Math 108B Homework 2

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1 Problem 1

Prove that if x, y are nonzero vectors in $\mathbf{R}^2$, then

$$\langle x, y \rangle = \|x\| \|y\| \cos\theta$$

where θ is the angle between x and y (thinking of x and y as arrows with initial point at the origin). *Hint:* draw the triangle formed by x, y, and $x-y$; then use the law of cosines.
Proof. By the law of cosines we know that

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\|x\|\|y\| \cos(\theta)$$

We know that $\|x-y\|^2 = (x-y) \cdot (x-y) = \|x\|^2 - 2x \cdot y + \|y\|^2$ so equating the two we have

$$\|x\|^2 - 2x \cdot y + \|y\|^2 = \|x^2\| + \|y\|^2 - 2\|x\|\|y\| \cos(\theta)$$

then subtracting $\|x\|^2$, $\|y\|^2$, and finally diving the by two to both sides we get

$$x \cdot y = \|x\|\|y\| \cos(\theta)$$
$$\langle x, y \rangle = \|x\|\|y\| \cos(\theta)$$

$\square$

2 Problem 2

Suppose $u, v \in V$. Prove that $\langle u, v \rangle = 0$ if and only if

$$\|u\| \leq \|u + av\|$$

for all $a \in \mathbf{F}$.
Proof. For one direction we assume that $\langle u, v \rangle = 0$, then for any $a \in \mathbf{F}$, we know that $\langle u, av \rangle = 0$. Then by Pythagorean theorem, $\|u + av\|^2 = \|u\|^2 + \|av\|^2 \geq \|u\|^2$, i.e. $\|u\| \leq \|u + av\|$. For the other direction, we assume that $\|u\| \leq \|u + av\|$. Squaring and then subtracting $\|u\|^2$ to both sides we have that $0 \leq \|u + av\|^2 - \|u\|^2$ $\square$

3 Problem 3

Prove that

$$(\sum_{j=1}^{n} a_j b_j)^2 \leq (\sum_{j=1}^{n} j a_j^2)(\sum_{j=1}^{n} \frac{b_j^2}{j})$$

for real numbers $a_1, ..., a_n$ and $b_1, ..., b_n$.
Proof. By the Cauchy-Schwarz inequality

$$(\sum_{j=1}^{n} a_j b_j)^2 = (\sum_{j=1}^{n} (\sqrt{j} a_j)(\frac{1}{\sqrt{j} b_j})^2$$

So we know that

$$(\sum_{j=1}^{n} (\sqrt{j} a_j)(\frac{1}{\sqrt{j} b_j})^2 \leq (\sum_{j=1}^{n} j a_j^2)(\sum_{j=1}^{n} \frac{b_j^2}{j})$$

$\square$

4 Problem 4

Suppose $u, v \in V$ are such that

$$\|u\| = 3, \qquad \|u + v\| = 4, \qquad \|u - v\| = 6.$$

What number must $\|v\|$ equal?
Proof. By the parallelogram inequality,

$$\|u+v\|^2 + \|u+v\|^2 = 2(\|u\|^2 + \|v\|^2)$$

$$4^2 + 6^2 = 2(3^2 + \|v\|^2$$

$$52 = 2(9 + \|v\|^2)$$

$$26 = 9 + \|v\|^2$$

$$\|v\| = \sqrt{17}$$

□

5 Problem 6

Prove that if V is a real inner-product space, then

$$\langle u, v\rangle = \frac{\|u+v\|^2 - \|u-v\|^2}{4}$$

for all $u, v \in V$
Proof. Expanding the terms give

$$\frac{\|u\| + 2\langle u, v\rangle + \|v\|^2 - \|v\|^2 + 2\langle u, v\rangle - \|v\|^2}{4}$$

$$\frac{4\langle u, v\rangle}{4} = \langle u, v\rangle$$

□

6 Problem 9 (for n=2 only)

Suppose n is a positive integer. Prove that

$$(\frac{1}{\sqrt{2\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, ..., \frac{\sin nx}{\sqrt{\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, ..., \frac{\cos nx}{\sqrt{\pi}})$$

is an orthonormal list of vectors in $C[-\pi, \pi]$, the vector space of continuous real-valued functions on $[-\pi, \pi]$ with inner product

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)dx.$$

Proof. Using the following formulas,

$$\int (\sin jt)^2 dt = \frac{2jt - \sin 2jt}{4j}$$

$$\int (\cos jt)^2 dt = \frac{2jt + 2jt}{4j}$$

we know that each of the element above has norm 1.Then, when $j \neq k$, two distinct elements in the list are orthogonal shown by

$$\int (\sin jt)(\sin kt)dt = \frac{j\sin(j-k)t + k\sin(j-k)t - j\sin(j+k)t + k\sin(j+k)t}{2(j-k)(j+k)}$$

$$\int (\sin jt)(\cos kt)dt = \frac{j\cos(j-k)t + k\cos(j-k)t - j\cos(j+k)t + k\cos(j+k)t}{2(k-j)(j+k)}$$

$$\int (\cos jt)(\cos kt)dt = \frac{j\sin(j-k)t + k\sin(j-k)t - j\sin(j+k)t + k\sin(j+k)t}{2(j-k)(j+k)}$$

$$\int (\sin jt)(\cos jt)dt = -\frac{(\cos jt)^2}{2j}$$

□

7 Problem 13

Suppose $(e_1, ..., e_n)$ is an orthonormal list of vectors in V. Let $v \in V$. Prove that

$$\|v\|^2 = |\langle v, e_1\rangle|^2 + ... + |\langle v, e_m\rangle|^2$$

if and only if $v \in \text{span}(e_1, ..., e_m)$.

Proof. By basis extension, we can extend $(e_1, ..., e_m)$ to an orthonormal basis $(e_1, ..., e_n)$. From this we get that $v = \langle v, e_1\rangle e_1 + ... + \langle v, e_n\rangle e_n$ and $\|v\|^2 = |\langle v, e_1\rangle|^2 + ...+$. This is only true of $\langle v, e_{m+1} = ... = \langle v, e_n\rangle = 0$. The equation is only true if and only if

$$\|v\|^2 = |\langle v, e_1\rangle|^2 + ... + |\langle v, e_m\rangle|^2$$

which only occurs when $v \in \text{span}(e_1, ..., e_m)$. □

8 Problem 17

Prove that if $P \in \mathcal{L}(V)$ is such that $P^2 = P$ and every vector in null P is orthogonal to every vector in range P, then P is an orthogonal projection.
Proof. By our assumption we want to show P is equal to the orthogonal projection P_U. We begin by letting $U = \text{range}(P)$, and suppose that $u \in V$. Then adding and subtracting Pv we know that $v = Pv + (v - Pv)$. Since $Pv \in \text{range}(P) = U$, $P(v - Pv) = Pv - P^2v = 0$ implying that $v - Pv \in \text{null}(P)$. Therefore, $v - Pv$ is orthogonal to all vectors in U. The vector in U is P_Uv, which we can conclude that $Pv = P_Uv$. $\square$

9 Problem 18

Prove that if $P \in \mathcal{L}(V)$ is such that $P^2 = P$ and

$$\|Pv\| \leq \|v\|$$

for every $v \in V$, then P is an orthogonal projection.
Proof. We suppose that $u \in \text{range}(P)$ and $w \in \text{null}(P)$, then $u \in \text{range}(P)$ implies that there exists $u' \in V$ such that $u = Pu'$. Multiplying P to both sides gives us that $Pu = P^2u' \Rightarrow Pu = u$. Now since $w \in \text{null}(P)$, we know that $P(u + aw) = u$. Therefore, $\|u\|^2 = \|P(u + aw)\|^2 \leq \|u + aw\|^2$ by **Problem 2**. $\square$

10 Problem 21

In $\mathbf{R}^4$ let

$$U = \text{span}((1, 1, 0, 0), (1, 1, 1, 2)).$$

Find $u \in U$ such that $\|u - (1, 2, 3, 4)\|$ is as small as possible.
Proof. Using the Gram-Schmidt on U, we get

$$e_1 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0)$$

and

$$e_2 = (0, 0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})$$

which are orthonormal basis of U. The closest point $u \in U$ to $(1,2,3,4)$ is $\langle (1,2,3,4), e_1 \rangle e_1 + \langle (1,2,3,4), e_2 \rangle e_2$ which is

$$(\frac{3}{2}, \frac{3}{2}, \frac{11}{5}, \frac{22}{5})$$

□

11 Problem 24

Find a polynomial $q \in \mathcal{P}_2(\mathbf{R})$ such that

$$p(\frac{1}{2}) = \int_0^1 p(x)q(x)dx$$

for every $p \in \mathcal{P}_2(\mathbf{R})$.
Proof. Let $e_1(x) = 1$, $e_2(x) = \sqrt{3}(-1+2x)$, $e_3(x) = \sqrt{5}(1-6x+6x^2)$ so that (e_1, e_2, e_3) are orthonormal basis of $P_2(\mathbf{R})$. Now defining $\phi(p) = p(\frac{1}{2})$, we want $q \in P_2(\mathbf{R})$ such that $\phi(p) = \langle p, q \rangle$ for every $p \in P_2(\mathbf{R})$. Now

$$q = \phi(e_1)e_1 + \phi(e_2)e_2 + \phi(e_3)e_3$$

$$q = e_1(\frac{1}{2})e_1 + e_2(\frac{1}{2})e_2 + e_3(\frac{1}{2}e_3$$

$$q = 1e_1 + 0e_2 + \sqrt{5}(\frac{6}{4} - \frac{6}{2} + 1)e_3$$

$$q = 1 - \frac{5}{2}(6x^2 - 6x + 1)$$

$$q = -15x^2 + 15x - \frac{3}{2}$$

□

12 Problem 25

Find a polynomial $q \in \mathcal{P}_2(\mathbf{R})$ such that

$$\int_0^1 p(x)(\cos \pi x)dx = \int_0^1 p(x)q(x)dx$$

for every $p \in \mathcal{P}_2(\mathbf{R})$.

Proof. Using same basis as **Problem 24**, we define

$$\phi(p) = \int_0^1 \cos(\pi x)dx$$

Then set $p(x) = 1$ and $\phi(x) = \int_0^1 \cos(\pi x)dx = 0$

$$\phi(x) = \int_0^1 \alpha + \beta(x - \frac{1}{2}) + \gamma(x - \frac{1}{2})^2 = \alpha + \gamma\frac{1}{2}$$

For $p(x) = x - \frac{1}{2}$

$$\phi(x - \frac{1}{2}) = \int_0^1 (x - \frac{1}{2}\cos(\pi x)dx$$

$$\phi(x - \frac{1}{2}) = \int_0^1 x\cos(\pi x)dx = -\frac{2}{\pi}$$

$$\phi(x - \frac{1}{2}) = \int_0^1 (x - \frac{1}{2})(\alpha + \beta(x - \frac{1}{2} + \gamma(x - \frac{1}{2})^2 = \beta\frac{1}{12}$$

implies that $\beta = -\frac{24}{\pi^2}$ Therefore, $q(x) = -\frac{24}{\pi^2}(x - \frac{1}{2})$

□