DSA3 seminar assignment

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1 Introduction

2 BFS and DFS

2.1 Purpose and theoretical background

Both BFS (breadth-first search) and DFS (depth-first search) are tree-search algorithms. They are both used to determine whether a graph is connected.

Theorem
A graph is connected if and only if, you can not split it into two such that there are no edges between them.

An approach to check wheter a graph is connected is by checking all the partitions (the problem is that there are 2^{n-1} of them so it is very inneficient). Another way to check whether a graph is connect is by checking all pairs of vertices - if there is a path connecting them (problem here again is that there are n^2 pairs of vertices and for each of them we have to check if there is a path which is also very inneficient on top of number of partitions that we would need to check).

Definition
A graph is connected if and only if it has a spanning tree.

Checking whether a graph is connected in such a way (by determining whether it has a spanning tree) is far and away the most efficient way to check graph connectedness.

Both DFS and BFS work in a similar way (they both produce spanning trees), but with differences that impact the priority of edges selected. Depending on the way we used to choose edges we get different trees.

A **tree-search** algorithm on G:
> Start with a single root vertex $r \ \varepsilon \ V(G)$. This is our initial tree (with just one vertex).
> Repeat (for as long as possible):

- Do we have a spanning tree?
- Is the tree edge cut empty?
- If not, add one edge from the cut to the tree.

If we want to check *connectedness* of G, that is the whole algorithm. As noted above, depending on the way we use to choose edges, different spanning tree will result (if the graph is connected of course).

Define: edge cut, rooted tree (also called r-tree), levels (distance from root to a specific vertex), ancestor, descendant, parent or predecessor and children.

Predecessor function: $p(v)$, for all $\begin{matrix} v \ \varepsilon V \\ \{r\} \end{matrix}$

2.2 BFS

2.2.1 BFS instroduction

In **breadth-first search** the adjacency lists of vertices are considered on a first-come-first serve basis. **BFS** is implemented with a *queue* data structure.

A *queue* data structure operates in a FIFO (first-in, first-out) principle. Meaning first element that entered the *queue*, will be the first one to leave the *queue* (elements are removed in the same order in which they were inserted).

In **BFS** algorithm we will first consider all of the neighbors (one-by-one) before we look through the neighbours of any of them.

Therefore, first edges incident to r are selected, and only after we have used (added them to the tree) we are looking at the neighbors of the neighbors of r.

This is called **breadth-first search** algorithm. This approach expands (spreads) the tree as much as possible.

We start with just the root r in the *queue* and we repeatedly pop the head of the *queue*, and push all its new neighbors to the *queue*.

For a connected graph, the algorithm will return:
- A spanning tree (**BFS** spanning tree) given by its predecessor function,
- the level function,
- the time function (order in which vertices are added to the tree)

2.2.2 BFS algorithm

> INPUT: a connected graph G, a vertex $r\ \varepsilon V(G)$
> OUTPUT: an r-tree $T \subseteq G$, its predecessor function p, its level function l, the time function t

BFS algorithm
$Q := \emptyset, \quad Q \leftarrow r, \quad l(r) := 0 \quad t(r) := 1, \quad \text{mark } r, i := 1$
while $Q \neq \emptyset$

  consider the head x of Q
  if x has unmarked neighbor y **then**

    $i++$
    $Q \leftarrow y, \quad \text{mark } y, \quad p(y) := x, \quad l(y) := l(x) + 1, \quad t(y) := i$

  else

    remove head of Q

  end if

end while
return everything

2.2.3 BFS properties

BFS-trees have two basic properties, the first of which justifies our referring to l as a level function.

Theorem Let T be a BFS tree of G, with root r.

  a.) $l(v) = d_T(r, v)$, , for every $v \in V$,

  b.) $|l(u) - l(v)| \leqslant 1$, for every $uv \in E(G)$.

Level of v is exactly the distance from root r to v.

Every edge of the graph connects only vertices of the same level of the tree or difference by most 1.

Theorem Let T be a BFS tree of G, with root r. Then

  $l(v) = d_G(r, v)$, for every $v \in V$

2.3 DFS

2.3.1 DFS introduction

In contrast to BFS, where we first scan the whole adjacency list of the vertex on top of the *queue*, in **depth-first search** we scan the adjacency list of the most recent vertex x added to the *stack* and we look for its neighbour not in T.

If there is such a neighbor, we add it to T. If not, we backtrack to the vertex which was added to T just before x and examine its neighbours, and so on.

For DFS to be implemented, we use a ***stack*** data structure. A *stack* is a linear data structure (like a simple list) which has a top element and basic operations such as placing a new item on top of the *stack*, or removing the top element from the *stack*. In contrast to *queue*, a *stack* operates in LIFO (last-in, first-out) principle, meaning elements are inserted and removed exclusively at the designated end of the structure, referred to as the top.

In **depth-first search**, the *stack* S is initially empty. We pick a root vertex r and scan its neighbours in its adjacency list. We pick one element from that list and place it on top of the *stack* S. We then look at this new vertex, now acting as the new top element of *stack* S and we inspect its adjacency list. If in that list exists a vertex y which is not already in our tree T we select it and add it to the top of *stack* S again. And so on, we continue until there are no suitable vertices in the top element of *stack* S. If there indeed aren't any, we remove the top element of *stack* S and check the vertex that is the new top of the *stack* S.

Again for a connected graph, the algorithm will return:

- A spanning tree (**DFS** spanning tree) given by its predecessor function,
- the level function,
- the time function (order in which vertices are added to the tree)

2.3.2 DFS algorithm

Completely the same as BFS, except that we use a ***stack*** instead of a queue.

> INPUT: a connected graph G, a vertex $r \ \varepsilon V(G)$
> OUTPUT: an r-tree $T \subseteq G$, its predecessor function p, its level function l, the time function t

DFS algorithm
$S := \emptyset, \quad S \leftarrow r, \quad l(r) := 0 \quad t(r) := 1, \quad \text{mark } r, i := 1$
while $S \neq \emptyset$

  consider the top vertex x of S
  if x has unmarked neighbor y **then**

    $i++$
    move y to the top of S, mark y, $p(y) := x$, $l(y) := l(x) + 1$, $t(y) := i$

  else

    remove x from S

  end if

end while
return everything

2.3.3 DFS properties

Theorem Let T be a DFS tree of G. Every edge of G joins vertices related in T.

Like we had for **BFS** a theorem that says every edge of the graph skips at most one level, which is the most important property of BFS. Here we have this very important property of **DFS** trees. Every edge goes vertically, from ancestor to predecessor.

The following proposition provides a link between the input graph G, its **DFS**-tree T, and the two time functions t and l returned by **DFS**.

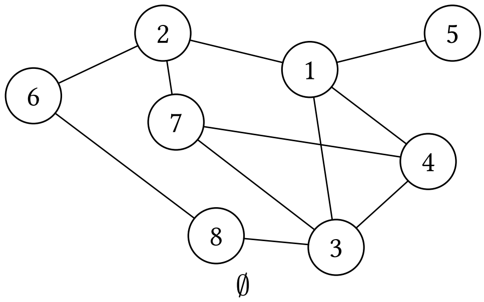
Proposition Let u and v be two vertices of G, with $f(u) < f(v)$.

a) If u and v are adjacent in G, then $l(v) < l(u)$.

b) u is an ancestor of v in T if and only if $l(v) < l(u)$.

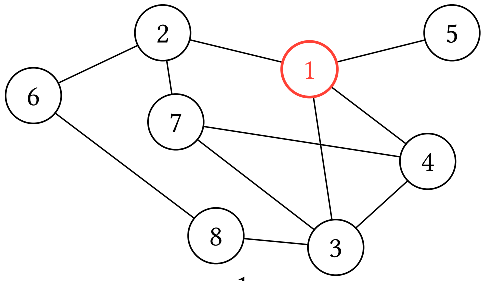
2.4 Algorithm examples

2.4.1 BFS example



∅

→



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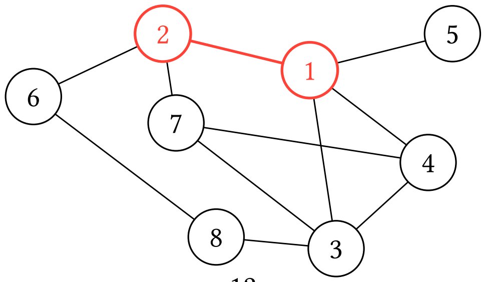

12

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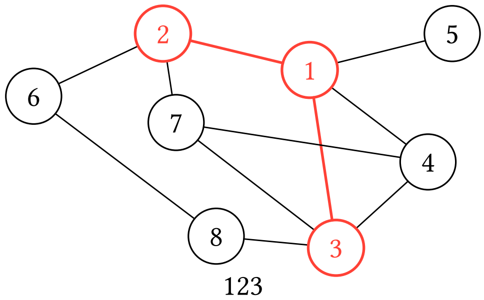

123

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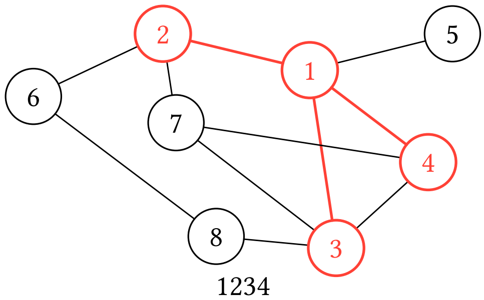

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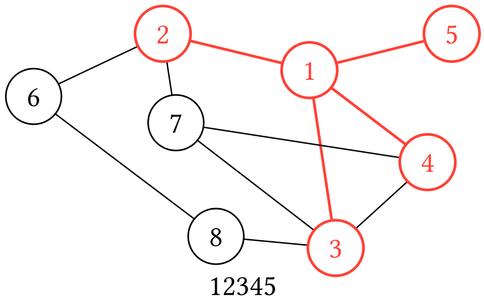
→



12345

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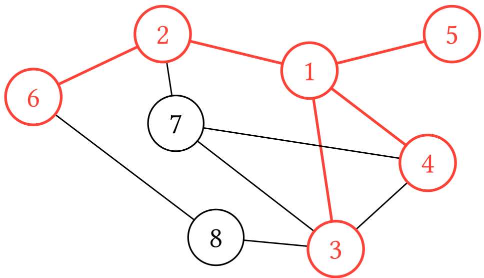
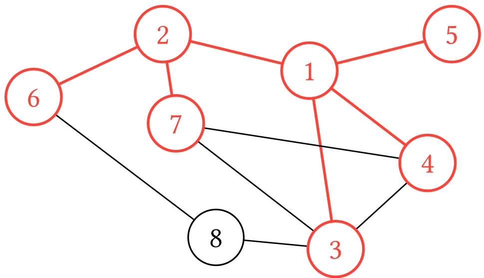


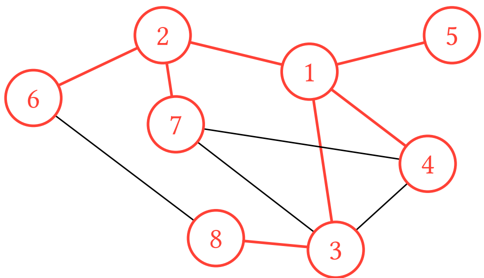
23456

→



234567

→

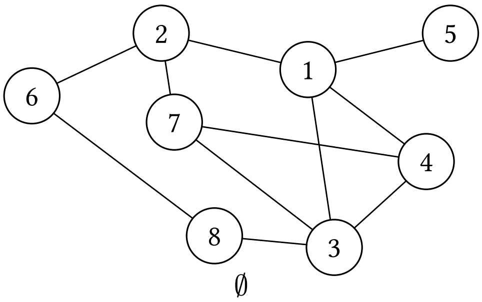


345678

detailed *(queue is used)*

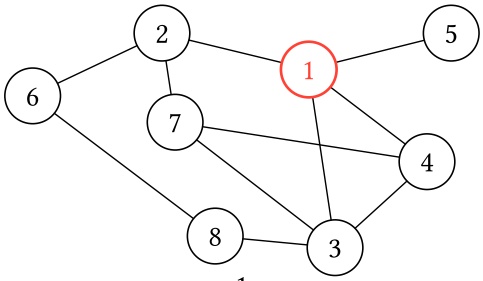
∅ → 1 → 12 → 123 →
1234 → 12345 → 2345 →
23456 → 234567 →
34567 → 345678 →
45678 → 5678 → 678 →
78 → 8 → ∅

2.4.2 DFS example



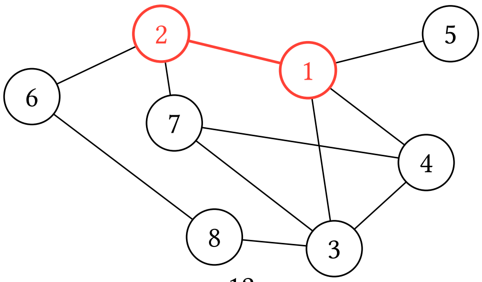
∅

→



1

→



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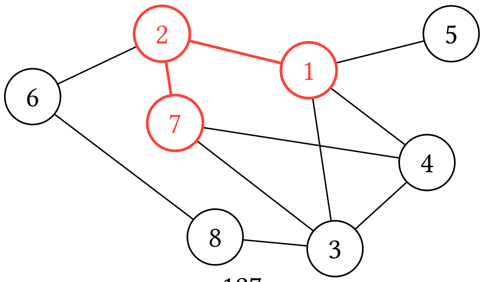
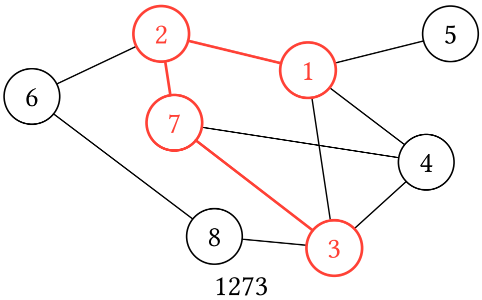


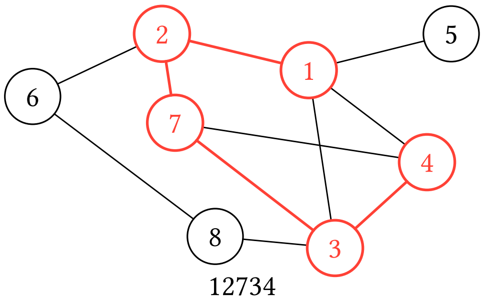
127

→



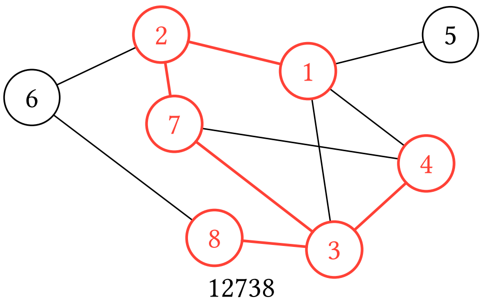
1273

→



12734

→



12738

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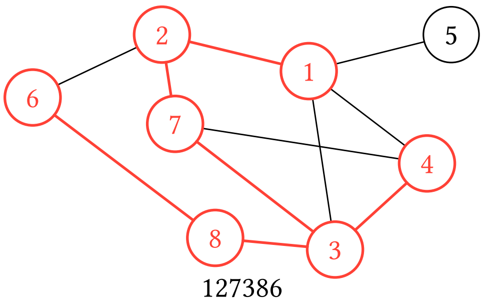

127386

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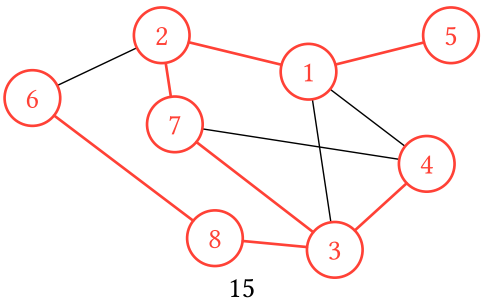

15

detailed *(stack is used)*

$\emptyset \to 1 \to 12 \to 127 \to$
$1273 \to 12734 \to 1273 \to$
$12738 \to 127386 \to$
$12738 \to 1273 \to 127 \to$
$12 \to 1 \to 15 \to 1 \to \emptyset$

3 Applications

4 Conculsion