Math 122A Homework 1

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1 Problem 1

Find all the cubic roots of $1 - i$.

Proof. First, we let $z = 1 - i$ and writing this as in polar form, we have $z = 1 - i = \sqrt{2}e^{-i\frac{\pi}{4}}$. Therefore $z^{\frac{1}{3}} = (1-i)^{\frac{1}{3}} = 2^{\frac{1}{6}}(e^{-i\frac{\pi}{12}+in\frac{2\pi}{3}})$ where $n \in \{0, 1, 2\}$ since multiples of 2π are valid. Therefore, we have roots $2^{\frac{1}{6}}(e^{-i\frac{\pi}{12}})$, $2^{\frac{1}{6}}(e^{-i\frac{\pi}{12}+i\frac{2\pi}{3}})$, and $2^{\frac{1}{6}}(e^{-i\frac{\pi}{12}+i\frac{4\pi}{3}})$. $\square$

2 Problem 2

Find

$$z = \frac{(\sqrt{3}+i)^{12}}{(1+i)^{10}}$$

Proof. Rewriting the numerator and denominator in polar coordinates, we have that $z = \frac{(2e^{i\frac{\pi}{6}})^{12}}{(\sqrt{2}e^{i\frac{\pi}{4}})^{10}}$. Distributing the exponentials, we have $z = \frac{2^{12}e^{i2\pi}}{2^5e^{i\frac{3\pi}{4}}}$. Which simplifies to $z = 2^7e^{i\frac{5\pi}{4}}$. $\square$

3 Problem 3

Let w be the n-th root of 1 (i.e. $w^n = 1$) different from 1 itself. Prove

(a) $1 + w + w^2 + ... + w^{n-1} = 0$

(b) $1 + 2w + 3w^2 + ... + nw^{n-1} = \frac{n}{w-1}$

Proof.

(a) To prove this we begin by proving the following lemma:

$$1 + z + z^2 + ... + z^k = \frac{1 - z^{k+1}}{1 - z}$$

Proceeding by way of induction, we look at the base case where $k = 0$ which results in $1 = \frac{1-z}{1-z} = 1$. Now assuming $1+z+z^2+...+z^k = \frac{1-z^{k+1}}{1-z}$ is true, we must prove that $k + 1$ follows, and this gives us that:

$$1 + z + z^2 + ... + z^k + z^{k+1} = \frac{1 - z^{k+1+1}}{1 - z}$$

By our assumption, we can the replace the first k terms to get

$$\frac{1 - z^{k+1}}{1 - z} + z^{k+1} = \frac{1 - z^{k+1+1}}{1 - z}$$

Multiplying by $(1 - z)$ to both sides we get that

$$1 - z^{k+1} + z^{k+1} - z^{k+1+1} = 1 - z^{k+1+1}$$

which implies $0 = 0$ thereby proving our lemma. Now we let $w = z$ and we have

$$1 + w + w^2 + ... + w^{n-1} + w^n = \frac{1 - w^{n+1}}{1 - w}$$

Since we were given that $w^n = 1$, we can substitute this to get

$$1 + w + w^2 + ... + w^{n-1} + 1 = \frac{1 - w}{1 - w} = 1$$

then subtracting 1 to both sides, we get $1 + w + w^2 + ... + w^{n-1} = 0$ thereby proving **(a)**.

(b) To prove this, we an algebraic method and we let $y = 1 + 2w + 3w^2 + ... + nw^{n-1}$. Now multiplying by w to both sides we get that $yw = w + 2w^2 + 3w3 + ... + nw^{n-1} + nw^n$. By subtracting y and yw, we get

$$y - yw = 1 + w + w^2 + ... + nw^{n-1} - (n - 1)w^{n-1} - nw^n$$

which simplifies to $y - yw = 1 + w + w^2 + ... + w^{n-1} - nw^n$. By **(a)**, we know that the terms up to $n-1$ is 0 and we are left with $y - yw = -nw^n$ which is equivalent to $y = \frac{n}{w-1}$. Since $y = 1 + 2w + 3w^2 + ... + nw^{n-1}$, we know that

$$1 + 2w + 3w^2 + ... + nw^{n-1} = \frac{n}{w-1}$$

$\square$

4 Problem 4

Let $a, b \in \mathbb{C}$

(a) Prove that if $|a| < 1$ and $|b| < 1$, then

$$\left|\frac{a-b}{1-\overline{a}b}\right| < 1$$

(b) Prove that if either $|a| = 1$ or $|b| = 1$, then

$$\left|\frac{a-b}{1-\overline{a}b}\right| = 1$$

Proof.

(a) We can rewrite our inequality to be

$$\left|\frac{a-b}{1-\overline{a}b}\right| = \frac{|a-b|}{|1-\overline{a}b|} < 1$$

Which implies that $|a-b| < |1-\overline{a}b|$. The modulus is always positive, we can square both sides of the inequality to get $|a-b|^2 < |1-\overline{a}b|^2$. This is equivalent to:

$$(a-b)(\overline{a-b}) < (1-ab)(\overline{1-\overline{a}b})$$

By properties of the conjugate we have that

$$(a-b)(\overline{a}-\overline{b}) < (1-\overline{a}b)(1-a\overline{b})$$

Then distributing we get

$$a\overline{a} - a\overline{b} - b\overline{a} + b\overline{b} < 1 - a\overline{b} - \overline{a}b - \overline{a}b + |a|^2|b|^2$$

$$a\overline{a} + b\overline{b} < 1 + |a|^2|b|^2$$
$$|a|^2 + |b|^2 < 1 + |a|^2|b|^2$$

Now letting $\alpha = |a|^2$ and $\beta = |b|^2$ where $\alpha, \beta \in [0, 1)$ we have that $\alpha + \beta \leq 1 + \alpha\beta$ which is equivalent to $\alpha - \alpha\beta \leq 1 - \beta$ which simplifies to $\alpha(1 - \beta) \leq 1 - \beta$

(b) By properties of modulus, we know that $|a| = a\overline{a} = 1$. Therefore, we can write our inequality as

$$\left|\frac{a-b}{1-\overline{a}b}\right| = \left|\frac{a-b}{a\overline{a}-\overline{a}b}\right| = \left|\frac{1}{a}\right|$$

$\square$

5 Problem 5

Express $\cos(4\theta)$ in terms of $\cos(\theta)$ and $\sin(\theta)$.
Proof. By DeMoivre's formula, we have that

$$\cos(4\theta) + i\sin(4\theta) = (\cos(\theta) + i\sin(\theta))^4$$

The right hand side simplifies to

$$(\cos(\theta)+i\sin(\theta))^4 = (\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta) + i(4\cos^3\sin\theta - 4\cos\theta\sin^3\theta)$$

$\square$

6 Problem 6

Establish the formula

$$\frac{1}{2} + \cos(\theta) + \cos(2\theta) + ... + \cos(n\theta) = \frac{\sin((\frac{n+1}{2})\theta)}{2\sin(\frac{\theta}{2})}$$

Proof. $\frac{1}{2} + \cos(\theta) + \cos(2\theta) + ... + \cos(n\theta)$ is equivalent to the real part of $1 + e^{i\theta} + e^{2i\theta} + ... + e^{in\theta} = 1 + e^{i\theta} + (e^{i\theta})^2 + ... + (e^{i\theta})^n$. Using the lemma proven in **Problem 3(a)**, we can simplify this to be $\mathrm{Re}[\frac{1-e^{i\theta(n+1)}}{1-e^{i\theta}}] = \mathrm{Re}[\frac{e^{i\theta(n+1)}-1}{e^{i\theta}-1}]$. Factoring out an $e^{i\frac{\theta}{2}}$ yields

$$Re\left[\frac{\frac{(e^{i(n+\frac{1}{2})\theta} - e^{-i\frac{\theta}{2}})}{2i}}{\frac{(e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}})}{2i}}\right]$$

$$= \frac{1}{\sin(\frac{\theta}{2})}\ \mathrm{Re}\left[\frac{e^{i(n+\frac{1}{2})\theta} - e^{-i\frac{\theta}{2}}}{2i}\right]$$

. Since $\sin\theta = \frac{e^{-i\theta} - e^{-i\theta}}{2i}$, we are left with

$$\frac{1}{2} + \cos(\theta) + \cos(2\theta) + ... + \cos(n\theta) = \frac{1}{\sin(\frac{\theta}{2})}\ \mathrm{Re}\left[\frac{\sin(\frac{n+1}{2}\theta)}{2i}\right] = \frac{\sin((\frac{n+1}{2})\theta)}{2\sin(\frac{\theta}{2})}$$

□

7 Problem 7

Give a necessary and sufficient condition for $z_1, z_2, z_3 \in \mathbb{C}$ to lie on a straight line.

Proof. z_3 belongs to a L line passing through z_1, z_2 and L is the set $\{z = \theta z_1 + (1-\theta)z_2 : \theta \in \mathbb{R}\}$. If $z_3 \in L$ then there exists $\theta \in \mathbb{R}$ such that $z_3 = z_1 + (1-\theta)z_2$ is true and the converse follows. □

8 Problem 8

**Prove that $z_1, z_2, z_3 \in \mathbb{C}$ are the vertices of an equilateral triangle if and only if

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$$

.

Proof. Suppose that we have z_1, z_2, z_3 that satisfy $z_1^2+z_2^2+z_3^2 = z_1z_2+z_2z_3+z_3z_1$, and we have a number $z_0 \in \mathbb{C}$. We claim that $w_1 = z_1 - z_0$, $w_2 = z_2 - z_0$, $w_3 = z_3 - z_0$. This implies that $w_1^2 + w_2^2 + w_3^2 = w_1w_2 + w_1w_3 + w_2w_3$, and $(z_1-z_0)^2+(z_2-z_0)^2+(z_3-z_0)^2 = (z_1-z_0)+(z_3-z_0)(z_1-z_0)+(z_2-z_0)(z_3-z_0)$. Furthermore, we claim that with the same previous condition, then for all $\theta \in \mathbb{R}$, $e^{i\theta}z_1, e^{i\theta}z_2, e^{i\theta}z_3$ also satisfy $z_1^2+z_2^2+z_3^2 = z_1z_2+z_2z_3+z_3z_1$. To prove this, we consider

$$w_1 = z_1 - \frac{z_1+z_2+z_3}{3}$$
$$w_2 = z_2 - \frac{z_1+z_2+z_3}{3}$$
$$w_3 = z_3 - \frac{z_1+z_2+z_3}{3}$$

We note that $w_1, w-2, w_3$ satisfy our condition and $w_1 + w_2 + w_3 = 0$ **(1)** and $w_1 = r_1, w_2 = r_2e^{i\theta_2}, w_3 = r_3e^{i\theta_3}$ **(2)**. Squaring $w_1 + w_2 + w_3 = 0$, we get that $w_1^2 + w_2^2 + w_3^2 + 2w_1w_2 + 2w_1w_3 + 2w_2w_3 = 0$ which implies that $w_1^2 + w_2^2 + w_3^2 = w_1w_2 + w_2w_3 + w_1w_3 = 0$ **(3)**. So by **(1)**, **(2)**, **(3)** we conclude that $r_1 = r_2 = r_3$ and $\theta_1 = \frac{2\pi}{3}$, $\theta_2 = \frac{4\pi}{3}$ which is an equilateral triangle. □

Prove $f : \mathbb{C} \sim \mathbb{R}^2 \to \mathbb{C} \sim \mathbb{R}^2$

Proof.

(a) Maps the line $x = a_0$ onto

$$u = a_0^2 - \frac{v^2}{4a_0^2}$$
$$u = a_0^2 - (\frac{v}{2a_0})^2 \text{ where } v = 2a_0y$$
$$u = a_0^2 - \frac{v^2}{4a_0^2}$$

□