Math 122B Homework 3

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1 Problem 1

Suppose f is a rational function of the form $f = P/Q$ where the polynomials P and Q satisfy $\deg Q - \deg P \geq 2$. Show that the sum of the residues of f is zero.

Proof. Letting R_0 be a radius such that all the poles of f are inside the circle $\gamma : |z| = R_0$. Then by the Residue Theorem we have:

$$\int_\gamma f(z)dz = 2\pi i \sum_{k=1}^{m} n(\gamma, z_k)\text{Res}(f; z_k)$$

for $\gamma > R_0$, and so $n(\gamma, z_k) = 1$. Now we let $Q(z) = \alpha_0 + \alpha_1 z + \cdots + \alpha_n z^n$, and $P(z) = \beta_0 + \beta_1 z + \cdots + \beta_m z^m$, and divide out γ to $|P(z)|$ and $|Q(z)|$ giving us:

$$\left| \int_\gamma f(z)dz \right| \leq \left(\frac{\gamma^m}{\gamma^n} \frac{\beta_m}{\alpha_n} \right) 4\pi\gamma$$

From here we see that if $n-(m+1) \geq 1 \Rightarrow n-m \geq 2$, then $\int_{C_R} f(z)dz \to 0$ as γ tends to infinity. Therefore we can conclude that if $\deg Q - \deg P \geq 2$ would result in $\int_{C_R} f(z)dz = 0 = 2\pi i\text{Res}(f; z_k)$. $\square$

2 Problem 2

Evaluate the following sums:

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$$

Proof.

(a) We can use

$$\sum_{\substack{n=-\infty \\ n \neq z_k}}^{\infty} f(n) = -\sum_{k} \mathrm{Res}(f(z)\pi \cot \pi z, z_k)$$

where C_N is a simple closed contour, n are integers inside C_N, z_k are poles of f. In our case, then, we have:

$$2\sum_{n=1}^{\infty} \frac{1}{n^2+1} = -\sum_{k=1}^{3} \mathrm{Res}\left(\frac{\pi \cot(\pi z)}{1+z^2}, z_k\right)$$

Before solving for the residues, we know that $\cot z$ has a Laurent expansion about 0 given by:

$$\cot z = \frac{1}{z} - \frac{z}{3} - \frac{1}{45}z^3 + ...$$

Using this, we can expand $\frac{\pi \cot \pi z}{1+z^2}$ about 0 and get:

$$\begin{aligned}
\frac{\pi \cot \pi z}{1+z^2} &= \frac{1}{1+z^2}\left(\frac{1}{z} - \frac{\pi^2}{3} - \frac{\pi^4 z^3}{45} + ...\right) \\
&= \frac{1}{z+z^2} - \frac{\pi^2 z}{3(1+z^2)} - \frac{\pi^4 z^3}{45(1+z^2)} - ... \qquad (1) \\
&= \frac{1}{z} + \frac{-1}{1+z} - \frac{\pi^2 z}{3(1+z^2)} - \frac{\pi^4 z^3}{45(1+z^2)} - ...
\end{aligned}$$

This implies that $\mathrm{Res}(\frac{\pi \cot(\pi z)}{1+z^2}, 0) = 1$ giving us our residue at $z_1 = 0$. Now for the poles at $z = \pm i$, since they are simple poles, we know that the residues is given by:

$$\mathrm{Res}(f, z) = \lim_{z \to z_0} (z - z_0) f(z)$$

This tells us that for $z_2 = i$ and $z_3 = -i$:

$$\mathrm{Res}(f, z_2) = \lim_{z \to i} \frac{\pi \cot(i\pi)}{z+i} = \frac{\pi \cot(i\pi)}{2i}$$

$$\text{Res}(f, z_3) = \lim_{z \to -i} \frac{\pi \cot(-i\pi)}{z - i} = \frac{\pi \cot(i\pi)}{2i}$$

Therefore,

$$2\sum_{n=1}^{\infty} \frac{1}{n^2+1} = -\left(1 + \frac{\pi \cot(\pi i)}{i}\right)$$

To simplify the second term, we recall

$$\cot(z) = \frac{ie^{iz} + ie^{-iz}}{e^{iz} - e^{-iz}}$$

Substituting this to our equation yields:

$$\begin{aligned} 2\sum_{n=1}^{\infty} \frac{1}{n^2+1} &= -\left(1 + \frac{\pi i \left(\frac{e^{-\pi}+e^{\pi}}{e^{-\pi}-e^{\pi}}\right)}{i}\right) \\ &= -\frac{1}{2}\left(1 + \pi \frac{e^{-\pi}+e^{\pi}}{e^{-\pi}-e^{\pi}}\right) \end{aligned} \tag{2}$$

(b) We can rewrite this summation as:

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{2} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1}{n^4}$$

Then by the same theorem as **(a)**, we have that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = -\frac{1}{2} \text{Res}\left(\frac{\pi \cot \pi z}{z^4}, 0\right)$$

We know that we have a singularity at $z = 0$ and the Laurent expansion about $z = 0$ for $\frac{\pi \cot \pi z}{z^4}$ is given by:

$$\frac{\pi \cot(\pi z)}{z^4} = \frac{1}{z^5} - \frac{\pi^2}{3z^3} - \frac{\pi^4}{45z} - \ldots$$

Therefore,

$$\text{Res}\left(\frac{\pi \cot \pi z}{z^4}, 0\right) = -\frac{\pi^4}{45}$$

and we can conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

(c) Lastly, by Residue Theorem, we know

$$\sum_{\substack{n=-\infty \\ n\neq z_k}}^{\infty} (-1)f(n) = -\sum_k \operatorname{Res}(\pi f(z)\csc \pi z, z_k)$$

Then in this case,

$$1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1} = -\sum_{z=\pm i} \operatorname{Res}\left(\frac{\pi}{(\sin \pi z)(z^2+1)}, z_k\right)$$

Finding our residues for $z_1 = i$, $z_2 = -i$, we have:

$$\begin{aligned} \operatorname{Res}\left(\frac{\pi}{(\sin \pi z)(z^2+1)}, z_1\right) &= (z-i)\lim_{z\to i} \frac{\pi}{(\sin \pi z)(z+i)(z-i)} \\ &= \frac{\pi}{(\sin i\pi)(2i)} \end{aligned} \tag{3}$$

and

$$\begin{aligned} \operatorname{Res}\left(\frac{\pi}{(\sin \pi z)(z^2+1)}, z_2\right) &= (z+i)\lim_{z\to -i} \frac{\pi}{(\sin \pi z)(z+i)(z-i)} \\ &= \frac{\pi}{(\sin i\pi)(2i)} \end{aligned} \tag{4}$$

So we know that:

$$\begin{aligned} -\sum_{z=\pm i} \operatorname{Res}\left(\frac{\pi}{(\sin \pi z)(z^2+1)}, z_k\right) &= \frac{\pi}{i(\sin i\pi)} \\ &= \frac{2\pi}{e^{\pi} - e^{-\pi}} \qquad \left(\text{by } \sin x = \frac{e^{ix} - e^{-ix}}{2i}\right) \end{aligned} \tag{5}$$

Therefore, we can conclude that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1} = -\frac{1}{2} + \frac{2\pi}{e^{\pi} - e^{-\pi}}$$

$\square$

3 Problem 3

Let U be an open set of the complex plane. Find conditions on U assuring that:

(a) The function $z \mapsto z^2$, $z \in U$, is one to one.

(b) The function $z \mapsto \cos(z)$, $z \in U$, is one to one.

Proof.

(a) We define a function f to be one to one if for all arbitrary z_1, z_2 in some region D, $f(z_1) \neq f(z_2)$. This condition only fails for $z = 0$, so any open set $U \subset \mathbb{C} \setminus 0$ is valid.

(b)

$\square$