1 Integral Basics

5.3 Average Value

Integrals are pretty much an infinite amount of Riemann sums. If f is integrable on $[a, b]$, then its average value (or mean) on $[a, b]$ is $\frac{1}{b-a}\int_a^b f(x)dx$.

5.5 U-Substitution

Since du can be substituted for $(\frac{du}{dx})dx$, u-substitution is possible.
Ex. Find $\int(x^3+x)^5(3x^2+1)dx$.

$$\int_a^b (x^3+x)^5(3x^2+1)dx$$
$$\int_a^b (u)^5(3x^2+1)dx$$
$$u = x^3 + x$$
$$\frac{du}{dx} = 3x^2+1$$
$$\int_{a^3+a}^{b^3+b} (u)^5(\frac{du}{dx})dx$$
$$\int_{a^3+a}^{b^3+b} (u)^5 du$$
$$\int_{a^3+a}^{b^3+b} (u)^5 du$$
$$\frac{(b^3+b)^6}{6} - \frac{(a^3+a)^6}{6}$$

5.6 Area Between Curves

If $f(x) \geq g(x)$ for $[a, b]$, then the area between both curves from a to b is equal to $A = \int_a^b [f(x) - g(x)]dx$.

6.1 Volume Using Integrals: Disk and Washer Methods

The volume of a solid of integrable cross-sectional area $A(x)$ from $x = a$ to $x = b$ is equal to $\int_a^b A(x)dx$.
The volume of a disk created by rotating a function $R(x)$ around $y = 0$ (the x-axis) for the same bounds is equal to $\int_a^b \pi[R(x)]^2dx$.
The volume of a disk created by rotating a function $R(y)$ around $x = 3$ for the same bounds is equal to $\int_{R((a)}^{R(b)} \pi[R(y)-3]^2dy$.
The reason 3 is subtracted from $R(y)$ is because the axis of rotation is closer to the area being rotated, so the radius of rotation will be less than if $x = 0$.
The volume of a washer created by rotating the space between $R(x)$ and $r(x)$ around $y = 1$ is equal to $\int_a^b \pi([R(x)-1]^2 - [r(x)-1]^2)dx$. The reason a 1 is subtracted is because the axis of rotation is closer to the area being rotated, so the radius of rotation will be less than if $y = 0$.
The volume of a washer created by rotating the space between R(y) and r(y) around x=0 (the y-axis) is equal to $\int_{R(a)}^{R(b)} \pi([R(y)]^2 - [r(y)]^2)dx$.

6.2 Volumes Using Integrals: Shell Method

$V = 2\pi\int_a bR(x)h(x)dx$
Start with drawing a rectangle parallel to the axis of rotation. The change in the height of this rectangle between a and b is modeled by $h(x)$. The change in the distance of the rectangle from the axis of rotation is modeled by $R(x)$.
Basically integrating an area $R(x) * h(x)$ over a certain interval times 2π.
When using the area between two curves, $h(x) = f(x) - g(x)$ where $f(x) > g(x)$ for [a, b].
For example, if you were trying to find the volume of the area enclosed by $y = \sqrt{x}$, $y = 0$ and $x = 4$ when rotated around

the y-axis,

- $a = 0$
- $b = 4$
- $R(x) = x$, as the distance between parallel rectangles to the y-axis would just be x.
- $h(x) = \sqrt{x}$, as the height of the area above the x-axis is $y = \sqrt{x}$.

So, the integral would read: $V = 2\pi \int_0^4 x\sqrt{x}dx$

2 Integral Applications and Sections

8.2 Integration by Parts

1. Pick a u to derive
2. Pick a dv to integrate
3. $\int u * dv = u * v - \int v * du$

8.3 Trigonometric Integrals

$sincos$ integals $\rightarrow sin^2(x) + cos^2(x) = 1$
$\int sin(x)cos^n(x)dx$
1. Case 1: m is odd $\rightarrow u = cosx$
- Make $sin^2(x) = (1 - cos^2(x))$
2. Case 2: n is odd $\rightarrow u = sinx$
- Make $cos^2(x) = (1 - sin^2(x))$
3. Case 3: m and n are even
- Use the following identities...
$sin^2(x) = \frac{1-cos2x}{2}$ and $cos^2(x) = \frac{1+cos2x}{2}$

Eliminating Square Roots
- Use following identities..

$$1. sin^2x + cos^2x = 1$$
$$2. cos2x = cos^2x - sin^2x$$
$$3. tan^2x + 1 = sec^2x$$
$$4. sin2x = 2sinxcosx$$
$$5. sinx^2 = \frac{1 - cos2x}{2}$$
$$6. cos^2x = \frac{1 + cos2x}{2}$$

Products of sin and cos

$$- sin(mx)sin(nx) = \frac{1}{2}(cos(m-n)x - cos(m+n)x)$$
$$- sin(mx)cos(nx) = \frac{1}{2}(sin(m-n)x + sin(m+n)x)$$
$$- cos(mx)cos(nx) = \frac{1}{2}(cos(m-n)x + cos(m+n)x)$$

8.4 Trigonometric Substitutions

$1.Case \quad 1 : a^2 + x^2$
$\rightarrow x = atan\theta$
$\rightarrow x = asec^2\theta d\theta$
$\rightarrow a^2 + x^2 = a^2 + a^2tan^2\theta = a^2(1 + tan^2\theta) = a^2sec^2\theta$
$2.Case \quad 2 : a^2 - x^2$
$\rightarrow x = asin\theta$
$\rightarrow dx = acos\theta d\theta$
$\rightarrow a^2 - x^2 = a^2 - a^2sin^2\theta = a^2(1 - sin^2\theta) = a^2cos^2\theta$
$3.Case \quad 3 : x^2 - a^2$
$\rightarrow x = asec\theta$
$\rightarrow dx = asec\theta tan\theta d\theta$
$\rightarrow x^2 - a^2 = a^2sec^2\theta - a^2 = a^2(sec^2\theta - 1) = a^2tan^2\theta$

8.5 Partial Fraction Decomposition

*Degree on top must be less than degree on bottom, or long divide

- If there is a linear factor $\rightarrow \frac{A}{x+1} + \frac{B}{(x+1)^2}$
- If there is a quadratic factor $\rightarrow \frac{Ax+B}{x^2+bx+c} + \frac{Cx+D}{(x^2+bx+c)^2}$
- Set $\frac{f(x)}{g(x)}$ equal to the sum of its partial fractions by multiplying
- Equate coefficients of powers of x to solve for variables

$$\frac{-x+3}{x^2-9x+20}$$
$$\frac{-x+3}{(x-4)(x-5)}$$
$$\frac{-x+3}{(x-4)(x-5)} = \frac{A}{x-4} + \frac{B}{x-5}$$
$$\frac{-x+3}{(x-4)(x-5)} * (x-4)(x-5) = \frac{A}{x-4}(x-4)(x-5) + \frac{B}{x-5}(x-4)(x-5)$$
$$-x+3 = A(x-5) + B(x-4)$$
$$x = 5 : -2 = A(0) + B(1) \rightarrow So, B = -2$$
$$x = 4 : -1 = A(-1) + B(0) \rightarrow So, A = 1$$
$$\frac{-x+3}{x^2-9x+20} = \frac{1}{x-4} - \frac{2}{x-5}$$

Density Problems

- 1-dimensional $\rightarrow$ need length or mass (δx meters or grams)

- 2-dimensional $\rightarrow$ need area of a section (rectangle bh where h is δx) (or other shape)
- 3-dimensional $\rightarrow$ need volume of a section (cylinder $= \pi r^2 h$ where h is δy / other shape)

3 Sequences, Series, and Tests

8.8 Improper Integrals

Convergent or Divergent: $\int_1^\infty \frac{1}{x} dx$

- If integral equals a finite number, the integral is convergent
- If integral equals infinity, it is divergent

$$\int_1^\infty \frac{1}{x} dx = \lim_{t\to\infty} \int_1^t \frac{1}{x} dx$$
$$= \lim_{t\to\infty} lnt - ln1$$
$$= \lim_{t\to\infty} \infty - 0 = \infty$$

$\int_1^\infty \frac{1}{x} dx$ is divergent since it equals infinity.

For $\int_1^\infty \frac{1}{x^p} dx$, if:

- $p > 1$, it is convergent
- $p \leq 1$, it is divergent
- Similar to p-series

For integrals with two improper bounds or a vertical asymptote, splitting integral into the sum of two integrals is helpful for integration.

10.1 Sequences

Arithmetic Sequences: $a_n = a_1 + (n-1)d$
Partial Sum of Arithmetic Sequence: $S_n = \frac{a_1+a_n}{2} n$
$d = a_2 - a_1 = a_3 - a_2$
Arithmetic Mean: $\frac{a+b}{2}$
Geometric Sequence: $a_n = a_1 * (r)^(n-1)$
Partial Sum of a Finite Geometric Sequence: $S_n = \frac{a_1(1-r^n)}{1-r}$
$r = \frac{a_2}{a_1} = \frac{a_3}{a_2}$
Sum of Infinite Geometric Sequence: $S = \frac{a_1}{1-R}$
Geometric Mean: $\sqrt{a * b}$
To find if a sequence converges or diverges:
$\lim_{n\to\infty} a_n = L \rightarrow$ converges
$\lim_{n\to\infty} a_n = DNE, \infty, -\infty \rightarrow$ diverges
To find if $a_n = \frac{sin(n)}{n}$ converges or diverges, use the Squeeze Theorem*
Since $-1 \leq sin(n) \leq 1$ and since when all are divided by n, the limit of both the left and right equations are equal to 0, the

limit of $\frac{sin(n)}{n}$ must also be 0.
*Squeeze Theorem states that if $g(x) \leq f(x) \leq h(x)$ and the limit of both $g(x)$ and $h(x) = L$, then the same limit of $f(x)$ must equal L.
Another useful strategy when finding the limit of a sequence is L'hopital's Rule, or taking the derivative of both the top and bottom of the equation and instead finding the limit of that. This helps to remove variables and simplify equations. L'hopital's Rule can be used when limits equal $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

10.2 Infinite Series

10.3 The Integral Test

To find $\sum_{n=1}^{\infty} \frac{1}{(n+2)^2}$, since a_n can be described as a $f(n)$, and since f is **positive**, **continuous**, and **decreasing** when $x \geq 1$, the Integral Test can be used.
For $\int_1^{\infty} f(x)dx$, if:
$= N$, Integral converges, Series converges
$= \pm\infty$, Integral diverges, Series diverges
For $f(x)\frac{1}{(x+2)^2}$:

- f is **positive** for every value except $x = -2$, but $x = -2$ does not count as it is below 1.
- f is **continuous** for every value except $x = -2$, but again, this does not count as it is below 1.
- f is **decreasing** as the first derivative of f is negative after $x = -2$.

Since $\lim_{b\to\infty} \int_1^b \frac{1}{(x+2)^2} dx = \frac{1}{3}$, a finite value, the original series converges.

10.4 Comparison Tests (Direct & Limit)

Direct Comparison Test
If $0 \leq a_n \leq b_n$ then the following can be used:
If $\sum b_n$ converges, $\sum a_n$ converges
If $\sum a_n$ diverges, $\sum b_n$ diverges

Limit Comparison Test
If $\lim_{n\to\infty} \frac{a_n}{b_n} = L \to$ both converge or both diverge

4 Sequences, Series, and Tests cont.

10.5 Ratio/Root Tests & Absolute Convergence

Ratio Test: $\lim_{n\to\infty} |\frac{a_n+1}{a_n}|$

- $< 1 \to$ converges
- $> 1 \to$ diverges
- $= 1 \to$ inconclusive

Root Test: $\lim_{n\to\infty} \sqrt[n]{|a_n|}$

- $< 1 \to$ converges

- $> 1 \rightarrow$ diverges

- $= 1 \rightarrow$ inconclusive

Absolute Value Test:
If $\sum |a_n|$ converges and $\sum a_n$ converges $\rightarrow$ absolutely convergent
If $\sum |a_n|$ diverges and $\sum a_n$ converges $\rightarrow$ conditionally convergent
Watch OGT Video on Absolute Convergence for some more notes

10.6 Alternating Series and Conditional Convergence

10.7 Power Series

$f(x) = \frac{1}{1-x}$ can be written as the series $\sum_{n=0} \infty x^n$

10.8-9 Taylor and Maclaurin Series and Polynomials

5 Parametric Curves

10.10

6.3 Arc Length

Pythagorean's Theoreom can be applied to find the length of a segment $f(x)$. If ds is equal to a single straight segment in $f(x)$, then dx is equal to the horizontal length and dy is equal to its vertical length.

$$\begin{aligned} (ds)^2 &= (dx)^2 + (dy)^2 \\ \sqrt{(ds)^2} &= \sqrt{(dx)^2 + (dy)^2} \\ ds &= dx\sqrt{(dx)^2/(dx)^2 + (dy)^2/(dx)^2} \\ ds &= \sqrt{1 + (dy/dx)^2} dx \end{aligned} \tag{1}$$

By taking the integral of this, you can get the total length of the segment.

$$s = \int_a^b \sqrt{1 + (f(x))^2}\, dx \tag{2}$$

6.4

11.1

11.2

6 Polar Coordinates

11.3

11.4

11.5

Complex Numbers

Standard form $= a + bi$. a is real portion, bi is imaginary. To graph: 3 units to the right, imaginary is 4 so 4 units up. To calculate absolute value, $/absa + bi = /sqrta^2 + b^2$
For -5 + 12i:
Graph 5 units to the left, and up 12 units.
$|-5 + 12i| = \sqrt{(-5)^2 + 12^2}$

$|-5+12i| = \sqrt{(25+144}$
$|-5+12i| = \sqrt{169}$
$|-5+12i| = 13$

An example of how a square root of negative looks: $\sqrt{-80} = i * \sqrt{16} * \sqrt{5} = 4i\sqrt{5}$

$$\begin{aligned} i &= \sqrt{-1} \\ i^2 &= -1 \\ i^3 &= -i \\ i^4 &= 1 \end{aligned} \tag{3}$$

Idea: Associate polar coordinates to complex numbers
$z = x + iy$
$r = |z| = \sqrt{x^2+y^2}$
if $x \geq 0$, $\theta = arctan\frac{y}{x}$
if $x \leq 0$, $\theta = arctan\frac{y}{x} + \pi$
if $x = 0, \theta = \pm\frac{\pi}{2}$
Find polar coordinates of $z = -2\sqrt{2} + 2\sqrt{2}i$
$r = |z| = \sqrt{(-2\sqrt{2})^2 + (2\sqrt{2})^2} = \sqrt{16} = 4$
Since $x \leq 0$, $\theta = arctan\frac{y}{x} + \pi = \frac{3\pi}{4}$
Polar Coordinates $= (4cos\frac{3\pi}{4}, 4sin\frac{3\pi}{4})$