Lukasiewicz ∞ - Valued Logic

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Contents

1 Introduction

This material has been prepared as a synthesis of the information presented in Chaptetr 4 of the *Algebraic Foundations of Many-Valued Reasoning*, by R.L.O Cignoli et. al, intended to be understood by any logic enthusiast with minimal knowledge in this field, by containing detailed proofs. The topic of this material regards the theoretical aspects of Lukasiewicz Infinite-Valued Logic and MV-Algebras in general, by focusing on the Completeness Theorem of Lukasiewicz Infinite-Valued Logic.

2 MV-Algebras

2.1 Preliminaries

Definition 2.1. *An MV-Algebra is an algebra $\langle A, \oplus, \neg, 0, \rangle$ with a binary operation $\oplus$, an unary operation $\neg$ and the constant 0, satisfying the following equations:*

MV1. $x \oplus (y \oplus z) = (x \oplus y) \oplus z$

MV2. $x \oplus y = y \oplus x$

MV3. $x \oplus 0 = x$

MV4. $\neg\neg x = x$

MV5. $x \oplus \neg 0 = \neg 0$

MV6. $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$

Remark 2.2. *We will denote an MV-Algebra by its universe A*

2.2 Term function

Definition 2.3 (Strings)**.** *A string (or a word) over a non-empty set S, is a finite list of elements of S.*

Definition 2.4 (MV-Terms)**.** *For each natural number $t \geq 1$, let:*

$$S_t = \{0, \neg, \oplus, v_1, v_2, ..., v_t, (,)\}$$

An MV-Term in the variables $v_1, v_2, ..., v_t$, is a string over S_t, which is defined inductively from a certain number of applications of the following rules:

T1. *The elements 0 and v_i, for each $i \in \{1, 2, ..., t\}$ (considered one-element strings) are MV-Terms*

T2. *If the string τ is a term, then $\neg\tau$ is a term*

T3. *If the strings τ and σ, then so is $(\tau \oplus \sigma)$*

T4. Only those defined by the previous rules are MV-Terms

Definition 2.5 (Term function)**.** *Let A be an MV-algebra, and τ an MV-Term in the variables $v_1, ..., v_t$ and assume $a_1, ..., a_t$ are elements of A. We will define the function:*

$$\tau : A^n \to A$$

by induction on the number of operation symbols. $\tau(a_1, ..., a_t)$ is defined as:

1. $v_i^A = a_i$, *for each i=1,...,t*
2. $(\neg\sigma)^A = \neg\left(\sigma^A\right)$
3. $(\sigma \oplus \rho)^A = \left(\sigma^A \oplus \rho^A\right)$

2.3 Wajsberg Algebras

Definition 2.6. *A Wajsberg algebra (for short, a W-algebra) is a system $A = \langle A, \to, \neg, 1\rangle$, where A is a nonempty set, and the binary operation $\to$, the unary operation $\neg$ and the distinguished element 1 satisfy the following equations:*

W1. $1 \to x = x$

W2. $(x \to y) \to ((y \to z) \to (x \to z)) = 1$

W3. $(x \to y) \to y = (y \to x) \to x$

W4. $(\neg x \to \neg y) \to (y \to x) = 1$

Theorem 2.7. *Let $\langle A, \to, \neg, 1\rangle$ be a Wajsberg algebra. Upon defining $x \oplus y \overset{def}{=} \neg x \to y$ and $0 \overset{def}{=} \neg 1$, the system $\langle A, \oplus, \neg, 0\rangle$ is an MV-algebra.*

Proof can be found in Cignoli et.al*, p. 82-85*

Remark 2.8. *By the previous theorem, we can define the implication on any MV-Algebra as:*

$$x \to y \overset{def}{=} \neg x \oplus y$$

3 Introduction to Lukasiewicz Logic

3.1 Generalities, Truth Values and Connectives

Jan Lukasiewicz introduced in the early twenties a system of logic where the truth values for propositions are numbers between 0 and 1. It started as a $3 - valued$ logic with the truth values 1, 0 and a value for uncertainty, but later it became an $n-valued$ logic from which the $\infty-valued$ form emerged. In the $n-valued$ form the truth values are represented in the following manner:

$$0, \frac{1}{n-1}, ..., \frac{n-2}{n-1}, 1$$

We will define $\oplus$ on $[0,1]$ as

$$x \oplus y = min(1, x+y)$$

As in classical propositional logic, 1 stands for $truth$ and 0 for $false$. The values in between model the degree of uncertainty. The main propositional connectives are:

- negation ($\neg$)
- implication ($\rightarrow$)

Let $x, y \in [0,1]$ be two arbitrary fixed values. We define implication and negation in the following way:

L1. $\neg x \overset{def}{=} 1-x$

L2. $x \rightarrow y \overset{def}{=} \min(1, 1-x+y)$

3.2 Language

The language in Lukasiewicz Logic is similar to that of the normal propositional logic.

Definition 3.1. *Let V be the set of variables which is defined as follows:*

$$V = \{v_n | n \in \mathbb{N}\}$$

Definition 3.2. *Let Σ be the alphabet, which is defined as follows:*

$$\Sigma = V \cup \{(,), \rightarrow, \neg\}$$

Definition 3.3. *The definition of* **Form** *(the set of formulas) is given inductively, as follows:*

F1. Each $X \in V$ is a propositional formula

F2. If ϕ is a formula, then $\neg\phi$ is a propositional formula

F3. If ϕ and ψ are formulas, then $\phi \rightarrow \psi$ is a propositional formula

F4. Only those defined by $F1, F2, F3$ are propositional formulas

Definition 3.4. *If $\phi \in$ Form, then* $\mathbf{Var}(\phi)$ *denotes the set of all propositional variables present in ϕ.*

3.3 Semantics

Let $\mathscr{A}$ be a MV-Algebra. $\phi, \psi \in$ **Form**, two arbitrary formulas. Let's consider the following set of variables $V = \{v_0, v_1, v_2, ..., v_n\}$, where $n \in \mathbb{N}$.

Definition 3.5. *An $\mathscr{A}$-valuation is a function e : **Form** $\to \mathscr{A}$ satisfying the following properties:*

E1. $e(\neg\phi) \stackrel{def}{=} \neg e(\phi)$

E2. $e(\phi \to \psi) \stackrel{def}{=} e(\phi) \to e(\psi)$

Lemma 3.6 (Unique readability)**.** *Any $\mathscr{A}$-valuation is uniquely defined by the values in the set of variables V:*

$$e(v_0), e(v_1), e(v_2), ..., e(v_n)$$

Definition 3.7 ($\mathscr{A}$-satisfiability)**.** *We say that an $\mathscr{A}$-valuation e satisfies a formula ϕ if, and only if $e(\phi) = 1$*

Definition 3.8 (Tautologies)**.** *We say that a formula ϕ is an $\mathscr{A}$-tautology if, and only if ϕ is satisfied by every $\mathscr{A}$-valuation. In other words, we say that ϕ is an $\mathscr{A}$-tautology iff for any $\mathscr{A}$-valuation $e, e(\phi) = 1$.*

Definition 3.9 (Semantic Equivalence)**.** *Two formulas ϕ, ψ are called to be $\mathscr{A}$-semantic equivalent if, and only if $e(\phi) = e(\psi)$, for each $\mathscr{A}$-valuation e.*

Lemma 3.10. *Two formulas ϕ, ψ are $\mathscr{A}$-semantic equivalent, iff the following formulas are $\mathscr{A}$-tautologies:*

- $\phi \to \psi$
- $\psi \to \phi$

Proof. **Direct implication:** Let's suppose that both $\phi \to \psi$ and $\psi \to \phi$ are $\mathscr{A}$-tautologies. In other words:

$$e(\phi \to \psi) = e(\psi \to \phi) = 1$$

Since e is an evaluation, proposition *E2* holds, so we can affirm that

$$e(\phi) \to e(\psi) = e(\psi) \to e(\phi) = 1 \tag{1}$$

From (1) we get that $e(\phi) = e(\psi)$ which by applying Definition 3.9, it translates to ϕ and ψ are $\mathscr{A}$-semantic equivalent

Converse implication: Since the equality *E2* applies both from left to right as well as right to left, the proof for the converse implication is just the proof of the direct implication, backwards. $\square$

Definition 3.11 (semantic $\mathscr{A}$-consequence)**.** *Let $\Omega \subset$ **Form**, a set of formulas. We say that ϕ is a semantic $\mathscr{A}$-consequence iff each $\mathscr{A}$-valuation e that $\mathscr{A}$-satisfies all the formulas in Ω satisfies ϕ as well.*

Proposition 3.12. *Let **Var**(ϕ) $= \{v_{i_1}, v_{i_2}, ..., v_{i_k}\}$. For each $\mathscr{A}$-valuation we have:*

$$e(\phi) = \phi^{\mathscr{A}}(e(v_{i_1}), e(v_{i_2}), ..., e(v_{i_k})) \tag{2}$$

where $\phi^{\mathscr{A}} : \mathscr{A}^k \to \mathscr{A}$, is the term function described in 2.5

Notation 3.13. *For each $\Omega \subseteq$ **Form**, $\Omega^{\models}$ denotes the set of semantic consequences of Ω.*

Notation 3.14. *$\emptyset^{\models}$ denotes the set of all tautologies.*

3.4 Syntax and Provability

Let $\phi, \psi, \chi \in$ **Form**.

Definition 3.15 (Axioms)**.** *An axiom of the Lukasiewicz infinite-valued propositional calculus is a formula that can be written in any one of the following ways:*

A1. $\phi \to (\psi \to \phi)$

A2. $(\phi \to \psi) \to ((\psi \to \chi) \to (\phi \to \chi))$

A3. $((\phi \to \psi) \to \psi) \to ((\psi \to \phi) \to \phi)$

A4. $(\neg\phi \to \neg\psi) \to (\psi \to \phi)$

Definition 3.16 (Proof from a set)**.** *We call a proof from a set of formulas Ω, a finite string of formulas $\phi_1, \phi_2, ..\phi_n \in$ **Form**, where $n \geq 1$, which satisfies one of the following properties, for each i, with $1 \leq i \leq n$:*

P1. *ϕ_i is an axiom*

P2. $\phi_i \in \Omega$

P3. *There are $j, k \in \{1, ..., i-1\}$, in such way that ϕ_k coincides with the formula $(\phi_j \to \phi_i)$. Thus, ϕ_i follows by modus ponens from ϕ_j and ϕ_k*

Definition 3.17. *A formula ϕ is provable from Ω ($\Omega \vdash \phi$), iff there is a proof $\phi_1\phi_2...\phi_n$ from Ω, such that $\phi_n = \phi$.*

Definition 3.18. *By a proof we mean a proof from the empty set.*

Definition 3.19. *A provable formula is a formula that is provable from the empty set.*

Notation 3.20. *By applying the Definition 3.19, the notation for a provable formula ϕ is:*

$$\emptyset \vdash \phi$$

But we can exclude the set and simply write:

$$\vdash \phi$$

Proposition 3.21. *Let ϕ, ψ, χ be three formulas. Then, the following are true:*

$$\vdash (\phi \to (\psi \to \chi) \to (\psi \to (\phi \to \chi))) \tag{3}$$

$$\vdash \phi \to \phi \tag{4}$$

$$\vdash (\psi \to \chi) \to ((\phi \to \psi) \to (\phi \to \chi)) \tag{5}$$

$$\vdash \neg\neg\phi \to (\psi \to \phi) \tag{6}$$

$$\vdash \neg\neg\phi \to \phi \tag{7}$$

$$\vdash (\phi \to \neg\psi) \to (\psi \to \neg\phi) \tag{8}$$

$$\vdash (\phi \to \neg\neg\psi) \to (\neg\psi \to \neg\phi) \tag{9}$$

$$\vdash \phi \to \neg\neg\phi \tag{10}$$

$$\text{If } \vdash \phi \text{ then, for each } \psi, \vdash \psi \to \phi \tag{11}$$

The proof for each, can be found in Cignoli et. al, p.89-92

3.5 Syntactic Equivalence Classes and Lindenbaum Algebra

Definition 3.22. *Let the binary relation $\equiv$ on **Form** be defined by $\phi \equiv \psi$ iff $\vdash \phi \to \psi$ and $\vdash \psi \to \phi$. $\equiv$ is an equivalence relation called the syntactic equivalence.*

Proposition 3.23. *The relation $\equiv$ satisfies the following:*

- *If $\phi \equiv \chi$ and $\psi \equiv \delta$ then $(\phi \to \psi) \equiv (\chi \to \delta)$*
- *If $\phi \equiv \psi$, then $\neg\phi \equiv \neg\psi$*

Proof can be found in Cignoli et. al, p.92-93

Definition 3.24 (syntactic equivalence class)**.** *For the relation $\equiv$ we would define the semantic equivalence class of $\phi \in Form$ as follows:*

$$|\phi| \overset{def}{=} \{\psi \in Form | \psi \equiv \phi\}$$

Lemma 3.25. *For each formula ϕ, $\phi \in \emptyset^{\vdash}$ iff $|\phi| = \emptyset^{\vdash}$.*

Proof. To prove the equivalence, we will prove the implication from left to right, and then from right to left.
Direct implication: Let's suppose that $|\phi| = \emptyset^{\vdash}$ and we want to prove that

$\phi \in \emptyset^{\vdash}$. It's clear that $\phi \in |\phi|$, so $\phi \in \emptyset^{\vdash}$.

Converse implication: Let's suppose that $\phi \in \emptyset^{\vdash}$ and we want to prove that $|\phi| = \emptyset^{\vdash}$. The proof is by *Reductio ab absurdum.* We assume the opposite of hour hypothesis, just to prove it wrong:

$$|\phi| \neq \emptyset^{\vdash}$$

Let ψ be a formula. Two cases emerge:

- $\psi \in \emptyset^{\vdash}$. By (11) we get that $\psi \equiv \phi$, which means that: $\emptyset^{\vdash} \subseteq |\phi|$.
- $\psi \in |\phi|$. Hence, $\vdash (\phi \to \psi)$. Considering the fact that $\vdash \phi$, we apply *Modus Ponens* to get that $\vdash \psi$, from which it immediately results that $|\phi| \subseteq \emptyset^{\vdash}$.

By combining the results of both cases we can conclude that $|\phi| = \emptyset^{\vdash}$, which contradicts the hypothesis of *Reductio ab absurdum.* Hence, the *Converse implication* is true.

□

Proposition 3.26. *The quotient set Form/≡: becomes a Wajsberg algebra, once equipped with the operations* $\neg$ *and* $\to$ *and the constant 1 as given by the following stipulations:*

C1. $|\phi| \to |\psi| \stackrel{def}{=} |\phi \to \psi|$

C2. $\neg|\phi| \stackrel{def}{=} |\neg\phi|$

C3. $1 \stackrel{def}{=} \emptyset^{\vdash}$

Proof. *C2* and *C3* yield well defined from Proposition 3.23. From Lemma 3.25 we know that $|\phi| = \emptyset^{\vdash}$, and since it is clear that $|\phi| \in \mathbf{Form}/\equiv$, it results that:

$$\emptyset^{\vdash} \in \mathbf{Form}/\equiv$$

By *C3* we get that:

$$1 \to |\phi| = \emptyset^{\vdash} \to |\phi| = |\psi \to \phi| \tag{12}$$

where $\psi \in \emptyset^{\vdash}$ We can apply axiom *A1* to obtain that:

$$\vdash \phi \to (\psi \to \phi)$$

By axiom *A3* we have:

$$(\psi \to \phi) \to \phi \equiv (\phi \to \psi) \to \psi$$

It follows from (11) that:

$$\vdash (\phi \to \psi) \to \psi$$

By Lemma 3.25 we get that $\vdash (\psi \to \phi) \to \phi$, so $\psi \to \phi \equiv \phi$

If we come back to (12) we can conclude that:

$$1 \to |\phi| = |\psi \to \phi| = |\phi|$$

Meaning that the quotient set satisfies *W1*

From *A2* we know that:

$$\vdash (\phi \to \psi) \to ((\psi \to \chi) \to (\phi \to \chi))$$

And by Lemma 3.25 we can can conclude that *W2* holds.

Similarly, *W3* and *W4* follow from the axioms *A3* and *A4* Since W1-W4 hold, **Form**/$\equiv$ is a Wajsberg Algebra. □

The immediate consequence of this proposition together with Theorem 2.7 is that the tuple $\langle \textbf{\textit{Form}}/\equiv, \oplus, \neg, 0\rangle$ *is a MV-Algebra.*

Definition 3.27 (Lindenbaum Algebra)**.** *The MV-algebra*

$$\mathscr{L} \stackrel{def}{=} \langle Form/\equiv, 0, \neg, \oplus\rangle$$

is called the Lindenbaum algebra of Lukasiewicz ∞*-valued calculus*

4 Useful Results

Theorem 4.1 (Chang's Completeness Theorem)**.** *An equation holds in [0,1] if and only if it holds in every MV-algebra.*

The proof can be found in *Cignoli et. al*, p.44-p48

Remark 4.2. *Our topic being the* ∞*-valued form of Lukasiewicz Logic, we should consider that the terms defined as* valuation, tautology, semantic equivalence, etc. *are related to the default MV-Algebra* $[0, 1]$ *because, by Chang Completeness Theorem, any eqution in [0,1] holds in any MV-Algebra.*

Notation 4.3. *Our topic being the* ∞*-valued form of Lukasiewicz Logic, we should consider that the terms defined in this section as* valuation, tautology, semantic equivalence, etc. *are related to the default MV-Algebra* $[0, 1]$.

Proposition 4.4. *For each* $\phi \in [0, 1]$ *we have that* $\phi \to \phi = 1$*. In other words* $\phi \to \phi$ *is a tautology.*

Proof. By *L2* we have that:

$$\phi \to \phi = min(1, 1 - \phi + \phi) = min(1, 1) = 1$$

□

Proposition 4.5. *All axioms are tautologies.*

Proof. By Chang's Completeness Theorem, it is enough to prove that the axioms are [0,1]-Tautologies, in order to prove that they are tautologies. Let ϕ, ψ, and χ, be three arbitrary formulas, and $e : \mathbf{Form} \to [0,1]$, an arbitrary valuation. We would prove that for each axiom A, $e(A) = 1$

A1. $\phi \to (\psi \to \phi)$

From *E2* we know that:

$$\begin{aligned} e(\phi \to (\psi \to \phi)) = e(\phi \to (\psi \to \phi)) &= e(\phi) \to e(\psi \to \phi)) \\ &= e(\phi) \to (e(\psi) \to e(\phi)) \end{aligned}$$

Moreover, by applying *L2*, we would get the following:

$$e(\phi \to (\psi \to \phi)) = min(1, 1 - e(\phi) + min(1, 1 - e(\psi) + e(\phi))) \tag{13}$$

We have two cases:

- $e(\phi) \geq e(\psi)$. Then, since $1 - e(\psi) + e(\phi) \geq 1$, we have that:

  $$min(1, 1 - e(\psi) + e(\phi)) = 1$$

  Considering this, relation (13) becomes:

  $$e(\phi \to (\psi \to \phi)) = min(1, 1 - e(\phi) + 1) = min(1, 2 - e(\phi))$$

  Since $e(\phi) \in [0,1]$, we have that $2 - e(\phi) \geq 1$, from which, it follows that:

  $$e(\phi \to (\psi \to \phi)) = 1$$

- $e(\psi) > e(\phi)$. Then, since $1 - e(\psi) + e(\phi) < 1$, we have that:

  $$min(1, 1 - e(\psi) + e(\phi)) = 1 - e(\psi) + e(\phi)$$

  Considering this, relation (13) becomes:

  $$e(\phi \to (\psi \to \phi)) = min(1, 1 - e(\phi) + 1 - e(\psi) + e(\phi)) = min(1, 2 - e(\psi))$$

  Since $e(\psi) \in [0,1]$, we have that $2 - e(\psi) > 1$, from which, it follows that:

  $$e(\phi \to (\psi \to \phi)) = 1$$

In both cases the conclusion was that $e(\phi \to (\psi \to \phi)) = 1$, and since ϕ, ψ and e, are all arbitrary, we can affirm that the last conclusion follows for all valuations, so, by Definition 3.8, we get that $\phi \to (\psi \to \phi)$ is a tautology.

A2. $(\phi \to \psi) \to ((\psi \to \chi) \to (\phi \to \chi))$

From *E2* we know that:

$$\begin{aligned} e((\phi \to \psi) \to ((\psi \to \chi) \to (\phi \to \chi))) &= e(\phi \to \psi) \to e((\psi \to \chi) \to (\phi \to \chi)) \\ &= (e(\phi) \to e(\psi)) \to ((e(\psi) \to e(\chi)) \to (e(\phi) \to e(\chi))) \end{aligned} \tag{14}$$

By applying *L2*, we would get:

$$e(\phi) \to e(\psi) = min(1, 1 - e(\phi) + e(\psi))$$

$$e(\psi) \to e(\chi) = min(1, 1 - e(\psi) + e(\chi))$$

$$e(\phi) \to e(\chi) = min(1, 1 - e(\phi) + e(\chi))$$

As in the proof for *A1*, we would have to analyse each case for the order of $e(\phi), e(\psi)$ and $e(\chi)$. The cases look as follows:

- $e(\phi) \le e(\psi) \le e(\chi)$. Since $1 - e(\phi) + e(\psi) \ge 1, 1 - e(\psi) + e(\chi) \ge 1$ and $1 - e(\phi) + e(\chi) \ge 1$, it follows that:

  $$e(\phi) \to e(\psi) = 1$$

  $$e(\psi) \to e(\chi) = 1$$

  $$e(\phi) \to e(\chi) = 1$$

  We can rewrite (14) as:

  $$e((\phi \to \psi) \to ((\psi \to \chi) \to (\phi \to \chi))) = 1 \to (1 \to 1) = 1$$

- $e(\phi) \le e(\chi) \le e(\psi)$. Since $1 - e(\phi) + e(\psi) \ge 1, 1 - e(\psi) + e(\chi) \le 1$, and $1 - e(\phi) + e(\chi) \ge 1$, it follows that:

  $$e(\phi) \to e(\psi) = 1$$

  $$e(\psi) \to e(\chi) = 1 - e(\psi) + e(\chi)$$

  $$e(\phi) \to e(\chi) = 1$$

  We can rewrite (14) as:

  $$e((\phi \to \psi) \to ((\psi \to \chi) \to (\phi \to \chi))) = 1 \to ((1 - e(\psi) + e(\chi)) \to 1)$$

  We can apply *L2*, once again, in the last relation to obtain that:

  $$\begin{aligned} &e((\phi \to \psi) \to ((\psi \to \chi) \to (\phi \to \chi))) \\ &= 1 \to min(1, 1 - 1 + e(\psi) - e(\chi) + 1) = 1 \to min(1, e(\psi) - e(\chi) + 1) \end{aligned}$$

  Since $e(\psi) \ge e(\chi)$, our relation becomes:

  $$e((\phi \to \psi) \to ((\psi \to \chi) \to (\phi \to \chi))) = 1 \to 1 = 1$$

- $e(\psi) \leq e(\phi) \leq e(\chi)$. Since $1 - e(\phi) + e(\psi) \leq 1, 1 - e(\psi) + e(\chi) \geq 1$, and $1 - e(\phi) + e(\chi) \geq 1$, it follows that:

$$e(\phi) \rightarrow e(\psi) = 1 - e(\phi) + e(\psi)$$

$$e(\psi) \rightarrow e(\chi) = 1$$
$$e(\phi) \rightarrow e(\chi) = 1$$

We can rewrite (14) as:

$$\begin{aligned} e((\phi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\phi \rightarrow \chi))) &= (1 - e(\phi) + e(\psi)) \rightarrow (1 \rightarrow 1) \\ &= (1 - e(\phi) + e(\psi)) \rightarrow 1 \end{aligned}$$

If we apply *L2*, once again, we get that:

$$\begin{aligned} e((\phi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\phi \rightarrow \chi))) &= min(1, 1 - 1 + e(\phi) - e(\psi) + 1) \\ &= min(1, 1 + e(\phi) - e(\psi)) \end{aligned}$$

Since $e(\phi) \geq e(\psi)$ our relation becomes:

$$e((\phi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\phi \rightarrow \chi))) = 1$$

- $e(\psi) \leq e(\chi) \leq e(\phi)$. Since $1 - e(\phi) + e(\psi) \leq 1, 1 - e(\psi) + e(\chi) \geq 1$, and $1 - e(\phi) + e(\chi) \leq 1$, it follows that:

$$e(\phi) \rightarrow e(\psi) = 1 - e(\phi) + e(\psi)$$

$$e(\psi) \rightarrow e(\chi) = 1$$
$$e(\phi) \rightarrow e(\chi) = 1 - e(\phi) + e(\chi)$$

We can rewrite (14) as:

$$e((\phi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\phi \rightarrow \chi))) = (1 - e(\phi) + e(\psi)) \rightarrow (1 \rightarrow (1 - e(\phi) + e(\chi)))$$

If we apply *L2*, once again, we get that:

$$\begin{aligned} &e((\phi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\phi \rightarrow \chi))) \\ &= (1 - e(\phi) + e(\psi)) \rightarrow min(1, 1 - 1 + 1 - e(\phi) + e(\chi)) \\ &= (1 - e(\phi) + e(\psi)) \rightarrow min(1, 1 - e(\phi) + e(\chi)) \end{aligned}$$

Since $e(\phi) \geq e(\chi)$ our relation becomes:

$$\begin{aligned} &e((\phi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\phi \rightarrow \chi))) \\ &= (1 - e(\phi) + e(\psi)) \rightarrow (1 - e(\phi) + e(\chi)) \\ &= min(1, 1 - 1 + e(\phi) - e(\psi) + 1 - e(\phi) + e(\chi)) = min(1, 1 - e(\psi) + e(\chi)) \end{aligned}$$

Since $e(\chi) > e(\psi)$, our relation becomes:

$$e((\phi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\phi \rightarrow \chi))) = 1$$

- $e(\chi) \leq e(\phi) \leq e(\psi)$. Since $1 - e(\phi) + e(\psi) \geq 1, 1 - e(\psi) + e(\chi) \leq 1$ and $1 - e(\phi) + e(\chi) \leq 1$, it follows that

$$e(\phi) \rightarrow e(\psi) = 1$$

$$e(\psi) \rightarrow e(\chi) = 1 - e(\psi) + e(\chi)$$

$$e(\phi) \rightarrow e(\chi) = 1 - e(\phi) + e(\chi)$$

We can rewrite (14) as:

$$e((\phi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\phi \rightarrow \chi))) = 1 \rightarrow ((1 - e(\psi) + e(\chi)) \rightarrow (1 - e(\phi) + e(\chi)))$$

If we apply *L2*, once again, we get that:

$$\begin{aligned} &e((\phi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\phi \rightarrow \chi))) \\ = 1 \rightarrow &min(1, 1 - 1 + e(\psi) - e(\chi) + 1 - e(\phi) + e(\chi)) \\ &= 1 \rightarrow min(1, 1 - e(\phi) + e(\psi)) \end{aligned}$$

Since $e(\phi) \leq e(\psi)$, it follows that:

$$e((\phi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\phi \rightarrow \chi))) = 1 \rightarrow 1 = 1$$

- $e(\chi) \leq e(\psi) \leq e(\phi)$. Since $1 - e(\phi) + e(\psi) \leq 1, 1 - e(\psi) + e(\chi) \leq 1$ and $1 - e(\phi) + e(\chi) \leq 1$, it follows that

$$e(\phi) \rightarrow e(\psi) = 1 - e(\phi) + e(\psi)$$

$$e(\psi) \rightarrow e(\chi) = 1 - e(\psi) + e(\chi)$$

$$e(\phi) \rightarrow e(\chi) = 1 - e(\phi) + e(\chi)$$

We can rewrite (14) as:

$$\begin{aligned} &e((\phi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\phi \rightarrow \chi))) \\ = (&e(\phi \rightarrow \psi)) \rightarrow ((1 - e(\psi) + e(\chi)) \rightarrow (1 - e(\phi) + e(\chi))) \end{aligned}$$

If we apply *L2*, once again, we get that:

$$\begin{aligned} &e((\phi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\phi \rightarrow \chi))) \\ &= e(\phi \rightarrow \psi) \rightarrow min(1, 1 - 1 + e(\psi) - e(\chi) + 1 - e(\phi) + e(\chi)) \\ &= e(\phi \rightarrow \psi) \rightarrow min(1, 1 - e(\phi) + e(\psi)) = e(\phi \rightarrow \psi) \rightarrow e(\phi \rightarrow \psi) \end{aligned}$$

By Proposition 4.4, it follows that:

$$e((\phi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\phi \rightarrow \chi))) = 1 \rightarrow 1 = 1$$

In each case the final result was:

$$e((\phi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\phi \rightarrow \chi))) = 1$$

By Definition 3.8, this means that $(\phi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\phi \rightarrow \chi))$ is a tautology.

A3. $((\phi \to \psi) \to \psi) \to ((\psi \to \phi) \to \phi)$
From *E2* we have that:

$$e(((\phi \to \psi) \to \psi) \to ((\psi \to \phi) \to \phi)) = e((\phi \to \psi) \to \psi) \to e((\psi \to \phi) \to \phi)$$

We can apply *L2* to get that:

$$e((\phi \to \psi) \to \psi) = min(1, 1 - e(\phi) + e(\psi)) \to e(\psi) \tag{15}$$

(15) will change according to the following cases:

- $e(\phi) \leq e(\psi)$. It follows that:

$$e((\phi \to \psi) \to \psi) = 1 \to e(\psi) = min(1, e(\psi)) = e(\psi)$$

- $e(\phi) > e(\psi)$. It follows that:

$$\begin{aligned} e((\phi \to \psi) \to \psi) &= (1 - e(\phi) + e(\psi)) \to e(\psi) \\ &= min(1, 1 - 1 + e(\phi) - e(\psi) + e(\psi)) = min(1, e(\phi)) = e(\phi) \end{aligned}$$

Analogously, for $e((\psi \to \phi) \to \phi))$, we will get the cases:

- $e(\phi) \leq e(\psi)$. It follows that:

$$e((\psi \to \phi) \to \phi)) = e(\psi)$$

- $e(\phi) > e(\psi))$. It follows that:

$$e((\psi \to \phi) \to \phi)) = e(\phi)$$

By merging all the cases get that:

- If $e(\phi) \leq e(\psi)$, then, by Proposition 4.5:

$$e(((\phi \to \psi) \to \psi) \to ((\psi \to \phi) \to \phi)) = e(\psi) \to e(\psi) = 1$$

- If $e(\phi) > e(\psi)$, then, by Proposition 4.5:

$$e(((\phi \to \psi) \to \psi) \to ((\psi \to \phi) \to \phi)) = e(\phi) \to e(\phi) = 1$$

A4. $(\neg\phi \to \neg\psi) \to (\psi \to \phi)$
From *E1* and *E2* we have that:

$$\begin{aligned} e((\neg\phi \to \neg\psi) \to (\psi \to \phi)) &= (e(\neg\phi \to \neg\psi) \to e(\psi \to \phi)) \\ = (e(\neg\phi) \to e(\neg\psi))) \to (e(\psi) \to e(\phi))) &= (\neg e(\phi) \to \neg e(\psi)) \to (e(\psi) \to e(\phi))) \end{aligned}$$

Now we can apply *L1* and *L2*:

$$\begin{aligned} \neg e(\phi) \to \neg e(\psi) = (1 - e(\phi)) \to (1 - e(\psi)) &= min(1, 1 - 1 + e(\phi) + 1 - e(\psi)) \\ &= min(1, 1 - e(\psi) + e(\phi)) = e(\psi) \to e(\phi) \end{aligned}$$

So we can say that:

$$e((\neg\phi \to \neg\psi) \to (\psi \to \phi)) = (e(\psi) \to e(\phi)) \to (e(\psi) \to e(\phi))$$

By Proposition 4.4, we can conclude that:

$$e((\neg\phi \to \neg\psi) \to (\psi \to \phi)) = 1$$

By Definition 3.8, this means that $(\neg\phi \to \neg\psi) \to (\psi \to \phi)$ is a tautology.

$\square$

Lemma 4.6. *Each formula provable from a set Ω of formulas is also a semantic consequence of this set. In symbols:*

$$\Omega^{\vdash} \subseteq \Omega^{\models}$$

Proof. Let $e : \mathbf{Form} \to [0,1]$ be a valuation with $e(\phi) = 1$ for all $\phi \in \Omega$.
We begin by proving that $\phi_1, \phi_2, ..\phi_n$ being a proof implies that $e(\phi_n) = 1$
The proof is by induction on n.

Base Case: $n = 1$.
Since ϕ_1 is a proof (by hypothesis), from Definition 3.16 two cases emerge:

- ϕ_1 is an axiom. Then, by Proposition 4.5, ϕ_1 is a tautology. From Definition 3.8 we get that $e(\phi_1) = 1$
- $\phi_1 \in \Omega$. From the hypothesis it follows that $e(\phi_1) = 1$

Induction hypothesis: Let $n > 1$ and for each proof from Ω, $\psi_1, \psi_2, ..\psi_n$, where $m < n$ we have $e(\psi_m) = 1$

Induction phase: Let $n > 1$, and $\phi_1, \phi_2, ..\phi_n$ be a proof from Ω. By Definition 3.16, three cases emerge:

- ϕ_n is an axiom. Then, by Proposition 4.5, ϕ_n is a tautology. From Definition 3.8 we get that $e(\phi_n) = 1$
- $\phi_n \in \Omega$. From the hypothesis it follows that $e(\phi_n) = 1$
- There exist $i, j \in \{1, ..., n-1\}$, such that ϕ_j coincides with the formula $(\phi_i \to \phi_n)$. By applying *E2*, we get that:

$$e(\phi_j) = e(\phi_i) \to e(\phi_n) \tag{16}$$

  Since $\phi_1, \phi_2, ..\phi_n$ is a proof from Ω and $i, j < n$, we have that both $\phi_1, \phi_2, ..\phi_i$ and $\phi_1, \phi_2, ..\phi_j$ are proofs from Ω. By applying the induction hypothesis we get that:

$$e(\phi_i) = e(\phi_j) = 1$$

Now we can rewrite relation (16) as:

$$1 = 1 \to e(\phi_n)$$

From *L2*, we get that:

$$1 = min(1, 1 - 1 + e(\phi_n)) = min(1, e(\phi_n)) = e(\phi_n)$$

□

5 Completeness Theorem

Lemma 5.1. *Let ϕ be a formula with $Var(\phi) \subset \{v_{i_1}, v_{i_2}, ..., v_{i_n}\}$, and $\mathscr{L}$ a Lindenbaum algebra. Then*

$$\phi^{\mathscr{L}}\left(|v_{i_1}|, |v_{i_2}|, ..., |v_{i_n}|\right) = |\phi|.$$

Proof. The proof is by induction on formulas. We have the following cases:

- ϕ is a variable, so $\phi = v_{i_j}$ for some $j = 1, \ldots, n$. Then $|\phi| = |v_{i_j}|$, and $\phi^{\mathscr{L}}\left(|v_{i_j}|\right) = |v_{i_j}|$.
- $\phi = \neg\psi$. From the induction hypothesis for ψ, we have that

  $$|\psi| = \psi^{\mathscr{L}}\left(|v_{i_1}|, |v_{i_2}|, ..., |v_{i_n}|\right).$$

  It follows that

  $$\begin{aligned}|\phi| = \neg|\psi| &= \neg\psi^{\mathscr{L}}\left(|v_{i_1}|, |v_{i_2}|, ..., |v_{i_n}|\right)\\ &= \phi^{\mathscr{L}}\left(|v_{i_1}|, |v_{i_2}|, ..., |v_{i_n}|\right).\end{aligned}$$

- $\phi = \psi \to \chi$. $|\phi| = |\psi \to \chi|$. From *C1* we obtain that $|\phi| = |\psi| \to |\chi|$ and by applying the induction hypothesis $\phi^{\mathscr{L}}\left(|v_{i_1}|, |v_{i_2}|, ..., |v_{i_n}|\right) = |\phi|$

□

Proposition 5.2. *Every tautology is provable. Thus, tautologies coincide with provable formulas. In symbols:*

$$\emptyset^{\models} = \emptyset^{\vdash}$$

Proof. Let ϕ be a formula with $\mathrm{Var}(\phi) \subset \{v_{i_1}, v_{i_2}, ..., v_{i_n}\}$. By Lemma 5.1, we have that

$$\phi^{\mathscr{L}}\left(|v_{i_1}|, |v_{i_2}|, ..., |v_{i_n}|\right) = |\phi|. \tag{17}$$

Let's suppose ϕ is not provable. By applying (17) and Lemma 3.25 and we get that that $|\phi| \neq 1$, so :

$$\phi^{\mathscr{L}}\left(|v_{i_1}|, |v_{i_2}|, ..., |v_{i_n}|\right) \neq 1$$

This means that $\mathscr{L}$ does not satisfy the equation $\phi = 1$. From Theorem 4.1 (Chang's Completeness Theorem) it results that the MV-Algebra $[0, 1]$ does not satisfy the equation $\phi = 1$, meaning that ϕ is not a tautology.

- We have so far proved that each formula that is not provable is not a tautology.
- From Lemma 4.6, in particular, we know that *all provable formulas are tautologies.*

By combing the last two results we can conclude that all tautologies are provable, meaning that tautologies coincide with provable formulas.

□