The cylinder at spatial infinity and asymptotic charges

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Introduction

- The Newman-Penrose (NP) constants serve as conserved quantities at null infinity in asymptotically flat gravitational fields.
- These constants present a comprehensive conservation system for various spins: spin-1 fields and spin-2 fields, with our research focusing on spin-0 fields linked to wave equation solutions.
- In the detailed context, while an infinite series of conserved quantities is identified in the linear theory, the non-linear General Relativity theory conserves only ten.

Conservation laws

- These charges are computed as 2-surface integrals at cuts $C \approx \mathbb{S}^2$ of null infinity $\mathscr{I}$.

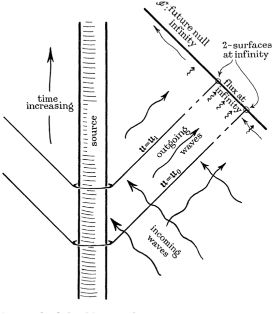

Figure: Visual representation of the behavior of the Newman-Penrose constants at null infinity.

Spin-1 (EM) Field

- A complex tetrad can be selected as follows:

$$l^\mu = \delta_1^\mu, \quad n^\mu = \delta_0^\mu - \delta_1^\mu,$$
$$m^\mu = \frac{1}{\tau}\left(\delta_2^\mu + \frac{1}{\sin\theta}\,\delta_3^\mu\right), \qquad \overline{m}^\mu = \frac{1}{\tau}\left(\delta_2^\mu - \frac{1}{\sin\theta}\,\delta_3^\mu\right). \quad (1)$$

- To describe the electromagnetic (EM) field, we make use of three complex tetrad components of the Maxwell field tensor denoted as $F_{\mu\nu}$:

$$\Phi_0 = F_{\mu\nu} l^\mu n^\nu,$$
$$\Phi_1 = \frac{1}{2} F_{\mu\nu}(l^\mu n^\nu + \overline{m}^\mu m^\nu),$$
$$\Phi_2 = F_{\mu\nu} \overline{m}^\mu n^\nu. \quad (2)$$

NP Constants Calculation

- The NP constants are calculated using the following formula:

$$F_m^{n,k} = \int {}_1\overline{Y}_{n-k+1,m} \Phi_0^{n+1} d\omega. \tag{3}$$

- The interpretation of charges, such as the Newman-Penrose constants, remains an open debate, yet their conservation is evident in general asymptotically flat spacetimes, even in events like black hole collisions.

The i^0 cylinder representation in Minkowski spacetime

- Consider spherical polar coordinates $(\tilde{t}, \tilde{\rho}, \vartheta^A)$ with $A = 1, 2$.
- The metric of physical Minkowski spacetime in this coordinate system is given by $\tilde{\boldsymbol{\eta}}$:

$$\tilde{\boldsymbol{\eta}} = -\mathbf{d}\tilde{t} \otimes \mathbf{d}\tilde{t} + \mathbf{d}\tilde{\rho} \otimes \mathbf{d}\tilde{\rho} + \tilde{\rho}^2 \boldsymbol{\sigma}. \tag{4}$$

- Introduce unphysical spherical polar coordinates (t, ρ, ϑ^A) as an intermediate step:

$$t = \frac{\tilde{t}}{\tilde{\rho}^2 - \tilde{t}^2}, \quad \rho = \frac{\tilde{\rho}}{\tilde{\rho}^2 - \tilde{t}^2}. \tag{5}$$

- The conformal metric in unphysical coordinates, $\boldsymbol{\eta} = \Xi^2 \tilde{\boldsymbol{\eta}}$:

$$\boldsymbol{\eta} = -\frac{1}{\tilde{\rho}^2 - \tilde{t}^2}(-\mathbf{d}\tilde{t} \otimes \mathbf{d}\tilde{t} + \mathbf{d}\tilde{\rho} \otimes \mathbf{d}\tilde{\rho} + \tilde{\rho}^2 \boldsymbol{\sigma}). \tag{6}$$

i^0 Representation in Unphysical Coordinates

- In this conformal representation ($t \in (-\infty, \infty)$, $\rho \in [0, \infty)$), spatial infinity and the origin interchange.
- i^0 is represented by the point $(t = 0, \rho = 0)$ in $(\mathbb{R}^4, \eta)$.
- Introduce coordinates $(\tau, \rho, \vartheta^A)$ with $t = \rho\tau$.
- Consider the conformal metric $\boldsymbol{g} = \rho^{-2}\boldsymbol{\eta}$.
- Express the unphysical metric $\boldsymbol{g}$ in F-coordinates:

$$\boldsymbol{g} = -\mathbf{d}\tau \otimes \mathbf{d}\tau + \frac{1-\tau^2}{\rho^2}\mathbf{d}\rho \otimes \mathbf{d}\rho - \frac{\tau}{\rho}(\mathbf{d}\rho \otimes \mathbf{d}\tau + \mathbf{d}\rho \otimes \mathbf{d}\tau) + \boldsymbol{\sigma}. \tag{7}$$

Conformal Factor and Lorentz Transformation

- The conformal factor Θ in F-coordinates and physical coordinates:

$$\Theta := \rho(1-\tau^2) = \frac{1}{\tilde{\rho}}. \tag{8}$$

- The boost parameter κ:

$$\kappa := \frac{1+\tau}{1-\tau} = -\frac{\tilde{v}}{\tilde{u}}. \tag{9}$$

- The Lorentz transformation that connects the NP and F-frames:

$$(\Lambda_+)^2 := \Theta^{-1}\kappa^{-1}, \quad (\Lambda_-)^2 := \Theta^{-1}\kappa. \tag{10}$$

i^0 Cylinder and Null Frames

- Identify future and past null infinity in the conformal representation:

$$
\begin{aligned}
\mathscr{I}^+ &\equiv \{p \in \mathcal{M} \mid \tau(p) = 1\},\\
\mathscr{I}^- &\equiv \{p \in \mathcal{M} \mid \tau(p) = -1\}.
\end{aligned}
$$

- The i^0-cylinder represents spatial infinity as an extended set $I \approx \mathbb{R} \times \mathbb{S}^2$:

$$
\begin{aligned}
I &\equiv \{p \in \mathcal{M} \mid \ |\tau(p)| \ 1,\ \rho(p) = 0\},\\
I^0 &\equiv \{p \in \mathcal{M} \mid \tau(p) = 0,\ \rho(p) = 0\}.
\end{aligned}
$$

F-Frame and Null Frames

- Introduce the F-frame:

$$\boldsymbol{e} = (1+\tau)\boldsymbol{\partial}_\tau - \rho\boldsymbol{\partial}_\rho, \quad \underline{\boldsymbol{e}} = (1-\tau)\boldsymbol{\partial}_\tau + \rho\boldsymbol{\partial}_\rho, \quad \boldsymbol{e_A} \quad \text{with} \quad \boldsymbol{A} = \{\uparrow,\downarrow\}. \tag{11}$$

- The NP-frame hinged at $\mathscr{I}^\pm$:

$$\begin{aligned} \text{NP hinged at } \mathscr{I}^+ &: \quad \boldsymbol{e}^+, \underline{\boldsymbol{e}}^+, \boldsymbol{e}^+_{\boldsymbol{A}}, \\ \text{NP hinged at } \mathscr{I}^- &: \quad \boldsymbol{e}^-, \underline{\boldsymbol{e}}^-, \boldsymbol{e}^-_{\boldsymbol{A}}. \end{aligned}$$

- Transformation between NP and F-frames:

$$\begin{aligned} \textit{NP hinged at } \mathscr{I}^+ &: \quad \boldsymbol{e}^+ = \Theta^{-2}L, \quad \underline{\boldsymbol{e}}^+ = \underline{L}, \quad \boldsymbol{e}^+_{\boldsymbol{A}} = \boldsymbol{e_A} = \Theta^{-1}\tilde{\boldsymbol{e}}_{\boldsymbol{A}} \\ \textit{NP hinged at } \mathscr{I}^- &: \quad \boldsymbol{e}^- = L, \quad \underline{\boldsymbol{e}}^- = \Theta^{-2}\underline{L}, \quad \boldsymbol{e}^-_{\boldsymbol{A}} = \boldsymbol{e_A} = \Theta^{-1}\tilde{\boldsymbol{e}}_{\boldsymbol{A}}. \end{aligned}$$

Spin-0 fields close to i^0 and $\mathscr{I}$

Results

Conclusion

References

Thank You