ELECENG 3CL4 Lab 2 Pre-lab

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1 Description of Laboratory Equipment

Pre-lab Question 1
The step response of the model $G(s)$ is in Equation 1. The derivation of the step response is shown below.

$$\begin{aligned}
\mathscr{L}^{-1}\left\{G(s)\frac{1}{s}\right\} &= \mathscr{L}^{-1}\left\{\frac{A}{s(s\tau_m+1)}\frac{1}{s}\right\} \\
&= A\ \mathscr{L}^{-1}\left\{\frac{1}{s^2(s\tau_m+1)}\right\} \\
&= A\ \mathscr{L}^{-1}\left\{\frac{{\tau_m}^2}{s\tau_m+1} - \frac{\tau_m}{s} + \frac{1}{s^2}\right\} \\
&= A\left[\mathscr{L}^{-1}\left\{\frac{{\tau_m}^2}{s\tau_m+1}\right\} - \mathscr{L}^{-1}\left\{\frac{\tau_m}{s}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\}\right] \\
&= A\left[\tau_m e^{-\frac{t}{\tau_m}} - \tau_m u(t) + t\right] \qquad (1)
\end{aligned}$$

Pre-lab Question 2
The step response is not bounded because the term t is not bounded.

2 Closed Loop System Identification

Pre-lab Question 3

(i) The output signal of the model can be expressed as:

$$\begin{aligned}
Y(s) = \frac{G_c(s)G(s)}{1+H(s)G_c(s)G(s)}R(s) + \frac{G(s)}{1+H(s)G_c(s)G(s)}T_d(s) \\
+ \frac{H(s)G_c(s)G(s)}{1+H(s)G_c(s)G(s)}N(s)
\end{aligned}$$

If we neglect the effects of the disturbance $T_d(s)$ and the noise $N(s)$, we are left with:

$$Y(s) = \frac{G_c(s)G(s)}{1+H(s)G_c(s)G(s)}R(s)$$

When $H(s) = 1$ and we let $\Theta(s) = Y(s)$, we can find the transfer function $T(s)$ in Equation 2 (idk why this isn't working), which is equal to Equation 4a from the lab document.

$$\begin{aligned}
T(s) &= \frac{\Theta(s)}{R(s)} \\
&= \frac{\frac{G_c(s)G(s)}{1+H(s)G_c(s)G(s)}R(s)}{R(s)} \\
&= \frac{G(s)G_c(s)}{1+G(s)G_c(s)} \qquad (2)
\end{aligned}$$

(ii) When we substitute $G(s) = \frac{A}{s(s\tau_m+1)}$ and $G_c(s) = K$ into Equation 2, and then substituting $\sqrt{\frac{KA}{\tau_m}} = \omega_n$ and $\frac{1}{2\omega_n\tau_m}$ into the result, we can derive Equation 3 which is equal to Equation 4b from the lab document:

$$\begin{aligned}
T(s) &= \frac{G(s)G_c(s)}{1 + G(s)G_c(s)} \\
&= \frac{\frac{A}{s(s\tau_m+1)}K}{1 + \frac{A}{s(s\tau_m+1)}K} \\
&= \frac{KA}{s(s\tau_m + 1)(1 + \frac{A}{s(s\tau_m+1)}K)} \\
&= \frac{KA}{s(s\tau_m + 1 + \frac{KA}{s})} \\
&= \frac{KA}{s^2\tau_m + s + KA} \\
&= \frac{KA/\tau_m}{s^2 + (1/\tau_m)s + KA/\tau_m} \\
&= \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}
\end{aligned} \tag{3}$$

Pre-lab Question 4
When given ζ and ω_n, we can derive Equation 4 for τ_m:

$$\begin{aligned}
\zeta &= \frac{1}{2\omega_n\tau_m} \\
\tau_m &= \frac{1}{2\omega_n\zeta}
\end{aligned} \tag{4}$$

By substituting Equation 4 into the equation for ω_n, we can derive Equation 5 for A:

$$\begin{aligned}
\omega_n &= \sqrt{\frac{KA}{\tau_m}} \\
\omega_n^2 &= \frac{KA}{\tau_m} \\
A &= \frac{\omega_n^2\tau_m}{K} \\
A &= \frac{\omega_n/2\zeta}{K}
\end{aligned} \tag{5}$$

2.1 Closed-loop System Identification from the Step Response

Pre-lab Question 5
The percent overshoot is determined at the peak time, T_p, of the step response. For the provided step response, this peak time occurs the first time $d\theta_{\text{step}}(t)/dt = 0$, which occurs

when $\omega_n\sqrt{1-\zeta^2}t = \pi$. Therefore, the peak time T_p is equal to $\frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$. At the peak time, the peak value of the step response is $1+\exp\left(-\zeta\pi/\sqrt{1-\zeta^2}\right)$. Therefore, the percent overshoot can be defined as:

$$\text{P.O.} = 100\frac{(1+\exp^{-(\zeta\pi/\sqrt{1-\zeta^2})})-1}{1} = 100\exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right) \tag{6}$$

Pre-lab Question 6

As discussed in the solution to Pre-lab Question 5, the peak time occurs the first time $d\theta_{\text{step}}(t)/dt = 0$, which occurs when $\omega_n\sqrt{1-\zeta^2}t = \pi$. Therefore, the peak time T_p is equal to $\frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$.

Pre-lab Question 7

Using the equation for percent overshoot derived in Pre-lab Question 5, you can calculate ζ as:

$$\begin{aligned}
P.O. &= 100\exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right)\\
\frac{P.O.}{100} &= \exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right)\\
\ln\left(\frac{P.O.}{100}\right) &= \frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\\
\ln^2\left(\frac{P.O.}{100}\right) &= \frac{\zeta^2\pi^2}{1-\zeta^2}\\
\ln^2\left(\frac{P.O.}{100}\right) - \zeta^2\ln^2\left(\frac{P.O.}{100}\right) &= \zeta^2\pi^2\\
\ln^2\left(\frac{P.O.}{100}\right) &= \zeta^2\pi^2 + \zeta^2\ln^2\left(\frac{P.O.}{100}\right)\\
\ln^2\left(\frac{P.O.}{100}\right) &= \zeta^2\left(\pi^2 + \ln^2\left(\frac{P.O.}{100}\right)\right)\\
\zeta &= \frac{\ln\left(\frac{P.O.}{100}\right)}{\sqrt{\pi^2+\ln^2\left(\frac{P.O.}{100}\right)}} \qquad (7)
\end{aligned}$$

We can use the equation relating peak time T_p with ω_n and ζ to solve for ω_n, substituting ζ with Equation 7.

$$\begin{aligned}
T_p &= \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}\\
\omega_n &= \frac{\pi}{T_p\sqrt{1-\zeta^2}}\\
\omega_n &= \frac{\pi}{T_p\sqrt{1-\frac{\ln^2\left(\frac{P.O.}{100}\right)}{\pi^2+\ln^2\left(\frac{P.O.}{100}\right)}}} \qquad (8)
\end{aligned}$$

2.2 Closed-Loop System Identification using the Frequency Response

Pre-lab Question 8

$$\begin{aligned}\frac{d}{d\omega}\left|\omega_n^2-\omega^2+j2\zeta\omega_n\omega\right|^2 &= \overbrace{2(\omega_n^2-\omega^2+j2\zeta\omega_n\omega)\frac{d}{d\omega}(\omega_n^2-\omega^2+j2\zeta\omega_n\omega)}^{\text{apply chain rule}}\\ &= 2(\omega_n^2-\omega^2+j2\zeta\omega_n\omega)(-2\omega+j2\zeta\omega_n)\\ 0 &= 4(\omega_n^2-\omega^2+j2\zeta\omega_n\omega)(-\omega+j\zeta\omega_n)\\ 0 &= (\omega_n^2-\omega^2+j2\zeta\omega_n\omega)(-\omega+j\zeta\omega_n)\\ &= (\omega_n^2-(\omega_n\sqrt{1-2\zeta^2})^2+j2\zeta\omega_n\sqrt{1-2\zeta^2})(-\sqrt{1-2\zeta^2}+j\zeta\omega_n)\end{aligned} \tag{9}$$

Pre-lab Question 9

The value of the peak is derived below in Equation 10.

$$\begin{aligned}M_p^2 &= \max_{\omega}\left|T(j\omega)\right|^2\\ &= \left|T(j\omega_p)\right|^2\\ &= \frac{\omega_n^4}{\left|\omega_n^2-(\omega_n\sqrt{1-2\zeta^2})^2+j2\zeta\omega_n^2\sqrt{1-2\zeta^2}\right|^2}\\ &= \frac{\omega_n^4}{\left|\omega_n^2-\omega_n^2(1-2\zeta^2)+j2\zeta\omega_n^2\sqrt{1-2\zeta^2}\right|^2}\\ &= \frac{\omega_n^4}{\left|\omega_n^2\left(j2\zeta\sqrt{1-2\zeta^2}+2\zeta^2\right)\right|^2}\\ &= \frac{\omega_n^4}{\omega_n^4(4\zeta^4+4\zeta^2(1-2\zeta^2))}\\ &= \frac{1}{4\zeta^2(\zeta^2+1-2\zeta^2)}\\ &= \frac{1}{4\zeta^2(1-\zeta^2)}\end{aligned} \tag{10}$$

Pre-lab Question 10

We can use the expression relating M_p to ζ to derive ζ.

$$
\begin{aligned}
M_p^2 &= \frac{1}{4\zeta^2(1-\zeta^2)} \\
4\zeta^2 M_p^2 - 4\zeta^4 M_p^2 - 1 &= 0 \\
\overbrace{-4M_p^2 u^2 + 4M_p^2 u - 1}^{u^2=\zeta^4,\ u=\zeta^2} &= 0 \\
u &= \frac{-4M_p^2 \pm \sqrt{16M_p^4 - 16M_p^2}}{-8M_p^2} \\
\zeta &= \sqrt{\frac{-4M_p^2 \pm \sqrt{16M_p^4 - 16M_p^2}}{-8M_p^2}}
\end{aligned} \tag{11}
$$

We can use the expression relating ω_p to ω_n and ζ to derive ω_n, after substituting Equation 11 for ζ.

$$
\begin{aligned}
\omega_p &= \omega_n \sqrt{1-2\zeta^2} \\
\omega_n &= \frac{\omega_p}{\sqrt{1-2\zeta^2}} \\
\omega_n &= \frac{\omega_p}{\sqrt{1-2\left(\frac{-4M_p^2 \pm \sqrt{16M_p^4 - 16M_p^2}}{-8M_p^2}\right)}}
\end{aligned} \tag{12}
$$