NOTES ON THE TOPOLOGICAL SUPER-STRING

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ABSTRACT. Some computations and remarks on topological string theory.

CONTENTS

1. INTRODUCTION

1.1. **Dreams.** One of the prevalent themes at play in both contemporary theoretical physics and mathematics is that of *duality*. In particular, we will adopt a form of *Koszul Duality* as a guiding light in our quest to understand some of the salient characteristics of **M-Theory**.

Remark 1.1. This introduction serves a motivational purpose, and subsequently lacks in rigour; Mathematicians may prefer to skip ahead to later sections.

Let X be a manifold of dimension $\mathbf{dim}(X) = n$. We will consider M-theory on $\mathbb{R} \times X$ in the presence of a stack of N 1-dimensional brane supported on $\mathbb{R} \times p$, for some point $p \in X$.

In this scenario one expects to be able to construct two algebras

- The algebra $\mathcal{A}_N$ of operators on the stack of N branes supported on $\mathbb{R} \times p$
- the algebra $\mathcal{B}$ of local operators of the gravitational theory on $\mathbb{R} \times X$. Operator product expansions in the direction of $\mathbb{R}$ equips $\mathcal{B}$ with the structure of associative algebra.

Holography then leads us to expect

Conjecture 1.2. In the limit as $\lim_{N\to\infty}$, $\mathcal{A}_N$ is the Koszul dual of $\mathcal{B}$.

Remark 1.3. One should be able to exploit Witten's proposal for holographic calculations of the OPE.

These notes will amount to a rigorous check of this form of holography in a few (relatively) simple examples.

1.2. **String Field Theory and Koszul Duality.** We expect Koszul duality for string field theories to be captured at the level of Calabi-Yau categories and their respective cyclic homologies, via the theorem of Loday-Quillen-Tsygan, which says

Theorem 1.4. ***Loday-Quillen-Tsygan*** *the Chevalley-Eilenberg-Lie homology of the lie algebra of infinite matrices over a unital associative algebra A is generated by the cyclic homology of A as an exterior algebra.*

The associative algebra A will coincide with the algebra of cochains arising in the large N limit, dual to the cyclic homology produced on the gravitational side.

Thus, for example, we will obtain an explicit matching between single string states and generators given by single trace operators[1] at the large N limit, while OPE's will coincide with string scattering amplitudes.

1.3. **Flavours of Topological Strings.** Topological strings manifest in various flavours. Common examples include

- The **A-model** where X is symplectic and branes are lagrangians $L \rightarrowtail X$
- The **B-model** where X is calabi-yau and branes are coherent sheaves.
- **Mixed A-B models** e.g. on $X \times X'$ with branes given by $L \times Z$ for $L \rightarrowtail X$ lagrangian and $Z \rightarrowtail Y$ holomorphic.

Remark 1.5. The above examples do not exhaust the full landscape of topological strings.

[1] At the classical level

Remark 1.6. The A-model is only interesting with world-sheet instantons taken into account.

Regardless of the context we will work in, we'll need to understand

- The open string theory, which is the theory on a Brane, and
- The closed string field theory, of a gravitational nature

1.4. **Open Strings.** If $\mathcal{C}$ is a Calabi-Yau category, with $\mathcal{F} \in \mathbf{Ob}\mathcal{C}$, one can construct a field theory with a space of fields[2]given by

$$\mathrm{RHom}(\mathcal{F}, \mathcal{F})[1],$$

also known as the open string states. The action for this field theory is built in terms of the A_∞ structure[3] on this derived Hom space.

2. First Example

2.1. **Our setup.** Let X be $\mathbb{R}^2_A \times \mathbb{C}^2_B$. Branes will be of the form $L \times Z$ where L is a (lagrangian) line in $\mathbb{R}^2_A$ and Z is the support of a coherent sheaf on $\mathbb{C}^2_B$. Consider, in particular, the Brane

$$\mathbb{R} \times p \rightarrowtail \mathbb{R}^2_A \times \mathbb{C}^2_B$$

for some point $p \in \mathbb{C}^2_B$.

2.2. **Open String Field States.** The open string states will be a tensor product of A-model and B-model states. If one consults the mathematical literature one would find the following prescription:

- $\mathbb{R} \rightarrowtail \mathbb{R}^2_A$ has open string states given by the Floer cohomology, which in this instance is uninteresting and reduces to the regular cohomology of $\mathbb{R}$. Since we need this in its sheaf incarnation, our open sring states will be the de Rahm complex $\Omega^*(\mathbb{R})$
- $\{p\} \rightarrowtail \mathbb{C}^2_B$ has open string states is given by $\mathbf{Ext}^*_{\mathcal{O}(\mathbb{C})}(\mathbb{C}, \mathbb{C})$ which has the form of an exterior algebra on 2 generators $= \mathbb{C}[\varepsilon_1, \varepsilon_2]$, each generator representing a normal direction.

[2]If we just produce a category, then the resulting field theory would be defined on a point. If instead we produce a sheaf of categories, we get a field theory on the support of this sheaf of categories i.e. a field theory on the given brane

[3]Via a standard procedure going back to Witten

2.3. **Open String Field Theory.** We hence find that the fields[4] are given by

$$\alpha \in \Omega^*(\mathbb{R})[\varepsilon_1, \varepsilon_2] \otimes \mathfrak{gl}_N[1]$$

with action

$$S(\alpha) = \int_{\mathbb{R}\times\mathbb{C}^{0|2}} \frac{1}{2}Tr(\alpha d\alpha) + \frac{1}{3}Tr\alpha^3$$

Remark 2.1. Note that in the action we perform a Berezin integral over $\mathbb{C}^{0|2}$

Remark 2.2. The shift of 1 in the space of fields above means that the 1-forms are now in ghost number 0, i.e. is a bosonic gauge field. Similarly, the scalars ε_i (representing the motions of brane) are also ghost number 0.

Remark 2.3. In the Non-BV setup, i.e. for a starting point from which we can follow the procedure outlined as in ?? to derive the above, we write down ghosts, the anti-fields, the anti-fields to the ghosts etc. to find a Quantum-Mechanical system with fields

$$A \in \Omega^1(\mathbb{R}) \otimes \mathfrak{gl}_N$$

a connection form and two scalars

$$\phi_1, \phi_2 \in \Omega^0(\mathbb{R}) \otimes \mathfrak{gl}_N$$

and action

$$S = \int Tr(\phi_1, d_A\phi_2)$$

with the usual gauge symmetry.

Remark 2.4. We can also proceed directly within the CY category. There we take a given object, tensor it with $\mathbb{C}^N$; passage to endomorphisms yields endomorphisms tensored with that of $\mathbb{C}^N$. More generally, one should take proceed via the cyclic cohomology of the category. What we really need is to be able to consider is an integral over world sheets decorated with cyclic cohomology classes. Then the 3-point function should be computable via the cup-product in cyclic cohomology together with the trace. However, the higher n-point functions are trickier.

Remark 2.5. This is literally 1-dimensional chern simons theory (via AKSZ formalism).

[4] in the Batalin-Vilkovisky formalism

Remark 2.6. we can also realize this via maps to $\mathbb{B}(G)$ from $\mathbb{R} \times \mathbb{C}^{0|2}$; this point of view makes manifest the origin of this theory as a reduction of chern simons on a 2-torus. (The two odd directions generate the cohomology of the 2-torus.) Elaborating on this point of view with an analysis of T-duality recovers Witten's prescriptions for Chern Simons theory via topological string theory.

2.4. **Closed String States.** Similarly, the closed string state space will be of the form

$$\mathrm{A-modelclosedstates} \otimes \mathrm{B-modelclosedstates}$$

For the A-model on $\mathbb{R}^2$ we have the de Rahm cohomology. For the B-model, following BCOV we consider

$$\mathbf{Ker}\partial \subset \mathbb{PV}^{*,*}(\mathbb{C}^2) \simeq \Omega^{0,*}(\mathbb{C}^2)[\partial_{z_1}, \partial_{z_2}]$$

Remark 2.7. The ∂_{z_i} are odd.

Remark 2.8. The constructions here are much more homologically involved. Updates coming soon.

Observe that the action of ∂ on PolyVector fields

$$\partial : PV^{i,*} \to PV^{i-1,*}$$

and that

$$PV^{0,*} \subset \mathbf{Ker}\partial$$

. Now since $\mathbb{C}^2$ is symplectic, we find

$$PV^{1,*}(\mathbb{C}^2) \simeq \Omega^{1,*}(\mathbb{C}^2)$$

where the operator ∂ becomes the holomorphic de Rahm operator. Hence we are considering the complex of closed[5] holomorphic 1-forms.

Remark 2.9. Following BCOV, we have stipulated that the fields must satisfy a differential equation, which inevitably produces a technical nightmare. Further, what is worse is that the action functional (coinciding with the inverse of ∂) is non-local.

Thus we are led to divergence free vector fields; equivalently symplectic vector fields in this case. Now we opt to replace **Ker**∂ by the image of ∂ i.e. by

$$\Omega^{0,*} - \mathrm{HamiltonianVectorFields}$$

[5]closed in the holomorphic sense

Remark 2.10. We have replaced one complex by another that is almost—but not quite—quasi-isomorphic: they differ by the cohomology of $\mathbb{C}$, which is (almost) trivial.

2.5. **Analysis of $N \to \infty$.** *E-mail address*: `root@raeez.com`
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