Math 119A Homework 5

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1 Problem 1

For the following operator T find bases for the general eigenspaces; give the atrices (for the standard basis) of the semisimple and nilpotent parts of T.

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Proof.

□

2 Problem 2

Identify R^{n+1} with the set P_n of polynomials of degree $\leq n$, via the correspondence

$$(a_n, ..., a_0) \leftrightarrow a_n t^n + ... + a_1 t + a_0$$

Let $D : P_n \to P_n$ be the differentiation operator. Prove D is nilpotent.
Proof.

□

3 Problem 3

Find the matrix of D in the standard basis in **Problem 2**
Proof.

□

4 Problem 4

Classify the following operators on $\mathbb{R}^4$ by similarity

(a) $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & -2 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & -1 & -1 & 0 \end{bmatrix}$

(e) $\begin{bmatrix} 0 & 0 & 0 & 100 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Proof.

□

5 Problem 5

Let A be a 3×3 real matrix which is not diagonal. If $(A + I)^2 = O$, find the real canonical form of A.

Proof.

□

6 Problem 6

Every $n \times n$ matrix is similar to its transpose.
Proof.

$\square$

7 Problem 7

Let $A \in L(\mathbb{R}^2)$. Suppose all solutions of $x' = Ax$ are periodic with the same period. Then A is semisimple and the characteristic polynomial is a power of $l^2 + a^2$, $a \in \mathbb{R}$.
Proof.

$\square$

8 Problem 8

Find a map $s : \mathbb{R} \to \mathbb{R}$ such that

$$s^{(3)} - s^{(2)} + 4s' - 4s = 0$$

$$(s0) = 1, \quad s'(0) = -1, \quad s''(0) = 1$$

Proof.

$\square$

9 Problem 9

Consider the equation

$$s^{(4)} + 4s^{(3)} + 5s^{(2)} + 4s' + 4s = 0$$

Find out for which initial conditions $s(0)$, $s'(0)$, $s''(0)$ there is a solution $s(t)$ such that $s(t)$ is periodic
Proof.

$\square$

10 Problem 10

If e^{tB} and e^{tA} are both contractions on $\mathbb{R}^n$, and $BA = AB$, then $e^{t(A+B)}$ is a contraction. Similarly for expansions.
Proof.

$\square$

11 Problem 11

Show that for **Problem 10** can be false if the assumption that $AB = BA$ is dropped.
Proof.

$\square$

12 Problem 12

e^{tA} is hyperbolic if and only if for each $x \neq 0$ either

$$|e^{tA}x| \to \infty \quad \text{as} \quad t \to \infty$$

or

$$|e^{tA}x| \to \infty \quad \text{as} \quad t \to -\infty$$

Proof.

$\square$

13 Problem 13

Show that a hyperbolic flow has no nontrivial periodic solutions.
Proof.

$\square$

14 Problem 14

For each of the following properties defines a set of real $n \times n$ matrices. Find out which sets are dense, and which are open in the space $L(\mathbb{R}^n)$ of all linear operators on $\mathbb{R}^n$:

(a) determinant $\neq 0$;

(b) trace is rational;

(c) entries are not integers;

(d) $e \leq$ determinant < 4

(e) $-1 < |\lambda| < 1$ for every eigenvalue λ;

(f) no real eigenvalues;

(g) each real eigenvalue has multiplicity one.

Proof.

(a)

$\square$

15 Problem 15

Which of the following properties of operators on $\mathbb{R}^n$ are generic?

(a) $|\lambda| \neq 1$ for every eigenvalue λ;

(b) $n = 2$; for some eigenvalue is not real;

(c) $n = 3$; some eigenvalue is not real;

(d) no solution of $x' = Ax$ is periodic (except the zero solution);

(e) there are n distinct eigenvalues with distinct imaginary parts;

(f) $Ax \neq x$ and $Ax \neq -x$ for all $x \neq 0$.

Proof.

(a)

$\square$