Homework 3

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January 21, 2014

1 Construct the Wald, LM, and LR test for this purpose

First, here are two important results that we will need:

A $\sqrt{n}(\hat{\beta} - \beta^0) \xrightarrow{d} N(0, \Sigma^{-1})$, where $\Sigma^{-1} \equiv (G'S^{-1}G)^{-1}$

Proof:

Population Objective Function: $Q(\beta) = \bar{g}(\beta)'S^{-1}\bar{g}(\beta)$ s.t. $\bar{g}(\beta) = \mathbb{E}[g(Z, \beta)]$, and $S^{-1} = \mathbb{E}[g(Z, \beta)g(Z, \beta)']$ and $G \equiv \frac{\partial}{\partial \beta}\bar{g}_n(\beta)$

Sample Objective Function: $Q_n(\beta) = \bar{g}_n(\beta)'S_n^{-1}\bar{g}_n(\beta)$ s.t. $\bar{g}_n(\beta) = \frac{1}{n}\sum_{i=1}^{n} g(Z_i, \beta)$ (and in the same way S_n^{-1} is the sample counterpart to S^{-1})

$\hat{\beta}$ minimizes Q_n, taking the F.O.C. w.r.t. β:

$\frac{\partial}{\partial \beta}Q_n(\beta) = \bar{G}_n'(\hat{\beta}_n)S_n^{-1}\bar{g}_n(\hat{\beta}_n) = 0,$ where $\bar{G}_n \equiv \frac{\partial}{\partial \beta}\bar{g}_n(\hat{\beta}_n)$

Taking a mean value expansion of $\bar{g}_n(\hat{\beta}_n)$ yields:

$\bar{g}_n(\hat{\beta}_n) = \bar{g}_n(\beta^0) + \bar{G}_n(\bar{\beta})(\hat{\beta}_n - \beta^0) + o_p(1)$

Plugging this result into the preceding equation:

$\bar{G}_n'(\hat{\beta}_n)S_n^{-1}\bar{g}_n(\hat{\beta}_n) = \bar{G}_n'(\hat{\beta}_n)S_n^{-1}\bar{g}_n(\beta^0) + \bar{G}_n'(\hat{\beta}_n)S_n^{-1}\bar{G}_n(\bar{\beta})(\hat{\beta}_n - \beta^0) + \bar{G}_n'(o_p(1))$

Note: $\bar{G}_n(\hat{\beta}_n) \xrightarrow{p} G$, $\bar{G}_n(\bar{\beta}) \xrightarrow{p} G$, $S_n^{-1} \xrightarrow{p} S^{-1}$, and $\bar{G}_n'(o_p(1)) \xrightarrow{p} 0$ So,

$G'S^{-1}\bar{g}_n(\beta^0) + G'S^{-1}G(\hat{\beta}_n - \beta^0) = 0$ Rearranging gives:

$(\hat{\beta}_n - \beta^0) = -\left(G'S^{-1}G\right)^{-1} G'S^{-1}\bar{g}_n(\beta^0)$, multiplying by $\sqrt{n}$:

$\sqrt{n}(\hat{\beta}_n - \beta^0) = -\left(G'S^{-1}G\right)^{-1} G'S^{-1}\sqrt{n}\bar{g}_n(\beta^0)$, and

$\sqrt{n}\bar{g}_n(\beta^0) \xrightarrow{d} N(0, S)$ (CLT) $\therefore$

$\sqrt{n}(\hat{\beta}_n - \beta^0) \xrightarrow{d} N(0, (G'S^{-1}G)^{-1})$ (Slutsky's) $\because$

$(G'S^{-1}G)^{-1}G'S^{-1}SS^{-1}G(G'S^{-1}G)^{-1} =$

$(G'S^{-1}G)^{-1}G'S^{-1}G(G'S^{-1}G)^{-1} =$

$(G'S^{-1}G)^{-1} \equiv \Sigma^{-1}$ (as defined above) So,

$\sqrt{n}(\hat{\beta}_n - \beta^0) \xrightarrow{d} N(0, (G'S^{-1}G)^{-1}) = N(0, \Sigma^{-1}) \quad \square$

B $\sqrt{n}\frac{\partial}{\partial\beta}Q_n(\beta^0) \xrightarrow{d} N(0, \Sigma)$

Proof:

Plug β^0 into $\frac{\partial}{\partial\beta}Q_n(\beta)$:

$\frac{\partial}{\partial\beta}Q_n(\beta^0) = \bar{G}_n'(\beta^0)S_n^{-1}\bar{g}_n(\beta^0)$ multiply by $\sqrt{n}$:

$\sqrt{n}\frac{\partial}{\partial\beta}Q_n(\beta^0) = \bar{G}_n'(\beta^0)S_n^{-1}\sqrt{n}\bar{g}_n(\beta^0)$

Again, $\bar{G}_n(\hat{\beta}_n) \xrightarrow{p} G$, $S_n^{-1} \xrightarrow{p} S^{-1}$

Note: $\sqrt{n}\bar{g}_n(\beta^0) \xrightarrow{d} N(0, S)$ (CLT) $\therefore$

$\bar{G}_n'(\beta^0)S^{-1}\sqrt{n}\bar{g}_n(\beta^0) \xrightarrow{d} N(0, G'S^{-1}SS^{-1}G) = N(0, G'S^{-1}G) = N(0, \Sigma) \quad \square$

1.1 Wald

(a) under $H_0 \ : \ a(\beta^0) = 0,\ \text{Compare}\ \ a(\hat{\beta})\ \&\ \ a(\beta^0)$

$$
\begin{aligned}
\sqrt{n}a(\hat{\beta}) &= \sqrt{n}[a(\beta^0) + A(\bar{\beta})(\hat{\beta} - \beta^0)],\ \text{where}:\ A(\beta) = \frac{\partial a}{\partial \beta'}(\beta) \\
&= A(\bar{\beta})\sqrt{n}(\hat{\beta} - \beta^0),\ \text{since under}\ H_0:\ a(\beta^0) = 0 \\
&\text{Define}\ A_0 = A(\beta^0) \\
&= A_0\sqrt{n}(\hat{\beta} - \beta^0) + op(1),\ \text{by}\ \text{A}\ A_0\sqrt{n}(\hat{\beta} - \beta^0) \xrightarrow{d} N(0, A_0\Sigma^{-1}A_0') \\
&\text{Define}\ \Omega = A_0\Sigma^{-1}A_0'\ \text{and let}\ LL' = \Omega\ \text{be a Cholesky decompositon} \\
&\implies L^{-1}\sqrt{n}a(\hat{\beta}) \xrightarrow{d} N(0, I)
\end{aligned}
$$

This gives our Wald under the null:

$$
\begin{aligned}
W &\equiv \left(\hat{L}^{-1}\sqrt{n}a(\hat{\beta})\right)' \left(\hat{L}^{-1}\sqrt{n}a(\hat{\beta})\right) \\
&= na(\hat{\beta})'(\hat{L}^{-1})'\hat{L}^{-1}a(\hat{\beta}) \\
&= na(\hat{\beta})'\hat{\Omega}^{-1}a(\hat{\beta}) \to \chi^2_r
\end{aligned}
$$

(b) under $H_l:\ a(\beta^0) = \frac{\delta}{\sqrt{n}}$

$$
\begin{aligned}
\sqrt{n}a(\hat{\beta}) &= \sqrt{n}\left[a(\beta^0) + A(\bar{\beta})(\hat{\beta} - \beta^0)\right] \\
&= \sqrt{n}\left[\frac{\delta}{\sqrt{n}} + A(\bar{\beta})(\hat{\beta} - \beta^0)\right] \\
&= \delta + A(\bar{\beta})\sqrt{n}(\hat{\beta} - \beta^0) \\
&\text{A} \implies \sqrt{n}a(\hat{\beta}) \xrightarrow{d} N(\delta, \Omega) \\
&\&\ L^{-1}\sqrt{n}a(\hat{\beta}) \xrightarrow{d} N(L^{-1}\delta, I)
\end{aligned}
$$

This gives our Wald under the local alternative:

$$
\begin{aligned}
W &\equiv \left(\hat{L}^{-1}\sqrt{n}a(\hat{\beta})\right)' \left(\hat{L}^{-1}\sqrt{n}a(\hat{\beta})\right) \\
&= na(\hat{\beta})'(\hat{L}^{-1})'\hat{L}^{-1}a(\hat{\beta}) \\
&= na(\hat{\beta})'\hat{\Omega}^{-1}a(\hat{\beta}) \to \chi^2_r(\kappa)
\end{aligned}
$$

Where κ is a non-centrality parameter s.t.:

$$
\kappa = \left(L^{-1}\delta\right)'\left(L^{-1}\delta\right) = \delta'\left(L^{-1}\right)'L^{-1}\delta = \delta'\left(L'\right)^{-1}L^{-1}\delta = \delta'\left(LL'\right)^{-1}\delta = \delta'\Omega^{-1}\delta
$$

(c) under $H_f: \ a(\beta^0) \neq 0$

Let's call our choice of a fixed alternative δ s.t. $a(\beta^0) = \delta$

$$\begin{aligned}
\sqrt{n}a(\hat{\beta}) &= \sqrt{n}\left[a(\beta^0) + A(\bar{\beta})(\hat{\beta} - \beta^0)\right] \\
&= \sqrt{n}\left[\delta + A(\bar{\beta})(\hat{\beta} - \beta^0)\right] \\
&= \sqrt{n}\delta + \sqrt{n}A(\bar{\beta})(\hat{\beta} - \beta^0) \\
&\text{A} \implies \sqrt{n}a(\hat{\beta}) \xrightarrow{d} N(\sqrt{n}\delta, \Omega) \\
&\& \ L^{-1}\sqrt{n}a(\hat{\beta}) \xrightarrow{d} N(L^{-1}\sqrt{n}\delta, I)
\end{aligned}$$

This gives our Wald under the fixed alternative:

$$\begin{aligned}
W &\equiv \left(\hat{L}^{-1}\sqrt{n}a(\hat{\beta})\right)' \left(\hat{L}^{-1}\sqrt{n}a(\hat{\beta})\right) \\
&= na(\hat{\beta})'(\hat{L}^{-1})'\hat{L}^{-1}a(\hat{\beta}) \\
&= na(\hat{\beta})'\hat{\Omega}^{-1}a(\hat{\beta}) \to \chi_r^2(\kappa)
\end{aligned}$$

Where κ is a non-centrality parameter s.t.:

$$\begin{aligned}
\kappa &= \left(L^{-1}\sqrt{n}\delta\right)' \left(L^{-1}\sqrt{n}\delta\right) = \delta'\left(L^{-1}\right)'\sqrt{n}\sqrt{n}L^{-1}\delta = n\delta'\left(L^{-1}\right)' L^{-1}\delta = n\delta'\left(L'\right)^{-1} L^{-1}\delta \\
&= n\delta'\left(LL'\right)^{-1}\delta = n\delta'\Omega^{-1}\delta
\end{aligned}$$

1.2 LM

(a) under $H_0 \ : \ a(\beta^0) = 0$

$\tilde{\beta} = \ argmin \ Q_n(\beta) \ s.t. \ a(\tilde{\beta}) = 0$

$\implies \tilde{\beta} = \ argmin \, [Q_n(\beta) + a(\beta)'\gamma_n]$, where γ_n is a L.M. & $\mathcal{L} = Q_n(\beta) + a(\beta)'\gamma_n$

F.O.C:

$\frac{\partial \mathcal{L}}{\partial \beta} = \frac{\partial}{\partial \beta}Q_n(\tilde{\beta}) + A'(\tilde{\beta})\gamma_n = 0$

and

$\frac{\partial \mathcal{L}}{\partial \gamma} = a'(\tilde{\beta}) = 0 \implies \sqrt{n}a'(\tilde{\beta}) = 0$

Now, applying a mean value expansion to the second F.O.C. yields:

$\sqrt{n}a(\tilde{\beta}) = \sqrt{n}a(\beta^0) + A(\bar{\beta})\sqrt{n}(\tilde{\beta} - \beta^0) + o_p(1)$ and under the null $H_0 \ : \ a(\beta^0) = 0$, So:

$\sqrt{n}a(\tilde{\beta}) = A(\bar{\beta})\sqrt{n}(\tilde{\beta} - \beta^0) + o_p(1)$ call this 1

Now, applying a mean value expansion to the first term in the first F.O.C. yields:

$\frac{\partial}{\partial \beta}Q_n(\tilde{\beta}) = \frac{\partial}{\partial \beta}Q_n(\beta^0) + \frac{\partial^2}{\partial \beta \partial \beta'}Q(\bar{\beta})(\tilde{\beta} - \beta^0) + o_p(1)$

$\implies \sqrt{n}\frac{\partial}{\partial \beta}Q_n(\tilde{\beta}) = \sqrt{n}\frac{\partial}{\partial \beta}Q_n(\beta^0) + \frac{\partial^2}{\partial \beta \partial \beta'}Q(\bar{\beta})\sqrt{n}(\tilde{\beta} - \beta^0) + o_p(1)$

$= \sqrt{n}\frac{\partial}{\partial \beta}Q_n(\beta^0) - \Sigma\sqrt{n}(\tilde{\beta} - \beta^0) + o_p(1)$ call this 2

Note: $-\Sigma = \frac{\partial^2}{\partial \beta \partial \beta'}Q(\bar{\beta})$ under the assumption that the model is correctly specified

Now, $A(\tilde{\beta}) \xrightarrow{p} A_0$:

$A(\tilde{\beta})'\gamma_n = A_0'\sqrt{n}\gamma_n + \left(A'(\tilde{\beta}) - A_0\right)'\sqrt{n}\gamma_n = A_0'\sqrt{n}\gamma_n + o_p(1)$ call this 3

Plugging 1, 2 & 3 into the F.O.C.s gives:

$\Sigma\sqrt{n}(\tilde{\beta} - \beta^0) + A_0'\sqrt{n}\gamma_n = \sqrt{n}\frac{\partial}{\partial \beta}Q_n(\beta^0)$

and

$A_0\sqrt{n}(\tilde{\beta} - \beta^0) = 0$

$\implies \sqrt{n}\gamma_n = (A_0\Sigma^{-1}A_0')^{-1}A\Sigma^{-1}\sqrt{n}\frac{\partial}{\partial \beta}Q_n(\beta^0) - (A_0\Sigma^{-1}A_0')^{-1}\sqrt{n}a(\beta^0) + o_p(1)$

and

$\implies \sqrt{n}(\tilde{\beta} - \beta^0) = \left[\Sigma^{-1}\Sigma^{-1}A_0^{-1}(A_0\Sigma^{-1})^{-1}A_0\Sigma^{-1}\right]\sqrt{n}\frac{\partial}{\partial \beta}Q_n(\beta^0) + o_p(1)$

$\sqrt{n}\gamma_n = \Omega^{-1}A\Sigma^{-1}\sqrt{n}\frac{\partial}{\partial \beta}Q_n(\beta^0) - \Omega^{-1}\sqrt{n}a(\beta^0) + o_p(1)$

$= \Omega^{-1}A\Sigma^{-1}\sqrt{n}\frac{\partial}{\partial \beta}Q_n(\beta^0) + o_p(1)$ (Since under the null $a(\beta^0) = 0$)

and B : $\sqrt{n}\frac{\partial}{\partial \beta}Q_n(\beta^0) \xrightarrow{d} N(0, \Sigma)$

$\implies \sqrt{n}\gamma_n \xrightarrow{d} N(0, \Omega^{-1}A_0\Sigma^{-1}\Sigma\Sigma^{-1}A_0'\Omega^{-1})$

$$\xrightarrow{d} N(0, \Omega^{-1} A_0 \Sigma^{-1} A_0' \Omega^{-1})$$

$$\xrightarrow{d} N(0, \Omega^{-1} \Omega \Omega^{-1})$$

$$\xrightarrow{d} N(0, \Omega^{-1})$$

$$\implies L' \sqrt{n} \gamma_n \xrightarrow{d} N(0, I)$$

Finally yielding our LM statistic under the null:

$$LM \equiv \left(\gamma_n' \sqrt{n} \tilde{L}'\right) \left(\gamma_n' \sqrt{n} \tilde{L}'\right)'$$

$$= n \gamma_n' \tilde{L} \tilde{L}' \gamma_n$$

$$= n \gamma_n' \tilde{\Omega} \gamma_n$$

$$= n \gamma_n' A(\tilde{\beta}) \tilde{\Sigma}^{-1} A'(\tilde{\beta}) \gamma_n$$

$$= n \left[A'(\tilde{\beta}) \gamma_n\right]' \tilde{\Sigma}^{-1} \left[A'(\tilde{\beta}) \gamma_n\right]$$

$$= \left[\sqrt{n} \tfrac{\partial}{\partial \beta} Q_n(\tilde{\beta})\right]' \tilde{\Sigma}^{-1} \left[\sqrt{n} \tfrac{\partial}{\partial \beta} Q_n(\tilde{\beta})\right] \to \chi_r^2$$

(b) $H_l: \ a(\beta^0) = \frac{\delta}{\sqrt{n}}$

from (a) we had:

$$\sqrt{n} \gamma_n = \Omega^{-1} A \Sigma^{-1} \sqrt{n} \tfrac{\partial}{\partial \beta} Q_n(\beta^0) - \Omega^{-1} \sqrt{n} a(\beta^0) + o_p(1)$$

and plugging in $a(\beta^0) = \frac{\delta}{\sqrt{n}}$ from H_l gives:

$$\sqrt{n} \gamma_n = \Omega^{-1} A \Sigma^{-1} \sqrt{n} \tfrac{\partial}{\partial \beta} Q_n(\beta^0) - \Omega^{-1} \sqrt{n} \tfrac{\delta}{\sqrt{n}} + o_p(1)$$

$$= \Omega^{-1} A \Sigma^{-1} \sqrt{n} \tfrac{\partial}{\partial \beta} Q_n(\beta^0) - \Omega^{-1} \delta + o_p(1)$$

and B : $\sqrt{n} \frac{\partial}{\partial \beta} Q_n(\beta^0) \xrightarrow{d} N(0, \Sigma)$

$$\implies \Omega^{-1} A \Sigma^{-1} \sqrt{n} \tfrac{\partial}{\partial \beta} Q_n(\beta^0) \xrightarrow{d} N\left(0, \Omega^{-1} A_0 \Sigma^{-1} \Sigma \Sigma^{-1} A_0' \Omega^{-1}\right)$$

$$\xrightarrow{d} N\left(0, \Omega^{-1} A_0 \Sigma^{-1} A_0' \Omega^{-1}\right)$$

$$\xrightarrow{d} N\left(0, \Omega^{-1}\Omega\Omega^{-1}\right)$$

$$\xrightarrow{d} N\left(0, \Omega^{-1}\right)$$

$$\implies \sqrt{n}\gamma_n \xrightarrow{d} N\left(-\Omega^{-1}\delta, \Omega^{-1})\right)$$

$$\implies L'\sqrt{n}\gamma_n \xrightarrow{d} N\left(-L'\Omega^{-1}\delta, I)\right)$$

Which gives us our LM under H_l :

$$LM = \left(\gamma_n'\sqrt{n}\tilde{L}'\right)\left(\tilde{L}'\sqrt{n}\gamma_n\right)' \to \chi_r^2(\kappa)$$

Where κ is a non-centrality parameter s.t.:

$$\kappa = \left(L'\Omega^{-1}\delta\right)'\left(L'\Omega^{-1}\delta\right) = \delta'\Omega^{-1}LL'\Omega^{-1}\delta = \delta'\Omega^{-1}\Omega\Omega^{-1}\delta = \delta'\Omega^{-1}\delta$$

(c) under $H_f:\ a(\beta^0) \neq 0$

again from (a) we had:

$$\sqrt{n}\gamma_n = \Omega^{-1}A\Sigma^{-1}\sqrt{n}\tfrac{\partial}{\partial\beta}Q_n(\beta^0) - \Omega^{-1}\sqrt{n}a(\beta^0) + o_p(1)$$

Let's call our choice of a fixed alternative δ s.t. $a(\beta^0) = \delta$

$$\implies \sqrt{n}\gamma_n = \Omega^{-1}A\Sigma^{-1}\sqrt{n}\tfrac{\partial}{\partial\beta}Q_n(\beta^0) - \Omega^{-1}\sqrt{n}\delta + o_p(1)$$

Again B : $\sqrt{n}\tfrac{\partial}{\partial\beta}Q_n(\beta^0) \xrightarrow{d} N(0, \Sigma)$

$$\implies \Omega^{-1}A\Sigma^{-1}\sqrt{n}\tfrac{\partial}{\partial\beta}Q_n(\beta^0) \xrightarrow{d} N\left(0, \Omega^{-1}\right)$$

$$\implies \sqrt{n}\gamma_n \xrightarrow{d} N\left(-\Omega^{-1}\sqrt{n}\delta, \Omega^{-1})\right)$$

$$\implies L'\sqrt{n}\gamma_n \xrightarrow{d} N\left(-L'\Omega^{-1}\sqrt{n}\delta, I)\right)$$

Which gives us our LM under H_f :

$$LM = \left(\gamma_n'\sqrt{n}\tilde{L}'\right)\left(\tilde{L}'\sqrt{n}\gamma_n\right)' \to \chi_r^2(\kappa)$$

Where κ is a non-centrality parameter s.t.:

$$\kappa = \left(L'\Omega^{-1}\delta\sqrt{n}\right)'\left(L'\Omega^{-1}\delta\sqrt{n}\right) = \delta'\Omega^{-1}L\sqrt{n}\sqrt{n}L'\Omega^{-1}\delta = n\delta'\Omega^{-1}\Omega\Omega^{-1}\delta = n\delta'\Omega^{-1}\delta$$

1.3 LR

(a) under $H_0 \; : \; a(\beta^0) = 0$

$$Q_n(\beta^0) = Q_n(\hat{\beta}) + \tfrac{\partial}{\partial\beta}Q_n(\hat{\beta})(\beta^0 - \hat{\beta}) + \tfrac{1}{2}(\beta^0 - \hat{\beta})'\tfrac{\partial^2}{\partial\beta\partial\beta'}Q_n(\bar{\beta})(\beta^0 - \hat{\beta})$$

$Q_n(\beta^0) = Q_n(\hat{\beta}) + \tfrac{1}{2}(\beta^0 - \hat{\beta})'\tfrac{\partial^2}{\partial\beta\partial\beta'}Q_n(\bar{\beta})(\beta^0 - \hat{\beta})$ Since F.O.C. $\implies \tfrac{\partial}{\partial\beta}Q_n(\hat{\beta})(\beta^0 - \hat{\beta}) = 0$

When correctly specified $\tfrac{\partial^2}{\partial\beta\partial\beta'}Q_n(\bar{\beta}) = -\Sigma$ So,

$$Q_n(\beta^0) = Q_n(\hat{\beta}) + \tfrac{1}{2}(\beta^0 - \hat{\beta})'\tfrac{\partial^2}{\partial\beta\partial\beta'}Q_n(\bar{\beta})(\beta^0 - \hat{\beta})$$

$$0 = Q_n(\hat{\beta}) - Q_n(\beta^0) + \tfrac{1}{2}(\beta^0 - \hat{\beta})'\tfrac{\partial^2}{\partial\beta\partial\beta'}Q_n(\bar{\beta})(\beta^0 - \hat{\beta})$$

$$Q_n(\hat{\beta}) - Q_n(\beta^0) = -\tfrac{1}{2}(\beta^0 - \hat{\beta})'\tfrac{\partial^2}{\partial\beta\partial\beta'}Q_n(\bar{\beta})(\beta^0 - \hat{\beta})$$

$$2\left[Q_n(\hat{\beta}) - Q_n(\beta^0)\right] = -(\beta^0 - \hat{\beta})'\tfrac{\partial^2}{\partial\beta\partial\beta'}Q_n(\bar{\beta})(\beta^0 - \hat{\beta})$$

$2\left[Q_n(\hat{\beta}) - Q_n(\beta^0)\right] = (\beta^0 - \hat{\beta})'\Sigma(\beta^0 - \hat{\beta})$ multiply by n

$$2n\left[Q_n(\hat{\beta}) - Q_n(\beta^0)\right] = \sqrt{n}(\hat{\beta} - \beta^0)'\Sigma\sqrt{n}(\hat{\beta} - \beta^0)$$

We need to find asymptotic distribution of $\sqrt{n}(\hat{\beta} - \beta^0)$
$\sqrt{n}(\hat{\beta} - \beta^0) = \Sigma^{-1}A_0'\Omega^{-1}A_0\Sigma^{-1}\sqrt{n}\tfrac{\partial}{\partial\beta}Q_n(\beta^0) + \Sigma^{-1}A_0'\Omega^{-1}\sqrt{n}a(\beta^0) + o_p(1)$

$H_0 \; : \; a(\beta^0) = 0 \implies \Sigma^{-1}A_0'\Omega^{-1}\sqrt{n}a(\beta^0) = 0$ so,

$$\sqrt{n}(\hat{\beta} - \beta^0) = \Sigma^{-1}A_0'\Omega^{-1}A_0\Sigma^{-1}\sqrt{n}\tfrac{\partial}{\partial\beta}Q_n(\beta^0) + o_p(1)$$

B $: \sqrt{n}\tfrac{\partial}{\partial\beta}Q_n(\beta^0) \xrightarrow{d} N(0, \Sigma)$

$$\implies \sqrt{n}(\hat{\beta} - \beta^0) \xrightarrow{d} N\left(0, \Sigma^{-1}A_0'\Omega^{-1}A_0\Sigma^{-1}\Sigma\Sigma^{-1}A_0'\Omega^{-1}A_0\Sigma^{-1}\right)$$

$$\xrightarrow{d} N\left(0, \Sigma^{-1}A_0'\Omega^{-1}A_0\Sigma^{-1}A_0'\Omega^{-1}A_0\Sigma^{-1}\right)$$

$$\xrightarrow{d} N\left(0, \Sigma^{-1}A_0'\Omega^{-1}\Omega\Omega^{-1}A_0\Sigma^{-1}\right)$$

$$\xrightarrow{d} N\left(0, \Sigma^{-1}A_0'\Omega^{-1}A_0\Sigma^{-1}\right)$$

$$\implies L^{-1}A_0\sqrt{n}(\hat{\beta}-\beta^0) \xrightarrow{d} N(0, I)$$

So,

$$LR(\beta^0) = 2n\left[Q_n(\hat{\beta}) - Q_n(\beta^0)\right]$$

$$= \sqrt{n}(\hat{\beta}-\beta^0)'\Sigma\sqrt{n}(\hat{\beta}-\beta^0)$$

$$= \left[\sqrt{n}\tfrac{\partial}{\partial\beta}Q_n(\beta^0\right]' \Sigma^{-1}A_0'\Omega^{-1}A_0\Sigma^{-1}\Sigma\Sigma^{-1}A_0'\Omega^{-1}A_0\Sigma^{-1}\left[\sqrt{n}\tfrac{\partial}{\partial\beta}Q_n(\beta^0\right]$$

$$= \left[\sqrt{n}\tfrac{\partial}{\partial\beta}Q_n(\beta^0\right]' \Sigma^{-1}A_0'\Omega^{-1}A_0\Sigma^{-1}A_0'\Omega^{-1}A_0\Sigma^{-1}\left[\sqrt{n}\tfrac{\partial}{\partial\beta}Q_n(\beta^0\right]$$

$$= \left[\sqrt{n}\tfrac{\partial}{\partial\beta}Q_n(\beta^0\right]' \Sigma^{-1}A_0'\Omega^{-1}\Omega\Omega^{-1}A_0\Sigma^{-1}\left[\sqrt{n}\tfrac{\partial}{\partial\beta}Q_n(\beta^0\right]$$

$$= \left[\sqrt{n}\tfrac{\partial}{\partial\beta}Q_n(\beta^0\right]' \Sigma^{-1}A_0'\Omega^{-1}A_0\Sigma^{-1}\left[\sqrt{n}\tfrac{\partial}{\partial\beta}Q_n(\beta^0\right]$$

$$= \left[\sqrt{n}\tfrac{\partial}{\partial\beta}Q_n(\beta^0\right]' \Sigma^{-1}A_0'L^{-1}(L{-}1)'A_0\Sigma^{-1}\left[\sqrt{n}\tfrac{\partial}{\partial\beta}Q_n(\beta^0\right]$$

$$\text{B} \implies \left[\sqrt{n}\tfrac{\partial}{\partial\beta}Q_n(\beta^0\right]' \Sigma^{-1}A_0'L^{-1} \xrightarrow{d} N(0, (L^{-1})'A_0'\Sigma^{-1}\Sigma\Sigma^{-1}A_0L^{-1})$$

$$\xrightarrow{d} N(0, (L^{-1})'A_0'\Sigma^{-1}A_0L^{-1})$$

$$\xrightarrow{d} N(0, (L^{-1})'\Omega L^{-1})$$

$$\xrightarrow{d} N(0, I)$$

$$\implies LR(\beta^0) \to \chi_r^2$$

(b) under $H_l:\ a(\beta^0) = \frac{\delta}{\sqrt{n}}$

From (a):

$\sqrt{n}(\hat{\beta}-\beta^0) = \Sigma^{-1}A_0'\Omega^{-1}A_0\Sigma^{-1}\sqrt{n}\frac{\partial}{\partial\beta}Q_n(\beta^0) + \Sigma^{-1}A_0'\Omega^{-1}\sqrt{n}a(\beta^0) + o_p(1)$

We showed the first term $\xrightarrow{d} N(0, \Sigma^{-1}A_0'\Omega^{-1}A_0\Sigma^{-1})$ So,

$\sqrt{n}(\hat{\beta}-\beta^0) \xrightarrow{d} N(\Sigma^{-1}A_0'\Omega^{-1}\delta, \Sigma^{-1}A_0'\Omega^{-1}A_0\Sigma^{-1})$ Since plugging in,

$a(\beta^0) = \frac{\delta}{\sqrt{n}}$, gives $\Sigma^{-1}A_0'\Omega^{-1}\sqrt{n}a(\beta^0)\delta = \Sigma^{-1}Ao'\Omega^{-1}\delta$

So, $L^{-1}A_0\sqrt{n}(\hat{\beta} - \beta^0) \xrightarrow{d} N(L^{-1}A_0\Sigma^{-1}A_0'\Omega^{-1}\delta, I)$

$$\xrightarrow{d} N(L^{-1}\Omega'\Omega^{-1}\delta, I)$$

$$\xrightarrow{d} N(L^{-1}\delta, I)$$

$\implies LR \to \chi^2_r(\kappa)$

Where κ is a non-centrality parameter s.t.:

$\kappa = \left(L^{-1}\delta\right)' \left(L^{-1}\delta\right) = \delta'\Omega^{-1}\delta$

(c) under $H_f:\ a(\beta^0) \neq 0$

From (a):
$\sqrt{n}(\hat{\beta} - \beta^0) = \Sigma^{-1}A_0'\Omega^{-1}A_0\Sigma^{-1}\sqrt{n}\frac{\partial}{\partial\beta}Q_n(\beta^0) + \Sigma^{-1}A_0'\Omega^{-1}\sqrt{n}a(\beta^0) + o_p(1)$

Again, we showed the first term $\xrightarrow{d} N(0, \Sigma^{-1}A_0'\Omega^{-1}A_0\Sigma^{-1})$ So,

$\sqrt{n}(\hat{\beta} - \beta^0) \xrightarrow{d} N(\Sigma^{-1}A_0'\Omega^{-1}\sqrt{n}\delta, \Sigma^{-1}A_0'\Omega^{-1}A_0\Sigma^{-1})$ Since plugging in,

$a(\beta^0) = \delta$, gives $\Sigma^{-1}A_0'\Omega^{-1}\sqrt{n}a(\beta^0)\delta = \Sigma^{-1}Ao'\Omega^{-1}\sqrt{n}\delta$

So, $L^{-1}A_0\sqrt{n}(\hat{\beta} - \beta^0) \xrightarrow{d} N(L^{-1}A_0\Sigma^{-1}A_0'\Omega^{-1}\sqrt{n}\delta, I)$

$$\xrightarrow{d} N(L^{-1}\Omega'\Omega^{-1}\sqrt{n}\delta, I)$$

$$\xrightarrow{d} N(L^{-1}\sqrt{n}\delta, I)$$

$\implies LR \to \chi^2_r(\kappa)$

Where κ is a non-centrality parameter s.t.:

$\kappa = \left(L^{-1}\delta\sqrt{n}\right)' \left(L^{-1}\delta\sqrt{n}\right) = n\delta'\Omega^{-1}\delta$

2 Show that the three tests are asymptotically equivalent under H_0 and H_l under standard GMM assumptions

(a) under H_0

$W, LM, LR \xrightarrow{d} \chi^2_r$ as shown in Part 1, so they are asymptotically equivalent under the H_0

(a) under H_l

$W, LM, LR \xrightarrow{d} \chi^2_r(\kappa)$, where $\kappa = \delta'\Omega^{-1}\delta$ as shown in Part 1, so they are asymptotically equivalent under H_l

3 Show that the three tests are consistent under H_f

$W, LM, LR \xrightarrow{d} \chi^2_r(\kappa)$, where $\kappa = n\delta'\Omega^{-1}\delta$ As shown in Part 1, and:

$$\underset{n\to\infty}{lim} P\left(W > \chi^2_r\right) = \underset{n\to\infty}{lim} P\left(LM > \chi^2_r\right) = \underset{n\to\infty}{lim} P\left(LR > \chi^2_r\right) = 1$$

So the three tests are consistent under H_f.

4 Suppose that we are interested in two null hypotheses: H_0^β : $\beta = 1$ and H_o^γ : $\gamma = 0$ in the context of Problem-1(c) of HW-2. Test these two null hypotheses separately using Wald, LM, and LR tests. Perform the tests for both sample periods.

W, LM, LR Results

W β Pre-Volcker	LM β Pre-Volcker	LR β Pre-Volcker
1.54e+03	5.34e-04	466.10
(0)	(.98)	(0)
W γ Pre-Volcker	LM γ Pre-Volcker	LR γ Pre-Volcker
9.34e+04	1.80	1.85e+03
(0)	(.18)	(0)
W β Volcker-Greenspan	LM β Volcker-Greenspan	LR β Volcker-greenspan
370.10	1.37e-05	583.44
(0)	(1.00)	(0)
W γ Volcker-Greenspan	LM γ Volcker-Greenspan	LR γ Volcker-Greenspan
9.73e+04	.19	3.20e+03
(0)	(.66)	(0)

Note: numbers in parenthesis are p-values for each statistic.

5 Ghysels, Hill, and Motegi(2013) Question

5.1 Preliminary

(a) $\mathbb{E}[x_t] = 0$
Proof:

$$\mathbb{E}[x_t] = \mathbb{E}\left[\phi x_{t-1} + \eta_t\right] = \phi\mathbb{E}[x_{t-1}] = \phi\mathbb{E}[x_t] \text{ by the stationarity of } \phi$$

$$\mathbb{E}[x_t] = \phi\mathbb{E}[x_t]$$

$$\mathbb{E}[x_t] - \phi\mathbb{E}[x_t] = 0$$
$$(1-\phi)\mathbb{E}[x_t] = 0 \implies \mathbb{E}[x_t] = 0$$

(b) $\mathbb{E}[x_t^2] = \frac{1}{1-\phi^2} \equiv \gamma_0$
Proof:

$$\mathbb{E}[x_t^2] = \mathbb{E}\left[(\phi x_{t-1} + \eta_t)(\phi x_{t-1} + \eta_t)\right] = \phi^2\mathbb{E}\left[x_{t-1}x_{t-1}\right] + 0 + \mathbb{E}[\eta_t\eta_t]$$

$$\mathbb{E}[x_t^2] = \phi^2\mathbb{E}\left[x_t x_t\right] + 1 \text{ again by stationarity of } \phi \text{ and since } \eta t \overset{i.i.d.}{\sim} (0,1)$$

$$\mathbb{E}[x_t^2] - \phi^2\mathbb{E}\left[x_t x_t\right] = 1$$

$$(1-\phi^2)\mathbb{E}[x_t^2] = 1$$

$$\mathbb{E}[x_t^2] = \frac{1}{(1-\phi^2)} \equiv \gamma_0$$

(c) $\mathbb{E}[x_t x_{t-1}] = \frac{\phi}{1-\phi^2} \equiv \gamma_1$
Proof:

$$\mathbb{E}[x_t x_{t-1}] = \mathbb{E}\left[(\phi x_{t-1} + \eta_t)(\phi x_{t-2} + \eta_{t-1})\right]$$

$$\mathbb{E}[x_t x_{t-1}] = \phi^2\mathbb{E}\left[(x_{t-1}x_{t-2}\right] + 0 + \mathbb{E}\left[\eta_t\eta_{t-1})\right]$$

$$\mathbb{E}[x_t x_{t-1}] = \phi^2\mathbb{E}\left[(x_t x_{t-1}\right] + \phi$$

$$\mathbb{E}[x_t x_{t-1}] - \phi^2\mathbb{E}\left[(x_t x_{t-1}\right] = \phi$$

$$(1-\phi^2)\mathbb{E}[x_t x_{t-1}] = \phi$$

$(1-\phi^2)\mathbb{E}[x_t x_{t-1}] = \frac{\phi}{(1-\phi^2)} \equiv \gamma_1$

5.2 Question (a)

What is the probability limit of $\hat{\beta}$?

$$
\begin{aligned}
\hat{\beta} &= \frac{1}{T}\sum_{t=1}^{T}(x_{t-1}^2)^{-1}(\frac{1}{T}\sum_{t=1}^{T}x_{t-1}y_t) \\
&= \gamma_0^{-1}\left(\frac{1}{T}\sum_{t=1}^{T}x_{t-1}y_t\right) + o_p(1) \text{ Because } \frac{1}{T}\sum_{t=1}^{T}x_{t-1}^2 \xrightarrow{p} \mathbb{E}(x_{t-1}^2)(byLLN) \\
&\ \&\ \mathbb{E}(x_{t-1}^2) = \mathbb{E}(x_t^2) = \gamma_0 \text{ by stationarity of } \phi \\
&= \gamma_0^{-1}\left[\frac{1}{T}\sum_{t=1}^{T}x_{t-1}\left(c_1x_{t-1}+c_2x_{t-2}+\epsilon_t\right)\right] + \ o_p(1) \\
&= \gamma_0^{-1}\left[c_1\frac{1}{T}\sum_{t=1}^{T}x_{t-1}^2 + c_2\frac{1}{T}\sum_{t=1}^{T}x_{t-1}x_{t-2} + \frac{1}{T}\sum_{t=1}^{T}x_{t-1}\epsilon_t\right] + o_p(1) \\
&= \gamma_0^{-1}\left(c_1\gamma_0 + c_2\gamma_1\right) + \ o_p(1) \xrightarrow{p} c1 + \frac{\gamma_1}{\gamma_0}c_2 \text{ where,}
\end{aligned}
$$

$$
\frac{1}{T}\sum_{t=1}^{T}x_{t-1}^2 \xrightarrow{p} \gamma_0 \ , \ \frac{1}{T}\sum_{t=1}^{T}x_{t-1}x_{t-2} \xrightarrow{p} \gamma_1 \ , \text{ and } \frac{1}{T}\sum_{t=1}^{T}x_{t-1}\epsilon_t \xrightarrow{p} 0
$$

$$
\text{because } \frac{1}{T}\sum_{t=1}^{T}x_{t-1}\epsilon_t \xrightarrow{p} \mathbb{E}(x_{t-1}\epsilon_t) \text{ (by LLN) } = \mathbb{E}(x_{t-1}\mathbb{E}(\epsilon_t)) = 0 \text{ , since } \mathbb{E}(\epsilon_t) = 0
$$

Under what conditions is $\hat{\beta}$ consistent for c_1?

$\hat{\beta} \xrightarrow{p} c_1$ when:

(1) $\phi = 0$ (x_{t-1} & x_{t-2} are uncorrelated)

(2) $c_2 = 0$ (x_{t-2} is irrelevant i.e. our model correctly specified)

5.3 Question (b)

Formulate the Wald statistic with respect to $H_0 \ : \ \beta = 0$ and call it W. Under non-causality what is the asymptotic distribution of W?

From above we have:

$$\hat{\beta} = \gamma_0^{-1}\left[c_1\frac{1}{T}\sum_{t=1}^{T} x_{t-1}^2 + c_2\frac{1}{T}\sum_{t=1}^{T} x_{t-1}x_{t-2} + \frac{1}{T}\sum_{t=1}^{T} x_{t-1}\epsilon_t\right] + o_p(1), \text{ multiply by} \sqrt{T}$$

$$\sqrt{T}\hat{\beta} = \sqrt{T}\gamma_0^{-1}\left[c_1\frac{1}{T}\sum_{t=1}^{T} x_{t-1}^2 + c_2\frac{1}{T}\sum_{t=1}^{T} x_{t-1}x_{t-2} + \frac{1}{T}\sum_{t=1}^{T} x_{t-1}\epsilon_t\right] + o_p(1)$$

$$\sqrt{T}\hat{\beta} = \sqrt{T}\gamma_0^{-1}\left[c_1\frac{1}{T}\sum_{t=1}^{T} x_{t-1}^2 + c_2\frac{1}{T}\sum_{t=1}^{T} x_{t-1}x_{t-2}\right] + \sqrt{T}\gamma_0^{-1}\left[\frac{1}{T}\sum_{t=1}^{T} x_{t-1}\epsilon_t\right] + o_p(1)$$

again: $\frac{1}{T}\sum_{t=1}^{T} x_{t-1}^2 \xrightarrow{p} \gamma_0$, and $\frac{1}{T}\sum_{t=1}^{T} x_{t-1}x_{t-2} \xrightarrow{p} \gamma_1$, so

$$\sqrt{T}\hat{\beta} = \sqrt{T}\gamma_0^{-1}\left(c_1\gamma_0 + c_2\gamma_1 + \frac{1}{T}\sum_{t=1}^{T} x_{t-1}\epsilon_t\right) + \ o_p(1)$$

$$\sqrt{T}\hat{\beta} = \sqrt{T}\beta^0 + \gamma_0^{-1}\frac{1}{\sqrt{T}}\sum_{t=1}^{T} x_{t-1}\epsilon_t + o_p(1) \text{ , where } \beta^0 \equiv c_1 + \frac{\gamma_1}{\gamma_0}c_2$$

note: $\frac{1}{\sqrt{T}}\sum_{t=1}^{T} x_{t-1}\epsilon_t \xrightarrow{d} N(0, \gamma_0)$, since:

$$Var \ = \mathbb{E}\left[(x_{t-1}\epsilon_t)^2\right] = \mathbb{E}\left[x_{t-1}^2\epsilon_t^2\right] = \mathbb{E}\left[x_{t-1}^2\mathbb{E}(\epsilon_t^2|\mathbf{I}_{t-1}\right] = \mathbb{E}\left[x_{t-1}^2 \times 1\right] = \mathbb{E}\left[x_{t-1}^2\right] = \gamma_0$$

$$\implies \sqrt{T}\hat{\beta} \xrightarrow{d} N(lim\ \sqrt{T}\beta^0,\ \gamma_0^{-1})$$

Now we need to standardize the variance so we pre-multiply $\sqrt{T}\hat{\beta}$ by $\sqrt{\gamma_0}$:

$$\sqrt{T}\sqrt{\gamma_0}\hat{\beta} \xrightarrow{d} N(\sqrt{\gamma_0}\ lim\ \sqrt{T}\beta^0, 1), \text{ yielding our Wald statistic:}$$

$$W = \left(\sqrt{T}\sqrt{\gamma_0}\hat{\beta}\right)^2 = T\gamma_0\hat{\beta}^2 \xrightarrow{d} \chi_1^2(\kappa), \text{ where } \kappa \text{ is our noncentrality parameter s.t.}$$

$$\kappa = (\sqrt{\gamma_0}\ lim\ \sqrt{T}\beta^0)^2, \text{ Note: } \kappa = 0 \iff \beta^0 = 0 \text{ (because } \gamma_0 \text{ can't be zero by definition)}$$

Under non-causality: $c_1 = c_2 = 0 \implies \beta^0 \implies \kappa = 0$

5.4 Question (c)

Suppose $c_1 = 0$ and $c_2 \neq 0$, a seemingly hard type of Granger causality to detect since our model only has one lag. Can we get power approaching one? (Put differently, does W diverge to infinity asymptotically?)

From above we note:

$$\kappa = 0 \iff \beta^0 = 0$$
$$\beta^0 = 0 \iff \phi = 0$$
$$\therefore$$
$$\kappa \neq 0 \iff \phi \neq 0$$

So we will still get power approaching one as long as $\phi \neq 0$

5.5 Question (d)

In general, under what conditions do you lose power? Are those conditions likely to hold in usual economic applications?

When $c_1, c_2 \neq 0$ we will lose power if $\beta^0 = 0$, which will be the case if $c_2 = -\frac{1}{\phi}c_1$, however in typical economic applications it is the case that : $\phi \in (0,1)$ & $|c_1| > |c_2|$. When these two things hold, we will never get $c_2 = -\frac{1}{\phi}c_1$ and therefore never have $\beta^0 = 0$. So for common economic applications this is not an issue.

5.6 Question (e)

Show that W converges to a non central chi-squared distribution. Characterize the non centrality parameter κ and show that $\kappa = 0$ under non-causality.

$\hat{\beta} = \frac{1}{T}\sum_{t=1}^{T}(x_{t-1}^2)^{-1}(\frac{1}{T}\sum_{t=1}^{T} x_{t-1}y_t)$ as before

and $\frac{1}{T}\sum_{t=1}^{T}(x_{t-1}^2)^{-1} \xrightarrow{p} \mathbb{E}\left[x_t^2\right]^{-1} = \gamma_0^{-1}$

$\implies \hat{\beta} = \gamma_0^{-1}\left[\frac{1}{T}\sum_{t=1}^{T} x_{t-1}\left(\frac{c_1}{\sqrt{T}}x_{t-1} + \frac{c_2}{\sqrt{T}}x_{t-2} + \epsilon_t\right)\right] + o_p(1)$

$\implies \sqrt{T}\hat{\beta} = \gamma_0^{-1}\left[c_1\frac{1}{T}\sum_{t=1}^{T} x_{t-1}^2 + c_2\frac{1}{T}\sum_{t=1}^{T} x_{t-1}x_{t-2} + \frac{1}{\sqrt{T}}\sum_{t=1}^{T} x_{t-1}\epsilon_t\right] + o_p(1)$

Note: $\frac{1}{T}\sum_{t=1}^{T} x_{t-1}^2 \xrightarrow{p} \gamma_0$, $\frac{1}{T}\sum_{t=1}^{T} x_{t-1}x_{t-2} \xrightarrow{p} \gamma_1$, and $\frac{1}{\sqrt{T}}\sum_{t=1}^{T} x_{t-1}\epsilon_t \xrightarrow{d} N(0,\gamma_0)$

$\implies \sqrt{T}\hat{\beta} \xrightarrow{d} N\left(\gamma_0 c_1 + \frac{\gamma 1}{\gamma_0}c_2, \gamma_0^{-1}\gamma_0\gamma_0^{-1}\right)$
$W = \left(\sqrt{T}\sqrt{\gamma_0}\hat{\beta}\right)^2 \to \chi_r^2(\kappa)$
Where κ is a non-centrality parameter s.t.:

$\kappa = \left[\sqrt{\gamma_0}c_1 + \frac{\gamma 1}{\sqrt{\gamma_0}}c_2\right]^2 = \gamma_0 c_1^2 + \frac{\gamma_1^2}{\gamma_0}c_2^2 + 2c_1c_2\gamma_1$

$= \frac{c_1^2}{1-\phi^2} + \left(\frac{\phi}{1-\phi^2}\right)^2(1-\phi^2)c_2^2 + \frac{2c_1c_2\phi}{1-\phi^2}$

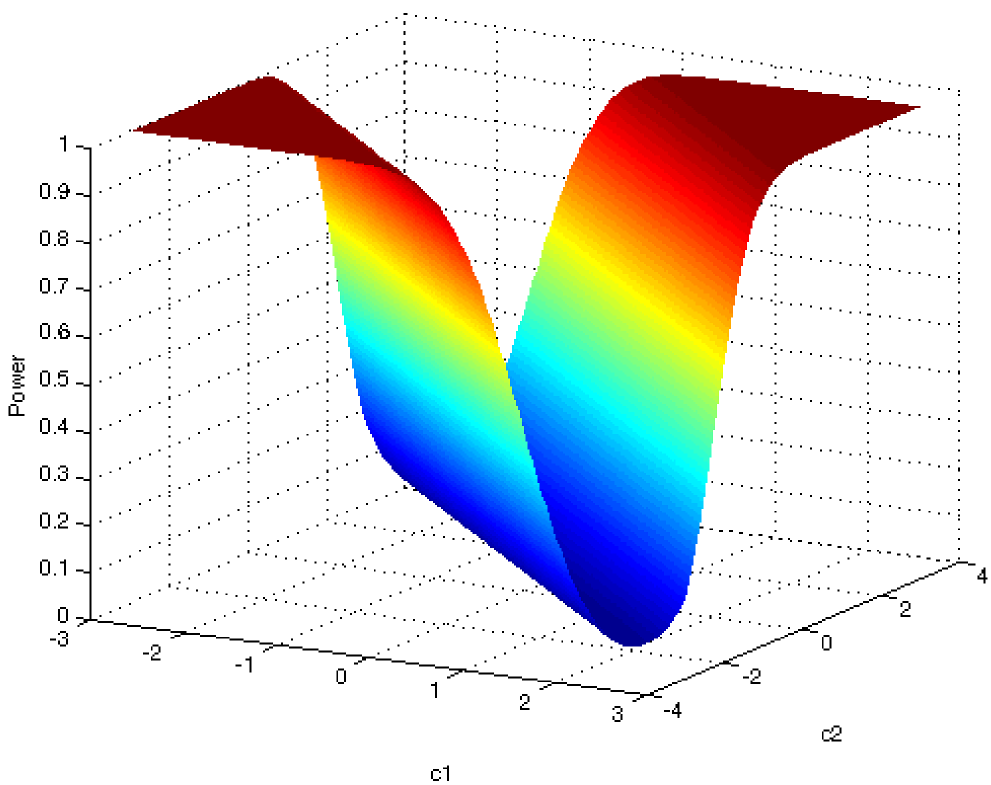
$= \frac{1}{1-\phi^2}\left[c_1^2 + \phi^2c_2^2 + 2c_1c_2\phi\right] = \frac{(c_1+\phi c_2)^2}{1-\phi^2}$

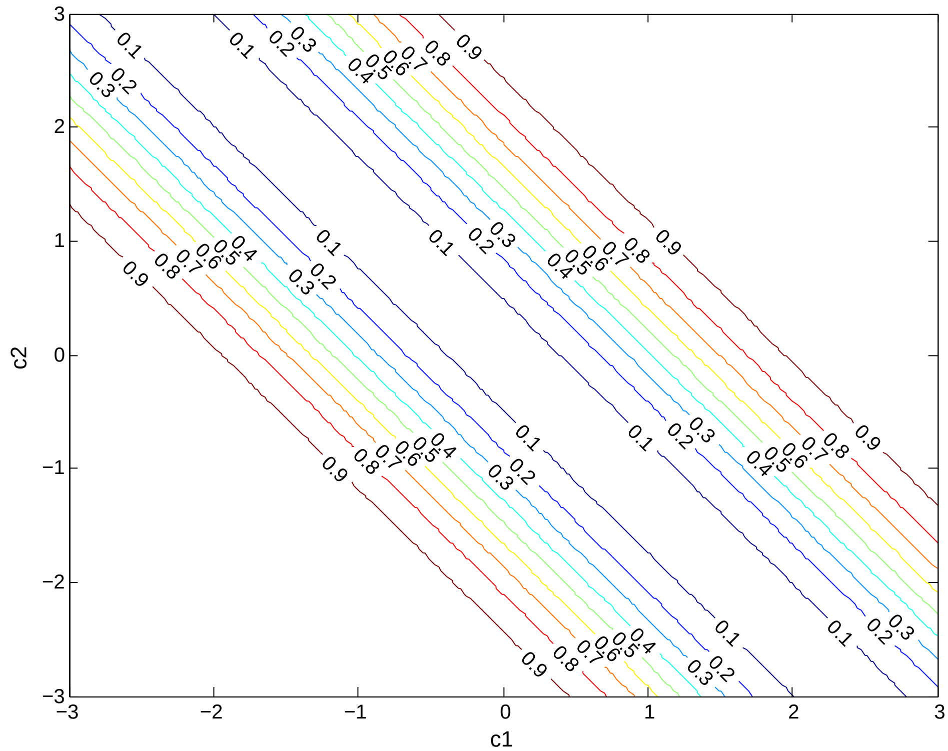
5.7 Question (f)

$$\begin{cases} F_0 = \chi_1^2, \text{ since } c_1 = c_2 = 0 \\ F_1 = \chi_1^2(\kappa), \text{ where } \kappa = \frac{(c_1+\phi c_2)^2}{1-\phi^2} \text{ from Question (e)} \end{cases}$$

$P(\alpha) = 1 - F_1\left(F_0^{-1}(1-\alpha)\right)$

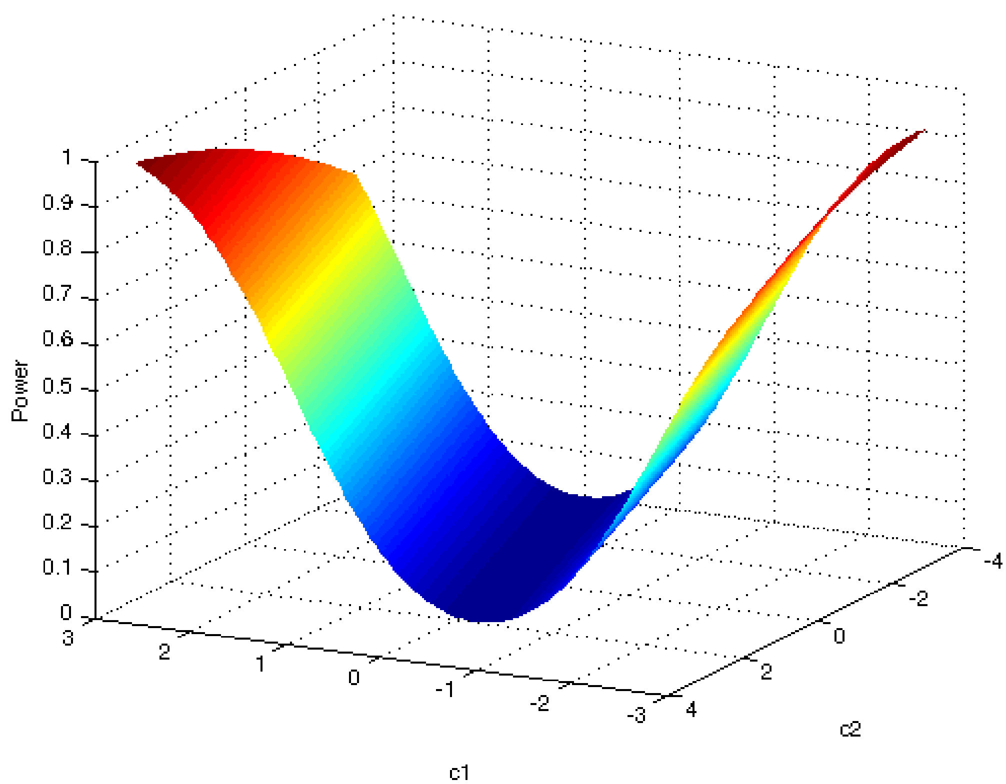
5.8 Question (g)

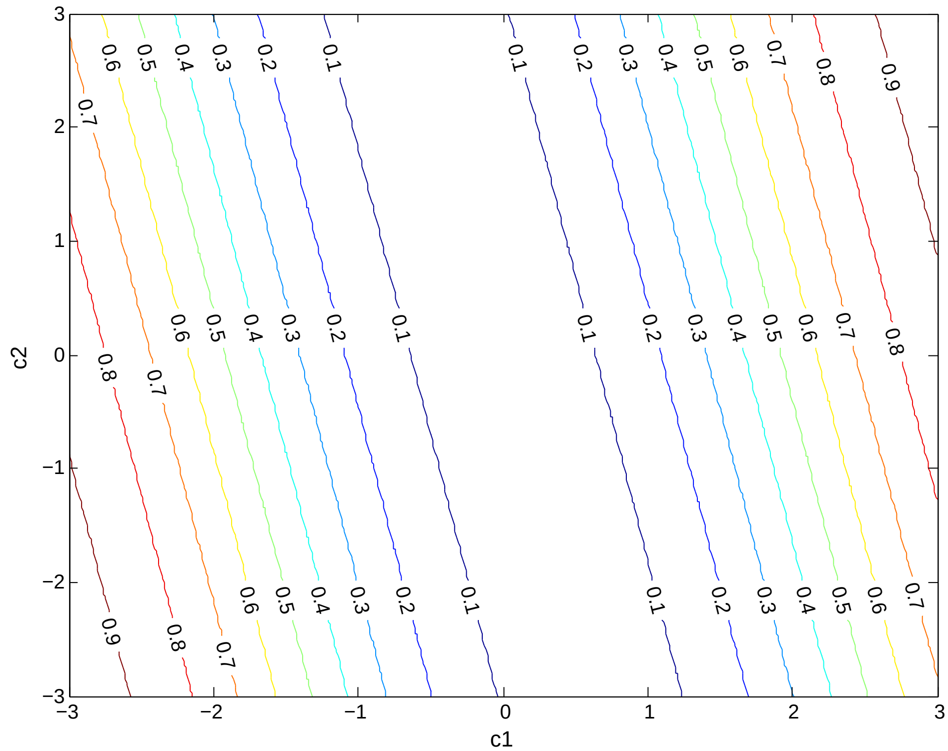



c1
c2
−3
−2
−1
0
1
2
3
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9

5.9 Question (h)

As we decreased the correlation between x_t and x_{t-1} from $\phi = 0.8$ to $\phi = 0.2$ the region of no local power is larger, but we still have local power going to one very quickly.





c1
c2
−3
−2
−1
0
1
2
3
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9