ELECENG 3CL4 Lab 3 Pre-lab

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1 Proportional Control of DC Motor

Pre-Lab Question 1
The closed-loop transfer function $T(s)$ is:

$$
\begin{aligned}
T(s) &= \frac{G(s)G_c(s)}{1+G(s)G_c(s)} \\
&= \frac{k_pG(s)}{1+k_pG(s)}
\end{aligned}
$$

We find the characteristic equation of the closed-loop transfer function $T(s)$ by equating the denominator of the transfer function to zero below:

$$
\begin{aligned}
0 &= 1 + k_pG(s) \\
0 &= 1 + \frac{k_pA}{s(s\tau_m+1)} \\
0 &= 1 + \frac{k_pA}{s^2\tau_m + s} \\
-1 &= \frac{k_pA}{s^2\tau_m + s} \\
-s^2\tau_m - s &= k_pA \\
0 &= s^2\tau_m + s + k_pA
\end{aligned}
$$

The closed-loop poles of the system are found by determining the poles of the characteristic equation, which is done in Equation 1. The poles have a constant real term, and a term that is real for $k_p \le \frac{1}{4A\tau_m}$ and imaginary for $k_p > \frac{1}{4A\tau_m}$.

$$
\begin{aligned}
p_{1,2} &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \\
a &= \tau_m,\ b = 1,\ c = k_pA \\
&= \frac{-1 \pm \sqrt{1-4\tau_m k_pA}}{2\tau_m} \\
p_{1,2} &= -\frac{1}{2\tau_m} \pm \frac{1}{2\tau_m}\sqrt{1-4k_pA\tau_m} \qquad (1)
\end{aligned}
$$

Pre-Lab Question 2
The path of the poles $p_{1,2}$ in the s-plane as k_p approaches infinity are shown in Figure 1. The two poles start wholly real, approaching $-\frac{1}{2\tau_m}$ as k_p approaches $\frac{1}{4A\tau_m}$. When the value of k_p is greater than $\frac{1}{4A\tau_m}$, the poles have a constant real part $-\frac{1}{2\tau_m}$ and an increasing/decreasing imaginary part which creates vertical paths for the poles in the s-plane.

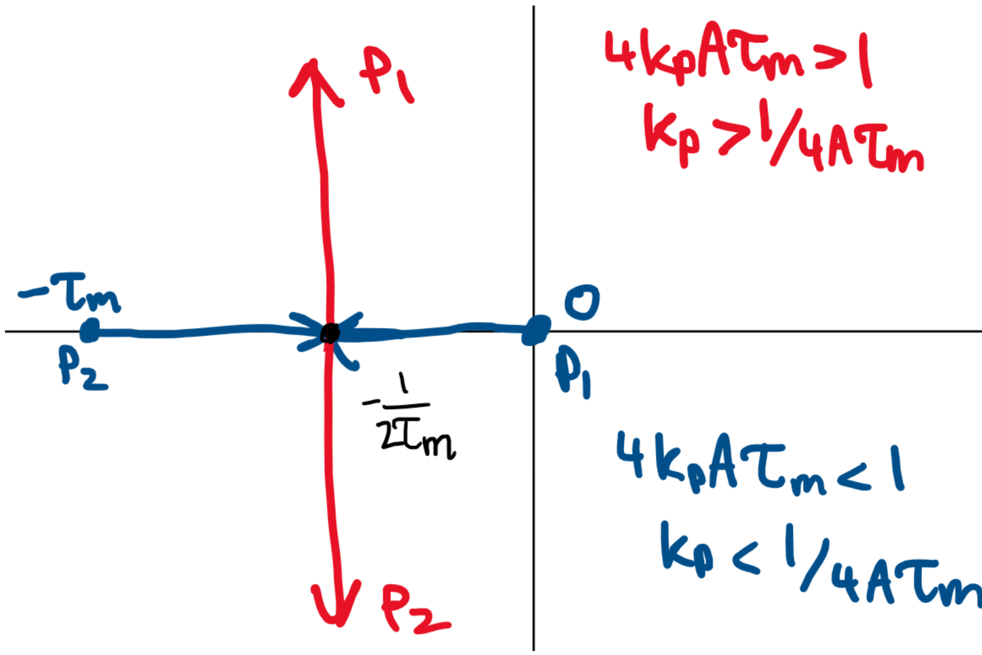

Figure 1: The path of the closed-loop poles as k_p approaches infinity

Pre-Lab Question 3

We can rewrite the closed-loop transfer function $T(s)$ to be in the standard form of a second order system:

$$\begin{aligned} T(s) &= \frac{k_pG(s)}{1+k_pG(s)} \\ &= \frac{\frac{k_pA}{s^2\tau_m+s}}{1+\frac{k_pA}{s^2\tau_m+s}} \\ &= \frac{k_pA}{s^2\tau_m+s+k_pA} \\ &= \frac{\frac{k_pA}{\tau_m}}{s^2+\frac{s}{\tau_m}+\frac{k_pA}{\tau_m}} \end{aligned}$$

For a second order system in the standard form:

$$F_2(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

We can determine the parameters $\zeta\omega_n$, ω_n, and ζ in Equation 2 from $T(s)$.

$$\begin{aligned} \omega_n^2 &= \frac{k_pA}{\tau_m} & 2\zeta\omega_n s &= \frac{s}{\tau_m} & \zeta &= \frac{1}{2\tau_m\omega_n} \\ \omega_n &= \sqrt{\frac{k_pA}{\tau_m}} & \zeta\omega_n &= \frac{1}{2\tau_m} & \zeta &= \frac{1}{2\sqrt{k_pA\tau_m}} \end{aligned} \tag{2}$$

Pre-Lab Question 4
When the controller gain $k_p = \frac{1}{4A\tau_m}$, we can show the system is critically damped by determining $\zeta = 1$ in Equation 3.

$$\begin{aligned}
\zeta &= \frac{1}{2\sqrt{k_p A\tau_m}} \\
&= \frac{1}{2\sqrt{\frac{A\tau_m}{4A\tau_m}}} \\
&= \frac{1}{2\sqrt{\frac{1}{4}}} \\
&= \frac{1}{2(0.5)} \\
\zeta &= 1
\end{aligned} \tag{3}$$

Pre-Lab Question 5
We can use the expression for the closed-loop poles found previously to write the pole positions as Equation 4 for $k_p > \frac{1}{4A\tau_m}$.

$$\begin{aligned}
p_{1,2} &= -\frac{1}{2\tau_m} \pm \frac{1}{2\tau_m}\sqrt{1 - 4k_p A\tau_m} \\
&= \frac{1}{2\tau_m}\left(-1 \pm \sqrt{1 - 4k_p A\tau_m}\right) \\
&= \frac{1}{2\tau_m}\left(-1 \pm j\sqrt{4k_p A\tau_m - 1}\right) \\
&= \frac{1}{2\tau_m}\left(-1 \pm j\sqrt{\frac{1 - \frac{1}{4k_p A\tau_m}}{\frac{1}{4k_p A\tau_m}}}\right) \\
&= \frac{1}{2\tau_m}\left(-1 \pm j\sqrt{\frac{1 - \left(\frac{1}{2\sqrt{k_p A\tau_m}}\right)^2}{\left(\frac{1}{2\sqrt{k_p A\tau_m}}\right)^2}}\right) \\
&= \frac{1}{2\tau_m}\left(-1 \pm j\sqrt{\frac{1-\zeta^2}{\zeta^2}}\right) \\
&= \frac{1}{2\tau_m}\left(-1 \pm j\frac{\sqrt{1-\zeta^2}}{\zeta}\right) \\
&= \frac{1}{2\tau_m}\left(-1 \pm j\tan(\cos^{-1}(\zeta))\right)
\end{aligned} \tag{4}$$

2 Trade-offs in Proportional Control of a Servomotor: Theoretical Insight

Pre-Lab Question 6
Using the provided expressions and values for ω_n and ζ, we can show that the closed-loop system in Figure 1 of the lab document will have the following T_s, $P.O.$ and T_{r1} when underdamped:

$$
\begin{aligned}
T_s &\approx \frac{4}{\zeta\omega_n} \\
&\approx \frac{4}{\frac{\omega_n}{2\omega_n\tau_m}} \\
&\approx \frac{4}{\frac{1}{2\tau_m}} \\
&\approx 8\tau_m \\
P.O. &= 100\exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right) \\
&= 100\exp\left(-\frac{\frac{\pi}{2\omega_n\tau_m}}{\sqrt{1-\left(\frac{1}{2\omega_n\tau_m}\right)^2}}\right) \\
&= 100\exp\left(-\frac{\frac{\pi}{2\omega_n\tau_m}}{\sqrt{1-\frac{1}{4\omega_n^2\tau_m^2}}}\right) \\
&= 100\exp\left(-\frac{\pi}{2\omega_n\tau_m\sqrt{1-\frac{1}{4\omega_n^2\tau_m^2}}}\right) \\
&= 100\exp\left(-\frac{\pi}{2\omega_n\tau_m\sqrt{\frac{4\omega_n^2\tau_m^2-1}{4\omega_n^2\tau_m^2}}}\right) \\
&= 100\exp\left(-\frac{\pi}{2\omega_n\tau_m\frac{\sqrt{4\omega_n^2\tau_m^2-1}}{\sqrt{4\omega_n^2\tau_m^2}}}\right) \\
&= 100\exp\left(-\frac{\pi}{\sqrt{4\omega_n^2\tau_m^2-1}}\right) \\
&= 100\exp\left(-\frac{\pi}{\sqrt{4\frac{k_pA}{\tau_m}\tau_m^2-1}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= 100 \exp\left(-\frac{\pi}{\sqrt{4k_pA\tau_m - 1}}\right) \\
T_{r1} &\approx \frac{2.16\zeta + 0.6}{\omega_n} \\
&\approx \frac{\frac{2.16}{2\omega_n\tau_m} + 0.6}{\omega_n} \\
&\approx \frac{\frac{2.16+1.2\omega_n\tau_m}{2\omega_n\tau_m}}{\omega_n} \\
&\approx \frac{2.16 + 1.2\omega_n\tau_m}{2\omega_n^2\tau_m} \\
&\approx \frac{2.16 + 1.2\sqrt{\frac{k_pA}{\tau_m}}\tau_m}{2\frac{k_pA}{\tau_m}\tau_m} \\
&\approx \frac{2.16 + 1.2\sqrt{k_pA\tau_m}}{2k_pA}
\end{aligned}
$$

Pre-Lab Question 7

T_s does not change with k_p. P.O. increases as k_p increases and it approaches a horizontal asymptote of 100%, which is the expected behaviour for an underdamped system. T_{r1} decreases as k_p decreases and approaches the horizontal asymptote of 0.

Pre-Lab Question 8

The 2% settling time T_s cannot be controlled through k_p, while the percent overshoot and the 10% to 90% rise time T_{r1} can.

Pre-Lab Question 9

From the block diagram from Figure 1 in the lab document, you can determine the total output response to the reference and disturbance inputs as Equation 5.

$$
\begin{aligned}
\Theta(s) &= \frac{k_pG(s)}{1 + k_pG(s)}R(s) + \frac{G(s)}{1 + k_pG(s)}T_d(s) \\
&= \frac{\frac{k_pA}{s(s\tau_m+1)}}{1 + \frac{k_pA}{s(s\tau_m+1)}}R(s) + \frac{\frac{A}{s(s\tau_m+1)}}{1 + \frac{k_pA}{s(s\tau_m+1)}}T_d(s) \\
&= \frac{\frac{k_pA}{s(s\tau_m+1)}}{\frac{s(s\tau_m+1)+k_pA}{s(s\tau_m+1)}}R(s) + \frac{\frac{A}{s(s\tau_m+1)}}{\frac{s(s\tau_m+1)+k_pA}{s(s\tau_m+1)}}T_d(s) \\
&= \frac{k_pA}{s(s\tau_m + 1) + k_pA}R(s) + \frac{A}{s(s\tau_m + 1) + k_pA}T_d(s) \\
&= \frac{k_pA}{s^2\tau_m + s + k_pA}R(s) + \frac{A}{s^2\tau_m + s + k_pA}T_d(s)
\end{aligned}
$$

$$= \frac{\frac{k_p A}{\tau_m}}{s^2 + \frac{1}{\tau_m}s + \frac{k_p A}{\tau_m}} R(s) + \frac{\frac{A}{\tau_m}}{s^2 + \frac{1}{\tau_m}s + \frac{k_p A}{\tau_m}} T_d(s) \tag{5}$$

Pre-Lab Question 10

To find the steady-state error to a step input $R(s) = \frac{\theta_d}{s}$ in absence of disturbance, calculate the steady-state error of the reference input term using the final limit theorem in Equation 6.

$$\begin{aligned}
e_{ss} &= \lim_{s \to 0} \; s \frac{1}{1 + G_c(s)G(s)} R(s) \\
&= \lim_{s \to 0} \; s \frac{1}{1 + G_c(s)G(s)} \frac{\theta_d}{s} \\
&= \lim_{s \to 0} \; \frac{\theta_d}{1 + G_c(s)G(s)} \\
&= \frac{\theta_d}{1 + \lim_{s \to 0} \; G_c(s)G(s)} \\
&= \frac{\theta_d}{1 + \lim_{s \to 0} \; \frac{k_p A}{s(s\tau_m + 1)}} \\
&= \frac{\theta_d}{1 + (\lim_{s \to 0} \; \frac{k_p A}{s(s\tau_m + 1)} \to \infty)} \\
e_{ss} &= 0
\end{aligned} \tag{6}$$

The feedback gain k_p does not have any effect on this error, because the system is type 1.

Pre-Lab Question 11

To find the steady-state error to a step input $R(s) = \frac{\theta_d}{s}$ in the presence of a constant disturbance $T_d(s) = \frac{\tau_d}{s}$, calculate the steady-state error of the reference input term using the final limit theorem in Equation 7.

$$\begin{aligned}
E(s) &= R(s) - Y(s) \\
&= R(s) - \left(\frac{k_p G(s)}{1 + k_p G(s)} R(s) + \frac{G(s)}{1 + k_p G(s)} T_d(s) \right) \\
&= R(s) - \frac{k_p G(s) R(s) - G(s) T_d(s)}{1 + k_p G(s)} \\
&= \frac{R(s) + k_p G(s) R(s)}{1 + k_p G(s)} - \frac{k_p G(s) R(s) - G(s) T_d(s)}{1 + k_p G(s)} \\
&= \frac{R(s) - G(s) T_d(s)}{1 + k_p G(s)} \\
e_{ss} &= \lim_{s \to 0} \; s \frac{1}{1 + k_p G(s)} \frac{\theta_d}{s} - \lim_{s \to 0} \; s \frac{G(s)}{1 + k_p G(s)} \frac{\tau_d}{s} \\
&= 0 - \lim_{s \to 0} \; \frac{G(s)}{1 + k_p G(s)} \tau_d \\
&= - \lim_{s \to 0} \; \frac{G(s)}{1 + k_p G(s)} \tau_d
\end{aligned}$$

$$
\begin{aligned}
&= -\lim_{s\to 0} \frac{\frac{A}{s(s\tau_m+1)}}{1+\frac{k_pA}{s(s\tau_m+1)}}\tau_d \\
&= -\lim_{s\to 0} \frac{A}{s(s\tau_m+1)+k_pA}\tau_d \\
&= -\frac{A\tau_d}{k_pA+\lim_{s\to 0}\ s(s\tau_m+1)} \\
&= -\frac{A\tau_d}{k_pA+(\lim_{s\to 0}\ s(s\tau_m+1)\to 0)} \\
&= -\frac{A\tau_d}{k_pA} \\
&= -\frac{\tau_d}{k_p} \qquad (7)
\end{aligned}
$$

A larger feedback gain k_p makes the error smaller, and vice versa, because k_p is inversely proportional to this error.

6 Proportional Controller with Velocity Feedback

Pre-Lab Question 12

The block diagram of the proportional control of the servomotor with additional velocity feedback can be transformed using block diagram transforms as shown in Figure 2. From the block diagram, we can see that the total output response when $F(s) \approx 1$ can be written as Equation 8.

$$
\begin{aligned}
\Theta(s) &= \frac{k_pG(s)}{1+k_vsG(s)+k_pG(s)}R(s) + \frac{G(s)}{1+k_vsG(s)+k_pG(s)}T_d(s) \\
&= \frac{\frac{k_pA}{s(s\tau_m+1)}}{1+s\frac{k_vA}{s(s\tau_m+1)}+\frac{k_pA}{s(s\tau_m+1)}}R(s) + \frac{\frac{A}{s(s\tau_m+1)}}{1+s\frac{k_vA}{s(s\tau_m+1)}+\frac{k_pA}{s(s\tau_m+1)}}T_d(s) \\
&= \frac{k_pA}{s(s\tau_m+1)+k_vAs+kpA}R(s) + \frac{A}{s(s\tau_m+1)+k_vAs+kpA}T_d(s) \\
&= \frac{\frac{k_pA}{\tau_m}}{s^2+\frac{1+k_vA}{\tau_m}s+\frac{k_pA}{\tau_m}}R(s) + \frac{\frac{A}{\tau_m}}{s^2+\frac{1+k_vA}{\tau_m}s+\frac{k_pA}{\tau_m}}T_d(s) \qquad (8)
\end{aligned}
$$

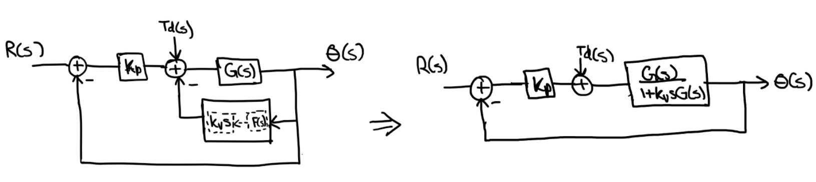

Figure 2: The block diagram transformation of the servomotor with additional velocity feedback

Pre-Lab Question 13
The steady-state error due to a step disturbance can be calculated through the final limit theorem as shown in Equation 9.

$$
\begin{aligned}
|e_{ss}| &= \lim_{s\to 0}\ s\frac{A}{s(s\tau_m+1)+k_vAs+k_pA}T_d(s) \\
&= \lim_{s\to 0}\ s\frac{A}{s(s\tau_m+1)+k_vAs+k_pA}\frac{\tau_d}{s} \\
&= \lim_{s\to 0}\ \frac{A\tau_d}{s(s\tau_m+1)+k_vAs+k_pA} \\
&= \frac{A\tau_d}{k_pA+\lim_{s\to 0}\ s(s\tau_m+1)+k_vAs} \\
&= \frac{A\tau_d}{k_pA+(\lim_{s\to 0}\ s(s\tau_m+1)+k_vAs\to 0)} \\
&= \frac{A\tau_d}{k_pA} \\
|e_{ss}| &= \frac{\tau_d}{k_p}
\end{aligned} \tag{9}
$$

Pre-Lab Question 14
The closed-loop transfer function can be written as:

$$
\begin{aligned}
T(s) &= \frac{k_pG(s)}{1+k_vsG(s)+k_pG(s)} \\
&= \frac{\frac{k_pA}{s(s\tau_m+1)}}{1+\frac{k_vA}{s(s\tau_m+1)}s+\frac{k_pA}{s(s\tau_m+1)}} \\
&= \frac{k_pA}{s^2\tau_m+(1+k_vA)s+k_pA} \\
&= \frac{\frac{k_pA}{\tau_m}}{s^2+\frac{1+k_vA}{\tau_m}s+\frac{k_pA}{\tau_m}}
\end{aligned}
$$

We can find ζ by writing the closed-loop transfer function in the form of a standard second order system and finding the parameters of the system in Equation 10.

$$F_2(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_ns+\omega_n^2}$$

$$
\begin{aligned}
\omega_n^2 &= \frac{k_pA}{\tau_m} & 2\zeta\omega_ns &= \frac{1+k_vA}{\tau_m}s & \zeta &= \frac{1+k_vvA}{2\tau_m\omega_n} \\
\omega_n &= \sqrt{\frac{k_pA}{\tau_m}} & \zeta\omega_n &= \frac{1+k_vA}{2\tau_m} & \zeta &= \frac{1+k_vA}{2\sqrt{k_pA\tau_m}}
\end{aligned} \tag{10}
$$

Pre-Lab Question 15

Increasing k_p will decrease ζ, which will lead to a decrease in the rise time and an increase in the maximum overshoot, while decreasing k_p will increase the rise time and decrease the maximum overshoot. Increasing k_v will increase ζ, which will lead to an increase in the rise time and a decrease in the maximum overshoot, while decreasing k_v will decrease the rise time and increase the maximum overshoot. The settling time is inversely proportional to $\zeta\omega_n$, which increases as k_v increases and decreases as k_v decreases. Therefore, the settling time increases as k_v decreases and decreases as k_v increases. The steady-state error to a constant disturbance is inversely proportional to k_p.