1 Fourier

Definition 1.1 (Fourier transform) The *Fourier transform* of a function $f(\mathbf{x})$ is defined by:

$$(\mathcal{F}f)(\boldsymbol{\nu}) = F(\boldsymbol{\nu}) = \int_{\mathbb{R}^n} f(\mathbf{x})e^{-i2\pi\boldsymbol{\nu}\cdot\mathbf{x}} \,\mathrm{d}\,\mathbf{x}. \tag{1}$$

Theorem 1.1 (Inverse Fourier transform) *The* inverse Fourier transform *is a function $\mathcal{F}^{-1}$ that satisfies $\mathcal{F}^{-1}(\mathcal{F}f) = f$ for any function f. The function $\mathcal{F}^{-1}$ of a function $F(\boldsymbol{\nu})$ is given by:*

$$(\mathcal{F}^{-1}F)(\mathbf{x}) = f(\mathbf{x}) = \int_{\mathbb{R}^n} F(\boldsymbol{\nu})e^{i2\pi\mathbf{x}\cdot\boldsymbol{\nu}} \,\mathrm{d}\,\boldsymbol{\nu}. \tag{2}$$

Theorem 1.2 (Fourier slice theorem) *Suppose $T_{\boldsymbol{\omega}}(r)$ is a tomogram of a function f. Then:*

$$(\mathcal{F}_1 T_{\boldsymbol{\omega}})(r) = (\mathcal{F}_n f)(r\boldsymbol{\omega}). \tag{3}$$

PROOF:

$$\begin{aligned}
(\mathcal{F}_1 T_{\boldsymbol{\omega}})(r) &= \int_{-\infty}^{\infty}\left(\int_{\mathbb{R}^n} f(\mathbf{z})\,\delta(s-\mathbf{z}\cdot\boldsymbol{\omega})\,\mathrm{d}\,\mathbf{z}\right)e^{-2\pi isr}\,\mathrm{d}\,s \\
&= \int_{\mathbb{R}^n} f(\mathbf{z})\left(\int_{-\infty}^{\infty}\delta(s-\mathbf{z}\cdot\boldsymbol{\omega})\,e^{-2\pi isr}\,\mathrm{d}\,s\right)\mathrm{d}\,\mathbf{z} \\
&= \int_{\mathbb{R}^n} f(\mathbf{z})\left(e^{-2\pi i\mathbf{z}\cdot\boldsymbol{\omega}r}(\mathcal{F}\delta)(r)\right)\mathrm{d}\,\mathbf{z} \\
&= \int_{\mathbb{R}^n} f(\mathbf{z})\,e^{-2\pi i\mathbf{z}\cdot(r\boldsymbol{\omega})}\,\mathrm{d}\,\mathbf{z} \\
&= (\mathcal{F}_n f)(r\boldsymbol{\omega})
\end{aligned}$$

This simple theorem can then be used to recover the data function f from its projections p.

2 DFT process

For the following, suppose that their are n sample points in each projection.

1. For each projection, take the DFT.
2. The DFT is not the same as the Fourier transform. So we need to do a shift. The first $(n/2) + 1$ (the division is rounded down to the nearest integer) elements need to be moved/wrapped to the end of the array.[1, pg 361]
3. Each of these transformed projections need to be put into a 2D plane, according to the inital angle of projection.

4. This 2D plane needs to be interpolated to create a set of values at Cartesian coordinates, as opposed to circular.

5. Cycle over this data set and find which x and y coordinate, (i, j), holds the largest value for the real component. This point corresponds to the DC component of the Fourier signal.

6. Do the reverse 2D DFT.

7. Again, the DFT is not the same at the Fourier transform. So we need to shift the points of the grid, firstly in the $-x$ direction so that the i coordinate is shifted to $x = 0$, and then secondly in the $-y$ direction so that the j coordinate is shifted to $y = 0$. Remember that this shift is to be wrapped.

8. Now we have the solution.

References

[1] Ronald N. Bracewell. *The Fourier Transform and Its Applications*. McGraw-Hill series in electrical engineering: Circuits and Systems. McGraw-Hill, New York, second edition, 1986.

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