Math 104B Homework 2

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1.) Let $f(x) = e^x$

(a) Compute the centered difference approximation of $f'(\frac{1}{2})$, i.e. $D_h^0 f(\frac{1}{2})$, for $h = \frac{0.1}{2^n}$, $n = 0, 1, \dots, 10$ and verify its quadratic rate of convergence.

(b) Determine approximately the optimal value of h_0 which gives the minimum total error (the sum of discretization error plus the round-off error) and verify this numerically.

(c) Construct and compute a fourth order approximation to $f'(\frac{1}{2})$ by applying Richardson's extrapolation to $D_h^0 f(\frac{1}{2})$.
Verify the rate of convergence numerically.
What is the optimal h_0 in this case?

Proof:

(a) Centered difference formula is defined as

$$D_h^0 f(x_0) = \frac{f(x_0+h) - f(x_0-h)}{2h}$$

Therefore, $D_h^0 f(\frac{1}{2})$ for $h = \frac{0.1}{2^n}$, $n = 0, 1, \dots, 10$ is given below:

In [18]:
```
from math import e
f = lambda x : e**x
x_0 = 1/2
error_list = {}
for n in range(10):
    h=0.1/(2**n)
    df_x = (f(x_0+h)-f(x_0-h))/(2*h)
    print(f"At h={h}, the derivative of f(x) at x=1/2 is: {df_x}, it's error is: {abs(f(1/2)-df_x)}")
    error_list[h] = abs(f(1/2)-df_x)
min_error = min(error_list, key=error_list.get)
print(f"Least h error: {min_error}")
```

```
At h=0.1, the derivative of f(x) at x=1/2 is: 1.6514705137461927, it's error is: 0.002749243046064498
At h=0.05, the derivative of f(x) at x=1/2 is: 1.6494083237722656, it's error is: 0.0006870530721374557
At h=0.025, the derivative of f(x) at x=1/2 is: 1.6488930178661754, it's error is: 0.00017174716604717588
At h=0.0125, the derivative of f(x) at x=1/2 is: 1.6487642064853159, it's error is: 4.293578518765884e-05
At h=0.00625, the derivative of f(x) at x=1/2 is: 1.6487320045835219, it's error is: 1.0733883393676535e-05
At h=0.003125, the derivative of f(x) at x=1/2 is: 1.6487239541670462, it's error is: 2.683466918007582e-06
At h=0.0015625, the derivative of f(x) at x=1/2 is: 1.6487219415666488, it's error is: 6.708665205579223e-07
At h=0.00078125, the derivative of f(x) at x=1/2 is: 1.6487214384167714, it's error is: 1.6771664324011226e-07
At h=0.000390625, the derivative of f(x) at x=1/2 is: 1.6487213126291067, it's error is: 4.192897851140742e-08
At h=0.0001953125, the derivative of f(x) at x=1/2 is: 1.6487212811830432, it's error is: 1.0482914980514124e-08
Least h_0 error: 0.0001953125
```

(b) The most optimal value for h_0 is 0.0001953125 since this returns the most minimal total error.

(c) Richardson's extrapolation of order four is:

$$D_h^{ext} f(x_0) = \frac{4D_{\frac{h}{2}}^0 f(x_0) - D_h^0 f(x_0)}{3} \tag{1}$$

Using this, we compute:

In [19]:
```
error_list.clear()
for n in range(10):
    h=0.1/(2**n)
    df_x_h_half = (f(x_0+(h/2))-f(x_0-(h/2)))/(2*(h/2))
    df_x = (f(x_0+h)-f(x_0-h))/(2*h)
    df_R_ext = ((4*df_x_h_half) - df_x)/3
    print(f"At h={h}, the derivative of f(x) using 4th order Richardson's extrapolation at x=1/2 is: {df_R_ext}, it's error is: {abs(f(1/2)-df_R_ext)}")
    error_list[h] = abs(f(1/2)-df_R_ext)

min_error = min(error_list, key=error_list.get)
print(f"Least h error: {min_error}")
```

```
At h=0.1, the derivative of f(x) using 4th order Richardson's extrapolation at x=1/2 is: 1.64872092711429, it's error is: 3.4358583822502453e-07
At h=0.05, the derivative of f(x) using 4th order Richardson's extrapolation at x=1/2 is: 1.648721249230812, it's error is: 2.1469316102695757e-08
At h=0.025, the derivative of f(x) using 4th order Richardson's extrapolation at x=1/2 is: 1.6487212693583626, it's error is: 1.341765587525856e-09
At h=0.0125, the derivative of f(x) using 4th order Richardson's extrapolation at x=1/2 is: 1.6487212706162573, it's error is: 8.387091021688775e-11
At h=0.00625, the derivative of f(x) using 4th order Richardson's extrapolation at x=1/2 is: 1.6487212706948877, it's error is: 5.240474720835664e-12
At h=0.003125, the derivative of f(x) using 4th order Richardson's extrapolation at x=1/2 is: 1.6487212706998495, it's error is: 2.786659791809143e-13
At h=0.0015625, the derivative of f(x) using 4th order Richardson's extrapolation at x=1/2 is: 1.6487212707001457, it's error is: 1.7541523789077473e-14
At h=0.00078125, the derivative of f(x) using 4th order Richardson's extrapolation at x=1/2 is: 1.648721270699885, it's error is: 2.431388423929093e-13
At h=0.000390625, the derivative of f(x) using 4th order Richardson's extrapolation at x=1/2 is: 1.648721270701022, it's error is: 8.93729534823251e-13
At h=0.0001953125, the derivative of f(x) using 4th order Richardson's extrapolation at x=1/2 is: 1.6487212707011167, it's error is: 9.885425811262394e-13
Least h error: 0.0015625
```

The most optimal h_0 in this case is 0.0015625.

2.) Use Taylor expansions to derive the error term of the sided difference approximation to $f'(x_0)$:

$$D_h f(x_0) = \frac{-f(x_0+2h) + 4f(x_0+h) - 3f(x_0)}{2h}$$

Proof:

By the notes, we know that the Taylor series expansion of $f(x_0+h)$ is:

$$f(x_0+h) = f(x_0) + f'(x_0)h + \frac{1}{2!}f''(x_0)h^2 + \frac{1}{3!}f^{(3)}(x_0)h^3 + \dots$$

This implies that $f(x_0+2h)$ is given by:

$$\begin{aligned} f(x_0+2h) &= f(x_0) + f'(x_0)(2h) + \frac{1}{2!}f''(x_0)(2h)^2 + \frac{1}{3!}f^{(3)}(x_0)(2h)^3 + \dots \\ &= f(x_0) + f'(x_0)2h + \frac{1}{2!}f''(x_0)4h^2 + \frac{1}{3!}f^{(3)}(x_0)8h^3 + \dots \end{aligned} \tag{2}$$

Using this, we know $D_h f(x_0)$ is:

$$\begin{aligned} D_h f(x_0) &= \frac{-f(x_0) - 2f'(x_0)h - \frac{4}{2!}f''(x_0)h^2 - \frac{8}{3!}f^{(3)}(x_0)h^3 - \dots + 4f(x_0) + 4f'(x_0)h + \frac{4}{2!}f''(x_0)h^2 + \frac{4}{3!}f^{(3)}(x_0)h^3 + \dots - 3f(x_0)}{2h} \\ &= \frac{2f'(x_0)h - \frac{4}{3!}f^{(3)}(x_0)h^3 - \dots}{2h} \\ &= f'(x_0) - \frac{2}{3!}f^{(3)}(x_0)h^2 - \dots \end{aligned} \tag{3}$$

Since the error is $|f'(x_0) - D_h f(x_0)|$, the error term is $\frac{2}{3!}f^{(3)}(x_0)h^2 - \dots$

3.) Consider the data points $(x_0, f_0), (x_1, f_1), \dots, (x_n, f_n)$, where the points $x_0, x_1, \dots, x_n$ are distinct but otherwise arbitrary (they could, for example, be Chebyshev nodes).
Then the derivative of the interpolating polynomial of these data is:

$$P_n'(x) = \sum_{j=0}^{n} l_j'(x) f_j,$$

where the l_j's are the elementary Lagrange polynomials:

$$l_j(x) = \frac{1}{\alpha_j}\prod_{\substack{k=0\\k\neq j}}^{n}(x - x_k), \quad \alpha_j = \prod_{\substack{k=0\\k\neq j}}^{n}(x_j - x_k)$$

We can evaluate the first equation at each nodes $x_0, x_1, \dots, x_n$ which will give us an approximation to the derivative of f at those points, i.e. $f'(x_i) \approx P_n'(x_i)$. We can write this as

$$\mathbf{f}' \approx D_n \mathbf{f}$$

where $\mathbf{f} = [f_0 f_1 \dots f_n]^T$, $\mathbf{f}' = [f(x_0) f(x_1) \dots f'(x_n)]^T$ and D_n is the $\mathbf{Differentiation\ Matrix}$, $(D_n)_{ij} = l_j'(x_i)$.

(a) Prove that:

$$l_j'(x) = l_j(x)\sum_{\substack{k=0\\k\neq j}}^{n}\frac{1}{x - x_k}$$

Hint: differentiate $\log l_j(x)$.

(b) Using the equation above, prove that

$$D(n_n)_{ij} = \frac{\alpha_i}{\alpha_j}\left(\frac{1}{x_i - x_j}\right), \quad i \neq j$$

$$D(n_n)_{ii} = \sum_{\substack{k=0\\k\neq i}}^{n}\frac{1}{x - x_k}$$

(c) Prove that:

$$\sum_{j=0}^{n}(D_n)_{ij} = 0 \quad \text{for all } i = 0, 1, \dots, n$$

(d) Obtain D_2 for the Chebyshev points $x_0 = -1, x_1 = 0, x_2 = 1$

Proof:

(a) Per the hint, taking the the $\log$ of $l_j(x)$ yields:

$$\begin{aligned} \log l_j(x) &= \log\left(\frac{\prod_{\substack{k=0\\k\neq j}}^{n}(x - x_k)}{\prod_{\substack{k=0\\k\neq j}}^{n}(x_j - x_k)}\right) \\ &= \log\left(\prod_{\substack{k=0\\k\neq j}}^{n}(x - x_k)\right) - \log\left(\prod_{\substack{k=0\\k\neq j}}^{n}(x_j - x_k)\right) \end{aligned} \tag{4}$$

Then differentiating with respect to x gives us:

$$\begin{aligned} \frac{l_j'(x)}{l_j(x)} &= \frac{1}{x - x_0} + \frac{1}{x - x_1} + \dots + \frac{1}{x - x_{j-1}} + \frac{1}{x - x_{j+1}} + \dots + \frac{1}{x - x_n} \\ &= \sum_{\substack{k=0\\k\neq j}}^{n}\frac{1}{x - x_k} \end{aligned} \tag{5}$$

Finally, multiplying by $l_j(x)$ to both sides:

$$l_j'(x) = l_j(x)\sum_{\substack{k=0\\k\neq j}}^{n}\frac{1}{x - x_k}$$

(b) We are given that $(D_n)_{ij} = l_j'(x_i)$, where D_n is the Differentiation Matrix.
Therefore, equating the given with the result from **(a)**:

$$(D_n)_{ij} = l_j(x_i)\sum_{\substack{k=0\\k\neq j}}^{n}\frac{1}{x_i - x_k}$$

Furthermore, we know

$$l_j(x_i) = \frac{\prod_{\substack{k=0\\k\neq j}}^{n}(x_i - x_k)}{\prod_{\substack{k=0\\k\neq j}}^{n}(x_j - x_k)} = \frac{\alpha_i}{\alpha_j}, \quad i \neq j$$

Using this, $(D_n)_{ij}$ becomes:

$$(D_n)_{ij} = \frac{\alpha_i}{\alpha_j}\left(\sum_{\substack{k=0\\k\neq j}}^{n}\frac{1}{x_i - x_k}\right) = \frac{\alpha_i}{\alpha_j}\left(\frac{1}{x_i - x_j}\right), \quad i \neq j$$

And if $i = j$, $l_j(x_i) = \frac{\alpha_i}{\alpha_i} = 1$, so $(D_n)_{ii}$ is simply:

$$(D_n)_{ii} = \sum_{\substack{k=0\\k\neq i}}^{n}\frac{1}{x_i - x_k}$$

(c) $\sum_{j=0}^{n}(D_n)_{ij} = 0 \quad \text{for all } i = 0, 1, \dots, n$ is given as:

$$(D_n)_{i0} + (D_n)_{i1} + (D_n)_{i2} + \dots =$$

(d) We can find this using

$$(D_2)_{ij} = \frac{\alpha_i}{\alpha_j}\left(\frac{1}{x_i - x_j}\right), \quad i \neq j$$

and

$$(D_2)_{ii} = \sum_{\substack{k=0\\k\neq i}}^{n}\frac{1}{x_i - x_k}, \quad \text{otherwise}$$

Now computing each entry of the Differentiation Matrix:

$$(D_2)_{00} = \sum_{\substack{k=0\\k\neq i}}^{2}\frac{1}{x_i - x_k} = -\frac{3}{2}$$

$$(D_2)_{01} = 2, (D_2)_{02} = -\frac{1}{2}, (D_2)_{10} = -\frac{1}{2}, (D_2)_{11} = 0, (D_2)_{12} = \frac{1}{2}, (D_2)_{20} = \frac{1}{2}, (D_2)_{21} = -2, (D_2)_{22} = \frac{3}{2}$$

And so our differentiation matrix D_2 is:

$$D_2 = \begin{bmatrix} -\frac{3}{2} & 2 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -2 & \frac{3}{2} \end{bmatrix}$$