Math 122B Homework 6

Rad Mallari

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1 Problem 1

Let $a > 0$ be a positive parameter. Find the image of the exterior of the unit circle by the comformal map

$$w = az + \frac{1}{z}$$

Proof. Letting $z = re^{it}$, we get that $w = a(re^{it}) + \frac{1}{re^{it}}$. Since $e^{it} = \cos(t) + i\sin(t)$, we can rewrite our comformal map as:

$$\begin{aligned} w &= ar(\cos(t) + i\sin(t)) + r(\cos(-t) + i\sin(-t)) \\ &= r\left[a(\cos(t) + i\sin(t)) + \cos(t) - i\sin(t)\right] \\ &= r\cos(t)(a+1) + ir\sin(t)(a-1) \\ &= r\left[\cos(a+1) + i\sin(a-1)\right] \end{aligned} \tag{1}$$

Which describes an ellipse. □

2 Problem 2

Let U be a simply connected domain, different than the entire complex plane. Let $z_0 \in U$. Let G denote the class of all analytic functions $g : U \to D$ satisfying $g'(z_0) > 0$. Show that

$$\sup_{g \in G} g'(z_0) < \infty$$

and that the supremum is attained by a function $f \in G$. Prove f is one-to-one.
Proof.

□

3 Problem 3

Let u be a harmonic function. Show that u^2 is harmonic if and only if u is constant.
Proof. u^2 implies that

$$(u^2)_{xx} + (u^2)_{yy} = 2u(u_{xx} + u_{yy}) + 2(u_x^2 + u_y^2) \tag{2}$$

Since it is harmonic, $(u^2)_{xx} + (u^2)_{yy} = 0$, therefore $u_{xx} + u_{yy} = 0$ and

$$\begin{aligned} u_x^2 + u_y^2 &= 0 \\ u_x = u_y &= 0 \end{aligned} \tag{3}$$

And we can san see that u is constant. $\square$

4 Problem 4

Suppose the function $f = u+iv$ is analytic. Prove uv is a harmonic function. Give example of two harmonic functions U, V with the properties that UV is not harmonic.
Proof. Using the definition of f, we have

$$f^2 = u^2 + 2iuv - v^2 \quad \text{and} \quad \frac{f^2}{2} = \frac{u^2}{2} + iuv - \frac{v^2}{2}$$

And that the imaginary part of $\frac{2}{2}$ is uv, while the real part is $u^2 - v^2$. By **Theorem 16.2**, the real and imaginary parts of a harmonic function are harmonic, therefore uv is harmonic.

Letting $u = x, v = x$ we have that $uv = x^2$. Since $u_{xx} = 2$ and $v_{xy} = 0$, we see that uv is *not* harmonic. Another is example is by letting $u = x$ and $v = x^2 - y^2$. This implies that $uv = x^3 - xy^2$, and $v_{xx} = 3$ while $v_{xy} = -2y$ so uv is *not* harmonic. $\square$