

Design and Analysis of Algorithms

Graphs 3: Depth-First Search

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3.4.2

Claim: A directed graph has a cycle *if and only if* its DFS reveals a back edge.

Proof:

By Induction. Consider a directed graph with two nodes, A and B ($A \rightarrow B \rightarrow A \dots$), in a cycle. **Base Case:** *If we execute a DFS from node $N = 0$ (node A), we will explore node B . Since the only other option is A , which has been marked as explored, we will create a back edge to include the cycle.* If we start exploring at $N = 1$ (node B), A will be marked as explored, and by the same logic we will create a back edge to include the cycle from $B \rightarrow A$. Here we can see the cyclic nature of back edges from $A \rightarrow B$ and $B \rightarrow A$ regardless of the starting node.

By Induction. Consider the same graph $A \rightarrow B \rightarrow A \dots$. **Base Case:** A graph was explored from $N = 0$ (node A) to generate a DFS tree that has the structure $A \rightarrow B \dots A$ with a back edge from $B \rightarrow A$. This is cyclic in nature. Now consider a DFS tree generated from $N = 1$ (node B) with the structure $B \rightarrow A$ and a back edge to from $B \rightarrow A$. We can see that a DFS tree generated from a cyclic graph is reproducible regardless of the starting node.