

# 1 Perpendicular drop point onto a line

line:

$$\mathbf{x} = \mathbf{m}t + \mathbf{c} \quad (1)$$

Given a point  $\mathbf{p}$  we want to know the value  $t_{\mathbf{p}}$  such that  $\mathbf{p} - \mathbf{x}_{\mathbf{p}}$  is perpendicular to the line 1, ie

$$(\mathbf{p} - \mathbf{x}_{\mathbf{p}}) \cdot \mathbf{m} = (\mathbf{p} - \mathbf{m}t_{\mathbf{p}} - \mathbf{c}) \cdot \mathbf{m} = 0$$

Solving this equation for  $t_{\mathbf{p}}$  yields:

$$t_{\mathbf{p}} = \frac{(\mathbf{p} - \mathbf{c}) \cdot \mathbf{m}}{\|\mathbf{m}\|^2}$$

## 2 Intersection of two lines in $\mathbb{R}^n$ ( $n \geq 2$ )

line 2:

$$\mathbf{x}_1 = \mathbf{m}_1 t_1 + \mathbf{c}_1 \quad (2)$$

line 3:

$$\mathbf{x}_2 = \mathbf{m}_2 t_2 + \mathbf{c}_2 \quad (3)$$

### 2.1 general ideas

Project lines onto a set of axes which are spanned by the projections of  $\mathbf{m}_2$  and  $\mathbf{m}_1$ . This will guarantee that if lines 2 and 3 intersect at all, then they will surely intersect in this axis plane. Then solve for  $t_1$  and  $t_2$ .

### 2.2 explicit steps

1. If  $\mathbf{m}_1 = \mathbf{m}_2$  then the lines don't intersect, finish.
2. Choose a pair of axes in which the projections of  $\mathbf{m}_1$  and  $\mathbf{m}_2$  *span* the axis plane. Call these axes  $i$  and  $j$ , ie  $(m_{1i}, m_{1j})$  and  $(m_{2i}, m_{2j})$  are neither parallel or equal to  $(0, 0)$ .
3. There *will* be a solutions for  $t_1$  and  $t_2$  in the following pair of simultaneous equations

$$\begin{cases} m_{1i}t_1 + c_{1i} &= m_{2i}t_2 + c_{2i} \\ m_{1j}t_1 + c_{1j} &= m_{2j}t_2 + c_{2j} \end{cases} \Rightarrow \begin{cases} t_1 &= \frac{m_{2i}(c_{1j}-c_{2j})-m_{2j}(c_{1i}-c_{2i})}{m_{1i}m_{2j}-m_{1j}m_{2i}} \\ t_2 &= \frac{m_{1i}(c_{1j}-c_{2j})-m_{1j}(c_{1i}-c_{2i})}{m_{1i}m_{2j}-m_{1j}m_{2i}} \end{cases}$$

4. Put  $t_1$  and  $t_2$  back into equations 2 and 3 respectively, to solve for the intersection point.
5. Check that  $\mathbf{x}_1 = \mathbf{x}_2$ , if not then finish.
6. The intersection exists and we are finished.

### 3 Perpendicular drop point onto a plane

plane:

$$(\mathbf{x} - \mathbf{c}) \cdot \mathbf{n} = 0 \quad (4)$$

Given any point  $\mathbf{p}$ , we wish to find the point on plane 4,  $\mathbf{x}_p$ , such that  $\mathbf{p} - \mathbf{x}_p$  is perpendicular to plane 4.  $\mathbf{x}_p = \mathbf{p} - k\mathbf{n}$ , for some number  $k$ . We substitute this into plane 4 to get

$$\begin{aligned} ((\mathbf{p} - k\mathbf{n}) - \mathbf{c}) \cdot \mathbf{n} &= 0 \\ \Rightarrow k &= \frac{(\mathbf{p} - \mathbf{c}) \cdot \mathbf{n}}{\|\mathbf{n}\|^2} \end{aligned}$$

### 4 Intersection of a line and a plane in $\mathbb{R}^3$

line:

$$\mathbf{x}_l = \mathbf{m}_l t_l + \mathbf{c}_l \quad (5)$$

plane:

$$(\mathbf{x}_p - \mathbf{c}_p) \cdot \mathbf{n}_p = 0 \quad (6)$$

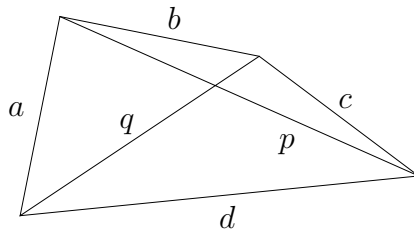
1. if  $\mathbf{m}_l \cdot \mathbf{n}_p = 0$  then the line is parallel to the plane, and there is no intersection
2. substitute  $\mathbf{x}_l$  into the plane 6

$$\begin{aligned} ((\mathbf{m}_l t_l + \mathbf{c}_l) - \mathbf{c}_p) \cdot \mathbf{n}_p &= 0 \\ \Rightarrow t_l (\mathbf{m}_l \cdot \mathbf{n}_p) &= \mathbf{c}_p \cdot \mathbf{n}_p - \mathbf{c}_l \cdot \mathbf{n}_p \\ \Rightarrow t_l &= \frac{\mathbf{c}_p \cdot \mathbf{n}_p - \mathbf{c}_l \cdot \mathbf{n}_p}{\mathbf{m}_l \cdot \mathbf{n}_p} \end{aligned}$$

3. put  $t_l$  back into 5 to find the intercept point

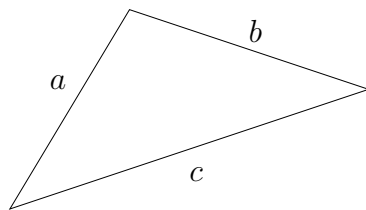
## 5 Useful geometric formulae

### 5.1 area of a quadrilateral in $\mathbb{R}^2$



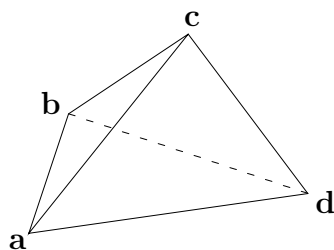
$$A = \frac{1}{4} \sqrt{4p^2q^2 - (b^2 + d^2 - a^2 - c^2)^2}$$

## 5.2 area of a triangle



$$A = \sqrt{s(s-a)(s-b)(s-c)} \text{ where } s = \frac{1}{2}(a+b+c)$$

## 5.3 volume of a tetrahedron



$$V = \frac{1}{6} \begin{vmatrix} (\mathbf{b} - \mathbf{a})_1 & (\mathbf{b} - \mathbf{a})_2 & (\mathbf{b} - \mathbf{a})_3 \\ (\mathbf{c} - \mathbf{a})_1 & (\mathbf{c} - \mathbf{a})_2 & (\mathbf{c} - \mathbf{a})_3 \\ (\mathbf{d} - \mathbf{a})_1 & (\mathbf{d} - \mathbf{a})_2 & (\mathbf{d} - \mathbf{a})_3 \end{vmatrix}$$