Quaternion Notes

Mac Radigan

Quaternions

ordered list representation

$\mathbf{q} = \left[\begin{array}{cccc} q_x & q_y & q_z & q_0 \end{array} \right]^\intercal$

scalar part

$R\left(\mathbf{q}\right) = q_0$

vector part

$I\left(\mathbf{q}\right) = \left[\begin{array}{ccc} q_x & q_y & q_z \end{array} \right]^\intercal$

conjugate

$\mathbf{q}^{*} = \left[\begin{array}{cccc} q_x & -q_y & -q_z & -q_0 \end{array} \right]^\intercal$

unit quaternion

$\|\mathbf{q}\| = \sqrt{\mathbf{q}\mathbf{q}^{*}} = \sqrt{q_0^2 + q_x^2 + q_y^2 + q_z^2}$

reciprocal

$\mathbf{q}^{-1} = \frac{\mathbf{q}^{*}}{\|\mathbf{q}\|}$

compact representation

$P\left(\mathbf{q}\right) = q_0 I\left(\mathbf{q}\right) = \left[\begin{array}{ccc} q_0 q_x & q_0 q_y & q_0 q_z \end{array} \right]^\intercal = \underset{\rightarrow}{q}$

complex matrix representation

$$\mathbf{q} = \begin{bmatrix} q_0 + q_x & q_y + q_z \\ -q_y + q_z & q_0 - q_x \end{bmatrix}$$

real matrix representation

$$\mathbf{q} = \begin{bmatrix} q_0 & -q_x & -q_y & -q_z \\ q_x & q_0 & -q_z & q_y \\ q_y & q_z & q_x & -q_y \\ q_z & -q_y & q_y & q_x \end{bmatrix}$$

product

$$\mathbf{pq} = R\left(\mathbf{p}\right) R\left(\mathbf{q}\right) + I\left(\mathbf{p}\right) I\left(\mathbf{q}\right) = \begin{bmatrix} p_0 \underset{\rightarrow}{q} + q_0 \underset{\rightarrow}{p} + \underset{\rightarrow}{p} \times \underset{\rightarrow}{q} \\ p_0 q_0 - \underset{\rightarrow}{p}\underset{\rightarrow}{q} \end{bmatrix}^{\mathsf{T}}$$

inverse

$qq^{-1} = q^{-1}q = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^{\mathsf{T}}$

$q^{-1} = \frac{q^*}{|q|^2}$ all non-zero cases

$q^{-1} = q^*$ for unit quaternions

covariance

$C_q = \frac{1}{N-1} \Sigma_{i=1}^{N} \left[q_i - \hat{q}\right] \left[q_i - \hat{q}\right]^{\mathsf{T}}$

axis-angle form

$$q\left(\underset{\rightarrow}{\hat{q}}, \theta\right) = \begin{bmatrix} q_x \\ q_y \\ q_z \\ q_0 \end{bmatrix} = \begin{bmatrix} \underset{\rightarrow}{\hat{q}} sin\left(\frac{\theta}{2}\right) \\ cos\left(\frac{\theta}{2}\right) \end{bmatrix}$$

antipodal symmetry

$$q\left(\hat{q}, \theta + 2\pi\right) = \begin{bmatrix} \underset{\rightarrow}{\hat{q}} sin\left(\pi + \frac{\theta}{2}\right) \\ cos\left(\pi + \frac{\theta}{2}\right) \end{bmatrix} = \begin{bmatrix} -\hat{q} sin\left(\frac{\theta}{2}\right) \\ -cos\left(\frac{\theta}{2}\right) \end{bmatrix}$$

quaternion vector rotation

rotate $\underset{\rightarrow}{v}$ by unit quaterion q

1. form quaternion v from $\underset{\rightarrow}{v}$ by setting

$$I(\mathbf{v}) = \underset{\rightarrow}{v} \text{ and } R(\mathbf{v}) = 0$$

$$\underset{\rightarrow}{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \rightarrow \begin{bmatrix} v_x \\ v_y \\ v_z \\ 0 \end{bmatrix}$$

2. form quaternion product $w = qvq^*$
3. find $\underset{\rightarrow}{w} = I(w)$