Taylor's Remainder Proof

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Lemma 1. Suppose $f(x)$ is an arbitrary function which is $n+1$ times differentiable.
Suppose there exists a where: $f(a) = f'(a) = f''(a) = \dots = f^n(a) = 0$.
Suppose there exists $x \neq a$ where $f(x) = 0$.
Then there exists c between x and a where $f^{n+1}(c) = 0$.

Proof. Let's call the interval $[x, a]$ or $[a, x]$, I.

$$f(a) = f(x) = 0 \xrightarrow{Rolle's\ Theorem} \exists a_1 \in I : f'(a_1) = 0$$

$$f'(a) = f'(a_1) = 0 \xrightarrow{Rolle's\ Theorem} \exists a_2 \in I : f''(a_2) = 0$$

$$\dots$$

$$f^n(a) = f^n(a_n) = 0 \xrightarrow{Rolle's\ Theorem} \exists c \in I : f^{n+1}(c) = 0$$

Taylor's Remainder Proof. Suppose $P_n(x)$ is Taylor's n-degree Polynomial for $f(x)$ with center $x = c$.
Then the function $g(x) = f(x) - P_n(x)$ has the property where $g(a) = g'(a) = g''(a) = \dots = g^n(a) = 0$.
Suppose there exists $y \neq a$. The error for Taylor's Approximation is $g(y)$. We define constant $C = \frac{g(y)}{(y-a)^{n+1}}$ and define function $h(x) = g(x) - C(x-a)^{n+1}$. This function has the properties for **Lemma 1** to be applied, where:

$$h(a) = h'(a) = h''(a) = \dots = h^n(a) = 0 \ and\ h(y) = 0$$

So there exists c between y and a where $h^{n+1}(c) = 0$.
We know $h^{n+1}(x) = f^{n+1}(x) - C(n+1)!$ following from the definition of $h(x)$. So it follows from the result of **Lemma 1** that $h^{n+1}(c) = f^{n+1}(c) - C(n+1)! = 0$. Therefore $C = \frac{f^{n+1}(c)}{(n+1)!}$. By the definition of C it follows that $g(y) = \frac{f^{n+1}(c)}{(n+1)!}(y-a)^{n+1}$ which $g(y)$ was the error of Taylor's Approximation of $f(y)$. Therefore we have proven the Taylor's Remainder Formula!