

ARLAB

ME/CprE/ComS 557

Computer Graphics and Geometric Modeling

Quaternions

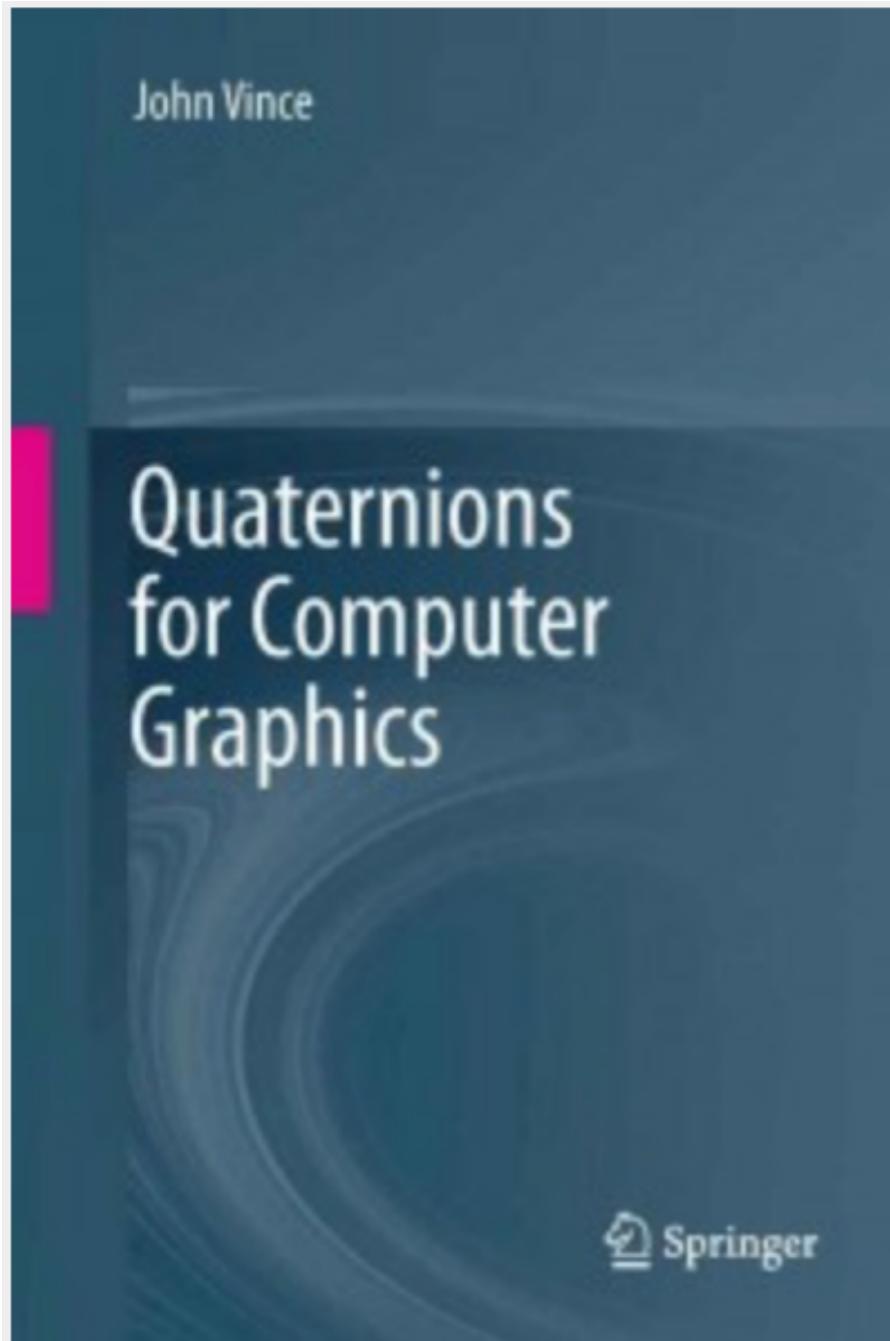
October 27th, 2015

Rafael Radkowski

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OF SCIENCE AND TECHNOLOGY

Book

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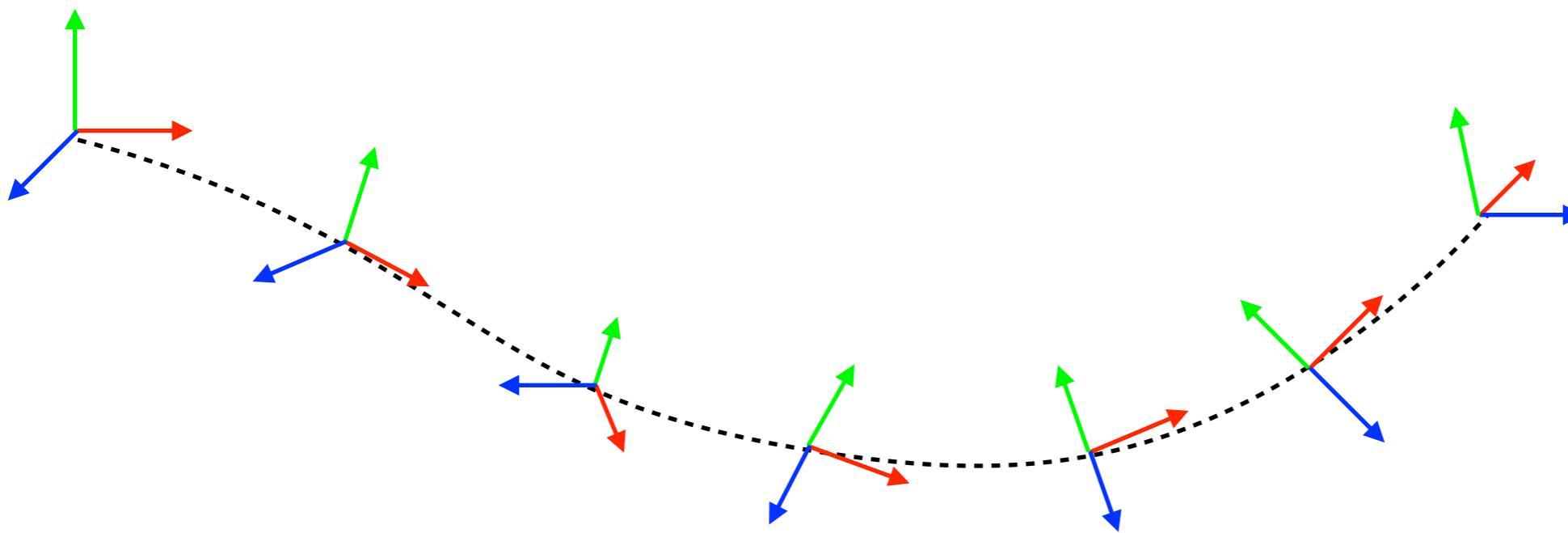


Vince, J (2011). Quaternions for Computer Graphics. 1st. ed. London: Springer

Why Quaternions

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Describing a path / locomotion as a sequence of matrices can result in an unexpected outcome due to the link of the rotation axis to each other in Cartesian space with Euler angles.

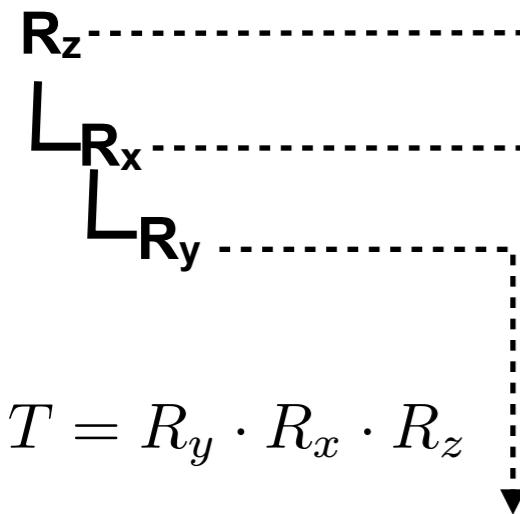


- Easier to "add" a new position / orientation to the current position / orientation
- Interpolation between points is possible.

Gimbal Lock Problem



The sequence of transformations (rotations in our case) can be considered as hierarchy of transformations.



$$T = R_y \cdot R_x \cdot R_z$$

$$T = \begin{bmatrix} \cos(\beta) & 0 & -\sin(\beta) & 0 \\ 0 & 1 & 0 & 0 \\ \sin(\beta) & 0 & \cos(\beta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ \cos(\alpha) & -\sin(\alpha) & 0 & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos(\gamma) & \sin(\gamma) & 0 & 0 \\ -\sin(\gamma) & \cos(\gamma) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

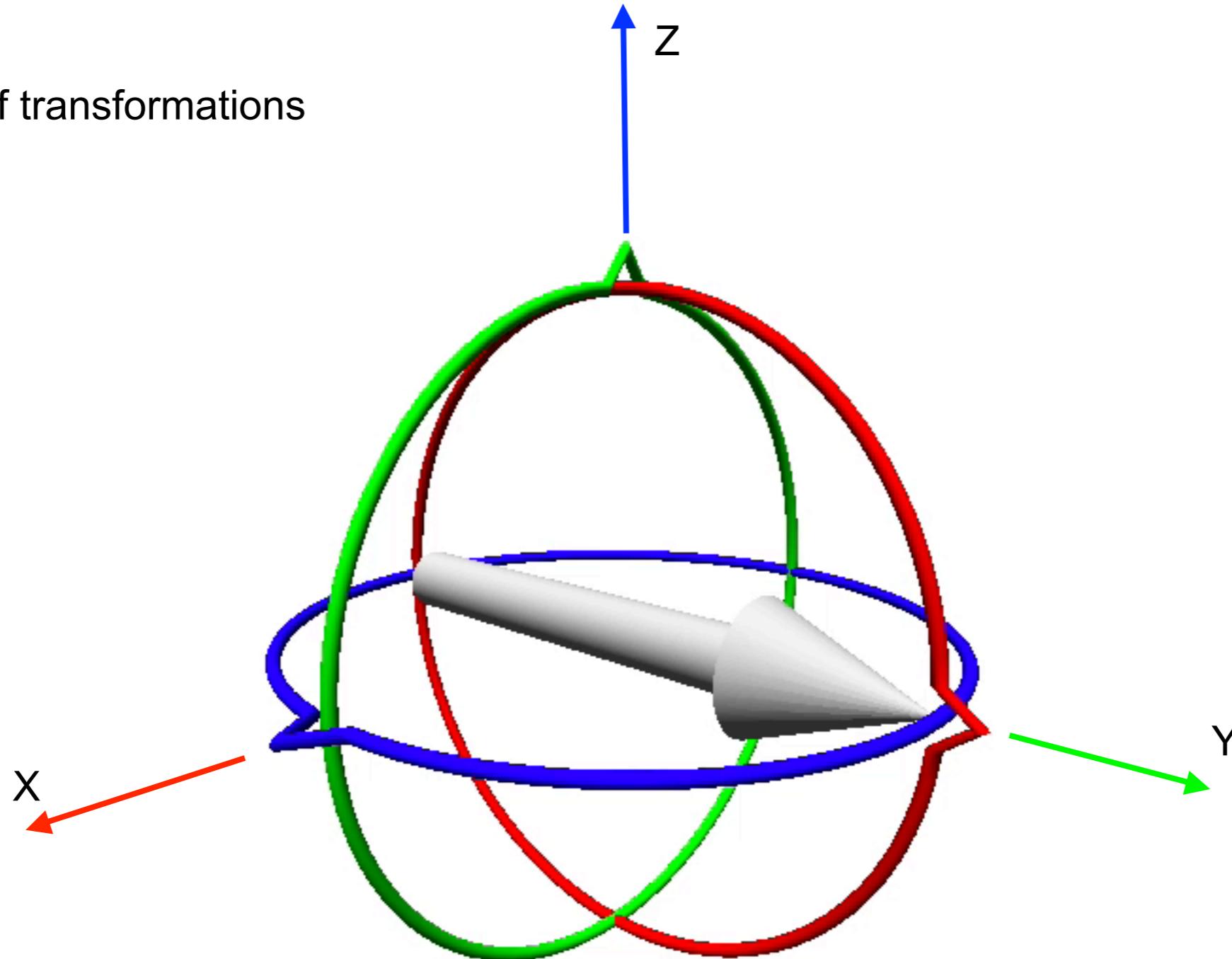
The transformations are linked: the axis y, and x also rotates when an object is rotated around the z-axis.

Gimbal Lock Problem



Hierarchy of transformations

z
└ x
 └ y

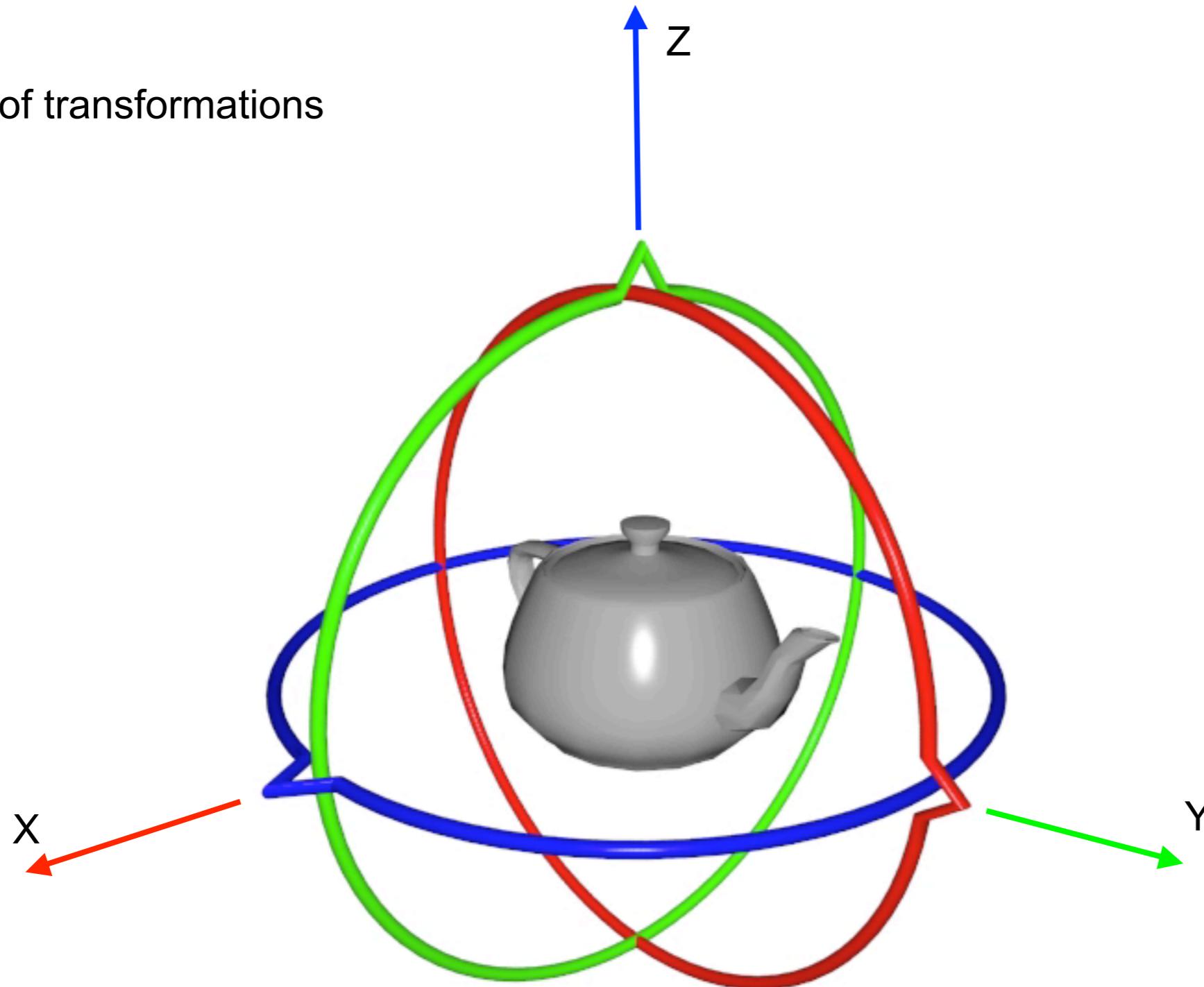


Gimbal Lock Problem



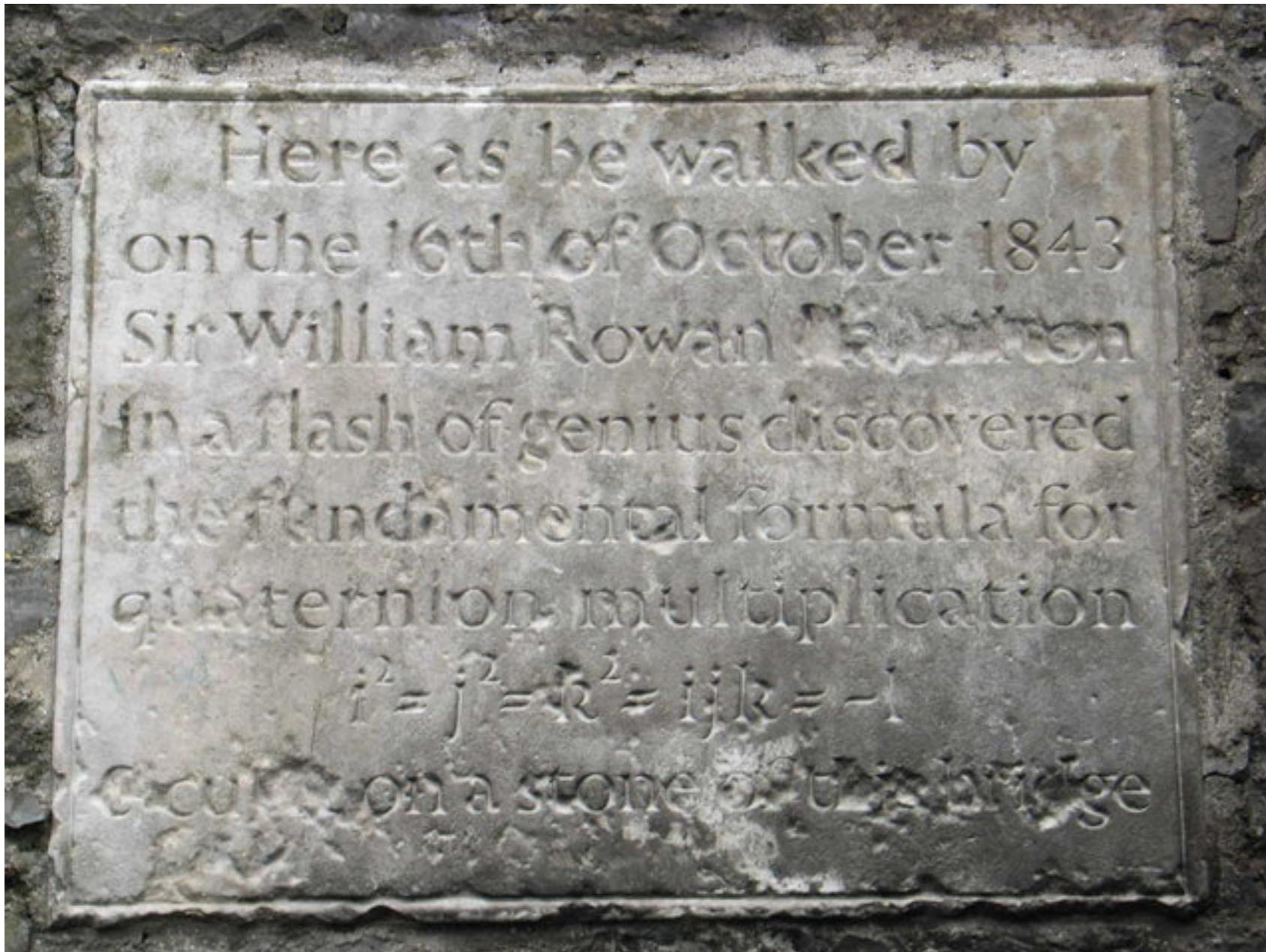
Hierarchy of transformations

z
└ x
 └ y



History

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Quaternions were discovered by Sir William Rowan Hamilton on Monday October 16th 1843 in Dublin, Ireland.

$$i^2 = j^2 = k^2 = -1$$

Complex Numbers

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Complex numbers introduce imaginary numbers to solve certain equations such as:

$$x + 1 = 0$$

$$x^2 = -1$$

$$x = \sqrt{-1}$$

which we know is not possible to solve with real numbers because the square of every number is always positive.

The **imaginary number** has been founded.

$$i^2 = -1$$

The set of complex numbers is the sum of a real number and an imaginary number

$$z = a + bi \quad a, b \in \mathbb{R}, i^2 = -1$$

- all real numbers are complex numbers with $b = 0$
- all imaginary numbers are complex numbers with $a = 0$

Some Rules

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Adding and Subtracting Complex Numbers

Addition:

$$(a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 + a_2) + (b_1 + b_2)i$$

Subtraction:

$$(a_1 + b_1 i) - (a_2 + b_2 i) = (a_1 - a_2) + (b_1 - b_2)i$$

Multiply a Complex Number by a Scalar:

$$\lambda(a + bi) = \lambda a + \lambda bi$$

Some Rules

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Product of Complex Numbers:

$$z_1 = (a_1 + b_1 i)$$

$$z_2 = (a_2 + b_2 i)$$

$$z_1 z_2 = (a_1 + b_1 i)(a_2 + b_2 i)$$

$$= a_1 a_2 + a_1 b_2 i + b_1 a_2 i + b_1 b_2 i$$

$$= (a_1 a_2 - b_1 b_2) + (a_1 b_2 + b_1 a_2) i$$

Square of Complex Numbers:

$$z^2 = (a + bi)(a + bi)$$

$$= (a^2 - b^2) + 2abi$$

Complex Conjugate:

$$z = (a + bi)$$

$$|z| = \sqrt{zz^*}$$

$$= \sqrt{(a + bi)(a - bi)}$$

$$= \sqrt{a^2 + b^2}$$

The conjugate of a complex number is a complex number with the imaginary part negated



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Some Rules

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Absolute Value of a Complex Number:

$$z = (a + bi)$$

$$|z| = \sqrt{zz^*}$$

$$= \sqrt{(a + bi)(a - bi)}$$

$$= \sqrt{a^2 + b^2}$$

We can use the conjugate of a complex number to compute the absolute value (or norm, or magnitude) of a complex number.

Power of i

$$i^0 = 1$$

$$i^1 = i$$

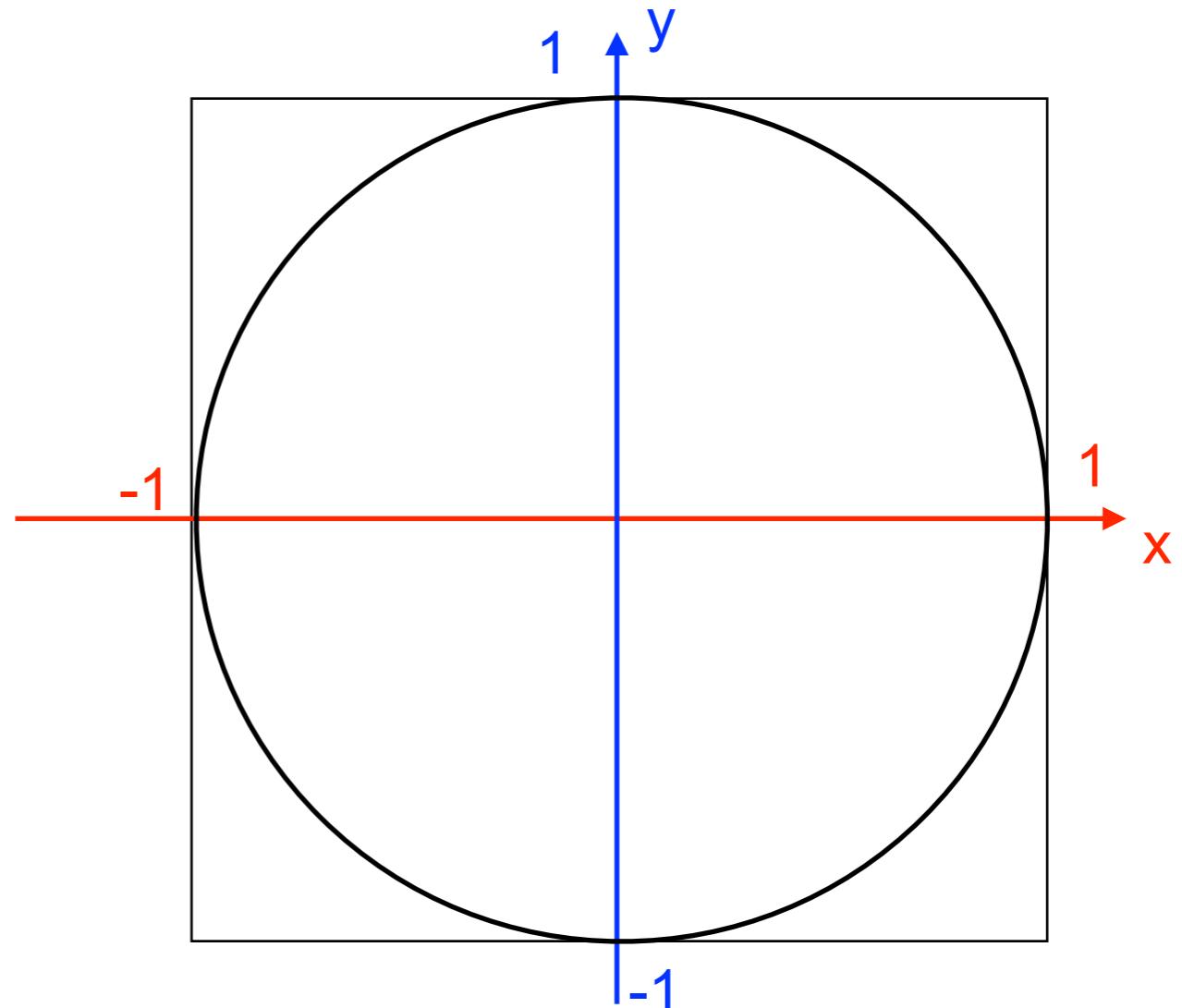
$$i^2 = -1$$

$$i^3 = ii^2 = -i$$

$$i^4 = i^2i^2 = 1$$

$$i^5 = ii^4 = i$$

$$i^6 = ii^5 = -1$$



This is a sequence $1, i, -1, -i, 1, i, -1, -i, 1, \dots$

The Complex Plane

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$$i^0 = 1$$

$$i^1 = i$$

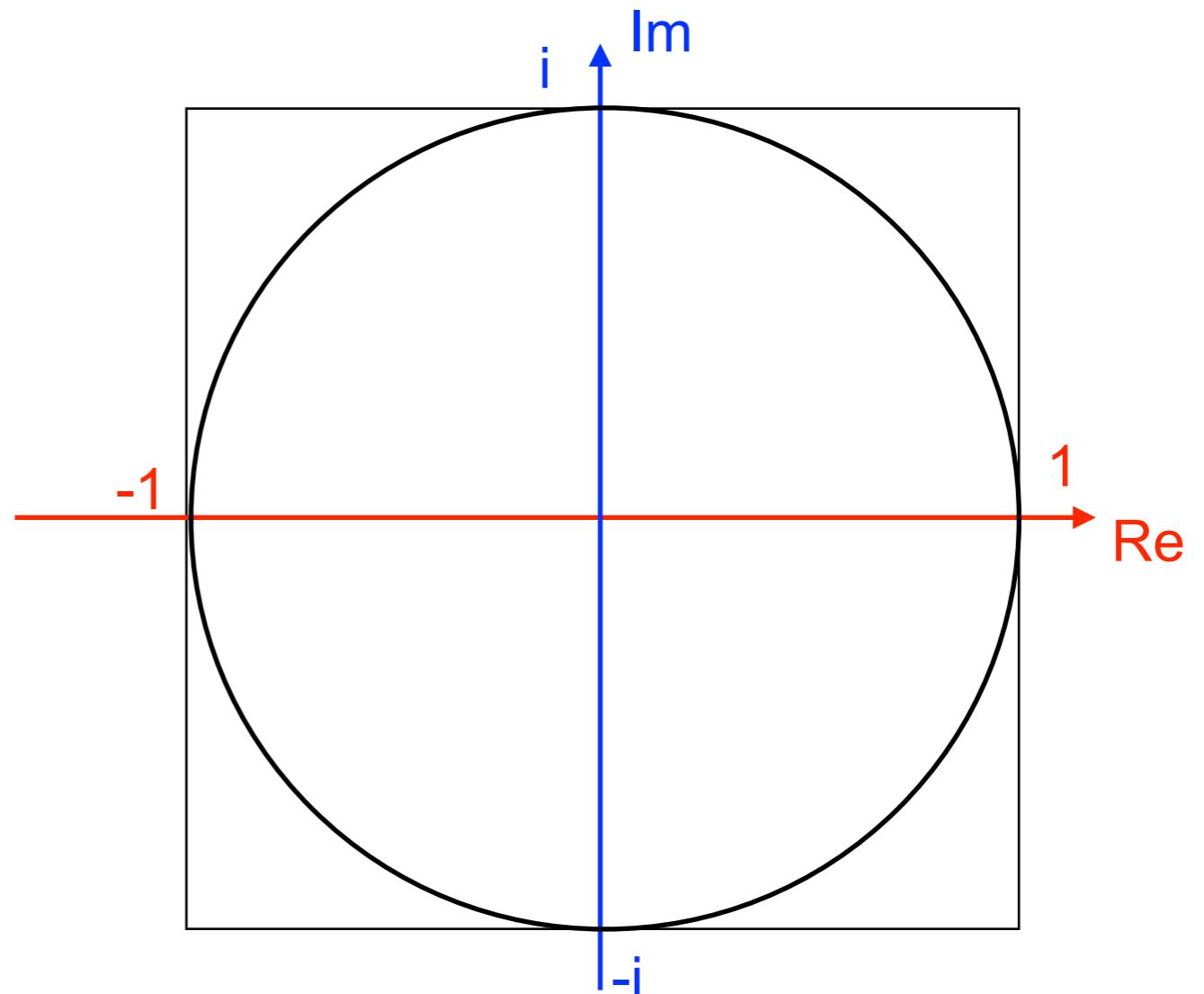
$$i^2 = -1$$

$$i^3 = ii^2 = -i$$

$$i^4 = i^2i^2 = 1$$

$$i^5 = ii^4 = i$$

$$i^6 = ii^5 = -1$$

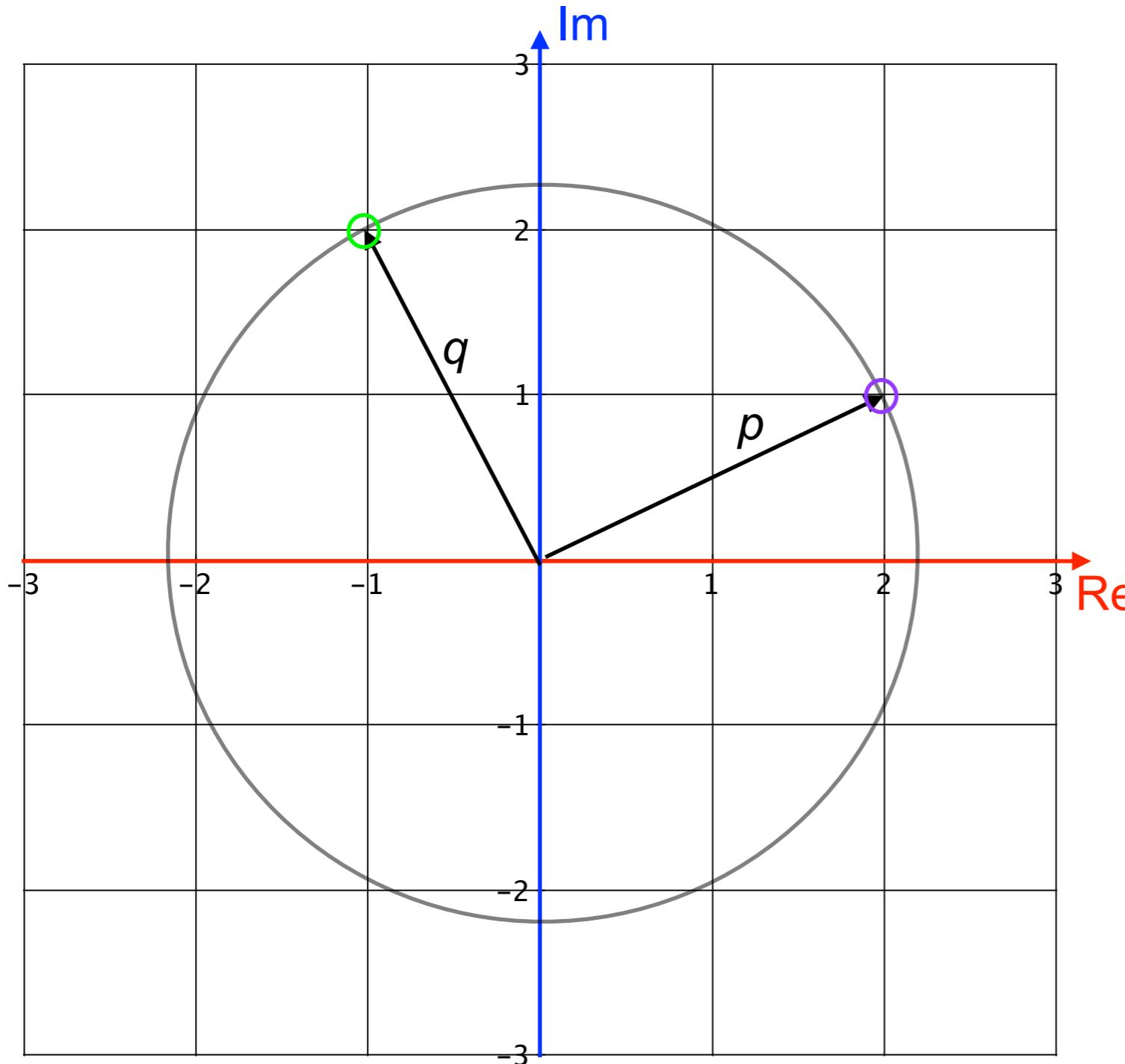


This is a sequence 1, i, -1, -i, 1, i, -1, -i, 1,

We can say that if we multiply a complex number by i, we can rotate the complex number through the complex plane at 90 degree increments.

Rotation with Complex Numbers

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Take an arbitrary point p in the complex plane:

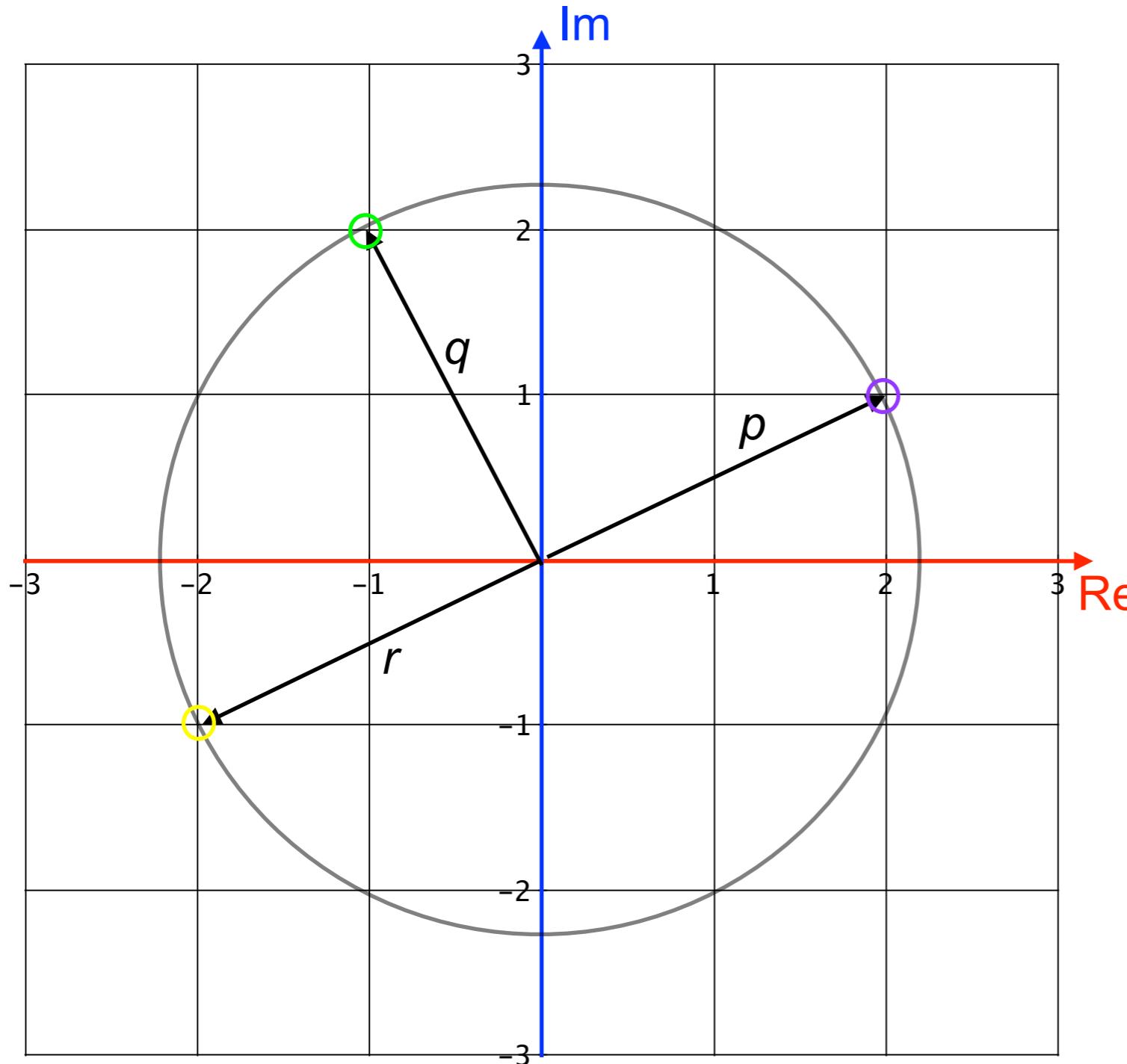
○ $p = 2 + i$

Multiply this point with i .

○
$$\begin{aligned} q &= p \cdot i \\ &= (2 + i)i \\ &= 2i + i^2 \\ &= -1 + 2i \end{aligned}$$

Rotation with Complex Numbers

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Take an arbitrary point p in the complex plane:

○ $p = 2 + i$

Multiply point q with i .

○ $q = -1 + 2i$

○ $r = q \cdot i$

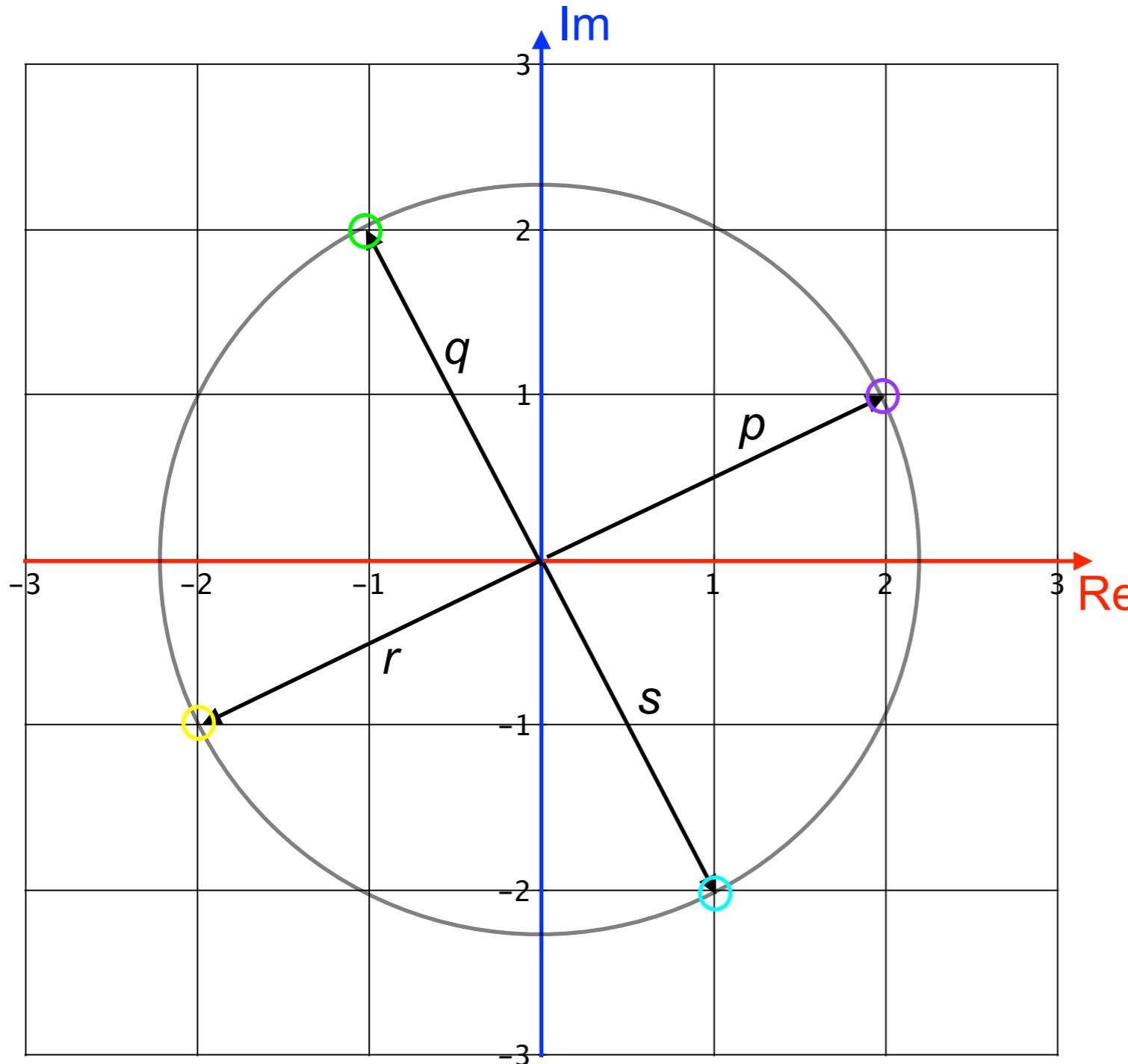
$$= (-1 + 2i)i$$

$$= -i + 2i^2$$

$$= -2 - i$$

Rotation with Complex Numbers

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Take an arbitrary point p in the complex plane:

$$p = 2 + i$$

Multiply point r with i .

$$r = -2 - i$$

$$s = r \cdot i$$

$$= (-2 - i)i$$

$$= -2i - i^2$$

$$= 1 - 2i$$

Rotation with Complex Numbers

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We can also perform arbitrary rotations in the complex plane by defining a complex number of the form:

$$q = \cos \theta + i \sin \theta$$

We obtain the general formula when multiplying a point p with q :

$$p = a + b i$$

$$q = \cos \theta + i \sin \theta$$

$$pq = (a + bi)(\cos \theta + i \sin \theta)$$

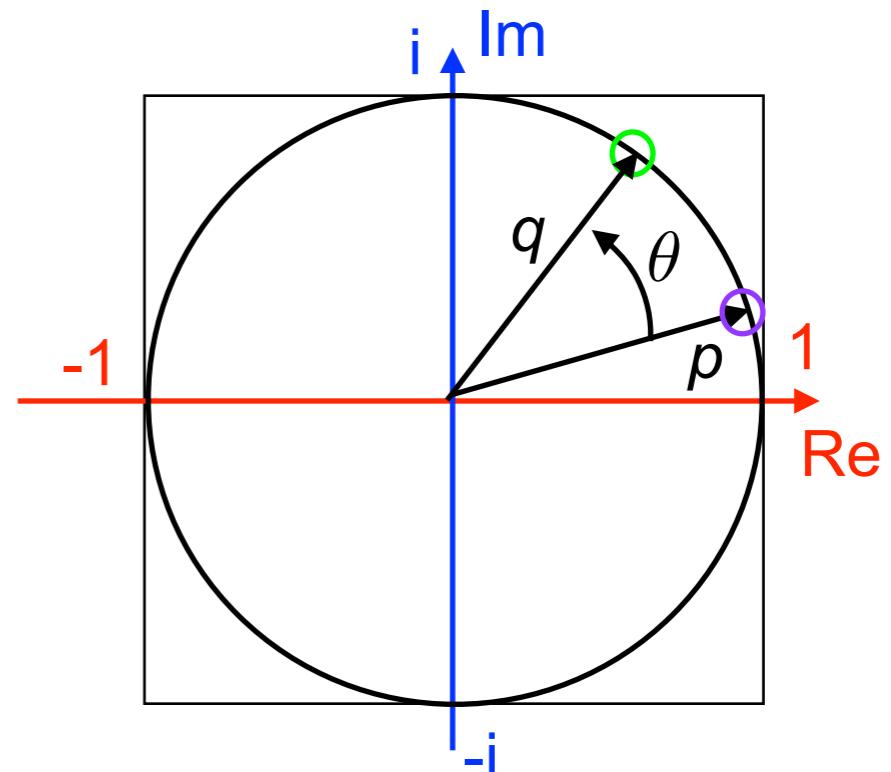
$$a' + b'i = a \cos \theta - b \sin \theta + (a \sin \theta + b \cos \theta)i$$

(pq rewritten)

or in matrix form:

$$\begin{bmatrix} a' & -b' \\ b' & a' \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

to rotate an arbitrary point counter-clockwise.



This was in 2D / one plane out or three (rotation of a point around one axis in a plane).

How do we extend this to 3D?



Quaternions

Quaternions

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We extend this to 3-dimensional space by adding two imaginary numbers to our number system in addition to i . One number for each additional plane.

The general form to express quaternions is:

$$q = s + xi + yj + zk \quad s, x, y, z \in \mathbb{R}, i, j, z \in \mathbb{I}$$

scalar value s

vector of complex numbers

with

$$i^2 + j^2 + k^2 = ijk = -1$$

and

$$\begin{array}{lll} ij = k & jk = i & ki = j \\ ji = -k & kj = -i & ik = -j \end{array}$$

Quaternions

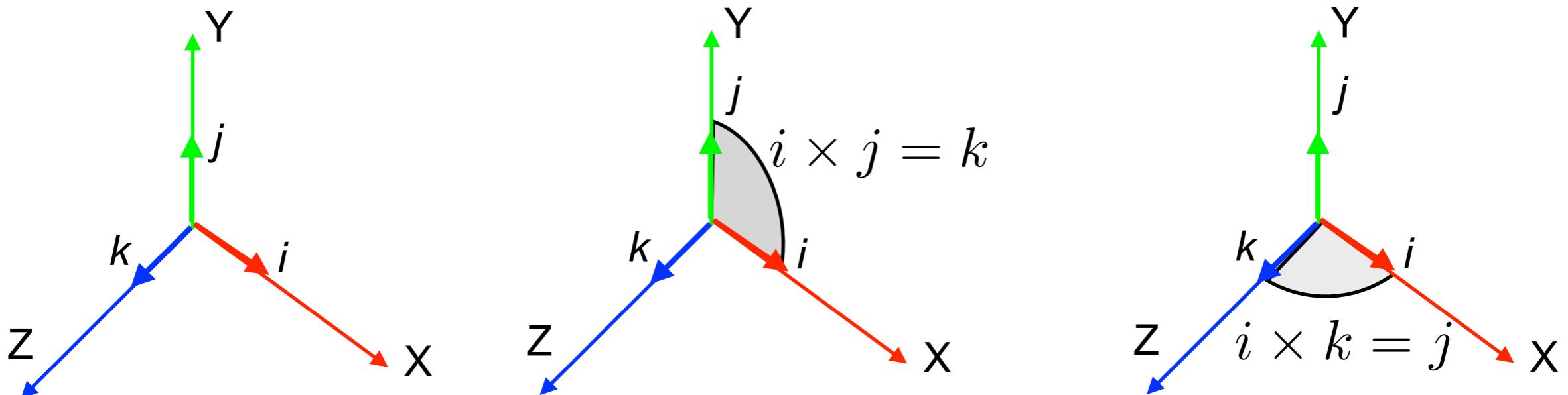
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When mapping these quaternion rules to a Cartesian coordinate system,

$$\begin{array}{lll} ij = k & jk = i & ki = j \\ ji = -k & kj = -i & ik = -j \end{array}$$

they behave similar to cross products.

$$\begin{array}{lll} \mathbf{x} \times \mathbf{y} = \mathbf{z}, & \mathbf{y} \times \mathbf{z} = \mathbf{x}, & \mathbf{z} \times \mathbf{x} = \mathbf{y} \\ \mathbf{y} \times \mathbf{x} = -\mathbf{z}, & \mathbf{z} \times \mathbf{y} = -\mathbf{x}, & \mathbf{x} \times \mathbf{z} = -\mathbf{y} \end{array}$$



Quaternions

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Hamilton recognized that the imaginary numbers i , j , and k can be used to represent three cartesian unit vectors i , j , and k with the same properties of imaginary numbers, such that

$$i^2 + j^2 + k^2 = ijk = -1$$

Quaternions can be represented as an **ordered pair**:

$$q = [s, \mathbf{v}] \quad s \in \mathbb{R}, \mathbf{v} \in \mathbb{R}^3$$

or with individual components

$$q = [s, x\mathbf{i} + y\mathbf{j} + z\mathbf{k}] \quad s, x, y, z \in \mathbb{R}$$

which allows us to show similarities to complex numbers.

Substitute to swap between notations.

Mathematical Rules

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Adding and subtracting quaternions:

$$q_a = [s_a, \mathbf{a}] \quad q_b = [s_b, \mathbf{b}]$$

with $\mathbf{a} = [x_a \mathbf{i} + y_a \mathbf{j} + z_a \mathbf{k}]$
similar for \mathbf{b}

$$q_a + q_b = [s_a + s_b, \mathbf{a} + \mathbf{b}]$$

$$q_a - q_b = [s_a - s_b, \mathbf{a} - \mathbf{b}]$$

but keep in mind:

$$q_a + q_b = (s_a + s_b) + (x_a + x_b)\mathbf{i} + (y_a + y_b)\mathbf{j} + (z_a + z_b)\mathbf{k}$$

$$q_a - q_b = (s_a - s_b) + (x_a - x_b)\mathbf{i} + (y_a - y_b)\mathbf{j} + (z_a - z_b)\mathbf{k}$$

Substitute to swap between notations.

Product of Quaternions

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Multiplying two quaternions takes more time, but the math is straight forward:

Use the quaternions q_a and q_b as starting point:

$$q_a = [s_a, \mathbf{a}] \quad q_b = [s_b, \mathbf{b}]$$

multiplying:

$$\begin{aligned} q_a q_b &= [s_a, \mathbf{a}] [s_b, \mathbf{b}] \\ &= (s_a + x_a i + y_a j + z_a k)(s_b + x_b i + y_b j + z_b k) \\ &= (s_a s_b - x_a x_b - y_a y_b - z_a z_b) \\ &\quad + (s_a x_b + s_b x_a + y_a z_b - y_b z_a)i \\ &\quad + (s_a y_b + s_b y_a + z_a x_b - z_b x_a)j \\ &\quad + (s_a z_b + s_b z_a + x_a y_b - x_b y_a)k \end{aligned}$$

keep in mind

$$i^2 = -1$$

$$i^2 + j^2 + k^2 = ijk = -1$$

$$\begin{array}{lll} ij = k & jk = i & ki = j \\ ji = -k & kj = -i & ik = -j \end{array}$$

Product of Quaternions

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Multiplying two quaternions takes more time, but the math is straight forward:

Use the quaternions q_a and q_b as starting point:

$$q_a = [s_a, \mathbf{a}] \quad q_b = [s_b, \mathbf{b}]$$

multiplying:

$$\begin{aligned} q_a q_b &= [s_a, \mathbf{a}] [s_b, \mathbf{b}] \\ &= (s_a + x_a i + y_a j + z_a k)(s_b + x_b i + y_b j + z_b k) \\ &= (s_a s_b - x_a x_b - y_a y_b - z_a z_b) \\ &\quad + (s_a x_b + s_b x_a + y_a z_b - y_b z_a)i \\ &\quad + (s_a y_b + s_b y_a + z_a x_b - z_b x_a)j \\ &\quad + (s_a z_b + s_b z_a + x_a y_b - x_b y_a)k \end{aligned}$$

note, change the complex number:

$$y_a j \cdot z_b k = (y_a z_b) i \rightarrow jk = i$$

keep in mind

$$i^2 = -1$$

$$i^2 + j^2 + k^2 = ijk = -1$$

$$\begin{array}{lll} ij = k & jk = i & ki = j \\ ji = -k & kj = -i & ik = -j \end{array}$$

Product of Quaternions



Now let's replace the imaginary number by ordered pairs

$$i = [0, \mathbf{i}] \ j = [0, \mathbf{j}] \ k = [0, \mathbf{k}] \text{ and } [1, \mathbf{0}] = 1$$

Pure quaternion

Real quaternion

gives

$$\begin{aligned} [s_a, \mathbf{a}][s_b, \mathbf{b}] &= (s_a s_b - x_a x_b - y_a y_b - z_a z_b)[1, \mathbf{0}] \\ &\quad + (s_a x_b + s_b x_a + y_a z_b - y_b z_a)[0, \mathbf{i}] \\ &\quad + (s_a y_b + s_b y_a + z_a x_b - z_b x_a)[0, \mathbf{j}] \\ &\quad + (s_a z_b + s_b z_a + x_a y_b - x_b y_a)[0, \mathbf{k}] \end{aligned}$$

rewriting as ordered pairs gives

$$\begin{aligned} [s_a, \mathbf{a}][s_b, \mathbf{b}] &= [s_a s_b - x_a x_b - y_a y_b - z_a z_b, \mathbf{0}] \\ &\quad + [0, (s_a x_b + s_b x_a + y_a z_b - y_b z_a) \mathbf{i}] \\ &\quad + [0, (s_a y_b + s_b y_a + z_a x_b - z_b x_a) \mathbf{j}] \\ &\quad + [0, (s_a z_b + s_b z_a + x_a y_b - x_b y_a) \mathbf{k}] \end{aligned}$$

Product of Quaternions

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We can extract the common components and further rewrite as:

$$\begin{aligned}[s_a, \mathbf{a}][s_b, \mathbf{b}] &= [s_a s_b - x_a x_b - y_a y_b - z_a z_b, \mathbf{0}] \\ &\quad + [0, s_a(x_b \mathbf{i} + y_b \mathbf{j} + z_b \mathbf{k}) + s_b(x_a \mathbf{i} + y_a \mathbf{j} + z_a \mathbf{k}) \\ &\quad + (y_z z_b - y_b z_a) \mathbf{i} + (z_a x_b - z_b x_a) \mathbf{j} + (x_a y_b - y_b x_a) \mathbf{k}]\end{aligned}$$

Product of Quaternions

ARLAB

We can extract the common components and further rewrite as:

$$[s_a, \mathbf{a}][s_b, \mathbf{b}] = [s_a s_b - x_a x_b - y_a y_b - z_a z_b, \mathbf{0}] + [0, s_a(x_b \mathbf{i} + y_b \mathbf{j} + z_b \mathbf{k}) + s_b(x_a \mathbf{i} + y_a \mathbf{j} + z_a \mathbf{k}) + (y_z z_b - y_b z_a) \mathbf{i} + (z_a x_b - z_b x_z) \mathbf{j} + (x_a y_b - y_b x_a) \mathbf{k}]$$

Real quaternion

Pure quaternion

We combine the two ordered pairs into a single ordered pair, because one component is always 0.

$$[s_a, \mathbf{a}][s_b, \mathbf{b}] = [s_a s_b - x_a x_b - y_a y_b - z_a z_b, s_a(x_b \mathbf{i} + y_b \mathbf{j} + z_b \mathbf{k}) + s_b(x_a \mathbf{i} + y_a \mathbf{j} + z_a \mathbf{k}) + (y_z z_b - y_b z_a) \mathbf{i} + (z_a x_b - z_b x_z) \mathbf{j} + (x_a y_b - y_b x_a) \mathbf{k}]$$

Product of Quaternions

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$$[s_a, \mathbf{a}][s_b, \mathbf{b}] = [s_a s_b - x_a x_b - y_a y_b - z_a z_b, \\ s_a(x_b \mathbf{i} + y_b \mathbf{j} + z_b \mathbf{k}) + s_b(x_a \mathbf{i} + y_a \mathbf{j} + z_a \mathbf{k}) \\ + (y_z z_b - y_b z_a) \mathbf{i} + (z_a x_b - z_b x_a) \mathbf{j} + (x_a y_b - y_b x_a) \mathbf{k}]$$

Product of Quaternions

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$$[s_a, \mathbf{a}][s_b, \mathbf{b}] = [s_a s_b - x_a x_b - y_a y_b - z_a z_b, \\ s_a(x_b \mathbf{i} + y_b \mathbf{j} + z_b \mathbf{k}) + s_b(x_a \mathbf{i} + y_a \mathbf{j} + z_a \mathbf{k}) \\ + (y_z z_b - y_b z_a) \mathbf{i} + (z_a x_b - z_b x_a) \mathbf{j} + (x_a y_b - y_b x_a) \mathbf{k}]$$

Dot product

pure quaternion

Cross product

with:

$$\mathbf{a} = x_a \mathbf{i} + y_a \mathbf{j} + z_a \mathbf{k}$$

$$\mathbf{b} = x_b \mathbf{i} + y_b \mathbf{j} + z_b \mathbf{k}$$

$$\mathbf{a} \cdot \mathbf{b} = x_a x_b \mathbf{i}^2 + y_a y_b \mathbf{j}^2 + z_a z_b \mathbf{k}^2$$

$$\mathbf{a} \times \mathbf{b} = (y_a z_b - y_b z_a) \mathbf{i} (z_a x_b - x_b z_a) \mathbf{j} + (x_a y_b - x_b y_a) \mathbf{k}$$

$$i^2 = j^2 = k^2 = -1$$

Product of Quaternions

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Substituting:

$$[s_a, \mathbf{a}][s_b, \mathbf{b}] = [s_a s_b - \mathbf{a} \cdot \mathbf{b}, s_a \mathbf{b} + s_b \mathbf{a} + \mathbf{a} \times \mathbf{b}]$$

Results in an equation for the product of two quaternions.

Real and Pure Quaternions

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A Real Quaternion is a quaternion with a vector term of 0:

$$q = [s, \mathbf{0}]$$

the Pure Quaternion as a quaternion that has a zero scalar term

$$q = [0, \mathbf{v}]$$

addition of both:

$$\begin{aligned} q &= [s, \mathbf{v}] \\ &= [s, \mathbf{0}] + [0, \mathbf{v}] \end{aligned}$$

Unit and Norm



Unit Quaternion

Given a vector \mathbf{v} , we can express this vector with its scalar magnitude and its direction:

$$\mathbf{v} = v\hat{\mathbf{v}} \quad \text{where } v = |\mathbf{v}| \quad \text{and } |\hat{\mathbf{v}}| = 1$$

Thus, we obtain a quaternion with length of the complex vector = 1.

Quaternion Norm

$$q = [s, \mathbf{v}]$$

$$|q| = \sqrt{s^2 + \mathbf{v}^2}$$

Normalization

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We can also normalize a quaternion:

$$q' = \frac{q}{|a|} = \frac{q}{\sqrt{s^2 + \mathbf{v}^2}}$$

Example:

$$q = [1, 4\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}]$$

Calculate the length:

$$|q| = \sqrt{1^2 + 4^2 + 4^2 + (-4)^2} = \sqrt{49} = 7$$

Divide the quaternion by the length (norm)

$$q' = \frac{q}{|q|} = \frac{1 + 4\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}}{7} = \frac{1}{7} + \frac{4}{7}\mathbf{i} + \frac{4}{7}\mathbf{j} - \frac{4}{7}\mathbf{k}$$

Quaternion Conjugate and Inverse

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The quaternion conjugate can be computed by negating the imaginary part of the quaternion:

$$q = [s, \mathbf{v}]$$

$$q^* = [s, -\mathbf{v}]$$

The inverse of a quaternion can be computed as:

$$q^{-1} = \frac{q^*}{|q|^2}$$

Quaternion Dot Product

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Similar to the vector dot product, we can also define a quaternion dot product by multiplying the scalar parts:

$$q_a = [s_a, x_a\mathbf{i} + y_a\mathbf{j} + z_a\mathbf{k}]$$

$$q_b = [s_b, x_b\mathbf{i} + y_b\mathbf{j} + z_b\mathbf{k}]$$

$$q_a \cdot q_b = s_a s_b + x_a x_b + y_a y_b + z_a z_b$$

Which also represents the angle between two quaternions:

$$\cos \theta = \frac{s_a s_b + x_a x_b + y_a y_b + z_a z_b}{|q_1||q_2|}$$

or with a unit-quaternion, length = 1 (similar to normalized vectors)

$$\cos \theta = s_a s_b + x_a x_b + y_a y_b + z_a z_b$$



Rotation with Quaternions

Rotation with Quaternions

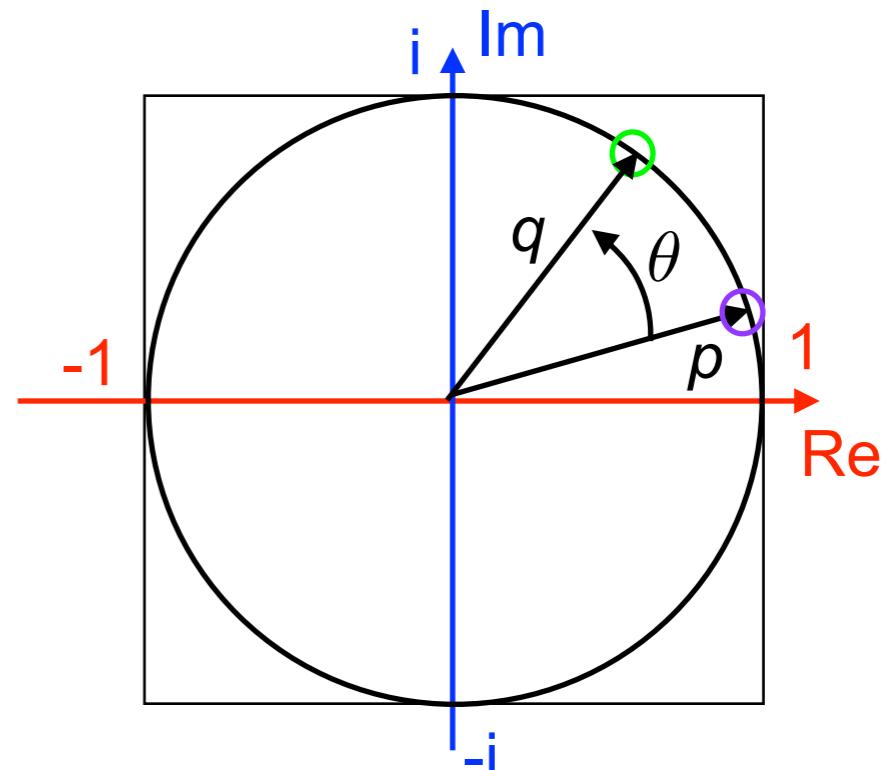
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Remember the Rotor we defined for complex numbers:

$$q = \cos \theta + i \sin \theta$$

Knowing the similarity to complex numbers, we can express a general Rotor for 3D space such as:

$$q = [\cos \theta, \sin \theta \mathbf{v}]$$



Let's test

Let's consider a vector p that we want to rotate, an pure quaternion:

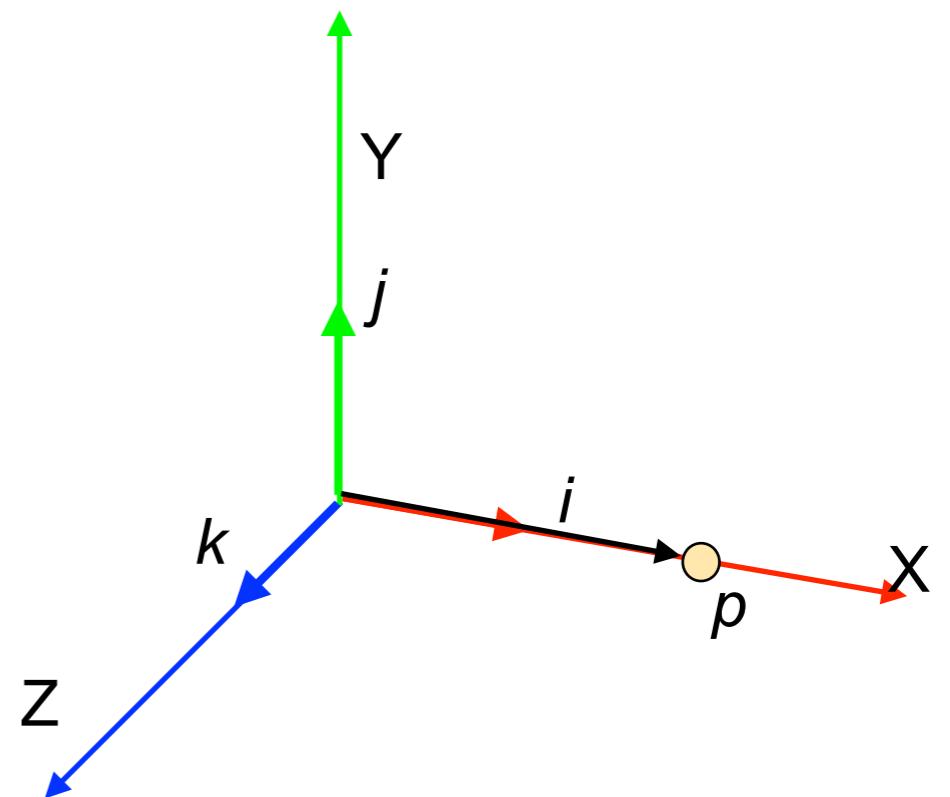
$$p = [0, 2\mathbf{i}]$$

we want to rotate $\theta = 45^\circ$ around \mathbf{k}

$$q = [\cos \theta, \sin \theta \mathbf{k}]$$

(I removed the components \mathbf{i}, \mathbf{j})

$$q = \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \mathbf{k} \right]$$

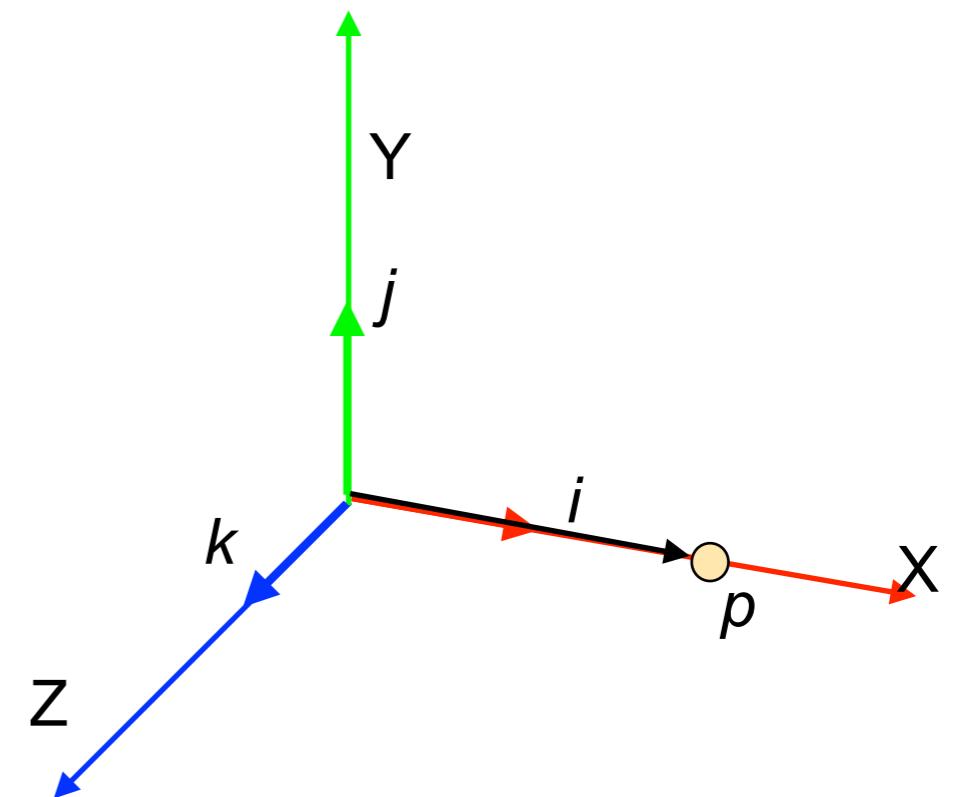


(Quaternions work in 4D,
drawing 3D is actually not correct.)

Let's test

and rotate:

$$\begin{aligned} p' &= qp \\ &= \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\mathbf{k} \right] [0, 2\mathbf{i}] \\ &= \left[0, 2\frac{\sqrt{2}}{2}\mathbf{i} + 2\frac{\sqrt{2}}{2}\mathbf{k} \times \mathbf{i} \right] \\ &= [0, \sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}] \end{aligned}$$



(Quaternions work in 4D,
drawing 3D is actually not correct.)

Let's test

and rotate:

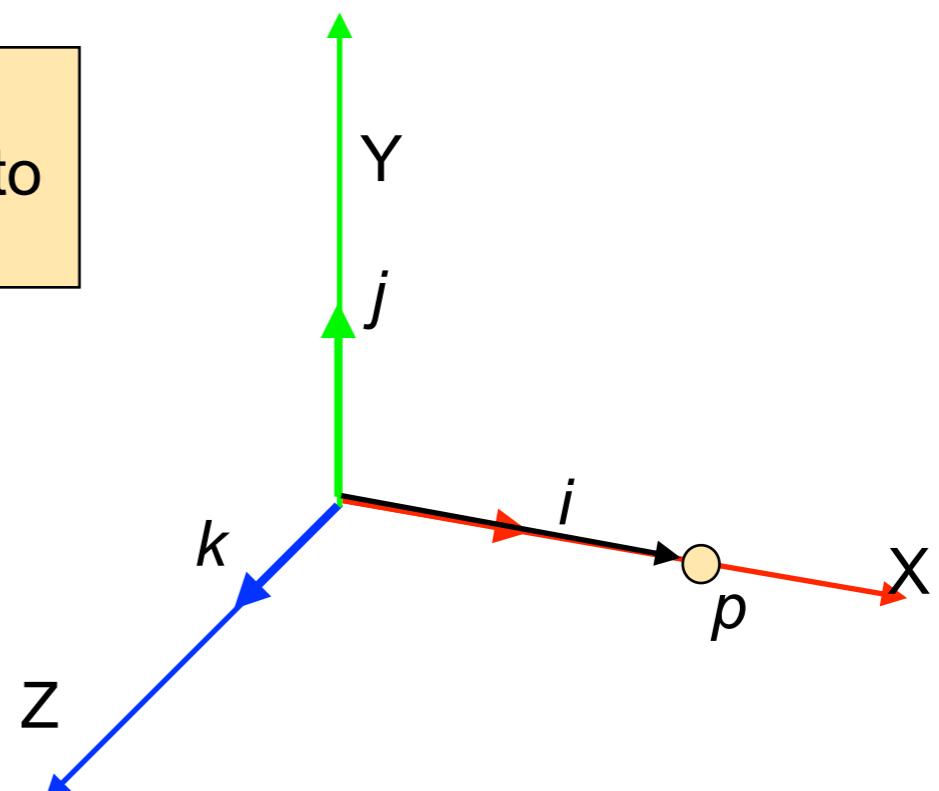
$$p' = qp$$

$$= \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \mathbf{k} \right] [0, 2\mathbf{i}]$$

Moving the scalar term to the front.

$$= \left[0, 2 \frac{\sqrt{2}}{2} \mathbf{i} + 2 \frac{\sqrt{2}}{2} \mathbf{k} \times \mathbf{i} \right]$$

$$= [0, \sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}]$$



Multiplication

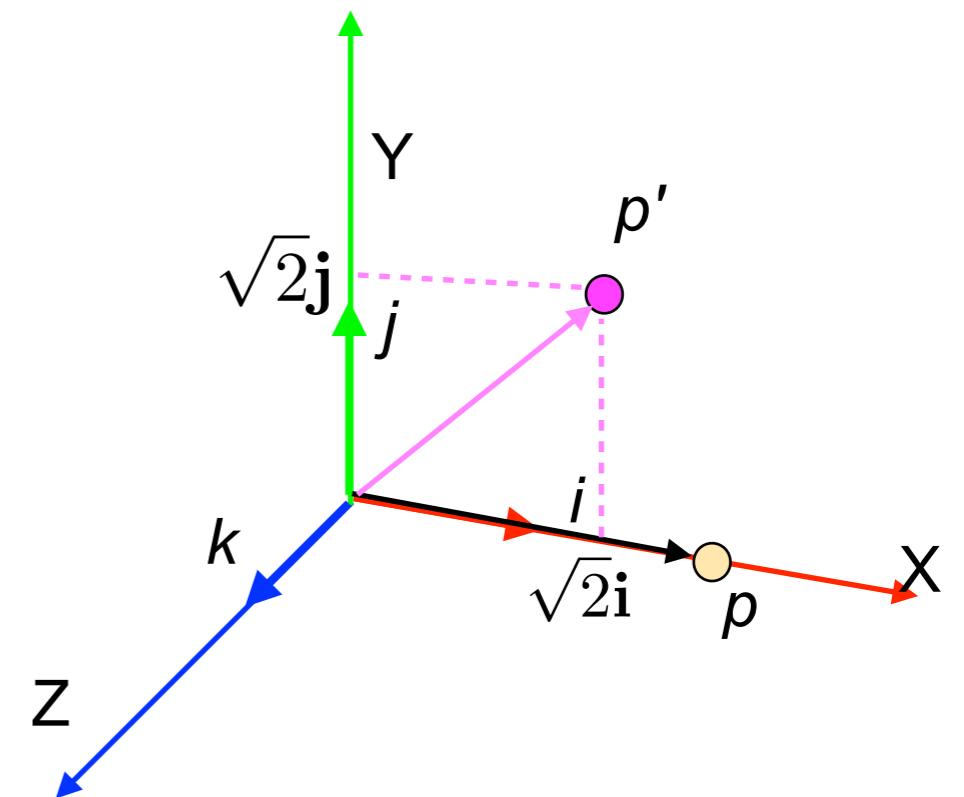
$$[s_a, \mathbf{a}][s_b, \mathbf{b}] = [s_a s_b - \mathbf{a} \cdot \mathbf{b}, s_a \mathbf{b} + s_b \mathbf{a} + \mathbf{a} \times \mathbf{b}]$$

(Quaternions work in 4D,
drawing 3D is actually not correct.)

Let's test

and rotate:

$$\begin{aligned} p' &= qp \\ &= \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\mathbf{k} \right] [0, 2\mathbf{i}] \\ &= \left[0, 2\frac{\sqrt{2}}{2}\mathbf{i} + 2\frac{\sqrt{2}}{2}\mathbf{k} \times \mathbf{i} \right] \\ &= [0, \sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}] \end{aligned}$$



We should maintain the magnitude

$$|p'| = \sqrt{\sqrt{2}^2 + \sqrt{2}^2} = 2$$

(Quaternions work in 4D,
drawing 3D is actually not correct.)

Arbitrary Rotation Vector

ARLAB

Let's consider a vector p that we want to rotate, an pure quaternion:

$$p = [0, 2\mathbf{i}]$$

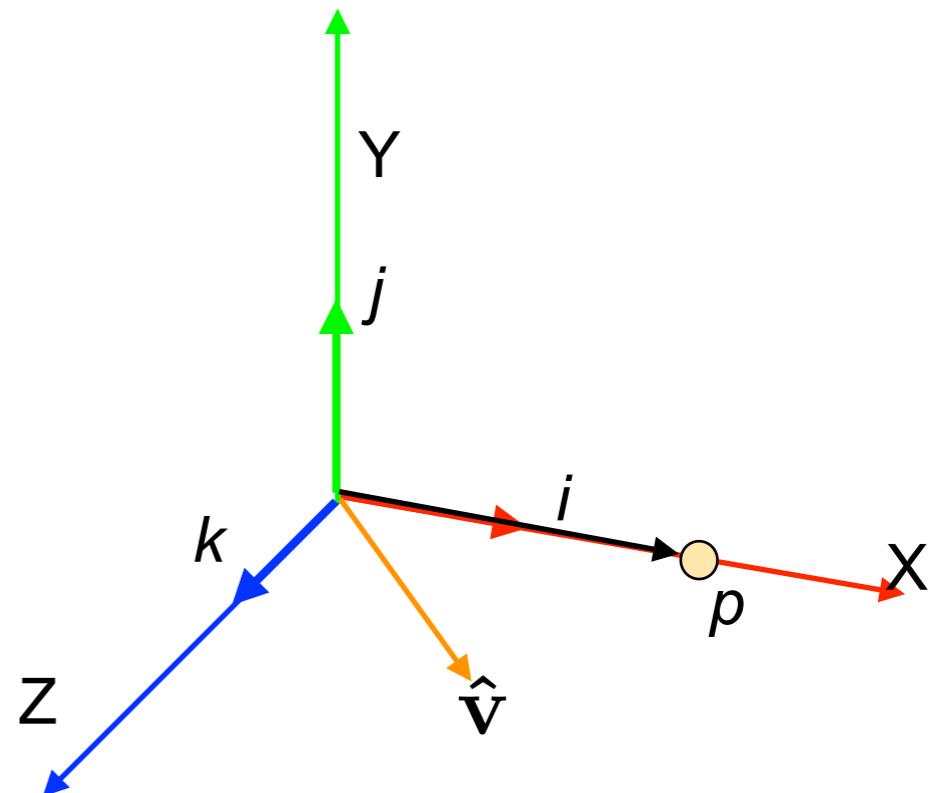
$$\mathbf{p} = 2\mathbf{i}$$

we want to rotate $\theta = 45^\circ$ around $\hat{\mathbf{v}}$

$$\hat{\mathbf{v}} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{k}$$

with

$$q = [\cos \theta, \sin \theta \hat{\mathbf{v}}]$$



Arbitrary Rotation Vector

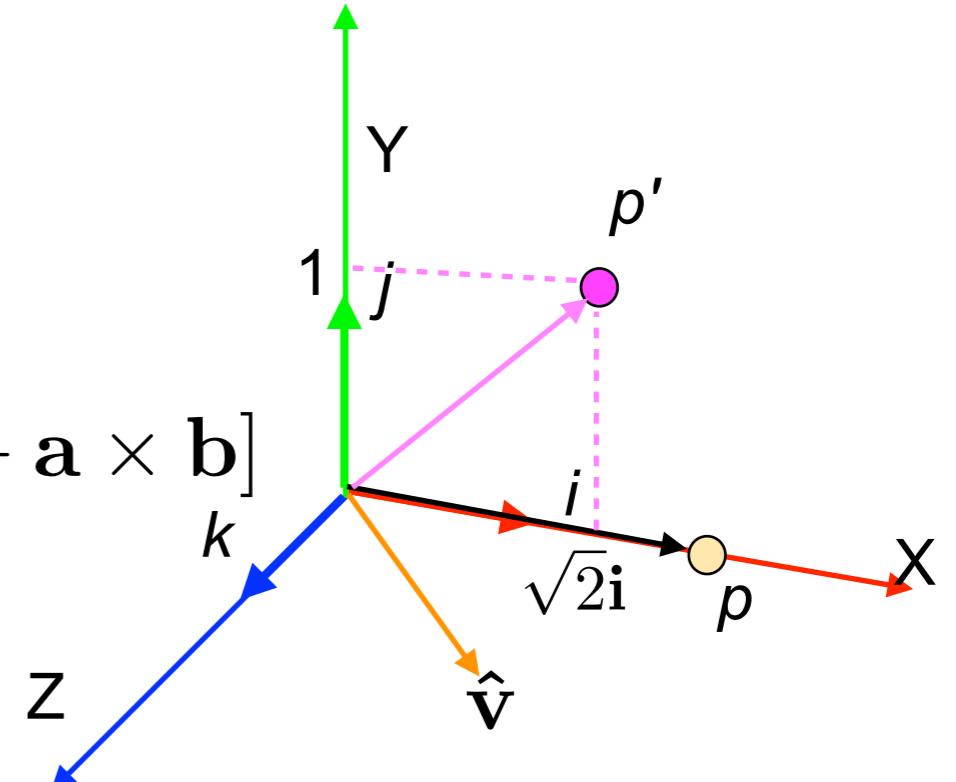
ARLAB

Calculate

$$\begin{aligned} p' &= qp \\ &= [\cos \theta, \sin \theta \hat{\mathbf{v}}][0, \mathbf{p}] \\ &= [-\sin \theta \hat{\mathbf{v}} \cdot \mathbf{p}, \cos \theta \mathbf{p} + \sin \theta \hat{\mathbf{v}} \times \mathbf{p}] \end{aligned}$$

Multiplication

$$[s_a, \mathbf{a}][s_b, \mathbf{b}] = [s_a s_b - \mathbf{a} \cdot \mathbf{b}, s_a \mathbf{b} + s_b \mathbf{a} + \mathbf{a} \times \mathbf{b}]$$



Substituting θ and \mathbf{p} and $\hat{\mathbf{v}}$

$$\begin{aligned} p' &= \left[-\frac{\sqrt{2}}{2} \left(\frac{\sqrt{2}}{2} \mathbf{i} + \frac{\sqrt{2}}{2} \mathbf{k} \right) \cdot (2\mathbf{i}), \frac{\sqrt{2}}{2} 2\mathbf{i} + \frac{\sqrt{2}}{2} \left(\frac{\sqrt{2}}{2} \mathbf{i} + \frac{\sqrt{2}}{2} \mathbf{k} \right) \times 2\mathbf{i} \right] \\ &= [-1, \sqrt{2}\mathbf{i} + \mathbf{j}] \end{aligned}$$

Arbitrary Rotation Vector

ARLAB

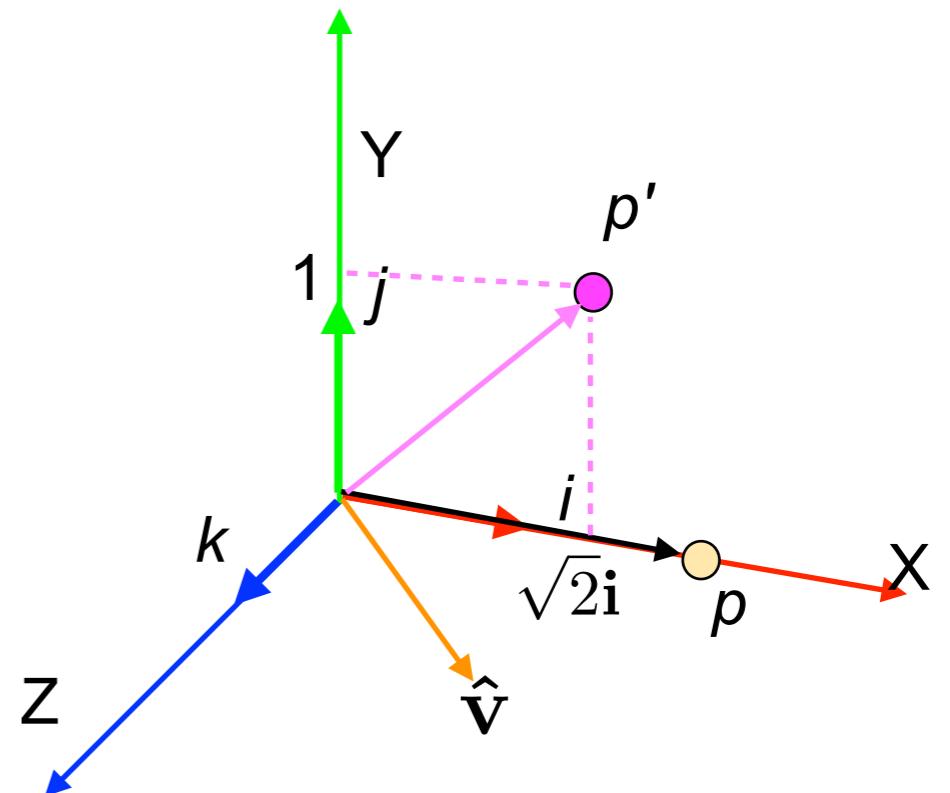
This does not look right to me.

$$p' = [-1, \sqrt{2}\mathbf{i} + \mathbf{j}]$$

The length of p' is also $\sqrt{3}$ instead of 2

now?

we have to get it back to 2.



Arbitrary Rotation Vector

ARLAB

We post-multiply the solution with the inverse of q

$$q = \left[\cos \theta, \sin \theta \left(\frac{\sqrt{2}}{2} \mathbf{i} + \frac{\sqrt{2}}{2} \mathbf{k} \right) \right]$$

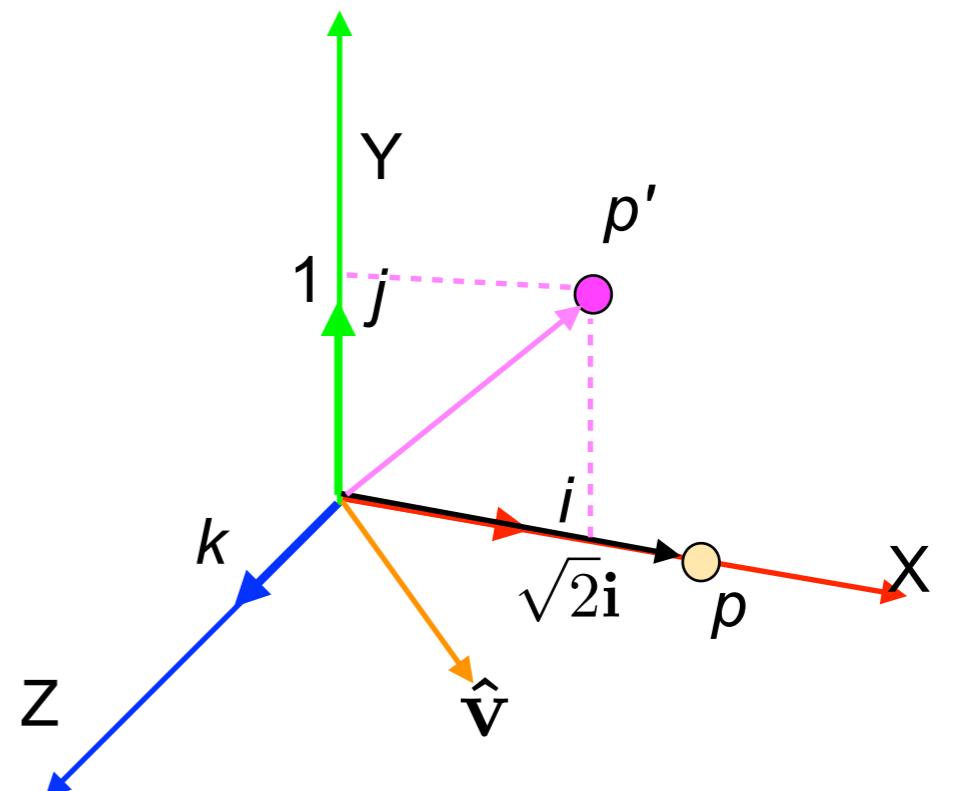
Let's compute the inverse

$$q^{-1} = \left[\cos \theta, -\sin \theta \left(\frac{\sqrt{2}}{2} \mathbf{i} + \frac{\sqrt{2}}{2} \mathbf{k} \right) \right]$$

for 45°

$$q^{-1} = \left[\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \left(\frac{\sqrt{2}}{2} \mathbf{i} + \frac{\sqrt{2}}{2} \mathbf{k} \right) \right]$$

$$= \frac{1}{2} [\sqrt{2}, -\mathbf{i} - \mathbf{k}]$$



Arbitrary Rotation Vector

ARLAB

Now we compute

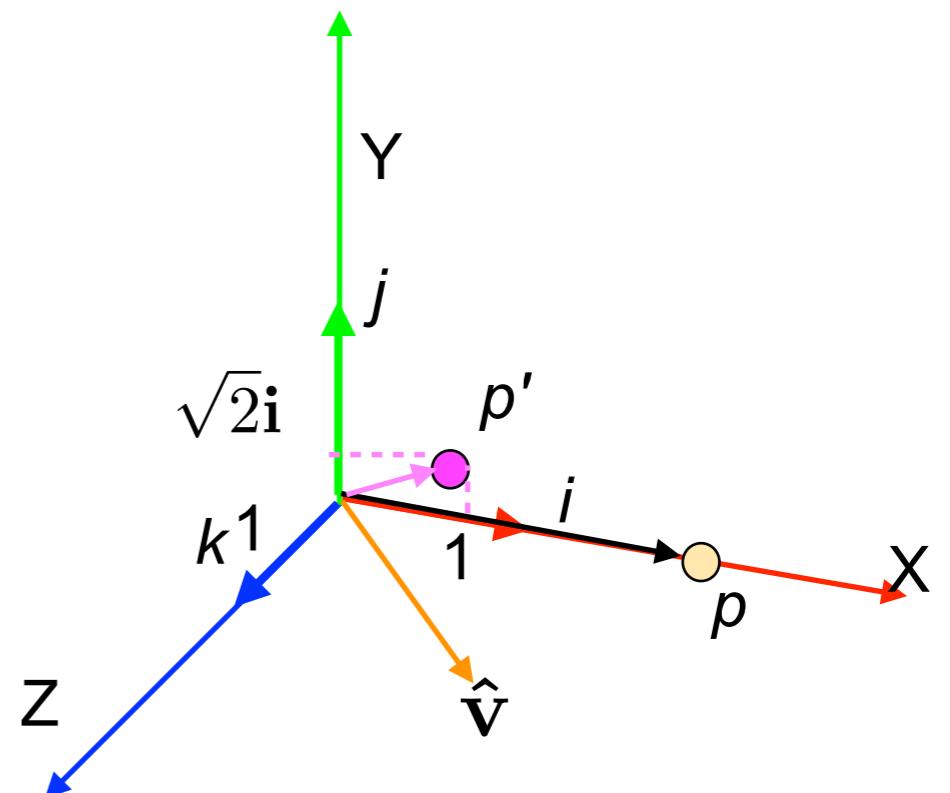
$$qp = [-1, \sqrt{2}\mathbf{i} + \mathbf{j}]$$

$$qpq^{-1} = [-1, \sqrt{2}\mathbf{i} + \mathbf{j}] \frac{1}{2} [\sqrt{2}, -\mathbf{i} - \mathbf{k}]$$

= ...

$$= [0, \mathbf{i} + \sqrt{2}\mathbf{j} + \mathbf{k}]$$

The length is 2 again



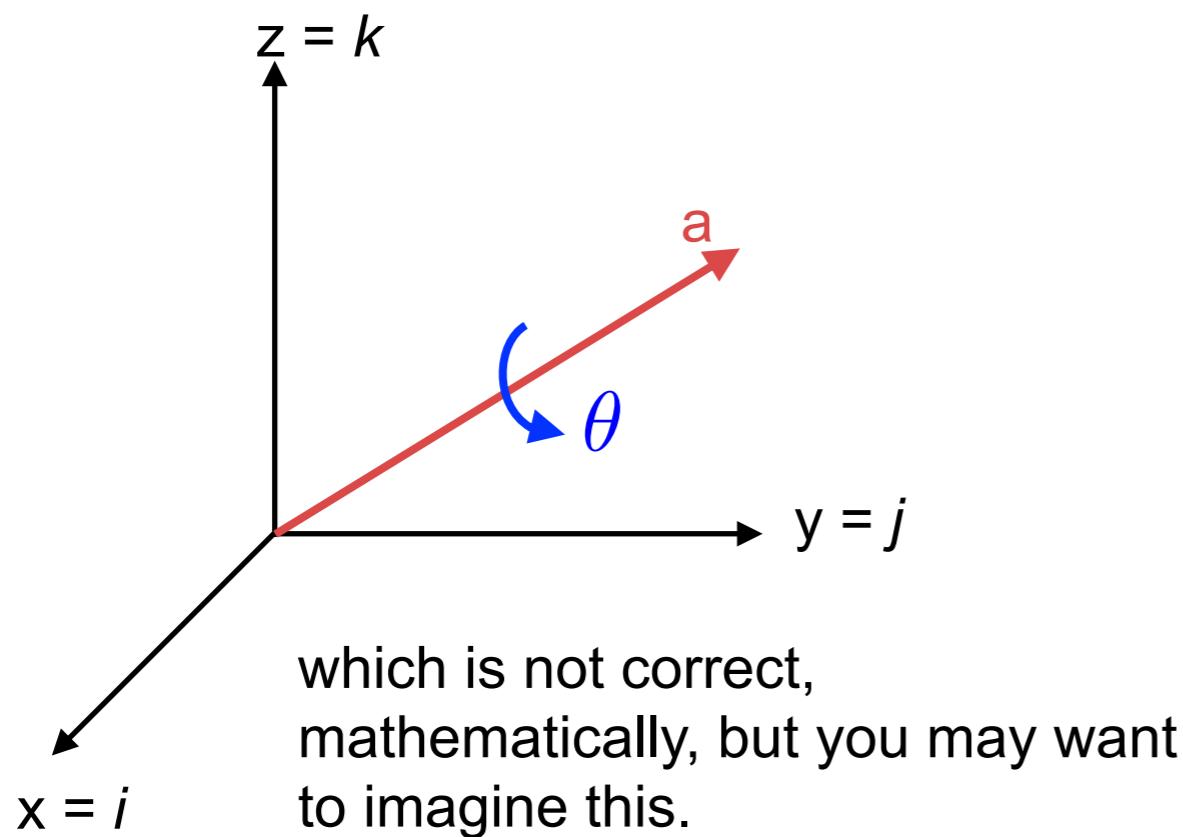
Quaternions and Rotations

ARLAB

A quaternion can represent a rotation by an angle θ around a unit axis \mathbf{a} :

$$q = \left[\cos \frac{\theta}{2}, \sin \frac{\theta}{2} \hat{\mathbf{v}} \right] = \left[\cos \frac{\theta}{2}, \underline{\sin \frac{\theta}{2} i + \sin \frac{\theta}{2} j + \sin \frac{\theta}{2} k} \right]$$

The “projections” of \mathbf{a} to x , y , and z



- the three complex numbers represent a vector in 3d space. A quaternion in \mathbb{R}^4 is a vector in \mathbb{R}^3 , if the scalar value is 0.
- the scalar value theta represents the rotation about this vector a.

Thank you!

Questions

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