Complex Signals

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- The spectrum of a time-domain signal $s(t)$ is given by its Fourier transform $S(\omega)$
$$S(\omega) = \mathbb{F}[s(t)] = \int_{-\infty}^{\infty} s(t)\, e^{\jmath\omega t} dt \tag{1}$$
where $\mathbb{F}[\cdot]$ is the Fourier operator and ω is the angular frequency in radians per second.

- From (1), the spectrum of a real-valued signal $s_r(t)$ is conjugate symmetric about DC (Hermitian)
$$\begin{aligned} \mathbb{F}[s_r(t)] &= S_r(\omega) \\ S_r^*(\omega) &= S_r(-\omega) \end{aligned} \tag{2}$$
That is, the magnitude of the spectrum is an even function of frequency, while the phase is an odd function of frequency.

  Hermitian symmetry between the positive and negative halves of the spectrum means that one can be deduced from the other. This information redundancy translates to wasted bandwidth.

  A real-valued signal represents a *single physical wire*.

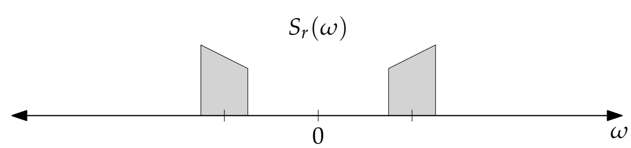

Figure 1: A real-valued signal spectrum.

- From (1), the spectrum of a complex-valued signal $s_c(t)$ is not Hermitian
$$\begin{aligned} \mathbb{F}[s_c(t)] &= S_c(\omega) \\ S_c^*(\omega) &\neq S_c(-\omega) \end{aligned} \tag{3}$$

That is, a complex signal does not contain redundant information, and, consequently, its bandwidth utilization efficiency (how much information can be packed in a given bandwidth) is twice that of a real signal.

A complex signal is carried over *two physical wires*.

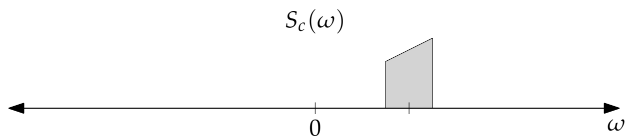

Figure 2: A complex-valued signal spectrum.

- From (1), the spectrum of the conjugate of a complex signal $s_c(t)$ is the conjugate mirror spectrum of that signal about DC

$$\begin{aligned} \mathbb{F}[s_c(t)] &= S_c(\omega) \\ \mathbb{F}[s_c^*(t)] &= S_c^*(-\omega) \end{aligned} \tag{4}$$

- From (4), a real signal can be obtained from a complex signal by conjugation and addition

$$\begin{aligned} \mathbb{F}[s_c(t)] &= S_c(\omega) \\ \mathbb{R}[s_c(t)] &= \frac{1}{2}\Big(s_c(t) + s_c^*(t)\Big) \\ \mathbb{R}[S_c(\omega)] &= \frac{1}{2}\Big(S_c(\omega) + S_c^*(-\omega)\Big) \end{aligned} \tag{5}$$

  where $\mathbb{R}[\cdot]$ is the real operator.

- From (1), multiplying a (real or complex-valued) signal $s(t)$ with a complex sinusoid of frequency $\pm\omega_0$ shifts the spectrum of that signal by $\mp\omega_0$

$$\begin{aligned} \mathbb{F}[s(t)] &= S(\omega) \\ \mathbb{F}\big[s(t) \cdot e^{\pm \jmath\omega_0 t}\big] &= S(\omega \mp \omega_0) \end{aligned} \tag{6}$$

  That is, the spectrum of $s(t)$ is frequency shifted by $\pm\omega_0$. In the context of a radio transceiver, this multiplication operation is referred to as mixing and is the basis for modulation and demodulation.

- Multiplication with a complex sinusoid is the ideal mixing operation because it frequency shifts the complete spectrum of the multiplicand in one direction only, either up or down in frequency. Consequently, (de)modulation using a complex sinusoid can discern between a signal and its image (Fig x).

- A real sinusoid, on the other hand, is the sum of two complex conjugate sinusoids, so when used for mixing, it frequency shifts the spectrum of the

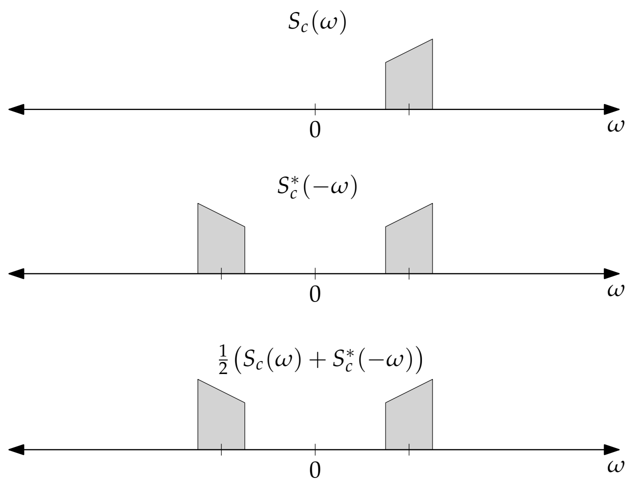

Figure 3: Using a complex-valued signal to build a real-valued signal.

multiplicand up and down simultaneously. This causes the positive and negative parts of the spectrum to fold on top of each other. Consequently, (de)modulation using a real sinusoid cannot discern between a signal and its image (Fig x).

- In a radio transmitter, a complex baseband signal $s_c(t)$ is up-converted to the desired carrier frequency ω_0 using a complex sinusoid and an image-free real signal is transmitted over the air

$$\begin{aligned} s_c(t) &= s_I(t) + \jmath s_Q(t) \\ s_{\text{TX}}(t) &= \mathbb{R}\left[s_c(t) \cdot e^{\jmath\omega_0 t}\right] \\ s_{\text{TX}}(t) &= s_I(t)\cos(\omega_0 t) - s_Q(t)\sin(\omega_0 t) \end{aligned} \tag{7}$$

  The operation in (7) is the well-known quadrature transmitter topology in Fig. 6 and the associated spectra shown along the chain are based on (3), (5) and (6).

- For a radio receiver, a real passband signal $s_r(t)$ is down-converted to baseband (usually to DC) using a complex sinusoid and an image-free complex signal is received

$$\begin{aligned} s_r(t) &= s_I(t)\cos(\omega_0 t) - s_Q(t)\sin(\omega_0 t) \\ s_{\text{RX}}(t) &= s_r(t) \cdot e^{-\jmath\omega_0 t} \\ s_{\text{RX}}(t) &= s_r(t)\cos(\omega_0 t) - \jmath s_r(t)\sin(\omega_0 t) \end{aligned} \tag{8}$$

  The operation in (8) is the well-known IQ receiver topology in Fig. 7 and the frequency spectra shown along the chain are based on (2) and (6).

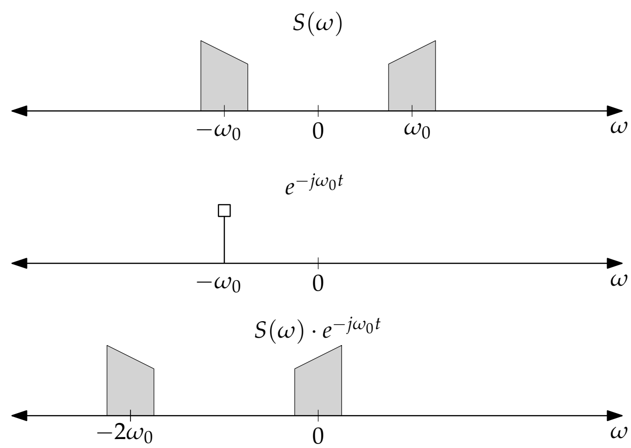

Figure 4: Mixing with a complex sinusoid.

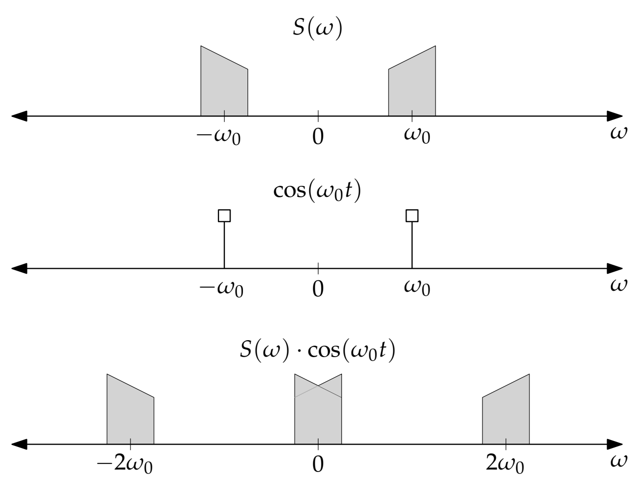

Figure 5: Mixing with a real sinusoid.

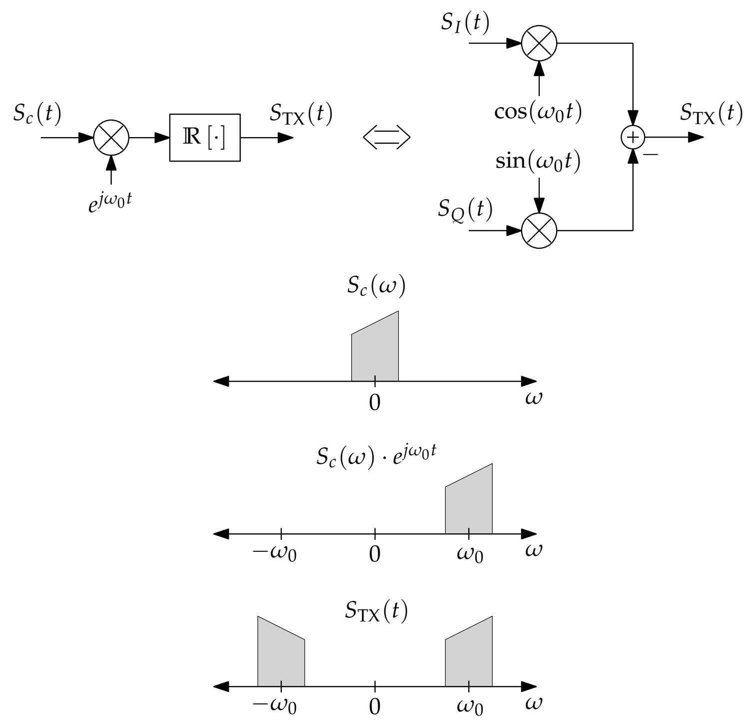

Figure 6: Quadrature transmitter topology.

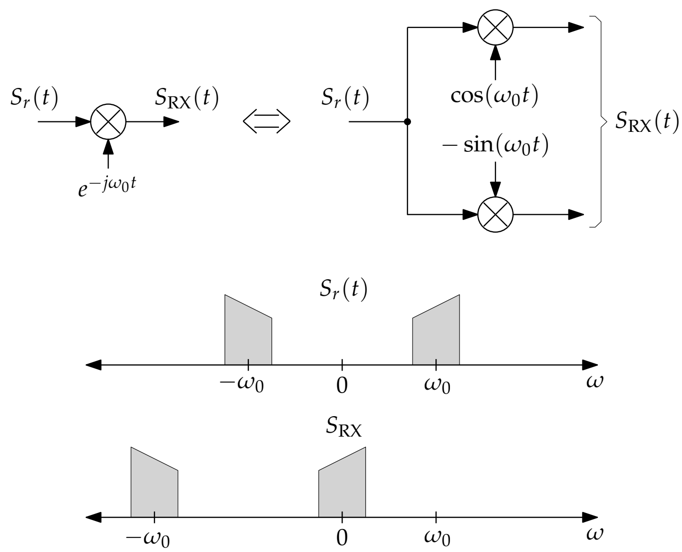

Figure 7: Quadrature receiver topology.