LPS ECON-103-601, Fall 2014
HW Due 10.27.14 Name: ______________________________

Show all work clearly and in order. Circle or box your final answer but points will be awarded based on a correct solution. A solution should always justify the steps taken and explain the assumptions needed to reach a final answer (e.g. how do you know you are not dividing by zero in the last step?).

Multivariate Transformations

Transformations of a Random Vector

A *random vector* is a vector of random variables. It is the multivariate version of a random variable. Just like each random variable has a distribution, the random vector has a distribution characterised by a joint PMF or PDF. We can transform this random vector just like we transform a random variable using a *one-to-one* transformation. Suppose (X, Y) is a continuous random vector with joint PDF $f_{X,Y}(x, y)$ and g is some transformation where $u = g_1(x, y)$ and $v = g_2(x, y)$. If the components of g (g_1 and g_2) are one-to-one transformations of $supp(X, Y)$ to $supp(U, V)$ where $supp(U, V) = \{(u, v) : u = g_1(x, y) \text{ and } g_2(x, y) \text{ for some } (x, y) \in supp(X, Y)\}$, then the inverse transformations $x = h_1(u, v)$ and $y = h_2(u, v)$ exist. Furthermore, we can define the Jacobian of this transformation H by

$$J = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}.$$

The Jacobian is the matrix of first derivatives. When J consists of only one component (e.g. x) instead of two or more (e.g. x and y), then it is a vector and we call it the *gradient*. In this case, H is consist of two one-to-one functions of u and v, so the Jacobian is a square matrix. Assuming J is not zero over $supp(U, V)$, the joint pdf of (U, V) is zero outside of its support and given by

$$f_{U,V}(u, v) = f_{X,Y}(h_1(u, v), h_2(u, v)) \cdot |J|$$

on its support. $|J|$ denotes the determinant of the J matrix.

Q1: Joint PMF without Jacobian

Show that if $X \sim Poisson(\theta)$ and $Y \sim Poission(\lambda)$ and X and Y are independent, then $X + Y \sim Poisson(\theta + \lambda)$. Recall that if $U \sim Poisson(\xi)$, then

$$f_U(u) = \frac{\xi^u e^{-\xi}}{u!}$$

and if U and V are independent, then

$$f_{U,V}(u,v) = f_U(u) \cdot f_V(v).$$

The support of the Poisson random variable is the set of non-negative integers. We use Poisson discrete random variables for counting events with outcomes like random timing or a random number of realizations. Let us define $Z = X+Y$ so that the event $\{Z = k\}$ (where $k \in \mathbb{Z}_+$) represented by the random variable Z is the union of disjoint events

$$\begin{aligned}
\{Z = k\} &= \{X = k \cap Y = 0\} \cup \{X = k-1 \cap Y = 1\} \cup ... \cup \{X = 0 \cap Y = k\} \\
&= \bigcup_{i=0}^{k} \{X = i \cap Y = k-i\} \\
\Rightarrow P(\{Z = k\}) &= P\left(\bigcup_{i=0}^{k} \{X = i \cap Y = k-i\}\right) \\
&= \sum_{i=0}^{k} P(\{X = i \cap Y = k-i\}) \textbf{ because the events are disjoint} \\
&= \sum_{i=0}^{k} P(\{X = i\}) \cdot P(\{Y = k-i\}) \textbf{ because } X \textbf{ and } Y \textbf{ are independent} \\
&= \sum_{i=0}^{k} p_X(i) \cdot p_Y(k-i) \textbf{ because } p_X(x) = P(\{X = x\}) \textbf{ by definition} \\
&= \sum_{i=0}^{k} \frac{\theta^i e^{-\theta}}{i!} \cdot \frac{\lambda^{(k-i)} e^{-\lambda}}{(k-i)!} \textbf{ substituting the definition of a Poisson PMF} \\
&= \sum_{i=0}^{k} \frac{1}{i!(k-i)!} \theta^i \lambda^{k-i} e^{-(\theta+\lambda)} \\
&= \sum_{i=0}^{k} \frac{k!}{k!} \cdot \frac{1}{i!(k-i)!} \cdot \theta^i \lambda^{k-i} \cdot e^{-(\theta+\lambda)} \\
&= \sum_{i=0}^{k} \frac{k!}{i!(k-i)!} \theta^i \lambda^{k-i} \frac{e^{-(\theta+\lambda)}}{k!} \\
&= \frac{e^{-(\theta+\lambda)}}{k!} \cdot \sum_{i=0}^{k} \frac{k!}{i!(k-i)!} \theta^i \lambda^{k-i} \\
&= \frac{e^{-(\theta+\lambda)}}{k!} \cdot \sum_{i=0}^{k} \binom{k}{i} \theta^i \lambda^{k-i} \\
&= \frac{e^{-(\theta+\lambda)}}{k!} \cdot (\theta+\lambda)^k \textbf{ reversing the binomial expansion} \\
\Rightarrow P(\{Z = k\}) &= \frac{(\theta+\lambda)^k e^{-(\theta+\lambda)}}{k!} \textbf{ which is the PMF of } Z \sim Poission(\theta+\lambda)
\end{aligned}$$

Q2: Joint PDF with Jacobian

Let X and Y be independent, standard normal variables (i.e. mean zero and variance one). Consider the transformation H where $(U,V) = H(X,Y)$, $U = X+Y$ and $V = X-Y$.

(a) What is the the joint PDF of X and Y?

$$f(x,y) = f(x)\cdot f(x) = \frac{1}{2\pi}\exp\left\{-\frac{1}{2}(x^2+y^2)\right\}$$

(b) Find $h_1(u,v)$ and $h_2(u,v)$.

$$\begin{aligned} U = X+Y,\ V &= X-Y \\ \Rightarrow X = \frac{U+V}{2},\ Y &= \frac{U-V}{2} \\ \Rightarrow h_1(u,v) = \frac{u+v}{2},\ h_2(u,v) &= \frac{u-v}{2} \end{aligned}$$

(c) Is the transformation H one-to-one? Why?

Yes, because linear transformations are one-to-one. Every (U,V) maps into a unique (X,Y).

(d) What is *supp*(X,Y) and *supp*(U,V)?

$$supp(X,Y) = supp(U,V) = (-\infty,\infty)\times(-\infty,\infty)$$

(e) Find the joint PDF of (U,V).

$$
\begin{aligned}
f_{U,V}(u,v) &= f_{X,Y}(h_1(u,v), h_2(u,v)) \cdot |J| \\
&= f_{X,Y}\left(\frac{u+v}{2}, \frac{u-v}{2}\right) \cdot |J| \\
J &= \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} \\
\Rightarrow |J| &= |-1/2 \cdot 1/2 - 1/2 \cdot 1/2| = |-1/2| = 1/2 \\
f_{U,V}(u,v) &= \frac{1}{2\pi} \exp\left\{-\frac{1}{2}\left[\left(\frac{u+v}{2}\right)^2 + \left(\frac{u-v}{2}\right)^2\right]\right\} \cdot \frac{1}{2} \\
&= \frac{1}{2\pi \cdot 2} \exp\left\{-\frac{1}{2}\left[\left(\frac{u}{2}\right)^2 + \left(\frac{v}{2}\right)^2\right]\right\}
\end{aligned}
$$

(f) Are U and V independent? How do you know?

Yes, they are independent, because we can express the joint as the product of the marginals.

$$
\begin{aligned}
f_{U,V}(u,v) &= \frac{1}{2\pi \cdot 2} \exp\left\{-\frac{1}{2}\left[\left(\frac{u}{2}\right)^2 + \left(\frac{v}{2}\right)^2\right]\right\} \\
&= \frac{1}{2\pi \cdot 2} \exp\left\{-\frac{1}{2}\left(\frac{u}{2}\right)^2\right\} \cdot \exp\left\{-\frac{1}{2}\left(\frac{v}{2}\right)^2\right\} \\
&= \underbrace{\frac{1}{\sqrt{2\pi \cdot 2}} \exp\left\{-\frac{1}{2}\left(\frac{u}{2}\right)^2\right\}}_{U \sim N(0,2)} \cdot \underbrace{\frac{1}{\sqrt{2\pi \cdot 2}} \exp\left\{-\frac{1}{2}\left(\frac{v}{2}\right)^2\right\}}_{V \sim N(0,2)}
\end{aligned}
$$

So for X and Y independent and standard normal, the sum/difference of X and Y is also a normal random variable.

$$\mathbb{E}[U] = \mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y] = 0 + 0 = 0$$

$$\mathbb{E}[V] = \mathbb{E}[X-Y] = \mathbb{E}[X] - \mathbb{E}[Y] = 0 - 0 = 0$$

$$Var(U) = Var(X+Y) = Var(X) + Var(Y) = 1 + 1 = 2$$

$$Var(V) = Var(X-Y) = Var(X) + Var(Y) = 1 + 1 = 2$$

We can also this case more generally that X and Y independent and normal, the sum (difference) of X and Y is also a normal random variable with mean $\mathbb{E}[X] \pm \mathbb{E}[Y]$ and variance $Var(X) + Var(Y)$.

Q3

Consider U as the random variable above and $V = Y$.

(a) What is $\mathbb{E}[U]$? $\mathbb{E}[V]$?

We know from above that $\mathbb{E}[U] = \mathbb{E}[X+Y] = 0$ and $\mathbb{E}[V] = \mathbb{E}[Y] = 0$.

(b) What is $Var(U)$? $Var(V)$?

We know from above that $Var[U] = Var[X]+Var[Y] = 2$ and $Var[V] = Var[Y] = 1$.

(c) What is $Cov(U, V)$?

$$Cov(U, V) = Cov(X+Y, Y) = Cov(X, Y) + Cov(Y, Y) = 0 + Var(Y) = 1$$

because X and Y are mean zero it is the case that

$$Cov(X+Y, Y) = \mathbb{E}[(X+Y)(Y)] = \mathbb{E}[XY] + \mathbb{E}[Y^2] = Cov(X, Y) + Var(Y)$$

Q4

Let X and Y be independent $N(0, 1)$ random variables. Consider the transformation $U = \frac{X}{Y}$. Find $f_U(u)$. (Hint: You want to define H such that it consists of two one-to-one transformations, so you can use the Jacobian transformation to obtain a joint PDF and then the marginal for U. Try $V = |Y|$ and note that $supp(X, Y) \neq supp(U, V)$).

Define the transformations $U = \frac{X}{Y}$ and $V = |Y|$, so that

$$supp(U, V) = (-\infty, \infty) \times [0, \infty)$$

The transformation is one-to-one, because $V = Y$ if $Y \geq 0$ and $V = -Y$ for $Y < 0$ and similarly $X = -UV$ for $Y < 0$ and $X = UV$ for $Y \geq 0$. Now $X = U \cdot V$ and $Y = V$ on the support of (U, V), so the Jacobian is

$$|J| = \left| \begin{pmatrix} v & u \\ 0 & 1 \end{pmatrix} \right| = |v| = v$$

Now

$$
\begin{aligned}
f_{U,V}(u,v) &= 2 \cdot f_{X,Y}(u \cdot v, v) \cdot v \text{ \textbf{we multiply 2 to add the density for} } Y < 0 \text{ \textbf{branch}} \\
&= \frac{1}{\pi} \exp\left\{-\frac{1}{2}\left[(uv)^2 + v^2\right]\right\} \cdot v \\
&= \frac{v}{\pi} \exp\left\{-\frac{1}{2} v^2 \left[u^2 + 1\right]\right\} \\
\Rightarrow f_U(u) &= \int_0^\infty \frac{v}{\pi} \exp\left\{-\frac{1}{2} v^2 \left[u^2 + 1\right]\right\} \mathrm{d}v \\
&= \frac{1}{\pi(u^2+1)}
\end{aligned}
$$

We call this marginal pdf the Cauchy distribution, which is the ratio of two standard normal random variables.

Q5

Prove that $Var(X \pm Y) = Var(X) + Var(Y) \pm 2 \cdot Cov(X, Y)$.

$$
\begin{aligned}
Var(X \pm Y) &= \mathbb{E}[(X \pm Y - \mathbb{E}[X \pm Y])^2] \\
&= \mathbb{E}[(X \pm Y - \mathbb{E}[X] \mp \mathbb{E}[Y])^2] \\
&= \mathbb{E}[((X - \mathbb{E}[X]) \pm (Y - \mathbb{E}[Y]))^2] \\
&= \mathbb{E}[(X - \mathbb{E}[X])^2 + (Y - \mathbb{E}[Y])^2 \pm (X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] \\
&= \mathbb{E}[(X - \mathbb{E}[X])^2] + \mathbb{E}[(Y - \mathbb{E}[Y])^2] \pm \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] \\
&= Var(X) + Var(Y) \pm 2Cov(X, Y) \quad \blacksquare
\end{aligned}
$$

Q6

Prove that $Cov(aX + b, cY + d) = a \cdot c \cdot Cov(X, Y)$ for any random variables X and Y and $a, b, c, d \in \mathbb{R}$.

$$
\begin{aligned}
Cov(aX + b, cY + d) &= \mathbb{E}\left[(aX + b - \mathbb{E}[aX + b])(cY + d - \mathbb{E}[cY + d])\right] \\
&= \mathbb{E}\left[(aX - a\mathbb{E}[X])(cY - c\mathbb{E}[Y])\right] \\
&= a \cdot c \cdot \mathbb{E}\left[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])\right] \\
&= a \cdot c \cdot Cov(X, Y) \quad \blacksquare
\end{aligned}
$$