ELECENG 3TR4 Lab 4: Random Processes

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April 4, 2021

Numerical Experiment #1: Evaluation of Autocorrelation and Power Spectral Density

i) The theoretical autocorrelation function of the output is derived in Equation 1.

$$\begin{aligned}
y(n) &= h(n) * w(n) \\
h(n) &= 2B\,\text{sinc}(2Bn) \\
&= 2 \cdot 250 sinc(2 \cdot 250n) \\
R_y(m) &= E\left[y(n) \cdot y(n+m)\right] \\
&= \sum_k \sum_j h(k)h(j)E[w(n-k)w(n+m-j)] \\
&= \sum_k h(k)h(k+m)\sigma_w^2 \\
&= \sum_k (500\,\text{sinc}(500k))(500\,\text{sinc}(500(k+m)))\sigma_w^2 \\
&= \sum_k 250000\,\text{sinc}(500k)\,\text{sinc}(500(k+m))\sigma_w^2
\end{aligned} \tag{1}$$

The PSD is just the Fourier transform of the autocorrelation function. The PSD is derived in Equation 2, where k is the gain of the noise.

$$\begin{aligned}
S_y(f) &= k\left|H(f)\right|^2 \\
&= k\,\text{rect}(\frac{f}{2 \cdot 250}) \\
&= k\,\text{rect}(\frac{f}{500})
\end{aligned} \tag{2}$$

The theoretical results from Equation 1 and Equation 2 can be compared to the results of the numerical experiment in Figure 1. We can see that both the autocorrelation and PSD match between the theoretical results and the numerical results. Both autocorrelation functions result in sinc functions with the same frequency. Similarly, both PSDs produce a rect function with a bandwidth of 250 Hz, although is noise in the passband of the MATLAB PSD, as the signal used for the MATLAB plot does not use an ideal white noise signal.

ii) We can see the impact of increasing the maxlag to 200 in Figure 2 and increasing the maxlag to 500 in Figure 3. We can see that there is more information in the PSD as we increase the maxlag, as doing so increases the frequency resolution of our PSD.

iii) We can estimate the bandwidth of the filter using the autocorrelation plot. We can see that in all three plots of the autocorrelation function, the zeros of the sinc function are spaced 0.002 seconds apart. This means the sinc function repeats peaks and troughs every 0.004 seconds. We can determine the bandwidth of the filter to be $\frac{1}{0.004\text{ s}}$ or 250 Hz.

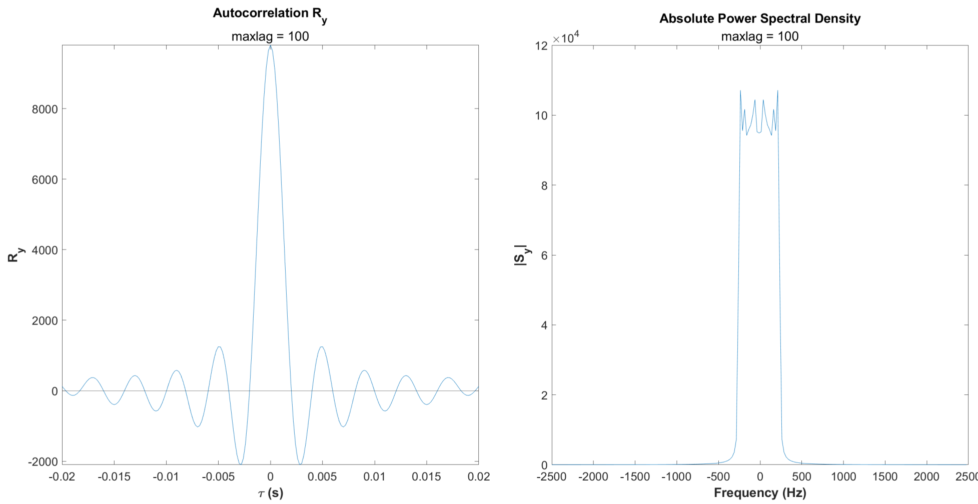

Figure 1: Autocorrelation and PSD for maxlag = 100

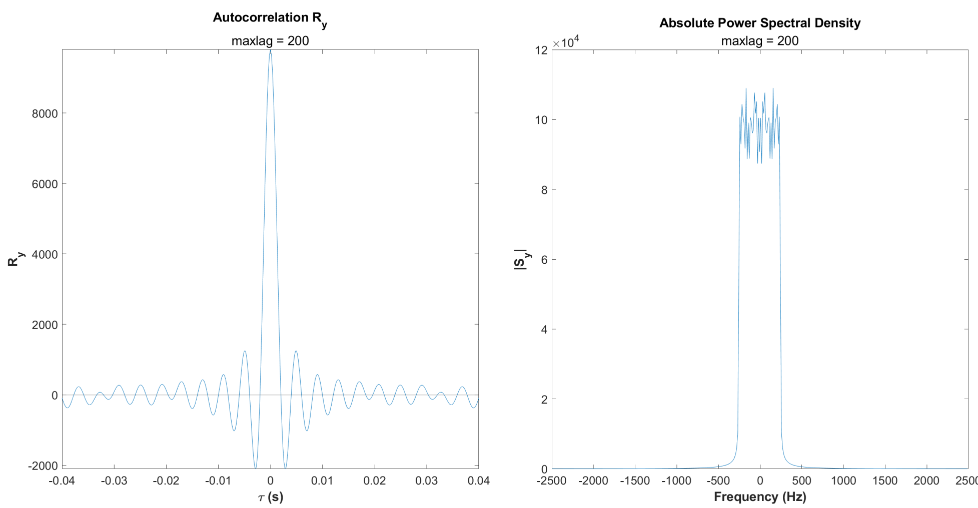

Figure 2: Autocorrelation and PSD for maxlag = 200

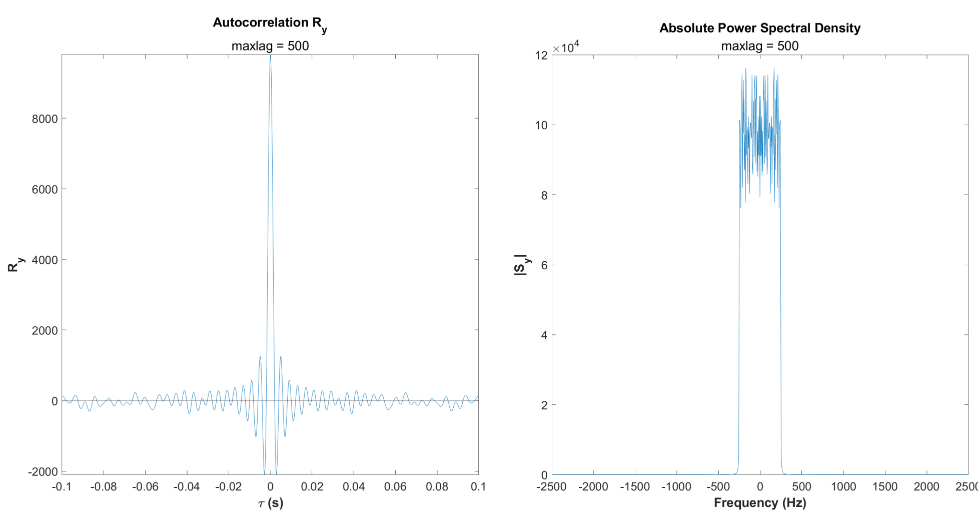

Figure 3: Autocorrelation and PSD for maxlag = 500

Numerical Experiment #2: A sinusoid buried in noise

The theoretical autocorrelation function of the output is derived in Equation 3.

$$\begin{aligned}
y(t) &= A\sin(2\pi f_c t + \theta) + w(t) \\
R_y(\tau) &= E\left\{y(t)y(t+\tau)\right\} \\
&= E\left\{\left[A\sin(\underbrace{2\pi f_c t + \theta}_{x}) + w(t)\right] \cdot \left[A\sin(\underbrace{2\pi f_c (t+\tau) + \theta}_{y}) + w(t+\tau)\right]\right\} \\
&= E\left\{A^2\sin(x)\sin(y) + A\sin(x)\cdot w(t+\tau) + w(t)\cdot A\sin(y) + w(t)\cdot w(t+\tau)\right\} \\
&= \frac{A^2}{2}\sin(2\pi f_c\tau) + 0 + 0 + \frac{N_0}{2}\delta(\tau) \\
&= \frac{A^2}{2}\sin(2\pi f_c\tau) + \frac{N_0}{2}\delta(\tau)
\end{aligned} \tag{3}$$

The PSD is just the Fourier transform of the autocorrelation function. The PSD is derived in Equation 4.

$$\begin{aligned}
R_y(\tau) &= \frac{A^2}{2}\sin(2\pi f_c\tau) + \frac{N_0}{2}\delta(\tau) \\
S_y(f) &= \left|F\left\{\frac{A^2}{2}\sin(2\pi f_c\tau)\right\} + F\left\{\frac{N_0}{2}\delta(\tau)\right\}\right| \\
&= \left|i\cdot\frac{A^2}{4}\left[\delta(f-f_c) - \delta(f+f_c)\right]\right| + \frac{N_0}{2} \\
&= \frac{A^2}{4}\left[\delta(f-f_c) - \delta(f+f_c)\right] + \frac{N_0}{2}
\end{aligned} \tag{4}$$

The theoretical results from Equation 3 and Equation 4 can be compared to the results of the numerical experiment in Figure 4. We can see that both the autocorrelation and PSD match between the theoretical results and the numerical results. Both autocorrelation functions result in sin functions with an impulse at $\tau = 0$. Similarly, both PSDs produce two impulses at what should be $-f_c$ and f_c, with a floor caused by the noise.

i) There is a peak at the zero lag in the autocorrelation plot. This peak is caused by the impulse in the autocorrelation function due to the white noise.

ii) We can see the impact of increasing the maxlag to 200 in Figure 5 and increasing the maxlag to 20000 in Figure 6. As we increase the maxlag the frequency resolution of the PSD increases, resulting in narrower peaks in the PSD plots. A table of measurements for the measured frequency estimates for f_c at each maxlag value is shown in Table 1. We can see that as the value of maxlag increases, the frequency estimate for f_c approaches 250 Hz due the the increase in frequency resolution. As maxlag increases, the size of the τ vector increases. The size of the frequency vector is dependent on the size of the τ vector. As the range of the frequency vector does not change, the increase in the number of frequency samples in the vector due to an increase maxlag increases the resolution of the frequency vector.

maxlag	f_c
100	236.32
200	243.14
20000	249.93

Table 1: Table of Measurements of maxlag and f_c

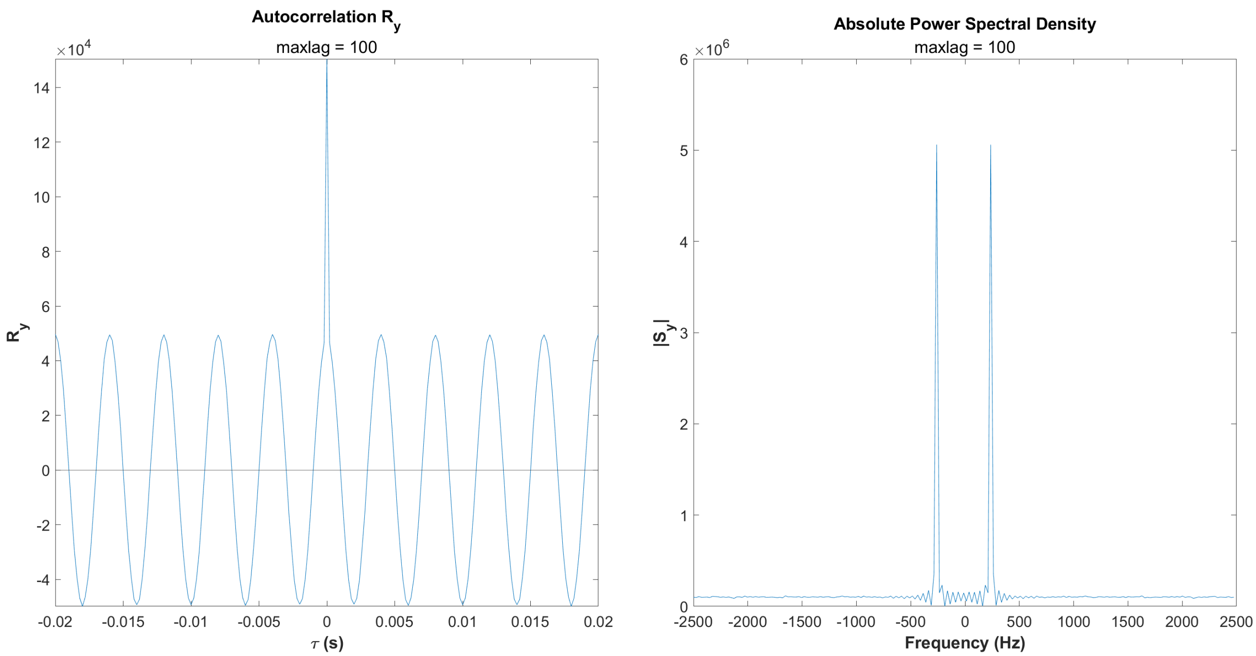

Figure 4: Autocorrelation and PSD for maxlag = 100

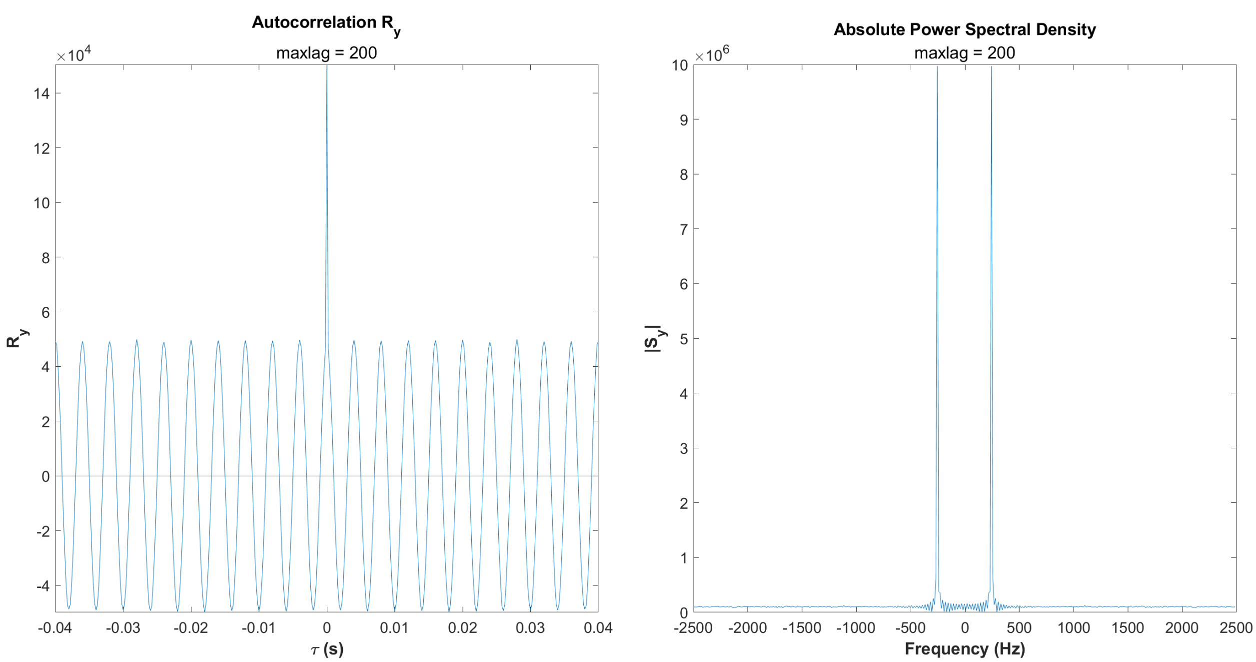

Figure 5: Autocorrelation and PSD for maxlag = 200

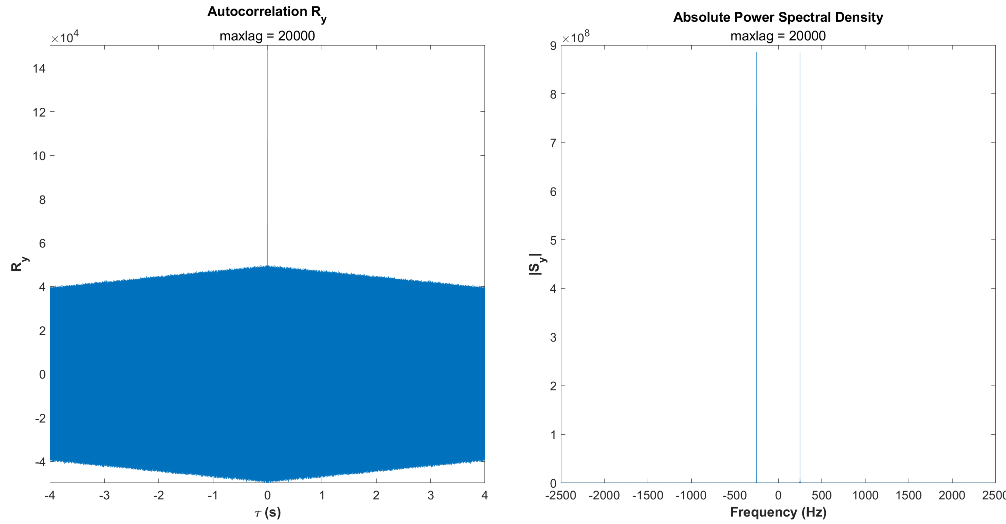

Figure 6: Autocorrelation and PSD for maxlag = 20000

iii) The signal yt is plotted as a function of time in Figure 7. We can see that the sinusoidal signal is buried in noise. We can estimate the frequency f_c directly from yt by taking the FFT of yt. We can clearly see in the FFT of yt shown in Figure 8 that the frequency f_c of the output signal yt is around 250 Hz, which matches our f_c estimates from after calculating the correlation. However, if the signal was buried in even more noise, it might not be possible to recover the signal with this methodology. In this case, it could even be possible to estimate the frequency of the signal by looking at the output signal yt in the time domain in Figure 7.

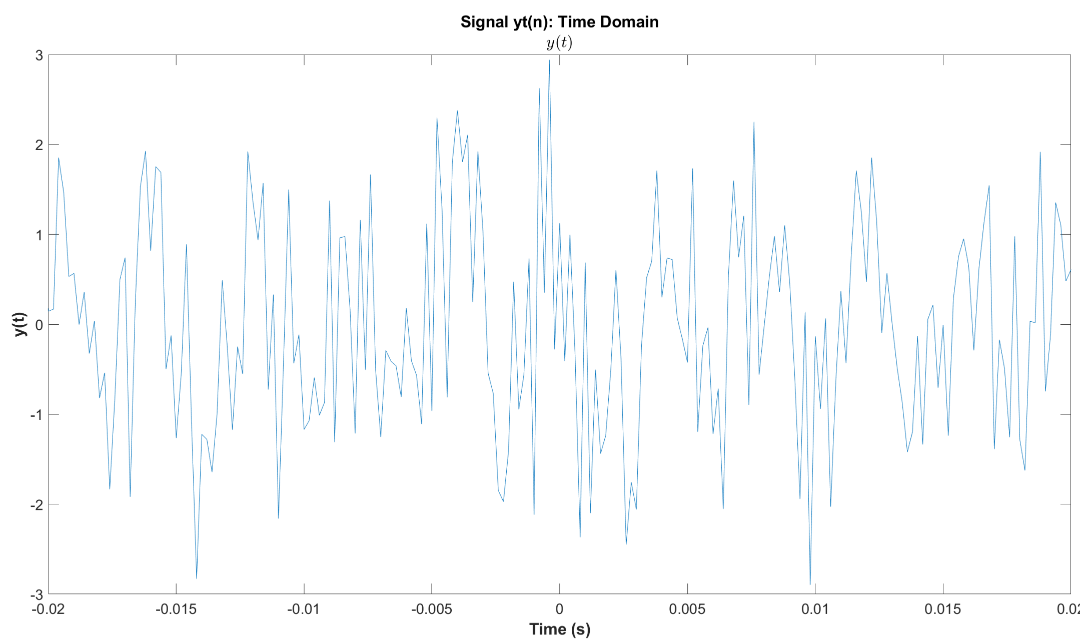

Figure 7: Signal yt: Time Domain

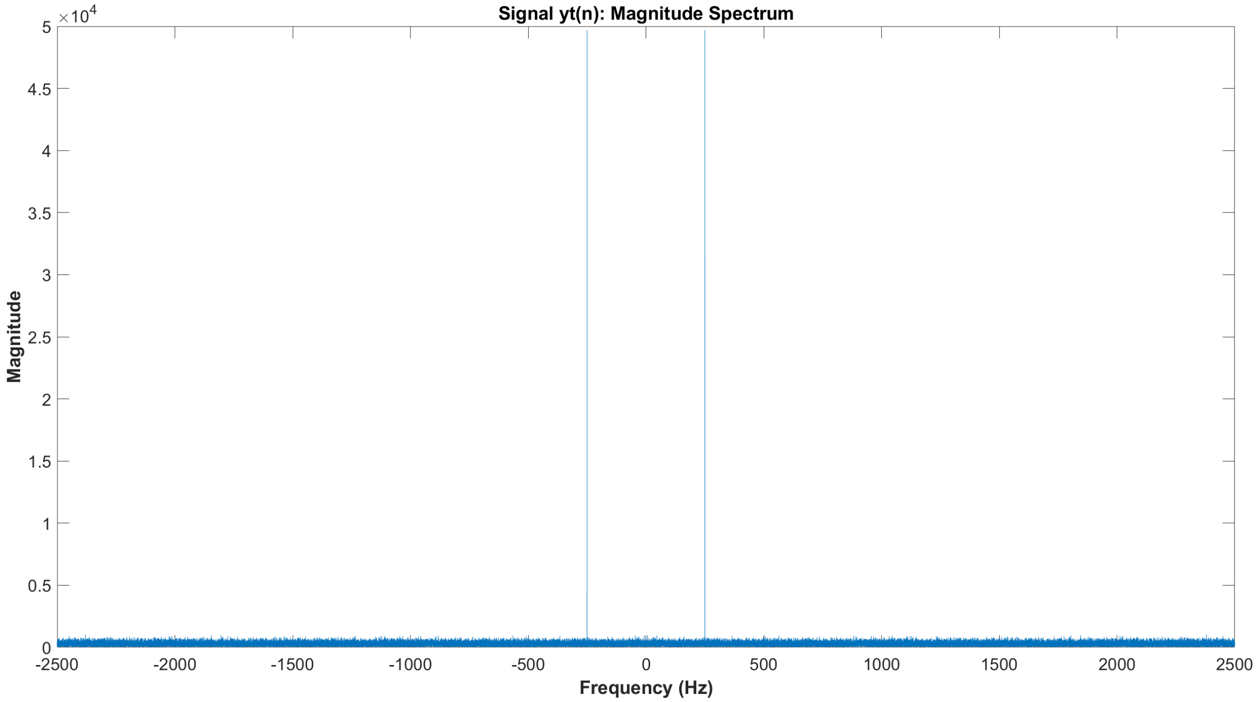

Figure 8: Signal yt: Magnitude Spectrum

Numerical Experiment #3: Delay Estimation

The signals $x(t)$ and $y(t)$ are plotted in the time domain in Figure 9. From the plot of $y(t)$ in the time domain, it is difficult to determine what the delay is as the signal is almost completely buried in noise. It is difficult to pinpoint the exact location of the peak of the delay due to the noise surrounding it.

An approach to determining the delay would be to plot the cross-correlation between x and y, and determining the value of τ where the autocorrelation function is at its maximum value. The cross-correlation between x and y is plotted in Figure 10. We can see that the maximum value of the cross-correlation function occurs at $\tau = 0.0424$ seconds. This delay roughly matches the delay found in the time domain plot of $y(t)$ in Figure 9.

The autocorrelation of y (xcorr(y,y)) cannot be used to estimate the delay as the maximum value of the autocorrelation function occurs at $\tau = 0$, which is not equal to the delay in the signal y.

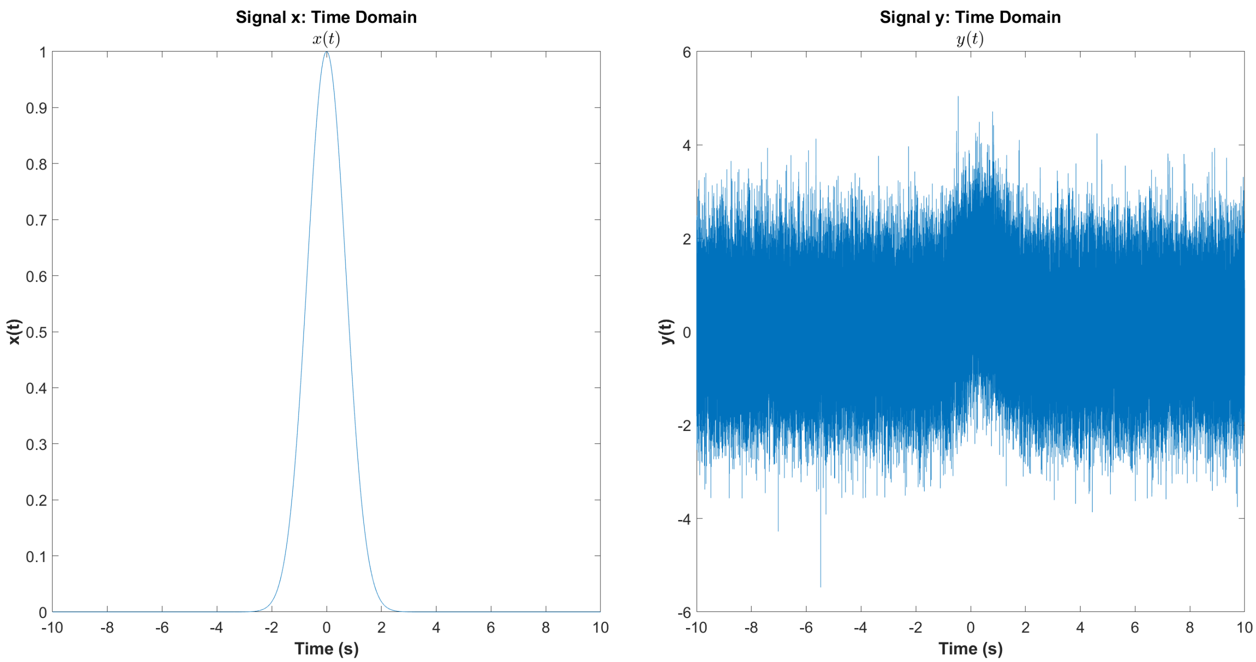

Figure 9: Signals $x(t)$ and $y(t)$: Time Domain

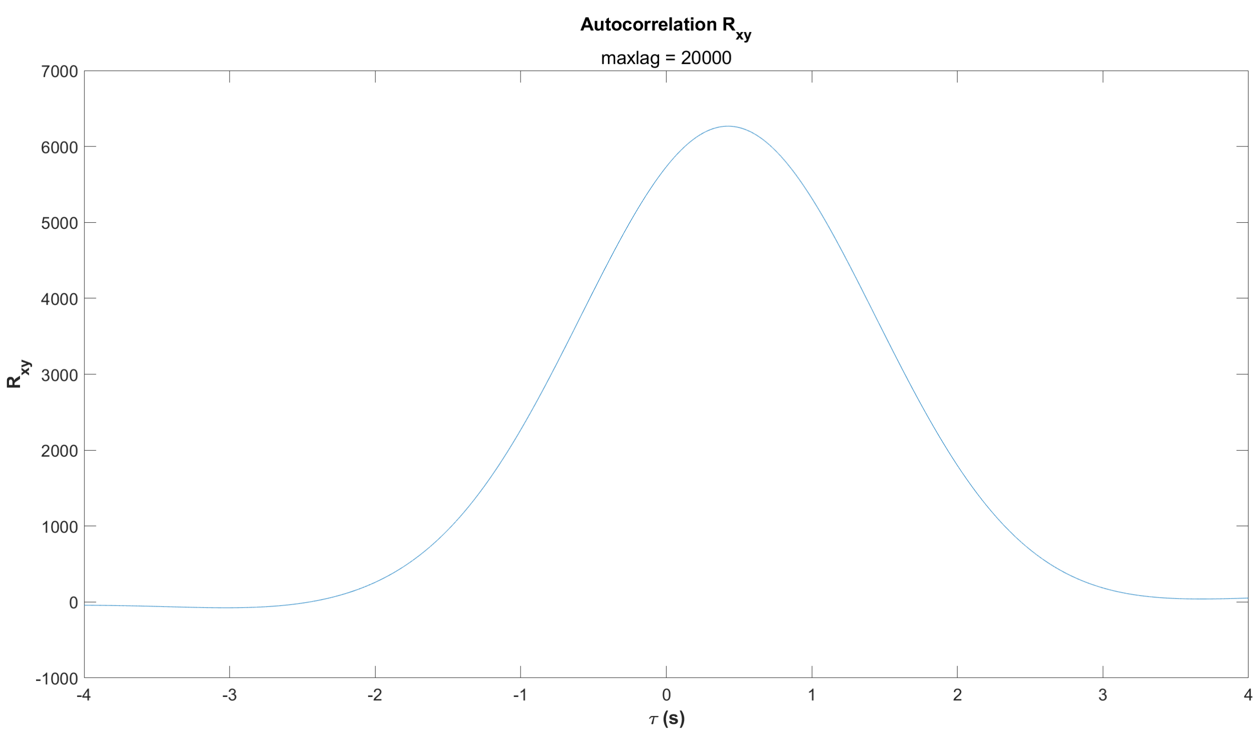

Figure 10: Cross-correlation between x and y