

Quaternion Notes

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Quaternions

ordered list representation

$$\mathbf{q} = \begin{bmatrix} q_x & q_y & q_z & q_0 \end{bmatrix}^T$$

scalar part

$$R(\mathbf{q}) = q_0$$

vector part

$$I(\mathbf{q}) = \begin{bmatrix} q_x & q_y & q_z \end{bmatrix}^T$$

conjugate

$$\mathbf{q}^* = \begin{bmatrix} q_x & -q_y & -q_z & -q_0 \end{bmatrix}^T$$

unit quaternion

$$\|\mathbf{q}\| = \sqrt{\mathbf{q}\mathbf{q}^*} = \sqrt{q_0^2 + q_x^2 + q_y^2 + q_z^2}$$

reciprocal

$$\mathbf{q}^{-1} = \frac{\mathbf{q}^*}{\|\mathbf{q}\|}$$

compact representation

$$P(\mathbf{q}) = q_0 I(\mathbf{q}) = \begin{bmatrix} q_0 q_x & q_0 q_y & q_0 q_z \end{bmatrix}^T = \underline{q}$$

complex matrix representation

$$\mathbf{q} = \begin{bmatrix} q_0 + q_x & q_y + q_z \\ -q_y + q_z & q_0 - q_x \end{bmatrix}$$

real matrix representation

$$\mathbf{q} = \begin{bmatrix} q_0 & -q_x & -q_y & -q_z \\ q_x & q_0 & -q_z & q_y \\ q_y & q_z & q_x & -q_y \\ q_z & -q_y & q_y & q_x \end{bmatrix}$$

product

$$\mathbf{pq} = R(\mathbf{p}) R(\mathbf{q}) + I(\mathbf{p}) I(\mathbf{q}) = \begin{bmatrix} p_0 \vec{q} + q_0 \vec{p} + \vec{p} \times \vec{q} \\ p_0 q_0 - \vec{p} \cdot \vec{q} \end{bmatrix}^T$$

inverse

$$qq^{-1} = q^{-1}q = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$$

$$q^{-1} = \frac{q^*}{|q|^2} \text{ all non-zero cases}$$

$$q^{-1} = q^* \text{ for unit quaternions}$$

covariance

$$C_q = \frac{1}{N-1} \sum_{i=1}^N [q_i - \hat{q}] [q_i - \hat{q}]^T$$

axis-angle form

$$q(\hat{q}, \theta) = \begin{bmatrix} q_x \\ q_y \\ q_z \\ q_0 \end{bmatrix} = \begin{bmatrix} \hat{q} \sin\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right) \end{bmatrix}$$

antipodal symmetry

$$q(\hat{q}, \theta + 2\pi) = \begin{bmatrix} \hat{q} \sin\left(\pi + \frac{\theta}{2}\right) \\ \cos\left(\pi + \frac{\theta}{2}\right) \end{bmatrix} = \begin{bmatrix} -\hat{q} \sin\left(\frac{\theta}{2}\right) \\ -\cos\left(\frac{\theta}{2}\right) \end{bmatrix}$$

quaternion vector rotation

rotate \underline{v} by unit quaterion q

1. form quaternion v from \underline{v} by setting

$$I(\mathbf{v}) = \underline{v} \text{ and } R(\mathbf{v}) = 0$$

$$\underline{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \rightarrow \begin{bmatrix} v_x \\ v_y \\ v_z \\ 0 \end{bmatrix}$$

2. form quaternion product $w = qvq^*$
3. find $\underline{w} = I(w)$