Math 104A Homework 6

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1.) Construct orthogonal polynomials of degree 0, 1, and 2 on the interval $(0,1)$ with respect to the weight function:

a.) $w(x) = \log\frac{1}{x}$
b.) $w(x) = \frac{1}{\sqrt{x}}$

Proof:

a.) Letting $\{g_1, g_2, \dots, g_n\}$ be an orthonormal system in an inner-product space E, we can construct orthogonal polynomials $p_0(x), p_1(x), \dots, p_n(x)$ defined as:

$$p_n(x) = (x - a_n)p_{n-1}(x) - b_n p_{n-2}(x) \quad (n \geq 2) \quad p_0(x) = 1, \quad p_1(x) = x - a_1$$

Where our unknowns are given by:

$$a_n = \frac{\langle xp_{n-1}\rangle, p_{n-1}\rangle}{\langle p_{n-1}, p_{n-1}\rangle}$$

$$b_n = \frac{\langle xp_{n-1}\rangle, p_{n-2}\rangle}{\langle p_{n-2}, p_{n-2}\rangle}$$

$$\langle f, g\rangle = \int_a^b f(x)g(x)w(x)dx$$

Therefore, we have that $p_0 = 1$, $p_1 = x - a_1$, and $p_2 = (x - a_2)p_1(x) - b_2 p_0(x) = (x-a_2)(x-a_1) - b_2$. Now solving for a_1, a_2, b_2:

$$a_1 = \frac{\langle x, 1\rangle}{\langle 1, 1\rangle} = \frac{\int_0^1 x\log\frac{1}{x}dx}{\int_0^1 \log\frac{1}{x}dx} = \frac{\frac{1}{4}}{1} = \frac{1}{4}$$

$$a_2 = \frac{\langle x(x-\frac{1}{4}), (x-\frac{1}{4})\rangle}{\langle (x-\frac{1}{4}), (x-\frac{1}{4})\rangle} = \frac{\int_0^1 x\left(x-\frac{1}{4}\right)^2\log\frac{1}{x}dx}{\int_0^1 \left(x-\frac{1}{4}\right)^2\log\frac{1}{x}dx} = \frac{\frac{13}{576}}{\frac{7}{144}} = \frac{13}{28}$$

$$b_2 = \frac{\langle x(x-\frac{1}{4}), 1\rangle}{\langle 1, 1\rangle} = \frac{\int_0^1 \left(x^2 - \frac{1}{4}x\right)\log\frac{1}{x}dx}{\int_0^1 \log\frac{1}{x}dx} = \frac{\frac{7}{144}}{1} = \frac{7}{144}$$

Altogether, for degree $0 \Rightarrow p_0(x) = 1$, for degree $1 \Rightarrow p_1(x) = x - \frac{1}{4}$, for degree $2 \Rightarrow p_2(x) = (x - \frac{13}{28})(x - \frac{1}{4}) - \frac{7}{144}$

b.) Similarly, we find our a_1, a_2, b_2:

$$a_1 = \frac{\langle x, 1\rangle}{\langle 1, 1\rangle} = \frac{\int_0^1 x\frac{1}{\sqrt{x}}dx}{\int_0^1 \frac{1}{\sqrt{x}}dx} = \frac{\frac{2}{3}}{2} = \frac{1}{3}$$

$$a_2 = \frac{\langle x\left(x-\frac{1}{3}\right), \left(x-\frac{1}{3}\right)\rangle}{\langle \left(x-\frac{1}{3}\right), \left(x-\frac{1}{3}\right)\rangle} = \frac{\int_0^1 x\left(x-\frac{1}{3}\right)^2\frac{1}{\sqrt{x}}dx}{\int_0^1 \left(x-\frac{1}{3}\right)^2\frac{1}{\sqrt{x}}dx} = \frac{\frac{88}{945}}{\frac{8}{45}} = \frac{11}{21}$$

$$b_2 = \frac{\langle x(x-\frac{1}{3}), 1\rangle}{\langle 1, 1\rangle} = \frac{\int_0^1 \left(x^2 - \frac{1}{3}x\right)\frac{1}{\sqrt{x}}dx}{\int_0^1 \frac{1}{\sqrt{x}}dx} = \frac{\frac{8}{45}}{2} = \frac{4}{45}$$

So we conclude with $p_0(x) = 1$, $p_1(x) = x - \frac{1}{3}$, $p_2(x) = \left(x - \frac{11}{21}\right)\left(x - \frac{1}{3}\right) - \frac{4}{45}$

2.) In each of the following, find the least square approximation of degrees 0, 1, and 2 for the function $f(x) = \sin(x)$ on the interval (a,b) with respect to the weight function $w(x) = 1$.

a.) (a,b) = $(-\pi, \pi)$
b.) (a,b) = $(-\frac{\pi}{2}, \frac{\pi}{2})$

Proof:

a.) Similar to **Homework 5, Problem 6**, we use:

$$E(a_0, a_1, \dots, a_n) = \sum_{j=1}^{m}(y_i - \sum_{k=0}^{n} a_k x_i^k)^2$$

Then for degree 0, we have the following:

$$E(a_0) = \int_{-\pi}^{\pi}(\sin(x) - a_0)^2 \Rightarrow \frac{\partial E}{a_0} = 2\int_{-\pi}^{\pi}(\sin(x) - a_0)dx = 0$$
$$\Rightarrow (-\cos(x) - a_0 x)\Big|_{-\pi}^{\pi} = 0$$
$$\Rightarrow (-\cos(\pi) - a_0\pi) - (\cos(-\pi) + a_0\pi) = 0$$
$$\Rightarrow 2a_0\pi = 0 \Rightarrow a_0 = 0$$

So for degree 0, $p_0(x) = 0$
For degree 1, we have:

$$E(a_0, a_1) = \int_{-\pi}^{\pi}(\sin(x) - (a_0 + a_1))^2$$

Solving for a_0:

$$\frac{\partial E}{\partial a_0} = -2\int_{-\pi}^{\pi}(\sin(x) - a_0 - a_1 x)dx = 0$$
$$\Rightarrow (-\cos(x) - a_0 x - \frac{1}{2}a_1 x^2)\Big|_{-\pi}^{\pi} = 0$$
$$\Rightarrow (1 - a_0\pi - \frac{1}{2}a_1\pi^2) - (1 + a_0\pi - \frac{1}{2}a_1\pi^2) = 0$$
$$\Rightarrow -2a_0\pi = 0 \Rightarrow a_0 = 0$$

Then for a_1:

$$\frac{\partial E}{\partial a_1} = -2\int_{-\pi}^{\pi}(\sin(x) - a_0 - a_1 x)(x)dx = 0$$
$$\Rightarrow \int_{-\pi}^{\pi} x\sin(x)dx - \int_{-\pi}^{\pi} a_0 x dx - \int_{-\pi}^{\pi} a_1 x^2 dx = 0$$
$$\Rightarrow (\sin(x) - x\cos(x))\Big|_{\pi}^{0} - \frac{1}{2}a_0 x^2\Big|_{\pi}^{0} - \frac{1}{3}a_1 x^3\Big|_{\pi}^{0} = 0$$
$$\Rightarrow 2\pi - 0 - \frac{2\pi^3}{9}a_1 = 0 \Rightarrow a_1 = \frac{9}{\pi^2}$$

Therefore, for degree 1, $p_0(x) = \frac{9}{\pi^2}x$
Then, for degree 2:

$$E(a_0, a_1, a_2) = \int_{-\pi}^{\pi}(\sin(x) - a_0 - a_1 x - a_2 x^2)dx = 0$$

For one equation,

$$\frac{\partial E}{\partial a_0} = -2\int_{-\pi}^{\pi}(\sin(x) - a_0 - a_1 x - a_2 x^2)dx = 0$$
$$\Rightarrow (-\cos(x) - a_0 x - \frac{1}{2}a_1 x^2 - \frac{1}{3}x^3)\Big|_{-\pi}^{\pi} = 0$$
$$\Rightarrow -2\pi a_0 - \frac{2}{3}\pi^3 a_2 = 0 \Rightarrow a_0 = \frac{1}{3}a_2$$

Then the next,

$$\frac{\partial E}{\partial a_1} = -2\int_{-\pi}^{\pi}(\sin(x) - a_0 - a_1 x - a_2 x^2)(x)dx = 0$$
$$\Rightarrow \int_{-\pi}^{\pi}(x\sin(x) - a_0 x - a_1 x^2 - a_2 x^3)dx = 0$$
$$\Rightarrow (\sin(x) - x\cos(x) - \frac{1}{2}a_0 x^2 - \frac{1}{3}a_1 x^3 - \frac{1}{4}a_2 x^4)\Big|_{-\pi}^{\pi} = 0$$
$$\Rightarrow 2\pi - \frac{2}{3}a_1\pi^3 = 0 \Rightarrow a_1 = \frac{3}{\pi^3}$$

Finally,

$$\frac{\partial E}{\partial a_2} = -2\int_{-\pi}^{\pi}(\sin(x) - a_0 - a_1 x - a_2 x^2)(x^2)dx = 0$$
$$\Rightarrow \int_{-\pi}^{\pi}(x^2\sin(x) - a_0 x^2 - a_1 x^3 - a_2 x^4)dx = 0$$
$$\Rightarrow (2x\sin(x) + (2 - x^2)\cos(x) - \frac{1}{3}a_0 x^3 - \frac{1}{4}a_1 x^4 - \frac{1}{5}a_2 x^5)\Big|_{-\pi}^{\pi} = 0$$
$$\Rightarrow -\frac{2}{3}\pi^3 a_0 - \frac{2}{5}\pi^5 a_2 = 0 \Rightarrow a_0 = \frac{3}{5}\pi^2 a_2$$

Therefore, the system of equations gives us that $a_0 = 0$, $a_1 = \frac{3}{\pi^3}$, $a_2 = 0$. So for degree 2, $p_2 = \frac{3}{\pi^3}x$

b.) Again, for degree 0,

$$E(a_0) = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}(\sin(x) - a_0)^2 \Rightarrow \frac{\partial E}{a_0} = 2\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}(\sin(x) - a_0)dx = 0$$
$$\Rightarrow (-\cos(x) - a_0 x)\Big|_{\frac{-\pi}{2}}^{\frac{\pi}{2}} = 0 \Rightarrow -\pi a_0 = 0 \Rightarrow a_0 = 0$$

So, $p_0(x) = 0$.
For degree 1:

$$E(a_0, a_1) = \int_{-\pi}^{\pi}(\sin(x) - (a_0 + a_1))^2$$
$$\Rightarrow \frac{\partial E}{\partial a_0} = -2\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}(\sin(x) - a_0 - a_1 x)dx = 0$$
$$\Rightarrow (x\sin(x) - a_0 x - \frac{1}{2}a_1 x^2)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 0$$
$$\Rightarrow \pi a_0 = 0 \Rightarrow a_0 = 0$$
$$\frac{\partial E}{\partial a_1} = -2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\sin(x) - a_0 - a_1 x)(x)dx = 0$$
$$\Rightarrow \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(x\sin(x) - a_0 x - a_1 x^2 - a_2 x^3)dx = 0$$
$$\Rightarrow (\sin(x) - x\cos(x) - \frac{1}{2}a_0 x^2 - \frac{1}{3}a_1 x^3)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 0$$
$$\Rightarrow 2 - \frac{\pi^3}{12}a_1 = 0 \Rightarrow a_1 = \frac{6}{\pi^3}$$

Therefore, $p_1 = \frac{6}{\pi^3}x$.
Lastly, for degree 2:

$$E(a_0, a_1, a_2) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\sin(x) - a_0 - a_1 x - a_2 x^2)dx = 0$$
$$\frac{\partial E}{\partial a_0} = -2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\sin(x) - a_0 - a_1 x - a_2 x^2)dx = 0$$
$$\Rightarrow (-\cos(x) - a_0 x - \frac{1}{2}a_1 x^2 - \frac{1}{3}x^3)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 0$$
$$\Rightarrow -\frac{1}{2}\pi a_0 - \frac{1}{12}\pi^3 a_1 \Rightarrow a_0 = \frac{1}{6}a_1$$
$$\frac{\partial E}{\partial a_1} = -2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\sin(x) - a_0 - a_1 x - a_2 x^2)(x)dx = 0$$
$$\Rightarrow \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(x\sin(x) - a_0 x - a_1 x^2 - a_2 x^3)dx = 0$$
$$\Rightarrow (\sin(x) - x\cos(x) - \frac{1}{2}a_0 x^2 - \frac{1}{3}a_1 x^3 - \frac{1}{4}a_2 x^4)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 0$$
$$\Rightarrow 2 - \frac{1}{12}a_1\pi^3 = 0 \Rightarrow a_1 = \frac{24}{\pi^3}$$
$$\frac{\partial E}{\partial a_2} = -2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\sin(x) - a_0 - a_1 x - a_2 x^2)(x^2)dx = 0$$
$$\Rightarrow \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(x^2\sin(x) - a_0 x^2 - a_1 x^3 - a_2 x^4)dx = 0$$
$$\Rightarrow (2x\sin(x) + (2 - x^2)\cos(x) - \frac{1}{3}a_0 x^3 - \frac{1}{4}a_1 x^4 - \frac{1}{5}a_2 x^5)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 0$$
$$\Rightarrow -\frac{1}{12}\pi^3 a_0 - \frac{1}{80}\pi^5 a_2 = 0 \Rightarrow a_0 = \frac{1}{20}\pi^2 a_2$$

We conclude, $p_2 = \frac{24}{\pi^3}x$

3.) Let $x_0, x_1, \dots, x_n$ be distinct points in $[a,b]$ and we interpolate a function $f(x)$ by a polynomial $p_n(x)$ by Lagrange interpolation method. Let $\{l_j(x)^{(n)}\}_{j=0}^n$ be the Lagrange polynomials used in Lagrange interpolation. Prove $\sum_{j=0}^n l_j(x)^{(n)} = 1$ for all x.

Proof:

We recall that $l_j(x)$ is defined as:

$$l_j(x) = \prod_{j=0, j\neq i}^{n} \frac{x - x_j}{x_i - x_j} \quad (0 \leq i \leq n)$$

And the Lagrange polynomial is:

$$p(x) = \sum_{j=0}^{n} y_j l_j(x)$$

Suppose we interpolate the function $y(x) = 1$, then $1 = \sum_{j=1}^n (1)l_j(x) = y(x)$, thereby showing what we want and interpolating our function perfectly. Furthermore, since $p(x)$ is unique, it must always be the case that $\sum_{j=0}^n l_j(x) = 1$ for all x.

4.) Let $x_0, x_1, \dots, x_n$ be distinct points in $[a,b]$. Show that if f and its first derivatives are defined respectively at points $x_0, x_1, \dots, x_n$, then there **exists a unique** polynomial q of degree at most $2n+1$ such that

$$q(x_j) = f(x_j), \quad \text{and} \quad q'(x_j) = f'(x_j)$$

for $j = 0, 1, \dots, n$.

Proof:

Assume there exists a different polynomial $t(x)$ where $\mathrm{Deg}(t(x)) \leq 2n+1$ such that

$$t(x_j) = f(x_j), \quad \text{and} \quad t'(x_j) = f'(x_j)$$

for $j = 0, 1, \dots, n$. Then the polynomial $r(x) = t(x) - q(x) = 0$ and $r'(x) = t'(x) - q'(x) = 0$ is a polynomial of degree $2n+1$ with roots of multiplicity 2 of $x_0, x_1, \dots, x_n$. To prove uniqueness, we let $s(x)$ be some polynomial, and $r(x) = s(x)(x-x_0)^2(x-x_1)^2 \dots (x-x_n)^2$. Then $s(x)$ must be 0 since $r(x)$ should have n zeros by the Fundamental Theorem of Algebra. If it is the case that $s(x) \neq 0$, then $\mathrm{Deg}(r(x)) > 2n+1$ and $r(x) = 0$. If $r(x) = 0$, it is implied that $q(x) = t(x)$ therefore, $q(x)$ is unique.

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