Phase Noise: Time Domain Model

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1 Ideal Timing Signal

An ideal (noiseless) local oscillator (LO) signal, $v_{\mathrm{LO}}(t)$, is periodic, so it can be expressed as a Fourier series

$$v_{\mathrm{LO}}(t) = \sum_{k=-\infty}^{\infty} a_k \, e^{\jmath k \omega_{\mathrm{LO}} t} \tag{1}$$

where ω_{LO} is the fundamental frequency of the LO signal in radians per second, k is the harmonic number, and a_k is the complex Fourier coefficient of the k-th harmonic.

2 Jitter Model

A noisy LO signal exhibits zero crossing variations and can be modeled by introducing a time-dependent jitter term $t_j(t)$ in (1)

$$v_{\mathrm{LO}}(t, t_j) = \sum_{k=-\infty}^{\infty} a_k e^{\jmath k \omega_{\mathrm{LO}} \left(t + t_j(t)\right)} = v_{\mathrm{LO}}(t) \cdot e^{\jmath k \omega_{\mathrm{LO}} t_j(t)} \tag{2}$$

The term $t_j(t)$ is a time series that captures the time difference between the actual and ideal zero crossings of the signal[1].

Alternatively, this jitter effect can be expressed in terms of the corresponding phase variation $\theta_{jk}(t)$

$$v_{\mathrm{LO}}(t, \theta_j) = \sum_{k=-\infty}^{\infty} a_k e^{\jmath k \omega_{\mathrm{LO}} t} \cdot e^{\jmath \theta_{jk}(t)} \tag{3}$$

$$\theta_{jk}(t) = k \omega_{\mathrm{LO}} t_j(t) \tag{4}$$

[1] As such, $t_j(t)$ represents absolute jitter, also known as time interval error (TIE). This is different from, although can be related to, other relative measures of jitter like period jitter or cycle-to-cycle jitter.

The expression in (4) makes sense because, for a given time shift $tj(t)$, the corresponding phase shift $\theta_{jk}(t)$ must be a linear function of frequency for every harmonic.

Usually, by design, $\theta_j(t) \ll 1$, and the effect of jitter can be reduced to an additive noise term $v_{nj}(t)$

$$v_{\mathrm{LO}}(t, \theta_j) \approx \sum_{k=-\infty}^{\infty} a_k \, e^{\jmath k \omega_{\mathrm{LO}} t} \big(1 + \jmath \theta_{jk}(t)\big) \approx v_{\mathrm{LO}}(t) + v_{nj}(t) \tag{5}$$

$$v_{nj}(t) \approx \jmath \sum_{k=-\infty}^{\infty} a_k \, e^{\jmath k \omega_{\mathrm{LO}} t} \cdot \theta_{jk}(t) \tag{6}$$

That is, jitter effectively introduces a quadrature copy of the LO signal, amplitude modulated by phase noise. The presence of this additional term is what degrades the signal-to-noise ratio (SNR) of the LO signal.

3 Signal-to-Noise Ratio (SNR)

We first calculate LO signal power using (1)

$$\begin{aligned} \overline{v_{\mathrm{LO}}^2(t)} &= \overline{v_{\mathrm{LO}}(t)\, v_{\mathrm{LO}}^*(t)} \\ &= \overline{\sum_{k=-\infty}^{\infty} a_k \, e^{\jmath k \omega_{\mathrm{LO}} t} \cdot \sum_{m=-\infty}^{\infty} a_m^* \, e^{-\jmath m \omega_{\mathrm{LO}} t}} \\ &= \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \overline{a_k \, a_m^* \, e^{\jmath (k-m) \omega_{\mathrm{LO}} t}} \end{aligned} \tag{7}$$

where $\bar{\cdot}$ denotes the average, and we have made use of the fact that the average of sums is the sum of averages. Furthermore, the average term in (7) is nonzero only for $k = m$. That is[2]

$$\overline{v_{\mathrm{LO}}^2(t)} = \sum_{k=-\infty}^{\infty} |a_k|^2 \tag{8}$$

which is Parseval's theorem for periodic signals[3].

[2]Since the LO signal is comprised of harmonically related sinusoids, and the product of two sinusoids of different frequencies has an average value of zero.

[3]Parseval's theorem links the time and frequency representations of a signal through its average power. For a periodic signal (also referred to as power signal), the theorem states that the mean square value (i.e. average power) of the signal is equal to the sum of squares of the signal's Fourier series coefficients.

In a similar manner, we evaluate the average noise power using (6)

$$\begin{aligned}
\overline{v_{nj}^2(t)} &= \overline{v_{nj}(t)v_{nj}^*(t)} \\
&= \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \overline{a_k\, a_m^*\, e^{\jmath(k-m)\omega_{\mathrm{LO}}t} \cdot \theta_{jk}(t)\theta_{jm}^*(t)} \qquad (9) \\
&= \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \overline{a_k\, a_m^*\, e^{\jmath(k-m)\omega_{\mathrm{LO}}t}} \cdot \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \overline{\theta_{jk}(t)\, \theta_{jm}^*(t)} \qquad (10)
\end{aligned}$$

where we LO and phase noise signals are statistically independent, so the average of their product is the product of their averages. We can then recognize the product of two terms in (9): the first term is the LO signal power given by (7), and the second term is the phase noise power $\overline{\theta_j^2(t)}$

$$\overline{v_{nj}^2(t)} = \overline{v_{\mathrm{LO}}^2(t)} \cdot \overline{\theta_j^2(t)} = \overline{v_{\mathrm{LO}}^2(t)} \cdot \theta_{j,rms}^2 \qquad (11)$$

where $\theta_{j,rms}$ is defined as the root-mean square phase noise.

The SNR is then readily available from (11)

$$SNR_{\mathrm{LO}} = \frac{1}{\theta_{j,rms}^2} \qquad (12)$$

The result in (12) tells us that the SNR is independent of the LO signal power, and is only a function of the rms phase noise $\theta_{j,rms}$. That is, the only way to improve the SNR of a timing signal is to lower its jitter, thus improving its timing accuracy.