Math 108B Homework 3

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Let V be an inner product space over F, and $T \in \mathcal{L}(V)$

1 Problem 1

Prove that if $U = \text{range } (T)$, then $U^{\perp} = \text{null } T^*$.
Proof.

$\square$

2 Problem 2

Suppose $P \in \mathcal{L}(V)$ is such that $P^2 = P$. Prove that P is an orthogonal projection if and only if P is self-adjoint.
Proof.

$\square$

3 Problem 3

Prove that if T is normal, then range $T =$ range T^*
Proof. Since T is normal we know that

$$\text{range } T = (\text{null } T^*)^{\perp} = (\text{null } T)^{\perp} = \text{range } T^*$$

. $\square$

4 Problem 4

Prove that if T is normal, then

$$\text{null } T^k = \text{null } T \quad \text{and} \quad \text{range } T^k = \text{range } T$$

for every positive integer k.
Proof. If $k = 1$, then this is trivial, and now since k is a positive integer, we consider $k \geq 2$. For the first, we know that if $v \in \text{null } T$, then $T^k v = T^{k-1}(Tv) = T^{k-1}0 = 0$. $\square$

5 Problem 5

Prove that there does not exist a self-adjoint operator $T \in \mathcal{L}(\mathbb{R}^3)$ such that $T(1, 2, 3) = (0, 0, 0)$ and $T(2, 5, 7) = (2, 5, 7)$
Proof.

$\square$

6 Problem 6

Give a counterexample to show that the product of two self-adjoint operators is not necessarily self-adjoint.
Proof.

$\square$

7 Problem 7

Suppose $F = \mathbb{C}$. Prove that a normal operator on V is self-adjoint if and only if all its eigenvalues are real.
Proof.

$\square$

8 Problem 8

Suppose $F = \mathbb{C}$ and T is a normal operator on V. Prove that there is a $S \in \mathcal{L}(V)$ such that $T = S^2$.
Proof.

$\square$

9 Problem 9

Prove that if T is a positive operator on V, then T^k is positive for every positive integer k.
Proof.

$\square$

10 Problem 10

Suppose T is a positive operator on V. Prove that T is invertible if and only if $\langle Tx, x \rangle$ is positive for $x \in V \setminus \{0\}$.
Proof.

$\square$

11 Problem 11

Prove that if $S \in \mathcal{L}(\mathbb{R}^3)$ is an isometry, then there exists a nonzero vector $x \in \mathbb{R}^3$ such that $S^2 x = x$.
Proof.

$\square$