ADVANCED BAYESIAN MODELING

Approximating the Integrals

Motivation

When parameter θ is jointly continuous (as usual), most Bayesian tools are based on integration.

In simple cases, explicit integration is possible (e.g., single binomial probability or simple normal sample).

But explicit integration is usually impossible even for the simplest hierarchical models (and most models we will see later).

Consider numerical approximations instead ...

Deterministic Numerical Integration

For one-dimensional integration over (a, b):

Decompose into intervals of lengths Δ_j with representative points x_j, so that

$$\int_a^b f(x)\,dx \quad \approx \quad \sum_j f(x_j)\,\Delta_j$$

(Various methods: midpoint rule, Simpson's rule ...)

Specialized weighted schemes also available (e.g., Gaussian quadrature)

For multidimensional integration over region A:

Decompose into regions of volumes Δ_j with representative points x_j, so that

$$\int_A f(x)\,dx \quad \approx \quad \sum_j f(x_j)\,\Delta_j$$

Problem: Breaks down quickly as dimension increases.

A d-dimensional integration grid using m values in each dimension needs a total of m^d integrand evaluations.

Another limitation: Integrand must be known exactly.

For most Bayesian tools, integrand includes posterior $p(\theta \mid y)$.

So the normalizing factor $p(y)$ must be known.

Approximating the normalizing factor is notoriously difficult (see BDA3, Sec. 13.10).

Numerical Integration by Simulation

Suppose random variates

$$\theta^1, \quad \theta^2, \quad \ldots, \quad \theta^S$$

are drawn directly from the posterior $p(\theta \mid y)$.

Then any finite posterior expectation can be approximated as

$$\mathrm{E}\big(h(\theta) \mid y\big) \;=\; \int h(\theta)\, p(\theta \mid y)\, d\theta \quad \approx \quad \frac{1}{S} \sum_{s=1}^{S} h(\theta^s)$$

Sometimes called **Monte Carlo** integration.

Advantages:

- No need to evaluate normalizing factor (unless random simulation requires it).
- Can reuse same simulation samples for many different integral approximations.

We seek methods of simulation that:

- Make most Bayesian tools easy to approximately compute
- Extend to high dimensions
- Don't need the normalizing factor $p(y)$