Math 108B Homework 3

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Let V be an inner product space over F, and $T \in \mathcal{L}(V)$

1 Problem 1

Prove that if $U = \text{range } (T)$, then $U^{\perp} = \text{null } T^*$.
Proof. Suppose $T \in \mathcal{L}(V, W)$, and let $U = \max T$. By a previous result,

$$w \in \text{null } T^* \Leftrightarrow T^*w = 0$$

where $w \in W$, then

$$w \in \text{null } T^* \Leftrightarrow \langle v, T^*w \rangle = 0 \quad \forall v \in V$$

$$w \in \text{null } T^* \Leftrightarrow \langle Tv, w \rangle = 0 \quad \forall v \in V$$

$$w \in \text{null } T^* \Leftrightarrow w \in (\text{range } T)^{\perp} = (U)^{\perp}$$

Thus we have $(U)^{\perp} = \text{null } T^*$. $\square$

2 Problem 2

Suppose $P \in \mathcal{L}(V)$ is such that $P^2 = P$. Prove that P is an orthogonal projection if and only if P is self-adjoint.
Proof. For one direction, we assume that P is a orthogonal projection. Then we use it's symmetry, denoted by P_u, where $u \in U$ and taking $w \in U^\perp$, $v \in V \Rightarrow v = u + w$. Taking two arbitrary v we have $v_1 = u_1 + w_1$ and $v_2 = u_2 + w_2$ which results in

$$\begin{aligned}
\langle Pv_1, v_2 \rangle &= \langle u_1, u_2 + w_2 \rangle \\
&= \langle u_1 + u_2 \rangle + \langle u_1 + w_2 \rangle \\
&= \langle u_1, u_2 \rangle \\
&= \langle u_1, u_2 \rangle + \langle w_1, u_2 \rangle \\
&= \langle u_1, w_1, u_2 \rangle \\
&= \langle u_1, Pv_2 \rangle
\end{aligned} \tag{1}$$

Hence $P = P^*$ and P is self-adjoint. For the other direction, we suppose that P is self-adjoint. Then taking $v \in V$, we know that $P(v - Pv) = Pv - P^2v = 0$ meaning

$$v - Pv \in \text{null } P = (\text{range } P^*)^\perp = (\text{range } P)^\perp$$

by the homework, and we can manipulate to get

$$v = Pv + (v - Pv)$$

□

3 Problem 3

Prove that if T is normal, then range $T =$ range T^*
Proof. Since T is normal we know that

$$\text{range } T = (\text{null } T^*)^\perp = (\text{null } T)^\perp = \text{range } T^*$$

□

4 Problem 4

Prove that if T is normal, then

$$\text{null } T^k = \text{null } T \quad \text{and} \quad \text{range } T^k = \text{range } T$$

for every positive integer k.

Proof. If $k = 1$, then this is trivial, and now since k is a positive integer, we consider $k \geq 2$. For the first, we know that if $v \in \text{null } T$, then $T^k v = T^{k-1}(Tv) = T^{k-1}0 = 0$. Therefore, $v \in \text{null } T^k$ and so $\text{null } T \subset \text{null } T^k$. Similarly, for $TT^* = T^*T$, we follow

$$\begin{aligned} \langle T^*T^{k-1}v, T^kT^{k-1}\rangle &= \langle T^*T^{k-1}v, T^{k-1}\rangle \\ &= \langle TvT^{k-1}\rangle \\ &= 0 \end{aligned} \tag{2}$$

This implies $T^*T^{k-1}v$ is orthogonal, meaning $T^{k-1}v$ is orthogonal by 7.4. By way of induction, $v \in \text{null } T^{k-1} \subset \text{null } T$ proves $\text{null } T^k = \text{null } T$. For range T, we know $\text{range } T = Tv \quad \forall v \in V$. So, for $v \in \text{range } T^k$, there exists $w \in V$ such that $T^k w = V$ implying

$$V = T^k w = (T^{k-1})Tw \Rightarrow v \in \text{range } T$$

This result shows that $\text{range } T^k \subset \text{range } T$, now using

$$\dim V = \dim \text{null } T + \dim \text{range } T$$

we get

$$\begin{aligned} \dim \text{range } T^k &= \dim V - \dim \text{null } T^k \\ &= \dim V - \dim \text{null } T \\ &= \dim \text{range } T \end{aligned} \tag{3}$$

So we conclude that $\text{range } T^k = \text{range } T$. $\square$

5 Problem 5

Prove that there does not exist a self-adjoint operator $T \in \mathcal{L}(\mathbb{R}^3)$ such that $T(1,2,3) = (0,0,0)$ and $T(2,5,7) = (2,5,7)$

Proof. By way of contradiction, we suppose $T \in \mathcal{L}(\mathbb{R}^3)$ where $T(1,2,3) = (0,0,0)$ and $T(2,5,3) = (2,5,7)$. Using $Tu = \lambda u$, we find the eigenvector $(1,2,3)$ has $\lambda_1 = 0$ and $(2,5,7)$ has $\lambda_2 = 1$. We assume that the corresponding eigenvectors of T are orthogonal such that for $\alpha = \lambda_1 = C$, $\beta = \lambda_2 = 1$, and $u = (1,2,3)$, $v = (2,5,7)$

$$(\alpha - \beta)\langle u, v\rangle = 0$$

Because $\langle u, v\rangle \neq 0$, T is not a self-adjoint operator. □

6 Problem 6

Give a counterexample to show that the product of two self-adjoint operators is not necessarily self-adjoint.

Proof. We begin by letting $S, T \in \mathcal{L}(V^2)$ be two operators with matrices

$$M(S) = \begin{bmatrix} 2 & 3 \\ 3 & 7 \end{bmatrix} \qquad M(T) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

So $M(S^*) = S$ and $M(J^*) = T$. So,

$$M(S) \cdot M(T) = \begin{bmatrix} 3 & 3 \\ 2 & 7 \end{bmatrix} \neq \begin{bmatrix} 3 & 2 \\ 3 & 7 \end{bmatrix}$$

and the product is not necessarily self-adjoint. □

7 Problem 7

Suppose $F = \mathbb{C}$. Prove that a normal operator on V is self-adjoint if and only if all its eigenvalues are real.

Proof. Letting $T \in \mathcal{L}(V)$ be normal. If T is self-adjoint then every corresponding eigenvalue must be real by 7.1. Now, suppose all the eigenvalues of a normal $T \in \mathcal{L}(V)$ are real. By 7.9, we know V gives us an orthonormal basis of the respective eigenvectors of T, specifically, $(e_1, ..., e_n) \in V$. From this we obtain a diagonal matrix for T, and T^* gives us a diagonal matrix as well. This implies that T and T^* commute and so T must be self-adjoint. □

8 Problem 8

Suppose $F = \mathbb{C}$ and T is a normal operator on V. Prove that there is a $S \in \mathcal{L}(V)$ such that $T = S^2$.
Proof. Letting $T \in \mathcal{L}(V)$ be a normal operator on the complex, inner-product space V. By 7.10, we know the orthonormal basis $(e_1, ..., e_n)$ of V consists of eigenvectors of T such that $T(e_j) = \lambda_{j^{\frac{1}{2}}} e_j$ such that each $\lambda_{j^{\frac{1}{2}}}$ represents the square root at every λ_j of T. Then,

$$\begin{aligned} T(e_j) &= (\lambda_{j^{\frac{1}{2}}} e_j)(\lambda_{j^{\frac{1}{2}}} e_j) \\ &= (\lambda_{j^{\frac{1}{2}}} e_j)^2 \end{aligned} \tag{4}$$

implies $T = S^2$. $\square$

9 Problem 9

Prove that if T is a positive operator on V, then T^k is positive for every positive integer k.
Proof. Letting $T \in \mathcal{L}(V)$ be a positive operator on V, for an arbitrary $k \in \mathbb{N}$ (positive integers), implies that T^k is self-adjoint by 7.24. Now looking at the case for when k is even we have that $k = 2n$ for $n \in \mathbb{N}$. By definition of positive operators, yields

$$\begin{aligned} \langle T^k v \rangle &= \langle T^{2n} v, v \rangle \\ &= \langle T^n \cdot T^n v, v \rangle \\ &= \langle T^n, T^n v \rangle \\ &\geq 0, \quad \forall v \in V \end{aligned} \tag{5}$$

Hence T^k is positive. Now for the case of when k is odd, we know $k = 2n+1$ for some $n \in \mathbb{N}$. By a similar process as the even case we have

$$\begin{aligned} \langle Tv, v \rangle &= \langle T^{2n+1} v, v \rangle \\ &= \langle T(T^{2n} v), v \rangle \\ &= \langle T(T^n \cdot T^n) v, v \rangle \\ &= \langle T(T^n) v, T^n v \rangle \\ &\geq 0 \quad \forall v \in V \end{aligned} \tag{6}$$

And again, $T > 0 \Rightarrow T^k$ is positive. So for both cases T^k is positive for all $k \in \mathbb{N}$. $\square$

10 Problem 10

Suppose T is a positive operator on V. Prove that T is invertible if and only if $\langle Tx, x\rangle$ is positive for $x \in V \setminus \{0\}$.

Proof. Letting T be a positive operator on V, we assume that T is invertible. Also, letting $x \in V \setminus \{0\}$ we have, by definition, a unique inverse of T denoted by $T^{-1} \neq 0$. Using 7.26d yields

$$\exists S \in \mathcal{L}(V) \Rightarrow T = S^*S$$

and it follows that

$$\begin{aligned} \langle Tx, x\rangle &= \langle S^*Sx, x\rangle \\ &= \langle Sx, Sx\rangle \end{aligned} \tag{7}$$

Since T is invertible, $Tx \neq 0$, hence

$$\langle Tx, x\rangle > 0$$

Now assuming the converse, $\langle Tx, x\rangle$ is positive for all $x \in V \setminus \{0\}$, and letting $u \in V \setminus \{0\}$. Then by 3.17, where $Tx = Tu$, we observe that

$$x = T^{-1}(Tx) = T^{-1}(Tu) = u \cdot x = u$$

and we can conclude that T is injective, and therefore invertible. $\square$

11 Problem 11

Prove that if $S \in \mathcal{L}(\mathbb{R}^3)$ is an isometry, then there exists a nonzero vector $x \in \mathbb{R}^3$ such that $S^2x = x$.

Proof. Assume $S \in \mathcal{L}(R^3)$ is an isometry, by definition $\|Sv\| = \|v\|$. 7.38 implies when $\mathbb{R}^3$ is an odd-dimnesional real inner-product space, there exists an orthonormal basis of V with respect to S. Hence S includes a block diagonal matrix where each block is of the form $|x|$ containing 1 or -1 or a 2x2 matrix. And so, at least 1 or -1 is an eigenvalue of S and we can conclude that in either case $Sx = \lambda x$, that is

$$S^2x = S(Sx) = S(\lambda x) = \lambda(\lambda x) = \lambda^2 x = x$$

$\square$