ELECENG 3TQ3 Assignment 3

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December 12, 2020

1. Let X be exponentially distributed random variable with $\lambda = 1$ and let Y be Gaussian distributed with mean μ and variance σ^2. Find:

   (a) distribution of Z = X + Y

$$
\begin{aligned}
f_Z(z) &= \int f_X(z-y)f_Y(y)dy = \int f_X(x)f_Y(z-x)dx \\
f_Z(z) &= \int f_X(x)f_Y(z-x)dx \\
f_Z(z) &= \int e^{-x}\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{z-x-\mu}{\sigma})^2}dx \\
f_Z(z) &= \frac{1}{\sigma\sqrt{2\pi}}\int e^{-x-\frac{1}{2}(\frac{z-x-\mu}{\sigma})^2)}dx \\
f_Z(z) &= \frac{1}{2}e^{\left(\mu+\frac{\sigma^2}{2}-z\right)}\operatorname{erf}\left(\frac{x-z+\sigma^2+\mu}{\sqrt{2}\sigma}\right)
\end{aligned}
$$

   (b) distribution $f_{X|Z}$

$$
\begin{aligned}
f_{X|Z}(x|u) &= \frac{f_{X,Z}(x,z)}{f_Z(z)} \\
f_{X|Z}(x|u) &= \frac{f_{X,Z}(x,z)}{\frac{1}{2}e^{\left(\mu+\frac{\sigma^2}{2}-z\right)}\operatorname{erf}\left(\frac{x-z+\sigma^2+\mu}{\sqrt{2}\sigma}\right)}
\end{aligned}
$$

2. Let X and Y be independent Gaussian distributed random variables with mean 0 and variance 1. Find:

   (a) Joint pdf of U = 2X + Y and V = -Y

$$
\begin{aligned}
E[U] &= E[2X+Y] = 0 \\
E[V] &= E[-Y] = 0 \\
Var(U) &= Var(2X+Y) = 2^2Var(X) + Var(Y) + 4Cov(X,Y) \\
Var(U) &= 4+1 = 5 \\
Var(V) &= Var(-Y) = Var(Y) = 1 \\
\mu_U &= 0,\ \mu_V = 0,\ \sigma_U = \sqrt{5},\ \sigma_V = 1,\ \rho_{U,V} = 0 \\
f_{U,V}(u,v) &= \frac{exp\left[-\frac{\left(\frac{u-\mu_U}{\sigma_U}\right)^2-\frac{2\rho_{U,V}(u-\mu_U)(v-\mu_V)}{\sigma_U\sigma_V}+\left(\frac{v-\mu_V}{\sigma_V}\right)^2}{2(1-\rho_{U,V}^2)}\right]}{2\pi\sigma_U\sigma_V\sqrt{1-\rho_{U,V}^2}} \\
f_{U,V}(u,v) &= \frac{1}{2\sqrt{5}\pi}e^{-(u^2/5+v^2)/2}
\end{aligned}
$$

(b) Conditional distribution of V|U

$$f_{V|U}(v|u) = \frac{f_{U,V}(u,v)}{f_U(u)}$$
$$= \frac{\frac{1}{2\sqrt{5}\pi} e^{-(u^2/5+v^2)/2}}{\frac{1}{\sqrt{10\pi}} e^{-(u/\sqrt{5})/2}}$$

(c) Find probability $P(V < -3|U > 1)$
$P(V < -3|U > 1) \approx 0.0036$. The probability was determined using MATLAB. Normal probability distributions for X and Y were generated, then put into a loop to generate a random number from each distribution. These random numbers would be used to create U and V, which are then tested to see if they meet the requirements ($U > 1$, then $V < -3$ if previous is true). The MATLAB code is shown is in Listing 1.

Listing 1: Question 2c

```
X = makedist('Normal','mu',0,'sigma',1);
Y = makedist('Normal','mu',0,'sigma',1);

success_given = 0; success = 0; num_trials = 5000000;
for i = 1:num_trials
   X_num = random(X); Y_num = random(Y);
   U = 2*X_num + Y_num; V = -Y_num;
   if (U > 1)
      success_given = success_given + 1;
      if (V < -3)
          success = success + 1;
      end
   end
end

result = success/success_given;
disp(result);
```

3. Let small powerhouse have two hydro-generators G1 and G2 and let life expectancy of stator coil be exponentially distributed with expected values 35 and 40 years respectively. Let cost of repair for generator 1 be Gaussian distributed with mean 35 million dollars and variance 4. Similarly, let the cost of repair for generator 2 be Gaussian distributed with mean 34.5 million and variance 1. Assuming that only one failure per generator is possible find the expected value of the repair cost in the first year.

   Let G1 be exponential random variable with $\lambda = \frac{1}{35}$
   The probability of G1 failing in the first year is:

$$P(G1 < 1) = 1 - e^{-\lambda x} = 1 - e^{-\frac{1}{35}}$$

Let G2 be exponential random variable with $\lambda = \frac{1}{40}$
The probability of G2 failing in the first year is:

$$P(G1 < 1) = 1 - e^{-\lambda x} = 1 - e^{-\frac{1}{40}}$$

The expected value for the first year is $P(G1 < 1)E[\text{cost of repair for G1}] + P(G2 < 1)E[\text{cost of repair for G2}] = (1 - e^{-\frac{1}{35}})(35) + (1 - e^{-\frac{1}{40}})(34.5) \approx 1.84$ million dollars.