The cylinder at spatial infinity and asymptotic charges

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Introduction

- The Newman-Penrose (NP) constants serve as conserved quantities at null infinity in asymptotically flat gravitational fields.
- These constants present a comprehensive conservation system for various spins: spin-1 fields and spin-2 fields, with our research focusing on spin-0 fields linked to wave equation solutions.
- In the detailed context, while an infinite series of conserved quantities is identified in the linear theory, the non-linear General Relativity theory conserves only ten.

Conservation laws

- These charges are computed as 2-surface integrals at cuts $C \approx \mathbb{S}^2$ of null infinity $\mathscr{I}$.

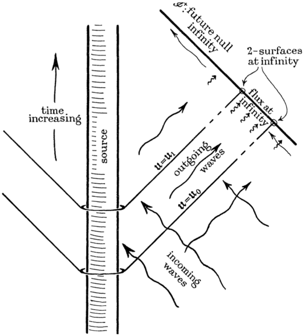

Figure: Visual representation of the behavior of the Newman-Penrose constants at null infinity.

Spin-1 (EM) Field

- A complex tetrad can be selected as follows:

$$l^\mu = \delta_1^\mu, \quad n^\mu = \delta_0^\mu - \delta_1^\mu,$$
$$m^\mu = \frac{1}{\tau}\left(\delta_2^\mu + \frac{1}{\sin\theta}\,\delta_3^\mu\right), \qquad \overline{m}^\mu = \frac{1}{\tau}\left(\delta_2^\mu - \frac{1}{\sin\theta}\,\delta_3^\mu\right). \tag{1}$$

- To describe the electromagnetic (EM) field, we make use of three complex tetrad components of the Maxwell field tensor denoted as $F_{\mu\nu}$:

$$\Phi_0 = F_{\mu\nu} l^\mu n^\nu,$$
$$\Phi_1 = \frac{1}{2} F_{\mu\nu}\left(l^\mu n^\nu + \overline{m}^\mu m^\nu\right),$$
$$\Phi_2 = F_{\mu\nu}\overline{m}^\mu n^\nu. \tag{2}$$

NP Constants Calculation

- The NP constants are calculated using the following formula:

$$F_m^{n,k} = \int {}_1\overline{Y}_{n-k+1,m} \Phi_0^{n+1} d\omega. \tag{3}$$

- The interpretation of charges, such as the Newman-Penrose constants, remains an open debate, yet their conservation is evident in general asymptotically flat spacetimes, even in events like black hole collisions.

The i^0 cylinder representation in Minkowski spacetime

- Consider spherical polar coordinates $(\tilde{t}, \tilde{\rho}, \vartheta^A)$ with $A = 1, 2$.
- The metric of physical Minkowski spacetime in this coordinate system is given by $\tilde{\boldsymbol{\eta}}$:

$$\tilde{\boldsymbol{\eta}} = -\mathbf{d}\tilde{t} \otimes \mathbf{d}\tilde{t} + \mathbf{d}\tilde{\rho} \otimes \mathbf{d}\tilde{\rho} + \tilde{\rho}^2 \boldsymbol{\sigma}. \tag{4}$$

- Introduce unphysical spherical polar coordinates (t, ρ, ϑ^A) as an intermediate step:

$$t = \frac{\tilde{t}}{\tilde{\rho}^2 - \tilde{t}^2}, \quad \rho = \frac{\tilde{\rho}}{\tilde{\rho}^2 - \tilde{t}^2}. \tag{5}$$

- The conformal metric in unphysical coordinates, $\boldsymbol{\eta} = \Xi^2 \tilde{\boldsymbol{\eta}}$:

$$\boldsymbol{\eta} = -\frac{1}{\tilde{\rho}^2 - \tilde{t}^2}\left(-\mathbf{d}\tilde{t} \otimes \mathbf{d}\tilde{t} + \mathbf{d}\tilde{\rho} \otimes \mathbf{d}\tilde{\rho} + \tilde{\rho}^2 \boldsymbol{\sigma}\right). \tag{6}$$

i^0 Representation in Unphysical Coordinates

- In this conformal representation $(t \in (-\infty, \infty),\ \rho \in [0, \infty))$, spatial infinity and the origin interchange.
- i^0 is represented by the point $(t = 0, \rho = 0)$ in $(\mathbb{R}^4, \eta)$.
- Introduce coordinates $(\tau, \rho, \vartheta^A)$ with $t = \rho\tau$.
- Consider the conformal metric $\boldsymbol{g} = \rho^{-2}\boldsymbol{\eta}$.
- Express the unphysical metric $\boldsymbol{g}$ in F-coordinates:

$$\boldsymbol{g} = -\mathbf{d}\tau \otimes \mathbf{d}\tau + \frac{1-\tau^2}{\rho^2}\mathbf{d}\rho \otimes \mathbf{d}\rho - \frac{\tau}{\rho}(\mathbf{d}\rho \otimes \mathbf{d}\tau + \mathbf{d}\rho \otimes \mathbf{d}\tau) + \boldsymbol{\sigma}. \tag{7}$$

Conformal Factor and Lorentz Transformation

- The conformal factor Θ in F-coordinates and physical coordinates:

$$\Theta := \rho(1 - \tau^2) = \frac{1}{\tilde{\rho}}. \tag{8}$$

- The boost parameter κ:

$$\kappa := \frac{1 + \tau}{1 - \tau} = -\frac{\tilde{v}}{\tilde{u}}. \tag{9}$$

- The Lorentz transformation that connects the NP and F-frames:

$$(\Lambda_+)^2 := \Theta^{-1}\kappa^{-1}, \quad (\Lambda_-)^2 := \Theta^{-1}\kappa. \tag{10}$$

i^0 Cylinder and Null Frames

- Identify future and past null infinity in the conformal representation:

$$
\begin{aligned}
\mathscr{I}^+ &\equiv \{p \in \mathcal{M} \mid \tau(p) = 1\}, \\
\mathscr{I}^- &\equiv \{p \in \mathcal{M} \mid \tau(p) = -1\}.
\end{aligned}
$$

- The i^0-cylinder represents spatial infinity as an extended set $I \approx \mathbb{R} \times \mathbb{S}^2$:

$$
\begin{aligned}
I &\equiv \{p \in \mathcal{M} \mid \ |\tau(p)| \ 1,\ \rho(p) = 0\}, \\
I^0 &\equiv \{p \in \mathcal{M} \mid \tau(p) = 0,\ \rho(p) = 0\}.
\end{aligned}
$$

F-Frame and Null Frames

- Introduce the F-frame:

$$\boldsymbol{e} = (1+\tau)\boldsymbol{\partial}_\tau - \rho\boldsymbol{\partial}_\rho, \quad \underline{\boldsymbol{e}} = (1-\tau)\boldsymbol{\partial}_\tau + \rho\boldsymbol{\partial}_\rho, \quad \boldsymbol{e_A} \quad \text{with} \quad \boldsymbol{A} = \{\uparrow,\downarrow\}. \tag{11}$$

- The NP-frame hinged at $\mathscr{I}^{\pm}$:

$$\begin{aligned} &\text{NP hinged at } \mathscr{I}^{+}: \quad \boldsymbol{e}^{+}, \underline{\boldsymbol{e}}^{+}, \boldsymbol{e}^{+}_{\boldsymbol{A}}, \\ &\text{NP hinged at } \mathscr{I}^{-}: \quad \boldsymbol{e}^{-}, \underline{\boldsymbol{e}}^{-}, \boldsymbol{e}^{-}_{\boldsymbol{A}}. \end{aligned}$$

- Transformation between NP and F-frames:

$$\begin{aligned} &\textit{NP hinged at } \mathscr{I}^{+}: \quad \boldsymbol{e}^{+} = \Theta^{-2}L, \quad \underline{\boldsymbol{e}}^{+} = \underline{L}, \quad \boldsymbol{e}^{+}_{\boldsymbol{A}} = \boldsymbol{e_A} = \Theta^{-1}\tilde{\boldsymbol{e}}_{\boldsymbol{A}} \\ &\textit{NP hinged at } \mathscr{I}^{-}: \quad \boldsymbol{e}^{-} = L, \quad \underline{\boldsymbol{e}}^{-} = \Theta^{-2}\underline{L}, \quad \boldsymbol{e}^{-}_{\boldsymbol{A}} = \boldsymbol{e_A} = \Theta^{-1}\tilde{\boldsymbol{e}}_{\boldsymbol{A}}. \end{aligned}$$

- For $(\tilde{M}, \tilde{\boldsymbol{g}})$ and $(M, \boldsymbol{g})$, the transformation of the D'Alembertian operator under conformal transformations is,

$$\square\phi - \frac{1}{6}\phi R = \Omega^{-3}\left(\tilde{\square}\tilde{\phi} - \frac{1}{6}\tilde{\phi}\tilde{R}\right). \tag{12}$$

- Using F-coordinates, the wave equation is represented by

$$\left(\tau^2 - 1\right)\partial_\tau^2\phi - 2\rho\tau\partial_\tau\partial_\rho\phi + \rho^2\partial_\rho^2\phi + 2\tau\partial_\tau\phi + \Delta_{\mathbb{S}^2}\phi = 0. \tag{13}$$

- We consider the Ansatz

$$\phi = \sum_{p=0}^{\infty}\sum_{\ell=0}^{p}\sum_{m=-\ell}^{m=\ell}\frac{1}{p!}a_{p;\ell,m}(\tau)\rho^p Y_{\ell m}. \tag{14}$$

- Solving (13) simplifies to solving the following (ODE) for every p, ℓ, and m:

$$(1-\tau^2)\ddot{a}_{p;\ell,m} + 2\tau(p-1)\dot{a}_{p,\ell,m} + (\ell+p)(\ell-p+1)a_{p;\ell,m} = 0. \tag{15}$$

Lemma

The solution to equation (15) *is given by:*

1. *For* $p \geq 1$ *and* $0 \leq \ell \leq p-1$

$$a(\tau)_{p;\ell,m} = A_{p,\ell,m}\left(\frac{1-\tau}{2}\right)^p P_\ell^{(p,-p)}(\tau) + B_{p,\ell,m}\left(\frac{1+\tau}{2}\right)^p P_\ell^{(-p,p)}(\tau) \tag{16}$$

2. *For* $p \geq 0$ *and* $\ell = p$*:*

$$a_{p;p,m}(\tau) = \left(\frac{1-\tau}{2}\right)^p \left(\frac{1+\tau}{2}\right)^p \left(C_{p,p,m} + D_{p,p,m}\int_0^\tau \frac{ds}{(1-s^2)^{p+1}}\right) \tag{17}$$

- When analyzing the hypergeometric function presented in Equation (17) for different values of p we can see the emergence of logarithmic terms. Let's consider the cases of $p = 0$ and $p = 1$, which yield the following expressions:

$$a_{0;0,0}(\tau) = C_{000} + \tfrac{1}{2} D_{000}(\log(1+\tau) - \log(1-\tau)) \tag{18}$$

$$\begin{aligned} a_{1;1,m}(\tau) &= \tfrac{1}{4}(1-\tau)(1+\tau)(C_{11m} + \tfrac{1}{4} D_{11m}(\log(1+\tau) - \\ &\quad - \log(1-\tau) + 2\tau(1-\tau^2))). \end{aligned} \tag{19}$$

- These logarithmic terms have implications for the linear version of the associated peeling property.

Remark

(Regularity condition). *Lemma 1* implies that expanding the integral in (17) results in logarithmic terms, hence $D_{p,p,m} = 0$ is called the regularity condition. The solutions for $a(\tau)$ are polynomic in τ, except for $\ell = p$ where one needs to impose the regularity condition to only have polynomic solutions.

- We can see how peeling is violated for the spin-0 field by expanding $\tilde{\phi}$ in terms of the F-frame. Making use of eqs (2.34), (2.35) and *Lemma 1*, we truncate (14) up tp $p = 1$ and get the following expression:

$$\tilde{\phi} = \frac{C_{000}}{\tilde{\rho}} + \frac{1}{2\tilde{\rho}} D_{000} \log\left(\frac{\tilde{\rho}+\tilde{t}}{\tilde{\rho}-\tilde{t}}\right) Y_{00} + \frac{1}{8\tilde{\rho}}\left(1 - \frac{\tilde{t}^2}{\tilde{\rho}^2}\right)$$
$$\left[D_{11-1} \log\left(\frac{\tilde{\rho}+\tilde{t}}{\tilde{\rho}-\tilde{t}}\right) Y_{1-1} + D_{110} \log\left(\frac{\tilde{\rho}+\tilde{t}}{\tilde{\rho}-\tilde{t}}\right) Y_{10}\right]$$
$$+ \left[D_{111} \log\left(\frac{\tilde{\rho}+\tilde{t}}{\tilde{\rho}-\tilde{t}}\right) Y_{11}\right] + \frac{1}{2\tilde{\rho}}\left(\tilde{t}^2 - \tilde{\rho}^2\right)\left(A_{100} + B_{100}\right) Y_{00} +$$
$$\frac{1}{4\tilde{\rho}}\left(1 - \frac{\tilde{t}^2}{\tilde{\rho}^2}\right)\left(C_{11-1} Y_{1-1} + C_{110} Y_{10} + C_{111} Y_{11}\right). \tag{20}$$

- The expantion of the physical field $\tilde{\phi}$ helps to understand the concept of peeling in the spin-0 case. In the spin-0 case, the peeling property is violated by the logarithmic terms that appear in the expansion of $\tilde{\phi}$.

The NP-constants for the spin-0 fields close to i^0 & $\mathscr{I}$

- The NP constants emerge from a set of asymptotic conservation laws. For the spin-0 field in flat spacetime one has:

$$\underline{L}(\tilde{\rho}^{-2\ell}L(e^+)^{\ell+1}\phi_{\ell m}) = 0, \qquad L(\tilde{\rho}^{-2\ell}\underline{L}(e^-)^{\ell+1}\phi_{\ell m}) = 0 \tag{21}$$

here, $\phi_{\ell m} = \int_{\mathbb{S}^2}\phi Y_{\ell m} d\sigma$ represents the integral of ϕ multiplied by the spherical harmonics $Y_{\ell m}$ over the surface $\mathbb{S}^2$, where $d\sigma$ denotes the area element on $\mathbb{S}^2$.

- One can introduce the $f(\tilde{\rho})$-modified NP constants in the following manner:

$$^{f}\mathcal{N}^{+}_{\ell,m} := f(\tilde{\rho})L(\boldsymbol{e}^+)^{\ell}\phi_{\ell m}\big|_{C^+}, \tag{22}$$

$$^{f}\mathcal{N}^{-}_{\ell,m} := f(\tilde{\rho})\underline{L}(\underline{\boldsymbol{e}}^-)^{\ell}\phi_{\ell m}\big|_{C^-}. \tag{23}$$

here, $C^{\pm} \approx \mathbb{S}^2$ represents a cut of $\mathscr{I}^{\pm}$.

- In the specific case of $f(\tilde{\rho}) = \tilde{\rho}^2$, these quantities are referred to as the "classical NP-constants" and are denoted as $\mathcal{N}^{\pm}_{\ell,m}$. They can be succinctly expressed as follows:

$$\mathcal{N}^{+}_{\ell,m} := (\boldsymbol{e}^{+})^{\ell+1}\phi_{\ell m}\big|_{C^{+}}, \tag{24}$$

$$\mathcal{N}^{-}_{\ell,m} := (\underline{\boldsymbol{e}}^{-})^{\ell+1}\phi_{\ell m}\big|_{C^{-}}. \tag{25}$$

In the notation for ${}^{f}\mathcal{N}^{+}_{\ell,m}$ presented below, it will be implicitly assumed that m takes values from $-\ell$ to ℓ.

The classical NP constants at $\mathscr{I}^+$

- This analysis is facilitated by the expression,

$$\phi_{\ell m} = \sum_{p=\ell}^{\infty} \frac{1}{p!} a_{p;\ell,m}(\tau)\rho^p. \tag{26}$$

- Considering $\ell = 0$, the computation of $\boldsymbol{e}^+(\phi_{00})$ is sufficient.

$$\boldsymbol{e}^+(\phi_{\ell m}) = 4\rho^{-1}(1+\tau)^{-2} \sum_{p=0}^{\infty} \frac{1}{p!}\rho^p((1+\tau)\dot{a}_{p;\ell,m} - pa_{p;\ell,m}). \tag{27}$$

With

$$Q^0_{p;\ell,m}(\tau) := (1+\tau)\dot{a}_{p;\ell,m} - pa_{p;\ell,m}. \tag{28}$$

- With this definition in place, we can express $\boldsymbol{e}^+(\phi_{\ell m})$ as follows:

$$\boldsymbol{e}^+(\phi_{\ell m}) = 4(\Lambda_+)^2 \sum_{p=0}^{\infty} \frac{1}{p!}\rho^p Q^0_{p,\ell,m}(\tau). \tag{29}$$

- To compute the $\ell = 0$ NP constant at $\mathscr{I}^+$, it is necessary to evaluate $\boldsymbol{e}^+(\phi_{00})$ at a specific cut C^+ of $\mathscr{I}^+$. By utilizing equation (29) we obtain the following expression:

$$\mathcal{N}^+_{0,0} = \lim_{\substack{\rho\to\rho_\star \\ \tau\to 1}} \boldsymbol{e}^+(\phi_{00}) = \sum_{p=0}^{\infty} \frac{1}{p!}\rho_\star^{p-1} Q^0_{p,0,0}|_{\mathscr{I}^+} = -A_{100}. \tag{30}$$

- To calculate the $\ell = 1$ NP constants, we must evaluate $(\boldsymbol{e}^+)^2(\phi_{1m})$.

$$(\boldsymbol{e}^+)^2(\phi_{\ell m}) = 4^2(\Lambda_+)^2 \boldsymbol{e}\Big((\Lambda_+)^2 \sum_{p=\ell}^{\infty} \frac{1}{p!}\rho^p Q^0_{p,\ell,m}(\tau)\Big), \tag{31}$$

where

$$\boldsymbol{e}\left(\Lambda_+^2\right) = -\Lambda_+^2. \tag{32}$$

- Utilizing the aforementioned equations yields:

$$(\boldsymbol{e}^+)^2(\phi_{\ell m}) =$$
$$= 4^2(\Lambda_+)^4 \sum_{p=\ell}^{\infty} \frac{1}{p!}\rho^p\big((1+\tau)\dot{Q}^0_{p;\ell,m}(\tau) - (p+1)Q^0_{p;\ell,m}(\tau)\big). \tag{33}$$

- Where

$$Q^1_{p,\ell,m}(\tau) := \left((1+\tau)\dot{Q}^0_{p,\ell,m}(\tau) - (p+1)Q^0_{p,\ell,m}(\tau)\right). \tag{34}$$

- Finally, one has:

$$\mathcal{N}^+_{1,m} = \lim_{\substack{\rho\to 0\\ \tau\to 1}} 2^{-4}\frac{1}{2!}Q^1_{2,1,m}\cdot(\tau) = 3A_{21m}. \tag{35}$$

- The NP constants for $\mathscr{I}^-$ can be calculated in a similar manner, where the time reversed version of the F-frame is used.

References

Thank You