Math 108B Homework 4

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For all problems:

- V is a vector space over $\mathbb{C}$
- $\dim V = n$
- $x, y \in V$
- $V, T, S, N \in \mathcal{L}(V)$
- $N \in \mathcal{L}(V)$ is nilpotent if $N^k = 0$ for some $k > 0$.

1 Problem 1

Prove that if $T^{m-1}x \neq 0$, $T^m x = 0$, then the set $\{x, Tx, ..., T^{m-1}x\}$ is linearly independent.

Proof. Taking $a_0, a_1, ..., a_{m-1} \in F$, where

$$a_0 x + a_1 Tx + a_2 T^2 v + ... + a_{m-1} T^{m-1} x = 0 \tag{1}$$

Due to our assumption that $T^{m-1} \neq 0$ we can apply it to both sides of (1). This gives us a $T^m x$ for all the terms except for the first and by the second assumption that we are left with:

$$T^{m-1} a_0 x = 0$$

Again, since $T^{m-1}x \neq 0$, we can deduce that $a_0 = 0$ which allows us to rewrite (1) as:

$$a_1 Tx + a_2 T^2 x + ... + a_{m-1} T^{m-1} x = 0 \tag{2}$$

Similarly, applying T^{m-2} leaves a T^m for every term except for the first giving us

$$T^{m-1}a_1x = 0$$

Therefore, $a_1 \neq 0$ by the same argument as before. We can repeat this process until the last term allowing us to conclude that $a_0 = a_1 = a_2 = ... = a_{m-1} = 0$. Since we only have the trivial solution, we know that the set $\{x, Tx, T^2x, ..., T^{m-1}x\}$ is linearly independent. $\square$

2 Problem 2

Prove that if ST is nilpotent, then TS is nilpotent.
Proof. We begin with our assumption that ST is nilpotent, there must be some $k > 0$ such that $(ST)^n0$. Now letting $(TS)^{n+1} = (TS)(TS)...(TS) = T(ST)(ST)...(ST)S = T(ST)^nS$ Then we see that the multiplier in the center is 0 by our assumption and we're left with

$$(TS)^{n+1} = T(0)S = 0$$

Therefore, TS is nilpotent as well. $\square$

3 Problem 3

Prove that if N is nilpotent, then 0 is the only eigenvalue of N.
Proof. Suppose that λ is an eigenvalue of N, then by definition, we have a nonzero vector $x \in V$ such that $\lambda x = Nx$. Applying N until k where $k > 0$, we have that $\lambda^k x = N^k x$. Since we aassumed that N is nilpotent, then with $\lambda^k x = 0$. Furthermore, we know $x \neq 0$ so we conclude that $\lambda = 0$. $\square$

4 Problem 4

Prove that if null $N^{n-1} \neq$ null N^n, then N is nilpotent.
Proof. **Proposition 8.5** in the book tells us that if n is a nonnegative integer such that null $T^n =$ null T^{n+1}, then

$$\text{null } T^0 \subset \text{null } T^1 \subset ... \subset \text{null } T^m = \text{null } T^{n+1} = \text{null } T^{n+2} = ...$$

Therefore by our assumption that null $N^{n-1} \neq$ null N^n, we can infer that null $N^{n-1} \neq$ null N^n for $0 \leq n \leq$ dim V. Therefore, $\{0\} =$ null $N^0 \subsetneq$ null $N^1 \subsetneq ... \subsetneq$ null $N^{n-1} \subsetneq$ null N^n. By **Proposition 8.6** n must increment by 1, in other words letting some $n = \dim V$ yields $\dim \text{null } N^n = n$. We can conclude that null $N^n = V$ which implies that $N^n = 0$, and finally N is nilpotent. □

5 Problem 5

Prove that if null $N^{n-2} \neq$ null N^{n-1}, then N has at most two distinct eigenvalues.
Proof. Similar to **(4)** by **Proposition 8.6** $\dim \text{null } T^n > \dim \text{null } T^{n-1}$ by at least 1 for $n = 1, ..., n-1$. From this we know that $\dim \text{null }^{n-1} \geq n-1$ which implies that 0 is an eigenvalue of T with multiplicity of at least $n-1$. Since V is a complex vector space and $T \in \mathcal{L}(V)$, then the sum of multiplicities of all eigenvalues of $T = \dim V$ by Proposition 8.18. Therefore, we can conclude that T can have at most one extra eigenvalue. □

6 Problem 6

Prove that for any T,

$$\text{null } T^n \cap \text{range } T^n = \{0\}$$

Proof. We begin by choosing some $v \in \text{null } T^n \cap \text{range } T^n$. Then for some $a \in T$, $T^n a = v$. Now by **Proposition 8.6** we know that:

$$\text{null } T^n = \text{null } T^{n+1} = \text{null } T^{n+2} = ...$$

and similarly, by **Proposition 8.9**,

$$\text{range } T^n = \text{range } T^{n+1} = \text{range } T^{n+2} = ...$$

which implies that $T^{n+1}a = T^n a = 0$ since $v \in \dim \text{null } T$. We know that $a \in \text{null } T^2$, and since $T^{n+1} = T^n$ by **8.6**, we deduce $T^n a = v = 0$ from which we can conclude that null $T^n \cap$ range $T^n = \{0\}$. $\square$

7 Problem 7

Find a 3×3 matrix whose minimal polynomial is z^2.
Proof. Since $V \in \mathbb{C}$, we take $T \in \mathcal{L}(\mathbb{C}^3)$ such that $T(z_1, z_2, z_3) = (z_2, 0, 0)$. By our construction, $T^2(z_1, z_2, z_3) = T(z_2, 0, 0) = (0, 0, 0)$. Our candidates therefore are: $1, z, z^2$ by **Theorem 8.34**. Applying our candidates shows us that

$$p(1) = 1 \neq 0$$
$$p(z) = z \implies p(T) = T \neq 0$$
$$p(z) = z^2 \implies p(T) = T^2 = 0$$

Since z^2 is the least monic polynomial where $p(T) = 0$, our minimal polynomial is z^2. $\square$

8 Problem 8

Find a 4×4 matrix whose minimimal polynomial is $z(z-1)^2$.
Proof. Similar to **Problem 7**, we want a matrix T such that $p(T) = T(T-1)^2 = 0$, where $p(z) = z(z-1)^2$. Therefore, we consider

$$T(z_1, z_2, z_3, z_4) = (0, z_2 + z_4, z_3, z_4)$$

Subtracting $I(z_1, z_2, z_3, z_4)$ to both sides yields:

$$T(z_1, z_2, z_3, z_4) - I(z_1, z_2, z_3, z_4) = (0, z_2 + z_4, z_3, z_4) - (z_1, z_2, z_3, z_4)$$

$$(T - I)(z_1, z_2, z_3, z_4) = (-z_1, z_4, 0, 0)$$

Applying this again we have that

$$(T - I)^2(z_1, z_2, z_3, z_4) = (z_1, 0, 0, 0) \implies T(T - I)^2 = 0$$

Since we want our minimal polynomial to be $z(z-1)^2$, which have the roots $0, 1$ (multiplicity 2), therefore the only monic polynomials that have the same roots are $z(z-1)$ and $z(z-1)^2$. Applying T to the former candidate, we have $T(T-1) = T(-z_1, z_4, 0, 0) = (0, z_4, 0, 0) \neq 0$, therefore $z(z-1)^2$ is the minimal polynomial of the 4×4 matrix T. $\square$

9 Problem 9

Suppose T is invertible. Prove that there is a polynomial p such that $T^{-1} = p(T)$.

Proof. Letting $a_0 + a_1 z + ... + a_{m-1} z^{z-1} + z^m$ be the minimal polynomial of T, then

$$a_0 I + a_1 T + ... + a_{m-1} T^{m-1} + T^m = 0$$

We know that $a_0 \neq 0$ since if it is, we can multiply both sides by T^{-1} and get

$$a_1 I + a_2 T + ... + a_{m-1} T^{m-2} + T^{m-1} = 0$$

which suggests that we have a monic polynomial that has a smaller degree contradicting the definition of minimal polynomial. This allows us to solve for the identity operator which is given by:

$$I = -\frac{a_1}{a_0} T - ... - \frac{a_{m-1}}{a_0} T^{m-1} - \frac{1}{a_0} T^m$$

Multiplying both sides by T^{-1} yields

$$T^{-1} = -\frac{a_1}{a_0} I - ... - \frac{a_{m-1}}{a_0} T^{m-2} - \frac{1}{a_0} T^{m-1}$$

So our polynomial is

$$p(z) = -\frac{a_1}{a_0} - ... - \frac{a_{m-1}}{a_0} z^{m-2} - \frac{1}{a_0} z^{m-1}$$

From which we can conclude that $T^{-1} = P(T)$. $\square$

10 Problem 10

Prove that V has a basis consisting of eigenvectors of T if and only if the minimal polynomial of T has no repeated roots.

Proof. For one direction, we begin with that we have a basis $(v_1, ..., v_n)$ of V consisting of T, where $\lambda_1, ..., \lambda_k$ are distinct eigenvalues. Then by our assumptions, using **Theorem 8.36**, we can construct a minimal polynomial where $p(z) = (z - \lambda_1)...(z - \lambda_k)$. Since our $\lambda_1, ..., \lambda_k$ are distinct eigenvalues, $p(z)$ must have distinct, i.e. non-repeating roots, thereby proving this direction.

On the other hand, we begin with the assumption that we have we minimal polynomial with distinct roots where our polynomial is defined as:

$$p(z) = (z - \lambda_1)...(z - \lambda_k)$$

Furthermore, we can let $\lambda_1...\lambda_k$ be the distinct eigenvalues of T. Then by definition of minimal polynomial, we have the following:

$$(T - \lambda_1 I)...(T - \lambda_k I)) = 0$$

Letting U_k be the corresponding subspaces of generalized eigenvectors for each λ_k, we can take a $v \in U_k$ where $u = (T - \lambda_k I)v \implies u \in U_k$. Furthermore, by definition, we know

$$\begin{aligned}(T\,|_{U_m} - \lambda_1 I)...(T\,|_{U_m} - \lambda_{m-1} I)u &= (T - \lambda_1 I)...(T - \lambda_m I)v \\ &= 0\end{aligned} \tag{3}$$

By **Theorem 8.23(c)**, $(T - \lambda_m I)\,|_{U_m}$ is nilpotent, and it's eigenvalue is only 0. Therefore, $T\,|_{U_m} - \lambda_k I$ is invertible in U_m for $k = 1, ..., m-1$, and (3) implies that $u = 0$. We can then conclude that v is an eigenvector T. Since v is some eigenvector of T with eigenvalue of λ_m is an eigenvector of T, we know that generalized v's of T is an eigenvector of T. Therefore, we have a basis of generalized eigenvectors of T which implies that there is a of V consisting of generalized eigenvectors of T by **Corollary 8.25**. This is the result proves the other direction of proof. □

11 Problem 11

Suppose $x \neq 0$. Let p be the monic polynomial of the smallest degree such that $p(T)x = 0$. Prove that p divides the minimal polynomial of T.

Proof. Letting q be the minimal polynomial of T, then there exists $s, r \in \mathcal{P}(F)$, where $q = sp + r$ and the degree of r is less than degree of p. Then we have that

$$q(T)x = s(T)p(T)v + r(T)x$$

By definition of minimal polynomial $q(T) = 0$ and $p(T)x = 0$ by our assumption. This results in $r(T)x = 0 \implies r = 0$, so the equation $q = sp + r$ becomes

$$q = sp$$

Here it's clear that $p \mid q$, which is the minimal polynomial of T. $\square$