B-splines

Definition 1 (spline space) Let points $a = x_0 < x_1 < \cdots < x_k < x_{k+1} = b$ and an integer $m \geq 1$ be given. We call

$$S_m(x_1, \ldots, x_k) = \left\{ f \in \mathcal{C}^{m-1}[a,b] : f|_{[x_i, x_{i+1}]} \in \mathcal{P}_m(\mathbb{R}), i = 0, \ldots, k \right\}$$

the space of *polynomial splines of degree m with k fixed knots $x_1, \ldots, x_k$*. For a given spline space $S_m(x_1, \ldots, x_k)$, we always associate further points $x_{-m} < \cdots < x_{-1} < a$ and $b < x_{k+2} < \cdots < x_{k+m+1}$, where these points may be chosen arbitrarily.

Theorem 1 (spline space dimension) *The dimension of $S_m(x_1, \ldots, x_k)$ is $k + m + 1$.*

Definition 2 (polynomial splines) Let points $x_{-m} < \cdots < x_{-1} < a = x_0 < x_1 < \cdots < x_k < x_{k+1} = b < x_{k+2} < \cdots < x_{k+m+1}$ be given. A function $f : (-\infty, \infty) \to \mathbb{R}$ is called a *polynomial spline of degree m with knots $x_{-m}, \ldots, x_{k+m+1}$* if f has $m-1$ continuous derivatives at x_i, $i = -m, \ldots, k+m+1$, and $f|_{(x_i, x_{i+1})} \in \mathcal{P}_m(\mathbb{R})$, $i = -m-1, \ldots, k+m+1$, where $x_{-m-1} = -\infty$ and $x_{k+m+2} = \infty$.

Theorem 2 *For each $i \in \{-m, \ldots, k\}$ there exists a unique spline B_i^m of degree m with knots $x_{-m}, \ldots, x_{k+m+1}$ such that*

$$\begin{aligned} B_i^m(t) &= 0, \quad t \in (-\infty, x_i] \cup [x_{i+m+1}, \infty), \\ B_i^m(t) &> 0, \quad t \in (x_i, x_{i+m+1}), \end{aligned}$$

and

$$\int_{x_i}^{x_{i+m+1}} B_i^m(t) \, \mathrm{d}\, t = 1.$$

Definition 3 (B-spline) The spline B_i^m in Theorem 2 is called the *B-spline* of degree m with support $[x_i, x_{i+m+1}]$.

Theorem 3 (B-spline basis) *The set of B-splines $\left\{B_{-m}^m, \ldots, B_k^m\right\}$ forms a basis of $S_m(x_1, \ldots, x_k)$ on $[a, b]$.*

Definition 4 (normalised B-spline) The spline N_i^m, defined by

$$N_i^m(x) = \frac{(x_{i+m+1} - x_i)}{(m+1)} B_i^m(x)$$

for all $x \in (-\infty, \infty)$, is called the *normalised B-spline of degree m with support (x_i, x_{i+m+1})*.

To uniquely determine a cubic B-spline through a set of $k+2$ knots there is a constraint that the second derivative at the 'boundary' knots ($i = 0, k+1$) be equal to zero. The fact that the cubic B-splines must be $\mathcal{C}^2$ demands the support of at least four consecutive intervals for internal knots, and two or three intervals at the boundaries. The following are an explicit set of cubic B-splines (from $S_3(x_1, \ldots, x_k)$):

$$\rho_0(x) = \begin{cases} \frac{1}{6}(x-x_0)^3 - (x-x_0) + 1 & x_0 \leq x < x_1 \\ -\frac{1}{6}(x-x_1)^3 + \frac{1}{2}(x-x_1)^2 - \frac{1}{2}(x-x_1) + \frac{1}{6} & x_1 \leq x < x_2 \\ 0 & x_2 \leq x \leq x_{k+1} \end{cases}$$

$$\rho_1(x) = \begin{cases} -\frac{1}{3}(x-x_0)^3 + (x-x_0) & x_0 \leq x < x_1 \\ \frac{1}{2}(x-x_1)^3 + (x-x_1)^2 + \frac{2}{3} & x_1 \leq x < x_2 \\ -\frac{1}{6}(x-x_2)^3 + \frac{1}{2}(x-x_2)^2 - \frac{1}{2}(x-x_2) + \frac{1}{6} & x_2 \leq x < x_3 \\ 0 & x_3 \leq x \leq x_{k+1} \end{cases}$$

$$\vdots$$

$$\rho_i(x) = \begin{cases} 0 & x_0 \leq x < x_{i-2} \\ \frac{1}{6}(x-x_{i-2})^3 & x_{i-2} \leq x < x_{i-1} \\ -\frac{1}{2}(x-x_{i-1})^3 + \frac{1}{2}(x-x_{i-1})^2 + \frac{1}{2}(x-x_{i-1}) + \frac{1}{6} & x_{i-1} \leq x < x_i \\ \frac{1}{2}(x-x_i)^3 - (x-x_i)^2 + \frac{2}{3} & x_i \leq x < x_{i+1} \\ -\frac{1}{6}(x-x_{i+1})^3 + \frac{1}{2}(x-x_{i+1})^2 - \frac{1}{2}(x-x_{i+1}) + \frac{1}{6} & x_{i+1} \leq x < x_{i+2} \\ 0 & x_{i+2} \leq x \leq x_{k+1} \end{cases}$$

$$\vdots$$

$$\rho_k(x) = \begin{cases} 0 & x_0 \leq x < x_{k-2} \\ \frac{1}{6}(x-x_{k-2})^3 & x_{k-2} \leq x < x_{k-1} \\ -\frac{1}{2}(x-x_{k-1})^3 + \frac{1}{2}(x-x_{k-1})^2 + \frac{1}{2}(x-x_{k-1}) + \frac{1}{6} & x_{k-1} \leq x < x_k \\ \frac{1}{3}(x-x_k)^3 - (x-x_k)^2 + \frac{2}{3} & x_k \leq x \leq x_{k+1} \end{cases}$$

$$\rho_{k+1}(x) = \begin{cases} 0 & x_0 \leq x < x_{k-1} \\ \frac{1}{6}(x-x_{k-1})^3 & x_{k-1} \leq x < x_k \\ -\frac{1}{6}(x-x_k)^3 + \frac{1}{2}(x-x_k)^2 + \frac{1}{2}(x-x_k) + \frac{1}{6} & x_k \leq x \leq x_{k+1} \end{cases}$$