Elementary Linear Algebra

Algebra

$A + B = B + A$
$A \cdot B \neq B \cdot A$
$a + B = B + a$
$A + (B + C) = (A + B) + C$
$A(BC) = (AB)C$
$a(bC) = (ab)C$
$A(B + C) = AB + AC$
$a(B + C) = aB + aC$

Operations

$A \pm B$ is possible if their sizes are the same.
$A \cdot B$ is possible if col(A) = row(B) and $AB_{ij} = A_i \cdot B_j$.
Power, $A^n = A \cdot A^2 \cdot A^3 \cdot \cdots \cdot A^n$ for $n > 0$. If A is invertible then $n < 0$.
Inverse, $A^{-1} = B$ iff $BA = AB = I$. Moreover $(AB)^{-1} = B^{-1}A^{-1}$.
Transpose, $A^T_{m\times n} = A_{n\times m}$ s.t. $(A^T)^T = A$ and $(AB)^T = B^T A^T$.

Common Matrices

Zero matrix, O s.t. $A \pm O = O \pm A = A$, $A - A = O$ and $OA = AO = O$.
Identity matrix, I_n s.t. all diagonal elements are 1 and all non-diagonal elements are 0.
Elementary Matrix, E is matrix formed by a single row/col operation on I. E transfers its operations when multiplied. E is invertible and E^{-1} is also an elementary matrix.

Invertibility

A is invertible iff:
A^{-1} is invertible and $(A^{-1})^{-1} = A$ or
A^n is invertible and $(A^n)^{-1} = (A^{-1})^n$ for $n \in \mathbb{N}$ or
kA is invertible and $(kA)^{-1} = k^{-1}A^{-1}$ for $k \neq 0$

Row Echelon Form of a Matrix

first non zero element in a row is 1
all rows with only zeros are grouped
lower leading 1 is right of the upper leading 1

Reduced Row Echelon Form of a Matrix: each column with a leading 1 has 0 in all rows

Row and inverse Row Operations

R_{ij} represents ith and jth row of matrix A
$aR_i = \frac{1}{a}R_i$ for $a \neq 0$ (multiples can be taken out or multiplied)
$R_{ij} = R_{ji}$ (rows can be interchanged)
$R_i + aR_j = R_{i+cj}$ (one row can me multiplied and added to another row)

Column and Inverse Column Operations are the same

Square Matrices

Equivalent Statements for $A_{n\times n}$:
A is invertible
$Ax = O$ has only the trivial solution, i.e. $x = O$
$Ax = b$ is consistent for every $b_{n\times 1}$ and has only one sol, i.e. $x = A^{-1}b$

Properties
reduced row echelon form $= I_{n\times n}$
$A = E_0 \cdot E_1 \cdot E_2 \cdot \cdots \cdot E_n$
determinant of $A = \det(A) \neq O$

Common Classes and their Properties
Diagonal matrix, D_n s.t. only diagonal elements are non-zero while diagonal elements are 0. Upper matrix, U s.t. the diagonal and upper elements are non-zero while other are 0. Lower matrix, U s.t. the diagonal and lower elements are non-zero while other are 0. Hence $U^T = L$ and $L^T = U$, $L_a \cdot L_b = L_c$ and $U_a \cdot U_b = U_c$. If U, L have non zero determinant, then they are invertible and $L^{-1} = L$ and $U^{-1} = U$
Determinant $= d_1 \cdot d_2 \cdot d_3 \ldots d_n$
Symmetric matrix is s.t. the the upper diagonal is reflected onto the lower diagonal.
$A^T = A$
If A, B are symmetrical, then $A \pm B$ and kA are also symmetrical
If A is invertible, then AA^T, A^TA and A^{-1} are also symmetrical

Determinant

M_{ij} is the matrix without the ith row and jth col.
$C_{ij} = (-1)^{i+j}M_{ij}$
Adjoint of A is a matrix s.t. $A_{ij} = C_{ij}$
Determinant of a 3 by 3 matrix, $\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{im}C_{im}$

Properties
$\det(A) = det(A^T)$
if $R_i \propto R_j$ or $C_i \propto C_j$, then det $= 0$
if size of $A = n \times n$, then $\det(kA) = k^n \det(A)$
if A, B, C have the same size $(n \times n)$ that differ by rth row s.t. $C_r = A_r + B_r$, then $\det(C) = \det(A) + \det(B)$
If A, B have the same size $(n \times n)$, then $\det(A \cdot B) = \det(A) \cdot \det(B)$
If A is invertible, then $\det(A^{-1}) = (\det(A))^{-1}$

Operations:
If A has the size $n \times n$,
If $A \to aR_i \to B$, then $\det(A) = a\det(B)$
If $A \to R_{ij} \to B$, then $\det(A) = -\det(B)$
If $A \to R_i + aR_j \to B$, then $\det(A) = \det(B)$

Determining Inverse Matrix using Determinant
$A^{-1} = \frac{1}{det(A)} \cdot \mathrm{Adj}(A)$
Crammer's Rule, $x_j = \frac{\det(A_j)}{\det(A)}$ where $A_j = A$ s.t. the jth col contains only x

Let V be an arbitrary non empty set on which two operations (addition and multiplication) are defined

Axioms of a Vector Space:

$u, v \in V$, then $u + v \in V$ (closure law of +)
$u + v = v + u$ (commutative)
$u + (v + w) = (u + v) + w$ (associative)
$\exists O \in V$ s.t. $O + u = u + O = u\ \forall u \in V$ (additive identity)
$\exists -u$ s.t. $-u + u = u + (-u) = O\ \forall u \in V$ (additive inverse)
if $u \in V$ and $k \in \mathbb{R}$, then $ku \in V$ (closure law of x)
$k(u + v) = ku + kv$ (distributive)
$(k + m)u = ku + mu$ (distributive)
$k(mu) = (km)u$ (associative)
$1u = u$

Subspaces

W is a subspace of V iff closure law of addition and multiplication hold and $W \subset V$.
Subspaces of $\mathbb{R}^n$ include $\mathbb{R}^2$ ($\{0\}$ and the set of all lines through the origin) and $\mathbb{R}^3$ ($\{0\}$, the set of all lines through the origin and the set of all planes passing through the origin)

Linear System:

If $Ax = 0$ is a homogeneous linear system with m equations and n unknown variables, then the set of solution is a subspace of $\mathbb{R}^n$.
Linear combination of vectors, $W = k_1v_1 + k_2v_2 + k_3v_3 + \ldots k_rv_r$ where k are scalars and v are vectors.
If $V = \{v_1, v_2, v_3, \ldots, v_r\}$, then the set W of linear combinations is a subspace of V. W is the smallest subspace of V that contains $v_1, v_2, v_3, \ldots v_r$ in the sense that every other subspace containing $v_1, v_2, v_3, \ldots v_r$ must also contain W.
If $S = \{v_1, v_2, v_3, \ldots, v_r\}$, then W of V is the space spanned by S, i.e. $W = \mathrm{span}(S)$, if det $\neq 0$.

$Ax = O$ has at least one solution, i.e. trivial sol. If it is the only solution, then the system is linearly independent (det $\neq 0$) and A is invertible. If there are any non-trivial solutions, the the system is linearly dependent (det $= 0$) and A is not invertible.
A set S with at least 2 vectors is linearly dependent iff at least one of the vectors can be expressed as a linear combination of the other vector (i.e. the vectors do not lie in the same axis or plane). Otherwise it is linearly independent.
If S spans V and is linearly independent, then S is the basis of V.

Dimension

$\dim(V)$ = number of vectors in a basis of V
$\dim(\mathbb{R}^n) = n$
$\dim(P_n) = n + 1$
$\dim(M_{mn}) = mn$

number of vectors in bases $= r$
if $r < \dim(V)$, it does not span V
if $r = \dim(V)$, it spans V and forms basis for V
if $r > \dim(V)$, the vectors are linearly dependent