ELECENG 3CL4 Lab 2 Pre-lab

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1 Description of Laboratory Equipment

Pre-lab Question 1
The step response of the model $G(s)$ is derived in Equation 1.

$$\begin{aligned}
\mathscr{L}^{-1}\left\{G(s)\frac{1}{s}\right\} &= \mathscr{L}^{-1}\left\{\frac{A}{s(s\tau_m+1)}\frac{1}{s}\right\} \\
&= A\ \mathscr{L}^{-1}\left\{\frac{1}{s^2(s\tau_m+1)}\right\} \\
&= A\ \mathscr{L}^{-1}\left\{\frac{{\tau_m}^2}{s\tau_m+1}-\frac{\tau_m}{s}+\frac{1}{s^2}\right\} \\
&= A\left[\mathscr{L}^{-1}\left\{\frac{{\tau_m}^2}{s\tau_m+1}\right\}-\mathscr{L}^{-1}\left\{\frac{\tau_m}{s}\right\}+\mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\}\right] \\
&= A\left[\tau_m e^{-\frac{t}{\tau_m}}-\tau_m u(t)+t\right] \qquad (1)
\end{aligned}$$

Pre-lab Question 2
The step response of the system, $\frac{G(s)}{s} = \frac{A}{s^2(s\tau_m+1)}$, is not bounded as there is a pole at $s = 0$, the origin.

2 Closed Loop System Identification

Pre-lab Question 3

(i) The output signal of the model can be expressed as:

$$Y(s) = \frac{G_c(s)G(s)}{1+H(s)G_c(s)G(s)}R(s) + \frac{G(s)}{1+H(s)G_c(s)G(s)}T_d(s) + \frac{H(s)G_c(s)G(s)}{1+H(s)G_c(s)G(s)}N(s)$$

If we neglect the effects of the disturbance $T_d(s)$ and the noise $N(s)$, we are left with:

$$Y(s) = \frac{G_c(s)G(s)}{1+H(s)G_c(s)G(s)}R(s)$$

When $H(s) = 1$ and we let $\Theta(s) = Y(s)$, we can find the transfer function $T(s)$ in Equation 2, which is equal to Equation 4a from the lab document.

$$\begin{aligned}
T(s) &= \frac{\Theta(s)}{R(s)} \\
&= \frac{\frac{G_c(s)G(s)}{1+H(s)G_c(s)G(s)}R(s)}{R(s)} \\
&= \frac{G(s)G_c(s)}{1+G(s)G_c(s)} \qquad (2)
\end{aligned}$$

(ii) When we substitute $G(s) = \frac{A}{s(s\tau_m+1)}$ and $G_c(s) = K$ into Equation 2, and then substitute $\sqrt{\frac{KA}{\tau_m}} = \omega_n$ and $\frac{1}{2\omega_n\tau_m}$ into the result, we can derive Equation 3 which is equal to Equation 4b from the lab document.

$$
\begin{aligned}
T(s) &= \frac{G(s)G_c(s)}{1+G(s)G_c(s)} \\
&= \frac{\frac{A}{s(s\tau_m+1)}K}{1+\frac{A}{s(s\tau_m+1)}K} \\
&= \frac{KA}{s(s\tau_m+1)(1+\frac{A}{s(s\tau_m+1)}K)} \\
&= \frac{KA}{s(s\tau_m+1+\frac{KA}{s})} \\
&= \frac{KA}{s^2\tau_m+s+KA} \\
&= \frac{KA/\tau_m}{s^2+(1/\tau_m)s+KA/\tau_m} \\
&= \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \qquad (3)
\end{aligned}
$$

Pre-lab Question 4

When given ζ and ω_n, we can derive Equation 4 for τ_m by using the expression relating τ_m to ζ and ω_n.

$$
\begin{aligned}
\zeta &= \frac{1}{2\omega_n\tau_m} \\
\tau_m &= \frac{1}{2\omega_n\zeta} \qquad (4)
\end{aligned}
$$

By substituting Equation 4 into the expression relating ω_n to A and τ_m, we can derive Equation 5 for A when given ζ and ω_n.

$$
\begin{aligned}
\omega_n &= \sqrt{\frac{KA}{\tau_m}} \\
\omega_n^2 &= \frac{KA}{\tau_m} \\
A &= \frac{\omega_n^2\tau_m}{K} \qquad (5) \\
A &= \frac{\omega_n/2\zeta}{K}
\end{aligned}
$$

2.1 Closed-loop System Identification from the Step Response

Pre-lab Question 5

The percent overshoot is determined at the peak time, T_p, of the step response. For the provided step response, this peak time occurs the first time $d\theta_{\text{step}}(t)/dt = 0$, which occurs when $\omega_n\sqrt{1-\zeta^2}t = \pi$. At the peak time, the peak value of the step response is $1 + \exp\left(-\zeta\pi/\sqrt{1-\zeta^2}\right)$. Therefore, the percent overshoot can be defined as Equation 6.

$$P.O. = 100\frac{(1+\exp^{-(\zeta\pi/\sqrt{1-\zeta^2})}) - 1}{1} = 100\exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right) \tag{6}$$

Pre-lab Question 6

As discussed in the solution to Pre-lab Question 5, the peak time occurs the first time $d\theta_{\text{step}}(t)/dt = 0$, which occurs when $\omega_n\sqrt{1-\zeta^2}t = \pi$. Therefore, the peak time T_p can be expressed as Equation 7.

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \tag{7}$$

Pre-lab Question 7

When given $P.O.$, we can derive Equation 8 for ζ by using Equation 6, which relates $P.O.$ to ζ.

$$\begin{aligned}
P.O. &= 100\exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right) \\
\frac{P.O.}{100} &= \exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right) \\
\ln\left(\frac{P.O.}{100}\right) &= \frac{-\zeta\pi}{\sqrt{1-\zeta^2}} \\
\ln^2\left(\frac{P.O.}{100}\right) &= \frac{\zeta^2\pi^2}{1-\zeta^2} \\
\ln^2\left(\frac{P.O.}{100}\right) - \zeta^2\ln^2\left(\frac{P.O.}{100}\right) &= \zeta^2\pi^2 \\
\ln^2\left(\frac{P.O.}{100}\right) &= \zeta^2\pi^2 + \zeta^2\ln^2\left(\frac{P.O.}{100}\right) \\
\ln^2\left(\frac{P.O.}{100}\right) &= \zeta^2\left(\pi^2 + \ln^2\left(\frac{P.O.}{100}\right)\right) \\
\zeta &= \frac{\ln\left(\frac{P.O.}{100}\right)}{\sqrt{\pi^2 + \ln^2\left(\frac{P.O.}{100}\right)}} \qquad (8)
\end{aligned}$$

By substituting Equation 8 into the expression relating T_p to ω_n and ζ, we can derive Equation 9 for ω_n when given T_p and $P.O.$.

$$\begin{aligned}
T_p &= \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \\
\omega_n &= \frac{\pi}{T_p\sqrt{1-\zeta^2}}
\end{aligned}$$

$$\omega_n = \frac{\pi}{T_p\sqrt{1-\frac{\ln^2\left(\frac{P.O.}{100}\right)}{\pi^2+\ln^2\left(\frac{P.O.}{100}\right)}}} \tag{9}$$

2.2 Closed-Loop System Identification using the Frequency Response

Pre-lab Question 8

In Equation 10, we differentiate the denominator with respect to ω, set the derivative equal to zero, then determine the value for $\omega = \omega_p$ for which $|T(j\omega)|^2$ reaches a maximum. As ω_p has to be a real number, ζ must be $\leq 1/\sqrt{2}$.

$$\begin{aligned}
\frac{d}{d\omega}\left(\left|\omega_n^2-\omega^2+j2\zeta\omega_n\omega\right|^2\right) &= \frac{d}{d\omega}\left((\omega_n^2-\omega^2)^2+(2\zeta\omega_n\omega)^2\right)\\
&= \frac{d}{d\omega}\left(\omega_n^4-2\omega_n^2\omega^2+\omega^4+4\zeta^2\omega_n^2\omega^2\right)\\
0 &= 4\omega^3-4\omega_n^2\omega+8\zeta^2\omega_n^2\omega\\
0 &= \omega\left(4\omega^2-4\omega_n^2+8\zeta^2\omega_n^2\right)\\
4\omega^2 &= 4\omega_n^2-8\zeta^2\omega_n^2\\
\omega^2 &= \omega_n^2\left(1-2\zeta^2\right)\\
\omega_p &= \omega_n\sqrt{1-2\zeta^2}
\end{aligned} \tag{10}$$

Pre-lab Question 9

The value of the peak is derived in Equation 11.

$$\begin{aligned}
M_p^2 &= \max_\omega |T(j\omega)|^2\\
&= |T(j\omega_p)|^2\\
&= \frac{\omega_n^4}{\left|\omega_n^2-(\omega_n\sqrt{1-2\zeta^2})^2+j2\zeta\omega_n^2\sqrt{1-2\zeta^2}\right|^2}\\
&= \frac{\omega_n^4}{\left|\omega_n^2-\omega_n^2(1-2\zeta^2)+j2\zeta\omega_n^2\sqrt{1-2\zeta^2}\right|^2}\\
&= \frac{\omega_n^4}{\left|\omega_n^2\left(j2\zeta\sqrt{1-2\zeta^2}+2\zeta^2\right)\right|^2}\\
&= \frac{\omega_n^4}{\omega_n^4(4\zeta^4+4\zeta^2(1-2\zeta^2))}\\
&= \frac{1}{4\zeta^2(\zeta^2+1-2\zeta^2)}\\
&= \frac{1}{4\zeta^2(1-\zeta^2)}
\end{aligned} \tag{11}$$

Pre-lab Question 10

When given M_p, we can derive Equation 12 for ζ by using the expression relating M_p to ζ.

$$\begin{aligned}
M_p^2 &= \frac{1}{4\zeta^2(1-\zeta^2)} \\
4\zeta^2 M_p^2 - 4\zeta^4 M_p^2 - 1 &= 0 \\
\overbrace{-4M_p^2 u^2 + 4M_p^2 u - 1}^{u^2=\zeta^4,\ u=\zeta^2} &= 0 \\
u &= \frac{-4M_p^2 \pm \sqrt{16M_p^4 - 16M_p^2}}{-8M_p^2} \\
\zeta &= \sqrt{\frac{-4M_p^2 \pm \sqrt{16M_p^4 - 16M_p^2}}{-8M_p^2}}
\end{aligned} \tag{12}$$

By substituting Equation 12 into the expression relating ω_p to ω_n and ζ, we can derive Equation 13 for ω_n when given ω_p and M_p.

$$\begin{aligned}
\omega_p &= \omega_n \sqrt{1-2\zeta^2} \\
\omega_n &= \frac{\omega_p}{\sqrt{1-2\zeta^2}} \\
\omega_n &= \frac{\omega_p}{\sqrt{1-2\left(\frac{-4M_p^2 \pm \sqrt{16M_p^4 - 16M_p^2}}{-8M_p^2}\right)}}
\end{aligned} \tag{13}$$