Oh did I talk about zero-forms before? Then now's a good time to say that a zero-form is just a function $f : \mathbb{R} \to \mathbb{R}$.

Now we can say what we want the **exterior derivative** to do to the various forms.

- on a zero-form f we have simply the usual derivative, df
- on a one-form $u = f dx$ we have the two-form $du = df \wedge dx$
- on a two-form $v = f dx \wedge dy$ we have the three-form $dv = df \wedge dx \wedge dy$.
- on a three-form we have $dw = df \wedge dx \wedge dy \wedge dz$ buuttt we don't want four-forms.

The exterior derivative is linear and satisfies a kind of Leibniz rule:

$$d(cv) = c * dv$$
$$d(v + w) = dv + dw$$
$$d(v \wedge w) = dv \wedge w + (-1)^{pq} v \wedge dw$$

where p is the *degree* (0, 1, 2, 3) of v and q the degree of w.

A *very* import property: $d^2 = 0$. This is related to the notion of cohomology and is maybe fundamental to all physics! At least people on the Internet will whisper that.

Now that we know about differential forms and the exterior derivative we can talk about contact forms!

3.4 Contact Forms

A one-form determines a field of planes through it's **kernel**: the set of vectors it maps to zero.

Consider $v = adx + bdy + cdz$. If a is in the kernel of v then $v(a) = 0$ implies $< a, (a, b, c) >= 0$ which is a tidy way to characterize a plane. TODO clean this up.

For a one-form v to be a **contact one-form** it must also satisfy this non-degeneracy condition:

$$dv \wedge v \neq= 0 \tag{1}$$

The standard contact form for $\mathbb{R}^3$ is $xdy + dz$. We can check this is a contact form:

$$\begin{aligned} d(xdy + dz) \wedge (xdy + dz) &= \\ (dx \wedge dy + d^2 z) \wedge (xdx + dz) &= \\ (dx \wedge dy) \wedge (xdy + dz) &= \\ (dx \wedge dy \wedge xdy) + (dx \wedge dy \wedge dz) &= \\ dx \wedge dy \wedge dz & \end{aligned}$$

It is important to note that a given contact form does not *uniquely* determine a plane field. We can "multiply" v by a function f without changing the kernel.

There is also a theorem of Darboux that says locally all contact forms have "the same" structure.

So the differences between contact structures only start to show up at the *global* level. That might make the non-integrability condition especially interesting since we can't "just integrate" the contact form to understand the global structure of the plane field!

But just because two contact forms have the same kernel *doesn't* mean they have the same Reeb field, our next and findal topic!

4 Reeb field

Examples of Reeb fields:

* talk about flows a bit * talk about what characterizes the Reeb field / flow * re: Huygens, talk about how the Reeb flow is maybe light rays? And there's some relationship to geodesic flow * talk about how contact forms that generate the same plane fields may *not* generate the same Reeb fields (I think)

References

[1] Andrew McInerney *First Steps in Differential Geometry: Riemannian, Contact, Symplectic*. Springer-Verlag New York, 2013.

[2] David Bachman *A Geometric Approach to Differential Forms*. Birkhuser Basel, 2006.

[3] Victor Andreevich Toponogov *Differential Geometry of Curves and Surfaces: A Concise Guide*. Birkhuser, 2006