سوال اول)

$\textbf{\textit{Interpolating polynomial}}$ *at most* $\textbf{\textit{degree n}}$ *of* $(x_0, y_0) \dots (x_n, y_n)$ *is* $\textbf{\textit{unique}}$, $P_n = x^n$ *interpolates the given points*

$\rightarrow P_n = x^n$ *is the interpolating polynomial*

$$1)\ Lagrange\ Multipliers: P_n(x) = \sum_{i=0}^{n} y_i \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$2)\ coefficient\ of\ x^i \equiv c_i$

$$c_n = 1 \rightarrow \sum_{i=0}^{n} y_i \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{1}{x_i - x_j} = 1$$

$$c_{n-1} = 0 \rightarrow \sum_{i=0}^{n} (y_i \sum_{\substack{j=0 \\ j\neq i}}^{n} -x_j \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{1}{x_i - x_j}) = \sum_{i=0}^{n} (y_i(x_i + \sum_{j=0}^{n} -x_j) \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{1}{x_i - x_j}) = \sum_{i=0}^{n} (x_i y_i \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{1}{x_i - x_j}) - \sum_{i=0}^{n} x_i \sum_{i=0}^{n} y_i \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{1}{x_i - x_j}$$

$$\rightarrow \sum_{i=0}^{n} x_i^{n+1} \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{1}{x_i - x_j} = \sum_{i=0}^{n} x_i$$

$(\mathbf{1})\ let\ \forall i\,; 0 \le i \le n : x_i = i$

$$\rightarrow \sum_{i=0}^{n} i^{n+1} \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{1}{i - j} = \sum_{i=0}^{n} i^{n+1} \left(\frac{1}{i} \times \frac{1}{i-1} \times \dots \times 1 \times -1 \times \dots \times \frac{1}{i-n}\right) = \sum_{i=0}^{n} \frac{i^{n+1}(-1)^{n-i}}{(n-i)!\,i!} = \sum_{i=0}^{n} \frac{i^{n+1}(-1)^{n-i}}{n!} \binom{n}{i} = \frac{n(n+1)}{2}$$

$$\rightarrow \boldsymbol{\sum_{i=0}^{n} i^{n+1}(-1)^{n-i} \binom{n}{i} = \ = \frac{n(n+1)!}{2}}$$

$(\mathbf{2})\ let\ \forall i\,; 0 \le i \le n : x_i = a - i$

$$\rightarrow \sum_{i=0}^{n} (a-i)^n \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{1}{(a-i)-(a-j)} = \sum_{i=0}^{n} (a-i)^n \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{1}{j-i} = \sum_{i=0}^{n} \frac{(a-i)^n(-1)^i}{(n-i)!\,i!} = \sum_{i=0}^{n} \frac{(a-i)^n(-1)^i}{n!} \binom{n}{i} = 1$$

$$\rightarrow \boldsymbol{\sum_{i=0}^{n} (a-i)^n(-1)^i \binom{n}{i} = n!}$$

سوال دوم)

$$Newton\ Multipliers:\ P_n(x) = \sum_{i=0}^{n} f[x_0, \dots, x_i] \prod_{j=0}^{i-1} (x - x_j)$$

$$let\ \forall i\,; x_i = i \rightarrow f[0, \dots, i] = \frac{\Delta^i f(0)}{i!} \quad (ref: section\ 3.3\ Numerical\ Analysis\ 9th\ edition)$$

$$P_4(x) = \sum_{i=0}^{4} \frac{\Delta^i P(0)}{i!} \prod_{j=0}^{i-1} (x - j))$$

$$= P(0) + \Delta P(0)(x-0) + \frac{\Delta^2 P(0)}{2!}(x-1)(x-0) + \frac{\Delta^3 P(0)}{3!}(x-2)(x-1)(x-0) + \frac{\Delta^4 P(0)}{4!}(x-3)(x-2)(x-1)(x-0)$$

$$= P(0) + \Delta P(0)x + (x-2)(x-1)x + (x-3)(x-2)(x-1)x$$

$$\Delta^2 P(10) = \Delta P(11) - \Delta P(10) = P(12) - 2P(11) - P(10)$$

$$P(12) = P(0) + 12\Delta P(0) + 1320 + 11880, \qquad P(11) = P(0) + 11\Delta P(0) + 990 + 7920, \qquad P(10) = P(0) + 10\Delta P(0) + 720 + 5040$$

$$\boldsymbol{\Delta^2 P(10) = 1140}$$

سوال سوم)

$First\ Order\ Spline\ \equiv Piecewise\ linear\ Interpolation$

$for\ every\ section\ the\ error\ can\ be\ found\ via\ the$ $\boldsymbol{linear\ interpolation\ error}$:

$$\forall i\,; 0 \le i < n\,; |f(x) - P_1(x)| = \left| \frac{f''(\alpha)}{2!}\left(x - \frac{i}{n}\right)\left(x - \frac{i+1}{n}\right) \right| = \frac{|f''(\alpha)|}{2}\left(x - \frac{i}{n}\right)\left(\frac{i+1}{n} - x\right);\ x \in \left[\frac{i}{n}, \frac{i+1}{n}\right]$$

$$Theorem: \forall i, 0 \le i < n, \forall x \in \left[\frac{i}{n}, \frac{i+1}{n}\right], \exists y \in \left[0, \frac{1}{n}\right]\ ;\ |f(x) - P_1(x)| \le |f(y) - P_1(y)|$$

$$Proof: let\ y = x - \frac{i}{n}:$$

$$|f(x) - P_1(x)| = \frac{|f''(\alpha)|}{2}\left(x - \frac{i}{n}\right)\left(\frac{i+1}{n} - x\right) = \frac{|f''(\alpha)|}{2}\left(\frac{1}{n} - y\right)y\,, \alpha \in \left[\frac{i}{n}, \frac{i+1}{n}\right]$$

$$|f(y) - P_1(y)| = \frac{|f''(\beta)|}{2}\left(y - \frac{i}{n}\right)\left(\frac{i+1}{n} - y\right) = \frac{|f''(\beta)|}{2}\left(\frac{1}{n} - y\right)y\,, \beta \in \left[0, \frac{1}{n}\right]$$

$$|f''|\ is\ strictly\ descending \rightarrow |f''(\beta)| \ge |f''(\alpha)| \rightarrow |f(x) - P_1(x)| \le |f(y) - P_1(y)|$$

$$\rightarrow for\ every\ x \in [0, 1], there\ exists\ y \in \left[0, \frac{1}{n}\right]\ where\ ;\ |f(x) - P_1(x)| \le |f(y) - P_1(y)| \rightarrow maximum\ error\ is\ in\ \left[0, \frac{1}{n}\right]$$

سوال چهارم)

$For\ natural\ cubic\ spline\ we\ have\ these\ equations\ for\ b_i, c_i, d_i: (ref: section\ 3.3\ Numerical\ Analysis\ 9th\ edition)$

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \\ h_0 & 2(h_0 + h_1) & h_1 & \ddots & & \vdots \\ 0 & h_1 & 2(h_1 + h_2) & h_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & h_{n-2} & 2(h_{n-2} + h_{n-1}) & h_{n-1} \\ 0 & \cdots & \cdots & 0 & 0 & 1 \end{bmatrix},$$

and **b** and **x** are the vectors

$$\mathbf{b} = \begin{bmatrix} 0 \\ \frac{3}{h_1}(a_2 - a_1) - \frac{3}{h_0}(a_1 - a_0) \\ \vdots \\ \frac{3}{h_{n-1}}(a_n - a_{n-1}) - \frac{3}{h_{n-2}}(a_{n-1} - a_{n-2}) \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

$$\boxed{b_{j-1} = \frac{1}{h_{j-1}}(a_j - a_{j-1}) - \frac{h_{j-1}}{3}(2c_{j-1} + c_j).}$$

$$\boxed{c_{j+1} = c_j + 3d_j h_j.}$$

$For\ clamped\ cubic\ spline\ we\ have\ these\ equations\ for\ c_i$:

$$A = \begin{bmatrix} 2h_0 & h_0 & 0 & \cdots & \cdots & 0 \\ h_0 & 2(h_0 + h_1) & h_1 & \ddots & & \vdots \\ 0 & h_1 & 2(h_1 + h_2) & h_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & h_{n-2} & 2(h_{n-2} + h_{n-1}) & h_{n-1} \\ 0 & \cdots & \cdots & 0 & h_{n-1} & 2h_{n-1} \end{bmatrix},$$

$$\mathbf{b} = \begin{bmatrix} \frac{3}{h_0}(a_1 - a_0) - 3f'(a) \\ \frac{3}{h_1}(a_2 - a_1) - \frac{3}{h_0}(a_1 - a_0) \\ \vdots \\ \frac{3}{h_{n-1}}(a_n - a_{n-1}) - \frac{3}{h_{n-2}}(a_{n-1} - a_{n-2}) \\ 3f'(b) - \frac{3}{h_{n-1}}(a_n - a_{n-1}) \end{bmatrix}, \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

$S_0(x) = a_0 + b_0(x+1) + c_0(x+1)^2 + d_0(x+1)^3$

$S_1(x) = a_1 + b_1 x + c_1 x^2 + d_1 x^3$

$S_2(x) = a_2 + b_2(x-1) + c_2(x-1)^2 + d_2(x-1)^3$

$\forall i\,, 0 \le i < 2\,; h_i = 1$

$a_0 = 0.5$

$a_1 = 1$

$a_2 = 2$

$a_3 = 4$

$(\mathbf{1})$ $\boldsymbol{Natural\ Cubic\ Spline}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3(2-1) - 3(1-0.5) = 1.5 \\ 3(4-2) - 3(2-1) = 3 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.2 \\ 0.7 \\ 0 \end{bmatrix}$$

$$b_2 = (4-2) - \frac{1}{3}(2 \times 0.7 + 0) = \frac{23}{15}$$

$$b_1 = (2-1) - \frac{1}{3}(2 \times 0.2 + 0.7) = \frac{19}{30}$$

$$b_0 = (1-0.5) - \frac{1}{3}(2 \times 0 + 0.2) = \frac{13}{30}$$

$$d_2 = \frac{c_3 - c_2}{3} = -\frac{7}{30}$$

$$d_1 = \frac{c_2 - c_1}{3} = -\frac{1}{6}$$

$$d_0 = \frac{c_1 - c_0}{3} = \frac{1}{15}$$

$$\boldsymbol{S_0(x) = 0.5 + \frac{13(x+1)}{30} + \frac{(x+1)^3}{15}}$$

$$\boldsymbol{S_1(x) = 1 + \frac{19x}{30} + \frac{x^2}{5} - \frac{x^3}{6}}$$

$$\boldsymbol{S_2(x) = 2 + \frac{23(x-1)}{15} + \frac{7(x-1)^2}{10} - \frac{7(x-1)^3}{30}}$$

$(\mathbf{2})$ $\boldsymbol{Clamped\ Cubic\ Spline}$

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3(1-0.5) - 3(1) = -1.5 \\ 3(2-1) - 3(1-0.5) = 1.5 \\ 3(4-2) - 3(2-1) = 3 \\ 3(-1) - 3(4-2) = -9 \end{bmatrix} \rightarrow \ c_0 = \frac{-23}{30}\,, c_1 = \frac{1}{30}\,, c_2 = \frac{32}{15}\,, c_4 = -\frac{167}{30}$$

$$b_2 = (4-2) - \frac{1}{3}\left(2 \times \frac{32}{15} - \frac{167}{30}\right) = \frac{73}{30}$$

$$b_1 = (2-1) - \frac{1}{3}\left(2 \times \frac{1}{30} + \frac{32}{15}\right) = \frac{4}{15}$$

$$b_0 = (1-0.5) - \frac{1}{3}\left(2 \times \frac{-23}{30} + \frac{1}{30}\right) = 1$$

$$d_2 = \frac{c_3 - c_2}{3} = \frac{-77}{30}$$

$$d_1 = \frac{c_2 - c_1}{3} = \frac{7}{10}$$
$$d_0 = \frac{c_1 - c_0}{3} = \frac{4}{15}$$

$$\boldsymbol{S_0(x) = 0.5 + (x+1) - \frac{23(x+1)^2}{30} + \frac{4(x+1)^3}{15}}$$
$$\boldsymbol{S_1(x) = 1 + \frac{4x}{15} + \frac{x^2}{30} + \frac{7x^3}{10}}$$
$$\boldsymbol{S_2(x) = 2 + \frac{73(x-1)}{30} + \frac{32(x-1)^2}{15} - \frac{77(x-1)^3}{30}}$$

سوال پنجم)

الف)

$$Newton\ Multipliers{:}\ P_n(x) = \sum_{i=0}^{n} f[x_0, \ldots, x_i] \prod_{j=0}^{i-1} (x - x_j)$$
$$data\ set{:}\ (x_0 = 0, x_1 = 1, x_2 = 1, x_3 = 2)$$
$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{ln2 - ln1}{1 - 0} = ln2$$
$$f[x_1, x_2] = f'(x_1) = \frac{1}{1+1} = 0.5$$
$$f[x_2, x_3] = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{ln3 - ln2}{3 - 2} = ln3 - ln2$$
$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{0.5 - ln2}{1 - 0} = 0.5 - ln2$$
$$f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1} = \frac{ln3 - ln2 - 0.5}{2 - 1} = ln3 - ln2 - 0.5$$
$$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0} = \frac{ln3 - ln2 - 0.5 - 0.5 + ln2}{2} = \frac{ln3}{2} - 0.5$$
$$P_3(x) = f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_1)(x - x_0) + f[x_0, x_1, x_2, x_3](x - x_2)(x - x_1)(x - x_0)$$
$$P_3(x) = \ln(2)\, x + (0.5 - \ln(2))(x - 1)x + \left(\frac{\ln(3)}{2} - 0.5\right)(x - 1)^2 x$$

ب)

$$Third\ order\ interpolation\ error \rightarrow \frac{f^4(\alpha(x))}{4!} \prod_{i=0}^{n} (x - x_i)$$
$$For\ the\ given\ interpolation, the\ error\ is\ \left|\frac{f^4(\alpha(x))}{4!}(x - 2)(x - 1)^2 x\right|$$

$$f(x) = \ln(x+1) \rightarrow f^4(x) = \frac{-6}{(x+1)^4} \rightarrow x \in [0,2]\, ; f^4(x) \le\ -6 \rightarrow \left|\frac{f^4\big(\alpha(x)\big)}{4!}\right| \le \frac{1}{4} \qquad (\mathbf{1})$$
$$\max \text{ of } (x-2)\,(x-1)^2 x \rightarrow \frac{d}{dx}\big((x-2)(x-1)^2 x\big) = (x-1)\big(x(x-1) + 2(x-2)x + (x-2)(x-1)\big) = (x-1)(2x^2 - 4x + 1) = 0$$
$$\rightarrow x = 1, 1 \pm \frac{\sqrt{2}}{2} \rightarrow \max(|(x-2)(x-1)^2 x|) = \frac{1}{4} \qquad (\mathbf{2})$$
$$(1), (2) \rightarrow \left|\frac{f^4(\alpha(x))}{4!}(x-2)(x-1)^2 x\right| \le \frac{1}{16}$$

سوال ششم)

$f[x_0, \dots, x_k] = f[\sigma(\boldsymbol{x})]$ *where* $\sigma(\boldsymbol{x})$ *is permutation of* $x_0, \dots, x_k$

$$f[x_0, \dots, x_n, x, x] = f[x, x_0, \dots, x_n, x] = \lim_{h \to 0} f[x, x_0, \dots, x_n, x+h] = \lim_{h \to 0} \frac{f[x_0, \dots, x_n, x+h] - f[x, x_0, \dots, x_n]}{h}$$

$$= \lim_{h \to 0} \frac{f[x_0, \dots, x_n, x+h] - f[x_0, \dots, x_n, x]}{h} \xrightarrow{g(x)=f[x_0,\dots,x_n,x]} \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = \frac{d}{dx} f[x_0, \dots, x_n, x]$$

سوال هفتم)

$$P(x) = \frac{(x_۳ - x)p_۲^{(۰,۲)}(x) + (x - x_۰)p_۲^{(۱,۳)}(x)}{x_۳ - x_۰}$$

degree of $P(x) = 3$

$$P(x_0) = \frac{(x_3 - x_0)p_2^{(0,2)}(x_0) + (x_0 - x_0)p_2^{(1,3)}(x_0)}{x_3 - x_0} = \frac{(x_3 - x_0)y_0 + 0}{x_3 - x_0} = y_0$$

$$P(x_1) = \frac{(x_3 - x_1)p_2^{(0,2)}(x_1) + (x_1 - x_0)p_2^{(1,3)}(x_1)}{x_3 - x_0} = \frac{(x_3 - x_1)y_1 + (x_1 - x_0)y_1}{x_3 - x_0} = y_1$$

$$P(x_2) = \frac{(x_3 - x_2)p_2^{(0,2)}(x_2) + (x_2 - x_0)p_2^{(1,3)}(x_2)}{x_3 - x_0} = \frac{(x_3 - x_2)y_2 + (x_2 - x_0)y_2}{x_3 - x_0} = y_2$$

$$P(x_3) = \frac{(x_3 - x_3)p_2^{(0,2)}(x_3) + (x_3 - x_0)p_2^{(1,3)}(x_3)}{x_3 - x_0} = \frac{0 + (x_3 - x_0)y_3}{x_3 - x_0} = y_3$$

$P(x)$ *interpolates* $n+1$ *points and is degree* $n \to$ *uniqueness theorem* $\to$ $P(x)$ *is the interpolating polynomial*

سوال هشتم)

let the dataset be: $\forall i, 0 \le i \le n; (x_i, y_i) = \left(\frac{i}{n}, P\left(\frac{i}{n}\right)\right)$

because we have $n+1$ *variables and the interpolating polynomial is unique*

$\to$ *the interpolating polynomial of degree n over dataset is* $P_n(x) = P(x)$

Lagrange Multipliers: $P_n(x) = \sum_{i=0}^{n} y_i \prod_{\substack{j=0 \\ j \ne i}}^{n} \frac{x - x_j}{x_i - x_j}$

$$= \sum_{i=0}^{n} P\left(\frac{i}{n}\right) \prod_{\substack{j=0 \\ j \ne i}}^{n} \frac{x - \frac{j}{n}}{\frac{i}{n} - \frac{j}{n}} = \sum_{i=0}^{n} P\left(\frac{i}{n}\right) \prod_{\substack{j=0 \\ j \ne i}}^{n} \frac{nx - j}{i - j}$$

$$\to P_n\left(-\frac{1}{n}\right) = \sum_{i=0}^{n} P\left(\frac{i}{n}\right) \prod_{\substack{j=0 \\ j \ne i}}^{n} \frac{-1 - j}{i - j}$$

$$\to \left|P\left(-\frac{1}{n}\right)\right| = \sum_{i=0}^{n} P\left(\frac{i}{n}\right) \prod_{\substack{j=0 \\ j \ne i}}^{n} \frac{1 + j}{|i - j|}$$

$$= \sum_{i=0}^{n} P\left(\frac{i}{n}\right)\left(\frac{1}{i} \times \frac{2}{i-1} \times \dots \times 1 \times 1 \times \dots \times \frac{n+1}{n-i}\right) = \sum_{i=0}^{n} P\left(\frac{i}{n}\right) \frac{(n+1)!}{(1+i)i!\,(n-i)!} \le \sum_{i=0}^{n} \binom{n+1}{i+1} = 2^{n+1} - 1$$