LPS ECON-103-601, Fall 2014
HW Due 11.03.14 Name: ____________________

Show all work clearly and in order. Circle or box your final answer but points will be awarded based on a correct solution. A solution should always justify the steps taken and explain the assumptions needed to reach a final answer (e.g. how do you know you are not dividing by zero in the last step?).

For Q1-Q3, suppose $X_i \sim i.i.d. N(\mu, \sigma^2)$

Q1

Prove $\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$.

We know that $X_i \sim i.i.d. N(\mu, \sigma^2)$, so $\bar{X}$ is distributed normal given the result that the sum of normals is also normal and knowing $\bar{X} = \frac{1}{n}\sum_{i=1} X_i$. $\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}$ is still normal but with an adjustment to the mean and variance, because (μ, σ^2) are not random variables. We just need to calculate the mean and variance.

$$\mathbb{E}\left[\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right] = \frac{1}{\sigma/\sqrt{n}}\mathbb{E}\left[\bar{X}-\mu\right] = \frac{1}{\sigma/\sqrt{n}}\left[\mathbb{E}\left[\bar{X}\right]-\mu\right] = \frac{1}{\sigma/\sqrt{n}}[\mu-\mu] = 0$$

$$Var\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right) = \frac{1}{\sigma^2/n}\cdot Var(\bar{X}-\mu) = \frac{1}{\sigma^2/n}\cdot Var(\bar{X}) = \frac{\sigma^2/n}{\sigma^2/n} = 1$$

Hence, $\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$. Note that we know $X_i \sim i.i.d. N(\mu, \sigma^2)$, so this distribution is *exact*. We did not apply the Central Limit Theorem.

Q2

Prove $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$. You may use the fact that $\bar{X}$ and S^2 are independent.

$\bar{X}$ and S^2 are independent is given.

$$\begin{aligned}
\sum_{i=1}^{n}(X_i-\mu)^2 &= \sum_{i=1}^{n}(X_i-\bar{X}+\bar{X}-\mu)^2 \\
&= \sum_{i=1}^{n}(X_i-\bar{X})^2 + n(\bar{X}-\mu)^2 \\
\Rightarrow \sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2 &= \frac{(n-1)S^2}{\sigma^2} + \left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right)^2,
\end{aligned}$$

dividing by σ^2 on both sides and substituting S^2 for $\sum_{i=1}^{n}(X_i-\bar{X})^2$. Now the right hand side is chi-squared random variables with n degrees of freedom, because $\frac{X_i-\mu}{\sigma}$ is a standard normal random variable. Similarly, $\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right)^2$ is a chi-squared random variable with 1 degree of freedom (because it is the square of a standard normal from Q1). From here, we can deduce that the left hand side is the sum of two independent chi-squared random variables (given $\bar{X} \perp S^2$) - one with 1 degree of freedom and hence the other must have $n-1$ degrees of freedom. When summing up independent chi-squared, the degrees of freedom add up. Why?

$$\sum_{i=1}^{k_1} Z_i^2 + \sum_{i=1}^{k_2} Z_i^2 = \sum_{i=1}^{k_1+k_2} Z_i^2 \sim \chi^2_{k_1+k_2} \textbf{ if } Z_i \sim N(0,1)$$

Q3

Prove $\frac{\bar{X}-\mu}{S/\sqrt{n}} \sim t_{n-1}$.

$$
\begin{aligned}
\frac{(\bar{X}-\mu)}{S/\sqrt{n}} &= \frac{\frac{(\bar{X}-\mu)}{\sigma/\sqrt{n}}}{\sqrt{\frac{S^2}{\sigma^2}}} \\
&= \frac{\frac{(\bar{X}-\mu)}{\sigma/\sqrt{n}}}{\sqrt{\frac{(n-1)S^2}{(n-1)\sigma^2}}} \\
&= \frac{\frac{(\bar{X}-\mu)}{\sigma/\sqrt{n}}}{\sqrt{\frac{1}{n-1}\frac{(n-1)S^2}{\sigma^2}}} \\
&= \frac{\frac{(\bar{X}-\mu)}{\sigma/\sqrt{n}}}{\sqrt{\frac{1}{n-1}\frac{(n-1)S^2}{\sigma^2}}} \\
Z &\equiv \frac{(\bar{X}-\mu)}{\sigma/\sqrt{n}} \sim N(0,1) \\
Y &\equiv \frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1) \\
Z \perp Y \Rightarrow &= \frac{Z}{\sqrt{Y/(n-1)}} \sim t_{n-1}
\end{aligned}
$$

$Z \perp Y$ **means** Z **and** Y **are independent.**

Q4

Suppose S^1 and S^2 are the sample variances of independent random sample of size n_1 and n_2 for normal populations (i.e. normal random variables) with variances σ_1^2 and σ_2^2. Prove $\sigma_2^2 S_1^2/\sigma_1^2 S_2^2 \sim F_{n_1-1,n_2-1}$.

$$
\begin{aligned}
(n_1-1)S_1^2/\sigma_1^2 &\sim \chi^2(n_1-1) \\
(n_2-1)S_2^2/\sigma_2^2 &\sim \chi^2(n_2-1) \\
\Rightarrow \frac{(n_1-1)S_1^2/\sigma_1^2}{(n_2-1)S_2^2/\sigma_2^2} &\sim \frac{\chi^2(n_1-1)}{\chi^2(n_2-1)} \\
\Rightarrow \frac{n_2-1}{n_1-1}\cdot\frac{(n_1-1)S_1^2/\sigma_1^2}{(n_2-1)S_2^2/\sigma_2^2} &\sim \frac{n_2-1}{n_1-1}\cdot\frac{\chi^2(n_1-1)}{\chi^2(n_2-1)} \\
\Leftrightarrow \frac{n_2-1}{n_1-1}\cdot\frac{(n_1-1)S_1^2/\sigma_1^2}{(n_2-1)S_2^2/\sigma_2^2} &\sim \frac{\chi^2(n_1-1)/(n_1-1)}{\chi^2(n_2-1)/(n_2-1)} \\
&= F_{n_1-1,n_2-1} \textbf{ because } S_1^2 \perp S_2^2 \\
\Rightarrow \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} &\sim F_{n_1-1,n_2-1}
\end{aligned}
$$