Examples of Implicitization of Hypersurfaces

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June 4, 2015

Let X be a smooth projective toric variety of dimension $n-1$. Let $S,\mathfrak{n}$ be its Cox ring and irrelevant ideal and $\mathscr{L}=\mathscr{O}_X(\mathbf{e})$ be a line bundle on X such that $h^0(\mathscr{L})>n$.
Let $\phi_0,\dots,\phi_n \in H^0(X,\mathscr{L})$ be linearly independent such that the induced

$$\phi=(\phi_0,\dots,\phi_n): X \longrightarrow \mathbb{P}^n_{x_0,\dots,x_n}$$

is generically finite onto its image. Then the closed image Y is irreducible hypersurface and so defined by a single polynomial over the x_j.

Definition

The implicitization problem, as we shall study it, is the problem of finding the *implicit equation* $P(\mathbf{x})$ defining $Y\subset\mathbb{P}^n$ when given the coordinate functions ϕ_j. More generally, it is concerned with the relation between the algebraic propeprties of the (ideal of the) ϕ_j in S and the geometric properties of Y in $\mathbb{P}^n$.

Problems with standard method:

- the problem can be solved by Gröbner bases but often this is not computationally feasible
- geometrically, GB are a black box which neither makes use of nor gives insight

Reincarnation: CAD

- Tomas Sederberg and Falai Chen, "Implicitization using moving curves and surfaces", 1995
- David Cox, Ronald Goldman, and Ming Zhang. "On the validity of implicitization by moving quadrics for rational surfaces with no base points", 2000
- Laurent Busé and Jean-Pierre Jouanolou. "On the closed image of a rational map and the implicitization problem", 2003

Fix a rational map ϕ as described before, and let

$$\phi_0, \dots, \phi_n \in H^0(X, \mathscr{O}_X(\mathbf{e}))$$

be its $n+1$ linearly independent coordinates.
We say that a degree-$(\mathbf{d}, i)$ form $g(s_0, \dots, s_m; x_0, \dots, x_n)$ in $S[\mathbf{x}]$ is a syzygy on ϕ over degree $\mathbf{d}$ if

$$g(s_0, \dots, s_m; \phi_0, \dots, \phi_n) = 0$$

as an identity in S.
The syzygies on ϕ over any $\mathbf{d}$ form a finite graded sub-$\mathbb{C}[\mathbf{x}]$-module of $S[\mathbf{x}]$. We denote this submodule by $\mathbb{I}_{\mathbf{d},\bullet}$.

Let

$$\{g_1, \ldots, g_\mu\}$$

be a minimal set of homogeneneous generators for the syzygies module $\mathbb{I}_{\mathbf{d},\bullet}$, for some given fixed ϕ and $\mathbf{d}$.
Let

$$\texttt{basis}(S_{\mathbf{d}})$$

be an r-row-matrix of the basis of $\mathbb{C}$-vector space basis of $S_{\mathbf{d}}$ in some fixed monomial order. For example, for $X = \mathbb{P}^2$ so that $S = \mathbb{C}[s_0, s_1, s_2]$, $\mathbf{d} = 2$ and the lexicographic order, we get

$$\texttt{basis}(S_3) = \begin{bmatrix} s_0^2 & s_0 s_1 & s_0 s_2 & s_1^2 & s_1 s_2 & s_2^2 \end{bmatrix}$$

which is an $(r = \dim_{\mathbb{C}}(S_{\mathbf{d}}) = 6)$-row vector over S and $S[\mathbf{x}]$.

Given the generating set $\{g_1, \ldots, g_\mu\}$ of $\mathbb{I}_{\mathbf{d},\bullet}$, we take N to be the coefficient matrix such that

$$\texttt{basis}(S_{\mathbf{d}}) \cdot N = \begin{bmatrix} g_1 & \ldots & g_\mu \end{bmatrix}$$

For example, given the syzygies

$$\{s_0x_1 - s_1x_3, s_2x_0x_1 - s_0x_3^2\}$$

over degree $\mathbf{d} = 1$ on a rational map with source $\mathbb{P}^2$, we get

$$\begin{bmatrix} s_0 & s_1 & s_2 \end{bmatrix} \begin{bmatrix} x_1 & -x_3^2 \\ -x_3 & 0 \\ 0 & x_0x_1 \end{bmatrix} = \begin{bmatrix} s_0x_1 - s_1x_3 & s_2x_0x_1 - s_0x_3^2 \end{bmatrix}$$

Theorem

Let X be a smooth projective toric variety of dimension $n-1$ and $\mathbf{e} \in \mathrm{Pic}(X)$ such that $h^0(\mathscr{O}_X(\mathbf{e})) > n$. Let $\phi : X \longrightarrow \mathbb{P}^n$ be a rational map given by $n+1$ linearly independent global sections of $\mathscr{O}_X(\mathbf{e})$ such that the base locus $Z \subset X$ is 0-dimensional. Let $\mathbf{d} \in \mathrm{reg}(B_P)$ be a degree in the regularity of the preimage of ϕ and N be the $r \times \mu$ matrix of syzygies of ϕ over $S_{\mathbf{d}}$. Then

$$\gcd(\texttt{minors}(r, N)) = P^{\deg(\phi)}$$

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$$\mathrm{rad}(\mathtt{minors}(r, N)) = P$$

Conjecture

Let X be a smooth projective toric variety of dimension $n-1$ and $\mathbf{e} \in \mathrm{Pic}(X)$ such that $h^0(\mathscr{O}_X(\mathbf{e})) > n$. Let $\phi : X \longrightarrow \mathbb{P}^n$ be a birational map given by $n+1$ linearly independent global sections of $\mathscr{O}_X(\mathbf{e})$. Let $\mathbf{d}$ be a degree such that $h^0(\mathscr{O}_X(\mathbf{d})) > 0$ and N be the $r \times \mu$ matrix of syzygies of ϕ over $S_{\mathbf{d}}$. Then the multiple-point locus of ϕ on its image $Y \subset \mathbb{P}^n$ is given, set-theoretically, by

$$\mathrm{rad}(\mathtt{minors}(r-1, N))$$

Corollary

Let X be a smooth projective toric variety of dimension $n-1$ and $\mathbf{e} \in \mathrm{Pic}(X)$ such that $h^0(\mathscr{O}_X(\mathbf{e})) > n$. Let $\phi : X \longrightarrow \mathbb{P}^n$ be a rational map given by $n+1$ linearly independent global sections of $\mathscr{O}_X(\mathbf{e})$ such that the base locus $Z \subset X$ is 0-dimensional and locally complete intersection. Let $\mathbf{d} \in \mathrm{Pic}(X)$ be large enough *in the poset of degrees. Then*

$$\gcd(\mathtt{minors}(r, N_1)) = P^{\deg(\phi)}$$

Corollary

Let $X = \mathbb{P}^2$ *and* $\phi : X \longrightarrow \mathbb{P}^3$ *be a morphism given by four linearly independent forms of degree* e. *Suppose that there are exactly* $e-1$ *linear syzygies on* ϕ *over* $e-1$. *Then* $N = (N_1 \mid N_2)$, N *is square and* ϕ *is birational. In particular, up to a unit,*

$$\det(N) = \det(N_1 \mid N_2) = P(\mathbf{x})$$

Corollary

Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ and $\phi : X \longrightarrow \mathbb{P}^3$ be a morphism given by four linearly independent forms of bidegree $\mathbf{e} = (e_1, e_2)$. Suppose that there are no linear syzygies on ϕ over $\mathbf{d} = (e_1 - 1, e_2 - 1)$. Then N_2 is square, $N = N_2$ and ϕ is birational. In particular, up to a unit,

$$\det(N) = \det(N_2) = P(\mathbf{x})$$

Example

Let $X = \mathbb{P}^2_{s,t,u}$ and $\phi = (su^2, t^2(s+u), st(s+u), tu(s+u))$. Then ϕ is birational onto its image; the basepoints are $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$, all ci of multiplicities 2, 3, and 1, respectively. For $\mathbf{d} = 1$ we get

$$N = \begin{bmatrix} -x_3 & 0 & x_1 & -x_3^2 \\ 0 & -x_3 & -x_2 & x_0x_2 + x_0x_3 \\ x_2 & x_1 & 0 & 0 \end{bmatrix}$$

and for $\mathbf{d} = 2$ we get a 6×9-matrix whose columns are all linear.
The implicit equation is $P(\mathbf{x}) = x_0x_1x_2 + x_0x_1x_3 - x_2x_3^2$ and

$$\gcd(\texttt{minors}(3, N)) = P(\mathbf{x})$$

as expected. However $\det(N_1) = 0$.

Example

Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ with Cox ring $S = \mathbb{C}[s, u; t, v]$ and let ϕ be the birational map to $\mathbb{P}^3$ given by

$$\phi = (s^2v^2, suv^2, u^2t^2 + u^2tv, sutv - 101u^2tv)$$

Then the base locus of ϕ consists of two points, given scheme-theoretically by $\langle s, t \rangle$ and $\langle u^2, uv, v^2 \rangle$. The former is a complete intersection point of multiplicity 1 and the latter is not—it is of degree 3 and multiplicity 4. The implicit equation, up to a unit, is

$$P(\mathbf{x}) = x_0^2x_2 - 202x_0x_1x_2 + 10201x_1^2x_2 - x_0x_1x_3 + 101x_1^2x_3 - x_0x_3^2$$

Example (cont)

Taking $\mathbf{d} = (1,1)$ we get

$$N = \begin{bmatrix} 0 & x_1 & 0 & 0 & x_0x_2 - x_3^2 \\ x_2 & -x_3 & -x_1 & -x_3 & -x_3^2 \\ -x_3 & -101x_1 & 0 & x_0 - 101x_1 & -10201x_1x_2 - 202x_3^2 \\ -101x_2 - x_3 & 0 & x_0 & 0 & 20402x_2x_3 - 202x_3^2 \end{bmatrix}$$

and

$$\gcd(\mathtt{minors}(4, N)) = P(\mathbf{x})$$

Note that N_1 is square and $\det(N_1) = x_1 P(\mathbf{x})$ so $P(\mathbf{x}) = \det(M)$ where M is the submatrix of N consisting of rows $2,3,4$ and columns $1,3,4$ but M is not a matrix of syzygies.

Example

Let $X = \mathbb{P}^2_{s,t,u}$ and $\phi = (s^5, t^5, su^4, st^2u^2)$. Then ϕ is generically 2-1 map with the single ci basepoint $(0,0,1)$ of multiplicity 5. We have

$$P(\mathbf{x}) = x_0x_1^4x_2^5 - x_3^{10}$$

Over $\mathbf{d} = 1$ we have

$$N = \begin{bmatrix} x_1x_2 & -x_3^8 & 0 \\ -x_2^3 & x_0x_1^3x_2^4 & 0 \\ 0 & 0 & x_0x_1^4x_2^5 - x_3^{10} \end{bmatrix}$$

and

$$\gcd(\mathtt{minors}(3,N)) = \det(N) = P(\mathbf{x})^2$$

Lemma

Let $Y \subset \mathbb{P}^3$ be a reduced surface of degree 4 whose singular locus contains 3 nondegenerate concurent lines. Then $Y = V(\det(M))$ for some 4×4-matrix M. M is not a matrix of syzygies.

Example

Let $X = \mathbb{P}^1_{s,u} \times \mathbb{P}^1_{t,v}$ and $\phi_0, \ldots, \phi_3$ be four general linearly independent biquadrics with common zero set $V(s^2, st, t^2)$. Let $\phi : X \longrightarrow \mathbb{P}^3$ be the map given by the ϕ_j. Then image(ϕ) is of degree 4 and singular along 3 nondegenerate concurent lines.

Example

Let $X = (\mathbb{P}^1)^3$, $\mathscr{L} = \mathscr{O}_X(2,2,1)$ and let $\phi_0, \dots, \phi_4$ be a 5 general linearly independent global sections, i.e.

$$\mathrm{Span}_{\mathbb{C}}\{\phi_0, \dots, \phi_4\} \in \mathbb{G}(4, \mathbb{P}H^0(X, \mathscr{L}))$$

Then ϕ is basepoint-free and birational. The implicit equation is of degree 24 on 5 variables, so a Gröbner basis calculation is unfeasible.
Over $\mathbf{d} = (2,1,1)$ we have $r = \dim_{\mathbb{C}}(S_{\mathbf{d}}) = 12$ rows and we expect N_1 to be empty while N_2 to be of size 12×12. Following our template proof, which now works almost automatically, we find that $N = N_2$ is square and $\det(N) = P(\mathbf{x})$.

Example (cont)

Computating the matrices N_1 and N_2 took less than 1s (0.702681s) on the machines in 103, while the computation of the equation itself, i.e. $\det(N)$, took 66m. I killed the Gröbner bases computataion after a little more than 24h.

Step 1.

Let $J = \langle \phi_0, \dots, \phi_n \rangle \subset S$ be the graded ideal of the coordinates of ϕ. Then $V(J)$ is the baselocus of ϕ. Let $R = S[\mathbf{x}] = S \otimes \mathbb{C}[\mathbf{x}]$ and $\mathrm{Rees}_S(J)$ be the Rees algebra of J. Then

$$\mathrm{Rees}_S(J) = R/\mathbb{I}$$

where

$$\mathbb{I} = \langle \sum_{|\alpha|=i} a_\alpha(\mathbf{s})\mathbf{x}^\alpha : \ \forall i \rangle$$

Geometrically, $\mathrm{Biproj}(\mathrm{Rees}_S(J)) = \Gamma(\phi)$ and we have the following

$$\begin{tikzcd}
\mathrm{Biproj}(B) = \Gamma \arrow[d, "\pi_1"] \arrow[rr, bend left, two heads, "\pi_2"] & \Gamma - E \arrow[l, hook'] \arrow[d, "\pi_1|_{\Gamma-E}"', "\cong"] \arrow[dr, two heads, "\pi_2|_{\Gamma-E}"] & \\
X \arrow[rr, bend right, dashed, two heads, "\phi"'] & X - V(J) \arrow[l, hook'] \arrow[r, two heads, "\phi|_{X-VJ}"] & Y \subset \mathbb{P}^n
\end{tikzcd}$$

commutative diagram

Step 2.

Show that

$$\mathrm{ann}_{\mathbb{C}[\mathbf{x}]}((R/\mathbb{I})_{\mathbf{d},\bullet}) = P$$

and that

$$\mathrm{coker}(N) \cong (R/\mathbb{I})_{\mathbf{d},\bullet}$$

as graded modules over $\mathbb{C}[\mathbf{x}]$.

Step 3.

Construct an isomorphism between

$$\mathrm{Proj}_{K(\mathbb{C}[\mathbf{x}]/P)}(\mathbb{C}[\mathbf{x}] - P)^{-1}(R/\mathbb{I})$$

and the scheme-theoretic inverse of the generic point γ of the image Y

$$\pi_2^{-1}(\gamma) = \mathrm{Spec}(\mathscr{O}_{\gamma,Y}) \times_Y \Gamma(\phi)$$

Step 4.

Use the previous steps and a result from [Gelfand et al. [1994], A, Theorem 30],

$$\mathrm{ord}_{Q(\mathbf{x})}(\det(\mathscr{C}_\bullet)) = \sum_i (-1)^i \,\mathrm{mult}_Q(H_i \mathscr{C}_\bullet)$$

to get that

$$\mathrm{div}(\det(\mathscr{C}_\bullet)) = \mathrm{length}_{T_P}(B_{\mathbf{d},\bullet})_P \cdot [Y]$$

and finally, relate the latter length to the constant Hilbert polynomial of the preimage.

Lemma

Let $\{g_1, \ldots, g_k\}$ *be a generating set for the Rees ideal* $\mathbb{I}$ *as an* $\mathbb{C}[\mathbf{x}]$*-module, and let* $(\mathbf{a}_k, i_k)$ *denote the bidegree of the* k*-th generator. Then the generators for* $\mathbb{I}_{\mathbf{d},\bullet}$ *are given by*

$$\{s^{\alpha} g_k : \mathbf{a}_k + \alpha = \mathbf{d} : \ \forall \alpha, k\}$$

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Algorithm (non-Algorithm)

- *compute a generating set for the Rees ideal* $\mathbb{I}$
- *use the above lemma to find the generators for* $\mathbb{I}_{\mathbf{d},\bullet}$
- *construct* N *out of the generators*
- *compute the minors*
- *compute the gcd*

Example (Problem with bullet 1 in Example 2)

$$\mathbb{I} = \left\langle \begin{cases} ux_0 - sx_1, (sv - 101uv)x_2 + (-ut - uv)x_3, \\ (t^2 + tv)x_1 - 101v^2x_2 + (-tv - v^2)x_3, \\ (st - 101ut)x_1 - svx_3, vx_0x_2 - 101vx_1x_2 + (-t - v)x_1x_3, \\ tx_0x_2 - 101tx_1x_2 - 101vx_2x_3 + (-t - v)x_3^2, \\ sx_0x_2 + (-202s + 10201u)x_1x_2 + (-s + 101u)x_1x_3 - sx_3^2, \\ tx_0x_1 - 101tx_1^2 - vx_0x_3, \\ P(x_0, x_1, x_2, x_3) \end{cases} \right\rangle$$

In particular, $\mathbb{I}$ contains a lot of information not relavant to us. Worse than that, $P(\mathbf{x})$ is always a generator of $\mathbb{I}$ in bidegree $(0, \deg(Y))$.

Example (Problem with bullet 3 in Example 1)

In the case of $\mathbf{d} = 2$, having a 6×9 one should **not** compute all the minors, 84 in this case, given that two random ones minors will often suffice.

Algorithm (`Main` - First Version)

input: *none*
output: *the implicit equation* P
Set $r = \dim_{\mathbb{C}}(S_{\mathbf{d}})$
Set $N' = r \times 0$ *matrix over* $\mathbb{C}[\mathbf{x}]$
while *(Condition) is not satisfied for* N' **do**
  Given $N_1, \ldots, N_{i-1}$, *use Algorithm* `LinearNi` *to compute* N_i
  Set $N' = N' \mid N_i$
end while
Report $P = \gcd(\mathtt{minors}(r, N'))$

(Condition) Given a matrix N' and expected degree k, N' is of full rank and there are nonzero minors of degree $\geq k$

Algorithm (LinearNi)

input: *a list of matrices of sygygy-generators* $N_1, \ldots, N_{i-1}$
output: *the syzygy-generators matrix* N_i
for $0 < j < i$ **do**
    Set $N_{ji} = \texttt{basis}(\mathbb{C}[\mathbf{x}]_{i-j}) \otimes N_j$
    Set K_{ji} *to be the linearization of* N_{ji}
end for
Set $K_i = \ker(\Phi^{(i)})$
Let K_i' *be such that* $\mathrm{Span}(K_i) = \mathrm{Span}(K_i') \oplus (\sum_j \mathrm{Span}(K_{ji}))$
Let N_i *be such that* $\texttt{basis}(R_{\mathbf{d},i}) \cdot K_i' = \texttt{basis}(S_{\mathbf{d}}) \cdot N_i$
Report N_i

Example

Let $X = (\mathbb{P}^1)^3$ and let $\phi : X \longrightarrow \mathbb{P}^4$ be given by 5 general tri-cubics. Then ϕ is generically 1-1 and basepoint-free. Its image has degree $48 = 3! \cdot 2^3$. This has far left the realm of Gröbner bases computations.

- do you really want the polynomial? — dense on 270'725 monomials;
- at first one may hope that $\mathbf{d} = (1, 1, 1)$ is a good choice mimicing the results over $\mathbb{P}^2$. Here N will have 8 rows and there will be no small-degree syzygies, so one can hope to get $N = N_6$ and $P(\mathbf{x}) = \det(N)$;
- there are no syzygies of degree i over $S_{\mathbf{d}}$ for i up to 10;
- any nonzero minor in the potential N (whatever it is) will have degree at least 88;

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- there are no syzygies of degree i over $S_{\mathbf{d}}$ for i up to 10;
- any nonzero minor in the potential N (whatever it is) will have degree at least 88;
- take another $\mathbf{d}$. for example, $\mathbf{d} = (2,2,1)$.

Example (cont)

- for $\mathbf{d} = (2, 2, 1)$ we have $r = 18$ rows;
- it took 169 $(0.03+1+19+147)$ seconds to find that $N_1 = N_2 = N_3 = 0$ and compute N_4;
- N_4 has 50 columns, so a huge number of maximal minors;
- each nonzero minor of N_4 has degree $72 > 48$, so chances are that any two nonzero minors will do, as long as N_4 has any;
- it takes 3 second to check that $\operatorname{rank} N_4 = 18$;
- I grabbed two square submatrices M_1 and M_2 uniformly at random and checked if they are nonsingular by evaluating over a finite field. This took 0.2 seconds;
- at this point one has to make sure that $\det(M_1)$ and $\det(M_2)$ do not share common factor. this can be easily checked by hand – see blackboard;
- it follows that $P(\mathbf{x}) = \gcd(\det(M_1), \det(M_2))$