Math 122A Homework 2

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1 Problem 1

Consider $f : \mathbb{C} \to \mathbb{C}$ defined as $f(z) = z^2$, i.e. $(x, y) \to (u(x, y), v(x, y))$ with

$$u(x, y) = \Re f(z) = \text{ (Real part of } f(z)) = x^2 - y^2$$

and

$$v(x, y) = \Im f(z) = \text{ (Imaginary part of } f(z)) = 2xy$$

Prove that $f : \mathbb{C} \sim \mathbb{R}^2 \to \mathbb{C} \sim \mathbb{R}^2$

(a) maps the line $x = a_0$ (constant) onto the parabola $u = a_0^2 - \frac{v^2}{4a_0^2}$.

(b) maps the line $y = b_0$ (constant) onto the parabola $u = -b_0^2 + \frac{v^2}{4b_0^2}$.

(c) maps the hyperbola $x^2 - y^2 = c_0$ (constant) onto the line $u = c_0$.

(d) maps the hyperbola $xy = d_0$ (constant) onto the line $v = 2d_0$.

Proof.

(a) Since $(x, y) \to (u(x, y), v(x, y))$ we have that $u = a_0^2 - y^2$ where we can find y by setting $v = 2xy = 2a_0y \Rightarrow y = \frac{v}{2a_0}$. Therefore, $u = a_0^2 - (\frac{v}{2_0})^2 = a_0^2 - \frac{v^2}{4a_0^2}$

(b) Similarly to **(a)**, we have that $u = x^2 - {b_0}^2$ where we can find using $v = 2xy = 2xb_0 \Rightarrow x = \frac{v}{2b_0}$. Therefore, $u = \frac{v^2}{4b_0^2} - b_0^2 = -b_0^2 + \frac{v^2}{4b_0^2}$

(c) Letting $u = c_0$, we have that $u(x, y) = c_0 = x^2 - y^2$

(d) Similarly to **(c)**, letting $v(x, y) = 2d_0$ then dividing by 2 to both sides we have that $v = 2d_0 = 2xy \Rightarrow v = xy$.

$\square$

2 Problem 2

Using that lines and circles in $\mathbb{R}^2$ are given by the equation

$$Ax + By + C(x^2 + y^2) = D, \qquad A, B, C, D \in \mathbb{R}$$

Prove that the function $f : \mathbb{C} - \{0\} \to \mathbb{C}$ defined as $f(z) = \frac{1}{z}$ maps any line and any circle onto a line or a circle.

Proof. Letting the equation of a line be $y = mx + b$ implies that $mx - y = -b$. Now letting $C = 0$, $B = -1$, $A = m$, and $D = -b$, we have $(m)x + (-1)y + 0 \cdot (x^2 + y^2) = -b \Rightarrow mx - y = -b \Rightarrow y = mx + b$. Now for the circle, we begin with the circle equation $(x - x_0)^2 + (y - y_0^2) = r^2$ where x_0, y_0 are the center and $r \geq 0$ is the radius. Expanding this equation we have that $x^2 - 2x_0x + x_0^2 + y^2 - 2y_0y + y_0^2 = r^2$. Moving the constants to the right side yields, $x^2 - 2x_0x + y_2 - 2y_0y = r^2 - x_0^2 - y_0^2$. Here the right side of the equation be D, $A = -2x_0$, $B = -2y_0$, $C = 1$. Now letting $f = \frac{1}{z}$ where $f : \mathbb{C} - \{0\} \to \mathbb{C}$, we can multiply with the conjugate to get $f = \frac{1}{z} \cdot \frac{\overline{z}}{\overline{z}} = \frac{x - iy}{x^2 + y^2}$. Splitting the terms we have that

$$f(z) = \underbrace{\frac{x}{x^2 + y^2}}_{u} + i\underbrace{\frac{-y}{x^2 + y^2}}_{v}$$

Now suppose (x, y) satisfy

$$Ax + By + C(x^2 + y^2) = D, \qquad A, B, C, D \in \mathbb{R}$$

and we let $\Omega = \{(x, y) : Ax + By + C(x^2 + y^2) = D\}$. Then there exists $f(\Omega) = \{(x, y) : A'u + B'v + C'(u^2 + v^2) = D'\}$. Now plugging in an arbitrary element of Ω we get that:

$$A'\frac{x}{x^2 + y^2} + B'\frac{-y}{x^2 + y^2} + C'(\frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2}) = D'$$

$$A'\frac{x}{x^2 + y^2} + B'\frac{-y}{x^2 + y^2} + C'(\frac{1}{x^2 + y^2}) = D'$$

This implies

$$A'x - B'y + C' = D'(x^2 + y^2)$$

and we see that $A' = A$, $B' = -B$, $C' = -D$, $D' = -C$. $\square$

3 Problem 3

Prove:

$$\text{(a)} \lim_{z\to 0} \frac{\overline{z}}{z} \textit{ does not exist} \quad \textit{and} \quad \text{(b)} \lim_{z\to 0} \frac{\overline{zz}}{z} = 0$$

Proof.

(a) Letting $z = x + iy$, we have that

$$\lim_{x+iy\to 0} \frac{x - iy}{x + iy}$$

Splitting into two cases, for $x \to 0$, we have that $\lim_{x\to 0} \frac{0-iy}{0-iy} = -1$, meanwhile for $y \to 0$, we have that $\lim_{y\to 0} \frac{x-i0}{x+i0} = 1$. Since we have two different limits, the limit cannot exist.

(b) Similarly, letting $z = x + iy$, we have

$$\lim_{x+iy\to 0} \frac{(x - iy)(x - iy)}{x + iy}$$

$$\lim_{x+iy\to 0} \frac{(x - y)^2 - 2ixy}{x + iy}$$

Now taking the limit for the real part, we have

$$\lim_{x\to 0} \frac{y^2 - 0}{0 - iy} = 0$$

And for the imaginary term, we have

$$\lim_{x\to 0} \frac{x^2 - 0}{x + 0} = 0$$

Therefore, $\lim_{z\to 0} \frac{\overline{zz}}{z} = 0$.

$\square$

4 Problem 4

Using induction and limit properties, prove:

$$\lim_{z \to w} z^n = w^n, \quad \forall n \in \mathbb{N}$$

Proof. By way of induction, we proceed with checking the base case $n = 1$. This gives us that

$$\lim_{z \to w} z = w, \quad \forall n \in \mathbb{N}$$

Using delta-epsilon definition of limits, we take some function $f(z)$ where $\lim_{z \to w} f(z) = L$. Then for some $\epsilon > 0$, there exists δ such that $L|z - w| < \delta$ implying $|f(z) - L| < \epsilon$ where $f(z) = z$ and $L = w$ which proves our case for $n = 1$. Now for inductive step, we assume that n holds, and we now check for

$$\lim_{n \to \infty} z^{n+1} = w^{n+1}$$

We can split this to be $\lim_{z \to w} z^n \cdot \lim_{z \to w} z$. From this we see that the multiplicand is our assumption, meanwhile the multiplier is our base case, thereby proving that

$$\lim_{z \to w} z^n = w^n, \quad \forall n \in \mathbb{N}$$

using induction. $\square$

5 Problem 5

For any $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. Define

$$T_A : \mathbb{C} - \left\{-\frac{d}{c}\right\} \to \mathbb{C} \ \textit{if}\ c \neq 0, \quad T_A : \mathbb{C} \to \mathbb{C} \ \textit{if}\ c = 0$$

with

$$T_A(z) = \frac{az+b}{cz+d}, \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1}$$

Prove:

(a) If $c \neq 0$, then

$$\lim_{z \to \infty} T_A(z) = \frac{a}{c}, \qquad \lim_{z \to -\frac{d}{c}} T_A(z) = \infty$$

(b) $T_A : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ is one-to-one and onto.

(c) $(T_A)^{-1} = T_{A^{-1}}$

(d) $T_A T_B = T_{AB}$

(e) T_A maps circles and lines onto circles or lines.

HINT: Prove that $T_A = T_4 T_3 T_2 T_1$, where

$$T_1(z) = z + \frac{d}{c}, T_2(z) = \frac{1}{z}, T_3(z) = \frac{(bc-ad)z}{c^2}, T_4(z) = \frac{z+a}{c}$$

and use problem 2.

Proof.

(a) Since $c \neq 0$, then T_A is given by

$$T_A : \mathbb{C} - \left\{-\frac{d}{c}\right\} \to \mathbb{C}$$

This gives us that $T_A(z) = \frac{az+b}{cz+d}$

(b)

(c)

(d)

(e)

$\square$