Algebraic $\mathcal{D}$-modules

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ABSTRACT. These notes are a work-in-progress!.

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20. More on the Riemann-Hilbert Map

Last lecture we considered the Riemann Hilbert map

$$\mathbf{RH} : \begin{array}{l}\text{Algebraic vector bundles on } X \\ \text{with an } \mathbf{RS} \text{ flat connection}\end{array} \longrightarrow \mathbf{Rep}\pi_1(X)$$

which assigns to each bundle (E, ∇) the monodromy representation of ∇. Note that both categories for fixed rank r have a moduli space of objects which generically is an algebraic variety, and so in particular, a complex manifold. However, the map **RH** is not algebraic; it is only holomorphic.

Let's consider two examples of what this map does.

21. Example 1: case of a complete curve

Let X be a projective curve of genus g, then the moduli space $\mathcal{M}_{dR}$ of line bundles with connection looks as follows: we have a map $\mathcal{M}_{dR} \longrightarrow \mathbf{Pic}_0(X) = \mathbf{Jac}(X)$ whose fiber is $\mathbf{A}^g$, an affine space bundle. Here $\mathbf{A}^g$ is a torsor over $H^0(X, \Omega)$, and in particular is an algebraic variety of dimension $2g$.

On the other hand, the betti moduli space $\mathcal{M}_b$ is the moduli space of representations of $\pi_1(X)$. Once we fix generators for $\pi_1(X)$:

$$a_1, \ldots, a_g, b_1, \ldots, b_g, \prod [a_i, b_i] = 1,$$

then we can identify

$$\mathcal{M}_b \simeq (\mathbf{C}^*)^{2g}.$$

Here the **RH** map is a holomorphic isomorphism

$$\mathcal{M}_{dR} \simeq \mathcal{M}_b.$$

Clearly it cannot be algebraic, since we have

LEMMA 21.1. *Any regular map $\mathbf{C}^* \to \mathbf{Jac}(X)$ is constant.*

PROOF. The map extends to

$$\mathbf{CP}^1 \to \mathbf{Jac}(X),$$

passage to the universal cover then yields

$$\mathbf{CP}^1 \to \mathbf{C}^n.$$

Now Liouville's theorem shows this map is constant. $\square$

In more detail, **RH** is inverse to a map

$$f : \mathcal{M}_b \to \mathcal{M}_{dR}.$$

To construct f, let's construct

$$\pi : (\mathbf{C}^*)^{2g} \to \mathbf{Jac}(X) = \mathbf{Pic}_0(X);$$

which is an affine space bundle. To define f, consider the $4g$-gon:

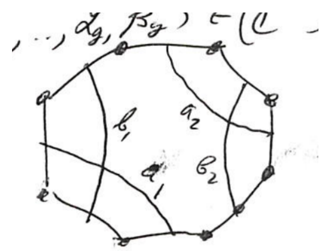

fig-4g-gon.png

Given

$$(\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g) \in (\mathbf{C}^*)^{2g}$$

we consider the trivial line bundle on the polygon and glue together a line bundle on X by using α_1 along a_1, β_1 along b_1 etc.

22. Example 2: the projective line minus four points

Let $X = \mathbf{P}^1 - \{0, 1, \lambda, \infty\}$ with $\lambda \neq 0, 1, \infty$ and let's restrict to connections with trivial determinant. Now $\mathcal{M}_{dR}$ has an open set $\mathcal{M}^\circ_{dR}$ of connections which have first order poles on the trivial bundle. Let's also restrict further to fixed monodromy; i.e. $\mathcal{M}^\circ_{dR}$ is the set of connections

$$\nabla = \partial - \frac{a_0}{z} - \frac{a_1}{z-1} - \frac{a_\lambda}{z-\lambda}, \mathbf{Tr}(a_j) = 0$$

and let $a_\infty = -a_0 - a_1 - a_\lambda$. Denote by

$$\mathcal{M}^\circ_{dR}(\alpha_0, \alpha_1, \alpha_\lambda, \alpha_\infty) = \{\nabla, a_j \sim \begin{pmatrix} \alpha_j & 0 \\ 0 & -\alpha_j \end{pmatrix}\}$$ then letting $A_j \sim \begin{pmatrix} e^{2\pi i \alpha_j} & 0 \\ 0 & e^{-2\pi i \alpha_j} \end{pmatrix}$ we see the map

$$\mathbf{RH} : \mathcal{M}_b(\alpha_0, \alpha_1, \alpha_\lambda, \alpha_\infty) \longrightarrow \{A_0, A_1, A_\lambda, A_\infty, A_0 A_1, A_\lambda A_\infty = 1\}$$

This map is highly transcendental.

Namely, let $\mathcal{P} \in \mathcal{M}_b(\alpha_0, \alpha_1, \alpha_\lambda, \alpha_\infty)$ and consider the point $Q_\lambda = \mathbf{RH}^{-1}_\lambda(\mathcal{P}) \in \mathcal{M}^\circ_{dR}(\alpha_0, \alpha_1, \alpha_\lambda, \alpha_\infty)$. This defines a flow on $\mathcal{M}^\circ_{dR}(\alpha_0, \alpha_1, \alpha_\lambda, \alpha_\infty)$ known as the Painlavé-6 flow.

23. Holonomic $\mathcal{D}$-modules with RS in higher dimensions

23.1. Constructible sheaves and complexes. Let X be a $\mathbf{C}$-algebraic variety. Denote by X^{an} the corresponding analytic variety considered with the classical topology. Let $\mathbf{C}_X$ be the constant sheaf on X^{an} and $\mathbf{Sh}(X^{an})$ the category of $\mathbf{C}_X$-modules i.e. sheaves of $\mathbf{C}$-vector spaces. The derived category of bounded $\mathbf{C}_X$-complexes will be denoted $\mathbf{D}(X^{an})$.

DEFINITION 23.1. A $\mathbf{C}_X$-module $\mathcal{F}$ is constructible if there exists a stratification

$$X = \cup_i X_i$$

of X by locally closed algebraic subvarieties such that $\mathcal{F}\big|_{X^{an}}$ is a locally constant complex of finite dimensional vector spaces.

REMARK 23.2. Note that a $\mathbf{C}_X$-complex is constructible if all of its cohomology sheaves are constructible as $\mathbf{C}_X$-vector spaces.

The full subcategory of $\mathbf{D}(X^{an})$ consisting of constructible complexes will be denoted by $\mathbf{D}_{con}(X^{an})$.

Any morphism $\pi : X \to Y$ of algebraic varieties induces a continuous map $\pi^{an} : X^{an} \to Y^{an}$, and we can consider the functors

$$\pi_!, \pi_* : \qquad \mathbf{D}(X^{an}) \longrightarrow \mathbf{D}(Y^{an})$$

$$\pi^*, \pi^! : \qquad \mathbf{D}(Y^{an}) \longrightarrow \mathbf{D}(A^{an})$$

We also have

$$\mathbb{D} : \qquad \mathbf{D}(X^{an}) \longrightarrow \mathbf{D}(X^{an})$$

We have

THEOREM 23.3. *These functors preserve the subcategory of derived constructible sheaves $\mathbf{D}_{con}$, and on them we have*

$$\mathbb{D}^2 = \mathbf{Id}$$
$$\mathbb{D}\pi^*\mathbb{D} = \pi^!$$
$$\mathbb{D}\pi_*\mathbb{D} = \pi_!$$

and $\mathbb{D}M = \mathbf{RHom}_{an}(M, \mathbf{C}_X)$.

23.2. De Rham Functor. Let $\mathcal{O}_X^{an}$ be the structure sheaf of X^{an}. We will assign to each $\mathcal{O}_X$-module M the corresponding *analytic sheaf* of $\mathcal{O}_X^{an}$-modules M^{an}, which is locally given by

$$M^{an} = \mathcal{O}_X^{an} \otimes_{\mathcal{O}_X} M$$

.

This defines an exact functor

$$an : M(\mathcal{O}_X) \to M(\mathcal{O}_{X^{an}})$$

and in particular an exact functor

$$an : M(\mathcal{D}_X) \to M(\mathbf{D}_X^{an})$$

, where $\mathcal{D}_X^{an}$ is the sheaf of analytic differential operators.

DEFINITION 23.4. The *De Rham Functor*

$$\mathbf{DR} : \mathbf{D}^b(\mathcal{D}_X) \to \mathbf{D}^b(X^{an}) = \mathbf{D}^b(\mathbf{Sh}(X^{an}))$$

is

$$\mathbf{DR}(M^\circ) = \Omega_X^{an} \otimes_{\mathcal{D}_X^{an}} (M^\circ)^{an}$$

REMARK 23.5. Since $\mathbf{dR}(\mathcal{D}_X)$ is a locally projective resolution of Ω_X we have

$$\mathbf{DR}(M^\circ) = \mathbf{dR}(\mathcal{D}_X^{an}) \otimes_{\mathcal{D}_X^{an}} (M^\circ)^{an}[\dim X]$$

In particular, if M is an $\mathcal{O}$-coherent $\mathcal{D}_X$-module corresponding to a vector bundle with a flat connection and $\mathcal{L} = M^{flat}$ is the local system of flat sections, then

$$\mathbf{DR}(M) = \mathcal{L}[dimX]$$

by Poincaré's lemma.

Here is the main theorem about the connection between $\mathcal{D}$-modules and constructible sheaves:

THEOREM 23.6.
- $\mathbf{DR}(\mathbf{D}_{hol}(\mathcal{D}_X)) \subset \mathbf{D}_{con}(X^{an})$, *and on* $\mathbf{D}_{hol}$ $\mathbf{DR}$ *commutes with both tensor product and* $\mathbf{D}$.
- *On the subcategory* $\mathbf{D}_{rs}$ *the functor* $\mathbf{DR}$ *commutes with all of the above functors*
- $\mathbf{DR} : \mathbf{D}_{rs}(\mathcal{D}_X) \to \mathbf{D}_{con}(X^{an})$ *is an equivalence.*

23.3. Definitions concerning vector bundles with flat connection. Let M be a vector bundle on X with a flat connection.

DEFINITION 23.7. M lies in $\mathbf{D}_{rs}$ if the restriction of M to any curve has regular singularities.

DEFINITION 23.8. An irreducible M in $\mathbf{D}_{hol}$ has *regular singularities* if $M = j_{!*}L$ for L a vector bundle with flat connection and regular singularities.

REMARK 23.9. An object $M \in \mathbf{D}_{hol}$ has regular singularities if and only if all composition factors have regular singularities.