The cylinder at spatial infinity and asymptotic charges

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- The Newman-Penrose (NP) constants are conserved quantities at infinity.
- These constants present a comprehensive conservation system for various spins.
- Infinite conserved quantities in the linear theory, and ten in the non-linear theory.
- Computed as 2-surface integrals at cuts $C \approx \mathbb{S}^2$ of null infinity $\mathscr{I}$.

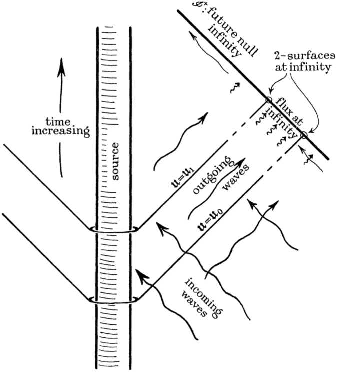

Figure: Visual representation of the behavior of the Newman-Penrose constants at null infinity.

The i^0 cylinder representation in Minkowski spacetime

- The physical metric is given by $\tilde{\boldsymbol{\eta}}$:

$$\tilde{\boldsymbol{\eta}} = -\mathbf{d}\tilde{t} \otimes \mathbf{d}\tilde{t} + \mathbf{d}\tilde{\rho} \otimes \mathbf{d}\tilde{\rho} + \tilde{\rho}^2 \boldsymbol{\sigma}. \tag{1}$$

- The conformal metric in unphysical coordinates, $\boldsymbol{\eta} = \Xi^2 \tilde{\boldsymbol{\eta}}$:

$$\boldsymbol{\eta} = -\frac{1}{\tilde{\rho}^2 - \tilde{t}^2}\left(-\mathbf{d}\tilde{t} \otimes \mathbf{d}\tilde{t} + \mathbf{d}\tilde{\rho} \otimes \mathbf{d}\tilde{\rho} + \tilde{\rho}^2 \boldsymbol{\sigma}\right). \tag{2}$$

- Introduce coordinates $(\tau, \rho, \vartheta^A)$ with $t = \rho\tau$.
- Consider the conformal metric $\boldsymbol{g} = \rho^{-2}\boldsymbol{\eta}$.
- Unphysical metric $\boldsymbol{g}$ in F-coordinates:

$$\boldsymbol{g} = -\mathbf{d}\tau \otimes \mathbf{d}\tau + \frac{1-\tau^2}{\rho^2}\mathbf{d}\rho \otimes \mathbf{d}\rho - \frac{\tau}{\rho}\left(\mathbf{d}\rho \otimes \mathbf{d}\tau + \mathbf{d}\rho \otimes \mathbf{d}\tau\right) + \boldsymbol{\sigma}. \tag{3}$$

Conformal Factor and Lorentz Transformation

- The conformal factor Θ:

$$\Theta := \rho(1-\tau^2) = \frac{1}{\tilde{\rho}}. \qquad (4)$$

- The boost parameter κ:

$$\kappa := \frac{1+\tau}{1-\tau} = -\frac{\tilde{v}}{\tilde{u}}. \qquad (5)$$

- Connection between NP and F-frames:

$$(\Lambda_+)^2 := \Theta^{-1}\kappa^{-1},$$
$$(\Lambda_-)^2 := \Theta^{-1}\kappa. \qquad (6)$$

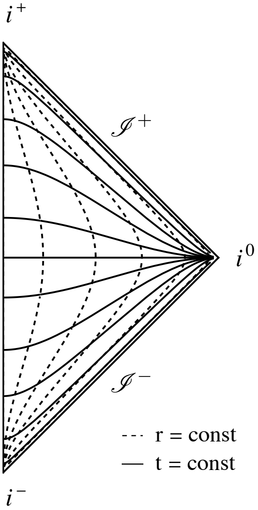

Figure: Representation of the compactified Minkowski spacetime.

i^0 Cylinder and Null Frames

- Identify future and past null infinity in the conformal representation:

$$\mathscr{I}^+ \equiv \{p \in \mathcal{M} \mid \tau(p) = 1\},$$
$$\mathscr{I}^- \equiv \{p \in \mathcal{M} \mid \tau(p) = -1\}.$$

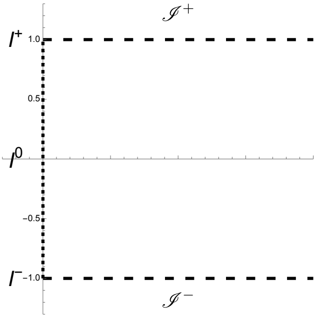

Figure: Representation of the Friedrich Cylinder.

- The i^0-cylinder represents spatial infinity:

$$\mathcal{I} \equiv \{p \in \mathcal{M} \mid \ |\tau(p)| \leq 1,\ \rho(p) = 0\}, \tag{7}$$

$$\mathcal{I}^0 \equiv \{p \in \mathcal{M} \mid \tau(p) = 0,\ \rho(p) = 0\}. \tag{8}$$

F-Frame and Null Frames

- Introduce the F-frame:

$$\boldsymbol{e} = (1+\tau)\boldsymbol{\partial}_\tau - \rho\boldsymbol{\partial}_\rho, \quad \underline{\boldsymbol{e}} = (1-\tau)\boldsymbol{\partial}_\tau + \rho\boldsymbol{\partial}_\rho, \quad \boldsymbol{e}_{\boldsymbol{A}} \quad \text{with} \quad \boldsymbol{A} = \{\uparrow, \downarrow\}. \tag{9}$$

- The NP-frame hinged at $\mathscr{I}^{\pm}$:

$$\begin{aligned} &\text{NP hinged at } \mathscr{I}^+ : \quad \boldsymbol{e}^+, \underline{\boldsymbol{e}}^+, \boldsymbol{e}^+_{\boldsymbol{A}}, \\ &\text{NP hinged at } \mathscr{I}^- : \quad \boldsymbol{e}^-, \underline{\boldsymbol{e}}^-, \boldsymbol{e}^-_{\boldsymbol{A}}. \end{aligned}$$

- Transformation between NP and F-frames:

$$\begin{aligned} &\textit{NP hinged at } \mathscr{I}^+ : \quad \boldsymbol{e}^+ = \Theta^{-2} L, \quad \underline{\boldsymbol{e}}^+ = \underline{L}, \quad \boldsymbol{e}^+_{\boldsymbol{A}} = \boldsymbol{e}_{\boldsymbol{A}} = \Theta^{-1}\tilde{\boldsymbol{e}}_{\boldsymbol{A}} \\ &\textit{NP hinged at } \mathscr{I}^- : \quad \boldsymbol{e}^- = L, \quad \underline{\boldsymbol{e}}^- = \Theta^{-2}\underline{L}, \quad \boldsymbol{e}^-_{\boldsymbol{A}} = \boldsymbol{e}_{\boldsymbol{A}} = \Theta^{-1}\tilde{\boldsymbol{e}}_{\boldsymbol{A}}. \end{aligned}$$

- The transformation of the D'Alembertian operator is,

$$\square\phi - \frac{1}{6}\phi R = \Omega^{-3}\left(\tilde{\square}\tilde{\phi} - \frac{1}{6}\tilde{\phi}\tilde{R}\right). \tag{10}$$

- Using F-coordinates, the wave equation is represented by

$$\left(\tau^2 - 1\right)\partial_\tau^2\phi - 2\rho\tau\partial_\tau\partial_\rho\phi + \rho^2\partial_\rho^2\phi + 2\tau\partial_\tau\phi + \Delta_{\mathbb{S}^2}\phi = 0. \tag{11}$$

- We consider the Ansatz

$$\phi = \sum_{p=0}^{\infty}\sum_{\ell=0}^{p}\sum_{m=-\ell}^{m=\ell}\frac{1}{p!}a_{p;\ell,m}(\tau)\rho^p Y_{\ell m}. \tag{12}$$

- Solving (11) simplifies to:

$$(1-\tau^2)\ddot{a}_{p;\ell,m} + 2\tau(p-1)\dot{a}_{p,\ell,m} + (\ell+p)(\ell-p+1)a_{p;\ell,m} = 0. \tag{13}$$

Lemma

The solution to equation (13) *is given by:*

1. *For* $p \geq 1$ *and* $0 \leq \ell \leq p-1$

$$a(\tau)_{p;\ell,m} = A_{p,\ell,m}\left(\frac{1-\tau}{2}\right)^p P_\ell^{(p,-p)}(\tau) + B_{p,\ell,m}\left(\frac{1+\tau}{2}\right)^p P_\ell^{(-p,p)}(\tau) \tag{14}$$

2. *For* $p \geq 0$ *and* $\ell = p$*:*

$$a_{p;p,m}(\tau) = \left(\frac{1-\tau}{2}\right)^p \left(\frac{1+\tau}{2}\right)^p \left(C_{p,p,m} + D_{p,p,m}\int_0^\tau \frac{ds}{(1-s^2)^{p+1}}\right) \tag{15}$$

- $p = 0$ and $p = 1$ cases:

$$a_{0;0,0}(\tau) = C_{000} + \tfrac{1}{2} D_{000}(\log(1+\tau) - \log(1-\tau)) \tag{16}$$

$$\begin{aligned} a_{1;1,m}(\tau) = \tfrac{1}{4}(1-\tau)(1+\tau)(C_{11m} + \tfrac{1}{4} D_{11m}(\log(1+\tau) - \\ - \log(1-\tau) + 2\tau(1-\tau^2))). \end{aligned} \tag{17}$$

- These logarithmic terms have implications for the linear version of the associated peeling property.

Remark

(Regularity condition). *Lemma 1* implies that expanding the integral in (15) results in logarithmic terms, hence $D_{p,p,m} = 0$ is called the regularity condition. The solutions for $a(\tau)$ are polynomic in τ, except for $\ell = p$ where one needs to impose the regularity condition to only have polynomic solutions.

- Expanding $\tilde{\phi}$ in terms of the F-frame:

$$\tilde{\phi} = \Theta\phi \Leftrightarrow \tilde{\phi} = \frac{C_{000}}{\tilde{\rho}} + \frac{1}{2\tilde{\rho}} D_{000} \log\left(\frac{\tilde{\rho}+\tilde{t}}{\tilde{\rho}-\tilde{t}}\right) Y_{00} + \frac{1}{16\tilde{\rho}^2}$$
$$\left[D_{11-1} \log\left(\frac{\tilde{\rho}+\tilde{t}}{\tilde{\rho}-\tilde{t}}\right) Y_{1-1} + D_{110} \log\left(\frac{\tilde{\rho}+\tilde{t}}{\tilde{\rho}-\tilde{t}}\right) Y_{10} + \right]$$
$$+ \left[D_{111} \log\left(\frac{\tilde{\rho}+\tilde{t}}{\tilde{\rho}-\tilde{t}}\right) Y_{11}\right] +$$
$$+ \frac{1}{2\tilde{\rho}^2} \left(A_{100} + B_{100}\right) Y_{00} + \frac{1}{4\tilde{\rho}^2} \left(C_{11-1} Y_{1-1} + C_{110} Y_{10} + C_{111} Y_{11}\right). \tag{18}$$

- In the spin-0 case, the peeling property is violated by the logarithmic terms that appear in the expansion of $\tilde{\phi}$.

The NP-constants for the spin-0 fields close to i^0 & $\mathscr{I}$

- Conservation laws:

$$\underline{L}(\tilde{\rho}^{-2\ell} L(e^{+})^{\ell+1}\phi_{\ell m}) = 0, \qquad L(\tilde{\rho}^{-2\ell} \underline{L}(e^{-})^{\ell+1}\phi_{\ell m}) = 0 \tag{19}$$

- Introducing the $f(\tilde{\rho})$-modified NP constants:

$${}^{f}\mathcal{N}^{+}_{\ell,m} := f(\tilde{\rho}) L(\boldsymbol{e}^{+})^{\ell}\phi_{\ell m}\big|_{C^{+}}, \tag{20}$$

$${}^{f}\mathcal{N}^{-}_{\ell,m} := f(\tilde{\rho}) \underline{L}(\underline{\boldsymbol{e}}^{-})^{\ell}\phi_{\ell m}\big|_{C^{-}}. \tag{21}$$

- For $f(\tilde{\rho}) = \tilde{\rho}^2$, we have the "classical NP-constants".

$$\mathcal{N}^{+}_{\ell,m} := (\boldsymbol{e}^{+})^{\ell+1}\phi_{\ell m}\big|_{C^{+}}, \tag{22}$$

$$\mathcal{N}^{-}_{\ell,m} := (\underline{\boldsymbol{e}}^{-})^{\ell+1}\phi_{\ell m}\big|_{C^{-}}. \tag{23}$$

The classical NP constants at $\mathscr{I}^+$

- This analysis is facilitated by the expression,

$$\phi_{\ell m} = \sum_{p=\ell}^{\infty} \frac{1}{p!} a_{p;\ell,m}(\tau)\rho^p. \tag{24}$$

- Considering $\ell = 0$, the computation of $\boldsymbol{e}^+(\phi_{00})$ is sufficient.

$$\boldsymbol{e}^+(\phi_{\ell m}) = 4\rho^{-1}(1+\tau)^{-2} \sum_{p=0}^{\infty} \frac{1}{p!}\rho^p\left((1+\tau)\dot{a}_{p;\ell,m} - p a_{p;\ell,m}\right). \tag{25}$$

With

$$Q^0_{p;\ell,m}(\tau) := (1+\tau)\dot{a}_{p;\ell,m} - p a_{p;\ell,m}. \tag{26}$$

- With this definition in place, we can express $\boldsymbol{e}^+(\phi_{\ell m})$ as follows:

$$\boldsymbol{e}^+(\phi_{\ell m}) = 4(\Lambda_+)^2 \sum_{p=0}^{\infty} \frac{1}{p!}\rho^p Q^0_{p,\ell,m}(\tau). \tag{27}$$

- To compute the $\ell = 0$ NP constant we evaluate at a cut C^+ and use equation (27):

$$\mathcal{N}^+_{0,0} = \lim_{\substack{\rho \to \rho_\star \\ \tau \to 1}} \boldsymbol{e}^+(\phi_{00}) = \sum_{p=0}^{\infty} \frac{1}{p!} \rho_\star^{p-1} Q^0_{p,0,0}(\tau)|_{\mathscr{I}^+} = -A_{100}. \qquad (28)$$

- To calculate the $\ell = 1$ NP constants, one has

$$\mathcal{N}^+_{1,m} = \lim_{\substack{\rho \to 0 \\ \tau \to 1}} 2^{-4} \frac{1}{2!} Q^1_{2,1,m}(\tau) = 3A_{21m}. \qquad (29)$$

- The NP constants for $\mathscr{I}^-$ can be calculated in a similar manner, where the time reversed version of the F-frame is used.

The i^0 cylinder logarithmic NP constants at $\mathscr{I}^-$

- Choice of $f(\tilde{\rho})$.
- The constants $D_{p;\ell,m}$ that define the regularity condition.
- We will compute the $\ell = 0$ and $\ell = 1$ modified NP constants.

$$\tilde{\rho}\underline{L}(\phi_{\ell m}) = \frac{(1+\tau)}{(1-\tau)} \sum_{p=0}^{\infty} \underline{Q}^{0}_{p;\ell,m}(\tau)\rho^{p}. \tag{30}$$

Therefore, for $\ell = 0$, we have:

$$^{\tilde{\rho}}\mathcal{N}^{-}_{0,0} = \lim_{\substack{\rho\to\rho_\star \\ \tau\to -1}} \kappa(\underline{\boldsymbol{e}}^{-})(\phi_{00}) = \sum_{p=0}^{\infty} \rho_\star^{p}\left[\frac{(1+\tau)}{(1-\tau)}\underline{Q}^{0}_{p;0,0}(\tau)\right]\Big|_{\mathscr{I}^-}. \tag{31}$$

- Evaluating at the critical set $\mathcal{I}^-$, we obtain:

$$^{\tilde{\rho}}\mathcal{N}^-_{0,0} = \lim_{\substack{\rho\to\rho_\star \\ \tau\to -1}} \sum_{p=0}^{\infty} \rho_\star^p \left[\frac{(1+\tau)}{(1-\tau)} \underline{Q}^0_{p;0,0}(\tau) \right] = \frac{1}{2} D_{000}. \tag{32}$$

- Similarly, for $\ell = 1$, the relevant quantity to evaluate is:

$$\tilde{\rho}\underline{L}(\underline{\boldsymbol{e}}^-)\phi_{1m} = 4\kappa(\Lambda_-)^2 \sum_{p=1}^{\infty} \frac{1}{p!} \rho^p \underline{Q}^1_{p;\ell,m}(\tau). \tag{33}$$

- Therefore, for $\ell = 1$, we have:

$$^{\tilde{\rho}}\mathcal{N}^-_{1,m} = \lim_{\substack{\rho\to\rho_\star \\ \tau\to -1}} \kappa\underline{\boldsymbol{e}}(\underline{\boldsymbol{e}}^-)\phi_{1m} = \sum_{p=1}^{\infty} \frac{1}{p!} \rho_\star^{p-1} (\kappa \underline{Q}^1_{p;1,m}(\tau))|_{\mathscr{I}^-}. \tag{34}$$

- Evaluating at the critical set $\mathcal{I}^-$, we obtain:

$$^{\tilde{\rho}}\mathcal{N}^-_{1,m} = -\frac{1}{4} D_{11m}. \tag{35}$$

The NP constants in terms of initial data

- Classical NP Constants at $\mathscr{I}^+$: $\mathcal{N}^+_{\ell,m} = q^+(\ell)\, A_{\ell+1,\ell,m}$.
- Modified NP Constants at $\mathscr{I}^+$: ${}^{\tilde{\rho}}\mathcal{N}^+_{\ell,m} = q^+(\ell)\, D_{\ell,\ell,m}$.
- Classical NP Constants at $\mathscr{I}^-$: $\mathcal{N}^-_{\ell,m} = q^-(\ell)\, B_{\ell+1,\ell,m}$.
- Modified NP Constants at $\mathscr{I}^-$: ${}^{\tilde{\rho}}\mathcal{N}^-_{\ell,m} = q^-(\ell)\, D_{\ell,\ell,m}$.

Conclusions

- Computed the NP constants for a spin-0 field near spatial and null infinity.
- Analytic initial data prescribed near i^0 results in solution irregularity at the convergence sets $\mathcal{I}^{\pm}$, controlled by a constant $D_{p;p,m}$.
- When the regularity condition, $D_{p;p,m} = 0$, is unmet, the classical NP constants become undefined.
- Upon satisfying the regularity condition, the classical NP constants at $\mathscr{I}^{\pm}$ arise from unique initial data components.
- While classical NP constants lack correspondence, the i^0 cylinder NP constants at consistently align due to shared origins in the initial data.