U = Set Theory

Relations

Cartesian Product, $A \times B = \{(x,y) : x \in A \wedge y \in B\}$

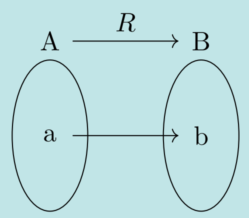
$|A \times B| = |A| \times |B|$

$A \times B \neq B \times A$ but $A \times B \approx B \times A$

Relation, $R \subseteq A \times B$

Correspondence: $aRb \iff (a,b) \in R$

Composition: if $R : A \to B$ and $S : B \to C$ then $R \circ S : A \to C$



Types of Relations and their Closures

Reflexive: $\forall a \in A, a \in R \Rightarrow$ closure $= R \cup \Delta_A$

Symmetric: if aRb then $bRa \Rightarrow$ closure $= R \cup R^{-1}$

Antisymmetric: if aRb and bRc then $a = c \Rightarrow$ no closure

Transitive: if aRb and bRc then $aRc \Rightarrow$ closure $= R \cup R \circ R = R^2 \cup R^3 \cup \cdots \cup R^n$

Partitions

A set can be partitioned into a collection of sets, α s.t. $\alpha \subseteq P(A)$, $\cap\alpha = \phi$ and $|\cup\alpha| = |A|$

Cross Partition: If $P_1 = \{A_i : i \in I\}$ and $P_2 = \{B_j : j \in J\}$, then $P_1 \times P_2 = \{A_i \cap B_j\} \backslash \phi$

System of Representation: $\{x_i : i \in A_i, i \in I\}$

Functions

$f : A \to B$ is a relation s.t.$\forall x \in A, \exists$ a unique image in B

$I(f) = f(A) = \{y : y \in B \wedge y = f(x) \forall x \in A\}$

$f : A \leftrightarrow B$ is a bijection if it is one to one and onto.

If it is not a bijection,its domain can be restricted to make it one to one and its co-domain can be restricted to make it onto.

If $f : A \leftrightarrow B$ then $\exists f^{-1}$ s.t. $f^{-1}(f(A)) = A$

Identity function, $I(x) = x$

Number of functions $= |B|^{|A|}$

Cardinal Numbers

Equipotent Sets $A \approx B$ if $\exists f : A \leftrightarrow B$

Countable sets are finite or denumerable (countably infinite).

Cardinality of finite sets is the collection of sets of that magnitude while cardinality of infinite sets are represented by transfinite numbers.

If $\exists f : \mathbb{N} \leftrightarrow A$, then A is denumerable and has cardinality $\aleph_0$

If $\exists f : [0,1] \leftrightarrow A$, then A is non denumerable and has cardinality $\mathbf{c}$

Continuum Theorem: $\nexists \alpha$ s.t. $\aleph_0 < \alpha < \mathbf{c}$

$\mathbf{c} = 2^{\aleph_0}$

Operations

$A \cup B = \{x : x \in A \vee x \in B\}$

$A \cap B = \{x : x \in A \wedge x \in B\}$

$A \backslash B = \{x : x \in A \wedge x \notin B\}$

$A \oplus B = \{x : x \in A, B \wedge x \notin A \cap B\}$

$A^c = \{x : x \in U \wedge x \notin A\}$

Algebra

Idempotent: $A \cup A = A$; $A \cap A = A$

Commutative: $A \cup B = B \cup A$; $A \cap B = B \cap A$

Distributive: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$; $A \cup (B \cup C) = (A \cap B) \cup (A \cap C)$

Identity: $A \cup \phi = A$; $A \cap \phi = \phi$; $A \cup U = U$; $A \cap U = A$

Involution: $A^{c^c} = A$

Compliment: $A \cup A^c = U$; $A \cap A^c = \phi$; $U^c = \phi$; $\phi^c = U$

DeMorgan's Law: $(A \cup B)^c = A^c \cap B^c$; $(A \cap B)^c = A^c \cup B^c$

Duality: $\cap \leftrightarrows \cup$; $U \leftrightarrows \phi$

Types of Relations Between Two Sets

Disjoint: $A \cap B = \phi$; Overlapping: $A \cap B \neq \phi$

Subset: $A \subseteq B$ if $A \cap B = A$ and $A \cup B = B$

Equal: $A = B$ if $A \subseteq B$ and $B \subseteq A$ (Schroeder Bernstein inequality)

Equivalent: $A \approx B$ if $|A| = |B|$

Collection of Sets

$\alpha = \{A_1, A_2, A_3, ...\}$

$\cup\alpha = \cup_{A \in \alpha}\alpha = \cup\{A : A \in \alpha\} = \{x : x \in A \text{ for some } A \in \alpha\}$

$\cap\alpha = \cap_{A \in \alpha}\alpha = \cap\{A : A \in \alpha\} = \{x : x \in A \forall A \in \alpha\}$

Power Set: $P(A)$ is the family of all subsets of A. $|A| = n \Rightarrow |P(A)| = 2^n$

Index Notation

$f : I \to \alpha$ s.t. $\{A_i\} = \{A_i : i \in I\}$

$\cup_{i \in I} A_i = \cup\{A_i : i \in I\} = \cup_i A_i$

$\cap_{i \in I} A_i = \cap\{A_i : i \in I\} = \cap_i A_i$

If $J \subseteq I$, then $\cup_{j \in J} \subseteq \cup_{i \in I}$ and $\cap_{j \in J} \supseteq \cap_{i \in I}$

$B \cap (\cup\{Ai\}) = \cup\{B \cap A_i\}$

$(U\{A_i\})^c = \cap\{A_i^c\}$

$U_\phi A_i = U$

Equivalence Relation

If R induced by A is reflexive, symmetric and transitive, it is an equivalence relation.

Equivalence Class: $[a] = \{x : (a,x) \in R\}$; $[a] \subseteq A$

Quotient Set, $A \backslash R = \{[a] : a \in A\}$ is always a partition of A

Ordered Relation

Comparable elements: $a \nparallel b \iff a \preceq b$ or $b \preceq a$

Lexicographic order: if $b \neq b'$ then $(a,b) \prec (a',b')$ if $b \prec b'$ and if $b = b'$ then $(a,b) \prec (a',b')$ if $a \prec a'$

Ordered-Isomorphic Set: $A \simeq T$ if $f : S \leftrightarrow T$ is similarity mapping (if $a \preceq b$ then $f(a) \preceq f(b)$).

Quasi Order set: irreflexive and transitive.

Partial Ordering set: reflexive, antisymmetric and transitive.

Linearly ordered set: set s.t. all elements are comparable

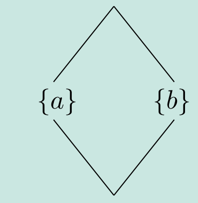
Ordered Set: If $(S, \preceq)$ and $A \subseteq S$, then A is an ordered set of S. Hence relation of A induced by S is $R_A = R \cap (A \times A)$.

Well ordered set: S is an ordered set s.t. every subset has a first element. Union of such sets is non commutative. The number of elements in such a set is known as its ordinal numbers and their algebra is non commutative.

Well ordering theorem: every set can be well ordered

Hasse Diagram of $(P(\{a,b\}), \subseteq)$

$\{a,b\}$ Maximal element, $a' \in S$ s.t.$\nexists x$ s.t. $x \succeq a$. Last element, $a \succeq x \forall x \in S$



ϕ Minimal element, $a \in S$ s.t. $\nexists x$ s.t. $x \preceq a$. First Element, $a \preceq x \forall x \in S$

Upper bound: $A \subseteq S$ and $M \in S$, then M is the upper bound if $x \preceq M \forall x \in A$. Supremum is the largest upper bound. Lower bound: $A \subseteq S$ and $m \in S$, then m is the lower bound if $x \succeq m \forall x \in A$. Infimum is the largest lower bound.

Limit element: $a \in S$ if $\nexists x$ s.t. $x \prec a$ and a is not the first element.

Initial segment: $a \in S$, then $S(a) = \{x : x \in S, x \prec a\}$. Hence A is shorter than B if A is similar to an initial segment of B.

Choice Function and Axiom of Choice

Choice function: if $\{A_i : i \in I\}$ is a non empty family of non empty subsets of X, then $\exists\, f : \{A_i : i \in I\} \to X$ s.t. $f(A_i) = a_i \in A_i$.

Cartesian Product: $\Pi\{A_i : i \in I\}$ is the collection of all choice functions defined on $\{A_i : i \in I\}$.

Axiom of choice: There exists at least 1 choice function for non empty family of non empty sets.

Zomelo's Postulate: Let $\{A_i : i \in I\}$ be a non empty family of disjoint non empty sets then $\exists$ a subset B, s.t. $B \cap A_i = \{a_i\} \forall i \in I$.

Zorn's Lemma: Let X be a non empty partial ordered set in which every chain has an upper bound in X, then X contains at least 1 maximal element.