INSTITUTO SUPERIOR TÉCNICO

PROJECTO MEFT

The Newman-Penrose constants for Spin-0 fields close to spatial and null infinity in Minkowski spacetime

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Research work performed for the Master in Engineering Physics

at

GRIT
CENTRA

January 14, 2023

1 Introduction

General relativity is a theory of gravitation developed by Albert Einstein that describes the force of gravity as a curvature of spacetime caused by mass and energy. In this theory, the force of gravity is not a force at all, but rather an effect of the way objects move through spacetime. In general relativity, the presence of a black hole causes the spacetime around it to bend and curve, resulting in strange effects such as time dilation and the deflection of light. A black hole, seen from an observer that is far away, is similar to a classical particle and is characterized by three quantities: total mass, electrical charge and spin. Given the fact that different black holes which have the same properties are identical, no measurements can be made to distinguish them [8]. 'Hair' will be the term which refers to features that help us to tell objects apart, thus black holes have almost none (No 'hair' theorem). To understand this theorem on how different objects interact with a black hole we need to look at the conservation laws, which black holes obey. When an object falls into a black hole, through conservation we are able to measure the initial and final black hole to deduce a few properties the object had. This has important consequences on how information is handled. Imagine stars and planets of all shapes and sizes, a black hole would reduce them to only three numbers, meaning plenty of information is lost. This is a problem known as the *information paradox* [8]. If a black hole gives no information on its consumption and then vanishes, then the final state would be completely independent of the initial state. In order to explain this, a recent theory was proposed - soft 'hair'- corresponding to non-trivial distortions in clocks that is sensitive to the consumption history of the black hole. Since every possible orientation of the distant rulers and clocks measure a different soft hair, in principle there are an infinite amount of properties, where the notion of distant can be idealised as infinity. For general relativity, the definition of infinity is complicated as it has to surpass the ambiguity when discussing coordinate dependent notions and because there exist different types of infinities. In these report we will focus on only two types: null-infinity denoted by $\mathscr{I}^+$ and spatial infinity denoted by the symbol i^0.

1.1 Newman-Penrose constants

The Newman-Penrose constants —originally introduced in [12]— are quantities defined on null-infinity. Such constants obey conservation laws for asymptotically flat gravitational fields. In flat spacetime, an infinite amount of those mentioned conservation laws exist for each spin value. For an isolated system in ordinary Electromagnetic (EM) theory, spin-1 field, the total charge is conserved, whereas in the linearized gravitational theory, spin-2 field, the total mass, linear and angular momentum are also conserved. In this case, we will study, spin-0 fields, which refer to solutions to the wave equation. The NP constants form an infinite hierarchy of conserved quantities for linear equations like the spin-1, spin-2, and spin-0 fields. Originally, Newman and Penrose showed that such constants have the form of:

$$G_m = (dipole)^2 - (monopole) \times (quadrupole)$$

—see [3]. In the full non-linear gravitational theory, the mass and momentum are no longer conserved, giving rise to ten different conserved quantities. It has been discussed whether the NP constants are zero for stationary spacetimes or not. For the Kerr solution they are zero [2], [1], which also happens for the Schwarzschild spacetime. In particular, it is argued that the magnitude of the constants gives information about the amount of residual radiation that is contained in the space-time after a black hole collision [3]. NP constants retain their value along null-infinity, and as a result of this one may be able to find information about the final time behaviour of the process of black hole collision. Therefore, the Newman-Penrose constants provide insights into how the system behaves at later times [3].

1.2 Friedrich cylinder at spatial infinity

Roger Penrose brought to the field of general relativity the notion of conformal transformation, which made a significant impact in the geometric understanding of infinity. This was crucial for the development of theory of asymptotics and gravitational radiation, taking a step forward in the mathematical understanding of gravitational waves. Although, customary in numerical approaches to general relativity, wave forms are computed at large radius, from first principles point of view they should be computed at null infinity $\mathscr{I}^+$. To do so, the Einstein field equations need to be expressed in terms of suitably rescaled fields, so one can evaluate the fields at $\mathscr{I}^+$. Technically this is done by a conformal transformation. In general relativity, conformal transformations are used to describe the behavior of physical systems under changes in the scale of spacetime. These transformations preserve the local structure of spacetime, but not necessarily its overall shape. The original metric, which we refer to as the physical metric, is denoted by $\tilde{g}$. We consider a transformation to an unphysical metric, g, which is given by

$$g_{ab} = \Xi^2 \tilde{g}_{ab}, \tag{1}$$

Ξ is a smooth function that approaches zero as the distance from the source increases. This transformation, denoted by (1), preserves angles, making it appropriate to describe it as conformal. H. Friedrich introduced the *conformal Einstein field equations* (CEFE), a formulation designed that is in accordance with the approach of R. Penrose.

A prototypical example is the conformal extension of the Minkowski spacetime which will be discussed in the following. One starts with the Minkowski metric line-element,

$$d\tilde{s}^2 = -d\tilde{t}^2 + d\tilde{r}^2 + \tilde{r}^2 d\Omega^2, \tag{2}$$

where $(\tilde{t}, \tilde{r}) \in (-\infty, +\infty) \times [0, +\infty)$ and $d\Omega^2$ represents the standard metric on $\mathbb{S}^2$. To get a conformal extension we need to do a coordinate transformation, corresponding to the advance and retarded times, $\tilde{u} = \tilde{t} - \tilde{r}$ & $\tilde{v} = \tilde{t} + \tilde{r}$, substituting equation (2).

$$d\tilde{s}^2 = -d\tilde{u}d\tilde{v} + \frac{(\tilde{u} - \tilde{v})^2}{4} d\Omega^2. \tag{3}$$

For the compactification, we need to introduce the following: $u = \tan U$ & $v = \tan V$, where U, $V \in (-\pi/2, \pi/2)$. Now, we are able to identify the conformal metric, ds. Using (2), we obtain

$$ds^2 = -4dUdV + \sin^2(V - U)d\Omega^2$$

where

$$ds^2 = \Xi^2 d\tilde{s}^2$$

with $\Xi = 2\cos U \cos V$. Given the domain of U and V, we introduce the following, $T = V + U$ & $\psi = V - U$. The domain of (T, ψ) is $(-\pi, \pi)$, with

$$ds^2 = -dT^2 + d\psi^2 + \sin^2\psi d\Omega^2, \tag{4}$$

where we define this as the standard Lorentzian metric on $\mathbb{R} \times \mathbb{S}^3$ or the Einstein Cylinder. The purpose of this report is to study what happens at infinity, and in order to do that we will focus on the region where $\Xi = 0$. This condition gives us the following regions, which are presented in the following table

Region	Name	Symbol
$\tilde{r} \to \infty$ with $\lvert\tilde{t}\rvert<\infty$	Spatial Infinity	i^0
$\tilde{t} \to \pm\infty$ with $\tilde{r}<\infty$	Future/Past Timelike Infinity	$i^{\pm}$
$\tilde{r} \to \infty,\ \tilde{t} \to \infty$ with $\lvert u\rvert<\infty$	Future Null-infinity	$\mathscr{I}^+$
$\tilde{r} \to \infty,\ \tilde{t} \to -\infty$ with $\lvert v\rvert<\infty$	Past Null-infinity	$\mathscr{I}^-$

The visual representation of this is a Penrose Diagram, depicted in Fig.1 Additionally, H. Friedrich

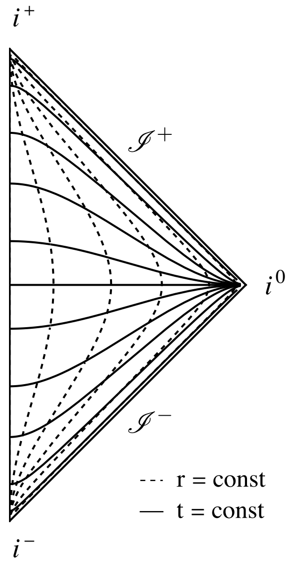

FIGURE 1: Penrose Diagram - Representation of the standard compactification of the Minkowski spacetime alongside the curves of constant time, solid black lines, and the curves of constant r, dotted black lines.

proposed another conformal representation of Minkowski spacetime specifically adapted for spatial infinity, which will be the one used in this work. By applying the following change of coordinates $\tilde{t} = \frac{\tau}{\rho(1-\tau^2)}$, $\tilde{\rho} = \frac{1}{\rho(1-\tau^2)}$ we arrive at this representation. Therefore,

$$\gamma = -d\tau^2 + \frac{(1-\tau^2)}{\rho^2}d\rho^2 - \frac{\tau}{\rho}(d\rho d\tau + d\tau d\rho) + d\Omega^2$$

We are now in the position to say that the conformal factor Θ is given by,

$$\gamma = \Theta^2 \tilde{\eta}. \tag{5}$$

$$\Theta = \rho(1-\tau^2). \tag{6}$$

As a result of the spacetime admitting only particles that travel slower than the speed of light, which in geometric units corresponds to ± 1, hence, we have

$$-1 \leq \tau \leq 1$$

with $\rho > 0$. Therefore, light travels towards infinity to the places where $\tau = \pm 1$ in the conformal extension. So, in this representation,

$$\mathscr{I}^+ \equiv \{\tau = 1\},\ \mathscr{I}^- \equiv \{\tau = -1\}$$

The sets where future and past null-infinities touch spatial infinity are named the critical sets and are given by,

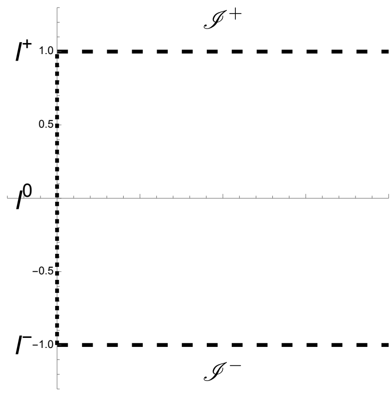

FIGURE 2: Representation of the Friedrich Cylinder. Past null infinity is represented by $\mathscr{I}^-$, Future null infinity as $\mathscr{I}^+$, spatial infinity as I^0 and the critical sets as I^+ and I^-.

$$\mathcal{I}^+ = \{\tau = 1,\ \rho = 0\},\ \mathcal{I}^- = \{\tau = -1,\ \rho = 0\}$$

2 NP - constants for spin-0 fields close to i^0 & $\mathscr{I}^+$ in flat spacetime

The i^0 cylinder representation has previously been used to calculate the NP constants for the spin-1 and spin-2 fields in flat space in [10] and [7]. In this report, we will perform a similar analysis for the solution to the wave equation, or spin-0 field. This calculation represents an extension of the analysis of linear fields propagating near spatial and null-infinity in flat spacetime. We are now in the position to calculate the Newman-Penrose constants for spin-0 fields, and by spin-0 fields we mean the wave equation. Therefore, the Newman-Penrose constants for the wave equation are obtained in the following way. First, we need to introduce the solution to the wave equation in Minkowski spacetime,

$$\tilde{\square}\tilde{\phi} = 0. \tag{7}$$

Following we need to introduce the radiation field, denoted by $\psi = \tilde{\rho}\tilde{\phi}$ and (7) reads as follows,

$$\partial_{\tilde{u}}\partial_{\tilde{v}}\psi = \frac{1}{\tilde{\rho}^2}\Delta_{\mathbb{S}^2}\psi, \tag{8}$$

where we have that $\partial_{\tilde{v}} = \partial_{\tilde{t}} + \partial_{\tilde{\rho}}$ & $\partial_{\tilde{u}} = \partial_{\tilde{t}} - \partial_{\tilde{\rho}}$ and $\Delta_{\mathbb{S}^2}$ being the Laplacian on $\mathbb{S}^2$.
It is of great advantage to make a harmonic decomposition, such that $\psi = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\psi_{lm}Y_{lm}$ and $\Delta_{\mathbb{S}^2}Y_{lm} = -l(l+1)$. As a result (8) reads:

$$\partial_{\tilde{u}}\partial_{\tilde{v}}\psi_{lm} = \frac{-l(l+1)}{\tilde{\rho}^2}\psi_{lm}, \tag{9}$$

And (9) implies that [6]

$$\partial_{\tilde{u}}(\tilde{\rho}^{-2l-2}(\tilde{\rho}^2\partial_{\tilde{v}})^{l+1}\psi_{lm}) = 0,\ \ \partial_{\tilde{v}}(\tilde{\rho}^{-2l-2}(\tilde{\rho}^2\partial_{\tilde{u}})^{l+1}\psi_{lm}) = 0$$

The higher-order NP-constants for (7) are defined at $\mathscr{I}^+$ [9],

$$\mathcal{I}^+_{lm} = \lim_{\tilde{v}\to\infty}\tilde{\rho}^{-2}\partial_{\tilde{v}}((\tilde{\rho}^2\partial_{\tilde{v}})^l\psi_{lm})$$

Our goal is to use conformal methods and the framework of the cylinder at spatial infinity to compute $\mathcal{I}^+_{lm}$. To do this, we need to recall that for two conformally related manifolds - which do not necessarily have to be the i^0 cylinder and Minkowski spacetime - $(\tilde{M}, \tilde{g})$ and (M, g), the D'Alembertian operator transforms under conformal transformations as follows [5]:

$$\square\phi - \frac{1}{6}\phi R = \Theta^{-3}\left(\tilde{\square}\tilde{\phi} - \frac{1}{6}\tilde{\phi}\tilde{R}\right), \tag{10}$$

where R and $\tilde{R}$ are the Ricci scalars of g and $\tilde{g}$ respectively and $\phi = \Theta^{-1}\tilde{\phi}$.

By applying the conformal transformation formula for the wave equation, given in equation (10), to the wave equation in (7) on the physical Minkowski spacetime $(\tilde{M}, \tilde{\eta})$ and selecting the target conformal extension - the unphysical spacetime - (M, g) to be Friedrich's cylinder at spatial infinity, we can obtain the following equation:

$$\square\phi = 0. \tag{11}$$

Thus, (11) is just the wave equation for the unphysical field propagating on the i^0 cylinder background.

A calculation shows that the conformal factor, in terms of physical coordinates is given simply by the inverse of the physical polar coordinate, additionally, further calculations show that:

$$\Theta^{-1} = \tilde{\rho}^{-1} \implies \phi = \tilde{\rho}\tilde{\phi} \implies \psi = \phi$$

$$L := \partial_{\tilde{v}}, \ \ \underline{L} := \partial_{\tilde{u}}, \ \ L = k^{-1}\Theta l, \ \ \underline{L} = k\Theta l$$

where $l = (1+\tau)\partial_\tau - \rho\partial_\rho$, $\underline{l} = (1-\tau)\partial_\tau + \rho\partial_\rho$ and $k = \frac{(1+\tau)}{(1-\tau)}$. To compute the NP constants, the following rescaled null derivative will play an important role $\hat{L} := \tilde{\rho}^2 L$ which explicitly reads

$$\hat{L} = \rho^{-1}(1+\tau)^{-1}\partial_\tau - (1+\tau)^{-2}\partial_\rho. \tag{12}$$

One can translate the expression above for the NP constants given above to the ones as follows,

$$l = 0 \rightarrow \tilde{\rho}^2 L\phi_{00} \rightarrow \hat{L}^1\phi_{00} = \langle \hat{L}^1\phi, Y_{00}\rangle,$$

$$l = 1 \rightarrow (\tilde{\rho}^2 L)^2\phi_{1m} \rightarrow \hat{L}^2\phi_{1m} = \langle \hat{L}^2\phi, Y_{1m}\rangle,$$
$$l = l \rightarrow (\tilde{\rho}^2 L)^{l+1}\phi_{lm} \rightarrow \hat{L}^{l+1}\phi_{lm} = \langle \hat{L}^{l+1}\phi, Y_{lm}\rangle,$$

where the symbol $\langle , \rangle$ represents the inner-product defined as

$$\langle \alpha, \beta\rangle = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \alpha\beta\sin(\theta)d\theta d\phi, \tag{13}$$

where α and β are functions of the coordinates spacetime coordinates. The orthogonality property of the spherical harmonics in this notation reads $\langle Y_{lm}, Y_{l'm'}\rangle = \delta_{ll'}\delta_{mm'}$. Following [11], we take as Ansatz for the solution to the wave equation:

$$\phi = \sum_{p=0}^{\infty}\sum_{l=0}^{p}\sum_{m=-l}^{m=l}\frac{1}{p!}a_{p;l,m}(\tau)\rho^p Y_{lm}. \tag{14}$$

Then, a calculation using the orthogonality of the spherical harmonics gives:

$$\phi_{lm} = \langle \phi, Y_{lm}\rangle = \sum_{p=0}^{\infty}\frac{1}{p!}a_{p;l,m}(\tau)\rho^p. \tag{15}$$

Equation (11) was solved in [11] and the solution is encoded in the following lemma.

Lemma 1. *(Minucci, Panosso Macedo & Valiente Kroon 2022 [11])*

1. *For $p \geq 1$ and $0 \leq \ell \leq p-1$*

$$a(\tau)_{p;\ell,m} = A_{p,\ell,m}\left(\frac{1-\tau}{2}\right)^p P_\ell^{(p,-p)}(\tau) + B_{p,\ell,m}\left(\frac{1+\tau}{2}\right)^p P_\ell^{(-p,p)}(\tau) \tag{16}$$

2. *For $p \geq 0$ and $\ell = p$:*

$$a_{p;p,m}(\tau) = \left(\frac{1-\tau}{2}\right)^p \left(\frac{1+\tau}{2}\right)^p \left(C_{p,p,m} + D_{p,p,m}\int_0^\tau \frac{ds}{(1-s^2)^{p+1}}\right) \tag{17}$$

where $A_{p,\ell,m}$, $B_{p,\ell,m}$, $C_{p,p,m}$ and $D_{p,p,m}$ are constants that can be written in terms of $a_{p;\ell,m}(0)$ and $\dot{a}_{p;\ell,m}(0)$ and $P_\gamma^{\alpha,\beta}(\tau)$ are the Jacobi polynomials.

Expanding the integral in (17) results in logarithmic terms, hence $D_{p,p,m} = 0$ is called the regularity condition. The solutions for $a(\tau)$ are polynomic in τ, except for $l = p$ where one needs to impose the regularity condition to only have polynomic solutions. [11].

Using the following notation:

$$\Lambda^2 := \Theta^{-1}k^{-1} = \rho^{-1}(1+\tau)^{-2}, \;\; e := (1+\tau)\partial_\tau - \rho\partial_\rho$$

we can rewrite the operator $\hat{L}$ as $\hat{L} = \Lambda^2 e$. Applying $\hat{L}$ to our scalar field one gets that

$$\hat{L}\phi = \Lambda^2 \sum_{p,l,m} Y_{l,m}\rho^p[(1+\tau)\dot{a}_{p;l,m} - pa_{p;l,m}]$$

where $(1+\tau)\dot{a}_{p;l,m} - pa_{p;l,m} = A^0_{p;l,m}(\tau)$ and $\sum_{p,l,m} = \sum_{p=0}^\infty \sum_{l=0}^p \sum_{m=-l}^{m=l}$
The general formula to compute the NP constants is given by,

$$\mathcal{I}^+_{lm} = \lim_{\substack{\rho\to 0\\ \tau\to 1}} \langle \hat{L}^{l+1}\phi, Y_{lm}\rangle$$

The NP constant is being evaluated at the critical set $\mathcal{I}^+$, hence $\rho = 0, \;\; \tau = 1$. We are now able to calculate the first NP constant, which is the case where $l = 0$. Thus, we have,

$$\mathcal{I}^+_{0m} = \lim_{\substack{\rho\to 0\\ \tau\to 1}} \langle \hat{L}\phi, Y_{00}\rangle = \lim_{\rho\to 0}\sum_{p=0}^\infty \frac{1}{p!}2^{-2}\rho^{p-1}A^0_{p,0,0}(1)$$

$$\Leftrightarrow \mathcal{I}^+_{0m} = \lim_{\rho\to 0}\{\frac{2^{-2}}{0!}\rho^{-1}A^0_{0,0,0}(1) + \frac{2^{-2}}{1!}A^0_{1,0,0}(1)\}$$

Now, we want to evaluate $A^0_{p,l,m}$:

$$A^0_{p;l,m}(\tau) = (1+\tau)\dot{a}_{p;l,m}(\tau) - pa_{p;l,m}(\tau) \to 2\dot{a}_{0,0,0}(\tau) = \frac{-D_{0,0,0}}{\tau - 1}$$

$$A^0_{1,0,0}(\tau) = 2\dot{a}_{1,0,0}(\tau) - 1 \times a_{1,0,0}(\tau) = -A_{1,0,0}$$

Imposing the regularity condition:

$$D_{0,0,0} = 0 \to \mathcal{I}^+_{0m} = \frac{-A_{100}}{4}$$

Being A_{100} determined by initial data as read from Lemma 1. To determine NP constants with $l = 1$, one needs to compute:

$$\mathcal{I}^+_{1m} = \lim_{\substack{\rho\to 0\\ \tau\to 1}} \langle \hat{L}^2\phi, Y_{1m}\rangle$$

The expression for $\hat{L}^2\phi$ is given by,

$$\hat{L}^2\phi = \Lambda^4 \sum_{p,l,m} Y_{lm}\rho^p[(1+\tau)\dot{A}^0_{p;l,m} - (p+1)A^0_{p;l,m}]$$

where,

$$A^1_{p;l,m} = (1+\tau)\dot{A}^0_{p;l,m} - (p+1)A^0_{p;l,m}$$

Thus,

$$\mathcal{I}^+_{1m} = \lim_{\substack{\rho\to 0\\ \tau\to 1}} 2^{-4}\{\rho^{-2}A^1_{0,1,m}(\tau) + \rho^{-1}A^1_{1,1,m}(\tau)\} + \lim_{\rho\to 0} 2^{-4}\frac{1}{2!}A^1_{2,1,m}(\tau)$$

Evaluating $A^1_{p;l,m}$ and imposing the regularity condition, one finds that the NP constant for $l = 1$ is given by,

$$\mathcal{I}^+_{1m} = 2^{-4}\frac{1}{2!}6A_{211}$$

The previous discussion suggests that, in principle, it should be possible to obtain a general formula $A_{p;l,m}(\tau)$. Revising the calculation of $\mathcal{I}^+_{0m}$ and $\mathcal{I}^+_{1m}$ one can obtain the following results concerning the overall structure of the spin-0 NP constants.
We start with,

$$A^n = k_0\binom{n+1}{0}(1+\tau)^n a^{(n)} - k_1\binom{n+1}{1}p(1+\tau)^{n-1}a^{(n-1)} + ... + (-1)^n\binom{n+1}{q}k_q a$$

where $a^{(n)}$ is the $n-th$ derivative, A^n is shorthand for $A^n_{p,l,m}$ and we define k_q to be,

$$k_q := \frac{p(p+q-1)!}{p!}$$

Therefore,

$$A^n_{p,l,m} = \sum_{q=0}^{n+1}(-1)^q\frac{(p+q-1)!}{(p-1)!}\binom{n+1}{q}(1+\tau)^{n-q+1}a^{(n-q+1)}. \tag{18}$$

The following lemma will be proved.

Lemma 2 (Gasperín & Pinto 2023)**.**

$$\hat{L}^n\phi = \Lambda^{2n}\sum_{p=0}^{\infty}\sum_{l=0}^{p}\sum_{m=-l}^{m=l}\frac{1}{p!}\rho^p A^{n-1}_{p,l,m}(\tau)Y_{0,l,m}, \tag{19}$$

with,

$$A^{n-1}_{p,l,m} = \sum_{q=0}^{n} (-1)^q k_q \binom{n}{q} (1+\tau)^{n-q} a^{(n-q)}. \tag{20}$$

Proof:
The basis of induction for arriving at the general formula for the NP constants will be the cases $n=0$ and $n=1$ and the induction hypothesis will be (19) and (20). The induction step is as follows,

$$\hat{L}^{n+1}\phi = \hat{L}(\hat{L}^n\phi) = \Lambda^{2(n+1)} \sum_{p,l,m} Y_{lm}\rho^p[(1+\tau)\dot{A}^{n-1}_{plm} - (p+n)A^{n-1}_{plm}]$$

Thus we have shown that,

$$\hat{L}^{n+1}\phi = \Lambda^{2(n+1)} \sum_{p,l,m} Y_{lm}\rho^p R^n_{plm}$$

where

$$R^n_{plm} := (1+\tau)\dot{A}^{n-1}_{plm} - (p+n)A^{n-1}_{plm}$$

Using the induction hypothesis (20) we can compute all the pieces to construct R^n_{plm}:

$$R^n_{plm} = \sum_{q=0}^{n} (-1)^q k_q \binom{n}{q} \{(1+\tau)^{n-q+1} a^{(n-q+1)} - (p+q)(1+\tau)^{n-q} a^{(n-q)}\}. \tag{21}$$

Expanding the first term in the first sum of (21) renders

$$R^n_{plm} = (-1)^0 k_0 \binom{n}{0} (1+\tau)^{n+1} a^{(n+1)} + \sum_{q=1}^{0} (-1)^q k_q \binom{n}{q} (1+\tau)^{n-q+1} a^{(n-q+1)} -$$

$$- \sum_{q=0}^{n} (-1)^q k_q \binom{n}{q} (p+q)(1+\tau)^{n-q} a^{(n-q)}$$

Separating the last term in the second sum and rearranging gives

$$R^n_{plm} = k_0(1+\tau)^{n+1} a^{(n+1)} + \sum_{q=1}^{n} (-1)^q k_q \binom{n}{q} (1+\tau)^{n-q+1} a^{(n-q+1)} +$$

$$+ \sum_{q=0}^{n-1} \left[(-1)^{q+1} k_{q+1} \binom{n}{q} (1+\tau)^{n-q} a^{(n-q)} \right] + (-1)^{n+1} k_{n+1} a$$

Relabelling $i = q-1 \implies q = i+1,\ \ i(q=1) = 0$ & $i(q=n+1) = n$

$$R^n_{plm} = k_0(1+\tau)^{n+1} a^{(n+1)} + \sum_{i=0}^{n-1} (-1)^{i+1} k_{i+1} \binom{n}{i+1} (1+\tau)^{n-i} a^{(n-i)} +$$

$$+ \sum_{q=0}^{n-1} \left[(-1)^{q+1} k_{q+1} \binom{n}{q} (1+\tau)^{n-q} a^{(n-q)} \right] + (-1)^{n+1} k_{n+1} a$$

For the second sum we simply relabel the dummy index $i = q$ and get

$$R^n_{plm} = k_0(1+\tau)^{n+1}a^{(n+1)} + \sum_{i=0}^{n-1}(-1)^{i+1}k_{i+1}\left[\binom{n}{i+1} + \binom{n}{i}\right](1+\tau)^{n-i}a^{(n-i)} + (-1)^{n+1}k_{n+1}a$$

Using the recursive identity of the binomial coefficients we get

$$R^n_{plm} = \sum_{q=0}^{n+1}(-1)^q k_q \binom{n+1}{q}(1+\tau)^{n-q+1}a^{(n-q+1)} = A^{n+1}_{plm}$$

Therefore, we have proven that (18) is indeed the general formula for calculating the NP constants. For completeness of this proof, we need to introduce one final lemma,

Lemma 3 (Gasperín & Pinto 2023)**.**
If the regularity condition is satisfied and the NP constants are finite then they are determined in terms of the initial data by

$$\mathcal{I}^+_{nm} = Q(n)A_{(n+1),n,m} \tag{22}$$

where $Q(n)$ is a numerical coefficient & $A_{(n+1),n,m}$ is determined by the initial data, for ϕ on $\tau = 0$.

Using Lemma 2,

$$\mathcal{I}^+_{nm} = \lim_{\substack{\rho\to 0\\ \tau\to 1}}\langle \hat{L}^{n+1}\phi, Y_{nm}\rangle = \lim_{\substack{\rho\to 0\\ \tau\to 1}}(1+\tau)^{-4(n+1)}\sum_{p=0}^{\infty}\frac{1}{p!}\rho^{p-2(n+1)}A^n_{pnm}(\tau) \Leftrightarrow$$

$$\Leftrightarrow \mathcal{I}^+_{nm} = \frac{2^{-4(n+1)}}{(n+1)!}A^n_{n+1,n,m}(\tau)$$

Evaluating $A^n_{n+1,n,m}$ at $\tau = 1$, one gets (22).

3 Conclusions and future work

General relativity is a theory of gravitation that explains the force of gravity as a curvature of spacetime caused by mass and energy. Black holes, which are objects with such strong gravitational forces that nothing, not even light, can escape them, are an important aspect of this theory. When an object falls into a black hole, it is reduced to just three numbers, leading to the loss of a large amount of information, a problem known as the "information paradox." A recent theory called "soft hair" has been proposed to explain this paradox by positing that non-trivial distortions in clocks, sensitive to the black hole's consumption history, can provide an infinite number of properties for a black hole in certain limits. The Newman-Penrose (NP) constants are quantities defined on null-infinity in general relativity that obey conservation laws for asymptotically flat gravitational fields. These constants can be used to study the residual radiation present in spacetime after a black hole collision, and have been shown to be zero for stationary spacetimes such as the Schwarzschild and Kerr solutions. However, it is still an open question whether the NP constants are zero for all stationary spacetimes.
In the future, we want to examine the NP constants of the critical set $\mathcal{I}^-_{nm}$ to see if there is a connection between the NP constants of the past and those of the future. Calculating the NP constants may allow us to retrieve the logarithmic solutions obtained through the conformal method using asymptotic expansions in a physical setting similar to those described in [4].

4 Bibliography

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