MATH 209: Linear Algebra

Contents

1 Vector Space

A vector space is a non-empty set whose elements are closed under vector addition and scalar multiplication.

1.1 Axioms

Suppose V is a non empty set on which two operations are defined: addition of vectors and multiplication by scalars. Then V is a vector space if $\forall\ \underline{u}, \underline{v}, \underline{w} \in V$ and $k, l \in \mathbb{R}$, the following properties hold:

- $\underline{u} + \underline{v} \in V$ (closure property under addition)
- $\underline{u} + \underline{v} = \underline{v} + \underline{u}$ (commutative property of vector addition)
- $\underline{u} + (\underline{v} + \underline{w}) = (\underline{u} + \underline{v}) + \underline{w}$ (associative property of vector addition)
- $\exists\ \underline{0} \in V$ st $\underline{u} + \underline{0} = \underline{0} + \underline{u} = \underline{u}$ (additive identity)
- $\forall\ \underline{u} \in V, \exists\ -u$ st $\underline{u} + (-\underline{u}) = -\underline{u} + \underline{u} = \underline{0}$ (additive inverse)
- $k\underline{u} \in V$ (closure property under scalar multiplication)
- $k(\underline{u} + \underline{v}) = k\underline{u} + k\underline{v}$ (distributive property of scalar multiplication over addition)
- $(k + l)\underline{u} = k\underline{u} + l\underline{u}$ (distributive property of scalar multiplication over addition)
- $(kl)\underline{u} = k(l\underline{u})$ (associative property of scalar multiplication)
- $1\underline{u} = \underline{u}$ (multiplicative identity)

Important results derived from these axioms include:

- $(-1)\underline{u} = -\underline{u}$

  $(-1)\underline{u} = -\underline{u} \iff \underline{u} + (-1)\underline{u} = \underline{0}$.
  LHS $= \underline{u} + (-1)\underline{u} = 1\underline{u} + (-1)\underline{u} = (1 + (-1))\underline{u} = 0\underline{u} = \underline{0} =$ RHS

- $0\underline{u} = \underline{0}$

  LHS $= 0\underline{u} = (1 + (-1))\underline{u} = 1\underline{u} + (-1\underline{u}) = \underline{u} - \underline{u} = \underline{0} =$ RHS

1.2 Subspace

$W \subseteq V$ is a subspace if and only if W is itself a vector space, i.e. closed under vector addition and scalar multiplication, i.e.

- $\underline{u} + \underline{v} \in V$ (closure property under addition)
- $k\underline{u} \in V$ (closure property under multiplication)

($\Rightarrow$) Let W be a subspace of V, then all axioms are satisfied including both closures.

($\Leftarrow$) Let $W \subseteq V$ be closed under vector addition and scalar multiplication.
Since $W \subseteq V$, all axioms except additive identity and additive inverse hold.
To show that these hold as well, consider the closure property under scalar multiplication, then:
If $k = 0 \Rightarrow 0\underline{u} = \underline{0} \in W$
If $k = -1 \Rightarrow -1\underline{u} = -\underline{u} \in W$

2 Linear Combination

Let $V = \{\underline{v_i} : i \in I\}$ be a vector space and $S = \{\underline{v_j} : j \in J\} \subseteq V$, where $I = \{1, 2, \ldots, n\}$ and $J = \{1, 2, \ldots, r\}$. Then $\underline{v} \in V$ can be expressed as a linear combination of vectors in S.

2.1 Linear Independent

Consider the additive identity as the linear combination of all vectors in S, i.e.

$$\underline{0} = \sum_{j=1}^{r} k_j \underline{v_j}$$

The trivial solution ($k_j = 0$) will always satisfy this expression. If this is the only solution, then S is linearly independent. However if there exists additional non-trivial solution(s), then S is linearly dependent.

If $|S| = 1$, then it is linearly dependent if and only if $\underline{v} = 0$. If $|S| \geq 2$, then it is linearly dependent if and only if any $\underline{v} \in S$ can be expressed as a linear combination of other vectors in S.

2.2 Span

Consider $\underline{v_i} \in V$ as a linear combination of vectors in S, i.e.

$$\underline{v_i} = \sum_{j=1}^{r} k_j \underline{v_j}$$

If all vectors in V can be written as a linear combination of S, then S is said to span V.

2.3 Basis

If S is linearly independent and spans V, then S is said to be a basis of V and the cardinality (size) of the basis is the dimension of that space, i.e. $\dim(V) = |S|$.

If V is a n-dimensional space and $|S| = n$, then S is a basis if it spans V or is linearly independent.

3 Linear Systems

A system of linear equations can be expressed as $A\underline{X} = \underline{B}$ where A is a $m \times n$ coefficient matrix, $\underline{X}$ is a $n \times 1$ variable vector and $\underline{B}$ is a $m \times 1$ solution vector.

$$A\underline{X} = \underline{B} \qquad \Rightarrow \qquad \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

3.1 Solving a Linear System

Solving a linear system requires elementary row operations, i.e.

- $R_i \to kR_i$, where $k \neq 0$.
- $R_i \leftrightarrow R_j$, where $i \neq j$.
- $R_i \to R_i + kR_j$, where $k \neq 0$ and $i \neq j$.

To solve a linear system $A\underline{X} = \underline{B}$,

1. Construct the augmented matrix,

$$[A|\underline{B}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right]$$

2. Evaluate the Row Echelon Form (Gaussian Elimination) using elementary row operations.
   - The first non-zero element in a row must be 1 (known as leading one).
   - All rows containing only 0 must be grouped at the bottom.
   - The leading one of each successive row must be to the right of the previous row's leading one.
3. Evaluate the Reduced Row Echelon Form (Gauss-Jordan Elimination) using elementary row operations.
   - The matrix must be in Row Echelon Form.
   - If a column contains a leading one, all other entries in that column must be 0.
4. Determine the free (not corresponding to leading ones) and leading variables (corresponding to leading ones), i.e. if a_{ij} is a leading one, then x_j is leading variable.
5. Rewrite the leading variables in terms of free variables.
6. Evaluate the reduced system to determine the infinite non-trivial solutions of the form

$$\underline{X} = \sum_{j=1}^{r} t_j \underline{v_j}$$

   where $v_j \in V$ and t_j is some variable.

3.2 Subspaces of a Homogeneous Linear System

Consider a homogeneous system ($\underline{B} = \underline{0}$) and let A be a square matrix of order n. Then there exists at least the trivial solution, i.e. $\underline{X} = \underline{0}$. If A is invertible ($\exists A^{-1}$ and $\det(A) \neq 0$), then this is the only solution of the system. However, if A is singular ($\nexists\, A^{-1}$ and $\det(A) = 0$), then the system has infinite non-trivial solutions as well.

For a homogeneous system with non-trivial solutions, the following subspaces exist:

- Null space of A: solution space of A,

$$N(A) = \{\underline{X} \in \mathbb{R}^n : A\underline{X} = \underline{0}\}$$

- Row space of A: space spanned by the row vectors of A,

$$R(A) = \left\{ \underline{u} \in \mathbb{R}^n : \underline{u} = \sum_{i=1}^{m} k_i \underline{r_i} \right\} \text{ where } \underline{r_i} = (a_{i1}, a_{i2}, \dots, a_{in}) \subseteq \mathbb{R}^n$$

- Column space of A: space spanned by the column vectors of A,

$$C(A) = \left\{ \underline{u} \in \mathbb{R}^m : \underline{u} = \sum_{i=1}^{n} k_i \underline{c_i} \right\} \text{ where } \underline{c_i} = (a_{1i}, a_{2i}, \dots, a_{mi}) \subseteq \mathbb{R}^m$$

Dimension Theorem for Matrices: For a square matrix $A_{n\times n}$, $nullity(A) + rank(A) = n$, where $nullity = \dim(N)$ or the number of free variables (rows without leading ones) in (reduced) row echelon form of A and $rank(A) = \dim(R) = \dim(C)$. Rank can also be determined by evaluating the

- number of linearly independent vectors in R or C.
- number of non-multiple vectors in R or C.
- highest order of the non-zero determinant of A.
- number of leading variables (rows with leading ones) in (reduced) row echelon form of A.

3.3 Inferences From Row Echelon Form

Let F be the (reduced) row echelon form of A, then:

- The rows with leading ones form a basis for $R(A)$ and $R(F)$. Because row operations consist of scalar multiplication and vector addition, therefore $F \in R(A) \subseteq \mathbb{R}^n$.
- The columns with leading ones form a basis for $C(F)$ but not for $C(A)$. Because row operations change the column space. However, the columns in A corresponding to the columns in F with leading ones form the basis for $C(A)$.

Given a homogeneous system, let $S = \{\underline{v_j} : j \in J\}$ be the set of vectors determined from the reduced system. Since the leading variables were written in terms of free variables, S is linearly independent. And since the linear combination of its vectors form a general vector, S spans the solution space as well. Hence S forms the basis for the null space and $\dim(N) = nullity(A) = |S|$.

4 Inner Product

4.1 Euclidean Inner Product

A vector $\vec{u} = |\vec{u}|\underline{u} = (u_1, u_2, \ldots, u_n)$, where $\underline{u}$ is the unit vector (length 1) and the length of a vector is

$$|\vec{u}| = \sqrt{(u_1^2 + u_2^2 + \cdots + u_n^2)} = (\vec{u}\cdot\vec{u})^{1/2}$$

Let $\vec{u}, \vec{v} \in \mathbb{R}^n$, then their dot product (aka euclidean inner product) is defined as

$$\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n = \begin{cases} 0 & \vec{u} = 0 \text{ or } \vec{v} = 0 \\ ||\vec{u}|| \cdot ||\vec{v}|| \cos\theta & \vec{u}, \vec{v} \neq 0 \end{cases}$$

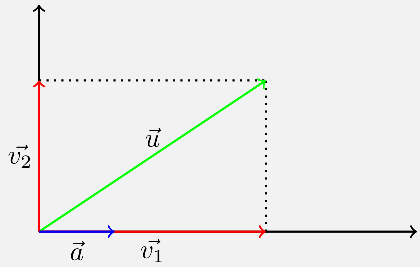
Moreover, properties of dot product include:

- $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$
- $(\vec{u}+\vec{v})\cdot\vec{w} = \vec{u}\cdot\vec{w} + \vec{u}\cdot\vec{w}$
- $k(\vec{u}\cdot\vec{v}) = (k\vec{u})\cdot\vec{v} = \vec{u}\cdot(k\vec{v})$
- $\vec{u}\cdot\vec{u} \geq 0$ and $\vec{u}\cdot\vec{u} = 0 \iff \vec{u} = \vec{0}$
- $\vec{u}\cdot\vec{v} = 0 \iff \vec{u}, \vec{v}$ are perpendicular to one another

All vectors in $\mathbb{R}^n$ have n perpendicular components (projections on n axis). For simplicity, consider $\vec{u} \in \mathbb{R}^2$, then $\vec{u} = \vec{v_1} + \vec{v_2}$ where $\vec{v_1}, \vec{v_2}$ are perpendicular components of u and $\vec{v_1}$ is a scalar vector of $\vec{a}$. Then $\vec{w_1}$ is the projection of $\vec{u}$ along $\vec{a}$, i.e.

$$\vec{w_1} = proj_{\underline{a}}\underline{u} = \frac{(\vec{u}\cdot\vec{a})\vec{a}}{|\vec{a}|^2}$$

and $\vec{w_2}$ (perpendicular to $\vec{a}$) is determined by $\vec{w_2} = \vec{u} - \vec{w_1}$



Let $\vec{u} = \vec{v_1} + \vec{v_2}$ where $\vec{v_1}$ is along $\vec{a}$ and $\vec{v_2}$ is perpendicular to $\vec{a}$

It is clear that $\vec{v_2} = \vec{u} - \vec{v_1}$

Let $\vec{v_1} = k\vec{a}$, where $k \in \mathbb{R}$

$\Rightarrow \vec{u} = k\vec{a} + \vec{v_2}$

$\Rightarrow \vec{u} \cdot \vec{a} = k\vec{a} \cdot \vec{a} + \vec{v_2} \cdot \vec{a}$ (dot product with $\vec{a}$)

$\Rightarrow \vec{a} \cdot \vec{u} = k||\vec{a}||^2 + 0 \quad \because \vec{v_2} \perp \vec{a} \iff \vec{v_2} \cdot \vec{a} = 0$ and $\vec{a} \cdot \vec{a} = ||\vec{a}||^2$

$\Rightarrow k = \frac{(\vec{a} \cdot \vec{u})}{||\vec{a}||^2}$

Substituting k into $\vec{v_1} = k\vec{a}$, we get

$$\vec{v_1} = \frac{(\vec{a} \cdot \vec{u})}{||\vec{a}||^2}\vec{a}$$

4.2 Inner Product Space

The dot product is an example of the more generalized concept of inner products. Hence, the notations and terms become generalized as well: $\vec{u} \to \underline{u}$, length $|\vec{u}| \to$ norm $||\underline{u}||$, perpendicular $\to$ orthogonal, dot product $\vec{u} \cdot \vec{v} \to$ inner product $\langle \underline{u}, \underline{v} \rangle$. This is necessary as dot product refers to ordinary vectors in $\mathbb{R}^n$ but inner product refers to generalized vectors which can matrices, polynomials, functions etc.

An inner product space is a real vector space with an inner product $\langle \underline{u}, \underline{v} \rangle \in \mathbb{R}$ which satisfies the following axioms $\forall\ \underline{u}, \underline{v} \in V$ and $k \in \mathbb{R}$:

- $\langle \underline{u}, \underline{v} \rangle = \langle \underline{v}, \underline{u} \rangle$ (symmetry)
- $\langle \underline{u} + \underline{v}, \underline{w} \rangle = \langle \underline{u}, \underline{w} \rangle + \langle \underline{u}, \underline{w} \rangle$ (additivity)
- $\langle k\underline{u}, \underline{v} \rangle = k \langle \underline{u}, \underline{v} \rangle$ (homogeneity)
- $\langle \underline{v}, \underline{v} \rangle \geq 0$ and $\langle \underline{v}, \underline{v} \rangle = 0 \iff \underline{v} = \underline{0}$ (positivity)

Important properties of inner products include

- Norm of a vector, $||\underline{u}|| = \langle \underline{u}, \underline{v} \rangle^{1/2}$
- $\underline{u}, \underline{v}$ are orthogonal vectors if and only if $\langle \underline{u}, \underline{v} \rangle = 0$.
  - if $\underline{u}, \underline{v} \in \mathbb{R}^n$, then they are orthogonal if and only if $\underline{u} \cdot \underline{v} = 0$.
  - if $A \in \mathbb{R}^{n \times n}$ (square matrix), then it is orthogonal if and only if all row/col vectors contained in it are orthogonal to one another, i.e. $\det(A) = \pm 1$. If, additionally, $A^t = A^{-1}$, then A is properly orthogonal.
- Orthonormal vectors are those orthogonal vectors whose norms are 1, i.e. $||\underline{u}|| = ||\underline{v}|| = 1$.
- $||\underline{u} + \underline{v}||^2 + ||\underline{u} - \underline{v}||^2 = 2||\underline{u}||^2 + 2||\underline{v}||^2$

4.3 Gram Schmidt Process

This is used to transform the basis of a vector space into orthogonal and then a orthonormal basis.

Let $B = \{\underline{u_i} : i \in I\}$ be a basis for a vector space V. Then, $B' = \{\underline{v_i} : i \in I\}$ is the orthogonal basis determined by evaluating the projections of basis vectors, i.e.

$$\underline{v_i} = \underline{u_i} - \sum_{j=1}^{i-1} \frac{\left\langle \underline{u_i}, \underline{v_j} \right\rangle \underline{v_j}}{||\underline{v_j}||^2} \quad \text{where} \quad \underline{v_1} = \underline{u_1}$$

And the orthonormal basis B'' is determined by dividing the orthogonal vectors by their norm, i.e.

$$B'' = \left\{ \frac{\underline{v_i}}{||\underline{v_i}||} : i \in I \right\}$$

Since we are considering generalized vectors and spaces, we prove this completely analytically.

Let $B = \{\underline{u_i}\}$ be a basis for V, then we must determine an orthogonal basis $B = \{\underline{v_i}\}$.

Let $\underline{v_1} = \underline{u_1}$.
Now $\underline{v_2}$ must be orthogonal to $\underline{v_1}$. A vector can be written as the sum of its orthogonal components, therefore consider $\underline{u_2} = k\underline{v_1} + \underline{v_2}$. Taking inner product with $\underline{v_1}$,

$$\left\langle \underline{u_2}, \underline{v_1} \right\rangle = k \left\langle \underline{v_1}, \underline{v_1} \right\rangle + \left\langle \underline{v_2}, \underline{v_1} \right\rangle \Rightarrow \left\langle \underline{u_2}, \underline{v_1} \right\rangle = k||\underline{v_1}||^2 + 0 \Rightarrow k = \frac{\left\langle \underline{u_2}, \underline{v_1} \right\rangle}{||\underline{v_1}||^2}$$

Hence $\underline{v_2}$ becomes

$$\underline{v_2} = \underline{u_2} - \frac{\left\langle \underline{u_2}, \underline{v_1} \right\rangle \underline{v_1}}{||\underline{v_1}||^2}$$

Now $\underline{v_3}$ must be orthogonal to $\underline{v_1}$ and $\underline{v_2}$. Similarly, $\underline{u_3} = k_1\underline{v_1} + k_2\underline{v_2} + \underline{v_3}$. Taking inner product with $\underline{v_1}, \underline{v_2}$ and substituting k_1, k_2 respectively into the equation, $\underline{v_3}$ becomes

$$\underline{v_3} = \underline{u_3} - \frac{\left\langle \underline{u_3}, \underline{v_1} \right\rangle \underline{v_1}}{||\underline{v_1}||^2} - \frac{\left\langle \underline{u_3}, \underline{v_2} \right\rangle \underline{v_2}}{||\underline{v_2}||^2}$$

Therefore, this is a recursively defined method where $\underline{v_i}$ must be orthogonal to the previous vectors and can be determined by

$$\underline{v_i} = \underline{u_i} - \sum_{j=1}^{i-1} \frac{\left\langle \underline{u_i}, \underline{v_j} \right\rangle \underline{v_j}}{||\underline{v_j}||^2}$$

Hence $B' = \{v_i\}$ is an orthogonal basis for V.

To determine an orthonormal basis, we simply divide all vectors in B' by their norm. Hence $\forall\ \underline{v_i} \in B'$,

$$\left|\left| \frac{\underline{v_i}}{||\underline{v_i}||} \right|\right| = \frac{||\underline{v_i}||}{||||\underline{v_i}||||} = \frac{||\underline{v_i}||}{||\underline{v_i}||} = 1$$

Therefore $B'' = \left\{ \frac{\underline{v_i}}{||\underline{v_i}||} \right\}$ is an orthonormal basis for V.

If a set $S = \{\underline{v_i} : i \in I\}$ is a set of orthogonal vectors in an inner product space, then S is linearly independent.

5 Eigen Space

If A is a square matrix of order n, then a non-empty vector $\underline{X} \in \mathbb{R}^n$ is an eigenvector of A if and only if $A\underline{X} = \lambda \underline{X}$ where $\lambda \in \mathbb{R}$ is an eigenvalue. Hence an eigenvector is a vector which when applied to a matrix has the same effect as multiplying the matrix by a scalar (eigenvalue). This effect can be contraction ($\lambda \in [0, 1)$), stretching ($\lambda > 1$) or reversal of direction ($\lambda < 0$).

The roots of the **characteristic equation of a matrix**, $\det(A - \lambda I) = 0$ are the eigenvalues of A and their corresponding eigenvectors are determined by evaluating $(A - I\lambda)\underline{X} = \underline{0}$

Consider the equation $A\underline{X} = \lambda\underline{X}$
$\Rightarrow A\underline{X} - \lambda\underline{X} = \lambda\underline{X} - \lambda\underline{X}$
$\Rightarrow \underline{X}(A - \lambda I) = \underline{0}$
$\because \underline{X} \neq \underline{0} \therefore (A - \lambda I) = \underline{0}$
$\Rightarrow \det(A - \lambda I) = 0$

Some interesting properties of eigenvalues include:

- Let eigenvector $\underline{X}$ corresponds to an eigenvalue λ for A, then $\underline{X}$ corresponds to λ^n for A^n.
- $\sum \lambda = \text{tr}(A)$ (trace is the sum of main diagonal entries)
- $\prod \lambda = \det(A)$

Consider the characteristic equation $\det(A - \lambda I) = 0$ for a matrix of order 2,

$$\begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = 0$$

$\Rightarrow (a - \lambda)(d - \lambda) - bc = 0$

$\Rightarrow \lambda^2 - \lambda(a + d) + ad - bc = 0$

$\Rightarrow \lambda^2 - \lambda tr(A) + \det(A) = 0$

Recall that for $ax^2 + bx + c = 0$ with roots α, β, it holds that $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$.

$\because \lambda_1, \lambda_2$ are roots of the characteristic equation $\therefore \lambda_1 + \lambda_2 = \text{tr}(A)$ and $\lambda_1\lambda_2 = \det(A)$

Eigen space of A corresponding to a λ are the vectors $\underline{X}$ that satisfy $A\underline{X} = \lambda\underline{X}$. The non-zero vectors in the space comprise the eigenvectors of A.
If an eigenvalue is repeated k times and the corresponding eigenspace is k-dimensional, then the basis vectors are linearly independent and correspond to the eigenvectors.

5.1 Diagonalization

Diagonalization is a process to transform a diagonalizable matrix A into a diagonal matrix D using an invertible matrix P, i.e. $D = PAP^{-1}$. The column vectors in P are the eigenvectors of A and their corresponding eigenvalues lie on the main diagonal entries of D. Hence A, D are said to be similar to one another.

A is diagonalizable if and only if it has n linearly independent eigenvectors.

($\Rightarrow$) If A is diagonalizable, then it has n linearly independent eigenvectors by definition which form a basis for $\mathbb{R}^n$.

($\Leftarrow$) Assume A has n linearly independent eigenvectors $\underline{P_i}$ corresponding to λ_i.
Let $P = [\underline{P_i}]$ be a matrix, then $AP = [A\underline{P_i}]$
$\because A\underline{P_i} = \lambda_i A \therefore AP = PD$ where D is a diagonal matrix whose entries are λ.
$\because P_i$ are linearly independent $\therefore rank(P) = n \Rightarrow \det(P) \neq 0 \Rightarrow P$ is invertible.
Hence $AP = PD \Rightarrow D = P^{-1}AP$.

5.2 Orthogonal Diagonalization

A square matrix A of order n is orthogonally diagonalizable if there exists an orthogonal matrix P st $P^{-1}AP = P^tAP$, i.e. $P^{-1} = P^t$. According to the Spectral theorem for matrices, A is orthogonally diagonalizable if and only if it is symmetric.

To determine such a P, determine the orthonormal basis of all eigenspaces individually since eigenvectors from different eigenspaces are orthogonal for a symmetric matrix.