Math 122A Homework 3

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1 Problem 1

Prove that $f(z) = |z|^2$ is not analytic in any open set $A \subset \mathbb{C}$.
Proof. Letting $z = x + iy$, we know that $|z|^2 = x^2 + y^2 + i0$. Taking the real part of $f(z)$ as u we have that $u(x, y) = x^2 + y^2$ and the imaginary as v, $v(x, y) = 0$. For $f(z)$ to be analytic, we know that $\forall z \in A$, $f'(z)$ must exist. So in order to check this, we take $\frac{\partial u}{\partial x} = 2x$, $\frac{\partial u}{\partial y} = 2y$, $\frac{\partial v}{\partial x} = 0$, $\frac{\partial v}{\partial y} = 0$. Now by Cauchy Riemann Equations, $f(z)$ would be differentiable for all $z \in A$ if $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, but using this we have that $2x \neq 0$ and $2y \neq -0$. Therefore, $f(z)$ is not analytic in any open set. $\square$

2 Problem 2

Let $A \subset \mathbb{C}$ be an open set. Assume that $f : A \to \mathbb{C}$ with $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ is analytic on A, i.e. $f'(z)$ exists for any point $z \in A$. By using change of variables, deduce the Cauchy-Riemann equations in polar coordinates

$$\frac{\partial u}{\partial r} = \partial_r u = \frac{1}{r}\partial_\theta v = \frac{\partial v}{\partial \theta}, \qquad \partial_r v = -\frac{1}{r}\partial_\theta u.$$

Proof. For f to be analytic, we must satisfy Cauchy-Riemann equations, similar to Problem 1. So, letting $z = x + iy = r(\cos\theta + i\sin\theta)$, and taking $u(x, y)$ be the real term and $v(x, y)$ be the imaginary term, we have that

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta$$

And by the Cauchy-Riemann we get

$$\frac{\partial u}{\partial r} = \frac{1}{r}(\frac{\partial v}{\partial y} r\cos\theta - \frac{\partial v}{\partial x} r\sin\theta) = \frac{1}{r}(\frac{\partial v}{\partial \theta})$$

Similarly, we using the chain rule,

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta$$

$$\frac{\partial v}{\partial r} = -\frac{1}{r}(\frac{\partial u}{\partial y} r\cos\theta - \frac{\partial u}{\partial x} r\sin\theta) = -\frac{1}{r}(\frac{\partial u}{\partial \theta})$$

Giving us that:

$$\frac{\partial u}{\partial r} = \partial_r u = \frac{1}{r}\partial_\theta v = \frac{\partial v}{\partial \theta}, \qquad \partial_r v = -\frac{1}{r}\partial_\theta u.$$

□

3 Problem 3

<u>DEFINITION</u> Let $A \subset \mathbb{R}^2$ be an open set. A function $h : A \subset \mathbb{R}^2 \to \mathbb{R}^2$ is said to be harmonic if:

1. h is twice differentiable in each variable in any point $(x, y) \in A$
2. $\partial_x^2 h(x, y) + \partial_y^2 h(x, y) = 0$ for any point $(x, y) \in A$

Given any $f : A \to \mathbb{C}$ with $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ analytic on A, and assuming that u, v are twice differentiable in each variable at any point $(x, y) \in A$, prove that $u(x, y)$ and $v(x, y)$ are harmonic.
Proof. Since f is analytic, we know that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

We also know that u, and v are twice differentiable so

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial xy}, \quad \frac{\partial^2 u}{\partial xy} = \frac{\partial^2 v}{\partial y^2} \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial xy}, \qquad \frac{\partial^2 u}{\partial yx} = -\frac{\partial^2 v}{\partial x^2}$$

□

4 Problem 4

In each of the following cases check if that given function $u = u(x, y)$ is harmonic (and in which domain), and if this is the case, find $v = v(x, y)$ such that $f : A \to \mathbb{C}$ with $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ is analytic on A, i.e. $v = v(x, y)$ is the harmonic conjugate of $u(x, y)$ (unique up to a constant). Wirte f as function of z, i.e. $f(z)$.

(a) $u(x, y) = e^y sin(x)$

(b) $u(x, y) = (x + y)^2$

(c) $u(x, y) = x + y^2$

(d) $u(x, y) = \ln(x^2 + y^2)$

(e) $u(x, y) = \tan^{-1}(\frac{y}{x})$

Proof.

(a) Taking the derivative twice, with respect to x and y we have that $\frac{\partial^2 u}{\partial x^2} = -e^y sin(x)$ and $\frac{\partial^2 u}{\partial y^2} = e^y sin(x)$ and we get that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -e^y sin(x) + e^y sin(x) = 0$$

So $u(x, y)$ is a harmonic. To find $v(x, y)$, such that $f(z)$ is analytic, we use the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = e^y \cos(x) \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = e^y \sin(x)$$

Taking the integrals we get that $v = e^y \sin(x) + c_0$ and $v = e^y \sin(x) + c_1$, which implies that $v(x, y) = e^y \sin(x) + c$ where $c \in \mathbb{R}$. So

$$f(x, y) = u(x, y) + iv(x, y) = e^y \sin(x) + i(e^y \cos(x) + c)$$

Writing f as a function of z,

(b) Taking the derivative for each variable, we have that $\frac{\partial^2 u}{\partial x^2} = 2 \cdot (1 + y)$ and $\frac{\partial^2 u}{\partial y^2} = 2 \cdot (1 + x)$. This gives us that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2(2 + x + y) \neq 0$$

Therefore, $u(x, y)$ is not harmonic.

(c) Similar to **(b)**, we have that $\frac{\partial^2 u}{\partial x^2} = 0$ and $\frac{\partial^2 u}{\partial y^2} = 2$ giving us that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 + 2 \neq 0$$

Therefore, u in this case is not harmonic.

(d) Again, we have $\frac{\partial^2 u}{\partial x^2} = -\frac{4x}{(x^2+y^2)^2} + \frac{2}{x^2+y^2}$ and $\frac{\partial^2 u}{\partial y^2} = -\frac{4y}{(x^2+y^2)^2} + \frac{2}{x^2+y^2}$, so u is not harmonic in this case because $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \neq 0$

(e) Finally, using the same method, we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{-1}{x^2 + y^2} + \frac{2xy}{(x^2 + y^2)^2}$$

and

$$\frac{\partial^2 u}{\partial y^2} = \frac{4y}{x(x + \frac{y^2}{x})^2}$$

which, again, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \neq 0$, therefore u is not harmonic.

$\square$

5 Problem 5

For which values of the real constants $a, b, c, d \in \mathbb{R}$ is

$$u(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$$

harmonic? In those cases, find its harmonic conjugate $v = v(x, y)$.
Write the analytic function $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ of z, i.e. $f(z)$.

Proof. For u to be harmonic, we want

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6ax + 2by + 2cx + 6dy = 0$$

Grouping the terms gives us

$$x(6a + 2c) + y(6d + 2b) = 0$$

Which implies that $b = -3d$ and $c = -3a$. So,

$$u(x, y) = ax^3 - 3dx^2y - 3axy^2 + dy^3$$

Now to find v we use the Cauchy-Riemann equations to get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 3ax^2 - 6dxy - 3ay^2$$

and

$$-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = 3dx^2 + 6axy - 3dy^2$$

Taking the integrals yields

$$v(x, y) = 3ax^2y - 3dxy^2 - ay^3 + c_0$$
$$v(x, y) = dx^3 - 3axy^2 - 3dxy^2 + c_1$$

which is equivalent to

$$v(x, y) = 3ax^2y - 3dxy^2 - ay^3 + dx^3 + c$$

where $c \in \mathbb{R}$. So, our adding u and v gives us f which is

$$f(z) = (ax^3 - 3dx^2y - 3axy^2 + dy^3) + i(3ax^2y - 3dxy^2 - ay^3 + dx^3) + c$$

□