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Soddy Sphere Packings' Construction Theorems and evidence of their Curvatures' Log-normality under the Local-to-Global Theorem

Abstract

This project deals with a gap in hyperbolic geometry (using hyperbolic group actions) and analytic number theory (analyzing integral curvatures' distributions) research addressing a three dimensional version of the Local-to-Global theorem for Soddy Sphere Packings (SSPs.) Several necessary construction theorems are proven and experimental evidence is obtained and confirmed as significant by probability tests for the log-normal behavior of the SSP's curvatures. The project also develops a new, upgraded algorithm necessary for construction and analysis of the curvatures' distribution. The proved construction theorems (as well as the methodology of proof) have a wide range of applications in hyperbolic geometry.

Soddy Sphere Packings' Construction Theorems and evidence of their Curvatures' Log-normality under the Local-to-Global Theorem

1. Introduction

1.1 Background

Apollonius of Perga studied the problem: Given three objects, any of which can be a circle, line or point, construct a circle tangent to each of the given three. Apollonius showed that given three mutually tangent circles that have disjoint points of tangency, there are exactly two circles that solve his problem (Apollonius's Theorem.) Rene Descartes formulated a theorem which gives the algebraic relationship among the three given and two constructed circles in the packing (discussed in detail in section 2.) After adding these two new circles to the configuration, there become many other triplets to which Apollonius's Theorem can be applied.

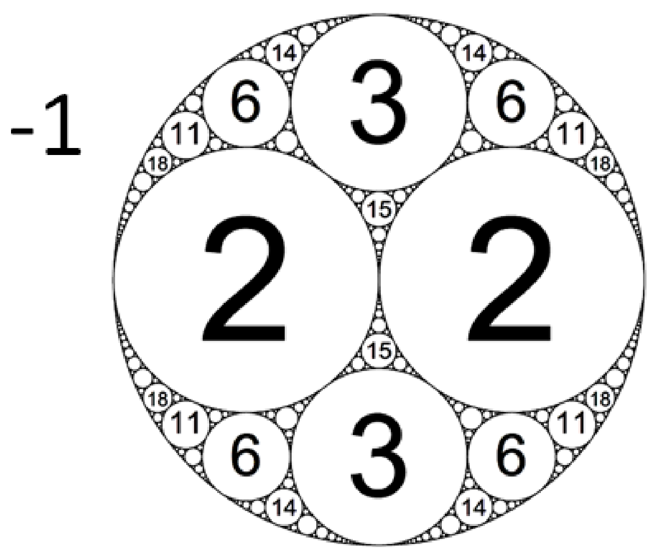

Figure 1: This is an example of an ACP; it is the Integral "Bugeye" ACP with root curvatures $\boldsymbol{a_1 = -1, a_2 = 2, a_3 = 2, a_4 = 3}$. $\boldsymbol{a_1}$ is negative in order to preserve the orientation of tangencies in the construction. [2]

Gottfried W. Leibniz was the first to iterate the above method *ad infinitum*, and thus create a packing of circles known as the Apollonian Circle Packing (ACP) [1, 2, 3], as shown in Figure 1.

Apollonius Circle Packings have received significant attention recently [1-14]. Frederick Soddy noted in 1937 that there were configurations of circle packings in which all curvatures (inverses of the radii) are integral; these Integral Apollonian Circle Packings became the backbone of pivotal research conducted by Ronald Graham, Jeffrey Lagarias, Colin Mallows, Allan Wilks, and Catherine Yan (GLMWY, as it is more commonly referred to.) [11]. Soddy also noted the possibility of creating such integral packings with *spheres*, which is a primary focus of this research.

Apollonius's Theorem is crucial to the construction of an ACP because it proves that only two circles can be tangent to each of three mutually tangent circles, therefore creating such a unique geometric configuration. A three circle configuration is enough to construct a specific ACP with strait edge and compass. All of the curvatures in an ACP, by convention, are positive except for the bounding circle, which is negative to designate its opposite orientation. Given a root quadruple configuration (first four circles, instead of three), it now becomes possible to algebraically determine the curvatures of the remainder of the packing. Apollonius's Theorem is applied to achieve the construction of ACPs, while Descartes' Theorem (discussed in section 2) is used to actually calculate the curvatures in an ACP.

1.2 Literature Review

The aforementioned GMLWY paper [11] is a critical turning point because it is the first to study number theory aspects of the ACP: specifically they stated the Local-to-Global conjecture (explained in the next paragraph,) the generalization of which is the focus of this paper. Research in the field of number theory concerning ACPs explodes post GLMWY [1-5, 7-8, 10-14]. Two more pivotal papers have been recently published on ACPs and SSPs that set the necessary framework for this research. First, Elena Fuchs and Katherine Sanden provide a major stepping stone by providing numerical evidence that analyzes the progression of curvatures following the Local-to-Global principle for ACPs[2]. Even more recently, Kontorovich studies Soddy Sphere Packings (SSPs) and proves the Local-to-Global theorem's variation in the third dimension. [8] There is an obvious gap in research—a newly established Local-to-Global theorem for spheres, but no understanding of the distribution of its progression (as there was with ACPs.) This research analyzes the distribution of admissible curvature progressions following the Local-to-Global conjecture.

The <u>Local-to-Global principle</u> states that *there is an effectively computable maximum curvature,*N_0*, so that if a given curvature, n, is greater than the maximum curvature,* N_0*, AND the given curvature, n, is admissible in the packing, then n is a curvature in the packing.* A number is admissible if it passes the mod 3 congruence condition (described in the next paragraph.) [1, 2, 3 ,7, 8, 14]

<u>1.2.1 Fuchs and Sanden Research that analyzes the distribution of curvatures among congruence classes following GLMWY's Local-to-Global Theorem [2,11]</u>

Elena Fuchs and Katherine Sanden investigate the previously mentioned "bug-eye" packing and a less symmetric "coins" packing. The first step is that they generate the bug-eye packing for all curvatures less than 10^8. After recording every curvature in the packing, they calculate the residues of each curvature mod 24 and validate that only if a curvature produces specific residues under mod 24, then that curvature can be part of the ACP. There are exactly 8 (out of 24) residue classes that should be occurring in the “bug-eye” ACP; if a curvature’s residue is one of the 8, then it is categorized as admissible in the packing.

The mod 24 congruence condition establishes the numbers that are admissible, which *can* be present in the specific packing and that every curvature in the packing will fall into one of the congruence classes mod 24. However, the condition does not establish that every number that passes the congruence condition *will* occur in the packing. That is where the aforementioned Local-to-Global conjecture becomes critical. By the conjecture, *there is an effectively computable maximum curvature,* N_0*, so that if a given curvature, n, is greater than the maximum curvature,* N_0*, AND the given curvature, n, is admissible (passes the congruence condition), then n is a curvature in the packing.* There are local obstructions to the rule in larger circles (smaller curvature) because there are fewer of them; however, there are progressively more curvatures as they get smaller, thus all admissible numbers start occurring as curvatures.

Fuchs and Sanden establish the possible congruence classes and their numerical analysis shows exactly how often each of the congruence classes occurs. Finally they build histograms that “illustrate the distribution of frequencies with which each [of the large] integer[s]… satisfying the specified congruence condition occurs as a curvature in the [bug-eye] packing.” There are thus 8 histograms, each with a different congruence condition. The number of exceptions, as well as the mean and variance are recorded and examined.

1.2.2 Kontorovich researches into the third dimension and develops the Local-to-Global Conjecture for Spheres [8].

In late 2012, Professor Kontorovich published “The Local-Global Principle for Integral Soddy Sphere Packings” [8]. He uses modifications of Apollonius’s and Descartes’ Theorems for three dimensions in order to construct the SSP. He shows that every sufficiently large number that is admissible also occurs in the packing, thus validating the Local-to-Global theorem. This proof shows that the density of numbers passing the Local-to-Global theorem approaches 100% as numbers increase:

$$\lim_{n\to\infty}\left(\frac{curvatures\ less\ than\ n\ that\ pass\ the\ Local-to-Global\ Theorem}{admissible\ curvatures\ (passing\ the\ mod\ 3\ condition)\ less\ than\ n}\right) = 1$$

1.3 Research Problem

Just like research opportunity on ACPs expanded post GLMWY, there is now a lot to investigate in SSPs. My research fills the gap by conducting a Fuchs/Sanden-esque numerical analysis on the Local-to-Global principle Professor Kontorovich proved for SSPs. This analysis will not only numerically validate the Local-to-Global principle, but will also show how the curvatures in the SSP are distributed among their various mod 3 congruence classes.

1.4 Conjecture (Research Hypothesis)

By using an arithmetic (matrix multiplication) adaptation of Descartes' Theorem, this research first aims to generate the curvatures of the same SSP that Professor Kontorovich uses (largest spheres with curvatures: -11, 21, 25, 27, 28) to the highest possible precision. By recording the curvatures as array indexes and the frequencies with which they occur as the values in the array, I should have a numerical set that is ready for experiments and analysis. After it is generated, the congruence conditions must be applied to the numbers to determine the congruence classes, verify the Local-to-Global Conjecture, see how the curvatures are distributed in terms of their residues, and plot histograms of the frequencies with which each number occurs. This should be enough empirics to begin validating the conjecture, and making more behavioral observations on the curvatures.

The 2 admissible residue classes under mod 3 for this packing are 0 and 1 (judging from the root quintuple's curvatures). Among the root quintuple 21 and 27 are 0 mod 3, and the other curvatures (-11, 25, 28) are 1 mod three. Thus it seems reasonable to hypothesize that the ratios of curvatures occurring in the two congruence classes (0 mod 3 and 1 mod 3) are 2/5ths and 3/5ths, respectively.

2. Methodology

The construction of ACPs and SSPs is impossible without the use of Apollonius's Theorem and Descartes' Theorem; therefore both must be proved. Proofs for their two dimensional forms have been developed [1], but not for their three dimensional analogues. Subsection 1 defines, constructs, and proves the principle of *inversion,* which is necessarily employed by both proofs. Subsection 2 explains the algorithm used to numerically analyze the Local-to-Global Conjecture. (The proofs of the two theorems are in the next section.)

2.1 Definition and proof of inversion, a prerequisite to the other proofs

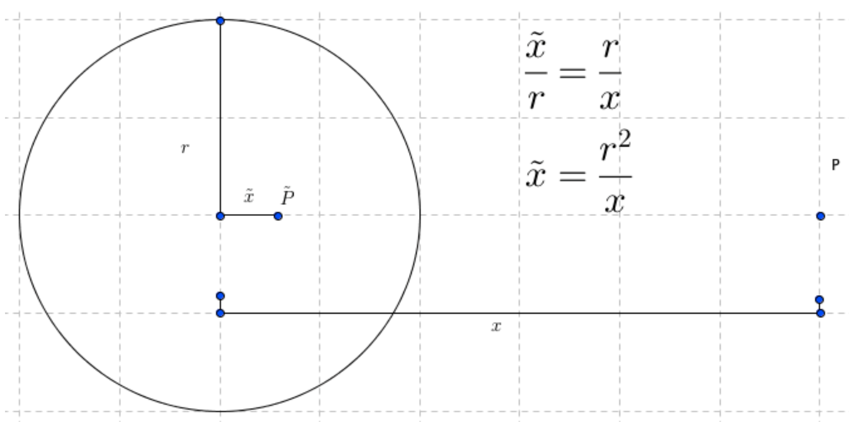

Figure 2: The Definition of Inversion. An inversion is a transformation of a point of set of points. Geometric inversion is always performed with respect to a circle. The simplest case of inversion is that of a point with respect to a circle. Given circle O with radius r, and any point on the plane, P (at a distance x away from the origin), Inversion of point P has the following relationship: $\frac{\tilde{x}}{r} = \frac{r}{x}$; $\tilde{x} = \frac{r^2}{x}$

This relationship can be proved in this geometrically:

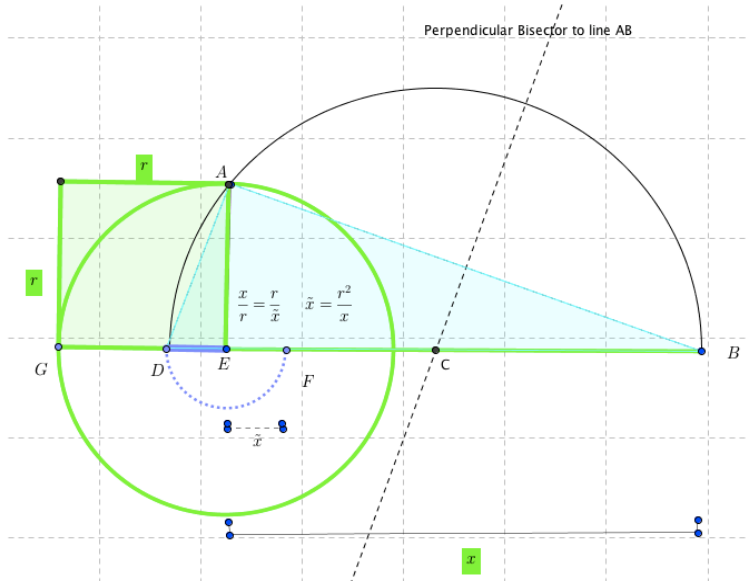

Figure 3: Construction/Derivation of Inversion Relationship $\tilde{x} = \frac{r^2}{x}$ is true because of similar right triangles AED, BEA, and BAD. The reflections of the legs DE and BE or triangle BAD, and the perpendicular to angle BAD lead to the proportion: DE/AE = AE/BE. Substituting the values r, x, and $\tilde{x}$ the equation becomes: $\frac{\tilde{x}}{r} = \frac{r}{x}$; $\tilde{x} = \frac{r^2}{x}$.

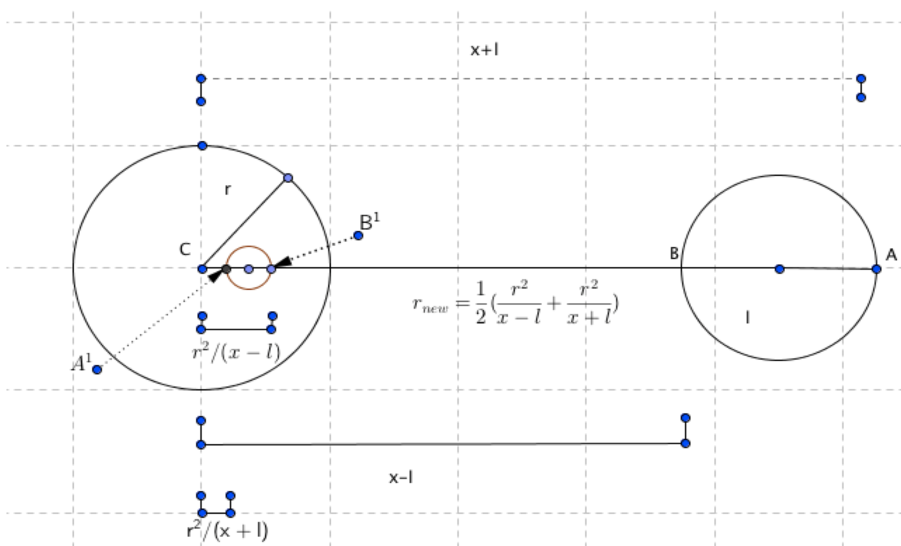

Figure 4: Here the same inversion relationship is employed; however this time we use ½ because we are looking for a radius, not a diameter. Thus we get the following values: $\frac{R^2}{x-l}$ is the inversion of the closest distance away from circle C which is point B, which is B^1inverted. $\frac{R^2}{x+l}$ is the inversion of the farthest distance away from circle C which is point A, which is A^1 inverted.$B^1 - A^1$ is the diameter, therefore half of it is the radius. Substituting the above we get:

$$r_{new} = \frac{1}{2}\left(\frac{r^2}{x-l} - \frac{r^2}{x+l}\right) \qquad r_{new} = \frac{r^2}{2}\left(\frac{x+l-(x-l)}{x^2-l^2}\right)$$

$$r_{new} = \frac{r^2}{2}\left(\frac{1}{x-l} - \frac{1}{x+l}\right) \qquad r_{new} = \frac{r^2}{\cancel{2}}\left(\frac{\cancel{2}l}{x^2-l^2}\right) \qquad r_{new} = \frac{r^2 l}{x^2-l^2}$$

2.2 Description of Algorithm for Local-to-Global theorem outputting curvatures of the SSP

A numerical analysis of the Local-to-Global conjecture would be unfeasible to perform using algebraic applications of Descartes' Theorem. (It would be impossibly consuming, even on a computer.) Thus, this procedure will employ a slight transformation of Descartes' Theorem that will involve matrix arithmetic.

The fastest method of applying Descartes' Theorem involves creating a group of vectors which contain the curvatures of an ACP and are generated by five 5x5 matrices (given in Figure 5.) During this operation, the 5 curvatures of the circles in a construction are denoted in vector form: $[a_1, a_2, a_3, a_4, a_5]$. An Identity Matrix is one that does not change the value of the vector you multiply it by, therefore keeping the vector values the same—similarly to any identity number in arithmetic. The generator matrices only differ from the identity matrix in one column: when $[a_1, a_2, a_3, a_4, a_5]$ is multiplied by one of the matrices, only one of the curvatures is changed. The non-identity column in each matrix transforms its respective number in the curvature vector into the next corresponding curvature in the SSP, effectively applying Descartes' Theorem.

It is important to note that different possibilities of vector multiplication create different variations on the resultant vector of curvature. The choice of matrix corresponds to the desired alteration in $[a_1, a_2, a_3, a_4, a_5]$. For example, altering a_3 means multiplying by the 3rd generator and finding the only other sphere tangent to $a_1, a_2, a_4,$ and a_5 other than a_3.

In this experiment, the vector $[a_1, a_2, a_3, a_4, a_5]$ is multiplied iteratively by the five matrices, in all possible permutations, to apply Descartes' Theorem and therefore the SSP.

$$M_1 = \begin{pmatrix} -1 & 1 & 1 & 1 & 1 \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, M_2 = \begin{pmatrix} 1 & & & & \\ 1 & -1 & 1 & 1 & 1 \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, M_3 = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ 1 & 1 & -1 & 1 & 1 \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, M_4 = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ 1 & 1 & 1 & -1 & 1 \\ & & & & 1 \end{pmatrix}, M_5 = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ 1 & 1 & 1 & 1 & -1 \end{pmatrix}$$

Figure 5: The five generator matrices used to apply Descartes' Theorem and create SSPs. The matrix's number corresponds to the sphere which is going to be altered post-multiplication.

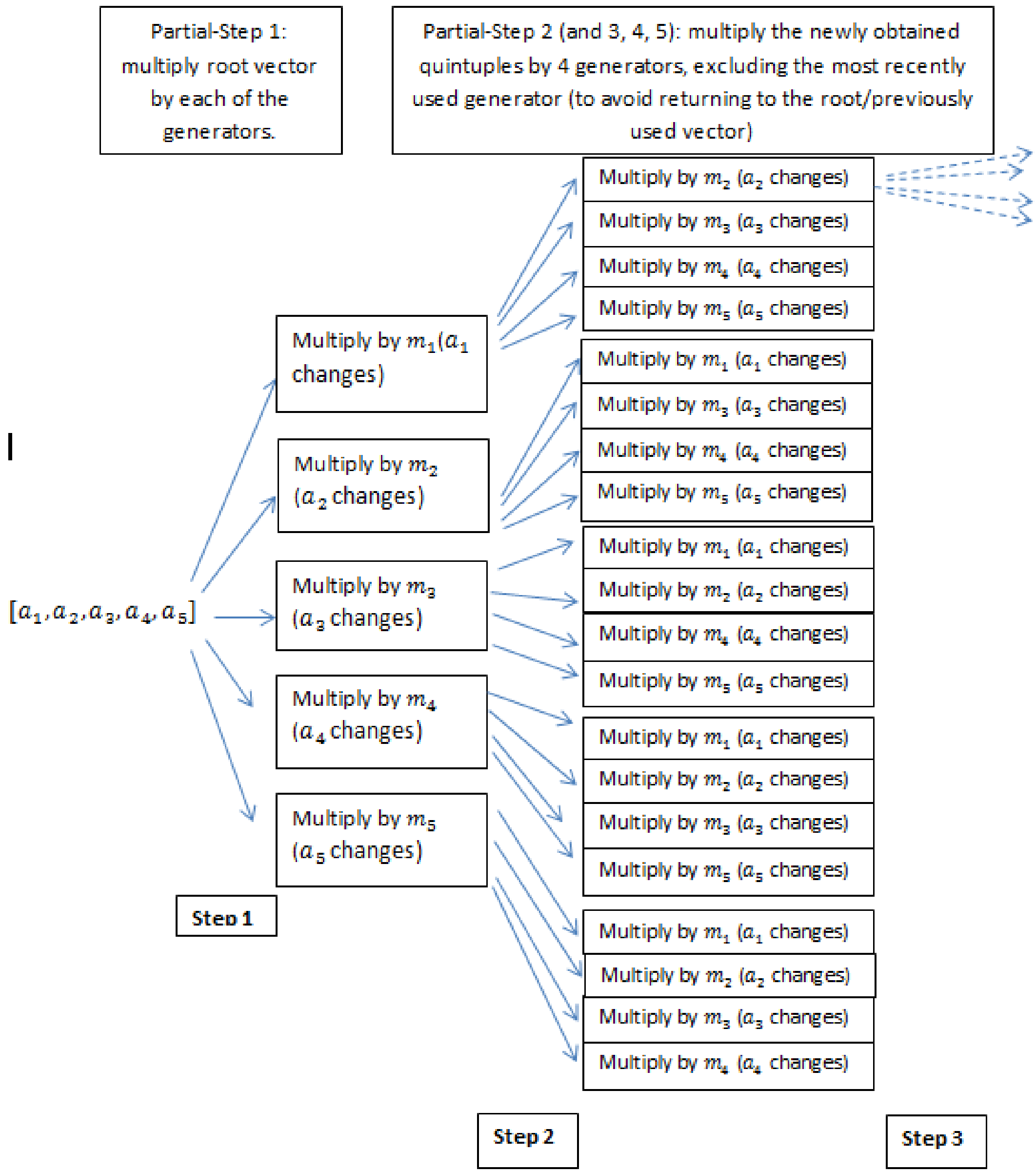

Figure 6: The 5 Generator Matrices that are used iteratively to apply Descartes' Theorem on root quintuples of spheres in vector form.

3. Results

3.1 Proof of Apollonius' Theorem, a necessary construction principle for SSPs.

Apollonius' Theorem states that for every 4 given mutually tangent spheres S_1, S_2, S_3, S_4 there are only two spheres $S_{5\alpha}$ and $S_{5\beta}$ that can be constructed such that they too are also tangent to each of the four original spheres. This proof will follow by inversion:

The starting configuration contains reference Sphere B, and pre-inversion spheres S_1, S_2, S_3, S_4. Reference circle B is not tangent to S_1, S_2, S_3, S_4 but is centered at a point where S_1 and S_2 are tangent to make the inverted diagram as simple as possible.

First employ inversion of S_1, S_2, S_3, S_4into reference sphere B to get $\widetilde{S_1}, \widetilde{S_2}, \widetilde{S_3}, \widetilde{S_4}$. The inverted figure appears like two tangent spheres ($\widetilde{S_3}, \widetilde{S_4}$), which are also both tangent to 2 parallel planes ($\widetilde{S_1}, \widetilde{S_2}$). Next, construct the possible tangent spheres to $\widetilde{S_1}, \widetilde{S_2}, \widetilde{S_3}, \widetilde{S_4}$ which are $\widetilde{S_{5\alpha}}$and $\widetilde{S_{4\beta}}$. Effectively, inversion preserves the tangencies of the spheres but lets us look at S_1, S_2, S_3, S_4 from a perspective in which it is clear that $\widetilde{S_{5\alpha}}$and $\widetilde{S_{4\beta}}$ are the only possible tangent spheres. Finally, invert back $\widetilde{S_{5\alpha}}$and $\widetilde{S_{4\beta}}$into circle B to get mutually tangent spheres to the original construction $S_{4\alpha}$and $S_{4\beta}$. Because $\widetilde{S_{5\alpha}}$and $\widetilde{S_{4\beta}}$ are the only possible mutually tangent spheres to $\widetilde{S_1}, \widetilde{S_2}, \widetilde{S_3}, \widetilde{S_4}$ and because tangencies are preserved under inversion, it must be true that $S_{4\alpha}$and$S_{4\beta}$ are the only possible mutually tangent circles toS_1, S_2, S_3, S_4. (Figure 7)

3.2 Proof of Descartes' Theorem, which constructs SSPs by outputting new curvatures

Descartes' Theorem: $3(a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2) = (a_1 + a_2 + a_3 + a_4 + a_5)^2$

Where $radius_{\boldsymbol{i}} = \frac{\boldsymbol{1}}{\boldsymbol{a_i}}$

The proof will follow by inversion, and because inversion is always performed on individual points, the same principles and formulae that where developed in the previous section, $\tilde{x} = \frac{r^2}{2x}$ and $r_{new} = \frac{r^2 l}{x^2 - l^2}$, are also applicable in three dimensional space. This can be visualized with the following: Start with a 3D pre-inversion configuration, and make a planar bisection. Use the planar bisection to construct any inversions (as they would be done in 2D). Rotate the planar bisection with the mapped out inversion to construct the inverted sphere or plane in 3D.

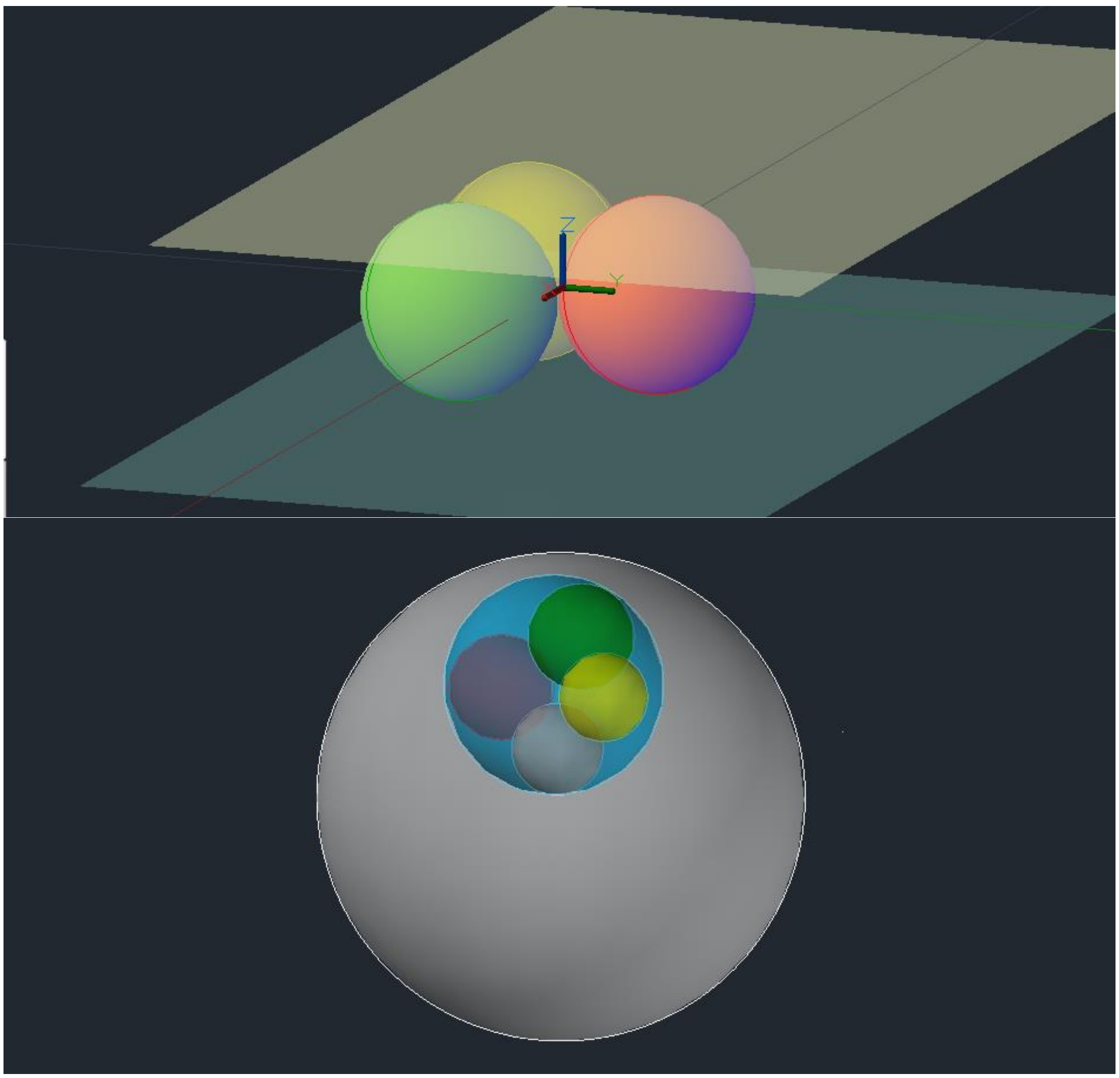

Figure 7: Proof of Apollonius's and Descartes' Theorems. Start with the inverted form of a quintuple of spheres centered on the origin, as shown in the upper picture. Re-inversion of those figures back into the reference spheres makes an SSP. There is clearly only one spot for a new sphere to be constructed that is mutually tangent to all but one of the spheres in the inverted image. Because points of tangency are preserved under inversion, Apollonius's Theorem is true. This diagram also depicts a proof of Apollonius's Theorem—using the coordinates of the inverted image we can generate equations which prove Descartes' Theorem.

Start with four mutually tangent spheres in the plane, labeled S_1, S_2, S_3, S_4. Consider their image under sphere inversion by a sphere B of radius R. The resulting objects are $\widetilde{S_1}, \widetilde{S_2}, \widetilde{S_3}, \widetilde{S_4}$. Because S_1 and S_2 pass through center B, $\widetilde{S_1}$ and $\widetilde{S_2}$ are planes. Furthermore, C_3 and C_4 do not pass through the center, so $\widetilde{S_3}$ and $\widetilde{S_4}$ are spheres.

Next we can substitute the previously derived formulas. First is the inversion $\tilde{x} = \frac{r^2}{2x}$ on the planes $\widetilde{S_1}$ and $\widetilde{S_2}$. Note that x is the distance between center B(x,y,z) and the closest point of $\widetilde{S_1}$ and $\widetilde{S_2}$ to B.

$\widetilde{S_1}$ has x: $\tilde{x} = \frac{r^2}{2x} = \frac{r^2}{2(z+1)}$ because the plane z=1 is z+1 away from B(x,y,z)

$\widetilde{S_2}$ has x: $\tilde{x} = \frac{r^2}{2x} = \frac{r^2}{2(z-1)}$ because the plane z=-1 is z-1 away from B(x,y,z)

Second, is the spherical inversion of $\widetilde{C_3}$ and $\widetilde{C_4}$ employing $r_{new} = \frac{r^2 l}{x^2-l^2}$.

In order to calculate the distance between B(x,y) and the centers of circles $\widetilde{C_3}$ and $\widetilde{C_4}$, we apply the distance formula $d = \sqrt{x^2+y^2+z^2}$

$\widetilde{S_3}$ has x: $r_{new} = \frac{r^2 l}{x^2-l^2} = \frac{r^2}{(\sqrt{x^2+(y+1)^2+z^2})^2-1} = \frac{r^2}{x^2+(y+1)^2+z^2-1}$ because $l = 1$ and the center is at (0,1,0).

$\widetilde{S_4}$ and $\widetilde{S_5}$ both have y-coordinates of $-\sqrt{3}+1$ because they are 1 away from the y-axis and 2 away from the center of $\widetilde{S_3}$(0,1,0). By right triangles the distance between then and $\widetilde{S_3}$ in y-coordinates is $\sqrt{3}$.

$\widetilde{S_4}$ has x: $r_{new} = \frac{r^2 l}{x^2-l^2} = \frac{r^2}{(\sqrt{(x+1)^2+(y-\sqrt{3}+1)^2+z^2})^2-1} = \frac{r^2}{(x+1)^2+(y-\sqrt{3}+1)^2+z^2-1}$ because $l = 1$ and the center is at $(1, -\sqrt{3}+1, 0)$

$\widetilde{S_5}$ has x: $r_{new} = \frac{r^2 l}{x^2-l^2} = \frac{r^2}{(\sqrt{(x-1)^2+(y-\sqrt{3}+1)^2+z^2})^2-1} = \frac{r^2}{(x-1)^2+(y-\sqrt{3}+1)^2+z^2-1}$ because $l = 1$ and the center is at $(-1, -\sqrt{3}+1, 0)$

Substitution of the above expressions as radii into Descartes' Theorem with $\boldsymbol{r_i} = \frac{\mathbf{1}}{\boldsymbol{a_i}}$ produces:

$$3(a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2) = (a_1 + a_2 + a_3 + a_4 + a_5)^2$$

The above statement (Descartes' Theorem) is true after long but simple algebraic simplification.

3.3 Distributions of frequencies with which curvatures appear in the packing

First, the program generated all numbers that would occur in the (-11, 21, 25, 27, 28), until a maximum curvature of 3000, which was by far the most time consuming part of the procedure, taking about a day after the code was debugged and upgraded to its simplest, fastest form. The frequencies of the occurring curvatures were the variables of interest, which were stored in the array which had curvatures as index. Sub-arrays of the 3 possible congruence classes mod three (0, 1, and 2) were then created. As anticipated, no curvatures existed that satisfied the 2 mod 3 congruence condition. Among the other two congruence classes the hypothesis held true—40.34% of all curvatures in the packing where 0 mod 3, while 59.66% were 1 mod 3. The histograms in Figure 8 show the distribution of their frequencies.

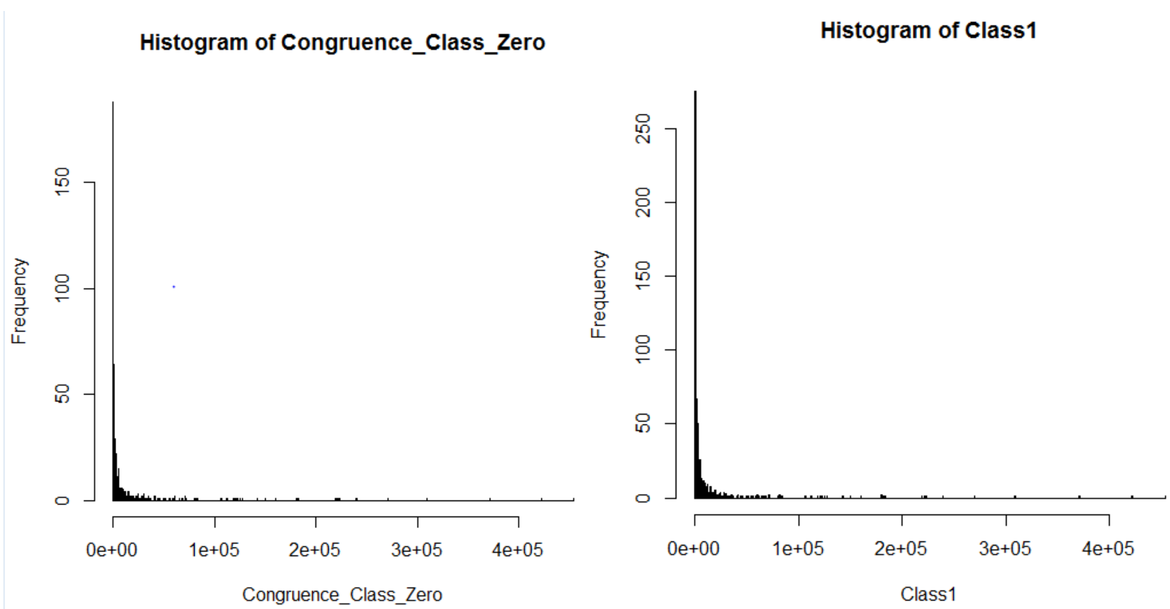

Figure 8: The distribution of frequencies with which each integer, m, from $0 < m < 3000$ satisfying the given congruence condition (either 0 mod 3 or 1 mod 3) occurs as a curvature in the SSP with root curvatures -11, 21, 25, 27, 28. The mean of the first histogram is 11322.94. The mean of the second histogram is 18431.62. There are 15 exceptions (of numbers that occur globally but not locally) in each dataset.

The distributions of the above frequencies are surprisingly unsatisfying. No obvious transformation of equation seems to relate with these distributions— until a log-frequency histogram is built (further investigated in the next section.) The log-of-frequency distributions are given in Figure 9.

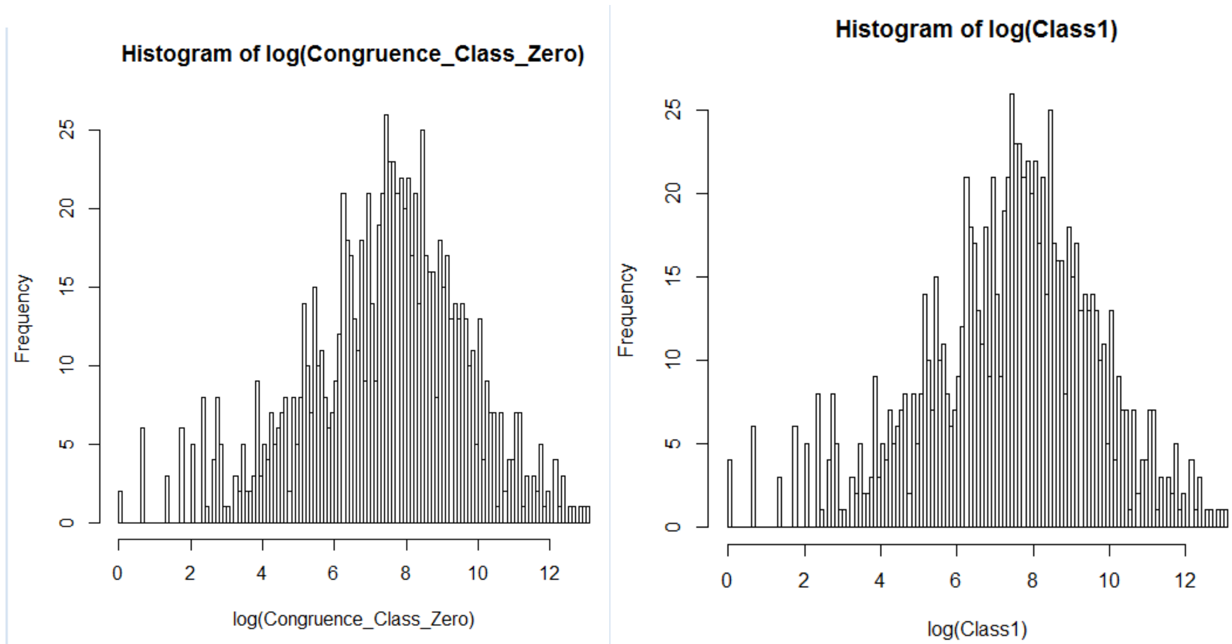

Figure 9: The distribution of *the logarithm* of the frequencies with which each integer, m, from $0 < m < 3000$ satisfying the given congruence condition (either 0 mod 3 or 1 mod 3) occurs as a curvature in the SSP with root curvatures -11, 21, 25, 27, 28. The mean of each histogram is 7.55 ± .2. The variance of each histogram is 5.35 ± .2.

Surprisingly, the log (frequency) histograms of both of the congruence classes seem rather normal. In order to further investigate the normality, a Q-Q plot was created for each of the distributions, depicted in Figure 10. Statistical probability tests confirm that this is indeed a log-normal distribution. Both Jargue-Bera and Shapiro-Wilk statistical goodness-of-fit tests in R confirm with a P value of point .12 that this distribution can be normal. Because by almost all conventional alpha (confidence) levels a distribution must have less than 10% probability of being rejected, this distribution cannot be rejected as non-normal.

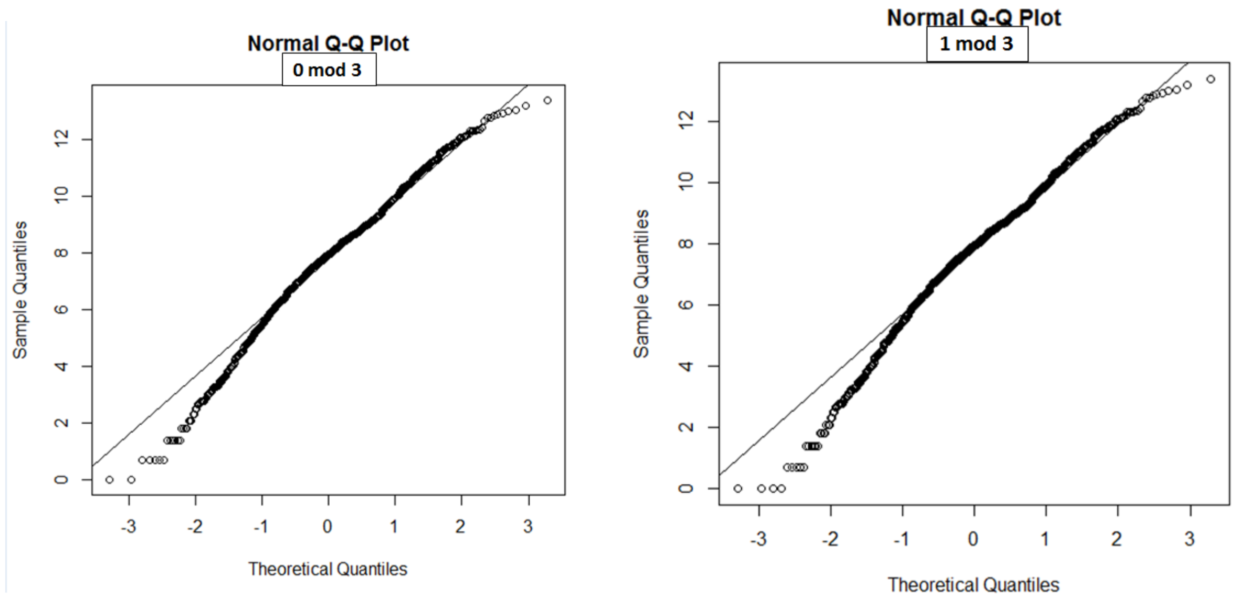

Figure 10: Q-Q plots of the distribution of *the logarithm* of the frequencies with which each integer, m, from $0 < m < 3000$ satisfying the given congruence condition (either 0 mod 3 or 1 mod 3) occurs as a curvature in the SSP with root curvatures -11, 21, 25, 27, 28.

4. Discussion

4.1 Unanticipated Descartes' Theorem Behavior

In order to verify its proper functionality, the algorithm suggested in the methodology was first used in its simplified form on a two-dimensional analogue of this study in order to check the code without running into problems. When the code was upgraded to a version that functionally applies to three dimensional space, an unexpected problem occurred. When using ACPs (with four 4x4 arrays), a variation of the above algorithm would produce proper results, however because SSPs occur in three dimensional space there is more room for error.

Specifically—in the first, highest branch on our algorithm, the root vector was multiplied by the following generators in order: 1, 2, 1, 2, 1. This procedure is allowed by the methodology (because no single generator is a repeat of its parent,) and its analog would have led to valid results in two dimensional space. What ends up occurring in the SSP, is that after the fifth generation is applied, the vector changes back into its root form: [-11, 21, 25, 27, 28]. This causes the tree diagram to fail as it gets stuck in a forever-lasting loop which always returns back to the root curvature after five steps. When one branch of the program fails, no other branches can be run.

The most probable explanation for this is the SSP form of Descartes' Theorem applies significantly smaller changes to curvatures in each of its steps. In two-dimensional space, the altered curvature would be subtracted from double the sum of each other curvature to get its new value; in three-dimensional space, however, it is subtracted from the single sum of each other curvature, thus outputting a much smaller curvature. This becomes especially problematic when only using generators that alter a_1 and a_2 because they as they rapidly increase the procedure becomes counterintuitive—a large number, a_1, and 3 insignificant small numbers, a_3, a_4, a_5, are subtracted by another large number, a_2, therefor potentially producing a curvature that is even smaller than a_2.

A reasonable solution to the above issue seems to be: do a check every time a new curvature is generated; if that curvature existed in the root quintuple, end the branch. Interestingly this revision also ran into problems fairly soon, on branch: 1, 2, 1, 2, 2, 1, 2, 1, 2, 1. After the final generator (#1) is applied to the vector it cycles back into the third reoccurrence of generator #3, causing the same type of cycle. The other option seemed to be: do a check on every step against previously generated curvatures (so that if they are the same, end the branch.) This, however, is unfeasible because every single combination of curvatures would have to be recorded, thus severely compromising the maximum curvature that could be investigated.

A satisfactory solution was a procedure in the program that would, after each step, check to see if the curvature decreased or increased. If it increased, the branch would be continued, if it decreased, the single number would still be recorded, but after doing so the program would call the next branch.

4.2 Significantly smaller maximum curvature

These results vary from the Fuchs-Sanden experiment in that they computed curvatures up to a maximum of10^8 in Matlab, while this algorithm took about a day to run to a maximum of $3 * 10^3$ in R Studio. The difference in computational software should be insignificant (or, if anything speed up this experiment.) There seems to be a few inherent factors in this experiment that lead to such a disparity. First, during every single generation in the diagram there is an extra step per branch. This means that there are exponentially more steps performed to get all of the curvatures until a maximum, which can also be visualized in that adding a third dimension adds more variability and more space to construct spheres. Second, operations with a 5 variables and 5 different, 5x5 matrices leads to a much longer computation than 4 variables and four, 4x4 matrices. Finally, the unexpected behavior of Descartes' Theorem and the need to detour the conventional methodology, increased the consumption of the operation.

4.3 Distribution of curvatures among congruence classes and exceptions

The hypothesis that 40% of curvatures in the packing would satisfy the 0 mod 3 congruence condition and 60% would satisfy the 1 mod 3 congruence condition is now empirically true. This leaves interesting room for future research to investigate whether the distribution of curvatures among curvatures can be predicted solely from the root-quintuple packing, as was possible in this case.

Every single number that was generated as a curvature in the SSP was locally *represented*, in that it satisfied either the 0 mod 3 or 1 mod 3 congruence condition. There were however, numbers that were admissible (such as 3 and 4) that satisfied one of the congruence conditions but were not represented. This can largely be explained by the fact that they were all very small numbers (compared to the whole set), smaller than the *maximum curvature* N_0 that Kontorovich wrote about. In conclusion, there were 15 such numbers for each congruence class.

4.4 Discovery of the logarithmic behavior

By far the most enlightening result of this paper, was the discovery that distribution of the frequencies with which each integer, m, from $0 < m < 3000$ satisfying the given congruence condition (either 0 mod 3 or 1 mod 3) occurred, were insignificant; however, the *logarithms of those frequencies were normally distributed*. The graphs in figure 8 are extremely unsatisfying, as they are very skewed right, with means that are overly-effected by outliers, and unbelievably large variances. A simply transformation of the graph—log (frequencies)—immediately made the graph unimodal and fairly-symmetric. Further analysis to investigate the normality of those graphs was done by Q-Q plots which showed even more convincing results. The distributions of both congruence classes didn't vary at all from an anticipated normal distribution, except when closer to their lower and upper extremes. This is probably because of the relatively small size; future research should definitely examine such distribution of logarithms, hopefully with greater sample size. It is possible that the distribution becomes progressively more normal as the sample size increases, eventually leading to a normally distributed set of frequencies.

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