Fréchet derivative

In Euclidean space the derivative of a function f at a point $\mathbf{x}$ is the best linear approximation of f at $\mathbf{x}$ (denoted $f'(\mathbf{x})$). In more general spaces the derivative of a function can be defined through notions of linearity in that space. There are many possible ways in which to define the derivative of a function in a general space, and one such definition is that of the Fréchet derivative [1, pp 6–9]:

Definition 0.1 (Fréchet derivative) Suppose X and Y are normed vector spaces. Let $a \in X$ and suppose a function T is defined on an open neighbourhood $A \subseteq X$ of a, ie. $T : A \to Y$. T is said to be Fréchet differentiable at a if there exists a bound linear transformation $u : X \to Y$ such that

$$\frac{T(a + \varepsilon b) - T(a)}{\varepsilon} - u(b) \xrightarrow[\text{uniformly}]{\varepsilon \to 0} \mathbf{0}$$

for each $b \in A$, and where $\mathbf{0}$ is the zero element of Y. If such a function u exists then it is called the *Fréchet derivative* of T and it is denoted T'.

In the study of tomography the Radon transform is a linear transformation from a space of functions on $\mathbb{R}^n$ (eg. $\mathcal{L}_2(\mathbb{R}^n)$) to $\mathbb{R}$. Using the above definition suppose that the Fréchet derivative of the Radon transform $\mathcal{R}$ exists, and let f and g be functions on $\mathbb{R}^n$:

$$\begin{aligned}
\frac{\mathcal{R}(f + \varepsilon g) - \mathcal{R}(f)}{\varepsilon} - u(g) &= \frac{\mathcal{R}(f) + \mathcal{R}(\varepsilon g) - \mathcal{R}(f)}{\varepsilon} - u(g) \\
&= \frac{\mathcal{R}(\varepsilon g)}{\varepsilon} - u(g) \\
&= \mathcal{R}(g) - u(g) \\
&\xrightarrow[\text{uniformly}]{\varepsilon \to 0} 0
\end{aligned}$$

and this must occur for all g. Since the expression $\mathcal{R}(g) - u(g)$ has no dependence on ε we conclude that that Fréchet derivative of $\mathcal{R}$ exists at f and is equal to

$$\mathcal{R}'(f) = u(f) = \mathcal{R}(f),$$

ie. the Radon transform is the best linear approximation to itself (which also follows from the fact that the Radon transform is itself a linear transformation).

By substituting the expression $f = c^j e_j$, where e_j are basis functions, we get

$$\begin{aligned}
\mathcal{R}'(f) &= \mathcal{R}'(c^j e_j) \\
&= c^j \mathcal{R}'(e_j) \\
&= c^j \mathcal{R}(e_j).
\end{aligned}$$

By breaking the Radon transform down into a set of integrals over the hyperplanes in $\mathbb{R}^n$ we can write $\mathcal{R}(f) = \left\{\mathcal{R}_i(f) : (\mathcal{R}_i(f))(\mathbf{x}) = \int_{\Gamma_i} f(\mathbf{x}) \, \mathrm{d}\,\mathbf{x}\right\}$, where Γ_i denotes a hyperplane. The Fréchet derivative matrix A_{ij} can then be defined as

$$A_{ij} = \mathcal{R}'_i \left(e_j\right)$$

from which $\mathcal{R}'_i(f) = A_{ij} c^j$.

References

[1] Sadayuki Yamamuro. *Differential Calculus in Topological Linear Spaces*, volume 374 of *Lecture Notes in Mathematics*. Springer-Verlag, New York, 1974.