Convex Optimization: Fall 2021, Homework #1 (Due: 10/08, 11 pm)

1. (10 points) Positive definite matrix A
   a. Show that the eigenvalues of A are positive
   b. Show that the determinant of A equals the product of its eigenvalues, i.e., $\det(A) = \prod_{i=1}^{n} \lambda_i$.
   c. Show that the positive definite matrix A is invertible.

2. (10 points) Show that the matrix $\mathrm{X} = \begin{bmatrix} x & y \\ y & z \end{bmatrix}$ (where $x \geq 0,\ z \geq 0,\ xz \geq y^2$) is positive semidefinite. (Hint: Use the fact that the summation of eigenvectors is the same as the summation of diagonal terms, i.e., $\mathrm{tr(X)} = \sum_i X_{ii} = \sum_i \lambda_i$.)

3. (10 points) Textbook (2.5)

4. (10 points) Textbook (2.12: a, b, c, d)

5. (10 points) Textbook (2.16)

6. (10 points) Let X_{opt} be the set of all optimal solutions of the following convex optimization:

$$X_{\mathrm{opt}} = \begin{array}{ll} \text{argmin} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

where $f_0(x)$ and $f_i(x)$ are convex functions. Show that X_{opt} is a convex set.

7. (10 points) Show that:

   a. Exponential e^{ax} is convex on **R**, for any $a \in \mathbf{R}$.
   b. Powers x^a is convex on $\mathbf{R}_{++}$ when $a \geq 1$ or $a \leq 0$, and concave for $0 \leq a \leq 1$.
   c. Entropy function (in information theory) $f(x) = -\sum_{i=1}^{n} x_i \log x_i$ with $\mathrm{dom}\, f(x) = \{x \in R_{+t}^{n} | \sum_{i=1}^{n} x_i = 1\}$ is strictly concave.
   d. Exponential e^{x^2} is convex on **R**.

8. (10 points) The Kullback-Leibler divergence is given by $D_{KL}(p, q) = \sum_{i=1}^{n} p_i \log \frac{p_i}{q_i}$ where p_i and q_i represents two distributions. Show that $D_{KL}(p, q) \geq 0$. (Hint: Use Jensen's inequality)

9. (10 points) Derive the conjugates of the following functions:

   a. $f(x) = |x|$
   b. $f(x) = \exp(x)$

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10. (10 points) Show that $f: \mathbf{R} \to \mathbf{R}$ is convex if and only if $f''(x) \geq 0$.

11. (10 points) Textbook (3.19)

12. (10 points) Textbook (3.21)