Math 119A Homework 3

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1 Problem 1

Prove or disprove that $x(t) = 0, y(t) = 3e^{2t}$ is a solution to the following initial value problem:

$$\begin{aligned} x' &= -x, \\ y' &= x + 2y; \\ x(0) &= 0, y(0) = 3 \end{aligned} \tag{1}$$

Proof. To check our solution we first check that the derivatives of $x(t)$ and $y(t)$ satisfy our differential equation:

$$\begin{aligned} x'(t) &= 0 = x \\ y'(t) &= 6e^{2t} = 0 + 2(3e^{2t}) = x + 2y \end{aligned} \tag{2}$$

Which shows that $x(t), y(t)$ are solutions to our differential equations. Now plugging in our initial values for $t = 0$, we get

$$\begin{aligned} x(0) &= 0 \\ y(0) &= 0 + e^0 = 3 \end{aligned} \tag{3}$$

Which also satisfies our initial values, therefore $x(t) = 0, y(t) = 3e^{2t}$ is a solution to our differential equation. $\square$

2 Problem 2

Prove or disprove that

$$A = \begin{bmatrix} \frac{5}{3} & \frac{1}{3} \\ 2 & 0 \end{bmatrix}$$

is one solution to $x' = Ax$ where $x(t) = (e^{2t} - e^{-t}, e^{2t} + 2e^{-t})$.
Proof. First, $x'(t)$ is

$$x'(t) = \begin{bmatrix} 2e^{2t} + e^{-t} \\ 2e^{2t} - 2e^{-t} \end{bmatrix}$$

Now Ax is

$$\begin{aligned} Ax &= \begin{bmatrix} \frac{5}{3} & \frac{1}{3} \\ 2 & 0 \end{bmatrix} \begin{bmatrix} e^{2t} - e^{-t} \\ e^{2t} + 2e^{-t} \end{bmatrix} \\ &= \begin{bmatrix} \frac{5}{3}\left(e^{2t} - e^{-t}\right) + \frac{1}{3}\left(e^{2t} + 2e^{-t}\right) \\ 2\left(e^{2t} - e^{-t}\right) + 0\left(e^{2t} + 2e^{-t}\right) \end{bmatrix} \\ &= \begin{bmatrix} \frac{5}{3}e^{2t} - \frac{5}{3}e^{-t} + \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t} \\ 2e^{2t} - 2e^{-t} \end{bmatrix} \\ &= \begin{bmatrix} \frac{6}{3}e^{2t} - \frac{3}{3}e^{-t} \\ 2e^{2t} - 2e^{-t} \end{bmatrix} \\ &= \begin{bmatrix} 2e^{2t} - e^{-t} \\ 2e^{2t} - 2e^{-t} \end{bmatrix} \end{aligned} \tag{4}$$

Showing that the book answer is incorrect because the first element of the resulting Ax has the incorrect sign in the second term. □

3 Problem 3

Show that all eigenvalues are positive is the condition on eigenvalues that is equivalent to $\lim_{t\to\infty} |x(t)| = \infty$ for every solution $x(t)$ to $x' = Ax$.
Proof.

□

4 Problem 4

Show that $b > 0$ is an assumption required to ensure that $\lim_{t\to\infty} x(t) = 0$ for every solution $x(t)$ if $b^2 - 4c > 0$.
Proof.

□

5 Problem 5

Prove or disprove that $x(t) = 3e^t \cos 2t + 9e^t \sin 2t, y(t) = 3e^t \sin 2t - 9e^t \cos 2t$ is a solution to the following initial value problem:

$$\begin{aligned} x' &= Ax, \\ x(0) &= (3, 9); \\ A &= \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \end{aligned} \tag{5}$$

Proof.

□

6 Problem 6

Prove or disprove that $\dim E = \dim E_c$ and $\dim F \geq \dim F_R$ are relations that exist between $\dim E$ and $\dim E_c$ and $\dim F$ and $\dim F_R$ given that $E \subset \mathbb{R}^n$ and $F \subset C^n$ are subspaces.
Proof.

□

7 Problem 7

Prove or disprove that $\dim \supset R_{CR}$ is a relation between F and F_{RC} given that $F \subset C^n$ is any subspace.
Proof.

□

8 Problem 8

Solve the following initial value problem

(a)

$$\begin{aligned} x' &= -y, \\ y' &= x; \\ x(0) &= 1, y(0) = 1 \end{aligned} \tag{6}$$

Proof.

(a)

□

9 Problem 9

Solve the initial value problem

$$\begin{aligned} x' &= -4y \\ y' &= x; \\ x(0) = 0, &\quad y(0) = -7 \end{aligned} \tag{7}$$

Proof.

□

10 Problem 10

Let $F \subset C^2$ be the subspace spanned by the vector $(1, i)$

(a) Prove that F is not invariant under conjugation and hence is not the complexification of any subspace of $\mathbb{R}^2$

(b) Find $F_{\mathbb{R}}$ and $(F_{\mathbb{R}})_C$.

Proof.

(a)

(b)

□

11 Problem 11

Let E be a real vector space and $T \in L(E)$. Show that $(\ker T)_C = \ker(T_C)$, $(\text{Im } T)_C = \text{Im } (T_C)$, and $(T^{-1})_C = (T_C)^{-1}$ if T is invertible.
Proof.

□