Extended Formulas to Compute Resultant and Contact Electromagnetic Force and Torque From Maxwell Stress Tensors

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Force density calculus in permanent magnets and other nonlinear magnetic media is still a challenge despite two hundred years of electromagnetic theory. While generally accepted formulas to compute the total electromagnetic force on a domain exist in the literature, calculating the local distribution of forces that are needed, for instance, to couple electromagnetic and mechanical finite-element method simulations is a more open subject. In particular, it is widely believed that the contact force between magnetic materials touching each other cannot be properly estimated and the insertion of thin virtual air gaps is required. In this paper, first, the formulas existing in the bibliography to compute the resultant electromagnetic force and torque on a bounded domain are revisited and extended. The new formulas can be applied to cases where the boundary of the domain is a discontinuity surface for the magnetic field. In particular, they allow computing the force and torque on a magnet totally or partially embedded in a ferromagnetic material without using any artificial air layer between them. Then, these formulas are checked by using numerical simulation of suitable experiments.

Index Terms— **Contact forces, magnetic force densities, Maxwell stress tensor, permanent magnets (PMs).**

I. Introduction

Accurate estimation of electromagnetic forces is a major concern in many engineering applications as far as these forces affect the mechanical and vibratory system behavior, so they are critical in such important fields as electrical machine control and design, among others.

Electromagnetic theory has been developed during the 19th century and the first half of the 20th century, so it is not surprising that nowadays a lot of analytical and numerical toolboxes are available in order to obtain the electric and magnetic fields with high accuracy, by solving Maxwell and material constitutive equations with proper boundary conditions.

Despite this fact, nowadays, there is great confusion about the correct calculus of the electromagnetic forces, and several equations, methods, and "equivalent formulas" are used in order to estimate these forces and their effect on material bodies. Unfortunately, most of these methods yield similar results only in simple cases, generally with all the bodies of interest surrounded by air, and they greatly diverge when applied to more complex problems. Even in basic cases, the correct force calculus is not a trivial matter [1] and becomes a real challenge when there are nonlinear ferromagnetic materials or permanent magnets (PMs) as far as there is a lack of sufficient knowledge of macroscopic-coupled constitutive laws [2].

It is quite surprising to realize this lack of knowledge, especially when the force over a charge (the Lorentz force) is perfectly known and described by

$$\mathbf{f} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{1}$$

where $\mathbf{f}$ is the electromagnetic force acting on a charge of value Q with speed $\mathbf{v}$, being in a region of the space with an electric field $\mathbf{E}$ and a magnetic flux density $\mathbf{B}$.

The main reason for the difficulty of applying (1) in practical cases is that it is not possible to perform a proper modeling of the position and speed of every charge in the region under study, so the use of constitutive laws linking the electromagnetic fields becomes necessary. However, these laws try to represent the sum of microscopic phenomena, and neglect many effects that could be of importance. In other words, the developed force model will be as good as the applied constitutive laws [3].

For the aforementioned reasons, theories based on different microscopic models exist, but the inconsistencies among them can be relevant when the study of forces in PMs is performed. More precisely, if the magnet is surrounded by air, most of the methods estimate similar values of the resultant force and torque on the PM, but the local forces are very different and magnet stress and strains remain uncertain.

In many cases, obtaining just the total force on the magnet is not enough. Rather, it is necessary to know exactly where they are applied and their specific distribution on the surface and on the volume, in order to predict the deformation of the magnet and the pressure on its container [e.g., the rotor in a PM synchronous machine (PMSM)]. Moreover, when the magnet is in contact with a ferromagnetic body, like the aforementioned rotor, each force calculus method yields different resultant forces becoming necessary to approximate these local forces through techniques based on virtual air gaps [4] or other equivalent methods [5], although they offer unclear results with doubtful accuracy.

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Many papers are devoted to the description of one or several particular methods such as the equivalent source models [6], the Maxwell stress tensor [7], or the virtual power principle (VPP) [8]. Due to its great practical interest, a lot of comparative studies between methods have been performed in the case of the interaction forces between two magnets or between a magnet and a ferromagnetic material, both by using numerical simulation [4], [9], [10] and even by carrying out practical experiments [11]. Some studies even try to develop a closed-form equation for the calculation of local forces in every point of the magnet [4], [12], [13]. However, to the best of our knowledge, a general, tested, and accurate formula capable of predicting the forces within the magnets or even the total force when the magnet is not completely surrounded by air has not been properly developed yet.

In this paper, given a Maxwell stress tensor (i.e., a tensor whose divergence is the magnetic force density), we rigorously obtain and generalize formulas to compute the resultant electromagnetic force and torque on a domain including discontinuity surfaces for the magnetic field. In this way, previous formulas widely accepted in the literature are verified and extended; even the domain surface can be itself a discontinuity surface for the field. By using these formulas, the force and torque on a magnet, totally or partially embedded in a ferromagnetic material, could be computed without using any artificial air layer between them.

Then, we make some computational experiments by using a simulation package to check the validity of these improved equations. In particular, it is shown that the usual formula for the magnetic torque on a domain is only true for symmetric Maxwell tensors.

The outline of this paper is as follows. In Section II, the classical Maxwell stress tensor is recalled, as well as the way of computing the magnetic force by taking its divergence in the sense of distributions. In particular, the classical formulas of total force and torque on a volume enclosed by a surface are validated and extended, and they are proved to be valid even if the volume is not completely surrounded by air. Later, the expression of the surface force density on a magnetically discontinuous surface is presented in the case where the magnetic field intensity and the magnetic flux density are collinear, thus excluding PMs. Afterward, a review of various expressions of Maxwell stress tensors from the bibliography is carried out. Next, formulas to compute the force density on PMs, both the volumetric and the surface ones, are given.

In Section III, several test cases involving PMs are numerically solved and the extended total force and torque expressions are checked. In particular, the suitability of having a symmetric Maxwell tensor is argued.

II. Magnetic Force Computation

Most authors consider that the primitive object is the electromagnetic force and then introduce, *a posteriori*, a Maxwell stress tensor as a tensor field whose divergence coincides with the force; thus, the first problem is how to determine this force. According to [2] and [3], the force should be obtained from the VPP by using the expression of the electromagnetic energy density.

But then, some major difficulties arise. First of all, the fields involved in any expression of the Maxwell stress tensors can be discontinuous across material interfaces and then the divergence must be understood in the sense of distributions. Second, from the mathematical point of view, many tensor fields exist whose divergence is a given vector field. In fact, several forms of Maxwell stress tensors are proposed in the literature [2], [11]. Although all of them coincide for linear isotropic materials, in which case the magnetic field and the magnetic induction are collinear, there are great differences when they are applied to magnetized materials. In this paper, several Maxwell stress tensors proposed in recent bibliography will be recalled and analyzed.

A. From Maxwell Stress Tensor to Electromagnetic Force Density

In the literature, it is assumed that a Maxwell stress tensor $\mathcal{M}$ must verify that the electromagnetic force density is the field defined from $\mathcal{M}$ by

$$\mathbf{f} = \mathrm{div}\mathcal{M}. \tag{2}$$

Let us adopt the convention that the divergence operator above is given in Cartesian coordinates by $(\mathrm{div}\mathcal{M})_i = \sum_{j=1}^{3} \mathcal{M}_{ij,j}$. Note that Maxwell tensor as defined by (2) is not unique (the curl of any vector field could be added). If $\mathcal{M}$ is smooth everywhere, this expression and Gauss' theorem can be used in order to compute the resultant electromagnetic force acting on a region $\boldsymbol{\Omega}$ by

$$\mathbf{F} = \int_{\Omega} \mathrm{div}\ \mathcal{M} dV = \int_{\partial\Omega} \mathcal{M}\mathbf{n} dA \tag{3}$$

where $\partial\boldsymbol{\Omega}$ denotes the boundary of $\boldsymbol{\Omega}$, $\mathbf{n}$ is the outward unit normal vector to $\partial\boldsymbol{\Omega}$, and dV and dA denote the (scalar) elements of volume and area, respectively. However, in many cases, we deal with nonhomogeneous media involving materials with different magnetic permeabilities (e.g., a ferromagnetic material in contact with air) or PMs. In these cases, Maxwell stress tensors are discontinuous across the interfaces and their divergence must be understood in the sense of distributions, that is, (2) means the following:

$$\langle \mathbf{f}, \mathbf{w} \rangle = -\int_{E} \mathcal{M} : \mathrm{grad}\ \mathbf{w} dV \tag{4}$$

where $\mathcal{E}$ denotes the whole affine space, $\mathbf{w}$ is any infinitely differentiable vector field with compact support in E, and ":" denotes the usual inner product of tensors, namely, $\mathcal{M} : \mathcal{F} = \sum_{i,j=1}^{3} \mathcal{M}_{ij} F_{ij}$. From now on, it will be assumed that the components of $\mathcal{M}$ are locally integrable functions, which guarantees that (4) makes sense for any $\mathbf{w}$.

It is worthwhile to notice that the right-hand side of (4) is ready to be introduced into the weak formulation of a mechanical model ($\mathbf{w}$ being a test function). In other words, for numerical mechanical computations by the finite-element method, the explicit expression of the force density field $\mathbf{f}$ is not needed. Actually, it is enough to compute the Maxwell stress tensor field from fields $\mathbf{B}$ and $\mathbf{H}$.

Moreover, it is noticed that, in principle, the Gauss theorem (3) cannot be directly applied if the involved fields

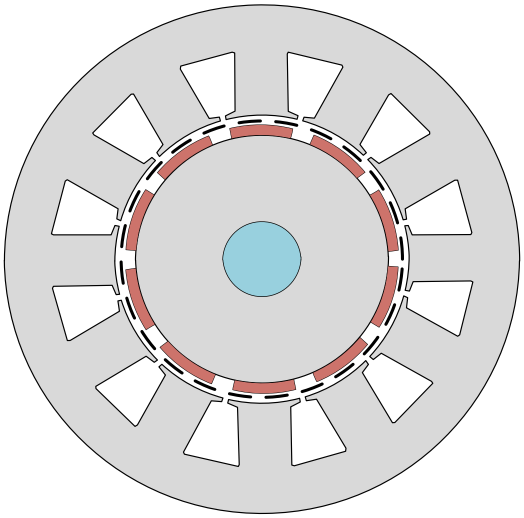

Fig. 1. Example of the integration air path selected in a 2-D-magnetic problem in order to apply (3).

are discontinuous on surfaces contained in the closure of $\mathbf{\Omega}$. Fortunately, a more general formula similar to (3) is shown below that is valid for the computation of the resultant force, even for these nonsmooth cases.

An important example will be considered. Fig. 1 shows a section of a PMSM. The goal is to compute the resultant force $\mathbf{F}$ (and, more important, the resultant torque $\mathbf{\Gamma}$) exerted on the rotor by the magnetic field, which is supposed to be already known from either analytical or numerical computations. Let us denote by S the imaginary surface in the air gap, as shown in Fig. 1. Since both $\mathbf{B}$ and $\mathbf{H}$ are continuous across this surface, then the same is true for any of the Maxwell stress tensors defined from these fields. Thus, it is a common practice to use the widespread formulas (5) and (6) below, even when they have not been strictly proved in cases where $\mathbf{B}$ and $\mathbf{H}$ exhibit discontinuities in the enclosed volume (which is, in fact, the case of the magnet/airgap and iron/airgap interfaces in a PMSM)

$$\mathbf{F} = \int_{\partial\Omega} \mathcal{M}\mathbf{n} dA \tag{5}$$

$$\mathbf{\Gamma} = \int_{\Omega} \mathbf{r} \times \mathrm{div}\mathcal{M} dV. \tag{6}$$

In this paper, formulas (5) and (6) are proved and their scope extended even when the selected enclosing surface is not completely contained in the air. In order to do so, the divergence of the Maxwell stress tensor in the sense of distributions has to be used.

B. Electromagnetic Force on Magnetically Discontinuous Surfaces

In principle, by using distributional calculus and any Maxwell stress tensor to be called generically $\mathcal{M}$, the expression of the magnetic surface force density can be deduced on surfaces where the magnetic properties jump as, for instance, on the boundary of a PM.

Let us call S a discontinuity surface. Locally, S divides space E into two parts to be called $+$ and $-$. Let us denote $\mathbf{n}^+$

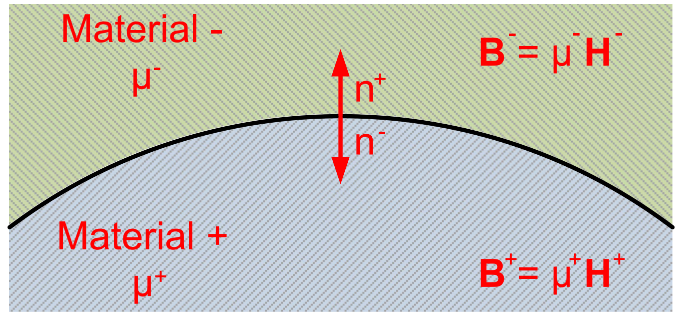

Fig. 2. Details of the boundary between two materials with different magnetic permeabilities.

(respectively, $\mathbf{n}^-$) the unit normal vector to S outward $+$ (respectively, $-$); of course, $\mathbf{n}^+ = -\mathbf{n}^-$ (see Fig. 2). The applied notation is as follows:

$$[a] = a^+ + a^-, \quad \{a\} = a^+ - a^-.$$

In particular

$$[\mathcal{M}\mathbf{n}] = \mathcal{M}^+\mathbf{n}^+ + \mathcal{M}^-\mathbf{n}^-. \tag{7}$$

From straightforward computations applied to (4), essentially using Gauss' theorem, it is possible to deduce that the surface force density on S is given by (see, for instance, [14] for detailed calculations)

$$f = -[\mathcal{M}\mathbf{n}]\delta_S \tag{8}$$

where δ_S denotes the Dirac measure on surface S. The above equality means

$$\langle \mathbf{f}, \mathbf{w} \rangle = -\int_S [\mathcal{M}\mathbf{n}] \cdot \mathbf{w} dA \tag{9}$$

for any vector field $\mathbf{w}$ indefinitely derivable with compact support in $\mathcal{E}$.

C. Computing the Resultant Magnetic Force on a Domain

If fields $\mathbf{B}$ and $\mathbf{H}$ are smooth everywhere, then (3), which has been directly obtained from Gauss' theorem, can be used to compute the total force. In this section, the case where some of these fields, and thereby the corresponding Maxwell stress tensors, exhibit jump discontinuities across a surface is considered. Let us suppose a domain $\mathbf{\Omega}$ including a closed discontinuity surface S for the magnetic field and such that its own boundary $\partial\mathbf{\Omega}$ may also be a discontinuity surface. Let $\mathcal{M}$ be a Maxwell stress tensor, which is supposed to be smooth in the rest of $\mathbf{\Omega}$. Let us call $\mathbf{m} = \mathrm{div}\mathcal{M}$ in the open set $\mathbf{\Omega}\backslash S$.

According to (4) and the distributional calculus, there are magnetic surface forces on S and $\partial\mathbf{\Omega}$ whose respective densities are given by $-[\mathcal{M}\mathbf{n}]\delta_S$ and $-[\mathcal{M}\mathbf{n}]\delta_{\partial\Omega}$. Hence, the whole force density including both volumetric and surface ones is the distribution

$$\mathbf{f} = \mathbf{m} - [\mathcal{M}\mathbf{n}]\delta_{\partial\Omega} - [\mathcal{M}\mathbf{n}]\delta_S. \tag{10}$$

Thus, the total force exerted on $\mathbf{\Omega}$ is given by

$$\mathbf{F} = \int_{\Omega} \mathbf{m} dV - \int_S [\mathcal{M}\mathbf{n}] dA - \int_{\partial\Omega} [\mathcal{M}\mathbf{n}] dA. \tag{11}$$

In order to obtain a formula for $\mathbf{F}$ similar to (5), the Gauss theorem is used. For this purpose, $\boldsymbol{\Omega}^+$ denotes the open set enclosed by S. Recall that $\mathbf{n}^+$ denotes the outward unit normal vector to $\partial\boldsymbol{\Omega}^+(=S)$. By using the Gauss theorem in $\boldsymbol{\Omega}^+$, the following equality is obtained:

$$\int_{\Omega^+} \mathbf{m} dV = \int_{\Omega^+} \operatorname{div}\mathcal{M} dV = \int_S \mathcal{M}^+\mathbf{n}^+ dA \tag{12}$$

where Ω^- is the open subset of Ω defined by: $\Omega^- = \Omega\backslash(\Omega^+ \cup S)$. By using Gauss' theorem in Ω^-, it follows that:

$$\begin{aligned}\int_{\Omega^-} \mathbf{m} dV = \int_{\Omega^-} \operatorname{div}\mathcal{M} dV &= \int_S \mathcal{M}^-\mathbf{n}^- dA \\ &+ \int_{\partial\Omega} \mathcal{M}^-\mathbf{n}^- dA.\end{aligned} \tag{13}$$

Let us replace these expressions in (11). Then

$$\begin{aligned}F &= \int_S \mathcal{M}^+\mathbf{n}^+ dA + \int_S \mathcal{M}^-\mathbf{n}^- dA + \int_{\partial\Omega} \mathcal{M}^-\mathbf{n}^- dA \\ &\quad - \int_S [\mathcal{M}\mathbf{n}] dA - \int_{\partial\Omega} [\mathcal{M}\mathbf{n}] dA \\ &= \int_{\partial\Omega} \mathcal{M}^-\mathbf{n}^- dA - \int_{\partial\Omega} [\mathcal{M}\mathbf{n}] dA \\ &= \int_{\partial\Omega} \mathcal{M}^{\text{ext}}\mathbf{n}^- dA\end{aligned} \tag{14}$$

where $\mathcal{M}^{\text{ext}}$ is the limit value of $\mathcal{M}$ on $\partial\boldsymbol{\Omega}$ coming from outside of $\boldsymbol{\Omega}$. It is worthwhile to note that this formula "does not see" the discontinuity across the internal surface S. Thus, formally, the same Gauss formula for smooth fields, i.e., (3), can be applied as far as the value of $\mathcal{M}$ in the boundary integral on $\partial\boldsymbol{\Omega}$ is taken as the outside limit value. Of course, if there is no discontinuity across $\partial\boldsymbol{\Omega}$, then $\mathcal{M}^{\text{ext}} = \mathcal{M}^-$ and (14) becomes exactly (3). In particular, if $\partial\boldsymbol{\Omega}$ is contained in the interior of a linear isotropic media as, for instance, air, then all Maxwell tensors having the same value in such media produce the same value of the total force $\mathbf{F}$.

It is worth noting that, as far as we know, all of the Maxwell tensors existing in the bibliography have this property, so they lead to the same value of the total force on domains surrounded by linear isotropic media (e.g., air). This is the mathematical reason why many Maxwell tensors are proposed through the years: all of them are consistent with the total force calculation but leading to great differences in the local forces, as noticed in [11].

D. Computing the Resultant Magnetic Torque on a Domain

Now, in the same general case previously introduced, it is possible to give a formula to compute the magnetic torque with respect to an origin o, exerted by the magnetic field on domain $\boldsymbol{\Omega}$. Let $\mathbf{r}$ be the position vector with respect to o. Then

$$\begin{aligned}\boldsymbol{\Gamma} = \int_\Omega \mathbf{r} \times \operatorname{div}\mathcal{M} dV &- \int_{\partial\Omega} \mathbf{r} \times [\mathcal{M}\mathbf{n}] dA \\ &- \int_S \mathbf{r}\times[\mathcal{M}\mathbf{n}] dA.\end{aligned} \tag{15}$$

In order to transform this expression, the following Green's formula will be used (see the Appendix):

$$\int_D \mathbf{r} \times \operatorname{div}T dV = \int_{\partial D} \mathbf{r}\times(\mathrm{T}\mathbf{n}) dA - 2\int_D \mathbf{t} dV \tag{16}$$

where T is a smooth tensor field defined in an open bounded set D and $\mathbf{t}$ denotes the axial vector of the skew part of tensor T, i.e., the only vector field such that

$$\frac{T - T^t}{2}\mathbf{a} = \mathbf{t} \times \mathbf{a} \forall \mathbf{a}. \tag{17}$$

Its coordinates are $t_i = -\varepsilon_{ijk}T_{jk}/2$, where ε_{ijk} is the alternating or Levi-Civita tensor, that is, $\mathbf{t} = (T_{32} - T_{23}, T_{13} - T_{31}, T_{21} - T_{12})/2$. Note that $\mathbf{t} = 0$ if and only if T is symmetric.

By using this formula and similar calculus to those in (14), expression (15) can be transformed into

$$\boldsymbol{\Gamma} = \int_{\partial\Omega} \mathbf{r} \times (\mathcal{M}^{\text{ext}}\mathbf{n}) dA - 2\int_\Omega \mathbf{t} dV \tag{18}$$

where now $\mathbf{t}$ denotes the axial vector of the skew part of $\mathcal{M}$.

Let us emphasize that if $\mathcal{M}$ is symmetric in $\boldsymbol{\Omega}$,then $\mathbf{t} = 0$ and (18) yields

$$\boldsymbol{\Gamma} = \int_{\partial\Omega} \mathbf{r}\times(\mathcal{M}^{\text{ext}}\mathbf{n}) dA. \tag{19}$$

Equation (19) is the widely accepted formula to compute the torque on a domain surrounded by air [15]. However, it is noticed that this formula only gives the right torque if the Maxwell tensor describing the magnetic force density inside $\boldsymbol{\Omega}$ is symmetric. Otherwise, (18) should be used.

E. Maxwell Stress Tensors

At this point, the main question is the following: leaving aside magnetostriction in a first stage is there a universally valid Maxwell stress tensor for any material, no matter whether it is linear or nonlinear, isotropic or anisotropic magnetized or not? In order to give an answer to this question, let us recall the procedure to get a Maxwell stress tensor. According to [2], the first step should consist in obtaining the right macroscopic constitutive law for the material in terms of an energy density function. Then, by using the VPP, one would obtain the expression of the electromagnetic force density, and finally, any tensor field whose divergence is equal to this force density would be a Maxwell stress tensor.

In order to write some Maxwell stress tensors proposed in the bibliography, the concepts of energy and co-energy are first recalled. It is assumed that there is a locally defined magnetic energy density $\psi(x, \mathbf{B})$, such that, at any point x, the magnetic induction and the magnetic intensity are related by

$$\mathbf{H} = \frac{\partial\psi}{\partial\mathbf{B}}(x, \mathbf{B}). \tag{20}$$

Note that this assumption excludes, for instance, materials with hysteresis. Moreover, in deformable media, ψ will be also dependent on the displacement field $\mathbf{u}$.

The conjugate (or Legendre–Fenchel transform) of ψ is called the magnetic co-energy density. It is defined at each point x in the affine space $\mathcal{E}$ by

$$\phi(x, \mathbf{H}) = \sup_{B\in\mathbb{R}^3} \{\mathbf{H}\cdot\mathbf{B} - \psi(x, \mathbf{B})\}. \tag{21}$$

Moreover, if $\psi(x,.)$ is convex, then the following properties (which show the interest of introducing ϕ) hold [16]:

1) $\mathbf{H}=\frac{\partial\psi}{\partial\mathbf{B}}(x,\mathbf{B})$ if and only if, $\mathbf{B}=\frac{\partial\phi}{\partial H}(x,\mathbf{H})$
2) If $\mathbf{H}=\frac{\partial\psi}{\partial\mathbf{B}}(x,\mathbf{B})$ then

$$\mathbf{B}\cdot\mathbf{H}=\psi(x,\mathbf{B})+\phi(x,\mathbf{H}). \tag{22}$$

Now, let us recall that the magnetic part of the classical Maxwell tensor is

$$\mathcal{M}_c(x)=\mathbf{B}(x)\otimes\mathbf{H}(x)-\frac{1}{2}\mathbf{B}(x)\cdot H(x)\mathcal{J} \tag{23}$$

where $\mathcal{J}$ denotes the identity tensor.

It is accepted that this tensor can be used to compute the magnetic force for (maybe nonhomogeneous) linear isotropic materials. As an example, let us compute the electromagnetic force assuming that μ is insensitive to strain and there is only one material discontinuity surface S, which is free of surface current. More precisely, it is supposed that the magnetic permeability jumps across S, but it changes smoothly elsewhere. Then the electromagnetic force density in $\mathcal{E}$ can be deduced by using (10) and presents the following vector distribution from [1]:

$$\begin{aligned}\mathbf{f}=&\,\mathbf{J}\times\mathbf{B}-\frac{\mathbf{1}}{\mathbf{2}}|\mathbf{H}|^{\mathbf{2}}\mathrm{grad}\mu\\&-\frac{1}{2}\left(\left\{\frac{1}{\mu}\right\}|\mathbf{B}^{+}\cdot\mathbf{n}^{+}|^{\mathbf{2}}-\{\mu\}|\mathbf{H_T}|^{\mathbf{2}}\right)\mathbf{n}^{+}\delta_S\end{aligned} \tag{24}$$

where $\mathbf{J}$ is the current density and $\mathbf{H}_T=\mathbf{n}\times\mathbf{H}\times\mathbf{n}$ is the (continuous across S) tangential component of $\mathbf{H}$ on S. In the above expression, $\mathbf{n}$ denotes either $\mathbf{n}^{+}$or $\mathbf{n}^{-}$, indistinctly. Let us notice that the first line of (24) is the regular (i.e., volumetric) part of the force density. In fact, gradμ denotes the gradient of the magnetic permeability, which is piecewise defined in the open set where it is smooth, i.e., in $\Omega\backslash$S.

Assuming that the magnetic energy density is given as a function of displacement $\mathbf{u}$ and flux density $\mathbf{B}$, $\psi(x,\mathbf{u},\mathbf{B})$, and by using the VPP, the following expression for the electromagnetic force is obtained in (see [2, Formula (7)]):

$$\mathbf{f}=\mathbf{J}\times\mathbf{B}-\frac{\partial\hat{\psi}}{\partial\mathbf{u}}(x,\mathbf{0},\mathbf{A}) \tag{25}$$

where $\mathbf{A}$ is a magnetic vector potential and $\{\hat{\psi}\}$ is also the magnetic energy density but defined as a function of A instead of B.

Then, assuming particular forms of ψ, this formula is applied in [2] to compute the force density in several cases of materials: linear and nonlinear, isotropic and nonisotropic, PMs. Assuming that the magnetic force is not sensitive to strain, in all of these cases, the deduced expression for the magnetic force density is the following (see [2, Formula (13)]):

$$\begin{aligned}\mathbf{f}&=\mathbf{J}\times\mathbf{B}-\frac{\partial\phi}{\partial x}(x,\mathbf{H})-\frac{1}{2}\mathrm{curl}(\mathbf{H}\times\mathbf{B})\\&=\mathbf{J}\times\mathbf{B}+\frac{\partial\psi}{\partial \mathrm{x}}(x,\mathbf{B})-\frac{1}{2}\mathrm{curl}(\mathbf{H}\times\mathbf{B})\end{aligned} \tag{26}$$

where the derivative in the second terms of the right-hand-sides (the gradients with respect to x) has possibly to be understood in the sense of distributions. Let us remark that, in the isotropic case (not necessarily homogeneous), $\mathbf{B}(x)$ and $\mathbf{H}(x)$ are collinear, so

$$\mathbf{H}(x)\otimes\mathbf{B}(x)=\mathbf{B}(x)\otimes\mathbf{H}(x) \tag{27}$$

and $\mathbf{H}(x)\times\mathbf{B}(x)=\mathbf{0}$. Thus, the last term in (26) vanishes for isotropic materials.

Moreover, for linear materials (not necessarily isotropic), it is fulfilled

$$\psi(x,\mathbf{B})=\phi(x,\mathbf{H})=\frac{1}{2}\mathbf{B}\cdot\mathbf{H}. \tag{28}$$

Then, it is easy to see that (26) yields (24) for linear isotropic materials. However, in more general situations, the classical Maxwell stress tensor (23) is not correct, because its divergence is no longer equal to the force in (26).

In [2] and [17], the following co-energy-based Maxwell stress tensor has been proposed:

$$\mathcal{M}_\phi(x)=\mathbf{H}(x)\otimes\mathbf{B}(x)-\phi(x,\mathbf{H}(x))\mathcal{J}. \tag{29}$$

Let us compute the divergence of this tensor

$$\begin{aligned}\mathrm{div}\mathcal{M}_\phi&=\mathrm{div}(\mathbf{H}(x)\otimes\mathbf{B}(x)-\phi(x,\mathbf{H}(x))\mathcal{J})\\&=\mathbf{J}\times\mathbf{B}-\frac{\partial\phi}{\partial\mathrm{x}}(x,\mathbf{H})\end{aligned} \tag{30}$$

(equality grad $\mathbf{H}$ $\mathbf{B}-$ grad $\mathbf{H^T B}$ $=$ curl$\mathbf{H}\times\mathbf{B}$ $=$ $\mathbf{J}\times\mathbf{B}$) has been used. A precise definition of the used differential operators, such as grad $\mathbf{HB}$, can be found at Appendix B [see (58)]. Let us emphasize that the divergence of $\mathcal{M}_\phi$ does not coincide with the force in (26) unless in the isotropic case where the last term in (26) is null. However, the symmetric part of tensor $\mathcal{M}_\phi$, namely

$$\mathcal{M}^s_\phi(x)=\frac{1}{2}(\mathbf{H}(x)\otimes\mathbf{B}(x)+\mathbf{B}(x)\otimes\mathbf{H}(x)-2\phi(x,\mathbf{H}(x))\mathcal{J}) \tag{31}$$

does satisfy

$$\mathrm{div}\mathcal{M}^s_\phi=\mathbf{f} \tag{32}$$

where $\mathbf{f}$ is the force given by (26). Indeed, since

$$\mathcal{M}^s_\phi=\mathcal{M}_\phi+\frac{1}{2}(\mathbf{B}\otimes\mathbf{H}-\mathbf{H}\otimes\mathbf{B}) \tag{33}$$

and

$$\mathrm{div}(\mathbf{B}\otimes\mathbf{H}-\mathbf{H}\otimes\mathbf{B})=\mathrm{curl}(\mathbf{BxH}) \tag{34}$$

then

$$\begin{aligned}\mathrm{div}\mathcal{M}^s_\phi&=\mathrm{div}\mathcal{M}_\phi+\frac{1}{2}\mathrm{curl}(\mathbf{B}\times\mathbf{H})\\&=\mathbf{J}\times\mathbf{B}-\frac{\partial\phi}{\partial x}(x,\mathbf{H})-\frac{1}{2}\mathrm{curl}(\mathbf{H}\times\mathbf{B})\end{aligned} \tag{35}$$

which is exactly (26). Therefore, $\mathcal{M}^s_\phi$ is a valid Maxwell tensor for materials for which the magnetic density force is given by (26). This tensor has been introduced in [18].

Some other Maxwell tensors are proposed in the bibliography as, for instance, the transpose of $\mathcal{M}_\phi$ [12], [13]

$$\mathcal{M}^t_\phi(x)=\mathbf{B}(x)\otimes\mathbf{H}(x)-\phi(x,\mathbf{H}(x))\mathcal{J} \tag{36}$$

but div$\mathcal{M}^t_\phi\neq\mathbf{f}$, and hence, it is not a correct Maxwell tensor.

Let us emphasize that in the interior of isotropic linear materials (e.g., in air), all of the above Maxwell stress tensors coincide

$$\mathcal{M}_c = \mathcal{M}_\phi = \mathcal{M}_\phi^s = \mathcal{M}_\phi^t. \tag{37}$$

This fact has an important well-known consequence: a domain Ω surrounded by a linear isotropic media (e.g., air) is considered. Then, the total force exerted onΩ can be computed by formula (14), regardless of the Maxwell stress tensor, as far as it coincides with the classical one in linear isotropic media. Note that this property is fulfilled for all of the above Maxwell tensors.

The same is true for the torque on Ω computed by formula (19), even for nonsymmetric Maxwell tensors, as far as they fulfill the above property.

F. Permanent Magnets

Fields $\mathbf{B}$ and $\mathbf{H}$ are not collinear for some materials. An important case corresponds to PMs for which the usual constitutive law has the form

$$\mathbf{B} = \mu\mathbf{H}+\mathbf{B^r} \tag{38}$$

where $\mathbf{B}^r$ is the remanent magnetization of the magnet. It is assumed that it is surrounded by a linear isotropic material. Since the boundary of the magnet is a discontinuity surface for the electromagnetic field, according to (10), there must be surface forces supported on this boundary.

Let us define the "+" region as the magnet and the "−" region as the outside of the magnet. Then, from (38)

$$\mathbf{B}^+ = \mu^+\mathbf{H}^+ + \mathbf{B^r} \tag{39}$$

in the magnet, while outside

$$\mathbf{B}^- = \mu^-\mathbf{H}^-. \tag{40}$$

Note that, since material in Ω^- is linear and isotropic (e.g., air), all of the above tensors lead to the same values of the total force on the magnet when formula (14) is applied. This is because these formulas only involve the values of Maxwell tensor in Ω^- (actually, the outside limit values on the magnet surface $S = \partial\Omega^-$), and in Ω^-, all of the above Maxwell tensors have the same values.

Let us notice that since $\mathbf{H}^+$ and $\mathbf{B}^+$ are not collinear, then

$$\mathbf{H}^+(x) \otimes \mathbf{B}^+(x) \neq \mathbf{B}^+(x) \otimes \mathbf{H}^+(x). \tag{41}$$

Moreover

$$\phi(x, \mathbf{H}^+) = \frac{|\mathbf{B}^+|^\mathbf{2}}{\mathbf{2}\mu^+} \tag{42}$$

$$\psi(x, \mathbf{B}^+) = \frac{|\mathbf{B}^+|^2}{2\mu^+} - \frac{1}{\mu^+}\mathbf{B}^+ \cdot \mathbf{B}^r \tag{43}$$

$$\frac{1}{2}\mathbf{B}^+ \cdot \mathbf{H}^+ = \frac{|\mathbf{B}^+|^2}{2\mu^+} - \frac{1}{2\mu^+}\mathbf{B}^+ \cdot \mathbf{B}^r. \tag{44}$$

For the sake of simplicity, it is assumed that μ^+ and μ^- are constant. Inside the magnet, formula (26) yields [2]

$$\mathbf{f} = \mathbf{J} \times \mathbf{B} - \frac{1}{\mu^+}(\text{grad}\mathbf{B}^r)^t\mathbf{B}^+ - \frac{1}{2}\text{curl}(\mathbf{H}\times\mathbf{B}^r). \tag{45}$$

Assuming $\mathbf{B}^r$ as constant in the magnet, this formula becomes

$$\begin{aligned}\mathbf{f} &= \mathbf{J} \times \mathbf{B} - \frac{1}{2}\text{curl}(\mathbf{H} \times \mathbf{B}^r) = \mathbf{J} \times \mathbf{B}\\ &\quad -\frac{1}{2}(\text{grad}\mathbf{H}\mathbf{B^r} + \mathbf{H}\text{div}\mathbf{B^r} - \text{grad}\mathbf{B^r}\mathbf{H} - \mathbf{B^r}\text{div}\mathbf{H})\\ &= \mathbf{JxB} - \frac{1}{2}\text{grad}\mathbf{H}\mathbf{B^r}\end{aligned} \tag{46}$$

because $\text{div}\mathbf{H} = (1/\mu^+)\text{div}(\mathbf{B}-\mathbf{B}^r) = \mathbf{0}$. Moreover, by tedious but straightforward computations, the following expression for the magnetic force density on the magnet surface can be obtained from (8) by taking $\mathcal{M} = \mathcal{M}_\phi^s$:

$$\text{f} = -\frac{1}{2}([\mathbf{B}.\mathbf{n}\mathbf{H}] + [\mathbf{H}.\mathbf{n}\mathbf{B}] - [\phi(x, \mathbf{H})\mathbf{n}])\delta_S. \tag{47}$$

Let us compute each term in (47)

$$[\mathbf{B} \cdot \mathbf{n}\mathbf{H}] = \left(\left\{\frac{1}{\mu}\right\} |\mathbf{B}^+ \cdot \mathbf{n}^+|^2 - \frac{1}{\mu^+}\mathbf{B}^+ \cdot \mathbf{n}^+\mathbf{B^r} \cdot \mathbf{n}^+\right)\mathbf{n}^+ \tag{48}$$

$$\begin{aligned}[\mathbf{H} \cdot \mathbf{n}\mathbf{B}] &= \left(\left\{\frac{1}{\mu}\right\} |\mathbf{B}^+ \cdot \mathbf{n}^+|^2 - \frac{1}{\mu^+}\mathbf{B}^+ \cdot \mathbf{n}^+\mathbf{B^r} \cdot \mathbf{n}^+\right)\mathbf{n}^+\\ &\quad +\frac{1}{\mu^+}(\mathbf{B}^+ \cdot \mathbf{n}^+\mathbf{B^r_T} - \mathbf{B^r} \cdot \mathbf{n}^+\mathbf{B^+_T})\end{aligned} \tag{49}$$

$$\begin{aligned}[\phi(x, \mathbf{H})\mathbf{n}] &= \left(\frac{1}{2}\left\{\frac{1}{\mu}\right\} |\mathbf{B}^+ \cdot \mathbf{n}^+|^2 + \frac{1}{2}\{\mu\}|\mathbf{H_T}|^2 + \mathbf{H_T}\mathbf{B^r_T}\right.\\ &\quad \left.+\frac{1}{2\mu^+}|\mathbf{B^r_T}|^2\right)\mathbf{n}^+.\end{aligned} \tag{50}$$

Therefore

$$\begin{aligned}\mathbf{f} &= \left(\left(-\frac{1}{2}\left\{\frac{1}{\mu}\right\} |\mathbf{B}^+ \cdot \mathbf{n}^+|^2 + \frac{1}{\mu^+}\mathbf{B}^+ \cdot \mathbf{n}^+\mathbf{B^r} \cdot \mathbf{n}^+\right.\right.\\ &\quad \left.+\frac{1}{2}\{\mu\}|\mathbf{H_T}|^2 + \mathbf{H_T}\mathbf{B^r_T} + \frac{|\mathbf{B^r_T}|^2}{2\mu^+}\right)\mathbf{n}^+\\ &\quad \left.-\frac{1}{2\mu^+}(\mathbf{B}^+ \cdot \mathbf{n}^+\mathbf{B^r_T} - \mathbf{B^r} \cdot \mathbf{n}^+\mathbf{B^+_T})\right)\delta_S.\end{aligned} \tag{51}$$

In the above expression, $\mathbf{H_T}$ denotes the tangential component of $\mathbf{H}$ on S, $\mathbf{H_T} = \mathbf{n} \times \mathbf{H} \times \mathbf{n}$, while $\mathbf{B^r_T}$ and $\mathbf{B^+_T}$ express the tangential components of $\mathbf{B^r}$ and $\mathbf{B}^+$, respectively. Notice that since there are no surface currents on S, then $\mathbf{H_T}$ is continuous across it. Let us remark that the first two terms represent normal forces to S, but the last two ones are tangent.

Finally, regarding the volumetric forces in the magnet, it is worth mentioning that several formulas can be found in the bibliography leading to different force densities. One of the most popular is the Kelvin force [8], [10]

$$\mathbf{f} = \mathbf{J} \times \mathbf{B} + \text{grad}\mathbf{H}\mathbf{B^r}. \tag{52}$$

In order to get (52), magnets are modeled as a set of magnetic dipoles without interference between them. However, this approach is seen as limited by many authors, who propose other alternative forms for $\mathbf{f}$ [3].

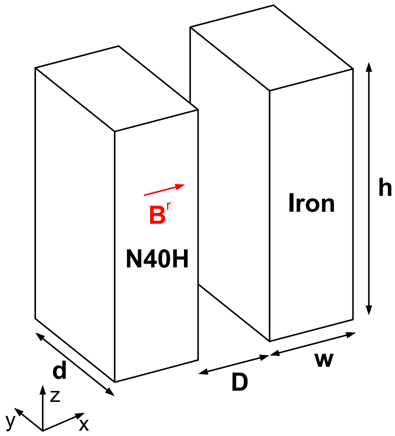

Fig. 3. N40H PM facing a ferromagnetic core—the simulation is carried out on a frontal slice (2-D).

TABLE I
PM N40H MAIN FEATURES

Feature	Value
Material	Nd-Fe-B
Magnetization	Parallel, x direction
Remanent magnetization (B^r)	1.13 T
Relative permeability (μ_r)	1.05
Height (h)	63.9 mm
Width (w)	13.4 mm
Depth (d)	31.0 mm

III. VALIDATION THROUGH 2-D-FEM

In order to show the usefulness of the extended total force and torque formulas (14) and (18) established in this paper, some numerical simulations have been carried out with the commercial electromagnetic finite-element method (FEM) software: FLUX-2-D. These numerical experiments have been done in two steps: in the first step, the total force between a magnet and a ferromagnetic material will be computed as a function of their distance. The main goal of this step is to numerically check the validity of (14) to compute the total force exerted on the magnet, even when the two materials are in contact as in PMSMs, i.e., without using any air layer between them. In the second step, formula (18) is applied in order to compare Maxwell stress tensors (29) and (31).

A. Total Force Calculus

This first simulation involves a PM (Nd–Fe–B, N40H) and a linear isotropic ferromagnetic core located at a variable distance (D). The ferromagnetic material, the magnet geometry, and the remanent magnetization vector are shown in Fig. 3, while in Table I, the main features and dimensions are summarized. The ferromagnetic material is supposed to be linear (no saturation is simulated); its relative permeability is equal to $\mu_r = 1300$, and its dimensions are exactly the same as the ones for the PM.

The resultant force on the magnet is calculated applying (14) on an air surface surrounding the magnet. Equation (14) is applied even in the limit case, when $D = 0$ mm and the ferromagnetic core and the magnet are in contact [in this case, the surface where (14) is applied includes the interface between the core and the magnet]. The results are shown in Figs. 4 and 5.

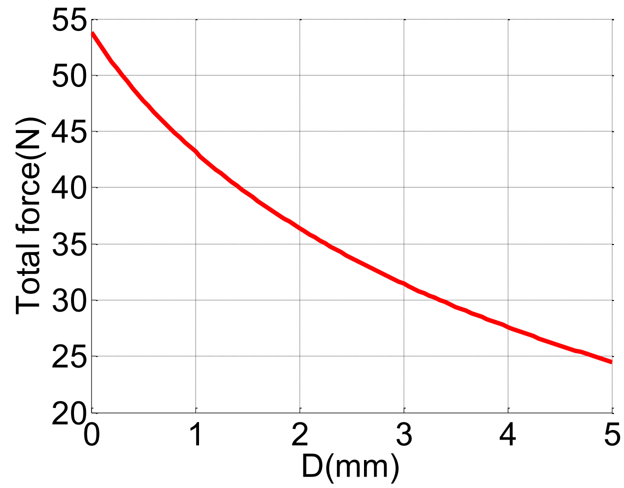

Fig. 4. Total force acting on the magnet as a function of distance.

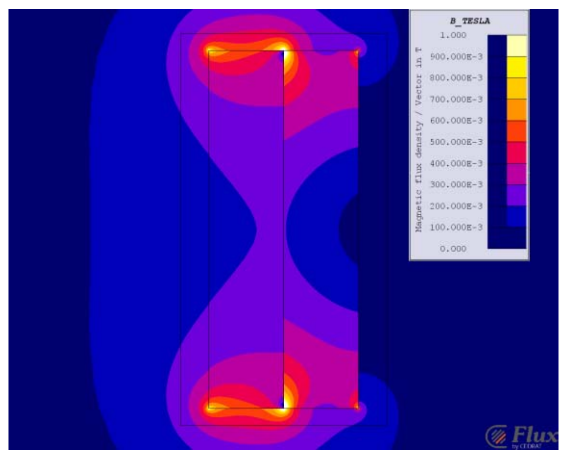

Fig. 5. Flux density obtained when the magnet and the iron are in contact.

As it is shown in Fig. 4, the results obtained in the limit case ($D = 0$ mm) are accurate and consistent with those obtained for a strictly positive distance in which case a surface of air can be used in the formula. Let us emphasize that for the limit case $D = 0$, neither virtual air layer nor any special treatment has to be used.

From these results, some important remarks should be highlighted: the first one is that no matter which of the above Maxwell stress tensor is used in (14), because they are all equal in linear isotropic media (i.e., both in the air and in the ferromagnetic core). The second result is that, in order to obtain the total force on a magnet in contact with a ferromagnetic core, the right way to apply (14) is using the *outside* limit values of the Maxwell tensor on its boundary; otherwise, the results obtained are not correct.

B. Total Torque Calculus Validation

Let us recall that all Maxwell tensors proposed above yield the same total force as far as (14) is applied to a surface totally

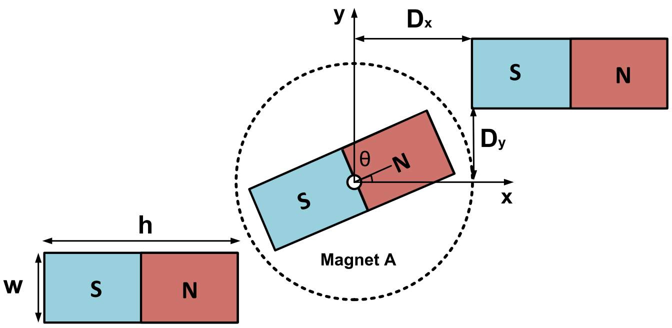

Fig. 6. Proposed 2-D-FEM simulation. Torque is calculated for different θ angles. Dotted line is used in order to apply (18).

TABLE II
GEOMETRIC PARAMETERS VALUES

Feature	Value
PM magnetization	Parallel, x direction
PMs remanent magnetization ($\overline{B^r}$)	1.13 T $\bar{x}$
PMs relative permeability (μ_r)	1.05
PMs height (h)	63.90 mm
PMs width (w)	13.40 mm
Magnet depth (d)	31.00 mm
External PMs x-displacement (D_x)	31.95 mm
External PMs y-displacement (D_y)	11.70 mm

included in a linear isotropic media. The same is true regarding formula (19) to compute the torque. This is because all of the above Maxwell stress tensors have the same values on a surface contained in such media. However, the conclusion is different regarding the computation of the total torque by using formula (18) which, let us emphasize, is the correct formula to compute the magnetic torque for any Maxwell stress tensor, no matter it is symmetric or nonsymmetric.

Indeed, this formula has two parts: the first one yields the same value for any of the aforementioned tensors as far as the bounding surface is chosen in a linear isotropic medium. However, this is no longer true for the second part, because the integration domain may include materials where **H** and **B** are not colinear (e.g., PMs).

In order to illustrate the above statements, a 2-D numerical simulation of the experiment shown in Fig. 6 is carried out.

In this test, the torque exerted by two external N40H PMs on a third one (called magnet A) is computed for different rotation angles. The three magnets are identical and very similar to the one used in Section A. Their main geometric features are shown in Table II.

In order to obtain a reliable value for the total torque exerted on magnet A, a well-known method based on the magnetic energy and the VPP will be used, namely

$$\boldsymbol{\Gamma} = -\frac{\partial W_m}{\partial \theta} \tag{53}$$

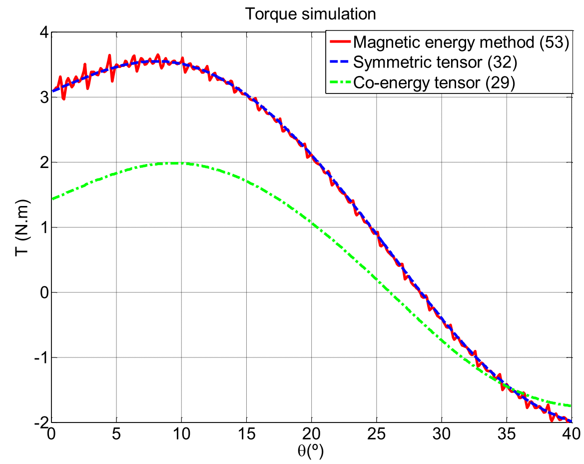

Fig. 7. Total torque obtained for different tensors and the VPP.

where W_m is the total magnetic energy contained in region $\boldsymbol{\Omega}$ (where the magnet is enclosed), that is

$$W_m = \int_{\boldsymbol{\Omega}} \psi(x, \theta, \mathbf{B}(x)) dV \tag{54}$$

and θ is the rotation angle with respect to the centre (see Fig. 6).

Two different Maxwell stress tensors are chosen to apply (18): the nonsymmetric co-energy-based tensor $\mathcal{M}_\phi$ (29) and its symmetric part $\mathcal{M}_\phi^s$ (31). The results obtained from these computations are shown in Fig. 7.

They show that, while $\mathcal{M}_\phi$ fails to predict the torque exerted on magnet A by means of formula (18), the symmetrized Maxwell stress tensor $\mathcal{M}_\phi^s$ calculates it to be very close to that obtained applying (53). Therefore, using (18), an extra proof that $\mathcal{M}_\phi$ is not a valid Maxwell stress tensor, when PMs are involved, is obtained. The same conclusion has been previously achieved in a completely theoretical manner, since $\mathcal{M}_\phi$ does not furnish the right magnetic force density [see (30)].

IV. CONCLUSION

Let us summarize the main contributions of this paper: the first one is the development and extension of a total force formula, capable to properly calculate forces between magnets and ferromagnetic media in contact without any special technique or approximation.

The second one is to prove and extend a formula to compute the total magnetic torque exerted by the magnetic field on an enclosure.

These formulas extend the ones existing in the bibliography to nonsmooth fields and are applied to some important cases.

Moreover, several expressions for Maxwell stress tensors from the bibliography are recalled and analyzed. All of them coincide for linear isotropic materials and lead to the same values of the total force on a domain surrounded by this kind of media. The main conclusion is that the symmetric tensor $\mathcal{M}_\phi^s$ is a Maxwell stress tensor for general nonmagnetostricitve and nonhysteric media, because its divergence is the magnetic force obtained in [2] for these media by using the VPP.

Therefore, it can be used to compute the force density at any point.

APPENDIX

A. Computing the Total Magnetic Torque on a Domain

In order to deduce (18), which allows computing the torque exerted by the magnetic field on a domain, a Gauss-like formula is needed. This formula will be proved below.

Given a smooth tensor field T defined in a bounded domain, D, the following equality holds:

$$\int_D \mathbf{r} \times \mathrm{div} T \, dV = \int_{\partial \mathrm{D}} \mathbf{r} \times (T\mathbf{n}) dA - 2\int_D \mathbf{t} dV \qquad (55)$$

where $\mathbf{t}$ is the axial vector field of the skew part of T, namely, the unique vector field satisfying

$$\frac{T - T^t}{2}\mathbf{a} = \mathbf{t} \times \mathbf{a}, \quad \forall \mathbf{a}. \qquad (56)$$

Its coordinates are $t_i = -\varepsilon_{ijk} T_{jk}/2$, where ε_{ijk} is the alternating or Levi–Civita tensor.

Indeed, by using Green's formula, it is deduced that

$$\begin{aligned}\int_D (\mathbf{r} \times \mathrm{div} T)_{\mathbf{i}} dV &= \int_D \varepsilon_{ijk} r_j (\mathrm{div} T)_{\mathbf{k}} dV = \int_D \varepsilon_{ijk} r_j T_{\mathrm{kl,l}} dV \\ &= \int_{\partial \mathrm{D}} \varepsilon_{ijk} r_j T_{kl} \mathrm{n}_l dA - \int_D \varepsilon_{ijk} r_{j,l} T_{kl} dV \\ &= \int_{\partial D} (\mathbf{r} \times \mathrm{T}\mathbf{n})_{\mathbf{i}} dA - \int_D \varepsilon_{ijk} T_{kj} dV \\ &= \int_{\partial D} (\mathbf{r} \times T\mathbf{n})_{\mathbf{i}} dA + \int_D \varepsilon_{ijk} T_{jk} dV \\ &= \int_{\partial \mathrm{D}} (\mathbf{r} \times T\mathbf{n})_{\mathbf{i}} dA - 2\int_D t_i dV. \end{aligned} \qquad (57)$$

B. Notations on Differential Operators

$$(\mathrm{grad}\mathbf{H}\mathbf{B})_i = \sum_{j=1}^{3} \frac{\partial H_i}{\partial x_j} B_j \qquad (58)$$

$$(\mathrm{grad}\mathbf{H}^{\mathbf{T}}\mathbf{B})_i = \sum_{j=1}^{3} \frac{\partial H_j}{\partial x_i} B_j. \qquad (59)$$

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