STAT 578: Advanced Bayesian Modeling

Week 1 – Lesson 4

Bayesian Tools for Inference

Fall 2019

Tools for Estimation and Testing

Classical (non-Bayesian) inference tools include point estimates, standard errors, confidence intervals, hypothesis tests, and point predictions and intervals.

Bayesian inference has corresponding tools:

- Posterior point estimates
- Measures of posterior spread
- Posterior (credible) intervals
- Posterior probabilities
- Posterior predictive distributions

Posterior Estimation

Single-number summaries of the posterior distribution of scalar parameter θ include:

- The posterior mean (if it exists)

$$\mathrm{E}(\theta \mid y)$$

- The posterior median
- The posterior mode (if well-defined)

In statistics, the posterior mean is the most popular.

For your dog-walking service, with flat $\mathrm{U}(0,1)$ prior on

$$\theta \quad = \quad \text{true proportion of potential clients}$$

the posterior was $\mathrm{Beta}(\alpha = 3, \beta = 19)$.

Then (BDA3, Table A.1)

- The posterior mean is

$$\mathrm{E}(\theta \mid y = 2) \quad = \quad \frac{3}{3+19} \quad \approx \quad 0.136$$

- The posterior mode is

$$\frac{3-1}{3+19-2} \quad \approx \quad 0.100$$

When not available analytically, the posterior mean and median can be approximated by simulation.

In R, function `rbeta` simulates from a beta distribution:

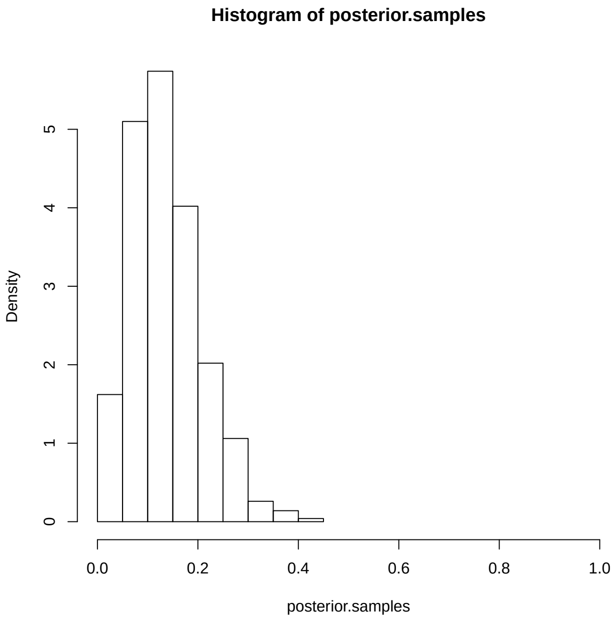
```
> posterior.samples <- rbeta(1000, 3, 19)  # 1000 independent samples

> mean(posterior.samples)
[1] 0.1370276

> median(posterior.samples)
[1] 0.128049
```

Compare with the true median:

```
> qbeta(0.5, 3, 19)
[1] 0.1253131
```

A histogram:

```
> hist(posterior.samples, freq=FALSE, xlim=c(0,1))
```



Histogram of posterior.samples
Density
0
1
2
3
4
5
0.0
0.2
0.4
0.6
0.8
1.0
posterior.samples

Approximating a posterior mode generally requires optimization:

```
> optimize(dbeta, c(0,1), 3, 19, maximum=TRUE)
$maximum
[1] 0.1000202

$objective
[1] 5.988776
```

Posterior Spread

To measure uncertainty, could use the posterior standard deviation (if it exists):

$$\sqrt{\mathrm{var}(\theta \mid y)}$$

For example,

```
> sd(posterior.samples)  # approx. posterior std. dev.
[1] 0.07048138
```

approximates the true posterior standard deviation (BDA3, Table A.1):

$$\sqrt{\frac{3 \cdot 19}{(3+19)^2\,(3+19+1)}} \approx 0.0716$$

Posterior Intervals

Instead of posterior location and spread, consider a range of values ...

A **posterior (credible) interval** is one that contains θ with specified posterior probability.

Could be defined in several ways ...

Most commonly, the $100(1-\alpha)$% posterior interval is the range between the $\alpha/2$ and $1-\alpha/2$ posterior quantiles.

This is called a **central** posterior interval.

Central posterior intervals are easily approximated by simulation.

E.g., an approximate 95% central posterior interval:

```
> quantile(posterior.samples, c(0.05/2, 1-0.05/2))
      2.5%      97.5%
0.02887084 0.29349264
```

This approximates the true interval:

```
> qbeta(c(0.05/2, 1-0.05/2), 3, 19)
[1] 0.03048897 0.30377441
```

So, with 95% posterior probability, between about 3% and 30% of households in your community might be interested in a dog-walking service.

Posterior Probabilities

Suppose you believe your dog-walking service will be successful if at least 15% of households in the community are interested.

How probable is that?

You could use the posterior probability

$$\Pr(\theta \geq 0.15 \mid y = 2)$$

Posterior probabilities are easily approximated by simulation:

```
> mean(posterior.samples >= 0.15)
[1] 0.377
```

This approximates the exact value

```
> 1 - pbeta(0.15, 3, 19)
[1] 0.3704955
```

So your dog-walking service has (at least) a 37% chance of success.

This bears similarity to a classical hypothesis test:

$$H_0 : \theta \leq 0.15 \qquad\qquad H_a : \theta > 0.15$$

The posterior probability $\Pr(H_0 \mid y)$ is sometimes used like a classical p-value.

However, there are more formal Bayesian approaches (later).