Notes on Spectral Graph Theory

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The Laplacian matrix is the product of an oriented incident matrix, say $\mathbb{M}$, and its transpose. Note that this matrix is Hermitian, which is a property we will exploit in the numerical computation of eigenvalues.

$$\mathbb{L} = \mathbb{M}\mathbb{M}^{\intercal} \tag{1}$$

We can determine if all nodes within a submatrix of an oriented incident matrix is connected from its algebraic connectivity. The second eigenvalue of the Laplacian matrix of a subgraph is the algebraic connectivity, and thus the subgraph is connected if the second eigenvalue is greater than zero.

$$\Delta_{\mathbb{L}}(s) = |s\mathbb{I} - \mathbb{L}| \tag{2}$$

$$\underline{\lambda} = \Delta_{\mathbb{L}}(0) \tag{3}$$

$$\lambda_2 > 0 \Leftrightarrow \text{ connected} \tag{4}$$

Of course, we may also compute the transitive closure from the adjacency matrix.

$$\mathbb{C}^* = (\mathbb{A} + \mathbb{I}_{N\times N})^N \tag{5}$$

We may exploit the fact that our Laplacian matrix is Hermitian, and use Kung's Algorithm for $\mathbb{QR}$ factorization (qrDecomp). We may then use the $\mathbb{QR}$ Algorithm (eigW) to compute the eigenvalues of the Laplacian matrix.

To support Kung's Algorithm, we introduce a notation for transvections, $T_{i,j}^{N}(x)$. The transvection is an $N \times N$ matrix with ones along the diaginal, the value x at row i and column j, and zeros elsewhere.

$$\mathbb{T}_{i,j}^{N}(x) = \mathbb{I}_N + x\mathbb{E}_{i,j} \tag{6}$$

Kung's $\mathbb{QR}$ Algorithm then computes a strictly positive diagonalization matrix, $\mathbb{E}$, and resultant diagonal matrix $\mathbb{D}$. We define $\mathbb{C} = \sqrt{\mathbb{D}}$, and can write the factorization of a matrix $\mathbb{M}_{N\times N}$ as follows.

$$\mathbb{E} = \prod_{i=1}^{N} \prod_{j \neq i} T_{i,j}^{N} \left(\mathbb{M}_{i,j} \right)^{*} \tag{7}$$

$$\mathbb{D} = \left(\mathbb{E}\mathbb{M} \right)^{*} \left(\mathbb{M}\mathbb{E} \right) = \mathbb{E}^{*}\mathbb{M}^{*}\mathbb{M}\mathbb{E} \tag{8}$$

$$\mathbb{C} = \sqrt{\mathbb{D}} \tag{9}$$

$$\mathbb{Q} = \mathbb{A}\mathbb{E}\mathbb{C}^{-1} \tag{10}$$

$$\mathbb{R} = \mathbb{C}\mathbb{E}^{-1} \tag{11}$$

Algorithm 1 Transvection

function TRANSVECTION($\mathbb{M}_{N\times N}, i, j$)

▷ Initialize an identity matrix

  allocate $\mathbb{T}_{N\times N} \leftarrow 0$

  for $k \leftarrow 1 \dots N$ **do**

    $\mathbb{T}_{k,k} \leftarrow 1$

  end for

▷ Conjugate and copy at i,j

  $\mathbb{T}_{i,j} \leftarrow \mathbb{M}_{i,j}^{*}$

▷ Return the transvection

  return $\mathbb{T}$

end function

The QR Algorithm iteratively computes the eigenvalues of a matrix using a QR decomposition. The algorithm converges when the elements in the subdiagonal approach zero.

Algorithm 2 Kung's Algorithm

function qrDecomp($\mathbb{M}_{N\ timesN}$)

▷ Initialize an identity matrix

allocate $\mathbb{E}_{N\times N} \leftarrow 0$

for $k \leftarrow 1 \ldots N$ **do**

$\mathbb{E}_{k,k} \leftarrow 1$

end for

▷ Create the diagonalizer by applying conjugate pairs of transvections to remove off-diagonal elements

for $i \leftarrow 1 \ldots N$ **do**

for $j \leftarrow 1 \ldots N$ **do**

if $i \neq j$ **then**

$\mathbb{E} = \mathbb{E} \cdot \textsc{Transvection}(\mathbb{M}, i, j)$

end if

end for

end for

▷ Apply the diagonalizer to create the diagonalization

$\mathbb{D} \leftarrow \mathbb{E}^*\mathbb{M}^*\mathbb{M}\mathbb{E}$

$\mathbb{C} \leftarrow \sqrt{\mathbb{D}}$

▷ Compute the $\mathbb{Q}$ and $\mathbb{R}$ matrices

$\mathbb{Q} \leftarrow \mathbb{A}\mathbb{E}\mathbb{C}^{-1}$

$\mathbb{R} \leftarrow \mathbb{C}\mathbb{E}^{-1}$

▷ Return the orthogonal and upper triangular decomposition

return $\mathbb{Q}, \mathbb{R}$

end function

Algorithm 3 QR Algorithm

function $\textsc{eig}W(\mathbb{A}_0^{N \times N})$

$\triangleright$ Set the initial conditions

$k \leftarrow 0$

$Q_0, \mathbb{R}_0 \leftarrow qrDecomp(A_0)$

$\triangleright$ Iterate until the subdiagonal converges to zero

while $\mathbb{A}_{k_{i,j}} < \epsilon, \forall j = i - 1 \forall i \in 1 \dots N$ **do**

$k \leftarrow k + 1$

$\mathbb{Q}_{k-1}, \mathbb{R}_{k-1} \leftarrow \textsc{qrDecomp}(\mathbb{A}_{k-1})$

$\mathbb{A}_k \leftarrow \mathbb{R}_{k-1}\mathbb{Q}_{k-1}$

end while

$\triangleright$ Create a vector from the diagonal elements of $\mathbb{A}_k$

$\underline{\lambda} \leftarrow \{a_{i,j} | \forall a_{i,j} \in \mathbb{A}_k \wedge i = j\}$

$\triangleright$ Return eigenvalues

return $\underline{\lambda}$

end function