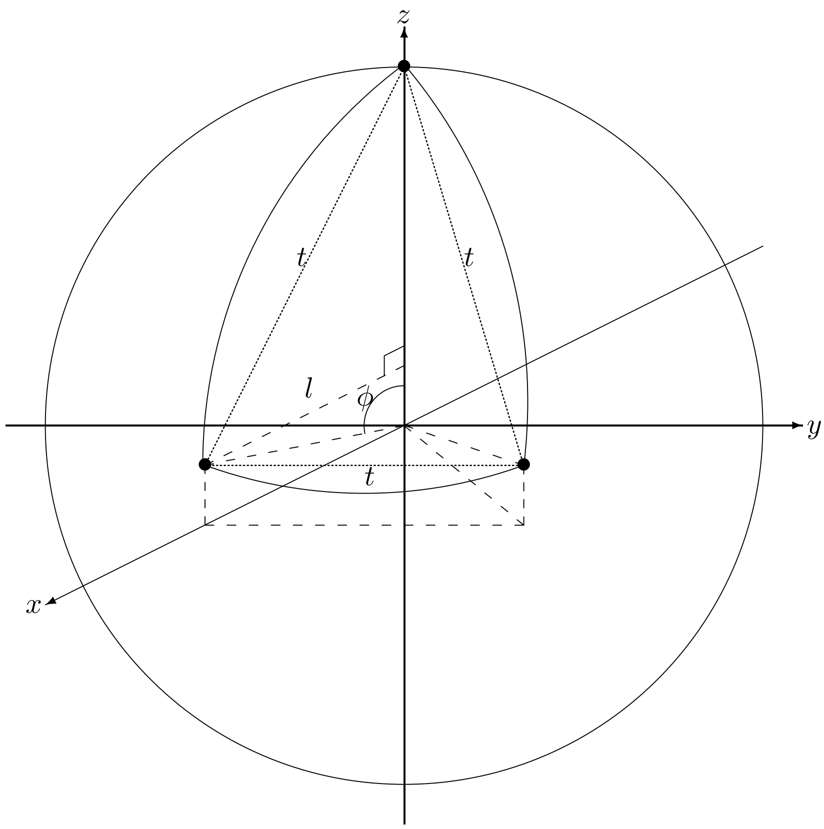
the Icosahedron

Theorem 1 *If an icosahedron is placed inside a sphere of unit radius then the triangles subtended from both poles have an angular width, at their base, of* $\frac{2\pi}{5}$ *and the bases are an angular distance of* ϕ *from the pole, where* ϕ *is given by* $\cos\phi = \frac{\cos\frac{2\pi}{5}}{1-\cos\frac{2\pi}{5}}$.

PROOF:



Firstly, there are five triangles around each pole, so the angular width of each triangle must be $\frac{2\pi}{5}$. Let t be the length of the straight line connecting two nodes of a triangle. Then, by the cosine rule,

$$t^2 = l^2 + l^2 - 2l^2\cos\frac{2\pi}{5} = 2l^2\left(1-\cos\frac{2\pi}{5}\right).$$

Since the sphere has unit radius we have $l = \sin\phi$ and $h = 1 - \cos\phi$. Pythagoras' theorem gives that

$$
\begin{aligned}
& t^2 = l^2 + h^2 \\
\Rightarrow\ & 2l^2 \left(1 - \cos\frac{2\pi}{5}\right) = l^2 + h^2 \\
\Rightarrow\ & 2\sin^2\phi \left(1 - \cos\frac{2\pi}{5}\right) = \sin^2\phi + (1 - \cos\phi)^2 \\
\Rightarrow\ & 2\sin^2\phi \left(1 - \cos\frac{2\pi}{5}\right) = \sin^2\phi + 1 - 2\cos\phi + \cos^2\phi \\
\Rightarrow\ & 2\sin^2\phi \left(1 - \cos\frac{2\pi}{5}\right) = 2\left(1 - \cos\phi\right) \\
\Rightarrow\ & \left(1 - \cos^2\phi\right)\left(1 - \cos\frac{2\pi}{5}\right) = (1 - \cos\phi) \\
\Rightarrow\ & (1 - \cos\phi)(1 + \cos\phi)\left(1 - \cos\frac{2\pi}{5}\right) = (1 - \cos\phi) \\
\Rightarrow\ & 1 + \cos\phi = \frac{1}{1 - \cos\frac{2\pi}{5}} \\
\Rightarrow\ & \cos\phi = \frac{\cos\frac{2\pi}{5}}{1 - \cos\frac{2\pi}{5}}.
\end{aligned}
$$

spherical co-ordinates for points on the icosahedron

	θ	ϕ
1	0	0
2	0	$\arccos\left(\frac{\cos\frac{2\pi}{5}}{1-\cos\frac{2\pi}{5}}\right)$
3	$\frac{2\pi}{5}$	$\arccos\left(\frac{\cos\frac{2\pi}{5}}{1-\cos\frac{2\pi}{5}}\right)$
4	$\frac{4\pi}{5}$	$\arccos\left(\frac{\cos\frac{2\pi}{5}}{1-\cos\frac{2\pi}{5}}\right)$
5	$\frac{6\pi}{5}$	$\arccos\left(\frac{\cos\frac{2\pi}{5}}{1-\cos\frac{2\pi}{5}}\right)$
6	$\frac{8\pi}{5}$	$\arccos\left(\frac{\cos\frac{2\pi}{5}}{1-\cos\frac{2\pi}{5}}\right)$
7	$\frac{9\pi}{5}$	$\pi - \arccos\left(\frac{\cos\frac{2\pi}{5}}{1-\cos\frac{2\pi}{5}}\right)$
8	$\frac{\pi}{5}$	$\pi - \arccos\left(\frac{\cos\frac{2\pi}{5}}{1-\cos\frac{2\pi}{5}}\right)$
9	$\frac{3\pi}{5}$	$\pi - \arccos\left(\frac{\cos\frac{2\pi}{5}}{1-\cos\frac{2\pi}{5}}\right)$
10	π	$\pi - \arccos\left(\frac{\cos\frac{2\pi}{5}}{1-\cos\frac{2\pi}{5}}\right)$
11	$\frac{7\pi}{5}$	$\pi - \arccos\left(\frac{\cos\frac{2\pi}{5}}{1-\cos\frac{2\pi}{5}}\right)$
12	0	π