Math 119A Homework 3

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1 Problem 1

Prove or disprove that $x(t) = 0, y(t) = 3e^{2t}$ is a solution to the following initial value problem:

$$\begin{aligned} x' &= -x, \\ y' &= x + 2y; \\ x(0) &= 0, y(0) = 3 \end{aligned} \tag{1}$$

Proof. To check our solution we first check that the derivatives of $x(t)$ and $y(t)$ satisfy our differential equation:

$$\begin{aligned} x'(t) &= 0 = x \\ y'(t) &= 6e^{2t} = 0 + 2(3e^{2t}) = x + 2y \end{aligned} \tag{2}$$

Which shows that $x(t), y(t)$ are solutions to our differential equations. Now plugging in our initial values for $t = 0$, we get

$$\begin{aligned} x(0) &= 0 \\ y(0) &= 0 + e^0 = 3 \end{aligned} \tag{3}$$

Which also satisfies our initial values, therefore $x(t) = 0, y(t) = 3e^{2t}$ is a solution to our differential equation. $\square$

2 Problem 2

Prove or disprove that

$$A = \begin{bmatrix} \frac{5}{3} & \frac{1}{3} \\ 2 & 0 \end{bmatrix}$$

is one solution to $x' = Ax$ where $x(t) = (e^{2t} - e^{-t}, e^{2t} + 2e^{-t})$.
Proof. First, $x'(t)$ is

$$x'(t) = \begin{bmatrix} 2e^{2t} + e^{-t} \\ 2e^{2t} - 2e^{-t} \end{bmatrix}$$

Now Ax is

$$\begin{aligned} Ax &= \begin{bmatrix} \frac{5}{3} & \frac{1}{3} \\ 2 & 0 \end{bmatrix} \begin{bmatrix} e^{2t} - e^{-t} \\ e^{2t} + 2e^{-t} \end{bmatrix} \\ &= \begin{bmatrix} \frac{5}{3}\left(e^{2t} - e^{-t}\right) + \frac{1}{3}\left(e^{2t} + 2e^{-t}\right) \\ 2\left(e^{2t} - e^{-t}\right) + 0\left(e^{2t} + 2e^{-t}\right) \end{bmatrix} \\ &= \begin{bmatrix} \frac{5}{3}e^{2t} - \frac{5}{3}e^{-t} + \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t} \\ 2e^{2t} - 2e^{-t} \end{bmatrix} \\ &= \begin{bmatrix} \frac{6}{3}e^{2t} - \frac{3}{3}e^{-t} \\ 2e^{2t} - 2e^{-t} \end{bmatrix} \\ &= \begin{bmatrix} 2e^{2t} - e^{-t} \\ 2e^{2t} - 2e^{-t} \end{bmatrix} \end{aligned} \tag{4}$$

Showing that the book answer is incorrect because the first element of the resulting Ax has the incorrect sign in the second term. □

3 Problem 3

Show that all eigenvalues are positive is the condition on eigenvalues that is equivalent to $\lim_{t\to\infty} |x(t)| = \infty$ for every solution $x(t)$ to $x' = Ax$.
Proof. Since every eigenvalues are positive, we would have a solution to our differential equation of the form

$$x_i(t) = c_{i1}e^{t\lambda_1} + ... + c_{in}e^{t\lambda_n}; \quad i = 1, ..., n$$

where λ_i is the eigenvalue, c_i constants. The limit of the norm then becomes:

$$\lim_{t\to\infty} \sqrt{(c_i e^{t\lambda_1})^2 + ... + (c_{in}e^{t\lambda_n})^2}$$

and since we have the summation of exponential terms raised to positive values, as $t \to \infty$, our limit as well goes to infinity. □

4 Problem 4

Show that $b > 0$ is an assumption required to ensure that $\lim_{t\to\infty} x(t) = 0$ for every solution $x(t)$ if $b^2 - 4c > 0$.
Proof.

$\square$

5 Problem 5

Prove or disprove that $x(t) = 3e^t \cos 2t + 9e^t \sin 2t, y(t) = 3e^t \sin 2t - 9e^t \cos 2t$ is a solution to the following initial value problem:

$$\begin{aligned} x' &= Ax, \\ x(0) &= (3, 9); \\ A &= \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \end{aligned} \tag{5}$$

Proof. Similar to Problem 2, we check that the initial value $t = 0$ yields $(3, 9)$. $x(0) = 3e^0 \cos(2 \cdot 0) + 9e^0 \sin(2 \cdot 0) = 3$ and $y(0) = 3e^0 \sin(2 \cdot 0) - 9e^0 \cos(2 \cdot 0) = 0$. So certainly, $x(0) = (3, 9)$. Now checking that $x' = Ax$, we find

$$\begin{aligned} Ax &= \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3e^t \cos(2t) + 9e^t \sin(2t) \\ 3e^t \sin(2t) - 9e^t \cos(2t) \end{bmatrix} \\ &= \begin{bmatrix} 3e^t \sin(2t) + 21e^t \cos(2t) \\ 21e^t \sin(2t) - 3e^t \cos(2t) \end{bmatrix} \end{aligned} \tag{6}$$

which is certainly equivalent to

$$x'(t) = 3e^t \sin(2t) + 21e^t \cos(2t)$$

and

$$y'(t) = 21e^t \sin(2t) - 3e^t \cos(2t)$$

therefore, our given $x(t)$ and $y(t)$ are solutions to our initial value problem.
$\square$

6 Problem 6

Prove or disprove that $\dim E = \dim E_{\mathbb{C}}$ and $\dim F \geq \dim F_{\mathbb{R}}$ are relations that exist between $\dim E$ and $\dim E_{\mathbb{C}}$ and $\dim F$ and $\dim F_{\mathbb{R}}$ given that $E \subset \mathbb{R}^n$ and $F \subset \mathbb{C}^n$ are subspaces.

Proof. The set E in this case is a set that contains vector spaces in R^n, meanwhile, $E_{\mathbb{C}}$ is the complexification of E which is obtained by taking all linear combinations of vectors in E with complex coefficients, i.e.

$$E_{\mathbb{C}} = \left\{ z \in \mathbb{C}^n | z = \sum_{i=1}^{k} \lambda_i z_i, z_i \in E, \lambda_i \in C \right\}$$

then we see that the number of elements in $E_{\mathbb{C}}$ is exactly the number of elements $z_i \in E$, therefore it must be the case that $\dim E = \dim E_{\mathbb{C}}$. Meanwhile, by the same arguement, we know that $\dim F \geq \dim F_{\mathbb{R}}$ because according to the book, $\mathbb{R}^n \subset \mathbb{C}^n$. □

7 Problem 7

Prove or disprove that $\dim F \supset R_{\mathbb{CR}}$ is a relation between F and $F_{\mathbb{RC}}$ given that $F \subset \mathbb{C}^n$ is any subspace.

Proof. Not sure how to answer this, where did $R_{\mathbb{CR}}$ come from? □

8 Problem 8

Solve the following initial value problem

(a)

$$\begin{aligned} x' &= -y, \\ y' &= x; \\ x(0) = 1&, y(0) = 1 \end{aligned} \tag{7}$$

Proof.

(a) The corresponding matrix for this differential equation is

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

The characteristic polynomial for this matrix is $\lambda^2 + 1$ giving us eigenvalues of $\pm i$. This gives us a solution of the form

$$\begin{cases} x(t) & = ue^{ta}\cos(tb) - ve^{ta}\sin(tb) \\ y(t) & = ue^{ta}\sin(tb) + ve^{ta}\cos(tb) \end{cases}$$

where $b = 1$, $a = 0$, and u, v are constants. Therefore, our general solution is

$$\begin{cases} x(t) = u\cos(t) - v\sin(t) \\ y(t) = u\sin(t) + v\cos(t) \end{cases}$$

Now solving for the constants yield

$$\begin{cases} x(0) = 1 = u\cos(0) - v\sin(0) \implies u = 1 \\ y(0) = 1 = u\sin(0) + v\cos(0) \implies v = 1 \end{cases}$$

giving us the final solution to our initial value problem which is

$$\begin{cases} x(t) = \cos(t) - \sin(t) \\ y(t) = \sin(t) + \cos(t) \end{cases}$$

□

9 Problem 9

Solve the initial value problem

$$\begin{aligned} x' &= -4y \\ y' &= x; \\ x(0) = 0, &\quad y(0) = -7 \end{aligned} \tag{8}$$

Proof. Similar to Problem 9, we have a corresponding matrix of

$$\begin{bmatrix} 0 & -4 \\ 1 & 0 \end{bmatrix}$$

Using a coordinate change, we can write this in a similar form of Problem 9, by using coordinates (u, v) where $x = 2v$, $y = u$. This yields:

$$\begin{cases} u' = y' = 2v \\ v' = \frac{1}{2}x' = -2u \end{cases}$$

Now this is exactly the same problem as Problem 9 which gives us the general solution of

$$\begin{cases} u(t) = u_0\cos(2t) - v_0\sin(2t) \\ v(t) = u_0\sin(2t) + v_0\cos(2t) \end{cases}$$

Where $(u(0), v(0)) = (u_0, v_0) \implies u_0 = -7, v_0 = 0$, giving us the solution to our initial value problem as

$$\begin{cases} x(t) = -\frac{7}{2}\sin(2t) \\ y(t) = -7\cos(2t) \end{cases}$$

□

10 Problem 10

Let $F \subset C^2$ be the subspace spanned by the vector $(1, i)$

(a) Prove that F is not invariant under conjugation and hence is not the complexification of any subspace of $\mathbb{R}^2$

(b) Find $F_{\mathbb{R}}$ and $(F_{\mathbb{R}})_{\mathbb{C}}$.

Proof.

(a) According to Invariant Subpaces, for a subspace F to be invariant under a transformation T, it must follow that for all vectors $v \in F$, $T(v) \in F$. The transformation, T, in our case is complex conjugation which takes an arbitrary vector $(a, bi) \in F$ to $(a, -bi)$.

(b) According to the book, for any subspace $F \subset C^n$, $F_{\mathbb{R}}$ is given by

$$F_{\mathbb{R}} = \{z \in F | \sigma(z) = z\}$$

where σ is the conjugation operator. Therefore, for our case $F_{\mathbb{R}}$ would be the set of vectors where elements of F are a 2 dimension pair of all real numbers since the complex conjugation of reals are reals, satisfying $\sigma(z) = z$? Not sure how to answer this.

□

11 Problem 11

Let E be a real vector space and $T \in L(E)$. Show that $(\ker T)_{\mathbb{C}} = \ker(T_{\mathbb{C}})$, $(\text{Im } T)_{\mathbb{C}} = \text{Im } (T_{\mathbb{C}})$, and $(T^{-1})_{\mathbb{C}} = (T_{\mathbb{C}})^{-1}$ if T is invertible.

Proof. Taking an arbitrarly element in $x \in (\ker T)_{\mathbb{C}}$, by definition of kernel we have $T(x) = 0$. Taking the complexification of $T(x)$ yields $(T(x))_{\mathbb{C}}=0$ which implies $(T_{\mathbb{C}})(x) = 0$ so $x \in \ker(T_{\mathbb{C}})$, therefore $(\ker T)_{\mathbb{C}} = \ker(T_{\mathbb{C}})$. Similarly, taking $x \in (\text{Im } T_{\mathbb{C}})$ there exists y in the vector space where $T(y) = x$, implies that $(T(y))_{\mathbb{C}} = x_{\mathbb{C}}$. This is equivalent to $(T_{\mathbb{C}})(y_{\mathbb{C}}) = x$ which implies $x \in \text{Im } (T_{\mathbb{C}})$ therefore $(\text{Im } T)_{\mathbb{C}} = (\text{Im } T_{\mathbb{C}})$. Finally, since we are given that T is invertible, it must be the case that $(T \cdot T^{-1})_{\mathbb{C}} = I_{\mathbb{C}}$. This is equivalent to $(T_{\mathbb{C}}) \cdot (T^{-1})_{\mathbb{C}} = I$. Taking $T_{\mathbb{C}}^{-1}$ to both sides yields $(T_{\mathbb{C}}^{-1}) \cdot (T_{\mathbb{C}}) \cdot (T^{-1})_{\mathbb{C}} = (T_{\mathbb{C}}^{-1}) \cdot I$ and simplifying gives us $(T^{-1})_{\mathbb{C}} = (T_{\mathbb{C}}^{-1})$. $\square$