Math 104B Homework 4

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1 Problem 1

Find α so that

$$A = \begin{bmatrix} \alpha & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 4 \end{bmatrix}$$

is positive definite.

Proof. Letting $\alpha = 4$, we have that a symmetric matrices, and using $A \cdot x = \lambda \cdot x$, we find that the eigenvalues are $\lambda = 5, 4, 1$. Therefore, since we have a symmetric matrix with positive eigenvalues, A is positive definite by definition. $\square$

2 Problem 2

Let A and B be $n \times n$ matrices. Prove that

$$||A + B||_\infty \leq ||A||_\infty + ||B||_\infty$$

Proof. We know by triangle inequality that

$$||u_i + v_i|| \leq ||u_i|| + ||v_i||$$

Meanwhile, definition of infinity norm is given by:

$$||v||_\infty := \lim_{p \to \infty} ||v||_p = \max \{|v_i|, i = 1, 2, ..., n\}$$

Furthermore, it is obvious that

$$||A_i + B_i|| \leq \max \{|A_i + B_i|, i \in \mathbb{N}\}$$

where A_i and B_i are the components of A, B respectively. Similarly,

$$||A_i|| + ||B_i|| \leq \max\{A_i, i \in \mathbb{N}\} + \max\{B_i, i \in \mathbb{N}\}$$

Therefore, by triangle inequality:

$$\begin{aligned} ||A_i + B_i|| &\leq \max\{|A_i + B_i|, i \in \mathbb{N}\} \\ &\leq ||A_i|| + ||B_i|| \\ &\leq \max\{A_i, i \in \mathbb{N}\} + \max\{B_i, i \in \mathbb{N}\} \end{aligned} \tag{1}$$

Which, by the second and last inequality, along with the definition of infinity norms, imply that

$$||A + B||_\infty \leq ||A||_\infty + ||B||_\infty$$

$\square$

3 Problem 3

Let

$$A = \begin{bmatrix} 2 & 1 & -10 \\ 1 & 2 & 1 \\ -5 & 1 & 4 \end{bmatrix}$$

Find $||A||_1$ and $||A||_\infty$.
Proof. We define $||A||_1 = \max_j \sum_{i=1}^n |a_{ij}|$ therefore:

$$\begin{aligned} ||A||_1 &= \max(|2| + |1| + |-5|, |1| + |2| + |1|, |-10| + |1| + |4|) \\ &= \max(8, 4, 15) \\ &= 15 \end{aligned} \tag{2}$$

And $||A||_\infty = \max_i \sum_{j=1}^n |a_{ij}|$, therefore:

$$\begin{aligned} ||A||_1 &= \max(|2| + |1| + |-10|, |1| + |2| + |1|, |-5| + |1| + |4|) \\ &= \max(13, 4, 11) \\ &= 13 \end{aligned} \tag{3}$$

$\square$

4 Problem 4

Let S be a nonsingular $n \times n$ matrix and let $||\cdot||$ be an induced matrix norm. Prove that $||S^{-1}AS||$ defines a matrix norm for all $n \times n$ matrices A.
Proof. If $||\cdot||$ is an induced matrix norm, then we know that

$$\|Ax\| \leq \|A\|\|x\|$$

and

$$\|AB\| \leq \|A\|\|B\|$$

Then letting x be an arbitraty matrix..? $\square$

5 Problem 5

Let I be the $n \times n$ identity matrix. Prove that $||I|| = 1$ for all induced norms.
Proof. We know that for some arbitrary vector x, $||Ix|| = ||x||$, and using the definition from the previous problem, clearly:

$$||I|| = \max_{x \neq 0} \frac{||x||}{||x||} = 1$$

and

$$||I|| = \max ||x|| = 1||Ix|| = ||I||$$

Thereby satisfying the definition in the previous problem. $\square$

6 Problem 6

Prove that the condition number of a nonsingular $n \times n$ matrix A is at least 1, i.e. $1 \leq ||A||||A^{-1}||$, for all induced matrix norms.
Proof. The condition number of a matrix A is defined as:

$$\kappa(A) = ||A||||A^{-1}||$$

By **Theorem 9.3(b)**, which states that if $||\cdot||$ is an induced matrix norm, then $||AB|| \leq ||A||||B||$, so we know that

$$\begin{aligned} ||A||||A^{-1}|| &\geq ||AA^{-1}|| \\ &= ||I|| \\ &= 1 \end{aligned} \tag{4}$$

$\square$

7 Problem 7

Let

$$A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

Find $||A||_2$.

Proof. We define $||A||_2$ as the square root of the largest eigenvalue of $A^T A$, but if $A^T = A$, then $||A||_2$ is simply the largest eigenvalue. Using the same equation in **Problem 1**, we find that the eigenvalues of A in this problem are $\lambda = 4, 2$. Since $A^T = A$, it must be the case that $\max \lambda = 4 = ||A||_2$. $\square$

8 Problem 8

Compute the condition number $\kappa_1(A) = ||A||_1||A^{-1}||_1$ for

$$A = \begin{bmatrix} 1 & 1+\epsilon \\ 1-\epsilon & 1 \end{bmatrix}$$

Proof. By the same process as **Problem 3**, we find $||A||_1$ using:

$$\begin{aligned} ||A||_1 &= \max(1 + |1-\epsilon|, 1+\epsilon+1) \\ &= \max(1+|1-\epsilon|, 2+\epsilon) = 2+\epsilon \quad (\text{for } \epsilon > 0) \end{aligned} \tag{5}$$

Now, A^{-1} is

$$A^{-1} = \frac{1}{\epsilon^2} \begin{bmatrix} 1 & -1-\epsilon \\ -1+\epsilon & 1 \end{bmatrix}$$

So, $||A^{-1}||$ is given by

$$\begin{aligned} ||A^{-1}||_1 &= \max\left(\left|\frac{1}{\epsilon^2}\right| + \left|\frac{1}{\epsilon^2} - \frac{1}{\epsilon}\right|, \left|\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\right| + \left|\frac{1}{\epsilon^2}\right| \right) \\ &= \frac{2}{\epsilon^2} + \frac{1}{\epsilon} \end{aligned} \tag{6}$$

Therefore, our condition number is

$$\kappa_1(A) = ||A||_1||A^{-1}||_1 = (2+\epsilon)\left(\frac{2}{\epsilon^2} + \frac{1}{\epsilon}\right)$$

$\square$

9 Problem 9

Prove that the condition number satisfies the property $\kappa(\lambda A) = \kappa(A)$ for all nonzero λ, scalar.
Proof. By definition, the left side of the equation is

$$\kappa(\lambda A) = \|\lambda A\| \|\frac{1}{\lambda} A^{-1}\|$$

Since λ is a scalar, by **9.5(iii)** in the textbook,

$$\begin{aligned} \|\lambda A\| \|\frac{1}{\lambda} A^{-1}\| &= |\lambda| \|A\| |\frac{1}{\lambda}| \|A^{-1}\| \\ &= |\frac{\lambda}{\lambda}| \|A\| \|A^{-1}\| \\ &= \|A\| \|A^{-1}\| \\ &= \kappa(A) \quad \text{(by definition of condition number)} \end{aligned} \tag{7}$$

Thereby proving the property $\kappa(\lambda A) = \kappa(A)$ $\square$