COMP50011 Coursework #1
Norms

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Question 1.

1) Take $x = \begin{pmatrix} x_1 & x_2 & ... & x_n \end{pmatrix}^T \in \mathbb{R}^n$ arbitrarily.

$$\|Ax\|_1 = \left\| \begin{matrix} a_{11} \cdot x_1 + a_{12} \cdot x_2 + ... + a_{1n} \cdot x_n \\ a_{21} \cdot x_2 + a_{12} \cdot x_2 + ... + a_{2n} \cdot x_n \\ ... \\ a_{m1} \cdot x_1 + a_{m2} \cdot x_2 + ... + a_{mn} \cdot x_n \end{matrix} \right\|_1 = \left\| \begin{matrix} \sum_{j=1}^n a_{1j} \cdot x_j \\ \sum_{j=1}^n a_{2j} \cdot x_j \\ ... \\ \sum_{j=1}^n a_{mj} \cdot x_j \end{matrix} \right\|_1 =$$

$$= \sum_{i=1}^m \left| \sum_{j=1}^n a_{ij} \cdot x_j \right| \leqslant \sum_{i=1}^m \left(\sum_{j=1}^n |a_{ij}| \cdot |x_j| \right) = \sum_{j=1}^n \left(\sum_{i=1}^m |a_{ij}| \cdot |x_j| \right) = \sum_{j=1}^n \left(|x_j| \cdot \sum_{i=1}^m |a_{ij}| \right)$$

Therefore, for all $x = \begin{pmatrix} x_1 & x_2 & ... & x_n \end{pmatrix}^T \in \mathbb{R}^n$, we have:

$$\|Ax\|_1 = \sum_{j=1}^n \left(|x_j| \cdot \sum_{i=1}^m |a_{ij}| \right)$$

2)

$$\|Ax\|_1 = \sum_{j=1}^n \left(|x_j| \cdot \sum_{i=1}^m |a_{ij}| \right) \leqslant \sum_{j=1}^n \left(|x_j| \cdot \max_{k \in \overline{1,n}} \sum_{i=1}^m |a_{ik}| \right) = \sum_{j=1}^n (|x_j| \cdot \|A\|_1) = \|A\|_1 \cdot \sum_{j=1}^n |x_j| = \|A\|_1 \cdot \|x\|_1$$

3) Let $k \in \overline{1,n}$, such as:

$$\sum_{i=1}^m |a_{ik}| = \max_{j=\overline{1,n}} \sum_{i=1}^m |a_{ik}| \implies \|A\|_1 = \sum_{i=1}^m |a_{ik}|$$

This means that the k$^{\text{th}}$ column in A gives its first norm. Take $x = \begin{pmatrix} 0 & ... & 0 & 1 & 0 & ... & 0 \end{pmatrix}^T \in \mathbb{R}^n$, formed by zeroes and a one in the k$^{\text{th}}$ row. Thus, $\|x\|_1 = 1$.

$$\|Ax\|_1 = \left\| \begin{matrix} a_{11} \cdot 0 + a_{12} \cdot 0 + ... + a_{1k} \cdot 1 + ... + a_{1n} \cdot 0 \\ a_{21} \cdot 0 + a_{22} \cdot 0 + ... + a_{2k} \cdot 1 + ... + a_{2n} \cdot 0 \\ ... \\ a_{m1} \cdot 0 + a_{m2} \cdot 0 + ... + a_{mk} \cdot 1 + ... + a_{mn} \cdot 0 \end{matrix} \right\|_1 = \left\| \begin{matrix} a_{1k} \\ a_{2k} \\ ... \\ a_{mk} \end{matrix} \right\|_1 = \sum_{i=1}^m |a_{ik}|$$

$$\frac{\|Ax\|_1}{\|x\|_1} = \frac{\sum_{i=1}^m |a_{ik}|}{1} = \|A\|_1$$

Since for any $x \in \mathbb{R}^n$, $\|A\|_1 \geqslant \frac{\|Ax\|_1}{\|x\|_1}$, and there exists $x \in \mathbb{R}^n$, such that $\|A\|_1 = \frac{\|Ax\|_1}{\|x\|_1}$, then $\|A\|_1 = \max_{x \neq 0} \frac{\|Ax\|_1}{\|x\|_1}$.

4)

$$\max_{x \neq 0} \frac{\|Ax\|_1}{\|x\|_1} = \max_{x \neq 0} \left\| \frac{Ax}{\|x\|_1} \right\|_1 = \max_{x \neq 0} \left\| A \cdot \frac{x}{\|x\|_1} \right\|_1$$

Take $y \in \mathbb{R}^n, y = \frac{x}{\|x\|_1}$. Thus, $\|y\|_1 = \left\| \frac{x}{\|x\|_1} \right\|_1 = \frac{\|x\|_1}{\|x\|_1} = 1$. Therefore,

$$\max_{x \neq 0} \left\| A \cdot \frac{x}{\|x\|_1} \right\|_1 = \max_{\|y\|_1 = 1} \|Ay\|_1$$

We get:

$$\max_{x \neq 0} \frac{\|Ax\|_1}{\|x\|_1} = \max_{\|x\|_1 = 1} \|Ax\|_1$$