Spin - 0 field NP constants in $\mathscr{I}^-$

Let $\hat{\underline{L}} := \tilde{\rho}^2 \underline{L}$, where $\tilde{\rho}^2 \underline{L} = \tilde{\rho}^{-2} \Theta \kappa \underline{l} = \Theta^{-2} \Theta \kappa \underline{l} =$

$$= \rho^{-1}(1-\tau^2)^{-1}\left(\frac{1+\tau}{1-\tau}\right)\left[(1-\tau)\partial_\tau + \rho\partial_\rho\right] = \rho^{-1}\left[(1-\tau)(1+\tau)\right]^{-1}\left(\frac{1+\tau}{1-\tau}\right)\left[(1-\tau)\partial_\tau + \rho\partial_\rho\right] =$$

$$= \rho^{-1}(1-\tau)^{-2}\left[(1-\tau)\partial_\tau + \rho\partial_\rho\right] = \rho^{-1}(1-\tau)^{-1}\partial_\tau + (1-\tau)^{-2}\partial_\rho$$

In order to compute the NP constants at past null-infinity, we perform a similar analyses:

$$l=0 \Rightarrow \tilde{\rho}^2 \underline{L}\, \phi_{00} \rightarrow \hat{\underline{L}}^1 \phi_{00} = \langle \hat{\underline{L}}^1 \phi, Y_{00} \rangle$$

$$l=1 \Rightarrow (\tilde{\rho}^2 \underline{L})^2 \phi_{1m} \rightarrow \hat{\underline{L}}^2 \phi_{1m} = \langle \hat{\underline{L}}^2 \phi, Y_{1m} \rangle$$

$$l=l \Rightarrow (\tilde{\rho}^2 \underline{L})^{l+1} \phi_{lm} \rightarrow \hat{\underline{L}}^{l+1} \phi_{lm} = \langle \hat{\underline{L}}^{l+1} \phi, Y_{lm} \rangle$$

The scalar field is given by,

$$\phi = \sum_{p=0}^{\infty} \sum_{l=0}^{p} \sum_{m=l}^{m=-l} \frac{1}{p!} a_{p;l,m}(\tau) \rho^p Y_{lm} \tag{1}$$

$$\phi_{lm} = \sum_{p=0}^{\infty} \sum_{l'=0}^{p'} \sum_{m'=-l'}^{l'} \frac{1}{p!} a_{p;l',m'}(\tau) \rho^p \delta_{ll'} \delta_{mm'}$$

$$= \sum_{p=0}^{\infty} \frac{1}{p!} a_{p;l',m'}(\tau) \rho^p \tag{2}$$

Introducing the following notation:

$$\left.\begin{aligned}\underline{\Lambda}^2 &:= \Theta^{-1} \kappa = \rho^{-1}(1-\tau)^{-2} \\ \underline{e} &:= (1-\tau)\partial_\tau + \rho\partial_\rho\end{aligned}\right\} \hat{\underline{L}} = \underline{\Lambda}^2 \underline{e}$$

$$\bullet\ \hat{\underline{L}}\phi = \underline{\Lambda}^2 \sum_{p=0}^{\infty}\sum_{l=0}^{p}\sum_{m=-l}^{l} \frac{1}{p!}\,\underline{e}\left(a_{p,l,m}(\tau)\rho^p Y_{lm}\right) = \underline{\Lambda}^2 \sum_{p,l,m} Y_{lm}\left[(1-\tau)\partial_\tau + \rho\partial_\rho\right]$$

$$\left(a_{p,l,m}(\tau)\rho^p\right) = \underline{\Lambda}^2 \sum_{p,l,m} Y_{lm}\left[(1-\tau)\dot{a}_{p,l,m}\rho^p + \rho a_{p,l,m} - p\rho^{p-1}\right] =$$

$$= \underline{\Lambda}^2 \sum_{p,l,m} Y_{lm}\rho^p\left[(1-\tau)\dot{a}_{p,l,m} + p a_{p,l,m}\right], \text{ where } (1-\tau)\dot{a} + pa = \underline{A}^{\circ}_{p,l,m}(\tau)$$

$\underline{l=0:}$

$$I^-_{om} = \lim_{\substack{\rho\to 0\\ \tau\to -1}} \langle \hat{\underline{L}}\phi, Y_{00}\rangle = \lim_{\substack{\rho\to 0\\ \tau\to -1}} \underline{\Lambda}^2 \sum_{p=0}^{\infty}\sum_{l=0}^{p}\sum_{-m}^{m} \frac{1}{p!}\underline{A}^{\circ}(\tau)\rho^p \delta_{l0}\delta_{m0} =$$

$$= \lim_{\substack{\rho\to 0\\ \tau\to -1}} \underline{\Lambda}^2 \sum_{p=0}^{\infty} \frac{1}{p!}(1-\tau)^{-2}\rho^{p-1}\underline{A}^{\circ}(\tau) = \lim_{\substack{\rho\to 0\\ \tau\to -1}} \sum_{p=0}^{\infty}\frac{1}{p!}2^{-2}\rho^{p-1}\underline{A}^{\circ}_{p,0,0}(-1) =$$

$$= \lim_{\rho\to 0}\left\{\frac{2^{-2}}{0!}\rho^{-1}\underline{A}^{\circ}_{0,0,0}(-1) + \frac{2^{-2}}{1!}\underline{A}^{\circ}_{1,0,0}(-1)\right\}$$

Evaluating $\underline{A}^{\circ}_{p,l,m}$:

$$\underline{A}^{\circ}_{p,l,m}(\tau) = (1-\tau)\dot{a}_{p,l,m}(\tau) + p\,a_{p,l,m}(\tau) \iff \underline{A}^{\circ}_{p,l,m}(-1) = 2\dot{a}_{p,l,m}(-1) + 0$$

$$\Rightarrow \underline{A}^{\circ}_{0,0,0}(\tau) = 2\dot{a}_{0,0,0}(\tau)$$

From Lemma 1, we can get $\mathring{a}_{0,p,0}(\tau)$:

(ii) => For $p \geq 0$ and $l = p$:

$$a_{p,p,m}(\tau) = \left(\frac{1-\tau}{2}\right)^p \left(\frac{1+\tau}{2}\right)^p \left[C_{p,p,m} + D_{p,p,m} \int_0^\tau \frac{ds}{(1-s^2)^{p+1}} \right]$$

$$\Rightarrow a_{0,0,0}(\tau) = \left(\frac{1-\tau}{2}\right)^0 \left(\frac{1+\tau}{2}\right)^0 \left[C_{0,0,m} + D_{0,0,m} \int_0^\tau \frac{ds}{(1-s^2)} \right] =$$

$$= C_{0,0,0} + \frac{1}{2} D_{0,0,0} \left[\log(1+\tau) - \log(1-\tau) \right]$$

- $$\mathring{a}_{0,0,0}(\tau) = \partial_\tau a_{0,0,0}(\tau) = -\frac{D_{0,0,0}}{\tau^2 - 1}$$

$$\rightarrow \underset{\sim}{A}^{\circ}_{0,0,0}(\tau) = -\frac{D_{0,0,0}}{\tau^2 - 1}$$

$$\rightarrow \underline{A}^{10}_{1,0,0}(\tau) = 2\,\mathring{a}_{1,0,0}(\tau) + a_{1,0,0}(\tau)$$

↳ From Lemma 1:

(i) For $p \geq 1$ and $0 \leq l \leq p-1$

$$a(\tau)_{p,l,m} = A_{p,l,m} \left(\frac{1-\tau}{2}\right)^p P_l^{(p,-p)}(\tau) + B_{p,l,m} \left(\frac{1+\tau}{2}\right)^p P_l^{(-p,p)}(\tau)$$

$$\Rightarrow a_{1,0,0}(\tau) = A_{1,0,0} \left(\frac{1-\tau}{2}\right)^1 P_0^{(1,-1)}(\tau) + B_{1,0,0} \left(\frac{1+\tau}{2}\right)^1 P_0^{(-1,1)}(\tau) =$$

$$= A_{1,0,0} \left(\frac{1-\tau}{2}\right) + B_{1,0,0} \left(\frac{1+\tau}{2}\right)$$

$$\dot{a}_{1,0,0}(\tau) = \partial_\tau a_{1,0,0}(\tau) = -\frac{A_{1,0,0}}{2} + \frac{B_{1,0,0}}{2}$$

$$\underline{A}^{\circ}_{1,0,0}(\tau) = 2\dot{a}_{1,0,0}(\tau) + a_{1,0,0}(\tau) = -A_{1,0,0} + B_{1,0,0} + A_{1,0,0}\left(\frac{1-(-1)}{2}\right)$$
$$+ B_{1,0,0}\left(\frac{1+(-1)}{2}\right) = -A_{1,0,0} + B_{1,0,0} + A_{1,0,0} + 0 =$$
$$= B_{1,0,0}$$

Imposing the regularity condition: $D_{0,0,0} = 0 \Rightarrow$

$$I^-_{0m} = \frac{\underline{A}^{\circ}_{1,0,0}}{4} = \frac{B_{1,0,0}}{4} \qquad (4)$$

$\underline{l=1}$

$$I^-_{1m} = \lim_{\substack{\rho\to 0\\ \tau\to -1}} \langle \hat{\underline{L}}^2\phi, Y_{1m}\rangle$$

$$\hat{\underline{L}}^2\phi = \underline{\Lambda}^2 \underline{e}(\hat{\underline{L}}\phi) = \underline{\Lambda}^2\underline{e}\left(\underline{\Lambda}^2\sum_{p,l,m} Y_{lm}\rho^p \underline{A}^{\circ}(\tau)\right) = \underline{\Lambda}^2\left[\underline{\Lambda}^2\sum_{p,l,m} Y_{lm}\underline{e}(\rho^p \underline{A}\right.$$
$$\left.+\sum_{p,l,m} Y_{lm}\rho^p\underline{A}^{\circ}_{p,l,m}(\tau)\,\underline{e}(\underline{\Lambda}^2)\right] \Rightarrow \underline{e}(\underline{\Lambda}^2) = \rho^{-1}(1-\tau)^{-2} = \underline{\Lambda}^2$$

$$\Leftrightarrow \hat{\underline{L}}^2\phi = \underline{\Lambda}^2\left[\underline{\Lambda}^2\sum_{p,l,m} Y_{lm}\,\underline{e}(\rho^p\underline{A}^{\circ}_{p,l,m}(\tau)) + \sum_{p,l,m} Y_{lm}\rho^p(\tau)\underline{\Lambda}^2\right] =$$
$$= \underline{\Lambda}^4\sum_{p,l,m}\left\{\left[(1-\tau)\rho^p\dot{\underline{A}}^{\circ}_{p,l,m} + \rho^p\underline{A}^{\circ}_{p,l,m}\,p\,\rho^{p-1}\right] + \rho^p\underline{A}^{\circ}_{p,l,m}\right\} =$$
$$= \underline{\Lambda}^4\sum_{p,l,m} Y_{lm}\rho^p\left[(1-\tau)\dot{\underline{A}}^{\circ}_{p,l,m} + (p+1)\underline{A}^{\circ}_{p,l,m}\right],$$

Where $(1-\tau)\dot{\underline{A}}^{0}_{p,l,m} + (p+1)\underline{A}^{0}_{p,l,m} = \underline{A}^{1}_{p,l,m}(\tau)$ (5)

- $\hat{\underline{L}}^2\phi = \underline{\Lambda}^4 \sum_{p,l,m} Y_{lm}\,\rho^p\,\underline{A}^1_{p,l,m}(\tau) \Rightarrow \bar{I}_{1m} = \lim_{\substack{\rho\to 0\\ \tau\to -1}} \langle \hat{\underline{L}}^2\phi, Y_{1m}\rangle =$

$$= \lim_{\substack{\rho\to 0\\ \tau\to -1}} \rho^{-2}(1-\tau)^{-4}\sum_{p=0}^{\infty}\frac{1}{p!}\underline{A}^1_{p,l,m}(\tau)\,\rho^p = \lim_{\substack{\rho\to 0\\ \tau\to -1}}(1-\tau)^{-4}\sum_{p=0}^{\infty}\frac{1}{p!}\rho^{p-2}\underline{A}^1_{p,l,m}(\tau)$$

$$= \lim_{\substack{\rho\to 0\\ \tau\to -1}}(1+\tau)^{-4}\left\{\frac{1}{0!}\rho^{-2}\underline{A}^1_{0,1,m}(\tau) + \frac{1}{1!}\rho^{-1}\underline{A}^1_{1,1,m}(\tau) + \frac{1}{2!}\rho^{0}\underline{A}^1_{2,1,m}(\tau) + \right.$$

$$\left. + \sum_{p=3}^{\infty}\frac{1}{p!}\rho^{p-3}\underline{A}^1_{p,1,m}(\tau)\right\} = \lim_{\rho\to 0} 2^{-4}\left\{\rho^{-2}\underline{A}^1_{0,1,m}(\tau) + \rho^{-1}\underline{A}^1_{1,1,m}(\tau)\right\} +$$

$$+ \lim_{\rho\to 0}\frac{2^{-4}}{2!}\underline{A}^1_{2,1,m}(\tau)$$

Evaluating $\underline{A}^1_{p,l,m}$:

$$\underline{A}^1_{p,l,m}(\tau) = (1-\tau)\dot{\underline{A}}^0_{p,l,m}(\tau) + (p+1)\underline{A}^0_{p,l,m}(\tau) = (1-\tau)\partial_\tau\left[(1-\tau)\dot{a}_{p,l,m} + p\,a_{p,l,m}\right] + (p+1)\left[(1-\tau)\dot{a}_{p,l,m} + p\,a_{p,l,m}\right] = (1-\tau)\left[(1-\tau)\ddot{a}_{p,l,m} - \dot{a}_{p,l,m} + p\,\dot{a}_{p,l,m}\right] + (p+1)\left[(1-\tau)\dot{a}_{p,l,m} + p\,a_{p,l,m}\right] \Leftrightarrow$$

$$\Leftrightarrow \underline{A}^1_{p,l,m}(\tau) = (1-\tau)^2\ddot{a}_{p,l,m} - \dot{a}_{p,l,m} + \tau\dot{a}_{p,l,m} + p\dot{a}_{p,l,m} - \tau p\,\dot{a}_{p,l,m} +$$

$$\cancel{+(P-1)\left[\dot{a}_{P,l,m} - \tau\dot{a}_{P,l,m} + P a_{P,l,m}\right]} \iff + (1-\tau)\dot{a} + Pa$$

$$\iff \underline{A}^{1}_{P,l,m}(\tau) = \cancel{(1-\tau)^2\ddot{a}_{P,l,m} - \dot{a}_{P,l,m} + \tau\dot{a}_{P,l,m} + P\dot{a}_{P,l,m} - \tau P\dot{a}_{P,l,m}}$$

$$\cancel{+ P\dot{a}_{P,l,m} - \tau P\dot{a}_{P,l,m}} + P(P-1)a_{P,l,m} \cancel{- \dot{a}_{P,l,m} + \tau\dot{a}_{P,l,m}} \iff$$

$$\cancel{\iff \underline{A}^{1}_{P,l,m}(\tau) = (1-\tau)^2\ddot{a}_{P,l,m} + 2P(1-\tau)\dot{a}_{P,l,m} + P(P-1)a_{P,l,m}} \qquad (6)$$

$$Q^{1}_{P,l,m}(\tau) = (1-\tau)^2\ddot{a} - (1-\tau)\dot{a} + 2P(1-\tau)\dot{a} + P^2 a + (1-\tau)\dot{a} + Pa$$

$$\underline{A}^{1}_{0,1,m}(\tau) = 0 = (1-\tau)^2\ddot{a} + 2P(1-\tau)\dot{a} + P(P+1)a$$

$$\underline{A}^{1}_{1,1,m}(\tau) = (1-\tau)^2\ddot{a}_{1,1,m}(\tau) + 2P(1-\tau)\dot{a}_{1,1,m} + P(P-1)a_{1,1,m} \iff$$

$$\iff \underline{A}^{1}_{1,1,m}(\tau) = (1-\tau)^2\ddot{a}_{1,1,m}(\tau) + 2(1-\tau)\dot{a}_{1,1,m} + 0$$

$$\cancel{a_{1,1,m}(\tau) = A_{1,1,m}\left(\frac{1-\tau}{2}\right)P_1^{(-1,1)}(\tau) + B_{1,1,m}\left(\frac{1+\tau}{2}\right)P_1^{(-1,1)}}$$

$$a_{1,1,m}(\tau) = \frac{1}{4}(1-\tau)(1+\tau)\left[C_{1,1,m} + \frac{1}{4}D_{1,1,m}\left(\log(1+\tau) - \log(1-\tau) + 2\tau(1-\tau^2)\right)\right]$$

$$\dot{a}_{1,1,m}(\tau) = -\frac{1}{8}\Big[D_{11m}\,\tau\log(1+\tau) - 5D_{11m}\tau^4 + 6D_{11m}\tau^2 + (4C_{11m} -$$

$$- D_{11m}\log(1-\tau))\tau - 2D_{11m}\Big]$$

$$\ddot{a}_{1,1,m}(\tau) = -\frac{1}{8}\Big[D_{11m}(1+\tau) + \frac{D_{11m}\tau}{1+\tau} - 20D_{11m}\tau^3 + \frac{D_{11m}\tau}{1-\tau} + 12D_{11m}\tau -$$

$$- D_{11m}\log(1-\tau) + 4C_{11m}\Big]$$

$$\underline{A}^{1}_{1,1,m}(\tau) = -\frac{(1-\tau)^2}{8}\Big[D_{11m}(1+\tau) + \frac{D_{11m}\tau}{(1+\tau)} - 20D_{11m}\tau^3 + \frac{D_{11m}\tau}{(1-\tau)} + 12D_{11m}\tau -$$

$$- D_{11m} \log(1-\tau) + 4C_{11m}\Big] \mp \frac{2}{8}(1-\tau)\Big[D_{11m}\tau \log(1+\tau) - 5D_{11m}\tau^4 + 6D_{11m}$$

$$+ \left(4C_{11m} - D_{11m}\log(1-\tau)\right)\tau - 2D_{11m}\Big] \Leftrightarrow$$

$$\Leftrightarrow \underline{A}^1_{1,1,m}(\tau) = -\frac{D_{11m}}{2(1-\tau)}$$

$$A^1_{2,1,m}(\tau) = -6B_{211}$$

Imposing the regularity condition: $D_{11m} = 0 \Rightarrow I^-_{1m} = -\frac{2^{-4}}{2!}6B_{2,1,1}$ (7)

$\underline{l=2}$

$$I_{2m} = \lim_{\substack{\rho\to 0\\ \tau\to -1}} \langle \underline{\hat{L}}^3 \phi, Y_{2m} \rangle$$

$$\underline{\hat{L}}^3\phi = \underline{\Lambda}^2 \underline{e}(\underline{\hat{L}}^2\phi) = \underline{\Lambda}^2\underline{e}\Big(\underline{\Lambda}^4 \sum_{p,l,m} Y_{lm}\rho^p \underline{A}^1_{p,l,m}\Big) =$$

$$= \underline{\Lambda}^2\Big[\underline{\Lambda}^4 \sum_{p,l,m} Y_{l,m}\,\underline{e}(\rho^p \underline{A}^1_{p,l,m}) + \sum_{p,l,m} Y_{lm}\rho^p \underline{A}^1_{p,l,m}\,\underline{e}(\underline{\Lambda}^4)\Big]$$

$$\Rightarrow \underline{e}(\underline{\Lambda}^{2n}) = n\underline{\Lambda}^{2n}$$

$$\underline{\hat{L}}^3\phi = \underline{\Lambda}^2\Big[\underline{\Lambda}^4\sum_{p,l,m} Y_{lm}\,\underline{e}(\rho^p\underline{A}^1_{p,l,m}) + 2\underline{\Lambda}^4\sum_{p,l,m} Y_{lm}\rho^p\underline{A}^1_{p,l,m}\Big] =$$

$$= \underline{\Lambda}^6 \sum_{p,l,m} Y_{lm}\Big\{\underline{e}(\rho^p\underline{A}^1_{p,l,m}) + 2\rho^p\underline{A}^1_{plm}\Big\} =$$

$$= \underline{\Lambda}^6 \sum_{p,l,m} \left\{ Y_{lm} \left[(1-\tau)\, \dot{\underline{A}}^1_{p,l,m}\, \rho^p + \rho\, \underline{A}^1_{p,l,m}\, p\, \rho^{p-1} \right] + 2\rho^p \underline{A}^1_{p,l,m} \right\} =$$

$$= \underline{\Lambda}^6 \sum_{p,l,m} \left\{ Y_{lm} \left[(1-\tau)\, \dot{\underline{A}}^1_{p,l,m}\, \rho^p + p\rho^{p-1} \underline{A}^1_{p,l,m} \right] + 2\rho^p \underline{A}^1_{p,l,m} \right\} =$$

$$= \underline{\Lambda}^6 \sum_{p,l,m} Y_{lm}\, \rho^p \left[(1-\tau)\, \dot{\underline{A}}^1_{p,l,m} + (p+2)\, \underline{A}^1_{p,l,m} \right]$$

$$\Rightarrow \hat{\underline{L}}^3 \phi = \underline{\Lambda}^6 \sum_{p,l,m} Y_{lm}\, \rho^p \underline{A}^2_{p,l,m}$$

$$I_{2m} = \lim_{\substack{\rho \to 0 \\ \tau \to -1}} \left\langle \hat{\underline{L}}^3 \phi, Y_{2m} \right\rangle = \lim_{\substack{\rho \to 0 \\ \tau \to -1}} \left\langle \underline{\Lambda}^6 \sum_{p,l',m'} Y_{l'm'}\, \rho^p \underline{A}^2_{p,l',m'}, Y_{2m} \right\rangle =$$

$$= \lim_{\substack{\rho \to 0 \\ \tau \to -1}} \underline{\Lambda}^6 \sum_{p=0}^{\infty} \rho^p \underline{A}^2_{p,2,m}(\tau) = \lim_{\substack{\rho \to 0 \\ \tau \to -1}} \rho^{-3} (1-\tau)^{-6} \sum_{p=0}^{\infty} \rho^p \underline{A}^2_{p,2,m} =$$

$$= \lim_{\substack{\rho \to 0 \\ \tau \to -1}} (1-\tau)^{-6} \sum_{p=0}^{\infty} \rho^{p-3} \underline{A}^2_{p,2,m}(\tau) =$$

$$= \lim_{\substack{\rho \to 0 \\ \tau \to -1}} (1-\tau)^{-6} \left\{ \underbrace{\sum_{p=0}^{2} \rho^{p-3} \underline{A}^2_{p,l,m}}_{\text{singular piece}} + \underbrace{\sum_{p=3}^{3} \rho^{0} \underline{A}^2_{p,2,m}}_{\text{regular piece}} + \underbrace{\sum_{p=4}^{\infty} \rho^{p-3} \underline{A}^2_{p,2,m}}_{\substack{\text{Vanishing piece} \\ \text{at } \rho = 0}} \right\} =$$

$$= \lim_{\substack{\rho \to 0 \\ \tau \to -1}} (1-\tau)^{-6} \sum_{p=0}^{2} \rho^{p-3} \underline{A}^2_{p,l,m} + \lim_{\substack{\rho \to 0 \\ \tau \to -1}} \underline{A}^2_{p,l,m} + 0$$

$$\Leftrightarrow I^{-}_{2m} = \lim_{\substack{\rho\to 0\\ \tau\to -1}} (1-\tau)^{-6} \Big\{ \bar{p}^{3}\, Q^{2}_{0;2,m} + \bar{p}^{2}\, Q^{2}_{1,2,m} + \bar{p}^{1}\, Q^{2}_{2;2,m} \Big\} +$$

$$+ \bar{\tau}^{6}\, Q^{2}_{3;2,m} = \lim_{\substack{\rho\to 0\\ \tau\to -1}} (1-\tau)^{-6} \Big\{ \bar{p}^{-3}\, Q^{2}_{0;2,m} + \bar{p}^{-2}\, Q^{2}_{1,2,m} + \bar{p}^{-1}\, Q^{2}_{2;2,m} \Big\} +$$

$$+ \bar{\tau}^{6}\, Q^{2}_{3;2,m}$$

Evaluating $Q^{2}_{3;2,m}$:

$$Q^{2}_{p;l,m} = (1-\tau)\,\dot{Q}^{1}_{p;l,m} + (p+2)\,Q^{1}_{p;l,m} = (1-\tau)\,\partial_\tau \Big[(1-\tau)^2 \ddot{a} + 2p(1-\tau)\dot{a}$$

$$+ p(p+1)a \Big] + (p+2)\Big[(1-\tau)^2 \ddot{a} + 2p(1-\tau)\dot{a} + p(p+1)a \Big] =$$

$$= (1-\tau)\Big[(1-\tau)^2 \dddot{a} - 2(1-\tau)\ddot{a} + 2p(1-\tau)\ddot{a} - 2p\,\dot{a} + p(p+1)\dot{a} \Big] +$$

$$+ (p+2)\Big[(1-\tau)^2 \ddot{a} + 2p(1-\tau)\dot{a} + p(p+1)a \Big] =$$

$$\cancel{= (1-\tau)^3 \dddot{a} - 2(1-\tau)^2 \ddot{a} + 2p(1-\tau)^2 \ddot{a} - 2p(1-\tau)\dot{a} + p(p+1)(1-\tau)\dot{a} +}$$

$$\cancel{+ p(1-\tau)^2 \ddot{a} + 2p^2(1-\tau)\dot{a} + p^2(p+1)a + 2(1-\tau)^2 \ddot{a} + 4p(1-\tau)\dot{a} + 2p(p-1}$$

$$\cancel{= (1-\tau)^3 \dddot{a}}$$

$$= (1-\tau)^3 \dddot{a} - 2(1-\tau)^2 \ddot{a} + 2p(1-\tau)^2 \ddot{a} - 2p(1-\tau)\dot{a} + (1-\tau)p(p+1)\dot{a} +$$

$$+ (p+2)(1-\tau)^2 \ddot{a} + 2p(p+2)(1-\tau)\dot{a} + p(p+1)(p+2)a =$$

$$= (1-\tau)^3 \dddot{a} + (1-\tau)^2 \ddot{a}\Big[2p - 2 + p + 2 \Big] + (1-\tau)\dot{a}\Big[-2p + p(p+1) + 2p(p+2) \Big]$$

$$+ p(p+1)(p+2)a =$$

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$$= (1-\tau)^3 \dddot{a} + 3p(1-\tau)^2 \ddot{a} + (1-\tau)\dot{a}\left[-2p + p^2 + p + 2p^2 + 4p\right] + p(p+1)(p+2)\, a =$$

$$= (1-\tau)^3 \dddot{a} + 3p(1-\tau)^2 \ddot{a} + (1-\tau)\dot{a}\,(3p^2 + 3p) + p(p+1)(p+2)\, a =$$

$$= (1-\tau)^3 \dddot{a} + 3p(1-\tau)^2 \ddot{a} + 3p(p+1)(1-\tau)\dot{a} + p(p+1)(p+2)\, a$$

$$\Rightarrow Q^2_{p;l,m}(\tau) = (1-\tau)^3 a^{(3)} + 3p(1-\tau)^2 a^{(2)} + 3p(p+1)(1-\tau)a^{(1)} + p(p+1)(p+2)\, a^{(0)} \quad (8$$

- $Q^2_{0;2,m}(\tau) = 0$; $Q^2_{1;2,m}(\tau) = 0$; $Q^2_{2;2,m}(\tau) = \dfrac{3 D_{222}}{4(1+\tau)}$; $Q^2_{3;2,m}(\tau) = 60\, B_{32}$

$\Rightarrow D_{222} = 0$

$$I^-_{2m} = 2^{-6} B^3_{322} \cdot 60 \qquad (9)$$

<u>Computing $l = 3$</u>

$$I^-_{3m} = \lim_{\substack{\rho \to 0 \\ \tau \to -1}} \langle \underline{L}^{14} \phi, Y_{3m} \rangle$$

$$\underline{L}^{14} \phi = \underline{\Lambda}^2 \underline{e}(\underline{L}^{13}\phi) = \underline{\Lambda}^2 \underline{e}\Big(\underline{\Lambda}^6 \sum_{p;l,m} Y_{lm}\, \rho^p\, Q^2_{p;l,m}\Big) =$$

$$= \underline{\Lambda}^2 \Big[\underline{\Lambda}^6 \sum_{p;l,m} Y_{lm}\, \underline{e}(\rho^p Q^2_{p;l,m}) + \sum_{p;l,m} Y_{lm}\, \rho^p Q^2_{p;l,m}\, \underline{e}(\underline{\Lambda}^6)\Big] =$$

$$= \underline{\Lambda}^2 \Big[\underline{\Lambda}^6 \sum_{p;l,m} Y_{lm}\, \underline{e}(\rho^p Q^2_{p;l,m}) + 3\underline{\Lambda}^6 \sum_{p;l,m} Y_{lm}\, \rho^p Q^2_{p;l,m}\Big] =$$

$$= \underline{\Lambda}^8 \sum_{p;l,m} Y_{lm} \Big\{\underline{e}(\rho^p Q^2_{p;l,m}) + 3\rho^p Q^2_{p;l,m}\Big\} = \underline{\Lambda}^8 \sum_{p;l,m} \Big\{ Y_{lm}\Big[(1-\tau)\dot{Q}^2 \rho^p$$

$$+ \rho Q^2 P \rho^{P-1} \Big] + 3\rho^P Q^2 \Big\} = \Lambda^8 \sum_{P,l,m} \Big\{ Y_{lm} \Big[(1-\tau)\dot{Q}^2 \rho^P + P\rho^{P} Q^2 \Big] +$$

$$+ 3\rho^P Q^2 \Big\} = \Lambda^8 \sum_{P,l,m} Y_{lm}\, \rho^P \Big[(1-\tau)\dot{Q}^2 + (P+3) Q^2 \Big]$$

$$(1-\tau)\dot{Q}^2 + (P+3) Q^2 = Q^3_{P;l,m}(\tau) \qquad (10)$$

$$I^{-}_{3m} = \lim_{\substack{\rho \to 0 \\ \tau \to -1}} (1-\tau)^{-8} \Big\{ \underbrace{\rho^{-4} Q^3_{0;3,m} + \rho^{-3} Q^3_{1;3,m} + \rho^{-2} Q^3_{2;3,m} + \rho^{-1} Q^3_{3;3,m}}_{\text{singular pieces}} \Big\} +$$

$$+ \underbrace{2^{-8} Q^3_{4;3,m}(-1)}_{\text{regular piece}}$$

<u>Evaluating $Q^3_{P;l,m}(\tau)$:</u>

$$Q^3_{P;l,m}(\tau) = (1-\tau)\ddot{Q}^2_{P;l,m} + (P+3) Q^2_{P;l,m}$$

$$\ddot{Q}^2_{P;l,m}(\tau) = -3(1-\tau)^2 a^{(3)} + (1-\tau)^3 a^{(4)} + 3P(1-\tau)^2 a^{(3)} - 6P(1-\tau) a^{(2)} +$$

$$+ 3P(P+1)(1-\tau) a^{(2)} - 3P(P+1) a^{(1)} =$$

$$= (1-\tau)^3 a^{(4)} - 3(1-\tau)^2 a^{(3)} + 3P(1-\tau)^2 a^{(3)} + (1-\tau) a^{(2)} \Big[-6P + 3P^2 + 3P \Big] -$$

$$- 3P(P+1) a^{(1)} + P(P+1)(P+2) a^{(1)} =$$

$$= (1-\tau)^3 a^{(4)} + (1-\tau)^2 a^{(3)} \Big[3P - 3 \Big] + 3P(P-1)(1-\tau) a^{(2)} -$$

$$- a^{(1)} \Big[-3P^2 - 3P + (P^2+P)(P+2) \Big] = \Big[-3P^2 - 3P + P^3 + 2P^2 + P^2 + 2P \Big]$$

$$= \Big[-\cancel{3P^2} + \cancel{3P^2} + P^3 - 3P + 2P \Big] =$$

$P^3 - P + \ldots$

$(P^2+P)(P-1) = P^3 + P^2 \ldots$

$P(P+1)(P-1)$

$P(P^2 + 2 - 3)$

$P(P^2-1)$

$(P^3 + 2P - 3P)$

$P^3 - P$

$$= (1-\tau)^3 a^{(4)} + 3(P-1)(1-\tau)^2 a^{(3)} + 3P(P-1)(1-\tau) a^{(2)} - a^{(1)} \Big[-\cancel{3P^2} - 3P + P^3 + 2\ldots$$

$$+ \cancel{P^2} + 2P \Big] =$$

$P^3 - P = P(P^2-1) = P(P+1)(1-P) = (P^2+P)(1-P) = P - P^3 \ldots$

$$= (1-\tau)^3 a^{(4)} + 3(P-1)(1-\tau)^2 a^{(3)} + 3P(P-1)(1-\tau) a^{(2)} - P(P+1)(P-1) a^{(1)}$$

$$Q^3_{P,l,m}(\tau) = (1-\tau)\dot{Q}^2 + (P+3)Q^2 = (1-\tau)^4 a^{(4)} + 3(P-1)(1-\tau)^3 a^{(3)} + 3P(P-1)(1-\tau$$

$$- P(P+1)(P-1)(1-\tau) a^{(1)} + (P+3)\Big[(1-\tau)^3 a^{(3)} + 3P(1-\tau)^2 a^{(2)} + 3P(P+1)(1-\tau) a^{(1)} +$$

$$+ P(P+1)(P+2) a^{(0)}\Big] =$$

$$= (1-\tau)^4 a^{(4)} + (1-\tau)^3 a^{(3)}\Big[3(P-1)+(P+3)\Big] + (1-\tau)^2 a^{(2)}\Big[\overset{3P(P-1)+(P+3)3P}{\cancel{\text{[illegible]}}}\Big] +$$

$$+ (1-\tau) a^{(1)}\Big[+3P(P+1) \mp P(P+1)(P-1)\Big] + P(P+1)(P+2)(P+3) a^{(0)} =$$

$$= (1-\tau)^4 a^{(4)} + 4P(1-\tau)^3 a^{(3)} + 6P(P+1)(1-\tau)^2 a^{(2)} + 4P(P+1)(P+2)(1-\tau) a^{(1)} +$$

$$+ P(P+1)(P+2)(P+3) a^{(0)}$$

$$\Rightarrow Q^3_{P,l,m}(\tau) = (1-\tau)^4 a^{(4)} + 4P(1-\tau)^3 a^{(3)} + 6P(P+1)(1-\tau)^2 a^{(2)} + 4P(P+1)(P+2)(1-\tau) a$$

$$+ P(P+1)(P+2)(P+3) a^{(0)} \qquad (12)$$

General Formula for $\mathcal{J}$ NP constants:

The previous discussion suggests that, in principle, it should be possible to obtain a general formula for $Q_{P,l,m}(\tau)$.

We start with,

$$Q^n = K_0\binom{n+1}{0}(1-\tau)^{(n)} a^{(n)} + K_1\binom{n+1}{1} P(1-\tau)^{(n-1)} a^{(n-1)} + K_2\binom{n+1}{2} P(P+1)(1-\tau)^{(n-2)} a^{(n-2)}$$

$$+ \dots + \binom{n+1}{q} K_q a$$

where we define k_q to be,

$$k_q := \frac{p(p+q-1)!}{p!}$$

and $\binom{n+1}{q}$ to be the binomial coefficients.

Therefore,

$$Q^{n}_{0;p,l,m} = \sum_{q=0}^{n+1} \frac{(p+q-1)!}{(p-1)!} \binom{n+1}{q} (1-\tau)^{(n-q+1)} \, a^{(n-q+1)} \qquad (13)$$

Substituting $n = 0,1,2,3$ we arrive at the same results:

- $$Q^{0}_{0;p,l,m}(\tau) = \sum_{q}^{1} \binom{1}{q} \frac{(p+q-1)!}{(p-1)!} (1-\tau)^{(1-q)} \, a^{(1-q)} = (1-\tau) a^{(1)} + p a^{(0)}$$

- $$Q^{1}_{0;p,l,m}(\tau) = \sum_{q}^{2} \binom{2}{q} \frac{(p+q-1)!}{(p-1)!} (1-\tau)^{q} \, a^{(q)} = (1-\tau)^{2} a^{(2)} + 2p(1-\tau) a^{(1)} + p(p+1)a$$

- $$Q^{2}_{0;p,l,m}(\tau) = \sum_{q}^{3} \binom{3}{q} \frac{(p+q-1)!}{(p-1)!} (1-\tau)^{(q-1)} \, a^{(q-1)} = (1-\tau)^{3} a^{(3)} + 3p(1-\tau)^{2} a^{(2)}$$

$$+ 3p(p+1)(1-\tau) a^{(1)} + p(p+1)(p+2) a^{(0)}$$

- $$Q^{3}_{0;p,l,m}(\tau) = \sum_{q}^{4} \binom{4}{q} \frac{(p+q-1)!}{(p-1)!} (1-\tau)^{(3-q+1)} \, a^{(3-q+1)} = (1-\tau)^{4} a^{(4)} + 4p(1-\tau)^{3} a$$

$$+ 6p(p+1)(1-\tau)^{2} a^{(2)} + 4p(p+1)(p+2)(1-\tau) a^{(1)} + p(p+1)(p+2)(p+3) a^{(0)}$$

Lemma

$$\hat{\underline{L}}^{(n)}\phi = \underline{\Lambda}^{2n}\sum_{p=0}^{\infty}\sum_{l=0}^{p}\sum_{m=l}^{m=-l}\frac{1}{p!}\rho^{p}Q^{n-1}_{0;p,l,m}(\tau)\,Y_{0,l,m} \tag{14}$$

$$Q^{n-1}_{0;p,l,m}(\tau) = \sum_{q}^{n} k_q\binom{n}{q}(1-\tau)^{(n-q)}\,a^{(n-q)} \tag{15}$$

Basis of induction : cases $n=0$, $n=1$, $n=2$, $n=3$

Induction hypothesis: eq (14) and eq (15)

Induction step

$$\hat{\underline{L}}^{(n+1)}(\phi) = \hat{\underline{L}}(\hat{\underline{L}}^{n}\phi) = \hat{\underline{L}}\left(\underline{\Lambda}^{2n}\sum_{p=0}^{\infty}\sum_{l=0}^{p}\sum_{m=l}^{m=-l}\frac{1}{p!}\rho^{p}Y_{lm}\,Q^{n-1}_{p,l,m}(\tau)\right) =$$

$$= \underline{\Lambda}^{2}\underline{e}\left(\underline{\Lambda}^{2n}\sum_{p,l,m}\rho^{p}Y_{lm}\,Q^{n-1}_{p,l,m}\right) = \underline{\Lambda}^{2}\left\{\underline{\Lambda}^{n}\sum_{p,l,m}\underline{e}\left(\rho^{p}Y_{lm}\,Q^{n-1}\right) + \sum_{p,l,m}\rho^{p}Q^{n-1}_{p,l,m}\,\underline{e}\left(\underline{\Lambda}^{2n}\right)\right\} = \underline{\Lambda}^{2}\left\{\underline{\Lambda}^{2n}\sum_{p,l,m}\underline{e}\left(\rho^{p}Y_{lm}\,Q^{n-1}\right) + n\sum_{p,l,m}\rho^{p}Y_{lm}\,Q^{n-1}_{p,l,m}\right.$$

$$= \underline{\Lambda}^{2(n+1)}\sum_{p,l,m}Y_{lm}\left[\underline{e}\left(\rho^{p}Q^{n-1}_{p,l,m}\right) + n\rho^{p}Q^{n-1}_{p,l,m}\right]$$

- $$\underline{e}\left(\rho^{p}Q^{n-1}_{p,l,m}\right) = \left[(1-\tau)\partial_\tau + \rho\partial_\rho\right]\left(\rho^{p}Q^{n-1}_{plm}\right) = (1-\tau)\rho^{p}\dot{Q}^{n-1}_{p,l,m} + \rho p\rho^{p-1}$$

$$= (1-\tau)\rho^{p}\dot{Q}^{n-1}_{p,l,m} + p\rho^{p}Q^{n-1}_{p,l,m}$$

$$\Rightarrow \underline{\hat{L}}^{(n+1)}(\phi) = \underline{\Lambda}^{z(n+1)} \sum_{p,l,m} Y_{lm} \left[(1-\tau)\rho^{p} \dot{Q}^{n-1} + p\rho^{p} Q^{n-1}_{plm} + n\rho^{p} \varphi^{n-1}_{plm} \right] =$$

$$= \underline{\Lambda}^{z(n+1)} \sum_{plm} Y_{lm}\, \rho^{p} \left[(1-\tau)\, \dot{\varphi}^{n-1}_{plm} + (p+n)\, \varphi^{n-1}_{p,l,m} \right],$$

We have proven that

$$\underline{\hat{L}}^{(n+1)}(\phi) = \underline{\Lambda}^{z(n+1)} \sum_{p,l,m} \rho^{p}\, T^{n}_{p,l,m}\, Y_{lm},$$

where $T^{n}_{plm} := (1-\tau)\,\dot{\varphi}^{n-1}_{p,l,m} + (p+n)\,\varphi^{n-1}_{p,l,m}$

with (15) we can compute the following:

$$\dot{\varphi}^{n-1}_{plm} = \partial_{\tau}\left[\sum_{q=0}^{n} K_q \binom{n}{q} (1-\tau)^{(n-q)} a^{(n-q)} \right] = \sum_{q=0}^{n} K_q \binom{n}{q} \left[(1-\tau)^{(n-q)} a^{(n-q+1)} \right.$$

$$\cancel{=} (n-q)(1-\tau)^{(n-q)} a^{(n-q)} = (1-\tau)\,\dot{\varphi}^{n-1}_{plm} = \sum_{q=0}^{n} K_q \binom{n}{q} \left[(1-\tau)^{(n-q+1)} a^{(n-q+1)} - \right.$$

$$\left. - (n-q)(1-\tau)^{(n-q)} a^{(n-q)} \right]$$

$$+ (p+n)\,\varphi^{n-1}_{plm} = \sum_{q=0}^{n} K_q \binom{n}{q} \left[(p+n)(1-\tau)^{(n-q)} a^{(n-q)} \right]$$

Adding the two, we get T^{n}_{plm}:

$$T^{n}_{plm} = \sum_{q=0}^{n} K_q \binom{n}{q} \left\{ (1-\tau)^{(n-q+1)} a^{(n-q+1)} + \left[-(n-q) + (p+n) \right] (1-\tau)^{(n-q)} a^{(n-q)} \right\}$$

$$= \sum_{q=0}^{n} K_q \binom{n}{q} \left\{ (1-\tau)^{(n-q+1)} a^{(n-q+1)} + (p+q)(1-\tau)^{(n-q)} a^{(n-q)} \right\} \qquad (16)$$

Expanding the first term in the first sum of eq. (16) renders

$$T^n_{p\ell m} = K_o \binom{n}{o} (1-\tau)^{(n+1)} a^{(n+1)} + \sum_{q=1}^{n} K_q \binom{n}{q} (1-\tau)^{(n-q+1)} a^{(n-q+1)} +$$

$$+ \sum_{q=0}^{n} K_q \binom{n}{q} (p+q)(1-\tau)^{(n-q)} a^{(n-q)}$$

Separating the last term in the second sum and rearranging gives

$$T^n_{p\ell m} = K_o (1-\tau)^{n+1} a^{(n+1)} + \sum_{q=1}^{n} K_q \binom{n}{q} (1-\tau)^{(n-q+1)} a^{(n-q+1)} + \left\{ \sum_{q=0}^{n-1} K_q \binom{n}{q} \right.$$

$$\left. (p+q)(1-\tau)^{(n-q)} a^{(n-q)} \right] + K_n (p+n) \binom{n}{n} a \Big\}$$

$$\left(K_{q+1} = (p+q) K_q \ ; \ K_{n+1} = [p+(n+1)] K_n \right)$$

$$\hookrightarrow T^n_{p\ell m} = K_o (1-\tau)^{(n+1)} a^{(n+1)} + \sum_{q=1}^{n} K_q \binom{n}{q} (1-\tau)^{(n-q+1)} a^{(n-q+1)} + \sum_{q=0}^{n-1} \left[K_{q+1} \binom{n}{q} (1-\tau)^{(n-q)} a^{(n-q)} \right] + K_{n+1} a$$

Renaming : $i = q-1 \Rightarrow q = i+1$

$i(q=1) = 0$ & $i(q=n+1) = n$

$$T^n_{p\ell m} = K_o (1-\tau)^{(n+1)} a^{(n+1)} + \sum_{i=0}^{n-1} K_{i+1} \binom{n}{i+1} (1-\tau)^{(n-i)} a^{(n-i)} + \sum_{q=0}^{n-1} K_{q+1} \binom{n}{q} (1-\tau)^{(n-q)}$$

$$a^{(n-q)} + K_{n+1} a$$

Regarding the second sum, we do $i=q$ and arrive at the following

$$T^{n}_{plm} = K_0 (1-\tau)^{(n+1)} a^{(n+1)} + \sum_{i=0}^{n-1} K_{i+1} \left[\binom{n}{i+1} + \binom{n}{i} \right] (1-\tau)^{(n-i)} a^{(n-i)} +$$

$$+ K_{n+1}\, a$$

We the help off recursive identity of the binomial coefficients:

$$T^{n}_{plm} = K_0 (1-\tau)^{(n+1)} a^{(n+1)} + \sum_{i=0}^{n-1} \left[K_{i+1} \binom{n+1}{i+1} (1-\tau)^{(n-i)} a^{(n-i)} \right] + K_{n+1}\, a =$$

$$= K_0 \binom{n+1}{0} (1-\tau)^{(n+1)} a^{(n+1)} + \sum_{i=0}^{n-1} K_{i+1} \binom{n+1}{i+1} (1-\tau)^{(n-i)} a^{(n-i)} + K_{n+1} \binom{n+1}{n+1}$$

$$T^{n}_{plm} = \sum_{q=0}^{n+1} K_q \binom{n+1}{q} (1-\tau)^{(n-q+1)} a^{(n-q+1)} = \varphi^{n+1}_{plm} \qquad \square$$

<u>Proposition:</u>

If the regularity condition $D_{nnm} = 0$ is satisfied and if the NP-Constants are finite then they are determined in terms of initial data via $I_{nm} = B(n)\, T_{1(n+1),n,m}$, (17)

Where $B(n)$ is a numerical coefficient and $T_{(n+1),n,m}$ is determined by the initial data for ϕ on $\tau = 0$.

$$\bar{I}_{nm} = \lim_{\substack{\rho \to 0 \\ \tau \to -1}} \langle \hat{\underline{L}}^{n+1} \phi, Y_{nm} \rangle \quad , \quad \text{Using Lemma}$$

$$= \lim_{\substack{\rho \to 0 \\ \tau \to -1}} \langle \underline{\Lambda}^{2(n+1)} \sum_{p,l,m} \rho^{p} T^{n}_{p,l,m} Y_{lm} , Y_{nm} \rangle =$$

$$= \lim_{\substack{\rho \to 0 \\ \tau \to -1}} \underline{\Lambda}^{2(n+1)} \sum_{p,l,m} \rho^{p} T^{n}_{p,l,m} \langle Y_{lm} , Y_{nm} \rangle =$$

$$= \lim_{\substack{\rho \to 0 \\ \tau \to -1}} \underline{\Lambda}^{2(n+1)} \sum_{p,l,m} \rho^{p} T^{n}_{p,l,m} \delta_{ln} \delta_{mm'} =$$

$$= \lim_{\substack{\rho \to 0 \\ \tau \to -1}} \underline{\Lambda}^{2(n+1)} \sum_{p=0}^{\infty} \sum_{l=0}^{p} \sum_{m=-l}^{l} \frac{1}{p!} \delta_{ln} \delta_{mm'} \rho^{p} T^{n}_{p,l,m'} =$$

$$= \lim_{\substack{\rho \to 0 \\ \tau \to -1}} \underline{\Lambda}^{2(n+1)} \sum_{p=0}^{\infty} \frac{1}{p!} \rho^{p} T^{n}_{p,n,m}(\tau) =$$

$$= \lim_{\substack{\rho \to 0 \\ \tau \to -1}} \left(\rho^{-1} (1-\tau)^{-2} \right)^{2(n+1)} \sum_{p=0}^{\infty} \frac{1}{p!} \rho^{p} T^{n}_{pnm}(\tau) =$$

$$= \lim_{\substack{\rho \to 0 \\ \tau \to -1}} (1-\tau)^{-4(n+1)} \rho^{-2(n+1)} \sum_{p=0}^{\infty} \frac{1}{p!} \rho^{p} T^{n}_{pnm}(\tau) =$$

$$= \lim_{\substack{\rho \to 0 \\ \tau \to -1}} (1-\tau)^{-4(n+1)} \sum_{p=0}^{\infty} \frac{1}{p!} \rho^{p-2(n+1)} T^{n}_{pnm}(\tau)$$

$$= \lim_{\substack{\rho \to 0 \\ \tau \to -1}} (1-\tau)^{-4(n+1)} \left[\sum_{p=0}^{2n+1} \frac{1}{p!} \rho^{p-2(n+1)} T^{n}_{pnm} + \frac{1}{(2n+2)!} \rho^{0} T^{n}_{2n+2,n,m} + \right.$$

$$\left. + \sum_{p=2n+3}^{\infty} \frac{1}{p!} \rho^{p-2(n+1)} T^{n}_{pnm} \right] =$$

$\Rightarrow$ The first sum is singular on $\rho = 0$ and the last vanishes on $\rho = 0$

$$= \lim_{\substack{\rho \to 0 \\ \tau \to -1}} (1-\tau)^{-4(n+1)} \left[\frac{T^{n}_{2n+2,n,m}(\tau)}{2(\cancel{2}n+\cancel{1})!} + \sum_{\substack{p=\cancel{0} \\ 2n+3}}^{2n+1} \frac{1}{p!} \rho^{p-2(n+1)} T^{n}_{pnm}(\tau) \right]$$

- $$I^{-}_{nm} = \frac{2^{-4(n+1)}}{2(\cancel{2}n+\cancel{1})!} T^{n}_{\cancel{2}n+\cancel{1},n,m} \,(\tau = -1)$$

$$= \frac{2^{-4(n+1)}}{2(\cancel{2}n+\cancel{1})!} \left\{ \sum_{q=0}^{n+1} K_q \binom{n+1}{q} (1-(-1))^{(n+1-q)} a^{(n+1-q)}_{2n+2,n,m} (-1) \right\}$$

$$\Rightarrow I^{+}_{nm} = \frac{2^{-4(n+1)}}{2(n+1)!} Q^{n}_{n+1,n,m}(\tau)$$

Evaluating $Q^{n}_{n+1,n,m}$ at $t=-1$ one gets (17).