Math 108C Homework 6

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1 Problem 1

Let S be a symmetric matrix and let λ_1 and λ_2 be two distinct eigenvalues of S. Show that the corresponding eigenspaces are orthogonal.
Proof. Taking two arbitrary eigenvectors u and v, corresponding to each eigenvalue, we have the following:

$$Su = \lambda_1 u \quad Sv = \lambda_2 v$$

For the eigenspaces to be orthogonal, it must be the case that $u \cdot v = 0$ and so:

$$\begin{aligned}
\lambda_1(u \cdot v) &= (\lambda_1 u) \cdot v \\
&= Su \cdot v = (Su)^T v \\
&= u^T S^T v \\
&= u^T S v \quad \text{(by symmetry of } S\text{)} \\
&= u^T \lambda_2 v \\
&= \lambda_2(u \cdot v)
\end{aligned} \tag{1}$$

From here, we see that $\lambda_1(u \cdot v) = \lambda_2(u \cdot v)$ which implies $(\lambda_1 - \lambda_2)(u \cdot v) = 0$. We know that $\lambda_1 \neq \lambda_2 \Rightarrow \lambda_1 - \lambda_2 = 0$, therefore $u \cdot v$ must be 0, implying they are orthogonal. □

2 Problem 5

Show that if S is positive definite, then S is invertible and its inverse is also positive definite.

Proof. Since S is a positive definite, we let $S = QDQ^T$ where D are the diagonal entries. QDQ^T here are invertible which implies that S is invertible as it is a product of invertible matrices. Now

$$S^{-1} = QD^{-1}Q^T$$

Is a postive definite matrix as well since D^{-1} has positive diagonal entries. $\square$

3 Problem 7

For what numbers b is the following matrix positive semi-definite?

$$A = \begin{bmatrix} 2 & -1 & b \\ -1 & 2 & -1 \\ b & -1 & 2 \end{bmatrix}$$

Proof. Letting $b = 2$, we have a semi-positive definite matrix. $\square$