Preliminary Theorems:

Rolle's Theorem: if f is diffrentiable on $[a, b]$ and $f(a) = f(b)$, then there exists $c \in [a, b]$ where $f'(c) = 0$

Mean Value Theorem: if f is diffrentiable on $[a, b]$, then there exists $c \in [a, b]$ where $f'(c) = \frac{f(b) - f(a)}{b - a}$

Intermediate Value Theorem: if f is continuous on $[a, b]$ then for every K between $f(a)$ and $f(b)$, there exists $c \in [a, b]$ where $f(c) = K$

Generalized Rolle's Theorem: if f is n times diffrentiable on $[a, b]$and there exists $n + 1$ distinct points on $[a, b]$ where $f(x_i) = 0$ for $\forall i \,;\, 0 \le i \le n$, then there exists $c \in [a, b]$ where $f^n(c) = 0$

Wheighted Mean Value Theorem: if f is continuous on $[a, b]$ and g does not change sign on $[a, b]$, then there exists $c \in [a, b]$ where $\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx$

Taylor's Equation:

$f(x) = P_n(x) + R_n(x) : \forall i \le n \,;\, f^{(i)}(x)$ is continuous on $[a, b]$; $f^{(n+1)}(x_0)$ exists for all $x_0 \in [a, b]$

$$P_n(x) = \sum_{i=0}^{n} (x - x_0)^i \frac{f^{(i)}(x_0)}{i!}$$

$$R_n(x) = (x - x_0)^{n+1} \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \text{ where } \xi(x) \in [a, b]$$

Taylor's equation for higher order of arguments:

$$f(x_0 + h, y_0 + k) = \sum_{i=0}^{n} \frac{1}{i!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^i f(x_0, y_0) + \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f\big(x_0 + h(x), y + k(y)\big)$$

Some famous taylor expansions:

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!} \qquad \ln(1 + x) = \sum_{i=0}^{\infty} \frac{(-1)^i x^{i+1}}{i + 1} \qquad \sin(x) = \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i+1}}{(2i + 1)!} \qquad \cos(x) = \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i}}{(2i)!} \qquad \frac{1}{1 + x} = \sum_{i=0}^{\infty} (-1)^i x^i$$

Interpolation:

Given the dataset $\{(x_0, y_0), \dots, (x_n, y_n)\}$ find the polynomial interpolating these points $(a \le x_i \le b)$

1) **Vandermonde Matrix**: $P_n(x) = a_0 + a_1 x + \cdots + a_n x^n$

$$\begin{bmatrix} 1 & x_0 & \dots & x_0^n \\ 1 & x_1 & \dots & x_1^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \dots & x_n^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix} \rightarrow \text{proof that the polynomial is found and unique}$$

$$\text{Error: } f(x) - P_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \prod_{i=0}^{n} (x - x_i) \text{ where } \big(\xi(x)\big) \in [a, b]$$

2) **Lagrange Multipliers**:

$$P(x) = \sum_{k=0}^{n} f(x_k) L_{n,k}(x) \text{ where } L_{n,k}(x) = \prod_{\substack{i=0 \\ i \ne k}}^{n} \frac{(x - x_i)}{(x_k - x_i)}$$

3) **Divided Differenves (Newton's Method)**:

$$P(x) = f[x_0] + \sum_{i=1}^{n} f[x_0, \dots, x_i] \prod_{j=0}^{i-1} (x - x_j)$$
$$f[x_0] = f(x_0)$$
$$f[x_0, \dots, x_i] = \frac{f[x_1, \dots, x_i] - f[x_0, \dots, x_{i-1}]}{x_i - x_0}$$

Theorem: $f[x_0, \dots, x_n] = \dfrac{f^{(n)}(\xi)}{n!}$ for some $\xi \in [a, b]$

4) **Backward and Forward Differences**:
$f(x_i) = f_i$
$\Delta f_i = f_{i+1} - f_i \rightarrow \Delta^n f_i = \Delta^{n-1}(\Delta f_i) = \Delta^{n-1}(f_{i+1} - f_i)$
$\nabla f_i = f_i - f_{i-1} \rightarrow \nabla^n f_i = \nabla^{n-1}(\nabla f_i) = \nabla^{n-1}(f_i - f_{i-1})$

Let x_i points be equally spaced with difference h:
$$P_n(x_0 + sh) = f_0 + \sum_{i=1}^{n} \binom{s}{i} \Delta^i f_0 \qquad P_n(x_n + sh) = f_n + \sum_{i=1}^{n} (-1)^i \binom{-s}{i} \nabla^i f_n$$

5) **Spline**:

5.1) Piecewise Linear Spline: $S_i(x) = f_i + (x - x_i)\Delta f_i$
→ Spline Error is the same as finding the maximum of n interpolating polynomials of the desired degree

5.2) Quadratic Spline → Find n parabolas interpolating each pair of points (Continuity of f')

5.3) Cubic Splice → Find n 3rd degree polynomials interpolating each pair of points (Continuity of f'')
$S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3$

finding the n 3rd degree polynomials requires solving these equations:

$$h_i = x_{i+1} - x_i$$
$$a_i = f_i$$
$$b_i = \frac{1}{h_i}(a_{i+1} - a_i) + \frac{h_{i-1}}{3}(c_{i+1} + c_i)$$
$$d_i = \frac{c_{i+1} - c_i}{3h_i}$$

5.3.1) Natural Cubic Spline: $S_0''(x_0) = S_{n-1}''(x_n) = 0$:
$$\begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ h_0 & 2(h_0 + h_1) & h_1 & 0 & \ddots & \vdots \\ 0 & h_1 & 2(h_1 + h_2) & h_2 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 0 & h_{n-2} & 2(h_{n-2} + h_{n-1}) & h_{n-1} \\ 0 & \dots & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{3}{h_1}(a_2 - a_1) - \frac{3}{h_0}(a_1 - a_0) \\ \vdots \\ \frac{3}{h_{n-1}}(a_n - a_{n-1}) + \frac{3}{h_{n-2}}(a_{n-1} - a_{n-2}) \\ 0 \end{bmatrix} = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix}$$

5.3.2) Clamped Cubic Spline: $S_0'(x_0) = f_0'$ and $S_{n-1}'(x_n) = f_n'$:
$$\begin{bmatrix} 2h_0 & h_0 & 0 & 0 & \dots & 0 \\ h_0 & 2(h_0 + h_1) & h_1 & 0 & \ddots & \vdots \\ 0 & h_1 & 2(h_1 + h_2) & h_2 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 & 0 \\ \vdots & \ddots & 0 & h_{n-2} & 2(h_{n-2} + h_{n-1}) & h_{n-1} \\ 0 & \dots & 0 & 0 & h_{n-1} & 2h_{n-1} \end{bmatrix} \begin{bmatrix} \frac{3}{h_0}(a_1 - a_0) - 3f_0' \\ \frac{3}{h_1}(a_2 - a_1) - \frac{3}{h_0}(a_1 - a_0) \\ \vdots \\ \frac{3}{h_{n-1}}(a_n - a_{n-1}) - \frac{3}{h_{n-2}}(a_{n-1} - a_{n-2}) \\ 3f_n' - \frac{3}{h_{n-1}}(a_n - a_{n-1}) \end{bmatrix} = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix}$$

Differentiation:

1) **Two Point Formula**:
From Taylor's Polynomial: $f(x_{i+1}) = f(x_i) + h_i f'(x_i) + \dfrac{h^2}{2} f''(\xi) \rightarrow f'(x_i) = \dfrac{\Delta f_i}{h_i} + \dfrac{hf''(\xi)}{2}$

From Lagrange's Multipliers:

$$f(x) = P_1(x) + R_1(x) = \frac{f_1}{h_0}(x - x_1) - \frac{f_0}{h_0}(x - x_0) + \frac{f''(\xi)}{2}(x - x_0)(x - x_1)$$

$$f'(x) = P_1'(x) + R_1'(x) = \frac{f_1 - f_0}{h_0} + \frac{f''(\xi)}{2}(2x - x_0 - x_1) \rightarrow x = x_i \rightarrow f'(x_i) = \frac{\Delta f_i}{h_i} + \frac{hf''(\xi)}{2}$$

2) **Three Point Formulas**:
Derived From Lagrange Multipliers:

$$f(x) = P_2(x) + R_2(x) = \frac{f_0(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} + \frac{f_1(x - x_2)(x - x_0)}{(x_1 - x_2)(x_1 - x_0)} + \frac{f_2(x - x_1)(x - x_0)}{(x_2 - x_1)(x_2 - x_0)} + \frac{f^{(3)}(\xi)(x - x_0)(x - x_1)(x - x_2)}{6}$$

$$f'(x_i) = \frac{f_0(2x_i - x_1 - x_2)}{2h^2} - \frac{f_1(2x_i - x_0 - x_2)}{h^2} + \frac{f_2(2x_i - x_0 - x_1)}{2h^2} + \frac{f^{(3)}(\xi)}{6}\prod_{\substack{j=0 \\ j\neq i}}^{2}(x_i - x_j)$$

$$2.1)\ f'(x_1) = \frac{-hf_0}{2h^2} + \frac{hf_2}{2h^2} - \frac{h^2 f^{(3)}(\xi)}{6} = \frac{f_2 - f_0}{2h} - \frac{h^2 f^{(3)}(\xi)}{6} \rightarrow \text{Mid Point Formula}$$

$$2.2)\ f'(x_0) = \frac{-3hf_0}{2h^2} + \frac{2hf_1}{h^2} - \frac{hf_2}{2h^2} + \frac{h^2 f^{(3)}(\xi)}{3} = \frac{-3f_0 + 4f_1 - f_2}{2h} + \frac{h^2 f^{(3)}(\xi)}{3} \rightarrow \text{End Point Formula}$$

3) **Five Point Formulas**:
Derived From Lagrange Multipliers:

$$3.1)\ f'(x_2) = \frac{f_0 - 8f_1 + 8f_3 - f_4}{12h} + \frac{h^4 f^{(5)}(\xi)}{30} \rightarrow \text{ Mid Point Formula}$$

$$3.2)\ f'(x_4) = \frac{-25f_0 + 48f_1 - 36f_2 + 16f_3 - 3f_4}{12h} + \frac{h^4 f^{(5)}(\xi)}{5} \rightarrow \text{End Point Formula}$$

4) **Second Derivative Mid Point Formula**:

$$f''(x_1) = \frac{f_0 - 2f_1 + f_2}{h^2} - \frac{h^2 f^{(4)}(\xi)}{12} \rightarrow \textit{Derived From Taylor's Polynomial}$$

5) **Round off Error Instability**:

Imagine $f(x + h) = \mathcal{F}(x + h) + e(x + h)$ where $e(x + h)$ is the round off error, let $|e(x)|$ be bounded by some ϵ

$$f'(x) = \frac{f(x + h) - f(x - h)}{2h} - \frac{h^2 f^{(3)}(\xi)}{6} \rightarrow \left|f'(x) - \frac{\mathcal{F}(x + h) - \mathcal{F}(x - h)}{2h}\right| = \left|\frac{e(x + h) - e(x - h)}{2h} + \frac{h^2 f^{(3)}(\xi)}{6}\right| \leq \frac{\epsilon}{h} + \frac{h^2 M}{6}$$

$\rightarrow$ not the best move to make h as small as possible

Integration:

1) **Trapezoidal Rule**: Use a linear approximation for the function

$$f(x) = \frac{f_1}{h_0}(x - a) - \frac{f_0}{h_0}(x - b) + \frac{f''(\xi(x))}{2}(x - a)(x - b) \rightarrow \int_a^b f(x)dx = \left[\frac{f_1(x - a)^2 - f_2(x - b)^2}{2(b - a)}\right]_a^b + \int_a^b \frac{f''(\xi(x))}{2}(x - a)(x - b)\,dx$$

$$\rightarrow \text{Weighted Mean Value Theorem: } \int_a^b \frac{f''(\xi(x))}{2}(x - a)(x - b)\,dx = \frac{f''(\xi)}{2}\int_a^b (x - a)(x - b)\,dx = -\frac{h^3 f''(\xi)}{12}$$

$$\rightarrow \int_a^b f(x)dx = \left[\frac{f_1(x - a)^2 - f_2(x - b)^2}{2(b - a)}\right]_a^b - \frac{h^3 f''(\xi)}{12} = \frac{f_1(b - a)^2 + f_2(b - a)^2}{2(b - a)} - \frac{h^3 f''(\xi)}{6} = \frac{h}{2}(f_1 + f_2) - \frac{h^3 f''(\xi)}{12}$$

$$\rightarrow \int_a^b f(x)dx = \frac{(b - a)(f(a) + f(b))}{2} - \frac{(b - a)^3 f''(\xi)}{12}$$

2) **Simpson's Rule**: Use a parabola approximation (2nd degree Interpolation) for the function

Taylor's Polynomial over the middle point:

$$f(x) = f_1 + (x - x_1)f_1' + \frac{(x - x_1)^2}{2}f_1'' + \frac{(x - x_1)^3}{6}f_1^{(3)} + \frac{(x - x_1)^4 f^{(4)}(\xi(x))}{24} \text{ where } x_1 = \frac{b + a}{2} \text{ and } h = \frac{b - a}{2}$$

$$\int_a^b f(x)dx = \left[f_1 x + \frac{(x - x_1)^2 f_1'}{2} + \frac{(x - x_1)^3 f_1''}{6} + \frac{(x - x_1)^4 f_1^{(3)}}{24}\right]_a^b + \int_a^b \frac{(x - x_1)^4 f^{(4)}(\xi(x))}{24}dx$$

$\rightarrow$ Weighted Mean Value Theorem: $\displaystyle\int_a^b \frac{(x-x_1)^4 f^{(4)}(\xi(x))}{24}dx = \frac{f^{(4)}(\xi)}{24}\int_a^b (x-x_1)^4\,dx = \frac{f^{(4)}(\xi)}{24}\left[\frac{(x-x_1)^5}{5}\right]_a^b = \frac{h^5 f^{(4)}(\xi)}{60}$

$$\rightarrow \left[f_1 x + \frac{(x-x_1)^2 f_1'}{2} + \frac{(x-x_1)^3 f_1''}{6} + \frac{(x-x_1)^4 f_1^{(3)}}{24}\right]_a^b = 2hf_1 + \frac{h^3 f_1''}{3}$$

(1) $\displaystyle\int_a^b f(x)dx = 2hf_1 + \frac{h^3 f_1''}{3} + \frac{h^5 f^{(4)}(\xi)}{60}$

(2) $\displaystyle f_1'' = \frac{f_0 - 2f_1 + f_2}{h^2} - \frac{h^2 f^{(4)}(\xi')}{12} \rightarrow \frac{h^3 f_1''}{3} = \frac{h}{3}(f_0 - 2f_1 + f_2) - \frac{h^5 f^{(4)}(\xi')}{36}$

$$\xrightarrow{(1),(2)} \int_a^b f(x)dx = \frac{h}{3}(f_a + 4f_1 + f_b) - \frac{h^5 f^{(4)}(\mu)}{90}$$

$$\int_a^b f(x)dx = \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right) - \frac{(b-a)f^{(4)}(\mu)}{2880}$$

3) Simpson's $\frac{3}{8}$ Rule:

$$\int_{x_0}^{x_3} f(x)dx = \frac{3h}{8}(f_0 + 3f_1 + 3f_2 + f_3) - \frac{3h^5 f^{(4)}(\mu)}{80}$$

$$\int_a^b f(x)dx = \frac{b-a}{8}\left(f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right) - \frac{(b-a)^5 f^{(4)}(\mu)}{6480}$$

4) Midpoint Rule:

$\displaystyle\int_a^b f(x)dx = (b-a)f\left(\frac{a+b}{2}\right) + \frac{(b-a)^3 f''(\mu)}{24} \rightarrow$ Error derived from Taylor's Polynomial

Composite Integration:

Break the interval to n pieces, use the methods mentioned above to calulate the integral

1) Composite Midpoint Rule:

Let points $\{x_1, \dots, x_n\}$ denote the middle of each smaller interval and $h = \dfrac{b-a}{n}$

$$\int_a^b f(x)dx = \sum_{i=1}^{n}\left(hf(x_i) + \frac{h^3 f''(\mu_i)}{24}\right) = h\sum_{i=1}^{n} f_i + \frac{nh^3 f''(\mu)}{24} = h\sum_{i=1}^{n} f_i + \frac{(b-a)h^2 f''(\mu)}{24}$$

2) Composite Trapezoidal Rule:

Let points $\{x_0, \dots, x_n\}$ denote the start and end of each smaller interval and $h = \dfrac{b-a}{n}$

$$\int_a^b f(x)dx = \sum_{i=1}^{n}\left(\frac{h(f_i + f_{i-1})}{2} - \frac{h^3 f''(\xi)}{12}\right) = \frac{h}{2}\left(f_0 + f_n + 2\sum_{i=1}^{n-1} f_i\right) - \frac{(b-a)h^2 f''(\mu)}{12}$$

3) Composite Simpson's Rule:

Let points $\{x_0, \dots, x_n\}$ denote the start and end of each smaller interval and $h = \dfrac{b-a}{n}$

Then find a parabola interpolating each three points in a row $\left(\frac{n}{2}\text{ parabolas needed}\right)$

$$\int_a^b f(x)dx = \sum_{i=1}^{\frac{n}{2}}\left(\frac{h}{3}(f_{2i-2} + 4f_{2i} + f_{2i+2}) - \frac{h^5 f^{(4)}(\mu)}{90}\right) = \frac{h}{3}\left(f_0 + f_n + 2\sum_{i=1}^{\frac{n}{2}-1} f_{2i} + 4\sum_{i=1}^{\frac{n}{2}} f_{2i-1}\right) - \frac{(b-a)h^4 f^{(4)}(\mu)}{180}$$

4) Round off Error $\mathcal{S}$tability:

Let $f(x) = \mathcal{F}(x) + e(x)$ where $e(x)$ is the round off error, bounded by some ϵ

Using Simpson's Rule to calculate the the integral, we get:

$e(h) \le \dfrac{h}{3}(3n\epsilon) = (b-a)\epsilon \rightarrow$ The error is independent of n