$f(x, y, \dots, u, u_x, u_y, \dots, u_{xx}, u_{yy}, u_{xy}, \dots) = 0$ Partial Differential Equations $f(x, y, z, z_x = p, z_y = q, z_{xx} = r, z_{yy} = s, z_{xy} = t) = 0$

Solutions of PDEs

Lagrange Solution of First Order PDE

If $Pp + Qq = R$, then the general solution is $F(u, v) = 0$, where u, v are independent solution of the ODEs formed from the following system:

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{P'dx + Q'dy + R'dz}{P'P + Q'Q + R'R} = \frac{0}{0} = \frac{df(x,y,z)}{f(x,y,z)}$$

To determine the particular solution or the integral surface passing a give curve, we parametrize the curve as functions of t, i.e. $x = x(t)$, $y = y(t)$ and $z = z(t)$ and substitute them into u, v to determine the integral solution.

Method of Separation of Variables

The solution of a PDE can be assumed to be

$$z(x, y) = X(x) \cdot Y(y) \neq 0 \quad \text{or} \quad z(x, y) = X(x) + Y(y)$$

Substituting the solution into the PDE and separate the variables such that the PDE becomes

$$f(x, X) = g(y, Y) = k \quad \rightarrow \quad f(x, X) = k \text{ and } g(y, Y) = k$$

where k is an arbitrary constant. Solving the ODE system and reverse substituting the solution into z will result in the general solution. To determine the particular solution, initial or boundary values are required.

First Order PDE

Quasi Linear: $P(x, y, z)p + Q(x, y, z)q = R(x, y, z)$

Semi Linear: $P(x, y)p + Q(x, y)q = R(z, y, z)$

Linear: $P(x, y)p + Q(x, y)q + R(x, y)z = S(x, y)$

Homogeneous: $P(x, y)p + Q(x, y)q + R(x, y)z = 0$

Formation of a PDE

Elimination of arbitrary constants: $F(x, y, z, a, b, \dots) = 0 \Rightarrow f(x, y, z, p, q, r, s, t) = 0$

Elimination of arbitrary functions: $F(x, y, z, g(x, y, z), h(x, y, z), \dots) = 0 \Rightarrow f(x, y, z, p, q, r, s, t) = 0$

Elimination of arbitrary functions: $F(u, v) = 0 \Rightarrow \frac{\partial(u, v)}{\partial(y, z)} + \frac{\partial(u, v)}{\partial(z, x)} = \frac{\partial(u, v)}{\partial(x, y)}; \quad \frac{\partial(u, v)}{\partial(y, z)} = \begin{vmatrix} u_y & v_y \\ u_z & v_y \end{vmatrix}$

Integral Transformations

Property	Fourier Complex Transform	Fourier Cosine Transform	Fourier Sine Transform	Laplace Transform
Transform Definition	$\hat{f}(\omega) = \mathcal{F}(f(x))(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx; \quad \omega \in (-\infty, \infty)$	$\hat{f}_c(\omega) = \mathcal{F}_c(f(x))(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos(\omega x) dx \quad ; \omega \geq 0$	$\hat{f}_s(\omega) = \mathcal{F}_s(f(x))(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin(\omega x) dx \quad ; \omega \geq 0$	$F(s) = \mathcal{L}(f(t))(s) = \int_0^{\infty} f(t) e^{-st} dt; \quad \lvert f(t) \rvert < Me^{at}$(exponential order)
Inverse Transform Definition	$f(x) = \mathcal{F}^{-1}(f(\omega))(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega; \quad x \in (-\infty, \infty)$	$f(x) = \mathcal{F}_c^{-1}(f(\omega))(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{f}_c(\omega) \cos(\omega x) d\omega \quad ; x \geq 0$	$f(x) = \mathcal{F}_s^{-1}(f(\omega))(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{f}_c(\omega) \sin(\omega x) d\omega \quad ; x \geq 0$	$f(t) = \mathcal{L}^{-1}(F(s))$
Linearity	$\mathcal{F}(af(x) + bg(x))(\omega) = a\mathcal{F}(f(x))(\omega) + b\mathcal{F}(g(x))(\omega)$	$\mathcal{F}_c(af(x) + bg(x))(\omega) = a\mathcal{F}_c(f(x))(\omega) + b\mathcal{F}_c(g(x))(\omega)$	$\mathcal{F}_s(af(x) + bg(x))(\omega) = a\mathcal{F}_s(f(x))(\omega) + b\mathcal{F}_s(g(x))(\omega)$	$\mathcal{L}(af(t) + bg(t))(s) = a\mathcal{L}(f(t))(s) + b\mathcal{L}(g(t))(s)$
Transform of Derivatives	$\mathcal{F}(f^{(n)}(x))(\omega) = (i\omega)^n \mathcal{F}(f(x))(\omega)$	$\mathcal{F}_c(f'(x))(\omega) = \omega \hat{f}_s(\omega) - \sqrt{\frac{2}{\pi}} f(0); \quad \mathcal{F}_c(f''(x))(\omega) = -\omega^2 \hat{f}_c(\omega) - \sqrt{\frac{2}{\pi}} f'(0)$	$\mathcal{F}_s(f'(x))(\omega) = -\omega \hat{f}_c(\omega); \quad \mathcal{F}_s(f'(x))(\omega) = -\omega^2 \hat{f}_s(\omega) + \omega \sqrt{\frac{2}{\pi}} f(0)$	$\mathcal{L}(f^{(n)}(t))(s) = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$
Derivative of Transforms	$\mathcal{F}(x^n f(x))(\omega) = i^n \frac{d^n}{d\omega^n} \hat{f}(\omega)$	$\mathcal{F}_c(xf(x))(\omega) = \frac{d}{d\omega} \hat{f}_s(\omega)$	$\mathcal{F}_s(xf(x))(\omega) = -\frac{d}{d\omega} \hat{f}_c(\omega)$	$\mathcal{L}(x^n f(t))(s) = (-1)^n \frac{d^n}{ds^n} F(s)$
Transform of Convolution	$\mathcal{F}(f(x) * g(x))(\omega) = \mathcal{F}(f(x))(\omega) \mathcal{F}(g(x))(\omega)$	-	-	$\mathcal{L}(f(t) * g(t))(s) = \mathcal{L}(f(t))(s) \mathcal{L}(g(t))(s)$
Shifting on the original variable	$\mathcal{F}(f(x - a))(\omega) = e^{ia\omega} \hat{f}(\omega)$	-	-	$\mathcal{L}(\mu_0(t - a) f(t - a))(s) = e^{-as} F(s)$
Shifting on the transformed variable	$\mathcal{F}(e^{(iax)} f(x)) = \hat{f}(\omega - a)$	-	-	$\mathcal{L}(e^{at} f(t))(s) = F(s - a)$