Phase Noise: Frequency Domain Model

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1 Recap

- A local oscillator (LO) signal, $v_{\mathrm{LO}}(t)$, is periodic, so it can be expressed as a Fourier series

$$v_{\mathrm{LO}}(t) = \sum_{k=-\infty}^{\infty} a_k \, e^{\jmath k \omega_{\mathrm{LO}} t} \tag{1}$$

  where ω_{LO} is the fundamental frequency of the signal in radians per second, k is the harmonic number, and a_k is the complex Fourier coefficient of the k-th harmonic.

- The effect of jitter on such a signal can be reduced to an additive noise term $v_{nj}(t)$

$$v_{nj}(t) \approx \jmath \sum_{k=-\infty}^{\infty} a_k \, e^{\jmath k \omega_{\mathrm{LO}} t} \cdot \theta_{jk}(t) \tag{2}$$

$$\theta_{jk}(t) = k\omega_{\mathrm{LO}} t_j(t) \tag{3}$$

  where $t_j(t)$ is random time series that captures the time difference between the actual and ideal zero crossings of the signal, and $\theta_{jk}(t)$ is the corresponding phase error.

- Given the rms phase noise value $\theta_{j,rms}$ of $\theta_{jk}(t)$, the LO signal-to-noise ratio (SNR) is given by

$$SNR_{\mathrm{LO}} = \frac{1}{\theta_{j,rms}^2} \tag{4}$$

2 Phase Noise Model

Since phase noise is a random process, its frequency domain properties are well captured through its power spectral density (PSD)[1]. Towards that end, we

[1] The PSD of a signal is the frequency domain representation of that signal based on its statistical properties, more specifically, its autocorrelation

first calculate the autocorrelation function $R_{nn}(\tau)$ of $v_{nj}(t)$ using (2)

$$\begin{aligned}
R_{nn}(\tau) &= \overline{v_{nj}(t) \cdot v_{nj}^*(t-\tau)} \\
&= \overline{\jmath \sum_{k=-\infty}^{\infty} a_k\, e^{\jmath k\omega_{\text{LO}} t}\, \theta_{jk}(t) \cdot -\jmath \sum_{m=-\infty}^{\infty} a_m^*\, e^{-\jmath m\omega_{\text{LO}}(t-\tau)}\, \theta_{jm}^*(t-\tau)} \\
&= \overline{\sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_k\, a_m^* e^{\jmath(k-m)\omega_{\text{LO}} t}\, e^{\jmath m\omega_{\text{LO}}\tau} \cdot \theta_{jk}(t)\, \theta_{jm}^*(t-\tau)} \\
&= \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \overline{a_k\, a_m^*\, e^{\jmath(k-m)\omega_{\text{LO}} t} e^{\jmath m\omega_{\text{LO}}\tau}} \cdot \overline{\theta_{jk}(t)\theta_{jm}^*(t-\tau)}
\end{aligned} \tag{5}$$

where $\bar{\cdot}$ is the expectation operator, and we made use of two of its properties: 1. that the expectation of the sum is the sum of expectations, and 2. that since the LO and phase noise signals are independent, the expectation of their product is the product of their expectations. Since the first term in (5) represents the product of sinusoids, its expected (i.e. average) value is non-zero only for $k = m$. That is

$$R_{nn}(\tau) = \sum_{k=-\infty}^{\infty} |a_k|^2\, R_{\theta\theta}(\tau)\, e^{\jmath k\omega_{\text{LO}}\tau} \tag{6}$$

$$R_{\theta\theta}(\tau) = \overline{\theta_{jk}(t)\, \theta_{jk}^*(t-\tau)} \tag{7}$$

where $R_{\theta\theta}(\tau)$ is the autocorrelation of the phase noise signal $\theta_{jk}(t)$.

The phase noise PSD, $S_{nn}(\omega)$, is then the Fourier transform of (6)

$$\begin{aligned}
S_{nn}(\omega) &= \mathbb{F}[R_{nn}(\tau)] \\
&= \sum_{k=-\infty}^{\infty} |a_k|^2\, \mathbb{F}\left[R_{\theta\theta}(\tau) e^{\jmath k\omega_{\text{LO}}\tau}\right] \\
&= \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} |a_k|^2\, \mathbb{F}[R_{\theta\theta}(\tau)]_{\omega - k\omega_{\text{LO}}}
\end{aligned} \tag{8}$$

where we made use of the modulation property of the Fourier transform and neglected the DC component of the LO signal[2]. Switching to a positive frequencies representation of the PSD and defining $S_{\theta\theta}(\omega)$ as the PSD of $\theta_{jk}(t)$, (8) can be written as

$$S_{nn}(\omega) = 2\sum_{k=1}^{\infty} |a_k|^2\, S_{\theta\theta}(\omega - k\omega_{\text{LO}}) \tag{9}$$

That is, the total noise PSD of an LO signal is equal to the phase noise PSD, replicated around each LO harmonic, and scaled by the Fourier coefficient of that harmonic.

[2]When it comes to timing errors, the DC component does not convey any useful information since it does not contribute to clock edges.

To further simplify (9), we express ω as a function of frequency offset $\Delta\omega$ from a desired harmonic $m\omega_{\mathrm{LO}}$

$$S_{nn}(m\omega_{\mathrm{LO}} + \Delta\omega) = 2\sum_{k=1}^{\infty} |a_k|^2 \, S_{\theta\theta}\big((m-k)\omega_{\mathrm{LO}} + \Delta\omega\big) \tag{10}$$

Typically $S_{\theta\theta}(\omega)$ exhibits a colored narrow-band spectrum. Under this assumption, the noise at a small frequency offset ($\Delta\omega \ll \omega_{\mathrm{LO}}$) away from a desired harmonic $m\omega_{\mathrm{LO}}$ is dominated by phase noise contributions of that harmonic. That is, to a first order, all $m \neq k$ terms can be neglected

$$S_{nn}(m\omega_{\mathrm{LO}} + \Delta\omega) \approx 2\,|a_m|^2 \, S_{\theta\theta}(\Delta\omega) \tag{11}$$

We can now define $L_\theta(\Delta\omega)$, a useful quantity

$$L_\theta(\Delta\omega) = S_{\theta\theta}(\Delta\omega) = \frac{S_{nn}(m\omega_{\mathrm{LO}} + \Delta\omega)}{2\,|a_m|^2} \tag{12}$$

$L_\theta(\Delta\omega)$ is the single-side band (SSB) phase noise PSD normalized to the m-th harmonic power $2\,|a_m|^2$.

3 Signal-to-Noise Ratio (SNR)

From Parseval's theorem, the mean square noise voltage $\overline{v_{nj}^2(t)}$ is given by integrating (11) across all frequencies

$$\overline{v_{nj}^2(t)} = 2\int_{-m\omega_{\mathrm{LO}}}^{\infty} S_{nn}(m\omega_{\mathrm{LO}} + \Delta\omega)d\Delta\omega \tag{13}$$

where the integration limits in (13) account for the positive frequencies representation adopted earlier. For frequency offsets of interest, $\Delta\omega \ll \omega_{\mathrm{LO}}$ (more on that in the next section), and the noise PSD can be well approximated as symmetric around each LO harmonic. This approximation, together with (12), allows us to express (13) as

$$\overline{v_{nj}^2(t)} = 2\int_{0}^{\infty} S_{nn}(m\omega_{\mathrm{LO}} + \Delta\omega)d\Delta\omega = 2\,|a_m|^2 \cdot 2\int_{0}^{\infty} L_\theta(\Delta\omega)d\Delta\omega \tag{14}$$

From (14), we find the signal-to-noise ratio (SNR)

$$SNR_{\mathrm{LO}} = \frac{2\,|a_m|^2}{\overline{v_{nj}^2(t)}} = \frac{1}{2\int_0^\infty L_\theta(\Delta\omega)d\Delta\omega} \tag{15}$$

4 Relating Time and Frequency Perspectives

By comparing (4) and (15)

$$\theta_{j,rms}^2 = 2 \int_0^{\infty} L_\theta(\Delta\omega) d\Delta\omega \tag{16}$$

The above relation shows that, unlike its time domain counterpart, the frequency domain model gives us the ability to discern the relevant phase noise contributions by adjusting the integration limit to account for the noise falling within the (baseband) bandwidth BW of our system

$$\theta_{j,rms}^2 = 2 \int_{\Delta\omega_0}^{\Delta\omega_0 + BW} L_\theta(\Delta\omega) d\Delta\omega \tag{17}$$

where $\Delta\omega_0$ is the offset between the desired LO harmonic and the center of our signal bandwidth. For example, in the context of frequency translation, $\Delta\omega_0$ would be zero in a direct conversion receiver, but non-zero in a low IF receiver.