EXAMPLES OF IMPLICITIZATION OF HYPERSURFACES THROUGH SYZYGIES

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write abstract here — only after you have an introduction

BIOGRAPHICAL SKETCH

Radoslav was born in Burgas, a city on the Black Sea cost. He attended the High School of Mathematics and the Sciences there, graduating in May 2005, and later Jacobs University Bremen, earning a Bachelor of Science degree in June 2008. He joined the Department of Mathematics at Cornell in August 2010.

To my parents.

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CHAPTER 1
INTRODUCTION

The implicitization problem asks for finding the equations defining the image of a rational map into some projective space $\mathbb{P}^n$. This is an old problem in algebraic geometry which was effectively solved in the 60s by Gröbner bases. However, this method has two intrinsic shortcomings. The first is its computational complexity. Even relatively small examples may require, using today's computational capacity, running time in the hundreds of years or more. The second problem is that geometrically it represents a black box.

CHAPTER 2

PRELIMINARIES

2.1. The goal of this chapter is to set up the notation and recall some basic facts which we shall need later on. The first section defines the main objects of interest and establishes some standard notation and terminology. Section 2 is a short and elementary treatment on strands of maps of free modules in the way we shall need them. Section 3 is solely devoted to the notion of multiplicity. It comes in many flavors and some care is needed when dealing with it.

Because notation is somewhat heavy and appears rather unmotivated at first, we present the gist of the first three sections through few examples in the last fourth section.

2.1 Notation

2.2. Define what a projective toric variety is.

2.3. Define the Cox ring of X as in **?**.

2.4. From now on we use the notation $\mathrm{Proj}(S, \mathfrak{n})$ for the construction of a projective toric variety from its Cox ring. In the cases where the irrelevant ideal $\mathfrak{n}$ is the usual one, for example for products of projective spaces, or is not relevant for the discussion, we shall simply write $\mathrm{Proj}(S)$. This is feasible because the construction coincides with the usual construction in case S is a quotient of a standard graded polynomial ring.

2.5. Let X be a smooth projective toric variety of dimension $n-1$ ($n > 1$). Let S be its Cox ring (2.3) and let S' be any homogeneous coordinate ring furnishing an embedding of X into projective space, i.e. $X = \mathrm{Proj}(S')$ with

$$S' = \mathbb{C}[s_0, \ldots, s_m]/\mathfrak{p}$$

for some homogeneous prime $\mathfrak{p}$ of height $m-n+1$. Let $T = \mathbb{C}[\boldsymbol{x}]$ be the homogeneous coordinate ring of $\mathbb{P}^n$.

Let $\mathscr{L}$ be a line bundle on X such that $h^0(\mathscr{L}) > n$. Because X is toric, there is a degree $\boldsymbol{e}$ on S such that $\mathscr{L} \cong \mathscr{O}_X(\boldsymbol{e})$. We shall sometimes write $\boldsymbol{e} \in \mathrm{Pic}(X)$.

Under the identification above, suppose that

$$\phi_0, \dots, \phi_n \in H^0(X, \mathscr{L}) = S_{\boldsymbol{e}}$$

are linearly independent and consider the rational map

$$\phi = (\phi_0, \dots, \phi_n) : X \longrightarrow \mathbb{P}^n = \mathrm{Proj}(T)$$

We denote by J the ideal of the coordinates ϕ_j,

$$J = \langle \phi_0, \dots, \phi_n \rangle \subset S$$

We assume from now on that the closed image of ϕ, which we denote by Y, is of dimension $n-1$. Since X is reduced and irreducible, so is Y and, in particular, $Y = V(P)$ for some principal homogeneous prime ideal $P \subset T$,

$$Y = \mathrm{image}(\phi) = V(P)$$

If we want to refer to a generator of P, necessarily up to a unit in T, we shall dereference the ideal by writing $P(\boldsymbol{x})$.

We refer to S as the Cox ring of source, to S' as its homogeneous coordinate ring, and to T as the homogeneous coordinate ring of the target.

Remark 2.6. Throughout this thesis, the term *image of a rational map*, as used in (2.5), means the scheme-theoretic image, also called the closed image. Formally,

$$Y = V(\ker \phi^{\#}), \quad \phi^{\#} : \mathscr{O}_{\mathbb{P}^n} \longrightarrow \phi_* \mathscr{O}_X$$

In our situation, is just the closure of the set-theoretic map on closed points of X.

2.7. Let $R = S \otimes T = S[\boldsymbol{x}]$. Then R is naturally bigraded by

$$\deg(af) = (\boldsymbol{d}, i)$$

whenever $a \in S_{\boldsymbol{d}} \subset R$ and $f \in T_i \subset R$. Let $S[t]$ be similarly graded, setting $\deg(t) = (-\boldsymbol{e}, 1)$. The blow-up algebras, $\mathrm{Rees}_S(J)$ and $\mathrm{Sym}_S(J)$, naturally become factor rings of R as follows.

The Rees algebra is the image of the bigraded map of S-algebras

$$\beta : R \longrightarrow S[t] : \quad x_j \mapsto \phi_j \cdot t$$

We set $I = \ker(\beta) \subset R$ to be the Rees ideal. It is the bigraded ideal of R describing the polynomial relations between the selected set of generators $\phi_0, \ldots, \phi_n$ for J, that is,

$$I = \langle \textstyle\sum_{|\alpha|=i} a(\boldsymbol{s})\boldsymbol{x}^\alpha \ : \ \textstyle\sum_{|\alpha|=i} a(\boldsymbol{s})\phi^\alpha = 0 \ ; \forall i \rangle \subset R$$

For the sake of brevity, we set $B = \mathrm{Rees}_S(J)$ as an S-algebra.

While $\mathrm{Rees}_S(J)$ captures all polynomial relations on the ϕ_j, the symmetric algebra $\mathrm{Sym}_S(J)$ captures only the linear ones. Specifically, we have

$$\mathrm{Sym}_S(J) = R / \langle \textstyle\sum_j a(\boldsymbol{s}) x_j \ : \ \textstyle\sum_j a(\boldsymbol{s}) \phi_j = 0 \rangle$$

We shall sometimes refer to R as the *ambient* ring of the blow-up algebras.

2.8. The ambient ring $R = S[\boldsymbol{x}]$ and the blow-up algebras just defined are in essence geometric objects. As $\mathbb{C}$-algebras, R corresponds to the bihomogeneous coordinate ring of the product of the source and the target varieties, and $B = \mathrm{Rees}_S(J)$ — to the coordinate ring of the graph of ϕ, $\Gamma(\phi)$ **??**, defined as the closure of

$$\{(q, \phi(q)) : q \in X\} \subset X \times \mathbb{P}^n$$

The natural surjective morphisms $R \longrightarrow \mathrm{Sym}_S(J) \longrightarrow \mathrm{Rees}_S(J)$ induce natural closed embeddings

$$\Gamma(\phi) = \mathrm{Biproj}(\mathrm{Rees}_S(J)) \longrightarrow \mathrm{Biproj}(\mathrm{Sym}_S(J)) \longrightarrow \mathrm{Biproj}(R) = X \times \mathbb{P}^n \qquad (2.8.1)$$

2.9. Let X be a variety and $q \in X$ be a smooth point, that is, such that the stalk $\mathscr{O}_q$ is regular local. Let $Z \subset X$ be a closed subscheme containing the point q. We say that q is a complete intersection (c.i.) point of Z if the stalk of q on Z, $\mathscr{O}_{q,Z}$, is a complete intersection factor ring of the regular local $\mathscr{O}_q$. Now suppose $Z \subset X$ is of codimension d at q and that the stalk $O_{q,Z}$ can be defined out of $\mathscr{O}_q$ by $d+1$ elements. In this case, we say that q is an almost complete intersection (a.c.i.) point of Z.

Note that the notion of a complete intersection ring does not require an ambient regular local ring but we shall not need this fact here.

2.10. Suppose that the base locus of the rational map ϕ, $V(J)$, is zero-dimensional. Then it is known that the first embedding in (2.8.1) is an isomorphism if and only if $V(J)$ is a locally complete intersection scheme (e.g. **?**, Proposition 4.14), that is, all its points are c.i. points.

The two equivalent conditions on J are sometimes expressed by saying that J is of linear type. We shall not use this notation.

2.11. Before we take the interpretation of the ideal I as the homogeneous polynomial syzygies on the ϕ_j any further, we note that its graded pieces for a fixed degree $\boldsymbol{d}$ on the source are finite as T-modules.

More generally, let M be any finite graded R-module. We denote by

$$M_{\boldsymbol{d},\bullet} = \bigoplus_{i \in \mathbb{Z}} M_{\boldsymbol{d},i}$$

the graded piece of M in degree $\boldsymbol{d}$ as a T-module, and note that

$$M = \bigoplus_{\boldsymbol{d}} M_{\boldsymbol{d},\bullet}$$

is a decomposition of M as a T-module. If $\{g_k : k\}$ is a finite generating set for M over R and $\{\boldsymbol{s}_\alpha : \alpha\}$ are the monomials in S, then

$$\{\boldsymbol{s}_\alpha \cdot g_k : \deg(\boldsymbol{s}_\alpha) + \deg_S(g_i) = \boldsymbol{d}; \quad \forall k, \alpha\}$$

is a finite generating set for $M_{\boldsymbol{d},\bullet}$ over T.

Since I is homogeneous in the bigraded noetherian R, its graded pieces $I_{\boldsymbol{d},\bullet}$ are finite T-modules. We shall sometimes call these graded pieces strands, akin to the equivalence between graded modules and complexes.

2.12. For a fixed degree $\boldsymbol{d}$ on the source, the elements of $I_{\boldsymbol{d},i}$ are degree-i syzygies of the ϕ_j with (module-)coefficients in $S_{\boldsymbol{d}}$. For specific degrees $\boldsymbol{d}$ and $i = 1,2$, these have been called moving planes and quadrics, respectively [references]. We shall only use this notation in Chapter 7 when we present a sort of a "template proof" and use prove two results relating our work to what is already known in the case of basepoint-free maps over $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$, as well as a couple of new results along those lines.

2.13. Throughout this thesis we work with a fixed monomial order on S (or rather S'). For example, this could be the graded lexicographic order but the specific choice is immaterial. With this convention, the statement

let $\boldsymbol{b}$ be a row-vector corresponding to the basis of $S_{\boldsymbol{d}}$

would mean a row vector having having as its coordinates the monomials of S, listed in the selected order.

In this sense, and fixing some basis on T too, we shall write

$$\texttt{basis}(S_{\boldsymbol{d}}), \quad \texttt{basis}(T_i), \quad \texttt{basis}(R_{\boldsymbol{d},i})$$

for the row vectors consisting of monomial bases in the chosen order for the $\mathbb{C}$-vector spaces $S_{\boldsymbol{d}}, T_i$ and $R_{\boldsymbol{d},i}$, respectively.

2.14. Let $r = \dim_{\mathbb{C}}(S_{\boldsymbol{d}})$ and let $\boldsymbol{b} = \texttt{basis}(S_{\boldsymbol{d}})$ be the row-r-vector of the monomial basis of S in degree $\boldsymbol{d}$. We shall mostly use r instead of $\dim_{\mathbb{C}}(S_{\boldsymbol{d}})$ when $\boldsymbol{d}$ has been fixed.

Given a form $g(\boldsymbol{x}) = g(\boldsymbol{s};\boldsymbol{x}) \in I_{\boldsymbol{d},i}$, that is, a syzygy of degree i over $S_{\boldsymbol{d}}$, we can write $g(\boldsymbol{x})$ as

$$g(\boldsymbol{x}) = \boldsymbol{b} \cdot C = \begin{bmatrix} x_0^i & x_0^{i-1}x_1 & \dots & x_n^i \end{bmatrix} \cdot C'$$

where C is an $r \times 1$ column vector with entries in T_i, and C' is a $\binom{n+i}{n} \times 1$ column vector with entries in $S_{\boldsymbol{d}}$.

We shall use the term syzygy both for the column vector C and the form g in the Rees ideal.

2.15. We now come to the most important bit of notation. Let us fix a degree $\boldsymbol{d}$ on the source. As already apperent from (2.14), identifying the generators of $I_{\boldsymbol{d}}$ with column-r-vectors, $I_{\boldsymbol{d}}$ becomes a sub-T-module of the free graded T^r.

Let N be the matrix built from those column-generators. Setting $\mu = \mu(I_{\boldsymbol{d},\bullet})$ to be the number of columns, N becomes an $r \times \mu$-matrix over T and more importantly, a graded T-linear map

$$N : \bigoplus_k T(-i_k) \longrightarrow T^r$$

whose image is just $I_{\boldsymbol{d},i}$.

By grouping the columns corresponding to the same degree i into submatrices N_i, for each valid i we get a single matrix whose columns are the degree i-syzygies over $S_{\boldsymbol{d}}$. Clearly, N_i is empty for any i larger than the maximum degree δ of a minimal generator of $I_{\boldsymbol{d},\bullet}$. By the assumption on the linear independence of the ϕ_j, N_0 is empty too. In any case, the N_i fit together to give N,

$$N = (N_1 \mid N_2 \mid \ldots \mid N_\delta)$$

In Chapter 7 we describe the close connection of N_1 and N_2 to the matrices used in the moving planes and quadrics results.

Finally, whenever useful, we write h_i for $\dim_{\mathbb{C}} \operatorname{Span}(N_i) = \dim_{\mathbb{C}}(I_{\boldsymbol{d},i})$ and h for the tuple $(h_1, \ldots, h_\delta)$.

2.16. Recall that for any ideal Q in a unique factorization domain T, $\gcd(Q)$ is defined to be the unique minimal principal ideal which contains Q, and this definition obviously generalizes the definition on elements when T is a Euclidean domain.

2.17. As a final bit of notation, we mention that we use $\operatorname{rad}(-)$ for the radial, and $\operatorname{sat}(-)$ for the saturation with respect to the irrelevant ideal.

2.2 Strands of Module Maps

2.18. Let us consider the coordinates of ϕ as a row vector over S. We get a graded S-linear map

$$\begin{bmatrix} \phi_0 & \phi_1 & \ldots & \phi_n \end{bmatrix} : S^{n+1} \longrightarrow S^1(\boldsymbol{e})$$

where $S(\boldsymbol{e})$ has the usual meaning of putting $1 \in S$ in degree $-\boldsymbol{e}$. Equally well, we can consider the graded S-linear map introduced by the quadratic monomials of the coordinates,

$$\begin{bmatrix} \phi_0^2 & \phi_0\phi_1 & \dots & \phi_n^2 \end{bmatrix} : S^{(n+2)(n+1)/2} \longrightarrow S^1(2\boldsymbol{e})$$

As already mentioned in (2.15), in the case $n = 3$, these two maps have been studied in relation to the methods of moving planes and quadrics. In this section we consider all such S-linear maps coming from ϕ, that is, not just those coming from monomials of degree 1 and 2.

2.19. Let k be a positive integer and $\boldsymbol{d}$ be a fixed degree on S such that $S_{\boldsymbol{d}} \neq 0$. Define $\phi^{(k)}$ to be the graded S-linear map formed by the coordinates of ϕ,

$$\phi^{(k)} = \begin{bmatrix} \phi_0^k & \phi_0^{k-1}\phi_1 & \dots & \phi_n^k \end{bmatrix} : S^{\binom{n+k}{n}} \longrightarrow S^1(k\boldsymbol{e})$$

and set $\Phi^{(k)}$ be the strand of $\phi^{(k)}$ in degree $\boldsymbol{d}$, that is,

$$\Phi^{(k)} : S_{\boldsymbol{d}}^{\binom{n+k}{n}} \longrightarrow S^1_{k\boldsymbol{e}+\boldsymbol{d}}$$

is a map of complex vector spaces.

Choosing bases (2.14), we can think of $\Phi^{(k)}$ as a matrix over $\mathbb{C}$ of size

$$\dim_{\mathbb{C}}(S_{k\boldsymbol{e}+\boldsymbol{d}}) \times r\binom{n+k}{n}$$

whose columns can be indexed by the monomials in $R_{\boldsymbol{d},k}$.

Example 2.20. Give a small example to clarify this.

2.21. The advantage of the matrices $\Phi^{(k)}$ over the matrices $\phi^{(k)}$ is that the kernel of $\Phi^{(i)}$ corresponds directly to the degree-i syzygies over $S_{\boldsymbol{d}}$. That is, for a fixed $\boldsymbol{d}$,

$$\boldsymbol{v} \mapsto \texttt{basis}(R_{\boldsymbol{d},i}) \cdot \boldsymbol{v} : \ker(\Phi^{(i)}) \longrightarrow I_{\boldsymbol{d},i}$$

is an isomorphism of vector spaces.

2.3 Multiplicity

2.22. We recall some notation from **?**. For a homogeneous prime ideal P in a graded ring T, we set $T_{(P)}$ to be the degree-0 graded piece of the localization of T at the homogeneous elements outside of P. If P is a minimal prime of a graded T-module M, we denote by $\mathrm{mult}_P(M)$ the length of the T_P-module M_P (see *loc. cit.*, I, Proposition 7.4).

2.23. Let Z be a zero-dimensional closed subscheme of a smooth projective variety X. Let $\mathscr{J} \subset \mathscr{O}_X$ be the ideal sheaf of Z and let $q \in Z$ be any point. Since Z is zero-dimensional, $\mathscr{J}_q \subset \mathscr{O}_{q,X}$ is an ideal of definition in the regular local $\mathscr{O}_{q,X}$.

Define the multiplicity of Z at q, denoted e_q, to be the Hilbert-Samuel multiplicity of $\mathscr{J}_q$. Define the degree of Z at q, denoted d_q, to be the length of the local ring,

$$\mathrm{length}(\mathscr{O}_{q,Z}) = \dim_{\mathbb{C}}(\mathscr{O}_{q,Z})$$

We have that $d_q \leq e_q$ with equality if and only if q is a local complete intersection point (**?**, Theorem 4.7.4).

2.24. We denote by $\widetilde{J}$ the ideal sheaf arizing from J.

2.25. Let $\mathscr{L}$ be a line bundle on X. We denote by $[\mathscr{L}]$ the class of $\mathscr{L}$ in the Chow ring of X. Since $\dim(X) = n-1$, we have that $[\mathscr{L}]^{n-1} \in \mathbb{Z}$. Suppose that the baselocus Z is zero-dimensional. Then by [Fulton, Proposition 4.4], see [Cox 2001] for details, we have the following

$$\deg(\phi)\deg(Y) = [\mathscr{O}(\boldsymbol{e})]^{n-1} - \sum_{q \in Z} e(\widetilde{J}_q, \mathscr{O}_q) \tag{2.25.1}$$

Example 2.26. The following cases are easy to analize: $\mathbb{P}^n$, $(\mathbb{P}^1)^n$.

Example 2.27. Here is a list of ways to compute the degree and multiplicity, and also to show how not to compute them.

- Both notions are geometric which means that we need to saturate our ideals before computing the degree and multiplicity.
- For both, the sum of the values at the points in a zero-dimensional scheme add to the corresponding notion for the scheme. This works better with degree.
- Just typing `degree`(J) and `multiplicity`(J) in Macaulay2 often has dissasterous results.
- If $X = \mathbb{P}^n$ and J is saturated, then `degree`(J) is in fact the degree of the zero-dimensional scheme. If the scheme is supported on a single point, then that's the degree of the point. Else, saturate with respect to the prime ideals of the other points.
- We can freely work over non-algebraically closed fields. Points in the same Galois closure are algebraically indistinguishable, so just assume they split equally to closed points over $\mathbb{C}$, or the algebraic closure.
- `multiplicity`(J) does work in projective space for an ideal supported on a single point. And the implementation uses a cool trick — the reference I have is due to Eisenbud.

2.4 Examples

In this section we try to summarize the notation and definitions introduced so far by detailing them in a couple of explicit examples.

Example 2.28 (`ex203`)**.** Let $X = \mathbb{P}^1_{s,u} \times \mathbb{P}^1_{t,v}$ be the product of two projective lines and let $S = \mathbb{C}[s,u;t,v]$ be its Cox ring—a homogeneous coordinate ring graded by $\mathrm{Pic}(X)$ such that $\deg(s) = \deg(u) = (1,0)$ and $\deg(t) = \deg(v) = (0,1)$. We write $X = \mathrm{Proj}(S)$ for the construction in the toric setting.

Let $\mathscr{L} = \mathscr{O}_X(2,2)$ be the line bundle on X given by twisting the structure sheaf by $\boldsymbol{p} = (2,2)$, either as an element in $\mathrm{Pic}(X) = \mathbb{Z}^2$ or as a degree on S. Since $h^0(\mathscr{L}) = 9$, we can choose linearly independent over $\mathbb{C}$ global section $\phi_0, \dots, \phi_3$ of $\mathscr{L}$ to get a rational map

$$\phi = (\phi_0, \dots, \phi_3) : X \longrightarrow \mathbb{P}^3 = \mathrm{Proj}(T)$$

where $T = \mathbb{C}[x_0, \dots, x_3]$.

Since the sections are just $(2,2)$-forms, we can form the graded ideal

$$J = \langle \phi_0, \dots, \phi_3 \rangle \subset S$$

and since we are only be interested of maps whose image is of full dimension, we can assume that the base locus of ϕ, $Z = V(J)$, is zero-dimensional. In particular, ϕ is a morphism of schemes away from $Z = V(J)$ as a set, but we remember the scheme structure on Z, the saturation of J with respect to the irrelevant ideal $\mathfrak{m} \subset S$ of X.

For an explicit example, we take

$$\phi = (s^2v^2, suv^2, u^2t^2 + u^2tv, sutv - 101u^2tv)$$

in which case

$$J = \langle s^2v^2, suv^2, u^2t^2 + u^2tv, sutv - 101u^2tv \rangle$$

is saturated and Z is zero-dimensional supported on the closed points

$$q_1 = (0,1) \times (0,1) \text{ and } q_2 = (1,0) \times (1,0)$$

The base locus Z looks quantatively different at q_1 and q_2, i.e. in the former case it its defining ideal looks like $\langle s,t \rangle$ in $\mathbb{A}^2_{s,t}$ while in the latter case like $\langle u^2, uv, v^2 \rangle$. This is formalized in in the following way. Let $\mathscr{J}$ be the ideal sheaf defining Z, then the geometric multiplicities are given by the length of the artinean locals,

$$d_{q_1} = \text{length}(\mathscr{O}_{q_1,Z}) = 1 \text{ and } d_{q_2} = \text{length}(\mathscr{O}_{q_2,Z}) = 3$$

while the algebraic multiplicities are given by the Samuel multiplicities of the ideals of finite colength $\mathscr{J}_{q_1}$ and $\mathscr{J}_{q_2}$,

$$e_{q_1} = e(\mathscr{J}_{q_1}, \mathscr{O}_{q_1,X}) = 1 \text{ and } e_{q_2} = e(\mathscr{J}_{q_2}, \mathscr{O}_{q_2,X}) = 4$$

which in turn reflects the fact that q_1 is a complete intersection (c.i.) point of Z while q_2 is not.

The closed image $Y \subset \mathbb{P}^3$ of ϕ is given by a single *implicit* equation

$$P(\boldsymbol{x}) = x_0^2x_2 - 202x_0x_1x_2 + 10201x_1^2x_2 - x_0x_1x_3 + 101x_1^2x_3 - x_0x_3^2$$

and we use P to also denote its principal prime ideal codimension-1 ideal in $T = \mathbb{C}[\boldsymbol{x}]$.

The Rees algebra $B = \text{Rees}_S(J)$ is the quotient of $R = S[\boldsymbol{x}]$ by the ideal $I \subset R$ of syzygies on the ϕ_j, and for any fixed degree $\boldsymbol{d}$, the graded piece $I_{\boldsymbol{d},\bullet}$ is a finite sub-T-module of T^r where

$$r = \dim_{\mathbb{C}}(S_{\boldsymbol{d}}) = h^0(\mathscr{O}_X(\boldsymbol{d}))$$

As such $I_{\boldsymbol{d},\bullet} = \text{image}(N)$ for an $r \times \mu$ graded matrix N, where μ is the size of a minimal homogeneous generating set. Specifically, in our example

$$I = \left\langle \begin{cases} ux_0 - sx_1, (sv - 101uv)x_2 + (-ut - uv)x_3, \\ (t^2 + tv)x_1 - 101v^2x_2 + (-tv - v^2)x_3, \\ (st - 101ut)x_1 - svx_3, vx_0x_2 - 101vx_1x_2 + (-t - v)x_1x_3, \\ tx_0x_2 - 101tx_1x_2 - 101vx_2x_3 + (-t - v)x_3^2, \\ sx_0x_2 + (-202s + 10201u)x_1x_2 + (-s + 101u)x_1x_3 - sx_3^2, \\ tx_0x_1 - 101tx_1^2 - vx_0x_3, \\ P(x_0, x_1, x_2, x_3) \end{cases} \right\rangle$$

and for $\boldsymbol{d} = (1,1)$ we get $r = 4$, $\mu = 5$ and

$$N = \begin{bmatrix} 0 & x_1 & 0 & 0 & x_0x_2 - x_3^2 \\ x_2 & -x_3 & -x_1 & -x_3 & -x_3^2 \\ -x_3 & -101x_1 & 0 & x_0 - 101x_1 & -10201x_1x_2 - 202x_3^2 \\ -101x_2 - x_3 & 0 & x_0 & 0 & 20402x_2x_3 - 202x_3^2 \end{bmatrix}$$

Each column C of N is a syzygy on the ϕ_j of some degree i on the x_j and degree $\boldsymbol{d} = (1,1)$ in the sense that

$$\begin{bmatrix} st & sv & ut & uv \end{bmatrix} \cdot C(\boldsymbol{x})$$

is a homogeneous degree $(\boldsymbol{d}, i)$-element of I, i.e.

$$\begin{bmatrix} st & sv & ut & uv \end{bmatrix} \cdot C(\phi) = 0$$

identically in S.

Finally, we note that ϕ is of degree 1, $\boldsymbol{d} \in \mathrm{reg}(J)$ and we have the equality of ideals

$$P^{\deg(\phi)} = \gcd(\texttt{minors}(4, N)) \quad \triangle$$

Example 2.29 (`ex204`)**.** add

CHAPTER 3

MAIN RESULTS

We are now ready to state our main results. While this chapter is supposed to be self-contained and most of the relevant notation and definitions are listed in (3.1) below, one should consult Chapter **??** for a more relaxed exposition and further definitions, for instance, for the notions of degree and multiplicity of a basepoint.

Examples 2.28 and 2.29 should serve as quick reference points.

3.1. Let X be a smooth projective toric variety of dimension $n-1$ ($n>1$) with Cox ring S and irrelevant ideal $\mathfrak{n} \subset S$. Let $T = \mathbb{C}[x_0, \ldots, x_n]$ and let

$$\phi = (\phi_0, \ldots, \phi_n) : X \longrightarrow \mathbb{P}^n = \mathrm{Proj}(T)$$

be a rational map given by linearly independent sections of the line bundle $\mathscr{O}_X(\boldsymbol{p})$ for some degree $\boldsymbol{p}$ on S. Let $J = \langle \phi_0, \ldots, \phi_n \rangle \subset S$ be ideal of the coordinate functions of ϕ.

Let $\boldsymbol{d}$ be a degree on S such that $r = \dim_{\mathbb{C}}(S_{\boldsymbol{d}}) > 0$ and let $I \subset R = S \otimes T$ be the Rees ideal of J. Let $I_{\boldsymbol{d},\bullet}$ be its degree-$\boldsymbol{d}$ bigraded piece, considered as a finite graded T-module. We denote by $B = R/I$ the Rees algebra of J.

Let N be the $r \times \mu$ matrix of a minimal set of homogeneous generators for $I_{\boldsymbol{d},\bullet}$ and let N_i be the submatrix of N corresponding to generators of degree i, that is,

$$N = (\ N_1 \mid \ldots \mid N_\delta\ )$$

3.2. Let $P \subset T$ be the prime ideal corresponding to the closed image of ϕ in $\mathbb{P}^n$. We denote this image by $Y = V(P)$. Let $Z = V(J) \subset X$ be the base locus of ϕ.

We are going to be interested the following three conditions:

1. The map ϕ is generically finite onto its image, that is, $Y \subset \mathbb{P}^n$ is of codimension 1 and in particular, the ideal P is principal. In this case, we denote a generator of P by $P(\boldsymbol{x})$.
2. The base locus Z is 0-dimensional, that is, consists of finitely many points. Note that those are necessarily closed over $\mathbb{C}$ but this is not needed for our results.
3. The map ϕ is birational.

Clearly, either of (2) and (3) implies (1).

We begin by making a few easy but important observations about the Rees ideal. These are needed later.

Proposition 3.3.

(a) *The ideal $I \subset R$ is prime, and so is I_P in the T-module localization R_P.*

(b) *The quotient B_P is naturally a finite-type graded $K(T/P)$-algebra with grading induced by S.*

(c) *The $K(T/P)$-algebra B_P is a homogeneous coordinate ring of a projective variety.*

3.4. By part (c) of Proposition 3.3, $\mathrm{Proj}(B_P)$ becomes a toric projective variety. Following either [Greg Smith and Maclagan] or [Hoffman and Wang], we can define its regularity, denoted $\mathrm{reg}(B_P)$. It is similar to the Castenlnuovo-Mumford regularity over $\mathbb{P}^n$ in sense that $\mathrm{reg}(B_P)$ is a finitely generated additively-closed subset of the semi-group of degrees on S, and for any $\boldsymbol{d} \in \mathrm{reg}(B_P)$, the scheme structure of $V(I_P)$ is already reflected in degree $\boldsymbol{d}$.

In light of Proposition 3.3, our first result becomes an easy exercise. However, it a step toward the goal of this paper—to exhibit a general relation between the algebra of the coordinates ϕ_j and the geometry of the image Y.

Theorem 3.5. *In the setup of* (3.1)*, one has*

$$\mathrm{rad}(\mathtt{minors}(r,N)) = P$$

The geometric interpetation of the theorem is clear — the nonzero minors of N define hypersurfaces in $\mathbb{P}^n$ whose intersection, at least set-theoretically, is the image Y.

Example [the twisted-cubic], for instance, shows that the radical is necessary.

Our next result is the main theorem of this thesis, unifying two currently popular non-Gröbner bases approaches to implicitization and setting the stage for both the ad-hoc template proofs in Chapter 7 and the fast implicitization method described in Chapter 6.

We point out that the following theorem is neither a stronger nor a weaker version of Theorem 3.5 but rather a specialization along the lines of [Buse-Juonolou].

Theorem 3.6. *Consider the setup of* (3.1) *and assume* (3.2.1)*. Fix a degree* $\mathbf{d} \in \mathrm{reg}(B_P)$ *as described in* (3.4)*. One has*

$$\gcd(\mathtt{minors}(r,N)) = P^{\deg\phi}$$

In particular, if there is a degree $\mathbf{d}$ *in the regularity for which* $\mu = r$*, then* N *is square and, up to a unit,*

$$\det(N) = P(\boldsymbol{x})^{\deg\phi}$$

Corollary 3.7. *In the setup of Theorem 3.6, let* M *be any* $r\times r$ *matrix of syzygies over* $S_{\mathbf{d}}$*. One has*

$$\det(M) = P(\boldsymbol{x})^{\deg\phi}\cdot H(\boldsymbol{x})$$

for a homogeneous $H(\boldsymbol{x})$ *of degree*

$$\deg\det(M) - \deg(\phi)\cdot\deg(Y) \tag{3.7.1}$$

Furthermore, there exist a list of such matrices $\{M_k\}$ *such that the corresponding* $H_k(\boldsymbol{x})$ *are nonzero and without a common factor.*

Geometrically, the former is a refinement of Theorem 3.5. Each of the maximal minors of N, in fact the determinant of any $r \times r$ matrix M of syzygies over $S_{\boldsymbol{d}}$, is either 0 or describes in $\mathbb{P}^n$ the union of a $\deg(\phi)$-fold Y and a hypersurface of degree (3.7.1). While an arbitrary collection M_k of such matrices may introduce hypersurfaces with an intersection that is strictly larger than Y, the maximal minors suffice to shave off any extraneous components.

The theme of extraneous factors is already apparent in **?** and **?** who use the approximation complex to show that, in our running notation, for a toric X, certain $\boldsymbol{d}$ and almost complete intersection baselocus Z,

$$\gcd(\texttt{minors}(r, N_1)) = P^{\deg\phi} \cdot \prod_{p \in Z} L_p(\boldsymbol{x})^{e_p - d_p}$$

where each $L_p(\boldsymbol{x})$ is a linear form.

In the case of complete intersection baselocus, the proof of Theorem 3.6 gives a special case of the above.

Corollary 3.8 (Buse-Jouanoulo)**.** *In the setup of Theorem 3.6, let* $\mathbf{d}$ *be* large enough *in the semigroup of degrees on S. Recall that* N_1 *is the matrix of linear syzygies. If* $\operatorname{Proj}(B)$ *is isomorphic to* $\operatorname{Proj}(\operatorname{Sym}(J))$ *in the sense of [equation from before], then*

$$\gcd(\texttt{minors}(r, N_1)) = P^{\deg\phi}$$

In particular, the result holds if the baselocus is 0*-dimensional and locally a complete intersection.*

It is a know fact that if M is a square matrix over T of size r, then the singular locus of $V(\det(M))$ is contained in the closed subset defined by the comaximal minors, that is, the $(r-1)$-minors. Although we failed to find a reference for the aforementioned result, we believe that this relation is more intrinsic and holds for all matrices N. Examples [at least 2-3 examples] explore this further. Specifically, we conjecture the following

Conjecture 3.9. *Let ϕ be birational and $\mathbf{d}$ be any degree on S. One has*

$$V(\texttt{minors}(r-1,N)) \subset \mathrm{Sing}(Y)$$

In the simplest cases of interest, when $X = \mathbb{P}^2$ or $X = \mathbb{P}^1 \times \mathbb{P}^1$ and ϕ is basepoint-free, we can chose $\boldsymbol{d}$ so that the matrix N becomes square. Next two theorems are slight generalizations of the results in **?**. More importantly, they show that our methods directly generalize the methods of moving planes and quadrics in the setting in which they are most useful.

Theorem 3.10. *Let $X = \mathbb{P}^2$, ϕ be basepoint-free, and suppose that there are exactly $p = \boldsymbol{e}$ linear syzyges over degree $\mathbf{d} = p-1$, that is, the minimal possible number. Then one has that $N = (N_1 \mid N_2)$, N is square and ϕ is birational. In particular,*

$$\det(N) = \det(N_1 \mid N_2) = P(\boldsymbol{x})$$

Theorem 3.11. *Let $X = \mathbb{P}^1 \times \mathbb{P}^1$, ϕ be basepoint-free with coordinates in degree $\boldsymbol{e} = (p,q)$, and suppose there are no linear syzyges over degree $\mathbf{d} = (p-1,q-1)$. Then one has that N_2 is square, $N = N_2$ and ϕ is birational. In particular,*

$$\det(N) = \det(N_2) = P(\boldsymbol{x})$$

Both of these theorems are examples of a template proof decribed in Chapter 7. While applying it in general requires elaborate choses of the degree $\boldsymbol{d}$ and regularity computations, in the case of basepoint-free maps we only used a type of Koszul-ness on the syzygies of low degree. This is the content of Theorem [reference the theorem in Chapter 7]

We conclude this list by a method to compute the degree of a rational map using Gröbner bases. While we are only going to use this in our examples, it helps expand our understanding about the object B_P.

The author wants to thank Mike Stillman for suggesting it.

Proposition 3.12. *Let $\mathbb{C}[s_0, \ldots, s_m]$ be the fixed ambient polynomial ring of S' as described in 2.5. Define the ideal I_B of $\mathbb{C}[s_0, \ldots, s_m; x_0, \ldots, x_n]$ by the equality*

$$B = \mathbb{C}[\boldsymbol{s};\boldsymbol{x}]/I_B$$

Let $>'$ be any product order in which the $\boldsymbol{s}$ variables come before the $\boldsymbol{x}$ variables. Then a reduced Gröbner basis for I_B with respect to $>'$ has the form

$$\begin{aligned}
g_1(\boldsymbol{s};\boldsymbol{x}) &= p_1(\boldsymbol{x})\boldsymbol{s}^{\alpha_1} + \textit{lower order terms} \\
&\cdots \\
g_r(\boldsymbol{s};\boldsymbol{x}) &= p_r(\boldsymbol{x})\boldsymbol{s}^{\alpha_r} + \textit{lower order terms} \\
g_{r+1}(\boldsymbol{s};\boldsymbol{x}) &= P(\boldsymbol{x})
\end{aligned}$$

and

$$\deg(\phi) = \deg\left(\langle \boldsymbol{s}^{\alpha_1}, \ldots, \boldsymbol{s}^{\alpha_r}\rangle \subset \mathbb{C}[\boldsymbol{s}]\right)$$

CHAPTER 4

EXAMPLES

This chapter consists of the heart of the thesis. In fact, carefully studied, this list of examples could probably make the proofs chapter obsolete.

Every example comes with a label of the form (`exXYZ`), which refers its a Macaulay2 test code. Those files can found at

```
http://www.math.cornell.edu/~rzlatev/phd-thesis/ex/XYZ.m2
```

Example 4.1 (`ex301`)**.** Let $X = \mathbb{P}^2_{s,t,u}$ and $J = \langle tu, su, st, s^2+t^2+u^2\rangle$. Then ϕ is basepoint-free and birational. The monic implicit equation is given by

$$P(\boldsymbol{x}) = x_0^2x_1^2 + x_0^2x_2^2 + x_1^2x_2^2 - x_0x_1x_2x_3$$

Setting $\boldsymbol{d} = 1$, we get

$$N = \begin{bmatrix} 0 & x_0 & x_1x_2 \\ x_1 & 0 & x_0x_2 \\ -x_2 & -x_2 & x_0x_1 - x_2x_3 \end{bmatrix}$$

whose determinant is $P(\boldsymbol{x})$. The results of Cox et al. [2000] apply and the matrix N is a variant of the matrix produced by the method of moving planes and quadrics.

Setting $\boldsymbol{d} = 2$, we get

$$N = \begin{bmatrix} x_2 & 0 & 0 & 0 & x_1 & 0 & 0 & 0 & x_0 \\ -x_3 & 0 & 0 & x_1 & 0 & 0 & 0 & x_0 & 0 \\ 0 & 0 & 0 & -x_2 & -x_3 & x_0 & x_2 & 0 & -x_2 \\ x_2 & 0 & x_1 & 0 & 0 & 0 & x_0 & 0 & 0 \\ 0 & x_1 & -x_2 & 0 & x_2 & 0 & -x_3 & -x_2 & 0 \\ x_2 & -x_2 & 0 & 0 & x_1 & -x_2 & x_0 & 0 & 0 \end{bmatrix}$$

which is a 6×9 matrix of linear forms. This was expected — the results of Busé and Jouanolou [2003] also apply and the method of the approximation complex guarantees a matrix of linear forms. Accordingly,

$$\gcd(\texttt{minors}(6,N)) = P$$

Note that the claim that ϕ is birational follows, a fortiori, from the degree formula (2.25.1). Indeed, we have

$$4\deg(\phi) = 2^2 - 0$$

We confirm this using Proposition 3.12 in Example 4.10.

Example 4.2 (ex302)**.** Clearly, if we replace s,t,u in Example 4.1 by general linear forms L_0, L_1, L_2, effectively changing coordinates on the source, we get the same answer. In this example we describe what happens if we take $X = \mathbb{P}^1_{s,u} \times \mathbb{P}^1_{t,v}$ instead and let the L_k be $(1,1)$-forms. Since the algebraic structure of the coordinates is the same, so is the equation of the image,

$$Y = V(x_0^2x_1^2 + x_0^2x_2^2 + x_1^2x_2^2 - x_0x_1x_2x_3)$$

and ϕ is again basepoint-free, basically because the L_k are three general forms on a 2-dimensional variety. However, the self-intersection of $\mathscr{O}(2,2)$ is now 8. It follows that ϕ is generically 2-1.

For $\boldsymbol{d} = (1,1)$ we get a square matrix of size 4 with $h = (2,1,0,1)$. As expected, up to a unit

$$\det(N) = P(\boldsymbol{x})^2$$

For $\boldsymbol{d} = (2,1)$ we get a square matrix of size 6 with $h = (4,2)$ for the which the last equality applies. For $\boldsymbol{d} = (2,2)$ we get a 9×12 matrix with $h = (11,1)$.

Example 4.3 (`ex303`)**.** Suppose that in the situation of Example 4.2 we took the forms L_k from $\langle st, sv, ut\rangle$ instead. Now ϕ has the unique basepoint $(0,1)\times(0,1)$ which is c.i. of degree 4. Indeed, on the affine local where $u = v = 1$, the point looks like $V(st, s^2+t^2)$. The equation of the image remains the same. Once again, (2.25.1) we know that ϕ must be birational. For $\boldsymbol{d} = (1,1)$ we get a 4×5-matrix. Below is the matrix resulting from $(L_0, L_1, L_2) = (st, sv, ut)$,

$$N = \begin{bmatrix} 0 & 0 & 0 & x_0 & x_1x_2 \\ x_1 & x_0 & 0 & 0 & 0 \\ -x_2 & 0 & x_0 & -x_2 & -x_2x_3 \\ 0 & -x_2 & -x_1 & 0 & x_1^2+x_2^2 \end{bmatrix}$$

and we have

$$\gcd(\texttt{minors}(4,N)) = P$$

Example 4.4 (`ex304`)**.** Let $X = \mathbb{P}^2$ and $J = \langle su^2, t^2(s+u), st(s+u), tu(s+u)\rangle$. Then ϕ is birational with three basepoints — $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$, all c.i. of degree 2, 3, and 1, respectively.

The implicit equation is given by

$$P(\boldsymbol{x}) = x_0x_1x_2 + x_0x_1x_3 - x_2x_3^2$$

As before, both the method of the moving planes and quadrics, and the method of the approximation complex apply. For $\boldsymbol{d} = 1$, we get

$$N = \begin{bmatrix} -x_3 & 0 & x_1 & -x_3^2 \\ 0 & -x_3 & -x_2 & x_0x_2+x_0x_3 \\ x_2 & x_1 & 0 & 0 \end{bmatrix}$$

and for $\boldsymbol{d} = 2$, we get a 6×9-matrix whose entries are all linear.

Example 4.5 (`ex305`)**.** Let $X = \mathbb{P}^2$ and $J = \langle s^3, tu^2, s^2t + u^3, stu\rangle$. The map is birational with a single c.i. basepoint of degree 2. For $\boldsymbol{d} = 1$ we have

$$N = \begin{bmatrix} x_1 & 0 & -x_3^2 \\ 0 & x_1x_2 - x_3^2 & x_0x_1 \\ x_3 & x_1^2 & 0 \end{bmatrix}$$

and $\det(N) = x_0x_1^4 - x_1x_2x_3^3 + x_3^5$, the implicit equation.

Starting from $\boldsymbol{d} = 2$, in which case we have

$$N = \begin{bmatrix} 0 & 0 & 0 & x_1 & -x_3 & 0 & 0 \\ x_3 & 0 & x_1 & 0 & 0 & -x_2 & 0 \\ 0 & x_1 & 0 & -x_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_0 & -x_1x_2 + x_3^2 \\ -x_2 & 0 & -x_3 & 0 & x_0 & 0 & x_1^2 \\ x_1 & -x_3 & 0 & 0 & 0 & x_3 & 0 \end{bmatrix}$$

we always have $\mu - 1$ linear columns and a single quadratic one.

Example 4.6 (`ex306`)**.** Let $X = \mathbb{P}^2$ and $J = \langle s^3, t^2u, s^2t + u^3, stu\rangle$. The map is birational with a single c.i. basepoint of degree 2. and $\det(N) = x_0^3x_1^4 - x_0^2x_1^3x_2x_3 + x_3^7$,

Example 4.7 (`ex307`)**.** Consider the twisted cubic curve C. It is the image of $X = \mathbb{P}^1_{s,t}$ under the map

$$\phi = (s^3, s^2t, st^2, t^3) : \mathbb{P}^1 \longrightarrow \mathbb{P}^3$$

which is birational onto its image. In light of Theorem **??**, we can carry out the same calculations even though the image is of codimension strictly bigger than 1.

Setting $\boldsymbol{d} = 1$, we get

$$N = \begin{bmatrix} -x_3 & -x_2 & -x_1 \\ x_2 & x_1 & x_0 \end{bmatrix}$$

and

$$\texttt{minors}(2,N) = \langle x_2^2 - x_1x_3, x_1x_2 - x_0x_3, x_1^2 - x_0x_2 \rangle$$

which is the usual equation for C in $\mathbb{P}^3$. Setting $\boldsymbol{d} = 2$, we get

$$N = \begin{bmatrix} 0 & -x_3 & -x_3 & -x_2 & -x_2 & -x_1 \\ -x_3 & x_2 & 0 & x_1 & 0 & x_0 \\ x_2 & 0 & x_1 & 0 & x_0 & 0 \end{bmatrix}$$

We have

$$\texttt{minors}(3,N) = \left\langle \begin{cases} x_2^2x_3 - x_1x_3^2, x_1x_2x_3 - x_0x_3^2, x_1^2x_3 - x_0x_2x_3, \\ x_2^3 - x_0x_3^2, x_1x_2^2 - x_0x_2x_3, x_0x_2^2 - x_0x_1x_3, \\ x_1^2x_2 - x_0x_1x_3, x_0x_1x_2 - x_0^2x_3, x_1^3 - x_0^2x_3, x_0x_1^2 - x_0^2x_2 \end{cases} \right\rangle$$

and

$$\mathrm{rad}(\texttt{minors}(3,N)) = \langle x_2^2 - x_1x_3, x_1x_2 - x_0x_3, x_1^2 - x_0x_2 \rangle$$

showing that the radical is necessary.

Example 4.8 (ex308). Let $X = \mathbb{P}^2_{s,t,u}$ and $J = \langle s^5, t^5, su^4, st^2u^2 \rangle$. Then ϕ is generically 2-1 map with the unique basepoint $(0,0,1)$ which is c.i. of degree 5. The image is defined by

$$P(\boldsymbol{x}) = x_0x_1^4x_2^5 - x_3^{10}$$

Setting $\boldsymbol{d} = 1$, we get

$$N = \begin{bmatrix} x_1x_2 & -x_3^8 & 0 \\ -x_3^2 & x_0x_1^3x_2^4 & 0 \\ 0 & 0 & x_0x_1^4x_2^5 - x_3^{10} \end{bmatrix}$$

and

$$\det(N) = P(\boldsymbol{x})^2$$

where one should note the principal 2-minor is just $P(\boldsymbol{x})$.

The alternative non-Gröbner bases approaches would require difficult computations; the method of the approximation complex would need to find the gcd of the maximal minors of a matrix of size 36×58, and none of the moving plane and quadrics methods will work since the map isn't birational (and (BP3) of Busé et al. [2003] fails in any case).

The practical gain of our method for this example, however, is arguable at best — we have to compute a degree 10 syzygy which is already the degree of the implicit equation. On the other hand, a better choice for $\boldsymbol{d}$ might help. For $\boldsymbol{d} = 3$ we get a 10×10-matrix with $h = (4,4,1,0,1)$, and for $\boldsymbol{d} = 4$ we get a 15×16-matrix with $h = (11,4,1)$.

Example 4.9 (`ex309`)**.** Let N_1 be the matrix of linear columns for $\boldsymbol{d} = 1$ in Example 4.5. We have that $\det(N_1) = 0$. This shows that not all maximal minors need to be nonzero.

This has nothing to do with the fact that N_1 is special. For another example, let us take $\boldsymbol{d} = 4$ in Example 4.8. Then N is a 15×16-matrix whose columns correspond to the syzygies

$$\begin{cases} t^2u^2x_2 - u^4x_3, t^3ux_2 - tu^3x_3, st^2ux_2 - su^3x_3, t^4x_2 - t^2u^2x_3, \\ st^3x_2 - stu^2x_3, s^2t^2x_2 - s^2u^2x_3, su^3x_1 - t^3ux_3, stu^2x_1 - t^4x_3, \\ s^2u^2x_1 - st^3x_3, u^4x_0 - s^4x_2, t^2u^2x_0 - s^4x_3, s^2tux_1x_2 - st^2ux_3^2, \\ s^3ux_1x_2 - s^2tux_3^2, s^3tx_1x_2 - s^2t^2x_3^2, s^4x_1x_2 - s^3tx_3^2, tu^3x_0x_1x_2 - s^3ux_3^3 \end{cases}$$

Let M be the square submatrix of the first 14 columns and the last column of N, that is, except the one corresponding to the syzygy $s^4x_1x_2 - s^3tx_3^2$. Then $\det(M) = 0$. The same is true for the square submatrix consisting of the first 15 columns but in that case $M = (N_1 \mid N_2)$ which we wanted to avoid.

Example 4.10 (`ex310`)**.** This example has been present elsewhere in the literature and is know to break all the available methods. Let $X = \mathbb{P}^2$ and take

$$
\begin{aligned}
J = \langle & -s^2t^3 + 3s^2t^2u + st^3u - 4st^2u^2 - stu^3 + 2t^2u^3 - tu^4 + u^5, \qquad \rangle \\
& s^2t^3 - 3s^2t^2u + st^3u + 3stu^3 - 2t^2u^3 + tu^4 - u^5, \\
& s^2t^3 - 3s^2t^2u - st^3u + 2s^2tu^2 + 4st^2u^2 - 3stu^3 - 2t^2u^3 + 3tu^4 - u^5, \\
& -s^2t^3 + 3s^2t^2u - st^3u - 3stu^3 + 3tu^4 - u^5
\end{aligned}
$$

so ϕ is generically 1-1 with 3 basepoints of total degree 17 and multiplicity 20. More precisely, the basepoints are the ci point $(1,1,1)$ of multiplicity 4, the aci point $(0,1,0)$ of degree 4 and multiplicity 5, and the aci point $(1,0,0)$ of degree 9 and multiplicity 11.

Irrelevant of any of the hideous basepoints, we get

$$
N = \begin{bmatrix} x_0 + x_1 & 0 & x_1^2 - x_3^2 \\ -x_0 - x_2 & 3x_0x_1 - x_1^2 + 4x_1x_2 + 3x_0x_3 - 3x_1x_3 + 4x_2x_3 - 2x_3^2 & x_0^2 + 7x_0x_1 - 3x_1^2 + x_0x_2 + 10x_1x_2 + \\ -x_2 + x_3 & x_0^2 - x_0x_1 - x_1^2 - 3x_0x_3 - 3x_1x_3 - x_3^2 & -5x_0x_1 - 2x_1^2 - 3x_0x_2 - 6x_1x_2 - \end{bmatrix}
$$

and

$$\det(N) = P(\boldsymbol{x}) \quad \triangle$$

Example 4.11 (`ex311`)**.** Let ϕ be the rational map from Example 4.10. We compute the degree and multiplicity of the base locus Z.

Set $q_1 = (1,0,0)$, $q_2 = (0,1,0)$ and $q_3 = (1,1,1)$, so that set-theoretically $Z = \{q_1, q_2, q_3\}$. Since J is saturated, the sum of the degrees of the basepoints is just the degree of the base locus,

$$\deg(Z \subset \mathbb{P}^2) = \deg(J \subset \mathbb{C}[s,t,u]) = 17$$

Because we are in projective space, this can be checked directly in Macaulay2. However, since Z is supported on multiple points, we cannot compute the total multiplicity in the same way. Indeed, `multiplicity`(J) gives 45.

Let Q_k be the prime ideals corresponding to the points q_k, and let $J_k = J : {Q_k}^{\infty}$. The the individual degree and multiplicity can be computed by computing the degree and multiplicity of the corresponding ideals J_k. Those capture the local structure of Z near q_k. We have

$$\begin{cases} J_1 = \langle tu^2, t^3 - 3t^2u, u^5 \rangle \\ J_2 = \langle su, s^2, u^3 \rangle \\ J_3 = \langle t^2 - 2tu + u^2, s^2 - 2su + u^2 \rangle \end{cases}$$

so in particular, q_3 is c.i., while q_1 and q_2 are a.c.i. points.

Example 4.12 (`ex312`)**.** Let $Y = V(P)$ for an irreducible form $P(\boldsymbol{x})$ of degree 4 on $\mathbb{P}^3$. Suppose further that $\mathrm{Sing}(Y)$, the singular locus Y, contains 3 concurrent non-degenerate lines (that is, passing trough a common point and spanning all of $\mathbb{P}^3$). Then $P(\boldsymbol{x})$ is the determinant of a square order-4 matrix M of linear forms.

We can prove this claim by a direct calculation. After a linear change of coordinates on $\mathbb{P}^3$, we can assume that the lines are the 3 coordinate axes in the distinguished $\{x_3 \neq 0\}$, that is, the lines are given by $V(x_0, x_1)$, $V(x_1, x_2)$ and $V(x_0, x_2)$.

On the level of ideals, using Euler's identity, the assumption translates to

$$\langle P_{x_0}(\boldsymbol{x}), \ldots, P_{x_3}(\boldsymbol{x}) \rangle \subset \langle x_0x_1, x_0x_2, x_1x_2 \rangle$$

so writing out $P(\boldsymbol{x}) = \sum_{|\alpha|=4} a_\alpha \boldsymbol{x}^\alpha$ and noting that each of the partials must be zero modulo $\langle x_0x_1, x_0x_2, x_1x_2 \rangle$, we only need to solve a linear system in the indeterminate coefficients. We get

$$P(\boldsymbol{x}) = a_1x_0^2x_1^2 + a_2x_0^2x_2^2 + a_3x_1^2x_2^2 + a_4x_0^2x_1x_2 + a_5x_0x_1^2x_2 + a_6x_0x_1x_2^2 + a_7x_0x_1x_2x_3$$

which is the determinant of

$$M = \begin{bmatrix} x_0 & & & x_1 \\ & x_1 & & x_2 \\ & & x_2 & x_0 \\ -a_3x_2 & -a_2x_0 & -a_1x_1 & a_4x_0 + a_5x_1 + a_6x_2 + a_7x_3 \end{bmatrix}$$

In fact, if $a_7 \neq 0$ we can do better. Setting $a_7 = 1$, a linear change of coordinates by the matrix

$$\begin{bmatrix} 1 & & & -a_4 \\ & 1 & & -a_5 \\ & & 1 & -a_6 \\ & & & 1 \end{bmatrix}$$

leaves the singular locus the same but simplifies the general form to

$$P'(\boldsymbol{x}) = a_1x_0^2x_1^2 + a_2x_0^2x_2^2 + a_3x_1^2x_2^2 + x_0x_1x_2x_3$$

and the matrix M to

$$M' = \begin{bmatrix} x_0 & & & x_1 \\ & x_1 & & x_2 \\ & & x_2 & x_0 \\ -a_3x_2 & -a_2x_0 & -a_1x_1 & x_3 \end{bmatrix}$$

where the entries are scaled variables.

Example 4.13 (313)**.** It turns out, however, that this M is not a matrix of syzygies, in the sense that only 3 out of its 4 columns are syzygies.

The example with 3 syzygies from the B exam.

Example 4.14 (`ex314`)**.** We apply Proposition 3.12 to compute the degree of the rational map in Example 4.1. We have

$$I = \left\langle \begin{cases} x_1 t - x_2 u, \\ x_0 s - x_2 u, \\ x_2 s^2 - x_3 st + x_2 t^2 + x_2 u^2, \\ x_1 s^2 - x_3 su + x_2 tu + x_1 u^2, \\ x_0 t^2 + x_2 su - x_3 tu + x_0 u^2, \\ x_1 x_2 s + x_0 x_2 t + (x_0 x_1 - x_2 x_3) u, \\ x_0^2 x_1^2 + x_0^2 x_2^2 + x_1^2 x_2^2 - x_0 x_1 x_2 x_3 \end{cases} \right\rangle$$

and we see $P(\boldsymbol{x})$ as the last generator. Next, we calculate a Gröbner basis with respect to a product monomial order where the s, t, u variables come before the x_j variables. Dropping the implicit equation, we get

$$\begin{cases} x_1 t - x_2 u, \\ x_0^2 x_2 t + (x_0^2 x_1 + x_1 x_2^2 - x_0 x_2 x_3) u, \\ x_0 s - x_2 u, \\ x_1 x_2 s + x_0 x_2 t + (x_0 x_1 - x_2 x_3) u, \\ x_0 t^2 + x_2 su - x_3 tu + x_0 u^2, \\ x_2 s^2 - x_3 st + x_2 t^2 + x_2 u^2, x_1 s^2 - x_3 su + x_2 tu + x_1 u^2 \end{cases}$$

Collecting the S-part of the leading terms, we get the ideal $\langle s, t \rangle$ whose degree is obviously 1.

We can carry out the same calculation for the map in Example 4.8. The Rees ideal and its Gröbner basis are as follow,

$$I = \left\langle \begin{cases} x_2t^2 - x_3u^2, \\ x_1x_2s - x_3^2t, \\ -x_3t^3 + x_1su^2, \\ -x_2s^4 + x_0u^4, \\ -x_3s^4 + x_0t^2u^2, \\ -x_1s^5 + x_0t^5, \\ -x_3^3s^3 + x_0x_1x_2tu^2, \\ -x_3^5s^2 + x_0x_1^2x_2^2u^2, \\ -x_3^8s + x_0x_1^3x_2^4t, \\ x_0x_1^4x_2^5 - x_3^{10} \end{cases} \right\rangle, \quad \begin{cases} x_1x_2s - x_3^2t, \\ x_3^8s - x_0x_1^3x_2^4t, \\ x_2t^2 - x_3u^2, \\ x_3^9t^2 - x_0x_1^4x_2^4u^2, \\ x_3^7st - x_0x_1^3x_2^3u^2, \\ x_3^5s^2 - x_0x_1^2x_2^2u^2, \\ x_3t^3 - x_1su^2, \\ x_3^3s^3 - x_0x_1x_2tu^2, \\ x_3^4s^2t^2 - x_0x_1^2x_2u^4, \\ x_3^2s^3t - x_0x_1u^4, \\ x_3s^4 - x_0t^2u^2, \\ x_2s^4 - x_0u^4, \\ x_1s^5 - x_0t^5 \end{cases}$$

This time the ideal of the S-part of the leading terms is $\langle s, t^2 \rangle$ which is of degree 2.

Example 4.15 (`ex315`)**.** While Examples 4.1, 4.6 and **??** support Conjecture 3.9, we point out that the latter is stated in its strongest possible form. For one, the claim is trivial in the case $\deg(\phi) > 1$ — in that case the singular locus is all of Y. This can be seen in Example 4.8.

On the other hand, while it is tempting to conjecture that

$$\mathrm{rad}(\langle P_{x_j} : j \rangle) \subset \mathrm{sat}(\texttt{minors}(r-1, N))$$

that is, that the inclusion of the conjecture is on the level of schemes, this is not true as illustrated by Example 4.4. In the case $\boldsymbol{d} = 1$, we get

$$\begin{aligned} \operatorname{rad}(\langle P_{x_j} : j \rangle) &= \langle x_3, x_1x_2, x_0x_2, x_0x_1 \rangle \\ \operatorname{sat}(\texttt{minors}(r-1, N)) &= \langle x_3^2, x_2x_3, x_1x_3, x_2^2, x_1x_2, x_1^2 \rangle \\ \operatorname{rad}(\texttt{minors}(r-1, N)) &= \langle x_1, x_2, x_3 \rangle \end{aligned}$$

and the only inclusion we have is

$$\operatorname{rad}(\langle P_{x_j} : j \rangle) \subset \operatorname{rad}(\texttt{minors}(r-1, N))$$

The results for other values of $\boldsymbol{d}$ are analogous.

Example 4.16 (316)**.** Structure of B_P

Example 4.17 (317)**.** Longer computation

CHAPTER 5

PROOFS OF THE MAIN RESULTS

While the ultimate goad of this chapter is to prove the results of Chapter 3, it is written in a way to help develop intuition about the interplay between matrices of syzygies and the geometry of Y, specifically, of course, its implicit equation.

For this reason we start with a few elementary resuls which are either new or else their proofs are new. For example, Lemma 5.1 is known [Soderberg and Chen prove it ad-hoc for linear and quadratic forms; find a good reference] while Lemma 5.2 is new.

We follow the notation of Chapter 2 and adopt setup of Chapter 3.

Lemma 5.1. *For any fixed* $\boldsymbol{d}$*, let* M *be a square* $r \times r$ *matrix of syzygies. Then*

$$\det(M) \in \langle P \rangle$$

In particular, if M *is any (not necessarily square) matrix of syzygies, then*

$$\texttt{minors}(r, M) \subset \langle P \rangle$$

Proof. The second statement clearly follows from the first, setting an empty minor to 0.

Let $\operatorname{adj}(M)$ be adjugate matrix and $\boldsymbol{b} = \texttt{basis}(S_{\boldsymbol{d}})$ be the row vector of the fixed basis for $S_{\boldsymbol{d}}$. Then

$$(\boldsymbol{b} \cdot M) \cdot \operatorname{adj}(M) = \boldsymbol{b} \cdot \det(M) \boldsymbol{1}_r = \det(M) \boldsymbol{b}$$

Since $\boldsymbol{b} \cdot M$ is a row vector of syzygies, the LHS vanishes identically in S under the substituion $x_0 = \phi_0(\boldsymbol{s}), \ldots, x_n = \phi_n(\boldsymbol{s})$. But then

$$\mathrm{RHS}|_{x_0=\phi_0,\ldots,x_n=\phi_n} = \det(M)(\phi_0, \ldots, \phi_n) b = \begin{bmatrix} 0 & 0 & \ldots & 0 \end{bmatrix}$$

over S, and since $\langle \boldsymbol{b} \rangle \subset S$ is the irrelevant ideal of a toric variety, truncated appropriately, it contains a nonzerodivisor on S, so $\det(M)(\phi_0, \ldots, \phi_n) = 0$. If follows that that $\det(M)$ is in the kernel $\langle P \rangle$ of $\phi^{\#}$, proving the first statement. □

Lemma 5.2. *For any fixed* $\mathbf{d}$*, N has at least as many columns as rows, i.e.*

$$\mu \geq r$$

and its ideal of maximal minors is nontrivial.

Proof. For each standard basis column vector $e_k \in \mathbb{C}^r$, $P(\boldsymbol{x})e_k$ is a graded syzygy. Let F be the $\mu \times r$ matrix of coefficients for getting the $P(\boldsymbol{x})e_k$ out of the generators of the syzygies over $S_{\boldsymbol{d}}$, i.e. the columns of N. Putting these together, we find

$$N \cdot F = P(\boldsymbol{x}) \boldsymbol{1}_r$$

The sizes of the matrices on the LHS are $r \times \mu$ and $\mu \times r$. Since the rank of the RHS as a T-matrix is r, we must have $\mu \geq r$.

The maximal minors are then of size $r \times r$. Since $\operatorname{rank}(N \cdot F) = r$, also $\operatorname{rank}(N) = r$ and so not all maximal minors vanish. □

Lemma 5.3. *For any fixed* $\mathbf{d}$*, there is an isomorphism of graded T-modules*

$$\operatorname{coker} N \cong B_{\boldsymbol{d},\bullet}$$

In particular, if $\mathscr{C}_\bullet$ is any graded resolution of $\operatorname{coker}(N)$ *over T, then $H_0\mathscr{C}_\bullet = B_{\boldsymbol{d},\bullet}$.*

Proof. This is obvious. We just note that the gradings are off by $\boldsymbol{d}$. □

Remark. The proof is actually commented out. Once I wrote it down it seemed tautology.

Lemma 5.4. *For any fixed* $\mathbf{d}$*,*

$$\mathrm{Supp}_T(B_{\boldsymbol{d},\bullet}) = V(P)$$

Proof. By identifying $T = 1 \otimes T \subset R$, we can think of T as a subring of R. If $1 \otimes Q(\boldsymbol{x}) \in I$ for any homogeneous $Q(\boldsymbol{x}) \in T$, then $Q(\phi) = 0$, so $Q(\boldsymbol{x}) \in \ker \phi^{\#} = P$. Specifically,

$$T \cap I = P$$

as subsets of R. If $Q \notin V(P)$ then $Q^c \cap I \neq \emptyset$, so $B_Q = 0$ as a T-module because some element of $P \subset I$ is inverted. This shows that $(B_{\boldsymbol{d},\bullet})_Q = 0$.

Conversely, let N' be the localization of N at P. Then

$$(B_{\boldsymbol{d},\bullet})_P = \mathrm{coker}(N)_P = \mathrm{coker}(N')$$

Since $\mathrm{Fitt}_0 \, \mathrm{coker}(N') = \mathtt{minors}(r, N') \subset PT_P \neq T_P$ by Lemma 5.1, the cokernel is nonzero, for example, by (**?**, Proposition 20.6). $\square$

Remark. For a geometric argument, see the proof of Lemma 5.7.

Proof of Proposition 3.3. The ideal I is prime in R as the kernel of a ring map into the domain $S[t]$. By Lemma 5.4,

$$P = I \cap T \subset R$$

so $B = R/I$ is naturally a T/P-algebra generated by the $\boldsymbol{s}$. The T- and T/P-module structure of B conicide and the T-module localization at P coincides with the localization of the ring R/I at the multiplicative set $(T - P)$. In particular, the localization B_P remains a domain and becomes a $K(T/P)$-algebra in a natural way. The remarks about its finite-type and the grading are obvious. This proves parts (a) and (b).

Let $\mathfrak{n}$ be the irrelevant ideal of S and let $g = g(\boldsymbol{s},\boldsymbol{x}) \in I : \mathfrak{n}^k \subset R = S[\boldsymbol{x}]$ for some $k > 0$. Then $\boldsymbol{s}^\alpha g \in I$ for all $|\alpha| = k$ and so

$$(\boldsymbol{s}^\alpha g)(\phi) = 0$$

in S. Since the latter is a domain, and X is nonempty, not all the $\boldsymbol{s}^\alpha$ are zero and so $g(\phi) = 0$. It follows, by the definition of I, that $g \in I$.

Noting that saturation commutes with localization finishes the proof of part (c). Part (d) follow immediately. $\square$

Proof of Theorem 3.5. By Lemma 5.4 and (**?**, Proposition 20.7), we have

$$\operatorname{rad}(\texttt{minors}(r,N)) = \operatorname{rad}(\operatorname{Fitt}_0(\operatorname{coker} N)) = \operatorname{rad}(\operatorname{ann}(B_{\mathbf{d},\bullet})) = P \qquad \square$$

Lemma 5.5. *For any fixed* $\mathbf{d}$*, let* $\mathscr{C}_\bullet$ *be a finite graded free resolution of* $\operatorname{coker} N$*. One has*

$$\operatorname{div}(\det(\mathscr{C}_\bullet)) = \operatorname{length}_{T_P}(B_{\mathbf{d},\bullet})_P \cdot [Y]$$

as a Weil divisor on $\operatorname{Proj}(T)$.

Proof. By (**?**, A, Theorem 30), applied to the factorial T, a principal prime $Q = \langle Q(\boldsymbol{x})\rangle$, and the generically exact $\mathscr{C}_\bullet$,

$$\operatorname{ord}_{Q(\boldsymbol{x})}(\det(\mathscr{C}_\bullet)) = \sum_i (-1)^i \operatorname{mult}_Q(H_i\mathscr{C}_\bullet)$$

Since $\mathscr{C}_\bullet$ is exact, all the higher homology vanishes, so

$$\operatorname{ord}_{Q(\boldsymbol{x})}(\det(\mathscr{C}_\bullet)) = \operatorname{mult}_Q(H_0\mathscr{C}_\bullet)$$

which is 0 outside of $\operatorname{ann}(H_0\mathscr{C}_\bullet)$. By Lemma 5.3 and 5.4 the latter is exactly P. Taking the sum over all non-associate irreducible homogeneous polynomials,

$$\operatorname{div}(\det(\mathscr{C}_\bullet)) = \sum_{Q(\boldsymbol{x})} \operatorname{ord}_{Q(\boldsymbol{x})}(\det(\mathscr{C}_\bullet)) \cdot [V(Q)]$$

$$= \sum_{Q(\boldsymbol{x})} \mathrm{mult}_Q(H_0\mathscr{C}_\bullet) \cdot [V(Q)]$$
$$= \mathrm{mult}_P(H_0\mathscr{C}_\bullet) \cdot [Y]$$
$$= \mathrm{length}_{T_P}(B_{\boldsymbol{d},\bullet})_P \cdot [Y]$$

establishing the claim. □

Lemma 5.6. *For any fixed* $\boldsymbol{d}$*,*

$$\mathrm{length}_{T_P}(B_{\boldsymbol{d},\bullet})_P = \dim_{K(T/P)}(B_{\boldsymbol{d},\bullet})_P$$

Proof. Note that $P(\boldsymbol{x}) \in I$, so $P(\boldsymbol{x})$ annihilates B as a T-module. Setting $\mathfrak{m}_P = PT_P \subset T_P$,

$$\mathrm{length}_{T_P}(B_{\boldsymbol{d},\bullet})_P = \sum_k \dim_{T_P/\mathfrak{m}_P} \mathfrak{m}_P^k(B_{\boldsymbol{d},\bullet})_P/\mathfrak{m}_P^{k+1}(B_{\boldsymbol{d},\bullet})_P$$
$$= \dim_{T_P/\mathfrak{m}_P}(B_{\boldsymbol{d},\bullet})_P$$

since $\mathfrak{m}_P$ in turn annihilates $(B_{\boldsymbol{d},\bullet})_P$. [A one-line argument is: the T- and T/P-module structure of B are the same.] □

Lemma 5.7. *Let* $\boldsymbol{d} \in \mathrm{reg}(B_P)$ *in the sense of* (**??**). *One has*

$$\dim_{K(T/P)}(B_{d,\bullet})_P = \deg\phi$$

Remark. This result requires neither $\mathrm{codim}(Y \subset \mathbb{P}^n) = 1$ nor $\dim V(J) = 0$.

Proof. Let $\Gamma = \mathrm{Biproj}(B)$ be the graph of the rational map ϕ, or equivalently the blow-up of X along the basepoints $V(J)$. Let $E \subset \Gamma$ be the exceptional locus. We point the reader to (**?**, II, Example 7.17.3) for the details, summarized in the following commutative diagram

$$\begin{tikzcd}
\operatorname{Biproj}(B) = \Gamma \arrow[d, "\pi_1"] \arrow[rr, bend left, "\pi_2"] & \Gamma - E \arrow[l, hook'] \arrow[d, "\pi_1|_{\Gamma - E}"', "\cong"] \arrow[dr, two heads, "\pi_2|_{\Gamma - E}"] & \\
X \arrow[rr, dashed, bend right, two heads, "\phi"'] & X - V(J) \arrow[l, hook'] \arrow[r, two heads, "\phi|_{X-VJ}"] & Y \subset \mathbb{P}^n
\end{tikzcd}$$

Since $\pi_2|_{\Gamma - E} = \phi|_{X-V(J)} \circ \pi_1|_{\Gamma - E}$ and $\pi_1|_{\Gamma - E}$ is an isomorphism, the morphism π_2 is generically finite onto its image and $\deg(\phi) = \deg(\pi_2)$. If $\gamma \in \mathbb{P}^n$ is the generic point of Y, then the scheme-theoretic fiber

$$\pi_2^{-1}(\gamma) = \operatorname{Spec}(\mathscr{O}_{\gamma,Y}) \times_Y \Gamma$$

is a closed zero-dimensional subscheme of Γ consisting of $\deg(\pi_2)$ points, counted with multiplicity.

The morphism $\pi_2 : \operatorname{Biproj}(B) \to \operatorname{Proj} T$ is induced by the graded map of $\mathbb{C}$-algebras

$$\pi_2^\sharp = (x_j \mapsto \overline{x_j \otimes 1}) : T \longrightarrow (T \otimes S)/I = B$$

and the fiber of $\gamma = [P] \in \operatorname{Proj} T$ corresponds to the set of bihomogeneous prime ideals of B which pull back to $P \subset T$ via $\pi_2^\sharp$. By an easy reduction, this set corresponds to the set of homogeneous in the S-grading prime ideals in the finite-type $K(T/P)$-algebra

$$K(T/P) \otimes_{T/P} B \cong B_P$$

The identification above presents the fiber as a $\operatorname{Proj}(-)$ over a field, i.e.

$$\operatorname{Proj}_{K(T/P)} B_P \xrightarrow{\sim} \pi_2^{-1}(\gamma) \subset \operatorname{Biproj}(B)$$

Since this is a finite projective scheme over the field $K(T/P)$, its degree is given by its constant Hilbert polynomial, i.e. by

$$\dim_{K(T/P)} (B_P)_{\boldsymbol{d}} = \dim_{K(T/P)} (B_{\boldsymbol{d},\bullet})_P$$

for all $\boldsymbol{d}$ in the regularity of the projective toric scheme above.

The result follows since by Proposition 3.3, B_P is the Cox ring of the said scheme. $\square$

Example 5.8. The construction of Lemma 5.7 is already apparent in Example **??**. Since $P \subset I$ in R, the bihomogeneous primes in $R/I = B$ which pull back to P are those in

$$(\mathbb{C}[x_0,x_1,x_2]/\langle x_0+x_1-x_2\rangle)[s,t]/\langle x_1 s^2 - x_0 t^2\rangle$$

which intersect the coefficient ring trivially, so in turn, those in

$$(\mathbb{C}[x_0,x_1,x_2] \setminus \langle x_0+x_1-x_2\rangle)^{-1}(\mathbb{C}[x_0,x_1,x_2]/\langle x_0+x_1-x_2\rangle)[s,t]/\langle x_1 s^2 - x_0 t^2\rangle$$

Proof of Theorem 3.6. Fix any such $\boldsymbol{d}$ and let $\mathscr{C}_\bullet$ be a minimal graded free resolution of $\operatorname{coker} N$. By (**?**, A, Theorem 34) which applies since $\mathscr{C}_\bullet$ is exact,

$$\det(\mathscr{C}_\bullet) = \gcd(\mathtt{minors}(r,N))$$

up to a unit of T. But then by Lemma 5.5, 5.6 and 5.7,

$$\begin{aligned}
\operatorname{div}(\gcd(\mathtt{minors}(r,N))) &= \operatorname{div}(\det(\mathscr{C}_\bullet)) \\
&= \operatorname{length}_{T_P}(B_{d,\bullet})_P \cdot [Y] \\
&= \dim_{K(T/P)}(B_{d,\bullet})_P \cdot [Y] \\
&= \deg\phi \cdot [Y]
\end{aligned}$$

Because this is just an equality of Weil divisors,

$$\gcd(\mathtt{minors}(r,N)) = P^{\deg\phi}$$ $\square$

Proof of Corollary 3.7. This follows directly from Theorem 3.6. $\square$

Proof of Corollary 3.8. Recall that the baselocus of ϕ is at worst locally complete intersection if and only if the natural morphism of schemes

$$\mathrm{Biproj}(\mathrm{Rees}_S(J)) \longrightarrow \mathrm{Biproj}(\mathrm{Sym}_S(J))$$

is an isomorphism.

We can now follow the proof of Theorem 3.6 verbatim replacing N by N_1 and B by $\mathrm{Sym}_S(J)$. Note that in this case $\boldsymbol{d}$ must be taken from the regularity of

$$(T-P)^{-1}(\mathrm{Sym}_S(J)/P\,\mathrm{Sym}_S(J))$$

as a $K(T/P)$-algebra instead. $\square$

We recall the statement of Proposition **??** [change the number below].

Proposition 5.9. *Let $\mathbb{C}[s_0,\ldots,s_m]$ be the fixed ambient polynomial ring of S' as described in 2.5. Define the ideal I_B of $\mathbb{C}[s_0,\ldots,s_m;x_0,\ldots,x_n]$ by the equality*

$$B = \mathbb{C}[\boldsymbol{s};\boldsymbol{x}]/I_B$$

Let $>'$ be any product order in which the $\boldsymbol{s}$ variables come before the $\boldsymbol{x}$ variables. Then a reduced Gröbner basis for I_B with respect to $>'$ has the form

$$\begin{aligned}
g_1(\boldsymbol{s};\boldsymbol{x}) &= p_1(\boldsymbol{x})\boldsymbol{s}^{\alpha_1} + \textit{lower order terms}\\
&\cdots\\
g_r(\boldsymbol{s};\boldsymbol{x}) &= p_r(\boldsymbol{x})\boldsymbol{s}^{\alpha_r} + \textit{lower order terms}\\
g_{r+1}(\boldsymbol{s};\boldsymbol{x}) &= P(\boldsymbol{x})
\end{aligned}$$

and

$$\deg(\phi) = \deg\left(\langle \boldsymbol{s}^{\alpha_1},\ldots,\boldsymbol{s}^{\alpha_r}\rangle \subset \mathbb{C}[\boldsymbol{s}]\right)$$

Proof. For any choice of $>'$, a reduced Gröebner basis for I_B will have the outlined general form except possibly for the term g_{r+1}. By Lemma **??** and its proof, $I \cap T = P$, so also $I' \cap T = P$. This establishes g_{r+1} and shows that $\alpha_k \neq 0$.

By reduceness, $P(\boldsymbol{x}) \nmid p_k(\boldsymbol{x})$, so the initial ideal of the preimage I'_B of I_P in $K(T/P)[\boldsymbol{s}]$—formed after factoring P and inverting $(T \setminus P)$—is generated by the $\boldsymbol{s}^{\alpha_k}$.

By Lemma 5.7, $\deg(\phi)$ is given by the constant Hilbert polynomial of B_P, so we only need to show that the latter coincides with the degree of I'_B. But this is clear since I'_B can be formed by taking any generators of I_P and adding generators for the extension of J to $K(T/P)[\boldsymbol{s}]$. □

Example 5.10. ... Give a good example.

Conjecture 5.11. *Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ and $J \subset S = \mathbb{C}[s,u;t,v]$ be given by 4 general $(2,2)$ forms in $\langle s^2, st, t^2 \rangle$. Then ϕ is generically 1-1 with the unique basepoint $(0,1) \times (0,1)$ of degree 3 and multiplicity 4, and the image is singular along 3 non-degenerate concurent lines.*

Lemma 5.12. *Let L_1, L_2, L_3 be general...*

Lemma 5.13 (`ex10`)**.** *Let $P(\boldsymbol{x})$ be a form of degree 4 on $\mathbb{P}^3$ and set $Y = V(P(\boldsymbol{x}))$. Suppose that $\mathrm{Sing}(Y)$ contains 3 non-degenerate concurent lines. Then there is a square 4×4 matrix of linear forms M such that $\det(M) = P(\boldsymbol{x})$.*

Proof. After a linear change of coordinates on $\mathbb{P}^3$ we can assume that the lines are the 3 coordinate axes in the distinguished $\{x_3 \neq 0\}$, i.e. $V(x_0, x_1)$, $V(x_1, x_2)$ and $V(x_0, x_2)$.

On the level of ideals, using Euler's identity, the assumption translates to

$$\langle P_{x_0}(\boldsymbol{x}), \ldots, P_{x_3}(\boldsymbol{x}) \rangle \subset \langle x_0x_1, x_0x_2, x_1x_2 \rangle$$

so writing out $P(\boldsymbol{x}) = \sum_{|\alpha|=4} a_\alpha \boldsymbol{x}^\alpha$ and noting that each of the partials must be zero modulo $\langle x_0x_1, x_0x_2, x_1x_2\rangle$, we only need to solve a linear system in the indeterminante coefficients. We get

$$P(\boldsymbol{x}) = a_0x_0^2x_1^2 + a_1x_0^2x_2^2 + a_3x_1^2x_2^2 + a_3x_0^2x_1x_2 + a_4x_0x_1^2x_2 + a_5x_0x_1x_2^2 + a_6x_0x_1x_2x_3$$

which is the determinant of

$$M = \begin{bmatrix} x_0 & & & x_1 \\ & x_1 & & x_2 \\ & & x_2 & x_0 \\ -a_2x_2 & -a_1x_0 & -a_0x_1 & a_6x_3 + a_3x_0 + a_4x_2 + a_5x_3 \end{bmatrix}$$

In fact, if $a_6 \neq 0$ we can do better. Setting $a_6 = 1$, a linear change of coordinates by the matrix

$$\begin{bmatrix} 1 & & & -a_3 \\ & 1 & & -a_4 \\ & & 1 & -a_5 \\ & & & 1 \end{bmatrix}$$

leaves the singular locus intact but simplifies the general form to

$$P'(\boldsymbol{x}) = a_0x_0^2x_1^2 + a_1x_0^2x_2^2 + a_2x_1^2x_2^2 + x_0x_1x_2x_3$$

The matrix M then becomes

$$M' = \begin{bmatrix} x_0 & & & x_1 \\ & x_1 & & x_2 \\ & & x_2 & x_0 \\ -a_2x_2 & -a_1x_0 & -a_0x_1 & x_3 \end{bmatrix}$$

each of whose coordinates is a scaled variable. $\square$

[Say how this is related to taking $\boldsymbol{d} = (1,1)$, what other methods apply, about the matrix N_1 can be taken as a submatrix of this matrix M and how this changes if I pre-compose the map with a map $\mathbb{P}^2 \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ instead.]

CHAPTER 6

A METHOD FOR FAST IMPLICITIZATION

This chapter is devoted to a description of a method for fast implicitization using the theory developed in Chapters 3 and 5. We present our main algorithm in Section 6.1 after we discuss the problems arising from the naive algorithm which we used for the examples of Chapter 4. In Section **??** and **??**, we employ the algorithm and its implementation to compute examples which are unfeasible to attack using the standard methods.

While some details on the Macaulay2 implementations follow the pseudo code presented, the code is available at

`http://www.math.cornell.edu/~rzlatev/phd-thesis/algos`

We continue to assume the notation of Chapter 2 and the setup of Chapter 3.

6.1 Algorithm

At first glance, an algorithm for finding the implicit equation is contained in the proof of the our main theorem, Theorem **??**. It consist of finding a T-module generating set for $I_{\boldsymbol{d},\bullet}$, which can be done using Lemma **??**. In fact, this is exactly what used to construct a naive algorithm for finding N in all explicit calculations so far. The pseudo code for the algorithm and its implementation follow.

Algorithm 6.1. NAIVE ALGORITHM.

Set $r = \dim_{\mathbb{C}}(S_{\boldsymbol{d}})$

Compute a generating set $\{g_k : k\}$ for $I_{\boldsymbol{d},\bullet}$ using Lemma **??**

Set N to be the coefficient matrix of the g_k with respect to $\texttt{basis}(S_{\boldsymbol{d}})$

Compute $P = \gcd(\texttt{minors}(r,N))$

Implementation 6.2. We first need an implementation of Lemma **??**. It takes an R-module M and degree $\boldsymbol{a}$ on S, and tries to produces the linearly independent elements in $M_{\boldsymbol{a},\bullet}$ each generator gives rise to.

```
procPushGensHigher = (a,r,M) -> (
  G := for g in M_* list (
    if all(a-((degree g)_r), Z->Z>=0) then
      basis((a-((degree g)_r))|{0},ring M)**g
    else continue);
  trim image fold(G,matrix {{0_(ring M)}},(a, b)->a|b)
)
```

The actual implementation of Algorithm 6.1 is pretty straight-forward. We only note that the ambient ring R is required as input for encapsulation.

```
ComputeNRees = method ()
ComputeNRees (Ideal, List, Ring) := Matrix => (J, d, R) -> (
  x := symbol x;
  I := reesIdeal(J, Variable=>x);
  AI := ring I;
  zm := 0*d;
  g := map(R,AI,first entries super basis(zm|{1},R));
  I = g(I);
  r := toList(0..#(degree J_0)-1);
  V := procPushGensHigher (d,r,I);
  matrix entries ( (gens V) // basis(d|{0}, R) )
)
```

6.3. The flaws of this algorithm as a method for fast implicitization are fatal.

1. Line 3 of the algorithm refers to Lemma **??** but unless we were given an R-generating set for the Rees ideal, we have to compute it from scratch. This is at least as hard a computation as the computation of the implicit equation in a very strict sense — the Rees ideal contains the implicit equation as its only generator in bidegree $(0,\bullet)$. This follows from Proposition 3.12 and is shown in Example

4.14. If we had a way to efficiently compute this, however, then we would only need an algorithm find the unique generator in the aforementioned degree.

2. In line 5, while computing the gcd of polynomials is fast, the number of minors to be computed can be very large. This happens even for reasonably small examples — the one described in Section **??** finds a matrix N of size 18×50. The number of minors to compute is

$$\binom{50}{18} = 18053528883775$$

so even if it took the unrealistic 0.001 seconds to compute each minor, a single machine would require 572 years to compute them all.

3. It gets worse. Coming back to the example referenced in 2, we note that each minor is a determinant of order 18, over 5 variables, and of total degree 72. Computing large determinants symbolically is time-consuming. The calculation of the determinant of a 13×13-matrix of quartic forms over 5 variables took about an hour to finish. Scaling this, we can expect this calculation to take around

$$14 \times 15 \times 16 \times 17 \times 18 = 1028160$$

hours or about 714 days.

4. Finally, suppose we have found the polynomial in question — by whatever means. It is a form of degree 48 over 5 variables, and very likely dense in those monomials. This suggests that the polynomial will be represented by

$$\binom{53}{5} = 2869685$$

coefficients.

Regrettably, (6.3.3) would be an issue for any algorithm relying on computing determinants of matrices of syzygies, while (6.3.4) would an issue for any implicitization algorithm whatsoever. However, this can be used as an argument *for* using syzygy matrices in place of the implicit equation altogether. The examples of Sections **??** and **??** further explore this theme.

In addition, we note that issue (6.3.2) could and should be attacked by parallel computing in addition to whatever method is used.

FINISH THIS TODAY!

[also mention that when $d = e$ this is just solving a system of linear equations for each of the coefficients of $P(\boldsymbol{x})$. See **?** for an article devoted to this idea.]

This chapter describes an algorithm for computing the closed image in the running setup. While somewhat similar algorithms have been proposed along the lines of the approximation complex (see [ref]), our algorithm utilizes the theory developed in Charpter **??** and allows for a general treatement regardless of the structure of the basepoints. Compared to the Gröbner basis calculations and even more naive approaches such as direct linear algebra calculations of the coefficients [], the method presented in Section 6.1 has two advantages:

1. its bottleneck parallelizes directly and, in fact, can be tackled separately;
2. it is not a blackbox in terms of the geometry, which allows for a probabilistic speedups;

In Section 6.3 we study a concrete example in which the second property manifests. This allows us to compute an image of degree 26 in 20 minutes as oppesed to 5 hours using Gröbner bases computations. In Section 6.4 we come back the failing example from Section **??** and compute its degree-48 image.

6.4. We continue to assume the setup of 2. We work over a fixed degree $\boldsymbol{d}$ in the regularity of B_P. To avoid distraction, we assume throughout this chapter that ϕ is birational. Most of what is said here holds for maps of degree greater than 1 essentially by substituting $P^{\deg \phi}$ for P.

6.5. At first glance, it seems that an algorithm for the computation of the image of ϕ can be extracted from our main result, Theorem 3.6, and in fact, is already outlined in the examples of Chapter 4. In its simplest form, it becomes

6.6. In reality, unless we figure out how to compute N efficiently, the algorithm boils down to computing a Gröbner basis again. In fact, one that is much more complex — we need to find a Groöbner basis for the Rees ideal I, which already contains $P(\boldsymbol{x})$ as the unique generator in degree $(0, \deg(Y))$. So while Algorithm **??** may be useful when studing the relation between the baselocus and the geometry of the image, it is rather useless as means of implicitization.

6.7. Another problem with Algorithm **??** is the computation of the minors. While computing the gcd of polynomials is fast, symbolicly computing determinants of large order is very time consuming even for a small number of variables.

6.8. Recall that

$$N = N_1 \mid \cdots \mid N_\delta$$

where N_i is the submatrix of N whose columns are degree-i syzygies. Loosely speaking, our plan is the following:

1. instead of computing the whole matrix N, we compute the N_i's and keep track of a partial syzygy matrix N';
2. instead of computing all the minors, we only compute as many as needed to determine the gcd correctly;

These two simple observations produce an immense speed up on average. In the case of the example described in the previous paragraph, we only need to compute 2 minors of N_4 [are you sure this is not all of N?]

Another feature of the algorithm that we are not going to pursue is

3. the setup should allow for an easy parallelization

Lemma 6.9. *Let*

$$N' = N'_i = N_1 \mid \cdots \mid N_i$$

be the partial matrix of syzygies up to degree i. If N' satisfies the following condition

$$\mathtt{minors}(r, N') \neq 0 \textit{ and } \deg(\gcd(\mathtt{minors}(r, N'))) = \deg(Y) \tag{C1}$$

then

$$\gcd(\mathtt{minors}(r, N')) = P$$

Proof. By Lemma **??**, we have that $\det(M) \in P$ for any square matrix of syzygies, so for every minors of N' in particular. It follows that the gcd of the minors is in P. The sufficiency follows since P is principal of the same degree as the gcd. □

6.10. After we detail the outline above, we prove correctness of the algorithm and then discuss some of its strong and weak points.

Algorithm 6.11. MAIN – FIRST VERSION.

input: none

output: the implicit equation P

Set $r = h^0(S_{\boldsymbol{d}})$

Set $N' = r \times 0$ matrix over T

while C1 is not satisfied for N' **do**

  Given $N_1, \ldots, N_{i-1}$, use Algorithm 6.12 to compute N_i

  Set $N' = N' \mid N_i$

end while

Report $P = \gcd(\mathtt{minors}(r, N'))$

Algorithm 6.12. COMPUTEPARTIALSYZYZIES.

input: a list of matrices of sygygy-generators $N_1, \ldots, N_{i-1}$

output: the syzygy-generators matrix N_i

for $0 < j < i$ **do**

Set $N_{ji} = \texttt{basis}(T_{i-j}) \otimes N_j$

Set K_{ji} to be the linearization of N_{ji}

end for

Set $K_i = \ker(\Phi^{(i)})$

Let K_i' be such that $\mathrm{Span}(K_i) = \mathrm{Span}(K_i') \oplus (\sum_j \mathrm{Span}(K_{ji}))$

Let N_i be such that $\texttt{basis}(R_{\boldsymbol{d},i}) \cdot K_i' = \texttt{basis}(S_{\boldsymbol{d}}) \cdot N_i$

Report N_i

6.13. A few remarks are in order. We prove the correctness of Algorithm 6.11 below and provide a complete realization in Macaulay2 in Implementation **??**. For example, finding a matrix "such that" is really finding a coefficient matrix for some vectors given some basis.

Theorem 6.14. *Let $\phi : X \longrightarrow \mathbb{P}^n$ be as before. Suppose that $\mathbf{d}$ is in the regularity of* $\mathrm{Proj}(B_P)$*. Then Algorithm 6.11 correctly computes $P^{\deg \phi}$ in at most δ steps, where δ is the maximal degree showing up in a minimal set of homogeneous generators for $I_{\boldsymbol{d},\bullet}$.*

Proof. Given that C1 is suffient, we only need to show N' is going to satisfy it after finitely many steps. But $N' = N$ at step δ and N satisfies the condition by Theorem 3.6. It follows that the matrix N' is going to satisfy C1 after at most δ iterations, in which case the gcd of the minors is P.

Remark: any nonzero minor is divisible by $P^{\deg \phi}$ by the main theorem. □

6.15. As presented, the algorithm has an obvious bottle neck—the number of minors to be computed is *very* large. For example, [show the example of the $\binom{72}{36}$ and use Stirling numbers to say how many matrices there are]. A probabilistic remedy is the following.

Make a generic change coordinates on the column-space of N. Now pick minors as along as they are nonzero (test rank) but vanish on the ϕ_j (test rank after substitute). Now you expect every next minor to knock the degree of the gcd down by at least one. Stop when the degree become $\deg(\phi)\deg(Y)$. This method obviously terminates—eventually all the minors are taken—but generally *much* quicker than computing all the minors. It is worth pointing out that the gcd computation is fast, computing determinants symbolically isn't.

6.16. We now present the final version of our algorithm using the probabilistic speed-ups described above. Because the algorithm's pseudo code closely follows typical computer algebra system's routines, we close the section with a realization in Macaulay2. In the next two sections, we explore the positives of even further, showing how it is sometimes possible to avoid even the determinant computations.

6.2 Implementation in Macaulay2

6.17. See `http://www.math.cornell.edu/~rzlatev/phd-thesis/`. Should include it here in one way or another or at least mention what I really do computationally.

6.3 First Example

As a first showcase example, we carry out the computation of a general degree-26 implicit equation with a preset baselocus.

Let $X = \mathbb{P}^1_{s,u} \times \mathbb{P}^1_{t,v}$ and $Q = \langle s^3, s^2t, t^2\rangle$. Let $\mathfrak{m} = \langle st, sv, ut, uv\rangle$ be the irrelevant ideal of X. Take J to be an ideal generated by 4 general biquartics $\phi_0,\dots,\phi_3$ in Q and set

$$\phi = (\phi_0,\dots,\phi_3) : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3$$

Note that here *general* refers to a choice of $\mathrm{Span}_{\mathbb{C}}\{\phi_0,\dots,\phi_3\}$ as an element in

$$\mathrm{Gr}(4, \mathtt{basis}(Q_{4,4})) \cong \mathbb{G}(3,19)$$

with respect to the resulting structure of I—see Theorem **??** for a precise statement. Over a finite field, however, this is equivalent to a *uniformly random* choice of the 80 coefficients.

A possible realization in Macaulay2 follows.

```
i1 : KK=ZZ/32009;
i2 : S=KK[s,u,t,v,Degrees=>{2:{1,0},2:{0,1}}];
i3 : T=KK[x_0..x_3];
i4 : Q=ideal(s^3,t^2,s^2*t);
o4 : Ideal of S
i5 : mm=ideal(s,u)*ideal(t,v);
o5 : Ideal of S
i6 : super basis({4,4},Q);
               1        20
o6 : Matrix S  <--- S
i7 : J=ideal(o6*random(S^20,S^4));
o7 : Ideal of S
```

Since Q is saturated with respect to $\mathfrak{m}$ and the saturation of J is Q, scheme-theoretically, ϕ's baselocus is just $V(Q)$. The latter is supported on a single point $q = (0,1)\times(0,1)$ of degree 5 and multiplicity 6.

```
i8 : decompose Q
o8 = {ideal (s, t)}
o8 : List
i9 : saturate(Q,mm)==Q
o9 = true
```

```
i10 : saturate(J,mm)==Q
o10 = true
i11 : multiplicity Q
o11 = 6
i12 : degree Q
o12 = 5
```

We refer the reader to Section **??** for an explanation why Macaulay2's routines compute the correct values (in biprojective space) and how to compute them by hand in the case of monomial baselocus.

We continue by setting up the ambient ring R.

```
i13 : R=KK[s,u,t,v,x_0..x_3,Degrees=>{2:{1,0,0},2:{0,1,0},4:{0,0,1}}];
i14 : J=sub(J, R);
o14 : Ideal of R
```

Fix $\boldsymbol{d}=(3,3)$. Because $\delta=2$ in this case, we can compute the N_{ji} matrices directly. Here is an alternative calculation to the one described in the previous section.

We start by computing N_1.

```
i15 : subs=toList((0..3)/(i->x_i=>J_i));
i16 : time G1=sub(super basis({3,3,1},R),subs)//(super basis({7,7,0}, R));
     -- used 0.013008 seconds
                64       64
o16 : Matrix R   <--- R
i17 : time K1=matrix entries gens ker G1;
     -- used 0.014648 seconds
                64       5
o17 : Matrix R   <--- R
i18 : time N1=(super basis({3,3,1},R))*K1//(super basis({3,3,0},R));
     -- used 0.000439 seconds
                16       5
o18 : Matrix R   <--- R
```

At this point $N'=N_1$. Because there are only 5 linear syzygies, N' cannot satisfy C1, so we continue.

```
i19 : time G2=sub(super basis({3,3,2},R),subs)//(super basis({11,11,0}, R));
     -- used 0.19233 seconds

              144       160
o19 : Matrix R    <--- R

i20 : time K22=matrix entries gens ker G2;
     -- used 0.177021 seconds

              160       32
o20 : Matrix R    <--- R

i21 : time K12=(((super basis({3,3,1},R))*K1)**(super basis({0,0,1}, R)))//(super basis({3,3,2},R));
     -- used 0.009606 seconds

              160       20
o21 : Matrix R    <--- R

i22 : time K2=gens trim image (K22%K12);
     -- used 0.009801 seconds

              160       12
o22 : Matrix R    <--- R

i23 : time N2=(super basis({3,3,2},R))*K2//(super basis({3,3,0},R));
     -- used 0.007547 seconds

              16       12
o23 : Matrix R   <--- R

i24 : time N=N1|N2;
     -- used 0.000043 seconds

              16       17
o24 : Matrix R   <--- R
```

We now have a partial matrix $N' = N_1 \mid N_2$ with 5 linear and 12 quadratic columns. This gives more columns than rows and, furthermore, any maximal minor would have degree 27 or 28, depending on whether it involves all the linear columns or not.

Because we actually have $N = N'$ in this case, N' would certainly satisfy C1. A probabilistic test for N' follows.

```
i25 : time Nf=sub(N, subs);
     -- used 0.447607 seconds

              16       17
o25 : Matrix R   <--- R

i26 : time rank N
     -- used 0.007021 seconds

o26 = 16

i27 : time rank Nf
     -- used 0.910507 seconds

o27 = 15
```

This shows us that the ideal of maximal minors of N is nonzero and that every minor is a multiple of the implicit equation. We randomly pick two minors of degree 27 and check if C1 is satisfied.

```
i28 : cols1=sort join({0,1,2,3,4},RandPerm(5,16,11))
o28 = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15}
o28 : List
i29 : time rank (N_cols1)
     -- used 0.00428 seconds
o29 = 16
i30 : time rank (Nf_cols1)
     -- used 5.32849 seconds
o30 = 15
i31 : cols2=sort join({0,1,2,3,4},RandPerm(5,16,11))
o31 = {0, 1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16}
o31 : List
i32 : cols1==cols2
o32 = false
i33 : time rank (N_cols2)
     -- used 0.005311 seconds
o33 = 16
i34 : time rank (Nf_cols2)
     -- used 5.15664 seconds
o34 = 15
```

We selected 11 random columns out of the 12 quadratic syzygies and added them to the list of linear syzygies. The (0-indexed) columns represented by the lists $cols_1$ and $cols_2$ correspond to nonzero minors vanishing on the implicit equation. Because both are of degree 27 and the equation of degree 26, those are going to be linear multiples of $P(\boldsymbol{x})$.

We have two possibilities for finding the equation. The first is direct computation.

```
i35 : time D1=det (N_cols1);
     -- used 475.552 seconds
i36 : time D2=det (N_cols2);
     -- used 485.142 seconds
i37 : time P=gcd (D1,D2);
     -- used 0.19819 seconds
```

Note that each of the determinants took about 8 minutes to get while the gcd — under 20 seconds. Computing determinants symbolically is another routine that is readily parallelizibe but very time consuming in general.

For a comparison the following – the machine, the time, etc, etc

[Say why this is needed and that after a general change of coordinates... the latter is part of the algorithm.]

[Sometimes inspecting N_{cols_1} we are able to find a 15×15 minor which would then be the implicit equation.]

Sometimes we can only give the answer as the gcd of two minors. Next example explores this situation.

6.4 Second Example

I have a few more examples in this spirit. Specifically, examples 604 and 605 in `./test` at https://github.com/radoslavraynov/ImplAlgos. Those show that wheather or not the numerics workout nicely, one should not study the syzygies separately, far less so, should one restric to linear and quadratic syzygies. Some of the matrices turn out to be square but also, what is apparent, is that there is no single recipe for what $\boldsymbol{d}$ to be used for a given $\boldsymbol{e}$ and some adjustments have to be made in order to balance the size of the matrices and the degrees of the entries (the degrees of the syzygies).

Include the example where we compute the image of 5 general $(2,2,2)$-forms on $\mathbb{P}^1 \times \mathbb{P}^1$. This example is hard. Here we have a basepoint-free birational map whose image is of degree 48. Over $\boldsymbol{d} = (1,1,1)$ we get $r = 8$, so one can hope that something along the lines of the result over $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ would work. Admittedly, I hoped that the image would be the determinant of an 8×8 matrix of sextics, and for this to be true, there has to be no syzygies in degree 5. Regrettably, things could not be further from the truth — there are no syzygies in degree up to 8, and computation for larger degrees would take even longer than the few hours or so I needed on my machine.

However, in degree $\boldsymbol{d} = (2,2,1)$, I get, easily or rather, quickly, an 18×50 matrix N_4 and $N_1 = N_2 = N_3 = 0$. Each of the maximal minors has degree 72 and has as a factor our degree-48 implicit equation.

Computing even a single determinant is very very hard—it took me an hour to compute a determinant of a 12×12 matrix in 5 variables, so it would probably take something in the magnitude of $13 * 14 * 15 * 16 * 17 * 18$ hours to compute a single 18×18 minor. The good news is that we only need two. In fact, by intersecting with a line in $\mathbb{P}^4$ we can detect if two minors share more than just the implicit equation—simply find the multiplicity of the zero-dimensional intersection. If it is 48, the gcd of those 2 is our guy. This is easy and quick—takes a minute or so.

CHAPTER 7

KOSZUL SYZYGIES AND BASEPOINT-FREE MAPS

7.1 Template Proofs

7.1. Most of the proofs of the "linear and quadratic syzygies"-type results consist, in one form or another, of the following steps

1. consider specific X and set some conditions on the base locus
2. choose $\boldsymbol{d}$ as a fixed function on $\boldsymbol{e}$
3. come up with a way to construct a square matrix by selecting a number of linear and quaratic syzygies
4. show that the determinant is in P or $P^{\deg\phi}$
5. show that the determinant is nonzero

Examples of this approach are [give a long list of refereces].

The conditions on the baselocus in (1) and (2) are dictated by making the numerics in (3) workout. Step (4) is often automatic, for example when map is assumed birational, while step (5) usually involves a cleverly chosen normal form for the syzygies, allowing the authors to assert that the determinant is nonzero.

Clearly, the above pattern is just a statement about the matrix N in a specific degree $\boldsymbol{d}$. By Theorem **??**, the h-vector of N depends only on the base locus of ϕ, but getting a hold of this relation is difficult. An example of this is the long lists of conditions in [the paper with B1-B6] in case the basepoints (over $\mathbb{P}^2$ and $\mathbb{P}^1\times\mathbb{P}^1$) are complete intersection points.

In the two simplest cases, however, of basepoint-free maps over $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$, there are very few extra conditions, and it is desirable to know if our matrices match the ones described elsewhere. We show that this is indeed the case in Sections 7.3 and 7.4. Because of the simplicity of the baselocus, or rather the absence there of, our proofs are streamlined by the fact that syzygies of low degree exhibit a property akin to Koszulness.

7.2 Koszul Syzygies in Low Degree

Proposition 7.2. *Let X be as in Section* **??**. *Let $\phi_0, \dots, \phi_n$ be n forms in degree $\boldsymbol{p}$ having no common zero on X, and let $\boldsymbol{q}$ be any degree such that*

$$H^1(X, \mathscr{O}_X(\boldsymbol{q} - (n-1)\boldsymbol{p})) = 0$$

Then for any $k \geq 1$ the syzygies of

$$M_0 = \begin{bmatrix} \phi_0^k & \phi_0^{k-1}\phi_1 & \dots & \phi_n^k \end{bmatrix}$$

in degree $\boldsymbol{q}$ are Koszul.

Proof. Let M_ℓ, $\ell = 1, \dots, n-1$, be matrices resolving M_0 when the ϕ_j are thought of as variables. Writing $\star$ for any positive integer which is not essential for the proof, we get a complex of coherent sheaves of modules on X,

$$\mathscr{O}_X^1 \xleftarrow{\widetilde{M}_0} \mathscr{O}_X(-k\boldsymbol{p})^\star \xleftarrow{\widetilde{M}_1} \mathscr{O}_X(-(k+1)\boldsymbol{p})^\star \xleftarrow{\widetilde{M}_2} \cdots \xleftarrow{\widetilde{M}_{n-1}} \mathscr{O}_X(-(k+n-1)\boldsymbol{p})^\star \longleftarrow 0$$

and tensoring with the locally free $\mathscr{O}_X(k\boldsymbol{p}+\boldsymbol{q})^1$, we get the following complex $\mathscr{K}_\bullet$

$$\mathscr{O}_X^1(k\boldsymbol{p}+\boldsymbol{q}) \xleftarrow{\widetilde{M}_0} \mathscr{O}_X(\boldsymbol{q})^\star \xleftarrow{\widetilde{M}_1} \mathscr{O}_X(\boldsymbol{q}-\boldsymbol{p})^\star \xleftarrow{\widetilde{M}_2} \cdots \xleftarrow{\widetilde{M}_{n-1}} \mathscr{O}_X(\boldsymbol{q}-(n-1)\boldsymbol{p})^\star \longleftarrow 0$$

Let U be a distinguished affine open in X. Restring $\mathscr{K}_\bullet$ to U is equivalent to first restricting the ϕ_j to U and then constructing the M_ℓ. But since the restrictions ϕ_j have no common zero on U, and $U \cong \mathbb{A}^{n-1}$, the n restrictions form a regular sequence there. It follows that the restriction of $\mathscr{K}_\bullet$ to U is acyclic. Since X has is covered by such opens U, $\mathscr{K}_\bullet$ itself must be acyclic.

By the last paragraph, $\mathscr{K}_\bullet$ is exact, so we can split it into two exact sequences, as the commutative diagram below shows,

$$\begin{tikzcd}
\mathscr{O}^1_X(k\boldsymbol{p}+\boldsymbol{q}) & \mathscr{O}_X(\boldsymbol{q})^\star \arrow[l, "\widetilde{M}_0"'] & & \mathscr{O}_X(\boldsymbol{q}-\boldsymbol{p})^\star \arrow[ll, dashed, "\widetilde{M}_1"'] \arrow[dl, "\pi"] & \dots \arrow[l, "\widetilde{M}_2"'] \\
& & \mathrm{Im}(\widetilde{M}_1) \arrow[ul, "\iota"] \arrow[dl] & & \\
& 0 & & 0 \arrow[ul] &
\end{tikzcd}$$

Consider the long exact sequences on cohomology of the exact sequences of sheaves above. For the one involving ι, the releavant part is

$$\dots \xleftarrow{H^0\widetilde{M}_0} H^0\mathscr{O}_X(\boldsymbol{q})^\star \xleftarrow{H^0\iota} H^0\mathscr{O}_X(\boldsymbol{q})^\star \longleftarrow 0$$

while for the other one, we look at

$$\dots \xleftarrow{H^1\widetilde{M}_2} H^1\mathscr{O}_X(\boldsymbol{q}-(n-1)\boldsymbol{p})^\star \xleftarrow{\delta} H^0\mathrm{Im}(\widetilde{M}_1)^\star \xleftarrow{H^0\pi} H^0\mathscr{O}_X(\boldsymbol{q}-\boldsymbol{p})^\star \xleftarrow{H^0\widetilde{M}_2} \dots$$

Finally, let $\boldsymbol{v} \in S_{\boldsymbol{q}}^{\binom{k+n-1}{n-1}} = H^0\mathscr{O}_X(\boldsymbol{q})^\star$ be a syzygy on M_0, and recall that $H^0\widetilde{M}_\ell = M_\ell$ and $\iota \circ \pi = \widetilde{M}_1$. Since $M_0\boldsymbol{v} = 0$, then $\boldsymbol{v} = H^0\iota(\boldsymbol{w})$ for some $\boldsymbol{w} \in \mathrm{Im}(\widetilde{M}_1)$. Since $H^1\mathscr{O}_X(\boldsymbol{q}-(n-1)\boldsymbol{p}) = 0$, so is $H^1\mathscr{O}_X(\boldsymbol{q}-(n-1)\boldsymbol{p})^\star$, and it follows that $H^0\pi$ is surjective. In particular, $\boldsymbol{w} = H^0\pi(\boldsymbol{u})$ for some $u \in H^0\mathscr{O}_X(\boldsymbol{q}-\boldsymbol{p})^\star$. But then

$$\boldsymbol{w} = H^0\iota \circ H^0\pi(\boldsymbol{u}) = H^0\widetilde{M}_1(\boldsymbol{u}) = M_1\boldsymbol{u}$$

establishing the claim. $\square$

Remark. Note that, in the last paragraph of the proof, we cannot use $\mathrm{Im}(\widetilde{M}_1) = \widetilde{\mathrm{Im}(M_1)}$ since X isn't affine—this is the whole point of the proof.

Example 7.3. In the case $n = 3$ and $k = 2$, the matrices M_ℓ can be taken as

$$M_0 = \begin{bmatrix} \phi_0^2 & \phi_0\phi_1 & \phi_1^2 & \phi_0\phi_2 & \phi_1\phi_2 & \phi_2^2 \end{bmatrix}$$

$$M_1 = \begin{bmatrix} -\phi_1 & 0 & -\phi_2 & 0 & 0 & 0 & 0 & 0 \\ \phi_0 & -\phi_1 & 0 & -\phi_2 & 0 & 0 & 0 & 0 \\ 0 & \phi_0 & 0 & 0 & 0 & -\phi_2 & 0 & 0 \\ 0 & 0 & \phi_0 & \phi_1 & -\phi_1 & 0 & -\phi_2 & 0 \\ 0 & 0 & 0 & 0 & \phi_0 & \phi_1 & 0 & -\phi_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & \phi_0 & \phi_1 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} \phi_2 & 0 & 0 \\ 0 & \phi_2 & 0 \\ -\phi_1 & 0 & 0 \\ \phi_0 & -\phi_1 & 0 \\ 0 & -\phi_1 & \phi_2 \\ 0 & \phi_0 & 0 \\ 0 & 0 & -\phi_1 \\ 0 & 0 & \phi_0 \end{bmatrix}$$

Note that the size of M_1 is $\binom{n-1+k}{n-1}\times$, and the size of M_{n-1} is $\binom{n-2+k}{n-1}\times$

Question 7.4. What is the size of M_ℓ in general? This should come from a cellular resolution of the ideal of all monomials in n variables in degree k, right? Also, it's important to say that this is a linear type resolution—all the maps are of degree 1, except for M_0, obviously.

Corollary 7.5. *Let $X = \mathbb{P}^{n-1}$ be the projective $(n-1)$-space or $X = \mathbb{P}^1 \times \cdots \mathbb{P}^1$ be the product of $n-1$ copies of $\mathbb{P}^1$, and let $\phi : X \to \mathbb{P}^n$ be basepoint-free. Let the coordinates of ϕ be in degree $\mathbf{p}$ and suppose that there are as few linear syzygies as possible. Then,* *after a linear change of coordinates on the target, $\phi_0, \dots, \phi_{n-1}$ satisfy the assumption of Proposition* **??**.

Proof. work out $\square$

7.6 (Template Proof)**.**

7.3 Basepoint-Free Maps over $X = \mathbb{P}^1 \times \mathbb{P}^1$

In this section and next section we workout the relation between our method and the ones described in **?**, whenever the latter can be applied. In particular, we first show that the matrices which our method constructs are the same, up to reformatting, as the matrices constructed by the referenced methods, and then show how our streamlined arguments apply to a more general situation.

Theorem 7.7. *Let $X = \mathbb{P}^1 \times \mathbb{P}^1$, ϕ be basepoint-free whose entries are of degree (p,q), and suppose there are no linear syzyges for $d = (p-1, q-1)$. Then N_2 is square, $N = N_2$ and ϕ is birational. In particular,*

$$\det(N) = \det(N_2) = P(\boldsymbol{x})$$

Remark 7.8. Theorem 7.7 implies that if the map ϕ is basepoint-free but not birational, then there must exist linear syzygies in degree $(p-1, q-1)$.

7.9. Let k be a positive integer. We set the matrix $\Phi^{(k)}$,

$$\Phi^{(k)} : S_{p-1,q-1}^{\binom{k+3}{3}} \longrightarrow S^1_{(k+1)p-1,(k+1)q-1}$$

to be a $\mathbb{C}$-linearization of the S-linear map

$$\phi^{(k)} = \begin{bmatrix} \phi_0^k & \phi_0^{k-1}\phi_1 & \cdots & \phi_3^k \end{bmatrix} : S^{\binom{k+3}{3}} \longrightarrow S^1$$

Specifically, $\Phi^{(k)}$ is of size $(k+1)^2 pq \times \binom{k+3}{3} pq$, and so

$$\dim_{\mathbb{C}} \ker(\Phi^{(k)}) \geq \binom{k+1}{3} pq \tag{7.9.1}$$

with equality if and only if $\Phi^{(k)}$ is of maximal rank, that is, of rank $(k+1)^2 pq$.

7.10. The columns of $\Phi^{(k)}$ are indexed by the monomials in $R_{p-1,q-1,k}$ and the elements of the kernel give rise to linear combinationations of those which are exactly the syzygies of degree k for the fixed $d = (p-1, q-1)$. We can write any such syzygy as

$$\sum_{|\alpha|=k} A_\alpha \cdot x_0^{\alpha_0} x_1^{\alpha_1} x_2^{\alpha_2} + \Big(\sum_{|j|=k-1} B_j \cdot x_0^{j_0} x_1^{j_1} x_2^{j_2} \Big) x_3 + \Big(\sum_{|\gamma|=k-2} C_\gamma x^\gamma \Big) x_3^2 \tag{7.10.1}$$

in which the first two summations corresponds to number of columns indexed by monomials involving x_3 at most linearly. This number is $\binom{k+2}{2} pq + \binom{k+1}{2} pq = (k+1)^2 pq$, which is the number of rows too. Set $\Psi^{(k)}$ to be the square submatrix of $\Phi^{(k)}$ formed by those columns.

Note that if the former is nonsingular, then the latter is of maximal rank. The proof only depends on $\Psi^{(k)}$ being of maximal rank though, not the fact that it is square, so we procede in this manner.

We begin by proving that the kernel elements of the S-linear version of $\Phi^{(k)}$ in degree $(2p-1, 2q-1)$ are Koszul, i.e. are $S_{p-1,q-1}$-linear combinations of the Koszul syzygies on the ϕ_j appropriately inflated to match $\Phi^{(k)}$.

While the proof is pretty much the same, as in **?**, we include it here fitted for our notation and generalized slightly.

Lemma 7.11. *The matrix $\Psi^{(k)}$ is of maximal rank for every positive k.*

Proof. The case $k = 1$ is just the assumption that the ϕ_j there are no linear syzygies in degree $(p-1, q-1)$ on the source, i.e. if

$$a_0\phi_0 + a_1\phi_1 + a_2\phi_2 + a_3\phi_3 = 0$$

in S where $a_j \in S_{p-1,q-1}$, then necessarily $a_j = 0$.

The case $k > 1$ is a direct consequence of the case $k = 1$ and the fact that ϕ_0, ϕ_1, ϕ_2 form a regular sequence. Let V,

$$V = \begin{bmatrix} V' \\ V'' \end{bmatrix}$$

be a kernel element of $\Psi^{(k)}$ with V' a column-$\binom{k+2}{2}$-vector over $\mathbb{C}$ and V'' is a column-$\binom{k+1}{2}$-vector over $\mathbb{C}$, corresponding to the coefficients not involving x_3 and the coefficients involving x_3, respectively. We can multiply out to get a k-syzygy on the ϕ_j over $S_{p-1,q-1}$, i.e.

$$\sum_{|\alpha|=k} A_\alpha \cdot \phi_0^{\alpha_0} \phi_1^{\alpha_1} \phi_2^{\alpha_2} + \Big(\sum_{|j|=k-1} B_j \cdot \phi_0^{j_0} \phi_1^{j_1} \phi_2^{j_2} \Big) \phi_3 = 0$$

as an equality in S, where the coefficients A_α, B_j are in $S_{p-1,q-1}$. We rewrite the above as

$$\sum_{|j|=k-1} \big(C_{0,j}\phi_0 + C_{1,j}\phi_1 + C_{2,j}\phi_2 + C_{3,j}\phi_3 \big) \phi_0^{j_0} \phi_1^{j_1} \phi_2^{j_2} = 0$$

which is again an equality in S, and again the coefficients $C_{j,j}$ are in $S_{p-1,q-1}$. Note also that $C_{3,j} = B_j$.

Since ϕ_0, ϕ_1, ϕ_2 form a regular sequence over S, we can rewrite the coefficients as linear combinations of their Koszul syzygies, i.e. for each j we get an equality in S of the form

$$C_{0,j}\phi_0 + C_{1,j}\phi_1 + C_{2,j}\phi_2 + C_{3,j}\phi_3 = D_{0,j}\phi_0 + D_{1,j}\phi_1 + D_{2,j}\phi_2$$

where D_0, D_1, D_2 are in $S_{p-1,q-1}$. By the assumption on the independence of the ϕ_j over $S_{p-1,q-1}$, we must have that $C_{3,j} = 0$. Equivalently, $B_j = 0$ for every j, so $V'' = 0$.

To conclude the proof, note further that any nontrivial syzygy of the monomials in ϕ_0, ϕ_1, ϕ_2 in any degree $k \geq 1$ must be of degree degree at least (p,q), i.e. must be a combination of Koszul syzygies again. Since $\deg(A_\alpha) = (p-1, q-1)$, it follows that $V' = 0$ too. $\square$

7.12. Paragraph **??** and Lemma 7.11 show that there are exactly $\binom{k+1}{3} pq$ linearly independent syzygies of degree k, so pq quadratic syzygies. Clearly degree-$(k-2)$ T-combinations of quadratic syzygies introduce degree-k syzygies. Next lemma shows that all syzygies of degree $k > 2$ arise in this way.

Lemma 7.13. *Any degree $k > 2$ syzygy of degree $d = (p-1, q-1)$ on the source is a degree-$(k-2)$ T-combination of quadratic syzygyes.*

Proof. Since $\Psi^{(k)}$ is square and of maximal rank by Lemma 7.11, it is nonsingular, so $\Phi^{(k)}$ is of maximal rank, $(k+1)^3 pq$. This means that there are pq linearly independent quadratic syzygies and by the same lemma any one of them must involve x_3^2 nontrivially. It follows that, up to scaling, any nonzero quadratic syzygy is of the form

$$g_\nu(\boldsymbol{s}, \boldsymbol{x}) = \ldots + \boldsymbol{s}^\nu x_3$$

where the $\boldsymbol{s}^\nu$ are the monomials of degree $(p-1, q-1)$ on the source.

We can rewrite an arbitrary degree-$(k > 2)$ syzygy as

$$g(\boldsymbol{s}, \boldsymbol{x}) = \sum_{|\alpha|=k} A'_\alpha \cdot x_0^{\alpha_0} x_1^{\alpha_1} x_2^{\alpha_2} + \Big(\sum_{|j|=k-1} B'_j \cdot x_0^{j_0} x_1^{j_1} x_2^{j_2} \Big) x_3 + \Big(\sum_\nu \boldsymbol{s}^\nu f_\nu \Big) x_3^2$$

where each f_ν is a form degree $k-2$ in T, so the syzygy

$$g(\boldsymbol{s}, \boldsymbol{x}) - \sum_\nu f_\nu \cdot g_\nu(\boldsymbol{s}, \boldsymbol{x})$$

has trivial x_3^k part in the sense of the discussion above. By Lemma 7.11 again, this syzygy must be zero, so every syzygy of degree k is a T-combination of quadratic syzygyes. $\square$

Proof of Theorem 7.7. Since there are no syzygies in degree 1 and all syzygyes in degree $k > 2$ are T-combinations of quadratic syzygyes by Lemma 7.13, we have $N = N_2$.

Since the number of linearly independent syzygies of degree 2 is pq, the matrix N_2 is square.

To finish the prove we note that necessarily $\operatorname{reg}(B_P) \leq (p-1, q-1)$ since the ideal of points I_P is generated in degree (p,q) over $K(T/P)$, so Theorem 3.6 applies.

Finally, birationallity follows by comparing the degree on both hand-sides of Theorem 3.6,

$$\deg(Y)\deg(\phi) = \deg\det(N) = \deg\det(N_2) = 2pq$$

and $\deg(Y) = 2pq$ since the parametrization is basepoint-free, so $\deg\phi = 1$. □

7.14. In the more general setting when X is an $(n-1)$-fold product of $\mathbb{P}^1$s and the coordinates of ϕ be of $(n-2)$-degree $(p_1, \ldots, p_{n-1})$. Now the sizes of $\Phi^{(k)}$ and $\Psi^{(k)}$ are

$$(k+1)^{n-1} p_1 \cdots p_{n-1} \times \binom{k+n}{n} p_1 \cdots p_{n-1}$$

and

$$(k+1)^{n-1} p_1 \cdots p_{n-1} \times (\binom{k+n-1}{n-1} + \binom{k-1+n-1}{n-1}) p_1 \cdots p_{n-1}$$

respectively. If $\phi_0, \ldots, \phi_{n-1}$ form a regular sequence, then the arguments of Lemma 7.11 apply verbatim, and $\Psi^{(k)}$ is of maximal rank, although not square for $n > 3$.

The arguments of Lemma 7.13 apply also, as long as the number of linearly independent quadratic syzygies is the number of monomials of degree $(p_1 - 1, \ldots, p_{n-1} - 1)$ on the source, $p_1 \cdots p_{n-1}$. Since this is just the difference between the number of columns of $\Phi^{(2)}$ and $\Psi^{(2)}$,

$$(\binom{n}{2} - \binom{n+1}{2} - \binom{n}{1}) p_1 \cdots p_{n-1}$$

the following result follows.

The problem to extend the result is however the fact that $\Psi^{(k)}$ is no longer square, more precisely, the number of columns is less than the number of rows for $n > 3$, so we cannot infer that the ranks of $\Psi^{(k)}$ and $\Phi^{(k)}$ are the same. The latter was necessary to find at least (and so exactly) $\dim_{\mathbb{C}}(S_{p_1-1,\ldots,p_{n-1}-1})$ many quadratic syzygyes, which in turn is integral for the proof of Lemma 7.13.

The following example shows this situation does occur.

Example 7.15. ... Then there are no quadratic syzygies.

Example 7.16. ... is there an example for which the matrix is square?

7.17. At any rate, the issue with the number of quadratic syzygies is the only problem to extending Theorem 7.7 to any product of $\mathbb{P}^1$s or to any degree-k syzygies, so by adding the latter as an assumption, we have the following provisional extension.

Remark 7.18. One can hope that something similar would work in higher dimension. For example, let $X = (\mathbb{P}^1)^n$ and $\mathscr{L} = \mathscr{O}(p_1,\ldots,p_n)$. Assuming that there are no degree-$(n!-1)$ syzygies over $S_{p_1-1,\ldots,p_n-1}$ we may expect to find $p_1\cdots p_n$ syzygies of degree $n!$ which to conviniently organize in an $p_1\cdots p_n$-square matrix. The determinant of this matrix would be homogeneous of degree $n!\cdot p_1\cdots p_n$ which is exactly the degree of the image $Y \subset \mathbb{P}^{n+1}$ in the absense of basepoints. While one would still need to argue that $N_k = 0$ for $k > n!$, this is the gist of what happend in the case $n = 2$.

However, already the case $n = 3$ and $\boldsymbol{p} = (2,2,2)$ shows otherwise. We get no sextic syzygies and, in fact, degree-k syzygies over $S_{1,1,1}$ for $k \leq 8$. The latter calculation is outlined in (`ex101`).

Theorem 7.19. *Let $X = \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ be an $(n-1)$-fold product and ϕ be basepoint-free with coordinates in degree $(p_1,\ldots,p_{n-1})$ such that there are no degree-$(m-1)$ syzygies in degree $d = (p_1-1,\ldots,p_{n-1}-1)$ on the source and such that the number of linearly independent degree-m syzygies is $p_1\cdots p_{n-1}$. Then $N = N_m$ is square and*

$$\det(N) = P^{\deg(\phi)}$$

7.4 Basepoint-Free Maps over $X = \mathbb{P}^2$

Lemma 7.20. *For every $k > 0$, the number of linearly independent degree-k syzygies over S_d, $d = p-1$, is at least*

$$\frac{k(k+1)p}{12}(kp - p + k + 5)$$

Proof. As before, we consider the S-linear map

$$\begin{bmatrix} \phi_0^k & \phi_0^{k-1}\phi_1 & \cdots & \phi_3^k \end{bmatrix} : S^{\binom{k+3}{3}} \longrightarrow S^1$$

and construct its $\mathbb{C}$-linearization, exactly as we did for $\Phi^{(k)}$ in Section **??**. It is a map of complex vector spaces

$$S_{p-1}^{\binom{k+3}{3}} \longrightarrow S^1_{(k+1)p-1}$$

so a matrix of size $\binom{kp+p+1}{2} \times \binom{k+3}{3}\binom{p+1}{2}$. Since there are at least as many columns as rows, the dimension of the kernel is at least

$$\binom{k+3}{3}\binom{p+1}{2} - \binom{kp+p+1}{2} = \frac{k(k+1)p}{12}(k+5+kp-p)$$

establishing the claim. $\square$

Remark 7.21. By Lemma 7.20 there are at least p linear syzygies and at least $p(p+7)/2$ quadratic syzygies.

The main result of this section states that the matrices which our method constructs are the same, up to formatting, as those constructed by the method of **?**.

Theorem 7.22. *Let $X = \mathbb{P}^2$, ϕ be basepoint-free whose entries are of degree p, and suppose there are exactly p linear syzyges for $d = p-1$, i.e. the minimal possible number. Then $N = N_1|N_2$, N is square and ϕ is birational. In particular,*

$$\det(N) = \det(N_1|N_2) = P(\boldsymbol{x})$$

Lemma 7.23. *Suppose that there are exactly p linearly independent linear syzygies, i.e. the minimal possible number. Then the $4p$ quadratic syzygies of the form $L_i(\boldsymbol{s},\boldsymbol{x})x_j$ for the p linear syzygies L_i are linearly independent. The number of linearly independent quadratic syzygies not emerging in this way is $\binom{p}{2}$. The vector subspace of S_{p-1} spanned by the coefficient of x_3^2 among all quadratic syzygies is all of S_{p-1}.*

Proof. Since ϕ_0, ϕ_1, ϕ_2 form a regular sequence in degree p, a nonzero linear syzygy over S_{p-1} must involve x_3 nontrivially. Let V_1 be the linear subspace of S_{p-1} spanned by the coefficient of x_3 among all linear syzygies, i.e.

$$V_1 = \mathrm{Span}\{a_3 : a_0x_1 + \ldots + a_3x_3 \text{ is a linear syzygy}\}$$

(and note that the *span* keyword isn't necessary). Since there are exactly p of those, by the observation just made, $\dim_{\mathbb{C}} V_1 = p$.

Any linear syzygy $L(\boldsymbol{s},\boldsymbol{x})$ gives rise to a quadratic syzygy of the form $L(\boldsymbol{s},\boldsymbol{x})x_j$ for each j. Let $g(\boldsymbol{s},\boldsymbol{x})$ be a quadratic syzygies not arising in this way. We know that g must involve x_3^2 nontrivially, for else, it is of the form

$$\sum_{j=0,1,2} (B_{j,0}x_0 + B_{j,1}x_1 + B_{j,2}x_2 + B_{j,3}x_3)x_j$$

and since ϕ_0, ϕ_1, ϕ_2 form a regular sequence in degree p, by the Koszul theorem (e.g. **??**), we have

$$B_{j,0}x_0 + B_{j,1}x_1 + B_{j,2}x_2 + B_{j,3}x_3 = C_{j,0}x_0 + C_{j,1}x_1 + C_{j,2}x_2$$

In particular,

$$L_j(\boldsymbol{s},\boldsymbol{x}) = B_{j,0}x_0 + B_{j,1}x_1 + B_{j,2}x_2 + B_{j,3}x_3 - C_{j,0}x_0 - C_{j,1}x_1 - C_{j,2}x_2$$

is a linear syzygies on the ϕ_j. But then $g - \sum_j L_jx_j$ is a quadratic syzygy only involving x_0, x_1, x_2, and so must be 0, contradicting the assumption that g was not T-generated by linear syzygies.

Let V_2 be the linear subspace of S_{p-1} spanned by the coefficient of x_3^2 for quadratic sygyzygies g as in the previous paragraph. The same argument shows that we cannot have $V_1 \cap V_2 \neq 0$.

We finish the proof by an easy dimension count. The discussion so far gives us that the number of *new* quadratic syzygies is at most $\dim_{\mathbb{C}} V_2 \leq \binom{p+1}{2} - p$, so even if all the $4p$ quadratic syzygies generated by the linear syzygies and the new quadratic syzygies are linearly independent altogether, we get at most $4p + \binom{p+1}{2} - p = p(p+7)/2$-many of them. On the other hand, the number of linearly independent quadratic syzygies must be at least $p(p+7)/2$ by Remark 7.21.

It follows that there are $\binom{p+1}{2} - p = \binom{p}{2}$ *new* quadratic syzygies, which along with the $4p$ pushed linear syzygies are linearly independent altogether. Also, the linear span of the S_{p-1} coefficient of x_3^2 among the quadratic syzygies is $V_1 \oplus V_2 = S_{p-1}$. $\square$

Lemma 7.24. *Let $g(\boldsymbol{s},\boldsymbol{x})$ be a syzygy of degree $k > 2$. Then g is a T-combination of linear and quadratic syzygies.*

Proof. The proof uses the same arguments as in the proof of Lemma 7.23. Since the S_{p-1}-coefficients of x_3^2 among the quadratic syzygies span S_{p-1}, we can find among them

$$g_\nu(\boldsymbol{s},\boldsymbol{x}) = (\text{terms involving } x_3 \text{ at most linearly}) + \boldsymbol{s}^\nu x_3^2$$

for all $|\nu| = p-1$. We can then rewrite

$$g(\boldsymbol{s},\boldsymbol{x}) = (\text{terms involving } x_3 \text{ at most linearly}) + (\sum_\nu \boldsymbol{s}^\nu \cdot h_\nu(\boldsymbol{x}))x_3^2$$

where $h_\nu(\boldsymbol{x})$ is a degree-$(k-2)$ homogeneous polynomial. But now

$$g - \sum_\nu h_\nu g_\nu = \sum_{|\beta|=k-1} (A_{\beta,0}x_0 + A_{\beta,1}x_1 + A_{\beta,2}x_2 + A_{\beta,3}x_3)x_0^{\beta_0}x_1^{\beta_1}x_2^{\beta_2}$$

where $A_{\beta,j}$ is a $(p-1)$-form in S. Lemma **??** and the same argument as in the proof of Lemma 7.23 show that the above must be T-generated by linear syzygies. The result now follows. $\square$

Proof of Theorem 7.22. By Lemma 7.24 we know that $N = N_1|N_2$ and since there are p linear and $\binom{p}{2}$ quadratic syzygies by Lemma 7.23, N is square. By Theorem 3.6,

$$\det(N) = \det(N_1|N_2) = P^{\deg\phi}$$

and comparing the degrees on both sides, we see that $\deg(\phi) = 1$. $\square$

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