LPS ECON-103-601, Fall 2014
HW Due 09.17.14 Name: ____________________

Show all work clearly and in order. Circle or box your final answer but points will be awarded based on a correct solution. A solution should always justify the steps taken and explain the assumptions needed to reach a final answer (e.g. how do you know you are not dividing by zero in the last step?).

Q1

Prove by induction that

$$P(E_1 \cup E_2 \cup ... \cup E_n) \leq \sum_{i=1}^{n} P(E_i)$$

for any finite sequence of events $E_1, E_2, ...$ and E_n.

Proof:

Step 1: Show that this statement holds for $n = 1$.

$$P(E_1) \leq P(E_1)$$

The statement holds trivially.

Step 2: Suppose that this statement holds for $n = k$.

$$P(E_1 \cup E_2 \cup ... \cup E_k) \leq \sum_{i=1}^{k} P(E_i)$$

Step 3: Show that this statement holds for $n = k + 1$.

$$\begin{aligned}
P\left((E_1 \cup E_2 \cup ... \cup E_k) \cup E_{k+1}\right) &= P\left(E_1 \cup E_2 \cup ... \cup E_k \cup E_{k+1}\right) \\
&= P\left(E_1 \cup E_2 \cup ... \cup E_k\right) + P(E_{k+1}) - \underbrace{P\left((E_1 \cup E_2 \cup ... \cup E_k) \cap E_{k+1}\right)}_{\in[0,1]} \\
&\leq P\left(E_1 \cup E_2 \cup ... \cup E_k\right) + P(E_{k+1}) \\
&\leq \sum_{i=1}^{k} P(E_i) + P(E_{k+1}) \text{ \textbf{by Step 2}} \\
&= \sum_{i=1}^{k+1} P(E_i)
\end{aligned}$$

$$\Rightarrow P(E_1 \cup E_2 \cup ... \cup E_k \cup E_{k+1}) \leq \sum_{i=1}^{k+1} P(E_i) \quad \textbf{Q.E.D.}$$

Q2

Prove $P(A \cap B^c) = P(A) - P(A \cap B)$.

Proof:

$$\begin{aligned}
&P(B^c) = P(B^c \cap A^c) + P(B^c \cap A) \textbf{ because } B^c \cap A^c \textbf{ and } B^c \cap A \textbf{ are disjoint}\\
\Leftrightarrow\ &P(B^c) = P((B \cup A)^c) + P(B^c \cap A) \textbf{ by DeMorgan's Law}\\
\Leftrightarrow\ &P(B^c) = 1 - P(B \cup A) + P(B^c \cap A)\\
\Leftrightarrow\ &P(B^c) = 1 - P(A \cup B) + P(A \cap B^c)\\
\Leftrightarrow\ &1 - P(B) = 1 - P(A \cup B) + P(A \cap B^c)\\
\Leftrightarrow\ &1 - P(B) = 1 - (P(A) + P(B) - P(A \cap B)) + P(A \cap B^c) \textbf{ using our first proposition}\\
\Leftrightarrow\ &1 - P(B) = 1 - P(A) - P(B) + P(A \cap B) + P(A \cap B^c)\\
\Leftrightarrow\ &1 - P(B) = 1 - P(B) - P(A) + P(A \cap B) + P(A \cap B^c)\\
\Leftrightarrow\ &0 = 0 - P(A) + P(A \cap B) + P(A \cap B^c)\\
\Leftrightarrow\ &P(A \cap B^c) = P(A) - P(A \cap B) \quad \textbf{Q.E.D.}
\end{aligned}$$

This is not necessarily the only proof.

Q3

Prove $P(A^c \cap B^c) = 1 - P(A) - P(B) + P(A \cap B)$

Proof:

$$\begin{aligned}
&P(A^c \cap B^c) = P((A \cup B)^c)) \textbf{ by DeMorgan's Law}\\
\Leftrightarrow\ &P(A^c \cap B^c) = 1 - P(A \cup B)\\
\Leftrightarrow\ &P(A^c \cap B^c) = 1 - (P(A) + P(B) - P(A \cap B)) \textbf{ using our first proposition}\\
\Leftrightarrow\ &P(A^c \cap B^c) = 1 - P(A) - P(B) + P(A \cap B)
\end{aligned}$$

This is not necessarily the only proof.

Q4

Ten fair coins are dropped on the floor. What is the probability that at least two of them show heads?

Solution:

$$\begin{aligned} P(\textbf{At least 2H}) &= 1 - P(\textbf{less than 2H}) \\ &= 1 - [P(\textbf{1H}) + P(\textbf{0H})] \\ &= 1 - P(\textbf{1H}) - P(\textbf{0H}) \end{aligned}$$

Dropping ten coins on the floor is equivalent to dropping one coin on the floor at a time, because the outcome of one toss does not affect the outcome of another. The coins are fair, so we will see {H} on the first coin half of the time. Fixing that we saw H on the first coin, we throw the second coin and suppose we obtain {T}, which occurs half of the time. So we see {HT} half of time we see H first, and we see H first half of the time. Informally, probability is the long run frequency of an event, so we should see {HT} half of half of the time or one-fourth of the time. We will see this formally when defining independence. Extending this logic to ten coins, we see that the probability of obtaining the sequence {TTTTTTTTTT} is $(1/2)^{10}$**, so** $P(\textbf{0H}) = (1/2)^{10}$**. However, 1H can occur many different ways (e.g. {TTTTTTTTTH}, {TTTTTTHTTT}, {HTTTTTTTTT}). How many ways?** ${}_{10}C_1 = 10$**. All of the sequences with a different way of obtaining 1H are disjoint events and the probability of obtaining any one sequence is** $(1/2)^{10}$**. By our definition,**

$$P(\textbf{1H}) = \sum_{i=1}^{10} P(\textbf{sequence with 1H}) = \sum_{i=1}^{10} (1/2)^{10} = 10 \cdot \frac{1}{2^{10}}.$$

Hence,

$$P(\textbf{At least 2H}) = 1 - 10 \cdot \frac{1}{2^{10}} - \frac{1}{2^{10}}$$

Q5

A fair coin is flipped ten times. What is the probability that heads comes up at least once?

Solution:

By Question 4, we see that

$$P(\textbf{At least 1H}) = 1 - \frac{1}{2^{10}}.$$

Q6

Three integers are picked randomly from the range 1 - 20, inclusive. What is the probability that the value of the second number lies exclusively in between the values of the first and third number?

<u>Solution:</u>

First note that for the second number to be exclusively in between the first and third, all three numbers must be different.

$$P(\textbf{all different numbers}) = \frac{20}{20} \cdot \frac{19}{20} \cdot \frac{18}{20} = \frac{19 \cdot 18}{20^2},$$

because we draw a number and then have 19 possibilities out of 20 and then 18 possibilities out of 20. As in Q4 and Q5, looking at the long run frequencies mean we multiply the probability to obtain the joint probability. Now, we need to compute the probability that these three numbers are also in the proper order, which is another event. There are three positions, and we must fix the middle number to be the 2nd position. So there are 2! valid permutations out of 3! possible ways to arrange these numbers, so the probability of the middle number being between the two (fixing the three numbers) is 2!/3! Thus,

$$\begin{aligned}
P(\textbf{2nd \# between 1st and 3rd}) = & \\
&= P(\textbf{all different numbers}) \cdot P(\textbf{middle \# in 2nd slot}) \\
&= \frac{19 \cdot 18}{20^2} \cdot \frac{2!}{3!} \\
&= \frac{57}{200}
\end{aligned}$$

Q7

David is dealt a hand consisting of five cards from a Salty Card Deck. A Salty Card Deck has fifty-four cards: a numberless silver card, a numberless golden card, and a standard deck of fifty-two playing cards. What is the probability that David gets dealt two pairs?

<u>Solution:</u>

There are $_{54}C_5$ ways to chose a group of five cards, so this is the number of possible events. First, we choose the two ranks (e.g. Ace, King, Queen, Two) of cards that will be our pairs, which can occur in $_{13}C_2$ ways. Note that the silver and gold cards cannot make pairs with any other cards. Among those two ranks, there are four cards that can form pairs for each of the two pairs (e.g. Ace of clubs, Ace of hearts, Ace of spades, Ace of diamonds) for $_4C_2$ combinations of pairs. So we have a total of $_{13}C_2 \cdot {}_4C_2$ ways to choose two pairs.

Finally, we can pick any remaining card to be the fifth card with ${}_{46}C_1$ chooses, since it cannot be any of the eight cards that might form a pair. Hence,

$$P(\textbf{two pairs}) = \frac{{}_{13}C_2 \cdot {}_4C_2 \cdot {}_4C_2 \cdot {}_{46}C_1}{{}_{54}C_5}$$

Q8

John is mailing letters to n friends, all of whom have different addresses. He has his n letters and n envelopes already addressed to his friends, but, in a fit of whimsy, John decides to randomly assign each letter to an envelope. Each envelope receives exactly one letter. Richard and Diane are two of Johns friends who are sent letters. What is the probability that Richard or Diane (but not both) receive the correct letter in the envelope addressed to him or her?

<u>**Solution:**</u>

Overall, there are

$${}_nC_1 \cdot {}_{n-1}C_1 \cdot {}_{n-2}C_1 \cdot {}_{n-3}C_1 \cdot {}_{n-4}C_1 \cdot ... \cdot 1 = n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot (n-4)!$$

ways for the letters to be sent. Fixing that Richard receives the correct letter, then there are

$${}_{n-1}C_1 \cdot {}_{n-2}C_1 \cdot {}_{n-3}C_1 \cdot {}_{n-4}C_1 \cdot ... \cdot 1 = (n-1) \cdot (n-2) \cdot (n-3) \cdot (n-4)!$$

ways for the letters to be sent. Fixing that Diane receives the correct letter, then there are

$${}_{n-1}C_1 \cdot {}_{n-2}C_1 \cdot {}_{n-3}C_1 \cdot {}_{n-4}C_1 \cdot ... \cdot 1 = (n-1) \cdot (n-2) \cdot (n-3) \cdot (n-4)!$$

ways for the letters to be sent. Hence,

$$P(\textbf{Richard}) = P(\textbf{Diane}) = \frac{(n-1) \cdot (n-2) \cdot (n-3) \cdot (n-4)!}{n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot (n-4)!} = \frac{1}{n}$$

Fixing that Richard and Diane receive the correct letters. Then there are

$${}_{n-2}C_1 \cdot {}_{n-3}C_1 \cdot {}_{n-4}C_1 \cdot ... \cdot 1 = (n-2) \cdot (n-3) \cdot (n-4)!$$

for the letters to be sent. Hence,

$$P(\textbf{Richard} \cap \textbf{Diane}) = \frac{(n-2) \cdot (n-3) \cdot (n-4)!}{n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot (n-4)!} = \frac{1}{n(n-1)}$$

$$\Rightarrow P(\textbf{Richard} \cup \textbf{Diane}) = P(\textbf{Richard}) + P(\textbf{Diane}) - P(\textbf{Richard} \cap \textbf{Diane})$$

$$= \frac{2}{n} - \frac{1}{n(n-1)} = \frac{2n-3}{n(n-1)}$$

Now, we know the probability that Richard or Diane receives the correct letter. However, the question says "not both", so we need to remove the extra overlap (you can see this in a Venn Diagram).

$$P(\textbf{Richard or Diane but not both}) = \frac{2n-3}{n(n-1)} - \frac{1}{n(n-1)} = \frac{2n-4}{n(n-1)}$$

Note that for $n = 2$, $P(\textbf{Richard or Diane but not both}) = 0$. **If Richard receives the correct letter, then Diane must receive the correct letter as well if there are only two of envelopes.**