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Numerical simulations of the ion-temperature-gradient-driven mode (η_i- mode) show that long-lived, large-scale coherent structures exist and play an important role in the study of the anomalous transport. In this work, the stationary solutions of the nonlinear equations describing the dynamics of the η_i-mode are investigated. The simplest solution is found to be a dipolar vortex in the shearless case. In the presence of the magnetic shear, a modified dipolar vortex solution is found through perturbation theory.

I. INTRODUCTION

Drift wave instabilities driven by the ion temperature gradient (η_i) mode ($\eta_i = d \ln T_i/d \ln N$) play an important role in explaining the anomalous transport in tokamak experiments.[1-3] Several studies have been made in order to assess to which extent the experimentally observed anomalous transport is correlated to the properties of turbulent drift wave spectra.[4,5] In particular, recent two-dimensional (2-D) numerical simulations[6,7] of the η_i-mode show that long-lived, large-scale coherent structures exist and considerably affect the level of magnitude of the anomalous transport. A significant reduction of the turbulent heat flux with respect to the quasilinear estimate and positive dependence on dissipations are indeed observed. Such phenomenology can be expected to be modified when the magnetic shear effects are accounted for. The presence of magnetic shear makes the problem, in general, 3D, therefore, for the sake of simplicity, we consider a simpler problem corresponding to a slab equilibrium with a single mode rational surface located at $x = 0$. In this work, we address the question of whether the η_i- mode nonlinear equations admit a coherent solution also in the presence of magnetic shear.

In the cold ion limit and slab model the nonlinear drift wave equation (Hasegawa–Mima equation[8]) have stationary solitary vortex solutions. These solutions correspond to a coherent fluid motion with vanishing anomalous thermal flux. The simplest form of these solutions is a dipolar vortex (modon) traveling in the direction perpendicular to the density gradient and first found by Larichev and Reznik.[9] Properties of this solution have been widely studied,[10-12] showing that it has a high degree of stability. The anomalous transport associated with the vortex–vortex inelastic collision was studied by Horton.[13] Similarly, the η_i-mode equations in the shearless limit admit a solution corresponding to a simple generalization of the dipolar vortex solution of the Hasegawa–Mima equation.[14,15]

This paper is organized as follows. In Sec. II, the fluid equations for a nonlinear η_i-mode in the presence of magnetic shear are presented. From the general stationary solution of the model equation we obtain, in Sec. III, the simplest stationary vortex solution with and without magnetic shear. In Sec. IV, the conclusion of this work is given.

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II. MODEL EQUATIONS

The hydrodynamic ion equations are used to describe the toroidal η_i instability and the electrons are assumed to satisfy the Boltzmann relation, $n_e/N_0 \cong e\Phi/T_e$. Two-dimensional slab geometry is considered, with the gradient in the direction of the equilibrium magnetic field approximated as $\nabla_\parallel \cong x/L_s \partial/\partial y \cong ik_y x/L_s$, where $L_s = qR/s$ is the shear length and $s = rq'/q$ and the curvature term evaluated at the outside of the torus. For the perturbed electrostatic potential φ, the parallel ion velocity v, and the ion pressure p, the following set of nonlinear fluid equations are obtained which describes the η_i-mode dynamics:[6]

$$(1 - \nabla_\perp^2) \frac{\partial \varphi}{\partial t} = -\left(1 - 2\epsilon_n + \frac{1+\eta_i}{\tau} \nabla_\perp^2\right) \frac{\partial \varphi}{\partial y} + 2\epsilon_n \frac{\partial p}{\partial y} - Sx \frac{\partial v}{\partial y} + [\varphi, \nabla_\perp^2 \varphi], \tag{1}$$

$$\frac{\partial v}{\partial t} = -Sx \frac{\partial}{\partial y} (\varphi + p) - [\varphi, v], \tag{2}$$

$$\frac{\partial p}{\partial t} = -\frac{1+\eta_i}{\tau} \frac{\partial \varphi}{\partial y} - [\varphi, p], \tag{3}$$

with $\epsilon_n = L_n/R$, $S = L_n/L_s$, $\tau = T_e/T_i$, and $L_n^{-1} = \nabla N/N$. In Eqs. (1)–(3), nomalized variables are used, with $(x,y) \to \rho_s(x,y)$, $z \to L_n z$, and $t \to L_n/c_s t$ with $c_s = (T_e/M_i)^{1/2}$, $\rho_s = c_s/\Omega_i$, and the fields scaled as

$$\left(\frac{e}{T_e}\phi, \frac{v_{\parallel,i}}{c_s}, \frac{\widetilde{P}_i}{P_e}\right) = \frac{\rho_s}{L_n} (\varphi, v, p).$$

The $\mathbf{E} \times \mathbf{B}$ convective nonlinearities are expressed in terms of the Poisson bracket operator by $\mathbf{v}_E \cdot \nabla f \equiv [\varphi, f]$.

In this work, the simplest fluid set of equations for the η_i-mode turbulence is used in order to focus attention on the problem of the existence of the coherent solution. Therefore the compressional terms in the ion pressure equation, which give a stabilizing contribution and are important in order to determine the critical ion temperature gradient η_c and the finite Larmor radius terms, have been neglected. It has been shown in Ref. 16 that the inclusion of nonlinear finite Larmor radius (FLR) terms gives a stabilizing contribution to η_i-mode turbulence.

In the shearless limit, the ion parallel velocity equation is decoupled from the potential and pressure equation, and

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the resulting equations have been studied through numerical simulations.[6,7] Also, in the case of cold ion limit and no toroidicity, Eq. (1) reduces to Hasegawa–Mima equation[8] (equivalent to the Rossby wave equation[17]).

III. STATIONARY SOLUTION

In this section, we look for the stationary solution of Eqs. (1)–(3) traveling with velocity u in the y direction, $\varphi(x,y,t) = \varphi(x,y-ut)$. Upon inserting this into Eqs. (1)–(3), Eq. (3) can be rewritten as a single Poisson bracket relation

$$[\varphi - ux, p - Kx] = 0, \tag{4}$$

where $K = (1+\eta_i)/\tau$. The general solution of Eq. (4) is

$$p = F(\varphi - ux) + Kx, \tag{5}$$

where F is an arbitrary function of $\varphi - ux$.

Using Eq. (5), the ion parallel velocity equation can be written as

$$[\varphi - ux, v] + [Sx, x(\varphi - ux + F)] = 0, \tag{6}$$

and the general solution is

$$v = G(\varphi - ux) + Sx^2(1+F')/2, \tag{7}$$

where G is an arbitrary function and F' is a derivative with respect to its argument.

With Eqs. (5) and (7), the continuity equation becomes

$$[\varphi - (u+K)x, \nabla_1^2\varphi] + \left(u - 1 + 2\epsilon_n + 2\epsilon_n F' - SxG' - \frac{S^2x^3}{2}F''\right)\frac{\partial\varphi}{\partial y} = 0. \tag{8}$$

The general solution of this equation is

$$\nabla_1^2\varphi = H(\varphi - (u+K)x) + P_0(\varphi - ux) + xP_1(\varphi - ux) + x^2P_2(\varphi - ux) + x^3P_3(\varphi - ux), \tag{9}$$

where H is an arbitrary function and $P_j(\varphi - ux)$, with $j = 0, 1, 2$, and 3, determined by

$$KP_3' = -(S^2/2)F'', \tag{10}$$

$$KP_2' = -3P_3, \tag{11}$$

$$KP_1' = -2P_2 - SG', \tag{12}$$

and

$$KP_0' = u - 1 + 2\epsilon_n + 2\epsilon_n F' - P_1. \tag{13}$$

Equation (9) is a nonlinear, partial differential equation for φ, to be solved with appropriate boundary conditions, if the arbitrary function F, G, and H are assigned. The most appropriate set of the boundary condition corresponds to fully localized solutions in both the x and y directions. In the case of a fully localized solution the unknown functions F, G, and H are determined by the condition that p, v, and φ vanish as $|x|$, $|y| \to \infty$, giving

$$F = A(\varphi - ux),$$

$$G = B(\varphi - ux)^2,$$

and

$$H = [C_1/K^3 - SBu^2(3K+u)/3(u+K)^3K^2] \times [\varphi - (u+K)x]^3 - C_2/K^2\cdot[\varphi - (u+K)x]^2 + (C_3 - uC)/K\cdot[\varphi - (u+K)x],$$

with

$$p = A\varphi, \tag{14}$$

$$v = B\varphi(\varphi - 2ux), \tag{15}$$

and

$$\nabla_1^2\varphi = C\varphi - (S^2/u^2)x^2\varphi - 3D(u+K)x\varphi^2 + D\varphi^3, \tag{16}$$

where C_1, C_2, and C_3 are integration constants of Eqs. (10), (11), and (12), respectively, and

$$A = K/u,$$

$$B = -S(1+A)/2u^2,$$

$$C = [u - 1 + 2\epsilon_n(1+A)]/(u+K),$$

$$D = SB(3u+K)/3(u+K)^3.$$

The localized solutions of Eqs. (14)–(16) are singular and the general solution is obtained by matching the solution of Eqs. (14)–(16) at some boundary corresponding to a closed curve in the (x,y) plane, with the solution in the region inside the boundary which is nonsingular at the origin $x = y = 0$. In the inner region, the form of the function H cannot be determined within the framework of the present analysis. In order to make further progress it is convenient to assume that the relation between vorticity $\nabla_1^2\varphi$ and potential φ is linear.

Equation (16) admits a solution that is periodic in y. However, it is difficult to solve Eq. (16). For a special value of $u = -K/3$, Eq. (16) reduces to a linear equation and the solution has a form of the parabolic cylinder function.

A. Solution in shearless case

The shearless case has been considered elsewhere[14] and we report here only the main results. In this case, the vortex boundary corresponds to a circle of radius r_0, with $x = r\cos\theta$, $y - ut = r\sin\theta$, and Eqs. (14)–(16) reduce to

$$p = A\varphi, \tag{17}$$

$$v = 0, \tag{18}$$

and

$$\nabla_1^2\varphi = C\varphi, \tag{19}$$

with $C > 0$ for localized solution. Figure 1 shows the regions in u,K space corresponding to vortices. Note that in the case $(1-2\epsilon_n)^2 < 8\epsilon_n K$ (the condition for linear instability when FLR terms are negligible), $C > 0$ yields $u > 0$ or $u < -K$. We note that Eq. (19) admits the linearly stable wave solution of the form $\exp(i\mathbf{k}\cdot\mathbf{x} - i\omega t)$ when $C < 0$. In this case the phase velocity is $u = \omega/k_y$. The regions in u,K space corresponding to linear waves are also shown in Fig. 1. With a linear relation between $\nabla_1^2\varphi$ and φ and p given by Eq. (17), the equation for φ in the inner region becomes

$$\nabla_1^2\varphi = -q^2\varphi + (k^2+q^2)(u+K)x \tag{20}$$

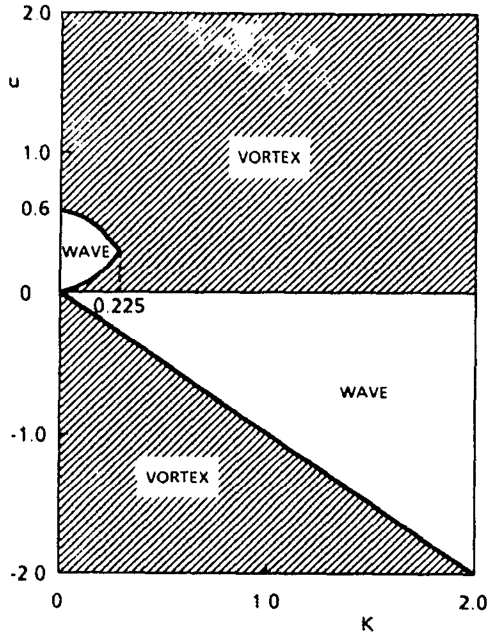

FIG. 1. Propagation regions of vortices and linear waves for $\epsilon_n = 0.2$. Vortices propagate in the hatched regions and waves propagate in the unhatched regions. The boundaries are $u = -K$, $u = 0$, $u = (1 - 2\epsilon_n - D^{1/2})/2$, and $u = (1 - 2\epsilon_n + D^{1/2})/2$, with $D = (1 - 2\epsilon_n)^2 - 8\epsilon_n K$.

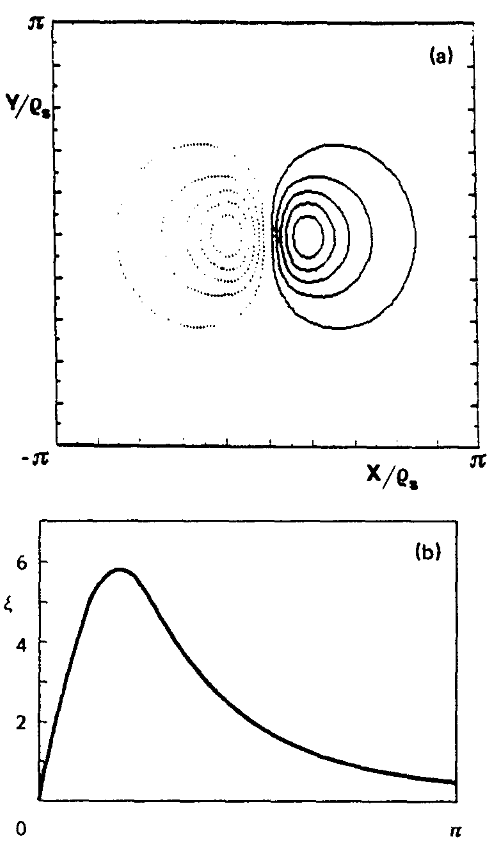

FIG. 2. Solution in the shearless case. (a) Contour plot of the potential for $u = 2.0$ and the physical parameters are $\eta_i = 1.0$, $\epsilon_n = 0.2$, and $\tau = 1$. The dotted line corresponds to the negative value. (b) The radial eigenfunction ξ versus radius.

and we obtain the usual dipolar vortex solution

$$\varphi = (u+K) r_0 \cos\theta \times \begin{cases} \dfrac{K_1(kr)}{K_1(kr_0)}, & r > r_0, \\[2ex] \left(1 + \dfrac{k^2}{q^2}\right)\dfrac{r}{r_0} - \dfrac{k^2}{q^2}\dfrac{J_1(qr)}{J_1(qr_0)}, & r < r_0, \end{cases} \tag{21}$$

where $k^2 = C$ and q is determined by

$$K_2(kr_0)/kK_1(kr_0) = -J_2(qr_0)/qJ_1(qr_0).$$

Solution (20) has two free parameters, the vortex speed u and the vortex radius r_0. The dipolar vortex solution in the (x,y) plane is shown in Fig. 2(a). In Fig. 2(b) the radial eigenfunction $\xi(r)\,[\varphi_0 = \xi(r)\cos\theta]$ is presented.

B. Solution in the presence of shear

In this case a new scale length appears in the problem, namely $x_T = uC^{1/2}/S$, corresponding to the ion sound turning point of the linear problem. It is convenient to distinguish between the case $r_0 \ll x_T$ and $r_0 > x_T$. In the first case the effect of the shear can be retained perturbatively, yielding a minor modification of the vortex shape. In the second case the eigenfunctions of Eq. (16) do not correspond to localized solutions because for $C > 0$ and $x > x_T$ the pseudopotential associated with the linear term of Eq. (16) changes from a well to an antiwell and localized solutions no longer exist. Here, we use perturbation theory to find a stationary localized solution for $r_0 \ll x_T$. In the lowest order in S, the solution is given in Eqs. (17)–(21). With $G = B(\varphi - ux)^2$, $P_3 = SBK/3$, $P_2 = -SB(\varphi - ux)$, $P_1 = 0$, and the simplest choice of $H = -[q^2 + k^2(u+K)/K][\varphi - (u+K)x]$ for the inner region, the perturbed equations are $(\varphi = \varphi_0 + \varphi_1)$

$$\nabla_\perp^2 \varphi_1 = \begin{cases} c\varphi_1 - \dfrac{S^2}{u^2} x^2 \varphi_0 - 3D(u+K)x\varphi_0^2 + D\varphi_0^3, \\ \qquad r > r_0, \\ -q^2\varphi_1 - SBx^2\varphi_0 + \dfrac{SB}{3}(3u+K)x^3, \quad r < r_0, \end{cases} \tag{22}$$

$$v_1 = B\varphi_0(\varphi_0 - 2ux), \tag{23}$$

and

$$p_1 = A\varphi_1. \tag{24}$$

We note that the magnetic shear yields a second-order contribution in vorticity Eq. (22.) The circular boundary is changed by the effect of shear, $[r_0 + r_1(\theta)]\mathbf{r}$. With the continuity condition of $\nabla_\perp^2 \varphi_1$ on the boundary, we obtain

$$\varphi_1(r_0) = \left(u + K - \frac{\partial \xi(r_0)}{\partial r}\right) r_1 \cos\theta. \tag{25}$$

The solutions for φ_1 are

$$\varphi_1 = \begin{cases} (a_1 J_1 + \varphi^{i}_{p,1})\cos\theta + (a_3 J_3 + \varphi^{i}_{p,3})\cos 3\theta, \\ \qquad r < r_0, \\ (b_1 K_1 + \varphi^{o}_{p,1})\cos\theta + (b_3 K_3 + \varphi^{o}_{p,3})\cos 3\theta, \\ \qquad r > r_0. \end{cases} \tag{26}$$

The particular solutions of Eq. (22) are given by ($m = 1$ and 3)

$$\varphi^{i}_{p,m} = \frac{\pi}{2}\Big(Y_m(qr)\int_0^r sG^i_m J_m(qs)ds + J_m(qr)\int_r^{r_0} sG^i_m Y_m(qs)ds - \frac{Y_m(qr_0)}{J_m(qr_0)}J_m(qr)\int_0^{r_0} sG^i_m J_m(qs)ds\Big) \quad (27)$$

and

$$\varphi^{o}_{p,m} = \Big(I_m(kr)\int_\infty^r sG^o_m K_m(ks)ds + K_m(kr)\int_r^{r_0} sG^o_m I_m(ks)ds - \frac{I_m(kr_0)}{K_m(kr_0)}K_m(kr)\int_\infty^{r_0} sG^o_m K_m(ks)ds\Big), \quad (28)$$

with

$$G^i_1 = 3G^i_3,$$

$$G^i_3 = \frac{1}{4}\Big(\frac{SB(3u+K)}{3}r^3 - SBr^2\xi\Big),$$

$$G^o_1 = 3G^o_3,$$

$$G^o_3 = \frac{1}{4}\Big(-\frac{S^2}{u^2}r^2\xi - 3D(u+K)r\xi^2 + D\xi^3\Big).$$

In Eqs. (26)–(28), J and Y are the Bessel functions of the first and second kind, and K and I are modified Bessel functions. From Eqs. (25) and (26), we obtain

$$r_1(\theta) = R\cos^2\theta. \quad (29)$$

The continuity condition of φ_1 gives

$$a_1 = 3Q/4J_1(qr_0), \quad (30)$$

$$a_3 = Q/4J_3(qr_0), \quad (31)$$

$$b_1 = 3Q/4K_1(kr_0), \quad (32)$$

and

$$b_3 = Q/4K_3(kr_0), \quad (33)$$

with R determined by the relation

$$Q = \Big(u + K - \frac{\partial\xi(r_0)}{\partial r}\Big)R = (u+K)\Big(1 - kr_0\frac{K_1'(kr)}{K_1(kr_0)}\Big)R.$$

The continuity condition of $\partial\varphi_1/\partial r$ yields

$$Q = \frac{4}{3}\frac{[(\partial\varphi^o_{p,1}/\partial r) - (\partial\varphi^i_{p,1}/\partial r)]_{(r=r_0)}}{q[J_1'(qr_0)/J_1(qr_0)] - k[K_1'(kr_0)/K_1(kr_0)]} \quad (34)$$

and

$$3\frac{q[J_1'(qr_0)/J_1(qr_0)] - k[K_1'(kr_0)/K_1(kr_0)]}{q[J_3'(qr_0)/J_3(qr_0)] - k[K_3'(kr_0)/K_3(kr_0)]} = \frac{(\partial\varphi^o_{p,1}/\partial r) - (\partial\varphi^i_{p,1}/\partial r)}{(\partial\varphi^o_{p,3}/\partial r) - (\partial\varphi^i_{p,3}/\partial r)}\Bigg|_{(r=r_0)}. \quad (35)$$

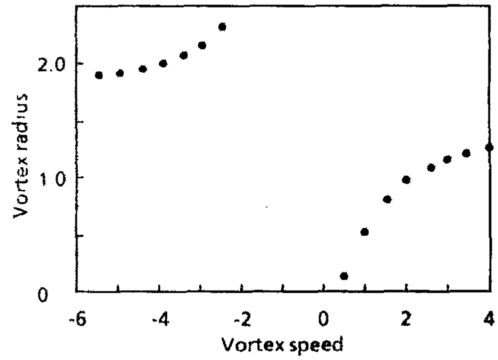

FIG. 3. The relation between the vortex speed r_0 and the speed u with parameter values as in Fig. 2.

We note that the shear reduces the free parameters from two to one. For a given value of the vortex speed u (or r_0), Eq. (35) determines the vortex radius r_0 (or u). The boundary which was a circle has a $m = 2$ ($\cos 2\theta$) component, i.e., becomes an ellipse. Figure 3 shows the relation between the vortex speed u and the radius r_0 for $\epsilon_n = 0.2$, $\eta_i = 1$, and $\tau = 1$. The vortex radius r_0 is less than the turning point x_T except in the case $u \to 0$ and $u \to -K$. Figure 4 shows the stationary solution given by Eqs. (26)–(35) for $u = 2.0$ and $S = 0.5$, with the other parameters the same as in Fig. 2.

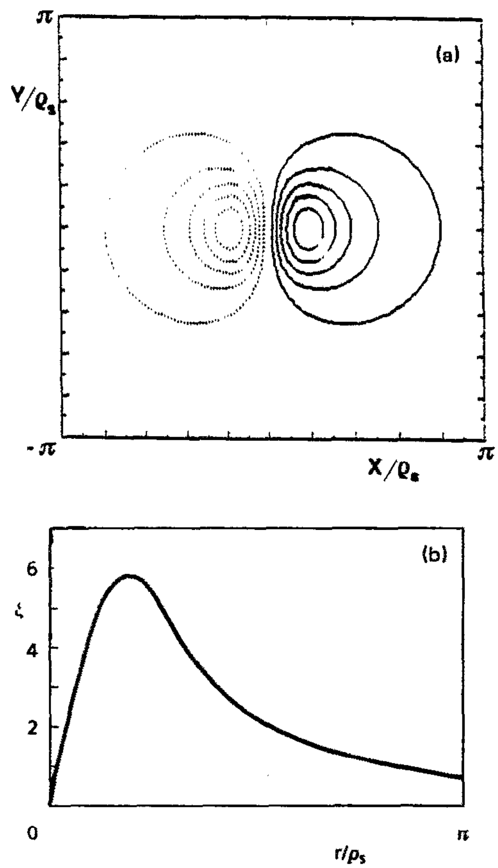

FIG. 4. Solution in the case with shear. (a) Contour plot of the potential for $u = 2.0$ and $S = 0.5$ with the other physical parameters as in Fig. 2. The dotted line corresponds to the negative value. (b) The radial eigenfunction ξ versus radius.

IV. CONCLUSION

We have shown that a localized stationary solution exists for the nonlinear equations describing the ion-temperature-gradient-driven turbulence with and without magnetic shear. In the shearless case the solution is a dipolar vortex that propagates in the y direction with velocity u, where u and r_0 are free parameters with $u>0$ or $u<-K$ for $(1-2\epsilon_n)^2<8\epsilon_n K$, which is the condition for η_i-mode instability when the FLR terms are neglected. When the magnetic shear is finite, the circular boundary of the shearless modon solution becomes an ellipse and the number of free parameters reduces to one with the vortex radius r_0 (or the speed u) determined by Eq. (35). This solution is self-consistent as long as the radius of the vortex is small compared with the position of the ion sound turning points, i.e., $r_0 \ll uC^{1/2}/S$. Unfolding the normalizations, the vortex radius has to be smaller than $\rho_i L_s/L_n$. This condition again corresponds to the existence of coherent structures on a scale which is large compared with the typical scale length of turbulence of the order of the ion Larmor radius. The work of Meiss and Horton[11] contains only the linear effect of magnetic shear and shows that shear damping is exponentially small in S. Our study includes a full nonlinear effect of shear and we have shown that there exists a stationary solution.

Although it is difficult to identify the coherent structures found in η_i-mode simulations as the simplest stationary solutions described in this work, the coherent structure may be a mixture of these solutions generated by the vortex–vortex or vortex–wave interactions.

[1]F. Romanelli, Plasma Phys. Controlled Fusion **31**, 1535 (1989).
[2]R. J. Fonck, R. Howell, K. Jachnig, L. Roquemore, G. Schilling, S. Scott, M. C. Zarnstorff, C. Bush, R. Goldston, H. Hsuan, D. Johnson, A. Ramsey, J. Schivell, and H. Towner, Phys. Rev. Lett. **63**, 520 (1989).
[3]F. X. Soldner, E. R. Muller, F. Wagner, H. S. Borsch, A. Eberhagen, H. U. Fahrbach, G. Fussmann, O. Gehre, J. Gernhardt, A. Gruber, W. Herrmann, G. Janeschitz, M. Kornherr, K. Krieger, H. M. Mayer, K. McCormick, H. D. Murmann, J. Neuhauser, R. Noltle, W. Poschenrieder, H. Rohr, K. H. Steuer, U. Stroth, N. Tsois, and H. Verbeek, Phys. Rev. Lett. **61**, 1105 (1989).
[4]W. Horton, Phys. Fluids **29**, 1491 (1986).
[5]G. S. Lee and P. H. Diamond, Phys. Fluids **29**, 3291 (1986).
[6]B. G. Hong and W. Horton, Phys. Fluids B **2**, 978 (1990).
[7]M. Ottaviani, F. Romanelli, R. Benzi, M. Briscolini, P. Santangelo, and S. Succi, Phys. Fluids B **2**, 67 (1990).
[8]A. Hasegawa and K. Mima, Phys. Rev. Lett. **39**, 205 (1977).
[9]V. D. Larichev and G. M. Reznik, Oceanology **16**, 547 (1976).
[10]G. R. Flierl, V. D. Larichev, J. C. McWilliams, and G. M. Larichev, Dyn. Atmos. Oceans **5**, 1 (1980).
[11]J. D. Meiss and W. Horton, Phys. Fluids **26**, 990 (1983).
[12]J. Nycander, Phys. Fluids B **1**, 1788 (1989).
[13]W. Horton, Phys. Fluids B **1**, 524 (1989).
[14]P. K. Shukla and J. Weiland, Phys. Lett. A **136**, 59 (1989).
[15]W. Horton and R. A. Smith, J. Geophys. Res. **93**, 12761 (1988).
[16]H. Nordman and J. Weiland, Nucl. Fusion **29**, 251 (1989).
[17]A. Hasegawa, C. G. Maclennan, and Y. Kodama, Phys. Fluids **22**, 2122 (1979).