رادین شاه دائی ۴۰۱۱۰۶۰۹۶ تمرین چهارم تئوری محاسبات عددی

سوال اول)

الف)

$\mathbf{Taylor's\ method\ for\ order\ 4}$:

$y_0 = 1$

$y_{i+1} = y_i + hT^4(x_i, y_i)$

$T^4(x_i, y_i) = f(x_i, y_i) + \frac{h}{2} f^{(1)}(x_i, y_i) + \frac{h^2}{6} f^{(2)}(x_i, y_i) + \frac{h^3}{24} f^{(3)}(x_i, y_i)$

$f(x, y) = x + y$

$f^{(1)}(x, y) = 1 + y' = 1 + x + y$

$f^{(2)}(x, y) = 1 + y' = 1 + x + y$

$f^{(3)}(x, y) = 1 + y' = 1 + x + y$

$hT^4(x_i, y_i) = (x_i + y_i)\left(h + \frac{h^2}{2} + \frac{h^3}{6} + \frac{h^4}{24}\right) + \frac{h^2}{2} + \frac{h^3}{6} + \frac{h^4}{24} = \alpha(x_i + y_i) + \alpha - 0.1$

$y_1 = y_0 + \alpha(x_0 + y_0) + \alpha - 0.1 = 1.1103$

$y_2 = y_1 + \alpha(x_1 + y_1) + \alpha - 0.1 = 1.1103 + \alpha(0.1 + 1.1103) + \alpha - 0.1 = 1.2428$

$y_3 = y_2 + \alpha(x_2 + y_2) + \alpha - 0.1 = 1.2428 + \alpha(0.2 + 1.2428) + \alpha - 0.1 = 1.3997$

$y_4 = y_3 + \alpha(x_3 + y_3) + \alpha - 0.1 = 1.4997 + \alpha(0.3 + 1.3997) + \alpha - 0.1 = 1.5836$

$y_5 = y_4 + \alpha(x_4 + y_4) + \alpha - 0.1 = 1.5836 + \alpha(0.4 + 1.5836) + \alpha - 0.1 = 1.7974$

ب)

$\mathbf{Euler's\ method}$:

$y_0 = 0$

$y_{i+1} = y_i + hf(x_i, y_i) = y_i + 0.2(1 + x_i)\cos(2y)$

$y_1 = y_0 + 0.2(1 + 0)\cos(0) = 0.2$

$y_2 = y_1 + 0.2(1 + 0.1)\cos(0.2 \times 2) = 0.4026$

$y_3 = y_2 + 0.2(1 + 0.2)\cos(0.4026 \times 2) = 0.5689$

ج)

$\mathbf{Runge - Kutta\ for\ order\ 4}$:

$y_1 = 1, h = 0.1, f(x, y) = xy^{1/3}$

$k_1 = hf(x_i, y_i)$

$k_2 = hf(x_i + 0.5h, y_i + 0.5k_2)$

$k_3 = hf(x_i + 0.5h, y_i + 0.5k_2)$

$k_4 = hf(x_{i+1}, y_i + k_3)$

$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

1) $y_{1.1} = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1.1068$

$k_1 = 0.1 \times 1 \times 1^{1/3} = 0.1$

$k_2 = 0.1 \times 1.05 \times 1.05^{1/3} = 0.10672$

$k_3 = 0.1 \times 1.05 \times 1.0534^{1/3} = 0.10684$

$k_4 = 0.1 \times 1.1 \times 1.10684^{1/3} = 0.11376$

2) $y_{1.2} = y_{1.1} + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1.2279$

$k_1 = 0.1 \times 1.1 \times y_1^{\frac{1}{3}} = 0.11378$

$k_2 = 0.1 \times 1.15 \times (y_1 + 0.5k_1)^{\frac{1}{3}} = 0.12096$

$k_3 = 0.1 \times 1.15 \times (y_1 + 0.5k_2)^{\frac{1}{3}} = 0.12109$

$k_4 = 0.1 \times 1.2 \times (y_1 + k_3)^{\frac{1}{3}} = 0.12850$

3) $y_{1.3} = 1.3641$

سوال دوم)

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$$y=\frac{x}{1+x^2}\rightarrow y'=\frac{(1+x^2)-x\times 2x}{(1+x^2)^2}=\frac{1}{1+x^2}-\frac{2x^2}{(1+x^2)^2}=\frac{1}{1+x^2}-2y^2$$

ب)

Heun's method:

$y_0=0, h=0.4,$

$$y_{i+1}=y_i+\frac{h}{4}\left(f(x_i,y_i)+3f\left(x_i+\frac{2h}{3},y_i+\frac{2h}{3}f\left(x_i+\frac{h}{3},y_i+\frac{h}{3}f(x_i,y_i)\right)\right)\right)$$

$$y_1=0+0.1\left(f(0,0)+3f\left(\frac{0.8}{3},\frac{0.8}{3}f\left(\frac{0.4}{3},\frac{0.4}{3}f(0,0)\right)\right)\right)=0.34182$$

$f(0,0)=1$

$f\left(\frac{0.4}{3},\frac{0.4}{3}f(0,0)\right)=0.94698$

$f\left(\frac{0.8}{3},\frac{0.8}{3}f\left(\frac{0.4}{3},\frac{0.4}{3}\right)\right)=0.80607$

$$y_2=y_1+0.1\left(f(0.4,y_1)+3f\left(0.4+\frac{0.8}{3},y_1+\frac{0.8}{3}f\left(0.4+\frac{0.4}{3},y_1+\frac{0.4}{3}f(0.4,y_1)\right)\right)\right)=0.48932$$

$f(0.4,y_1)=f(0.4,0.34182)=0.62839$

$f\left(0.4+\frac{0.4}{3},y_1+\frac{0.4}{3}f(0.4,y_1)\right)=f(0.53333,0.42561)=0.41627$

$f\left(0.4+\frac{0.8}{3},y_1+\frac{0.8}{3}f\left(0.4+\frac{0.4}{3},y_1+\frac{0.4}{3}f(0.4,y_1)\right)\right)=f(0.66667,0.45286)=0.28221$

$y_3=0.4958, y_4=0.4523, y_5=0.4016$

Euler's method:

$y_0=0$

$y_{i+1}=y_i+hf(x_i,y_i)$

$y_1=y_0+hf(0,0)=0.4$

$y_2=y_1+hf(0.4,0.4)=61682$

$y_3=0.5563, y_4=0.4726, y_5=0.4063$

ج)

point	Real value	Heun's method	Euler's method	Heun's absolute E	Euler's absolute E
y1	0.3448	0.3418	0.4	0.003	0.055
y2	0.4878	0.4893	0.61682	0.0015	0.129
y3	0.4918	0.4958	0.5563	0.004	0.065
y4	0.4494	0.4523	0.4726	0.0029	0.023
y5	0.4	0.4016	0.4063	0.0016	0.0063

Heun's error $\rightarrow O(h^3)$

Euler's error $\rightarrow O(h^2)$

سوال سوم)

$$y_{i+1} = y_i + \frac{h}{4}\left(f(x_i, y_i) + 3f\left(x_i + \frac{2h}{3}, y_i + \frac{2h}{3}f(x_i, y_i)\right)\right)$$

Taylor's expansion:

$$f(x_0 + h, y_0 + k) = \sum_{i=0}^{n} \frac{1}{i!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^i f(x_0, y_0) + \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f\big(x_0 + h(x), y + k(y)\big)$$

$$\rightarrow\ f\left(x_i + \frac{2h}{3}, y_i + \frac{2h}{3}f(x_i, y_i)\right) = f(x_i, y_i) + \frac{2h}{3}\frac{\partial f(x_i, y_i)}{\partial x} + \frac{2hf(x_i, y_i)}{3}\frac{\partial f(x_i, y_i)}{\partial y} + \frac{1}{2!}\left(\frac{2h}{3}\frac{\partial}{\partial x} + \frac{2hf(x_i, y_i)}{3}\frac{\partial}{\partial y}\right)^2 f(x', y')$$
$$= f(x_i, y_i) + \frac{2h}{3}\left(\frac{\partial f(x_i, y_i)}{\partial x} + \frac{dy(x_n)}{dx}\frac{\partial f(x_i, y_i)}{\partial y}\right) + \frac{2h^2}{9}\left(\frac{\partial^2}{\partial x^2} + 2\frac{\partial^2}{\partial x\,\partial y} + \frac{\partial^2}{\partial y^2}\right) f(x', y')$$
$$= f(x_i, y_i) + \frac{2h}{3}\frac{df(x_i, y_i)}{dx} + O(h^2)$$

$$\rightarrow y_{i+1} = y_i + \frac{h}{4}\left(f(x_i, y_i) + 3f(x_i, y_i) + 2h\frac{df(x_i, y_i)}{dx} + O(h^2)\right) = y_i + hf(x_i, y_i) + \frac{h^2}{2}\frac{df(x_i, y_i)}{dx} + O(h^3)$$

$$y_{i+1} = y_i + h\Phi(x_i, y_i)$$
$$\Phi(x_i, y_i) = f(x_i, y_i) + \frac{h}{2}\frac{df(x_i, y_i)}{dx} + O(h^2)$$

taylor's expansion for y_{i+1}:

$$y_{i+1} = y_i + hy'(x_i) + \frac{h^2}{2}y''(x_i) + O(h^3)$$
$$\frac{y_{i+1} - y_i}{h} = y'(x_i) + \frac{h}{2}y''(x_i) + O(h^2) = f(x_i, y_i) + \frac{h}{2}\frac{df(x_i, y_i)}{dx} + O(h^2)$$

local truncation error:

$$\frac{y_{i+1} - y_i}{h} - \Phi(x_i, y_i) = f(x_i, y_i) + \frac{h}{2}\frac{df(x_i, y_i)}{dx} + O(h^2) - \left(f(x_i, y_i) + \frac{h}{2}\frac{df(x_i, y_i)}{dx} + O(h^2)\right) = O(h^2) \rightarrow \text{Order 2 Runge} - \text{Kutta}$$

سوال چهارم)

Adams Bashford two step method:
$y_0 = 6, y_1 \rightarrow \text{find via euler's method}$
$h = 0.1$
$y_{i+1} = y_i + \frac{h}{2}\left(3f(x_i, y_i) - f(x_{i-1}, y_{i-1})\right)$

```python
import numpy as np

def f(t, y):
    return 0.2 * y - 0.1 * y**2

def adams_bashforth(y0, y1, t0, t1, h):
    num_steps = int((t1 - t0) / h)
    t_values = np.linspace(t0, t1, num_steps + 1)
    y_values = np.zeros(num_steps + 1)
    y_values[0] = y0
    y_values[1] = y1

    for i in range(1, num_steps):
        y_values[i + 1] = y_values[i] + h / 2 * (
            3 * f(t_values[i], y_values[i]) - f(t_values[i - 1], y_values[i - 1])
        )

    return y_values

y0 = 6
t0 = 0
t_end = 20
h = 0.1

y1 = y0 + h * f(t0, y0)

y_adams = adams_bashforth(y0, y1, t0, t_end, h)

print(y_adams[-1])
```

$y(20) = 2.0247649200739883$

سوال پنجم)

Taylor's method for order n:

$y_0 = 1$

$y_{i+1} = y_i + hT^n(x_i, y_i)$

$T^n(x_i, y_i) = f(x_i, y_i) + \frac{h}{2}f^{(1)}(x_i, y_i) + \cdots + \frac{h^{n-1}}{n!}f^{(n-1)}(x_i, y_i)$

$\to y_{i+1} = y_i + h\Phi(x_i, y_i)$ where $\Phi(x_i, y_i) = T^n(x_i, y_i)$

Taylor exapansion of y_{i+1} *over* $y_i \to y_{i+1} = y_i + hy'(x_i) + \frac{h^2}{2}y^{(2)}(x_i) + \cdots + \frac{h^n}{n!}y^{(n)}(x_i) + \frac{h^{n+1}}{(n+1)!}y^{(n+1)}(\xi_{x_i})$

$$\frac{y_{i+1} - y_i}{h} = y'(x_i) + \frac{h}{2}y^{(2)}(x_i) + \cdots + \frac{h^{n-1}}{n!}y^{(n)}(x_i) + \frac{h^n}{(n+1)!}y^{(n+1)}(\xi_{x_i})$$

$$= f(x_i, y_i) + \frac{h}{2}f^{(1)}(x_i, y_i) + \cdots + \frac{h^{n-1}}{n!}f^{(n-1)}(x_i, y_i) + \frac{h^n}{(n+1)!}f^{(n)}\left(\xi_{x_i}, y(\xi_{x_i})\right) \to y^{(k)}(x_i) = f^{(k-1)}(x_i, y_i)$$

$$\to local\ truncation\ error = \frac{y_{i+1} - y_i}{h} - \Phi(x_i, y_i) = \frac{h^n}{(n+1)!}f^{(n)}\left(\xi_{x_i}, y(\xi_{x_i})\right) = O(h^n)$$

سوال ششم)

Definition 5.22 Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ denote the (not necessarily distinct) roots of the characteristic equation

$$P(\lambda) = \lambda^m - a_{m-1}\lambda^{m-1} - \cdots - a_1\lambda - a_0 = 0$$

associated with the multistep difference method

$$w_0 = \alpha, \quad w_1 = \alpha_1, \quad \ldots, \quad w_{m-1} = \alpha_{m-1}$$

$$w_{i+1} = a_{m-1}w_i + a_{m-2}w_{i-1} + \cdots + a_0 w_{i+1-m} + hF(t_i, h, w_{i+1}, w_i, \ldots, w_{i+1-m}).$$

If $|\lambda_i| \le 1$, for each $i = 1, 2, \ldots, m$, and all roots with absolute value 1 are simple roots, then the difference method is said to satisfy the **root condition**. ■

Definition 5.23
(i) Methods that satisfy the root condition and have $\lambda = 1$ as the only root of the characteristic equation with magnitude one are called **strongly stable**.
(ii) Methods that satisfy the root condition and have more than one distinct root with magnitude one are called **weakly stable**.
(iii) Methods that do not satisfy the root condition are called **unstable**. ■

الف)

$w_{i+1} = 3w_i - 2w_{i-1} + hF(\ldots) \rightarrow P(\lambda) = \lambda^2 - 3\lambda + 2 = (\lambda-2)(\lambda-1) \rightarrow \lambda_1 = 2, \lambda_2 = 1$
$\rightarrow$ doesn't statsify root condition $\rightarrow$ unstable

$$\text{Local truncation error} = \frac{y_{i+1} - w_{i+1}}{h} = \frac{y_{i+1} - 3y_i + 2y_{i-1}}{h} - F(\ldots) = \frac{y_{i+1} - 3y_i + 2y_{i-1}}{h} - \frac{13f_{i+1} - 20f_i - 5f_{i-1}}{12}$$
$$= \frac{y_{i+1} - 3y_i + 2y_{i-1}}{h} - \frac{13y'_{i+1} - 20y'_i - 5y'_{i-1}}{12}$$

$$\text{taylor expansion for } y_{i+1} \rightarrow y_{i+1} = y_i + hy'_i + \frac{h^2}{2}y_i^{(2)} + \frac{h^3}{6}y_i^{(3)} + \frac{h^4}{24}y^{(4)}(x_1)$$
$$\text{taylor expansion for } y_{i-1} \rightarrow y_{i-1} = y_i - hy'_i + \frac{h^2}{2}y_i^{(2)} - \frac{h^3}{6}y_i^{(3)} + \frac{h^4}{24}y^{(4)}(x_2)$$
$$\frac{y_{i+1} - 3y_i + 2y_{i-1}}{h} = \frac{y_i + hy'_i + \frac{h^2}{2}y_i^{(2)} + \frac{h^3}{6}y_i^{(3)} + \frac{h^4}{24}y^{(4)}(x_1) + 2y_i - 2hy'_i + h^2y_i^{(2)} - \frac{h^3}{3}y_i^{(3)} + \frac{h^4}{12}y^{(4)}(x_2) - 3y_i}{h}$$
$$= -y'_i + \frac{3h}{2}y_i^{(2)} - \frac{h^2}{6}y_i^{(3)} + O(h^3)$$

$$\text{taylor expansion for } y'_{i+1} = y'_i + hy_i^{(2)} + \frac{h^2}{2}y_i^{(3)} + \frac{h^3}{6}y^{(4)}(x_3)$$
$$\text{taylor expansion for } y'_{i-1} = y'_i - hy_i^{(2)} + \frac{h^2}{2}y_i^{(3)} - \frac{h^3}{6}y^{(4)}(x_4)$$
$$\frac{13y'_{i+1} - 20y'_i - 5y'_{i-1}}{12} = \frac{13y'_i + 13hy_i^{(2)} + \frac{13h^2}{2}y_i^{(3)} + \frac{13h^3}{6}y^{(4)}(x_3) - 5y'_i + 5hy_i^{(2)} - \frac{5h^2}{2}y_i^{(3)} + \frac{5h^3}{6}y^{(4)}(x_4) - 20y'_i}{12}$$
$$= \frac{-12y'_i + 18hy_i^{(2)} + 8h^2y_i^{(3)}}{12} + O(h^3) = -y'_i + \frac{3h}{2}y_i^{(2)} + \frac{2h^2}{3}y_i^{(3)} + O(h^3)$$

$$\rightarrow \text{Local truncation error} = \frac{y_{i+1} - 3y_i + 2y_{i-1}}{h} - \frac{13y'_{i+1} - 20y'_i - 5y'_{i-1}}{12} = -\frac{5h^2}{6}y_i^{(3)} + O(h^3) = O(h^2)$$

ب)

$$w_{i+1}=\frac{4}{3}w_i-\frac{1}{3}w_{i-1}+hF(\dots)\to P(\lambda)=\lambda^2-\frac{4}{3}\lambda+\frac{1}{3}=\left(\lambda-\frac{1}{3}\right)(\lambda-1)\to\ \lambda_1=\frac{1}{3},\lambda_2=1$$

$\to$ statsifies root condition, more than 1 root $\to$ weakly stable

$$\text{Local truncation error}=\frac{y_{i+1}-w_{i+1}}{h}=\frac{y_{i+1}-\frac{4}{3}y_i+\frac{1}{3}y_{i-1}}{h}-\frac{2f_{i+1}}{3}=\frac{y_{i+1}-\frac{4}{3}y_i+\frac{1}{3}y_{i-1}}{h}-\frac{2y'_{i+1}}{3}$$

$$\text{taylor expansion for } y_{i+1}\to y_{i+1}=y_i+hy'_i+\frac{h^2}{2}y_i^{(2)}+\frac{h^3}{6}\ y_i^{(3)}+\frac{h^4}{24}y^{(4)}(x_1)$$
$$\text{taylor expansion for } y_{i-1}\to y_{i-1}=y_i-hy'_i+\frac{h^2}{2}y_i^{(2)}-\frac{h^3}{6}\ y_i^{(3)}+\frac{h^4}{24}y^{(4)}(x_2)$$

$$\frac{y_{i+1}-\frac{4}{3}y_i+\frac{1}{3}y_{i-1}}{h}=\frac{3y_{i+1}-4y_i+y_{i-1}}{3h}$$
$$=\frac{3y_i+3hy'_i+\frac{3h^2}{2}y_i^{(2)}+\frac{h^3}{2}\ y_i^{(3)}+\frac{h^4}{8}y^{(4)}(x_1)+y_i-hy'_i+\frac{h^2}{2}y_i^{(2)}-\frac{h^3}{6}\ y_i^{(3)}+\frac{h^4}{24}y^{(4)}(x_2)-4y_i}{3h}$$
$$=\frac{2hy'_i+2h^2y_i^{(2)}+\frac{h^3}{3}y_i^{(3)}+O(h^4)}{3h}=\frac{2y'_i}{3}+\frac{2h}{3}y_i^{(2)}+\frac{h^2}{9}y_i^{(3)}+O(h^3)$$

$$\text{taylor expansion for } y'_{i+1}=y'_i+hy_i^{(2)}+\frac{h^2}{2}y_i^{(3)}+\frac{h^3}{6}\ y^{(4)}(x_3)\to\frac{2y'_{i+1}}{3}=\ \frac{2y'_i}{3}+\frac{2h}{3}y_i^{(2)}+\frac{h^2}{3}y_i^{(3)}+O(h^3)$$

$$\to\ \text{Local truncation error}=\frac{y_{i+1}-\frac{4}{3}y_i+\frac{1}{3}y_{i-1}}{h}-\frac{2y'_{i+1}}{3}=\ -\frac{2h^2}{9}y_i^{(3)}+O(h^3)=O(h^2)$$