Chapter 1

Counting, Combinations and Permutations

Counting serves as the basis to do elementary probability. Understanding some simple principles of counting is crucial to applying the probability to the problems that later follow, notability Bayes' Rule. If you can master these simple techniques for counting, then probability will be quite straightforward.

1.1 Fundamental Counting Principle

The basic principle for counting remains the same regardless of order or replacement of the object we count. This principle is sometimes known as the *Fundamental Counting Principle*. It says that if we have K events and n_k outcomes for each event, then there exists $n_1 \cdot n_2 \cdot n_3 \cdot \ldots \cdot n_K$ ways for the events to occur. Equivalently, we may assign n_k values to each of the k items. In the case that the number of the outcomes we can assign to each item or event is the same, the number of outcomes is n^K. We will appeal to this principle over and over again in what follows.

Example Consider a deck of 52 card. How many ways can we draw 3 cards without putting the drawn cards back in the deck?

$$\underline{52} \cdot \underline{51} \cdot \underline{50} = 132,600$$

Suppose we put the drawn cards back in the deck, what is the number of outcomes now?

$$\underline{52} \cdot \underline{52} \cdot \underline{52} = 140,608$$

The first case is an example of a combination without replacement, and the second shows a combination with replacement. In either case, the exact order of the cards is irrelevant. The (Jack, Queen, King) triple is the same as the (King, Jack, Queen) triple. We impose the importance of order, because we treat each triplet as a different outcomes. We call this case a *k-permutation* where $k = 3$. There are six way to order the 3 cards we draw.

$$\underline{3} \cdot \underline{2} \cdot \underline{1} = 3! = 6$$

Alternatively, we can look at all the alternatives for our (Jack, Queen, King) triple to see there are 6 ways to order our 3 cards.

Card 1	Card 2	Card 3
King	Queen	Jack
King	Jack	Queen
Queen	Jack	King
Queen	King	Jack
Jack	Queen	King
Jack	King	Queen

There are 6 ways to order every triple and there are $51 \times 50 \times 49$ way to draw 3 cards in order without replacement and 52^3 ways to draw 3 cards in order with replacement. If we want to ignore the order of the cards, then we need to reduce the number of possible draws by a factor of $\frac{1}{6}$. Why? We need to account for the fact that our (King, Queen, Jack) triplet can occur 6 times, but we only want to count it once. Already, we see there are two key dimensions to counting that we must consider $-$ repetition and order. Counting the number of possible outcomes becomes straightforward once we determine these dimensions in most problems.

1.2 Permutations

A *permutation* is an ordering of a collection of objects. For instance, the (King, Queen, Jack) triple is one permutation and the (Queen, Jack, King) permutation is another. So the question "how many ways can we order a deck of cards?" is equivalent to "how many permutations exist in a deck of cards?" From the counting principle we established, the total number of permutations without replacing the cards is

$$\underline{52} \cdot \underline{51} \cdot ... \cdot \underline{1} = 52!$$

The "!" denotes evaluating the factorial of a number. If we replace the cards in the deck each draw and draw a new deck with 52 cards, then there are 52^{52} permutations!

Definition We define the factorial of a number $N \in \mathbb{N}$ as follows:

$$N! = \Pi_{j=0}^{N}(N-j),$$

$$0! \equiv 1.$$

where Π denotes the product of the sequence defined by $N-j$ for $j = 0, 1, ..., N$.

We can now generalize the means of finding the number of permutations of a collection of objects using the factorial. The number of permutations possible for a collection of N objects is $N!$ At this point, we evaluate the permutation over the entire set of objects. Usually, we only want to look at subset of possible events or objects, so instead we are interested in a *k-permutation*. A *k-permutation* is an ordering of k objects from a set of K objects. Obviously, a K-permutation is the usual permutation and 1-permutation is K. Returning to the 3 card without replacement draws, we saw $52 \cdot 51 \cdot 50$ the permutations. Note that

$$52 \cdot 51 \cdot 50 = \frac{52!}{49!} = \frac{52!}{(52-3)!}$$

This observation leads to the following Lemma.

Lemma The number of *k-permutations* from a collection of N objects or outcomes without replacement is

$${}_NP_k = \frac{N!}{(N-k)!}$$

Now suppose, we ignore the different suits of cards. The number of permutations for the 52 card deck we calculated before now includes permutations that are not distinct. In the deck, we have a multiplicity of 4 for each card. We must reduce the number of permutations to eliminate this multiplicity for each card in order to count the number of *distinct permutations.*

Lemma The number of *distinct permutations* of N objects with M distinct subsets of identical objects repeating with multiplicities of $k_1,\ k_2, ..., k_M$, respectively, is

$$\frac{N!}{k_1!k_2!k_3!...k_M!}$$

Hence, there are $\frac{52!}{(4!)^{13}}$ distinct permutations in a deck of 52 cards. How many distinct permutation exist in the word MISSISSIPPI? Determining the number of *distinct k-permutations* is a much harder problem.