Real-world algorithms
Semester 1, 2011

Homework 3

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This assignment is **due on Apr 7** in class. You are allowed (even encouraged) to discuss these problems with your fellow classmates. All submitted work, must be *written individually* without consulting someone else's solutions.

Problem 1: Let G be an undirected graph. A matching is a subset of the edges of G such that no two edges share a common endpoint. The maximum matching problem is to find a maximum cardinality matching in a given input graph.

1. Give an IP formulation for the maximum matching problem.
2. Show that the relaxation of the above program is **integral if G is bipartite**.

   Hint: Show that the edge-vertex incidence matrix is totally unimodular.
3. Derive the dual of the above LP and come up with a combinatorial interpretation of the resulting optimization problem.
4. State the consequences of the Strong Duality Theorem in terms of the combinatorial interpretation of the primal and dual LPs.
5. Does the last statement hold for general graphs?

Problem 2: Prove the Hoffman-Kruskal Theorem:

Theorem 1 *Let $\mathbf{A}$ be a matrix with integer coefficients, then $\mathbf{A}$ is totally unimodular if and only if for every integral vectors $\mathbf{b}$ and $\mathbf{c}$ the linear program*

$$\max \mathbf{c} \cdot \mathbf{x} : \mathbf{A}\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}$$

has an optimal integral solution.

Break your proof into two steps:

1. Show that if $\mathbf{A}$ is integral but not totally unimodular then there exists an integral $\mathbf{b}$ such that $\mathbf{A}\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}$ has a basic feasible solution with at least one fractional coordinate.

   Hint: Build the desired basic solution using the smallest square submatrix $\tilde{\mathbf{A}}$ of $\mathbf{A}$ such that $\det \tilde{\mathbf{A}} \notin \{0, 1, -1\}$
2. Let $\mathbf{x}^*$ be a basic feasible solution of some set of inequalities $\mathbf{A}\mathbf{x}^* \leq \mathbf{b}$ where $\mathbf{A}$ and $\mathbf{b}$ are integral. Show that there is an integral vector $\mathbf{c}$ such that $\mathbf{x}^*$ is the unique optimal solution of $\max \mathbf{c} \cdot \mathbf{x} : \mathbf{A}\mathbf{x} \leq \mathbf{b}$. (Notice the lack of non-negativity constraints!)

   Hint: Let $\tilde{\mathbf{A}}$ be the basic matrix associated with the basis defining $\mathbf{x}^*$. Set $\mathbf{c}$ to be the sum of the rows of $\tilde{\mathbf{A}}$. Use strong duality to argue the optimality of $\mathbf{x}^*$.