Rotation

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1 2-dimensions

Given a vector $(x, y) = (r\cos\theta, r\sin\theta) \in \mathbb{R}^2$, we can rotate this vector through an angle θ' by applying the following matrix:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta' & -\sin\theta' \\ \sin\theta' & \cos\theta' \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

PROOF:

$$\begin{aligned} x' &= r\cos(\theta + \theta') & y' &= r\sin(\theta + \theta') \\ &= r(\cos\theta\cos\theta' - \sin\theta\sin\theta') & &= r(\cos\theta\sin\theta' + \sin\theta\cos\theta') \\ &= r\cos\theta\cos\theta' - r\sin\theta\sin\theta' & &= r\cos\theta\sin\theta' + r\sin\theta\cos\theta' \\ &= x\cos\theta' - y\sin\theta' & &= x\sin\theta' + y\cos\theta' \end{aligned}$$

2 3-dimensions

Given a vector $(x, y, z) = (r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi) \in \mathbb{R}^3$, we can rotate this vector through angles θ' and ϕ' by applying the following non-linear transformation:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} x\cos\phi'\cos\theta' - y\cos\phi'\sin\theta' + z\sin\phi'\left(\cos\theta\cos\theta' - \sin\theta\sin\theta'\right) \\ x\cos\phi'\sin\theta' + y\cos\phi'\cos\theta' + z\sin\phi'\left(\cos\theta\sin\theta' + \sin\theta\cos\theta'\right) \\ z\cos\phi' - r\sin\phi\sin\phi' \end{bmatrix}$$

PROOF:

$$\begin{aligned}
x' &= r\sin\left(\phi + \phi'\right)\cos\left(\theta + \theta'\right) \\
&= r\left(\sin\phi\cos\phi' + \cos\phi\sin\phi'\right)\left(\cos\theta\cos\theta' - \sin\theta\sin\theta'\right) \\
&= r\sin\phi\cos\phi'\cos\theta\cos\theta' - r\sin\phi\cos\phi'\sin\theta\sin\theta' \\
&\qquad + r\cos\phi\sin\phi'\cos\theta\cos\theta' - r\cos\phi\sin\phi'\sin\theta\sin\theta' \\
&= x\cos\phi'\cos\theta' - y\cos\phi'\sin\theta' + z\sin\phi'\left(\cos\theta\cos\theta' - \sin\theta\sin\theta'\right) \\
y' &= r\sin\left(\phi + \phi'\right)\sin\left(\theta + \theta'\right) \\
&= r\left(\sin\phi\cos\phi' + \cos\phi\sin\phi'\right)\left(\sin\theta\cos\theta' + \cos\theta\sin\theta'\right) \\
&= r\sin\phi\cos\phi'\sin\theta\cos\theta' + r\sin\phi\cos\phi'\cos\theta\sin\theta' \\
&\qquad + r\cos\phi\sin\phi'\sin\theta\cos\theta' + r\cos\phi\sin\phi'\cos\theta\sin\theta' \\
&= y\cos\phi'\cos\theta' + x\cos\theta'\sin\theta' + z\sin\theta'\left(\sin\theta\cos\theta' + \cos\theta\sin\theta'\right) \\
&= x\cos\theta'\sin\theta' + y\cos\phi'\cos\theta' + z\sin\theta'\left(\cos\theta\sin\theta' + \sin\theta\cos\theta'\right) \\
z' &= r\cos\left(\phi + \phi'\right) \\
&= r\cos\phi\cos\phi' - r\sin\phi\sin\phi' \\
&= z\cos\phi' - r\sin\phi\sin\phi'
\end{aligned}$$