

1 Integral Basics

5.3 Average Value

Integrals are pretty much an infinite amount of Riemann sums. If f is integrable on $[a, b]$, then its average value (or mean) on $[a, b]$ is $\frac{1}{b-a} \int_a^b f(x) dx$.

5.5 U-Substitution

Since du can be substituted for $(\frac{du}{dx})dx$, u-substitution is possible.

Ex. Find $\int (x^3 + x)^5 (3x^2 + 1) dx$.

$$\begin{aligned} \int_a^b (x^3 + x)^5 (3x^2 + 1) dx \\ \int_a^b (u)^5 (3x^2 + 1) dx \\ u = x^3 + x \\ \frac{du}{dx} = 3x^2 + 1 \\ \int_{a^3+a}^{b^3+b} (u)^5 \left(\frac{du}{dx}\right) dx \\ \int_{a^3+a}^{b^3+b} (u)^5 du \\ \int_{a^3+a}^{b^3+b} (u)^5 du \\ \frac{(b^3 + b)^6}{6} - \frac{(a^3 + a)^6}{6} \end{aligned} \tag{1}$$

5.6 Area Between Curves

If $f(x) \geq g(x)$ for $[a, b]$, then the area between both curves from a to b is equal to $A = \int_a^b [f(x) - g(x)] dx$.

6.1 Volume Using Integrals: Disk and Washer Methods

The volume of a solid of integrable cross-sectional area $A(x)$ from $x = a$ to $x = b$ is equal to $\int_a^b A(x) dx$.

The volume of a disk created by rotating a function $R(x)$ around $y = 0$ (the x-axis) for the same bounds is equal to $\int_a^b \pi[R(x)]^2 dx$.

The volume of a disk created by rotating a function $R(y)$ around $x = 3$ for the same bounds is equal to $\int_{R(a)}^{R(b)} \pi[R(y) - 3]^2 dy$.

The reason 3 is subtracted from $R(y)$ is because the axis of rotation is closer to the area being rotated, so the radius of rotation will be less than if $x = 0$.

The volume of a washer created by rotating the space between $R(x)$ and $r(x)$ around $y = 1$ is equal to $\int_a^b \pi([R(x) - 1]^2 - [r(x) - 1]^2) dx$. The reason a 1 is subtracted is because the axis of rotation is closer to the area being rotated, so the radius of rotation will be less than if $y = 0$.

The volume of a washer created by rotating the space between $R(y)$ and $r(y)$ around $x=0$ (the y-axis) is equal to $\int_{R(a)}^{R(b)} \pi([R(y)]^2 - [r(y)]^2) dy$.

6.2 Volumes Using Integrals: Shell Method

Summarize Shell Method

2 Integral Applications and Sections

8.2 Integration by Parts

1. Pick a u to derive
2. Pick a dv to integrate
3. $\int u * dv = u * v - \int v * du$

8.3 Trigonometric Integrals

sincos integrals $\rightarrow \sin^2(x) + \cos^2(x) = 1$

$\int \sin(x)\cos^n(x)dx$

1. Case 1: m is odd $\rightarrow u = \cos x$

- Make $\sin^2(x) = (1 - \cos^2(x))$

2. Case 2: n is odd $\rightarrow u = \sin x$

- Make $\cos^2(x) = (1 - \sin^2(x))$

3. Case 3: m and n are even

- Use the following identities...

$$\sin^2(x) = \frac{1 - \cos 2x}{2} \text{ and } \cos^2(x) = \frac{1 + \cos 2x}{2}$$

Eliminating Square Roots

- Use following identities..

$$1. \sin^2 x + \cos^2 x = 1$$

$$2. \cos 2x = \cos^2 x - \sin^2 x$$

$$3. \tan^2 x + 1 = \sec^2 x$$

$$4. \sin 2x = 2 \sin x \cos x$$

$$5. \sin x^2 = \frac{1 - \cos 2x}{2}$$

$$6. \cos^2 x = \frac{1 + \cos 2x}{2}$$

Products of \sin and \cos

$$- \sin(mx)\sin(nx) = \frac{1}{2}(\cos(m-n)x - \cos(m+n)x)$$

$$- \sin(mx)\cos(nx) = \frac{1}{2}(\sin(m-n)x + \sin(m+n)x)$$

$$- \cos(mx)\cos(nx) = \frac{1}{2}(\cos(m-n)x + \cos(m+n)x)$$

8.4 Trigonometric Substitutions

1. *Case 1* : $a^2 + x^2$

$$\rightarrow x = a \tan \theta$$

$$\rightarrow dx = a \sec^2 \theta d\theta$$

$$\rightarrow a^2 + x^2 = a^2 + a^2 \tan^2 \theta = a^2(1 + \tan^2 \theta) = a^2 \sec^2 \theta$$

2. *Case 2* : $a^2 - x^2$

$$\rightarrow x = a \sin \theta$$

$$\rightarrow dx = a \cos \theta d\theta$$

$$\rightarrow a^2 - x^2 = a^2 - a^2 \sin^2 \theta = a^2(1 - \sin^2 \theta) = a^2 \cos^2 \theta$$

3. *Case 3* : $x^2 - a^2$

$$\rightarrow x = a \sec \theta$$

$$\rightarrow dx = a \sec \theta \tan \theta d\theta$$

$$\rightarrow x^2 - a^2 = a^2 \sec^2 \theta - a^2 = a^2(\sec^2 \theta - 1) = a^2 \tan^2 \theta$$

8.5 Partial Fraction Decomposition

*Degree on top must be less than degree on bottom, or long divide

- If there is a linear factor $\rightarrow \frac{A}{x+1} + \frac{B}{(x+1)^2}$
- If there is a quadratic factor $\rightarrow \frac{Ax+B}{x^2+bx+c} + \frac{Cx+D}{(x^2+bx+c)^2}$
- Set $\frac{f(x)}{g(x)}$ equal to the sum of its partial fractions by multiplying
- Equate coefficients of powers of x to solve for variables

$$\begin{aligned} & \frac{-x+3}{x^2-9x+20} \\ & \frac{-x+3}{(x-4)(x-5)} \\ & \frac{-x+3}{(x-4)(x-5)} = \frac{A}{x-4} + \frac{B}{x-5} \\ & \frac{-x+3}{(x-4)(x-5)} * (x-4)(x-5) = \frac{A}{x-4}(x-4)(x-5) + \frac{B}{x-5}(x-4)(x-5) \\ & -x+3 = A(x-5) + B(x-4) \\ & x=5 : -2 = A(0) + B(1) \rightarrow \text{So, } B = -2 \\ & x=4 : -1 = A(-1) + B(0) \rightarrow \text{So, } A = 1 \\ & \frac{-x+3}{x^2-9x+20} = \frac{1}{x-4} - \frac{2}{x-5} \end{aligned}$$

Density Problems

- 1-dimensional \rightarrow need length or mass (δx meters or grams)

- 2-dimensional \rightarrow need area of a section (rectangle bh where h is δx) (or other shape)
- 3-dimensional \rightarrow need volume of a section (cylinder $= \pi r^2 h$ where h is δy / other shape)

3 Sequences, Series, and Tests

8.8

10.1

10.2

10.3

10.4

4 Sequences, Series, and Tests cont.

10.5

10.6

10.7

10.8

10.9

5 Parametric Curves

10.10

6.3 Arc Length

Pythagorean's Theorem can be applied to find the length of a segment $f(x)$. If ds is equal to a single straight segment in $f(x)$, then dx is equal to the horizontal length and dy is equal to its vertical length.

$$\begin{aligned} (ds)^2 &= (dx)^2 + (dy)^2 \\ \sqrt{(ds)^2} &= \sqrt{(dx)^2 + (dy)^2} \\ ds &= dx \sqrt{(dx)^2 / (dx)^2 + (dy)^2 / (dx)^2} \\ ds &= \sqrt{1 + (dy/dx)^2} dx \end{aligned} \tag{2}$$

By taking the integral of this, you can get the total length of the segment.

$$s = \int_a^b \sqrt{1 + (f'(x))^2} dx \tag{3}$$

6.4

11.1

11.2

6 Polar Coordinates

11.3

11.4

11.5

Complex Numbers

Standard form = $a + bi$. a is real portion, bi is imaginary. To graph: 3 units to the right, imaginary is 4 so 4 units up. To calculate absolute value, $|a + bi| = \sqrt{a^2 + b^2}$

For $-5 + 12i$:

Graph 5 units to the left, and up 12 units.

$$|-5 + 12i| = \sqrt{(-5)^2 + 12^2}$$

$$|-5 + 12i| = \sqrt{25 + 144}$$

$$|-5 + 12i| = \sqrt{169}$$

$$|-5 + 12i| = 13$$

An example of how a square root of negative looks: $\sqrt{-80} = i * \sqrt{16} * \sqrt{5} = 4i\sqrt{5}$

$$i = \sqrt{-1}$$

$$i^2 = -1$$

$$i^3 = -i$$

$$i^4 = 1$$

(4)

Idea: Associate polar coordinates to complex numbers

$$z = x + iy$$

$$r = |z| = \sqrt{x^2 + y^2}$$

$$\text{if } x \geq 0, \theta = \arctan \frac{y}{x}$$

$$\text{if } x \leq 0, \theta = \arctan \frac{y}{x} + \pi$$

$$\text{if } x = 0, \theta = \pm \frac{\pi}{2}$$

Find polar coordinates of $z = -2\sqrt{2} + 2\sqrt{2}i$

$$r = |z| = \sqrt{(-2\sqrt{2})^2 + (2\sqrt{2})^2} = \sqrt{16} = 4$$

$$\text{Since } x \leq 0, \theta = \arctan \frac{y}{x} + \pi = \frac{3\pi}{4}$$

$$\text{Polar Coordinates} = (4\cos \frac{3\pi}{4}, 4\sin \frac{3\pi}{4})$$