The cylinder at spatial infinity and asymptotic charges

Rafael Pinto

Instituto Superior Técnico

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Advisors: **Dr. Edgar Gasperín** and **Dr. Alex Vañó Viñuales**

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Introduction

- The Newman-Penrose (NP) constants serve as conserved quantities at null infinity in asymptotically flat gravitational fields.
- These constants present a comprehensive conservation system for various spins: spin-1 fields and spin-2 fields, with our research focusing on spin-0 fields linked to wave equation solutions.
- In the detailed context, while an infinite series of conserved quantities is identified in the linear theory, the non-linear General Relativity theory conserves only ten.

Conservation laws

- These charges are computed as 2-surface integrals at cuts $C \approx \mathbb{S}^2$ of null infinity $\mathscr{I}$.

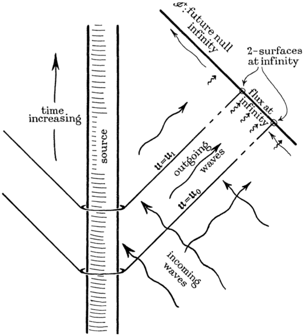

Figure: Visual representation of the behavior of the Newman-Penrose constants at null infinity.

The i^0 cylinder representation in Minkowski spacetime

- Consider spherical polar coordinates $(\tilde{t}, \tilde{\rho}, \vartheta^A)$ with $A = 1, 2$.
- The metric of physical Minkowski spacetime in this coordinate system is given by $\tilde{\boldsymbol{\eta}}$:

$$\tilde{\boldsymbol{\eta}} = -\mathbf{d}\tilde{t} \otimes \mathbf{d}\tilde{t} + \mathbf{d}\tilde{\rho} \otimes \mathbf{d}\tilde{\rho} + \tilde{\rho}^2 \boldsymbol{\sigma}. \tag{1}$$

- Introduce unphysical spherical polar coordinates (t, ρ, ϑ^A) as an intermediate step:

$$t = \frac{\tilde{t}}{\tilde{\rho}^2 - \tilde{t}^2}, \quad \rho = \frac{\tilde{\rho}}{\tilde{\rho}^2 - \tilde{t}^2}. \tag{2}$$

- The conformal metric in unphysical coordinates, $\boldsymbol{\eta} = \Xi^2 \tilde{\boldsymbol{\eta}}$:

$$\boldsymbol{\eta} = -\frac{1}{\tilde{\rho}^2 - \tilde{t}^2}\left(-\mathbf{d}\tilde{t} \otimes \mathbf{d}\tilde{t} + \mathbf{d}\tilde{\rho} \otimes \mathbf{d}\tilde{\rho} + \tilde{\rho}^2 \boldsymbol{\sigma}\right). \tag{3}$$

i^0 Representation in Unphysical Coordinates

- In this conformal representation $(t \in (-\infty, \infty),\ \rho \in [0, \infty))$, spatial infinity and the origin interchange.
- i^0 is represented by the point $(t = 0, \rho = 0)$ in $(\mathbb{R}^4, \eta)$.
- Introduce coordinates $(\tau, \rho, \vartheta^A)$ with $t = \rho\tau$.
- Consider the conformal metric $\boldsymbol{g} = \rho^{-2}\boldsymbol{\eta}$.
- Express the unphysical metric $\boldsymbol{g}$ in F-coordinates:

$$\boldsymbol{g} = -\mathbf{d}\tau \otimes \mathbf{d}\tau + \frac{1-\tau^2}{\rho^2}\mathbf{d}\rho \otimes \mathbf{d}\rho - \frac{\tau}{\rho}\left(\mathbf{d}\rho \otimes \mathbf{d}\tau + \mathbf{d}\rho \otimes \mathbf{d}\tau\right) + \boldsymbol{\sigma}. \tag{4}$$

Conformal Factor and Lorentz Transformation

- The conformal factor Θ in F-coordinates and physical coordinates:

$$\Theta := \rho(1-\tau^2) = \frac{1}{\tilde{\rho}}. \tag{5}$$

- The boost parameter κ:

$$\kappa := \frac{1+\tau}{1-\tau} = -\frac{\tilde{v}}{\tilde{u}}. \tag{6}$$

- The Lorentz transformation that connects the NP and F-frames:

$$(\Lambda_+)^2 := \Theta^{-1}\kappa^{-1}, \quad (\Lambda_-)^2 := \Theta^{-1}\kappa. \tag{7}$$

i^0 Cylinder and Null Frames

- Identify future and past null infinity in the conformal representation:

$$\mathscr{I}^+ \equiv \{p \in \mathcal{M} \mid \tau(p) = 1\},$$
$$\mathscr{I}^- \equiv \{p \in \mathcal{M} \mid \tau(p) = -1\}.$$

- The i^0-cylinder represents spatial infinity as an extended set $\mathcal{I} \approx \mathbb{R} \times \mathbb{S}^2$:

$$\mathcal{I} \equiv \{p \in \mathcal{M} \mid \ |\tau(p)| \ 1,\ \rho(p) = 0\},$$
$$\mathcal{I}^0 \equiv \{p \in \mathcal{M} \mid \tau(p) = 0,\ \rho(p) = 0\}.$$

F-Frame and Null Frames

- Introduce the F-frame:

$$\boldsymbol{e} = (1+\tau)\boldsymbol{\partial}_\tau - \rho\boldsymbol{\partial}_\rho, \quad \underline{\boldsymbol{e}} = (1-\tau)\boldsymbol{\partial}_\tau + \rho\boldsymbol{\partial}_\rho, \quad \boldsymbol{e_A} \quad \text{with} \quad \boldsymbol{A} = \{\uparrow, \downarrow\}. \tag{8}$$

- The NP-frame hinged at $\mathscr{I}^\pm$:

$$\begin{aligned} &\text{NP hinged at } \mathscr{I}^+ : \quad \boldsymbol{e}^+, \underline{\boldsymbol{e}}^+, \boldsymbol{e}^+_{\boldsymbol{A}}, \\ &\text{NP hinged at } \mathscr{I}^- : \quad \boldsymbol{e}^-, \underline{\boldsymbol{e}}^-, \boldsymbol{e}^-_{\boldsymbol{A}}. \end{aligned}$$

- Transformation between NP and F-frames:

$$\begin{aligned} &\textit{NP hinged at } \mathscr{I}^+ : \quad \boldsymbol{e}^+ = \Theta^{-2}L, \quad \underline{\boldsymbol{e}}^+ = \underline{L}, \quad \boldsymbol{e}^+_{\boldsymbol{A}} = \boldsymbol{e_A} = \Theta^{-1}\tilde{\boldsymbol{e}}_{\boldsymbol{A}} \\ &\textit{NP hinged at } \mathscr{I}^- : \quad \boldsymbol{e}^- = L, \quad \underline{\boldsymbol{e}}^- = \Theta^{-2}\underline{L}, \quad \boldsymbol{e}^-_{\boldsymbol{A}} = \boldsymbol{e_A} = \Theta^{-1}\tilde{\boldsymbol{e}}_{\boldsymbol{A}}. \end{aligned}$$

- For $(\tilde{M}, \tilde{\boldsymbol{g}})$ and $(M, \boldsymbol{g})$, the transformation of the D'Alembertian operator under conformal transformations is,

$$\Box\phi - \frac{1}{6}\phi R = \Omega^{-3}\left(\tilde{\Box}\tilde{\phi} - \frac{1}{6}\tilde{\phi}\tilde{R}\right). \tag{9}$$

- Using F-coordinates, the wave equation is represented by

$$\left(\tau^2 - 1\right)\partial_\tau^2\phi - 2\rho\tau\partial_\tau\partial_\rho\phi + \rho^2\partial_\rho^2\phi + 2\tau\partial_\tau\phi + \Delta_{\mathbb{S}^2}\phi = 0. \tag{10}$$

- We consider the Ansatz

$$\phi = \sum_{p=0}^{\infty}\sum_{\ell=0}^{p}\sum_{m=-\ell}^{m=\ell}\frac{1}{p!}a_{p;\ell,m}(\tau)\rho^p Y_{\ell m}. \tag{11}$$

- Solving (10) simplifies to solving the following (ODE) for every p, ℓ, and m:

$$(1-\tau^2)\ddot{a}_{p;\ell,m} + 2\tau(p-1)\dot{a}_{p,\ell,m} + (\ell+p)(\ell-p+1)a_{p;\ell,m} = 0. \tag{12}$$

Lemma

The solution to equation (12) is given by:

1. *For $p \geq 1$ and $0 \leq \ell \leq p-1$*

$$a(\tau)_{p;\ell,m} = A_{p,\ell,m}\left(\frac{1-\tau}{2}\right)^{p} P_{\ell}^{(p,-p)}(\tau) + B_{p,\ell,m}\left(\frac{1+\tau}{2}\right)^{p} P_{\ell}^{(-p,p)}(\tau) \tag{13}$$

2. *For $p \geq 0$ and $\ell = p$:*

$$a_{p;p,m}(\tau) = \left(\frac{1-\tau}{2}\right)^{p}\left(\frac{1+\tau}{2}\right)^{p}\left(C_{p,p,m} + D_{p,p,m}\int_{0}^{\tau}\frac{ds}{(1-s^{2})^{p+1}}\right) \tag{14}$$

- When analyzing the hypergeometric function presented in Equation (14) for different values of p we can see the emergence of logarithmic terms. Let's consider the cases of $p=0$ and $p=1$, which yield the following expressions:

$$a_{0;0,0}(\tau) = C_{000} + \tfrac{1}{2} D_{000}(\log(1+\tau) - \log(1-\tau)) \tag{15}$$

$$\begin{aligned} a_{1;1,m}(\tau) = {} & \tfrac{1}{4}(1-\tau)(1+\tau)(C_{11m} + \tfrac{1}{4} D_{11m}(\log(1+\tau) - \\ & - \log(1-\tau) + 2\tau(1-\tau^2))). \end{aligned} \tag{16}$$

- These logarithmic terms have implications for the linear version of the associated peeling property.

Remark

(Regularity condition). *Lemma 1* implies that expanding the integral in (14) results in logarithmic terms, hence $D_{p,p,m} = 0$ is called the regularity condition. The solutions for $a(\tau)$ are polynomic in τ, except for $\ell = p$ where one needs to impose the regularity condition to only have polynomic solutions.

- We can see how peeling is violated for the spin-0 field by expanding $\tilde{\phi}$ in terms of the F-frame. Making use of eqs (2.34), (2.35) and *Lemma 1*, we truncate (11) up tp $p = 1$ and get the following expression:

$$\tilde{\phi} = \frac{C_{000}}{\tilde{\rho}} + \frac{1}{2\tilde{\rho}} D_{000} \log\left(\frac{\tilde{\rho}+\tilde{t}}{\tilde{\rho}-\tilde{t}}\right) Y_{00} + \frac{1}{8\tilde{\rho}}\left(1-\frac{\tilde{t}^2}{\tilde{\rho}^2}\right)$$
$$\left[D_{11-1}\log\left(\frac{\tilde{\rho}+\tilde{t}}{\tilde{\rho}-\tilde{t}}\right) Y_{1-1} + D_{110}\log\left(\frac{\tilde{\rho}+\tilde{t}}{\tilde{\rho}-\tilde{t}}\right) Y_{10}\right]$$
$$+\left[D_{111}\log\left(\frac{\tilde{\rho}+\tilde{t}}{\tilde{\rho}-\tilde{t}}\right) Y_{11}\right] + \frac{1}{2\tilde{\rho}}\left(\tilde{t}^2-\tilde{\rho}^2\right)\left(A_{100}+B_{100}\right) Y_{00} +$$
$$\frac{1}{4\tilde{\rho}}\left(1-\frac{\tilde{t}^2}{\tilde{\rho}^2}\right)\left(C_{11-1}Y_{1-1} + C_{110}Y_{10} + C_{111}Y_{11}\right). \tag{17}$$

- The expantion of the physical field $\tilde{\phi}$ helps to understand the concept of peeling in the spin-0 case. In the spin-0 case, the peeling property is violated by the logarithmic terms that appear in the expansion of $\tilde{\phi}$.

The NP-constants for the spin-0 fields close to i^0 & $\mathscr{I}$

- The NP constants emerge from a set of asymptotic conservation laws. For the spin-0 field in flat spacetime one has:

$$\underline{L}(\tilde{\rho}^{-2\ell}L(e^{+})^{\ell+1}\phi_{\ell m})=0, \qquad L(\tilde{\rho}^{-2\ell}\underline{L}(e^{-})^{\ell+1}\phi_{\ell m})=0 \tag{18}$$

here, $\phi_{\ell m}=\int_{\mathbb{S}^2}\phi Y_{\ell m}d\sigma$ represents the integral of ϕ multiplied by the spherical harmonics $Y_{\ell m}$ over the surface $\mathbb{S}^2$, where $d\sigma$ denotes the area element on $\mathbb{S}^2$.

- One can introduce the $f(\tilde{\rho})$-modified NP constants in the following manner:

$${}^{f}\mathcal{N}^{+}_{\ell,m}:=f(\tilde{\rho})L(\boldsymbol{e}^{+})^{\ell}\phi_{\ell m}\big|_{C^{+}}, \tag{19}$$

$${}^{f}\mathcal{N}^{-}_{\ell,m}:=f(\tilde{\rho})\underline{L}(\underline{\boldsymbol{e}}^{-})^{\ell}\phi_{\ell m}\big|_{C^{-}}. \tag{20}$$

here, $C^{\pm}\approx\mathbb{S}^2$ represents a cut of $\mathscr{I}^{\pm}$.

- In the specific case of $f(\tilde{\rho}) = \tilde{\rho}^2$, these quantities are referred to as the "classical NP-constants" and are denoted as $\mathcal{N}^{\pm}_{\ell,m}$. They can be succinctly expressed as follows:

$$\mathcal{N}^{+}_{\ell,m} := (\boldsymbol{e}^{+})^{\ell+1} \phi_{\ell m}\big|_{C^{+}}, \tag{21}$$

$$\mathcal{N}^{-}_{\ell,m} := (\underline{\boldsymbol{e}}^{-})^{\ell+1} \phi_{\ell m}\big|_{C^{-}}. \tag{22}$$

In the notation for ${}^{f}\mathcal{N}^{+}_{\ell,m}$ presented below, it will be implicitly assumed that m takes values from $-\ell$ to ℓ.

The classical NP constants at $\mathscr{I}^+$

- This analysis is facilitated by the expression,

$$\phi_{\ell m} = \sum_{p=\ell}^{\infty} \frac{1}{p!} a_{p;\ell,m}(\tau)\rho^p. \tag{23}$$

- Considering $\ell = 0$, the computation of $\boldsymbol{e}^+(\phi_{00})$ is sufficient.

$$\boldsymbol{e}^+(\phi_{\ell m}) = 4\rho^{-1}(1+\tau)^{-2} \sum_{p=0}^{\infty} \frac{1}{p!}\rho^p((1+\tau)\dot{a}_{p;\ell,m} - p a_{p;\ell,m}). \tag{24}$$

With

$$Q^0_{p;\ell,m}(\tau) := (1+\tau)\dot{a}_{p;\ell,m} - p a_{p;\ell,m}. \tag{25}$$

- With this definition in place, we can express $\boldsymbol{e}^+(\phi_{\ell m})$ as follows:

$$\boldsymbol{e}^+(\phi_{\ell m}) = 4(\Lambda_+)^2 \sum_{p=0}^{\infty} \frac{1}{p!}\rho^p Q^0_{p,\ell,m}(\tau). \tag{26}$$

- To compute the $\ell=0$ NP constant at $\mathscr{I}^+$, it is necessary to evaluate $\boldsymbol{e}^+(\phi_{00})$ at a specific cut C^+ of $\mathscr{I}^+$. By utilizing equation (26) we obtain the following expression:

$$\mathcal{N}^+_{0,0} = \lim_{\substack{\rho\to\rho_\star \\ \tau\to 1}} \boldsymbol{e}^+(\phi_{00}) = \sum_{p=0}^{\infty} \frac{1}{p!}\rho_\star^{p-1} Q^0_{p,0,0}|_{\mathscr{I}^+} = -A_{100}. \tag{27}$$

- To calculate the $\ell=1$ NP constants, we must evaluate $(\boldsymbol{e}^+)^2(\phi_{1m})$.

$$(\boldsymbol{e}^+)^2(\phi_{\ell m}) = 4^2(\Lambda_+)^2\boldsymbol{e}\Big((\Lambda_+)^2\sum_{p=\ell}^{\infty}\frac{1}{p!}\rho^p Q^0_{p,\ell,m}(\tau)\Big), \tag{28}$$

where

$$\boldsymbol{e}\left(\Lambda_+^2\right) = -\Lambda_+^2. \tag{29}$$

- Utilizing the aforementioned equations yields:

$$(\boldsymbol{e}^+)^2(\phi_{\ell m}) = 4^2(\Lambda_+)^4\sum_{p=\ell}^{\infty}\frac{1}{p!}\rho^p\big((1+\tau)\dot{Q}^0_{p;\ell,m}(\tau) - (p+1)Q^0_{p;\ell,m}(\tau)\big). \tag{30}$$

- Where

$$Q^1_{p,\ell,m}(\tau) := \big((1+\tau)\,\dot{Q}^0_{p,\ell,m}(\tau) - (p+1)\,Q^0_{p,\ell,m}(\tau)\big). \tag{31}$$

- Finally, one has:

$$\mathcal{N}^+_{1,m} = \lim_{\substack{\rho\to 0\\ \tau\to 1}} 2^{-4}\frac{1}{2!}\,Q^1_{2,1,m}\cdot(\tau) = 3A_{21m}. \tag{32}$$

- The NP constants for $\mathscr{I}^-$ can be calculated in a similar manner, where the time reversed version of the F-frame is used.

The i^0 cylinder logarithmic NP constants at $\mathscr{I}^-$

- There exists a choice of $f(\tilde{\rho})$ that allows the calculation of the associated modified NP constants without imposing the regularity condition.
- The constants $D_{p;\ell,m}$ that define the regularity condition precisely correspond to the $f(\tilde{\rho}) = \tilde{\rho}$-modified NP constants.
- We will compute the $\ell = 0$ and $\ell = 1$ modified NP constants ${}^{f}\mathcal{N}^{-}_{\ell,m}$.

$$\tilde{\rho}\underline{L}(\phi_{\ell m}) = \frac{(1+\tau)}{(1-\tau)} \sum_{p=0}^{\infty} \underline{Q}^{0}_{p;\ell,m}(\tau)\rho^{p}. \tag{33}$$

Therefore, for $\ell = 0$, we have:

$$ {}^{\tilde{\rho}}\mathcal{N}^{-}_{0,0} = \lim_{\substack{\rho\to\rho_\star \\ \tau\to -1}} \kappa(\underline{\boldsymbol{e}}^{-})(\phi_{00}) = \sum_{p=0}^{\infty} \rho_\star^{p} \left[\frac{(1+\tau)}{(1-\tau)} \underline{Q}^{0}_{p;0,0} \right] \Big|_{\mathscr{I}^-}. \tag{34}$$

- Evaluating at the critical set $\mathcal{I}^-$, we obtain:

$$ {}^{\tilde{\rho}}\mathcal{N}^-_{0,0} = \lim_{\substack{\rho\to\rho_\star \\ \tau\to -1}} \sum_{p=0}^{\infty} \rho_\star^p \left[\frac{(1+\tau)}{(1-\tau)} \underline{Q}^0_{p;0,0} \right] = \frac{1}{2} D_{000}. \tag{35} $$

- Similarly, for $\ell = 1$, the relevant quantity to evaluate is:

$$ \tilde{\rho}\underline{L}(\underline{\boldsymbol{e}}^-)\phi_{1m} = 4\kappa(\Lambda_-)^2 \sum_{p=1}^{\infty} \frac{1}{p!} \rho^p \underline{Q}^1_{p;\ell,m}. \tag{36} $$

- Therefore, for $\ell = 1$, we have:

$$ {}^{\tilde{\rho}}\mathcal{N}^-_{1,m} = \lim_{\substack{\rho\to\rho_\star \\ \tau\to -1}} \kappa\underline{\boldsymbol{e}}(\underline{\boldsymbol{e}}^-)\phi_{1m} = \sum_{p=1}^{\infty} \frac{1}{p!} \rho_\star^{p-1} (\kappa \underline{Q}^1_{p;1,m})|_{\mathscr{I}^-}. \tag{37} $$

- Evaluating at the critical set $\mathcal{I}^-$, we obtain:

$$ {}^{\tilde{\rho}}\mathcal{N}^-_{1,m} = -\frac{1}{4} D_{11m}. \tag{38} $$

The general case: Induction proof

Proposition

$$(\boldsymbol{e}^+)^n \phi_{\ell m} = (\Lambda_+)^{2n} \sum_{p=\ell}^{\infty} \frac{1}{p!} Q^{n-1}_{p;\ell,m}(\tau)\rho^p \tag{39}$$

with

$$Q^{n-1}_{p;\ell,m}(\tau) = \sum_{q=0}^{n} (-1)^q p^{[q]} \binom{n}{q} (1+\tau)^{n-q} a^{(n-q)}_{p;\ell,m}(\tau) \tag{40}$$

where $p^{[q]}$ denotes the rising factorial.

- Assuming that equations (39) and (40) are valid (induction hypothesis), we can apply $\boldsymbol{e}^+$ to equation (39), which yields the following expression:

$$(\boldsymbol{e}^+)^{n+1}\phi_{\ell m} = (\Lambda_+)^2 \boldsymbol{e}\Big((\Lambda_+)^{2n}\sum_{p=\ell}^{\infty}\rho^p Q^{n-1}_{p;\ell,m}(\tau)\Big). \tag{41}$$

- Expanding expression (41), we can get the following result:

$$(\boldsymbol{e}^+)^{n+1}\phi_{\ell m} = (\Lambda_+)^{2(n+1)}\sum_{p=\ell}^{\infty}\rho^p\left[(1+\tau)\dot{Q}^{n-1}_{p;\ell,m} - (p+n)Q^{n-1}_{p;\ell,m}\right]. \tag{42}$$

- Thus we have shown that:

$$(\boldsymbol{e}^+)^{n+1}\phi_{\ell m} = (\Lambda_+)^{2(n+1)}\sum_{p=\ell}^{\infty}\rho^p R^n_{p;\ell,m} \tag{43}$$

where

$$R^n_{p;\ell,m} = (1+\tau)\dot{Q}^{n-1}_{p;\ell,m} - (p+n)Q^{n-1}_{p;\ell,m}. \tag{44}$$

- Using the induction hypothesis (40), we can compute all the pieces to construct $R^{n}_{p;\ell,m}$:

$$\dot{Q}^{n-1}_{p;\ell,m} = \frac{d}{d\tau}\left[\sum_{q=0}^{n}(-1)^{q}p^{[q]}\binom{n}{q}(1+\tau)^{n-q}a^{(n-q)}_{p;\ell,m}(\tau)\right] =$$

$$= \sum_{q=0}^{n}(-1)^{q}p^{[q]}\binom{n}{q}$$

$$\left[(1+\tau)^{n-q}a^{(n-q+1)}_{p;\ell,m} + (n-q)(1+\tau)^{n-q-1}a^{(n-q)}_{p;\ell,m}\right]. \qquad (45)$$

- By substituting equation (40) (induction hypothesis) and rearranging, we obtain the following:

$$R^{n}_{p;\ell,m} = \sum_{q=0}^{n}(-1)^{n}p^{[q]}\binom{n}{q}$$

$$\left\{(1+\tau)^{n-q+1}a^{(n-q+1)} - (p+q)(1+\tau)^{n-q}a^{(n-q)}\right\}. \qquad (46)$$

- It follows from the definition of the rising factorial that $p^{[q+1]} = (p+q)p^{[q]}$, in particular $p^{[n+1]} = (p+n+1)p^{[n]}$. Using this fact and relabeling according to $i = q-1 \rightarrow q = i+1$ and $i(q=1) = 0$ & $i(q=n+1) = n$, we have:

$$R^n_{p;\ell,m} = p^{[0]}(1+\tau)^{n+1}a^{(n+1)} + \\ + \sum_{i=0}^{n-1}(-1)^{i+1}p^{[i+1]}\left[\binom{n}{i+1} + \binom{n}{i}\right](1+\tau)^{n-i}a^{n-i} + \\ + (-1)^{n+1}p^{[n+1]}a. \tag{47}$$

- Using the recursive identity of the binomial coefficients, the latter can be rewritten more compactly as

$$R^n_{p;\ell,m} = \sum_{q=0}^{n+1}(-1)^q p^{[q]}\binom{n+1}{q}(1+\tau)^{n-q+1}a^{(n-q+1)}. \tag{48}$$

- And recalling equation (40) then one concludes that

$$R^n_{p;\ell,m} = Q^n_{p;\ell,m}. \tag{49}$$

The NP constants in terms of initial data

- Classical NP Constants at $\mathscr{I}^+$:

$$\mathcal{N}^{+}_{\ell,m} = q^{+}(\ell)\, A_{\ell+1,\ell,m}$$

- Modified NP Constants at $\mathscr{I}^+$ (with $f(\tilde{\rho}) = \tilde{\rho}$):

$$^{\tilde{\rho}}\mathcal{N}^{+}_{\ell,m} = q^{+}(\ell)\, D_{\ell,\ell,m}$$

- Classical NP Constants at $\mathscr{I}^-$:

$$\mathcal{N}^{-}_{\ell,m} = q^{-}(\ell)\, B_{\ell+1,\ell,m}$$

- Modified NP Constants at $\mathscr{I}^-$ (with $f(\tilde{\rho}) = \tilde{\rho}$):

$$^{\tilde{\rho}}\mathcal{N}^{-}_{\ell,m} = q^{-}(\ell)\, D_{\ell,\ell,m}$$

- Key Observations for Spin-0 Field Near i^0:
  1. Regularity condition is essential for well-defined NP constants.
  2. Classical NP constants differ between $\mathscr{I}^{\pm}$.
  3. Logarithmic NP constants at $\mathscr{I}^{\pm}$ are equivalent (up to a numerical constant) with the same initial data.

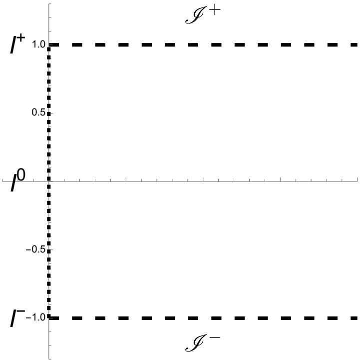

Figure: Representation of the Friedrich Cylinder. Past null infinity is represented by $\mathscr{I}^{-}$, Future null infinity $\mathscr{I}^{+}$, spatial infinity as i^0 and the critical sets as $\mathcal{I}^{+}$ and $\mathcal{I}^{-}$.

Conclusions

- Computed the NP constants for a spin-0 field near spatial and null infinity using the i^0 cylinder framework, exploring the link between NP constants at future and past null infinity based on initial data.
- Analytic initial data prescribed near i^0 results in solution irregularity at the convergence sets $\mathcal{I}^{\pm}$, controlled by a constant $D_{p;p,m}$.
- When the regularity condition, $D_{p;p,m} = 0$, is unmet, the classical NP constants become undefined, allowing a polyhomogeneous expansion for the physical field $\tilde{\phi}$ around null infinities.
- Upon satisfying the regularity condition, the classical NP constants at $\mathscr{I}^{\pm}$ arise from unique initial data components.
- Conclusively, while classical NP constants at $\mathscr{I}^{\pm}$ usually lack correspondence, the i^0 cylinder NP constants at $\mathscr{I}^{\pm}$ consistently align due to shared origins in the initial data.