

ARLAB

ME/CprE/ComS 557

Computer Graphics and Geometric Modeling

Spline Modeling

December 3rd, 2015

Rafael Radkowski

IOWA STATE UNIVERSITY
OF SCIENCE AND TECHNOLOGY

Final Presentation

ARLAB

Thursday, Dec 10th, 2015, 2:10 - 3:30

- Show what you have to this points.
- Add sketches, concept drawings, etc. if cannot render anything at this point.

8 (min) -10 minutes presentation per team

- Introduce your team
 - Introduce your application, especially the goal of the application. What is the user intend to do.
 - Explain the objects in your scene and their behavior
 - Explain the user interaction
 - Highlight the most sophisticated techniques you used.
 - Describe your experience.
-
- Test your presentations / computer in class next Tuesday.
 - Online students: please record a video

Due date is still:

Due date: Friday, Dec. 18th, 2015, 8 pm



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COURSE ANNOUNCEMENT

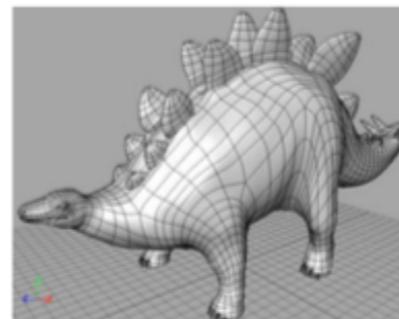
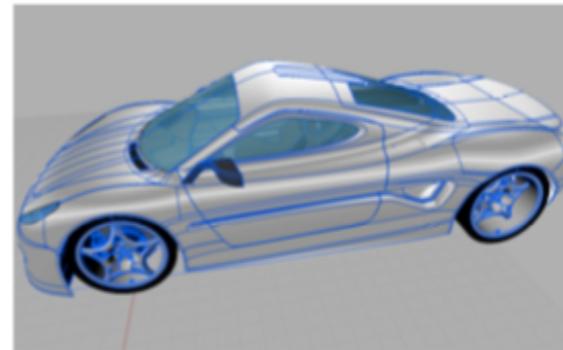
ME 625
Surface Modeling

Spring 2016

Instructor:
James Oliver

Surface modeling technologies are at the core of all contemporary computer shape modeling tools, spanning applications as diverse as mechanical, industrial, aerospace, naval, architectural, and even apparel design. This course explores the theory and practice of contemporary parametric sculptured surface design. Topics include:

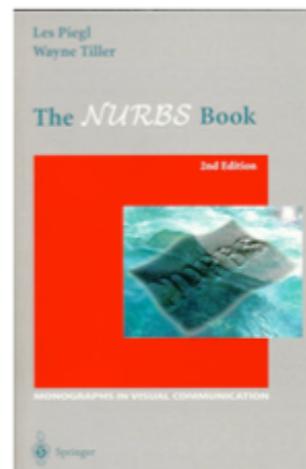
- Curve and surface basics
- B-spline curves and surfaces
 - Basis functions
 - Derivatives
- Rational B-spline curves and surfaces
 - Conics
 - Re-parameterization
- Geometric tools
 - Knot insertion
 - Degree elevation
- Construction Techniques
- Trimmed surfaces
- Interpolation and Fitting
- Point Inversion and projection
- Shape modification
- Surface/surface intersection
- Applications – design, mesh generation, NC milling, integration with solid modeling systems, high-dimensional NURBS, others.



Text: *The NURBS Book*, by Les Piegl and Wayne Tiller, Springer-Verlag

Meeting Time: T, TH 11:00 – 12:15

Also available via Engineering/LAS Online Education



Content

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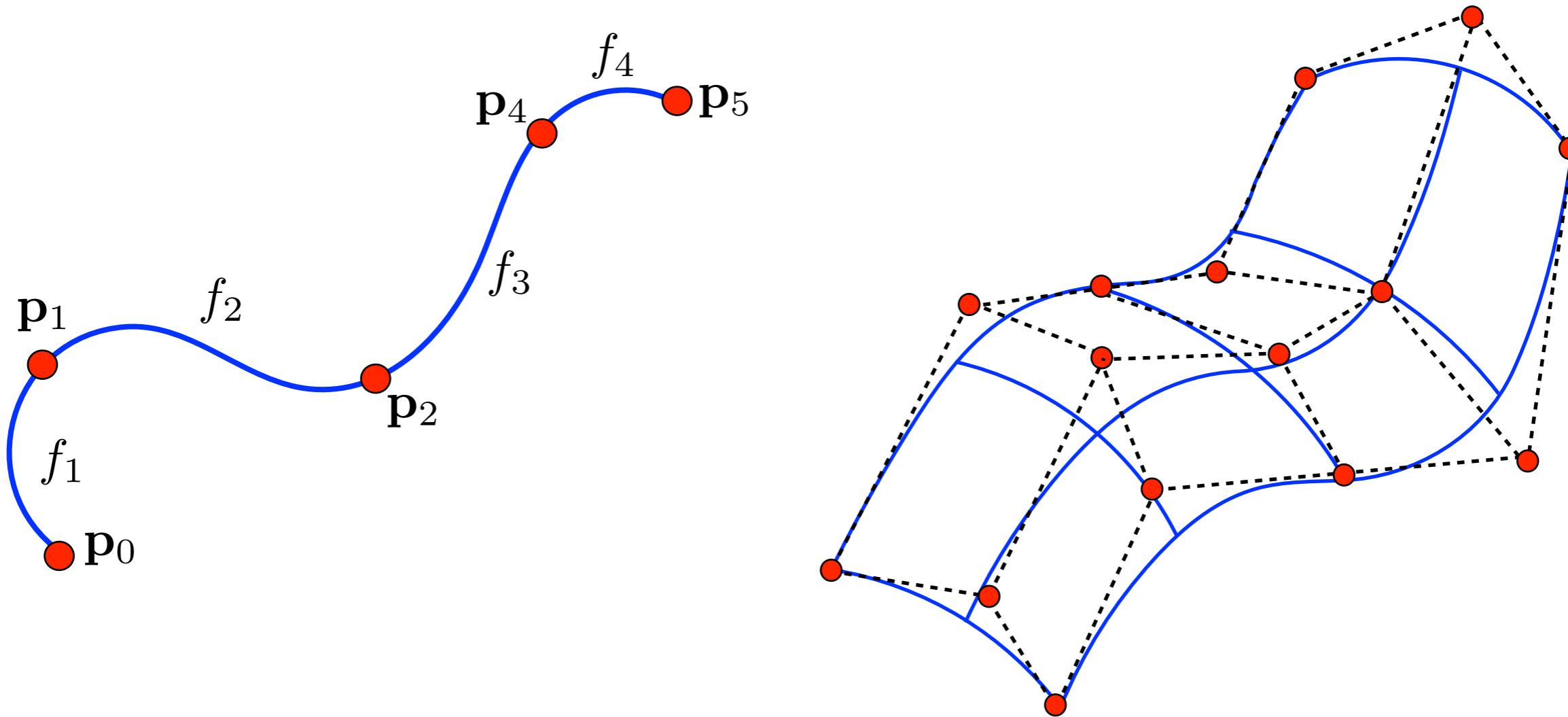
- Splines
- Terminology
- Cubic Spline
- Bezier Splines

next time:

- surfaces
- drawing

In modeling terminology, a spline curve is a flexible strip used to produce a smooth curve through a designated set of points. Mathematically, we can describe this curve with piecewise cubic polynomial functions.

A spline surface can be described with two sets of orthogonal spline curves.

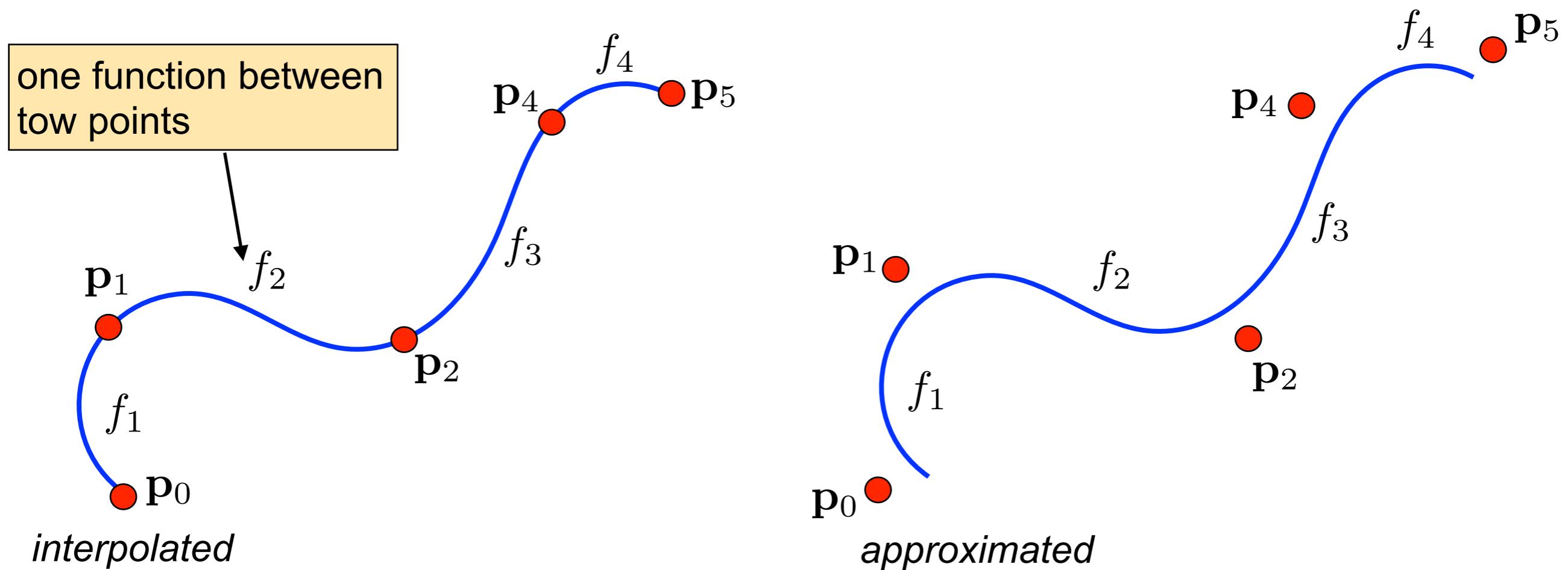


Interpolated and Approximated

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We specify a spline using **control points**

If the resulting curve fits with the control points, we talk about an **interpolated** curve.
If the control points are not part of the curve but determine the general direction, we talk about an **approximated** curve.



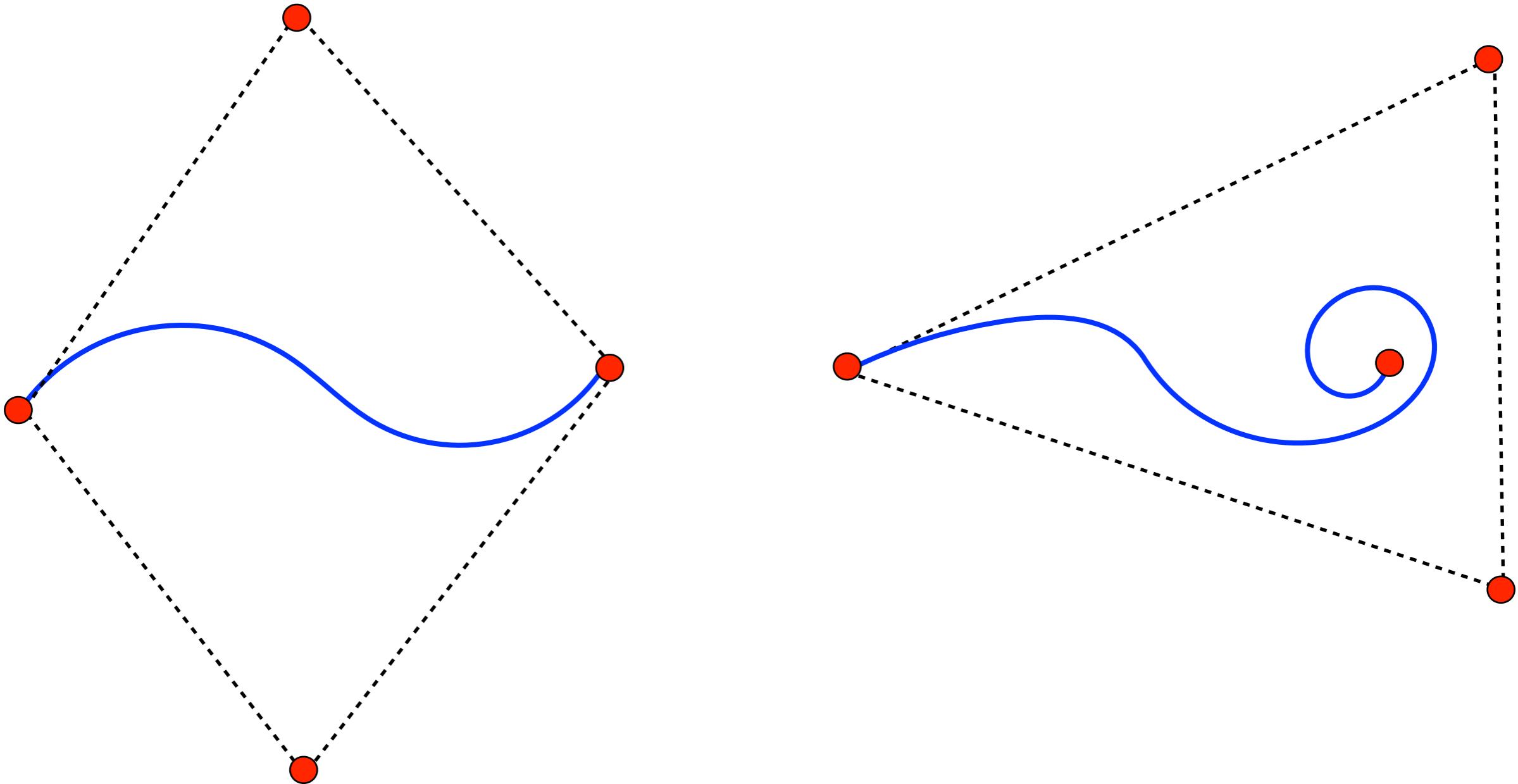
interpolated

f_1 :function

Convex Hull

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If a set of control points forms the boundaries for a region in space, we call this a **convex hull**.



Parametric Continuity Conditions

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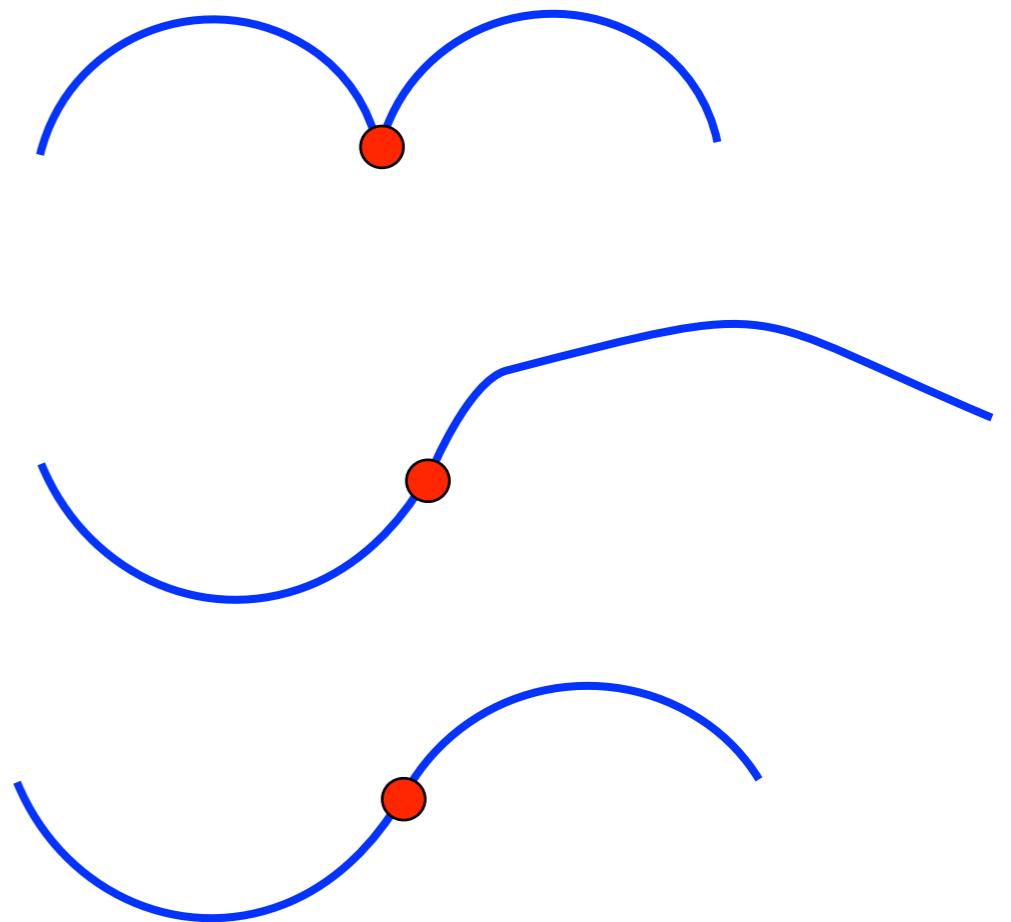
We must ensure a smooth transition between the different spline functions.

Therefore, we impose **continuity conditions**. Let's presume each section of the spline is represented by a set of parametric condition of the form

$$x = x(u), \quad y = y(u), \quad z = z(u), \quad u_1 \leq u \leq u_2$$

Zero-order parametric continuity C^0

means, the curves meet



First-order parametric continuity C^1

means, the first derivative (tangent lines) are equal at the meeting point

Second-order parametric continuity C^2

means, the first and second derivative are equal at the meeting point

Geometric Continuity Conditions

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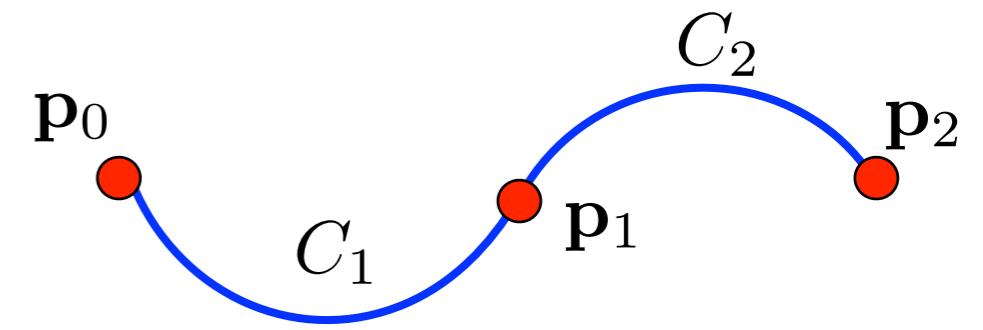
Geometric continuity conditions require that the parametric derivate are **proportional** to each other (instead of equality).

Zero-order geometric continuity G^0
means, the curves meet

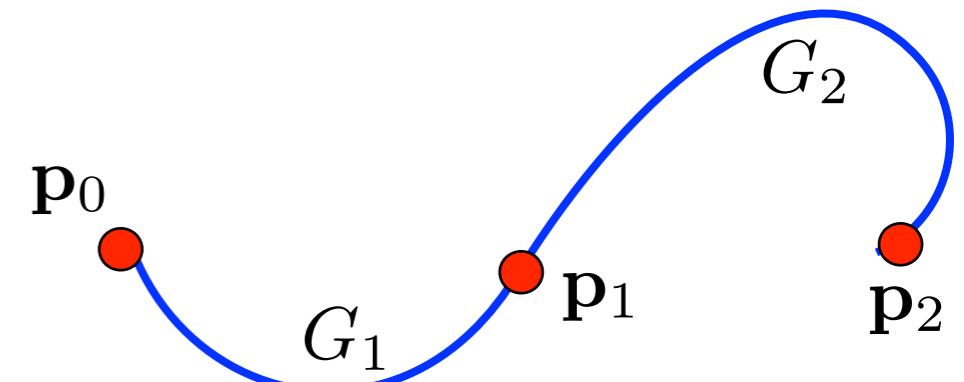
First-order geometric continuity G^1
means, the first derivative (tangent lines) are
proportional at the intersection

Second-order geometric continuity G^2
means, the first and second derivative are
proportional at the intersection point

The magnitude of all derivate can vary.



Parametric continuity



Geometric continuity

Spline Specification

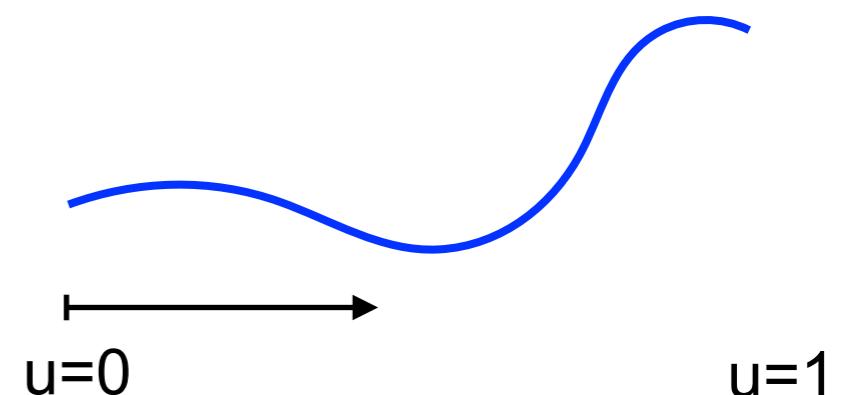
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Let's start with a polynomial representation of a curve

$$x(u) = a_x u^3 + b_x u^2 + c_x u + d_x, \quad 0 \leq u \leq 1$$

Rewritten in matrix form.

$$x(u) = [\begin{array}{cccc} u^3 & u^2 & u & 1 \end{array}] \begin{bmatrix} a_x \\ b_x \\ c_x \\ d_x \end{bmatrix}$$
$$= \mathbf{U} \cdot \mathbf{C}$$



We always run from
 $u = [0, 1]$

Let's split C into

$$\mathbf{C} = \mathbf{M}_{\text{spline}} \cdot \mathbf{M}_{\text{geom}}$$

where \mathbf{M}_{geom} is a four-element column matrix containing the geometric constraints (boundary conditions).

and $\mathbf{M}_{\text{spline}}$ is a 4x4 matrix that transforms the geometric constraints into the polynomial coefficients a_x, b_x, c_x, d_x .

Spline Specification



Thus, we finally represent a spline as

$$x(u) = \mathbf{U} \cdot \mathbf{M}_{spline} \cdot \mathbf{M}_{geom}$$

The matrix \mathbf{M}_{spline} characterizes the spline representation. It is often called the **basis matrix**.

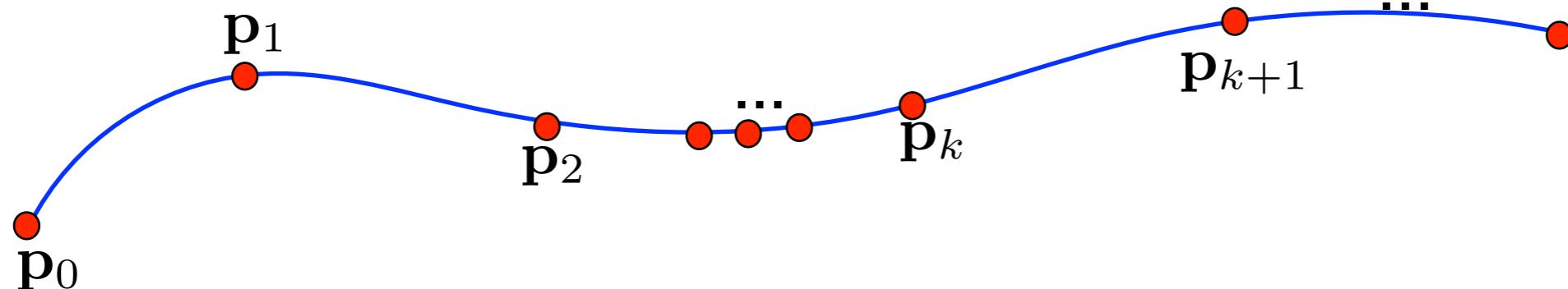
In other words: a **spline basic matrix** is a function that allows us to determine the coefficients a, b, c, and d, using control points / boundary conditions in \mathbf{M}_{geom}

Cubic Spline Interpolation Methods

Given a set of control points, cubic splines are obtained by fitting the input points into with a piecewise cubic polynomial curve that passes through every control point.

With n curve sections, suppose we have $n+1$ points with the coordinates

$$\mathbf{p}_k = \{x_k, y_k, z_k\}, \quad k = 0, 1, 2, \dots, n$$



We can describe a parametric curve that fits between each pair of control points with the following equation

$$x(u) = a_x u^3 + b_x u^2 + c_x u + d_x$$

$$y(u) = a_y u^3 + b_y u^2 + c_y u + d_y \quad 0 \leq u \leq 1$$

$$z(u) = a_z u^3 + b_z u^2 + c_z u + d_z$$

Cubic Spline Interpolation Methods

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- For each of these three equations, we need to compute the values for the coefficients a, b, c, d .
- We do this by setting enough boundary conditions.

Advantages

of cubic splines in comparison to quadratic polynomial splines

- flexible
- fast computation
- less memory
- more stable

Different Methods

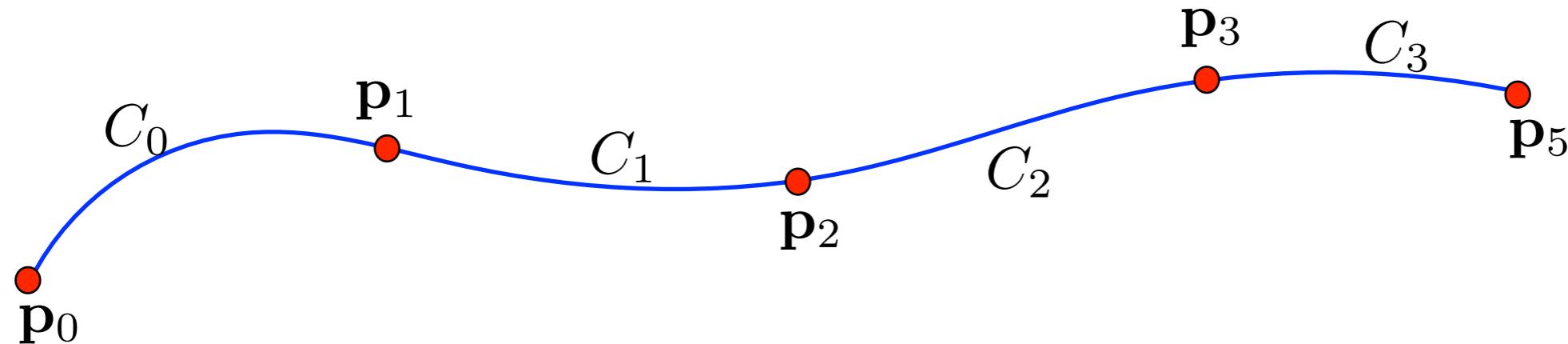
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- Natural Cubic Spline
- Hermite Spline
- Cardinal Splines
- Kochanek-Bartels Splines

Natural Cubic Spline

It was the first, but obsolete today.

- Mathematic representation of the original draft spline
- Suppose we have four spline sections n, and n+1 control points



- This results in $4n$ polynomial coefficients a, b, c , and d .
- We have four boundary conditions. Each intersecting curve at each control point must have the same boundary condition.
- This gives us 3 equations + 1 equation for the start point.
- Result in $4n - 4$ equations.
- Disadvantage: change of one control point requires recalculating everything along the entire system of equations.

We do not want to
recalculate everything
because we move one
control point.

Hermite Spline

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A Hermite (after the French mathematician Charles Hermite) spline is a piecewise cubic polynomial function with a specified tangent at each control point.

The spline can be adjusted locally because it only depends on its endpoint constraints.

$\mathbf{P}(u)$ represents a parametric cubic point function for the curve section between the points p_k and p_{k+1}

Then, the boundary conditions are defined by:

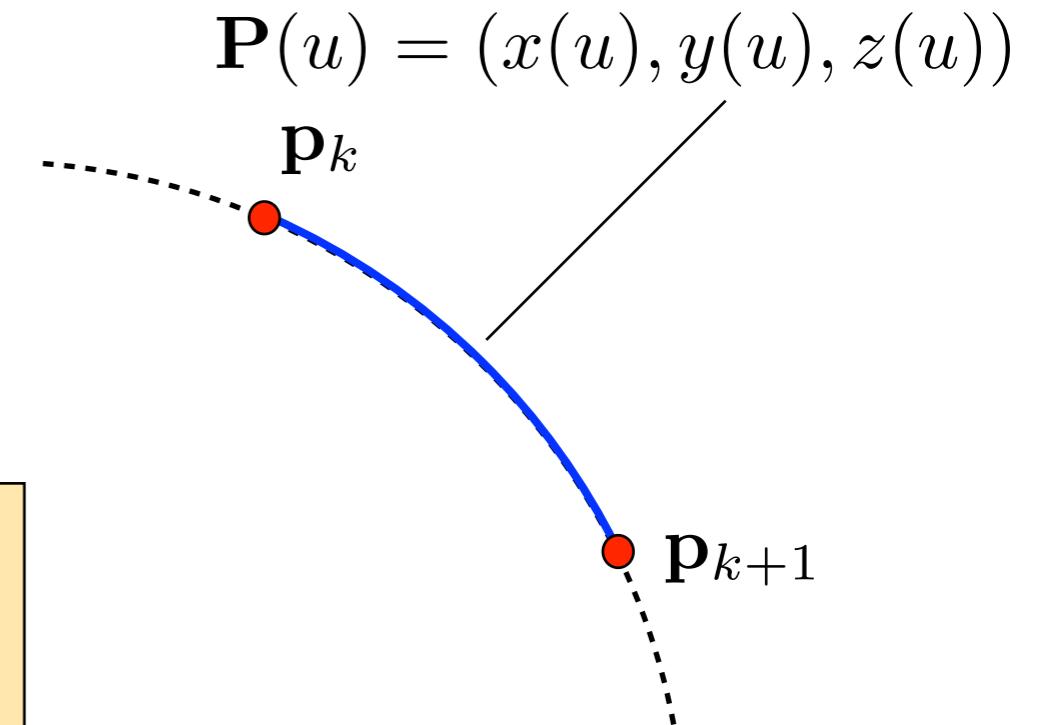
$$\mathbf{P}(0) = \mathbf{p}_k$$

$$\mathbf{P}(1) = \mathbf{p}_{k+1}$$

$$\mathbf{P}'(0) = \mathbf{D} \mathbf{p}_k$$

$$\mathbf{P}'(1) = \mathbf{D} \mathbf{p}_{k+1}$$

Note, we need to specify these conditions = points + endpoint slopes



with $\mathbf{D} \mathbf{p}_k$, $\mathbf{D} \mathbf{p}_{k+1}$ the first derivative, the slope of the curve.

Hermite Spline

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Add the polynomial curve

$$\mathbf{P}(u) = \mathbf{a} u^3 + \mathbf{b} u^2 + \mathbf{c} u + \mathbf{d}, \quad 0 \leq u \leq 1$$

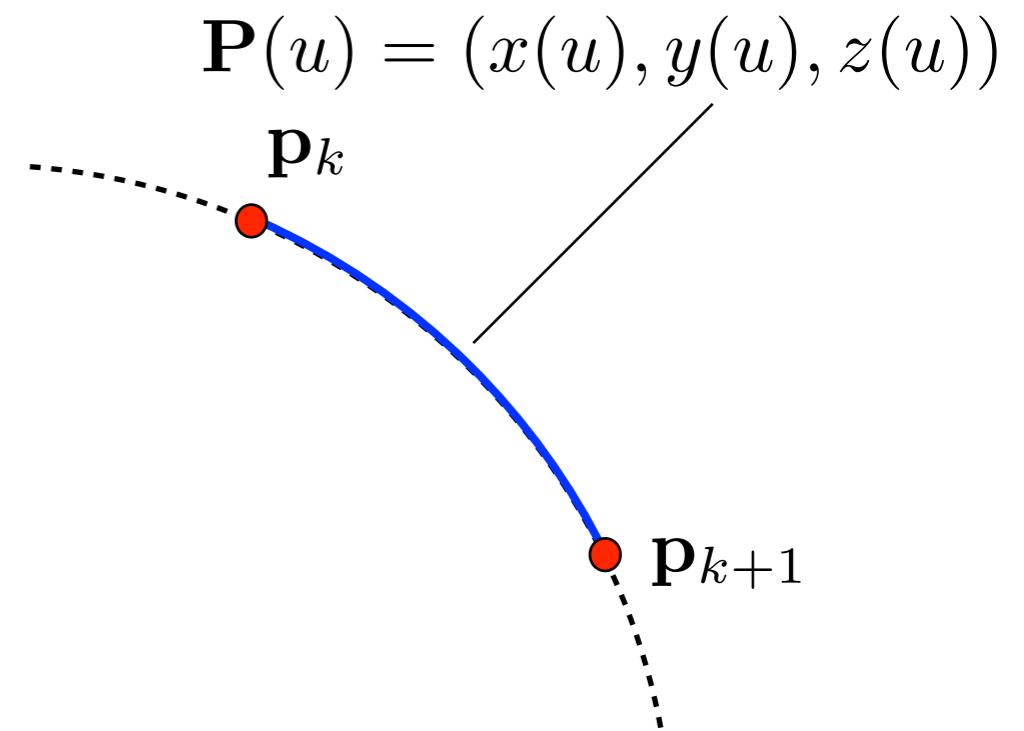
note, **a**, **b**, **c**, and **d** in vector form to represent the components for x, y, z

In matrix form

$$\mathbf{P}(u) = [\begin{array}{cccc} u^3 & u^2 & u & 1 \end{array}] \cdot \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \end{bmatrix}$$

and the first derivative

$$\mathbf{P}'(u) = [\begin{array}{cccc} 3u^2 & 2u^1 & 1 & 0 \end{array}] \cdot \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \end{bmatrix}$$



Hermite Spline

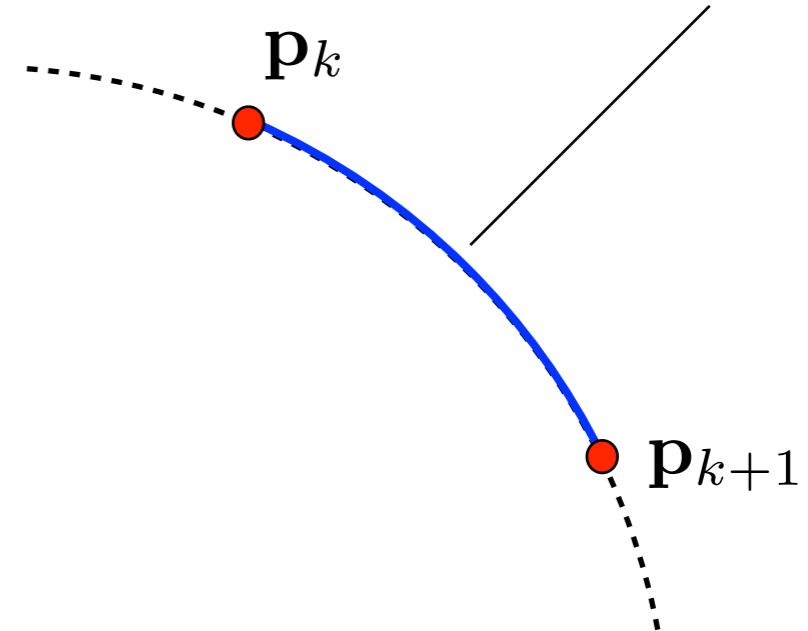
Substitute the endpoint values 0 and 1 for the parameter u , we can express the boundary conditions in matrix form.

$$\begin{bmatrix} p_k \\ p_{k+1} \\ D p_k \\ D p_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

solving for the parameters

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix}^{-1} \cdot \begin{bmatrix} p_k \\ p_{k+1} \\ D p_k \\ D p_{k+1} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} p_k \\ p_{k+1} \\ D p_k \\ D p_{k+1} \end{bmatrix}$$

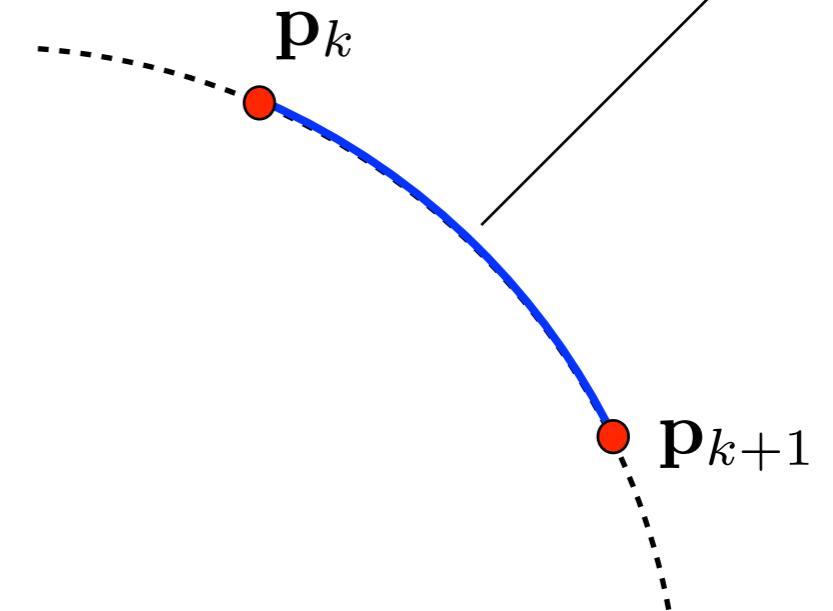


Hermite Spline

$$= \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{p}_k \\ \mathbf{p}_{k+1} \\ \mathbf{D} \mathbf{p}_k \\ \mathbf{D} \mathbf{p}_{k+1} \end{bmatrix}$$

$$= \mathbf{M}_H \cdot \begin{bmatrix} \mathbf{p}_k \\ \mathbf{p}_{k+1} \\ \mathbf{D} \mathbf{p}_k \\ \mathbf{D} \mathbf{p}_{k+1} \end{bmatrix}$$

$$\mathbf{P}(u) = (x(u), y(u), z(u))$$



with \mathbf{M}_H , the **Hermite matrix**, which is the inverse of the boundary constraint matrix, thus can be rewritten as:

$$\mathbf{P}(u) = [u^3 \quad u^2 \quad u \quad 1] \cdot \mathbf{M}_H \cdot \begin{bmatrix} \mathbf{p}_k \\ \mathbf{p}_{k+1} \\ \mathbf{D} \mathbf{p}_k \\ \mathbf{D} \mathbf{p}_{k+1} \end{bmatrix}$$

before, we called this: $\mathbf{C} = \mathbf{M}_{\text{spline}} \cdot \mathbf{M}_{\text{geom}}$

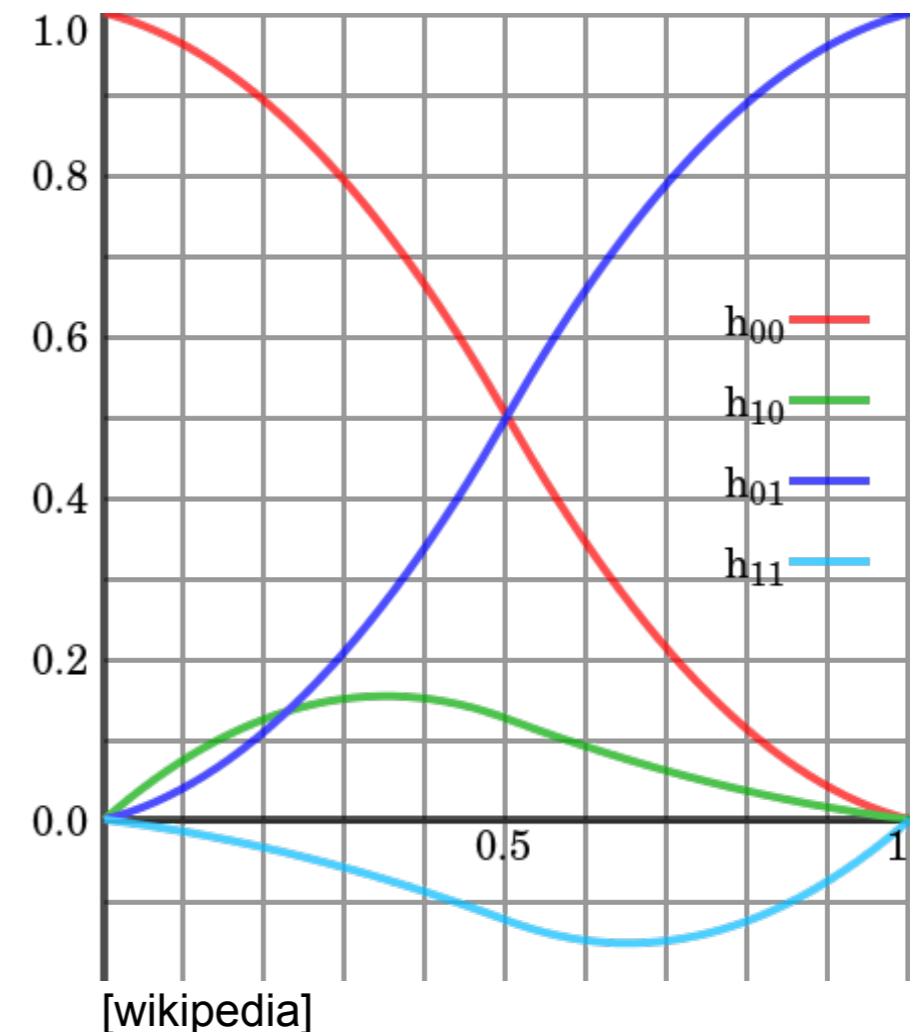
Hermite Spline

We can express this a polynomial function:

$$\begin{aligned}\mathbf{P}(u) &= \mathbf{p}_k(2u^3 - 3u^2 + 1) + \mathbf{p}_{k+1}(-2u^3 + 3u^2) + \\ &\quad \mathbf{D} \mathbf{p}_k(u^3 - 2u^2 + u) + \mathbf{D} \mathbf{p}_{k+1}(u^3 - u^2) \\ &= \mathbf{p}_k H_0 + \mathbf{p}_{k+1} H_1 + \mathbf{D} \mathbf{p}_k H_2 + \mathbf{D} \mathbf{p}_{k+1} H_3\end{aligned}$$

with H_i , the **Hermite basis function**

The result in each subinterval is a linear combination of these four functions (u)



To draw the line:

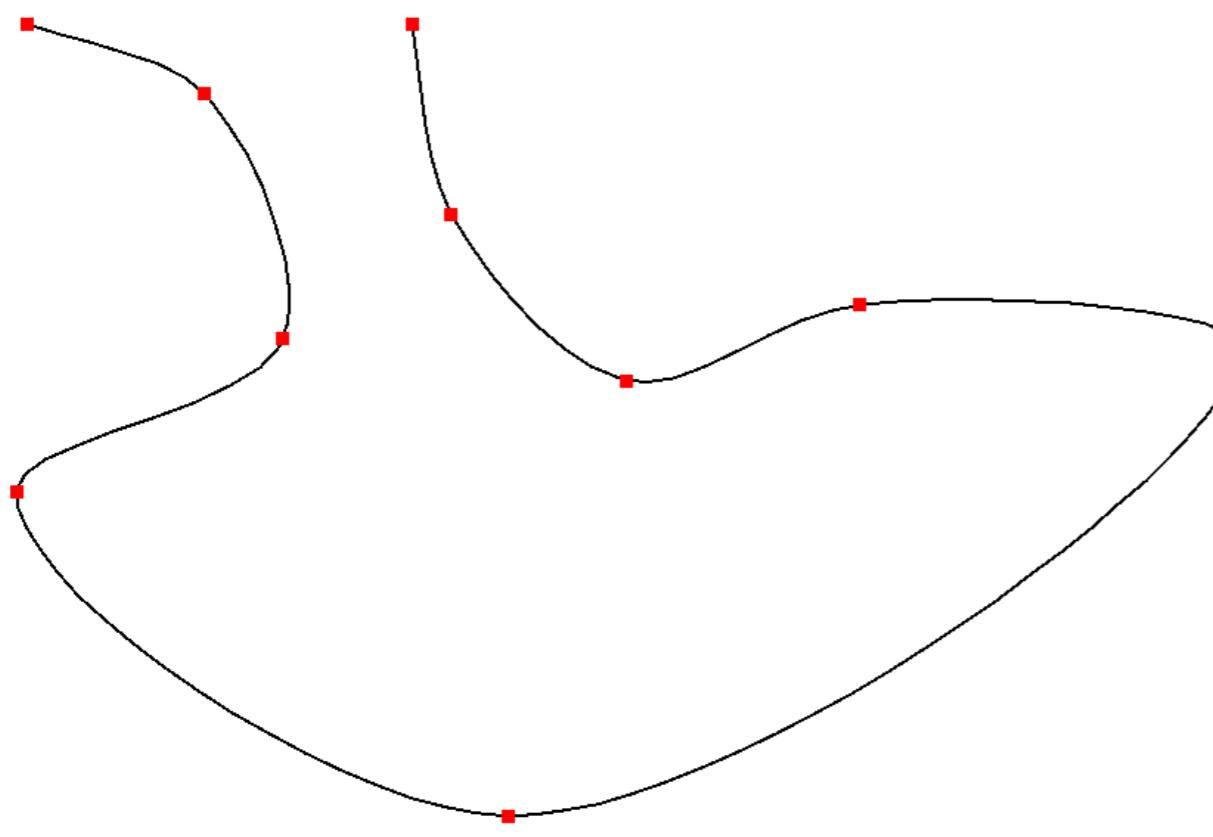
- Set the control points
- calculate for all u : $0 \leq u \leq 1$
- draw each result -> line

What does this chart tell us?

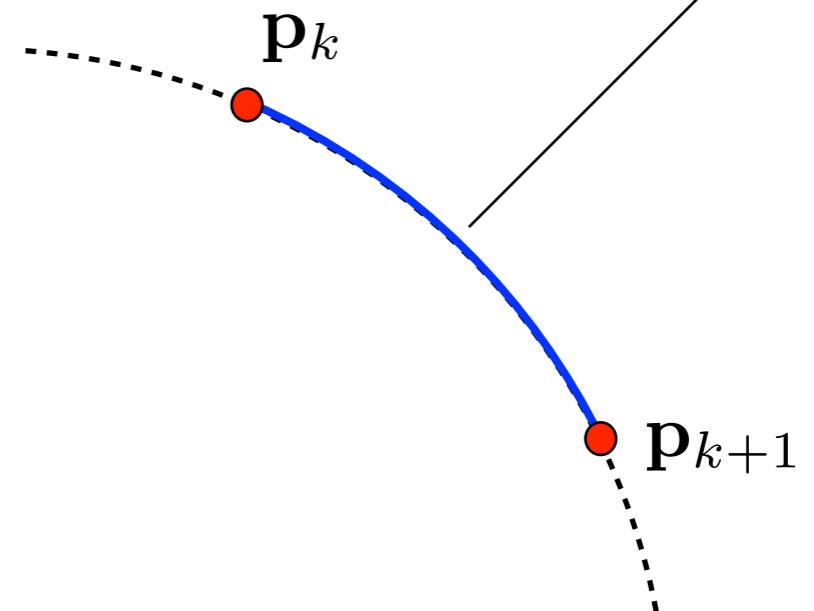
Hermite Spline

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Useful for all curves where one can specify the curve slope



$$\mathbf{P}(u) = (x(u), y(u), z(u))$$



We need to specify the **parametric** conditions for all control points.

$$\mathbf{P}(0) = \mathbf{p}_k$$

$$\mathbf{P}(1) = \mathbf{p}_{k+1}$$

$$\mathbf{P}'(0) = \mathbf{D} \mathbf{p}_k$$

$$\mathbf{P}'(1) = \mathbf{D} \mathbf{p}_{k+1}$$

Cardinal Splines

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Similar to the Hermite spline, the Cardinal Spline is a piecewise cubic polynomial function with a specified tangent at each control point. The difference is that we do not have to specify the endpoint tangents, the slope at the control point is calculated from the two adjacent control points.

We use the boundary conditions

$$\mathbf{P}(0) = \mathbf{p}_k$$

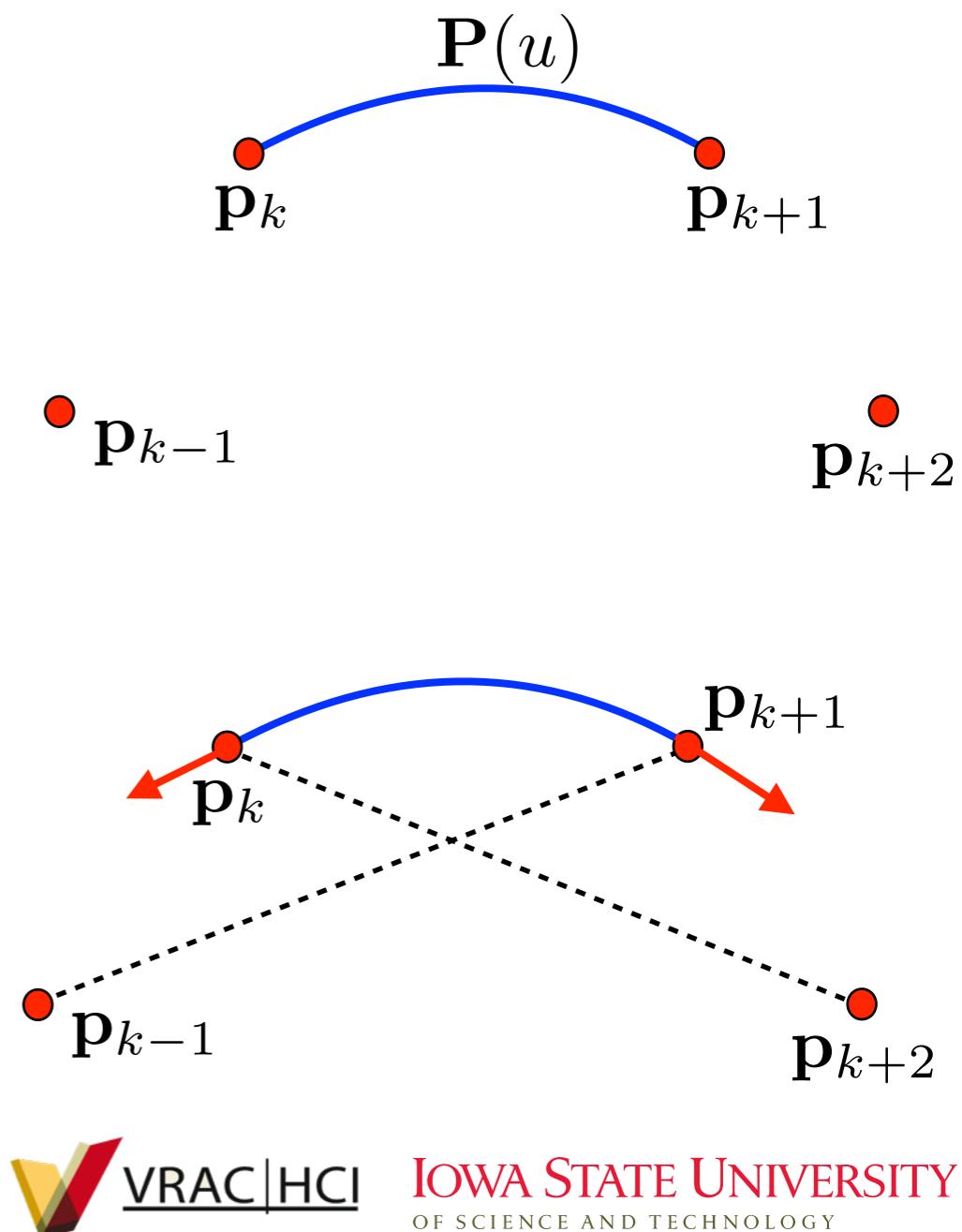
$$\mathbf{P}(1) = \mathbf{p}_{k+1}$$

$$\mathbf{P}'(0) = \frac{1}{2}(1-t)(\mathbf{p}_{k+1} - \mathbf{p}_{k-1})$$

$$\mathbf{P}'(1) = \frac{1}{2}(1-t)(\mathbf{p}_{k+2} - \mathbf{p}_k)$$

Thus, the slopes at the control points are proportional to the chords

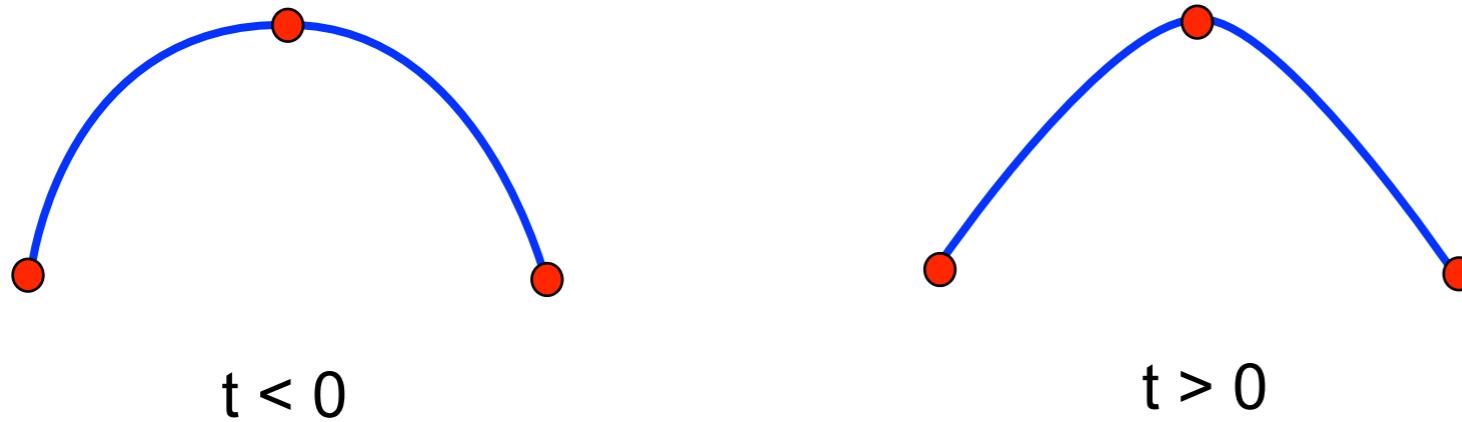
$$\overline{\mathbf{p}_{k+1} \mathbf{p}_{k-1}} \quad \text{and} \quad \overline{\mathbf{p}_k \mathbf{p}_{k+2}}$$



Cardinal Splines

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The parameter t is called the tension parameter: tight corners to round corners



with $t = 0$: Catmull-Rom splines

Cardinal Splines

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Starting with the equation

$$\mathbf{P}(u) = [\ u^3 \quad u^2 \quad u \quad 1 \] \cdot \mathbf{M}_C \cdot \begin{bmatrix} \mathbf{p}_{k-1} \\ \mathbf{p}_k \\ \mathbf{p}_{k+1} \\ \mathbf{p}_{k+2} \end{bmatrix}$$

the Cardinal matrix \mathbf{M}_C is:

$$\mathbf{M}_C = \begin{bmatrix} -s & 2-s & s-2 & s \\ 2s & s-3 & 3-2s & -s \\ -s & 0 & s & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

with $s = (1-t)/2$

Cardinal Splines

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Rewritten in polynomial form

$$\begin{aligned}\mathbf{P}(u) &= \mathbf{p}_{k-1}(-s u^3 + 2s u^2 - s) + \mathbf{p}_k[(2 - s)u^3 + (s - 3)u^2 + 1] + \\ &\quad \mathbf{p}_{k+1}[(s - 2)u^3 + (3 - 2s)u^2 + s u] + \mathbf{p}_{k+2}(s u^3 - s u^2) \\ &= \mathbf{p}_{k-1}\text{CAR}_0(u) + \mathbf{p}_k\text{CAR}_1(u)\mathbf{p}_{k+1}\text{CAR}_2(u) + \mathbf{p}_{k+2}\text{CAR}_3(u)\end{aligned}$$

with the Cardinal spline function CAR_k

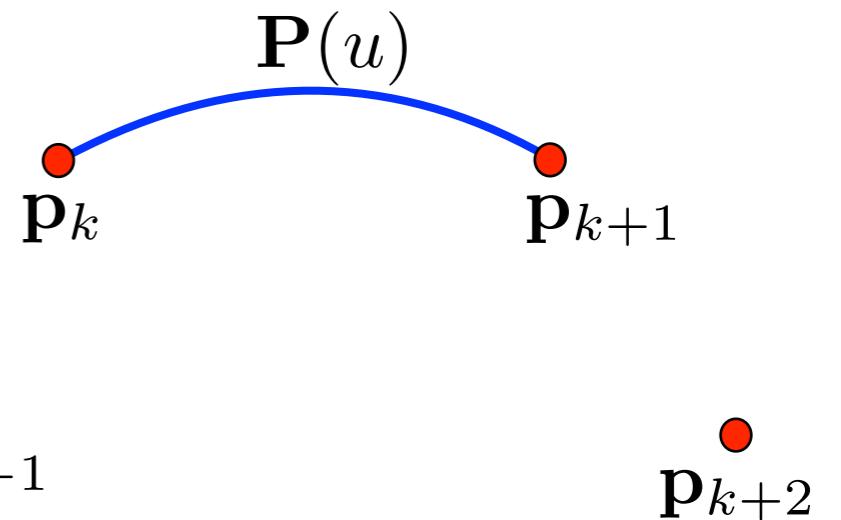
Kochanek-Bartels Splines

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This is an extension of the Cardinal spline.

Two additional parameters are introduced to allow more shape flexibility.

Given the four control points



$$\mathbf{P}(0) = \mathbf{p}_k$$

$$\mathbf{P}(1) = \mathbf{p}_{k+1}$$

$$\begin{aligned} \mathbf{P}'(0)_{in} &= \frac{1}{2}(1-t) [(1+b)(1-c)(\mathbf{p}_k - \mathbf{p}_{k-1}) \\ &\quad + (1-b)(1+c)(\mathbf{p}_{k+1} - \mathbf{p}_k)] \end{aligned}$$

$$\begin{aligned} \mathbf{P}'(1)_{in} &= \frac{1}{2}(1-t) [(1+b)(1+c)(\mathbf{p}_{k+1} - \mathbf{p}_k) \\ &\quad + (1-b)(1-c)(\mathbf{p}_{k+2} - \mathbf{p}_{k+1})] \end{aligned}$$

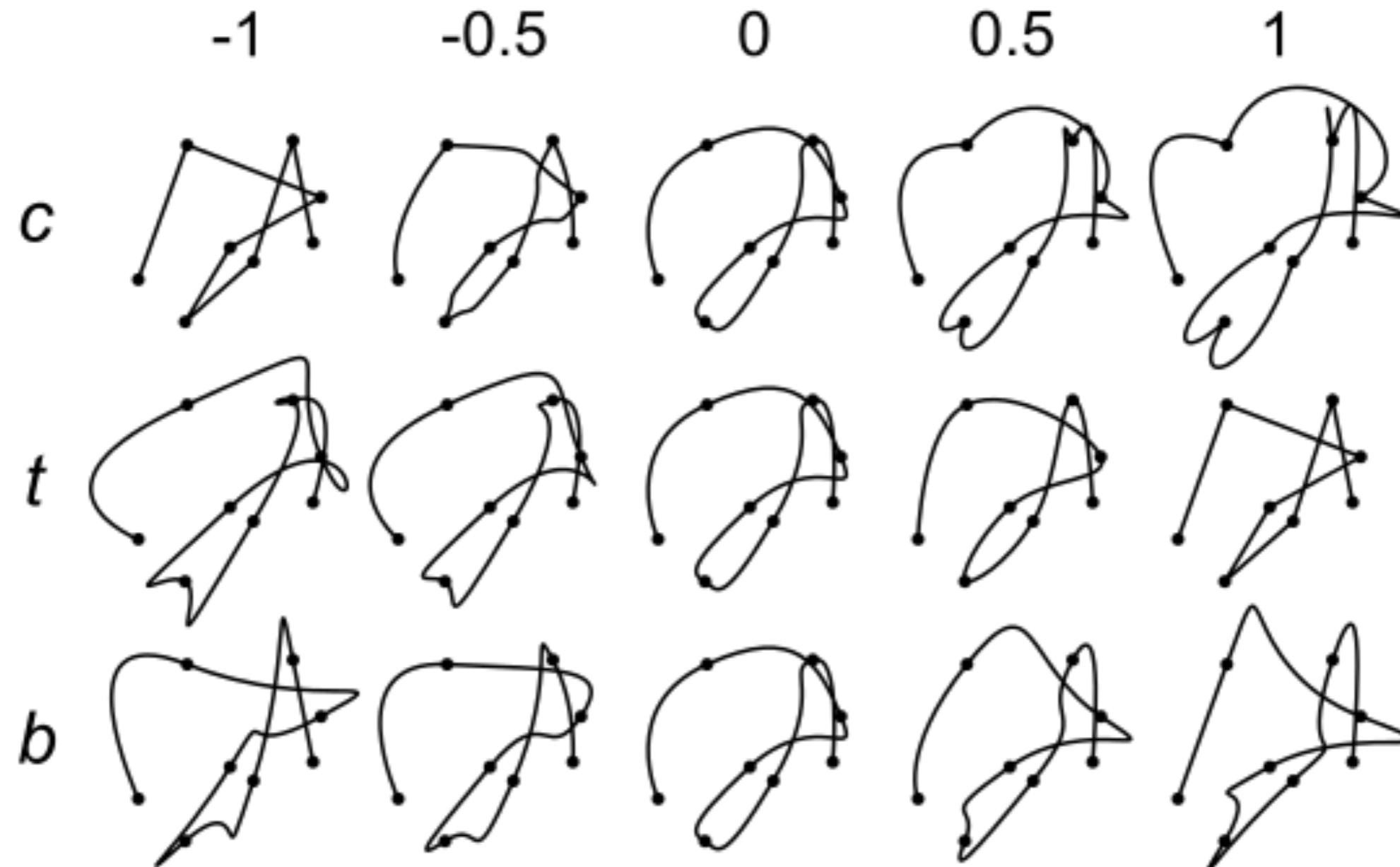
Kochanek-Bartels Splines

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Two new parameters:

b: bias

c: continuity



[wikipedia]

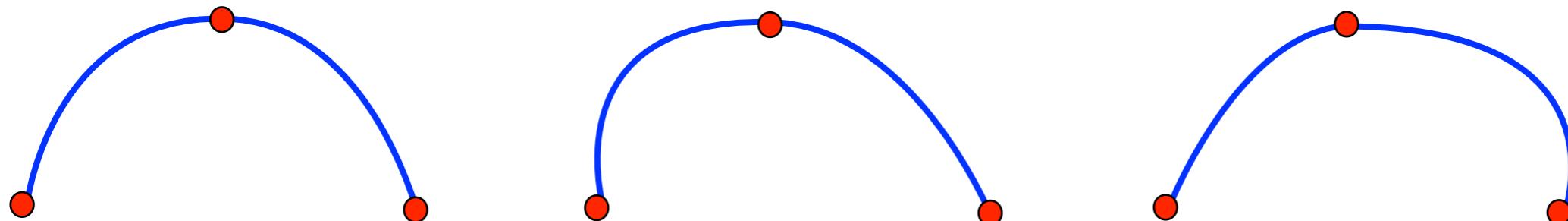
Kochanek-Bartels Splines

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Continuity: round corners to box corners



Bias: post - pre magnitude

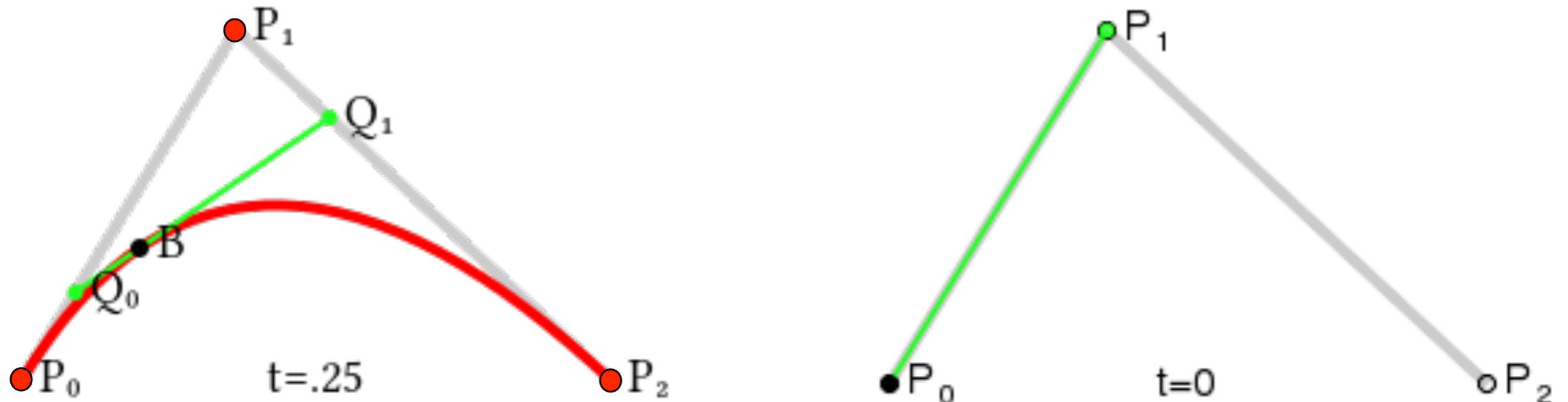


Bezier Spline Curves

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Bezier splines was developed by the French engineer Pierre Bezier for the car design at Renault.

- They can be fitted to any number of control points.
- Bridge computer graphics and computer-aided design
- Can be modified with blending functions, approximated, etc.
- They are very easy to implement.



Quadratic curves, [wikipedia]

Bezier Spline Equation

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We start with the general equation for Bezier Splines of every order!

Given a set of control points for n curve sections, with $n+1$ control points

$$\mathbf{p}_k = \{x_k, y_k, z_k\}, \quad k = 0, 1, 2, \dots, n$$

k, index of the point
n, number of points

We define a position vector to draw a line as

$$\mathbf{P}(u) = \sum_{k=0}^n \mathbf{p}_k \text{BEZ}_{k,n}(u), \quad 0 \leq u \leq 1$$

It describes the path between \mathbf{p}_0 and \mathbf{p}_1

The **Bezier blending function** is the *Bernstein polynomials*

$$\text{BEZ}_{k,n}(u) = C(n, k)u^k(1 - u)^{n-k}$$

with the parameter C:

$$C(n, k) = \frac{n!}{k!(n - k)!}$$



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Bezier Spline Equation

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The vector $\mathbf{P}(u)$ represents the three parametric equations:

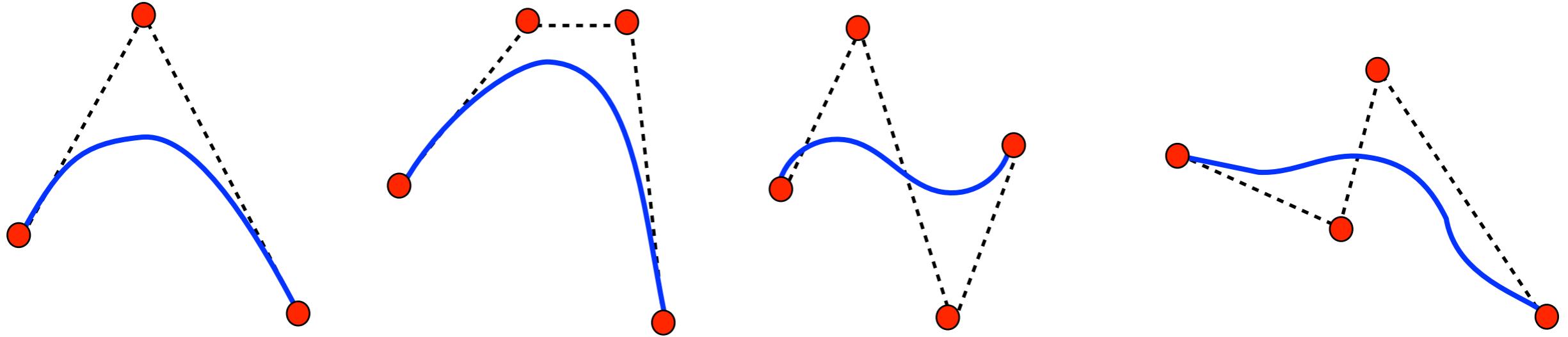
$$x(u) = \sum_{k=0}^n x_k \text{BEZ}_{k,n}(u)$$

$$y(u) = \sum_{k=0}^n y_k \text{BEZ}_{k,n}(u)$$

$$z(u) = \sum_{k=0}^n z_k \text{BEZ}_{k,n}(u)$$

Bezier Spline Equation

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The order of the Bezier curve is $n-1$ (n , number of control points)

- quadratic or 2nd order curve: three points generate a parabola
- cubic or 3rd order curve: four points a cubic curve

etc.

Cubic Bezier Curve

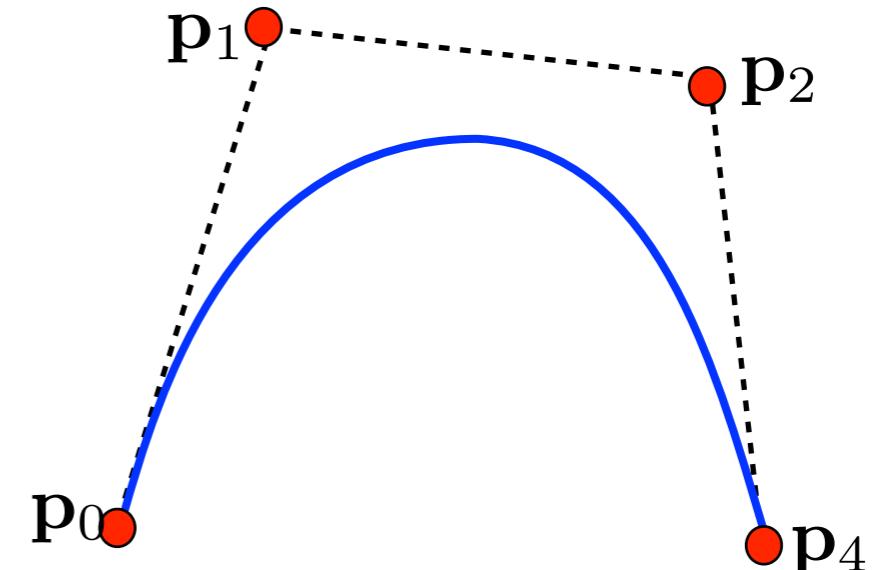
For a Cubic Bezier Function, we use the following blending functions

$$BEZ_{0,3}(u) = (1 - u)^3$$

$$BEZ_{1,3}(u) = 3u(1 - u)^2$$

$$BEZ_{2,3}(u) = 3u^2(1 - u)$$

$$BEZ_{3,3}(u) = u^3$$



The blending function determine how the control points influence the shape.

for $u = 0$: only the first blending function has an impact. Thus, we always begin with p_0

for $u = 1$: only the last blending function has an impact, Thus, we will always end with p_3

Each of this function is non zero in the range between $0 < u < 1$

This also means, we have no local control points. The entire curve is always affected.

Cubic Bezier Curve

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Back to the basic function:

$$\mathbf{P}(u) = \sum_{k=0}^n \mathbf{p}_k \text{BEZ}_{k,n}(u), \quad 0 \leq u \leq 1$$

We need to follow the boundary conditions:

$$\mathbf{P}(0) = \mathbf{p}_0$$

$$\mathbf{P}(1) = \mathbf{p}_n$$

Along with the hire order boundary conditions

$$\mathbf{P}'(0) = -n \mathbf{p}_0 + n \mathbf{p}_1$$

$$\mathbf{P}'(1) = -n \mathbf{p}_{n-1} + n \mathbf{p}_n$$

$$\mathbf{P}''(0) = n(n-1)[(\mathbf{p}_2 - \mathbf{p}_1) - (\mathbf{p}_1 - \mathbf{p}_0)]$$

$$\mathbf{P}''(1) = n(n-1)[(\mathbf{p}_{n-2} - \mathbf{p}_{n-1}) - (\mathbf{p}_{n-1} - \mathbf{p}_n)]$$

Cubic Bezier Curve

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For our cubic curve at the endpoints

$$\mathbf{P}'(0) = 3(\mathbf{p}_1 - \mathbf{p}_0)$$

$$\mathbf{P}'(1) = 3(\mathbf{p}_3 + \mathbf{p}_2)$$

$$\mathbf{P}''(0) = 6(\mathbf{p}_0 - 2\mathbf{p}_1\mathbf{p}_2)$$

$$\mathbf{P}''(1) = 6(\mathbf{p}_1 - 2\mathbf{p}_2 + \mathbf{p}_3)$$

which we can transform into a matrix:

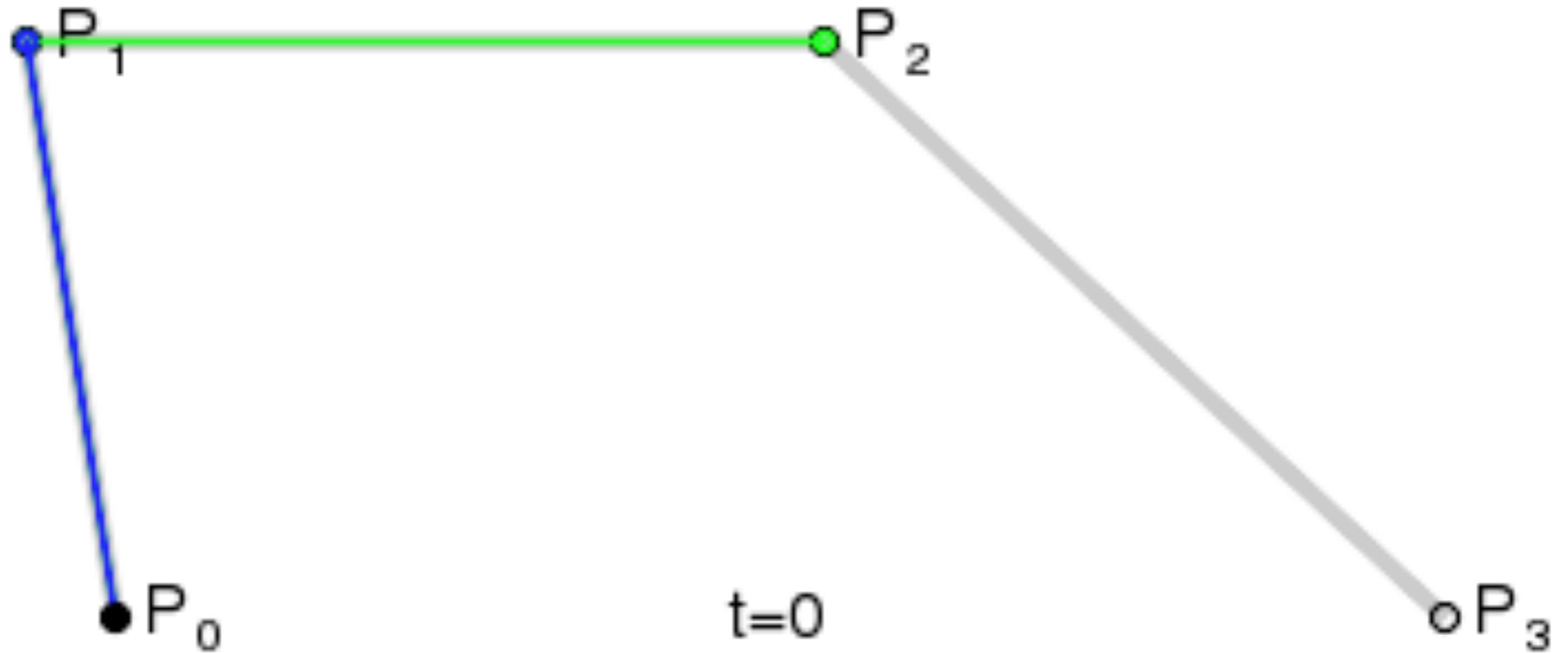
$$\mathbf{P}(u) = [u^3 \quad u^2 \quad u \quad 1] \cdot \mathbf{M}_{Bez} \cdot \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}$$

with the **Bezier matrix**:

$$\mathbf{M}_{Bez} = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Cubic Bezier Curve

ARLAB



Cubic curves, [wikipedia]

Thank you!

Questions

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