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Finite-Resistivity Instabilities of a Sheet Pinch

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The stability of a plane current layer is analyzed in the hydromagnetic approximation, allowing for finite isotropic resistivity. The effect of a small layer curvature is simulated by a gravitational field. In an incompressible fluid, there can be three basic types of "resistive" instability: a long-wave "tearing" mode, corresponding to breakup of the layer along current-flow lines; a short-wave "rippling" mode, due to the flow of current across the resistivity gradients of the layer; and a low-g gravitational interchange mode that grows in spite of finite magnetic shear. The time scale is set by the resistive diffusion time τ_R and the hydromagnetic transit time τ_H of the layer. For large $S = \tau_R/\tau_H$, the growth rate of the "tearing" and "rippling" modes is of order $\tau_R^{-3/5}\tau_H^{-2/5}$, and that of the gravitational mode is of order $\tau_R^{-1/3}\tau_H^{-2/3}$. As $S \to \infty$, the gravitational effect dominates and may be used to stabilize the two nongravitational modes. If the zero-order configuration is in equilibrium, there are no overstable modes in the incompressible case. Allowance for plasma compressibility somewhat modifies the "rippling" and gravitational modes, and may permit overstable modes to appear. The existence of overstable modes depends also on increasingly large *zero-order* resistivity gradients as $S \to \infty$. The three unstable modes merely require increasingly large gradients of the *first-order* fluid velocity; but even so, the hydromagnetic approximation breaks down as $S \to \infty$. Allowance for isotropic viscosity increases the effective mass density of the fluid, and the growth rates of the "tearing" and "rippling" modes then scale as $\tau_R^{-2/3}\tau_H^{-1/3}$. In plasmas, allowance for thermal conductivity suppresses the "rippling" mode at moderately high values of S. The "tearing" mode can be stabilized by conducting walls. The transition from the low-g "resistive" gravitational mode to the familiar high-g infinite conductivity mode is examined. The extension of the stability analysis to cylindrical geometry is discussed. The relevance of the theory to the results of various plasma experiments is pointed out. A nonhydromagnetic treatment will be needed to achieve rigorous correspondence to the experimental conditions.

I. INTRODUCTION

A PRINCIPAL result of pinch[1,2] and stellarator[3] research has been the observed instability of configurations that the hydromagnetic theory[4,5] would predict to be stable in the limit of high electrical conductivity. In order to establish the cause of this observed instability, the extension of the hydromagnetic analysis to the case of finite conductivity becomes of considerable interest.

A number of particular "resistive" instability modes have been discussed in previous publications. Dungey[6] has shown that, at an x-type neutral point of a magnetic-field structure in plasma, finite conductivity can give rise to an unstably growing current concentration. By Dungey's mechanism, a sheet pinch can tear along current-flow lines, so as

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to form discrete parallel filaments.[7,8] This "tearing" mode is purely growing and is symmetric about the midplane of the sheet pinch.

Murty[9] has analyzed the case of a very-low-conductivity incompressible fluid slab of finite thickness, and has found two purely growing modes: the symmetric "tearing" mode; and an asymmetric "rippling" mode. In the latter case, the conductivity gradient at the edge of the slab permits current channeling into first-order "ripples" that run at an angle with respect to both the zero-order current and the zero-order magnetic field. The resultant motor force amplifies the ripples.

Aitken *et al.*[2,10] have treated cylindrical geometry, and have found a purely growing (helical) "rippling" mode in the very-low-conductivity limit. In the high-conductivity limit, they find an overstable "rippling" mode.[2,11] The ripples in the latter case run in the direction of the mean zero-order current. The existence of overstability depends on the compressibility of the fluid and on large resistivity gradients.

The instability of the positive column[12,13] is somewhat related to the instability of fully ionized plasmas of finite conductivity. Kadomtsev and Nedospasov[14] have demonstrated a "rippling" mode, which is purely growing in the rest frame of the electrons, but is overstable in the laboratory frame. The extension of this mode to fully ionized plasmas has been considered by Hoh,[15] Kuckes,[16] and Kadomtsev.[17]

In the present analysis, general equations are derived for the plane resistive current layer in the incompressible hydromagnetic approximation. A dispersion relation is obtained in the limit of high conductivity that describes purely growing modes of the "tearing" and "rippling" types. An interchange mode driven by a gravitational field perpendicular to the plane layer is also included.

The analysis for the plane current layer is particularly significant in the high-conductivity limit, since the problem then separates into the analysis of two regions: (1) a narrow central region, where finite conductivity permits relative motions of field and fluid, and where geometric curvature may be neglected; (2) an outer region, where field and fluid are coupled as in the infinite-conductivity case, and where generalizations to nonplanar geometry can be introduced as desired.

In Sec. II the problem is delineated and the basic assumptions and equations displayed. In Secs. III–V a formal mathematical solution is developed. In Sec. VI the basic physical mechanisms are discussed and a simple heuristic derivation and summary of the results is given. For those not interested in mathematical details or preferring a preliminary physical discussion, it is suggested that Sec. VI be read prior to Secs. III–V. Section VII is devoted to a comparison with experiment. The effects of various generalizations and extensions of the basic problem are considered in the appendixes as follows: Appendix A. Compressibility; Appendix B. Low-Conductivity Limit; Appendix C. Short Wavelength; Appendix D. Long-Wavelength Limit; Appendix E. The Transition to the ∞-Conductivity Limit of the Rayleigh–Taylor Instability; Appendix F. Thermal Conductivity; Appendix G. External Conductors; Appendix H. Viscosity; Appendix I. Cylindrical Geometry.

II. ASSUMPTIONS AND BASIC EQUATIONS

We treat an infinite plane current layer specified by

$$\mathbf{B}_0 = \hat{x}B_{x0}(y) + \hat{z}B_{z0}(y) \tag{1}$$

The following assumptions are made.

1. The hydromagnetic approximation is assumed to be valid, and the ion pressure and inertia terms are neglected in Ohm's law.

$$\partial\mathbf{B}/\partial t = \nabla\times(\mathbf{v}\times\mathbf{B}) - \nabla\times[(\eta/4\pi)\nabla\times\mathbf{B}]. \tag{2}$$

As the analysis will show, these assumptions are violated in the treatment of a plasma of sufficiently high conductivity, since the "resistive" modes then develop increasingly sharp discontinuities, and we must expect "finite-Larmor-radius" effects. Plasma stability in the limit of high but finite conductivity (the limit of maximum practical interest) thus depends critically on nonhydromagnetic effects. An isotropic resistivity η is assumed in Eq. (2), and the mass of the electrons is neglected. It is of interest to note that inclusion of the electron-inertia term in Ohm's law gives rise to a "tearing" mode in

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the collisionless limit[18] that is analogous to the "resistive tearing" mode considered here.

2. The fluid is assumed to be incompressible.

$$\nabla \cdot \mathbf{v} = 0. \tag{3}$$

In the high-conductivity limit the effect of compressibility on the fluid dynamics is negligible. (See Appendix A.)

3. Viscosity is neglected, so that the equation of motion may be written as

$$\nabla \times (\rho\, d\mathbf{v}/dt) = \nabla \times [(1/4\pi)(\nabla \times \mathbf{B}) \times \mathbf{B} + \mathbf{g}\rho] \tag{4}$$

where ρ is the mass density and $\mathbf{g}$ the acceleration due to gravity. As usual, the $\mathbf{g}\rho$ term may be interpreted as resulting from acceleration of the current layer, or from the interaction of a plasma pressure gradient and a slight curvature of the current layer.[19] The effect of viscosity is discussed in Appendix H.

4. Perturbations in plasma resistivity are assumed to result only from convection.

$$\partial\eta/\partial t + \mathbf{v} \cdot \nabla\eta = 0. \tag{5}$$

The neglect of thermal conductivity along magnetic field lines, however, becomes important for a high-temperature plasma,[17] and we must then use the equation

$$\frac{\partial\eta}{\partial t} + \mathbf{v} \cdot \nabla\eta = \frac{-\eta}{nT}\, \mathbf{B} \cdot \nabla\left(\frac{\kappa \mathbf{B} \cdot \nabla T}{B^2}\right), \tag{5a}$$

where κ is the coefficient of thermal conductivity along magnetic field, n is the particle density, and T the temperature. The associated stabilizing effect against the "rippling" mode is discussed in Appendix F. The neglect of Ohmic heating in Eq. (5) is unimportant in the high-conductivity short-wavelength limit. For low-conductivity plasma, a small amount of Ohmic heating due to first-order currents tends to accelerate the "rippling" mode and retard the "tearing" mode. If the Ohmic heating is sufficiently strong to *reduce* the local electric field at a current concentration, a trivial type of "tearing" instability results that depends primarily on thermal rather than fluid transport effects. The effect of plasma compressibility on Eq. (5) is discussed in Appendix A.

5. Perturbations in $\mathbf{g}\rho$ are assumed to result only from convection

$$\partial(\mathbf{g}\rho)/\partial t + \mathbf{v} \cdot \nabla(\mathbf{g}\rho) = 0. \tag{6}$$

In the presence of a neutral background gas, the first-order currents may, however, give rise to an additional density perturbation by an increase in the local rate of ionization. This mechanism may have a destabilizing effect.[20,21] The effect of compressibility is discussed in Appendix A.

6. The zero-order distribution will be assumed to have $\mathbf{v}_0 = 0$. Strictly speaking, this condition implies

$$\nabla \times (\eta_0 \nabla \times \mathbf{B}_0) = 0 \tag{7}$$

which will be referred to as the standard case. For modes with sufficiently large growth rates and wavelengths, however, the approximation of null zero-order velocity is valid even if Eq. (7) is not strictly satisfied, since the values of $\mathbf{v}_0$ to be expected are those of ordinary resistive diffusion. Some of our results will therefore be presented in their most general form, without invoking Eq. (7).

Denoting perturbed quantities by the subscript 1,

$$f_1(\mathbf{r}, t) = f_1(y) \exp\,[i(k_x x + k_z z) + \omega t]$$

we obtain to first order the set of equations

$$\omega \mathbf{B}_1 = \nabla \times (\mathbf{v}_1 \times \mathbf{B}_0) - (1/4\pi)\nabla \times [\eta_0 \nabla \times \mathbf{B}_1 + \eta_1 \nabla \times \mathbf{B}_0], \tag{8}$$

$$\omega \nabla \times \rho_0 \mathbf{v}_1 = \nabla \times \{(1/4\pi)[(\mathbf{B}_0 \cdot \nabla)\mathbf{B}_1 + (\mathbf{B}_1 \cdot \nabla)\mathbf{B}_0] + (\mathbf{g}\rho)_1\}, \tag{9}$$

$$\nabla \cdot \mathbf{v}_1 = \nabla \cdot \mathbf{B}_1 = 0, \tag{10}$$

$$\omega\eta_1 + (\mathbf{v}_1 \cdot \nabla)\eta_0 = 0, \tag{11}$$

$$\omega(\mathbf{g}\rho)_1 + (\mathbf{v}_1 \cdot \nabla)(\mathbf{g}\rho)_0 = 0. \tag{12}$$

From this set of equations, we may separate two that involve only B_{y1} and v_{y1}. Equations governing the remaining first-order quantities (not needed in the present analysis) are given in Appendix A. In dimensionless form, we have

$$\frac{\psi''}{\alpha^2} = \psi\left(1 + \frac{p}{\bar{\eta}\alpha^2}\right) + \frac{W}{\alpha^2}\left(\frac{F}{\bar{\eta}} + \frac{\bar{\eta}'F'}{\bar{\eta}p}\right), \tag{13}$$

$$\frac{(\bar{\rho}W')'}{\alpha^2} = W\left[\bar{\rho} - \frac{S^2G}{p^2} + \frac{FS^2}{p}\left(\frac{F}{\bar{\eta}} + \frac{\bar{\eta}'F'}{\bar{\eta}p}\right)\right] + \psi S^2\left(\frac{F}{\bar{\eta}} - \frac{F''}{p}\right), \tag{14}$$

where

$$\psi = B_{y1}/B, \qquad W = -iv_{y1}k\tau_{\mathrm{R}},$$

$$F = (k_x B_{x0} + k_z B_{z0})/kB, \qquad k = (k_x^2 + k_z^2)^{\frac{1}{2}},$$

$$\alpha = ka, \qquad \tau_{\mathrm{R}} = 4\pi a^2/\langle\eta\rangle, \qquad \tau_{\mathrm{H}} = a(4\pi\langle\rho\rangle)^{\frac{1}{2}}/B,$$

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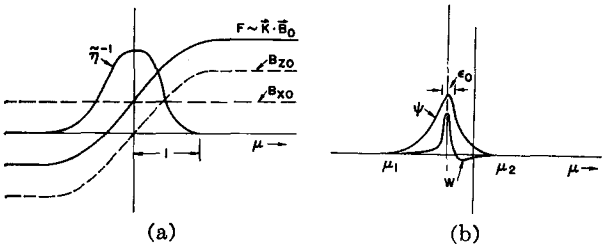

FIG. 1(a) Equilibrium configuration of sheet pinch. (b) Form of the perturbation.

$$S = \tau_R/\tau_H, \qquad p = \omega\tau_R, \qquad \tilde{\eta} = \eta_0/\langle\eta\rangle,$$
$$\tilde{\rho} = \rho_0/\langle\rho\rangle, \qquad G = \tau_H^2 A_1.$$

The primes denote differentiation with respect to a dimensionless variable $\mu = y/a$, where a is a measure of the thickness of the current layer. The quantities B, $\langle\eta\rangle$, and $\langle\rho\rangle$ are measures of the field strength, resistivity, and mass density respectively. The quantity A_1 may be interpreted as $-(g/\rho_0)\,\partial\rho_0/\partial y$ for the gravitational-field case; or as $-(1/\rho_0)\,\partial(\rho_0\dot{v}_0)/\partial y$ for a current layer with zero-order acceleration $\dot{v}_0$; or very roughly as $-(1/\tau_H^2)(a^2/4R_c)\,\partial\beta_0/\partial y$ for a current-layer of mean radius of magnetic curvature R_c and plasma pressure P_0, where $\beta_0 = 4\pi P_0/B^2$. The latter application is discussed in detail in Appendix I.

For thermal plasmas, we have approximately $S \sim 0.1aT^2\beta_0^{-\frac{1}{2}}$, with T in eV. The parameter S exceeds a hundred for most present-day hot-plasma experiments and must become much larger yet in experiments of thermonuclear interest. Accordingly, our primary interest will be in the case $S \rightarrow \infty$. Note that the growth rate p is expressed in units of the resistive diffusion time.

Only one component of the $\mathbf{B}_0$ field, namely $\mathbf{k}(\mathbf{k}\cdot\mathbf{B}_0)$ appears in Eqs. (13) and (14). For any given $\mathbf{B}_0$ field having finite shear, we may choose $\mathbf{k}$ so that $\mathbf{k}\cdot\mathbf{B}_0 \sim F$ passes through a null. The typical μ-dependence of F and η that will be considered here is illustrated in Fig. 1(a).

The zero-order equilibrium condition of Eq. (7) may be written as

$$(\tilde{\eta}'/\tilde{\eta})F' = -F''. \tag{15}$$

The usual boundary conditions are that both ψ and W should vanish at infinity or at conducting boundaries, located at $\mu = \mu_1, \mu_2$.

III. GENERAL REMARKS

Nonexistence of Overstable Modes

Equations (13) and (14) can be solved in the limit $S \rightarrow \infty$, to give an oscillatory mode [Re $(p) = 0$]. Expanding about this solution in powers of S^{-1}, one finds that, when Im $(p) \neq 0$, either Re $(p) = O(1)$ (so that the growth rate is insignificant); or else the zero-order current layer must have sharp resistivity gradients, which become increasingly so as $S \rightarrow \infty$. In a more general analysis, including plasma compressibility, etc., we would expect the same result. This follows since the equations can always be expanded in powers of S^{-1}, as long as the zero-order conductivity is large everywhere and has finite gradients. Thus the modes of greatest practical interest are new modes that do not exist at all for $S = \infty$. The situation is similar to that for hydrodynamic shear-flow stability at high Reynolds number.[22]

In the incompressible case, we can show that no overstable modes exist at all, provided that non-equilibrium zero-order configurations are excluded by requiring Eq. (7) (or 15) to be satisfied. This condition is appropriate for configurations with sharp resistivity gradients, since the zero-order diffusion velocity could not otherwise be neglected. For convenience, we will use a definition of the quantity $\langle\eta\rangle$ such that $\tilde{\eta}F' = 1$. Equations (13), (14) can then be rewritten in the form

$$\frac{p^2}{\alpha^2 S^2 F}\left[(\tilde{\rho}W')' + \alpha^2 W\left(\frac{S^2 G}{p^2} - \tilde{\rho}\right)\right]$$
$$= (p\psi + WF)\left(pF' - \frac{F''}{F}\right) \tag{16}$$
$$= p\psi'' - p\psi\left(\alpha^2 + \frac{F''}{F}\right). \tag{17}$$

Equations (16) and (17) yield the condition

$$\int_{\mu_1}^{\mu_2} d\mu \left\{\frac{p^2}{|p|^2\alpha^2 S^2}\left[\tilde{\rho}\,|W'|^2 + \alpha^2\,|W|^2\left(\tilde{\rho} - \frac{S^2 G}{p^2}\right)\right] + \frac{pF' - F''/F}{|pF' - F''/F|^2}\left|\psi'' - \psi\left(\alpha^2 + \frac{F''}{F}\right)\right|^2 + |\psi'|^2 + |\psi|^2\left(\alpha^2 + \frac{F''}{F}\right)\right\} = 0. \tag{18}$$

Taking the imaginary part of Eq. (18), we find that if Im $(p) \neq 0$, then Re $(p) \leq 0$.

Characteristics of Unstable Modes

We will devote primary attention to those unstable modes for which $S \rightarrow \infty$ and $p \sim S^{\zeta}$ where $0 < \zeta < 1$. The lower limit on ζ corresponds to a growth rate that is of the same order as the rate of resistive diffusion, and is therefore insignificant. The upper limit on ζ is reached only by modes that exist also in the standard infinite-conductivity treatment.

Since the growth rates of the modes to be considered are slow compared with the hydromagnetic rates, the flow is subsonic; i.e., the incompressibility approximation is satisfactory (cf. Appendix A). On the other hand, since the growth rates are fast compared with resistive diffusion rates, the effect of Ohmic heating is negligible.

A discussion of unstable modes in the limit $S \to 0$ is given in Appendix B. It is shown that in this limit the growth rates approach hydromagnetic rates.

For unstable modes, all quantities in Eq. (18) are real. In the limit $S \to \infty$, Eq. (18) can be satisfied in three distinct ways, each corresponding to a negative contribution from one of the three terms: (1) if $G > 0$, there can be gravitationally driven modes; (2) if ψ is peaked near the point $F = 0$, and if we can have $F''/F > 0$ at this point (i.e., if $\eta' \neq 0$ there), then there are modes corresponding to the "rippling" instability; (3) since F''/F is predominantly negative, for sufficiently small α^2 there are modes corresponding to the "tearing" instability.

The behavior of the solutions over most of the range in μ can be established on a general basis. As $S \to \infty$, we must have

$$p\psi \approx -FW \tag{19}$$

everywhere except in a small interval near $F = 0$. This condition follows from the consideration that either W or ψ would diverge strongly at large μ if the right-hand term in Eq. (16) were either negative or positive except in a small interval. Eq. (19) is, of course, the condition that the fluid remains "frozen" to magnetic field lines.

Using Eq. (19), we then see from Eqs. (16) and (17) that the (infinite-conductivity) equation

$$\psi'' - \psi(\alpha^2 + F''/F - G/F^2) = 0 \tag{20}$$

must be satisfied everywhere except in a small interval. The general procedure in the $S \to \infty$, $0 < \zeta < 1$ limit is therefore as follows. We obtain solutions to Eq. (20) that vanish at $\mu = \mu_1, \mu_2$, the external boundaries. These solutions cannot, in general, be joined without a discontinuity in ψ',

$$\Delta' = \psi_2'/\psi_2 - \psi_1'/\psi_1, \tag{21}$$

where the subscripts refer to values on either side of the point of juncture. The typical behavior of ψ is illustrated in Fig. 1(b). The discontinuity in ψ'/ψ corresponds to large local values of ψ''. From Eqs. (16) and (17) we see that such values can be obtained only near the point $F = 0$. Equation (13) implies that large local values of W are also obtained near the same point and only there. The second stage of the general solution therefore consists in solving for ψ and W in a small region R_0 about the point $F = 0$, with the boundary conditions that ψ'/ψ matches the solutions of Eq. (20), and that W is well behaved outside the region R_0.

In more formal terms, we may say that Eqs. (19) and (20) provide an asymptotic solution of Eqs. (16) and (17), which breaks down near $F = 0$. We note that if $F \neq 0$ everywhere, then Eq. (20) applies throughout, and there is no solution unless $G/(F')^2 = O(1)$, in which case the layer is unstable even in the $S = \infty$ limit.

The argument of this section has, for reasons of convenience, made use of Eqs. (16) and (17), which refer specifically to the standard case [i.e., Eq. (15) holds]. The conclusions can, however, be extended to more general choices of F, if desired.

IV. SOLUTIONS IN THE OUTER REGION

We assume that Eq. (20) holds everywhere outside a small region R_0 with a width of order ϵ_0 around the point μ_0 at which $F = 0$. Eq. (20) is to be solved subject to the boundary condition $\psi = 0$ at the points μ_1, μ_2, which we will take for convenience at $\mp\infty$. We will calculate the quantity Δ' of Eq. (21) for the case $\psi_1 = \psi_2$ which is of principal interest in Sec. V. Equation (20) yields the expression

$$\Delta' = -2\alpha - \frac{1}{\psi_1}\int_{-\infty}^{\infty} d\mu\, e^{-\alpha|\mu|}\psi\,\frac{F''}{F} + O\left[\frac{G}{(F')^2\epsilon_0}\right]. \tag{22}$$

Note that when $F'' \neq 0$ at μ_0, there is a singularity in the integrand on the right side of Eq. (22). Difficulties arising from the corresponding logarithmic singularity in ψ' are avoided here, since we consider only Δ' instead of the individual values of ψ_1' and ψ_2'. In this and the following section we will restrict ourselves to the case where $|G|/(F')^2$ is sufficiently small so that the G term in Eq. (22) can be neglected. The case of larger G is discussed in Appendix E.

For the case $\alpha^2 \gg 1$, one obtains from Eq. (22)

$$\Delta' = -2\alpha + O(1/\alpha). \tag{23}$$

For the case $\alpha^2 \ll 1$, we expand

$$\psi = e^{-\alpha|\mu-\mu_0|}(\psi_{(0)} + \alpha\psi_{(1)} + \alpha^2\psi_{(2)} \cdots)$$

and find

$$-(\psi_{(n)}F' - \psi_{(n)}'F)' = [2|\mu-\mu_0|/(\mu-\mu_0)]F\psi_{(n-1)}'.$$

The well-behaved solution is characterized by

$$\psi_{(0)} = |F| \tag{24}$$

and near the point μ_0 by

$$\psi_{(1)} = F_{-\infty}^2/F', \ \mu < \mu_0; \ \psi_{(1)} = F_{\infty}^2/F', \ \mu > \mu_0. \tag{25}$$

In calculating Δ', the derivative of $\psi_{(1)}$ may be neglected, though it has a logarithmic singularity at μ_0, since the contributions it makes near μ_0 cancel out, and the rest is of order α. Thus we obtain

$$\Delta' = (1/\alpha)(F')^2(1/F_{-\infty}^2 + 1/F_{\infty}^2). \tag{26}$$

In the case of symmetric F''/F, it will be of interest to obtain Δ' for arbitrary α. For this purpose it is convenient to choose specific models. When

$$F = \tanh \mu, \tag{27}$$

then Eq. (20) may be solved explicitly in terms of associated Legendre functions, and we have

$$\Delta' = 2(1/\alpha - \alpha). \tag{28}$$

When

$$F = \mu, \quad |\mu| < 1; \qquad F = 1, \quad \mu > 1; \qquad F = -1, \quad \mu < -1; \tag{29}$$

we have

$$\Delta' = 2\alpha\left[\frac{(1-\alpha) - \alpha \tanh \alpha}{\alpha - (1-\alpha)\tanh \alpha}\right]. \tag{30}$$

Note that Δ' goes monotonically from ∞ to $-\infty$ as α goes from 0 to ∞. There is a null of Δ' at the point $\alpha = \alpha_c$, which occurs at 1 and 0.64 respectively for the models of Eqs. (27) and (29).

V. SOLUTIONS IN THE REGION OF DISCONTINUITY

Basic Equations

In the small region R_0 about the point μ_0 we may take the quantities F', F'', $\tilde{\eta}$, $\tilde{\eta}'$, G, and $\tilde{\rho}$ in Eqs. (13) and (14) to be constant. We may approximate F as $F'(\mu - \mu_0)$ and neglect the term $\tilde{\rho}'W'$ relative to $\tilde{\rho}W''$.

Defining a new independent variable

$$\theta = (1/\epsilon)(\mu - \mu_0 + \tilde{\eta}'/2p), \tag{31}$$

Eqs. (13) and (14) may be written as

$$d^2\psi/d\theta^2 - \epsilon^2\alpha^2\psi = \epsilon\Omega[4\psi + U(\theta + \delta_1)], \tag{32}$$

$$d^2U/d\theta^2 + U(\Lambda - \tfrac{1}{4}\theta^2) = \psi(\theta - \delta), \tag{33}$$

where

$$\epsilon = [p\tilde{\eta}\tilde{\rho}/4\alpha^2S^2(F')^2]^{\frac{1}{4}}, \tag{34}$$

$$U = W(4\epsilon F'/p), \tag{35}$$

$$\Omega = p\epsilon/4\tilde{\eta}, \tag{36}$$

$$\delta = (1/4\Omega)(F''/F' + \tilde{\eta}'/2\tilde{\eta}), \tag{37}$$

$$\delta_1 = (1/8\Omega)(\tilde{\eta}'/\tilde{\eta}), \tag{38}$$

$$\Lambda = (\tilde{\eta}')^2/16\epsilon^2p^2 + S^2\alpha^2\epsilon^2G/p^2\tilde{\rho} - \alpha^2\epsilon^2. \tag{39}$$

Note that in the standard case [cf. Eq. (15)] we have $\delta = -\delta_1$.

Let us expand U in terms of normalized Hermite functions

$$U = \sum_{n=0}^{\infty} a_n u_n, \tag{40}$$

where

$$d^2u_n/d\theta^2 + (n + \tfrac{1}{2} - \tfrac{1}{4}\theta^2)u_n = 0 \tag{41}$$

and

$$u_n = \frac{(-1)^n}{(2\pi)^{\frac{1}{4}}(n!)^{\frac{1}{2}}} e^{\frac{1}{4}\theta^2} \frac{d^n}{d\theta^n} e^{-\frac{1}{2}\theta^2}. \tag{42}$$

Then Eq. (33) can be written in the form

$$a_n = \frac{1}{\Lambda - (n + \frac{1}{2})} \int_{-\infty}^{\infty} d\theta_1 \, u_n\psi(\theta_1 - \delta). \tag{43}$$

Equations (32) and (33) are valid over the range $(\mu - \mu_0)^2 \ll 1$. We will apply these equations over a region R_0 of width ϵ_0, outside of which Eqs. (19) and (20) are to be valid. From Eqs. (32) and (33) it follows then that we must require $\epsilon_0 > \epsilon$.

The most important class of unstable modes corresponds to the approximation $\psi =$ const in R_0. For this case we obtain the "tearing" and "rippling" modes over a range of α consistent with $\epsilon_0 \,|\,\psi'/\psi\,| < 1$ in R_0, or roughly

$$\epsilon_0 \,|\Delta'| < 1. \tag{44}$$

Using the requirement $\epsilon_0 > \epsilon$ and the results of Eqs. (23) and (26), we may rewrite Eq. (44) as

$$\epsilon(F')^2(1/F_{-\infty}^2 + 1/F_{\infty}^2) < \alpha < 1/2\epsilon \tag{44a}$$

For $\psi =$ const in R_0 we also obtain the low-G gravitational interchange mode. Sufficient conditions for the constancy of ψ and for the negligibility of the G term in Eq. (20) are provided by Eq. (44a) together with the requirement that $|G|(F')^{-2}$ be small compared with ϵ, or compared with $\epsilon\,|\Delta'|$ when $|\Delta'| \gg 1$. In Appendix E, we will show that the condition on G can in general be relaxed considerably. We may take ψ to be constant over $\epsilon_0 = (1 + |\Delta'|)^{-1}$. Then the G term in Eq. (20) will be small (when $G > 0$), provided that we have

$$G/(F')^2 < \tfrac{1}{4} \tag{45}$$

and

$$G/(F_{\pm\infty})^2 < \alpha^2. \quad (45a)$$

We note that Eq. (45) is equivalent to the well-known Suydam criterion for instability of an infinite-conductivity plasma at short wavelengths.

Solutions with Constant ψ

If ψ = const and Eqs. (44) and (45) are satisfied, we obtain from Eqs. (32) and (43)

$$\Delta' = \Omega \sum_{n=0}^{\infty} 4\left[\int_{-\infty}^{\infty} d\theta_1\, u_n\right]^2 + \frac{1}{\Lambda - (n + \frac{1}{2})} \cdot\left[\int_{-\infty}^{\infty} d\theta_1\, u_n(\theta_1 + \delta_1)\right]\left[\int_{-\infty}^{\infty} d\theta_2\, u_n(\theta_2 - \delta)\right]\cdot \quad (46)$$

Using the integrals

$$\int_{-\infty}^{\infty} d\theta_1\, u_n = 2^{\frac{3}{4}}\left[\frac{\Gamma(\frac{1}{2}n + \frac{1}{2})}{\Gamma(\frac{1}{2}n + 1)}\right]^{\frac{1}{2}} \quad (n \text{ even});$$
$$= 0 \quad (n \text{ odd});$$
$$\int_{-\infty}^{\infty} d\theta_1\, \theta_1 u_n = 0 \quad (n \text{ even}),$$
$$= 2^{9/4}\left[\frac{\Gamma(\frac{1}{2}n + 1)}{\Gamma(\frac{1}{2}n + \frac{1}{2})}\right]^{\frac{1}{2}} \quad (n \text{ odd});$$

we obtain the eigenvalue equation

$$\Delta' = 2^{7/2}\Omega \sum_{m=0}^{\infty} \frac{\Gamma(m + \frac{1}{2})}{\Gamma(m + 1)} \cdot\left[\frac{\Lambda - \frac{1}{2}}{\Lambda - (2m + \frac{3}{2})} - \frac{\delta\delta_1/4}{\Lambda - (2m + \frac{1}{2})}\right], \quad (47)$$

where Δ' is determined by the "outside" solutions (cf. Sec. IV). We have replaced even n by $2m$ and odd n by $2m + 1$. The series is convergent, since terms for large m go like $m^{-\frac{3}{2}}$. [Note, however, that if we had calculated $\psi' \,|_0^\infty$, we would have obtained a divergent result, proportional to

$$(\delta_1 - \delta) \sum_{m=0}^{\infty} \frac{1}{m}\cdot$$

Since $(\delta_1 - \delta) \sim F''/F'$, and $\theta^2 \sim m$, we may identify this divergence with the logarithmic singularity indicated by Eq. (22).]

The sums can be evaluated as hypergeometric series of argument 1 to give

$$\Delta' = 2^{7/2}\pi\Omega\left[\frac{\Gamma(\frac{3}{4} - \frac{1}{2}\Lambda)}{\Gamma(\frac{1}{4} - \frac{1}{2}\Lambda)} + \frac{\delta\delta_1}{8}\,\frac{\Gamma(\frac{1}{4} - \frac{1}{2}\Lambda)}{\Gamma(\frac{3}{4} - \frac{1}{2}\Lambda)}\right]\cdot \quad (47a)$$

A form that is sometimes more convenient is

$$\Delta' \pm \left[(\Delta')^2 - \pi^2\frac{\tilde{\eta}'}{\tilde{\eta}}\left(\frac{2F''}{F'} + \frac{\tilde{\eta}'}{\tilde{\eta}}\right)\right]^{\frac{1}{2}} = \pi 2^{9/2}\Omega\,\frac{\Gamma(\frac{3}{4} - \frac{1}{2}\Lambda)}{\Gamma(\frac{1}{4} - \frac{1}{2}\Lambda)}\cdot \quad (47b)$$

The following general remarks can be made about the solutions of Eq. (47).

1. If $\delta\delta_1 < 0$ (the most common case), then Δ'/Ω goes from ∞ to $-\infty$ as Λ goes from $\frac{1}{2}$ to $\frac{3}{2}$, from $\frac{3}{2}$ to $\frac{5}{2}$, etc. The quantity Ω, related to Λ by Eqs. (36) and (39), is finite for finite Λ. Hence for any given Δ', as obtained from Eq. (21), there is an infinite sequence of eigenvalues $\Lambda \sim 1, 2, 3, \cdots$. There is also an eigenvalue below $\frac{1}{2}$, which moves to 0 as $\Delta' \to \infty$, while Ω becomes large.

2. If $\delta\delta_1 = 0$ then Δ'/Ω goes from ∞ to $-\infty$ as Λ goes from $2m + \frac{3}{2}$ to $2m + \frac{7}{2}$. The sequence of eigenvalues is $\Lambda \sim 2, 4, 6 \cdots$. There is also an eigenvalue below $\frac{3}{2}$.

3. If $0 < \delta\delta_1 \ll \Lambda$ then Δ'/Ω covers almost the entire range from ∞ to $-\infty$ as Λ goes from $2m + \frac{3}{2}$ to $2m + \frac{7}{2}$. The excluded interval is

$$|\Delta'/\Omega| < 4\pi(\delta\delta_1)^{\frac{1}{2}}. \quad (48)$$

For Δ'/Ω outside the excluded interval, there is also an eigenvalue below $\frac{3}{2}$.

4. When $|\Lambda| \ll \frac{1}{2}$, Eq. (47) reduces to

$$\Delta' = \Omega(12 + 13\,\delta\delta_1). \quad (49)$$

In that case Ω is to be determined by the value of Δ' in Eq. (21), and the condition on Λ is to be verified by means of Eqs. (36) and (39). Evidently this case arises only for positive Δ'.

We can now identify a number of basic modes.

The "Rippling" Mode

The "rippling" mode is characterized by the finiteness of Λ and the predominance of the $(\eta')^2$ term on the right side of Eq. (39). In that case

$$p = \left[\frac{(\tilde{\eta}')^2\alpha S\,|F'|}{8\Lambda\tilde{\eta}^{\frac{1}{2}}\tilde{\rho}^{\frac{1}{2}}}\right]^{2/5}, \quad (50)$$

$$\Omega = |\tilde{\eta}'|/16\tilde{\eta}\Lambda^{\frac{1}{2}}, \quad (51)$$

$$\delta_1 = 2\Lambda^{\frac{1}{2}}, \quad (52)$$

$$\epsilon = |\tilde{\eta}'|/4p\Lambda^{\frac{1}{2}}. \quad (53)$$

In the standard case we have $\delta\delta_1 = -4\Lambda$, and the remarks made above in paragraph 1 then apply to the Λ spectrum. (For other reasonable choices of $\delta\delta_1 < 0$, the Λ spectrum is modified only slightly.)

In the limit $\alpha \gg 1$, which according to Eq. (23) corresponds to large negative Δ', we find that the eigenvalues Λ lie slightly below the points $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, $\cdots$. For the fastest growing mode, which corresponds to a solution U that is basically symmetric near μ_0, we have $\Lambda \approx \frac{1}{2}$. As we move towards the other limit, $\alpha \ll 1$, (i.e., large positive Δ') the eigenvalues that were slightly below $\frac{3}{2}$, $\frac{5}{2}$, $\cdots$ move to points slightly above $\frac{1}{2}$, $\frac{3}{2}$, $\cdots$. The fastest growing mode of this series again occurs for $\Lambda \approx \frac{1}{2}$, and corresponds to a basically symmetric U in the neighborhood of μ_0. From Eq. (50) we see that the growth rates of these modes become small as $\alpha \to 0$. The eigenvalue lying below $\frac{1}{2}$ moves toward 0 as $\alpha \to 0$. This mode goes over into the "tearing" mode (see below); the associated U becomes antisymmetric, and the growth rate becomes large as $\alpha \to 0$.

If we depart from the standard case and consider the limit $\delta = 0$, $\delta_1 \neq 0$ (cf. paragraph 2), the eigenvalues near $\frac{1}{2}$, $\frac{5}{2}$, $\frac{9}{2}$, $\cdots$ disappear. If $\delta\delta_1 > 0$ (cf. paragraph 3), there is no solution for $|\Delta'| \ll 1$, but for large or small α there are eigenvalues near $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, $\cdots$.

Using Eqs. (50) and (53), we may express the condition of Eq. (44) as

$$\left[\frac{\tilde{\eta}^6\tilde{\rho}(F')^8}{16S^2\Lambda^{\frac{1}{2}}(\tilde{\eta}')^4}\left(\frac{1}{F_{-\infty}^2}+\frac{1}{F_{\infty}^2}\right)^5\right]^{1/7} < \alpha < \left[\frac{S^2(F')^2\Lambda^{\frac{1}{2}}}{2|\tilde{\eta}'|\tilde{\eta}\tilde{\rho}}\right]^{\frac{1}{3}}.$$

The behavior of the "rippling" mode for larger α is discussed in Appendix C.

As we have noted in Sec. II, paragraph 4, the use of Eq. (5) to give the first-order resistivity becomes inaccurate for a high-temperature plasma, where thermal conductivity along magnetic field lines is highly effective. For the "rippling" mode, which depends critically on the nature of the resistivity perturbation, the growth rate is then actually much smaller than would be indicated by Eq. (50). An estimate of the correction factor is given in Appendix F.

The stabilization of the "rippling" mode by gravitational effects is discussed in the section on the gravitational mode.

The "Tearing" Mode

The "tearing" mode is characterized by the condition $|\Lambda| \ll \frac{1}{2}$ (cf. paragraph 4). Eq. (49) shows that this mode is limited to positive Δ', i.e., to $\alpha \leq \alpha_0 \sim 1$ [cf. Eqs. (28) and (30)]. The growth rate is obtained from Eqs. (34) and (36):

$$p = 4(\alpha S\Omega^2\tilde{\eta}^{\frac{3}{2}}\tilde{\rho}^{-\frac{1}{2}}\,|F'|)^{2/5}; \tag{54}$$

and the condition on Λ can be expressed by means of Eq. (39):

$$\Lambda = \frac{1}{\Omega^2}\left[\left(\frac{\tilde{\eta}'}{16\tilde{\eta}}\right)^2 + \frac{pG}{64\tilde{\eta}(F')^2}\right] \ll 1. \tag{55}$$

For $\alpha \ll 1$, we have from Eqs. (26) and (49)

$$\Omega = \frac{1}{12\alpha}(F')^2\left(\frac{1}{F_{-\infty}^2}+\frac{1}{F_{\infty}^2}\right) \tag{56}$$

so that

$$p = (F')^2\left(\frac{2S\tilde{\eta}^{\frac{3}{2}}}{9\alpha\tilde{\rho}^{\frac{1}{2}}}\right)^{2/5}\left(\frac{1}{F_{-\infty}^2}+\frac{1}{F_{\infty}^2}\right)^{4/5}. \tag{57}$$

The fastest growing mode is generally obtained for the "symmetric case" where $F'' = 0$ at $F = 0$. A lower limit to α is set by Eq. (44)

$$\alpha > \left(\frac{1}{F_{-\infty}^2}+\frac{1}{F_{\infty}^2}\right)\left(\frac{\tilde{\eta}\tilde{\rho}^{\frac{1}{2}}\,|F'|^9}{3.3S}\right)^{\frac{1}{4}}. \tag{58}$$

The maximal growth rate p_m thus goes as $S^{\frac{1}{3}}$. (Appendix D treats this limit by a method that avoids the constant $-\psi$ approximation but confirms the present result for p_m.)

If the current layer is perfectly symmetric, so that the nulls of $\tilde{\eta}'$ and G occur at the same point $\mu = 0$, Eq. (55) is always satisfied for a mode where $\mu_0 = 0$. More generally, we see that the $(\eta')^2$ term in Eq. (55) is always negligible for $\alpha \ll 1$. The effect of the gravitational term when $G \neq 0$ at μ_0 is discussed in the next section.

For modes of the "tearing" type, the solution U is basically antisymmetric, since the $\tilde{\eta}'$ terms can usually be neglected by symmetry or because $\alpha \ll 1$.

The Gravitational Interchange Mode

The gravitational interchange mode is characterized by the finiteness of Λ and the predominance of the G term on the right side of Eq. (39). Instability is obtained for $G > 0$, and the appropriate growth rate is

$$p = (S\alpha G\tilde{\eta}^{\frac{1}{2}}/2\Lambda\,|F'|\,\tilde{\rho}^{\frac{1}{2}})^{\frac{2}{3}}. \tag{59}$$

The magnitude of G for which Eq. (59) holds is restricted by Eqs. (45) and (45a).

To evaluate Λ, we note from Eqs. (36) and (39) that Ω is given by

$$\Omega = G/16\epsilon\Lambda(F')^2.$$

For $\delta\delta_1 < 0$, Eq. (47) gives a series of eigenvalues Λ lying in the intervals $\frac{1}{2}$ to $\frac{3}{2}$, $\frac{3}{2}$ to $\frac{5}{2}$, etc. There may also be an eigenvalue in the interval 0 to $\frac{1}{2}$, if Δ'/Ω is not too large. Unless $G(F')^{-2}$ is of order ϵ or less, the quantity Ω is generally very large, so that $|\Delta'|/\Omega$, $|\delta\delta_1| \ll 1$. In that case the eigenvalues lie at $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, etc.

If $\delta\delta_1 = 0$, there is a series of eigenvalues lying in the intervals $\frac{3}{2}$ to $\frac{7}{2}$, $\frac{7}{2}$ to $\frac{11}{2}$, etc., and there may also be an eigenvalue in the interval 0 to $\frac{3}{2}$, if Δ'/Ω is not too large. For $|\Delta'|/\Omega \ll 1$, we have $\Lambda = \frac{1}{2}$, $\frac{5}{2}$, $\frac{9}{2}$, etc. These eigenvalues are obtained for example, for pure gravitational modes with $G(F')^{-2}$ finite as $S \to \infty$.

We now consider the effect of the G term on the growth of the "rippling" mode. The condition that the resistivity-gradient term should be dominant on the right side of Eq. (39) is

$$p < (\tilde{\eta}')^2(F')^2/4\tilde{\eta}\ |G| \tag{60}$$

so that, unless $G = 0$, the gravitational term always predominates in the limit $S \to \infty$, $p \to \infty$. If $G > 0$, then instability continues above the limit set by Eq. (60) in the form of gravitational modes or mixed "gravitational-rippling" modes. If $G < 0$, there are no gravitationally driven modes, and Eq. (60) then sets an upper limit to the growth of all short-wave interchange instabilities.

We next consider the effect of the G-term on the growth of the "tearing" mode, neglecting the resistivity gradient terms (cf. the preceding section). Using Eqs. (34), (36), and (39) we find that

$$p = 4^3\Lambda\Omega^2\tilde{\eta}(F')^2/G. \tag{61}$$

If $G > 0$, then the "tearing" mode, which is characterized by $\Lambda \ll \frac{1}{2}$ and by the consequent applicability of Eq. (49), is restricted by the condition

$$p < 2\tilde{\eta}(\Delta')^2(F')^2/9G. \tag{62}$$

As $S \to \infty$, $p \to \infty$, Eq. (62) is violated, Λ moves up toward $\frac{3}{2}$, and we have the gravitational or mixed "gravitational-tearing" mode. If $G < 0$, then Λ is negative, and Eqs. (47) and (61) yield the condition

$$p < \tilde{\eta}(\Delta')^2(F')^2/17\ |G| \tag{62a}$$

in the limit $S \to \infty$. Eq. (62a) then sets an upper limit to the growth of long-wave instabilities. [We note that in the present analysis the quantity $G(F')^{-2}$ is limited by Eq. (45a), so that for extremely small α Eq. (62a) is not very restrictive.]

We may summarize the gravitational effects qualitatively in terms of four characteristic ranges of G.

I. $G < 0$. The gravitational force is stabilizing. For finite $|G|(F')^{-2}$ essentially all the resistive instabilities are suppressed in the limit $S \to \infty$; i.e., we have $p \sim S^0$ for the "rippling" and "tearing" modes.

II. $G = 0$, or at least $GS^{2/5} < 1$. In this case we may have the pure "rippling" or "tearing" modes with $p \sim S^{2/5}$.

III. $G > 0$, but not large enough for infinite-conductivity instabilities. In this case we have $p \sim S^{\frac{2}{3}}$.

IV. $G > 0$ and large enough for infinite-conductivity instabilities; i.e., $G(F')^{-2} > \frac{1}{4}$ for short-wave modes. In that case, of course, $p \sim S$.

VI. SUMMARY AND ELUCIDATION OF PRINCIPAL RESULTS

In the high-S limit, a current layer with finite gradients has three basic unstable modes and no overstable modes. The approximate properties of the unstable modes in their characteristic parametric range are summarized in Table I. Here it has been assumed that the dimensionless quantities F', F'', etc., are all of order unity. References are given to the more exact equations of the main text, and to supplementary material that more clearly defines the range of validity of the analysis and extends it somewhat. We will now discuss and rederive the modes of Table I in heuristic terms.

The existence of the three "resistive" instabilities depends on the local relaxation of the constraint that fluid must remain attached to magnetic field. For a zero-order field that is not a vacuum field, possibilities of lowering potential energy are always present; the introduction of finite conductivity makes some energetically possible modes topologically accessible. In the case of the infinite-conductivity modes, lines of force that are initially distinct must remain so during the perturbation. For the three "resistive" modes, lines of force that are initially distinct link up during the perturbation. These modes have no counterpart in the infinite-conductivity limit and disappear altogether, their

TABLE I. Summary of approximate properties of unstable modes in the high-S limit.

Mode	Range of Instability	Growth Rate p	Region of Disc. ϵ	Relevant Equations	Valid Range of Equations	Supplementary Equations
"Rippling"	$\tilde{\eta}' \neq 0$	$\alpha^{2/5}S^{2/5}$	$\alpha^{-2/5}S^{-2/5}$	(50)–(53)	$S^{-2/7} < \alpha < S^{2/3}$ $\lvert G\rvert < \alpha^{-2/5}S^{-2/5}$	(60) Appendixes A, C, F, H
"Tearing"	$\alpha < 1$	$\alpha^{-2/5}S^{2/5}$	$\alpha^{-3/5}S^{-2/5}$	(54)–(58)	$S^{-1/4} < \alpha$ $\lvert G\rvert < \alpha^{-8/5}S^{-2/5}$	(62) Appendixes D, G, H, I
Gravitational Interchange	$G > 0$	$\alpha^{2/3}S^{2/3}G^{2/3}$	$\alpha^{-1/3}S^{-1/3}G^{1/6}$	(59)	$S^{-1/4}G^{1/8} < \alpha < S^{1/2}G^{-1/4}$ $G < 1, \alpha^2$ $G > \alpha^{-2/5}S^{-2/5}, \alpha^{-8/5}S^{-2/5}$	(60)–(62) Appendixes A, C, E, H, I

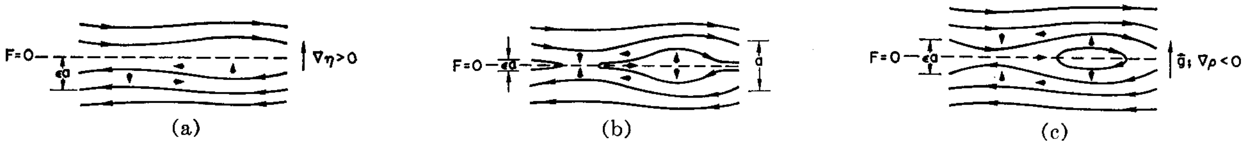

FIG. 2(a) Perturbed fields and velocities—"rippling" mode. Solid arrows indicate fluid velocity. (b) Perturbed fields and velocities—"tearing" mode. (c) Perturbed fields and velocities—gravitational mode.

characteristic times becoming infinite. The situation is quite analogous to the new modes which occur in hydrodynamics when the constraint of conservation of vorticity is removed by the presence of finite viscosity.

The growth rates of the "resistive" modes are sufficiently small on the hydromagnetic time scale so that the fluid motion is subsonic, i.e., incompressible. This feature is of critical importance in simplifying the analysis of the plane current layer: it permits us to consider the magnetic-field and velocity components within the ky plane independently of the components in the direction $\hat{n}$ normal to the ky plane. The reasons for this decoupling effect are readily seen.

The coordinate along $\hat{n}$ is ignorable; therefore, the field lines of the component $B_\perp$ in the $\hat{n}$ direction are not distorted during the perturbation. The only manner in which the magnitude of $B_\perp$ could affect the motion in the ky plane is by way of the magnetic pressure $B_\perp^2/8\pi$. The gradients of this pressure, however, merely tend to induce plasma compression or expansion. An incompressible fluid automatically provides compensating hydrostatic pressure gradients, so that there is no net effect on the dynamics. As for Ohm's law, there the resistive diffusion term does not couple the field components if the resistivity is isotropic, and the convective term couples the magnetic field and velocity components within the ky plane to each other. Finally, the two equations specifying $\mathbf{B}$ and $\mathbf{v}$ to be solenoidal hold as well for the vector components in the ky plane taken alone. Thus we have four equations for two unknown two-component vectors, and we may restrict ourselves in what follows to the analysis of the two-dimensional problem. Typical field and velocity components in the ky plane are illustrated in Fig. 2. We note parenthetically that the convenient reducibility of the three-dimensional finite-resistivity stability problem is wholly analogous to the reducibility of the finite-viscosity stability problem of ordinary hydrodynamics.[22]

[22] The similarity of the finite-resistivity and finite-viscosity problems was first pointed out to us by E. Reshotko. For a discussion of finite-viscosity instabilities, see C. C. Lin, *The Theory of Hydrodynamic Stability* (Cambridge University Press, New York, 1955).

To understand the basic character of the unstable modes, let us consider the mechanism whereby the fluid resists detachment from flux lines. Starting with Ohm's law

$$\eta\mathbf{j} = \mathbf{E} + \mathbf{v}\times\mathbf{B}, \tag{63}$$

let us suppose that the fluid is moving but the flux lines are not, i.e.. $\mathbf{E} \equiv 0$. Then we find $\mathbf{j} = (\mathbf{v}\times\mathbf{B})/\eta$, with a resultant motor force

$$\mathbf{F}_s = \mathbf{j}\times\mathbf{B} = [\mathbf{B}(\mathbf{v}\cdot\mathbf{B}) - \mathbf{v}B^2]/\eta \tag{64}$$

that opposes the fluid motion. In the limit $\eta \to 0$, this force, of course, prevents any separate fluid motion from taking place. We note, however, that the restraining force becomes arbitrarily weak near the point where $\mathbf{B}$ vanishes, and this is the key to the situation. Since the quantity $\mathbf{B}$ in the present discussion refers only to the magnetic-field components in the ky plane, we can generally select $\mathbf{k}$ so that $\mathbf{B}$ has a null at any desired value of y. We may expect that detached fluid motion can take place within a region of order ϵa about such a null point. For each unstable mode, we will find a driving force $\mathbf{F}_d$ that dominates the restraining force $\mathbf{F}_s$ within the inner region, and that is itself dominated by $\mathbf{F}_s$ outside this region.

We can relate the "skin depth" ϵa to the growth rate of the instability. Since $\mathbf{F}_d$ is comparable in magnitude to $\mathbf{F}_s$, the rate at which work is done on the fluid is given by

$$P \sim -\mathbf{v}\cdot\mathbf{F}_s \sim v_y^2(B')^2(\epsilon a)^2/\eta, \tag{65}$$

where we have used $B \sim B'\epsilon a$. The driving force gives rise to motion both in the $\hat{y}$ and $\mathbf{k}$ directions, since $\nabla\cdot\mathbf{v} = 0$. In general the instability wavelength will be much larger than ϵa, and therefore the fluid kinetic energy in the $\mathbf{k}$ direction is dominant. Equating the rate of change of this energy to the driving power, we have

$$\omega\rho\, v_y^2/k^2(\epsilon a)^2 = v_y^2(B')^2(\epsilon a)^2/\eta.$$

The skin depth is then given by

$$\epsilon a \sim \left\{\frac{\omega\rho\eta}{k^2(B')^2}\right\}^{\frac{1}{4}} \tag{66}$$

which agrees with Eq. (34) when expressed in the appropriate dimensionless variables. To arrive at the instability growth rates, we must next determine ϵa by comparison of $\mathbf{F}_d$ with $\mathbf{F}_s$. For this purpose we turn to consideration of specific modes.

In the "rippling" mode of Fig. 2(a), the circulatory motion of the fluid creates a ridge of lower-resistivity fluid into which the local current is channeled. In other words, when a resistivity gradient exists, Ohm's law in its linearized form has an extra term

$$\eta_0 \mathbf{j}_1 = -\eta_1 \mathbf{j}_0 + \mathbf{v} \times \mathbf{B}, \tag{67}$$

where η_1 is given by the convective law

$$\eta_1 = -\mathbf{v} \cdot \nabla \eta_0 / \omega, \tag{68}$$

and where $\mathbf{E} \equiv 0$ has again been used, as is appropriate within the small region of decoupled flow. The η_1 term in Eq. (67) gives rise to a motor force

$$\begin{aligned} \mathbf{F}_{dr} &= \mathbf{j}_1 \times \mathbf{B} \\ &= [(\mathbf{v} \cdot \nabla \eta_0)/\omega \eta_0] \mathbf{j}_0 \times \mathbf{B} \end{aligned} \tag{69}$$

that changes sign as $\mathbf{B}$ passes from one side of the null point to the other. Hence, $\mathbf{F}_{dr}$ is a stabilizing force on the side of higher resistivity and a destabilizing force on the side of lower resistivity. An unstable mode is obtained if the region of decoupled flow lies on the lower-resistivity side and has a width ϵa such that the driving power

$$\mathbf{v} \cdot \mathbf{F}_{dr} \sim v_y^2 \eta_0' (B')^2 (\epsilon a)/4\pi \eta_0 \omega \tag{70}$$

just dominates $\mathbf{v} \cdot \mathbf{F}_s$ inside the region. Comparison of Eqs. (65) and (70) yields

$$\epsilon a \sim \eta_0' / 4\pi\omega. \tag{71}$$

From Eqs. (66) and (71) we can then obtain a growth rate that agrees with Eq. (50) and Table I. We note incidentally that the fluid flow and the perturbation current density are strongly peaked in the decoupled region, while the magnetic-field perturbation falls off over a region in y that is of order k^{-1}. In this outer region the fluid and field are well coupled, and a fluid motion of small magnitude accompanies the field perturbation.

We turn next to the gravitational interchange mode, which is quite similar in character to the "rippling" mode. In the presence of a mass-density gradient, and a y-directed gravitational field, the fluid motion gives rise to a force

$$\mathbf{F}_{dg} = \rho_1 \mathbf{g} = (-v_y \rho_0'/\omega)\mathbf{g} \tag{72}$$

which is destabilizing if $\mathbf{g}$ points toward decreasing density. Comparison of $\mathbf{v} \cdot \mathbf{F}_{dg}$ with Eq. (65) gives

$$\epsilon a \sim [\rho_0' g \eta/(B')^2 \omega]^{\frac{1}{3}}. \tag{73}$$

From Eqs. (66) and (73) we then obtain a growth rate that agrees with Eq. (59) and Table I. The mode that decouples the fluid and field most effectively in this case is the counter-circulatory mode shown in Fig. 2(c). In the infinite-conductivity case, such a fluid motion would lead to local compressions of $\mathbf{B}$, and so could not proceed unless $\mathbf{g}$ is large. This phenomenon is known as shear stabilization. In the mode of Fig. 2(c), the opposing flux components brought together at the null in $\mathbf{B}$ can cancel out, and so the mode can grow for arbitrarily small $\mathbf{g}$. As $\mathbf{g}$ increases, the region of substantial motion becomes wider, until conditions for infinite-conductivity instability are reached. We note that if $\mathbf{g}$ points in the stabilizing direction, the possibility exists of using $\mathbf{F}_{dg}$ to overcome the driving force $\mathbf{F}_{dr}$, thus stabilizing the "rippling" mode.

The "tearing" mode of Fig. 2(b) differs from the other two modes in that it is typically a long-wave rather than a short-wave mode relative to the dimension of the current layer. The driving force is due to the structure of the magnetic field outside the region of decoupled flow; i.e., the tendency of the sheet current to break up into a set of parallel pinches. [The nature of this force is readily perceived by applying the "rubber-band" argument to the diagram of Fig. 2(b), but as we are not dealing with a localized perturbation the argument is not quite so simple.] Even in the inner region, the flow is not perfectly decoupled, and a term

$$\mathbf{E} \sim (\omega B_y/k)\hat{n} \tag{74}$$

must be taken into account in Ohm's law. This term corresponds to the generation of the perturbation flux that links the field regions on either side of $\mathbf{B} = 0$. We have then

$$\eta_0 \mathbf{j}_1 = \mathbf{E}_1 + \mathbf{v} \times \mathbf{B} \tag{75}$$

and we must select ϵa so that the first term on the right in Eq. (75) dominates the second in the region of the partly decoupled flow. Using $\nabla \cdot \mathbf{B} = 0$, we have

$$\mathbf{j}_1 \sim (B_y''/4\pi k)\hat{n}. \tag{76}$$

For wavelengths that are much greater than the current-layer thickness a, we find

$$B_y'' \sim B_y'/\epsilon a \approx B_y/\epsilon k a \tag{77}$$

(cf. Eq. 28). If we now choose ϵa so that $\eta_0 \mathbf{j}_1 \sim \mathbf{E}_1$, we find

$$\epsilon a \sim \eta_0/4\pi k \omega. \tag{78}$$

The growth rate obtained from Eqs. (66) and (78) approximates the result of Eq. (57) and Table I. This analysis is applicable only for $ka \ll 1$, since otherwise Eq. (77) breaks down, B_y''/B_y becoming negative. Similarly, if B_y must vanish at a finite distance, Eq. (77) is altered, B_y''/B_y being diminished or made negative. The significance of these features in regard to stability is suggested also by Fig. 2(b): the closed lines of force cannot drive the instability by the "rubber-band" effect unless they are stretched out along the $\mathbf{B} = 0$ line, i.e., have small ka; and if conducting walls were introduced at finite values of y the lateral crowding of the lines of force would impede the driving mechanism.

It is of interest to note that the basic driving force for the "tearing" instability also exists in the infinite-conductivity equation (20). The displacement $\xi = \psi/F$ (and thus the instability itself) is precluded in the ordinary theory by the requirement that ξ be finite where $F = 0$.

VII. RELATION TO EXPERIMENT

Owing to the approximations made in Sec. II, the present results cannot be expected to provide a general basis for the prediction of instability phenomena in experimental plasmas. In particular, the use of the hydromagnetic approximation is not suitable for high-temperature plasmas, while for low-temperature plasmas the negelect of Ohmic heating, ionization effects, etc., becomes unjustifiable.

In spite of these shortcomings in rigor, the present analysis appears to be consistent with a wide range of experimental results and therefore permits speculation about the causes and remedies of observed instabilities. We discuss in this section a number of observed instability phenomena that are uncorrelated or inversely correlated with predictions of the infinite-conductivity hydromagnetic theory, but that can be accounted for at least qualitatively in terms of the present analysis. Further corrections and generalizations (some of which are discussed in the appendices) may provide a quantitative description of the stability behavior of current layers in experimental plasmas.

The "Tearing" Mode

Since the "tearing" mode is a long-wavelength instability that involves a considerable disturbance of the magnetic field of a current layer, it is particularly suitable for detailed experimental study. We will consider first the simple sheet pinch, which is characterized by $B_{z0} \equiv 0$.

In theta pinches where an initial B_z field is entrapped in plasma and compressed by a fast-rising B_z field of opposite sign,[23,24] a cylindrical current layer results that is fairly well represented by the plane-sheet-pinch model of the present analysis. Typical τ_R and τ_H values are 1–10 μsec and 0.01 μsec respectively, so that S is of order 100–1000. The "tearing" mode in this case would consist of a breakup of the cylindrical current layer into adjacent rings. The fastest growing wavelength [cf. Eq. (58)] is given by $\alpha \sim 0.2$, a value that is not sensitive to the exact magnitude of S. The corresponding growth rate [cf. Eq. (57)] is $p \sim 20$. Thus the predicted e-folding time is in the range 0.05–0.5 μsec.

In those experiments where the plasma volume is short in the z direction,[25] the current layer is found to collapse into a single ring, presumably because there is not adequate room and time for a full wavelength of the "tearing" mode to establish itself. A more satisfactory test of the theory is expected for theta pinches that are sufficiently long to accommodate a number of wavelengths at $\alpha \sim 0.2$. Under these conditions, recent experiments at Aldermaston[26] have demonstrated plasma breakup into as many as six rings, with an instability growth time of about 0.3 μsec. Even long reverse-field theta pinches are found stable under certain conditions,[26] which may be related to the effect of cylindrical geometry and external conductors (cf. Appendix G).

The gyro-orbits of particles in theta pinches are not very small compared with the dimension a of the current layer itself, let alone the dimension of discontinuity ϵa of the "tearing" mode. We note, however (cf. Sec. II, paragraph 1), that the "tearing" mode exists not only in the hydromagnetic limit but also in the collisionless limit, where the Vlasov equation is used directly.[18] It seems likely, therefore, that allowance for nonhydromagnetic effects is not crucial in the case of the "tearing" mode.

A second experimental embodiment of the simple sheet pinch is the Triax or tubular dynamic pinch.[27] In this case a reverse-B_θ layer is created. In the high-density and highly dynamic forms of this pinch (3 megamperes, 300 μD_2) that are usually employed, the "tearing" mode has not been seen, though an

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[24] H. A. B. Bodin, T. S. Green, G. B. F. Niblett, N. J. Peacock, J. M. P. Quinn, and J. A. Reynolds, Nuclear Fusion Suppl., Pt. 2, 521 (1962).

[25] V. Josephson, M. H. Dazey, and R. Wuerker, Phys. Rev. Letters 5, 416 (1960).

[26] H. A. B. Bodin (private communication, 1962).

[27] O. A. Anderson, W. A. Baker, J. Ise, Jr., W. B. Kunkel, R. V. Pyle, and J. M. Stone, in *Proceedings of the Second International Conference on Peaceful Uses of Atomic Energy* (United Nations, Geneva, 1958), Vol. 32, p. 150.

effort has been made to induce it.[28] This study is currently being extended. In a slower and weaker pinch (450 kiloamperes), the expected tearing along (axial) current-flow lines has been observed[29] at pressures of 30–300μ in deuterium and argon, with wavelengths and growth times that agree well with the present analysis.

The addition of a B_{x0} field to the simple reverse-B_{z0} sheet pinch evidently has no effect at all on the "tearing" mode if $k_x = 0$ (the "symmetric" case, where $F'' = 0$ at $F = 0$). Thus the "tearing" mode may occur in the Triax configuration even in the presence of an axial magnetic field. Evidence for such an instability has been found by magnetic probe measurements.[1]

In more general current layers, such as those of the "stabilized" and "inverse stabilized" pinches,[1] we can always choose our coordinates so as to transform the current layer into the basic model that is obtained for the Triax plus axial field. If we wish to look at the "symmetric case" with $k_x = 0$, we orient the x axis of the plane model along $\mathbf{B}$ at the midpoint of the current layer. If the thickness a of the current layer is small compared with its radius R, the outer solution of Sec. IV is readily adapted to cylindrical geometry (see Appendix I). In this manner we can show that a "stabilized pinch" with a sharp current layer is essentially always unstable against the "tearing" mode. A hard-core pinch with a large vacuum B_θ field and no null in the B_z field would have an advantage here, since in this limit k is forced to become large by the periodicity requirement along the axial coordinate, the limiting α for which the "tearing" mode can exist being reached when $B_\theta/B_z \sim R/a$.

A second advantage that is realizable in the hard-core pinch relates to the use of magnetic fields produced largely by external conductors. In extending the present results to nonplanar current layers, the quantities F', F'' require special interpretation. From the derivation of Eqs. (13) and (14), it is clear that F', F'' relate to the zero-order current in the plasma. In the planar case, a zero-order vacuum-field component has constant B_{x0}, B_{z0}, and thus cannot contribute to F', F''. In the nonplanar case, where a zero-order vacuum-field component may have nonzero derivatives of B_{x0}, B_{z0}, the contribution of the vacuum-field component to F', F'' must be specifically excluded. If the zero-order plasma current is small relative to currents in rigid conductors (e.g., currents in the central core and B_z-winding of a hard-core pinch), then the F', F'' terms in Eqs. (13), (14) tend to become small relative to the F terms. Accordingly, the "rippling" and "tearing" modes, which depend on the F' and F'' terms respectively, tend to be inhibited. The behavior of the gravitational interchange mode is given in Appendix I.

For the "stabilized pinch," an $m = 1$ mode conforming with the magnetic-field direction could be obtained even in the infinite-conductivity limit, so that for this configuration a detailed experimental study would be necessary to establish the occurrence of the resistive "tearing" mode. For an "inverse stabilized pinch" with $B_\theta \sim B_z$, an $m = 1$ mode conforming with the field has been found experimentally,[2] contrary to the prediction of the infinite-conductivity theory, and consistent with the present analysis. For $\tau_R \sim 20$ μsec, $\tau_H \sim 0.2$ μsec, the e-folding time of the "tearing" mode [cf. Eqs. (57), (58)] is about 2 μsec.

A striking feature of magnetic-probe traces taken on the "stabilized"[30] and "inverse stabilized"[2] pinch discharges is that magnetic turbulence is suppressed during the initial dynamic phase. The present analysis provides a possible explanation. In Sec. V we note that a sufficiently strong gravitational effect (i.e., an accelerational effect in the present case) will suppress the "tearing" mode in favor of the gravitational interchange mode. Especially in the presence of an oscillating gravitational field, only short-wave gravitational interchange modes tend to grow, with a resultant minimal disturbance of the magnetic field.

The Interchange Modes

In the limit of high S and small G, the "rippling" and gravitational modes grow preferentially at short wavelengths and with $\mathbf{k}\cdot\mathbf{B} = 0$, so that there is a minimal disturbance of the magnetic field. The main effects to be looked for experimentally are a fluctuating electric field transverse to $\mathbf{B}$ and a loss of hot plasma out of the current layer. The "rippling" mode interchanges high-conductivity against low-conductivity plasma, and the gravitational mode interchanges high-pressure against low-pressure plasma or permits decelerating plasma to pass across magnetic field.

Recent studies on Zeta[31] have shown that the dominant nonradiative energy loss takes place by

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[29] C. E. Kuivinen, Bull. Am. Phys. Soc. 8, 150 (1963).

[30] L. C. Burkhardt and R. H. Lovberg, in *Proceedings of the Second International Conference on Peaceful Uses of Atomic Energy* (United Nations, Geneva, 1958), Vol. 32, p. 29.

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convection of hot plasma across magnetic field to the tube wall. The plasma convection is accompanied by fluctuating transverse electric fields with frequencies in the 10–100 kc range. The wavelengths are of order 20 cm in the direction across magnetic field, and are much greater in the direction along magnetic field. The parameters τ_R and τ_H are typically 3000 μsec and 1 μsec. For the "rippling" mode, we thus obtain $p \sim 100$ (cf. Eq. (50)). The resultant e-folding time of 30 μsec is perhaps somewhat too long to fit the data. The β value is of order 0.1, so that the G term and the $(\eta')^2$ term in Eq. (39) are of the same magnitude. The cooperation of the two driving mechanisms leads to a slightly enhanced growth rate. The stabilizing effect of heat conductivity on the "rippling" mode in Zeta (cf. Appendix F) begins to play an important role at electron temperatures above 10 eV.

The main difficulty in accounting for the Zeta results lies in the extremely short dimension of the region of discontinuity ($\epsilon a \sim 1$ cm) that is called for by the present analysis. The ion gyroradii in Zeta tend to be of this size or even larger. A nonhydromagnetic treatment of the region of discontinuity is therefore necessary to provide a rigorously valid model.

The attribution of the plasma loss in Zeta primarily to the "rippling" mode would have one especially engaging feature that deserves mention. Contrary to expectation from the ordinary theory of the interchange mode, the plasma in Zeta is *most* stable when the field lines in the central region of null shear come back on themselves on going once around the major circumference of the torus.[32] At these "magic number" points—the higher harmonics of the Kruskal limit—the periodicity condition around the major circumference permits an interchange mode to align itself perfectly with the null-shear magnetic field in the central region, which is advantageous for the growth of gravitational modes. We note, however, from Eq. (31) that the "rippling" mode is not perfectly aligned with the local magnetic field, but rather with the magnetic field at a point that is slightly displaced from the point of interchange. To generate the basic motor force of the instability, the perturbed current channel must make a small angle with respect to the field in the hot plasma. Thus the "magic number" regimes are generally unfavorable to the growth of the "rippling" mode.

A number of authors[15–17] have pointed out that the "rippling" mode is well suited to account for the pump-out[33] phenomenon in discharge tubes of the stellarator type. Using the present results for the growth rate [cf. Eq. (50)] and assuming typical parameters $\tau_R = 100$ μsec $\tau_H = 0.01$ μsec, we obtain $p \sim 100$. Due to the heat-conductivity effect (cf. Appendix F) the expected growth times of ~ 1 μsec become longer at electron temperatures above 10 eV. Again we note that the dimension ϵa of the discontinuity has an unrealistically small value: less than a millimeter. A nonhydromagnetic treatment is needed to give the true growth rate.

The "rippling" mode is not an inherent threat to the stellarator plasma-confinement scheme, since the discharge current can always be replaced, at least conceptually, by other heating methods. But even in the absence of current, the stellarator generally has some tendency toward gravitational interchanges, which is supposed to be suppressed by the shear of the magnetic field. For typical parameters of present experiments, the G term in Eq. (39) is much smaller than the η'^2 term but once the heating current is removed the pressure-driven resistive mode may become the dominant source of difficulty. The same remark applies to the stellaratorlike "Levitron"[1] (toroidal hard-core pinch). Hopefully, the nonhydromagnetic effects will serve to suppress the interchange mode in the limit of high conductivity. In particular, in reference 34 it is shown that, for a certain class of perturbations, a stabilizing charge-separation occurs due to finite Larmor radius. This leads to stability if $\omega_H/\omega_c < (kR_L)^2$ where ω_c is the cylotron frequency, and ω_H is the growth rate without correction for finite-Larmor-radius R_L. Since ω_H tends to be small for resistive instabilities, one might expect a strong stabilizing effect.

While the gravitational and resistivity-gradient effects are usually such as to collaborate in promoting instability in pinch and stellarator-type devices, the possibility exists of designing special regimes where the two effects are balanced against each other. For example, a stellarator discharge might be stabilized against the "rippling" mode if the absolute magnetic field strength were made to increase everywhere with radius. Such an effect can be achieved by giving the stabilizing windings an appropriate pitch. The required magnitude of stabilizing field is indicated by Eq. (60).

Direct observation of interchange modes has been

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[34] M. N. Rosenbluth, N. A. Krall, and N. Rostoker, Nuclear Fusion Suppl., Pt. 1, 143 (1962).

possible in a linear "stabilized pinch" experiment.[35] Stereoscopic Kerr-cell photography through a screen electrode reveals luminous "streamers" that have various orientations during the dynamic and quasistatic phases of the pinch cycle. During the dynamic phase, the streamers are aligned with the magnetic field in the regions of highest g stress (i.e., nearly pure B_z field and nearly pure B_θ) field. During the quasistatic phase, helical streamers are seen, which are more nearly aligned with the mean magnetic field in the current layer, and which can be accounted for as a mixture of the "rippling" and "tearing" modes.

VIII. COMPUTATIONAL PROGRAM

To supplement the analytical treatment in the range of intermediate S and to obtain accurate results in general for specific choices of F and of the boundary conditions, an IBM 709 code is available. This code is based on Eqs. (2)–(6) in linearized form and makes use of Fourier analysis in space but not in time. Accordingly, the development of specific initial disturbances can be studied. The code is applicable to both unstable and overstable modes, and will be capable of incorporating Ohmic heating, ionization, and similar effects.

Preliminary results have been reported,[8] and a more exhaustive study is under way.

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APPENDIX A. EFFECT OF FLUID COMPRESSIBILITY

Allowance for fluid compressibility has two principal effects on the stability analysis. The equation of motion of the fluid is altered, and the first-order quantities η_1 and $(\mathbf{g}\rho)_1$ receive nonconvective contributions.

We begin by demonstrating that the dynamic effect of compressibility is generally negligible for the modes of Sec. V. The equation $\nabla\cdot\mathbf{v}_1 = 0$ is no longer valid now, and we replace W' by $W' + i\alpha\tau_R\nabla\cdot\mathbf{v}_1$ in Eq. (14) [Equation (13) is unaltered]. The extra term arises from the fact that $\nabla\cdot\mathbf{v}_1 \neq 0$ when one eliminates $(\mathbf{k}\cdot\mathbf{v}_1)$ from the equations of motion.

To find $\nabla\cdot\mathbf{v}_1$, we use the equation describing the pressure perturbation

$$\omega P_1 + \mathbf{v}_1\cdot\nabla P_0 + \gamma P_0\nabla\cdot\mathbf{v}_1 = [(\gamma - 1)/(4\pi)^2] \cdot[\eta_1(\nabla\times\mathbf{B}_0)^2 + 2\eta_0(\nabla\times\mathbf{B}_0)\cdot(\nabla\times\mathbf{B}_1)]. \quad (A.1)$$

The effect of Ohmic heating has been included. Heat losses due to conduction and radiation have been neglected.

To determine the first-order pressure contribution P_1, we use the equation of motion in the form

$$\mathbf{k}\cdot\{\rho_0\omega\mathbf{v}_1 + \nabla P_1 - (4\pi)^{-1}[(\nabla\times\mathbf{B}_0)\times\mathbf{B}_1 + (\nabla\times\mathbf{B}_1)\times\mathbf{B}_0] - (\rho\mathbf{g})_1\} = 0 \quad (A.2)$$

which reduces to

$$P_1 + \frac{B_{\perp 0}B_{\perp 1}}{4\pi} + i\,\frac{B_{y1}}{4\pi}\,\frac{dB_{\parallel 0}}{k\,dy} - i\,\frac{\rho_0\omega}{k}\,v_\parallel = 0, \quad (A.3)$$

where

$$B_{\perp 0} = \hat{y}\cdot(\mathbf{k}\times\mathbf{B}_0)/k = BH,$$
$$B_{\perp 1} = \hat{y}\cdot(\mathbf{k}\times\mathbf{B}_1)/k = (B/i\alpha)\,\chi,$$
$$B_{\parallel 0} = \mathbf{k}\cdot\mathbf{B}_0/k = BF,$$
$$B_{\parallel 1} = \mathbf{k}\cdot\mathbf{B}_1/k = (iB/\alpha)\psi'$$
$$v_\parallel = \mathbf{k}\cdot\mathbf{v}_1/k = -(1/k\alpha\tau_R)(W' + i\alpha\tau_R\nabla\cdot\mathbf{v}_1).$$

We will show below that in the "region of discontinuity" of Sec. V, which is the region of critical interest, the B_{y1} and $v_\parallel$ terms are negligible. One finds also $|B_{\perp 0}B_{\perp 1}| \gg |B_{\parallel 0}B_{\parallel 1}|$.

Equation (A.3) then yields

$$P_1 = -B_{\perp 0}B_{\perp 1}/4\pi = -\mathbf{B}_0\cdot\mathbf{B}_1/4\pi \quad (A.4)$$

as would be expected for a subsonic motion: the total (fluid plus magnetic) pressure remains approximately constant.

To evaluate the $B_{\perp 0}B_{\perp 1}$ term in Eq. (A.3), and to prove its predominance, we must find the perturbation-field amplitude χ. In the analysis of the incompressible case, it was unnecessary to obtain χ explicitly in order to find the dispersion relation. The solution was obtained in terms of ψ

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and W, and the quantity χ could then be derived, if desired, by means of Eqs. (8) and (9). Since we will show that compressibility has a negligible effect, we may proceed in precisely the same manner, first obtaining χ for the incompressible case, and then using the result in Eqs. (A.3) and (A.1) to verify the effect of compressibility. The terms involving $\nabla\cdot\mathbf{v} \neq 0$ may be easily checked subsequently to be small.

From Eqs. (8) and (9) we obtain

$$\begin{aligned}(\tilde{\eta}\chi')' = \chi[\alpha^2\tilde{\eta} + p + (\alpha^2 S^2/\tilde{\rho}p)F^2] - WH' \\ - (1/p)[(W\tilde{\eta}'H') - \alpha^2 W\tilde{\eta}'H] \\ + (\alpha^2S^2/\tilde{\rho}p)FH'\psi,\end{aligned} \quad \text{(A.5)}$$

an equation that is similar in structure to Eqs. (13) and (14). In the "outer region" of Sec. IV, we have

$$\chi F = -H'\psi \quad \text{(A.6)}$$

which may be used with Eqs. (19), (A.3), and (A.1) to demonstrate strictly incompressible flow, as might be expected. In the region of discontinuity, the terms of Eq. (A.5) involving $\tilde{\eta}'$ are of order $p^{\frac{1}{2}}S^{-\frac{1}{2}}$ relative to the ψ term, and may be neglected. Similarly, the term WH' is of order $p^{\frac{1}{2}}S^{-1}$ relative to the ψ term and may be neglected except in the case $G \sim 1$, in which case the two terms are comparable. Transforming to the variable $\theta_1 = (\mu - \mu_0)/\epsilon$, and making the usual approximations in the region of discontinuity (cf. Sec. V), we obtain

$$d^2\chi/d\theta_1^2 - \tfrac{1}{4}\theta_1^2\chi = (H'/4\epsilon F')\psi\theta_1. \quad \text{(A.7)}$$

Thus χ behaves much like W. From Eq. (A.7) we infer

$$\chi = O[(H'/\epsilon F')\psi] \quad \text{(A.8)}$$

in the region of discontinuity.

We may now return to evaluate the magnitude of terms in Eq. (A.3). From Eq. (A.8), we see directly that the $B_{\nu 1}$ term is negligible. From Eqs. (A.8) and (35), it follows that the $v_\parallel$ term is of order $p^{7/4}S^{-\frac{3}{4}}$, and is therefore negligible. We have now proved Eq. (A.4). Using Eqs (A.4), (A.8), and the zero-order relation $P_0' \approx -B^2HH'/4\pi$, we may write

$$P_1 = iO(\psi P_0'/\alpha\epsilon F'). \quad \text{(A.9)}$$

We may now estimate the magnitude of the terms of Eq. (A.1) in the region of discontinuity. The η_1 term is negligible relative to the ∇P_0 term, since

$$\eta_1(\nabla\times\mathbf{B}_0)^2 = O(\mathbf{v}_1\cdot\nabla P_0/p)$$

where we have used Eq. (11). For the η_0 term we find

$$\eta_0(\nabla\times\mathbf{B}_0)\cdot(\nabla\times\mathbf{B}_1) = O(\omega P_1/p\epsilon)$$

so that it is of the same magnitude as the P_1 term for the "rippling" mode [cf. Eq. (53)], but is negligible for the low-α "tearing" mode [cf. Eqs. (36) and (56)] and the gravitational interchange mode. Next we compare the P_1 and ∇P_0 terms. From Eq. (A.9) we have

$$\omega P_1 = iO(p\psi P_0'/\epsilon\alpha F'\tau_R) \quad \text{(A.10)}$$

and we see directly that

$$\mathbf{v}_1\cdot\nabla P_0 = iO(WP_0'/\alpha\tau_R). \quad \text{(A.11)}$$

Using Eq. (35) and remembering that $U = 0(\psi)$, we conclude that the P_1 and ∇P_0 terms are of the same order.

Finally we may evaluate the correction to W' in Eq. (14) that results from compressibility. As we have noted in connection with Eq. (A.6), this correction is insignificant in the outer region. In the region of discontinuity, we have from Eq. (A.1), and from the remarks on the relative magnitude of its terms

$$\begin{aligned}|i\alpha\tau_R\nabla\cdot\mathbf{v}_1| &= O[(P_0'/\gamma P_0)W] \\ &= O[(\epsilon P_0'/\gamma P_0)W'] \\ &\ll |W'|.\end{aligned} \quad \text{(A.12)}$$

Thus, allowance for fluid compressibility does not directly affect the fluid motion involved in the modes of Sec. V.

The "tearing" mode is thus completely unaffected. For the "rippling" and gravitational interchange modes, which depend on the nature of Eqs. (11) and (12), there will generally be indirect compressional effects, since we see from Eq. (A.12) that the compressional changes in ρ and η are comparable to the purely convective changes. Accordingly, we write the equations

$$\omega\rho_1 + \mathbf{v}_1\cdot\nabla\rho_0 = -\rho_0\nabla\cdot\mathbf{v}_1, \quad \text{(A.13)}$$

$$\begin{aligned}\omega\eta_1 + \mathbf{v}_1\cdot\nabla\eta_0 = \tfrac{3}{2}(\gamma - 1)\eta_0 \\ \cdot\left[\nabla\cdot\mathbf{v}_1 - \frac{\eta_1(\nabla\times\mathbf{B}_0)^2 + 2\eta_0(\nabla\times\mathbf{B}_0)\cdot(\nabla\times\mathbf{B}_1)}{(4\pi)^2P_0}\right],\end{aligned} \quad \text{(A.14)}$$

where we have used the plasma properties $\eta \sim T^{-\frac{3}{2}}$ and $\rho T \sim P$.

We begin with the standard gravitational-interchange case, i.e., $\tilde{\eta}' = F'' = 0$, and $|G/(F')^2| \ll 1$. From Eq. (A.1) (neglecting the Ohmic-heating terms) and Eqs. (A.4) and (A.13), we obtain as the modified version of Eq. (33)

$$\begin{aligned}d^2U/d\theta^2 + U\{\Lambda(1 - L_g) - \tfrac{1}{4}\theta^2\} \\ = \psi\theta - (4\epsilon F'\Lambda L_g/H')\chi\end{aligned} \quad \text{(A.15)}$$

where

$$L_g = \rho_0 P_0'/\gamma\rho_0' P_0.$$

Equations (32), (A.7), and (A.15) can readily be solved by the method of Sec. V, and yield the eigenvalue equation

$$\Delta' = \frac{\pi 2^{7/2}\Omega}{1 - L_g} \cdot \left[\frac{\Gamma\{\frac{3}{4} - \frac{1}{2}\Lambda(1 - L_g)\}}{\Gamma\{\frac{1}{4} - \frac{1}{2}\Lambda(1 - L)_g\}} - L_g\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}\right]. \quad \text{(A.16)}$$

We note that Λ is not much altered for $L_g < 1$; in the case $L_g > 1$, however, we now find unstable modes for both positive and negative G. (As will emerge in Appendix I, the dependence on compressibility is a peculiarity of the "true" gravitational mode, which disappears when G is to be interpreted as arising from a pressure gradient and curved field.)

For the "rippling" mode we proceed in a similar manner, now setting $G = 0$ and including Ohmic-heating effects. We obtain

$$\frac{d^2U}{d\theta_c^2} + U\{\Lambda(1 - L_r)^2 - \tfrac{1}{4}\theta_c^2\} = \psi(\theta_c - \delta_c) - \frac{4\epsilon F'\Lambda L_2}{H'}\chi - 3\tilde{\eta}^2\frac{(\gamma - 1)H'}{\gamma\beta_0\epsilon p^2}\frac{d\chi}{d\theta_c}(\theta_c - \delta_{1c}) \quad \text{(A.17)}$$

where

$$\beta_0 = 4\pi P_0/B^2,$$

$$\theta_c = \frac{1}{\epsilon}\left[\mu - \mu_0 + \frac{\tilde{\eta}'}{2p}(1 - L_r)\right],$$

$$\delta_c = \frac{1}{4\Omega}\left\{\frac{F''}{F'} + \frac{\tilde{\eta}'}{2\tilde{\eta}}(1 - L_r)\right\},$$

$$\delta_{1c} = \frac{1}{8\Omega}\frac{\tilde{\eta}'}{\eta}(1 - L_r),$$

$$L_r = -\frac{3(\gamma - 1)\eta_0 P_0'}{2\gamma\eta_0' P_0}.$$

The equations for the "rippling" mode are more difficult to solve formally, because the homogeneous part of Eq. (A.17) involves θ_c, while that of Eq. (A.7) involves θ_1. The main features of the result are evident, however. Since $\epsilon \sim 1/p$ for the "rippling" mode, we see that the χ and $d\chi/d\theta$ terms in Eq. (A.17) are of the same order in S and α as the ψ term. For $\alpha \gg 1$, we know from Sec. V that the U term becomes of order α relative to the ψ term, because $(n + \frac{1}{2} - \Lambda) \to 1/\alpha$ in Eq. (43). Thus, for the case of maximum interest, where S and α are large, we have simply

$$\Lambda = (m + \tfrac{1}{2})/(1 - L_r)^2, \quad m = 1, 2, 3 \cdots. \quad \text{(A.18)}$$

Since generally $P_0'\eta_0' < 0$, we will have $L_r > 0$, so that the growth-rate is reduced. For $L_r > 1$, the pressure-gradient effect dominates the resistivity-gradient effect in Eq. (A.14), and we obtain a new "rippling" mode that resembles the old one in every respect, except that η_0'/η_0 is to be replaced by $\frac{3}{2}[(\gamma - 1)/\gamma]P_0'/P_0$. Note that, in the case of maximum interest considered here, the Ohmic-heating effects do not play a role.

The analysis in this appendix has been carried out for H, $H' \neq 0$ at $F = 0$. If either H or H' has a null at $F = 0$, then it is easy to see that the compressibility effects become completely negligible as $S \to \infty$. If H' has a null, the Ohmic-heating effects become completely negligible as $S \to \infty$. (These remarks, of course, cover the important special case of unsheared field, where $H \equiv 0$.)

APPENDIX B. LOW-CONDUCTIVITY LIMIT

For $S \ll 1$, unstable modes cannot grow faster than ordinary resistive diffusion. In order that a zero-order equilibrium may exist, we therefore require that Eq. (15) be satisfied. For convenience we let $\tilde{\eta}F' = 1$. We will treat the case $G \equiv 0$.

To make a general estimate of p, we note from Eq. (18) that W^2 must be of order $S^2\psi^2$ or less. If we had $p \gg S$, Eq. (13) would reduce to

$$\psi'' - (\alpha^2 + pF')\psi = 0 \quad \text{(B.1)}$$

from which follows

$$\int_{\mu_1}^{\mu_2} d\mu\, [(\psi')^2 + (\alpha^2 + pF')\psi^2] = 0 \quad \text{(B.2)}$$

so that $p < 0$. Unstable modes are therefore characterized by $p \leq 0(S)$, which means that they grow on the hydromagnetic rather than on the resistive time scale.

The case $S < 1$ may be applied to liquid-metal experiments and to some experiments with dense, low-temperature plasmas of heavy ions. The former application has been investigated exhaustively by Murty,[9] who includes surface-tension and gravitational forces, and uses the slab model

$$F' = 1, \quad |\mu| < 1; \qquad F' = 0, \quad |\mu| > 1, \quad \text{(B.3)}$$

together with the specification $\tilde{\rho} = F'$.

We will begin with a more general treatment. As $S \to 0$, $p \to 0$, Eqs. (13) and (16) reduce to

$$\psi'' - \alpha^2\psi + vF'' = 0, \quad \text{(B.4)}$$

$$(\tilde{\rho}v')' - \alpha^2\tilde{\rho}v + (\alpha^2S^2/p^2)F''(vF + \psi) = 0, \quad \text{(B.5)}$$

where $W = pv$. There are two characteristic cases: $\alpha^2 \gg |F''/F|$, for which the ψ term in Eq. (B.5) is negligible; and $\alpha^2 \ll |F''/F|_{\max}$, which includes Murty's model.

For $\alpha^2 \gg |F''/F|$, we will set $\tilde{\rho} = 1$, and Eq. (B.5) then becomes

$$v'' + \alpha^2 v[-1 + (S^2/p^2)FF''] = 0. \qquad \text{(B.6)}$$

We note that short-wave perturbations are now localized near the point μ_r, where $(FF'')' = 0$ rather than near $F = 0$; and we expand

$$FF'' = f_0 + f_2\mu_2^2$$

where $\mu_2 = \mu - \mu_r$. Equation (B.6) then has the same form as Eq. (41). The fastest growing mode is given by

$$v = \exp[-\tfrac{1}{2}\alpha(-f_2/f_0)^{\frac{1}{2}}\mu_2^2]$$

and the corresponding eigenvalue relation is

$$p = Sf_0^{\frac{1}{2}}. \qquad \text{(B.7)}$$

The condition for instability is $f_0 > 0 > f_2$. To illustrate what this condition means, let us consider the model

$$F = F_0 + \tanh \mu_0$$

where F_0 is an arbitrary constant. Then we have

$$FF'' = -2(\tanh \mu/\cosh^2 \mu)(F_0 + \tanh \mu).$$

For $F_0 = 0$, we find $FF'' < 0$; therefore there are no modes of the symmetric "tearing" type. For $F_0^2 \ll 1$, we find $\mu_r = -\frac{1}{2}F_0$, $f_0 = \frac{1}{2}F_0^2$, $f_2 = -2$. For $F_0^2 \gg 1$, we find $\mu_r = \mp\sinh^{-1}(2^{-\frac{1}{2}})$, with the sign of μ_r to be taken opposite to that of F_0 for instability, in which case $f_0 = 2\,|F_0|\,3^{-\frac{3}{2}}$, $f_2 = 8\,|F_0|\,3^{-\frac{3}{2}}$. Thus the "rippling" mode exists for $F_0 \neq 0$, and grows most rapidly for $F_0^2 \gg 1$.

We turn now to the second characteristic case, $\alpha^2 \ll |F''/F|_{\max}$. The model of Eq. (B.3) is typical of this case, and will be adopted here. For simplicity, we will specify $\tilde{\rho} = 1$ everywhere. This density profile is somewhat more suitable for the plasma application than Murty's, and is well suited to describe liquid-metal layers suspended in a density-matching oil.[36] Except at the two points where $F'' \neq 0$, the solution to Eqs. (B.4) and (B.5) have the form $\psi, v \sim e^{\pm\alpha\mu}$, so that the problem is a purely algebraic one. To obtain the dispersion relation, it is convenient to write Eq. (B.3) in the form

$$F'' = -\delta(\mu - 1) + \delta(\mu + 1)$$

$$F = F_1 = F_0 - 1, \qquad \mu < -1;$$

$$F = F_0 + \mu, \qquad |\mu| < 1;$$

$$F = F_2 = F_0 + 1, \qquad \mu > 1.$$

The resultant equations involve ψ_1, ψ_2, v_1, and v_2, where the subscripts refer to values of ψ and v at $\mu = -1$ and $+1$ respectively. Eliminating ψ_1, ψ_2, one obtains the two-dimensional homogeneous vector equation

$$v_2[2p^2/\alpha S^2 - (1/2\alpha)(1 - e^{-4\alpha}) + F_2] - v_1F_1e^{-2\alpha} = 0, \qquad \text{(B.8)}$$

$$v_1[2p^2/\alpha S^2 - (1/2\alpha)(1 - e^{-4\alpha}) - F_1] + v_2F_2e^{-2\alpha} = 0, \qquad \text{(B.9)}$$

so that

$$p^2/S^2 = \tfrac{1}{4}(1 - e^{-4\alpha}) - \tfrac{1}{2}\alpha\{1 \pm [1 + (F_0^2 - 1)(1 - e^{-4\alpha})]^{\frac{1}{2}}\}. \qquad \text{(B.10)}$$

The eigenmodes are described by

$$v_2 = v_1 \frac{(F_0 - 1)e^{-2\alpha}}{F_0 \mp [1 + (F_0^2 - 1)(1 - e^{-4\alpha})]^{\frac{1}{2}}}. \qquad \text{(B.11)}$$

The typical "tearing" mode is found for $F_0 = 0$, where Eq. (B.11) yields $v_2 = \pm v_1$, and may be identified with the antisymmetric-v eigenmode. The typical "rippling" mode is found for $F_0^2 \gg 1$, so that $v_2 = v_1e^{-2\alpha}[1 \mp (1 - e^{-4\alpha})^{\frac{1}{2}}]^{-1}$, and may be identified with the eigenmode involving the positive square root, when $F_0 > 0$.

For $\alpha \ll 1$, the eigenvalues are

$$p^2/S^2 = 0, \alpha, \qquad \text{(B.12)}$$

and the eigenmodes are characterized by

$$v_2/v_1 = 1, \qquad (F_0 - 1)/(F_0 + 1). \qquad \text{(B.13)}$$

The first of these modes, describing a simple displacement of the current layer, is the low-α limit of the "rippling" mode; the second covers the "tearing" mode.

For $\alpha \gg 1$, we have

$$p^2/S^2 = \tfrac{1}{4} - \tfrac{1}{2}\alpha(1 \pm F_0). \qquad \text{(B.14)}$$

In the special case $F_0 = 0$, Eq. (B.14) gives a single solution

$$p^2/S^2 = \tfrac{1}{4} - \tfrac{1}{2}\alpha \qquad \text{(B.15)}$$

so that both the antisymmetric-v "tearing" mode and the symmetric-v mode are stable. Equation (B.14) indicates that instability can be obtained only for $F_0^2 > 1$, and that the growth rate increases with F_0^2. For $F_0^2 \to \infty$, we have

[36] S. A. Colgate, H. P. Furth, and F. O. Halliday, Revs. Mod. Phys. **32**, 744 (1960).

$$p^2/S^2 = \mp\tfrac{1}{2}\alpha F_0, \tag{B.16}$$

$$v_2/v_1 = 2e^{2\alpha}, \qquad \tfrac{1}{2}e^{-2\alpha}. \tag{B.17}$$

For positive F_0, the second of these eigenmodes is unstable and corresponds to the "rippling" mode.

The results we have obtained for the $\alpha^2 \ll |F''/F|_{\max}$ case are substantially similar to those of Murty.[9] We note that, for the "tearing" mode, p has a maximum of order S. For the "rippling" mode, Eq. (B.16) indicates that p is of order $\alpha^{\frac{1}{2}}S$, in contrast with Eq. (B.7), where p is found to be of order S. Thus the increase in growth rate at short wavelengths predicted by Eq. (B.16) is seen to be dependent on the discontinuities of F' assumed in the zero-order configuration. The long-wave "tearing" mode is not dependent on the exact structure of F, and similar growth rates would be expected for continuous-F' models.

APPENDIX C. THE LIMIT $\alpha^2\epsilon^2 > 1$

We consider here the case where the instability wavelength is smaller than the region of discontinuity. This limit is more of mathematical than of physical interest, since ordinary resistive diffusion proceeds more rapidly at such small wavelengths than does the instability. For $\alpha^2\epsilon^2 > 1$, it is convenient to write Eq. (32) in the form

$$\psi = \frac{-\Omega}{2\alpha}\int_{-\infty}^{\infty} d\theta_1\, e^{-\epsilon\alpha|\theta-\theta_1|}\, U(\theta_1 + \delta_1)$$

$$\approx \frac{-\Omega}{\epsilon\alpha^2}\, U(\theta + \delta_1) \tag{C.1}$$

valid if $\Omega/\epsilon\alpha^2 \ll 1$. Thus we see that ψ is no longer constant in the region of discontinuity, but varies as strongly as U.

The ψ term in Eq. (33) becomes negligible, and the eigenvalues Λ lie very close to $\frac{1}{2}, \frac{3}{2}, \frac{5}{2} \cdots$.

For the "rippling" mode we now have from Eq. (39), when $\alpha^2\epsilon^2 \gg 1$,

$$p = (S\tilde{\eta}'\,|F'|/2\tilde{\eta}^{\frac{1}{2}}\tilde{\rho}^{\frac{1}{2}})^{\frac{2}{3}}. \tag{C.2}$$

For the gravitational interchange mode, we may drop the restriction Eq. (45) on the magnitude of G, and obtain from Eq. (39), when $\alpha^2\epsilon^2 \gg \Lambda$,

$$p = SG^{\frac{1}{2}}/\tilde{\rho}^{\frac{1}{2}}. \tag{C.3}$$

Using Eqs. (C.2) and (C.3), we may verify the validity of the initial assumption that $\Omega/\epsilon\alpha^2 \ll 1$ if $\alpha^2\epsilon^2 > 1$.

APPENDIX D. LOW-α LIMIT FOR THE "TEARING" MODE

Neglecting terms of order α^2 in Eqs (13) and (14), we will analyze the "tearing" mode without invoking the constant-ψ approximation which was used in Sec. V, and which is inapplicable for $\alpha \to 0$. We will neglect G and $\tilde{\eta}'$, and restrict ourselves to the symmetric "tearing" mode, where $F''' = 0$ at $F = 0$.

In the region of discontinuity, we will set $\tilde{\eta} = \tilde{\rho} = F' = 1$ for convenience, so that $F = \mu$. Eqs. (13) and (14) then yield

$$z''' = pz' + (\alpha^2S^2/p)(\mu^2z' + 4\mu z) \tag{D.1}$$

where

$$z = \psi'' = p\psi + WF'\mu. \tag{D.2}$$

Introducing a new independent variable

$$\theta_1 = \mu(\alpha^2S^2/p)^{\frac{1}{4}}$$

and defining

$$\lambda = p^{\frac{3}{2}}/\alpha S$$

we obtain

$$d^3z/d\theta_1^3 = (\lambda + \theta_1^2)\, dz/d\theta_1 + 4\theta_1 z, \tag{D.3}$$

to be solved subject to the boundary conditions

$$\left.\begin{aligned} z &= 1, \\ z' &= 0, \end{aligned}\right\}\ \theta_1 = 0,$$

$$z = 0, \qquad \theta_1 = \infty.$$

We observe that there are two well-behaved solutions at $\theta_1 = \infty$, namely $z \sim \theta_1^{-4}$ and $z \sim \exp(-\frac{1}{2}\theta_1^2)$, so that Eq. (D.2) can always be solved. The relationship between the solution z and the quantity Δ' of Eq. (21) is given by

$$\Delta' = 2\lim_{\theta_1\to\infty}\left(\frac{\psi'}{\psi - \mu\psi'}\right) \tag{D.4}$$

where the denominator represents the intersection of the asymptote of ψ with the ψ axis at $\mu = 0$. Using Eq. (D.2), we have

$$\Delta' = 2p\int_0^\infty z\, d\mu \bigg/ \left(1 - p\int_0^\infty \mu z\, d\mu\right). \tag{D.5}$$

If we let $\Delta' = 2/\alpha$ (cf. Eq. 28) and define $p_1 = S^{-\frac{1}{2}}p$, $\alpha_1 = S^{\frac{1}{4}}\alpha$, we obtain from Eq. (D.5) the eigenvalue relation

$$p_1^{7/4}H(p_1/\alpha_1) = 1 \tag{D.6}$$

where

$$H(\lambda) = \lambda^{-\frac{1}{4}}\int_0^\infty z\, d\theta_1 \bigg/ \left(1 - \lambda\int_0^\infty \theta_1 z\, d\theta_1\right) \tag{D.7}$$

with the integrals to be determined by means of the solution of Eq. (D.3). We note that near $\lambda = 0$, $H \approx \lambda^{-\frac{1}{4}}$, so that $p_1 \approx \alpha_1^{-2/5}$, or $p \approx (S/\alpha)^{2/5}$, the

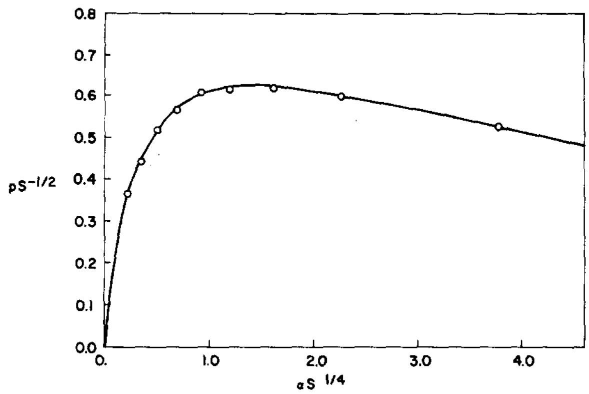

FIG. 3. Growth rates for long wavelength "tearing" mode.

result obtained in Sec. V. It is also easy to verify by substitution that for $\lambda = 1$, we have $z = \exp\,(-\frac{1}{2}\theta_1^2)$ and $H(\lambda) = \infty$. Thus p_1 is zero both for $\lambda = 0$ and $\lambda = 1$, and has a maximum value of order unity somewhere in between. This result verifies the conjecture made in Sec. V that the maximum of p with respect to α is of order $S^{\frac{1}{2}}$. Eq. (D.4) has been integrated numerically on the UCSD-CDC 1604. The solution was used to construct Fig. 3, which describes the detailed behavior of p in the long wavelength limit.

APPENDIX E. GENERAL ANALYSIS OF GRAVITATIONAL INSTABILITIES

Basic Equations

In Sec. V we have treated the gravitational interchange instability for the case where $G(F')^{-2}$ is sufficiently small so that ψ is constant in a region R_0 of width $\epsilon_0 \sim (1 + |\Delta'|)^{-1}$ about the point μ_0, where $F = 0$. The object of this appendix is to derive conditions on $G(F')^{-2}$ for which the constant-ψ approximation is justified, and to extend the analysis to the case of stronger gravitational fields.

As in Eqs. (32) and (33), we will treat a region about the point μ_0 in which F', $\tilde{\eta}$, G, and $\tilde{\rho}$ are constant, while F is approximated by $F'\mu_1$, where $\mu_1 = \mu - \mu_0$. For simplicity we will treat the pure gravitational-instability case where F'', $\tilde{\eta}' = 0$, so that the "tearing" and "rippling" modes are absent.

The analysis is valid in the range $\mu_1^2 \ll 1$. Thus it applies to high-G modes of short wavelength, i.e., $\alpha > 1$, and it also permits us to assess the constancy of ψ near μ_0 for low-G modes of arbitrary wavelength. High-G modes of long wavelength are not covered, but these are of lesser practical interest.

Eqs. (13) and (14) now reduce to the form

$$\psi''/\alpha^2 = \psi(1 + p/\alpha^2) + (W/\alpha^2)F'\mu_1, \qquad \text{(E.1)}$$

$$W''/\alpha^2 = W[1 - S^2G/p^2 + (F')^2\mu_1^2S^2/p] + \psi S^2F'\mu_1, \qquad \text{(E.2)}$$

where we have set $\tilde{\eta} = \tilde{\rho} = 1$.

We make the Fourier transform

$$\psi = \int_{-\infty}^{\infty} dk_0\, \psi_t e^{ik_0\mu_1},$$

$$W = \int_{-\infty}^{\infty} dk_0\, W_t e^{ik_0\mu_1},$$

and obtain

$$\psi_t\left(\frac{k_0^2}{\alpha^2} + 1 + \frac{p}{\alpha^2}\right) + i\,\frac{F'}{\alpha^2}\frac{dW_t}{dk_0} = 0, \qquad \text{(E.3)}$$

$$W_t\left(\frac{k_0^2}{\alpha^2} + 1 - \frac{S^2G}{p^2}\right) - \frac{(F')^2S^2}{p}\frac{d^2W_t}{dk_0^2} + iF'S^2\frac{d\psi_t}{dk_0} = 0. \qquad \text{(E.4)}$$

Eliminating ψ_t, we have

$$\frac{(F')^2S^2}{p}\frac{d}{dk_0}\left(\frac{k_0^2 + \alpha^2}{k_0^2 + \alpha^2 + p}\frac{dW_t}{dk_0}\right) - W_t\left(\frac{k_0^2}{\alpha^2} + 1 - \frac{S^2G}{p^2}\right) = 0. \qquad \text{(E.5)}$$

In our usual limit $S \rightarrow \infty$, where $p \gg 1$, α^2, we may reduce Eq. (E.5) to standard form

$$\frac{d}{dk_1}\left\{\frac{k_1^2 + \sigma}{1 + k_1^2}\frac{dW_t}{dk_1}\right\} - (Ak_1^2 - D)W_t = 0, \qquad \text{(E.6)}$$

where

$$k_0 = p^{\frac{1}{2}}k_1, \qquad \sigma = \alpha^2/p \ll 1,$$

$$A = p^3/(F')^2S^2\alpha^2, \qquad D = G/(F')^2 - A\sigma.$$

We will confine ourselves to the case $G > 0$. Since $\sigma \ll 1$, we may split the analysis of Eq. (E.5) into two overlapping ranges: $k_1^2 < 1$ and $k_1^2 > \sigma$.

For $k_1^2 < 1$, we let $k_2 = \sigma^{-\frac{1}{2}}k_1 = \alpha^{-1}k_0$, and obtain

$$\frac{d}{dk_2}\left[(1 + k_2^2)\frac{dW_t}{dk_2}\right] - (A\sigma k_2^2 - D)W_t = 0. \qquad \text{(E.7)}$$

Low-G Case

At this point we turn specifically to consideration of the case where $G(F')^{-2} < \frac{1}{4}$. As we will see, the approximation $A\sigma \ll 1$ is appropriate to this case. Eq. (E.7) then reduces to the Legendre equation. The general solution is

$$W_t = C_1P_h(ik_2) + C_2P_h(-ik_2) \qquad \text{(E.8)}$$

where

$$h = \tfrac{1}{2}[-1 + (1 - 4D)^{\frac{1}{2}}] < 0. \qquad \text{(E.9)}$$

We note that the region $k_2 < 1$ is related to the "outer" region of Sec. IV, and that the choice of C_1 and C_2 will reflect the outside boundary conditions, e.g., for a symmetric layer we would have $C_1 = -C_2$. For large k_2 the asymptotic form of the Legendre function gives

$$W_t \approx L_1 k_2^h + L_2 k_2^{-(h+1)} \tag{E.10}$$

where the constants $L_{1,2}$ are determined from the C's. For $D < \frac{1}{4}$, W_t decays at large k_2 (with the first term predominating), while for $D > \frac{1}{4}$, W_t is oscillatory. Since the oscillatory behavior is not acceptable, it follows that the $A\sigma \ll 1$ approximation is consistent only with the case $D < \frac{1}{4}$, [i.e., the case $G(F')^{-2} < \frac{1}{4}$].

Continuing with the analysis of the $D < \frac{1}{4}$ case, we proceed to solve Eq. (E.6) in the range $k_1^2 > \sigma$, using $W_t \approx k_1^h$ to give the behavior of W_t for $\sigma < k_1^2 \ll 1$. The appropriate form of Eq. (E.6) is

$$\frac{d}{dk_1}\left\{\frac{k_1^2}{1+k_1^2}\frac{dW_t}{dk_1}\right\} - (Ak_1^2 - D)W_t = 0 \tag{E.11}$$

and a solution is

$$W_t = k_1^h e^{(\frac{1}{2}h)k_1^2} \tag{E.12}$$

with eigenvalue

$$A = h^2, \tag{E.13}$$

clearly the lowest eigenvalue. Hence, for the case $G(F')^{-2} < \frac{1}{4}$ we have

$$p = \left(S\alpha F'\left\{\frac{1 - [1 - 4G(F')^{-2}]^{\frac{1}{2}}}{2}\right\}\right)^{\frac{1}{2}}. \tag{E.14}$$

In the low-G limit, Eq. (E.14) reduces to Eq. (59) for $\Lambda = \frac{1}{2}$, thus verifying our use of the constant-ψ approximation. However, this verification of the results of Sec. V is limited to the case $\alpha \gg 1$, since only in this limit is the instability localized, so that we may neglect F'', set $F = F'\mu$, etc.

For $\alpha < 1$, ψ increases away from μ_0 [cf. Eq. (26)], and the G-term in Eq. (20) may become important. In particular, if G is constant, we must require

$$G/(F_{\pm\infty})^2 \ll \alpha^2 \tag{E.15}$$

in order to be able to neglect the contribution of the G-term at large μ.

High-G Case

To study the case $G(F')^{-2} > \frac{1}{4}$, i.e., where the Suydam criterion is violated, we return to the range $k_1^2 < 1$ and to Eq. (E.7) with $A\sigma$ finite. We note that the finiteness of $A\sigma$ implies $p \sim S$, as we would expect for an instability that exists in the infinite-conductivity limit. Equation (E.16) has been integrated numerically, and the results are given in Fig. 4. The main features of the solution may be deduced analytically.

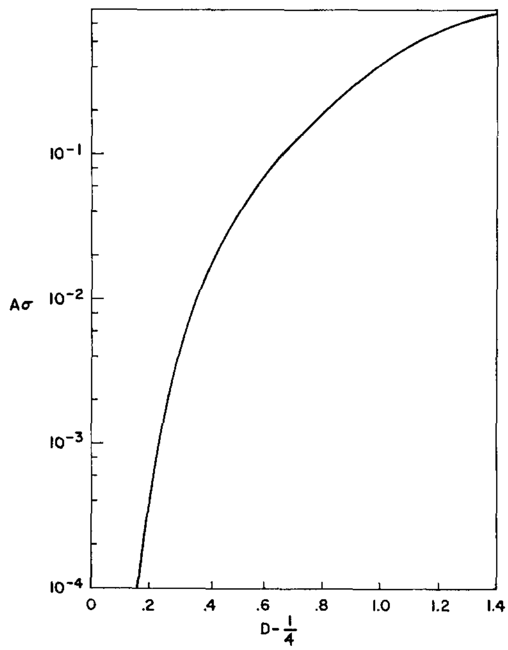

FIG. 4. Plot of eigenvalues of Eq. (E.16) for growth rates of hydromagnetically unstable gravitational mode.

For $G(F')^{-2} > \frac{1}{4}$ the complete solution W_t is obtained within the range $k_1^2 < 1$. (The range $k_1^2 > \sigma$ corresponds to the "region of discontinuity" of Section V, and disappears for infinite-conductivity modes.) Thus we have the boundary condition that the solution of Eq. (E.7) should vanish at $\pm\infty$. If we define $k_2 = \sinh z$, then Eq. (E.7) can be reduced to the form

$$d^2W_t/dz^2 + [(D - \tfrac{1}{4}) - \tfrac{1}{4}\operatorname{sech}^2 z - A\sigma \sinh^2 z]W_t = 0. \tag{E.16}$$

For small $A\sigma$, the resulting "potential well" is almost a square well, and we derive the eigenvalue

$$A\sigma \approx (4D - 1)\exp[-2\pi/(D - \tfrac{1}{4})^{\frac{1}{2}}]. \tag{E.17}$$

Thus, for small $D - \frac{1}{4}$, $A\sigma$ is extremely small. The growth rate p does not effectively become of order S until $D \sim \frac{1}{2}$.

For large $A\sigma$, Eq. (16) reduces to the harmonic oscillator equation, and we have the eigenvalue

$$A\sigma = \tfrac{1}{4} + (D - \tfrac{1}{2})^2. \tag{E.18}$$

For $G(F')^{-2} \gg \frac{1}{2}$, this may be written

$$p^2/S^2 = G - (F')^2[G/(F')^2 - \tfrac{1}{2}]^{\frac{1}{2}}. \tag{E.19}$$

Note that, since we have assumed $|\mu_1| < 1$ in deriving Eqs. (E.1) and (E.2), the discussion of the high-G case is valid only for $\alpha > 1$. Plasma compressibility, which has been neglected throughout, presumably also affects the high-G case.

APPENDIX F. EFFECT OF THERMAL CONDUCTIVITY

From Eq. (5a) we obtain to first order

$$\omega\eta_1 + \mathbf{v}\cdot\nabla\eta_0 = +(2\kappa/3n_0B_0^2)$$
$$\cdot[\mathbf{B}_0\cdot\nabla(\mathbf{B}_0\cdot\nabla\eta_1) + \mathbf{B}_0\cdot\nabla(\mathbf{B}_1\cdot\nabla\eta_0)] \qquad (\text{F.1})$$

so that approximately (setting $B_0^2 = B^2$),

$$\eta_1(p + KF^2\alpha^2) + (i/\alpha)\eta_0'(W - KF\alpha^2\psi) = 0, \qquad (\text{F.2})$$

$$K = 2\kappa\tau_R/3na^2 = 8\pi\kappa/3n\langle\eta\rangle, \qquad (\text{F.3})$$
$$\approx (10^{14}/n)T^4,$$

where T is in eV.

Note that outside the small skin depth of thickness ϵ, around the point where F vanishes, Eq. (19) tells us that $\psi = -WF/p$, so that the correction terms cancel in Eq. (F.2). This is reasonable since in the outside region material is moving with the field lines, and the condition $\mathbf{B}\cdot\nabla T = 0$ is maintained.

If we were to use Eq. (F.2) in the calculations of Sec. V, only the terms proportional to η' would be affected. In the limit $K \to \infty$ we could simply set $\tilde{\eta}' = 0$ in Eq. (39). This would not alter the results for the gravitational and "tearing" modes. For our previous results on the "rippling" mode to be valid, we should require

$$KF'^2\epsilon^2\alpha^2/p < 1 \qquad (\text{F.4})$$

where we have set $F = F'\epsilon$, its value at the edge of the region of the discontinuity. Using Eqs. (34) and (50) and putting $\tilde{\eta}'$, F', $\tilde{\eta}$, $\tilde{\rho} \approx 1$ we obtain as a condition of validity

$$K(\alpha^{4/5}/S^{6/5}) < 1. \qquad (\text{F.5})$$

Since $K \sim T^4$, $S \sim T^2$, the correction term evidently becomes dominant as $T \to \infty$. In cases of practical interest,[3,31] the critical value of T is of order 10 eV. At higher temperatures a mode of the "rippling" type still exists, but its growth rate depends on K and is greatly diminished relative to that of the ordinary "rippling" mode.

We have assumed here that the classical values of η and κ may be used. In experimental situations where η is enhanced by cooperative phenomena, the magnitude of K may differ from the estimate given in Eq. (F.3).

APPENDIX G. STABILIZATION BY CONDUCTING WALLS

Whether the end-points μ_1, μ_2 are located at finite or infinite μ is important only for low-α modes, in particular for the "tearing" mode. In this appendix we derive a marginal stability condition for the "tearing" mode in the presence of conducting walls located at μ_1, μ_2.

From Eq. (49), we have that the marginal stability condition is characterized by $\Delta' = 0$. From Eqs. (20) and (21) we see then that the stability condition is equivalent to the requirement that $\alpha^2 > \alpha_c^2$, where α_c^2 is the eigenvalue of the equation

$$\psi'' - \psi(\alpha_c^2 + F''/F) = 0 \qquad (\text{G.1})$$

with $\psi = 0$ at μ_1, μ_2. As $|\mu_1|$, $|\mu_2|$ become smaller, we have $\alpha_c^2 \to 0$, and the current layer is then completely stable against the "tearing" mode.

We begin by considering the simple model $F = \tanh\mu$, for which $\alpha_c = 1$ when $\mu_1, \mu_2 = \mp\infty$ (cf. Eq. 28). Equation (G.1) then becomes

$$\psi'' - \psi(\alpha_c^2 - 2/\cosh^2\mu) = 0. \qquad (\text{G.2})$$

The solutions of this equation have been discussed in reference 18. For $\mu_1 = -\mu_2$, one finds $\alpha_c = 0$, 0.50, 0.95 when $\mu_2 = 1.20$, 1.36, 2.20. For $\mu_2 < 1.20$, absolute stability is achieved.

For the general symmetric layer we restrict ourselves to writing down the value of μ_2 which completely stabilizes the tearing mode. In this case $\alpha_c = 0$ and Eq. (G.1) is trivially soluble to give

$$\int_0^{\mu_2} \frac{F''}{FF'^2}\,d\mu + \frac{1}{F(\mu_2)F'(\mu_2)} = 0. \qquad (\text{G.3})$$

The generalization to cylindrical geometry is discussed in Appendix I.

APPENDIX H. EFFECT OF FINITE VISCOSITY

We will consider the case of isotropic fluid viscosity, simply adding a term $\rho\nu\nabla^2\mathbf{v}$ to the inertial term $\rho\, d\mathbf{v}/dt$ in the equation of motion [Eq. (4)]. This treatment indicates the general character and magnitude of viscous effects, but gives only a first approximation to the case of a hot plasma, which is well known to have an extremely complicated viscosity tensor. We will defer consideration of the full viscosity tensor and the Hall-effect terms in Ohm's law (which correspond to finite-Larmor-radius effects) to a later paper, where the present instabilities are approached from the point of view of the full set of plasma equations.

The most appropriate value of ν for the isotropic-viscosity analysis is probably that corresponding to motion transverse to magnetic-field lines. In the case of the modes of Sec. V, this motion exhibits extremely steep transverse velocity gradients in the region of discontinuity near $F = 0$. For the purpose of estimating the order of magnitude of ν in a plasma, we note then that

$$\nu \approx \tau_i v_i^2/(1 + \omega_i^2 \tau_i^2),$$

where v_i, ω_i, and τ_i are respectively the ion thermal velocity, gyrofrequency, and collision time. We are usually interested in the case of singly charged ions and $\omega_i \tau_i \gg 1$. For comparison purposes, the resistivity may be written

$$\eta \approx m_e c^2 / n e^2 \tau_e$$

so that

$$\frac{\nu}{\eta} \approx \frac{\beta_i}{4\pi} \frac{\tau_e m_i}{\tau_i m_e}, \tag{H.1}$$

where β_i is the ratio of the ion thermal pressure to the magnetic pressure, and m_i, m_e refer to the ion and electron masses. For a fully ionized plasma, we have $\tau_e/\tau_i \approx (m_e/m_i)^{\frac{1}{2}}(T_e/T_i)^{\frac{3}{2}}$. In what follows, we will use the expression $\tilde{\rho}\nu = [\langle\eta\rangle/4\pi]q$, where q is generally slightly larger than unity, except for very-low-β or low-(T_e/T_i) plasmas, when it is small, or for $|B_0|$ very small, in which case q may become very large.

Using the modified form of Eq. (4), we now obtain instead of Eq. (14)

$$\tilde{\rho} W'' - \frac{q}{p} W'''' = \alpha^2 W \left[\tilde{\rho} - \frac{S^2 G}{p^2} + \frac{F S^2}{p} \cdot \left(\frac{F}{\tilde{\eta}} + \frac{\tilde{\eta}' F'}{\tilde{\eta} p} \right) \right] + \psi \alpha^2 S^2 \left(\frac{F}{\tilde{\eta}} - \frac{F''}{p} \right). \tag{H.2}$$

We have retained only the highest derivatives of W in the inertial and viscous terms. As has been noted previously, for $S \to \infty$ the left-hand side of Eq. (14) is important only in the region of discontinuity. Since $q/p \to 0$ in the high-S limit, this remark is equally true for Eq. (H.2). The predominance of the highest derivatives of W follows from the same consideration.

To calculate the effect of viscosity in the region of discontinuity, we may proceed as in Sec. V, using Eqs. (13) and (H.2). We note that the viscous term is of order $q/\tilde{\rho} p \epsilon^2$ relative to the inertial term. Therefore (unless $q \ll 1$) the viscous term will predominate, and this is the situation that we will consider here. Since the $\tilde{\rho} W''$ term is negligible except in the limit of Appendix C, we note that the mass density now completely disappears from the equations, and is replaced by an "effective mass density" $\rho_\nu = q/p\epsilon^2$. Thus we may adapt the analysis of Sec. V simply by replacing $\tilde{\rho}$ with ρ_ν. The basic scale unit ϵ of Eq. (34) now becomes

$$\epsilon = (q^{\frac{1}{2}} \tilde{\eta}^{\frac{1}{2}} / 2\alpha S\, |F'|)^{\frac{1}{3}} \tag{H.3}$$

so that

$$\rho_\nu = (1/p)(2\alpha S\, |F'|\, q/\tilde{\eta}^{\frac{1}{2}})^{\frac{2}{3}}. \tag{H.4}$$

Equations (31) and (32) are unaltered except in the interpretation of ϵ, and Eq. (33) becomes

$$U'''' - U(\Lambda - \tfrac{1}{4}\theta^2) = -\psi(\theta - \delta). \tag{H.5}$$

We note that the homogeneous equation

$$U'''' - U(\Lambda - \tfrac{1}{4}\theta^2) = 0$$

is derived from the variational form

$$\Lambda = \int_{-\infty}^{\infty} d\theta\, [(U'')^2 + \tfrac{1}{4}\theta^2 U^2] \bigg/ \int_{-\infty}^{\infty} d\theta\, U^2 \tag{H.6}$$

so that there is a set of positive eigenvalues Λ, with a lowest eigenvalue of order unity. Thus the solution of Eqs. (32) and (H.5) proceeds in a manner very similar to the solution of Eqs. (32) and (33). In the growth rates of Eqs. (50), (57), and (59), we may simply replace $\tilde{\rho}$ by ρ_ν and obtain approximately for the "rippling" mode,

$$p \approx \frac{|\tilde{\eta}'|}{3\Lambda^{\frac{1}{2}}} \left(\frac{\alpha S\, |F'|}{q^{\frac{1}{2}} \tilde{\eta}^{\frac{1}{2}}} \right)^{\frac{1}{3}}, \tag{H.7}$$

for the "tearing" mode,

$$p \approx \frac{1}{3} \left(\frac{2 S \tilde{\eta}^{-5/2}\, |F'|^7}{\alpha^2 q^{\frac{1}{2}}} \right)^{\frac{1}{3}} \left(\frac{1}{F_{-\infty}^2} + \frac{1}{F_{\infty}^2} \right), \tag{H.8}$$

and for the gravitational interchange mode

$$p \approx \frac{G}{\Lambda} \left(\frac{\alpha S \tilde{\eta}}{4\, |F'|^2\, q^{\frac{1}{2}}} \right)^{\frac{2}{3}}, \tag{H.9}$$

where the fastest growing modes have $\Lambda = O(1)$.

We note that our previous results are left qualitatively unaltered. For the "rippling" and "tearing modes, the effective mass density ρ_ν becomes large as $S \to \infty$, and the thickness ϵ of the region of discontinuity then increases, while the growth rates are depressed somewhat. For the gravitational interchange mode, we have

$$\rho_\nu = (q\Lambda/\tilde{\eta}G)(4F')^{4/3}, \tag{H.10}$$

which is independent of S. Therefore ϵ and p are altered only by constant factors.

APPENDIX I. EFFECTS OF CYLINDRICAL GEOMETRY

When applied to nonplanar current layers, the stability analysis of the plane resistive current layer must be extended in two major aspects.

(1) Allowance must be made for the destabilizing force associated with a negative plasma-pressure gradient along the radius of curvature; this effect has been simulated approximately in the planar analysis by means of a gravitational field, but an exact interpretation of G remains to be given.

(2) For long-wave modes, i.e., particularly for the "tearing" mode, which is always long-wave in the present sense, the cylindrical geometry modifies the solution in the "outer region," and therefore affects the value of Δ' to be used in the dispersion relation. We will treat the usual high-S limit.

Interpretation of G

To generalize the planar analysis, we will use a cylindrical coordinate system where $x \rightarrow r\theta$, $y \rightarrow r$, and $z \rightarrow z$. Thus the zero-order configuration is given by

$$\mathbf{B}_0 = \hat{\theta}B_{\theta 0}(r) + \hat{z}B_{z0}(r) \tag{I.1}$$

and the perturbations are given by

$$f_1(\mathbf{r}, t) = f_1(r) \exp [i(m\theta + k_z z) + \omega t].$$

In analogy with quantities defined in Sec. II and Appendix A, we will write

$$\psi = B_{r1}/B, \qquad W = -iv_{r1}k\tau_{\mathrm{R}},$$
$$\chi = (ia/B)[k_zB_{\theta 1} - (m/r)B_{z1}],$$
$$V = \tau_{\mathrm{R}}[k_zv_{\theta 1} - (m/r)v_{z1}],$$
$$F = [(m/r)B_{\theta 0} + k_zB_{z0}]/kB,$$
$$H = [k_zB_{\theta 0} - (m/r)B_{z0}]/kB,$$
$$k = (k_z^2 + m^2/R^2)^{\frac{1}{2}},$$

where R is the radius of $F = 0$. We will use $\tilde{k}_z = k_z/k$, $\tilde{R} = R/a$.

As in the planar analysis the effect of the destabilizing mechanism appears only in a small region $r \approx R$, and for convenience we will specialize our equations to hold in this region. Since we are not concerned here with the "rippling" and "tearing" modes, we may neglect the η_1 and F'' terms in what follows. Thus we obtain from the pressure-balance equation [Eq. (9)]

$$(p\tilde{\rho}/\alpha^2S^2)W'' = F\psi'' - 2\tilde{k}_z(H_\theta/\tilde{R})\chi, \tag{I.2}$$

where $H_\theta = B_{\theta 0}/B$. The independent variable is $\mu = r/a$. In deriving Eq. (I.2), we have made the usual approximation (cf. Sec. V) that for zero-order quantities $\epsilon f_0' \ll f_0$. In the present context, this includes $\epsilon \ll \tilde{R}$. We have also neglected terms in $\alpha\epsilon$, which was found to be appropriate in Sec. V. Finally, we have used $\psi' \ll \chi$, which is justified in Appendix A. [From the latter remark and from Eq. (10), it follows incidentally that $\mathbf{k}\cdot\mathbf{B}_1 = 0$ in the region of discontinuity, so that $B_{\theta 1}$, B_{z1}, and χ all have the same r-dependence.] From Ohm's Law (Eq. 8), we obtain

$$\psi'' = (p/\tilde{\eta})\psi + (F/\tilde{\eta})W \tag{I.3}$$

as before, and we now need the additional component

$$\chi'' = \frac{p}{\tilde{\eta}}\chi + \frac{\alpha F}{\tilde{\eta}}V - \frac{W}{\tilde{\eta}}\left[\tilde{k}_z\left(H_\theta' - \frac{H_\theta}{\tilde{R}}\right) - \frac{m}{\tilde{R}}H_z'\right]. \tag{I.4}$$

Equation (I.4) in turn introduces the dimensionless velocity component V in the $\mathbf{k}\times\mathbf{B}$ direction, so that we must make use of the appropriate component of the pressure-balance equation, and obtain

$$\frac{p\tilde{\rho}}{S^2}V = \alpha F\chi + \psi\left[\tilde{k}_z\alpha\left(H_\theta' + \frac{H_\theta}{\tilde{R}}\right) - \frac{m}{\tilde{R}}H_z'\right]. \tag{I.5}$$

(The χ and W terms of Eq. (I.4) are now seen to be of order $p^2/\alpha^2S^2\epsilon^2(F')^2 \sim p^{\frac{3}{2}}/\alpha SF' \sim G/(F')^2 \ll 1$ [cf. Eq. (59)] relative to the V term, and are negligible except in the special limit of Case 2, discussed below.)

Using Eqs. (I.2–I.5), we may now carry out an expansion procedure like that of Sec. V using

$$\theta_1 = (\mu - \tilde{R})/\epsilon.$$

It is convenient to introduce $\beta_0 = 4\pi P_0/B^2$, for which the zero-order pressure-balance equation gives

$$\beta_0' + H_\theta H_\theta' + H_\theta^2/\tilde{R} + H_zH_z' = 0. \tag{I.6}$$

We also use $\chi = \Omega X$, where $\Omega = \epsilon p/4\tilde{\eta}$, and we recall $U = W4\epsilon F'/p$. The equations may then be written

$$\frac{d^2U}{d\theta_1^2} - \tfrac{1}{4}\theta_1^2U = \theta_1\psi - \frac{\tilde{k}_zH_\theta}{2\tilde{R}F'}X, \tag{I.7}$$

$$d^2\psi/d\theta_1^2 = \epsilon\Omega(4\psi + \theta_1U), \tag{I.8}$$

$$\frac{d^2X}{d\theta_1^2} - X\left(\frac{\theta_1^2}{4} + 4\epsilon\Omega\right) = -\theta_1\frac{\tilde{k}_z\beta_0'}{4\epsilon\Omega F'H_\theta}\psi + \frac{\tilde{k}_z(\beta_0' + 2H_\theta^2/\tilde{R})}{F'H_\theta}U. \tag{I.9}$$

It is of incidental interest to note that F' is related to the "magnetic shear" by the equation

$$F' = -\tilde{k}_zH_z[\log (H_\theta/\mu H_z)]'.$$

We next eliminate X from Eqs. (I.7) and (I.9), obtaining a fourth-order equation, and we solve by expanding U as in Eq. (40). We obtain

$$a_n\left[(n+\tfrac{1}{2})(n+\tfrac{1}{2}+4\epsilon\Omega)+\frac{\tilde{k}_z^2(\beta_0'+2H_\theta^2/\tilde{R}}{2\tilde{R}(F')^2}\right]=\left[\frac{\tilde{k}_z^2\beta_0'}{8\tilde{R}(F')^2\epsilon\Omega}-(n+\tfrac{1}{2}+4\epsilon\Omega)\right]\int_{-\infty}^{\infty}d\theta_1\ \theta_1 u_n\psi. \qquad (\mathrm{I.10})$$

Making the constant-ψ approximation, as in section V, and using the integrals listed below Eq. (46), we obtain

$$\Delta' = 2^{\frac{3}{2}}\Omega\sum_{m=0}^{\infty}\left\{\frac{m+\frac{3}{4}+2\epsilon\Omega+[\tilde{k}_z^2/4\epsilon\Omega\tilde{R}(F')^2][(m+\frac{1}{2})\beta_0'+2\epsilon\Omega(\beta_0'+2H_\theta^2/\tilde{R})]}{(m+\frac{3}{4})(m+\frac{3}{4}+2\epsilon\Omega)+[\tilde{k}_z^2/8\tilde{R}(F')^2](\beta_0'+2H_\theta^2/R)}\right\}\frac{\Gamma(m+\frac{1}{2})}{\Gamma(m+1)}. \qquad (\mathrm{I.11})$$

We will be interested mainly in short-wave instabilities, so that, as in the plane case, we have $\Delta' = -2\alpha$, where $\alpha \gg 1$. We note immediately that $\beta_0' \geq 0$ is sufficient for stability. (If $B_{\theta 0} \equiv 0$, then the point $F = 0$ occurs at $\beta_0' = 0$, so that there is no instability.) We note also that $\epsilon\Omega$ cannot be greater than of order $\beta_0'/(F')^2$, if the right-hand side of Eq. (I.11) is to be negative. Thus we may neglect $2\epsilon\Omega$ relative to $m + \frac{3}{4}$.

We now obtain the growth rate for $\beta_0' < 0$. Since $\epsilon \to 0$ as $S \to \infty$, the β_0'/ϵ term in the numerator will predominate on the right-hand side of Eq. (I.11), unless Ω becomes large. Thus the finiteness of Δ' implies $\Omega \to \infty$, which in turn implies that the sum of the series in Eq. (I.11) goes to zero.

As has been noted in Sec. V, in the present analysis of finite-conductivity interchange modes of the gravitational type, we must restrict ourselves to a range of parameters such that the infinite-conductivity modes are stable. Thus the Suydam criterion, $8\beta_0'\tilde{k}_z^2/\tilde{R}(F')^2 < 1$, must hold. In that case, the β_0'-contribution to the $(\beta_0' + 2H_\theta^2/\tilde{R})$- terms in Eq. (I.11) is seen to be negligible relative to the $(m + \frac{3}{4})$ terms. The $(2H_\theta^2/\tilde{R})$ contribution is also negligible when $(H_\theta\tilde{k}_z/\tilde{R}F')^2 \ll 1$. This is the usual case, which we will refer to as Case 1. The opposite condition is satisfied for Case 2.

Case 1 is the case of large shear. This appears more clearly if we write the defining condition as

$$[\log (H_\theta/\mu H_z)]'^2 \gg (H_\theta/\mu H_z)^2. \qquad (\mathrm{I.12})$$

In order that the series should be near a null we require

$$\tilde{k}_z^2\beta_0'/\epsilon\Omega\tilde{R}(F')^2 \approx 5. \qquad (\mathrm{I.13})$$

Evidently there is only a single null, corresponding to the growth rate

$$p = (2S\alpha\tilde{k}_z^2\ |\beta_0'|\ \tilde{\eta}^{\frac{1}{2}}/5\ |F'|\ \tilde{R}\tilde{\rho}^{\frac{1}{2}})^{\frac{2}{3}}. \qquad (\mathrm{I.14})$$

Thus we may identify the quantity G of Eq. (59) somewhat loosely with $-\tilde{k}_z^2\beta_0'/\tilde{R}$. We note, however, that the pressure-gradient-destabilization term in Eq. (I.7) is not effectively identical with the gravitational-force term in Eq. (33)—for example, it gives rise to only a single unstable mode instead of to a whole spectrum. Also, allowing for finite compressibility does not affect the present result, while we have seen in Appendix A that the true gravitational mode is somewhat modified.

Case 2 is the case of small shear, where the opposite of Eq. (I.12) holds. If the Suydam criterion is to be satisfied also, Case 2 can occur only for $8\ |\beta_0'| \ll H_\theta^2/\tilde{R}$. From Eq. (I.11), one estimates then that

$$p \sim (S\alpha\tilde{k}_z\ |\beta_0'|\ \tilde{\eta}^{\frac{1}{2}}/H_\theta\tilde{\rho}^{\frac{1}{2}})^{\frac{2}{3}}. \qquad (\mathrm{I.15})$$

The Tearing Mode

Only the solution in the "outer region" is affected by cylindrical geometry. From Eq. (9), with $\mathbf{v} \equiv 0$, one obtains a second-order differential equation for ψ, similar to Eq. (20), but somewhat less tractable. Given m and α_z, one may calculate Δ' as in Sec. III; or else one may set $\Delta' = 0$, as in Appendix G, and obtain a stability condition on m and α_z. To do the complete analysis, goes beyond the scope of this paper, but several points perhaps deserve comment.

1. Except for the $m = 0$ mode, the quantity α can no longer be made arbitrarily small, since

$$\alpha^2 = \frac{m^2}{\tilde{R}^2}\left(1 + \frac{B_{\theta 0}^2}{B_{z0}^2}\bigg|_{\mu_0}\right). \qquad (\mathrm{I.16})$$

From the plane results (Appendix G) we know that small α is most unstable, and similarly we expect small m and large $\tilde{R}$ to be most unstable here. We note also that large $B_{\theta 0}/B_{z0}$ prevents low-α modes for $m > 0$.

2. A plausible approximation[18] is to treat the layer itself as being approximately plane ($a \ll R$), so that Eq. (20) applies, and to use the familiar Bessel-function solutions in the vacuum regions. This is a useful method for proving instability in the case of the more unstable configurations, (for example, most "stabilized pinches"). From the point of view of obtaining exact stability criteria, this approach is unfortunately not wholly satisfactory, since one finds that stability *cannot* be achieved under the conditions where the approximation is both valid and useful. That is to say, stability requires either $a \sim R$, or else $B_{\theta 0}/B_{z0}\ |_{\mu_0} \sim R/a$, (for either of which Eq. (20) is inadequate); or else,

there must be close-fitting conducting walls, so that the plane approximation holds in the vacuum region also, if it holds in the current layer itself.

Note added in proof. The marginal stability problem for the "tearing" mode in cylindrical geometry is closely related to the problem of "neighboring equilibria" investigated by Rebut.[37] The neighboring-equilibrium analysis, however, necessarily confines itself to $P_0' = 0$ at $\mu = \mu_0$, whereas we have seen that the "tearing" mode exists more generally. If $P_0' \neq 0$ at $\mu = \mu_0$, then χ becomes discontinuous as $p \rightarrow 0$; but for large p there is no such difficulty. Further results on the cylindrical stability problem with $P_0' = 0$ at $\mu_0 = 0$ are given in references 38 and 39.

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Green's Function for the Linearized One-Dimensional Krook Equation with Electric Forces

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An integral representation is obtained for the Green's function for the linearized one-dimensional Krook equation with the induced electric field of the medium included. Various asymptotic expansions in time are then obtained. When the plasma frequency is set to zero, slightly modified hydrodynamic modes appear. For nonzero plasma frequency, only plasma oscillations unaffected by the collisions are present. Finally, the initial value problem corresponding to an initial wave packet of approximate wavenumber k is considered. For times long, but not too long, plasma oscillations are present for which the frequency and wavenumber satisfy the usual Landau dispersion relation for small wavenumber. After a sufficiently long time, the solution behaves like the Green's function itself and exhibits Landau damping.

I. INTRODUCTION

IN an earlier paper[1] we obtained the Green's function for the linearized Vlasov equation and then derived various asymptotic expansions of the electric field. While we proved that the proposed expression for the Green's function was correct, we could not offer a direct derivation that might be applicable to other problems. In this paper we wish to consider another related problem and obtain the Green's function in a more natural way. The derivation presented should be applicable to other integro-partial-differential equations similar to the linearized Krook equation studied here. In particular, we shall study the linearized Krook equation[2] for a gas of charged particles, so that the electric field produced by the gas will also appear, as in the Vlasov equation. If we set the plasma frequency equal to zero, then we are considering the ordinary linearized Krook equation. Thus we shall be able to examine the effects of collisions on plasma oscillations, and we shall also be able to consider the transition from a kinetic model of a gas to a hydrodynamic model as studied by other authors.[3,4]

After a formulation of the problem and a derivation of an integral representation of the perturbed mass density of the gas we attack the problem of obtaining asymptotic expansions of the answer. When the plasma frequency vanishes, we obtain the various expansions on the basis of reasonable assumptions on the behavior of the answer. We can then present a picture of the decaying hydrodynamical waves and the extent to which the hydrodynamical picture is valid. For the case of nonzero plasma frequency, we again rely on reasonable assumptions to obtain the expansions. We find no hydrodynamical waves, and the plasma oscillations are essentially unaffected by the collision terms. It is clear from the results that the plasma oscillations are qualitatively different from the hydrodynamical waves. We then consider the initial value

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