Math 108B Final Review

Rad Mallari

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1. Prove that if T has $k+1$ eigenvalues, then dim range $T \geq k$

   Proof. We have $k+1$ distinct eigenvalues which imply that we have k linearly independent vectors. Then there is at least k nonzero eigenvalues, say $\lambda_1, ..., \lambda_k = 0$. Letting v_j be the eigenvector corresponding to λ_j (where $1 \leq j \leq k$), then $T(\frac{v_j}{\lambda_j}) = \frac{1}{\lambda_j} T_j v_j = v_j \Rightarrow v_j \in$ range $T \Rightarrow v_1, ..., v_k$ are l. ind. $\square$

2. Prove that if $ST = TS$, then for every constant λ, the subspace null $(S - \lambda I)$ is invariant under T

   Proof. Let $\lambda \in \mathbb{F}$, suppose $v \in$ null $(S - \lambda I)$. Then $(S - \lambda I)(Tv) = (STv - \lambda Tv) = (TSv - \lambda Tv) = T(Sv - \lambda v) = 0$. Therefore, $Tv \in \text{null}(S - \lambda I)$. Hence $\text{null}(S - \lambda I)$ is invariant under T. $\square$

3. Prove that every nonzero vector is an eigenvalue of T, then $T = \lambda I$ for some constant λ.

   Proof. For each $v \in V$, $\exists a_v \in \mathbb{F}$ s.t. $Tv = a_v v$. Since $T \cdot 0 = 0$, we choose a_0 to be in $\mathbb{F}$, but for $v \in V \setminus \{0\}$, a_v is unique. To show that T is a scalar multiple of I, we sho that a_v is indep. of v for $v \in V \setminus \{0\}$. Suppose $v, w \in V \setminus \{0\}$, we w.t.s $a_v = a_w$. Case 1: (v, w) is lin. dep., then $\exists \lambda \in \mathbb{F}$ s.t. $w = \lambda v$ which implies $a_w w = Tw = T(\lambda w) = \lambda Tv = \lambda(a_v w) = a_v w \Rightarrow a_v = a_w$. Case 2: (v, w) lin. ind., we have $a_{v+w}(v + w) = T(v + w) = Tv + Tw = a_v v + a_w w$. This implies $(a_{v+w} - a_v)v + (a_{v+w} - a_w)w = 0$, since (v, w) lin. ind. this implies $a_{v+w} = a_v$ and $a_{v+w} = a_w$ and we conclude $a_v = a_w$. $\square$

4. Prove $P^2 = P$, then $\text{null}P + \text{range}P = V \quad \text{null}P \cap \text{range}P = \{0\}$.

   Proof. Suppose $v \in \text{null}P \cap \text{range}P$. Then $Pv = 0$ and $\exists w \in V$ s.t. $v = Pw$. Applying P to both sides, $Pv = P^2 w = Pw$, but $Pv = 0 \Rightarrow Pw = 0$. Because $v = Pw \Rightarrow v = 0$. Since v is arbitrary $\text{null}P \cap \text{range}P = 0$. Suppose $v \in V$, then $v = (v - Pv) + Pv$. $P(v - Pv) = Pv - P^2 v = 0$, so $(v - Pv) \in \text{null}P$, hence $Pv \in \text{range}P$. Therefore, $v \in \text{null}P + \text{range}P$. $v \in V$ being arbitrary implies $v = \text{null}P + \text{range}P$. $\square$

5. Let $V = \mathbb{R}^4$, $x_1 = (1,0,4,2), x_2 = (2,3,7,6)$. Find an orthonormal basis of $\text{span}(x_1, x_2)$.

$$x_4 = x_2 - \frac{\langle x_2, x_1\rangle}{\|x_1\|^2}x_1 \quad \Rightarrow \quad x_4 = (0,3,-1,2) \quad \Rightarrow \quad (\frac{1}{\sqrt{21}}x_1, \frac{1}{\sqrt{14}}x_4)$$

6. Let $V = \mathbb{R}^5$, $x_1 = (3,0,0,2,1)$, $x_2 = (9,3,5,6,3)$. Same as 5.

$$x_4 = x_2 - 3x_1 = (0,3,5,0,0) \quad \Rightarrow \quad \langle \frac{x_1}{\|x_1\|}, \frac{x_4}{\|x_4\|}\rangle \quad \Rightarrow \quad \left(\frac{x_1}{\sqrt{14}}, \frac{x_4}{\sqrt{34}}\right)$$

7. Let $V = \mathbb{R}^4$, $e_1 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, $e_2 = (\frac{1}{\sqrt{2}}, 0, 0, -\frac{1}{\sqrt{2}})$, $U = \text{span}(e_1, e_2)$.

   (i) Verify (e_1, e_2) is an orthonormal basis of U.
   Proof. Take the inner product, if 0, then orthonormal. □

   (ii) Find $x \in U$ s.t. $\|(1,2,3,4) - x\| = \text{minimal}$.
   Proof. $\|y - x\| = \min$

$$\begin{aligned} x &= \langle y, e_1\rangle e_1 + \langle y, e_2\rangle e_2 \\ &= \langle (1,2,3,4), (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})\rangle (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \\ &+ \langle (1,2,3,4), (\frac{1}{\sqrt{2}}, 0, 0, -\frac{1}{\sqrt{2}})\rangle (\frac{1}{\sqrt{2}}, 0, 0, -\frac{1}{\sqrt{2}}) \\ &= (1, \frac{5}{2}, \frac{5}{2}, 1) \quad \text{(after number crunching)} \end{aligned} \tag{1}$$

□

8. Let V the space consisting of real polynomials with the inner product $\langle p, q\rangle = \int_0^1 p(x)q(x)dx$. Let $U = \text{span}(1, x, x^2)$. Given that (p_0, p_1, p_2) orthonormal basis of U where $p_0 = 1$, $p_1 = \sqrt{12}\left(x - \frac{1}{2}\right)$, $p_2 = \sqrt{180}\left(x^2 - x + \frac{1}{6}\right)$. Find $p \in U$ such that $\|x^4 - p\| = \text{minimal}$.

   Proof. Let $x^4 = p \Rightarrow p = \langle q, p_0\rangle p_0 + \langle q, p_1\rangle p_1 + \langle q, p_2\rangle p_2$

$$\langle x^4, p_0\rangle = \int_0^4 x^4 dx = \left[\frac{x^5}{5}\right]_0^1 = \frac{1}{5} \quad \text{(repeat for } \langle x^4, p_1\rangle, \langle x^4, p_2\rangle)$$

   Then plug each inner product into p. □

9. Suppose T is normal. Prove that $\|Tx\| = \|T^*x\|$ for every x.

   Proof. T normal implies

$$\begin{aligned} T^*T - T^*T = 0 &\iff \langle (T^*T - TT^*)v, v\rangle = 0 \quad \forall v \in V \\ &\iff \langle T^*Tv, v\rangle = \langle TT^*v, v\rangle \quad \forall v \in V \\ &\iff \|Tv\|^2 = \|T^*v\|^2 \quad \forall v \in V \end{aligned}$$

□

10. Suppose T is self-adjoint. Let $\alpha, \beta \in \mathbb{R}$ such that $\alpha^2 < \beta$. Prove the operator $T^2 + 2\alpha T + \beta I$ is invertible.

Proof. Let v a nonzero in V. Then

$$\begin{aligned}\langle (T^2 + \alpha T + \beta I)v, v\rangle &= \langle T^2 v + v\rangle + \alpha\langle Tv, v\rangle + \beta\langle v, v\rangle \\ &= \langle Tv, Tv\rangle + \alpha\langle Tv, v\rangle + \beta\|v\|^2 \\ &\geq \|Tv\|^2 - |\alpha|\|Tv\|\|v\| + \beta\|v\|^2 \quad \text{(By Cauchy-Schwarz)} \\ &= \left(\|Tv\| - \frac{\|\alpha\|\|v\|}{2}\right)^2 + \left(\beta - \frac{\alpha^2}{4}\right)\|v\|^2 \\ &> 0 \quad \text{(implies } (T^2 + \alpha T + \beta I)v \neq 0) \end{aligned} \tag{2}$$

So we can conclude that $T^2+\alpha T+\beta I$ is injective which implies $T^2+\alpha T+\beta I$ is invertible. $\square$

11. Prove that for every S, the S^*S is positive.

Proof. Let $T = S^*S$, then $T^* = (S^*S)^* = S^*(S^*)^* = S^*S = T$, and therefore, T is self-adjoint. Note that $\langle Tv, v\rangle = \langle S^*Sv, v\rangle = \langle Sv, Sv\rangle \geq 0 \quad \forall v \in V$. Therefore, T is positive. $\square$

12. Suppose that S is an isometry. Prove that $\langle Sx, Sy\rangle = \langle x, y\rangle$ for x, y.

Proof. Since S is an isometry, $\forall u, v \in V$.

$$\begin{aligned}\langle Su, Sv\rangle &= \frac{(\|Su + Sv\|^2 - \|Su - Sv\|^2)}{4} \\ &= \frac{(\|S(u + v)\|^2 - \|S(u - v)\|^2)}{4} \\ &= \frac{(\|u + v\|^2 - \|u - v\|^2)}{4} \\ &= \langle u, v\rangle \end{aligned} \tag{3}$$

Where second line is due to linearity of S and third is due to isometry of S. $\square$

13. Suppose N is self-adjoint and nilpotent. Prove that $N = 0$

Proof. Since N is self-adjoint, $\exists$ an orthonormal basis $(e_1, ..., e_n)$ of V consisting of eigenvectors of N by the spectral theorem. N being nilpotent implies that 0 is the only eigenvalue of N. Therefore, the eigenvalues corresponding to eache $e_j = 0$ which implies $Ne_j = 0 \quad \forall e_j$. Because $(e_1, ..., e_n)$ is the basis of V, $N = 0$. $\square$

14. Suppose that null$T^3 \neq$ nullT^4. Prove that null$T \neq$ nullT^2.

Proof. Suppose null$T$ = nullT^2. We know the **Proposition 8.5** in the book that if $T \in \mathcal{L}(V)$ and m is nonnegative integer s.t. nullT^m = nullT^{m+1} then

$$\text{null}T^0 \subset \text{null}T^1 \subset ... \subset \text{null}T^m = \text{null}T^m = \text{null}T^{m+1} = \text{null}T^{m+1} = ...$$

therefore, by this proposition, nullT^2 = nullT^4 which is a contradiction. So it must be the case that null$T^2 \neq$ null$T^4 \Rightarrow$ null$T \neq$ nullT^2 □

15. Find minimal matrix that is

(a) 4×4 with minimal polynomial $(z+1)^2(z-1)$.

(b) 5×5 with minimal polynomial $z(z-3)^2(z+4)$.

Proof.

(a) $\begin{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} & \cdots & 0 \\ \vdots & \begin{bmatrix} 1 \end{bmatrix} & 0 \\ 0 & 0 & \begin{bmatrix} 0 \end{bmatrix} \end{bmatrix}$ **(b)** $\begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & 0 & -4 \end{bmatrix}$

□

16. Find the minimal polynomial of:

(a) $\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$

Proof.

(a) $(z-2)^3(z-1)$

(b) $(z-3)^2(z+3)^2(z-4)$

□

Gram-Schmidt: For $\dim V = 2$, $x_1, x_2 \in U$ are lin. ind.

$$x_3 = x_1 - \frac{\langle x_1, x_2 \rangle}{\|x_2\|^2} x_2$$

then $\langle x_3, x_2 \rangle = 0$(this is orthogonal) and

$$x_4 = x_2 - \frac{\langle x_2, x_1 \rangle}{\|x_1\|^2} x_1$$

then $\langle x_4, x_1 \rangle = 0$ is orthogonal. To find orthonormal basis (y_1, y_2) of U:

$$\text{Either} \quad \begin{cases} y_1 = \frac{1}{\|x_3\|} x_3 \\ y_2 = \frac{1}{\|x_2\|} x_2 \end{cases} \qquad \text{Or} \quad \begin{cases} y_1 = \frac{1}{\|x_1\|} x_1 \\ y_2 = \frac{1}{\|x_4\|} x_4 \end{cases}$$