ADVANCED BAYESIAN MODELING

Monte Carlo Approximations

Monte Carlo methods are approximations based on random simulation.

Most Bayesian tools can be well approximated by Monte Carlo.

Assume random variates

$$\theta^1, \quad \theta^2, \quad \ldots, \quad \theta^S$$

are drawn from posterior $p(\theta \mid y)$.

(Methods for drawing random variates considered later.)

Posterior Expectations

If $h(\theta)$ has finite posterior expectation,

$$\mathrm{E}\big(h(\theta) \mid y\big) \;=\; \int h(\theta)\, p(\theta \mid y)\, d\theta \;\approx\; \frac{1}{S} \sum_{s=1}^{S} h(\theta^s)$$

Examples include a scalar parameter's posterior mean or variance, and a posterior probability (using an indicator function).

Suppose scalar variates $h(\theta^s)$ are in R vector `h.sim`.

Approximation to $\mathrm{E}\big(h(\theta) \mid y\big)$ in R:

```
mean(h.sim)
```

Approximations to $\mathrm{var}\big(h(\theta) \mid y\big)$ and its square root:

```
var(h.sim)
sd(h.sim)
```

Approximation to $\Pr\big(h(\theta) > 0 \mid y\big)$:

```
mean(h.sim > 0)
```

Posterior Marginals

Let θ_j be a part of parameter vector θ.

Then the corresponding elements of the random variates

$$\theta_j^1, \quad \theta_j^2, \quad \ldots, \quad \theta_j^S$$

are already drawn from the posterior for θ_j.

Posterior marginal summaries for θ_j (means, variances, quantiles, ...) can be based on these variates.

The posterior density of a continuous θ_j can be approximated using histograms or density estimation techniques (such as *kernel density estimation*).

If θ_j is one-dimensional, with variates in R vector `thetaj.sim`, a histogram is produced by

```
hist(thetaj.sim)
```

A kernel density estimate is plotted with

```
plot(density(thetaj.sim))
```

Other functions in R packages (like `coda`) will produce these automatically.

Posterior Quantiles and Intervals

Posterior quantiles of scalar $h(\theta)$ can be approximated with *empirical* quantiles.

To approximate the α (lower) posterior quantile q_α, first put the variates

$$h(\theta^1), \quad h(\theta^2), \quad \ldots, \quad h(\theta^S)$$

in *increasing* order, then use the $[\alpha S]$th value, where $[\alpha S]$ is αS rounded to the nearest integer.

(More sophisticated schemes exist, but all are very similar when S is large.)

If R vector `h.sim` contains the variates $h(\theta^s)$, then the `alpha` quantile q_α is approximately

```
quantile(h.sim, alpha)
```

The central $(1-\alpha)100\%$ posterior interval for $h(\theta)$, which is $(q_{\alpha/2},\ q_{1-\alpha/2})$, can similarly be approximated.

For example, the endpoints of the central 95% posterior interval:

```
quantile(h.sim, c(0.025, 0.975))
```

Posterior Prediction

A predictive quantity $\tilde{y}$ can often be drawn directly from its conditional density

$$p(\tilde{y} \mid \theta) \qquad \text{(conditionally independent of } y\text{)}$$

If variate $\tilde{y}^s$ is drawn from $p(\tilde{y} \mid \theta^s)$, then

$$\tilde{y}^1, \quad \tilde{y}^2, \quad \ldots, \quad \tilde{y}^S$$

are variates as if drawn from posterior predictive density $p(\tilde{y} \mid y)$.

Use these as before to approximate posterior predictive means, variances, quantiles, intervals, etc.

For example, if $S = 1000$ posterior draws of

$$\theta = (\mu, \sigma)$$

are in corresponding elements of R vectors `mu` and `sigma`, and

$$\tilde{y} \mid \mu, \sigma \sim \mathrm{N}(\mu, \sigma^2)$$

then $S = 1000$ posterior predictive draws of $\tilde{y}$ are generated by

```
rnorm(1000, mu, sigma)
```

Prior Simulation

If the prior is proper, the same methods can be used to approximate *prior* means, variances, quantiles, etc. Just draw the variates

$$\theta^1, \quad \theta^2, \quad \ldots, \quad \theta^S$$

from the prior instead of the posterior.

(However, many prior-based quantities are usually available analytically.)