

The cylinder at spatial infinity and asymptotic charges

Rafael Pinto

Instituto Superior Técnico

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Advisors: **Dr. Edgar Gasperín** and **Dr. Alex Vañó Viñuales**

Table of Contents

- 1 Newman-Penrose constants
- 2 Cylinder at i^0
- 3 Spin-0 fields close to i^0 and \mathcal{I}
- 4 The NP-constants for the spin-0 fields close to i^0 & \mathcal{I}
- 5 The classical NP constants at \mathcal{I}^+
- 6 The i^0 cylinder logarithmic NP constants at \mathcal{I}^-
- 7 The NP constants in terms of initial data
- 8 Conclusions

- The Newman-Penrose (NP) constants serve as conserved quantities at null infinity in asymptotically flat gravitational fields.
- These constants present a comprehensive conservation system for various spins: spin-1 fields and spin-2 fields, with our research focusing on spin-0 fields linked to wave equation solutions.
- In the detailed context, while an infinite series of conserved quantities is identified in the linear theory, the non-linear General Relativity theory conserves only ten.

Conservation laws

- These charges are computed as 2-surface integrals at cuts $C \approx \mathbb{S}^2$ of null infinity \mathcal{I} .

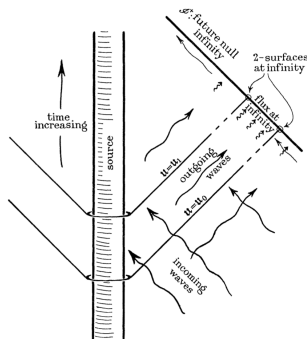


Figure: Visual representation of the behavior of the Newman-Penrose constants at null infinity.

The i^0 cylinder representation in Minkowski spacetime

- Consider spherical polar coordinates $(\tilde{t}, \tilde{\rho}, \vartheta^A)$ with $A = 1, 2$.
- The metric of physical Minkowski spacetime in this coordinate system is given by $\tilde{\eta}$:

$$\tilde{\eta} = -\mathbf{d}\tilde{t} \otimes \mathbf{d}\tilde{t} + \mathbf{d}\tilde{\rho} \otimes \mathbf{d}\tilde{\rho} + \tilde{\rho}^2 \sigma. \quad (1)$$

- Introduce unphysical spherical polar coordinates (t, ρ, ϑ^A) as an intermediate step:

$$t = \frac{\tilde{t}}{\tilde{\rho}^2 - \tilde{t}^2}, \quad \rho = \frac{\tilde{\rho}}{\tilde{\rho}^2 - \tilde{t}^2}. \quad (2)$$

- The conformal metric in unphysical coordinates, $\eta = \Xi^2 \tilde{\eta}$:

$$\eta = -\frac{1}{\tilde{\rho}^2 - \tilde{t}^2} (-\mathbf{d}\tilde{t} \otimes \mathbf{d}\tilde{t} + \mathbf{d}\tilde{\rho} \otimes \mathbf{d}\tilde{\rho} + \tilde{\rho}^2 \sigma). \quad (3)$$

i^0 Representation in Unphysical Coordinates

- In this conformal representation ($t \in (-\infty, \infty)$, $\rho \in [0, \infty)$), spatial infinity and the origin interchange.
- i^0 is represented by the point ($t = 0, \rho = 0$) in (\mathbb{R}^4, η) .
- Introduce coordinates $(\tau, \rho, \vartheta^A)$ with $t = \rho\tau$.
- Consider the conformal metric $\mathbf{g} = \rho^{-2}\eta$.
- Express the unphysical metric \mathbf{g} in F -coordinates:

$$\mathbf{g} = -\mathbf{d}\tau \otimes \mathbf{d}\tau + \frac{1 - \tau^2}{\rho^2} \mathbf{d}\rho \otimes \mathbf{d}\rho - \frac{\tau}{\rho} (\mathbf{d}\rho \otimes \mathbf{d}\tau + \mathbf{d}\rho \otimes \mathbf{d}\tau) + \sigma. \quad (4)$$

Conformal Factor and Lorentz Transformation

- The conformal factor Θ in F -coordinates and physical coordinates:

$$\Theta := \rho(1 - \tau^2) = \frac{1}{\tilde{\rho}}. \quad (5)$$

- The boost parameter κ :

$$\kappa := \frac{1 + \tau}{1 - \tau} = -\frac{\tilde{v}}{\tilde{u}}. \quad (6)$$

- The Lorentz transformation that connects the NP and F-frames:

$$(\Lambda_+)^2 := \Theta^{-1}\kappa^{-1}, \quad (\Lambda_-)^2 := \Theta^{-1}\kappa. \quad (7)$$

- Identify future and past null infinity in the conformal representation:

$$\mathcal{I}^+ \equiv \{p \in \mathcal{M} \mid \tau(p) = 1\},$$

$$\mathcal{I}^- \equiv \{p \in \mathcal{M} \mid \tau(p) = -1\}.$$

- The i^0 -cylinder represents spatial infinity as an extended set $\mathcal{I} \approx \mathbb{R} \times \mathbb{S}^2$:

$$\mathcal{I} \equiv \{p \in \mathcal{M} \mid |\tau(p)| \leq 1, \rho(p) = 0\}, \quad (8)$$

$$\mathcal{I}^0 \equiv \{p \in \mathcal{M} \mid \tau(p) = 0, \rho(p) = 0\}. \quad (9)$$

F-Frame and Null Frames

- Introduce the F -frame:

$$\mathbf{e} = (1 + \tau)\partial_\tau - \rho\partial_\rho, \quad \underline{\mathbf{e}} = (1 - \tau)\partial_\tau + \rho\partial_\rho, \quad \mathbf{e}_A \quad \text{with} \\ \mathbf{A} = \{\uparrow, \downarrow\}. \quad (10)$$

- The NP-frame hinged at \mathcal{I}^\pm :

$$\text{NP hinged at } \mathcal{I}^+ : \quad \mathbf{e}^+, \underline{\mathbf{e}}^+, \mathbf{e}_A^+,$$

$$\text{NP hinged at } \mathcal{I}^- : \quad \mathbf{e}^-, \underline{\mathbf{e}}^-, \mathbf{e}_A^-.$$

- Transformation between NP and F-frames:

$$\text{NP hinged at } \mathcal{I}^+ : \quad \mathbf{e}^+ = \Theta^{-2}L, \quad \underline{\mathbf{e}}^+ = \underline{L}, \quad \mathbf{e}_A^+ = \mathbf{e}_A = \Theta^{-1}\tilde{\mathbf{e}}_A$$

$$\text{NP hinged at } \mathcal{I}^- : \quad \mathbf{e}^- = L, \quad \underline{\mathbf{e}}^- = \Theta^{-2}\underline{L}, \quad \mathbf{e}_A^- = \mathbf{e}_A = \Theta^{-1}\tilde{\mathbf{e}}_A.$$

- For $(\tilde{M}, \tilde{\mathbf{g}})$ and (M, \mathbf{g}) , the transformation of the D'Alembertian operator under conformal transformations is,

$$\square\phi - \frac{1}{6}\phi R = \Omega^{-3} \left(\tilde{\square}\tilde{\phi} - \frac{1}{6}\tilde{\phi}\tilde{R} \right). \quad (11)$$

- Using F -coordinates, the wave equation is represented by

$$(\tau^2 - 1)\partial_\tau^2\phi - 2\rho\tau\partial_\tau\partial_\rho\phi + \rho^2\partial_\rho^2\phi + 2\tau\partial_\tau\phi + \Delta_{\mathbb{S}^2}\phi = 0. \quad (12)$$

- We consider the Ansatz

$$\phi = \sum_{p=0}^{\infty} \sum_{\ell=0}^p \sum_{m=-\ell}^{m=\ell} \frac{1}{p!} a_{p;\ell,m}(\tau) \rho^p Y_{\ell m}. \quad (13)$$

- Solving (12) simplifies to solving the following (ODE) for every p , ℓ , and m :

$$(1 - \tau^2)\ddot{a}_{p;\ell,m} + 2\tau(p-1)\dot{a}_{p;\ell,m} + (\ell+p)(\ell-p+1)a_{p;\ell,m} = 0. \quad (14)$$

Lemma

The solution to equation (14) is given by:

- ① For $p \geq 1$ and $0 \leq \ell \leq p - 1$

$$\begin{aligned} a(\tau)_{p;\ell,m} &= A_{p,\ell,m} \left(\frac{1-\tau}{2} \right)^p P_\ell^{(p,-p)}(\tau) + \\ &B_{p,\ell,m} \left(\frac{1+\tau}{2} \right)^p P_\ell^{(-p,p)}(\tau) \end{aligned} \quad (15)$$

- ② For $p \geq 0$ and $\ell = p$:

$$a_{p;p,m}(\tau) = \left(\frac{1-\tau}{2} \right)^p \left(\frac{1+\tau}{2} \right)^p \left(C_{p,p,m} + D_{p,p,m} \int_0^\tau \frac{ds}{(1-s^2)^{p+1}} \right) \quad (16)$$

- When analyzing the hypergeometric function presented in Equation (16) for different values of p we can see the emergence of logarithmic terms. Let's consider the cases of $p = 0$ and $p = 1$, which yield the following expressions:

$$a_{0;0,0}(\tau) = C_{000} + \frac{1}{2}D_{000}(\log(1 + \tau) - \log(1 - \tau)) \quad (17)$$

$$a_{1;1,m}(\tau) = \frac{1}{4}(1 - \tau)(1 + \tau)(C_{11m} + \frac{1}{4}D_{11m}(\log(1 + \tau) - \log(1 - \tau) + 2\tau(1 - \tau^2))). \quad (18)$$

- These logarithmic terms have implications for the linear version of the associated peeling property.

Remark

(Regularity condition). *Lemma 1* implies that expanding the integral in (16) results in logarithmic terms, hence $D_{p,p,m} = 0$ is called the regularity condition. The solutions for $a(\tau)$ are polynomial in τ , except for $\ell = p$ where one needs to impose the regularity condition to only have polynomial solutions.

- We can see how peeling is violated for the spin-0 field by expanding $\tilde{\phi}$ in terms of the F -frame. Making use of eqs (2.34), (2.35) and *Lemma 1*, we truncate (13) up to $p = 1$ and get the following expression:

$$\begin{aligned}
\tilde{\phi} = & \frac{C_{000}}{\tilde{\rho}} + \frac{1}{2\tilde{\rho}} D_{000} \log \left(\frac{\tilde{\rho} + \tilde{t}}{\tilde{\rho} - \tilde{t}} \right) Y_{00} + \frac{1}{8\tilde{\rho}} \left(1 - \frac{\tilde{t}^2}{\tilde{\rho}^2} \right) \\
& \left[D_{11-1} \log \left(\frac{\tilde{\rho} + \tilde{t}}{\tilde{\rho} - \tilde{t}} \right) Y_{1-1} + D_{110} \log \left(\frac{\tilde{\rho} + \tilde{t}}{\tilde{\rho} - \tilde{t}} \right) Y_{10} \right] \\
& + \left[D_{111} \log \left(\frac{\tilde{\rho} + \tilde{t}}{\tilde{\rho} - \tilde{t}} \right) Y_{11} \right] + \frac{1}{2\tilde{\rho}} (\tilde{t}^2 - \tilde{\rho}^2) (A_{100} + B_{100}) Y_{00} + \\
& \frac{1}{4\tilde{\rho}} \left(1 - \frac{\tilde{t}^2}{\tilde{\rho}^2} \right) (C_{11-1} Y_{1-1} + C_{110} Y_{10} + C_{111} Y_{11}). \tag{19}
\end{aligned}$$

- The expansion of the physical field $\tilde{\phi}$ helps to understand the concept of peeling in the spin-0 case. In the spin-0 case, the peeling property is violated by the logarithmic terms that appear in the expansion of $\tilde{\phi}$.

The NP-constants for the spin-0 fields close to i^0 & \mathcal{I}

- The NP constants emerge from a set of asymptotic conservation laws. For the spin-0 field in flat spacetime one has:

$$\underline{L}(\tilde{\rho}^{-2\ell} L(e^+)^{\ell+1} \phi_{\ell m}) = 0, \quad L(\tilde{\rho}^{-2\ell} \underline{L}(e^-)^{\ell+1} \phi_{\ell m}) = 0 \quad (20)$$

here, $\phi_{\ell m} = \int_{S^2} \phi Y_{\ell m} d\sigma$ represents the integral of ϕ multiplied by the spherical harmonics $Y_{\ell m}$ over the surface S^2 , where $d\sigma$ denotes the area element on S^2 .

- One can introduce the $f(\tilde{\rho})$ -modified NP constants in the following manner:

$${}^f \mathcal{N}_{\ell, m}^+ := f(\tilde{\rho}) L(e^+)^{\ell} \phi_{\ell m} |_{C^+}, \quad (21)$$

$${}^f \mathcal{N}_{\ell, m}^- := f(\tilde{\rho}) \underline{L}(e^-)^{\ell} \phi_{\ell m} |_{C^-}. \quad (22)$$

here, $C^{\pm} \approx S^2$ represents a cut of \mathcal{I}^{\pm} .

- In the specific case of $f(\tilde{\rho}) = \tilde{\rho}^2$, these quantities are referred to as the "classical NP-constants" and are denoted as $\mathcal{N}_{\ell,m}^{\pm}$. They can be succinctly expressed as follows:

$$\mathcal{N}_{\ell,m}^{+} := (\mathbf{e}^{+})^{\ell+1} \phi_{\ell m} |_{C^{+}}, \quad (23)$$

$$\mathcal{N}_{\ell,m}^{-} := (\mathbf{e}^{-})^{\ell+1} \phi_{\ell m} |_{C^{-}}. \quad (24)$$

In the notation for ${}^f\mathcal{N}_{\ell,m}^{+}$ presented below, it will be implicitly assumed that m takes values from $-\ell$ to ℓ .

The classical NP constants at \mathcal{I}^+

- This analysis is facilitated by the expression,

$$\phi_{\ell m} = \sum_{p=\ell}^{\infty} \frac{1}{p!} a_{p;\ell,m}(\tau) \rho^p. \quad (25)$$

- Considering $\ell = 0$, the computation of $\mathbf{e}^+(\phi_{00})$ is sufficient.

$$\mathbf{e}^+(\phi_{\ell m}) = 4\rho^{-1}(1+\tau)^{-2} \sum_{p=0}^{\infty} \frac{1}{p!} \rho^p ((1+\tau)\dot{a}_{p;\ell,m} - p a_{p;\ell,m}). \quad (26)$$

With

$$Q_{p;\ell,m}^0(\tau) := (1+\tau)\dot{a}_{p;\ell,m} - p a_{p;\ell,m}. \quad (27)$$

- With this definition in place, we can express $\mathbf{e}^+(\phi_{\ell m})$ as follows:

$$\mathbf{e}^+(\phi_{\ell m}) = 4(\Lambda_+)^2 \sum_{p=0}^{\infty} \frac{1}{p!} \rho^p Q_{p,\ell,m}^0(\tau). \quad (28)$$

- To compute the $\ell = 0$ NP constant at \mathcal{I}^+ , it is necessary to evaluate $\mathbf{e}^+(\phi_{00})$ at a specific cut C^+ of \mathcal{I}^+ . By utilizing equation (28) we obtain the following expression:

$$\mathcal{N}_{0,0}^+ = \lim_{\substack{\rho \rightarrow \rho_\star \\ \tau \rightarrow 1}} \mathbf{e}^+(\phi_{00}) = \sum_{p=0}^{\infty} \frac{1}{p!} \rho_\star^{p-1} Q_{p,0,0}^0|_{\mathcal{I}^+} = -A_{100}. \quad (29)$$

- To calculate the $\ell = 1$ NP constants, we must evaluate $(\mathbf{e}^+)^2(\phi_{1m})$.

$$(\mathbf{e}^+)^2(\phi_{\ell m}) = 4^2(\Lambda_+)^2 \mathbf{e} \left((\Lambda_+)^2 \sum_{p=\ell}^{\infty} \frac{1}{p!} \rho^p Q_{p,\ell,m}^0(\tau) \right), \quad (30)$$

where

$$\mathbf{e}(\Lambda_+^2) = -\Lambda_+^2. \quad (31)$$

- Utilizing the aforementioned equations yields:

$$(\mathbf{e}^+)^2(\phi_{\ell m}) = 4^2(\Lambda_+)^4 \sum_{p=\ell}^{\infty} \frac{1}{p!} \rho^p \left((1+\tau) \dot{Q}_{p;\ell,m}^0(\tau) - (p+1) Q_{p;\ell,m}^0(\tau) \right). \quad (32)$$

- Where

$$Q_{p,\ell,m}^1(\tau) := ((1 + \tau)\dot{Q}_{p,\ell,m}^0(\tau) - (p + 1)Q_{p,\ell,m}^0(\tau)). \quad (33)$$

- Finally, one has:

$$\mathcal{N}_{1,m}^+ = \lim_{\substack{\rho \rightarrow 0 \\ \tau \rightarrow 1}} 2^{-4} \frac{1}{2!} Q_{2,1,m}^1 \cdot (\tau) = 3A_{21m}. \quad (34)$$

- The NP constants for \mathcal{J}^- can be calculated in a similar manner, where the time reversed version of the F -frame is used.

The i^0 cylinder logarithmic NP constants at \mathcal{I}^-

- There exists a choice of $f(\tilde{\rho})$ that allows the calculation of the associated modified NP constants without imposing the regularity condition.
- The constants $D_{p;\ell,m}$ that define the regularity condition precisely correspond to the $f(\tilde{\rho}) = \tilde{\rho}$ -modified NP constants.
- We will compute the $\ell = 0$ and $\ell = 1$ modified NP constants ${}^f\mathcal{N}_{\ell,m}^-$.

$$\tilde{\rho}\underline{L}(\phi_{\ell m}) = \frac{(1+\tau)}{(1-\tau)} \sum_{p=0}^{\infty} \underline{Q}_{p;\ell,m}^0(\tau) \rho^p. \quad (35)$$

Therefore, for $\ell = 0$, we have:

$$\tilde{\rho}\mathcal{N}_{0,0}^- = \lim_{\substack{\rho \rightarrow \rho_* \\ \tau \rightarrow -1}} \kappa(\underline{\mathbf{e}}^-)(\phi_{00}) = \sum_{p=0}^{\infty} \rho_*^p \left[\frac{(1+\tau)}{(1-\tau)} \underline{Q}_{p;0,0}^0 \right] |_{\mathcal{I}^-}. \quad (36)$$

- Evaluating at the critical set \mathcal{I}^- , we obtain:

$$\tilde{\rho} \mathcal{N}_{0,0}^- = \lim_{\substack{\rho \rightarrow \rho_* \\ \tau \rightarrow -1}} \sum_{p=0}^{\infty} \rho_*^p \left[\frac{(1+\tau)}{(1-\tau)} Q_{p;0,0}^0 \right] = \frac{1}{2} D_{000}. \quad (37)$$

- Similarly, for $\ell = 1$, the relevant quantity to evaluate is:

$$\tilde{\rho} \underline{L}(\underline{\mathbf{e}}^-) \phi_{1m} = 4\kappa(\Lambda_-)^2 \sum_{p=1}^{\infty} \frac{1}{p!} \rho_*^p \underline{Q}_{p;\ell,m}^1. \quad (38)$$

- Therefore, for $\ell = 1$, we have:

$$\tilde{\rho} \mathcal{N}_{1,m}^- = \lim_{\substack{\rho \rightarrow \rho_* \\ \tau \rightarrow -1}} \kappa \underline{\mathbf{e}}(\underline{\mathbf{e}}^-) \phi_{1m} = \sum_{p=1}^{\infty} \frac{1}{p!} \rho_*^{p-1} (\kappa \underline{Q}_{p;1,m}^1)|_{\mathcal{I}^-}. \quad (39)$$

- Evaluating at the critical set \mathcal{I}^- , we obtain:

$$\tilde{\rho} \mathcal{N}_{1,m}^- = -\frac{1}{4} D_{11m}. \quad (40)$$

The NP constants in terms of initial data

- Classical NP Constants at \mathcal{I}^+ :

$$\mathcal{N}_{\ell,m}^+ = q^+(\ell) A_{\ell+1,\ell,m}$$

- Modified NP Constants at \mathcal{I}^+ (with $f(\tilde{\rho}) = \tilde{\rho}$):

$$\tilde{\rho} \mathcal{N}_{\ell,m}^+ = q^+(\ell) D_{\ell,\ell,m}$$

- Classical NP Constants at \mathcal{I}^- :

$$\mathcal{N}_{\ell,m}^- = q^-(\ell) B_{\ell+1,\ell,m}$$

- Modified NP Constants at \mathcal{I}^- (with $f(\tilde{\rho}) = \tilde{\rho}$):

$$\tilde{\rho} \mathcal{N}_{\ell,m}^- = q^-(\ell) D_{\ell,\ell,m}$$

- Key Observations for Spin-0 Field Near i^0 :
 - ① Regularity condition is essential for well-defined NP constants.
 - ② Classical NP constants differ between \mathcal{I}^\pm .
 - ③ Logarithmic NP constants at \mathcal{I}^\pm are equivalent (up to a numerical constant) with the same initial data.

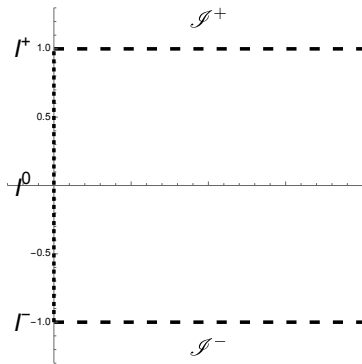


Figure: Representation of the Friedrich Cylinder. Past null infinity is represented by \mathcal{I}^- , Future null infinity \mathcal{I}^+ , spatial infinity as i^0 and the critical sets as \mathcal{I}^+ and \mathcal{I}^- .

Conclusions

- Computed the NP constants for a spin-0 field near spatial and null infinity using the i^0 cylinder framework, exploring the link between NP constants at future and past null infinity based on initial data.
- Analytic initial data prescribed near i^0 results in solution irregularity at the convergence sets \mathcal{I}^\pm , controlled by a constant $D_{p;p,m}$.
- When the regularity condition, $D_{p;p,m} = 0$, is unmet, the classical NP constants become undefined, allowing a polyhomogeneous expansion for the physical field $\tilde{\phi}$ around null infinities.
- Upon satisfying the regularity condition, the classical NP constants at \mathcal{I}^\pm arise from unique initial data components.
- Conclusively, while classical NP constants at \mathcal{I}^\pm usually lack correspondence, the i^0 cylinder NP constants at \mathcal{I}^\pm consistently align due to shared origins in the initial data.