Math 122B Homework 1

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1 Problem 1

Prove that for any positive integer n,

$$\text{Res}_0(1 - e^{-z})^{-n} = 1$$

Proof. Letting $f(z) = \frac{1}{(1-e^{-z})^n}$, we have singularities at

$$e^{-z} = 1 \implies z = 2\pi ik, \quad k \in \mathbb{Z}$$

By using the using the Residue Theorem, we can compute the integral:

$$\frac{1}{2\pi i}\int_C \frac{dz}{(1 - e^{-z})^n} = \text{Res}_0\left(\frac{1}{(1 - e^{-z})^n}\right)$$

By way of substitution, we can let $w = (1 - e^{-z})$ which implies $dw = e^{-z}$, and our equation becomes

$$\frac{1}{2\pi i}\int_C \frac{dz}{(1 - e^{-z})^n} = \frac{1}{2\pi i}\int_C \frac{dw}{w^n(1 - w)} = \text{Res}_0\left(\frac{dw}{w^n(1 - w)}\right)$$

Here it's clear that we have a pole of order n at $w = 0$. Solving for the right hand side, we have that:

$$\text{Res}_0\left(\frac{dw}{w^n(1 - w)}\right) = 1$$

So we can conclude:

$$\frac{1}{2\pi i}\int_C \frac{dz}{(1 - e^{-z})^n} = \text{Res}_0\left(\frac{1}{(1 - e^{-z})^n}\right) = \text{Res}_0\left(\frac{dw}{w^n(1 - w)}\right) = 1$$

$\square$

2 Problem 2

Show that Rouche's Theorem remains valid if the condition: $|f| > |g|$ on γ is replaced by: $|f| \geq |g|$ and $f + g$ does not vanish on γ.
Proof. Restating Rouche's Theorem as:

$$|f(z) - g(z)| < |g(z)|$$

Would tell us that there are no zeros in γ. Furthermore, if $|f(z)| \leq |g(z)|$, then we know that the number of zeros in $f = \alpha \cdot g$ where α is some constant. Replacing $|f| > |g|$ with $|f| \geq |g|$ tell us $|f| > |g|$ or $|f| = |g|$. Since $f + g$ does not vanish on γ, f, g have an equal number of zeros in γ. Therefore Rouche's Theorem is valid with the condition $|f| \geq |g|$ $\square$

3 Problem 3

Show that the function $\sqrt{z^2 - 1}$ can be defined, and it is analytic, on the complex plane minus the closed interval $[-1, 1]$.
Proof. Letting $f(z) = e^{\frac{1}{2}\log(z^2-1)}$, we must prove that f is continuous in $\mathbb{C} \setminus (-\infty, 1)$. To do this, we use Morera's Theorem which states that if f is continuous on an open set D, and $\int_\Gamma f(z)dz = 0$ where Γ is the boundary of a closed rectangle in D, then f is continuous on D. Since f have singularities along ± 1, then $f(z)$ is continuous along $\mathbb{C} \setminus (-\infty, 1)$. Furthermore, f is analytic in the domain since $\square$