Math 122A Homework 7 and 8

Rad Mallari

March 9, 2022

1 Problem 1

Let $D_1(z_0) = \{z \in \mathbb{C} : |z - z_0| < 1\}$. Let $f, g : D_1(z_0) \to \mathbb{C}$ be two analytic functions on $D_1(z_0)$. Prove that if

$$f^{(n)}(z_0) = g^{(n)}(z_0), \quad n = 0, 1, 2, 3, ...$$

then $f(z) = g(z)$, $\forall z \in D_1(z_0)$.

Proof. By our given, we know there is a unique Taylor Series expansion of $f(z)$ and $g(z)$ centered around z_0 such that

$$f(z) = \sum_{n=0}^{\infty} (z - z_0)^n \frac{f^{(n)}(z_0)}{n!} \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} (z - z_0)^n \frac{g^{(n)}(z_0)}{n!}$$

where $n = 0, 1, 2, 3, ...$ therefore, equating the two we have that

$$\sum_{n=0}^{\infty} (z - z_0)^n \frac{f^{(n)}(z_0)}{n!} = \sum_{n=0}^{\infty} (z - z_0)^n \frac{g^{(n)}(z_0)}{n!}$$

This reduces to

$$f^{(n)}(z_0) = g^{(n)}(z_0)$$

Which is exactly what we want. $\square$

2 Problem 2

Let $D_1(z_0) = \{z \in \mathbb{C} : |z - z_0| < 1\}$. Let $f : D_1(z_0) \to \mathbb{C}$ be an analytic function on $D_1(z_0)$ such that is has a zero of $N \in \mathbb{N}$ at z_0, i.e.

$$f(z_0) = f'(z_0) = ... = f^{N-1}(z_0) = 0, \quad f^n(z_0) \neq 0$$

(i) Prove that there exists $g : D_1(z_0) \to \mathbb{C}$ analytic on $D_1(z_0)$ with $g(z_0) \neq 0$ and

$$f(z) = (z - z_0)^N g(z)$$

(ii) There exists $\delta > 0$ such that if $0 < |z - z_0| < \delta$ such that $f(z) \neq 0$. (The zeros of a non-trivial analytic function are isolated)

Proof.

(i) Since we are given that

$$f(z_0) = f'(z_0) = ... = f^{N-1}(z_0) = 0, \quad f^n(z_0) \neq 0$$

and letting $z_0 = 0$, we know that we can Taylor expand $f(z)$ such that:

$$f(z) = \underbrace{\sum_{n=0}^{N-1} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n}_{=0 \text{ (by definition)}} + \sum_{k=N}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k$$

where the remaining nonzero sum terms consists of analytic functions. Factoring out a $(z - z_0)^N$ yields:

$$f(z) = (z - z_0)^N \cdot \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k$$

Finally, letting $g(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k$ we conclude:

$$f(z) = (z - z_0)^N \cdot g(z_0)$$

(ii) Taking $f(z)$ in **Problem 2(i)**, we know that after the first zero terms of the Taylor expansion, we have

$$f(z) = (z - z_0)^N \cdot g(z_0)$$

where $g(z)$ is analytic, therefore continuous. Clearly, the first term of $g(0) \neq 0$ and is a constant and the following terms are nonzero by definition. So, it follows that there must exist a nonzero $\delta > 0$ such that $|z - z_0| < \delta$ which implies that $|g(z)| \neq 0$. Clearly, $(z - z_0)^N \neq 0$ so the zeros of a non-trivial analytic function are isolated.

$\square$

3 Problem 3

Let $f(z) = \sin(\frac{\pi}{z})$. Thus $f(\frac{1}{n}) = 0$. Does this contradict the result in **Problem 2**?

Proof. We notice that $\frac{\pi}{z}$ is not analytic for any disk $|z - z_0| < 1$. Therefore, we fail the condition of **Problem 2(i)**. $\square$

4 Problem 4

Find the order of each of the zeros of the given functions:

(a) $(z^2 - 4z + 4)^2$

(b) $z^2(1 - \cos(z))$

(c) $e^{2z} - 3e^z - 4$

Proof. Functions f that are analytic at a point z_0 has a zero of order m at z_0 if and only if there is a function g, which is analytic and nonzero at z_0 such that

$$f(z) = (z - z_0)^m g(z)$$

(a) Therefore, we can factor simplify this to get

$$((z-2)^2)^2 = (z-2)^4$$

which makes it clear that we have a $g(z) = 0$ and $z_0 = 2$, from which we can conlude we have a zero $m = 4$.

(b) Using the Taylor exapnsion of $\cos z$ about $z_0 = 0$, we have that:

$$\begin{aligned} z^2(1-\cos(z)) &= z^2\left[1 - \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + ...\right)\right] \\ &= z^2\left(\frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} + ...\right) \\ &= z^4\left(\frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!} + ...\right) \quad \text{(factoring out a } z^2\text{)} \end{aligned} \tag{1}$$

From here, we have the form we wanted where we let our multiplicand be $(z - z_0) = (z - 0)^4$, and letting $g(z)$ be the multiplier which is $\frac{1}{2!}$ when $z_0 = 0$, i.e. nonzero. Therefore, our m or the order of zero is 4. Furthermore, we have a zero of order 2 at $z = 2\pi n$ where $n \in \mathbb{Z}$ since the derivative of $(1 - \cos(z))$ is 0 at $z = 2\pi n$ where $n \in \mathbb{Z}$

(c) Similar to **(a)**, we can factor this to get $(e^z - 4)(e^z + 1)$. Here we can solve for z individually, and get $e^z = 4 \Rightarrow z = \ln(4)$, so we have a zero of order 1 at $\ln(4)$. Also, $e^z = -1 \Rightarrow z = \ln(-1) = i\pi + 2\pi n$ where $n \in \mathbb{Z}$ giving us a zero of order 1 at $i\pi$.

$\square$

5 Problem 5

Locate the isolated singularity of the given function and tell whether it is a removable singularity, a pole, or an essential singularity.

(a) $\dfrac{e^z - 1}{z}$

(b) $\dfrac{z^2}{\sin(z)}$

(c) $\dfrac{e^z - 1}{e^{2z} - 1}$

(d) $\dfrac{z^4 - 2z^2 + 1}{(z-1)^2}$

Proof. If a function f has an isolated singular point at z_0, then it's Laurent series form is:

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0) + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + ... + \frac{b_n}{(z - z_0)^n} + ...$$

When all $b_n = 0$, then we have a removable singular point z_0. If we have $n \geq 1$, where the b_n terms are nonzero, and n is finite, then we have a pole of order n. Finally if we have an infinite number of b_n, which are nonzero, then z_0 is an essential singular point of f.

(a) This has a singularity at $z_0 = 0$, therefore taking the Taylor expansion of e^z about z_0 gives:

$$\begin{aligned}\frac{e^z - 1}{z} &= \frac{\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + ...\right) - 1}{z} \quad \text{(subtracting 1)} \\ &= \frac{z + \frac{z^2}{2!} + \frac{z^3}{3!} + ...}{z} \quad \text{(dividing by } z\text{)} \\ &= 1 + \frac{z}{2!} + \frac{z^2}{3!} + \frac{z^3}{4} + ...\end{aligned} \tag{2}$$

Here it's clear that we do not have b terms since we do not have terms where $(z - z_0)$ is the denominator. Therefore, $z_0 = 0$ is a removable singular point.

(b) We know that $z = 0$ for $z_0 = 0$. Therefore, expanding about z_0, we get

$$\begin{aligned}\frac{z^2}{\sin(z)} &= \frac{z^2}{z - \frac{z^3}{3!} + \frac{z^5}{5!} - ...} \\ &= \frac{z^2}{z\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - ...\right)} \quad \text{(factoring a } z \text{ in the denominator)} \\ &= z \cdot \frac{1}{\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - ...\right)} \\ &= z \cdot (1 + \frac{z^2}{3!} + \frac{z^4}{5!} + ...) \\ &= z + \frac{z^2}{3!} + \frac{z^4}{5!} + ...\end{aligned} \tag{3}$$

And again, since the our Laurent expansion contains no b_n terms where $(z - z_0)$ is in the denominator, so $z = 0$ is a a removable singular point. Additionally, when we have $z_0 = \pi n$ where $n \in \mathbb{Z} \setminus \{0\}$, we have a pole of order 1 since $\sin(\pi n)$ is zero by a degree greater than the numerator.

(c) For this, we have a pole of order 1 at $z_0 = 2i\pi n$ where $n \in \mathbb{Z}$ for $e^z - 1$ since $e^z = 1$ at $x = 0$ and $y = 2i\pi n$. Similarly, we have a pole of order 1 for $e^{2z} - 1$ at $z_0 = i\pi n$ where $n \in \mathbb{Z}$ since $e^{2z} = 1$ at $x = 0$ and $2y = 2i\pi n$.

(d) Factoring out z^2 from the first two terms in the numerator yields:

$$\begin{aligned}\frac{z^4 - 2z^2 + 1}{(z-1)^2} &= \frac{z^2(z^2 - 2 + 1)}{(z-1)^2} \\ &= \frac{z^2(z-1)(z+1)}{(z-1)^2} \\ &= \frac{z^2(z+1)}{(z-1)}\end{aligned} \tag{4}$$

Here it's clear that we have a we have a removable point at $z = 1$.

□

6 Problem 6

Find the Laurent series for a given function about the point $z = 0$ and find the residue at that point.

(a) $\dfrac{e^z - 1}{z}$

(b) $\dfrac{z}{(\sin(z))^2}$

(c) $\dfrac{1}{e^z - 1}$

(d) $\dfrac{1}{1 - \cos(z)}$

In **(c)** and **(d)** compute only three terms of the Laurent series.
Proof.

(a) We can rewrite this as:

$$\frac{e^z - 1}{z} = \frac{1}{z}(e^z - 1)$$

The Laurent series of e^z at $z = 0$ is:

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + ...$$

Therefore the Taylor expansion of e^z:

$$\begin{aligned} \frac{1}{z}(e^z - 1) &= \frac{1}{z}\left(z + \frac{z^2}{2!} + \frac{z^3}{3!} + ...\right) \\ &= 1 + \frac{z}{2!} + \frac{z^2}{3!} + ... \end{aligned} \tag{5}$$

Since the principal part of the series is 0, our $\text{Res}(f, 0) = 0$

(b) Multiplying by $\frac{z}{z}$ to our equation give us:

$$\frac{z}{(\sin(z))^2} = \frac{z}{\sin(z)} \cdot \frac{z}{\sin(z)} \cdot \frac{1}{z}$$

We notice $\frac{z}{\sin(z)}$ is analytic about 0, and so there exists a Taylor expansion where:

$$\frac{z}{\sin(z)} = a_0 + a_1 z + a_2 z^2 + ...$$

Multiplying $\sin(z)$ to get:

$$z = \sin(z)(a_0 + a_1 z + a_2 z^2 + ...)$$

Expanding $\sin(z)$ yields:

$$z = (z - \frac{z^3}{3!} + \frac{z^5}{5!} - ...)(a_0 + a_1 z + a_2 z^2 + ...)$$

When multiplying out, we note that for the coefficients of each power: at power of $0 \Rightarrow 0 = 0$, at power of $1 \Rightarrow 1 = a_0$, at power of $2 \Rightarrow 0 = a_1$, at power of $3 \Rightarrow 0 = a_2 - \frac{a_0}{3!} \Rightarrow a_2 = \frac{1}{3!}$, at power of $4 \Rightarrow 0 = a_3 - \frac{a_1}{3!}$,... This gives us that:

$$\frac{z}{\sin(z)} = (1 + \frac{1}{6} z^2 + ...)$$

Going back to our original equation, we get:

$$\begin{aligned} \frac{z}{\sin(z)} \cdot \frac{z}{\sin(z)} \cdot \frac{1}{z} &= \left(1 + \frac{1}{6} z^2 + ...\right) \cdot \left(1 + \frac{1}{6} z^2 + ...\right) \cdot \frac{1}{z} \\ &= (1 + \frac{1}{3} z^2 + \frac{1}{36} z^4 + ...) \cdot \frac{1}{z} \\ &= \frac{1}{z} + \frac{1}{3} z^2 + \frac{1}{36} z^3 + ... \end{aligned} \tag{6}$$

From here we see, that the first term of the principal part is $\frac{1}{z}$, therefore our residue is the coffecient 1, i.e. $\text{Res}(f, 0) = 1$

(c) e^z has a Taylor expansion:

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + ... + \frac{z^{(n)}}{n!} + ...$$

Therefore, $e^z - 1$ is given by:

$$\begin{aligned} e^z - 1 &= z + \frac{z^2}{2!} + \frac{z^3}{3!} + ... + \frac{z^{(n)}}{n!} + ... \\ &= z \left(1 + \frac{z}{2!} + \frac{z^2}{3!} + ... + \frac{z^{(n-1)}}{n!} + ...\right) \end{aligned} \tag{7}$$

And so $\frac{1}{1-e^z}$ can be rewritten as:

$$\frac{1}{e^z - 1} = \frac{1}{z} \cdot \underbrace{\frac{1}{\left(1 + \frac{z}{2!} + \frac{z^2}{3!} + ... + \frac{z^{(n-1)}}{n!} + ...\right)}}_{g(0) = 1 \text{ therefore analytic at } 0}$$

Therefore, the $g(z)$ has some Taylor expansion given by:

$$\frac{1}{\left(1 + \frac{z}{2!} + \frac{z^2}{3!} + ... + \frac{z^{(n-1)}}{n!} + ...\right)} = (a_0 + a_1 z + a_2 z^2 + ...)$$

$$1 = \left(1 + \frac{z}{2!} + \frac{z^2}{3!} + ... + \frac{z^{(n-1)}}{n!} + ...\right) \cdot (a_0 + a_1 z + a_2 z^2 + ...)$$

Then similar to **Problem 5(b)**, we can expand by matching the coefficients with respect to their power on the left side. We list this as: at power $0 \Rightarrow 1 = a_0 1 \Rightarrow a_0 = 1$, at power $1 \Rightarrow 0 = a_0 \frac{1}{2!} + a_1 \Rightarrow 0 = \frac{1}{2} + a_1$ at power $0 = a_2 + a_1 \frac{1}{2!} + a_0 \frac{1}{3!} \Rightarrow a_2 = \frac{1}{12}$, ... And so we get that:

$$\begin{aligned} \frac{1}{e^z - 1} &= \frac{1}{z} \cdot \left(1 - \frac{1}{2} z + \frac{1}{12} z^2 + ...\right) \\ &= \frac{1}{z} - \frac{1}{2} + \frac{1}{12} z^2 + ... \end{aligned} \tag{8}$$

And again, the only principal part term of the principal part of our Laurent series has a coefficient of 1 so our $\text{Res}(f, 0) = 1$

(d) The Taylor expansion of

$$\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + ...$$

Therefore, our equation becomes:

$$\begin{aligned} \frac{1}{1 - \cos(z)} &= \frac{1}{1 - \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + ...\right)} \\ &= \frac{1}{\frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} - ...} \\ &= \frac{2}{z^2} \cdot \underbrace{\frac{1}{\left(1 - \frac{2! z^2}{4!} + \frac{2! z^4}{6!} - ...\right)}}_{g(z) \neq 0 \text{ at } z = 0, \text{ so } g(z) \text{ is analytic}} \end{aligned} \tag{9}$$

Since $g(z)$ is analytic, there exists some Taylor expansion where:

$$1 = \left(1 - \frac{2! z^2}{4!} + \frac{2! z^4}{6!} - ...\right) (a_0 + a_1 z + a_2 z^2 + ...)$$

By the same method as **Problem (b) and (c)**, we can expand by matching the coefficients with respect to their power on the left side: for power $0 \Rightarrow 1 = 1a_0 \Rightarrow a_0 = 1$, for power $1 \Rightarrow 0 = a_1 \Rightarrow a_1 = 0$, for power $2 \Rightarrow 0 = \left(\frac{2!}{4!}\right) a_0 + a_2 \Rightarrow a_2 = -\frac{2!}{4!}$, ... Therefore, **Equation (9)** becomes:

$$\begin{aligned} \frac{1}{1-\cos(z)} &= \frac{2!}{z^2}(1 - \frac{2!z^2}{4!} + ...) \\ &= \frac{2!}{z^2} - 1 + ... \end{aligned} \tag{10}$$

Here, we do no have a principal term of $\frac{1}{z}$, therefore, $\text{Res}(f, 0) = 0$.

$\square$

7 Problem 7

Find the residue of $f(z) = \frac{1}{1+z^n}$ at the point $z_0 = e^{i\frac{\pi}{n}}$
Proof. $f(z)$ have singularities at $1 + z^n = 0 \Rightarrow z^n = -1$. Using polar coordinates, we know that $z = -1 = e^{i\pi}$, therefore $z^n = e^{\frac{i\pi}{n}+\frac{2\pi}{n}}$. It follows that we have singularities at $z_1 = e^{\frac{i\pi}{n}}$, $z_2 = e^{i(\frac{\pi}{n}+\frac{2\pi}{n})}$, ..., $z_n = e^{i(\frac{\pi}{n}+\frac{2\pi(n-1)}{n})}$
Therefore, our $f(z)$ is

$$f(z) = \frac{1}{(z-z_1)(z-z_2)...(z-z_n)} = \frac{1}{(z-z_1)} \cdot \underbrace{\frac{1}{(z-z_{2)(z-z_3)...(z-z_n)}}}_{g(z)\neq 0 \text{ at } z_1\text{, therefore analytic}}$$

Where from here, it's clear that $z_1, ..., z_n$ are all simple poles. The power series of $g(z)$ about z_1 is of the form:

$$g(z) = g(z_1) + g'(z_1)(z-z_1) + g''(z_1)\frac{(z-z_1)^2}{2!} + ...$$

By definition, the residue is given by:

$$\text{Res}(f, z_1) = \lim_{z\to z_1}(z-z_1)f(z) = b_1$$

Therefore, plugging in our values:

$$\begin{aligned}
\text{Res}(f, z_1) &= \lim_{z\to z_1}(z-z_1)f(z) \\
&= \lim_{z\to z_1}\frac{(z-z_1)}{1+z^n} \\
&= \lim_{z\to z_1}\frac{1}{\frac{1+z^n}{z-z_1}} \\
&= \frac{1}{nz_1^{n-1}} \quad \text{(By L'Hospital's Rule)}
\end{aligned} \tag{11}$$

□

8 Problem 8

Calculate:

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)(4+x^2)} dx$

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} (= \frac{\pi}{2})$

(c) $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin(ax)}{x^2+b^2} dx (= \pi e^{-ab})$

(d) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin(x)}{x} dx (= \pi)$

(e) $\displaystyle\int_{0}^{2\pi} \frac{dt}{2+\cos^2(t)}$

Proof.

(a) This can rewritten as the improper integral:

$$\int_{-R}^{R} \frac{x^2}{(1+x^2)(4+x^2)} dx + \int_{C_R} \frac{z^2}{(1+z^2)(4+z^2)} dz$$

It follows that:

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)(4+x^2)} dx &= 2i\pi \sum_{k=1}^{n} \mathrm{Res}_{z=z_k} f(z) \\
&= 2i\pi(\mathrm{Res}_{z=i} f(z_i) + \mathrm{Res}_{z=2i} f(z_{2i})) \\
&= 2i\pi(\lim_{z\to i} \frac{(z-1)z^2}{(z-1)(z+1)(z^2+4)} + \lim_{z\to 2i} \frac{(z-2i)z^2}{(z^2+1)(z+2i)(z-2i)}) \\
&= 2i\pi(-\frac{1}{6i} + \frac{1}{3i}) \\
&= 2i\pi(-\frac{1}{6i} + \frac{2}{6i}) \\
&= \frac{\pi}{3}
\end{aligned} \tag{12}$$

(b) Similarly, we have:

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} &= \text{Res}_{z=i} f(z) \\
&= \lim_{z\to i} \frac{d}{dz}\left(\frac{(z-i)^2}{(z+i)^2(z-i)^2}\right) \\
&= 2i\pi \lim_{z\to i} \frac{-2}{(z+i)^3} \\
&= 2i\pi \left(\frac{-2}{4i}\right) \\
&= \frac{\pi}{2}
\end{aligned} \tag{13}$$

(c) This is given by:

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{x\sin(ax)}{x^2+b^2} dx &= \text{Res}_{z=bi} \frac{(z-bi)e^{iaz}}{(z+bi)(z-bi)} \\
&= 2i\pi \lim_{z\to bi} \frac{(z-bi)e^{iaz}}{(z+bi)(z-bi)} \\
&= 2i\pi \frac{e^{-ab}}{2ib} \\
&= \frac{\pi e^{-ab}}{b}
\end{aligned} \tag{14}$$

**Not sure why there's a b in the denominator, I followed the same steps in class 3/8/22.

(d) Letting $\sin(z) = \frac{e^{ix}-e^{-ix}}{2i}$, then using the same technique:

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\sin(x)}{x} dx &= \text{Res}_{z=0} \frac{1}{2i}\frac{ze^{iz}}{z} - \text{Res}_{z=0} \frac{1}{2i}\frac{ze^{-iz}}{z} \\
&= 2i\pi \left[\lim_{z\to 0} \frac{1}{2i}\frac{ze^{iz}}{z} - \lim_{z\to 0} \frac{1}{2i}\frac{ze^{-iz}}{z}\right] \\
&= 2i\pi \left[\frac{1}{2i}e^0 - 0\right] \\
&= \pi
\end{aligned} \tag{15}$$

(e) We know that $\cos(t) = \frac{e^{it}+e^{-it}}{2}$. Parameterizing z implies $z(t) = \gamma(t) = e^{it}$ where $\gamma \in [0, 2\pi]$, and $dz(t) = ie^{it}dt \Rightarrow dt = \frac{dz}{ie^{it}} =$

$fracdziz$.Substituting $\cos(z)$ turns our function into:

$$\begin{aligned}
\int_0^{2\pi} \frac{dt}{2+\cos^2(t)} &= \oint_{|z|=1} \frac{1}{2+\left(\frac{z+\frac{1}{z}}{2}\right)^2} \frac{dz}{iz} \\
&= \oint_{|z|=1} \frac{4}{8+\left(z+\frac{1}{z}\right)^2} \frac{dz}{iz} \\
&= \frac{1}{i} \oint_{|z|=1} \frac{4}{8+z^2+2+\frac{1}{z^2}} \frac{dz}{z} \\
&= -i \oint_{|z|=1} \frac{4z}{z^4+10z^2+1} dz
\end{aligned} \tag{16}$$

Letting $u = z^2$, our equation becomes:

$$\begin{aligned}
-i \oint_{|z|=1} \frac{4z}{z^4+10z^2+1} dz &= -i \oint_{|z|=1} \frac{4u^{-1}}{u^2+10u+1} dz \\
&= -i \oint_{|z|=1} \frac{4u^{-1}}{[u+(-5+2\sqrt{6})][u+(-5-2\sqrt{6})]} dz
\end{aligned} \tag{17}$$

$\square$