Math 122A Homework 6

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1 Problem 1

Let C be a closed, positive, and simple curve. Using Green's Theorem prove that

$$\frac{1}{2i}\int_C \bar{z}dz = \text{area enclosed by } C$$

Proof. Since $\bar{z} = x - iy$, this is equivalent to

$$\frac{1}{2i}\int_C (x-iy)(dx+idy) = \frac{1}{2i}\left[\int_C (xdx+ydy) + i\int_C (xdy - ydx)\right]$$

Where D is the area boundead by C. Now using Green's Theorem, twice we have that

$$\begin{aligned}\frac{1}{2i}&\left[\iint_D (0-0)dxdy + i\iint_D (1-(-1))dxdy\right]\\ &= \iint_D dxdy\end{aligned}$$

Which after evaluating the integrals is exactly the area enclosed by C. $\square$

2 Problem 2

Consider the function $f(z) = (z+1)^2$ and the region R bounded by the triangle with vertices $0, 2, i$ (its boundary and interior). Find the points where $|f(z)|$ reaches its maximum and minimum value of R.

Proof.

$\square$

3 Problem 3

Find the maximum of $|\sin(z)|$ on $[0, 2\pi] \times [0, 2\pi]$.
Proof.

$\square$

4 Problem 4

Calculate:

(a)

$$\int_0^{2\pi} \frac{d\theta}{a + b\cos(\theta)}, \qquad 0 < b < a$$

HINT: Work backwards using $\cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2}$ to convert the integral into a complex integral along the curve $|z| = 1$

(b)

$$\int_0^{2\pi} \frac{d\theta}{(a + b\cos(\theta))^2}$$

(c)

$$\int_0^{2\pi} \frac{\sin(\theta)d\theta}{(a + b\cos(\theta))^2}, \qquad 0 < b < a$$

Proof.

(a)

(b)

(c)

$\square$

5 Problem 5

Prove that if $f : \mathbb{C} \to \mathbb{C}$ is entire such that for some $n \in \mathbb{N}$

$$\lim_{|z|\to\infty} \frac{|f(z)|}{|z|^n} = M < \infty,$$

then f is a polynomial of degree at most n.
Proof.

$\square$

6 Problem 6

Let $A \subset \mathbb{C}$ be an open set and $f : A \to \mathbb{C}$ be an analytic function on A. Assuming that $z_0 \in A$ such that

$$\{z \in \mathbb{C} : |z - z_0| \leq R\}, \quad R > 0$$

prove that

$$f(z_0) = \frac{1}{\pi R^2} \iint_{|z-z_0|\leq R} f(x + iy) dx dy$$

Proof.

$\square$

7 Problem 7

Let $f : R \to R$ be defined as

$$f(x) = e^{\frac{-1}{x^2}} \quad \text{if} \quad x \neq 0, \quad f(0) = 0$$

Show that f is infinitely differentiable $\forall n \in \mathbb{N}$, $f^{(n)}(0) = 0$. Verify that the power series of f at $x = 0$ does not agree with f in any neighborhood of 0.
Proof.

$\square$