Problem 2, Homework 3

Due on November 7, 2013

1 Motivation

This extra problem is based on the classic literature of time series analysis and hypothesis testing you learned in our current course and before, but a recent extension of this problem into the Mixed Data Sampling (MIDAS) literature can be found at Ghysels, Hill, and Motegi (2013), "Regression-based Mixed Frequency Granger Causality Tests", Working Paper at the University of North Carolina at Chapel Hill.

Suppose that you are interested in the impact of a time series $\{x_t\}$ on another time series $\{y_t\}$. Suppose that the true DGP is:

$$y_t = \sum_{i=1}^{p} c_i x_{t-i} + \epsilon_t.$$

We are interested in testing $c_i = 0$ for all $i \in \{1, \dots, p\}$. If this condition holds, we say that x *does not Granger cause* y. If it does not hold (i.e. if at least one of c_i is nonzero), we say that x *Granger causes* y.

A natural way of formulating a model is:

$$y_t = \sum_{i=1}^{q} \beta_i x_{t-i} + u_t, \quad t = 1, \dots, T.$$

Assuming $q \geq p$, we can easily imagine (and prove) that the usual Wald test with respect to $H_0 : \beta_1 = \cdots = \beta_q = 0$ will perform perfectly (i.e. no size distortions and power approaching 1) *in large sample*. However, we may get size distortions when q is large and T is small. It is therefore worth considering a parsimonious model with $q < p$ to control size, but an important question here is how much power we can get. The main goal of this exercise is to analyze that point in the simplest possible setting.

2 Questions

Assume that the true DGP is:

$$
\begin{aligned}
y_t &= \sum_{i=1}^{2} c_i x_{t-i} + \epsilon_t, \quad \epsilon_t \overset{i.i.d.}{\sim} (0,1), \\
x_t &= \phi x_{t-1} + \eta_t, \quad \eta_t \overset{i.i.d.}{\sim} (0,1), \quad \phi \in (-1,1).
\end{aligned}
$$

Your model is:

$$
y_t = \beta x_{t-1} + u_t.
$$

Let $\hat{\beta}$ be the OLS estimator for β.

Question (a) What is the probability limit of $\hat{\beta}$? Under what conditions is $\hat{\beta}$ consistent for c_1?

Question (b) Formulate the Wald statistic with respect to $H_0 : \beta = 0$ and call it W. Under non-causality (i.e. $c_1 = c_2 = 0$), what is the asymptotic distribution of W?

Question (c) Suppose that $c_1 = 0$ and $c_2 \neq 0$, a seemingly hard type of Granger causality to detect since our model has only one lag. Can we get power approaching one? (Put differently, does W diverge to infinity asymptotically?)

Question (d) In general, under what conditions do you lose power? Are those conditions likely to hold in usual economic applications? (Hint: Find conditions where W does not diverge to infinity even though there is causality.)

Next we replace our DGP with the following in order to consider local power:

$$
\begin{aligned}
y_t &= \sum_{i=1}^{2} \frac{c_i}{\sqrt{T}} x_{t-i} + \epsilon_t, \quad \epsilon_t \overset{i.i.d.}{\sim} (0,1), \\
x_t &= \phi x_{t-1} + \eta_t, \quad \eta_t \overset{i.i.d.}{\sim} (0,1), \quad \phi \in (-1,1).
\end{aligned}
$$

Question (e) Show that W converges to a noncentral chi-squared distribution. Characterize the noncentrality parameter κ and show that $\kappa = 0$ under non-causality.

Question (f) In Question (e) you have derived the limit distribution under non-causality, F_0, and the limit distribution under causality, F_1. Using F_0 and F_1, show the formula for computing power $\mathcal{P}(\alpha)$, where α is a nominal size. (Hint: Recall the definition of size and power. These concepts are extremely important!)

Question (g) Now turn on Matlab. Assume $\alpha = 0.05$ and $\phi = 0.8$. Draw a 3D figure that plots $\mathcal{P}(\alpha)$ against $c_1 \in (-3, 3)$ and $c_2 \in (-3, 3)$. Also draw a contour plot for that figure.

Question (h) Redo Question (g) with $\phi = 0.2$. Explain how your results have changed and provide some intuitive reason for the change.