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ME/CprE/ComS 557

Computer Graphics and Geometric Modeling

Quaternions

October 27th, 2015

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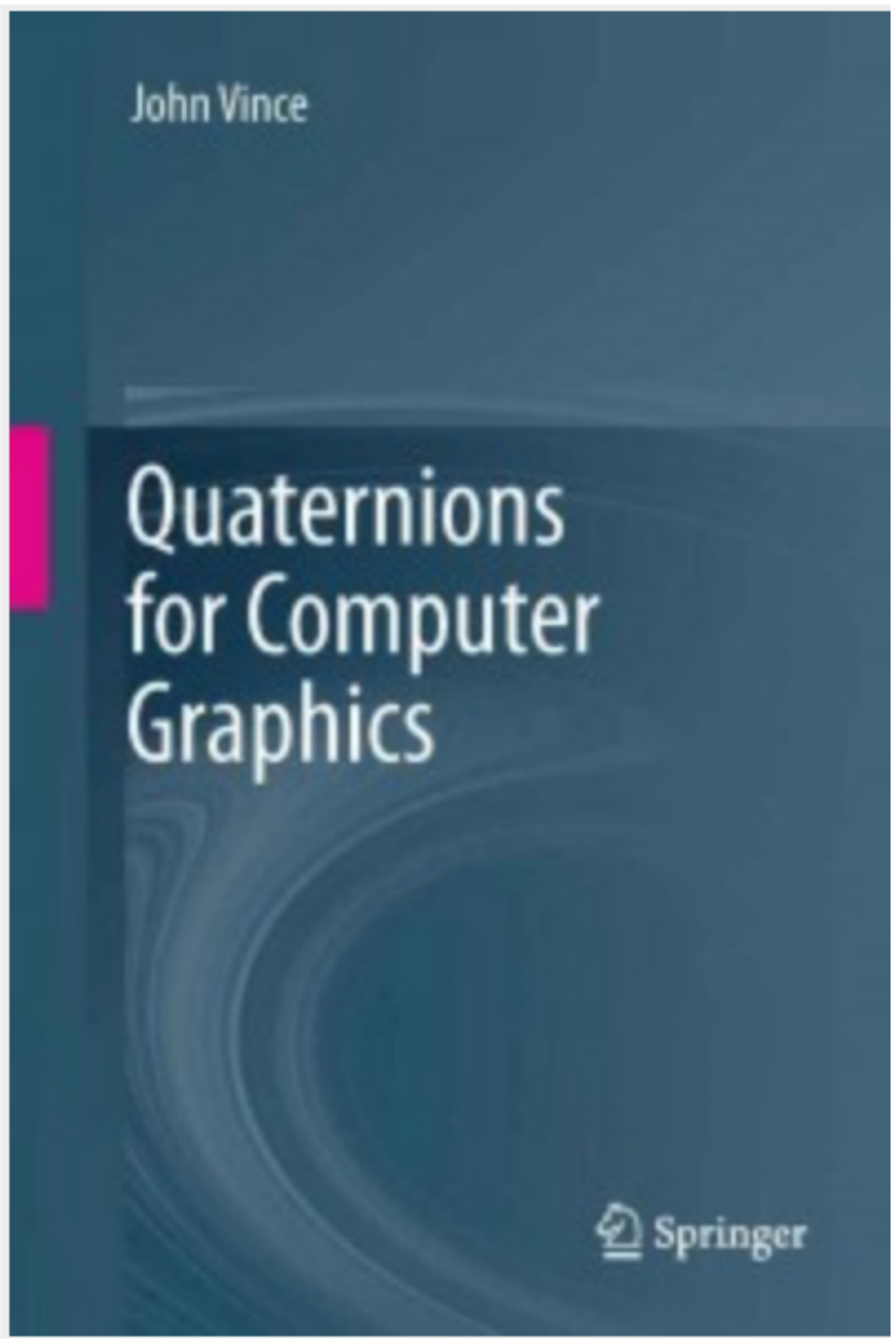
Book



Vince, J (2011). Quaternions for Computer Graphics. 1st. ed. London: Springer

Why Quaternions

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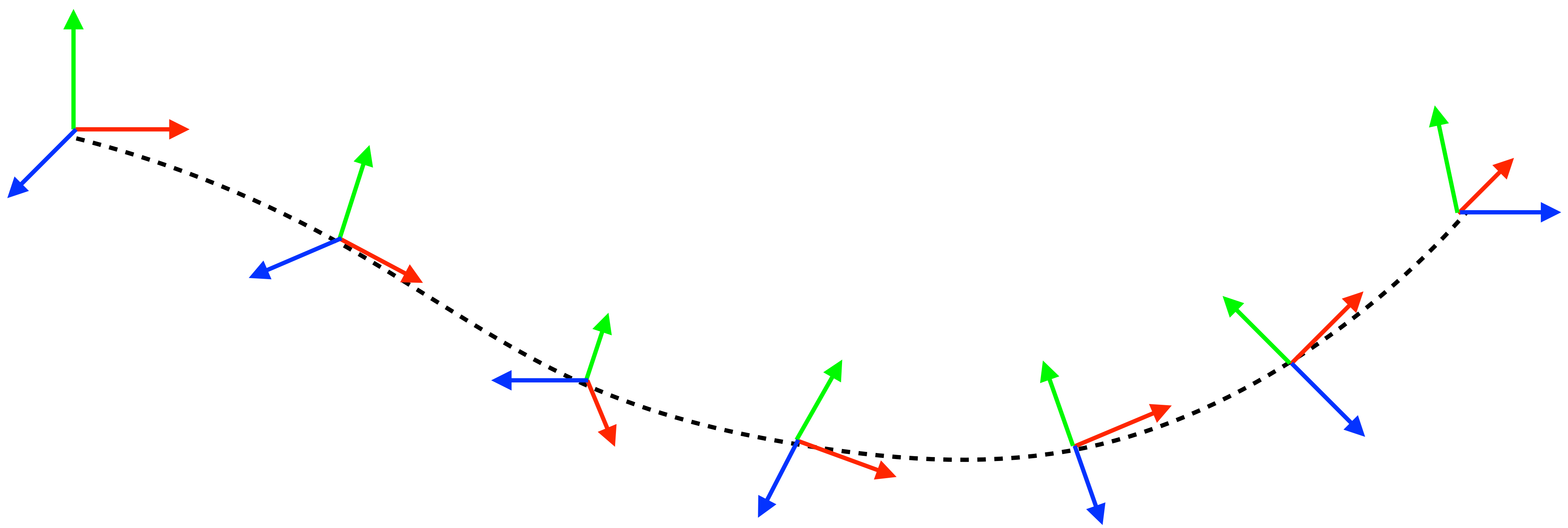
Describing a path / locomotion as a sequence of matrices can result in an unexpected outcome due to the link of the rotation axis to each other in Cartesian space with Euler angles.



- Easier to "add" a new position / orientation to the current position / orientation
- Interpolation between points is possible.

Gimble Lock Problem

The sequence of transformations (rotations in our case) can be considered as hierarchy of transformations.

- R_z
  - R_x
    - R_y

$$T = R_y \cdot R_x \cdot R_z$$

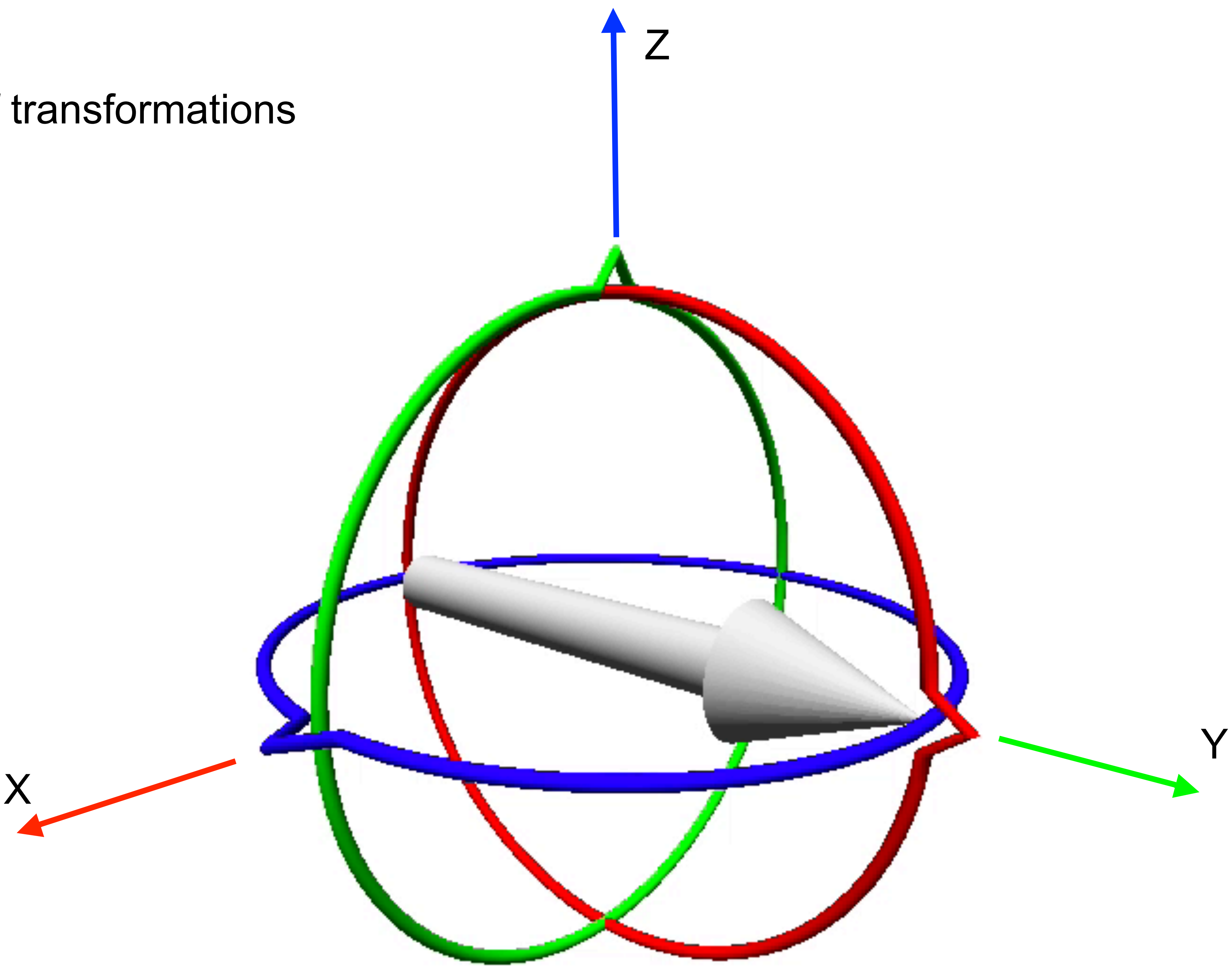
$$T = \begin{bmatrix} cos(\beta) & 0 & -sin(\beta) & 0 \\ 0 & 1 & 0 & 0 \\ sin(\beta) & 0 & cos(\beta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ cos(\alpha) & -sin(\alpha) & 0 & 0 \\ sin(\alpha) & cos(\alpha) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} cos(\gamma) & sin(\gamma) & 0 & 0 \\ -sin(\gamma) & cos(\gamma) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The transformations are linked: the axis y, and x also rotates when an object is rotated around the z-axis.

Gimble Lock Problem

Hierarchy of transformations

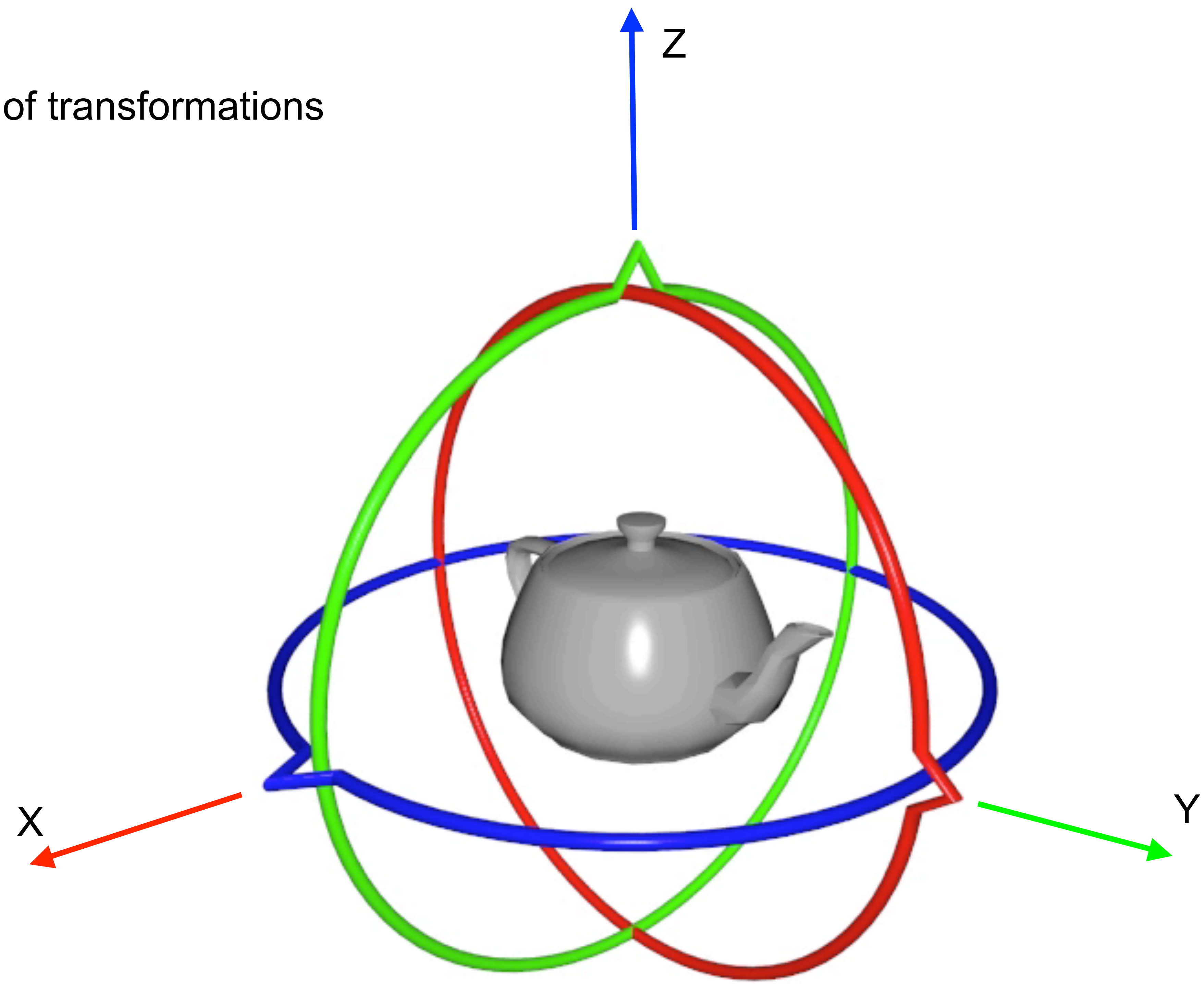
Z
└X
  └Y



Gimble Lock Problem

Hierarchy of transformations

Z
└ **X**
  └ **Y**



History

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$$i^2 = j^2 = k^2 = -1$$

Quaternions where discovered by Sir William Rowan Hamilton on Monday October 16th 1843 in Dublin, Ireland.

Complex Numbers

Complex numbers introduce imaginary numbers to solve certain equations such as:

$$x + 1 = 0$$

$$x^2 = -1$$

$$\cancel{x = \sqrt{-1}}$$

which we know is not possible to solve with real numbers because the square of every number is always positive.

The **imaginary number** has been founded.

$$i^2 = -1$$

The set of complex numbers is the sum of a real number and an imaginary number

$$z = a + bi \quad a, b \in \mathbb{R}, \; i^2 = -1$$

- all real numbers are complex numbers with b = 0
- all imaginary numbers are complex numbers with a = 0

Some Rules

Adding and Subtracting Complex Numbers

Addition:

$$(a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 + a_2) + (b_1 + b_2)i$$

Subtraction:

$$(a_1 + b_1 i) - (a_2 + b_2 i) = (a_1 - a_2) + (b_1 - b_2)i$$

Multiply a Complex Number by a Scalar:

$$\lambda(a + bi) = \lambda a + \lambda bi$$

Some Rules

Product of Complex Numbers:

$$\begin{aligned} z_1 &= (a_1 + b_1 i) \\ z_2 &= (a_2 + b_2 i) \\ z_1 z_2 &= (a_1 + b_1 i)(a_2 + b_2 i) \\ &= a_1 a_2 + a_1 b_2 i + b_1 a_2 i + b_1 b_2 i \\ &= (a_1 a_2 - b_1 b_2) + (a_1 b_2 + b_1 a_2) i \end{aligned}$$

Square of Complex Numbers:

$$\begin{aligned} z^2 &= (a + bi)(a + bi) \\ &= (a^2 - b^2) + 2abi \end{aligned}$$

Complex Conjugate:

$$\begin{aligned} z &= (a + bi) \\ |z| &= \sqrt{zz^*} \\ &= \sqrt{(a + bi)(a - bi)} \\ &= \sqrt{a^2 + b^2} \end{aligned}$$

The conjugate of a complex number is a complex number with the imaginary part negated

Some Rules

Absolute Value of a Complex Number:

$$
\begin{aligned}
z &= (a + bi) \\
|z| &= \sqrt{zz^*} \\
&= \sqrt{(a+bi)(a-bi)} \\
&= \sqrt{a^2 + b^2}
\end{aligned}
$$

We can use the conjugate of a complex number to compute the absolute value (or norm, or magnitude) of a complex number.

Power of i

$$i^0 = 1$$

$$i^1 = i$$

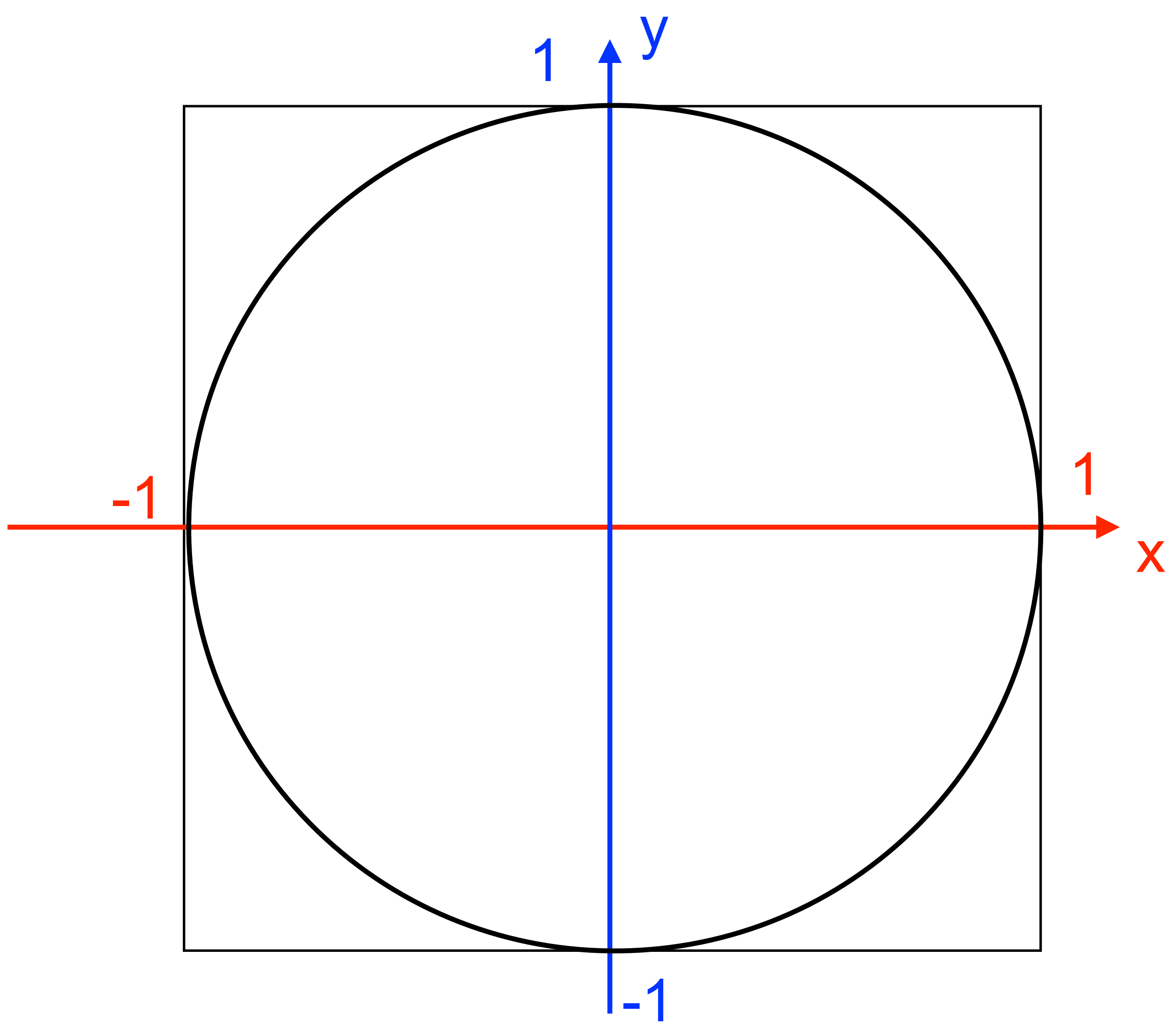
$$i^2 = -1$$

$$i^3 = ii^2 = -i$$

$$i^4 = i^2 i^2 = 1$$

$$i^5 = ii^4 = i$$

$$i^6 = ii^5 = -1$$



This is a sequence 1, i, -1, -i, 1, i, -1, -i, 1, ...

The Complex Plane

$$i^0 = 1$$

$$i^1 = i$$

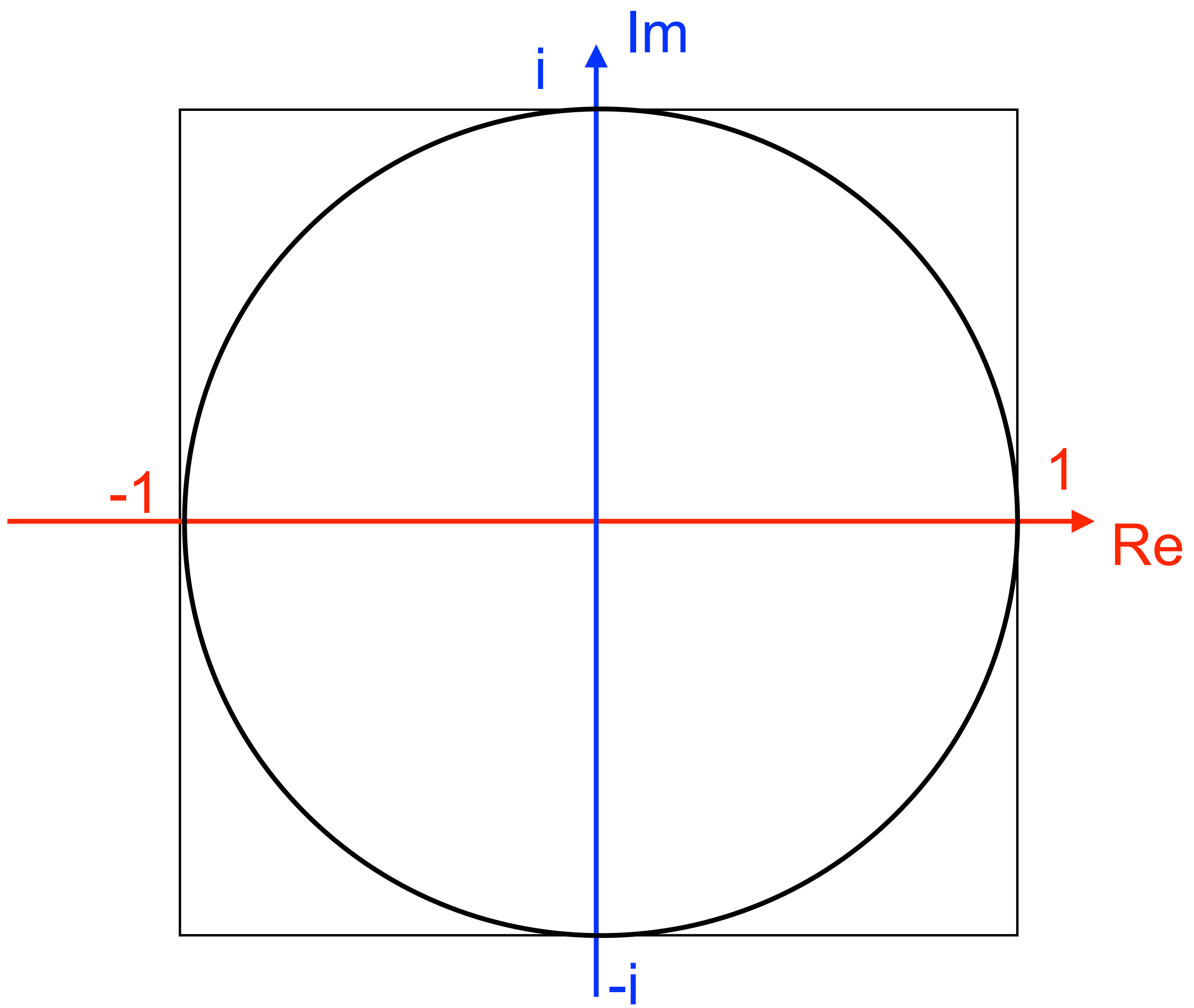
$$i^2 = -1$$

$$i^3 = ii^2 = -i$$

$$i^4 = i^2 i^2 = 1$$

$$i^5 = ii^4 = i$$

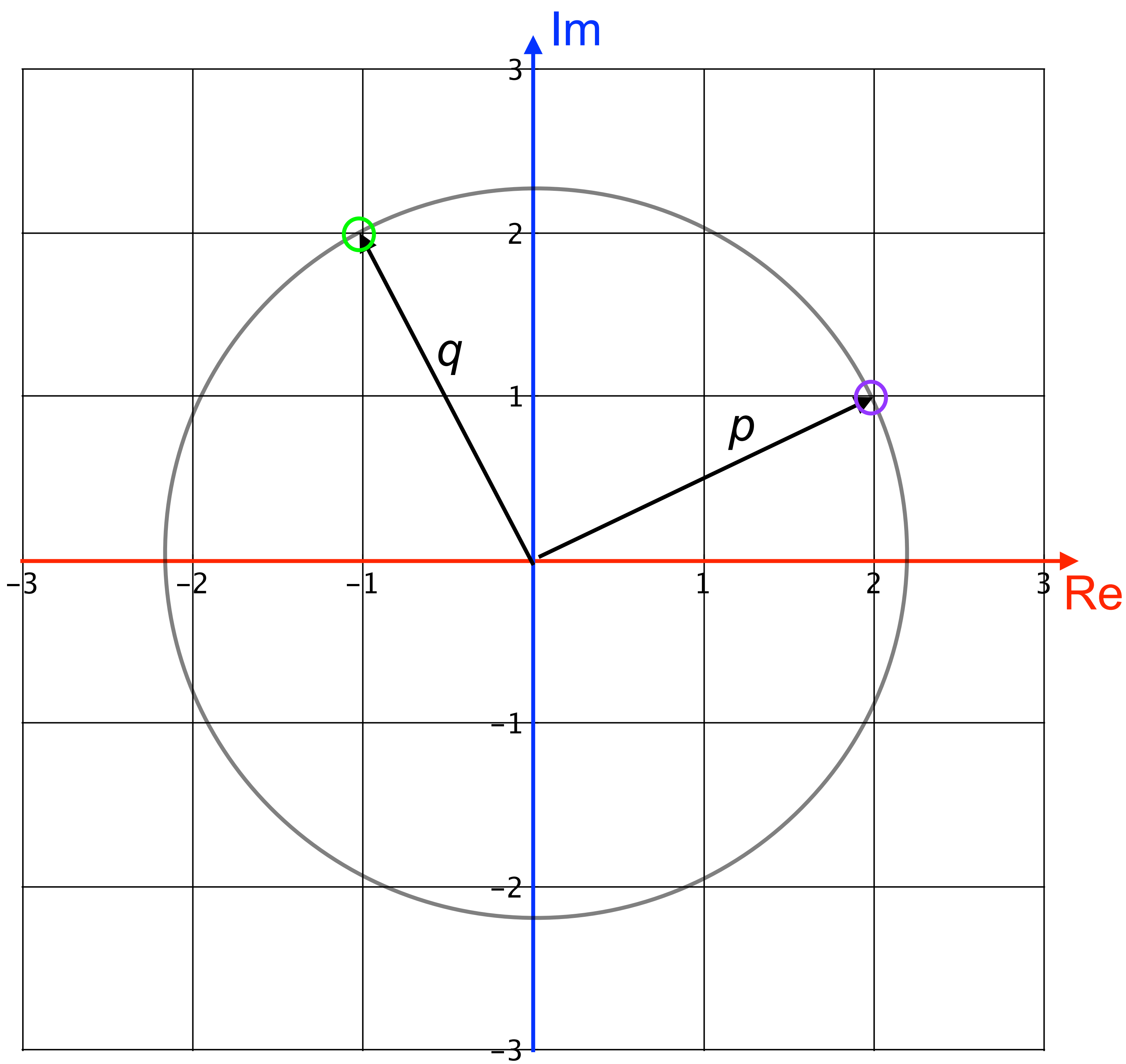
$$i^6 = ii^5 = -1$$



This is a sequence 1, i, -1, -i, 1, i, -1, -i, 1, ...

We can say that if we multiply a complex number by i, we can rotate the complex number through the complex plane at 90 degree increments.

Rotation with Complex Numbers



Take an arbitrary point p in the complex plane:

$$p = 2 + i$$

Multiply this point with *i*.

$$\begin{aligned} q &= p\,i \\ &= (2+i)i \\ &= 2i + i^2 \\ &= -1 + 2i \end{aligned}$$

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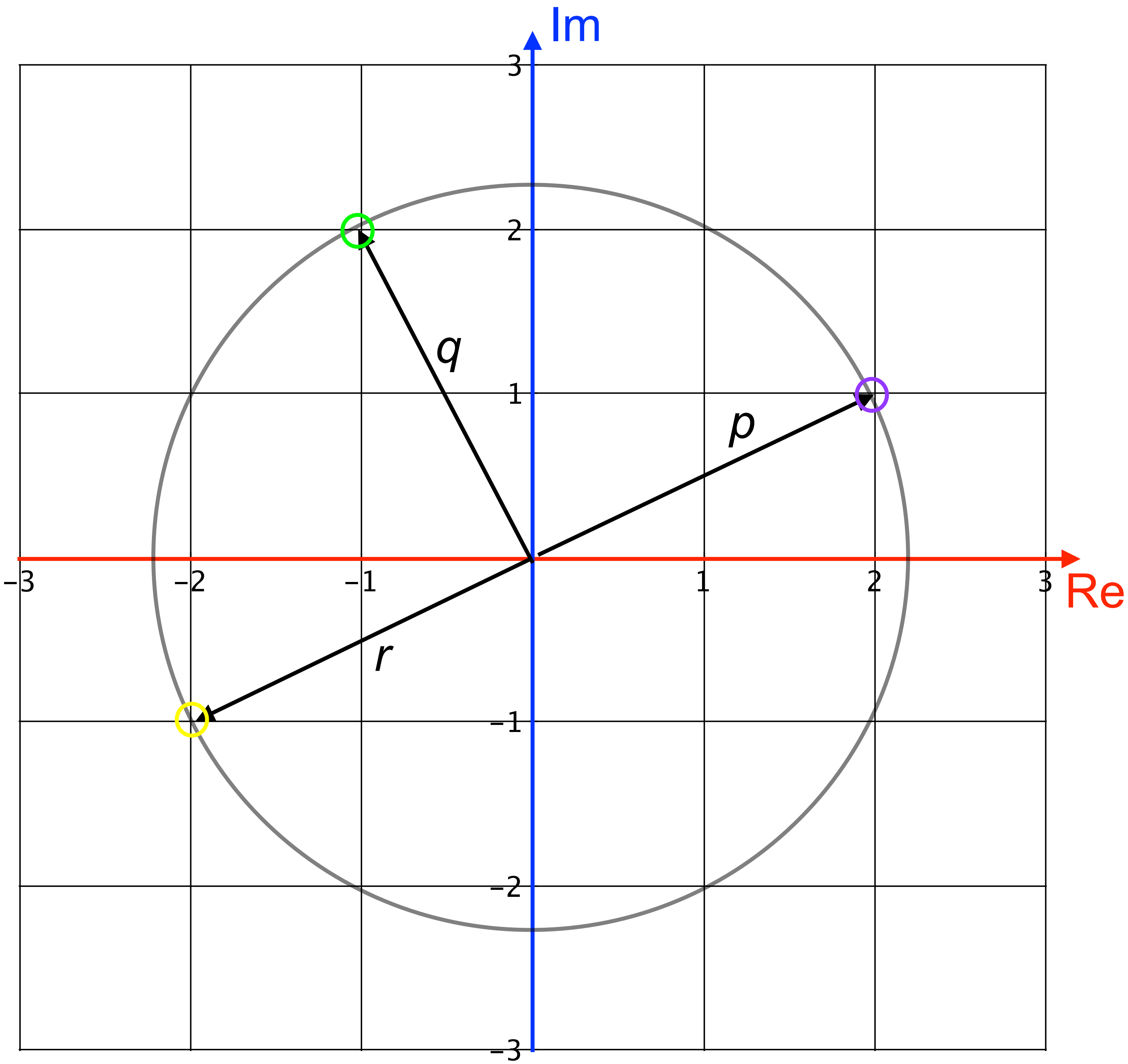
Rotation with Complex Numbers



Take an arbitrary point p in the complex plane:

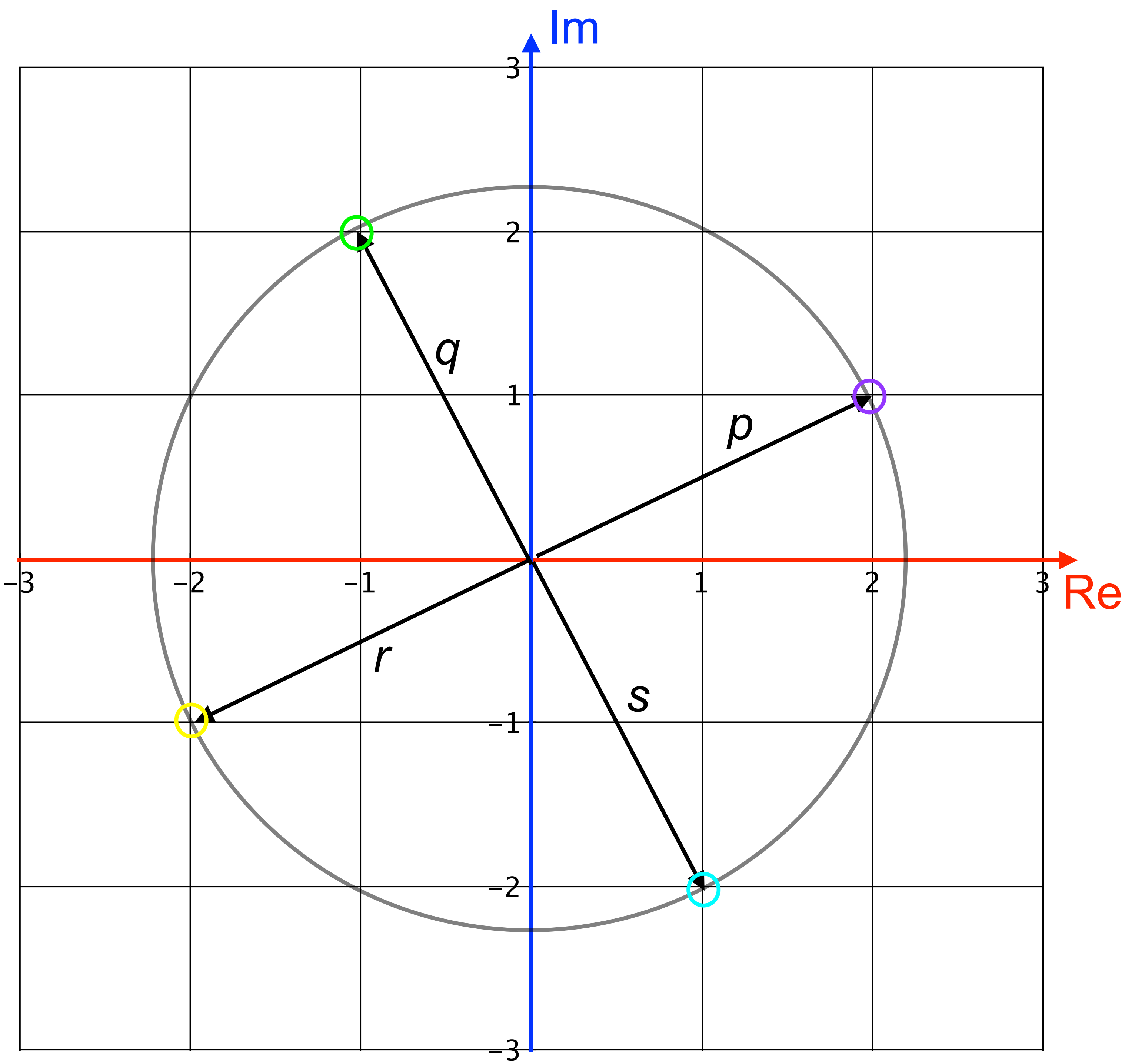
$$p = 2 + i$$

Multiply point *q* with *i.*

$$q = -1 + 2i$$

$$\begin{aligned} r &= q\, i \\ &= (-1 + 2i)i \\ &= -i + 2i^2 \\ &= -2 - i \end{aligned}$$

Rotation with Complex Numbers



Take an arbitrary point p in the complex plane:

$$p = 2 + i$$

Multiply point *r* with *i*.

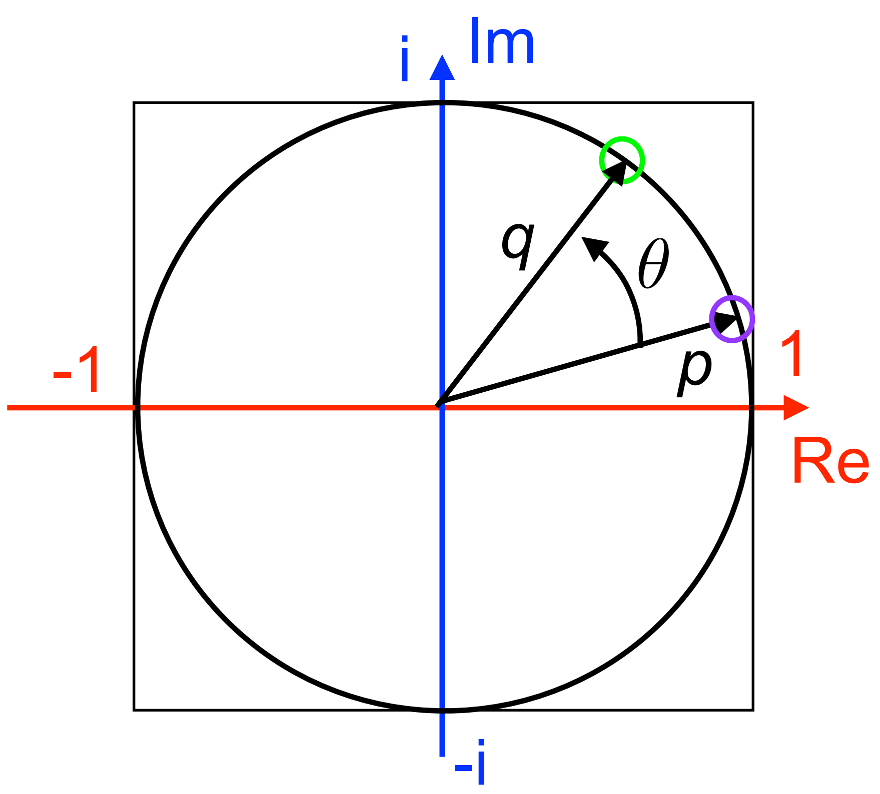
$$r = -2 - i$$

$$\begin{aligned} s &= r\, i \\ &= (-2 - i)i \\ &= -2i - i^2 \\ &= 1 - 2i \end{aligned}$$

Rotation with Complex Numbers

We can also perform arbitrary rotations in the complex plane by defining a complex number of the form:

$$\boxed{q = \cos\theta + i\sin\theta}$$



We obtain the general formula when multiplying a point *p* with *q*:

$$p = a + b\,i$$

$$q = \cos\theta + i\ \sin\theta$$

$$pq = (a+bi)(\cos\theta + i\ \sin\theta)$$

$$a' + b'i = a\ \cos\theta - b\ \sin\theta + (a\ \sin\theta + b\ \cos\theta)i$$

(*pq* rewritten)

or in matrix form:

$$\begin{bmatrix} a' & -b' \\ b' & a' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

to rotate an arbitrary point counter-clockwise.

This was in 2D / one plane out or three (rotation of a point around one axis in a plane).

How do we extend this to 3D?

Quaternions

Quaternions

We extend this to 3-dimensional space by adding two imaginary numbers to our number system in addition to *i*. One number for each additional plane.

The general form to express quaternions is:

$$\boxed{q = s + xi + yj + zk \qquad s, x, y, z \in \mathbb{R},\ i, j, z \in \mathbb{I}}$$

scalar value s

vector of complex numbers

with

$$i^2 + j^2 + k^2 = ijk = -1$$

and

$$\begin{array}{ccc} ij = k & jk = i & ki = j \\ ji = -k & kj = -i & ik = -j \end{array}$$

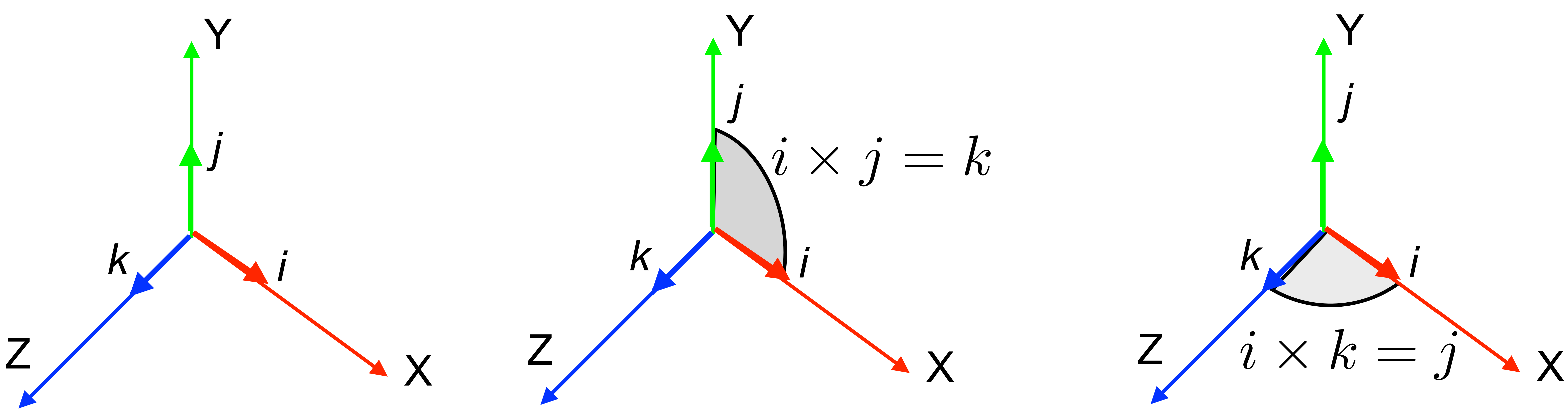
Quaternions

When mapping these quaternion rules to a Cartesian coordinate system,

$$\begin{array}{ccc} ij = k & jk = i & ki = j \\ ji = -k & kj = -i & ik = -j \end{array}$$

they behave similar to cross products.

$$\begin{array}{ccc} \mathbf{x} \times \mathbf{y} = \mathbf{z}, & \mathbf{y} \times \mathbf{z} = \mathbf{x}, & \mathbf{z} \times \mathbf{x} = \mathbf{y} \\ \mathbf{y} \times \mathbf{x} = -\mathbf{z}, & \mathbf{z} \times \mathbf{y} = -\mathbf{x}, & \mathbf{x} \times \mathbf{z} = -\mathbf{y} \end{array}$$



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Quaternions

Hamilton recognized that the imaginary numbers *i*, *j*, and *k* can be used to represent three cartesian unit vectors *i*, *j*, and *k* with the same properties of imaginary numbers, such that

$$i^2 + j^2 + k^2 = ijk = -1$$

Quaternions can be represented as an **ordered pair**:

$$q = [s, \mathbf{v}] \quad s \in \mathbb{R}, \mathbf{v} \in \mathbb{R}^3$$

or with individual components

$$q = [s, x\mathbf{i} + y\mathbf{y} + z\mathbf{k}] \quad s, x, y, z \in \mathbb{R}$$

which allows us to show similarities to complex numbers.

Substitute to swap between notations.

Mathematical Rules

Adding and subtracting quaternions:

$$q_a = [s_a, \mathbf{a}] \quad q_b = [s_b, \mathbf{b}]$$

with $\mathbf{a} = [x_a\mathbf{i} + y_a\mathbf{y} + z_a\mathbf{k}]$

similar for **b**

$$q_a + q_b = [s_a + s_b, \mathbf{a} + \mathbf{b}]$$

$$q_a - q_b = [s_a - s_b, \mathbf{a} - \mathbf{b}]$$

but keep in mind:

$$q_a + q_b = (s_a + s_b) + (x_a + x_b)\mathbf{i} + (y_a + y_b)\mathbf{j} + (z_a + z_b)\mathbf{k}$$

$$q_a - q_b = (s_a - s_b) + (x_a - x_b)\mathbf{i} + (y_a - y_b)\mathbf{j} + (z_a - z_b)\mathbf{k}$$

Substitute to swap between notations.

Product of Quaternions

Multiplying two quaternions takes more time, but the math is straight forward:

Use the quaternions q_a and q_b as starting point:

$$q_a = [s_a, \mathbf{a}] \quad q_b = [s_b, \mathbf{b}]$$

multiplying:

$$\begin{aligned} q_a q_b &= [s_a, \mathbf{a}][s_b, \mathbf{b}] \\ &= (s_a + x_a i + y_a j + z_a k)(s_b + x_b i + y_b j + z_b k) \\ &= (s_a s_b - x_a x_b - y_a y_b - z_a z_b) \\ &\quad + (s_a x_b + s_b x_a + y_a z_b - y_b z_a) i \\ &\quad + (s_a y_b + s_b y_a + z_a x_b - z_b x_a) j \\ &\quad + (s_a z_b + s_b z_a + x_a y_b - x_b y_a) k \end{aligned}$$

keep in mind

$$i^2 = -1$$

$$i^2 + j^2 + k^2 = ijk = -1$$

$$ij = k \quad jk = i \quad ki = j$$

$$ji = -k \quad kj = -i \quad ik = -j$$

Product of Quaternions

Multiplying two quaternions takes more time, but the math is straight forward:

Use the quaternions q_a and q_b as starting point:

$$q_a = [s_a, \mathbf{a}] \quad q_b = [s_b, \mathbf{b}]$$

multiplying:

$$\begin{aligned}
q_a q_b &= [s_a, \mathbf{a}][s_b, \mathbf{b}] \\
&= (s_a + x_a i + \boxed{y_a j} + z_a k)(s_b + x_b i + y_b j + \boxed{z_b k}) \\
&= (s_a s_b - x_a x_b - y_a y_b - z_a z_b) \\
&\quad + (s_a x_b + s_b x_a + \boxed{y_a z_b} - y_b z_a) i \\
&\quad + (s_a y_b + s_b y_a + z_a x_b - z_b x_a) j \\
&\quad + (s_a z_b + s_b z_a + x_a y_b - x_b y_a) k
\end{aligned}$$

keep in mind

$$i^2 = -1$$

$$i^2 + j^2 + k^2 = ijk = -1$$

$$ij = k \qquad \boxed{jk = i} \qquad ki = j$$

$$ji = -k \qquad kj = -i \qquad ik = -j$$

note, change the complex number:

$$y_a j \cdot z_b k = (y_a z_b) i \ \rightarrow jk = i$$

Product of Quaternions

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Now let's replace the imaginary number by ordered pairs

$$i = [0, \mathbf{i}] \; j = [0, \mathbf{j}] \; k = [0, \mathbf{k}] \quad \text{and} \quad [1, \mathbf{0}] = 1$$

Pure quaternion

Real quaternion

gives

$$\begin{aligned}[s_a, \mathbf{a}][s_b, \mathbf{b}] = &(s_a s_b - x_a x_b - y_a y_b - z_a z_b)[1, \mathbf{0}] \\ &+ (s_a x_b + s_b x_a + y_a z_b - y_b z_a)[0, \mathbf{i}] \\ &+ (s_a y_b + s_b y_a + z_a x_b - z_b x_a)[0.\mathbf{j}] \\ &+ (s_a z_b + s_b z_a + x_a y_b - x_b y_a)[0, \mathbf{k}]\end{aligned}$$

rewriting as ordered pairs gives

$$\begin{aligned}[s_a, \mathbf{a}][s_b, \mathbf{b}] = &[s_a s_b - x_a x_b - y_a y_b - z_a z_b, \mathbf{0}] \\ &+ [0, (s_a x_b + s_b x_a + y_a z_b - y_b z_a)\mathbf{i}] \\ &+ [0, (s_a y_b + s_b y_a + z_a x_b - z_b x_a)\mathbf{j}] \\ &+ [0, (s_a z_b + s_b z_a + x_a y_b - x_b y_a)\mathbf{k}]\end{aligned}$$

Product of Quaternions

We can extract the common components and further rewrite as:

$$
\begin{aligned}
[s_a, \mathbf{a}][s_b, \mathbf{b}] = {} & [s_a s_b - x_a x_b - y_a y_b - z_a z_b, \mathbf{0}] \\
& + [0, s_a(x_b\mathbf{i} + y_b\mathbf{j} + z_b\mathbf{k}) + s_b(x_a\mathbf{i} + y_a\mathbf{j} + z_a\mathbf{k}) \\
& + (y_z z_b - y_b z_a)\mathbf{i} + (z_a x_b - z_b x_z)\mathbf{j} + (x_a y_b - y_b x_a)\mathbf{k}]
\end{aligned}
$$

Product of Quaternions

We can extract the common components and further rewrite as:

$$
\begin{aligned}
[s_a, \mathbf{a}][s_b, \mathbf{b}] = & \boxed{[s_a s_b - x_a x_b - y_a y_b - z_a z_b, \mathbf{0}]} \\
& + [0, s_a(x_b\mathbf{i} + y_b\mathbf{j} + z_b\mathbf{k}) + s_b(x_a\mathbf{i} + y_a\mathbf{j} + z_a\mathbf{k}) \\
& + (y_z z_b - y_b z_a)\mathbf{i} + (z_a x_b - z_b x_z)\mathbf{j} + (x_a y_b - y_b x_a)\mathbf{k}]
\end{aligned}
$$

Real quaternion

Pure quaternion

We combine the two ordered pairs into a single ordered pair, because one component is always 0.

$$
\begin{aligned}
[s_a, \mathbf{a}][s_b, \mathbf{b}] = & [s_a s_b - x_a x_b - y_a y_b - z_a z_b, \\
& s_a(x_b\mathbf{i} + y_b\mathbf{j} + z_b\mathbf{k}) + s_b(x_a\mathbf{i} + y_a\mathbf{j} + z_a\mathbf{k}) \\
& + (y_z z_b - y_b z_a)\mathbf{i} + (z_a x_b - z_b x_z)\mathbf{j} + (x_a y_b - y_b x_a)\mathbf{k}]
\end{aligned}
$$

Product of Quaternions

$$
\begin{aligned}
[s_a, \mathbf{a}][s_b, \mathbf{b}] = [s_a s_b & \boxed{- x_a x_b - y_a y_b - z_a z_b,} \\
& \boxed{s_a(x_b\mathbf{i} + y_b\mathbf{j} + z_b\mathbf{k})} \boxed{+} \boxed{s_b(x_a\mathbf{i} + y_a\mathbf{j} + z_a\mathbf{k})} \\
& \boxed{+ (y_z z_b - y_b z_a)\mathbf{i} + (z_a x_b - z_b x_z)\mathbf{j} + (x_a y_b - y_b x_a)\mathbf{k}]}
\end{aligned}
$$

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Product of Quaternions

Dot product

$$[s_a, \mathbf{a}][s_b, \mathbf{b}] = [s_a s_b \boxed{- x_a x_b - y_a y_b - z_a z_b,}$$

pure quaternion

$$\boxed{s_a(x_b\mathbf{i} + y_b\mathbf{j} + z_b\mathbf{k})} + \boxed{s_b(x_a\mathbf{i} + y_a\mathbf{j} + z_a\mathbf{k})}$$

$$\boxed{+ (y_z z_b - y_b z_a)\mathbf{i} + (z_a x_b - z_b x_z)\mathbf{j} + (x_a y_b - y_b x_a)\mathbf{k}]}$$

Cross product

with:

$$\mathbf{a} = x_a\mathbf{i} + y_a\mathbf{j} + z_a\mathbf{k}$$

$$\mathbf{b} = x_b\mathbf{i} + y_b\mathbf{j} + z_b\mathbf{k}$$

$$\mathbf{a} \cdot \mathbf{b} = x_a x_b \mathbf{i}^2 + y_a y_b \mathbf{j}^2 + z_a z_b \mathbf{k}^2$$

$$\mathbf{a} \times \mathbf{b} = (y_a z_b - y_b z_a)\mathbf{i}(z_a x_b - x_b z_a)\mathbf{j} + (x_a y_b - x_b y_a)\mathbf{k}$$

$$i^2 = j^2 = k^2 = -1$$

Product of Quaternions

Substituting:

$$\boxed{[s_a, \mathbf{a}][s_b, \mathbf{b}] = [s_a s_b - \mathbf{a} \cdot \mathbf{b}, s_a \mathbf{b} + s_b \mathbf{a} + \mathbf{a} \times \mathbf{b}]}$$

Results in an equation for the product of two quaternions.

Real and Pure Quaternions

A Real Quaternion is a quaternion with a vector term of 0:

$$q = [s, \mathbf{0}]$$

the Pure Quaternion as a quaternion that has a zero scalar term

$$q = [0, \mathbf{v}]$$

addition of both:

$$\begin{aligned} q &= [s, \mathbf{v}] \\ &= [s, \mathbf{0}] + [0, \mathbf{v}] \end{aligned}$$

Unit and Norm

Unit Quaternion

Given a vector **v**, we can express this vector with its scalar magnitude and its direction:

$$\mathbf{v} = v\hat{\mathbf{v}} \qquad \text{where} \quad v = |\mathbf{v}| \qquad \text{and} \ |\hat{\mathbf{v}}| = 1$$

Thus, we obtain a quaternion with length of the complex vector = 1.

Quaternion Norm

$$q = [s, \mathbf{v}]$$

$$|q| = \sqrt{s^2 + \mathbf{v^2}}$$

Normalization

We can also normalize a quaternion:

$$q' = \frac{q}{|a|} = \frac{q}{\sqrt{s^2 + \mathbf{v}^2}}$$

Example:

$$q = [1, 4\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}]$$

Calculate the length:

$$|q| = \sqrt{1^2 + 4^2 + 4^2 + (-4)^2} = \sqrt{49} = 7$$

Divide the quaternion by the length (norm)

$$q' = \frac{q}{|q|} = \frac{1 + 4\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}}{7} = \frac{1}{7} + \frac{4}{7}\mathbf{i} + \frac{4}{7}\mathbf{j} - \frac{4}{7}\mathbf{k}$$

Quaternion Conjugate and Inverse

The quaternion conjugate can be computed by negating the imaginary part of the quaternion:

$$q = [s, \mathbf{v}]$$
$$q* = [s, -\mathbf{v}]$$

The inverse of a quaternion can be computed as:

$$q^{-1} = \frac{q^*}{|q|^2}$$

Quaternion Dot Product

Similar to the vector dot product, we can also define a quaternion dot product by multiplying the scalar parts:

$$q_a = [s_a, x_a\mathbf{i} + y_a\mathbf{j} + z_a\mathbf{k}]$$

$$q_b = [s_b, x_b\mathbf{i} + y_b\mathbf{j} + z_b\mathbf{k}]$$

$$q_a \cdot q_b = s_a s_b + x_a x_b + y_a y_b + z_a z_b$$

Which also represents the angle between two quaternions:

$$\cos\theta = \frac{s_a s_b + x_a x_b + y_a y_b + z_a z_b}{|q_1||q_2|}$$

or with a unit-quaternion, length = 1 (similar to normalized vectors)

$$\cos\theta = s_a s_b + x_a x_b + y_a y_b + z_a z_b$$

Rotation with Quaternions

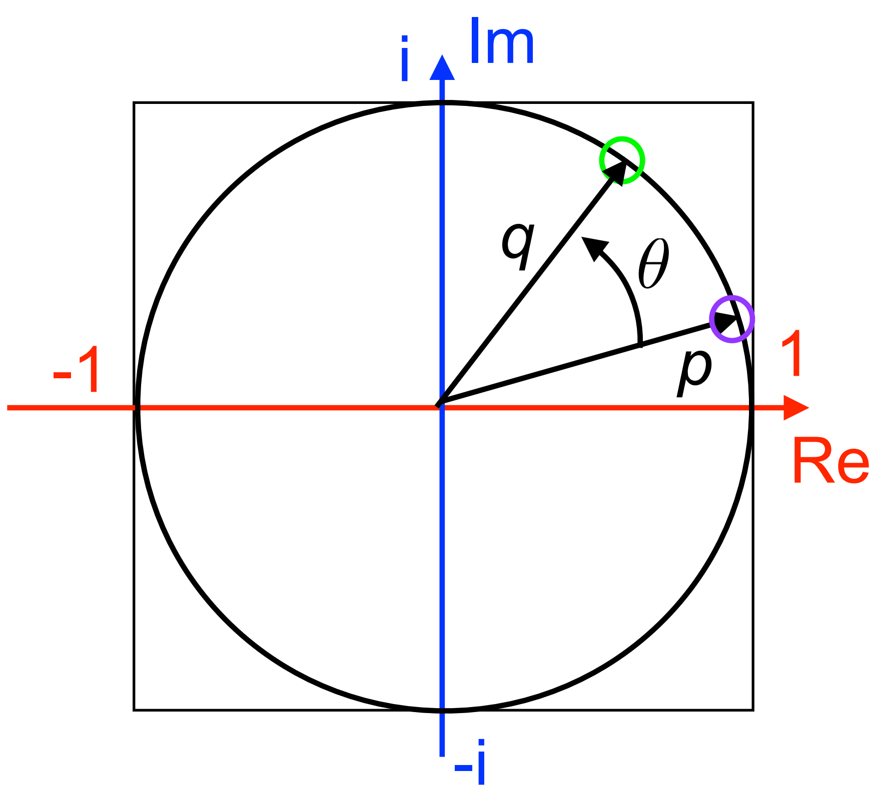
Rotation with Quaternions

Remember the Rotor we defined for complex numbers:

$$q = \cos\theta + i\sin\theta$$

Knowing the similarity to complex numbers, we can express a general Rotor for 3D space such as:

$$\boxed{q = [\cos\theta, \sin\theta\mathbf{v}]}$$



Let's test

Let's consider a vector *p* that we want to rotate,
an pure quaternion:

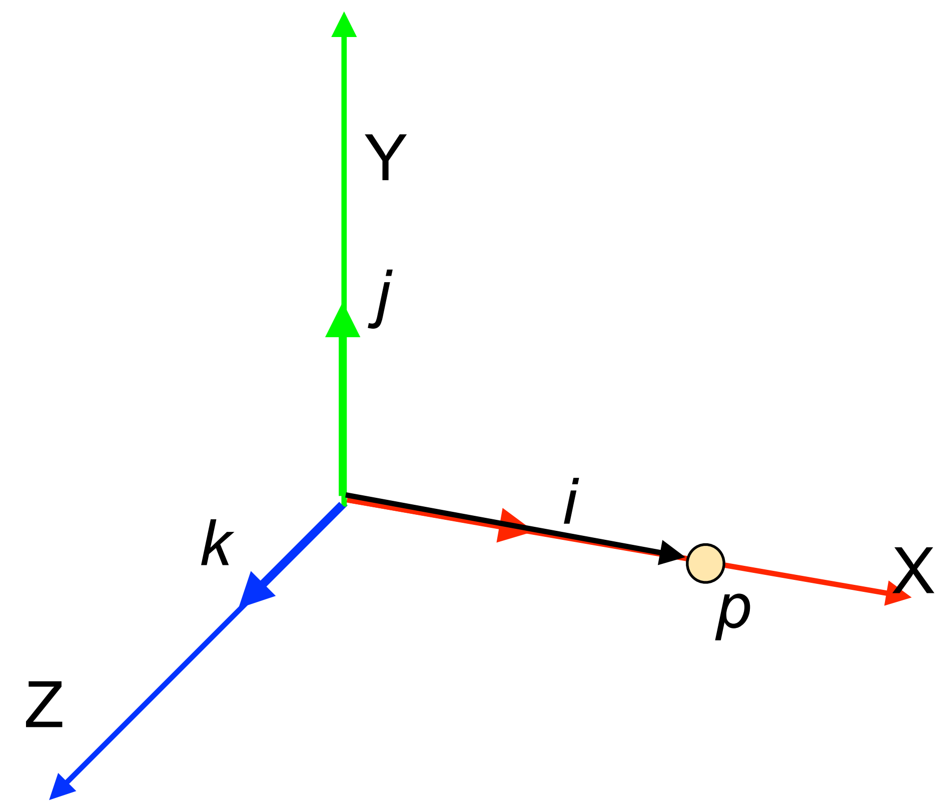
$$p = [0, 2\mathbf{i}]$$

we want to rotate θ = 45° around **k**

$$q = [\cos\theta, \sin\theta\mathbf{k}]$$

(I removed the components **i**,**j)**

$$q = \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\mathbf{k}\right]$$

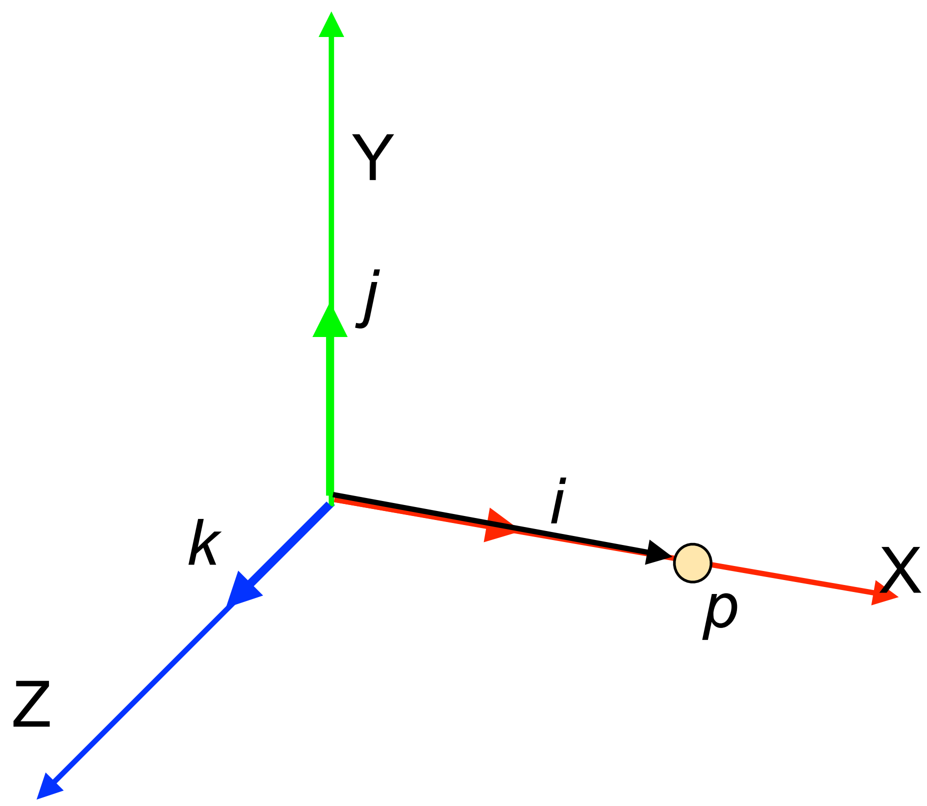


(Quaternions work in 4D,
drawing 3D is actually not correct.)

Let's test

and rotate:

$$
\begin{aligned}
p' &= qp \\
&= \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\mathbf{k}\right][0, 2\mathbf{i}] \\
&= \left[0, 2\frac{\sqrt{2}}{2}\mathbf{i} + 2\frac{\sqrt{2}}{2}\mathbf{k} \times \mathbf{i}\right] \\
&= [0, \sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}]
\end{aligned}
$$



(Quaternions work in 4D,
drawing 3D is actually not correct.)

Let's test

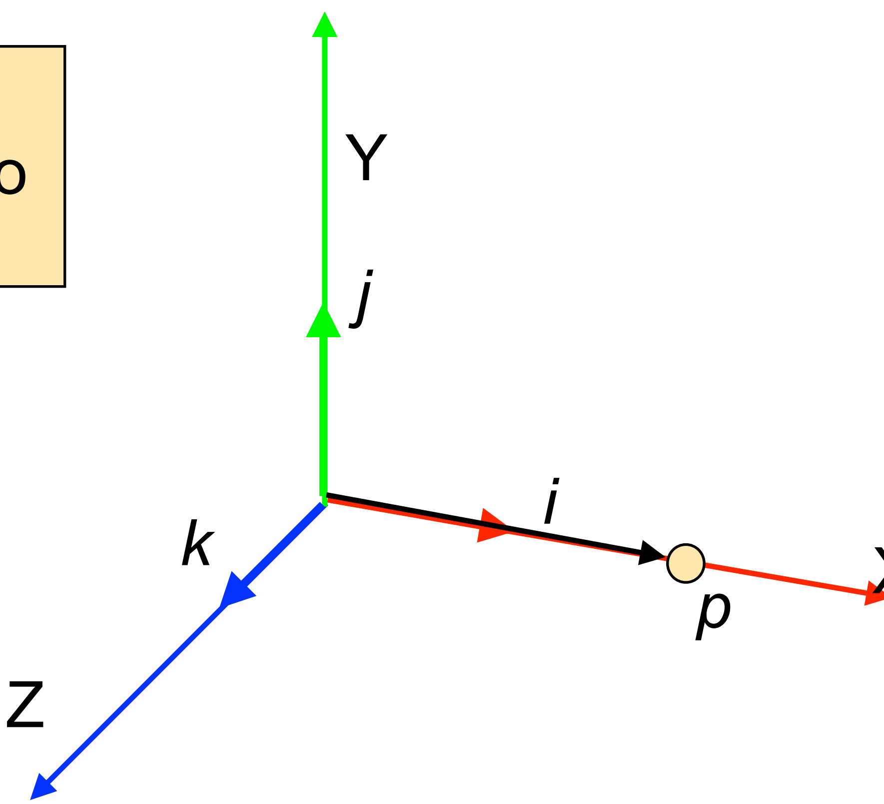
ARLAB

and rotate:

Merging pure quaternion with quaternion.

Move the scalar term to the front.

$$
\begin{aligned}
p' &= qp \\
&= \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\mathbf{k}\right][0, 2\mathbf{i}] \\
&= \left[0, \boxed{2\frac{\sqrt{2}}{2}\mathbf{i}} + \boxed{2\frac{\sqrt{2}}{2}\mathbf{k} \times \mathbf{i}}\right] \\
&= [0, \sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}]
\end{aligned}
$$



Multiplication

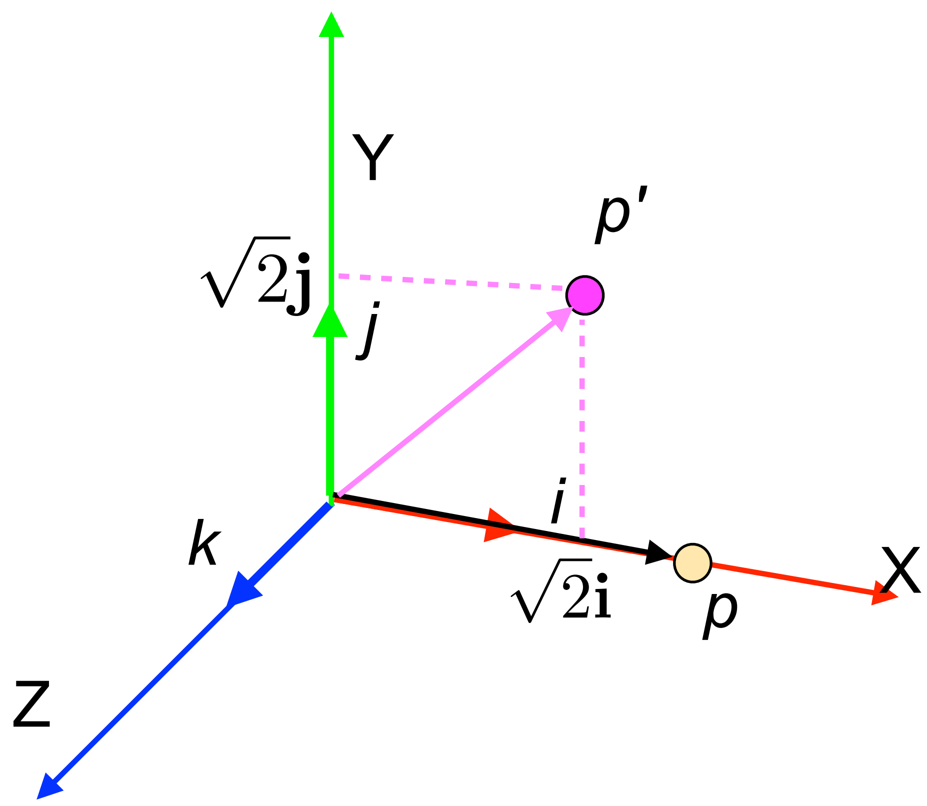
$$[s_a, \mathbf{a}][s_b, \mathbf{b}] = [s_a s_b - \mathbf{a} \cdot \mathbf{b}, \boxed{s_a \mathbf{b}} + s_b \mathbf{a} + \boxed{\mathbf{a} \times \mathbf{b}}]$$

(Quaternions work in 4D, drawing 3D is actually not correct.)

Let's test

and rotate:

$$
\begin{aligned}
p' &= qp \\
&= \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\mathbf{k}\right][0, 2\mathbf{i}] \\
&= \left[0, 2\frac{\sqrt{2}}{2}\mathbf{i} + 2\frac{\sqrt{2}}{2}\mathbf{k} \times \mathbf{i}\right] \\
&= [0, \sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}]
\end{aligned}
$$



We should maintain the magnitude

$$|p'| = \sqrt{\sqrt{2}^2 + \sqrt{2}^2} = 2$$

(Quaternions work in 4D,
drawing 3D is actually not correct.)

Arbitrary Rotation Vector

Let's consider a vector *p* that we want to rotate,
an pure quaternion:

$$p = [0, 2\mathbf{i}]$$

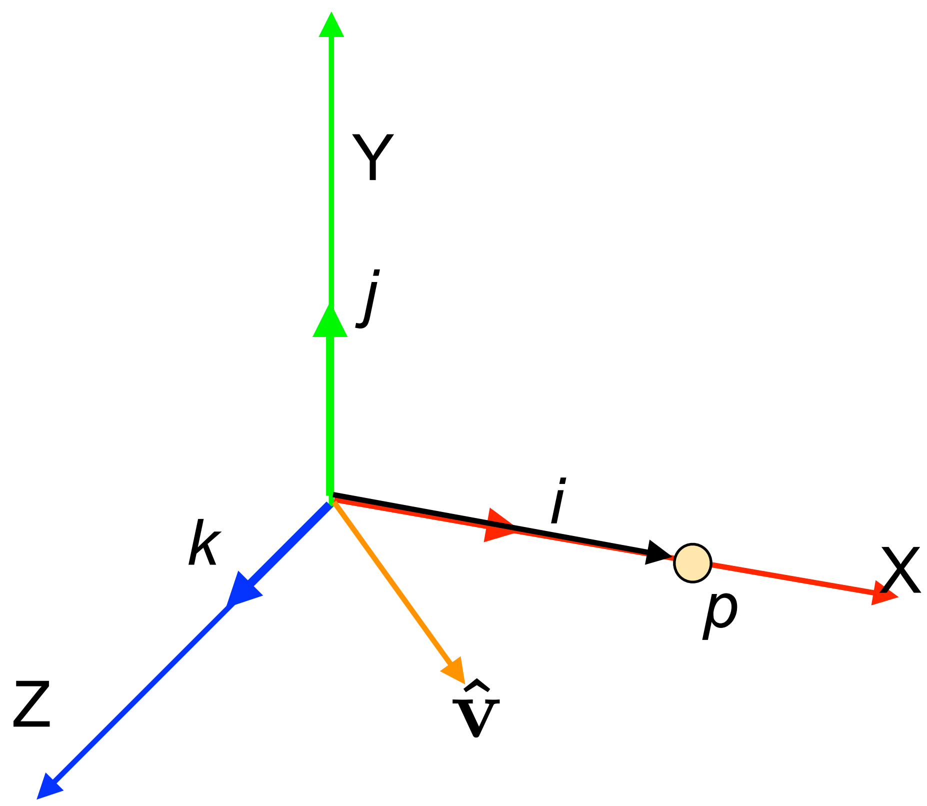
$$\mathbf{p} = 2\mathbf{i}$$

we want to rotate θ = 45° around $\hat{\mathbf{v}}$

$$\hat{\mathbf{v}} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{k}$$

with

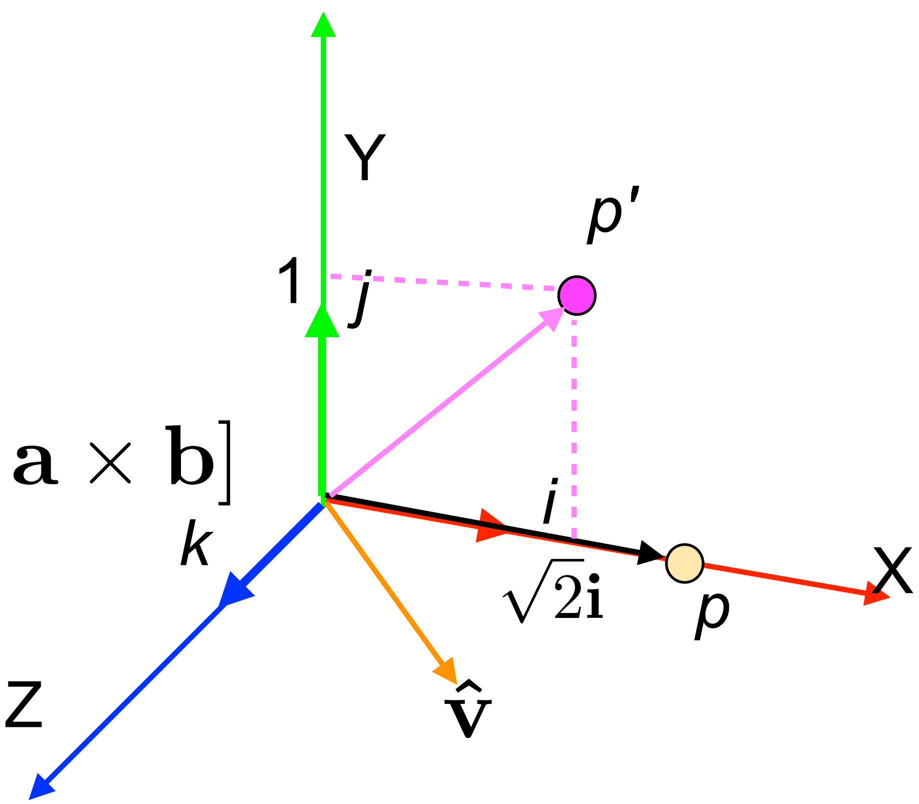
$$q = [\cos\theta, \sin\theta\hat{\mathbf{v}}]$$



Arbitrary Rotation Vector

Calculate

$$
\begin{aligned}
p' &= qp \\
&= [\cos\theta, \sin\theta\hat{\mathbf{v}}][0, \mathbf{p}] \\
&= [-\sin\theta\hat{\mathbf{v}}\cdot\mathbf{p}, \cos\theta\mathbf{p} + \sin\theta\hat{\mathbf{v}}\times\mathbf{p}]
\end{aligned}
$$



Multiplication

$$[s_a, \mathbf{a}][s_b, \mathbf{b}] = [s_a s_b - \mathbf{a}\cdot\mathbf{b}, s_a\mathbf{b} + s_b\mathbf{a} + \mathbf{a}\times\mathbf{b}]$$

Substituting θ and **p** and $\hat{\mathbf{v}}$

$$
\begin{aligned}
p' &= \left[-\frac{\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{k}\right)\cdot(2\mathbf{i}), \frac{\sqrt{2}}{2}2\mathbf{i} + \frac{\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{k}\right)\times 2\mathbf{i}\right] \\
&= [-1, \sqrt{2}\mathbf{i} + \mathbf{j}]
\end{aligned}
$$

Arbitrary Rotation Vector

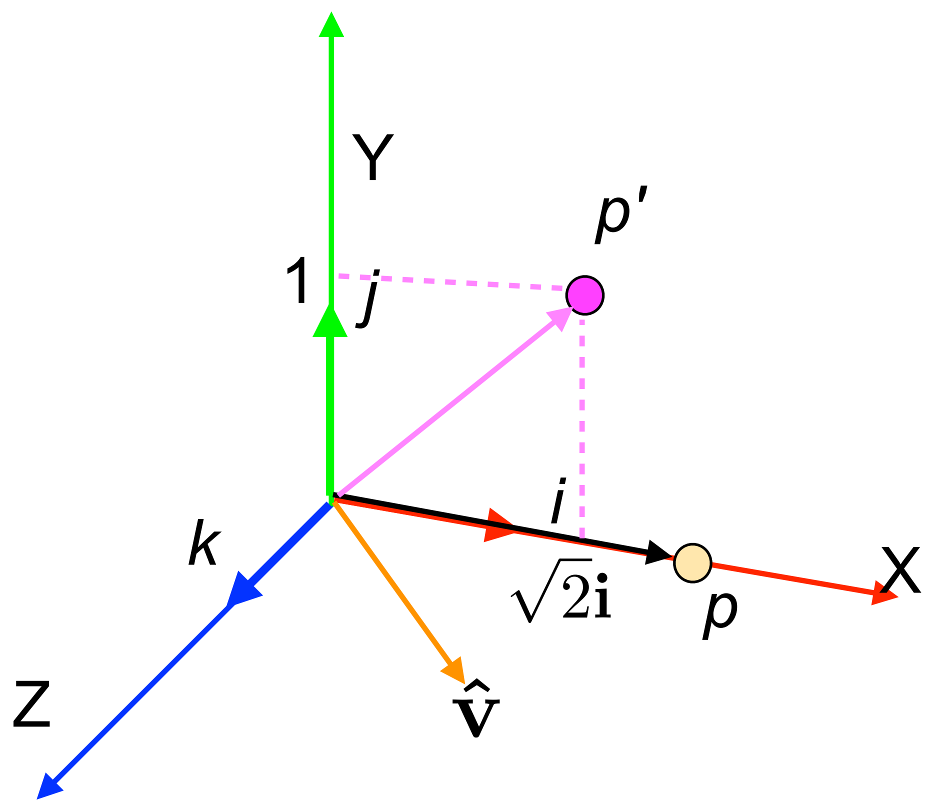
This does not look right to me.

$$p' = [-1, \sqrt{2}\mathbf{i} + \mathbf{j}]$$

The length of p' is also $\sqrt{3}$ instead of 2

now?

we have to get it back to 2.



Arbitrary Rotation Vector

We post-multiply the solution with the inverse of *q*

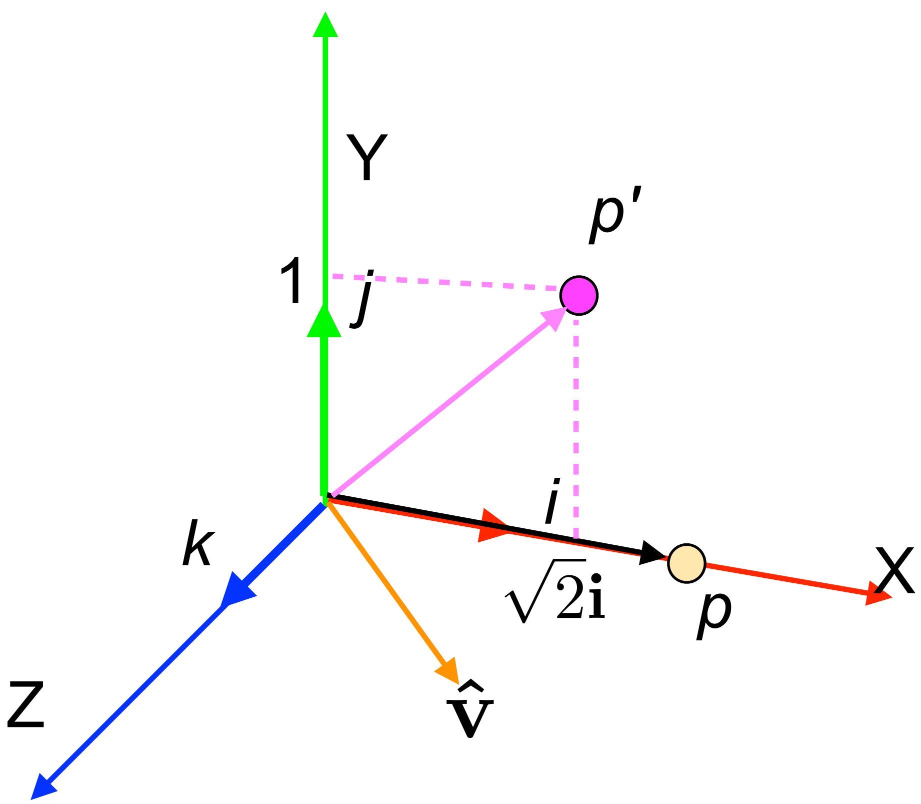
$$q = \left[\cos\theta, \sin\theta\left(\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{k}\right)\right]$$

Let's compute the inverse

$$q^{-1} = \left[\cos\theta, -\sin\theta\left(\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{k}\right)\right]$$

for 45º

$$q^{-1} = \left[\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{k}\right)\right]$$
$$= \frac{1}{2}[\sqrt{2}, -\mathbf{i} - \mathbf{k}]$$



Arbitrary Rotation Vector

Now we compute

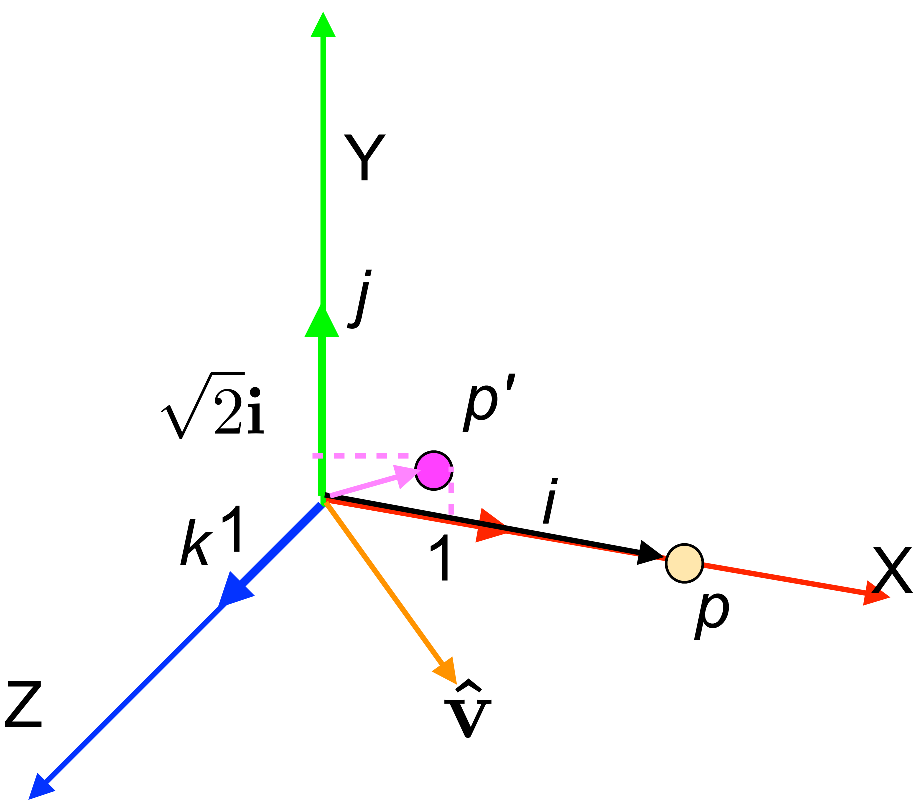
$$qp = [-1, \sqrt{2}\mathbf{i} + \mathbf{j}]$$

$$qpq^{-1} = [-1, \sqrt{2}\mathbf{i} + \mathbf{j}]\frac{1}{2}[\sqrt{2}, -\mathbf{i} - \mathbf{k}]$$

$$= \ldots$$

$$= [0, \mathbf{i} + \sqrt{2}\mathbf{j} + \mathbf{k}]$$

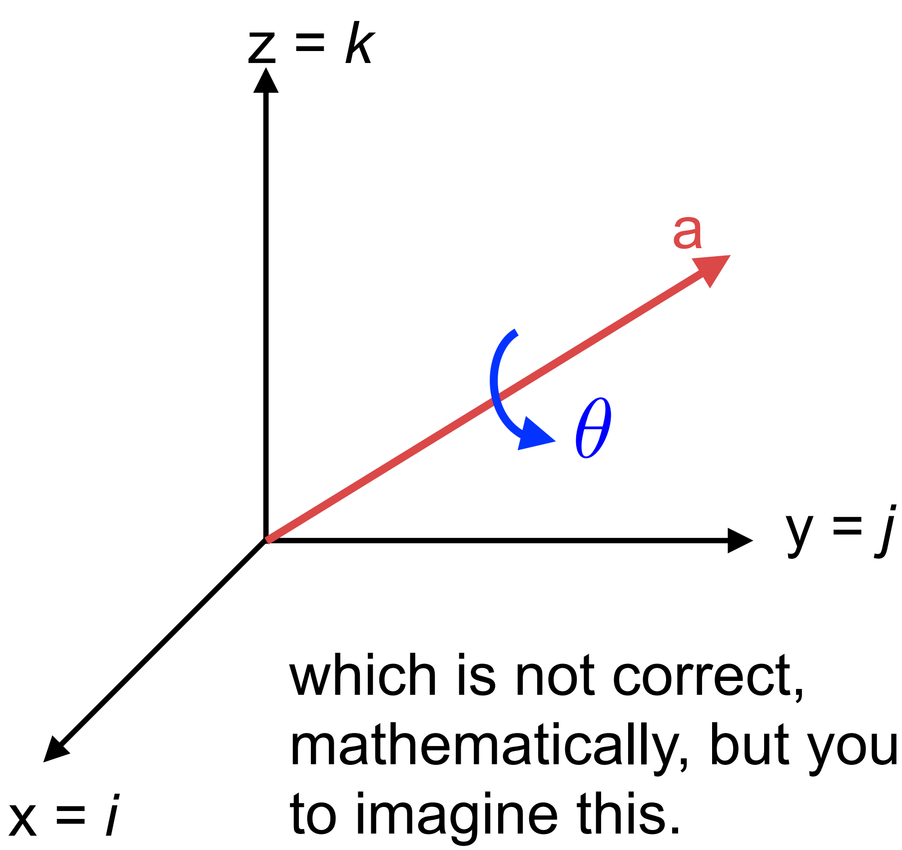
The length is 2 again



Quaternions and Rotations

A quaternion can represent a rotation by an angle θ around a unit axis **a**:

$$q = \left[\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\hat{\mathbf{v}}\right] = \left[\cos\frac{\theta}{2}, \underbrace{\sin\frac{\theta}{2}i + \sin\frac{\theta}{2}j + \sin\frac{\theta}{2}k}_{\text{The “projections“ of a to x, y, and z}}\right]$$



which is not correct, mathematically, but you may want to imagine this.

- the three complex numbers represent a vector in 3d space. A quaternion in is a vector in $\mathbb{R}^4$ is a vector in $\mathbb{R}^3$, if the scalar value is 0.
- the scalar value theta represents the rotation about this vector a.

Thank you!

Questions

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