Math 122A Homework 7 and 8

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1 Problem 1

Let $D_1(z_0) = \{z \in \mathbb{C} : |z - z_0| < 1\}$. Let $f, g : D_1(z_0) \to \mathbb{C}$ be two analytic functions on $D_1(z_0)$. Prove that if

$$f^{(n)}(z_0) = g^{(n)}(z_0), \quad n = 0, 1, 2, 3, ...$$

then $f(z) = g(z)$, $\forall z \in D_1(z_0)$.
Proof. By our given, we know there is a Taylor Series expansion of $f(z)$ and $g(z)$ centered around z_0 such that

$$f(z) = \sum_{n=0}^{\infty}(z - z_0)^n \frac{f^{(n)}(z_0)}{n!} \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty}(z - z_0)^n \frac{g^{(n)}(z_0)}{n!}$$

where $n = 0, 1, 2, 3, ...$ therefore, equating the two we have that

$$\sum_{n=0}^{\infty}(z - z_0)^n \frac{f^{(n)}(z_0)}{n!} = \sum_{n=0}^{\infty}(z - z_0)^n \frac{g^{(n)}(z_0)}{n!}$$

This reduces to

$$f^{(n)}(z_0) = g^{(n)}(z_0)$$

Which is exactly what we want. $\square$

2 Problem 2

Let $D_1(z_0) = \{z \in \mathbb{C} : |z - z_0| < 1\}$. Let $f : D_1(z_0) \to \mathbb{C}$ be an analytic function on $D_1(z_0)$ <u>such that is has a zero of $N \in \mathbb{N}$ at z_0</u>, i.e.

$$f(z_0) = f'(z_0) = ... = f^{N-1}(z_0) = 0, \quad f^n(z_0) \neq 0$$

(i) Prove that there exists $g : D_1(z_0) \to \mathbb{C}$ analytic on $D_1(z_0)$ with $g(z_0) \neq 0$ and

$$f(z) = (z - z_0)^N g(z)$$

(ii) There exists $\delta > 0$ such that if $0 < |z - z_0| < \delta$ such that $f(z) \neq 0$. (The zeros of a non-trivial analytic function are isolated)

Proof.

(i) Since we are given that

$$f(z_0) = f'(z_0) = ... = f^{N-1}(z_0) = 0, \quad f^n(z_0) \neq 0$$

and letting $z_0 = 0$, we know that we can Taylor expand $f(z)$ such that:

$$f(z) = \underbrace{\sum_{n=0}^{N-1} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n}_{=0 \text{ (by definition)}} + \sum_{k=N}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k$$

where the remaining nonzero sum terms consists of analytic functions. Factoring out a $(z - z_0)^N$ yields:

$$f(z) = (z - z_0)^N \cdot \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k$$

Finally, letting $g(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k$ we conclude:

$$f(z) = (z - z_0)^N \cdot g(z_0)$$

(ii) Taking $f(z)$ in **Problem 2(i)**, we know that after the first zero terms of the Taylor expansion, we have

$$f(z) = (z - z_0)^N \cdot g(z_0)$$

Clearly, the first term of $g(0) \neq 0$ and is a constant and the following terms are nonzero by definition. So, it follows that there must exist a nonzero $\delta > 0$ such that $|z - z_0| < \delta$ which implies that $|g(z)| \neq 0$. Clearly, $(z - z_0)^N \neq 0$ so the zeros of a non-trivial analytic function are isolated.

$\square$

3 Problem 3

Let $f(z) = \sin(\frac{\pi}{z})$. Thus $f(\frac{1}{n}) = 0$. Does this contradict the result in **Problem 2**?

Proof. We notice that for all possible of $\frac{1}{n}$, $n \in \mathbb{N}$, we have $f(\frac{1}{n}) = \sin(n\pi)$ which is 0 for all n. Furthermore as the limit approaches infinity, $\frac{1}{n}$ approaches 0, and therefore $f(\frac{1}{n}) = \sin(0) = 0 \Rightarrow f(\frac{1}{n}) = f'(\frac{1}{n}) = ... = f^n(\frac{1}{n}) = 0$. This fails our assumption that

$$f(z_0) = f'(z_0) = ... = f^{N-1}(z_0) = 0, \quad f^n(z_0) \neq 0$$

and so this does not contradict the result of **Problem 2**. $\square$

4 Problem 4

Find the order of each of the zeros of the given functions:

(a) $(z^2 - 4z + 4)^2$

(b) $z^2(1 - \cos(z))$

(c) $e^{2z} - 3e^z - 4$

Proof. Functions f that are analytic at a point z_0 has a zero of order m at z_0 if and only if there is a function g, which is analytic and nonzero at z_0 such that

$$f(z) = (z - z_0)^m g(z)$$

(a) Therefore, we can factor simplify this to get

$$((z-2)^2)^2 = (z-2)^4$$

which makes it clear that we have a $g(z) = 0$ and $z_0 = 2$, from which we can conlude we have a zero $m = 4$.

(b) Using the Taylor exapnsion of $\cos z$ about $z_0 = 0$, we have that:

$$\begin{aligned} z^2(1-\cos(z)) &= z^2\left[1 - \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + ...\right)\right] \\ &= z^2\left(\frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} + ...\right) \\ &= z^4\left(\frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!} + ...\right) \quad \text{(factoring out a } z^2\text{)} \end{aligned} \tag{1}$$

From here, we have the form we wanted where we let our multiplicand be $(z - z_0) = (z - 0)^4$, and letting $g(z)$ be the multiplier which is $\frac{1}{2!}$ when $z_0 = 0$, i.e. nonzero. Therefore, our m or the order of zero is 4.

(c) Similar to **(a)**, we can factor this to get $(e^z - 4)(e^z - 1)$. Now, similar to **(b)**, we Taylor expand our e^z to get:

□

5 Problem 5

Locate the isolated singularity of the given function and tell whether it is a removable singularity, a pole, or an essential singularity.

(a) $\dfrac{e^z - 1}{z}$

(b) $\dfrac{z^2}{\sin(z)}$

(c) $\dfrac{e^z - 1}{e^{2z} - 1}$

(d) $\dfrac{1}{1 - cos(z)}$

Proof. If a function f has an isolated singular point at z_0, then it's Laurent series form is:

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0) + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + ... + \frac{b_n}{(z - z_0)^n} + ...$$

When all $b_n = 0$, then we have a removable singular point z_0. If we have $n \geq 1$, where the b_n terms are nonzero, and n is finite, then we have a pole of order n. Finally if we have an infinite number of b_n, which are nonzero, then z_0 is an essential singular point of f.

(a) This has a singularity at $z_0 = 0$, therefore taking the Taylor expansion of e^z about z_0 gives:

$$\begin{aligned}
\frac{e^z - 1}{z} &= \frac{\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + ...\right) - 1}{z} \quad \text{(subtracting 1)} \\
&= \frac{z + \frac{z^2}{2!} + \frac{z^3}{3!} + ...}{z} \quad \text{(dividing by } z\text{)} \\
&= 1 + \frac{z}{2!} + \frac{z^2}{3!} + \frac{z^3}{4} + ...
\end{aligned} \tag{2}$$

Here it's clear that we do not have b terms since we do not have terms where $(z - z_0)$ is the denominator. Therefore, $z_0 = 0$ is a removable singular point.

(b) We know that $\sin(z) = 0$ for $z_0 = \pi k$ where $k \in \mathbb{Z}$. Therefore, expanding about z_0, we get

$$\begin{aligned}
\frac{z^2}{\sin(z)} &= \frac{z^2}{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots} \\
&= \frac{z^2}{z\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots\right)} \quad \text{(factoring a } z \text{ in the denominator)} \\
&= z \cdot \frac{1}{\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots\right)} \\
&= z \cdot (1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \dots) \\
&= z + \frac{z^2}{3!} + \frac{z^4}{5!} + \dots
\end{aligned} \tag{3}$$

And again, since the our Laurent expansion contains no b_n terms where $(z - z_0)$ is in the denominator, $z = \pi n$ is a a removable singular point.

(c) For this, we have an isolated singularity for when $e^{2z} - 1 = 0$. To find these points, we rewrite e^{2z} as

$$e^{2x+2iy} = e^{2x} \cdot e^{2iy} = e^{2x} \left(\cos(2y) + i \sin(2y)\right)$$

$e^{2x} \left(\cos(2y) + i \sin(2y)\right) = 1$ is true for when $x = 0$, and $2y = 2\pi k$ where $k \in \mathbb{Z}$ which implies that we have singularties for $z = x + iy = i\pi k$. Now Taylor expanding e^z:

$$\begin{aligned}
\frac{e^z - 1}{e^{2z} - 1} &= \frac{\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots\right) - 1}{\left(1 + 2z + \frac{2^2 z^2}{2!} + \frac{2^3 z^3}{3!} + \dots\right) - 1} \\
&= \frac{z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots}{2z + \frac{2^2 z^2}{2!} + \frac{2^3 z^3}{3!} + \dots} \quad \text{(after the } \pm 1 \text{ cancel)}
\end{aligned} \tag{4}$$

(d)

□

6 Problem 6

Find the Laurent series for a given function about the point $z = 0$ and find the residue at that point.

(a) $\dfrac{e^z - 1}{z}$

(b) $\dfrac{z}{(\sin(z))^2}$

(c) $\dfrac{1}{e^z - 1}$

(d) $\dfrac{1}{1 - \cos(z)}$

In **(c)** and **(d)** compute only three terms of the Laurent series.
Proof.

(a)

(b)

(c)

(d)

$\square$

7 Problem 7

Find the residue of $f(z) = \frac{1}{1+z^n}$ at the point $z_0 = e^{i\frac{\pi}{n}}$

Proof.

$\square$

8 Problem 8

Calculate:

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)(4+x^2)}dx$

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2}(=\frac{\pi}{2})$

(c) $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin(ax)}{x^2+b^2}dx(=\pi e^{-ab})$

(d) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin(x)}{x}dx(=\pi)$

(e) $\displaystyle\int_{0}^{2\pi} \frac{dt}{2+\cos^2(t)}dx$

Proof.

(a)

(b)

(c)

(d)

(e)

□