

# EXAMPLES OF IMPLICITIZATION OF HYPERSURFACES THROUGH SYZYGIES

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## 1. INTRODUCTION

The main goal of this thesis is to develop a framework for both efficiently computing the image  $Y$  of a rational map  $\phi : X \longrightarrow \mathbb{P}^n$ , and studying the relation between algebraic properties of the coordinates  $\phi_j$  on one hand, and geometric properties of the image  $Y$ , on the other. We are most interested in the case when  $X$  is a smooth projective toric variety of dimension  $n - 1$  and  $\phi$  is generically finite, so finding the image entails computing a single equation  $P$ , the implicit equation.

This is (a version of) the implicitization problem which is classical in algebraic geometry. It has an effective solution through elimination using Gröbner bases, for instance, but those techniques have two drawbacks. First, they become unfeasible even in modest complexity. Second, they represent a black box in terms of geometry—one which takes the  $\phi_j$  and returns  $P$ , in our Sun's lifetime or not. Motivated by the advance of computer aided design (CAD), interest in new approaches to the implicitization problem was reincarnated in the 1990s, with the initiating papers being Sederberg and Chen [1995].

Our approach follows the theme of moving surfaces but sets to remove the intrinsic ad-hoc constructions. In this sense, we drew inspiration by the method of the approximation complex, initiated by Busé and Jouanolou [2003]. Our main results are a blend of two. Specifically, let  $S$  be the Cox ring of  $X$  graded by  $\text{Pic}(X)$ , and  $J = \langle \phi_0, \dots, \phi_n \rangle \subset S$  be the ideal of the coordinates. We construct, for any degree  $\mathbf{d}$  on  $S$ , a matrix  $N$  representing a generating set for the polynomial relations on the  $\phi_j$  with coefficients from  $S_{\mathbf{d}}$  only. This matrix generalizes the types of representation matrices studied by the said methods. One of our main results is an analogue to the results for approximation complex. We show that if  $N$  is of size  $r \times \mu$ , then  $\text{rank}(N) = r$  and

$$\gcd(\text{minors}(r, N)) = P^{\deg(\phi)}$$

Relaxing the requirement that  $X$  be of dimension  $n - 1$ , we find that

$$\text{rad}(\text{minors}(r, N)) = P$$

Results of form: if conditions (1)–(k) hold, then there exists a square matrix  $M$  such that  $\det(M) = P$ , which are the main theme of the moving surfaces method, are now equivalent to saying the degree- $(\mathbf{d}, \bullet)$  ideal of the coordinates  $J$  has  $r$  relations over  $S_{\mathbf{d}}$  as described above. This suggest a template for proofs allowing one to focus on the various conditions only.

Moreover, the matrix  $N$  can be computed incrementally using only tools from linear algebra, allowing calculation in situations previously out of reach.

## 2. RESULTS

We are now ready to state our main results. While this chapter is supposed to be self-contained and most of the relevant notation and definitions are listed in (1) below, one should consult Chapter ?? for a more relaxed exposition and further definitions, for instance, for the notions of degree and multiplicity of a basepoint.

Examples ?? and ?? should serve as quick reference points.

**1.** Let  $X$  be a smooth projective toric variety of dimension  $n - 1$  ( $n > 1$ ) with Cox ring  $S$  and irrelevant ideal  $\mathfrak{n} \subset S$ . Let  $T = \mathbb{C}[x_0, \dots, x_n]$  and let

$$\phi = (\phi_0, \dots, \phi_n) : X \longrightarrow \mathbb{P}^n = \text{Proj}(T)$$

be a rational map given by linearly independent sections of the line bundle  $\mathcal{O}_X(\mathbf{e})$  for some degree  $\mathbf{e}$  on  $S$ . Let  $J = \langle \phi_0, \dots, \phi_n \rangle \subset S$  be the ideal of the coordinates  $\phi_j$ .

Let  $\mathbf{d}$  be a degree on  $S$  such that  $r = \dim_{\mathbb{C}}(S_{\mathbf{d}}) > 0$  and let  $I \subset R = S \otimes T$  be the Rees ideal of  $J$ . Let  $I_{\mathbf{d}, \bullet}$  be its degree- $(\mathbf{d}, \bullet)$  bigraded piece, considered as a finite graded  $T$ -module. We denote by  $B = R/I$  the Rees algebra of  $J$ .

Let  $N$  be the  $r \times \mu$  coefficient matrix of a minimal set of homogeneous generators for  $I_{\mathbf{d}, \bullet}$  with respect to  $\text{basis}(S_{\mathbf{d}})$ , and let  $N_i$  be the submatrix of  $N$  corresponding to generators of degree  $i$ , i.e.

$$N = (N_1 \mid \dots \mid N_{\delta})$$

**2.** Let  $P \subset T$  be the prime ideal corresponding to the closed image of  $\phi$  in  $\mathbb{P}^n$ . We denote this image by  $Y = V(P)$ . Let  $Z = V(J) \subset X$  be the base locus of  $\phi$ .

We are going to be interested the following three conditions:

- (1) The map  $\phi$  is generically finite onto its image, that is,  $Y \subset \mathbb{P}^n$  is of codimension 1 and in particular, the ideal  $P$  is principal. In this case, we denote a generator of  $P$  by  $P(\mathbf{x})$ .
- (2) The base locus  $Z$  is zero-dimensional, that is, consists of finitely many points. Note that those are necessarily closed over  $\mathbb{C}$ .
- (3) The map  $\phi$  is birational onto its image.

Clearly, either of (2) and (3) implies (1).

A few easy but important observations about the Rees ideal follow.

**Proposition 3.** *Consider the setup of (1). One has*

- (1) *The ideal  $I \subset R$  is prime, and so is  $I_P$  in the  $T$ -module localization  $R_P$ .*
- (2) *The quotient  $B_P$  is naturally a finite-type graded  $K(T/P)$ -algebra with grading induced by  $S$ .*
- (3) *The  $K(T/P)$ -algebra  $B_P$  is a homogeneous coordinate ring of a projective variety.*

**4.** Let  $V(I_P)$  be the closed subset of the projective toric variety  $\text{Biproj}(R_P)$ . Note that the former is a variety over  $K(T/P)$  with the grading of  $S$ . Following Maclagan and Smith [2004] and using Proposition 3, we consider the regularity of the defining ideal  $I_P$ , denoted by  $\text{reg}(I_P)$ .

Recall that  $\text{reg}(I_P)$  is a finitely generated additively-closed subset of the semigroup of degrees on  $S$ , and that for any  $\mathbf{d} \in \text{reg}(I_P)$ , we have  $\langle (I_P)_{\mathbf{d}} \rangle = I_P$ . This parallels the usual Castelnuovo-Mumford regularity for  $\mathbb{P}^n$  and is the content of Theorem 1.3 in the referenced paper.

In light of Proposition 3, our first result becomes an easy exercise. However, it is a step toward the goal of this paper — to exhibit a general relation between the algebra of the coordinates  $\phi_j$  and the geometry of the image  $Y$ .

**Theorem 5.** *In the setup of (1), one has*

$$\text{rad}(\text{minors}(r, N)) = P$$

The geometric interpretation of the theorem is clear — the nonzero minors of  $N$  define hypersurfaces in  $\mathbb{P}^n$  whose intersection, at least set-theoretically, is the image  $Y$ .

Example 18, for instance, shows that the radical is necessary.

Our next result is the main theorem of this thesis, unifying two currently popular non-Gröbner basis approaches to implicitization and setting the stage for both the ad-hoc template proofs in Chapter ?? and the fast implicitization method described in Chapter 5.

**Theorem 6.** *Consider the setup of (1) and assume (2.1). Fix a degree  $\mathbf{d} \in \text{reg}(I_P)$  as described in (4). One has*

$$\text{gcd}(\text{minors}(r, N)) = P^{\deg \phi}$$

*In particular, if  $\mu = r$ , then  $N$  is square and, up to a unit,*

$$\det(N) = P(\mathbf{x})^{\deg \phi}$$

**Corollary 7.** *In the setup of Theorem 6, let  $M$  be any  $r \times r$  matrix of syzygies over  $S_{\mathbf{d}}$ . One has*

$$\det(M) = P(\mathbf{x})^{\deg \phi} \cdot H(\mathbf{x})$$

*for a homogeneous  $H(\mathbf{x})$  of degree*

$$\deg(\det(M)) - \deg(\phi) \cdot \deg(Y) \tag{7.1}$$

*Furthermore, there exist a list of such matrices  $\{M_k\}$  whose corresponding  $H_k(\mathbf{x})$  are nonzero and have common factor 1.*

Geometrically, the former is a refinement of Theorem 5. Each of the maximal minors of  $N$ , in fact, the determinant of any  $r \times r$  matrix  $M$  of syzygies over  $S_{\mathbf{d}}$ , is either zero or describes the union of a  $\deg(\phi)$ -fold  $Y$  and a hypersurface of degree (7.1). While an arbitrary collection  $M_k$  of such matrices may introduce hypersurfaces with an intersection that is strictly larger than  $Y$ , the maximal minors suffice to shave off any extraneous components.

The theme of extraneous factors is already apparent in Busé et al. [2003], Busé et al. [2009] and Botbol et al. [2009]. In our notation, they used the approximation complex to show that for a toric  $X$ , certain  $\mathbf{d}$  and empty or zero-dimensional almost complete intersection base locus  $Z$ ,

$$\text{gcd}(\text{minors}(r, N_1)) = P^{\deg \phi} \cdot \prod_{q \in Z} L_q(\mathbf{x})^{e_q - d_q}$$

where each  $L_q(\mathbf{x})$  is a linear form, and  $e_q$  and  $d_q$  are the multiplicity and degree of  $q$ .

In the case of complete intersection base locus, the proof of Theorem 6 gives a special case of the above.

It is known that if  $M$  is a square matrix over  $T$  of size  $r$ , then the singular locus of  $V(\det(M))$  is contained in the closed subset defined by the comaximal minors, that is, the  $(r-1)$ -minors. Although we failed to find a reference, we believe that this relation is more intrinsic and holds for all representation matrices  $N$ . However, what this ought to correspond to is the multiple-point locus of the image. See Example 27 for details. We conjecture the following

**Conjecture 8.** *Consider the setup of (1) and assume (2.3). On the level of closed points, one has*

$$V(\text{minors}(r-1, N)) \subset \text{Sing}(Y)$$

In the simplest cases of interest, when  $X = \mathbb{P}^2$  or  $X = \mathbb{P}^1 \times \mathbb{P}^1$  and  $\phi$  is basepoint-free, we can chose  $\mathbf{d}$  so that the matrix  $N$  becomes square. Next two theorems are slight generalizations of the results in Cox et al. [2000]. More importantly, they show that our methods directly generalize the methods of moving planes and quadrics in the setting in which they are most useful.

**Theorem 9.** *Let  $X = \mathbb{P}^2$ ,  $\phi$  be basepoint-free, and suppose that there are exactly  $p = \mathbf{e}$  linear syzygies over degree  $\mathbf{d} = p - 1$ , that is, the minimal possible number. One has that  $N$  is square and  $N = (N_1 \mid N_2)$ . In particular,*

$$\det(N) = \det(N_1 \mid N_2) = P(\mathbf{x})^{\deg(\phi)}$$

**Theorem 10.** *Let  $X = \mathbb{P}^1 \times \mathbb{P}^1$ ,  $\phi$  be basepoint-free with coordinates in degree  $\mathbf{e} = (p, q)$ , and suppose there are no linear syzygies over degree  $\mathbf{d} = (p - 1, q - 1)$ . Then one has that  $N_2$  is square and  $N = N_2$ . In particular,*

$$\det(N) = \det(N_2) = P(\mathbf{x})^{\deg(\phi)}$$

Both of these theorems are examples of a template proof described in Chapter ???. While applying it in general requires elaborate choses of the degree  $\mathbf{d}$  and regularity computations, for example Adkins et al. [2005], in the case of Theorems 9 and 10, we only use a type of Koszul-ness on the syzygies of low degree. This is the content of Section ???.

We conclude this list by a method to compute the degree of a rational map using Gröbner bases. While we are only going to use this in our examples, it helps expand our understanding about the object  $B_P$ .

The author wants to thank Mike Stillman for suggesting the following

**Proposition 11.** *Let  $\mathbb{C}[s_0, \dots, s_m]$  be the fixed ambient polynomial ring of  $S'$  as described in (1). Define the ideal  $I_B$  of  $\mathbb{C}[s_0, \dots, s_m; x_0, \dots, x_n]$  by the equality*

$$B = \mathbb{C}[\mathbf{s}; \mathbf{x}] / I_B$$

*Let  $>'$  be any product order in which the  $\mathbf{s}$  variables come before the  $\mathbf{x}$  variables. Then a reduced Gröbner basis for  $I_B$  with respect to  $>'$  has the form*

$$g_1(\mathbf{s}; \mathbf{x}) = p_1(\mathbf{x})\mathbf{s}^{\alpha_1} + \text{lower order terms}$$

...

$$g_r(\mathbf{s}; \mathbf{x}) = p_r(\mathbf{x})\mathbf{s}^{\alpha_r} + \text{lower order terms}$$

$$g_{r+1}(\mathbf{s}; \mathbf{x}) = P(\mathbf{x})$$

*Further, one has*

$$\deg(\phi) = \deg(\langle \mathbf{s}^{\alpha_1}, \dots, \mathbf{s}^{\alpha_r} \rangle \subset \mathbb{C}[\mathbf{s}])$$

### 3. EXAMPLES

**Example 12 (ex301).** Let  $X = \mathbb{P}_{s,t,u}^2$  and  $J = \langle tu, su, st, s^2 + t^2 + u^2 \rangle$ . Then  $\phi$  is basepoint-free and generically 1-1. The monic implicit equation is given by

$$P(\mathbf{x}) = x_0^2 x_1^2 + x_0^2 x_2^2 + x_1^2 x_2^2 - x_0 x_1 x_2 x_3$$

Setting  $d = 1$ , we get

$$N = \begin{bmatrix} 0 & x_0 & x_1x_2 \\ x_1 & 0 & x_0x_2 \\ -x_2 & -x_2 & x_0x_1 - x_2x_3 \end{bmatrix}$$

whose determinant is just  $P(\mathbf{x})$ . The results of Cox et al. [2000] apply and the matrix  $N$  is a variant of the matrix produced by the method of moving planes and quadrics.

Setting  $d = 2$ , we get

$$N = \begin{bmatrix} x_2 & 0 & 0 & 0 & x_1 & 0 & 0 & 0 & x_0 \\ -x_3 & 0 & 0 & x_1 & 0 & 0 & 0 & x_0 & 0 \\ 0 & 0 & 0 & -x_2 & -x_3 & x_0 & x_2 & 0 & -x_2 \\ x_2 & 0 & x_1 & 0 & 0 & 0 & x_0 & 0 & 0 \\ 0 & x_1 & -x_2 & 0 & x_2 & 0 & -x_3 & -x_2 & 0 \\ x_2 & -x_2 & 0 & 0 & x_1 & -x_2 & x_0 & 0 & 0 \end{bmatrix}$$

which is a  $6 \times 9$  matrix of linear forms. This was expected — the results of Busé and Jouanolou [2003] also apply and the method of the approximation complex guarantees a matrix of linear forms. Accordingly,

$$\gcd(\text{minors}(6, N)) = P$$

Note that the claim that  $\phi$  is of degree 1 follows, a fortiori, from the degree formula (??). Indeed, we have

$$4 \deg(\phi) = 2^2 - 0$$

We confirm this using Proposition 11 in Example 26.

**Example 13 (ex302).** Clearly, if we replace  $s, t, u$  in Example 12 by general linear forms  $L_0, L_1, L_2$ , effectively changing coordinates on the source, we get the same equation. In this example we describe what happens if we take  $X = \mathbb{P}_{s,u}^1 \times \mathbb{P}_{t,v}^1$  instead and let the  $L_k$  be  $(1, 1)$ -forms. Since the algebraic structure of the coordinates is the same, so is the equation of the image,

$$Y = V(x_0^2x_1^2 + x_0^2x_2^2 + x_1^2x_2^2 - x_0x_1x_2x_3)$$

and  $\phi$  is again basepoint-free, basically for the same reason — the  $L_k$  are general. However, the self-intersection of the divisor corresponding to the coordinates, now  $[\mathcal{O}(2, 2)]$ , is 8. It follows that  $\phi$  is generically 2-1.

For  $d = (1, 1)$  we get a square matrix of size 4 with  $h = (2, 1, 0, 1)$ . As expected, up to a unit

$$\det(N) = P(\mathbf{x})^2$$

For  $d = (2, 1)$  we get a square matrix of size 6 with  $h = (4, 2)$ , for which the last equality again applies. For  $d = (2, 2)$  we get a  $9 \times 12$ -matrix with  $h = (11, 1)$  such that

$$\gcd(\text{minors}(9, N)) = P^2$$

**Example 14 (ex303).** Suppose that in the situation of Example 13 we took the forms  $L_k$  from  $\langle st, sv, ut \rangle$  instead. Now  $\phi$  has the unique basepoint  $(0, 1) \times (0, 1)$  which is c.i. of degree 4. Indeed, on the affine open where  $u = v = 1$ , the point looks like  $V(st, s^2 + t^2)$ . The equation of the image remains the same. Once again, by (??) we know that  $\phi$  must be generically 1-1.

For  $\mathbf{d} = (1, 1)$  we get a  $4 \times 5$ -matrix. Below is the matrix resulting from  $(L_0, L_1, L_2) = (st, sv, ut)$ ,

$$N = \begin{bmatrix} 0 & 0 & 0 & x_0 & x_1x_2 \\ x_1 & x_0 & 0 & 0 & 0 \\ -x_2 & 0 & x_0 & -x_2 & -x_2x_3 \\ 0 & -x_2 & -x_1 & 0 & x_1^2 + x_2^2 \end{bmatrix}$$

and, of course,

$$\gcd(\text{minors}(4, N)) = P$$

**Example 15 (ex304).** Let  $X = \mathbb{P}^2$  and  $J = \langle su^2, t^2(s+u), st(s+u), tu(s+u) \rangle$ . Then  $\phi$  is generically 1-1 with three basepoints —  $(1, 0, 0)$ ,  $(0, 1, 0)$  and  $(0, 0, 1)$ , all c.i. of degree 2, 3, and 1, respectively.

The implicit equation is given by

$$P(\mathbf{x}) = x_0x_1x_2 + x_0x_1x_3 - x_2x_3^2$$

As before, both the method of the moving planes and quadrics, and the method of the approximation complex apply. For  $\mathbf{d} = 1$ , we get

$$N = \begin{bmatrix} -x_3 & 0 & x_1 & -x_3^2 \\ 0 & -x_3 & -x_2 & x_0x_2 + x_0x_3 \\ x_2 & x_1 & 0 & 0 \end{bmatrix}$$

and for  $\mathbf{d} = 2$ , we get a  $6 \times 9$ -matrix whose entries are all linear.

**Example 16 (ex305).** Let  $X = \mathbb{P}^2$  and  $J = \langle s^3, tu^2, s^2t + u^3, stu \rangle$ . Then  $\phi$  is generically 1-1 with a single c.i. basepoint of degree 2. For  $\mathbf{d} = 1$  we have

$$N = \begin{bmatrix} x_1 & 0 & -x_3^2 \\ 0 & x_1x_2 - x_3^2 & x_0x_1 \\ x_3 & x_1^2 & 0 \end{bmatrix}$$

and  $\det(N) = x_0x_1^4 - x_1x_2x_3^3 + x_3^5$ , the implicit equation.

Starting from  $\mathbf{d} = 2$ , in which case we have

$$N = \begin{bmatrix} 0 & 0 & 0 & x_1 & -x_3 & 0 & 0 \\ x_3 & 0 & x_1 & 0 & 0 & -x_2 & 0 \\ 0 & x_1 & 0 & -x_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_0 & -x_1x_2 + x_3^2 \\ -x_2 & 0 & -x_3 & 0 & x_0 & 0 & x_1^2 \\ x_1 & -x_3 & 0 & 0 & 0 & x_3 & 0 \end{bmatrix}$$

we always have  $\mu - 1$  linear columns and a single quadratic one.

**Example 17 (ex306).** Let  $X = \mathbb{P}^2$  and  $J = \langle s^3, t^2u, s^2t + u^3, stu \rangle$ . This is the situation of Example ???. We have

$$P(\mathbf{x}) = x_0^3x_1^4 - x_0^2x_1^3x_2x_3 + x_3^7$$

For  $\mathbf{d} = 1$ , we find that  $N$  is square of order 3 with  $h = (1, 1, 0, 1)$ . For  $\mathbf{d} = 2$ , we get a square  $6 \times 6$  matrix with  $h = (5, 1)$ , and for  $\mathbf{d} \geq 3$ , we get a non-square matrix of linear forms only.

**Example 18 (ex307).** Consider the twisted cubic curve  $C$ . It is the image of  $X = \mathbb{P}_{s,t}^1$  under the map

$$\phi = (s^3, s^2t, st^2, t^3) : \mathbb{P}^1 \longrightarrow \mathbb{P}^3$$

which is birational onto its image. In light of Theorem 5, we can carry out the same calculations as before, even though the image is of codimension strictly bigger than 1.

Setting  $\mathbf{d} = 1$ , we get

$$N = \begin{bmatrix} -x_3 & -x_2 & -x_1 \\ x_2 & x_1 & x_0 \end{bmatrix}$$

and

$$\text{minors}(2, N) = \langle x_2^2 - x_1x_3, x_1x_2 - x_0x_3, x_1^2 - x_0x_2 \rangle$$

which are the usual equations for  $C$  in  $\mathbb{P}^3$ . Setting  $\mathbf{d} = 2$ , we get

$$N = \begin{bmatrix} 0 & -x_3 & -x_3 & -x_2 & -x_2 & -x_1 \\ -x_3 & x_2 & 0 & x_1 & 0 & x_0 \\ x_2 & 0 & x_1 & 0 & x_0 & 0 \end{bmatrix}$$

We have

$$\text{minors}(3, N) = \left\langle \begin{array}{l} x_2^2x_3 - x_1x_3^2, x_1x_2x_3 - x_0x_3^2, x_1^2x_3 - x_0x_2x_3, \\ x_2^3 - x_0x_3^2, x_1x_2^2 - x_0x_2x_3, x_0x_2^2 - x_0x_1x_3, \\ x_1^2x_2 - x_0x_1x_3, x_0x_1x_2 - x_0^2x_3, x_1^3 - x_0^2x_3, x_0x_1^2 - x_0^2x_2 \end{array} \right\rangle$$

and

$$\text{rad}(\text{minors}(3, N)) = \langle x_2^2 - x_1x_3, x_1x_2 - x_0x_3, x_1^2 - x_0x_2 \rangle$$

showing that the radical is necessary.

**Example 19 (ex308).** Let  $X = \mathbb{P}_{s,t,u}^2$  and  $J = \langle s^5, t^5, su^4, st^2u^2 \rangle$ . Then  $\phi$  is generically 2-1 map with the unique basepoint  $(0, 0, 1)$  which is c.i. of degree 5. The image is defined by

$$P(\mathbf{x}) = x_0x_1^4x_2^5 - x_3^{10}$$

Setting  $\mathbf{d} = 1$ , we get

$$N = \begin{bmatrix} x_1x_2 & -x_3^8 & 0 \\ -x_3^2 & x_0x_1^3x_2^4 & 0 \\ 0 & 0 & x_0x_1^4x_2^5 - x_3^{10} \end{bmatrix}$$

and

$$\det(N) = P(\mathbf{x})^2$$

One should note that the principal 2-minor of  $N$  is just  $P(\mathbf{x})$ .

The alternative non-Gröbner bases approaches would require difficult computations; the method of the approximation complex would need to find the gcd of the maximal minors of a matrix of size  $36 \times 58$ , and none of the moving plane and quadrics methods will work since the map isn't birational (and (BP3) of Busé et al. [2003] fails in any case).

The practical gain of our method for this example, however, is arguable at best — we have to compute a degree 10 syzygy which is already the degree of the implicit equation. On the other hand, a better choice for  $\mathbf{d}$  might help. For  $\mathbf{d} = 3$  we get a  $10 \times 10$ -matrix with  $h = (4, 4, 1, 0, 1)$ , and for  $\mathbf{d} = 4$  we get a  $15 \times 16$ -matrix with  $h = (11, 4, 1)$ .

**Example 20** (ex309). Let  $N_1$  be the matrix of linear columns for  $\mathbf{d} = 1$  in Example 15. We have that  $\det(N_1) = 0$ . This shows that not all maximal minors need to be nonzero.

This has nothing to do with the fact that  $N_1$  is special. For another example, let us take  $\mathbf{d} = 4$  in Example 19. Then  $N$  is a  $15 \times 16$ -matrix whose columns correspond to the syzygies

$$\begin{cases} t^2u^2x_2 - u^4x_3, t^3ux_2 - tu^3x_3, st^2ux_2 - su^3x_3, t^4x_2 - t^2u^2x_3, \\ st^3x_2 - stu^2x_3, s^2t^2x_2 - s^2u^2x_3, su^3x_1 - t^3ux_3, stu^2x_1 - t^4x_3, \\ s^2u^2x_1 - st^3x_3, u^4x_0 - s^4x_2, t^2u^2x_0 - s^4x_3, s^2tux_1x_2 - st^2ux_3^2, \\ s^3ux_1x_2 - s^2tux_3^2, s^3tx_1x_2 - s^2t^2x_3^2, s^4x_1x_2 - s^3tx_3^2, tu^3x_0x_1x_2 - s^3ux_3^3 \end{cases}$$

Let  $M$  be the square submatrix of the first 14 columns and the last column of  $N$ , that is, leaving out the column corresponding to the syzygy  $s^4x_1x_2 - s^3tx_3^2$ . Then  $\det(M) = 0$ . The same is true for the square submatrix consisting of the first 15 columns but in that case  $M = (N_1 \mid N_2)$  which we wanted to avoid.

**Example 21** (ex310). The following example has been presented elsewhere in the literature as a difficult one to handle. Let  $X = \mathbb{P}^2$  and take

$$\begin{aligned} J = \langle & -s^2t^3 + 3s^2t^2u + st^3u - 4st^2u^2 - stu^3 + 2t^2u^3 - tu^4 + u^5, \\ & s^2t^3 - 3s^2t^2u + st^3u + 3stu^3 - 2t^2u^3 + tu^4 - u^5, \\ & s^2t^3 - 3s^2t^2u - st^3u + 2s^2tu^2 + 4st^2u^2 - 3stu^3 - 2t^2u^3 + 3tu^4 - u^5, \\ & -s^2t^3 + 3s^2t^2u - st^3u - 3stu^3 + 3tu^4 - u^5 \end{aligned} \rangle$$

so  $\phi$  is generically 1-1 with base locus of total degree 17 and multiplicity 20. More precisely, the basepoints are the c.i. point  $(1, 1, 1)$  of multiplicity 4, the a.c.i. point  $(0, 1, 0)$  of degree 4 and multiplicity 5, and the a.c.i. point  $(1, 0, 0)$  of degree 9 and multiplicity 11.

Unlike other examples in this section, already in degree  $\mathbf{d} = 1$ , we find only linear and quadratic syzygies. In this sense the example is rather simple. We find

$$N = \begin{bmatrix} x_0 + x_1 & 0 & \dots \\ -x_0 - x_2 & 3x_0x_1 - x_1^2 + 4x_1x_2 + 3x_0x_3 - 3x_1x_3 + 4x_2x_3 - 2x_3^2 & \dots \\ -x_2 + x_3 & x_0^2 - x_0x_1 - x_1^2 - 3x_0x_3 - 3x_1x_3 - x_3^2 & \dots \\ \dots & x_1^2 - x_3^2 & \dots \\ \dots & x_0^2 + 7x_0x_1 - 3x_1^2 + x_0x_2 + 10x_1x_2 + 6x_0x_3 - 9x_1x_3 + 9x_2x_3 - 6x_3^2 & \dots \\ \dots & -5x_0x_1 - 2x_1^2 - 3x_0x_2 - 6x_1x_2 - 8x_0x_3 - 5x_1x_3 - 3x_2x_3 & \dots \end{bmatrix}$$

and

$$\det(N) = P(\mathbf{x})$$

**Example 22** (ex311). Let  $\phi$  be the rational map from Example 21. We compute the degree and multiplicity of the base locus  $Z$ .

Set  $q_1 = (1, 0, 0)$ ,  $q_2 = (0, 1, 0)$  and  $q_3 = (1, 1, 1)$ , so that set-theoretically  $Z = \{q_1, q_2, q_3\}$ . Since  $J$  is saturated, the sum of the degrees of the basepoints is just the degree of the base locus,

$$\deg(Z \subset \mathbb{P}^2) = \deg(J \subset \mathbb{C}[s, t, u]) = 17$$

Because we are in projective space, this can be checked directly in Macaulay2. However, since  $Z$  is supported on multiple points, we cannot compute the total multiplicity in the same way. Indeed, running `multiplicity(J)` gives 45, not the correct 20.



This is because the multiplicity is computed as the degree of the normal cone over the closed subscheme, but this is a well-behaved projective variety over a field only in case the support is a single point.

Let  $Q_k$  be the prime ideals corresponding to the points  $q_k$ , and set

$$J_k = J : (J : Q_k^\infty)$$

The degree and multiplicity of  $q_k$  can be computed from the ideal  $J_k$  — those capture the local structure of  $Z$  near  $q_k$ . We have

$$\begin{cases} J_1 = \langle tu^2, t^3 - 3t^2u, u^5 \rangle \\ J_2 = \langle su, s^2, u^3 \rangle \\ J_3 = \langle t^2 - 2tu + u^2, s^2 - 2su + u^2 \rangle \end{cases}$$

so in particular,  $q_3$  is c.i., while  $q_1$  and  $q_2$  are a.c.i. points.

**Example 23** (ex312). Let  $Y = V(P)$  for an irreducible form  $P(\mathbf{x})$  of degree 4 on  $\mathbb{P}^3$ . Suppose further that  $\text{Sing}(Y)$ , the singular locus of  $Y$ , contains 3 concurrent non-degenerate lines (that is, passing through a common point and spanning all of  $\mathbb{P}^3$ ). Then  $P(\mathbf{x})$  is the determinant of a square order-4 matrix  $M$  of linear forms.

We can prove this claim by a direct calculation. After a linear change of coordinates on  $\mathbb{P}^3$ , we can assume that the lines are the 3 coordinate axes in the distinguished  $\{x_3 \neq 0\}$ , that is, the lines are given by  $V(x_0, x_1)$ ,  $V(x_1, x_2)$  and  $V(x_0, x_2)$ .

On the level of ideals, using Euler's identity, the assumption translates to

$$\langle P_{x_0}(\mathbf{x}), \dots, P_{x_3}(\mathbf{x}) \rangle \subset \langle x_0x_1, x_0x_2, x_1x_2 \rangle$$

so writing out  $P(\mathbf{x}) = \sum_{|\alpha|=4} a_\alpha \mathbf{x}^\alpha$  and noting that each of the partials must be zero modulo  $\langle x_0x_1, x_0x_2, x_1x_2 \rangle$ , we only need to solve a linear system in the indeterminate coefficients. We get

$$P(\mathbf{x}) = a_1x_0^2x_1^2 + a_2x_0^2x_2^2 + a_3x_1^2x_2^2 + a_4x_0^2x_1x_2 + a_5x_0x_1^2x_2 + a_6x_0x_1x_2^2 + a_7x_0x_1x_2x_3$$

which is the determinant of

$$M = \begin{bmatrix} x_0 & & & x_1 \\ & x_1 & & x_2 \\ & & x_2 & x_0 \\ -a_3x_2 & -a_2x_0 & -a_1x_1 & a_4x_0 + a_5x_1 + a_6x_2 + a_7x_3 \end{bmatrix}$$

In fact, if  $a_7 \neq 0$  we can do better. Setting  $a_7 = 1$ , a linear change of coordinates by the matrix

$$\begin{bmatrix} 1 & & & -a_4 \\ & 1 & & -a_5 \\ & & 1 & -a_6 \\ & & & 1 \end{bmatrix}$$

leaves the singular locus the same but simplifies the general form to

$$P'(\mathbf{x}) = a_1x_0^2x_1^2 + a_2x_0^2x_2^2 + a_3x_1^2x_2^2 + x_0x_1x_2x_3$$

and the matrix  $M$  to

$$M' = \begin{bmatrix} x_0 & & & x_1 \\ & x_1 & & x_2 \\ & & x_2 & x_0 \\ -a_3x_2 & -a_2x_0 & -a_1x_1 & x_3 \end{bmatrix}$$

whose entries are scaled variables.

**Example 24** (ex313). In the situation of Example ??, we had

$$N = \begin{bmatrix} 0 & x_1 & 0 & 0 & x_0x_2 - x_3^2 \\ x_2 & -x_3 & -x_1 & -x_3 & -x_3^2 \\ -x_3 & -101x_1 & 0 & x_0 - 101x_1 & -10201x_1x_2 - 202x_3^2 \\ -101x_2 - x_3 & 0 & x_0 & 0 & 20402x_2x_3 - 202x_3^2 \end{bmatrix}$$

where  $N = N_1|N_2$  and  $\det(N_1) = x_1P(\mathbf{x})$ . Let  $M$  be the submatrix corresponding to rows  $\{2, 3, 4\}$  and columns  $\{1, 2, 3\}$ . Then  $\det(M) = P(\mathbf{x})$  but  $M$  is not technically a representation matrix — its rows cannot be indexed by the monomials of  $S_d$ .

The situation of the previous example, 23, is very similar. While we were lucky and found a determinantal representation for  $P(\mathbf{x})$ , it does not come in the form of a matrix of syzygies. There is only 3 linear syzygies, corresponding to three of the columns of  $M$ , but the last one is not.

**Example 25** (ex319). Let  $X = \mathbb{P}^1 \times \mathbb{P}^1$  and  $J \subset S = \mathbb{C}[s, u; t, v]$  be given by 4 general  $(2, 2)$  forms in  $\langle s^2, st, t^2 \rangle$ . Then  $\phi$  is generically 1-1 with the unique basepoint  $(0, 1) \times (0, 1)$  of degree 3 and multiplicity 4. After a linear change of coordinates on the target, we can assume that the  $\phi_j$  are of the form outlined in Example 13, so the representation of Example 23 applies.

**Example 26** (ex314). We apply Proposition 11 to compute the degree of the rational map in Example 12. We have

$$I = \left\langle \begin{cases} x_1t - x_2u, \\ x_0s - x_2u, \\ x_2s^2 - x_3st + x_2t^2 + x_2u^2, \\ x_1s^2 - x_3su + x_2tu + x_1u^2, \\ x_0t^2 + x_2su - x_3tu + x_0u^2, \\ x_1x_2s + x_0x_2t + (x_0x_1 - x_2x_3)u, \\ x_0^2x_1^2 + x_0^2x_2^2 + x_1^2x_2^2 - x_0x_1x_2x_3 \end{cases} \right\rangle$$

and we see  $P(\mathbf{x})$  as the last generator. Next, we calculate a Gröbner basis with respect to a product monomial order where the  $s, t, u$  variables come before the  $x_j$  variables. Dropping the implicit equation, we get

$$\begin{cases} x_1t - x_2u, \\ x_0^2x_2t + (x_0^2x_1 + x_1x_2^2 - x_0x_2x_3)u, \\ x_0s - x_2u, \\ x_1x_2s + x_0x_2t + (x_0x_1 - x_2x_3)u, \\ x_0t^2 + x_2su - x_3tu + x_0u^2, \\ x_2s^2 - x_3st + x_2t^2 + x_2u^2, \\ x_1s^2 - x_3su + x_2tu + x_1u^2 \end{cases}$$

Collecting the  $S$ -part of the leading terms, we get the ideal  $\langle s, t \rangle$  whose degree is obviously 1.

We can carry out the same calculation for the map in Example 19. The Rees ideal and its Gröbner basis are as follow,

$$I = \left\langle \begin{array}{l} x_2 t^2 - x_3 u^2, \\ x_1 x_2 s - x_3^2 t, \\ -x_3 t^3 + x_1 s u^2, \\ -x_2 s^4 + x_0 u^4, \\ -x_3 s^4 + x_0 t^2 u^2, \\ -x_1 s^5 + x_0 t^5, \\ -x_3^3 s^3 + x_0 x_1 x_2 t u^2, \\ -x_3^5 s^2 + x_0 x_1^2 x_2^2 u^2, \\ -x_3^8 s + x_0 x_1^3 x_2^4 t, \\ x_0 x_1^4 x_2^5 - x_3^{10} \end{array} \right\rangle, \left\langle \begin{array}{l} x_1 x_2 s - x_3^2 t, \\ x_3^8 s - x_0 x_1^3 x_2^4 t, \\ x_2 t^2 - x_3 u^2, \\ x_3^9 t^2 - x_0 x_1^4 x_2^4 u^2, \\ x_3^7 s t - x_0 x_1^3 x_2^3 u^2, \\ x_3^5 s^2 - x_0 x_1^2 x_2^2 u^2, \\ x_3 t^3 - x_1 s u^2, \\ x_3^3 s^3 - x_0 x_1 x_2 t u^2, \\ x_3^4 s^2 t^2 - x_0 x_1^2 x_2 u^4, \\ x_3^2 s^3 t - x_0 x_1 u^4, \\ x_3 s^4 - x_0 t^2 u^2, \\ x_2 s^4 - x_0 u^4, \\ x_1 s^5 - x_0 t^5 \end{array} \right\rangle,$$

This time the ideal of the  $S$ -part of the leading terms is  $\langle s, t^2 \rangle$  which is of degree 2.

**Example 27 (ex315).** While Examples 12, 17 and 18 support Conjecture 8, we point out that the latter is stated in its strongest possible form. For one, the claim is trivial in the case  $\deg(\phi) > 1$  for then the comaximal minors vanish on all of  $Y$  again. This can be seen in Example 19.

On the other hand, while it is tempting to conjecture that

$$\text{rad}(\langle P_{x_j} : j \rangle) \subset \text{sat}(\text{minors}(r-1, N))$$

that is, that the inclusion of the conjecture is on the level of schemes, this is not true as illustrated by Example 15. In the case  $\mathbf{d} = 1$ , we get

$$\begin{aligned} \text{rad}(\langle P_{x_j} : j \rangle) &= \langle x_3, x_1 x_2, x_0 x_2, x_0 x_1 \rangle \\ \text{sat}(\text{minors}(r-1, N)) &= \langle x_3^2, x_2 x_3, x_1 x_3, x_2^2, x_1 x_2, x_1^2 \rangle \\ \text{rad}(\text{minors}(r-1, N)) &= \langle x_1, x_2, x_3 \rangle \end{aligned}$$

and the only inclusion we have is

$$\text{rad}(\langle P_{x_j} : j \rangle) \subset \text{rad}(\text{minors}(r-1, N))$$

The results for other values of  $\mathbf{d}$  are analogous.

**Example 28.** It may seem at first that taking the smallest possible degree is always our best option, but Example 19 provides a good counterexample. The examples in Chapter 5, which are computationally more complex, often have the first few  $N_i$  being zero, for low  $\mathbf{d}$ , so nonzero matrices tend to be larger in size and degree of their minors.

It is precisely this flexibility to choose a good  $\mathbf{d}$ , that is one of the contributions of our results to the methods of moving hypersurfaces.

#### 4. PROOFS

While the ultimate goal of this chapter is to prove the results of Chapter ??, it is written in a way to help develop intuition about the interplay between representation matrices on one hand and the geometry of  $Y$  on the other.

For this reason we start with a few elementary results with two-fold purpose. Firstly, they put together a some easy facts about our matrices  $N$ . Secondly, they highlight, when compared to other ad-hoc proofs, the advantage of our point of view.

We follow the notation of Chapter ?? and adopt the setup of (1). In particular, we work over a fix degree  $\mathbf{d}$  with  $S_{\mathbf{d}} \neq 0$ , and do not yet require that  $\phi$  be generically finite.

**Lemma 29.** *Let  $M$  be a square  $r \times r$  matrix of syzygies. Then*

$$\det(M) \in P$$

*In particular, if  $M$  is any (not necessarily square) matrix of syzygies, then*

$$\text{minors}(r, M) \subset P$$

*Proof.* The second statement clearly follows from the first, setting an empty minor to zero.

Let  $\text{adj}(M)$  be adjugate matrix and set  $\mathbf{b} = \text{basis}(S_{\mathbf{d}})$ . Then

$$(\mathbf{b} \cdot M) \cdot \text{adj}(M) = \mathbf{b} \cdot \det(M) \mathbf{1}_r = \det(M) \mathbf{b}$$

Since  $\mathbf{b} \cdot M$  is a row vector of syzygies, the LHS vanishes identically in  $S$  under the substitution  $x_0 = \phi_0(\mathbf{s}), \dots, x_n = \phi_n(\mathbf{s})$ . But then

$$\text{RHS}|_{x_0=\phi_0, \dots, x_n=\phi_n} = \det(M)(\phi_0, \dots, \phi_n) \mathbf{b} = [0 \ 0 \ \dots \ 0]$$

over the domain  $S$ . It follows that  $\det(M)(\phi_0, \dots, \phi_n) = 0$ , so that  $\det(M)$  is in the kernel  $P$  of  $\phi^\#$ , proving the first statement.  $\square$

**Lemma 30.** *Any representation matrix  $N$  has at least as many columns as rows, i.e.*

$$\mu \geq r$$

*and its ideal of maximal minors is nonzero.*

*Proof.* For each standard basis column vector  $e_k \in \mathbb{C}^r$ ,  $P(\mathbf{x})e_k$  is a graded syzygy. Let  $F$  be the  $\mu \times r$  matrix of coefficients for getting  $P(\mathbf{x})e_k$  out of the generators of the syzygies over  $S_{\mathbf{d}}$ , that is

$$N \cdot F = P(\mathbf{x}) \mathbf{1}_r$$

The sizes of the matrices on the LHS are  $r \times \mu$  and  $\mu \times r$ . Since the rank of the RHS as a  $T$ -matrix is  $r$ , we must have  $\mu \geq r$ .

The maximal minors are then of size  $r \times r$ . Since  $\text{rank}(N \cdot F) = r$ , also  $\text{rank}(N) = r$  and so not all maximal minors vanish.  $\square$

**Lemma 31.** *There is an isomorphism of graded  $T$ -modules*

$$\text{coker } N \cong B_{\mathbf{d}, \bullet}$$

*In particular, if  $\mathcal{C}_\bullet$  is any graded resolution of  $\text{coker}(N)$  over  $T$ , then  $H_0 \mathcal{C}_\bullet = B_{\mathbf{d}, \bullet}$ .*

*Proof.* This is obvious. The sequence

$$\bigoplus_k T(-i_k) \xrightarrow{N} T^r \xrightarrow{\mathbf{b} \cdot} B_{\mathbf{d}, \bullet} \longrightarrow 0 \quad (31.1)$$

is exact by definition, proving the claim.  $\square$

**Lemma 32.** *One has*

$$\text{ann}_T(B_{\mathbf{d}, \bullet}) = P$$

*Proof.* Identifying  $T = 1 \otimes T = R_{0,\bullet}$ , we can think of  $T$  as a subring of  $R$ . Since  $R$  is a graded domain, we have

$$\text{ann}_T(B_{d,\bullet}) = T \cap I$$

By the definition of  $I$ , any form  $Q(\mathbf{x}) \in T$  with  $Q(\mathbf{x}) \in I$  is the kernel of the ring map  $\phi^\#$ . It follows that

$$T \cap I = P$$

completing the proof.  $\square$

*Remark.* The lemma above shows that  $\text{Supp}_T(B_{d,\bullet}) = V(P)$  and there is a cool way to see that the  $T$ -module localization  $(B_{d,\bullet})_P$  is nonzero. Let  $N'$  be the localization of  $N$  at  $P$ . Then

$$(B_{d,\bullet})_P = \text{coker}(N)_P = \text{coker}(N')$$

Since  $\text{Fitt}_0 \text{coker}(N') = \text{minors}(r, N') \subset PT_P \neq T_P$  by Lemma 29, the cokernel is nonzero, for example, by (Eisenbud [1995], Proposition 20.6).

For a geometric argument, see the proof of Lemma 35.

**Proof of Proposition 3.** Since  $I$  is the kernel of a ring map into a domain,  $I$  is prime. By Lemma 32,

$$P = T \cap I \subset R$$

so  $B = R/I$  is naturally a finite-type  $S$ -graded  $T/P$ -algebra. The  $T$ -module localization of  $B$  at  $P$  is just the localization of the ring  $R/I$  at the multiplicative set  $(T - P)$ . In particular, the localization  $B_P$  remains a domain, now as a  $K(T/P)$ -algebra. This proves parts (a) and (b).

Since  $I$  is prime in  $R$ ,  $I$  is saturated with respect to the irrelevant ideal  $\mathfrak{n} \subset S \subset R$ . Now (c) follows because saturation commutes with localization.  $\square$

**Proof of Theorem 5.** By Lemma 31, Lemma 32 and (Eisenbud [1995], Proposition 20.7), we have

$$\text{rad}(\text{minors}(r, N)) = \text{rad}(\text{Fitt}_0(\text{coker } N)) = \text{rad}(\text{ann}(B_{d,\bullet})) = P \quad \square$$

From now on, we assume that  $\phi$  is generically finite, or equivalently, that  $P$  is principal.

**Lemma 33.** *Let  $\mathcal{C}_\bullet$  be a finite graded free resolution of  $\text{coker } N$ . One has*

$$\text{div}(\det(\mathcal{C}_\bullet)) = \text{length}_{T_P}(B_{d,\bullet})_P \cdot [Y]$$

as Weil divisors on  $\mathbb{P}^n$ .

*Proof.* By (Gelfand et al. [1994], A, Theorem 30), applied to the factorial  $T$ , a principal prime  $Q = \langle Q(\mathbf{x}) \rangle$ , and the generically exact  $\mathcal{C}_\bullet$ ,

$$\text{ord}_{Q(\mathbf{x})}(\det(\mathcal{C}_\bullet)) = \sum_i (-1)^i \text{mult}_Q(H_i \mathcal{C}_\bullet)$$

Since  $\mathcal{C}_\bullet$  is exact, all the higher homology vanishes, so

$$\text{ord}_{Q(\mathbf{x})}(\det(\mathcal{C}_\bullet)) = \text{mult}_Q(H_0 \mathcal{C}_\bullet)$$

The RHS above is zero outside  $\text{ann}(H_0\mathcal{C}_\bullet)$ , so by Lemma 31 and 32, the RHS is nonzero only for  $Q = P$ . Summing over all non-associate irreducible homogeneous polynomials, we get

$$\begin{aligned} \text{div}(\det(\mathcal{C}_\bullet)) &= \sum_{Q(\mathbf{x})} \text{ord}_{Q(\mathbf{x})}(\det(\mathcal{C}_\bullet)) \cdot [V(Q)] \\ &= \sum_{Q(\mathbf{x})} \text{mult}_Q(H_0\mathcal{C}_\bullet) \cdot [V(Q)] \\ &= \text{mult}_P(H_0\mathcal{C}_\bullet) \cdot [Y] \\ &= \text{length}_{T_P}(B_{\mathbf{d},\bullet})_P \cdot [Y] \end{aligned}$$

establishing the claim.  $\square$

**Lemma 34.** *One has*

$$\text{length}_{T_P}(B_{\mathbf{d},\bullet})_P = \dim_{K(T/P)}(B_{\mathbf{d},\bullet})_P$$

*Proof.* Note that  $P(\mathbf{x}) \in I$ , so  $P(\mathbf{x})$  annihilates  $B$  as a  $T$ -module. Setting  $\mathfrak{m}_P = PT_P \subset T_P$ ,

$$\begin{aligned} \text{length}_{T_P}(B_{\mathbf{d},\bullet})_P &= \sum_k \dim_{T_P/\mathfrak{m}_P} \mathfrak{m}_P^k(B_{\mathbf{d},\bullet})_P / \mathfrak{m}_P^{k+1}(B_{\mathbf{d},\bullet})_P \\ &= \dim_{T_P/\mathfrak{m}_P}(B_{\mathbf{d},\bullet})_P \end{aligned}$$

since  $\mathfrak{m}_P$  in turn annihilates  $(B_{\mathbf{d},\bullet})_P$ .  $\square$

*Remark.* A one-line argument would be: the  $T$ - and  $T/P$ -module structure of  $B$  are the same.

**Lemma 35.** *Let  $\mathbf{d} \in \text{reg}(I_P)$  in the sense of (4). One has*

$$\dim_{K(T/P)}(B_{\mathbf{d},\bullet})_P = \deg(\phi)$$

*Remark.* This result requires neither  $\text{codim}(Y \subset \mathbb{P}^n) = 1$  nor  $\dim V(J) = 0$ .

*Proof.* Let  $\Gamma = \text{Biproj}(B)$  be the graph of the rational map  $\phi$ , or equivalently the blow-up of  $X$  along the basepoints  $V(J)$ . Let  $E \subset \Gamma$  be the exceptional locus. We point the reader to (Hartshorne [1977], II, Example 7.17.3) for the details, summarized in the following commutative diagram

$$\begin{array}{ccccc} & & \xrightarrow{\pi_2} & & \\ \text{Biproj}(B) = \Gamma & \xleftarrow{\quad} & \Gamma - E & & \\ \downarrow \pi_1 & & \downarrow \pi_1|_{\Gamma-E} \cong & \searrow \pi_2|_{\Gamma-E} & \\ X & \xleftarrow{\quad} & X - V(J) & \xrightarrow{\phi|_{X-V(J)}} & Y \subset \mathbb{P}^n \\ & \searrow \phi & & & \end{array}$$

Since  $\pi_2|_{\Gamma-E} = \phi|_{X-V(J)} \circ \pi_1|_{\Gamma-E}$  and  $\pi_1|_{\Gamma-E}$  is an isomorphism, the morphism  $\pi_2$  is generically finite onto its image and  $\deg(\phi) = \deg(\pi_2)$ . If  $\gamma \in \mathbb{P}^n$  is the generic point of  $Y$ , then the scheme-theoretic fiber

$$\pi_2^{-1}(\gamma) = \text{Spec}(\mathcal{O}_{\gamma,Y}) \times_Y \Gamma$$

is a closed zero-dimensional subscheme of  $\Gamma$  consisting of  $\deg(\pi_2)$  points, counted with multiplicity.

The morphism  $\pi_2 : \text{Biproj}(B) \rightarrow \text{Proj}(T)$  is induced by the graded map of  $\mathbb{C}$ -algebras

$$\pi_2^\# = (x_j \mapsto \overline{x_j \otimes 1}) : T \longrightarrow (T \otimes S)/I = B$$

and the fiber of  $\gamma = [P] \in \text{Proj}(T)$  corresponds to the set of bihomogeneous prime ideals of  $B$  which pull back to  $P \subset T$  via  $\pi_2^\sharp$ . By an easy reduction, for example (Vakil [2013], Exercise 7.3.H, 7.3.K and 9.3.A), this set corresponds to the set of  $S$ -grading-homogeneous prime ideals of the  $K(T/P)$ -algebra

$$K(T/P) \otimes_{T/P} B \cong B_P$$

The identification above presents the fiber as a  $\text{Proj}(-)$  over a field, that is,

$$\text{Proj}_{K(T/P)} B_P \xrightarrow{\sim} \pi_2^{-1}(\gamma) \subset \text{Biproj}(B)$$

Since this is a finite projective scheme over the field  $K(T/P)$ , its degree is well-defined and given by

$$\dim_{K(T/P)}(B_P)_d = \dim_{K(T/P)}(B_{d,\bullet})_P$$

for all  $d \in \text{reg}(I_P)$ .  $\square$

**Proof of Theorem 6.** Let  $\mathcal{C}_\bullet$  be a minimal graded free resolution of  $\text{coker } N$ . By (Gelfand et al. [1994], A, Theorem 34) which applies since  $\mathcal{C}_\bullet$  is exact,

$$\det(\mathcal{C}_\bullet) = \gcd(\text{minors}(r, N))$$

up to a unit of  $T$ . But then by Lemma 33, 34 and 35,

$$\begin{aligned} \text{div}(\gcd(\text{minors}(r, N))) &= \text{div}(\det(\mathcal{C}_\bullet)) \\ &= \text{length}_{T_P}(B_{d,\bullet})_P \cdot [Y] \\ &= \dim_{K(T/P)}(B_{d,\bullet})_P \cdot [Y] \\ &= \deg(\phi) \cdot [Y] \end{aligned}$$

Because this is just an equality of Weil divisors,

$$\gcd(\text{minors}(r, N)) = P^{\deg(\phi)} \quad \square$$

**Proof of Corollary 7.** This follows directly from Theorem 6.  $\square$

**Proof of Proposition 11.** For any choice of  $>'$ , a reduced Gröebner basis for  $I_B$  will have the outlined general form except possibly for the term  $g_{r+1}$ . Since  $I \cap T = P$  by Lemma 32, a Gröbner basis must include  $P(\mathbf{x})$  as its unique generator involving just the  $x_j$ . This proves the first part.

Define the ideal  $I'_B$  by the identity

$$B_P = \frac{(\mathbb{C}[\mathbf{x}] - P)^{-1} \mathbb{C}[\mathbf{s}; \mathbf{x}]}{I'_B}$$

By reduce-ness,  $P(\mathbf{x}) \nmid p_k(\mathbf{x})$ , so  $I'_B$  can be obtained from the generating set for  $I_B$  by removing  $P(\mathbf{x})$ .

By the proof of Lemma 35, we know that  $\text{Proj}_{K(T/P)} B_P$  corresponds to the scheme-theoretic preimage of the generic point of  $Y$ , and in turn, this gives

$$\deg(\phi) = H^0(\text{Proj}_{K(T/P)} B_P, \mathcal{O}) = \deg(I'_B)$$

where  $\mathcal{O}$  denotes the structure sheaf of  $\text{Proj}_{K(T/P)} B_P$ , and the degree on the far-right is the degree in  $K(T/P)\mathbb{C}[\mathbf{s}]$ . The result now follows because the initial ideal of  $I'_B$  has the same degree as  $I'_B$ .  $\square$

## 5. A METHOD FOR FAST IMPLICITIZATION

This chapter is devoted to the computational aspects of our results. Section 5.1 describes two algorithms for implicitization. The first one is simple and robust, and is used for studying the matrices  $N$  when Gröbner basis calculations can be carried out efficiently. The second one is more involved and is used when direct computations are unfeasible. In those cases, the second algorithm's lead is significant. Section 5.3 provides details about the algorithms and support to the latter claim in the form of a few worked examples.

Both, as means to illustrate that our algorithms are effective, say, in the sense of computational algebraic geometry, and as a setup for the examples to follow, in Section 5.2 we implement those algorithms in the Macaulay2 system.

The code is available at

<http://www.math.cornell.edu/~rzlatev/phd-thesis>

We continue to follow the notation of Chapter ?? and the setup of (1). However, to avoid distraction, we assume throughout the chapter that  $\phi$  is generically 1-1.

### 5.1. The Algorithms.

At first glance, an algorithm for finding the implicit equation is contained in the proof of our main theorem, Theorem 6. In its simplest form, it becomes

**Algorithm 36.** NAIVE ALGORITHM.

**input:**  $J, d$   
**output:**  $N, P$   
 Set  $r = \dim_{\mathbb{C}}(S_d)$   
 Compute an  $R$ -generating set  $\{h_\ell : \ell\}$  for the Rees ideal  $I$   
 Compute a  $T$ -generating set  $\{g_k : k\}$  for  $I_{d,\bullet}$  from the  $h_\ell$  using (??)  
 Set  $N$  to be the coefficient matrix of the  $g_k$  with respect to  $\text{basis}(S_d)$   
 Compute  $P = \gcd(\text{minors}(r, N))$   
 Return  $N, P$

**37.** The conciseness and robustness of Algorithm 36 made it our preferred tool for testing the theory. In fact, all calculations presented so far, including all examples of Chapter 3, were carried out using this algorithm.

At the same time, its simplicity allows us to spot some of its drawbacks. We distinguish four major ones.

- (1) Computing an  $R$ -generating set for the Rees ideal is at least as hard as computing the implicit equation itself — we have  $I_{0,\bullet} = P$ . This follows from Proposition 11 and shows up in Examples ?? and 26.
- (2) While computing the gcd of two polynomials is fast, computing all minors could be difficult since their number can be very large. This happens even for reasonably small examples. For instance, the smallest nonzero matrix  $N$  for  $d = (2, 2, 1)$  in Example 50 is of size  $18 \times 50$ . The number of maximal minors is

$$\binom{50}{18} = 18'053'528'883'775$$

so even if it takes the unrealistic 0.001 seconds to compute each minor, a single machine would require 572 years to compute them all.



- (3) Continuing with Example 50, we note that each maximal minor is a determinant of an  $18 \times 18$ -matrix of quartic forms in 5 variables. Computing large determinants symbolically is time-consuming. We did not manage to compute any nonzero minor.

Example 48 involves a somewhat similar calculation — the determinant of a  $12 \times 12$ -matrix of quadratic forms in 5 variables took about an hour to compute. Extrapolating, we can speculate that our  $18 \times 18$  determinant would take somewhere in the order of

$$13 \times 14 \times 15 \times 16 \times 17 \times 18 = 13'366'080$$

hours. That is about 1525 years.

- (4) Finally, suppose we have found the polynomial in question — by whatever means. It is a form of degree 48 in 5 variables, and very likely dense in the monomials of that degree. This suggests that the polynomial will be represented by

$$\binom{53}{5} = 2'869'685$$

coefficients.

Regrettably, (37.3) would be an issue for any algorithm relying on computing determinants of representation matrices, while (37.4) would be an issue for any implicitization algorithm whatsoever. Rather than seeing these as obstacles, we point them out as an argument *for* the idea of using representation matrices in place of the implicit equation altogether. We explore this theme further in the examples of Section 5.3.

**38.** Fix a degree  $d$  as before and recall that

$$N = (N_1 \mid \cdots \mid N_\delta)$$

Consider the following.

- (1) Instead of computing the whole matrix  $N$ , one can compute the  $N_i$ 's separately, keeping track of a partial representation matrix  $N'$ .
- (2) Instead of computing all the minors, one only needs to compute sufficiently many to determine the gcd correctly.

These two simple observations produce an immense speed up on average. The advantage of (38.1) over computing an  $R$ -generating set for the Rees ideal is that it uses only linear algebraic routines. The advantage of computing only sufficiently many, rather than all, of the minors is obvious.

**39.** Let

$$N' = N'_i = (N_1 \mid \cdots \mid N_i)$$

be the partial matrix of syzygies up to degree  $i$ . Recall that  $h_i$  denotes the number of columns in  $N_i$  and set  $p = \deg(P)$ . Consider the following condition

$$\begin{cases} \sum h_i i = p & \text{if } \sum h_i = r \\ \sum h_i i \geq p & \text{if } \sum h_i > r \end{cases} \quad (\text{C1})$$

This is a necessary condition for  $N'$  to be used instead of  $N$ , since if  $\deg(\det(M)) < \deg(P)$ , then  $M$  must be singular. It is almost certainly not a sufficient condition to conclude that  $N' = N$ , but often this is the case.

**40.** Let  $M_1, M_2$  be nonsingular matrices of syzygies, as in (29), and let  $Y_1, Y_2 \subset \mathbb{P}^n$  be the hypersurfaces they define. Let  $L$  be a general line in  $\mathbb{P}^n$ . If  $M_1, M_2$  satisfy the condition

$$L \cap Y_1 \cap Y_2 \subset L \cap Y \quad (\text{C2})$$

then  $\gcd(\det(M_1), \det(M_2)) = P$ .

The condition can be used for testing (38.2). To prove the claim, note that we always have

$$Y_1 \cap Y_2 \supset V(\gcd(\det(M_1), \det(M_2))) \supset Y$$

which together with (C2) gives  $L \cap Y_1 \cap Y_2 = L \cap Y$ , which is true exactly when  $Y_1 \cap Y_2$  does not contain any other hypersurfaces besides  $Y$ . Indeed, if the gcd is a proper multiple of  $P$ , then the intersection of  $Y_1$  and  $Y_2$  contains another hypersurface, whose intersection with the general  $L$  is not going to be on  $Y$ .

Furthermore, the condition itself can be tested by computing the multiplicity of  $Y_1 \cap Y_2 \cap L$  and comparing it to the multiplicity of  $Y \cap L$ . If the two are equal, then the condition is satisfied.

Summarizing the discussion so far, we propose

**Algorithm 41.** PROPOSED ALGORITHM.

**input:**  $J, d, p = \deg(P)$   
**output:** a list of matrices  $M_k$  such that  $\gcd(\{M_k : k\}) = P$   
 Set  $r = \dim_{\mathbb{C}}(S_d)$   
 Set  $N' = r \times 0$  matrix over  $T$

**while** (C1) is not satisfied for  $N'$  **do**  
   Given  $N_1, \dots, N_{i-1}$ , use Algorithm 42 to compute  $N_i$   
   Set  $N' = N' \mid N_i$   
**end while**  
 Set  $M_1, M_2$  to be the zero  $r \times r$  matrix  
 Let  $\mathcal{C}$  be some stopping criterion  
 Let  $i$  be the index of the last computed matrix and let  $\ell = r - h_1 - \dots - h_{i-1}$   
**while**  $\mathcal{C}$  is not satisfied and  $M_1, M_2$  is do not satisfy (C2) **do**  
   Let  $c_1, c_2$  be a random sets of  $\ell$  columns of  $N_i$   
   Set  $M_1 = N_1 \mid \dots \mid N_{i-1} \mid (N_i)_{c_1}$   
   Set  $M_2 = N_1 \mid \dots \mid N_{i-1} \mid (N_i)_{c_2}$   
**end while**  
 Return  $P = \gcd(\det(M_1), \det(M_2))$

**Algorithm 42.** COMPUTE PARTIAL SYZYGIES.

**input:** a list of the already computed  $N_1, \dots, N_{i-1}$   
**output:**  $N_i$   
**for**  $0 < j < i$  **do**  
   Set  $N_{ji} = \text{basis}(T_{i-j}) \otimes N_j$   
   Set  $K_{ji}$  to be the linearization of  $N_{ji}$   
**end for**  
 Set  $K_i = \ker(\Phi^{(i)})$   
 Let  $K'_i$  be such that  $\text{Span}(K_i) = \text{Span}(K'_i) \oplus (\sum_j \text{Span}(K_{ji}))$   
 Let  $N_i$  be such that  $\text{basis}(R_{d,i}) \cdot K'_i = \text{basis}(S_d) \cdot N_i$   
 Return  $N_i$

## 5.2. Implementation in Macaulay2.

43. We start with a realization for (??).

```
PushGens = (d,I) -> (
  r := toList(0..(#d-1));
  G := for g in I_* list (
    if all(d-((degree g)_r), Z->Z>=0)
    then basis((d-((degree g)_r))|{0},ring I)**g
    else continue;
  trim image fold(G,matrix {{0_(ring I)}},(a, b)->a|b)
)
```

44. Using (43), Algorithm 36 is straight-forward to implement. We require  $R$  for encapsulation.

```
ComputeNRees = method ()
ComputeNRees (Ideal, List, Ring) := Matrix => (J, d, R) -> (
  x := symbol x;
  I := reesIdeal(J, Variable=>x);
  AI := ring I;
  zm := 0*d;
  g := map(R,AI,first entries super basis(zm|{1},R));
  I = g(I);
  V := PushGens (d,I);
  matrix entries ( (gens V) // basis(d|{0}, R) )
)
```

45. Algorithm 42 is the one we make most use of in Section 5.3. Its fourth argument is the list of already computed matrices  $N_1, \dots, N_{i-1}$ . If this list's size is not  $i-1$ , then we just compute all linearly independent syzygies of degree  $i$  —  $\text{basis}(I_{d,i})$ . The ideal  $J$  is supplied in the form of the matrix  $F = \phi^{(1)} \otimes R$  (see ??).

```
ComputeNi = method ()
ComputeNi (Matrix, ZZ, List, List) := Matrix => (F, i, d, lst) -> (
  R := ring F;
  m := #d;
  d0 := d|{0};
  di := d|{i};
  zm := 0*d;
  fj := flatten entries matrix F;
  xj := flatten entries super basis (zm|{1}, R);
  n := #fj;
  r := numcols super basis(d0, R);
  subs := apply(n,j->xj_j=>fj_j);
  e0 := (degree fj_0);
  G := sub(super basis(di, R), subs) // (super basis(i*e0+d0, R));
  K := matrix entries gens ker G;
  Nii := (super basis(di, R))*K // (super basis(d0, R));
  Nii = sub(matrix entries Nii,R);
  Nji := random(R^r,R^0);
  if #lst==i-1
  then Nji = fold (for j from 1 to #lst list
    (super basis(zm|{i-j}, R)**(lst_(j-1))), random(R^r,R^0), (m1,m2)->m1|m2);
  gens trim image (Nii%Nji)
)
```

**46.** We now implement the first part of the proposed Algorithm 41. We omit the second part simply because of (37.3). Besides, after a general change of coordinates, any two nonzero minors are likely to suffice and we can compute them manually when necessary. See Examples 50 and 49 for details.

```

ComputeNConj = method ()
ComputeNConj (Matrix, ZZ, List) := Matrix => (F, p, d) -> (
  R := ring F;
  q := 0;
  i := 1;
  lst := {};
  r := numcols super basis (d|{0},R);
  N := random(R^r,R^0);
  zm := 0*d;
  fj := flatten entries matrix F;
  xj := flatten entries super basis (zm|{1}, R);
  n := #fj;
  subs := apply(n,j->xj_j=>fj_j);
  while q<p or not procTestCondN(N, subs) do (
    Ni := ComputeNi(F,i,d,lst);
    q = q + (numcols Ni)*i;
    lst = append(lst, Ni);
    N = N|Ni;
    i = i+1; );
  N
)

```

### 5.3. Examples.

We now field-test our algorithms and code on several examples of somewhat higher computational complexity than those in Chapter 3.

The running times can vary a lot from one machine to another, so the numbers below should not be treated as benchmarks. We include them only to provide a general idea how the different methods perform relative to each other.

The machine that we used was a MacBook Pro laptop with a 2.9 GHz Intel Core i7 processor and 8 GB 1600 MHz DDR3 memory, running Macaulay2 version 1.8.

**Example 47** (ex601). Let  $\phi : (\mathbb{P}^1)^3 \rightarrow \mathbb{P}^4$  be given by 5 generic  $(2, 1, 1)$ -forms. The base locus is empty, so by (??) and (??), the degree of the image is 12.

Consider  $\mathbf{d} = (1, 1, 1)$ . Our method computes a candidate matrix  $N'$  in a little more than 0.1s. The matrix  $N'$  is square and we have  $\det(N') = P$ . Computing the determinant of the  $8 \times 8$ -matrix  $N$  takes 3s. The standard Gröbner basis computation takes 131s.

The details follow.

```

i1 : loadPackage "ImplicitizationAlgos";
i2 : KK=ZZ/32009;
i3 : S=KK[s_0,s_1,t_0,t_1,u_0,u_1,
      Degrees=>{2:{1,0,0},2:{0,1,0},2:{0,0,1}}];
i4 : T=KK[x_0..x_4];
i5 : B=super basis({2,1,1},S);
o5 : Matrix S  1 ---- 12 S

```

```

i6 : J=ideal(B*random(S^12,S^5));
o6 : Ideal of S
i7 : R=KK[s_0,s_1,t_0,t_1,u_0,u_1,x_0..x_4,
      Degrees=>{2:{1,0,0,0},2:{0,1,0,0},2:{0,0,1,0},5:{0,0,0,1}}];
i8 : F=sub(gens J,R);
o8 : Matrix R  <--- R
      1      5
i9 : d={1,1,1};
i10 : time N1=ComputeNi(F,1,d,{});
      -- used 0.0119639 seconds
o10 : Matrix R  <--- R
      8      4
i11 : time N2=ComputeNi(F,2,d,{N1});
      -- used 0.0920771 seconds
o11 : Matrix R  <--- R
      8      4

```

The partial matrix  $N' = N_1|N_2$  is square. The degree of its determinant is  $12 = 4 \cdot 1 + 4 \cdot 2$ , so this is our candidate matrix. We check at the end that  $\det(N') = P(x)$ .

```

i12 : time ComputeNi(F,2,d,{ });
      -- used 0.0867573 seconds
o12 : Matrix R  <--- R
      8      24
i13 : time N3=ComputeNi(F,3,d,{N1,N2});
      -- used 0.846328 seconds
o13 : Matrix R  <--- 0
      8
i14 : time ComputeNi(F,3,d,{ });
      -- used 0.903612 seconds
o14 : Matrix R  <--- R
      8      80
i15 : time ComputeNi(F,4,d,{ });
      -- used 6.38165 seconds
o15 : Matrix R  <--- R
      8      200

```

Note that even though  $N'$  is square and giving the implicit equation right off the bat, we cannot be sure that  $N = N_1|N_2$ . Lines 12–15 support this claim. For instance, because the columns of  $N_1$  and  $N_2$  form a nonsingular matrix, the degree-3 syzygies they give rise to are going to be linearly independent. There are exactly  $80 = 4 \cdot \binom{4+2}{2} + 4 \cdot \binom{4+1}{1}$  of them, so  $N_3 = 0$ . The same argument shows that  $N_4 = 0$ .

```

i16 : N=sub(N1|N2,T);
o16 : Matrix T  <--- T
      8      8
i17 : time N'=ComputeNConj(F,12,d);
      -- used 0.138608 seconds
o17 : Matrix R  <--- R
      8      8
i18 : N'=sub(N',T);

```

```

      8      8
o18 : Matrix T <--- T
i19 : image N==image N'
o19 = true
i20 : time P'=ideal det N;
      -- used 3.0447 seconds
o20 : Ideal of T

```

Line 20 shows that  $\det(N') = P(\mathbf{x})$ .

```

i21 : time P=ker map(S,T,J_*);
      -- used 131.05 seconds
o21 : Ideal of T
i22 : P==P'
o22 = true

```

The standard method to compute the implicit equation takes more than 40 times longer.

**Example 48** (ex602). We compute the implicit equation of five general  $(2,2,1)$ -forms over  $(\mathbb{P}^1)^3$ . The base locus is empty, so the degree of the equation is 24. It takes our method less than a second to find it in the form of a determinant of an  $18 \times 18$ -matrix of quadratic forms. It take a little less than an hour to compute the actual equation. The standard Gröbner basis calculation did not finish in 24 hours.

```

i1 : loadPackage "ImplicitizationAlgos";
i2 : KK=ZZ/32009;
i3 : S=KK[s_0,s_1,t_0,t_1,u_0,u_1,
      Degrees=>{2:{1,0,0},2:{0,1,0},2:{0,0,1}}];
i4 : T=KK[x_0..x_4];
i5 : B=super basis({2,2,1},S);
      1      18
o5 : Matrix S <--- S
i6 : J=ideal(B*random(S^18,S^5));
o6 : Ideal of S
i7 : R=KK[s_0,s_1,t_0,t_1,u_0,u_1,x_0..x_4,
      Degrees=>{2:{1,0,0,0},2:{0,1,0,0},2:{0,0,1,0},5:{0,0,0,1}}];
i8 : F=sub(gens J,R);
      1      5
o8 : Matrix R <--- R
i9 : d={2,1,1};
i10 : time N1=ComputeNi(F,1,d,{});
      -- used 0.017498 seconds
      12
o10 : Matrix R <--- 0
i11 : time N2=ComputeNi(F,2,d,{});
      -- used 0.298143 seconds
      12      12
o11 : Matrix R <--- R
i12 : time N3=ComputeNi(F,3,d,{N1,N2});
      -- used 4.01669 seconds
      12

```

```

o12 : Matrix R  <--- 0
i13 : time N'=ComputeNConj(F,24,d);
      -- used 0.762765 seconds

      12      12
o13 : Matrix R  <--- R
i14 : N'==N2
o14 = true
i15 : rank N2
o15 = 12

```

Since we know that  $N' = N_2$  is square, its determinant is a form of degree  $12 \cdot 2 = 24$ , so  $\det(N') = P(\mathbf{x})$ . Note that we do not have to compute the determinant to make sure that  $N'$  is nonsingular. We can compute the rank by substituting random numbers for the  $x_j$  and computing over a finite field.

```

i17 : N=sub(N2,T);

      12      12
o17 : Matrix T  <--- T
i18 : time P'=det N;
      -- used 3310.29 seconds

```

While we actually can compute the determinant, so also the implicit equation itself, we could have answered questions like "Is this point on the image" using  $N'$  only.

**Example 49** (ex604). Consider a map given by four general bi-quartics in the ideal  $\langle s^3, s^2t, t^2 \rangle$ . Then the base locus is the point  $(0,0,1)$  of degree 5 and multiplicity 6. The degree of the image is 26. Our method finds it in explicit form in about 6 minutes, while it takes us less than a minute to find two square matrices  $M_1, M_2$  for which

$$\gcd(\det(M_1), \det(M_2)) = P(\mathbf{x})$$

This highlights the interplay between the algebra and geometry — one of our initial goals. The standard method takes more than 6 hours.

```

i1 : loadPackage "ImplicitizationAlgos";
i2 : KK=ZZ/32009;
i3 : S=KK[s,u,t,v,Degrees=>{2:{1,0},2:{0,1}}];
i4 : T=KK[x_0..x_3];
i5 : Z=ideal(s^3,s^2*t,t^2);
o5 : Ideal of S
i6 : B=super basis({4,4},Z);

      1      20
o6 : Matrix S  <--- S
i7 : J=ideal(B*random(S^20,S^4));
o7 : Ideal of S
i8 : decompose J
o8 = {ideal (s, t), ideal (s, u), ideal (v, t)}
o8 : List
i9 : multiplicity Z
o9 = 6

```

The base locus is supported on a single point, so the Macaulay2 command computes the correct quantity.

```
i10 : R=KK[s,u,t,v,x_0..x_3,
      Degrees=>{2:{1,0,0},2:{0,1,0},4:{0,0,1}}];
i11 : F=sub(gens J,R);
o11 : Matrix R   $\begin{smallmatrix} 1 & & & 4 \\ & & & \end{smallmatrix}$ 
i12 : d={2,2};
i13 : time N1=ComputeNi(F,1,d,{});
      -- used 0.00821136 seconds
o13 : Matrix R   $\begin{smallmatrix} 9 \\ & & & \end{smallmatrix}$ 
i14 : time N2=ComputeNi(F,2,d,{N1});
      -- used 0.0901831 seconds
o14 : Matrix R   $\begin{smallmatrix} 9 \\ & & & \end{smallmatrix}$ 
```

The degree  $\mathbf{d} = (2, 2)$  does not look promising. Take  $\mathbf{d} = (3, 3)$  instead.

```
i15 : d={3,3};
i16 : time N1=ComputeNi(F,1,d,{});
      -- used 0.0208163 seconds
o16 : Matrix R   $\begin{smallmatrix} 16 & & 5 \\ & & \end{smallmatrix}$ 
i17 : time N2=ComputeNi(F,2,d,{N1});
      -- used 0.221774 seconds
o17 : Matrix R   $\begin{smallmatrix} 16 & & 12 \\ & & \end{smallmatrix}$ 
i18 : N=N1|N2;
o18 : Matrix R   $\begin{smallmatrix} 16 & & 17 \\ & & \end{smallmatrix}$ 
i19 : time rank N
      -- used 0.009011 seconds
o19 = 16
```

This is our candidate matrix  $N'$  — it has more columns than rows and any nonzero minor would have degree 27 or 28. Having  $N'$  be of full rank is not enough by itself.

```
i20 : c1=sort join({0,1,2,3,4}, RandPerm(5,16,11));
i21 : time rank N_c1
      -- used 0.00241363 seconds
o21 = 16
i22 : c2=sort join({0,1,2,3,4}, RandPerm(5,16,11));
i23 : c1==c2
o23 = false
i24 : time rank N_c2
      -- used 0.00881294 seconds
o24 = 16
```

At this point we found two distinct length-12 sets of columns,  $c_1, c_2$ , such that the square matrices  $M_1, M_2$  they give rise to are nonsingular. We can show that these satisfy the



equality described above. However, we can actually compute the implicit equation in this case.

```
i25 : time D1=det N_c1;
      -- used 164.479 seconds
i26 : time D2=det N_c2;
      -- used 162.898 seconds
i27 : time P=gcd(D1,D2);
      -- used 0.0646495 seconds
i28 : deg P
o28 : 26
```

**Example 50** (ex603). We enhance Example 48 and compute the implicit equation of five general  $(2,2,2)$ -forms on  $(\mathbb{P}^1)^3$ . The base locus is again empty, so the degree of the image is 48.

In this setup there is no hope of computing any determinant explicitly, let alone computing  $P(\mathbf{x})$  using Gröbner bases directly. We can, however, repeat what we did in Example 49. We work over  $\mathbf{d} = (2,2,1)$  and find two square matrices  $M_1, M_2$  of order 18 such that

$$\gcd(\det(M_1), \det(M_2)) = P(\mathbf{x})$$

While testing if the matrices are nonsingular is easy, checking whether their determinants do not have any other prime common factor is much harder. We follow the discussion in (40) and find that the common intersection locus of the two hypersurfaces with a general line has multiplicity 48.

```
i1 : loadPackage "ImplicitizationAlgos";
i2 : KK=ZZ/32009;
i3 : S=KK[s_0,s_1,t_0,t_1,u_0,u_1,Degrees=>{2:{1,0,0},2:{0,1,0},2:{0,0,1}}];
i4 : T=KK[x_0..x_4];
i5 : B=super basis({2,2,2},S);
      1      27
o5 : Matrix S <--- S
i6 : J=ideal(B*random(S^27,S^5));
o6 : Ideal of S
i7 : R=KK[s_0,s_1,t_0,t_1,u_0,u_1,x_0..x_4,
      Degrees=>{2:{1,0,0,0},2:{0,1,0,0},2:{0,0,1,0},5:{0,0,0,1}}];
i8 : F=sub(gens J,R);
      1      5
o8 : Matrix R <--- R
i9 : d={1,1,1};
i10 : N1=ComputeNi(F,1,d,{});
      8
o10 : Matrix R <--- 0
i11 : N2=ComputeNi(F,2,d,{});
      8
o11 : Matrix R <--- 0
i12 : N3=ComputeNi(F,2,d,{});
      8
```

```
o12 : Matrix R <--- 0
```

We sneak in a remark here. One might expect to find that  $N_6$  is square, so that  $\deg(\det(N_6)) = 48$ . In fact  $N_i = 0$  for up to at least  $i = 8$ . In general, smaller  $\mathbf{d}$  tend to produce the few matrices  $N_i$  zero. This follows from the sort of Koszul-ness of small syzygies. Take  $\mathbf{d} = (2, 2, 1)$  instead.

```
i13 : d = {2,2,1};
i14 : time N4=ComputeNi(F,4,d,{});
      -- used 151.652 seconds
      18      50
o14 : Matrix R <--- R
i15 : time rank N4
      -- used 2.2373 seconds
o15 = 18
i16 : c1=RandPerm(0,49,18);
i17 : M1=N4_c1;
      18      18
o17 : Matrix R <--- R
i18 : rank M1
o18 = 18
i19 : c2=RandPerm(0,49,18);
i20 : M2=N4_c2;
      18      18
o20 : Matrix R <--- R
i21 : rank M2
o21 = 18
i22 : sort c1==sort c2
o22 = false
```

The two matrices are nonsingular.

```
i23 : M1=sub(M1,T);
      18      18
o23 : Matrix T <--- T
i24 : M2=sub(M2,T);
      18      18
o24 : Matrix T <--- T
i25 : L=ideal random(T^{1,1,1},T^1);
o25 : Ideal of T
```

Here  $L$  or rather  $V(L)$  is the generic line.

```
i26 : TL=T/L;
i27 : super basis(1,TL)
o27 = | x_3 x_4 |
      1      2
o27 : Matrix TL <--- TL
i28 : U=KK[y_0,y_1];
i29 : g=map(U,TL,{0,0,0,y_0,y_1});
o29 : RingMap U <--- TL
```

```

i30 : M1'=g(sub(M1,TL));
      18      18
o30 : Matrix U  <--- U
      18      18
i31 : M2'=g(sub(M2,TL));
      18      18
o31 : Matrix U  <--- U

```

This last bit is for computational purposes only. Since we are on a  $\mathbb{P}^1$ , we should make use of this and work over 2 variables only.

```

i32 : time D1=det(M1');
      -- used 0.317001 seconds
i33 : time D2=det(M2');
      -- used 0.334676 seconds
i34 : degree gcd(D1,D2)
o34 = {48}
o34 : List

```

The multiplicity of the common intersection is 48! We are done.

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