EXAMPLES OF IMPLICITIZATION OF HYPERSURFACES THROUGH SYZYGIES

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Let X be a smooth projective toric variety of dimension $n-1$ and let $\phi : X \longrightarrow \mathbb{P}^n$ be a generically finite rational map. The closed image Y can be defined by a single equation $P(\boldsymbol{x})$, called the implicit equation. The implicitization problem asks for techniques for finding the implicit equation.

This is an old problem in algebraic geometry, and can be solved effectively through elimination using Gröbner bases. However, this solution represents a black box in terms of the geometry on the base locus $Z \subset X$ and Y, and is unfeasible even for reasonably small examples.

In this thesis, we use ideas from the two most popular non-Gröbner basis approaches, the method of the approximation complex and the method of moving surfaces, to construct a family of matrices N, one for each element in $\mathrm{Pic}(X)$, capturing determinantal representations for $P(\boldsymbol{x})$

We design and implement an efficient algorithm for computing the representation matrices and the implicit equation which allow us to calculate in previously intractable situations.

BIOGRAPHICAL SKETCH

Radoslav was born in Burgas, on the Bulgarian Black Sea cost. He attended the High School of Mathematics and the Sciences in his hometown, graduating in May 2005. He then studied mathematics and computer science at Jacobs University Bremen, earning a Bachelor of Science degree in June 2008. He joined the Department of Mathematics at Cornell in August 2010.

To my parents.

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CHAPTER 1

INTRODUCTION

The main goal of this thesis is to develop a framework for efficiently computing the closed image of a rational map $\phi : X \longrightarrow \mathbb{P}^n$ by making use of the relation between algebraic properties of the coordinates ϕ_j on one hand, and geometric properties of the image Y, on the other. We consider the setup where X is a smooth projective toric variety and are most interested in the case when X of dimension $n-1$ and ϕ is generically finite. In this case, the image is defined by a single equation $P(\boldsymbol{x})$, called the implicit equation.

This is a version of the implicitization problem, which is classical in algebraic geometry. It has an effective solution through elimination using Gröbner bases, for instance, but those techniques have two drawbacks. First, they become unfeasible even on examples of very modest complexity. Second, they represent a black box in terms of geometry—a tool which takes the ϕ_j and returns P, and whether the computation finishes before our Sun dies or not, no geometric feature can be used or inferred from the process. Motivated partially by the advance of computer aided design (CAD), interest in new approaches to the implicitization problem reappeared in the 1990s. One reference often cited as the initiating work is Sederberg and Chen [1995], a SIGGRAPH paper using moving curves and surfaces for implicitization. One of the first works building on this idea from the standing point of commutative algebra is Cox et al. [2000].

Our approach follows the theme of moving surfaces but sets to remove the intrinsic ad-hoc constructions it requires. In this sense, we draw inspiration by the method of the approximation complex, initiated by Busé and Jouanolou [2003]. Our main results are a blend of the results in the two papers.

Let S be the Cox ring of X graded by $\mathrm{Pic}(X)$, and $J = \langle \phi_0, \dots, \phi_n \rangle \subset S$ be the ideal of the coordinates. Then the Rees ideal I of J can be thought of as an algebraic object through the Rees algebra, but also a geometric object through the graph of ϕ. We construct, for any degree $\boldsymbol{d}$ on S, a matrix N representing a generating set for the polynomial relations on the ϕ_j with coefficients from $S_{\boldsymbol{d}}$ only. This is just a bigraded piece of the Rees ideal. The matrix N generalizes the types of representation matrices studied by the aforementioned papers. One of our main results, Theorem 3.6, is an analogue of a result for the approximation complex. We show that if N is of size $r \times \mu$, then $\operatorname{rank}(N) = r$ and

$$\gcd(\operatorname{minors}(r, N)) = P(\boldsymbol{x})^{\deg(\phi)}$$

Relaxing the requirement that X be of dimension $n-1$, we find that

$$\operatorname{rad}(\operatorname{minors}(r, N)) = P$$

Furthermore, results of form: if conditions (1)–(k) hold, then there exists a square matrix M such that $\det(M) = P(\boldsymbol{x})$, which are the main theme of the moving surfaces methods, now become essentially equivalent to saying the degree-$(\boldsymbol{d}, \bullet)$ graded piece of the Rees ideal is minimally generated by r elements. This suggest a template for proofs allowing one to focus the specifics of the conditions.

Moreover, the matrix N can be computed incrementally using only tools from linear algebra. This is important because on average entails a significant speed up, allowing us to calculate examples which are out of reach via Gröbner basis techniques. Most importantly, this allows us to use geometric tricks to confirm say, of the form

$$P(\boldsymbol{x}) = \gcd(\det(M_1), \det(M_2))$$

for some selected matrices M_1 and M_2.

The thesis is organized as follows. In Chapter 2, we establish the notation and define the main objects. Section 2.3 is devoted to the notion of multiplicity of a base point and introduces less-standard notation. We state our main results in Chapter 3 in a rather self-contained form. As the title suggests, examples are the heart of this thesis. Chapter 4 presents some 16 examples describing our main results. In fact, we believe that a careful read renders some, or all, of the proofs of the main results, presented in Chapter 5, unnecessary. Chapter 7 relates our construction to the results of Cox et al. [2000]. We take another stand point, though, outlining a template for such a proof in Section **??**, and then filling-in the details in Sections **??** and **??**.

We consider this thesis a work in computational algebraic geometry. As such we field-test our approach and in Chapter 6 we develop and implement, in Macaulay2, Grayson and Stillman [1993-2014], algorithms for finding N and P. We compare computation time with the standard techniques and show how one case use geometric insight to leverage computation.

CHAPTER 2

PRELIMINARIES

The goal of this chapter is to set up the notation and recall some basic facts which we shall need later on. The first section defines the main objects of interest and establishes some standard notation and terminology. Section 2 is a short and elementary treatment on strands of maps of free modules in the way we shall need them. Section 3 is solely devoted to the notion of multiplicity. It comes in many flavors and some care is needed when dealing with it.

Section 2.4, we present two examples with the sole purpose of working out the definitions of the previous three sections.

2.1 Notation

2.1. Recall the definition of a projective toric variety.

2.2. Define the Cox ring of X as in Cox [1993].

2.3. From now on we use the notation $\mathrm{Proj}(S,\mathfrak{n})$ for the construction of a projective toric variety from its Cox ring. In the cases where the irrelevant ideal $\mathfrak{n}$ is the usual one, for example for products of projective spaces, or is not relevant to the discussion, we shall simply write $\mathrm{Proj}(S)$. This is within reason because the construction coincides with the usual construction in case S is standardly graded.

2.4. Let X be a smooth projective toric variety of dimension $n-1$ $(n>1)$. Let S be its Cox ring (2.2) and let S' be any homogeneous coordinate ring furnishing an embedding of X into projective space, i.e. $X = \mathrm{Proj}(S')$ with

$$S' = \mathbb{C}[s_0,\ldots,s_m]/\mathfrak{p}$$

for some homogeneous prime $\mathfrak{p}$ of height $m-n+1$. Let $T = \mathbb{C}[\boldsymbol{x}]$ be the homogeneous coordinate ring of $\mathbb{P}^n$.

Let $\mathscr{L}$ be a line bundle on X such that $h^0(\mathscr{L}) > n$. Because X is toric, there is a degree $\boldsymbol{e}$ on S such that $\mathscr{L} \cong \mathscr{O}_X(\boldsymbol{e})$.

Under the identification above, suppose that

$$\phi_0, \dots, \phi_n \in H^0(X, \mathscr{L}) = S_{\boldsymbol{e}}$$

are linearly independent and consider the rational map

$$\phi = (\phi_0, \dots, \phi_n) : X \longrightarrow \mathbb{P}^n = \operatorname{Proj}(T)$$

We denote by J the ideal of the coordinates ϕ_j,

$$J = \langle \phi_0, \dots, \phi_n \rangle \subset S$$

We assume from now on that the closed image of ϕ, which we denote by Y, is of dimension $n-1$. Since X is reduced and irreducible, so is Y and, in particular, $Y = V(P)$ for some principal homogeneous prime ideal $P \subset T$,

$$Y = \operatorname{image}(\phi) = V(P)$$

If we want to refer to a generator of P, necessarily up to a unit in T, we shall dereference the ideal by writing $P(\boldsymbol{x})$.

We sometimes refer to S as the ring of source and to T as the ring of the target.

2.5. Throughout this thesis, the term *image of a rational map*, as used in (3.1), means the scheme-theoretic image, also called the closed image. Formally,

$$Y = V(\ker \phi^{\#}), \quad \phi^{\#} : \mathscr{O}_{\mathbb{P}^n} \longrightarrow \phi_* \mathscr{O}_X$$

In our situation, this is just the closure of the set-theoretic map on closed points.

2.6. Let $R = S \otimes T = S[\boldsymbol{x}]$. Then R is naturally bigraded by

$$\deg(af) = (\boldsymbol{d}, i)$$

whenever $a \in S_{\boldsymbol{d}} \subset R$ and $f \in T_i \subset R$. Let $S[t]$ be similarly graded, setting $\deg(t) = (-\boldsymbol{e}, 1)$. The blow-up algebras, $\mathrm{Rees}_S(J)$ and $\mathrm{Sym}_S(J)$, naturally become factor rings of R as follows.

The Rees algebra is the image of the bigraded map of S-algebras

$$\beta : R \longrightarrow S[t] : \quad x_j \mapsto \phi_j \cdot t$$

The Rees ideal of J is the bigraded ideal $I = \ker(\beta) \subset R$. It is generated by the the polynomial relations on the generators $\phi_0, \ldots, \phi_n$ of J, that is,

$$I = \langle \textstyle\sum_{|\alpha|=i} a(\boldsymbol{s})\boldsymbol{x}^{\alpha} \ : \ \sum_{|\alpha|=i} a(\boldsymbol{s})\phi^{\alpha} = 0 \, ; \forall i \rangle \subset R$$

For the sake of brevity, we denote the Rees algebra by B.

While $\mathrm{Rees}_S(J)$ captures all polynomial relations on the ϕ_j, the symmetric algebra $\mathrm{Sym}_S(J)$ captures only the linear ones. Specifically, we have

$$\mathrm{Sym}_S(J) = R / \langle \textstyle\sum_j a(\boldsymbol{s}) x_j \ : \ \sum_j a(\boldsymbol{s})\phi_j = 0 \rangle$$

We shall sometimes refer to R as the ambient ring of the blow-up algebras.

2.7. The ring $R = S[\boldsymbol{x}]$ and the blow-up algebras just defined are in essence geometric objects. As $\mathbb{C}$-algebras, R corresponds to the bihomogeneous coordinate ring of the product of the source and the target varieties, and $B = \mathrm{Rees}_S(J)$ — to the coordinate ring of the graph of ϕ, $\Gamma(\phi)$, defined (e.g. Harris [1992]) as the closure of

$$\{(q, \phi(q)) : q \in X\} \subset X \times \mathbb{P}^n$$

The natural surjective morphisms $R \longrightarrow \mathrm{Sym}_S(J) \longrightarrow \mathrm{Rees}_S(J)$ induce natural closed embeddings

$$\Gamma(\phi) = \mathrm{Biproj}(\mathrm{Rees}_S(J)) \longrightarrow \mathrm{Biproj}(\mathrm{Sym}_S(J)) \longrightarrow \mathrm{Biproj}(R) = X \times \mathbb{P}^n \qquad (2.7.1)$$

2.8. Denote the subscheme $V(J) \subset X$ by Z. Then Z is the base locus of the rational map. Geometrically, $\mathrm{Biproj}(B)$ is the blow-up of the variety X along the closed Z. See Lemma 5.7 for details.

2.9. Let X be a variety and $q \in X$ be a smooth point, that is, such that the stalk $\mathscr{O}_q$ is regular local. Let $Z \subset X$ be a closed subscheme containing the point q. We say that q is a complete intersection (c.i.) point of Z if the stalk of q on Z, $\mathscr{O}_{q,Z}$, is a complete intersection factor ring of the regular local $\mathscr{O}_q$. Now suppose $Z \subset X$ is of codimension d at q and that the stalk $O_{q,Z}$ can be defined from $\mathscr{O}_q$ by $d+1$ elements. In this case, we say that q is an almost complete intersection (a.c.i.) point of Z.

2.10. Suppose that the base locus of the rational map ϕ, $V(J)$, is zero-dimensional. The first embedding in (2.7.1) is an isomorphism if and only if $V(J)$ is a locally complete intersection scheme. For a proof in our setting, see (Busé and Jouanolou [2003], Proposition 4.14).

Loosely speaking, this follows because either condition is equivalent to J being of linear type. For future reference, this does not mean that $N = N_1$.

2.11. The ideal I is bigraded in R, so its S-graded pieces are finite T-modules. We denote the graded piece in degree $\boldsymbol{d}$ on S by $I_{\boldsymbol{d},\bullet}$, and sometimes call it a (T-)strand of I.

More generally, let $\boldsymbol{M}$ be a finite bigraded R-module generated by some finite set $\{h_\ell : \ell\}$. Let $\boldsymbol{d}$ be any degree on S. Setting $(\boldsymbol{a}_\ell, i_\ell) = \deg(h_\ell)$, one has

$$\sum_{\ell,\ \boldsymbol{b}_\ell\ :\ \boldsymbol{a}_\ell + \boldsymbol{b}_\ell = \boldsymbol{d}} (R_{\boldsymbol{b}_\ell,\bullet}) h_\ell = \boldsymbol{M}_{\boldsymbol{d},\bullet}$$

so

$$\bigcup_{\ell} \{\boldsymbol{s}_k \cdot h_\ell : \mathrm{Span}_{\mathbb{C}}\{\boldsymbol{s}_k : k\} = S_{\boldsymbol{b}_\ell}\}$$

is finite and a T-module generating set for $\boldsymbol{M_d}$.

2.12. For a fixed degree $\boldsymbol{d}$ on the source, the elements of $I_{\boldsymbol{d},i}$ are degree-i syzygies of the ϕ_j with (module-)coefficients in $S_{\boldsymbol{d}}$. For specific degrees $\boldsymbol{d}$ and $i = 1,2$, these have been called moving planes and quadrics, respectively [references]. We shall only use this notation in Chapter 7 when we present a sort of a "template proof" and use prove two results relating our work to what is already known in the case of basepoint-free maps over $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$, as well as a couple of new results along those lines.

2.13. Throughout this thesis we work with a fixed monomial order on S (or rather S'). For example, this could be the graded lexicographic order but the specific choice is immaterial. With this convention, the statement

> let $\boldsymbol{b}$ be a row-vector corresponding to the basis of $S_{\boldsymbol{d}}$

would mean a row vector having having as its coordinates the monomials of S, listed in the selected order.

In this sense, and fixing some basis on T too, we shall write

$$\texttt{basis}(S_{\boldsymbol{d}}), \quad \texttt{basis}(T_i), \quad \texttt{basis}(R_{\boldsymbol{d},i})$$

for the row vectors consisting of monomial bases in the chosen order for the $\mathbb{C}$-vector spaces $S_{\boldsymbol{d}}, T_i$ and $R_{\boldsymbol{d},i}$, respectively.

2.14. Let $r = \dim_{\mathbb{C}}(S_{\boldsymbol{d}})$ and let $\boldsymbol{b} = \texttt{basis}(S_{\boldsymbol{d}})$ be the row-r-vector of the monomial basis of S in degree $\boldsymbol{d}$. We shall mostly use r instead of $\dim_{\mathbb{C}}(S_{\boldsymbol{d}})$ when $\boldsymbol{d}$ has been fixed.

Given a form $g(\boldsymbol{x}) = g(\boldsymbol{s};\boldsymbol{x}) \in I_{\boldsymbol{d},i}$, that is, a syzygy of degree i over $S_{\boldsymbol{d}}$, we can write $g(\boldsymbol{x})$ as

$$g(\boldsymbol{x}) = \boldsymbol{b} \cdot C = \begin{bmatrix} x_0^i & x_0^{i-1}x_1 & \dots & x_n^i \end{bmatrix} \cdot C'$$

where C is an $r \times 1$ column vector with entries in T_i, and C' is a $\binom{n+i}{n} \times 1$ column vector with entries in $S_{\boldsymbol{d}}$.

We shall use the term syzygy both for the column vector C and the form g in the Rees ideal.

2.15. We now come to the most important bit of notation. Let us fix a degree $\boldsymbol{d}$ on the source. As already apperent from (2.14), identifying the generators of $I_{\boldsymbol{d}}$ with column-r-vectors, $I_{\boldsymbol{d}}$ becomes a sub-T-module of the free graded T^r.

Let N be the matrix built from those column-generators. Setting $\mu = \mu(I_{\boldsymbol{d},\bullet})$ to be the number of columns, N becomes an $r \times \mu$-matrix over T and more importantly, a graded T-linear map

$$N : \bigoplus_k T(-i_k) \longrightarrow T^r$$

whose image is just $I_{\boldsymbol{d},i}$.

By grouping the columns corresponding to the same degree i into submatrices N_i, for each valid i we get a single matrix whose columns are the degree i-syzygies over $S_{\boldsymbol{d}}$. Clearly, N_i is empty for any i larger than the maximum degree δ of a minimal generator of $I_{\boldsymbol{d},\bullet}$. By the assumption on the linear independence of the ϕ_j, N_0 is empty too. In any case, the N_i fit together to give N,

$$N = (N_1 \mid N_2 \mid \dots \mid N_\delta)$$

In Chapter 7 we describe the close connection of N_1 and N_2 to the matrices used in the moving planes and quadrics results.

Finally, whenever useful, we write h_i for $\dim_{\mathbb{C}} \mathrm{Span}(N_i) = \dim_{\mathbb{C}}(I_{\boldsymbol{d},i})$ and h for the tuple $(h_1, \dots, h_\delta)$.

2.16. Recall that for any ideal Q in a unique factorization domain T, $\gcd(Q)$ is defined to be the unique minimal principal ideal which contains Q, and this definition obviously generalizes the definition on elements when T is a Euclidean domain.

2.17. As a final bit of notation, we mention that we use $\mathrm{rad}(-)$ for the radial, and $\mathrm{sat}(-)$ for the saturation with respect to the irrelevant ideal.

2.2 Strands of Module Maps

2.18. Let us consider the coordinates of ϕ as a row vector over S. We get a graded S-linear map

$$\begin{bmatrix} \phi_0 & \phi_1 & \dots & \phi_n \end{bmatrix} : S^{n+1} \longrightarrow S^1(\boldsymbol{e})$$

where $S(\boldsymbol{e})$ has the usual meaning of putting $1 \in S$ in degree $-\boldsymbol{e}$. Similarly, we can consider the graded S-linear map given by the quadratic monomials of the coordinates,

$$\begin{bmatrix} \phi_0^2 & \phi_0\phi_1 & \dots & \phi_n^2 \end{bmatrix} : S^{(n+2)(n+1)/2} \longrightarrow S^1(2\boldsymbol{e})$$

These two maps have a central role in the methods of moving planes and quadrics. However, there is no reason to stop at degree 2, so next we describe the general situation.

2.19. Let k be a positive integer and $\boldsymbol{d}$ be a fixed degree on S such that $S_{\boldsymbol{d}} \neq 0$. Define $\phi^{(k)}$ to be the graded S-linear map formed by the coordinates of ϕ,

$$\phi^{(k)} = \begin{bmatrix} \phi_0^k & \phi_0^{k-1}\phi_1 & \cdots & \phi_n^k \end{bmatrix} : S^{\binom{n+k}{n}} \longrightarrow S^1(k\boldsymbol{e})$$

and set $\Phi^{(k)}$ be the strand of $\phi^{(k)}$ in degree $\boldsymbol{d}$, that is,

$$\Phi^{(k)} : S_{\boldsymbol{d}}^{\binom{n+k}{n}} \longrightarrow S^1_{k\boldsymbol{e}+\boldsymbol{d}}$$

is a map of complex vector spaces.

Choosing bases (2.14), we can think of $\Phi^{(k)}$ as a matrix over $\mathbb{C}$ of size

$$\dim_{\mathbb{C}}(S_{k\boldsymbol{e}+\boldsymbol{d}}) \times r\binom{n+k}{n}$$

whose columns can be indexed by the monomials in $R_{\boldsymbol{d},k}$.

2.20. The advantage of the matrices $\Phi^{(k)}$ over the matrices $\phi^{(k)}$ is that the kernel of $\Phi^{(i)}$ corresponds directly to the degree-i syzygies over $S_{\boldsymbol{d}}$. That is, for a fixed $\boldsymbol{d}$,

$$\boldsymbol{v} \mapsto \texttt{basis}(R_{\boldsymbol{d},i}) \cdot \boldsymbol{v} : \ker(\Phi^{(i)}) \longrightarrow \boldsymbol{I}_{\boldsymbol{d},i}$$

is an isomorphism of vector spaces.

2.3 Multiplicity

2.21. Recall the following notation from Hartshorne [1977]. For a homogeneous prime ideal P in a graded ring T, we set $T_{(P)}$ to be the degree-0 graded piece of the localization of T at the homogeneous elements outside of P. If P is a minimal prime of a graded T-module $\boldsymbol{M}$, we denote by $\mathrm{mult}_P(\boldsymbol{M})$ the length of the T_P-module $\boldsymbol{M}_P$ (see *loc. cit.*, I, Proposition 7.4).

2.22. Let Z be a zero-dimensional closed subscheme of a smooth projective variety X. Let $\mathscr{I} \subset \mathscr{O}_X$ be the ideal sheaf of Z and let $q \in Z$ be any point. Since Z is zero-dimensional, $\mathscr{I}_q \subset \mathscr{O}_{q,X}$ is an ideal of definition in the regular local ring $\mathscr{O}_{q,X}$.

Define the multiplicity of Z at q, denoted e_q, to be the Hilbert-Samuel multiplicity of $\mathscr{I}_q$, denoted $e(\mathscr{I}_q, \mathscr{O}_q)$ (see Eisenbud [1995] or Bruns and Herzog [1998]).

Define the degree of Z at q, denoted d_q, to be the length of the local ring,

$$\mathrm{length}(\mathscr{O}_{q,Z}) = \dim_{\mathbb{C}}(\mathscr{O}_{q,Z})$$

We have that $d_q \leq e_q$ with equality if and only if q is a c.i. point (Bruns and Herzog [1998], Theorem 4.7.4).

2.23. We shall mostly be interested in the degree and multiplicity of points on the base locus Z. Since by assumption X is toric, the ideal sheaf $\mathscr{J}$ of Z in (2.22) is the ideal sheaf $\widetilde{J}$. We stick to this notation for the rest of the paper.

2.24. Let $\mathscr{L}$ be a line bundle on X. We denote by $[\mathscr{L}]$ the class of $\mathscr{L}$ in the Chow ring. Since $\dim(X) = n-1$, we can identify $[\mathscr{L}]^{n-1}$ with an integer — its degree. Suppose that the base locus Z is zero-dimensional. Then by (Fulton [1984], Proposition 4.4), see Cox [2001] for details, we have the formula

$$\deg(\phi)\deg(Y) = [\mathscr{L}]^{n-1} - \sum_{q \in Z} e(\widetilde{J}_q, \mathscr{O}_q) \tag{2.24.1}$$

2.25. The self-intersection $[\mathscr{L}]^{n-1}$ is obvious when $X = \mathbb{P}^{n-1}$. Then $\mathscr{L} = \mathscr{O}(d)$ for some integer d, and

$$[\mathscr{O}(d)]^{n-1} = d^{n-1}$$

Similarly, let $X = (\mathbb{P}^1)^{n-1}$. Then $\mathscr{L} = \mathscr{O}(e_1, \ldots, e_{n-1})$ for integers e_k, and

$$[\mathscr{O}(e_1, \ldots, e_{n-1})]^{n-1} = (n-1)! \cdot e_1 \cdots e_{n-1} \tag{2.25.1}$$

The formula above can be easily proved by remembering that the rulings of X have self-intersection zero, so the only nonzero term in the power of $[\mathscr{L}]$ in the Chow ring, is the multiplication of all rulings.

Example 2.26. Here is a list of ways to compute the degree and multiplicity, and also to show how not to compute them.

- Both notions are geometric which means that we need to saturate our ideals before computing the degree and multiplicity.

- For both, the sum of the values at the points in a zero-dimensional scheme add to the corresponding notion for the scheme. This works better with degree.
- Just typing `degree`(J) and `multiplicity`(J) in Macaulay2 often has dissasterous results.
- If $X = \mathbb{P}^n$ and J is saturated, then `degree`(J) is in fact the degree of the zero-dimensional scheme. If the scheme is supported on a single point, then that's the degree of the point. Else, saturate with respect to the prime ideals of the other points.
- We can freely work over non-algebraically closed fields. Points in the same Galois closure are algebraically indistinguishable, so just assume they split equally to closed points over $\mathbb{C}$, or the algebraic closure.
- `multiplicity`(J) does work in projective space for an ideal supported on a single point. And the implementation uses a cool trick — the reference I have is due to Eisenbud.

2.4 Examples

Example 2.27 (`ex201`)**.** Let $X = \mathbb{P}^1_{s,u} \times \mathbb{P}^1_{t,v}$ be the product of two projective lines, and let $S = \mathbb{C}[s,u;t,v]$ be its Cox ring — a bihomogeneous ring graded by $\mathrm{Pic}(X)$ such that $\deg(s) = \deg(u) = (1,0)$ and $\deg(t) = \deg(v) = (0,1)$. The irrelevant ideal $\mathfrak{n}$ is the product of the irrelevant ideals of the components,

$$\mathfrak{n} = \langle s,u\rangle \cap \langle t,v\rangle = \langle st, sv, ut, uv\rangle$$

We write $X = \mathrm{Proj}(S,\mathfrak{n})$ or simply $X = \mathrm{Proj}(S)$ for the construction in the toric setting.

Consider the rational map $\phi : X \longrightarrow \mathbb{P}^3$ given by

$$\phi = (s^2v^2, suv^2, u^2t^2 + u^2tv, sutv - 101u^2tv)$$

In this case, the coordinates $\phi_0, \ldots, \phi_3$ are global sections of the line bundle $\mathscr{O}_X(2,2)$, and are linearly independent over the base field $\mathbb{C}$. Note that

$$h^0(\mathscr{O}_X(2,2), X) = \dim_{\mathbb{C}}(S_{(2,2)}) = 9$$

The ideal of the coordinates,

$$J = \langle s^2v^2, suv^2, u^2t^2 + u^2tv, sutv - 101u^2tv \rangle \subset S$$

defines the base locus, $Z = V(J)$. In this case, Z is supported on the points

$$q_1 = (0,1) \times (0,1) \text{ and } q_2 = (1,0) \times (1,0)$$

so the base locus is zero-dimensional and the map is generically finite.

The points are qualitatively different. Near q_1 the scheme Z looks like $V(s,t)$ in $\mathbb{A}^2_{s,t}$, while near q_2, Z looks like $V(u^2, uv, v^2)$ in $\mathbb{A}^2_{u,v}$. This allows us to compute their degrees and multiplicities. Let $\mathscr{O}_{q_i,Z}$ denote the stalk at q_i on $Z \subset X$ ($i = 1,2$), and let $\widetilde{J}$ denote the ideal sheaf arising form J. The the degrees are

$$d_{q_1} = \text{length}(\mathscr{O}_{q_1,Z}) = 1 \text{ and } d_{q_2} = \text{length}(\mathscr{O}_{q_2,Z}) = 3$$

while the multiplicities are

$$e_{q_1} = e(\widetilde{J}_{q_1}, \mathscr{O}_{q_1,X}) = 1 \text{ and } e_{q_2} = e(\widetilde{J}_{q_2}, \mathscr{O}_{q_2,X}) = 4$$

where $e(\boldsymbol{I}, \boldsymbol{R})$, for a Noetherian local $\boldsymbol{R}$ and an ideal of definition $\boldsymbol{I}$, denotes the Hilbert-Samuel multiplicity.

Note further that $d_{q_1} = e_{q_1}$ reflects the fact that q_1 is a complete intersection (c.i.) point, while $d_{q_2} < e_{q_2}$ so q_2 is not. It is, however, an almost complete intersection (a.c.i.) point, since it is of codimension-2 and is (stalk-, affine-)locally defined by 3 equations.

The closed image $Y \subset \mathbb{P}^3$ of ϕ is given by a single equation,

$$P(\boldsymbol{x}) = x_0^2x_2 - 202x_0x_1x_2 + 10201x_1^2x_2 - x_0x_1x_3 + 101x_1^2x_3 - x_0x_3^2$$

The equation $P(\boldsymbol{x})$, called the implicit equation, show up in the Rees ideal.

Let $B = \mathrm{Rees}_S(J)$ be realized as the quotient ring of $R = S[\boldsymbol{x}]$ by the aforementioned Rees ideal I. In our case,

$$I = \left\langle \begin{cases} ux_0 - sx_1, (sv - 101uv)x_2 + (-ut - uv)x_3, \\ (t^2 + tv)x_1 - 101v^2x_2 + (-tv - v^2)x_3, \\ (st - 101ut)x_1 - svx_3, vx_0x_2 - 101vx_1x_2 + (-t - v)x_1x_3, \\ tx_0x_2 - 101tx_1x_2 - 101vx_2x_3 + (-t - v)x_3^2, \\ sx_0x_2 + (-202s + 10201u)x_1x_2 + (-s + 101u)x_1x_3 - sx_3^2, \\ tx_0x_1 - 101tx_1^2 - vx_0x_3, \\ P(x_0, x_1, x_2, x_3) \end{cases} \right\rangle$$

and $P(\boldsymbol{x})$ can be seen as the unique linear generator in bidegree $((0,0),3)$. In fact, it is the unique $\mathbb{C}[\boldsymbol{x}]$-generator of $I_{(0,0),\bullet}$.

Generalizing slightly, let $\boldsymbol{d} = (1,1)$ be a degree on S. Then $I_{\boldsymbol{d},\bullet}$ is minimally generated by 5 elements as a $\mathbb{C}[\boldsymbol{x}]$-module. Let those be $g_1(\boldsymbol{s};\boldsymbol{x}), \ldots, g_5(\boldsymbol{s};\boldsymbol{x})$. Writing

$$\texttt{basis}(S_{\boldsymbol{d}}) = \begin{bmatrix} st & sv & ut & uv \end{bmatrix}$$

to be a row vector over R consisting of a linear basis for $S_{\boldsymbol{d}}$, we can define the matrix N

over $\mathbb{C}[\boldsymbol{x}]$ by the identity

$$\texttt{basis}(S_{\boldsymbol{d}}) \cdot N = \begin{bmatrix} g_1 & \dots & g_5 \end{bmatrix}$$

In this case, N is a 4×5-matrix each of whose columns contains forms of the same degree. Specifically,

$$N = \begin{bmatrix} 0 & x_1 & 0 & 0 & x_0x_2 - x_3^2 \\ x_2 & -x_3 & -x_1 & -x_3 & -x_3^2 \\ -x_3 & -101x_1 & 0 & x_0 - 101x_1 & -10201x_1x_2 - 202x_3^2 \\ -101x_2 - x_3 & 0 & x_0 & 0 & 20402x_2x_3 - 202x_3^2 \end{bmatrix}$$

In this sense, N is a representation matrix as well as a matrix of syzygies. It represents a generating set for the graded syzygies of the ϕ_j with coefficients in $S_{(1,1)}$.

Example 2.28 (`ex202`)**.** Let $X = \mathbb{P}^2$. Its Cox ring is just its standard homogeneous coordinate ring, $\mathbb{C}[s,t,u]$, and the irrelevant ideal $\mathfrak{n} = \langle s,t,u \rangle$.

Let $J \subset S$ be a graded ideal generated by 4 linearly independent forms of the same degree $\boldsymbol{e}$. For concreteness we can take

$$J = \langle s^3, t^2u, s^2t + u^3, stu \rangle$$

so that its generators are all 3-forms. We can think of J as defining a rational map to $\mathbb{P}^3$,

$$\phi = (s^3, t^2u, s^2t + u^3, stu) : X \longrightarrow \mathbb{P}^3$$

Then ϕ is a morphism on the open set away from $Z = V(J)$. Note that J is not saturated, so there is a better representative for the base locus scheme, i.e.

$$\mathrm{sat}(J) = J : \mathfrak{n}^\infty = \langle s^2, u \rangle$$

From this we know that Z is supported on $q = (0,0,1)$ only, and that q is a complete intersection point, so its degree and multiplicity coincide,

$$\boldsymbol{e}_q = d_q = \text{length}(\mathscr{O}_q/\widetilde{J_q}) = 2$$

Let P be the principal ideal generated by the implicit equation for ϕ. By the formulas in Section 2.3, we have

$$\deg(\phi)\deg(P) = 3^2 - e_q = 7$$

Since the coordinates ϕ_j are not linearly dependent, and they never will be in this thesis, $\deg(P) \neq 1$, so $\deg(\phi) = 1$ and the map is generically 1-1. We then have that the implicit equation is a septic. Indeed,

$$P(\boldsymbol{x}) = x_0^3 x_1^4 - x_0^2 x_1^3 x_2 x_3 + x_3^7$$

Besides a rational map, or a ring map in the form of $\phi^{\#}$, ϕ can be made into a map of free S-modules. For example, we have that

$$\phi^{(1)} = \begin{bmatrix} \phi_0 & \dots & \phi_3 \end{bmatrix} : S^4 \longrightarrow S^1(3)$$

is a graded map of S-modules.

We can linearize this map in this map in the following way. Take a degree on S, for concreteness, take $\boldsymbol{d} = 1$. We take a the strand of the map $\phi^{(1)}$ in degree $\boldsymbol{d} = 1$. To this end, note that the basis on the LHS can be indexed by the monomials in $R_{1,1}$, for example, by putting

$$\begin{bmatrix} 0 \\ s \\ 0 \\ 0 \end{bmatrix} = sx_1$$

and the basis on the RHS can be indexed by the monomials in S_4. Multiplying out $\phi^{(1)}$ with an element S^4 is then equivalent to evaluation $x_j = \phi_j$. Reading off the coefficients we get for the linearization

$$\Phi^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

We end this example noting that $\ker(\Phi^{(1)})$ is isomorphic to $I_{1,1}$. In our situation this is easy to check. The kernel is spanned by

$$\boldsymbol{v} = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}^\top$$

which corresponds to $I_{1,1} = \mathrm{Span}_{\mathbb{C}}(sx_1 - tx_3)$ after multiplication with the indexing set for the columns,

$$\texttt{basis}(R_{1,1}) = \begin{bmatrix} sx_0 & sx_1 & sx_2 & sx_3 & tx_0 & tx_1 & tx_2 & tx_3 & ux_0 & ux_1 & ux_2 & ux_3 \end{bmatrix}$$

CHAPTER 3

MAIN RESULTS

We are now ready to state our main results. While this chapter is supposed to be self-contained and most of the relevant notation and definitions are listed in (3.1) below, one should consult Chapter 2 for a more relaxed exposition and further definitions, for instance, for the notions of degree and multiplicity of a basepoint.

Examples 2.27 and 2.28 should serve as quick reference points.

3.1. Let X be a smooth projective toric variety of dimension $n-1$ ($n>1$) with Cox ring S and irrelevant ideal $\mathfrak{n} \subset S$. Let $T = \mathbb{C}[x_0, \dots, x_n]$ and let

$$\phi = (\phi_0, \dots, \phi_n) : X \longrightarrow \mathbb{P}^n = \mathrm{Proj}(T)$$

be a rational map given by linearly independent sections of the line bundle $\mathscr{O}_X(\boldsymbol{e})$ for some degree $\boldsymbol{e}$ on S. Let $J = \langle \phi_0, \dots, \phi_n \rangle \subset S$ be the ideal of the coordinates ϕ_j.

Let $\boldsymbol{d}$ be a degree on S such that $r = \dim_{\mathbb{C}}(S_{\boldsymbol{d}}) > 0$ and let $I \subset R = S \otimes T$ be the Rees ideal of J. Let $I_{\boldsymbol{d},\bullet}$ be its degree-$(\boldsymbol{d},\bullet)$ bigraded piece, considered as a finite graded T-module. We denote by $B = R/I$ the Rees algebra of J.

Let N be the $r \times \mu$ coefficient matrix of a minimal set of homogeneous generators for $I_{\boldsymbol{d},\bullet}$ with respect to $\mathtt{basis}(S_{\boldsymbol{d}})$, and let N_i be the submatrix of N corresponding to generators of degree i, i.e.

$$N = (\ N_1 \mid \dots \mid N_\delta\ )$$

3.2. Let $P \subset T$ be the prime ideal corresponding to the closed image of ϕ in $\mathbb{P}^n$. We denote this image by $Y = V(P)$. Let $Z = V(J) \subset X$ be the base locus of ϕ.

We are going to be interested the following three conditions:

(1) The map ϕ is generically finite onto its image, that is, $Y \subset \mathbb{P}^n$ is of codimension 1 and in particular, the ideal P is principal. In this case, we denote a generator of P by $P(\boldsymbol{x})$.

(2) The base locus Z is zero-dimensional, that is, consists of finitely many points. Note that those are necessarily closed over $\mathbb{C}$.

(3) The map ϕ is birational onto its image.

Clearly, either of (2) and (3) implies (1).

A few easy but important observations about the Rees ideal follow.

Proposition 3.3. *Consider the setup of* (3.1). *One has*

(1) *The ideal $I \subset R$ is prime, and so is I_P in the T-module localization R_P.*

(2) *The quotient B_P is naturally a finite-type graded $K(T/P)$-algebra with grading induced by S.*

(3) *The $K(T/P)$-algebra B_P is a homogeneous coordinate ring of a projective variety.*

3.4. Let $V(I_P)$ be the closed subset of the projective toric variety $\mathrm{Biproj}(R_P)$. Note that the former is a variety over $K(T/P)$ with the grading of S. Following Maclagan and Smith [2004] and using Proposition 3.3, we consider the regularity of the defining ideal I_P, denoted by $\mathrm{reg}(I_P)$.

Recall that $\mathrm{reg}(I_P)$ is a finitely generated additively-closed subset of the semigroup of degrees on S, and that for any $\boldsymbol{d} \in \mathrm{reg}(I_P)$, we have $\langle (I_P)_{\boldsymbol{d}} \rangle = I$. This parallels the usual Castelnuovo-Mumford regularity for $\mathbb{P}^n$ and is the content of Theorem 1.3 in the referenced paper.

In light of Proposition 3.3, our first result becomes an easy exercise. However, it is a step toward the goal of this paper — to exhibit a general relation between the algebra of the coordinates ϕ_j and the geometry of the image Y.

Theorem 3.5. *In the setup of* (3.1)*, one has*

$$\operatorname{rad}(\mathtt{minors}(r,N)) = P$$

The geometric interpretation of the theorem is clear — the nonzero minors of N define hypersurfaces in $\mathbb{P}^n$ whose intersection, at least set-theoretically, is the image Y.

Example 4.7, for instance, shows that the radical is necessary.

Our next result is the main theorem of this thesis, unifying two currently popular non-Gröbner bases approaches to implicitization and setting the stage for both the ad-hoc template proofs in Chapter 7 and the fast implicitization method described in Chapter 6.

Theorem 3.6. *Consider the setup of* (3.1) *and assume* (3.2.1)*. Fix a degree* $\mathbf{d} \in \operatorname{reg}(I_P)$ *as described in* (3.4)*. One has*

$$\gcd(\mathtt{minors}(r,N)) = P^{\deg\phi}$$

In particular, if there is a degree $\mathbf{d}$ *in the regularity for which* $\mu = r$*, then* N *is square and, up to a unit,*

$$\det(N) = P(\boldsymbol{x})^{\deg\phi}$$

Corollary 3.7. *In the setup of Theorem 3.6, let* M *be any* $r \times r$ *matrix of syzygies over* $S_{\boldsymbol{d}}$*. One has*

$$\det(M) = P(\boldsymbol{x})^{\deg\phi} \cdot H(\boldsymbol{x})$$

for a homogeneous $H(\boldsymbol{x})$ *of degree*

$$\deg(\det(M)) - \deg(\phi) \cdot \deg(Y) \tag{3.7.1}$$

Furthermore, there exist a list of such matrices $\{M_k\}$ *whose corresponding* $H_k(\boldsymbol{x})$ *are nonzero and have common factor* 1.

Geometrically, the former is a refinement of Theorem 3.5. Each of the maximal minors of N, in fact, the determinant of any $r \times r$ matrix M of syzygies over $S_{\boldsymbol{d}}$, is either zero or describes the union of a $\deg(\phi)$-fold Y and a hypersurface of degree (3.7.1). While an arbitrary collection M_k of such matrices may introduce hypersurfaces with an intersection that is strictly larger than Y, the maximal minors suffice to shave off any extraneous components.

The theme of extraneous factors is already apparent in Busé et al. [2003], Busé et al. [2009] and Botbol et al. [2009]. In our notation, they used the approximation complex to show that for a toric X, certain $\boldsymbol{d}$ and empty or zero-dimensional almost complete intersection base locus Z,

$$\gcd(\mathtt{minors}(r, N_1)) = P^{\deg\phi} \cdot \prod_{q \in Z} L_q(\boldsymbol{x})^{e_q - d_q}$$

where each $L_q(\boldsymbol{x})$ is a linear form, and e_q and d_q are the multiplicity and degree of q.

In the case of complete intersection base locus, the proof of Theorem 3.6 gives a special case of the above.

Corollary 3.8 (Busé et al. [2003])**.** *In the setup of Theorem 3.6, let* $\boldsymbol{d}$ *be* large enough *in* $\mathrm{reg}(I_P)$. *Let* $\mathrm{Biproj}(B)$ *and* $\mathrm{Biproj}(\mathrm{Sym}(J))$ *be naturally isomorphic in the sense of* (2.7.1). *One has*

$$\gcd(\mathtt{minors}(r, N_1)) = P^{\deg\phi}$$

In particular, the result holds if the base locus is empty or zero-dimensional and locally a complete intersection.

It is known that if M is a square matrix over T of size r, then the singular locus of $V(\det(M))$ is contained in the closed subset defined by the comaximal minors, that is, the $(r-1)$-minors. Although we failed to find a reference, we believe that this relation is more intrinsic and holds for all representation matrices N. However, what this ought to correspond to is the multiple-point locus of the image. See Example 4.17 for details. We conjecture the following

Conjecture 3.9. *Consider the setup of* (3.1) *and assume* (3.2.3). *On the level of closed points, one has*

$$V(\mathtt{minors}(r-1,N)) \subset \mathrm{Sing}(Y)$$

In the simplest cases of interest, when $X=\mathbb{P}^2$ or $X=\mathbb{P}^1\times\mathbb{P}^1$ and ϕ is basepoint-free, we can chose $\boldsymbol{d}$ so that the matrix N becomes square. Next two theorems are slight generalizations of the results in Cox et al. [2000]. More importantly, they show that our methods directly generalize the methods of moving planes and quadrics in the setting in which they are most useful.

Theorem 3.10. *Let $X=\mathbb{P}^2$, ϕ be basepoint-free, and suppose that there are exactly $p=\boldsymbol{e}$ linear syzygies over degree $\boldsymbol{d}=p-1$, that is, the minimal possible number. One has that $N=(N_1 \mid N_2)$, N is square and ϕ is birational. In particular,*

$$\det(N)=\det(N_1 \mid N_2)=P(\boldsymbol{x})$$

Theorem 3.11. *Let $X=\mathbb{P}^1\times\mathbb{P}^1$, ϕ be basepoint-free with coordinates in degree $\boldsymbol{e}=(p,q)$, and suppose there are no linear syzygies over degree $\boldsymbol{d}=(p-1,q-1)$. Then one has that N_2 is square, $N=N_2$ and ϕ is birational. In particular,*

$$\det(N)=\det(N_2)=P(\boldsymbol{x})$$

Both of these theorems are examples of a template proof described in Chapter 7. While applying it in general requires elaborate choses of the degree $\boldsymbol{d}$ and regularity computations, for example Adkins et al. [2005], in the case of Theorems 7.22 and 7.7, we only use a type of Koszul-ness on the syzygies of low degree. This is the content of Section 7.2.

We conclude this list by a method to compute the degree of a rational map using Gröbner bases. While we are only going to use this in our examples, it helps expand our understanding about the object B_P.

The author wants to thank Mike Stillman for suggesting the following

Proposition 3.12. *Let* $\mathbb{C}[s_0,\dots,s_m]$ *be the fixed ambient polynomial ring of* S' *as described in* (3.1)*. Define the ideal* I_B *of* $\mathbb{C}[s_0,\dots,s_m;x_0,\dots,x_n]$ *by the equality*

$$B = \mathbb{C}[\boldsymbol{s};\boldsymbol{x}]/I_B$$

Let $>'$ *be any product order in which the* $\boldsymbol{s}$ *variables come before the* $\boldsymbol{x}$ *variables. Then a reduced Gröbner basis for* I_B *with respect to* $>'$ *has the form*

$$\begin{aligned} g_1(\boldsymbol{s};\boldsymbol{x}) &= p_1(\boldsymbol{x})\boldsymbol{s}^{\alpha_1} + \textit{lower order terms} \\ &\cdots \\ g_r(\boldsymbol{s};\boldsymbol{x}) &= p_r(\boldsymbol{x})\boldsymbol{s}^{\alpha_r} + \textit{lower order terms} \\ g_{r+1}(\boldsymbol{s};\boldsymbol{x}) &= P(\boldsymbol{x}) \end{aligned}$$

Further, one has

$$\deg(\phi) = \deg\left(\langle \boldsymbol{s}^{\alpha_1},\dots,\boldsymbol{s}^{\alpha_r}\rangle \subset \mathbb{C}[\boldsymbol{s}]\right)$$

CHAPTER 4

EXAMPLES

This chapter is the heart of the thesis. It consists of examples highlighting the results of Chapter 3 and motivating the results of Chapters 6 and 7.

We recall that the Macaulay2 code for the examples is available at

`http://www.math.cornell.edu/~rzlatev/phd-thesis/`

Example 4.1 (`ex301`)**.** Let $X = \mathbb{P}^2_{s,t,u}$ and $J = \langle tu, su, st, s^2+t^2+u^2 \rangle$. Then ϕ is basepoint-free and generically 1-1. The monic implicit equation is given by

$$P(\boldsymbol{x}) = x_0^2x_1^2 + x_0^2x_2^2 + x_1^2x_2^2 - x_0x_1x_2x_3$$

Setting $\boldsymbol{d} = 1$, we get

$$N = \begin{bmatrix} 0 & x_0 & x_1x_2 \\ x_1 & 0 & x_0x_2 \\ -x_2 & -x_2 & x_0x_1 - x_2x_3 \end{bmatrix}$$

whose determinant is just $P(\boldsymbol{x})$. The results of Cox et al. [2000] apply and the matrix N is a variant of the matrix produced by the method of moving planes and quadrics.

Setting $\boldsymbol{d} = 2$, we get

$$N = \begin{bmatrix} x_2 & 0 & 0 & 0 & x_1 & 0 & 0 & 0 & x_0 \\ -x_3 & 0 & 0 & x_1 & 0 & 0 & 0 & x_0 & 0 \\ 0 & 0 & 0 & -x_2 & -x_3 & x_0 & x_2 & 0 & -x_2 \\ x_2 & 0 & x_1 & 0 & 0 & 0 & x_0 & 0 & 0 \\ 0 & x_1 & -x_2 & 0 & x_2 & 0 & -x_3 & -x_2 & 0 \\ x_2 & -x_2 & 0 & 0 & x_1 & -x_2 & x_0 & 0 & 0 \end{bmatrix}$$

which is a 6×9 matrix of linear forms. This was expected — the results of Busé and Jouanolou [2003] also apply and the method of the approximation complex guarantees a matrix of linear forms. Accordingly,

$$\gcd(\texttt{minors}(6,N)) = P$$

Note that the claim that ϕ is of degree 1 follows, a fortiori, from the degree formula (2.24.1). Indeed, we have

$$4\deg(\phi) = 2^2 - 0$$

We confirm this using Proposition 3.12 in Example 4.16.

Example 4.2 (`ex302`)**.** Clearly, if we replace s,t,u in Example 4.1 by general linear forms L_0, L_1, L_2, effectively changing coordinates on the source, we get the same equation. In this example we describe what happens if we take $X = \mathbb{P}^1_{s,u}\times\mathbb{P}^1_{t,v}$ instead and let the L_k be $(1,1)$-forms. Since the algebraic structure of the coordinates is the same, so is the equation of the image,

$$Y = V(x_0^2x_1^2 + x_0^2x_2^2 + x_1^2x_2^2 - x_0x_1x_2x_3)$$

and ϕ is again basepoint-free, basically for the same reason — the L_k are general. However, the self-intersection of the divisor corresponding to the coordinates, now $[\mathscr{O}(2,2)]$, is 8. It follows that ϕ is generically 2-1.

For $\boldsymbol{d} = (1,1)$ we get a square matrix of size 4 with $h = (2,1,0,1)$. As expected, up to a unit

$$\det(N) = P(\boldsymbol{x})^2$$

For $\boldsymbol{d} = (2,1)$ we get a square matrix of size 6 with $h = (4,2)$, for which the last equality again applies. For $\boldsymbol{d} = (2,2)$ we get a 9×12-matrix with $h = (11,1)$ such that

$$\gcd(\texttt{minors}(9,N)) = P^2$$

Example 4.3 (`ex303`)**.** Suppose that in the situation of Example 4.2 we took the forms L_k from $\langle st, sv, ut\rangle$ instead. Now ϕ has the unique basepoint $(0,1)\times(0,1)$ which is c.i. of degree 4. Indeed, on the affine open where $u = v = 1$, the point looks like $V(st, s^2+t^2)$. The equation of the image remains the same. Once again, by (2.24.1) we know that ϕ must be generically 1-1.

For $\boldsymbol{d} = (1,1)$ we get a 4×5-matrix. Below is the matrix resulting from $(L_0, L_1, L_2) = (st, sv, ut)$,

$$N = \begin{bmatrix} 0 & 0 & 0 & x_0 & x_1x_2 \\ x_1 & x_0 & 0 & 0 & 0 \\ -x_2 & 0 & x_0 & -x_2 & -x_2x_3 \\ 0 & -x_2 & -x_1 & 0 & x_1^2+x_2^2 \end{bmatrix}$$

and, of course,

$$\gcd(\texttt{minors}(4,N)) = P$$

Example 4.4 (`ex304`)**.** Let $X = \mathbb{P}^2$ and $J = \langle su^2, t^2(s+u), st(s+u), tu(s+u)\rangle$. Then ϕ is generically 1-1 with three basepoints — $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$, all c.i. of degree 2, 3, and 1, respectively.

The implicit equation is given by

$$P(\boldsymbol{x}) = x_0x_1x_2 + x_0x_1x_3 - x_2x_3^2$$

As before, both the method of the moving planes and quadrics, and the method of the approximation complex apply. For $\boldsymbol{d} = 1$, we get

$$N = \begin{bmatrix} -x_3 & 0 & x_1 & -x_3^2 \\ 0 & -x_3 & -x_2 & x_0x_2 + x_0x_3 \\ x_2 & x_1 & 0 & 0 \end{bmatrix}$$

and for $\boldsymbol{d} = 2$, we get a 6×9-matrix whose entries are all linear.

Example 4.5 (ex305)**.** Let $X=\mathbb{P}^2$ and $J=\langle s^3,tu^2,s^2t+u^3,stu\rangle$. Then ϕ is generically 1-1 with a single c.i. basepoint of degree 2. For $\boldsymbol{d}=1$ we have

$$N=\begin{bmatrix} x_1 & 0 & -x_3^2 \\ 0 & x_1x_2-x_3^2 & x_0x_1 \\ x_3 & x_1^2 & 0 \end{bmatrix}$$

and $\det(N)=x_0x_1^4-x_1x_2x_3^3+x_3^5$, the implicit equation.

Starting from $\boldsymbol{d}=2$, in which case we have

$$N=\begin{bmatrix} 0 & 0 & 0 & x_1 & -x_3 & 0 & 0 \\ x_3 & 0 & x_1 & 0 & 0 & -x_2 & 0 \\ 0 & x_1 & 0 & -x_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_0 & -x_1x_2+x_3^2 \\ -x_2 & 0 & -x_3 & 0 & x_0 & 0 & x_1^2 \\ x_1 & -x_3 & 0 & 0 & 0 & x_3 & 0 \end{bmatrix}$$

we always have $\mu-1$ linear columns and a single quadratic one.

Example 4.6 (ex306)**.** Let $X=\mathbb{P}^2$ and $J=\langle s^3,t^2u,s^2t+u^3,stu\rangle$. The map is birational with a single c.i. basepoint of degree 2. and $\det(N)=x_0^3x_1^4-x_0^2x_1^3x_2x_3+x_3^7$,

Example 4.7 (ex307)**.** Consider the twisted cubic curve C. It is the image of $X=\mathbb{P}^1_{s,t}$ under the map

$$\phi=(s^3,s^2t,st^2,t^3):\mathbb{P}^1\longrightarrow\mathbb{P}^3$$

which is birational onto its image. In light of Theorem 3.5, we can carry out the same calculations as before, even though the image is of codimension strictly bigger than 1.

Setting $\boldsymbol{d}=1$, we get

$$N=\begin{bmatrix} -x_3 & -x_2 & -x_1 \\ x_2 & x_1 & x_0 \end{bmatrix}$$

and

$$\texttt{minors}(2,N) = \langle x_2^2 - x_1x_3, x_1x_2 - x_0x_3, x_1^2 - x_0x_2 \rangle$$

which is the usual equation for C in $\mathbb{P}^3$. Setting $\boldsymbol{d} = 2$, we get

$$N = \begin{bmatrix} 0 & -x_3 & -x_3 & -x_2 & -x_2 & -x_1 \\ -x_3 & x_2 & 0 & x_1 & 0 & x_0 \\ x_2 & 0 & x_1 & 0 & x_0 & 0 \end{bmatrix}$$

We have

$$\texttt{minors}(3,N) = \left\langle \begin{cases} x_2^2x_3 - x_1x_3^2, x_1x_2x_3 - x_0x_3^2, x_1^2x_3 - x_0x_2x_3, \\ x_2^3 - x_0x_3^2, x_1x_2^2 - x_0x_2x_3, x_0x_2^2 - x_0x_1x_3, \\ x_1^2x_2 - x_0x_1x_3, x_0x_1x_2 - x_0^2x_3, x_1^3 - x_0^2x_3, x_0x_1^2 - x_0^2x_2 \end{cases} \right\rangle$$

and

$$\mathrm{rad}(\texttt{minors}(3,N)) = \langle x_2^2 - x_1x_3, x_1x_2 - x_0x_3, x_1^2 - x_0x_2 \rangle$$

showing that the radical is necessary.

Example 4.8 (`ex308`). Let $X = \mathbb{P}^2_{s,t,u}$ and $J = \langle s^5, t^5, su^4, st^2u^2 \rangle$. Then ϕ is generically 2-1 map with the unique basepoint $(0,0,1)$ which is c.i. of degree 5. The image is defined by

$$P(\boldsymbol{x}) = x_0x_1^4x_2^5 - x_3^{10}$$

Setting $\boldsymbol{d} = 1$, we get

$$N = \begin{bmatrix} x_1x_2 & -x_3^8 & 0 \\ -x_3^2 & x_0x_1^3x_2^4 & 0 \\ 0 & 0 & x_0x_1^4x_2^5 - x_3^{10} \end{bmatrix}$$

and

$$\det(N) = P(\boldsymbol{x})^2$$

One should note that the principal 2-minor of N is just $P(\boldsymbol{x})$.

The alternative non-Gröbner bases approaches would require difficult computations; the method of the approximation complex would need to find the gcd of the maximal minors of a matrix of size 36×58, and none of the moving plane and quadrics methods will work since the map isn't birational (and (BP3) of Busé et al. [2003] fails in any case).

The practical gain of our method for this example, however, is arguable at best — we have to compute a degree 10 syzygy which is already the degree of the implicit equation. On the other hand, a better choice for $\boldsymbol{d}$ might help. For $\boldsymbol{d} = 3$ we get a 10×10-matrix with $h = (4,4,1,0,1)$, and for $\boldsymbol{d} = 4$ we get a 15×16-matrix with $h = (11,4,1)$.

Example 4.9 (`ex309`)**.** Let N_1 be the matrix of linear columns for $\boldsymbol{d} = 1$ in Example 4.4. We have that $\det(N_1) = 0$. This shows that not all maximal minors need to be nonzero.

This has nothing to do with the fact that N_1 is special. For another example, let us take $\boldsymbol{d} = 4$ in Example 4.8. Then N is a 15×16-matrix whose columns correspond to the syzygies

$$\begin{cases} t^2u^2x_2 - u^4x_3, t^3ux_2 - tu^3x_3, st^2ux_2 - su^3x_3, t^4x_2 - t^2u^2x_3, \\ st^3x_2 - stu^2x_3, s^2t^2x_2 - s^2u^2x_3, su^3x_1 - t^3ux_3, stu^2x_1 - t^4x_3, \\ s^2u^2x_1 - st^3x_3, u^4x_0 - s^4x_2, t^2u^2x_0 - s^4x_3, s^2tux_1x_2 - st^2ux_3^2, \\ s^3ux_1x_2 - s^2tux_3^2, s^3tx_1x_2 - s^2t^2x_3^2, s^4x_1x_2 - s^3tx_3^2, tu^3x_0x_1x_2 - s^3ux_3^3 \end{cases}$$

Let M be the square submatrix of the first 14 columns and the last column of N, that is, leaving out the column corresponding to the syzygy $s^4x_1x_2 - s^3tx_3^2$. Then $\det(M) = 0$. The same is true for the square submatrix consisting of the first 15 columns but in that case $M = (N_1 \mid N_2)$ which we wanted to avoid.

Example 4.10 (`ex310`)**.** This example has been present elsewhere in the literature and is know to break all the available methods. Let $X = \mathbb{P}^2$ and take

$$
\begin{aligned}
J = \langle & -s^2t^3 + 3s^2t^2u + st^3u - 4st^2u^2 - stu^3 + 2t^2u^3 - tu^4 + u^5, \\
& s^2t^3 - 3s^2t^2u + st^3u + 3stu^3 - 2t^2u^3 + tu^4 - u^5, \\
& s^2t^3 - 3s^2t^2u - st^3u + 2s^2tu^2 + 4st^2u^2 - 3stu^3 - 2t^2u^3 + 3tu^4 - u^5, \\
& -s^2t^3 + 3s^2t^2u - st^3u - 3stu^3 + 3tu^4 - u^5 \rangle
\end{aligned}
$$

so ϕ is generically 1-1 with 3 basepoints of total degree 17 and multiplicity 20. More precisely, the basepoints are the ci point $(1,1,1)$ of multiplicity 4, the aci point $(0,1,0)$ of degree 4 and multiplicity 5, and the aci point $(1,0,0)$ of degree 9 and multiplicity 11.

Irrelevant of any of the hideous basepoints, we get

$$
N = \begin{bmatrix} x_0 + x_1 & 0 & x_1^2 - x_3^2 \\ -x_0 - x_2 & 3x_0x_1 - x_1^2 + 4x_1x_2 + 3x_0x_3 - 3x_1x_3 + 4x_2x_3 - 2x_3^2 & x_0^2 + 7x_0x_1 - 3x_1^2 + x_0x_2 + 10x_1x_2 + \\ -x_2 + x_3 & x_0^2 - x_0x_1 - x_1^2 - 3x_0x_3 - 3x_1x_3 - x_3^2 & -5x_0x_1 - 2x_1^2 - 3x_0x_2 - 6x_1x_2 - \end{bmatrix}
$$

and

$$\det(N) = P(\boldsymbol{x}) \quad \triangle$$

Example 4.11 (`ex311`)**.** Let ϕ be the rational map from Example 4.10. We compute the degree and multiplicity of the base locus Z.

Set $q_1 = (1,0,0)$, $q_2 = (0,1,0)$ and $q_3 = (1,1,1)$, so that set-theoretically $Z = \{q_1, q_2, q_3\}$. Since J is saturated, the sum of the degrees of the basepoints is just the degree of the base locus,

$$\deg(Z \subset \mathbb{P}^2) = \deg(J \subset \mathbb{C}[s,t,u]) = 17$$

Because we are in projective space, this can be checked directly in Macaulay2. However, since Z is supported on multiple points, we cannot compute the total multiplicity in the same way. Indeed, running `multiplicity`(J) gives 45, not the correct 20.

Let Q_k be the prime ideals corresponding to the points q_k, and set

$$J_k = J : (J : {Q_k}^{\infty})$$

The degree and multiplicity of q_k can be computed from the ideal J_k — those capture the local structure of Z near q_k. We have

$$\begin{cases} J_1 = \langle tu^2, t^3 - 3t^2u, u^5 \rangle \\ J_2 = \langle su, s^2, u^3 \rangle \\ J_3 = \langle t^2 - 2tu + u^2, s^2 - 2su + u^2 \rangle \end{cases}$$

so in particular, q_3 is c.i., while q_1 and q_2 are a.c.i. points.

Example 4.12 (`ex312`)**.** Let $Y = V(P)$ for an irreducible form $P(\boldsymbol{x})$ of degree 4 on $\mathbb{P}^3$. Suppose further that $\mathrm{Sing}(Y)$, the singular locus of Y, contains 3 concurrent non-degenerate lines (that is, passing trough a common point and spanning all of $\mathbb{P}^3$). Then $P(\boldsymbol{x})$ is the determinant of a square order-4 matrix M of linear forms.

We can prove this claim by a direct calculation. After a linear change of coordinates on $\mathbb{P}^3$, we can assume that the lines are the 3 coordinate axes in the distinguished $\{x_3 \neq 0\}$, that is, the lines are given by $V(x_0, x_1)$, $V(x_1, x_2)$ and $V(x_0, x_2)$.

On the level of ideals, using Euler's identity, the assumption translates to

$$\langle P_{x_0}(\boldsymbol{x}), \ldots, P_{x_3}(\boldsymbol{x}) \rangle \subset \langle x_0x_1, x_0x_2, x_1x_2 \rangle$$

so writing out $P(\boldsymbol{x}) = \sum_{|\alpha|=4} a_\alpha \boldsymbol{x}^\alpha$ and noting that each of the partials must be zero modulo $\langle x_0x_1, x_0x_2, x_1x_2 \rangle$, we only need to solve a linear system in the indeterminate coefficients. We get

$$P(\boldsymbol{x}) = a_1x_0^2x_1^2 + a_2x_0^2x_2^2 + a_3x_1^2x_2^2 + a_4x_0^2x_1x_2 + a_5x_0x_1^2x_2 + a_6x_0x_1x_2^2 + a_7x_0x_1x_2x_3$$

which is the determinant of

$$M = \begin{bmatrix} x_0 & & & x_1 \\ & x_1 & & x_2 \\ & & x_2 & x_0 \\ -a_3x_2 & -a_2x_0 & -a_1x_1 & a_4x_0+a_5x_1+a_6x_2+a_7x_3 \end{bmatrix}$$

In fact, if $a_7 \neq 0$ we can do better. Setting $a_7 = 1$, a linear change of coordinates by the matrix

$$\begin{bmatrix} 1 & & & -a_4 \\ & 1 & & -a_5 \\ & & 1 & -a_6 \\ & & & 1 \end{bmatrix}$$

leaves the singular locus the same but simplifies the general form to

$$P'(\boldsymbol{x}) = a_1x_0^2x_1^2 + a_2x_0^2x_2^2 + a_3x_1^2x_2^2 + x_0x_1x_2x_3$$

and the matrix M to

$$M' = \begin{bmatrix} x_0 & & & x_1 \\ & x_1 & & x_2 \\ & & x_2 & x_0 \\ -a_3x_2 & -a_2x_0 & -a_1x_1 & x_3 \end{bmatrix}$$

whose entries are scaled variables.

Example 4.13 (313)**.** It turns out, however, that this M is not a matrix of syzygies, in the sense that only 3 out of its 4 columns are syzygies.

The example with 3 syzygies from the B exam.

Example 4.14 (318)**.** Change to $\mathbb{P}^1 \times \mathbb{P}^1$ source.

Example 4.15 (319)**.** Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ and $J \subset S = \mathbb{C}[s,u;t,v]$ be given by 4 general $(2,2)$ forms in $\langle s^2, st, t^2\rangle$. Then ϕ is generically 1-1 with the unique basepoint $(0,1)\times(0,1)$ of degree 3 and multiplicity 4, and the image is singular along 3 non-degenerate concurrent lines.

Example 4.16 (`ex314`)**.** We apply Proposition 3.12 to compute the degree of the rational map in Example 4.1. We have

$$I = \left\langle \begin{cases} x_1t - x_2u, \\ x_0s - x_2u, \\ x_2s^2 - x_3st + x_2t^2 + x_2u^2, \\ x_1s^2 - x_3su + x_2tu + x_1u^2, \\ x_0t^2 + x_2su - x_3tu + x_0u^2, \\ x_1x_2s + x_0x_2t + (x_0x_1 - x_2x_3)u, \\ x_0^2x_1^2 + x_0^2x_2^2 + x_1^2x_2^2 - x_0x_1x_2x_3 \end{cases} \right\rangle$$

and we see $P(\boldsymbol{x})$ as the last generator. Next, we calculate a Gröbner basis with respect to a product monomial order where the s,t,u variables come before the x_j variables. Dropping the implicit equation, we get

$$\begin{cases} x_1t - x_2u, \\ x_0^2x_2t + (x_0^2x_1 + x_1x_2^2 - x_0x_2x_3)u, \\ x_0s - x_2u, \\ x_1x_2s + x_0x_2t + (x_0x_1 - x_2x_3)u, \\ x_0t^2 + x_2su - x_3tu + x_0u^2, \\ x_2s^2 - x_3st + x_2t^2 + x_2u^2, x_1s^2 - x_3su + x_2tu + x_1u^2 \end{cases}$$

Collecting the S-part of the leading terms, we get the ideal $\langle s,t\rangle$ whose degree is obviously 1.

We can carry out the same calculation for the map in Example 4.8. The Rees ideal and its Gröbner basis are as follow,

$$I = \left\langle \begin{cases} x_2t^2 - x_3u^2, \\ x_1x_2s - x_3^2t, \\ -x_3t^3 + x_1su^2, \\ -x_2s^4 + x_0u^4, \\ -x_3s^4 + x_0t^2u^2, \\ -x_1s^5 + x_0t^5, \\ -x_3^3s^3 + x_0x_1x_2tu^2, \\ -x_3^5s^2 + x_0x_1^2x_2^2u^2, \\ -x_3^8s + x_0x_1^3x_2^4t, \\ x_0x_1^4x_2^5 - x_3^{10} \end{cases} \right\rangle, \quad \begin{cases} x_1x_2s - x_3^2t, \\ x_3^8s - x_0x_1^3x_2^4t, \\ x_2t^2 - x_3u^2, \\ x_3^9t^2 - x_0x_1^4x_2^4u^2, \\ x_3^7st - x_0x_1^3x_2^3u^2, \\ x_3^5s^2 - x_0x_1^2x_2^2u^2, \\ x_3t^3 - x_1su^2, \\ x_3^3s^3 - x_0x_1x_2tu^2, \\ x_3^4s^2t^2 - x_0x_1^2x_2u^4, \\ x_3^2s^3t - x_0x_1u^4, \\ x_3s^4 - x_0t^2u^2, \\ x_2s^4 - x_0u^4, \\ x_1s^5 - x_0t^5 \end{cases}$$

This time the ideal of the S-part of the leading terms is $\langle s, t^2\rangle$ which is of degree 2.

Example 4.17 (`ex315`)**.** While Examples 4.1, 4.6 and 4.7 support Conjecture 3.9, we point out that the latter is stated in its strongest possible form. For one, the claim is trivial in the case $\deg(\phi) > 1$ for then the comaximal minors vanish on all of Y again. This can be seen in Example 4.8.

On the other hand, while it is tempting to conjecture that

$$\operatorname{rad}(\langle P_{x_j} : j\rangle) \subset \operatorname{sat}(\texttt{minors}(r-1, N))$$

that is, that the inclusion of the conjecture is on the level of schemes, this is not true as illustrated by Example 4.4. In the case $\boldsymbol{d} = 1$, we get

$$\begin{aligned}
\mathrm{rad}(\langle P_{x_j} : j\rangle) &= \langle x_3, x_1x_2, x_0x_2, x_0x_1\rangle \\
\mathrm{sat}(\texttt{minors}(r-1, N)) &= \langle x_3^2, x_2x_3, x_1x_3, x_2^2, x_1x_2, x_1^2\rangle \\
\mathrm{rad}(\texttt{minors}(r-1, N)) &= \langle x_1, x_2, x_3\rangle
\end{aligned}$$

and the only inclusion we have is

$$\mathrm{rad}(\langle P_{x_j} : j\rangle) \subset \mathrm{rad}(\texttt{minors}(r-1, N))$$

The results for other values of $\boldsymbol{d}$ are analogous.

Example 4.18 (316)**.** The construction of Lemma 5.7 is already apparent in Example **??**. Since $P \subset I$ in R, the bihomogeneous primes in $R/I = B$ which pull back to P are those in

$$(\mathbb{C}[x_0, x_1, x_2]/\langle x_0 + x_1 - x_2\rangle)[s, t]/\langle x_1s^2 - x_0t^2\rangle$$

which intersect the coefficient ring trivially, so in turn, those in

$$(\mathbb{C}[x_0, x_1, x_2] \setminus \langle x_0 + x_1 - x_2\rangle)^{-1}(\mathbb{C}[x_0, x_1, x_2]/\langle x_0 + x_1 - x_2\rangle)[s, t]/\langle x_1s^2 - x_0t^2\rangle$$

Example 4.19. It may seem at first that taking the smallest possible degree is always our best option, but Example 4.8 provides a good counterexample. The examples in Chapter 6, which are computationally more complex, often have the first few N_i being zero, for low $\boldsymbol{d}$, so nonzero matrices tend to be larger in size and degree of their minors.

It is the possibility to choose a good $\boldsymbol{d}$ that is one of the contributions of our results to the methods of moving hypersurfaces.

CHAPTER 5

PROOFS OF THE MAIN RESULTS

While the ultimate goal of this chapter is to prove the results of Chapter 3, it is written in a way to help develop intuition about the interplay between representation matrices on one hand and the geometry of Y on the other.

For this reason we start with a few elementary results with two-fold purpose. Firstly, they put together a some easy facts about our matrices N. Secondly, they highlight, when compared to other ad-hoc proofs, the advantage of our point of view.

We follow the notation of Chapter 2 and adopt the setup of (3.1). In particular, we work over a fix degree $\boldsymbol{d}$ with $S_{\boldsymbol{d}} \neq 0$, and do not yet require that ϕ be generically finite.

Lemma 5.1. *Let M be a square $r \times r$ matrix of syzygies. Then*

$$\det(M) \in P$$

In particular, if M is any (not necessarily square) matrix of syzygies, then

$$\mathtt{minors}(r, M) \subset P$$

Proof. The second statement clearly follows from the first, setting an empty minor to zero.

Let $\operatorname{adj}(M)$ be adjugate matrix and set $\boldsymbol{b} = \mathtt{basis}(S_{\boldsymbol{d}})$. Then

$$(\boldsymbol{b} \cdot M) \cdot \operatorname{adj}(M) = \boldsymbol{b} \cdot \det(M) \boldsymbol{1}_r = \det(M) \boldsymbol{b}$$

Since $\boldsymbol{b} \cdot M$ is a row vector of syzygies, the LHS vanishes identically in S under the substitution $x_0 = \phi_0(\boldsymbol{s}), \ldots, x_n = \phi_n(\boldsymbol{s})$. But then

$$\mathrm{RHS}|_{x_0=\phi_0,\ldots,x_n=\phi_n} = \det(M)(\phi_0, \ldots, \phi_n) \boldsymbol{b} = \begin{bmatrix} 0 & 0 & \ldots & 0 \end{bmatrix}$$

over the domain S. It follows that $\det(M)(\phi_0,\dots,\phi_n)=0$, so that $\det(M)$ is in the kernel P of $\phi^\#$, proving the first statement. □

Lemma 5.2. *Any representation matrix N has at least as many columns as rows, i.e.*

$$\mu \geq r$$

and its ideal of maximal minors is nonzero.

Proof. For each standard basis column vector $e_k \in \mathbb{C}^r$, $P(\boldsymbol{x})e_k$ is a graded syzygy. Let F be the $\mu \times r$ matrix of coefficients for getting $P(\boldsymbol{x})e_k$ out of the generators of the syzygies over $S_{\boldsymbol{d}}$, that is

$$N \cdot F = P(\boldsymbol{x})\boldsymbol{1}_r$$

The sizes of the matrices on the LHS are $r \times \mu$ and $\mu \times r$. Since the rank of the RHS as a T-matrix is r, we must have $\mu \geq r$.

The maximal minors are then of size $r \times r$. Since $\operatorname{rank}(N \cdot F) = r$, also $\operatorname{rank}(N) = r$ and so not all maximal minors vanish. □

Lemma 5.3. *There is an isomorphism of graded T-modules*

$$\operatorname{coker} N \cong B_{\boldsymbol{d},\bullet}$$

In particular, if $\mathscr{C}_\bullet$ is any graded resolution of $\operatorname{coker}(N)$ *over T, then $H_0\mathscr{C}_\bullet = B_{\boldsymbol{d},\bullet}$.*

Proof. This is obvious. The sequence

$$\bigoplus_k T(-i_k) \xrightarrow{N} T^r \xrightarrow{\boldsymbol{b}\,\cdot} B_{\boldsymbol{d},\bullet} \longrightarrow 0 \tag{5.3.1}$$

is exact by definition, proving the claim. □

Lemma 5.4. *One has*

$$\operatorname{ann}_T(B_{\boldsymbol{d},\bullet}) = P$$

Proof. Identifying $T = 1 \otimes T = R_{0,\bullet}$, we can think of T as a subring of R. Since R is a graded domain, we have

$$\mathrm{ann}_T(B_{\boldsymbol{d},\bullet}) = T \cap I$$

By the definition of I, any form $Q(\boldsymbol{x}) \in T$ with $Q(\boldsymbol{x}) \in I$ is the the kernel of the ring map $\phi^{\#}$. It follows that

$$T \cap I = P$$

completing the proof. □

Remark. The lemma above shows that $\mathrm{Supp}_T(B_{\boldsymbol{d},\bullet}) = V(P)$ and there is a cool way to see that the T-module localization $(B_{\boldsymbol{d},\bullet})_P$ is nonzero. Let N' be the localization of N at P. Then

$$(B_{\boldsymbol{d},\bullet})_P = \mathrm{coker}(N)_P = \mathrm{coker}(N')$$

Since $\mathrm{Fitt}_0 \, \mathrm{coker}(N') = \mathtt{minors}(r, N') \subset PT_P \neq T_P$ by Lemma 5.1, the cokernel is nonzero, for example, by (Eisenbud [1995], Proposition 20.6).

For a geometric argument, see the proof of Lemma 5.7.

Proof of Proposition 3.3. Since I is the kernel of the ring map into a domain, I is prime. By Lemma 5.4,

$$P = T \cap I \subset R$$

so $B = R/I$ is naturally a finite-type S-graded T/P-algebra. The T-module localization of B at P is just the localization of the ring R/I at the multiplicative set of homogeneous elements in $(T - P)$. In particular, the localization B_P remains a domain, now as a $K(T/P)$-algebra. This proves parts (a) and (b).

Since I is prime in R, I is saturated with respect to the irrelevant ideal $\mathfrak{n} \subset S \subset R$. Now (c) follows because saturation commutes with localization. □

Proof of Theorem 3.5. By Lemma 5.3, Lemma 5.4 and (Eisenbud [1995], Proposition 20.7), we have

$$\operatorname{rad}(\mathtt{minors}(r,N)) = \operatorname{rad}(\operatorname{Fitt}_0(\operatorname{coker} N)) = \operatorname{rad}(\operatorname{ann}(B_{\boldsymbol{d},\bullet})) = P \qquad \square$$

From now on, we assume that ϕ is generically finite, or equivalently, that P is principal.

Lemma 5.5. *Let $\mathscr{C}_\bullet$ be a finite graded free resolution of* $\operatorname{coker} N$*. One has*

$$\operatorname{div}(\det(\mathscr{C}_\bullet)) = \operatorname{length}_{T_P}(B_{\boldsymbol{d},\bullet})_P \cdot [Y]$$

as Weil divisors on $\mathbb{P}^n$.

Proof. By (Gelfand et al. [1994], A, Theorem 30), applied to the factorial T, a principal prime $Q = \langle Q(\boldsymbol{x}) \rangle$, and the generically exact $\mathscr{C}_\bullet$,

$$\operatorname{ord}_{Q(\boldsymbol{x})}(\det(\mathscr{C}_\bullet)) = \sum_i (-1)^i \operatorname{mult}_Q(H_i \mathscr{C}_\bullet)$$

Since $\mathscr{C}_\bullet$ is exact, all the higher homology vanishes, so

$$\operatorname{ord}_{Q(\boldsymbol{x})}(\det(\mathscr{C}_\bullet)) = \operatorname{mult}_Q(H_0 \mathscr{C}_\bullet)$$

The RHS above is zero outside $\operatorname{ann}(H_0 \mathscr{C}_\bullet)$, so by Lemma 5.3 and 5.4, the RHS is nonzero only for $Q = P$. Summing over all non-associate irreducible homogeneous polynomials, we get

$$\begin{aligned}
\operatorname{div}(\det(\mathscr{C}_\bullet)) &= \sum_{Q(\boldsymbol{x})} \operatorname{ord}_{Q(\boldsymbol{x})}(\det(\mathscr{C}_\bullet)) \cdot [V(Q)] \\
&= \sum_{Q(\boldsymbol{x})} \operatorname{mult}_Q(H_0 \mathscr{C}_\bullet) \cdot [V(Q)] \\
&= \operatorname{mult}_P(H_0 \mathscr{C}_\bullet) \cdot [Y] \\
&= \operatorname{length}_{T_P}(B_{\boldsymbol{d},\bullet})_P \cdot [Y]
\end{aligned}$$

establishing the claim. $\square$

Lemma 5.6. *One has*

$$\operatorname{length}_{T_P}(B_{\boldsymbol{d},\bullet})_P = \dim_{K(T/P)}(B_{\boldsymbol{d},\bullet})_P$$

Proof. Note that $P(\boldsymbol{x}) \in I$, so $P(\boldsymbol{x})$ annihilates B as a T-module. Setting $\mathfrak{m}_P = PT_P \subset T_P$,

$$\begin{aligned}\operatorname{length}_{T_P}(B_{\boldsymbol{d},\bullet})_P &= \sum_k \dim_{T_P/\mathfrak{m}_P} \mathfrak{m}_P^k(B_{\boldsymbol{d},\bullet})_P/\mathfrak{m}_P^{k+1}(B_{\boldsymbol{d},\bullet})_P \\ &= \dim_{T_P/\mathfrak{m}_P}(B_{\boldsymbol{d},\bullet})_P\end{aligned}$$

since $\mathfrak{m}_P$ in turn annihilates $(B_{\boldsymbol{d},\bullet})_P$. □

Remark. A one-line argument would be: the T- and T/P-module structure of B are the same.

Lemma 5.7. *Let* $\mathbf{d} \in \operatorname{reg}(I_P)$ *in the sense of* (3.4)*. One has*

$$\dim_{K(T/P)}(B_{d,\bullet})_P = \deg(\phi)$$

Remark. This result requires neither $\operatorname{codim}(Y \subset \mathbb{P}^n) = 1$ nor $\dim V(J) = 0$.

Proof. Let $\Gamma = \operatorname{Biproj}(B)$ be the graph of the rational map ϕ, or equivalently the blow-up of X along the basepoints $V(J)$. Let $E \subset \Gamma$ be the exceptional locus. We point the reader to (Hartshorne [1977], II, Example 7.17.3) for the details, summarized in the following commutative diagram

$$\begin{tikzcd}
\operatorname{Biproj}(B) = \Gamma \arrow[d, "\pi_1"] \arrow[rrrd, bend left, two heads, "\pi_2"] & \Gamma - E \arrow[l, hook'] \arrow[d, "\pi_1|_{\Gamma-E}"', "\cong"] \arrow[rrd, two heads, "\pi_2|_{\Gamma-E}"] & & \\
X \arrow[rrr, bend right, dashed, "\phi"] & X - V(J) \arrow[l, hook'] \arrow[rr, two heads, "\phi|_{X-VJ}"] & & Y \subset \mathbb{P}^n
\end{tikzcd}$$

Since $\pi_2|_{\Gamma - E} = \phi|_{X - V(J)} \circ \pi_1|_{\Gamma - E}$ and $\pi_1|_{\Gamma - E}$ is an isomorphism, the morphism π_2 is generically finite onto its image and $\deg(\phi) = \deg(\pi_2)$. If $\gamma \in \mathbb{P}^n$ is the generic point of Y, then the scheme-theoretic fiber

$$\pi_2^{-1}(\gamma) = \operatorname{Spec}(\mathscr{O}_{\gamma,Y}) \times_Y \Gamma$$

is a closed zero-dimensional subscheme of Γ consisting of $\deg(\pi_2)$ points, counted with multiplicity.

The morphism $\pi_2 : \operatorname{Biproj}(B) \to \operatorname{Proj}(T)$ is induced by the graded map of $\mathbb{C}$-algebras

$$\pi_2^\sharp = (x_j \mapsto \overline{x_j \otimes 1}) : T \longrightarrow (T \otimes S)/I = B$$

and the fiber of $\gamma = [P] \in \operatorname{Proj}(T)$ corresponds to the set of bihomogeneous prime ideals of B which pull back to $P \subset T$ via $\pi_2^\sharp$. By an easy reduction, for example (Vakil [2013], Exercise 7.3.H, 7.3.K and 9.3.A), this set corresponds to the set of S-grading-homogeneous prime ideals of the $K(T/P)$-algebra

$$K(T/P) \otimes_{T/P} B \cong B_P$$

The identification above presents the fiber as a $\operatorname{Proj}(-)$ over a field, that is,

$$\operatorname{Proj}_{K(T/P)} B_P \xrightarrow{\sim} \pi_2^{-1}(\gamma) \subset \operatorname{Biproj}(B)$$

Since this is a finite projective scheme over the field $K(T/P)$, its degree is well-defined and given by

$$\dim_{K(T/P)} (B_P)_{\boldsymbol{d}} = \dim_{K(T/P)} (B_{\boldsymbol{d},\bullet})_P$$

for all $\boldsymbol{d} \in \operatorname{reg}(I_P)$. $\square$

Proof of Theorem 3.6. Let $\mathscr{C}_\bullet$ be a minimal graded free resolution of coker$N$. By (Gelfand et al. [1994], A, Theorem 34) which applies since $\mathscr{C}_\bullet$ is exact,

$$\det(\mathscr{C}_\bullet) = \gcd(\texttt{minors}(r,N))$$

up to a unit of T. But then by Lemma 5.5, 5.6 and 5.7,

$$\begin{aligned} \operatorname{div}(\gcd(\texttt{minors}(r,N))) &= \operatorname{div}(\det(\mathscr{C}_\bullet)) \\ &= \operatorname{length}_{T_P}(B_{d,\bullet})_P \cdot [Y] \\ &= \dim_{K(T/P)}(B_{d,\bullet})_P \cdot [Y] \\ &= \deg(\phi) \cdot [Y] \end{aligned}$$

Because this is just an equality of Weil divisors,

$$\gcd(\texttt{minors}(r,N)) = P^{\deg(\phi)} \qquad \square$$

Proof of Corollary 3.7. This follows directly from Theorem 3.6. $\square$

Proof of Corollary 3.8. $\square$

Proof of Proposition 3.12. For any choice of $>'$, a reduced Gröebner basis for I_B will have the outlined general form except possibly for the term g_{r+1}. Since $I \cap T = P$ by Lemma 5.4, a Gröbner basis must include $P(\boldsymbol{x})$ as its unique generator involving just the x_j. This proves the first part.

Define the ideal $\boldsymbol{I}$ by the identity

$$B_P = \frac{(\mathbb{C}[\boldsymbol{x}] - P)^{-1}\mathbb{C}[\boldsymbol{s};\boldsymbol{x}]}{I'_B}$$

By reduce-ness, $P(\boldsymbol{x}) \nmid p_k(\boldsymbol{x})$, so I'_B can be obtained from the generating set for I_B by removing $P(\boldsymbol{x})$.

By the proof of Lemma 5.7, we know that $\operatorname{Proj}_{K(T/P)} B_P$ corresponds to the scheme-theoretic preimage of the generic point of Y, and in turn, this gives

$$\deg(\phi) = H^0(\operatorname{Proj}_{K(T/P)} B_P, \mathscr{O}) = \deg(I'_B)$$

where $\mathscr{O}$ denotes the structure sheaf of $\operatorname{Proj}_{K(T/P)} B_P$, and the degree on the far-right is the degree in $K(T/P)\mathbb{C}[\boldsymbol{s}]$. The result now follows because the initial ideal of I'_B has the same degree. $\square$

CHAPTER 6

A METHOD FOR FAST IMPLICITIZATION

This chapter is devoted to the computational aspects of our results. Section 6.1 describes two algorithms for implicitization. The first one is simple and robust, and is used for studying the matrices N when Gröbner basis calculations can be carried out efficiently. The second one is more involved and is used when direct computations are unfeasible. In those cases, the second algorithm's lead is significant. Section 6.3 provides details about the algorithms and support to the claim in the form of a few worked examples.

Both, as means to illustrate that our algorithms are effective, say, in the sense of computational algebraic geometry, and as a setup for the examples to follow, in Section 6.2 we implement those algorithms in the Macaulay2 system.

The code is available at

`http://www.math.cornell.edu/~rzlatev/phd-thesis`

We continue to follow the notation of Chapter 2 and the setup of (3.1). However, to avoid distraction, we assume throughout the chapter that ϕ is generically 1-1.

In the final section, we workout a few examples. Not only do those examples highlight the performance of the algorithms, they show how implicitization is no longer a black box but geometrically expired and driven.

6.1 The Algorithms

At first glance, an algorithm for finding the implicit equation is contained in the proof of the our main theorem, Theorem 3.6. In its simplest form, it becomes

Algorithm 6.1. NAIVE ALGORITHM.

input: J, $\boldsymbol{d}$

output: N, P

Set $r = \dim_{\mathbb{C}}(S_{\boldsymbol{d}})$

Compute an R-generating set $\{h_\ell : \ell\}$ for the Rees ideal I

Compute a T-generating set $\{g_k : k\}$ for $I_{\boldsymbol{d},\bullet}$ from the h_ℓ using (2.11)

Set N to be the coefficient matrix of the g_k with respect to $\texttt{basis}(S_{\boldsymbol{d}})$

Compute $P = \gcd(\texttt{minors}(r,N))$

Return N, P

6.2. The conciseness and robustness of Algorithm 6.1 made it our preferred tool for testing the theory. In fact, all calculations presented so far, including all examples of Chapter 4, were carried out using this algorithm.

At the same time, its simplicity allows us to spot some of its drawbacks. We distinguish four major ones.

(1) Computing an R-generating set for the Rees ideal is at least as hard as computing the implicit equation itself — we have $I_{0,\bullet} = P$. This follows from Proposition 3.12 and shows up in Examples 2.28 and 4.16.

(2) While computing the gcd of two polynomials is fast, computing all minors could be difficult since their number can be very large. This happens even for reasonably small examples. For instance, the smallest nonzero matrix N for $\boldsymbol{d} = (2,2,1)$ in Example 6.15 is of size 18×50. The number of maximal minors is

$$\binom{50}{18} = 18'053'528'883'775$$

so even if it took the unrealistic 0.001 seconds to compute each minor, a single machine would require 572 years to compute them all.

(3) Continuing with Example 6.15, we note that each maximal minor is a determinant of an 18×18-matrix of quartic forms in 5 variables. Computing large determinants symbolically is time-consuming. We did not manage to compute any nonzero minor.

Example 6.13 involves a somewhat similar calculation — the determinant of a 12×12-matrix of quadratic forms in 5 variables took about an hour to compute. Extrapolating, we can speculate that our 18×18 determinant would take somewhere in the order of

$$13 \times 14 \times 15 \times 16 \times 17 \times 18 = 13'366'080$$

hours. That is about 1525 years.

(4) Finally, suppose we have found the polynomial in question — by whatever means. It is a form of degree 48 in 5 variables, and very likely dense in the monomials of that degree. This suggests that the polynomial will be represented by

$$\binom{53}{5} = 2'869'685$$

coefficients.

Regrettably, (6.2.3) would be an issue for any algorithm relying on computing determinants of representation matrices, while (6.2.4) would be an issue for any implicitization algorithm whatsoever. Rather than seeing these as obstacles, we point them out as an argument *for* the idea of using representation matrices in place of the implicit equation altogether. We explore this theme further in the examples of Section 6.3.

6.3. Fix a degree $\boldsymbol{d}$ as before and recall that

$$N = (\ N_1 \mid \cdots \mid N_\delta\ )$$

Consider the following.

(1) Instead of computing the whole matrix N, one can compute the N_i's separately, keeping track of a partial representation matrix N'.

(2) Instead of computing all the minors, one only needs to compute sufficiently many to determine the gcd correctly.

These two simple observations produce an immense speed up on average. The advantage of (6.3.1) over computing an R-generating set for the Rees ideal is that it uses only linear algebraic routines. The advantage of computing only sufficiently many, rather than all, of the minors is obvious.

6.4. Let

$$N' = N_i' = (\ N_1 \mid \cdots \mid N_i\ )$$

be the partial matrix of syzygies up to degree i. If N' satisfies the condition

$$\deg(\gcd(\mathtt{minors}(r, N'))) = \deg(P) \tag{C1}$$

then $\gcd(\mathtt{minors}(r, N')) = P$.

This is true for any collection of matrices of syzygies M_k, as in (5.1). Indeed, $P(\boldsymbol{x})$ divides each $\det(M_k)$, so $P(\boldsymbol{x})$ divides their gcd, but both forms have the same degree by assumption.

In particular, this shows that if $\deg(\det(M)) < \deg(P)$, then M must be singular.

6.5. Let M_1, M_2 be matrices of syzygies, as in (5.1), and let $Y_1, Y_2 \subset \mathbb{P}^n$ be the hypersurfaces they define. Let L be a general line in $\mathbb{P}^n$. If M_1, M_2 satisfy the condition

$$L \cap Y_1 \cap Y_2 \subset L \cap Y \tag{C2}$$

then $\gcd(\det(M_1), \det(M_2)) = P$.

The condition can be used for testing (6.3.2). To prove the claim, note that we always have

$$Y_1 \cap Y_2 \supset V(\gcd(\det(M_1), \det(M_2))) \supset Y$$

which together with (C2) gives $L \cap Y_1 \cap Y_2 = L \cap Y$, which is true exactly when $Y_1 \cap Y_2$ does not contain any other hypersurfaces besides Y.

Indeed, if the gcd is a proper multiple of P, then the intersection of Y_1 and Y_2 contains another hypersurface, whose intersection with the general L is not going to be on Y.

Summarizing the discussion so far, we propose

Algorithm 6.6. PROPOSED ALGORITHM.

input: J, $\boldsymbol{d}$, $p = \deg(P)$

output: a list of matrices M_k such that $\gcd(\{M_k : k\}) = P$

Set $r = \dim_{\mathbb{C}}(S_{\boldsymbol{d}})$

Set $N' = r \times 0$ matrix over T

while ?? is not satisfied for N' **do**

Given $N_1, \ldots, N_{i-1}$, use Algorithm 6.7 to compute N_i

Set $N' = N' \mid N_i$

end while

Report $P = \gcd(\texttt{minors}(r, N'))$

Algorithm 6.7. COMPUTE PARTIAL SYZYGIES.

input: a list of the already computed $N_1, \ldots, N_{i-1}$

output: N_i

for $0 < j < i$ **do**

Set $N_{ji} = \texttt{basis}(T_{i-j}) \otimes N_j$

Set K_{ji} to be the linearization of N_{ji}

end for

Set $K_i = \ker(\Phi^{(i)})$

Let K_i' be such that $\mathrm{Span}(K_i) = \mathrm{Span}(K_i') \oplus (\sum_j \mathrm{Span}(K_{ji}))$

Let N_i be such that $\mathtt{basis}(R_{\boldsymbol{d},i}) \cdot K_i' = \mathtt{basis}(S_{\boldsymbol{d}}) \cdot N_i$

Return N_i

6.2 Implementation in Macaulay2

6.8. We start with a realization for (2.11).

```
PushGens = (d,I) -> (
  r := toList(0..(#d-1));
  G := for g in I_* list (
    if all(d-((degree g)_r), Z->Z>=0)
    then basis((d-((degree g)_r))|{0},ring I)**g
    else continue);
  trim image fold(G,matrix {{0_(ring I)}},(a, b)->a|b)
  )
```

6.9. Using (6.8), Algorithm 6.1 is straight-forward to implement. We require R for encapsulation.

```
ComputeNRees = method ()
ComputeNRees (Ideal, List, Ring) := Matrix => (J, d, R) -> (
  x := symbol x;
  I := reesIdeal(J, Variable=>x);
  AI := ring I;
  zm := 0*d;
  g := map(R,AI,first entries super basis(zm|{1},R));
  I = g(I);
  V := PushGens (d,I);
  matrix entries ( (gens V) // basis(d|{0}, R) )
  )
```

6.10. Algorithm 6.7 is the one we make most use of in Section 6.3. Its fourth argument is the list of already computed matrices $N_1, \ldots, N_{i-1}$. If this list's size is not $i-1$, then we just compute all linearly independent syzygies of degree i — $\mathtt{basis}(I_{\boldsymbol{d},i})$. The ideal J is supplied in the form of the matrix $F = \phi^{(1)} \otimes R$ (see 2.19).

```
ComputeNi = method ()
ComputeNi (Matrix, ZZ, List, List) := Matrix => (F, i, d, lst) -> (
  R := ring F;
  m := #d;
  d0 := d|{0};
  di := d|{i};
  zm := 0*d;
  fj := flatten entries matrix F;
  xj := flatten entries super basis (zm|{1}, R);
  n := #fj;
  r := numcols super basis(d0, R);
  subs := apply(n,j->xj_j=>fj_j);
  e0 := (degree fj_0);
  G := sub(super basis(di, R), subs) // (super basis(i*e0+d0, R));
  K := matrix entries gens ker G;
  Nii := (super basis(di, R))*K // (super basis(d0, R));
  Nii = sub(matrix entries Nii,R);
  Nji := random(R^r,R^0);
  if #lst==i-1
  then Nji = fold (for j from 1 to #lst list
    (super basis(zm|{i-j}, R)**(lst_(j-1))), random(R^r,R^0), (m1,m2)->m1|m2);
  gens trim image (Nii%Nji)
  )
```

6.11. This only implement the first part of the proposed Algorithm 6.6. We omit the minors speed-up simply because of (6.2.3). Further, after general change of coordinates, two minors ought to suffice, and we can compute them manually when necessary. See Examples 6.15 and 6.14 for details.

```
ComputeNConj = method ()
ComputeNConj (Matrix, ZZ, List) := Matrix => (F, p, d) -> (
  R := ring F;
  q := 0;
  i := 1;
  lst := {};
  r := numcols super basis (d|{0},R);
  N := random(R^r,R^0);
  zm := 0*d;
  fj := flatten entries matrix F;
  xj := flatten entries super basis (zm|{1}, R);
  n := #fj;
  subs := apply(n,j->xj_j=>fj_j);
  while q<p or not procTestCondN(N, subs) do (
    Ni := ComputeNi(F,i,d,lst);
    q = q + (numcols Ni)*i;
    lst = append(lst, Ni);
    N = N|Ni;
    i = i+1; );
  N
  )
```

6.3 Examples

We now field-test our algorithms and code on several examples of somewhat higher computational complexity than those in Chapter 4.

The running times can vary a lot from one machine to another, so the numbers below should not be treated as benchmarks. We include them only to provide a general idea how the different methods preform relative to each other.

The machine that we used was a MacBook Pro laptop with a 2.9 GHz Intel Core i7 processor and 8 GB 1600 MHz DDR3 memory, running Macaulay2 version 1.8.

Example 6.12 (ex601)**.** Let $\phi : (\mathbb{P}^1)^3 \longrightarrow \mathbb{P}^4$ be given by 5 generic $(2,1,1)$-forms. The base locus is empty, so by (2.25.1) and (2.24.1), the degree of the image is 12.

Consider $\boldsymbol{d} = (1,1,1)$. Our method computes a candidate matrix N' in a little more than 0.1s. The matrix N' is square and we have $\det(N') = P$. Computing the determinant of the 8×8-matrix N takes 3s. The standard Gröbner basis computation takes 131s.

The details follow.

```
i1 : loadPackage "ImplicitizationAlgos";
i2 : KK=ZZ/32009;
i3 : S=KK[s_0,s_1,t_0,t_1,u_0,u_1,
          Degrees=>{2:{1,0,0},2:{0,1,0},2:{0,0,1}}];
i4 : T=KK[x_0..x_4];
i5 : B=super basis({2,1,1},S);
                1       12
o5 : Matrix S  <--- S
i6 : J=ideal(B*random(S^12,S^5));
o6 : Ideal of S
i7 : R=KK[s_0,s_1,t_0,t_1,u_0,u_1,x_0..x_4,
          Degrees=>{2:{1,0,0,0},2:{0,1,0,0},2:{0,0,1,0},5:{0,0,0,1}}];
i8 : F=sub(gens J,R);
                1       5
o8 : Matrix R  <--- R
i9 : d={1,1,1};
```

```
i10 : time N1=ComputeNi(F,1,d,{});
     -- used 0.0119639 seconds

               8       4
o10 : Matrix R  <--- R

i11 : time N2=ComputeNi(F,2,d,{N1});
     -- used 0.0920771 seconds

               8       4
o11 : Matrix R  <--- R
```

The partial matrix $N' = N_1|N_2$ is square. The degree of its determinant is $12 = 4\cdot 1 + 4\cdot 2$, so this is our candidate matrix. We check at the end that $\det(N') = P(\boldsymbol{x})$.

```
i12 : time ComputeNi(F,2,d,{});
     -- used 0.0867573 seconds

               8       24
o12 : Matrix R  <--- R

i13 : time N3=ComputeNi(F,3,d,{N1,N2});
     -- used 0.846328 seconds

               8
o13 : Matrix R  <--- 0

i14 : time ComputeNi(F,3,d,{});
     -- used 0.903612 seconds

               8       80
o14 : Matrix R  <--- R

i15 : time ComputeNi(F,4,d,{});
     -- used 6.38165 seconds

               8       200
o15 : Matrix R  <--- R
```

Note that even though N' is square and giving the implicit equation right off the bat, we cannot be sure that $N = N_1|N_2$. Lines 12–15 support this claim. For instance, because the columns of N_1 and N_2 form a nonsingular matrix, the degree-3 syzygies they give rise to are going to be linearly independent. There are exactly $80 = 4\cdot\binom{4+2}{2} + 4\cdot\binom{4+1}{1}$ of them, so $N_3 = 0$. The same argument shows that $N_4 = 0$.

```
i16 : N=sub(N1|N2,T);

               8       8
o16 : Matrix T  <--- T

i17 : time N'=ComputeNConj(F,12,d);
     -- used 0.138608 seconds

               8       8
o17 : Matrix R  <--- R
```

```
i18 : N'=sub(N',T);
                8       8
o18 : Matrix T  <--- T
i19 : image N==image N'
o19 = true
i20 : time P'=ideal det N;
     -- used 3.0447 seconds
o20 : Ideal of T
```

Line 20 shows that $\det(N') = P(\boldsymbol{x})$.

```
i21 : time P=ker map(S,T,J_*);
     -- used 131.05 seconds
o21 : Ideal of T
i22 : P==P'
o22 = true
```

The standard method to compute the implicit equation takes more than 40 times longer.

Example 6.13 (`ex602`)**.** We compute the implicit equation of five general $(2,2,1)$-forms over $(\mathbb{P}^1)^3$. The base locus is empty, so the degree of the equation is 24. It takes our method less than a second to find it in the form of a determinant of an 18×18-matrix of quadratic forms. It take a little less than an hour to compute the actual equation. The standard Gröbner basis calculation did not finish in 24 hours.

```
i1 : loadPackage "ImplicitizationAlgos";
i2 : KK=ZZ/32009;
i3 : S=KK[s_0,s_1,t_0,t_1,u_0,u_1,
          Degrees=>{2:{1,0,0},2:{0,1,0},2:{0,0,1}}];
i4 : T=KK[x_0..x_4];
i5 : B=super basis({2,2,1},S);
              1       18
o5 : Matrix S  <--- S
i6 : J=ideal(B*random(S^18,S^5));
o6 : Ideal of S
i7 : R=KK[s_0,s_1,t_0,t_1,u_0,u_1,x_0..x_4,
          Degrees=>{2:{1,0,0,0},2:{0,1,0,0},2:{0,0,1,0},5:{0,0,0,1}}];
i8 : F=sub(gens J,R);
              1       5
o8 : Matrix R  <--- R
i9 : d={2,1,1};
```

```
i10 : time N1=ComputeNi(F,1,d,{});
     -- used 0.017498 seconds

                12
o10 : Matrix R   <--- 0

i11 : time N2=ComputeNi(F,2,d,{});
     -- used 0.298143 seconds

                12       12
o11 : Matrix R   <--- R

i12 : time N3=ComputeNi(F,3,d,{N1,N2});
     -- used 4.01669 seconds

                12
o12 : Matrix R   <--- 0

i13 : time N'=ComputeNConj(F,24,d);
     -- used 0.762765 seconds

                12       12
o13 : Matrix R   <--- R

i14 : N'==N2

o14 = true

i15 : rank N2

o15 = 12
```

Since we know that $N' = N_2$ is square, its determinant is a form of degree $12 \cdot 2 = 24$, so $\det(N') = P(\boldsymbol{x})$. Note that we do not have to compute the determinant to make sure that N' is nonsingular. We can compute the rank by substituting random numbers for the x_j and computing over a finite field.

```
i17 : N=sub(N2,T);

                12       12
o17 : Matrix T   <--- T

i18 : time P'=det N;
     -- used 3310.29 seconds
```

While we actually can compute the determinant, so also the implicit equation itself, we could have answered questions like "Is this point on the image" using N' only.

Example 6.14 (ex604)**.** Consider a map given by four general bi-quartics in the ideal $\langle s^3, s^2t, t^2 \rangle$. Then the base locus is the point $(0,0,1)$ of degree 5 and multiplicity 6. The degree of the image is 26. Our method finds it in explicit form in about 6 minutes, while it takes us less than a minute to find two square matrices M_1, M_2 for which

$$\gcd(\det(M_1), \det(M_2)) = P(\boldsymbol{x})$$

This highlights the interplay between the algebra and geometry — one of our initial goals. The standard method takes more than 6 hours.

```
i1 : loadPackage "ImplicitizationAlgos";
i2 : KK=ZZ/32009;
i3 : S=KK[s,u,t,v,Degrees=>{2:{1,0},2:{0,1}}];
i4 : T=KK[x_0..x_3];
i5 : Z=ideal(s^3,s^2*t,t^2);
o5 : Ideal of S
i6 : B=super basis({4,4},Z);
              1       20
o6 : Matrix S  <--- S
i7 : J=ideal(B*random(S^20,S^4));
o7 : Ideal of S
i8 : decompose J
o8 = {ideal (s, t), ideal (s, u), ideal (v, t)}
o8 : List
i9 : multiplicity Z
o9 = 6
```

The base locus is supported on a single point, so the Macaulay2 command computes the correct quantity.

```
i10 : R=KK[s,u,t,v,x_0..x_3,
          Degrees=>{2:{1,0,0},2:{0,1,0},4:{0,0,1}}];
i11 : F=sub(gens J,R);
               1       4
o11 : Matrix R  <--- R
i12 : d={2,2};
i13 : time N1=ComputeNi(F,1,d,{});
     -- used 0.00821136 seconds
               9
o13 : Matrix R  <--- 0
i14 : time N2=ComputeNi(F,2,d,{N1});
     -- used 0.0901831 seconds
               9
o14 : Matrix R  <--- 0
```

The degree $\boldsymbol{d} = (2,2)$ doesn't look promising. We choose $\boldsymbol{d} = (3,3)$ for no specific reason.

```
i15 : d={3,3};

i16 : time N1=ComputeNi(F,1,d,{});
     -- used 0.0208163 seconds

                 16       5
o16 : Matrix R   <--- R

i17 : time N2=ComputeNi(F,2,d,{N1});
     -- used 0.221774 seconds

                 16       12
o17 : Matrix R   <--- R

i18 : N=N1|N2;

                 16       17
o18 : Matrix R   <--- R

i19 : time rank N
     -- used 0.009011 seconds

o19 = 16
```

This is our candidate matrix N' — it has more columns than rows and any nonzero minor would have degree 27 or 28. Having N' be of full rank is not enough by itself.

```
i20 : c1=sort join({0,1,2,3,4}, RandPerm(5,16,11));

i21 : time rank N_c1
     -- used 0.00241363 seconds

o21 = 16

i22 : c2=sort join({0,1,2,3,4}, RandPerm(5,16,11));

i23 : c1==c2

o23 = false

i24 : time rank N_c2
     -- used 0.00881294 seconds

o24 = 16
```

At this point we found two distinct length-12 sets of columns, c_1, c_2, such that the square matrices M_1, M_2 they give rise to are nonsingular. We can show that these satisfy the equality described above. However, we can actually compute this.

```
i25 : time D1=det N_c1;
     -- used 164.479 seconds

i26 : time D2=det N_c2;
     -- used 162.898 seconds

i27 : time P=gcd(D1,D2);
     -- used 0.0646495 seconds
```

Example 6.15 (`ex603`)**.** We enhance Example 6.13 and compute the implicit equation of five general $(2,2,2)$-forms on $(\mathbb{P}^1)^3$. The base locus is again empty, so the degree of the image is 48.

In this setup there is no hope of computing any determinant explicitly, let alone computing $P(\boldsymbol{x})$ using Gröbner bases directly. We can, however, repeat what we did in Example 6.14. We work over $\boldsymbol{d} = (2,2,1)$ and find two square matrices M_1, M_2 of order 18 such that

$$\gcd(\det(M_1), \det(M_2)) = P(\boldsymbol{x})$$

While testing if the matrices are nonsingular is easy, checking whether their determinants do not have any other prime common factor is much harder. We follow the discussion in (6.5) and find that the common intersection locus of two hypersurfaces with a general line has multiplicity 48.

```
i1 : loadPackage "ImplicitizationAlgos";
i2 : KK=ZZ/32009;
i3 : S=KK[s_0,s_1,t_0,t_1,u_0,u_1,Degrees=>{2:{1,0,0},2:{0,1,0},2:{0,0,1}}];
i4 : T=KK[x_0..x_4];
i5 : B=super basis({2,2,2},S);
             1       27
o5 : Matrix S  <--- S
i6 : J=ideal(B*random(S^27,S^5));
o6 : Ideal of S
i7 : R=KK[s_0,s_1,t_0,t_1,u_0,u_1,x_0..x_4,
         Degrees=>{2:{1,0,0,0},2:{0,1,0,0},2:{0,0,1,0},5:{0,0,0,1}}];
i8 : F=sub(gens J,R);
             1       5
o8 : Matrix R  <--- R
i9 : d={1,1,1};
i10 : N1=ComputeNi(F,1,d,{});
              8
o10 : Matrix R  <--- 0
i11 : N2=ComputeNi(F,2,d,{});
              8
o11 : Matrix R  <--- 0
i12 : N3=ComputeNi(F,2,d,{});
              8
o12 : Matrix R  <--- 0
```

This is only half the story. One might expect to find that N_6 is square, so that $\deg(\det(N_6)) = 48$. In fact $N_i = 0$ for up to $i = 8$. We work over $\boldsymbol{d} = (2,2,1)$.

```
i13 : d = {2,2,1};

i14 : time N4=ComputeNi(F,4,d,{});
     -- used 151.652 seconds

              18       50
o14 : Matrix R   <--- R

i15 : time rank N4
     -- used 2.2373 seconds

o15 = 18

i16 : c1=RandPerm(0,49,18);

i17 : M1=N4_c1;

              18       18
o17 : Matrix R   <--- R

i18 : rank M1

o18 = 18

i19 : c2=RandPerm(0,49,18);

i20 : M2=N4_c2;

              18       18
o20 : Matrix R   <--- R

i21 : rank M2

o21 = 18

i22 : sort c1==sort c2

o22 = false
```

The two matrices corresponding the selected (distinct) sets of columns are nonsingular.

```
i23 : M1=sub(M1,T);

              18       18
o23 : Matrix T   <--- T

i24 : M2=sub(M2,T);

              18       18
o24 : Matrix T   <--- T

i25 : L=ideal random(T^{1,1,1},T^1);

o25 : Ideal of T
```

Here L or rather $V(L)$ is the generic line.

```
i26 : TL=T/L;

i27 : super basis(1,TL)

o27 = | x_3 x_4 |

               1        2
o27 : Matrix TL  <--- TL

i28 : U=KK[y_0,y_1];

i29 : g=map(U,TL,{0,0,0,y_0,y_1});

o29 : RingMap U <--- TL

i30 : M1'=g(sub(M1,TL));
```

```
                18       18
o30 : Matrix U   <--- U
i31 : M2'=g(sub(M2,TL));
                18       18
o31 : Matrix U   <--- U
```

This last bit is for computational purposes only. Since we are on a $\mathbb{P}^1$, we should use this and work over 2 variables only.

```
i32 : time D1=det(M1');
     -- used 0.317001 seconds
i33 : time D2=det(M2');
     -- used 0.334676 seconds
i34 : degree gcd(D1,D2)
o34 = {48}
o34 : List
```

The degree of the common intersection is 48! We are done.

CHAPTER 7

KOSZUL SYZYGIES AND BASEPOINT-FREE MAPS

7.1 Template Proofs

7.1. Most of the proofs of the "linear and quadratic syzygies"-type results consist, in one form or another, of the following steps

(1) consider specific X and set some conditions on the base locus

(2) choose $\boldsymbol{d}$ as a fixed function on $\boldsymbol{e}$

(3) come up with a way to construct a square matrix by selecting a number of linear and quaratic syzygies

(4) show that the determinant is in P or $P^{\deg \phi}$

(5) show that the determinant is nonzero

Examples of this approach are [give a long list of refereces].

The conditions on the baselocus in (1) and (2) are dictated by making the numerics in (3) workout. Step (4) is often automatic, for example when map is assumed birational, while step (5) usually involves a cleverly chosen normal form for the syzygies, allowing the authors to assert that the determinant is nonzero.

Clearly, the above pattern is just a statement about the matrix N in a specific degree $\boldsymbol{d}$. By Theorem **??**, the h-vector of N depends only on the base locus of ϕ, but getting a hold of this relation is difficult. An example of this is the long lists of conditions in [the paper with B1-B6] in case the basepoints (over $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$) are complete intersection points.

In the two simplest cases, however, of basepoint-free maps over $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$, there are very few extra conditions, and it is desirable to know if our matrices match the ones described elsewhere. We show that this is indeed the case in Sections 7.3 and 7.4. Because of the simplicity of the baselocus, or rather the absence there of, our proofs are streamlined by the fact that syzygies of low degree exhibit a property akin to Koszulness.

7.2 Koszul Syzygies in Low Degree

Proposition 7.2. *Let X be as in Section* **??**. *Let $\phi_0, \ldots, \phi_n$ be n forms in degree $\boldsymbol{p}$ having no common zero on X, and let $\boldsymbol{q}$ be any degree such that*

$$H^1(X, \mathscr{O}_X(\boldsymbol{q} - (n-1)\boldsymbol{p})) = 0$$

Then for any $k \geq 1$ the syzygies of

$$M_0 = \begin{bmatrix} \phi_0^k & \phi_0^{k-1}\phi_1 & \ldots & \phi_n^k \end{bmatrix}$$

in degree $\boldsymbol{q}$ are Koszul.

Proof. Let M_ℓ, $\ell = 1, \ldots, n-1$, be matrices resolving M_0 when the ϕ_j are thought of as variables. Writing $\star$ for any positive integer which is not essential for the proof, we get a complex of coherent sheaves of modules on X,

$$\mathscr{O}_X^1 \xleftarrow{\widetilde{M}_0} \mathscr{O}_X(-k\boldsymbol{p})^\star \xleftarrow{\widetilde{M}_1} \mathscr{O}_X(-(k+1)\boldsymbol{p})^\star \xleftarrow{\widetilde{M}_2} \cdots \xleftarrow{\widetilde{M}_{n-1}} \mathscr{O}_X(-(k+n-1)\boldsymbol{p})^\star \longleftarrow 0$$

and tensoring with the locally free $\mathscr{O}_X(k\boldsymbol{p}+\boldsymbol{q})^1$, we get the following complex $\mathscr{K}_\bullet$

$$\mathscr{O}_X^1(k\boldsymbol{p}+\boldsymbol{q}) \xleftarrow{\widetilde{M}_0} \mathscr{O}_X(\boldsymbol{q})^\star \xleftarrow{\widetilde{M}_1} \mathscr{O}_X(\boldsymbol{q}-\boldsymbol{p})^\star \xleftarrow{\widetilde{M}_2} \cdots \xleftarrow{\widetilde{M}_{n-1}} \mathscr{O}_X(\boldsymbol{q}-(n-1)\boldsymbol{p})^\star \longleftarrow 0$$

Let U be a distinguished affine open in X. Restring $\mathscr{K}_\bullet$ to U is equivalent to first restricting the ϕ_j to U and then constructing the M_ℓ. But since the restrictions ϕ_j have no common zero on U, and $U \cong \mathbb{A}^{n-1}$, the n restrictions form a regular sequence there. It follows that the restriction of $\mathscr{K}_\bullet$ to U is acyclic. Since X has is covered by such opens U, $\mathscr{K}_\bullet$ itself must be acyclic.

By the last paragraph, $\mathscr{K}_\bullet$ is exact, so we can split it into two exact sequences, as the commutative diagram below shows,

$$\begin{tikzcd}
\mathscr{O}^1_X(k\boldsymbol{p}+\boldsymbol{q}) & \mathscr{O}_X(\boldsymbol{q})^\star \arrow[l, "\widetilde{M}_0"'] & & \mathscr{O}_X(\boldsymbol{q}-\boldsymbol{p})^\star \arrow[ll, dashed, "\widetilde{M}_1"'] \arrow[dl, "\pi"] & \ldots \arrow[l, "\widetilde{M}_2"'] \\
& & \mathrm{Im}(\widetilde{M}_1) \arrow[ul, "\iota"'] \arrow[dl] & & \\
& 0 & & 0 \arrow[ul] &
\end{tikzcd}$$

Consider the long exact sequences on cohomology of the exact sequences of sheaves above. For the one involving ι, the releavant part is

$$\ldots \xleftarrow{H^0\widetilde{M}_0} H^0\mathscr{O}_X(\boldsymbol{q})^\star \xleftarrow{H^0\iota} H^0\mathscr{O}_X(\boldsymbol{q})^\star \longleftarrow 0$$

while for the other one, we look at

$$\ldots \xleftarrow{H^1\widetilde{M}_2} H^1\mathscr{O}_X(\boldsymbol{q}-(n-1)\boldsymbol{p})^\star \xleftarrow{\delta} H^0\,\mathrm{Im}(\widetilde{M}_1)^\star \xleftarrow{H^0\pi} H^0\mathscr{O}_X(\boldsymbol{q}-\boldsymbol{p})^\star \xleftarrow{H^0\widetilde{M}_2} \ldots$$

Finally, let $\boldsymbol{v} \in S_{\boldsymbol{q}}^{\binom{k+n-1}{n-1}} = H^0\mathscr{O}_X(\boldsymbol{q})^\star$ be a syzygy on M_0, and recall that $H^0\widetilde{M}_\ell = M_\ell$ and $\iota \circ \pi = \widetilde{M}_1$. Since $M_0\boldsymbol{v} = 0$, then $\boldsymbol{v} = H^0\iota(\boldsymbol{w})$ for some $\boldsymbol{w} \in \mathrm{Im}(\widetilde{M}_1)$. Since $H^1\mathscr{O}_X(\boldsymbol{q}-(n-1)\boldsymbol{p}) = 0$, so is $H^1\mathscr{O}_X(\boldsymbol{q}-(n-1)\boldsymbol{p})^\star$, and it follows that $H^0\pi$ is surjective. In particular, $\boldsymbol{w} = H^0\pi(\boldsymbol{u})$ for some $u \in H^0\mathscr{O}_X(\boldsymbol{q}-\boldsymbol{p})^\star$. But then

$$\boldsymbol{w} = H^0\iota \circ H^0\pi(\boldsymbol{u}) = H^0\widetilde{M}_1(\boldsymbol{u}) = M_1\boldsymbol{u}$$

establishing the claim. $\square$

Remark. Note that, in the last paragraph of the proof, we cannot use $\mathrm{Im}(\widetilde{M}_1) = \widetilde{\mathrm{Im}(M_1)}$ since X isn't affine—this is the whole point of the proof.

Example 7.3. In the case $n = 3$ and $k = 2$, the matrices M_ℓ can be taken as

$$M_0 = \begin{bmatrix} \phi_0^2 & \phi_0\phi_1 & \phi_1^2 & \phi_0\phi_2 & \phi_1\phi_2 & \phi_2^2 \end{bmatrix}$$

$$M_1 = \begin{bmatrix} -\phi_1 & 0 & -\phi_2 & 0 & 0 & 0 & 0 & 0 \\ \phi_0 & -\phi_1 & 0 & -\phi_2 & 0 & 0 & 0 & 0 \\ 0 & \phi_0 & 0 & 0 & 0 & -\phi_2 & 0 & 0 \\ 0 & 0 & \phi_0 & \phi_1 & -\phi_1 & 0 & -\phi_2 & 0 \\ 0 & 0 & 0 & 0 & \phi_0 & \phi_1 & 0 & -\phi_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & \phi_0 & \phi_1 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} \phi_2 & 0 & 0 \\ 0 & \phi_2 & 0 \\ -\phi_1 & 0 & 0 \\ \phi_0 & -\phi_1 & 0 \\ 0 & -\phi_1 & \phi_2 \\ 0 & \phi_0 & 0 \\ 0 & 0 & -\phi_1 \\ 0 & 0 & \phi_0 \end{bmatrix}$$

Note that the size of M_1 is $\binom{n-1+k}{n-1} \times$, and the size of M_{n-1} is $\binom{n-2+k}{n-1} \times$

Question 7.4. What is the size of M_ℓ in general? This should come from a cellular resolution of the ideal of all monomials in n variables in degree k, right? Also, it's important to say that this is a linear type resolution—all the maps are of degree 1, except for M_0, obviously.

Corollary 7.5. *Let $X = \mathbb{P}^{n-1}$ be the projective $(n-1)$-space or $X = \mathbb{P}^1 \times \cdots \mathbb{P}^1$ be the product of $n-1$ copies of $\mathbb{P}^1$, and let $\phi : X \to \mathbb{P}^n$ be basepoint-free. Let the coordinates of ϕ be in degree $\mathbf{p}$ and suppose that there are as few linear syzygies as possible. Then,* *after a linear change of coordinates on the target, $\phi_0, \ldots, \phi_{n-1}$ satisfy the assumption of Proposition* **??**.

Proof. work out $\square$

7.6 (Template Proof)**.**

7.3 Basepoint-Free Maps over $X = \mathbb{P}^1 \times \mathbb{P}^1$

In this section and next section we workout the relation between our method and the ones described in **?**, whenever the latter can be applied. In particular, we first show that the matrices which our method constructs are the same, up to reformatting, as the matrices constructed by the referenced methods, and then show how our streamlined arguments apply to a more general situation.

Theorem 7.7. *Let $X = \mathbb{P}^1 \times \mathbb{P}^1$, ϕ be basepoint-free whose entries are of degree (p,q), and suppose there are no linear syzyges for $d = (p-1, q-1)$. Then N_2 is square, $N = N_2$ and ϕ is birational. In particular,*

$$\det(N) = \det(N_2) = P(\boldsymbol{x})$$

Remark 7.8. Theorem 7.7 implies that if the map ϕ is basepoint-free but not birational, then there must exist linear syzygies in degree $(p-1, q-1)$.

7.9. Let k be a positive integer. We set the matrix $\Phi^{(k)}$,

$$\Phi^{(k)} : S_{p-1,q-1}^{\binom{k+3}{3}} \longrightarrow S^1_{(k+1)p-1,(k+1)q-1}$$

to be a $\mathbb{C}$-linearization of the S-linear map

$$\phi^{(k)} = \begin{bmatrix} \phi_0^k & \phi_0^{k-1}\phi_1 & \cdots & \phi_3^k \end{bmatrix} : S^{\binom{k+3}{3}} \longrightarrow S^1$$

Specifically, $\Phi^{(k)}$ is of size $(k+1)^2pq \times \binom{k+3}{3}pq$, and so

$$\dim_{\mathbb{C}} \ker(\Phi^{(k)}) \geq \binom{k+1}{3}pq \tag{7.9.1}$$

with equality if and only if $\Phi^{(k)}$ is of maximal rank, that is, of rank $(k+1)^2pq$.

7.10. The columns of $\Phi^{(k)}$ are indexed by the monomials in $R_{p-1,q-1,k}$ and the elements of the kernel give rise to linear combinationations of those which are exactly the syzygies of degree k for the fixed $d = (p-1, q-1)$. We can write any such syzygy as

$$\sum_{|\alpha|=k} A_\alpha \cdot x_0^{\alpha_0} x_1^{\alpha_1} x_2^{\alpha_2} + \Big(\sum_{|j|=k-1} B_j \cdot x_0^{j_0} x_1^{j_1} x_2^{j_2} \Big) x_3 + \Big(\sum_{|\gamma|=k-2} C_\gamma x^\gamma \Big) x_3^2 \tag{7.10.1}$$

in which the first two summations corresponds to number of columns indexed by monomials involving x_3 at most linearly. This number is $\binom{k+2}{2}pq + \binom{k+1}{2}pq = (k+1)^2pq$, which is the number of rows too. Set $\Psi^{(k)}$ to be the square submatrix of $\Phi^{(k)}$ formed by those columns.

Note that if the former is nonsingular, then the latter is of maximal rank. The proof only depends on $\Psi^{(k)}$ being of maximal rank though, not the fact that it is square, so we procede in this manner.

We begin by proving that the kernel elements of the S-linear version of $\Phi^{(k)}$ in degree $(2p-1, 2q-1)$ are Koszul, i.e. are $S_{p-1,q-1}$-linear combinations of the Koszul syzygies on the ϕ_j appropriately inflated to match $\Phi^{(k)}$.

While the proof is pretty much the same, as in **?**, we include it here fitted for our notation and generalized slightly.

Lemma 7.11. *The matrix $\Psi^{(k)}$ is of maximal rank for every positive k.*

Proof. The case $k = 1$ is just the assumption that the ϕ_j there are no linear syzygies in degree $(p-1, q-1)$ on the source, i.e. if

$$a_0\phi_0 + a_1\phi_1 + a_2\phi_2 + a_3\phi_3 = 0$$

in S where $a_j \in S_{p-1,q-1}$, then necessarily $a_j = 0$.

The case $k > 1$ is a direct consequence of the case $k = 1$ and the fact that ϕ_0, ϕ_1, ϕ_2 form a regular sequence. Let V,

$$V = \begin{bmatrix} V' \\ V'' \end{bmatrix}$$

be a kernel element of $\Psi^{(k)}$ with V' a column-$\binom{k+2}{2}$-vector over $\mathbb{C}$ and V'' is a column-$\binom{k+1}{2}$-vector over $\mathbb{C}$, corresponding to the coefficients not involving x_3 and the coefficients involving x_3, respectively. We can multiply out to get a k-syzygy on the ϕ_j over $S_{p-1,q-1}$, i.e.

$$\sum_{|\alpha|=k} A_\alpha \cdot \phi_0^{\alpha_0} \phi_1^{\alpha_1} \phi_2^{\alpha_2} + \Big(\sum_{|j|=k-1} B_j \cdot \phi_0^{j_0} \phi_1^{j_1} \phi_2^{j_2} \Big) \phi_3 = 0$$

as an equality in S, where the coefficients A_α, B_j are in $S_{p-1,q-1}$. We rewrite the above as

$$\sum_{|j|=k-1} \big(C_{0,j}\phi_0 + C_{1,j}\phi_1 + C_{2,j}\phi_2 + C_{3,j}\phi_3 \big) \phi_0^{j_0} \phi_1^{j_1} \phi_2^{j_2} = 0$$

which is again an equality in S, and again the coefficients $C_{j,j}$ are in $S_{p-1,q-1}$. Note also that $C_{3,j} = B_j$.

Since ϕ_0, ϕ_1, ϕ_2 form a regular sequence over S, we can rewrite the coefficients as linear combinations of their Koszul syzygies, i.e. for each j we get an equality in S of the form

$$C_{0,j}\phi_0 + C_{1,j}\phi_1 + C_{2,j}\phi_2 + C_{3,j}\phi_3 = D_{0,j}\phi_0 + D_{1,j}\phi_1 + D_{2,j}\phi_2$$

where D_0, D_1, D_2 are in $S_{p-1,q-1}$. By the assumption on the independence of the ϕ_j over $S_{p-1,q-1}$, we must have that $C_{3,j} = 0$. Equivalently, $B_j = 0$ for every j, so $V'' = 0$.

To conclude the proof, note further that any nontrivial syzygy of the monomials in ϕ_0, ϕ_1, ϕ_2 in any degree $k \geq 1$ must be of degree degree at least (p,q), i.e. must be a combination of Koszul syzygies again. Since $\deg(A_\alpha) = (p-1, q-1)$, it follows that $V' = 0$ too. $\square$

7.12. Paragraph **??** and Lemma 7.11 show that there are exactly $\binom{k+1}{3}pq$ linearly independent syzygies of degree k, so pq quadratic syzygies. Clearly degree-$(k-2)$ T-combinations of quadratic syzygies introduce degree-k syzygies. Next lemma shows that all syzygies of degree $k > 2$ arise in this way.

Lemma 7.13. *Any degree $k > 2$ syzygy of degree $d = (p-1, q-1)$ on the source is a degree-$(k-2)$ T-combination of quadratic syzygyes.*

Proof. Since $\Psi^{(k)}$ is square and of maximal rank by Lemma 7.11, it is nonsingular, so $\Phi^{(k)}$ is of maximal rank, $(k+1)^3 pq$. This means that there are pq linearly independent quadratic syzygies and by the same lemma any one of them must involve x_3^2 nontrivially. It follows that, up to scaling, any nonzero quadratic syzygy is of the form

$$g_\nu(\boldsymbol{s}, \boldsymbol{x}) = \ldots + \boldsymbol{s}^\nu x_3$$

where the $\boldsymbol{s}^\nu$ are the monomials of degree $(p-1, q-1)$ on the source.

We can rewrite an arbitrary degree-$(k > 2)$ syzygy as

$$g(\boldsymbol{s}, \boldsymbol{x}) = \sum_{|\alpha|=k} A'_\alpha \cdot x_0^{\alpha_0} x_1^{\alpha_1} x_2^{\alpha_2} + \big(\sum_{|j|=k-1} B'_j \cdot x_0^{j_0} x_1^{j_1} x_2^{j_2}\big) x_3 + \big(\sum_\nu \boldsymbol{s}^\nu f_\nu\big) x_3^2$$

where each f_ν is a form degree $k-2$ in T, so the syzygy

$$g(\boldsymbol{s}, \boldsymbol{x}) - \sum_\nu f_\nu \cdot g_\nu(\boldsymbol{s}, \boldsymbol{x})$$

has trivial x_3^k part in the sense of the discussion above. By Lemma 7.11 again, this syzygy must be zero, so every syzygy of degree k is a T-combination of quadratic syzygyes. $\square$

Proof of Theorem 7.7. Since there are no syzygies in degree 1 and all syzygyes in degree $k > 2$ are T-combinations of quadratic syzygyes by Lemma 7.13, we have $N = N_2$.

Since the number of linearly independent syzygies of degree 2 is pq, the matrix N_2 is square.

To finish the prove we note that necessarily $\operatorname{reg}(B_P) \leq (p-1, q-1)$ since the ideal of points I_P is generated in degree (p,q) over $K(T/P)$, so Theorem 3.6 applies.

Finally, birationallity follows by comparing the degree on both hand-sides of Theorem 3.6,

$$\deg(Y)\deg(\phi) = \deg\det(N) = \deg\det(N_2) = 2pq$$

and $\deg(Y) = 2pq$ since the parametrization is basepoint-free, so $\deg\phi = 1$. □

7.14. In the more general setting when X is an $(n-1)$-fold product of $\mathbb{P}^1$s and the coordinates of ϕ be of $(n-2)$-degree $(p_1, \ldots, p_{n-1})$. Now the sizes of $\Phi^{(k)}$ and $\Psi^{(k)}$ are

$$(k+1)^{n-1} p_1 \cdots p_{n-1} \times \binom{k+n}{n} p_1 \cdots p_{n-1}$$

and

$$(k+1)^{n-1} p_1 \cdots p_{n-1} \times (\binom{k+n-1}{n-1} + \binom{k-1+n-1}{n-1}) p_1 \cdots p_{n-1}$$

respectively. If $\phi_0, \ldots, \phi_{n-1}$ form a regular sequence, then the arguments of Lemma 7.11 apply verbatim, and $\Psi^{(k)}$ is of maximal rank, although not square for $n > 3$.

The arguments of Lemma 7.13 apply also, as long as the number of linearly independent quadratic syzygies is the number of monomials of degree $(p_1 - 1, \ldots, p_{n-1} - 1)$ on the source, $p_1 \cdots p_{n-1}$. Since this is just the difference between the number of columns of $\Phi^{(2)}$ and $\Psi^{(2)}$,

$$(\binom{n}{2} - \binom{n+1}{2} - \binom{n}{1}) p_1 \cdots p_{n-1}$$

the following result follows.

The problem to extend the result is however the fact that $\Psi^{(k)}$ is no longer square, more precisely, the number of columns is less than the number of rows for $n > 3$, so we cannot infer that the ranks of $\Psi^{(k)}$ and $\Phi^{(k)}$ are the same. The latter was necessary to find at least (and so exactly) $\dim_{\mathbb{C}}(S_{p_1-1,\ldots,p_{n-1}-1})$ many quadratic syzygyes, which in turn is integral for the proof of Lemma 7.13.

The following example shows this situation does occur.

Example 7.15. ... Then there are no quadratic syzygies.

Example 7.16. ... is there an example for which the matrix is square?

7.17. At any rate, the issue with the number of quadratic syzygies is the only problem to extending Theorem 7.7 to any product of $\mathbb{P}^1$s or to any degree-k syzygies, so by adding the latter as an assumption, we have the following provisional extension.

Remark 7.18. One can hope that something similar would work in higher dimension. For example, let $X = (\mathbb{P}^1)^n$ and $\mathscr{L} = \mathscr{O}(p_1, \ldots, p_n)$. Assuming that there are no degree-$(n!-1)$ syzygies over $S_{p_1-1,\ldots,p_n-1}$ we may expect to find $p_1 \cdots p_n$ syzygies of degree $n!$ which to conviniently organize in an $p_1 \cdots p_n$-square matrix. The determinant of this matrix would be homogeneous of degree $n! \cdot p_1 \cdots p_n$ which is exactly the degree of the image $Y \subset \mathbb{P}^{n+1}$ in the absense of basepoints. While one would still need to argue that $N_k = 0$ for $k > n!$, this is the gist of what happend in the case $n = 2$.

However, already the case $n = 3$ and $\boldsymbol{p} = (2,2,2)$ shows otherwise. We get no sextic syzygies and, in fact, degree-k syzygies over $S_{1,1,1}$ for $k \leq 8$. The latter calculation is outlined in (`ex101`).

Theorem 7.19. *Let $X = \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ be an $(n-1)$-fold product and ϕ be basepoint-free with coordinates in degree $(p_1, \ldots, p_{n-1})$ such that there are no degree-$(m-1)$ syzygies in degree $d = (p_1 - 1, \ldots, p_{n-1} - 1)$ on the source and such that the number of linearly independent degree-m syzygies is $p_1 \cdots p_{n-1}$. Then $N = N_m$ is square and*

$$\det(N) = P^{\deg(\phi)}$$

7.4 Basepoint-Free Maps over $X = \mathbb{P}^2$

Lemma 7.20. *For every $k > 0$, the number of linearly independent degree-k syzygies over S_d, $d = p-1$, is at least*

$$\frac{k(k+1)p}{12}(kp-p+k+5)$$

Proof. As before, we consider the S-linear map

$$\begin{bmatrix} \phi_0^k & \phi_0^{k-1}\phi_1 & \cdots & \phi_3^k \end{bmatrix} : S^{\binom{k+3}{3}} \longrightarrow S^1$$

and construct its $\mathbb{C}$-linearization, exactly as we did for $\Phi^{(k)}$ in Section **??**. It is a map of complex vector spaces

$$S_{p-1}^{\binom{k+3}{3}} \longrightarrow S^1_{(k+1)p-1}$$

so a matrix of size $\binom{kp+p+1}{2} \times \binom{k+3}{3}\binom{p+1}{2}$. Since there are at least as many columns as rows, the dimension of the kernel is at least

$$\binom{k+3}{3}\binom{p+1}{2} - \binom{kp+p+1}{2} = \frac{k(k+1)p}{12}(k+5+kp-p)$$

establishing the claim. □

Remark 7.21. By Lemma 7.20 there are at least p linear syzygies and at least $p(p+7)/2$ quadratic syzygies.

The main result of this section states that the matrices which our method constructs are the same, up to formatting, as those constructed by the method of **?**.

Theorem 7.22. *Let $X = \mathbb{P}^2$, ϕ be basepoint-free whose entries are of degree p, and suppose there are exactly p linear syzyges for $d = p-1$, i.e. the minimal possible number. Then $N = N_1|N_2$, N is square and ϕ is birational. In particular,*

$$\det(N) = \det(N_1|N_2) = P(\boldsymbol{x})$$

Lemma 7.23. *Suppose that there are exactly p linearly independent linear syzygies, i.e. the minimal possible number. Then the* $4p$ *quadratic syzygies of the form* $L_i(\boldsymbol{s},\boldsymbol{x})x_j$ *for the p linear syzygies* L_i *are linearly independent. The number of linearly independent quadratic syzygies not emerging in this way is* $\binom{p}{2}$*. The vector subspace of* S_{p-1} *spanned by the coefficient of* x_3^2 *among all quadratic syzygies is all of* S_{p-1}*.*

Proof. Since ϕ_0, ϕ_1, ϕ_2 form a regular sequence in degree p, a nonzero linear syzygy over S_{p-1} must involve x_3 nontrivially. Let V_1 be the linear subspace of S_{p-1} spanned by the coefficient of x_3 among all linear syzygies, i.e.

$$V_1 = \text{Span}\{a_3 : a_0x_1 + \ldots + a_3x_3 \text{ is a linear syzygy}\}$$

(and note that the *span* keyword isn't necessary). Since there are exactly p of those, by the observation just made, $\dim_{\mathbb{C}} V_1 = p$.

Any linear syzygy $L(\boldsymbol{s},\boldsymbol{x})$ gives rise to a quadratic syzygy of the form $L(\boldsymbol{s},\boldsymbol{x})x_j$ for each j. Let $g(\boldsymbol{s},\boldsymbol{x})$ be a quadratic syzygies not arising in this way. We know that g must involve x_3^2 nontrivially, for else, it is of the form

$$\sum_{j=0,1,2} (B_{j,0}x_0 + B_{j,1}x_1 + B_{j,2}x_2 + B_{j,3}x_3)x_j$$

and since ϕ_0, ϕ_1, ϕ_2 form a regular sequence in degree p, by the Koszul theorem (e.g. **??**), we have

$$B_{j,0}x_0 + B_{j,1}x_1 + B_{j,2}x_2 + B_{j,3}x_3 = C_{j,0}x_0 + C_{j,1}x_1 + C_{j,2}x_2$$

In particular,

$$L_j(\boldsymbol{s},\boldsymbol{x}) = B_{j,0}x_0 + B_{j,1}x_1 + B_{j,2}x_2 + B_{j,3}x_3 - C_{j,0}x_0 - C_{j,1}x_1 - C_{j,2}x_2$$

is a linear syzygies on the ϕ_j. But then $g - \sum_j L_j x_j$ is a quadratic syzygy only involving x_0, x_1, x_2, and so must be 0, contradicting the assumption that g was not T-generated by linear syzygies.

Let V_2 be the linear subspace of S_{p-1} spanned by the coefficient of x_3^2 for quadratic sygyzygies g as in the previous paragraph. The same argument shows that we cannot have $V_1 \cap V_2 \neq 0$.

We finish the proof by an easy dimension count. The discussion so far gives us that the number of *new* quadratic syzygies is at most $\dim_{\mathbb{C}} V_2 \leq \binom{p+1}{2} - p$, so even if all the $4p$ quadratic syzygies generated by the linear syzygies and the new quadratic syzygies are linearly independent altogether, we get at most $4p + \binom{p+1}{2} - p = p(p+7)/2$-many of them. On the other hand, the number of linearly independent quadratic syzygies must be at least $p(p+7)/2$ by Remark 7.21.

It follows that there are $\binom{p+1}{2} - p = \binom{p}{2}$ *new* quadratic syzygies, which along with the $4p$ pushed linear syzygies are linearly independent altogether. Also, the linear span of the S_{p-1} coefficient of x_3^2 among the quadratic syzygies is $V_1 \oplus V_2 = S_{p-1}$. □

Lemma 7.24. *Let $g(\boldsymbol{s},\boldsymbol{x})$ be a syzygy of degree $k > 2$. Then g is a T-combination of linear and quadratic syzygies.*

Proof. The proof uses the same arguments as in the proof of Lemma 7.23. Since the S_{p-1}-coefficients of x_3^2 among the quadratic syzygies span S_{p-1}, we can find among them

$$g_\nu(\boldsymbol{s},\boldsymbol{x}) = (\text{terms involving } x_3 \text{ at most linearly}) + \boldsymbol{s}^\nu x_3^2$$

for all $|\nu| = p-1$. We can then rewrite

$$g(\boldsymbol{s},\boldsymbol{x}) = (\text{terms involving } x_3 \text{ at most linearly}) + (\sum_\nu \boldsymbol{s}^\nu \cdot h_\nu(\boldsymbol{x}))x_3^2$$

where $h_\nu(\boldsymbol{x})$ is a degree-$(k-2)$ homogeneous polynomial. But now

$$g - \sum_\nu h_\nu g_\nu = \sum_{|\beta|=k-1} (A_{\beta,0}x_0 + A_{\beta,1}x_1 + A_{\beta,2}x_2 + A_{\beta,3}x_3)x_0^{\beta_0}x_1^{\beta_1}x_2^{\beta_2}$$

where $A_{\beta,j}$ is a $(p-1)$-form in S. Lemma **??** and the same argument as in the proof of Lemma 7.23 show that the above must be T-generated by linear syzygies. The result now follows. $\square$

Proof of Theorem 7.22. By Lemma 7.24 we know that $N = N_1|N_2$ and since there are p linear and $\binom{p}{2}$ quadratic syzygies by Lemma 7.23, N is square. By Theorem 3.6,

$$\det(N) = \det(N_1|N_2) = P^{\deg\phi}$$

and comparing the degrees on both sides, we see that $\deg(\phi) = 1$. $\square$

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