

ELECENG 3CL4 Lab 2 Pre-lab

Aaron Pinto
pintoa9
L02

Raeed Hassan
hassam41
L02

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1 Description of Laboratory Equipment

Pre-lab Question 1

The step response of the model $G(s)$ is derived in Equation 1.

$$\begin{aligned}
 \mathcal{L}^{-1} \left\{ G(s) \frac{1}{s} \right\} &= \mathcal{L}^{-1} \left\{ \frac{A}{s(s\tau_m + 1)} \frac{1}{s} \right\} \\
 &= A \mathcal{L}^{-1} \left\{ \frac{1}{s^2(s\tau_m + 1)} \right\} \\
 &= A \mathcal{L}^{-1} \left\{ \frac{\tau_m^2}{s\tau_m + 1} - \frac{\tau_m}{s} + \frac{1}{s^2} \right\} \\
 &= A \left[\mathcal{L}^{-1} \left\{ \frac{\tau_m^2}{s\tau_m + 1} \right\} - \mathcal{L}^{-1} \left\{ \frac{\tau_m}{s} \right\} + \mathcal{L}^{-1} \left\{ \frac{1}{s^2} \right\} \right] \\
 &= A \left[\tau_m e^{-\frac{t}{\tau_m}} - \tau_m u(t) + t \right]
 \end{aligned} \tag{1}$$

Pre-lab Question 2

The step response is not bounded because the term At is not bounded.

2 Closed Loop System Identification

Pre-lab Question 3

(i) The output signal of the model can be expressed as:

$$\begin{aligned}
 Y(s) &= \frac{G_c(s)G(s)}{1 + H(s)G_c(s)G(s)} R(s) + \frac{G(s)}{1 + H(s)G_c(s)G(s)} T_d(s) \\
 &\quad + \frac{H(s)G_c(s)G(s)}{1 + H(s)G_c(s)G(s)} N(s)
 \end{aligned}$$

If we neglect the effects of the disturbance $T_d(s)$ and the noise $N(s)$, we are left with:

$$Y(s) = \frac{G_c(s)G(s)}{1 + H(s)G_c(s)G(s)} R(s)$$

When $H(s) = 1$ and we let $\Theta(s) = Y(s)$, we can find the transfer function $T(s)$ in Equation 2, which is equal to Equation 4a from the lab document.

$$\begin{aligned}
 T(s) &= \frac{\Theta(s)}{R(s)} \\
 &= \frac{\frac{G_c(s)G(s)}{1+H(s)G_c(s)G(s)} R(s)}{R(s)} \\
 &= \frac{G(s)G_c(s)}{1 + G(s)G_c(s)}
 \end{aligned} \tag{2}$$

- (ii) When we substitute $G(s) = \frac{A}{s(s\tau_m+1)}$ and $G_c(s) = K$ into Equation 2, and then substitute $\sqrt{\frac{KA}{\tau_m}} = \omega_n$ and $\frac{1}{2\omega_n\tau_m}$ into the result, we can derive Equation 3 which is equal to Equation 4b from the lab document.

$$\begin{aligned}
T(s) &= \frac{G(s)G_c(s)}{1 + G(s)G_c(s)} \\
&= \frac{\frac{A}{s(s\tau_m+1)}K}{1 + \frac{A}{s(s\tau_m+1)}K} \\
&= \frac{KA}{s(s\tau_m+1)(1 + \frac{A}{s(s\tau_m+1)}K)} \\
&= \frac{KA}{s(s\tau_m+1 + \frac{KA}{s})} \\
&= \frac{KA}{s^2\tau_m + s + KA} \\
&= \frac{KA/\tau_m}{s^2 + (1/\tau_m)s + KA/\tau_m} \\
&= \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}
\end{aligned} \tag{3}$$

Pre-lab Question 4

When given ζ and ω_n , we can derive Equation 4 for τ_m by using the expression relating τ_m to ζ and ω_n .

$$\begin{aligned}
\zeta &= \frac{1}{2\omega_n\tau_m} \\
\tau_m &= \frac{1}{2\omega_n\zeta}
\end{aligned} \tag{4}$$

By substituting Equation 4 into the expression relating ω_n to A and τ_m , we can derive Equation 5 for A when given ζ and ω_n .

$$\begin{aligned}
\omega_n &= \sqrt{\frac{KA}{\tau_m}} \\
\omega_n^2 &= \frac{KA}{\tau_m} \\
A &= \frac{\omega_n^2\tau_m}{K} \\
A &= \frac{\omega_n/2\zeta}{K}
\end{aligned} \tag{5}$$

2.1 Closed-loop System Identification from the Step Response

Pre-lab Question 5

The percent overshoot is determined at the peak time, T_p , of the step response. For the

provided step response, this peak time occurs the first time $d\theta_{\text{step}}(t)/dt = 0$, which occurs when $\omega_n \sqrt{1 - \zeta^2} t = \pi$. At the peak time, the peak value of the step response is $1 + \exp\left(-\zeta\pi/\sqrt{1 - \zeta^2}\right)$. Therefore, the percent overshoot can be defined as Equation 6.

$$P.O. = 100 \frac{(1 + \exp^{-(\zeta\pi/\sqrt{1-\zeta^2})}) - 1}{1} = 100 \exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right) \quad (6)$$

Pre-lab Question 6

As discussed in the solution to Pre-lab Question 5, the peak time occurs the first time $d\theta_{\text{step}}(t)/dt = 0$, which occurs when $\omega_n \sqrt{1 - \zeta^2} t = \pi$. Therefore, the peak time T_p can be expressed as Equation 7.

$$T_p = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}} \quad (7)$$

Pre-lab Question 7

When given $P.O.$, we can derive Equation 8 for ζ by using Equation 6, which relates $P.O.$ to ζ .

$$\begin{aligned} P.O. &= 100 \exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right) \\ \frac{P.O.}{100} &= \exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right) \\ \ln\left(\frac{P.O.}{100}\right) &= \frac{-\zeta\pi}{\sqrt{1-\zeta^2}} \\ \ln^2\left(\frac{P.O.}{100}\right) &= \frac{\zeta^2\pi^2}{1-\zeta^2} \\ \ln^2\left(\frac{P.O.}{100}\right) - \zeta^2 \ln^2\left(\frac{P.O.}{100}\right) &= \zeta^2\pi^2 \\ \ln^2\left(\frac{P.O.}{100}\right) &= \zeta^2\pi^2 + \zeta^2 \ln^2\left(\frac{P.O.}{100}\right) \\ \ln^2\left(\frac{P.O.}{100}\right) &= \zeta^2 \left(\pi^2 + \ln^2\left(\frac{P.O.}{100}\right)\right) \\ \zeta &= \frac{\ln\left(\frac{P.O.}{100}\right)}{\sqrt{\pi^2 + \ln^2\left(\frac{P.O.}{100}\right)}} \end{aligned} \quad (8)$$

By substituting Equation 8 into the expression relating T_p to ω_n and ζ , we can derive Equation 9 for ω_n when given T_p and $P.O.$.

$$\begin{aligned} T_p &= \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}} \\ \omega_n &= \frac{\pi}{T_p \sqrt{1 - \zeta^2}} \end{aligned}$$

$$\omega_n = \frac{\pi}{T_p \sqrt{1 - \frac{\ln^2\left(\frac{P.O.}{100}\right)}{\pi^2 + \ln^2\left(\frac{P.O.}{100}\right)}}} \quad (9)$$

2.2 Closed-Loop System Identification using the Frequency Response

Pre-lab Question 8

In Equation 10, we differentiate the denominator with respect to ω , set the derivative equal to zero, then determine the value for $\omega = \omega_p$ for which $|T(j\omega)|^2$ reaches a maximum.

$$\begin{aligned} \frac{d}{d\omega} \left(|\omega_n^2 - \omega^2 + j2\zeta\omega_n\omega|^2 \right) &= \frac{d}{d\omega} ((\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2) \\ &= \frac{d}{d\omega} (\omega_n^4 - 2\omega_n^2\omega^2 + \omega^4 + 4\zeta^2\omega_n^2\omega^2) \\ 0 &= 4\omega^3 - 4\omega_n^2\omega + 8\zeta^2\omega_n^2\omega \\ 0 &= \omega (4\omega^2 - 4\omega_n^2 + 8\zeta^2\omega_n^2) \\ 4\omega^2 &= 4\omega_n^2 - 8\zeta^2\omega_n^2 \\ \omega^2 &= \omega_n^2 (1 - 2\zeta^2) \\ \omega_p &= \omega_n \sqrt{1 - 2\zeta^2} \end{aligned} \quad (10)$$

Pre-lab Question 9

The value of the peak is derived in Equation 11.

$$\begin{aligned} M_p^2 &= \max_{\omega} |T(j\omega)|^2 \\ &= |T(j\omega_p)|^2 \\ &= \frac{\omega_n^4}{\left| \omega_n^2 - (\omega_n \sqrt{1 - 2\zeta^2})^2 + j2\zeta\omega_n^2 \sqrt{1 - 2\zeta^2} \right|^2} \\ &= \frac{\omega_n^4}{\left| \omega_n^2 - \omega_n^2(1 - 2\zeta^2) + j2\zeta\omega_n^2 \sqrt{1 - 2\zeta^2} \right|^2} \\ &= \frac{\omega_n^4}{\left| \omega_n^2 \left(j2\zeta \sqrt{1 - 2\zeta^2} + 2\zeta^2 \right) \right|^2} \\ &= \frac{\omega_n^4}{\omega_n^4 (4\zeta^4 + 4\zeta^2(1 - 2\zeta^2))} \\ &= \frac{1}{4\zeta^2(\zeta^2 + 1 - 2\zeta^2)} \\ &= \frac{1}{4\zeta^2(1 - \zeta^2)} \end{aligned} \quad (11)$$

Pre-lab Question 10

When given M_p , we can derive Equation 12 for ζ by using the expression relating M_p to ζ .

$$\begin{aligned}
 M_p^2 &= \frac{1}{4\zeta^2(1-\zeta^2)} \\
 4\zeta^2 M_p^2 - 4\zeta^4 M_p^2 - 1 &= 0 \\
 \underbrace{-4M_p^2 u^2 + 4M_p^2 u - 1}_{u^2=\zeta^4, u=\zeta^2} &= 0 \\
 u &= \frac{-4M_p^2 \pm \sqrt{16M_p^4 - 16M_p^2}}{-8M_p^2} \\
 \zeta &= \sqrt{\frac{-4M_p^2 \pm \sqrt{16M_p^4 - 16M_p^2}}{-8M_p^2}} \tag{12}
 \end{aligned}$$

By substituting Equation 12 into the expression relating ω_p to ω_n and ζ , we can derive Equation 13 for ω_n when given ω_p and M_p .

$$\begin{aligned}
 \omega_p &= \omega_n \sqrt{1 - 2\zeta^2} \\
 \omega_n &= \frac{\omega_p}{\sqrt{1 - 2\zeta^2}} \\
 \omega_n &= \frac{\omega_p}{\sqrt{1 - 2 \left(\frac{-4M_p^2 \pm \sqrt{16M_p^4 - 16M_p^2}}{-8M_p^2} \right)}} \tag{13}
 \end{aligned}$$