CS 205 Homework 3

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March 7, 2021

1. Prove the following distributive law: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   If the left side were a set, it would contain all of A, as well as the intersection of B and C. So for the set L that is the left side, $A \subseteq L$, and $(B \cap C) \subseteq L$.
   For the set R that is the right side, the union of everything in $A + B$ and everything in $A + C$ would contain everything in A since everything in A is also in A. It would also contain all elements at the intersection of B and C. So, like L, $A \subseteq R$, and $(B \cap C) \subseteq R$. Since both sides have the exact same components, they must be equal.

2. Suppose $A \subseteq B$ and $C \subseteq D$. Show that $A \times C \subseteq B \times D$.
   Since A is a subset of B and C is a subset of D, $\forall x(x \in A \implies x \in B) \land \forall y(y \in C \implies y \in D)$. Therefore, any products that can be created between A and C will also be products that can be created between B and D. Therefore, $(A \times C \subseteq B \times D)$.

3. Prove that $A - (B \cap C) = (A - B) \cup (A - C)$.
   The left side is A minus any elements of B that are also in C. The right side is two operations, $A - B$ and $A - C$. After doing both of these and creating a set of the union of both, what will remain is A minus the elements of B that are also in C because for example, for $x \in A \land x \in B \land x \notin C$, x will be subtracted from A in $A - B$ but not in $A - C$, so $(A - B) \cup (A - C)$ will still contain x becuase it will not be removed from A in $A - C$. Only elements that are in both B and C will not be included, alike the left side.

4. An ordered pair (a, b) can be defined as the set a, a, b. Show that $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$. For an ordered pair to be equal to a, a, b, by the definition of an ordered pair, the first term of the ordered pair must be equal to a and the second must be equal to b. There are no exceptions to this. Therefore, for $(a, b) = (c, d)$, c must equal a and d must equal b.

5. Let $f : R \to R$ be defined by $f(x) = 4x^3 - 2$.
   (a) Is f injective?
   f is injective since it maps to each member of the codomain once and never more than once.
   (b) Is f surjective?
   f is surjective since it maps to each member of the codomain once.

6. Let $S = P(R)$. Let $f : R \to S$ be defined by $f(x) = y \in R : y^2 < x$.
   (a) Is f injective?
   f is not injective because for any given value of x, there are multiple values of y that can satisfy the equation.
   (b) Is f surjective? f is not surjective. While all members of the reals are mapped to, the empty set is not and cannot be mapped to with the given function.

7. Suppose $f : A \to B$ and $g : B \to C$.
   (a) Prove that if f is surjective and g is not injective, $g \circ f$ is not injective.
   While every element of B is mapped to by an element of A, since g is not injective, there are elements of C that are mapped to more than once by values of B. Therefore, for $g \circ f = A \to B \to C$, there will be multiple elements of A that will map to the same element in C, making $g \circ f$ not injective.
   (b) Prove that if f is not surjective and g is injective, $g \circ f$ is not surjective.
   While there are no elements of C mapped to more than once by elements of B, not all elements of B are mapped to by elements of A. Therefore, since each element in B maps to a unique element in C, but not every element in B is mapped to by an element of A, there will be elements of C not mapped to by elements of A, making $g \circ f$ not surjective.

8. Suppose $f : B \to C$, $g : A \to B$, and $h : A \to B$. If f is injective, prove that if $f \circ g = f \circ h$, then $g = h$.
   f being injective means that no elements of C are mapped to more than once by B. So, only one element of A can map to an element of B. Since this is true, g and h have to be equal in order for $f \circ g = f \circ h$ to be true as g and h both map $A \to B$.

9. Suppose $f : A \to B$ is injective. Show that there exists a $B' \subseteq B$ such that $f^-1 : B' \to A$.
   Since f is injective, there are no two elements of A that map to the same element in B. Since it was not stated that f is surjective, we can assume that A is surjective to a certain subset of B that we can call B'. This subset B' will be able to map to every element of A, making f^-1 valid.

10. Is $\{A_0, A_1, A_2\}$ a partition of Z?
    The given set is a partition of Z because A_0 does not have any elements in common with A_1 or A_2, which also do not have any common elements between one another. For any integers used for k, all three sets will still have completely unique elements between one another. They will combine to form S without any overlap.