Math 122A Homework 7 and 8

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1 Problem 1

Let $D_1(z_0) = \{z \in \mathbb{C} : |z - z_0| < 1\}$. Let $f, g : D_1(z_0) \to \mathbb{C}$ be two analytic functions on $D_1(z_0)$. Prove that if

$$f^{(n)}(z_0) = g^{(n)}(z_0), \quad n = 0, 1, 2, 3, ...$$

then $f(z) = g(z)$, $\forall z \in D_1(z_0)$.
Proof.

$\square$

2 Problem 2

Let $D_1(z_0) = \{z \in \mathbb{C} : |z - z_0| < 1\}$. Let $f : D_1(z_0) \to \mathbb{C}$ be an analytic function on $D_1(z_0)$ such that is has a zero of $N \in \mathbb{N}$ at z_0, i.e.

$$f(z_0) = f'(z_0) = ... = f^{N-1}(z_0) = 0, \quad f^n(z_0) \neq 0$$

(i) Prove that there exists $g : D_1(z_0) \to \mathbb{C}$ analytic on $D_1(z_0)$ with $g(z_0) \neq 0$ and

$$f(z) = (z - z_0)^N g(z)$$

(ii) There exists $\delta > 0$ such that if $0 < |z - z_0| < \delta$ such that $f(z) \neq 0$. (The zeros of a non-trivial analytic function are isolated)

Proof.

(i)

(ii)

□

3 Problem 3

Let $f(z) = \sin(\frac{\pi}{2})$. Thus $f(\frac{1}{n}) = 0$. Does this contradict the result in **Problem 2**?
Proof.

$\square$

4 Problem 4

Find the order of each of the zeros of the given functions:

(a) $(z^2 - 4z + 4)^2$

(b) $z^2(1 - \cos(z))$

(c) $e^{2z} - 3e^z - 4$

Proof.

(a)

(b)

(c)

$\square$

5 Problem 5

Locate the isolated singularity of the given function and tell whether it is a removeable singularity, a pole, or an essential singularity.

(a) $\dfrac{e^z - 1}{z}$

(b) $\dfrac{z^2}{\sin(z)}$

(c) $\dfrac{e^z - 1}{e^{2z} - 1}$

(d) $\dfrac{1}{1 - cos(z)}$

Proof.

(a)

(b)

(c)

(d)

$\square$

6 Problem 6

Find the Laurent series for a given function about the point $z = 0$ and find the residue at that point.

(a) $\dfrac{e^z - 1}{z}$

(b) $\dfrac{z}{(\sin(z))^2}$

(c) $\dfrac{1}{e^z - 1}$

(d) $\dfrac{1}{1 - \cos(z)}$

In **(c)** and **(d)** compute only three terms of the Laurent series.
Proof.

(a)

(b)

(c)

(d)

$\square$

7 Problem 7

Find the residue of $f(z) = \frac{1}{1+z^n}$ at the point $z_0 = e^{i\frac{\pi}{n}}$
Proof.

$\square$

8 Problem 8

Calculate:

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)(4+x^2)} dx$

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} (= \frac{\pi}{2})$

(c) $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin(ax)}{x^2+b^2} dx (= \pi e^{-ab})$

(d) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin(x)}{x} dx (= \pi)$

(e) $\displaystyle\int_{0}^{2\pi} \frac{dt}{2+\cos^2(t)} dx$

Proof.

(a)

(b)

(c)

(d)

(e)

□