Math 122A Homework 4

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1 Problem 1

In each case find all the values of z such that:

(a) $z = i^i$

(b) $z = (1-i)^{1+i}$

(c) $e^{\frac{1}{z}} = 1 + i\sqrt{3}$

Proof.

(a) By DeMoivre's formula we know that $i = e^{i(\frac{\pi}{2}+2\pi k)}$ where $k \in \mathbb{Z}$, since this occurs every 2π. Therefore substituting this to the right side of our equation, we have that $z = (e^{i(\frac{\pi}{2}+2\pi k)})^i \Rightarrow z = e^{i^2(\frac{\pi}{2}+2\pi k)} = e^{-(\frac{\pi}{2}+2\pi k)}$.

(b) Similarly, using DeMoivre's formula, we know that $1-i = \sqrt{2}e^{-i(\frac{\pi}{4}+2\pi k)}$ where $k \in \mathbb{Z}$, which implies $z = (\sqrt{2}e^{-i(\frac{\pi}{4}+2\pi k)})^{(1+i)} = \sqrt{2}^{(1+i)}e^{-i(\frac{\pi}{4}+2\pi k)(1+i)}$. Distributing the exponents gives us $z = \sqrt{2}^{(1+i)}e^{(\frac{\pi}{4}+2\pi k)} \cdot e^{-i(\frac{\pi}{4}+2\pi k)}$. Again, by DeMoivre's formula we have that:

$$z = \sqrt{2}^{(1+i)}\left(e^{(\frac{\pi}{4}+2\pi k)}\cos(-\frac{\pi}{4} - 2\pi k) + i\sin(-\frac{\pi}{4} - 2\pi k)\right)$$

$$z = \sqrt{2}^{(1+i)} \cdot e^{(\frac{\pi}{4}+2\pi k)}\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)$$

$$z = (\sqrt{2})^i \cdot e^{(\frac{\pi}{4}+2\pi k)} \cdot (1-i)$$

(c) Finally, by the same formula we have that $1+i\sqrt{3} = 2e^{i(\frac{\pi}{3}+2\pi k)}$ where $k \in \mathbb{Z}$, so our equation becomes $e^{\frac{1}{z}} = 2e^{i(\frac{\pi}{3}+2\pi k)}$. Taking the natural log of both sides gives us $\frac{1}{z} = \ln(2e^{i(\frac{\pi}{3}+2\pi k)}) \Rightarrow \frac{1}{z} = \ln(2) + i(\frac{\pi}{3}+2\pi k)$. Multiplying by z and dividing by the right side to both sides yields,

$$z = \frac{1}{\ln(2) + i(\frac{\pi}{3}+2\pi k)}$$

$$z = \frac{\ln(2)}{(\ln 2)^2 + (\frac{\pi}{3}+2\pi k)^2} - i\left(\frac{\frac{\pi}{3}+2\pi k}{(\ln 2)^2 + (\frac{\pi}{3}+2\pi k)^2}\right)$$

$\square$

2 Problem 2

For any $z \in \mathbb{C}$ define:

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos(z) = \frac{e^{iz} + e^{-iz}}{2}, \quad \sinh(z) = \frac{e^{z} - e^{-z}}{2}, \quad \cosh(z) = \frac{e^{z} + e^{-z}}{2}$$

Prove:

(a) $\sin(-z) = -\sin(z), \qquad \cos(-z) = \cos(z)$

(b) $\sin^2(z) + \cos^2(z) = 1, \qquad \cosh^2(z) - \sinh^2(z) = 1$

(c) $\cos(z_1 + z_2) = \cos(z_1)\cos(z_2) - \sin(z_1)\sin(z_2)$

(d) $\cosh(z_1 + z_2) = \cosh(z_1)\cosh(z_2) + \sinh(z_1)\sinh(z_2)$

(e) $(\sin(z))' = \cos(z), \qquad (\cos(z))' = -\sin(z)$

(f) $(\sinh(z))' = \cosh(z) \qquad (\cosh(z))' = \sinh(z)$

Proof.

(a) By definition, we know that

$$\sin(-z) = \frac{e^{-iz} - e^{iz}}{2i}.$$

Meanwhile,

$$-\sin(z) = -\left(\frac{e^{iz} - e^{-iz}}{2i}\right) = \frac{e^{-iz} - e^{iz}}{2i},$$

which is exactly $\sin(-z)$ therefore $\sin(-z) = -\sin(z)$.

Now,

$$\cos(-z) = \frac{e^{-iz} + e^{iz}}{2} = \frac{e^{iz} + e^{-iz}}{2},$$

by definition which is exactly $\cos(z)$.

(b) By definition we have that:

$$\sin^2(z) = \frac{-e^{2iz} - e^{-2iz}}{4} + \frac{1}{2}$$

and

$$\cos^2(z) = \frac{e^{2iz} + e^{-2iz}}{4} + \frac{1}{2}$$

Therefore, adding $\sin^2(z) + \cos^2(z)$ yields

$$\sin^2(z) + \cos^2(z) = \frac{-e^{2iz} - e^{-2iz}}{4} + \frac{1}{2} + \frac{e^{2iz} + e^{-2iz}}{4} + \frac{1}{2}$$

$$\sin^2(z) + \cos^2(z) = 1$$

Similarly, by definition, we know

$$-\sinh^2(z) = -\left(\frac{e^{2z} + e^{-2z}}{4} - \frac{1}{2}\right) = -\frac{-e^{2z} - e^{-2z}}{4} + \frac{1}{2}$$

and

$$\cosh^2(z) = \frac{e^{2z} + e^{-2z}}{4} + \frac{1}{2}$$

So adding them together gives us

$$\cosh^2(z) + -\sinh^2(z) = \frac{e^{2z} + e^{-2z}}{4} + \frac{1}{2} - \frac{-e^{2z} - e^{-2z}}{4} + \frac{1}{2}$$

$$\cosh^2(z) + -\sinh^2(z) = 1$$

(c) We know that, by definition:

$$\cos(z_1)\cos(z_2) = \frac{e^{i(z_1+z_2)} + e^{i(z_1-z_2)} + e^{i(z_2-z_1)} + e^{i(-z_1-z_2)}}{4}$$

and

$$-\sin(z_1)\sin(z_2) = \frac{e^{i(z_1+z_2)} - e^{i(z_1-z_2)} - e^{i(z_2-z_1)} + e^{i(-z_1-z_2)}}{4}$$

Therefore,

$$\cos(z_1)\cos(z_2) - \sin(z_1)\sin(z_2) = \frac{2e^{i(z_1+z_2)} + 2e^{i(-z_1-z_2)}}{4}$$

$$\cos(z_1)\cos(z_2) - \sin(z_1)\sin(z_2) = \frac{e^{i(z_1+z_2)} + e^{i(-z_1-z_2)}}{2}$$

Now

$$\cos(z_1 + z_2) = \frac{e^{i(z_1+z_2)} + e^{i(-z_1-z_2)}}{2}$$

which is exactly our result from $\cos(z_1)\cos(z_2) - \sin(z_1)\sin(z_2)$.

(d) Taking the derivative of $\sin(z)$

$$(\sin(z))' = \frac{d}{dz}\left(\frac{e^{iz}}{2i}\right) - \frac{d}{dz}\left(\frac{e^{-iz}}{2i}\right)$$

$$(\sin(z))' = \frac{ie^{iz}}{2i} - \frac{-ie^{-iz}}{2i}$$

$$(\sin(z))' = \frac{e^{iz} + e^{-iz}}{2}$$

Which is exactly the definition of $\cos(z)$ And for $\cos(z)$

$$(\cos(z))' = \frac{d}{dz}\left(\frac{e^{iz}}{2}\right) + \frac{d}{dz}\left(\frac{e^{-iz}}{2}\right)$$

$$(\cos(z))' = \frac{ie^{iz}}{2} + \frac{-ie^{-iz}}{2}$$

Multiplying by $\frac{i}{i}$, we get

$$(\cos(z))' = \frac{-e^{iz} + e^{-iz}}{2i}$$

Which is exactly the definition of $-\sin(z)$

(e) Again, taking the derivative of $\sinh(z)$

$$(\sinh(z))' = \frac{d}{dz}\left(\frac{e^{z}}{2}\right) - \frac{d}{dz}\left(\frac{e^{-z}}{2}\right)$$

$$(\sinh(z))' = \frac{e^{z}}{2} - \frac{-e^{-z}}{2} = \frac{e^{z} + e^{-z}}{2}$$

which is the definition of $\cosh(z)$. Finally, taking the derivative of $\cosh(z)$ gives us

$$(\cosh(z))' = \frac{d}{dz}\left(\frac{e^{z}}{2}\right) + \frac{d}{dz}\left(\frac{e^{-z}}{2}\right)$$

$$(\cosh(z))' = \frac{e^{z}}{2} + \frac{-e^{-z}}{2} = \frac{e^{z} - e^{-z}}{2}$$

which is the definition of $\sinh(z)$.

(f) Finally, taking the derivative of $\sinh(z)$, we have that

$$(\sinh(z))' = \frac{e^z}{2} + \frac{e^{-z}}{2} = \frac{e^z + e^{-z}}{2} = \cosh(z)$$

Furthermore, taking the derivative of $\cosh(z)$ yields:

$$(\cosh(z))' = \frac{e^z}{2} + \frac{-e^{-z}}{2} = \frac{e^z - e^{-z}}{2} = \sinh(z)$$

□

3 Problem 3

Evaluate the following integrals ($k \in \mathbb{Z}$):

(a) $\int_1^2 (\frac{1}{t} + i)^2 dt$

(b) $\int_0^{\frac{\pi}{3}} e^{it} dt$

(c) $\int_0^{2\pi} e^{ikt} dt$

Proof.

(a)

$$\int_1^2 \left(\frac{1}{t} + i\right)^2 dt = \int_1^2 \left(\frac{1}{t^2} + 2i\frac{1}{t} - 1\right) dt$$
$$= -(-\frac{1}{2}) + 2i(\ln(2) - \ln(1)) - 1$$
$$= -\frac{1}{2} + 2i\ln(2)$$

(b)

$$\int_0^{\frac{\pi}{3}} e^{it} dt = \frac{1}{i}\left(e^{i\frac{\pi}{3}} - 1\right)$$
$$= \frac{1}{i}(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right) - 1)$$
$$= -\frac{1}{2i}\left(\frac{i}{i}\right) + \frac{\sqrt{3}}{2}$$
$$= \frac{i + \sqrt{3}}{2}$$

(c)

$$\int_0^{2\pi} e^{ikt} dt = \frac{i}{ik}(e^{i2\pi k} - 1)$$
$$= \frac{1}{ik}(1 - 1)$$
$$= 0$$

$\square$

4 Problem 4

In each case write the equation of the curve representing:

(a) The segment joining 1 and i

(b) The circumference of center $1-i$ and radius 2, in the counter clockwise direction

(c) The triangle with vertices $1, i, -2$.

Proof.

(a) We know that $z_0 = 1$ and $z_1 = 1 + i$, then parameterizing our z we have

$$\gamma : [0, 1] \to \mathbb{C}$$

$$t : [0, 1] \to ti + (1 - t)$$

(b) Similarly, we have a circle of radius 2, centered at $1-i$, so our equation is

$$|z - (1 - i)| = 2$$

Parameterizing this we have

$$z(t) = (1 - i) + 2e^{it} \qquad \text{for } 0 \leq t \leq 2\pi$$

(c) For this we split the parameterization to three pieces $\gamma = \gamma_1 + \gamma_2 + \gamma_3$ where $\gamma_1 : [0, 1] \to \mathbb{C}$, $\gamma_2 : [0, 1] \to \mathbb{C}$, $\gamma_3 : [0, 1] \to \mathbb{C}$ and $\gamma_1(t) = ti + (1 - t)$, $\gamma_2(t) = (1 - t)i - 2t$, $\gamma_3(t) = 3t - 2$.

□

5 Problem 5

Evaluate the following integrals:

(a) $\int_\gamma x dx$, γ the boundary on the unit square.

(b) $\int_\gamma e^z dz$, γ the portion of the unit circle joining 1 and i in the counter clockwise direction.

(c) $\int_\gamma x dy$, γ is the boundary of a bounded region $A \subset \mathbb{R}^2$ (without holes) in the counter clockwise direction.
HINT: Use Green's Theorem.

Proof.

(a) Similar to **Problem 5(c)**, we have four parts we parameterize our curve which is $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$ where each γ are given by:

$$\begin{aligned}
\gamma_1 : [0,1] \to \mathbb{C} \quad &\Rightarrow \quad \gamma_1(t) = (1-t) + (1+i)t = 1 + it \\
\gamma_2 : [0,1] \to \mathbb{C} \quad &\Rightarrow \quad \gamma_2(t) = 1 - t + i \\
\gamma_3 : [0,1] \to \mathbb{C} \quad &\Rightarrow \quad \gamma_3(t) = (1-t)i = i - ti \\
\gamma_4 : [0,1] \to \mathbb{C} \quad &\Rightarrow \quad \gamma_4(t) = t
\end{aligned}$$

Now parameterize the integral $\int_\gamma x dz = \int_0^1 f(\gamma_n(t))\dot{\gamma_n} dt$

$$\begin{aligned}
\int_0^1 f(\gamma_1(t))\dot{\gamma_1} dt &= \int_0^1 i dt = i \\
\int_0^1 f(\gamma_2(t))\dot{\gamma_2} dt &= \int_0^1 (1-t)(-1) dt = -\frac{1}{2} \\
\int_0^1 f(\gamma_3(t))\dot{\gamma_3} dt &= \int_0^1 (0)(-i) dt = 0 \\
\int_0^1 f(\gamma_4(t))\dot{\gamma_4} dt &= \int_0^1 t dt = \frac{1}{2}
\end{aligned}$$

Therefore, $\int_\gamma x dz = i$.

(b) By the theorem in section 48 and lecture 7, we know that $\int e^z dz$ can be given by $F(b) - F(a)$, therefore $\int_\gamma e^z dz = e^i - e$

(c) Green's Theorem states that:

$$\oint_C P_{(x,y)}dx + Q_{(x,y)}dy \equiv \iint\limits_{\Omega} (\partial_x Q - \partial_y P)dxdy$$

Where in our case, $\Omega = A$, $P = 0$, and $Q = x$ giving us

$$\oint xdy = \iint\limits_{A} (\partial_x x)dA = A$$

$\square$