Math 108B Homework 3

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Let V be an inner product space over F, and $T \in \mathcal{L}(V)$

1 Problem 1

Prove that if $U = \text{range } (T)$, then $U^{\perp} = \text{null } T^*$.
Proof.

$\square$

2 Problem 2

Suppose $P \in \mathcal{L}(V)$ is such that $P^2 = P$. Prove that P is an orthogonal projection if and only if P is self-adjoint.
Proof.

$\square$

3 Problem 3

Prove that if T is normal, then range T = range T^*
Proof.

$\square$

4 Problem 4

Prove that if T is normal, then

$$\text{null } T^k = \text{null } (T) \quad \text{and} \quad \text{range } T^k = \text{range } T$$

for every positive integer k.
Proof.

$\square$

5 Problem 5

Prove that there does not exist a self-adjoint operator $T \in \mathcal{L}(\mathbb{R}^3)$ such that $T(1,2,3) = (0,0,0)$ and $T(2,5,7) = (2,5,7)$
Proof.

$\square$

6 Problem 6

Give a counterexample to show that the product of two self-adjoint operators is not necessarily self-adjoint.
Proof.

$\square$

7 Problem 7

Suppose $F = \mathbb{C}$. Prove that a normal operator on V is self-adjoint if and only if all its eigenvalues are real.
Proof.

$\square$

8 Problem 8

Suppose $F = \mathbb{C}$ and T is a normal operator on V. PRove that there is a $S \in \mathcal{L}(V)$ such that $T = S^2$.

Proof.

$\square$

9 Problem 9

Prove that if T is a positive operator on V, then T^k is positive for every positive integer k.

Proof.

$\square$

10 Problem 10

Suppose T is a positive operator on V. Prove that T is invertible if and only if $\langle Tx, x\rangle$ is positive for $x \in V \setminus \{0\}$.

Proof.

$\square$

11 Problem 11

Prove that if $S \in \mathcal{L}(\mathbb{R}^3)$ is an isometry, then there exists a nonzero vector $x \in \mathbb{R}^3$ such that $S^2x = x$.

Proof.

$\square$