The Chain Rule

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Consider the composite function $y = f(g(x))$. The derivative of a composite function is the derivative of the outer function evaluated at the inner function, times the derivative of the inner function. This is the Chain Rule.

Chain Rule
Let $y = f(u)$ and $u = g(x)$. If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $f \circ g = f(g(x))$ is differentiable at x.

$$\frac{\mathrm{d}f(g(x))}{\mathrm{d}x} = (f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

Problem 1

Find the derivative of $y = (1+x)^{15}$,
where $f(u) = u^{15}$ and $u = g(x) = (1+x)$

$$(f \circ g)'(x) = f'(u) \cdot g'(x) \qquad \Rightarrow 15 \cdot u^{14} \cdot (0+1)$$

$$\Rightarrow 15 \cdot u^{14} \cdot 1$$

$$\boxed{\Rightarrow (f \circ g)'(x) = 15(1+x)^{14}}$$

Problem 2

Find the derivative of

$$y = \left(\frac{3x-2}{x+5}\right)^3, where f(u) = u^3 \ \ and \ \ u = g(x) = \left(\frac{3x-2}{x+5}\right)$$

$$\frac{\mathrm{d}f(g(x))}{\mathrm{d}x} = f'(u) \cdot g'(x) \ \ = 3 \cdot u^2 \cdot \left(\frac{(x+5)(3-0) - (3x-2)(1+0)}{(x+5)^2}\right)$$

$$\Rightarrow 3 \cdot \left(\frac{3x-2}{x+5}\right)^2 \cdot \left(\frac{(3x+15) - (3x-2)}{(x+5)^2}\right)$$

$$\boxed{\Rightarrow \frac{\mathrm{d}f(g(x))}{\mathrm{d}x} = 3 \cdot \left(\frac{3x-2}{x+5}\right)^2 \cdot \left(\frac{17}{(x+5)^2}\right)}$$