Due: 8th of June 2011 at 1pm.

COMP 5045 – Assignment 3

For full points you need to prove the correctness and complexity of your algorithms.

1. In this question, we consider finite point sets in the Euclidean plane.
   (a) Prove that the travelling salesperson tour TSP of a set S of points is monotone in the following sense: If $S \subseteq S'$, then the length of $TSP(S)$ is less than or equal to the length of $TSP(S')$. [2 points]
   (b) Prove that the minimum spanning tree MST of a set S of points is not monotone: If $S \subseteq S'$, then the length of $MST(S)$ is not necessarily less than or equal to the length of $MST(S')$. [2 points]
   (c) Prove that the length of an optimal TSP tour of a set S of points in the Euclidean plane is greater than the total length of $MST(S)$ and smaller than twice the total length of $MST(S)$. [6 points]
2. Consider a set S of n points in the plane and let $V(S)$ be the Voronoi diagram of S. Prove that a Voronoi cell $V(p)$ of a point $p \in S$ is unbounded if and only if p lies on the convex hull of S. [10 points]
3. (a) Prove that the maximal degree of a vertex in a minimum spanning tree is at most six. [6 points]
   (b) Show an example point set with n points for which *every* Delaunay triangulation has at least one vertex of degree $n - 1$. Argue about the correctness of your example. [4 points]
4. Let I be a set of intervals on the real line. We want to store these intervals such that we can efficiently determine those intervals that are completely contained in a given query interval $[x, x']$. Describe a data structure that uses $O(n \log n)$ storage and solves such queries in $O(\log n + k)$ time, where k is the number of answers. [10 points]
5. The goal of this problem is to describe an alternative way of constructing a t-spanner. Let P be a set of n points in the plane and let $t > 1$ be the desired stretch factor. Let $k = \lceil 2\pi/\tau \rceil$, where τ is chosen such that $\frac{1}{(\cos\tau - \sin\tau)} \leq t$. (For efficiency, we would like τ to be as large as possible.) Surround each point p_i of P with a collection of k sectors, where the angle of each sector is at most τ, and for each of these k sectors, create an edge from p_i to its closest point within this sector, see Fig. 1a. Let $G(t)$ be the resulting graph.
   (a) To analyse the spanner properties of this graph, we first prove a lemma. Let u be a point, and let S be a sector of angle $\tau < 45°$ (as defined above) whose apex is at u. Let v and w be points within S, such that w is closer to u than v (That is, $|uw| < |uv|$). Prove that $|uw| + t \cdot |wv| \leq t \cdot |uv|$. (Hint: This reduces to basic trigonometry. It may help to consider the point w', which is the orthogonal projection of w onto the segment uv. See figure 1b.)

(b) Prove by induction on the length of paths that $G(t)$ is a t-spanner for P. As part of your solution, explain how to construct a spanner path between any two points of P in $G(t)$. (Hint: Use part (a) to prove the spanner bound.)

(c) Show that for every constant $t > 1$, the number of edges of $G(t)$ is $O(n)$.

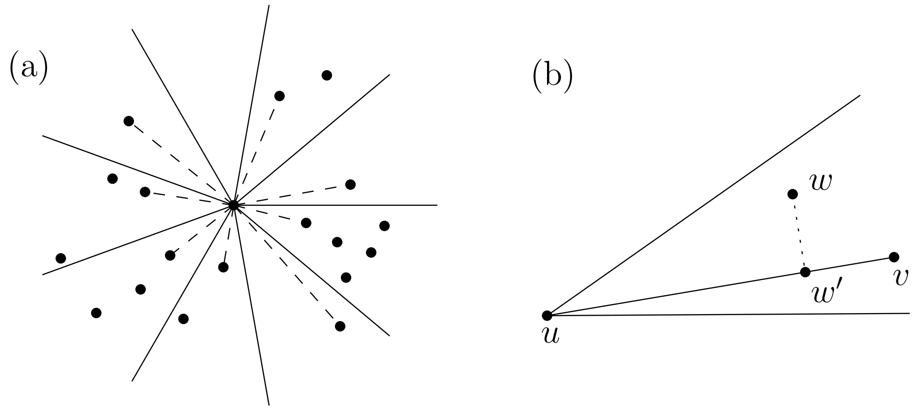

Figure 1: Illustrations of question 5.