LPS ECON-103-601, Fall 2014
Midterm Practice Solutions

Exam Instructions: You will have two hours to complete this exam (17.00-19.00). No calculators or notes are permitted. Write your name on each bluebook. Show all work clearly and labelled in your bluebook. Circle or box your final answer but points will be awarded based on a correct solution. A solution or proof should always justify the steps taken and explain the assumptions needed to reach a final answer (e.g. how do you know you are not dividing by zero in the last step?). Failure to follow instructions will result in a lose of marks.

Counting, Permutuations & Combinations

Q1

When the following binomial expression $(x+y)^{12}$ is fully expanded, what is the sum of the coefficients?

We know

$$(x+y)^{12} = {}_{12}C_0 \cdot x^{12} + {}_{12}C_1 \cdot x^{11}y + {}_{12}C_{02} \cdot x^{10}y^2 + ... + {}_{12}C_1 \cdot xy^{11} + {}_{12}C_{12} \cdot y^{12}$$

To obtain the sum of the coefficients, we can set $x = 1, y = 1$. Hence, the sum of the coefficients is $(1+1)^{12} = 2^{12}$

Q2

Let X equal the number of ways that a group of ten ducks can be chosen from twenty distinct ducks and let Y equal the number of ways that an eleven person committee can be chosen from twenty candidates. What is the value of X + Y?

Let's look at the general case

$$\begin{aligned}
\frac{n!}{(n-k)!k!} + \frac{n!}{(n-k+1)!(k-1)!} &= \left(\frac{n!}{(n-k+1)!k!}\right) \cdot \left(\frac{(n-k+1)!}{(n-k)!} + k\right) \\
&= \frac{n!}{(n-k+1)!k!} \cdot (n+1) \\
&= \frac{(n+1)!}{(n+1-k)!k!} \\
&= {}_{n+1}C_k
\end{aligned}$$

So in our specific case, the value of $X + Y$ is ${}_{21}C_{11}$.

Q3

One general, two captains, two majors, and three sergeants are seated at a fixed round table for a meeting. How many distinct arrangements based on rank can they be seated around the table?

This question ask for the number of distinct permutations but now we have to account for the fact that we arranged them around a circle. Recall that the number of distinct permutation in a line is

$$\frac{8!}{1!2!2!3!} = 8 \cdot 7 \cdot 6 \cdot 5$$

We can think of permutation as being indistinguishable in a circle if we move the first position to the end. We can move the positions from 1, 2, 3, 4, 5, 6, 7, 8 to 2, 3, 4, 5, 6, 7, 8, 1 and have the same permutation. So there is a multiplicity of 8 in the number permutations when we connect the ends to form a circle. To remove the multiplicity, we divide by 8. Hence, the number of distinct arrangements is $7 \cdot 6 \cdot 5 = 210$**.**

Q4

Thomas, John, Lindsey, Luke, and Josh are waiting in line at the movie theatre. In how many different ways can they queue up if John must be in front of Thomas?

John can only be in position 1, 2, 3 or 4. If John is first in the queue, then there are 4! ways to arrange the others. If John is second in the queue, then there are $3 \cdot 3!$ **ways to arrange the others.**

$$\underline{3} \cdot \underline{J} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1}$$

If John is third in the queue, then there are $3 \cdot 2 \cdot 2$ **ways to arrange the others.**

$$\underline{3} \cdot \underline{2} \cdot \underline{J} \cdot \underline{2} \cdot \underline{1}$$

If John is fourth in the queue, then there are $3!$ **ways to arrange the others.**

$$\underline{3} \cdot \underline{2} \cdot \underline{1} \cdot \underline{J} \cdot \underline{1}$$

Hence, there are $24 + 18 + 12 + 6 = 60$ **ways they can queue up.**

Q5

How many 3-digit numbers have exactly one zero?

$$\underline{9} \cdot \underline{1} \cdot \underline{9} + \underline{9} \cdot \underline{9} \cdot \underline{1} = 2 \cdot 81 = 162$$

Descriptive Statistics

Q1

What is the difference between a parameter and a statistic? **A parameter is a characteristic of a population whereas a statistic is a characteristic of sample.**

Q2

Which of the following measures are robust to outliers? Why?
(I) Mean (II) Median (III) Standard Deviation (IV) Range (V) Interquartile Range

(II) Median and (V) IQR, because moving a outlier further away does not change their value.

Q3

Suppose some data is skewed-right. Which is likely greater, the median or the mean?

$Mean > Median$. **Why?**

Q4

True or False: We can establish a cause and effect relationship using a complete census of the population. Why or Why not?

False but why? Think about confounders

Q5

Why does sampling error occur and how it is different from nonsampling error? Give an example.

Each sample differs so sampling error occurs due to randomness, but nonsampling error is a systematic error. Can you think of an example of nonsampling error aside from the one given in class?

Probability Theory

Q1

Define a conditional probability space.

See Chapter 3 of Lecture Notes

Q2

State and Prove Bayes' Rule.

See Chapter 3 of Lecture Notes and use the defintion of conditional probability

Q3

Prove $P(X \cup Y) = P(X) \cdot P(Y^c|X) + P(Y)$.

$$\begin{aligned} P(X \cup Y) &= P(X) + P(Y) - P(X \cap Y) \\ &= P(X) + P(Y) - P(X) \cdot P(Y|X) \\ &= P(X) \cdot (1 - P(Y|X)) + P(Y) \\ &= P(X) \cdot P(Y^c|X) + P(Y) \ \blacksquare \end{aligned}$$

Q4

Prove that if X and Y are independent, then X and Y^c are independent.

$$\begin{aligned} P(X) &= P((X \cap Y) \cup (X \cap Y^c)) \\ &= P(X \cap Y) + P(X \cap Y^c) \\ P(X) &= P(X) \cdot P(Y) + P(X \cap Y^c) \\ P(X \cap Y^c) &= P(X)(1 - P(Y)) \\ P(X \cap Y^c) &= P(X) \cdot P(Y^c) \ \blacksquare \end{aligned}$$

Q5

Prove $P(X \cap Y \cap Z) = -P(Z)P(X \cup Y|Z) + P(Z)P(X|Z) + P(Z)P(Y|Z)$.

$$\begin{aligned} P(X \cap Y \cap Z) &= P(Z) \cdot P(X \cap Y|Z) \\ &= P(Z)[-P(X \cup Y|Z) + P(X|Z) + P(Y|Z)] \\ &= -P(Z)P(X \cup Y|Z) + P(Z)P(X|Z) + P(Z)P(Y|Z) \ \blacksquare \end{aligned}$$

Probability Applications

Q1

Bill rolls two fair, standard 6-sided dice. What is the probability that sum of the two numbers he rolls equals 7?

The ways to sum to 7 are $\{(1,6),(6,1),(2,5),(5,2),(3,4),(4,3)\}$ **out of** (6×6) **36 possible sums. Hence,** $P(sum=7)=\frac{1}{6}$**.**

Q2

The words "color" and "colour" are two spellings of the same word. Out of the worlds English speakers. 60% of them spell the word as color. Call such people American. 40% of the worlds English speakers spell the word as colour. Call such people European. A random English speaker is asked to write the word, and a letter is randomly chosen from his spelling. Given that a vowel is chosen, what is the probability that the speaker is American?

$$P(\textbf{American}|\textbf{vowel}) =$$
$$P(\textbf{American}|\textbf{vowel ``u''})\cdot P(\textbf{vowel ``u''})\ +\ P(\textbf{American}|\textbf{vowel ``o''})\cdot P(\textbf{vowel ``o''})$$

$$\begin{aligned}&P(\textbf{American}|\textbf{vowel ``u''})=0\\ &\quad\Rightarrow P(\textbf{American}|\textbf{vowel})=P(\textbf{American}|\textbf{vowel ``o''})\cdot P(\textbf{vowel ``o''})\end{aligned}$$

$$P(\textbf{American}|\textbf{vowel ``o''})=\frac{P(\textbf{American})\cdot P(\textbf{vowel ``o''}|\textbf{American})}{P(\textbf{vowel ``o''})}\ \textbf{by Bayes' Rule}$$
$$\Rightarrow P(\textbf{American}|\textbf{vowel})=P(\textbf{American})\cdot P(\textbf{vowel ``o''}|\textbf{American})=0.6\cdot 0.4=0.24$$

Q3

Felix, Paul-Julien, and Sean are being put in a line with seven additional students (everyone in line is distinguishable). If the order of the line is chosen at random, what is the probability that the group of Felix, Paul-Julien, and Sean (in any order) is at the front of the line (i.e. the first three people in line must be some arrangement of Felix, Paul-Julien, and Sean)?

There are $3!$ **ways to arrange Felix, Paul-Julien and Sean at the front of the line and** $7!$ **ways to arrange to everyone else. Hence, the probability that those three are at the front of the line is** $\frac{3!7!}{10!}=\frac{1}{120}$**.**

Q4

A point is randomly chosen on a hundred foot pole and the pole is cut at that point into two pieces. What is the probability that the shorter piece is less than 1/3 the size of the longer piece?

Given $X + Y = 100$ and $X < Y$, the question ask what is $P(X < Y/3)$?

$P(X < Y/3) = P(100-Y < Y/3) = P(300-3Y < Y) = P(300 < 4Y) = P(Y > 75) = 0.25$

Q5

In Mirrielees, 30% of people speak German and 16% of people who speak German speak Chinese. Given that 20% of people in Mirrielees speak Chinese, what is the probability that a randomly selected person speaks German if he or she speaks Chinese?

Using Bayes' Rule,

$$\begin{aligned} P(\textbf{Speak German} \mid \textbf{Speak Chinese}) &= \frac{P(\textbf{Speak German and Speak Chinese})}{P(\textbf{Speak Chinese})} \\ &= \frac{0.3 \cdot 0.16}{0.2} \\ &= 0.24 \end{aligned}$$

Q6

Dustin downloads two programs onto his computer. Each program independently has an 80% chance that its a virus. Given that at least one of the two programs is a virus, what is the probability that both programs are viruses?

$$
\begin{aligned}
P(\textbf{program with virus}) &= 0.8 \\
P(\textbf{both have viruses}) &= 0.8 \cdot 0.8 \\
&= 0.64 \\
P(\textbf{at least a virus}) &= 1 - P(\textbf{no viruses}) \\
&= 1 - 0.2 \cdot 0.2 = 0.96 \\
P(\textbf{both have viruses}|\textbf{at least a virus}) &= \frac{P(\textbf{both have viruses} \cap \textbf{at least a virus}))}{P(\textbf{at least a virus})} \\
&= \frac{P(\textbf{both have viruses})}{P(\textbf{at least a virus})} \\
&= \frac{0.64}{0.96} \\
&= \frac{2}{3}
\end{aligned}
$$

Q7

Joyce has fifty turtles that will eat anything they can and a magical kale leaf that only one specific turtle in her group can eat. If she randomly picks turtles (without replacement) from her group to eat the magical kale leaf, what is the probability that the ninth turtle she picks will eat the kale leaf?

She needs to pick 8 wrong turtles and then the right turtle, so $49/50 \cdot 48/49 \cdot \cdot 1/42 = 1/50$**.**

Q8

A machine produces parts that are either good (90%), slightly defective (2%), or obviously defective (8%). Produced parts get passed through an automatic inspection machine, which is able to detect any part that is obviously defective and discard it. What is the quality of the parts that make it through the inspection machine and get shipped?

The quality of the parts is just the probability that the part is good given that the part is not obviously defective (OD).

$$P(G) = 0.9, P(SD) = 0.02, P(OD) = 0.08$$

$$P(G|OD^c) = \frac{P(G \cap OD^c)}{P(OD^c)} = \frac{P(G)}{1 - P(OD)} = \frac{0.9}{1 - 0.08} = 0.978$$

Q9

Suppose Aziz mixed up five good fuses and two defective ones. To find the defective fuses, he tests them one-by-one, at random and without replacement. What is the probability that he is lucky and finds both of the defective fuses in the first two tests?

Let D_t be the event of detecting the defective fuse in the t^{th} test. We want to compute

$$P(D_1 \cap D_2) = P(D_1) \cdot P(D_2|D_1) = \frac{2}{7} \cdot \frac{1}{6} = \frac{1}{21}$$

Q10

One half percent of the population has a particular disease. A test is developed for the disease. The test gives a false positive 3% of the time and a false negative 2% of the time.
a) What is the probability that a random person tests positive?
b) Jack just got the bad news that the test came back positive, what is the probability that Jack has the disease?

a)

$$\begin{aligned} P(T = +) &= P(T = +|D) \cdot P(D) + P(T = +|D^c) \cdot P(D^c) \\ &= 0.98 \cdot 0.005 + 0.03 \cdot 0.995 \end{aligned}$$

b)

$$\begin{aligned} P(D|T = +) &= \frac{P(D) \cdot P(T = +|D)}{P(T = +)} \\ &= \frac{0.005 \cdot 0.98}{0.98 \cdot 0.005 + 0.03 \cdot 0.995} \\ &\approx 0.14 \end{aligned}$$

Probability Distributions and Density

Q1

What is the upper bound on the support of f such that f is proper density function where $f(x) = x^2$ and $x > 0$?

For a proper density function,

$$\int_0^c x^2 \mathrm{d}x = 1$$
$$\Rightarrow \frac{c^3}{3} = 1$$
$$\Leftrightarrow c = 3^{1/3}$$

Hence, the support of f is $[0, 3^{1/3}]$.

Q2

Calculate $\mathbb{E}[Z]$ where Z is a binomial random variable over n trials with probability p (Recall: $P^{(n)}(Z = k) = {}_nC_k \cdot p^k \cdot (1-p)^{n-k}$).

By definition,

$$\begin{aligned}
\mathbb{E}[Z] &= \sum_{k=0}^{n} k \cdot {}_nC_k \cdot p^k \cdot (1-p)^{n-k} \\
&= \sum_{k=1}^{n} k \cdot {}_nC_k \cdot p^k \cdot (1-p)^{n-k} \\
&= p \sum_{k=1}^{n} k \cdot {}_nC_k \cdot p^{k-1} \cdot (1-p)^{n-k} \\
&= p \sum_{k=1}^{n} k \cdot {}_nC_k \cdot p^{k-1} \cdot (1-p)^{(n-1)-(k-1)} \\
&= p \sum_{k=0}^{\tilde{n}} k \cdot {}_nC_k \cdot p^{\tilde{k}} \cdot (1-p)^{\tilde{n}-\tilde{k}}
\end{aligned}$$

$$\begin{aligned}
k \cdot {}_nC_k &= k \frac{n!}{(n-k)!k!} \\
&= n \cdot \frac{(n-1)!}{(k-1)!(n-k)!} \\
&= n \cdot {}_{n-1}C_{k-1} \\
&= n \cdot {}_{\tilde{n}}C_{\tilde{k}}
\end{aligned}$$

$$
\begin{aligned}
\Rightarrow \mathbb{E}[Z] &= p \sum_{k=0}^{\tilde{n}} n \cdot {}_{\tilde{n}}C_{\tilde{k}} \cdot p^{\tilde{k}} \cdot (1-p)^{\tilde{n}-\tilde{k}} \\
&= np \sum_{k=0}^{\tilde{n}} \cdot {}_{\tilde{n}}C_{\tilde{k}} \cdot p^{\tilde{k}} \cdot (1-p)^{\tilde{n}-\tilde{k}}
\end{aligned}
$$

$$
\begin{aligned}
\sum_{k=0}^{\tilde{n}} \cdot {}_{\tilde{n}}C_{\tilde{k}} \cdot p^{\tilde{k}} \cdot (1-p)^{\tilde{n}-\tilde{k}} &= \sum_{k=0}^{\tilde{n}} P^{(\tilde{n})}(Z = \tilde{k}) \\
&= 1
\end{aligned}
$$

$$
\Rightarrow \mathbb{E}[Z] = np
$$

Q3

Prove $Var(aX + b) = a \cdot Var(X)$.

$$
\begin{aligned}
&Var(aX+b) = \mathbb{E}\left([(aX+b) - \mathbb{E}(aX+b)]^2\right) \\
&Var(aX+b) = \mathbb{E}\left([(aX+b) - a\mathbb{E}(X) - b]^2\right) \ \textbf{by Linearity of Expectations} \\
&Var(aX+b) = \mathbb{E}\left([(aX - a\mathbb{E}(X)]^2\right) \\
&Var(aX+b) = a^2 \underbrace{\mathbb{E}\left([(X - \mathbb{E}(X)]^2\right)}_{=Var(X)} \ \textbf{by Linearity of Expectations} \\
&Var(aX+b) = a^2 \cdot Var(X) \ \blacksquare
\end{aligned}
$$

Q4

Calculate the variance of a continuous random variable $X \sim \text{Uniform}(0,1)$.

$$
\begin{aligned}
Var(X) &= \int_0^1 (\theta - \mathbb{E}(X))^2 f(\theta) d\theta \\
&= \int_0^1 \left(\theta - \frac{1}{2}\right)^2 d\theta \\
&= \frac{\left(1 - \frac{1}{2}\right)^3}{3} - \frac{\left(0 - \frac{1}{2}\right)^3}{3} \\
&= \frac{\left(\frac{1}{2}\right)^3}{3} + \frac{\left(\frac{1}{2}\right)^3}{3} \\
&= \frac{2}{3} \cdot \frac{1}{8} = \frac{1}{12}
\end{aligned}
$$

Q5

Calculate $\mathbb{E}[W]$ where W is a Poisson discrete random variable where $P(W = k) = \frac{\lambda^k e^{-\lambda}}{k!}$.

First, note that $k \in [0, \infty)$. By definition,

$$
\begin{aligned}
\mathbb{E}[W] &= \sum_{k=0}^{\infty} k \cdot \frac{\lambda^k e^{-\lambda}}{k!} \\
&= \lambda e^{-\lambda} \sum_{k=0}^{\infty} \cdot \frac{\lambda^{k-1}}{(k-1)!} \\
&= \lambda e^{-\lambda} \sum_{\tilde{k}=0}^{\infty} \cdot \frac{\lambda^{\tilde{k}}}{\tilde{k}!} \\
&= \lambda e^{-\lambda} \sum_{\tilde{k}=0}^{\infty} \cdot \frac{\lambda^{\tilde{k}}}{\tilde{k}!}
\end{aligned}
$$

Recall,

$$
\sum_{k=0}^{\infty} \cdot \frac{x^k}{k!} = e^x
$$

$$
\Rightarrow \mathbb{E}[W] = \lambda
$$