FINITE ELEMENT IMPLEMENTATION OF VIRTUAL WORK PRINCIPLE FOR MAGNETIC OR ELECTRIC FORCE AND TORQUE COMPUTATION

J.L. Coulomb and G. Meunier
Laboratoire d'Electrotechnique LA CNRS
ENSIEG BP 46 38402 St Martin d'Hères FRANCE

Abstract - In a recent paper [1], we described a methodology for the numerical evaluation of global magnetic or electric force and torque acting on a movable part of an electromagnetic device. The method is based on application of virtual work principle. Force and torque are obtained as the derivatives of the energy versus the displacement of the moving part.

The paper [1] was essentially theoritical. In the present paper, we give more practical informations about the implementation of the method in a finite element context and illustrate it by a 3D application.

INTRODUCTION

To compute magnetic or electric force and torque acting on a movable but rigid body, the virtual work approach is an alternative to Maxwell's tensor formulation. The force and the torque are obtained as the derivatives of the energy (or coenergy) versus the displacement of the movable part. The theoritical aspects of this method may be found in [1]. In the present paper, we shall discuss some practical points not given in [1].

The force or torque algorithm in finite element field computation is the following :

a) Computation of the nodal values of the potential using classical finite element method

b) Determination of the derivatives of the coordinates of the nodes versus the virtual displacement of the movable part.

c) Computation of the force or torque by integration over virtually distorted finite elements.

The step (a) is classical and will not be discussed here. The step (b) is one of the keys of the power of the implementation. Here, we express the coordinates of the nodes in an homogeneous mode. This allows us to describe very easily any simple or complicated virtual moving and its derivatives. During the step (c), the local derivative method [1] is applied. It is important to notice that exactly the same algorithm acts either for 2D or 3D and force or torque computations.

After a brief explanation on the virtual work method, we will develop the points (c) and (b) and then illustrate it by a 3D application.

VIRTUAL WORK PRINCIPLE

We consider a magnetostatic problem governed by a magnetic scalar potential $\Psi(x,y,z)$ such as the magnetic field vector $H(x,y,z)$ is

$$H = -\text{grad}\ \Psi \qquad (1)$$

The magnetic coenergy is given by

$$W' = \int_V \int_0^H B\,dH\,dv \qquad (2)$$

where $B(x,y,z)$ the magnetic flux density vector is related to H by the constitutive law :

$$B = B(H) \qquad (3)$$

The global force F_s and the global torque C_θ acting on a movable but rigid part of the domain are :

$$F_s = \frac{\partial W'}{\partial s} \qquad C_\theta = \frac{\partial W'}{\partial \theta} \qquad (4)(5)$$

Here, s measures the virtual translation of the movable part along a given direction and θ measures the virtual rotation arround a given axis. In these virtual displacements, the scalar potential is kept unchanged [1]. We remark that the movable part must be strictly surrounded by the empty space to allow virtual moving.

Similar results can be obtained starting from the magnetic vector potential $A(x,y,z)$ and the virtual derivatives of the magnetic energy instead of the scalar potential and the coenergy. The method applies also for electrostatic and magnetodynamic problems.

FINITE ELEMENT CONTEXT

We suppose now that an approximate magnetic field has been computed by the finite element method [2]. Let $\Psi_1, \Psi_2, \ldots, \Psi_i, \ldots, \Psi_n$ the set of nodal values of the potential solution of the finite element process. On each finite element, the approximate scalar potential and the approximate magnetic field are :

$$\Psi = \sum \alpha_i \Psi_i \qquad H = -\sum \text{grad}\,\alpha_i \Psi_i \qquad (6)(7)$$

where $\alpha_i(x,y,z)$ are the shape functions. Then, the approximate magnetic coenergy is :

$$W' = \sum_e \int_{V_e} \int_0^H B\,dH\,dv \qquad (8)$$

here the sum acts on all of the finite elements.

FINITE DIFFERENCE APPROXIMATIONS

Now the aim is to deduce from (4), (5) and (8) some approximate values for force and torque. The finite element solution has been obtained for a given position s_o, respectively θ_o of the movable part. The movable part could be actually displaced at a closed position s_1, respectively θ_1. A finite difference approximation of (4), respectively (5), should give

$$F_s = \frac{W'_1 - W'_o}{s_1 - s_o} \qquad C_\theta = \frac{W'_1 - W'_o}{\theta_1 - \theta_o} \qquad (9)(10)$$

In fact, we DO NOT USE this method because it introduces round off errors.

EXACT DERIVATIVES

The derivatives versus the virtual displacement of the coordinates (x_i, y_i, z_i) of every node can be expressed in any case. For instance, if we are interested in the computation of the x-componant of the force, the derivatives along the x-direction of the nodal coordinates are :

$$\frac{\partial x_i}{\partial s} = p\ , \quad \frac{\partial y_i}{\partial s} = 0\ , \quad \frac{\partial z_i}{\partial s} = 0$$

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with P = 1 for the nodes of movable part,
p = 0 for the nodes of fixed part,
0 ≤ p ≤ 1 arbitrary for the intermediate nodes belonging to the virtually distorted area (empty space).

The coenergy expression (8) is a sum of integrals over all of the finite elements. Each integral depends on the coordinates of the nodes of the element. Knowing the derivative of the nodal coordinates, it is possible to evaluate the derivative of each element integral. Their sum gives an approximate value for the force or torque.

A systematic way to obtain the derivative of (8) is given by the parametric representation which is available for all types of finite elements (2D or 3D, rectilinear or curvilinear, first or second order, ...). Over each element, the global coordinates (x,y,z) are expressed in term of local coordinates (u,v,w)

$$[x\ y\ z] = \sum_i \beta_i . [x_i\ y_i\ z_i] \qquad (12)$$

Where the $\beta_i(u,v,w)$ functions are identical to the $\alpha_i(u,v,w)$ ones (6) for isoparametric elements. The (x_i,y_i,z_i) are the coordinates of the nodes of the element. For instance, for a classical four nodes tetrahedron, we have

$$\alpha_1 = \beta_1 = u \ , \ \alpha_2 = \beta_2 = v$$
$$\alpha_3 = \beta_3 = w \ , \ \alpha_4 = \beta_4 = 1-u-v-w \qquad (13)$$

After some mathematical treatments involving the local jacobian derivative method (see [11), we obtain from (4) and (8) an approximate force

$$F_s = \sum_e \left[\int_{V_e} B \frac{\partial H}{\partial s}\, dv + \int_{V_e} \int_0^H BdH \frac{\partial}{\partial s}(dv) \right] \qquad (14)$$

where we define

$$\frac{\partial}{\partial s}(dv) = /G/^{-1} . \frac{\partial /G/}{\partial s} . dv \qquad (15)$$

/G/ is the determinant of G the jacobian matrix of the transformation (12). The derivative of the field (7) is

$$\frac{\partial H}{\partial s} = - \sum_i \frac{\partial}{\partial s}(\mathrm{grad}\,\alpha_i) . Y_i \qquad (16)$$

with [1]

$$\frac{\partial}{\partial s}(\mathrm{grad}\,\alpha_i) = - G^{-1} . \frac{\partial G}{\partial s} . \mathrm{grad}\,\alpha_i \qquad (17)$$

Similar expressions arise for the torque.

Starting from the relation (12), $\beta_i(u,v,w)$ functions, nodes coordinates and their derivatives versus displacement s or e, it is obvious to build the expressions of G, /G/ and their derivatives [1].

Because the sum (14) is limited to elements belonging to empty space (the undistorted elements give zero contribution and the distorted elements must be in empty space), it is always possible to make analytical integration of (14) over rectilinear elements (even for high order element). Numerical quadrature must be used for curvilinear elements.

COORDINATE DERIVATIVES

The keys of the method of force and torque computation are the derivatives of the nodal coordinates versus the virtual displacement. They are three parts in the domain : the fixed, the movable and the intermediate parts.

The nodes of the fixed part have their derivatives equal to zero.

The coordinate derivatives of the nodes of the movable part can be obtained using the homogeneous coordinates technic :

$$\left[\frac{\partial x_i}{\partial s} \frac{\partial y_i}{\partial s} \frac{\partial z_i}{\partial s} \right] = [\, x_i\ y_i\ z_i\ 1 \,]\ D \qquad (18)$$

where D is a 4x3 matrix which describes the virtual derivative operator. The three operators (allowing F_x, F_y and F_z computations) for virtual translations along the x, y and z axis are

$$\begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} \quad \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} \quad \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} \qquad (19)$$

The three operators (allowing torque computation) for virtual rotations around axis passing by the point (a,b,c) and parallel to the x, y and z axis are

$$\begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & c & -b \end{vmatrix} \quad \begin{vmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ -c & 0 & a \end{vmatrix} \quad \begin{vmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \\ b & -a & 0 \end{vmatrix} \qquad (20)$$

The coordinate derivatives of the nodes of the intermediate area (the free and distorted area between the fixed and the movable parts) can be select arbitrary. We choose smooth, monotone and continuous distribution between zero values for the nodes belonging to the fixed boundary and the total virtual displacement for the nodes common with them of the movable part. A simple way, the one we use, is to restrict this distorted area to only one layer of finite elements (for instance the nearest layer) surrounding the movable part. This layer can be deduced from data very easily by only topological considerations (either for 2D or 3D problems). More sophisticated choice including more than one layer of finite element (the limit is the entire free space) can be done.

TETRAHEDRAL ANALYTICAL INTEGRATION

To give a more precise idea of the method, we will develop the analytical quadrature of (14) in the case of the popular four nodes tetrahedron.

Starting from the shape functions of the tetrahedron (13) and using the isoparametric coordinate expressions (12), we obtain the jacobian matrix (constant inside the element) :

$$G = \begin{bmatrix} x_1-x_4 & y_1-y_4 & z_1-z_4 \\ x_2-x_4 & y_2-y_4 & z_2-z_4 \\ x_3-x_4 & y_3-y_4 & z_3-z_4 \end{bmatrix} \qquad (21)$$

To make easier the matrix handling, we define 3 column vectors X, Y and Z functions of the nodal coordinates :

$$X = \begin{bmatrix} x_1-x_4 \\ x_2-x_4 \\ x_3-x_4 \end{bmatrix} \quad Y = \begin{bmatrix} y_1-y_4 \\ y_2-y_4 \\ y_3-y_4 \end{bmatrix} \quad Z = \begin{bmatrix} z_1-z_4 \\ z_2-z_4 \\ z_3-z_4 \end{bmatrix} \qquad (22)$$

Now G becomes

$$G = [\, X\ Y\ Z \,] \qquad (23)$$

The determinant /G/ of the jacobian matrix G is (for a direct oriented tetrahedron)

$$/G/ = X.(YxZ) = Y.(ZxX) = Z.(XxY) \qquad (24)$$

here the operators . and x mean scalar and inner vectorial products.
The volume V_e of the tetrahedron is related to /G/ by

$$V_e = \frac{/G/}{6} \qquad (25)$$

The inverse matrix G^{-1} of the matrix G is

$$G^{-1} = /G/^{-1} . \left[YxZ \quad ZxX \quad XxY \right]^t \qquad (26)$$

With the derivatives of the coordinates versus the virtual displacement, it is possible to define the derivatives of X, Y and Z from (22). Then, the derivatives of G (23) and /G/ (24) are

$$\frac{\partial G}{\partial s} = \left[\frac{\partial X}{\partial s} \quad \frac{\partial Y}{\partial s} \quad \frac{\partial Z}{\partial s} \right] \qquad (27)$$

$$\frac{\partial /G/}{\partial s} = \frac{\partial X}{\partial s}.(YxZ) + \frac{\partial Y}{\partial s}.(ZxX) + \frac{\partial Z}{\partial s}.(XxY) \quad (28)$$

(same things for θ).

We notice that for undistorted element, the derivatives of X, Y and Z are equal to zero (the four nodes have equal virtual displacements). In this case, the derivatives of G and /G/ are zero and this element has no contribution to the sum (14).

The shape functions and their gradients are

$$[\ \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4\ ] = [\ u\ v\ w\ 1-u-v-w\ ]$$

$$[\mathrm{grad}\alpha_1\ \mathrm{grad}\alpha_2\ \mathrm{grad}\alpha_3\ \mathrm{grad}\alpha_4] = G^{-1}\begin{bmatrix}1&0&0&-1\\0&1&0&-1\\0&0&1&-1\end{bmatrix} \quad (29)\ (30)$$

For a given tetrahedral element, knowing
- .the coordinates of the nodes,
- .theirs derivatives versus virtual displacement,
- .the nodal values of the potential,

we are able to evaluate

G, $/G/$, G^{-1}, $\frac{\partial G}{\partial s}$, $\frac{\partial /G/}{\partial s}$ and V_e

and after

H with (7) and (30) and $\frac{\partial H}{\partial s}$ with (17) (30).

Taking into account that distorted element must be located in empty space, we have

$$B = \mu_o H \quad (31)$$

$$\int_o^H BdH = \frac{\mu_o H^2}{2} \quad (32)$$

Inside a first order tetrahedron, these quantities are independant of space. So the contribution of the element to the sum (14) becomes

$$\left(\mu_o H \frac{\partial H}{\partial s} + \frac{\mu_o H^2}{2} /G/^{-1} \frac{\partial /G/}{\partial s} \right) . V_e \quad (33)$$

(same for θ).

VECTOR POTENTIAL FORMULATION

In this section, we consider the dual case of the scalar potential formulation. Let a magnetic vector potential A(x,y,z) such as the magnetic flux density vector B(x,y,z) is

$$B = \mathrm{curl}\ A \quad (34)$$

The magnetic energy, force and torque are

$$W = \int_V \int_o^H HdB\ dv \quad (35)$$

$$F_s = -\frac{\partial W}{\partial s} \qquad C_\theta = -\frac{\partial W}{\partial \theta} \quad (36)\ (37)$$

The finite element implementation uses the following interpolation (dual of (6) and (7))

$$A = \sum \alpha_i A_i \quad (38)$$

$$B = \sum \mathrm{grad}\alpha_i \times A_i \quad (39)$$

where A_i are the values of the vector potential at the nodes. To obtain these nodal values we use an energy variational formulation added to a penalty term to insure the unicity of the solution (div A = 0) [3].

The force acting on a movable part is (14)

$$F_s = -\frac{\Sigma}{e}\left(\int_{V_e} H \frac{\partial B}{\partial s}\, dv + \int_{V_e} \int_o^B HdB\, \frac{\partial}{\partial s}(dv) \right) \quad (40)$$

The procedure of integration of (40) is the same as for its of (14) exept that

$$\frac{\partial B}{\partial s} = \frac{\partial}{\partial s}(\mathrm{grad}\alpha_i) \times A_i \quad (41)$$

and that in distorted elements (empty space)

$$H = \frac{B}{\mu_o} \quad (42)$$

$$\int_o^B HdB = \frac{B^2}{2\mu_o} \quad (43)$$

For a first order tetrahedron, the analytical quadrature of (40) gives

$$-\left(\frac{B}{\mu_o} \frac{\partial B}{\partial s} + \frac{B^2}{2\mu_o} /G/^{-1} \frac{\partial /G/}{\partial s} \right) . V_e \quad (44)$$

Similar results arise for the torque.

3D EXAMPLE

A comparison between theoritical and finite element force and torque computations has already been given in [1] for a 2D magnetostatic problem solved by scalar potential. In this paper, we illustrate the method with a 3D magnetostatic problem solved with a vector potential formulation.

We consider the two identical cylindrical coils carrying the same current and embedded in empty space (fig. 1). It is a linear axisymmetric problem treated as a 3D one. The domain is discretized into second order tetrahedron using an automatic mesh generator [4]. After the finite element solution has been found, the axial force acting on a coil has been computed. Numerical result and exact one are in good agreement. We point out that the method can also be applied on any rigid body (magnetic material linear or not, carrying current or not).

CONCLUSION

A method allowing electric and magnetic force and torque computations has been exposed. This method is based on energy or coenergy consideration. So, it is easily implemented in a finite element package working on the same principles.

It makes use of derivatives of nodal coordinates versus virtual displacements. We have seen how these derivatives can be deduced from the nodal coordinates using differential matricial operators for the virtual translation and rotation. Some practical aspects concerning the finite element formulae have been discussed in the case of scalar or vectorial magnetic potential. Analytical quadrature for tetrahedral elements have been given.

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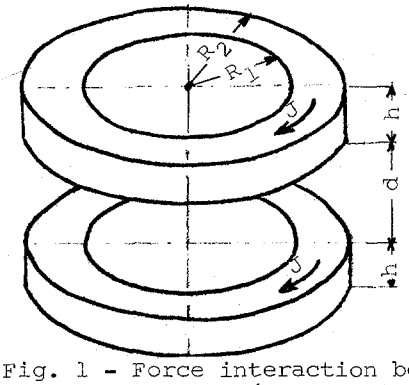

R_1 = 50 mm
R_2 = 60 mm
h = 10 mm
d = 40 mm
J = 3 A/mm^2

F_{approx} = 74 10^{-3} N
F_{exact} = 78 10^{-3} N

Fig. 1 - Force interaction between two identical rings of current (exact and virtual work values) embedded in a cylindrical box (height=240 mm radius=200 mm). The 1/8 is discretized into 135 second order tetrahedron (250 nodes).