Math 108B Homework 4

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For all problems:

- V is a vector space over $\mathbb{C}$
- $\dim V = n$
- $x, y \in V$
- $V, T, S, N \in \mathcal{L}(V)$
- $N \in \mathcal{L}(V)$ is nilpotent if $N^k = 0$ for some $k > 0$.

1 Problem 1

Prove that if $T^{m-1}x \neq 0$, $T^m x = 0$, then the set $\{x, Tx, ..., T^{m-1}x\}$ is linearly independent.
Proof.

$\square$

2 Problem 2

Prove that if ST is nilpotent, then TS is nilpotent.
Proof.

$\square$

3 Problem 3

Prove that if N is nilpotent, then 0 is the only eigenvalue of N.
Proof.

$\square$

4 Problem 4

Prove that if null $N^{n-1} \neq$ null N^n, then N is nilpotent.
Proof.

$\square$

5 Problem 5

Prove that if null $N^{n-2} \neq$ null N^{n-1}, then N has at most two distinct eigenvalues.
Proof.

$\square$

6 Problem 6

Prove that for any T,

$$\text{null } T^n \cap \text{range } T^n = \{0\}$$

Proof.

$\square$

7 Problem 7

Find a 3×3 matrix whose minimal polynomial is z^2.
Proof.

$\square$

8 Problem 8

Find a 4×4 matrix whose minimimal polynomial is $z(z-1)^2$.
Proof.

$\square$

9 Problem 9

Suppose T is invertible. Prove that there is a polynomial p such that $T^{-1} = p(T)$.
Proof.

$\square$

10 Problem 10

Prove that V has a basis consisting of eigenvectors of T if and only if the minimal polynomial of T has no repeated roots.
Proof.

$\square$

11 Problem 11

Suppose $x \neq 0$. Let p be the monic polynomial of the smallest degree such that $p(T)x = 0$. Prove that p divides the minimal polynomial of T.
Proof.

$\square$