$${}^{+}I^{+}_{2m} = \lim_{\substack{\tau \to 1 \\ \rho \to 0}} -\frac{10\, A_{32m}}{(1+\tau)^6} = -\frac{10\, A_{32m}}{2^6} = -\frac{10\, A_{32m}}{64} = -\frac{5\, A_{32m}}{32}$$

Spin-0 field NP constants

Let $\hat{L} := \tilde{\rho}^2 L$

$$\tilde{\rho}^2 L = \tilde{\rho}^2\, \Theta\, \kappa^{-1} l = \Theta^{-2}\, \Theta\, \kappa^{-1} l = \Theta^{-1} \kappa^{-1} l = \rho^{-1}(1-\tau^2)^{-1}\left(\frac{1-\tau}{1+\tau}\right)\left[(1+\tau)\partial_\tau - \rho\partial_\rho\right]$$

$$= \rho^{-1}\left[(1+\tau)(1-\tau)\right]^{-1}\left(\frac{1-\tau}{1+\tau}\right)\left[(1+\tau)\partial_\tau - \rho\partial_\rho\right] = \rho^{-1}(1+\tau)^{-2}\left[(1+\tau)\partial_\tau - \rho\partial_\rho\right] =$$

$$= \rho^{-1}(1+\tau)^{-1}\partial_\tau - (1+\tau)^{-2}\partial_\rho$$

In order to compute the NP constants we have to do the following:

$$l = 0 \Rightarrow \tilde{\rho}^2 L\, \phi_{00} \longrightarrow \hat{L}^1 \phi_{00} = \langle \hat{L}^1 \phi, Y_{00} \rangle$$

$$l = 1 \Rightarrow (\tilde{\rho}^2 L)^2 \phi_{1m} \longrightarrow \hat{L}^2 \phi_{1m} = \langle \hat{L}^2 \phi, Y_{1m} \rangle$$

$$l = l \Rightarrow (\tilde{\rho}^2 L)^{l+1} \phi_{lm} \longrightarrow \hat{L}^{l+1} \phi_{lm} = (\hat{L}^{l+1} \phi, Y_{lm})$$

where our scalar field ϕ is given by:

$$\phi = \sum_{p=0}^{\infty} \sum_{l=0}^{p} \sum_{m=l}^{m=-l} \frac{1}{p!} a_{p;l,m}(\tau)\, \rho^p\, Y_{lm} \qquad (1)$$

$$\phi_{lm} = \langle \phi, Y_{lm} \rangle = \int_{S^2} \sum_{p=0}^{\infty} \sum_{l'=0}^{p} \sum_{m'=-l'}^{l'} \frac{1}{p!} a_{p;l',m'}(\tau)\, \rho^p\, Y_{l'm'}\, Y_{lm}\, dS$$

$$= \sum_{p=0}^{\infty} \sum_{l'=0}^{p'} \sum_{m'=-l'}^{l'} \frac{1}{p!} a_{p;l',m'}(\tau)\, \rho^p\, \delta_{ll'}\, \delta_{mm'}$$

$$\Rightarrow \phi_{lm} = \sum^{\infty} \frac{1}{p!} a_{p;l,m}(\tau)\, \rho^p \qquad (2)$$

Using the following notation:

$$\left.\begin{aligned} \Lambda^2 &:= \Theta^{-1} K^{-1} = \rho^{-1}(1+\tau)^{-2} \\ e &:= (1+\tau)\partial_\tau - \rho\partial_\rho \end{aligned}\right\} \Rightarrow \hat{\mathcal{L}} = \Lambda^2 e$$

$$\cdot\ \hat{\mathcal{L}}\phi = \Lambda^2 \sum_{p=0}^{\infty}\sum_{l=0}^{p}\sum_{m=-l}^{l} \frac{1}{p!}\, e\left(a_{p;l,m}(\tau)\rho^p Y_{lm}\right) = \Lambda^2 \sum_{p,l,m} Y_{lm}\, e\left(a_{p;l,m}(\tau)\rho^p\right) =$$

$$= \Lambda^2 \sum_{p,l,m} Y_{lm}\left[(1+\tau)\partial_\tau - \rho\partial_\rho\right]\left(a_{p;l,m}(\tau)\rho^p\right) =$$

$$= \Lambda^2 \sum_{p,l,m} Y_{lm}\left[(1+\tau)\dot{a}_{p;l,m}\rho^p - \rho\, a_{p;l,m}\cdot p\rho^{p-1}\right] =$$

$$= \Lambda^2 \sum_{p,l,m} Y_{lm}\rho^p\left[(1+\tau)\dot{a}_{p;l,m} - p\, a_{p;l,m}\right], \text{ where } (1+\tau)\dot{a}_{p;l,m} - p\, a_{p;l,m} = A_{p}$$

$(\equiv)$

$l=0$

$$I_{0m} = \lim_{\substack{\rho\to 0\\ \tau\to 1}} \langle \hat{L}\phi, Y_{00}\rangle = \lim_{\substack{\rho\to 0\\ \tau\to 1}} \langle \Lambda^2 \sum_{p,l,m} Y_{lm} A^0_{p,l,m}(\tau)\rho^p, Y_{00}\rangle =$$

$$= \lim_{\substack{\rho\to 0\\ \tau\to 1}} \Lambda^2 \sum_{p,l,m} A^0_{p,l,m}(\tau)\rho^p \langle Y_{lm}, Y_{00}\rangle = \lim_{\substack{\rho\to 0\\ \tau\to 1}} \Lambda^2 \sum_{p=0}^{\infty}\sum_{l=0}^{p}\sum_{-m}^{m} \frac{1}{p!} A^0_{p,l,m}(\tau)\rho^p \delta_{l0}\delta_{m0} =$$

$$= \lim_{\substack{\rho\to 0\\ \tau\to 1}} \Lambda^2 \sum_{p=0}^{\infty} \frac{1}{p!} A^0_{p,0,0}(\tau)\rho^p = \lim_{\substack{\rho\to 0\\ \tau\to 1}} \rho^{-1}(1+\tau)^{-2} \sum_{p=0}^{\infty} \frac{1}{p!} A^0_{p,0,0}(\tau)\rho^p =$$

$$= \lim_{\substack{\rho\to 0\\ \tau\to 1}} \sum_{p=0}^{\infty} \frac{1}{p!}(1+\tau)^{-2}\rho^{p-1} A^0_{p,0,0}(\tau) = \lim_{\rho\to 0} \sum_{p=0}^{\infty} \frac{1}{p!} 2^{-2}\rho^{p-1} A^0_{p,0,0}(1) =$$

$$= \lim_{\rho\to 0}\left[\frac{2^{-2}}{0!}\rho^{-1} A_{0,0,0}(1) + \frac{2^{-2}}{p!}\rho^0 A^0_{1,0,0}(1) + \sum_{p=2}^{\infty} \frac{2^{-2}}{p!}\rho^{p-1} A^0_{p,0,0}(\tau)\right] =$$

$$\Rightarrow I_{0m} = \lim_{\rho\to 0} \left\{ \frac{\tau^{-2}}{0!} \rho^{-1} A^{0}_{0,0,0}(1) + \frac{\tau^{-2}}{1!} A^{0}_{1,0,0}(1) \right\}$$

Now, we want to evaluate $A^{0}_{p,l,m}$:

$$A^{0}_{p,l,m}(\tau) = (1+\tau)\dot{a}_{p,l,m}(\tau) - p\, a_{p,l,m}(\tau) \Rightarrow A^{0}_{0,0,0}(\tau) = 2\dot{a}_{0,0,0}(\tau) - 0\cdot a_{0,0,0}(\tau) \quad (3$$

$$= 2\dot{a}_{0,0,0}(\tau) = -\frac{D_{000}}{\tau-1}$$

$$A^{0}_{1,0,0}(\tau) = 2\dot{a}_{1,0,0}(1) - 1\cdot a_{1,0,0}(1) = -A_{100}$$

Imposing the regularity condition : $D_{000} = 0 \Rightarrow I_{0m} = -\frac{A_{100}}{4}$ (4)

<u>Computing the $l=1$ constant</u>

$$I_{1m} = \lim_{\substack{\rho\to 0\\ \tau\to 1}} \langle \hat{L}^2\phi, Y_{1m}\rangle$$

$$\hat{L}^2\phi = \Lambda^2 e(\hat{L}\phi) = \Lambda^2 e\Big(\Lambda^2 \sum_{p,l,m} Y_{lm}\, \rho^p A^{0}_{p,l,m}(\tau)\Big) =$$

$$= \Lambda^2\Big[\Lambda^2 \sum_{p,l,m} Y_{lm}\, e\big(\rho^p A^{0}_{p,l,m}(\tau)\big) + \sum_{p,l,m} Y_{lm}\, \rho^p A^{0}_{p,l,m}(\tau)\, e(\Lambda^2)\Big]$$

We see that :

$$e(\Lambda^2) = [(1+\tau)\partial_\tau - \rho\partial_\rho][\rho^{-1}(1+\tau)^{-2}] = [(1+\tau)\rho^{-1}(-2)(1+\tau)^{-3} - \rho(1+\tau)^{-2}(-1)\rho^{-2}$$

$$= -2\rho^{-1}(1+\tau)^{-2} + \rho^{-1}(1+\tau)^{-2} = -\rho^{-1}(1+\tau)^{-2} = -\Lambda^2$$

$$\Rightarrow e(\Lambda^2) = -\Lambda^2$$

$$\hat{L}^2\phi = \Lambda^2\left[\Lambda^2\sum_{p,l,m} Y_{lm}\, e\left(\rho^p A^0_{p,l,m}(\tau)\right) + \sum_{p,l,m} Y_{lm}\,\rho^p\,(\tau)(-\Lambda^2)\right] =$$

$$= \Lambda^2\left\{\Lambda^2\sum_{p,l,m} Y_{lm}\left[e\left(\rho^p A^0_{p,l,m}\right) - A^0_{p,l,m}\,\rho^p\right]\right\} =$$

$$= \Lambda^4\sum_{p,l,m} Y_{lm}\left\{\left[(1+\tau)\rho^p \dot{A}^0_{p,l,m} - \rho^p A^0_{p,l,m}\, p\,\rho^{p-1}\right] - \rho^p A^0_{p,l,m}\right\} =$$

$$= \Lambda^4\sum_{p,l,m} Y_{lm}\,\rho^p\left[(1+\tau)\dot{A}^0_{p;l,m} - (p+1)A^0_{p;l,m}\right], \text{ where}$$

$$(1+\tau)\dot{A}^0_{p;l,m} - (p+1)A^0_{p;l,m} = A^1_{p;l,m}(\tau) \qquad (5)$$

$$\hat{L}^2\phi = \Lambda^4\sum_{p,l,m} Y_{lm}\,\rho^p A^1_{p,l,m}(\tau) \Rightarrow$$

$$I_{1m} = \lim_{\substack{\rho\to 0\\ \tau\to 1}}\langle \hat{L}^2\phi, Y_{1m}\rangle = \lim_{\substack{\rho\to 0\\ \tau\to 1}}\left\langle \Lambda^4\sum_{p,l,m} Y_{lm'}\,\rho^p A^1_{p,l,m'}(\tau),\, Y_{1m}\right\rangle =$$

$$= \lim_{\substack{\rho\to 0\\ \tau\to 1}}\Lambda^4\sum_{p=0}^{\infty}\sum_{l=0}^{p}\sum_{m'=-l}^{l}\frac{1}{p!}A^1_{p,l,m}(\tau)\,\delta_{l1}\,\delta_{mm'} = \lim_{\substack{\rho\to 0\\ \tau\to 1}}\Lambda^4\sum_{p=0}^{\infty}\frac{1}{p!}A^1_{p,1,m}(\tau)\,\rho^p$$

$$= \lim_{\substack{\rho\to 0\\ \tau\to 1}}\rho^{-2}(1+\tau)^{-4}\sum_{p=0}^{\infty}\frac{1}{p!}A^1_{p,l,m}(\tau)\,\rho^p = \lim_{\substack{\rho\to 0\\ \tau\to 1}}(1+\tau)^{-4}\sum_{p=0}^{\infty}\frac{1}{p!}\rho^{p-2}A^1_{p,1,m}(\tau) =$$

$$= \lim_{\substack{\rho\to 0\\ \tau\to 1}}(1+\tau)^{-4}\left\{\sum_{p=0}^{1}\frac{1}{p!}\rho^{p-2}A^1_{p,l,m}(\tau) + \sum_{p=2}^{2}\frac{1}{p!}\rho^0 A^1_{p,1,m}(\tau) + \sum_{p=3}^{\infty}\frac{1}{p!}\rho^{p-2}A^1_{p,1,m}(\tau)\right.$$

$$= \lim_{\substack{\rho\to 0\\ \tau\to 1}}(1+\tau)^{-4}\left\{\frac{1}{0!}\rho^{-2}A^1_{0,1,m}(\tau) + \frac{1}{1!}\rho^{-1}A^1_{1,1,m}(\tau) + \frac{1}{2!}\rho^0 A^1_{2,1,m}(\tau) + \sum_{p=3}^{\infty}\frac{1}{p!}\rho^{p-3}A^1_{p,1,m}(\tau)\right.$$

$$I_{1m} = \lim_{\substack{\rho \to 0 \\ \tau \to 1}} z^{-4} \left\{ \rho^{-2} A^1_{0,1,m}(\tau) + \rho^{-1} A^1_{1,1,m}(\tau) \right\} + \lim_{\rho \to 0} z^{-4} \frac{1}{2!} A^1_{2,1,m}(\tau)$$

Evaluating $A^1_{p,l,m}$:

$$A^1_{p,l,m}(\tau) = (1+\tau)\dot{A}^0_{p,l,m} - (p+1)A^0_{p,l,m} = (1+\tau)\frac{d}{d\tau}\left[(1+\tau)\dot{a}_{p,l,m} - p\,a_{p,l,m}\right] -$$

$$-(p+1)\left[(1+\tau)\dot{a}_{p,l,m} - p\,a_{p,l,m}\right] = (1+\tau)\left[(1+\tau)\ddot{a}_{p,l,m} + \dot{a}_{p,l,m} - p\,\dot{a}_{p,l,m}\right] +$$

$$-(p+1)\left[(1+\tau)\dot{a}_{p,l,m} - p\,a_{p,l,m}\right] = (1+\tau)^2\ddot{a}_{p,l,m} - 2p(1+\tau)\dot{a}_{p,l,m} + p(p+1)a_{p,l,m}$$

$$A^1_{p,l,m} = (1+\tau)^2\ddot{a}_{p,l,m} - 2p(1+\tau)\dot{a}_{p,l,m} + p(p+1)a_{p,l,m} \qquad (6)$$

$$A^1_{0,1,m}(\tau) = 0, \quad A^1_{1,1,m}(\tau) = \frac{D_{111}}{2(\tau-1)}, \quad A^1_{2,1,m}(\tau) = 6A_{211}$$

$$\Rightarrow D_{111} = 0 \Rightarrow I_{1m} = z^4 \frac{1}{2!} 6A_{211} \qquad (7)$$

<u>Computing $l=2$ constant</u>

$$I_{2m} = \lim_{\substack{\rho \to 0 \\ \tau \to 1}} \langle \hat{L}^3 \phi, Y_{2m} \rangle$$

$$\hat{L}^3 \phi = \Lambda^2 e(\hat{L}^2 \phi) = \Lambda^2 e\left(\Lambda^4 \sum_{p,l,m} Y_{lm} \rho^p A^1_{p,l,m}\right) =$$

$$= \Lambda^2 \left[\Lambda^4 \sum_{p,l,m} Y_{lm}\, e\left(\rho^p A^1_{p,l,m}\right) + \sum_{p,l,m} Y_{lm} \rho^p A^1_{p,l,m}\, e(\Lambda^4)\right]$$

$$\Rightarrow e(\Lambda^{2n}) = -n\Lambda^{2n}. \quad \text{In particular } e(\Lambda^4) = -2\Lambda^4$$

$$\hat{L}^3\phi = \Lambda^2\left[\Lambda^4 \sum_{p,l,m} Y_{lm}\, e\left(\rho^p A^1_{p,l,m}\right) - 2\Lambda^4 \sum_{p,l,m} Y_{lm}\, \rho^p A^1_{p,l,m}\right] =$$

$$= \Lambda^6 \sum_{p,l,m} Y_{lm} \left\{ e\left(\rho^p A^1_{p,l,m}\right) - 2\rho^p A^1_{p,l,m} \right\} =$$

$$= \Lambda^6 \sum_{p,l,m} \left\{ Y_{lm}\left[(1+\tau)\dot{A}^1_{p,l,m}\rho^p - \rho A^1_{p,l,m}\, p\, \rho^{p-1}\right] - 2\rho^p A^1_{p,l,m} \right\} =$$

$$= \Lambda^6 \sum_{p,l,m} \left\{ Y_{lm}\left[(1+\tau)\dot{A}^1_{p,l,m}\rho^p - p\rho^p A^1_{p,l,m}\right] - 2\rho^p A^1_{p,l,m} \right\} =$$

$$= \Lambda^6 \sum_{p,l,m} Y_{lm}\, \rho^p \left[(1+\tau)\dot{A}^1_{p,l,m} - (p+2) A^1_{p,l,m}\right]$$

$$\Rightarrow \hat{L}^3\phi = \Lambda^6 \sum_{p,l,m} Y_{lm}\, \rho^p A^2_{p,l,m}$$

$$I_{2m} = \lim_{\substack{\rho\to 0\\ \tau\to 1}} \langle \hat{L}^3\phi, Y_{2m}\rangle = \lim_{\substack{\rho\to 0\\ \tau\to 1}} \left\langle \Lambda^6 \sum_{p,l',m'} Y_{l'm'}\, \rho^p A^2_{p,l',m'}, Y_{2m}\right\rangle =$$

$$= \lim_{\substack{\rho\to 0\\ \tau\to 1}} \Lambda^6 \sum_{p,l',m'} \rho^p A^2_{p,l',m'} \langle Y_{l'm'}, Y_{2m}\rangle = \lim_{\substack{\rho\to 0\\ \tau\to 1}} \Lambda^6 \sum_{p=0}^{\infty}\sum_{l'=0}^{p}\sum_{m'=-l'}^{l'} \rho^p A^2_{p,l',m'}\, \delta_{l'2}\delta_{m'm}$$

$$= \lim_{\substack{\rho\to 0\\ \tau\to 1}} \Lambda^6 \sum_{p=0}^{\infty} \rho^p A^2_{p,2,m}(\tau) = \lim_{\substack{\rho\to 0\\ \tau\to 1}} \rho^{-3}(1+\tau)^{-6} \sum_{p=0}^{\infty} \rho^p A^2_{p,2,m} =$$

$$= \lim_{\substack{\rho\to 0\\ \tau\to 1}} (1+\tau)^{-6} \sum_{p=0}^{\infty} \rho^{p-3} A^2_{p,2,m}(\tau) =$$

$$= \lim_{\substack{\rho\to 0\\ \tau\to 1}} (1+\tau)^{-6} \Big\{ \underbrace{\sum_{p=0}^{2} \rho^{p-3} A^{2}_{p;\ell,m}}_{\text{singular piece}} + \underbrace{\sum_{p=3}^{3} \rho^{0} A^{2}_{p;2,m}}_{\substack{\text{regular}\\ \text{piece}}} + \underbrace{\sum_{p=4}^{\infty} \rho^{p-3} A^{2}_{p;2,m}}_{\substack{\text{Vanishing piece}\\ \text{at } \rho=0}} \Big\} =$$

$$= \lim_{\substack{\rho\to 0\\ \tau\to 1}} (1+\tau)^{-6} \sum_{p=0}^{2} \rho^{p-3} A^{2}_{p;2,m} + \lim_{\substack{\rho\to 0\\ \tau\to 1}} A^{2}_{3;2,m} + 0$$

$$I_{2m} = \lim_{\substack{\rho\to 0\\ \tau\to 1}} (1+\tau)^{-6} \Big\{ \underbrace{\rho^{-3} A^{2}_{0;2,m} + \rho^{-2} A^{2}_{1;2,m} + \rho^{-1} A^{2}_{2;2,m}}_{\text{singular pieces}} \Big\} + \underbrace{2^{-6} A^{2}_{3;2,m}}_{\text{regular piece}}$$

Evaluating $A^{2}_{p;\ell,m}$

$$A^{2}_{p;\ell,m} = (1+\tau)\dot{A}^{1}_{p;\ell,m} - (p+2) A^{1}_{p;\ell,m} = \frac{d}{d\tau}(1+\tau)\left[(1+\tau)^2 \ddot{a} - 2p(1+\tau)\dot{a} + p(p+1)a\right] - (p+2)\left[(1+\tau)^2\ddot{a} - 2p(1+\tau)\dot{a} + p(p+1)a\right]$$

$$A^{2}_{p;\ell,m} = (1+\tau)^3 \dddot{a} + 3p(1+\tau)^2\ddot{a} + 3p(1+p)(1+\tau)\dot{a} - p(p+1)(2+p)a$$

- $A^{2}_{0;2,m}(\tau) = 0$
- $A^{2}_{1;2,m}(\tau) = 0$
- $A^{2}_{2;2,m}(\tau) = \dfrac{3D_{222}}{4(1-\tau)}$
- $A^{2}_{3;2,m}(\tau) = -60A_{322}$

$\Rightarrow D_{222} = 0$

$$I_{2m} = 2^{-6} A^{2}_{3;2,m}(1) = 2^{-6} \cdot 60 A_{322}$$

Computing $\ell = 3$

$$\underline{I}_{3m} = \lim_{\substack{\rho \to 0 \\ \tau \to 1}} \langle \hat{L}^4 \phi, Y_{3m} \rangle$$

$$\hat{L}^4 \phi = \Lambda^2 e(\hat{L}^3 \phi) = \Lambda^2 e\left(\Lambda^6 \sum_{p,\ell,m} Y_{\ell m} \rho^p A^2_{p,\ell,m}\right) =$$

$$= \Lambda^2 \left[\Lambda^6 \sum_{p,\ell,m} Y_{\ell m}\, e\left(\rho^p A^2_{p,\ell,m}\right) + \sum_{p,\ell,m} Y_{\ell m} \rho^p A^2_{p,\ell,m}\, e(\Lambda^6)\right]$$

Recall that : $e(\Lambda^2) = -\Lambda^2 \Rightarrow e(\Lambda^{2n}) = e\left[(\Lambda^2)^n\right] = n(\Lambda^2)^{n-1} e(\Lambda^2)$

$\Rightarrow$ In particular $e(\Lambda^6) = -3\Lambda^6$

$$\hat{L}^4 \phi = \Lambda^2 \left[\Lambda^6 \sum_{p,\ell,m} Y_{\ell m}\, e\left(\rho^p A^2_{p,\ell,m}\right) - 3\Lambda^6 \sum_{p,\ell,m} Y_{\ell m} \rho^p A^2_{p,\ell,m}\right] =$$

$$= \Lambda^8 \sum_{p,\ell,m} Y_{\ell m} \left\{ e\left(\rho^p A^2_{p,\ell,m}\right) - 3\rho^p A^2_{p,\ell,m} \right\} = \Lambda^8 \sum_{p,\ell,m} \left\{ Y_{\ell m} \left[(1+\tau)\dot{A}^2_{p,\ell,m} \rho^p - \right.\right.$$

$$\left.\left. - \rho A^2_{p,\ell,m}\, p \rho^{p-1}\right] - 3\rho^p A^2_{p,\ell,m} \right\} = \Lambda^8 \sum_{p,\ell,m} \left\{ Y_{\ell m} \left[(1+\tau)\dot{A}^2_{p,\ell,m} \rho^p - p\rho^p A^2_{p,\ell,m}\right.\right.$$

$$\left.\left. - 3\rho^p A^2_{p,\ell,m} \right\} = \Lambda^8 \sum_{p,\ell,m} Y_{\ell m} \rho^p \left[(1+\tau)\dot{A}^2_{p,\ell,m} - (p+3) A^2_{p,\ell,m}\right]$$

$$(1+\tau)\dot{A}^2_{p,\ell,m} - (p+3) A^2_{p,\ell,m} = A^3_{p,\ell,m} \qquad (10)$$

$$\Rightarrow \hat{L}^4 \phi = \Lambda^8 \sum_{p,\ell,m} Y_{\ell m} \rho^p A^3_{p,\ell,m}$$

We compute the following NP constant :

$$\underline{I}_{3m} = \lim_{\substack{\rho \to 0 \\ \tau \to 1}} \langle \hat{L}^4 \phi, Y_{3m} \rangle = \lim_{\substack{\rho \to 0 \\ \tau \to 1}} \left\langle \Lambda^8 \sum_{p,\ell,m} Y_{\ell' m'} \rho^p A^3_{p,\ell',m'}, Y_{3m} \right\rangle =$$

$$= \lim_{\substack{\rho\to 0\\ \tau\to 1}} \Lambda^8 \sum_{p',l',m'} \rho^p A^3_{p;l',m'} \langle Y_{l'm'}, Y_{3m} \rangle = \lim_{\substack{\rho\to 0\\ \tau\to 1}} \Lambda^8 \sum_{p=0}^{\infty} \sum_{l'=0}^{p} \sum_{m'=-l'}^{l'} \rho^p A^3_{p;l',m'} \delta\delta$$

$$= \lim_{\substack{\rho\to 0\\ \tau\to 1}} \Lambda^8 \sum_{p=0}^{\infty} \rho^p A^3_{p;3,m}(\tau) = \lim_{\substack{\rho\to 0\\ \tau\to 1}} \rho^{-4}(1+\tau)^{-8} \sum_{p=0}^{\infty} \rho^p A^3_{p;3,m} =$$

$$= \lim_{\substack{\rho\to 0\\ \tau\to 1}} (1+\tau)^{-8} \sum_{p=0}^{\infty} \rho^{p-4} A^3_{p;l,m}(\tau)$$

$$= \lim_{\substack{\rho\to 0\\ \tau\to 1}} (1+\tau)^{-8} \Big\{ \underbrace{\sum_{p=0}^{3} \rho^{p-4} A^3_{p;l,m}}_{\text{singular piece}} + \underbrace{\sum_{p=4}^{4} \rho^{0} A^3_{p;3,m}}_{\text{regular piece}} + \underbrace{\sum_{p=5}^{\infty} \rho^{p-4} A^3_{p;3,m}}_{\text{Vanishig piece at } \rho=0} \Big\} =$$

$$= \lim_{\substack{\rho\to 0\\ \tau\to 1}} (1+\tau)^{-8} \sum_{p=0}^{3} \rho^{p-4} A^3_{p;l,m} + \lim_{\substack{\rho\to 0\\ \tau\to 1}} A^3_{p;3,m} + 0$$

$$\Rightarrow \quad I_{3m} = \lim_{\substack{\rho\to 0\\ \tau\to 1}} (1+\tau)^{-8} \Big\{ \underbrace{\rho^{-4} A^3_{0;3,m} + \rho^{-3} A^3_{1;3,m} + \rho^{-2} A^3_{2;3,m} + \rho^{-1} A^3_{3;3,m}}_{\text{Singular pieces}} \Big\}$$

$$+ \underbrace{2^{-8} A^3_{4;3,m}(1)}_{\text{regular piece}}$$

Evaluating $A^3_{p;l,m}(\tau)$:

$$A^3_{p;l,m} = (1+\tau)\dot{A}^2_{p;l,m} - (p+3) A^2_{p;l,m} \qquad (10)$$

$$\dot{A}^2_{p;l,m} = (1+\tau)^3 a^{(4)} + 3(1+\tau)^2 a^{(3)} - 3p(1+\tau)^2 a^{(3)} - 6p(1+\tau) a^{(2)} +$$

$$+ 3p(1+p)(1+\tau) a^{(2)} + 3p(1+p) a^{(1)} - p(p+1)(p+2) a^{(1)} =$$

$$= (1+\tau)^3 a^{(4)} + 3(1+\tau)^2 (1-p) a^{(3)} + 3p(1+\tau)\left[-2+p+1\right] a^{(2)} + p(p+1)(3-(p+2)) a^{(1)}$$

$$= (1+\tau)^3 a^{(4)} + 3(1+\tau)^2 (p-1) a^{(3)} + 3p(p-1)(1+\tau) a^{(2)} + p(p+1)(1-p) a^{(1)}$$

$$A^3_{p;l,m}(\tau) = (1+\tau)\dot{A}^2 - (p+3)A^2 = (1+\tau)^4 a^{(4)} - 4p(1+\tau)^3 a^{(3)} + 3p(1+\tau)^2(2p+2)a^{(2)} +$$

$$+ (1+\tau)(1+p)(-4p^2-8p)a^{(1)} + p(p+1)(p+2)(p+3)a$$

$$A^3_{p;l,m}(\tau) = (1+\tau)^4 a^{(4)} - 4p(1+\tau)^3 a^{(3)} + 6p(p+1)(1+\tau)^2 a^{(2)} - 4p(p+2)(p+1)(1+\tau)a^{(1)} +$$

$$+ p(p+1)(p+2)(p+3)a \qquad (12)$$

The previous discussion suggests that, in principle, it shoud be possible to obtain a general formula $A_{p;l,m}(\tau)$. Revising the calculation of I_{0m}, I_{1m}, I_{2m}, I_{3m} one can obtain the following results concerning the overall structure of the spin-0 N-P constants:

We start with,

$$A^n = K_0\binom{n+1}{0}(1+\tau)^n a^{(n)} - K_1\binom{n+1}{1}p(1+\tau)^{n-1}a^{(n-1)} + K_2\binom{n+1}{2}p(p+1)(1+\tau)^{n-2}a^{(n-2)}$$

$$+ \dots + (-1)^n\binom{n+1}{q}K_q\, a.$$

where we define K_q to be,

$$K_q := \frac{p(qp+q-1)!}{p!}$$

and $\binom{n+1}{q}$ to be the binomial coefficients

Therefore,

$$A^n_{0;p,l,m} = \sum_{q=0}^{n+1}(-1)^q\frac{(p+q-1)!}{(p-1)!}\binom{n+1}{q}(1+\tau)^{n-q+1}a^{(n-q+1)} \qquad (13)$$

If we substitute $n=0,1,2,3$ we see that we arrive at the desired results.

$$A^{0}_{0;p,l,m} = \sum_{q=0}^{1} (-1)^{q} \binom{1}{q} \frac{(p+q-1)!}{(p-1)!} (1+\tau)^{-q+1} a^{(1-q)} = (1+\tau)\, a^{(1)} - p\, a^{(0)}$$

$$A^{1}_{0;p,l,m} = \sum_{q=0}^{2} (-1)^{q} \binom{2}{q} \frac{(p+q-1)!}{(p-1)!} (1+\tau)^{-q} a^{(q)} = (1+\tau)^{2} a^{(2)} - 2p(1+\tau)\, a^{(1)} + p(p+1)\, a^{(0)}$$

$$A^{2}_{0;p,l,m} = \sum_{q=0}^{3} (-1)^{q} \frac{(p+q-1)!}{(p-1)!} \binom{3}{q} (1+\tau)^{q-1} a^{(q-1)} = (1+\tau)^{3} a^{(3)} + 3p(1+\tau)^{2} a^{(2)} +$$

$$+ 3p(p+1)(1+\tau)\, a^{(1)} - p(p+1)(p+2)\, a^{(0)}$$

$$A^{3}_{0;p,l,m} = \sum_{q=0}^{4} (-1)^{q} \binom{4}{q} \frac{(p+q-1)!}{(p-1)!} (1+\tau)^{3-q+1} a^{(3-q+1)} = (1+\tau)^{4} a^{(4)} - 4p(1+\tau)^{3} a^{(3)}$$

$$+ 6p(p+1)(1+\tau)^{2} a^{(2)} - 4p(p+1)(p+2)(1+\tau)\, a^{(1)} + p(p+1)(p+2)(p+3)\, a$$

Lemma:

$$\hat{L}^{(n)}(\phi) = \Lambda^{2n} \sum_{p=0}^{\infty} \sum_{l=0}^{p} \sum_{m=l}^{m=-l} \frac{1}{p!} \rho^{p} A^{n-1}_{0;p,l,m}(\tau)\, Y_{0,l,m} \qquad (14)$$

With

$$A^{n-1}_{0;p,l,m} = \sum_{q=0}^{n} (-1)^{q} k_{q} \binom{n}{q} (1+\tau)^{n-q} a^{(n-q)} \qquad (15)$$

Basis of induction: cases $n=0$, $n=1$, $n=2$, $n=3$

Induction hypothesis: eq (14) & eq (15)

Induction step:

$$\hat{L}^{n+1}(\phi) = \hat{L}\left(\hat{L}^{n}(\phi)\right) = \hat{L}\left(\Lambda^{2n}\sum_{p=0}^{\infty}\sum_{l=0}^{p}\sum_{m=l}^{m=-l}\frac{1}{p!}\rho^{p}Y_{lm}A^{n-1}_{p,l,m}\right) =$$

$$= \Lambda^{2} e\left(\Lambda^{2n}\sum_{p,l,m}\rho^{p}Y_{lm}A^{n-1}_{p,l,m}\right) = \Lambda^{2}\left\{\Lambda^{2n}\sum_{p,l,m} e\left(\rho^{p}Y_{lm}A^{n-1}\right) + \right.$$

$$\left. + \sum_{p,l,m}\rho^{p}A^{n-1}_{p,l,m}\, e\left(\Lambda^{2n}\right)\right\} = \Lambda^{2}\left\{\Lambda^{2n}\sum_{p,l,m} e\left(\rho^{p}Y_{lm}A^{n-1}\right) - n\sum_{p,l,m}\rho^{p}Y_{lm}A \right.$$

$$= \Lambda^{2(n+1)}\sum_{p,l,m} Y_{lm}\left[e\left(\rho^{p}A^{n-1}_{p,l,m}\right) - n\rho^{p}A^{n-1}_{p,l,m}\right]$$

$$\Rightarrow e\left(\rho^{p}A^{n-1}_{plm}\right) = \left[(1+\tau)\partial_{\tau} - \rho\partial_{\rho}\right]\left(\rho^{p}A^{n-1}_{plm}\right) = (1+\tau)\rho^{p}\dot{A}^{n-1}_{plm} - \rho p \rho^{p-1}A^{n-1}_{plm}$$

$$= (1+\tau)\rho^{p}\dot{A}^{n-1}_{plm} - p\rho^{p}A^{n-1}_{plm}$$

$$\Rightarrow \hat{L}^{(n+1)}(\phi) = \Lambda^{2(n+1)}\sum_{p,l,m} Y_{lm}\left[(1+\tau)\rho^{p}\dot{A}^{n-1}_{plm} - p\rho^{p}A^{n-1}_{plm} - n\rho^{p}A^{n-1}_{plm}\right] =$$

$$= \Lambda^{2(n+1)}\sum_{p,l,m} Y_{lm}\rho^{p}\left[(1+\tau)\dot{A}^{n-1}_{plm} - (p+n)A^{n-1}_{plm}\right]$$

Thus we have shown that:

$$\hat{L}^{n+1}(\phi) = \Lambda^{2(n+1)}\sum_{p,l,m}\rho^{p}R^{n}_{plm}, \quad \text{where}$$

$$R^{n}_{plm} := (1+\tau)\dot{A}^{n-1}_{p,l,m} - (p+n)A^{n-1}_{p,l,m}$$

Using the induction hypothesis eq. (15) we can compute all the pieces to construct R^n_{plm}:

$$\dot{A}^{n-1}_{plm} = \frac{d}{d\tau}\left[\sum_{q=0}^{n} (-1)^q K_q \binom{n}{q} (1+\tau)^{n-q} a^{(n-q)}\right] = \sum_{q=0}^{n} (-1)^q K_q \binom{n}{q} \Big[(1+\tau)^{n-q} a^{(n-q+1)}$$

$$+ (n-q)(1+\tau)^{n-q-1} a^{(n-q)}\Big] \Rightarrow (1+\tau)\dot{A}^{n-1}_{plm} = \sum_{q=0}^{n} (-1)^q K_q \binom{n}{q} \Big[(1+\tau)^{n-q+1} a^{(n-q+1)} +$$

$$+ (n-q)(1+\tau)^{n-q} a^{(n-q)}$$

$$-(p+n)A^{n-1}_{plm} = \sum_{q=0}^{n} (-1)^q K_q \binom{n}{q} \Big[-(p+n)(1+\tau)^{n-q} a^{(n-q)}\Big]$$

adding up these two pieces, we get R^n_{plm}:

$$R^n_{plm} = \sum_{q=0}^{n} (-1)^q K_q \binom{n}{q} \Big\{(1+\tau)^{n-q+1} a^{(n-q+1)} + \big[n-q-(p+n)\big](1+\tau)^{n-q} a^{(n-q)}\Big\} =$$

$$= \sum_{q=0}^{n} (-1)^q K_q \binom{n}{q} \Big\{(1+\tau)^{n-q+1} a^{(n-q+1)} - (p+q)(1+\tau)^{n-q} a^{(n-q)}\Big\} \qquad (16)$$

Expanding the first term in the first sum of eq. (16) renders

$$R^n_{plm} = (-1)^0 K_0 \binom{n}{0} (1+\tau)^{n+1} a^{(n+1)} + \sum_{q=1}^{n} (-1)^q K_q \binom{n}{q} (1+\tau)^{n-q+1} a^{(n-q+1)} -$$

$$- \sum_{q=0}^{n} (-1)^q K_q \binom{n}{q} (p+q)(1+\tau)^{n-q} a^{(n-q)}$$

Separating the last term in the second sum and rearranging gives

$$R^n_{plm} = K_0 (1+\tau)^{n+1} a^{(n+1)} + \sum_{q=1}^{n} (-1)^q K_q \binom{n}{q} (1+\tau)^{n-q+1} a^{(n-q+1)} +$$

$$+ - \left\{ \sum_{q=0}^{n-1} \left[(-1)^q K_q \binom{n}{q} (p+q)(1+\tau)^{n-q} a^{(n-q)} \right] + (-1)^n K_n (p+n) \binom{n}{n} a \right.$$

$$\Rightarrow K_{q+1} = (p+q) K_{q\cancel{+m}}$$

$$K_{n+1} = (p+(n+1)) K_n \Rightarrow$$

$$R^n_{plm} = K_0 (1+\tau)^{n+1} a^{(n+1)} + \sum_{q=1}^{n} (-1)^q K_q \binom{n}{q} (1+\tau)^{n-q+1} a^{(n-q+1)} + \sum_{q=0}^{n-1} \left[(-1)^{q+1} K_{q+1} \binom{n}{q} \right.$$

$$\left. (1+\tau)^{n-q} a^{(n-q)} \right] + (-1)^{n+1} K_{n+1} a .$$

Relabelling $i = q-1 \Rightarrow q = i+1$
$i(q=1) = 0$ & $i(q=n+1) = n$

$$R^n_{plm} = K_0 (1+\tau)^{n+1} a^{(n+1)} + \sum_{i=0}^{n-1} (-1)^{i+1} K_{i+1} \binom{n}{i+1} (1+\tau)^{n-i} a^{(n-i)} +$$

$$+ \sum_{q=0}^{n-1} (-1)^{q+1} K_{q+1} \binom{n}{q} (1+\tau)^{n-q} a^{(n-q)} + (-1)^{n+1} K_{n+1} a$$

For the second sum we simply relabel the dumie index $i = q$ and get

$$R^n_{plm} = K_0 (1+\tau)^{n+1} a^{(n+1)} + \sum_{i=0}^{n-1} (-1)^{i+1} K_{i+1} \left[\binom{n}{i+1} + \binom{n}{i} \right] (1+\tau)^{n-i} a^{(n-i)} + (-1)^{n+1} K_{n+1} a$$

Using the recursive identity of the binomial coefficients

$$\binom{i}{j} = \binom{i-1}{j} + \binom{i-1}{j-1}$$

$$\binom{n+1}{i+1} = \binom{n}{i+1} + \binom{n}{i} \Rightarrow$$

$$\cancel{R^{n}_{plm} = K_0 (1+\tau)^{n+1} a^{(n+1)} + \sum_{q=0}^{n-1} (-1)^{q} K_q \binom{n}{q} (1+\tau)^{n-q+1} a^{(n-q+1)} +}$$

$$\cancel{+ \sum_{q=0}^{n-1} \left[(-1)^{q+1} K_{q+1} \binom{n}{q} (1+\tau)^{n-q} a^{(n-q)} \right] + (-1)^{n+1} K_{n+1} a}$$

$$R^{n}_{plm} = K_0 (1+\tau)^{n+1} a^{(n+1)} + \sum_{i=0}^{n-1} \left[(-1)^{i+1} K_{i+1} \binom{n+1}{i+1} (1+\tau)^{n-i} a^{(n-i)} \right] + (-1)^{n+1} K_{n+1}$$

$$= (-1)^{0} K_0 \binom{n+1}{0} (1+\tau)^{n+1} a^{(n+1)} + \sum_{i=0}^{n-1} (-1)^{i+1} K_{i+1} \binom{n+1}{i+1} (1+\tau)^{n-i} a^{(n-i)} +$$

$$+ (-1)^{n+1} K_{n+1} \binom{n+1}{n+1} a$$

$$\Rightarrow R^{n}_{plm} = \sum_{q=0}^{n+1} (-1)^{q} K_q \binom{n+1}{q} (1+\tau)^{n-q+1} a^{(n-q+1)} = A^{n+1}_{plm} \quad \square$$