Math 104A Homework 1

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1 Problem 1

Review and state the following theorems of Calculus:

(a) The Intermediate Value Theorem

(b) The Mean Value Theorem

(c) Rolle's Theorem

(d) The Mean Value Theorem for Integrals

(e) The Weighted Mean Value Theorem for Integrals

(f) The Taylor's theorem with Lagrange remainder (Alternative for of Taylor's theorem)

Proof.

(a) If f is a continuous real-valued function on an interval I, then f has the intermediate value property on I: Whenever $a, b \in I$, $a < b$ and y lies between $f(a)$ and $f(b)$ [i.e., $f(a) < y < f(b)$ or $f(b) < y < f(a)$], there exists at least one x in (a, b) such that $f(x) = y$.

(b) Let f be continuous function on $[a, b]$ that is differentiable on (a, b) and satisfies $f(a) = f(b)$. There exists at least one x in (a, b) such that

$$f'(x) = \frac{f(b) - f(a)}{b - a}$$

(c) Let f be continuous function on $[a, b]$ that is differentiable on (a, b) and satisfies $f(a) = f(b)$. There exists at least one x in (a, b) such that $f'(x) = 0$.

(d) Let f be continuous function on $[a, b]$ that is differentiable on (a, b) and satisfies $f(a) = f(b)$. There exists at least one x in (a, b) such that

$$f(x) = \frac{1}{b-a} \int_a^b f(t)dt$$

(e) Suppose $f(x), g(x) \in [a, b]$, is integrable, does not change sign, and $f(x)$ is continuous. Then there exists $\eta \in (a, b)$ such that

$$\int_a^b f(x)g(x) = f(\eta) \int_a^b g(x)dx$$

(f) If $f(x)$ is $n+1$ times continuously differentiable $f \in C^{n+1}$ on an interval containing a, then

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + R_{n+1}(x)$$

where the remainder is

$$R_{n+1}(x) = \int_a^x \int_a^{x_1} ... \int_a x_n f^{n+1}(x_{n+1}) dx_{n+1} ... dx_2 dx_1$$

$\square$

2 Problem 2

For the function $f(x) = x^4 - 4x^3 - 1$ and the interval $[a, b] = [0, 2]$, find the number ξ that occurs in the mean value theorem.
Proof. Since our function is clearly continuous and f' exists, i.e. $f'(x) = 4x^3 - 12x^2$, we can use $f(x) - f(c) = f'(\xi)(x - c)$ where in this case, $x = 2$ and $c = 0$. Therefore, using this formula we have that $f(2) - f(0) = f'(\xi)(2 - 0)$ which is equal to $f'(\xi) = -9$. $\square$

3 Problem 3

Prove that if $f \in \mathbb{C}^n(\mathbb{R})$ and $f(x_0) = f(x_1) = ... = f(x_n)$ for $x_0 < x_1 < ... < x_n$, then $f^{(n)}(\xi) = 0$ for some $\xi \in (x_0, x_n)$.
Proof. Using induction, we have the proposition that $P(n)$: "if $f \in \mathbb{C}^n(\mathbb{R})$ and $f(x_0) = f(x_1) = ... = f(x_n)$ for $x_0 < x_1 < ... < x_n$, then $f^{(n)}(\xi) = 0$". So beginning with the base case where $n = 1$, we have that $f(x_0) = f(x_1) = 0$ where f is continuous. Then by Rolle's theorem, there exists $\xi_1 \in (0, 1)$ where $f'(\xi_1) = 0$. Now assuming our proposition $P(n)$ is true for all n, we check that $P(n + 1)$ holds. So now our proposition is $\square$

4 Problem 4

Derive the Taylor series with remainder term for $\ln(1 + x)$ about 1. Derive an inequality that gives the number of terms that must be taken to yield $\ln 4$ with error tems less than 2^{-10}.
Proof. Letting $f(x) = \ln(1 + x)$, we find the first few derivatives f which are $f'(x) = \frac{1}{1+x}$, $f''(x) = \frac{-1}{(1+x)^2}$, $f'''(x) = \frac{2}{(1+x)^3}$. For this problem we are given that $a = 1$ so $f'(1) = \frac{1}{2}$, $f''(1) = -\frac{1}{2^2}$, $f'''(1) = \frac{2}{2^3}$. So our Taylor series is

$$f(x) = \ln(2) + \frac{1}{2}(x - 1) + \frac{-1}{2^2 * 2!}(x - 1)^2 + \frac{2}{2^3 * 3!}(x - 1)^3 + ...$$

$\square$