Algebraic $\mathcal{D}$-modules

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ABSTRACT. These notes are a work-in-progress!.

Contents

Part 1

Lecture 1: Algebraic $\mathcal{D}$-modules

lec1

1. Lecture 1

Part 2

Lecture 2: Hilbert Polynomials of filtered algebras

lec2

2. Lecture 2

Part 3

Lecture 3: Further properties of the algebra $\mathcal{D}$

$\boxed{\texttt{lec3}}$

3. section Plan

Part 4

Lecture 4: $\mathcal{O}$-coherent $\mathcal{D}$-modules, 1

lec4

4. section Plan

Part 5

Lecture 5: $\mathcal{O}$-coherent $\mathcal{D}$-modules, 2

$\boxed{\texttt{lec5}}$

5. section Plan

Part 6

Lecture 6: Functional dimension and homological algebra

lec6

6. section Plan

Part 7

Lecture 7: $\mathcal{D}$-modules on general affine varieties

lec7

7. section Plan

Part 8

Lecture 8: Functiorial Yoga, 1

lec8

8. section Plan

Part 9

Lecture 9: Proof of Kashiwara's theorem

lec9

9. section Plan

Part 10

Lecture 10: Theorem on Preservation of Holonomicity

lec10

10. section Plan

Part 11

Lecture 11: $\mathcal{D}$-modules on general varieties

lec11

11. section Plan

Part 12

Lecture 12: Equivariant $\mathcal{D}$-modules

lec12

12. section Plan

Part 13

Lecture 13: Derived Categories, 1

lec13

13. section Plan

Part 14

Lecture 14: Derived Categories, 2

lec14

14. section Plan

Part 15

Lecture 15: Derived Functors

lec15

15. section Plan

Part 16

Lecture 16: Functorial Yoga, 2

`lec16`

16. section Plan

Part 17

Lecture 17: The derived category of holonomic $\mathcal{D}$-modules

lec17

17. section Plan

Part 18

Lecture 18: Functorial Yoga, 3

lec18

18. The functors $\pi^!$,π_*, $\mathbb{D}$ and $\boxtimes$

THEOREM 18.1. *The functors $\pi^!$,π_*, $\mathbb{D}$ and $\boxtimes$ preserve the derived category of holonomic $\mathcal{D}$-modules.*

PROOF. The theorem is clear for $\boxtimes$. For $\mathbb{D}$ this is a local statement, so it reduces to the affine case where we already know it. For $\pi : X \to Y$, the case of $\pi^!$ reduces to the cases when both X and Y are affine, and again we already know it. It remains to show the property for π_*. Any π is a composition of a closed embedding, an open embedding and a projection

$$\mathbf{P}^N \times Y \to Y.$$

The case of a closed embedding is local, so again reduces to the affine case. In the case of an open embedding

$$j : U \to X$$

we may assume that X is affine. Our strategy will be to again reduce to the affine case; cover U by affine open sets

$$U = \bigcup_{\alpha=1}^{n} U_\alpha,$$

and consider the Cech complex

$$C_k = \bigoplus_{\{\alpha_1,\ldots,\alpha_k\}} j_{\alpha_1,\ldots,\alpha_k *} M|_{U_{\alpha_1}\cap\cdots\cap U_{\alpha_k}}$$

then by definition C represents $j_*(M)$. We know that C_k are holonomic, so C is holonomic, i.e. j_*M is holonomic. Now in the case of the projection

$$\pi : \mathbf{P}^N \times Y \to Y$$

, we may assume that Y is affine, and we already know the statement for $\pi' : \mathbf{A}^N \times Y \to Y$, so we proceed by induction. Let M on $\mathbf{P}^N \times Y$ be holonomic, and $j : \mathbf{A}^N \times Y \to \mathbf{P}^N \times Y$, then we have an exact triangle

$$M \to j_*M \to N$$

where N is supported on $\mathbf{P}^{N-1} \times Y$, so we have

$$\pi_*M \to \pi_*j_*M \to \pi_*N.$$

Now π_*N is holonomic by our induction hypothesis, and $\pi_*j_*M = (\pi \circ j)_*M$ is holonomic by reduction to the affine case, so π_*M is holonomic, as desired. $\square$

REMARK 18.2. Let $\pi : X \to \mathbf{pt}$ where X is smooth of dimension n, then

$$H^i(\pi_* \mathcal{O}) = H^{i+n}(X, \mathbf{C})$$

so by poincaré duality

$$H^i(\pi_! \mathcal{O}) = H^{-i}(\pi_* \mathcal{O})^* = H^{n-i}(X, \mathbf{C})^* = H_c^{n+i}(X, \mathbf{C}).$$

Therefore, $\pi_!$ is called the direct image with compact support, and its right adjoint $\pi^!$ is called inverse image with compact support. If instead we work on a singular variety, instead of $\mathcal{O}$ we can take $\mathbf{IC}_X$, then we will get the intersection cohomology $\mathbf{IH}^*$ (resp. the intersection cohomology with compact support $\mathbf{IH}_c^*$) with the appropriate shift.

18.1. An example. Let $X \to \mathbf{A}^2$ be given by $xy = 0$, then $X = X_1 \cup X_2$ with $X_1 = \{x = 0\}$ and $X_2 = \{y = 0\}$. We have

$$\mathbf{IC}_X = i_{1*}\mathcal{O} \oplus i_{2*}\mathcal{O}$$

for

$$i_j : X_j \to X.$$

Note that this scenario satisfies the required conditions. So

$$\dim \mathbf{IH}^i(X) = \begin{cases} 2 & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

So we see this cohomology theory is different from the ordinary cohomology.

18.2. another example. Let Γ be a finite group and let X be a smooth Γ-variety such that Γ acts on X freely at the generic point. Assume that $Y = X/\Gamma$ is a variety, then Γ acts on $\pi_* \mathcal{O}_X$. Let $V \to Y$ be the open set of points coming from the locus of trivial stabilizers

LEMMA 18.3.

$$j_{!*}(\pi_* \mathcal{O}_X\big|_V) = \pi_* \mathcal{O}_X$$

PROOF. Let $\hat{j} : U \to X$ where $U = \pi^{-1}(V)$. Since π is a finite morphism, it is proper, hence

$$j_!(\pi_* \mathcal{O}_X\big|_V) = \pi_*(\hat{j}_!(\mathcal{O}_X\big|_U)$$

$$j_*(\pi_* \mathcal{O}_X\big|_V) = \pi_*(\hat{j}_* \mathcal{O}_X\big|_U)$$

so

$$\mathbf{Im}(j_!(\pi_* \mathcal{O}_X\big|_V) \to j_*(\pi_* O_X\big|_V) = \pi_*(\mathbf{Im}(\hat{j}_! \mathcal{O}_X\big|_U \to \hat{j}_* \mathcal{O}_X\big|_U)) = \pi_* \mathcal{O}_X$$

since X is smooth. This means that

$$j_{!*}(\pi_*\mathcal{O}_X\big|_V) = \pi_*\mathcal{O}_X$$

$\square$

COROLLARY 18.4.

$$\mathbf{IC}_Y = (\pi_*\mathcal{O}_X)^\Gamma$$

COROLLARY 18.5.

$$H^i(\mathbf{IC}_Y) = \mathbf{IH}^i(Y)$$

so

$$H^{\dim X+i}(Y, \mathbf{C}) = H^{\dim X+i}(X, \mathbf{C})^\Gamma$$

18.3. another example. Let $Y \to \mathbf{C}^3$ be defined by $xy - z^2 = 0$, then

$$Y = \mathbf{C}^2/\mathbb{Z}/2\mathbb{Z}$$

and so

$$\mathbf{IH}^*(Y) = H^*(Y)$$

i.e. coincides with the usual cohomology.

18.4. another example. Let $\pi : X \to Y$ be a morphism of irreducible algebraic varieties. We say that π is small if

$$\operatorname{codim}\{y \in Y | \dim \pi^{-1}(y) \geq m\} \geq 2m + 1$$

PROPOSITION 18.6. *Suppose $\pi : X \to Y$ is a small resolution of singularities, then*

$$\pi_*\mathcal{O}_X = \mathbf{IC}_Y$$

so

$$\mathbf{IH}^*(Y) = H^*(X)$$

EXAMPLE 18.7. Let $\mathfrak{g}$ be a simple lie algebra, and

$$\widetilde{\mathfrak{g}} = \{(x, \mathfrak{b}) | x \in \mathfrak{g}, x \in \mathfrak{b}, \mathfrak{b} \text{ a borel subalgebra}\}$$

then we have a map

$$\pi : \widetilde{\mathfrak{g}} \to \mathfrak{g} \times_{\mathfrak{h}/W} \mathfrak{h}$$

given by $\pi(x, \mathfrak{b}) = (x, \hat{x})$ where $\hat{x}$ is the image of x under the canonical quotient

$$\mathfrak{b}/[\mathfrak{b}, \mathfrak{b}] = \mathfrak{h}$$

i.e. the standard cartan. It is known that π is a small resolution called the *Grothendieck simultaneous resolution*, thus

$$\mathbf{IH}^*(\mathfrak{g} \times_{\mathfrak{h}/W} \mathfrak{h}) = H^*(\widetilde{\mathfrak{g}}) = H^*(G/B)$$

EXAMPLE 18.8. $\mathfrak{g} = \mathfrak{sl}_2$, then $\mathfrak{g} \times_{\mathfrak{h}/W} \mathfrak{h}$ is a double cover of $\mathfrak{g}$ branched over the nilpotent cone $xy + z^2 = 0$, hence it is the surface $t^2 = xy + z^2$—a homogenous quadric Q in $\mathbf{C}^4$. Also, note that $\widetilde{g}$ in this case is the total space of $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ as a bundle over $\mathbf{P}^1$. It follows that

$$\mathbf{IH}^j(Q) = \begin{cases} \mathbf{C} & \text{if } j = 0 \text{ or } j = 2 \\ 0 & \text{otherwise} \end{cases}$$

while

$$H^j(Q) = \begin{cases} \mathbf{C} & \text{if } j = 0 \\ 0 & \text{otherwise} \end{cases}$$

Part 19

Lecture 19: $\mathcal{D}$-modules with regular singularities, 1

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19. section Plan

Part 20

Lecture 20: $\mathcal{D}$-modules with regular singularities, 2

lec20

20. More on the Riemann-Hilbert Map

Last lecture we considered the Riemann Hilbert map

$$\mathbf{RH} : \begin{array}{l}\text{Algebraic vector bundles on } X \\ \text{with an } \mathbf{RS} \text{ flat connection}\end{array} \longrightarrow \mathbf{Rep}\pi_1(X)$$

which assigns to each bundle (E, ∇) the monodromy representation of ∇. Note that both categories for fixed rank r have a moduli space of objects which generically is an algebraic variety, and so in particular, a complex manifold. However, the map **RH** is not algebraic; it is only holomorphic.

Let's consider two examples of what this map does.

21. Example 1: case of a complete curve

Let X be a projective curve of genus g, then the moduli space $\mathcal{M}_{dR}$ of line bundles with connection looks as follows: we have a map $\mathcal{M}_{dR} \longrightarrow \mathbf{Pic}_0(X) = \mathbf{Jac}(X)$ whose fiber is $\mathbf{A}^g$, an affine space bundle. Here $\mathbf{A}^g$ is a torsor over $H^0(X, \Omega)$, and in particular is an algebraic variety of dimension $2g$.

On the other hand, the betti moduli space $\mathcal{M}_b$ is the moduli space of representations of $\pi_1(X)$. Once we fix generators for $\pi_1(X)$:

$$a_1, \ldots, a_g, b_1, \ldots, b_g, \prod [a_i, b_i] = 1,$$

then we can identify

$$\mathcal{M}_b \simeq (\mathbf{C}^*)^{2g}.$$

Here the **RH** map is a holomorphic isomorphism

$$\mathcal{M}_{dR} \simeq \mathcal{M}_b.$$

Clearly it cannot be algebraic, since we have

LEMMA 21.1. *Any regular map* $\mathbf{C}^* \to \mathbf{Jac}(X)$ *is constant.*

PROOF. The map extends to

$$\mathbf{CP}^1 \to \mathbf{Jac}(X),$$

passage to the universal cover then yields

$$\mathbf{CP}^1 \to \mathbf{C}^n.$$

Now Liouville's theorem shows this map is constant. □

In more detail, **RH** is inverse to a map

$$f : \mathcal{M}_b \to \mathcal{M}_{dR}.$$

To construct f, let's construct

$$\pi : (\mathbf{C}^*)^{2g} \to \mathbf{Jac}(X) = \mathbf{Pic}_0(X);$$

which is an affine space bundle. To define f, consider the $4g$-gon:

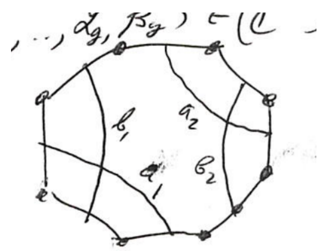

fig-4g-gon.png

Given

$$(\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g) \in (\mathbf{C}^*)^{2g}$$

we consider the trivial line bundle on the polygon and glue together a line bundle on X by using α_1 along a_1, β_1 along b_1 etc.

22. Example 2: the projective line minus four points

Let $X = \mathbf{P}^1 - \{0, 1, \lambda, \infty\}$ with $\lambda \neq 0, 1, \infty$ and let's restrict to connections with trivial determinant. Now $\mathcal{M}_{dR}$ has an open set $\mathcal{M}^\circ_{dR}$ of connections which have first order poles on the trivial bundle. Let's also restrict further to fixed monodromy; i.e. $\mathcal{M}^\circ_{dR}$ is the set of connections

$$\nabla = \partial - \frac{a_0}{z} - \frac{a_1}{z-1} - \frac{a_\lambda}{z-\lambda}, \mathbf{Tr}(a_j) = 0$$

and let $a_\infty = -a_0 - a_1 - a_\lambda$. Denote by

$$\mathcal{M}^\circ_{dR}(\alpha_0, \alpha_1, \alpha_\lambda, \alpha_\infty) = \{\nabla, a_j \sim \begin{pmatrix} \alpha_j & 0 \\ 0 & -\alpha_j \end{pmatrix}\}$$ then letting $A_j \sim \begin{pmatrix} e^{2\pi i \alpha_j} & 0 \\ 0 & e^{-2\pi i \alpha_j} \end{pmatrix}$ we see the map

$$\mathbf{RH} : \mathcal{M}_b(\alpha_0, \alpha_1, \alpha_\lambda, \alpha_\infty) \longrightarrow \{A_0, A_1, A_\lambda, A_\infty, A_0 A_1, A_\lambda A_\infty = 1\}$$

This map is highly transcendental.

Namely, let $\mathcal{P} \in \mathcal{M}_b(\alpha_0, \alpha_1, \alpha_\lambda, \alpha_\infty)$ and consider the point $Q_\lambda = \mathbf{RH}^{-1}_\lambda(\mathcal{P}) \in \mathcal{M}^\circ_{dR}(\alpha_0, \alpha_1, \alpha_\lambda, \alpha_\infty)$. This defines a flow on $\mathcal{M}^\circ_{dR}(\alpha_0, \alpha_1, \alpha_\lambda, \alpha_\infty)$ known as the Painlavé-6 flow.

23. Holonomic $\mathcal{D}$-modules with RS in higher dimensions

23.1. Constructible sheaves and complexes. Let X be a $\mathbf{C}$-algebraic variety. Denote by X^{an} the corresponding analytic variety considered with the classical topology. Let $\mathbf{C}_X$ be the constant sheaf on X^{an} and $\mathbf{Sh}(X^{an})$ the category of $\mathbf{C}_X$-modules i.e. sheaves of $\mathbf{C}$-vector spaces. The derived category of bounded $\mathbf{C}_X$-complexes will be denoted $\mathbf{D}(X^{an})$.

DEFINITION 23.1. A $\mathbf{C}_X$-module $\mathcal{F}$ is constructible if there exists a stratification

$$X = \cup_i X_i$$

of X by locally closed algebraic subvarieties such that $\mathcal{F}\big|_{X^{an}}$ is a locally constant complex of finite dimensional vector spaces.

REMARK 23.2. Note that a $\mathbf{C}_X$-complex is constructible if all of its cohomology sheaves are constructible as $\mathbf{C}_X$-vector spaces.

The full subcategory of $\mathbf{D}(X^{an})$ consisting of constructible complexes will be denoted by $\mathbf{D}_{con}(X^{an})$.

Any morphism $\pi : X \to Y$ of algebraic varieties induces a continuous map $\pi^{an} : X^{an} \to Y^{an}$, and we can consider the functors

$$\pi_!, \pi_* : \qquad \mathbf{D}(X^{an}) \longrightarrow \mathbf{D}(Y^{an})$$

$$\pi^*, \pi^! : \qquad \mathbf{D}(Y^{an}) \longrightarrow \mathbf{D}(A^{an})$$

We also have

$$\mathbb{D} : \qquad \mathbf{D}(X^{an}) \longrightarrow \mathbf{D}(X^{an})$$

We have

THEOREM 23.3. *These functors preserve the subcategory of derived constructible sheaves $\mathbf{D}_{con}$, and on them we have*

$$\mathbb{D}^2 = \mathbf{Id}$$
$$\mathbb{D}\pi^*\mathbb{D} = \pi^!$$
$$\mathbb{D}\pi_*\mathbb{D} = \pi_!$$

and $\mathbb{D}M = \mathbf{RHom}_{an}(M, \mathbf{C}_X)$.

23.2. De Rham Functor. Let $\mathcal{O}_X^{an}$ be the structure sheaf of X^{an}. We will assign to each $\mathcal{O}_X$-module M the corresponding *analytic sheaf* of $\mathcal{O}_X^{an}$-modules M^{an}, which is locally given by

$$M^{an} = \mathcal{O}_X^{an} \otimes_{\mathcal{O}_X} M$$

.

This defines an exact functor

$$an : M(\mathcal{O}_X) \to M(\mathcal{O}_{X^{an}})$$

and in particular an exact functor

$$an : M(\mathcal{D}_X) \to M(\mathbf{D}_X^{an})$$

, where $\mathcal{D}_X^{an}$ is the sheaf of analytic differential operators.

DEFINITION 23.4. The *De Rham Functor*

$$\mathbf{DR} : \mathbf{D}^b(\mathcal{D}_X) \to \mathbf{D}^b(X^{an}) = \mathbf{D}^b(\mathbf{Sh}(X^{an}))$$

is

$$\mathbf{DR}(M^\circ) = \Omega_X^{an} \otimes_{\mathcal{D}_X^{an}} (M^\circ)^{an}$$

REMARK 23.5. Since $\mathbf{dR}(\mathcal{D}_X)$ is a locally projective resolution of Ω_X we have

$$\mathbf{DR}(M^\circ) = \mathbf{dR}(\mathcal{D}_X^{an}) \otimes_{\mathcal{D}_X^{an}} (M^\circ)^{an}[\dim X]$$

In particular, if M is an $\mathcal{O}$-coherent $\mathcal{D}_X$-module corresponding to a vector bundle with a flat connection and $\mathcal{L} = M^{flat}$ is the local system of flat sections, then

$$\mathbf{DR}(M) = \mathcal{L}[dimX]$$

by Poincaré's lemma.

Here is the main theorem about the connection between $\mathcal{D}$-modules and constructible sheaves:

THEOREM 23.6.
- $\mathbf{DR}(\mathbf{D}_{hol}(\mathcal{D}_X)) \subset \mathbf{D}_{con}(X^{an})$, *and on* $\mathbf{D}_{hol}$ $\mathbf{DR}$ *commutes with both tensor product and* $\mathbf{D}$.
- *On the subcategory* $\mathbf{D}_{rs}$ *the functor* $\mathbf{DR}$ *commutes with all of the above functors*
- $\mathbf{DR} : \mathbf{D}_{rs}(\mathcal{D}_X) \to \mathbf{D}_{con}(X^{an})$ *is an equivalence.*

23.3. Definitions concerning vector bundles with flat connection. Let M be a vector bundle on X with a flat connection.

DEFINITION 23.7. M lies in $\mathbf{D}_{rs}$ if the restriction of M to any curve has regular singularities.

DEFINITION 23.8. An irreducible M in $\mathbf{D}_{hol}$ has *regular singularities* if $M = j_{!*}L$ for L a vector bundle with flat connection and regular singularities.

REMARK 23.9. An object $M \in \mathbf{D}_{hol}$ has regular singularities if and only if all composition factors have regular singularities.