MATH 311: Topology and Metric Spaces

Contents

1 Metric Spaces

(X, d) is a metric space if $X \neq \phi$ and $d : X \to \mathbb{R}$ is a metric satisfying the following axioms $\forall\, x, y, z \in X$:

M_1 $d(x, y) \geq 0$

M_2 $d(x, y) = 0 \iff x = y$

M_3 $d(x, y) = d(y, x)$

M_4 $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality)

Common metric spaces include:

- $(\mathbb{R}, d_1)$ with absolute/usual distance, i.e. $d_1(x, y) = |x - y|$
- $(\mathbb{R}^n, d_n)$ with euclidean distance, i.e. $d_n(\underline{x}, \underline{y}) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^n}$
- $(\mathbb{R}^n, d_p)$ with postal distance, i.e. $d_p(\underline{x}, \underline{y}) = \sum_{i=1}^{n} |x_i - y_i|$
- $(\mathbb{R}^n, d_{max})$ with Chebyshev distance, i.e. $d_{max}(\underline{x}, \underline{y}) = \max\{|x_i - y_i|\}$
- (X, d_d) with discrete distance, i.e. $d_d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$

1.1 Neighbourhood

A r-neighbourhood (also known as a open ball) is a set st $N_r(a; d) = \{x \in X : d(x, a) < r\}$ where $r > 0$. The shape of r-neighbourhoods varies depending on the metric space, for example:

- $N_r(a; d_1) = (a - r, a + r)$ (open interval)
- $N_r(\underline{a}; d_2) = (x_1 - a_1)^2 + (x_2 - a_2)^2 < r^2$ (open circle)
- $N_r(\underline{a}; d_p) = |x_1 - a_1| + |x_2 - a_2| < r$ (open rhombus)
- $N_r(\underline{a}; d_{max}) = (a_1 - r, a_1 + r) \times (a_2 - r, a_2 + r)$ (open square)
- $N_r(a, d_d) = \begin{cases} \{a\} & r \in (0, 1] \\ X & r > 1 \end{cases}$

$N \subseteq X$ is a neighbourhood of a point a if $\exists\, N_r(a) \subseteq N$. For a collection of all neighbourhoods of a point, $\aleph(a)$, the following properties hold:

- $X \in \aleph(a)$
- $N \in \aleph(a), N \subseteq M \Rightarrow M \in \aleph(a)$
- $N_1, N_2 \in \aleph(a) \Rightarrow N_1 \cap N_2 \in \aleph(a)$
- $N \in \aleph(a) \Rightarrow \exists\, L \in \aleph(b)$ st $N \in \aleph(x)\ \forall x \in L$

1.2 Open Sets

If (X, d) is a metric space, then $G \subseteq X$ is an open set $\iff \forall x \in G, \exists\, N_r(x; d) \subseteq G$. The following properties hold for open sets:

- X, ϕ are open.
- If $\{G_i : i \in I\}$ is collection of open sets, then $\cup G_{i \in I}$ is open.
- If G_i, G_j are open sets, then $G_i \cap G_j$ is open.

1.3 Continuity

A function $f : (X, d_x) \to (Y, d_y)$ is continuous at $x = a \in X \iff \forall \varepsilon > 0, \exists\, \delta > 0$ st $f(N_\delta(a)) \subseteq N_\varepsilon(f(a))$.

1.4 Complete Metric Spaces

A metric space (X, d) is complete if and only if every cauchy sequence in X is convergent in X.

A sequence $(x_n : n \in \mathbb{N})$ in a metric space (X, d) is cauchy if and only if $\forall \varepsilon > 0, \exists\, k(\varepsilon) \in \mathbb{N}$ st $d(x_n, x_m) < \varepsilon \;\forall n, m \geq k(\varepsilon)$.

A sequence $(x_n : n \in \mathbb{N})$ in a metric space (X, d) converges to $b \in X$ if and only if $\forall \varepsilon > 0, \exists\, n(\varepsilon) \in \mathbb{N}$ st $x_n \in N_r(b; d) \;\forall n \geq n(k)$.

Theorem: $\mathbb{R}$ with usual metric is complete.

Let (x_n) be a cauchy sequence in $\mathbb{R}$, then we need to show that it converges in $\mathbb{R}$.

Let $S_m = \cup\{x_n : n \geq m\}$

$\because (x_n)$ is bounded $\therefore (S_m)$ is bounded $\Rightarrow \sup(S_m) = t_m$

$\because S_{m+1} \subseteq S_m \therefore t_{m+1} \leq t_m$

$\Rightarrow (t_m)$ is a decreasing sequence st $\lim\limits_{m\to\infty}(t_m) = b$

Now, we need to only show that $\lim\limits_{n\to\infty}(x_n) = b$

$\because (x_n)$ is cauchy $\therefore \forall \varepsilon > 0, \exists\, n_0 \in \mathbb{N}$ st $|x_n - x_m| < \varepsilon \;\forall n, m \geq n_0$

$\because \lim\limits_{m\to\infty}(t_m) = b \therefore \forall \varepsilon > 0, \exists m_0 \in \mathbb{N}$ st $|t_m - b| < \varepsilon \;\forall m \geq m_0$

Let $N = \max\{m_0, n_0\}$, then $t_N - \varepsilon$ is not an upper bound of S_N

$\Rightarrow \forall \varepsilon > 0, \exists\, M \geq N$ st $t_N - \varepsilon < x_M \leq t_N < t_N + \varepsilon \Rightarrow |t_n - x_m| < \varepsilon$

$\Rightarrow \forall n \geq N, |x_n - b| = |x_N - b + (x_M - x_M) + (t_N - t_N)| \leq |x_N - x_M| + |x_M - t_N| + |t_N - b| < 3\varepsilon$

$\rightarrow \forall n \geq N, |x_n - b| < 3\varepsilon$

$\Rightarrow \lim\limits_{n\to\infty}(x_n) \to b \in \mathbb{R}$

$\because$ all cauchy sequences in $\mathbb{R}$ converge in $\mathbb{R} \therefore \mathbb{R}$ is complete

From this theorem, several properties can be derived, including:

- Every closed interval in $\mathbb{R}$ is complete.
- Let $A \neq \phi$ be a subset of a metric space. Then $z \in \bar{A} \iff \exists\,(x_n) \to z$ in A.
- A subset of a metric space A is closed $\iff$ all convergent sequence in A converge in A.
- All cauchy sequences in a metric space are convergent but the converse need not be true.

Theorem: Let (X, d) be a metric space, then if a subspace A is complete, then it is closed in X.

Let (X, d) be a metric space and A a complete subspace, then we need to show A is closed, i.e. $A = \bar{A}$ but $\because A \subseteq \bar{A} \therefore$ we only need to show that $\bar{A} \subseteq A$

Let $x \in \bar{A}$

$\because X$ is a metric space $\therefore \exists (x_n) \to x$ in A

$\Rightarrow (x_n)$ is cauchy

$\because A$ is complete $\therefore x \in A$

$\Rightarrow \bar{A} \subseteq A$

Hence A is closed in X

Theorem: If (X, d) is a complete metric space, then any closed subset $B \subseteq X$ is also complete.

Let (x_n) be a cauchy sequence in B

$\Rightarrow (x_n)$ is cauchy in X and $\lim(x_n) = x \in X$

$\because B$ is closed $\therefore x \in B$

$\Rightarrow$ every cauchy sequence in B converges in B

Hence B is complete

Let $A \neq \phi$ is a subset of a metric space (X, d) and $z \in X$. Then $z \in \bar{A}$ if and only if $\exists\,(x_n)$ in A st $\lim(x_n) = z$.

2 Topology

(X, τ) is a topological space if $X \neq \phi$ and τ is a collection of open subsets of X st

T_1 $X, \phi \in \tau$

T_2 $\{A_i : i \in I\} \subseteq \tau \Rightarrow \cup \cup_{i \in I} A_i \in \tau$

T_3 $A_i, A_j \in \tau \Rightarrow A_i \cap A_j \in \tau$

If a metric space is used to form τ, then the topological space is said to be induced by that metric space. Hence every metric space defines a topological space, but the converse need not be true. Common topologies include:

- Usual topology, τ contains all neighbourhoods of that metric space
- Trivial topology, $\tau = \{\phi, X\}$
- Discrete topology, $\tau = P(X)$
- Cofinite topology, $\tau = \{G : |G^c| \in \mathbb{R}\}$

Comparing topologies: For topologies τ_1 and τ_2, only one of the following relations hold:

- $\tau_1 \leq \tau_2$ (coarser/weaker/smaller), i.e. $\forall\, x \in \tau_1, x \in \tau_2$
- $\tau_1 \geq \tau_2$ (finer/stronger/larger), i.e. $\forall\, x \in \tau_2, x \in \tau_1$
- $\tau_1 = \tau_2$, i.e. $\tau_1 \leq \tau_2$ and $\tau_1 \geq \tau_2$
- $\tau_1 \parallel \tau_2$ (non-comparable), i.e. $\tau_1 \nleq \tau_2$ and $\tau_1 \ngeq \tau_2$

For a set X, the trivial topology is weakest topology while the discrete topology is the strongest topology that can be defined.

If $x \in G \subseteq N$ - for some $G \in \tau$ and $N \in X$ - then N is a **neighbourhood** of τ. Hence it follows that every open set is a neighbourhood of each of its point. The **system of a point** is the collection of all neighbourhoods of a point, i.e. $\aleph(x) = \{N \subseteq X : x \in G \subseteq N\}$.

$B \subseteq \tau$ is a **basis** of τ if and only if $\forall x \in G$ and $\forall G \in \tau$, $\exists\, B_x \in B$ st $x \in B_x \subseteq G$.

($\Rightarrow$) Suppose B is a basis of τ.

Let $G \in \tau$, then $G = \cup_{\alpha \in I} B_\alpha$ st $B_\alpha \subseteq B$.

Let $x \in G$, then $x \in V_\alpha$ for some $B_\alpha \subseteq B$.

$\Rightarrow B_\alpha \subseteq G$.

($\Leftarrow$) Suppose $\forall\ x \in G\ \forall\ G \in \tau$, $\exists\ B_x \in B$ st $x \in B_x \subset G$, then we need to show that $\forall G \in \tau$ may be expressed as a union of sub-collections of B.

The statement implies $\cup_{x\ inG}\{x\} \in \cup_{x \in G}\{B_x\} \subseteq G$

$\Rightarrow G \subseteq \cup_{x \in G}\{B_x\} \subseteq G$

$\Rightarrow G = \cup_{x \in G}\{B_x\}$

If τ_1, τ_2 be topologies on X st B_1 is basis for τ_1. If $B_1 \subseteq \tau_2$, then $\tau_1 \leq \tau_2$.

A collection of open sets is a **sub-basis** if the finite intersection of its elements forms a basis for τ.

If $A \subseteq X$, then (A, τ_A) is a **subspace** st $\tau_A = \{A \cap G : G \in \tau\}$. Similarly, if B is a basis for τ, then $B \cap A$ is a basis for τ_A.

Let $A \subseteq X$ and $x \in X$. Then x is a **limit point** of A if $\forall G \in \tau$ st $x \in G$, $(G \backslash \{x\}) \cap A \neq \phi$. The **derived set** A' contains all limit points of A. The **closure** of a set, $\bar{A} = A \cup A' = \{G \in \tau : A \cap G \neq \phi\}$. Important properties of closure include:

- $A \subseteq X$ is dense if $\bar{A} = X$, e.g. $\mathbb{Q}$ is dense in $\mathbb{R}$
- $A \subseteq B \Rightarrow \bar{A} \subseteq \bar{B}$
- $\bar{A} = \bar{\bar{A}}$
- In discrete topology, $A = \bar{A}$, $\forall A \subseteq X$
- In trivial topology, $\bar{A} = X$, $\forall A \subseteq X$

2.1 Closed Sets

There are several methods to define a closed set, including the following:

- $F \subseteq X$ is closed if and only if F^c is open.
- $F \subseteq X$ is closed if and only if it contains all of its limits points, ie. $F' \subseteq F$.

  ($\Rightarrow$) Let F be closed and suppose $x \in F'$ st $x \notin F$

  $\Rightarrow x \in F^c$

  $\because F^c$ is open $\therefore F^c \backslash \{x\} \cap F = \phi$ which is a contradiction

  $\Rightarrow x \in F$

  $\Rightarrow F' \subseteq F$

  ($\Leftarrow$) Suppose $F' \subseteq F$.

  Let $x \notin F$, then $\exists G_x$ st $G_x \cap F = \phi$.

  $\Rightarrow \cup_{x \in F^c}\{G_x\} = F^c$ is open $\because G_x$ is open and arbitrary unions of open sets is open.

  $\Rightarrow (F^c)^c = F$ is closed.

- $F \subseteq X$ is closed if and only if $\bar{F} = F$.

($\Rightarrow$) Suppose $\bar{F} = F$. Since $F \subseteq \bar{F}$, we only need to show that $\bar{F} =\subseteq F$, i.e. $F^c \subseteq \bar{F}^c$

let $x \in F^c$

$\because F \cap F^c = \phi \therefore x \notin \bar{F}$

$\Rightarrow x \in \bar{F}^c$

$\Rightarrow F^c \subseteq \bar{F}^c$

($\Leftarrow$) Suppose $F^c \subseteq \bar{F}^c$.

Let $x \in F^c \Rightarrow x \in \bar{F}^c \Rightarrow x \notin \bar{F}$.

$\exists\, G_x \in \tau$ st $F \cap G = \phi$.

$\Rightarrow \cup_{x \in F^c} G_x = F^c$ is open $\because G_x$ is open and arbitrary unions of open sets is open.

$\Rightarrow (F^c)^c = F$ is closed.

Closed sets have numerous important properties, including:

- X, ϕ are closed.
- If $\{G_i : i \in I\}$ is collection of closed sets, then $\cap G_{i \in I}$ is closed.
- If G_i, G_j are closed sets, then $G_i \cup G_j$ is closed.
- $\bar{A}$ is closed.
- If $A \subseteq B \subseteq \bar{A}$ and B is closed, then $B = \bar{A}$.

A set is **clopen** if it is both open and closed (eg. ϕ, X).

2.2 Boundary

If $A \subseteq X$, then

- Interior, $Int(A) = \{x \in A : x \in G \subseteq A\}$ for some $G \in \tau$.
- Exterior, $Ext(A) = \{x \in A^c : x \in G \subseteq A^c\}$ for some $G \in \tau$.
- Boundary, $Fr(A) = \{x \in X : G \cap A \neq \phi \wedge G \cap A^c \neq \phi\ \forall G \in \tau \text{ st } x \in G\}$.

Important properties include:

- $A \subseteq B \iff Int(A) \subseteq Int(B)$,
- $Int(A)$ and $Ext(A)$ are the largest open subsets of A and $X \backslash A$ respectively,
- A is open $\iff Int(A) = A$
- $Ext(A) = Int(A^c)$ and $Int(A) = Ext(A^c)$
- $Int(A) \cap Ext(A) = \phi$

2.3 Continuity

A function $f : X \to Y$ is continuous at $x \in X$ if and only if

$$\forall H \in \tau_Y \text{ st } f(x) \in H \subseteq Y,\ \exists G \in \tau_X \text{ st } x \in G \text{ and } f(G) \subseteq H$$

A function is continuous overall if $\forall H \in Y, \exists f^{-1}(H) \in \tau_x$. Examples include:

- $f(G) = a \in \mathbb{R}\ \forall G \in \tau_X$
- $f : (X, P(X)) \to (Y, \tau)$
- $f : X \to (Y, \{Y, \phi\})$

- If $f : X \to Y$ is continuous, then $f^{-1}(Int(A)) \subseteq Int(f^{-1}(A))$ where $A \subseteq Y$
- If f is an identity function, then $\tau_2 \leq \tau_1 \iff f(x)$ is continuous

Theorem: Let X, Y, Z be topological spaces st $g : X \to Y$ and $f : Y \to Z$ are continuous functions. Then $f \circ g : X \to Z$ is a continuous function.

We must show that $(f \circ g)^{-1}(H) \in \tau_X \ \forall H \in \tau_Z$

Let $H \in \tau_Z$

$\Rightarrow f^{-1}(H) \in \tau_Y \ \because f$ is continuous

$\Rightarrow g^{-1}(f^{-1}(H)) \in \tau_X \ \because g$ is continuous

$\Rightarrow (g^{-1} \circ f^{-1})(H) \in \tau_X$

$(f \circ g)^{-1}(H) \in \tau_X$

Theorem: $f : X \to Y$ is continuous if and only if $\forall$ closed subsets $F \subseteq Y$, $f^{-1}(F)$ is closed in X, i.e. $(f^{-1}(F))^c \in \tau_X \ \forall F^c \in \tau_Y$

($\Rightarrow$) Let f be a continuous function and $F^c \in \tau_Y$, then we have to show $(f^{-1}(F))^c \in \tau_X$

$\because F^c \in \tau_Y \therefore f^{-1}(F^c) \in \tau_X \because f$ is continuous

$\Rightarrow (f^{-1}(F))^c \in \tau_X \because f^{-1}(F^c) = f^{-1}(Y \backslash F) = f^{-1}(Y) \backslash f^{-1}(F) = X \backslash f^{-1}(F) = (f^{-1}(F))^c$

$\Rightarrow f^{-1}(F)$ is closed in X

$\Rightarrow (f^{-1}(F))^c \in \tau_X$

($\Leftarrow$) Let $(f^{-1}(F))^c \in \tau_X$, $\forall F^c \in \tau_Y$, then we have to show that $f^{-1}(F^c) \in \tau_X$, $\forall F^c \in \tau_Y$

$\because (f^{-1}(F))^c \in \tau_X \therefore f^{-1}(F^c) \in \tau_X$, $\forall F^c \in \tau_Y$

Hence f is continuous.

Theorem: Let (X, τ_X) and (Y, τ_Y) be topological spaces and B a basis for τ_Y. Then a function $f : X \to Y$ is continuous if and only if $f^{-1}(\beta) \in \tau_X \ \forall \beta \in B$

($\Rightarrow$) Let f be a continuous function, then we have to show that $f^{-1}(\beta) \in \tau_X \ \forall \beta \in B$

Let $H \in \tau_Y$, then $f^{-1}(H) \in \tau_X \because f$ is continuous

$\Rightarrow f^{-1}(\cup_{\beta \in B} \beta) \in \tau_X \because \forall G \in \tau_x$ and B is a basis for τ_X

$\Rightarrow \cup_{\beta \in B} f^{-1}(\beta) \in \tau_X$

$f^{-1}(\beta) \in \tau_X$, $\forall \beta \in B$

($\Leftarrow$) Let $f^{-1}(\beta) \in \tau_X$, $\forall \beta \in B$, then we have to show that $f^{-1}(H) \in \tau_X$, $\forall H \in \tau_Y$

Let $f^{-1}(H) = f^{-1}(\cup_{\beta \in B} \beta)$ for some arbitrary $H \in \tau_Y$

$\Rightarrow f^{-1}(H) = \cup_{\beta \in B} f^{-1}(\beta) \in \tau_X$

$\Rightarrow f^{-1}(H) \in \tau_X$

Hence f is continuous

A **homeomorphism** is a special type of function f st f is bijective and f, f^{-1} are continuous. If $f : X \leftrightarrow Y$ is a homeomorphism, then the two spaces are said to be **homeomorphic**, i.e. $X \cong Y$. Examples: In usual topology, $[0, 1] \cong [a, b]$ and $(0, 1) \cong (a, b)$

3 Topological Spaces

While there are infinite topological spaces, they can be categorized based on certain properties. Here we discuss some of the most important types of spaces.

3.1 Product Spaces

If (X, τ_X) and (Y_X, τ_X) are topological spaces, then $(X \times Y, \tau_{X\times Y})$ is their product space. The basis for the smallest product topology can be determined by taking all cartesian products of open sets of both spaces, i.e. $B = \{G \times H : G \in \tau_X, H \in \tau_Y\}$. Similarly, the sub-basis for the smallest product topology is $S = \{\pi_X^{-1}(G), \pi_Y^{-1}(H) : G \in \tau_X, H \in \tau_Y\}$ where $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are continuous projections.

3.2 Quotient Spaces

Suppose (X, τ_X) is a topological space and let $\{A_k : k \in K\}$ be a collection of disjoint subsets. Then $({}^{X}/_{A_k}, {}^{\tau_X}/_{\{A_k\}})$ is a quotient space where ${}^{X}/_{A_k}$ is the image set of p and $p^{-1}(H) \in \tau_X \ \forall H \in \tau_X/\{A_k\}$ where p is a mapping, i.e.

$$p(x) = \begin{cases} x & x \notin \cup_{k\in K} A_k \\ a_k & x \in A_k \end{cases}$$

3.3 Compact Spaces

A collection of open sets $G = \{G_i : i \in I\}$ is an open cover of $A \iff A \subseteq \cup G$. A topological space is compact if and only if every open cover of A has a finite sub-covering.

Theorem: Let $f : X \to Y$ is continuous if X is compact, $f(X)$ is compact.

Suppose $U = \{G_i\}$ is an open cover of $f(X)$

$\because f$ is continuous $\therefore f^{-1}(U) \subseteq \tau_X$

$\Rightarrow \{f^{-1}(G_i) : G_i \in U\} \subseteq \tau_X$ is an open covering of X

$\because X$ is compact $\therefore$ a finite subcollection $\{f^{-1}(G_i) : G_i \in U, i \in \{1, 2, \dots, n\}\}$ will cover X

$\Rightarrow \{G_i : i \in \{1, 2, \dots, n\}\}$ covers $f(X)$

Hence $f(X)$ is compact

It follows that if $X \cong Y$, then X is compact $\iff Y$ is compact

Theorem: Every closed subset of a compact space is compact.

Let X be a compact space and $F \subseteq X$ a closed subset

Suppose $F \subseteq G = \cup\{G_i : G_i \in \tau\}$

$\because F$ is closed $\therefore F^c$ is open

$\Rightarrow G \cup F^c$ is an open cover of X

$\because X$ is compact $\therefore$ a finite subcollection covers X

$\because F \nsubseteq F^c \therefore F$ must be covered by a finite subcollection of G

$\Rightarrow F \subseteq \cup\{G_i : i \in \{1, 2, \dots, n\}\}$

Hence F is compact

Theorem: Every compact subset of a T2 space is closed.

Suppose A is a compact subset of a T2 space X. Then we need to show A is closed, i.e. $A^c \in \tau$

Hence it is sufficient to show that $\forall x \in A^c, \exists\, G_x \in \tau$ st $x \in G_x \subseteq A^c$

Let $x \in A^c$ and $\{y\} \subseteq A$

$\because X$ is T2 $\therefore \forall y \in A, \exists\, G_{xy}, H_{xy} \in \tau$ which are disjoint and contain x, y respectively

$\Rightarrow \{H_{xy} : y \in A\}$ covers A

$\because A$ is compact $\therefore A \subseteq H_x = \cup\{H_{xyi} : i \in \{1, 2, \ldots, n\}\}$

$\Rightarrow$ the intersection of the corresponding sets $G_x = \cap\{G_{xyi}\}$ is open

$\because H_x \cap G_x = \phi \therefore G_x \cap A = \phi$

$\Rightarrow \forall x \in A^c, \exists\, G_x \in \tau$ st $x \in G_x \subseteq A^c$

A is closed

In a compact T2 space, a subset is compact if and only if it closed

Theorem: If $f : X \leftrightarrow Y$ is a continuous bijection from compact X to T2 Y, then f is a homeomorphism.

$\because f$ is bijective and continuous $\therefore$ we only need to show f^{-1} is continuous

$\because f$ is continuous if and only if $\forall$ closed sets F of Y, $f^{-1}(F)$ is closed in X

$\therefore$ we only need to show that $\forall$ closed sets $(f^{-1})^{-1}(F) = f(F)$ in Y are closed

Let $F \subseteq X$ be closed, then F is compact $\because$ every closed subset of a compact space is compact

$\Rightarrow f(F)$ is compact $\because$ compactness is preserved under continuous functions

$\Rightarrow f(F)$ is closed $\because Y$ is T2 and every compact subset of a T2 space is closed

Hence f is a homeomorphism

A topological basis is compact if and only if for all basis B, every open cover of X by B contains a finite sub-covering.

Theorem: If Y_1, Y_2 are topological spaces, then $Y_1 \times Y_2$ with product topology is compact if and only if Y_1 and Y_2 are compact.

($\Rightarrow$) Suppose (Y_1, τ_1) and (Y_2, τ_2) are compact topological spaces with product topology $Y_1 \times Y_2$

$\Rightarrow \pi_1(Y_1 \times Y_2)$ and $\pi_2(Y_1 \times Y_2)$ are compact $\because$ compactness is preserved under continuous maps

$\because \pi_1, \pi_2$ are continuous under product topology $\therefore \pi_1(Y_1 \times Y_2) = Y_1$ and $\pi_2(Y_1 \times Y_2) = Y_2$

$\Rightarrow Y_1, Y_2$ are compact

($\Leftarrow$) We need to show that for all open covers of $Y_1 \times Y_2$, say U, by members of basis B have a finite sub-covering.

Let $B = \{G \times H : G \in \tau_{Y1}, H \in \tau_{Y2}\}$

$\forall x \in Y_1, \exists$ a subcollection of U which covers $\{x\} \times Y_2$

$\because \{x\} \times Y_2 \cong Y_2 \therefore \{x\} \times Y_2$ is compact

Let $G_x = \cap\{G_i \times H_i : i \in \{1, 2, \ldots, n\}\}$ be the intersection of the finite sub-covering of $\{x\} \times Y_2$

$\Rightarrow G_x \times Y_2$ is an open subset of $Y_1 \times Y_2$ containing $\{x\} \times Y_2$

$\Rightarrow G_x \times Y_2$ is covered by the same subcollection as $\{x\} \times Y_2$ by definition

$\Rightarrow \{G_x \times Y_2 : \forall x \in Y_1\}$ covers $Y_1 \times Y_2$

$\because Y_1$ is compact $\therefore$ a finite subcollection $\{G_x : x \in Y_1\}$ covers Y_1

$\Rightarrow$ a finite subcollection $\{G_x \times Y_2\}$ covers $Y_1 \times Y_2$

$\because$ each $G_x \times Y_2$ is covered by a finite number of members of U

Hence $Y_1 \times Y_2$ is compact

3.4 Connected Spaces

A topological space (X, τ) is connected if and only if X is not the union of two non-empty disjoint open sets, i.e.

$$\forall G, H \in \tau,\ X \neq G \cup H \text{ and } G \cap H \neq \phi$$

If ϕ, X are the only clopen sets in τ, then (X, τ) is connected, as otherwise $X = A \cup A^c$. Examples include:

- Trivial topological space is connected since X does not contain any disjoint sets.
- Discrete topological space is disconnected since $X = A \cup A^c$.
- An infinite set with cofinite topology is connected.
- $\mathbb{R}$ with usual topology is connected.

Important properties and results of connected spaces include:

- If a set is connected, its subsets/subspaces need not be connected.
- X is connected if and only if there are no continuous functions from X *onto* a discrete space with two elements.
- X is connected if and only if every continuous function from X *into* a discrete space with two elements is constant.
- If $X \cong Y$, then X is connected if and only if Y is connected.

Theorem: If f is a continuous function from X into Y where X is connected, then $f(X)$ is connected.

Suppose $f(X)$ is disconnected, then $\exists\, G, H \in \tau_Y$ st $G \cup H = f(X)$ and $G \cap H = \phi$.

$\because G, H \in \tau_Y \therefore f^{-1}(G), f^{-1}(H) \in \tau_X$ by continuity of f

$f^{-1}(G) \cup f^{-1}(H) = f^{-1}(G \cup H) = f^{-1}(f(X)) = X$

$f^{-1}(G) \cap f^{-1}(H) = f^{-1}(G \cap H) = f^{-1}(\phi) = \phi$

$\Rightarrow X$ is disconnected which is a contradiction

Hence $f(X)$ is connected

Theorem: Suppose that $\{A_i : i \in I\}$ is a family of connected subsets of topological space X st $A_i \cap A_j \neq \phi \; \forall i, j \in I (i \neq J)$. Then $A = \cup\{A_i : i \in I\}$ is connected.

Let $f : A \to \{a, b\}$ be a continuous function from A into $\{a, b\}$ with discrete topology

$\because A_i$ is connected $\therefore f$ can not map onto $\{a, b\}$, i.e $f(A_i) \neq \{a, b\}$

$\because A_i$ is connected $\therefore f$ is constant.

Let $f(A_i) = a$

$\because A_i \cap A_j \neq \phi \therefore G \cap H \neq \phi$ for some $G \in A_i$ and $H \in A_j$

$\Rightarrow f(H) = f(G) = a$

$\Rightarrow \forall i \in I,\ f(A_i) = a$

Hence A is connected

It follows that if $\{A_i : i \in I\}$ is a family of connected subsets of a topological space X and if there exists connected subset $B \subseteq X$ st $B \cap A_i \neq \phi$, then $B \cup (\cup_{i \in I} A_i)$ is connected.

Theorem: The topological spaces X and Y are connected if and only if $X \times Y$ with the product topology is connected.

$(\Rightarrow)$ Suppose X, Y are connected, then we need to show $X \times Y$ is connected

Let $B = X \times \{y_0\}$ where $y_0 \in Y$. Then B is connected $\because B \cong X$

Let $C_x = \{x\} \times Y,\ \forall x \in X$. Then C_x are connected $\because C_x \cong Y$

$\because (x, y_0) \in B \cap C_x,\ \forall x \in X \therefore B \cap C_x \neq \phi$

$\Rightarrow B \cup (\cup_{x \in X} C_x)$ is connected by common point criterion

$\because B \cup (\cup_{x \in X} C_x) \cong X \times Y \therefore X \times Y$ is connected

$(\Leftarrow)$ Suppose $X \times Y$ with product topology is connected, then $\because \pi_X(X \times Y) = X, \pi_Y(X \times Y) = Y$

and connectedness is preserved under continuous functions $\therefore X, Y$ are connected

4 Separation Axioms

While the exact definition various, these are generally used to distinguish between disjoint sets and distinct points. Stronger axioms correlate to a more "stricter" sense of separation which can be considered in terms of open and closed sets. The separation axioms are labelled as T_i where $i \in \{0, 1, 2, 2\frac{1}{2}, 3, 3\frac{1}{2}, 4, 5, 6\}$ and a larger i corresponds to stronger properties. Hence stronger axioms will satisfy the weaker axioms as well, i.e. $T4$ is $T3$ which is $T2$ which is $T1$.

4.1 T1 - T4 Spaces

A space is **accessible/T1** if and only if all pairs of distinct points in X are separated, i.e

$$\forall x, y \in X, \exists G_x, G_y \in \tau \text{ st } x \in G_x, x \notin G_y \text{ and } y \in G_y, y \notin G_x$$

Theorem: A space is T1 id and only if all singletons are closed.

($\Rightarrow$) Let $y \in \{x\}^c$

$\because X$ is $T1 \therefore \exists G_x, G_y \in \tau$ st $x \in G_x, x \notin G_y$ and $y \in G_y, y \notin G_x$

$\Rightarrow y \in G_y \subseteq \{x\}^c \because x \notin G_y$

$\Rightarrow \{y\} \subseteq G_y \subseteq \{x\}^c$

$\Rightarrow \cup_{y \in \{x\}^c}\{y\} \subseteq \cup_{y \in \{x\}^c} G_y \subseteq \{x\}^c$

$\Rightarrow \{x\}^c \subseteq \cup_{y \in \{x\}^c} G_y \subseteq \{x\}^c$

$\Rightarrow \cup_{y \in \{x\}^c} G_y = \{x\}^c$

$\because G_y \in \tau \therefore \{x\}^c \in \tau$

Hence $\{x\}$ is closed

A space is **hausdorff/T2** if and only if all distinct points are separated by closed neighbourhoods, i.e.

$$\forall x, y \in X, \exists G_x, G_y \in \tau \text{ st } x \in G_x, y \in G_y \text{ and } G_x \cap G_y = \phi$$

A space is **regular/T3** or regular T1 if only if it any point and a closed subset (not containing that point) are separated by neighbourhoods, i.e.

$$F \subseteq X \text{ st } F^c \in \tau, \exists G, H \in \tau \text{ st } G \cap H = \phi, x \in G, \text{ and } F \subseteq H$$

A space is **normal/T4** or normal T1 if and only if all disjoint closed sets are separated by neighbourhoods, i.e.

$$F_1, F_2 \subseteq X \text{ st } F_1^c, F_2^c \in \tau, \exists G_1, G_2 \in \tau \text{ st } G_1 \cap G_2 = \phi, F_1 \subseteq G_1 \text{ and } F_2 \subseteq G_2$$

Theorem: A closed continuous image of normal space is normal.

Let X be a normal space and $f : X \to Y$ a closed continuous function.

Let $F, G \subseteq Y$ be disjoint closed subsets

$\Rightarrow f^{-1}(F), f^{-1}(G)$ are closed in X by continuity of f

$f^{-1}(F) \cap f^{-1}(G) = f^{-1}(F \cap G) = f^{-1}(\phi) = \phi$

$\because X$ is T4 $\therefore \exists U, V \in \tau_X$ st $f^{-1}(G) \subseteq U$ and $f^{-1}(H) \subseteq V$

$\Rightarrow U^c, V^c$ are closed in X

$\because f$ is closed continuous $\therefore f(U^c), f(V^c)$ are closed in Y

$\Rightarrow (f(U^c))^c = U, (f(V^c))^c = V \in \tau_Y$

$\because f(U) \cap f(V) = f(U \cap V) = f(\phi) = \phi$ and $F \subseteq f(U), G \subseteq f(V) \therefore f(X)$ is a normal space.

A compact subset of a T2 space is T3.

Theorem: Every compact Hausdorff space is normal.

Let X be a compact T2 space and $F_1, F_2 \subseteq X$ be disjoint closed subsets

$\because$ a closed subset of a compact space is compact $\therefore F_1, F_2$ are compact

$\because$ a compact subset of a T2 is T3 $\therefore \forall x \in F_2, \exists$ disjoint open sets G_x, H_x st $x \in G_x$ and $F_1 \subseteq H_x$

$\because F_2$ is compact $\therefore$ it is covered by a finite covering, say $F_2 \subseteq G = \cup\{G_x : x \in F_2\}$

$\because G_x \in \tau \therefore G \in \tau$

Let $H = \cap\{H_x : x \in F_2\}$

$\because H_x \cap G_x = \phi$, $\forall x \in F_2 \therefore H \cap G = \phi$

Similarly $\forall$ disjoint closed sets $\in X, \exists$ disjoint open sets containing them

Hence X is T4

4.2 Relationship Between Spaces

Every T2 satisfies T1

Let X be T2 and $x, y \in X$ st $x \neq y$

$\Rightarrow \exists G_x, G_y \in X$ st $x \in G_x$, $y \in G_y$ and $G_x \cap G_y = \phi$

$\Rightarrow x \in G_x, x \notin G_y$ and $y \in G_y, y \notin G_x$

Hence T2 is T1

Every T3 satisfies T2

Let X be T3 and $x, y \in X$ st $x \neq y$

$\Rightarrow \exists G, H \in \tau$ st $G \cap H = \phi$ and $x \in G$ and $\{y\} \subseteq H$ where $\{y\}$ is closed $\because$ T3 is T1

$\Rightarrow x \in G$ and $y \in H$ where $G \cap H = \phi$

Hence T3 is T2

Every T4 satisfies T3

Let X be T4 and let F be ca closed subset st $x \notin F$

$\because$ T4 is T1 $\therefore \{x\}$ is closed

$\Rightarrow \{x\} \cap F = \phi$

$\exists G, H \in \tau$ which are disjoint and contain $\{x\}, F$ respectively

$\Rightarrow x \in G$ and $F \subseteq H$ where G, H are disjoint and F is closed

Hence T4 is T3

4.3 Metric Spaces

All metric spaces satisfy all separation axioms.

Accessible (T1) Space

Let (X, d) be a metric space, then $\because$ all singletons in (X, d) are closed $\therefore (X, d)$ is T1

Hausdorff (T2) Space

Let (X, d) be a metric space and $x, y \in X$ st $x \neq y$

$\because x \neq y \therefore d(x, y) = r > 0$

Let $x \in N_{r/3}(x)$ and $y \in N_{r/3}(y)$

Now we only need to show that $N_{r/3}(x) \cap N_{r/3}(y) = \phi$

Suppose $N_{r/3}(x) \cap N_{r/3}(y) \neq \phi$

Let $z \in N_{r/3}(x) \cap N_{r/3}(y)$, then $z \in N_{r/3}(x)$ and $z \in N_{r/3}(y)$

$\Rightarrow d(x, y) < \frac{r}{3}$ and $d(y, z) < \frac{r}{3}$

$\because r = d(x, y) \leq d(x, z) + d(z, y) \therefore d(x, y) < \frac{r}{3} + \frac{r}{3}$

$\Rightarrow r < \frac{2}{3}r$

$\because r > 0 \therefore r \nless \frac{2}{3}r$ which is a contradiction

$\Rightarrow N_{r/3}(x) \cap N_{r/3}(y) = \phi$

Hence (X, d) is T2

Regular (T3) Space

Suppose (X, d) is a metric space, F is a closed subset and $x \notin F$

$\Rightarrow x \in F^c$

$\Rightarrow \exists\, N_r(x) \subseteq F^c$

$\Rightarrow N_{r/2}(x) \subseteq \bar{N_{r/2}}(x) \subseteq N_r(x)$

$\Rightarrow N_{r/2}(x) \cap (N_r(x))^c = \phi$ st $x \in N_{r/2}$ and $F \subseteq (N_r(x))^c$

Hence (X, d) is T3