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MODELLING OF DRIFT WAVE TURBULENCE WITH A FINITE ION TEMPERATURE GRADIENT

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Abstract—With the use of consistent orderings in $\varepsilon \sim \rho_s/a$ and $\delta \sim k_\perp \rho_s$ model equations are derived for the drift instabilities from the electrostatic two-fluid equations. The electrical resistivity η included in the system allows the dynamics of both the collisional drift wave instability ($\eta \neq 0$) and the collisionless ion temperature gradient driven instability ($\eta = 0$). The model equations also include effects of sheared velocity flows in the equilibrium plasma. The model equations used extensively in earlier nonlinear studies are obtained as appropriate limits of the model equations derived in the present work. The effects of electron temperature fluctuations are also discussed.

1. INTRODUCTION

IN THIS WORK we consider a derivation of reduced nonlinear fluid equations for the description of drift wave turbulence and vortices in low beta confinement systems with a finite ion temperature gradient and magnetic shear. Numerous earlier works on drift waves contain more specialized derivations depending on a particular ordering of the several small parameters in the system. Here we generalize these earlier results in several aspects being careful to distinguish between the equilibrium expansion parameters and perturbation expansion parameters. Furthermore, inclusion of finite resistivity and electron temperature fluctuation enables us to study characteristics of the ion temperature gradient driven drift instability (i.e. the so-called η_i-mode) under more general conditions. In order to consider situations relevant to current Tokamak experiments, we also assume that ambient sheared ion flows may exist in the plasma. On the other hand, for the reasons of simplicity, we neglect effects associated with trapped particles. Although kinetic effects such as those associated with ion Landau damping and trapped electrons are likely to play a prominent role in some circumstances, we only discuss the fluid model here, assuming that the fluid model approximates the dynamics of the strongly destabilized mode reasonably well, as is commonly believed.

One of the earliest accounts of the instability caused by ion temperature gradients is found in the paper by RUDAKOV and SAGDEEV (1961), where it is shown that the growth of the "ionic electrostatic" wave is caused by "a continuous inflow of heat from a region with a high unperturbed temperature into the region where the temperature is rising on account of the compression due to the plasma wave" under the conditions of zero density gradient [$n(x) = \text{const}$] and finite temperature gradient [$T_i(x) = T_e(x) \neq \text{const}$]. More detailed discussions on the fluid and kinetic models of this instability, the dispersion relation, the critical value of η_i for the marginal stability and localization of the mode are presented by KADOMTSEV and POGUTSE (1965), based on the local approximation. COPPI *et al.* (1967) have derived the integral equation of

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the instability due to ion temperature gradients from the Vlasov equation, using the normal mode analysis. The fluid limit of this kinetic model is also discussed therein. On the other hand, the model equations for instabilities of the collisional drift wave due to ion and electron temperature gradients are derived and discussed by HINTON and HORTON (1971) and also by HORTON and VARMA (1972), based on the two-fluid equations (BRAGINSKII, 1965) with the effects of resistivity, viscosity and thermal conductivity. The ion temperature gradient driven drift instability, modelled in a slightly different way from previous model equations and called the ion-mixing mode by COPPI and SPIGHT (1978), is used to explain the rate of density rise observed when a neutral gas is fed into a Tokamak plasma during a stable discharge (COPPI and SPIGHT, 1978; ANTONSEN *et al.*, 1979).

Simple fluid model equations of drift waves in the absence of the ion and electron temperature gradients are derived in the collisionless limit by HASEGAWA and MIMA (1978) and in the collisional limit by HASEGAWA and WAKATANI (1983) and WAKATANI and HASEGAWA (1984). These sets of equations provide simple models of plasma turbulence, from which one can relatively easily perform mode-coupling analyses and study plasma turbulence properties such as wavenumber spectra. Horton *et al.* (1980) have also presented a simple set of fluid equations of the ion temperature gradient driven turbulence in order to assess the anomalous ion thermal transport. In their model, the electron temperature gradient effects are excluded and only the three scalar fields of fluctuations, the electric potential $\tilde{\phi}$, the ion pressure $\tilde{p}$ and the parallel ion velocity $\tilde{v}_{\parallel}$, are involved. Several other simple fluid models have been proposed for the study of drift wave turbulence under various conditions [for example, see HORTON *et al.* (1978), TERRY and HORTON (1983), and also HORTON *et al.* (1981) and BROCK and HORTON (1982)].

There are two different branches of the ion temperature gradient driven mode. One is called "slab type," which is the drift wave coupled with the ion acoustic wave that is destabilized by the local ion temperature gradient. The other is called "interchange type" (HORTON *et al.*, 1981; BROCK and HORTON, 1982), which is destabilized by bad curvature of the magnetic field lines in the presence of the finite ion temperature gradients. Since several experimental results suggested that the turbulence associated with these two branches of the ion temperature gradient driven mode was likely to be the cause of the anomalous thermal transport observed in Tokamaks and stellarators [for example, see BROWER *et al.* (1987), SÖLDER *et al.* (1988) and FONK *et al.* (1989)], numerous detailed studies of the ion temperature gradient driven mode have been presented.

The goal of this work is to derive a set of reduced equations governing the slab-type ion temperature gradient driven mode, which is generalization of the model equations of HORTON *et al.* (1980). Although it is possible to derive the interchange-type ion temperature gradient mode equations from the reduced equations (50)–(53) derived in Section 2, we do not present the final form of the interchange-type equations in the present work. The readers who are interested in the interchange-type equations are suggested to refer to, for example, HORTON *et al.* (1981) and BROCK and HORTON (1982). When the effect of magnetic shear is stronger than the effect of magnetic field curvature, the slab-type ion temperature gradient driven instability is predicted to be excited and become a dominant source of the experimentally observed anomalous heat transport (HAMAGUCHI and HORTON, 1990a,b).

In deriving the reduced equations, we start from the compressible two-fluid equations and ignore fluctuations of magnetic field and electron temperature, as in HORTON *et al.* (1980). It is also assumed that the mode is localized on a particular magnetic field line (i.e. $k_\perp \gg k_\parallel$) and that the typical frequency and growth rate of the mode are much smaller than the ion cyclotron frequency. The background fields such as the mean ion temperature gradient and the ambient magnetic field vary slowly in time and space, compared to the fluctuations. The specific ordering of physical quantities of this mode is given in Subsection 2.2 as the ε-ordering. This ordering significantly simplifies the model equations as summarized in Section 3.

2. BASIC EQUATIONS

2.1. *Electrostatic two-fluid transport equations*

We start from the electrostatic two-fluid equations (BRAGINSKII, 1965). For low-frequency modes with wavelengths longer than the Debye length, we may assume charge neutrality and discard the Poisson equations (the plasma approximation). Namely, we take $n_i = n_e$ and allow $\nabla \cdot \mathbf{E} \neq 0$. We also consider the case of zero electron mass limit (i.e. $m_e \to 0$) and constant electron temperature T_e. Effects of electron temperature fluctuations will be discussed in Section 4. The set of equations then becomes

$$m_i n_i \left(\frac{\partial}{\partial t} + \mathbf{v}_i \cdot \nabla\right)\mathbf{v}_i = -\nabla p_i + e n_i \left(-\nabla\Phi + \frac{\mathbf{v}_i}{c} \times \mathbf{B}\right) - e n_i \eta \mathbf{j} - \nabla \cdot \mathbf{\Pi}_i \tag{1}$$

$$0 = -T_e \nabla n_i - e n_i \left(-\nabla\Phi + \frac{\mathbf{v}_e}{c} \times \mathbf{B}\right) + e n_i \eta \mathbf{j} \tag{2}$$

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \mathbf{v}_i) = 0 \tag{3}$$

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{v}_e) = 0 \quad (n_i = n_e) \tag{4}$$

$$\frac{\partial p_i}{\partial t} + \mathbf{v}_i \cdot \nabla p_i + \gamma p_i \nabla \cdot \mathbf{v}_i = -(\gamma - 1)(\nabla \cdot \mathbf{q}_i + \mathbf{\Pi}_i : \nabla \mathbf{v}_i) \tag{5}$$

where

$$p_i = n_i T_i \qquad \mathbf{j} = e n_i (\mathbf{v}_i - \mathbf{v}_e) \tag{6}$$

$$\mathbf{q}_i = -\frac{\boldsymbol{\kappa}}{\gamma - 1} \nabla T_i + \frac{\gamma}{(\gamma - 1)} \frac{n_i T_i}{\omega_{ci} m_i} (\hat{\mathbf{b}} \times \nabla T_i) \tag{7}$$

and

$$\mathbf{\Pi}_i = -\nu_{\|i}\mathbf{W}^{(0)} - \nu_{\perp i}^{(1)}\mathbf{W}^{(1)} - \nu_{\perp i}^{(2)}\mathbf{W}^{(2)} + \nu_i^{FLR}\mathbf{W}^{FLR}. \tag{8}$$

Equations (3) and (4) yield

$$\nabla \cdot \mathbf{j} = 0. \tag{9}$$

In the equations above, the subscripts i and e denote the corresponding quantities of ions and electrons, respectively. The magnitude of the electron charge is denoted as e, ω_{ci} is the ion cyclotron frequency $\omega_{ci} = ZeB/m_i c$, Z is the ratio of the ion charge to e, γ is the ratio of the specific heats, c, n_i, m, p, T, Φ and $\mathbf{v}$ are the light velocity, the number density, mass, pressure, temperature, electrostatic potential and velocity, respectively. The magnetic field $\mathbf{B}$ is assumed to be time-independent, satisfying $\nabla \cdot \mathbf{B} = 0$, and $\hat{\mathbf{b}} = \mathbf{B}/|\mathbf{B}|$ is the unit vector in the $\mathbf{B}$-direction. Since the electrostatic limit is considered, we may ignore the Maxwell equations. Therefore the current density $\mathbf{j}$ given by equation (6) and satisfying equation (9) need not satisfy $\mathbf{j} = c\nabla \times \mathbf{B}/4\pi$. We note that the electron thermal energy balance equation is not included in system (1)–(8) since a constant electron temperature T_e has been assumed. In the expression for the ion heat flux $\mathbf{q}_i$ of equation (7), $\boldsymbol{\kappa}$ denotes the heat conductivity tensor.

The traceless tensor $\mathbf{\Pi}_i$ consists of the gyroviscous tensor $\mathbf{\Pi}_i^{FLR} = -\nu_i^{FLR}\mathbf{W}^{FLR}$ and the collisional stress tensors $\mathbf{\Pi}_i^{col} = -\nu_{\|i}\mathbf{W}^{(0)} - \nu_{\perp i}^{(1)}\mathbf{W}^{(1)} - \nu_{\perp i}^{(2)}\mathbf{W}^{(2)}$. In Cartesian coordinates, the (α, β) component of $\mathbf{\Pi}_i$ defined in equation (8) is given (BRAGINSKII, 1965) by

$$W_{\alpha\beta}^{(0)} = \tfrac{3}{2}(b_\alpha b_\beta - \tfrac{1}{3}\delta_{\alpha\beta})(b_\mu b_\nu - \tfrac{1}{3}\delta_{\mu\nu})\sigma_{\mu\nu} \tag{10}$$

$$W_{\alpha\beta}^{(1)} = (\delta_{\alpha\mu}^{\perp}\delta_{\beta\nu}^{\perp} - \tfrac{1}{2}\delta_{\alpha\beta}^{\perp}\delta_{\mu\nu}^{\perp})\sigma_{\mu\nu} \tag{11}$$

$$W_{\alpha\beta}^{(2)} = (\delta_{\alpha\mu}^{\perp}b_\beta b_\nu + \delta_{\beta\nu}^{\perp}b_\alpha b_\mu)\sigma_{\mu\nu} \tag{12}$$

$$W_{\alpha\beta}^{(3)} = \tfrac{1}{2}(\delta_{\alpha\mu}^{\perp}\varepsilon_{\beta\gamma\nu} + \delta_{\beta\nu}^{\perp}\varepsilon_{\alpha\gamma\mu})b_\gamma\sigma_{\mu\nu} \tag{13}$$

$$W_{\alpha\beta}^{(4)} = (b_\alpha b_\mu \varepsilon_{\beta\gamma\nu} + b_\beta b_\nu \varepsilon_{\alpha\gamma\mu})b_\gamma\sigma_{\mu\nu}, \tag{14}$$

and

$$W_{\alpha\beta}^{FLR} = W_{\alpha\beta}^{(3)} + 2W_{\alpha\beta}^{(4)}, \tag{15}$$

where b_α ($\alpha = 1, 2, 3$) is the α component of $\hat{\mathbf{b}}$, $\sigma_{\mu\nu}$ is the (μ, ν) component of the rate-of-strain tensor

$$\sigma_{\mu\nu} = \partial_\nu v_{i\mu} + \partial_\mu v_{i\nu} - \tfrac{2}{3}\delta_{\mu\nu}\nabla \cdot \mathbf{v}_i,$$

$\partial_\nu = \partial/\partial x_\nu$, $\delta_{\mu\nu}$ is the unit or the Kronecker delta, $\delta_{\mu\nu}^{\perp} = \delta_{\mu\nu} - b_\mu b_\nu$, $\varepsilon_{\alpha\beta\gamma}$ is an anti-symmetric unit tensor and $\nu_i^{FLR} = p_i/2\omega_{ci}$. In the limit of a strong magnetic field ($\omega_{ci}\tau_i \gg 1$, where τ_i is the ion–ion collision time), the viscosity coefficients are given by $\nu_{\|i} = 0.96 n_i T_i \tau_i$, $\nu_{\perp i}^{(2)} = 0.30 n_i T_i/\omega_{ci}^2 \tau_i$ and $\nu_{\perp i}^{(1)} = \nu_{\perp i}^{(2)}/4$. The divergence of the tensor in equation (1) is defined as $(\nabla \cdot \mathbf{\Pi}_i)_\alpha = \partial_\beta \mathbf{\Pi}_{i\alpha\beta}$ and the contraction in equation (5) is defined as $\mathbf{\Pi}_i : \nabla \mathbf{v}_i = \mathbf{\Pi}_{i\alpha\beta} \partial_\beta v_{i\alpha}$. We note that $\mathbf{\Pi}_i : \nabla \mathbf{v}_i$ may be calculated from the formula

$$\mathbf{W}^{(l)} : \nabla \mathbf{v}_i = \tfrac{1}{2} \operatorname{tr} (\mathbf{W}^{(l)})^2 \quad (l = 1, 2, 3)$$

and, in particular,

$$\mathbf{W}^{\mathrm{FLR}} : \nabla \mathbf{v}_i = 0,$$

Namely, the gyroviscosity does not produce heat.

The parallel stress tensor $\mathbf{\Pi}_{\|i} = -\nu_{\|i} \mathbf{W}^{(0)}$ can be written in an explicit vector form as

$$\mathbf{\Pi}_{\|i} = -3\nu_{\|i} (\hat{\mathbf{b}}\hat{\mathbf{b}} - \tfrac{1}{3}\mathbf{I})\lambda, \tag{16}$$

where

$$\lambda = \hat{\mathbf{b}} \cdot ((\hat{\mathbf{b}} \cdot \nabla)\mathbf{v}_i) - \tfrac{1}{3} \nabla \cdot \mathbf{v}_i \tag{17}$$

and $\mathbf{I}$ is the unit tensor. Using equations (16) and (17), we obtain

$$\mathbf{\Pi}_{\|i} : \nabla \mathbf{v}_i = -3\nu_{\|i} \lambda^2.$$

The other parts of the stress tensor generally have no simple vector expressions if $\hat{\mathbf{b}}$ is a function of space. However, if the dependence of $\hat{\mathbf{B}}$ on the space coordinates $\mathbf{x}$ is weak in the sense that $|\partial_\alpha v_{i\beta}|/|v_{i\beta}| \gg |\partial_\mu b_\nu|$ (α, β, μ, $\nu = 1, 2, 3$) or, in other words, the space derivatives of the velocity $\mathbf{v}_i$ are much larger than the space derivatives of the unit vector $\hat{\mathbf{b}}$, then the divergence of the collisional stress tensor $\nabla \cdot \mathbf{\Pi}_i^{\mathrm{col}}$ in equation (1) may be calculated from the following expressions:

$$\nabla \cdot \mathbf{W}^{(0)} = 3\hat{\mathbf{b}}(\hat{\mathbf{b}} \cdot \nabla)\lambda - \nabla\lambda + \mathcal{O}(\varepsilon^*) \tag{18}$$

$$\nabla \cdot \mathbf{W}^{(1)} = \nabla_\perp \cdot \nabla \mathbf{v}_\perp + \mathcal{O}(\varepsilon^*) \tag{19}$$

$$\nabla \cdot \mathbf{W}_\|^{(2)} = \hat{\mathbf{b}}(\partial_\| (\nabla_\perp \cdot \mathbf{v}) + \nabla_\perp \cdot \nabla v_\|) + \partial_\| (\nabla_\perp v_\|) + \partial_\|^2 \mathbf{v}_\perp + \mathcal{O}(\varepsilon^*). \tag{20}$$

Here $\mathcal{O}(\varepsilon^*)$ denotes terms smaller than the leading terms by order of $\varepsilon^* \simeq |\partial_\mu b_\nu|/(|\partial_\alpha v_\beta|/|v_\beta|)$. We have also used $\mathbf{v}_\perp = \delta^\perp_{\alpha\beta} v_\beta = \mathbf{v} - \hat{\mathbf{b}}(\hat{\mathbf{b}} \cdot \mathbf{v})$, $v_\| = \hat{\mathbf{b}} \cdot \mathbf{v}$, $\nabla_\perp = \nabla - \hat{\mathbf{b}}(\hat{\mathbf{b}} \cdot \nabla)$ and $\partial_\| = \hat{\mathbf{b}} \cdot \nabla$. We note that equations (18)–(20) hold exactly if the unit vector $\hat{\mathbf{b}}$ is constant (i.e. independent of $\mathbf{x}$). In equations (18)–(20), the subscripts i were omitted for simplicity. The divergence of the gyroviscous tensor $\nabla \cdot \mathbf{\Pi}_i^{\mathrm{FLR}}$ is discussed in Appendix A.

In the system of equations (1)–(5), we retain the resistivity $\eta = m_e/n_i e^2 \tau_e$, where τ_e [$\tau_e^{-1} = 0.51 \nu_{ei}^{\mathrm{Br}}$ in terms of the collision frequency ν_{ei}^{Br} given by Braginskii (1965)] is the electron collision time, as a possible nonzero coefficient while the electron mass and the electron diffusion coefficients are all set to be zero. The condition that the friction force $\mathbf{R} = n_i e\eta \mathbf{j}$ in equations (1) and (2) has a significant effect on the dynamics is given by the following argument. In the parallel component of equation (2) to the magnetic field $\mathbf{B}$, balancing the first term to the last term on the right-hand side yields the relation $k_\parallel n_i T_e \sim n_i e\eta j_\parallel \sim (n_i e)^2 \eta v_{\parallel i}$. We also expect from equation (3) that the time derivative of the density is of the same order as the parallel density flux, or $\omega n_i \sim k_\parallel n_i v_\parallel$. Here $k_\parallel$ and ω denote the parallel component of the wavenumber vector $\mathbf{k}$ to $\mathbf{B}$ and a typical frequency of the mode, respectively. Eliminating $v_{\parallel i}$ from these two relations, we obtain the condition for the collisional drift mode

$$\omega \sim \frac{k_\parallel^2 T_e}{m_e \nu_e},$$

where $\nu_e = \tau_e^{-1}$ is the electron collision frequency. Since, as will be shown later, the typical frequency ω of drift waves is given by the electron diamagnetic frequency $\omega_e^* = (T_e/m_i\omega_{ci})(k_y/L_n)$ with L_n being the ion density gradient scale length, the condition above may be written as

$$\omega_e^* \sim \frac{k_\parallel^2 T_e}{m_e \nu_e}. \tag{21}$$

Since $T_e/m_e = \lambda_{mfp}^2 \nu_e^2$, where $\lambda_{mfp} = \nu_e^{-1}\sqrt{T_e/m_e}$ is the mean free path of the electrons, the right-hand side of condition (21) represents a frequency of electron diffusion in the length $k_\parallel^{-1}$ along the magnetic field lines. On the other hand, the condition for the collisionless drift mode [i.e. $\eta \to 0$ in equations (1) and (2)] is given by

$$\omega_e^* \ll \frac{k_\parallel^2 T_e}{m_e \nu_e}, \tag{22}$$

that is, the wave motion is sufficiently slower than the electron diffusion along the magnetic field lines. These two limiting cases—the collisional drift mode and the collisionless drift mode—will be discussed in more detail later. We note that, in the case of a flat density profile ($L_n \to \infty$), the ion temperature gradient scale length L_T should be taken as a macroscopic scale length (HAMAGUCHI and HORTON, 1990b), instead of L_n.

2.2. *The ε-ordering*

We now simplify equations (1)–(9) by specifying a certain ordering of physical quantities. Since we are concerned with low-level fluctuations of the plasma governed by equations (1)–(9), each physical quantity may be split into two parts: the mean part, which varies slowly in time and space, and the perturbed part, which fluctuates in time and varies rapidly in space. For example, $p_i = p_{i0} + \tilde{p}_i$, where the subscript 0 denotes the mean part and the tilde denotes the fluctuating part. As typical scales (HORTON *et al.*, 1990), we choose the ion cyclotron frequency ω_{ci} to measure the frequency and the sound speed $c_s = (T_e/m_i)^{1/2}$ to measure the velocity. The lengths are then measured with the ion inertial scale length $\rho_s = c_s\omega_{ci}^{-1}$ and the electric potential is measured with T_e/e.

Since typical fusion plasmas satisfy the condition that $\rho_s \ll a$, where a denotes a macroscopic scale length such as the minor radius of a toroidal confinement device, we use $\varepsilon = \rho_s/a$ as an essential small parameter in the system. The smallnesses of fluctuating quantities are then assumed to be of order ε or

$$\tilde{\mathbf{v}}_i/c_s \sim \tilde{\mathbf{v}}_e/c_s \sim \tilde{n}_i/n_{i0} \sim \tilde{p}_i/p_{i0} \sim e\tilde{\Phi}/T_e \sim \varepsilon$$

whereas the mean quantities are of order 1, such as $p_{i0}/n_{i0}T_e = \mathcal{O}(1)$. The variation of the fluctuating and mean quantities in space may be characterized by the following orderings of space derivatives: the perpendicular derivative $\nabla_\perp$ is of order 1 [i.e. $\rho_s\nabla_\perp = \mathcal{O}(1)$] and the parallel derivatives $\partial_\parallel$ is of order ε [i.e. $\rho_s\partial_\parallel = \mathcal{O}(\varepsilon)$] when these operators are applied to fluctuating quantities, whereas $\rho_s\partial_\perp = \mathcal{O}(\varepsilon)$ and $\rho_s\partial_\parallel = \mathcal{O}(\varepsilon^2)$ when these operators are applied to the mean quantities.

For fluctuations localized on a particular rational surface at $\psi = \psi_s$, where ψ denotes an appropriate magnetic flux coordinate, the mean ion velocity $\mathbf{v}_i$ may be expanded around the rational surface as

$$\mathbf{v}_{i0}(\psi) = \mathbf{v}_{i0}(\psi_s) + (\psi - \psi_s)\,\partial\mathbf{v}_{i0}/\partial\psi_s + \dots.$$

The second term of this expansion may be regarded as being of order ε; i.e. $|\psi-\psi_s|/|\psi_s| = \mathcal{O}(\varepsilon)$ in the neighborhood of the rational surface. On the moving frame with constant velocity $\mathbf{V}_c \equiv \mathbf{v}_{i0}(\psi_s)$, the mean velocity $\mathbf{v}_{i0}$ may be expressed as $\mathbf{v}_{i0} = (\psi-\psi_s)\;\partial\mathbf{v}_i/\partial\psi_s + \cdots$ of $\mathcal{O}(\varepsilon)$, which significantly simplifies the system of equations by eliminating terms involving $\mathbf{V}_c$. From now on, therefore, we always use this moving (inertial) frame to describe the system of equations. On this moving coordinate system, we may assume that $\mathbf{v}_{i0}/c_s \sim \rho_s\nabla_\perp\mathbf{v}_{i0}/c_s = \mathcal{O}(\varepsilon)$, $\mathbf{v}_{i0}(\psi = \psi_s) = 0$, $\rho_s\partial_\parallel\mathbf{v}_{i0}/c_s = \mathcal{O}(\varepsilon^2)$, $e\Phi_0/T_e = \mathcal{O}(\varepsilon)$ and time derivatives $\omega_{ci}^{-1}\;\partial/\partial t$ of the fluctuating and mean quantities are of order ε and ε^3, respectively.

We now consider the relative sizes of the diffusion terms of equations (1)–(5), assuming that the diffusion coefficients are given in the limit of high collisionality and strong magnetic field. With the use of the natural units of the mode introduced above, the scales of the diffusion coefficients of equations (7) and (8) are then given by

$$\eta\frac{e^2 n_i}{m_i\omega_{ci}} \simeq (\omega_{ce}\tau_e)^{-1} \tag{23}$$

$$\frac{\nu_{\parallel i}}{m_i n_i \rho_s c_s} \simeq \left(\frac{T_i}{T_e}\right)\omega_{ci}\tau_i \tag{24}$$

$$\frac{\nu_{\perp i}^{(1)}}{m_i n_i \rho_s c_s} \simeq \left(\frac{T_i}{T_e}\right)(\omega_{ci}\tau_i)^{-1} \tag{25}$$

$$\frac{\nu_i^{\mathrm{FLR}}}{m_i n_i \rho_s c_s} \simeq \left(\frac{T_i}{T_e}\right) \tag{26}$$

$$\frac{\kappa_{\parallel i}}{n_i \rho_s c_s} \simeq \left(\frac{T_i}{T_e}\right) \omega_{ci} \tau_i \tag{27}$$

$$\frac{\kappa_{\perp i}}{n_i \rho_s c_s} \simeq \left(\frac{T_i}{T_e}\right) (\omega_{ci} \tau_i)^{-1} \tag{28}$$

$$\frac{n_i T_i}{\omega_{ci} m_i} \Big/ n_i \rho_s c_s \simeq \left(\frac{T_i}{T_e}\right), \tag{29}$$

where $\nu_{\parallel i} = n_i T_i \tau_i$, $\nu_{\perp i}^{(2)} \simeq 4\nu_{\perp i}^{(1)} \simeq n_i T_i/\omega_{ci}^2 \tau_i$, $\kappa_{\parallel i} \simeq n_i T_i \tau_i / m_i$, $\kappa_{\perp i} \simeq n_i T_i / m_i \omega_{ci}^2 \tau_i$ and $\boldsymbol{\kappa}\nabla = \kappa_{\perp i}\nabla_\perp + \kappa_{\parallel i}\nabla_\parallel$. From equations (21) and (22), the ratio of the electron collision frequency τ_e^{-1} to the electron cyclotron frequency $\omega_{ce} = eB/m_e c$ satisfies

$$(\omega_{ce}\tau_e)^{-1} \sim \frac{(k_\parallel L_n)^2}{k_y L_n} \sim \varepsilon \quad \text{for the collisional drift mode} \tag{30}$$

and

$$(\omega_{ce}\tau_e)^{-1} \ll \frac{(k_\parallel L_n)^2}{k_y L_n} \sim \varepsilon \quad \text{for the collisionless drift mode.} \tag{31}$$

Here we have used $k_\parallel = \mathcal{O}(L_n^{-1})$ and $k_y L_n = \mathcal{O}(\varepsilon^{-1})$. [In the case of a flat density profile ($L_n \to \infty$), we should take (HAMAGUCHI and HORTON, 1990b) $k_\parallel = \mathcal{O}(L_T^{-1})$ and $k_y L_T = \mathcal{O}(\varepsilon^{-1})$.]

For the collisional drift mode, therefore, the diffusion coefficients have the following scalings under the relevant normalization specified above:

Collisional drift mode

$$\eta \sim \varepsilon,$$

$$\nu_{\parallel i} \sim \kappa_{\parallel i} \sim \frac{1}{\varepsilon Z^2}\left(\frac{m_e}{m_i}\right)^{1/2}\left(\frac{T_i}{T_e}\right)^{5/2},$$

$$\nu_{\perp i}^{(1)} \sim \kappa_{\perp i} \sim \varepsilon Z^2 \left(\frac{m_i}{m_e}\right)^{1/2}\left(\frac{T_e}{T_i}\right)^{1/2},$$

$$\nu_i^{\mathrm{FLR}} \sim \frac{n_i T_i}{\omega_{ci} m_e} \sim \frac{T_i}{T_e}.$$

Here we have used equation (30) and the relation $(\omega_{ci}\tau_i)/(\omega_{ce}\tau_e) \simeq (m_e/m_i)^{1/2} \times (T_i/T_e)^{3/2} Z^{-2}$.

When the collisionality of the plasma is low, some expressions of the diffusion coefficients used for a collisional plasma need to be modified. In particular, the parallel diffusion coefficients $\nu_{\parallel i}$ and $\kappa_\parallel$ should be chosen to model collisionless ion Landau damping effects (HAMAGUCHI and HORTON, 1990a,b; HAMMETT and PERKINS, 1990).

For the collisionless drift mode, therefore, we assume that the resistivity and all the perpendicular diffusion coefficients are given by the classical collision theory as in equations (25) and (28), whereas $\nu_{\|i}$ and $\kappa_{\|i}$ are given as quantities of order 1. From equation (30), we have the following conditions for the collisionless drift mode:

Collisionless drift mode

$$\eta \ll \varepsilon, \qquad \nu_{\perp i}^{(1)}, \nu_{\perp i}^{(2)}, \kappa_{\perp i} \ll \varepsilon Z^2 \left(\frac{m_i}{m_e}\right)^{1/2} \left(\frac{T_e}{T_i}\right)^{1/2}, \qquad \nu_i^{\mathrm{FLR}} \sim \frac{n_i T_i}{\omega_{ci} m_i} \sim \frac{T_i}{T_e}.$$

Under the scaling assumptions of fluctuation magnitude described above, equations (18)–(20) may be further simplified and the following expressions for the stress tensor are obtained with the use of the formulae (B-2) and (B-3) in Appendix B:

$$\begin{aligned} -(\nabla\cdot\mathbf{\Pi})_\perp &= -\nu_\|\nabla_\perp\lambda + \nu^{\mathrm{FLR}}\nabla_\perp\chi + \mathcal{O}(\varepsilon^2) \\ &= -\nu_\|\nabla_\perp\lambda + \nu_\perp^{(1)}\nabla_\perp^2\mathbf{v}_\perp + \nu^{\mathrm{FLR}}\nabla_\perp\chi \\ &\quad -\nu^{\mathrm{FLR}}(\hat{\mathbf{b}}\times\nabla(\nabla\cdot\mathbf{v}_\perp) - 2\partial_\|(\nabla\times\mathbf{v})_\perp) + 2(\hat{\mathbf{b}}\times(\nabla_\perp\nu^{\mathrm{FLR}}\cdot\nabla)\mathbf{v}) \\ &\quad + (\hat{\mathbf{b}}\cdot(\nabla\times\mathbf{v}))\nabla_\perp\nu^{\mathrm{FLR}} + 2\hat{\mathbf{b}}(\nabla\nu^{\mathrm{FLR}}\times\hat{\mathbf{b}})\cdot\nabla v_\| + \mathcal{O}(\varepsilon^3), \\ -(\nabla\cdot\mathbf{\Pi})_\| &= 2\nu_\|\,\partial_\|\lambda + \nu_\perp^{(2)}\nabla_\perp^2 v_\| + 2\nu^{\mathrm{FLR}}\partial_\|\chi + 2(\nabla\nu^{\mathrm{FLR}}\times\hat{\mathbf{b}})\cdot\nabla v_\| + \mathcal{O}(\varepsilon^3), \end{aligned} \tag{33}$$

where

$$\chi = \hat{\mathbf{b}}\cdot(\nabla\times\mathbf{v}) = \hat{\mathbf{b}}\cdot(\nabla\times\mathbf{v}_\perp) + \mathcal{O}(\varepsilon^*), \qquad \lambda = \partial_\| v_\| - \tfrac{1}{3}(\nabla\cdot\mathbf{v}),$$

and the subscripts i are omitted for simplicity. As we will show shortly afterwards the divergence of $\mathbf{v}_i$ is of $\mathcal{O}(\varepsilon^2)$ and, therefore, $\lambda \sim \mathcal{O}(\varepsilon^2)$. In estimating the order of each term in equations (32) and (33), we have used the collisional drift mode scalings of the diffusion coefficients without taking into account the magnitude of the factor $(m_e/m_i)^{1/2}$; i.e. $\nu_\| \sim \varepsilon^{-1}$, $\nu_\perp^{(1)} \sim \nu_\perp^{(2)} \sim \varepsilon$, and $\nu^{\mathrm{FLR}} \sim 1$. Therefore, the first term of equation (32) $\nu_{\|i}\nabla_\perp\lambda \sim \varepsilon$, the second term $\nu_{\perp i}^{(1)}\nabla_\perp^2\mathbf{v}_{\perp i} \sim \varepsilon^2$, the third term $\nu_i^{\mathrm{FLR}}\nabla_\perp\chi \sim \varepsilon$ and the fourth term is of order ε^2. For equation (33) all the terms are of $\mathcal{O}(\varepsilon^2)$.

We now derive equations for fluctuating quantities, assuming that the mean quantities are given. We first note that taking the lowest order of the continuity equation (3) yields $\nabla\cdot\mathbf{v}_i = \mathcal{O}(\varepsilon^2)$. Therefore, we need to determine the perpendicular ion flow velocity $\mathbf{v}_{\perp i}$ up to $\mathcal{O}(\varepsilon^2)$ as well as $v_{\|i}$ up to $\mathcal{O}(\varepsilon)$ in order to estimate $\nabla\cdot\mathbf{v}_i$ to lowest order. Writing $\mathbf{v}_{\perp i} = \mathbf{v}_\perp^{(0)} + \mathbf{v}_\perp^{(1)} + \mathcal{O}(\varepsilon^3)$ with $\mathbf{v}^{(0)} = \mathcal{O}(\varepsilon)$ and $\mathbf{v}^{(1)} = \mathcal{O}(\varepsilon^2)$, we require that $\mathbf{v}^{(0)}$ satisfies

$$-\nabla_\perp p_i + en_i\left(-\nabla_\perp\Phi + \frac{\mathbf{v}_\perp^{(0)}}{c}\times\mathbf{B}\right) - \nu_{\|i}\nabla_\perp\lambda^{(0)} + \nu_0^{\mathrm{FLR}}\nabla_\perp\hat{\mathbf{b}}\cdot(\nabla\times\mathbf{v}_\perp^{(0)}) = 0, \tag{34}$$

where $\nu_0^{\mathrm{FLR}} = p_0/2\omega_{ci}$ and $\lambda^{(0)} = \partial_\| v_{\|i} - (1/3)\nabla\cdot(\mathbf{v}^{(0)} + \mathbf{v}^{(1)})$. Equation (34) is the lowest order contribution from the perpendicular components of equation (1). It follows from equation (34) that we may write

$$\mathbf{v}_{\perp}^{(0)} = \mathbf{v}_{\mathrm{E}}+\mathbf{v}_{\mathrm{d}}+\mathbf{v}_{\mathrm{F}}, \tag{35}$$

where

$$\mathbf{v}_{\mathrm{E}} = \mathrm{c}\frac{\hat{\mathbf{b}}\times\nabla_{\perp}\Phi}{B}, \tag{36}$$

$$\mathbf{v}_{\mathrm{d}} = \mathrm{c}\frac{\hat{\mathbf{b}}\times\nabla_{\perp}p_i}{\mathrm{e}n_{\mathrm{i}}B} \tag{37}$$

and $\mathbf{v}_{\mathrm{F}}$ satisfies

$$\mathrm{e}n_{\mathrm{i}}\left(\frac{\mathbf{v}_{\mathrm{F}}}{\mathrm{c}}\times\mathbf{B}\right) = v_{\parallel\mathrm{i}}\nabla_{\perp}\lambda^{(0)}-v_0^{\mathrm{FLR}}\nabla_{\perp}\hat{\mathbf{b}}\cdot(\nabla\times\mathbf{v}_{\perp}^{(0)}). \tag{38}$$

In the case of the collisionless drift mode where $v_{\parallel\mathrm{i}}\sim 1$, the term $v_{\parallel\mathrm{i}}\nabla_{\perp}\lambda^{(0)}$ should be dropped from equations (34) and (38). It should be noted that $\mathbf{v}_{\perp}^{(0)}$ contains non-fluctuating mean flows $\mathbf{v}_{\mathrm{E}0}$ and $\mathbf{v}_{\mathrm{d}0}$.

In Appendix B, it is shown that the following identity [equation (B-8)] holds:

$$\frac{1}{\omega_{\mathrm{ci}}}\hat{\mathbf{b}}\times\nabla_{\perp}p_{\mathrm{i}}\cdot\nabla v_{\parallel\mathrm{i}}+(\nabla\cdot\mathbf{\Pi}_{\mathrm{i}}^{\mathrm{FLR}})_{\parallel} = -\frac{p_{\mathrm{i}}}{\omega_{\mathrm{ci}}}\partial_{\parallel}(\hat{\mathbf{b}}\cdot\nabla\times\mathbf{v}_{\mathrm{i}})+\mathcal{O}(\varepsilon^3). \tag{39}$$

The parallel component of equation (1) then becomes to lowest order

$$m_{\mathrm{i}}n_{\mathrm{i}0}\left(\frac{\partial\tilde{v}_{\parallel\mathrm{i}}}{\partial t}+(\mathbf{v}_{\mathrm{E}}+\mathbf{v}_{\mathrm{F}})\cdot\nabla v_{\parallel\mathrm{i}}\right) = -\partial_{\parallel}p_{\mathrm{i}}-\mathrm{e}n_{\mathrm{i}0}\,\partial_{\parallel}\Phi-\frac{p_{\mathrm{i}0}}{\omega_{\mathrm{ci}}}\partial_{\parallel}(\hat{\mathbf{b}}\cdot\nabla\times\mathbf{v}_{\perp}^{(0)})$$
$$-\mathrm{e}n_{\mathrm{i}0}\eta j_{\parallel}+2v_{\parallel\mathrm{i}}\,\partial_{\parallel}\lambda^{(0)}+v_{\perp\mathrm{i}}^{(2)}\Delta_{\perp}v_{\parallel\mathrm{i}}. \tag{40}$$

The electron momentum equation yields

$$0 = -T_{\mathrm{e}}\nabla_{\perp}n_{\mathrm{i}}-\mathrm{e}n_{\mathrm{i}}\left(-\nabla_{\perp}\Phi+\frac{\mathbf{v}_{\perp\mathrm{e}}}{\mathrm{c}}\times\mathbf{B}\right), \tag{41}$$

$$0 = -T_{\mathrm{e}}\,\partial_{\parallel}n_{\mathrm{i}}+\mathrm{e}n_{\mathrm{i}}\,\partial_{\parallel}\Phi+\mathrm{e}n_{\mathrm{i}}\eta j_{\parallel}. \tag{42}$$

From equations (3), (5) and (9), we have

$$\frac{\partial\tilde{n}_{\mathrm{i}}}{\partial t}+\mathbf{v}_{\mathrm{E}}\cdot\nabla n_{\mathrm{i}}+n_{\mathrm{i}0}\nabla\cdot\mathbf{v}_{\mathrm{E}}+\nabla\cdot(n_{\mathrm{i}}\mathbf{v}_{\mathrm{d}}+n_{\mathrm{i}}\mathbf{v}_{\mathrm{F}})+n_{\mathrm{i}0}(\nabla\cdot\mathbf{v}_{\perp}^{(1)})+n_{\mathrm{i}0}\,\partial_{\parallel}v_{\parallel\mathrm{i}} = 0, \tag{43}$$

$$\frac{\partial\tilde{p}_{\mathrm{i}}}{\partial t}+(\mathbf{v}_{\mathrm{E}}+\mathbf{v}_{F})\cdot\nabla p_{i}+\gamma p_{i0}\nabla\cdot(\mathbf{v}_{\mathrm{E}}+\mathbf{v}_{F})$$
$$+\gamma p_{\mathrm{i}0}\nabla\cdot\mathbf{v}_{\perp}^{(1)}+\gamma p_{\mathrm{i}0}\,\partial_{\parallel}v_{\parallel\mathrm{i}}+\gamma\nabla\cdot\frac{\hat{\mathbf{b}}\times\nabla(p_{\mathrm{i}}T_{\mathrm{i}})}{m_{\mathrm{i}}\omega_{\mathrm{ci}}} = \nabla\cdot(\boldsymbol{\kappa}\nabla\tilde{T}_{\mathrm{i}}) \tag{44}$$

and

$$\nabla \cdot \mathbf{j} = 0. \tag{45}$$

In deriving equation (44), the following identity is used:

$$\gamma p_i \nabla \cdot \mathbf{v}_d + (\gamma - 1)\nabla \cdot \mathbf{q}^{\mathrm{FLR}} = \gamma \nabla \cdot \frac{\hat{\mathbf{b}} \times \nabla (p_i T_i)}{m_i \omega_{ci}},$$

where

$$\mathbf{q}^{\mathrm{FLR}} = \frac{\gamma}{\gamma - 1} \cdot \frac{p_i}{m_i \omega_{ci}} (\hat{\mathbf{b}} \times \nabla T_i).$$

From equation (41), the perpendicular component of the electron flow velocity is given by

$$\mathbf{v}_{\perp e} = \mathbf{v}_E - \mathbf{v}_{de},$$

where

$$\mathbf{v}_{de} = c\frac{\hat{\mathbf{b}} \times T_e \nabla n_i}{e n_i B}.$$

Since $\mathbf{j} = en_i(\mathbf{v}_i - \mathbf{v}_e)$, equation (45) may be written as

$$\nabla \cdot (n_i(\mathbf{v}_d + \mathbf{v}_{de})) + \nabla \cdot (n_i \mathbf{v}_F) + n_{i0} \nabla \cdot \mathbf{v}_\perp^{(1)} + e^{-1} \partial_\parallel j_\parallel = 0 \tag{46}$$

to lowest order. This equation gives a relationship between $\mathbf{v}_\perp^{(1)}$ and $j_\parallel$.

The higher order correction $\mathbf{v}_\perp^{(1)}$ to the velocity field may be calculated from equation (1). Writing down the terms of equation (1) up to $\mathcal{O}(\varepsilon^2)$ with the use of equations (35)–(37), we obtain

$$m_i n_i \left(\frac{\partial}{\partial t} + \mathbf{v}_\perp^{(0)} \cdot \nabla\right)\mathbf{v}_\perp^{(0)} + \frac{en_i}{c}\eta \mathbf{j}_\perp^{(0)} + (\nabla \cdot \mathbf{\Pi}_i)_\perp = \frac{en_i}{c}(\mathbf{v}_F + \mathbf{v}_\perp^{(1)}) \times \mathbf{B}, \tag{47}$$

where $\mathbf{j}_\perp^{(0)} = en(\mathbf{v}_d + \mathbf{v}_{de}) = c\hat{\mathbf{b}} \times \nabla_\perp (p_i + n_i T_e)/B$. Applying $\nabla \cdot \hat{\mathbf{b}}/m_i\omega_{ci} \times$ to equation (47) yields

$$\nabla \cdot \hat{\mathbf{b}} \times \left(\frac{n_i}{\omega_{ci}}\left(\frac{\partial}{\partial t} + \mathbf{v}_\perp^{(0)} \cdot \nabla\right)(\mathbf{v}_E + \mathbf{v}_F) + \frac{1}{m_i \omega_{ci}}\nabla_\perp \zeta + \frac{1}{m_i \omega_{ci}}\nabla \cdot \mathbf{\Pi}_i^{\mathrm{col}}\right)$$
$$= \nabla \cdot (n_i \mathbf{v}_F) + n_{i0} \nabla \cdot \mathbf{v}_\perp^{(1)}, \tag{48}$$

where we use the following relation

$$m_i n_i \left(\frac{\partial}{\partial t} + \mathbf{v}_\perp^{(0)} \cdot \nabla\right)\mathbf{v}_d + (\nabla \cdot \mathbf{\Pi}^{\mathrm{FLR}})_\perp = \frac{1}{\omega_{ci}} \hat{\mathbf{b}} \times \nabla \left(\frac{\partial}{\partial t}\tilde{p}_i + \mathbf{v}_\perp^{(0)} \cdot \nabla p_i\right) + \nabla_\perp \zeta \tag{49}$$

with

$$\zeta = -\frac{p_i}{2\omega_{ci}} \hat{\mathbf{b}} \cdot \nabla \times \mathbf{v}_\perp^{(0)}.$$

Equation (49) is shown to hold up to $\mathcal{O}(\varepsilon^2)$ in Appendix B [equation (B-7)]. Using equation (46) as well as the relation

$$j_\parallel = \frac{T_e}{e\eta}\left(\frac{\partial_\parallel n_i}{n_{i0}} - \frac{e}{T_e}\partial_\parallel \Phi\right)$$

obtained from equation (42), we rewrite equation (48) as

$$\nabla \cdot \hat{\mathbf{b}} \times \left(\frac{n_i}{\omega_{ci}}\left(\frac{\partial}{\partial t}\mathbf{v}_\perp^{(0)} \cdot \nabla\right)(\mathbf{v}_E + \mathbf{v}_F) + c\frac{n_i \eta}{B}\mathbf{j}_\perp^{(0)} + \frac{1}{m_i \omega_{ci}}\nabla_\perp \zeta + \frac{1}{m_i \omega_{ci}}\nabla \cdot \mathbf{\Pi}_i^{\mathrm{col}}\right)$$

$$- \frac{1}{\omega_{ci}^2 m_i}\Delta_\perp\left(\frac{\partial}{\partial t}\tilde{p}_i + (\mathbf{v}_E + \mathbf{v}_F)\cdot \nabla p_i\right)$$

$$= -\nabla \cdot (n(\mathbf{v}_d + \mathbf{v}_{de})) + \frac{1}{e\eta}\left(\partial_\parallel^2 \tilde{\Phi} - \frac{T_e}{e n_{i0}}\partial_\parallel^2 \tilde{n}_i\right). \tag{50}$$

Similarly, equations (40), (43) and (44) may be written as

$$m_i n_{i0}\left(\frac{\partial \tilde{v}_{\parallel i}}{\partial t} + (\mathbf{v}_E + \mathbf{v}_F)\cdot \nabla v_{\parallel i}\right)$$

$$= -\partial_\parallel p_i - T_e \partial_\parallel n_i - \frac{p_{i0}}{\omega_{ci}}\partial_\parallel(\hat{\mathbf{b}} \cdot \nabla \times \mathbf{v}_\perp^{(0)}) + 2 v_{\parallel i}\partial_\parallel \lambda^{(0)} + v_{\perp i}^{(2)}\Delta_\perp \tilde{v}_{\parallel i} \tag{51}$$

$$\frac{\partial \tilde{n}_i}{\partial t} + \mathbf{v}_E \cdot \nabla n_i + n_{i0}(\nabla \cdot \mathbf{v}_E) - \nabla \cdot (n_i \mathbf{v}_{de}) + n_{i0}\partial_\parallel v_{\parallel i} + \frac{1}{e\eta}\left(\partial_\parallel^2 \tilde{\Phi} - \frac{T_e}{e n_{i0}}\partial_\parallel^2 \tilde{n}_i\right) = 0 \tag{52}$$

$$\frac{\partial \tilde{p}_i}{\partial t} + (\mathbf{v}_E + \mathbf{v}_F)\cdot \nabla p_i + \gamma p_{i0}(\nabla \cdot \mathbf{v}_E + \partial_\parallel v_{\parallel i}) - \gamma T_i \nabla \cdot (n_i(\mathbf{v}_d + \mathbf{v}_{de})) - \gamma T_i \mathbf{v}_F \cdot \nabla n_i$$

$$+ \frac{\gamma T_i}{e\eta}\left(\partial_\parallel^2 \tilde{\Phi} - \frac{T_e}{e n_{i0}}\partial_\parallel^2 \tilde{n}_i\right) + \gamma \nabla \cdot \frac{\hat{\mathbf{b}} \times \nabla (p_i T_i)}{m_i \omega_{ci}} = \nabla \cdot (\boldsymbol{\kappa} \nabla \tilde{T}_i). \tag{53}$$

Equations (50)–(53) form the evolution equations for the fluctuating quantities $\tilde{\Phi}$, $\tilde{n}_i$, $\tilde{p}_i$ and $\tilde{v}_{\parallel i}$.

The perturbed parallel current $\tilde{j}_\parallel$ is related to $\tilde{n}_i$ and $\tilde{\Phi}$ in the following manner. Dividing equation (42) by n_i and integrating the resulting equation, we obtain

$$n_i = n_{i0}(\alpha, \beta) \exp\left(\frac{e\tilde{\Phi}}{T_e} + \frac{e}{T_e}\eta \int_{s_0}^{s} \tilde{j}_\parallel \, ds'\right),$$

where α and β denote general magnetic coordinates and s' denotes a distance along the field line $\mathbf{B} = \nabla\alpha \times \nabla\beta$. The integration constant is chosen in such a way that $n_{i0}(\alpha, \beta)$ represents the mean number density. In the case where $e\tilde{\Phi}/T_e$, $e\eta \int \tilde{j}_\parallel \, ds'/T_e \ll 1$, we obtain

$$\frac{\tilde{n}_i}{n_{i0}} = \frac{e\tilde{\Phi}}{T_e} + \frac{e\eta}{T_e} \int_{s_0}^{s} \tilde{j}_\parallel \, ds'.$$

2.3. *Long wavelength approximation*

In this section we introduce a subsidiary ordering, assuming that wavelengths of the modes are much longer than ρ_s, that is, $k_\perp \rho_s \ll 1$. More precisely, using a small parameter δ satisfying $\varepsilon \ll \delta \ll 1$, we assume that $\rho_s \nabla_\perp = \mathcal{O}(\delta)$, $\rho_s \nabla_\parallel = \mathcal{O}(\delta^2\varepsilon)$ and $\omega_{ci}^{-1}\, \partial/\partial t = \mathcal{O}\,(\delta^2\varepsilon)$ when these operators are applied to fluctuating quantities and $\rho_s \nabla_\perp = \mathcal{O}(\delta\varepsilon)$ and $\rho_s \nabla_\parallel = \mathcal{O}(\delta^2\varepsilon^2)$ when these operators are applied to mean quantities. Since $(k_\parallel L_n)^2/(k_y L_n) = \mathcal{O}(\varepsilon\delta^2)$ under these assumptions, we have, from equation (23)–(31), $\eta \sim \delta^3\varepsilon$, $\nu_{\perp i}^{(1)} \sim \nu_{\perp i}^{(2)} \sim \varepsilon\delta^2$ and $\nu_{\parallel i} \sim (\varepsilon\delta^2)^{-1}$ for the collisional drift mode, where $(m_e/m_i)^{1/2} = \mathcal{O}(\delta)$ is assumed. It follows from equation (36)–(38) that $\mathbf{v}_\perp^{(1)} \sim \mathbf{v}_E \sim \mathbf{v}_d \sim \delta\varepsilon$, $\lambda^{(0)} \sim \delta^2\varepsilon^2$ and $\mathbf{v}_F \sim \delta^3\varepsilon$ and we obtain $\mathbf{v}_\perp^{(1)} \sim \delta^3\varepsilon^2$ from equation (47). Since $\nabla \cdot \mathbf{v}_i \sim \delta^2\varepsilon^2$ from equation (3), it follows that $\tilde{v}_{\parallel i} \sim \tilde{j}_\parallel \sim \varepsilon$. The terms $\nabla \cdot \mathbf{v}_\perp^{(1)}$ and $\nabla \cdot \mathbf{v}_F$, which are of $\mathcal{O}(\delta^4\varepsilon^2)$, are neglected in equation (43) and (44) under the subsidiary orderings in this section. Taking the lowest order contributions from equations (42), (51), (43) and (44), we obtain

$$\partial_\parallel \left(\frac{\tilde{n}_i}{n_{i0}} - \frac{e\tilde{\Phi}}{T_e}\right) = 0 \tag{54}$$

$$m_i n_{i0} \left(\frac{\partial \tilde{v}_{\parallel i}}{\partial t} + \mathbf{v}_E \cdot \nabla v_{\parallel i}\right) = -\partial_\parallel p_i - T_e \partial_\parallel n_i + 2\nu_{\parallel i} \partial_\parallel \lambda^{(0)} + \nu_{\perp i}^{(2)} \Delta_\perp \tilde{v}_{\parallel i} \tag{55}$$

$$\frac{\partial \tilde{n}_i}{\partial t} + \mathbf{v}_E \cdot \nabla n_i + n_{i0} \nabla \cdot \mathbf{v}_E + \nabla \cdot (n_i \mathbf{v}_d) + n_{i0}\, \partial_\parallel v_{\parallel i} = 0 \tag{56}$$

$$\frac{\partial \tilde{p}_i}{\partial t} + \mathbf{v}_E \cdot \nabla p_i + \gamma p_{i0} (\nabla \cdot \mathbf{v}_E + \partial_\parallel v_{\parallel i}) + \gamma \nabla \cdot \frac{\hat{\mathbf{b}} \times \nabla(p_i T_i)}{eB} = \nabla \cdot (\boldsymbol{\kappa} \nabla \tilde{T}_i), \tag{57}$$

where

$$\lambda^{(0)} = \tfrac{2}{3}\partial_\| v_{\|i} - \tfrac{1}{3}\nabla\cdot(\mathbf{v}_E+\mathbf{v}_d). \tag{58}$$

2.4. *Cold ion approximation*

In this section we consider the case where the ion temperature is significantly lower than the electron temperature, i.e. $\varepsilon \ll \delta = T_i/T_e \ll 1$. In this case, the ion Larmor radius $\rho_i = (T_i/m_i)^{1/2}/\omega_{ci}$ becomes much smaller than ρ_s, and therefore we assume that the perpendicular wavenumber $k_\perp$ satisfies $k_\perp\rho_s = \mathcal{O}(1)$. We also assume that the ion temperature gradient can be much greater than the typical gradient of a mean quantity or $a/L_p = \mathcal{O}(\delta^{-1})$ $(L_p^{-1} = \partial \ln p_{i0}/\partial r)$, so that the perturbed ion pressure $\tilde{p}$ could scale as $\tilde{p}/p_{i0} = \mathcal{O}(\varepsilon/\delta) \ll 1$. In this scaling, we have $\nabla_\perp p_{i0} \sim \nabla_\perp \tilde{p}_i \sim \nabla_\perp \tilde{T}_i \sim \varepsilon$, $p_{i0} \sim v_i^{\mathrm{FLR}} \sim \delta$ and $\zeta \sim \delta\varepsilon$. Although the diffusion coefficients $v_{\|i}$, $v_{\perp i}^{(1)}$, $v_{\perp i}^{(2)}$, $k_\|$ and $k_\perp$ have a T_i/T_e dependence, we ignore this dependence and assume that $\eta \sim \varepsilon$, $v_{\|i} \sim \kappa_\| \sim \varepsilon^{-1}$ and $v_{\perp i}^{(1)} \sim v_{\perp i}^{(2)} \sim \kappa_\perp \sim \varepsilon$ as in Section 2.2 so that the terms involving the diffusion coefficients are still kept in the lowest order equations. Under these assumptions, it follows from equations (35)–(38) that $\mathbf{v}_F$ becomes of order $\delta\varepsilon$ whereas $\mathbf{v}_E$ and $\mathbf{v}_d$ are of order ε. The polarized drift velocity $\mathbf{v}_p = \mathbf{v}_F + \mathbf{v}_\perp^{(1)}$ is then obtained from equations (47) and (49) to the lowest order as

$$\mathbf{v}_p = \hat{\mathbf{b}} \times \left(\frac{1}{\omega_{ci}}\left(\frac{\partial}{\partial t} + (\mathbf{v}_E+\mathbf{v}_d)\cdot\nabla\right)\mathbf{v}_E + \frac{c\eta}{B}\mathbf{j}_\perp^{(0)} + \frac{1}{n_i m_i \omega_{ci}^2}\hat{\mathbf{b}}\times\nabla\left(\frac{\partial}{\partial t}\tilde{p}_i + \mathbf{v}_E\cdot\nabla p_i\right) + \frac{1}{e n_i m_i \omega_{ci}}\nabla\cdot\mathbf{\Pi}_i^{\mathrm{col}}\right).$$

Taking the lowest order components of equations (50)–(53), we obtain

$$\begin{aligned}\nabla\cdot\hat{\mathbf{b}}\times\left(\frac{n_i}{\omega_{ci}}\left(\frac{\partial}{\partial t} + (\mathbf{v}_E+\mathbf{v}_d)\cdot\nabla\right)\mathbf{v}_E + \frac{c n_i}{B}\eta\mathbf{j}_\perp^{(0)} + \frac{1}{m_i\omega_{ci}}\nabla\cdot\mathbf{\Pi}_i^{\mathrm{col}}\right)\\ + \frac{1}{m_i\omega_{ci}^2}\Delta_\perp\left(\gamma\nabla\cdot\frac{\hat{\mathbf{b}}\times\nabla(p_i T_i)}{m_i\omega_{ci}} - \nabla\cdot(\boldsymbol{\kappa}\nabla\tilde{T}_i)\right)\\ = -\nabla\cdot(n_i(\mathbf{v}_d+\mathbf{v}_{de})) + \frac{1}{e\eta}\left(\partial_\|^2\tilde{\Phi} - \frac{T_e}{e n_{i0}}\partial_\|^2\tilde{n}_i\right),\end{aligned} \tag{59}$$

$$m_i n_{i0}\left(\frac{\partial\tilde{v}_{\|i}}{\partial t} + \mathbf{v}_E\cdot\nabla v_{\|i}\right) = -\partial_\| p_i - T_e\,\partial_\| n_i + 2v_{\|i}\,\partial_\|\lambda^{(0)} + v_\perp^{(2)}\Delta_\perp\tilde{v}_{\|i}, \tag{60}$$

$$\frac{\partial\tilde{n}_i}{\partial t} + \mathbf{v}_E\cdot\nabla n_i + n_{i0}(\nabla\cdot\mathbf{v}_E) - \nabla\cdot(n_i\mathbf{v}_{de}) + n_{i0}\partial_\| v_\| + \frac{1}{e\eta}\left(\partial_\|^2\tilde{\Phi} - \frac{T_e}{e n_{i0}}\partial_\|^2\tilde{n}_i\right) = 0, \tag{61}$$

$$\frac{\partial\tilde{p}_i}{\partial t} + \mathbf{v}_E\cdot\nabla p_i + \gamma\nabla\cdot\frac{\hat{\mathbf{b}}\times\nabla(p_i T_i)}{m_i\omega_{ci}} = \nabla\cdot(\boldsymbol{\kappa}\nabla\tilde{T}_i). \tag{62}$$

Equation (62) has been used in deriving equation (59) from equation (50).

3. SLAB MODELS WITH MAGNETIC SHEAR

In this section we consider a simple geometry of magnetic field and further simplify the basic equation of the ion temperature gradient driven mode obtained in the previous section. In the usual Cartesian coordinate system (x,y,z) with the unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$, we assume that the magnetic field is given by

$$\mathbf{B}(x) = B_z\left(\hat{\mathbf{z}}+\frac{x-x_0}{L_s}\hat{\mathbf{y}}\right),$$

where L_s denotes the shear scale length and the equation $x = x_0$ gives the magnetic surface on which the modes are considered to be localized. This sheared slab magnetic field models a local magnetic field configuration near a mode rational surface in a fusion device, such as Tokamaks and reversed field pinches. In this section we are particularly concerned with the weak shear case ($a \ll L_s$, where a denotes a typical macroscopic length such as the minor radius of a Tokamak), which is appropriate for a Tokamak with small toroidal curvature. Under this weak shear assumption, we have $B \equiv |\mathbf{B}(x)| \simeq B_z$ and all the terms which have the form $\nabla\cdot(\hat{\mathbf{b}}\times\nabla f)$ such as $\nabla\cdot\mathbf{v}_E$ and $\nabla\cdot n\mathbf{v}_d$ drop from the mode equations since these terms can be shown to be too small. Here we also assume that all the mean quantities are functions of only x and evaluated at the mode rational surface $x = x_0$. In particular, we use the following space scales

$$L_n = -\left(\frac{\mathrm{d}}{\mathrm{d}x}\ln n_{i0}\right)^{-1}, \qquad L_T = -\left(\frac{\mathrm{d}}{\mathrm{d}x}\ln T_{i0}\right)^{-1}$$

and the nondimensional parameters

$$\eta_i = \frac{L_n}{L_T}, \qquad K = \frac{T_i}{T_e}(1+\eta_i), \qquad \Gamma = \frac{\gamma T_i}{T_e},$$

$$S_\perp = \frac{L_n V_0'}{c_s}, \qquad S_\parallel = \frac{L_n}{c_s}\frac{\mathrm{d}v_{\parallel 0}}{\mathrm{d}x},$$

where the perpendicular shear flow V_0 is assumed to be given by the $\mathbf{E}\times\mathbf{B}$ flow caused by the mean electric potential

$$\Phi_0 = \frac{1}{2c}(x-x_0)^2 B V_0'$$

with $V_0' = \mathrm{d}V_0/\mathrm{d}x$ evaluated at $x = x_0$. The appropriate nondimensional space–time variables are

$$\tilde{x} = \frac{x-x_0}{\rho_s}, \quad \tilde{y} = \frac{y}{\rho_s}, \quad \tilde{z} = \frac{z}{L_n}, \qquad \tilde{t} = \frac{tc_s}{L_n}$$

and the nondimensional dependent variables are

$$\phi = \frac{e\tilde{\Phi}}{T_e}\frac{L_n}{\rho_s}, \qquad n = \frac{\tilde{n}_i}{n_{i0}}\frac{L_n}{\rho_s},$$

$$v = \frac{\tilde{v}_{\parallel i}}{c_s}\frac{L_n}{\rho_s}, \qquad p = \frac{\tilde{p}_i T_i}{p_{i0} T_e}\frac{L_n}{\rho_s}.$$

In the case of a flat density profile ($L_n \to \infty$), L_T should be taken as a typical macroscopic length, and the normalization stated above should be changed accordingly. For a detailed discussion on normalization appropriate for a flat density profile, see HAMAGUCHI and HORTON (1990b).

With the use of the nondimensional variables defined above, equations (54)–(58) of the long wavelength drift waves may be further simplified in slab geometry. Equation (54), which becomes $\partial_\parallel(\phi - n) = 0$ in slab geometry, yields

$$\phi = n. \tag{63}$$

Using this relation, we derive from equations (55)–(58)

$$\frac{\partial \phi}{\partial \tilde{t}} = -(1+S_\perp \tilde{x})\frac{\partial \phi}{\partial \tilde{y}} - \tilde{\partial}_\parallel v \tag{64}$$

$$\frac{\partial v}{\partial \tilde{t}} + \{\phi, v\} = -S_\perp \tilde{x}\frac{\partial v}{\partial \tilde{y}} + S_\parallel \frac{\partial \phi}{\partial \tilde{y}} - \tilde{\partial}_\parallel(p+n) + \mu_\perp^{(2)}\tilde{\Delta}_\perp v + \mu_\parallel \tilde{\partial}_\parallel \lambda^{(0)}, \tag{65}$$

$$\frac{\partial p}{\partial \tilde{t}} + \{\phi, p\} = -S_\perp \tilde{x}\frac{\partial p}{\partial \tilde{y}} - K\frac{\partial}{\partial \tilde{y}} - \Gamma\,\tilde{\partial}_\parallel v + \chi_\perp \tilde{\Delta}_\perp p + \chi_\parallel \tilde{\partial}_\parallel^2 p, \tag{66}$$

where

$$\tilde{\partial}_\parallel = s\tilde{x}\frac{\partial}{\partial \tilde{y}} + \frac{\partial}{\partial \tilde{z}}, \qquad \tilde{\mathbf{\nabla}}_\perp = \hat{\mathbf{x}}\frac{\partial}{\partial \tilde{x}} + \hat{\mathbf{y}}\frac{\partial}{\partial \tilde{y}}, \qquad \tilde{\Delta}_\parallel = \tilde{\mathbf{\nabla}}_\perp^2$$

and

$$\{f, g\} = \frac{\partial f}{\partial \tilde{x}}\frac{\partial g}{\partial \tilde{y}} - \frac{\partial f}{\partial \tilde{y}}\frac{\partial g}{\partial \tilde{x}}.$$

In deriving equation (66), the term $\mathbf{\nabla}\cdot(\boldsymbol{\kappa}\mathbf{\nabla}\tilde{T}_i)$ of equation (62) is replaced by with $V_0' = \mathrm{d}V_0/\mathrm{d}x$ evaluated at $x = x_0$. The appropriate nondimensional space–time variables are

$$\tilde{x} = \frac{x - x_0}{\rho_s}, \quad \tilde{y} = \frac{y}{\rho_s}, \quad \tilde{z} = \frac{z}{L_n}, \qquad \tilde{t} = \frac{tc_s}{L_n}$$

and the nondimensional dependent variables are

In the case of the low ion temperature drift waves, we obtain the following non-dimensional form of equations (59)–(62):

$$\frac{\partial}{\partial \tilde{t}}\tilde{\Delta}_\perp \phi + \{\phi, \tilde{\Delta}_\perp \phi\} = (K - S_\perp \tilde{x})\frac{\partial}{\partial \tilde{y}}\tilde{\Delta}_\perp \phi + S_\perp \frac{\partial^2 p}{\partial \tilde{x}\,\partial \tilde{y}} + \{\tilde{\Delta}_\perp \phi, p\}$$
$$+ \left\{\frac{\partial \phi}{\partial \tilde{x}}, \frac{\partial p}{\partial \tilde{x}}\right\} + \left\{\frac{\partial \phi}{\partial \tilde{y}}, \frac{\partial p}{\partial \tilde{y}}\right\} - \frac{1}{\sigma_\parallel}\tilde{\Delta}_\perp (p+n) - \tilde{\Delta}_\perp(\chi_\perp \tilde{\Delta}_\perp p + \chi_\parallel \tilde{\partial}_\parallel^2 p)$$
$$- \sigma_\parallel \tilde{\partial}_\parallel^2(\phi - n) + \mu_\perp^{(1)}(\phi + p) \quad (67)$$

$$\frac{\partial n}{\partial \tilde{t}} + \{\phi, n\} = -S_\perp \tilde{x}\frac{\partial n}{\partial \tilde{y}} - \frac{\partial \phi}{\partial \tilde{y}} - \tilde{\partial}_\parallel v - \sigma_\parallel \tilde{\partial}_\parallel^2(\phi - n) \quad (68)$$

$$\frac{\partial v}{\partial \tilde{t}} + \{\phi, v\} = -S_\perp \tilde{x}\frac{\partial v}{\partial \tilde{y}} + S_\parallel \frac{\partial \phi}{\partial \tilde{y}} - \partial_\parallel (p+n) + \mu_\perp^{(2)}\tilde{\Delta}_\perp v + \mu_\parallel \tilde{\partial}_\parallel \lambda^{(0)} \quad (69)$$

$$\frac{\partial p}{\partial t} + \{\phi, p\} = -S_\perp \tilde{x}\frac{\partial p}{\partial \tilde{y}} - K\frac{\partial \phi}{\partial \tilde{y}} + \chi_\perp \tilde{\Delta}_\perp p + \chi_\parallel \tilde{\partial}_\parallel^2 p. \quad (70)$$

Here the normalized electric conductivity $\sigma_\parallel$ and perpendicular viscosity $\mu_\perp^{(1)}$ are given by

$$\sigma_\parallel = \frac{m_i \omega_{ci} \rho_s}{n_{i0}\eta e^2 L_n} \qquad \text{and} \qquad \mu_\perp^{(1)} = \frac{\nu_\perp^{(1)} L_n}{m_i n_{i0} c_s \rho_s^2}.$$

We note that equations (63) and (64) may be reduced from equations (67) and (68), respectively, under the long wavelength approximation, whereas equation (70) of the perturbed pressure p is the $\Gamma \to 0$ limit of equation (66). The equations for the parallel velocity v [equations (65) and (69)] are the same in both of the approximations. Exploiting these similarities of both the sets of equations, we combine these equations and propose the following set of equations, which holds both in the long wavelength approximation and the low ion temperature approximation in the sheared slab magnetic field. From now on, we will drop the tildes from the independent variables for simplicity in this section:

$$\frac{\partial}{\partial t}\Delta_\perp \phi + \{\phi, \Delta_\perp \phi\} = (K - S_\perp x)\frac{\partial}{\partial y}\Delta_\perp \phi + S_\perp \frac{\partial^2 p}{\partial x\,\partial y}$$
$$+ \{\Delta_\perp \phi, p\} + \left\{\frac{\partial \phi}{\partial x}, \frac{\partial p}{\partial x}\right\} + \left\{\frac{\partial \phi}{\partial y}, \frac{\partial p}{\partial y}\right\} + \partial_\parallel j + \mu_\perp^{(1)}\phi, \quad (71)$$

$$\frac{\partial n}{\partial t} + \{\phi, n\} = -S_\perp x\frac{\partial n}{\partial y} - \frac{\partial \phi}{\partial y} - \partial_\parallel v + \partial_\parallel j, \quad (72)$$

$$\frac{\partial v}{\partial t} + \{\phi, v\} = -S_\perp x\frac{\partial v}{\partial y} - S_\parallel \frac{\partial \phi}{\partial y} - \partial_\parallel (p+n) + \mu_\perp^{(2)}\Delta_\perp v + \mu_\parallel \partial_\parallel^2 v, \quad (73)$$

$$\frac{\partial p}{\partial t}+\{\phi,p\} = -S_{\perp}x\frac{\partial p}{\partial y}-K\frac{\partial \phi}{\partial y}-\Gamma\,\partial_{\parallel}v+\chi_{\perp}\Delta_{\perp}p+\chi_{\parallel}\,\partial_{\parallel}^{2}p \tag{74}$$

where the nondimensionalized parallel electric current is given by

$$j/\sigma_{\parallel} = \partial_{\parallel}(n-\phi). \tag{75}$$

In equation (71), the diffusion terms $\sigma_{\parallel}^{-1}\Delta_{\perp}(p+n)$, $\Delta_{\perp}(\chi_{\perp}\Delta_{\perp}p+\chi_{\parallel}\partial_{\parallel}^{2}p)$ and $\mu_{\perp}^{(1)}\Delta_{\perp}^{2}p$, which are kept in equation (67), have been dropped for reasons of simplicity. Although dropping these diffusion terms from equation (71) is not consistent from the point of view of the collisional drift wave ordering discussed in Section 2.2, it does not change the relevant dynamics of strongly destabilized modes, which tend to have smaller wavenumbers $k_{\perp}$ and $k_{\parallel}$. We also note that since plasmas in most fusion devices are either collisionless or marginally collisional, the diffusion coefficients $\mu_{\perp}^{(1)}$, $\mu_{\perp}^{(2)}$, $\chi_{\perp}$ and $\sigma_{\parallel}^{-1}$ are small whereas the parallel diffusion coefficients $\mu_{\parallel}$ and $\chi_{\parallel}$ are of order unity, which model the ion Landau damping effects. The diffusion terms $\mu_{\perp}^{(1)}$ $\Delta_{\perp}^{2}\phi$ in equation (71), $\mu_{\perp}^{(2)}\Delta_{\perp}v$ in equation (73) and $\chi_{\perp}\Delta_{\perp}p$ in equation (74) are, however, retained as the energy sinks of high $k_{\perp}$ modes in a turbulence state of the mode. In equation (73), $\lambda^{(0)}$ is replaced by $\partial_{\parallel}v_{\parallel}$ and the term $\mu_{\parallel}\,\partial_{\parallel}^{2}v$, together with the term $\chi_{\parallel}\,\partial_{\parallel}^{2}p$ in equation (74), is retained as a simple model of the ion Landau damping effects associated with the parallel motion of the plasma. One would need to use the kinetic equations in order to study the dynamics which could be strongly affected by the diffusion, such as the dynamics of the marginally stable mode. For a discussion on the stability threshold of collisionless ion temperature gradient driven mode, see HAMAGUCHI and HORTON (1990a,b) and HAMMETT and PERKINS (1990).

The domain on which equations (71)–(74) are solved may be given by the cubic box $-L_x \leqslant x \leqslant L_x$, $0 \leqslant y \leqslant L_y$ and $0 \leqslant z \leqslant L_z$. Here L_y and L_z are constants of order unity (note that x, y and z are normalized here: $x/\rho_s \to x$, $y/\rho_s \to y$ and $z/L_n \to z$) whereas L_x is taken to be large enough such that when there is magnetic shear ($L_s \neq \infty$), single helicity modes localized at $x = 0$ decay sufficiently at $|x| \to L_x$. In the case of zero magnetic shear $2L_x$ represents the width of the constant background fields. The boundary conditions of equation (71)–(74) for this domain are that all the dependent variables are assumed to vanish at $x = L_x$ and to be periodic in the y- and z-directions.

The energy balance equation associated with the set of equations (71)–(74) is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}E_t = S_{\perp}\left\langle\frac{\partial p}{\partial x}\frac{\partial \phi}{\partial y}\right\rangle - S_{\parallel}\left\langle v\frac{\partial \phi}{\partial y}\right\rangle - \left\langle n\frac{\partial \phi}{\partial y}\right\rangle - \frac{K}{\Gamma}\left\langle p\frac{\partial \phi}{\partial y}\right\rangle - W_D, \tag{76}$$

where the energy E_t of the fluctuations is given by

$$E_t = \frac{1}{2}\left(\langle n^2\rangle+\langle|\nabla_{\perp}\phi|^2\rangle+\langle v^2\rangle+\frac{1}{\Gamma}\langle p^2\rangle\right) \tag{77}$$

and the energy sink by

$$W_D = \mu_\perp^{(1)}\langle|\Delta_\perp\phi|^2\rangle + \mu_\perp^{(2)}\langle|\nabla_\perp v|^2\rangle + \mu_\parallel^{(2)}\langle|\partial_\parallel v|^2\rangle + \frac{\chi_\perp}{\Gamma}\langle|\nabla_\perp p|^2\rangle + \frac{\chi_\parallel}{\Gamma}\langle|\partial_\parallel p|^2\rangle + \sigma_\parallel\langle|\partial_\parallel(\phi-n)|^2\rangle.$$

Here $\langle\ \rangle$ denotes the space average of the contained quantity over the domain or

$$\langle\ \rangle = \frac{1}{2L_xL_yL_z}\int_{-L_x}^{L_x}\mathrm{d}x\int_0^{L_y}\mathrm{d}y\int_0^{L_z}\mathrm{d}z.$$

If L_x is taken to be ∞, the normalization factor $1/2L_x$ needs to be chosen appropriately (HAMAGUCHI and HORTON, 1990a). The four transport fluxes in equation (76)

$$\left\langle\frac{\partial p}{\partial x}\frac{\partial\phi}{\partial x}\right\rangle \propto \langle\tilde{v}_{\mathrm{d}y}\tilde{v}_{\mathrm{E}x}\rangle, \qquad \left\langle v\frac{\partial\phi}{\partial y}\right\rangle \propto \langle\tilde{v}_\parallel\tilde{v}_{\mathrm{E}x}\rangle, \qquad \left\langle n\frac{\partial\phi}{\partial y}\right\rangle \propto \langle\tilde{n}_\mathrm{i}\tilde{v}_{\mathrm{E}x}\rangle,$$

and

$$\left\langle p\frac{\partial\phi}{\partial y}\right\rangle \propto \langle\tilde{p}_\mathrm{i}\tilde{v}_{\mathrm{E}x}\rangle,$$

where $\tilde{v}_{Ex} = -\mathrm{c}B^{-1}\,\partial\tilde{\Phi}/\partial y$ and $\tilde{v}_{dy} = \mathrm{c}(en_\mathrm{i}B)^{-1}\,\partial\tilde{p}_\mathrm{i}/\partial x$, are proportional to the transverse transports of the y component of the perturbed diamagnetic flow $v_{\mathrm{d}y}$, the perturbed parallel flow $v_{\parallel\mathrm{i}}$, the perturbed density $\tilde{n}_\mathrm{i}$ and the perturbed ion pressure $\tilde{p}_\mathrm{i}$, respectively.

In the collisionless limit or $\sigma_\parallel\to\infty$, the set of equations (71)–(75) may be further simplified. From equations (71), (72) and (75), the adiabatic electron relation $\phi = n$ (or $\tilde{n}_\mathrm{i}/n_{\mathrm{i}0} = \mathrm{e}\tilde{\Phi}/T_\mathrm{e}$) is obtained in the limit $\sigma_\parallel\to\infty$. Eliminating the term $\partial_\parallel j$ from equations (71) and (72) and setting $\phi = n$ yields the evolution equation for ϕ. The set of equations thus obtained provides the fluid model of the ion temperature gradient driven mode or the η_i-mode in sheared magnetic fields with finite sheared flows:

$$\frac{\partial}{\partial t}(1-\Delta_\perp)\phi - \{\phi,\Delta_\perp\phi\} = -(1+K\Delta_\perp)\frac{\partial\phi}{\partial y} - S_\perp x(1-\Delta_\perp)\frac{\partial\phi}{\partial y} - S_\perp\frac{\partial p}{\partial x\,\partial y} - \partial_\parallel v + \{p,\Delta_\perp\phi\} + \left\{\frac{\partial p}{\partial x},\frac{\partial\phi}{\partial x}\right\} + \left\{\frac{\partial p}{\partial y},\frac{\partial\phi}{\partial y}\right\} - \mu_\perp^{(1)}\Delta_\perp^2\phi \quad (78)$$

$$\frac{\partial v}{\partial t} + \{\phi,v\} = -S_\perp x\frac{\partial v}{\partial y} - S_\parallel\frac{\partial\phi}{\partial y} - \partial_\parallel(p+\phi) + \mu_\perp^{(2)}\Delta_\perp v + \mu_\parallel\,\partial_\parallel^2 v \quad (79)$$

$$\frac{\partial p}{\partial t} + \{\phi,p\} = -S_\perp x\frac{\partial p}{\partial y} - K\frac{\partial\phi}{\partial y} - \Gamma\,\partial_\parallel v + \chi_\perp\Delta_\perp p + \chi_\parallel\,\partial_\parallel^2 p. \quad (80)$$

The same domain and the boundary conditions as those for equations (71)–(75) may also be used for equations (78)–(80).

The energy balance equation associated with the set of equations (78)–(80) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{\mathrm{T}} = S_{\perp}\left\langle\frac{\partial p}{\partial x}\frac{\partial \phi}{\partial y}\right\rangle - S_{\parallel}\left\langle v\frac{\partial \phi}{\partial y}\right\rangle - \frac{K}{\Gamma}\left\langle p\frac{\partial \phi}{\partial y}\right\rangle - W_D^0,$$

where

$$E_T = \frac{1}{2}\left(\langle\phi^2\rangle + \langle|\nabla_{\perp}\phi|^2\rangle + \langle v^2\rangle + \frac{1}{\Gamma}\langle p^2\rangle\right)$$

and

$$W_D^0 = \mu_{\perp}^{(1)}\langle|\Delta_{\perp}\phi|^2\rangle + \mu_{\perp}^{(2)}\langle\nabla_{\perp}v|^2\rangle + \mu_{\parallel}^{(2)}\langle|\partial_{\parallel}v|^2\rangle + \frac{\chi_{\perp}}{\Gamma}\langle|\nabla_{\perp}p|^2\rangle + \chi_{\parallel}\Gamma\langle|\partial_{\parallel}p|^2\rangle.$$

We note that in the limit $\sigma_{\parallel} \to \infty$ there is no particle flux ($\langle n\,\partial\phi/\partial y\rangle = 0$) since we have assumed an adiabatic electron response ($n = \phi$).

The instability threshold given by a critical value K_c of K is an important characteristic of the linear instability. Obtaining an expression for K_c from general equations such as equations (71)–(74) or (78)–(80) is a rather formidable task and we do not discuss it in the present work. However, we note that the threshold K_c based on the fluid equations of equations (78)–(80) in the absence of sheared flow ($S_{\perp} = S_{\parallel} = 0$) is obtained in HAMAGUCHI and HORTON (1990a,b), and is compared to the corresponding kinetic calculations. We also note that the perpendicular sheared flow $S_{\perp}$ is shown to have a stabilizing effect on the ion temperature gradient driven instability and the critical value of $S_{\perp,c}$, below which the mode is completely stabilized, is discussed in HAMAGUCHI and HORTON (1991).

4. EFFECTS OF ELECTRON TEMPERATURE FLUCTUATIONS

Since the introduction of the Hasegawa–Wakatani model (HASEGAWA and WAKATANI, 1983) it is common to take the electron temperature as constant. With the neglect of electron temperature fluctuations it is only the parallel electrical resistivity that drives the drift wave branch unstable. As shown by HINTON and HORTON (1971) and HORTON and VARMA (1972), however, in the presence of electron temperature fluctuations the processes of electron parallel diffusivities $\nu_{\parallel e}$ and $\kappa_{\parallel e}$ and the thermo-electron effects $\alpha_{\text{thermo}} n_e\,\partial_{\parallel} T_e$ in the electron momentum equation and $\alpha_{\text{thermo}} T_e\,\partial_{\parallel} j_{\parallel}/e$ in the electron energy balance equation ($\alpha_{\text{thermo}} = 0.71$) also contribute to the growth rate of the collisional drift wave. In particular, it is known that positive η_e ($= L_n/L_{T_e}$) can be a strong stabilizing effect on the collisional drift wave (HORTON, 1990). In this section, we include these effects in the formulation presented in the previous section.

The inclusion of the thermo-electric effect and the electron parallel viscosity generalizes equation (2) to

$$0 = -\partial_{\parallel} n_e T_e + e n_e\,\partial_{\parallel}\Phi + e n_e \eta j_{\parallel} - \beta n_e\,\partial_{\parallel} T_e + \tfrac{4}{3}\nu_{\parallel e}\,\partial_{\parallel}^2 v_{\parallel e} \tag{81}$$

where the electron parallel viscosity $\nu_{\parallel e} = \alpha_{visc} n_e T_e/\nu_e$ with $\alpha_{visc} = 0.4$. The thermal energy balance equation is

$$\frac{3}{2} n_e \left[\frac{\partial}{\partial t} + (\mathbf{v}_E + \mathbf{v}_{\parallel e})\cdot\nabla\right] T_e + n_e T_e \,\partial_\parallel v_{\parallel e} = \partial_\parallel(\kappa_{\parallel e}\,\partial_\parallel T_e) + \frac{\beta}{e} T_e\,\partial_\parallel j_\parallel + \eta j_\parallel^2 \tag{82}$$

with the electron parallel thermal diffusivity

$$\kappa_{\parallel e} = \alpha_{heat}\frac{n_e T_e}{m_e \nu_e}, \qquad \alpha_{heat} = 1.6,$$

where the ratio of specific heats is assumed to be $\gamma = 5/3$. We note that $n_e = n_i$ from charge neutrality.

As in Section 3, we write the electron temperature as $T_e = T_{e0} + \tilde{T}_e$ and the electron parallel flow velocity as $v_{\parallel e} = v_{\parallel e0} + \tilde{v}_{\parallel e}$. For simplicity, the mean flow velocities are taken to be zero ($\mathbf{v}_{e0} = \mathbf{v}_{i0} = 0$) in this section. With the use of dimensionless dependent variables

$$\mathscr{T}_e = \frac{\tilde{T}_e}{T_0}\frac{L_n}{\rho_s}, \qquad v_e = \frac{\tilde{v}_{\parallel e}}{c_s}\frac{L_n}{\rho_s}, \qquad j = \frac{L_n}{e n_0 c_s \rho_s} j_\parallel$$

and dimensionless parameters

$$\chi_{\parallel e} = \frac{\kappa_{\parallel e}}{L_n c_s n_0} = \frac{\alpha_{heat} T_{e0}}{m_e \nu_e L_n c_s}, \qquad \mu_{\parallel e} = \frac{4 c_s \nu_{\parallel e}}{3 n_0 T_{e0} L_n} = \frac{4 \nu_{\parallel e}}{3 m_i n_0 c_s L_n}, \qquad \eta_e = \mathrm{d}\,(\ln T_{e0})/(\ln n_0),$$

equation (81) can be written as

$$j/\sigma_\parallel = \tilde{\partial}_\parallel(n - \phi + (1 + \alpha_{thermo})\mathscr{T}_e) - \mu_{\parallel e}\tilde{\partial}_\parallel^2 v_e, \tag{83}$$

where j also satisfies

$$j = v - v_e. \tag{84}$$

Similarly, the nondimensional form of equation (82) becomes

$$\frac{3}{2}\left(\frac{\partial}{\partial \tilde{t}}\mathscr{T}_e + \{\phi, \mathscr{T}_e\} + \eta_e\frac{\partial \phi}{\partial \tilde{y}}\right) + \tilde{\partial}_\parallel v_e = \chi_{\parallel e}\tilde{\partial}_\parallel^2 \mathscr{T}_e + \alpha_{thermo}\tilde{\partial}_\parallel j + \frac{1}{\sigma_\parallel} j^2. \tag{85}$$

This electron thermal balance equation brings in the parameter η_e from the $\tilde{v}_{Ex}\,\mathrm{d}T_e/\mathrm{d}x$ convection. Replacing equation (75) by (83), we obtain the set of equations (71)–(74), (83)–(85) (with $S_\perp = S_\parallel = 0$ since the mean flow velocity is assumed to be zero in this section), which governs collisional drift waves under the influence of electron temperature fluctuations.

5. LINEAR DISPERSION RELATIONS

In this section, we discuss linear dispersion relations of the systems derived in Sections 3 and 4. Here the local approximation is employed, in which the parallel derivative $\tilde{\partial}_\parallel$ is replaced by a constant i $\tilde{k}_\parallel$. It is also assumed that the mean shear flows are zero ($S_\perp = S_\parallel = 0$) for simplicity. Under these assumptions, the space and time dependence of the normal mode may be given by $\exp \mathrm{i}(\tilde{k}_x\tilde{x}+\tilde{k}_y\tilde{y}+\tilde{k}_z\tilde{z}-\tilde{\omega}\tilde{t})$ with $\tilde{k}_\perp^2 = \tilde{k}_x^2+\tilde{k}_y^2$ and the constant $\tilde{k}_\parallel$ models the operator $\tilde{\partial}_\parallel$ with the relations $\tilde{k}_\parallel = s\Delta_x\tilde{k}_y+\tilde{k}_z$, where Δ_x denotes a typical mode width in the x-direction. In the case of zero magnetic shear ($s = 0$), the local approximation gives the exact dispersion relation of the linearized systems with the relations $\tilde{k}_\parallel = \tilde{k}_z$. With the use of $\Omega = \tilde{\omega}/\tilde{k}_y$, the dispersion relation of the system of equations (71)–(75) is given by

$$(1+k_\perp^2)\Omega^2-(1-k_\perp^2K)\Omega-\left(\frac{k_\parallel}{k}\right)^2\frac{1+\dfrac{K}{\Omega}}{1-\dfrac{\Gamma}{\Omega^2}\left(\dfrac{k_\parallel}{k}\right)^2}$$

$$= \mathrm{i}\,\epsilon_\eta\Omega^2(\Omega+K)-\mathrm{i}\left(\frac{k_\parallel}{k}\right)^2\epsilon_\eta\frac{K+\Omega}{1-\dfrac{\Gamma}{\Omega^2}\left(\dfrac{k_\parallel}{k}\right)^2}, \quad (86)$$

where

$$\epsilon_\eta = \frac{kk_\perp^2}{\sigma_\parallel k_\parallel^2},$$

$k = \tilde{k}_y$ and all the tildes were dropped for simplicity. We also note that all the diffusion coefficients in equations (71)–(75) are ignored for simplicity.

5.1. *Collisionless ion temperature gradient driven modes*

The collisionless limit ($\epsilon_\eta = 0$) of equation (86) gives the well-known dispersion relation of the slab-type ion temperature gradient driven mode [equations (67)–(70) with $S_\perp = S_\parallel = 0$]. In the presence of strong ion pressure gradient ($K \gg 1$), the fastest growing mode is given (HAMAGUCHI and HORTON, 1990a) by the relation $\tilde{k}_\perp^2 = 1/K$ ($\ll 1$) with the eigenfrequency

$$\Omega = \frac{-1+\mathrm{i}\sqrt{3}}{2}\left(\left(\frac{\tilde{k}_\parallel}{\tilde{k}_y}\right)^2 K\right)^{1/3}. \quad (87)$$

There are also two stable branches with $\tilde{k}_\perp^2 = 1/K$ given by

$$\Omega = \left(\left(\frac{\tilde{k}_\parallel}{\tilde{k}_y}\right)^2 K\right)^{1/3} \quad \text{and} \quad \frac{-1-\mathrm{i}\sqrt{3}}{2}\left(\left(\frac{\tilde{k}_\parallel}{\tilde{k}_y}\right)^2 K\right)^{1/3}.$$

Thus the unstable branch travels in the ion diamagnetic direction and the nondamping stable branch travels in the electron diamagnetic direction. In the dimensional form, the unstable eigenfrequency may be written in terms of $\omega = (c_s/L_n)\Omega\tilde{k}_y$ as

$$\omega = \frac{-1+i\sqrt{3}}{2}(-\omega^*_{pi}k^2_\parallel c^2_s)^{1/3}$$

when

$$k_\perp \rho_s = (T_i(1+\eta_i)/T_e)^{-1/2},$$

where $\omega^*_{pi} = -(cT_i/eB)(k_y/L_n)(1+\eta_i) = -\omega^*_e K$, $\omega^*_e = (cT_e/eB)k_y$, $k_y = \tilde{k}_y/\rho_s$, $k_\parallel = \tilde{k}_\parallel/L_n$ and $k_\perp = \tilde{k}_\perp/\rho_s$. This instability results from the coupling of the ion acoustic wave with the thermal mode arising from ω^*_{pi} (Hamaguchi and Horton, 1990a). A more detailed physical interpretation of this mode is given in Section 6.

5.2. *Collisional modification of ion temperature gradient driven modes*

The presence of small resistivity ϵ_η in equation (86) affects the growth rate of the ion temperature gradient driven mode given by equation (87). Taking the ordering assumption that $\tilde{k}^2_\perp = 1/K \ll 1$, $(\tilde{k}_\parallel/\tilde{k}_y)^2 \ll 1$, $K/\Omega \gg 1$ and $\epsilon_\eta K \ll 1$, we obtain $\Omega = ((1+i\sqrt{3})/2)K^{1/3}(\tilde{k}_\parallel/\tilde{k}_y)^{2/3}(1+i\epsilon_\eta K/3)$, or in terms of dimensional form,

$$\gamma_G = (-\omega^*_{pi}k^2_\parallel c^2_s)^{1/3}\left(\frac{\sqrt{3}}{2} - \epsilon_\eta \frac{T_i}{3T_e}(1+\eta_i)\right),$$

where

$$\epsilon_\eta = \frac{\omega^*_e \nu_e}{k^2_\parallel v^2_{th,e}} k^2_\perp \rho^2_s$$

where γ_G denotes the growth rate $\gamma_G = \mathrm{Im}\,\omega$ and $v_{th,e} = \sqrt{T_e/m_e}$. It is shown that the finite resistivity reduces the growth of the ion temperature gradient driven mode.

5.3. *Collisional drift wave*

Hasegawa and Wakatani have shown that finite resistivity destabilizes the drift wave (Hasegawa and Wakatani, 1983; Wakatani and Hasegawa, 1984). This is a different branch of instability from the instability discussed in the previous subsection and occurs even in the absence of ion temperature gradients. Under the ordering assumption that $(\tilde{k}_\parallel/\tilde{k}_y)^2 = \mathcal{O}(\epsilon_\eta)$ and $\Omega = \Omega_0 + \Omega_1$ with $|\Omega_1/\Omega_0| = \mathcal{O}(\epsilon_\eta)$, we obtain to lowest order

$$\Omega_0 = \frac{1-\tilde{k}^2_\perp K}{1+\tilde{k}^2_\perp} \quad [= \mathcal{O}(1)].$$

Here, unlike Subsections 5.1 and 5.2, it is assumed that $\tilde{k}_\perp \sim K = \mathcal{O}(1)$. From equation (86), it is easy to obtain the next-order expression

$$\Omega_1 = \frac{(1+K)}{(1-k_\perp^2 K)^2}\left(\frac{\tilde{k}_\parallel}{\tilde{k}_y}\right)^2 + i\epsilon_\eta \frac{(1+K)(1-k_\perp^2 K)}{(1+k_\perp^2)^3}.$$

The growth rate is then given in dimensional form by

$$\gamma_G = \frac{\omega_e^{*2}\nu_e}{k_\parallel^2 v_{th,e}^2}\frac{k_\perp^2\rho_s^2}{(1+k_\perp^2\rho_s^2)^3}\left(1+\frac{T_i}{T_e}(1+\eta_i)\right)\left(1-\frac{T_i}{T_e}(1+\eta_i)k_\perp^2\rho_s^2\right). \tag{88}$$

If the wavelength is long ($k_\perp\rho_s \ll 1$), finite ion pressure gradients ($1+\eta_i >$) are shown to further destabilize the unstable collisional drift wave in equation (88).

5.4. *Effects of electron temperature fluctuations*

Finite electron temperature fluctuations, together with electron diffusivity and thermo-electron effects, alter the growth rate of the collisional drift wave (HINTON and HORTON, 1971; HORTON and VARMA; 1972, HORTON, 1990). Based on equations (83)–(85) which describe these effects, we examine the growth rate of the collisional drift wave in this subsection. The dispersion relation obtained from the system of equations (71)–(74), (83)–(85) is given by

$$(1+k_\perp^2)\Omega^2-(1-k_\perp^2K)\Omega-i\epsilon_\eta\Omega^2(\Omega+K)\left(1+\sigma_\parallel\mu_{\parallel e}k_\parallel^2+\frac{\sigma_\parallel}{\chi_{\parallel e}}\frac{(1+\alpha_{thermo})^2}{\left(1-i\frac{3}{2}\epsilon_\eta\frac{\sigma_\parallel\Omega^2}{\chi_{\parallel e}k_\perp^2}\right)}\right)$$

$$+i\epsilon_\eta\frac{\sigma_\parallel\Omega^2}{\chi_{\parallel e}k_\perp^2}\frac{(1+\alpha_{thermo})}{\left(1-i\frac{3}{2}\epsilon_\eta\frac{\sigma_\parallel\Omega^2}{\chi_{\parallel e}k_\perp^2}\right)}\left[\frac{3}{2}\eta_e+\left(\frac{k_\parallel}{k}\right)^2\frac{K+\Omega(1-i\epsilon_\eta(K+\Omega))}{\Omega^2\left(1-\frac{\Gamma}{\Omega^2}\left(\frac{k_\parallel}{k}\right)^2\right)}\right]$$

$$=\left(1-i\epsilon_\eta\left(\frac{k_\parallel}{k_\perp}\right)^2\Omega\sigma_\parallel\mu_{\parallel e}\right)\left(\frac{k_\parallel}{k}\right)^2\frac{1+K/\Omega-i\epsilon_\eta(K+\Omega)}{1-\frac{\Gamma}{\Omega^2}\left(\frac{k_\parallel}{k}\right)^2}, \tag{89}$$

where $k=\tilde{k}_y$ and all the tildes were dropped for simplicity. It is easy to see that equation (86) is obtained from equation (89) in the limit $\chi_{\parallel e}\to\infty$ and $\mu_{\parallel e}\to 0$. Here the large $\chi_{\parallel e}$ limit prevents electron temperature fluctuations from being excited by allowing fast heat transport along the magnetic field line while the small $\mu_{\parallel e}$ limit corresponds to the assumption made in deriving equation (86) that the electron parallel diffusion $\mu_{\parallel e}$ is small.

The eigenfrequency Ω of the collisional drift wave may be obtained from equation (89) under the following ordering assumptions: $\epsilon_\eta=\tilde{k}_y\tilde{k}_\perp/\sigma_\parallel\tilde{k}_\parallel^2 \ll 1$, $\sigma_\parallel\sim\chi_{\parallel e}$, $\sigma_\parallel\chi_{\parallel e}\tilde{k}_\parallel^2=\mathcal{O}(1)$, $\tilde{k}_\perp^2\sim(\tilde{k}_\parallel/\tilde{k}_y)^2=\mathcal{O}(\epsilon_\eta^{1/2})$. Writing $\Omega=\Omega_0+\Omega_1$ with $|\Omega_1/\Omega_0|\ll 1$, we obtain the lowest order frequency Ω_0 of the long wavelength collisional drift wave as $\Omega_0=1$. Calculating the higher order contribution Ω_1, we obtain the normalized growth rate

$$\mathrm{Im}\,\Omega = \epsilon_\eta\left[(1+K)f_{k_\parallel} - \frac{\sigma_\parallel(1+\alpha_{\mathrm{thermo}})}{\chi_{\parallel e}\tilde{k}_\perp^2}\left(\frac{3}{2}\eta_e + \left(\frac{\tilde{k}_\parallel}{\tilde{k}}\right)^2(1+K)\right) - \frac{\tilde{k}_\parallel^4\sigma_\parallel\mu_{\parallel e}}{\tilde{k}_\perp^2\tilde{k}_y^2}(1+K)\right],$$

where

$$f_{k_\parallel} = 1 + \tilde{k}_\parallel^2\sigma_\parallel\mu_{\parallel e} + \frac{\sigma_\parallel}{\chi_{\parallel e}}(1+\alpha_{\mathrm{thermo}})^2.$$

In terms of dimensional form, the growth rate is given by

$$\gamma_G = \frac{\omega_e^{*2}\nu_e}{k_\parallel^2 v_{\mathrm{th},e}^2}\left\{k_\perp^2\rho_s^2\left(1+\frac{T_i}{T_e}(1+\eta_i)\right)f_{k_\parallel} - \left(\frac{1+\alpha_{\mathrm{thermo}}}{\alpha_{\mathrm{heat}}}\right)\left[\frac{3}{2}\eta_e + \frac{k_\parallel^2 c_s^2}{\omega_e^{*2}}\left(1+\frac{T_i}{T_e}(1+\eta_i)\right)\right]\right.$$
$$\left. - \frac{4}{3}\alpha_{\mathrm{visc}}\frac{v_{\mathrm{th},e}^2 k_\parallel^2}{\nu_e^2}\frac{c_s^2 k_\parallel^2}{\omega_e^{*2}}\left(1+\frac{T_i}{T_e}(1+\eta_i)\right)\right\}, \quad (90)$$

where

$$f_{k_\parallel} = 1 + \frac{4}{3}\alpha_{\mathrm{visc}}\frac{T_e k_\parallel^2}{m_e\nu_e^2} + \frac{(1+\alpha_{\mathrm{thermo}})^2}{\alpha_{\mathrm{heat}}}.$$

Here we have used the classical expressions for the parallel electron diffusivities $\kappa_{\parallel e} = \alpha_{\mathrm{heat}} n_e T_e/m_e\nu_e(\alpha_{\mathrm{heat}} = 1/6)$ and $\nu_{\parallel e} = \alpha_{\mathrm{visc}} n_e T_e/\nu_e(\alpha_{\mathrm{visc}} = 0.4)$ and the thermo-electron effect $\alpha_{\mathrm{thermo}} = 0.71$, as given in Section 4. We note that the first term in { } in the expression for γ_G in equation (90) corresponds to the growth rate of the collisional drift wave [equation (88) with $k_\perp\rho_s \ll 1$] enhanced by $f_{k_\parallel}$ over the value obtained by only including electrical conductivity ($f_{k_\parallel} \to 1$). It is shown in equation (90) that a substantial, positive η_e can be a strong stabilizing effect on the collisional drift wave. The role of the parallel electron viscosity α_{visc} is more complicated since it contributes both to increasing the enhancement factor $f_{k_\parallel}$ and to the direct damping in the last term of the γ_G formula.

6. DISCUSSION

In this paper, we have derived from the electrostatic two-fluid equations the sets of equations governing the nonlinear dynamics of the drift waves in the presence of ion temperature gradients. The derivation is based on consistent orderings, in which the modes are assumed to be localized on a particular magnetic field line and to fluctuate much faster than the evolution of the mean fields but much more slowly than ion gyromotion. The effects of mean shear flows and electron temperature fluctuations are also discussed. The final equations for the collisional drift wave [equations (71)–(75)] and for the collisionless drift wave [equations (78)–(80)] provide the basis for a nonlinear analysis of the collisional drift wave instability and the collisionless ion temperature gradient driven instability, respectively.

It is worth while exercising our intuition to draw physical pictures of the instabilities discussed in the previous sections. For this purpose, we consider the simplest possible case, namely the case of no mean sheared flows ($S_\perp = S_\parallel = 0$), no magnetic shear

($s = 0$) and no diffusion. In order to understand the physical mechanism of the collisionless ion temperature gradient driven mode, however, we need a finite ion pressure gradient [$p_i(x) \neq$ const], which is the free energy source of the mode. On the other hand, for the collisional drift instability, we consider the effects of a finite density gradient [$n(x) \neq$ const], which is the driving force of the drift wave, and finite resistivity, which gives rise to the breakdown of the adiabatic electron response ($n \neq \phi$). A finite ion pressure gradient plays a secondary role in the collisional drift instability, modifying its growth rate as shown in equation (88).

Starting with the collisionless ion temperature gradient driven mode, we consider the fastest growing mode whose perpendicular wavenumber $k_\perp$ satisfies $k_\perp^2 \simeq K^{-1}$. Suppose we have a small positive perturbation of the electrostatic potential ϕ in the plasma. This potential perturbation induces a $\mathbf{E} \times \mathbf{B}$ flow circulating around the perturbation, as shown in Fig. 1a. The high pressure ions and low pressure ions are mixed by the $\mathbf{E} \times \mathbf{B}$ flow and this mixture creates a high pressure spot and a low pressure spot on each side of the potential perturbations as shown in Fig. 1b. This convection of the ion pressure is described by equation (80) or

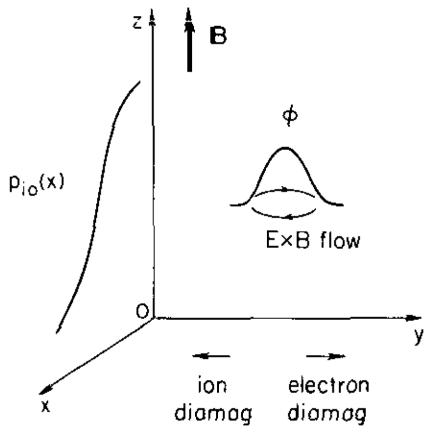

FIG. 1a.—A small positive perturbation of the electrostatic potential Φ causing a $\mathbf{E} \times \mathbf{B}$ flow.

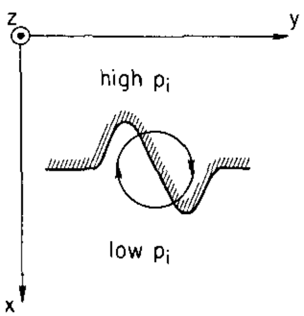

FIG. 1b.—The $\mathbf{E} \times \mathbf{B}$ flow viewed from the z-direction.

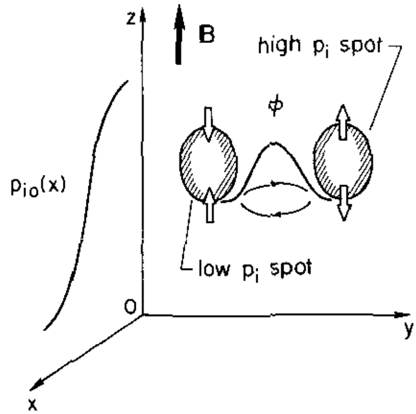

FIG. 1c.—High and low p_i spots created by the $\mathbf{E}\times\mathbf{B}$ *flow.* White arrows indicate plasma flows along the magnetic field lines.

$$\frac{\partial p}{\partial \tilde{t}} = -K\frac{\partial \phi}{\partial \tilde{y}},$$

where the effect of Γ is ignored. The parallel dynamics then plays an important role: into the low pressure spot, plasmas flow from the outside along the magnetic field lines whereas plasmas in the high pressure spot are pushed away along the field lines (Fig. 1c). The left-hand side and the third term of the right-hand side of equation (79) or

$$\frac{\partial v}{\partial \tilde{t}} = -\tilde{\partial}_{\parallel} p$$

describe this process. This process, coupled with the effect of nonzero Γ (or the parallel compressibility) of equation (80), induces the ion acoustic wave, which is destabilized by the ion pressure gradient in this case. Finally, through the balance between the first term of the left-hand side and the parallel compression term of the right-hand side of equation (78) or

$$\frac{\partial \phi}{\partial \tilde{t}} = -\tilde{\partial}_{\parallel} v,$$

the parallel motion of the plasmas increases the electrostatic potential at the low pressure spot and decreases it at the high pressure spot.

The increased electrostatic potential at the low pressure spot repeats the same sequence of processes and generates more potential perturbations in the negative y-direction or the ion diamagnetic direction. This mechanism thus induces a growing mode travelling in the ion diamagnetic direction [equation (87)], which is the ion temperature gradient driven instability. It should be noted that this instability exists even in the absence of the density gradient and magnetic shear. On the other hand, the high pressure spot in Fig. 1b, which induces the decrease of the potential, generates a stable branch of the mode travelling in the electron diamagnetic direction.

The collisional drift instability is a higher order correction to the stable drift wave caused by finite resistivity. Ignoring the ion parallel flow and the small contribution from the parallel current $\partial_{\|}j$, we have the linear equation of density fluctuation from equation (72)

$$\frac{\partial n}{\partial \tilde{t}} = -\frac{\partial \phi}{\partial \tilde{y}}.$$

This is a simple situation described in Figs 1a and b, namely, the $\mathbf{E}\times\mathbf{B}$ flow convects the density, instead of the ion pressure. In the case of the adiabatic electrons, the equation above gives the stable drift wave $\Omega = 1$ (or $\omega = \omega_e^*$). However, in the presence of finite resistivity, the electron density fluctuation is related to the potential fluctuation through equation (75) or $n = \phi + j/(\mathrm{i}\tilde{k}_{\|}\sigma_{\|})$. The parallel current j is then related to the polarization current through the relation $\nabla\cdot\mathbf{j} = 0$ [equation (9)] or

$$\frac{\partial}{\partial \tilde{t}}\tilde{\Delta}_{\perp}\phi = \tilde{\partial}_{\|}j, \tag{91}$$

which is a simplified form of equation (71). Here we note that the right-hand side of equation (91) is proportional to the divergence of the polarization current. Therefore the difference between $\tilde{n}$ and $\tilde{\Phi}$ has a 90° phase shift from $\tilde{\Phi}$ or $n = \phi + \mathrm{i}\omega\tilde{k}_{\perp}^2\phi/\sigma_{\|}\tilde{k}_{\|}^2$, which destabilizes the drift wave. When a finite ion pressure gradient is present $(K \neq 0)$, equation (91) is replaced by $(\partial/\partial\tilde{t} - K\partial/\partial\tilde{y})\tilde{\Delta}_{\perp}\phi = \tilde{\partial}_{\|}j$, as in equation (71). This modification by finite $K = T_i(1+\eta_i)/T_e$ means the inclusion of the diamagnetic flow convecting the polarization current. As shown in equation (88), a finite ion pressure gradient increases the growth rate of the long wavelength $(k_{\perp}\rho_s \ll 1)$ collisional drift wave instability.

Although the linear analysis based on the local approximation presented in Section 5 reveals the generic properties of the instabilities, the eigenvalue analysis is necessary for studying the cross-field mode structure of the instabilities in the presence of sheared magnetic and flow fields. In fact, it has been shown (HAMAGUCHI and HORTON, 1990a,b) that strong magnetic shear has a stabilizing effect on the collisionless ion temperature gradient drift instability, which cannot be shown from the local analysis. Our more recent study (HAMAGUCHI and HORTON, 1991) also shows that finite $\mathbf{E}\times\mathbf{B}$ mean shear flows $(S_{\perp} \neq 0)$ reduce the growth rate and consequently reduce turbulent transport in the plasma. These stabilizing effects by magnetic and flow shear are of significant importance in practical applications; for example, it is widely believed that improved energy confinement observed in H-mode discharges (WAGNER *et al.*, 1982) is related to reduction of instabilities by the $\mathbf{E}\times\mathbf{B}$ shear flows near the plasma edge. An analysis of such effects by using the linear eigenmode analysis and nonlinear numerical simulations is beyond the goal of the present paper and will be presented elsewhere.

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APPENDIX A: THE GYROVISCOUS TENSOR

In this Appendix we calculate the divergence of the gyroviscous stress tensor $\nabla\cdot\boldsymbol{\Pi}_i^{\rm FLR}$ in equation (1). In what follows, the subscripts i denoting ion quantities are omitted for simplicity. As assumed in Section 2.1, if the dependence of $\hat{\mathbf{b}}$ on the space coordinates $\mathbf{x}$ is weak in the sense that $|\partial_\alpha v_\beta|/|v_\beta| \gg |\partial_\mu b_\nu|$ (α, β, μ, $\nu = 1, 2, 3$), the divergence of the gyrovoscous stress tensor $\nabla\cdot\boldsymbol{\Pi}^{\rm col}$ may be calculated from the following expressions:

$$\nabla\cdot\mathbf{W}^{(3)} = \nabla_\perp^2(\hat{\mathbf{b}}\times\mathbf{v})+\mathcal{O}(\varepsilon^*) \tag{A-1}$$

$$= -\nabla_\perp(\hat{\mathbf{b}}\cdot(\nabla\times\mathbf{v}))+\hat{\mathbf{b}}\times\nabla(\nabla\cdot\mathbf{v}_\perp)+\mathcal{O}(\varepsilon^*), \tag{A-2}$$

$$\nabla\cdot\mathbf{W}^{(4)} = \partial_\parallel^2(\hat{\mathbf{b}}\times\mathbf{v})+\hat{\mathbf{b}}\times\nabla(\partial_\parallel v_\parallel)-\hat{\mathbf{b}}\,\partial_\parallel(\hat{\mathbf{b}}\cdot(\nabla\times\mathbf{v}))+\mathcal{O}(\varepsilon^*) \tag{A-3}$$

$$= 2\,\partial_\parallel^2(\hat{\mathbf{b}}\times\mathbf{v})-\partial_\parallel(\nabla\times\mathbf{v})+\mathcal{O}(\varepsilon^*). \tag{A-4}$$

Here $\mathcal{O}(\varepsilon^*)$ denotes the terms smaller than the leading terms by order of $\varepsilon^*\simeq|\partial_\mu b_\nu|/(|\partial_\alpha v_\beta|/|v_\beta|)$. We have also used $\mathbf{v}_\perp = \delta_{\alpha\beta}^\perp v_\beta = \mathbf{v}-\hat{\mathbf{b}}(\hat{\mathbf{b}}\cdot\mathbf{v})$, $v_\parallel = \hat{\mathbf{b}}\cdot\mathbf{v}$, $\nabla_\perp = \nabla-\hat{\mathbf{b}}(\hat{\mathbf{b}}\cdot\nabla)$ and $\partial_\parallel = \hat{\mathbf{b}}\cdot\nabla$, as in Section 2.1. In deriving equation (A-4) from (A-3), the relation

$$\hat{\mathbf{b}}\times\nabla(\partial_\parallel v_\parallel) = \partial_\parallel^2(\hat{\mathbf{b}}\times\mathbf{v})+\hat{\mathbf{b}}\,\partial_\parallel(\hat{\mathbf{b}}\cdot(\nabla\times\mathbf{v}))-\partial_\parallel(\nabla\times\mathbf{v})+\mathcal{O}(\varepsilon^*)$$
$$= \partial_\parallel^2(\hat{\mathbf{b}}\times\mathbf{v})-\partial_\parallel(\nabla\times\mathbf{v})_\perp+\mathcal{O}(\varepsilon^*) \qquad \text{(A-5)}$$

is used. From equations (A-2) and (A-4) we obtain the expression for $\mathbf{W}^{\mathrm{FLR}} = \mathbf{W}^{(3)}+2\mathbf{W}^{(4)}$ as

$$\nabla\cdot\mathbf{W}^{\mathrm{FLR}} = -\nabla_\perp(\hat{\mathbf{b}}\cdot(\nabla\times\mathbf{v}))+\hat{\mathbf{b}}\times\nabla(\nabla\cdot\mathbf{v}_\perp)+4\,\partial_\parallel^2(\hat{\mathbf{b}}\times\mathbf{v})-2\,\partial_\parallel(\nabla\times\mathbf{v})+\mathcal{O}(\varepsilon^*). \qquad \text{(A-6)}$$

Since $v_{\mathrm{i}}^{\mathrm{FLR}} = p/2\omega_{\mathrm{ci}}$, we also need the following expressions to calculate $\nabla\cdot\mathbf{\Pi}^{\mathrm{FLR}}$:

$$\nabla p\cdot\mathbf{W}^{(3)} = \hat{\mathbf{b}}\times(\nabla_\perp p\cdot\nabla)\mathbf{v}+((\nabla p\times\hat{\mathbf{b}})\cdot\nabla)\mathbf{v}_\perp+\mathcal{O}(\varepsilon^*)$$
$$= 2(\hat{\mathbf{b}}\times(\nabla_\perp p\cdot\nabla)\mathbf{v})+(\hat{\mathbf{b}}\cdot(\nabla\times\mathbf{v}))\nabla_\perp p+(\nabla p\times\hat{\mathbf{b}})(\nabla\cdot\mathbf{v}_\perp)+\mathcal{O}(\varepsilon^*) \qquad \text{(A-7)}$$

$$\nabla p\cdot\mathbf{W}^{(4)} = \partial_\parallel p(\hat{\mathbf{b}}\times\nabla v_\parallel+\hat{\mathbf{b}}\times\partial_\parallel\mathbf{v})+\hat{\mathbf{b}}(\nabla p\times\hat{\mathbf{b}})\cdot(\nabla v_\parallel+\partial_\parallel\mathbf{v})+\mathcal{O}(\varepsilon^*). \qquad \text{(A-8)}$$

Using the identity

$$(\hat{\mathbf{b}}\times\nabla v_\alpha)\,\partial_\alpha p = (\hat{\mathbf{b}}\cdot(\nabla\times\mathbf{v}))\nabla_\perp p-(\nabla\times\mathbf{v})_\perp\,\partial_\parallel p+\hat{\mathbf{b}}\times(\nabla_\perp p\cdot\nabla)\mathbf{v}, \qquad \text{(A-9)}$$

we obtain from (A-7) and (A-8)

$$-2(\hat{\mathbf{b}}\times\nabla v_\alpha)\,\partial_\alpha p+\nabla p\cdot\mathbf{W}^{\mathrm{FLR}}$$
$$= 2(\partial_\parallel p)(\hat{\mathbf{b}}\times\nabla_\perp v_\parallel+\hat{\mathbf{b}}\times\partial_\parallel\mathbf{v}_\perp)+2\hat{\mathbf{b}}((\nabla_\perp p\times\hat{\mathbf{b}})\cdot\nabla v_\parallel+(\nabla_\perp p\times\hat{\mathbf{b}})\partial_\parallel\mathbf{v})$$
$$-(\hat{\mathbf{b}}\cdot(\nabla\times\mathbf{v}))\nabla_\perp p+2(\nabla\times\mathbf{v})_\perp\,\partial_\parallel p+(\nabla_\perp p\times\hat{\mathbf{b}})(\nabla\cdot\mathbf{v}_\perp)+\mathcal{O}(\varepsilon^*) \qquad \text{(A-10)}$$
$$= 2\hat{\mathbf{b}}((\nabla_\perp p\times\hat{\mathbf{b}})\cdot\nabla v_\parallel+(\nabla_\perp p\times\hat{\mathbf{b}})\,\partial_\parallel\mathbf{v})$$
$$+4\partial_\parallel p(\hat{\mathbf{b}}\times\partial_\parallel\mathbf{v}_\perp)-(\hat{\mathbf{b}}\cdot(\nabla\times\mathbf{v}))\nabla_\perp p+(\nabla_\perp p\times\hat{\mathbf{b}})(\nabla\cdot\mathbf{v}_\perp)+\mathcal{O}(\varepsilon^*). \qquad \text{(A-11)}$$

The following identity is used to derive equation (A-11) from (A-10):

$$(\nabla\times\mathbf{v})_\perp = -\hat{\mathbf{b}}\times(\nabla v_\parallel-\partial_\parallel\mathbf{v}-(\nabla\hat{\mathbf{b}})\cdot\mathbf{v}) \qquad \text{(A-12)}$$

where the tensor $(\nabla\hat{\mathbf{b}})_{\alpha\beta} = \partial_\alpha b_\beta = \mathcal{O}(\varepsilon^*)$.

Taking the parallel component of equation (A-11) yields

$$(\nabla p\cdot\mathbf{W}^{\mathrm{FLR}})_\parallel = 2((\nabla_\perp p\times\hat{\mathbf{b}})\cdot\nabla v_\parallel+(\nabla_\perp p\times\hat{\mathbf{b}})\,\partial_\parallel\mathbf{v})+\mathcal{O}(\varepsilon^*). \qquad \text{(A-13)}$$

We do not discuss derivation of equations (A-1)–(A-8) from the general expressions of equations (13) and (14). However, we note that (A-1)–(A-8), (A-10), (A-11) and (A-13) hold exactly [i.e. without the error of $\mathcal{O}(\varepsilon^*)$] if the unit vector $\hat{\mathbf{b}}$ is independent of $\mathbf{x}$. This is easily confirmed by choosing a coordinate system in which the unit vector $\hat{\mathbf{b}}$ is expressed as $\hat{\mathbf{b}} = (0, 0, 1)$ and compare components of both sides of equations (A-1)–(A-8). The expressions for $\mathbf{W}^{(3)}$ and $\mathbf{W}^{(4)}$ in such a coordinate system is given in BRAGINSKII (1965).

APPENDIX B: DERIVATION OF EQUATIONS (39) AND (49)

In this Appendix we derive equations (39) and (49) from the formulae summarized in Appendix A. As in Appendix A, the subscripts i denoting ion quantities are omitted for simplicity. It is easy to show that the following identity holds exactly:

$$m_i n\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla\right)\frac{\hat{\mathbf{b}}\times\nabla p}{neB} = -\frac{1}{\omega_{\mathrm{ci}}n}(\hat{\mathbf{b}}\times\nabla p)\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla\right)n$$
$$+\frac{1}{\omega_{\mathrm{ci}}}\hat{\mathbf{b}}\times\nabla\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla\right)p+\frac{1}{\omega_{\mathrm{ci}}}((\mathbf{v}\cdot\nabla)\hat{\mathbf{b}}\times\nabla p)-\frac{1}{\omega_{\mathrm{ci}}}(\hat{\mathbf{b}}\times\nabla v_\alpha)\,\partial_\alpha p, \qquad \text{(B-1)}$$

where the repeated indices are summed from 1 to 3 as before. We also assume that the unit vector $\hat{\mathbf{b}} = \mathbf{B}/|\mathbf{B}|$ is a function of space $\mathbf{x}$ only but $B = |\mathbf{B}| = \text{const}$ in time and space. We note that if the dependence of $\hat{\mathbf{b}}$ on $\mathbf{x}$ is weak in the sense that $|\partial_\alpha v_\beta|/|v_\beta| \gg |\partial_\mu b_\nu|$ (α, β, μ, $\nu = 1, 2, 3$), as assumed in Appendix A, the term $(\mathbf{v}\cdot\nabla)\hat{\mathbf{b}}\times\nabla p/\omega_{\mathrm{ci}} = \mathcal{O}(\varepsilon^*)$ of equation (B-1) is ignorable.

We now use the ordering assumptions made in Section 2.2 in order to further simplify Eq. (B-1):

$$\mathbf{v}/c_s = \mathcal{O}(\varepsilon), \qquad \nabla_\perp\cdot\mathbf{v}(\rho_s/c_s) = \mathcal{O}(\varepsilon^2), \qquad \nabla_\perp p_0(\rho_s/n_{i0}T_e) = \mathcal{O}(\varepsilon),$$

$$\nabla_\perp\tilde{p}(\rho_s/n_{i0}T_e) = \mathcal{O}(\varepsilon), \qquad \nabla_\perp v_\alpha(\rho_s/c_s) = \mathcal{O}(\varepsilon), \qquad \partial_\parallel/|\nabla_\perp| = \mathcal{O}(\varepsilon), \qquad \omega^{-1}\partial/\partial t = \mathcal{O}(\varepsilon).$$

Under these assumptions, we have from equation (A-6)

$$\nabla\cdot\mathbf{W}^{\mathrm{FLR}} = -\nabla_\perp(\hat{\mathbf{b}}\cdot(\nabla\times\mathbf{v}))+\mathcal{O}(\varepsilon^2) \tag{B-2}$$

$$= -\nabla_\perp(\hat{\mathbf{b}}\cdot(\nabla\times\mathbf{v}))+\hat{\mathbf{b}}\times\nabla(\nabla\cdot\mathbf{v}_\perp)-2\partial_\parallel(\nabla\times\mathbf{v})+\mathcal{O}(\varepsilon^3) \tag{B-3}$$

and from equations (A-7) and (A-8)

$$\nabla p\cdot\mathbf{W}^{\mathrm{FLR}} = 2(\hat{\mathbf{b}}\times(\nabla_\perp p\cdot\nabla)\mathbf{v})+(\hat{\mathbf{b}}\cdot(\nabla\times\mathbf{v}))\nabla_\perp p+2\hat{\mathbf{b}}(\nabla p\times\hat{\mathbf{b}})\cdot\nabla v_\parallel+\mathcal{O}(\varepsilon^3). \tag{B-4}$$

We note that the third term $4\partial_\parallel^2\,(\hat{\mathbf{b}}\times\mathbf{v})$ of equation (A-6) is of order ε^3 and the contribution from this term to equation (B-3) is ignorable.

The perpendicular component of equation (B-1) becomes, under these assumptions,

$$m_i n\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla\right)\mathbf{v}_d = \frac{1}{\omega_{ci}}\hat{\mathbf{b}}\times\nabla\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla\right)p = \frac{1}{\omega_{ci}}(\hat{\mathbf{b}}\times\nabla v_\alpha)\,\partial_\alpha p \tag{B-5}$$

up to $\mathcal{O}(\varepsilon^2)$. Here $\mathbf{v}_d = (\hat{\mathbf{b}}\times\nabla p)/neB = \mathcal{O}(\varepsilon)$ and the first term on the right-hand side of equation (A-1), which becomes $-(\hat{\mathbf{b}}\times\nabla p)(\nabla\cdot\mathbf{v})/\omega_{ci} = \mathcal{O}(\varepsilon^3)$ with the use of the continuity equation ($\partial n/\partial t+\nabla\cdot(n\mathbf{v}) = 0$), is dropped. We also obtain from equation (A-11)

$$-\frac{1}{\omega_{ci}}(\hat{\mathbf{b}}\times\nabla v_\alpha)\,\partial_\alpha p+(\nabla\nu^{\mathrm{FLR}}\cdot\mathbf{W}^{\mathrm{FLR}})_\perp = -\hat{\mathbf{b}}\cdot(\nabla\times\mathbf{v})\nabla_\perp\nu^{\mathrm{FLR}} \tag{B-6}$$

up to $\mathcal{O}(\varepsilon^2)$, where $\nu^{\mathrm{FLR}} = p/2\omega_{ci}$.

Combining equations (B-5) and (B-6), we obtain

$$\begin{aligned} m_i n&\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla\right)\mathbf{v}_d+(\nabla\cdot\mathbf{\Pi}^{\mathrm{FLR}})_\perp \\ &= \frac{1}{\omega_{ci}}\hat{\mathbf{b}}\times\nabla\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla\right)p-\hat{\mathbf{b}}\cdot(\nabla\times\mathbf{v})\nabla_\perp\nu^{\mathrm{FLR}}+\nu^{\mathrm{FLR}}(\nabla\cdot\mathbf{W}^{\mathrm{FLR}})_\perp+\mathcal{O}(\varepsilon^3) \\ &= \frac{1}{\omega_{ci}}\hat{\mathbf{b}}\times\nabla\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla\right)p+\nabla_\perp\zeta+\mathcal{O}(\varepsilon^2), \end{aligned} \tag{B-7}$$

where $\mathbf{\Pi}^{\mathrm{FLR}} = \nu^{\mathrm{FLR}}\mathbf{W}^{\mathrm{FLR}}$, $\zeta = -\nu^{\mathrm{FLR}}\hat{\mathbf{b}}\cdot(\nabla\times\mathbf{v})$ and equation (B-2) has been used. Equation (B-7) is equivalent to (49).

The parallel ion momentum equation can also be simplified under these ordering assumptions. From equation (B-3) and (B-4), we obtain

$$(\hat{\mathbf{b}}\times\nabla_\perp p)\cdot\nabla v_\parallel+\left(\nabla\frac{p}{2}\right)\cdot\mathbf{W}_\parallel^{\mathrm{FLR}}+\frac{p}{2}(\nabla\cdot\mathbf{W}^{\mathrm{FLR}})_\parallel = -p\,\partial_\parallel(\hat{\mathbf{b}}\cdot\nabla\times\mathbf{v})+\mathcal{O}(\varepsilon^3), \tag{B-8}$$

which is equivalent to equation (39).