# The cylinder at spatial infinity and asymptotic charges

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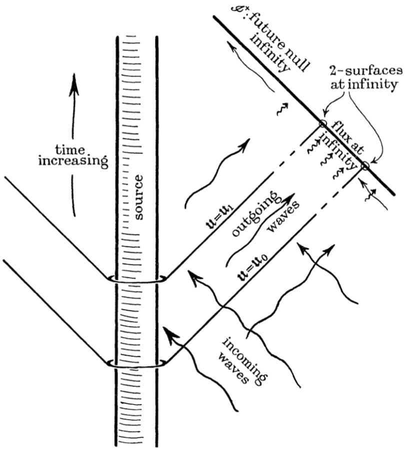
# Introduction

- NP constants conserved at $\mathscr{I}$.
- Linear theory: infinite quantities
  Non-linear theory: ten quantities.
- Computed at cuts $C \approx \mathbb{S}^2$ of $\mathscr{I}$.



# The $i^0$ cylinder representation in Minkowski spacetime

- The physical metric is given by $\tilde{\boldsymbol{\eta}}$:

$$\tilde{\boldsymbol{\eta}} = -\mathbf{d}\tilde{t} \otimes \mathbf{d}\tilde{t} + \mathbf{d}\tilde{\rho} \otimes \mathbf{d}\tilde{\rho} + \tilde{\rho}^2 \boldsymbol{\sigma}.$$

- The conformal metric in unphysical coordinates, $\boldsymbol{\eta} = \Xi^2 \tilde{\boldsymbol{\eta}}$:

$$\boldsymbol{\eta} = -\frac{1}{\tilde{\rho}^2 - \tilde{t}^2}\left(-\mathbf{d}\tilde{t} \otimes \mathbf{d}\tilde{t} + \mathbf{d}\tilde{\rho} \otimes \mathbf{d}\tilde{\rho} + \tilde{\rho}^2 \boldsymbol{\sigma}\right).$$

- Introduce coordinates $(\tau, \rho, \vartheta^A)$ with $t = \rho\tau$.
- Consider the conformal metric $\boldsymbol{g} = \rho^{-2}\boldsymbol{\eta}$.
- Unphysical metric $\boldsymbol{g}$ in $F$-coordinates:

$$\boldsymbol{g} = -\mathbf{d}\tau \otimes \mathbf{d}\tau + \frac{1-\tau^2}{\rho^2}\mathbf{d}\rho \otimes \mathbf{d}\rho - \frac{\tau}{\rho}\left(\mathbf{d}\rho \otimes \mathbf{d}\tau + \mathbf{d}\rho \otimes \mathbf{d}\tau\right) + \boldsymbol{\sigma}.$$

# Conformal Factor and Lorentz Transformation

- The conformal factor $\Theta$:

$$\Theta := \rho(1-\tau^2) = \frac{1}{\tilde{\rho}}.$$

- The boost parameter $\kappa$:

$$\kappa := \frac{1+\tau}{1-\tau} = -\frac{\tilde{v}}{\tilde{u}}.$$

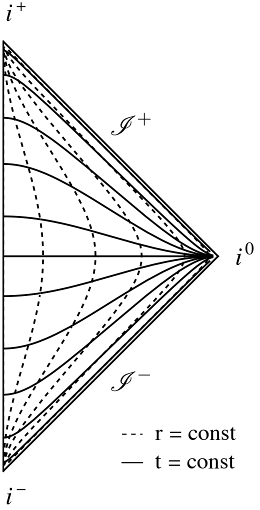



Figure: Minkowski spacetime.

# $i^0$ Cylinder and Null Frames

- Identify $\mathscr{I}^+$ and $\mathscr{I}^-$:

$$\mathscr{I}^+ \equiv \{p \in \mathcal{M} \mid \tau(p) = 1\},$$
$$\mathscr{I}^- \equiv \{p \in \mathcal{M} \mid \tau(p) = -1\}.$$

- The $i^0$-cylinder represents spatial infinity:

$$\mathcal{I} \equiv \{p \in \mathcal{M} \mid \ |\tau(p)| \leq 1,\, \rho(p) = 0\},$$
$$\mathcal{I}^0 \equiv \{p \in \mathcal{M} \mid \tau(p) = 0,\, \rho(p) = 0\}.$$

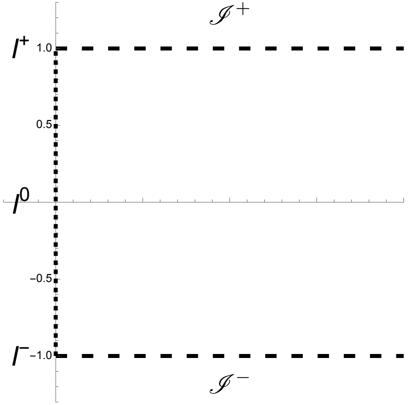



Figure: Friedrich Cylinder.

# $F$-Frame and Null Frames

- Introduce the $F$-frame:

$$\boldsymbol{e} = (1+\tau)\boldsymbol{\partial}_\tau - \rho\boldsymbol{\partial}_\rho, \quad \underline{\boldsymbol{e}} = (1-\tau)\boldsymbol{\partial}_\tau + \rho\boldsymbol{\partial}_\rho, \quad \boldsymbol{e_A} \quad \text{with} \quad \boldsymbol{A} = \{\uparrow, \downarrow\}.$$

- The NP-frame hinged at $\mathscr{I}^\pm$:

$$\begin{aligned} &\text{NP hinged at } \mathscr{I}^+ : \quad \boldsymbol{e}^+, \underline{\boldsymbol{e}}^+, \boldsymbol{e}^+_{\boldsymbol{A}}, \\ &\text{NP hinged at } \mathscr{I}^- : \quad \boldsymbol{e}^-, \underline{\boldsymbol{e}}^-, \boldsymbol{e}^-_{\boldsymbol{A}}. \end{aligned}$$

- NP and physical frames:

$$\begin{aligned} &\textit{NP hinged at } \mathscr{I}^+ : \quad \boldsymbol{e}^+ = \Theta^{-2} L, \quad \underline{\boldsymbol{e}}^+ = \underline{L}, \quad \boldsymbol{e}^+_{\boldsymbol{A}} = \boldsymbol{e_A} = \Theta^{-1}\tilde{\boldsymbol{e}}_{\boldsymbol{A}} \\ &\textit{NP hinged at } \mathscr{I}^- : \quad \boldsymbol{e}^- = L, \quad \underline{\boldsymbol{e}}^- = \Theta^{-2}\underline{L}, \quad \boldsymbol{e}^-_{\boldsymbol{A}} = \boldsymbol{e_A} = \Theta^{-1}\tilde{\boldsymbol{e}}_{\boldsymbol{A}}. \end{aligned}$$

- The transformation of the D'Alembertian operator is,

$$\Box\phi - \frac{1}{6}\phi R = \Omega^{-3}\left(\tilde{\Box}\tilde{\phi} - \frac{1}{6}\tilde{\phi}\tilde{R}\right).$$

- Using $F$-coordinates, the wave equation is represented by

$$\left(\tau^2 - 1\right)\partial_\tau^2\phi - 2\rho\tau\partial_\tau\partial_\rho\phi + \rho^2\partial_\rho^2\phi + 2\tau\partial_\tau\phi + \Delta_{\mathbb{S}^2}\phi = 0. \qquad (1)$$

- We consider the Ansatz

$$\phi = \sum_{p=0}^{\infty}\sum_{\ell=0}^{p}\sum_{m=-\ell}^{m=\ell}\frac{1}{p!}a_{p;\ell,m}(\tau)\rho^p Y_{\ell m}. \qquad (2)$$

- Substituting (2) in (1) simplifies to:

$$(1-\tau^2)\ddot{a}_{p;\ell,m} + 2\tau(p-1)\dot{a}_{p,\ell,m} + (\ell+p)(\ell-p+1)a_{p;\ell,m} = 0. \qquad (3)$$

## Lemma

*The solution to equation (3) is given by:*

1. *For $p \geq 1$ and $0 \leq \ell \leq p-1$*

$$a(\tau)_{p;\ell,m} = A_{p,\ell,m}\left(\frac{1-\tau}{2}\right)^{p} P_{\ell}^{(p,-p)}(\tau) + B_{p,\ell,m}\left(\frac{1+\tau}{2}\right)^{p} P_{\ell}^{(-p,p)}(\tau)$$

2. *For $p \geq 0$ and $\ell = p$:*

$$a_{p;p,m}(\tau) = \left(\frac{1-\tau}{2}\right)^{p}\left(\frac{1+\tau}{2}\right)^{p}\left(C_{p,p,m} + D_{p,p,m}\int_{0}^{\tau}\frac{ds}{(1-s^{2})^{p+1}}\right) \tag{4}$$

- $p = 0$ and $p = 1$ cases:

$$a_{0;0,0}(\tau) = C_{000} + \frac{1}{2}D_{000}(\log(1+\tau) - \log(1-\tau))$$
$$a_{1;1,m}(\tau) = \frac{1}{4}(1-\tau)(1+\tau)(C_{11m} + \frac{1}{4}D_{11m}(\log(1+\tau) -$$
$$- \log(1-\tau) + 2\tau(1-\tau^2))).$$

- Log terms violate peeling.

## Remark

(Regularity condition). The solutions for $a(\tau)$ are polynomic in $\tau$: $D_{p,p,m} = 0$. Otherwise we need to impose the regularity condition.

- Expanding $\tilde{\phi}$:

$$\tilde{\phi} = \Theta\phi \Leftrightarrow \tilde{\phi} = \frac{C_{000}}{\tilde{\rho}} + \frac{1}{2\tilde{\rho}} D_{000} \log\left(\frac{\tilde{\rho}+\tilde{t}}{\tilde{\rho}-\tilde{t}}\right) Y_{00} + \frac{1}{16\tilde{\rho}^2}$$
$$\left[D_{11-1} \log\left(\frac{\tilde{\rho}+\tilde{t}}{\tilde{\rho}-\tilde{t}}\right) Y_{1-1} + D_{110} \log\left(\frac{\tilde{\rho}+\tilde{t}}{\tilde{\rho}-\tilde{t}}\right) Y_{10} + \right]$$
$$+ \left[D_{111} \log\left(\frac{\tilde{\rho}+\tilde{t}}{\tilde{\rho}-\tilde{t}}\right) Y_{11}\right] +$$
$$+ \frac{1}{2\tilde{\rho}^2} \left(A_{100} + B_{100}\right) Y_{00} + \frac{1}{4\tilde{\rho}^2} \left(C_{11-1} Y_{1-1} + C_{110} Y_{10} + C_{111} Y_{11}\right).$$

- In the spin-0 case, the peeling property is violated.

# The NP-constants for the spin-0 fields close to $i^0$ & $\mathscr{I}$

- Conservation laws:

$$\underline{L}(\tilde{\rho}^{-2\ell}L(e^+)^{\ell+1}\phi_{\ell m}) = 0, \qquad L(\tilde{\rho}^{-2\ell}\underline{L}(e^-)^{\ell+1}\phi_{\ell m}) = 0$$

- $f(\tilde{\rho})$-modified NP constants:

$${}^{f}\mathcal{N}^{+}_{\ell,m} := f(\tilde{\rho})L(\boldsymbol{e}^+)^{\ell}\phi_{\ell m}\big|_{C^+}$$

$${}^{f}\mathcal{N}^{-}_{\ell,m} := f(\tilde{\rho})\underline{L}(\underline{\boldsymbol{e}}^-)^{\ell}\phi_{\ell m}\big|_{C^-}$$

- For $f(\tilde{\rho}) = \tilde{\rho}^2$, classical NP-constants.

$$\mathcal{N}^{+}_{\ell,m} := (\boldsymbol{e}^+)^{\ell+1}\phi_{\ell m}\big|_{C^+}$$

$$\mathcal{N}^{-}_{\ell,m} := (\underline{\boldsymbol{e}}^-)^{\ell+1}\phi_{\ell m}\big|_{C^-}$$

# The classical NP constants at $\mathscr{I}^+$

- This analysis is facilitated by the expression,

$$\phi_{\ell m} = \sum_{p=\ell}^{\infty} \frac{1}{p!} a_{p;\ell,m}(\tau)\rho^p.$$

- Considering $\ell = 0$, the computation of $\boldsymbol{e}^+(\phi_{00})$ is sufficient.

$$\boldsymbol{e}^+(\phi_{\ell m}) = 4\rho^{-1}(1+\tau)^{-2} \sum_{p=0}^{\infty} \frac{1}{p!} \rho^p ((1+\tau)\dot{a}_{p;\ell,m} - p a_{p;\ell,m}).$$

With

$$Q^0_{p;\ell,m}(\tau) := (1+\tau)\dot{a}_{p;\ell,m} - p a_{p;\ell,m}.$$

- With this definition in place, we can express $\boldsymbol{e}^+(\phi_{\ell m})$ as follows:

$$\boldsymbol{e}^+(\phi_{\ell m}) = 4(\Lambda_+)^2 \sum_{p=0}^{\infty} \frac{1}{p!} \rho^p Q^0_{p,\ell,m}(\tau).$$

- To compute the $\ell = 0$ NP constant we evaluate at a cut $C^+$:

$$\mathcal{N}^+_{0,0} = \lim_{\substack{\rho\to\rho_\star \\ \tau\to 1}} \boldsymbol{e}^+(\phi_{00}) = \sum_{p=0}^{\infty} \frac{1}{p!}\rho_\star^{p-1} Q^0_{p,0,0}(\tau)\big|_{\mathscr{I}^+} = -A_{100}.$$

- To calculate the $\ell = 1$ NP constants, one has

$$\mathcal{N}^+_{1,m} = \lim_{\substack{\rho\to 0 \\ \tau\to 1}} 2^{-4}\frac{1}{2!} Q^1_{2,1,m}(\tau) = 3A_{21m}.$$

- The NP constants for $\mathscr{I}^-$ can be calculated in a similar manner, where the time reversed version of the $F$-frame is used.

# The $i^0$ cylinder logarithmic NP constants at $\mathscr{I}^-$

- Choice of $f(\tilde{\rho})$.

- We will compute the $\ell = 0$ and $\ell = 1$ modified NP constants.

$$\tilde{\rho}\underline{L}(\phi_{\ell m}) = \frac{(1+\tau)}{(1-\tau)} \sum_{p=0}^{\infty} \underline{Q}^{0}_{p;\ell,m}(\tau)\rho^{p}.$$

Therefore, for $\ell = 0$, we have:

$$^{\tilde{\rho}}\mathcal{N}^{-}_{0,0} = \lim_{\substack{\rho \to \rho_\star \\ \tau \to -1}} \kappa(\underline{\boldsymbol{e}}^{-})(\phi_{00}) = \sum_{p=0}^{\infty} \rho_\star^{p} \left[ \frac{(1+\tau)}{(1-\tau)} \underline{Q}^{0}_{p;0,0}(\tau) \right] |_{\mathscr{I}^-}.$$

- Evaluating at the critical set $\mathcal{I}^-$, we obtain:

$$^{\tilde{\rho}}\mathcal{N}^-_{0,0} = \lim_{\substack{\rho\to\rho_\star \\ \tau\to -1}} \sum_{p=0}^{\infty} \rho_\star^p \left[ \frac{(1+\tau)}{(1-\tau)} \underline{Q}^0_{p;0,0}(\tau) \right] = \frac{1}{2} D_{000}.$$

- Similarly, for $\ell = 1$, the relevant quantity to evaluate is:

$$\tilde{\rho}\underline{L}(\underline{\boldsymbol{e}}^-)\phi_{1m} = 4\kappa(\Lambda_-)^2 \sum_{p=1}^{\infty} \frac{1}{p!} \rho^p \underline{Q}^1_{p;\ell,m}(\tau).$$

- Therefore, for $\ell = 1$, we have:

$$^{\tilde{\rho}}\mathcal{N}^-_{1,m} = \lim_{\substack{\rho\to\rho_\star \\ \tau\to -1}} \kappa\underline{\boldsymbol{e}}(\underline{\boldsymbol{e}}^-)\phi_{1m} = \sum_{p=1}^{\infty} \frac{1}{p!} \rho_\star^{p-1} (\kappa \underline{Q}^1_{p;1,m}(\tau))|_{\mathscr{I}^-}.$$

- Evaluating at the critical set $\mathcal{I}^-$, we obtain:

$$^{\tilde{\rho}}\mathcal{N}^-_{1,m} = -\frac{1}{4} D_{11m}.$$

# The NP constants in terms of initial data

- Classical NP Constants at $\mathscr{I}^+$: $\mathcal{N}^+_{\ell,m} = q^+(\ell)\, A_{\ell+1,\ell,m}$.
- Modified NP Constants at $\mathscr{I}^+$: ${}^{\tilde{\rho}}\mathcal{N}^+_{\ell,m} = q^+(\ell)\, D_{\ell,\ell,m}$.
- Classical NP Constants at $\mathscr{I}^-$: $\mathcal{N}^-_{\ell,m} = q^-(\ell)\, B_{\ell+1,\ell,m}$.
- Modified NP Constants at $\mathscr{I}^-$: ${}^{\tilde{\rho}}\mathcal{N}^-_{\ell,m} = q^-(\ell)\, D_{\ell,\ell,m}$.

- Reminder:

$$a(\tau)_{p;\ell,m} = A_{p,\ell,m}\left(\frac{1-\tau}{2}\right)^p P_\ell^{(p,-p)}(\tau) + B_{p,\ell,m}\left(\frac{1+\tau}{2}\right)^p P_\ell^{(-p,p)}(\tau)$$

$$a_{p;p,m}(\tau) = \left(\frac{1-\tau}{2}\right)^p\left(\frac{1+\tau}{2}\right)^p\left(C_{p,p,m} + D_{p,p,m}\int_0^\tau \frac{ds}{(1-s^2)^{p+1}}\right)$$

# Conclusions

- Computed the NP constants for a spin-0 field at $\mathscr{I}$.
- $D_{p;p,m} \neq 0$: classical NP constants become undefined.
- $D_{p;p,m} = 0$: classical NP constants at $\mathscr{I}^{\pm}$ become defined.
- Classical NP constants $\neq$ correspondence, the $i^0$ cylinder NP constants share initial data.
- These results were published in: J. Math. Phys. 64, 082502 (2023) https://doi.org/10.1063/5.0158746