LPS ECON-103-601, Fall 2014
HW Due 10.27.14 Name: ____________________

Show all work clearly and in order. Circle or box your final answer but points will be awarded based on a correct solution. A solution should always justify the steps taken and explain the assumptions needed to reach a final answer (e.g. how do you know you are not dividing by zero in the last step?).

Multivariate Transformations

Transformations of a Random Vector

A *random vector* is a vector of random variables. It is the multivariate version of a random variable. Just like each random variable has a distribution, the random vector has a distribution characterised by a joint PMF or PDF. We can transform this random vector just like we transform a random variable using a *one-to-one* transformation. Suppose (X,Y) is a continuous random vector with joint PDF $f_{X,Y}(x,y)$ and g is some transformation where $u = g_1(x,y)$ and $v = g_2(x,y)$. If the components of g (g_1 and g_2) are one-to-one transformations of $supp(X,Y)$ to $supp(U,V)$ where $supp(U,V) = \{(u,v) : u = g_1(x,y) \text{ and } g_2(x,y) \text{ for some } (x,y) \in supp(X,Y)\}$, then the inverse transformations $x = h_1(u,v)$ and $y = h_2(u,v)$ exist. Furthermore, we can define the Jacobian of this transformation H by

$$J = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}.$$

The Jacobian is the matrix of first derivatives. When J consists of only one component (e.g. x) instead of two or more (e.g. x and y), then it is a vector and we call it the *gradient*. In this case, H is consist of two one-to-one functions of u and v, so the Jacobian is a square matrix. Assuming J is not zero over $supp(U,V)$, the joint pdf of (U,V) is zero outside of its support and given by

$$f_{U,V}(u,v) = f_{X,Y}(h_1(u,v), h_2(u,v)) \cdot |J|$$

on its support. $|J|$ denotes the determinant of the J matrix.

Q1: Joint PMF without Jacobian

Show that if $X \sim Poisson(\theta)$ and $Y \sim Poission(\lambda)$ and X and Y are independent, then $X + Y \sim Poisson(\theta + \lambda)$. Recall that if $U \sim Poisson(\xi)$, then

$$f_U(u) = \frac{\xi^u e^{-\xi}}{u!}$$

and if U and V are independent, then

$$f_{U,V}(u,v) = f_U(u) \cdot f_V(v).$$

Q2: Joint PDF with Jacobian

Let X and Y be independent, standard normal variables (i.e. mean zero and variance one). Consider the transformation H where $(U,V) = H(X,Y)$, $U = X+Y$ and $V = X-Y$.

(a) What is the the joint PDF of X and Y?

(b) Find $h_1(x,y)$ and $h_2(x,y)$.

(c) Is the transformation H one-to-one? Why?

(d) What is $supp(X,Y)$ and $supp(U,V)$?

(e) Find the joint PDF of (U,V).

(f) Are U and V independent? How do you know?

Q3

Consider U as the random variable above and $V = Y$.

(a) What is $\mathbb{E}[U]$? $\mathbb{E}[V]$?

(b) What is $Var(U)$? $Var(V)$?

(c) What is $Cov(U,V)$?

Q3

Let X and Y be independent $N(0,1)$ random variables. Consider the transformation $U = \frac{X}{Y}$. Find $f_U(u)$. (Hint: You want to define H such that it consists of two one-to-one transformations, so you can use the Jacobian transformation to obtain a joint PDF and then the marginal for U. Try $V = |Y|$ and note that $supp(X,Y) \neq supp(U,V)$).

Q4

Prove that $Var(X \pm Y) = Var(X) + Var(Y) \pm 2 \cdot Cov(X,Y)$

Q5

Prove that $Cov(aX+b, cY+d) = a \cdot b \cdot Cov(X,Y)$ for any random variables X and Y and $a,b,c,d \in \mathbb{R}$.