AIP | Physics of Fluids

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Citation: Phys. Fluids 1, 253 (1958); doi: 10.1063/1.1705883
View online: http://dx.doi.org/10.1063/1.1705883
View Table of Contents: http://pof.aip.org/resource/1/PFLDAS/v1/i4
Published by the American Institute of Physics.

THE

PHYSICS OF FLUIDS

VOLUME 1, NUMBER 4 JULY-AUGUST, 1958

The Stellarator Concept*

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(Received May 27, 1958)

The basic concepts of the controlled thermonuclear program at Project Matterhorn, Princeton University are discussed. In particular, the theory of confinement of a fully ionized gas in the magnetic configuration of the stellarator is given, the theories of heating are outlined, and the bearing of observational results on these theories is described.

Magnetic confinement in the stellarator is based on a strong magnetic field produced by solenoidal coils encircling a toroidal tube. The configuration is characterized by a "rotational transform," such that a single line of magnetic force, followed around the system, intersects a cross-sectional plane in points which successively rotate about the magnetic axis. A theorem by Kruskal is used to prove that each line of force in such a system generates a toroidal surface; ideally the wall is such a surface. A rotational transform may be generated either by a solenoidal field in a twisted, or figure-eight shaped, tube, or by the use of an additional transverse multipolar helical field, with helical symmetry.

Plasma confinement in a stellarator is analyzed from both the macroscopic and the microscopic points of view. The macroscopic equations, derived with certain simplifying assumptions, are used to show the existence of an equilibrium situation, and to discuss the limitations on material pressure in these solutions. The single-particle, or microscopic, picture shows that particles moving along the lines of force remain inside the stellarator tube to the same approximation as do the lines of force. Other particles are presumably confined by the action of the radial electric field that may be anticipated.

Theory predicts and observation confirms that initial breakdown, complete ionization, and heating of a hydrogen or helium gas to about 10^6 degrees K are possible by means of a current parallel to the magnetic field (ohmic heating). Flow of impurities from the tube walls into the heated gas, during the discharge, may be sharply reduced by use of an ultra-high vacuum system; some improvement is also obtained with a divertor, which diverts the outer shell of magnetic flux away from the discharge. Experiments with ohmic heating verify the presence of a hydromagnetic instability predicted by Kruskal for plasma currents greater than a certain critical value and also indicate the presence of other cooperative phenomena. Heating to very much higher temperatures can be achieved by use of a pulsating magnetic field. Heating at the positive-ion cyclotron resonance frequency has been proposed theoretically and confirmed observationally by Stix. In addition, an appreciable energy input to the positive ions should be possible, in principle, if the pulsation period is near the time between ion-ion collisions or the time required for a positive ion to pass through the heating section (magnetic pumping).

1. INTRODUCTION

THE controlled release of thermonuclear power requires[1] the confinement of a hydrogen plasma at a temperature exceeding 10^8 degrees for an appreciable fraction of a second. A strong magnetic field would appear to offer the only practical hope of achieving this objective. While an enormous variety of magnetic configurations are possible, two relatively simple geometries may be set apart at the outset, both involving infinite cylinders, with axial symmetry. In the first of these, the magnetic field is produced by an axial current flowing through the gas. This configuration is the so-called "pinch-

* Supported by the U. S. Atomic Energy Commission under Contract AT(30-1)-1238 with Princeton University.
[1] R. F. Post, Revs. Modern Phys. 28, 338 (1956).

effect," which has been extensively discussed in the recent literature. In the second simple geometry, the magnetic field is parallel to the axis, and is produced by external currents, flowing in solenoidal windings encircling the plasma. Such a straight cylinder, with an externally produced field, forms the basis of the magnetic mirror or "pyrotron" device, proposed by Post.[2]

The stellarator, like the pyrotron, utilizes an external magnetic field, produced by coils encircling a tube containing the heated gas. However, instead of a finite cylindrical tube, the stellarator employs a tube bent into a configuration topologically like a torus, without ends. Such a tube will be referred to as "toroidal." Relative to the pyrotron, this configuration has the advantage, in principle, of permitting more complete confinement, since end losses are eliminated. Relative to the pinch discharge the stellarator offers the advantage, again in principle, of permitting equilibrium in a steady state. Both these advantages might be of importance in a controlled thermonuclear reactor. The present paper outlines the basic concepts involved in the confinement and heating of a gas in a stellarator.

One basic complexity in the stellarator results from the fact that the simple torus, in which the magnetic lines of force are circles centered at the axis of symmetry, does not permit equilibrium confinement of a plasma in a straightforward manner. Microscopically this result follows at once from the particle drifts associated with the inhomogeneity of the magnetic field. These drifts, pointed out some 50 years ago by Thomson,[3] subsequently discussed by Gunn,[4] and recently analyzed in detail by Alfvén,[5] produce motions perpendicular both to $\mathbf{B}$ and to ∇B; these motions are in opposite directions for electrons and positive ions. The resultant separation of charges produces electric fields which sweep the ionized gas towards the wall. Macroscopically, this same result is obtained from the fluid equations in Sec. 3.1.

Basically, the confinement scheme in the stellarator consists of modifying the magnetic field so that a single line of force, followed indefinitely, generates not a single circle but rather an entire toroidal surface, called a "magnetic surface." The tube enclosing the gas is, ideally, one of these surfaces, enclosing an entire family of such magnetic surfaces. It is shown in Sec. 2 that such surfaces can be produced, to a high approximation. Section 3 discusses the confinement of a plasma in a system characterized by magnetic surfaces. Heating of the gas is treated in Sec. 4; the techniques considered involve heating to intermediate temperatures (about 10^6 degrees K) by currents flowing parallel to the confining magnetic field, and subsequent heating to very high temperatures by means of a pulsating magnetic field, with a wide variety of possible pulsation frequencies.

These analyses do not take into account the complex time-dependent cooperative processes that normally occur in laboratory plasmas. Any attempt to predict theoretically how well a gas at thermonuclear temperatures might be confined within a stellarator is hindered at the present time by lack of information or understanding of such cooperative phenomena.

The present paper constitutes an introduction to the series of papers from Project Matterhorn printed in this Journal. Subsequent theoretical papers discuss the macroscopic equilibrium of plasma in the stellarator, treat in great detail the problem of hydromagnetic stability, and obtain quantitative estimates of the amount of heating to be anticipated with various methods. The experimental papers, to appear in the next issue of this Journal, present observational material on ionization and heating of a gas in the stellarator, with particular emphasis on the evidence for instabilities and other cooperative phenomena.

2. ROTATIONAL TRANSFORM AND MAGNETIC SURFACES

In this section we consider certain properties of a stellarator magnetic field which do not depend directly on the presence of a plasma in the system. In particular, we are interested in how far a line of force in a stellarator tube may be followed before it intersects the tube wall. If we could show in all rigor that one and only one magnetic surface passed through each point within the tube, then no line of force would ever intersect the tube wall. While this result is not exactly true, it is apparently true to a high order of approximation. The demonstration of this result depends on a certain abstract property of the stellarator magnetic field, called a "rotational transform." We first discuss the properties of such transforms, and postpone until later a discussion of how they are produced.

[2] R. F. Post, (to be published).
[3] J. J. Thomson, *Conduction of Electricity through Gases* (Cambridge University Press, New York, 1906), second edition, p. 109.
[4] R. Gunn, Phys. Rev. **33**, 832 (1929).
[5] H. Alfvén, *Cosmical Electrodynamics* (Clarendon Press, Oxford, England, 1950).

2.1 Properties of Rotational Transforms

Let us pass a plane through the stellarator tube at some point. For convenience in representation, we shall take the plane to be perpendicular to the magnetic field, **B**, in the central region of the tube, although this restriction is not required. We assume that the magnetic field is nonzero at all points in the cross-sectional plane. Then through any point, such as P_1, for example, a line of force will pass. This line of force may now be followed, in the direction of the magnetic field, along the stellarator tube, until it has completed one circuit of the toroidal tube and intersects the cross-sectional plane at a point P_2, as shown in Fig. 1. In the ideal torus P_2 coincides with P_1 and every line of force is closed after one circuit. If the degeneracy of the torus is removed, P_2 no longer coincides with P_1.

One might raise some question as to whether any of the points P_2 lie inside the tube wall. It is not difficult to construct a toroidal tube with a solenoidal winding such that most of the lines of force stay within the tube for at least one circuit. The interesting question is rather which lines intersect the tube wall after many circuits, and we may safely assume that for most points P_1, P_2 will exist and will lie inside the tube wall. If we exclude, for the moment, those areas for which P_2 does not lie within the tube wall, then for every point P_1, we have a point P_2; the transformation of the set of points P_1 into the set of points P_2 is called a "magnetic transform" of the cross-sectional plane.

A magnetic transform of this type is characterized by an important property. Let us take the density of points P_1 to be proportional to the component of B normal to the cross-sectional plane. Since the plane is transformed into itself by the magnetic transform, the density of these points must be the same function of position before the transform as after. A transform with this property has been called "measure preserving" by Kruskal.

Let us now assume that the magnetic transform is also "primarily rotational." This condition states that at least the outer portions of the plane all rotate in the same direction in a single transformation. As we shall see subsequently, there are several ways of achieving this result in a toroidal system. According to Kruskal,[6] it follows from this assumption and from the Brouwer fixed-point theorem that there must be at least one point in the plane which is transformed into itself. In some types of stellarator the magnetic transform involves only small deformations of the plane, in addition to a general rotation. In such systems there will be only one point that transforms into itself, and only one line of force that is closed after a single circuit around the stellarator. This line is called the "magnetic axis."

[6] M. D. Kruskal (informal communication).

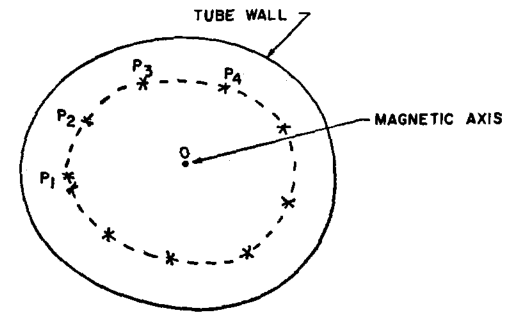

FIG. 1. Cross-sectional plane, showing intersection points produced by a single line of force.

A second basic result on measure-preserving rotational transforms, also established by Kruskal,[7] is that any point, other than one of the fixed points, when followed through successive transformations, will not move far from a single closed curve. This result is illustrated in Fig. 1, where the points P_2, P_3, P_4 generated by successive magnetic transforms of the point P_1, all lie close to a single closed curve. Thus a single line of force, after many circuits around the tube, generates a magnetic surface.

More precisely, let us introduce coordinates r, θ in the cross-sectional plane depicted in Fig. 1; r may be measured from the magnetic axis, denoted by the point 0. The value of $\Delta\theta$ between P_1 and P_2 is denoted by ι, and is called the "rotational transform angle". Let θ equal 0 at P_1, and let us assume that θ equals 2π for the point P_n. The distance Δr from P_1 to P_n is called the "deviation from closure" of the point n. Evidently Δr measures how far the line of force has strayed from a closed curve. Kruskal has shown that Δr decreases more rapidly than any power of $1/n$. Hence, one may surmise that Δr varies about as exp $(-Kn)$, where K is some dimensionless constant.

The physical reason for this result can best be understood in the special case that the normal component of the magnetic field is constant over the cross-section plane. The analysis of more general systems may be reduced mathematically to a consideration of this special situation. In this case the density of points in the plane must remain constant in successive transformations. Let us now draw in the cross-section plane a closed curve connecting

[7] M. D. Kruskal, U. S. Atomic Energy Commission Report No. NYO-998 (PM-S-5), 1952.

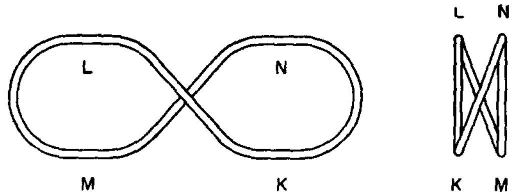

FIG. 2. Top and end views of a figure-eight stellarator.

point P_1 and its successive transformed points as smoothly as possible. Since the magnetic transform now preserves areas, the total area enclosed within this curve must remain constant in successive transformations. Hence all points on the curve cannot move inwards with successive transformations. If some move in, others must move out.

In the special case that the θ coordinate of every point returns to its original value after n transforms, it is possible for some points on the curve, together with all their transformed points, to move steadily in, while the points between move steadily out. Thus the closed curve develops wrinkles in successive transformations; this rate of wrinkling decreases very rapidly with increasing n. In the more general case that the θ coordinate of a point never returns exactly to its initial value (to within a multiple of 2π), one would expect a further averaging out of these radial motions to occur. Even if the transform angle, ι, is not small, the deviation of a line of force from a magnetic surface should be small; if necessary, a group of p transforms, which give an ι very nearly a multiple of 2π, can be taken as the basic transform, and Kruskal's theorem used to demonstrate that a line of force never departs very far from a single closed surface. We shall therefore assume in the following discussions that each line of force does in fact generate a single magnetic surface, and that a single line, if followed sufficiently far, comes arbitrarily close to any point on this surface.

2.2 Methods for Producing a Rotational Transform

To produce a rotational transform in a vacuum field it suffices to twist a torus out of a single plane. Virtually any such distortion will remove the degeneracy of the ideal torus and produce a rotational transform.

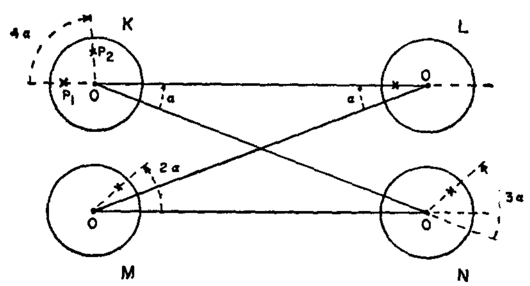

FIG. 3. Rotational transform in figure-eight stellarator.

The simplest such system is the figure-eight, historically the first geometry proposed[8,9] for a stellarator. The topography is indicated in Fig. 2. The 180° curving end sections, LM and KN, are in planes each tilted at an angle, α, to the parallel planes in which the reverse-curvature sections LK and MN are placed. Figure 3 indicates that a rotational transform is present. This figure represents cross-sectional planes at K, L, M, and N, all as seen from one end of the device. The point 0 represents the magnetic axis, while P_1 and P_2 denotes the successive intersections of a single line of force with a cross-sectional plane at K. Intersections of this line of force with the other three planes are denoted by crosses. The solid lines represent the path followed by the magnetic axis. From K to L and from M to N there occur simple translations of the cross-sectional plane, while from L to M and from N to K, a rotation about an axis, inclined at an angle α to the vertical, is involved.

Evidently the line of force which passes through P_1 in plane K, and is then followed through one circuit, through planes L, M, and N, intersects plane K again in a point P_2, rotated by a rotational transform angle ι. Examination of the figure shows that for this geometry,

$$\iota = 4\alpha. \tag{1}$$

Moreover, ι is independent both of distance, r, from the magnetic axis, and of angle, θ. In an actual system, mutual interference between the stray fields of the curving sections LM and MN will modify these results slightly, but the general features remain unchanged. In more general twisted systems of this type, with a purely solenoidal field, ι is simply the integral of the torsion around the magnetic axis, a result first pointed out by Kruskal.[6]

A rotational transform angle may be produced in a variety of other ways. When a plasma current is flowing around the simple torus, a rotational transform appears, despite its absence in the vacuum field. If steady-state confinement is envisaged, however, a rotational transform must be present in the vacuum field, since a plasma current along the magnetic field cannot be maintained in a steady

[8] L. Spitzer, Jr., U. S. Atomic Energy Commission Report No. NYO-993 (PM-S-1), 1951.

[9] L. Spitzer, Jr., U. S. Atomic Energy Commission Report No. NYO-995, (PM-S-3), 1951.

state. The most important alternative method for producing such a rotational transform is the use of a transverse magnetic field, whose direction rotates with distance along the magnetic axis. We shall follow a line of force and show that a transform angle appears. Let us consider an infinite cylinder, with coordinates r, θ, and z. We consider the r and θ coordinates of a single line of force as z increases. The coordinates of points along such a line of force are related by the differential expression

$$\frac{dz}{B_z} = \frac{dr}{B_r} = \frac{r\,d\theta}{B_\theta}. \qquad (2)$$

Suppose now that B_r and B_θ are produced by $2l$ wires, wound helically on the outside of the cylinder, such that currents flow oppositely in adjacent wires, and with a pitch $2\pi l/h$. If we denote the tube radius by r_1, and if hr is small compared to unity, then for small r/r_1 we have, from the appropriate solutions of Laplace's equation

$$B_r = Ar^{l-1}\sin(l\theta - hz), \qquad (3)$$

$$B_\theta = Ar^{l-1}\cos(l\theta - hz), \qquad (4)$$

where A is a constant characterizing the strength of the transverse field. There is also a component of B_z associated with the current in the helical wires, but its magnitude is less than B_r by a factor hr. We assume that B_{z0}, the component of B_z produced by a separate solenoidal winding, is the dominant axial field.

Let us follow a line of force whose coordinates are r_0 and θ_0 in the absence of the transverse field. Equation (2) may now be integrated by means of a power-series expansion in A, the coefficient in Eqs. (3) and (4). To first order in A, $r - r_0$ and $\theta - \theta_0$ vary as the cosine and sine, respectively, of $l\theta - hz$; to this order, the line of force is a helix, and its intersection with a plane moving along the z direction is a circle. Solving to second order in A, we must take into account that for l equal to 2 or more, B_θ is larger on the outside of the circle (r greater than r_0), where B_θ is positive, than on the inside (r less than r_0), where B_θ is negative. As a result, the positive values of $d\theta/dz$ in Eq. (2) more than offset the negative ones, and θ increases systematically with increasing z. A detailed integration by Johnson and Oberman[10] shows that ι_h, the increase of θ in one period $2\pi/h$ of the helical field, is given by

$$\iota_h = \frac{\pi A^2 r^{2l-4}}{h^2 B_{z0}^2}\{2(l-1) + h^2r^2 + O(h^4r^4)\}. \qquad (5)$$

[10] Johnson, Oberman, Kulsrud, and Frieman, Phys. Fluids 1, 281 (1958).

The term in $(hr)^2$ is included to give results for l equal to unity; in this case a rotational transform arises from the variation of B_z with r. The configuration for which l is unity, with a helical magnetic axis, and its use for confining a plasma were proposed by Koenig,[11] who first studied the use of helical fields in connection with the stellarator. For small hr, an appreciable ι is more readily obtained with transverse fields of higher multiplicity for which the transverse field vanishes at the magnetic axis. The properties of such fields, and of the magnetic surfaces associated with them, have been extensively studied[10] by the Matterhorn theoretical group, under E. A. Frieman.

In the experimental program at Matterhorn, described in the subsequent papers,[12–16] rotational transforms have been produced both with the figure-eight geometry and with transverse fields, with l equal to 3. In either case, the existence of a rotational transform is readily confirmed experimentally by observation of a narrow electron beam, which follows the line of force. As pointed out subsequently, the chief advantage of the multipolar transverse field over the figure-eight is the greater hydromagnetic stability which, in theory, it should yield.

3. CONFINEMENT

The objective of confinement theory is to demonstrate that the number of particles striking the tube wall is negligibly small. An exact proof would presumably require a detailed solution of the Boltzmann equation, together with the field equations. For the complex magnetic configuration of the stellarator this would be a difficult task indeed. An approximate treatment will be followed here.

First we shall use the macroscopic equations for the plasma, based on a number of simplifying assumptions. With these equations, together with the field equations, we can show that an equilibrium situation is possible in which the plasma is macroscopically confined. Since particles moving with some particular velocity might conceivably escape, even though the plasma as a whole were confined, we

[11] H. R. Koenig, U. S. Atomic Energy Commission Report No. NYO-7310 (PM-S-20), 1956.
[12] Coor, Cunningham, Ellis, Heald, and Kranz, Phys. Fluids (to be published).
[13] Kruskal, Johnson, Gottlieb, and Goldman, Phys. Fluids (to be published).
[14] Bernstein, Chen, Heald, and Kranz, Phys. Fluids (to be published).
[15] Burnett, Grove, Palladino, Stix, and Wakefield, Phys. Fluids (to be published).
[16] T. H. Stix and R. W. Palladino, Phys. Fluids (to be published).

shall next consider the trajectories of single particles in the electric and magnetic fields determined from the macroscopic equations. These two approximate methods, taken together, indicate nearly perfect confinement, if collision and cooperative phenomena are ignored. The hydromagnetic stability of these equilibria has been extensively analyzed by Frieman, Kruskal, and their collaborators. The result of this analysis, summarized briefly at the end of Sec. 3.2, indicate stable equilibrium under certain conditions. All these results, taken together, encourage the belief that magnetic confinement in a stellarator may be adequate for a controlled thermonuclear reactor.

3.1 Macroscopic Equations

The two-fluid equations for the velocity and electric current of a fully ionized gas are well known.[17,18] For the derivation of these equations and their application here the following nontrivial assumptions are made:

(a) The electrical resistivity η is negligibly small, and the mean free path very long.

(b) Over a distance of one radius of gyration the relative change of all macroscopic quantities is small.

(c) The transverse and longitudinal pressures, $p_\perp$ and $p_\parallel$ are equal.

(d) All macroscopic quantities are independent of time at each position.

(e) The mean macroscopic velocity, $\mathbf{v}$, vanishes.

Assumption (a) is clearly a good approximation for a rarefied gas at very high temperatures. Collisions between particles will produce some diffusion across the lines of magnetic force,[18] but this rate is so slow compared both with the diffusion rate observed[12] and with the diffusion rate that could be tolerated that we may neglect collisions entirely in most discussions of confinement. Once this first assumption has been made, assumption (b) is required for use of the macroscopic equations. This second assumption has a number of important consequences. Since the sheath thickness is generally much less than the radius of the gyration, the plasma must be characterized by approximate electrical neutrality, with the electron density n_e closely equal to Z times the ion density n_i, where Z is the mean ionic charge. In addition, this assumption leads[19,20] to the result that the material stress tensor is diagonal, provided that the principal axis is parallel to the magnetic field; the three components are then $p_\parallel$ parallel to the field and $p_\perp$ in the two directions perpendicular to the field. In any device much larger than the radius of gyration, assumption (b) should be approximately valid in a steady state, except in a thin layer near the wall where this assumption must break down. Assumption (c) should also be valid in a system where any changes are even slower than the time between collisions, since collisions will clearly tend to equalize $p_\parallel$ and $p_\perp$.

Assumption (d) is of critical importance in the analysis. The assumption of a steady state immediately implies that the plasma is quiescent, and excludes turbulence, oscillations, instabilities, and other cooperative phenomena of the sort normally present in gaseous discharges. Any equilibria obtained on the basis of this assumption may not necessarily be stable against various types of disturbances. The last assumption replaces the more usual one (which partly results from assumption (b)) that quadratic terms in $\mathbf{v}$ and $\mathbf{j}$ are negligible. This more stringent condition is not so arbitrary as might first appear. In fact, the near vanishing of $\mathbf{v}$ is a simple consequence[21] of the equation of motion. The argument is that in most heating methods there are no appreciable forces tending to produce any momentum, and hence the macroscopic velocity must be vanishingly small. A fuller analysis of effects associated with macroscopic velocities would be desirable.

On the basis of these assumptions, the equations of equilibrium become, in emu,

$$\mathbf{j} \times \mathbf{B} = \nabla p, \tag{6}$$

$$\nabla \times \mathbf{B} = 4\pi \mathbf{j}, \tag{7}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{8}$$

In addition, the generalized Ohm's law determines the electric field, in terms of the pressure gradient for the positive ions, while Poisson's law then gives the charge density. Since neither of these quantities is of particular significance in the present analysis these equations may be omitted.

On the basis of these equations it was shown[22] several years ago that no simple equilibrium is possible in a torus if the lines of force are assumed

[17] A. Schlüter, Z. Naturforsch. **5a**, 72 (1950).

[18] L. Spitzer, Jr., *Physics of Fully Ionized Gases* (Interscience Publishers, Inc., New York, 1956).

[19] K. M. Watson, Phys. Rev. **102**, 12 (1956).

[20] Chew, Goldberger, and Low, Proc. Roy. Soc. (London) **A236**, 112 (1956).

[21] See reference 18, pp. 44–45.

[22] L. Spitzer, Jr., U. S. Atomic Energy Commission Report No. NYO-997 (PM-S-4), 1952.

circles centered at the axis of symmetry. We introduce cylindrical coordinates R, φ, and z, with z taken along the axis of symmetry of the torus; we assume that only B_φ differs from zero, and that all quantities are independent of φ. If we now take the curl of Eq. (6), eliminating $\mathbf{j}$ by means of Eq. (7), we obtain

$$\partial B_\varphi / R\partial z = 0. \tag{9}$$

Thus for equilibrium either R must be infinite or the magnetic field (and pressure) must be independent of z. Equation (9) for toroidal fields ($B_R = B_z = 0$) corresponds to a theorem by Ferraro[23] for poloidal fields ($B_\varphi = 0$) in connection with the theory of magnetic stars. It may be remarked that the left-hand side of Eq. (9) is simply $-4\pi\nabla\cdot\mathbf{j}$; i.e., the currents required by Eq. (6) for the simple torus possess a nonvanishing divergence.

3.2 Plasma Equilibrium in the Stellarator

We next apply these equations to a stellarator, characterized by the existence of toroidal magnetic surfaces. From Eq. (6) it is evident that ∇p is perpendicular to $\mathbf{B}$ and hence p is constant along a line of force and must therefore be constant over an entire magnetic surface. Similarly $\mathbf{j}$ is perpendicular to ∇p and, therefore, $\mathbf{j}$ must be parallel to the isobaric surfaces, and hence to the magnetic surfaces. We denote by $\mathbf{j}_\perp$ the component of $\mathbf{j}$ perpendicular to $\mathbf{B}$ (and to ∇p) and by $\mathbf{j}_\parallel$ the component of $\mathbf{j}$ parallel to $\mathbf{B}$.

An important result, which permits plasma equilibrium in the stellarator, will now be established. Equation (6) determines $\mathbf{j}_\perp$ in terms of $\mathbf{B}$ and $\mathbf{p}$. In general, $\nabla\cdot\mathbf{j}_\perp$ will not be zero. In the torus the divergence of this current gives rise to charge separation which destroys equilibrium, since conditions are uniform along each line of force and an accumulation of electric charge cannot readily leak out across the lines of force. In the stellarator, currents along the lines of force are possible, and charges may flow from a region where $\nabla\cdot\mathbf{j}_\perp$ is positive to another where $\nabla\cdot\mathbf{j}_\perp$ is negative. The total divergence of $\mathbf{j}_\perp$, integrated over the volume element between two adjacent magnetic surfaces, must vanish, as may be seen by use of Gauss's theorem, together with the fact that $j_\perp$ is parallel to the magnetic surface. Hence currents flowing along the lines of force can cancel out the divergences in $\mathbf{j}_\perp$, and will lead to a current system in which $\nabla\cdot\mathbf{j}$ vanishes.

[23] V. C. A. Ferraro, Astrophys. J. **119**, 407 (1954).

The existence of solutions to Eqs. (6)–(8), in a system characterized by magnetic surfaces, has been analyzed in an elegant manner by Kruskal and Kulsrud,[24] who also take diffusion into account. Here we follow an earlier and simpler treatment,[22] and demonstrate the existence of solutions to Eqs. (6)–(8) by the simple artifice of showing how to construct such solutions. We assume that p is a small quantity, and obtain a solution by successive iteration. The zero-order solution is taken as the vacuum solution, with j_0 the current in the external coils, and B_0 the vacuum field, with its magnetic surfaces. The solution of order n is then defined by the equations

$$\mathbf{j}_n \times \mathbf{B}_{n-1} = \nabla p_n, \tag{10}$$

$$\nabla \times \mathbf{B}_n = 4\pi \mathbf{j}_n, \tag{11}$$

$$\nabla\cdot\mathbf{B}_n = 0. \tag{12}$$

Evidently p_n must be assumed constant on the magnetic surface obtained in the previous iteration, but is otherwise arbitrary. It may generally be assumed that p_n is in each case a monotonically decreasing function of distance from the magnetic axis. If the solution converges, it must evidently yield a solution of Eqs. (6)–(8).

In principle this iteration scheme is straightforward, but in practice the algebra is cumbersome. It turns out that the chief obstacle to convergence is the distortion of the magnetic surfaces by the magnetic fields associated with the plasma current $j_\parallel$, along the magnetic field. Even though this current density is low, the currents must travel an appreciable distance, and even the weak magnetic field associated with these currents may distort the magnetic surfaces out of all recognition.

The approximate condition for convergence in a sinple figure-eight device may be derived in the simplest manner. The transverse current $j_\perp$ is evidently about equal to p/B_0r, where B_0 is the axial vacuum field and r is the distance of the tube wall, or plasma boundary, from the magnetic axis. The divergence of $j_\perp$ results from the inverse proportionality between B_0 and R, and is about equal in magnitude to p/B_0rR, where we take R to be the radius of curvature of the magnetic axis. The divergence of $\mathbf{j}_\parallel$ is also equal to this quantity, and over a tube length equal to R, $j_\parallel$ will build up to about p/B_0r. Hence $j_\parallel$ is of the same order of magnitude as $j_\perp$. Over a tube cross section, $\mathbf{j}_\parallel$ will have opposite directions on opposite sides. The

[24] M. D. Kruskal and R. M. Kulsrud, Phys. Fluids **1**, 265 (1958).

magnetic field on the axis due to this plasma current will be of the order $2\pi r j_{\parallel}$, or $2\pi p/B_0$. The new magnetic axis therefore will be inclined at an angle $2\pi p/B_0^2$ relative to its direction *in vacuo*. The condition for negligible distortion of the magnetic surfaces is that the deviation of the magnetic axis be small compared to r. If the currents j flow along half the axial length, L, of the machine, on the average, before cancellation, this condition yields

$$\beta \ll 8r/L; \tag{13}$$

β, the ratio of material to magnetic pressure, is defined by

$$\beta = 8\pi p/B_0^2, \tag{14}$$

where p is evaluated on the magnetic axis. A more precise discussion,[22] taking into account the detailed variation of $j_{\parallel}$ over the cross section, yields a coefficient $16/\pi$ instead of 8 in Eq. (13); this computation assumes that L much exceeds $2\pi R$ and that the transform angle 4α (see Fig. 3) is small. If inequality (13) is not satisfied, the method of iteration fails, and it is not known what type of solution, if any, may exist.

In an infinite cylinder, values of β as great as unity might be envisaged. In the figure-eight stellarator, of the type shown in Fig. 2, r/L can scarcely exceed 0.02, and β must therefore be small compared to 0.1. If the rotational transform is produced by transverse fields, however, the transform angle, ι, for the device may much exceed 2π, in principle. It is readily shown that the upper limit on β is proportional to $\iota^2 r/R$, in this situation, and hence equilibrium values of β substantially greater than 0.1 should be possible, although at the cost of somewhat greater over-all axial length.

The question of the hydromagnetic stability of such configurations has been extensively studied by the Matterhorn theoretical group, under E. Frieman. Basic concepts and methods of analysis have been published by Bernstein, Frieman, Kruskal, and Kulsrud,[25] with application to the stellarator in the paper by Johnson, Oberman, Kulsrud, and Frieman.[10] Because of the importance of this work, the results will be summarized briefly here.

Instabilities tend to be most marked if the lines of force can interchange positions with the least possible bending. In the case of an axially symmetric field, if B_θ vanishes everywhere, the lines of force can interchange positions without bending, and if bulges are present in the field the plasma is unstable. However, if a B_θ component is present, so that lines of force are helices about the cylinder axis, and if B_θ/r increases with r so that the pitch of the helices decreases with increasing r, then the outer and inner lines of force cannot be interchanged without appreciable bending, and the plasma tends to be stable. In the same way, if the transform angle, ι, varies with r in a stellarator, the outer lines of force are topologically different from the inner ones, and the plasma is stable against all hydromagnetic disturbances, provided that β is less than some critical value β_c. Computations of β_c for a cylinder, with helical transverse fields, with $l = 3$, added to an axial confining field, indicate[10] that values of β_c as great as 0.1 could be obtained if the approximate theory could be trusted somewhat beyond its range of validity. There is some reason to believe that corrections for finite radius of gyration may increase β_c by a factor of about two, although the theory is still very incomplete in this respect. An experimental test of this theory has not yet been obtained, although the corresponding theory applied to kink instability in the stellarator (see Sec. 4) is apparently in close agreement with the observations. The maximum value of β for which a stellarator plasma is stable can probably best be determined by experiment.

3.3 Single Particles

In the absence of collisions, confinement of single particles will be shown to follow quite generally from the existence of a rotational transform and from the asymptotic behavior of a gyrating particle in a strong magnetic field. It has been shown by Kruskal[26] that the magnetic moment, μ (about equal by $mw_\perp^2/2$), of a gyrating particle is constant to all orders of ak, where a is the gyration radius and k is about $|\nabla(\ln B)|$. Constancy of μ to first order in ak had previously been demonstrated by Alfvén,[5] and to second order by Hellwig.[27] Similarly, Kruskal has also shown[26] that the motion of the guiding center is independent of the phase of gyration to all orders of ak. Kruskal's theory does not yield either a simple definition of μ, to all orders in ak, nor yet a simple definition of the guiding center, but shows that such definitions must exist and how, in principle, to construct them. In consequence of these results we may assume that for each particle the total energy, W, is not the only integral of the

[25] Bernstein, Frieman, Kruskal, and Kulsrud, Proc. Roy. Soc. (London) **A244**, 17 (1958).

[26] M. D. Kruskal, *Proceedings of the Third International Conference on Ionization Phenomena in Gases*, Venice, 1957 (to be published).

[27] G. Hellwig, Z. Naturforsch. **10a**, 508 (1955).

motion, but there exists also a second integral, the magnetic moment, μ.

Thanks to these results it may now be shown that successive intersections of particles with a particular cross-section plane, similar to that discussed in Sec. 2.1, produce a transformation similar in its properties to the magnetic transform generated by successive intersections of a line of force with this same plane. We restrict consideration to particles within ranges dW and $d\mu$ centered at some energy, W, and some magnetic moment, μ, and let the density in phase space, within these narrow ranges of W and μ, be constant everywhere. Within an interval of time Δt, the guiding centers of these particles will intersect the cross-sectional plane at a number of points, P_1. The density of such points will be a known function of position in the plane, depending on the magnetic field B, and the electric potential Φ, through the two integrals of motion.

Each particle whose guiding center has intersected the plane at a point P_1 will ultimately cross the plane again with the guiding center intersecting at a point P_2. Normally, if a particle passes through a point, the three components of its velocity, $\mathbf{w}$ are arbitrary, and its subsequent trajectory is not determined. In the present case, the two integrals of motion determine $w_\perp$ and $w_\parallel$, and the third velocity component, the phase of gyration, has no effect on the motion. Hence to each point P_1, there corresponds one and only point P_2. Moreover, the particles which have produced all the points P_1 in the time Δt will all produce points P_2 within the same time interval, and hence the density of intersection points in the transformed plane will be the same function of position as in the original plane. Thus this "particle transform" is measure preserving in the same sense as is the magnetic transform discussed earlier.

From the same arguments as before it follows that free particles are confined in a stellarator to a very high approximation, provided that the particle transform is primarily rotational. Such a transform will be assured for particles whose velocity is mostly parallel to $\mathbf{B}$ ($w_\parallel > w_\perp$), so that no reflection can occur from regions of relatively high field. The rotational magnetic transform guarantees a rotational particle transform for such particles. Qualitative experimental confirmation of this prediction is obtained from observations of runaway electrons reported[12] by the Matterhorn experimental group under M. B. Gottlieb. Electrons traveling at speeds near the velocity of light are observed to make about 5×10^5 circuits around a stellarator, during the ten milliseconds or so after the applied voltage is reduced to zero, but the confining field is still moderately high.

For particles which are trapped between two regions of relatively high field, or which are moving at a relatively very slow rate along the magnetic field, further arguments must be invoked to guarantee a primarily rotational particle transform. Two separate mechanisms are important. Firstly, the diamagnetic effect of the plasma produces a radial gradient of the axial field, and this inhomogeneity produces a drift of guiding centers about the magnetic axis. Secondly, a radial electric field will produce a similar rotation. Such an electric field is required by the assumption that the macroscopic velocity vanish, since only a radial electric field can cancel out, in a steady state, the velocity associated with a radial pressure gradient. It has been shown by Spitzer[22,28] that such a radial field arises naturally when the gas is ionized and heated. A more detailed analysis[29] of these phenomena indicates that these two effects produce adequate rotation about the magnetic axis to guarantee confinement for most particles with relatively low $w_\parallel$.

One important exception should be noted. For some particles the different effects producing rotation may cancel out, leaving only the unidirectional drift produced by the curvature of the field. The seriousness of this effect is reduced by two factors. As distance from the magnetic axis changes, the different mechanisms producing rotation will change in different ways, and the cancellation will disappear. Moreover, the cancellation will be exact only for a particular particle energy, W, and a particular magnetic moment, μ; collisions will change these quantities, and restrict the time during which a unidirectional drift will occur. These effects have been discussed elsewhere[9,29]; the analysis, while admittedly approximate and incomplete, indicates that this process is not of great importance, although it may increase the diffusion rate somewhat above the value given by electron-ion collisions.

4. HEATING

A gas can be ionized and heated, in general, by energetic particles, by photons, or by electric fields. Consideration of heating in a stellarator has been limited to electric fields. Two general types of

[28] L. Spitzer, Jr., Astrophys. J. **116**, 299 (1952); see pp. 308–309.
[29] L. Spitzer, Jr., U. S. Atomic Energy Commission Report No. NYO-7316 (PM-S-26), 1957.

heating by electric fields may be distinguished, depending on whether the electric field, **E**, is parallel to or perpendicular to the magnetic field, **B**. In the configuration of the stellarator, an electric field parallel to **B** can be produced only by induction, by changing the magnetic flux threading the toroidal tube. Since the current induced by this field heats the gas by ohmic, or Joule, losses, this process is known as "ohmic heating." An electric field perpendicular to **B** could be produced electrostatically. However, a plasma shields itself so effectively against electrostatic fields that primary consideration has been given to electric fields induced by pulsating the magnetic field. Heating by this method is called "magnetic pumping" or, in the special case that the pulsation frequency is chosen close to the ion cyclotron frequency, "ion cyclotron resonance heating." The general principles involved in these heating methods are discussed in the following.

4.1 Ohmic Heating

In ohmic heating the only function of the magnetic field, in principle, is to prevent lateral diffusion, and the heating can be analyzed, to a first approximation, without regard to the magnetic field. The electrical breakdown of a gas is well known, and has been extensively analyzed elsewhere. Theoretical consideration at Princeton[30–33] has been devoted to the final ionization and heating of helium or hydrogen gas by means of a unidirectional pulse of constant voltage; the initial ionization level was assumed to be about ten percent.

The problem of changes within a gas, when an electric field is applied, is very complicated in the general case, since both the radiation field and the electron velocity distribution may alter in complex ways. Under conditions of interest in the laboratory the radiation field is generally weak enough to be negligible. To make the problem tractable the electron velocity distribution may be assumed Maxwellian. The chief processes that need be considered other than straight heating, are excitation, ionization, charge exchange, etc., and in principle, at least, the rates of these processes are simple functions of the electron temperature.

The assumptions made in the theoretical work, and the results obtained, are given in the accompanying paper by Berger, Bernstein, Frieman, and Kulsrud.[33] The results indicate clearly that ionization and heating of hydrogen or helium by this technique should be entirely feasible, and temperatures of 10^6 degrees K should be obtainable. At higher temperatures the plasma resistivity becomes so low that heating with practical currents becomes difficult. The detailed predictions of the theory are somewhat uncertain because the basic assumption of a Maxwellian distribution may not be entirely realistic. The mean free path of an electron increases so rapidly with increasing energy that electrons which are in the tail of the Maxwellian distribution and which are sufficiently energetic to excite and ionize atoms may gain very appreciable energies in one free path.

Before any experiments had been carried out on ohmic heating, it was pointed out by Kruskal[34] that discharges in the stellarator should be subject to kink instability. This hydromagnetic instability was predicted for heating currents greater than the critical current that will reduce the transform angle to 0 (or increase it to 2π). This critical current is now generally known as the "Kruskal limit."

Extensive observations of ohmic heating have been carried out by the Matterhorn experimental group, under M. Gottlieb, and are reported in several subsequent experimental papers.[12–15] The data indicate clearly that nearly complete ionization is attained, with electron and ion temperatures in the neighborhood of 5×10^5 degrees K. The occurrence of the predicted kink instability, at currents above the Kruskal limit, is fully verified experimentally. However, the detailed predictions of the ohmic heating theory are not substantiated, presumably because of the non-Maxwellian distribution of electron velocities. In support of this hypothesis, intense x-rays from runaway electrons are observed, with energies up to 10^6 ev.

These data emphasize the very great importance of impurities from the walls streaming into the discharge. In the early observations the carbon and oxygen ions presumably outnumbered the helium ions during the later stages of the discharge, and sharply reduced the electron temperature. With the use of ultra-high vacuum techniques, resulting in base pressures below 10^{-9} mm of Hg and relatively clean surfaces, the efflux of wall impurities has been reduced by more than an order of magnitude.

Another method of reducing the impurity level has been use of a divertor. This device was proposed

[30] J. M. Berger, and E. A. Frieman, U. S. Atomic Energy Commission Report No. NYO-6046 (PM-S-16), 1954.
[31] J. M. Berger and L. M. Goldman, U. S. Atomic Energy Commission Report No. NYO-7311 (PM-S-21), 1956.
[32] J. M. Berger, U. S. Atomic Energy Commission Report No. NYO-7312 (PM-S-22), 1956.
[33] Berger, Bernstein, Frieman, and Kulsrud, Phys. Fluids 1, 297 (1958).
[34] M. D. Kruskal, U. S. Atomic Energy Commission Report No. NYO-6045 (PM-S-12), 1954.

relatively early[8] to take away from the discharge the particles nearest the wall and to avert bombardment of the discharge tube by charged particles. In the divertor, an outer shell of flux is diverted or bent away from the main discharge into a large auxiliary chamber. Any impurities produced by wall bombardment in the divertor chamber return relatively slowly into the main discharge tube. A schematic diagram of a divertor is shown in Fig. 4; the device has cylindrical symmetry about the magnetic axis. The theory of this device, together with observations on its effectiveness in reducing the impurity level, without an ultra-high vacuum, is reported in the subsequent paper by Burnett, Grove, Palladino, Stix, and Wakefield.[15] Apparently the divertor reduces the ratio of impurity ions to helium ions by a factor of about one-fifth.

The most important new observational result that has emerged from these ohmic heating studies is the evidence on cooperative phenomena. During ohmic heating the plasma is anything but quiescent. Runaway particles start abruptly to hit the tube wall, producing x-rays, sometimes in short bursts, at times dependent on the magnitude of the confining field. When voltage is applied around the stellarator, the plasma is not well confined by the magnetic field, and reaches the wall in about 10^{-4} sec, presumably because of cooperative phenomena of some sort. After ohmic heating, a current of runaway electrons, amounting to some 10 amp/cm^2 may persist for several milliseconds after the voltage is turned off, and then abruptly disappear, producing a burst of x-rays and additional ionization and excitation in the plasma. The detailed study of these phenomena should increase our understanding of plasma physics and enable one to predict how an ionized gas might behave in a full-scale thermonuclear reactor.

4.2 Magnetic Pumping

Pulsation of an axial confining field produces an oscillating electrical field, encircling the tube axis. This electric field can increase the energy of gyrating charged particles. If the pulsation frequency is much less than the cyclotron frequency, however, this increase in energy is computed most simply from the magnetic moment, μ, which, according to the results by Kruskal[26] should be very accurately constant. Under these conditions, the increase in kinetic energy of motion, transverse to the field, is given by the usual formula for adiabatic compression of a gas, provided γ is set equal to 2, corresponding to the presence of two degrees of freedom. Instead of thinking about the electric fields induced by the pulsating magnetic field, we may think of the external lines of force as constituting a piston, and the pulsation of the external field as providing a pumping action.

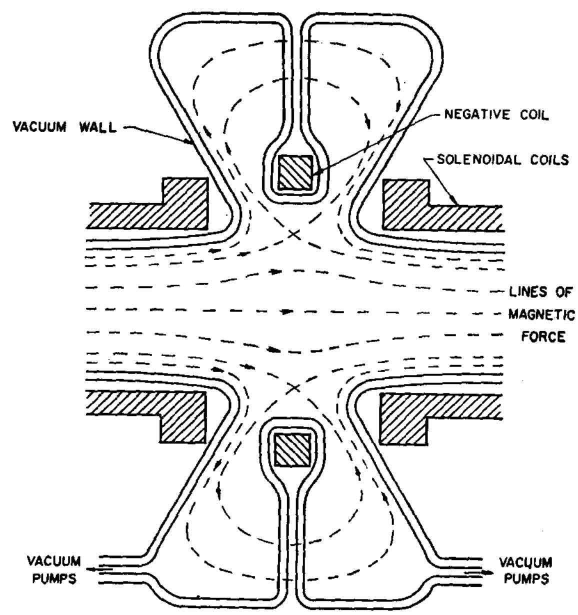

FIG. 4. Divertor.

Evidently if the plasma were entirely adiabatic, or isothermal, the work done on the gas during the compression would just equal the work done on the piston during expansion. No heating would result. To obtain net heating in a pumping cycle there must be a phase lag between temperature and density. Such a phase lag may be produced in a variety of different ways, and hence there are many frequencies at which magnetic pumping can be effective.

One mechanism for producing such a phase lag is the effect of collisions in exchanging energy between motions parallel and perpendicular to the lines of force. This effect was analyzed early[35] at Project Matterhorn, as a possible substitute for ohmic heating, but was dropped because it was not an effective method for completing the ionization of a gas. More recently, this mechanism has been analyzed by Berger and Newcomb,[36] and, independently, by Schlüter,[37] with identical results. The analysis at Princeton is given in the subsequent paper by the Matterhorn theoretical group.[38] It is

[35] L. Spitzer, Jr., and L. Witten, U. S. Atomic Energy Commission Report No. NYO-999 (PM-S-6), 1953.
[36] J. M. Berger and W. A. Newcomb, U. S. Atomic Energy Commission Report No. NYO-6046 (PM-S-13), 1954.
[37] A. Schlüter, Z. Naturforsch. **12a**, 822 (1957).
[38] Berger, Newcomb, Dawson, Frieman, Kulsrud, and Lenard, Phys. Fluids **1**, 301 (1958).

clear that magnetic pumping at the positive-ion collisional frequency can, in principle, heat a fully ionized gas to very high temperatures; however, the rate of heating falls off as $T^{-\frac{1}{2}}$ with increasing temperature, an inconvenient drop.

Another method of producing the desired phase lag is to pump in a short section of tube, with a pulsation period about equal to the time required for a positive ion to travel through the pumping section. In this situation the temperature lags because of loss of heat out the ends. If the mean free path is short compared to the length of the pumping section, magnetic pumping at this frequency produces acoustic waves, which travel along the magnetic field. For long mean free paths, the particles may be treated as free, and the energy is effectively thermalized by fine-scale mixing. The analysis[36] of this "transit-time heating," reported in a subsequent Matterhorn paper[38] shows that this method should be an effective means for heating a plasma to very high temperatures, particularly since the rate of energy input increases as $T^{\frac{3}{2}}$ with increasing temperature, if the frequency is optimized at each temperature.

Experimental verification of heating by magnetic pumping has not yet been possible at Project Matterhorn, since the radio-frequency power available has not been sufficient to balance the losses from the plasma, due to inadequate confinement and too high an impurity level.

4.3 Ion-Cyclotron Resonance Heating

Pulsation of the confining field at the cyclotron frequency of the positive ions should give very rapid heating at very low ion densities. The effect is most conveniently understood not as a macroscopic pumping but as a microscopic resonance between the oscillating electric field and the gyration of the positive ions. However, this type of heating produces separation of charges, and at appreciable plasma densities the resultant electrostatic fields prevent any appreciable heating in this way. It was pointed out by Stix that use of two adjacent heating sections, with identical pulsation frequencies, but differing in phase by 180°, would make it possible for electrons to cancel out this separation of charges by flowing back and forth along the lines of force, and thus permit heating at the ion cyclotron frequency even at substantial plasma densities.

The axisymmetric free oscillations of a cylindrical plasma column, in a strong axial field, were analyzed in detail by Stix,[39] who found that indeed for sufficiently short wavelengths a plasma resonance existed close to the ion cyclotron frequency. Later analyses both by Stix[40] and by Kulsrud and Lenard[38] have considered the input of energy into the gas both at the exact ion cyclotron frequency and at the adjacent plasma resonance frequency. It appears that a substantial amount of energy can be fed into the gas at the plasma resonance frequency, and that thermalization of the energy can readily be achieved in a small system. At low β (low ratio of material pressure to magnetic pressure in the vacuum field), this technique offers the great advantage, in principle, over magnetic pumping that the coupling between the external circuits and the plasma is very much better. For a large system, at moderate β, it is not obvious from theory alone which system would be most effective.

Detailed observational results on plasma heating at frequencies adjacent to ion cyclotron resonance are reported in the subsequent experimental paper by Stix and Palladino.[16] Measures on the external circuits at low power indicate that, in fact, resonant loading is observed at frequencies at and below the cyclotron frequencies both for hydrogen and helium. The energy fed into the gas exceeds the energy dissipated in the external circuit, a prerequisite for efficient heating. Measures at high power appear to be generally consistent with expectations, although direct evidence of plasma heating has not, as yet, been obtained.

[39] T. H. Stix, Phys. Rev. **106**, 1146 (1957).
[40] T. H. Stix, Phys. Fluids 1, 308 (1958).