A BORCHERDS LIFT OF THE WEAK JACOBI FORM $\phi_{0,1}$, GENERALIZED BORCHERDS-KAC-MOODY SUPERALGEBRAS AND THE IGUSA CUSP FORM Δ_5

RAEEZ LORGAT

ABSTRACT. Motivated by considerations in arithmetic algebraic geometry, we investigate a relationship between the siegel modular form known as the Igusa cusp form Δ_5 of weight 5 and a certain weak Jacobi form $\phi_{0,1}$ of weight 0 and index 1. Following Kac-Moody, Borcherds, Gritsenko and Nikulin et. al., we work out in elementary detail the derivation of an infinite product formula for Δ_5 as an expansion at the cusp. In doing so, we give an elementary construction of a pair of an infinite dimensional generalized kac-moody superalgebra $\mathfrak{g}$ along with its *automorphic correction* $\mathfrak{g} \subset \mathfrak{g}_{\Delta_5}$, both constructed from an underlying real (in the case of $\mathfrak{g}$) (respectively real and imaginary in the case of $\mathfrak{g}_{\Delta_5}$) root datum. Both of these sets of root datum are realized in the geometry of the lattice $\Lambda^{3,2}$ of signature $(3,2)$, its hyperbolic sublattices $\Lambda^{2,1}, \Lambda^{2,1}_{II}$,the associated Weyl group(s) and Weyl vector. The macdonald identity for the automorphic correction algebra witnesses the infinite product formula for the cusp form Δ_5 via the Weyl-Kac character formula applied to the trivial one dimensional representation $\mathbb{C}$. The weight 0 index 1 weak Jacobi form $\phi_{0,1}$ is related to Δ_5 as a Jacobi form counting the super dimensions of weight spaces in the automorphic correction $\mathfrak{g}_{\Delta_5}$.

CONTENTS

1. A MOTIVATING CONJECTURE

Fix the genus $g = 2$ and consider the integral symplectic groups Γ_t, or so-called *paramodular groups*, and their corresponding congruence subgroups of Hecke type

$$t, n \in \mathbb{Z}_+ : \Gamma_t(N) = \{ \begin{pmatrix} * & *t & * & * \\ & * & * & *t^{-1} \\ N & *Nt & * & * \\ Nt & *Nt & *t & * \end{pmatrix} \in \mathbf{Sp_4}(\mathbb{Q}) | * \in \mathbb{Z}\}.$$

The group $\Gamma_1 = \Gamma_1(1) = \mathbf{Sp}_4(\mathbb{Z})$ while $\Gamma_t = \Gamma_t(1)$ can be conjugated to the integral symplectic group of integral skew-symmetric form with elementary divisor $(1,t)$. The corresponding quotients $\mathbb{H}_2/\Gamma_t$ of the genus 2 Siegel-upper space are the moduli spaces of $(1,t)$-polarized abelian surfaces.

Gritsenko-Clery[1] answered the following question

Question 1. *Which Siegel modular forms with respect to $\Gamma_t(N)$ (with character/multiplier system) vanish exactly along $\Gamma_t(N)$-translates of the diagonal $\{\begin{pmatrix} a & \\ & b \end{pmatrix} | a, b \in \mathbb{H}_1\} \subset \mathbb{H}_2$ precisely to order 1?*

their answer comes in the form of

Theorem 1. *For all possible congruence subgroups $\Gamma_t(N) < \Gamma_t$ there are exactly 8 diagonal-divisor modular forms.*

See [1] for the full treatment. This writeup concerns the simplest specimen from this class, the weight 5 cuspidal modular form for the group $\Gamma_1 = \mathbf{Sp}_4(\mathbb{Z})$ given by Igusa's cusp form Δ_5, defined in the following section.

One of our motivations for investigating the detailed structure of Δ_5, and indeed, each modular form from this class, comes from the arithmetic geometry of certain Calabi-Yau threefolds.

Let (E, e_0) be an elliptic curve with an N-torsion point e_0 and let S be an elliptically fibered CY3

$$\pi : S \to \mathbb{P}^1$$

Date: April 2 2020.

admitting sections

$$s_1, s_2 : \mathbb{P}^1 \to S$$

with s_2 of order N relative to s_1, and where we treat s_1 as the zero-section. Then we can form the N sections

$$s_1, s_2, \ldots, s_k = s + s_2(\pi(s_{k-1})), \ldots, s_N.$$

The product

$$S \times E$$

admits a free action by the finite group $\mathbb{Z}/N\mathbb{Z}$ given by

$$(s, e) \mapsto (s + s_2(\pi(s)), e + e_0);$$

and so the quotient X is a projective Calabi-Yau threefold. Now consider the class of the fiber $F \in Pic(S)$ of the map π, and form the curve class

$$\beta_h = \frac{1}{N}(s_1 + \ldots + s_N + hF), \text{ for } h \geq 0.$$

We are interested in a certain zeta function of X formed out of virtual volumes of the moduli stacks of 1-dimensional sheaves supported on β_h. To define this zeta function, consider $\mathrm{Hilb}^n(X, \beta_h)$ the Hilbert scheme of 1-dimensional subschemes $Z \subset X$ such that

$$[Z] = \beta_h$$
$$\chi(\mathcal{O}_Z) = n$$

i.e. with given algebro-topological invariants of cycle class β_h and euler characteristic n. E acts on $S \times E$, and this action descends to X. We can then define the reduced numerical Donaldson-Thomas invariant associtaed to (β_h, n) [2],[3] by virtual integration over this quotient stack

$$\int_{\mathrm{Hilb}^n(X,\beta_h)/E} \nu de = \sum_{k \in \mathbb{Z}} k \cdot e(\nu^{-1}(k))$$

where $\nu : \mathrm{Hilb}^n(X, \beta_h) \to \mathbb{Z}$ is Behrend's constructible function. Let $< \alpha, \beta >: \int_S \alpha \cup \beta$ be the intersection form on S, then we have [3]

Theorem 2. *For $X = S \times E$ i.e. order $N = 1$ we have*

$$Z^X(q, t, p) = \sum_{h=0}^{\infty} \sum_{d=0}^{\infty} \sum_{n \in \mathbb{Z}} \mathbf{DT}^X_{n,(\beta_h,d)} q^{d-1} t^{\frac{1}{2}<\beta_h,\beta_h>} (-p)^n = \frac{C}{(\Delta_5)^2}$$

for a constant C.

Moreover, as will become relevant from later discussion, it is interesting to note that the *elliptic genus* of a K3 surface is given by the weak jacobi form $\phi_{0,1}$, defined in section 6. Generalizing slightly, then we may consider in addition to (E, e_0) the data (S, L, g_N, h_M) of a K3 surface S together with lattice polarization L, and finite order symplectic automorphisms g_N and h_M of orders N, M necessarily less than or equal to 8 by a theorem of Nikulin[4]. Then we may analogously define the twisted zeta functions Z^X_{L,h_M} counting $\frac{1}{M}$-fractional sheaves on $X = (S \times E)/\mathbb{Z}/N\mathbb{Z}$ via numerical (or if we wish, refined) Donaldson-Thomas theory. In the numerical case, we have

Conjecture 1. *all eight diagonal divisor modular froms of Gritsenko-Clery arise, up to constant C,as reciprocal-square roots of Z^X_{L,h_M}. Moreover, these Siegel paramodular forms all arise as denominator functions of generalized Borcherds-Kac-Moody superalgebras, with root multiplicities specified by $g_N - h_M$-twisted twined elliptic genera of K3 surfaces.*

Our treatment follows [5], while most of the fundamental ideas concerning automorphic corrections of generalized Borcherds-Kac-Moody algebras follows Borcherds[6]. We make no claims to originality, except in the formulation of the conjecture, in which case we accept all responsibility for lack of precision.

2. Preamble on the cusp form Δ_5

We freely use results from Freitag[7] and Van der Geer[8]. Recall the ring of Siegel modular forms

$$\mathcal{SM}(\mathbf{Sp}_4(\mathbb{Z})) = \mathbb{C}[E_4, E_6, \Delta_{10}, \Delta_{12}]$$

generated by two eisenstein series (E_4, E_6) of weights 4 and 6 and the two siegel cusp forms Δ_{10}, Δ_{12} of weights 10 and 12 respectively. Note that Δ_{10} is the square of a cusp form Δ_5 of weight 5 with a non-trivial multiplier system ν_{Δ_5}. Δ_5 may be explicitly expressed as the product of all ten even theta constants

$$\Delta_5 = \prod_{\substack{(a,b) \in (\mathbb{Z}/2\mathbb{Z})^2 \\ {}^t ab \equiv 0 \mod 2}} \nu_{a,b}$$

where

$$\nu_{ab}(z) = \sum_{l \in \mathbb{Z}^2} \exp(\pi i (z[l + \frac{1}{2}a] + {}^t bl))$$

using

$$z[x] = {}^t xzx.$$

Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbf{Sp}_4(\mathbb{Z})$, then the explicit form of the non-trivial multiplier system

$$\nu_{\Delta_5} : \mathbf{Sp}_4(\mathbb{Z}) \to \mathbb{C}$$

for Δ_5 such that $|\nu_{\Delta_5}(g)| = 1$ for all $g \in \mathbf{Sp}_4(\mathbb{Z})$ (found by Maass [9]) is given by

$$\nu_{\Delta_5}\begin{pmatrix} 0 & \mathbf{I}_2 \\ -\mathbf{I}_2 & 0 \end{pmatrix} = 1,$$

$$\nu_{\Delta_5}\begin{pmatrix} \mathbf{I}_2 & B \\ 0 & \mathbf{I}_2 \end{pmatrix} = (-1)^{b_1+b_2+b_3},$$

$$\nu_{\Delta_5}\begin{pmatrix} {}^tA^{-1} & 0 \\ 0 & A \end{pmatrix} = (-1)^{(1+a_1+a_4)(1+a_2+a_3)+a_1a_4}$$

where $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in \mathbf{GL}_2(\mathbb{Z})$ and $B = \begin{pmatrix} b_1 & b_2 \\ b_2 & b_3 \end{pmatrix} \in M_{2\times 2}(\mathbb{Z})$.

Using the expression in terms of even theta constants as well as the explicit form of the multiplier system, we can show that in terms of the matrix $Z = \begin{pmatrix} z_0 & z_1 \\ z_2 & z_3 \end{pmatrix}$ that we can express the fourier expansion

$$\Delta_5(Z) = \sum_{\substack{n,l,m=1\text{mod}2, \\ 4nm-l^2>0, \\ n,m>0}} f(n,l,m)\exp(\pi i(nz_1 + lz_2 + mz_3))$$

It is easy to show that $f(1,1,1) = 64$ as well as that $64|f(n,l,m)$ for all n,l,m. Finally, we will need the following identity of power series

$$1 + \frac{1}{64}\sum_{n\in\mathbb{N}} f(1+2t,1,1)q^t = \prod_{k\in\mathbb{N}}(1-q^k)^9$$

Proof. Recall the fourier-Jacobi expension of

$$\Delta_5(Z) = \sum_{\substack{m>0, \\ m=1 \mod 2}} \phi_{5,m}(z_1,z_2)\exp(\pi i m z_3)$$

then the first Fourier-Jacobi coefficient is a Jacobi cusp form of index $\frac{1}{2}$ and non-trivial character. We'll need the Jacobi theta-series

$$\nu_{11}(z_1,z_2) = \sum_{n\in\mathbb{Z}}(-1)^n\exp(\frac{\pi i}{4}(2n+1)^2 z_1 + \pi i(2n+1)z_2),$$

a variant of the Jacobi triple-product formula yields a product expansion for

$$\nu_{11} = q^{\frac{1}{8}}r^{-\frac{1}{2}}\prod_{n\geq 1}(1-q^{n-1}r)(1-q^n r^{-1})(1-q^n)$$

where $q = \exp(2\pi i z_1)$ and $r = \exp(2\pi i z_2)$, but then $\psi_{5,\frac{1}{2}} = \eta(z_1)^9\nu_{11}(z_1,z_2)$ is another Jacobi cusp form of index $\frac{1}{2}$ and the same character, with

$$\eta(z_1) = \exp(\frac{\pi i z_1}{12})\prod_{n\geq 1}(1-\exp(2\pi i n z_1);$$

the squares of these Jacobi cusp forms are Jacobi cusp forms of weight 10 and index 1; up to a constant, there is only one of these, and it is the first Fourier-Jacobi coefficient of $\Delta_{10} = \Delta_5^2$; comparing fourier coefficients of the product expansion we obtain

$$\frac{1}{64}\phi_{5,1}(z_1,z_2) = \psi_{5,\frac{1}{2}}(z_1,z_2) = -q^{\frac{1}{2}}r^{-\frac{1}{2}}\prod_{n\geq 1}(1-q^{n-1}r)(1-q^n r^{-1})(1-q^n)^{10}$$

the desired identity as an application of the Jacobi triple-product identity applied to the coefficient of $r^{\frac{1}{2}}$. □

Together these fundamental properties of the cusp form Δ_5 will be used to construct a generalized kac moody lie superalgebra with denominator identity in terms of Δ_5.

3. An isomorphism between the symplectic group $\mathbf{Sp}_4(\mathbb{Z})/\{\pm\mathbf{I}_5\}$ and the orthogonal group $\mathbf{O}(\Lambda^{3,2})_+/\pm\mathbf{I}_5$

Proofs for results of this section can be found in [10]. We obtain an isomorphism relating relevant low rank symplectic and orthogonal groups.

Consider the rank 4 free $\mathbb{Z}$-module

$$\Lambda^4 = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3 \oplus \mathbb{Z}e_4.$$

Any $\mathbb{Z}$-linear map $g : \Lambda^4 \to \Lambda^4$ induces a linear map $\wedge^2 g : \Lambda^4 \wedge \Lambda^4 \to \Lambda^4 \wedge \Lambda^4$. In particular, we have an induced action of $\mathbf{SL}_4(\mathbb{Z})$.

We have a (pfaffian) scalar product $(,) : \Lambda^4 \wedge \Lambda^4 \to \mathbb{C}$ defined by $u \wedge v = (u,v)e_1 \wedge e_2 \wedge e_3 \wedge e_4 \in \wedge^4\Lambda^4$. This is an $\mathbf{SL}_4(\mathbb{Z})$ invariant even unimodular integral symmetric bilinear form of signature $(3,3)$.

Observe that the for $q = e_1 \wedge e_3 + e_2 \wedge e_4 \in \Lambda^4 \wedge \Lambda^4$ we have

$$-x \wedge y \wedge q = B_q(x,y)e_1 \wedge e_2 \wedge e_3 \wedge e_4$$

and hence the elements $q \in \Lambda^4 \wedge \Lambda^4$ can be identified with integral skew-symmetric bilinear forms on Λ^4.

It follows that

$$\{g : \Lambda^4 \to \Lambda^4 | (g \wedge g)(e_1 \wedge e_3 + e_2 \wedge e_4) = e_1 \wedge e_3 + e_2 \wedge e_4\} \simeq \mathbf{Sp}_4(\mathbb{Z}).$$

Hence the lattice

$$\Lambda^{3,2} = (e_1 \wedge e_3 + e_2 \wedge e_4)^{\perp} \subset \Lambda^4 \wedge \Lambda^4$$

is isomorphic to

$$\Lambda^{3,2} \simeq \Lambda^{(1,1)} \oplus \Lambda^{(1,1)} \oplus [2]$$

with $[2]$ the one dimensional $\mathbb{Z}$ lattice with inner product given by the matrix (2) and $\Lambda^{(1,1)}$ the standard integral hyperbolic plane i.e. a lattice with quadratic form $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

Next we fix a basis $(f_i)_{\{1,2,3,-2,-1\}}$ in $\Lambda^{3,2}$ given by

$$\begin{aligned} (f_1 &= e_1 \wedge e_2, \\ f_2 &= e_2 \wedge e_3, \\ f_3 &= e_1 \wedge e_3 - e_2 \wedge e_4, \\ f_{-2} &= e_4 \wedge e_1, \\ f_{-1} &= e_4 \wedge e_3) \end{aligned}$$

Now the real orthogonal group $\mathbf{O}_{\mathbb{R}}(\Lambda^{3,2}) = \mathbf{O}_{\mathbb{R}}(\Lambda^{3,2} \otimes \mathbb{R})$ acts on the domain

$$\mathbb{H}^{\mathbf{IV}} = \{Z \in \mathbb{P}(\Lambda^{3,2} \otimes \mathbb{C}) | (Z,Z) = 0, (Z,\overline{Z}) < 0\} = \mathbb{H}^{\mathbf{IV}}_+ \cup \overline{\mathbb{H}^{\mathbf{IV}}_+}$$

where we have (in the basis of the $(f_i)_{\{1,2,3,-2,-1\}}$ given above

$$\mathbb{H}^{\mathbf{IV}}_+ = \{Z = {}^t((z_2^2 - z_1z_3), z_3, z_2, z_1, 1) \cdot z_0 \in \mathbb{H}^{\mathbf{IV}} | \mathrm{Im}(z_1) > 0\},$$

is the classical homogenous domain of type **IV**. Note that the condition $(Z,\overline{Z}) < 0$ is equivalent to $y_1y_3 - y_2^2 > 0$ where the $y_i = \mathrm{Im}(z_i)$.

Then it is easy to see that the domain $\mathbb{H}^{\mathbf{IV}}_+$ coincides with the Siegel upper half plane, $\mathbb{H}_2$ after we identify points of $\mathbb{H}^{\mathbf{IV}}_+$ with symmetric matrices $\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix}$.

The real orthogonal group $\mathbf{O}_{\mathbb{R}}(\Lambda^{3,2})$ has four connected components. We denote by $\mathbf{O}_{\mathbb{R}}(\Lambda^{3,2})_+$ the subgroup of $\mathbf{O}_{\mathbb{R}}(\Lambda^{3,2})$ of index 2 consisting of those elements which leave $\mathbb{H}^{\mathbf{IV}}_+$ invariant. The kernel of the action of $\mathbf{O}_{\mathbb{R}}(\Lambda^{3,2})_+$ on $\mathbb{H}^{\mathbf{IV}}_+$ is given by $\pm\mathbf{I}_5$. Since $\Lambda^{3,2}$ is odd-dimensional, the group $\mathbf{O}_{\mathbb{R}}(\Lambda^{3,2})_+ = \pm\mathbf{I}_5\mathbf{SO}_{\mathbb{R}}(\Lambda^{3,2})_+$ where $\mathbf{SO}_{\mathbb{R}}(\Lambda^{3,2})_+$ is the subgroup of elements with real spin-norm equal to 1. Then we denote

$$\mathbf{O}(\Lambda^{3,2})_+ = \mathbf{O}_{\mathbb{R}}(\Lambda^{3,2})_+ \cap \mathbf{O}(\Lambda^{3,2})$$

and

$$\mathbf{SO}(\Lambda^{3,2})_+ = \mathbf{SO}_{\mathbb{R}}(\Lambda^{3,2})_+ \cap \mathbf{O}(\Lambda^{3,2})$$

It is now an elementary exercise to realize concretely the images of the generators of $\mathbf{Sp}_4(\mathbb{Z})$ given for $M = \begin{pmatrix} m_1 & m_2 \\ m_2 & m_3 \end{pmatrix}$ such that $M \in M_{2\times 2}(\mathbb{Z})$,

$$\wedge^2(g_0) = \wedge^2(\begin{pmatrix} 0 & \mathbf{I}_2 \\ -\mathbf{I}_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\wedge^2(g_{M\in M_{2\times 2}(\mathbb{Z})}) = \wedge^2(\begin{pmatrix} \mathbf{I}_2 & M \\ 0 & \mathbf{I}_2 \end{pmatrix}) = \begin{pmatrix} 1 & -m_1 & 2m_2 & -m_3 & m^2-m_1m_2 \\ 0 & 1 & 0 & 0 & m_3 \\ 0 & 0 & 1 & 0 & m_2 \\ 0 & 0 & 0 & 0 & m_1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\wedge^2((g_A)_{A\in GL_2(\mathbb{Z})}) = \wedge^2(\begin{pmatrix} {}^tA^{-1} & 0 \\ 0 & A \end{pmatrix}) = \det(A)\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & a_1^2 & -2a_1a_2 & a_2^2 & 0 \\ 0 & -a_1a_3 & a_1a_4+a_2a_3 & -a_2a_4 & 0 \\ 0 & a_3^2 & -2a_3a_4 & a_4^2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Hence we have

Lemma 1. *The correspondence $\wedge^2$ defines an isomorphism*

$$\wedge^2 : \mathbf{Sp}_4(\mathbb{Z})/\{\pm\mathbf{I}_5\} \to \mathbf{SO}_+(\Lambda^{3,2}) \simeq \mathbf{O}(\Lambda^{3,2})_+/\{\pm\mathbf{I}_5\}$$

yielding a commutative square

$$\begin{array}{ccc} \mathbb{H}_2 & \longrightarrow & \mathbb{H}_2 \\ \downarrow & & \downarrow \\ \mathbb{H}_+^{\mathbf{IV}} & \xrightarrow{g\wedge g} & \mathbb{H}_+^{\mathbf{IV}} \end{array}$$

involving isomorphisms $\mathbb{H}_2 \to \mathbb{H}_+^{\mathbf{IV}}$ and arbitrary $g \in \mathbf{Sp}_4(\mathbb{Z})$.

4. Δ_5 AND THE LATTICE $\Lambda^{3,2}$

Consider some fixed primitive hyperbolic sublattice

$$\Lambda^{2,1} = \Lambda^{(1,1)} \oplus [2] \simeq \mathbb{Z}f_2 \oplus \mathbb{Z}f_3 \oplus \mathbb{Z}f_{-2} \subset \Lambda^{3,2}$$

Extending automorphisms $\phi \in \mathbf{O}(\Lambda^{2,1})$ to be the identity on $(\Lambda^{2,1})^\perp$ yields an embedding $\mathbf{O}(\Lambda^{2,1}) \to \mathbf{O}(\Lambda^{3,2})$, hence we can investigate the automorphy of Δ_5 with respect to the subgroup $\mathbf{O}(\Lambda^{2,1})$.

Recall for example from Kac [11] that to every primitive element $(\alpha \in \Lambda^{2,1})$ satisfying $(\alpha,\alpha) > 0$ and $(\alpha,\alpha)|2(\Lambda^{2,1},\alpha)$ defines a reflection

$$s_\alpha : x \mapsto 2\frac{(x,\alpha)}{(\alpha,\alpha)}\alpha$$

for all $x \in \Lambda^{2,1}$. Then $s_\alpha(\alpha) = -\alpha$ and $s_\alpha|_{\alpha^\perp}$ is the identity. Hence for all $\alpha \in \Lambda^{2,1}$ satisfying $(\alpha,\alpha) = 2$ we get a reflection

$$s_\alpha : x \mapsto (x,\alpha)\alpha.$$

Now observe

Lemma 2. *Consider α an element with square 2, then if $\alpha \in \{\delta_1 = 2f_2 - f_3, \delta_2 = 2f_{-2} - f_3, \delta_3 = f_3\}$ we have*

$$\Delta_5(s_\alpha Z) = -\Delta_5(Z)$$

while if $\alpha \in \{f_{-2} - f_2, f_2 - f_3, f_2 + f_3\}$ then

$$\Delta_5(s_\alpha Z) = \Delta_5(Z)$$

Proof. Denoting by $\overline{U} = \wedge^2(\begin{pmatrix} {}^tU^{-1} & 0 \\ 0 & U \end{pmatrix})$ with $U \in \mathbf{GL}_2(\mathbb{Z})$ then we have

$$s_{f_{-2}-f_2} = -\overline{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}, s_{f_3} = -\overline{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}, s_{f_2-f_3} = -\overline{\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}}.$$

Then the result follows from Maas' explicit formula for the multiplier of Δ_5. □

Now observe that since $\Lambda^{2,1}$ is hyperbolic of signature $(2,1)$ then $\Lambda^{2,1}$ defines a cone $\mathcal{C}(\Lambda^{2,1}) = \{x \in \Lambda^{2,1} \otimes \mathbb{R} | (x,x) < 0\}$. Then $\mathcal{C}(\Lambda^{2,1})$ is a union of two half-cones; We select one of these cones $\mathcal{C}(\Lambda^{2,1})_+$ by the constraint that the complexified cone

$$\Omega(\mathcal{C}(\Lambda^{2,1})_+) = \Lambda^{2,1} \otimes \mathbb{R} + i\mathcal{C}(\Lambda^{2,1})_+ \subset \mathbb{H}_+^{\mathbf{IV}}$$

hence for

$$z = z_3f_2 + z_2f_3 + z_1f_{-2} \in \Omega(\mathcal{C}(\Lambda^{2,1}))$$

then the corresponding point

$$Z = {}^t((z_2^2 - z_1z_3), z_3, z_2, z_1, 1) \cdot z_0 \in \mathbb{H}_+^{\mathbf{IV}}.$$

Denote by $\mathbf{O}(\Lambda^{2,1})_+$ the subgroup of $\mathbf{O}(\Lambda^{2,1})$ of index 2 fixing the half-cone $\mathcal{C}(\Lambda^{2,1})_+$. Then since $\Lambda^{2,1}$ is a hyperbolic lattice, the group $\mathbf{O}(\Lambda^{2,1})_+$ is discrete in the corresponding hyerbolic space

$$\mathcal{C}(\Lambda^{2,1})_+/\mathbb{R}_{>0}$$

where we take the qotient by the strictly positive real numbers' scaling action. Hyperbolicity also implies that the hyperbolic space $\mathcal{C}(\Lambda^{2,1})_+/\mathbb{R}_{>0}$ has a fundamental domain of finite volume. It follows that any reflection s_α with $\alpha \in \Lambda^{2,1}$ satisfying $(\alpha, \alpha) > 0$ is a reflection in the hyperplane

$$\mathbb{H}_\alpha = \{\mathbb{R}_{>0}x \in \mathcal{C}(\Lambda^{2,1})_+/\mathbb{R}_{>0} | (x, \alpha) = 0\}.$$

Hence this reflection maps the half-space

$$\mathbb{H}_{\alpha,+} = \{\mathbb{R}_{>0}x \in \mathcal{C}(\Lambda^{2,1})_+/\mathbb{R}_{>0} | (x, \alpha) \leq 0\}$$

to the opposite half-space $\mathbb{H}_{-\alpha}$ which are both bounded by the hyperplane $\mathbb{H}_\alpha$. We call α *orthogonal* to both $\mathbb{H}_\alpha$ and $\mathbb{H}_{\alpha,+}$. Taken together, all the reflections of $\Lambda^{2,1}$ generate a reflection subgroup

$$W(\Lambda^{2,1}) \subset \mathbf{O}(\Lambda^{2,1})_+.$$

The lattice $\Lambda^{2,1}$ is special, in particular because the automorphism group is known explicitly. Here we list the facts we will need from [4] The group $\mathbf{O}(\Lambda^{2,1})_+ = W^{(2)}(\Lambda^{2,1})$ where the index (2) indicates the subgroup generated by reflections in all elements of $= \Lambda^{2,1}$ with square 2. Analagously, we define $\Delta^{(k)}(\Lambda^{2,1})$ the set of all primitive elements $\delta \in \Lambda^{2,1}$ with $(\delta, \delta) = k$ which define reflections s_δ of $\Lambda^{2,1}$ and similarly $W^{(k)}$ denotes the reflection group generated by all these reflections s_δ. Thus, we can reformulate what we have seen so far in this language, by stating that $\mathbf{O}(\Lambda^{2,1})_+$ is generated by reflections in $\Delta^{(2)}(\Lambda^{2,1})$ and any element of this set δ corresponds to a reflection s_δ of one of two types:

- Type I: $(\delta, \Lambda^{2,1}) = \mathbb{Z}$
- Type II: $(\delta, \Lambda^{2,1}) = 2\mathbb{Z}$

We introduce sublattices $\Lambda_I^{2,1}$ and $\Lambda_{II}^{2,1}$ which are generated by elements δ_I or δ_{II} of type I and II respectively. Then we have

$$\Lambda_I^{2,1} = \{mf_2 + lf_3 + nf_{-2} \in \Lambda^{2,1} | m + l + n = 0 \mod 2\}$$

and

$$\Lambda_{II}^{2,1} = \{mf_2 + lf_3 + nf_{-2} \in \Lambda^{2,1} | m = n = 0 \mod 2\}$$

and an element $\delta \in \Delta^{(2)}(\Lambda^{2,1})$ has type I (respectively type II) if and only if $\delta \in \Lambda_I^{2,1}$ (respectively $\delta \in \Lambda_{II}^{2,1}$). It follows that the subgroups of $\mathbf{O}(\Lambda^{2,1})_+$ generated by all reflections of type I (respectively type II) are given by $W^{(2)}(\Lambda_I^{2,1})$ (respectively $W^{(2)}(\Lambda_{II}^{2,1})$.) All lattices $\Lambda^{2,1}, \Lambda_I^{2,1}$ and $\Lambda_{II}^{2,1}$ are $W^{(2)}(\Lambda^{2,1})$ invariant and both subgroups of reflections of type I and II are normal in $W^{(2)}(\Lambda^{2,1})$. The index of $W^{(2)}(\Lambda_I^{2,1})$ as a subgroup is 2, while the index of $W^{(2)}(\Lambda_{II}^{2,1})$ is 6. We have fundamental polyhedra $\mathcal{P}, \mathcal{P}_I$ and $\mathcal{P}_{II}$ for each respective reflection group; denoting by $\mathcal{P}' \in \{\mathcal{P}, \mathcal{P}_I, \mathcal{P}_{II}\}$ we can express these polyhedra explictly in each case by the intersection of all hyperplanes

$$\bigcap_{\delta \in \mathcal{P}'_{prim}} \mathbb{H}_{\delta,+}$$

determined by minimal sets of primitive elements of positive square (in this case, square 2) in $\Lambda^{2,1}$. These sets are given explicitly by $\mathcal{P}'_{prim}$ of (primitive) orthogonal vectors to the given polyhedra by

$$\mathcal{P}_{prim} = \{f_2 - f_3, f_{-2} - f_2, f_3\}$$
$$\mathcal{P}_{I,prim} = \{f_2 - f_3, f_{-2} - f_2, f_2 + f_3\}$$
$$\mathcal{P}_{II,prim} = \{\delta_1, \delta_2, \delta_3\}$$

It follows that the three types of reflection groups $W^{(2)}(\Lambda^{2,1}), W^{(2)}(\Lambda_I^{2,1})$ and $W^{(2)}(\Lambda_{II}^{2,1})$ are generated by reflections in faces of each of the fundamental polyhedra respectively. We denote by

$$Aut(\mathcal{P}') = \{g \in \mathbf{O}(\Lambda^{2,1})_+ | g\mathcal{P}' = \mathcal{P}'\}$$

the group of symmetries of each fundamental polyhedron $\mathcal{P}'$, then the group $Aut(\mathcal{P}_{prim})$ is trivial, while the group $Aut(\mathcal{P}_{I,prim})$ has order 2 and is generated by s_{f_3}, while the group $Aut(\mathcal{P}_{II,prim}) \simeq S_3$ and is generated by $s_{f_2 - f_3}, s_{f_{-2} - f_2}$. So we can realize

$$\mathbf{O}(\Lambda^{2,1})_+ \simeq W^{(2)}(\Lambda^{2,1}) \simeq W^{(2)}(\Lambda_I^{2,1}) \rtimes Aut(\mathcal{P}_I) \simeq W^{(2)}(\Lambda_{II}^{2,1}) \rtimes Aut(\mathcal{P}_{II})$$

To summarize, the automorphy of Δ_5 with respect to subgroups of $\mathbf{O}(\Lambda^{2,1})$ can be expressed as

Lemma 3. *Δ_5 is either invariant or anti-invariant with respect to elements of the group $\mathbf{O}(\Lambda^{2,1})_+$. By our explicit classification above, we can distinguish two cases:*

- *when $w \in W^{(2)}(\Lambda_I^{2,1})$ and $a \in Aut(\mathcal{P}_{I,prim})$ we have*
$$\Delta_5(w \cdot aZ) = det(a)\Delta_5(Z)$$
- *when $w \in W^{(2)}(\Lambda_{II}^{2,1})$ and $a \in Aut(\mathcal{P}_{II,prim})$ we have*
$$\Delta_5(w \cdot aZ) = det(w)\Delta_5(Z)$$

Next let us investigate the cone $\Delta(\Lambda_{II}^{2,1})_+ = \mathbb{R}_{\geq 0}\delta_1 + \mathbb{R}_{\geq 0}\delta_2 + \mathbb{R}_{\geq 0}\delta_3$ along with its dual cone
$$\Delta(\Lambda_{II}^{2,1})_+^* = \{x \in \Lambda^{2,1} \otimes \mathbb{R} | (x, \delta_i) \leq 0\}.$$
Since $\mathcal{P}_{II} \subset \mathcal{C}(\Lambda^{2,1})_+/\mathbb{R}_{>0}$ has finite volume in the hyperbolic space and since the cone $\overline{\mathcal{C}(\Lambda^{2,1})_+} = \overline{\mathcal{C}(\Lambda^{2,1})_+^*}$ is self-dual, the above is equivalent to the sequence of embeddings of cones
$$\Delta(\Lambda_{II}^{2,1})_+^* \subset \overline{\mathcal{C}(\Lambda^{2,1})_+} \subset \Delta(\Lambda_{II}^{2,1})_+$$
We stress this property is key to our construction, and is actually equivalent to finite volume of $\mathcal{P}_{II}$.

Another important property of the group $W^{(2)}(\mathcal{P}_{II})$ is the existence of a *Lattice Weyl vector*. This is an element $\rho \in \mathcal{P}_{II} \otimes \mathbb{Q}$ satisfying
$$(\rho, \delta_i) = -\frac{(\delta_i, \delta_i)}{2} = -1$$
for any $\delta_i \in \mathcal{P}_{II}$. But then by the Gram matrix of the δ_i
$$(\delta_i, \delta_j) = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}$$
we have
$$\rho = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_2 + \frac{1}{2}\delta_3 = f_2 - \frac{1}{2}f_3 + f_{-2}$$

Identifying $\Lambda^{2,1} \otimes \mathbb{Q} \simeq \Lambda_{II}^{2,1} \otimes \mathbb{Q}$, then it is clear that $\rho \in \Delta(\Lambda_{II}^{2,1})_+^*$ hence by the sequence of embeddings of cones above we have $\rho \in \mathcal{C}(\Lambda^{2,1})_+ = \mathcal{C}(\Lambda_{II}^{2,1})_+$.

Now we use the reflection group $W^{(2)}(\Lambda_{II}^{2,1})$ to study the fourier coefficients of Δ_5. Again fix $z = z_1 f_{-2} + z_2 f_3 + z_3 f_2 \in \Lambda^{2,1} \otimes \mathbb{R} + i\mathcal{C}(\Lambda^{2,1})_+ = \Lambda_{II}^{2,1} \otimes \mathbb{R} + i\mathcal{C}(\Lambda_{II}^{2,1})_+$ as above. Then the lattice
$$(\Lambda_{II}^{2,1})^* = \mathbb{Z}\frac{1}{2}f_2 + \mathbb{Z}\frac{1}{2}f_3 + \mathbb{Z}\frac{1}{2}f_{-2} = \frac{1}{2}\Lambda^{2,1}$$

Thus for $n, l, m \in \mathbb{Z}$ we have $n \equiv m \equiv l \equiv 1 \mod 2$, $n, m > 0$ and $4nm - l^2 > 0$, and hence we can express
$$\begin{aligned} &\frac{1}{64}f(n,l,m)\exp(\pi i(nz_1 + lz_2 + mz_3)) \\ &= \frac{1}{64}f(n,l,m)\exp(-\pi i(nf_2 - lf_3\frac{1}{2} + mf_{-2}, z)) \\ &= m(a)\exp(-\pi i(\rho + a, z)), \end{aligned}$$
where
$$a = (n-1)f_2 - (l-1)\frac{1}{2}f_3 + (m-1)f_2 \in (\Lambda^{2,1})^* = \frac{1}{2}\Lambda_{II}^{2,1}$$
and
$$m(a) = -\frac{1}{64}f(n,l,m).$$
By the properties at the end of the pre-amble on the cusp form Δ_5, we see $\rho + a \in \mathcal{C}(\Lambda^{2,1})_+$, $m(a) \in \mathbb{Z}$ and $m(0) = -1$.

Expressing the type II case in the third lemma above in terms of the lattice weyl vector ρ, the reflection group $W^{(2)}(\Lambda_{II}^{2,1})$ and the $m(a)$, we see
$$\frac{1}{64}\Delta_5(Z) = \sum_{w \in W^{(2)}(\Lambda_{II}^{2,1})} \det(w)(\sum_{\rho + a \in (\Lambda_{II}^{2,1})^* \cap \Lambda_{II}^{2,1}} m(a)\exp(-\pi i(w(\rho + a), z)))$$
since for this sum we have $\rho + a \in (\Lambda_{II}^{2,1})^* \cap \Lambda_{II}^{2,1}$ we see $(\rho + a, \delta_i) \leq 0$ for all i. If $(\rho + a, \delta_i) = 0$ then the corresponding fourier coefficient $m(a) = 0$ since Δ_5 is anti-invariant with respect to s_{δ_i}. It follows that $(\rho + a, \delta_i)$ is integral and $(\rho + a, \delta_i) < 0$ for all i. By the construction of ρ we have $(a, \delta_i) \leq 0$ and it follows then that $a \in \Delta(\Lambda_{II}^{2,1})_+^*$ but then by the sequence of cone embeddings above we have
$$a \in \mathbb{R}_{\geq 0}\mathcal{C}(\Lambda_{II}^{2,1})_+^* \cap \overline{\mathcal{C}(\Lambda^{2,1})_+} = \mathbb{R}_{\geq 0}\mathcal{P}_{II}.$$
It follows that $a \in \mathbb{R}_{>0}\mathcal{P}_{II}$ if $a \neq 0$. If $a = 0$ we have $m(a) = -1$, so by the congruences $m \equiv n \equiv l \equiv 1 \mod 2$ we have $a \in 2(\Lambda^{2,1})^* = \Lambda_{II}^{2,1}$.

Now, putting this all together, and changing the range of summation to $a \in \Lambda_{II}^{2,1} \cap \mathbb{R}_{>0}\mathcal{P}_{II}$, we find

Lemma 4.
$$\sum_{w \in W^{(2)}(\Lambda^{2,1})} det(w)(\exp(-\pi i(w(\rho), z)) - \sum_{a \in \Lambda_{II}^{2,1} \cap \mathbb{R}_{>0}\mathcal{P}_{II}} m(a) \exp(-\pi i(w(\rho + a), z)))$$

Now we investigate the behaviour of a certain triple of primitive elements at *infinity*. Their relevance will become clear shortly. Consider the primitive elements $a_0 \in \Lambda_{II}^{2,1} \cap \mathbb{R}_{>0}\mathcal{P}_{II}$ with $(a_0, a_0) = 0$ corresponding to the three vertices at infinity of the hyperbolic plane; by the third lemma above, the group $Aut(\mathcal{P}_{II})$ is transitive on these three vertices and the corresponding primitive elements are given explicitly by $\{2f_2, 2f_{-2}, 2f_2 - 2f_3 + 2f_{-2}\}$; furthermore, by the third lemma above, the group $Aut(\mathcal{P}_{II})$ preserves the fourier expansion just obtained. Thus given say $a_0 = 2f_2$ one of the three primitive elements of $\Lambda_{II}^{2,1} \cap \mathbb{R}_{>0}\mathcal{P}_{II}$ we have the identity of formal power series

$$1 + \frac{1}{64} \sum_{t \in \mathbb{N}} f(1+2t, 1, 1) q^t = \prod_{k \in \mathbb{N}} (1 - q^k)^9$$

is equivalent to an equality

$$1 - \sum_{t \in \mathbb{N}} m(ta_0) q^t = \prod_{t \in \mathbb{N}} (1 - q^t)^{\tau(ta_0) = 9}$$

where $\tau(a) = 9$ for any $a \in \Lambda_{II}^{2,1} \cap \mathbb{R}_{>0}\mathcal{P}_{II}$ with $(a, a) = 0$. Then transitivity of the group S_3 means it is true for all three primitive elements.

The just-derived expression for the fourier transform of $\frac{1}{64}\Delta_5$, as well as the generating series and product in q^t arising from multiples of the primitive elements at infinity sharply recalls to mind the Weyl-Kac denominator formula in the theory of generalized kac-moody algebras. Indeed, the fundamental polyhedron $\mathcal{P}_{II}$ along with the set of orthogonal vectors $\mathcal{P}_{II,prim} = \{\delta_1, \delta_2, \delta_3\}$ can be considered as a (real) root datum for the the Gram matrix

$$(\delta_i, \delta_j) = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}$$

5. The generalized Kac-Moody algebra $\mathfrak{g}$ and its automorphic correction $\mathfrak{g}_{\Delta_5}$

The Gram matrix of the elements $\mathcal{P}_{II,prim}$ is integral, has only 2 on the diagonal and only non-positive integers off the diagonal, and hence it is symmetric generalized Cartan matrix. The theory of Kac-Moody algebras associates to this matrix an infinite dimensional lie algebra we will denote $\mathfrak{g}$. However, the structure of the lattices $\Lambda^{3,2}, \Lambda^{2,1}$ and $\Lambda_{II}^{2,1}$ as well as the fourier coefficients of Δ_5 strongly suggests the presence of a strictly larger *corrected* lie algebra. Constructions of Borcherds along this line inform the construction of this *automorphic correction* $\mathfrak{g}_{\Delta_5}$.

Using the coefficients $m(a)$ and $\tau(a)$ of the last section, we introduce the following sets of *simple imaginary roots* for $a \in \Lambda_{II}^{2,1} \cap \mathbb{R}_{>0}\mathcal{P}_{II}$:

$$\Delta_{\bar{0}}^{im} = \{\tau(a)a | (a, a) = 0, \tau(a) > 0\}$$

where ka for $k \in \mathbb{N}$ means that we repeat a exactly k times and

$$\Delta_{\bar{1}}^{im} = \{m(a)a | (a, a) < 0, m(a) < 0\}$$

where ka for $-k \in \mathbb{N}$ means we repeat the element a exactly $-k$ times. The negative sign here corresponds to all of the $-k$ elements being *odd*, in the sense of superalgebras, hence the elements $\Delta_{\bar{0}}^{im}$ and $\Delta_{\bar{1}}^{im}$ are the *even and odd imaginary simple roots* of a new, strictly larger algebra $\mathfrak{g}_{\Delta_5}$. Then we denote the imaginary simple roots by

$$\Delta^{im} = \Delta_{\bar{0}}^{im} \cup \Delta_{\bar{1}}^{im}$$

Following from the kac-moody construction of $\mathfrak{g}$, we have

$$\Delta_{\bar{0}}^{re} = \Delta^{re} = \mathcal{P}_{II,prim} = \mathbb{Z}\delta_1 \oplus \mathbb{Z}\delta_2 \oplus \mathbb{Z}\delta_3$$

which are the *real even simple roots*. Hence $\mathfrak{g}_{\Delta_5}$ is a superalgebra *without real odd roots*. By construction, elements of Δ^{re} correspond to elements of $\Lambda_{II}^{2,1} \subset \Lambda_{II}^{2,1} \otimes \mathbb{R}$. Furthermore we observe $(\alpha, \alpha) > 0$ if $\alpha \in \Delta^{re}$ and $(\alpha, \alpha) \le 0$ if $\alpha \in \Delta^{im}$ and $(\alpha, \alpha') \le 0$ for all distinct $\alpha, \alpha' \in \Delta$ if $(\alpha, \alpha) > 0$ we also have

$$2\frac{(\alpha, \alpha')}{(\alpha, \alpha)} \in \mathbb{Z}$$

which is valid because here $(\alpha, \alpha) = 2$ and $\alpha' \in \Lambda_{II}^{2,1}$ where $\Lambda_{II}^{2,1} = \{\delta_1, \delta_2, \delta_3\} = \Delta^{re}$. Then our generalized kac-moody lie superalgebra $\mathfrak{g}_{\Delta_5}$ is generated by $h_\alpha, e_\alpha, f_\alpha$ with $\alpha \in \Delta$.

Then the map $\alpha \mapsto h_\alpha$ gives an embedding of $\Lambda_{II}^{2,1} \otimes \mathbb{R}$ into $\mathfrak{g}_{\Delta_5}$ as an even abelian subalgebra. We have the relations

- $[h_\alpha, e_{\alpha'}] = (\alpha, \alpha')e_{\alpha'}$ and $[h_\alpha, f_{\alpha'}] = -(\alpha, \alpha')f_{\alpha'}$
- $[e_\alpha, f_{\alpha'}] = h_\alpha$ if $\alpha = \alpha'$ and is 0 otherwise
- $(\mathrm{ad} e_\alpha)^{1 - 2\frac{(\alpha, \alpha')}{\alpha, \alpha)}} e_{\alpha'} = (\mathrm{ad} f_\alpha)^{1 - 2\frac{(\alpha, \alpha')}{\alpha, \alpha)}} f_{\alpha'} = 0$ if $\alpha \in \Delta^{re}$

- if $(\alpha,\alpha')=0$ then $[e_\alpha,e_{\alpha'}]=[f_\alpha,f_{\alpha'}]=0$

The superalgebra $\mathfrak{g}_{\Delta_5}$ is graded by $\Lambda_{II}^{2,1}$. Let

$$\widetilde{\mathcal{C}(\Lambda_{II}^{2,1})_+}=\sum_{\alpha\in\Delta}\mathbb{Z}_+\alpha\subset\Lambda_{II}^{2,1}$$

be the integral cone generated by all simple roots, which happens in this instance to coincide with the integral cone of all simple real roots (recall the sequence of embeddings of cones in the previous section). Now we have the triangular decomposition

$$\mathfrak{g}_{\Delta_5}=(\oplus_{\alpha\in\widetilde{\mathcal{C}(\Lambda_{II}^{2,1})_+}}\mathfrak{g}_\alpha)\oplus(\Lambda_{II}^{2,1}\otimes\mathbb{R})\oplus(\oplus_{\alpha\in-\widetilde{\mathcal{C}(\Lambda_{II}^{2,1})_+}}\mathfrak{g}_\alpha)$$

then e_α and f_α have degree α and $-\alpha$ respectively, $\mathfrak{g}_0=\Lambda_{II}^{2,1}\otimes\mathbb{R}$ and we call all elements $\alpha\in\pm\widetilde{\mathcal{C}(\Lambda_{II}^{2,1})_+}$ *roots* if $\mathfrak{g}_\alpha$ is nonzero. We define *positive* (respectively *negative*) roots by $\Delta_\pm=\Delta\cap\pm\widetilde{\mathcal{C}(\Lambda_{II}^{2,1})_+}$. For every root α we denote $\mathbf{mult}_{\bar{0}}\alpha=\dim\mathfrak{g}_{\alpha,\bar{0}}$ and $\mathbf{mult}_{\bar{1}}\alpha=-\dim\mathfrak{g}_{\alpha,\bar{1}}$ then

$$\mathbf{mult}\alpha=\mathbf{mult}_{\bar{0}}\alpha+\mathbf{mult}_{\bar{1}}\alpha=\dim\mathfrak{g}_{\alpha,\bar{0}}-\dim\mathfrak{g}_{\alpha,\bar{1}}.$$

Finally, we arrive at the denominator identity, which is the Weyl-Kac-Borcherds character formula applied to the complex one-dimensional representation of our generalized kac-moody superalgebra; it reads

$$\begin{aligned}\Phi&=\sum_{w\in W^{(2)}(\Lambda_{II}^{2,1})}\det(w)(\exp(-2\pi i(w(\rho),z))-\sum_{a\in\Lambda_{II}^{2,1}\cap\mathbb{R}_{>0}\mathcal{P}_{II}}m(a)\exp(-2\pi i(w(\rho+a),z)))\\&=\exp(-2\pi i(\rho,z))\prod_{\alpha\in\Delta_+}(1-\exp(-2\pi i(\alpha,z)))^{\mathbf{mult}\alpha}\end{aligned}$$

valid for $z\in\Omega(\Lambda_{II}^{2,1})=\Lambda_{II}^{2,1}\otimes\mathbb{R}+i\mathcal{C}(\Lambda_{II}^{2,1})_+$ and the function Φ is called the denominator function. Thus applying the results of the last section we arrive at the infinite product expansion for the form Δ_5:

Theorem 3.

$$\frac{1}{64}\Delta_5(2Z)=\Phi(z)$$

and hence the denominator of the corrected generalized kac-moody lie superalgebra $\mathfrak{g}_{\Delta_5}$ is the siegel modular form of genus 2 and weight 5.

The denominator function Φ is well-defined on the complexified cone $\Omega(\mathcal{C}(\Lambda_{II}^{2,1})_+)$ which admits embeddings as a cusp into the type **IV** domain. The embedding is not canonical, as it is defined up to changing $z\mapsto tz$ for $t\in\mathbb{N}$.

6. Super dimensions of root spaces and the weight 0 index 1 weak Jacobi form $\phi_{0,1}$

The last section demonstrates that there is a product formula for Δ_5:

$$\begin{aligned}\Phi&=\sum_{w\in W^{(2)}(\Lambda_{II}^{2,1})}\det(w)(\exp(-2\pi i(w(\rho),z))-\sum_{a\in\Lambda_{II}^{2,1}\cap\mathbb{R}_{>0}\mathcal{P}_{II}}m(a)\exp(-2\pi i(w(\rho+a),z)))\\&=\exp(-2\pi i(\rho,z))\prod_{\alpha\in\Delta_+}(1-\exp(-2\pi i(\alpha,z)))^{\mathbf{mult}\alpha}\end{aligned}$$

we will now combine automorphy of Δ_5 with knowledge of certain related Jacobi forms to determine the integers $\mathbf{mult}\alpha$ for roots α. Throughout we freely reference Eichler-Zagier[12]. Consider the Fourier-Jacobi expansion of Δ_{12} and denote by $\phi_{12,1}$ the Jacobi cusp form of weight 12 appearing as its first fourier-Jacobi coefficient. Its fourier coefficients may be calculated in terms of explicitly given Eisenstein series for the groups $\mathbf{SL}_2$ and the Jacobi group. The form has integral and coprime coefficients

$$\phi_{12,1}(z_1,z_2)=(r^{-1}+10+r)q+(10r^{-2}-88r^{-1}-132-88r+10r^2)q^2+\ldots$$

we introduce another function with integral coefficients

$$\phi_{0,1}(z_1,z_2)=\frac{\phi_{12,1}}{\delta_{12}}=\sum_{\substack{n\geq0\\l\in\mathbb{Z}}}f(n,l)\exp(2\pi i(nz_1+lz_2)$$

where

$$\delta_{12}(z_1)=q\prod_{n\geq1}(1-q^n)^{24}$$

is the $\mathbf{SL}_2(\mathbb{Z})$ cusp form of weight 12. Then $\phi_{0,1}$ is a weak Jacobi form of weight 0 and index 1. It satisfies the same functional equations as holomorphic Jacobi forms and has nonzero coefficients only with indices $(n,l)\in\mathbb{Z}^2$ such that $n\geq0$ (as $\phi_{12,1}$ is a cusp form) and $4n-l^2\geq-1$. The weight is even, hence $f(n,l)=f(n,-l)$ and $f(n,l)$ only depends on $4n-l^2$. Explicitly

$$\phi_{0,1}(z_1,z_2)=(r^{-1}+10+r)+q(10r^{-2}-64r^{-1}+108-64r+10r^2)+\ldots$$

then we have the product expansion

Theorem 4.
$$\frac{1}{64}\Delta_5 = \exp(\pi i(z_1+z_2+z_3)) \prod_{n,l,m\in\mathbb{Z},(n,l,m)>0} (1-\exp(2\pi i(nz_1+lz_2+mz_3)))^{f(nm,l)}$$
where the condition $(n,l,m)>0$ means the product is taken over the set of positive roots Δ_+

Remark 1. *Explicitly, the condition $(n,l,m)>0$ means that $n\geq 0, m\geq 0$ and l is aritrary integral if $n>0$ or $m>0$ and $l<0$ if $n=m=0$.*

Proof. An analysis of the fourier coefficients of the form $\phi_{12,1}$ as well as its expression as a linear combination of standard Jacobi theta functions allows one to prove that the fourier coefficients of $\phi_{0,1}$ have the asymptotic behavior of
$$f(n,l) = O(\exp(\sqrt{4n-l^2}).$$
This estimate together with the methodology of Kac[11] allows us to prove that the product in the theorem converges on any neighborhood of the zero dimensional cusp of $\mathbf{Sp}_4(\mathbb{Z})$.

We express the product in the theorem as
$$\exp(\pi i(z_1+z_2+z_3)) \prod_{n>0,l\in\mathbb{Z}\text{or}n=0,l<0} (1-\exp(2\pi i(nz_1+lz_2))^{f(0,l)}$$
$$\times \prod_{n\geq 0,m>0,l\in\mathbb{Z}} (1-\exp(2\pi i(nz_1+lz_2+mz_3)))^{f(nm,l)}$$
where we have $f(0,0)=10, f(0,-1)=1, f(0,l)=0$ if $l<-1$.

Consider next the minus embedding of the usual hecke operators $T_-(m)$ for $\mathbf{GL}_2(\mathbb{Z})$ then for each Jacobi form ϕ of weight k and index $t\in\mathbb{Q}$ we have a function $(\phi|_k T_-(m))$ which is a Jacobi form of index mt. One can proceed to show the factors of the above product admits an expression of its logarithm as
$$\log(\prod_{n\geq 0,m>0,l\in\mathbb{Z}} (\cdots) = -\sum_{m\geq 1} m^2(\phi_{0,1}\exp^{2\pi i t z_3}|_0 T_-(m)))$$
The expansion shows that this particular factor is invariant with respect to the maximal parabolic subgroup Γ_∞ of $\mathbf{Sp}_4(\mathbb{Z})$.

Now observe that the other factor in the product is equal to $\psi_{5,\frac{1}{2}}\exp(2\pi i t z_3)$. Together these facts imply the product transforms like a modular form for the Jacobi group $\Gamma_\infty/\pm\mathbf{I}_4$ with a certain character $\nu_\infty:\Gamma_\infty/\pm\mathbf{I}_4\to\pm 1$. Testing against the matrix $I=\begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & 0 \\ & & 0 & 1 \\ & 0 & 1 & 0\end{pmatrix}$

yields antiinvariance of the product, while the multiplier system returns 1. It follows that since the Jacobi group and I together generate $\mathbf{PSp}_4(\mathbb{Z})$ we get that the product is a Siegel modular form of weight 5 with the same multiplier system as Δ_5. Comparing the first fourier coefficient finishes the proof.

□

References

[1] Gritsenko-Clery, *The Siegel modular forms of genus 2 with the simplest divisor*. arXiv, 2008.
[2] Thomas, *A holomorphic Casson invariant for Calabi-Yau 3-folds, and bundles on K3 fibrations*. arXiv, 1998.
[3] G.Oberdieck-A.Pixton, *Holomorphic anomaly equations and the Igusa cusp form conjecture*. arXiv, 2018.
[4] V. Nikulin, *Surfaces of type K3 with finite automorphism group and Picard group of rank three*. Proc. Steklov. Math. Inst. 165, 1984.
[5] Gritsenko-Nikulin, *Siegel automorphic form corrections of some Lorentzian Kac-Moody Lie algebras*. arXiv, 1995.
[6] R. Borcherds, *Automorphic forms with singularities on Grassmannians*. arXiv, 1996.
[7] E. Freitag, *Siegelsche Modulfunktionen*. Springer, 1983.
[8] V. der Geer, *Siegel Modular Forms*. arXiv, 2007.
[9] H. Maass, *Die Multiplikatorsysteme zur Siegelschen Modulgruppe*. Nachrichten der Akad. der Wissen. Gottingen (II. Math. Phys.Klasse) Nr 11, 1964.
[10] A. Knapp, *Lie Groups Beyond an Introduction*. Progress in Math. 140, Birkhauser, 2002.
[11] V. Kac, *Infinite Dimensional Lie Algebras*. Cambridge University Press, 1990.
[12] M. Eichler and D. Zagier, *The theory of Jacobi forms*. Progress in Math. 55, Birkhauser, 1985.