A finite element time domain method for Maxwell's equations

Herbert Egger, **Bogdan Radu**

Johann Radon Institute for Computational and Applied Mathematics (RICAM)
Austrian Academy of Sciences (ÖAW)
Linz, Austria

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RICAM Group seminar

Abstract

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We present two discretization methods for solving Maxwell's equations on unstructured grids efficiently.

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- The first method is based on a finite-element approximation by Nédélec elements of type II, which allows for mass-lumping by numerical quadrature. The resulting scheme has two degrees of freedom per edge and a block diagonal mass matrix, and hence allows for an efficient time integration by explicit methods.

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- The first method is based on a finite-element approximation by Nédélec elements of type II, which allows for mass-lumping by numerical quadrature. The resulting scheme has two degrees of freedom per edge and a block diagonal mass matrix, and hence allows for an efficient time integration by explicit methods.

- The second method is obtained by an algebraic reduction of the first, leading to a scheme with only one degree of freedom per edge most of the time. This scheme has a sparse inverse mass matrix, again enabling an efficient time integration

Maxwell's equations

$$
\begin{aligned}
\operatorname{curl} \boldsymbol{H}(t) &= \quad \partial_t \boldsymbol{D}(t) + \boldsymbol{j}(t) && \text{Ampere/Maxwell Law} \\
\operatorname{curl} \boldsymbol{E}(t) &= -\partial_t \boldsymbol{B}(t) && \text{Faraday Law} \\
\operatorname{div} \boldsymbol{D}(t) &= \varrho && \text{Gauss Law} \\
\operatorname{div} \boldsymbol{B}(t) &= 0 && \text{Magnetic Gauss Law}
\end{aligned}
$$

Material laws for linear and non-dispersive but inhomogeneous/anisotropic media

$$
\boldsymbol{D}(t) = \varepsilon \boldsymbol{E}(t) \qquad \text{and} \qquad \boldsymbol{B}(t) = \mu \boldsymbol{H}(t) \qquad \text{and} \qquad \boldsymbol{j}(t) = \boldsymbol{j}_s(t) + \sigma \boldsymbol{E}(t)
$$

Substituting the material laws yields

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\begin{aligned}
\varepsilon \partial_t \boldsymbol{E}(t) + \sigma \boldsymbol{E}(t) - \operatorname{curl} \boldsymbol{H}(t) &= -\boldsymbol{j}_s(t) \\
\mu \partial_t \boldsymbol{H}(t) + \operatorname{curl} \boldsymbol{E}(t) &= 0
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Initial and boundary conditions will be discussed in a bit.

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Goal: systematic and flexible space discretization

- stable: no artificial energy production
- accurate: provable convergence rates
- efficient: appropriate for explicit time-stepping methods

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Methods: FDTD/FIT, FEM, FVM, DG, ...

- 1966 - Yee - Numerical solution of initial boundary value problems involving Maxwell's equations in isotropic media
- 1977 - Weiland - Eine Methode zur Lösung der Maxwell'schen Gleichungen für sechskomponentige Felder auf diskreter Basis
- 1980 - Taflove - Application of the Finite-Difference Time-Domain method to sinusoidal steady-state electromagnetic penetration problems

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Finite differences : TE case

$$\boldsymbol{E} = \begin{pmatrix} E_x \\ E_y \\ 0 \end{pmatrix} \qquad \boldsymbol{H} = \begin{pmatrix} 0 \\ 0 \\ H_z \end{pmatrix}$$

$$\begin{cases} \varepsilon\partial_t E_x + \sigma E_x - \partial_y H_z = -j_{s,1}, \\ \varepsilon\partial_t E_y + \sigma E_y + \partial_x H_z = -j_{s,2}, \\ -\mu\,\partial_t H_z - \partial_x E_y + \partial_y E_x = 0. \end{cases}$$

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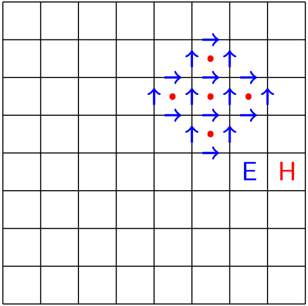

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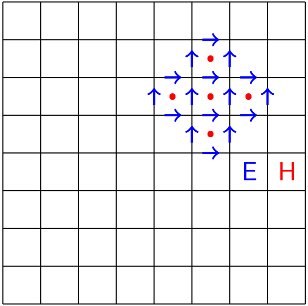

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Pros

- Easy to implement
- stable, $h^2 + \tau^2$ accurate, efficient

Cons

- Complex domains

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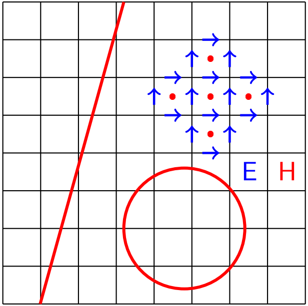

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\boldsymbol{n} \times \boldsymbol{E}(t) &= \boldsymbol{G}(t) && \text{on } \partial\Omega, t > 0 \\
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Finite element method

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\begin{aligned}
\varepsilon\partial_{tt}\boldsymbol{E}(t) + \sigma\partial_t\boldsymbol{E}(t) + \operatorname{curl}(\mu^{-1}\operatorname{curl}\boldsymbol{E}(t)) &= -\partial_t\boldsymbol{j}_s(t) && \text{in } \Omega, t > 0\\
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Galerkin method: For $t>0$ find $\boldsymbol{E}_h(t) \in \boldsymbol{V}_h \subseteq H(\operatorname{curl},\Omega)$ such that

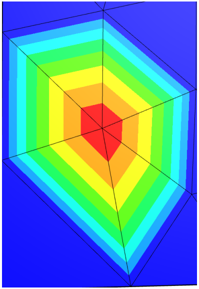
$$(\varepsilon\partial_{tt}\boldsymbol{E}_h(t), \boldsymbol{v}_h) + (\operatorname{curl}\boldsymbol{E}_h(t), \operatorname{curl}\boldsymbol{v}_h) = 0$$

for all test functions $\boldsymbol{v}_h \in \boldsymbol{V}_h$.

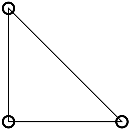
Finite element spaces: What are proper spaces for discretizing $H(\operatorname{curl},\Omega)$?

Basis functions

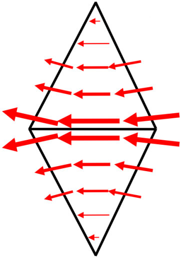
- Global basis functions for H^1



- Local degrees of freedom



- Global basis functions for $H(\mathrm{curl})$



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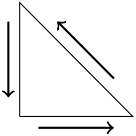

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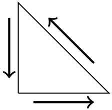
Finite element spaces on reference elements.

- 1980 - Nedelec - Mixed Finite Elements in $\mathbb{R}^3$



$$\boldsymbol{V}_h(T) = \mathcal{N}_0(T)$$

$$\phi_1 = (1-y, x) \qquad \phi_3 = (y, 1-x)$$
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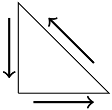

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Lemma (accuracy) If $\boldsymbol{E}$ is sufficiently smooth, then

$$\|\boldsymbol{E}(t) - \boldsymbol{E}_h(t)\|_{H(\mathrm{curl})} \leq Ch$$

- 1992 - Monk - Analysis of a finite element method for Maxwell's equations
- 1993 - Monk - An analysis of Nedelec's method for spatial discretization of Maxwell's equations

Stability and accuracy.
Lowest order MFEM yields stable and accurate approximation in space.

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Numerical solution. Time integration of resulting ode system

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by explicit schemes requires application of $\mathsf{M}_\varepsilon^{-1}$.

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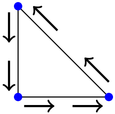
Remedy – Mass-lumping: replace M_ε by approximation M_ε^L such that

- M_ε^L corresponds to positive definite matrix (stability)
- M_ε^L is good approximation for M_ε (accuracy)
- $(\mathsf{M}_\varepsilon^L)^{-1}$ can be applied efficiently (efficiency)

construction of M_ε^L usually via numerical quadrature.

First order elements... with mass lumping!

Use a larger polynomial space



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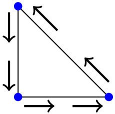

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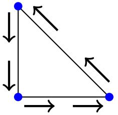

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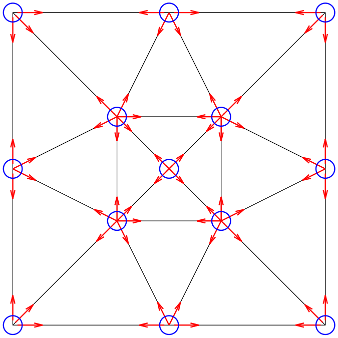

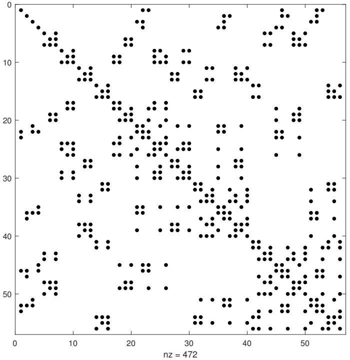

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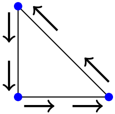

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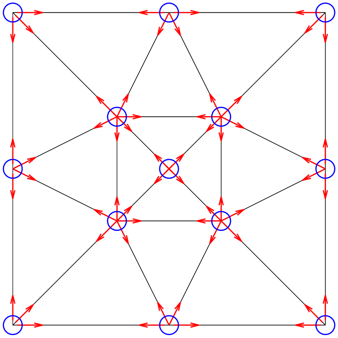

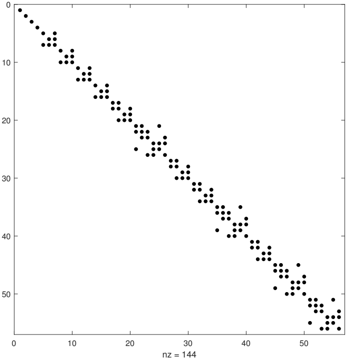

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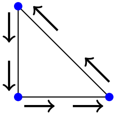

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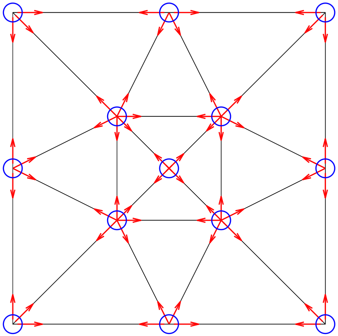

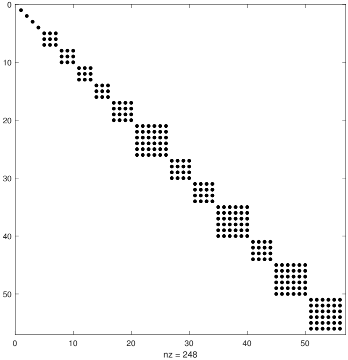

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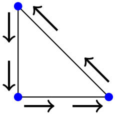

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- 2020 - Egger, Radu - A mass-lumped mixed finite element method for Maxwell's equations

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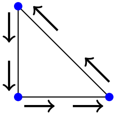

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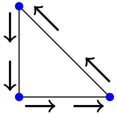

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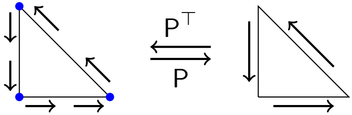
Requirement : The quadrature rule must be exact for $P_0(T)^2 \times \widetilde{\boldsymbol{V}}_h(T)$

A projection matrix and its properties

Consider the projection $\Pi_\mathsf{P} : \mathcal{NC}_1 \to \mathcal{N}_0$ realized algebraically by the matrix P:



such that $\mathsf{P}\mathsf{P}^\top = \mathsf{Id}$. Then

$$\mathsf{M} := \mathsf{P}\widetilde{\mathsf{M}}\mathsf{P}^\top \qquad \text{and} \qquad \mathsf{K} := \mathsf{P}\widetilde{\mathsf{K}}\mathsf{P}^\top$$

are mass and stiffness matrices for $\mathcal{N}_0$ w.r.t. the projected basis. Moreover,

$$\widetilde{\mathsf{K}} = \mathsf{P}^\top\mathsf{K}\mathsf{P}$$

since both $\mathcal{NC}_1$ and $\mathcal{N}_0$ produce the same curls, i.e. $\operatorname{curl}\Pi_\mathsf{P} v_h = \operatorname{curl} v_h$.
This is a consequence of the commuting diagram property

Let $\widetilde{\mathbf{e}}(t)$ be the coefficient vector to $\boldsymbol{E}_h(t)$, and it satisfies

$$\widetilde{\mathsf{M}}_\varepsilon^L \partial_{tt}\widetilde{\mathbf{e}}(t) + \widetilde{\mathsf{K}}\widetilde{\mathbf{e}}(t) = 0$$

Then $\mathbf{e}(t) \coloneqq \mathsf{P}\widetilde{\mathbf{e}}(t)$ satisfies

$$\begin{aligned}\partial_{tt}\mathbf{e}(t) &= \mathsf{P}\partial_{tt}\widetilde{\mathbf{e}}(t) = -\mathsf{P}(\widetilde{\mathsf{M}}_\varepsilon^L)^{-1}\widetilde{\mathsf{K}}\widetilde{\mathbf{e}}(t) = -\mathsf{P}(\widetilde{\mathsf{M}}_\varepsilon^L)^{-1}\mathsf{P}^\top\mathsf{K}\mathsf{P}\widetilde{\mathbf{e}}(t)\\ &= -\mathsf{P}(\widetilde{\mathsf{M}}_\varepsilon^L)^{-1}\mathsf{P}^\top\mathsf{K}\mathbf{e}(t)\end{aligned}$$

Thus $\mathbf{e}(t)$ satisfies

$$\mathsf{M}_\varepsilon^L \partial_{tt}\mathbf{e}(t) + \mathsf{K}\mathbf{e}(t) = 0$$

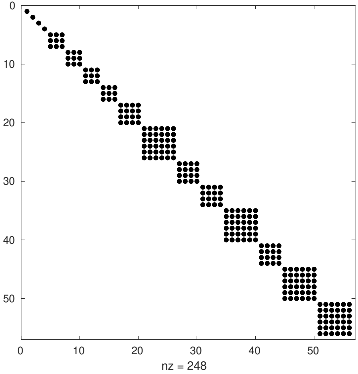
where $(\mathsf{M}_\varepsilon^L)^{-1} = \mathsf{P}\ (\widetilde{\mathsf{M}}_\varepsilon^L)^{-1}\ \mathsf{P}^\top$. **Note:** M_ε^L is not a FEM matrix.

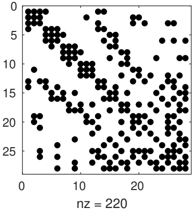
Theorem (accuracy) If $\boldsymbol{E}$ is sufficiently smooth, then $\boldsymbol{E}_h(t) \coloneqq \Pi_{\mathsf{P}}\widetilde{\boldsymbol{E}}_h(t)$ satisfies

$$\|\boldsymbol{E}(t) - \boldsymbol{E}_h(t)\|_{H(\mathrm{curl})} \le Ch$$

Impact on the matrices

Let us investigate the impact of the reduction (visually)





Inverse of $\widetilde{\mathrm{M}}_\varepsilon$ on the left, the inverse of M_ε on the right.

Complications...

$$\boxed{(\varepsilon\partial_{tt}\widetilde{\boldsymbol{E}}_h(t), \widetilde{\boldsymbol{v}}_h)_h + (\operatorname{curl}\widetilde{\boldsymbol{E}}_h(t), \operatorname{curl}\widetilde{\boldsymbol{v}}_h) = 0}$$

$$\Downarrow$$

$$\mathbf{e}(t) := \mathsf{P}\widetilde{\mathbf{e}}(t)$$

$$\Downarrow$$

$$\partial_{tt}\mathbf{e}(t) = -(\mathsf{P}(\widetilde{\mathsf{M}}_\varepsilon^L)^{-1}\mathsf{P}^\top)\mathsf{K}\mathbf{e}(t)$$

$$\boxed{(\varepsilon d_{\tau\tau}\widetilde{\boldsymbol{E}}_h^n, \widetilde{\boldsymbol{v}}_h)_h + (\operatorname{curl}\widetilde{\boldsymbol{E}}_h^n, \operatorname{curl}\widetilde{\boldsymbol{v}}_h) = 0}$$

$$\Downarrow$$

$$\mathbf{e}^n := \mathsf{P}\widetilde{\mathbf{e}}^n$$

$$\Downarrow$$

$$\frac{\mathbf{e}^{n+1} - 2\mathbf{e}^n + \mathbf{e}^{n-1}}{\tau^2} = -(\mathsf{P}(\widetilde{\mathsf{M}}_\varepsilon^L)^{-1}\mathsf{P}^\top)\mathsf{K}\mathbf{e}^n$$

$$\boxed{(\varepsilon d_{\tau\tau}\widetilde{\boldsymbol{E}}_h^n, \widetilde{\boldsymbol{v}}_h)_h + (\operatorname{curl}\widetilde{\boldsymbol{E}}_h^n, \operatorname{curl}\widetilde{\boldsymbol{v}}_h) = 0}$$

$$\Downarrow$$

$$\mathbf{e}^n := \mathsf{P}\widetilde{\mathbf{e}}^n$$

$$\Downarrow$$

$$\mathbf{e}^{n+1} = 2\mathbf{e}^n - \mathbf{e}^{n-1} - \tau^2(\mathsf{P}(\widetilde{\mathsf{M}}_\varepsilon^L)^{-1}\mathsf{P}^\top)\mathsf{K}\mathbf{e}^n$$

Complications...

$$(\varepsilon d_{\tau\tau}\widetilde{\boldsymbol{E}}_h^n, \widetilde{\boldsymbol{v}}_h)_h + (\sigma d_\tau \widetilde{\boldsymbol{E}}_h^n, \widetilde{\boldsymbol{v}}_h)_h + (\operatorname{curl} \widetilde{\boldsymbol{E}}_h^n, \operatorname{curl} \widetilde{\boldsymbol{v}}_h) = 0$$

$$\Downarrow$$

$$\mathbf{e}^n := \mathsf{P}\widetilde{\mathbf{e}}^n$$

$$\Downarrow$$

$$\begin{aligned}\mathbf{e}^{n+1} = 2\mathbf{e}^n - \mathbf{e}^{n-1} &- \tau^2(\mathsf{P}(\widetilde{\mathsf{M}}_\varepsilon^L + \tfrac{\tau}{2}\widetilde{\mathsf{M}}_\sigma^L)^{-1}\mathsf{P}^\top)\mathsf{K}\mathbf{e}^n \\ &- \tau^2(\mathsf{P}(\widetilde{\mathsf{M}}_\varepsilon^L + \tfrac{\tau}{2}\widetilde{\mathsf{M}}_\sigma^L)^{-1}\left(\widetilde{\mathsf{M}}_\sigma^L \frac{\widetilde{\mathbf{e}}^n - \widetilde{\mathbf{e}}^{n-1}}{\tau}\right)\end{aligned}$$

$$(\varepsilon d_{\tau\tau}\widetilde{\boldsymbol{E}}_h^n, \widetilde{\boldsymbol{v}}_h) + (\sigma d_\tau \widetilde{\boldsymbol{E}}_h^n, \widetilde{\boldsymbol{v}}_h)_h + (\operatorname{curl}\widetilde{\boldsymbol{E}}_h^n, \operatorname{curl}\widetilde{\boldsymbol{v}}_h) = 0$$

$$\cancel{\Downarrow}$$

$$\mathbf{e}^n := \mathsf{P}\widetilde{\mathbf{e}}^n$$

$$\cancel{\Downarrow}$$

$$\begin{aligned}\mathbf{e}^{n+1} := 2\mathbf{e}^n - \mathbf{e}^{n-1} &- \tau^2(\mathsf{P}(\widetilde{\mathsf{M}}_\varepsilon^L + \tfrac{\tau}{2}\widetilde{\mathsf{M}}_\sigma^L)^{-1}\mathsf{P}^\top)\mathsf{K}\mathbf{e}^n \\ &- \tau^2(\mathsf{P}(\widetilde{\mathsf{M}}_\varepsilon^L + \tfrac{\tau}{2}\widetilde{\mathsf{M}}_\sigma^L)^{-1}\left(\mathsf{P}^\top\mathsf{M}_\sigma\mathsf{P}\frac{\widetilde{\mathbf{e}}^n - \widetilde{\mathbf{e}}^{n-1}}{\tau}\right)\end{aligned}$$

Recall:

$$\widetilde{\mathsf{K}} = \mathsf{P}^\top\mathsf{K}\mathsf{P} \qquad \text{but} \qquad \widetilde{\mathsf{M}}_\sigma^L \neq \mathsf{P}^\top\mathsf{M}_\sigma\mathsf{P}$$

$$\boxed{(\varepsilon d_{\tau\tau}\widetilde{\boldsymbol{E}}_h^n, \widetilde{\boldsymbol{v}}_h)_h + (\sigma d_\tau \widetilde{\boldsymbol{E}}_h^n, \widetilde{\boldsymbol{v}}_h)_h + (\operatorname{curl}\widetilde{\boldsymbol{E}}_h^n, \operatorname{curl}\widetilde{\boldsymbol{v}}_h) = 0}$$

$$\not\Downarrow$$

$$\mathbf{e}^n := \mathsf{P}\widetilde{\mathbf{e}}^n$$

$$\not\Downarrow$$

$$\begin{aligned}\mathbf{e}^{n+1} := 2\mathbf{e}^n - \mathbf{e}^{n-1} &- \tau^2(\mathsf{P}(\widetilde{\mathsf{M}}_\varepsilon^L + \tfrac{\tau}{2}\widetilde{\mathsf{M}}_\sigma^L)^{-1}\mathsf{P}^\top)\mathsf{K}\mathbf{e}^n \\ &- \tau^2(\mathsf{P}(\widetilde{\mathsf{M}}_\varepsilon^L + \tfrac{\tau}{2}\widetilde{\mathsf{M}}_\sigma^L)^{-1}\mathsf{P}^\top\mathsf{M}_\sigma \frac{\mathbf{e}^n - \mathbf{e}^{n-1}}{\tau}\end{aligned}$$

$$\boxed{\left(\left(\varepsilon+\tfrac{\tau}{2}\sigma\right)d_{\tau\tau}\widetilde{\boldsymbol{E}}_h^n,\widetilde{\boldsymbol{v}}_h\right)_h+\left(\sigma\Pi_{\mathsf{P}}d_\tau\widetilde{\boldsymbol{E}}_h^{n-1/2},\Pi_{\mathsf{P}}\widetilde{\boldsymbol{v}}_h\right)+\left(\operatorname{curl}\widetilde{\boldsymbol{E}}_h^n,\operatorname{curl}\widetilde{\boldsymbol{v}}_h\right)=0}$$

$$\Downarrow$$

$$\mathbf{e}^n := \mathsf{P}\widetilde{\mathbf{e}}^n$$

$$\Downarrow$$

$$\begin{aligned}\mathbf{e}^{n+1} := 2\mathbf{e}^n-\mathbf{e}^{n-1}&-\tau^2\left(\mathsf{P}\left(\widetilde{\mathsf{M}}_\varepsilon^L+\tfrac{\tau}{2}\widetilde{\mathsf{M}}_\sigma^L\right)^{-1}\mathsf{P}^\top\right)\mathsf{K}\mathbf{e}^n\\&-\tau^2\left(\mathsf{P}\left(\widetilde{\mathsf{M}}_\varepsilon^L+\tfrac{\tau}{2}\widetilde{\mathsf{M}}_\sigma^L\right)^{-1}\mathsf{P}^\top\mathsf{M}_\sigma\frac{\mathbf{e}^n-\mathbf{e}^{n-1}}{\tau}\right.\end{aligned}$$

Let's do some numerics...

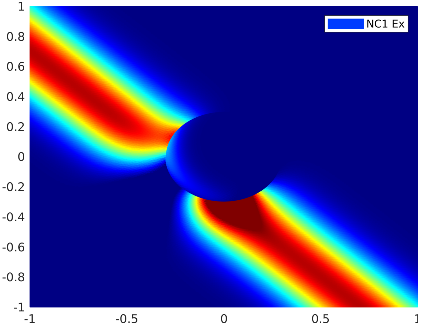


Figure: $\mathcal{NC}_1$ method with mass-lumping

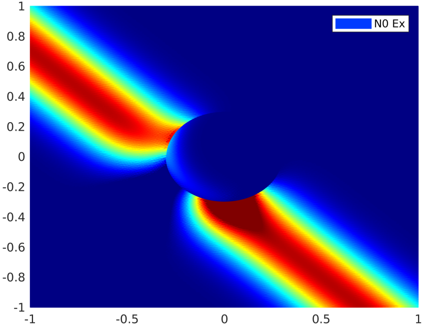
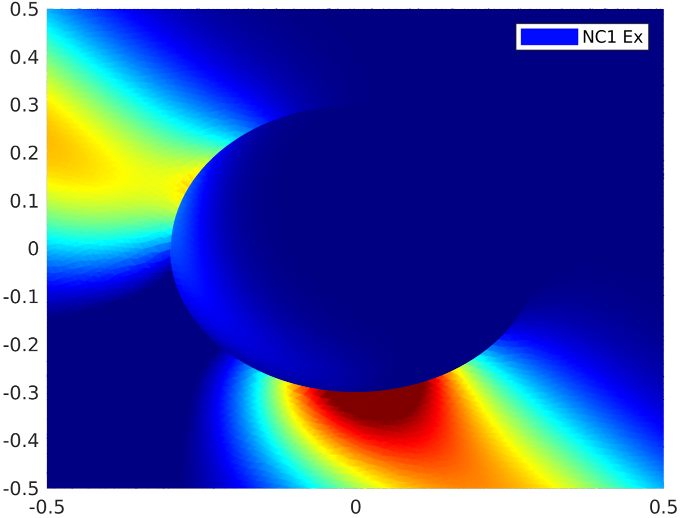
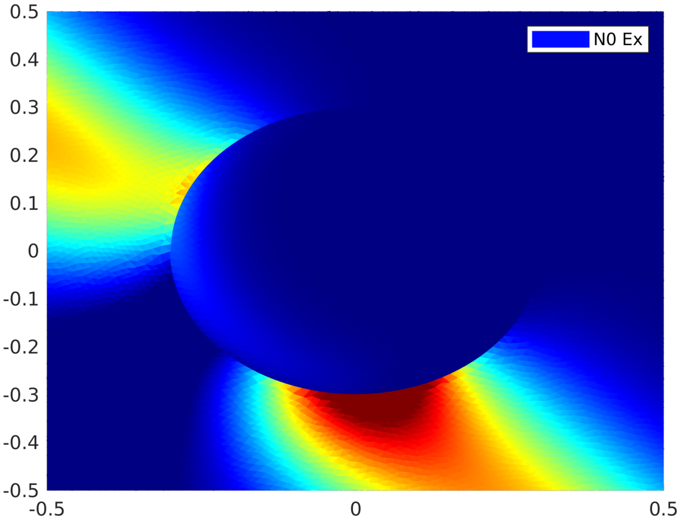


Figure: Reduced $\mathcal{NC}_1$ method (to $\mathcal{N}_0$)

Let's do some numerics...




0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.5
0
0.5
N0 Ex

$$\boxed{((\varepsilon + \tfrac{\tau}{2}\sigma) d_{\tau\tau} \widetilde{\boldsymbol{E}}_h^n, \widetilde{\boldsymbol{v}}_h)_h + (\sigma \Pi_{\mathsf{P}} d_\tau \widetilde{\boldsymbol{E}}_h^{n-1/2}, \Pi_{\mathsf{P}} \widetilde{\boldsymbol{v}}_h) + (\operatorname{curl} \widetilde{\boldsymbol{E}}_h^n, \operatorname{curl} \widetilde{\boldsymbol{v}}_h) = 0}$$

Theorem (accuracy) If $\boldsymbol{E}$ is sufficiently smooth, then $\boldsymbol{E}_h^n := \Pi_{\mathsf{P}} \widetilde{\boldsymbol{E}}_h^n$ satisfies

$$\|\boldsymbol{E}^n - \boldsymbol{E}_h^n\|_{H(\mathrm{curl})} \leq C(h^{1/2} + \tau^2)$$

ÖAW RICAM

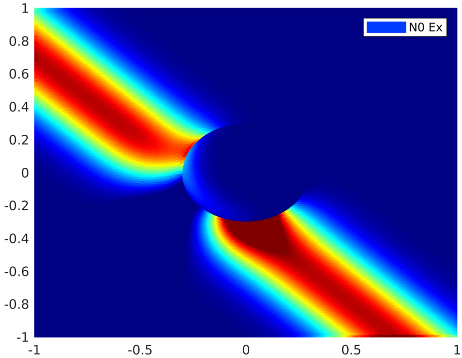
$$\boxed{((\varepsilon + \tfrac{\tau}{2}\sigma)d_{\tau\tau}\widetilde{\boldsymbol{E}}_h^n, \widetilde{\boldsymbol{v}}_h)_h + (\sigma\Pi_{\mathsf{P}} d_\tau \widetilde{\boldsymbol{E}}_h^{n-1/2}, \Pi_{\mathsf{P}}\widetilde{\boldsymbol{v}}_h) + (\operatorname{curl}\widetilde{\boldsymbol{E}}_h^n, \operatorname{curl}\widetilde{\boldsymbol{v}}_h) = 0}$$

Theorem (accuracy) If $\boldsymbol{E}$ is sufficiently smooth, then $\boldsymbol{E}_h^n := \Pi_{\mathsf{P}}\widetilde{\boldsymbol{E}}_h^n$ satisfies

$$\|\boldsymbol{E}^n - \boldsymbol{E}_h^n\|_{H(\mathrm{curl})} \leq C(h^{1/2} + \tau^2)$$

Further complications for non-trivial boundary conditions $\boldsymbol{n} \times \boldsymbol{E}(t) = G(t)$. Again:

$$\|\boldsymbol{E}^n - \boldsymbol{E}_h^n\|_{H(\mathrm{curl})} \leq C(h^{1/2} + \tau^2)$$



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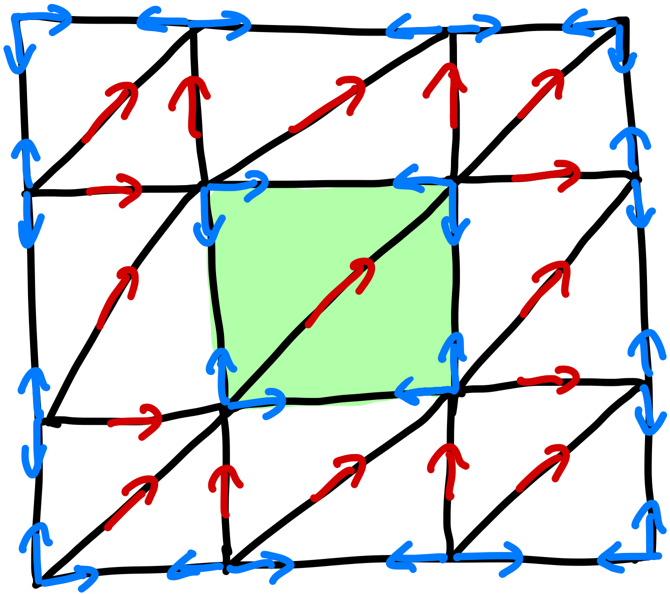

$\sigma \neq 0$

Let's fix this!

$$\mathcal{N}_0^\dagger := \operatorname{span}\left\{ \begin{array}{ll} \boldsymbol{\Phi}_0^e & \text{if e is an interior edge and } \sigma \text{ is smooth across it} \\ \boldsymbol{\Phi}_1^{e,1}, \Phi_1^{e,2} & \text{if e is a boundary edge and } \sigma \text{ jumps across it} \end{array} \right\}$$

Define the projection $\Pi_{\mathsf{P}}^\dagger : \mathcal{NC}_1 \to \mathcal{N}_0^\dagger$ and the method:

$$((\varepsilon + \tfrac{\tau}{2}\sigma) d_{\tau\tau} \widetilde{\boldsymbol{E}}_h^n, \widetilde{\boldsymbol{v}}_h)_h + (\sigma \Pi_{\mathsf{P}}^\dagger d_\tau \widetilde{\boldsymbol{E}}_h^{n-1/2}, \Pi_{\mathsf{P}}^\dagger \widetilde{\boldsymbol{v}}_h) + (\operatorname{curl} \widetilde{\boldsymbol{E}}_h^n, \operatorname{curl} \widetilde{\boldsymbol{v}}_h) = 0$$

Theorem (accuracy) If $\boldsymbol{E}$ is sufficiently smooth, then $\boldsymbol{E}_h^n := \Pi_{\mathsf{P}}^\dagger \widetilde{\boldsymbol{E}}_h^n$ satisfies

$$\|\boldsymbol{E}^n - \boldsymbol{E}_h^n\|_{H(\mathrm{curl})} \leq C(h + \tau^2)$$

Summary

- Derived 2 methods