STAT 578: Advanced Bayesian Modeling

Week 1 – Lesson 2

Random Variables, Distributions, and Densities

Fall 2019

One Random Variable

We review some essentials from probability, including

- Random variables
- Densities
- Notation for distributions
- Expected values and other distribution features

We will *not* cover these in depth. For more details, consult an undergraduate-level probability textbook or course.

Random Variables

Recall that a **random variable** is a quantity whose uncertainty can be measured in terms of probability.

Its **distribution** defines the collection of probability statements that may be made about it alone.

E.g., a random variable U might have a 50% chance of exceeding 2:

$$\Pr(U > 2) \;=\; 0.5$$

Densities

A **discrete** random variable U has countably (perhaps infinitely) many possible values.

It has a **discrete distribution**, with **discrete density**

$$p(u) \;=\; \Pr(U = u)$$

The collection of such probabilities must sum to 1, and we write

$$\sum_u p(u) \;=\; 1$$

A **continuous** random variable U takes values on a continuum and has a **continuous distribution**: one that has a **continuous density**

$$p(u) \;\geq\; 0$$

satisfying

$$\Pr(U \in D) \;=\; \int_D p(u)\, du$$

We note that $p(u)$ integrates to 1, and write

$$\int p(u)\, du \;=\; 1$$

Unlike a discrete density, a continuous density may exceed 1.

Note: We use the same notation $p(\cdot)$ for both discrete and continuous densities.

Commonly used distributions are usually of a named type, with particular cases defined by parameters:

- The **binomial** distribution with parameters n and $p \in (0,1)$ is discrete:

$$p(u) \;=\; \binom{n}{u} p^u (1-p)^{n-u} \qquad u = 0, 1, \ldots, n$$

- The **normal** distribution with parameters μ and $\sigma^2 > 0$ is continuous:

$$p(u) \;=\; \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{1}{2\sigma^2}(u-\mu)^2} \qquad \text{(all } u\text{)}$$

Notation

When random variable U has a distribution of a named type, we write

$$U \quad \sim \quad name(parameters)$$

For example:

$$\begin{array}{lll} U & \sim & \mathrm{Bin}(n, p) \\ U & \sim & \mathrm{N}(\mu, \sigma^2) \end{array}$$

U	$\sim$	$\mathrm{Bin}(n, p)$	for a binomial
U	$\sim$	$\mathrm{N}(\mu, \sigma^2)$	for a normal

See BDA3, Tables A.1 and A.2 for naming conventions.

Distribution Features

The **expected value** (or **mean**, or **expectation**) of random variable U is

$$\mathrm{E}(U) = \begin{cases} \displaystyle\sum_u u\, p(u) & U \text{ discrete} \\ \displaystyle\int u\, p(u)\, du & U \text{ continuous} \end{cases}$$

when it exists. More generally,

$$\mathrm{E}\big(g(U)\big) = \begin{cases} \displaystyle\sum_u g(u)\, p(u) & U \text{ discrete} \\ \displaystyle\int g(u)\, p(u)\, du & U \text{ continuous} \end{cases}$$

The **variance** of random variable U is

$$\mathrm{var}(U) \;=\; \mathrm{E}\Big(\big(U - \mathrm{E}(U)\big)^2\Big)$$

when it exists.

Its **standard deviation** is $\sqrt{\mathrm{var}(U)}$.

For example, if $U \sim \mathrm{Bin}(n, p)$, then

$$\mathrm{E}(U) = np \qquad \mathrm{var}(U) = np(1-p)$$

Or, if $U \sim \mathrm{N}(\mu, \sigma^2)$, then

$$\mathrm{E}(U) = \mu \qquad \mathrm{var}(U) = \sigma^2$$

which explains why we call these parameters the *mean* and *variance*.

Other features include the **quantiles**.

If U is continuous, an α quantile q_α of U satisfies

$$\alpha \;=\; \Pr(U \leq q_\alpha) \;=\; \int_{-\infty}^{q_\alpha} p(u)\, du$$

A **median** is a 0.5 quantile.

For example, $U \sim \mathrm{N}(\mu, \sigma^2)$ has median μ since $p(u)$ is symmetric around μ.

If a density has a maximizer, it is a **mode**.

Local maximizers are sometimes also called modes.

If there is just one local maximizer, and it is also a (global) maximizer, the density is *unimodal*. Otherwise, it may be *bimodal* or *multimodal*.