Math 119A Homework 4

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1 Problem 1

Prove or disprove that matrix E given by

$$\begin{bmatrix} 0 & -\sqrt{2} \\ \sqrt{2} & 0 \end{bmatrix}$$

with basis $(0, -\sqrt{2}, \sqrt{2})$, and $(1, -2, -1)$ is a two-dimensional matrix $E \subset \mathbb{R}^2$ that satisfies $T|E$ of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. T is given by

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -2 \\ 0 & 1 & 0 \end{bmatrix}$$

Proof. Applying T to our first basis yields

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -\sqrt{2} \\ \sqrt{2} \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ -2\sqrt{2} \\ -\sqrt{2} \end{bmatrix} = \sqrt{2} \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

Similarly, for the second basis we have

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix} = 2 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$\square$

2 Problem 2

Prove or disprove that

$$\begin{aligned} x_1 &= Ce^t - B\cos(\sqrt{2}t) + A\sin(\sqrt{2}t) \\ x_2 &= (2B - A\sqrt{2})\cos(\sqrt{2}t) - B(\sqrt{2} + 2A)\sin(\sqrt{2}t) \\ x_3 &= (B + A\sqrt{2})\cos(\sqrt{2}t) + (B\sqrt{2} - A)\sin(\sqrt{2}t) \end{aligned} \tag{1}$$

is the solution to $x' = Tx$ for the operator T given in Problem 1.
Proof. Taking the derivative of x with respect to t yields

$$\begin{aligned} x_1' &= Ce^t + \sqrt{2}B\sin(\sqrt{2}t) + A\sqrt{2}\cos(\sqrt{2}t) \\ x_2' &= (-2\sqrt{2}B + 2A)\sin(\sqrt{2}t) - (2B + 2\sqrt{2}A)\sin(\sqrt{2}A) \\ x_3' &= (-\sqrt{2}B - 2A)\sin(\sqrt{2}t) + (2B - \sqrt{2}A)\cos(\sqrt{2}t) \end{aligned} \tag{2}$$

Meanwhile Tx yields

$$\begin{aligned} Tx &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} Ce^t - B\cos(\sqrt{2}t) + A\sin(\sqrt{2}t) \\ (2B - A\sqrt{2})\cos(\sqrt{2}t) - (B\sqrt{2} + 2A)\sin(\sqrt{2}t) \\ (B + A\sqrt{2})\cos(\sqrt{2}t) + (B\sqrt{2} - A)\sin(\sqrt{2}t) \end{bmatrix} \\ &= \begin{bmatrix} Ce^t - B\cos(\sqrt{2}t) + A\sin(\sqrt{2}t) + (B + A\sqrt{2})\cos(\sqrt{2}t) + (B\sqrt{2} - A)\sin(\sqrt{2}t) \\ -2\left((B + A\sqrt{2})\cos(\sqrt{2}t) + (B\sqrt{2} - A)\sin(\sqrt{2}t)\right) \\ (2B - A\sqrt{2})\cos(\sqrt{2}t) - (B\sqrt{2} + 2A)\sin(\sqrt{2}t) \end{bmatrix} \\ &= \begin{bmatrix} Ce^t + A\sqrt{2}\cos(\sqrt{2}t) + B\sqrt{2}\sin(\sqrt{2}t) \\ -(2B + 2\sqrt{2}A)\cos(\sqrt{2}t) + (-2B\sqrt{2} + 2A)\sin(\sqrt{2}t) \\ (2B - A\sqrt{2})\cos(\sqrt{2}t) - (B\sqrt{2} + 2A)\sin(\sqrt{2}t) \end{bmatrix} \end{aligned} \tag{3}$$

Which is exactly equation (2), therefore, our system of equations x is the general solution to our differential equation. $\square$

3 Problem 3

Prove or disprove that $A = 1$, $B = \sqrt{n}$ satisfies the largest $A > 0$ and smallest $B > 0$ such that

$$A|x| \leq |x|_{sum} \leq B|x|$$

for all $x \in \mathbb{R}^n$.

Proof. Substituting A and B gives us

$$|x| \leq |x|_{sum} \leq \sqrt{n}|x|$$

The left side of our inequality implies that for $x \in \mathbb{R}^n$:

$$\begin{aligned} |x| &\leq |x|_{sum} \\ \sqrt{x_1^2 + ... + x_n^2} &\leq |x_1| + ... + |x_n| \\ x_1 + ... + x_n &\leq (|x_1| + ... + |x_n|)^2 \quad \textbf{(squaring both sides)} \end{aligned} \tag{4}$$

Which is clearly true. Now for the other side we get that

$$\begin{aligned} |x_1| + ... + |x_n| &\leq \sqrt{n}\sqrt{x_1^2 + ... + x_n^2} \\ (|x_1| + ... + |x_n|)^2 &\leq n(x_1^2 + ... + x_n^2) \end{aligned} \tag{5}$$

Therefore, $B = \sqrt{n}$ is valid. □

4 Problem 4

Prove or disprove that the following

(a) $\sqrt{2}$

(b) $\frac{1}{2}$

(c) 1

(d) $\frac{1}{2}$

are norms to the vector $(1, 1) \in \mathbb{R}^2$ under following

(a) The Euclidean norm;

(b) The Euclidean B-norm, where B, is the basis $\{(1, 2), (2, 2)\}$;

(c) The max norm;

(d) The B-max norm

Proof.

(a) The Euclidean norm of a vector is define as $\sqrt{x_1^2 + ... + x_n^2}$. Therefore, we have that $\sqrt{1^2 + 1^2} = \sqrt{2}$, proving (a).

(b) The Euclidean B-norm is defined as $\|x\|_B = (t_1^2 + ... + t_n^2)^{\frac{1}{2}}$ if $x = \sum_{i=1}^{n} t_j f_j$ where B is a basis in $\mathbb{R}^n$, i.e. $B = \{f_1, ..., f_n\}$. Therefore using this definition, we $n = 2$ and $f_1 = (1, 2)$ and $f_2 = (2, 2)$. I do not know what t_j is? Is this the components of the vector $(1, 1)$?

(c) The max norm is defined as $\|x\|_{\max} = \max\{\|x_1\|, ..., \|x_n\|\}$. Therefore, since we only have one vector, $\|x\|_{\max} = 1$.

(d) The B-max norm is defined as $\|x\|_{B\max} = \max\{\|t_1\|, ..., \|t_n\|\}$.

□

5 Problem 5

Prove or disprove that $(x^2 + xy + y^2)^{\frac{1}{2}}$ and $\frac{1}{2}(|x| + |y|) + \frac{2}{3}(x^2 + y^2)^{\frac{1}{2}}$ are norms defined in $\mathbb{R}^2$.
Proof. Norms must be functions $N : \mathbb{R}^n \to \mathbb{R}$ that satisfy:

(1) $N(x) \geq 0$ and $N(x) = 0$ if and only if $x = 0$

(2) $N(x + y) \leq N(x) + N(y)$

(3) $N(\alpha x) = |\alpha| N(x)$.

Therfore, first checking (1) by letting $(x, y) = (0, 0)$, we have $N(0, 0) = (0^2 + 0 \cdot 0 + 0^2)^{\frac{1}{2}} = 0 \geq 0$ and $N(0, 0) = \frac{1}{2}(|0| + |0|) + \frac{2}{3}(0^2 + 0^2)^{\frac{1}{2}} = 0 \geq 0$ showing that both norms satisfy (1). Now taking arbitrary values (x_1, y_1) and (x_2, y_2) gives us

$$N(x_1 + x_2, y_1 + y_2) = \left((x_1 + x_2)^2 + (x_1 + x_2)(y_1 + y_2) + (y_1 + y_2)^2\right)^{\frac{1}{2}}$$

Meanwhile, we have

$$N(x_1, y_2) + N(x_2, y_2) = (x_1^2 + x_1 y_1 + y_1^2)^{\frac{1}{2}} + (x_2^2 + x_2 y_2 + y_2^2)^{\frac{1}{2}}$$

Squaring both equations yields:

$$(x_1 + x_2)^2 + (x_1 + x_2)(y_1 + y_2) + (y_1 + y_2)^2$$

and

$$(x_1^2 + x_1 y_1 + y_1^2) + (x_1^2 + x_1 y_1 + y_1^2)(x_2^2 + x_2 y_2 + y_2^2) + (x_2^2 + x_2 y_2 + y_2^2)$$

For the other norm we have that

$$N(x_1+x_2,y_1+y_2)=\frac{1}{2}(|x_1+x_2|+|y_1+y_2|)+\frac{2}{3}((x_1+x_2)^2+(y_1+y_2)^2)^{\frac{1}{2}}$$

Finally, taking some arbitrary scalar α, we get that

$$\begin{aligned} N(\alpha x,\alpha y) &= \left((\alpha x)^2+\alpha^2 xy+(\alpha y)^2\right)^{\frac{1}{2}} \\ &= \left(\alpha^2(x^2+xy+y^2)\right)^{\frac{1}{2}} \\ &= \alpha\left(x^2+xy+y^2\right)^{\frac{1}{2}} = |\alpha|N(x,y) \end{aligned} \tag{6}$$

Similarly for the second norm,

$$\begin{aligned} N(\alpha x,\alpha y) &= \frac{1}{2}(|\alpha x|+|\alpha y|)+\frac{2}{3}((\alpha x)^2+(\alpha y)^2)^{\frac{1}{2}} \\ &= \frac{1}{2}(\alpha(|x|+|y|))+\frac{2}{3}(\alpha^2(x^2+y^2))^{\frac{1}{2}} \\ &= \alpha\frac{1}{2}(|x|+|y|)+\alpha\frac{2}{3}(x^2+y^2)^{\frac{1}{2}} \\ &= \alpha\left(\frac{1}{2}(|x|+|y|)+\frac{2}{3}(x^2+y^2)^{\frac{1}{2}}\right) = |a|N(x,y) \end{aligned} \tag{7}$$

Satisfying (3). $\square$

6 Problem 6

Prove or disprove that 1 is the uniform norm of the following operator in $\mathbb{R}^2$

$$\begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix}$$

Proof. Uniform norm of T is defined as $\|T\| = \max\{Tx \mid |x| \le 1\}$. Therefore, taking an arbitrary vector $x=(x_1,x_2)$ and applying T, we have

$$\begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ -4x_2 \end{bmatrix}$$

Letting $x=(1,1)$, we get $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$ so certainly, 1 is the maximum x that would satisfy our definition. $\square$

7 Problem 7

In the vector space $L(\mathbb{R}^2)$, let T be the transformation defined by

$$T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c-3a & d-3b \end{bmatrix}$$

(a) is T linear?

(b) Does there exist a 2×2 matrix A such that $AB = T(B)$ for all 2×2 matrices B?

(c) Does there exist a 2×2 matrix A such that $BA = T(B)$ for all 2×2 matrices B?

Proof.

(a) For T to be a linear transformation, we must satisfy $T(u+v) = T(u) + T(v)$ where $u, v \in L(\mathbb{R}^2)$ and for some arbitrary scalar c, $T(cu) = cT(u)$. Therefore, taking two arbitrary vectors $u = (u_1, u_2) \in L(\mathbb{R}^2)$ and $v = (v_1, v_2) \in L(\mathbb{R}^2)$ and we get that for

$$\begin{aligned} T(u+v) &= \begin{bmatrix} a & b \\ c-3a & d-3b \end{bmatrix} \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \end{bmatrix} \\ &= \begin{bmatrix} a(u_1+v_1)+b(u_2+v_2) \\ (c-3a)(u_1+v_1)+(d-3b)(u_2+v_2) \end{bmatrix} \\ &= \begin{bmatrix} au_1+bu_2 \\ (c-3a)u_1+(d-3b)u_2 \end{bmatrix} \begin{bmatrix} av_1+bv_2 \\ (c-3a)v_1+(d-3b)v_2 \end{bmatrix} \\ &= \begin{bmatrix} a & b \\ c-3a & d-3b \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} a & b \\ c-3a & d-3b \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\ &= T(u) + T(v) \end{aligned} \tag{8}$$

satisfying our first requirement. Now for the second requirement, we an arbitrary scalar λ and vector $u = (u_1, u_2) \in L(\mathbb{R}^2)$, we see

$$\begin{aligned} cT(u) &= \lambda \begin{bmatrix} a & b \\ c-3a & d-3b \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ &= \begin{bmatrix} \lambda a u_1 + \lambda b u_2 \\ \lambda(c-3a)u_1 + \lambda(d-3b)u_2 \end{bmatrix} \\ &= \begin{bmatrix} a & b \\ c-3a & d-3b \end{bmatrix} \begin{bmatrix} \lambda u_1 \\ \lambda u_2 \end{bmatrix} \\ &= \begin{bmatrix} a & b \\ c-3a & d-3b \end{bmatrix} \lambda \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ &= T(\lambda u) \end{aligned} \tag{9}$$

satisfying our second requirement, so our our transformation T is linear.

(b)

□

8 Problem 8

Show that

$$\|T\| \cdot \|T^{-1}\| \geq 1$$

for every invertible operator T.

Proof. We can prove this by letting x be a non-zero vector. Then we let $u = \frac{x}{\|x\|}$, and we have that $\|T^{-1}x\| = \|T^{-1}\frac{x}{\|x\|}\| = \|T^{-1}U\| \leq \|T^{-1}\| \cdot \|u\| = \|T^{-1}\|$. On the other hand, taking $v = \frac{x}{\|x\|}$, we have $\|Tx\| = \|T(\frac{x}{\|x\|})\| = \|Tv\| \leq \|T\| \cdot \|v\| = \|T\|$. Therefore, since T is invertible, it implies that $\|x\| = \|T^{-1}Tx\| \leq \|T^{-1}\| \cdot \|T\|x \leq \|T^{-1}\|\|T\| \cdot \|x\|$. Now dividing both sides by $\|x\|$, since x is non-zero, we have $\|T\| \cdot \|T^{-1}\| \geq 1$ proving our statement. □

9 Problem 9

Let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be an operator that leaves a subspace $E \subset \mathbb{R}^2$ invariant. Let $x : \mathbb{R} \to \mathbb{R}^2$ be a solution of $x' = Ax$. If $x(t_0) \in E$ for some $t_0 \in \mathbb{R}$, show that $x(t) \in E$ for all $t \in \mathbb{R}$.

Proof.

□

10 Problem 10

Suppose $A \in L(\mathbb{R}^2)$ has a real eigenvalue $\lambda < 0$. Then the equation $x' = Ax$ has at least one nontrivial solution $x(t)$ such that

$$\lim_{t\to\infty} x(t) = 0$$

.

Proof. Is this a question? $\square$