1 Integral Basics

5.3 Average Value

Integrals are pretty much an infinite amount of Riemann sums. If f is integrable on $[a, b]$, then its average value (or mean) on $[a, b]$ is $\frac{1}{b-a}\int_a^b f(x)dx$.

5.5 U-Substitution

Since du can be substituted for $(\frac{du}{dx})dx$, u-substitution is possible.
Ex. Find $\int (x^3+x)^5(3x^2+1)dx$.

$$\int_a^b (x^3+x)^5(3x^2+1)dx$$
$$\int_a^b (u)^5(3x^2+1)dx$$
$$u = x^3+x$$
$$\frac{du}{dx} = 3x^2+1$$
$$\int_{a^3+a}^{b^3+b} (u)^5(\frac{du}{dx})dx$$
$$\int_{a^3+a}^{b^3+b} (u)^5du$$
$$\int_{a^3+a}^{b^3+b} (u)^5du$$
$$\frac{(b^3+b)^6}{6} - \frac{(a^3+a)^6}{6}$$

5.6 Area Between Curves

If $f(x) \geq g(x)$ for $[a, b]$, then the area between both curves from a to b is equal to $A = \int_a^b [f(x) - g(x)]dx$.

6.1 Volume Using Integrals: Disk and Washer Methods

The volume of a solid of integrable cross-sectional area $A(x)$ from $x = a$ to $x = b$ is equal to $\int_a^b A(x)dx$.
The volume of a disk created by rotating a function $R(x)$ around $y = 0$ (the x-axis) for the same bounds is equal to $\int_a^b \pi[R(x)]^2dx$.
The volume of a disk created by rotating a function $R(y)$ around $x = 3$ for the same bounds is equal to $\int_{R((a)}^{R(b)} \pi[R(y)-3]^2dy$. The reason 3 is subtracted from $R(y)$ is because the axis of rotation is closer to the area being rotated, so the radius of rotation will be less than if $x = 0$.
The volume of a washer created by rotating the space between $R(x)$ and $r(x)$ around $y = 1$ is equal to $\int_a^b \pi([R(x)-1]^2 - [r(x)-1]^2)dx$. The reason a 1 is subtracted is because the axis of rotation is closer to the area being rotated, so the radius of rotation will be less than if $y = 0$.
The volume of a washer created by rotating the space between R(y) and r(y) around x=0 (the y-axis) is equal to $\int_{R(a)}^{R(b)} \pi([R(y)]^2 - [r(y)]^2)dx$.

6.2 Volumes Using Integrals: Shell Method

$V = 2\pi \int_a bR(x)h(x)dx$
Start with drawing a rectangle parallel to the axis of rotation. The change in the height of this rectangle between a and b is modeled by $h(x)$. The change in the distance of the rectangle from the axis of rotation is modeled by $R(x)$.
Basically integrating an area $R(x) * h(x)$ over a certain interval times 2π.
When using the area between two curves, $h(x) = f(x) - g(x)$ where $f(x) > g(x)$ for [a, b].
For example, if you were trying to find the volume of the area enclosed by $y = \sqrt{x}$, $y = 0$ and $x = 4$ when rotated around

the y-axis,

- $a = 0$
- $b = 4$
- $R(x) = x$, as the distance between parallel rectangles to the y-axis would just be x.
- $h(x) = \sqrt{x}$, as the height of the area above the x-axis is $y = \sqrt{x}$.

So, the integral would read: $V = 2\pi \int_0^4 x\sqrt{x}dx$

2 Integral Applications and Sections

8.2 Integration by Parts

1. Pick a u to derive
2. Pick a dv to integrate
3. $\int u * dv = u * v - \int v * du$

8.3 Trigonometric Integrals

$sincos$ integals $\rightarrow sin^2(x) + cos^2(x) = 1$
$\int sin(x)cos^n(x)dx$
1. Case 1: m is odd $\rightarrow u = cosx$
- Make $sin^2(x) = (1 - cos^2(x))$
2. Case 2: n is odd $\rightarrow u = sinx$
- Make $cos^2(x) = (1 - sin^2(x))$
3. Case 3: m and n are even
- Use the following identities...
$sin^2(x) = \frac{1-cos2x}{2}$ and $cos^2(x) = \frac{1+cos2x}{2}$

Eliminating Square Roots
- Use following identities..

$$1.sin^2x + cos^2x = 1$$
$$2.cos2x = cos^2x - sin^2x$$
$$3.tan^2x + 1 = sec^2x$$
$$4.sin2x = 2sinxcosx$$
$$5.sinx^2 = \frac{1 - cos2x}{2}$$
$$6.cos^2x = \frac{1 + cos2x}{2}$$

Products of sin and cos

$$-sin(mx)sin(nx) = \frac{1}{2}(cos(m-n)x - cos(m+n)x)$$
$$-sin(mx)cos(nx) = \frac{1}{2}(sin(m-n)x + sin(m+n)x)$$
$$-cos(mx)cos(nx) = \frac{1}{2}(cos(m-n)x + cos(m+n)x)$$

8.4 Trigonometric Substitutions

$1.Case \quad 1 : a^2 + x^2$
$\rightarrow x = atan\theta$
$\rightarrow x = asec^2\theta d\theta$
$\rightarrow a^2 + x^2 = a^2 + a^2tan^2\theta = a^2(1 + tan^2\theta) = a^2sec^2\theta$
$2.Case \quad 2 : a^2 - x^2$
$\rightarrow x = asin\theta$
$\rightarrow dx = acos\theta d\theta$
$\rightarrow a^2 - x^2 = a^2 - a^2sin^2\theta = a^2(1 - sin^2\theta) = a^2cos^2\theta$
$3.Case \quad 3 : x^2 - a^2$
$\rightarrow x = asec\theta$
$\rightarrow dx = asec\theta tan\theta d\theta$
$\rightarrow x^2 - a^2 = a^2sec^2\theta - a^2 = a^2(sec^2\theta - 1) = a^2tan^2\theta$

8.5 Partial Fraction Decomposition

*Degree on top must be less than degree on bottom, or long divide

- If there is a linear factor $\rightarrow \frac{A}{x+1} + \frac{B}{(x+1)^2}$
- If there is a quadratic factor $\rightarrow \frac{Ax+B}{x^2+bx+c} + \frac{Cx+D}{(x^2+bx+c)^2}$
- Set $\frac{f(x)}{g(x)}$ equal to the sum of its partial fractions by multiplying
- Equate coefficients of powers of x to solve for variables

$$\frac{-x+3}{x^2-9x+20}$$
$$\frac{-x+3}{(x-4)(x-5)}$$
$$\frac{-x+3}{(x-4)(x-5)} = \frac{A}{x-4} + \frac{B}{x-5}$$
$$\frac{-x+3}{(x-4)(x-5)} * (x-4)(x-5) = \frac{A}{x-4}(x-4)(x-5) + \frac{B}{x-5}(x-4)(x-5)$$
$$-x+3 = A(x-5) + B(x-4)$$
$$x = 5 : -2 = A(0) + B(1) \rightarrow So, B = -2$$
$$x = 4 : -1 = A(-1) + B(0) \rightarrow So, A = 1$$
$$\frac{-x+3}{x^2-9x+20} = \frac{1}{x-4} - \frac{2}{x-5}$$

Density Problems

- 1-dimensional $\rightarrow$ need length or mass (δx meters or grams)

- 2-dimensional $\rightarrow$ need area of a section (rectangle bh where h is δx) (or other shape)

- 3-dimensional $\rightarrow$ need volume of a section (cylinder $= \pi r^2 h$ where h is δy / other shape)

3 Sequences, Series, and Tests

8.8 Improper Integrals

Convergent or Divergent: $\int_1^\infty \frac{1}{x} dx$

- If integral equals a finite number, the integral is convergent

- If integral equals infinity, it is divergent

$$\int_1^\infty \frac{1}{x} dx = \lim_{t\to\infty} \int_1^t \frac{1}{x} dx$$
$$= \lim_{t\to\infty} lnt - ln1$$
$$= \lim_{t\to\infty} \infty - 0 = \infty$$

$\int_1^\infty \frac{1}{x} dx$ is divergent since it equals infinity.

For $\int_1^\infty \frac{1}{x^p} dx$, if:

- $p > 1$, it is convergent

- $p \leq 1$, it is divergent

- Similar to p-series

For integrals with two improper bounds or a vertical asymptote, splitting integral into the sum of two integrals is helpful for integration.

10.1 Sequences

Arithmetic Sequences: $a_n = a_1 + (n-1)d$
Partial Sum of Arithmetic Sequence: $S_n = \frac{a_1+a_n}{2} n$
$d = a_2 - a_1 = a_3 - a_2$
Arithmetic Mean: $\frac{a+b}{2}$
Geometric Sequence: $a_n = a_1 * (r)^(n-1)$
Partial Sum of a Finite Geometric Sequence: $S_n = \frac{a_1(1-r^n)}{1-r}$
$r = \frac{a_2}{a_1} = \frac{a_3}{a_2}$
Sum of Infinite Geometric Sequence: $S = \frac{a_1}{1-R}$
Geometric Mean: $\sqrt{a * b}$
To find if a sequence converges or diverges:
$\lim_{n\to\infty} a_n = L \rightarrow$ converges
$\lim_{n\to\infty} a_n = DNE, \infty, -\infty \rightarrow$ diverges
To find if $a_n = \frac{sin(n)}{n}$ converges or diverges, use the Squeeze Theorem*
Since $-1 \leq sin(n) \leq 1$ and since when all are divided by n, the limit of both the left and right equations are equal to 0, the

limit of $\frac{sin(n)}{n}$ must also be 0.
*Squeeze Theorem states that if $g(x) \leq f(x) \leq h(x)$ and the limit of both $g(x)$ and $h(x) = L$, then the same limit of $f(x)$ must equal L.
Another useful strategy when finding the limit of a sequence is L'hopital's Rule, or taking the derivative of both the top and bottom of the equation and instead finding the limit of that. This helps to remove variables and simplify equations. L'hopital's Rule can be used when limits equal $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

10.2 Infinite Series

An infinite series is simply the sum of an infinite sequence of numbers.
The nth-Term Test for Divergence
$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to]infty} a_n$ fails to exist or is different from zero.
Combining Series
If $\sum a_n = A$ and $\sum b_n = B$ are convergent series, then:

- Sum Rule: $\sum(a_n + b_n) = \sum a_n + \sum b_n = A + B$
- Difference Rule: $\sum(a_n - b_n) = \sum a_n - \sum b_n = A - B$
- Constant Multiple Rule: $\sum ka_n = k \sum a_n = kA$ (any number k)

For Telescoping Series:
-Write out terms to find equation for Partial Sums S_n
-Take limit of value containing n. -If a number can be found, then the series converges. -Partial fraction decomposition can also be used to get terms in Telescoping Series Form.

10.3 The Integral Test

To find $\sum_{n=1}^{\infty} \frac{1}{(n+2)^2}$, since a_n can be described as a $f(n)$, and since f is **positive**, **continuous**, and **decreasing** when $x \geq 1$, the Integral Test can be used.
For $\int_1^{\infty} f(x)dx$, if:
$= N$, Integral converges, Series converges
$= \pm\infty$, Integral diverges, Series diverges
For $f(x)\frac{1}{(x+2)^2}$:

- f is **positive** for every value except $x = -2$, but $x = -2$ does not count as it is below 1.
- f is **continuous** for every value except $x = -2$, but again, this does not count as it is below 1.
- f is **decreasing** as the first derivative of f is negative after $x = -2$.

Since $\lim_{b\to\infty} \int_1^b \frac{1}{(x+2)^2} dx = \frac{1}{3}$, a finite value, the original series converges.

10.4 Comparison Tests (Direct & Limit)

Direct Comparison Test
If $0 \leq a_n \leq b_n$ then the following can be used:
If $\sum b_n$ converges, $\sum a_n$ converges
If $\sum a_n$ diverges, $\sum b_n$ diverges

Limit Comparison Test
If $\lim_{n\to\infty} \frac{a_n}{b_n} = L \to$ both converge or both diverge

4 Sequences, Series, and Tests cont.

10.5 Ratio/Root Tests & Absolute Convergence

Ratio Test: $\lim_{n\to\infty}|\frac{a_n+1}{a_n}|$

- $< 1 \to$ converges
- $> 1 \to$ diverges
- $= 1 \to$ inconclusive

Root Test: $\lim_{n\to\infty}\sqrt[n]{|a_n|}$

- $< 1 \to$ converges
- $> 1 \to$ diverges
- $= 1 \to$ inconclusive

Absolute Value Test:
If $\sum|a_n|$ converges and $\sum a_n$ converges $\to$ absolutely convergent
If $\sum|a_n|$ diverges and $\sum a_n$ converges $\to$ conditionally convergent
Watch OGT Video on Absolute Convergence for some more notes

10.6 Alternating Series and Conditional Convergence

The Alternating Series Test:
The series $\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4 + ...$ converges if the following is satisifed:

- The u_n's are all positive.
- The u_n's are eventually nonincreasing: $u_n \leq$ for all $n \leq N$, for some integer N.
- $u_n \to 0$.

10.7 Power Series

General Form: $\sum_{n=0}^{\infty} a_n(x-c)^n$ where c is the center.
$g(x) = \sum_{n=0}^{\infty} ax^n = \frac{a}{1-x}$
$f(x) = \frac{1}{1-x}$ can be written as the series $\sum_{n=0}^{\infty} x^n$
To find center, set the value being raised to n and solve for x.
Based on the result of $\lim_{n\to\infty}|\frac{U_{n+1}}{U_n}|$:

- $0 \to$ Series converges for all x-values $R = \infty$ $I = (-\infty, \infty)$
- $\infty \to$ Converges when $x = c$ $R = 0$ $I = c$
- $\frac{1}{R}|x-c| < 1$. Converges when $|x-c| < R$. Diverges when $|x-c| > R$. $R = R$ $I = (-R+c, R+c)$.

10.8-9 Taylor and Maclaurin Series and Polynomials

Finding the taylor series of $f(x) = ln(x)$ centered at $c = 1$. First, find the first couple derivates:
$f'(x) = \frac{1}{x}$
$f''(x) = \frac{-1}{x^2}$
$f'''(x) = \frac{2}{x^3}$
$f^4(x) = \frac{-6}{x^4}$
$f(1) = 0, f'(1) = 1, f''(1) = -1, f'''(1) = 2, f^4(1) = -6$
Now, by putting derivates into expanded form: $f(x) = f(c) + f'(c)(x-c)^1 + \frac{f''(c)(x-c)^2}{2!} + \frac{f'''(c)(x-c)^3}{3!}$ it will be easier to put $f(x)$ into summation notation.
$lnx = \sum_{n=0}^{\infty} \frac{(-1)^n(x-1)^{n+1}}{n+1}$
Taylor's Remainder: $Rn(x) = \frac{f^{n+1}(2)(x-c)^{n+1}}{(n+1)!}$

5 Parametric Curves

10.10 Application of Taylor Series

Binomial Series: $(1+x)^k = 1 + \frac{k}{1!}x + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + ...$
Ex. $f(x) = \frac{1}{(1+x)^2} = (1+x)^{-2}$
$(1+x)^{-2} = 1 - 2x + \frac{(-2)(-3)}{2!}x^2 + \frac{(-2)(-3)(-4)}{3!}x^3 + ...$
$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 + ...$
$(1+x)^{-2} = \sum_{n=0}^{\infty}(-1)^n(n+1)x^n$

6.3 Arc Length

Pythagorean's Theoreom can be applied to find the length of a segment $f(x)$. If ds is equal to a single straight segment in $f(x)$, then dx is equal to the horizontal length and dy is equal to its vertical length.

$$\begin{aligned} (ds)^2 &= (dx)^2 + (dy)^2 \\ \sqrt{(ds)^2} &= \sqrt{(dx)^2 + (dy)^2} \\ ds &= dx\sqrt{(dx)^2/(dx)^2 + (dy)^2/(dx)^2} \\ ds &= \sqrt{1 + (dy/dx)^2}dx \end{aligned} \tag{1}$$

By taking the integral of this, you can get the total length of the segment.

$$s = \int_a^b \sqrt{1 + (f(x))^2}\, dx \tag{2}$$

6.4 Surface Area

If the function $f(x) \geq 0$ is **continuously differentiable** on $[a, b]$, the area of the surface generated by revolving $f(x)$ around the x-axis is $S = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}dx$. Similarly, the area of the surface generated when rotating $g(y)$ around the y-axis is $S = \int_c^d 2\pi g(y)\sqrt{1 + (g'(y))^2}dy$.

11.1 Parametric Curves

If x and y are given as functions $x = f(t)$ and $y = g(t)$ over an interval I then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a parametric curve. The equations are the parametric equations.
For $x = t^2$ and $y = t + 1$:

t	x	y
-3	9	-2
-2	4	-1
-1	1	0
0	0	1
1	1	2
2	4	3
3	9	4

These points can then be plotted to graph the parametric curve.
Parametric Curves can also be combined into Cartesian Equations. However, since parametric curves are only on certain intervals of t, when t is removed to find the Cartesian Equation, the curve will usually continue infintely.

11.2 Calculus of Parametric Curves

After finding $\frac{dy}{dt}$ and $\frac{dx}{dt}$, $\frac{dy}{dx}$ can be found by dividing $\frac{dy}{dt}$ by $\frac{dx}{dt}$.
To find second derivative, find the derivative of the first derivative ($\frac{dy}{dx}$) and divide this by $\frac{dx}{dt}$. : $\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}}$.
A positive second derivative means the curve is concave up, while a negative second derivative means the curve is concave down.
To find the length of curve C defined parametrically by $x = f(t)$ and $y = g(t)$, $a \leq t \leq b$, where f' and g' are continuous and not simultaneously zero on $[a, b]$, and C is traversed exactly once as t increase from $t = a$ to $t = b$, then the length of C can be found to be $\int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}dt$
Area bounded by parametric equation $(f(t), g(t))$ derivation:
$A = \int_a^b f(x)dx$
Since $f(x) = y$, $y = g(t)$, and $dx = f'(t)dt$, the equation can be rewritten as $A = \int_\alpha^\beta g(t)f'(t)dt$, where α and β are t-values.
Surface Area:
Revolution around the x-axis ($y \geq 0$): $S = \int_a^b 2\pi y\sqrt{[f'(t)]^2 + [g'(t)]^2}dt$
Revolution around the y-axis ($x \geq 0$): $S = \int_a^b 2\pi x\sqrt{[f'(t)]^2 + [g'(t)]^2}dt$

6 Polar Coordinates

11.3 Polar Coordinates

Method of describing lines originating from the origin by giving (r, θ) where r is the length of the segment and θ is the angle from the positive x-axis. Points in polar coordinates can be represented in multiple ways. For example, $P(2, \frac{\pi}{4}) = P(-2, \frac{5\pi}{4})$. The following equations relate Polar and Cartesian Coordinates and simplify moving between the two: $x = rcos\theta, y = rsin\theta, r^2 = x^2 + y^2, tan\theta = \frac{y}{x}$.

11.4 Graphing Polar Coordinate Equations

To find the slope of a polar curve $r = f(\theta)$, the following equation must be used: $\frac{dy}{dx} = \frac{f'(\theta)sin\theta + f(\theta)cos\theta}{f'(\theta)cos\theta - f(\theta)sin\theta}$. Notice that when $f(\theta) = 0$, $\frac{dy}{dx} = tan\theta$.

11.5 Areas and Lengths in Polar Coordinates

For the length of a polar curve, $L = \int_\alpha^\beta \sqrt{r^2 + (\frac{dr}{d\theta})^2}d\theta$ can be used for the interval α to β. For finding the area of polar coordinates, $A = \int_\alpha^\beta \frac{1}{2}r^2 d\theta$ where α and β are angular bound that can be found by setting r to 0.

Complex Numbers

Standard form = $a + bi$. a is real portion, bi is imaginary. To graph: 3 units to the right, imaginary is 4 so 4 units up. This is called an Argand Diagram. To calculate absolute value, or **modulus**, $|a + bi| = \sqrt{a^2 + b^2}$
For -5 + 12i:
Graph 5 units to the left, and up 12 units.
$|-5 + 12i| = \sqrt{(-5)^2 + 12^2}$
$|-5 + 12i| = \sqrt{(25 + 144}$
$|-5 + 12i| = \sqrt{169}$
$|-5 + 12i| = 13$

For addition and subtraction of complex numbers, simply add or subtract the real and imaginary parts separately: $(5+4i)+(3-7i) = 8-3i$. For multiplying a real number and imaginary number, apply the distributive property: $5*(6-7i) = 30-35i$. For multiplying two complex numbers, make sure to FOIL: $(1-i)*(-3+2i) = -3+2i+3i-2i^2 = -1+5i$. The **complex conjucate** ($\overline{z}$) of a complex number has the same real part but opposite imaginary part: the conjucate of $5+5i$ is $5-5i$. The rectangular form of numbers such as $\frac{1}{i}$ can be found by multiplying and dividing by the complex conjucate. Therefore

when dividing complex number z_1 by z_2, $\frac{z_1}{z_2} = z_1 * \frac{\overline{z_2}}{|z_2|^2}$.
An example of how a square root of negative looks: $\sqrt{-80} = i * \sqrt{16} * \sqrt{5} = 4i\sqrt{5}$
$i = i^5 = \sqrt{-1}$
$i^2 = i^6 = -1$
$i^3 = i^7 = -i$
$i^4 = i^8 = 1$
Idea: Associate polar coordinates to complex numbers
$z = x + iy$
$r = |z| = \sqrt{x^2 + y^2}$
if $x \geq 0$, $\theta = arctan\frac{y}{x}$
if $x \leq 0$, $\theta = arctan\frac{y}{x} + \pi$
if $x = 0, \theta = \pm\frac{\pi}{2}$
Find polar coordinates of $z = -2\sqrt{2} + 2\sqrt{2}i$
$r = |z| = \sqrt{(-2\sqrt{2})^2 + (2\sqrt{2})^2} = \sqrt{16} = 4$
Since $x \leq 0$, $\theta = arctan\frac{y}{x} + \pi = \frac{3\pi}{4}$
Polar Coordinates $= (4cos\frac{3\pi}{4}, 4sin\frac{3\pi}{4})$
Euler's Formula for any θ is $e^{i\theta} = cos\theta + isin\theta$. Using this, the **exponential form** of a complex number with polar coordinates (r, θ) is $z = |z|e^{i\theta}$.
Therefore, $z_1 z_2 = r_1 r_2(cos(\theta_1 + \theta_2) + isin(\theta_1 + \theta_2))$. This also shows that $\frac{z_1}{z_2} = \frac{r_1}{r_2}e^{i(\theta_1 - \theta_2)}$. De Moivre's Formula: If $z \neq 0$ has polar coordinates $(|z|, \theta)$, for any integer $n \geq 0$: $z^n = |z|^n e^{in\theta} = |z|^n(cos(n\theta) + isin(n\theta))$ and $z^{-n} = |z|^{-n}e^{-in\theta} = |z|^{-n}(cos(n\theta) - isin(n\theta))$. So, if $z = cos\theta + isin\theta$, $(|z| = 1)$, then $(cos\theta + isin\theta)^n = (cos(n\theta) + isin(n\theta))$. For roots of complex numbers, $z_k = \sqrt[n]{r}[cos\frac{\theta + 2\pi k}{n} + isin\frac{\theta + 2\pi k}{n}]$ where k is the amount of roots.