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Assortativity in complex networks

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We survey the concept of assortativity, starting from its original definition by Newman in 2002. Degree assortativity is the most commonly used form of assortativity. Degree assortativity is extensively used in network science. Since degree assortativity alone is not sufficient as a graph analysis tool, assortativity is usually combined with other graph metrics. Much of the research on assortativity considers undirected, non-weighted networks. The research on assortativity needs to be extended to encompass also directed links and weighted links. In addition, the relation between assortativity and line graphs, complementary graphs and graph spectra needs further work, to incorporate directed graphs and weighted links. The present survey paper aims to summarize the work in this area and provides a new scope of research.

Keywords: assortativity; network; graph; graph spectrum; robustness; degree; degree correlation.

1. Introduction

In this survey paper, we provide an overview of *assortative mixing* in complex networks. The concept of assortativity was introduced by Newman [1] in 2002 and is extensively studied since then. Assortativity is a graph metric. It represents to what extent nodes in a network associate with other nodes in the network, *being of similar sort* or *being of opposing sort*. Generally, the assortativity of a network is determined for the *degree* (number of direct neighbours) of the nodes in the network. The concept of assortativity may, however, be applied to other characteristics of a node as well, such as node weight, coreness, node betweenness, kth level node degree (number of nodes that can be reached in no more than k hops; also known as *expansion*) etc. In addition, assortativity may be applied to node characteristics that are not directly topology-related, such as race or language (see, e.g. Quayle *et al.* [2] and Nagoshi *et al.* [3]).

Assortativity is expressed as a scalar value, ρ, in the range $-1 \leqslant \rho \leqslant 1$. Degree assortativity is identified as ρ_D. A network is said to be *assortative* when high-degree nodes are, on average, connected to other nodes with high-degree and low-degree nodes are, on average, connected to other nodes with low degree. A network is said to be disassortative[1] when, on average, high-degree nodes are connected to nodes with low(er) degree and, on average, low-degree nodes are connected to nodes with high(er)

[1] The term *disassortative* is also used in some publications.

degree. Assortativity provides information about the structure of a network, but also about dynamic behaviour of the network and robustness of the network, such as random or targeted attack and virus spread. Considering only the degree distribution of the network does, generally, not provide sufficient information about the network.

The original definition of assortativity (Newman [1]), for non-weighted, non-directed networks, is based on the correlation between random variables. We define the linear correlation coefficient between two random variables X and Y as follows:

$$\rho(X,Y) = \frac{E[XY] - \mu_X \mu_Y}{\sigma_X \sigma_Y} = \frac{E[(X - E[X])(Y - E[X])]}{\sqrt{E[(X - E[X)^2]E[(Y - E[Y])^2]}} \tag{1}$$

where μ_X and μ_Y are the mean of X and Y, respectively, $E[XY] - \mu_X \mu_Y$ is the covariance of X and Y and σ_X and σ_Y are their respective standard deviation. To derive a definition of degree correlation, assortativity, we apply the following approach. We randomly select a link l in the graph. The link connects two nodes: a start node, denoted l_- and an end node, denoted l_+. The degree D of l_- is denoted by D_{l-} and the degree of l_+ is denoted by D_{l+}. Newman [1] derived, from Equation (1), the linear degree correlation coefficient:

$$\rho_D = \frac{\sum_{jk} jk(e_{jk} - q_j q_k)}{\sigma_q^2} \tag{2}$$

where

e_{jk} is the joint-remaining degree probability for remaining degree j and remaining degree k (the remaining degree of a node is equal to the degree of that node minus one);

$q_k = \frac{(k+1)p_{k+1}}{\sum_j jp_j}$ is the normalized distribution of the remaining degree D_r of a randomly selected node;

σ_q is the standard deviation of q_k.

The assortativity is quantified by the Pearson correlation coefficient (Van Mieghem [4], Chapters 2 and 4) of X and Y, whereby X and Y are the remaining degree at the end of a link. The division by σ_q^2 (variance of q) serves to normalize the assortativity in the range $[-1, 1]$. The rationale for basing the degree assortativity on the *remaining degree* $(= d_i - 1)$ of node i rather than on the *degree* of node i, d_i, is that the tendency for two nodes i and j to connect is determined from the moment that these nodes i and j are not yet connected, i.e. have degree $d_i - 1$ and $d_j - 1$. We also refer to Van Mieghem [4] for a further description of the linear correlation coefficient.

Assortativity has, since its inception in 2002, been studied extensively, notably by Newman [5,6], D'Agostino *et al.* [7], Chang *et al.* [8], Estrada [9], Holme *et al.* [10], König *et al.* [11], Leung *et al.* [12], Litvak *et al.* [13], Liu *et al.* [14], Manka-Krason and Kulakowski [15], Nguyen [16], Piraveenan *et al.* [17–23], Van Mieghem *et al.* [24,25], Wang *et al.* [26], Winterbach *et al.* [27], Xia *et al.* [28], Xu *et al.* [29], Xulvi-Brunet *et al.* [30], J. Zhou *et al.* [31] and D. Zhou *et al.* [32].

The paper is organized as follows. After a brief introduction on graph analysis in Section 2, Section 3 presents a literature survey. Section 4 lays the theoretical foundation for the concept of assortativity, including various examples. Although the foundation considers directed networks (networks with directed links), the examples in this section comprise undirected networks only. Section 5 applies assortativity to networks with non-weighted, directed links. This was studied by, among others, Foster *et al.* [33] and Piraveenan *et al.* [20]. Assortativity for *weighted* networks is, thus far, not sufficiently explored. Section 6 studies the relation between assortativity and graph spectra. Here, we refer to the work by Van Mieghem *et al.* [24,25]. Section 7 studies methods for influencing the network's assortativity, such as degree-preserving rewiring (DPR). A network's assortativity may be modified for a specific reason, such as increasing the network's robustness (e.g. resilience against link removal) or to mitigate the effect of virus spread through the network. Section 8 looks into the relation between assortativity and line graphs. This relation was studied by, among others, Liu *et al.* [14] and Manka-Krason and Kulakowski [15]. Section 9 studies the relation between assortativity and complementary graphs, studied by Wang *et al.* [26]. In Section 10, we study the concept of local assortativity. We refer to the work of Piraveenan *et al.* [18]. Section 11 contains conclusions and provides directions for future work.

2. Graph analysis

Extensive description of graph analysis can be found in Van Mieghem [4], part III, and in Van Mieghem [34]. We represent a network as a graph $G(\mathcal{N},\mathcal{L})$, with node set $\mathcal{N}$ of $N = |\mathcal{N}|$ nodes and with link set $\mathcal{L}$ of $L = |\mathcal{L}|$ links.[2] The graph may be represented through its adjacency matrix A of size $N \times N$, with elements

$$\begin{aligned} a_{ij} &= 1, \text{ when a link exists between node } i \text{ and node } j \\ &= 0, \text{ when no link exists between node } i \text{ and node } j \end{aligned}$$

We assume that no self-loops exist (hence $a_{ii} = 0$) and no overlapping links, i.e. there cannot be more than one link between i and j. Such a graph is known as a *simple graph*. For undirected graphs, links have no direction. For undirected graphs A is symmetrical, i.e. $a_{ij} = a_{ji}$ and $A = A^\top$. For directed graphs, where links have a direction, we have $a_{ij} \neq a_{ji}$, for $i \neq j$. For example, when a link exists from node i to node j, but not in the other direction, then $a_{ij} = 1$ and $a_{ji} = 0$. Graphs may comprise *weighted links*, in which case a link has a *weight* associated with it. Usually, this *weight* w_{ij} is expressed by a non-negative real number.[3] The corresponding weighted adjacency matrix W contains the weights w_{ij} instead of a_{ij}.

We study assortativity in graphs of various classes, whereby the class constitutes a description of the topology of the network. The degree sequence of a network alone cannot be considered as a comprehensive characterization of that network, which is one of the reasons for Newman to introduce the concept of assortativity. Well-known network models (classes) include:

- Erdős–Rényi (ER) random graph [35,36]. A graph of the class ER comprises a set of N nodes. Nodes in the network are connected by a link with probability p. The presence of a link between a node pair is stochastically independent of the presence of a link between any other node pair. The ER network has binomial degree (D) distribution:

$$\Pr[D = k] = \binom{N-1}{k} p^k (1-p)^{N-1-k} \tag{3}$$

[2] Literature also uses the terminology *vertex* to refer to a node and *edge* to refer to a link; the notation $G(V, E)$ is used, were V is the set of vertices and E is the set of edges.
[3] When the graph represents an electrical circuit comprising resistive components, capacitive components and inductive components, then w_{ij} may be a complex variable.

For large N and $\lambda = p(N-1)$ is independent of N, the degree distribution Equation (3) evolves into a Poisson distribution (Van Mieghem [4], Chapter 2):

$$\Pr[D=k] = e^{-\lambda}\frac{\lambda^k}{k!} \tag{4}$$

- Barabási–Albert (BA) random graph [37]. A graph of the class BA is a *growing* network model. The network is built from a starting graph, normally a complete graph with N_0 nodes, to which iteratively a node is added. Each newly added node is connected to m existing, randomly selected nodes. The probability of attachment to a randomly selected node is proportional to the degree probability of that existing node, which explains the term *preferential attachment*.

  For large N, a BA graph has power law (scale-free) degree distribution:

  $$\Pr[D=k] = ck^{-3} \tag{5}$$

  A scale-free graph generalizes Equation (5) to a power-law degree distribution:

  $$\Pr[D=k] = ck^{-\gamma} \tag{6}$$

  whereby $\gamma > 1$, although, for most real-world networks, $1.2 \leqslant \gamma \leqslant 3.5$ and where c is a normalization factor so that $\sum_{k=1}^{N-1}\Pr[D=k] = 1$.

- Watts–Strogatz (WS) small world graph [38]. A WS graph is an *evolving* graph. The graph is constructed from a regular graph, such as (typically) a ring lattice. An iterative rewiring process is applied to rewire each link, with probability p_r, to a randomly selected other node. The term 'small world' is used since networks of this class exhibit shortest paths with small average hop count $E[H]$. At the same time, the clustering coefficient, C, of the graph remains high. (C is a measure of the number of triangles in the graph relative to the number of triplets in the graph [4].) When p_r increases towards 1, the resulting network will be close to an ER network.

  The degree of the WS graph is centred around the degree of the non-rewired nodes. When we consider a WS graph constructed from a ring lattice without rewiring ($p_r = 0$), the degree of each node is identical, as each node is connected to the same number of neighbours. When the rewiring probability p_r increases or when the number of rewiring cycles c increases, the graph becomes more random and the degree distribution evolves towards a binomial distribution (or Poisson distribution, depending on N and L). For $p_r = 0$, we have the average hop count of the shortest path as a function of N, $E[H_N] = O(N)$. When p_r becomes sufficiently large, the average hop count of the shortest path becomes $E[H_N] = O(\log N)$.

- Callaway growing network [39]. The Callaway graph class is, just like the BA graph class, a growing network model. The graph is built from an initial complete graph with small N_0, e.g. 5. Then nodes are added iteratively. For every added node, a link is added between two nodes. The two nodes are selected randomly from the set of existing nodes and the newly added node. As such, a Callaway network model may be considered a variant of the BA network model. The degree distribution has an exponential form.

- Lattice (e.g. ring lattice, spherical lattice, square lattice, cubic lattice). The lattice is a regular structure, whereby each node is connected to a defined number of neighbours. However, for a square lattice, for example, nodes at the edge of the network structure have smaller degree.

- Bi-partite network (generalized to k-partite network). The bi-partite network comprises one subnetwork N with n nodes and another subnetwork M with m nodes. Links exist between nodes in N and nodes in M, but not between nodes in N or between nodes in M.
- A star network is a special case of a (N, M) bi-partite network; it has $N = 1$.
- A tree network is any connected graph with $L = N - 1$ links.

Complex networks of a particular class have certain qualifying characteristics, one of them being the degree distribution. Network rewiring modifies the topology of the network in order to change certain characteristic of the network. The rewiring may have the effect that the network is transformed from one graph class to another graph class. When we consider, for example, an observed network that is classified as a BA graph, and we apply random rewiring, the network will gradually become a network possessing the characteristics of an ER random network. Rewiring is described in more detail in Section 7.

The degree D of a randomly selected node in the graph represents the number of direct neighbours. The *degree vector* is represented by $[d_1, d_2, \ldots, d_N]^{\mathrm{T}}$. The degree probability density function, PDF, is defined as $f_D(k) = \Pr[D = k]$. The joint degree PDF, being the probability that two randomly selected nodes have two specific degree values, is defined as $f_{D1D2}(k, l) = \Pr[D_1 = k, D_2 = l]$. When the random selection of the first node is stochastically independent of the random selection of the second node, then $f_{D1D2}(k, l) = f_D(k) \cdot f_D(l)$. Assortativity relates to the joint degree distribution for links in the network. When randomly selecting a link l in a graph, the degree of the node on one end of the link, d_{l-}, is not stochastically independent of the degree of the node on the other end of the link, d_{l+}. Assortativity is a measure of the extent to which D_{l-} and D_{l+} are correlated for a network.

The study of assortativity often comprises network transformation. The assortativity of a network may be increased or decreased by a network modification such as link addition, link removal, link rewiring and DPR. DPR is described in Section 7.

3. Literature survey

When we consider an ER graph, the existence of a link between two nodes has no relation to the degree of these respective nodes. When considering a BA graph, we observe that the preferential attachment of a link between a newly added node and an existing node is a function of the degree of that existing node. For a BA graph with sufficiently large network size, we observe that, on average, for each node i, the distribution of the degree of the nodes connected to node i, is independent of the degree of node i itself; i.e. no degree correlation exists.

Newman observed that in many real-world networks the degree of the nodes connected to a randomly selected node i has a relation with the degree of that node i itself. In other words $\Pr[D_j = k|_{j\sim i}] \neq \Pr[D = k]$, $j \sim i$ indicating the existence of a link between i and j and D_j being the degree of a randomly selected neighbour j. Two networks with identical degree vector may have different assortativity. Some empirically found assortativity values, as well as simulation results, are shown in Table 1; [1] and [5] provide further assortativity values of actual networks.

Physics authors and mathematics authors, albeit the latter to a lesser extent, tend to publish articles with others that have equally high or equally low number of publications, i.e. forming an assortative network. For connections on the Internet, we note that highly connected Autonomous Systems (AS) are connected to other AS's that themselves have few(er) connections, i.e. forming a disassortative network. The same observation applies for undirected hyperlinks between Web pages. For experimental (i.e. generated) ER and BA networks, the assortativity is ~ 0.

TABLE 1 *Assortativity for value for different networks*

Network	Assortativity
Physics coauthorship [40]	0.363
Mathematics coauthorship [41]	0.120
Company directors (see [42] for reference)	0.276
Connections between autonomous systems on the Internet [43]	−0.189
World Wide Web (see [5] for reference)	−0.067
Undirected hyperlinks between Web pages in a single domain [37]	−0.065
Neural network (see [42] for reference)	−0.163
Experimental ER graph (for sufficiently large network size)	∼0
Experimental BA graph (for sufficiently large network size)	∼0

Assortativity has a direct relation with the robustness of the network, in terms of connectivity of the network. A failure of, or targeted attack on a high-degree node in an assortative network would leave other high-degree nodes connected to one another. This minimizes the chance of the network as a whole to become disconnected. In a disassortative network, high-degree nodes are less connected to one another. Many paths between nodes in the network are dependent on high degree node(s). Failure of a high-degree node in a disassortative network would hence have more impact on the connectedness of the network.

Newman [5] provides a general exploration of assortativity, applied on various kinds of network and using various node characteristics based on which assortative mixing may occur.

Extensive exploration of the concept of assortativity has been done by Piraveenan [21]. Piraveenan studies, among others, the existence of networks that are *perfectly assortative* or *perfectly disassortative*. In a perfectly assortative network, all nodes are connected to other node(s) of the same type, e.g. same degree. One example is a complete network, where all nodes are connected to all other nodes and all nodes have degree $d_i = N - 1$. For such network, degree assortativity is maximal, $\rho_D = 1$. If the network comprises nodes with different degree, then perfect degree assortativity is still possible. Perfect degree assortativity would be reached when the network is fragmented in sub-networks, whereby each subnetwork itself constitutes a complete network. Perfect disassortativity is more difficult to achieve. One class of network that is determined [25] to be perfectly disassortative is the complete bipartite graph, $K_{m,n}$, with $m \neq n$. A star graph is an example of complete bipartite graph, $K_{m,1}$. Piraveenen *et al.* [18] associate assortativity also with *information content* of a network. Networks which are degree assortative or degree disassortative have higher information content than networks which are degree non-assortative. When considering a random node i of a degree assortative or degree disassortative network, we know what degree(s) to expect for the nodes connected to node i. When considering a random node i of a degree non-assortative network, we have no expectation of the degree(s) of the nodes connected to node i. This relation between assortativity and network information was also observed by Sole and Valverde [44]. The information content related to the link set of a graph is defined in [19] as follows:

$$I(q) = \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} e_{j,k} \log \frac{e_{j,k}}{q_j q_k} \tag{7}$$

where

$j, k \in N$;

q_j, q_k probability of remaining degree j, k (the 'remaining degree' is the number of links leaving the node other than the link on which we arrived, i.e. for node i, the remaining degree is $d_i - 1$);

$e_{j,k}$ probability that a randomly selected link l connects a node with remaining degree j and a node with remaining degree k; hence $e_{j,k} = \Pr[\{D_{l-} - 1 = j\} \cap \{D_{l+} - 1 = k\}]$.

For scale-free networks, we replace in Equation (7) N by N_p, N_p being the cut-off point for scale-freeness of the degree distribution. It follows from Equation (7) that the information content of a network is influenced by the ratio between the joint degree probability $e_{j,k}$ and the product of the respective probabilities q_j and q_k. By changing the topology of a network through, e.g. rewiring, we can increase or decrease the information content of that network. Piraveenan *et al.* [19] have observed a direct relation between the information content of a network and that network's (dis)assortativity.

Assortativity does not reveal information about individual nodes. A network with a given assortativity comprises nodes that contribute to this assortativity. Not all nodes contribute equally to the network's assortativity. A network that is non-assortative *overall* may comprise nodes that are themselves highly assortative, e.g. comprise high-degree nodes that connect to other high-degree nodes. Networks with equal assortativity may have different distribution of the *assortativity contribution* per node. The assortativity contribution is referred to as *local assortativity* [21] or *node assortativity*. Local assortativity is further explored in Section 10.

Assortativity for networks with weighted links is studied by Chang *et al.* [8]. The rationale is that in many networks, the link between two nodes may be weak or strong, i.e. may have a weight, reflecting aspects such as data transfer capability, data transfer cost or length. Open shortest path first (OSPF)-based IP networks is one practical example where links have a defined weight, namely the data transfer capacity between two connected OSPF routers.[4] The degree of a node is no longer a discrete quantity, but rather a set of real variables. Chang *et al.* [8] propose that the *Strength* of a node is the sum of the weights of the links connected to that node. Hence, assortativity for these networks relates to the tendency of nodes with the same or opposing strength to be connected to one another. This definition of node strength is also applied by Wang *et al.* [45], which studies, among others, the distribution of the link weight for a single node. Assortativity for networks with weighted links[5] is also studied by Leung and Chau [12]; the weighted assortativity, ρ_w, is proposed in [12]. When considering a network with weighted links, we can still calculate the (non-weighted) assortativity by ignoring the weight of the links. It is then observed (e.g. in [12]) that ρ_w and ρ_D can differ substantially for a network. One may, however, question the validity of ignoring the link weights in a weighted network. A link with small weight has in that case the same connection value as a link with high weight, while these links may contribute significantly differently towards the network's robustness and other network characteristics.

König *et al.* [11] present a network *transformation model* whereby a stationary, non-assortative or disassortative network migrates towards an assortative network. The network transformation includes a combination of link addition and link removal (decay of existing links). The motivation of this model is that nodes in the network, e.g. a human interaction network or a technological network, are constrained in the number of links that it can maintain. This model is distinctively different from the network growth models from, e.g. Catanzaro *et al.* [40] and Piraveenan *et al.* [23], for generating a network with a given assortativity. For these models, a network grows through the addition of nodes and links, according to some rule. These models do not define a constraint in the number of links incident to a node. Neither do they consider the removal of links.

[4] This data transfer capability is not only dependent on the physical characteristics of the Ethernet cable, but also on the capability of the IP interfaces on the OSPF router.

[5] Newman's original definition of assortativity does not consider weighted links.

Litvak and Van der Hofstad [13] observe that networks which are inherently degree disassortative, such as the Internet, show a decreasing degree disassortativity as the network size N increases, i.e. ρ_D moves towards 0. It is shown that the assortativity of the network is influenced by the distribution of the degrees of the nodes in the network. A broad distribution of degrees (range of degree values) has a decreasing effect on the assortativity value. For large networks, the degree distribution will on average be broader than for small networks. To mitigate this apparent shortcoming of Newman's assortativity definition, Litvak and Van der Hofstad [13] propose the rank correlation as an alternative method for calculating degree–degree correlation. The rank correlation is defined as follows:

$$\rho_n^{rank} = \frac{\sum_{i=1}^{n}(r_i^X - (n+1)/2)(r_i^Y - (n+1)/2)}{\sqrt{\sum_{i=1}^{n}(r_i^X - (n+1)/2)^2 \sum_{i}^{n}(r_i^Y - (n+1)/2)^2}} \tag{8}$$

whereby X and Y are random variables, representing the degree at either end of a randomly selected link in the network. The variables r_i^X and r_i^Y are the rank of an observation X_i and Y_i, respectively, for the case that the sample values $(X_i)_{|i=1,\ldots,n}$ and $(Y_i)_{|i=1,\ldots,n}$ are ranked in descending order. The rank correlation is based on the classical Spearman's rho[6] measure [46] (as opposed to assortativity, which is based on the Pearson correlation coefficient (Van Mieghem [4], Chapter 2). Litvak and Van der Hofstad [13] show that the ranking definition allows for uncovering disassortativity in networks even for large N and continue to propose that the rank correlation should, along with assortativity, be a standard tool for complex network analysis, specifically networks that are by nature scale-free in terms of degree distribution.

Note For Newman's definition of assortativity Equation (1), Equation (2), based on the Pearson correlation coefficient, graph theoretic notation exists Equation (14), Equation (15), Van Mieghem [25]. For the rank correlation, based on the Spearman rho measure, such graph theoretic notation does not exist. When performing graph analysis, the theoretic notation of the assortativity is far easier to work with than the original definition. This makes the assortativity, for practical purposes, a more attractive metric than the rank correlation.

Holme and Zhao [10] view assortativity from a different perspective. Holme and Zhao [10] define a null model for a network, formed by the ensemble $G(G)$, whereby each element of G constitutes a network G' with the same degree sequence as G. G comprises all G' that may be formed from G, with the same degree sequence. All elements from G are placed in two-dimensional space formed by the assortativity and the clustering coefficient [4,47]:

$$C = 3\frac{n_{\text{triangle}}}{n_{\text{triplet}}} \tag{9}$$

where n_{triangle} is the number of triangles (=closed triplet, also denoted as $\blacktriangle_G$) and n_{triplet} is the number of connected triplets (subgraph consisting of three nodes, connected through two or three links). Traversing the entire G, i.e. generating all elements G' of G, is non-trivial. Several techniques exist for transforming a network for increasing or decreasing assortativity. Winterbach *et al.* [27] study a *greedy* DPR approach. Van Mieghem *et al.* [25] and Noldus and Van Mieghem [48] apply a targeted approach for selecting links to be rewired in order to affect the assortativity, without affecting the degree sequence.

[6] We write 'rho' instead of ρ, to prevent confusion with assortativity.

Placing the elements from G into said two-dimensional space formed by ρ_D (horizontal axis) and C (vertical axis) provides the null model. When analysing an observed network G, whereby assortativity and clustering coefficient of that network are calculated, the network can be pegged into its corresponding position in the assortativity-clustering space. We can then determine how far the assortativity of the network G may be increased, or decreased, and what clustering coefficient may be attained.

A different two-dimensional metric space may be devised, such as assortativity versus average shortest path or assortativity versus effective graph resistance. Each such two-dimensional metric space, serving as null model for the observed network, may be used to interpret the potential assortativity range of the network. Put differently, it visualizes the extent to which an increase or decrease in assortativity of the network may affect the other metric.

The concept of a null-model in a graph is applied also by Maslov and Sneppen [49]. Their null-model G_{null} of a graph G is formed by a node set N, whereby each node in N has the same degree as the corresponding node in G. Hence, the degree of each node in G_{null} is identical to the degree of the corresponding node in G. Links, however, are randomly distributed. G_{null} may be generated by random rewiring of all the links. G_{null} will have a probability $\mathrm{P}_{\text{null}}[D_i = k, D_j = l]$, whereby i and j are the nodes at the end of a randomly selected link and k and l their respective degrees. For an observed network, the joint-degree probability of a randomly selected link is defined as $\mathrm{P}[D_i = k, D_j = l]$. Visualization (colour plotting) of the ratio $\mathrm{P}[D_i = k, D_j = l]/\mathrm{P}_{\text{null}}[D_i = k, D_j = l]$ shows (dis)assortative, or non-assortative, trend for nodes of varying degree. A further interesting observation by Maslov and Sneppen [49] is that when considering all nodes j that are neighbour of a specific node i, certain kinds of networks exhibit a power-law ratio between $\mathrm{E}[D_j]$ and D_i for a node i with $D_i = k$. Specifically, nodes i with high D_i have low $\mathrm{E}[D_j]$. This is observed for the Internet and for protein networks. This characteristic of a network will, on average, curb the spread of deleterious information/data.

Estrada [9] introduces a method for determining whether a network is assortative, by inspecting the following structural characteristics of the network: (a) clustering coefficient (also known as transitivity), (b) intermodular connectivity and (c) branching. Assortativity is rewritten in [9] into an expression containing, among others, the clustering coefficient, the intermodular connectivity and the branching. By inspecting the relation between these three network metrics, it can be determined whether the network is assortative, disassortative or non-assortative. It is shown that both clustering coefficient and intermodular connectivity have a positive effect on assortativity, while branching has a negative effect on assortativity. Estrada's result [9] corresponds to the fact that a correlation is observed, empirically, between clustering coefficient and assortativity, as well as between modularity and assortativity. Networks with high modularity and high clustering coefficient are normally assortative. This is also observed by Youssef *et al.* [50].

Whereas assortativity is presented as a single value in the range $[-1, 1]$, the assortativity of a network is, in a way, a representation of a characteristic that may differ for each node in the graph. Each node i in the graph has a certain connectivity value d_i, i.e. the degree of the node. Each neighbour j of node i itself also has a connectivity d_j. For each node i, the average degree of its neighbours can be determined. In this manner, we can calculate the average of d_j, $\mathrm{E}[D_j]$ as a function of d_i. This approach is studied by Pastor-Satorras *et al.* [42]. For assortative networks, $\mathrm{E}[D_j]$ will increase for increasing d_i, whilst for disassortative networks, $\mathrm{E}[D_j]$ will decrease for increasing d_i. For non-assortative networks, $\mathrm{E}[D_j]$ will remain constant for increasing or decreasing d_i.

Li *et al.* [51] propose a network metric that is related to assortativity, namely *Likelihood*. Likelihood,[7] which we denote as Lh, considers the degree of adjacent nodes. The likelihood of a graph

[7] Li *et al.* use L to denote likelihood, but that will confuse with number of links in a graph, so we use Lh instead.

$G(N, L)$ is defined as follows:

$$\mathrm{Lh}(G) = \sum_{(i,j)\in L} d_i d_j \tag{10}$$

The likelihood corresponds to the second Zagreb index, as defined by Gutman and Trinajstic [52]. To compare graphs of different class and size, the likelihood definition as given in Equation (10) is normalized within the range of $\mathrm{Lh}(G)_{\min}$ and $\mathrm{Lh}(G)_{\max}$, yielding the *normalized likelihood* $\mathrm{Lh}_{\mathrm{norm}}(G)$.

$$\mathrm{Lh}_{\mathrm{norm}}(G) = \frac{\mathrm{Lh}(G) - \mathrm{Lh}(G)_{\min}}{\mathrm{Lh}(G)_{\max} - \mathrm{Lh}(G)_{\min}} \tag{11}$$

The extreme values $\mathrm{Lh}(G)_{\min}$ and $\mathrm{Lh}(G)_{\max}$ relate to a graph of particular size N and degree set. The normalized likelihood has a value in the range [0,1]. Calculating $\mathrm{Lh}_{\mathrm{norm}}(G)$ requires that $\mathrm{Lh}(G)_{\min}$ and $\mathrm{Lh}(G)_{\max}$ are calculated. Li *et al.* [51] show how $\mathrm{Lh}(G)_{\min}$ and $\mathrm{Lh}(G)_{\max}$ can be calculated. For a change in network topology, such as DPR, the absolute change in $\mathrm{Lh}_{\mathrm{norm}}(G)$ can easily be recalculated, as the product $d_i d_j$ changes only for the nodes involved in the rewiring, and hence $\mathrm{Lh}(G)$ can be recalculated. Since $\mathrm{Lh}(G)_{\max}$ and $\mathrm{Lh}(G)_{\min}$ are constant for constant degree sequence, the $\mathrm{Lh}_{\mathrm{norm}}(G)$ for the rewired graph follows from the $\mathrm{Lh}(G)$ from the rewired graph.

Li *et al.* [51] show a relation between likelihood and assortativity. However, no further experiments are conducted to study how the likelihood and assortativity correlate for a network of particular class and degree sequence. By means of network rewiring, we can vary the assortativity of the network between its minimum and maximum value.

4. Degree assortativity of non-weighted, undirected graphs

Networks may be classified through their degree distribution. When randomly selecting a node of a network, the probability of that node having degree k is defined by the degree PDF, $F_D(k) = \Pr[D = k]$ of that network. Figure 1 shows a number of examples of degree PDF for different classes of graphs.

When we consider a single node in a graph, the probability of that node having a particular degree follows from the degree distribution of that graph. A particular node i with degree $d_i = k$ is connected to k other nodes. Each one of these k other nodes has its own degree, $d_{j|i\sim j} = l$. When taking the average degree of the nodes incident to node i, $\overline{d_j} = \frac{\sum_k d_{j|i\sim j}}{k}$, we can define the relation between d_i and $\overline{d_j}$. Considering that there may be multiple nodes with a specific degree k, we average this relation for all nodes with a specific k. Visualizing this relation reveals the degree correlation in a graph. Specifically, it reveals that for certain graph classes, the average degree of nodes adjacent to i is dependent or not dependent on the degree of i. This is shown in Fig. 2 for an ER graph with $N = 10\,000$ and $p = 0.1$.

In this example, the average degree of nodes j adjacent to node i is stable around 1000, for different k. We may consider (at least) the following degree distributions for a graph:

(1) degree distribution, $\Pr[D = k]$, defining the probability for a randomly selected node to have degree k;

(2) combined-degree distribution, $\Pr[D_1 = j, D_2 = k] = \Pr[D_1 = j|D_2 = k] \cdot \Pr[D_2 = k] = \Pr[D_2 = k|D_1 = j] \cdot \Pr[D_1 = j]$, defining the probability for a randomly selected pair of nodes that do not have to be connected to each other, to have degrees j and k, respectively; presuming that the selection of node 2 is stochastically independent of the selection of node 1, the combined degree distribution can be written as $\Pr[D_1 = j, D_2 = k] = \Pr[D_1 = j] \cdot \Pr[D_2 = k]$;

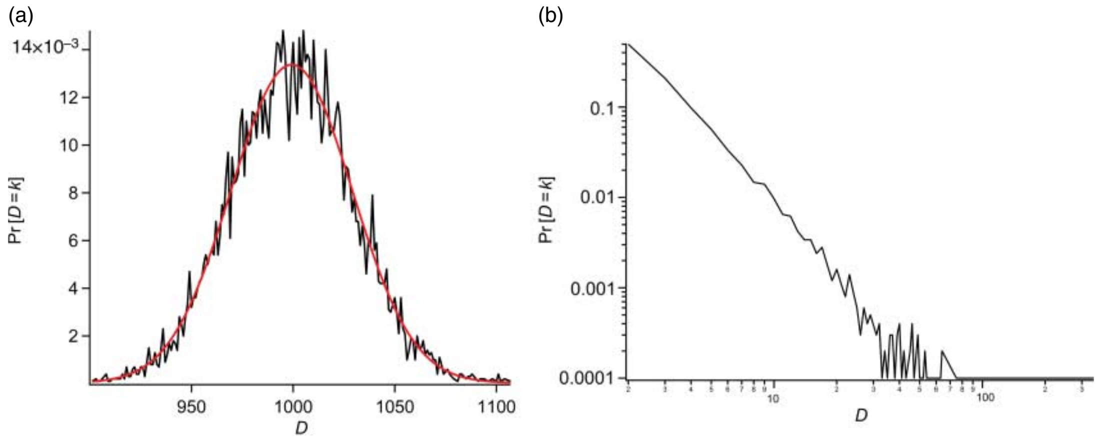

FIG. 1. Examples of degree PDF. (a) Binomial degree distribution. Erdös–Rényi random graph; $N = 10\,000$, $p = 0.1$. (b) Power-law degree distribution. Scale-free graph; $N = 10\,000$, $N_0 = 4$, 2 links added for each new node.

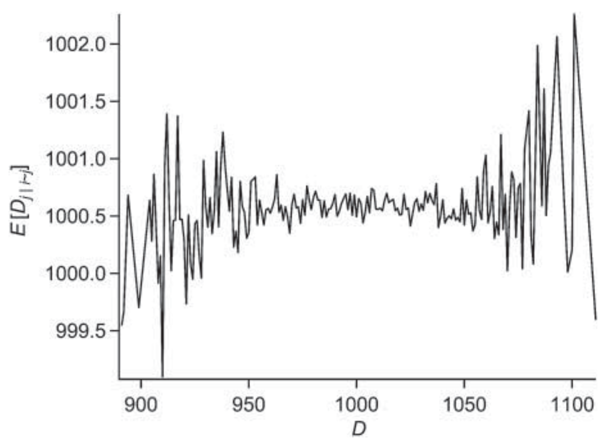

FIG. 2. Degree–degree correlation for ER graph, $N = 10\,000$, $p = 0.1$ and $E[D] = (N-1)p \approx 1000$.

(3) joint degree distribution, $\Pr[D_{l-} = j, D_{l+} = k]$, defining the probability for a randomly selected link l to have degree j on one end of the link (denoted l_-) and to have degree k on the other end of the link (denoted l_+). This relation represents assortativity (degree correlation).

We observe the following:

– *Non-assortative network*. The distribution of the degree of the nodes j connected to a randomly selected node i follows the same distribution as the degree distribution of the network as a whole.

– *Assortative network*. For a randomly selected node i, there will be a concentration of high(er) joint-degree probability for connected nodes j having the *same or similar* degree as i.

– *Disassortative network*. For a randomly selected node i, there will be a concentration of high(er) joint-degree probability for connected nodes j having a degree *different from* the degree of i.

TABLE 2 *Assortativity for random graph, $N = 1000$, varying p*

	Assortativity		
p	$r_{\min}$	$r_{\max}$	r_{average}
0.0009210 ($= p_c$)	−0.163	0.181	−0.005
0.009210 ($= 10p_c$)	−0.052	0.047	−0.002
0.09210 ($= 100p_c$)	−0.018	0.015	−0.002
0.1	−0.018	0.013	−0.002
0.2	−0.013	0.008	−0.002
0.5	−0.007	0.003	−0.002

Whereas the degree distribution of a network is considered a first-order metric for characterizing the network, assortativity is considered as a second-order metric. The relevance of assortativity is strongly related to the assortativity range, $\rho_{\max} - \rho_{\min}$, for that network, whilst keeping the degree distribution of that network unaffected. A network may be transformed, through link rewiring, whereby the degree distribution is not changed (DPR). With DPR, the degree of the involved nodes is not affected. When applying DPR on a dense network, the assortativity of that network will vary between $\rho_{\min} - \rho_{\max}$, whereby $\rho_{\max}$ and $\rho_{\max}$ will be specific for this network, with this specific degree distribution. The range of ρ_D for the sparse network is found to be larger than the range of ρ_D for the dense network. A large range of ρ_D implies greater relevance of assortativity as second-order metric for the network, compared with a small range of ρ_D.

We will now study the assortativity of various classes of networks. The four classes of graphs that are considered are: (a) ER random graph, (b) BA scale-free graph random graph, (c) WS small-world random graph and (d) Callaway random growth model.

4.1 *ER random graph*

We expect a degree assortativity $\rho_D \approx 0$ for the ER class of random graph. The reason is that the presence of a link between two nodes is independent of the presence of links between these nodes and other nodes. There is no dependency between the degree of a node and the probability that there is a link between that node and another node of particular degree. Table 2 provides the assortativity for an ER class of graph with varying p (link probability). We vary p from the threshold link density[8] p_c to 0.5. For each ER graph $G_p(N)$, the assortativity has been calculated, using Equation (15), for 10 000 instances of that $G_p(N)$.

Figure 3 shows the PDF (including curve fitting), distributed over 250 bins, of the assortativity for some of the graph models.

Figure 3 shows that the assortativity for the ER network is distributed around 0. Hence $\mathrm{E}[\rho_D] = 0$. For a higher value of p, the ρ_D is distributed over a smaller range.

4.2 *BA scale-free graph random graph*

We consider the growth model as defined by Barabási & Albert (BA) [37]. For the BA graph, a network is constructed from a small (e.g. $N_0 = 3$) complete network. The nodes in the starting graph, as well

[8] $p_c \approx \ln(N)/N$ (asymptotically), for large N, is the critical link probability for ER graph, above which the graph will almost surely be connected (Erdös & Rényi [36], [9]; [4]).

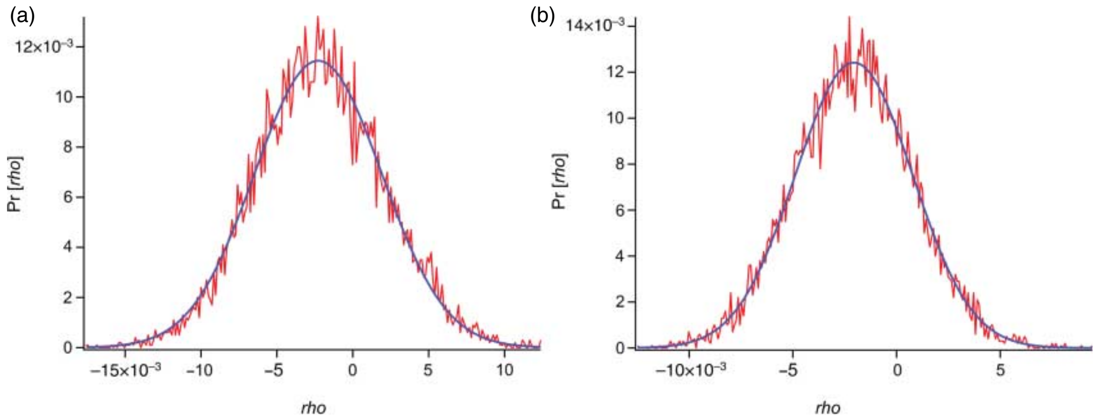

FIG. 3. Assortativity (rho) PDF for ER random graph. (a) $N = 1000$, $p = 0.1$. (b) $N = 1000$, $p = 0.2$.

TABLE 3 *Assortativity for BA graph, $N = 1000$*

Graph definition		Assortativity		
N_0	m	$r_{\min}$	$r_{\max}$	r_{average}
3	2	−0.147	−0.038	−0.092
4	2	−0.158	−0.038	−0.089
5	2	−0.135	−0.038	−0.084
10	2	−0.116	−0.006	−0.064
10	3	−0.093	−0.018	−0.055
10	5	−0.078	−0.008	−0.046

as the nodes that were added in the earlier part of the growth process have a higher expected degree. During the growth process, these nodes have a higher *attraction* for the new nodes to be added, due to the higher degree of these nodes. The degree of these nodes, therefore, tends to increase more than the degree of the nodes that were added later in the growth process. Put differently, when one node has, through the random node selection of the growth model, a higher degree than the other nodes in the network, that node tends to continue to attract more new connections than the other nodes.

An intuitive thought (1) might be that BA graphs would show disassortativity. The reason is that every time a new link is added, the link is placed between the newly added node n, which has degree $0 \leqslant d_n < m$ (m being the number of links that are added for each newly added node) just prior to the attachment, which is by definition a low degree, and the existing node, which is likely a node with relatively high degree, resulting from the preferential attachment. Placing a link between a low-degree node and a high-degree node is expected to make the graph as a whole more disassortative.

Table 3 provides the assortativity range for various BA graphs. The network is built up to $N = 1000$, starting from a complete graph with $3 \leqslant N_0 \leqslant 10$ and $2 \leqslant m \leqslant 5$. The assortativity has been calculated for 10 000 instances of each graph definition.

Figure 4 shows the PDF (including curve fitting), distributed over 250 bins, of the assortativity for the different graphs models.

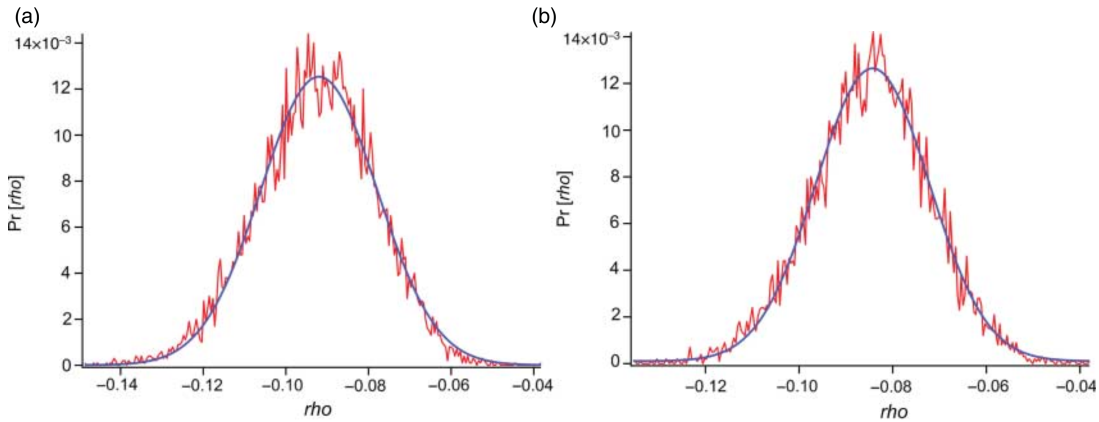

FIG. 4. Assortativity (rho) PDF for BA graphs. (a) $N = 1000$, $N_0 = 3$, $m = 2$. (b) $N = 1000$, $N_0 = 5$, $m = 2$.

Figure 4 and Table 3 show that the simulated BA network overall is slightly disassortative, ρ_D distributed around −0.10 and −0.08, respectively. Experiments show that for large N, ρ_D for BA graph will be ≈ 0.

4.3 *WS small world random graph*

The Watts & Strogatz [38] small-word graph is obtained by applying n steps of random rewiring on all links of a ring-lattice. However, the principle of constructing a small-world graph may also be applied to other regular graph classes, such as square lattice, as for example explored by Makowiec [53], cubic lattice or spherical lattice. A WS graph obtained from a square lattice may not mimic a *small-world* (so we might not want to call it a 'WS graph'). A WS graph may, alternatively, be generated through *random link addition*, as opposed to *random link rewiring*. This is proposed by Dorogovtsev [54].

When constructing a WS graph from a ring lattice, the following parameters are used:

- N Size of the lattice
- d Degree of each node; it shall be an even value; each node is connected to $(d/2)$ adjacent neighbours on either side
- n Number of rewiring cycles; in each rewiring cycle, all links in the graph are visited and are randomly rewired with probability p_r
- p_r The probability for a link to be rewired, if not already rewired, during one rewiring cycle

Figure 5 shows a number of ring lattices that are rewired into a WS graph model. All lattices have $N = 100$.

Figure 6 shows the assortativity (including curve fitting) distribution for two of the WS graph models shown in Fig. 5, with a difference that N is set to 1000. The assortativity distribution is obtained by generating each graph model 10 000 times with the same characteristic parameters (N, d, n, p_r). For reference, the assortativity for the non-rewired ring lattice is 1.0, since all nodes have equal degree.

Figure 6 illustrates that the iterative rewiring process makes the network slightly disassortative. This is to be expected, since each rewiring step 'breaks' the chain of nodes that are equally connected to a

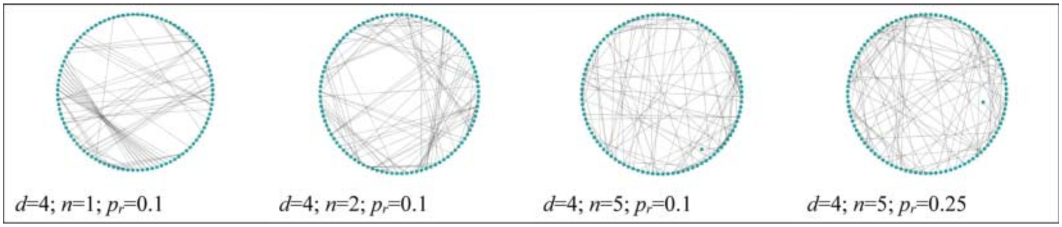

FIG. 5. Rewired ring lattices (WS graph model).

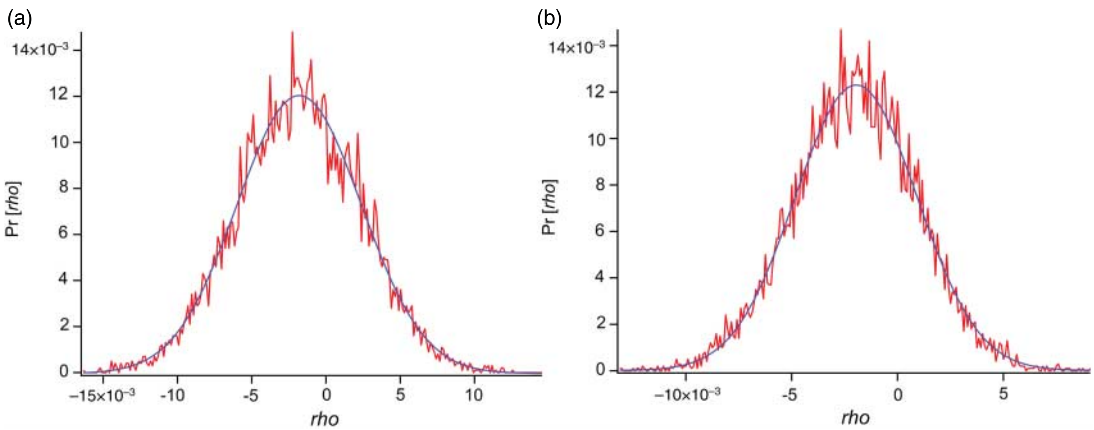

FIG. 6. Assortativity (rho) PDF for rewired ring lattices (WS graph model). (a) $d=4$; $n=1$; $p_r=0.1$. (b) $d=4$; $n=5$; $p_r=0.25$.

designated number of their respective neighbouring nodes. The rewiring steps obviously break that. With increasing number of rewiring steps, the network becomes random and so ρ_D approaches 0.

We will now explore the assortativity of the WS graph model when applying the rewiring on a square lattice. All nodes that are not corner node or edge node have $d=4$. It is conceivable that a square lattice may also be devised with larger d, e.g. $d=8$, whereby also diagonal connections are included. Figure 7 shows WS graph models obtained from this square lattice ($N=64, d=4$), for constant p_r ($p_r=0.1$) and various n.

Fig. 8 shows assortativity distribution (including curve fitting), distributed over 250 bins, associated with WS graph models as shown in Fig. 7, with $N=1024$. The test is repeated 10 000 times.

The assortativity of the square lattice is <1.0, since not all nodes have the same degree (nodes at the graph boundary have lower degree). For increasing N, the assortativity of the non-rewired square lattice approaches 1.0, since the effect of the lower degree of nodes at the graph boundary decreases. As more rewiring cycles are performed, the assortativity decreases, as expected. The rewiring introduces randomness, so assortativity will (eventually) approach 0.

A further variant of the traditional WS model is the following. Iteratively, an existing link from node $i \in [1,N]$ is rewired with probability p_r or a new link is added to node i with probability p_r. When an existing link is rewired or a new link is added, instead of selecting the target node *randomly* over all other nodes, a *preferential rewiring/preferential attachment* is applied. The rewiring/attachment is done by randomly selecting a node and then applying a further probability proportional to the degree of that

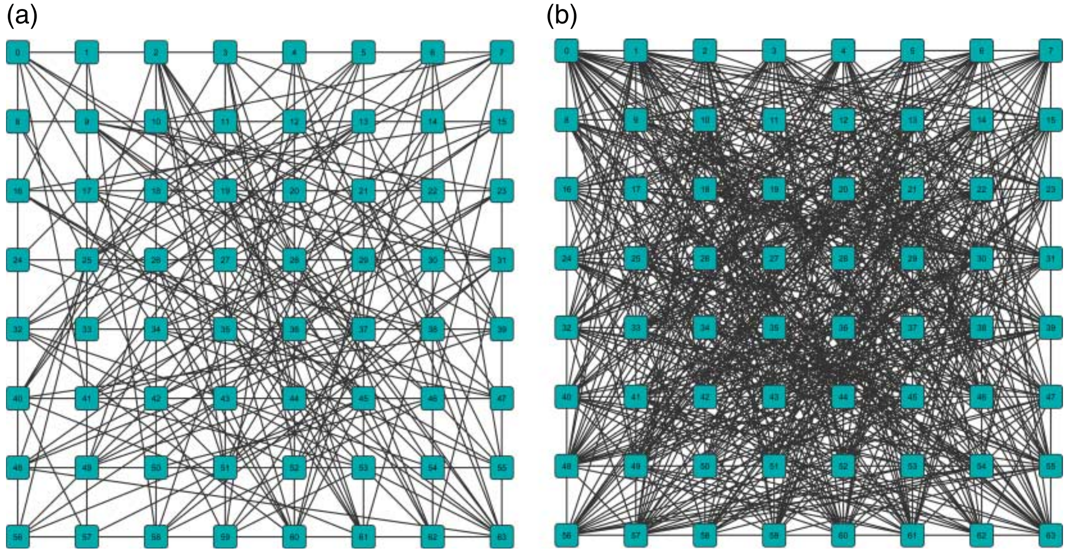

FIG. 7. WS graph models from square lattice. (a) $n = 1$; Average shortest path = 2.01. (b) $n = 5$; Average shortest path = 1.56.

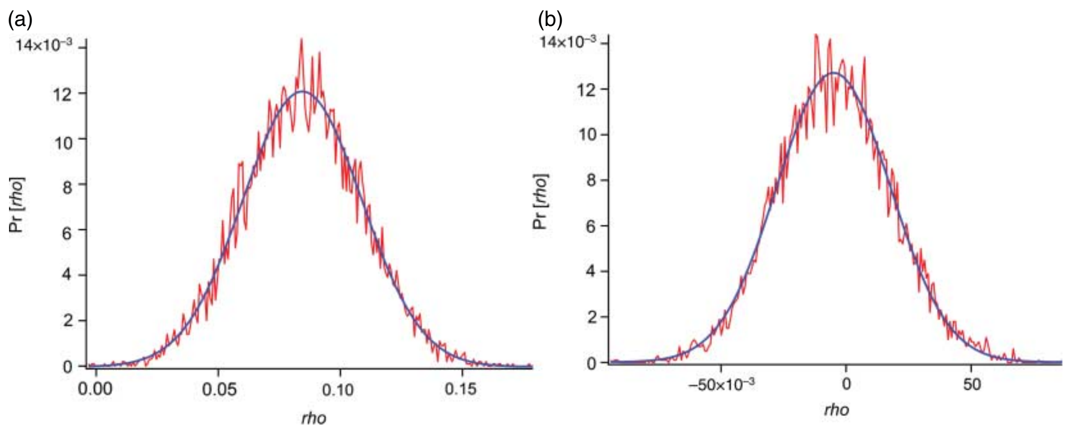

FIG. 8. Assortativity (rho) distribution for WS graph models from square lattice. (a) $n = 1$. (b) $n = 10$.

node. The rationale of this model is that the regular lattice represents an 'imposed network structure', such as a (created) closed community. Within the community structure, starting off as a regular structure, preferential attachment starts to occur. With an increasing number of rewirings or link additions, the graph is expected to approach the BA model, with power law degree distribution.

4.4 *Callaway random growth model*

The Callaway model [39] forms part of the class of 'growth models'. For every newly added node, a link is added with probability δ, between two randomly selected nodes (selected uniformly from all nodes, including the newly added node), provided that no link exists yet between the two selected nodes. The Callaway model differs from the ER model in that the nodes that are added at a later moment have a smaller chance of having been selected once or multiple times for creating a link from that node, to another node. So, these nodes will, on average, have a lower degree. The Callaway growth model also differs from the BA growth model, since for the BA growth model, new links are always created between the newly added node and an existing node. The Callaway growth model could, however, create a network with isolated nodes. A further difference between the Callaway growth model and the BA growth model is that for the Callaway model, the degree of nodes does not influence their likelihood of attracting more links. Hence, there is no *preference* in the attachment. The rationale of the Callaway growth model, when comparing with the ER model, is that the degree of a node becomes a function of the age of that node.

Callaway *et al.* [39] observe that their network growth model exhibits a distinctive degree correlation, as a function of the degree of a node. It is shown in [39] that a positive correlation coefficient exists, for nodes to connect with other nodes of similar degree. This differs clearly from the ER random graph, whereby degree correlation is, on average 0. Positive degree correlation is in Callaway graphs prevalent especially for higher degree nodes, rather than for lower degree nodes. High-degree nodes tend to connect to other high-degree nodes. For lower degree nodes, the degree of connected nodes is more evenly distributed between low(er) degree nodes and high(er) degree nodes.

Callaway *et al.* [39] define the following formula for degree correlation:

$$\rho = \frac{\sum_{i\sim j|(d_i=q,d_j=r)}[q-(1+4\delta)][r-(1+4\delta)]e_{qr}}{4\delta(1+2\delta)} \tag{12}$$

where by

$i \sim j$ link l with start node $i = l_-$ having $d_{l-} = q$ and end node $j = l_+$ having $d_{l+} = r$;

δ probability of adding a link between the randomly selected pair of nodes;

e_{qr} $\frac{2\delta}{1+4\delta}(e_{q-1,r} + e_{q,r-1}) + \frac{p_{q-1}p_{r-1}}{1+4\delta}$, which is the fraction of links l with $d_{l-} = q$ and $d_{l+} = r$.

Clearly, Equation (12) for ρ is recursive; ρ will therefore have to be calculated during the growth process of the graph. According to Equation (12), the value of δ ranges between 0.0 and 1.0. Experiment and calculation shows, see Fig. 9, that ρ will range between 0.0 and 0.4.

Newman [1] has conveniently transformed Equation (12) into:

$$\rho = \frac{\delta}{(1+2\delta)} \tag{13}$$

4.5 *Assortativity of specific classes of graph*

In the present section, we compute the assortativity of selected graph classes. For this purpose, we make use of the reformulation of assortativity by Van Mieghem *et al.* [25]:

$$\rho = \frac{N_1N_3 - N_2^2}{N_1\sum_{i=1}^{N} d_i^3 - N_2^2} \tag{14}$$

where $N_k = u^TA^ku$ equals the total number of walks with k hops. Thus, $N_0 = N$, $N_1 = 2L$, $N_2 = \sum(d_i)^2 = d^Td$ and $N_3 = d^TAd = u^TA^3u$ (Van Mieghem [4]).

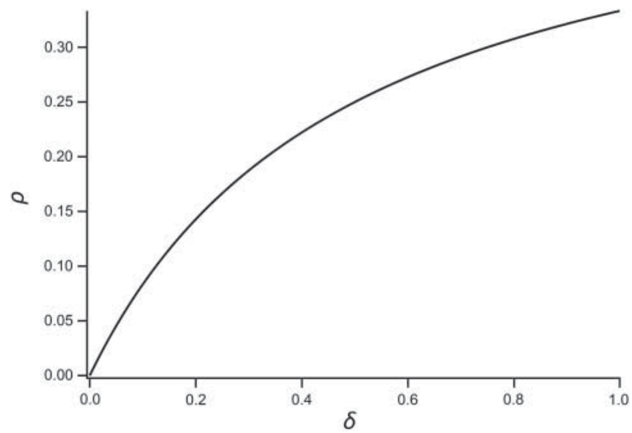

FIG. 9. Degree correlation, ρ, as a function of link density, δ.

Equation (14) is further transformed into (Van Mieghem *et al.* [25]):

$$\rho_D(G) = 1 - \frac{\sum_{i\sim j}(d_i - d_j)^2}{\sum_{i=1}^N d_i^3 - \frac{1}{2L}\left(\sum_{i=1}^N d_i^2\right)^2} \tag{15}$$

where d_i represents the degree of node i. Equation (15) for assortativity constitutes a *graph theoretic* formula, since it expresses the assortativity in terms of the degree of the nodes in the graph. Equation (15) can be written as follows:

$$\rho_D(G) = 1 - 2\frac{\sum_{i=1}^N (d_i)^3 - \sum_{i\sim j} d_i d_j}{\sum_{i=1}^N d_i^3 - \frac{1}{2L}\left(\sum_{i=1}^N d_i^2\right)^2} \tag{16}$$

Combining Equation (16) with Equation (10) results in:

$$\rho_D(G) = 1 - 2\frac{\sum_{i=1}^N (d_i)^3 - \mathrm{Lh}(G)}{\sum_{i=1}^N d_i^3 - \frac{1}{2L}\left(\sum_{i=1}^N d_i^2\right)^2} \tag{17}$$

We conclude from Equation (17) that a linear relation exists between the assortativity and the likelihood of a graph.

4.5.1 *Path graph.* A path graph with $N = 2$ is perfectly assortative, since all nodes have equal degree. For a path graph of $N > 2$, all the nodes have $d_i = 2$, except for $i = 1$ and $i = N$, which have $d_i = 1$. The term $\sum_{i\sim j}(d_i - d_j)^2$ in Equation (15) equals 2, since the outermost links have $d_1 - d_2 = -1$ and $d_{N-1} - d_N = 1$ while all other links have $d_i - d_j = 0$. The term $\sum_{i=1}^N d_i^k$ in Equation (15) for $k = 2$

yields $4N-6$, while for $k=3$ it yields $8N-14$. Using (15) yields

$$\rho_D(G) = -\frac{1}{N-2}. \tag{18}$$

The path graph with $N>2$ is always disassortative. For $N=3$, $\rho_D=-1$ and with increasing N, ρ_D increases towards 0.

4.5.2 *Star graph.* The central node in the star graph with $N>2$ has $d_i = N-1$, all other nodes have $d_i = 1$. The star graph is a special case of complete bipartite graph $K_{m,n}$, namely $m=1$ and $n=N-1$. The complete bipartite graph has $\rho_D=-1$, as will be shown in a next section. So, a star graph always has $\rho_D=-1$.

4.5.3 *Lattice.* We consider a two-dimensional lattice of size $N\times M$, with the number of links equal to $L=(N-1)M+(M-1)N=2NM-N-M$. The nodes in the graph can be divided in the following groups;

- $(N-2)(M-2)$ inner nodes with $d_i=4$;
- $2(N-2)+2(M-2)=2N+2M-8$ nodes along the edge with $d_i=3$;
- 4 corner nodes with $d_i=2$.

The links can be grouped as follows:

- each corner node has two links, for which $d_i-d_j=1$;
- apart from the links from the corner nodes, there are $2(N-3)+2(M-3)$ links residing at the perimeter of the lattice, for which $d_i-d_j=0$;
- along the perimeter of the lattice, there are $2(N-2)+2(M-2)=2N+2M-8$ links connecting to inner nodes, for which $d_i-d_j=1$;
- the remaining links in the lattice connect nodes of equal degree, so $d_i-d_j=0$.

Figure 10 provides graphical representation for assortativity of a lattice with $N=2,\ldots,35$ and $M=2,\ldots,35$.

For a 2×2 lattice, $\rho_D=1$, since all nodes have degree 2.

4.5.4 *Complete bi-partite graph.* For a complete bi-partite graph $K_{m,n}$, with $n\neq m$, we have perfect disassortativity, i.e. $\rho_D=-1$. The m nodes in set M all have degree equal to $|N|$ and the n nodes in group N all have degree equal to $|M|$. Since we assume $n\neq m$, all low-degree nodes are connected to high-degree nodes and all high-degree nodes are connected to low-degree nodes. The $\rho_D=-1$ for the complete bi-partite graph can also be shown as follows. The term $\sum_{i\sim j}(d_i-d_j)^2$ in Equation (15) is equal to $nm(n-m)^2$, since there are n^*m links, connecting nodes which have degree n and m, respectively. The term $\sum_{i=1}^{N} d_i^3$ is equal to $mn^3+nm^3=nm(n^2+m^2)$, since each node in set M has degree n

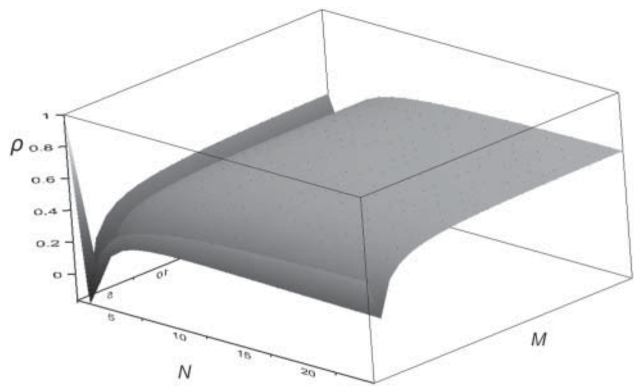

FIG. 10. Assortativity for lattice graph, $N = 2\cdots 35$ and $M = 2\cdots 35$.

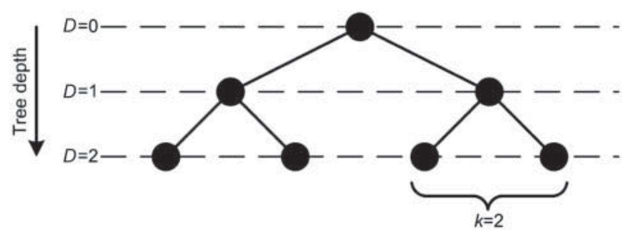

FIG. 11. Tree graph with $D = 2$ and $k = 2$.

and each node in set N has degree m. Likewise, the term $\sum_{i=1}^{N} d_i^2$ equals $nm(n+m)$, while the number of links L is equal to n^*m. Filling these terms into Equation (15) yields $\rho_D = -1$.

4.5.5 *Tree graph.* We consider a k-ary tree as shown in Fig. 11. Each node has equal number of branches, denoted k. The depth of the tree is denoted by D.

For a depth $D = 1$, we have perfect disassortativity, $\rho_D = -1$; the tree resembles a path graph with $N = 3$. As the depth D increases, the tree becomes less disassortative. This is shown in Fig. 12.

For trees with $k > 2$, the rise of ρ_D will be less than for trees with $k = 2$. This is attributed to the relative larger number of end-branches for which $|d_i - d_j| = k$. The difference in degree for the nodes connected by the end-branches, makes the graph more disassortative.

5. Degree assortativity of non-weighted, directed graphs

Degree assortativity for directed networks follows the same principle as degree assortativity for non-directed networks. Equation (2) can be generalized for directed networks. Newman [5] defines the degree assortativity for directed networks as follows:

$$\rho = \frac{\sum_{jk} jk(e_{jk} - q_j^{\text{in}} q_k^{\text{out}})}{\sigma_{\text{in}}\sigma_{\text{out}}} \tag{19}$$

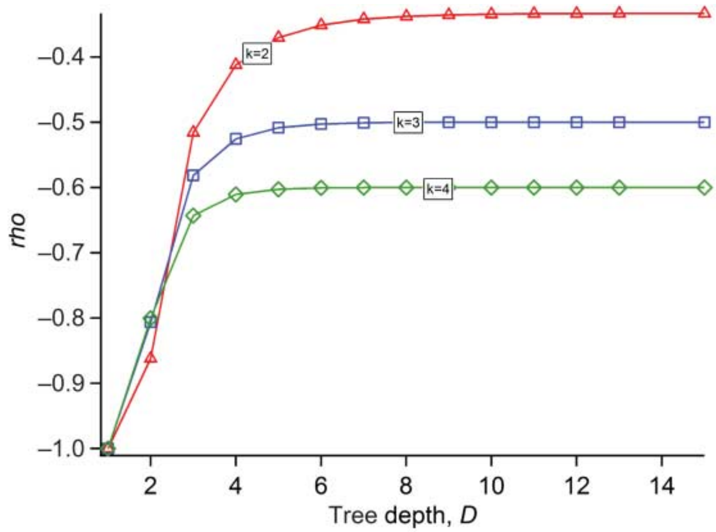

FIG. 12. Assortativity for tree graph with $k=2$, $k=3$ and $k=4$.

where

$^{\text{in, out}}$ refer to in-degree and out-degree, respectively;

e_{jk} fraction of links connecting a node with out-degree k to a node with in-degree j, whereby $j, k \in N$;

$q_j^{\text{in}} = \frac{(j+1)p_{j+1}^{\text{in}}}{\sum_j jp_j^{\text{in}}} = \frac{(j+1)\Pr[D_{\text{in}}=j+1]}{E[D_{\text{in}}]}$ normalized distribution of in-degree, where D_{in} is the in-degree of a randomly selected node;

$q_k^{\text{out}} = \frac{(k+1)p_{k+1}^{\text{out}}}{\sum_k kp_k^{\text{out}}} = \frac{(k+1)\Pr[D_{\text{out}}=k+1]}{E[D_{\text{out}}]}$ normalized distribution of out-degree, where D_{out} is the out-degree of a randomly selected node;

$\sigma_{\text{in}}, \sigma_{\text{out}}$ standard deviation of q_j^{in}, standard deviation of q_k^{out}

Newman's assortativity Equation (19) for directed networks considers the correlation between a node's out-degree and the adjacent nodes' respective in-degree, and the correlation between a node's in-degree and the adjacent nodes' respective out-degree. It may, however, be more logical if the assortativity for directed networks would measure the correlation between nodes considering their respective in-degree or their respective out-degree. The rationale is that in-degree and out-degree represent a characteristic of a node and the correlation coefficient should be based on *comparable characteristics*. Piraveenan *et al.* [20] propose a modified definition of assortativity for directed network, taking this (intuitive) more logical approach. Some examples of directed networks are (Newman [1]) e-mail address books, World Wide Web and software dependencies. The graph creation models that are commonly used for undirected networks are not directly usable for directed network. The models describe the 'adding of a link', but do not describe how the direction of that link is chosen. More specifically, the processes for generating a graph of class ER, BA or WS do not take link direction into account. In order to be able to study directed networks of different class, and specifically study the assortativity of these directed networks, such models would be needed. Li *et al.* [55] describe a method for generating a graph with required directionality. Chen and Olvera [43] describe methods for creating directed graphs with a given in-degree and out-degree distribution. Also, rewiring methods need to be enhanced for considering directed links. A network growth model should also decide on the weight of the added link. In addition, a growth model should increase the weight of existing links. So, not only the network size (node set, N) and the number of links (link set, L) grows, but also the weights of the links in the network will change over time. Leung and Chau [12] propose a growth model for a network composed of weighted links.

When analysing directed networks through their assortativity, a multi-layered approach may be applied. For a directed network, the assortativity may be differentiated in three classes:

(i) in-degree assortativity, which measures the tendency of nodes with particular in-degree to connect to other nodes with the same in-degree or with different in-degree;

(ii) out-degree assortativity, which measures the tendency of nodes with particular out-degree to connect to other nodes with the same out-degree or with different out-degree; and

(iii) overall assortativity, which forms an aggregation of the in-degree assortativity and the out-degree assortativity.

A network may be assortative for its in-degree, but disassortative for its out-degree or vice versa. Piraveenan *et al.* [20] have studied this differentiated form of assortativity. The degree distribution and the degree correlation for directed graphs are also studied by Myers [56] in ‘networks’ built from software dependencies. Myers [56] observes, indeed, that directed networks may exhibit different assortativity when taking the direction into account.

6. Relation between assortativity and graph spectra

The spectrum of a graph [34] is defined as the set of eigenvalues of a particular connectivity matrix of that network, together with the corresponding eigenvectors. The Laplacian matrix Q of G is defined as $Q = \Delta - A$, where A is the adjacency matrix and $\Delta = \mathrm{diag}(d_i)$. The following graph spectra are commonly used in spectral graph analysis:

Adjacency eigenvalues	The set of N eigenvalues $\lambda_N \leqslant \lambda_{N-1} \leqslant \cdots \leqslant \lambda_1$ of the adjacency matrix A. The highest eigenvalue of A, λ_1, is known as spectral radius. The difference between λ_1 and λ_2 is known as spectral gap
Laplacian eigenvalues	This spectrum is derived from the Laplacian matrix Q. It is formed by the set of N eigenvalues, $0 \leqslant \mu_{N-1} \leqslant \cdots \leqslant \mu_1$ of Q. The second smallest eigenvalues of Q, μ_{N-1}, is known as the algebraic connectivity (coined by Fiedler [57])

Graph spectra have been extensively studied and have proven to be a useful tool to evaluate essential properties of a network. We refer the reader to Cvetkovic *et al.* [58] for background on Adjacency spectrum and to Mohar [59] for a survey on the Laplacian spectrum, and to Van Mieghem [34] for extensive analysis on the use of graph spectra for complex networks.

The spectral radius is of particular interest for a network. We are interested to know how λ_1 relates to the assortativity. Is there a relation between assortativity and λ_1? Generally, we notice that with increasing ρ_D, there is also an increase in λ_1. Hence, if we would like to have a large λ_1, then one way of achieving that is increasing the assortativity of the graph. The increase in assortativity also leads to faster information (including virus!) spread through the network, due to the high inter-connectivity of the high-degree nodes. This faster information spread with increasing assortativity is observed despite the fact that increasing assortativity also leads to higher average hop count. It is furthermore shown by Van Mieghem *et al.* [25] that an increase of λ_1 may lead to disconnectivity of a graph.

We are also interested in the relation between the algebraic connectivity and assortativity. It is shown in [25] that with decreasing assortativity of a graph, towards disassortativity, λ_1 decreases but μ_{N-1} increases. A higher μ_{N-1} translates into increased topological robustness of the network. Generally, we notice that with increasing ρ_D, μ_{N-1} decreases. Hence, by influencing the assortativity of the graph, we can affect μ_{N-1}. Especially, by making the network more disassortative, we can improve the topological robustness. But this will, at the same time, decrease the speed of information spread through the network.

The Laplacian spectrum of the graph may be used to express the effective graph resistance. Effective graph resistance is initially defined as (Klein & Randić [60]):

$$R_G = \sum_{1 \leqslant i \leqslant j \leqslant N} R_{ij} \tag{20}$$

Where R_{ij} is the effective resistance between node i and node j in the graph. The effective graph resistance is also commonly expressed as:

$$R_G = N \sum_{i=1}^{N-1} \frac{1}{\mu_i} \tag{21}$$

where μ_i is the ith eigenvalue of the Laplacian matrix of G. Van Mieghem *et al.* [24] show that an increase in assortativity leads to an increase in R_G. Depending on the actual network, the increase of assortativity above a certain value leads to sharp rise of R_G and eventually R_G approaches ∞. This can be explained by the aforementioned fact that the increase in assortativity will eventually lead to graph disconnectivity, hence $R_G = \infty$.

7. Influencing assortativity through network topology changes

It is shown in earlier sections that the assortativity of a network represents a specific structural aspect of that network and that we may want to influence that specific aspect of the network. For example, we may want to increase the spectral radius, λ_1, or increase the algebraic connectivity, μ_{N-1}. The present section shows that we can change these spectral values by changing the assortativity of the graph.

Zhou *et al.* [32] consider assortativity for single, stand-alone networks versus assortativity for interdependent networks. They observed that increasing the assortativity of a network makes the network more robust against node removal, but at the same time makes the network less stable. It is, however, further observed in [32] that for interdependent networks, an increase in assortativity decreases the robustness of the interdependent networks. Generally, when changing the assortativity of a network through topological changes to that network, we change other metrics of that network as well, such as the effective graph resistance. Increasing ρ_D through DPR leads to an increase in R_G. When applying link addition instead of DPR, both ρ_D and R_G may potentially be improved. Ellens *et al.* [61] investigate methods to improve R_G through link addition.

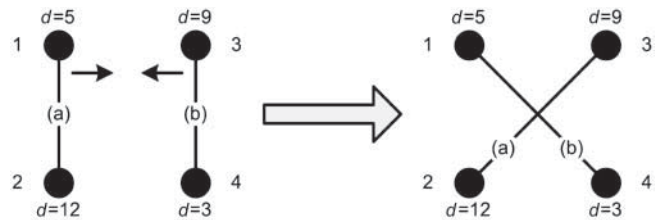

FIG. 13. Degree-preserving rewiring.

D'Agostino *et al.* [7] have also studied the effect of assortativity on the robustness of a network. A general conclusion is that disassortative networks are easier to immunize due to a higher epidemiological threshold. Although assortative networks are more prone to the propagation of failures, these networks require a longer intervention time to prevent further failures (epidemic spreading). Assortativity of a network relates both to the network's robustness against node failure and to the network's dynamic behaviour with respect to failure spreading.

Trajanovski *et al.* [62] have introduced the concept of robustness envelope and have determined that the relation between assortativity and robustness against node failure depends on the type of failure. A moderate increase in assortativity increases the network's robustness against targeted node attacks, while a moderate decrease in assortativity increases the network's robustness against random uniform node attacks (or node failure).

The network topology change can take various forms. We identify: (i) link addition, (ii) link removal, (iii) link rewiring and (iv) DPR. The rationale of DPR is that the degree sequence (first-order graph metric) is unaffected, whilst the assortativity or other second-order graph metric changes. The impact on assortativity, by network modification, is highly dependent on the kind of modification, on the link(s) and nodes involved in the modification and on the class of the network. The topology modification will affect also other metrics of the network than assortativity. For example, link removal will increase the average shortest path in the network or may disconnect the graph; it will also increase the effective graph resistance. Link removal, link addition and link rewire (if not pair-wise) will affect the degree vector of the graph. Hence, network topology modifications will in practice be bound by certain restrictions.

DPR entails that the network is rewired in such way that the degree vector $d = [d_1, d_2, d_3, \ldots, d_N]^\top$ is preserved. The degree probability distribution is unchanged, but the joint degree distribution $F_{D1D2}(k,l)$ for two randomly selected nodes may be affected. DPR is a common technique for network modification without altering d. DPR has the practical characteristic that a node keeps the number of traffic connections. One prominent example is formed by IP routers, which have a defined number of interfaces. An IP sub-network may be rewired, while keeping the number of cables per router constant. By keeping the degree vector unchanged, we can influence the assortativity of the network through DPR. This is visualized in Fig. 13.

The rewiring depicted in Fig. 13 relates to graphs with undirected, non-weighted links. By rewiring node 2 to node 3 and node 4 to node 1, nodes 2 and 3 both become more assortative, as they are now connected to other nodes (nodes 3 and 2, respectively) that have degree closer to their own degree. Node 1 and 4 also become more assortative, being rewired to other nodes with degree closer to their own degree. The basis of assortativity change due to DPR is provided in Lemma 1 in Van Mieghem *et al.* [25].

DPR increases or decreases the assortativity with multitudes of a deterministic value. Equation (15) is rewritten as

$$\rho(G) = 1 - \frac{(d_{l_{1-}} - d_{l_{1+}})^2}{S} - \ldots - \frac{(d_{l_{L-}} - d_{l_{L+}})^2}{S} \tag{22}$$

where

$$S = \sum_{i=1}^{N} d_i^3 - \frac{1}{2L}\left(\sum_{i=1}^{N} d_i^2\right)^2$$

l_{1_-}, l_{1_+} start node ($-$), end node ($+$) of link 1 of link set L.

Every link $i \sim j|_{di \neq dj}$ pulls $\rho_D(G)$ from 1 further towards -1, in multiples of unit steps S^{-1}. The change in $\rho_D(G)$ resulting from DPR can hence easily be calculated, since one set of links $[i \sim j, k \sim l]$ is replaced by another set of links $[i \sim k, j \sim l]$. Their respective contributions to $\rho_D(G)$ equal $-((d_i - d_j)^2/S)$, $-((d_k - d_l)^2/S)$ before rewiring, and $-((d_i - d_k)^2/S)$, $-((d_j - d_l)^2/S)$ after rewiring. The change in $\rho_D(G)$ from this rewiring equals $2(d_i d_k + d_j d_l - d_i d_j - d_k d_l)/S$.

Different strategies may be devised, in order to let the network topology converge in as few rewiring steps as possible towards the desired state. In addition, the rewiring strategy should be such that a link pair that is suitable for rewiring can be found in as few attempts possible. Winterbach *et al.* [27] outline a methodology for exhaustively rewiring a network, studying whether a greedy approach yields the optimum assortativity for a network within reasonable time. The approach in Winterbach *et al.* [27] is based on constructing a set R of all rewirable link pairs in the network. Rewiring is done by selecting a link pair from R and verifying that rewiring that link pair will increase the assortativity. The rewiring action leads to a change of R; some of the link pairs in R are no longer rewirable and new rewirable link pairs are added. So, R would have to be updated after every rewiring step. In fact, especially when rewiring is applied on large networks, in access of 1000 nodes, an optimized algorithm for finding rewirable links becomes crucial in order to curb the computation time. The need for such optimized algorithm further depends on the practical use of the network rewiring. When the aim of the rewiring is to transform the network towards its optimum assortativity (or other required metric), then many repeated rewiring steps are required. Alternatively, the aim may be to increase the assortativity to a defined absolute value or to increase the assortativity with a defined factor. In such cases, fewer rewiring steps may be needed. Winterbach *et al.* [27] derive also an *exact* method for calculating the maximum assortativity that may be reached through DPR. This method has computational complexity C of order $C = \mathrm{O}(N^6)$, so the method is, practically, suitable for small networks only.

Noldus and Van Mieghem [48] describe a method for assortative rewiring by ordering nodes according to their degree. The first link of the node pair is formed by selecting the node with highest degree and selecting the link connecting that node with another node having the lowest degree of all nodes connected to this first node. The second link is found by selecting the node with lowest degree and selecting the link connecting that node with another node having the highest degree of all nodes connected to this second node. The rewiring step, depicted in Fig. 13, follows that approach. When a link pair is found, a check can be done whether the conditions for rewiring are fulfilled, namely: the links shall not share a node, rewiring shall not result in overlapping links and rewiring shall not result in graph disconnection.

For DPR, link pairs can be found that will increase the assortativity of the network (Van Mieghem *et al.* [25], Noldus and Van Mieghem [48]). What is not yet explored is how to deterministically find the link pair that will provide the largest absolute increase or decrease in assortativity.

When applying DPR, a graph G is transformed to G'. The graphs G and G' belong to a set of graphs G with the same degree vector d. It must, however, yet be proven whether all $G' \in G$ can be reached through DPR, when starting from G. The set G may in fact comprise several clusters of G', each cluster comprising graphs that may be created through iterative rewiring from any other graph within that cluster, but not from a graph residing in another cluster. The set G may be reflected through a graph itself, $G(N, L)$, whereby each node $\in N$ represents a (rewired) graph G and each link $\in L$ represents a rewiring action. When G is not connected, then apparently it is not possible to rewire between all G'.

There is currently no known methodology to efficiently determine the absolute bounds for assortativity that can be achieved through DPR, other than exhaustive analysis: rewiring G to every possible G', then rewiring every obtained G' to every possible G'', with G'' not being isomorphic to G or any of the already obtain G' or G''. This process is then to be continued until no further rewiring is possible. Then the complete set G is determined and the assortativity range of $G' \in G$ can be determined, but only for the cluster in G that G belongs to. Furthermore, instead of using rewiring to achieve a G' with minimal or maximum assortativity, other network topology changes may be applied, such as link addition or link upgrade (increasing the link transmission capacity). There is no literature available that investigates the effect on assortativity from these other network topology changes.

The changing of assortativity of a network, whilst keeping the degree vector unmodified, is studied also by Xia *et al.* [28]. Their method is based on random selection of link pairs for rewiring the network, for the case that the rewiring leads to increase or decrease of assortativity, as appropriate, and otherwise discard the rewiring and (randomly) select a next link pair. This method is due to its randomness deemed to be less efficient than the method applied by Noldus and Van Mieghem [48], where a targeted link set selection is applied. Xia *et al.* show that by decreasing the assortativity of a scale-free network, whilst keeping the degree vector unmodified, the packet drop rate of the network decreases, i.e. the network traffic performance improves. The test was done for a traffic model that represents typical traffic model in the Internet. The packet drop rate for a single node is related, obviously, to the capacity of that node and to the amount of traffic that is scheduled to pass through that node. We learn from [28] that rewiring may also be applied to optimize the network's performance for traffic throughput. The rewiring is done to balance the traffic over the nodes in the network. We observe here that the optimum network configuration, in terms of balanced traffic flow through the respective nodes in the network, is dependent on the traffic model. For a given network instance, any traffic model will lead to a certain load on each of the nodes. This load may be expressed as the 'traffic betweenness' of the node. The betweenness value for a node considers the shortest paths, for any node pair in the network, traversing that node. The traffic betweenness takes also the traffic flow for each path into account. A shortest path carrying more traffic contributes proportionally more to the node's betweenness than a shortest path carrying less traffic.

Further work is needed to identify the relation between assortativity and betweenness. As also pointed out by Martin-Hernandez *et al.* [47], nodes or links with high betweenness would cause more traffic disruption when subject to failure compared with nodes or links with low betweenness. Rewiring the network will affect the betweenness of individual nodes and individual links. Not all possible DPR steps may therefore be feasible, since some rewiring steps may lead to an increase of betweenness of particular node or link beyond a threshold value.

Xulvi-Brunet & Sokolov [30] apply DPR to a non-assortative, scale-free graph to either increase the assortativity to a maximum value (network becomes increasingly assortative) or to decrease the assortativity to a minimum value (network becomes increasingly disassortative). Various metrics are compared against the increase assortativity/disassortativity. The following effects are observed:

Increasing assortativity when $\rho_D > 0$:

- Average hop count $E[H]$ grows rapidly. The logarithmic relation between $E[H]$ and N, which we normally observe for scale-free networks, is preserved when ρ_D increases.
- Average clustering coefficient C increases.
- Information spread through the network, when considering each individual node in the network as origin of information, becomes slower. The increased assortativity will have the effect that information is rapidly arriving at high-degree nodes, as high(er)-degree nodes are connected to other high(er)-degree nodes. But the further distribution to (remote) nodes with lower degree will be slower, as these low(er) degree nodes are connected to other low(er) degree nodes.
- Node percolation is affected. When the assortativity of the network increases, the network will more easily break up resulting from removal of a fraction, q, of nodes in the network ($0 \leqslant q \leqslant 1$).

Decreasing assortativity when $\rho_D < 0$:

- The average hop count $E[H]$ grows marginally.
- Average clustering coefficient C decreases and will eventually become 0. Compare: a star network has clustering coefficient $= 0$ (no loops in the network) and is maximum disassortative.
- Information spread through the network, when considering each individual node in the network as origin of information, shows stark 'peaks and valleys'. Information spreads fast when the information has reached a high(er) degree node, since that high(er) degree node reaches out to a large number of connected nodes. From the respective connected nodes, however, information spreads slower, since these connected nodes will typically be low(er) degree nodes, following from the disassortative nature of the network. The low(er) degree nodes are, however, connected to a (small) number of high(er) degree nodes, so information will start spreading faster again etc.
- Node percolation is minimally affected. When the disassortativity of the network increases, the network will slightly more easily break up resulting from removal of a fraction, q, of nodes in the network ($0 \leqslant q \leqslant 1$).

8. Relation between assortativity and line graphs

A graph $G(N, L)$ can be represented as a line graph $H(N_H, L_H)$, whereby N_H represents the number of nodes in H and L_H represents the number of links in H (Liu *et al.* [14]). Line graphs are often used as network model for certain real-world network structures. The line graph is created as follows:

- each link in the graph G is represented through a node in H; hence, $|N_H| = |L|$;
- two nodes in H are connected if and only if the corresponding links in G share a node in G. $|L_H|$ can be expressed as $|L_H| = \sum_i (d_i(d_i - 1)/2) = 1/2 \sum_i d_i^2 - L$, where i is a node in G (Van Mieghem [34]).

Figure 14 shows a few examples of graph G and their respective corresponding line graph H.

We are considering here only graphs with undirected, non-weighted links.

Manka-Krason and Kulakowski [15] show that line graphs of simple, non-directed graphs, always show positive linear degree–degree correlation, which translates into assortativity, i.e. $\rho_D > 0$. This result from [15] seems to be in disagreement with Liu *et al.* [14]. For the networks G generated in [14],

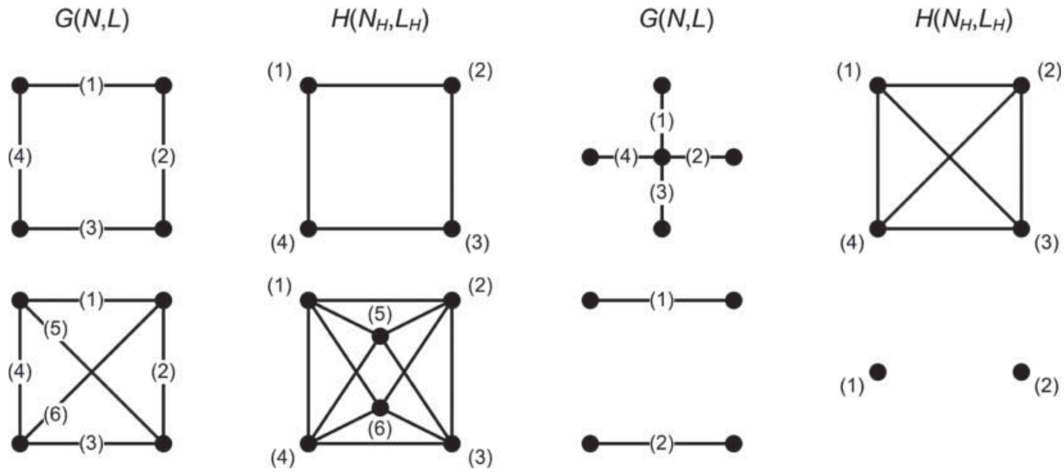

FIG. 14. Example of graphs and corresponding line graphs.

$\rho_D(G)$ is stepwise increased towards 0; the $\rho_D(H)$ for each G varies around 0, including $\rho_D(H) < 0$. Liu *et al.* [14,63] construct the root graph from a line graph. An initial line graph is constructed, comprising a set of cliques, as well as the corresponding root graph. The root graph corresponding to the initial line graph is a collection of star graphs, each star corresponding to a clique in the initial line graph. Since a star graph has $\rho_D = -1$, a graph G consisting of a collection of stars also has $\rho_D = -1$. The line graph is iteratively transformed by merging two randomly selected nodes for which $h > 1$, i.e. belong to different cliques from the initial line graph. Each such transformation of the line graph translates to a reduction of the number of nodes, in the corresponding root graph, with $k = 1$. The root graph becomes more assortative at every step, up to $\rho_D = 0$. By varying the number of cliques in the initial line graph and the size of the cliques, a network with a desired ρ_D in the range $-1 \leqslant \rho_D \leqslant 0$ can be created. (to be precise: the process can be repeated until $\rho_D \geqslant \rho_{\text{desired}}$.) Hence, the model presented in [14] may be used to generate graphs with a desired assortativity. Manka-Krason and Kulakowski [15] have taken a different approach for generating line graphs. A large number of realizations of a particular graph class (ER and BA) are generated. For each graph realization, the corresponding line graph is generated. This is a different way of constructing a graph than the manner in which graphs are generated in [14], which explains the difference in results regarding the assortativity value for the line graphs.

It is yet to be explored how the analysis of the assortativity of a network, such as increasing the assortativity of the network through DPR, may be facilitated by analysing the line graph of the network, instead of analysing the original graph of the network. For example, increasing the assortativity through DPR may be accomplished by applying an algorithm to find suitable link pairs in G. Such algorithm may be derived from the *linear law for assortativity*, devised by Liu *et al.* [14]. The algorithm shall preferably deterministically find the link pair that yields maximum assortativity increase (or decrease) when degree-preserving rewired.

9. Relation between assortativity and complementary graphs

The adjacency matrix of the complement G^c of G is denoted A^c, whereby $A^c = J - I - A$, whereby $J = uu^\top$ (the all one matrix) and I is the identity matrix (Van Mieghem [34]). Wang *et al.* [26] show that a linear relation exists between the degree assortativity of the original graph, $\rho_D(G)$, and the degree assortativity of the complementary graph, $\rho_D(G^c)$. The assortativity for the complementary graph is expressed as [26]:

$$\rho_D(G^c) = -\rho_D(G)\frac{2L\sigma^2[D_{l+}(G)]}{(N(N-1)-2L)\sigma^2[D_{l+}(G^c)]} + 1 - \frac{N^2\sigma^2[D(G)] - 2L\sigma^2[D_{l+}(G)]}{(N(N-1)-2L)\sigma^2[D_{l+}(G^c)]} \tag{23}$$

whereby $\sigma^2[D_{l+}(G)]$ is the variance of the degree at one side of an arbitrary link in G and $\sigma^2[D_{l+}(G^c)]$ is the variance of the degree at one side of an arbitrary link in G^c. It follows from Equation (23) that $\rho_D(G^c)$ is linearly related to $\rho_D(G)$, resulting from the observation that except for $\rho_D(G)$, all terms and factors in Equation (23) are constant for a particular degree vector. This linear relation enables us to study the assortativity in a graph by considering the complementary graph. When applying changes to the topology of the complementary graph whereby the degree vector is not affected, a resulting change in assortativity relates linearly to the corresponding change in assortativity of the original graph. DPR applied on a graph can be modelled to a corresponding action in the complementary graph. The search for the link pair in a graph that provides the largest increase or decrease in assortativity of the graph amounts in finding the link pair in the complementary graph that provides largest decrease or increase, respectively, in assortativity.

We derive the degree assortativity for G's complement, $\rho_D(G^c)$, from Equation (15).

$$\rho_D(G^c) = 1 - \frac{\sum_{i/\sim/j}(d_j - d_i)^2}{\sum_{i=1}^{N}(N-1-d_i)^3 - (1/2L_c)\left(\sum_{i=1}^{N}(N-1-d_i)\right)^2} \tag{24}$$

whereby

(1) $i/\sim/j$ no link exists between nodes i and j;

(2) $L^c = \frac{N(N-1)}{2} - L$, the number of links in G^c.

When applying DPR, $\rho_D(G)$ is increased or decreased with a deterministic multiple of

$$S = \frac{1}{\sum_{i=1}^{N} d_i^3 - (1/2L)\left(\sum_{i=1}^{N} d_i^2\right)^2}.$$

The $\rho_D(G^c)$ is decreased or increased with a deterministic multiple of

$$S_c = \frac{1}{\sum_{i=1}^{N}(N-1-d_i)^3 - (1/2L_c)\left(\sum_{i=1}^{N}(N-1-d_i)\right)^2}$$

S and S_c are not affected by the DPR. Hence there exists a linear relation between $\rho_D(G)$ and $\rho_D(G^c)$ during DPR.

It is further observed in [26] that the assortativity range, $\rho_{\min} - \rho_{\max}$, in a sparse network is generally larger than in dense networks. This means that in a sparse network, the assortativity has more relevance as second-order characterizer (metric) than in a dense network. Figure 15 shows the distribution (fitted curves) of $\rho_D(G)$ and the distribution of $\rho_D(G^c)$ taken over 10^5 realizations of an ER random graph with $N = 1000$ and $p = 0.1$.

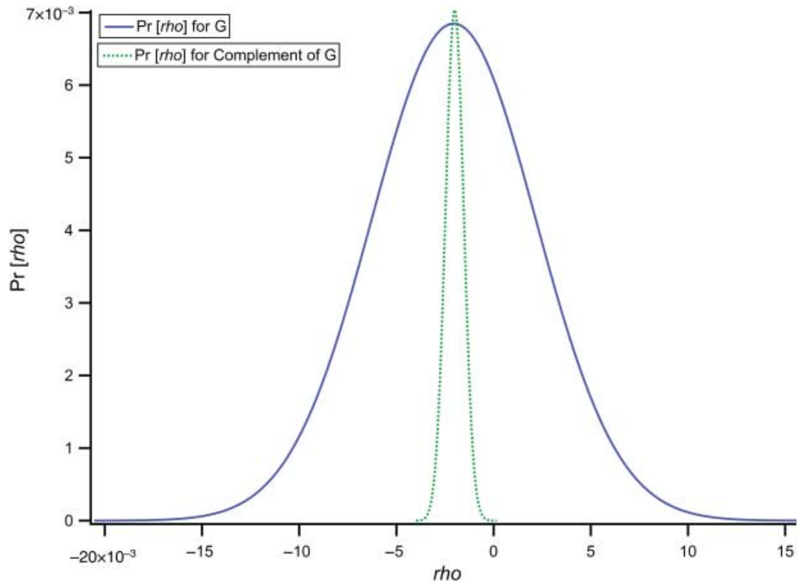

FIG. 15. Distribution of assortativity (rho) of G versus distribution of assortativity of G^c.

Since $p < 0.5$ graph G^c has a higher link density than G. This manifests itself in a smaller assortativity range for G^c than for G. When p approaches 0.5, G and G^c will have approximately equal assortativity distribution. When p increases above 0.5, then G^c will have a larger assortativity range than G.

10. Local assortativity

Piraveenan *et al.* [18] introduce the concept of *Local assortativity*. Local assortativity provides an additional dimension to network analysis. We have already earmarked assortativity as *second-order* network metric, as opposed to degree distribution, being a *first-order* network metric. Following this line of thinking, local assortativity may be considered a *third-order* network metric, as it provides further differentiation in graphs with equal degree distribution and equal assortativity.

When considering a network with a certain degree distribution and certain assortativity, it is observed that the individual nodes in that network contribute differently to the assortativity of that network as a whole. An assortative network may comprise nodes that contribute positively to the network's overall assortativity, as well as nodes that contribute negatively to the network's overall assortativity. Since the network as a whole is assortative, there is effectively more *positive* contribution to the network's assortativity than *negative* contribution to the network's assortativity.

Local assortativity is defined such that each node in the network has its own assortativity value, which is dependent on local properties of the node. Namely its degree and the degree of its neighbours. Local assortativity ρ_i of node i is defined as [18]:

$$\rho_i = \frac{(j+1)(j\bar{k} - \mu_q^2)}{2L\sigma_q^2} \tag{25}$$

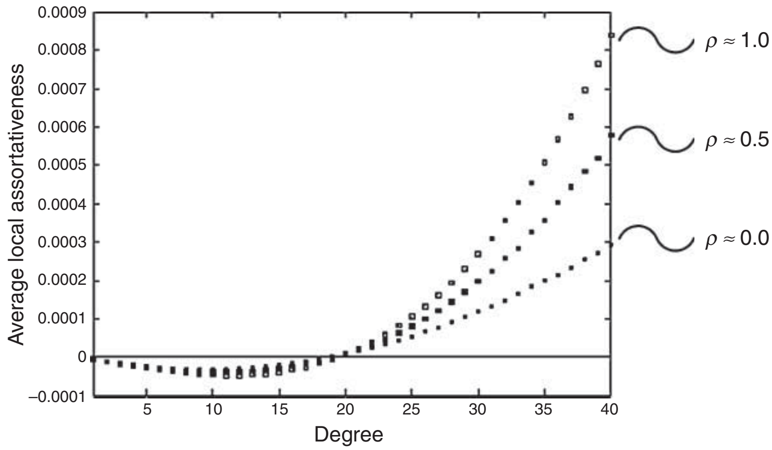

FIG. 16. Average local assortativity, as a function of the degree of a node (source: (18)).

where

$j = d_i - 1$, the remaining degree of node i;

$\bar{k} = \frac{1}{d_i}\sum_{i\sim j}(d_j - 1)$, the average remaining degree of the neighbours of node i;

σ_q is the standard deviation of the distribution of j (the remaining degree) over all nodes in the network;

$\mu_q = E[D-1]$, the mean of j (the remaining degree) over all nodes in the network

L is the number of links in the network.

Equation (25) satisfies the requirement:

$$\rho = \sum_{i=1}^{N} \rho_i \tag{26}$$

Figure 16 shows the average local assortativity value, as a function of the degree of a node, for example networks with different ρ_D.

Figure 16 reflects for nodes with equal degree the average local assortativity. We observe that for an assortative network, high-degree nodes contribute substantially more towards the network's assortativity than low-degree nodes. In addition, up to a certain degree value, nodes have a *negative* contribution to the network's assortativity. We call this degree the *critical degree*, k_c.

The local assortativity, as defined in [18], is also very convenient for recalculating assortativity when DPR is applied. There are four nodes and two links involved in a DPR action (Fig. 13). It follows from Equation [25] that the local assortativity for these four nodes is affected, but is unaffected for the other nodes in the network. The change in local assortativity of the four nodes involved in the rewiring can be calculated (by calculating the new $\bar{k}$ for these four nodes) and with that the new assortativity.

We briefly describe scalar assortativity. The scalar assortativity, L, is a generalization of Newman's assortativity; it measures the tendency of nodes to be connected to other nodes having the same or having opposing (scalar) value (Newman [5]). The 'value' of a node may be a Boolean or a (continuous) scalar value, hence the term 'scalar assortativity'. The scalar value of a node may change over time; nodes in a network may be subject to change, imposed by arbitrary stimulus or influence. A change in a node's scalar value hence affects the scalar assortativity of the network. The scalar assortativity of a network may hence be expressed as a function of time, L^t. A scalar assortativity value of $L^t = 1$ means that at time t, all links in the network have identical node state at either side of the link. For a network with $L^t = -1$, all links in the network have, at time t, nodes with dissimilar node state at either side of the link. As expected, $L^t = 0$ means that at time t, nodes with a particular state are equally likely connected to nodes with the same state as connected to nodes with dissimilar states. Scalar assortativity is a useful network metric when studying dynamic networks whereby the state (scalar value) of a node changes over time.

Piraveenan *et al.* [17] introduce the concept of node congruity, l. The node congruity is defined as the contribution of a particular node to the network's scalar assortativity. We may regard the node congruity as *local scalar assortativity*. The sum of the node congruity for all nodes in the network equals the scalar assortativity of that network. Considering that the scalar value of the nodes in a network may change over time, the node congruity will also change over time. At time t, a node's congruity is denoted as l^t. The congruity of node i represents a scaled difference between the average state (or scalar value) of the neighbours of node i and the average state (or scalar value) of the network as a whole. When the neighbours of node i have on average a higher scalar value than the expected value for the entire network, then node i has a positive congruity. A negative congruity for node i occurs when the neighbours of node i have on average a lower scalar value than the expected value for the entire network. Piraveenan *et al.* [17] continue to show that the distribution of node congruity for a network provides additional tool to study a network's dynamic behaviour.

Xu *et al.* [29] have also observed that within a network, the overall assortativity of that network may be (strongly) influenced by a small number of highly connected nodes, commonly referred to as *rich nodes* or *superrich nodes*, as appropriate. Nodes with low degree may exhibit a different mixing pattern than nodes with high degree in that same network. For networks with power-law degree distribution, superrich nodes are defined as nodes whose degree exceeds the cut-off degree k_c [29]. For a network that is overall scale-free and that contains one or more nodes with $k \gg k_c$, these (super)rich nodes are apparently not the result of the growth process of that network. They may, instead, have been artificially added. A (small) number of superrich nodes may therefore give a false overall assortativity value for the network as a whole. Xu *et al.* [29] propose a modified method for calculating assortativity, namely by excluding the superrich nodes. This modified definition of assortativity is denoted as ρ_c. Xu *et al.* [29] state that ρ_c constitutes a more realistic descriptor of the network than ρ_D. To determine from what degree onwards the (super)rich nodes in the network start to distort the assortativity, ρ_c should be calculated over $n = \sum_{i=1}^{i=N} 1|_{k_i \leqslant p}$ nodes, with p ranging from k_c to $N-1$. Assortativity is then reflected as a graph, with ρ_c a function of p. As p becomes larger, starting at k_c, the assortativity becomes more and more affected by the superrich nodes. Rationale of this approach is that not *every* node with $k \geqslant k_c$ is necessarily a superrich node. A node with degree slightly higher than k_c may be the natural result of the growth process, so ought to be taken into account.

A final form of local assortativity we study is the Universal Assortativity Coefficient (UAC) defined by Zhang *et al.* [64]. The UAC assigns an assortativity contribution value to a single link or to a group of links. The assortativity contribution value for a single link is formed by the relation between the remaining degree of the nodes on the end-points of the link and the expected remaining degree of the network as a whole. The UAC for a link l, ρ_l, is defined as:

$$\rho_l = \frac{(j - U_q)(k - U_q)}{L\sigma_q^2} \tag{27}$$

where

j, k the remaining degree of the respective nodes at the end of link l; item$U_q = E[D-1]$, the expected value of the remaining degree of the entire network;

σ_q standard deviation of the remaining degree distribution of the entire network; item$L$ number of links in the network.

The following holds:

$$\rho = \sum_{l=1}^{L} \rho_l \tag{28}$$

Analogous to the general definition of assortativity, when $\rho_l > 0$, the link is said to be assortative, when $\rho_l < 0$, the link is said to be disassortative and when $\rho_l = 0$, the link is said to be non-assortative. According to Equation (27), a link is assortative when for both ends of the link, the remaining degree is higher or lower than the expected degree remaining degree of the entire network. Otherwise, the link will be disassortative, unless for both ends of the link, the remaining degree is equal to the expected degree remaining degree of the entire network. The absolute value $|\rho_l|$ is the assortative (for $\rho_l > 0$) or disassortative (for $\rho_l < 0$) strength of the link. Equation (27) can be used on a group of links; the UAC for a group of links is the sum of the individual ρ_l. Zhang *et al.* [64] argue that their definition of local assortativity is advantageous compared with the definition from Piraveenan *et al.* [18]. Their main argument is that their definition pertains to links, whereas the definition form Piraveenan *et al.* [18] pertains to nodes. The link local assortativity can be applied to an arbitrary set of links, e.g. the links of a single node.

11. Conclusions and future work

The concept of assortativity has been extensively studied since its introduction by Newman [1]. The assortativity, being a second-order metric, adds insight in the characteristics of a network. Although assortativity as a concept may be applied to any qualification of a node, it is most often applied on the degree of a node, yielding *degree assortativity*. Various adapted forms of assortativity have been proposed over time. Assortativity for directed networks and for networks with weighted links needs further study. Social networks, for example, exhibit connections that have *direction*. Communication networks contain links that may have *weight*, expressing, e.g. link capacity or transmission cost.

Networks may be rewired for changing static or dynamic behaviour of that network, e.g. to increase or decrease the assortativity of that network. DPR is studied extensively as a means to rewire a network without changing its degree vector d. When links are directional, DPR must distinguish between in-degree and out-degree of the nodes involved in the rewiring and keep both constant. When links have a weight, rewiring may be done degree-preserving, but then the nodes involved in the rewiring end up with a different *connectivity*. This follows from the fact that links with different weight contribute different to the connectivity of a node.

Assortativity alone is not always sufficiently representative of the network as a whole. A network may be assortative overall, but some nodes would qualify as particularly disassortative, or the other way around. Each node or each link in that network contributes in some portion to the overall assortativity. This leads to *local assortativity* (local node assortativity, local link assortativity). The assortativity of a network may be reflected through a node-assortativity PDF or link-assortativity density function. Many aspects of local assortativity remain unexplored.

The relation between line graphs and assortativity, as well as the relation between complementary graphs and assortativity, has been studied in various papers. Constructing line graph and constructing complementary graph is generally defined for simple graphs containing undirected, non-weighted links. Finally, we propose the following areas for further research in assortativity:

– Graph theoretic definitions of assortativity for networks containing weighted links.

– Network rewiring methodology for networks containing weighted links and/or directed links.

– Relation between assortativity and betweenness, also in combination with network rewiring.

– Using the line graph H of a root graph G as a tool for applying (degree-preserving) rewiring in the root graph G.

– Using the complementary graph G^c as a tool for applying (degree-preserving) rewiring in the original graph G.

– Distribution of local node assortativity and local link assortativity, also in combination with network rewiring. Can we apply DPR to a network to alter the local node assortativity distribution or local link assortativity distribution, whilst keeping the assortativity of the network as whole unaffected.

– Relation between assortativity, betweenness and effective graph resistance.

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References

1. Newman, M. (2002) Assortative mixing in networks. *Phys. Rev. Lett.*, **89**, 208701.
2. Quayle, A., Siddiqui, A. & Jones, A. (2006) Modeling network growth with assortative mixing. *Eur. Phys. J. B* **50**, 617–630.
3. Nagoshi, C., Johnson, R. & Danko, G. (1990) Assortative mating for cultural identification as indicated by language use. *Behav. Genet.*, **20**, 23–31.
4. Van Mieghem, P. (2014) *Performance Analysis of Complex Networks and Systems*. Cambridge: Cambridge University Press.
5. Newman, M. (2003) Mixing patterns in networks. *Phys. Rev. E*, **67**, 026126.
6. Newman, M. (2003) The structure and function of complex networks. *SIAM Rev.*, **45**, 167–256.
7. D'Agostino, G., Scala, A., Zlatic, V. & Caldarelli, G. (2012) Assortativity effects on diffusion-like processes in scale-free networks. *Eur. Phys. Lett.*, **97**, 68006.
8. Chang, H., Su, B., Zhou, Y. & He, D. (2007) Assortativity and act degree distribution of some collaboration networks. *Phys. A: Stat. Mech. Appl.*, **383**, 687–702.
9. Estrada, E. (2011) Combinatorial study of degree assortativity in networks. *Phys. Rev. E*, **84**, 047101.
10. Holme, P. & Zhao, J. (2007) Exploring the assortativity-clustering space of a network's degree sequence. *Phys. Rev. E*, **75**, 046111.
11. Konig, M., Tessone, C. & Zenou, Y. (2008) From Assortative to Dissortative Networks: The Role of Capacity. CCSS Working Paper Series.

12. Leung, C. & Chau, H. (2007) Weighted assortative and disassortative networks Model. *Phys. A: Stat. Mech. Appl.*, **378**, 591–602.

13. Litvak, N. & van der Hofstad, R. (2013) Uncovering disassortativity in large scale-free networks. *Phys. Rev. E*, **87**, 1–11.

14. Liu, D., Trajanovski, S. & Van Mieghem, P. (2013) Random line graphs and a linear law for assortativity. *Phys. Rev. E*, **87**, 012816, 1–10.

15. Manka-Krason, A. & Kulakowski, K. (2010) Assortativity in random line graphs. *Acta Phys. Pol. B*, **3**, 259–268.

16. Nguyen, D. & Tran, D. (2010) On generating power-law networks with assortative mixing. *International Conference on Communications and Electronics*, Nha Trang, pp. 30–35.

17. Piraveenan, M., Prokopenko, M. & Zomaya, A. (2011) On congruity of nodes and assortative information content in complex networks. *School of information technologies, Sydney University*, Technical Report 668.

18. Piraveenan, M., Prokopenko, M. & Zomaya, A. (2008) Local assortativeness in scale-free networks. *Europhys. Lett.*, **84**, 28002.

19. Piraveenan, M., Prokopenko, M. & Zomaya, A. (2008) Assortativeness and information in scale-free networks. *Eur. Phys. J. B*, **67**, 291–300.

20. Piraveenan, M., Prokopenko, M. & Zomaya, A. (2009) Assortative Mixing in Directed Networks. *School of information technologies, Sydney University*, Technical Report 637.

21. Piraveenan, M. (2010) Topological analysis of complex networks using assortativity. PhD thesis.

22. Piraveenan, M., Prokopenko, M. & Zomaya, A. (2010) Assortative mixing in directed biological networks. *IEEE/ACM Trans. Comput. Biol. Bioinf.*, **9**, 66–78.

23. Piraveenan, M., Prokopenko, M. & Zomaya, A. (2006) Information-cloning of scale-free networks. *Proceedings of the 9th European conference on Advances in artificial life*, Lisbon, pp. 925–935.

24. Van Mieghem, P., Ge, X., Schumm, P., Trajanovski, S. & Wang, H. (2010) Spectral graph analysis of modularity and assortativity. *Phys. Rev. E*, **82**, 056113.

25. Van Mieghem, P., Wang, H., Ge, X., Tang, S. & Kuipers, F. (2010) Influence of assortativity and degree-preserving rewiring on the spectra of networks. *Eur. Phys. J. B*, **76**, 643–652.

26. Wang, H., Winterbach, W. & Van Mieghem, P. (2011) Assortativity of complementary graphs. *Eur. Phys. J. B*, **83**, 203–214.

27. Winterbach, W., Ridder, D., Wang, H., Reinders, M. & Van Mieghem, P. (2012) Do greedy assortativity optimization algorithms produce good results? *Eur. Phys. J. B*, **85**, article 151.

28. Xia, Y., Tse, C. & Lau, F. (2012) Effect of assortativity on traffic performance in scale-free networks. *IEEE Int. Symp. on Circuits Syst.*, 906–909. http://ieeexplore.ieee.org/xpl/articleDetails.jsp?arnumber=6272190&sortType%3Dasc_p_Sequence%26filter%3DAND(p_IS_Number%3A6270389)%26pageNumber%3D10.

29. Xu, X., Zhang, J., Sun, J. & Small, M. (2009) Revising the simple measures of assortativity in complex networks. *Phys. Rev. E*, **80**, 056106.

30. Xulvi-Brunet, R. & Sokolov, I. (2005) Changing correlations in networks: assortativity and dissortativity. *Acta Phys. Pol. B*, **36**, 1431–1455.

31. Zhou, J., Xu, X., Zhang, J., Sun, J., Small, M. & Lu, J. (2008) Generating an assortative network with a given degree distribution. *Int. J. Bifurcation Chaos*, **18**, 3495–3502.

32. Zhou, D., Stanley, H., D'Agostino, G. & Scala, A. Assortativity decreases the robustness of interdependent networks. *Phys. Rev. E*, **86**, 066103, 1–7.

33. Foster, J., Foster, D., Grassberger, P. & Paczuski, M. (2010) Edge direction and the structure of networks. *Proc. Natl Acad. Sci. USA*, **107**, 10815–10820.

34. Van Mieghem, P. (2010) *Graph Spectra for Complex Networks*. Cambridge: Cambridge University Press.

35. Erdős, P. & Rényi, A. (1959) On random graphs. *Publicationes Mathematicae*, **6**, 290–297.

36. Erdős, P. & Rényi A. (1960) On the evolution of random graphs. *Publ. Math. Inst. the Hung. Acad. Sci.*, **5**, 17–61.

37. Barabási, A. & Albert, R. (1999) Emergence of scaling in random networks. *Science*, **286**, 509–512.

38. Watts, D. & Strogatz, S. (1998) Collective dynamics of 'small-world' networks. *Nature*, **393**, 440–442.
39. Callaway, D., Hopcroft, J., Kleinberg, J., Newman, M. & Strogatz, S. (2001) Are randomly grown graphs really random. *Phys. Rev. E*, **64**, 041902.
40. Catanzaro, M., Caldarelli, G. & Pietronero, L. (2004) Assortative model for social networks. *Phys. Rev. E*, **70**, 037101.
41. Grossman, J. & Ion, P. (1995) On a portion of the well-known collaboration graph. *Conf. Comb. Graph Theory Comput.*, **108**, 129–131.
42. Pastor-Satorras, R., Vázquez, A. & Vespignani, A. (2001) Dynamical and correlation properties of the internet. *Phys. Rev. Lett.*, **87**, 258701, 1–4.
43. Chen, N. & Olvera-Cravioto, M. (2013) Directed random graphs with given degree distributions. *Stoch. Syst.*, **3**, 147–186.
44. Sole, R. & Valverde, S. (2004) Information theory of complex networks?: on evolution and architectural constraints. *Complex Netw., Lect. Notes Phys.*, **650**, 189–207.
45. Wang, H., Douw, L., Hernández, J., Reijneveld, J., Stam, C. & Van Mieghem, P. (2010) Effect of tumor resection on the characteristics of functional brain networks. *Phys. Rev. E*, **82**, 021924.
46. Spearman, C. Refer to http://udel.edu/∼mcdonald/statspearman.html for explanation of Spearman's rho.
47. Martin-Hernandez, J., Li, Z. & Van Mieghem, P. (2014) Weighted betweenness and algebraic connectivity. *J. Complex Netw.*, **2**, 272–287.
48. Noldus, R. & Van Mieghem, P. (2013) Effect of degree-preserving, assortative rewiring on OSPF router configuration. *25th International Teletraffic Congress (ITC)*, Shanghai, 10–12 September 2013.
49. Maslov, S. & Sneppen, K. (2002) Specificity and stability in topology of protein networks. *Science* **296**, 910–913.
50. Youssef, M., Khorramzadeh, Y. & Eubank, S. (2013) Network reliability: the effect of local network structure on diffusive processes. *Phys. Rev. E*, **88**, 052810(10).
51. Li, L., Alderson, D., Willinger, W. & Doyle, J. (2004) A first-principles approach to understanding the internet's router-level topology. *ACM SIGCOMM Comput. Commun. Rev.*, **34**, 3–14.
52. Gutman, I. & Trinajstic, N. (1972) Graph theory and molecular orbitals, Total π electron energy of alternant hydrocarbons. *Chem. Phys. Lett.*, **17**, 535–538.
53. Makowiec, D. (2005) Evolving network—simulation study. From regular lattice to scale free network. *Eur. Phys. J. B – Condens. Matter Complex Syst.*, **48**, 547–555.
54. Dorogovtsev, N. & Mendes, J. (2002) Evolution of networks. *Adv. Phys.*, **51**, 1079–1187.
55. Li, C., Wang, H. & Van Mieghem, P. (2013) Epidemic threshold in directed networks. *Phys. Rev. E*, **88**, 062802, 1–11.
56. Myers, C. (2003) Software systems as complex networks: structure, function, and evolvability of software collaboration graphs. *Phys. Rev. E* **68**, 046116.
57. Fiedler, M. (1973) Algebraic connectivity of graphs. *Czech. Math. J.*, **23**, 298–305.
58. Cvetkoviæ, D., Doob, M. & Sachs, H. (1980) *Spectra of Graphs: Theory and Application*. New York: Academic Press.
59. Mohar, B. (1991) The Laplacian spectrum of graphs. Graph theory, combinatorics and applications", *Wiley*, **2**, 871–898.
60. Klein, D. & Randiæ, M. (1993) Resistance distance. *J. Math. Chem.*, **12**, 81–95.
61. Ellens, W., Spieksma, F., Van Mieghem, P., Jamakovic, A. & Kooij, R. (2011) Effective graph resistance. *Linear Algebra Appl.*,**435**, 2491–2506.
62. Trajanovski, S., Martín-Hernández, J., Winterbach, W. & Van Mieghem, P. (2013) Robustness envelopes of networks. *J. Complex Netw.*, **1**, 44–62.
63. Liu, D., Trajanovski, S. & Van Mieghem, P. (2014) ILIGRA: an efficient inverse line graph algorithm. *J. Math. Model. Algorithms Oper. Res.* (to appear).
64. Zhang, G., Cheng, S. & Zhang, G. (2011) A universal assortativity measure for network analysis. *Phys. Rev. E.* http://arxiv.org/abs/1212.6456.