Algorithmic game theory, 2011 sem 2

Problem Set 1
Due September 19, 2011 (Submit a pdf on blackboard)

Collaboration for solving these problems is very much encouraged.
However

- you should write down answers on your own
- you should mainly discuss the problems you have not solved yet

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1. Two identical firms – let's call them Firm 1 and Firm 2 – must decide simultaneously and independently whether to enter a new market and what product to produce if they do enter the market. Each firm, if it enters, can develop and produce either product A or product B. If both firms enter and produce product A they each lose ten million dollars. If both firms enter and both produce product B, they each make a profit of five million dollars. If both enter and one produces A while the other produces B, then they each make a profit of ten million dollars. Any firm that does not enter makes a profit of zero. Finally, if one firm does not enter and the other firm produces A it makes a profit of fifteen million dollars, while if the single entering firm produces B it makes a profit of thirty million dollars.

   You are the manager of firm 1 and you have to choose a strategy for your firm.

   (a) Set this situation up as a game with two players, Firms 1 and 2, and three strategies for each firm: produce A, produce B, or do not enter.
   (b) One of your employees argues that you should enter the market (although he is not sure what product you should produce) because no matter what Firm 2 does, entering and producing product B is better than not entering. Evaluate this argument.
   (c) Another employee agrees with the person in part (b) and argues that as strategy A could result in a loss (if the other firm also produces A) you should enter and produce B. If both firms reason this way, and thus enter and produce product B, will their play of the game form a Nash equilibrium? Explain.
   (d) Find all the pure-strategy Nash equilibria of this game.
   (e) Another employee of your firm suggests merging the two firms and deciding cooperatively on strategies so as to maximize the sum of profits. Ignoring whether this merger would be allowed by the regulators, do you think it's a good idea? Explain.

   [problem 15 page 187 from Networks Markets and Crowds]

2. Consider a two-player game given in matrix form where each player has n strategies. Assume that the payoffs for each player are in the range [0, 1] and are selected independently and uniformly at random. Show that the probability that this random game has a pure (deterministic) Nash equilibrium approaches $1 - 1/e$ as n goes to infinity. You may use the fact that $\lim(1 - 1/n)^n = 1/e$ as n goes to infinity.

[Problem 1.2 page 26 of "Algorithmic game theory" textbook]

3. Consider an n person game in which each player has only two strategies. This game has 2^n possible outcomes (for the 2^n ways the n players can play), therefore the game in matrix form is exponentially large. To circumvent this, in Chapter 7 we will consider a special class of games called graphical games. The idea is that the value (or payoff) of a player can depend only on a subset of players. We will define a dependence graph G, whose nodes are the players, and an edge between two players i and j represents the fact that the payoff of player i depends on the strategy of player j or vice versa. Thus, if node i has k neighbors, then its payoff depends only on its own strategy and the strategies of its k neighbors.

Consider a game where the players have 2 pure strategies each and assume that the graph G is a tree with maximum degree 3. Give a polynomial time algorithm to decide if such a game has a pure Nash equilibrium. (Recall that there are 2^n possible pure strategy vectors, yet your algorithm must run in time polynomial in n.)

[Problem 1.4 page 26 of "Algorithmic game theory" textbook]

4. Recall Example 17.3 p449 of the textbook:

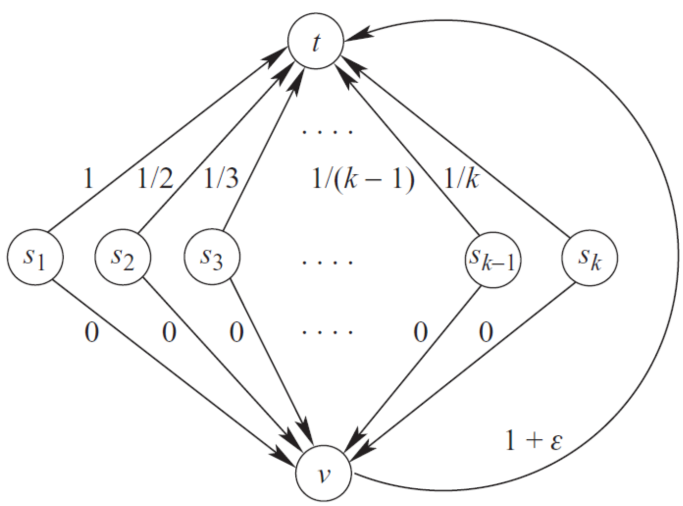

Figure 17.3. The price of stability in Shapley network design games can be at least $\mathcal{H}_k$ (Example 17.3).

Suppose we modify the $\mathcal{H}_k$ example (Example 17.3) so that all of the network edges are undirected. In other words, each player i can choose a path from s_i to t that traverses each edge in either direction. What is the price of stability in the resulting Shapley network design game?

[Problem 17.2 page 459 of "Algorithmic game theory" textbook]

5. This problem is related to the atomic selfish routing model as discussed in chapter 18 of the textbook. Note that in the lectures we discussed only the non-atomic case.

Atomic selfish routing games are defined as follows (The following paragraph is from the textbook, page 465):

"…

An *atomic selfish routing game* or *atomic instance* is defined by the same ingredients as a nonatomic one: a directed graph $G = (V, E)$, k source–sink pairs $(s_1, t_1), \ldots, (s_k, t_k)$, a positive amount r_i of traffic for each pair (s_i, t_i), and a nonnegative, continuous, nondecreasing cost function $c_e : \mathcal{R}^+ \to \mathcal{R}^+$ for each edge e. We also denote an atomic instance by a triple (G, r, c). The intuitive difference between a nonatomic and an atomic instance is that in the former, each commodity represents a large population of individuals, each of whom controls a negligible amount of traffic; in the latter, each commodity represents a single player who must route a significant amount of traffic on a single path.

…"

In the *unweighted* case, all traffic rates r_i are the same: All players need to route the same amount of flow (therefore all players have the same "weight" in the game). In the weighted case however, players have different amounts of flow that needs to be routed.

This exercise explores lower bounds on the price of anarchy in atomic selfish routing games with affine cost functions.

(a) Modify the players' weights in the AAE example (Example 18.6) so that the price of anarchy in the resulting weighted atomic instance is precisely $(3 + \sqrt{5})/2 \approx 2.618$.

(b) Can you devise an unweighted atomic instance with 3 players, affine cost functions, and price of anarchy equal to 5/2? Can you achieve a price of anarchy of $(3 + \sqrt{5})/2$ using 3 players and variable weights?

(c) What is the largest price of anarchy in an atomic instance with affine cost functions and only 2 players?

[Problem 18.2 page 484 of "Algorithmic game theory" textbook]

Example 18.6 is shown below (atomic, unweighted case).

Example 18.6 (AAE example) Consider the bidirected triangle network shown in Figure 18.3. We assume that there are four players, each of whom needs to route one unit of traffic. The first two have source u and sinks v and w, respectively; the third has source v and sink w; and the fourth has source w and sink v. Each player has two strategies, a one-hop path and a two-hop path. In the optimal flow, all players route on their one-hop paths, and the cost of this flow is 4. This flow is also an equilibrium flow. On the other hand, if all players route on their two-hop paths, then we obtain a second equilibrium flow. Since the first two players each incur three units of cost and the last two players each incur two units of cost, this equilibrium flow has a cost of 10. The price of anarchy of this instance is therefore $10/4 = 2.5$.

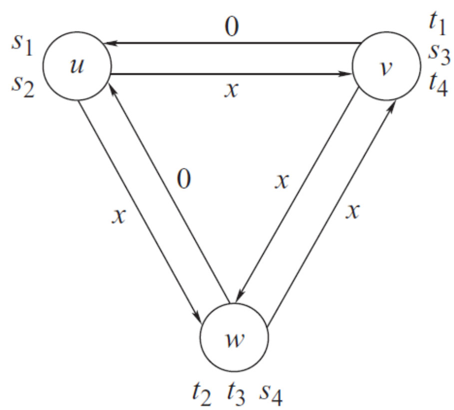

Figure 18.3. The AAE example (Example 18.6). In atomic instances with affine cost functions, different equilibrium flows can have different costs, and the price of anarchy can be as large as 5/2.