In [1]:

```python
import math
from typing import List

from matplotlib import pyplot as plt
```

In [2]:

```python
"""
1.) Write computer codes to compute the coefficients c_0, c_1,...,c_n and to
evaluate the corresponding interpolation polynomial at an arbitrary point x.
Test your code and turn in a run of your test.
"""
def newton_coefficients(x: List[float], f_x: List[float]):
    """Finds the coefficients for Newton's interpolation form.
    :param x: List of nodes.
    :param f_x: List of values at x nodes.
    :return: List of coefficients used for Newton's interpolation polynomial.
    """
    n = len(x)
    c = f_x
    for k in range(1, n):
        for j in range(n - 1, k - 1, -1):
            c[j] = (c[j] - c[j - 1]) / (x[j] - x[j - k])
    return c


def interpolation_polynomial(x: float, x_j: List[float], c: List[float]):
    """Uses Horner-like scheme to create Newton's polynomial.
    :param x: Point to evaluate the polynomial at
    :param x_j: List of nodes
    :param c: List of polynomial coefficients
    :return:
    """
    n = len(c) - 1
    p = c[n]

    for j in range(n, 0, -1):
        p = (x - x_j[j - 1]) * p + c[j - 1]
    return p
```

In [3]:

```python
#Test your codes and turn in a run of your test.
print(" -------------------------------------------------")
print("|Test your codes and turn in a run of your test.|")
print(" -------------------------------------------------")
x = [1, 3, 5, 8]
y = [1, 4, 16, 32]

coeffs = newton_coefficients(x, y)
print(coeffs)
print(interpolation_polynomial(7, x, coeffs))

# consider f(x) = xe^{-x^{2}} for x in [-1, 1]
# with nodes x_j = -1 + j(2/10) for j = 0, 1, ..., 100
print("\n\n ------------------------------------------------------------------------------------------------
-------")
print("|Consider f(x) = xe^{-x^{2}} for x in [-1, 1] with nodes x_j = -1 + j(2/10) for j = 0, 1, ..., 1
00|")
print(" ------------------------------------------------------------------------------------------------
---")
f = lambda x: x * math.e ** ((-x) ** 2)
x_j = [-1 + j * (2 / 10) for j in range(11)]
f_x = [f(x) for x in x_j]
p_10_coeffs = newton_coefficients(x_j, f_x)
print(p_10_coeffs)

# evaluate p_10_coeffs(x) at points x^{bar} = -1 + j(2/10) for j = 0, 1, ..., 100
```

```python
x_j = [-1 + j * (2 / 100) for j in range(101)]
f_x = [f(x) for x in x_j]
p_100_coeffs = newton_coefficients(x_j, f_x)

print("\n\n --------------------------------------------------------------------------------")
print("|Evaluate p_10_coeffs(x) at points x^{bar} = -1 + j(2/10) for j = 0, 1, ..., 100|")
print(" --------------------------------------------------------------------------------")
print(p_100_coeffs)

# plot error f(x) - p_100_coeffs(x)
print("\n\n ----------------------------------")
print("|Plot error f(x) - p_100_coeffs(x)|")
print(" ----------------------------------")
error = []
for x in x_j:
    error.append(f(x) - interpolation_polynomial(x, x_j, p_100_coeffs))
print(error)
```

```
 --------------------------------------------------
|Test your codes and turn in a run of your test.|
 --------------------------------------------------
[1, 1.5, 1.125, -0.17976190476190476]
28.37142857142857



 ---------------------------------------------------------------------------------------------------
|Consider f(x) = xe^{-x^{2}} for x in [-1, 1] with nodes x_j = -1 + j(2/10) for j = 0, 1, ..., 100|
 ---------------------------------------------------------------------------------------------------
[-2.718281828459045, 6.005485625075421, -6.798875878841591, 5.777423248730219, -3.6477517800001955, 2.0
59950229854935, -0.8967347795691796, 0.4143727957645485, -0.1176379681123087, 0.06535442672907697, -3.4
22262473407045e-14]



 --------------------------------------------------------------------------------
|Evaluate p_10_coeffs(x) at points x^{bar} = -1 + j(2/10) for j = 0, 1, ..., 100|
 --------------------------------------------------------------------------------
[-2.718281828459045, 7.889764735125053, -12.606112981153295, 15.243832440142329, -14.830360505893443, 1
2.432649445315537, -9.162655302082728, 6.1026052640632065, -3.7118689152117406, 2.0918726627835875, -1.
1018747816963461, 0.5629894996242295, -0.36044963616763637, 0.5221862990450042, -0.6526851585859442, -6
.222887133712091, 70.33176559914924, -426.033384318273, 1676.8258986230778, -2435.680052795681, -27204.
406989024526, 324188.6436838271, -2281371.2289189333, 12738467.981469905, -61008935.23490221, 259052026
.5868144, -992068366.8398452, 3459538530.859048, -11040443682.21639, 32285802698.603836, -86269920854.0
7645, 208711185935.31577, -447324261961.3897, 804230805070.761, -1005209792994.6742, -193223423165.3785
7, 6746831790837.391, -29064292219838.715, 90599062525990.69, -235775520319654.0, 528066737311073.75, -
998333117304781.4, 1431718066140862.2, -723099491934963.0, -4781796487401692.0, 2.509993718997923e+16,
-8.461142975765483e+16, 2.3754394592906755e+17, -5.957793776958326e+17, 1.37531602912691e+18, -2.968592
0711262807e+18, 6.049461436644929e+18, -1.1715700364312578e+19, 2.167183632444103e+19, -3.8457349802894
57e+19, 6.574156672188006e+19, -1.0875313384729225e+20, 1.7501155898352386e+20, -2.7570655138050186e+20
, 4.283484940114444e+20, -6.616936383007434e+20, 1.024401598790698e+21, -1.599336105754854e+21, 2.52574
7485472814e+21, -4.031608540685484e+21, 6.477926814070286e+21, -1.0415757532937283e+22, 1.6654790653953
647e+22, -2.633979028267126e+22, 4.1027975890425495e+22, -6.2756960615785795e+22, 9.408956641797405e+22
, -1.381193203783847e+23, 1.9841940556473974e+23, -2.7891904150533197e+23, 3.836953935119684e+23, -5.16
67663120995794e+23, 6.812627208214578e+23, -8.798843277222843e+23, 1.1135353673742848e+24, -1.381328941
3627597e+24, 1.6801366345763847e+24, -2.0043742643254685e+24, 2.345990561607477e+24, -2.694697715725043
e+24, 3.038455086565372e+24, -3.3641829207473196e+24, 3.6586469480632893e+24, -3.909423872635325e+24, 4
.10583849098171e+24, -4.239760318575955e+24, 4.3061631022276817e+24, -4.303382698566521e+24, 4.23305223
2479602e+24, -4.0997403803314273e+24, 3.9103603037378124e+24, -3.6734454437046414e+24, 3.3983990406074
774e+24, -3.0948156444286504e+24, 2.7719477914053645e+24, -2.438355482117769e+24]



 ----------------------------------
|Plot error f(x) - p_100_coeffs(x)|
 ----------------------------------
[0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, -2.220446049250313e-16, 0.0, 2.220446
049250313e-16, -2.220446049250313e-16, 0.0, -3.3306690738754696e-16, -1.1102230246251565e-16, -3.330669
0738754696e-16, 0.0, -2.220446049250313e-16, 0.0, -2.220446049250313e-16, -2.220446049250313e-16, -4.44
0892098500626e-16, -3.3306690738754696e-16, 2.220446049250313e-16, -3.3306690738754696e-16, 1.665334536
9377348e-16, 5.551115123125783e-17, -6.661338147750939e-16, -3.885780586188048e-16, -4.996003610813204e
-16, 2.7755575615628914e-16, -4.440892098500626e-16, 7.771561172376096e-16, -2.220446049250313e-16, 9.4
3689570931383e-16, 9.936496070395151e-15, 2.275957200481571e-15, -3.366751322175787e-14, -1.21708199074
53279e-13, 4.991285162958548e-13, 7.561729020721941e-13, 1.1802642196911961e-12, 2.70571065552616e-12,
2.2648272146597037e-12, -9.343022883134822e-12, -3.33510996597397e-13, 2.839700107004628e-10, -3.624003
0421019753e-10, -8.832594516894332e-10, 3.4892238126582598e-09, -7.29711999158944e-09, -2.6272101244506
63e-08, -6.662684931546536e-09, -4.08418883851569e-07, -8.154833719964394e-07, -1.6307467751164317e-06,
1.9686870261770295e-07, 1.1047541748077094e-07, -6.578210556706932e-05, -4.087650389461572e-05, 7.34132
```

```
8788379453e-06, 0.00078241796885980 54, 0.0035455132558080416, 0.008914204867678577, 0.0020821470527089 2
7, 0.17548445191438633, 0.21128236015936797, 1.542427510253138, 4.765560727111014, 18.233425509413113,
-6.1980283325464045, -276.5214072848528, -333.9244205259848, -1598.7304723737645, -3337.257406706855, 1
4920.61472870076, 23645.781120149742, 45803.95777667443, -16411.578856151827, -1209301.93024623, -46711
11.490718087, 47019.7033094692, -18506603.45376617, 8252317.106046082, 8337131.489840948, -303708622.85
347605, -757187703.6942973, -47719577.43960556, -5886563141.831871, 20227075226.188244, 122802216082.53
265, 187950988868.49536, 1013060895496.484, 10023901962772.307, 11845091382674.354, 99816762687726.31]
```

In [4]:

```python
"""
2.) Obtain the Hermite interpolation polynomial corresponding to the data
f(0) = 0, f'(0) = 0, f(1) = 2, f'(1) = 3.
"""

print("""
    In order to do this we first find our coefficients given by:

    x_j         0th         1st                                 2nd
3rd
     0        f(0)=0    f[0,0]=f'(0)/1=0             f[0,0,1]=(f[0,1]-f[0,0])/(1-0)=2     f[0,0,1,1]=(f[
0,1,1]-f[0,0,1])/(1-0)=-1
     0        f(0)=0    f[0,1]=(f(1)-f(0))/(1-0)=2   f[0,1,1]=(f[1,1]-f[0,1])/(1-0)=3
     1        f(1)=2    f[1,1]=f'(1)/1=3
     1        f(1)=2

    Then using the Newton interpolation polynomial:
    p_k(x)=f(x_0)+f[x_0,x_1](x-x_0)+...+f[x_0,x_1,...,x_k](x-x_0)...(x-x_k-1)
    we can obtain
    p_3(x)=f(0)+f[0,0](x-x_0)+f[0,0,1](x-x_0)(x-x_1)+f[0,0,1,1](x-x_0)(x-x_1)(x-x_2)
          =0+0(x-0)+2(x-0)(x-0)+(-1)(x-0)(x-0)(x-1)
          =2x^(2)+(-1)x^(3)-x^(2)
          =-x^(3)+x^(2)
""")
```


```
    In order to do this we first find our coefficients given by:

    x_j         0th         1st                                 2nd
3rd
     0        f(0)=0    f[0,0]=f'(0)/1=0             f[0,0,1]=(f[0,1]-f[0,0])/(1-0)=2     f[0,0,1,1]=(f[
0,1,1]-f[0,0,1])/(1-0)=-1
     0        f(0)=0    f[0,1]=(f(1)-f(0))/(1-0)=2   f[0,1,1]=(f[1,1]-f[0,1])/(1-0)=3
     1        f(1)=2    f[1,1]=f'(1)/1=3
     1        f(1)=2

    Then using the Newton interpolation polynomial:
    p_k(x)=f(x_0)+f[x_0,x_1](x-x_0)+...+f[x_0,x_1,...,x_k](x-x_0)...(x-x_k-1)
    we can obtain
    p_3(x)=f(0)+f[0,0](x-x_0)+f[0,0,1](x-x_0)(x-x_1)+f[0,0,1,1](x-x_0)(x-x_1)(x-x_2)
          =0+0(x-0)+2(x-0)(x-0)+(-1)(x-0)(x-0)(x-1)
          =2x^(2)+(-1)x^(3)-x^(2)
          =-x^(3)+x^(2)
```


In [5]:

```python
"""
3.) In class, we learned to use piecewise cubic splines that interpolate a
function. Find a piecewise linear function that interpolates
(0, 2), (1, 2),(2, 1),(3, 9).
"""
print("""
A piecewise linear function, say f(x), that interpolates the points is:
      /  x+2 for 0<x<=1
f(x)= | -x+3 for 1<x<=2
      \ 3x-5 for 2<x<=3
""")
```


```
A piecewise linear function, say f(x), that interpolates the points is:
      /  x+2 for 0<x<=1
f(x)= | -x+3 for 1<x<=2
      \ 3x-5 for 2<x<=3
```

In [6]:

```
"""
4.) Write a code to compute a natural spline S(x) which interpolates a collection
of given points (x0, y0),(x1, y1), ··· ,(xn, yn) where x0 < x1 < ··· < xn (do not assume
they are equidistributed). Your code should have a triadiagonal solver for the resulting
linear system of equations.
"""

def tridiagonal_matrix_solver(a, b, c, d):
    n = len(a)
    # Initialize variables
    m, l, u, y, x = [], [], [], [], [0 for i in range(n + 1 )]

    m.append(a[0])
    for j in range(0, n):
        l.append(c[j - 1] / m[j - 1])
        u.append(b[j - 1])
        m.append(a[j] - l[j - 1] * b[j - 1])

    # Forward substitution
    y.append(d[0])
    for j in range(0, n):
        y.append(d[j] - l[j - 1] * y[j - 1])

    # Backward substitution
    for j in range(n, -1, -1):
        if j == n:
            x[j] = y[j] / m[j]
            continue
        x[j] = (y[j] - b[j] * x[j + 1]) / m[j]
    return x


def natural_cubic_spline_coeffs(x: List[int], y: List[int]):
    n = len(x)
    h = [x[i + 1] - x[i] for i in range(n - 1)]
    matrix_a_coeffs = [2 * (h[i] + h[i + 1]) for i in range(n - 2)]
    matrix_b_coeffs = matrix_c_coeffs = [h[i] for i in range(n - 2)]
    matrix_d_coeffs = []

    for i in range(n - 2):
        matrix_d_coeffs.append(
            -6 / h[i] * (y[1] - y[0]) + 6 / h[i + 1] * (y[i + 2] - y[i - 1])
        )

    z = tridiagonal_matrix_solver(
        matrix_a_coeffs, matrix_b_coeffs, matrix_c_coeffs, matrix_d_coeffs
    )

    polynomial_coeffs = []
    for j in range(n - 2):
        a_j = (1 / h[j] * (z[j + 1] - z[j]))
        b_j = z[j] / 2
        c_j = 1 / h[j] * (y[j + 1] - y[j]) - (h[j] / 6 * (z[j + 1] + 2 * z[j]))
        d_j = y[j]
        polynomial_coeffs.append((a_j, b_j, c_j, d_j))
    return polynomial_coeffs
```

In [7]:

```
x = [0, 1, 3, 6, 8, 12]
y = [5, 8, 12, 44, 60, 87]
coeffs = natural_cubic_spline_coeffs(x, y)

n = len(x)
for i in range(n - 2):
    print(f"S_0({i}<x<={i + 1}) = {coeffs[i][0]} + {coeffs[i][1]} (x-{x[i]})+ {coeffs[i][2]} (x-{x[i]})
^2+ {coeffs[i][3]} (x-{x[i]})^3")
```

```
S_0(0<x<=1) = -0.5711488785928083 + -17.31634650867194 (x-0)+ 20.41153798843741 (x-0)^2+ 5 (x-0)^3
S_0(1<x<=2) = 22.64382508507934 + -17.601920947968345 (x-1)+ 22.107958505883797 (x-1)^2+ 8 (x-1)^3
S_0(2<x<=3) = 2.4746402066505393 + 5.0419041371109925 (x-3)+ -8.171006054642119 (x-3)^2+ 12 (x-3)^3
S_0(3<x<=4) = -6.581450653983353 + 8.753864447086801 (x-6)+ -5.1200951248513675 (x-6)^2+ 44 (x-6)^3
```

In [8]:

```python
"""
5.) Use the values in Table 1 to construct a smooth parametric
representation of a curve passing through the points (xj , yj ), j = 0,
1, ··· , 8 by finding the two natural cubic splines interpolating and (
tj , yj ),j = 0, 1, ··· , 8, respectively. Tabulate the coefficients of
the splines and plot the resulting parametric curve.

j       t_j     x_j     y_j
0        0      1.5     0.75
1       0.618   0.9     0.9
2       0.935   0.6     1.0
3       1.255   0.35    0.8
4       1.636   0.2     0.45
5       1.905   0.1     0.2
6       2.317   0.5     0.1
7       2.827   1.0     0.2
8       3.330   1.5     0.25
"""
# Cubic spline of (x_j, y_j) for j = 0, ..., 8
print("Cubic spline of (x_j, y_j) for j = 0, ..., 8")
x = [1.5, 0.9, 0.6, 0.35, 0.2, 0.1, 0.5, 1.0, 1.5]
y = [0.75, 0.9, 1.0, 0.8, 0.45, 0.2, 0.1, 0.2, 0.25]
n = len(x)
coeffs_xj_yj = natural_cubic_spline_coeffs(x, y)
for i in range(n - 2):
    print(f"S_0({i}<x<={i + 1}) = {coeffs_xj_yj[i][0]} + {coeffs_xj_yj[i][1]} (x-{x[i]})+ {coeffs_xj_yj[i][2]} (x-{x[i]})^2+ {coeffs_xj_yj[i][3]} (x-{x[i]})^3")


# Cubic spline of (t_j, y_j) for j = 0, ..., 8
print("\n\nCubic spline of (t_j, y_j) for j = 0, ..., 8")
y = [0.75, 0.9, 1.0, 0.8, 0.45, 0.2, 0.1, 0.2, 0.25]
t = [0, 0.618, 0.935, 1.255, 1.636, 1.905, 2.317, 2.827, 3.330]
n = len(t)
coeffs_tj_yj = natural_cubic_spline_coeffs(t, y)
for i in range(n - 2):
    print(f"S_0({i}<t<={i + 1}) = {coeffs_tj_yj[i][0]} + {coeffs_tj_yj[i][1]} (t-{t[i]})+ {coeffs_tj_yj[i][2]} (t-{t[i]})^2+ {coeffs_tj_yj[i][3]} (t-{t[i]})^3")
```

```
Cubic spline of (x_j, y_j) for j = 0, ..., 8
S_0(0<x<=1) = 7.147318416959333 + 3.3485488812719493 (x-1.5)+ 1.3302902237456093 (x-1.5)^2+ 0.75 (x-1.5)^3
S_0(1<x<=2) = -2.2161777236624447 + 1.2043533561841495 (x-0.9)+ 0.061215339376848366 (x-0.9)^2+ 0.9 (x-0.9)^3
S_0(2<x<=3) = 43.38505671506665 + 1.5367800147335162 (x-0.6)+ 0.7322673295681013 (x-0.6)^2+ 1.0 (x-0.6)^3
S_0(3<x<=4) = 739.3007690585034 + -3.8863520746498157 (x-0.35)+ -1.0219973618335247 (x-0.35)^2+ 0.8 (x-0.35)^3
S_0(4<x<=5) = -1149.0606208661065 + -59.33390975403756 (x-0.2)+ -1.5182899406269117 (x-0.2)^2+ 0.45 (x-0.2)^3
S_0(5<x<=6) = -13.837045167041207 + -1.8808787107322298 (x-0.1)+ 0.8713393554139908 (x-0.1)^2+ 0.2 (x-0.1)^3
S_0(6<x<=7) = 18.737766180000055 + -4.648287744140471 (x-0.5)+ 1.743403614570233 (x-0.5)^2+ 0.1 (x-0.5)^3


Cubic spline of (t_j, y_j) for j = 0, ..., 8
S_0(0<t<=1) = 0.8656869965550764 + 2.493699104476398 (t-0)+ -1.3534919819511888 (t-0)^2+ 0.75 (t-0)^3
S_0(1<t<=2) = -27.497651009657726 + 2.7611962924119164 (t-0.618)+ -0.09930656939378368 (t-0.618)^2+ 0.9 (t-0.618)^3
S_0(2<t<=3) = -8.032665295591125 + -1.597181392618834 (t-0.935)+ 0.023188866682781417 (t-0.935)^2+ 1.0 (t-0.935)^3
S_0(3<t<=4) = -21.00237788552037 + -2.882407839913413 (t-1.255)+ 0.6876832457766728 (t-1.255)^2+ 0.8 (t-1.255)^3
S_0(4<t<=5) = 21.71990671209232 + -6.883360827105044 (t-1.636)+ 0.6603103378191958 (t-1.636)^2+ 0.45 (t-1.636)^3
```

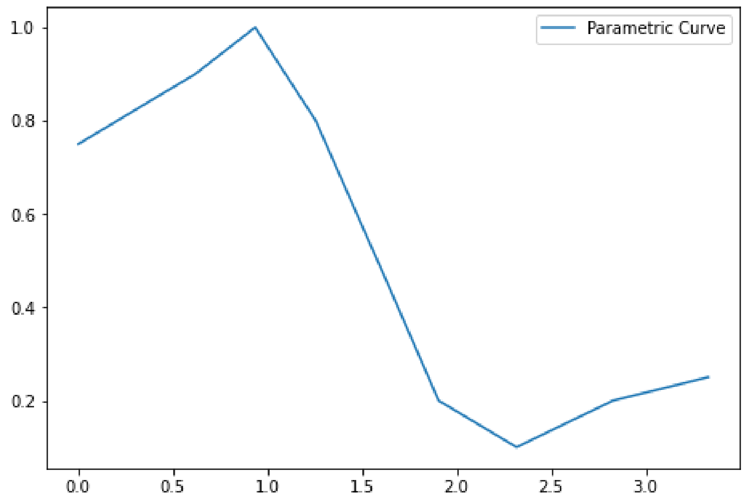
```
S_0(5<t<=6) = 19.597506705263296 + -3.9620333743286253 (t-1.905)+ 0.8352127739250836 (t-1.905)^2+ 0.2 (
t-1.905)^3
S_0(6<t<=7) = 0.6852968698977455 + 0.07505300695561452 (t-2.317)+ 0.12809377851511847 (t-2.317)^2+ 0.1
(t-2.317)^3
```

In [9]:

```
 y = [0.75, 0.9, 1.0, 0.8, 0.45, 0.2, 0.1, 0.2, 0.25]
t = [0, 0.618, 0.935, 1.255, 1.636, 1.905, 2.317, 2.827, 3.330]
fig = plt.figure()
ax = fig.add_axes([0, 0, 1, 1])
ax.plot(t, y, label="Parametric Curve")
ax.legend()
```

Out[9]:

```
<matplotlib.legend.Legend at 0x224215fdfc8>
```



In [ ]: