- The sample space S of an experiment E is the set of all possible outcomes s of the experiment E
- An event A is a subset of the sample space S, and we say that event A occurred if the actual outcome s_{actual} is in event A

Intro

Definitions
- Experiment E : all pebbles in the garden, perfectly round ones
- Sample space S : pebbles in the garden, the super set
- A possible outcome s (s $\in$ S) : a pebble
- Actual outcome s_{actual} : a perfectly round pebble
- Event A (A $\subseteq$ S) : all blue pebbles in the garden,
- Event A occurred ($s_{actual} \in$ A) : the perfectly round pebble s_{actual} is also blue
- Event $\emptyset$: all pebbles in the garden that can fly
- Something must happen ($s_{actual} \in$ S) : there exist a perfectly round pebble s_{actual} in the sample space S
- Event A or event B (inclusive) (A $\cup$ B) : pebbles that are either blue or square shaped (event B)
- Event A or event B (inclusive) (A $\cap$ B) : blue and square-shaped pebbles
- Event not A (A^c) : pebbles that are not blue
- Event C implies event A (A $\subseteq$ C) : example if event C is the event that a pebble is blue and the its perimeter is larger than 7cm
- Event D or event B mutually exclusive (A $\cap$ B $= \emptyset$) : event D for triangular pebbles
- A_1 , ... , A_n are a partition of the sample space S
($A_1 \cup ... \cup A_n =$ S) , $A_i \cup A_j = \emptyset$ for $i \neq j$: let each A_k a pebble of a different shape

Naive probability $P_{naive}(A) = \frac{|A|}{|S|}$ where $|A|$ cardinality of event A

Binomial Coefficient : $\binom{n}{k}$

——Binomial Formula :

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

—–The Team Captain :

$$n\binom{n}{k} = k\binom{n-1}{k-1}$$

—–Vandermonde's identity :

$$\binom{m+n}{k} = \sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i}$$

—–General Definition of Probability A probability space consists of: - sample space S - probability function P - function P input: an event A such that event $A \subseteq S$ - function P output: P(A) is the output where $P(A) \in [0,1]$

—– 1) $P(\emptyset) = 0$, $P(S) = 1$

——— 2) if A_1, A_2 , ... are disjoint events ($A_i \cap A_j = \emptyset for i \neq j$) then

$$P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$$

——— 3) properties of probability:

- P(A^c) = 1 - P(A)
- $A \subseteq B \Rightarrow P(A) \leq P(B)$
- $P(A \cup B)$ = P(A) + P(B) - $P(A \cap B)$

A and A^c disjoint hence
1) $P(S) = P(A \cup A^c) = P(A) + P(A^c)$
—
— let $A \subseteq B$,
— then A and (B - A) disjoint,
— then:
2) $P(B) = P(A \cup (B - A)) = P(A \cup (B \cap A^c)) = P(A) + P(B \cap A^c)$
3) $(B \cap A) \cup (B \cap A^c) = B$
4) $P(B \cap A) + P(B \cap A^c) = P(B)$
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Inclusion Exclusion

5) $P(A \cup B \cup C) = P(A) + P(B) + P(C)$

________________ $-P(A \cap B) - P(B \cap C) - P(A \cap C)$

________________ $+P(A \cap B \cap C)$

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Inclusion Exclusion

6) for any event A_1, A_2, ..., A_n

$$P(\bigcup_{i=1}^{\infty} A_i) =$$

____________________________ $\sum_i$ P(A_i) -

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____________________________ $\sum_{i<j}$ P$(A_i \cap A_j)$ +

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____________________________ $\sum_{i<j<k}$ P$(A_i \cap A_j \cap A_k)$ -

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____________________________ ... +

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____________________________ $(-1)^{n+1}$ P$(A_1 \cap ... \cap A_n)$

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— Example: Consider a well-shuffled deck of n cards, labeled 1 through n. You flip over the cards one by one, saying the numbers 1 through n as you do so. You win the game if, at some point, the number you say aloud is the same as the number on the card being flipped over (for example, if the 7th card in the deck has the label 7). What is the probability of winning?

Solution: Let A_i be the event that the ith card in the deck has the number i written on it. We are interested in the probability of the union $A_1 \cup A_2 \cup ... \cup A_n$

$$P(\bigcup_{i=1}^{n} A_i) = (-1)^2 \frac{\binom{n}{1}}{n} + (-1)^3 \frac{\binom{n}{2}}{n(n-1)} + (-1)^4 \frac{\binom{n}{3}}{n(n-1)(n-2)} + ... + (-1)^{n+1} \frac{\binom{n}{n}}{n!}$$

$$\Rightarrow \quad P(\bigcup_{i=1}^{n} A_i) = 1 - \frac{1}{2!} + \frac{1}{3!} - \ ... \ + \ (-1)^{n+1} \frac{1}{n!}$$

$$and \qquad \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ ... = e^{-1} \qquad \Rightarrow \qquad P(\bigcup_{i=1}^{\infty} A_i) = 1 - e^{-1}$$