Math 108B Homework 1

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March 21, 2022

1 Problem 1

Prove that if $U_1, U_2, ..., U_m$ are subspaces of V invariant under T, then $U_1 + U_2 + ... + U_m$ is invariant under T. Here

$$U_1 + U_2 + ... + U_m = \{y_1 + y_2 + ... + y_m : y_j \in U_j, 1 \leq j \leq m\}$$

Proof. Setting $x = U_1 + U_2 + ... + U_m$, then $T(x) = T(U_1 + U_2 + ... + U_m)$. Since we know that $T \in \mathcal{L}(V)$ we can write this as $T(x) = T(U_1) + T(U_2) + ... + T(U_m)$. Furthermore, we know that $U_1, U_2, ..., U_m$ are subspaces of V invariant under T, therefore, $T(x) = T(U_1) + T(U_2) + ... + T(U_m) \subseteq U_1 + U_2 + ... + U_m = x$. And so, we can conclude that $T(x) \subseteq x$, i.e. invariant under T. $\square$

2 Problem 2

Prove that the intersection of any collection of subspaces of V invariant under T is invariant under T.
Proof. If we have a set of subspaces of V, say $\{U_m\}$, that are invariant under T and we take an element u in the intersection of all U_m, then similar to **Problem 1** since $T \in \mathcal{L}(V)$, $Tu \in U_m$ for all m. Therefore, the all intersection of U_m is invariant under T. $\square$

3 Problem 3

Suppose U is a subspace of V that is invariant under every T. Prove that $U = \{0\}$ or V.
Proof. By way of contradiction, suppose we have $U \neq \{0\}$ or V, and we let $x_1 \in U \setminus \{0\}$ and $x_2 \notin U$. Extending a basis $\{x_1, b_1, b_2, ..., b_n\}$ of V and defining T as $T(x_1) = x_2$ where $T(b_m) = 0$ for $m = 1, ..., n$. Then $T \in \mathcal{L}(V)$ and T maps $x_1 \in U$ to an element not in U and we conclude that U is not invariant under T. $\square$

4 Problem 4

Suppose $S, T \in \mathcal{L}(V)$ such that $ST = TS$. Prove that the subspace

$$\text{null}(\lambda I - T) = \{x \in V : (\lambda I - T)x = 0\}$$

is invariant under S for every $\lambda \in F$.
Proof. Taking an element $u \in \text{null}(T - \lambda I)$, we have that

$$(T - \lambda I)(Su) = TSu - \lambda Su = 0$$

$$TSu - \lambda Su = STu - \lambda Su = 0$$

$$S(Tu - \lambda u) = 0$$

by linearity of T. Therefore, $Su \in \text{null}(T - \lambda I)$ and $\text{null}(T - \lambda I)$ is invariant of S for every $\lambda \in F$. $\square$

5 Problem 5

Let

$$V = \{(a, b) : a, b \in F\}$$

so $n = 2$. Define T by $T(a, b) = (b, a)$. Find eigenvalues and eigenvectors of T.
Proof. Taking the standard basis $v = \{(1, 0), (0, 1)\}$, then $T(0, 1) = 0(1, 0) + 1(0, 1)$ and $T(1, 0) = 1(1, 0) + 0(0, 1)$, and so $T = \{(0, 1), (1, 0)\}$. The characteristic equation then is

$$\begin{vmatrix} 0 - \lambda & 1 \\ 1 & 0 - \lambda \end{vmatrix}$$

Which is equal to $\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$. For $\lambda = 1$, we can let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Using $(T - I\lambda)x = 0$ and using row reduction, we will get a free variable. Letting the free variable be x_2 results in our eigenvector $x = \left\{ x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} : x_2 \in \mathbb{R} \right\}$. Doing the similar process for $\lambda = -1$, we get the eigenvector $x = \left\{ x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} : x_2 \in \mathbb{R} \right\}$ $\square$

6 Problem 6

Let

$$V = \{(a, b, c) : a, b, c \in F\}$$

so $n = 3$. Define T by $T(a, b, c) = (2b, 0, 5c)$. Find eigenvalues and eigenvectors of T.

Proof. Similarly to **Problem 5**, we have the standard basis

$$v = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$$

We then have our matrix for $T = \{(0, 2, 0), (0, 0, 0), (0, 0, 5)\}$. To find the eigenvalues then is given by $|T - \lambda I|$ which is

$$\begin{vmatrix} 0-\lambda & 2 & 0 \\ 0 & 0-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{vmatrix}$$

This yields the characteristic polynomial $\lambda^3 - 5\lambda^2 = \lambda^2(\lambda - 5) = 0 \Rightarrow \lambda = 0$ (multiplicity 2), 5. Now for $\lambda = 5$, we again use $(T - I\lambda)x = 0$ where $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Again, we get a free variable and let $x_2 = x_3 = 0$, which results in our eigenvector $x = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. For $\lambda = 0$ $\square$

7 Problem 7

Suppose

$$Tx_1 = Tx_2 = ... = Tx_n = x_1 + x_2 + ... + x_n$$

Find all eigenvalues and eigenvectors of T. (Hint: When is $Tx = 0$?)
Proof. We have that:

$$Tx_1 = x_1 + x_2 + ... + x_n$$

$$Tx_2 = x_1 + x_2 + ... + x_n$$

$$...$$

$$Tx_n = x_1 + x_2 + ... + x_n$$

and we have the matrix M for the transformation is

$$\begin{bmatrix} 1 & 1 & ... & 1 \\ 1 & 1 & ... & 1 \\ & & ... & \\ 1 & 1 & ... & 1 \end{bmatrix}$$

Therefore, our eigenvalues are $\lambda_1 = 0$, $\lambda_2 = 0$, ..., $\lambda_{n-1} = 0$,$\lambda_n = n$. And so our eigenvectors are:

$$(A - \lambda_i)x_i = 0$$

That is $x_1 = (1, -1, 0, ..., 0)$, $x_2 = (1, 0, -1, 0, ..., 0)$, ..., $x_{n-1} = (1, 0, ..., 0, -1)$, and $x_n = (1, 1, ..., 1)$ □

8 Problem 8

Suppose the dimension of the subspace range(T) $= k$. Prove that T has at most $k + 1$ distinct eigenvalues.
Proof. Assume T has $k + 2$ distinct eigenvalues, we claim that range(T) $\geq k + 1$. Let $\lambda_1, \lambda_2, ..., \lambda_{k+2}$ be distinct eigenvalues of T and $y_1, y_2, ..., y_{k+2}$ be the vectors of T which are linearly independent. Since at most one of the eigenvalues is 0, there are at least $k+1$ of the vectors is in range(T) implying that T has at most $k + 1$ distinct eigenvalues. □

9 Problem 9

Suppose T is invertible and $0 \neq \lambda \in F$. Prove that λ is an eigenvalue of T if and only if $1/\lambda$ has an eigenvalue of T^{-1}.

Proof. Since T is invertible, we know that it is injective, and so $(\lambda I - T)$ is invertible. Therefore, every eigenvalue of $T \neq 0$. Furthermore, there exists an $x \neq 0$ such that $Tx = \lambda x \Rightarrow T^{-1}Tx = \lambda x T^{-1} \Rightarrow \frac{x}{\lambda} = xT^{-1}$ □

10 Problem 10

Suppose $S, T \in \mathcal{L}(V)$. Prove that ST and TS have the same eigenvalues.

Proof. To prove this, suppose we have an eigenvalue λ of ST, then there exists a vector $x \neq 0$ such that $STx = \lambda x$. Now, letting $y = Tx$, we have that $TSy = \lambda y$. □

11 Problem 11

Suppose every non-zero vector in V has and eigenvector of T. Prove that $T = \lambda I$ for some $\lambda \in F$.

Proof.

□

12 Problem 12

Suppose that T has n distinct eigenvalues and that $S \in \mathcal{L}(V)$ has the same eigenvectors as T. Prove that $ST = TS$.

Proof.

□