Chapter 1

Counting, Combinations and Permutations

Counting serves as the basis to do elementary probability. Understanding some simple principles of counting is crucial to applying the probability to the problems that later follow, notability Bayes' Rule. If you can master these simple techniques for counting, then probability will be quite straightforward.

1.1 Fundamental Counting Principle

The basic principle for counting remains the same regardless of order or replacement of the object we count. This principle is sometimes known as the *Fundamental Counting Principle*. It says that if we have K events and n_k outcomes for each event, then there exists $n_1 \cdot n_2 \cdot n_3 \cdot \ldots \cdot n_K$ ways for the events to occur. Equivalently, we may assign n_k values to each of the k items. In the case that the number of the outcomes we can assign to each item or event is the same, the number of outcomes is n^K. We will appeal to this principle over and over again in what follows.

Example Consider a deck of 52 card. How many ways can we draw 3 cards without putting the drawn cards back in the deck?

$$\underline{52} \cdot \underline{51} \cdot \underline{50} = 132,600$$

Suppose we put the drawn cards back in the deck, what is the number of outcomes now?

$$\underline{52} \cdot \underline{52} \cdot \underline{52} = 140,608$$

The first case is an example of a combination without replacement, and the second shows a combination with replacement. In either case, the exact order of the cards is irrelevant. The (Jack, Queen, King) triple is the same as the (King, Jack, Queen) triple. We impose the importance of order, because we treat each triplet as a different outcomes. We call this case a *k-permutation* where $k = 3$. There are six way to order the 3 cards we draw.

$$\underline{3} \cdot \underline{2} \cdot \underline{1} = 3! = 6$$

Alternatively, we can look at all the alternatives for our (Jack, Queen, King) triple to see there are 6 ways to order our 3 cards.

Card 1	Card 2	Card 3
King	Queen	Jack
King	Jack	Queen
Queen	Jack	King
Queen	King	Jack
Jack	Queen	King
Jack	King	Queen

There are 6 ways to order every triple and there are $51 \times 50 \times 49$ way to draw 3 cards in order without replacement and 52^3 ways to draw 3 cards in order with replacement. If we want to ignore the order of the cards, then we need to reduce the number of possible draws by a factor of $\frac{1}{6}$. Why? We need to account for the fact that our (King, Queen, Jack) triplet can occur 6 times, but we only want to count it once. Already, we see there are two key dimensions to counting that we must consider $-$ repetition and order. Counting the number of possible outcomes becomes straightforward once we determine these dimensions in most problems.

1.2 Permutations

A *permutation* is an ordering of a collection of objects. For instance, the (King, Queen, Jack) triple is one permutation and the (Queen, Jack, King) permutation is another. So the question "how many ways can we order a deck of cards?" is equivalent to "how many permutations exist in a deck of cards?" From the counting principle we established, the total number of permutations without replacing the cards is

$$\underline{52} \cdot \underline{51} \cdot ... \cdot \underline{1} = 52!$$

The "!" denotes evaluating the factorial of a number. If we replace the cards in the deck each draw and draw a new deck with 52 cards, then there are 52^{52} permutations!

Definition We define the factorial of a number $N \in \mathbb{N}$ as follows:

$$N! = \Pi_{j=0}^{N}(N-j),$$
$$0! \equiv 1.$$

where Π denotes the product of the sequence defined by $N - j$ for $j = 0, 1, ..., N$.

We can now generalize the means of finding the number of permutations of a collection of objects using the factorial. The number of permutations possible for a collection of N objects is $N!$ At this point, we evaluate the permutation over the entire set of objects. Usually, we only want to look at subset of possible events or objects, so instead we are interested in a *k-permutation.* A *k-permutation* is an ordering of k objects from a set of K objects. Obviously, a K-permutation is the usual permutation and 1-permutation is K. Returning to the 3 card without replacement draws, we saw $52 \cdot 51 \cdot 50$ the permutations. Note that

$$52 \cdot 51 \cdot 50 = \frac{52!}{49!} = \frac{52!}{(52-3)!}$$

This observation leads to the following Lemma.

Proposition 1.2.1 *The number of* k-permutations *from a collection of N objects or outcomes without replacement is*

$$_NP_k = \frac{N!}{(N-k)!}$$

Now suppose, we ignore the different suits of cards. The number of permutations for the 52 card deck we calculated before now includes permutations that are not distinct. In the deck, we have a multiplicity of 4 for each card. We must reduce the number of permutations to eliminate this multiplicity for each card in order to count the number of *distinct permutations.*

Proposition 1.2.2 *The number of* distinct permutations *of N objects with M distinct subsets of identical objects repeating with multiplicities of $k_1,\ k_2, ..., k_M$, respectively, is*

$$\frac{N!}{k_1!k_2!k_3!...k_M!}$$

Hence, there are $\frac{52!}{(4!)^{13}}$ distinct permutations in a deck of 52 cards. How many distinct permutation exist in the word MISSISSIPPI? Determining the number of *distinct k-permutations* is a much harder problem.

1.3 Binomial Coefficients & Combinations

We introduce binomial coefficients and combinations together because of their intimate relationship. Combinations deal with cases where order is irrelevant. In our deck of cards, we have to reduce the number of counts when we ignore the order of the 3 cards drawn. From Section 1.1, we can deduce that a *combination* is a collection of order irrelevant k objects or outcomes from a collection of N objects or outcomes and the following:

Proposition 1.3.1 *The number of combinations when drawing k objects from a collection of N without replacement is*

$$_NC_k = \left(\begin{array}{c} N \\ k \end{array} \right) = \frac{N!}{k!(N-k)!}$$

and

$$\left(\begin{array}{c} N+k-1 \\ k \end{array} \right) = \frac{(N+k-1)!}{k!(N-1)!}$$

when drawing with replacement.

We can also see that $_NC_k = {}_NC_{N-k}$ by substituting $N-k$ for k in the formula above. We saw why multiplying by $\frac{1}{k!}$ changed the number of permutations into the number of combinations in our cards example. It reduces the multiplicity that comes from drawing three of the same cards in different orders. However, our cards example draws without replacement. Let us construct a new example to understand how many combinations of size k we obtain with replacement. Now we will draw 3 cards with replacement and count the number of possible combinations. Suppose we lined up the cards in no particular order and imagine some machine goes down the line of cards. The machine starts at the first card in line and decides either to *choose* (C) the card or *skip* (S) it. If the machine decided to skip, then it moves onto the next card. If the machine chooses, then it repeat the choose-skip decision. It repeats this decision at most k times (Why?). So some examples of the choosing the first card include:

$$\begin{array}{l} S \\ C-S \\ C-C-S \\ C-C-C \end{array}$$

Suppose the machine chose $C-C-S$ for the first card. Then it only has one C left. So every card will receive a skip except one. By default, the machine skips the rest of the cards once it picks 3 cards. The machine's choices may look something like

$$C-C-\underbrace{S-S-S-S-...-S-C-S-...-S}_{51\ skips\ +\ 1\ choice}$$

or

$$S-S-\underbrace{C-S}_{3^{rd}\ card}-\underbrace{C-S}_{4^{th}\ card}-...-S-C-S-...-S$$

The machine uses 51 skips and 3 choices in the first pattern. In the second pattern, the machine uses again 3 choices and 51 skips. The machine only skips each card at most once until each reaches the end of the line. So how many decisions does the machine make? 54 or $N+k-1$. The order of the cards is arbitrary, so suppose the machine reaches the last card every time with only choosing 2 cards. Then it must choose the last card and cannot use a skip, hence the machine has k choices (C) and $N-1$ skips (S). We can always reorder the cards to see that the machine cannot make a skip on the last card since order does not matter. In essence, the machine fills $N+k-1$ positions with k choices. How many different ways can it do this? $_{N+k-1}C_k$. The machine represents the choice we make when taking combinations with replacement. Note that choosing S or C remains the same decision because $_{N+k-1}C_k = {}_{N+k-1}C_{N-1}$.

For binomial coefficients, we consider the following binomial expansions

$$\begin{aligned}
(x+y)^0 &= 1 \\
(x+y)^1 &= x+y \\
(x+y)^2 &= x^2+2xy+y^2 \\
(x+y)^3 &= x^3+3x^2y+3xy^2+y^3.
\end{aligned}$$

Writing out the coefficients only, we see *Pascal's Triangle* where one is the first and last entry on the rows and each entry is the sum of the two entries above it.

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 1 & & 1 & & \\
 & 1 & & 2 & & 1 & \\
1 & & 3 & & 3 & & 1
\end{array}$$

We can calculate the entry for the k^{th} position (going left to right) in the n^{th} row by ${}_{n-1}C_{k-1}$. For example, the third position in the fourth row is ${}_3C_2 = 3$. In our binomial expansions, the n^{th} row corresponds to raising our two numbers to the $n-1$ power. n corresponds to the first power of the first number, $n-1$ corresponds to the second power of the first number and so on. So we have the following proposition, which we can prove by mathematical induction.

Proposition 1.3.2 *The m-order binomial expansion of $x+y$ is*

$$(x+y)^m = \sum_{j=1}^{m+1} {}_mC_{j-1} \cdot x^{m-j+1} \cdot y^{j-1}$$

or

$$(x+y)^m = \sum_{j=0}^{m} {}_mC_j \cdot x^{m-j} \cdot y^j$$

where the coefficients correspond to the $m+1$ row of Pascal's Triangle.