MATH 309: Real Analysis

Contents

1 Real Numbers

1.1 Algebraic Properties

Algebraic properties of real numbers, $\forall a, b, c \in \mathbb{R}$

- closure: $a + b \in \mathbb{R}$ and $a \cdot b \in \mathbb{R}$
- commutative: $a + b = b + a$ and $a \cdot b = b \cdot a$
- associative: $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
- identity: $a + 0 = 0 + a = a$ and $1 \cdot a = a \cdot 1 = a$
- inverse: $a + (-a) = -a + a = 0$ and $a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1$
- distributive: $a \cdot (b + c) = a \cdot b + a \cdot c = (b + c) \cdot a$

These axioms are used to define numerous results, including:

- $a + z = a \Rightarrow z = 0$
- $ab = b$ st $b \neq 0 \Rightarrow a = 1$
- $ab = 0 \Rightarrow a$ or $b = 0$

1.2 Order Properties

Order property of real numbers (aka law of Trichotomy): Let $\mathbb{P} = \{x \in \mathbb{R} : x > 0\}$ be closed under scalar addition and multiplication. Then for $a \in \mathbb{R}$, only one of the following holds:

- $a \in \mathbb{P} \quad \Rightarrow a > 0$
- $-a \in \mathbb{P} \quad \Rightarrow a < 0$
- $a, -a \notin \mathbb{P} \quad \Rightarrow a = 0$

The order property is used to define numerous results, including:

- $a > b, b > c \Rightarrow a > c$
- $a > b \Rightarrow a + c > b + c$
- $a > b, c < 0 \Rightarrow ac < bc$
- $a \neq 0 \Rightarrow a^2 > 0$
- $0 \leq a < \varepsilon, \quad \forall \varepsilon > 0 \Rightarrow a = 0$
- $|a + b| \leq |a| + |b|$ (triangular identity)
- $||a| - |b|| \leq |a - b|$

1.3 Neighbourhood

Th neighbourhood of a point, $V_r(a) = \{x \in \mathbb{R} : |x - a| < r\} = (a - r, a + r)$. Some results include:

- Let $U = \{x \in \mathbb{R} : 0 < x < 1\}$. If $\varepsilon < a, 1 - a$, then $V_r(a) \subseteq U$
- Let $x \in V_r(a)$ and $y \in V_r(b)$, then $x + y \in V_{2r}(a + b)$

2 Boundaries

Let S be a non-empty set, then it is

- Bounded above if $\exists u \in \mathbb{R}$ st $s \leq u, \forall s \in S$.
  - The set U containing all such u is known as the upper bound of S.
  - The supremum, $\sup(S) = \alpha$ if $\alpha \in U$ st $\alpha \leq u, \forall u \in U$.

- Bounded below if $\exists l \in \mathbb{R}$ st $s \geq l, \forall s \in S$.
    - The set L containing all such w is known as the lower bound of S.
    - The infimum, $\inf(S) = \beta$ if $\beta \in L$ st $\beta \geq l, \forall l \in L$.
- Bounded if it is bounded above and below and unbounded if it is not bounded.

Another method to define a supremum (and similarly infimum) is to let S be non-empty set. Then $u \in \mathbb{R}$ is the supremum if $s \leq u, \forall s \in S$ and if $v < u \Rightarrow \exists s' \in S$ st $v < s'$.

Boundaries are used to prove numerous properties and theorems, including

- The completeness property of $\mathbb{R}$: Every non-empty subset of $\mathbb{R}$ that has an upper bound must also have a supremum in $\mathbb{R}$.
- Archimedean Property: If $x \in \mathbb{R} \Rightarrow \exists n_x \in \mathbb{N}$ st $x \leq n_x$.

    Let $x \in \mathbb{R}$ st $x > n, \forall n \in \mathbb{N}$

    By the completeness theorem, $\sup(N) = u$

    $\Rightarrow u - 1$ is not an upper bound of $\mathbb{N}$ as $u - 1 < u$

    $\Rightarrow \exists m \in \mathbb{N}$ st $u - 1 < m$

    It follows $u < m + 1$ which implies u is not an upper bound which is a contradiction.

- Density Theorem: $\mathbb{Q}$ is dense in $\mathbb{R}$

    Let $x, y \in \mathbb{R}$ st $x < y$, then we need to show that $\exists r \in \mathbb{Q}$ st $x < r < y$

    $x < y \Rightarrow y - x > 0$

    By Archimedean property, $\exists n \in \mathbb{N}$ st $0 < \frac{1}{n} < y - x \Rightarrow nx + 1 < ny$

    Let $x > 0$, then $\exists m \in \mathbb{N}$ st $m - 1 \leq nx \leq m \Rightarrow m < nx + 1$

    Hence $nx < m < nx + 1 < ny \Rightarrow m < ny$

    $\Rightarrow nx < m < ny \Rightarrow x < \frac{m}{n} < ny$

    $\Rightarrow x < r < y$ where $r = \frac{m}{n} \in \mathbb{Q}$

3 Intervals

Let $S \subseteq \mathbb{R}$. If $x, y \in S$ st $x < y$ and $[x, y] \subseteq S$, then S is an interval (Characterization Theorem).

Nested Intervals Property: If $I_n = [a_n, b_n], n \in \mathbb{N}$ is a nested sequence of closed intervals, then $\exists \alpha \in \mathbb{R}$ st $\alpha \in I_n, \forall n \in \mathbb{N}$.

Let $I_n = [a_n, b_n], \forall n \in \mathbb{N}$ st $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots \supseteq I_n \supseteq \ldots$

$I_n = [a_n, b_n] \Rightarrow b_n \geq a_n$

$\because I_1 \supseteq In \therefore a_1 \leq a_n \leq b_n \leq b_1 \Rightarrow a_n \leq b_1 \Rightarrow A = \{a_n : n \in \mathbb{N}\} \leq b_1$

$\because b_1$ is an upper bound of $A \therefore \exists \alpha \in \mathbb{R}$ st $\sup(A) = \alpha$ and $a_n \leq \alpha, \forall n \in \mathbb{N} \quad - (1)$

Let $A = \{a_k : k \in \mathbb{N}\}$ and $n \in \mathbb{N}$. Suppose

- $k < n$, then $I_k \supseteq I_n \Rightarrow a_k \leq a_n \leq b_n \leq b_k \Rightarrow a_k \leq b_n, \forall k < n$.
- $k \geq n$, then $I_n \supseteq I_k \Rightarrow a_n \leq a_k \leq b_k \leq b_n \Rightarrow a_k \leq b_n, \forall k \geq n$.

Hence $a_k \leq b_n, \forall k \in \mathbb{N} \Rightarrow b_n$ is an upper bound and $\sup(A) = \alpha$ st $\alpha \leq b_n, \forall n \in \mathbb{N} \quad - (2)$

From (1) and (2), $a_n \leq \alpha \leq b_n \Rightarrow \alpha \in [a_n, b_n] = I_n, \forall n \in \mathbb{N}$

Using this property, we can show that $\mathbb{R}$ is uncountable.

Let $\mathbb{R}$ be countable, then $I_0 = [0,1] \subseteq \mathbb{R}$ is countable and can be written as a sequence $I_0 = \{x_1, x_2, \dots, x_n, \dots\}$

Let $I_i \subset I_{i-1}$ st $x_i \notin I_i$, then $I_1 \supset I_2 \supseteq \dots \supseteq I_n \supseteq \dots$

By the nested interval property, $\exists \alpha \in I_0$ st $\alpha \in I_n, \forall n \in \mathbb{N}$

$\because x_i \notin I_i \therefore \alpha = x_n \notin I_n, \forall n \in \mathbb{N}$

$\Rightarrow \alpha \notin I_0$ which is a contradiction. Hence I_0 is uncountable, i.e. $\mathbb{R}$ is uncountable.

4 Sequences

A sequence $X : \mathbb{N} \to \mathbb{R}$ such that $X = (x_n : n \in \mathbb{N}) = (x_n)$. A subsequence X' is derived from elements of X while preserving their order. A m-tail of $X = (x_{n+m})$ is a subsequence which contains all elements of X except the initial m terms.

4.1 Limits

The limit of a sequence, $\lim\limits_{n\to\infty} X = x$ (also denoted as $X \to x$) st $\forall \varepsilon > 0, \exists k \in \mathbb{N}$ st $|x_n - x| < \varepsilon, \forall n \geq k$.

A sequence is convergent if and only if $\overline{\lim}(x_n) = \underline{\lim}(x_n)$ where

- limit superior $\overline{\lim}(x_n) = \lim\sup(x_n) = \sup\{z : z$ is a limit point of some subsequence of $(x_n)\}$.
- limit inferior $\underline{\lim}(x_n) = \lim\inf(x_n) = \inf\{z : z$ is a limit point of some subsequence of $(x_n)\}$.

If subsequences of X have different limits, then X is divergent (divergence criteria). A sequence is properly divergent if:

- $(x_n) \to +\infty$ if $\forall \alpha \in \mathbb{R}, \exists k \in \mathbb{N}$ st $s_n > \alpha \forall n \geq k$
- $(x_n) \to -\infty$ if $\forall \alpha \in \mathbb{R}, \exists k \in \mathbb{N}$ st $s_n < \alpha \forall n \geq k$

Let $X = (x_n)$ be a sequence in $\mathbb{R}$ convergent on $x \in \mathbb{R}$. Then important properties and theorems related to limits of sequences include:

- X is bounded if $\exists M > 0$ st $|x_n| \leq M, \forall n \in \mathbb{N}$.
- Every convergent sequence is bounded but converse is not true.
- If $x_n \geq 0, \forall n \in N$, then $x \geq 0$.
- Let $Y = (y_n) \to y$. If $x_n \leq y_n, \forall n \in N$, then $x \leq y$.
- Uniqueness theorem: each sequence in $\mathbb{R}$ can have at most one limit.

  Let $X = (x_n)$ be a sequence st $\lim x_n = x_1, x_2$

  Then $\forall \varepsilon > 0, \exists k_1, k_2 \in \mathbb{N}$ st $|x_n - x_1| < \frac{\varepsilon}{2}, \forall n \geq k_1$ and $|x_n - x_2| < \frac{\varepsilon}{2}, \forall n \geq k_2$

  Consider $|x_1 - x_2|$

  $= |x_1 - x_2 + x_n - x_n| = |(x_1 - x_n) + (x_n - x_2)| \leq |x_n - x_1| + |x_n - x_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$

  $\Rightarrow |x_1 - x_2| < \varepsilon, \forall n \geq k$ where $k = \max(k_1, k_2)$

  $\Rightarrow |x_1 - x_2| = 0 \Rightarrow x_1 = x_2$

- Squeeze Theorem: let X, Y, Z be convergent st $x_n \leq y_n \leq z_n$ $\forall n \in \mathbb{N}$. If $x = z = w$, then $y = w$.

Let $x = z = w$

$\Rightarrow \forall \varepsilon > 0, \exists k \in \mathbb{N}$ st $|x_n - w|, |z_n - w| < \varepsilon \; \forall n \geq k$

$\Rightarrow -\varepsilon < x_n - w < \varepsilon$ and $-\varepsilon < z_n - w < \varepsilon$

$\Rightarrow -\varepsilon < x_n - w < y_n - w < z_n - w < \varepsilon$

$\Rightarrow -\varepsilon < y_n - w < \varepsilon$

$\Rightarrow |y_n - w| < \varepsilon$

$\Rightarrow Y \to w$

4.2 Monotone Sequences

A monotonic sequence is a sequence which either increases or decreases.

Monotone Convergence Theorem: a monotone sequence $X = (x_n)$ on $\mathbb{R}$ is convergent if and only it is bounded. From this, it follows that

- X is bounded increasing, then $X \to x = \sup\{x_n\}$
- X is bounded decreasing, then $X \to x = \inf\{x_n\}$

($\Rightarrow$) Let $X = (x_n)$ be a monotone convergent sequence. Since every convergent sequence is bounded, therefore X is bounded.

($\Leftarrow$) Let X be bounded, then $\exists M \in \mathbb{R}$ st $|x_n| \leq M \forall n \in \mathbb{N}$

$\Rightarrow M$ is an upper bound for $\{x_n\}$

By completeness property, $\exists \sup\{x_n\} = x' \in \mathbb{R}$

$\Rightarrow \forall \varepsilon > 0, x' - \varepsilon < x' \Rightarrow x' - \varepsilon$ is not an upper bound for $\{x_n\}$

$\Rightarrow x_k \in \{x_n\}$ st $x_k > x' - \varepsilon$.

Suppose X is increasing, then $\forall n \geq k$, $x_n > x_k \Rightarrow x_n > x' - \varepsilon$

$\Rightarrow x' + \varepsilon > x' > x_n > x' - \varepsilon$

$\Rightarrow x' + \varepsilon > x_n > x' - \varepsilon$

$\Rightarrow \varepsilon > x_n - x' > \varepsilon$

$\Rightarrow |x_n - x'| < \varepsilon$

$\Rightarrow (x_n) \to x'$

Suppose X is decreasing, then $Y = -X$ is increasing and $\lim(Y) = -sup(Y)$.

$\Rightarrow \lim(X) = \inf\{x_n\}$

Monotone Subsequence Theorem: Every sequence in $\mathbb{R}$ has a monotone sequence.

x_m is peak term of X st $x_m \geq x_n$ $\forall n \geq m$

Assume X is decreasing, then it has infinite peak terms $x_{m_1} \geq x_{m_2} \geq \cdots \geq x_{m_k} \geq$. Hence (x_{m_k}) is a decreasing subsequence of X.

Assume X is not decreasing, then it has finite peak terms. Order them st $x_{m_1} \geq x_{m_2} \geq \cdots \geq x_{m_k}$.
Since x_{m_k} is the last peak term, hence $x_{m_{k+1}} = x_{s_1}$ is not a peak term.
Then $\exists s_2$ st $x_{s_1} < x_{s_2}$ where $s_2 > s_1$. And $\because$ s_2 is not a peak term, then there $\exists s_3$ st $x_{s_2} < x_{s_3}$ where $s_3 > s_2$.
Hence (x_{s_k}) is a recursively defined increasing subsequence of X

4.3 Cauchy Sequence

$X = (x_n)$ is a cauchy sequence if $\forall \varepsilon > 0, \exists k(\varepsilon) \in N$ st $\forall n, m \geq k(\varepsilon)$, $|x_n - x_m| < \varepsilon$.

let $X \to x$, then $\forall \varepsilon > 0, \exists k \in \mathbb{N}$ st $\forall n \geq k$, $|x_n - x| < \varepsilon/2$.
Consider $n, m > k(\varepsilon)$, then $|x_n - x_m| = |x_n - x + x - x_m| \leq |x_n - x| + |x_m - x| < \varepsilon$
$\Rightarrow |x_n - x_m| < \varepsilon$

Hence, A sequence is convergent on $\mathbb{R}$ if and only if it is cauchy.

4.4 Series

A series, $\sum_{n=1}^{\infty} x_n$ is the sum of all terms of a sequence (x_n). A series is convergent if the sequence (S_k) is convergent but the the converse is not true. (S_k) is a recursively defined sequence where $S_k = \sum_{n=1}^{k} x_n = S_{k-1} + x_k$ is the partial sum of a series.

5 Functions

Let $A \subseteq \mathbb{R}$ and $c \in \mathbb{R}$. Then c is a cluster point of A if either of the following hold:

- $\forall \delta > 0, \exists A' \subseteq A$ st $|A'| > 0, c \notin A'$ and $|x - c| < \delta$, i.e. $x \in V_\delta(c)$
- $\exists (a_n)$ in A st $(a_n) \to c$ where $c \notin \{a_n\}$

5.1 Limit

For a function $f : A \to \mathbb{R}$, L is a limit of f at c (cluster point), i.e. $L = \lim\limits_{x \to c} f(x)$, if either of the following hold:

- $\forall \varepsilon > 0, \exists \delta > 0$ st if $x \in A$, then $|f(x) - L| < \varepsilon$ when $|x - c| < \delta$.
- $\forall (x_n) \in A$ st $x_n \neq c, \forall n \in \mathbb{N}$, $\lim(f(x_n)) = L$ when $\lim(x_n) = c$.

$f(x)$ is bounded on $V_r(c)$ if $\exists V_\delta(c)$ and $M > 0$ st $|f(x)| \leq M$ $\forall x \in A \cap V_\delta(s)$. Hence it follows that if f has a limit at c (cluster point), then it is bounded on some neighbourhood of that c.

5.2 Continuity

$f : A = [a, b] \to \mathbb{R}$ is continuous at c (cluster point) if and only if either of the following hold

- $\forall$ $V_\varepsilon(f(c)), \exists V_\delta(c)$ st if $x \in V_\delta(c) \cap A$, then $f(x) \in V_\varepsilon(f(c))$, i.e. $f(A \cap V_\delta(c)) \subseteq V_\varepsilon(f(c))$.
- Sequential criteria: If $\forall (x_n) \in A$ st $x_n \to c$, $f(x_n) \to f(c)$.

- $j_f(c) = 0$ where $j_f(c) = \lim\limits_{x \to c^+} f(x) - \lim\limits_{x \to c^-} f(x)$ is a jump of a function where
    - $\lim\limits_{x \to c^-} f(x) = \sup\{f(x) : x \in (a, c)\}$
    - $\lim\limits_{x \to c^+} f(x) = \inf\{f(x) : x \in (c, b)\}$

Uniformity: Let $f : A \to \mathbb{R}$ and $u, v \in A$, then f is

- Uniform continuous: $\forall \varepsilon > 0, \exists \delta > 0$ st $|f(u) - f(v)| \leq \varepsilon$ when $|u - v| < \delta$
- Uniform discontinuous: $(x_n), (u_n) \in A$ st $(x_n), (u_n) \to 0$ and $\lim |f(x_n) - f(u_n)| \not\to 0$

5.3 Calculus

$f'(c)$ is derivative of $f(x)$ at $x = c$ if $\forall \varepsilon > 0, \exists \delta > 0$ st $\left|\frac{f(x)-f(c)}{x-c} - f'(c)\right| < \varepsilon$ when $|x - c| < \delta$.

If $f : I = [a, b] \to \mathbb{R}$ has a derivative at $x = c$, then it is continuous at c.

Let f be differentiable on $c \in I$, then $f'(c) = \lim\limits_{x \to c} \frac{f(x)-f(c)}{x-c}$ exists.

Let $x \neq c, \forall x \in I$ and consider $f(x) - f(c)$

$= f(x) - f(c) = \frac{f(x)-f(c)}{x-c}(x - c)$

Applying limits, $\lim\limits_{x \to c}(f(x) - f(c)) = \lim\limits_{x \to c}(\frac{f(x)-f(c)}{x-c}(x - c))$

$\Rightarrow \lim f(x) - \lim f(c) = \lim(\frac{f(x)-f(c)}{x-c}) \lim(x - c)$

$\because \lim(\frac{f(x)-f(c)}{x-c})$ exists and $\lim(x - c) = 0 \therefore \lim f(x) - \lim f(c) = 0$

$\lim f(x) = \lim f(c)$. Hence $f(x)$ is continuous at c

Caratheodory theorem: if $f : I \to \mathbb{R}$ and $c \in I$, then f is differentiable at c if and only if $\exists\ \phi(x) : I \to \mathbb{R}$ continuous at c and satisfies the equation $f(x) - f(c) = \phi(x)(x - c)$

($\Rightarrow$) Let $\phi(x) = \begin{cases} f(x) - f(c) & x \neq c \\ f'(c) & x = c \end{cases}$

Verifying continuity at $x = c$,

$\lim\limits_{x \to c} \phi(x) = \lim\limits_{x \to c} \frac{f(x)-f(c)}{x-c} = f'(c) = \phi(c)$

Verifying if $\phi(x)$ satisfies the equation,

if $x = c$, then $f(c) - f(c) = f'(c)(c - c) \Rightarrow 0 = 0$

if $x \neq c$, then $f(x) - f(c) = \frac{f(x)-f(c)}{x-c}(x - c) \Rightarrow 0 = 0$

($\Leftarrow$)$\exists\ \phi(x) : I \to \mathbb{R}$ continuous at c and satisfies the equation $f(x) - f(c) = \phi(x)(x - c)$. We have to show $f'(c)$ exists.

$f'(c) = \lim\limits_{x \to c} \frac{f(x)-f(c)}{x-c} = \lim \phi(x) = \phi(c) = f'(c)$

Chain rule: Let $I, J \subseteq \mathbb{R}$ st $g : I \to \mathbb{R}$ and $f : J \to \mathbb{R}$ st $f(J) \subseteq I$ and $c \in J$, then $(g \cdot f)'(c) = g(f(c)) \cdot f'(c)$

Rolle's Theorem: Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) st $f(a) = f(b) = 0$, then $\exists$ at least one point $x \in (a, b)$ st $f'(c) = 0$

Mean Value Theorem: $f : I \to \mathbb{R}$ continuous on $[a, b]$ and differentiable on (a, b), then $\exists c \in (a, b)$ st $f(b) - f(a) = f'(c)(b - a)$

Consider the straight line joining $(a, f(a))$ and $(b, f(b))$, i.e.
$(y - y_1) = m(x - x_1)$ where $m = \frac{f(b)-f(a)}{b-a}$
$\Rightarrow y - f(a) = \frac{f(b)-f(a)}{b-a}(x - a)$
$\Rightarrow y = f(a) + \frac{f(b)-f(a)}{b-a}(x - a)$
Let $\phi(x) = f(x) - y$. Since $f(x), y$ are continuous and differentiable, then $\phi(x)$ is continuous and differentiable on their respective intervals.
$\phi(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x - a)$
Since $\phi(a) = \phi(b) = 0$, therefore we can apply Rolle's theorem, i.e $\phi'(c) = 0$. Evaluating,
$\phi'(c) = f'(c) - 0 - \frac{f(b)-f(a)}{b-a} = 0$
$f(b) - f(a) = f'(c)(b - a)$

Taylor Series: Let $f : I \to \mathbb{R}$ be a function on $I = [a, b]$ st $f^{(n)}, \forall n \in \mathbb{N}$ are continuous on I and $f^{(n+1)}$, exists on (a, b). If $x_0 \in I$, then $\forall x \in I, \exists c \in (x, x_0)$ st $f(x) = P(x) + R_2(x)$ where

$$P(x) = f(x_0) + \cdots + \frac{f^{(n)}(x_0)(x - x_0)^n}{n!}$$

and

$$R_2(x) = \frac{f^{(n)}(c)(x - x_0)^{n+1}}{(n+1)!}$$