Math 122A Homework 6

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1 Problem 1

Let C be a closed, positive, and simple curve. Using Green's Theorem prove that

$$\frac{1}{2i}\int_C \bar{z}dz = \text{area enclosed by } C$$

Proof. Since $\bar{z} = x - iy$, this is equivalent to

$$\frac{1}{2i}\int_C (x-iy)(dx+idy) = \frac{1}{2i}\left[\int_C (xdx+ydy) + i\int_C (xdy-ydx)\right]$$

Where D is the area boundead by C. Now using Green's Theorem, twice we have that

$$\frac{1}{2i}\left[\iint_D (0-0)dxdy + i\iint_D (1-(-1))dxdy\right]$$
$$= \iint_D dxdy$$

Which after evaluating the integrals is exactly the area enclosed by C. $\square$

2 Problem 2

Consider the function $f(z) = (z+1)^2$ and the region R bounded by the triangle with vertices $0, 2, i$ (its boundary and interior). Find the points where $|f(z)|$ reaches its maximum and minimum value of R.

Proof. By the Maximum Modulus Theorem, we know that $|f(z)| = |z+1|^2$. Then the minimum would be at the boundary where $|f(z)|$ is the closest to -1 and maximum is the furthest. Then we know that since $|f(z)| = 1$ at $z = 0$, then this is the minimum, and since $|f(z)| = 9$ at $z = 2$, then this is the maximum. $\square$

3 Problem 3

Find the maximum of $|\sin(z)|$ on $[0, 2\pi] \times [0, 2\pi]$.

Proof. We can rewrite $\sin(z)$ as:

$$\begin{aligned}
\sin(z) &= \frac{e^{iz} - e^{-iz}}{2i} = \frac{e^{x+iy} - e^{-i(x+iy)}}{2i} \\
&= \frac{e^{ix}e^{-y} - e^{-ix}e^{y}}{2i} \\
&= \frac{(\cos x + i\sin x)e^{y} - (\cos x - i\sin x)e^{y}}{2i} \\
&= -\frac{1}{i}\cos(x)\left(\frac{e^{-y} - e^{y}}{2}\right) + \sin(x)\left(\frac{e^{y} + e^{-y}}{2}\right) \\
&= i\cos(x)\sinh(y) + \sin(x)\cosh(y)
\end{aligned} \tag{1}$$

And using Maximum Modulus Theorem, and **(1)**, we have that:

$$|\sin(z)|^2 = \cos^2(x)\sinh^2(y) + \sin^2(x)\cosh^2(y)$$

Using the identities that states $\cosh^2(y) - \sinh^2(y) = 1$ and $\cos^2(x) + \sin^2(x) = 1$, we can rewrite this as:

$$|\sin(z)|^2 = \cos^x \sinh^2(y) + \sin^2(x) \cdot (1 + \sinh^2(y))$$

Leaving us with

$$|\sin(z)|^2 = \sinh^2(y) + \sin^2(x)$$

We know that that maximum of $\sin^2(x) = 1$ which is at $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$, meanwhile the maximum of $\sinh^2(y)$ is located at $y = 2\pi$. Therefore, out maximum is at the boundaries 2π. $\square$

4 Problem 4

Calculate:

(a)

$$\int_0^{2\pi} \frac{d\theta}{a + b\cos(\theta)}, \quad 0 < b < a$$

HINT: Work backwards using $\cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2}$ to convert the integral into a complex integral along the curve $|z| = 1$

(b)

$$\int_0^{2\pi} \frac{d\theta}{(a + b\cos(\theta))^2}$$

(c)

$$\int_0^{2\pi} \frac{\sin(\theta)d\theta}{(a + b\cos(\theta))^2}, \quad 0 < b < a$$

Proof.

(a) By the hint, we work backwards and get

$$\int_0^{2\pi} \frac{d\theta}{a + b\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)}$$

Factoring out a 2 gives us

$$2\int_0^{2\pi} \frac{d\theta}{2a + b\left(e^{i\theta} + e^{-i\theta}\right)}$$

Letting $z(\theta) = e^{i\theta}$, where $z : [0, 2\pi] \to \mathbb{C}$, implies $d\theta = \frac{dz}{iz}$ and our line integral becomes

$$I = \frac{2}{i}\oint_{|z|=1} \frac{dz}{bz^2 + 2az + b}$$

Here our $f(z) = \frac{1}{bz^2+2az+b}$ is analytic except at $\frac{a\pm\sqrt{a^2-b^2}}{b}$. These two points are:

$$z_1 = \frac{a + a\sqrt{1 - \frac{b^2}{a^2}}}{b} \quad \text{and} \quad z_2 = \frac{a - a\sqrt{1 - \frac{b^2}{a^2}}}{b}$$

By our condition that $0 < b < a$, we know z_2 must be outside our z, therefore letting $h(z) = \frac{1}{z - \left(\frac{a - a\sqrt{1-\frac{b^2}{a^2}}}{b}\right)}$ and by Cauchy Theorem we get

$$I = \frac{2}{i} \oint_{|z|=1} \frac{h(z_2)dz}{z - \left(\frac{a + a\sqrt{1-\frac{b^2}{a^2}}}{b}\right)} = \frac{2\pi b}{\sqrt{a^2 - b^2}}$$

(b) Expanding out our denominator yields

$$\int_0^{2\pi} \frac{d\theta}{a^2 + 2ab\cos\theta + b^2\cos^2\theta}$$

Using brute force method, and letting $u = a + b\cos\theta$, $\frac{d}{du}\left(\frac{1}{u}\right) = -\frac{1}{u^2}$ Then by Fundamental Theorem of Calculus, then plugging back our u:

$$\begin{aligned} \int \frac{1}{u^2} &= -\int \frac{d}{du}\left(\frac{1}{u}\right) \\ &= -\frac{1}{a+b} + \frac{1}{a+b} = 0 \end{aligned} \tag{2}$$

(c) Similar to **(b)**, letting $u = a + b\cos\theta$ and $du = -b\sin\theta d\theta$ yields:

$$-\frac{1}{b}\int_0^{2\pi} \frac{1}{u^2}du = \frac{1}{b}\left(\frac{1}{u}\right)\Bigg|_0^{2\pi}$$

Plugging in our u becomes

$$\begin{aligned} -\frac{1}{b}\int_0^{2\pi} \frac{1}{u^2}du &= \frac{1}{b}\left(\frac{1}{a + b\cos\theta}\right)\Bigg|_0^{2\pi} \\ &= \frac{1}{b}\left[\frac{1}{a+b} - \frac{1}{a+b}\right] \\ &= 0 \end{aligned} \tag{3}$$

$\square$

5 Problem 5

Prove that if $f : \mathbb{C} \to \mathbb{C}$ is entire such that for some $n \in \mathbb{N}$

$$\lim_{|z| \to \infty} \frac{|f(z)|}{|z|^n} = M < \infty,$$

then f is a polynomial of degree at most n.
Proof. Since f is analytic, by a theorem in **Lecture 16** which states that f has a power series expansion in the neighborhood of analyticity that is

$$f(z_0) = \sum_{n=0}^{\infty} (z_0 - z_1)^n \left(\frac{f^n(z_1)}{n!} \right)$$

In our case, $z_1 = 0$ so this becomes

$$f(z_0) = \sum_{n=0}^{\infty} (z_0)^n \left(\frac{f^n(0)}{n!} \right)$$

By **Section 49** of the book, we get

$$|f^n(0)| \leq \frac{n! M_R}{R^n}$$

Where M_R denotes the maximum value of $|f(z)|$. □

6 Problem 6

Let $A \subset \mathbb{C}$ be an open set and $f : A \to \mathbb{C}$ be an analytic function on A. Assuming that $z_0 \in A$ such that

$$\{z \in \mathbb{C} : |z - z_0| \leq R\}, \quad R > 0$$

prove that

$$f(z_0) = \frac{1}{\pi R^2} \iint_{|z-z_0| \leq R} f(x+iy)dxdy$$

Proof. By the Cauchy Theorem, we know that $f(z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z-z_0}$ Using polar coordinates and parameterizing γ yields $\int_0^R f(z)dr = \int_0^R \int_0^{2\pi} f(z+re^{ie\theta})$ Multiplying by r to both sides results in:

$$\int_0^R rf(z)dr = \frac{1}{2\pi} \int_0^R \int_0^{2\pi} rf(z+re^{i\theta})drd\theta$$

$$f(z) \int_0^R rdr = \frac{1}{2\pi} \int_0^R \int_0^{2\pi} rf(z+re^{i\theta})drd\theta$$

$$\frac{1}{2}R^2 f(z) = \frac{1}{2\pi} \int_0^R \int_0^{2\pi} rf(z+re^{i\theta})drd\theta$$

$$\frac{1}{2}R^2 f(z) = \frac{1}{2\pi} \int_{|(x+iy)|-z \leq R} f(x+iy)drd\theta$$

$$f(z) = \frac{1}{\pi R^2} \int_{|(x+iy)|-z \leq R} f(x+iy)drd\theta$$

$\square$

7 Problem 7

Let $f : R \to R$ be defined as

$$f(x) = e^{\frac{-1}{x^2}} \quad \text{if} \quad x \neq 0, \quad f(0) = 0$$

Show that f is infinitely differentiable $\forall n \in \mathbb{N}$, $f^{(n)}(0) = 0$. Verify that the power series of f at $x = 0$ does not agree with f in any neighborhood of 0.
Proof. We are given that

$$f(x) = \begin{cases} e^{\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Furthermore, we know that $f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h}$ which implies

$$f'(0) = \lim_{h\to 0} \frac{e^{-\frac{1}{h^2}} - 0}{h} = \lim_{h\to 0} \frac{e^{\frac{1}{h^2}}}{h}$$

Then as $x = \frac{1}{h}$ approaches ∞ we know

$$f'(0) = \lim_{x\to\infty} e^{-x^2} x = \lim_{x\to\infty} \frac{x}{e^{x^2}} = 0$$

So $f'(0) = 0$ Similarly,

$$f''(h) = \lim_{h\to 0} \frac{f'(h) - f'(0)}{h} = \frac{2}{h^3} e^{-\frac{1}{h^2}}$$

We know that the second term is 0 so we are left with

$$f''(h) = \lim_{h\to 0} \frac{f'(h)}{h} = \lim_{h\to 0} 2\frac{1}{h^4} e^{\frac{1}{h^2}}$$

Then as $x = \frac{1}{h}$ approaches ∞ we get that

$$f''(h) = \lim_{t\to\infty} \frac{2t^4}{e^{t^2}} = 0$$

$\square$