Chapter 9

Interval Estimation

So far we only looked at (closed form) point estimators like MME and MLE. But the point estimate $\hat{\theta}$ tells us nothing about the accuracy of the estimate. We want to know how informative a statistic is about the true value. We also want to know what values of the true parameter θ are consistent with the data. For these purposes, we turn to interval estimation. Interval estimation takes the statistic and perhaps other information and constructs bound for the true parameter. These bounds are also random variables, because they are based a random variable. For a different sample, we may obtain different bounds. We call these bounds a *confidence interval.*

Definition The interval $[\hat{\theta}_1, \hat{\theta}_2]$ is called a confidence interval for θ if $\hat{\theta}_1$ and $\hat{\theta}_2$ are realisations of random variables $\hat{\Theta}_1$ and $\hat{\Theta}_2$ such that

$$\mathbb{P}r(\hat{\Theta}_1 < \theta < \hat{\Theta}_2) = 1 - \alpha$$

where $\alpha \in [0, 1]$. Meaning over repeated sample, the confidence interval contains the true parameter (θ) $(1-\alpha)100\%$ of time. We call $(1-\alpha)100\%$ the *degree of confidence.* We call the lower bound and upper bound of the interval *confidence limits.*

Remark Note that this definition does not mean $\theta \in [\hat{\theta}_1, \hat{\theta}_2]$ with probability $1-\alpha$. This concept is entirely different altogether and known as a *credible set.* Credible sets appear in Bayesian econometrics, which is beyond the scope of this course.

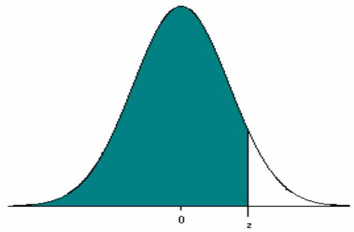
In order to construct confidence intervals, we need to know the sampling distribution of our statistic. From the sampling distribution, we obtain bounds for the normalised statistic and then we invert the statistic to obtain confidence limits.

Example Confidence Interval for μ, σ known

We know when $X_i \sim i.i.d.N(\mu, \sigma^2)$

$$Z \equiv \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$$

Z-TABLE



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

We can look at the Z-Table above to determine the bounds $\{\beta_1, \beta_2\}$ such that

$$\mathbb{P}r(\beta_1 \leq Z \leq \beta_2) = 0.95$$

to construct a 95% confidence interval for $\bar{X}$. In this class, we construct symmetric confidence intervals, so we want the probability in the upper and lower tails of the sampling distribution to be 2.5%. In the row with 1.9 and column with 0.06, we see

$$\mathbb{P}r(Z \leq 1.96) = 0.975.$$

The row number gives the first and tenths digit of the value of β and the column gives the hundredths digits where $\mathbb{P}r(Z \leq \beta)$ is the number in the table. The normal distribution is symmetric, so we know

$$\mathbb{P}r(Z \leq -1.96) = 0.025.$$

and

$$\mathbb{P}r(Z \leq 1.96) - \mathbb{P}r(Z \leq -1.96) = \mathbb{P}r(-1.96 \leq Z \leq 1.96) = 0.95,$$

hence $\{\beta_1, \beta_2\} = \{-1.96, 1.96\}$. Now we are ready to invert our test statistic.

$$\begin{aligned}
\mathbb{P}r(-1.96 \leq Z \leq 1.96) &= 0.95 \\
\mathbb{P}r\left(-1.96 \leq \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \leq 1.96\right) &= 0.95 \\
\mathbb{P}r\left(-1.96\frac{\sigma}{\sqrt{n}} \leq \bar{X}-\mu \leq 1.96\frac{\sigma}{\sqrt{n}}\right) &= 0.95 \\
\mathbb{P}r\left(-\bar{X}-1.96\frac{\sigma}{\sqrt{n}} \leq -\mu \leq -\bar{X}+1.96\frac{\sigma}{\sqrt{n}}\right) &= 0.95 \\
\mathbb{P}r\left(\bar{X}-1.96\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{X}+1.96\frac{\sigma}{\sqrt{n}}\right) &= 0.95
\end{aligned}$$

The second step substitutes in the normalised statistic, which we defined above. The last step results from multiplying the inequality by -1 which flips the direction of the inequality. Hence, our 95% CI is

$$\left[\bar{X}-1.96\frac{\sigma}{\sqrt{n}}, \bar{X}+1.96\frac{\sigma}{\sqrt{n}}\right]$$

Remark Notice that the confidence interval depends on $\bar{X}$ which is a random variable. $\bar{X}$ may take on different values depending on the random sample taken. Also notice the confidence interval is symmetric by construction. The confidence interval is centred on $\bar{X}$. As $\bar{X}$ moves around in a different sample so too does the confidence interval. However, its width remains fixed. So we look at the width of the confidence interval to learn about the statistic. A wider the confidence interval indicates a less informative statistic, because we need to a larger interval to capture the true parameter $(1-\alpha)100\%$ of the time. Usually, we want to know whether the confidence interval contains zero. If it does not, then we conclude with 95% confidence or allowing 5% error that the true parameter value is not zero.

Example Confidence Interval for μ, σ unknown

We know when $X_i \sim i.i.d.N(\mu, \sigma^2)$

$$t \equiv \frac{\bar{X}-\mu}{S/\sqrt{n}} \sim t_{n-1}$$

Suppose $n = 15$ so that

$$\mathbb{P}r(-2.145 \leq t \leq 2.145) = 0.95.$$

The upper for t comes from the T-Table, and the Student-t distribution (with 14 degrees of freedom) is symmetric like the normal distribution so the lower confidence limits is the -1 times the upper confidence limit. Now we can invert the test statistic, which are the same steps as before, so the 95% CI for μ is

$$\left[\bar{X}-2.145\frac{S}{\sqrt{n}}, \bar{X}+2.145\frac{S}{\sqrt{n}}\right]$$

This confidence interval may be wider than the confidence interval with σ known. A statistic is less informative when estimate its variance $\left(\sigma^2/\sqrt{n}\text{ in this case}\right)$, because we do not know the true precision of the estimator. We need a wider confidence interval to contain μ 95% of the time over repeated samples, so the confidence interval is wider.

Example Confidence Interval for σ^2

Recall

$$Y = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$$

Suppose $n = 15$ so that $\{\chi^2_{0.025}(14), \chi^2_{0.975}(14)\} = \{4.660, 26.119\}$ and

$$\mathbb{P}r(4.660 \leq Y \leq 26.119) = 0.95$$

Now we invert the statistic to obtain the confidence interval for σ^2.

$$\begin{aligned}
\mathbb{P}r(4.660 \le Y \le 26.119) &= 0.95 \\
\mathbb{P}r\left(4.660 \le \frac{(n-1)S^2}{\sigma^2} \le 26.119\right) &= 0.95 \\
\mathbb{P}r\left(\frac{4.660}{(n-1)S^2} \le \frac{1}{\sigma^2} \le \frac{26.119}{(n-1)S^2}\right) &= 0.95 \\
\mathbb{P}r\left(\frac{(n-1)S^2}{4.660} \le \sigma^2 \le \frac{(n-1)S^2}{26.119}\right) &= 0.95
\end{aligned}$$

Hence, the 95% CI is

$$\left[\frac{(n-1)S^2}{4.660}, \frac{(n-1)S^2}{26.119}\right].$$

The width of the confidence interval depends directly on S^2. Why? Think about what S^2 is.

Example Confidence Interval for $\mu_1 - \mu_2$, σ_1^2, σ_2^2 known

Suppose we have two normal, independent samples of size n_1 and n_2. We know from Chapter 7 that the sum or difference of two independent normal is also a normal random variable with a mean equal to the sum or difference of their means and variance equal to the sum of their variances. Hence,

$$Z = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \sim N(0,1)$$

Now we can construct the 95% CI

$$\left[(\bar{X}_1 - \bar{X}_2) - \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}, (\bar{X}_1 - \bar{X}_2) - \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}\right].$$