LPS ECON-103-601, Fall 2014
HW Due 10.22.14 Name: ____________________

Show all work clearly and in order. Circle or box your final answer but points will be awarded based on a correct solution. A solution should always justify the steps taken and explain the assumptions needed to reach a final answer (e.g. how do you know you are not dividing by zero in the last step?).

Q1

Let $X_1 \sim Bernoulli(1/2)$ independently of $X_2 \sim Bernoulli(2/3)$ and define $Y = 2X_1$ and $Z = X_2 - X_1$.

(a) Express the joint PMF of Y and Z in tabular form.

First note that $suppY = \{0, 2\}$ and $suppZ = \{-1, 0, 1\}$. Then $p(y, z)$ is tabular form is

$Y \setminus Z$	-1	0	1
0	0	1/6	1/3
2	1/6	1/3	0

If you let $X_2 \sim Bernoulli(1/2)$. Then $p(y, z)$ is tabular form is

$Y \setminus Z$	-1	0	1
0	0	1/4	1/4
2	1/4	1/4	0

(b) Find the marginal PMF of Y.

$$p_Y(0) = 1/2$$
$$p_Y(2) = 1/2$$

(c) Find the conditional PMF of Z?

$$p_{Z|Y}(z|y) = p(y, z)/p(y)$$

$$p_{Z|Y}(z|0) = \begin{cases} 0 & if\ z = -1 \\ 1/3 & if\ z = 0 \\ 2/3 & if\ z = 1 \end{cases}$$

$$p_{Z|Y}(z|2) = \begin{cases} 1/3 & if\ z = -1 \\ 2/3 & if\ z = 0 \\ 0 & if\ z = 1 \end{cases}$$

If instead, you used $X_2 \sim Bernoulli(1/2)$,

$$p_{Z|Y}(z|0) = \begin{cases} 0 & if\ z = -1 \\ 1/2 & if\ z = 0 \\ 1/2 & if\ z = 1 \end{cases}$$

$$p_{Z|Y}(z|2) = \begin{cases} 1/2 & if\ z = -1 \\ 1/2 & if\ z = 0 \\ 0 & if\ z = 1 \end{cases}$$

(d) Calculate $Cov(Y, Z)$. Are Y and Z independent? What is the intuition?

$$Cov(Y, Z) = \mathbb{E}[(Y - \mathbb{E}[Y])(Z - \mathbb{E}[Z])] = \mathbb{E}[YZ] - \mathbb{E}[Z]\mathbb{E}[Y]$$

$$\begin{aligned} \mathbb{E}[Z] &= 1/6 \cdot -1 + 1/2 \cdot 0 + 1 \cdot 1/3 = 1/6 \\ \mathbb{E}[Y] &= 1/2 \cdot 0 + 1/2 \cdot 1 = 1 \\ supp(YZ) &= \{-2, 0, 2\} \\ \mathbb{E}[YZ] &= -2 \cdot 1/6 + 0 \cdot \mathbb{P}r(YZ = 0) + 2 \cdot 0 = -1/3 \\ \Rightarrow Cov(Y, Z) &= \mathbb{E}[YZ] - \mathbb{E}[X]\mathbb{E}[Y] = -1/3 - 1/6 = -1/2 \end{aligned}$$

Y and Z are negatively correlated so they cannot be independent. Intuitively, both random variables depend on X_1, so knowing Y or Z tells us something about X_1 which in turn tells us something about Z or Y, respectively.

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$$Cov(Y, Z) = \mathbb{E}[(Y - \mathbb{E}[Y])(Z - \mathbb{E}[Z])] = \mathbb{E}[YZ] - \mathbb{E}[Z]\mathbb{E}[Y]$$

$$\begin{aligned} \mathbb{E}[Z] &= 1/4 \cdot -1 + 1/2 \cdot 0 + 1 \cdot 1/4 = 0 \\ \Rightarrow Cov(Y, Z) &= \mathbb{E}[YZ] \end{aligned}$$

$$\begin{aligned} supp(YZ) &= \{-2, 0, 2\} \\ \mathbb{E}[YZ] &= -2 \cdot 1/4 + 0 \cdot \mathbb{P}r(YZ = 0) + 2 \cdot 0 = -1/2 \\ \Rightarrow Cov(Y, Z) &= \mathbb{E}[YZ] = -1/2 \end{aligned}$$

Y and Z are negatively correlated so they cannot be independent. Intuitively, both random variables depend on X_1, so knowing Y or Z tells us something about X_1 which in turn tells us something about Z or Y, respectively.

We can also see this directly

$$Cov(Y, Z) = Cov(2X_1, X_2 - X_1) = 2\underbrace{Cov(X_1, X_2)}_{=0} - Cov(X_1, X_1) = -2Var(X_1).$$

We know the variance of a Bernoulli random variable is $p(1 - p) = 1/4$, hence $-2Var(X_1) = -1/2$.

Q2

Let X be a continuous random variable CDF $F(x) = \log_c(x)$ where $c \in \mathbb{R}_{++}$.

(a) What is the support of X?

Let the lower bound of the support be $\hat{a}$ and the upper bound be $\hat{b}$. We know $F(\hat{a}) = 0$ and $F(\hat{b}) = 1$. Hence, $x \in [1, c]$.

(b) What is the $P(a \leq x \leq b)$ where $a < e$ and $b > e$?

We have to construct different cases where $[a, b]$ overlaps with the support.

Case I: $[a, b] \subseteq [1, c]$

$$P(a \leq x \leq b) = F(b) - F(a) = \log_c(b) - \log_c(a) = \log_c(b/a)$$

Case II: $1 < c \leq a$

$$P(a \leq x \leq b) = 0$$

Case III: $1 < a < c$ **and** $b > c$

$$P(a \leq x \leq b) = F(c) - F(a) = 1 - \log_c(a) = \log_c(c/a)$$

Case IV: $1 < b < c$ **and** $a < 1$

$$P(a \leq x \leq b) = F(b) - F(0) = \log_c(b)$$

(c) Calculate $\mathbb{E}[X]$.

$$\begin{aligned}
\mathbb{E}[X] &= \int_1^c x \cdot F'(x) \mathrm{d}x \\
&= \int_1^c x \cdot \frac{1}{x \log(c)} \mathrm{d}x \\
&= \int_1^c \frac{1}{\log(c)} \mathrm{d}x \\
&= \frac{c-1}{\log(c)}
\end{aligned}$$

(d) Calculate $\mathbb{E}[X^2]$.

$$\begin{aligned}
\mathbb{E}[X^2] &= \int\limits_1^c x^2 \cdot F'(x)\mathrm{d}x \\
&= \int\limits_1^c x^2 \cdot \frac{1}{x\log(c)}\mathrm{d}x \\
&= \int\limits_1^c \frac{x}{\log(c)}\mathrm{d}x \\
&= \frac{1}{\log(c)}\int\limits_1^c x\mathrm{d}x \\
&= \frac{c^2-1}{2\log(c)}
\end{aligned}$$

Note that now we can calculate $Var(X)$.

$$Var(X) = \frac{c^2-1}{2\log(c)} - \left(\frac{c-1}{\log(c)}\right)^2$$