Math 119A Homework 4

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1 Problem 1

Prove or disprove that matrix E given by

$$\begin{bmatrix} 0 & -\sqrt{2} \\ \sqrt{2} & 0 \end{bmatrix}$$

with basis $(0, -\sqrt{2}, \sqrt{2})$, and $(1, -1, -1)$ is a two-dimensional matrix $E \subset \mathbb{R}^2$ that satisfies $T|E$ of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.
Proof.

$\square$

2 Problem 2

Prove or disprove that

$$\begin{aligned} x_1 &= Ce^t - B\cos(\sqrt{2}t) + A\sin(\sqrt{2}t) \\ x_2 &= (2B - A\sqrt{2})\cos(\sqrt{2}t) - B(\sqrt{2} + 2A)\sin(\sqrt{2}t) \\ x_3 &= (B + A\sqrt{2})\cos(\sqrt{2}t) + (B\sqrt{2} - A)\sin(\sqrt{2}t) \end{aligned} \tag{1}$$

is the solution to $x' = Tx$ for the operator T given in Problem 1.
Proof.

$\square$

3 Problem 3

Prove or disprove that $A = 1,\ B = \sqrt{n}$ satisfies the largest $A > 0$ and smallest $B > 0$ such that

$$A|x| \leq |x|_{sum} \leq B|x|$$

for all $x \in \mathbb{R}^n$.
Proof.

□

4 Problem 4

Prove or disprove that the following

(a) $\sqrt{2}$

(b) $\frac{1}{2}$

(c) 1

(d) $\frac{1}{2}$

are norms to the vector $(1, 1) \in \mathbb{R}^2$ under following

(a) The Eucledian norm;

(b) The Eucledian B-norm, where B, is the basis $\{(1, 2), (2, 2)\}$;

(c) The max norm;

(d) The B-max norm

Proof.

(a)

□

5 Problem 5

Prove or disprove that $(x^2 + xy + y^2)^{\frac{1}{2}}$ and $\frac{1}{2}(|x| + |y|) + \frac{2}{3}(x^2 + y^2)^{\frac{1}{2}}$ are norms defined in $\mathbb{R}^2$.
Proof.

□

6 Problem 6

Prove or disprove that 1 is the uniform norm of the following operator in $\mathbb{R}^2$

$$\begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix}$$

Proof.

$\square$

7 Problem 7

In the vector space $L(\mathbb{R}^2)$, let T be the transformation defined by

$$T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c - 3a & d - 3b \end{bmatrix}$$

(a) is T linear?

(b) Does there exist a 2×2 matrix A such that $AB = T(B)$ for all 2×2 matrices B?

(c) Does there exist a 2×2 matrix A such that $BA = T(B)$ for all 2×2 matrices B?

Proof.

(a)

$\square$

8 Problem 8

Show that

$$\|T\| \cdot \|T^{-1}\| \geq 1$$

for every invertible operator T.
Proof.

$\square$

9 Problem 9

Let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be an operator that leaves a subspace $E \subset \mathbb{R}^2$ invariant. Let $x : \mathbb{R} \to \mathbb{R}^2$ be a solution of $x' = Ax$. If $x(t_0) \in E$ for some $t_0 \in \mathbb{R}$, show that $x(t) \in E$ for all $t \in \mathbb{R}$.

Proof.

$\square$

10 Problem 10

Suppose $A \in L(\mathbb{R}^2)$ has a real eigenvalue $\lambda < 0$. Then the equation $x' = Ax$ has at least one nontrivial solution $x(t)$ such that

$$\lim_{t \to \infty} x(t) = 0$$

.

Proof. Is this a question? $\square$