سوال اول)
الف)

$2n+1$ points are defined from $x_0 = a$ to $x_{2n} = b \to$ between them are $n+1$ intervals of size $\frac{h}{2}$

$T(2n)$ contains the sum of the area of $2n$ trapezoids as the result of the integral $\to$ contains all indeces:

$$\to T(2n) = \sum_{i=0}^{2n-1} \frac{h\big(f(x_i) + f(x_{i+1})\big)}{4} = \frac{h}{4}\left(f(x_0) + f(x_{2n}) + 2\sum_{i=1}^{2n-1} f(x_i)\right)$$

$T(n)$ contains the sum of the area of n trapezoids as the result of the integral $\to$ contains points with even indeces:

$$\to T(n) = \sum_{i=0}^{n-1} \frac{h\big(f(x_{2i}) + f(x_{2i+2})\big)}{2} = \frac{h}{2}\left(f(x_0) + f(x_{2n}) + 2\sum_{i=1}^{n-1} f(x_{2i})\right)$$

$M(n)$ contains the sum of the area of n rectangles as the result of the integral $\to$ contains points with odd indeces:

$$\to M(n) = \sum_{i=0}^{n-1} hf(x_{2i+1}) = h\sum_{i=0}^{n-1} f(x_{2i+1})$$

$$\frac{1}{2}\big(M(n) + T(n)\big)$$

$$\begin{aligned}\frac{1}{2}\big(M(n) + T(n)\big) &= \frac{h}{4}\left(f(x_0) + f(x_{2n}) + 2\sum_{i=1}^{n-1} f(x_{2i})\right) + \frac{h}{4}\left(2\sum_{i=0}^{n-1} f(x_{2i+1})\right) \\ &= \frac{h}{4}\left(f(x_0) + f(x_{2n}) + 2\sum_{i=1}^{n-1} f(x_{2i}) + 2\sum_{i=0}^{n-1} f(x_{2i+1})\right) \\ &= \frac{h}{4}\left(f(x_0) + f(x_{2n}) + 2\sum_{i=1}^{2n-1} f(x_i)\right) = T(2n)\end{aligned}$$

ب)

$$E_{T(2n)} = \sum_{i=0}^{2n-1} E_{T_i} \rightarrow E_{T_i} = \text{Error of Trapezoidal Integral from } x_i \text{ to } x_{i+1} = \frac{h^3 f''(\xi_{x_i})}{96} \text{ with } h = \frac{b-a}{n}$$

$$E_{T(2n)} = \sum_{i=0}^{2n-1} E_{T_i} = \sum_{i=0}^{2n-1} \frac{h^3 f''(\xi_{x_i})}{96} = \frac{h^3}{96} \sum_{i=0}^{2n-1} f''(\xi_{x_i})$$

$$\min(f''(x)) \le f''(\xi_{x_i}) \le \max(f''(x)) \text{ for } \forall \xi_{x_i}, x \in [a,b]$$

$$2n \times \min(f''(x)) \le \sum_{i=0}^{2n-1} f''(\xi_{x_i}) \le 2n \times \max(f''(x)) \rightarrow \min(f''(x)) \le \frac{1}{2n} \sum_{i=0}^{2n-1} f''(\xi_{x_i}) \le \max(f''(x))$$

Intermediate value theorem: $\exists \mu \in [a,b]$ such that $f''(\mu) = \frac{1}{2n} \sum_{i=0}^{2n-1} f''(\xi_{x_i}) \rightarrow$

$$\rightarrow E_{T(2n)} = \sum_{i=0}^{2n-1} E_{T_i} = \frac{h^3}{96} \sum_{i=0}^{2n-1} f''(\xi_{x_i}) = \frac{h^3 n}{48} f''(\mu) = \frac{(b-a)^3}{48n^2} f''(\mu) = \frac{b-a}{12} \times \frac{(b-a)^2}{4n^2} \times f''(\mu) = \frac{b-a}{12} \times \left(\frac{h}{2}\right)^2 \times f''(\mu)$$

lemma 1: $f''(x_i) \approx \frac{1}{h^2}\left(f(x_{i-1}) + f(x_{i+1}) - 2f(x_i)\right)$

lemma 2: $f'(x_i) \approx \frac{1}{h}\left(f(x_{i+1}) - f(x_i)\right)$

$$\frac{T(2n) - T(n)}{3} = \frac{h}{12}\left(f(x_0) + f(x_{2n}) + 2\sum_{i=1}^{2n-1} f(x_i) - 2f(x_0) - 2f(x_{2n}) - 4\sum_{i=1}^{n-1} f(x_{2i})\right)$$

$$= \frac{h}{12}\left(\sum_{i=0}^{n-1} 2f(x_{2i+1}) - \sum_{i=1}^{n-1} 2f(x_{2i}) - f(x_0) - f(x_{2n})\right)$$

$$= \frac{h}{12}\left(\sum_{i=1}^{n-1}\left(f(x_{2i+1}) + f(x_{2i-1}) - 2f(x_{2i})\right) + f(x_1) + f(x_{2n-1}) - f(x_0) - f(x_{2n})\right)$$

$$= \frac{h^3}{48}\sum_{i=1}^{n-1} \frac{f(x_{2i+1}) + f(x_{2i-1}) - 2f(x_{2i})}{\frac{h^2}{4}} + \frac{h}{12}\left(f(x_1) + f(x_{2n-1}) - f(x_0) - f(x_{2n})\right)$$

$$\approx \frac{h^3}{48}\sum_{i=1}^{n-1} f''(x_{2i}) + \frac{h^2}{12}\left(\frac{f(x_1) - f(x_0)}{\frac{h}{2}} - \frac{f(x_{2n}) - f(x_{2n-1})}{\frac{h}{2}}\right)$$

$$\approx \frac{h^3}{48}\sum_{i=1}^{n-1} f''(x_{2i}) + \frac{h^3}{48} \frac{f'(x_0) - f'(x_{2n-1})}{\frac{h}{2}}$$

$$\approx \frac{h^3}{48}\sum_{i=0}^{n-1} f''(\mu_i) \text{ where } \mu_i \text{ are nearly equally spaced points } \in [a,b]$$

$$\min(f''(x)) \le f''(\mu_i) \le \max(f''(x)) \text{ for } \forall \xi_{x_i}, x \in [a,b]$$

$$n \times \min(f''(x)) \le \sum_{i=0}^{n-1} f''(\mu_i) \le n \times \max(f''(x)) \rightarrow \min(f''(x)) \le \frac{1}{n}\sum_{i=0}^{n-1} f''(\mu_i) \le \max(f''(x))$$

Intermediate value theorem: $\exists \mu \in [a,b]$ such that $f''(\mu') = \frac{1}{n}\sum_{i=0}^{2n-1} f''(\mu_i) \rightarrow$

$$\frac{h^3}{48}\sum_{i=0}^{n-1} f''(\mu_i) = \frac{h^3 n}{48} f''(\mu') = \frac{(b-a)^3}{48n^2} f''(\mu') = \frac{b-a}{12} \times \frac{(b-a)^2}{4n^2} \times f''(\mu') = \frac{b-a}{12} \times \left(\frac{h}{2}\right)^2 \times f''(\mu')$$

$\mu' \approx \mu$ because they're the intermediate value theorem for n and 2n points nearly equally spaced between a and b

سوال دوم)

Trapezoidal Integration:

$$\int_a^b f(x)dx \approx \frac{h}{2}\left(f(a) + f(b) + 2\sum_{i=1}^{n-1} f(x_i)\right) \rightarrow n = 4 \text{ and } h = 0.1$$

$$\int_{0.1}^{0.5} \frac{\cos x}{x} dx \approx \frac{0.1}{2}\left(\frac{\cos(0.1)}{0.1} + \frac{\cos(0.5)}{0.5} + \frac{2\cos(0.2)}{0.2} + \frac{2\cos(0.3)}{0.3} + \frac{2\cos(0.4)}{0.4}\right) = 1.624$$

Simpson Integration:

$$\int_a^b f(x)dx \approx \frac{h}{3}\left(f(a) + f(b) + 2\sum_{i=1}^{\frac{n}{2}-1} f(x_{2i}) + 4\sum_{i=1}^{\frac{n}{2}} f(x_{2i-1})\right) \rightarrow n = 4 \text{ and } h = 0.1$$

$$\int_{0.1}^{0.5} \frac{\cos x}{x} dx \approx \frac{0.1}{3}\left(\frac{\cos(0.1)}{0.1} + \frac{\cos(0.5)}{0.5} + \frac{2\cos(0.3)}{0.3} + \frac{4\cos(0.2)}{0.2} + \frac{4\cos(0.4)}{0.4}\right) = 1.563$$

سوال سوم)

$f(x) = \cosh(x^2)$
$f'(x) = 2x\sinh(x^2)$
$f''(x) = 2\sinh(x^2) + 4x^2\cosh(x^2)$
$f^3(x) = 12x\cosh(x^2) + 8x^3\sinh(x^2)$
$f^4(x) = 48x^2\sinh(x^2) + 16x^4\cosh(x^2) + 12\cosh(x^2)$

$\max(\sinh(x^2) + 4x^2\cosh(x^2)) = 8.523, x \in [0,1]$
$\max(48x^2 sinh(x) + 16x^4 cosh(x) + 12 cosh(x)) = 99.62, x \in [0,1]$

Trapezoidal Error:

$$|E_T| = \frac{(b-a)h^2 f''(\xi)}{12} \le \frac{8.523}{12n^2} \le 0.001 \rightarrow n \ge 27$$

Simpson Error:

$$|E_C| = \frac{(b-a)h^4 f^{(4)}(\xi)}{180} \le \frac{99.62}{180n^4} \le 0.001 \rightarrow n \ge 5$$

سوال چهارم)
الف)

Taylor Expansions over $x_0 = 4.5$

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2}f''(x_0) + O(h^3)$$

$$f(x_0 - h) = f(x_0) - hf'(x_0) + \frac{h^2}{2}f''(x_0) - O(h^3)$$

$$\rightarrow f'(x_0) = \frac{f(x_0 + h) - f(x_0 - h)}{2h} + O(h^2)$$

$$\rightarrow f'(4.5) = \frac{f(4.6) - f(4.4)}{0.2} + O(h^2) = 0.1048 + O(h^2)$$

ب)

Taylor Expansion over $x_0 = 3$

$$f(x_0 + h) = f(x_0) + hf'(x_0) + O(h^2)$$

$$\rightarrow f'(x_0) = \frac{f(x_0 + h) - f(x_0)}{h} + O(h)$$

$$\rightarrow f'(3) = \frac{f(3.1) - f(3)}{0.1} + O(h) = 0.5083 + O(h)$$

سوال پنجم)

$Taylor\ Expansions$:

$$f(x_0+2h)=f(x_0)+2hf^{'}(x_0)+2h^2f^{''}(x_0)+\frac{4h^3f^3(x_0)}{3}+\frac{2h^4f^4(x_0)}{3}+O(h^5)$$

$$f(x_0+h)=\ f(x_0)+hf^{'}(x_0)+\frac{h^2f^{''}(x_0)}{2}+\frac{h^3f^3(x_0)}{6}+\frac{h^4f^4(x_0)}{24}+O(h^5)$$

$$f(x_0-h)=\ f(x_0)-hf^{'}(x_0)+\frac{h^2f^{''}(x_0)}{2}-\frac{h^3f^3(x_0)}{6}+\frac{h^4f^4(x_0)}{24}+O(h^5)$$

$$f(x_0-2h)=f(x_0)-2hf^{'}(x_0)+2h^2f^{''}(x_0)-\frac{4h^3f^3(x_0)}{3}+\frac{2h^4f^4(x_0)}{3}+O(h^5)$$

$$\rightarrow\ f(x_0+2h)-2f(x_0+h)+2f(x_0-h)-f(x_0-2h)=\ 2h^3f^3(x_0)+O(h^5)$$

$$\rightarrow\ f^3(x_0)=\frac{f(x_0+2h)-2f(x_0+h)+2f(x_0-h)-f(x_0-2h)}{2h^3}+O(h^2)$$

سوال ششم)

الف)

$$\mathrm{P_3(x)=a_3x^3+a_2x^2+a_1x+a_0}\rightarrow\int_0^h \mathrm{P_3(x)dx}=\left[\frac{a_3x^4}{4}+\frac{a_2x^3}{3}+\frac{a_1x^2}{2}+a_0x\right]\begin{matrix}h\\0\end{matrix}=\frac{a_3h^4}{4}+\frac{a_2h^3}{3}+\frac{a_1h^2}{2}+a_0h$$

$$\mathrm{P'(x)=3a_3x^2+2a_2x+a_1}$$

$$\frac{h}{2}(f(0)+f(h))+\frac{h^2}{12}\big(f'(0)-f'(h)\big)=\frac{h}{2}(a_0+a_3h^3+a_2h^2+a_1h+a_0)+\frac{h^2}{12}(a_1-3a_3h^2-2a_2h-a_1)$$

$$=a_0h+\frac{a_1h^2}{2}+\frac{a_2h^3}{3}+\frac{a_3h^4}{4}=\int_0^h P_3(x)dx$$

ب)

$$\int_0^{nh}f(x)dx=\sum_{i=0}^{n-1}\int_{ih}^{(i+1)h}f(x)dx=\sum_{i=0}^{n-1}\int_0^h f(x+ih)dx=\sum_{i=0}^{n-1}\frac{h}{2}(f(ih)+f((i+1)h)+\frac{h^2}{12}\Big(f'(ih)-f'\big((i+1)h\big)\Big)$$

$$=\frac{h}{2}\left(f(0)+f(nh)+2\sum_{i=0}^{n-1}f(ih)\right)+\frac{h^2}{12}(f'(0)-f'(nh))$$