Markowitz Portfolio

Mac Radigan

Introduction

Markowitz portfolio theory is used in investment management as a tool for diversifying away risk through portfolio balancing. Given a set of investment assets, it finds the most efficient allocation (portfolio weights). Here, efficiency is defined as minimizing the expected variance. Markowitz portfolios may be subject to specified constraints, such as a specific return on investment (expected price), non-negativity constraints (restricting short selling), and may include risk-free assets or market indexes [**?**].

Implementation

This optmization problem may be expressed as minimizing the portfolio variance, subject to the constraint that the individual allocation weights of the stocks add to unity, and the expected return is equal to the specified target.

$$\begin{aligned} \underset{w}{\text{minimize}} \quad & \sigma_{p,w}^2 = \mathbf{w}^T \mathbf{\Sigma} \mathbf{w} \\ \text{subject to} \quad & \mu_{opt} = \mathbf{w}^T \mu \\ & \mathbf{w}^T \underline{\mathbf{1}} = 1 \end{aligned}$$

Where

$$\begin{array}{cl} \sigma_{p,w}^2 & \text{portfolio variance of weighted assets} \\ \mathbf{w} & \text{individual asset weights} \\ \mathbf{\Sigma} & \text{covariance matrix} \\ \mu & \text{individual asset returns} \\ \mu_{opt} & \text{target portfolio return} \end{array} \tag{1a}$$

Forming the Lagrangian function for the constrained minimization, we have

$$L(w, \lambda_1, \lambda_2) = \mathbf{w}^T \mathbf{\Sigma} \mathbf{w} + \lambda_1 (\mathbf{w}^T \mu - \mu_{opt}) \tag{2}$$

So the first order conditions are

$$2\partial L(w, \lambda_1, \lambda_2) = 2\mathbf{\Sigma}\mathbf{w} + \lambda_1 \mu + \lambda_2 \underline{\mathbf{1}} = \underline{\mathbf{0}} \tag{3}$$

$$\frac{2\partial L(w, \lambda_1, \lambda_2)}{\partial \lambda_1} = \mathbf{w}^T \mu - \mu_{opt} = 0 \tag{4}$$

$$\frac{2\partial L(w, \lambda_1, \lambda_2)}{\partial \lambda_2} = \mathbf{w}^T \underline{\mathbf{1}} - 1 = 0 \tag{5}$$

Expressed in matrix form

$$\mathbf{A}\mathbf{x} = \begin{bmatrix} 2\mathbf{\Sigma} & \mu & \underline{\mathbf{1}} \\ \mu^T & 0 & 0 \\ \underline{\mathbf{1}}^T & 0 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{w} \\ \lambda_1 \\ \lambda_2 \end{bmatrix} \begin{bmatrix} \underline{\mathbf{0}}^T \\ \mu_{opt} \\ 1 \end{bmatrix} = \mathbf{b} \tag{6}$$

This may be solved for $\mathbf{b}$, where the first $n-2$ elements of $\mathbf{b}$ are the portfolio weights ($\mathbf{w}_n = \mathbf{b}_n$).

$$\mathbf{b} = \mathbf{A}^{-1}\mathbf{x} \tag{7}$$