Finite element mass lumping in H(curl)

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PhD Defense

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Maxwell's equations

Electromagnetic wave propagation in linear and non-dispersive but possibly inhomogeneous and anisotropic media

$$\begin{aligned} \varepsilon\,\partial_t E(t) &= \quad \mathrm{curl}\; H(t) - \sigma E(t) && \text{in } \Omega \\ \mu\,\partial_t H(t) &= -\mathrm{curl}\; E(t) && \text{in } \Omega \end{aligned}$$

in Ω, with $E(0) = E_0$ and $H(0) = H_0$ in Ω and $n \times E(t) = 0$ on $\partial\Omega$

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Goal: systematic and flexible space discretization

- stable: no artificial energy production
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Methods: FDTD/FIT, FEM, FVM, DG, ...

Finite differences (FDTD/FIT)

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- 1966 - Yee - Numerical solution of initial boundary value problems involving Maxwell's equations in isotropic media
- 1977 - Weiland - Eine Methode zur Lösung der Maxwell'schen Gleichungen für sechskomponentige Felder auf diskreter Basis
- 1980 - Taflove - Application of the Finite-Difference Time-Domain method to sinusoidal steady-state electromagnetic penetration problems

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Finite differences : TE case

$$
E = \begin{pmatrix} E_x \\ E_y \\ 0 \end{pmatrix} \qquad H = \begin{pmatrix} 0 \\ 0 \\ H_z \end{pmatrix}
$$

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\begin{cases}
\varepsilon \partial_t E_x = \partial_y H_z - \sigma E_x, \\
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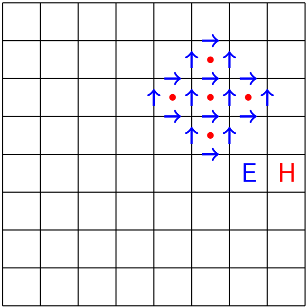

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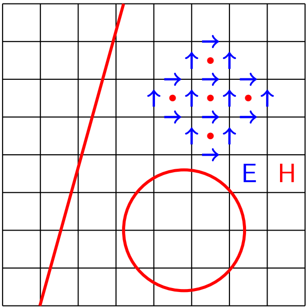

Pros

- Easy to implement
- stable, accurate $O(h^2 + \tau^2)$, efficient

Cons

- Difficulties in dealing with complex domains

Finite element method

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Approximation spaces: $V_h \subset H_0(\mathrm{curl}, \Omega)$ and $Q_h \subset L^2(\Omega)$

Galerkin method: For $t > 0$, find $E_h(t) \in V_h$ and $H_h(t) \in Q_h$ such that

$$
\begin{aligned}
(\varepsilon \partial_t E_h(t), v_h)_\Omega - (H_h(t), \mathrm{curl}\; v_h)_\Omega &= 0 \\
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for all test functions $v_h \in V_h$ and $q_h \in Q_h$, and for all $t > 0$.

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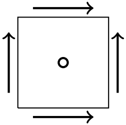
Algebraic realization. For a choice of basis functions, we have

$$\begin{aligned} \mathbf{M}_\varepsilon \partial_t \mathbf{e}(t) - \mathbf{C}^\top \mathbf{h}(t) &= 0 \\ \mathbf{D}_\mu \partial_t \mathbf{h}(t) + \mathbf{C}\, \mathbf{e}(t) &= 0 \end{aligned}$$

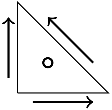
First order elements

Finite element spaces on reference elements.

- 1980 - Nedelec - Mixed Finite Elements in $\mathbb{R}^3$



$$V_h(Q) = \mathcal{N}_0(Q) \qquad \phi_1 = (1-y,0) \qquad \phi_3 = (0,1-x)$$
$$Q_h(Q) = P_0(Q) \qquad \phi_2 = (y,0) \qquad \phi_4 = (0,x)$$



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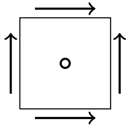

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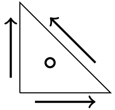

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Lemma (accuracy) If E and H are sufficiently smooth, then

$$\|E(t) - E_h(t)\|_{L^2} + \|H(t) - H_h(t)\|_{L^2} \leq Ch$$

- 1992 - Monk - Analysis of a finite element method for Maxwell's equations
- 1993 - Monk - An analysis of Nedelec's method for spatial discretization of Maxwell's equations

First order elements

Stability and accuracy.
Lowest order MFEM yields stable and accurate approximation in space.

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Numerical solution. Time integration of resulting ode system

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by explicit schemes requires application of $\mathbf{M}_\varepsilon^{-1}$ and $\mathbf{D}_\mu^{-1}$.

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Thus, explicit time-stepping for standard MFEM is not efficient.

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Remedy – Mass-lumping: replace $\mathbf{M}_\varepsilon$ by approximation $\mathbf{M}_\varepsilon^L$ such that

- $\mathbf{M}_\varepsilon^L$ corresponds to positive definite matrix (stability)
- $\mathbf{M}_\varepsilon^L$ is good approximation for $\mathbf{M}_\varepsilon$ (accuracy)
- $(\mathbf{M}_\varepsilon^L)^{-1}$ can be applied efficiently (efficiency)

construction of $\mathbf{M}_\varepsilon^L$ usually via numerical quadrature.

Mass lumping literature

- 1975 - Fried, Malkus - Finite element mass matrix lumping by numerical integration with no convergence rate loss
- 1999 - Kong, Mulder, Veldhuizen - Higher-order triangular and tetrahedral finite elements with mass lumping for solving the wave equation
- 2000 - Becache, Joly, Tsogka - An analysis of new mixed finite elements for the approximation of wave propagation models
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- 2018 - Geevers, Mulder, Vegt - New higher-order mass-lumped tetrahedral elements for wave propagation modelling

Mass-lumping in H^1

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Mass-lumping in H^1

Mass-lumping in $H(\mathrm{div})$ and $H(\mathrm{curl})$

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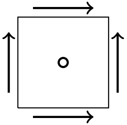
- 2020 - Egger, Radu - A mass-lumped mixed finite element method for acoustic wave propagation.
- 2020 - Egger, Radu - A mass-lumped mixed finite element method for Maxwell's equations
- 2021 - Egger, Radu - A second order finite element method with mass lumping for wave equations in H(div).
- 2021 - Egger, Radu - A Second-Order Finite Element Method with Mass Lumping for Maxwell's Equations on Tetrahedra.

Observation for the lowest order case



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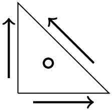

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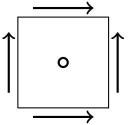

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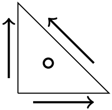

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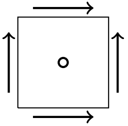

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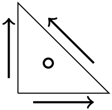

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General rule of thumb: Exactly two basis functions are necessary for each quadrature point in order to achieve local orthogonalization.

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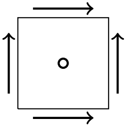

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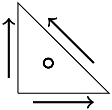

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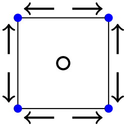
Some existing methods: Acute mesh lumping (triangles)

- 1996 - Baranger - Connection between finite volume and mixed finite element methods

First order elements... with mass lumping!

Use a larger polynomial space **[WheelerYotov'06]**



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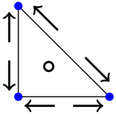

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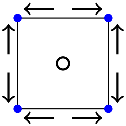

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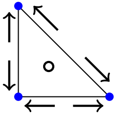

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Lemma. $\widetilde{\mathbf{M}}_\epsilon^L$ is block diagonal and thus also $(\widetilde{\mathbf{M}}_\epsilon^L)^{-1}$.

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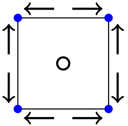

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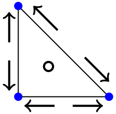

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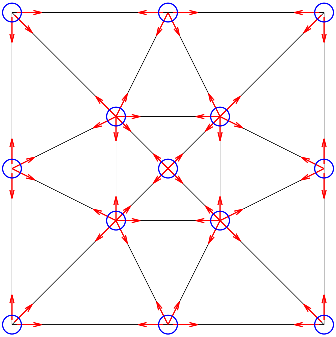
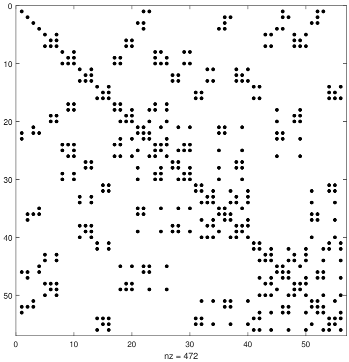
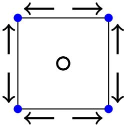




First order elements... with mass lumping!

Use a larger polynomial space **[WheelerYotov'06]**



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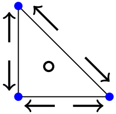

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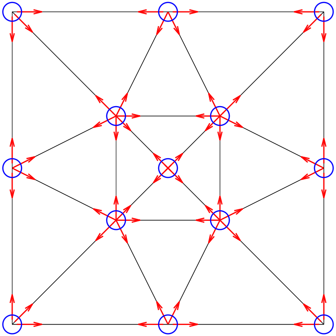

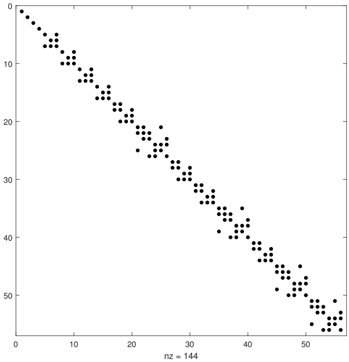

First order elements... with mass lumping!

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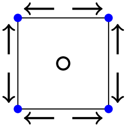

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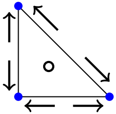

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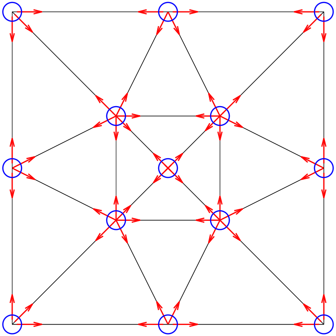

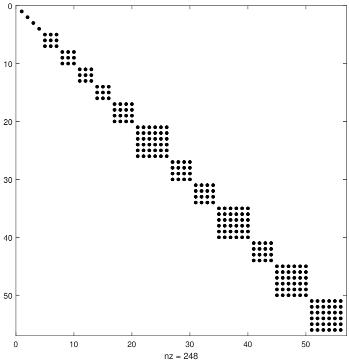

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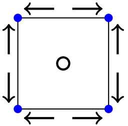

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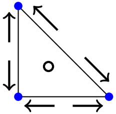

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- 2020 - Egger, Radu - A mass-lumped mixed finite element method for Maxwell's equations

First order elements... with mass lumping!

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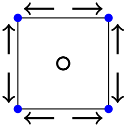

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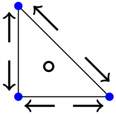

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Proof Idea: Error splitting in discrete and projection error, discrete stability, energy estimates, consistency error, analysis of the quadrature error (Strang).

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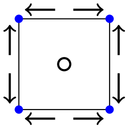

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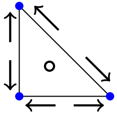

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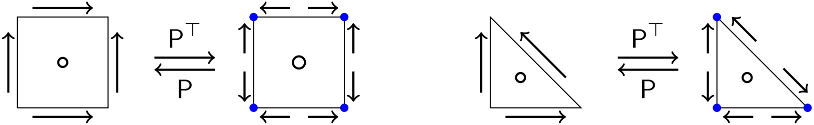
First order elements... a Yee-like method

Idea : Use lowest order space to represent solution, compute update in the enriched space, and then project back to the lowest order space.



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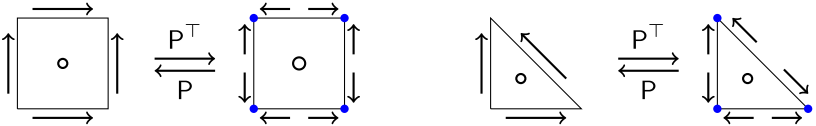

Formal representation of the inverse mass matrix :

$$(\mathbf{M}_\epsilon^L)^{-1} = \mathsf{P}\ (\widetilde{\mathbf{M}}_\epsilon^L)^{-1}\ \mathsf{P}^\top$$

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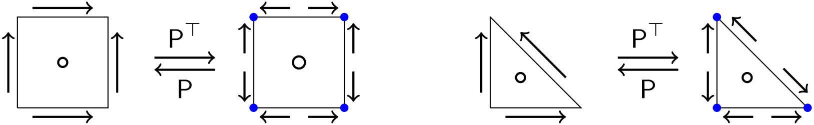

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$$(\mathbf{M}_\epsilon^L)^{-1} = \mathsf{P}\ (\widetilde{\mathbf{M}_\epsilon^L})^{-1}\ \mathsf{P}^\top$$

Note :

➔ The inverse is sparse, the corresponding mass matrix is full

➔ We have equivalence to FDTD on square elements!

➔ Similar idea in :

- 2008 - Codecasa, Politi - Explicit, consistent and conditionally stable extension of FDTD to tetrahedral grids by FIT

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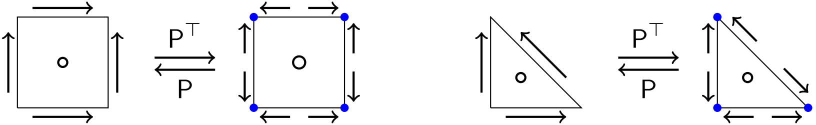

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First order elements on tetrahedral meshes

The same concept also applies in 3D on tetrahedral meshes

$V_h(T) = \mathcal{N}_0(T)$
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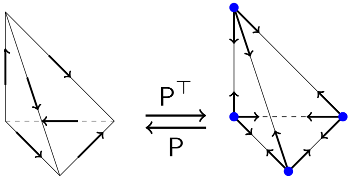

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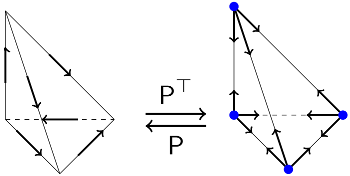

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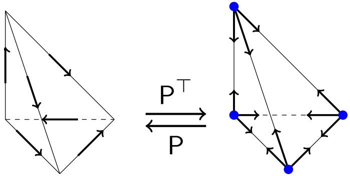

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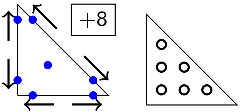
Next task : Extension to second order elements.

Second order elements

Extension to second order elements

- 1997 - Elmkies, Joly - Elements finis d'arete et condensation de masse pour les equations de Maxwell - le cas 2D



$$\widehat{V}_h(T) = \mathcal{N}_1(T) \oplus B = \mathcal{E}\mathcal{J}_1(T) \subseteq P_3(T)$$
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Second order elements

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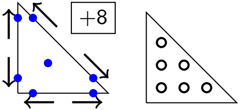

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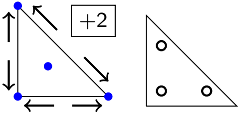
New proposal :



$$\widehat{V}_h(T) = \mathcal{N}_1(T) \subseteq P_2(T)$$
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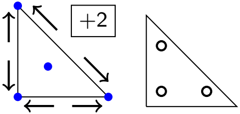
The quadrature rule is exact for P_2 polynomials ... but is this enough ?

Short notes on the analysis

New proposal :



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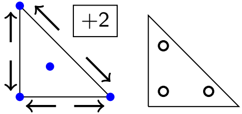

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Proof Idea: Discrete stability, energy estimates, Galerkin orthogonality, consistency error, Strang analysis of the quadrature error.

Classic requirement : The quadrature rule has to be exact for $P_1(T)^d \times \widehat{V}_h(T)$

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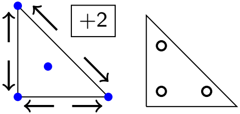

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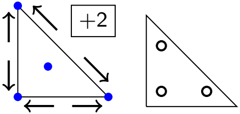
New requirements

(i) There exists a splitting $\widehat{V}_h(T) = W(T) \oplus B(T)$ such that $\dim(B(T)) = \dim(\mathrm{curl}(B(T)))$ and $\mathrm{curl}(B(T)) \cap \mathrm{curl}(W(T)) = \{0\}$

(ii) The quadrature rule is exact for $P_1(T)^2 \times W(T)$

Short notes on the analysis

New proposal :



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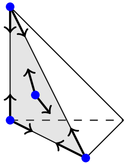
(ii) The quadrature rule is exact for $P_1(T)^2 \times \mathcal{NC}_1(T) = P_2(T)^2$

Second order method - 3D

- 1997 - Elmkies, Joly - Elements finis d'arete et condensation de masse pour les equations de Maxwell - le cas 3D



$$\widehat{V}_h(T) = \mathcal{N}_1(T)$$

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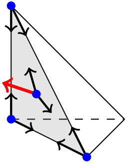

$$\widehat{V}_h(T) = \mathcal{N}_1(T) \oplus B(T) \qquad B(T) = \text{span}\left\{\begin{array}{l}\Phi_1 = \lambda_2\lambda_3\lambda_4\nabla\lambda_1 \\ \Phi_2 = \lambda_1\lambda_3\lambda_4\nabla\lambda_2 \\ \Phi_3 = \lambda_1\lambda_2\lambda_4\nabla\lambda_3 \\ \Phi_4 = \lambda_1\lambda_2\lambda_3\nabla\lambda_4\end{array}\right\}$$

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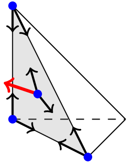

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But $\nabla(\lambda_1\lambda_2\lambda_3\lambda_4) = \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 \Rightarrow \mathrm{curl}(\Phi_1 + \Phi_2 + \Phi_3 + \Phi_4) = 0.$

Thus $\dim(B(T)) \neq \dim(\mathrm{curl}\, B(T))$!

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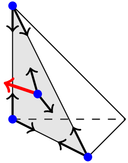

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Theorem [EggerRadu21]. If (and only if) $\mathrm{div}(\boldsymbol{E}) = 0$, then

$$\|\boldsymbol{E}(t) - \widehat{\boldsymbol{E}}_h(t)\| + \|\boldsymbol{H}(t) - \widehat{\boldsymbol{H}}_h(t)\| \leq Ch^2$$

Note. In general, second order convergence is lost!

Second order method - 3D

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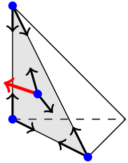

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But $\nabla(\lambda_1\lambda_2\lambda_3\lambda_4) = \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 \Rightarrow \text{curl}(\Phi_1 + \Phi_2 + \Phi_3 + \Phi_4) = 0.$

Thus $\dim(B(T)) \neq \dim(\text{curl}\, B(T))$!

Theorem [EggerRadu21]. If (and only if) $\text{div}(\boldsymbol{E}) = 0$, then

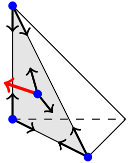
$$\|\boldsymbol{E}(t) - \widehat{\boldsymbol{E}}_h(t)\| + \|\boldsymbol{H}(t) - \widehat{\boldsymbol{H}}_h(t)\| \leq Ch^2$$

Note. In general, second order convergence is lost!

Solution. Modify one basis function, for example $\widehat{\Phi}_4 = \lambda_1\lambda_2\lambda_3(\lambda_2 - \lambda_1 + 1)\nabla\lambda_4$

Second order method - 3D

- 1997 - Elmkies, Joly - Elements finis d'arete et condensation de masse pour les equations de Maxwell - le cas 3D



$$\widehat{V}_h(T) = \mathcal{N}_1(T) \oplus B(T) \qquad B(T) = \text{span} \left\{ \begin{array}{l} \Phi_1 = \lambda_2\lambda_3\lambda_4\nabla\lambda_1 \\ \Phi_2 = \lambda_1\lambda_3\lambda_4\nabla\lambda_2 \\ \Phi_3 = \lambda_1\lambda_2\lambda_4\nabla\lambda_3 \\ \qquad\widehat{\Phi}_4 \end{array} \right\}$$

But $\nabla(\lambda_1\lambda_2\lambda_3\lambda_4) = \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 \Rightarrow \operatorname{curl}(\Phi_1 + \Phi_2 + \Phi_3 + \Phi_4) = 0.$

$$\text{Thus } \dim(B(T)) \neq \dim(\operatorname{curl} B(T)) \text{ !}$$

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Further results

Acoustic wave equation

Dual mixed formulation of

$$
\begin{aligned}
\partial_t u(t) &= \nabla p(t) && \text{in } \Omega \\
\partial_t p(t) &= -\operatorname{div} u(t) && \text{in } \Omega
\end{aligned}
$$

- 1988 - Geveci - On the application of mixed finite element methods to the wave equations

$$u \in H(\operatorname{div}, \Omega) \quad \text{and} \quad p \in L^2(\Omega)$$

Acoustic wave equation

Galerkin method: For $t > 0$, find $u_h(t) \in V_h$ and $p_h(t) \in Q_h$ such that

$$\begin{aligned}(\partial_t u_h(t), v_h)_h - (p_h(t), \operatorname{div} v_h) &= 0\\ (\partial_t p_h(t), q_h) + (\operatorname{div} u_h(t), q_h) &= 0\end{aligned}$$

for all $v_h \in V_h \subseteq H(\operatorname{div}, \Omega)$ and $q_h \in Q_h \subseteq L^2(\Omega)$, and for all $t > 0$.

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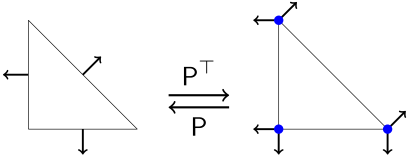
for all $v_h \in V_h \subseteq H(\mathrm{div}, \Omega)$ and $q_h \in Q_h \subseteq L^2(\Omega)$, and for all $t > 0$.

First order methods

$V_h(T) = \mathsf{RT}_0(T)$
$Q_h(T) = P_0(T)$



$\widetilde{V}_h(T) = \mathsf{BDM}_1(T)$
$\widetilde{Q}_h(T) = P_0(T)$

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Galerkin method: For $t > 0$, find $u_h(t) \in V_h$ and $p_h(t) \in Q_h$ such that

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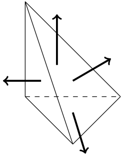
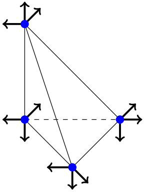


$$\underset{\mathsf{P}}{\overset{\mathsf{P}^\top}{\rightleftarrows}}$$



$$
\begin{aligned}
\widetilde{V}_h(T) &= \mathsf{BDM}_1(T) \\
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Theorem [EggerRadu20]. If u and p are sufficiently smooth, then

$$\|u(t) - u_h(t)\| + \|p(t) - p_h(t)\| \leq Ch$$

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for all $v_h \in V_h \subseteq H(\operatorname{div}, \Omega)$ and $q_h \in Q_h \subseteq L^2(\Omega)$, and for all $t > 0$.

Second order methods

$$V_h(T) = \mathsf{RT}_1(T)$$
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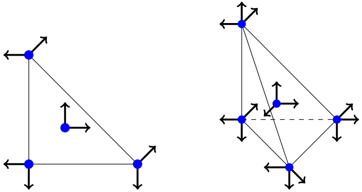

Theorem [EggerRadu20]. If u and p are sufficiently smooth, then

$$\|u(t) - u_h(t)\| + \|p(t) - p_h(t)\| \leq Ch^2$$

Extensions to H(div) : Porous media flow

$$\begin{aligned} K^{-1}u + \nabla p &= 0 && \text{in } \Omega \\ \operatorname{div} u &= f && \text{in } \Omega \\ p &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Discrete variational formulation

$$\begin{aligned} (K^{-1}u_h, v_h) - (p_h, \operatorname{div} v_h) &= 0 && \forall v_h \in V_h \subseteq H(\operatorname{div}, \Omega) \\ (\operatorname{div} u_h, q_h) &= (f, q_h) && \forall q_h \in Q_h \subseteq L^2(\Omega) \end{aligned}$$

Problem : we have to solve a full (indefinite) saddle point system ...

Extensions to H(div) : Porous media flow

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Discrete variational formulation via *mass lumping* (MFMFE) **[WheelerYotov06]**

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(\operatorname{div} u_h, q_h) &= (f, q_h) && \forall q_h \in Q_h \subseteq L^2(\Omega)
\end{aligned}
$$

For appropriate spaces V_h, Q_h and $(\cdot,\cdot)_h$, the *lumped mass matrix* $\mathsf{M_h}$ is block-diagonal, and the variable u_h can be eliminated efficiently.

$$
\begin{pmatrix} \mathsf{M_h} & -\mathsf{C}^\top \\ \mathsf{C} & 0 \end{pmatrix} \begin{pmatrix} \mathsf{u} \\ \mathsf{p} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathsf{f} \end{pmatrix} \qquad \Longrightarrow \qquad \mathsf{C}\mathsf{M_h}^{-1}\mathsf{C}^\top \mathsf{p} = \mathsf{f}
$$

The problem reduces to symmetric, positive definite cell-centered system for the pressure (CCFD)

Extensions in H(div) : Poroelasticity

Further extension to poroelasticity

$$
\begin{aligned}
-\mathrm{div}(2\mu\epsilon(u) + \lambda\mathrm{div}(u)I) + \nabla p &= f && \text{on } \Omega,\ t > 0,\\
\mathrm{div}\,\partial_t u + c_s\partial_t p + \mathrm{div}\, w &= g && \text{on } \Omega,\ t > 0,\\
\mathcal{K}^{-1} w + \nabla p &= 0 && \text{on } \Omega,\ t > 0.
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Discrete spaces. $u_h \in P_2^+(\mathcal{T}_h) \cap H^1(\Omega)$, $p_h \in P_1(\mathcal{T}_h)$, $w_h \in \mathsf{RT}_1$

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Algebraic structure:

$$
\begin{aligned}
\mathsf{A}\mathsf{u}(t) \quad -\mathsf{B}^\top \mathsf{p}(t) \qquad\qquad &= \mathsf{f}(t),\\
\mathsf{B}\partial_t \mathsf{u}(t) + \mathsf{C}\partial_t \mathsf{p}(t) + \mathsf{D}\mathsf{w}(t) \; &= \mathsf{g}(t),\\
-\mathsf{D}^\top \mathsf{p}(t) + \mathsf{M_h} \mathsf{w}(t) &= 0.
\end{aligned}
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Mass-lumping. Local elimination of the Darcy velocity w

$$
\begin{aligned}
\mathsf{A}\mathsf{u}(t) \quad -\mathsf{B}^\top \mathsf{p}(t) \qquad\qquad\qquad &= \mathsf{f}(t),\\
\mathsf{B}\partial_t \mathsf{u}(t) + \mathsf{C}\partial_t \mathsf{p}(t) + \mathsf{D}\mathsf{M}_\mathsf{h}^{-1}\mathsf{D}^\top \mathsf{p}(t) &= \mathsf{g}(t),
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\end{aligned}
$$

Analysis. Existence of weak solutions

Extensions in H(div) : Poroelasticity

Further extension to poroelasticity

$$\begin{aligned} -\mathrm{div}(2\mu\epsilon(u) + \lambda\mathrm{div}(u)I) + \nabla p &= f && \text{on } \Omega,\ t > 0, \\ \mathrm{div}\,\partial_t u + c_s\partial_t p + \mathrm{div}\,w &= g && \text{on } \Omega,\ t > 0, \\ \mathcal{K}^{-1}w + \nabla p &= 0 && \text{on } \Omega,\ t > 0. \end{aligned}$$

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$$\begin{aligned} \mathsf{A}\mathsf{u}(t) \quad -\mathsf{B}^\top\mathsf{p}(t) \qquad\qquad &= \mathsf{f}(t), \\ \mathsf{B}\partial_t\mathsf{u}(t) + \mathsf{C}\partial_t\mathsf{p}(t) + \mathsf{D}\mathsf{M}_\mathsf{h}^{-1}\mathsf{D}^\top\mathsf{p}(t) &= \mathsf{g}(t), \end{aligned}$$

Analysis. Existence of weak solutions

Theorem. Together with an appropriate time discretization, we obtain

$$\|u(t) - u_h(t)\| + \|p(t) - p_h(t)\| + \|w(t) - w_h(t)\| \leq C(h^2 + \tau^2)$$

Closing remarks

Further topics:

- Parameter-robust preconditioning for poroelasticity
- Reduced symmetry methods for elasticity with mass lumping
- Higher order extension

Closing remarks

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- Parameter-robust preconditioning for poroelasticity
- Reduced symmetry methods for elasticity with mass lumping
- Higher order extension

Key ingredients for mass lumping:

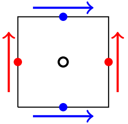
- Start with a basis space V_h that contains all $P_k(T)^d$ polynomials (for approximation).
- V_h dictates the number of continuity conditions on the boundary
- Find a quadrature rule that has sufficiently many quadrature points on the boundary and has the desired accuracy
- Extend V_h by appropriate "bubble" functions such that we have exactly d-many functions for each quadrature point.

List of relevant publications

- 2020 - Egger, Radu - A mass-lumped mixed finite element method for acoustic wave propagation.
- 2020 - Egger, Radu - A mass-lumped mixed finite element method for Maxwell's equations
- 2021 - Egger, Radu - A second order finite element method with mass lumping for wave equations in H(div).
- 2021 - Egger, Radu - A Second-Order Finite Element Method with Mass Lumping for Maxwell's Equations on Tetrahedra.

Thank you for your attention!

Strategy 1: Directional quadrature



$$V_h(\widehat{Q}) = \mathcal{N}_0^I(\widehat{Q}) \qquad \phi_1 = (1-y, 0) \qquad \phi_3 = (0, 1-x)$$
$$Q_h(\widehat{Q}) = P_0(\widehat{Q}) \qquad \phi_2 = (y, 0) \qquad \phi_4 = (0, x)$$

Assumption (A). $\epsilon = \mathrm{diag}(\epsilon^x, \epsilon^y)$ and every element Q is rectangular
i.e., Piola-transform and multiplication by ϵ do not change orientation

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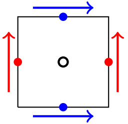

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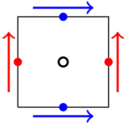
Directional quadrature.

$$\begin{aligned} \mathbf{M}_\epsilon \leftarrow (\epsilon\phi_i, \phi_j) &\overset{(A)}{=} (\epsilon^x \phi_i^x, \phi_j^x) + (\epsilon^y \phi_i^y, \phi_j^y) \\ &\approx (\epsilon^x \phi_i^x, \phi_j^x)_{Q,x} + (\epsilon^y \phi_i^y, \phi_j^y)_{Q,y} \to \mathbf{M}_\epsilon^L \end{aligned}$$

Strategy 1: Directional quadrature



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Lemma. If (A) holds for all elements Q, then $\mathbf{M}_\epsilon^L$ is diagonal.
Moreover, the MFEM with directional lumping equivalent to FDTD/FIT;

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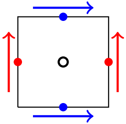

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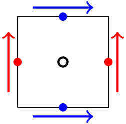

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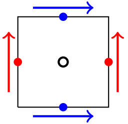
Lemma. If (A) holds for all elements Q, then $\mathbf{M}_\epsilon^L$ is diagonal. Moreover, the MFEM with directional lumping equivalent to FDTD/FIT;

Lemma (accuracy). Let (A) hold for all elements, then

$$\|\boldsymbol{E}(t) - \boldsymbol{E}_h(t)\| + \|\boldsymbol{H}(t) - \boldsymbol{H}_h(t)\| \leq Ch \qquad \text{and}$$
$$\|\Pi_h \boldsymbol{E}(t) - \boldsymbol{E}_h(t)\| + \|\Pi_h^0 \boldsymbol{H}(t) - \boldsymbol{H}_h(t)\| \leq Ch^2$$

Strategy 1: Directional quadrature



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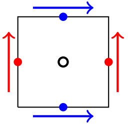
Remark. Not applicable for non-orthogonal grids or anisotropic media!

Strategy 1: Directional quadrature



$V_h(\widehat{Q}) = \mathcal{N}_0^I(\widehat{Q})$ $\qquad \phi_1 = (1-y, 0) \qquad \phi_3 = (0, 1-x)$

$Q_h(\widehat{Q}) = P_0(\widehat{Q})$ $\qquad \phi_2 = (y, 0) \qquad \phi_4 = (0, x)$

Assumption (A). $\epsilon = \mathrm{diag}(\epsilon^x, \epsilon^y)$ and every element Q is rectangular i.e., Piola-transform and multiplication by ϵ do not change orientation

Directional quadrature.

$$\begin{aligned}\mathbf{M}_\epsilon \leftarrow (\epsilon\phi_i, \phi_j) &\overset{(A)}{=} (\epsilon^x \phi_i^x, \phi_j^x) + (\epsilon^y \phi_i^y, \phi_j^y) \\ &\approx (\epsilon^x \phi_i^x, \phi_j^x)_{Q,x} + (\epsilon^y \phi_i^y, \phi_j^y)_{Q,y} \to \mathbf{M}_\epsilon^L\end{aligned}$$

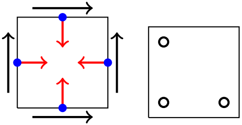
Lemma. If (A) holds for all elements Q, then $\mathbf{M}_\epsilon^L$ is diagonal.
Moreover, the MFEM with directional lumping equivalent to FDTD/FIT;

Remark. Not applicable for non-orthogonal grids or anisotropic media!

Goal. provide generalization to unstructured grids and anisotropic media

Strategy 2 : Extended finite element space

Add additional interior basis functions **[ElmkiesJoly'93]**.



$$V_h(\widehat{Q}) = \mathcal{N}_0^I(\widehat{Q}) \oplus B = \mathsf{EJ}_1(\widehat{Q}) \subseteq P_2(\widehat{Q})$$
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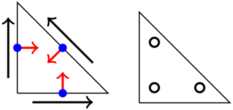

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Numerical Integration

Use the midpoint rule, which is exact for P_2 functions.

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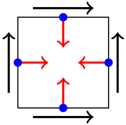
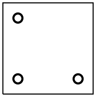

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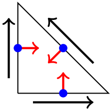
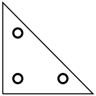

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Exactness requirement
The quadrature rule should be exact for $P_k \times V_h$, $(k = 0)$ for the first order case

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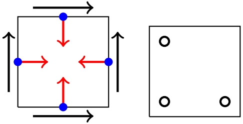

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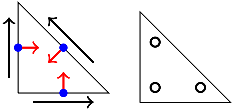

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Lemma (accuracy)

If $\boldsymbol{E}$ and $\boldsymbol{H}$ are sufficiently smooth. Then

$$\|\boldsymbol{E}(t) - \boldsymbol{E}_h(t)\| + \|\boldsymbol{H}(t) - \boldsymbol{H}_h(t)\| \leq Ch$$

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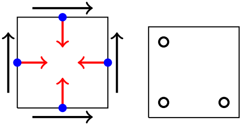

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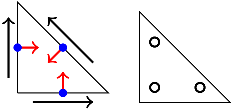

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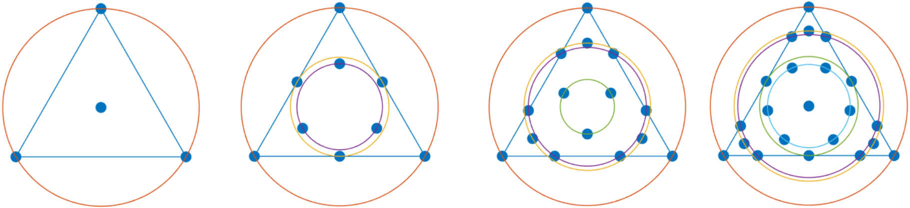
Note: Space enrichment allows to utilize standard numerical quadrature. Resulting method works for unstructure grids and anisotropic coefficients.

But: Enrichment substantially increases the number of dof's.

Extension to even higher orders

We look for Gauss-Lobatto type quadrature rules !



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