Regular Grammars for Sets of Graphs of Tree-Width 2

Radu Iosif (CNRS/Verimag)
joint work with Marius Bozga (Verimag) and Florian Zuleger (TU Wien)

Recognisable Sets

Finite representations of infinite sets (words, trees)

- closed under boolean operations (union, intersection, complement)
- decidable emptiness (inclusion) problem (words, trees)
- equivalence with MSO-definability (words, trees)

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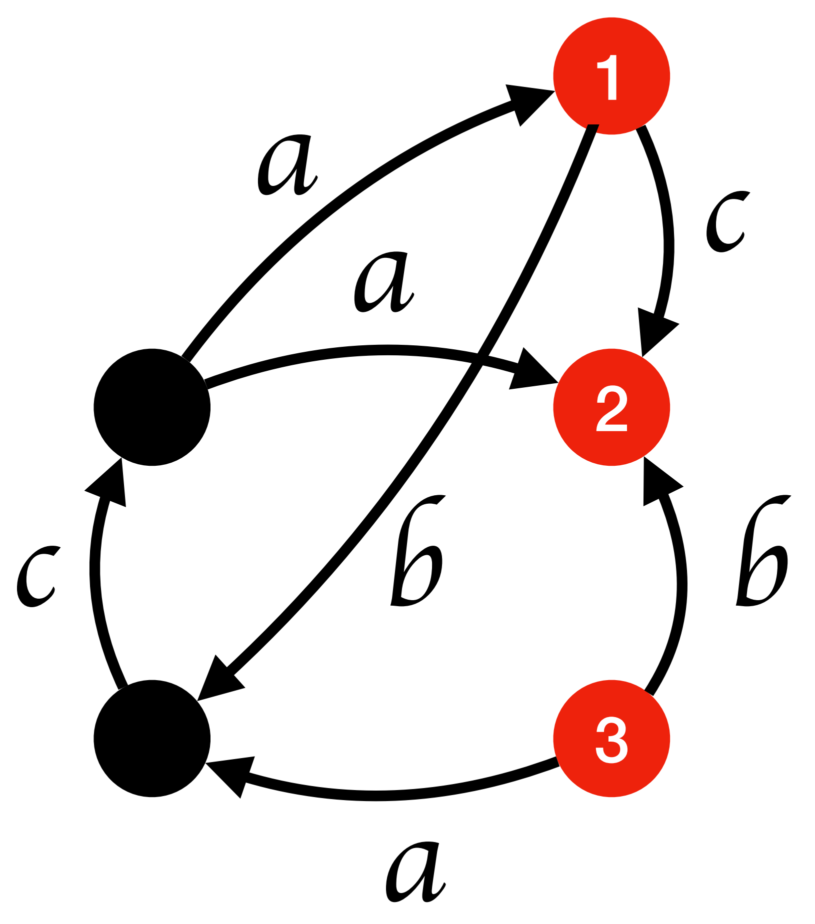
What about graphs ?

An Algebra of Graphs

$\mathbb{B} \stackrel{\text{def}}{=} \{a, b, c, \ldots\}$ finite alphabet of (binary) edge labels

$\mathcal{G}_\tau(\mathbb{B}) \stackrel{\text{def}}{=}$ set of graphs of sort $\boldsymbol{\tau}$ with labels in $\mathbb{B}$



sources

sort(G) = {1,2,3}

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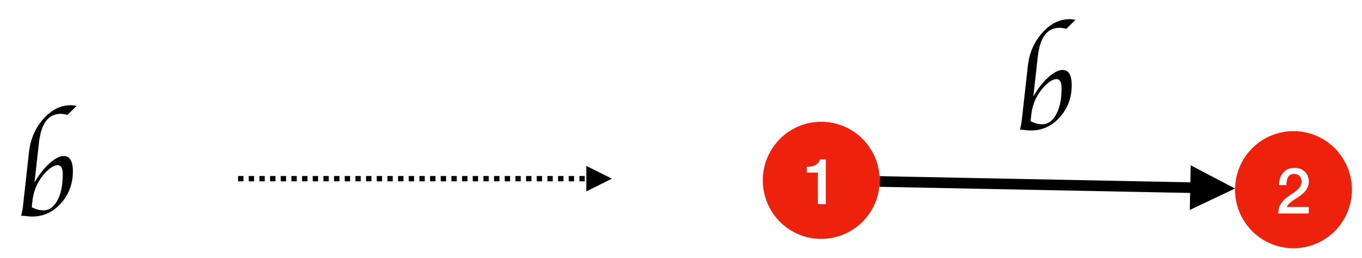

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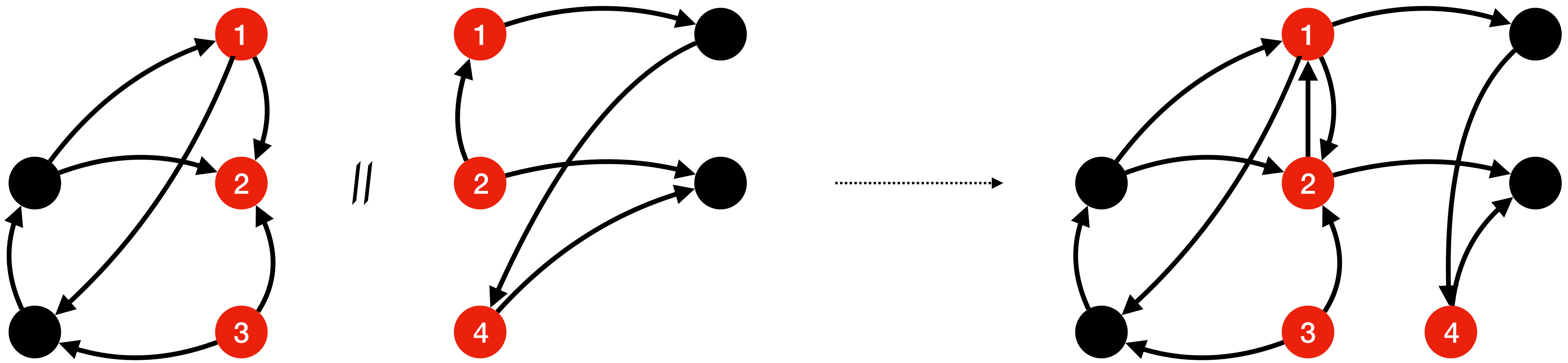

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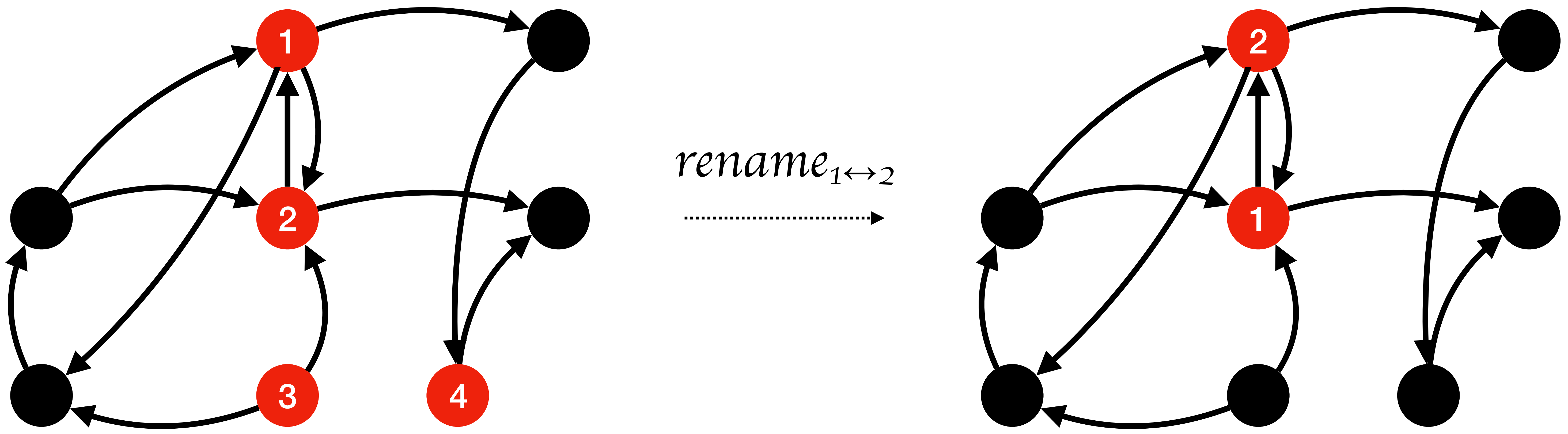

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h **homomorphism** $\iff$ $\begin{array}{l} \mathrm{sort}(a) = \mathrm{sort}(\mathrm{h}(a)) \\ \mathrm{h}(f^{\mathcal{G}(\Sigma)}(a_1,\ldots,a_n)) = f^{\mathbb{A}}(\mathrm{h}(a_1),\ldots,\mathrm{h}(a_n)) \end{array}$

Recognisable Sets of Graphs

$$(\mathcal{G}_\tau(\mathbb{B}),\ \|^{\mathcal{HR}},\ \{rename^{\mathcal{HR}}\}_{\alpha\,:\,\Sigma\to\Sigma},\ \{a^{\mathcal{HR}}\}_{a\in\Sigma}) \xrightarrow{\quad h \quad} (\mathrm{A}_\tau,\ \|^{\mathbb{A}},\ \{rename^{\mathbb{A}}\}_{\alpha\,:\,\Sigma\to\Sigma},\ \{a^{\mathbb{A}}\}_{a\in\Sigma})$$

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\rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} \quad \text{locally finite} \\
L \ \text{recognisable} & = & h^{-1}(F)
\end{array}$$

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Why locally finite and not just finite ?

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$\supseteq$ (left), $\supseteq$ (right)   locally finite

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Grids are MSO-definable but building $\{\mathcal{G}_{n,n} \mid n \in \mathbb{N}\}$ requires infinitely many sorts

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(Graphs) CMSO-definability $\Longrightarrow$ recognisability, but not viceversa [Courcelle'90]

Tree-width (algebraic)

$\mathcal{T}(\mathcal{HR}) \stackrel{\text{def}}{=}$ set of $\mathcal{HR}$-terms

$tw(\mathrm{G}) \stackrel{\text{def}}{=} min\{$ (number of constants occurring in t) - 1 $\mid t \in \mathcal{T}(\mathcal{HR}), t^{\mathcal{HR}} = \mathrm{G}\}$

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Recognisable sets of bounded tree-width graphs have finitely many non-empty sorts:

- recognisability can be defined in terms of finite algebras (of practical importance)
- recognisability $\Longleftrightarrow$ CMSO-definability [Bojańczyk & Pilipczuk'16]

Context-Free Sets

$\mathcal{L} \subseteq \mathcal{A}$ context-free $\Leftrightarrow \mathcal{L} = \{ t^{\mathcal{HR}} \mid t \in \mathcal{T} \}$, where $\mathcal{T} \subseteq \mathcal{T}(\mathcal{HR})$ is recognisable

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A context-free grammar $\Gamma = (\mathcal{N}, \leftarrow)$ consists of finitely many rules:

$X \leftarrow t[Y_1, \ldots Y_n]$	$X, Y_1, \ldots Y_n \in \mathcal{N}, t \in \mathcal{T}(\mathcal{HR})$
$\leftarrow X$	axiom

$$\mathcal{L}(\Gamma) \stackrel{\text{def}}{=} \{ t^{\mathcal{HR}} \mid \leftarrow X \leftarrow \ldots \leftarrow t \text{ complete derivation} \}$$

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(Graphs) Recognisable and context-free sets are incomparable

- (words) recognisable $\Longrightarrow$ context-free
- (terms) recognisable $\Longleftrightarrow$ context-free

Regular Sets of Graphs

Regular grammars are context-free grammars that define recognisable sets

- is there a simple syntactic definition ?
- regularity $\stackrel{?}{\Longleftrightarrow}$ recognisability for bounded tree-width sets
  - regularity $\stackrel{?}{\Longleftrightarrow}$ CMSO-definability (recognisability $\Longleftrightarrow$ CMSO-definability)
- is the construction of a (minimal) recogniser algebra effective ?
- is there a regular/recognisable equivalent of MSO-definability ?

Stratified Grammars

$\Gamma = (\mathcal{N}, \leftarrow)$, where $(\mathcal{X}, \mathcal{Y})$ is a partition of $\mathcal{N}$ and the rules have the forms:

A. $X \leftarrow X \parallel Y \underbrace{\parallel \ldots \parallel}_{\text{q times} \stackrel{\text{def}}{=} Y^{\#q}} Y$

B. $X \leftarrow Y^{\#q_1} \parallel \ldots \parallel Y^{\#q_k}$

C. $Y \leftarrow t[Z_1, \ldots, Z_m]$

D. $X \leftarrow t$, if t is a ground term

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$\mathbf{X}_\tau \leftarrow \mathbf{Y}_\tau^{\#\geq m}$, $m = \min\{q_1,\ldots,q_k\}$ for all (B) rules

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$\mathtt{FP}(\Gamma)$

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$\mathcal{X} \quad \mathcal{Y}$
$\uplus \quad \uplus$

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$\subseteq \mathbb{N}$

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$\leftarrow \mathrm{Z}_\tau$

FP(Γ)

A Refinement Theorem

$\mathcal{A} = (\{\mathrm{A}_\tau\}_{\tau \subseteq \mathbb{N}}, \|^{\mathcal{A}}, \{rename^{\mathcal{A}}\}_{\alpha\,:\,\mathbb{B}\to\mathbb{B}}, \{b^{\mathcal{A}}\}_{b \in \mathbb{B}})$ locally finite

$idem(a) = a \|^{\mathcal{A}} \ldots \|^{\mathcal{A}} a$ is the (unique) element such that $idem(a) \|^{\mathcal{A}} idem(a) = idem(a)$

$\mathcal{A}$ is aperiodic $\iff$ $idem(a) \|^{\mathcal{A}} a = a$, for all $a \in \mathrm{A}_{\mathrm{sort}(a)}$

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Theorem [Refinement]

1. For each stratified grammar Γ and each recognisable set L, one can build a stratified grammar Γ' such that FP(Γ') $\subseteq$ FP(Γ) and $\mathcal{L}(\Gamma') = \mathcal{L}(\Gamma) \cap$ L.

2. Γ' is aperiodic if Γ and $\mathcal{A}$ are both aperiodic.

A Completeness Theorem

A class is a derived algebra of $\mathcal{HR}$ whose finite signature includes $/\!/$

- we consider the classes of trees, (disoriented) series-parallel graphs, graphs of tree-width ≤ 2

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A class C is (aperiodic) universally stratified if there exists a (aperiodic) stratified grammar Γ_C such that $\mathcal{L}(\Gamma_C)$ is the domain of C

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A class C is (aperiodic) universally stratified if there exists a (aperiodic) stratified grammar Γ_C such that $\mathcal{L}(\Gamma_C)$ is the domain of C

Corollary Let C be an (aperiodic) universally stratified class and L a set recognisable in C (by an aperiodic recogniser). Then, there exists a (aperiodic) stratified grammar Γ such that FP(Γ) ⊆ FP(Γ_C) and $\mathcal{L}(\Gamma) = \mathrm{L}$.

Regular Grammars

A regular grammar Γ for a class C is a stratified grammar such that $\mathtt{FP}(\Gamma) \subseteq \mathtt{FP}_C$

- each refinement Γ' of Γ is a regular grammar **(RT)**

Theorem [Meta] Let C be the class.

1. L is (aperiodic) recognisable in C ⇔ L is the language of a (aperiodic) regular grammar for C
2. Given grammars Γ and Γ', the problem $\mathcal{L}(\Gamma) \subseteq \mathcal{L}(\Gamma')$ is …, when Γ' is regular for C

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Proofs for trees, (disoriented) series-parallel graphs, graphs of tree-width ≤ 2

Unranked and Unordered Trees

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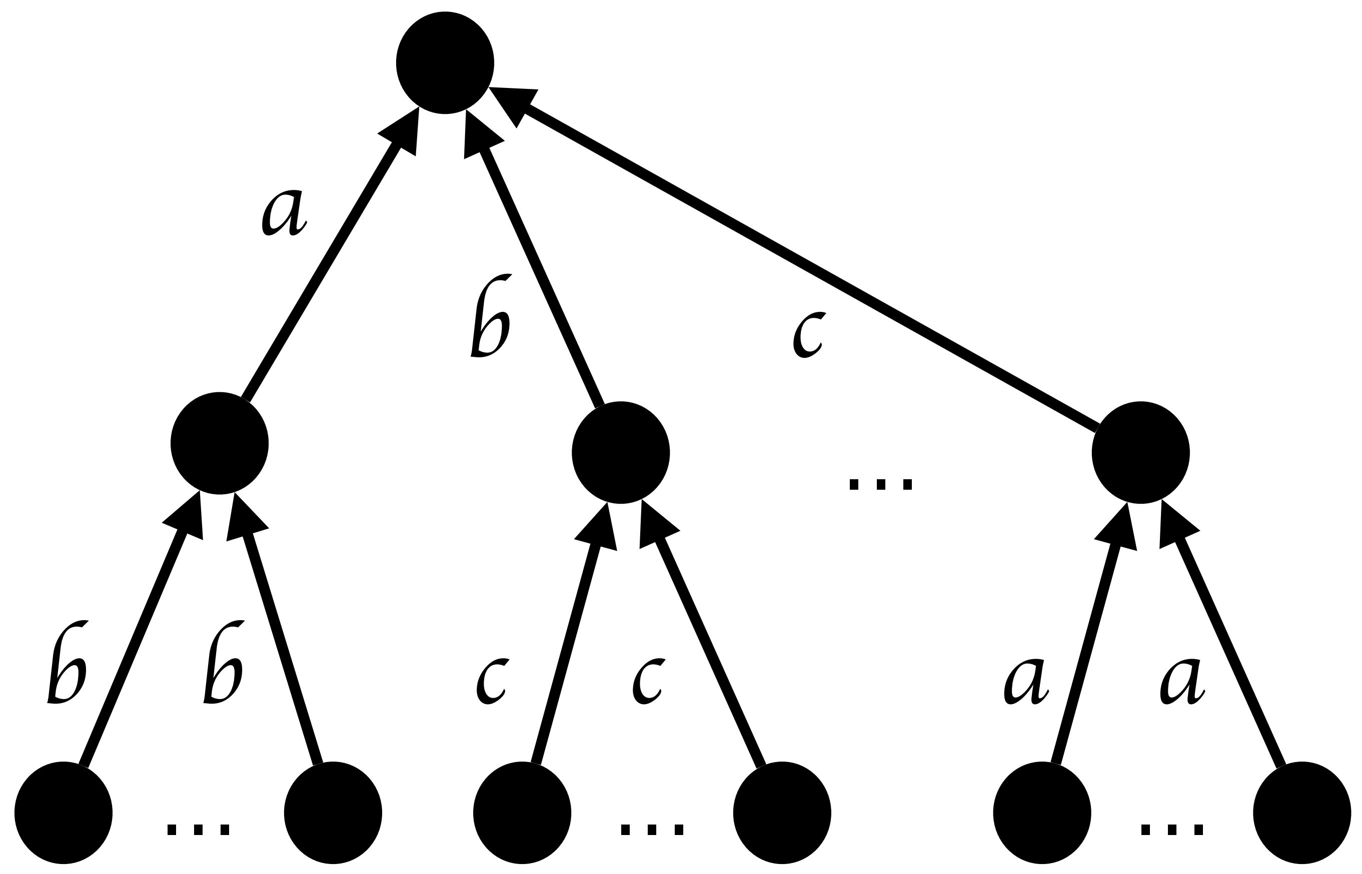

number and order of children are not fixed

Unranked and Unordered Trees

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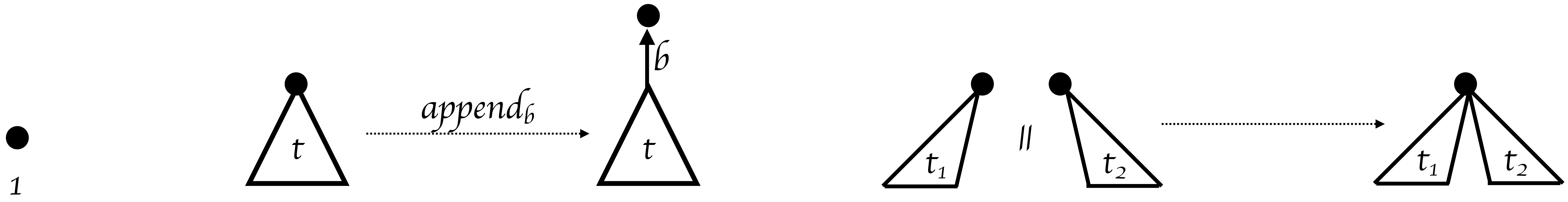
$\mathcal{T}=(\mathrm{T}(\mathbb{B}), \{append^{\mathcal{T}}_b\}_{b \in \mathbb{B}}, 1^{\mathcal{T}}, \|^{\mathcal{T}})$ derived $\mathcal{HR}$-algebra (finite signature, one sort $\{1\}$)



(Trees) Recognisability $\implies$ Regularity

A regular tree grammar Γ is a stratified grammar with nonterminals $\mathcal{X} \uplus \mathcal{Y}$ such that

$$\mathtt{FP}(\Gamma) \subseteq \{\mathtt{X} \leftarrow \mathtt{Y}^{\# \geq 0}\} \cup \{\mathtt{Y} \leftarrow \mathit{append}_b(\mathtt{X}) \mid b \in \mathbb{B}\} \cup \{\leftarrow \mathtt{X}\}$$

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Lemma [Universality] The following aperiodic stratified grammar produces $\mathrm{T}(\mathbb{B})$:

$$\mathrm{X} \leftarrow \mathrm{X} \,||\, \mathrm{Y} \qquad \mathrm{X} \leftarrow 0 \qquad \mathrm{Y} \leftarrow \mathit{append}_b(\mathrm{X}) \qquad \leftarrow \mathrm{X}$$

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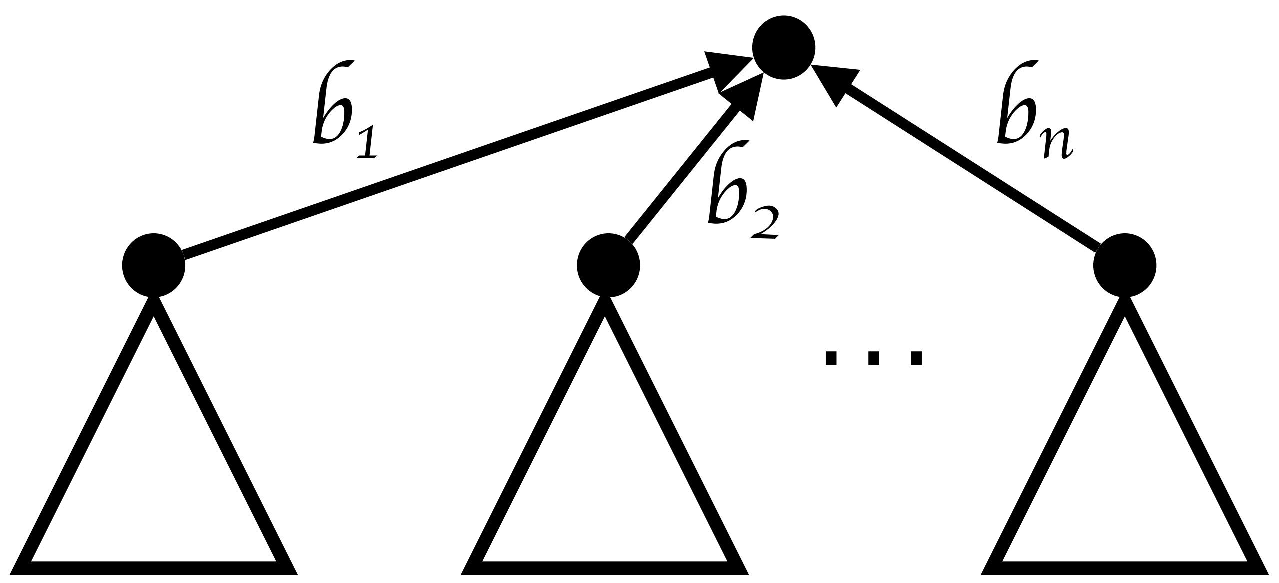
$\Longrightarrow$ each set recognisable in $\mathcal{T}$ is the language of a regular tree grammar

(Trees) Regularity $\implies$ Recognisability

Γ = regular tree grammar



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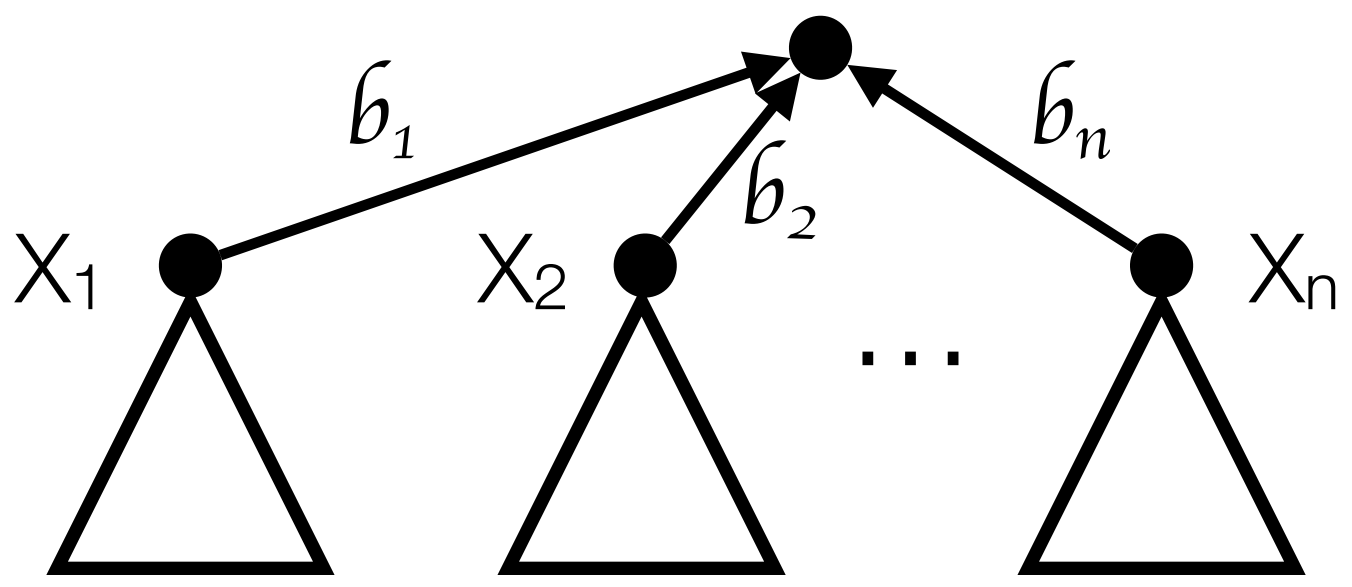

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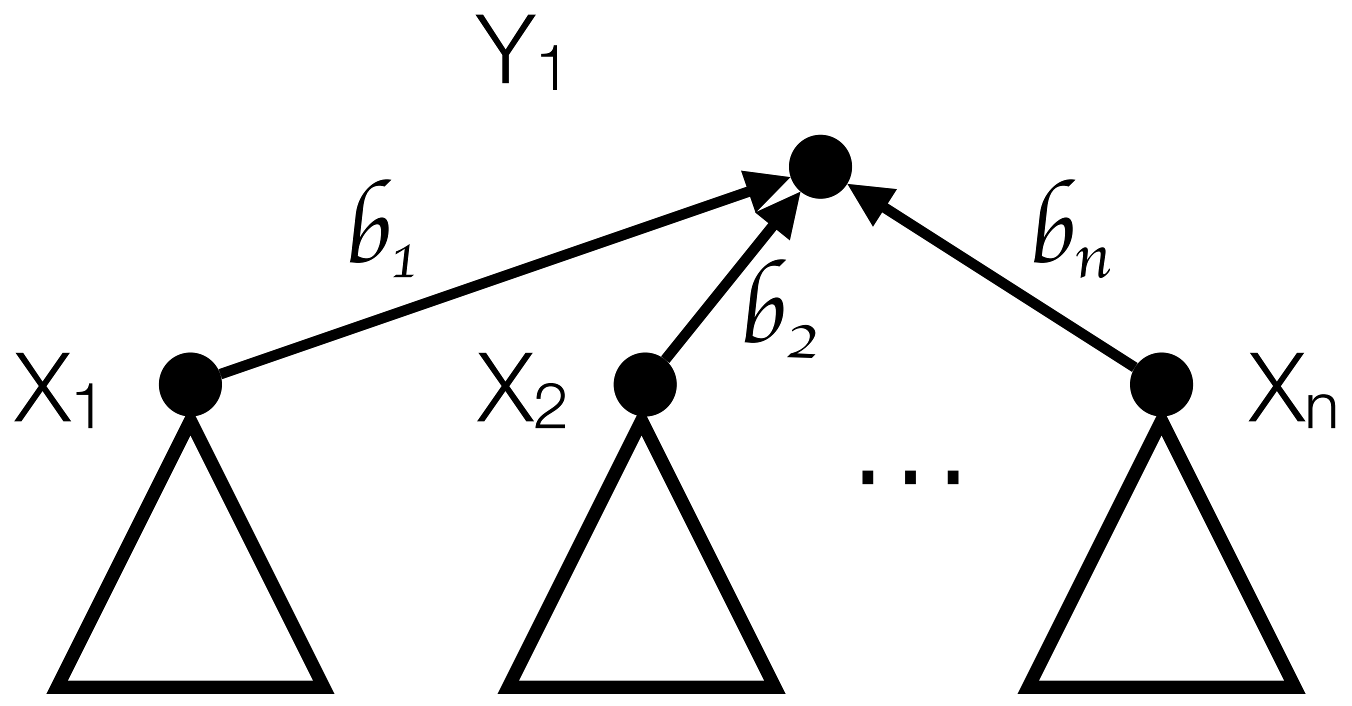
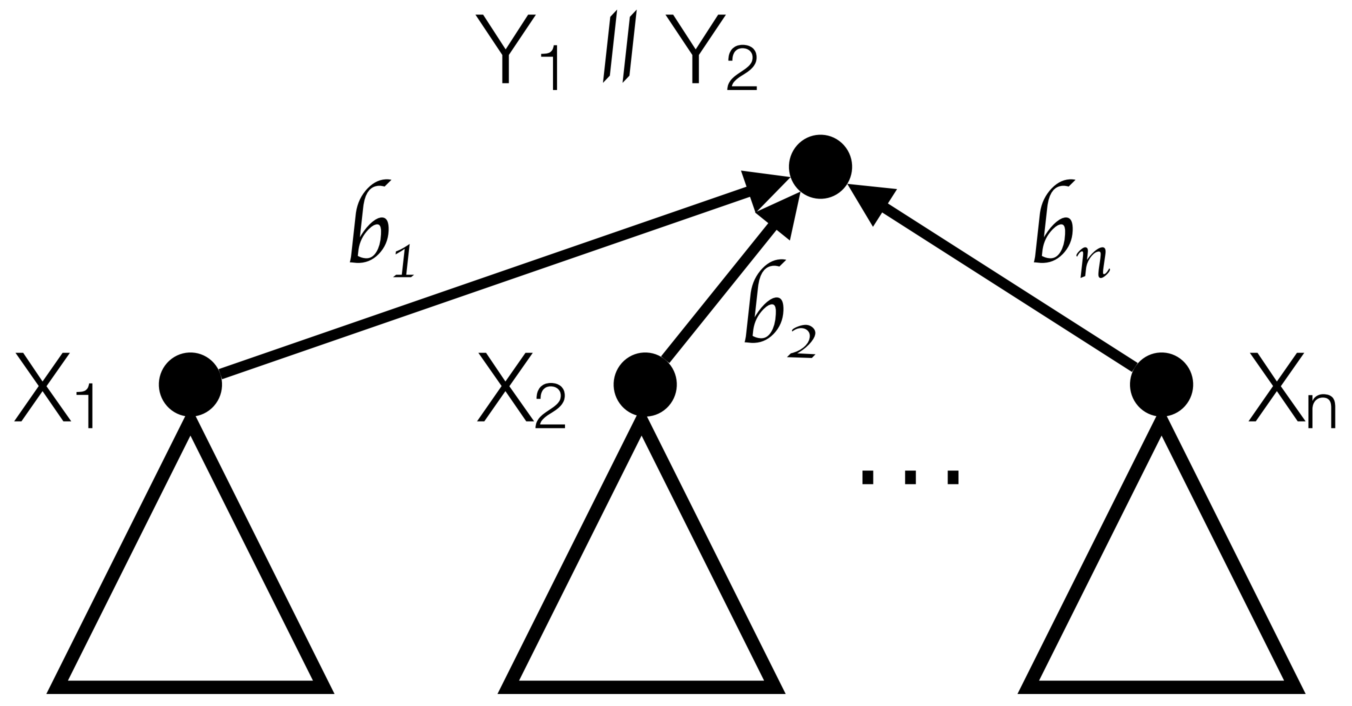


$Y_1 \leftarrow append_{b_1}(X_1)$

(Trees) Regularity $\Longrightarrow$ Recognisability

Γ = regular tree grammar



$Y_1 \leftarrow append_{b_1}(X_1)$

$Y_2 \leftarrow append_{b_2}(X_2)$

(Trees) Regularity $\Longrightarrow$ Recognisability

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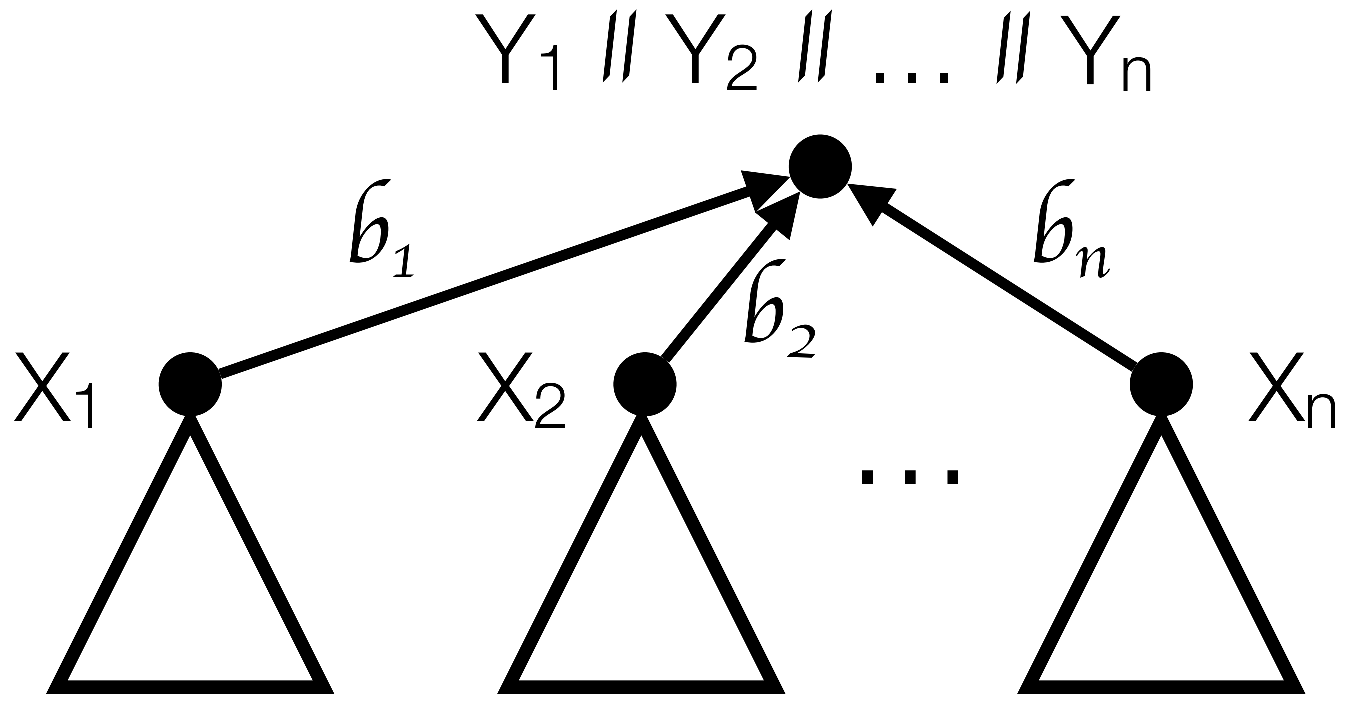


$Y_1 \leftarrow \mathit{append}_{b_1}(X_1)$

$Y_2 \leftarrow \mathit{append}_{b_2}(X_2)$

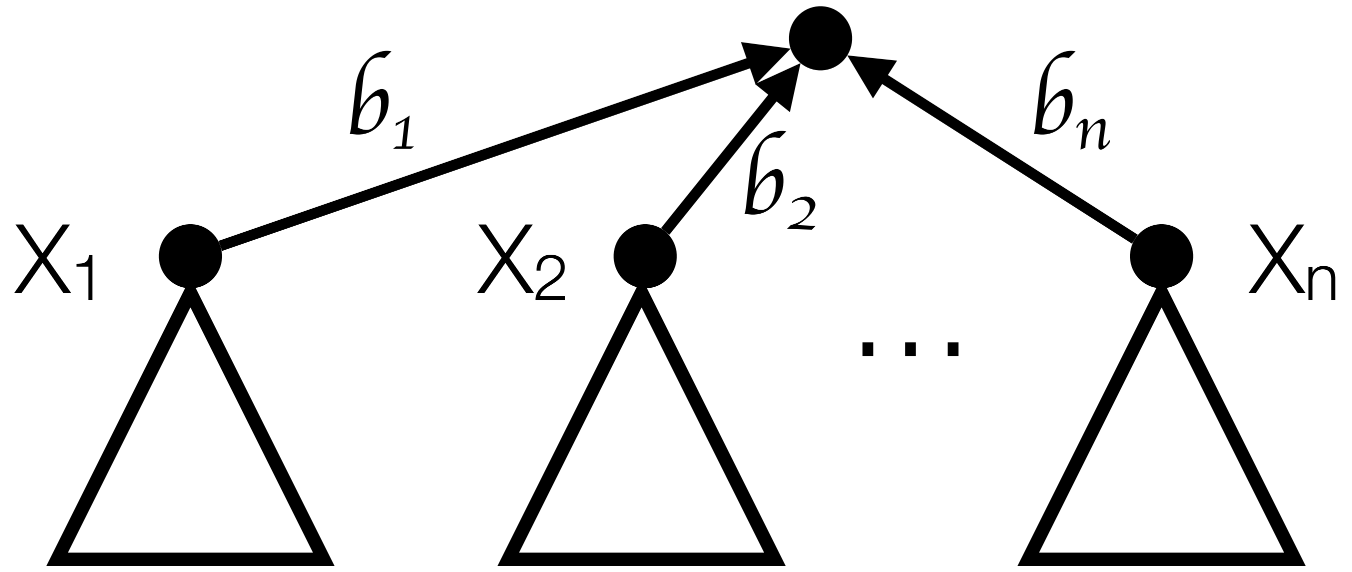
$\ldots$

$Y_n \leftarrow \mathit{append}_{b_n}(X_n)$

(Trees) Regularity $\implies$ Recognisability

Γ = regular tree grammar

$$X \leftarrow Y_1 \mathbin{/\!/} Y_2 \mathbin{/\!/} \ldots \mathbin{/\!/} Y_n$$



$$Y_1 \leftarrow \mathit{append}_{b_1}(X_1)$$

$$Y_2 \leftarrow \mathit{append}_{b_2}(X_2)$$

$$\ldots$$

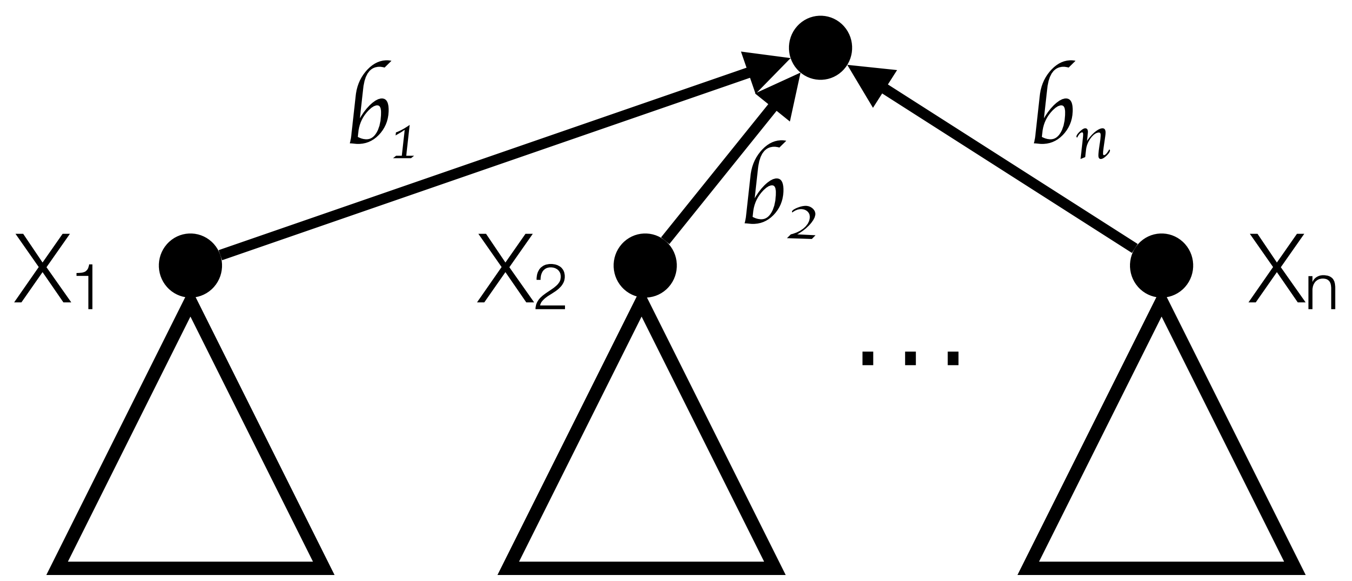
$$Y_n \leftarrow \mathit{append}_{b_n}(X_n)$$

(Trees) Regularity $\Longrightarrow$ Recognisability

Γ = regular tree grammar

view

$$X \rightsquigarrow \{\{Y_1 , Y_2 , \ldots , Y_n\}\}$$



$$Y_1 \leftarrow \mathit{append}_{b1}(X_1)$$

$$Y_2 \leftarrow \mathit{append}_{b2}(X_2)$$

$$\ldots$$

$$Y_n \leftarrow \mathit{append}_{bn}(X_n)$$

(Trees) Regularity $\Longrightarrow$ Recognisability

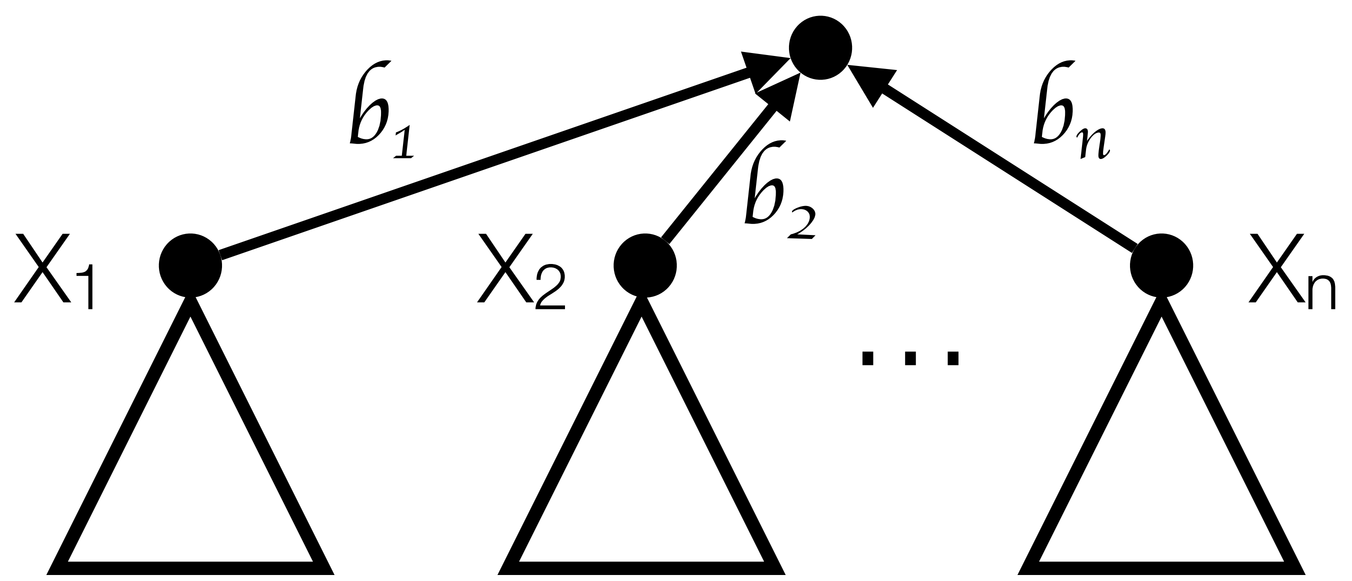
Γ = regular tree grammar

view

$X \rightsquigarrow \ulcorner \{\{Y_1, Y_2, \ldots, Y_n\}\} \urcorner$

reduced view

sufficient to consider multiplicities $\leq$ size(Γ)



$Y_1 \leftarrow append_{b_1}(X_1)$

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$\ldots$

$Y_n \leftarrow append_{b_n}(X_n)$

A Recogniser Algebra for Trees

Γ = regular tree grammar

$\mathcal{A} \overset{\text{def}}{=} (\mathrm{A}, \{append^{\mathcal{A}}{}_{b}\}_{b \in \mathbb{B}}, 1^{\mathcal{A}}, \|^{\mathcal{A}})$

$\mathrm{A} \overset{\text{def}}{=}$ set of reduced views

$$\frac{\mathrm{X} \rightsquigarrow \mathrm{m} \in a \qquad \mathrm{Y} \leftarrow append_{b}(\mathrm{X})}{\{\{\mathrm{Y}\}\} \in append_{b}{}^{\mathcal{A}}(a)}$$

$$\frac{\mathrm{m}_1 \in a_1 \qquad \mathrm{m}_2 \in a_2}{\ulcorner \mathrm{m}_1 + \mathrm{m}_2 \urcorner \in a_1 \|^{\mathcal{A}} a_2}$$

$$\frac{}{\varnothing \in 1^{\mathcal{A}}}$$

A Recogniser Algebra for Trees

Γ = regular tree grammar

$\mathcal{A} \stackrel{\text{def}}{=} (\mathrm{A}, \{\mathit{append}^{\mathcal{A}}{}_{b}\}_{b \in \mathbb{B}}, 1^{\mathcal{A}}, \|^{\mathcal{A}})$

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$$\frac{\mathrm{X} \rightsquigarrow \mathrm{m} \in a \qquad \mathrm{Y} \leftarrow \mathit{append}_b(\mathrm{X})}{\{\{\mathrm{Y}\}\} \in \mathit{append}_b{}^{\mathcal{A}}(a)}$$

$$\frac{\mathrm{m}_1 \in a_1 \qquad \mathrm{m}_2 \in a_2}{\ulcorner \mathrm{m}_1 + \mathrm{m}_2 \urcorner \in a_1 \|^{\mathcal{A}} a_2}$$

$$\frac{}{\varnothing \in 1^{\mathcal{A}}}$$

$$\begin{array}{ccc} (\mathrm{T}(\mathbb{A}), \{\mathit{append}^{\mathcal{T}}{}_{b}\}_{b \in \mathbb{B}}, 1^{\mathcal{T}}, \|^{\mathcal{T}}) & \xrightarrow{\mathrm{h}(\mathrm{T}) \stackrel{\text{def}}{=} \{ \ulcorner \mathrm{m} \urcorner \mid \text{ m view of T} \}} & (\mathrm{A}, \{\mathit{append}^{\mathcal{A}}{}_{b}\}_{b \in \mathbb{B}}, 1^{\mathcal{A}}, \|^{\mathcal{A}}) \\ \cup\!\!| & & \cup\!\!| \\ \mathcal{L}(\Gamma) & = & \mathrm{h}^{-1}(\{\mathrm{h}(\mathrm{T}) \mid \mathrm{T} \in \mathcal{L}(\Gamma)\}) \end{array}$$

A Recogniser Algebra for Trees

Γ = regular tree grammar $\qquad \mathcal{A} \stackrel{\text{def}}{=} (A, \{append^{\mathcal{A}}_{b}\}_{b \in \mathbb{B}}, 1^{\mathcal{A}}, \|^{\mathcal{A}}) \qquad$ A $\stackrel{\text{def}}{=}$ set of reduced views

$$\frac{X \rightsquigarrow m \in a \qquad Y \leftarrow append_b(X)}{\{\{Y\}\} \in append_b^{\mathcal{A}}(a)} \qquad \frac{m_1 \in a_1 \qquad m_2 \in a_2}{\ulcorner m_1 + m_2 \urcorner \in a_1 \|^{\mathcal{A}} a_2} \qquad \frac{}{\varnothing \in 1^{\mathcal{A}}}$$

$$\begin{array}{ccc} (T(\mathbb{A}), \{append^{\mathcal{T}}_{b}\}_{b \in \mathbb{B}}, 1^{\mathcal{T}}, \|^{\mathcal{T}}) & \xrightarrow{h(T) \stackrel{\text{def}}{=} \{\ulcorner m \urcorner \mid m \text{ view of } T\}} & (A, \{append^{\mathcal{A}}_{b}\}_{b \in \mathbb{B}}, 1^{\mathcal{A}}, \|^{\mathcal{A}}) \\ \cup\!\shortparallel & & \cup\!\shortparallel \\ \mathcal{L}(\Gamma) & & = h^{-1}(\{h(T) \mid T \in \mathcal{L}(\Gamma)\}) \end{array}$$

$\mathcal{A}$ is aperiodic whenever Γ is aperiodic

(Trees) Regularity $\iff$ Recognisability

Theorem

1. L is (aperiodic) recognisable in $\mathcal{T}$ ⇔ L is the language of a (aperiodic) regular grammar for $\mathcal{T}$
2. Given grammars Γ and Γ', the problem $\mathcal{L}(\Gamma) \subseteq \mathcal{L}(\Gamma')$ is in 2EXP ∩ EXP-hard, if Γ' is regular for $\mathcal{T}$
   - the size of the recogniser algebra for Γ' is at most doubly exponential in its size

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Theorem

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   - the size of the recogniser algebra for Γ' is at most doubly exponential in its size

Is the double exponential really needed ?

What if the regular grammar is aperiodic ?

Series Parallel Graphs

$\mathbb{B} \stackrel{\text{def}}{=} \{a, b, c, \ldots\}$ finite alphabet of (binary) edge labels

Series Parallel Graphs

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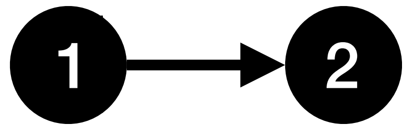

b

Series Parallel Graphs

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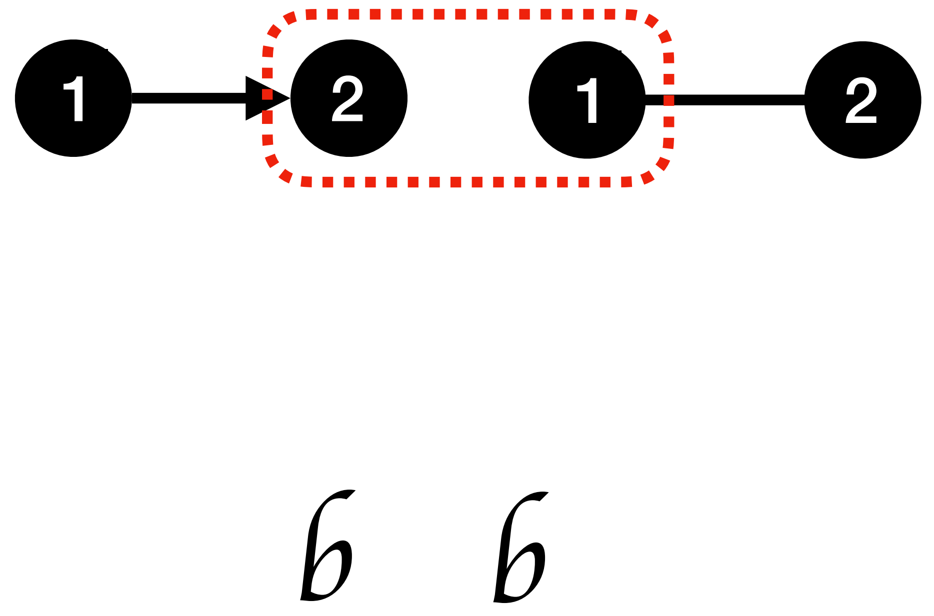

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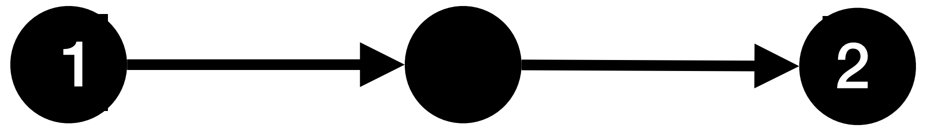

$b \circ b$

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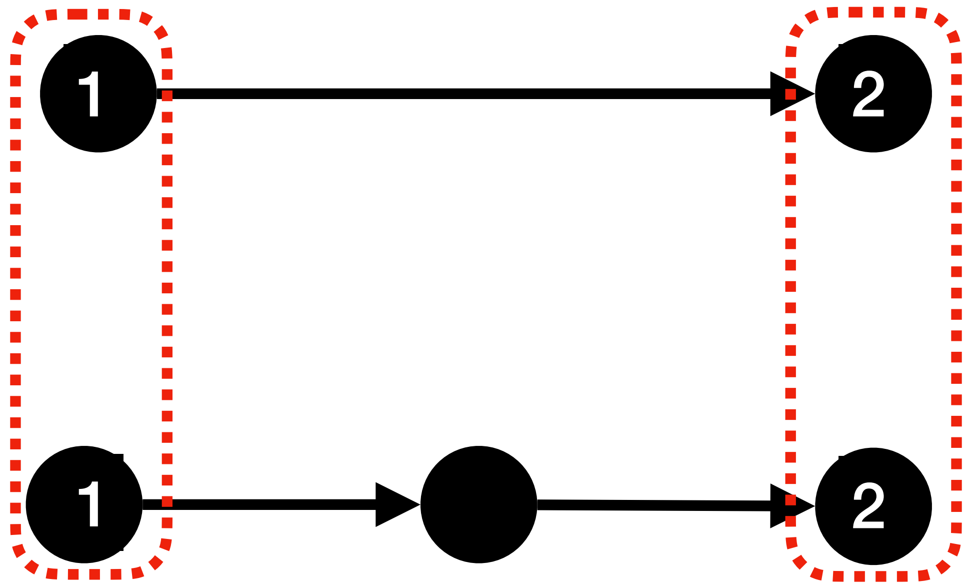

$b \circ b \quad b$

Series Parallel Graphs

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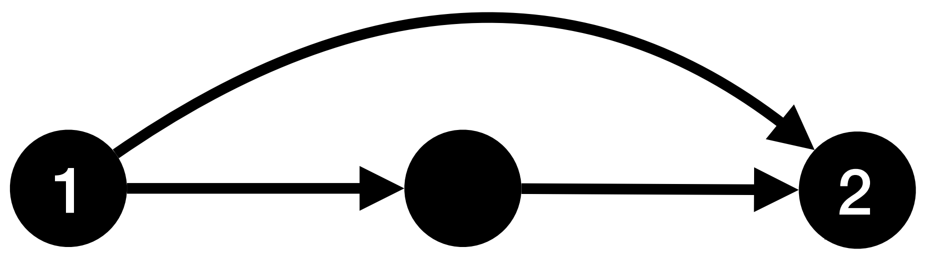
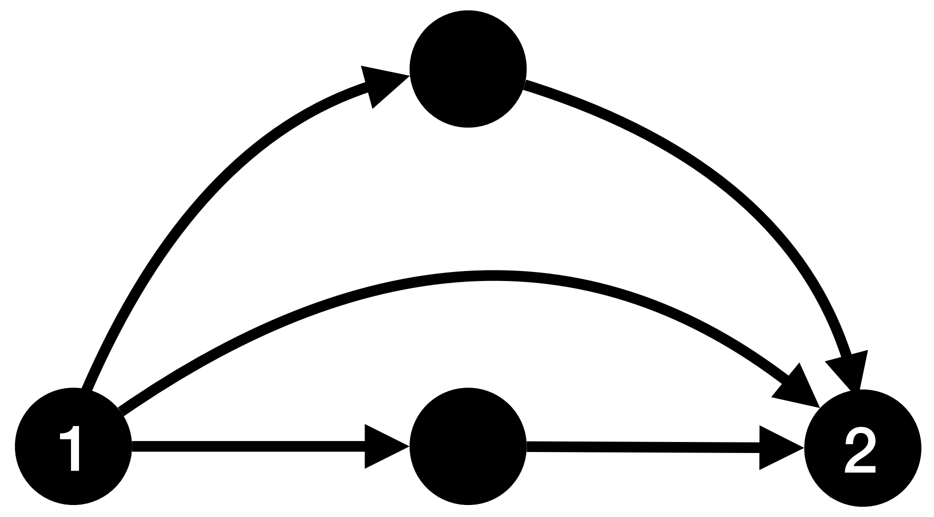


$b \circ b \parallel b$

Series Parallel Graphs

$\mathbb{B} \stackrel{\text{def}}{=} \{a, b, c, \ldots\}$ finite alphabet of (binary) edge labels

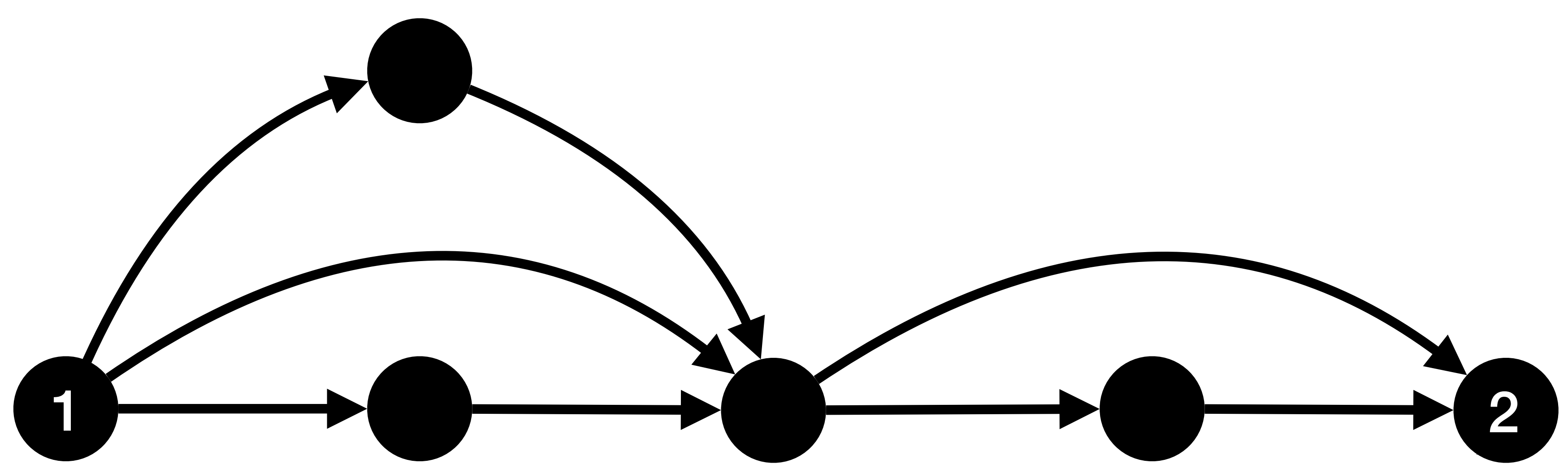


$$(b \circ b \parallel b) \parallel b \circ b$$

Series Parallel Graphs

$\mathbb{B} \stackrel{\text{def}}{=} \{a, b, c, \ldots\}$ finite alphabet of (binary) edge labels

$\mathcal{SP} = \left(\mathrm{SP}(\mathbb{B}), \{b^{\mathcal{SP}}\}_{b \in \mathbb{B}}, \circ^{\mathcal{SP}}, \|^{\mathcal{SP}}\right)$



$$\left((b \circ b \parallel b) \parallel b \circ b\right) \circ (b \circ b \parallel b)$$

Series Parallel Graphs

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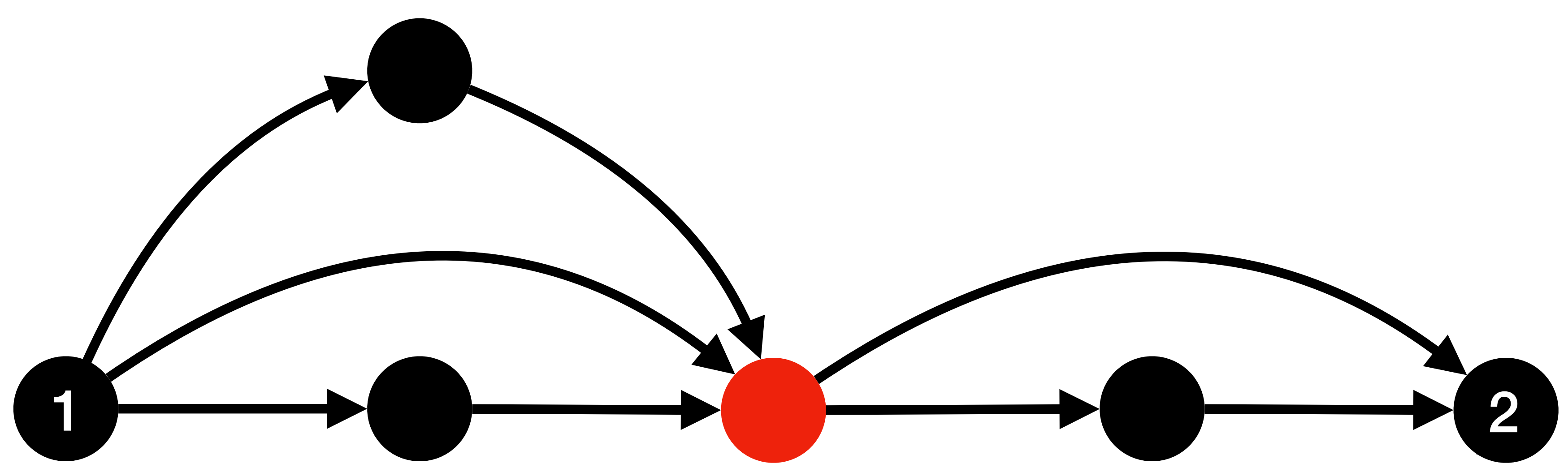
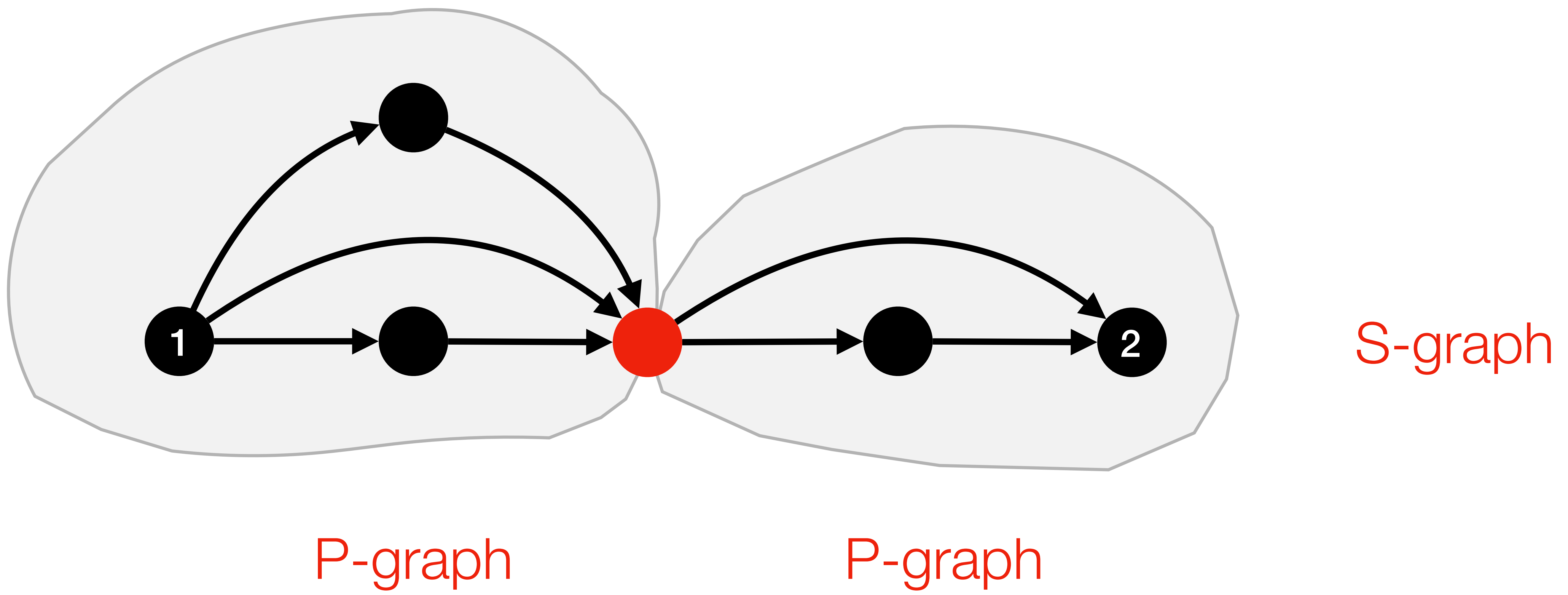


S-graph

Series Parallel Graphs

$\mathbb{B} \stackrel{\text{def}}{=} \{a, b, c, \ldots\}$ finite alphabet of (binary) edge labels

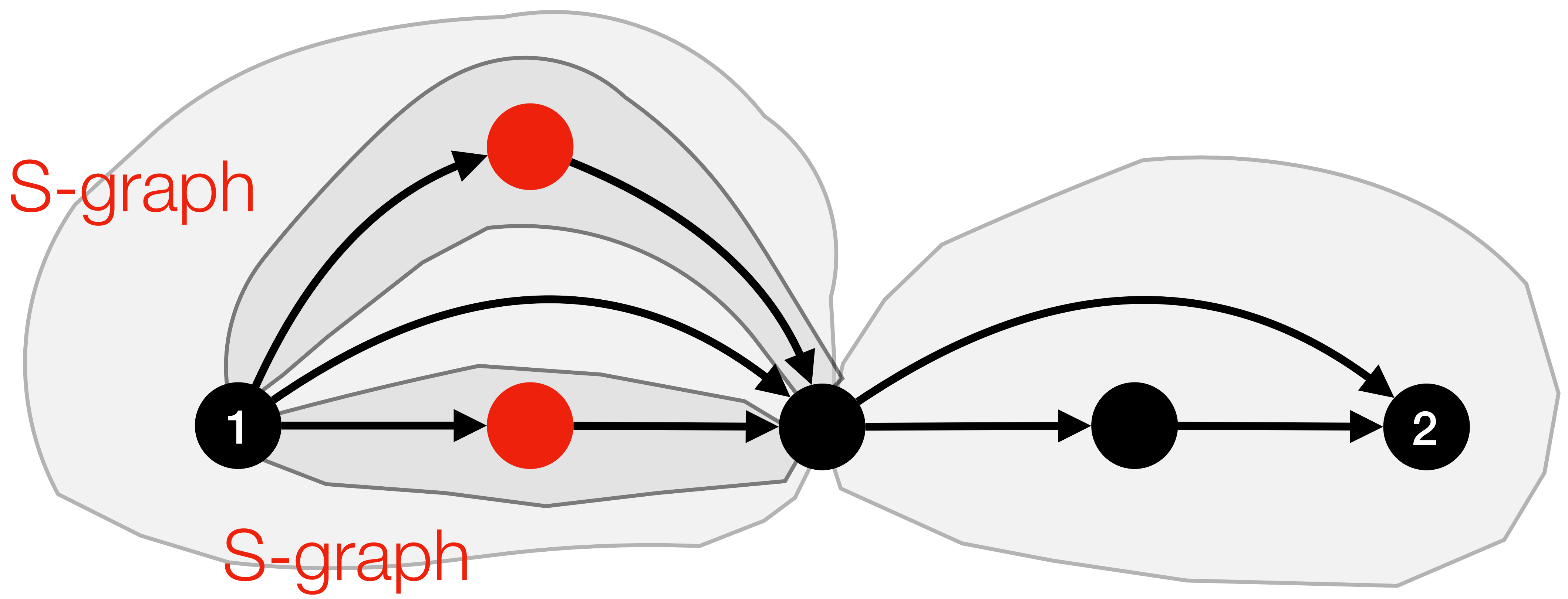
$\mathcal{SP}$=(SP($\mathbb{B}$), $\{b^{\mathcal{SP}}\}_{b\,\in\mathbb{B}}$, $\circ^{\mathcal{SP}}$, $||^{\mathcal{SP}}$)



Series Parallel Graphs

$\mathbb{B} \stackrel{\text{def}}{=} \{a, b, c, \ldots\}$ finite alphabet of (binary) edge labels

$\mathcal{SP} = (\mathrm{SP}(\mathbb{B}), \{b^{\mathcal{SP}}\}_{b \in \mathbb{B}}, \circ^{\mathcal{SP}}, \|^{\mathcal{SP}})$



(SP) Recognisability $\Longrightarrow$ Regularity

$\mathbb{B} \stackrel{\text{def}}{=} \{a, b, c, \ldots\}$ finite alphabet of (binary) edge labels

$\mathcal{SP}$=(SP($\mathbb{B}$), $\{b^{\mathcal{SP}}\}_{b \in \mathbb{B}}$, $\circ^{\mathcal{SP}}$, $\|^{\mathcal{SP}}$)

A regular SP grammar Γ is a stratified grammar with nonterminals $\mathcal{P} \uplus \mathcal{S}$ such that

$$\mathrm{FP}(\Gamma) \subseteq \{\mathtt{P} \leftarrow \mathtt{S}^{\#\geq 2}, \mathtt{S} \leftarrow \mathtt{P} \circ \mathtt{S}, \mathtt{S} \leftarrow \mathtt{P} \circ \mathtt{P}, \leftarrow \mathtt{P}, \leftarrow \mathtt{S}\} \cup \{\mathtt{P} \leftarrow b, \mathtt{S} \leftarrow b \mid b \in \mathbb{B}\}$$

(SP) Recognisability $\Longrightarrow$ Regularity

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Lemma [Universality] The following aperiodic stratified grammar produces SP($\mathbb{B}$):

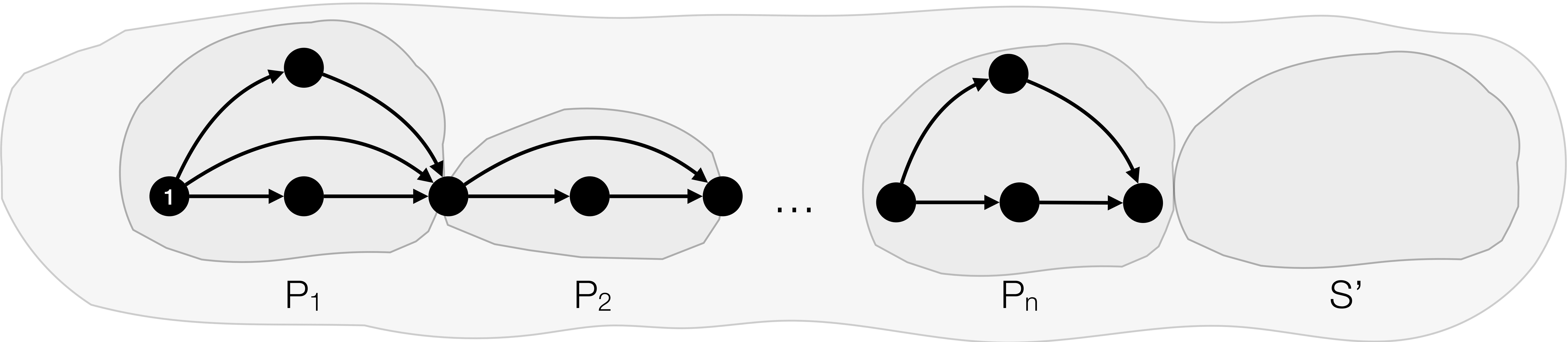
$$\begin{array}{llll} \mathrm{P} \leftarrow \mathrm{P} \,\|\, \mathrm{S} & \mathrm{S} \leftarrow \mathrm{P} \circ \mathrm{S} & \mathrm{S} \leftarrow \mathrm{b} & \leftarrow \mathrm{P} \\ \mathrm{P} \leftarrow \mathrm{S} \,\|\, \mathrm{S} & \mathrm{S} \leftarrow \mathrm{P} \circ \mathrm{P} & \mathrm{P} \leftarrow \mathrm{b} & \leftarrow \mathrm{S} \end{array}$$

$\Longrightarrow$ each set recognisable in $\mathcal{SP}$ is the language of a regular SP grammar

(SP) Regularity $\Longrightarrow$ Recognisability

Γ = regular SP grammar

$\mathcal{A} = (A, \{b^{\mathcal{A}}\}_{b \in \mathbb{B}}, \circ^{\mathcal{A}}, \|^{\mathcal{A}})$



(SP) Regularity $\Longrightarrow$ Recognisability

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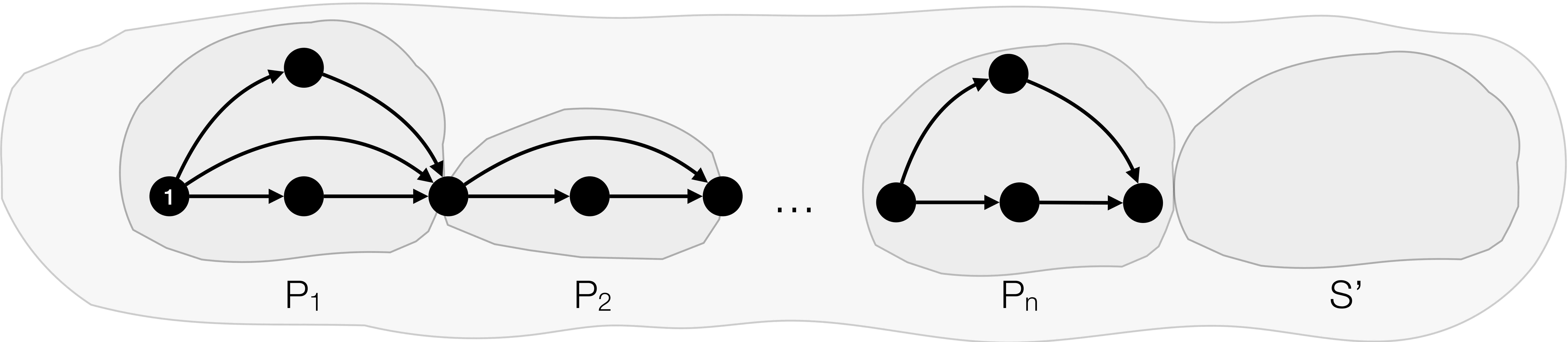

$S \leftarrow P_1 \circ S_1 \leftarrow \ldots \leftarrow P_1 \circ \ldots \circ P_n \circ S'$

(SP) Regularity $\Longrightarrow$ Recognisability

Γ = regular SP grammar

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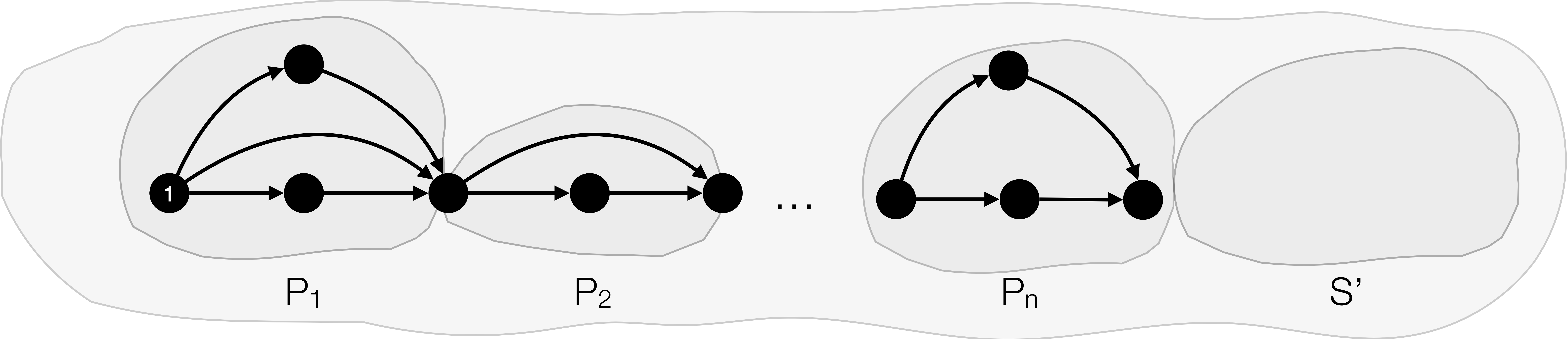


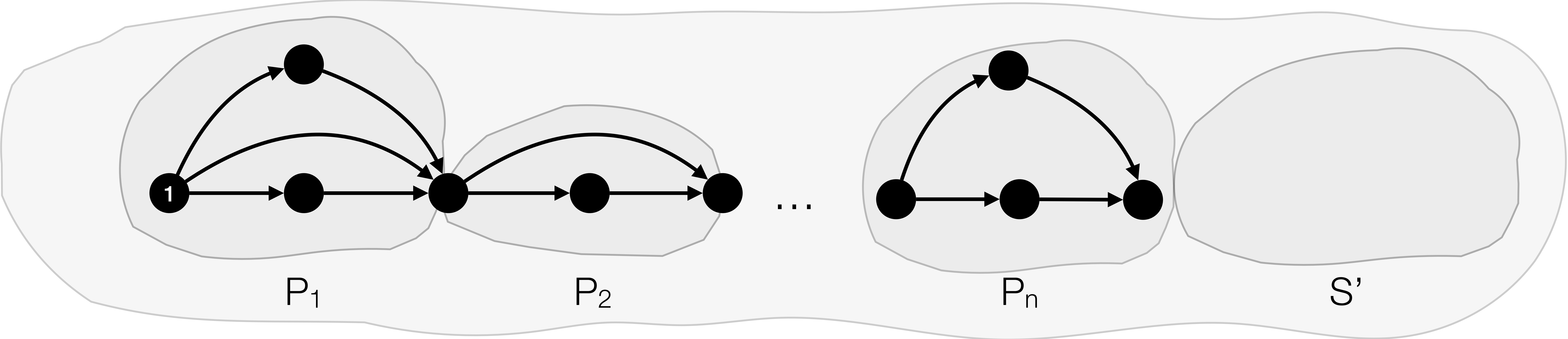
$S \leftarrow P_1 \circ S_1 \leftarrow \ldots \leftarrow P_1 \circ \ldots \circ P_n \circ S'$

(S,S') view of S-graph $G = t_1^{SP} \circ \ldots \circ t_n^{SP}$ (each $P_i \leftarrow \ldots \leftarrow t_i$ is a complete derivation)

(SP) Regularity $\Longrightarrow$ Recognisability

Γ = regular SP grammar

$\mathcal{A} = (A, \{b^{\mathcal{A}}\}_{b \in \mathbb{B}}, \circ^{\mathcal{A}}, \|^{\mathcal{A}})$



$S \leftarrow P_1 \circ S_1 \leftarrow \ldots \leftarrow P_1 \circ \ldots \circ P_n \circ S'$

(S,S') view of S-graph G = $t_1^{SP} \circ \ldots \circ t_n^{SP}$ (each $P_i \leftarrow \ldots \leftarrow t_i$ is a complete derivation)

{{S_1, …, S_n}} view of P-graph G = $t_1^{SP} \| \ldots \| t_n^{SP}$ (each $S_i \leftarrow \ldots \leftarrow t_i$ is a complete derivation)

A Recogniser Algebra for SP

Γ = regular SP grammar

$\mathcal{A} = (A, \{b^{\mathcal{A}}\}_{b \in \mathbb{B}}, \circ^{\mathcal{A}}, ||^{\mathcal{A}})$

set of reduced S/P-views

$$\frac{P \rightsquigarrow m \in a_1 \quad (S_1,Q) \in a_2 \quad S \leftarrow P \circ S_1}{(S,Q) \in a_1 \circ^{\mathcal{A}} a_2}$$

$$\frac{(S,S_1) \in a_1 \quad (S_1,Q) \in a_2}{(S,Q) \in a_1 \circ^{\mathcal{A}} a_2}$$

$$\frac{(S,S_1) \in a_1 \quad P \rightsquigarrow m \in a_2 \quad S_1 \leftarrow P \circ Q}{(S,Q) \in a_1 \circ^{\mathcal{A}} a_2}$$

A Recogniser Algebra for SP

Γ = regular SP grammar

$\mathcal{A} = (A, \{\mathit{b}^{\mathcal{A}}\}_{\mathit{b} \in \mathbb{B}}, \circ^{\mathcal{A}}, \|^{\mathcal{A}})$

set of reduced S/P-views

$$\frac{P \rightsquigarrow m \in a_1 \quad (S_1,Q) \in a_2 \quad S \leftarrow P \circ S_1}{(S,Q) \in a_1 \circ^{\mathcal{A}} a_2} \qquad \frac{(S,S_1) \in a_1 \quad (S_1,Q) \in a_2}{(S,Q) \in a_1 \circ^{\mathcal{A}} a_2} \qquad \frac{(S,S_1) \in a_1 \quad P \rightsquigarrow m \in a_2 \quad S_1 \leftarrow P \circ Q}{(S,Q) \in a_1 \circ^{\mathcal{A}} a_2}$$

$$\begin{array}{ccc} (SP(\mathbb{B}), \{\mathit{b}^{SP}\}_{\mathit{b} \in \mathbb{B}}, \circ^{SP}, \|^{SP}) & \xrightarrow{h(G) \overset{\text{def}}{=} \{m \mid \ m \text{ reduced view of } G\}} & (A, \{\mathit{b}^{\mathcal{A}}\}_{\mathit{b} \in \mathbb{B}}, \circ^{\mathcal{A}}, \|^{\mathcal{A}}) \\ \cup\!| & & \cup\!| \\ \mathcal{L}(\Gamma) & & = h^{-1}(\{h(G) \mid G \in \mathcal{L}(\Gamma)\}) \end{array}$$

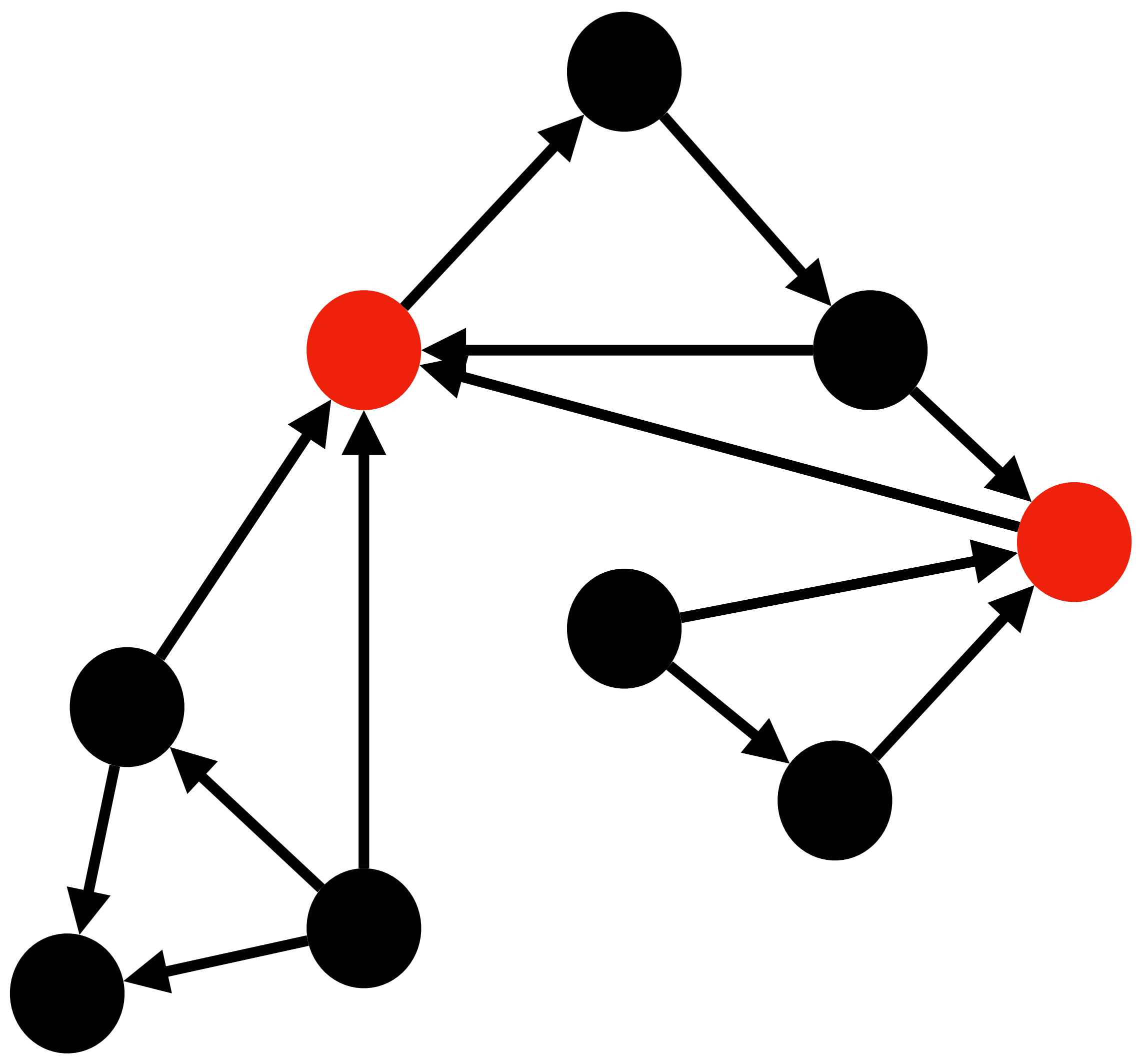
(SP) Regularity $\Longleftrightarrow$ Recognisability

Theorem

1. L is (aperiodic) recognisable in $\mathcal{SP}$ ⇔ L is the language of a (aperiodic) regular grammar for $\mathcal{SP}$
2. Given grammars Γ and Γ', the problem $\mathcal{L}(\Gamma) \subseteq \mathcal{L}(\Gamma')$ is in 2EXP ∩ EXP-hard, if Γ' is regular for $\mathcal{SP}$

Graphs of Tree-Width ≤ 2

$\mathbb{B} \stackrel{\text{def}}{=} \{a, b, c, \ldots\}$ finite alphabet of (binary) edge labels



Graphs of Tree-Width ≤ 2

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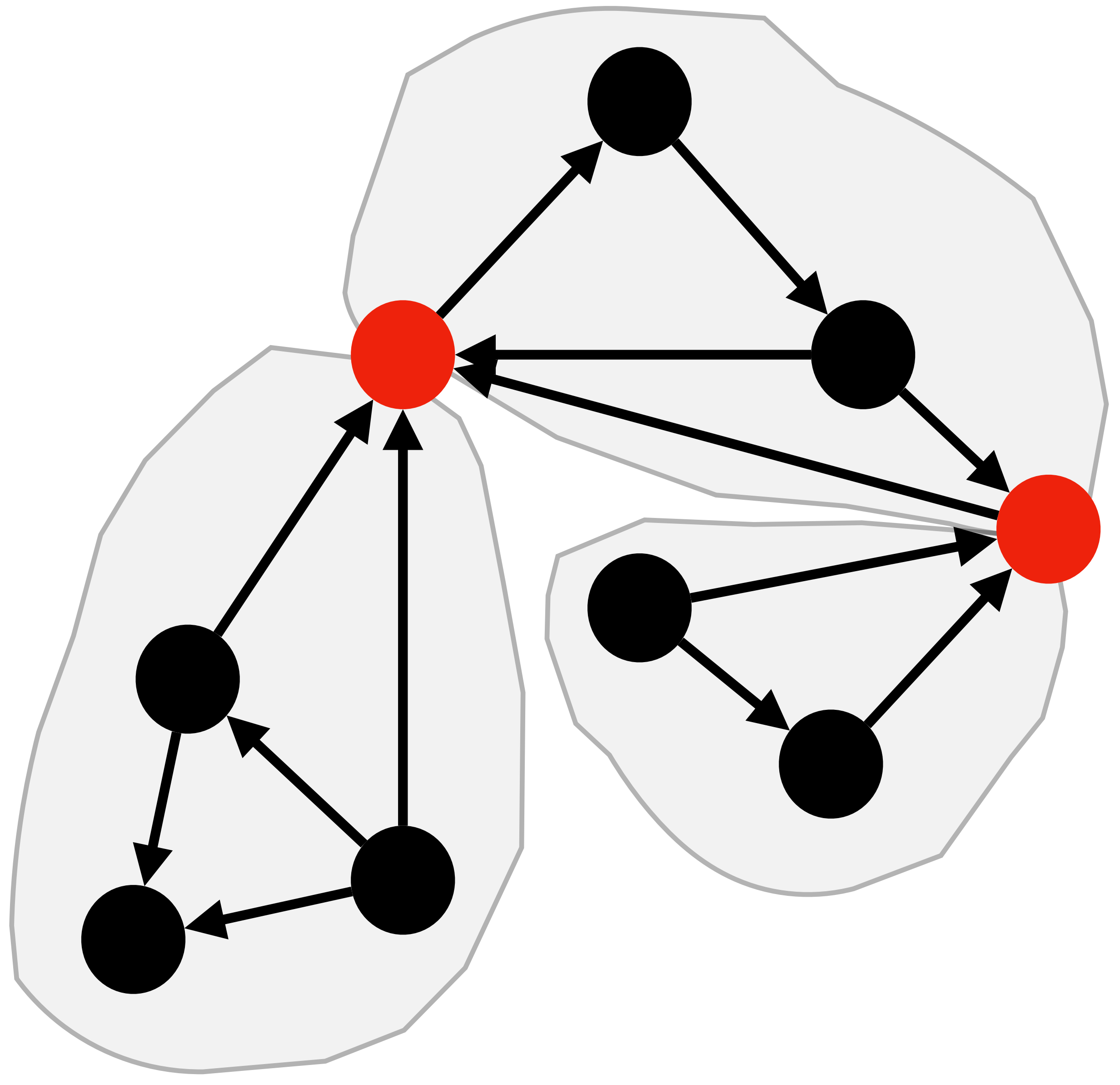

blocks = disoriented SP graphs

Graphs of Tree-Width ≤ 2

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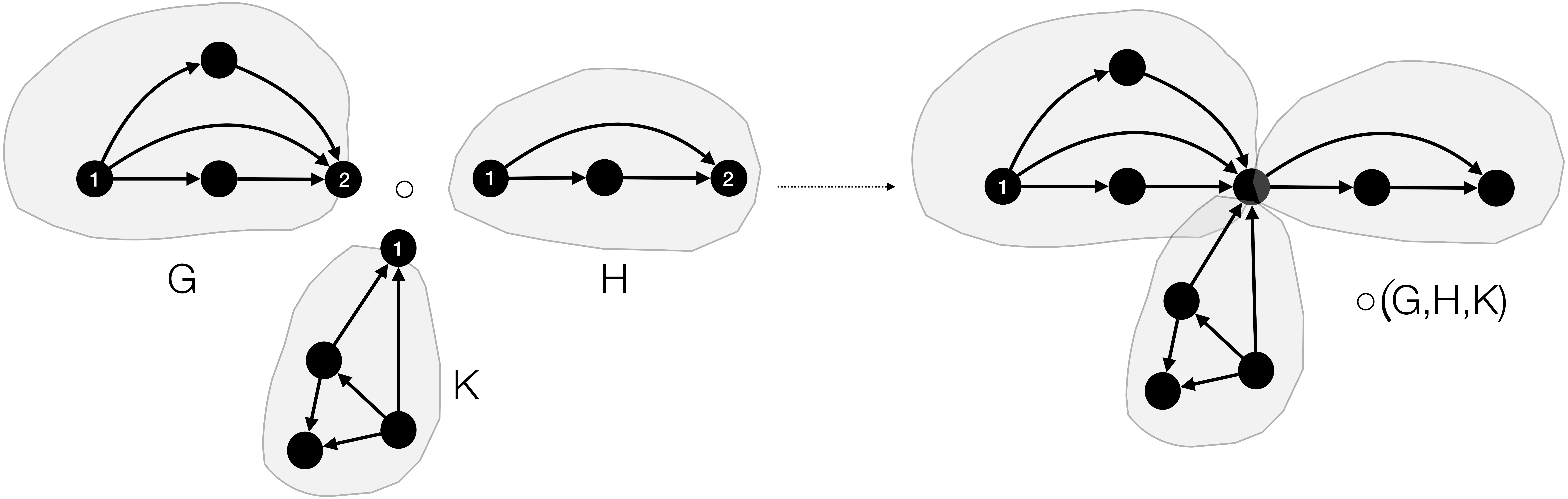
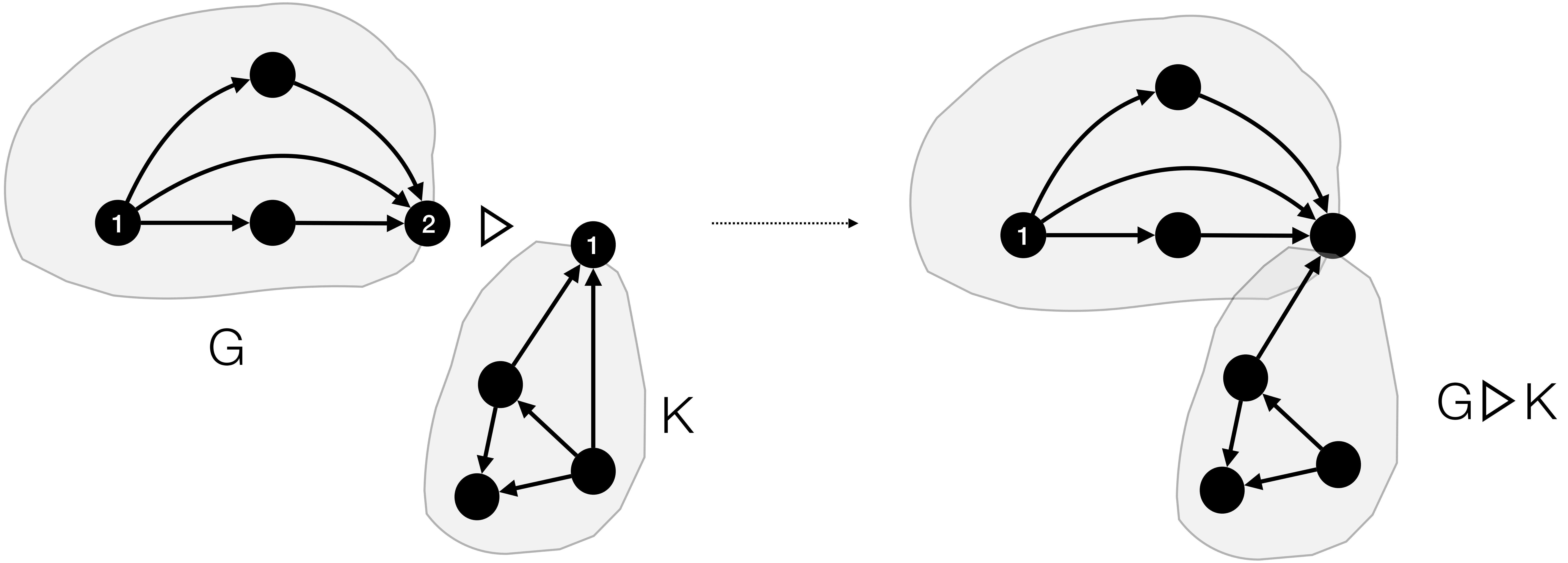


Graphs of Tree-Width ≤ 2

$\mathbb{B} \stackrel{\text{def}}{=} \{a, b, c, \ldots\}$ finite alphabet of (binary) edge labels

$\mathcal{G}_2 = (\{\mathcal{G}_\tau(\mathbb{B})\}_{\tau \in \{\{1\},\{1,2\}\}}, \{b^{\mathcal{G}_2}{}_{1,2}, b^{\mathcal{G}_2}{}_{2,1}\}_{b \in \mathbb{B}}, \circ^{\mathcal{G}_2}, \triangleright^{\mathcal{G}_2}, /\!/^{\mathcal{G}_2})$



(Tw≤2) Recognisability $\Longrightarrow$ Regularity

$\mathbb{B} \overset{\text{def}}{=} \{a, b, c, \ldots\}$ finite alphabet of (binary) edge labels

$\mathcal{G}_2 = (\{\mathcal{G}^{\leq 2}{}_{\tau}(\mathbb{B})\}_{\tau \in \{\{1\},\{1,2\}\}}, \{b^{\mathcal{G}_2}{}_{1,2}, b^{\mathcal{G}_2}{}_{2,1}\}_{b \in \mathbb{B}}, \circ^{\mathcal{G}_2}, \triangleright^{\mathcal{G}_2}, \|^{\mathcal{G}_2})$

A regular tw≤2 grammar Γ is a stratified grammar with nonterminals $\overbrace{\mathcal{X} \uplus \mathcal{Y}}^{\{1\}} \uplus \overbrace{\mathcal{P} \uplus \mathcal{S}}^{\{1,2\}}$:

$$\mathrm{FP}(\Gamma) \subseteq \{X \leftarrow Y^{\# \geq 0}, P \leftarrow S^{\# \geq 2}, Y \leftarrow P \triangleright X\} \cup \{S \leftarrow \circ\,(P,S,X), S \leftarrow \circ\,(P,P,X), \leftarrow X\}$$

$$\{P \leftarrow b_{1,2}, P \leftarrow b_{2,1}, S \leftarrow b_{1,2}, S \leftarrow b_{2,1} \mid b \in \mathbb{B}\}$$

(Tw≤2) Recognisability $\Longrightarrow$ Regularity

$\mathbb{B} \stackrel{\text{def}}{=} \{a, b, c, \ldots\}$ finite alphabet of (binary) edge labels

$\mathcal{G}_2 = (\{\mathcal{G}^{\leq 2}{}_{\tau}(\mathbb{B})\}_{\tau \in \{\{1\},\{1,2\}\}}, \{b^{\mathcal{G}_2}{}_{1,2}, b^{\mathcal{G}_2}{}_{2,1}\}_{b \in \mathbb{B}}, \circ^{\mathcal{G}_2}, \rhd^{\mathcal{G}_2}, \|^{\mathcal{G}_2})$

A regular tw≤2 grammar Γ is a stratified grammar with nonterminals $\underbrace{\mathcal{X} \uplus \mathcal{Y}}_{\{1\}} \uplus \underbrace{\mathcal{P} \uplus \mathcal{S}}_{\{1,2\}}$:

$$FP(\Gamma) \subseteq \{X \leftarrow Y^{\#\geq 0}, P \leftarrow S^{\#\geq 2}, Y \leftarrow P \rhd X\} \cup \{S \leftarrow \circ\,(P,S,X), S \leftarrow \circ\,(P,P,X), \leftarrow X\}$$
$$\{P \leftarrow b_{1,2}, P \leftarrow b_{2,1}, S \leftarrow b_{1,2}, S \leftarrow b_{2,1} \mid b \in \mathbb{B}\}$$

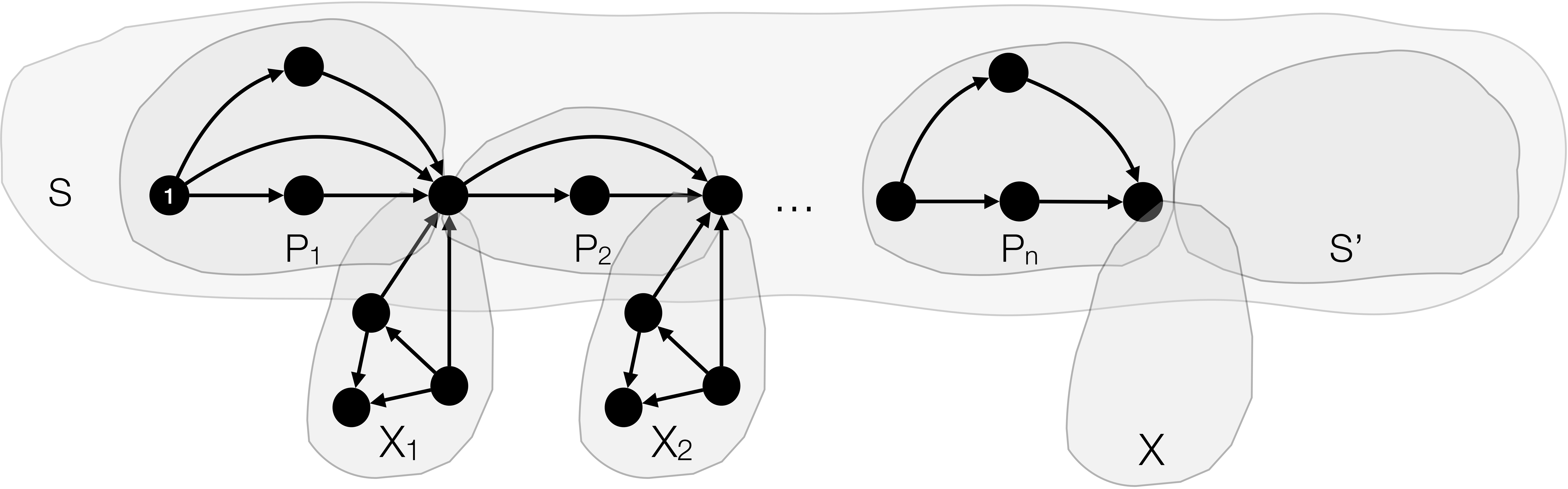
Lemma [Universality] The following aperiodic stratified grammar produces $\mathcal{G}^{\leq 2}(\mathbb{B})$:

$P \leftarrow P \parallel S$	$S \leftarrow \circ(P,S,X)$	$S \leftarrow b_{i,3-i}$	$X \leftarrow X \parallel Y$	$Y \leftarrow P \rhd X$
$P \leftarrow S \parallel S$	$S \leftarrow \circ(P,P,X)$	$P \leftarrow b_{i,3-i}$	$X \leftarrow 1$	$\leftarrow X$

$\Longrightarrow$ each set recognisable in $\mathcal{G}_2$ is the language of a regular tw≤2 grammar

(Tw≤2) Regularity $\Longrightarrow$ Recognisability

Γ = regular tw≤2 grammar $\mathcal{A}=(\{A_\tau\}_{\tau\in\{\{1\},\{1,2\}\}}, \{b^{\mathcal{A}}_{1,2}, b^{\mathcal{A}}_{2,1}\}_{b\in\mathbb{B}}, \circ^{\mathcal{A}}, \triangleright^{\mathcal{A}}, \|^{\mathcal{A}})$



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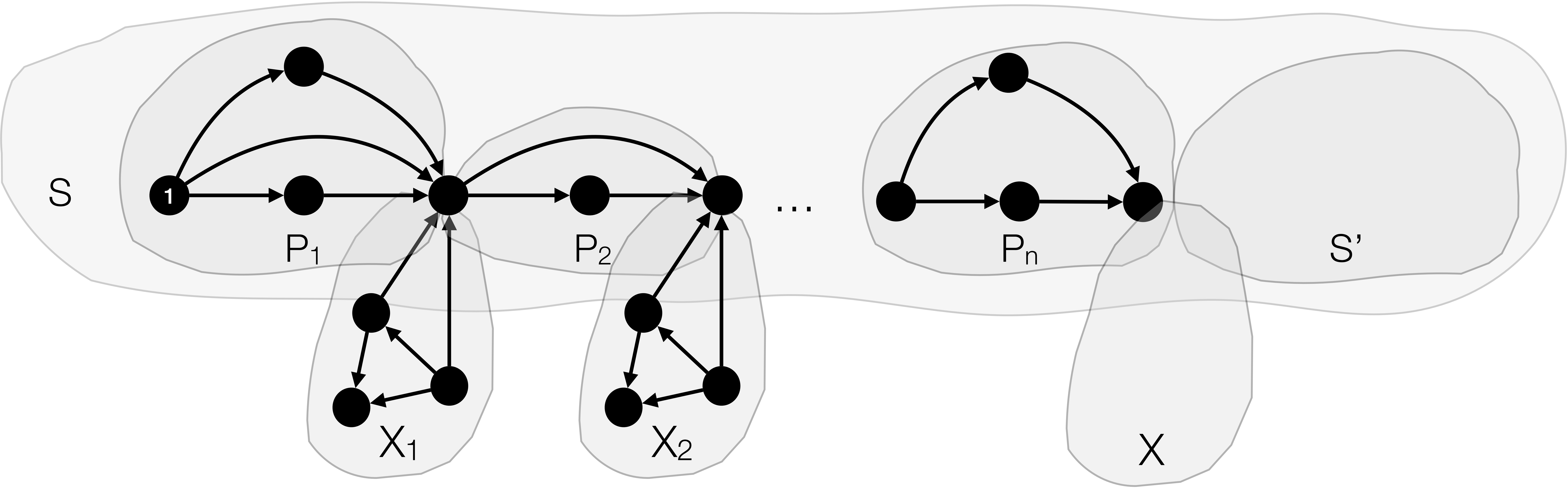

(S,S',X) view of $S_{\{1,2\}}$-graph G = $\circ(t_1^{\mathcal{G}_2}, \circ(t_2^{\mathcal{G}_2}, \ldots \circ(t_{n-1}^{\mathcal{G}_2}, u_{n-1}^{\mathcal{G}_2}, t_n^{\mathcal{G}_2}) \ldots))$ $\quad$ $P_i \leftarrow \ldots \leftarrow t_i$, $X_i \leftarrow \ldots \leftarrow u_i$ complete derivations

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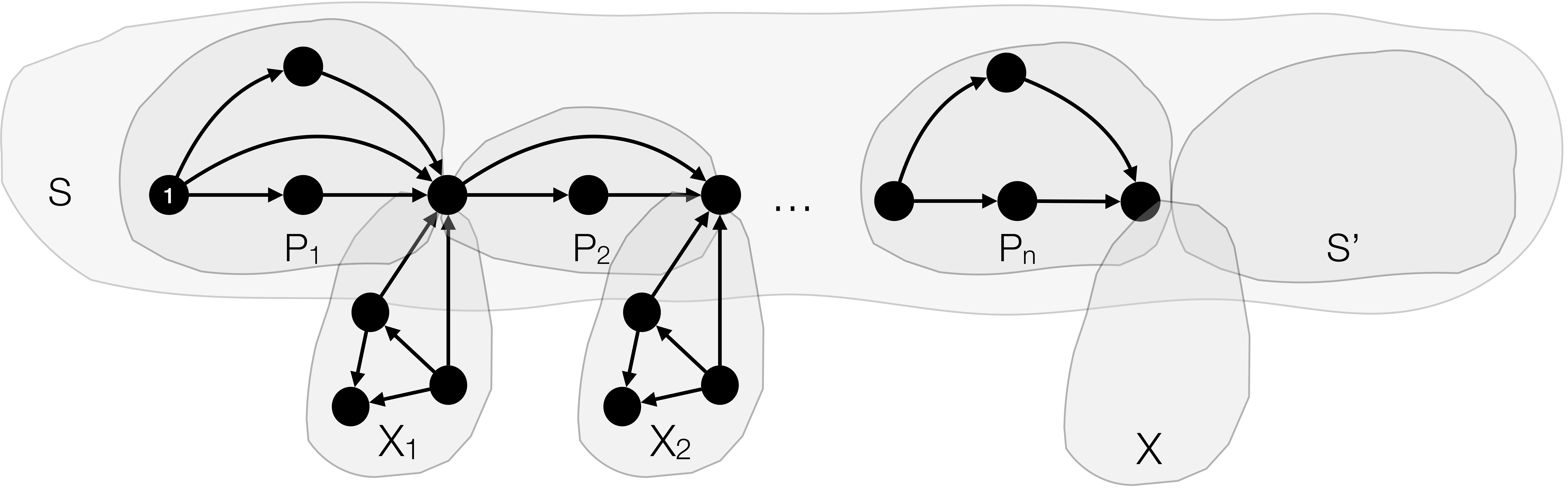

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$\{\{S_1, \ldots, S_n\}\}$ view of $P_{\{1,2\}}$-graph G = $t_1{}^{\mathcal{G}_2} \| \ldots \| t_n{}^{\mathcal{G}_2}$ $S_i \leftarrow \ldots \leftarrow t_i$ complete derivations

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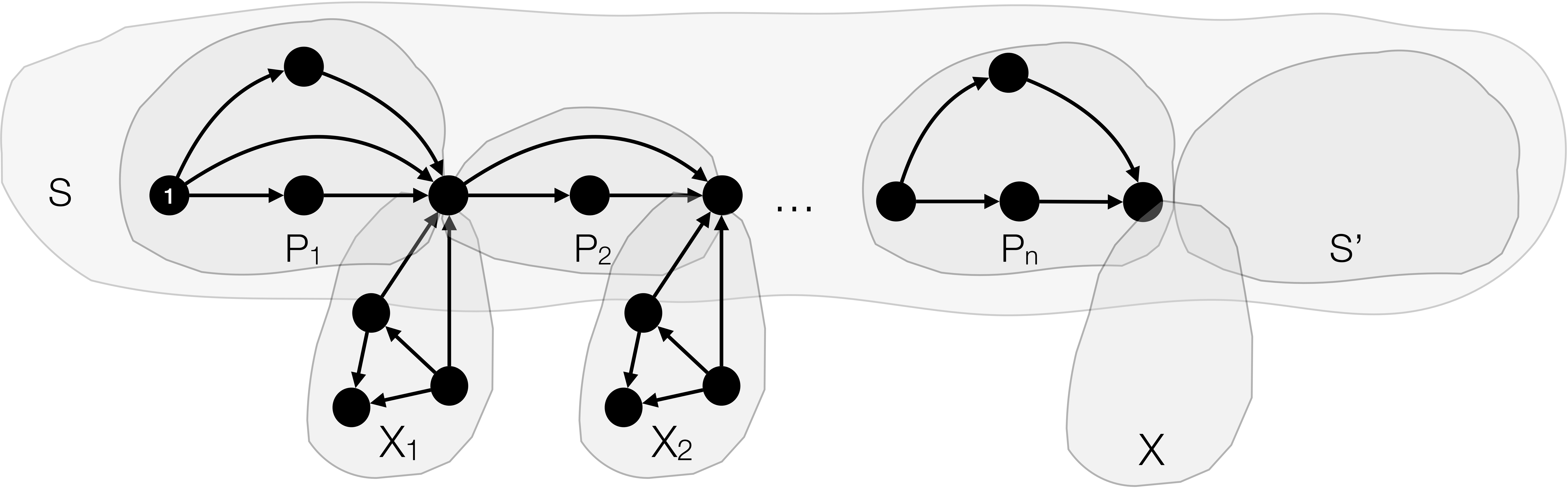

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{{Y_1, …, Y_n}} view of $\{1\}$-graph $\quad G = t_1^{\mathcal{G}_2} \| \ldots \| t_n^{\mathcal{G}_2}$ $\quad X_i \leftarrow \ldots \leftarrow t_i$ complete derivations

A Recogniser Algebra for Tw≤2

Γ = regular tw≤2 grammar

$\mathcal{A}=(\{A_\tau\}_{\tau\in\{\{1\},\{1,2\}\}}, \{b^{\mathcal{A}}{}_{1,2}, b^{\mathcal{A}}{}_{2,1}\}_{b\in\mathbb{B}}, \circ^{\mathcal{A}}, \triangleright^{\mathcal{A}}, //^{\mathcal{A}})$

$$\frac{P \rightsquigarrow m \in a_1 \qquad (S_1,Q,X) \in a_2 \qquad S \leftarrow \circ(P,S_1,X_1) \qquad X_1 \rightsquigarrow k \in a_3}{(S,Q,X) \in \circ^{\mathcal{A}}(a_1,a_2,a_3)}$$

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$$\mathcal{G}_2 \xrightarrow{h(G) \overset{\text{def}}{=} \{m \mid \text{ m reduced view of } G\}} \mathcal{A}$$

$$\mathcal{G}_2 \supseteq \mathcal{L}(\Gamma) \qquad \mathcal{A} \supseteq$$

$$\mathcal{L}(\Gamma) = h^{-1}(\{h(G) \mid G \in \mathcal{L}(\Gamma)\})$$

(Tw≤2) Regularity $\iff$ Recognisability

Theorem

1. L is (aperiodic) recognisable in $\mathcal{G}_2$ ⇔ L is the language of a (aperiodic) regular grammar for $\mathcal{G}_2$
2. Given grammars Γ and Γ', the problem $\mathcal{L}(\Gamma) \subseteq \mathcal{L}(\Gamma')$ is in 2EXP ∩ EXP-hard, if Γ' is regular for $\mathcal{G}_2$