Convex Optimization: Fall 2021, Homework #2 (Due: 12/10)

1. (20 points) Show that the following the epigraph problem form is equivalent to the standard problem by using KKT conditions.

   (Standard problem) $\underset{x}{\text{minimize}}\ \ f_0(x)$

   (Epigraph problem) $\underset{x,t}{\text{minimize}}\ \ t$ subject to $f_0(x) \le t$

2. (40 points) We have the following norm approximation problem: $\underset{x}{\text{maximize}}\ \ \|Ax - b\|$

   (a) For $\|Ax - b\| = \|Ax - b\|_\infty$, show that the following LP is an equivalent problem.

   $$\underset{x,t}{\text{minimize}}\ \ t \qquad \text{subject to } -t\mathbf{1} \le Ax - b \le t\mathbf{1}$$

   where $x \in R^n$, $t \in R, A \in R^{m\times n}$, and $\mathbf{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$.

   (b) For $\|Ax - b\| = \|Ax - b\|_1$, show that the following LP is an equivalent problem.

   $$\underset{x,y}{\text{minimize}}\ \ \mathbf{1}^T y \qquad \text{subject to } -y \le Ax - b \le y$$

   where $x \in R^n$, $y \in R^m$, $A \in R^{m\times n}$, and $\mathbf{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$.

3. (20 points) Suppose that $f(x)$ is *concave* of $x = (x_1, \dots, x_k)$ when x is a probability vector (probability mass function). Consider the following optimization problem.

$$\underset{x}{\text{maximize}}\ \ f(x)$$

$$\text{subject to } \textstyle\sum_{i=1}^{k} x_i = 1,\ x_i \ge 0 \text{ for } 1 \le i \le k$$

Then, derive the following condition for the optimal x^* by using KKT conditions:

$$\frac{\partial f(x)}{\partial x_i} = \nu \text{ for } i \text{ such that } x_i > 0;$$

$$\frac{\partial f(x)}{\partial x_i} \le \nu \text{ for } i \text{ such that } x_i = 0;$$

4. (20 points) For an underdetermined linear equation: $Ax = b$ where $A \in R^{p\times n}$ where $p < n$ and $\text{rank}(A) = p$. In order to transform an equality-constrained optimization problem into an unconstrained optimization problem, we parametrize the affine feasible set: $\{x|Ax = b\} = \{Fz + \hat{x}|z \in R^{n-p}\}$.

Suppose that the first p columns of A are independent, i.e., $A = [A_1\ \ A_2]$ where $A_1 \in R^{p\times p}$ is nonsingular. Then, show that $F = \begin{bmatrix} -A_1^{-1}A_2 \\ I \end{bmatrix}$ and $\hat{x} = \begin{bmatrix} A_1^{-1}b \\ 0 \end{bmatrix}$.

5. (Optional) Boyd 10.1 (a) (p. 557) [Nonsingularity of the KKT matrix]