Automorphic Corrections

and diagonal divisor genus 2 Siegel modular forms

Raeez Lorgat

Game Plan

- Diagonal Divisor Siegel Modular Forms
- Igusa's Cusp form Δ_5 of Weight 5
- Trivial transfer of automorphic forms from $\mathbf{Sp}_4(\mathbb{Z})$ to $\mathbf{O}(\Lambda^{3,2})_+$
- Reflection groups $W_I^{(2)}, W_{II}^{(2)}$
- The Weyl vector ρ
- Fourier-coefficients of Δ_5 and the hyperbolic lattice $\Lambda_{II}^{2,1}$
- *Automorphic correction* and a generalized Borcherds-Kac-Moody super algebra $\mathfrak{g}_{\Delta_5}$
- Root super-multiplicities of $\mathfrak{g}_{\Delta_5}$ and the weak Jacobi form $\phi_{0,1}$ of index 1
- Arithmetic geometry of some Calabi-Yau 3-folds

Diagonal Divisor Siegel Modular Forms

- Integral symplectic groups Γ_t, and corresponding congruence subgroups of Hecke Type (*Paramodular groups of genus 2*)

$$t, n \in \mathbb{Z}_+ : \Gamma_t(N) = \{ \begin{pmatrix} * & *\,t & * & * \\ * & * & * & *\,t^{-1} \\ *\,N & *\,Nt & * & * \\ *\,Nt & *\,Nt & *\,t & * \end{pmatrix} \in \mathbf{Sp_4}(\mathbb{Q}) \mid * \in \mathbb{Z} \}$$

- *Which Siegel modular forms with respect to $\Gamma_t(N)$ (with character/multiplier system) vanish exactly along $\Gamma_t(N)$-translates of the diagonal $\{ \begin{pmatrix} a & \\ & b \end{pmatrix} \mid a, b \in \mathbb{H}_1 \} \subset \mathbb{H}_2$ precisely to order 1?*

Diagonal Divisor Siegel Modular Forms

- **Theorem** *For congruence subgroups* $\Gamma_t(N) < \Gamma_t$ *there are exactly eight dd-modular forms*

- $M_5(\Gamma_1, \nu_2) = \Delta_5$ $\quad M_2(\Gamma_2, \nu_4)$ $\quad M_3(\Gamma_1(2), \nu_2)$

  $M_1(\Gamma_3, \nu_6)$ $\quad M_2(\Gamma_1(3), \nu_2)$

  $M_{\frac{1}{2}}(\Gamma_4, \nu_8)$ $\quad M_{\frac{3}{2}}(\Gamma_1(4), \nu_4)$ $\quad M_1(\Gamma_2(2), \nu_4)$

Igusa's cusp form Δ_5 of weight 5

- $\mathcal{S}\mathcal{M}(\mathbf{Sp}_4(\mathbb{Z})) = \mathbb{C}[E_4, E_6, \Delta_{10}, \Delta_{12}]$
- $\Delta_{10} = (\Delta_5)^2$
- Δ_5 cusp form weight 5, non-trivial multiplier system $\nu_{\Delta_5} : \mathbf{Sp}_4(\mathbb{Z}) \to \mathbb{C}$
- $\Delta_5 = \prod_{(a,b)\in(\mathbb{Z}/2\mathbb{Z})^2, {}^t ab \equiv 0 \mod 2} \nu_{a,b}$

$$\nu_{ab}(z) = \sum_{l\in\mathbb{Z}^2} \exp(\pi i(z(l + \frac{1}{2}a)^t z(l + \frac{1}{2}a) + {}^t bl))$$

Igusa's cusp form Δ_5 of weight 5

- $\nu_{\Delta_5} : \mathbf{Sp}_4(\mathbb{Z}) \to \mathbb{C}$ found explicitly by Maas
- $|\nu_{\Delta_5}(g)| = 1$
- $\nu_{\Delta_5}\begin{pmatrix} & \mathbf{I}_2 \\ -\mathbf{I}_2 & \end{pmatrix} = 1 \qquad \nu_{\Delta_5}\begin{pmatrix} \mathbf{I}_2 & B \\ & \mathbf{I}_2 \end{pmatrix} = (-1)^{b_1+b_2+b_3}$

$$\nu_{\Delta_5}\begin{pmatrix} {}^tA^{-1} & \\ & A \end{pmatrix} = (-1)^{(1+a_1+a_4)(1+a_2+a_3)+a_1a_4}$$

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in \mathbf{GL}_2(\mathbb{Z}) \quad \text{and} \quad B = \begin{pmatrix} b_1 & b_2 \\ b_2 & b_3 \end{pmatrix} \in M_{2\times 2}(\mathbb{Z})$$

Igusa's cusp form Δ_5 of weight 5

- Expression in terms of even theta constants + Maas' multiplier system

$$\Longrightarrow \Delta_5(z) = \sum_{\substack{n,l,m = 1 \text{ mod } 2, \\ 4nm - l^2 > 0, \\ n,m > 0}} f(n,l,m) \exp(\pi i (n z_1 + l z_2 + m z_3))$$

$$z = \begin{pmatrix} z_0 & z_1 \\ z_2 & z_3 \end{pmatrix}$$

- $f(1,1,1) = 64$ and $64 \,|\, f(n,l,m)$

Igusa's cusp form Δ_5 of weight 5

- **Lemma** $$1 + \frac{1}{64}\sum_{n\in\mathbb{N}} f(1+2t,1,1)q^t = \prod_{k\in\mathbb{N}}(1-q^k)^9$$

- ***proof (sketch).*** $$\Delta_5(Z) = \sum_{\substack{m>0,\\ m=1 \mod 2}} \phi_{5,m}(z_1,z_2)\exp(\pi i m z_3)$$

- First Fourier-Jacobi coefficient $\phi_{5,m}$ is a Jacobi cusp form of index $\frac{1}{2}$ and non-trivial character

Igusa's cusp form Δ_5 of weight 5

- Jacobi theta-series

$$\nu_{11}(z_1, z_2) = \sum_{n\in\mathbb{Z}} (-1)^n \exp(\frac{\pi i}{4}(2n+1)^2 z_1 + \pi i(2n+1)z_2)$$

- Jacobi-triple-product-formula $\Longrightarrow$

$$\nu_{11} = q_1^{\frac{1}{8}} q_2^{-\frac{1}{2}} \prod_{n\geq 1} (1 - q_1^{n-1} q_2)(1 - {q_1}^n {q_2}^{-1})(1 - {q_1}^n)$$

- $\psi_{5,\frac{1}{2}} \exp(2\pi i t z_3) = \eta(z_1)^9 \nu_{11}(z_1, z_2)$ is another Jacobi cusp form of index $\dfrac{1}{2}$

$\eta = \exp(\dfrac{\pi i \tau}{12}) \prod_{n\geq 1} (1 - \exp(2\pi i n \tau))$ is Dedekind's η-function

Igusa's cusp form Δ_5 of weight 5

- Pass to squares → get Jacobi cusp forms of weight 10 index 1

- Up to a constant, unique and coincides with first Fourier-Jacobi-coefficient of $\Delta_{10} = (\Delta_5)^2$

- $$\frac{1}{64}\phi_{5,1} = \psi_{5,\frac{1}{2}} = -\, q_1^{\frac{1}{2}} q_2^{-\frac{1}{2}} \prod_{n\geq 1} (1 - q_1^{n-1} q_2)(1 - {q_1}^n {q_2}^{-1})(1 - {q_1}^n)^{10}$$

- Compare Fourier-coeffients + apply Jacobi-triple-product identity to coefficient of ${q_2}^{\frac{1}{2}}$ Q.E.D.

Transfer of automorphic forms from $\mathbf{Sp}_4(\mathbb{Z})$ to $\mathbf{O}(\Lambda^{3,2})_+$

- $\Lambda^4 = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3 \oplus \mathbb{Z}e_4$

- $\mathbb{Z}$-linear maps $g : \Lambda^4 \to \Lambda^4$ induce $\wedge^2 g : \Lambda^4 \wedge \Lambda^4 \to \Lambda^4 \wedge \Lambda^4$

- Induces action of $\mathbf{SL}_4(\mathbb{Z})$

- Pfaffian scalar product $(,) : \Lambda^4 \wedge \Lambda^4 \to \mathbb{C}$

$$u \wedge v = (u, v)e_1 \wedge e_2 \wedge e_3 \wedge e_4 \in \wedge^4 \Lambda^4$$

- $\mathbf{SL}_4(\mathbb{Z})$-invariant signature $(3,3)$ even unimodular integral symmetric binilear form

Transfer of automorphic forms from $\mathbf{Sp}_4(\mathbb{Z})$ to $\mathbf{O}(\Lambda^{3,2})_+$

- Take $q = e_1 \wedge e_3 + e_2 \wedge e_4 \in \Lambda^4 \wedge \Lambda^4$
- $-x \wedge y \wedge q = B_q(x, y) e_1 \wedge e_2 \wedge e_3 \wedge e_4$ defines integral skew-symmetric bilinear form (elements $q \in \Lambda^4 \wedge \Lambda^4$ define such forms)
- $\{g : \Lambda^4 \to \Lambda^4 \,|\, (g \wedge g)(e_1 \wedge e_3 + e_2 \wedge e_4) = e_1 \wedge e_3 + e_2 \wedge e_4\} \simeq \mathbf{Sp}_4(\mathbb{Z})$

Transfer of automorphic forms from $\mathbf{Sp}_4(\mathbb{Z})$ to $\mathbf{O}(\Lambda^{3,2})_+$

- $\Lambda^{3,2} = (e_1 \wedge e_3 + e_2 \wedge e_4)^{\perp} \subset \Lambda^4 \wedge \Lambda^4$
- $\Lambda^{3,2} \simeq \Lambda^{1,1} \oplus \Lambda^{1,1} \oplus \mathbb{Z} \simeq \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \oplus [2]$
- Fix the basis
  $f_1 = e_1 \wedge e_2,\ f_2 = e_2 \wedge e_3,\ f_3 = e_1 \wedge e_3 - e_2 \wedge e_4,\ f_{-2} = e_4 \wedge e_1,\ f_{-1} = e_4 \wedge e_3$

Transfer of automorphic forms from $\mathbf{Sp}_4(\mathbb{Z})$ to $\mathbf{O}(\Lambda^{3,2})_+$

- $\mathbf{O}_{\mathbb{R}}(\Lambda^{3,2}) = \mathbf{O}_{\mathbb{R}}(\Lambda^{3,2} \otimes \mathbb{R})$ acts on

$$\mathbb{H}^{IV} = \{Z \in \mathbb{P}(\Lambda^{3,2} \otimes \mathbb{C}) \mid (Z,Z) = 0, (Z,\overline{Z}) < 0\} = \mathbb{H}^{IV}_+ \cup \overline{\mathbb{H}^{IV}_+}$$

- In (f_i)-basis

$$\mathbb{H}^{IV}_+ = \{Z = {}^t((z_2^2 - z_1 z_3), z_3, z_2, z_1, 1) \cdot z_0 \in \mathbb{H}^{IV} \mid \mathrm{Imag}(z_1) > 0\}$$

- Observe $(Z,\overline{Z}) < 0$ equivalent to $y_1 y_3 - y_2^2 > 0$ for $y_i = \mathrm{Imag}\ z_i$

- $\mathbb{H}^{IV}_+ = \{ \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} \mid \text{positive def. imaginary part}\}$

Transfer of automorphic forms from $\mathbf{Sp}_4(\mathbb{Z})$ to $\mathbf{O}(\Lambda^{3,2})_+$

- $\mathbf{O}_{\mathbb{R}}(\Lambda^{3,2})$ has four connected components
- $\mathbf{O}_{\mathbb{R}}(\Lambda^{3,2})_+$ subgroup of index 2 fixing $\mathbb{H}_+^{IV}$
- Kernel of action of $\mathbf{O}_{\mathbb{R}}(\Lambda^{3,2})_+$ on $\mathbb{H}_+^{IV}$ is $\pm\mathbf{I}_5$
- $\Lambda^{3,2}$ odd-dimensional $\Longrightarrow \mathbf{O}_{\mathbb{R}}(\Lambda^{3,2})_+ = \pm\,\mathbf{I}_5\mathbf{SO}_{\mathbb{R}}(\Lambda^{3,2})_+$
- Notations: $\mathbf{O}(\Lambda^{3,2})_+ = \mathbf{O}_{\mathbb{R}}(\Lambda^{3,2})_+ \cap \mathbf{O}(\Lambda^{3,2})$
  $\mathbf{SO}(\Lambda^{3,2})_+ = \mathbf{SO}_{\mathbb{R}}(\Lambda^{3,2})_+ \cap \mathbf{O}(\Lambda^{3,2})$

Transfer of automorphic forms from $\mathbf{Sp}_4(\mathbb{Z})$ to $\mathbf{O}(\Lambda^{3,2})_+$

- Elementary exercise: lift generators of $\mathbf{Sp}_4(\mathbb{Z})$ to $\wedge^2(\Lambda^4)$

- **Lemma** The map $\wedge^2$ defines an isomorphism

$$\wedge^2 : \mathbf{Sp}_4(\mathbb{Z})/\{\pm\mathbf{I}_5\} \to \mathbf{SO}_+(\Lambda^{3,2}) \simeq \mathbf{O}(\Lambda^{3,2})_+/\{\pm\mathbf{I}_5\}$$

$$\begin{array}{ccc} \mathbb{H}_2 & \longrightarrow & \mathbb{H}_2 \\ \downarrow & & \downarrow \\ \mathbb{H}_+^{IV} & \xrightarrow{g\wedge g} & \mathbb{H}+^{IV} \end{array}$$

Reflection groups $W_I^{(2)}, W_{II}^{(2)}$

- Fix some primitive hyperbolic sub lattice
$$\Lambda^{2,1} = \Lambda^{1,1} \oplus [2] \simeq \mathbb{Z}f_2 \oplus \mathbb{Z}f_3 \oplus \mathbb{Z}f_{-2}$$

- Extending $g \in \mathbf{O}(\Lambda^{2,1})$ to the identity on $(\Lambda^{2,1})^\perp$ yields embedding

$$\mathbf{O}(\Lambda^{2,1}) \to \mathbf{O}(\Lambda^{3,2})$$

- Automorphy of Δ_5 wrt $\mathbf{O}(\Lambda^{2,1})$?

Reflection groups $W_I^{(2)}, W_{II}^{(2)}$

- To every primitive element $\alpha \in \Lambda^{2,1}$ with $(\alpha, \alpha) > 0$ and $(\alpha, \alpha) | 2(\Lambda^{2,1}, \alpha)$

$$s_\alpha : x \mapsto 2\frac{(x, \alpha)}{(\alpha, \alpha)}\alpha$$

- $s_\alpha(\alpha) = -\alpha$ and $s_\alpha|_{\alpha^\perp}$ is the identity

- For any $\alpha, (\alpha, \alpha) = 2$ we get $s_\alpha : x \mapsto (x, \alpha)\alpha$

Reflection groups $W_I^{(2)}, W_{II}^{(2)}$

- **Lemma** Consider α of square 2:

- if $\alpha \in \{\delta_1 = 2f_2 - f_3, \delta_2 = 2f_{-2} - f_3, \delta_3 = f_3\}$

$$\Delta_5(s_\alpha z) = -\Delta_5(z)$$

- If $\alpha \in \{f_{-2} - f_2, f_2 - f_3, f_2 + f_3\}$

$$\Delta_5(s_\alpha z) = \Delta_5(z)$$

Reflection groups $W_I^{(2)}, W_{II}^{(2)}$

- **Proof** Let $\overline{U} = \wedge^2 \left(\begin{pmatrix} {}^tU^{-1} & 0 \\ 0 & U \end{pmatrix}\right)$ with $U \in \mathbf{GL}_2(\mathbb{Z})$ then

- $s_{f_{-2}-f_2} = -\overline{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}, s_{f_3} = -\overline{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}, s_{f_2-f_3} = -\overline{\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}}$

apply Maas' explicit formula for multiplier; Q.E.D.

Reflection groups $W_I^{(2)}, W_{II}^{(2)}$

- $\Lambda^{2,1}$ hyperbolic; defines cone $\mathscr{C}(\Lambda^{2,1}) = \{x \in \Lambda^{2,1} \otimes \mathbb{R} \,|\, (x,x) < 0\}$
- $\mathscr{C}(\Lambda^{2,1})$ union of two half-cones; pick $\mathscr{C}(\Lambda^{2,1})_+$ by constraining complex cone

$$\Omega(\mathscr{C}(\Lambda^{2,1})_+) = \Lambda^{2,1} \otimes \mathbb{R} + i\mathscr{C}(\Lambda^{2,1})_+ \subset \mathscr{H}_+^{IV}$$

- Let $\mathbf{O}(\Lambda^{2,1})_+$ be the subgroup of $\mathbf{O}(\Lambda^{2,1})$ of index 2 fixing the above positive cone
- $\Lambda^{2,1}$ hyperbolic $\Longrightarrow \mathbf{O}(\Lambda^{2,1})_+$ discrete in the hyperbolic space
- **Lemma** $\Lambda^{2,1}$ *hyperbolic* $\Longrightarrow \mathscr{C}(\Lambda^{2,1})/\mathbb{R}_{>0}$ *has fundamental domain of finite volume*

Reflection groups $W_I^{(2)}, W_{II}^{(2)}$

- any $\alpha \in \Lambda^{2,1}, (\alpha, \alpha) > 0$ is a reflection in a hyperplane

$$\mathscr{H}_\alpha = \{\mathbb{R}_{>0} x \in \mathscr{C}(\Lambda^{2,1})_+ / \mathbb{R}_{>0} \,|\, (x, \alpha) = 0\}$$

- And the reflection maps the *half-space*

$$\mathscr{H}_{+\alpha} = \{\mathbb{R}_{>0} x \in \mathscr{C}(\Lambda^{2,1})_+ / \mathbb{R}_{>0} \,|\, (x, \alpha) \leq 0\}$$

- To the *opposite* half-space $\mathscr{H}_{-\alpha}$

- Call α *orthogonal* to both $\mathscr{H}_\alpha, \mathscr{H}_{-\alpha}$

Reflection groups $W_I^{(2)}, W_{II}^{(2)}$

- All reflections generate a reflection subgroup

$$W(\Lambda^{2,1}) \subset \mathbf{O}(\Lambda^{2,1})_+ \simeq PGL_2(\mathbb{Z})$$

- The lattice $\Lambda^{2,1}$ is special: we know $Aut(\Lambda^{2,1})$ explicitly.

- Define $\Delta^{(k)}(\Lambda^{2,1}) = \{\delta \in \Lambda^{2,1} \,|\, (\delta, \delta) = k, \delta \text{ primitive}\}$

$$W^{(k)} = \{s_\delta \,|\, \delta \in \Delta^{(k)}(\Lambda^{2,1})\}$$

- **Lemma** $W(\Lambda^{2,1}) = W^{(2)}$ and every reflection s_δ is one of two types

  Type I: $(\delta, \Lambda^{2,1}) = \mathbb{Z}$

  Type II: $(\delta, \Lambda^{2,1}) = 2\mathbb{Z}$

Reflection groups $W_I^{(2)}, W_{II}^{(2)}$

- Sub lattices $\Lambda_I^{2,1}, \Lambda_{II}^{2,1}$ generated by primitive squares δ_I, δ_{II}
  normal subgroups $W^{(2)}(\Lambda_I^{2,1}), W^{(2)}(\Lambda_{II}^{2,1})$ of index 2 and 6

- $\Lambda_I^{2,1} = \{mf_2 + lf_3 + nf_{-2} \in \Lambda^{2,1} \,|\, m + l + n = 0 \mod 2\}$
  $\Lambda_{II}^{2,1} = \{mf_2 + lf_3 + nf_{-2} \in \Lambda^{2,1} \,|\, m = n = 0 \mod 2\}$

- Fundamental polyhedra $\{\mathscr{P}, \mathscr{P}_I, \mathscr{P}_{II}\} \ni \mathscr{P}' = \bigcap_{\delta \in \mathscr{P}'_{prim}} \mathscr{H}_{+\delta}$, $A(\mathscr{P}') = \{g \in \mathbf{O}(\Lambda^{2,1})_+ \,|\, g\mathscr{P}' = \mathscr{P}'\}$

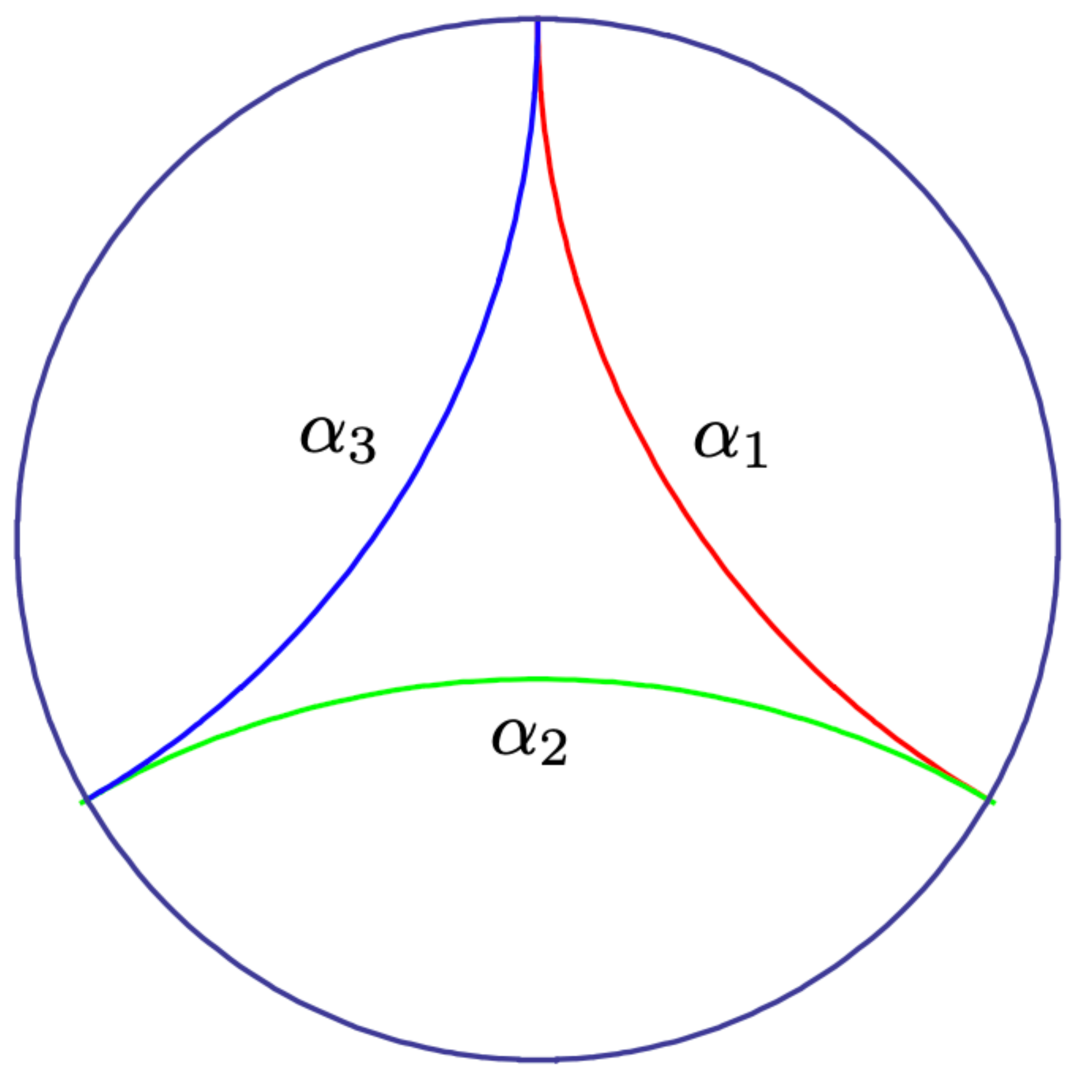
  $\mathscr{P}'_{prim}$ set of square 2 primitive orthogonal vectors to $\mathscr{P}$

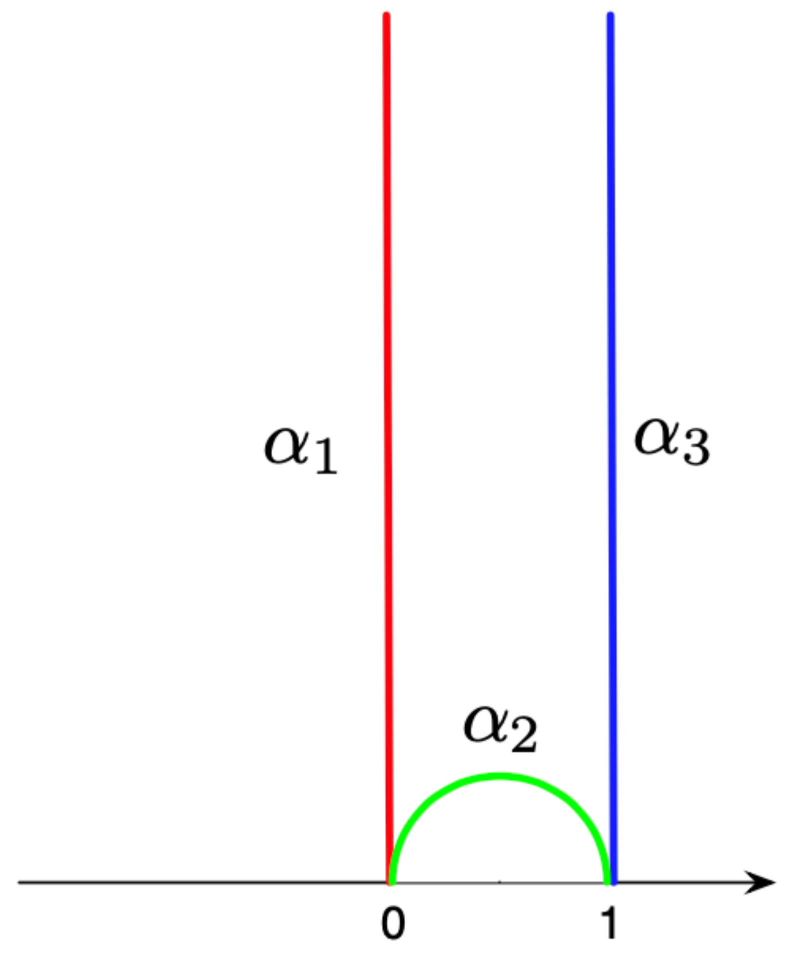
- $\mathscr{P}_{prim} = \{f_2 - f_3, f_{-2} - f_2, f_3\}$ $\quad A(\mathscr{P}) = < 1 >$
  $\mathscr{P}_{I,prim} = \{f_2 - f_3, f_{-2} - f_2, f_2 + f_3\}$ $\quad A(\mathscr{P}_I) = < s_{f_3} > \simeq S_2$
  $\mathscr{P}_{II,prim} = \{\delta_1 = 2f_2 - f_3, \delta_2 = 2f_{-2} - f_3, f_3\}$ $\quad A(\mathscr{P}_{II}) = < s_{f_2 - f_3}, s_{f_{-2}-f_2} > \simeq S_3$

Reflection groups $W_I^{(2)}, W_{II}^{(2)}$

- $\mathbf{O}(\Lambda^{2,1})_+ \simeq W^{(2)}(\Lambda^{2,1}) \simeq W^{(2)}(\Lambda_I^{2,1}) \rtimes Aut(\mathscr{P}_I) \simeq W^{(2)}(\Lambda_{II}^{2,1}) \rtimes Aut(\mathscr{P}_{II})$

$$PGL_2(\mathbb{Z}) \simeq W^{(2)}(\Lambda_{II}^{2,1}) \rtimes S_3$$





$\mathscr{P}_{II}$ and hyperbolic space $\mathscr{C}(\Lambda^{2,1})_+/\mathbb{R}_{>0}$

Reflection groups $W_I^{(2)}, W_{II}^{(2)}$

- **Lemma** Δ_5 *is either invariant or ant-invariant wrt* $\mathbf{O}(\Lambda^{2,1})_+$
  - $w \in W^{(2)}(\Lambda_I^{2,1})$ *and* $a \in Aut(\mathscr{P}_{I,prim})$

$$\Delta_5(w \cdot az) = \mathrm{det}(a)\Delta_5(z)$$

  - $w \in W^{(2)}(\Lambda_{II}^{2,1})$ *and* $a \in Aut(\mathscr{P}_{II,prim})$

$$\Delta_5(w \cdot az) = \mathrm{det}(w)\Delta_5(z)$$

The Weyl vector ρ

- Consider cone $\Delta(\Lambda_{II}^{2,1})_+ = \mathbb{R}_{\geq 0}\delta_1 \oplus \mathbb{R}_{\geq 0}\delta_2 \oplus \mathbb{R}_{\geq 0}\delta_3$

  dual cone $\Delta(\Lambda_{II}^{2,1})_+^* = \{x \in \Lambda^{2,1} \otimes \mathbb{R} \,|\, (x, \delta_i) \leq 0\}$

- Cone $\overline{\mathscr{C}(\Lambda^{2,1})_+}^* = \overline{\mathscr{C}(\Lambda^{2,1})_+}$ self-dual

  $\mathscr{P}_{II} \subset \mathscr{C}(\Lambda^{2,1})_+/\mathbb{R}_{>0}$ has finite volume $\Longleftrightarrow$

$$\Delta(\Lambda_{II}^{2,1})_+^* \subset \overline{\mathscr{C}(\Lambda^{2,1})_+}^* = \overline{\mathscr{C}(\Lambda^{2,1})_+} \subset \Delta(\Lambda_{II}^{2,1})_+$$

The Weyl vector ρ

- Recall *lattice Weyl vector ρ satisfies* $(\rho, \delta_i) = -\frac{1}{2}(\delta_i, \delta_i) = -1$

- Consider cone $\Delta(\Lambda_{II}^{2,1})_+ = \mathbb{R}_{\geq 0}\delta_1 \oplus \mathbb{R}_{\geq 0}\delta_2 \oplus \mathbb{R}_{\geq 0}\delta_3$
  dual cone $\Delta(\Lambda_{II}^{2,1})_+^* = \{x \in \Lambda^{2,1} \otimes \mathbb{R} \,|\, (x, \delta_i) \leq 0\}$

- Cone $\overline{\mathscr{C}(\Lambda^{2,1})_+}^* = \overline{\mathscr{C}(\Lambda^{2,1})_+}$ self-dual
  and $\mathscr{P}_{II} \subset \mathscr{C}(\Lambda^{2,1})_+/\mathbb{R}_{>0}$ has finite volume $\Longrightarrow$

$$\Delta(\Lambda_{II}^{2,1})_+^* \subset \overline{\mathscr{C}(\Lambda^{2,1})_+}^* = \overline{\mathscr{C}(\Lambda^{2,1})_+} \subset \Delta(\Lambda_{II}^{2,1})_+$$

The Weyl vector ρ

- Gram matrix of $\Lambda_{II}^{2,1} = \mathbb{Z}\delta_1 \oplus \mathbb{Z}\delta_1 \oplus \mathbb{Z}\delta_1$

$$(\delta_i, \delta_j) = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}$$

- so $W^{(2)}(\Lambda_{II}^{2,1})$ has $\rho = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_2 + \frac{1}{2}\delta_3 = f_2 - \frac{1}{2}f_3 + f_{-2}$

- Identifying $\Lambda^{2,1} \otimes \mathbb{Q} \simeq \Lambda_{II}^{2,1} \otimes \mathbb{Q}$ then $\rho \in \Delta(\Lambda_{II}^{2,1})_+^*$

Fourier coefficients of χ_5 and the lattice $\Lambda_{II}^{2,1}$

- Use $W^{(2)}(\Lambda^{2,1})$ to study Fourier-coefficients of Δ_5

- Let $z = z_1 f_{-2} + z_2 f_3 + z_3 f_2 \in \Lambda^{2,1} \otimes \mathbb{R} + i\mathscr{C}(\Lambda^{2,1})_+$
$$= \Lambda_{II}^{2,1} \otimes \mathbb{R} + i\mathscr{C}(\Lambda_{II}^{2,1})_+$$

- Recall $(\Lambda_{II}^{2,1})^* = \mathbb{Z}\frac{1}{2}f_2 + \mathbb{Z}\frac{1}{2}f_3 + \mathbb{Z}\frac{1}{2}f_{-2} = \frac{1}{2}\Lambda^{2,1}$
$$\rho = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_2 + \frac{1}{2}\delta_3 = f_2 - \frac{1}{2}f_3 + f_{-2}$$

Fourier coefficients of χ_5 and the lattice $\Lambda_{II}^{2,1}$

- For $n, l, m \in \mathbb{Z}$ s.t. $n, m > 0, n \equiv m \equiv l \equiv 1 \mod 2, 4nm - l^2 > 0$

- $\frac{1}{64} f(n,l,m) \exp(\pi i(nz_1 + lz_2 + mz_3))$
  $= \frac{1}{64} f(n,l,m) \exp(-\pi i(nf_2 - lf_3 \frac{1}{2} + mf_{-2}, z))$
  $= m(a) \exp(-\pi i(\rho + a, z))$

- $a = (n-1)f_2 - (l-1)\frac{1}{2} f_3 + (m-1)f_2 \in (\Lambda^{2,1})^* = \frac{1}{2}\Lambda_{II}^{2,1}$
  $m(a) = -\frac{1}{64} f(n,l,m)$

- $\rho + a \in \mathscr{C}(\Lambda^{2,1})_+, m(a) \in \mathbb{Z}$ and $m(0) = -1$

Fourier coefficients of χ_5 and the lattice $\Lambda_{II}^{2,1}$

- **Lemma** Δ_5 *is either invariant or anti-invariant wrt* $\mathbf{O}(\Lambda^{2,1})_+$
  - $w \in W^{(2)}(\Lambda_{II}^{2,1})$ *and* $a \in Aut(\mathscr{P}_{II,prim})$

$$\Delta_5(w \cdot az) = \mathrm{det}(w)\Delta_5(z)$$

- $$\frac{1}{64}\Delta_5(z) = \sum_{w \in W^{(2)}(\Lambda_{II}^{2,1})} \mathrm{det}(w)\Big(\sum_{\rho + a \in (\Lambda_{II}^{2,1})^* \cap \Lambda_{II}^{2,1}} m(a)\exp(-\pi i(w(\rho + a), z))\Big)$$

Fourier coefficients of χ_5 and the lattice $\Lambda_{II}^{2,1}$

- $\rho + a \in (\Lambda_{II}^{2,1})^* \cap \Lambda_{II}^{2,1} \implies (\rho + a, \delta_i) \leq 0$
- When $(\rho + a, \delta_i) = 0$ then $m(a) = 0$
  - $\implies (\rho + a, \delta_i) \in \mathbb{Z}$ and $(\rho + a, \delta_i) < 0$
- Construction of $\rho \implies (a, \delta_i) \leq 0$

$$\implies a \in \Delta(\Lambda_{II}^{2,1})_+^* = \text{dual of}$$

$$\Delta(\Lambda_{II}^{2,1})_+ = \mathbb{R}_{\geq 0}\delta_1 \oplus \mathbb{R}_{\geq 0}\delta_2 \oplus \mathbb{R}_{\geq 0}\delta_3$$

Fourier coefficients of χ_5 and the lattice $\Lambda_{II}^{2,1}$

- Cone embeddings

$$\Delta(\Lambda_{II}^{2,1})_+^* \subset \overline{\mathscr{C}(\Lambda^{2,1})_+}^* = \overline{\mathscr{C}(\Lambda^{2,1})_+} \subset \Delta(\Lambda_{II}^{2,1})_+ \implies$$

$$a \in \mathbb{R}_{\geq 0}\mathscr{C}(\Lambda_{II}^{2,1})_+^* \cap \overline{\mathscr{C}(\Lambda^{2,1})_+} = \mathbb{R}_{\geq 0}\mathscr{P}_{II}$$

- So $a \in \mathbb{R}_{>0}\mathscr{P}_{II}$ if $a \neq 0$ and if $a = 0$ $m(a) = -1$
- Hence $a \in 2(\Lambda^{2,1})^* = \Lambda_{II}^{2,1}$!

Fourier coefficients of χ_5 and the lattice $\Lambda_{II}^{2,1}$

- Cone embeddings

$$\Delta(\Lambda_{II}^{2,1})_+^* \subset \overline{\mathscr{C}(\Lambda^{2,1})_+}^* = \overline{\mathscr{C}(\Lambda^{2,1})_+} \subset \Delta(\Lambda_{II}^{2,1})_+ \implies$$

$$a \in \mathbb{R}_{\geq 0}\mathscr{C}(\Lambda_{II}^{2,1})_+^* \cap \overline{\mathscr{C}(\Lambda^{2,1})_+} = \mathbb{R}_{\geq 0}\mathscr{P}_{II}$$

- So $a \in \mathbb{R}_{>0}\mathscr{P}_{II}$ if $a \neq 0$ and if $a = 0$ $m(a) = -1$

- Hence $a \in 2(\Lambda^{2,1})^* = \Lambda_{II}^{2,1}$!

Fourier coefficients of χ_5 and the lattice $\Lambda_{II}^{2,1}$

- Change range of summation to $a \in \Lambda_{II}^{2,1} \cap \mathbb{R}_{>0}\mathscr{P}_{II}$ to get $\dfrac{\Delta_5(z)}{64} =$

$$\sum_{w \in W^{(2)}(\Lambda^{2,1})} \text{det}(w)(\exp(-\pi i(w(\rho), z)) - \sum_{a \in \Lambda_{II}^{2,1} \cap \mathbb{R}_{>0}\mathscr{P}_{II}} m(a)\exp(-\pi i(w(\rho + a), z)))$$

Fourier coefficients of χ_5 and the lattice $\Lambda_{II}^{2,1}$

- Consider $a_0 \in \Lambda_{II}^{2,1} \cap \mathbb{R}_{>0}\mathscr{P}_{II}$ satisfying $(a_0, a_0) = 0$
- $Aut(\mathscr{P}_{II})$ is transitive on the three $a_0 \in \{2f_2, 2f_{-2}, 2f_2 - 2f_3 + 2f_{-2}\}$ *vertices at* ∞
- $Aut(\mathscr{P}_{II})$ preserves the Fourier expansion
- So pick, say, $a_0 = 2f_2$

$$1 + \frac{1}{64}\sum_{t\in\mathbb{N}} f(1+2t,1,1)q^t = \prod_{k\in\mathbb{N}}(1-q^k)^9$$ similarly holds for all three vertices at ∞

- i.e. $1 - \sum_{t\in\mathbb{N}} m(ta_0)q^t = \prod_{t\in\mathbb{N}}(1-q^t)^{\tau(ta_0)}$ implicitly defines some coefficients $\tau(a)$ for multiples of a_0

Automorphic correction and a GBKM super algebra $\mathfrak{g}_{\Delta_5}$

- Borcherds-Kac-Moody theory:

$$(\delta_i, \delta_j) = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix} \mapsto \mathfrak{g}$$

- $\mathfrak{g}$ infinite dimensional

- But Fourier expansion, and esp. coefficients $m(a)$ and $\tau(a) = 9$ suggest a modification (originally due to Borcherds)

Automorphic correction and a GBKM super algebra $\mathfrak{g}_{\Delta_5}$

- Let $\Delta^{im}_{\bar{0}} = \{\tau(a)a \,|\, (a,a) = 0, \tau(a) > 0\}$
  $\Delta^{im}_{\bar{1}} = \{m(a)a \,|\, (a,a) < 0, m(a) < 0\}$

  $\Delta^{im} = \Delta^{im}_{\bar{0}} \cup \Delta^{im}_{\bar{1}}$

- Let $\Delta^{re} = \Delta^{re}_{\bar{0}} = \mathscr{P}_{II,prim} = \mathbb{Z}\delta_1 \oplus \mathbb{Z}\delta_2 \oplus \mathbb{Z}\delta_3$

- Then $\Delta = \Delta^{re} \cup \Delta^{im}$ is a root system for a strictly larger algebra

Automorphic correction and a GBKM super algebra $\mathfrak{g}_{\Delta_5}$

- By construction $\alpha \in \Delta^{re} \implies (\alpha, \alpha) > 0$

  $\alpha \in \Delta^{im} \implies (\alpha, \alpha) \leq 0$

- $(\alpha, \alpha') \leq 0$ for all distinct $\alpha, \alpha' \in \Delta$

- $2\dfrac{(\alpha, \alpha')}{(\alpha, \alpha)} \in \mathbb{Z}$

- Δ is indeed a root system

Automorphic correction and a GBKM super algebra $\mathfrak{g}_{\Delta_5}$

- Generate a super-lie algebra $\mathfrak{g}_{\Delta_5}$ by $\{e_\alpha, h_\alpha, f_\alpha \,|\, \alpha \in \Delta\}$
  - Subject to
    $[h_\alpha, e_{\alpha'}] = (\alpha, \alpha')e_{\alpha'}$ $\mathfrak{sl}_2$ *relations*
    $[h_\alpha, f_{\alpha'}] = -(\alpha, \alpha')f_{\alpha'}$
    $[e_\alpha, f_{\alpha'}] = h_\alpha$ if $\alpha = \alpha'$, else 0
    if $(\alpha, \alpha') = 0$ then $[e_\alpha, e_{\alpha'}] = [f_\alpha, f_{\alpha'}] = 0$
    $\alpha \in \Delta^{re}$
    $(\mathrm{ad} e_\alpha)^{1-2\frac{(\alpha, \alpha')}{\alpha, \alpha)}} e_{\alpha'} = (\mathrm{ad} f_\alpha)^{1-2\frac{(\alpha, \alpha')}{\alpha, \alpha)}} f_{\alpha'} = 0$ *Serre relations*

Automorphic correction and a GBKM super algebra $\mathfrak{g}_{\Delta_5}$

- The algebra $\mathfrak{g}_{\Delta_5}$ is graded by $\Lambda_{II}^{2,1}$
- Integral cone of simple roots $\mathscr{C}_\Delta = \sum_{\alpha \in \Delta} \mathbb{Z}_+ \alpha \subset \Lambda_{II}^{2,1}$
- Triangular decomposition $\mathfrak{g}_{\Delta_5} = \oplus_{\alpha \in \mathscr{C}_\Delta} \mathfrak{g}_\alpha \oplus \Lambda_{II}^{2,1} \oplus_{\alpha \in -\mathscr{C}_\Delta} \mathfrak{g}_\alpha$

Automorphic correction and a GBKM super algebra $\mathfrak{g}_{\Delta_5}$

- The algebra $\mathfrak{g}_{\Delta_5}$ is graded by $\Lambda_{II}^{2,1}$
- Integral cone of simple roots $\mathscr{C}_\Delta = \sum_{\alpha \in \Delta} \mathbb{Z}_+ \alpha \subset \Lambda_{II}^{2,1}$
- Triangular decomposition $\mathfrak{g}_{\Delta_5} = \oplus_{\alpha \in \mathscr{C}_\Delta} \mathfrak{g}_\alpha \oplus \Lambda_{II}^{2,1} \oplus_{\alpha \in -\mathscr{C}_\Delta} \mathfrak{g}_\alpha$
- $\Delta_\pm = \pm \mathscr{C}_\Delta \cap \Delta$
- For every root α denote $\mathbf{mult}_\alpha = \mathbf{mult}_{\bar{0}} + \mathbf{mult}_{\bar{1}} = \mathbf{dim} \mathfrak{g}_{\bar{0}} - \mathbf{dim} \mathfrak{g}_{\bar{1}}$

Automorphic correction and a GBKM super algebra $\mathfrak{g}_{\Delta_5}$

- GBKM theory allows us to apply the Weyl-Kac-Borcherds character formula to $\mathbb{C}$ trivial module

$$\Phi = \sum_{w \in W^{(2)}(\Lambda_{II}^{2,1})} \mathrm{det}(w)(\exp(-2\pi i(w(\rho), z)) - \sum_{a \in \Lambda_{II}^{2,1} \cap \mathbb{R}_{>0}\mathscr{P}_{II}} m(a)\exp(-2\pi i(w(\rho + a), z)))$$

$$= \exp(-2\pi i(\rho, z)) \prod_{\alpha \in \Delta_+} (1 - \exp(-2\pi i(\alpha, z)))^{\mathrm{mult}\alpha}$$

- Valid for $z \in \Omega(\Lambda_{II}^{2,1}) = \Lambda_{II}^{2,1} \otimes \mathbb{R} + i\mathscr{C}(\Lambda_{II}^{2,1})_+$

Automorphic correction and a GBKM super algebra $\mathfrak{g}_{\Delta_5}$

- **Theorem (Borcherds-Gritsenko-Nikulin)**

$$\frac{1}{64}\Delta_5(2z) = \Phi(z)$$

The denominator of the automorphic corrected $\mathfrak{g}_{\Delta_5}$ is the Siegel modular cusp form of genus 2 and weight 5

Root super-multiplicities of $\mathfrak{g}_{\Delta_5}$ and the weak Jacobi form $\phi_{0,1}$

- $\Delta_5 = \sum_{w \in W^{(2)}(\Lambda_{II}^{2,1})} \mathrm{det}(w)(\exp(-2\pi i(w(\rho), z)) - \sum_{a \in \Lambda_{II}^{2,1} \cap \mathbb{R}_{>0}\mathscr{P}_{II}} m(a)\exp(-2\pi i(w(\rho + a), z)))$

$$= \exp(-2\pi i(\rho, z)) \prod_{\alpha \in \Delta_+} (1 - \exp(-2\pi i(\alpha, z)))^{\mathbf{mult}\alpha}$$

- We determine $\mathbf{mult}\alpha$ in terms of a weak Jacobi form

Root super-multiplicities of $\mathfrak{g}_{\Delta_5}$ and the weak Jacobi form $\phi_{0,1}$

- Fourier-Jacobi expansion of $\Delta_{12} \to \phi_{12,1}$ Jacobi cusp form of weight 12

- $\phi_{12,1} = (r^{-1} + 10 + r)q + (10r^{-2} - 88r^{-1} - 132 - 88r + 10r^2)q^2 + \ldots$

- Let $\phi_{0,1} = \dfrac{\phi_{12,1}}{\delta_{12}} = \sum\limits_{n \geq 0, l \in \mathbb{Z}} f(n,l) \exp(2\pi i(nz_1 + lz_2))$

  $\delta_{12} = q \prod\limits_{n \geq 1} (1 - q^n)^{24}$ is a $\mathbf{SL}_2(\mathbb{Z})$ cusp form

- $\phi_{0,1} = (r^{-1} + 10 + r) + q(10r^{-2} - 64r^{-1} + 108 - 64r + 10r^2) + \ldots$

Root super-multiplicities of $\mathfrak{g}_{\Delta_5}$ and the weak Jacobi form $\phi_{0,1}$

- **Theorem**

$$\frac{1}{64}\Delta_5 = \exp(\pi i(z_1 + z_2 + z_3)) \prod_{n,l,m \in \mathbb{Z},(n,l,m)>0} (1 - \exp(2\pi i(nz_1 + lz_2 + mz_3)))^{f(nm,l)}$$

- $(n, l, m) > 0$ means that $n \geq 0$,$m \geq 0$ and l is arbitrary integral if $n > 0$ or $m > 0$ and $l < 0$ if $n = m = 0$.

Root super-multiplicities of $\mathfrak{g}_{\Delta_5}$ and the weak Jacobi form $\phi_{0,1}$

- Analyze Fourier coefficients of $\phi_{12,1}$

  express as linear combination of Jacobi-theta functions

- $\implies \phi_{0,1}$ has asymptotic behaviour of $f(n,l) \sim O(\exp(\sqrt{4n - l^2}))$

- **Kac** *Infinite Dimensional Lie Algebras* $\implies$ product converges on any neighborhood of the zero-dimensional cusp of $\mathbf{Sp_4}(\mathbb{Z})$

Root super-multiplicities of $\mathfrak{g}_{\Delta_5}$ and the weak Jacobi form $\phi_{0,1}$

- $$\exp(\pi i(z_1 + z_2 + z_3)) \prod_{n>0, l\in\mathbb{Z} \text{ or } n=0, l<0} (1 - \exp(2\pi i(nz_1 + lz_2))^{f(0,l)} \\ \times \prod_{n\geq 0, m>0, l\in\mathbb{Z}} (1 - \exp(2\pi i(nz_1 + lz_2 + mz_3)))^{f(nm,l)}$$
- $f(0,0) = 10, f(0,-1) = 1, f(0,l) = 0$ if $l < -1$
- Denote big products A & B

Root super-multiplicities of $\mathfrak{g}_{\Delta_5}$ and the weak Jacobi form $\phi_{0,1}$

- $T_{-}(m)$ minus embeddings $\mathbf{GL_2}(\mathbb{Z})$-Hecke operators
  - Jacobi form ϕ weight k index $t \ \mapsto (\phi\,|_k T_{-}(m))$ Jacobi form index mt
- $$\log(B) = -\sum_{m\geq 1} m^2(\phi_{0,1}\exp^{2\pi i t z_3}\,|_0\, T_{-}(m)))$$
  - Invariant wrt maximal parabolic subgroup $\Gamma_\infty \subset \mathbf{Sp_4}(\mathbb{Z})$

Root super-multiplicities of $\mathfrak{g}_{\Delta_5}$ and the weak Jacobi form $\phi_{0,1}$

- $A = \psi_{5,\frac{1}{2}} \exp(2\pi i t z_3) = \eta(z_1)^9 \nu_{11}(z_1, z_2)$

- $\eta = \exp(\frac{\pi i \tau}{12}) \prod_{n \geq 1} (1 - \exp(2\pi i n \tau))$

  Dedekind's η function

- $\nu_{11}(z_1, z_2) = \sum_{n \in \mathbb{Z}} (-1)^n \exp(\frac{\pi i}{4}(2n+1)^2 z_1 + \pi i (2n+1) z_2)$

  Jacobi-theta series

Root super-multiplicities of $\mathfrak{g}_{\Delta_5}$ and the weak Jacobi form $\phi_{0,1}$

- Properties of A & B taken together $\Longrightarrow$
  product transforms as modular form for Jacobi group $\Gamma_\infty / \pm \mathbf{I}$ with determined character
  $\nu_\infty : \Gamma_\infty / \pm \mathbf{I} \to \pm 1$

- Test against $S = \begin{pmatrix} \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & 0 \\ 0 & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \end{pmatrix}$ yields anti-invariance of product

- Jacobi group and S generate $\mathbf{PSp_4}(\mathbb{Z}) \Longrightarrow$
  - Product transforms as a Siegel modular form of weight 5 with the same character as Δ_5

- Compare first fourier coefficient. Q.E.D.

Arithmetic Geometry of some CY3's

- Let (S, g_N, h_M) be a K3 surface S with two symplectic automorphisms of finite order $N, M \leq 8$

- Let (E, e_0) be a non-singular elliptic curve with an N-torsion point e_0

- $\mathbb{Z}/N\mathbb{Z}$ acts freely on $S \times E$ preserving the CY form by

$$(s, e) \mapsto (gs, e + e_0)$$

- The quotient $X = (S \times E)/\mathbb{Z}/N\mathbb{Z}$ is a smooth projective CY3

Arithmetic Geometry of some CY3's

- Let $(S, s \mapsto s + s_2(\pi(s)), id)$ an elliptically fibered K3: $\pi : S \to \mathbb{P}^1$ admitting sections

$$s_1, s_2 : \mathbb{P}^1 \to S$$

  such that s_2 is of order N relative to s_1, treated as zero-section

- Consider the N sections

$$s_1, s_2, \ldots, s_k = s + s_2(\pi(s_{k-1})), \ldots, s_N$$

- Let $F \in Pic(S)$ be a class of a fiber of $\pi : S \to \mathbb{P}^1$. Consider

$$\beta_h = \frac{1}{N}(s_1 + \ldots + s_N + hF), \text{ for } h \geq 0$$

Arithmetic Geometry of some CY3's

- Let $<\alpha,\beta>: \int_S \alpha\cup\beta$ be the intersection pairing on S and form

$$Z^X(q,t,p) = \sum_{h=0}^{\infty}\sum_{d=0}^{\infty}\sum_{n\in\mathbb{Z}} \mathbf{DT}^X_{n,(\beta_h,d)} q^{d-1} t^{\frac{1}{2}<\beta_h,\beta_h>}(-p)^n$$

- Defined as follows:
  - let $\mathrm{Hilb}^n(X,\beta_h)$ the Hilbert scheme of 1-dimensional subschemes $Z\subset X$, such that

$$[Z]=\beta_h, \qquad \chi(\mathcal{O}_Z)=n\in\mathbb{Z}$$

  - E acts on X inducing an action on $\mathrm{Hilb}^n(X,\beta_h)$
  - $\mathbf{DT}^X_{n,(\beta_h,d)} = \int_{\mathrm{Hilb}^n(X,\beta_h)/E} \nu de = \sum_{k\in\mathbb{Z}} k\cdot e(\nu^{-1}(k))$ where. $\nu:\mathrm{Hilb}^n(X,\beta_h)\to\mathbb{Z}$ is Behrend's function

Arithmetic Geometry of some CY3's

- **Theorem** Let $N = 1$ i.e. consider $X = S \times E$ a smooth projective CY3 arising from an elliptically fibered K3 surface and an elliptic curve

$$Z^{S\times E} = -\frac{C}{(\Delta_5)^2}$$

- Moreover, the elliptic genus of a K3 surface is $2\phi_{0,1}$

Arithmetic Geometry of some CY3's

- Let X come from (S, g_N^r, g_N^s) and (E, e_0) where S is elliptically fibered, more generally (S, g_N, h_M) with a lattice polarization L

- We can define the $L-(h_M = g_N^s)$-twisted series $Z^X_{L,h_M}(q,t,p)$

- **Conjecture** all dd Siegel modular forms arise, up to constant, as

$$\sqrt{-\frac{1}{Z^X_{L,h_M}(q,t,p)}}$$

  and there is a GBKM algebra with correction $\mathfrak{g} \subset \mathfrak{g}_L$, depending on L, given by a(g_N, h_M) -twisted elliptic genus of a K3-surface $F_L^{(g_N,h_M)}$, a Jacobi form of weight 0, index 1

Arithmetic Geometry of some CY3's

- Let $A = \frac{1}{4}\phi_{0,1}$ B = $\phi_{-2,1} = \frac{\nu_1^2}{\eta^6}$
- Case $N \in \{1,2,3,5,7\}$ For all $1 \leq s, r \leq N-1$ and $0 \leq k \leq N-1$

$$F^{(0,0)}(\tau,z) = \frac{8}{N}A(\tau,z)$$

$$F^{(0,s)}(\tau,z) = \frac{8}{N(N+1)}A(\tau,z) - \frac{2}{N+1}B(\tau,z)\mathcal{E}_N(\tau)$$

$$F^{(r,rk)}(\tau,z) = \frac{8}{N(N+1)}A(\tau,z) + \frac{2}{N(N+1)}B(\tau,z)\mathcal{E}_N\left(\frac{\tau+k}{N}\right)$$

Arithmetic Geometry of some CY3's

- Case $N = 4$ For all $s \in \{0,1,2,3\}$

$$F^{(0,0)}(\tau,z) = 2A(\tau,z),$$

$$F^{(0,1)}(\tau,z) = F^{(0,3)}(\tau,z) = \frac{1}{4}\left[\frac{4A}{3} - B\left(-\frac{1}{3}\mathcal{E}_2(\tau) + 2\mathcal{E}_4(\tau)\right)\right],$$

$$F^{(1,s)}(\tau,z) = F^{(3,3s)} = \frac{1}{4}\left[\frac{4A}{3} + B\left(-\frac{1}{6}\mathcal{E}_2(\frac{\tau+s}{2}) + \frac{1}{2}\mathcal{E}_4(\frac{\tau+s}{4})\right)\right],$$

$$F^{(2,1)}(\tau,z) = F^{(2,3)} = \frac{1}{4}\left(\frac{4A}{3} - \frac{B}{3}(5\mathcal{E}_2(\tau) - 6\mathcal{E}_4(\tau)\right),$$

$$F^{(0,2)}(\tau,z) = \frac{1}{4}\left(\frac{8A}{3} - \frac{4B}{3}\mathcal{E}_2(\tau)\right),$$

$$F^{(2,2s)}(\tau,z) = \frac{1}{4}\left(\frac{8A}{3} + \frac{2B}{3}\mathcal{E}_2(\frac{\tau+s}{2})\right).$$

Arithmetic Geometry of some CY3's

- Case $N = 6$ For all $s \in \{0,1,2,3\}$

$$
\begin{aligned}
F^{(0,0)} &= \frac{4}{3}A \\
F^{(0,1)} &= F^{(0,5)} = \frac{1}{6}\left[\frac{2A}{3} - B\left(-\frac{1}{6}\mathcal{E}_2(\tau) - \frac{1}{2}\mathcal{E}_3(\tau) + \frac{5}{2}\mathcal{E}_6(\tau)\right)\right], \\
F^{(0,2)} &= F^{(0,4)} = \frac{1}{6}\left[2A - \frac{3}{2}B\mathcal{E}_3(\tau)\right], \\
F^{(0,3)} &= \frac{1}{6}\left[\frac{8A}{3} - \frac{4}{3}B\mathcal{E}_2(\tau)\right]. \\
F^{(1,k)} &= F^{(5,5k)} = \frac{1}{6}\left[\frac{2A}{3} + B\left(-\frac{1}{12}\mathcal{E}_2(\frac{\tau+k}{2}) - \frac{1}{6}\mathcal{E}_3(\frac{\tau+k}{3}) + \frac{5}{12}\mathcal{E}_6(\frac{\tau+k}{6})\right)\right], \\
F^{(2,2k+1)} &= \frac{A}{9} + \frac{B}{36}\left[\mathcal{E}_3(\frac{\tau+2+k}{3}) + \mathcal{E}_2(\tau) - \mathcal{E}_2\left(\frac{\tau+k+2}{3}\right)\right], \\
F^{(4,4k+1)} &= \frac{A}{9} + \frac{B}{36}\left[\mathcal{E}_3(\frac{\tau+1+k}{3}) + \mathcal{E}_2(\tau) - \mathcal{E}_2\left(\frac{\tau+k+1}{3}\right)\right],
\end{aligned}
$$

Arithmetic Geometry of some CY3's

- Case $N = 6$ For all $s \in \{0,1,2,3\}$

$$
\begin{aligned}
F^{(3,1)} = F^{(3,5)} &= \frac{A}{9} - \frac{B}{12}\mathcal{E}_3(\tau) - \frac{B}{72}\mathcal{E}_2(\frac{\tau+1}{2}) + \frac{B}{8}\mathcal{E}_2(\frac{3\tau+1}{2}), \\
F^{(3,2)} = F^{(3,4)} &= \frac{A}{9} - \frac{B}{12}\mathcal{E}_3(\tau) - \frac{B}{72}\mathcal{E}_2(\frac{\tau}{2}) + \frac{B}{8}\mathcal{E}_2(\frac{3\tau}{2}), \\
F^{(2r,2rk)} &= \frac{1}{6}\left[2A + \frac{1}{2}B\mathcal{E}_3(\frac{\tau+k}{3})\right], \\
F^{(3,3k)} &= \frac{1}{6}\left[\frac{8A}{3} + \frac{2}{3}B\mathcal{E}_2(\frac{\tau+k}{2})\right].
\end{aligned}
$$

Arithmetic Geometry of some CY3's

- Case $N = 8$

$$\begin{aligned}
F^{(0,0)} &= A, \\
F^{(0,1)} &= F^{(0,3)} = F^{(0,5)} = F^{(0,7)}, \\
&= \frac{1}{8}\left[\frac{2A}{3} - B\left(-\frac{1}{2}\mathcal{E}_4(\tau) + \frac{7}{3}\mathcal{E}_8(\tau)\right)\right]. \\
F^{(r,rk)} &= \frac{1}{8}\left[\frac{2A}{3} + \frac{B}{8}\left(-\mathcal{E}_4(\frac{\tau+k}{4}) + \frac{7}{3}\mathcal{E}_8(\frac{\tau+k}{8})\right)\right], r = 1,3,5,7 \\
F^{(2,1)} &= F^{(6,3)} = F^{(2,5)} = F^{(6,7)}, \\
&= \frac{1}{8}\left[\frac{2A}{3} + \frac{B}{3}\left(-\mathcal{E}_2(2\tau) + \frac{3}{2}\mathcal{E}_4(\frac{2\tau+1}{4})\right)\right]; \\
F^{(2,3)} &= F^{(6,5)} = F^{(2,7)} = F^{(6,1)}, \\
&= \frac{1}{8}\left[\frac{2A}{3} + \frac{B}{3}\left(-\mathcal{E}_2(2\tau) + \frac{3}{2}\mathcal{E}_4(\frac{2\tau+3}{4})\right)\right].
\end{aligned}$$

Arithmetic Geometry of some CY3's

- Case $N = 8$

$$
\begin{aligned}
F^{(0,2)} = F^{(0,6)} &= \frac{1}{8}\left(\frac{4A}{3} - B\left(-\frac{1}{3}\mathcal{E}_2(\tau) + 2\mathcal{E}_4(\tau)\right)\right), \\
F^{(0,4)} &= \frac{1}{8}\left(\frac{8A}{3} - \frac{4B}{3}\mathcal{E}_2(\tau)\right), \\
F^{(2,2s)} = F^{(6,6s)} &= \frac{1}{8}\left(\frac{4A}{3} + B\left(-\frac{1}{6}\mathcal{E}_2(\frac{\tau+s}{2}) + \frac{1}{2}\mathcal{E}_4(\frac{\tau+s}{4})\right)\right), \\
F^{(4,4s)} &= \frac{1}{8}\left(\frac{8A}{3} + \frac{2B}{3}\mathcal{E}_2(\frac{\tau+s}{2})\right), \\
F^{(4,2)} = F^{(4,6)} &= \frac{1}{8}\left(\frac{4A}{3} - \frac{B}{3}(3\mathcal{E}_2(\tau) - 4\mathcal{E}_2(2\tau)\right), \\
F^{(4,2k+1)} &= \frac{1}{8}\left(\frac{2A}{3} + B\left(\frac{4}{3}\mathcal{E}_2(4\tau) - \frac{2}{3}\mathcal{E}_2(2\tau) - \frac{1}{2}\mathcal{E}_4(\tau)\right)\right).
\end{aligned}
$$