Math 108C Homework 4

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May 3rd, 2022

1.) Show that a projection matrix P is orthogonal if $P = P^T$

Proof:

Taking $x \in \R^n$, we can write it as

$$x = Px + (I_n - P)x$$

It is clear that $Px \in \operatorname{col}(P)$, and to show that $(I - P)x \in \operatorname{col}(P)^{\perp} = \operatorname{null}(P^T)$:

$$P^T(I_n - P)x = P(I_n - P)x = (P - P^2)x = P(-Px) = 0$$

4.) Show that the dot product of orthogonal matrices is orthogonal.
Is it true that the sum of orthogonal matrices is orthogonal?

Proof:

Let $U_1, U_2, \ldots, U_k$ be orthogonal matrices of $n \times n$.

Then taking the dot product of $U_1, U_2, \ldots, U_k$ is $n \times n$ as well and

$$(U_1 \cdot U_2 \cdot \ldots \cdot U_k)^T (U_1 \cdot U_2 \cdot \ldots \cdot U_k) = U_1^T \cdot U_2^T \cdot \ldots \cdot U_k^T \cdot U_1 \cdot U_2 \cdot \ldots \cdot U_k = I$$

So the dot product of orthogonal matrices is orthogonal. No, the sum of orthogonal matrices may not always be orthogonal.

8.) What values can the determinant of an orthogonal matrix have? Prove.

Proof:

Letting U be an orthogonal matrix, we know that

$$UU^T = I_n$$

Therefore, we know that $\det(UU^T) = \det(U)\det(U^T) = \det(U^1) = 1$.

From here it's easy to conclude then that the values can be either 1 or -1.

15.) Show that the product of Householder matrices is an orthogonal matrix

Proof:

We recall that if $u^T u = 1$ where u is a unit vector.

Then the Householder matrix is defined as $H = I_n - 2u \cdot u^T$ Further, we see that

$$H^T H = H^2 = I - 4u \cdot u^T + 4u \cdot u^T \cdot u \cdot u^T = I$$

Therefore it is orthogonal.