M98/510/H(1)M

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MARKSCHEME

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MATHEMATICS

Higher Level

Paper 1

13 pages

1. Since, $\sin\theta < 0$, $\cos\theta = \frac{2}{5}$, $\sin\theta = -\sqrt{1-\cos^2\theta} = -\sqrt{\left(1-\frac{4}{25}\right)} = -\frac{\sqrt{21}}{5}$ *(M1)(A1)*

Hence, $\tan\theta = \frac{\sin\theta}{\cos\theta} = -\frac{\sqrt{21}}{2}$ and $\sec\theta = \frac{1}{\cos\theta} = \frac{5}{2}$ *(A1)(A1)*

Answers: $\sin\theta = -\frac{\sqrt{21}}{5}$, $\tan\theta = -\frac{\sqrt{21}}{2}$, $\sec\theta = \frac{5}{2}$ *(C2)(C1)(C1)*

2. (a) $\frac{1}{8} + 3k + \frac{1}{6}k + \frac{1}{4} + \frac{1}{6}k = 1$ *(M1)*

Thus, $\frac{10k}{3} = \frac{5}{8}$ and $k = \frac{3}{16}$ *(A1)*

(b)

x	0	1	2	3	4
$p(X = x)$	$\frac{1}{8}$	$\frac{9}{16}$	$\frac{1}{32}$	$\frac{1}{4}$	$\frac{1}{32}$

$p(0 < X < 4) = \frac{9}{16} + \frac{1}{32} + \frac{1}{4} = \frac{27}{32}$ *(M1)(A1)*

Answers: (a) $k = \frac{3}{16}$ *(C2)*

(b) $p(0 < X < 4) = \frac{27}{32}$ *(C2)*

3. $(\sqrt{3})^{126} = 3^{63}$ *(M1)*

Hence, $3^{x^2-1} = 3^{63}$ *(A1)*

Therefore, $x^2 - 1 = 63$ or $x = \pm 8$ *(M1)(A1)*

Answers: $x = \pm 8$ *(C4)*

4. Let $-2+\mathrm{i}2\sqrt{3} = r(\cos\theta + \mathrm{i}\sin\theta)$ *(M1)*

Then $r = \left|-2+\mathrm{i}2\sqrt{3}\right| = \sqrt{4+12} = 4$ *(A1)*

and $\tan\theta = \dfrac{2\sqrt{3}}{-2} = -\sqrt{3}$

Hence $\theta = \dfrac{2}{3}\pi \quad (0 \le \theta \le \pi)$ *(A1)*

Thus $z = 4\mathrm{e}^{\mathrm{i}(2/3\pi + 2k\pi)}, \quad k = 0 \pm 1, \pm 2, \ldots$ *(R1)*

Answer: $z = 4\mathrm{e}^{\mathrm{i}(2/3\pi + 2k\pi)}, \quad k = 0 \pm 1, \pm 2, \ldots$ *(C4)*

Note: Award ***(C4)*** for $z = 4\mathrm{e}^{(2/3\pi)\mathrm{i}}$

5. $p(A \cap B) = p(A)p(B) = \left(\dfrac{1}{4}\right)\left(\dfrac{1}{8}\right) = \dfrac{1}{32}$ *(M1)(A1)*

$p(B) = \dfrac{p(A \cap B)}{p(A|B)} = \dfrac{1/32}{1/4} = \dfrac{1}{8}$ *(M1)(A1)*

$p(A) = \dfrac{1/32}{1/8} = \dfrac{1}{4}$

Answers: $p(A) = \dfrac{1}{4}, \quad p(B) = \dfrac{1}{8}$ *(C2)(C2)*

6. Let X be the mean test score

$p(X > 80) = p\left(Z > \dfrac{80-60}{10}\right) = p(Z > 2)$ *(M1)(A1)*

$= 1 - 0.9773 = 0.0227$ *(M1)(A1)*

Answer: $p(X > 80) = 0.0227$ *(C4)*
(Also accept 0.0228 which is obtainable through calculator)

Note: Some candidates may use a continuity correction as follows:

$X \sim N(60, 10^2)$

Hence $p(X > 80) = p\left(Z > \dfrac{80.5-60}{10}\right) = p(Z > 2.05)$ *(M1)(A1)*

$= 1 - 0.9798 = 0.0202$ *(M1)(A1)*

Answer: $p(X > 80) = 0.0202$ *(C4)*

7. (a) $2+4(n-1)=58$ or $4n-2=58 \Rightarrow n=15$ *(M1)(A1)*

(b) Sum of 15 terms of a geometric sequence with first term 2 and common ratio $\frac{1}{2}$ is $2\left(\frac{1-(1/2)^{15}}{1-1/2}\right)=4\left(1-\frac{1}{2^{15}}\right)$ *(M1)(A1)*

Answers: (a) $n=15$ *(C2)*

(b) $4\left(1-\frac{1}{2^{15}}\right)$ or $\frac{32\,767}{8192}$ *(C2)*

8. $E(X)=(1)\frac{2}{9}+2\left(\frac{1}{9}\right)+3\left(\frac{2}{9}\right)+4\left(\frac{1}{9}\right)+5\left(\frac{2}{9}\right)+(6)\left(\frac{1}{9}\right)$ *(M1)*

$=\frac{2}{9}+\frac{2}{9}+\frac{6}{9}+\frac{4}{9}+\frac{10}{9}+\frac{6}{9}=\frac{30}{9}=3\frac{3}{9}=3\frac{1}{3}$ *(A1)*

$E(X^2)=(1)^2\frac{2}{9}+(2)^2\frac{1}{9}+(3)^2\frac{2}{9}+(4)^2\frac{1}{9}+(5)^2\frac{2}{9}+(6)^2\frac{1}{9}$

$=\frac{2}{9}+\frac{4}{9}+\frac{18}{9}+\frac{16}{9}+\frac{50}{9}+\frac{36}{9}=\frac{126}{9}=14$

$\mathrm{Var}(X)=E(X^2)-(E(X))^2=14-\left(\frac{10}{3}\right)^2$ *(M1)*

$=14-\frac{100}{9}=\frac{126-100}{9}=\frac{26}{9}$ *(A1)*

Answers: $E(X)=\frac{10}{3}$, $\mathrm{Var}(X)=\frac{26}{9}$ *(C2)(C2)*

9. $\sin x\tan x=\sin x \Rightarrow \sin x(\tan x-1)=0$ *(M1)*

$\sin x=0$ when $x=0$, $x=\pi$, or $x=2\pi$ *(A1)*

$\tan x-1=0$ when $x=\frac{\pi}{4}$ or $x=\frac{5\pi}{4}$ *(M1)(A1)*

The solutions are $x=0,\frac{\pi}{4},\pi,\frac{5\pi}{4},2\pi$

Answers: $x=0,\frac{\pi}{4},\pi,\frac{5\pi}{4},2\pi$ *(C4)*

10. The normal to the planes are $\vec{n}_1 = 2\vec{i} + 3\vec{j} - \vec{k}$ and $\vec{n}_2 = 7\vec{i} - \vec{j} + 3\vec{k}$ *(A1)(A1)*

Angle between the two planes is given by

$$\arccos^{-1}\frac{\vec{n}_1 \cdot \vec{n}_2}{\left|\vec{n}_1\right|\left|\vec{n}_2\right|} = \arccos\frac{14-3-3}{\sqrt{14}\sqrt{59}} = \arccos\frac{8}{\sqrt{826}} \qquad \textit{(M1)}$$

$$= 73.8° \qquad \textit{(A1)}$$

Answer: 73.8° *(C4)*

11. $$f'(x) = \frac{\dfrac{\ln x}{\sqrt{1-x^2}} - \dfrac{\arcsin x}{x}}{(\ln x)^2} \qquad \textit{(M1)(M1)(M1)(A1)}$$

$$= \frac{x\ln x - \sqrt{1-x^2}\arcsin x}{x\sqrt{1-x^2}(\ln x)^2}$$

Answer: $f'(x) = \dfrac{x\ln x - \sqrt{1-x^2}\arcsin x}{x\sqrt{1-x^2}(\ln x)^2}$ *(C4)*

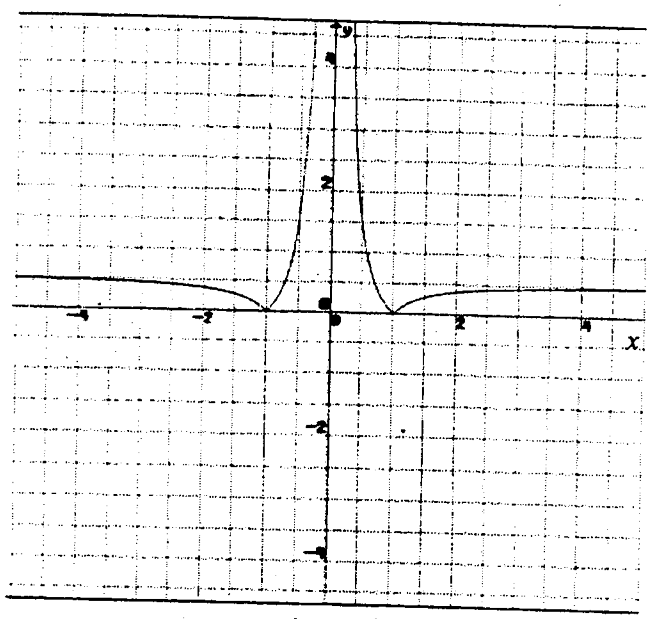
or any equivalent form. (Simplification of the final answer is not required.)

12. Area $= 4\int_0^1 y\,\mathrm{d}x = 4\int_0^1 \sqrt{x^2 - x^4}\,\mathrm{d}x = 4\int_0^1 x\sqrt{1-x^2}\,\mathrm{d}x$ *(M1)(M1)*

$$= \left[\left(-\frac{4}{2}\right)\left(\frac{2}{3}\right)\left(1-x^2\right)^{3/2}\right]_0^1 = \left(-\frac{4}{3}\right)(-1) = \frac{4}{3} \qquad \textit{(M1)(A1)}$$

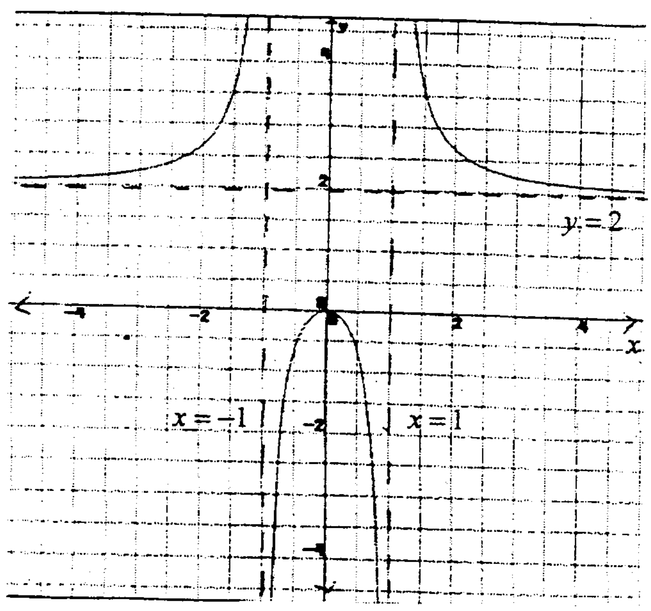
Answer: Area $= \dfrac{4}{3}$ *(C4)*

13.



$|f(x)|$

(C1)



$\dfrac{1}{f(x)}$

Asymptotes *(C1)*.
Curves *(C2)*.
Deduct 1 mark for each mistake.

14. $\int \frac{\mathrm{d}y}{y} = \int \cos x \,\mathrm{d}x, \quad 0 < x < \infty \quad \Rightarrow \ln|y| = \sin x + C$

Since $y > 0$, $y = A\mathrm{e}^{\sin x}$, A being a constant *(M1)(A1)*

Since, $y = 1$ when $x = \frac{\pi}{2}$, we get,

$A\mathrm{e}^{\sin \pi/2} = 1$ or $A = \frac{1}{\mathrm{e}}$ *(M1)*

Hence, $y = \left(\frac{1}{\mathrm{e}}\right)\mathrm{e}^{\sin x} = \mathrm{e}^{\sin x - 1}$ *(A1)*

Answer: $y = \mathrm{e}^{\sin x - 1}$ *(C4)*

Note:	Some students may solve the problem by using integrating factor. For $\mathrm{e}^{-\int \cos x \mathrm{d}x} = \mathrm{e}^{-\sin x}$ as the integrating factor award *(C1)* and proceed according to the markscheme above.

15. (a) $6\int_0^k (x^2 + x)\mathrm{d}x = 6\left(\frac{k^3}{3} + \frac{k^2}{2}\right) = 2k^3 + 3k^2 = 1$ *(M1)*

$\Rightarrow 2k^3 + 3k^2 - 1 = 0 \;\Rightarrow\; (k+1)(2k^2 + k - 1) = 0$

$\Rightarrow (k+1)(k+1)(2k-1) = 0$

Therefore, $k = -1$ or $k = \frac{1}{2}$

Since $k > 0$, $k = \frac{1}{2}$ *(A1)*

(b) $E(X) = 6\int_0^{1/2} (x^2 + x)x \,\mathrm{d}x = 6\left[\frac{x^4}{4} + \frac{x^3}{3}\right]_0^{1/2}$ *(M1)*

$= 6\left[\frac{1}{64} + \frac{1}{24}\right] = \frac{11}{32}$ *(A1)*

Answers: (a) $k = \frac{1}{2}$ *(C2)*

(b) $E(X) = \frac{11}{32}$ *(C2)*

16. Differentiating $x^3 + y^3 = 6xy$ implicitly with respect to x, we get

$$3x^2 + 3y^2 y' = 6y + 6xy' \Rightarrow y' = \frac{2y - x^2}{y^2 - 2x}$$ *(M1)(A1)*

Slope at $(3, 3)$ is $(y')_{(3,3)} = -1$ *(A1)*

Tangent has equation $y - 3 = (-1)(x - 3)$ i.e. $x + y = 6$ *(A1)*

Answer: $x + y = 6$ *(C4)*

17.
$$\int \arctan x \, dx = x \arctan x - \int \frac{x}{1 + x^2} dx$$ *(M1)(A1)*

$$= x \arctan x - \frac{1}{2} \int \frac{2x}{1 + x^2} dx$$

$$= x \arctan x - \frac{1}{2} \ln(1 + x^2) + C$$ *(M1)(A1)*

Answer: $x \arctan x - \frac{1}{2} \ln(1 + x^2) + C$ *(C4)*

18. There is a non-zero solution if and only if

$$\begin{vmatrix} 2 & -2 & k \\ 1 & 0 & 4 \\ k & 1 & 1 \end{vmatrix} = 0$$ *(R1)*

$\Rightarrow 2(-4) + 2(1 - 4k) + k = 0$ *(M1)(A1)*

$\Rightarrow -7k = 6$ or $k = -\frac{6}{7}$ *(A1)*

Answer: $k = -\frac{6}{7}$ *(C4)*

19. $f(x)$ is defined so long as $x^2 - 4 \geq 0$

But $x^2 - 4 \geq 0$ if and only if $|x| \geq 2 \quad \Rightarrow x \leq -2$ or $x \geq 2$ *(M1)*

So the domain is $\{x \in \mathbb{R} | x \leq -2 \text{ or } x \geq 2\}$ *(A1)*

Since, $f(x) = e^{3x^2} + \sqrt{x^2 - 4}$, we find that $f(-2) = f(2) = e^{12}$

Further, we observe that e^{3x^2} and $\sqrt{x^2 - 4}$ increase as $x \geq 2$ or $x \leq -2$ *(M1)*

Also $\lim_{x \to \infty} f(x) = \infty$ and $\lim_{x \to -\infty} f(x) = \infty$

So the range of f is $\{x \in \mathbb{R} | e^{12} \leq x\}$ *(A1)*

Answer: Domain: $\{x \in \mathbb{R} | x \leq -2 \text{ or } x \geq 2\}$ *(C2)*

Range: $\{y \in \mathbb{R} | e^{12} \leq y\}$ *(C2)*

20. (a) Since $z = x + iy$ and $z^* = x - iy$,

$\left|z - 2 - i\sqrt{3}\right| = (\sqrt{2})\left|z^* - 1 + i\sqrt{3}\right|$ is equivalent to

$\left|(x-2) + i(y - \sqrt{3})\right| = (\sqrt{2})\left|(x-1) - i(y - \sqrt{3})\right|$

Thus, we get $\left\{(x-2)^2 + (y-\sqrt{3})^2\right\}^{1/2} = (\sqrt{2})\left\{(x-1)^2 + (y-\sqrt{3})^2\right\}^{1/2}$ *(M1)*

On squaring both sides, we obtain,

$$(x-2)^2 + (y-\sqrt{3})^2 = 2(x-1)^2 + 2(y-\sqrt{3})^2$$

$$\Rightarrow x^2 - 4x + 4 = 2x^2 - 4x + 2 + (y-\sqrt{3})^2 \qquad \textit{(M1)}$$

$$\Rightarrow x^2 + (y-\sqrt{3})^2 = 2 \text{ or } x^2 + y^2 - 2\sqrt{3}y + 1 = 0 \qquad \textit{(A1)}$$

(b) This is a circle of radius $\sqrt{2}$ with its centre at $(0, \sqrt{3})$. *(A1)*

Answers: (a) Equation of the circle is $x^2 + (y-\sqrt{3})^2 = 2$ *(C3)*

or $x^2 + y^2 - 2\sqrt{3}y + 1 = 0$

(b) Centre of the circle is $(0, \sqrt{3})$, radius is $\sqrt{2}$ *(C1)*