Algebraic $\mathcal{D}$-modules

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ABSTRACT. These notes are a work-in-progress!.

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13. Derived Categories

Let $\mathcal{A}$ be an abelian category, and $\mathcal{C}(\mathcal{A})$ be the category of all complexes over $\mathcal{A}$. Let $\mathcal{C}^+(\mathcal{A})$ be the category of complexes K with $K^i = 0$ for $i << 0$, and analogously let $\mathcal{C}^-(\mathcal{A})$ be the category of complexes with $K^i = 0$ for $i >> 0$. Let $\mathcal{C}^b(\mathcal{A})$ be the intersection of these two categories; i.e. the category of bounded complexes. Let $\mathcal{C}_0(\mathcal{A})$ be the category of complexes with zero differential; we have a functor of cohomology

$$H : \mathcal{C}(\mathcal{A}) \to \mathcal{C}_0(\mathcal{A}) = \bigoplus_{i \in \mathbb{Z}} \mathcal{A}$$

which attaches to every complex its cohomology.

Recall that $f : \mathcal{C} \to \mathcal{D}$ is a quasi-isomorphism if

$$H(f) : H(\mathcal{C}) \to H(\mathcal{D})$$

is an isomorphism.

THEOREM 13.1. *There exists a unique (up to canonical equivalence) category $\mathbf{D}(\mathcal{A})$, called the derived category, together with a functor*

$$Q : \mathcal{C}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$$

such that

- *For $f : K^* \to L^*$ a quasi-isomorphism, $Q(f)$ is an isomorphism.*
- *The pair $(\mathbf{D}, Q)$ is universal for this property, i.e. any functor*

  $$F : \mathcal{C}(\mathcal{A}) \to \mathbf{D}'$$

  sending quasi-isomorphisms to isomorphisms factors through $\mathbf{D}(\mathcal{A})$ i.e. there exists

  $$G : \mathbf{D}(\mathcal{A}) \to \mathbf{D}'$$

  satisfying

  $$F = G \circ Q$$

  .

The objects of $\mathbf{D}(\mathcal{A})$ are the objects of $\mathcal{C}(\mathcal{A})$.

REMARK 13.2. More generally, if $\mathcal{C}$ is any category, and $\mathcal{S}$ is any class of morphisms, we can define the category $\mathcal{C}[\mathcal{S}^{-1}]$ and a functor

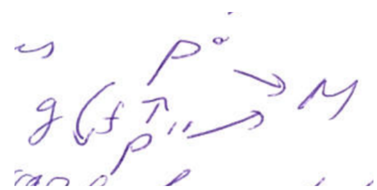
$$\mathcal{C} \to \mathcal{C}[\mathcal{S}^{-1}]$$

which satisfy both conditions outlined in the theorem above. In our case $\mathbf{D}(\mathcal{A}) = \mathcal{C}(\mathcal{A})[\mathcal{S}^{-1}]$ where $\mathcal{S}$ is the class of quasi-isomorphisms.

14. Motivation

Suppose $\mathcal{A} = A - \mathbf{Mod}$, then any $M \in \mathcal{A}$ has a projective resolution, and any two resolutions are homotopy equivalent: there is a pair of morphisms

fig-lec13-motiv.png



such that $f \circ g$ and $g \circ f$ are homotopic to the identity, i.e.

$$f \circ g - 1 = hd + dh$$

etc. We want all these resolutinos to be one object, defined canonically. In particular, we want this to be the case for $M = 0$, i.e. we want any exact complex of projectives to be zero.

14.1. Structure of derived categories.

- Shift functor: $K^* \to K^*[i]$, $K[i]^j = K^{j+i}$.
- Distinguished triangles.

For a left exact functor $F : \mathcal{A} \to \mathcal{B}$ we'll define

$$\mathbf{R}F : \mathbf{D}(\mathcal{A}) \to \mathbf{D}(\mathcal{B})$$

and similarly for a right exact functor $G : \mathcal{A} \to \mathcal{B}$ we'll define

$$\mathbf{L}G : \mathbf{D}(\mathcal{A}) \to \mathbf{D}(\mathcal{B}).$$

We will want to say that it is exact, so we'll need an analogue of short exact sequences in the usual sense[1]We define the notion of the *cone of a morphism of complexes*: let $f : K^* \to L^*$ be a morphism of complexes; we define the complex $C(f)^*$ called the *cone* of f, as follows:

$$C(f)^* = K^*[1] \oplus L^*$$

i.e.

$$C(f)^i = K^{i+1} \oplus L^i$$

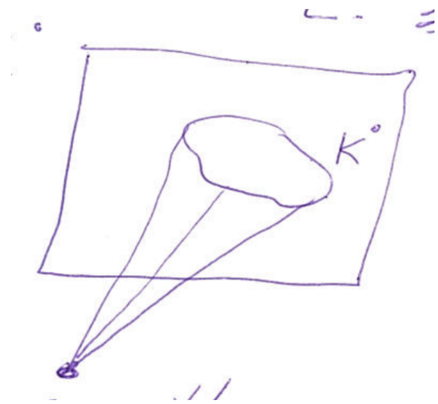
with differential

$$d(k^{i+1}, l^i) = (-d_K k^{i+1}, f(k^{i+1} + d_L l^i)$$

[1]Since the category $\mathbf{D}(\mathcal{A})$ is not abelian, and we don't have the notion of kernel and cokernel of morphisms, so we have to replace these notions with something else.

REMARK 14.1. Suppose that K^* and L^* come from actual simplicial complexes, and we are given a simplicial map $K^* \to L^*$, then we can consider the L^* space obtained by gluing a cone over K to L along the map f

fig-lec13-cone.png



Then teh complex $C^*(f)$ computes the (reduced) homology of this space, and so the cohomology of $C^*(f)$ is the reduced homology of the space obtained by contracting the image of f to a point (if f is an inclusion).

EXERCISE 14.2. Show that if f is an embedding $C^*(f)$ is isomorphic to L^*/K^*.

LEMMA 14.3. *The sequence*

$$\cdots \to H^i(K) \to H^i(L) \to H^i(C(f)) \to H^{i+1}(K) \to \cdots$$

is exact (where the connecting map correspondings to the projection $C(f) \to K[1]$

PROOF. If $f : K^* \to L^*$, then $C(f) \simeq L^*/K^*$ (quasi-isomorphism), so this is the well-known long exact sequence. More generally, we define the *cylinder* of f:

$$\mathbf{Cyl}(f) = K^* \oplus K^*[1] \oplus L^*$$

with

$$d(k^i, k^{i+1}, l^i) = (d_K k^i - k^{i+1}, -d_K k^{i+1}, f(k^{i+1} - d_L l^i)$$

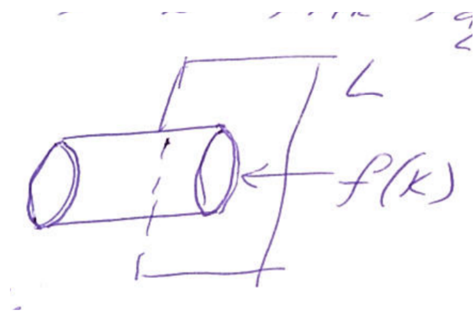


fig-lec13-cyl.png

The natural inclusion

$$L^* \to \mathbf{Cyl}(f)$$

is a quasi-isomorphism, and the sequence

$$K^* \to \mathbf{Cyl}(f) \to C(f)$$

is an exact sequence of complexes, as desired. □

DEFINITION 14.4. A *distinguished triangle* in $\mathbf{D}(\mathcal{A})$ is a triangle

$$X \to Y \to Z \to X[1]$$

which is the image under Q of

$$K^* \to L^* \to C(f) \to K[1]$$

15. Main problem

The cone of $f : X \to Y$ for $X, Y \in \mathbf{D}(\mathcal{A})$ is not canonically defined. It is unique, but only up to a non-canonical isomorphism, because in order to construct the cone we had to pick complexes representing X and Y.

DEFINITION 15.1. Let $\mathcal{A}, \mathcal{B}$ be categories. A functor $F : \mathbf{D}(\mathcal{A}) \to \mathbf{D}(\mathcal{B})$ is called exact if it commutes with the shift functor and maps distinguished triangles to distinguished triangles.

DEFINITION 15.2. $X \in \mathbf{D}(\mathcal{A})$ is an H^0-complex if $H^i(X) = 0$ for $i \neq 0$.

LEMMA 15.3. *The inclusion $\mathcal{A} \to \mathbf{D}(\mathcal{A})$ induces an equivalence between $\mathcal{A}$ and the full subcategory of H^0 complexes.*

Let $X, Y \in \mathcal{A}$ then we can define **Ext** by

$$\mathbf{Ext}^i(X, Y) = \mathbf{Hom}_{\mathbf{D}(\mathcal{A})}(X^*, Y^*[i]).$$

(Note that $\mathbf{Hom}_{\mathcal{A}}(X, Y) = \mathbf{Hom}_{\mathbf{D}(\mathcal{A})}(X^*, Y^*)$). If $\mathcal{A}$ has enough projectives or injectives, then one can show that it is the same definition as before.

16. Another way of thinking about the derived category

It is not hard to prove that $\mathbf{D}(\mathcal{A})$ exists and is unique, but not easy to understand what morphisms are.

EXAMPLE 16.1. Suppose $\mathcal{C}$ is a category with one object X and $\mathbf{End}X = A$, a monoid. Let $S \to A$ be a multiplicative subset, then $A[S^{-1}]$ is the monoid consisting of words with letters $a \in A$ and $s^{-1}, s \in S$ with equivalence relations

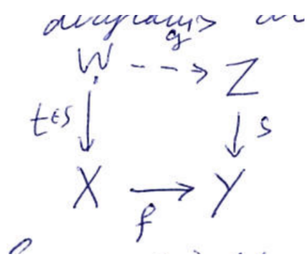
$$\begin{cases} \cdots(a_1)(a_2)\cdots = \cdots(a_1a_2)\cdots \\ \cdots s_1^{-1}s_2^{-1}\cdots = \cdots(s_1s_2)^{-1}\cdots \\ \cdots s^{-1}s\cdots = \cdots = \cdots ss^{-1}\cdots \end{cases}$$

So it's hard to understand what morphisms are, but things are better if we have "the condition", i.e. the class S is *localizable*.

DEFINITION 16.2. A multiplicative closed set S is localizable if

- $\forall s : Z \to Y, s \in S$ and $f : X \to Y$ there exists $t \in S$ and g such that the following diagrams are commutative

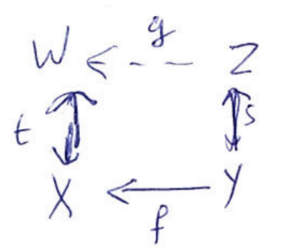
fig-lec13-comm1.png



i.e.

$$s^{-1}f = gt^{-1}$$

fig-lec13-comm2.png



i.e.

$$fs^{-1} = t^{-1}g$$

So a right fraction can be rewritten as a left fraction, and vice-versa, and

- Let $f : X \to Y$ then $\exists s \in S$ such that $sf = sg$ if and only if $\exists t \in S$ with
$$ft = gt$$
(this will be a condition for f to equal g in the localization).

If S is localizable, then the localization $\mathcal{C}[S^{-1}]$ has a nice description: morphisms can be represented by *roofs* (left or right)

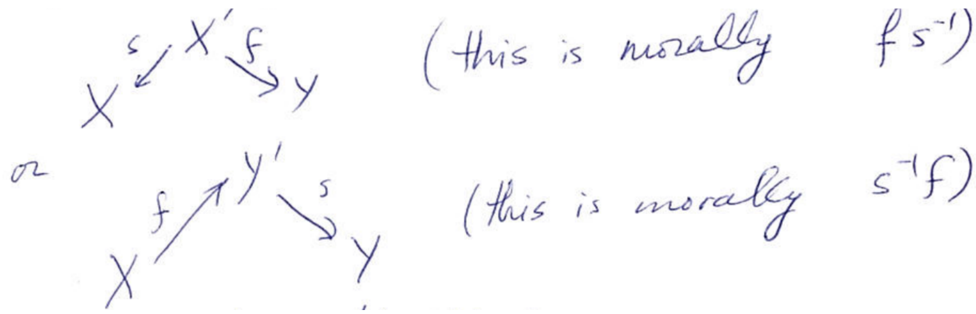
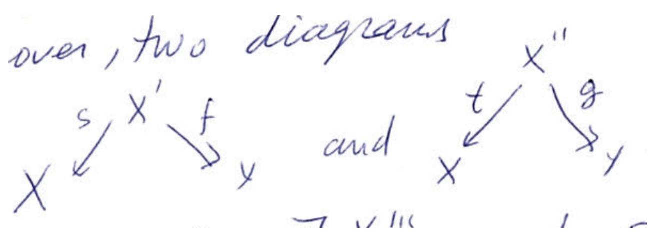


fig-lec13-roofs.png

Moreover, two diagrams



ec13-two-diagrams.png

define the same morphism if $\exists X'''$ and $r : X''' \to X'$, $h : X''' \to X''$, $r, h \in S$ such that the following diagram is commutative

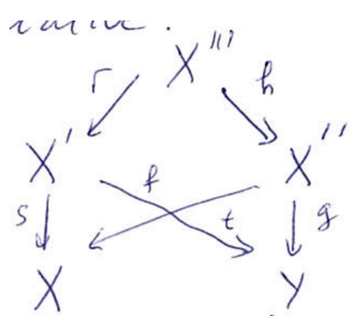


fig-lec13-equiv.png

i.e. to show that $fs^{-1} = gt^{-1}$, we find r, h such that
$$fr = gh, sr = th$$
then
$$fs^{-1} = frr^{-1}s^{-1} = ghr^{-1}s^{-1} = ghh^{-1}t^{-1} = gt^{-1}$$

PROPOSITION 16.3. *If $\mathcal{S}$ is a localizable class, then $\mathcal{C}[\mathcal{S}^{-1}]$ is the category with objects given by objects of $\mathcal{C}$ and morphisms given by roofs with the equivalence given above.*

Part 14

Lecture 14: Derived Categories, 2

lec14

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Part 15

Lecture 15: Derived Functors

`lec15`

18. section Plan

Part 16

Lecture 16: Functorial Yoga, 2

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Part 17

Lecture 17: The derived category of holonomic $\mathcal{D}$-modules

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Part 18

Lecture 18: Functorial Yoga, 3

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21. The functors $\pi^!$,π_*, $\mathbb{D}$ and $\boxtimes$

THEOREM 21.1. *The functors $\pi^!$,π_*, $\mathbb{D}$ and $\boxtimes$ preserve the derived category of holonomic $\mathcal{D}$-modules.*

PROOF. The theorem is clear for $\boxtimes$. For $\mathbb{D}$ this is a local statement, so it reduces to the affine case where we already know it. For $\pi : X \to Y$, the case of $\pi^!$ reduces to the cases when both X and Y are affine, and again we already know it. It remains to show the property for π_*. Any π is a composition of a closed embedding, an open embedding and a projection

$$\mathbf{P}^N \times Y \to Y.$$

The case of a closed embedding is local, so again reduces to the affine case. In the case of an open embedding

$$j : U \to X$$

we may assume that X is affine. Our strategy will be to again reduce to the affine case; cover U by affine open sets

$$U = \bigcup_{\alpha=1}^{n} U_\alpha,$$

and consider the Cech complex

$$C_k = \bigoplus_{\{\alpha_1,\dots,\alpha_k\}} j_{\alpha_1,\dots,\alpha_k *} M|_{U_{\alpha_1} \cap \dots \cap U_{\alpha_k}}$$

then by definition C represents $j_*(M)$. We know that C_k are holonomic, so C is holonomic, i.e. j_*M is holonomic. Now in the case of the projection

$$\pi : \mathbf{P}^N \times Y \to Y$$

, we may assume that Y is affine, and we already know the statement for $\pi' : \mathbf{A}^N \times Y \to Y$, so we proceed by induction. Let M on $\mathbf{P}^N \times Y$ be holonomic, and $j : \mathbf{A}^N \times Y \to \mathbf{P}^N \times Y$, then we have an exact triangle

$$M \to j_*M \to N$$

where N is supported on $\mathbf{P}^{N-1} \times Y$, so we have

$$\pi_*M \to \pi_*j_*M \to \pi_*N.$$

Now π_*N is holonomic by our induction hypothesis, and $\pi_*j_*M = (\pi \circ j)_*M$ is holonomic by reduction to the affine case, so π_*M is holonomic, as desired. $\square$

REMARK 21.2. Let $\pi : X \to \mathbf{pt}$ where X is smooth of dimension n, then

$$H^i(\pi_*\mathcal{O}) = H^{i+n}(X, \mathbf{C})$$

so by poincaré duality

$$H^i(\pi_!\mathcal{O}) = H^{-i}(\pi_*\mathcal{O})^* = H^{n-i}(X, \mathbf{C})^* = H_c^{n+i}(X, \mathbf{C}).$$

Therefore, $\pi_!$ is called the direct image with compact support, and its right adjoint $\pi^!$ is called inverse image with compact support. If instead we work on a singular variety, instead of $\mathcal{O}$ we can take $\mathbf{IC}_X$, then we will get the intersection cohomology $\mathbf{IH}^*$ (resp.the intersection cohomology with compact support $\mathbf{IH}_c^*$) with the appropriate shift.

21.1. An example. Let $X \to \mathbf{A}^2$ be given by $xy = 0$, then $X = X_1 \cup X_2$ with $X_1 = \{x = 0\}$ and $X_2 = \{y = 0\}$. We have

$$\mathbf{IC}_X = i_{1*}\mathcal{O} \oplus i_{2*}\mathcal{O}$$

for

$$i_j : X_j \to X.$$

Note that this scenario satisfies the required conditions. So

$$\dim \mathbf{IH}^i(X) = \begin{cases} 2 & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

So we see this cohomology theory is different from the ordinary cohomology.

21.2. another example. Let Γ be a finite group and let X be a smooth Γ-variety such that Γ acts on X freely at the generic point. Assume that $Y = X/\Gamma$ is a variety, then Γ acts on $\pi_*\mathcal{O}_X$. Let $V \to Y$ be the open set of points coming from the locus of trivial stabilizers

LEMMA 21.3.

$$j_{!*}(\pi_*\mathcal{O}_X\big|_V) = \pi_*\mathcal{O}_X$$

PROOF. Let $\hat{j} : U \to X$ where $U = \pi^{-1}(V)$. Since π is a finite morphism, it is proper, hence

$$j_!(\pi_*\mathcal{O}_X\big|_V) = \pi_*(\hat{j}_!(\mathcal{O}_X\big|_U)$$

$$j_*(\pi_*\mathcal{O}_X\big|_V) = \pi_*(\hat{j}_*\mathcal{O}_X\big|_U)$$

so

$$\mathbf{Im}(j_!(\pi_*\mathcal{O}_X\big|_V) \to j_*(\pi_*O_X\big|_V) = \pi_*(\mathbf{Im}(\hat{j}_!\mathcal{O}_X\big|_U \to \hat{j}_*\mathcal{O}_X\big|_U)) = \pi_*\mathcal{O}_X$$

since X is smooth. This means that

$$j_{!*}(\pi_*\mathcal{O}_X\big|_V) = \pi_*\mathcal{O}_X$$

$\square$

COROLLARY 21.4.

$$\mathbf{IC}_Y = (\pi_*\mathcal{O}_X)^\Gamma$$

COROLLARY 21.5.

$$H^i(\mathbf{IC}_Y) = \mathbf{IH}^i(Y)$$

so

$$H^{\dim X+i}(Y, \mathbf{C}) = H^{\dim X+i}(X, \mathbf{C})^\Gamma$$

21.3. another example. Let $Y \to \mathbf{C}^3$ be defined by $xy - z^2 = 0$, then

$$Y = \mathbf{C}^2/\mathbb{Z}/2\mathbb{Z}$$

and so

$$\mathbf{IH}^*(Y) = H^*(Y)$$

i.e. coincides with the usual cohomology.

21.4. another example. Let $\pi : X \to Y$ be a morphism of irreducible algebraic varieties. We say that π is small if

$$\operatorname{codim}\{y \in Y | \dim \pi^{-1}(y) \geq m\} \geq 2m + 1$$

PROPOSITION 21.6. *Suppose $\pi : X \to Y$ is a small resolution of singularities, then*

$$\pi_*\mathcal{O}_X = \mathbf{IC}_Y$$

so

$$\mathbf{IH}^*(Y) = H^*(X)$$

EXAMPLE 21.7. Let $\mathfrak{g}$ be a simple lie algebra, and

$$\widetilde{\mathfrak{g}} = \{(x, \mathfrak{b}) | x \in \mathfrak{g}, x \in \mathfrak{b}, \mathfrak{b} \text{ a borel subalgebra}\}$$

then we have a map

$$\pi : \widetilde{\mathfrak{g}} \to \mathfrak{g} \times_{\mathfrak{h}/W} \mathfrak{h}$$

given by $\pi(x, \mathfrak{b}) = (x, \hat{x})$ where $\hat{x}$ is the image of x under the canonical quotient

$$\mathfrak{b}/[\mathfrak{b}, \mathfrak{b}] = \mathfrak{h}$$

i.e. the standard cartan. It is known that π is a small resolution called the *Grothendieck simultaneous resolution*, thus

$$\mathbf{IH}^*(\mathfrak{g} \times_{\mathfrak{h}/W} \mathfrak{h}) = H^*(\widetilde{\mathfrak{g}}) = H^*(G/B)$$

EXAMPLE 21.8. $\mathfrak{g} = \mathfrak{sl}_2$, then $\mathfrak{g} \times_{\mathfrak{h}/W} \mathfrak{h}$ is a double cover of $\mathfrak{g}$ branched over the nilpotent cone $xy + z^2 = 0$, hence it is the surface $t^2 = xy + z^2$—a homogenous quadric Q in $\mathbf{C}^4$. Also, note that $\widetilde{g}$ in this case is the total space of $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ as a bundle over $\mathbf{P}^1$. It follows that

$$\mathbf{IH}^j(Q) = \begin{cases} \mathbf{C} & \text{if } j = 0 \text{ or } j = 2 \\ 0 & \text{otherwise} \end{cases}$$

while

$$H^j(Q) = \begin{cases} \mathbf{C} & \text{if } j = 0 \\ 0 & \text{otherwise} \end{cases}$$

Part 19

Lecture 19: $\mathcal{D}$-modules with regular singularities, 1

lec19

22. section Plan

Part 20

Lecture 20: $\mathcal{D}$-modules with regular singularities, 2

`lec20`

23. More on the Riemann-Hilbert Map

Last lecture we considered the Riemann Hilbert map

$$\mathbf{RH} : \begin{array}{l}\text{Algebraic vector bundles on } X \\ \text{with an } \mathbf{RS} \text{ flat connection}\end{array} \longrightarrow \mathbf{Rep}\pi_1(X)$$

which assigns to each bundle (E, ∇) the monodromy representation of ∇. Note that both categories for fixed rank r have a moduli space of objects which generically is an algebraic variety, and so in particular, a complex manifold. However, the map **RH** is not algebraic; it is only holomorphic.

Let's consider two examples of what this map does.

24. Example 1: case of a complete curve

Let X be a projective curve of genus g, then the moduli space $\mathcal{M}_{dR}$ of line bundles with connection looks as follows: we have a map $\mathcal{M}_{dR} \longrightarrow \mathbf{Pic}_0(X) = \mathbf{Jac}(X)$ whose fiber is $\mathbf{A}^g$, an affine space bundle. Here $\mathbf{A}^g$ is a torsor over $H^0(X, \Omega)$, and in particular is an algebraic variety of dimension $2g$.

On the other hand, the betti moduli space $\mathcal{M}_b$ is the moduli space of representations of $\pi_1(X)$. Once we fix generators for $\pi_1(X)$:

$$a_1, \ldots, a_g, b_1, \ldots, b_g, \prod [a_i, b_i] = 1,$$

then we can identify

$$\mathcal{M}_b \simeq (\mathbf{C}^*)^{2g}.$$

Here the **RH** map is a holomorphic isomorphism

$$\mathcal{M}_{dR} \simeq \mathcal{M}_b.$$

Clearly it cannot be algebraic, since we have

LEMMA 24.1. *Any regular map $\mathbf{C}^* \to \mathbf{Jac}(X)$ is constant.*

PROOF. The map extends to

$$\mathbf{CP}^1 \to \mathbf{Jac}(X),$$

passage to the universal cover then yields

$$\mathbf{CP}^1 \to \mathbf{C}^n.$$

Now Liouville's theorem shows this map is constant. □

In more detail, **RH** is inverse to a map

$$f : \mathcal{M}_b \to \mathcal{M}_{dR}.$$

To construct f, let's construct

$$\pi : (\mathbf{C}^*)^{2g} \to \mathbf{Jac}(X) = \mathbf{Pic}_0(X);$$

which is an affine space bundle. To define f, consider the $4g$-gon:

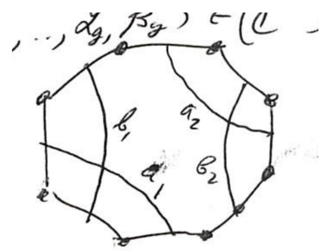

fig-4g-gon.png

Given

$$(\alpha_1, \beta_1, \dots, \alpha_g, \beta_g) \in (\mathbf{C}^*)^{2g}$$

we consider the trivial line bundle on the polygon and glue together a line bundle on X by using α_1 along a_1, β_1 along b_1 etc.

25. Example 2: the projective line minus four points

Let $X = \mathbf{P}^1 - \{0, 1, \lambda, \infty\}$ with $\lambda \neq 0, 1, \infty$ and let's restrict to connections with trivial determinant. Now $\mathcal{M}_{dR}$ has an open set $\mathcal{M}^\circ_{dR}$ of connections which have first order poles on the trivial bundle. Let's also restrict further to fixed monodromy; i.e. $\mathcal{M}^\circ_{dR}$ is the set of connections

$$\nabla = \partial - \frac{a_0}{z} - \frac{a_1}{z-1} - \frac{a_\lambda}{z-\lambda}, \mathbf{Tr}(a_j) = 0$$

and let $a_\infty = -a_0 - a_1 - a_\lambda$. Denote by

$$\mathcal{M}^\circ_{dR}(\alpha_0, \alpha_1, \alpha_\lambda, \alpha_\infty) = \{\nabla, a_j \sim \begin{pmatrix} \alpha_j & 0 \\ 0 & -\alpha_j \end{pmatrix}\}$$ then letting $A_j \sim \begin{pmatrix} e^{2\pi i \alpha_j} & 0 \\ 0 & e^{-2\pi i \alpha_j} \end{pmatrix}$ we see the map

$$\mathbf{RH} : \mathcal{M}_b(\alpha_0, \alpha_1, \alpha_\lambda, \alpha_\infty) \longrightarrow \{A_0, A_1, A_\lambda, A_\infty, A_0 A_1, A_\lambda A_\infty = 1\}$$

This map is highly transcendental.

Namely, let $\mathcal{P} \in \mathcal{M}_b(\alpha_0, \alpha_1, \alpha_\lambda, \alpha_\infty)$ and consider the point $Q_\lambda = \mathbf{RH}^{-1}_\lambda(\mathcal{P}) \in \mathcal{M}^\circ_{dR}(\alpha_0, \alpha_1, \alpha_\lambda, \alpha_\infty)$. This defines a flow on $\mathcal{M}^\circ_{dR}(\alpha_0, \alpha_1, \alpha_\lambda, \alpha_\infty)$ known as the Painlavé-6 flow.

26. Holonomic $\mathcal{D}$-modules with RS in higher dimensions

26.1. Constructible sheaves and complexes. Let X be a $\mathbf{C}$-algebraic variety. Denote by X^{an} the corresponding analytic variety considered with the classical topology. Let $\mathbf{C}_X$ be the constant sheaf on X^{an} and $\mathbf{Sh}(X^{an})$ the category of $\mathbf{C}_X$-modules i.e. sheaves of $\mathbf{C}$-vector spaces. The derived category of bounded $\mathbf{C}_X$-complexes will be denoted $\mathbf{D}(X^{an})$.

DEFINITION 26.1. A $\mathbf{C}_X$-module $\mathcal{F}$ is constructible if there exists a stratification

$$X = \cup_i X_i$$

of X by locally closed algebraic subvarieties such that $\mathcal{F}\big|_{X^{an}}$ is a locally constant complex of finite dimensional vector spaces.

REMARK 26.2. Note that a $\mathbf{C}_X$-complex is constructible if all of its cohomology sheaves are constructible as $\mathbf{C}_X$-vector spaces.

The full subcategory of $\mathbf{D}(X^{an})$ consisting of constructible complexes will be denoted by $\mathbf{D}_{con}(X^{an})$.

Any morphism $\pi : X \to Y$ of algebraic varieties induces a continuous map $\pi^{an} : X^{an} \to Y^{an}$, and we can consider the functors

$$\pi_!, \pi_* : \qquad \mathbf{D}(X^{an}) \longrightarrow \mathbf{D}(Y^{an})$$

$$\pi^*, \pi^! : \qquad \mathbf{D}(Y^{an}) \longrightarrow \mathbf{D}(A^{an})$$

We also have

$$\mathbb{D} : \qquad \mathbf{D}(X^{an}) \longrightarrow \mathbf{D}(X^{an})$$

We have

THEOREM 26.3. *These functors preserve the subcategory of derived constructible sheaves $\mathbf{D}_{con}$, and on them we have*

$$\mathbb{D}^2 = \mathbf{Id}$$

$$\mathbb{D}\pi^*\mathbb{D} = \pi^!$$

$$\mathbb{D}\pi_*\mathbb{D} = \pi_!$$

and $\mathbb{D}M = \mathbf{RHom}_{an}(M, \mathbf{C}_X)$.

26.2. De Rham Functor. Let $\mathcal{O}_X^{an}$ be the structure sheaf of X^{an}. We will assign to each $\mathcal{O}_X$-module M the corresponding *analytic sheaf* of $\mathcal{O}_X^{an}$-modules M^{an}, which is locally given by

$$M^{an} = \mathcal{O}_X^{an} \otimes_{\mathcal{O}_X} M$$

.

This defines an exact functor

$$an : M(\mathcal{O}_X) \to M(\mathcal{O}_{X^{an}})$$

and in particular an exact functor

$$an : M(\mathcal{D}_X) \to M(\mathbf{D}_X^{an})$$

, where $\mathcal{D}_X^{an}$ is the sheaf of analytic differential operators.

DEFINITION 26.4. The *De Rham Functor*

$$\mathbf{DR} : \mathbf{D}^b(\mathcal{D}_X) \to \mathbf{D}^b(X^{an}) = \mathbf{D}^b(\mathbf{Sh}(X^{an}))$$

is

$$\mathbf{DR}(M^\circ) = \Omega_X^{an} \otimes_{\mathcal{D}_X^{an}} (M^\circ)^{an}$$

REMARK 26.5. Since $\mathbf{dR}(\mathcal{D}_X)$ is a locally projective resolution of Ω_X we have

$$\mathbf{DR}(M^\circ) = \mathbf{dR}(\mathcal{D}_X^{an}) \otimes_{\mathcal{D}_X^{an}} (M^\circ)^{an}[\dim X]$$

In particular, if M is an $\mathcal{O}$-coherent $\mathcal{D}_X$-module corresponding to a vector bundle with a flat connection and $\mathcal{L} = M^{flat}$ is the local system of flat sections, then

$$\mathbf{DR}(M) = \mathcal{L}[dim X]$$

by Poincaré's lemma.

Here is the main theorem about the connection between $\mathcal{D}$-modules and constructible sheaves:

THEOREM 26.6.
- $\mathbf{DR}(\mathbf{D}_{hol}(\mathcal{D}_X)) \subset \mathbf{D}_{con}(X^{an})$, *and on* $\mathbf{D}_{hol}$ $\mathbf{DR}$ *commutes with both tensor product and* $\mathbf{D}$.
- *On the subcategory* $\mathbf{D}_{rs}$ *the functor* $\mathbf{DR}$ *commutes with all of the above functors*
- $\mathbf{DR} : \mathbf{D}_{rs}(\mathcal{D}_X) \to \mathbf{D}_{con}(X^{an})$ *is an equivalence.*

26.3. Definitions concerning vector bundles with flat connection. Let M be a vector bundle on X with a flat connection.

DEFINITION 26.7. M lies in $\mathbf{D}_{rs}$ if the restriction of M to any curve has regular singularities.

DEFINITION 26.8. An irreducible M in $\mathbf{D}_{hol}$ has *regular singularities* if $M = j_{!*}L$ for L a vector bundle with flat connection and regular singularities.

REMARK 26.9. An object $M \in \mathbf{D}_{hol}$ has regular singularities if and only if all composition factors have regular singularities.