EXAMPLES OF IMPLICITIZATION OF HYPERSURFACES THROUGH SYZYGIES

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ABSTRACT. Let X be a smooth projective toric variety of dimension $n-1$ and let $\phi : X \longrightarrow \mathbb{P}^n$ be a generically finite rational map. The closed image Y can be defined by a single equation $P(\boldsymbol{x})$, called the implicit equation. The implicitization problem asks for techniques for finding the implicit equation. This is an old problem in algebraic geometry and can be solved effectively through elimination using Gröbner bases, for instance. However, this solution represents a black box in terms of geometry and is unfeasible for even rather modest examples.

In this thesis, we develop a framework for both efficiently computing the image Y of a rational map and studying the relation between algebraic properties of the coordinates ϕ_j on one hand, and geometric properties of the image Y, on the other. We follow the theme of moving surfaces by Cox et al. [2000] and the method of the approximation complex by Busé and Jouanolou [2003] to construct a family of matrices N, one for each element in $\text{Pic}(X)$, capturing determinantal representations for the implicit equation.

An algorithm for computing N and $P(\boldsymbol{x})$ is described and implemented in the Macaulay2 system. It is benchmarked against standard approaches and later used to carry out computations in previously intractable situations.

1. INTRODUCTION

The goal of this paper is to develop a framework for efficiently computing the closed image of a rational map $\phi : X \longrightarrow \mathbb{P}^n$ by making use of the relation between algebraic properties of the coordinates ϕ_j on one hand, and geometric properties of the image Y, on the other. We consider the setup where X is a smooth projective toric variety and are most interested in the case when X is of dimension $n-1$ and ϕ is generically finite onto its closed image. In this case, the image is defined by a single equation $P(\boldsymbol{x})$, called the implicit equation.

This is a version of the implicitization problem, which is classical in algebraic geometry. It has an effective solution through elimination using Gröbner bases, for instance, but those techniques have two drawbacks. First, they become unfeasible even on examples of very modest complexity. Second, they represent a black box in terms of geometry—a tool which takes the ϕ_j and returns P, but whether the computation finishes in our Sun's lifetime or not, no geometric feature can be used for or inferred from the process. Motivated partially by the advance of computer aided design (CAD), interest in new approaches to the implicitization problem reappeared in the 1990s. One reference often cited as the initiating paper is Sederberg and Chen [1995] — a SIGGRAPH paper using moving curves and surfaces for implicitization in the case $X = \mathbb{P}^1 \times \mathbb{P}^1$ and $X = \mathbb{P}^2$. One of the first works building on this idea from the standing point of commutative algebra and algebraic geometry is Cox et al. [2000].

Our approach follows the theme of moving surfaces but sets to remove the intrinsic ad-hoc constructions it requires. In this sense, we draw inspiration from the method of the

approximation complex, initiated by Busé and Jouanolou [2003]. Our main results are a blend of the results in the two papers.

Let S be the Cox ring of X and let $J = \langle \phi_0, \ldots, \phi_n \rangle \subset S$ be the ideal of the coordinates. Then the Rees ideal I of J can be thought of as an algebraic object through the Rees algebra and a geometric object through the graph of ϕ. This duality helps us present concise and well-motivated proofs. We construct, for any degree $\boldsymbol{d}$ on S, a matrix N representing a generating set for the polynomial relations on the ϕ_j with coefficients from $S_{\boldsymbol{d}}$ only. This is just a bigraded piece of the Rees ideal. The matrix N generalizes the types of representation matrices studied by the aforementioned papers. One of our main results, Theorem 6, is an analogue of a result for the approximation complex. If we let $r \times \mu$ be the size of N, we show that $\mathrm{rank}(N) = r$ and, under mild assumptions on $\boldsymbol{d}$, that

$$\gcd(\mathrm{minors}(r, N)) = P(\boldsymbol{x})^{\deg(\phi)}$$

Relaxing the requirement that X be of dimension $n - 1$, we find in Theorem 5 that

$$\mathrm{rad}(\mathrm{minors}(r, N)) = P$$

Furthermore, results of the form: if conditions (1)–(k) hold, then there exists a square matrix M such that $\det(M) = P(\boldsymbol{x})$ — which are the main theme of the moving surfaces methods (e.g. Busé et al. [2003] and Adkins et al. [2005]), now become essentially equivalent to saying that the degree-$(\boldsymbol{d}, \bullet)$ graded piece of the Rees ideal is minimally generated by r elements. This suggest a template for proofs allowing one to more clearly address the specifics of the presented conditions.

Moreover, the matrix N can be computed incrementally using only tools from linear algebra. This is important because it entails a significant on-average speed up, allowing us to calculate examples which are out of reach via Gröbner basis techniques. More importantly, this allows us to use geometric tricks, for instance, to confirm that

$$P(\boldsymbol{x}) = \gcd(\det(M_1), \det(M_2))$$

for some selected matrices M_1 and M_2. See Examples 50 and 51.

We state our results in Section 2, after spending some time to setup the notation and define the main objects of interests. As the title suggests, examples are the heart of this paper, so in Section 3 we presents 20 examples describing these results. In fact, we believe that a careful read of the section renders some, if not all, of the proofs, presented in Section 4, unnecessary.

The paper was conceived as a work in computational algebraic geometry. As such we field-test our approach and in Section 5 we develop and implement, in Macaulay2, Grayson and Stillman [1993-2015], algorithms for computing N and P. We then use this implementation them to benchmark our algorithms against the standard approach and workout several computationally more difficult examples.

2. RESULTS

1. Let X be a smooth projective toric variety of dimension $n - 1$ ($n > 1$) with Cox ring S and irrelevant ideal $\mathfrak{n} \subset S$. We write $\mathrm{Proj}(S, \mathfrak{n})$ for the construction of the variety from the ring, or just $\mathrm{Proj}(S)$ when $\mathfrak{n}$ is apparent. See Cox [1993] for details. Let $T = \mathbb{C}[x_0, \ldots, x_n]$ and let

$$\phi = (\phi_0, \ldots, \phi_n) : X \longrightarrow \mathbb{P}^n = \mathrm{Proj}(T)$$

be a rational map given by linearly independent sections of the line bundle $\mathscr{O}_X(\boldsymbol{e})$ for some degree $\boldsymbol{e}$ on S. Let $J = \langle \phi_0, \dots, \phi_n \rangle \subset S$ be the ideal of the coordinates ϕ_j.

Let $\boldsymbol{d}$ be a degree on S such that $r = \dim_{\mathbb{C}}(S_{\boldsymbol{d}}) > 0$ and let $I \subset R = S \otimes T$ be the Rees ideal of J. Let $I_{\boldsymbol{d},\bullet}$ be its degree-$(\boldsymbol{d},\bullet)$ bigraded piece, considered as a finite graded T-module. We denote by $B = R/I$ the Rees algebra of J.

Let us fix a monomial order on S and T. We write $\mathtt{basis}(-)$ for the row vector of a monomial basis of a vector space.

Suppose a degree $\boldsymbol{d}$ is fixed. We let N be the $r \times \mu$ coefficient matrix of a minimal set of homogeneous generators for $I_{\boldsymbol{d},\bullet}$ with respect to $\mathtt{basis}(S_{\boldsymbol{d}})$, that is,

$$\mathtt{basis}(S_{\boldsymbol{d}}) \cdot N = \begin{bmatrix} g_1 & \cdots & g_\mu \end{bmatrix}$$

where the g_k form a minimal set of homogeneous generators for $I_{\boldsymbol{d},\bullet}$. Further, let N_i be the submatrix of N corresponding to generators of degree i, i.e.

$$N = (\, N_1 \mid \dots \mid N_\delta \,)$$

2. Let $P \subset T$ be the prime ideal defining to the closed image of ϕ in $\mathbb{P}^n$. We denote this image by $Y = V(P)$. Let $Z = V(J) \subset X$ be the base locus of ϕ.

We are going to be interested the following three conditions:

(1) The map ϕ is generically finite onto its image, that is, $Y \subset \mathbb{P}^n$ is of codimension 1 and in particular, the ideal P is principal. In this case, we denote a generator of P by $P(\boldsymbol{x})$.
(2) The base locus Z is zero-dimensional, that is, consists of finitely many points. Note that those are necessarily closed over $\mathbb{C}$.
(3) The map $\phi : X \longrightarrow Y$ is birational.

Clearly, either of (2) and (3) implies (1).

Proposition 3. *Consider the setup of Paragraph 1. One has*

(1) *The ideal $I \subset R$ is prime, and so is I_P in the T-module localization R_P.*
(2) *The quotient B_P is naturally a finite-type graded $K(T/P)$-algebra with grading induced by S.*
(3) *The $K(T/P)$-algebra B_P is a homogeneous coordinate ring of a projective variety.*

4. Let $V(I_P)$ be the closed subset of the projective toric variety $\mathrm{Biproj}(R_P)$. Following Maclagan and Smith [2004] and using Proposition 3, we consider the regularity of the defining ideal I_P, denoted by $\mathrm{reg}(I_P)$.

Recall that $\mathrm{reg}(I_P)$ is a finitely generated additively-closed subset of the semigroup of degrees on S, and that for any $\boldsymbol{d} \in \mathrm{reg}(I_P)$, we have $\langle (I_P)_{\boldsymbol{d}} \rangle = I_P$. This parallels the usual Castelnuovo-Mumford regularity for $\mathbb{P}^n$ and is the content of Theorem 1.3 in the referenced paper.

In light of Proposition 3, our first result becomes an easy exercise. However, it is a step toward the goal of this paper—to exhibit a general relation between the algebra of the coordinates ϕ_j and the geometry of the image Y.

Theorem 5. *In the setup of Paragraph 1, one has*

$$\mathtt{rad}(\mathtt{minors}(r, N)) = P$$

The geometric interpretation of the theorem is clear—the nonzero minors of N define hypersurfaces in $\mathbb{P}^n$ whose intersection, at least set-theoretically, is the image Y.

Example 19, for instance, shows that the radical is necessary.

Our next result is the main theorem of this paper, paralleling the corresponding results in the setting of the approximation complex and moving surfaces, and setting the stage for the fast implicitization method described in Section 5.

Theorem 6. *Consider the setup of Paragraph 1 and assume* (2-1). *Fix a degree* $\boldsymbol{d} \in \mathrm{reg}(I_P)$ *as described in Paragraph 4. One has*

$$\gcd(\mathtt{minors}(r,N)) = P^{\deg\phi}$$

In particular, if $\mu = r$, *then N is a square matrix and, up to a unit,*

$$\det(N) = P(\boldsymbol{x})^{\deg\phi}$$

Corollary 7. *In the setup of Theorem 6, let M be any* $r \times r$ *representation matrix of syzygies on the* ϕ_j *with coefficients in* $S_{\boldsymbol{d}}$, *that is,* $\mathtt{basis}(S_{\boldsymbol{d}}) \cdot M$ *is an r-vector of forms in* $I_{\boldsymbol{d},\bullet}$. *One has*

$$\det(M) = P(\boldsymbol{x})^{\deg\phi} \cdot H(\boldsymbol{x})$$

for a homogeneous $H(\boldsymbol{x})$ *of degree*

$$\deg(\det(M)) - \deg(\phi)\deg(Y) \tag{7.1}$$

Furthermore, there exist a list of such matrices $\{M_k\}$ *whose corresponding* $H_k(\boldsymbol{x})$ *are nonzero and have common factor* 1.

Geometrically, the former is a refinement of Theorem 5. Each of the maximal minors of N, in fact, the determinant of any $r \times r$ matrix M of syzygies over $S_{\boldsymbol{d}}$, is either zero or describes the union of a $\deg(\phi)$-fold copy of the image Y and a hypersurface of degree as in (7.1). While an arbitrary collection M_k of such matrices may introduce hypersurfaces with an intersection that is strictly larger than Y, the maximal minors suffice to shave off any extraneous components.

The theme of extraneous factors is already apparent in Busé et al. [2003], Busé et al. [2009] and Botbol et al. [2009]. In our notation, they used the approximation complex to show that for a toric X, certain $\boldsymbol{d}$ and empty or zero-dimensional almost complete intersection base locus Z,

$$\gcd(\mathtt{minors}(r,N_1)) = P^{\deg\phi} \cdot \prod_{q \in Z} L_q(\boldsymbol{x})^{e_q - d_q}$$

where each $L_q(\boldsymbol{x})$ is a linear form, and e_q and d_q are the multiplicity and degree of q.

In the case of complete intersection base locus, the proof of Theorem 6 gives a special case of the above.

It is known that if M is a square matrix over T of size r, then the singular locus of $V(\det(M))$ is contained in the closed subset defined by the comaximal minors, that is, the $(r-1)$-minors. This fact is mentioned, without a reference, in Blekherman et al. [2012]. We believe that this relation is more intrinsic and holds for all representation matrices N. However, what this ought to correspond to is the multiple-point locus of the image. See Example 28 for details. We conjecture the following

Conjecture 8. *Consider the setup of Paragraph 1 and assume* (2-3). *On the level of closed points, one has*

$$V(\mathtt{minors}(r-1,N)) \subset \mathrm{Sing}(Y)$$

In the simplest cases of interest, when $X = \mathbb{P}^2$ or $X = \mathbb{P}^1 \times \mathbb{P}^1$ and ϕ is basepoint-free, we can chose $\boldsymbol{d}$ so that the matrix N becomes square. Next two theorems are slight generalizations of the results in Cox et al. [2000] in the sense that they show that there are only linear and quadratic syzygies. More importantly, they show that our methods directly generalize the methods of moving planes and quadrics in the setting in which they are most useful.

Theorem 9. *Let $X = \mathbb{P}^2$, ϕ be basepoint-free, and suppose that there are exactly $p = \boldsymbol{e}$ linear syzygies over degree $\boldsymbol{d} = p - 1$, that is, the minimal possible number. One has that N is square and $N = (N_1 \mid N_2)$. In particular,*

$$\det(N) = \det(N_1 \mid N_2) = P(\boldsymbol{x})^{\deg(\phi)}$$

Theorem 10. *Let $X = \mathbb{P}^1 \times \mathbb{P}^1$, ϕ be basepoint-free with coordinates in degree $\boldsymbol{e} = (p, q)$, and suppose there are no linear syzygies over degree $\boldsymbol{d} = (p - 1, q - 1)$. One has that N_2 is square and $N = N_2$. In particular,*

$$\det(N) = \det(N_2) = P(\boldsymbol{x})^{\deg(\phi)}$$

Both of these theorems are examples of a template proof described in Paragraph **??**. While applying it in general requires elaborate choses of the degree $\boldsymbol{d}$ and regularity computations, see for example Adkins et al. [2005], in the case of Theorems 9 and 10, we only use a type of Koszul-ness on the syzygies of low degree.

In fact, motivated by these two results and the examples of Section 3, we propose the following

Conjecture 11. *In the setup of Theorem 6, suppose $i > 0$ be such that*

$$N' = (N_1 \mid \ldots \mid N_i)$$

is square and $\deg(\det(N')) = \deg(\phi) \deg(Y)$. Then $N = N'$ and so $\det(N') = P(\boldsymbol{x})^{\deg(\phi)}$.

We conclude this list by a method to compute the degree of a rational map using Gröbner bases. While we are only going to use this in our examples, it helps expand our understanding about the object B_P.

Proposition 12. *Let $\mathbb{C}[s_0, \ldots, s_m]$ be the fixed ambient polynomial ring of S. Define the ideal I_B of $\mathbb{C}[s_0, \ldots, s_m; x_0, \ldots, x_n]$ by the equality*

$$B = \mathbb{C}[\boldsymbol{s}; \boldsymbol{x}] / I_B$$

Let $>'$ be any product monomial order in which the $\boldsymbol{s}$ variables come before the $\boldsymbol{x}$ variables. Then a reduced Gröbner basis for I_B with respect to $>'$ has the form

$$g_1(\boldsymbol{s}; \boldsymbol{x}) = p_1(\boldsymbol{x}) \boldsymbol{s}^{\alpha_1} + \text{lower order terms}$$
$$\cdots$$
$$g_r(\boldsymbol{s}; \boldsymbol{x}) = p_r(\boldsymbol{x}) \boldsymbol{s}^{\alpha_r} + \text{lower order terms}$$
$$g_{r+1}(\boldsymbol{s}; \boldsymbol{x}) = P(\boldsymbol{x})$$

Further, one has

$$\deg(\phi) = \deg\left(\langle \boldsymbol{s}^{\alpha_1}, \ldots, \boldsymbol{s}^{\alpha_r} \rangle \subset \mathbb{C}[\boldsymbol{s}] \right)$$

3. EXAMPLES

1

Example 13 (ex301). Let $X = \mathbb{P}^2_{s,t,u}$ and $J = \langle tu, su, st, s^2 + t^2 + u^2 \rangle$. Then ϕ is basepoint-free and generically 1-1. The monic implicit equation is given by

$$P(\boldsymbol{x}) = x_0^2 x_1^2 + x_0^2 x_2^2 + x_1^2 x_2^2 - x_0 x_1 x_2 x_3$$

Setting $\boldsymbol{d} = 1$, we get

$$N = \begin{bmatrix} 0 & x_0 & x_1 x_2 \\ x_1 & 0 & x_0 x_2 \\ -x_2 & -x_2 & x_0 x_1 - x_2 x_3 \end{bmatrix}$$

whose determinant is just $P(\boldsymbol{x})$. The results of Cox et al. [2000] apply and the matrix N is a variant of the matrix produced by the method of moving planes and quadrics.

Setting $\boldsymbol{d} = 2$, we get

$$N = \begin{bmatrix} x_2 & 0 & 0 & 0 & x_1 & 0 & 0 & 0 & x_0 \\ -x_3 & 0 & 0 & x_1 & 0 & 0 & 0 & x_0 & 0 \\ 0 & 0 & 0 & -x_2 & -x_3 & x_0 & x_2 & 0 & -x_2 \\ x_2 & 0 & x_1 & 0 & 0 & 0 & x_0 & 0 & 0 \\ 0 & x_1 & -x_2 & 0 & x_2 & 0 & -x_3 & -x_2 & 0 \\ x_2 & -x_2 & 0 & 0 & x_1 & -x_2 & x_0 & 0 & 0 \end{bmatrix}$$

which is a 6×9 matrix of linear forms. This was expected — the results of Busé and Jouanolou [2003] also apply and the method of the approximation complex guarantees a matrix of linear forms. Accordingly,

$$\gcd(\texttt{minors}(6, N)) = P$$

Note that the claim that ϕ is of degree 1 follows, a fortiori, from the degree formula (**??**). Indeed, we have

$$4 \deg(\phi) = 2^2 - 0$$

We confirm this using Proposition 12 in Example 27.

Example 14 (ex302). Clearly, if we replace s, t, u in Example 13 by general linear forms L_0, L_1, L_2, effectively changing coordinates on the source, we get the same equation. In this example we describe what happens if we take $X = \mathbb{P}^1_{s,u} \times \mathbb{P}^1_{t,v}$ instead and let the L_k be $(1, 1)$-forms. Since the algebraic structure of the coordinates is the same, so is the equation of the image,

$$Y = V(x_0^2 x_1^2 + x_0^2 x_2^2 + x_1^2 x_2^2 - x_0 x_1 x_2 x_3)$$

and ϕ is again basepoint-free, basically for the same reason — the L_k are general. However, the self-intersection of the divisor corresponding to the coordinates, now $[\mathscr{O}(2,2)]$, is 8. It follows that ϕ is generically 2-1.

For $\boldsymbol{d} = (1, 1)$ we get a square matrix of size 4 with $h = (2, 1, 0, 1)$. As expected, up to a unit

$$\det(N) = P(\boldsymbol{x})^2$$

For $\boldsymbol{d} = (2, 1)$ we get a square matrix of size 6 with $h = (4, 2)$, for which the last equality again applies. For $\boldsymbol{d} = (2, 2)$ we get a 9×12-matrix with $h = (11, 1)$ such that

$$\gcd(\texttt{minors}(9, N)) = P^2$$

Example 15 (ex303). Suppose that in the situation of Example 14 we took the forms L_k from $\langle st, sv, ut \rangle$ instead. Now ϕ has the unique basepoint $(0,1) \times (0,1)$ which is c.i. of degree 4. Indeed, on the affine open where $u = v = 1$, the point looks like $V(st, s^2 + t^2)$. The equation of the image remains the same. Once again, by (**??**) we know that ϕ must be generically 1-1.

For $\boldsymbol{d} = (1,1)$ we get a 4×5-matrix. Below is the matrix resulting from $(L_0, L_1, L_2) = (st, sv, ut)$,

$$N = \begin{bmatrix} 0 & 0 & 0 & x_0 & x_1 x_2 \\ x_1 & x_0 & 0 & 0 & 0 \\ -x_2 & 0 & x_0 & -x_2 & -x_2 x_3 \\ 0 & -x_2 & -x_1 & 0 & x_1^2 + x_2^2 \end{bmatrix}$$

and, of course,

$$\gcd(\texttt{minors}(4, N)) = P$$

Example 16 (ex304). Let $X = \mathbb{P}^2$ and $J = \langle su^2, t^2(s+u), st(s+u), tu(s+u) \rangle$. Then ϕ is generically 1-1 with three basepoints — $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$, all c.i. of degree 2, 3, and 1, respectively.

The implicit equation is given by

$$P(\boldsymbol{x}) = x_0 x_1 x_2 + x_0 x_1 x_3 - x_2 x_3^2$$

As before, both the method of the moving planes and quadrics, and the method of the approximation complex apply. For $\boldsymbol{d} = 1$, we get

$$N = \begin{bmatrix} -x_3 & 0 & x_1 & -x_3^2 \\ 0 & -x_3 & -x_2 & x_0 x_2 + x_0 x_3 \\ x_2 & x_1 & 0 & 0 \end{bmatrix}$$

and for $\boldsymbol{d} = 2$, we get a 6×9-matrix whose entries are all linear.

Example 17 (ex305). Let $X = \mathbb{P}^2$ and $J = \langle s^3, tu^2, s^2 t + u^3, stu \rangle$. Then ϕ is generically 1-1 with a single c.i. basepoint of degree 2. For $\boldsymbol{d} = 1$ we have

$$N = \begin{bmatrix} x_1 & 0 & -x_3^2 \\ 0 & x_1 x_2 - x_3^2 & x_0 x_1 \\ x_3 & x_1^2 & 0 \end{bmatrix}$$

and $\det(N) = x_0 x_1^4 - x_1 x_2 x_3^3 + x_3^5$, the implicit equation.

Starting from $\boldsymbol{d} = 2$, in which case we have

$$N = \begin{bmatrix} 0 & 0 & 0 & x_1 & -x_3 & 0 & 0 \\ x_3 & 0 & x_1 & 0 & 0 & -x_2 & 0 \\ 0 & x_1 & 0 & -x_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_0 & -x_1 x_2 + x_3^2 \\ -x_2 & 0 & -x_3 & 0 & x_0 & 0 & x_1^2 \\ x_1 & -x_3 & 0 & 0 & 0 & x_3 & 0 \end{bmatrix}$$

we always have $\mu - 1$ linear columns and a single quadratic one.

Example 18 (ex306). Let $X = \mathbb{P}^2$ and $J = \langle s^3, t^2 u, s^2 t + u^3, stu \rangle$. This is the situation of Example **??**. We have

$$P(\boldsymbol{x}) = x_0^3 x_1^4 - x_0^2 x_1^3 x_2 x_3 + x_3^7$$

For $\boldsymbol{d} = 1$, we find that N is square of order 3 with $h = (1,1,0,1)$. For $\boldsymbol{d} = 2$, we get a square 6×6 matrix with $h = (5,1)$, and for $\boldsymbol{d} \geq 3$, we get a non-square matrix of linear forms only.

Example 19 (ex307)**.** Consider the twisted cubic curve C. It is the image of $X = \mathbb{P}^1_{s,t}$ under the map

$$\phi = (s^3, s^2t, st^2, t^3) : \mathbb{P}^1 \longrightarrow \mathbb{P}^3$$

which is birational onto its image. In light of Theorem 5, we can carry out the same calculations as before, even though the image is of codimension strictly bigger than 1.

Setting $\boldsymbol{d} = 1$, we get

$$N = \begin{bmatrix} -x_3 & -x_2 & -x_1 \\ x_2 & x_1 & x_0 \end{bmatrix}$$

and

$$\mathtt{minors}(2,N) = \langle x_2^2 - x_1x_3, x_1x_2 - x_0x_3, x_1^2 - x_0x_2 \rangle$$

which are the usual equations for C in $\mathbb{P}^3$. Setting $\boldsymbol{d} = 2$, we get

$$N = \begin{bmatrix} 0 & -x_3 & -x_3 & -x_2 & -x_2 & -x_1 \\ -x_3 & x_2 & 0 & x_1 & 0 & x_0 \\ x_2 & 0 & x_1 & 0 & x_0 & 0 \end{bmatrix}$$

We have

$$\mathtt{minors}(3,N) = \left\langle \begin{cases} x_2^2x_3 - x_1x_3^2, x_1x_2x_3 - x_0x_3^2, x_1^2x_3 - x_0x_2x_3, \\ x_2^3 - x_0x_3^2, x_1x_2^2 - x_0x_2x_3, x_0x_2^2 - x_0x_1x_3, \\ x_1^2x_2 - x_0x_1x_3, x_0x_1x_2 - x_0^2x_3, x_1^3 - x_0^2x_3, x_0x_1^2 - x_0^2x_2 \end{cases} \right\rangle$$

and

$$\mathtt{rad}(\mathtt{minors}(3,N)) = \langle x_2^2 - x_1x_3, x_1x_2 - x_0x_3, x_1^2 - x_0x_2 \rangle$$

showing that the radical is necessary.

Example 20 (ex308)**.** Let $X = \mathbb{P}^2_{s,t,u}$ and $J = \langle s^5, t^5, su^4, st^2u^2 \rangle$. Then ϕ is generically 2-1 map with the unique basepoint $(0,0,1)$ which is c.i. of degree 5. The image is defined by

$$P(\boldsymbol{x}) = x_0x_1^4x_2^5 - x_3^{10}$$

Setting $\boldsymbol{d} = 1$, we get

$$N = \begin{bmatrix} x_1x_2 & -x_3^8 & 0 \\ -x_3^2 & x_0x_1^3x_2^4 & 0 \\ 0 & 0 & x_0x_1^4x_2^5 - x_3^{10} \end{bmatrix}$$

and

$$\det(N) = P(\boldsymbol{x})^2$$

One should note that the principal 2-minor of N is just $P(\boldsymbol{x})$.

The alternative non-Gröbner bases approaches would require difficult computations; the method of the approximation complex would need to find the gcd of the maximal minors of a matrix of size 36×58, and none of the moving plane and quadrics methods will work since the map isn't birational (and (BP3) of Busé et al. [2003] fails in any case).

The practical gain of our method for this example, however, is arguable at best — we have to compute a degree 10 syzygy which is already the degree of the implicit equation.

On the other hand, a better choice for $\boldsymbol{d}$ might help. For $\boldsymbol{d} = 3$ we get a 10×10-matrix with $h = (4, 4, 1, 0, 1)$, and for $\boldsymbol{d} = 4$ we get a 15×16-matrix with $h = (11, 4, 1)$.

Example 21 (ex309). Let N_1 be the matrix of linear columns for $\boldsymbol{d} = 1$ in Example 16. We have that $\det(N_1) = 0$. This shows that not all maximal minors need to be nonzero.

This has nothing to do with the fact that N_1 is special. For another example, let us take $\boldsymbol{d} = 4$ in Example 20. Then N is a 15×16-matrix whose columns correspond to the syzygies

$$\begin{cases} t^2u^2x_2 - u^4x_3, t^3ux_2 - tu^3x_3, st^2ux_2 - su^3x_3, t^4x_2 - t^2u^2x_3, \\ st^3x_2 - stu^2x_3, s^2t^2x_2 - s^2u^2x_3, su^3x_1 - t^3ux_3, stu^2x_1 - t^4x_3, \\ s^2u^2x_1 - st^3x_3, u^4x_0 - s^4x_2, t^2u^2x_0 - s^4x_3, s^2tux_1x_2 - st^2ux_3^2, \\ s^3ux_1x_2 - s^2tux_3^2, s^3tx_1x_2 - s^2t^2x_3^2, s^4x_1x_2 - s^3tx_3^2, tu^3x_0x_1x_2 - s^3ux_3^3 \end{cases}$$

Let M be the square submatrix of the first 14 columns and the last column of N, that is, leaving out the column corresponding to the syzygy $s^4x_1x_2 - s^3tx_3^2$. Then $\det(M) = 0$. The same is true for the square submatrix consisting of the first 15 columns but in that case $M = (N_1 \mid N_2)$ which we wanted to avoid.

Example 22 (ex310). The following example has been presented elsewhere in the literature as a difficult one to handle. Let $X = \mathbb{P}^2$ and take

$$J = \big\langle -s^2t^3 + 3s^2t^2u + st^3u - 4st^2u^2 - stu^3 + 2t^2u^3 - tu^4 + u^5, \qquad \big\rangle$$
$$s^2t^3 - 3s^2t^2u + st^3u + 3stu^3 - 2t^2u^3 + tu^4 - u^5,$$
$$s^2t^3 - 3s^2t^2u - st^3u + 2s^2tu^2 + 4st^2u^2 - 3stu^3 - 2t^2u^3 + 3tu^4 - u^5,$$
$$-s^2t^3 + 3s^2t^2u - st^3u - 3stu^3 + 3tu^4 - u^5$$

so ϕ is generically 1-1 with base locus of total degree 17 and multiplicity 20. More precisely, the basepoints are the c.i. point $(1,1,1)$ of multiplicity 4, the a.c.i. point $(0,1,0)$ of degree 4 and multiplicity 5, and the a.c.i. point $(1,0,0)$ of degree 9 and multiplicity 11.

Unlike other examples in this section, already in degree $\boldsymbol{d} = 1$, we find only linear and quadratic syzygies. In this sense the example is rather simple. We find

$$N = \begin{bmatrix} x_0 + x_1 & 0 & \cdots \\ -x_0 - x_2 & 3x_0x_1 - x_1^2 + 4x_1x_2 + 3x_0x_3 - 3x_1x_3 + 4x_2x_3 - 2x_3^2 & \cdots \\ -x_2 + x_3 & x_0^2 - x_0x_1 - x_1^2 - 3x_0x_3 - 3x_1x_3 - x_3^2 & \cdots \end{bmatrix}$$

$$\begin{matrix} \cdots & x_1^2 - x_3^2 \\ \cdots & x_0^2 + 7x_0x_1 - 3x_1^2 + x_0x_2 + 10x_1x_2 + 6x_0x_3 - 9x_1x_3 + 9x_2x_3 - 6x_3^2 \\ \cdots & -5x_0x_1 - 2x_1^2 - 3x_0x_2 - 6x_1x_2 - 8x_0x_3 - 5x_1x_3 - 3x_2x_3 \end{matrix}$$

and

$$\det(N) = P(\boldsymbol{x})$$

Example 23 (ex311). Let ϕ be the rational map from Example 22. We compute the degree and multiplicity of the base locus Z.

Set $q_1 = (1,0,0)$, $q_2 = (0,1,0)$ and $q_3 = (1,1,1)$, so that set-theoretically $Z = \{q_1, q_2, q_3\}$. Since J is saturated, the sum of the degrees of the basepoints is just the degree of the base locus,

$$\deg(Z \subset \mathbb{P}^2) = \deg(J \subset \mathbb{C}[s,t,u]) = 17$$

Because we are in projective space, this can be checked directly in Macaulay2. However, since Z is supported on multiple points, we cannot compute the total multiplicity in the same way. Indeed, running $\mathtt{multiplicity}(J)$ gives 45, not the correct 20.

This is because the multiplicity is computed as the degree of the normal cone over the closed subscheme, but this is a well-behaved projective variety over a field only in case the support is a single point.

Let Q_k be the prime ideals corresponding to the points q_k, and set

$$J_k = J : (J : Q_k^\infty)$$

The degree and multiplicity of q_k can be computed from the ideal J_k — those capture the local structure of Z near q_k. We have

$$\begin{cases} J_1 = \langle tu^2, t^3 - 3t^2u, u^5 \rangle \\ J_2 = \langle su, s^2, u^3 \rangle \\ J_3 = \langle t^2 - 2tu + u^2, s^2 - 2su + u^2 \rangle \end{cases}$$

so in particular, q_3 is c.i., while q_1 and q_2 are a.c.i. points.

Example 24 (ex312). Let $Y = V(P)$ for an irreducible form $P(\mathbf{x})$ of degree 4 on $\mathbb{P}^3$. Suppose further that $\mathrm{Sing}(Y)$, the singular locus of Y, contains 3 concurrent non-degenerate lines (that is, passing trough a common point and spanning all of $\mathbb{P}^3$). Then $P(\mathbf{x})$ is the determinant of a square order-4 matrix M of linear forms.

We can prove this claim by a direct calculation. After a linear change of coordinates on $\mathbb{P}^3$, we can assume that the lines are the 3 coordinate axes in the distinguished $\{x_3 \neq 0\}$, that is, the lines are given by $V(x_0, x_1)$, $V(x_1, x_2)$ and $V(x_0, x_2)$.

On the level of ideals, using Euler's identity, the assumption translates to

$$\langle P_{x_0}(\mathbf{x}), \dots, P_{x_3}(\mathbf{x}) \rangle \subset \langle x_0 x_1, x_0 x_2, x_1 x_2 \rangle$$

so writing out $P(\mathbf{x}) = \sum_{|\alpha|=4} a_\alpha \mathbf{x}^\alpha$ and noting that each of the partials must be zero modulo $\langle x_0 x_1, x_0 x_2, x_1 x_2 \rangle$, we only need to solve a linear system in the indeterminate coefficients. We get

$$P(\mathbf{x}) = a_1 x_0^2 x_1^2 + a_2 x_0^2 x_2^2 + a_3 x_1^2 x_2^2 + a_4 x_0^2 x_1 x_2 + a_5 x_0 x_1^2 x_2 + a_6 x_0 x_1 x_2^2 + a_7 x_0 x_1 x_2 x_3$$

which is the determinant of

$$M = \begin{bmatrix} x_0 & & & x_1 \\ & x_1 & & x_2 \\ & & x_2 & x_0 \\ -a_3 x_2 & -a_2 x_0 & -a_1 x_1 & a_4 x_0 + a_5 x_1 + a_6 x_2 + a_7 x_3 \end{bmatrix}$$

In fact, if $a_7 \neq 0$ we can do better. Setting $a_7 = 1$, a linear change of coordinates by the matrix

$$\begin{bmatrix} 1 & & & -a_4 \\ & 1 & & -a_5 \\ & & 1 & -a_6 \\ & & & 1 \end{bmatrix}$$

leaves the singular locus the same but simplifies the general form to

$$P'(\mathbf{x}) = a_1 x_0^2 x_1^2 + a_2 x_0^2 x_2^2 + a_3 x_1^2 x_2^2 + x_0 x_1 x_2 x_3$$

and the matrix M to

$$M' = \begin{bmatrix} x_0 & & & x_1 \\ & x_1 & & x_2 \\ & & x_2 & x_0 \\ -a_3x_2 & -a_2x_0 & -a_1x_1 & x_3 \end{bmatrix}$$

whose entries are scaled variables.

Example 25 (ex313). In the situation of Example **??**, we had

$$N = \begin{bmatrix} 0 & x_1 & 0 & 0 & x_0x_2 - x_3^2 \\ x_2 & -x_3 & -x_1 & -x_3 & -x_3^2 \\ -x_3 & -101x_1 & 0 & x_0 - 101x_1 & -10201x_1x_2 - 202x_3^2 \\ -101x_2 - x_3 & 0 & x_0 & 0 & 20402x_2x_3 - 202x_3^2 \end{bmatrix}$$

where $N = N_1|N_2$ and $\det(N_1) = x_1P(\boldsymbol{x})$. Let M be the submatrix corresponding to rows $\{2,3,4\}$ and columns $\{1,2,3\}$. Then $\det(M) = P(\boldsymbol{x})$ but M is not technically a representation matrix — its rows cannot be indexed by the monomials of S_d.

The situation of the previous example, 24, is very similar. While we were lucky and found a determinantal representation for $P(\boldsymbol{x})$, it does not come in the form of a matrix of syzygies. There is only 3 linear syzygies, corresponding to there of the columns of M, but the last one is not.

Example 26 (ex319). Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ and $J \subset S = \mathbb{C}[s,u;t,v]$ be given by 4 general $(2,2)$ forms in $\langle s^2, st, t^2 \rangle$. Then ϕ is generically 1-1 with the unique basepoint $(0,1) \times (0,1)$ of degree 3 and multiplicity 4. After a linear change of coordinates on the target, we can assume that the ϕ_j are of the form outlined in Example 14, so the representation of Example 24 applies.

Example 27 (ex314). We apply Proposition 12 to compute the degree of the rational map in Example 13. We have

$$I = \left\langle \begin{cases} x_1t - x_2u, \\ x_0s - x_2u, \\ x_2s^2 - x_3st + x_2t^2 + x_2u^2, \\ x_1s^2 - x_3su + x_2tu + x_1u^2, \\ x_0t^2 + x_2su - x_3tu + x_0u^2, \\ x_1x_2s + x_0x_2t + (x_0x_1 - x_2x_3)u, \\ x_0^2x_1^2 + x_0^2x_2^2 + x_1^2x_2^2 - x_0x_1x_2x_3 \end{cases} \right\rangle$$

and we see $P(\boldsymbol{x})$ as the last generator. Next, we calculate a Gröbner basis with respect to a product monomial order where the s,t,u variables come before the x_j variables. Dropping the implicit equation, we get

$$\begin{cases} x_1t - x_2u, \\ x_0^2x_2t + (x_0^2x_1 + x_1x_2^2 - x_0x_2x_3)u, \\ x_0s - x_2u, \\ x_1x_2s + x_0x_2t + (x_0x_1 - x_2x_3)u, \\ x_0t^2 + x_2su - x_3tu + x_0u^2, \\ x_2s^2 - x_3st + x_2t^2 + x_2u^2, \\ x_1s^2 - x_3su + x_2tu + x_1u^2 \end{cases}$$

Collecting the *S*-part of the leading terms, we get the ideal $\langle s,t \rangle$ whose degree is obviously 1.

We can carry out the same calculation for the map in Example 20. The Rees ideal and its Gröbner basis are as follow,

$$
I = \left\langle
\begin{cases}
x_2 t^2 - x_3 u^2, \\
x_1 x_2 s - x_3^2 t, \\
-x_3 t^3 + x_1 s u^2, \\
-x_2 s^4 + x_0 u^4, \\
-x_3 s^4 + x_0 t^2 u^2, \\
-x_1 s^5 + x_0 t^5, \\
-x_3^3 s^3 + x_0 x_1 x_2 t u^2, \\
-x_3^5 s^2 + x_0 x_1^2 x_2^2 u^2, \\
-x_3^8 s + x_0 x_1^3 x_2^4 t, \\
x_0 x_1^4 x_2^5 - x_3^{10}
\end{cases}
\right\rangle,
\quad
\begin{cases}
x_1 x_2 s - x_3^2 t, \\
x_3^8 s - x_0 x_1^3 x_2^4 t, \\
x_2 t^2 - x_3 u^2, \\
x_3^9 t^2 - x_0 x_1^4 x_2^4 u^2, \\
x_3^7 s t - x_0 x_1^3 x_2^3 u^2, \\
x_3^5 s^2 - x_0 x_1^2 x_2^2 u^2, \\
x_3 t^3 - x_1 s u^2, \\
x_3^3 s^3 - x_0 x_1 x_2 t u^2, \\
x_3^4 s^2 t^2 - x_0 x_1^2 x_2 u^4, \\
x_3^2 s^3 t - x_0 x_1 u^4, \\
x_3 s^4 - x_0 t^2 u^2, \\
x_2 s^4 - x_0 u^4, \\
x_1 s^5 - x_0 t^5
\end{cases}
$$

This time the ideal of the *S*-part of the leading terms is $\langle s, t^2 \rangle$ which is of degree 2.

Example 28 (`ex315`). While Examples 13, 18 and 19 support Conjecture 8, we point out that the latter is stated in its strongest possible form. For one, the claim is trivial in the case $\deg(\phi) > 1$ for then the comaximal minors vanish on all of Y again. This can be seen in Example 20.

On the other hand, while it is tempting to conjecture that

$$
\text{rad}(\langle P_{x_j} : j \rangle) \subset \text{sat}(\texttt{minors}(r-1,N))
$$

that is, that the inclusion of the conjecture is on the level of schemes, this is not true as illustrated by Example 16. In the case $\boldsymbol{d} = 1$, we get

$$
\text{rad}(\langle P_{x_j} : j \rangle) = \langle x_3, x_1 x_2, x_0 x_2, x_0 x_1 \rangle
$$
$$
\text{sat}(\texttt{minors}(r-1,N)) = \langle x_3^2, x_2 x_3, x_1 x_3, x_2^2, x_1 x_2, x_1^2 \rangle
$$
$$
\text{rad}(\texttt{minors}(r-1,N)) = \langle x_1, x_2, x_3 \rangle
$$

and the only inclusion we have is

$$
\text{rad}(\langle P_{x_j} : j \rangle) \subset \text{rad}(\texttt{minors}(r-1,N))
$$

The results for other values of $\boldsymbol{d}$ are analogous.

Example 29. It may seem at first that taking the smallest possible degree is always our best option, but Example 20 provides a good counterexample. The examples in Chapter **??**, which are computationally more complex, often have the first few N_i being zero, for low $\boldsymbol{d}$, so nonzero matrices tend to be larger in size and degree of their minors.

It is precisely this flexibility to choose a good $\boldsymbol{d}$, that is one of the contributions of our results to the methods of moving hypersurfaces.

4. PROOFS

While the ultimate goal of this chapter is to prove the results of Chapter **??**, it is written in a way to help develop intuition about the interplay between representation matrices on one hand and the geometry of Y on the other.

For this reason we start with a few elementary results with two-fold purpose. Firstly, they put together a some easy facts about our matrices N. Secondly, they highlight, when compared to other ad-hoc proofs, the advantage of our point of view.

We follow the notation of Chapter **??** and adopt the setup of (1). In particular, we work over a fix degree $\boldsymbol{d}$ with $S_{\boldsymbol{d}} \neq 0$, and do not yet require that ϕ be generically finite.

Lemma 30. *Let M be a square $r \times r$ matrix of syzygies. Then*

$$\det(M) \in P$$

In particular, if M is any (not necessarily square) matrix of syzygies, then

$$\mathtt{minors}(r, M) \subset P$$

Proof. The second statement clearly follows from the first, setting an empty minor to zero.
   Let $\mathrm{adj}(M)$ be adjugate matrix and set $\boldsymbol{b} = \mathtt{basis}(S_{\boldsymbol{d}})$. Then

$$(\boldsymbol{b} \cdot M) \cdot \mathrm{adj}(M) = \boldsymbol{b} \cdot \det(M)\boldsymbol{1}_r = \det(M)\boldsymbol{b}$$

Since $\boldsymbol{b} \cdot M$ is a row vector of syzygies, the LHS vanishes identically in S under the substitution $x_0 = \phi_0(\boldsymbol{s}), \ldots, x_n = \phi_n(\boldsymbol{s})$. But then

$$\mathrm{RHS}|_{x_0 = \phi_0, \ldots, x_n = \phi_n} = \det(M)(\phi_0, \ldots, \phi_n)\boldsymbol{b} = \begin{bmatrix} 0 & 0 & \ldots & 0 \end{bmatrix}$$

over the domain S. It follows that $\det(M)(\phi_0, \ldots, \phi_n) = 0$, so that $\det(M)$ is in the kernel P of $\phi^{\#}$, proving the first statement. $\qquad\square$

Lemma 31. *Any representation matrix N has at least as many columns as rows, i.e.*

$$\mu \geq r$$

and its ideal of maximal minors is nonzero.

Proof. For each standard basis column vector $e_k \in \mathbb{C}^r$, $P(\boldsymbol{x})e_k$ is a graded syzygy. Let F be the $\mu \times r$ matrix of coefficients for getting $P(\boldsymbol{x})e_k$ out of the generators of the syzygies over $S_{\boldsymbol{d}}$, that is

$$N \cdot F = P(\boldsymbol{x})\boldsymbol{1}_r$$

The sizes of the matrices on the LHS are $r \times \mu$ and $\mu \times r$. Since the rank of the RHS as a T-matrix is r, we must have $\mu \geq r$.
   The maximal minors are then of size $r \times r$. Since $\mathrm{rank}(N \cdot F) = r$, also $\mathrm{rank}(N) = r$ and so not all maximal minors vanish. $\qquad\square$

Lemma 32. *There is an isomorphism of graded T-modules*

$$\mathrm{coker}\, N \cong B_{\boldsymbol{d}, \bullet}$$

In particular, if $\mathscr{C}_{\bullet}$ is any graded resolution of $\mathrm{coker}(N)$ over T, then $H_0 \mathscr{C}_{\bullet} = B_{\boldsymbol{d}, \bullet}$.

Proof. This is obvious. The sequence

$$\bigoplus_k T(-i_k) \xrightarrow{N} T^r \xrightarrow{\boldsymbol{b}\,\cdot} B_{\boldsymbol{d}, \bullet} \longrightarrow 0 \tag{32.1}$$

is exact by definition, proving the claim. $\qquad\square$

Lemma 33. *One has*

$$\mathrm{ann}_T(B_{\boldsymbol{d},\bullet}) = P$$

Proof. Identifying $T = 1 \otimes T = R_{0,\bullet}$, we can think of T as a subring of R. Since R is a graded domain, we have

$$\mathrm{ann}_T(B_{\boldsymbol{d},\bullet}) = T \cap I$$

By the definition of I, any form $Q(\boldsymbol{x}) \in T$ with $Q(\boldsymbol{x}) \in I$ is the the kernel of the ring map $\phi^\#$. It follows that

$$T \cap I = P$$

completing the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark. The lemma above shows that $\mathrm{Supp}_T(B_{\boldsymbol{d},\bullet}) = V(P)$ and there is a cool way to see that the T-module localization $(B_{\boldsymbol{d},\bullet})_P$ is nonzero. Let N' be the localization of N at P. Then

$$(B_{\boldsymbol{d},\bullet})_P = \mathrm{coker}(N)_P = \mathrm{coker}(N')$$

Since $\mathrm{Fitt}_0 \, \mathrm{coker}(N') = \mathtt{minors}(r, N') \subset PT_P \ne T_P$ by Lemma 30, the cokernel is nonzero, for example, by (Eisenbud [1995], Proposition 20.6).

For a geometric argument, see the proof of Lemma 36.

Proof of Proposition 3. Since I is the kernel of a ring map into a domain, I is prime. By Lemma 33,

$$P = T \cap I \subset R$$

so $B = R/I$ is naturally a finite-type S-graded T/P-algebra. The T-module localization of B at P is just the localization of the ring R/I at the multiplicative set $(T - P)$. In particular, the localization B_P remains a domain, now as a $K(T/P)$-algebra. This proves parts (a) and (b).

Since I is prime in R, I is saturated with respect to the irrelevant ideal $\mathfrak{n} \subset S \subset R$. Now (c) follows because saturation commutes with localization. $\qquad\qquad\qquad\qquad\square$

Proof of Theorem 5. By Lemma 32, Lemma 33 and (Eisenbud [1995], Proposition 20.7), we have

$$\mathrm{rad}(\mathtt{minors}(r, N)) = \mathrm{rad}(\mathrm{Fitt}_0(\mathrm{coker}\, N)) = \mathrm{rad}(\mathrm{ann}(B_{\boldsymbol{d},\bullet})) = P \qquad\square$$

From now on, we assume that ϕ is generically finite, or equivalently, that P is principal.

Lemma 34. *Let $\mathscr{C}_\bullet$ be a finite graded free resolution of $\mathrm{coker}\, N$. One has*

$$\mathrm{div}(\det(\mathscr{C}_\bullet)) = \mathrm{length}_{T_P}(B_{\boldsymbol{d},\bullet})_P \cdot [Y]$$

as Weil divisors on $\mathbb{P}^n$.

Proof. By (Gelfand et al. [1994], A, Theorem 30), applied to the factorial T, a principal prime $Q = \langle Q(\boldsymbol{x}) \rangle$, and the generically exact $\mathscr{C}_\bullet$,

$$\mathrm{ord}_{Q(\boldsymbol{x})}(\det(\mathscr{C}_\bullet)) = \sum_i (-1)^i \mathrm{mult}_Q(H_i \mathscr{C}_\bullet)$$

Since $\mathscr{C}_\bullet$ is exact, all the higher homology vanishes, so

$$\mathrm{ord}_{Q(\boldsymbol{x})}(\det(\mathscr{C}_\bullet)) = \mathrm{mult}_Q(H_0 \mathscr{C}_\bullet)$$

The RHS above is zero outside $\mathrm{ann}(H_0\mathscr{C}_\bullet)$, so by Lemma 32 and 33, the RHS is nonzero only for $Q = P$. Summing over all non-associate irreducible homogeneous polynomials, we get

$$
\begin{aligned}
\mathrm{div}(\det(\mathscr{C}_\bullet)) &= \sum_{Q(\boldsymbol{x})} \mathrm{ord}_{Q(\boldsymbol{x})}(\det(\mathscr{C}_\bullet)) \cdot [V(Q)] \\
&= \sum_{Q(\boldsymbol{x})} \mathrm{mult}_Q(H_0\mathscr{C}_\bullet) \cdot [V(Q)] \\
&= \mathrm{mult}_P(H_0\mathscr{C}_\bullet) \cdot [Y] \\
&= \mathrm{length}_{T_P}(B_{\boldsymbol{d},\bullet})_P \cdot [Y]
\end{aligned}
$$

establishing the claim. $\qquad\square$

Lemma 35. *One has*

$$
\mathrm{length}_{T_P}(B_{\boldsymbol{d},\bullet})_P = \dim_{K(T/P)}(B_{\boldsymbol{d},\bullet})_P
$$

Proof. Note that $P(\boldsymbol{x}) \in I$, so $P(\boldsymbol{x})$ annihilates B as a T-module. Setting $\mathfrak{m}_P = PT_P \subset T_P$,

$$
\begin{aligned}
\mathrm{length}_{T_P}(B_{\boldsymbol{d},\bullet})_P &= \sum_k \dim_{T_P/\mathfrak{m}_P} \mathfrak{m}_P^k(B_{\boldsymbol{d},\bullet})_P/\mathfrak{m}_P^{k+1}(B_{\boldsymbol{d},\bullet})_P \\
&= \dim_{T_P/\mathfrak{m}_P}(B_{\boldsymbol{d},\bullet})_P
\end{aligned}
$$

since $\mathfrak{m}_P$ in turn annihilates $(B_{\boldsymbol{d},\bullet})_P$. $\qquad\square$

Remark. A one-line argument would be: the T- and T/P-module structure of B are the same.

Lemma 36. *Let $\boldsymbol{d} \in \mathrm{reg}(I_P)$ in the sense of* (4). *One has*

$$
\dim_{K(T/P)}(B_{d,\bullet})_P = \deg(\phi)
$$

Remark. This result requires neither $\mathrm{codim}(Y \subset \mathbb{P}^n) = 1$ nor $\dim V(J) = 0$.

Proof. Let $\Gamma = \mathrm{Biproj}(B)$ be the graph of the rational map ϕ, or equivalently the blow-up of X along the basepoints $V(J)$. Let $E \subset \Gamma$ be the exceptional locus. We point the reader to (Hartshorne [1977], II, Example 7.17.3) for the details, summarized in the following commutative diagram

Since $\pi_2|_{\Gamma-E} = \phi|_{X-V(J)} \circ \pi_1|_{\Gamma-E}$ and $\pi_1|_{\Gamma-E}$ is an isomorphism, the morphism π_2 is generically finite onto its image and $\deg(\phi) = \deg(\pi_2)$. If $\gamma \in \mathbb{P}^n$ is the generic point of Y, then the scheme-theoretic fiber

$$
\pi_2^{-1}(\gamma) = \mathrm{Spec}(\mathscr{O}_{\gamma,Y}) \times_Y \Gamma
$$

is a closed zero-dimensional subscheme of Γ consisting of $\deg(\pi_2)$ points, counted with multiplicity.

The morphism $\pi_2 : \mathrm{Biproj}(B) \to \mathrm{Proj}(T)$ is induced by the graded map of $\mathbb{C}$-algebras

$$
\pi_2^\sharp = (x_j \mapsto \overline{x_j \otimes 1}) : T \longrightarrow (T \otimes S)/I = B
$$

and the fiber of $\gamma = [P] \in \mathrm{Proj}(T)$ corresponds to the set of bihomogeneous prime ideals of B which pull back to $P \subset T$ via $\pi_2^{\sharp}$. By an easy reduction, for example (Vakil [2013], Exercise 7.3.H, 7.3.K and 9.3.A), this set corresponds to the set of S-grading-homogeneous prime ideals of the $K(T/P)$-algebra

$$K(T/P) \otimes_{T/P} B \cong B_P$$

The identification above presents the fiber as a $\mathrm{Proj}(-)$ over a field, that is,

$$\mathrm{Proj}_{K(T/P)} B_P \xrightarrow{\sim} \pi_2^{-1}(\gamma) \subset \mathrm{Biproj}(B)$$

Since this is a finite projective scheme over the field $K(T/P)$, its degree is well-defined and given by

$$\dim_{K(T/P)}(B_P)_{\boldsymbol{d}} = \dim_{K(T/P)}(B_{\boldsymbol{d},\bullet})_P$$

for all $\boldsymbol{d} \in \mathrm{reg}(I_P)$. $\square$

Proof of Theorem 6. Let $\mathscr{C}_{\bullet}$ be a minimal graded free resolution of $\mathrm{coker}\, N$. By (Gelfand et al. [1994], A, Theorem 34) which applies since $\mathscr{C}_{\bullet}$ is exact,

$$\det(\mathscr{C}_{\bullet}) = \gcd(\texttt{minors}(r,N))$$

up to a unit of T. But then by Lemma 34, 35 and 36,

$$
\begin{aligned}
\mathrm{div}(\gcd(\texttt{minors}(r,N))) &= \mathrm{div}(\det(\mathscr{C}_{\bullet})) \\
&= \mathrm{length}_{T_P}(B_{d,\bullet})_P \cdot [Y] \\
&= \dim_{K(T/P)}(B_{d,\bullet})_P \cdot [Y] \\
&= \deg(\phi) \cdot [Y]
\end{aligned}
$$

Because this is just an equality of Weil divisors,

$$\gcd(\texttt{minors}(r,N)) = P^{\deg(\phi)}$$ $\square$

Proof of Corollary 7. This follows directly from Theorem 6. $\square$

Proof of Proposition 12. For any choice of $>'$, a reduced Gröebner basis for I_B will have the outlined general form except possibly for the term g_{r+1}. Since $I \cap T = P$ by Lemma 33, a Gröbner basis must include $P(\boldsymbol{x})$ as its unique generator involving just the x_j. This proves the first part.

Define the ideal I'_B by the identity

$$B_P = \frac{(\mathbb{C}[\boldsymbol{x}] - P)^{-1}\mathbb{C}[\boldsymbol{s};\boldsymbol{x}]}{I'_B}$$

By reduce-ness, $P(\boldsymbol{x}) \nmid p_k(\boldsymbol{x})$, so I'_B can be obtained from the generating set for I_B by removing $P(\boldsymbol{x})$.

By the proof of Lemma 36, we know that $\mathrm{Proj}_{K(T/P)} B_P$ corresponds to the scheme-theoretic preimage of the generic point of Y, and in turn, this gives

$$\deg(\phi) = H^0(\mathrm{Proj}_{K(T/P)} B_P, \mathscr{O}) = \deg(I'_B)$$

where $\mathscr{O}$ denotes the structure sheaf of $\mathrm{Proj}_{K(T/P)} B_P$, and the degree on the far-right is the degree in $K(T/P)\mathbb{C}[\boldsymbol{s}]$. The result now follows because the initial ideal of I'_B has the same degree as I'_B. $\square$

5. A Method for Fast Implicitization

This chapter is devoted to the computational aspects of our results. Section 5.1 describes two algorithms for implicitization. The first one is simple and robust, and is used for studying the matrices N when Gröbner basis calculations can be carried out efficiently. The second one is more involved and is used when direct computations are unfeasible. In those cases, the second algorithm's lead is significant. Section 5.3 provides details about the algorithms and support to the latter claim in the form of a few worked examples.

Both, as means to illustrate that our algorithms are effective, say, in the sense of computational algebraic geometry, and as a setup for the examples to follow, in Section 5.2 we implement those algorithms in the Macaulay2 system.

The code is available at

https://github.com/radoslavraynov/ImplicitizationAlgos

We continue to follow the notation of Chapter **??** and the setup of (1). However, to avoid distraction, we assume throughout the chapter that ϕ is generically 1-1.

5.1. **The Algorithms.**

At first glance, an algorithm for finding the implicit equation is contained in the proof of the our main theorem, Theorem 6. In its simplest form, it becomes

Algorithm 37. Naive Algorithm.
> **input:** $J, \boldsymbol{d}$
> **output:** N, P
> Set $r = \dim_{\mathbb{C}}(S_{\boldsymbol{d}})$
> Compute an R-generating set $\{h_\ell : \ell\}$ for the Rees ideal I
> Compute a T-generating set $\{g_k : k\}$ for $I_{\boldsymbol{d},\bullet}$ from the h_ℓ using (**??**)
> Set N to be the coefficient matrix of the g_k with respect to $\mathtt{basis}(S_{\boldsymbol{d}})$
> Compute $P = \gcd(\mathtt{minors}(r, N))$
> Return N, P

38. The conciseness and robustness of Algorithm 37 made it our preferred tool for testing the theory. In fact, all calculations presented so far, including all examples of Chapter **??**, were carried out using this algorithm.

At the same time, its simplicity allows us to spot some of its drawbacks. We distinguish four major ones.

(1) Computing an R-generating set for the Rees ideal is at least as hard as computing the implicit equation itself — we have $I_{0,\bullet} = P$. This follows from Proposition 12 and shows up in Examples **??** and 27.

(2) While computing the gcd of two polynomials is fast, computing all minors could be difficult since their number can be very large. This happens even for reasonably small examples. For instance, the smallest nonzero matrix N for $\boldsymbol{d} = (2, 2, 1)$ in Example 51 is of size 18×50. The number of maximal minors is

$$\binom{50}{18} = 18'053'528'883'775$$

so even if it takes the unrealistic 0.001 seconds to compute each minor, a single machine would require 572 years to compute them all.

(3) Continuing with Example 51, we note that each maximal minor is a determinant of an 18×18-matrix of quartic forms in 5 variables. Computing large determinants symbolically is time-consuming. We did not manage to compute any nonzero minor.

Example 49 involves a somewhat similar calculation — the determinant of a 12×12-matrix of quadratic forms in 5 variables took about an hour to compute. Extrapolating, we can speculate that our 18×18 determinant would take somewhere in the order of

$$13 \times 14 \times 15 \times 16 \times 17 \times 18 = 13'366'080$$

hours. That is about 1525 years.

(4) Finally, suppose we have found the polynomial in question — by whatever means. It is a form of degree 48 in 5 variables, and very likely dense in the monomials of that degree. This suggests that the polynomial will be represented by

$$\binom{53}{5} = 2'869'685$$

coefficients.

Regrettably, (38-3) would be an issue for any algorithm relying on computing determinants of representation matrices, while (38-4) would be an issue for any implicitization algorithm whatsoever. Rather than seeing these as obstacles, we point them out as an argument *for* the idea of using representation matrices in place of the implicit equation altogether. We explore this theme further in the examples of Section 5.3.

39. Fix a degree d as before and recall that

$$N = (\, N_1 \mid \cdots \mid N_\delta \,)$$

Consider the following.

(1) Instead of computing the whole matrix N, one can compute the N_i's separately, keeping track of a partial representation matrix N'.
(2) Instead of computing all the minors, one only needs to compute sufficiently many to determine the gcd correctly.

These two simple observations produce an immense speed up on average. The advantage of (39-1) over computing an R-generating set for the Rees ideal is that it uses only linear algebraic routines. The advantage of computing only sufficiently many, rather than all, of the minors is obvious.

40. Let

$$N' = N'_i = (\, N_1 \mid \cdots \mid N_i \,)$$

be the partial matrix of syzygies up to degree i. Recall that h_i denotes the number of columns in N_i and set $p = \deg(P)$. Consider the following condition

$$\begin{cases} \sum h_i i = p & \text{if } \sum h_i = r \\ \sum h_i i \geq p & \text{if } \sum h_i > r \end{cases} \tag{C1}$$

This is a necessary condition for N' to be used instead of N, since if $\deg(\det(M)) < \deg(P)$, then M must be singular. It is almost certainly not a sufficient condition to conclude that $N' = N$, but often this is the case.

41. Let M_1, M_2 be nonsingular matrices of syzygies, as in (30), and let $Y_1, Y_2 \subset \mathbb{P}^n$ be the hypersurfaces they define. Let L be a general line in $\mathbb{P}^n$. If M_1, M_2 satisfy the condition

$$L \cap Y_1 \cap Y_2 \subset L \cap Y \tag{C2}$$

then $\gcd(\det(M_1), \det(M_2)) = P$.

The condition can be used for testing (39-2). To prove the claim, note that we always have

$$Y_1 \cap Y_2 \supset V(\gcd(\det(M_1), \det(M_2))) \supset Y$$

which together with (C2) gives $L \cap Y_1 \cap Y_2 = L \cap Y$, which is true exactly when $Y_1 \cap Y_2$ does not contain any other hypersurfaces besides Y. Indeed, if the gcd is a proper multiple of P, then the intersection of Y_1 and Y_2 contains another hypersurface, whose intersection with the general L is not going to be on Y.

Furthermore, the condition itself can be tested by computing the multiplicity of $Y_1 \cap Y_2 \cap L$ and comparing it to the multiplicity of $Y \cap L$. If the two are equal, then the condition is satisfied.

Summarizing the discussion so far, we propose

Algorithm 42. PROPOSED ALGORITHM.

   input: $J, \boldsymbol{d}, p = \deg(P)$
   output: a list of matrices M_k such that $\gcd(\{M_k : k\}) = P$
   Set $r = \dim_{\mathbb{C}}(S_{\boldsymbol{d}})$
   Set $N' = r \times 0$ matrix over T

   while (C1) is not satisfied for N' **do**
       Given $N_1, \ldots, N_{i-1}$, use Algorithm 43 to compute N_i
       Set $N' = N' \mid N_i$
   end while
   Set M_1, M_2 to be the zero $r \times r$ matrix
   Let $\mathscr{C}$ be some stopping criterion
   Let i be the index of the last computed matrix and let $\ell = r - h_1 - \ldots - h_{i-1}$
   while $\mathscr{C}$ is not satisfied and M_1, M_2 is do not satisfy (C2) **do**
       Let c_1, c_2 be a random sets of ℓ columns of N_i
       Set $M_1 = N_1 \mid \cdots \mid N_{i-1} \mid (N_i)_{c_1}$
       Set $M_2 = N_1 \mid \cdots \mid N_{i-1} \mid (N_i)_{c_2}$
   end while
   Return $P = \gcd(\det(M_1), \det(M_2))$

Algorithm 43. COMPUTE PARTIAL SYZYGIES.

   input: a list of the already computed $N_1, \ldots, N_{i-1}$
   output: N_i
   for $0 < j < i$ **do**
       Set $N_{ji} = \mathtt{basis}(T_{i-j}) \otimes N_j$
       Set K_{ji} to be the linearization of N_{ji}
   end for
   Set $K_i = \ker(\Phi^{(i)})$
   Let K_i' be such that $\mathrm{Span}(K_i) = \mathrm{Span}(K_i') \oplus (\sum_j \mathrm{Span}(K_{ji}))$
   Let N_i be such that $\mathtt{basis}(R_{\boldsymbol{d},i}) \cdot K_i' = \mathtt{basis}(S_{\boldsymbol{d}}) \cdot N_i$
   Return N_i

5.2. **Implementation in Macaulay2.**

44. We start with a realization for (**??**).

```
PushGens = (d,I) -> (
  r := toList(0..(#d-1));
  G := for g in I_* list (
    if all(d-((degree g)_r), Z->Z>=0)
    then basis((d-((degree g)_r))|{0},ring I)**g
    else continue);
  trim image fold(G,matrix {{0_(ring I)}},(a, b)->a|b)
  )
```

45. Using (44), Algorithm 37 is straight-forward to implement. We require R for encapsulation.

```
ComputeNRees = method ()
ComputeNRees (Ideal, List, Ring) := Matrix => (J, d, R) -> (
  x := symbol x;
  I := reesIdeal(J, Variable=>x);
  AI := ring I;
  zm := 0*d;
  g := map(R,AI,first entries super basis(zm|{1},R));
  I = g(I);
  V := PushGens (d,I);
  matrix entries ( (gens V) // basis(d|{0}, R) )
  )
```

46. Algorithm 43 is the one we make most use of in Section 5.3. Its fourth argument is the list of already computed matrices $N_1, \ldots, N_{i-1}$. If this list's size is not $i - 1$, then we just compute all linearly independent syzygies of degree i — $\mathrm{basis}(I_{d,i})$. The ideal J is supplied in the form of the matrix $F = \phi^{(1)} \otimes R$ (see **??**).

```
ComputeNi = method ()
ComputeNi (Matrix, ZZ, List, List) := Matrix => (F, i, d, lst) -> (
  R := ring F;
  m := #d;
  d0 := d|{0};
  di := d|{i};
  zm := 0*d;
  fj := flatten entries matrix F;
  xj := flatten entries super basis (zm|{1}, R);
  n := #fj;
  r := numcols super basis(d0, R);
  subs := apply(n,j->xj_j=>fj_j);
  e0 := (degree fj_0);
  G := sub(super basis(di, R), subs) // (super basis(i*e0+d0, R));
  K := matrix entries gens ker G;
  Nii := (super basis(di, R))*K // (super basis(d0, R));
  Nii = sub(matrix entries Nii,R);
  Nji := random(R^r,R^0);
  if #lst==i-1
  then Nji = fold (for j from 1 to #lst list
    (super basis(zm|{i-j}, R)**(lst_(j-1))), random(R^r,R^0), (m1,m2)->m1|m2);
  gens trim image (Nii%Nji)
  )
```

47. We now implement the first part of the proposed Algorithm 42. We omit the second part simply because of (38-3). Besides, after a general change of coordinates, any two nonzero minors are likely to suffice and we can compute them manually when necessary. See Examples 51 and 50 for details.

```
ComputeNConj = method ()
ComputeNConj (Matrix, ZZ, List) := Matrix => (F, p, d) -> (
  R := ring F;
  q := 0;
  i := 1;
  lst := {};
  r := numcols super basis (d|{0},R);
  N := random(R^r,R^0);

  zm := 0*d;
  fj := flatten entries matrix F;
  xj := flatten entries super basis (zm|{1}, R);
  n := #fj;
  subs := apply(n,j->xj_j=>fj_j);

  while q<p or not procTestCondN(N, subs) do (
    Ni := ComputeNi(F,i,d,lst);
    q = q + (numcols Ni)*i;
    lst = append(lst, Ni);
    N = N|Ni;
    i = i+1; );
  N
  )
```

5.3. **Examples.**

We now field-test our algorithms and code on several examples of somewhat higher computational complexity than those in Chapter **??**.

The running times can vary a lot from one machine to another, so the numbers below should not be treated as benchmarks. We include them only to provide a general idea how the different methods preform relative to each other.

The machine that we used was a MacBook Pro laptop with a 2.9 GHz Intel Core i7 processor and 8 GB 1600 MHz DDR3 memory, running Macaulay2 version 1.8.

Example 48 (ex601). Let $\phi : (\mathbb{P}^1)^3 \longrightarrow \mathbb{P}^4$ be given by 5 generic $(2,1,1)$-forms. The base locus is empty, so by (**??**) and (**??**), the degree of the image is 12.

Consider $\boldsymbol{d} = (1,1,1)$. Our method computes a candidate matrix N' in a little more than 0.1s. The matrix N' is square and we have $\det(N') = P$. Computing the determinant of the 8×8-matrix N takes 3s. The standard Gröbner basis computation takes 131s.

The details follow.

```
i1 : loadPackage "ImplicitizationAlgos";

i2 : KK=ZZ/32009;

i3 : S=KK[s_0,s_1,t_0,t_1,u_0,u_1,
         Degrees=>{2:{1,0,0},2:{0,1,0},2:{0,0,1}}];

i4 : T=KK[x_0..x_4];

i5 : B=super basis({2,1,1},S);

              1       12
o5 : Matrix S  <--- S
```

```
i6 : J=ideal(B*random(S^12,S^5));

o6 : Ideal of S

i7 : R=KK[s_0,s_1,t_0,t_1,u_0,u_1,x_0..x_4,
          Degrees=>{2:{1,0,0,0},2:{0,1,0,0},2:{0,0,1,0},5:{0,0,0,1}}];

i8 : F=sub(gens J,R);

             1      5
o8 : Matrix R  <--- R

i9 : d={1,1,1};

i10 : time N1=ComputeNi(F,1,d,{});
     -- used 0.0119639 seconds

             8      4
o10 : Matrix R  <--- R

i11 : time N2=ComputeNi(F,2,d,{N1});
     -- used 0.0920771 seconds

             8      4
o11 : Matrix R  <--- R
```

The partial matrix $N' = N_1|N_2$ is square. The degree of its determinant is $12 = 4 \cdot 1 + 4 \cdot 2$, so this is our candidate matrix. We check at the end that $\det(N') = P(\mathbf{x})$.

```
i12 : time ComputeNi(F,2,d,{});
     -- used 0.0867573 seconds

             8      24
o12 : Matrix R  <--- R

i13 : time N3=ComputeNi(F,3,d,{N1,N2});
     -- used 0.846328 seconds

             8
o13 : Matrix R  <--- 0

i14 : time ComputeNi(F,3,d,{});
     -- used 0.903612 seconds

             8      80
o14 : Matrix R  <--- R

i15 : time ComputeNi(F,4,d,{});
     -- used 6.38165 seconds

             8      200
o15 : Matrix R  <--- R
```

Note that even though N' is square and giving the implicit equation right off the bat, we cannot be sure that $N = N_1|N_2$. Lines 12–15 support this claim. For instance, because the columns of N_1 and N_2 form a nonsingular matrix, the degree-3 syzygies they give rise to are going to be linearly independent. There are exactly $80 = 4 \cdot \binom{4+2}{2} + 4 \cdot \binom{4+1}{1}$ of them, so $N_3 = 0$. The same argument shows that $N_4 = 0$.

```
i16 : N=sub(N1|N2,T);

             8      8
o16 : Matrix T  <--- T

i17 : time N'=ComputeNConj(F,12,d);
     -- used 0.138608 seconds

             8      8
o17 : Matrix R  <--- R

i18 : N'=sub(N',T);
```

```
                 8        8
o18 : Matrix T  <--- T

i19 : image N==image N'

o19 = true

i20 : time P'=ideal det N;
     -- used 3.0447 seconds

o20 : Ideal of T
```

Line 20 shows that $\det(N') = P(\boldsymbol{x})$.

```
i21 : time P=ker map(S,T,J_*);
     -- used 131.05 seconds

o21 : Ideal of T

i22 : P==P'

o22 = true
```

The standard method to compute the implicit equation takes more than 40 times longer.

Example 49 (ex602). We compute the implicit equation of five general $(2,2,1)$-forms over $(\mathbb{P}^1)^3$. The base locus is empty, so the degree of the equation is 24. It takes our method less than a second to find it in the form of a determinant of an 18×18-matrix of quadratic forms. It take a little less than an hour to compute the actual equation. The standard Gröbner basis calculation did not finish in 24 hours.

```
i1 : loadPackage "ImplicitizationAlgos";

i2 : KK=ZZ/32009;

i3 : S=KK[s_0,s_1,t_0,t_1,u_0,u_1,
         Degrees=>{2:{1,0,0},2:{0,1,0},2:{0,0,1}}];

i4 : T=KK[x_0..x_4];

i5 : B=super basis({2,2,1},S);

              1        18
o5 : Matrix S  <--- S

i6 : J=ideal(B*random(S^18,S^5));

o6 : Ideal of S

i7 : R=KK[s_0,s_1,t_0,t_1,u_0,u_1,x_0..x_4,
         Degrees=>{2:{1,0,0,0},2:{0,1,0,0},2:{0,0,1,0},5:{0,0,0,1}}];

i8 : F=sub(gens J,R);

              1        5
o8 : Matrix R  <--- R

i9 : d={2,1,1};

i10 : time N1=ComputeNi(F,1,d,{});
     -- used 0.017498 seconds

             12
o10 : Matrix R    <--- 0

i11 : time N2=ComputeNi(F,2,d,{});
     -- used 0.298143 seconds

             12        12
o11 : Matrix R    <--- R

i12 : time N3=ComputeNi(F,3,d,{N1,N2});
     -- used 4.01669 seconds

             12
```

```
o12 : Matrix R   <--- 0

i13 : time N'=ComputeNConj(F,24,d);
      -- used 0.762765 seconds

              12        12
o13 : Matrix R    <--- R

i14 : N'==N2

o14 = true

i15 : rank N2

o15 = 12
```

Since we know that $N' = N_2$ is square, its determinant is a form of degree $12 \cdot 2 = 24$, so $\det(N') = P(\mathbf{x})$. Note that we do not have to compute the determinant to make sure that N' is nonsingular. We can compute the rank by substituting random numbers for the x_j and computing over a finite field.

```
i17 : N=sub(N2,T);

              12        12
o17 : Matrix T    <--- T

i18 : time P'=det N;
      -- used 3310.29 seconds
```

While we actually can compute the determinant, so also the implicit equation itself, we could have answered questions like "Is this point on the image" using N' only.

Example 50 (ex604). Consider a map given by four general bi-quartics in the ideal $\langle s^3, s^2t, t^2 \rangle$. Then the base locus is the point $(0,0,1)$ of degree 5 and multiplicity 6. The degree of the image is 26. Our method finds it in explicit form in about 6 minutes, while it takes us less than a minute to find two square matrices M_1, M_2 for which

$$\gcd(\det(M_1), \det(M_2)) = P(\mathbf{x})$$

This highlights the interplay between the algebra and geometry — one of our initial goals. The standard method takes more than 6 hours.

```
i1 : loadPackage "ImplicitizationAlgos";

i2 : KK=ZZ/32009;

i3 : S=KK[s,u,t,v,Degrees=>{2:{1,0},2:{0,1}}];

i4 : T=KK[x_0..x_3];

i5 : Z=ideal(s^3,s^2*t,t^2);

o5 : Ideal of S

i6 : B=super basis({4,4},Z);

              1       20
o6 : Matrix S  <--- S

i7 : J=ideal(B*random(S^20,S^4));

o7 : Ideal of S

i8 : decompose J

o8 = {ideal (s, t), ideal (s, u), ideal (v, t)}

o8 : List

i9 : multiplicity Z

o9 = 6
```

The base locus is supported on a single point, so the Macaulay2 command computes the correct quantity.

```
i10 : R=KK[s,u,t,v,x_0..x_3,
           Degrees=>{2:{1,0,0},2:{0,1,0},4:{0,0,1}}];

i11 : F=sub(gens J,R);

              1      4
o11 : Matrix R  <--- R

i12 : d={2,2};

i13 : time N1=ComputeNi(F,1,d,{});
     -- used 0.00821136 seconds

              9
o13 : Matrix R  <--- 0

i14 : time N2=ComputeNi(F,2,d,{N1});
     -- used 0.0901831 seconds

              9
o14 : Matrix R  <--- 0
```

The degree $d = (2,2)$ does not look promising. Take $d = (3,3)$ instead.

```
i15 : d={3,3};

i16 : time N1=ComputeNi(F,1,d,{});
     -- used 0.0208163 seconds

              16      5
o16 : Matrix R   <--- R

i17 : time N2=ComputeNi(F,2,d,{N1});
     -- used 0.221774 seconds

              16       12
o17 : Matrix R   <--- R

i18 : N=N1|N2;

              16       17
o18 : Matrix R   <--- R

i19 : time rank N
     -- used 0.009011 seconds

o19 = 16
```

This is our candidate matrix N' — it has more columns than rows and any nonzero minor would have degree 27 or 28. Having N' be of full rank is not enough by itself.

```
i20 : c1=sort join({0,1,2,3,4}, RandPerm(5,16,11));

i21 : time rank N_c1
     -- used 0.00241363 seconds

o21 = 16

i22 : c2=sort join({0,1,2,3,4}, RandPerm(5,16,11));

i23 : c1==c2

o23 = false

i24 : time rank N_c2
     -- used 0.00881294 seconds

o24 = 16
```

At this point we found two distinct length-12 sets of columns, c_1, c_2, such that the square matrices M_1, M_2 they give rise to are nonsingular. We can show that these satisfy the

equality described above. However, we can actually compute the implicit equation in this case.

```
i25 : time D1=det N_c1;
      -- used 164.479 seconds

i26 : time D2=det N_c2;
      -- used 162.898 seconds

i27 : time P=gcd(D1,D2);
      -- used 0.0646495 seconds

i28 : deg P

o28 : 26
```

Example 51 (ex603). We enhance Example 49 and compute the implicit equation of five general $(2,2,2)$-forms on $(\mathbb{P}^1)^3$. The base locus is again empty, so the degree of the image is 48.

In this setup there is no hope of computing any determinant explicitly, let alone computing $P(\boldsymbol{x})$ using Gröbner bases directly. We can, however, repeat what we did in Example 50. We work over $\boldsymbol{d} = (2,2,1)$ and find two square matrices M_1, M_2 of order 18 such that

$$\gcd(\det(M_1), \det(M_2)) = P(\boldsymbol{x})$$

While testing if the matrices are nonsingular is easy, checking whether their determinants do not have any other prime common factor is much harder. We follow the discussion in (41) and find that the common intersection locus of the two hypersurfaces with a general line has multiplicity 48.

```
i1 : loadPackage "ImplicitizationAlgos";

i2 : KK=ZZ/32009;

i3 : S=KK[s_0,s_1,t_0,t_1,u_0,u_1,Degrees=>{2:{1,0,0},2:{0,1,0},2:{0,0,1}}];

i4 : T=KK[x_0..x_4];

i5 : B=super basis({2,2,2},S);

              1        27
o5 : Matrix S  <--- S

i6 : J=ideal(B*random(S^27,S^5));

o6 : Ideal of S

i7 : R=KK[s_0,s_1,t_0,t_1,u_0,u_1,x_0..x_4,
         Degrees=>{2:{1,0,0,0},2:{0,1,0,0},2:{0,0,1,0},5:{0,0,0,1}}];

i8 : F=sub(gens J,R);

              1        5
o8 : Matrix R  <--- R

i9 : d={1,1,1};

i10 : N1=ComputeNi(F,1,d,{});

              8
o10 : Matrix R  <--- 0

i11 : N2=ComputeNi(F,2,d,{});

              8
o11 : Matrix R  <--- 0

i12 : N3=ComputeNi(F,2,d,{});

              8
```

```
o12 : Matrix R  <--- 0
```

We sneak in a remark here. One might expect to find that N_6 is square, so that $\deg(\det(N_6)) = 48$. In fact $N_i = 0$ for up to at least $i = 8$. In general, smaller $\boldsymbol{d}$ tend to produce the few matrices N_i zero. This follows from the sort of Koszul-ness of small syzygies. Take $\boldsymbol{d} = (2,2,1)$ instead.

```
i13 : d = {2,2,1};

i14 : time N4=ComputeNi(F,4,d,{});
      -- used 151.652 seconds

                18        50
o14 : Matrix R    <--- R

i15 : time rank N4
      -- used 2.2373 seconds

o15 = 18

i16 : c1=RandPerm(0,49,18);

i17 : M1=N4_c1;

                18        18
o17 : Matrix R    <--- R

i18 : rank M1

o18 = 18

i19 : c2=RandPerm(0,49,18);

i20 : M2=N4_c2;

                18        18
o20 : Matrix R    <--- R

i21 : rank M2

o21 = 18

i22 : sort c1==sort c2

o22 = false
```

The two matrices are nonsingular.

```
i23 : M1=sub(M1,T);

                18        18
o23 : Matrix T    <--- T

i24 : M2=sub(M2,T);

                18        18
o24 : Matrix T    <--- T

i25 : L=ideal random(T^{1,1,1},T^1);

o25 : Ideal of T
```

Here L or rather $V(L)$ is the generic line.

```
i26 : TL=T/L;

i27 : super basis(1,TL)

o27 = | x_3 x_4 |

                 1        2
o27 : Matrix TL    <--- TL

i28 : U=KK[y_0,y_1];

i29 : g=map(U,TL,{0,0,0,y_0,y_1});

o29 : RingMap U <--- TL
```

```
i30 : M1'=g(sub(M1,TL));

                18        18
o30 : Matrix U     <--- U

i31 : M2'=g(sub(M2,TL));

                18        18
o31 : Matrix U     <--- U
```

This last bit is for computational purposes only. Since we are on a $\mathbb{P}^1$, we should make use of this and work over 2 variables only.

```
i32 : time D1=det(M1');
     -- used 0.317001 seconds

i33 : time D2=det(M2');
     -- used 0.334676 seconds

i34 : degree gcd(D1,D2)

o34 = {48}

o34 : List
```

The multiplicity of the common intersection is 48! We are done.

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