Math 104B Homework 3

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April 18th, 2022

1.) Derive directly the Simpson rule on $[-1,1]$ by approximating the integrand f with the Hermite-interpolation polynomial, which interpolates $f(-1)$, $f(0)$, $f'(0)$, and $f(1)$.

Proof:

Simpson's rule is given by

$$\int_a^b f(x)dx = \frac{1}{6}(b-a)\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$$

Meanwhile, Hermite Interpolation for polynomial with degree 3 (since we have three givens) is

$$f(x) \approx p_3(x) = f(x_0) + f[x_0,x_1](x-x_0)(x-x_1) + [x_0,x_1,x_2](x-x_0)(x-x_1)(x-x_2) + f[x_0,x_1,x_2,x_3](x-x_0)(x-x_1)(x-x_2)(x-x_3)$$

where $f(x_0)=f(-1)$, $f[x_0,x_1]=\frac{f(0)-f(-1)}{0+1}$, $f[x_0,x_1,x_2]=\frac{f'(0)-(f(0)-f(-1))}{0-(-1)}$, and $f[x_0,x_1,x_2,x_3]=\frac{-f'(0)-f(0)+f(-1)-(f'(0)-f(0)+f(1))}{1-(-1)}$

Therefore, our Hermite polynomial interpolating the points $f(-1)$, $f(0)$, $f'(0)$, $f(1)$ is:

$$p_3(x) = x^2(x+1) + (-2f'(0)+f(1)-f(-1))x^2(x^2-1)$$

Then the integral of $p_3(x)$ from $[-1,1]$ would approximately be equal to

2.) Let

$$f(x) = \begin{cases} 1+x, & -1 \le x \le 0 \\ 1-x, & 0 \le x \le 1 \end{cases}$$

Find an approximation of the integral of f over $[-1,1]$ by using

(a) the simple Trapezoidal rule over $[-1,1]$.

(b) the simple Simpson rule over $[-1,1]$.

(c) the simple Trapezoidal rule over $[-1,0]$ and then the Trapezoidal rule over $[0,1]$.

Compute the error in each case and explain the differences in the accuracy of the results.

Proof:

(a) The simple Trapezoidal rule is defined as:

$$\int_a^b f(x)dx \approx \frac{(b-a)}{2}[f(a)+f(b)]$$

Therefore, the approximation of the integral of f over $[-1,1]$ is:

$$\begin{aligned}\int_{-1}^1 f(x)dx &\approx \frac{(1-(-1))}{2}[f(-1)+f(1)] \\ &= 0\end{aligned} \tag{1}$$

The integral of $f(x)$ is 1, therefore the error is $|1-0|=1$. This error may be due to the Trapezoidal rule not accounting that we have a piecewise function.

(b) By the definition of Simpson's rule in **(1)**, the integral of f is given by:

$$\begin{aligned}\int_{-1}^1 f(x)dx &\approx \frac{(1-(-1)}{6}(1-(-1))\left[f(-1)+4f\left(\frac{-1+1}{2}\right)+f(1)\right] \\ &= \frac{1}{3}[f(-1)+4f(0)+f(1)] \\ &= \frac{1}{3}[0+4+0] = \frac{4}{3}\end{aligned} \tag{2}$$

The error in this case is $|1-\frac{4}{3}| = \frac{1}{3}$.

(c) Using the same formula as **(a)** we have:

$$\begin{aligned}\int_{-1}^1 f(x)dx &= \int_{-1}^0 f(x)dx + \int_0^1 f(x)dx \\ &\approx \frac{(0-(-1))}{2}[f(0)+f(-1)] + \frac{(1-0)}{2}[f(0)+f(1)] \\ &= \frac{1}{2}[1+0] + \frac{1}{2}[1+(1-1)] \\ &= \frac{1}{2}+\frac{1}{2} = 1\end{aligned} \tag{3}$$

Here the error is $|1-1|=0$

3.) **(a)** Construct a quadrature rule of the form

$$\int_{-1}^1 f(x)dx \approx A_0f(-1) + A_1f(0) + A_2f(1)$$

which is exact for polynomials of degree ≤ 2.

(b) Derive the 3-point (Legendre) Gaussian quadrature to approximate $\int_{-1}^1 f(x)dx$ (i.e you need to obtain the nodes x_0, x_1, x_2 and the corresponding weights A_0, A_1, A_2).

(c) Verify its degree of precision.

(d) Compare the accuracy of this 3-point Gaussian quadrature with that of the simple Simpson rule for approximating $\int_{-1}^1 e^x dx$.

(e) Show that the 3-point Gaussian quadrature can be used for approximating $\int_a^b f(x)dx$ by doing a simple change of variables and apply this to approximate

$$\int_0^4 \frac{\sin x}{x}dx$$

Proof:

(a) Using the basis $(1, x, x^2)$, we have

$$\begin{aligned}\text{for } f(x)=1 \implies \int_{-1}^1 (1)dx &= A_0f(-1)+A_1f(0)+A_2f(1) \\ (1-(-1)) &= A_0+A_1+A_2 \\ 2 &= A_0+A_1+A_2 \\ \text{for } f(x)=x \implies \int_{-1}^1 xdx &= A_0f(-1)+A_1f(0)+A_2f(1) \\ \frac{x^2}{2}\Big|_{-1}^1 &= -A_0+0+A_2 \\ 0 &= -A_0+A_2 \\ A_0 &= A_2 \\ \text{for } f(x)=x^2 \implies \int_{-1}^1 x^2dx &= A_0f(-1)+A_1f(0)+A_2f(1) \\ \frac{x^3}{3}\Big|_{-1}^1 &= A_0+A_2 \\ \frac{2}{3} &= A_0+A_2\end{aligned} \tag{4}$$

Solving for the system of equation above yields our quadrature rule:

$$\int_{-1}^1 f(x)dx = \frac{1}{3}f(-1)+\frac{2}{3}f(0)+\frac{1}{3}f(1) = \frac{2}{3}(f(-1)+2f(0)+2f(1))$$

(b) Now finding A_0, A_1, A_2 and $x_0, x_1, x_2 \in [-1,1]$ that satisfies the equation above using the basis $(1, x, x^2, x^3, x^4, x^5)$.

By symmetry, $x_1 = 0 \implies x_2 = -x_0$

$$\begin{aligned}\text{for } f(x)=1 \implies \int_{-1}^1 (1)dx &= A_0f(x_0)+A_{x_1}f(x_1)+A_2f(x_2) \\ 2 &= A_0+A_1+A_2 \\ \text{for } f(x)=x \implies \int_{-1}^1 xdx &= A_0f(x_0)+A_{x_1}f(x_1)+A_2f(x_2) \\ 0 &= A_0x_0+A_1x_1+A_2x_2 \\ 0 &= A_0x_0+A_1\cdot 0-A_2x_0 \\ A_0 &= A_2\end{aligned} \tag{5}$$

(c)

(d)

(e)