Math 122A Homework 5

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1 Problem 1

Let $z_0 \in \mathbb{C}$ be any interior point to any positive oriented simple closed curve C. Prove

$$\oint_C \frac{dz}{z - z_0} = 2\pi i, \quad \oint_C \frac{dz}{(z - z_0)^{n+1}} = 0, \quad n = 0, 1, 2, 3, ...$$

Proof. Suppose we have an f in $A \subseteq \mathbb{C} \to \mathbb{C}$, is analytic on A. Letting $C = \gamma$ and be defined as $\gamma : [a, b] \to A$, and assuming f is analytic on and inside γ, then parameterizing it as

$$\oint_\gamma f(z)dz = 0 = \int_a^b f(\gamma(t))\dot{\gamma}(t)dt$$

Then letting γ be a circle centered at $z = z_0$ with radius $R > 0$, we can get our wanted result. To do this we let $\gamma : [0, 2\pi] \to \mathbb{C}$, where $t \to z_0 + Re^{it}$, so $\gamma(t) = z_0 + Re^{it}$ and $\dot{\gamma(t)} = iRe^{it}$. Therefore,

$$\oint_\gamma f(z)dz = \int_0^2 \frac{\dot{\gamma}(t)}{\gamma(t) - z_0} = \int_0^{2\pi} \frac{iRe^{it}}{Re^{it}} dt = i \int_0^{2\pi} dt = 2\pi i$$

For the second one, we use the same suppositions and get

$$\oint_\gamma \frac{dz}{(z - z_0)^{n+1}} = \oint_{|z-z_0|=r} = \int_0^{2\pi} \frac{ie^{it}dt}{(re^{it})^{n+1}} = \frac{i}{r^n} \int_0^{2\pi} e^{-i(n+1)t}dt$$

$$= \frac{i}{r^n} \int_0^{2\pi} e^{-i(n+1)t}dt = \frac{i}{r^n} \left[\frac{e^{2\pi i(n+1)}}{i(n+1)} - \frac{e^0}{i(n+1)} \right] = 0$$

$\square$

2 Problem 2

Let C be the contour of the circle $|z-i|=2$ in the positive sense. Find

(a) $\oint_C \frac{dz}{z^2+4}$

(b) $\oint_C \frac{e^z}{z-\frac{\pi i}{2}}$

(c) $\oint_C \frac{\cos(z)}{(z^2+16)z}$

(d) $\oint_c \frac{dz}{2z+1}$

Proof. To prove these, we use the Cauchy theorem

$$f(z_0) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-z_0}dz$$

Where f must be analytic inside and z_0 must be inside the curve. Therefore,

(a) Letting $f(z) = \frac{1}{z+2i}$, we have that

$$f(2i) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-2i}dz$$

$$2\pi i\left(\frac{1}{4i}\right) = \oint_\gamma \frac{f(z)}{z-2i}dz$$

$$\frac{\pi}{2} = \oint_\gamma \frac{f(z)}{z-2i}dz$$

(b) For this we let $f(z) = 1$ and $z_0 = \frac{\pi}{2}$ giving us

$$f(\frac{\pi}{2}) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-i\frac{\pi}{2}}dz$$

$$2\pi i e^{i\frac{\pi}{2}} = \oint_\gamma \frac{f(z)}{z-i\frac{\pi}{2}}dz$$

By DeMoivre's formula, we know that $e^{i\frac{\pi}{2}} = \cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2}) = i$, therefore we can further simplify and get

$$2\pi i^2 = \oint_\gamma \frac{f(z)}{z-i\frac{\pi}{2}}dz$$

$$-2\pi = \oint_\gamma \frac{f(z)}{z-i\frac{\pi}{2}}dz$$

(c) Factoring out the denominator gives us

$$\oint_C \frac{\cos(z)}{(z^2+16)z} = \oint_\gamma \frac{\cos(z)}{(z+4i)(z-4i)z} dz$$

Therefore, we know we have singularities at $z = \pm 4i, 0$. It's clear that since our circles has radius 2, we only need to worry about $z = 0$, therefore we can let $f(z) = \frac{\cos(z)}{(z-4i)(z+4i)}$ and now the formula becomes

$$f(0) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{(0-4i)(0+4i)} dz$$

$$\frac{1}{16} = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z-0} dz$$

$$\frac{\pi i}{8} = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z-0} dz$$

(d) Finally, for this we let $f(z) = -\frac{1}{2}$, then

$$f(\frac{1}{2}) = \frac{1}{2\pi i} \oint_\gamma \frac{dz}{z-z_0}$$

$$2\pi i \frac{1}{2} = \oint_\gamma \frac{dz}{z-z_0}$$

$$\pi i = \oint_\gamma \frac{dz}{z-z_0}$$

$\square$

3 Problem 3

For $z \in \mathbb{C}$ and $|z| \neq 3$, denote C the contour of the circle $|z| = 3$ in the positive sense and define

$$g(z) = \oint_C \frac{2w^2 - w - 2}{w - z} dw$$

Find values of $g(2)$ and $g(3 + 2i)$.
Proof. Similar to the suppositions of **Problem 2**, we let $w = z$, $z = z_0$, $g(z) = 2z^2 - z - 2$ giving us $g(2)$ as

$$g(2) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0} dz$$

$$8\pi i = \oint_\gamma \frac{f(z)}{z - z_0} dz$$

and for $g(3 + 2i)$, we get

$$g(3 + 2i) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0} dz$$

$$0 = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0} dz$$

$$0 = \oint_\gamma \frac{f(z)}{z - z_0} dz$$

□

4 Problem 4

Assuming that the given contour is positive oriented, compute

(a) $\oint_{|z|=3} \frac{(e^z+z)}{z-2} dz$

(b) $\oint_{|z|=3} \frac{e^z}{z^2}$

(c) $\oint_{|z|=3} \frac{dz}{z^2+z+1}$

(d) $\oint_{|z|=3} \frac{dz}{z^2-1}$

DEFINITION: A $f : \mathbb{C} \to \mathbb{C}$ is an ENTIRE function if f is analytic in all $\mathbb{C}$
Proof. To prove these, we use the Cauchy theorem

$$f(z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0} dz$$

Where f must be analytic inside and z_0 must be inside the curve. Therefore,

(a) Letting $f(z) = e^z + z$, we get that

$$f(2) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z-2} dz$$

$$(2\pi i)(e^2 + 2) = \oint_\gamma \frac{f(z)}{z-2} dz$$

(b) Using

$$f'(z_0) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(z)}{(z - z_0)^{n+1}} dz$$

If we let $f(z) = e^z$, this implies that

$$f'(0) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z^2} dz$$

$$1 = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z^2} dz$$

$$2\pi i = \oint_\gamma \frac{f(z)}{z^2} dz$$

(c) Using the quadratic formula, we can factor the denominator to get

$$\oint_{|z|=3} \frac{dz}{(z - (\frac{-1+i\sqrt{3}}{2}))(z - (\frac{-1-i\sqrt{3}}{2}))}$$

Here it's clear that our singularities are at $z = \frac{-1\pm i\sqrt{3}}{2}$ which are both in the curve. Now by deformation, we can split this curve into 4 curves γ_1, γ_2, γ_3, and γ_4 as half circles centered around our singualrities. Since γ_2 and γ_3 are vertical lines going in the opposite directions, we know that these part of of the curve adds to zero. This leaves us with

$$\frac{1}{2\pi i}\left[\oint_{\gamma_1} \frac{f_1(z)}{z - \frac{-1+i\sqrt{3}}{2}} + \oint_{\gamma_4} \frac{f_4(z)}{z - \frac{-1-i\sqrt{3}}{2}}\right]$$

Where $f_1(z) = \frac{1}{z - \frac{-1-i\sqrt{3}}{2}}$ and $f_4(z) = \frac{1}{z - \frac{-1+i\sqrt{3}}{2}}$. Solving the integrals individually using the same process as **Problem 2**, we get

$$\frac{1}{2\pi i}\left[\frac{1}{i\sqrt{3}} - \frac{1}{i\sqrt{3}}\right] = 0$$

(d) Since the singularites are on the contour, we have not yet learned the tools to solve this problem.

□

5 Problem 5

Prove that if f is entire and there exists $z_0 \in \mathbb{C}$ and $r > 0$ such that

$$f(\mathbb{C}) \cap \{z \in \mathbb{C} : |z - z_0| < r\} = \emptyset$$

then f is a constant function.
Proof. To show this, we can consider the function $g(z) = \frac{1}{f(z)-z_0}$. Since $f(z) - z_0 \neq 0$, and $f(\mathbb{C}) \cap \{z \in \mathbb{C} : |z - z_0| < r\} = \emptyset$, we know that $g(z)$ is entire. Furthermore, $|f(z) - z_0| \geq r$ implies that $|g(z)| = \left|\frac{1}{f(z)-z_0}\right| = \frac{1}{|f(z)-z_0|} \leq \frac{1}{r}$ and therefore, $g(z)$ is bounded. Then, by Liouville's Thorem, we know that $g(z)$ is constant and we can solve for $f(z)$ to get that

$$g \cdot f(z) - g \cdot z_0 = 1$$

$$f(z) = \frac{1 + (g \cdot z_0)}{g}$$

Hence $f(z)$ is constant. $\square$

6 Problem 6

Identify all entire functions f such that $\forall z \in \mathbb{C} : |f(z)| \leq 2\,|z|$.
Proof. Similar to the proof of Liouville's Theorem, it is enough to show that $f''(z_0) = 0$. Then we using Cauchy's Third Theorem

$$f''(z_0) = \frac{2!}{2\pi i} \oint_{\mathbb{C}_R} \frac{f(z)}{(z - z_0)^3}$$

Parameterizing our curve using

$$\gamma(t) = z_0 + Re^{eit} \qquad \dot{\gamma}(t) = iRe^{et} \qquad \gamma : [0, 2\pi] \to \mathbb{C}$$

Giving us

$$f''(z_0) = \frac{2!}{2\pi i} \int_0^{2\pi} \frac{f(\gamma(t))}{(\gamma(t) - z_0)^3} \dot{\gamma}(t) dt$$

$$f''(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + Re^{it})}{R^3 e^{3it}} iRe^{it} dt$$

$$f''(z_0) = \frac{1}{\pi} \int_0^{2\pi} \frac{f(z_0 + Re^{it})}{R^2 e^{2it}} dt$$

Taking the absolute value yields

$$|f''(z_0)| = \left| \frac{2}{\pi} \int_0^{2\pi} \frac{f(z_0 + Re^{it})}{R^2 e^{2it}} dt \right|$$

$$\leq \frac{2}{\pi R^2} \int_0^{2\pi} \left| f(z_0 + Re^{it}) dt \right|$$

Then for our problem we want to show that

$$\frac{2}{\pi R^2} \int_0^{2\pi} \left| f(z_0 + Re^{it}) dt \right| \leq \frac{2}{\pi R^2} \int_0^{2\pi} 2 \left| z + Re^{it} \right| dt$$

$$\leq \frac{4}{\pi R^2} \int_0^{2} (|z| + R) dt = \frac{4|z|}{r} + \frac{4}{R}$$

So

$$f''(z) \leq \frac{4|z|}{R^2} + \frac{4}{R}$$

Moving all the terms to the left side and dividing by $|z|$ yields

$$\left(f''(z) - \frac{4}{R} \right) \frac{1}{|z|} \leq \frac{4}{R^2} = 0$$

Since R is arbitrary and $|f''(z_0)|$ is independent of R, we know that $R \to \infty$ implies that $|f''(z_0)| = 0$ and that $f''(z_0) = 0 \quad \forall z_0 \in \mathbb{C}$. Therefore, we can conclude that $f'(z_0) = c_0$ which implies that $f(z_0)$ is of the form $f(z_0) = c_0 + b$. $\square$