CS 205 Homework 3

Keith Lehman, kpl56@scarletmail.rutgers.edu

March 5, 2021

1. Prove the following distributive law: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
If the left side were a set, it would contain all of A, as well as the intersection of B and C. So for the set L that is the left side, $A \subseteq L$, and $(B \cap C) \subseteq L$.
For the set R that is the right side, the union of everything in $A+B$ and everything in $A+C$ would contain everything in A since everything in A is also in A. It would also contain all elements at the intersection of B and C. So, like L, $A \subseteq R$, and $(B \cap C) \subseteq R$. Since both sides have the exact same components, they must be equal.

2. Suppose $A \subseteq B$ and $C \subseteq D$. Show that $A \times C \subseteq B \times D$.
Since A is a subset of B and C is a subset of D, $\forall x(x \in A \implies x \in B) \wedge \forall y(y \in C \implies y \in D)$. Therefore, any products that can be created between A and C will also be products that can be created between B and D. Therefore, $(A \times C \subseteq B \times D)$.

3. Prove that $A - (B \cap C) = (A - B) \cup (A - C)$.
The left side is A minus any elements of B that are also in C. The right side is two operations, $A - B$ and $A - C$. After doing both of these and creating a set of the union of both, what will remain is A minus the elements of B that are also in C because for example, for $x \in A \wedge x \in B \wedge x \notin C$, x will be subtracted from A in $A - B$ but not in $A - C$, so $(A - B) \cup (A - C)$ will still contain x becuase it will not be removed from A in $A - C$. Only elements that are in both B and C will not be included, alike the left side.

4. An ordered pair (a, b) can be defined as the set a, a, b. Show that $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$. For an ordered pair to be equal to a, a, b, by the definition of an ordered pair, the first term of the ordered pair must be equal to a and the second must be equal to b. There are no exceptions to this. Therefore, for $(a, b) = (c, d)$, c must equal a and d must equal b.

5. Let $f : R \to R$ be defined by $f(x) = 4x^3 - 2$.
   (a) Is f injective?
   f is injective since it maps to each member of the codomain once and never more than once.
   (b) Is f surjective?
   f is surjective since it maps to each member of the codomain once.

6. Let $S = P(R)$. Let $f : R \to S$ be defined by $f(x) = y \in R : y^2 < x$.
   (a) Is f injective?
   f is not injective because for any given value of x, there are multiple values of y that can satisfy the equation.
   (b) Is f surjective? f is not surjective. While all members of the reals are mapped to, the empty set is not and cannot be mapped to with the given function.

7. Suppose $f : A \to B$ and $g : B \to C$.
   (a) Prove that if f is surjective and g is not injective, $g \circ f$ is not injective.
   While every element of B is mapped to by an element of A, since g is not injective, there are elements of C that are mapped to more than once by values of B. Therefore, for $g \circ f = A \to B \to C$, there will be multiple elements of A that will map to the same element in C, making $g \circ f$ not injective.
   (b) Prove that if f is not surjective and g is injective, $g \circ f$ is not surjective.
   While there are no elements of C mapped to more than once by elements of B, not all elements of B are mapped to by elements of A. Therefore, since each element in B maps to a unique element in C, but not every element in B is mapped to by an element of A, there will be elements of C not mapped to by elements of A, making $g \circ f$ not surjective.

8. Suppose $f : B \to C$, $g : A \to B$, and $h : A \to B$. If f is injective, prove that if $f \circ g = f \circ h$, then $g = h$.
   f being injective means that no elements of C are mapped to more than once by B. So, only one element of A can map to an element of B. Since this is true, g and h have to be equal in order for $f \circ g = f \circ h$ to be true as g and h both map $A \to B$.

9. Suppose $f : A \to B$ is injective. Show that there exists a $B' \subseteq B$ such that $f^-1 : B' \to A$.
   Since f is injective, there are no two elements of A that map to the same element in B. Since it was not stated that f is surjective, we can assume that A is surjective to a certain subset of B that we can call B'. This subset B' will be able to map to every element of A, making f^-1 valid.