SOME CALCULATIONS CONCERNING THE TOPOLOGICAL STRING

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ABSTRACT. Some computations and remarks on Kodaira-Spencer Gravity and quantum string amplitudes in topological string theory. Much of the analysis has its origin in the work of Bershadsky-Cecotti-Ooguri-Vafa.

CONTENTS

1. INTRODUCTION

1.1. **Dreams.** One of the prevalent themes at play in both contemporary theoretical physics and mathematics is that of *duality*. In particular, we will adopt a form of *Koszul Duality* as a guiding light in our quest to understand some of the salient characteristics of **M-Theory**.

Remark 1.1. This introduction serves a motivational purpose, and subsequently lacks in rigour; Mathematicians may prefer to skip ahead to later sections.

Let X be a manifold of dimension $\mathbf{dim}(X) = n$. We will consider M-theory on $\mathbb{R} \times X$ in the presence of a stack of N 1-dimensional branes supported on $\mathbb{R} \times p$, for some point $p \in X$.

In this scenario one expects to be able to construct two algebras

- The algebra $\mathcal{A}_N$ of operators on the stack of N branes supported on $\mathbb{R} \times p$
- the algebra $\mathcal{B}$ of local operators of the gravitational theory on $\mathbb{R} \times X$. Operator product expansions in the direction of $\mathbb{R}$ equips $\mathcal{B}$ with the structure of associative algebra.

Holography then leads us to expect

Conjecture 1.2. In the limit as $\lim_{N \to \infty}$, $\mathcal{A}_N$ is the Koszul dual of $\mathcal{B}$.

Remark 1.3. One should be able to exploit Witten's proposal for holographic calculations of the OPE.

These notes will amount to a rigorous check of this form of holography in a few (relatively) simple examples.

1.2. **String Field Theory and Koszul Duality.** We expect Koszul duality for string field theories to be captured at the level of Calabi-Yau categories and their respective cyclic homologies, via analogues of the theorem of Loday-Quillen-Tsygan, which says

Theorem 1.4. ***Loday-Quillen-Tsygan*** *the Chevalley-Eilenberg-Lie homology of the lie algebra of infinite matrices over a unital associative algebra A is generated by the cyclic homology of A as an exterior algebra.*

The associative algebra A will coincide with the algebra of cochains arising in the large N limit, dual to the cyclic homology produced on the gravitational side.

Thus, for example, we will obtain an explicit matching between single string states and generators given by single trace operators[1] at the large N limit, while OPE's will coincide with string scattering amplitudes.

1.3. **Flavours of Topological Strings.** Topological strings manifest in various flavours. Common examples include

- The **A-model** where X is symplectic and branes are lagrangians $L \rightarrowtail X$

[1]At the classical level

- The **B-model** where X is calabi-yau and branes are coherent sheaves.
- **Mixed A-B models** e.g. on $X \times X'$ with branes given by $L \times Z$ for $L \rightarrowtail X$ lagrangian and $Z \rightarrowtail Y$ holomorphic.

Remark 1.5. The above examples do not exhaust the full landscape of topological strings.

Remark 1.6. The A-model is only interesting with world-sheet instantons taken into account.

Regardless of the context we will work in, we'll need to understand

- The open string theory, which is the theory on a Brane, and
- The closed string field theory, of a gravitational nature

1.4. **Open Strings.** If $\mathcal{C}$ is a Calabi-Yau category, with $\mathcal{F} \in \mathbf{Ob}\mathcal{C}$, one can construct a field theory with a space of fields[2]given by

$$\mathrm{RHom}(\mathcal{F}, \mathcal{F})[1],$$

also known as the open string states. The action for this field theory is built in terms of the A_∞ structure[3] on this derived Hom space.

2. First Example

2.1. **Our setup.** Let X be $\mathbb{R}^2_A \times \mathbb{C}^2_B$. Branes will be of the form $L \times Z$ where L is a (lagrangian) line in $\mathbb{R}^2_A$ and Z is the support of a coherent sheaf on $\mathbb{C}^2_B$. Consider, in particular, the Brane

$$\mathbb{R} \times p \rightarrowtail \mathbb{R}^2_A \times \mathbb{C}^2_B$$

for some point $p \in \mathbb{C}^2_B$.

2.2. **Open String Field States.** The open string states will be a tensor product of A-model and B-model states. If one consults the mathematical literature one would find the following prescription:

- $\mathbb{R} \rightarrowtail \mathbb{R}^2_A$ has open string states given by the Floer cohomology, which in this instance is uninteresting and reduces to the regular cohomology of $\mathbb{R}$. Since we need this in its sheaf incarnation, our open sring states will be the de Rahm complex $\Omega^*(\mathbb{R})$

[2]If we just produce a category, then the resulting field theory would be defined on a point. If instead we produce a sheaf of categories, we get a field theory on the support of this sheaf of categories i.e. a field theory on the given brane

[3]Via a standard procedure going back to Witten

- $\{p\} \rightarrowtail \mathbb{C}^2_B$ has open string states is given by $\mathbf{Ext}^*_{\mathcal{O}(\mathbb{C})}(\mathbb{C},\mathbb{C})$ which has the form of an exterior algebra on 2 generators $= \mathbb{C}[\varepsilon_1, \varepsilon_2]$, each generator representing a normal direction.

2.3. **Open String Field Theory.** We hence find that the fields[4] are given by

$$\alpha \in \Omega^*(\mathbb{R})[\varepsilon_1, \varepsilon_2] \otimes \mathfrak{gl}_N[1]$$

with action

$$S(\alpha) = \int_{\mathbb{R}\times\mathbb{C}^{0|2}} \frac{1}{2} Tr(\alpha d\alpha) + \frac{1}{3} Tr\alpha^3$$

Remark 2.1. Note that in the action we perform a Berezin integral over $\mathbb{C}^{0|2}$

Remark 2.2. The shift of 1 in the space of fields above means that the 1-forms are now in ghost number 0, i.e. is a bosonic gauge field. Similarly, the scalars ε_i (representing the motions of brane) are also ghost number 0.

2.3.1. *De-BV-fying the open string field theory.* In the Non-BV setup, i.e. for a starting point from which we can follow the procedure outlined as in ?? to derive the above, we write down ghosts, the anti-fields, the anti-fields to the ghosts etc. to find a Quantum-Mechanical system with fields

$$A \in \Omega^1(\mathbb{R}) \otimes \mathfrak{gl}_N$$

a connection form and two scalars

$$\phi_1, \phi_2 \in \Omega^0(\mathbb{R}) \otimes \mathfrak{gl}_N$$

and action

$$S = \int Tr(\phi_1, d_A\phi_2)$$

with the usual gauge symmetry.

Remark 2.3. We can also proceed directly within the CY category. There we take a given object, tensor it with $\mathbb{C}^N$; passage to endomorphisms yields endomorphisms tensored with that of $\mathbb{C}^N$. More generally, one should take proceed via the cyclic cohomology of the category. What we really need is to be able to consider is an integral over world sheets decorated with cyclic cohomology classes. Then the 3-point function should be computable via the cup-product in cyclic cohomology together with the trace. However, the higher n-point functions are trickier.

[4]in the Batalin-Vilkovisky formalism

Remark 2.4. This is literally 1-dimensional chern simons theory (via AKSZ formalism).

Remark 2.5. we can also realize this via maps to $\mathbb{B}(G)$ from $\mathbb{R} \times \mathbb{C}^{0|2}$; this point of view makes manifest the origin of this theory as a reduction of chern simons on a 2-torus. (The two odd directions generate the cohomology of the 2-torus.) Elaborating on this point of view with an analysis of T-duality recovers Witten's prescriptions for Chern Simons theory via topological string theory.

2.4. **Closed String States.** Similarly, the closed string state space will be of the form

$$\mathrm{A-modelclosedstates} \otimes \mathrm{B-modelclosedstates}$$

For the A-model on a symplectic manifold, as usual, we get the cohomology of that manifold; but again as we are constructing a field tehory on $\mathbb{R}^2$ we take the de Rahm cohomology complex. For the B-model, following BCOV's analysis on a CY 3-fold, we consider

$$\mathbf{Ker}\partial \subset \mathbb{PV}^{*,*}(\mathbb{C}^2) \simeq \Omega^{0,*}(\mathbb{C}^2)[\partial_{z_1}, \partial_{z_2}]$$

i.e. the kernel of a certain operator on the space of poly-vector fields.

Remark 2.6. The ∂_{z_i} are odd.

Remark 2.7. The constructions here are much more homologically involved. Updates coming soon.

Observe that the action of ∂ on PolyVector fields

$$\partial : \mathbb{PV}^{i,*} \to \mathbb{PV}^{i-1,*}$$

and that

$$\mathbb{PV}^{0,*} \subset \mathbf{Ker}\partial.$$

Now since $\mathbb{C}^2$ is symplectic, we find

$$\mathbb{PV}^{1,*}(\mathbb{C}^2) \simeq \Omega^{1,*}(\mathbb{C}^2)$$

with the restriction of the operator ∂ acting as the holomorphic de Rahm operator. Hence we are considering the complex of closed[5] holomorphic 1-forms.

Remark 2.8. Following BCOV, we have stipulated that the fields must satisfy a differential equation, which inevitably produces a technical nightmare. Further, what is worse is that the action functional (coinciding with the inverse of ∂) is non-local.

[5]closed in the holomorphic sense

Also, note that when restricting to poly-2-vector-fields

$$\mathbb{PV}^{2,*}(\mathbb{C}^2) \supset \mathbf{Ker}\partial$$

$\mathbf{Ker}\partial$ picks out poly-2-vector-fields that are independent of z, hence

$$\mathbb{PV}^{2,*}(\mathbb{C}^2) \supset \mathbf{Ker}\partial \simeq \mathbb{C}$$

(the cohomology is $\mathbb{C}$), which is again (almost) trivial. What is more, by the definition of a propagator in BCOV theory, this does not propagate—hence form background fields which we'll ignore.

Remark 2.9. this statement generalizes the sheafy setting.

2.5. **Closed String Field Theory.** Combining this analysis, for our candidate for the closed string field theory we are led to divergence free vector fields; equivalently, in this case, symplectic vector fields. Now we opt to replace $\mathbf{Ker}\partial$ by the image of ∂ i.e. by

$$\Omega^{0,*} - \text{HamiltonianVectorFields}$$

Remark 2.10. We have replaced one complex by another that is almost—but not quite—quasi-isomorphic: they differ by the cohomology of $\mathbb{C}$, which is (almost) trivial.

2.5.1. *The Action Functional.* We can now write down the action, for

$$\alpha \in \Omega^*(\mathbb{R}^2)\hat{\otimes}\Omega^{0,*}(\mathbb{C}^2)[1]$$

$$\beta \in \Omega^*(\mathbb{R}^2)\hat{\otimes}\Omega^{0,*}(\mathbb{C}^2)[2]$$

the action is an integral over real 6-dimensional space, cubic in the fields

$$\int_{\mathbb{R}^2\times\mathbb{C}^2} \beta\bar{\partial}\alpha dz_1 dz_2 + \int_{\mathbb{R}^2\times\mathbb{C}^2} \beta\partial\alpha\partial\alpha$$

Remark 2.11. Observe that the second term of the action can be expressed as

$$\int_{\mathbb{R}^2\times\mathbb{C}^2} \beta\partial\alpha\partial\alpha = \int_{\mathbb{R}^2\times\mathbb{C}^2} \beta\{\alpha,\alpha\} dz_1 dz_2$$

where the expression $\partial\alpha\partial\alpha$ can be expressed as a poisson bracket, hence we see the lie bracket on hamiltonian vector fields entering the action functional.

Remark 2.12. Before, the quantum mechanical system lived on a line (extent of the brane), while this gravitational theory occupies all of space-time.

2.5.2. *De-BV-fying the closed string field theory.*

Remark 2.13. This theory is closely related to a 4-dimensional BF theory. TODO more

Consider space-time

$$\mathbb{R}^2_{x_i} \times \mathbb{C}^2_{z_i}$$

where the subscripts indicate coordinates. Paying attention to ghost number zero components

$$\beta = \beta_{x_1 x_2} dx_1 dx_2 + \beta_{x_i \bar{z}_j} dx_i d\bar{z}_j + \beta_{\bar{z}_1 \bar{z}_2 d\bar{z}_1 d\bar{z}_2}$$

$$\alpha = \alpha_{x_i} dx_i + \alpha_{\bar{z}_j} d\bar{z}_j$$

and considering the action as given above, we see that β has 1-form gauge symmetry, while α has 0-form gauge symmetry.

Remark 2.14. As in the previously alluded to 4-dimensional BF theory, β—having 1-form gauge symmetry—also has secondary gauge symmetry, which is why we find a large complex. TODO more.

Analysing α, as a Hamiltonian vector field on $\mathbb{C}^2$, we see the term $\alpha_{\bar{z}_j} d\bar{z}_j$ is the Beltrami differential, where we deform $\mathbb{C}^2$ as a symplectic surface. The term $\alpha_{x_i} dx_i$ is a connection on $\mathbb{R}^2$ valued in Hamiltonian vector fields; the terms together imply that we have a flat bundle of $\mathbb{C}^2$'s over $\mathbb{R}^2$. In fact, the equations of motion determine an integrable deformation of complex structure. Analysing β in the action functional, it only appears linearly. This implies they're just lagrangian multiplier fields in enforcing the equations resulting in the integrable deformation of complex structure + flatness of the bundle of $\mathbb{C}^2$'s over $\mathbb{R}^2$.

Remark 2.15. This final statement is analogous to BF theory where B is a lagrangian multiplier and F is zero, hence why we see in the next section that the phase space is a cotangent bundle.

2.5.3. *Phase Space.* If we put this on

$$\mathbb{R} \times S^1 \times X$$

where X is homolomorphic symplectic, then the phase space is

$$T^*(\mathrm{Moduliofholomorphic} - \mathrm{symplecticsurfacesfiberedover} S^1 + \mathrm{flatconnection})$$

Remark 2.16. If we work in perturbation theory near X we will find the above moduli space near this configuration.

2.6. **Closed String Fields.** Given a closed string field, we should obtain a deformation of the gauge theory on the brane. For example, if the brane is on the line x_1 and the closed string field is

$$dx_1 z_1^k z_2^l$$

we ought to obtain a deformation of our quantum mechanical system on the line x_1. We can identify this deformation by deforming the action to

$$\int Tr\phi_1 d_A \phi_2 + \int dx_1 Tr\phi_1^k \phi_2^l$$

by adding on a Hamiltonian. Many matrix models of hamiltonians can be engineered by these backgrounds.

Remark 2.17. Geometrically, this just amounts to saying that the $\mathbb{C}^2$ has a connection, allowing parallel transport.

2.7. **Computation of Operators in the gauge theory, and the large N limit.** TODO more In any theory where we find the de Rahm complex as a space of fields, all the local operators cannot have any derivatives (in X); anything involving a derivative in X can be killed by the BRST operator.

It follows that we will only care about values of fields at a point, but also of anti-fields etc.

2.7.1. *Ghost* 0 *operators.* These will be generated by elements of the form

$$Tr(p(\phi_1, \phi_2))$$

where P is a non-commutative polynomial[6]. As $N \lim \infty$, there are no relations, these are freely generated.

2.7.2. *Ghost* -1 *operators.* Ghost number -1 operators will be generated by elements of the form

$$Tr(\psi(p(\phi_1, \phi_2))$$

where

$$\psi = A^*$$

is the anti-field to the field given by a connection A. TODO write up rest of fields

[6]Here we are appealing to invariant theory, where the algebra of invariant matrices is generated by elements of the above form.

2.7.3. *Algebra of local operators on the brane.* By cyclic symmetry of the trace

$$\int Tr(\phi_1[A, \phi_2]) = \int Tr(A[\phi_1, \phi_2])$$

Since the BRST operator acts on the anti-field of the connection, and it appears at ϕ_1, ϕ_2, it turns it into a commutator, hence

$$Q(Tr(\psi p(\phi_1, \phi_2)) = Tr([\phi_1, \phi_2] p(\phi_1, \phi_2))$$

Corollary 2.18. *In Q-cohomology, expressions involving words in ϕ_1, ϕ_2 can be commuted past one-another. We have a basis of ghost* #0 *operators given by*

$$Tr(\phi_1^k \phi_2^l)$$

and ghost $\#-1$ *operators are given by*[7]

$$Tr(\mathrm{Sym}\{\psi \phi_1^k \phi_2^l)$$

Similarly, the classical local operators are

$$\mathbf{S}^*(\mathbb{C}[z_1, z_2] \oplus \mathbb{C}[z_1, z_2][1]$$

Remark 2.19. The ghost #0 operators in the corollary above correspond on the string-gravitational side to polynomial observables

$$z_1^k z_2^l$$

2.8. **Computation of operators in the gravitational theory.** TODO write up full argument. Similarly, we can build local operators through a limited series of operations: we can evaluate θ and α at a point, we can differentiate etc. By a standard cohomological argument, only the zero-form parts of α and β will yield physical operators in the Q-cohomology, while only the z_1 and z_2 derivatives can contribute, as the other derivatives can be cancelled by the image of Q.

The ring of local operators is generated by the operators of ghost #1:

$$\alpha \mapsto \partial_{z_1}^k \partial_{z_2}^l \alpha$$

and operators of ghost #2:

$$\beta \mapsto \partial_{z_1}^k \partial_{z_2}^l \beta$$

These operators have the opposite transformation properties as those from the gauge theory.

[7] $\mathrm{Sym}\{\cdots\}$ denotes the symmetrized expression: i.e. the sum of permutations of factors of the product of the expression

We find similarly that the ring of classical local operators is given by

$$C^*(\mathbb{C}[z_1, z_2] \ltimes \mathbb{C}[z_1, z_2][1])$$

Chevalley cochains of a semi-direct product of (shifted) polynomial rings. The first $\mathbb{C}[z_1, z_2]$ in the semi-direct product, via the equations of motion, corresponds to the α field and is a lie algebra with poisson bracket, while the second $\mathbb{C}[z_1, z_2][1]$ as the adjoint module acted upon by the first, corresponds to the β field.

Remark 2.20. Recall again that think of α's as hamiltonian vector fields, while the β's are functions admitting an action of hamiltonian vector fields, when we can consider $\alpha \in \mathbb{C}[z_1, z_2]$ and $\beta \in \mathbb{C}[z_1, z_2][1]$. The action is then given by

$$[\alpha, \beta] = \{\alpha, \beta\}$$

where the poisson bracket is given by the constant poisson tensor $dz_1 dz_2$.

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