LPS ECON-103-601, Fall 2014
HW Due 11.07.14 Name: ______________________________

Show all work clearly and in order. Circle or box your final answer but points will be awarded based on a correct solution. A solution should always justify the steps taken and explain the assumptions needed to reach a final answer (e.g. how do you know you are not dividing by zero in the last step?).

$$\text{Suppose } X_1, ..., X_n \sim i.i.d. N(\mu, \sigma^2)$$

Q1

Derive the likelihood function $L(\mu, \sigma^2; X)$.

$$\begin{aligned}
L(\mu, \sigma^2; X) &= f(X_1, ..., X_n; \mu, \sigma^2) \\
&= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(X_i - \mu)^2\right\} \\
&= \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i - \mu)^2\right\}
\end{aligned}$$

Q2

Derive the maximum likelihood estimator for μ. Is it unbiased? Is it consistent? Justify your answers.

$$\log L(\mu, \sigma^2; X) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i - \mu)^2$$

$$\text{FOC}(\mu): \ \frac{\partial \log L}{\partial \mu} = \frac{1}{\hat{\sigma}^2_{MLE}}\sum_{i=1}^{n}(X_i - \hat{\mu}_{MLE}) = 0 \Rightarrow \hat{\mu}_{MLE} = \bar{X}$$

Q3

Derive the maximum likelihood estimator for σ^2. Is it unbiased? Justify your answer.

$$\log L(\mu, \sigma^2; X) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i - \mu)^2$$

$$\text{FOC}(\sigma^2): \ \frac{\partial \log L}{\partial \sigma^2} = -\frac{n}{2\hat{\sigma}^2_{MLE}} + \frac{1}{2[\hat{\sigma}^2_{MLE}]^2}\sum_{i=1}^{n}(X_i - \hat{\mu}_{MLE})^2 = 0$$

$$\Rightarrow \hat{\sigma}^2_{MLE} = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2$$

Q4

Construct a 95% confidence interval for σ^2 using $\hat{\sigma}^2_{MLE}$.

First note that $\hat{\sigma}^2_{MLE} = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2 = \frac{n-1}{n}S^2$ and we know

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1).$$

Hence, we know that

$$\frac{n\hat{\sigma}^2_{MLE}}{\sigma^2} \sim \chi^2(n-1)$$

Now we proceed with constructing the CI.

$$\mathbb{P}r\left(\chi^2_{0.025}(n-1) < \frac{n\hat{\sigma}^2_{MLE}}{\sigma^2} < \chi^2_{0.975}(n-1)\right) = 0.95$$

Inverting,

$$\mathbb{P}r\left(\frac{\chi^2_{0.025}(n-1)}{n\hat{\sigma}^2_{MLE}} < \frac{1}{\sigma^2} < \frac{\chi^2_{0.975}(n-1)}{n\hat{\sigma}^2_{MLE}}\right) = 0.95$$

$$\mathbb{P}r\left(\frac{n\hat{\sigma}^2_{MLE}}{\chi^2_{0.975}(n-1)} < \sigma^2 < \frac{n\hat{\sigma}^2_{MLE}}{\chi^2_{0.025}(n-1)}\right) = 0.95$$

Hence, the 95% CI is $\left[\frac{n\hat{\sigma}^2_{MLE}}{\chi^2_{0.975}(n-1)}, \frac{n\hat{\sigma}^2_{MLE}}{\chi^2_{0.025}(n-1)}\right]$.

Q5

Derive a method of moments estimator for μ and σ^2. How do the MME estimators for μ and σ^2 compare to the MLE estimators?

$$\mathbb{E}[X] = \mu$$
$$\mathbb{E}[X^2] = \mu^2 + \sigma^2$$

$$\Rightarrow \hat{\mu}_{MME} = \bar{x}$$

$$\Rightarrow \hat{\sigma}^2_{MME} = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}^2$$

$$\begin{aligned}
\hat{\sigma}^2_{MME} &= \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}^2 \\
&= \frac{1}{n}\left[\sum_{i=1}^{n} x_i^2 - n\bar{x}^2\right] \\
&= \frac{1}{n}\left[\sum_{i=1}^{n} x_i^2 - 2n\bar{x}^2 + n\bar{x}^2\right] \\
&= \frac{1}{n}\left[\sum_{i=1}^{n} x_i^2 - 2\bar{x}\cdot n\bar{x} + n\bar{x}^2\right] \\
&= \frac{1}{n}\left[\sum_{i=1}^{n} x_i^2 - 2\bar{x}\cdot \sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\bar{x}^2\right] \\
&= \frac{1}{n}\left[\sum_{i=1}^{n} \left(x_i^2 - 2\bar{x}x_i + \bar{x}^2\right)\right] \\
&= \frac{1}{n}\left[\sum_{i=1}^{n} (x_i - \bar{x})^2\right] \\
&= \hat{\sigma}^2_{MLE}
\end{aligned}$$

Hence, MME estimators are the same as the MLE estimators.